The Lyapunov exponents of generic volume preserving and symplectic systems

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To Jacob Palis, with friendship and admiration.

Abstract

We show that the integrated Lyapunov exponents of $C^1$ volume preserving diffeomorphisms are simultaneously continuous at a given diffeomorphism only if the corresponding Oseledets splitting is trivial (all Lyapunov exponents equal to zero) or else dominated (uniform hyperbolicity in the projective bundle) almost everywhere.

We deduce a sharp dichotomy for generic volume preserving diffeomorphisms on any compact manifold: almost every orbit either is projectively hyperbolic or has all Lyapunov exponents equal to zero.

Similarly, for a residual subset of all $C^1$ symplectic diffeomorphisms on any compact manifold, either the diffeomorphism is Anosov or almost every point has zero as a Lyapunov exponent, with multiplicity at least 2.

Finally, given any closed group $G \subset \text{GL}(d)$ that acts transitively on the projective space, for a residual subset of all continuous $G$-valued cocycles over any measure preserving homeomorphism of a compact space, the Oseledets splitting is either dominated or trivial.

1 Introduction

Lyapunov exponents describe the asymptotic evolution of a linear cocycle over a transformation: positive or negative exponents correspond to exponential growth or decay of the norm, respectively, whereas vanishing exponents mean lack of exponential behavior.

In this work we address two basic, a priori unrelated problems. One is to understand how frequently do Lyapunov exponents vanish on typical orbits. The other, to analyze the dependence of Lyapunov exponents as functions of the system. We are especially interested in dynamical cocycles, i.e. given by the derivatives of conservative diffeomorphisms, but we discuss the general situation as well.

Several approaches have been proposed for proving existence of non-zero Lyapunov exponents. Let us mention Furstenberg [10], Herman [12], Kotani [13], among others. In contrast, we show here that vanishing Lyapunov exponents are actually

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very frequent: for a residual (dense $G_\delta$) subset of all volume-preserving $C^1$ diffeomorphisms, and for almost every orbit, all Lyapunov exponents are equal to zero or else the Oseledets splittings is dominated. This extends to generic continuous $G$-valued cocycles over any transformation, for any matrix group $G$ that acts transitively on the projective space.

Domination, or uniform hyperbolicity in the projective bundle, means that each Oseledets subspace is more expanded/less contracted than the next, by a definite uniform factor. This is a very strong property. In particular, domination implies that the angles between the Oseledets subspaces are bounded from zero, and the Oseledets splitting extends to a continuous splitting on the closure. For this reason, it can often be excluded a priori:

**Example 1.** Let $f : S^1 \to S^1$ be a homeomorphism and $\mu$ be any invariant ergodic measure with $\text{supp} \, \mu = S^1$. Let $\mathcal{N}$ be the set of all continuous $A : S^1 \to \text{SL}(2, \mathbb{R})$ non-homotopic to a constant. For a residual subset of $\mathcal{N}$, the Lyapunov exponents of the corresponding cocycle over $(f, \mu)$ are zero. That is because the cocycle has no invariant continuous subbundle if $A$ is non-homotopic to a constant.

These results generalize to arbitrary dimension the work of Bochi [3], where it was shown that generic area preserving $C^1$ diffeomorphisms on any compact surface either are uniformly hyperbolic (Anosov) or have no hyperbolicity at all: both Lyapunov exponents equal to zero almost everywhere. This fact had been announced by Mañé [15, 16] in the early eighties.

The high dimensional setting requires a conceptually different approach. That is partly because of the difficulty involved in handling several subbundles, with variable dimensions, and partly because one has to deal with projectively hyperbolic, instead of uniformly hyperbolic, sets. The properties of projectively hyperbolic sets are much weaker (e.g. they need not be robust) and not yet understood.

Our strategy is to analyze the dependence of Lyapunov exponents on the dynamics. We obtain the following characterization of the continuity points of Lyapunov exponents in the space of volume preserving $C^1$ diffeomorphisms on any compact manifold: they must have all exponents equal to zero or else the Oseledets splitting must be dominated, over almost every orbit. Similarly for continuous linear cocycles over any transformation, and in this setting the necessary condition is known to be sufficient.

The issue of continuous or differentiable dependence of Lyapunov exponents on the underlying system is subtle, and not well understood. See Ruelle [23] and also Bourgain, Jitomirskaya [7] for a discussion and further references. We also mention the following simple application of the result we just stated, in the context of quasi-periodic Schrödinger cocycles:

**Example 2.** Let $f : S^1 \to S^1$ be an irrational rotation, $\mu$ be Lebesgue measure, and $A : S^1 \to \text{SL}(2, \mathbb{R})$ be given by

$$A = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

for some $E \in \mathbb{R}$ and $V : S^1 \to \mathbb{R}$ continuous. Then $A$ is a point of discontinuity for the Lyapunov exponents, among all continuous cocycles over $(f, \mu)$, if and only if
the exponents are non-zero and $E$ is in the spectrum of the associated Schrödinger operator. Compare [7]. This is because $E$ is in the complement of the spectrum if and only if the cocycle is uniformly hyperbolic, which for $\text{SL}(2, \mathbb{R})$ cocycles is equivalent to domination.

We extend the two-dimensional result of Mañé–Bochi also in a different direction, namely to symplectic diffeomorphisms on any compact symplectic manifold. Firstly, we prove that continuity points for the Lyapunov exponents either are uniformly hyperbolic or have at least 2 Lyapunov exponents equal to zero at almost every point. Consequently, generic symplectic $C^1$ diffeomorphisms either are Anosov or have vanishing Lyapunov exponents with multiplicity at least 2 at almost every point.

Topological results in the vein of our present theorems were obtained by Millionshchikov [18], in the early eighties, and by Bonatti, Díaz, Pujals, Ures [6, 9], in their recent characterization of robust transitivity for diffeomorphisms. A counterpart of the latter for symplectic maps had been obtained by Newhouse [20] in the seventies, recently extended by Arnaud [1].

1.1 Dominated splittings

Let $M$ be a compact manifold of dimension $d \geq 2$. Let $f : M \to M$ be a diffeomorphism and $\Gamma \subset M$ be an $f$-invariant set. Suppose for each $x \in \Gamma$ one is given non-zero subspaces $E^1_x$ and $E^2_x$ such that $T_x M = E^1_x \oplus E^2_x$, the dimensions of $E^1_x$ and $E^2_x$ are constant, and the subspaces are $Df$-invariant: $Df_x(E^i_x) = E^i_{f(x)}$ for all $x \in \Gamma$ and $i = 1, 2$.

**Definition 1.1.** Given $m \in \mathbb{N}$, we say that $T_\Gamma M = E^1 \oplus E^2$ is an $m$-dominated splitting if for every $x \in \Gamma$ we have

$$\|Df^m_x|_{E^2_x}\| \cdot \|(Df^m_x|_{E^1_x})^{-1}\| \leq \frac{1}{2}.$$  \hspace{1cm} (1.1)

We call $T_\Gamma M = E^1 \oplus E^2$ a dominated splitting if it is $m$-dominated for some $m \in \mathbb{N}$. Then we write $E^1 \succ E^2$.

Condition (1.1) means that, for typical tangent vectors, their forward iterates converge to $E^1$ and their backward iterates converge to $E^2$, at uniform exponential rates. Thus, $E^1$ acts as a global hyperbolic attractor, and $E^2$ acts as a global hyperbolic repeller, for the dynamics induced by $Df$ on the projective bundle.

More generally, we say that a splitting $T_\Gamma M = E^1 \oplus \cdots \oplus E^k$, into any number of sub-bundles, is dominated if

$$E^1 \oplus \cdots \oplus E^j \succ E^j+1 \oplus \cdots \oplus E^k \quad \text{for every } 1 \leq j < k.$$

We say that a splitting $T_\Gamma M = E^1 \oplus \cdots \oplus E^k$, is dominated at $x$, for some point $x \in \Gamma$, if it is dominated when restricted to the orbit $\{f^n(x); n \in \mathbb{Z}\}$ of $x$.

1.2 Dichotomy for volume preserving diffeomorphisms

Let $f \in \text{Diff}^1_\mu(M)$. By the theorem of Oseledets [21], for $\mu$-almost every point $x \in M$, there exists $k(x) \in \mathbb{N}$, real numbers $\hat{\lambda}_1(f, x) > \cdots > \hat{\lambda}_{k(x)}(f, x)$, and a splitting
$T_x M = E^1_x \oplus \cdots \oplus E^k_x$ of the tangent space at $x$, all depending measurably on the point $x$, such that

$$
limit_{n \to \pm \infty} \frac{1}{n} \log \| Df^n_x(v) \| = \hat{\lambda}_j(f, x) \quad \text{for all } v \in E^j_x \setminus \{0\}.
$$

Let $\lambda_1(f, x) \geq \lambda_2(f, x) \geq \cdots \geq \lambda_d(f, x)$ be the numbers $\hat{\lambda}_j(x)$, in non-increasing order and each repeated with multiplicity $\dim E^j_x$. They are called the Lyapunov exponents of $f$ at $x$. Note that $\lambda_1(f, x) + \cdots + \lambda_d(f, x) = 0$, because $f$ preserves volume. We say that the Oseledets splitting is trivial at $x$ when $k(x) = 1$, that is, when all Lyapunov exponents vanish.

**Theorem 1.** There exists a residual set $R \subset \text{Diff}^1_\mu(M)$ such that, for each $f \in R$ and $\mu$-almost every $x \in M$, the Oseledets splitting of $f$ is either trivial or dominated at $x$.

For $f \in R$ the ambient manifold $M$ splits, up to zero measure, into disjoint invariant sets $Z$ and $D$ corresponding to trivial splitting and dominated splitting, respectively. Moreover, $D$ may be written as an increasing union $D = \cup_{m \in \mathbb{N}} D_m$ of compact $f$-invariant sets, each admitting a dominated splitting of the tangent bundle.

If $f \in R$ is ergodic then either $\mu(Z) = 1$ or there is $m \in \mathbb{N}$ such that $\mu(D_m) = 1$. The first case means that all the Lyapunov exponents vanish almost everywhere. In the second case, the Oseledets splitting extends continuously to a dominated splitting of the tangent bundle over the whole ambient manifold $M$.

**Example 3.** Let $f_t : N \to N$, $t \in S^1$, be a smooth family of volume preserving diffeomorphisms on some compact manifold $N$, such that $f_t = \text{id}$ for $t$ in some interval $I \subset S^1$, and $f_t$ is partially hyperbolic for $t$ in another interval $J \subset S^1$. Such families may be obtained, for instance, using the construction of partially hyperbolic diffeomorphisms isotopic to the identity in [5]. Then $f : S^1 \times N \to S^1 \times N$, $f(t, x) = (t, f_t(x))$ is a volume preserving diffeomorphism for which $D \supset S^1 \times J$ and $Z \supset S^1 \times I$.

Thus, in general we may have $0 < \mu(Z) < 1$. However, we ignore whether such examples can be made generic (see also section [3]):

**Problem 1.** Is there a residual subset of $\text{Diff}^1_\mu(M)$ for which invariant sets with a dominated splitting have either zero or full measure?

**Theorem [1]** is a consequence of the following result about continuity of Lyapunov exponents as functions of the dynamics. For $j = 1, \ldots, d - 1$, define

$$
\text{LE}_j(f) = \int_M [\lambda_1(f, x) + \cdots + \lambda_j(f, x)] d\mu(x).
$$

It is well-known that the functions $f \in \text{Diff}^1_\mu(M) \mapsto \text{LE}_j(f)$ are upper semi-continuous. Our next main theorem shows that lower semi-continuity is much more delicate:
Theorem 2. Let $f_0 \in \text{Diff}^1_{\mu}(M)$ be such that the map
\[ f \in \text{Diff}^1_{\mu}(M) \mapsto (\text{LE}_1(f), \ldots, \text{LE}_{d-1}(f)) \in \mathbb{R}^{d-1} \]
is continuous at $f = f_0$. Then for $\mu$-almost every $x \in M$, the Oseledec splitting of $f_0$ is either dominated or trivial at $x$.

The set of continuity points of a semi-continuous function on a Baire space is always a residual subset of the space (see e.g. [14, §31.X]); therefore theorem 1 is an immediate corollary of theorem 2.

Problem 2. Is the necessary condition in theorem 2 also sufficient for continuity?

Diffeomorphisms with all Lyapunov exponents equal to zero almost everywhere, or else whose Oseledec splitting extends to a dominated splitting over the whole manifold, are always continuity points. Moreover, the answer is affirmative in the context of linear cocycles, as we shall see.

1.3 Dichotomy for symplectic diffeomorphisms

Now we turn ourselves to symplectic systems. Let $(M^q, \omega)$ be a compact symplectic manifold without boundary. We denote by $\mu$ the volume measure associated to the volume form $\omega^q = \omega \wedge \cdots \wedge \omega$. The space $\text{Sympl}^1_{\omega}(M)$ of all $C^1$ symplectic diffeomorphisms is a subspace of $\text{Diff}^1_{\mu}(M)$. We also fix a Riemannian metric on $M$, the particular choice being irrelevant for all purposes.

The Lyapunov exponents of symplectic diffeomorphisms have a symmetry property: $\lambda_j(f, x) = -\lambda_{2q-j}(f, x)$ for all $1 \leq j \leq q$. In particular, $\lambda_q(x) \geq 0$ and $\text{LE}_q(f)$ is the integral of the sum of all non-negative exponents. Consider the splitting
\[ T_x M = E^+_x \oplus E^0_x \oplus E^-_x, \]
where $E^+_x$, $E^0_x$, and $E^-_x$ are the sums of all Oseledec spaces associated to positive, zero, and negative Lyapunov exponents, respectively. Then $\dim E^+_x = \dim E^-_x$ and $\dim E^0_x$ is even.

Theorem 3. Let $f_0 \in \text{Sympl}^1_{\omega}(M)$ be such that the map
\[ f \in \text{Sympl}^1_{\omega}(M) \mapsto \text{LE}_q(f) \in \mathbb{R} \]
is continuous at $f = f_0$. Then for $\mu$-almost every $x \in M$, either $\dim E^0_x \geq 2$ or the splitting $T_x M = E^+_x \oplus E^-_x$ is hyperbolic along the orbit of $x$.

In the second alternative, what we actually prove is that the splitting is dominated at $x$. This is enough because, for symplectic diffeomorphisms, dominated splittings into two subspaces of the same dimension are uniformly hyperbolic.

As in the volume preserving case, the function $f \mapsto \text{LE}_q(f)$ is continuous on a residual subset $\mathcal{R}_1$ of $\text{Sympl}^1_{\omega}(M)$. Also, we show that there is a residual subset $\mathcal{R}_2 \subset \text{Sympl}^1_{\omega}(M)$ such that for every $f \in \mathcal{R}_2$ either $f$ is an Anosov diffeomorphism or all its hyperbolic sets have zero measure. Taking $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$, we obtain:

Theorem 4. There exists a residual set $\mathcal{R} \subset \text{Sympl}^1_{\omega}(M)$ such that every $f \in \mathcal{R}$ either is Anosov or has at least two zero Lyapunov exponents at almost every point.

For $d = 2$ one recovers the two-dimensional result of Mañé-Bochi.
1.4 Linear cocycles

Now we comment on corresponding statements for linear cocycles. Let $M$ be a compact Hausdorff space, $\mu$ a Borel regular probability measure, and $f : M \to M$ a homeomorphisms that preserves $\mu$.

Let $G \subset \text{GL}(d, \mathbb{R})$ be a closed group and $C(M, G)$ represent the space of all continuous maps $M \to G$, endowed with the $C^0$-topology. To each $A \in C(M, G)$ one associates the linear cocycle

$$F_A : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x, v) = (f(x), A(x)v).$$

(1.2)

Oseledets theorem extends to this setting, and so does the concept of dominated splitting; see sections 2.1 and 2.2.

**Theorem 5.** Let $G$ be a closed subgroup of $\text{GL}(d, \mathbb{R})$ acting transitively on $\mathbb{R}P^{d-1}$. Then $A_0 \in C(M, G)$ is a point of continuity of

$$C(M, G) \ni A \mapsto (\text{LE}_1(A), \ldots, \text{LE}_{d-1}(A)) \in \mathbb{R}^{d-1}$$

if and only if the Oseledets splitting of the cocycle $F_A$ at $x$ is either dominated or trivial at $\mu$-almost every $x \in M$.

Consequently, there exists a residual subset $\mathcal{R} \subset C(M, G)$ such that for every $A \in \mathcal{R}$ and almost every $x \in X$, the Oseledets splitting of $F_A$ at $x$ is either trivial or dominated.

The most common matrix groups satisfy the hypothesis of the theorem, e.g., $\text{GL}(d, \mathbb{R})$, $\text{SL}(d, \mathbb{R})$, $\text{Sp}(2q, \mathbb{R})$, as well as $\text{SL}(d, \mathbb{C})$, $\text{GL}(d, \mathbb{C})$ (which are isomorphic to subgroups of $\text{GL}(2d, \mathbb{R})$). Notice that compact groups are not of interest in this context, because all Lyapunov exponents vanish identically.

**Corollary 1.** Assume $(f, \mu)$ is ergodic. For any $G$ as in Theorem 5, there exists a residual subset $\mathcal{R} \subset C(M, G)$ such that every $A \in \mathcal{R}$ either has all exponents equal at almost every point, or there exists a dominated splitting of $M \times \mathbb{R}^d$ which coincides with the Oseledets splitting almost everywhere.

1.5 Extensions and related problems

**Problem 3.** For generic smooth families $\mathbb{R}^p \to \text{Diff}^1_\mu(M)$, $\text{Sympl}^1_\omega(M)$, $C(M, G)$, what can be said of the Lebesgue measure of the subset of parameters corresponding to zero Lyapunov exponents ?

**Problem 4.** What are the continuity points of Lyapunov exponents in $\text{Diff}^{1+r}_\mu(M)$ or $C^r(M, G)$ for $r > 0$ ?

Most of the results stated above were announced in [4]. Actually, our theorems 3 and 4 do not quite give the full strength of theorem 4 in [4]. The difficulty is that the symplectic analogue of our construction of realizable sequences is less satisfactory, unless the subspaces involved have the same dimension; see remark 5.2.

**Problem 5.** The Oseledets splitting of generic symplectic $C^1$ diffeomorphisms is either trivial or partially hyperbolic at almost every point, Theorem 5 and the corollary remain true if one replaces $C(M, G)$ by $L^\infty(M, G)$. We only need $f$ to be an invertible measure preserving transformation.
2 Preliminaries

2.1 Lyapunov exponents, Oseledets splittings

Let $M$ be a compact Hausdorff space and $\pi : E \to M$ be a continuous finite-dimensional vector bundle endowed with a continuous Riemann structure. A cocycle over a homeomorphism $f : M \to M$ is a continuous transformation $F : E \to E$ such that $\pi \circ F = f \circ \pi$ and $E_x : E_x \to E_{f(x)}$ is a linear isomorphism on each fiber $E_x = \pi^{-1}(x)$. Notice that (1.2) corresponds to the case when the vector bundle is trivial.

2.1.1 Oseledets theorem

Let $\mu$ be any $f$-invariant Borel probability measure in $M$. The theorem of Oseledets \cite{21} states that for $\mu$-almost every point $x$ there exists a splitting

$$E_x = E_x^1 \oplus \cdots \oplus E_x^k(x),$$

(2.1)

and real numbers $\hat{\lambda}_1(x) > \cdots > \hat{\lambda}_k(x)$ such that $F_x(E_x^j) = E_x^j(f(x))$ and

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \| F_x^n(v) \| = \hat{\lambda}_j(x)$$

for $v \in E_x^j \setminus \{0\}$ and $j = 1, \ldots, k(x)$. Moreover, if $J_1$ and $J_2$ are any disjoint subsets of the set of indices $\{1, \ldots, k(x)\}$, then

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \left( \bigoplus_{j \in J_1} E_x^j(f^n(x)) \bigoplus_{j \in J_2} E_x^j(f^n(x)) \right) = 0.$$  (2.2)

Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x)$ be the numbers $\hat{\lambda}_j(x)$, each repeated with multiplicity $\dim E_x^j$ and written in non-increasing order. When the dependence on $F$ matters, we write $\lambda_i(F, x) = \lambda_i(x)$. In the case when $F = Df$, we write $\lambda_i(f, x) = \lambda_i(F, x) = \lambda_i(x)$.

2.1.2 Exterior products

Given a vector space $V$ and a positive integer $p$, let $\wedge^p(V)$ be the $p$:th exterior power of $V$. This is a vector space of dimension $\binom{d}{p}$, whose elements are called $p$-vectors. It is generated by the $p$-vectors of the form $v_1 \wedge \cdots \wedge v_p$ with $v_j \in V$, called the decomposable $p$-vectors. A linear map $L : V \to W$ induces a linear map $\wedge^p(L) : \wedge^p(V) \to \wedge^p(W)$ such that

$$\wedge^p(L)(v_1 \wedge \cdots \wedge v_p) = L(v_1) \wedge \cdots \wedge L(v_p).$$

If $V$ has an inner product, then we always endow $\wedge^p(V)$ with the inner product such that $\| v_1 \wedge \cdots \wedge v_p \|$ equals the $p$-dimensional volume of the parallelepiped spanned by $v_1, \ldots, v_p$. See \cite{2} section 3.2.3.

More generally, there is a vector bundle $\wedge^p(E)$, with fibers $\wedge^p(E_x)$, associated to $E$, and there is a vector bundle automorphism $\wedge^p(F)$, associated to $F$. If the
vector bundle $\mathcal{E}$ is endowed with a continuous inner product, then $\wedge^p(\mathcal{E})$ also is. The Oseledets data of $\wedge^p(F)$ can be obtained from that of $F$, as shown by the proposition below. For a proof, see [2, theorem 5.3.1].

**Proposition 2.1.** The Lyapunov exponents (with multiplicity) of the automorphism $\wedge^p(F)$ at a point $x$ are the numbers

$$\lambda_{i_1}(x) + \cdots + \lambda_{i_p}(x), \quad \text{where } 1 \leq i_1 < \cdots < i_p \leq d.$$ 

Let $\{e_1(x), \ldots, e_d(x)\}$ be a basis of $\mathcal{E}_x$ such that

$$e_i(x) \in E^\ell_x \quad \text{for } \dim E^1_x + \cdots + \dim E^{\ell-1}_x < i \leq \dim E^1_x + \cdots + \dim E^\ell_x.$$

Then the Oseledets space $E^j_{\wedge^p}$ of $\wedge^p(F)$ corresponding to the Lyapunov exponent $\hat{\lambda}_j(x)$ is the sub-space of $\wedge^p(\mathcal{E}_x)$ generated by

$$e_{i_1} \wedge \cdots \wedge e_{i_p}, \quad \text{with } 1 \leq i_1 < \cdots < i_p \leq d \text{ and } \lambda_{i_1}(x) + \cdots + \lambda_{i_p}(x) = \hat{\lambda}_j(x).$$

2.1.3 Semi-continuity of integrated exponents

Let us indicate $\Lambda_p(F,x) = \lambda_1(F,x) + \cdots + \lambda_p(F,x)$, for $p = 1, \ldots, d - 1$. We define the integrated Lyapunov exponent

$$\text{LE}_p(F) = \int_M \Lambda_p(F,x) \, d\mu(x).$$

More generally, if $\Gamma \subset M$ is a measurable $f$-invariant subset, we define

$$\text{LE}_p(F,\Gamma) = \int_{\Gamma} \Lambda_p(F,x) \, d\mu(x).$$

By proposition 2.1, $\Lambda_p(F,x) = \lambda_1(\wedge^p F,x)$ and so $\text{LE}_p(F,\Gamma) = \text{LE}_1(\wedge^p(F),\Gamma)$. When $F = Df$, we write $\Lambda_p(f,x) = \Lambda_i(F,x)$ and $\text{LE}_p(f,\Gamma) = \text{LE}_p(F,\Gamma)$.

**Proposition 2.2.** If $\Gamma \subset M$ is a measurable $f$-invariant subset then

$$\text{LE}_p(F,\Gamma) = \inf_{n \geq 1} \frac{1}{n} \int_{\Gamma} \log \|\wedge^p(F^m)\| \, d\mu(x).$$

*Proof.* The sequence $a_n = \int_{\Gamma} \log \|\wedge^p(F^m)\| \, d\mu$ is subadditive ($a_{n+m} \leq a_n + a_m$), therefore $\lim \frac{a_n}{n} = \inf \frac{a_n}{n}$. \qed

As a consequence of proposition 2.2, the map $f \in \text{Diff}^1\mu(M) \mapsto \text{LE}_p(f)$ is upper semi-continuous, as mentioned in the introduction.

2.2 Dominated splittings

Let $\Gamma \subset M$ be an $f$-invariant set. A splitting $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is *dominated* for $F$ if it is $F$-invariant, the dimensions of $E^i_x$ are constant on $\Gamma$, and there exists $m \in \mathbb{N}$ such that, for every $x \in \Gamma$,

$$\frac{\|F^m_x|_{E^2_x}\|}{m(F^m_x|_{E^1_x})} \leq \frac{1}{2}, \quad (2.3)$$
We denote $m(L) = \|L^{-1}\|^{-1}$ the co-norm of a linear isomorphism $L$. The dimension of the space $E^1$ is called the index of the splitting.

A few elementary properties of dominated decompositions follow. The proofs are left to the reader.

**Transversality:** If $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is a dominated splitting then the angle $\angle (E^1_x, E^2_x)$ is bounded away from zero, over all $x \in \Gamma$.

**Uniqueness:** If $\mathcal{E}_\Gamma = E^1 \oplus E^2$ and $\hat{\mathcal{E}}_\Gamma = \hat{E}^1 \oplus \hat{E}^2$ are dominated decompositions with $\dim E^i = \dim \hat{E}^i$ then $E^i = \hat{E}^i$ for $i = 1, 2$.

**Continuity:** A dominated splitting $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is continuous, and extends continuously to a dominated splitting over the closure of $\Gamma$.

### 2.3 Dominance and hyperbolicity for symplectic maps

Let $(V, \omega)$ be a symplectic vector space of dimension $2q$. Given a subspace $W \subset V$, its *symplectic orthogonal* is the space (of dimension $2q - \dim W$)

$$W^\omega = \{ w \in W; \ \omega(v, w) = 0 \text{ for all } v \in V \}.$$ 

The subspace $W$ is called *symplectic* if $W^\omega \cap W = \{0\}$, that is, $\omega|_{W \times W}$ is a non-degenerate form. $W$ is called *isotropic* if $W \subset W^\omega$, that is, $\omega|_{W \times W} \equiv 0$. The subspace $W$ is called *Lagrangian* if $W = W^\omega$, that is, it is isotropic and $\dim W = q$.

