The blocker postulates for measures of voting power

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Received: 25 January 2022 / Accepted: 11 August 2022 / Published online: 30 September 2022
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Abstract
A proposed measure of voting power should satisfy two conditions to be plausible: first, it must be conceptually justified, capturing the intuitive meaning of what voting power is; second, it must satisfy reasonable postulates. This paper studies a set of postulates, appropriate for a priori voting power, concerning blockers (or vetoers) in a binary voting game. We specify and motivate five such postulates, namely, two subadditivity blocker postulates, two minimum-power blocker postulates, each in weak and strong versions, and the added-blocker postulate. We then test whether three measures of voting power, namely the classic Penrose–Banzhaf measure, the classic Shapley–Shubik index, and the newly proposed recursive measure, satisfy these postulates. We find that the first measure fails four of the postulates, the second fails two, while the third alone satisfies all five postulates. This work consequently adds to the plausibility of the recursive measure as a reasonable measure of voting power.

1 Introduction
A proposed measure of voting power should satisfy two conditions to be plausible: first, it must be conceptually justified, in the sense that it captures the intuitive meaning of what voting power is; second, it must satisfy reasonable postulates for measures of voting power. Numerous postulates have been defended in the voting-power literature: most are for a priori voting power (i.e., voting power solely in virtue of the formal voting structure itself, constituted by the agenda of potential outcomes, the sets of actors, their action profiles, and the decision function mapping vote configurations onto outcomes), while others are for a posteriori voting power (i.e.,

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voting power also in virtue of the distribution of preferences, and consequent incentives for strategic interaction, within the voting structure) (Felsenthal and Machover 1998; Laruelle and Valenciano 2005).

This paper studies a set of postulates, appropriate for a priori voting power, concerning blockers (or vetoers) in a binary voting game. Our aim is two-fold. First, to specify and motivate five postulates concerning blockers, namely, two subadditivity blocker postulates, two minimum-power blocker postulates, each in weak and strong versions, and the added-blocker postulate. Second, to test whether three measures of voting power, namely the classic Penrose–Banzhaf measure (PB), the classic Shapley–Shubik index (SS), and the newly proposed recursive measure (RM), satisfy these postulates. We find that PB fails the first four postulates, SS fails the strong subadditivity blocker postulate and the added-blocker postulate, while RM alone satisfies all five postulates. Further, it is already known (Abizadeh and Vetta 2021) that RM satisfies a plethora of other reasonable postulates, including the iso-invariance, dummy, dominance, donation, minimum-power bloc, and quarrel postulates. This work consequently adds to the plausibility of RM as a reasonable measure of voting power.

2 Three measures of voting power

In this section, we present the three measures of voting power studied in this paper. The first are the two classic measures of voting power, namely, the Penrose–Banzhaf measure and Shapley–Shubik index. The third is the aforementioned recursive measure proposed by Abizadeh and Vetta (2021). Before formally defining these three measures, we introduce the notions of a simple voting game and a measure of voting power.

2.1 Simple voting games

Here we present the class of voting games, called simple voting games (SVGs), for which it would be reasonable to expect measures of voting power to satisfy our postulates. Denote by $N = \{1, 2, \ldots, n\}$ a nonempty, finite set of players with $z = 2$ strategies, voting YES or voting NO. Let $\mathcal{O} = \{\text{YES, NO}\}$ be the set of alternative outcomes. A division or complete vote configuration $\mathcal{S} = (S, \overline{S})$ of the set $N$ is an ordered partition of players where the first element in the ordered pair is the set of YES-voters and the second element is the set of NO-voters. Thus, for $\mathcal{S} = (S, \overline{S})$, the subset $S \subseteq N$ comprises the set of YES-voters and the subset $\overline{S} = N \setminus S$ comprises the set of NO-voters. Note the convention of representing a bipartitioned division by its first element in blackboard bold.

Let $\mathcal{D}$ be the set of all logically possible divisions $\mathcal{S}$ of $N$. A binary voting game, in which each player has two possible strategies, is a function $\mathcal{G}(\mathcal{S})$ mapping the set of all possible divisions $\mathcal{D}$ to the two outcomes in $\mathcal{O}$. A monotonic binary voting game is one satisfying the condition:

(i) Monotonicity. If $\mathcal{G}(\mathcal{S}) = \text{YES}$ and $S \subseteq T$, then $\mathcal{G}(\mathcal{T}) = \text{YES}$. 

