On the Low Surface Magnetic Field Structure of Quark Stars

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Abstract

Following some of the recent articles on hole super-conductivity and related phenomena by Hirsch [1, 2, 3], a simple model is proposed to explain the observed low surface magnetic field of the expected quark stars. It is argued that the diamagnetic moments of the electrons circulating in the electro-sphere induce a magnetic field, which forces the existing quark star magnetic flux density to become dilute. We have also analysed the instability of normal-superconducting interface due to excess accumulation of magnetic flux lines, assuming an extremely slow growth of superconducting phase through a first order bubble nucleation type transition.

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1. INTRODUCTION

More than two decades ago, Witten had conjectured in an outstanding article [4] (see also [5]) that a flavor symmetric quark matter at zero temperature and at zero pressure would be energetically the most stable configuration. This exotic phase is known as the strange quark matter (SQM). Since then a large number of articles have been published on the theoretical studies of the properties of bulk SQM as well as droplets (known as strange-lets) of SQM [6, 7, 8]. In those investigations various kinds of confinement models were used. It has further been shown that SQM phase can exist in compact stellar objects, known as strange quark stars or strange stars [9, 10]. The finite size strange-lets are expected to be formed both in ultra-relativistic heavy ion collisions and also during QCD phase transition in the early universe [11, 12]. The strange-lets expected to have formed in the early universe, are also called the strange nuggets, believed to be the relics from primordial quark-hadron phase transition. These relics are also one of the viable candidates for baryonic dark matter. Inside the strange stars, three light flavors: up ($u$), down ($d$), strange ($s$), along with small quantity of electrons are in $\beta$-equilibrium. The presence of electrons make the system electrically charge neutral. So far the gross properties of such compact objects are concerned, it is almost impossible to distinguish strange stars from the conventional neutron stars. Except, it is believed that these compact objects, which are expected to be strange stars, are fast rotating (milli second or sub-milli second pulsars) and the surface magnetic field is quite low ($\leq 10^8$ G) [13]. It is further assumed that strange stars are very old and also extremely cold objects, formed by the accretion of matter from the companion binary counter part. However, for a strange star, the compositional structure in the microscopic scale is extremely complex in nature. Since the density at the core region is very high ($3-4$ times normal nuclear density), the corresponding quark matter is almost flavour symmetric. As one goes radially outward, the density of matter decreases and since $s$-quarks are much more heavier than $u$ and $d$-quarks (current masses for $u$ and $d$-quarks are $5-10$MeV, whereas, for $s$-quark, it is $\approx 150$MeV), in this region, it is therefore energetically not favorable to produce enough $s$-quarks through weak processes. The abundance of $s$-quark is therefore steadily decreases as one moves towards the surface region. The charge neutrality of quark matter near the outer core or the crustal region is therefore mainly maintained by electrons instead of strange quarks. The density of electron is negligibly small at the core and is maximum at the crustal...
region. Of course it is not very high ($\leq 10^{-4}$ times normal nuclear density). These electrons are bound to the positively charged quark matter by electromagnetic force (of which the Coulomb force is the non-relativistic scenario) and are allowed to move freely across the strong Coulomb field. Because of this strong electromagnetic force, they can not move far away from the quark matter surface. The electron gas at the surface of strange stars, form a very thin layer of width a few thousand Fermi, known as electro-sphere. Therefore, the gross compositional structure of a strange star is very simple: a positively charged quark matter region covered by a thin electro-sphere. Since the strange quark matter is energetically more favorable, in the case of strange stars, it is unlikely to have a crust of dense iron like matter, which is expected to be present in neutron stars. Further, depending on the internal temperature and density of strange star, the quark matter may be in normal phase or in super-conducting state or in color flavor locked phase (CFL) [14]. The CFL phase at the core region is expected if the density is extremely high. Since the kinetic energies of the electrons are much larger than the corresponding super-conducting gap energy, they never show super-conductivity, either inside the star or at the electro-sphere. Which actually means that the critical temperature for superconducting transition of electron is much less than the corresponding critical temperature for quark matter.

In this article we shall propose a mechanism by which very low surface magnetic field of strange stars can be obtained. We have assumed a type-I superconducting transition and consider two possible flux expulsion processes. We shall show that extremely slow expulsion mechanism leads to an instability at the normal-superconductor interface of quark matter. In this article we have also analysed the interface instability. It is further noticed that by a slow expulsion process, at several points on the interface, since the strength of magnetic field becomes $> B_c$, the critical strength to destroy type-I superconductivity of quark matter, the growth of superconducting quark matter phase will stop abruptly. On the other hand, if it is an extremely fast process, we ultimately get a stable quark star of very low surface magnetic field.

The paper is organized in the following manner: In the next section we shall compare a strange star with a giant atom. In this section we shall discuss qualitatively the physical processes which may take place at the surface of strange star and compare them with a giant ultra-heavy atom. In section 3. we shall make a comparative study of superconducting quark matter in strange star and the theory of hole superconductivity. The comparison is again
purely qualitative. The mathematical formalism will be given in section 6. We shall discuss the mechanism of slow flux expulsion and give a mathematical formalism in sections 5 and 6 respectively. In the last section we have given the conclusion of the present work.

2. A GIANT ATOM MODEL

In this section we shall try to explain the possible low magnetic surface structure of quark stars. In this model calculation, we assume a prompt phase transition to superconducting phase and the strange star is treated as an equivalent giant atom with positively charged quark matter (along with some admixture of electrons) as the nucleus and the electrons in the electro-sphere are treated as orbital electrons. Unlike normal atoms, here size of the nucleus is \( \gg \) than the volume occupied by the electrons in the electro-sphere. Now it is very easy to show from the \( \beta \)-equilibrium and charge neutrality conditions that the net positive charge content of the quark matter nucleus is \( \gg 172 \), the critical value of \( Z \), the atomic number of a typical super-heavy Dirac-atom. At the quark matter surface, the vacuum will therefore become unstable, and spontaneous \( e^+e^- \)-pairs will be created if there are unoccupied energy shell \([15]\) (see also \([16]\)).