Now let $(M, \omega)$ be a symplectic manifold of dimension $d = 2q$. We also fix in $M$ a Riemannian structure. For each $x \in M$, let $J_x : T_x M \to T_x M$ be the anti-symmetric isomorphism defined by $\omega(v, w) = \langle J_x v, w \rangle$ for all $v, w \in T_x M$. Denote

$$C_\omega = \sup_{x \in M} \| J_x^{\pm 1} \|. \quad (2.4)$$

In particular, we have

$$|\omega(v, w)| \leq C_\omega \|v\| \|w\| \quad \text{for all } v, w \in T_x M. \quad (2.5)$$

**Lemma 2.3.** If $E, F \subset T_x M$ are two Lagrangian subspaces with $E \cap F = \{0\}$ and $\alpha = \angle (E, F)$ then:

1. For every $v \in E \setminus \{0\}$ there exists $w \in F \setminus \{0\}$ such that
   $$|\omega(v, w)| \geq C_\omega^{-1} \sin \alpha \|v\| \|w\|.$$

2. If $S : T_x M \to T_y M$ is any symplectic linear map and $\beta = \angle (S(E), S(F))$ then
   $$C_\omega^{-2} \sin \alpha \leq m(S|_E) \|S|_F\| \leq C_\omega^2 (\sin \beta)^{-1}.$$
Proof. To prove part 1, let \( p : T_xM \to F \) be the projection parallel to \( E \). Given a non-zero \( v \in E \), take \( w = p(J_xv) \). Since \( E \) is isotropic, \( \omega(v, w) = \|J_xv\|^2 \geq C_{\omega}^{-1} \|v\| \|J_xv\| \). Also \( \|w\| \leq \|p\| \|J_xv\| \) and \( \|p\| = 1/\sin \alpha \), so the claim follows.

To prove part 2, take a non-zero \( v \in E \) such that \( \|Sw\|/\|v\| = m(S|E) \) and let \( w \) be given by part 1. Then

\[
C_{\omega}^{-1} \sin \alpha \|v\| \|w\| \leq |\omega(v, w)| = |\omega(Sv, Sw)| \leq C_{\omega} \|Sv\| \|Sw\|.
\]

Thus \( m(S|E) \|Sw\|/\|w\| \geq C_{\omega}^{-2} \sin \alpha \), proving the lower inequality in part 2. The upper inequality follows from the lower one applied to \( S(F) \), \( S(E) \) and \( S^{-1} \) in the place of \( E \), \( F \), and \( S \), respectively. \( \square \)

**Lemma 2.4.** Let \( f \in \text{Symp}^1(M) \), and let \( x \) be a regular point. Assume that \( \lambda_q(f, x) > 0 \), that is, there are no zero exponents. Let \( E^+_x \) and \( E^-_x \) be the sum of all Oseledets subspaces associated to positive and to negative Lyapunov exponents, respectively. Then

1. The subspaces \( E^+_x \) and \( E^-_x \) are Lagrangian.

2. If the splitting \( E^+ \oplus E^- \) is dominated at \( x \) then \( E^+ \) is uniformly expanding and \( E^- \) is uniformly contracting along the orbit of \( x \).

**Proof.** To prove part 1, we only have to show that the spaces \( E^+_x \) and \( E^-_x \) are isotropic. Take vectors \( v_1, v_2 \in E^-_x \). Take \( \varepsilon > 0 \) with \( \varepsilon < \lambda_q(f, x) \). For every large \( n \) and \( i = 1, 2 \), we have \( \|Df^n_xv_i\| \leq e^{-n\varepsilon} \|v_i\| \). Hence, by (2.5),

\[
|\omega(v_1, v_2)| = |\omega(Df^n_xv_1, Df^n_xv_2)| \leq C_{\omega} e^{-2n\varepsilon} \|v_1\| \|v_2\|,
\]

that is, \( \omega(v_1, v_2) = 0 \). A similar argument, iterating backwards, gives that \( E^+_x \) is isotropic.

Now assume that \( E^+_x > E^-_x \) at \( x \). Let \( \alpha > 0 \) be a lower bound for \( \langle E^+_x, E^-_x \rangle \) along the orbit of \( x \), and let \( C = C_{\omega}^2(\sin \alpha)^{-1} \). By domination, there exists \( m \in \mathbb{N} \) be such that

\[
\frac{\|Df^m_{f^n(x)}|E^-\|}{m(Df^m_{f^n(x)}|E^+\|)} < \frac{1}{4C}, \quad \text{for all } n \in \mathbb{Z}.
\]

By part 2 of lemma 2.4, we have \( C^{-1} \leq m(Df^m_{f^n(x)}|E^+\|) \|Df^m_{f^n(x)}|E^-\| \leq C \). Therefore

\[
m(Df^m_{f^n(x)}|E^+\|) > 2 \quad \text{and} \quad \|Df^k_{f^n(x)}|E^-\| \leq \frac{1}{2} \quad \text{for all } n \in \mathbb{Z}.
\]

This proves part 2. \( \square \)

**Remark 2.5.** More generally, existence of a dominated splitting implies partial hyperbolicity: If \( E \oplus \hat{F} \) is a dominated splitting, with \( \dim E \leq \dim \hat{F} \), then \( \hat{F} \) splits invariantly as \( \hat{F} = C \oplus F \), with \( \dim F = \dim E \). Moreover, \( E \) is uniformly expanding and \( F \) is uniformly contracting. This fact was pointed out by Mañé in [14]. A proof in dimension 4 was given recently by Arnaud [11]. Since the present paper does not use this result, we omit the proof.
2.4 Angle estimation tools

Here we collect a few useful facts from elementary linear algebra. We begin by noting that, given any one-dimensional subspaces \(A, B, \) and \(C\) of \(\mathbb{R}^d\), then

\[
\sin \angle(A, B) \sin \angle(A + B, C) = \sin \angle(C, A) \sin \angle(C + A, B) = \sin \angle(B, C) \sin \angle(B + C, A).
\]

Indeed, this quantity is the 3-dimensional volume of the parallelepiped with unit edges in the directions \(A, B\) and \(C\). As a corollary, we get:

**Lemma 2.6.** Let \(A, B\) and \(C\) be subspaces (of any dimension) of \(\mathbb{R}^d\). Then

\[
\sin \angle(A, B + C) \geq \sin \angle(A, B) \sin \angle(A + B, C).
\]

Let \(v, w\) be non-zero vectors. For any \(\alpha \in \mathbb{R}\), \(\|v + \alpha w\| \geq \|v\| \sin \angle(v, w)\), with equality when \(\alpha = \langle v, w \rangle / \|w\|^2\). Given \(L \in \text{GL}(d, \mathbb{R})\), let \(\beta = \langle Lv, Lw \rangle / \|Lw\|^2\) and \(z = v + \beta w\). By the previous remark, \(\|z\| \geq \|v\| \sin \angle(v, w)\) and \(\|Lz\| = \|Lv\| \sin \angle(Lv, Lw)\). Therefore

\[
\sin \angle(Lv, Lw) = \frac{\|Lz\|}{\|Lv\|} \geq \frac{m(L) \|v\|}{\|Lv\|} \sin \angle(v, w).
\]

(2.6)

As a consequence of (2.6), we have:

**Lemma 2.7.** Let \(L : \mathbb{R}^d \to \mathbb{R}^d\) be a linear map and let \(v, w\) be non-zero vectors. Then

\[
\frac{m(L)}{\|L\|} \leq \frac{\sin \angle(Lv, Lw)}{\sin \angle(v, w)} \leq \frac{\|L\|}{m(L)}.
\]

Thus \(\|L\| / m(L)\) measures how much angles can be distorted by \(L\). At last, we give a bound for this quantity when \(d = 2\).

**Lemma 2.8.** Let \(L : \mathbb{R}^2 \to \mathbb{R}^2\) be an invertible linear map and let \(v, w \in \mathbb{R}^2\) be linearly independent unit vectors. Then

\[
\frac{\|L\|}{m(L)} \leq 4 \max \left\{ \frac{\|Lv\|}{\|Lw\|}, \frac{\|Lw\|}{\|Lv\|} \right\} \frac{1}{\sin \angle(v, w)} \frac{1}{\sin \angle(Lv, Lw)}.
\]

**Proof.** We may assume that \(L\) is not conformal, for in the conformal case the left hand side is 1 and the inequality is obvious. Let \(Rs\) be the direction most contracted by \(L\), and let \(\theta, \phi \in [0, \pi]\) be the angles that the directions \(Rv\) and \(Rw\), respectively, make with \(Rs\). Suppose that \(\|Lv\| \geq \|Lw\|\). Then \(\phi \leq \theta\) and so \(\angle(v, w) \leq 2\theta\). Hence

\[
\|Lv\| \geq \|L\| \sin \theta \geq \frac{1}{2} \|L\| \sin 2\theta \geq \frac{1}{2} \|L\| \sin \angle(v, w).
\]

Moreover, \(|\det L| = m(L) \|L\|\) and

\[
\|Lv\| \|Lw\| \sin \angle(Lv, Lw) = |\det L| \sin \angle(v, w).
\]

The claim is an easy consequence of these relations.
2.5 Coordinates, metrics, neighborhoods

Let \((M, \omega)\) be a symplectic manifold of dimension \(d = 2q \geq 2\). According to Darboux’s theorem, there exists an atlas \(\mathcal{A}^* = \{ \varphi_i : V_i^* \to \mathbb{R}^d \}\) of canonical local coordinates, that is, such that

\[(\varphi_i)_* \omega = dx_1 \wedge dx_2 + \cdots + dx_{2q-1} \wedge dx_{2q}\]

for all \(i\). Similarly, cf. [3], given any volume structure \(\beta\) on a \(d\)-dimensional manifold \(M\), one can find an atlas \(\mathcal{A}^* = \{ \varphi_i : V_i^* \to \mathbb{R}^d \}\) consisting of charts \(\varphi_i\) such that

\[(\varphi_i)_* \beta = dx_1 \wedge \cdots \wedge dx_d.

In either case, assuming \(M\) is compact one may choose \(\mathcal{A}^*\) finite. Moreover, we may always choose \(\mathcal{A}^*\) so that every \(V_i^*\) contains the closure of an open set \(V_i\), such that the restrictions \(\varphi_i : V_i \to \mathbb{R}^d\) still form an atlas of \(M\). The latter will be denoted \(\mathcal{A}\). Let \(\mathcal{A}^*\) and \(\mathcal{A}\) be fixed once and for all.

By compactness, there exists \(r_0 > 0\) such that for each \(x \in M\), there exists \(i(x)\) such that the Riemannian ball of radius \(r_0\) around \(x\) is contained in \(V_{i(x)}\). For definiteness, we choose \(i(x)\) smallest with this property. For technical convenience, when dealing with the point \(x\) we express our estimates in terms of the Riemannian metric \(\| \cdot \| = \| \cdot \|_x\) defined on that ball of radius \(r_0\) by \(\| v \| = \| D\varphi_{i(x)} v \|\). Observe that these Riemannian metrics are (uniformly) equivalent to the original one on \(M\), and so there is no inconvenience in replacing one by the other.

We may also view any linear map \(A : T_{x_1}M \to T_{x_2}M\) as acting on \(\mathbb{R}^d\), using local charts \(\varphi_{i_1}(x_1)\) and \(\varphi_{i_2}(x_2)\). This permits us to speak of the distance \(\|A - B\|\) between \(A\) and another linear map \(B : T_{x_3}M \to T_{x_4}M\) whose base points are different:

\[\|A - B\| = \| D_2 A D_1^{-1} - D_4 B D_3^{-1} \|, \quad \text{where} \quad D_j = (D\varphi_{i_j})(x_j).\]

For \(x \in M\) and \(r > 0\) small (relative to \(r_0\)), \(B_r(x)\) will denote the ball of radius \(r\) around \(x\) relative to the new metric. In other words, \(B_r(x) = \varphi_{i(x)}^{-1}(B(\varphi_{i(x)}(x), r))\). We assume that \(r\) is small enough so that the closure of \(B_r(x)\) is contained in \(V_{i(x)}^*\).

**Definition 2.9.** Let \(\varepsilon_0 > 0\). The \(\varepsilon_0\)-basic neighborhood \(U(id, \varepsilon_0)\) of the identity in \(\text{Diff}_\mu^1(M)\), or in \(\text{Sympl}_\mu^1(M)\), is the set \(U(id, \varepsilon_0)\) of all \(h \in \text{Diff}_\mu^1(M)\), or \(h \in \text{Sympl}_\mu^1(M)\), such that \(h^\pm(V_i) \subset V_i^*\) for each \(i\) and

\[h(x) \in B(x, \varepsilon_0) \quad \text{and} \quad \| Dhx - I \| < \varepsilon_0 \quad \text{for every} \ x \in M.\]

For a general \(f \in \text{Diff}_\mu^1(M)\), or \(f \in \text{Sympl}_\mu^1(M)\), the \(\varepsilon_0\)-basic neighborhood \(U(f, \varepsilon_0)\) is defined by: \(g \in U(f, \varepsilon_0)\) if and only if \(f^{-1} \circ g \in U(id, \varepsilon_0)\) or \(g \circ f^{-1} \in U(id, \varepsilon_0)\).

2.6 Realizable sequences

The following notion, introduced in [3], is crucial to the proofs of theorems [4] through [6]. It captures the idea of sequence of linear transformations that can be (almost) realized on subsets with large relative measure as tangent maps of diffeomorphisms close to the original one.
Definition 2.10. Given \( f \in \text{Diff}^1_{\mu}(M) \) or \( f \in \text{Symp}^1_{\omega}(M) \), constants \( \varepsilon_0 > 0 \), and \( 0 < \kappa < 1 \), and a non-periodic point \( x \in M \), we call a sequence of linear maps (volume preserving or symplectic)

\[
T_xM \xrightarrow{L_0} T_{f_0}f_xM \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} T_{f^{n-1}x}M
\]

an \((\varepsilon_0, \kappa)\)-realizable sequence of length \( n \) at \( x \) if the following holds:

For every \( \gamma > 0 \) there is \( r > 0 \) such that the iterates \( f^j(B_r(x)) \) are two-by-two disjoint for \( 0 \leq j \leq n \), and given any non-empty open set \( U \subset B_r(x) \), there are \( g \in \mathcal{U}(f, \varepsilon_0) \) and a measurable set \( K \subset U \) such that

(i) \( g \) equals \( f \) outside the disjoint union \( \bigcup_{j=0}^{n-1} f^j(U) \);

(ii) \( \mu(K) > (1 - \kappa)\mu(U) \);

(iii) if \( y \in K \) then \( \|Dg_{y^j} - L_j\| < \gamma \) for every \( 0 \leq j \leq n - 1 \).

Some basic properties of realizable sequences are collected in the following

Lemma 2.11. Let \( f \in \text{Diff}^1_{\mu}(M) \) or \( f \in \text{Symp}^1_{\omega}(M) \), \( x \in M \) not periodic and \( n \in \mathbb{N} \).

1. The sequence \( \{Df_x, \ldots, Df_{f^{n-1}(x)}\} \) is \((\varepsilon_0, \kappa)\)-realizable for every \( \varepsilon_0 \) and \( \kappa \) (we call this a trivial realizable sequence).

2. Let \( \kappa_1, \kappa_2 \in (0,1) \) be such that \( \kappa = \kappa_1 + \kappa_2 < 1 \). If \( \{L_0, \ldots, L_{n-1}\} \) is \((\varepsilon_0, \kappa_1)\)-realizable at \( x \), and \( \{L_n, \ldots, L_{n+m-1}\} \) is \((\varepsilon_0, \kappa_2)\)-realizable at \( f^n(x) \), then \( \{L_0, \ldots, L_{n+m-1}\} \) is \((\varepsilon_0, \kappa)\)-realizable at \( x \).

3. If \( \{L_0, \ldots, L_{n-1}\} \) is \((\varepsilon_0, \kappa)\)-realizable at \( x \), then \( \{L_{n-1}, \ldots, L_0\} \) is an \((\varepsilon_0, \kappa)\)-realizable sequence at \( f^n(x) \) for the diffeomorphism \( f^{-1} \).

Proof. The first claim is obvious. For the second one, fix \( \gamma > 0 \). Let \( r_1 \) be the radius associated to the \((\varepsilon_0, \kappa_1)\)-realizable sequence, and \( r_2 \) be the radius associated to the \((\varepsilon_0, \kappa_2)\)-realizable sequence. Fix \( 0 < r < r_1 \) such that \( f^n(B_r(x)) \subset B(f^n(x), r_2) \). Then the \( f^j(B_r(x)) \) are two-by-two disjoint for \( 0 \leq j \leq n + m \). Given an open set \( U \subset B_r(x) \), the realizability of the first sequence gives us a diffeomorphism \( g_1 \in \mathcal{U}(f, \varepsilon_0) \) and a measurable set \( K_1 \subset U \). Analogously, for the open set \( f^n(U) \subset B(f^n(x), r_2) \) we find \( g_2 \in \mathcal{U}(f, \varepsilon_0) \) and a measurable set \( K_2 \subset f^n(U) \). Then define a diffeomorphism \( g \) as \( g = g_1 \) inside \( U \cup \cdots \cup f^{n-1}(U) \) and \( g = g_2 \) inside \( f^n(U) \cup \cdots \cup f^{n+m-1}(U) \), with \( g = f \) elsewhere. Consider also \( K = K_1 \cap g^{-n}(K_2) \). Using that \( g \) preserves volume, one checks that \( g \) and \( K \) satisfy the conditions in definition 2.10. For claim 3, notice that \( \mathcal{U}(f, \varepsilon_0) = \mathcal{U}(f^{-1}, \varepsilon_0) \).

The next lemma makes it simpler to verify that a sequence is realizable: we only have to check the conditions for certain open sets \( U \subset B_r(x) \).

Definition 2.12. A family of open sets \( \{W_\alpha\} \) in \( \mathbb{R}^d \) is a Vitali covering of \( W = \bigcup_\alpha W_\alpha \) if there is \( C > 1 \) and for every \( y \in W \), there are sequences of sets \( W_\alpha \ni y \) and positive numbers \( s_n \to 0 \) such that

\[
B_{s_n}(y) \subset W_\alpha \subset B_{Cs_n}(y)
\]

for all \( n \in \mathbb{N} \).
A family of subsets \( \{U_\alpha\} \) of \( M \) is a Vitali covering of \( U = \cup_\alpha U_\alpha \) if each \( U_\alpha \) is contained in the domain of some chart \( \varphi_i(\alpha) \) in the atlas \( A \), and the images \( \{\varphi_i(\alpha)(U_\alpha)\} \) form a Vitali covering of \( W = \varphi(U) \), in the previous sense.

**Lemma 2.13.** Let \( f \in \Diff^1(M) \) or \( f \in \Symp^1(M) \), and let \( \varepsilon_0 > 0 \) and \( \kappa > 0 \). Consider any sequence \( L_j : T_{f^j(x)} M \to T_{f^{j+1}(x)} M, \) \( 0 \leq j \leq n - 1 \) of linear maps at a non-periodic point \( x \), and let \( \varphi : V \to \mathbb{R}^d \) be a chart in the atlas \( A \), with \( V \ni x \). Assume the conditions in definition 2.10 are valid for every element of some Vitali covering \( \{U_\alpha\} \) of \( B_r(x) \). Then the sequence \( L_j \) is \((\varepsilon_0, \kappa)\)-realizable.

**Proof.** Let \( U \) be an arbitrary open subset of \( B_r(x) \). By Vitali’s covering lemma (see [17]), there is a countable family of two-by-two disjoint sets \( U_\alpha \) covering \( U \) up to a zero Lebesgue measure subset. Thus we can find a finite family of \( U_\alpha \) with disjoint closures and such that \( \mu(U - \bigcup_\alpha U_\alpha) \) is as small as we please. For each \( U_\alpha \) there are, by hypothesis, a perturbation \( g_\alpha \in U(f, \varepsilon_0) \) and a measurable set \( K_\alpha \subseteq U_\alpha \) with the properties (i)-(iii) of definition 2.10. Let \( K = \bigcup K_\alpha \) and define \( g \) as being equal to \( g_\alpha \) on each \( f^j(U_\alpha) \) with \( 0 \leq j \leq n - 1 \). Then \( g \in U(f, \varepsilon_0) \) and the pair \((g, K)\) have the properties required by definition 2.10. \( \square \)

### 3 Geometric consequences of non-dominance

The aim of this section is to prove the following key result, from which we shall deduce theorem 2 in section 4:

**Proposition 3.1.** Given \( f \in \Diff^1(M) \), \( \varepsilon_0 > 0 \) and \( 0 < \kappa < 1 \), if \( m \in \mathbb{N} \) is sufficiently large then the following holds: Let \( y \in M \) be a non-periodic point and suppose one is given a non-trivial splitting \( T_y M = E \oplus F \) such that

\[
\frac{\|Df_y^m|_F\|}{m(Df_y^m|_E)} \geq \frac{1}{2}.
\]

Then there exists an \((\varepsilon_0, \kappa)\)-realizable sequence \( \{L_0, \ldots, L_{m-1}\} \) at \( y \) of length \( m \) and there are non-zero vectors \( v \in E \) and \( w \in Df_y^m(F) \) such that

\[
L_{m-1} \cdots L_0(v) = w.
\]

#### 3.1 Nested rotations

Here we present some tools for the construction of realizable sequences. The first one yields sequences of length 1:

**Lemma 3.2.** Given \( f \in \Diff^1(M) \), \( \varepsilon_0 > 0 \), \( \kappa > 0 \), there exists \( \varepsilon > 0 \) with the following properties:

Suppose we are given a non-periodic point \( x \in M \), a splitting \( \mathbb{R}^d = X \oplus Y \) with \( X \perp Y \) and \( \dim Y = 2 \), and an elliptic linear map \( \hat{R} : Y \to Y \) with \( \|\hat{R} - I\| < \varepsilon \). Consider the linear map \( R : T_x M \to T_x M \) given by \( R(u + v) = u + \hat{R}(v) \), for \( u \in X \), \( v \in Y \). Then \( \{Df_x R\} \) is an \((\varepsilon_0, \kappa)\)-realizable sequence of length 1 at \( x \) and \( \{R Df_{f^{-1}(x)}\} \) is an \((\varepsilon_0, \kappa)\)-realizable sequence of length 1 at the point \( f^{-1}(x) \).
We also need to construct long realizable sequences. Part 2 of lemma 2.11 provides a way to do this, by concatenation of shorter sequences. However, simple concatenation is far too crude for our purposes because it worsens $\kappa$: the relative measure of the set where the sequence can be (almost) realized decreases when the sequence increases. This problem is overcome by lemma 3.3 below, which allows us to obtain certain non-trivial realizable sequences with arbitrary length while keeping $\kappa$ controlled.

In short terms, we do concatenate several length 1 sequences, of the type given by lemma 3.2, but we also impose that the supports of successive perturbations be mapped one to the other. More precisely, there is a domain $C_j$ by lemma 3.2, but we also impose that the supports of successive perturbations be controlled.

To obtain certain non-trivial realizable sequences with arbitrary length while keeping the sequence increases. This problem is overcome by lemma 3.3 below, which allows us to construct long realizable sequences. Part 2 of lemma 2.11 provides a way to do this, by concatenation of shorter sequences. However, simple concatenation is far too crude for our purposes because it worsens $\kappa$: the relative measure of the set where the sequence can be (almost) realized decreases when the sequence increases. This problem is overcome by lemma 3.3 below, which allows us to construct long realizable sequences with arbitrary length while keeping $\kappa$ controlled.

In short terms, we do concatenate several length 1 sequences, of the type given by lemma 3.2, but we also impose that the supports of successive perturbations be mapped one to the other. More precisely, there is a domain $C_0$ invariant under the sequence, in the sense that $L_j \cdots L_0(C_0) = Df_j^j(C_0)$ for all $j$. Following 3, where a similar notion was introduced for the 2-dimensional setting, we call such $L_j$ nested rotations. When $d > 2$ the domain $C_0$ is not compact, indeed it is the product $C_0 = X_0 \oplus B_0$ of a codimension 2 subspace $X_0$ by an ellipse $B_0 \subset X_0^\perp$.

Let us fix some terminology to be used in the sequel. If $E$ is a vector space with an inner product and $F$ is a subspace of $E$, we endow the quotient space $E/F$ with the inner product that makes $v \in F^\perp \mapsto (v+F) \in E/F$ an isometry. If $E'$ is another vector space, any linear map $L : E \to E'$ induces a linear map $L/F : E/F \to E'/F'$, where $F' = L(F)$. If $E'$ has an inner product, then we indicate by $\|L/F\|$ the usual operator norm.

**Lemma 3.3.** Given $f \in \text{Diff}^1_\mu(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, there exists $\varepsilon > 0$ with the following properties: Suppose we are given a non-periodic point $x \in M$ and, for $j = 0, 1, \ldots, n - 1$,

- codimension 2 spaces $X_j \subset T_{f^j(x)}M$ such that $X_j = Df_j^j(X_0)$;
- ellipses $B_j \subset (T_{f^j(x)}M)/X_j$ centered at zero with $B_j = (Df_j^j/X_0)(B_0)$.
- linear maps $\hat{R}_j : (T_{f^j(x)}M)/X_j \to (T_{f^j(x)}M)/X_j$ such that $\hat{R}_j(B_j) \subset B_j$ and $\|\hat{R}_j - I\| < \varepsilon$.

Consider the linear maps $R_j : T_{f^j(x)}M \to T_{f^j(x)}M$ such that $R_j$ restricted to $X_j$ is the identity, $R_j(X_j^\perp) = X_j^\perp$ and $R_j/X_j = \hat{R}_j$. Define

$$L_j = Df_{f^j(x)}R_j : T_{f^j(x)}M \to T_{f^{j+1}(x)}M \quad \text{for} \quad 0 \leq j \leq n - 1.$$  

Then $\{L_0, \ldots, L_{n-1}\}$ is an $(\varepsilon_0, \kappa)$-realizable sequence of length $n$ at $x$.

We shall prove lemma 3.3 in section 3.1.2. Notice that lemma 3.2 is contained in lemma 3.3: take $n = 1$ and use also part 3 of lemma 2.11. Actually, lemma 3.2 also follows from the forthcoming lemma 3.4.

### 3.1.1 Cylinders and rotations

We call a cylinder any affine image $C$ in $\mathbb{R}^d$ of a product $B^{d-i} \times B^i$, where $B^j$ denotes a ball in $\mathbb{R}^j$. If $\psi$ is the affine map, the axis $A = \psi(B^{d-i} \times \{0\})$ and the base $B = \psi(\{0\} \times B^i)$ are ellipsoids. We also write $C = A \oplus B$. The cylinder is called right if $A$ and $B$ are perpendicular. The case we are most interested in is when $i = 2$.  

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The present section contains three preliminary lemmas that we use in the proof of lemma 3.3. The first one explains how to rotate a right cylinder, while keeping the complement fixed. The assumption \( a > \tau b \) means that the cylinder \( C \) is thin enough, and it is necessary for the \( C^1 \) estimate in part (ii) of the conclusion.

**Lemma 3.4.** Given \( \varepsilon_0 > 0 \) and \( 0 < \sigma < 1 \), there is \( \varepsilon > 0 \) with the following properties: Suppose we are given a splitting \( \mathbb{R}^d = X \oplus Y \) with \( X \perp Y \) and \( \dim Y = 2 \), a right cylinder \( \mathcal{A} \oplus \mathcal{B} \) centered at the origin with \( \mathcal{A} \subset X \) and \( \mathcal{B} \subset Y \), and a linear map \( \hat{R} : Y \to Y \) such that \( \hat{R}(\mathcal{B}) = \mathcal{B} \) and \( \|\hat{R} - I\| < \varepsilon \). Then there exists \( \tau > 1 \) such that the following holds:

Let \( R : \mathbb{R}^d \to \mathbb{R}^d \) be the linear map defined by \( R(u + v) = u + \hat{R}v \), for \( u \in X \), \( v \in Y \). For \( a, b > 0 \) consider the cylinder \( C = a \mathcal{A} \oplus b \mathcal{B} \). If \( a > \tau b \) and \( \text{diam} \ C < \varepsilon_0 \) then there is a \( C^1 \) volume preserving diffeomorphism \( h : \mathbb{R}^d \to \mathbb{R}^d \) satisfying

(i) \( h(z) = z \) for every \( z \notin C \) and \( h(z) = R(z) \) for every \( z \in \sigma C \);

(ii) \( \|h(z) - z\| < \varepsilon_0 \) and \( \|Dh_z - I\| < \varepsilon_0 \) for all \( z \in \mathbb{R}^d \).

**Proof.** We choose \( \varepsilon > 0 \) small enough so that

\[
\frac{18\varepsilon}{1 - \sigma} < \varepsilon_0. \tag{3.1}
\]

Let \( \mathcal{A}, \mathcal{B}, X, Y, \hat{R}, R \) be as in the statement of lemma. Let \( \{e_1, \ldots, e_d\} \) be an orthonormal basis of \( \mathbb{R}^d \) such that \( e_1, e_2 \in Y \) are in the directions of the axes of the ellipse \( \mathcal{B} \) and \( e_j \in X \) for \( j = 3, \ldots, d \). We shall identify vectors \( v = xe_1 + ye_2 \in Y \) with the coordinates \( (x, y) \). Then there are constants \( \lambda \geq 1 \) and \( \rho > 0 \) such that \( \mathcal{B} = \{(x, y); \lambda^{-2}x^2 + \lambda^2y^2 \leq \rho^2\} \). Relative to the basis \( \{e_1, e_2\} \), let

\[
H_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \]

The assumption \( \hat{R}(\mathcal{B}) = \mathcal{B} \) implies that \( \hat{R} = H_\lambda R_\alpha H_\lambda^{-1} \) for some \( \alpha \). Besides, the condition \( \|\hat{R} - I\| < \varepsilon \) implies

\[
\lambda^2|\sin \alpha| \leq \|\hat{R} - I\|(0, 1) < \varepsilon. \tag{3.2}
\]

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \varphi(t) = 1 \) for \( t \leq \sigma \), \( \varphi(t) = 0 \) for \( t \geq 1 \), and \( 0 \leq -\varphi'(t) \leq 2/(1 - \sigma) \) for all \( t \). Define smooth maps \( \psi : Y \to \mathbb{R} \) and \( \tilde{g}_t : Y \to Y \) by

\[
\psi(x, y) = \alpha \varphi(\sqrt{x^2 + y^2}) \quad \text{and} \quad \tilde{g}_t(x, y) = R_{\varphi(t)\psi(x, y)}(x, y).
\]

On the one hand, \( \tilde{g}_t(x, y) = (x, y) \) if either \( t \geq 1 \) or \( x^2 + y^2 \geq 1 \). On the other hand, \( \tilde{g}_t(x, y) = R_\alpha(x, y) \) if \( t \leq \sigma \) and \( x^2 + y^2 \geq \sigma^2 \). We are going to check that the derivative of \( \tilde{g}_t \) is close to the identity if \( \varepsilon \) is close to zero; note that \( |\sin \alpha| \) is also close to zero, by (3.2). We have

\[
D(\tilde{g}_t)(x, y) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} + \begin{pmatrix} -x \sin(\psi) - y \cos(\psi) \\ x \cos(\psi) - y \sin(\psi) \end{pmatrix} \cdot (t \partial_x \psi \quad t \partial_y \psi)
\]

\[
= R_{t\varphi(x, y)} + t \left[ R_{\pi/2 + t\psi(x, y)} \right] \cdot D\psi(x, y)
\]

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Consider \(0 \leq t \leq 1\) and \(x^2 + y^2 \leq 1\). Then
\[
\|D(\tilde{g}_t)(x,y) - I\| = \|R_{t\psi}(x,y) - I\| + \|R_{\pi/2+t\psi}(x,y)\cdot D\psi(x,y)\| \\
\leq |\sin(t\psi(x,y))| + \|(2\alpha x \varphi'(x^2 + y^2), 2\alpha y \varphi'(x^2 + y^2))\|
\]
Taking \(\varepsilon\) small enough, we may suppose that \(\alpha \leq 2|\sin \alpha|\). In view of the choice of \(\varphi\) and \(\psi\), this implies
\[
\|D(\tilde{g}_t)(x,y) - I\| \leq |\sin \alpha| + 4|\alpha|/(1 - \sigma) \leq 9|\sin \alpha|/(1 - \sigma).
\]
(3.3)

We also need to estimate the derivative with respect to \(t\):
\[
\|\partial_t \tilde{g}(x,y)\| \leq \|\varphi'(t)\psi(x,y)R_{\pi/2+t\psi}(x,y)\| \leq 4|\sin \alpha|/(1 - \sigma).
\]
(3.4)

Now define \(g_t : Y \to Y\) by \(g_t = H_\lambda \circ \tilde{g}_t \circ H_\lambda^{-1}\). Each \(g_t\) is an area preserving diffeomorphism equal to the identity outside \(\mathcal{B}\). Thus
\[
\|g_t(x,y) - (x,y)\| < \text{diam} \mathcal{B},
\]
(3.5)

for every \((x,y) \in \mathcal{B}\). Moreover, \(g_t = \tilde{R} = H_\lambda R_\alpha H_\lambda^{-1}\) on \(\sigma \mathcal{B}\) for all \(t \leq \sigma\). By (3.3),
\[
\|D(g_t)(x,y) - I\| = \|H_\lambda(D(\tilde{g}_t)(\lambda^{-1}x,\lambda y) - I)\cdot H_\lambda^{-1}\| \leq \lambda^2 \left(\frac{9|\sin \alpha|}{1 - \sigma}\right),
\]
and, applying (3.2) and (3.1), we deduce that
\[
\|D(g_t)(x,y) - I\| < \frac{9\varepsilon}{1 - \sigma} < \frac{\varepsilon_0}{2}
\]
(3.6)

for all \((x,y) \in \mathcal{B}\). Similarly, by (3.4),
\[
\|
\partial_t g_t(x,y)\| \leq \lambda^2 \|
\partial_t \tilde{g}_t(\lambda^{-1}x, \lambda y)\| \leq \lambda^2 \left(\frac{4|\sin \alpha|}{1 - \sigma}\right) < \frac{\varepsilon_0}{2}.
\]
(3.7)

Now let \(Q : X \to \mathbb{R}\) be a quadratic form such that \(\mathcal{A} = \{u \in X; Q(u) \leq 1\}\), and let \(q : \mathbb{R}^d \to X\) and \(p : \mathbb{R}^d \to Y\) be the orthogonal projections. Given \(a, b > 0\), define \(h : \mathbb{R}^d \to \mathbb{R}^d\) by
\[
h(z) = z' + bg_{a^{-2}Q(z')}(b^{-1}z''), \quad \text{where} \ z' = q(z) \text{ and } z'' = p(z).
\]
It is clear that \(h\) is a volume preserving diffeomorphism. The subscript \(t = a^{-2}Q(z')\) is designed so that \(t \leq 1\) if and only if \(z' \in a\mathcal{A}\). Then \(h(z) = z\) if either \(z' \notin a\mathcal{A}\) or \(z'' \notin b\mathcal{B}\). Moreover, \(h(z) = z' + \tilde{R}(z'') = R(z)\) if \(z' \in a\mathcal{A}\) and \(z'' \in \sigma b\mathcal{B}\). This proves property (i) in the statement. The hypothesis \(\text{diam} \mathcal{C} < \varepsilon_0\) and (3.5) give
\[
\|h(z) - z\| = b\|g_{a^{-2}Q(z')}(b^{-1}z'') - b^{-1}z''\| \\
< b\text{diam} \mathcal{B} \leq \text{diam}(a\mathcal{A} \oplus b\mathcal{B}) < \varepsilon_0
\]
which is the first half of (ii). Finally, fix \(\tau > 1\) such that \(\|DQ_u\| \leq \tau\|u\|\) for all \(u \in \mathbb{R}^d\), and assume that \(a > \tau b\). Clearly,
\[
Dh = q + \frac{b}{a\tau}(\partial_t g)(DQ)q + (Dg)p.
\]
Using (3.6), (3.7), and the fact that \(\|q\| = \|p\| = 1\) (these are orthogonal projections),

\[
\|Dh - I\| \leq \frac{b}{a^2} \|\partial_h g\| (DQ)q + \|(Dg - I)p\| \\
\leq \frac{b}{a^2} \|\partial_h g\| \tau a \|q\| + \|Dg - I\| \|p\| < \varepsilon_0.
\]

This completes the proof of property (ii) and the lemma. 

The second of our auxiliary lemmas says that the image of a small cylinder by a \(C^1\) diffeomorphism \(h\) contains the image by \(Dh\) of a slightly shrunk cylinder. Denote \(C(y, \rho) = \rho C + y\), for each \(y \in \mathbb{R}^d\) and \(\rho > 0\).

**Lemma 3.5.** Let \(h : \mathbb{R}^d \to \mathbb{R}^d\) be a \(C^1\) diffeomorphism with \(h(0) = 0\), \(C \subset \mathbb{R}^d\) be a cylinder centered at 0, and \(0 < \lambda < 1\). Then there exists \(r > 0\) such that for any \(C(y, \rho) \subset B_r(0)\),

\[h(C(y, \rho)) \supset Dh_0(C(0, \lambda \rho)) + h(y)\]

**Proof.** Fix a norm \(\|\cdot\|\) in \(\mathbb{R}^d\) for which \(C = \{z \in \mathbb{R}^d; \|z\|_0 < 1\}\). Such a norm exists because \(C\) is convex and \(\mathcal{C} = -\mathcal{C}\). Let \(H = Dh_0\) and \(g : \mathbb{R}^d \to \mathbb{R}^d\) be such that \(h = H \circ g\). Since \(g\) is \(C^1\) and \(Dg_0 = I\), we have

\[g(z) - g(y) = z - y + \xi(z, y)\quad\text{with}\quad\lim_{(z, y) \to (0, 0)} \frac{\xi(z, y)}{\|z - y\|_0} = 0.
\]

Choose \(r > 0\) such that \(\|z\|, \|y\| \leq r \Rightarrow \|\xi(z, y)\|_0 < (1 - \lambda)\|z - y\|_0\) (where \(\|\cdot\|\) denotes the Euclidean norm in \(\mathbb{R}^d\)). Now suppose \(C(y, \rho) \subset B_r(0)\), and let \(z \in \partial C(y, \rho)\). Then \(\|z - y\|_0 = \rho\) and

\[\|g(z) - g(y)\|_0 \geq \|z - y\|_0 - \|\xi(z, y)\|_0 > \lambda \rho.
\]

This proves that the sets \(g(\partial C(y, \rho)) - g(y)\) and \(\lambda C\) are disjoint. Applying the linear map \(H\), we find that \(h(\partial C(y, \rho)) - h(y)\) and \(\lambda HC\) are disjoint. From topological arguments, \(h(C(y, \rho)) - h(y) \supset \lambda HC\). 

The third lemma says that a linear image of a sufficiently thin cylinder contains some right cylinder with almost the same volume. The idea is contained in figure 1. The proof of the lemma is left to the reader.

**Lemma 3.6.** Let \(A \oplus B\) be a cylinder centered at the origin, \(L : \mathbb{R}^d \to \mathbb{R}^d\) be a linear isomorphism, \(A_1 = L(A)\) and \(B_1 = p(L(B))\), where \(p\) is the orthogonal projection onto the orthogonal complement of \(A_1\). Then, given any \(0 < \lambda < 1\), there exists \(\tau > 1\) such that if \(a > \tau b\),

\[L(aA + bB) \supset \lambda aA_1 \oplus bB_1.
\]
3.1.2 Proof of the nested rotations lemma 3.3

Proof. Let \( f, \epsilon_0, \) and \( \kappa \) be given. Define \( \sigma = (1 - \kappa)^{1/2d} \) and then take \( \epsilon > 0 \) as given by lemma 3.4. Now let \( x, n, X_j, B_j, \tilde{R}_j, R_j, L_j \) be as in the statement. We want to prove that \( \{L_0, \ldots, L_n\} \) is an \((\epsilon_0, \kappa)\)-realizable sequence of length at \( x \), cf. definition 2.10.

In short terms, we use lemma 3.4 to construct the realization \( g \) at each iterate. The subset \( U \setminus K \), where we have no control on the approximation, has two sources: lemma 3.4 gives \( h = R \) only on a slightly smaller cylinder \( \sigma C \); and we need to straighten out (lemma 3.5) and to “rightify” (lemma 3.6) our cylinders at each stage. These effects are made small by considering cylinders that are small and very thin. That is how we get \( U \setminus K \) with relative volume less than \( \kappa \), independently of \( n \).

For clearness we split the proof into three main steps:

**Step 1:** Fix any \( \gamma > 0 \). We explain how to find \( r > 0 \) as in definition 2.10.

We consider local charts \( \varphi_j : V_j \rightarrow \mathbb{R}^d \) with \( \varphi_j = \varphi_{i(f^jx)} \) and \( V_j = V_{i(f^jx)} \), as introduced in section 2.5. Let \( r' > 0 \) be small enough so that

- \( f_j(B_{r'}(x)) \subseteq V_j^* \) for every \( j = 0, 1 \ldots, n \);
- the sets \( f_j(B_{r'}(x)) \) are two-by-two disjoint;
- \( \|Df_z - Df_{f_j(x)}\|\|R_j\| < \gamma \) for every \( z \in f_j(B_{r'}(x)) \) and \( j = 0, 1 \ldots, n \).

We use local charts to translate the situation to \( \mathbb{R}^d \). Let \( f_j = \varphi_{j+1} \circ f \circ \varphi_j^{-1} \) be the expression of \( f \) in local coordinates near \( f^j(x) \) and \( f^{j+1}(x) \). To simplify the notations, we suppose that each \( \varphi_j \) has been composed with a translation to ensure \( \varphi_j(f^j(x)) = 0 \) for all \( j \). Up to identification of tangent spaces via the charts \( \varphi_j \) and \( \varphi_{j+1} \), we have \( L_j = (Df_j)_0 R_j \).

Let \( A_0 \subseteq X_0 \) be any ellipsoid centered at the origin (a ball, for example), and let \( A_j = Df_{f_j} A_0 \) for \( j \geq 1 \). We identify \( (T_{f^j(x)} M)/X_j \) with \( X_j^\perp \), so that we may consider \( B_j \subseteq X_j^\perp \). In these terms, the assumption \( B_j = (Df_{f_j}/X_0)(B_0) \) means that \( B_j \) is the orthogonal projection of \( Df_{f_j}(B_0) \) onto \( X_j^\perp \).

Fix \( 0 < \lambda < 1 \) close enough to 1 so that \( \lambda^{4n(d-1)} > 1 - \kappa \). Let \( \tau_j > 1 \) be associated to the data \((A_j \oplus B_j, (Df_j)_0, \lambda)\) by lemma 3.6: if \( a > \tau_j b \) then

\[
(Df_j)_0(aA_j \oplus bB_j) \supset \lambda aA_{j+1} \oplus bB_{j+1} \quad (3.8)
\]
and let $\tau'_j > 1$ be associated to the data $(\varepsilon_0, \sigma, X_j \oplus X_j^\perp, A_j \oplus B_j, \bar{R}_j)$ by lemma 3.4.

Fix $a_0 > 0$ and $b_0 > 0$ such that
\[ a_0 > b_0 \lambda^{-n} \max \{\tau_j, \tau'_j; 0 \leq j \leq n - 1\}. \]  
(3.9)

For $0 \leq j \leq n$, define $C_j = \lambda^j a_0 A_j \oplus \lambda^{-j} b_0 B_j$. For $z \in \mathbb{R}^d$ and $\rho > 0$, denote $C_j(z, \rho) = \rho C_j + z$. Applying lemma 3.3 to the data $(f_j, C_j, \lambda)$ we get $r_j > 0$ such that
\[ C(z, \rho) \subset B_{r_j}(0) \Rightarrow f_j(C(z, \rho)) \supset (Df_j)_0(C_j(0, \lambda\rho)) + f_j(z). \]  
(3.10)

Now take $r > 0$ such that $r < r'$ and, for each $j = 1, \ldots, m - 1$,
\[ f_{j-1} \cdots f_0(B_r(0)) \subset B_{r_j}(0). \]  
(3.11)

**Step 2:** Let $U$ be fixed. We find $g \in \mathcal{U}(f, \varepsilon_0)$ and $K \subset U$ as in definition 2.10.

For this we take advantage of lemma 2.13: it suffices to consider open sets of the form $U = \varphi_0^{-1}(C_0(y_0, \rho))$, because the cylinders $C_0(y_0, \rho)$ contained in $B_r(0)$ constitute a Vitali covering.

We claim that, for each $j = 0, 1, \ldots, m - 1$, and every $t \in [0, \rho]$,
\[ C_j(y_j, t) \subset f_{j-1} \cdots f_0(B_r(0)) \]  
(3.12)

and
\[ f_j(C_j(y_j, t)) \supset C_{j+1}(y_{j+1}, t). \]  
(3.13)

For $j = 0$, relation (3.12) means $C_0(y_0, t) \subset B_r(0)$, which is true by assumption. We proceed by induction. Assume (3.12) holds for some $j \geq 0$. Then, by (3.11) and (3.10),
\[ f_j(C_j(y_j, t)) \supset (Df_j)_0(C_j(0, \lambda t)) + y_{j+1} \]
\[ = (Df_0)_0 \left[ (\lambda^{2j+1} ta_0 A_j) \oplus (\lambda^{j+1} t b_0 B_j) \right] + y_{j+1}. \]

Relation (3.9) implies that $\lambda^{2j+1} ta_0 > \tau_j(\lambda^{j+1} t b_0)$. So, we may use (3.8) to conclude that
\[ f_j(C_j(y_j, t)) \supset (\lambda^{2j+2} ta_0 A_j) \oplus (\lambda^{j+1} t b_0 B_j) + y_{j+1} = C_{j+1}(y_{j+1}, t). \]

This proves that (3.13) holds for the same value of $j$. Moreover, it is clear that if (3.13) holds for all $0 \leq i \leq j$ then (3.12) is true with $j + 1$ in the place of $j$. This completes the proof of (3.12) and (3.13).

Condition (3.9) also implies $\lambda^{2j} a_0 > \tau'_j(\lambda^{j} b_0)$. So, we may use lemma 3.4 (centered at $y_j$) to find a volume preserving diffeomorphism $h_j : \mathbb{R}^d \to \mathbb{R}^d$ such that

1. $h_j(z) = z$ for all $z \notin C_j(y_j, \rho)$ and $h_j(z) = y_j + R_j(z - y_j)$ for all $z \in C_j(y_j, \sigma \rho)$

and, consequently,
\[ h_j(C_j(y_j, \sigma \rho)) = C_j(y_j, \sigma \rho) \quad \text{and} \quad h_j(C_j(y_j, \rho)) = C_j(y_j, \rho). \]  
(3.14)
2. \( \|h_j(z) - z\| < \varepsilon_0 \) and \( \|(Dh_j)_z - I\| < \varepsilon_0 \) for all \( z \in \mathbb{R}^d \).

\( R_j \) is the linear map \( T_{fj(x)} \to T_{fj+1(x)} \) in the statement of the theorem or, more precisely, its expression in local coordinates \( \varphi_j \). Let \( S_j = \varphi_j^{-1}(\{z; h(z) \neq z\}) \subset M \). By property 1 above and the inclusion (3.12),
\[
S_j \subset \varphi_j^{-1}(f_{j-1} \cdots f_0(B_r(0))) = f^j(B_r(x)).
\]

In particular, the sets \( S_j \) have pairwise disjoint closures. This permits us to define a diffeomorphism \( g \in \text{Diff}_\mu^1(M) \) by
\[
g = \begin{cases} 
\varphi_j^{-1} \circ (f_j \circ h_j) \circ \varphi_j & \text{on } S_j \text{ for each } 0 \leq j \leq n - 1 \\
\text{constant} & \text{outside } S_0 \sqcup \cdots \sqcup S_{n-1}.
\end{cases}
\]

Property 2 above gives that \( f^{-1} \circ g \in \mathcal{U}(\text{id}, \varepsilon_0) \), and so \( g \in \mathcal{U}(f, \varepsilon_0) \).

**Step 3:** Now we define \( K \subset U \) and check the conditions (i)–(iii) in definition 2.10.

By construction, \( h_j = \text{id} \) outside \( C_j(y_j, \rho) \), and so
\[
\varphi_j^{-1} \circ (f_j \circ h_j) \circ \varphi_j = f \quad \text{outside } \varphi^{-j}(C_j(y_j, \rho)).
\]

Using (3.13) and (3.14), we have \( \varphi_j^{-1}(C_j(y_j, \rho)) \subset f^j(U) \) for all \( 0 \leq j \leq n - 1 \). Recall that \( U = \varphi_0^{-1}(C_0(y_0, \rho)) \). Hence, \( g = j \) outside the disjoint union \( \bigsqcup_{j=0}^{n-1} f^j(U) \). This proves condition (i).

Define \( K = g^{-n}(\varphi_n^{-1}(C_n(y_n, \sigma \rho))) \). Using (3.13) and (3.14) in the same way as before, we see that \( K \subset U \). Also, since all the maps \( f, g, h_j, \varphi_j \) are volume preserving, and all the cylinders \( C_j(y_j, \rho), C_j(y_j, \sigma \rho) \), are right

\[
\frac{\text{vol } K}{\text{vol } U} = \frac{\text{vol}(\sigma \rho \lambda^{2n} a \mathcal{A}_n \oplus \sigma \rho \lambda^n b \mathcal{B}_n)}{\text{vol}(\rho a \mathcal{A}_0 \oplus \rho b \mathcal{B}_0)} = \frac{(\lambda^{2n} \sigma)^{d-2} \text{vol } \mathcal{A}_n (\lambda^n \sigma)^2 \text{vol } \mathcal{B}_n}{\text{vol } \mathcal{A}_0 \text{ vol } \mathcal{B}_0}.
\]

Notice also that \( \text{vol } \mathcal{A}_n \text{ vol } \mathcal{B}_n = \text{vol } \mathcal{A}_0 \text{ vol } \mathcal{B}_0 \), since the cylinders \( Df^{n_k}_x(\mathcal{A}_0 \oplus \mathcal{B}_0) \) and \( \mathcal{A}_n \oplus \mathcal{B}_n \) differ by a shear. So, the right hand side is equal to \( \lambda^{2n(d-1)} \sigma^d \). Now, this expression is larger than \( 1 - \kappa \), because we have chosen \( \sigma = (1 - \kappa)^{1/2} \) and \( \lambda > (1 - \kappa)^{1/(4n(d-1))} \). This gives condition (ii).

Finally, let \( z \in K \). Recall that \( L_j = Df_{fj(x)}R_j \). Moreover, \( (Dh_j)_j \varphi_j g^j(z) = R_j \) (we continue to identify \( R_j \) with its expression in the local chart \( \varphi_j \)), because
\[
g^j(z) \in g^{-n+j}(\varphi_n^{-1}(C_n(y_n, \sigma \rho))) \subset \varphi_j^{-1}(C_j(y_j, \sigma \rho)).
\]

Therefore, writing \( z_j = h_j(\varphi_j(g^j(z))) \) for simplicity,
\[
\|Dg^j(z) - L_j\| = \|D(f_j)z_j R_j - D(f_j)0 R_j\| \leq \|D(f_j)z_j - D(f_j)0\| \|R_j\| < \gamma.
\]

The last inequality follows from our choice of \( r' \). This gives condition (iii) in definition 2.10. The proof of lemma 3.3 is complete. \( \square \)

**Remark 3.7.** This last step explains why it is technically more convenient to require \( \|Dg^j(z) - L_j\| < \gamma \), rather than \( Dg^j(z) = L_j \), when defining realizable sequence.
3.2 Proof of the directions interchange proposition 3.1

*Proof.* First, we define some auxiliary constants. Fix $0 < \kappa' < \frac{1}{4}\kappa$. Let $\varepsilon_1 > 0$, depending on $f$, $\varepsilon_0$ and $\kappa'$, be given by lemma 3.2. Let $\varepsilon_2 > 0$, depending on $f$, $\varepsilon_0$ and $\kappa$, be given by lemma 3.3. Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Fix $\alpha > 0$ such that $\sqrt{2} \sin \alpha < \varepsilon$. Take

$$K \geq (\sin \alpha)^{-2} \quad \text{and} \quad K \geq \max \left\{ \| Df_x \| / m(Df_x); \ x \in M \right\}.$$  \hfill (3.15)

Let $\beta > 0$ be such that

$$8\sqrt{2} K \sin \beta < \varepsilon \sin^6 \alpha. \hfill (3.16)$$

Finally, assume $m \in \mathbb{N}$ satisfies $m \geq 2\pi/\beta$.