Now for a many body fermion system, the microscopic theory of superconductivity suggests that if the interaction favors the formation of pairs at low temperature, the system may undergo a phase transition to a super-fluid state. In astrophysics, this is expected to occur in the dense neutron matter of cold neutron stars. Whereas the small percentage of protons (\( \sim 4\% \)) inside neutron stars undergo a transition to type-II superconducting phase \([17, 18]\). In the case of a many body system of fermions the well known BCS theory is generally used to study the superconducting properties due to fermion pairing. One fermion of momentum \( \vec{p} \) and spin \( \vec{s} \) combines with another one of momentum \(-\vec{p}\) and spin \(-\vec{s}\) and form a Cooper pair. In the case of type-I electronic super-conductors the coupling is mediated by the electron-phonon interaction. In the case of quark matter, however, the basic quark-quark interaction at large distance favors the formation of Cooper pairs. In the case of quark matter, since the force is mediated via gluons, it gives rise to what is known as color superconductivity. For a highly degenerate fermion system, which is true for strange star matter, the pairing takes place near the Fermi surface. The other important condition that must be satisfied for the formation of Cooper pairs is that the temperature \((T)\) of the system
should be much less than the super-conducting energy gap ($\Delta$). In the case of strange stars, since the Fermi levels are not identical for $u$, $d$ and $s$-quarks, only same type of quarks can form Cooper pairs at the Fermi surface and give rise to color super-conductivity [19]. It is interesting to note that the electrons, whose density is extremely low compared to quark matter, may also be treated as highly degenerate relativistic plasma, but are unlikely to form Cooper pairs.

In the next section we have assumed a type-I super-conducting phase of quark matter within the strange stars and develop a mechanism by which the expelled magnetic field is reduced at the electro-sphere. Since we are not investigating any of the properties of super-conducting quark matter, rather, we are interested to study some of the important features of normal electron gas layer at the surface region / inside the electro-sphere, which are essential to have low surface magnetic field, then instead of standard relativistic version of BCS theory [19], here, we have followed the interesting idea of "theory of hole superconductivity", proposed by Hirsch in a series of articles [1, 2, 3, 20, 21]. In the case of strange stars the charge separation or the charge asymmetry takes place by the combined effect of charge neutrality and $\beta$-equilibrium conditions; completely different physical mechanism. Further, the properties of both positively charged quark matter inside the star and the electrons in the electro-sphere are just opposite to the predicted nature of hole states and the electrons for the hole super-conductor. In fact, we have noticed that the theory of hole super-conductivity is not applicable for super-conducting quark matter of strange stars. The reason is the strong interaction which bind the quarks within the strange stars.

3. THE HOLE SUPERCONDUCTIVITY AND COMPACT STRANGE STARS

To develop a mechanism of strange star magnetic field suppression at the electro-sphere, we consider the charge asymmetric strange star as an equivalent hole super-conductor. We have noticed, that there are a lot of similarities and also dissimilarities between the laboratory samples and the strange stars within the theory of hole super-conductivity. In the case of laboratory superconducting samples, according to the theory proposed by Hirsch, the inner part is positively charged holes in normal phase. The electrons at the surface region are super-conducting. The surface layer is bound to the inner region by strong electromagnetic force. In the case of strange stars, the inner region is positively charged quark matter.
But unlike the laboratory superconductors, it is in super-conducting phase. Whereas, the outer layer, the electro-sphere, which is a degenerate electron gas, is in normal phase. The force between these two regions is again electromagnetic in nature. Going a step further, we assume following the model by Hirsch, that the electrons in the electro-sphere gyrate about the magnetic lines of force, expelled from the super-conducting interior of the strange star. The origin of such orbital motion of these normal electrons in electro-sphere about the magnetic flux lines is the well known Lorentz force. This orbital motion of electrons will generate a magnetic field at the surface of the strange star. According to Lenz's law, the induced field must be in the direction opposite to that of existing one. At the surface or in the electro-sphere, therefore, the magnetic field from the tiny magnets produced by the gyration of electrons about the existing lines of forces reduce the strange star magnetic field by diamagnetic effect. Therefore, unlike the hole super-conductor, in the case of strange stars, the magnetic field expelled from the super-conducting quark matter are suppressed by the electron gas, which is in normal phase. By the repulsive diamagnetic action, there will be an effective dilution of magnetic flux lines inside the electro-sphere. The dilution of magnetic flux will naturally increase the size (width) of the electro-sphere. Analogous to the phenomenon of frozen-in magnetic field, here, the "frozen-in electron gas" will be pulled away by the flux lines. The increase in size of the electro-sphere will depend both on the repulsive magnetic force and the attractive coulomb force and in equilibrium configuration, these two will balance each other. The width of the electro-sphere of a typical quark star of surface magnetic field $\lesssim 10^8$G can be a few tens of km, which is at least an order of magnitude less than the radius of the light cylinder, $\sim 100$km, for a milli second pulsar. Now this increase in size of the electro-sphere will reduce the electron density within the system. The relation between the induced field and the existing field is given by eqn.(39). Further, because of the motion of electrons around the quark matter nucleus, the actual trajectory of an electron in the electro-sphere is more or less like a closed helical spring produced by the motion of circular orbit along the lines of force. Since the magnetic field strengths at the poles are very high, the electrons in the helical trajectory will be reflected back from these region. The two ends will therefore behave like magnetic mirrors. It is also obvious from eqn.(39) that the radius of gyration is larger at the equatorial region compared to the polar values. Since the diamagnetism of the electrons inside the electro-sphere is caused by Lenz's law, the sign of $\vec{L}.\vec{p}/|\vec{L}.\vec{p}|$, which may be called as the effective "orbital-helicity" will change
sign after each reflection by the magnetic mirrors at the poles. Here $\vec{p}$ and $\vec{L}$ are respectively the linear momentum and the angular momentum of the electrons. Further, when a balance between the existing field and the induced field will be established, the electrons within each half of the electro-sphere will undergo steady helical motion.