Let $y \in M$ be a non-periodic point and $T_y M = E \oplus F$ be a splitting as in the hypothesis:

$$\frac{\| Df_y^{m+1}(y) \|}{m(Df_y^{m+1}(y))} \geq \frac{1}{2}. \hfill (3.17)$$

We write $E_j = Df_j^j(E)$ and $F_j = Df_j^j(F)$ for $j = 0, 1, \ldots, m$. The proof is divided in three cases. Lemma 3.2 suffices for the first two, in the third step we use the full strength of lemma 3.3.

**First case:** Suppose there exists $\ell \in \{0, 1, \ldots, m\}$ such that

$$\langle (E_\ell, F_\ell) \rangle < \alpha. \hfill (3.18)$$

Fix $\ell$ as above. Take unit vectors $\xi \in E_\ell$ and $\eta \in F_\ell$ such that $\langle (\xi, \eta) \rangle < \alpha$. Let $Y = \mathbb{R}\xi \oplus \mathbb{R}\eta$ and $X = Y^\perp$. Let $\hat{R} : Y \to Y$ be a rotation such that $\hat{R}(\xi) = \eta$. Then $\| \hat{R} - I \| = \sqrt{2} \sin \langle (\xi, \eta) \rangle < \varepsilon$. Let $R : T_{f^\ell(y)} M \to T_{f^\ell(y)} M$ be such that $R$ preserves both $X$ and $Y$, $R|X = I$ and $R|Y = \hat{R}$.

Consider first $\ell < m$. By lemma 3.2, the length 1 sequence $\{ Df_{f^\ell(y)} R \}$ is $(\kappa', \varepsilon_0)$-realizable at $f^\ell(y)$. Using part 2 of lemma 2.11 we conclude that

$$\{ L_0, \ldots, L_{m-1} \} = \{ Df_y, \ldots, Df_{f^{\ell-1}(y)}, Df_{f^\ell(y)} R, Df_{f^{\ell+1}(y)} \}, \ldots, Df_{f^{m-1}(y)} \}$$

is a $(\kappa, \varepsilon_0)$-realizable sequence of length $m$ at $y$. The case $\ell = m$ is similar. By lemma 3.2, the length 1 sequence $\{ R Df_{f^{m-1}(y)} \}$ is $(\kappa', \varepsilon_0)$-realizable at $f^{m-1}(y)$. Then, by part 2 of lemma 2.11,

$$\{ L_0, \ldots, L_{m-1} \} = \{ Df_y, \ldots, Df_{f^{m-2}(y)}, R Df_{f^{m-1}(y)} \}$$

is a $(\kappa, \varepsilon_0)$-realizable sequence of length $m$ at $y$. In either case, $L_{m-1} \cdots L_0$ sends the vector $v = Df^{-\ell}(\xi) \in E_0$ to a vector $w$ collinear to $Df_{m-\ell}(\eta) \in F_m.$
Second case: Assume there exist $k, \ell \in \{0, \ldots, m\}$, with $k < \ell$, such that
\[ \frac{\|Df_{f_k(y)}^\ell \xi\|_{F_k}}{m(Df_{f_k(y)}^\ell |_{E_k})} > K. \quad (3.19) \]

Fix $k$ and $\ell$ as above. Let $\xi \in E_k, \eta \in F_k$ be unit vectors such that
\[ \|Df_{f_k(y)}^\ell (\xi)\| = m(Df_{f_k(y)}^\ell |_{E_k}) \quad \text{and} \quad \|Df_{f_k(y)}^\ell (\eta)\| = m(Df_{f_k(y)}^\ell |_{F_k}), \]
($Df_{f_k(y)}^\ell$ is always meant at the point $f_k(y)$). Define also unit vectors
\[ \xi' = \frac{Df_{f_k(y)}^\ell (\xi)}{\|Df_{f_k(y)}^\ell (\xi)\|} \in E_\ell \quad \text{and} \quad \eta' = \frac{Df_{f_k(y)}^\ell (\eta)}{\|Df_{f_k(y)}^\ell (\eta)\|} \in F_\ell. \]

Let $\xi_1 = \xi + (\sin \alpha) \eta$. Then $\theta = \angle(\xi, \xi_1) \leq \alpha$, simply because $\|\xi\| = \|\eta\| = 1$. In particular, if $\tilde{R} : \mathbb{R} \xi \oplus \mathbb{R} \eta \to \mathbb{R} \xi \oplus \mathbb{R} \eta$ is a rotation of angle $\pm \theta$, sending $\mathbb{R} \xi$ to $\mathbb{R} \xi_1$ then
\[ \|\tilde{R} - I\| = \sqrt{2} \sin \theta < \varepsilon. \]

Let $Y = \mathbb{R} \xi \oplus \mathbb{R} \eta$ and $X = Y^\perp$. Let $R : T_{f_k(y)} M \to T_{f_k(y)} M$ be such that $R$ preserves both $X$ and $Y$, with $R|_X = I$ and $R|_Y = \tilde{R}$. By lemma 3.1, the length 1 sequence $\{Df_{f_k(y)} R\}$ is $(\kappa', \varepsilon_0)$-realizable at $f(k)$. Let $\eta'_1 = s \xi_1' + \eta'$, where
\[ s = \frac{1}{\sin \alpha} \frac{\|Df_{f_k(y)}^\ell (\xi)\|}{\|Df_{f_k(y)}^\ell (\eta)\|} = \frac{1}{\sin \alpha} \frac{m(Df_{f_k(y)}^\ell |_{E_k})}{m(Df_{f_k(y)}^\ell |_{F_k})}. \]

Then the vectors $Df_{f_k(y)}^\ell \xi_1$ and $\eta_1$ are collinear. Besides, $s < 1/(K \sin \alpha) < \sin \alpha$, because of (3.13) and (3.19). Hence, $\theta' = \angle(\eta'_1, \eta') < \alpha$. Then, as before, there exists $R' : T_{f_k(y)} M \to T_{f_k(y)} M$ such that $R'(\mathbb{R} \eta'_1) = \mathbb{R} \eta$ and $\{R' Df_{f_k(y)}^\ell \eta_1\}$ is a $(\kappa', \varepsilon_0)$-realizable sequence of length 1 at $f(k)$.

Notice that (3.13) and (3.19) imply $\ell - 1 > k$. Then we may define a sequence $\{L_0, \ldots, L_{m-1}\}$ of linear maps as follows:
\[ L_j = \begin{cases} Df_{f_k(y)} R & \text{for } j = k \\ R' Df_{f_{\ell-1}(y)} & \text{for } j = \ell - 1 \\ Df_{f_j(y)} & \text{for all other } j. \end{cases} \]

By parts 1 and 2 of lemma 2.11, this is a $(\kappa, \varepsilon_0)$-realizable sequence of length $m$ at $y$. By construction, $L_{m-1} \cdots L_0$ sends $v = Df_{f_k(y)}(\xi) \in E_0$ to a vector $w$ collinear to $Df_{f_{m-\ell}(y)}(\eta') \in F_m$.

Third case: We suppose that we are not in the previous cases, that is, we assume
\[ \text{for every } j \in \{0, 1, \ldots, m\}, \quad \angle(E_j, F_j) \geq \alpha. \quad (3.20) \]

and
\[ \text{for every } i, j \in \{0, \ldots, m\} \text{ with } i < j, \quad \frac{\|Df_{f_{i}(y)}^{j-i} (\cdot)\|_{F_i}}{m(Df_{f_{i}(y)}^{j-i} |_{E_i})} \leq K. \quad (3.21) \]
We now use the assumption (3.17), and the choice of \( m \) in (3.16). Take unit vectors \( \xi \in E_0 \) and \( \eta \in F_0 \) such that \( \|Df^m \xi\| = m \|Df^m|_{E_0}\) and \( \|Df^m \eta\| = \|Df^m|_{F_0}\) \((Df^m \) is always computed at \( y \)). Let also \( \eta' = Df^{m}(\eta)/\|Df^{m}(\eta)\| \in F_m \).

Define \( G_0 = E_0 \cap \xi^\perp \) and \( G_j = Df^j (G_0) \subset E_j \) for \( 0 < j \leq m \). Dually, define \( H_m = F_m \cap \eta'^\perp \) and \( H_j = Df^{j-m}(H_m) \subset F_j \) for \( 0 \leq j < m \). In addition, consider unit vectors \( v_j \in E_j \cap G_j^\perp \) and \( w_j \in F_j \cap H_j^\perp \) for \( 0 \leq j \leq m \). These vectors are uniquely defined up to a choice of sign, and \( v_0 = \pm \xi \) and \( w_m = \pm \eta' \). See Figure 2.

For \( j = 0, \ldots, m \), define
\[
X_j = G_j \oplus H_j \quad \text{and} \quad Y_j = \mathbb{R}v_j \oplus \mathbb{R}w_j.
\]
The spaces \( X_j \) are invariant: \( Df^j (y)(X_j) = X_{j+1} \) \((\text{the } Y_j \text{ are not})\). We shall prove, using (3.20) and (3.21), that the maps \( Df^j (y)/X_0 : T_y M/X_0 \to T_{f^j(y)} M/X_j \) do not distort angles too much:

**Lemma 3.8.** For every \( j = 0, 1, \ldots, m \),
\[
\frac{\|Df^j (y)/X_0\|}{m(Df^j (y)/X_0)} \leq \frac{8K}{\sin^{\beta} \alpha}.
\]

![Figure 2: Setup for application of the nested rotations lemma](image)

Let us postpone the proof of this fact for a while, and proceed preparing the application lemma 3.3. Let \( B_0 \subset (T_y M)/X_0 \) be a ball and \( B_j = (Df^j (y)/X_0)(B_0) \) for \( 0 < j \leq m \). Since \( m\beta \geq 2\pi \), it is possible to choose numbers \( \theta_0, \ldots, \theta_{m-1} \) such that \( |\theta_j| \leq \beta \) for all \( j \) and
\[
\sum_{j=0}^{m-1} \theta_j = \angle(v_0 + X_0, w_0 + X_0).
\]

Let \( P_j : (T_y M)/X_0 \to (T_y M)/X_0 \) be the rotation of angle \( \theta_j \). Define linear maps \( \hat{R}_j : (T_{f^j(y)} M)/X_j \to (T_{f^j(y)} M)/X_j \) by
\[
\hat{R}_j = (Df^j (y)/X_0) P_j (Df^j (y)/X_0)^{-1}.
\]
Since $P_j$ preserves the ball $B_0$, we have $\hat{R}_j(B_j) = B_j$ for all $j$. Moreover,

$$\|\hat{R}_j - I\| \leq \frac{\|DF_y^j/X_0\|}{m(DF_y^j/X_0)} \|P_j - I\| \leq \frac{8K}{\sin^6 \alpha} \sqrt{2} \sin \beta < \varepsilon,$$

by lemma 3.3, the relation $\|P_j - I\| \leq \sqrt{2} \sin \beta$, and our choice (3.16) of $\beta$.

Applying lemma 3.3 to these data $(\varepsilon_0, \kappa, x = y, n = m, X_j, B_j, \hat{R}_j)$ we obtain an $(\varepsilon_0, \kappa)$-realizable sequence $\{L_0, \ldots , L_{m-1}\}$ at the point $y$, with $L_j|_{X_j} = Df_{f(y)}|_{X_j}$ and

$$L_j/X_j = (DF_y^j)/X_j \hat{R}_j = (DF_y^{j+1}/X_j)P_j(DF_y^j/X_0)^{-1}.$$

Let $L = L_{m-1} \cdots L_0$. Then $L/X_0 = (DF_y^m/X_0)P_{m-1} \cdots P_0$. In particular, by (3.22),

$$L(w_0 + X_0) = (DF_y^m/X_0)(w_0 + X_0) = Df_{f(y)}^m(w_0) + X_m.$$

Recall that $X_m = G_m \oplus H_m$ by definition. Then we may write

$$L(w_0) = Df_{f(y)}^m(w_0) + u_m + u'_m$$

with $u_m \in G_m$ and $u'_m \in H_m$. Let $u_0 = (Df_{f(y)}^m)^{-1}(u_m) \in G_0 \subset X_0 \cap E_0$. Since $L$ equals $Df_{f(y)}^m$ on $X_0$, we have $L(u_0) = u_m$. This means that the vector $v = v_0 - u_0 \in E_0$ is sent by $L$ to the vector $Df_{f(y)}^m(w_0) + u'_m \in F_m$. This finishes the third and last case of proposition 3.1.

Now we are left to give the proof of lemma 2.8. Recall that $X_j = G_j \oplus H_j$, $G_j \subset F_j$, and $H_j \subset F_j$, $v_j \in E_j$ and $w_j \in F_j$, and $v_j \perp G_j$ and $w_j \perp H_j$. Hence, using (3.20),

$$\langle (X_j, v_j) = \langle (H_j, v_j) \geq \langle (F_j, v_j) \geq \alpha \quad \text{and} \quad \langle (X_j \oplus \mathbb{R}v_j, w_j) = \langle (\mathbb{R}v_j \oplus G_j, w_j) \geq \alpha \langle (E_j, F_j) \geq \alpha$$

Using lemma 2.6 with $A = X_j$, $B = \mathbb{R}v_j$, $C = \mathbb{R}w_j$, we deduce the following lower bound for the angle between the spaces $X_j$ and $Y_j = \mathbb{R}v_j \oplus \mathbb{R}w_j$:

$$\sin \langle (X_j, Y_j) \geq \sin \langle (X_j, v_j) \sin \langle (\mathbb{R}v_j \oplus X_j, w_j) \geq \sin^2 \alpha.$$

Let $\pi_j : Y_j \to (T_{f(y)}^j)M/X_j$ be the canonical map $\pi_j(w) = w + X_j$. Then $\pi_j$ is an isomorphism, $\|\pi_j\| = 1$ and

$$\|\pi_j^{-1}\| = 1/\sin \langle (Y_j, X_j) \leq 1/\sin^2 \alpha.$$ (3.23)

(the quotient space has the norm that makes $X_j^j \ni w \mapsto w + X_j$ an isometry).

Now let $p_j : T_{f(y)}^jM \to Y_j$ be the projection onto $Y_j$ associated to the splitting $T_{f(y)}^jM = X_j \oplus Y_j$. Let $D_j : Y_j \to Y_{j+1}$ be given by $D_j = p_{j+1} \circ (Df_{f(y)}^j|_{Y_j})$. Define

$$D^{(j)} : Y_0 \to Y_j \quad \text{by} \quad D^{(j)} = D_{j-1} \circ \cdots \circ D_0 = p_j \circ (Df_{f(y)}^j|_{Y_0}).$$

We claim that the following inequalities hold:

$$\frac{1}{2K} \leq \frac{\|D^{(j)}(w_0)\|}{\|D^{(j)}(v_0)\|} \leq K \quad \text{for every} \ j \geq 0 \ j \leq m.$$ (3.24)
To prove this, consider the matrix of $D_j$ relative to bases $\{v_j, w_j\}$ and $\{v_{j+1}, w_{j+1}\}$:

$$D_j = \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}.$$ 

Then $\|D^{(j)}(v_0)\| = |a_{j-1} \cdots a_0|$ and $\|D^{(j)}(w_0)\| = |b_{j-1} \cdots b_0|$, since $v_j$ and $w_j$ are unit vectors. Moreover, for $0 \leq i < j \leq m$ we have

$$|a_{j-1} \cdots a_i| = \|p_j \circ D f^{j-i}_f(y)(v_i)\| = \|p_i \circ D f^{j-i}_f(y)(v_j)\|^{-1},$$

$$|b_{j-1} \cdots b_i| = \|p_j \circ D f^{j-i}_f(y)(w_i)\| = \|p_i \circ D f^{j-i}_f(y)(w_j)\|^{-1}.$$ 

Recall that $v_s \in E_s$ and $w_s \in F_s$ for all $s$. When restricted to $E_s$ (or $F_s$), the map $p_s$ is the orthogonal projection to the direction of $v_s$ (or $w_s$). In particular, $\|p_i|_{E_i}\| = \|p_j|_{F_j}\| = 1$ and so

$$|a_{j-1} \cdots a_i| \geq \|D f^{j-i}_f(y)|_{E_i}\|^{-1} = m(D f^{j-i}_f|_{E_i}) \quad \text{and} \quad |b_{j-1} \cdots b_i| \leq \|D f^{j-i}_f|_{F_i}\|.$$ 

Using (3.21), we obtain that

$$\frac{|b_{j-1} \cdots b_i|}{|a_{j-1} \cdots a_i|} \leq K \quad \text{for all } 0 \leq i \leq k \leq m. \quad (3.25)$$

Taking $i = 0$ gives the upper inequality in (3.24). For the same reasons, and the definitions of $v_0 = \xi$ and $w_m = \eta' = D f^m_y(\eta)/\|D f^m_y(\eta)\|$, we also have

$$|a_{m-1} \cdots a_0| \leq \|D f^m_y(v_0)\| = \|D f^m_y(\xi)\| = m(D f^m_y|_{E_0}),$$

$$|b_{m-1} \cdots b_0| \geq \|D f^m_y(w_m)\|^{-1} = \|D f^m_y(\eta)\| = \|D f^m_y|_{F_0}\|.$$ 

Now (3.17) translates into

$$\frac{|b_{m-1} \cdots b_0|}{|a_{m-1} \cdots a_0|} > \frac{1}{2}.$$ 

Combining this inequality and (3.25), with $i, j$ replaced by $j, m$, we find

$$\frac{|b_{j-1} \cdots b_0|}{|a_{j-1} \cdots a_0|} = \frac{|b_{m-1} \cdots b_0|}{|a_{m-1} \cdots a_0|} / \frac{|b_{m-1} \cdots b_j|}{|a_{m-1} \cdots a_j|} > \frac{1/2}{K},$$

which is the remaining inequality in (3.24).

Now, combining lemma 2.8 with (3.24) and $\langle\langle v_s, w_s \rangle\rangle \geq \alpha$, we get

$$\frac{\|D^{(j)}\|}{m(D^{(j)})} \leq \frac{8K}{\sin^2 \alpha}.$$ 

Moreover, $D f^j_y/X_0 = \pi_j \circ D^{(j)}\circ \pi^{-1}_0$. So, using the relation (3.23),

$$\frac{\|D f^j_y/X_0\|}{m(D f^j_y/X_0)} \leq \frac{\|\pi_j\|}{m(\pi_j)} \cdot \frac{\|D^{(j)}\|}{m(D^{(j)})} \cdot \frac{\|\pi_0\|}{m(\pi_0)} \leq \frac{8K}{\sin^6 \alpha}.$$ 

This finishes the proof of lemma 3.8. \[
\square
\]

The proof of proposition 3.1 is now complete. \[
\square
\]
4 Proof of theorems 1 and 2

Let us define some useful invariant sets. Given \( f \in \text{Diff}_\mu^1(M) \), let \( \mathcal{O}(f) \) be the set of the regular points, in the sense of the theorem of Oseledec. Given \( p \in \{1, \ldots, d-1\} \) and \( m \in \mathbb{N} \), let \( \mathcal{D}_p(f, m) \) be the set of points \( x \) such that there is an \( m \)-dominated splitting of index \( p \) along the orbit of \( x \). That is, \( x \in \mathcal{D}_p(f, m) \) if and only if there exists a splitting \( T_{f^n_x}M = E_n \oplus F_n \) \((n \in \mathbb{Z})\) such that for all \( n \in \mathbb{Z} \), \( \dim E_n = p \), \( DF_{f^n_x}(E_n) = E_{n+1} \) and

\[
\frac{\|DF_{f^n_x}(x)|E_n\|}{m(DF_{f^n_x}(x)|E_n)} \leq \frac{1}{2}.
\]

By section 2.2 \( \mathcal{D}_p(f, m) \) is a closed set. Let

\[
\Gamma_p(f, x) = \bigcap_{m \in \mathbb{N}} \Gamma_p(f, m), \quad \Gamma_p^x(f, x) = \bigcap_{m \in \mathbb{N}} \Gamma_p^x(f, m).
\]

It is clear that all these sets are invariant under \( f \).

**Lemma 4.1.** For every \( f \) and \( p \), the set \( \Gamma_p^x(f, \infty) \) contains no periodic points of \( f \). In other words, \( \bigcap_{m \in \mathbb{N}} (\Gamma_p^x(f, m) \setminus \Gamma_p^x(f, m)) = \emptyset \).

**Proof.** Suppose that \( x \in \mathcal{O}(f) \) is periodic, of period \( n \), and \( \lambda_p(f, x) > \lambda_{p+1}(f, x) \). The eigenvalues of \( DF_{f^n_x} \) are \( \nu_1, \ldots, \nu_d \), with \( |\nu_i| = e^{n\lambda_i(f, x)} \). Let \( E \) (resp. \( F \)) be the sum of the eigenspaces of \( DF_{f^n_x} \) associated to the eigenvalues \( \nu_1, \ldots, \nu_p \) (resp. \( \nu_{p+1}, \ldots, \nu_d \)). Then the splitting \( T_{f^n_x}M = E \oplus F \) is \( DF_{f^n_x} \)-invariant. Spreading it along the orbit of \( x \), we obtain a dominated splitting. That is, \( x \in \mathcal{D}_p(f, m) \) for some \( m \in \mathbb{N} \), and so \( x \not\in \Gamma_p^x(f, \infty) \).

4.1 Lowering the norm along an orbit segment

Recall that we write \( \Lambda_p(f, x) = \lambda_1(x) + \cdots + \lambda_p(x) \) for each \( p = 1, \ldots, d \).

**Proposition 4.2.** Let \( f \in \text{Diff}_\mu^1(M) \), \( \varepsilon_0 > 0 \), \( \kappa > 0 \), \( \delta > 0 \) and \( p \in \{1, \ldots, d-1\} \). Then, for every sufficiently large \( m \in \mathbb{N} \), there exists a measurable function \( N : \Gamma_p^x(f, m) \to \mathbb{N} \) with the following properties: For almost every \( x \in \Gamma_p^x(f, m) \) and every \( n \geq N(x) \) there exists an \((\varepsilon_0, \kappa)\)-realizable sequence \( \{\tilde{L}^{(x,n)}_0, \ldots, \tilde{L}^{(x,n)}_{n-1}\} \) at \( x \) of length \( n \) such that

\[
\frac{1}{n} \log \|\bigwedge^p \tilde{L}^{(x,n)}_{n-1} \cdots \tilde{L}^{(x,n)}_0\| \leq \frac{\Lambda_{p-1}(f, x) + \lambda_{p+1}(f, x)}{2} + \delta.
\]

**Proof.** Fix \( f \), \( \varepsilon_0 \), \( \kappa \), \( \delta \) and \( p \). For clearness, we divide the proof into two parts:
Proposition 3.1 holds for $f$ and $\bigcap_{n \in \mathbb{N}}$ decreasing sequence and their union is a full measure subset of $\Gamma$. Define $q$ for there exists $\ell/n$ define the function $N$ and let $\Gamma = N \setminus \{\}$. Let $\Gamma = \Gamma_{p,m}$. We may suppose that $\mu(\Gamma) > 0$, otherwise there is nothing to prove. Consider the splitting $T_{\Gamma} M = E \oplus F$, where $E$ is the sum of the Oseledets subspaces corresponding to the first $p$ Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_p$ and $F$ is the sum of the subspaces corresponding to the other exponents $\lambda_{p+1} \geq \cdots \geq \lambda_d$. This makes sense since $\lambda_p > \lambda_{p+1}$ on $\Gamma$. Let $A \subset \Gamma$ be the set of points $y$ such that the non-domination condition (3.17) holds. By definition of $\Gamma = \Gamma_{p,m}$, we have

$$\Gamma = \bigcup_{n \in \mathbb{Z}} f^n(A).$$

Let $\lambda_i^p(x), 1 \leq i \leq p$ denote the Lyapunov exponents of the cocycle $\wedge^p(Df)$ over $f$, in non-increasing order. Let $V_x$ denote the Oseledets subspace associated to the upper exponent $\lambda_1^p(x)$ and let $H_x$ be the sum of all other Oseledets subspaces. This gives us a splitting $\wedge^p(TM) = V \oplus H$. By proposition 2.1, we have

$$\lambda_1^p(x) = \lambda_1(x) + \cdots + \lambda_{p-1}(x) + \lambda_p(x),$$
$$\lambda_2^p(x) = \lambda_1(x) + \cdots + \lambda_{p-1}(x) + \lambda_{p+1}(x).$$

If $x \in \Gamma$ then $\lambda_p(x) > \lambda_{p+1}(x)$ and so $\lambda_1^p(x) > \lambda_2^p(x)$. That is, the subspace $V_x$ is one-dimensional.

For almost every $x \in \Gamma$, Oseledets’ theorem gives $Q(x) \in \mathbb{N}$ such that for all $n \geq Q(x), x$ have:

1. $\frac{1}{n} \log \|\wedge^p(Df^x) v\| < \lambda_1^p(x) + \delta$ for every $v \in V_x \setminus \{0\}$;
2. $\frac{1}{n} \log \|\wedge^p(Df^x) w\| < \lambda_2^p(x) + \delta$ for every $w \in H_x \setminus \{0\}$;
3. $\frac{1}{n} \log \sin \langle V_{f^n x}, H_{f^n x} \rangle > -\delta$.

For $q \in \mathbb{N}$, let $B_q = \{x \in \Gamma; Q(x) \leq q\}$. Then $B_q \uparrow \Gamma$, that is, the $B_q$ form a non-decreasing sequence and their union is a full measure measure subset of $\Gamma$. Define $C_0 = \emptyset$ and

$$C_q = \bigcup_{n \in \mathbb{Z}} f^n(A \cap f^{-m}(B_q)).$$

Since $f^{-m}(B_q) \uparrow \Gamma$ and $\bigcup_{n \in \mathbb{Z}}$, we have $C_q \uparrow \Gamma$. To prove the proposition we must define the function $N$ on $\Gamma$. We are going to define it on each of the sets $C_q \setminus C_{q-1}$ separately. From now on, let $q \in \mathbb{N}$ be fixed.

We need the following recurrence result, proved in lemma 3.12.

**Lemma 4.3.** Let $f \in \text{Diff}_+^1(M)$. Let $A \subset M$ be a measurable set with $\mu(A) > 0$, and let $\Gamma = \bigcup_{n \in \mathbb{Z}} f^n(A)$. Fix any $\gamma > 0$. Then there exists a measurable function $N_0 : \Gamma \to \mathbb{N}$ such that for almost every $x \in \Gamma$, and for all $n \geq N_0(x)$ and $t \in (0,1)$, there exists $\ell \in \{0,1,\ldots,n\}$ such that $t - \gamma \leq \ell/n \leq t + \gamma$ and $f^\ell(x) \in A$.  

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Let $c$ be a strict upper bound for $\log \|\wedge^p(Df)\|$ and $\gamma = \min\{c^{-1}\delta, 1/10\}$. Using (1.2) and lemma 4.3, we find a measurable function $N^0(x) : C_q \to \mathbb{N}$ such that for almost every $x \in C_q$, every $n \geq N^0(x)$ and every $t \in (0, 1)$ there is $\ell \in \{0, 1, \ldots, n\}$ with $|\ell/n - t| < \gamma$ and $f^\ell(x) \in A \cap f^{-m}(B_q)$. We define $N(x)$ for $x \in C_q \setminus C_{q-1}$ as the least integer such that

$$N(x) \geq \max\{N^0(x), 10Q(x), m\gamma^{-1}, \delta^{-1}\log[4/\sin<(V_z, H_x)>]\}.$$ 

Now fix a point $x \in C_q \setminus C_{q-1}$ and $n \geq N(x)$. We will now construct the sequence $\{\hat{L}^{(x,n)}\}$. Since $n \geq N^0(x)$, there exists $\ell \in \mathbb{N}$ such that

$$\left|\frac{\ell}{n} - \frac{1}{2}\right| < \gamma \quad \text{and} \quad y = f^\ell(x) \in A \cap f^{-m}(B_q).$$

Since $y \in A$, where the non-domination condition (3.17) holds, proposition 3.1 gives a sequence $\{L_0, \ldots, L_{m-1}\}$ which is $(\varepsilon_0, 1/\kappa)$-realizable, such that there are non-zero vectors $v_0 \in E_y$, $w_0 \in F_{f^m(y)}$ for which

$$L_{m-1} \ldots L_0(v_0) = w_0.$$ 

We form the sequence $\{\hat{L}^{(x,n)}_0, \ldots, \hat{L}^{(x,n)}_{n-1}\}$ of length $n$ by concatenating

$$\{Df^i(x); \ 0 \leq i < \ell\}, \quad \{L_0, \ldots, L_{m-1}\}, \quad \{Df^{i+m}(x); \ \ell + m \leq i < m\}.$$ 

According to parts 1 and 2 of lemma 2.1, the concatenation is an $(\varepsilon_0, \kappa)$-realizable sequence at $x$.

**Part 2:** Estimation of $\|\wedge^p(\hat{L}^{(x,n)}_{n-1} \ldots \hat{L}^{(x,n)}_0)\|$.

Write $\wedge^p(\hat{L}^{(x,n)}_{n-1} \ldots \hat{L}^{(x,n)}_0) = D_1LD_0$, with $D_0 = \wedge^p(Df^\ell)$, $D_1 = \wedge^p(Df^{\ell-m})$, and $L = \wedge^p(L_{m-1} \ldots L_0)$. The key observation is:

**Lemma 4.4.** The map $L : \wedge^p(T_y M) \to \wedge^p(T_{f^m(y)} M)$ satisfies $L(V_y) \subset H_{f^m(y)}$.

**Proof.** Proposition 2.1 describes the spaces $V$ and $H$. Let $z \in G$ and consider a basis $\{e_1(z), \ldots, e_d(z)\}$ of $T_z M$ such that

$$e_i(z) \in E^1_x \quad \text{for dim} \ E^1_x + \cdots + \text{dim} \ E^j_x < i \leq \text{dim} \ E^1_x + \cdots + \text{dim} \ E^j_x.$$ 

Then $V_z$ is the space generated by $e_1(z) \wedge \cdots \wedge e_p(z)$ and $H_z$ is generated by the vectors $e_{i_1}(z) \wedge \cdots \wedge e_{i_p}(z)$ with $1 \leq i_1 < \cdots < i_p \leq d$, $i_p > p$. Also notice that $\{e_1(z), \ldots, e_p(z)\}$ and $\{e_{p+1}(z), \ldots, e_d(z)\}$ are bases for the spaces $E_z$ and $F_z$, respectively. Consider the vectors $v_0 \in E_y$ and $w_0 = L(v_0) \in F_{f^m(y)}$, where $L = L_{m-1} \ldots L_0$. There is $\nu \in \{1, \ldots, p\}$ such that

$$\{v_0, e_1(y), \ldots, e_{\nu-1}(y), e_{\nu+1}(y), \ldots, e_p(y)\}$$

is a basis for $E_y$. Therefore $V_y$ is generated by the vector

$$v_0 \wedge e_1(y) \wedge \cdots \wedge e_{\nu-1}(y) \wedge e_{\nu+1}(y) \wedge \cdots \wedge e_p(y),$$
which is mapped by \( \mathcal{L} = \wedge^p(L) \) to
\[
 w_0 \wedge Le_1(y) \wedge \cdots \wedge Le_{\nu-1}(y) \wedge Le_{\nu+1}(y) \wedge \cdots \wedge Le_p(y),
\]
(4.3)
Write \( w_0 \) as a linear combination of vectors \( e_{p+1}(f^m(y)), \ldots, e_d(f^m(y)) \) and write each \( Le_i(y) \) as a linear combination of vectors \( e_1(f^m(y)), \ldots, e_d(f^m(y)) \). Substituting in (4.3), we get a linear combination of \( e_{i_1}(f^m(y)) \wedge \cdots \wedge e_p(f^m(y)) \) where \( e_1(f^m(y)) \wedge \cdots \wedge e_p(f^m(y)) \) does not appear. This proves that the vector in (4.3) belongs to \( H_{f^m(y)} \).

To carry on the estimates, we introduce a more convenient norm: For \( x_0, x_1 \in \Gamma \) we represent a linear map \( T : \wedge^p(T_{x_0}M) \to \wedge^p(T_{x_1}M) \) by its matrix
\[
 T = \begin{pmatrix} T^{++} & T^{+-} \\ T^{-+} & T^{--} \end{pmatrix}
\]
with respect to the splittings \( T_{x_0}M = V_{x_0} \oplus H_{x_0} \) and \( T_{x_1}M = V_{x_1} \oplus H_{x_1} \). Then we define
\[
 \|T\|_{\text{max}} = \max \left\{ \|T^{++}\|, \|T^{+-}\|, \|T^{-+}\|, \|T^{--}\| \right\}.
\]
The following elementary lemma relates this norm with the original one \( \|T\| \) (that comes from the metric in \( \wedge^p(T_{\Gamma}M) \)).

**Lemma 4.45.** Let \( \theta_{x_0} = \angle(V_{x_0}, H_{x_0}) \) and \( \theta_{x_1} = \angle(V_{x_1}, H_{x_1}) \). Then:
1. \( \|T\| \leq 4 (\sin \theta_{x_0})^{-1} \|T\|_{\text{max}} \);
2. \( \|T\|_{\text{max}} \leq (\sin \theta_{x_1})^{-1} \|T\| \).

**Proof.** Let \( v = v_+ + v_- \in V_{x_0} \oplus H_{x_0} \). We have \( \|v_*\| \leq \|v\|/\sin \theta_{x_0} \) for \( * = + \) and \( * = - \). So
\[
 \|Tv\| \leq \|T^{++}v_+\| + \|T^{+-}v_-\| + \|T^{-+}v_+\| + \|T^{--}v_-\| \leq 4\|T\|_{\text{max}}\|v\|/\sin \theta_{x_0}.
\]
This proves part 1. The proof of part 2 is similar. Let \( v_+ \in V_{x_0} \). Its image splits as \( Tv_+ = T^{++}v_+ + T^{-+}v_+ \in V_{x_1} \oplus H_{x_1} \). Hence,
\[
 \|T^{*+}v_+\| \leq \|Tv_+\| (\sin \theta_{x_1})^{-1} \leq \|T\|\|v_+\| (\sin \theta_{x_1})^{-1}
\]
for \( * = + \) and \( * = - \). Together with a corresponding estimate for \( T^{*-}v_- \), this gives part 2.

For the linear maps we were considering, the matrices have the form:
\[
 D_i = \begin{pmatrix} D_i^{++} & 0 \\ 0 & D_i^{--} \end{pmatrix}, \ i = 0, 1, \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 0 & D_0^{++} \\ D_1^{-+} & D_1^{--} \end{pmatrix}:
\]
\( D_i^{+-} = 0 = D_i^{-+} \) because \( V \) and \( H \) are \( \wedge^p(Df) \)-invariant, and \( D^{++} = 0 \) because of Lemma 4.4. Then
\[
 \wedge^p(\tilde{L}_{n-1} \cdots \tilde{L}_0) = \begin{pmatrix} 0 & D_0^{+0} \\ D_1^{-0} & D_1^{0+} \end{pmatrix}.
\]
(4.4)
Lemma 4.6. For \( i = 0, 1, \ x \in C_q \setminus C_{q-1} \) and \( n \geq N(x) \),
\[
\log \| D_i^{++} \| < \frac{1}{2} n (\lambda_1^{\lambda_p}(x) + 5\delta) \quad \text{and} \quad \log \| D_i^{--} \| < \frac{1}{2} n (\lambda_2^{\lambda_p}(x) + 5\delta).
\]

Proof. Since \( \ell > (\frac{1}{2} - \gamma)n > \frac{1}{10} n \geq \frac{1}{10} N(x) \geq Q(x) \), we have
\[
\log \| D_0^{++} \| = \log \| \wedge^p (D f_x^L)|_{V_x} \| < \ell (\lambda_1^{\lambda_p}(x) + \delta),
\]
\[
\log \| D_0^{--} \| = \log \| \wedge^p (D f_x^L)|_{H_x} \| < \ell (\lambda_2^{\lambda_p}(x) + \delta).
\]

Let \( \lambda \) be either \( \lambda_1^{\lambda_p}(x) \) or \( \lambda_2^{\lambda_p}(x) \). Using \( \gamma \lambda < \gamma c \leq \delta \) and \( \gamma < 1 \), we find
\[
\ell (\lambda + \delta) < n(\frac{1}{2} + \gamma)(\lambda + \delta) < n(\frac{1}{2} \lambda + \frac{1}{2} \delta + \delta + \delta) = \frac{1}{2} n (\lambda + 5\delta).
\]
This proves the case \( i = 0 \). We have \( n - \ell - m > n(\frac{1}{2} - \gamma) - \gamma n > \frac{1}{10} n \geq Q(x) \geq q \).

Also \( f^{\ell}(x) \in f^{-m}(B_q) \), and so \( Q(f^{\ell+m}(x)) \leq q \). Therefore
\[
\log \| D_1^{++} \| = \log \| \wedge^p (D f_{f^{\ell+m}})|_{V_{f^{\ell+m}}} \| < (n - \ell - m)(\lambda_1^{\lambda_p}(x) + \delta),
\]
\[
\log \| D_1^{--} \| = \log \| \wedge^p (D f_{f^{\ell+m}})|_{H_{f^{\ell+m}}} \| < (n - \ell - m)(\lambda_2^{\lambda_p}(x) + \delta).
\]
As before, \( (n - \ell - m)(\lambda + \delta) < n(\frac{1}{2} + \gamma)(\lambda + \delta) = \frac{1}{2} n (\lambda + 5\delta) \). This proves case \( i = 1 \). \( \square \)

Lemma 4.7. \( \log \| \mathcal{L} \|_{\text{max}} < 2n\delta. \)

Proof. Since the sequence \( \{L_0, \ldots , L_{m-1}\} \) is realizable, each \( L_j \) is close to the value of \( Df \) at some point. Therefore we may assume that \( \log \| \wedge^p (L_j) \| < c \). In particular,
\( \log \| \mathcal{L} \| < mc \leq n c \gamma \leq n \delta \). We have \( \ell + m \geq n(\frac{1}{2} - \gamma) \geq \frac{1}{10} n \geq Q(x) \). So
\( \log [1/ \sin \angle (V_{f^{\ell+m}}, H_{f^{\ell+m}})] < \delta \) and, by part 2 of lemma 4.4, \( \log \| \mathcal{L} \|_{\text{max}} < 2n\delta. \) \( \square \)

Using lemmas 4.4 and 4.7, we bound each of the entries in (4.4):
\[
\log \| D_1^{++} \mathcal{L}^{++} D_0^{--} \| < \frac{1}{2} n (\lambda_1^{\lambda_p}(x) + \lambda_2^{\lambda_p}(x) + 14\delta)
\]
\[
\log \| D_1^{--} \mathcal{L}^{--} D_0^{++} \| < \frac{1}{2} n (\lambda_1^{\lambda_p}(x) + \lambda_2^{\lambda_p}(x) + 14\delta)
\]
\[
\log \| D_1^{--} \mathcal{L}^{--} D_0^{--} \| < \frac{1}{2} n (2\lambda_2^{\lambda_p}(x) + 14\delta)
\]
The third expression is smaller than either of the first two, so we get
\[
\log \| \wedge^p (\hat{L}_{n-1} \cdots \hat{L}_0) \|_{\text{max}} < n \left( \frac{\lambda_1^{\lambda_p}(x) + \lambda_2^{\lambda_p}(x)}{2} + 7\delta \right).
\]
Therefore, by part 1 of lemma 4.3 and \( \log [4/ \sin \angle (V_x, H_x)] < n\delta \),
\[
\log \| \wedge^p (\hat{L}_{n-1} \cdots \hat{L}_0) \| < n \left( \frac{\lambda_1^{\lambda_p}(x) + \lambda_2^{\lambda_p}(x)}{2} + 8\delta \right).
\]
We also have \( \lambda_1^{\lambda_p}(x) + \lambda_2^{\lambda_p}(x) = \Lambda_{p-1}(f, x) + \Lambda_{p+1}(f, x) \). This proves proposition 4.2 (replace \( \delta \) with \( \delta/8 \) along the proof). \( \square \)
4.2 Globalization

The following proposition renders global the construction of proposition 4.2:

**Proposition 4.8.** Let \( f \in \text{Diff}^1_{\mu}(M) \), \( \varepsilon_0 > 0 \), \( p \in \{1, \ldots, d-1\} \) and \( \delta > 0 \). Then there exist \( m \in \mathbb{N} \) and a diffeomorphism \( g \in \mathcal{U}(f, \varepsilon_0) \) that equals \( f \) outside the open set \( \Gamma_p(f, m) \) and such that

\[
\int_{\Gamma_p(f, m)} \Lambda_p(g, x) \, d\mu(x) < \delta + \int_{\Gamma_p(f, m)} \frac{\Lambda_{p-1}(f, x) + \Lambda_{p+1}(f, x)}{2} \, d\mu(x).
\]

We need some preparatory terminology:

**Definition 4.9.** Let \( f \in \text{Diff}^1_{\mu}(M) \). An \( f \)-tower (or simply tower) is a pair of measurable sets \((T, T_b)\) such that there is a positive integer \( n \), called the height of the tower, such that the sets \( T_b, f(T), \ldots, f^{n-1}(T_b) \) are pairwise disjoint and their union is \( T \). \( T_b \) is called the base of the tower.

An \( f \)-castle (or simply castle) is a pair of measurable sets \((Q, Q_b)\) such that there exists a finite or countable family of pairwise disjoint towers \((T_i, T_{b_i})\) such that \( Q = \bigcup T_i \) and \( Q_b = \bigcup T_{b_i} \). \( Q_b \) is called the base of the castle.

A castle \((Q, Q_b)\) is a sub-castle of a castle \((Q', Q_b')\) if \( Q_b \subset Q_b' \) and for every point \( x \in Q_b \), if \( n \) (respectively \( n' \)) denotes the height of a tower of \((Q, Q_b)\) (respectively \((Q', Q_b')\)) that contains \( x \), then \( n = n' \). In particular, \( Q \subset Q' \).

We shall frequently omit reference to the base of a castle \( Q \) in our notations.

**Definition 4.10.** Given \( f \in \text{Diff}^1_{\mu}(M) \) and a positive measure set \( A \subset M \), consider the return time \( \tau : A \to \mathbb{N} \) defined by \( \tau(x) = \inf\{n \geq 1; f^n(x) \in A\} \). If we denote \( A_n = \tau^{-1}(n) \) then \( T_n = A_n \cup f(A_n) \cup \cdots \cup f^{n-1}(A_n) \) is a tower. Consider the castle \( Q \), with base \( A \), given by the union of the towers \( T_n \). \( Q \) is called the Kakutani castle with base \( A \).

Note that \( Q = \bigcup_{n \in \mathbb{Z}} f^n(A) \mod 0 \), in particular the set \( Q \) is invariant.

**Proof of proposition 4.8.** Let \( f, \varepsilon_0, p \) and \( \delta \) be given. For simplicity, we write

\[
\phi(x) = \frac{\Lambda_{p-1}(f, x) + \Lambda_{p+1}(f, x)}{2}.
\]

**Step 1:** Construction of families of castles \( \hat{Q}_i \supset Q_i \).

Let \( \kappa = \delta^2 \). Take \( m \in \mathbb{N} \) large enough so that the conclusion of proposition 4.2 holds: there exists a measurable function \( N : \Gamma^*_p(f, m) \to \mathbb{N} \) such that for a.e. \( x \in \Gamma^*_p(f, m) \) and every \( n \geq N(x) \) there exists a \((\varepsilon_0, \kappa)\)-realizable sequence \( \{\hat{L}^{(x,n)}_{0}, \ldots, \hat{L}^{(x,n)}_{n-1}\} \) at \( x \) of length \( n \) such that

\[
\frac{1}{n} \log \|\Lambda^p(\hat{L}^{(x,n)}_{n-1} \cdots \hat{L}^{(x,n)}_{0})\| \leq \phi(x) + \delta.
\]  \hspace{1cm} (4.5)

We shall also (see lemma 4.1) assume that \( m \) is large enough so that

\[
\mu(\Gamma^*_p(f, m) \setminus \Gamma^*_p(f, m)) < \delta.
\]  \hspace{1cm} (4.6)
Let $C > \sup_{g \in \mathcal{U}(f, \varepsilon_0)} \sup_{g \in \mathcal{M}} \log \| \wedge^p (Dg_y) \|$ and $\ell = \lceil C/\delta \rceil$. For $i = 1, 2, \ldots, \ell$, let

$$Z^i = \{ x \in \Gamma_p^*(f, m); \ (i - 1)\delta \leq \phi(x) < i\delta \}.$$ 

Each $Z_i$ is an $f$-invariant set. Since $\phi < C$, we have $\Gamma_p^*(f, m) = \bigsqcup_{i=1}^\ell Z^i$. Define the sets $Z_n^i = \{ x \in Z^i; \ N(x) \leq n \}$ for $n \in \mathbb{N}$ and $1 \leq i \leq \ell$. Obviously, $Z_n^i \uparrow Z^i$ when $n \to \infty$. Fix $H \in \mathbb{N}$ such that, for all $i = 1, 2, \ldots, \ell$,

$$\mu(Z^i \setminus Z_H^i) < \delta^2 \mu(Z^i). \quad (4.7)$$ 

Using the fact that $\Lambda_p(f)$ equals $\phi$ in the $f$-invariant set $\Gamma_p(f, m) \setminus \Gamma_p^*(f, m)$, and proposition [23], we may also assume that $H$ is large enough so that

$$\int_{\Gamma_p(f, m) \setminus \Gamma_p^*(f, m)} \frac{1}{n} \log \| \wedge^p (Df^n) \| < \delta + \int_{\Gamma_p(f, m) \setminus \Gamma_p^*(f, m)} \phi \quad (4.8)$$

for all $n \geq H$.

A measure preserving transformation is **aperiodic** if the set of periodic points has zero measure. The following result was proved in [3, Lemma 4.1]:

**Lemma 4.11.** For every aperiodic invertible measure preserving transformation $f$ on a probability space $X$, every subset $U \subset X$ of positive measure, and every $n \in \mathbb{N}$, there exists a positive measure set $V \subset U$ such that the sets $V, f(V), \ldots, f^n(V)$ are two-by-two disjoint. Besides, $V$ can be chosen maximal in the measure-theoretical sense: no set that includes $V$ and has larger measure than $V$ has the stated properties.

By definition of the set $\Gamma_p^*(f, m)$, the map $f : \Gamma_p^*(f, m) \to \Gamma_p^*(f, m)$ is aperiodic. So, by lemma [4.11], for each $i$ there is $B^i \subset Z^i_H$ such that $B^i, f(B^i), \ldots, f^{H-1}(B^i)$ are two-by-two disjoint and such that $B^i$ is maximal for these properties (in the measure-theoretical sense). Consider the following $f$-invariant set:

$$\hat{Q}^i = \bigsqcup_{n \in \mathbb{Z}} f^n(B^i).$$

$\hat{Q}^i$ is the Kakutani castle with base $B^i$. It is contained in $Z^i$ and, by the maximality of $B^i$, it contains $Z^i_H$ up to a zero measure subset. Thus, by (4.7),

$$\mu(Z^i \setminus \hat{Q}^i) < \delta^2 \mu(Z^i). \quad (4.9)$$

Let $Q^i \subset \hat{Q}^i$ be the sub-castle consisting of all the towers of $\hat{Q}^i$ with heights at most $3H$ floors. The following is a key property of the construction:

**Lemma 4.12.** For each $i = 1, 2, \ldots, \ell$, we have $\mu(\hat{Q}^i \setminus Q^i) \leq 3\mu(Z^i \setminus Z^i_H)$.

**Proof.** We split the castle $\hat{Q}^i$ into towers as $\hat{Q}^i = \bigsqcup_{k=H}^\infty T^i_k$ where $B^i = \bigsqcup_{k=H}^\infty B^i_k$ is the base $\hat{Q}^i_0$, and $T^i_k = \bigsqcup_{j=0}^{j-1} f^j(B^i_k)$ is the tower with base $B^i_k$ and of height $k$ floors. Take $k \geq 2H$ and $H \leq j \leq k - H$. The sets $f^j(B^i_k), \ldots, f^{j+H-1}(B^i_k)$ are disjoint and do not intersect $B^i \sqcup \cdots \sqcup f^{H-1}(B^i)$. Since $B^i$ is maximal, we conclude that

$$k \geq 2H \quad \text{and} \quad H \leq j \leq k - H \quad \Rightarrow \quad \mu(f^j(B^i_k) \cap Z^i_H) = 0.$$
(otherwise we could replace $B^i$ with $B^i \cup (f^j(B^i)) \cap Z_H^i$), contradicting the maximality of $B^i$. Thus

$$k \geq 2H \implies \mu(T_k^i \setminus Z_H^i) \geq \frac{k - 2H + 1}{k} \mu(T_k^i).$$

In particular,

$$k \geq 3H + 1 \implies \mu(T_k^i \setminus Z_H^i) > \frac{1}{3} \mu(T_k^i)$$

and so

$$\mu(\hat{Q}^i \setminus Q^i) = \sum_{k=3H+1}^{\infty} \mu(T_k^i) \leq \sum_{k=3H+1}^{\infty} 3\mu(T_k^i \setminus Z_H^i) = 3\mu \left( \bigcup_{k=3H+1}^{\infty} T_k^i \setminus Z_H^i \right) \leq 3\mu(Z^i \setminus Z_H^i),$$

as claimed.

**Step 2:** Construction of the diffeomorphism $g$.

**Lemma 4.13.** For almost every $x \in \Gamma^*_p(f, m)$ and every $n \geq N(x)$, there exists $r > 0$ such that for every ball $U = B_r(x)$ with $0 < r' < r$ there exist $h \in \mathcal{U}(f, \varepsilon_0)$ and a measurable set $K \subset B_r(x)$ such that

1. $h$ equals $f$ outside $\bigcup_{j=0}^{n-1} f^j(B^i_r(x))$;
2. $\mu(K) > (1 - \kappa)\mu(B_r(x))$;
3. if $y \in K$ then $\frac{1}{n} \log \|\wedge^p(Dh_{y})\| < \phi(x) + 2\delta$.

**Proof.** Fix $x$ and $n \geq N(x)$. Recall the point $x$ is not periodic. Let $\gamma > 0$ be very small. Since the sequence $\{\hat{L}_j^{(x,n)}\}$ given by proposition 4.2 is $(\kappa, \varepsilon_0)$-realizable, there exists $r > 0$ such that for every ball $U = B_r(x)$ with $0 < r' < r$ there exists $h \in \mathcal{U}(f, \varepsilon_0)$ satisfying condition (i) above and there exists $K \subset B_r(x)$ satisfying condition (ii) and

$$y \in K \text{ and } 0 \leq j \leq n - 1 \implies \|Dh_{h^jy} - \hat{L}_j^{(x,n)}\| < \gamma.$$ 

Taking $\gamma$ small enough, this inequality and (4.3) imply

$$y \in K \implies \frac{1}{n} \log \|\wedge^p(Dh_{y})\| < \frac{1}{n} \log \|\wedge^p(\hat{L}_{n-1}^{(x,n)} \cdots \hat{L}_0^{(x,n)})\| + \delta \leq \phi(x) + 2\delta,$$

as claimed in the lemma.

**Lemma 4.14.** Fix $\gamma > 0$. There exists $g \in \mathcal{U}(f, \varepsilon_0)$ and for each $i = 1, 2, \ldots, \ell$ there exist a $g$-castle $U^i$ and a $g$-sub-castle $K^i$ such that:

1. the $U^i$ are open, pairwise disjoint, and contained in $\Gamma^*_p(f, m)$;
(ii) \( \mu(U^i \setminus Q^i) < 2\gamma \mu(Z^i) \) and \( \mu(Q^i \setminus U^i) < 2\gamma \mu(Z^i) \);

(iii) \( \mu(U^i \setminus K^i) < 2\kappa \mu(Z^i) \);

(iv) \( g(U^i) = f(U^i) \) and \( g \) outside \( \bigcup_{i=1}^f U^i \);

(v) if \( y \) is in base of \( K^i \) and \( n(y) \) is the height of the tower of \( K^i \) that contains \( x \) then

\[
\frac{1}{n(y)} \log \|\wedge^p(Dg^n(y))\| < i\delta + 2\delta.
\]

Proof. By the regularity of the measure \( \mu \), one can find a compact sub-castle \( J^i \subset Q^i \) such that

\[
\mu(Q^i \setminus J^i) < \gamma \mu(\hat{Q}^i). \tag{4.10}
\]

Since the \( J^i \) are compact and disjoint we can find open pairwise disjoint castles \( V^i \) such that each \( V^i \) contains \( J^i \) as a sub-castle, is contained in the open and invariant set \( \Gamma_p(f, m) \), and

\[
\mu(V^i \setminus J^i) < \gamma \mu(\hat{Q}^i). \tag{4.11}
\]

For each \( x \in J_b^i \), let \( n(x) \) be the height of the tower that contains \( x \). \( J_b^i \) is contained in \( Z_H^i \), so \( N(x) \leq H \leq n(x) \). Let \( r(x) > 0 \) be the radius given by lemma 4.13, with \( n = n(x) \). This is defined for almost every \( x \in J_b^i \). Reducing \( r(x) \) if needed, we suppose that the ball \( \overline{B}_{r(x)}(x) \) is contained in the base of a tower in \( V^i \) (with the same height).