4. SLOW EXPULSION OF MAGNETIC FLUX LINES

In this section we shall give a qualitative picture of very slow magnetic flux expulsion from the superconducting region. It has been shown in ref. [19] in a relativistic version of BCS theory, that if a normal quark matter system undergoes a superconducting phase transition, the newly produced quark matter phase will be a type-I superconductor. They have also shown in that the critical magnetic field to destroy such pairing is $\sim 10^{16}$G for $n \sim 3 - 4n_0$, with $n_0 = 0.17\text{fm}^{-3}$, the normal nuclear density. This magnetic field strength is indeed much larger than the typical pulsar magnetic field. The corresponding critical temperature is $\sim 10^9 - 10^{10}$K, which is again high enough for strange stars, which are expected to be extremely cold objects. In this section, instead of investigating the superconducting properties of quark matter inside strange stars, we shall give a possible mechanism of flux expulsion by Meissner effect, assuming that the growth of superconducting phase is extremely slow. Further, we assume that the magnetic field strength at the core region of a strange star are much less than the corresponding critical value for the destruction of superconducting property and the temperature is also low enough. Then during such a type-I superconducting phase transition, the magnetic flux lines from the superconducting quark matter of the strange star will be pushed out towards the normal crustal region. Now for a small type-I superconducting laboratory sample placed in an external magnetic field less than the corresponding critical value, the expulsion of magnetic field takes place instantaneously. Whereas in the bulk strange star scenario, the picture may be completely different. It may take several thousands of years for the magnetic flux lines to get expelled from the superconducting core. Which further means, that the growth of superconducting phase in strange stars may not be instantaneous. A simple estimate shows that the expulsion time due to ohmic diffusion is $\sim 10^4$ yrs [22]. It was shown by Chau using Ginzberg-Landau formalism that the time for expulsion of magnetic lines of force accompanied by the enhancement of magnetic field non-uniformities at the crustal region gets prolonged to $10^7$ yrs [23]. Alford
et. al. investigated the expulsion of magnetic lines of force from the colour superconducting region by considering the pairing of like and unlike quarks and obtained the expulsion time much larger than the age of the Universe [24].

In our investigation of magnetic flux expulsion from growing superconducting core of a strange star, the idea of impurity diffusion in molten alloys or the transport of baryon numbers from hot quark matter soup to hadronic matter during quark-hadron phase transition in the early universe, expected to occur micro-second after big bang (the first mechanism is used by the material scientists and metallurgists [25], whereas the later one is used by cosmologists working in the field of big bang nucleo-synthesis [26, 27]) are assumed. In the present section, we shall further show the possibility of Mullins-Sekerka normal-superconducting interface instability [28, 29] in quark matter. This is generally observed in the case of solidification of pure molten metals at the solid-liquid interface, if there is a temperature gradient. The interface will always be stable if the temperature gradient is positive, otherwise it will be unstable. In alloys, the criteria for stable / unstable behaviour is more complicated. It is seen that, during solidification of an alloy, there is a substantial change in the concentration ahead of the interface. Here solute diffusion as well as the heat flow effects must be considered simultaneously. The particular problem we are going to investigate here is analogous to solute diffusion during solidification of an alloy.

5. A FORMALISM FOR SLOW NUCLEATION

It has been assumed that the growth of superconducting quark bubble started from the centre of the star and the nomenclature controlled growth for such phenomenon has been used. If the magnetic field strength and the temperature of the star are a few orders of magnitude less than their critical values, the normal quark matter phase is thermodynamically unstable relative to the corresponding superconducting one. Then due to fluctuation, a droplet of superconducting quark matter bubble may be nucleated in metastable normal quark matter medium. If the size of this superconducting bubble is greater than the corresponding critical value, it will act as the nucleating centre for the growth of superconducting quark core. The critical radius can be obtained by minimising the free energy. Then
following the work of Mullins and Sekerka, we have

$$r_c = \frac{16\pi\alpha}{B^{(c)}^2 \left[ 1 - \left( \frac{B}{B^{(c)}} \right)^2 \right]},$$

where $\alpha$ is the surface tension or the surface energy per unit area of the critical superconducting bubbles, (from this expression it is possible to obtain the critical size of the quark matter bubble by considering $10^{-3} \leq \alpha \leq 1$ (in MeV/fm$^{-3}$) as the range of surface tension) which is greater than zero for a type-I superconductor-normal interface, $B^{(c)}$ is the critical magnetic field. In presence of a magnetic field $B < B^{(c)}$, the normal to superconducting transition is first order in nature. As the superconducting phase grows continuously, the magnetic field lines will be pushed out into the normal quark matter crust. This is the usual Meissner effect observed in type-I superconductor. We compare this phenomenon of magnetic flux expulsion from a growing superconducting quark matter core with the diffusion of impurities from the frozen phase of molten metal or the transport of baryon numbers from hot quark matter soup during quark-hadron phase transition in the early universe. The formation of superconducting zone is compared with the solidification of molten metal or with the transition to hadronic phase with almost zero baryon number. It is known from the simple thermodynamic calculations that if the free energy of molten phase decreases in presence of impurity atoms, then during solidification they prefer to reside in the molten phase otherwise they go to the solid phase. In this particular case the magnetic field lines play the role of impurity atoms and because of less free energy, they prefer to remain in normal quark matter phase. The normal quark matter phase plays the role of molten metal or the hot quark soup. Whereas the superconducting phase can be compared with the frozen solid phase or the hadronic phase. This idea was applied to baryon number transport during first order quark-hadron phase transition in the early Universe, where baryon number replaces impurity, quark phase replaces molten metal and hadronic matter replaces that of solid metal. Of course the baryon number prefers to stay in the quark phase because of Boltzmann suppression factor in the hadronic phase. Since the magnetic flux lines prefer to reside in the normal phase, the well known Meissner effect can therefore be restated as the solubility of magnetic flux lines in the superconducting phase is zero with a finite penetration depth.

The dynamical equation for the flux expulsion can be obtained from the simplified model of sharp normal-superconducting interface. The expulsion equation is given by the well
known diffusion equation \[30\]
\[
\frac{\partial B}{\partial t} = D \nabla^2 B
\] (2)

where \(B\) is the magnetic field intensity and \(D\) is the diffusion coefficient, given by

\[D = \frac{c^2}{4\pi \sigma_n},\] (3)

\(\sigma_n\) is the electrical conductivity of the normal quark matter and \(c\) is the velocity of light. The electrical conductivity of quark matter for \(B = 0\) is given by \[31, 32\]
\[
\sigma_n \sim \alpha_s^{-3/2} T_{10}^{-2}
\]
or \[33\]
\[
\sigma_n \sim (\alpha_s T_{10})^{-5/3}
\] (4)

expressed in sec\(^{-1}\), where \(\alpha_s\) is the strong coupling constant and \(T_{10} = T / 10^{10}\)K, the numerical value for this electrical conductivity in the case of quark matter relevant for quark star density is \(\sim 10^{26}\) sec\(^{-1}\). In the order of magnitude estimate, we have used this numerical value for electrical conductivity. In this context we must mention that in presence of strong quantizing magnetic field, \(\sigma_n\) may not be a scalar quantity, magnetic field destroys the isotropy of the electromagnetic properties of the medium. In particular, for extremely large \(B\), the components of electric current vector orthogonal to \(B\) become extremely small. Which indicates that the quarks can move only along the direction of magnetic field or in other words, across the field the resistivity becomes extremely high. When the conditions for charge neutrality and \(\beta\)-equilibrium are considered together, since the mass of \(s\)-quark is assumed to be 150Mev, the electron density can not become exactly zero, but it is a few orders of magnitude less than the \(s\)-quark density. Therefore, one can neglect the electron contribution to electrical conductivity.

A solution of eqn.(2) with spherical symmetry can be obtained by Greens’ function technique, and is given by (for a general topological structure, no analytical solution is possible)

\[
B(r, t) = \frac{1}{2r(\pi Dt)^{1/2}} \int_0^\infty B^{(0)}(r') \left[ \exp(-u_-^2) - \exp(-u_+^2) \right] r' dr'
\] (5)

where \(u_\pm = (r \pm r')/2(Dt)^{1/2}\) and \(B^{(0)}(r)\) is the magnetic field distribution within the star at \(t = 0\), which is of course an entirely unknown function of radial coordinate \(r\). To obtain an
estimate of magnetic field diffusion time scale ($\tau_D$), we assume $B^{(0)}(r) = B^{(0)} = \text{constant}$. Then we have from eqn.(5)

$$B(r, t) = B^{(0)} \left[ 1 - \frac{2}{r} (\pi D t)^{1/2} \right]$$

(6)

Hence, if we put $B(r, t) = 0$ (field free condition), the estimated time scale for the expulsion of magnetic flux lines is $\sim 10^5 - 10^6$ yrs. Which is of the same order of magnitude as the Ohmic decay scale. From this simple estimate, this is the approximate time scale for complete expulsion of magnetic field lines. Which is of course quite large. Unfortunately, nothing else can be inferred about the growth of superconducting zone and the associated expulsion of magnetic flux lines from this region. The reason behind such uncertainty is our lack of knowledge or definite ideas on the numerical values of the parameters present in eqn.(5).

To get some idea of the effect of magnetic field on the structure of growing superconducting zone, we shall now investigate the morphological instability of normal-superconducting interface of quark matter. The motion of normal-superconducting interface is extremely important in this case and has to be taken into consideration. Then instead of eqn.(2) which is valid in the rest frame, an equation expressed in a coordinate system which is moving with an element of the boundary layer is the correct description of such superconducting growth, known as Directional Growth. The equation is called Directional Growth Equation, and is given by

$$\frac{\partial B}{\partial t} - v \frac{\partial B}{\partial z} = D \nabla^2 B$$

(7)

where the motion of the plane interface is assumed to be along $z$-axis and $v$ is the velocity of the front. This diffusion equation must be supplemented by the boundary conditions at the interface. The first boundary condition is obtained by combining Ampere’s and Faraday’s laws at the interface, and is given by

$$Bv \big|_s = -D(\nabla B) \cdot \hat{n} \big|_s$$

(8)

where $\hat{n}$ is the unit vector normal to the interface directed from the normal phase to the superconducting phase. This is nothing but the continuity equation for magnetic flux diffusion. For the normal growth of superconducting zone, the rate at which excess magnetic field lines are rejected from the interior of the phase is balanced by the rate at which magnetic flux lines diffuses ahead of the two-phase interface. Therefore the boundary layer between
superconducting-normal quark matter phases will become unstable if excess magnetic field lines are present on the surface of the growing superconducting bubble, i.e., if the rate of diffusion of magnetic flux lines is slow enough compared to the rate at which they are expelled from the superconducting zone. Local thermodynamic equilibrium at the interface gives (Gibbs-Thompson criterion)

\[ B \approx B^{(c)} \left( 1 - \frac{4\pi \alpha}{RB^{(c)}R} \right) = B^{(c)} (1 - \delta C) \]  
(9)

where \( \delta \) is called capillary length with \( \alpha \) the surface tension, \( C \) is the curvature = \( 1/R \) (for a spherical surface), and \( B^{(c)} \) is the thermodynamic critical field.