Using Vitali’s covering lemma \(^1\), we can find a finite collection of disjoint balls \( U_k^i = B_{r_{k,i}}(x_{k,i}) \), with \( x_{k,i} \in J_b^i \) and \( 0 < r_{k,i} < r(x_{k,i}) \), such that

\[
\mu(J_b^i \setminus \bigcup_k \overline{U_k^i}) < \gamma \mu(J_b^i). \tag{4.12}
\]

Let \( n_{k,i} = n(x_{k,i}) \). Notice that \( n(x) = n_{k,i} \) for all \( x^i \in U_k^i \).

Now we apply lemma 4.13 to each ball \( U_k^i \). We get, for each \( k \), a measurable set \( K_b^i \subset U_k^i \) and a diffeomorphism \( h_{k,i} \in \mathcal{U}(f, \varepsilon_0) \) such that (in 3 we use that \( x_{k,i} \in Z^i \))

1. \( h_{k,i} \) equals \( f \) outside the set \( \bigcup_{j=0}^{i-1} f^j(U_k^i) \);
2. \( \mu(K_b^i) > (1 - \kappa) \mu(U_k^i) \);
3. if \( y \in K_b^i \) then
\[
\frac{1}{n_{k,i}} \log \|\wedge^p(Dh_{k,i}^{n_{k,i}})\| < \phi(x_{k,i}) + 2\delta < i\delta + 2\delta.
\]

Let \( g \) be equal to \( h_{k,i} \) in the set \( \bigcup_{j=0}^{i-1} f^j(U_k^i) \), for each \( i \) and \( k \), and be equal to \( f \) outside. Since those sets are disjoint, \( g \in \text{Diff}_1^b(M) \) is a well-defined diffeomorphism. Each \( h_{k,i} \) belongs to \( \mathcal{U}(f, \varepsilon_0) \) and so \( g \) also does.

\(^1\) First, cover the basis \( J_b^i \) of the castle by chart domains.
Since each $U_k^i$ is contained in the base of a tower in the castle $V^i$, $V^i$ is also a castle for $g$. Let $U_j^i$ be the $g$-sub-castle of $V^i$ with base $\cup_k U_k^i$. Analogously, let $K^i$ be the $g$-sub-castle of $U^i$ with base $\cup_k K_k^i$.

It remains to prove claims (ii) and (iii) in the lemma. Making use of the castle structures, relation (4.12) and item 2 above imply, respectively,

$$\mu(J^i \setminus U^i) < \gamma \mu(J^i) \quad \text{and} \quad \mu(U^i \setminus K^i) < \kappa \mu(U^i).$$  \hspace{1cm} (4.13)

By (4.11) and $\hat{Q}^i \subset Z^i$,

$$\mu(U^i \setminus Q^i) < \mu(V^i \setminus J^i) < \gamma \mu(\hat{Q}^i) \leq \gamma \mu(Z^i).$$  \hspace{1cm} (4.14)

This implies the first part of item (ii). Combining the first part of (4.13) with (4.10),

$$\mu(Q^i \setminus U^i) < \mu(Q^i \setminus J^i) + \mu(J^i \setminus U^i) < 2\gamma \mu(\hat{Q}^i) \leq 2\gamma \mu(Z^i).$$

This proves the second part of item (ii). Finally, second inequality in (4.13) and

$$\mu(U^i) < \mu(Q^i) + \mu(U^i \setminus Q^i) < (1 + \gamma) \mu(\hat{Q}^i) < 2\mu(Z^i).$$

imply item (iii). The lemma is proved. \hfill \Box

**Step 3:** Conclusion of the proof of proposition 4.8.

Let $U = \bigsqcup_{i=1}^\ell U^i$ and $Q = \bigsqcup_{i=1}^\ell Q^i$ and $\hat{Q} = \bigsqcup_{i=1}^\ell \hat{Q}^i$. Set $N = H \delta^{-1}$. (Of course, we can assume that $\delta^{-1} \in \mathbb{N}$.) Let

$$G = \bigsqcup_{i=1}^\ell G^i \quad \text{where} \quad G^i = Z^i \cap \bigcap_{j=0}^{N-1} g^{-j}(K^i)$$

for each $i = 1, 2, \ldots, \ell$. The next lemma means that on $G$ we managed to reduce some time $N$ exponent:

**Lemma 4.15.** If $x \in G^i$ then

$$\frac{1}{N} \log \| \wedge^p (Dg_x^N) \| < i \delta + (6C + 2) \delta.$$

**Proof.** For $y \in K_b^i$, let $n(y)$ be the height of the $g$-tower containing $y$. Take $x \in G^i$, say $x \in G^i$. Since the heights of towers of $K^i$ are less than $3H$, we can write

$$N = k_1 + n_1 + n_2 + \cdots + n_j + k_2$$

so that $0 \leq k_1, k_2 < 3H$, $1 \leq n_1, \ldots, n_j < 3H$, and the points

$$x_1 = g^{k_1}(x), \quad x_2 = g^{k_1+n_1}(x), \quad \ldots, \quad x_{j+1} = g^{k_1+n_1+\cdots+n_j}(x)$$

are exactly the points of the orbit segment $x, g(x), \ldots, g^{N-1}(x)$ which belong to $K_b^i$.

Write

$$\| \wedge^p (Dg_x^N) \| \leq \| \wedge^p (Dg_x^{k_1}) \| \| \wedge^p (Dg_{x_1}^{n_1}) \| \cdots \| \wedge^p (Dg_{x_{j+1}}^{n_j}) \| \| \wedge^p (Dg_{x_{j+1}}^{k_2}) \|. \hspace{1cm} 36$$
Using item (v) of lemma 4.14, and our choice of \( N = H\delta^{-1} \), we get
\[
\log \| \wedge^p (Dg^N_z) \| < k_1 C + (n_1 + \cdots + n_j)(i\delta + 2\delta) + k_2 C \\
< 6HC + N(i\delta + 2\delta) < N(6C\delta + i\delta + 2\delta),
\]
as claimed. \( \square \)

We also use that \( G \) covers most of \( U \cup \Gamma^*_p(f,m) \), as asserted by the next lemma. Let us postpone the proof of this lemma for a while.

**Lemma 4.16.** Let \( \gamma = \delta^2/(\ell H) \) in lemma 4.14. Then \( \mu(U \cup \Gamma^*_p(f,m) \setminus G) < 12\delta \).

Continuing with the proof of proposition 4.8, write \( \psi(x) = \frac{1}{N} \log \| \wedge^p (Dg^N_z) \| \). Since \( g \) leaves invariant the set \( \Gamma^*_p(f,m) \), proposition 2.2 gives
\[
\int_{\Gamma^*_p(f,m)} \Lambda_p(g) \leq \int_{\Gamma^*_p(f,m)} \psi.
\]
We split the integral on the right hand side as
\[
\int_{\Gamma^*_p(f,m)} \psi = \int_{\Gamma^*_p(f,m) \setminus (U \cup \Gamma^*_p(f,m))} \psi + \int_{(U \cup \Gamma^*_p(f,m)) \setminus G} \psi + \int_G \psi = (I) + (II) + (III).
\]

Outside \( U \), \( g \) equals \( f \) and so \( \psi \) equals \( \frac{1}{N} \log \| \wedge^p (Df^N_z) \| \). Thus
\[
(I) \leq \int_{\Gamma^*_p(f,m) \setminus \Gamma^*_p(f,m)} \frac{1}{N} \log \| \wedge^p (Df^N_z) \| < \delta + \int_{\Gamma^*_p(f,m) \setminus \Gamma^*_p(f,m)} \phi,
\]
by (4.8). By lemma 4.16 and (4.8), \( \mu((U \cup \Gamma^*_p(f,m)) \setminus G) < 13\delta \). Since \( \psi < C \), we have \( (II) \leq 13C\delta \).

By lemma 4.15,
\[
(III) = \sum_{i=1}^{\ell} \int_{G^i} \psi \leq \sum_{i=1}^{\ell} (i\delta + (6C + 2)\delta) \mu(G^i) < (6C + 3)\delta + \sum_{i=1}^{\ell} (i - 1)\delta \mu(G^i).
\]

Since \( \phi \geq (i - 1)\delta \) inside \( Z^i \supset G^i \), we have
\[
(III) < (6C + 3)\delta + \int_{\Gamma^*_p(f,m)} \phi.
\]

Summing the three terms, we get the conclusion of proposition 4.8 (replace \( \delta \) with \( \delta/(18C + 4) \) throughout the arguments):
\[
\int_{\Gamma^*_p(f,m)} \Lambda_p(g) < (18C + 4)\delta + \int_{\Gamma^*_p(f,m)} \phi.
\]

This completes the proof of the proposition, modulo proving lemma 4.16. \( \square \)
**Step 4:** Proof of lemma 4.16.

The following observations will be useful in the proof: If $X \subset M$ is a measurable set and $N \in \mathbb{N}$, then

$$
\mu \left( \bigcup_{j=0}^{N-1} g^{-j}(X) \right) \leq \mu(X) + (N-1)\mu(g^{-1}(X) \setminus X). \tag{4.15}
$$

Moreover, $\mu(g^{-1}(X) \setminus X) = \mu(X \setminus g^{-1}(X))$.

**Proof of lemma 4.16.** We shall prove first that

$$
\mu(\hat{Q}^i \setminus G^i) < 10\delta \mu(Z^i). \tag{4.16}
$$

Since $\hat{Q}^i \subset Z^i$, we have $\hat{Q}^i \setminus G^i \subset \hat{Q}^i \cap \bigcup_{j=0}^{N-1} g^{-j}(M \setminus K^i)$. Substituting $M \setminus K^i \subset (U^i \setminus K^i) \cup (Q^i \setminus U^i) \cup (\hat{Q}^i \setminus Q^i) \cup (M \setminus \hat{Q}^i)$, we obtain

$$
\hat{Q}^i \setminus G^i \subset \bigcup_{j=0}^{N-1} g^{-j}(U^i \setminus K^i) \cup \bigcup_{j=0}^{N-1} g^{-j}(Q^i \setminus U^i) \cup \bigcup_{j=0}^{N-1} g^{-j}(\hat{Q}^i \setminus Q^i) \cup \left[ \hat{Q}^i \cap \bigcup_{j=1}^{N-1} g^{-j}(M \setminus \hat{Q}^i) \right] = (I) \cup (II) \cup (III) \cup (IV).
$$

Let us bound the measure of each of these sets. The second one is easy: by lemma 4.14(ii) and our choices $\gamma = \delta^2/\ell H$ and $N = H/\delta$,

$$
\mu(II) \leq N \mu(Q^i \setminus U^i) < 2N\gamma \mu(Z^i) < \delta \mu(Z^i).
$$

The other terms are more delicate.

The set $X_1 = U^i \setminus K^i$ is a $g$-castle whose towers have heights at least $H$. Hence its base, which contains $X_1 \setminus g(X_1)$, measures at most $\frac{1}{H} \mu(X_1)$. By (1.13), we get

$$
\mu(I) < \left( 1 + \frac{N}{H} \right) \mu(X_1) < 2\delta^{-1} \mu(X_1).
$$

By lemma 4.14(iii), we have $\mu(X_1) < 2\kappa \mu(Z^i) = 2\delta^2 \mu(Z^i)$. So, $\mu(I) < 4\delta \mu(Z^i)$. 

Let $X_3 = \hat{Q}^i \setminus Q^i$. By lemma 4.12 and (4.7), we have $\mu(X_3) < \delta^2 \mu(Z_i)$. Since $f$ and $g$ differ only in $U$, we have

$$
g(X_3) \setminus X_3 \subset [f(X_3) \setminus X_3] \cup g(X_3 \cap U) = (V) \cup (VI).
$$

Since $X_3$ is an $f$-castle whose towers have heights of at least $3H$,

$$
\mu(V) = \mu(X_3 \setminus f(X_3)) \leq \frac{1}{3H} \mu(X_3).
$$

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Since \( X_3 \cap U \subset \bigcup_k (U^k \setminus Q^k) \), lemma \[4.14\](ii) gives \( \mu(\text{VI}) \leq 2\ell \gamma \mu(Z_i) \). Combining the estimates of \( \mu(\text{V}), \mu(\text{VI}), \mu(X_3) \) with \[4.15\] and the definitions of \( N \) and \( \gamma \),

\[
\mu(\text{III}) < \mu(X_3) + N \left( \frac{1}{3H} \mu(X_3) + 2\ell \gamma \mu(Z_i) \right) < (1 + \frac{1}{3\delta}) \mu(X_3) + 2\delta \mu(Z_i) < 3\delta \mu(Z_i).
\]

We also have

\[
(IV) = \hat{Q}^i \setminus \bigcap_{j=1}^{N-1} g^{-j}(\hat{Q}^i) \subset \bigcup_{j=1}^{N-1} (g^{j-1}(\hat{Q}^i) \setminus g^{-j}(\hat{Q}^i))
\]

In particular, \( \mu(IV) \leq (N-1) \mu(\hat{Q}^i \setminus g^{-1}(\hat{Q}^i)) \). Notice that \( \hat{Q}^i \) is \( f \)-invariant. Therefore

\[
\hat{Q}^i \setminus g^{-1}(\hat{Q}^i) \subset [\hat{Q}^i \cap \bigcup_{k \neq i} U^k] \cup [U^i \setminus g^{-1}(\hat{Q}^i)] = (VII) \cup (VIII).
\]

Combining

\[
(VII) \subset \bigcup_{k \neq i} (U^k \setminus \hat{Q}^k) \subset \bigcup_{k \neq i} (U^k \setminus Q_k)
\]

with lemma \[4.14\](ii) we obtain \( \mu(VII) \leq 2(\ell - 1) \gamma \mu(Z_i) \). Using that \( g(U^i) = f(U^i) \) and \( \hat{Q}^i = f(\hat{Q}^i) \), we also get

\[
\mu(VIII) = \mu(g(U^i) \setminus \hat{Q}^i) = \mu(U^i \setminus \hat{Q}^i) \leq \mu(U^i \setminus Q_i) < 2\gamma \mu(Z_i).
\]

Altogether, \( \mu(\hat{Q}^i \setminus g^{-1}(\hat{Q}^i)) < 2\ell \gamma \mu(Z_i) \) and \( \mu(IV) \leq 2N \ell \gamma \mu(Q^i) < 2\delta \mu(Z_i) \).

Summing the four parts, we obtain \[4.16\]. Now

\[
\mu(U \cup \Gamma^*_p(f, m) \setminus G) \leq \mu(\Gamma^*_p(f, m) \setminus \hat{Q}) + \mu(U \setminus \hat{Q}) + \mu(\hat{Q} \setminus G)
\]

\[
= \mu(IX) + \mu(X) + \mu(XI).
\]

Using \[4.3\], lemma \[4.14\], and \[4.16\], respectively, we get

\[
\mu(IX) \leq \sum_i \mu(Z^i \setminus \hat{Q}^i) < \delta^2 < \delta,
\]

\[
\mu(X) \leq \mu(U \setminus Q) \leq \sum_i \mu(U^i \setminus \hat{Q}^i) < 2\gamma < \delta,
\]

\[
\mu(XI) \leq \sum_i \mu(\hat{Q}^i \setminus G) < 10\delta.
\]

Summing the three parts, we conclude the proof of lemma \[4.16\].
4.3 End of the proof of theorems 1 and 2

We give an explicit lower bound for the discontinuity “jump” of the semi-continuous function $\text{LE}_p(\cdot)$. Denote, for each $p = 1, \ldots, d$,

$$J_p(f) = \int_{\Gamma_p(f, \infty)} \frac{\lambda_p(f, x) - \lambda_{p+1}(f, x)}{2} \, d\mu(x).$$

**Proposition 4.17.** Given $f \in \text{Diff}^1(M)$ and $p \in \{1, \ldots, d - 1\}$, and given any $\varepsilon_0 > 0$ and $\delta > 0$, there exists a diffeomorphism $g \in \mathcal{U}(f, \varepsilon_0)$ such that

$$\int_M \Lambda_p(g, x) \, d\mu(x) < \int_M \Lambda_p(f, x) \, d\mu(x) - J_p(f) + \delta.$$

**Proof.** Let $f$, $p$, $\varepsilon_0$ and $\delta$ be as in the statement. Using proposition 4.8, we find $m \in \mathbb{N}$ and $g \in \mathcal{U}(f, \varepsilon_0)$ such that $g = f$ outside $\Gamma_p(f, m)$ and

$$\int_{\Gamma_p(f, m)} \Lambda_p(g) < \delta + \int_{\Gamma_p(f, m)} \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2}.$$

Then

$$\int_M \Lambda_p(g) = \int_{\Gamma_p(f, m)} \Lambda_p(g) + \int_{M \setminus \Gamma_p(f, m)} \Lambda_p(g) \leq \delta + \int_{\Gamma_p(f, m)} \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2} + \int_{M \setminus \Gamma_p(f, m)} \Lambda_p(f).$$

Since $\Gamma_p(f, \infty) \subset \Gamma_p(f, m)$, and the integrand is non-negative,

$$\int_{\Gamma_p(f, m)} \left( \Lambda_p(f) - \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2} \right) \geq \int_{\Gamma_p(f, \infty)} \left( \Lambda_p(f) - \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2} \right) = J_p(f).$$

Therefore, the previous inequality implies

$$\int_M \Lambda_p(g) < \delta - J_p(f) + \int_M \Lambda_p(f),$$

as we wanted to prove. $\square$

**Theorem 2 follows easily from proposition 4.17.**

**Proof of theorem 2.** Let $f \in \text{Diff}^1(M)$ be a point of continuity of $\text{LE}_p(\cdot)$ for all $p = 1, \ldots, d - 1$. Then $J_p(f) = 0$ for every $p$. This means that $\lambda_p(f, x) = \lambda_{p+1}(f, x)$ for almost every $x \in \Gamma_p(f, \infty)$. Let $x \in M$ be an Oseledets regular point. If all Lyapunov exponents of $f$ at $x$ vanish, there is nothing to prove. Otherwise, for any $p$ such that $\lambda_p(f, x) > \lambda_{p+1}(f, x)$, the point $x \notin \Gamma_p(f, \infty)$ (except for a zero measure set of $x$). This means that $x \in D_p(f, m)$ for some $m$: there is a dominated splitting
of index $p$, $T_{f^n}M = E_n \oplus F_n$, $n \in \mathbb{Z}$ along the orbit of $x$. Clearly, domination implies that $E_n$ is the sum of the Oseledets subspaces of $f$, at the point $f^n x$, associated to the Lyapunov exponents $\lambda_1(f, x), \ldots, \lambda_p(f, x)$, and $F_n$ is the sum of the spaces associated to the other exponents. Since this holds whenever $\lambda_p(f, x)$ is bigger than $\lambda_{p+1}(f, x)$, it proves that the Oseledets splitting is dominated at $x$. \hfill \square

Theorem 1 is an immediate consequence:

**Proof of theorem 1.** The function $f \mapsto LE_j(f)$ is semi-continuous for every $j = 1, \ldots, d - 1$, see section 2.1.3. Hence, there exists a residual subset $\mathcal{R}$ of Diff$_1^1\mu(M)$ such that any $f \in \mathcal{R}$ is a point of continuity for $f \mapsto (LE_1(f), \ldots, LE_{d-1}(f))$. By theorem 2, every point of continuity satisfies the conclusion of theorem 1. \hfill \square

5 Consequences of non-dominance for symplectic maps

Here we prove a symplectic analogue of proposition 3.1:

**Proposition 5.1.** Given $f \in \text{Sympl}_1^1(M), \varepsilon_0 > 0$ and $0 < \kappa < 1$, if $m \in \mathbb{N}$ is large enough then the following holds:

Let $y \in M$ be a non-periodic point and suppose we are given a non-trivial splitting $T_y M = E \oplus F$ into two Lagrangian spaces such that

$$\frac{\|Df^m y\|_F}{m(Df^m y|_E)} \geq \frac{1}{2}. \quad (5.1)$$

Then there exists a $(\varepsilon_0, \kappa)$-realizable sequence $\{L_0, \ldots, L_{m-1}\}$ at $y$ of length $m$ and there are non-zero vector $v \in E, w \in Df^m y(F)$ such that $L_{m-1} \cdots L_0(v) = w$.

**Remark 5.2.** The hypothesis that $E$ and $F$ are Lagrangian subspaces in proposition 5.1 is the sole reason why theorem 3 is weaker than what is stated in [4].

In subsections 5.1 and 5.2 we prove two results, namely, lemmas 5.3 and 5.8, that are used in subsection 5.3 to prove proposition 5.1. In section 6 we prove theorems 3 and 4 using proposition 5.1.

5.1 Symplectic realizable sequences of length 1

First, we recall some elementary facts: Let $(\cdot, \cdot)$ denote the usual hermitian inner product in $\mathbb{C}^q$. Up to identification of $\mathbb{C}^q$ with $\mathbb{R}^{2q}$, the standard inner product in $\mathbb{R}^{2q}$ is $\text{Re}(\cdot, \cdot)$ and the standard symplectic form in $\mathbb{R}^{2q}$ is $\text{Im}(\cdot, \cdot)$. The unitary group $U(q)$ is subgroup of GL$(q, \mathbb{C})$ formed by the linear maps that preserve the hermitian product. Viewing $R \in U(q)$ as a map $R : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$, then $R$ is both symplectic and orthogonal.

If $R : T_x M \to T_y M$ is a $\omega$-preserving linear map, we shall call $R$ unitary if it preserves the inner product in $T_x M$ induced from the Euclidean inner product in $\mathbb{R}^{2q}$ by the chart $\varphi_i(x)$ (recall subsection 2.5).

The next lemma constructs realizable sequences of length 1:
Lemma 5.3. Given $f \in \Sympl^1(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, there exists $\varepsilon > 0$ with the following properties: Suppose we are given a non-periodic point $x \in M$ and an unitary map $R : T_x M \to T_x M$ with $|R - I| < \varepsilon$. Then $\{Df_x R\}$ is an $(\varepsilon_0, \kappa)$-realizable sequence of length 1 at the point $x$ and $\{R Df_{f^{-1}(x)}\}$ is an $(\varepsilon_0, \kappa)$-realizable sequence of length 1 at the point $f^{-1}x$.

We need the following elementary lemma, whose proof is left to the reader:

Lemma 5.4. Let $H : \mathbb{R}^q \to \mathbb{R}$ be a smooth function such that the corresponding Hamiltonian flow $\varphi^t : \mathbb{R}^q \to \mathbb{R}^q$ is globally defined for every $t \in \mathbb{R}$. Let $\tau : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $\psi$ be a primitive of $\tau$. Define $\tilde{H} = \psi \circ H$. Then the Hamiltonian flow $(\tilde{\varphi}^t)$ of $\tilde{H}$ is globally defined and it is given by $\tilde{\varphi}^t(x) = \varphi^{\tau(H(x))t}(x)$.

If $R \in U(q)$ then all its eigenvalues belong to the unit circle in $\mathbb{C}$. Moreover, there exists an orthonormal basis of $C^q$ formed by eigenvectors of $R$. If $J \subset \mathbb{R}$ is an interval, we define $S_J$ as the set of matrices $R \in U(q)$ whose eigenvalues can be written as $e^{i\theta_1}, \ldots, e^{i\theta_q}$, with all $\theta_k \in J$. There is $C_0 > 0$, depending only on $q$, such that if $\varepsilon > 0$ and $R \in S_{(-\varepsilon, \varepsilon)}$ then $|R - I| < C_0 \varepsilon$. It is convenient to consider first the case where the arguments of the eigenvalues of $R$ have all the same sign:

Lemma 5.5. Given $\varepsilon_0 > 0$ and $0 < \sigma < 1$, there exists $\varepsilon > 0$ with the following properties: Given $R \in S_{(-\varepsilon, 0)} \cup S_{(0, \varepsilon)}$, there exists a bounded open set $U \subset \mathbb{R}^{2q}$ such that $\sigma U \subset U$, and there exists a $C^1$ symplectomorphism $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ such that

(i) $h(z) = z$ for every $z \notin U$ and $h(z) = R(z)$ for every $z \in \sigma U$;

(ii) $\|Dh_z - I\| < \varepsilon_0$ for all $z \in \mathbb{R}^{2q}$.

Proof of lemma 5.3. Let $\varepsilon_0$ and $\sigma$ be given. Let $\varepsilon > 0$ be a small number, to be specified later. Take $R \in S_{(0, \varepsilon)}$: the other possibility is tackled in a similar way. Let $\{v_1, \ldots, v_q\}$ be an orthonormal basis of eigenvectors of $R$, with associated eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_q}$, and all $0 < \theta_k < \varepsilon$. Up to replacing $R$ with $SRS^{-1}$, for some $S \in U(q)$, we may assume that the basis $\{v_1, \ldots, v_q\}$ coincides with the standard basis of $C^q$. Therefore $R$ assumes the form

$$R(z_1, \ldots, z_q) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_q} z_q)$$

Let $H : C^q \to \mathbb{R}$ be given by $H(z) = \frac{1}{2} \sum \theta_k |z_k|^2$. Then $R$ is the time 1 map of the Hamiltonian flow of $H$. Besides, there is $C_1 = C_1(q)$ such that

$$\|z\| \|DH_z\| \leq C_1 H(z) \quad \text{for all } z \in C^q. \quad (5.2)$$

Let $\tau : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\tau(s) = 1$ for $s \leq \sigma^2$, $\tau(t) = 0$ for $t \geq 1$, and $0 \leq -\tau'(t) \leq 2/(1 - \sigma^2)$ for all $t$. Let $\psi(s) = \int_0^s \tau(u) \, du$ and let $\tilde{H} = \psi \circ H$. By lemma 5.4, the time 1 map $h$ of the Hamiltonian flow of $\tilde{H}$ is

$$h(z) = (e^{i\theta_1 \tau(H(z))} z_1, \ldots, e^{i\theta_q \tau(H(z))} z_q)$$

Then $h(z) = R(z)$ if $H(z) \leq \sigma^2$ and $h(z) = z$ if $H(z) \geq 1$. Moreover, a direct calculation and (5.2) give

$$\|Dh_z - I\| \leq C_2 \varepsilon (1 - \sigma^2)^{-1} + \varepsilon \quad \text{for every } z \in C^q \text{ with } H(z) \leq 1,$$
where $C_2 = C_2(q)$. Take $\varepsilon = \varepsilon(\varepsilon_0, \sigma)$ such that the right hand side is less than $\varepsilon_0$. Since $H$ is definite positive, the set $U = \{ z \in \mathbb{C}^q; H(z) < 1 \}$ is bounded. \hfill \Box

**Remark 5.6.** We may assume that the set $U$ in lemma 5.3 has arbitrarily small diameter. Indeed, if $a > 0$ then we may replace $U$ with $\tilde{U} = aU$ and $h$ with $\tilde{h}(z) = ah(a^{-1}z)$. Notice $D\tilde{h}_z = Dh_{a^{-1}z}$, so $\tilde{h}$ is a symplectomorphism and satisfies property (ii) of the lemma.