To investigate the stability of superconducting-normal interface, we shall follow the original work by Mullins and Sekerka [28, 29], and consider a steady state growth of superconducting core. Then the time derivative in eqn.(7) will not appear. Introducing \( r_\perp = \left(x^2 + y^2\right)^{1/2} \) as the transverse coordinate at the interface, we have after rearranging eqn.(7)

\[ \left[ \frac{\partial^2}{\partial r_\perp^2} + \frac{1}{r_\perp} \frac{\partial}{\partial r_\perp} + \frac{\partial^2}{\partial z^2} + \frac{v \partial}{D \partial z} \right] B = 0 \]  
(10)

The approximation that the solidification is occurring under steady state condition used in the freezing of molten material will be followed in the present case of normal to superconducting phase transition. Now if it is assumed that these two phenomena taking place in two completely separate physical world are almost identical natural processes, then the concentration of magnetic flux lines and normal-superconducting interface morphology will be independent of time. The main disadvantage of this assumption is that there will be no topological evolution of the interface shape. As a consequence of this constraint the solution to the basic diffusion problem is indeterminate and a whole range of morphologies is permissible from the mathematical point of view. In order to distinguish the solution which is the most likely to correspond to reality, it is necessary to find some additional criteria. The examination of the stability of a slightly perturbed growth form is probably the most reasonable manner in which this situation may be treated. In the following we shall investigate the morphological instability of normal-superconducting interface from eqn.(10). Assuming a solution of this equation expressed as the product of separate functions of \( r_\perp \) and \( z \) and setting the separation constant equal to zero and using the boundary condition given by eqn.(9), we have for an unperturbed boundary layer moving along \( z \)-axis

\[ B = B^{(s)} \exp(-zv/D) = B^{(s)} \exp(-2z/l) \]  
(11)
where \( l = 2D/v \) is the layer thickness, which is very small for small \( D \). Mathematically, the thickness of this layer is infinity. For practical purpose an effective value \( l \) can be taken. The order of magnitude estimates or limiting values for the three quantities \( D, v \) and \( l \) can be obtained from the stability condition of planer interface.

Due to excess magnetic flux lines at the interface, the form of the planer normal-superconducting interface described by the equation \( z = 0 \) is assumed to be changed by a small perturbation represented by the simple sine function

\[
    z = \epsilon \sin(\vec{k} \cdot \vec{r}_\perp) \tag{12}
\]

where \( \epsilon \) is very small amplitude and \( \vec{k} \) is the wave vector of the perturbation. Then the perturbed solution of the magnetic field distribution near the interface can be written as

\[
    B = B^{(s)} \exp(-vz/D) + A\epsilon \sin(\vec{k} \cdot \vec{r}_\perp) \exp(-bz) \tag{13}
\]

where \( A \) and \( b \) are two unknown constants. Since the solution should satisfy the diffusion equation (eqn.(10)), we have

\[
    b = \frac{v}{2D} + \left[ \left( \frac{v}{2D} \right)^2 + k^2 \right]^{1/2} \tag{14}
\]

To evaluate \( A \), we utilise the assumption that both \( \epsilon \) and \( \epsilon \sin(\vec{k} \cdot \vec{r}_\perp) \) are small enough so that we can keep only the linear terms in the expansion of exponentials present in eqn.(13). Then at the interface, we have after some straightforward algebraic manipulation

\[
    A = \frac{v}{D} B^{(s)} \tag{15}
\]

The expression describing the magnetic field distribution ahead of the slightly perturbed interface then reduces to

\[
    B = B^{(s)} \left[ \exp(-vz/D) + \frac{v}{D} \epsilon \sin(\vec{k} \cdot \vec{r}_\perp) \exp(-bz) \right] \tag{16}
\]

Now from the other boundary condition (eqn.(9)) we have

\[
    B^{(s)} = B^{(c)} - \frac{4\pi\alpha B^{(c)}}{B^{(s)}B^{(c)}} C \tag{17}
\]

where \( C = z''/(1 + z'^2)^{3/2} \) is the curvature at \( z = \epsilon \sin(\vec{k} \cdot \vec{r}_\perp) \) and prime indicates derivative with respect to \( r_\perp \).
Neglecting $z'^2$, which is small for small perturbation, we have

$$B^{(s)} = B^{(c)} + \Gamma k^2 S$$  \hspace{1cm} (18)

where $\Gamma = 4\pi \alpha B^{(c)}/B^{(s)2}$ and we have replaced $\epsilon \sin(\vec{k}.\vec{r}_\perp)$ by $S$. Since the amplitude of perturbation $\epsilon$ is extremely small, the quantity $S$ is also negligibly small.

Now eqn.(18) can also be written as

$$B^{(s)} = B^{(c)} + GS$$  \hspace{1cm} (19)

where

$$G = \frac{dB}{dz} |_{z=S} = -\frac{v}{D} \left(1 - \frac{vS}{D}\right) B^{(s)} - bAS(1 - bS)$$  \hspace{1cm} (20)

Combining these two equations, we have

$$k^2 \Gamma + \frac{v}{D} \left(1 - \frac{vS}{D}\right) B^{(s)} - \frac{bvB^{(s)}S}{D} (1 - bS) = 0$$  \hspace{1cm} (21)

This expression determines the form (values of $k$) which the perturbed interface must assume in order to satisfy all of the conditions of the problem. To analyse the behaviour of the roots, we replace right hand side of eqn.(21) by some parameter $-P$. (We have taken $-P$ in order to draw a close analogy with the method given in refs. [26, 27]). Then rearranging eqn.(21), we have

$$- k^2 \Gamma - \frac{v}{D} \left(1 - \frac{vS}{D}\right) B^{(s)} + \frac{bvB^{(s)}S}{D} (1 - bS) = P$$  \hspace{1cm} (22)

(in refs. [26, 27] the parameter $P$ is related to the time derivative of $\epsilon$, the amplitude of small perturbation). If the parameter $P$ is positive for any value of $k$, the distortion of the interface will increase, whereas, if it is negative for all values of $k$, the perturbation will disappear and the interface will be stable. In order to derive a stability criterion, it only needs to know whether eqn.(20) has roots for positive values of $k$. If it has no roots, then the interface is stable because the $P - k$ curve never rises above the positive $k$-axis and $P$ is therefore negative for all wavelengths. We have used Descartes’s theorem to check how many positive roots are there. It is more convenient to express $k$ in terms of $b$ and then replacing $b$ by $\omega + v/D$. Then we have from eqn.(22)

$$- \omega^2 \left( \Gamma + \frac{vB^{(s)}S^2}{D} \right) - \omega \left( \Gamma + \frac{2vB^{(s)}S^2}{D} - B^{(s)} S \right) \frac{v}{D}$$

$$- \frac{v}{D} B^{(s)} \left(1 - \frac{v}{D} S\right)^2 = P$$  \hspace{1cm} (23)
This is a quadratic equation for $\omega$. The first and the third terms are always negative. The second term will also be negative if

$$\Gamma + \frac{2\nu B^{(s)} S^2}{D} - B^{(s)} S > 0 \quad (24)$$

Then it follows from Decart’s rule that if the condition (24) is satisfied, there can not be any positive root. Which implies that the small perturbation of the interface will disappear. Since the amplitude of perturbation is assumed to be extremely small, the quantity $S = \epsilon \sin(\mathbf{k} \cdot \mathbf{r}_\perp)$ is also negligibly small. Under such circumstances the middle term of eqn. (23) is much smaller than rest of the terms. The Decart’s rule given by the condition (24) can be re-written as

$$\Gamma > B^{(s)} S \quad (25)$$

Which after some simplification gives the stability criterion for the plane unperturbed interface, given by

$$\alpha > \frac{B^{(s)3} S}{4\pi B^{(c)}} \quad (26)$$

From the stability criterion, it follows that the normal-superconducting interface energy/area of quark matter has a lower bound, which depends on the interface magnetic field strength, critical field strength and also on the perturbation term $S$. An order of magnitude of this lower limit can be obtained by assuming $B^{(s)} = 10^{-3} B^{(c)}$. (Since the critical field $B^{(c)} \sim 10^{16}$G, and the neutron star magnetic field strength $B \sim 10^{13}$G, we may use this equality). Then the lower limit is given by

$$\alpha_L \approx 10^{-9} \text{ MeV/fm}^2 \left( \frac{S}{\text{fm}} \right) \quad (27)$$

On the other hand for $B^{(s)} = 0.1B^{(c)}$, we have

$$\alpha_L \approx 10^{-3} \text{ MeV/fm}^2 \left( \frac{S}{\text{fm}} \right) \quad (28)$$

The approximate general expression for the lower limit is given by

$$\alpha_L \approx \hbar^3 \text{ MeV/fm}^2 \left( \frac{S}{\text{fm}} \right) \quad (29)$$

where $\hbar = B^{(s)}/B^{(c)}$. Therefore the maximum value of this lower limit is

$$\alpha_L^{\text{max.}} \approx 1 \text{ MeV/fm}^2 \left( \frac{S}{\text{fm}} \right) \quad (30)$$


when the two phase are in thermodynamic equilibrium. Of course for such a strong magnetic field, as we have seen [34, 35] that there can not be a first order quark-hadron phase transition.

On the other hand if we do not have control on the interface energy, which can in principle be obtained from Landau-Ginzberg model, we can re-write the stability criteria in terms of interface concentration of magnetic field strength $B^{(s)}$, and is given by

$$B^{(s)} < \left[ \frac{4\pi \alpha B^{(c)}}{S \left(1 - \frac{2v}{D}S\right)} \right]^{1/3} \tag{31}$$

This is more realistic than the condition imposed on the surface tension $\alpha$. Now for a type-I superconductor, the surface tension $\alpha > 0$, which implies $1 - 2vS/D > 0$. Therefore, we have $2vS/D < 1$. For the typical value of $\sigma_n \sim 10^{26}$ sec$^{-1}$ for the electrical conductivity of normal quark matter, the profile velocity $v < D/2S \sim 10^{-6}/S$ cm/sec $\sim 1$ cm/sec for $S \sim 10^{-6}$ cm. Therefore the interface velocity $< 1$ cm/sec for such typical values of $\sigma_n$ and $S$ to make the planer interface stable under small perturbation. Now the thickness of the layer at the interface is $l = 2D/v > 10^{-6}$ cm for such values of $D$ (or $\sigma_n$) and $v$. Here $S$ is always greater than 0, otherwise, the magnetic field strength at the normal-superconductor interface becomes unphysical. As before, if the second term of eqn.(23) is negligibly small compared to other two terms, we have