**Lemma 5.7.** Given $\varepsilon_0 > 0$ and $0 < \sigma < 1$, there exists $\varepsilon > 0$ with the following properties: Given $R \in S_{(-\varepsilon, \varepsilon)}$, there exists a bounded open set $U \subset \mathbb{R}^{2q}$ such that $\sigma U \subset U$, a measurable set $K \subset U$ with $\text{vol}(U \setminus K) < 3(1 - \sigma^d) \text{vol}(U)$, and a $C^1$ symplectomorphism $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ such that

(i) $h(z) = z$ for every $z \notin U$ and $Dh_z = I$ for every $z \in K$;

(ii) $\|Dh_z - I\| < \varepsilon_0$ for all $z \in \mathbb{R}^{2q}$.

**Proof.** Any $R \in S_{(-\varepsilon, \varepsilon)}$ can be written as a product $R = R_+ R_-$, with $R_+ \in S_{(0, \varepsilon)}$ and $R_- \in S_{(-\varepsilon, 0)}$, in fact we may take $R_+$ and $R_-$ with the same eigenbasis as $R$. Applying lemma 5.3 to $R_+$, with $\varepsilon_0$ replaced by $\varepsilon_0/2$, we obtain sets $U_+$ and symplectomorphisms $h_+$, Let $U = U_+$. Consider the family $\mathcal{F}$ of all sets of the form $aU_- + b$, with $a > 0$ and $b \in \mathbb{R}^{2q}$, that are contained in $U$. This is a Vitali covering of $U$, so we may find a finite number of disjoint sets $U_i = a_iU_- + b_i \in \mathcal{F}$ that cover $U$ except for a set of volume $(1 - \sigma^d) \text{vol}(U)$. Using lemma 5.3 and remark 5.4, for each $i$ we find a symplectomorphism $h_i$ such that $h_i = \text{id}$ outside $U_i$ and $D(h_i)_z = R_-$ for all $z \in K_i = a_i\sigma U_- + b_i$, and $D(h_i)_z \text{ is uniformly close to } I$. Let $K = (\sigma U) \cap \bigcup_i K_i$. Define $h = h_+ \circ h_i$ inside each $U_i$, and $h = h_+$ outside. Then $K$ and $h$ have the desired properties. \hfill \Box

**Proof of lemma 5.3.** Given $\varepsilon_0$ and $\kappa$, choose $\sigma$ close to 1 so that $3(1 - \sigma^d) < \kappa$. Remark 5.6 also applies to lemma 5.3, so the set $U$ may be taken with arbitrarily small diameter. Using lemma 2.13, we conclude that the sequences $\{Df_{x,R}\}$ and $\{R Df_{f^{-1}(x)}\}$ are $(\varepsilon_0, \kappa)$-realizable as stated. \hfill \Box

### 5.2 Symplectic nested rotations

In this subsection we prove an analogue of lemma 3.3 for symplectic maps:

**Lemma 5.8.** Given $f \in \text{Symp}^1(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, $E > 1$, and $0 < \gamma \leq \pi/2$, there exists $\beta > 0$ with the following properties: Suppose we are given a non-periodic point $x \in M$, an iterate $n \in \mathbb{N}$, and a two-dimensional symplectic subspace $Y_0 \subset T_xM$ such that:

- $\|Df_j|_{Y_0}\|/\text{m}(Df_j|_{Y_0}) \leq E^2$ for every $j = 1, \ldots, n$;
- $< \langle X_j, Y_j \rangle \geq \gamma$ for each $j = 0, \ldots, n - 1$ where $X_0 = Y_0^\omega$, $X_j = Df^j_{x}(X_0)$, and $Y_j = Df^j_{x}(Y_0)$. 

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Let \( \theta_0, \ldots , \theta_{n-1} \in [-\beta, \beta] \) and let \( S_0, \ldots , S_{n-1} : Y_0 \to Y_0 \) be the rotations of the plane \( Y_0 \) by angles \( \theta_0, \ldots , \theta_{n-1} \). Let linear maps
\[
T_x M \xrightarrow{L_0} T_{f(x)} M \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} T_{f^n(x)} M
\]
be defined by \( L_j(v) = Df_{f^j(x)}(v) \) for \( v \in X_j \) and \( L_j(w) = (Df_{f^j(y)})^{-1}(w) \) for \( w \in Y_j \). Then \( \{L_0, \ldots , L_{n-1}\} \) is an \((\epsilon_0, \kappa)\)-realizable sequence of length \( n \) at the point \( x \).

We begin by proving a perturbation lemma that corresponds to Lemma 3.3.

**Lemma 5.9.** Given \( \epsilon_0 > 0 \) and \( 0 < \sigma < 1 \), there is \( \epsilon > 0 \) with the following properties: Suppose we are given: a splitting \( \mathbb{R}^2q = X \oplus Y \) with \( \dim Y = 2 \), \( X^w = Y \) and \( X \perp Y \), an ellipsoid \( A \subset X \) centered at the origin, and a unitary map \( R \in U(q) \) with \( R|_X = I \) and \( \|R-I\| < \epsilon \).

Then there exists \( \tau > 1 \) such that the following holds. Let \( B \) be the unit ball in \( Y \). For \( a, b > 0 \) consider the cylinder \( C = C_{a,b} = aA \oplus bB \). If \( a > \tau b \) and \( \text{diam} C < \epsilon_0 \) then there is a \( C^1 \) symplectomorphism \( h : \mathbb{R}^2q \to \mathbb{R}^2q \) satisfying:

1. \( h(z) = z \) for every \( z \notin C \) and \( h(z) = R(z) \) for every \( z \in \sigma C \);
2. \( \|h(z) - z\| < \epsilon_0 \) and \( \|Dh_z - I\| < \epsilon_0 \) for all \( z \in \mathbb{R}^{2q} \).

**Remark 5.10.** If \( H : \mathbb{R}^{2q} \to \mathbb{R} \) is a smooth function with bounded \( \|DH\| \) and \( \|D^2 H\| \), then the associated Hamiltonian flow \( \varphi^t : \mathbb{R}^{2q} \to \mathbb{R}^{2q} \) is defined for every time \( t \in \mathbb{R} \), and
\[
\|\varphi^t(z) - z\| \leq |t| \sup \|DH\|, \quad \|(D\varphi^t)_z - I\| \leq \exp(|t| \sup \|D^2 H\|) - 1.
\]
for every \( z \in \mathbb{R}^{2q} \) and \( t \in \mathbb{R} \).

**Proof of lemma 5.9.** Given \( \epsilon_0 \) and \( \sigma \), let
\[
K = 10(1 - \sigma)^{-2} + 20\sigma^{-1}(1 - \sigma)^{-1} + 30(1 - \sigma)^{-1} + 3.
\]
Fix \( \bar{t} > 0 \) such that \( e^{\bar{t}K} - 1 < \epsilon_0 \), and let \( \epsilon > 0 \) be such that \( \epsilon < \sqrt{2} \sin \bar{t} \). Let \( X, Y, A, B, \) and \( R \) be as in the statement. Let \( A : X \to X \) be a linear map such that \( A(A) \) is the unit ball in \( X \). We define \( \tau = \|A\| \).

Let \( H : \mathbb{R}^{2q} \to \mathbb{R} \) be defined by \( H(x, y) = H(y) = \frac{1}{2}y^2 \), where \( (x, y) \) are coordinates with respect to the splitting \( X \oplus Y \). The Hamiltonian flow of \( H \) is a linear flow \( (R_t)_t \), where \( R_t \) is a rotation of angle \( t \) in the plane \( Y \), with axis \( X \). In particular, \( \|R_t - I\| = \sqrt{2} \sin t \) and there exists \( t_0 \) with |\( t_0 | < \bar{t} \) such that \( R_{t_0} = R \).

Take numbers \( a, b > 0 \) such that \( a/b > \tau \) and the cylinder \( C = aA \oplus bB \) has diameter less than \( \epsilon_0 \). We are going to construct another Hamiltonian \( \tilde{H} \) which is equal to \( H \) inside \( \sigma C \) and constant outside \( C \). The symplectomorphism \( h \) will be defined as the time \( t_0 \) of the Hamiltonian flow associated to \( \tilde{H} \).

For this we need a few auxiliary functions. Let \( \zeta : \mathbb{R} \to [0,1] \) be a smooth function such that:
• \( \zeta(t) = 1 \) for \( t \leq \sigma \) and \( \zeta(t) = 0 \) for \( t \geq 1 \);

• \( |\zeta'(t)| \leq 10/(1 - \sigma) \) and \( |\zeta''(t)| \leq 10/(1 - \sigma)^2 \).

Let \( \hat{\psi} : X \to [0,1] \) be defined by \( \hat{\psi}(x) = \zeta(a^{-1}\|x\|) \), and \( \psi : X \to [0,1] \) be defined by \( \psi = \hat{\psi} \circ A \). It is clear that

\[
\psi(x) = 1 \text{ for } x \in \sigma aA \text{ and } \psi(x) = 0 \text{ for } x \notin aA. \quad (5.3)
\]

We estimate the derivatives:

\[
D\hat{\psi}_x(v) = a^{-1}\zeta'(a^{-1}\|x\|)\frac{\langle x, v \rangle}{\|x\|},
\]

\[
D^2\hat{\psi}_x(v, w) = a^{-2}\zeta''(a^{-1}\|x\|)\frac{\langle x, v \rangle \langle x, w \rangle}{\|x\|^2} + a^{-1}\zeta'(a^{-1}\|x\|)\left(\frac{\langle v, w \rangle - \langle x, v \rangle \langle x, w \rangle}{\|x\|^2}\right).
\]

Since \( |\zeta'(a^{-1}\|x\|)|/\|x\| \leq 10a(1 - \sigma)^{-1}\sigma^{-1} \), we get the bounds

\[
\|D\hat{\psi}\| \leq 10(1 - \sigma)^{-1}a^{-1} \text{ and } \|D^2\hat{\psi}\| \leq [10(1 - \sigma)^{-2} + 20\sigma^{-1}(1 - \sigma)^{-1}]a^{-2}.
\]

Moreover,

\[
\|D\hat{\psi}_x\| \leq \|A\| \|D\hat{\psi}_x\| \text{ and } \|D^2\hat{\psi}_x\| \leq \|A\|^2\|D^2\hat{\psi}_x\|.
\]

Now define \( \rho : \mathbb{R} \to \mathbb{R} \) by \( \rho(t) = \int_0^t \zeta \) and then let \( \phi : Y \to \mathbb{R} \) be given by \( \phi(y) = \frac{1}{2}b^2\rho(b^{-1}\|y\|)^2 \). Then

\[
\phi(y) = H(y) \text{ for } y \in \sigma bB \text{ and } \phi(y) = c \text{ for } y \notin bB, \quad (5.4)
\]

where \( 0 < c < \frac{1}{2}b^2 \) is a constant. The first and second derivatives of \( \phi \) are:

\[
D\phi_y(v) = b\rho(b^{-1}\|y\|)\rho'(b^{-1}\|y\|)\frac{\langle y, v \rangle}{\|y\|},
\]

\[
D^2\phi_y(v, w) = \left[\rho'(b^{-1}\|y\|)^2 + \rho(b^{-1}\|y\|)\rho''(b^{-1}\|y\|)\right]\frac{\langle y, v \rangle}{\|y\|} + b\rho(b^{-1}\|y\|)\frac{\rho'(b^{-1}\|y\|)}{\|y\|}\left(\frac{\langle v, w \rangle - \langle y, v \rangle \langle y, w \rangle}{\|y\|^2}\right).
\]

Since \( |\rho| \leq 1, |\rho'| \leq 1, |\rho''| \leq 10(1 - \sigma)^{-1} \), and \( |\rho'(b^{-1}\|y\|)|/\|y\| \leq b^{-1} \), we have

\[
\|D\phi\| \leq b \text{ and } \|D^2\phi\| \leq 3 + 10(1 - \sigma)^{-1}.
\]

Define \( \bar{H} : \mathbb{R}^2 \to \mathbb{R} \) by \( \bar{H}(x, y) = c - \psi(x)(c - \phi(y)) \). Then, by (5.3) and (5.4),

\[
x \in \sigma aA \quad \text{and} \quad y \in \sigma bB \Rightarrow \bar{H}(x, y) = H(y),
\]

\[
x \notin aA \text{ or } y \notin bB \Rightarrow \bar{H}(x, y) = c. \quad (5.5)
\]

The derivatives of \( \bar{H} \) are (write \( v = v_x + v_y \in X \oplus Y \) and analogously for \( w \))

\[
D\bar{H}_{(x,y)}(v) = -(c - \phi(y))D\hat{\psi}_x(v_x) + \psi(x)D\phi_y(v_y),
\]

\[
D^2\bar{H}_{(x,y)}(v, w) = -(c - \phi(y))D^2\hat{\psi}_x(v_x, w_x) + D\hat{\psi}_x(v_x)D\phi_y(w_y) + D\psi_x(w_x)D\phi_y(v_y) + \psi(x)D^2\phi_y(v_y, w_y).
\]
Using the previous bounds we obtain
\[ \|D^2 \tilde{H}\| \leq [10(1 - \sigma)^{-2} + 20\sigma^{-1}(1 - \sigma)^{-1}] \|A\|^2 (b/a)^2 + 20(1 - \sigma)^{-1}\|A\|(b/a) + 3 + 10(1 - \sigma)^{-1}. \]

Since \( a/b > \|A\| \), we conclude that \( \|D^2 \tilde{H}\| \leq K \).

Take \( h : \mathbb{R}^{2q} \to \mathbb{R}^{2q} \) to be the time \( t_0 \) map of the Hamiltonian flow associated to \( \tilde{H} \). Property (i) in the lemma follows from (5.5). Since diam\( C < \varepsilon_0 \), we have \( \|h(z) - z\| < \varepsilon_0 \) for all \( z \). And, by remark 5.10, \( \|Dh_z - I\| \leq e^{t_0 K} - 1 \leq e^{tK} - 1 < \varepsilon_0 \), proving (ii) and the lemma.

An ellipse \( B \) contained in a 2-dimensional symplectic subspace \( Y \subset \mathbb{R}^{2q} \) and centered at the origin has eccentricity \( E \) if it is the image of the unit ball under a linear transformation \( \tilde{B} : Y \to Y \) with \( \|\tilde{B}\|/m(\tilde{B}) = E^2 \). If a map \( \tilde{R} : Y \to Y \) preserves the ellipse \( B \), then \( \tilde{R}^{-1} \tilde{R} B \) is a rotation of the plane \( Y \) of some angle \( \theta \). In this case we say that \( \tilde{R} \) rotates the ellipse \( B \) through angle \( \theta \).

The following statement is a more flexible version of lemma 5.9. In fact, it follows from lemma 5.9 just by a change of the inner product.

**Lemma 5.11.** Given \( \varepsilon_0 > 0, 0 < \sigma < 1, \gamma > 0 \) and \( E > 1 \), there is \( \beta > 0 \) with the following properties: Suppose we are given:

- a splitting \( \mathbb{R}^{2q} = X \oplus Y \) with \( \dim Y = 2 \), \( X' = Y \) and \( \alpha(X, Y) \geq \gamma \);
- an ellipsoid \( A \subset X \) centered at the origin;
- an ellipse \( B \subset Y \) centered at the origin and with eccentricity at most \( E \);
- a map \( \tilde{R} : Y \to Y \) that rotates \( B \) through angle \( \theta \), with \( |\theta| < \beta \).

Then there exists \( \tau > 1 \) such that the following holds. Let \( R : \mathbb{R}^{2q} \to \mathbb{R}^{2q} \) be the linear map defined by \( R(v) = v \) if \( v \in X \) and \( R(w) = \tilde{R}(w) \) if \( w \in Y \). For \( a, b > 0 \) consider the cylinder \( C = C_{a,b} = aA \oplus bB \). If \( a > \tau b \) and \( \text{diam} C < \varepsilon_0 \) then there is an \( \text{C}^1 \) symplectomorphism \( h : \mathbb{R}^{2q} \to \mathbb{R}^{2q} \) satisfying:

(i) \( \tilde{h}(z) = z \) for every \( z \notin C \) and \( \tilde{h}(z) = R(z) \) for every \( z \in \sigma C \);

(ii) \( \|\tilde{h}(z) - z\| < \varepsilon_0 \) and \( \|Dh_z - I\| < \varepsilon_0 \) for all \( z \in \mathbb{R}^{2q} \).

Now lemma 5.8 is proved in the same way as we proved lemma 3.3 using lemmas 5.11 and 3.5 instead. The argument is even a bit simpler since no truncation (like in lemma 3.6) is necessary, as we assume that the angles \( \alpha(X_j, Y_j) \) are bounded from zero. The details are left to the reader.

### 5.3 Proof of proposition 5.1

We use the following lemma, which will also be needed in section 5.

**Lemma 5.12.** Let \( G \subset GL(d, \mathbb{R}) \) be a closed group which acts transitively in \( \mathbb{R}^{d-1} \). Then for every \( \varepsilon_1 > 0 \) there exists \( \alpha > 0 \) such that if \( v_1, v_2 \in \mathbb{R}^d \) satisfy \( \alpha(v_1, v_2) < \alpha \) then there exists \( R \in G \) such that \( \|R - I\| < \varepsilon_1 \) and \( R(\bar{v}_1) = \bar{v}_2 \).
Proof. For $\delta > 0$, let $U_{\delta} = \{ R; R \in G, \| R-I \| < \delta \}$. Given $\varepsilon > 0$, fix $\delta > 0$ such that if $R_1, R_2 \in U_\delta$ then $R_2R_1^{-1} \in U_\varepsilon$. The hypothesis on the group implies that for any $w \in \mathbb{R}P^{d-1}$, the map $G \to \mathbb{R}P^{d-1}$ given by $A \mapsto A(w)$ is open (this follows from [11, Theorem II.3.2]). Therefore, for any $\delta > 0$, the set $U_{\delta}(w) = \{ Rw; R \in U_\delta \}$ is an open neighborhood of $w$. Now take two directions $v_1, v_2 \in \mathbb{R}P^{d-1}$ sufficiently close. Then both belong to some $U_{\delta}(w_1)$, and so there are $R_1, R_2 \in U_\delta$ such that $v_1 = R_1w_1$ and $v_2 = R_2w_1$. Therefore $R = R_2R_1^{-1}$ belongs to $U_{\varepsilon_1}$ and $Rv_1 = v_2$. 

Proof. Let $f, \varepsilon_0, \kappa$ be given. Fix $0 < \kappa' < \frac{1}{2}\kappa$. Let $\varepsilon > 0$, depending on $f, \varepsilon_0, \kappa'$, be given by lemma $5.3$. Let $\alpha > 0$, depending on $\varepsilon_1 = \varepsilon$ and $G = U(q)$, be given by lemma $5.12$. Take $K$ satisfying $K \geq (\sin \alpha)^{-2}$ and $K \geq \max_x \|Df_x\|/m(Df_x)$. Let $E > 1$ and $\gamma > 0$ be given by

$$E^2 = 8C_\omega^4K(\sin \alpha)^{-4} \quad \text{and} \quad \sin \gamma = \frac{1}{2}C_\omega^{-14}K^{-2}\sin^9 \alpha,$$

where $C_\omega$ is as in (2.4). Let $\beta > 0$ be given by lemma $5.8$. Finally, let $m \geq 2\pi/\beta$. The proof is divided into three cases.

**First case:** Suppose that there exists $\ell \in \{0, 1, \ldots, m\}$ such that

$$\angle(E_\ell, F_\ell) < \alpha. \quad (5.6)$$

Fix $\ell$ as above and take unit vectors $\xi \in E_\ell, \eta \in F_\ell$ such that $\angle(\xi, \eta) < \alpha$. By lemma $5.12$, there exists a unitary transformation $R : T_{f(y)}m \to T_{f(y)}m$ such that $\|R-I\| < \varepsilon$ and $R(\xi) = \eta$. By lemma $5.3$, the sequences $\{Df_{f(\xi)}R\}$ and $\{RDF_{f^{-1}(\eta)}\}$ are $(\kappa', \varepsilon_0)$-realizable. Define $\{L_0, \ldots, L_{m-1}\}$ as

$$\{Df_{y}, \ldots, Df_{f^{-1}(y)}, Df_{f(y)}R, Df_{f^{+1}(y)}, \ldots, Df_{f^{m-1}(y)}\}$$

if $\ell < m$ and as $\{Df_{y}, \ldots, Df_{f^{m-2}(y)}, RDF_{f^{m-1}(y)}\}$ if $\ell = m$. In either case, this is a $(\kappa, \varepsilon_0)$-realizable sequence of length $m$ at $y$, whose product $L_{m-1} \cdots L_0$ sends the direction $\mathbb{R}Df^{-\ell}(\xi) \subset E_0$ to the direction $\mathbb{R}Df^{-m}(\eta) \subset F_m$.

**Second case:** Assume that there exist $k, \ell \in \{0, \ldots, m\}$ with $k < \ell$ and

$$\frac{\|Df_{f(k)}^\ell\|}{m(Df_{f(k)}^\ell)} > K. \quad (5.7)$$

The proof of this case is easily adapted from the second case in the proof of proposition $3.1$. We leave the details to the reader.

**Third case:** We suppose that we are not in the previous cases, that is,

for every $j \in \{0, 1, \ldots, m\}$, $\angle(E_j, F_j) \geq \alpha. \quad (5.8)$

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and

for every $i, j \in \{0, \ldots, m\}$ with $i < j$, \[
\frac{\|Df_{j-i}^m\|}{m(Df_{j-i}^m|F_0)} \leq K. \tag{5.9}
\]

By (5.8) and lemma 2.3, we have, for all $i, j \in \{0, \ldots, m\}$ with $i < j$,

\[
C_\omega^{-2} \sin \alpha \leq m(Df_{j-i}^m|F_0) \|Df_{j-i}^m|F_0\| \leq C_\omega^2 (\sin \alpha)^{-1}. \tag{5.10}
\]

This, together with (5.9), gives

\[
m(Df_{j-i}^m|F_0) \geq C_\omega^{-1} K^{-1/2} (\sin \alpha)^{1/2}, \tag{5.11}
\]

\[
\|Df_{j-i}^m|F_0\| \leq C_\omega K^{1/2} (\sin \alpha)^{-1/2}. \tag{5.12}
\]

Also, by (5.10) and the main assumption (5.1),

\[
m(Df^m|E_0) \leq 2^{1/2} C_\omega (\sin \alpha)^{-1/2}, \tag{5.13}
\]

\[
\|Df^m|F_0\| \geq 2^{-1/2} C_\omega^{-1} (\sin \alpha)^{1/2}. \tag{5.14}
\]

Let $v_0 \in E_0$ be such that $\|v_0\| = 1$ and $\|Df^m(v_0)\| = m(Df^m|E_0)$. Using lemma 2.3, take $w_0 \in F_0$ with $\|w_0\| = 1$ such that $|\omega(v_0, w_0)| \geq C_\omega^{-1} \sin \alpha$. Let $G_0 = E_0 \cap w_0^\perp$ and $H_0 = F_0 \cap w_0^\perp$. (By $w^\perp$ we mean $(\mathbb{R}w)^\perp$.) Let $X_0 = G_0 \oplus H_0$ and $Y_0 = \mathbb{R}v_0 \oplus \mathbb{R}w_0$. Then $X_0 = Y_0^\perp$. Let, for $j = 1, \ldots, m$,

\[
v_j = Df^j(v_0)/\|Df^j(v_0)\|, \quad G_j = Df^j(G_0), \quad X_j = Df^j(X_0), \tag{5.15}
\]

\[
w_j = Df^j(w_0)/\|Df^j(w_0)\|, \quad H_j = Df^j(H_0), \quad Y_j = Df^j(Y_0)
\]

(all the derivatives are at $y$). By (2.5),

\[
C_\omega^{-1} \sin \alpha \leq |\omega(v_0, w_0)| = |\omega(Df^m v_0, Df^m w_0)| \leq C_\omega \|Df^m v_0\| \|Df^m w_0\|.
\]

Thus $\|Df^m w_0\| \geq C_\omega^{-2} \sin \alpha \cdot m(Df^m|E_0)^{-1}$ and, using (5.10),

\[
\|Df^m w_0\| \geq C_\omega^{-4} \sin^2 \alpha \cdot \|Df^m|F_0\| \geq C_\omega^{-4} \sin^2 \alpha. \tag{5.16}
\]

That is, $w_0$ is “almost” the most expanded vector by $Df^m$ in $F_0$. By (5.1) and (5.15),

\[
\frac{\|Df^m w_0\|}{\|Df^m v_0\|} \geq C_\omega^{-4} \sin^2 \alpha \frac{\|Df^m|F_0\|}{m(Df^m|E_0)} \geq \frac{1}{2} C_\omega^{-4} \sin^2 \alpha.
\]

This and (5.9) imply that, for each $j = 1, \ldots, m$,

\[
K \geq \frac{\|Df^j w_0\|}{\|Df^j v_0\|} \geq \frac{\|Df^m w_0\|/\|Df^{m-j} w_0\|}{\|Df^m w_0\|/m(Df^{m-j} w_0)} \geq \frac{1}{2} C_\omega^{-4} K^{-1} \sin^2 \alpha.
\]