$$B^{(s)} < \left[ \frac{4\pi \alpha B^{(c)}}{S} \right]^{1/3} \tag{32}$$

For the sake of illustration, we have shown in fig.(1), the distribution of magnetic field in a very small portion of horizontal plane at the perturbed interface. We have solved eqn.(17) numerically and use the typical parameter set as given above. Along z-axis, we have plotted the ratio of the strength of surface magnetic field and the critical strength for the destruction of type I quark matter superconductivity and x and y axes represent the x-y coordinates of this tiny horizontal plane. The distribution is extremely chaotic in nature and some of the magnetic field peaks are stronger that the critical strength to destroy type-I superconductivity. Then analogous to the case of crystal growth in an impure molten alloy, where the process is stopped because of high density of accumulated impurities at the interface, here also, further growth of superconducting phase will be stopped. Therefore, it is a kind of instability at the interface, which forces the growth process to stop abruptly.
Since the slow growth of superconducting phase does not give a stable quark star with very low surface magnetic field, is therefore not a physically acceptable phenomenon.

6. THE PROMPT PROCESS OF FLUX EXPULSION

We have noticed that the slow expulsion process, in which diffusion mechanism of magnetic lines of force may be applicable, leads to a kind of instability at the interface. It ultimately stops abruptly the growth process of superconducting zone. As a consequence we will not get a stable quark star with very low surface magnetic field.

In this section, we assume that the superconducting phase of quark matter grows by a faster process; so that the diffusion model for the magnetic lines of force is not applicable. Which actually means, that the expulsion also occurs with a faster rate. Now during this process, since magnetic flux changes rapidly with time, an emf will be induced in the system. Although, the superconducting quark Cooper pairs do not feel electromagnetic force, a Lorentz force will be exerted on the electrons near the surface region, which are in normal phase. These electrons will try to nullify the force following Lenz’s law. As a consequence they will start gyrating about the magnetic lines of force at the surface / in the electro-sphere. We have already argued that these circulation of the electrons about the magnetic field lines will produce a diamagnetic effect to oppose the Lorentz force by reducing the strength of magnetic field at the surface.

To get an estimate for induced magnetic field strength at the strange star surface / electro-sphere and the corresponding reduction in magnetic field strength, we follow the recent article by Hirsch [20]. The magnetic dipole moment per electron in the electro-sphere, due to the tiny orbital motion is given by

\[
\mu = \frac{e u}{2} a \quad \text{(we have assumed that } c = 1) \quad (33)
\]

where \( u \) is the orbital velocity and \( a \) is the radius of the orbit. The induced emf in the electronic orbit is then given by

\[
E = \frac{a}{2} \frac{\partial B}{\partial t} \quad (34)
\]

Then the corresponding change in orbital speed is given by

\[
\Delta u = \frac{ea}{2m_e} B \quad (35)
\]
Hence the change in magnetic moment per electron is

$$\Delta \mu = \frac{e^2 a^2}{4m_e} B,$$

and the associated average value of the induced magnetization per unit volume in the electro-sphere is given by

$$\mathcal{M} = \frac{n_e e^2 a^2}{4m_e} B$$

where $n_e$ is the average electron density in the electro-sphere (in reality, $n_e$ must be maximum near the quark matter surface and is minimum at the outer surface of the electro-sphere). Hence one can obtain the strength of induced magnetic field at the surface region, which is given by

$$B_{\text{ind}} = 4\pi \mathcal{M}$$

Therefore the ratio of induced magnetic field by the electrons in the electro-sphere to the existing strange star magnetic field is given by

$$\frac{B_{\text{ind}}}{B} = \frac{n_e e^2}{m_e} \pi a^2$$

The strength of induced magnetic field is therefore a monotonically increasing function of electron density. It is very easy to verify that for complete suppression of magnetic field of a strange star, with a typical average electron density $n_e = 10^{-5} n_B$, where $n_B = 0.17 \text{fm}^{-3}$, the normal nuclear density, the orbital radius $a \sim 7 \text{fm}$. Which is about three orders of magnitude less than the width of the electro-sphere. Therefore it is quite obvious that the electro-sphere of width $\sim 1000 \text{fm}$ can accommodate a large number of tiny magnets. It may be argued that the actual form of trajectories for such tiny magnets in each half of the electro-sphere are closed helical spring.

Now, if there is no reduction, the magnetic field strength at the surface could be as high as $10^{30}$ times the magnetic field of a typical neutron star. One can obtain this number by considering the ratio $R^2/d^2$, where $R = 10 \text{Km}$, the radius of the star and $d = 1000 \text{fm}$, the width of the electro-sphere and assuming that the magnetic flux remains conserved during Meissner expulsion from super-conducting quark matter. This value is far above the upper limit predicted by Shabad and Usov [36].

Now it has been discussed in the literature that if the density of quark matter, particularly, at the core region is sufficiently high, the color super-conducting quark matter undergoes a
phase transition to what we call the CFL phase \[37, 38, 39\]. It is therefore expected that at the ultra-dense interior, there will be a color as well as charge neutral CFL phase. The number of quarks in the CFL cluster is even and divisible by three. It is found that in this new phase all the three flavors $u$, $d$ and $s$ can form pairs with the same flavor as well as with other components, i.e., their Fermi levels are identical. In this case the excess $u$-quarks will therefore go to the outer region. The gross structure as mentioned at the beginning of this article may not therefore be correct for a strange star with extremely high core density; at the interior, it could be the CFL phase, at the outer core or inner crust region, it is the positively charged color neutral quark matter in non-CFL color super-conducting phase and finally the thin outer surface, the electro-sphere, is the normal electron gas. Whether the formation of CFL phase at the inner part of a super-conducting strange star is an equivalent QCD picture of the so called Tau-effect \[40, 41\] needs further investigation.