Therefore, using (5.8) and lemma 2.8,

\[
\frac{\|Df^j|Y_0\|}{m(Df^j|Y_0)} \leq 8 C_\omega^4 K (\sin \alpha)^{-4} = E^2. \tag{5.16}
\]
We now deduce some angle estimates. First, we claim that
\[
\sin \angle(v_0, G_0) \geq C_\omega^{-2} \sin \alpha \quad \text{and} \quad \sin \angle(w_0, H_0) \geq C_\omega^{-2} \sin \alpha.
\] (5.17)
Indeed, write \(v_0 = u + u'\) with \(u' \in G_0\) and \(u \perp G_0\). Since \(G_0\) is skew-orthogonal to \(w_0\),
\[
C_\omega^{-1} \sin \alpha \leq |\omega(v_0, w_0)| = |\omega(u, w_0)| \leq C_\omega \|u\|.
\]
That is, \(\sin \angle(v_0, G_0) = \|u\| \geq C_\omega^{-2} \sin \alpha\). Analogously we prove the other inequality in (5.17). Next, we estimate \(\sin \angle(v_j, G_j)\) and \(\sin \angle(w_j, H_j)\) for \(j = 1, \ldots, m\). For this we use relation (2.6) from subsection 2.4, which gives:
\[
\sin \angle(v_0, G_0) \geq \frac{m(Df_j|_{E_0})}{\|Df_jv_0\|} \sin \angle(v_0, G_0),
\] (5.18)
\[
\sin \angle(w_0, H_0) \geq \frac{\|Df_jw_0\|}{\|Df_j|_{F_0}\|} \sin \angle(w_0, H_0).
\] (5.19)
By (5.11) and (5.13),
\[
\|Df_jv_0\| = \frac{\|Df^m v_0\|}{\|Df^{m-j} v_0\|} \leq \frac{m(Df^m|_{E_0})}{m(Df^{m-j}|_{E_0})} \leq 2^{1/2} C_\omega^2 K^{1/2} (\sin \alpha)^{-1}.
\]
for each \(j = 1, \ldots, m\). So, using (5.11) again,
\[
\|Df_jv_0\| \leq 2^{1/2} C_\omega^3 K (\sin \alpha)^{-3/2}.
\]
This, together with (5.17) and (5.18), gives
\[
\sin \angle(v_j, G_j) \geq 2^{-1/2} C_\omega^{-5} K^{-1} (\sin \alpha)^{5/2}.
\] (5.20)
Similarly, by (5.13), (5.12), and (5.14),
\[
\|Df_jw_0\| = \frac{\|Df^m w_0\|}{\|Df^{m-j} w_0\|} \geq C_\omega^4 \sin^2 \alpha \frac{\|Df^m|_{F_0}\|}{\|Df^{m-j}|_{F_0}\|} \geq 2^{-1/2} C_\omega^{-6} K^{-1/2} \sin^3 \alpha.
\]
Then, using (5.12) again,
\[
\|Df_jw_0\| \geq 2^{-1/2} C_\omega^{-7} K^{-1} (\sin \alpha)^{7/2}.
\]
By (5.17) and (5.19),
\[
\sin \angle(w_j, H_j) \geq 2^{-1/2} C_\omega^{-9} K^{-1} (\sin \alpha)^{9/2}.
\] (5.21)
Now we use lemma 2.6 three times:
\[
\sin \angle(Y_j, X_j) \geq \sin \angle(v_j, X_j) \sin \angle(w_j, \mathbb{R} v_j \oplus X_j)
\]
\[
\geq \sin \angle(v_j, G_j) \sin \angle(E_j, H_j) \sin \angle(w_j, \mathbb{R} v_j \oplus X_j)
\]
\[
\geq \sin \angle(v_j, G_j) \sin \angle(w_j, H_j) \sin \angle(E_j, H_j)^2
\]
So, using (5.20), (5.21), and \(\angle(E_j, H_j) \geq \alpha\), we obtain
\[
\sin \angle(X_j, Y_j) \geq \frac{1}{2} C_\omega^{-14} K^{-2} \sin^9 \alpha = \sin \gamma.
\] (5.22)
Relations (5.10) and (5.22) permit us to apply lemma 5.8. Since \(m \beta \geq 2 \pi\), it is possible to choose numbers \(\theta_0, \ldots, \theta_{m-1}\) such that \(0 \leq \theta_j \leq \beta\) and \(\sum \theta_j = \angle(v_0, w_0)\). Let \(S_j\) and \(L_j\) be as in lemma 5.8. We have \(L_{m-1} \cdots L_0|_{Y_0} = (Df^m|_{Y_0}) S_{m-1} \cdots S_0\), so \(L_{m-1} \cdots L_0(\mathbb{R} v_0) = \mathbb{R} w_m\). This completes the proof of proposition 5.1. \[\square\]
6 Proof of theorems 3 and 4

Given \( f \in \text{Diff}_C^1(M) \) and \( m \in \mathbb{N} \), let \( \mathcal{D}(f,m) \) be the (closed) set of points \( x \) such that there is a \( m \)-dominated splitting of index \( q = d/2 \) along the orbit of \( x \). Let \( \Gamma(f,m) = M \setminus \mathcal{D}(f,m) \) and let \( \Gamma^*(f,m) \) be the set of points \( x \in \Gamma(f,m) \) which are regular, not periodic and satisfy \( \lambda_q(f,x) > 0 \). Let also \( \Gamma(f,\infty) = \bigcap_{m \in \mathbb{N}} \Gamma(f,m) \).

We have the following symplectic analogues of propositions 4.2, 4.8 and 4.17.

**Proposition 6.1.** Let \( f \in \text{Sympl}_C^1(M) \), \( \varepsilon_0 > 0 \), \( \delta > 0 \), and \( 0 < \kappa < 1 \). If \( m \in \mathbb{N} \) is sufficiently large, then there exists a measurable function \( N : \Gamma^*(f,m) \to \mathbb{N} \) such that for a.e. \( x \in \Gamma^*(f,m) \) and every \( n \geq N(x) \) there exists a \((\varepsilon_0,\kappa)\)-realizable sequence \( \{\hat{L}_0, \ldots, \hat{L}_{n-1}\} \) at \( x \) of length \( n \) such that

\[
\frac{1}{n} \log \| \wedge^q(\hat{L}_{n-1} \cdots \hat{L}_0) \| \leq \Lambda_{q-1}(f,x) + \delta.
\]

**Proposition 6.2.** Let \( f \in \text{Sympl}_C^1(M) \), \( \varepsilon_0 > 0 \) and \( \delta > 0 \). Then there exist \( m \in \mathbb{N} \) and a diffeomorphism \( g \in \mathcal{U}(f,\varepsilon_0) \) that equals \( f \) outside the open set \( \Gamma(f,m) \) and such that

\[
\int_{\Gamma(f,m)} \Lambda_q(g,x) \, d\mu(x) < \delta + \int_{\Gamma(f,m)} \Lambda_{q-1}(f,x) \, d\mu(x).
\]

**Proposition 6.3.** Given \( f \in \text{Sympl}_C^1(M) \), let

\[
J(f) = \int_{\Gamma(f,\infty)} \lambda_q(f,x) \, d\mu(x).
\]

Then for every \( \varepsilon_0 > 0 \) and \( \delta > 0 \), there exists a diffeomorphism \( g \in \mathcal{U}(f,\varepsilon_0) \) such that

\[
\int_M \Lambda_q(g,x) \, d\mu(x) < \int_M \Lambda_q(f,x) \, d\mu(x) - J(f) + \delta.
\]

The proofs of these propositions are exactly the same as those of the corresponding results in section 4 in the following logical order:

- proposition 5.1 \( \Rightarrow \) proposition 6.1 \( \Rightarrow \) proposition 6.2 \( \Rightarrow \) proposition 6.3.

Concerning the first implication, notice that if \( x \in \Gamma^*(f,m) \) then, by lemma 2.4, the spaces \( E^+_{\omega} \) and \( E^-_{\omega} \) (that correspond to positive and negative Lyapunov exponents) are Lagrangian, so proposition 5.1 applies.

6.1 Conclusion of the proof of theorems 3 and 4

**Proof of theorem 3.** Let \( f \in \text{Sympl}_C^1(M) \) be a point of continuity of the map \( \text{LE}_q(\cdot) \).

By proposition 6.3, \( J(f) = 0 \), that is, \( \lambda_q(f,x) = 0 \) for a.e. \( x \in \Gamma(f,\infty) \). Let \( x \in M \) be a regular point. If \( \lambda_q(f,x) > 0 \), we have (if we exclude a zero measure set of \( x \) \( \not\in \Gamma(f,\infty) \)). This means that there is a dominated splitting, \( T^{f^n(x)} M = E_n \oplus F_n \), \( n \in \mathbb{Z} \) of index \( q \), along the orbit of \( x \). Then \( E_n \) is the sum of the Oseledets spaces of \( f \), at the point \( f^n(x) \), associated to the Lyapunov exponents \( \lambda_1(f,x) \), \( \ldots \), \( \lambda_q(f,x) \), and \( F_n \) is the sum of the spaces associated to the other exponents. By part 2 of lemma 2.4, the splitting \( T^{f^n(x)} M = E_n \oplus F_n \), \( n \in \mathbb{Z} \) is hyperbolic. \( \square \)
The next proposition is used to deduce theorem 3 from theorem 2.

**Proposition 6.4.** There is a residual subset \( \mathcal{R}_2 \subset \text{Sympl}_1^\omega (M) \) such that if \( f \in \mathcal{R}_2 \) then either \( f \) is Anosov or every hyperbolic set of \( f \) has measure 0.

**Proof.** This is a modification of an argument from [16]. We use the fact, proved in [24], that \( C^2 \) diffeomorphisms are dense in the space \( \text{Sympl}_1^\omega (M) \). Another key ingredient is that the hyperbolic sets of any \( C^2 \) non-Anosov diffeomorphism have zero measure. We comment on the latter near the end.

For each open set \( U \subset M \) with \( U \neq M \) and each \( f \in \text{Diff}^1_\mu(M) \), consider the maximal \( f \)-invariant set inside \( U \),

\[
\Lambda_f(U) = \bigcap_{n \in \mathbb{Z}} f^n(U).
\]

For \( \varepsilon > 0 \), let \( D(\varepsilon, U) \) be the set of diffeomorphisms \( f \in \text{Sympl}_1^\omega (M) \) such that at least one of the following properties is satisfied:

(i) There is a neighborhood \( U \) of \( f \) such that \( \Lambda_g(U) \) is not hyperbolic for all \( g \in U \);

(ii) \( \mu(\Lambda_f(U)) < \varepsilon \).

Clearly, the set \( D(\varepsilon, U) \) is open. Moreover, it is dense. Indeed, if \( f \) does not satisfy (i) then there is \( g \) close to \( f \) such that \( \Lambda_g(U) \) is hyperbolic. Take \( f_1 \in C^2 \) close to \( g \) in \( \text{Sympl}_1^\omega (M) \). Then \( \Lambda_{f_1}(U) \) is hyperbolic with measure zero, and so \( f_1 \in D(\varepsilon, U) \).

This proves denseness. Hence the set

\[
D(U) = \cap_{\varepsilon > 0} D(\varepsilon, U) \supset \{ f \in \text{Sympl}_1^\omega (M); \ \Lambda_f(U) \text{ is hyperbolic} \Rightarrow \mu(\Lambda_f(U)) = 0 \}
\]

is residual. Now take \( \mathcal{B} \) a countable basis of open sets of \( M \) and let \( \hat{\mathcal{B}} \) be the set of all finite unions of sets in \( \mathcal{B} \). The set

\[
\mathcal{R}_2 = \bigcap_{U \in \mathcal{B}, U \neq M} D(U)
\]

is residual in \( \text{Sympl}_1^\omega (M) \) and the hyperbolic sets for every non-Anosov \( f \in \mathcal{R} \) have zero measure.

Finally, we explain why all hyperbolic sets of a \( C^2 \) non-Anosov diffeomorphism have zero measure. This is well-known for hyperbolic basic sets, see [8]. We just outline the arguments in the general case. Suppose \( f \) has a hyperbolic set \( \Lambda \) with \( \mu(\Lambda) > 0 \). Using absolute continuity of the unstable lamination, we get that \( \mu_u(W_\varepsilon(x) \cap \Lambda) > 0 \) for some \( x \in \Lambda \), where \( \mu_u \) denotes Lebesgue measure along unstable manifolds. By bounded distortion and a density point argument, we find points \( x_k \in \Lambda \) such that \( \mu_u(W_\varepsilon(x_k) \setminus \Lambda) \) converges to zero. Taking an accumulation point \( x_0 \) we get that \( W_\varepsilon^u(x_0) \subset \Lambda \). We may suppose that every point of \( \Lambda \) is in the support of \( \mu|\Lambda \). In particular, there are recurrent points of \( \Lambda \) close to \( x_0 \). Applying the shadowing lemma, we find a hyperbolic periodic point \( p_0 \) close to \( x_0 \). In particular, \( W^s(p_0) \) intersects \( W^u_\varepsilon(x_0) \) transversely. Using the \( \lambda \)-lemma we conclude that the whole \( W^u(p_0) \) is contained in \( \Lambda \). Define \( \Lambda_0 \) as the closure of the unstable
manifold of the orbit of $p_0$. This is a hyperbolic set contained in $\Lambda$, and it consists of entire unstable manifolds. Hence, $W^s(\Lambda_0)$ is an open neighborhood of $\Lambda_0$. Using that $f$ preserves volume, we check that $f^{-1}(W^s(\Lambda_0)) = W^u_s(\Lambda_0)$. This implies that $W^s(\Lambda_0) = \Lambda_0$ and so, by connectedness, $\Lambda_0$ must be the whole $M$. Consequently, $f$ is Anosov.

**Proof of theorem 4.** It suffices to take $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ with $\mathcal{R}_1$ a residual set of continuity points of $f \mapsto \text{LE}_q(f)$, and $\mathcal{R}_2$ as in proposition 5.4.

7 Proof of theorem 5

Let $M$ be a compact Hausdorff space, $\mu$ a Borel regular measure and $f : M \to M$ a homeomorphism preserving the measure $\mu$. Let also $G \subset \text{GL}(d, \mathbb{R})$ be a closed group which acts transitively on $\mathbb{R}P^{d-1}$.

The following result provides an analogue of proposition 3.1:

**Proposition 7.1.** Given $A \in C(M, G)$ and $\varepsilon > 0$, if $m \in \mathbb{N}$ is large enough then the following holds:

Let $y \in M$ be a non-periodic point and suppose it is given a non-trivial splitting $\mathbb{R}^d = E \oplus F$ such that

$$\frac{\|A^m(y)|_F\|}{\mu(A^m(y)|_E)} \geq \frac{1}{2}. \quad (7.1)$$

Then there exists, for each $j = 0, 1, \ldots, m-1$, some $L_j \in G$ with $\|L_j - A(f^jy)\| < \varepsilon_0$, and there are non-zero vectors $v \in E$ and $w \in A^m(y)(F)$ with $L_{m-1} \cdots L_0(v) = w$.

**Proof.** Let $\varepsilon_1 = \|A\|^{-1}_\infty \varepsilon$, where $\|A\|_\infty = \sup_{x \in M} \|A(x)\|$. Let $\alpha > 0$, depending on $\varepsilon_1$, be given by lemma 5.12. Let

$$K = \max\{1/\sin^2 \alpha, \|A\|_\infty\|A^{-1}\|_\infty\}, \quad C = \frac{8K}{\sin^2 \alpha},$$

and $m \geq 2C/\alpha$. Now take $y$, $E$ and $F$ as in the statement. For $j = 0, 1, \ldots, m-1$, indicate $A_j = A(f^jy)$, $E_j = A^j(x)(E)$, $F_j = A^j(x)(F)$. As before, we divide the rest of the proof into three cases:

**First case:** We assume that there exists $\ell \in \{0, \ldots, m\}$ such that

$$\prec (E_\ell, F_\ell) < \alpha. \quad (7.2)$$

Fix $\ell$ as above and take $\xi \in E_\ell$, $\eta \in F_\ell$ such that $\prec (\xi, \ell) < \alpha$. Let $R \in G$ be such that $\|R - I\| < \varepsilon$ and $R(\mathbb{R}\xi) = \mathbb{R}\eta$. If $\ell \neq m$, then we define $L_j$ as $L_\ell = A_\ell R$ and $L_j = A_j$ for $j \neq \ell$. If $\ell = m$, then we define $L_j$ as $L_\ell = RA_\ell$ and $L_j = A_j$ for $j \neq m$. In either case, the sequence $\{L_0, \ldots, L_{m-1}\}$ has the required properties.

**Second case:** Assume that there exist $k, \ell \in \{0, \ldots, m\}$ such that $k < \ell$ and

$$\frac{\|A_{\ell-1} \cdots A_k|_{F_\ell}\|}{\mu(A_{\ell-1} \cdots A_k|_{E_\ell})} > K. \quad (7.3)$$

Once more, this is similar to the second case in propositions 3.1 and 5.1. We leave it to the reader to spell-out the details.
Third case: We suppose that we are not in the previous cases, that is, we assume
\[ \langle E_j, F_j \rangle \geq \alpha. \]  
(7.4)
and
\[ \text{for every } i, j \in \{0, \ldots, m\} \text{ with } i < j, \quad \frac{\|A_j \cdots A_i F_i\|}{m(A_j \cdots A_i | E_i)} \leq K. \]  
(7.5)
Take unit vectors \( \xi \in E_0 \) and \( \eta \in F_0 \) such that
\[ \|A_{m-1} \cdots A_0 (\xi)\| = \|A_{m-1} \cdots A_0 | E_0\| \] and \( \|A_{m-1} \cdots A_0 (\eta)\| = m(A_{m-1} \cdots A_0 | F_0). \)
Let \( \xi_j = A_{j-1} \cdots A_0 (\xi), \eta_j = A_{j-1} \cdots A_0 (\eta) \) and \( Y_j = \mathbb{R} \xi_j \oplus \mathbb{R} \eta_j. \) By the assumption (7.1), we have \( \|A_{m-1} \cdots A_0 (\eta)\|/\|A_{m-1} \cdots A_0 (\xi)\| \geq 1/2. \) Also, using (7.4), we have that for each \( j \in \{1, \ldots, m\}, \)
\[ K \geq \frac{\|A_{j-1} \cdots A_0 (\eta)\|}{\|A_{j-1} \cdots A_0 (\xi)\|} \geq \frac{\|A_{m-1} \cdots A_0 (\eta)\|/\|A_{m-1} \cdots A_0 (\xi)\|}{\|A_{m-1} \cdots A_0 \|/m(A_{m-1} \cdots A_0)} \geq \frac{1}{2K}. \]
This, together with (7.4) and lemma 2.8 implies that, for all \( j \in \{1, \ldots, m\}, \)
\[ \frac{\|A_{j-1} \cdots A_0 | y_0\|}{m(A_{j-1} \cdots A_0 | y_0)} < C. \]  
(7.6)
Now assign orientations to the planes \( Y_j \) such that each \( A_j | y_j : Y_j \rightarrow Y_{j+1} \) is orientation-preserving. Let \( P_j \) be the projective space of \( Y_j, \) with the induced orientation. Let \( v_j = \mathbb{R} \xi_j \) and \( w_j = \mathbb{R} \eta_j \in P_j; \) For each \( z \in P_j, \) let \( z \rightarrow [z] \) be the oriented angle between \( z \) and \( v_j. \) So \( z \rightarrow [z] \) is a bijection and \( [z] \rightarrow [A_j z] \) is monotonic. If \( L : Y_0 \rightarrow Y_j \) is any linear map then, by lemma 2.7,
\[ 0 < |z_2| - |z_1| \leq \frac{\pi}{2} \implies \frac{|Lz_2| - |Lz_1|}{|z_2| - |z_1|} \leq \frac{2}{\pi} \cdot \frac{\|L\|}{m(L)}. \]  
(7.7)
We define directions \( u_0 \in P_0, \ldots, u_m \in P_m \) by recurrence as follows: Let \( [u_0] = 0 \) and
\[ [u_{j+1}] = [A_j u_j] + \min\{[w_{j+1}] - [A_j u_j], \alpha\}. \]  
(7.8)
Then, for each \( j < m, \) \( [A_j u_j] \leq [u_{j+1}] \leq [w_{j+1}]. \) Therefore, defining \( [z_j] = [(A_{j-1} \cdots A_0)^{-1} u_j, \]
we have
\[ 0 = [z_0] \leq [z_1] \leq \cdots \leq [z_m] \leq [w_0] < \pi. \]
In particular, for some \( i = 0, \ldots, m-1, \) \( [z_{i+1}] - [z_i] < \pi/m. \) Therefore, by (7.6) and (7.7),
\[ [u_{i+1}] - [A_i u_i] = [A_{j-1} \cdots A_0 z_{i+1}] - [A_{j-1} \cdots A_0 z_i] < 2C/m < \alpha. \]
By (7.5), \( [u_{i+1}] = [w_{j+1}]. \) We conclude that \( [u_m] = [w_{m}]. \) Now for each \( j, \) let \( R_j \in G \) be such that \( \|R_j - I\| < \varepsilon \) and \( R_j(A_j u_j) = u_{j+1}. \) Let also \( L_j = R_j A_j. \) Then \( L_{m-1} \cdots L_0 (v_0) = w_m. \) \( \square \)
Next we define sets $\Gamma_p(A,m)$, $\Gamma_p^*(A,m)$, $\Gamma_p^0(A,m)$ for $p \in \{1, \ldots, d-1\}$ and $m \in \mathbb{N}$, in the same way as in section 4, with the obvious adaptations. Lemma 4.3 also applies in the present context.

**Proposition 7.2.** Given $A \in C(M,G)$, $\varepsilon > 0$, $\delta > 0$, and $p \in \{1, \ldots, d-1\}$, if $m \in \mathbb{N}$ is sufficiently large then there exists a measurable function $N : \Gamma_p(A,m) \to \mathbb{N}$ such that for a.e. $x \in \Gamma_p^*(A,m)$ and every $n \geq N(x)$ there exist matrices $\hat{L}_0, \ldots, \hat{L}_{n-1} \in G$ such that $\|\hat{L}_j - A(f^j x)\| < \varepsilon$ and

$$\frac{1}{n} \log \|A^p(\hat{L}_{n-1} \cdots \hat{L}_0)\| \leq \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} + \delta.$$

The proof is the same as proposition 4.2.

**Proposition 7.3.** Let $A \in C(M,G)$, $\varepsilon_0 > 0$, $p \in \{1, \ldots, d-1\}$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$ and a cocycle $B \in C(M,G)$, with $\|B - A\|_\infty < \varepsilon_0$, that equals $A$ outside the open set $\Gamma_p(A,m)$ and such that

$$\int_{\Gamma_p(A,m)} \Lambda_p(B,x) \, d\mu(x) < \delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} \, d\mu(x).$$

The proof of proposition 7.3 is not just an adaptation of that of proposition 4.8, because Vitali’s lemma may not apply to $M$. We begin by proving a weaker statement, in lemma 4.4. Let $L^\infty(M,G)$ denote the set of bounded measurable functions from $M$ to $G$. Oseledets theorem also applies for cocycles in $L^\infty(M,G)$.

**Lemma 7.4.** Let $A \in C(M,G)$, $\varepsilon_0 > 0$, $p \in \{1, \ldots, d-1\}$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$ and a cocycle $\tilde{B} \in L^\infty(M,G)$, with $\|\tilde{B} - A\|_\infty < \varepsilon_0/2$, that equals $A$ outside the open set $\Gamma_p(A,m)$ and such that

$$\int_{\Gamma_p(A,m)} \Lambda_p(\tilde{B},x) \, d\mu(x) < \delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} \, d\mu(x).$$

**Sketch of proof.** We shall explain the necessary modifications of the proof of proposition 4.8. The sets $Z^i$, $Q^i$ and $Q^e$ are defined as before. In lemma 4.14, the castles $U^i$ and $K^i$ become equal to $Q^i$ (as $\kappa$ and $\gamma$ were 0). We decompose each base $Q^i_k$ into finitely many disjoint measurable sets $U^i_k$ with small diameter. In each tower with base $U^i_k$ we construct the perturbation $\tilde{B}$ using proposition 7.2, taking $\tilde{B}$ constant in each floor. The definitions of $N$ and $G^i$ are the same. In lemma 4.16 several bounds (those involving $\kappa$ or $\gamma$) become trivial. Then one concludes the proof in the same way as before.

**Proof of proposition 7.3.** Let $A$, $\varepsilon_0$, $p$ and $\delta$ be as in the statement. Let $m$ and $\tilde{B}$ be given by lemma 7.4. Let $N \in \mathbb{N}$ be such that

$$\int_{\Gamma_p(A,m)} \frac{1}{N} \log \|A^p(\tilde{B}^N(x))\| \, d\mu < 2\delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} \, d\mu.$$

Let $\gamma = N^{-1}\delta$. Using Lusin’s theorem (see [22]) and the fact that $G$ is a manifold (see [11]), one finds a continuous $B : M \to G$ such that $B = \tilde{B} = A$ outside the
open set $\Gamma_p(A, m)$, the norm $\|B - \tilde{B}\|_\infty < \varepsilon_0/2$, and the set $E = \{x \in M; B(x) \neq \tilde{B}(x)\}$ has measure $\mu(E) < \gamma$. Let $G = \bigcap_{j=0}^{N-1} f^{-j}(\Gamma_p(A, m) \setminus E) \subset \Gamma_p(A, m)$. Then $\mu(\Gamma_p(A, m) \setminus G) \leq N \mu(E) < \delta$. Then, letting $C$ be an upper bound for $\log \|\Lambda^p(B(x))\|$, we have

$$\int_{\Gamma_p(A, m)} \Lambda_p(B, x) \, d\mu \leq \int_{\Gamma_p(A, m)} \frac{1}{N} \log \|\Lambda^p(B^N(x))\| \, d\mu$$

$$< C \delta + 2 \delta + \int_{\Gamma_p(A, m)} \frac{\Lambda_{p-1}(A, x) + \Lambda_{p+1}(A, x)}{2} \, d\mu.$$ 

Up to replacing $\delta$ with $\delta/(C + 2)$, this completes the proof.

Using proposition 7.3, one concludes the proof of theorem 5 exactly as in subsection 4.3. The fact that either vanishing of the exponents or dominance of the splitting is also a sufficient condition for continuity is an easy consequence of semi-continuity of Lyapunov exponents and robustness of dominated splittings under small perturbations of the cocycle.

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