Therefore we expect that depending on the density of quark matter, the inner part of a strange star is either a type-I super-conductor or in CFL phase, the outer region is color neutral type-I super-conducting non-CFL phase, mainly dominated by $u$-quarks and some electrons, and finally the outer surface is a thin layer of electron gas in normal state. The magnetic field lines are expelled from quark matter phase to the electron gas layer with a small penetration at the outer surface of the quark matter. We have conjectured that the classical diamagnetism of the normal electron gas suppresses the magnetic field of a strange star by diluting the existing magnetic flux lines in the electro-sphere by the repulsive action of the induced magnetic field. If $B \approx B_{\text{ind}}$, we have almost total suppression. In this ideal situation, so far the emission due to magnetic activities are concerned, the object becomes dark to the observers. However, because of vacuum instability, non-thermal surface $\gamma$ emission is possible. Since $Z$ is very high, the Coulomb field at the quark matter surface will be extremely strong, then $e^+e^-$ pairs may be produced from vacuum by Schwinger mechanism \[42\]. The created $e^+$ gets annihilated with one of the electrons in the electro-sphere and $e^-$ will occupy one of the empty energy levels. Therefore, $\gamma$ emission will also be forbidden if the star is extremely cold, when there will be no more vacant states for the produced electrons \[43\].

An alternative explanation for low magnetic field of strange stars can also be obtained from a very simple model of magnetic circuits. The combination of electro-sphere and the magneto-sphere may be treated as an equivalent magnetic circuit with varying reluctance.
Of course, there are some open flux lines goes out after cutting the light cylinder. In that case, we may assume that the whole universe is a complicated network of magnetic circuits with varying reluctance and the strength of sources of magneto motive force. From the definition, we have total flux
\[ N = \frac{\text{MMF}}{R} \] (40)
where MMF is the effective magneto motive force and \( R \) is the reluctance, which is given by
\[ R = \oint \frac{dl}{\alpha \mu} \] (41)
where \( dl \) is the length of some flux carrying element of cross section \( \alpha \) and magnetic permeability \( \mu \). Let us consider the passage of expelled magnetic flux lines through surface region, when there is no gyrating motion of electrons. Here we may take \( \mu \approx 1 \), the reluctance is then given by.
\[ R_1 = \int \frac{dl}{\alpha} \mid_{es} + R_{ms} \] (42)
where \( es \) indicates the electro-sphere and \( ms \) is the magneto-sphere. On the other hand, with the gyrating electrons, since there is a diamagnetic effect at the surface / electro-sphere, we must have the permeability \( \mu < 1 \) in this region. In Gaussian unit, we have \( \mu = 1 - 4\pi \chi \).
Where from eqn.(37) the susceptibility is given by
\[ \chi = \frac{\partial M}{\partial B} = \frac{n_e e^2 a^2}{4m_e} \] (43)
The reluctance is then given by
\[ R_2 = \int \frac{dl}{\mu \alpha'} \mid_{es} + R_{ms} \] (44)
where \( \alpha' \) is now the new cross sectional area of the flux carrying element. Since the MMF source and the total flux does not change by the gyration of electrons at the surface, we have effectively
\[ R_1 = R_2 \] (45)
which gives
\[ \alpha' = \alpha \frac{1}{\mu} \] (46)
Since \( \mu \ll 1 \), it is therefore quite obvious that the area of cross section of the electro-sphere will increase by several orders of magnitude. As a result the magnetic flux density will also be reduced by the same factor. Now it is well known that Larmor radius \( a \sim B^{-1} \),
here $B$ is the surface magnetic field. Therefore it is quite possible in the parameter space (electron density and the corresponding Larmor radius) for the physically acceptable values for electron density and Larmor radius, the surface magnetic field can be as low as $10^8$G.

For the typical values of electron density $n_e \sim 10^{-12} n_0$ and Larmor radius $a \sim 10 \, \text{Å}$, the magnetic permeability becomes $\sim 0.012$. Which gives surface magnetic field a few times $10^7$G. Since the Larmor radius increases with the decrease in magnetic field strength, it will further increase the induced magnetic field (see eqn.(39)), if it dominates over the electron density. The induced magnetic field, which is diamagnetic in nature will further reduce the existing field. So the solution must be self-consistent in nature. We believe that ultimately a steady state will be established in the electro-sphere. This is true for both the models.

7. CONCLUSION

From this investigation, we may conclude in a straight forward way that if strange stars really exists with very low surface magnetic field, the type-I superconducting transition in the quark matter phase must have occurred within the star. Further, the transition process must be very prompt, so that diffusion model for the expulsion of magnetic flux lines is not applicable, where the later gives rise to some kind of instability at the interface and finally stops the growth of superconducting phase abruptly. Further, the slow transition model does have any mechanism to reduce the surface field strength, which is $\leq 10^8$G.

Our next conclusion is that the transition is quite fast. The classical diamagnetism of gyrating electrons at the surface / electro sphere may be the possible source of opposing field to reduce the expelled surface magnetic field by several orders of magnitude. It is quite obvious that with this prompt transition model, using the idea of Hirsch, one can obtain a surface magnetic field for the expected quark stars as low as $10^8$G.

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FIG. 1: The variation of magnetic field along z-axis (plotted as a ratio of $B/B^{cr}$, where $B^{cr} \sim 10^{16}$ G, the critical strength to destroy the type-I quark matter superconductivity) with the orthogonal coordinates $x$ and $y$. 