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A SVG is a monotonic binary voting game that also satisfies non-triviality:

(ii) Non-Triviality. \( \exists S | G(S) = \text{YES} \) and \( \exists T | G(T) = \text{NO} \).

Monotonicity and non-triviality jointly ensure that SVGs also have the following property:

(iii) Unanimity. \( G(\emptyset, N) = \text{NO} \) and \( G(N, \emptyset) = \text{YES} \).

Note that unanimity itself implies non-triviality. Thus conditions (i) and (iii) also characterize the class of SVGs.

Call any player whose vote corresponds to the division outcome a successful player.

Let \( W \) be the collection of all sets of players \( S \) such that \( G(S) = \text{YES} \) (that is, if each member of \( S \) were to vote YES, they would be successful YES-voters). We call this the collection of YES-successful subsets of \( N \), also commonly called winning coalitions. We can now alternatively characterize conditions (i)–(iii) as:

(i) Monotonicity. If \( S \in W \) and \( S \subseteq T \) then \( T \in W \).

(ii) Non-Triviality. \( \exists S \in W \) and \( \exists T \not\in W \).

(iii) Unanimity. \( N \in W \) and \( \emptyset \not\in W \).

In the discussion and proofs that follow, as is standard in the voting-power literature, we assume that our voting games are SVGs.

2.2 Voting power

We define a measure of voting power for SVGs as a function \( \Psi \) that assigns to each player \( i \) a nonnegative real number \( \Psi_i \geq 0 \) and that satisfies two sets of basic adequacy postulates: the iso-invariance postulate, according to which the a priori voting power of any player according to that measure remains the same between two isomorphic games; and the dummy postulates, according to which a player has zero a priori voting power if and only if it is a dummy, and the addition of a dummy to a voting structure leaves other players’ a priori voting power unchanged (Felsenthal and Machover 1998: 236).

A dummy is a voter who is not decisive in any division, where being decisive means being in a position in which one could have effected a different outcome in a given division by (unilaterally) voting differently than one did. The concept can be formalized for SVGs as follows: a player \( i \) is YES-decisive in division \( S \) if and only if \( i \in S \in W \) but \( S \setminus \{i\} \not\in W \); is NO-decisive if and only if \( i \not\in S \not\in W \) but \( S \cup \{i\} \in W \); and is decisive if and only if it is either YES-decisive or NO-decisive.

A measure of voting power can be represented as assigning to each player \( i \) a value

\[
\Psi_i = \sum_{S \in D} \alpha_i(S) \cdot \gamma(S)
\]

where \( \alpha_i(S) \) is the division efficacy score of player \( i \) in division \( S \) and \( \gamma(S) \) is the division weight assigned to \( S \) for any division \( S \in D \). The defining characteristic of a given measure of voting power is therefore its specification of a player’s division efficacy score for each division and each division’s weight. We shall label the a priori voting power of a player according to a measure \( \Psi \), i.e., in abstraction from any information about the distribution of preferences, using the lower case \( \psi \).

Before proceeding, we introduce one further set of concepts. Let a player \( i \)’s YES-efficacy score \( \alpha_i^+ \) be equal to \( \alpha_i \) in divisions in which \( i \) votes YES, equal to 0 otherwise;
and i’s no-efficacy score $\alpha^-_i$ be equal to $\alpha_i$ in divisions in which $i$ votes no, equal to 0 otherwise. We say that a player’s yes-voting power $\Psi^+$ sums over its weighted yes-efficacy scores, and its no-voting power $\Psi^-$ sums over its weighted no-efficacy scores. That is, for SVGs,

$$\Psi^+_i = \sum_{S \in D} \alpha^+_i(S) \cdot \gamma(S) \quad \Psi^-_i = \sum_{S \in D} \alpha^-_i(S) \cdot \gamma(S) \quad \Psi_i = \Psi^+_i + \Psi^-_i$$

We may now present the three aforementioned measures of voting power.

### 2.3 The Penrose–Banzhaf Measure

The Penrose–Banzhaf (PB) measure (Penrose 1946; Banzhaf 1965, 1966; Felsenthal and Machover 1998) bases a voter’s division efficacy score $\alpha_i(S)$ on being decisive. In particular, PB, which was originally conceived as a measure of a priori voting power, equates a player’s voting power with the proportion of logically possible divisions in which it is decisive. This, in turn, is typically taken to represent the ex ante probability that a player $i$ will be decisive in a voting structure, under the assumptions of voting independence (votes are not correlated) and equiprobable voting (the probability a player votes for one alternative equals the probability it votes for any other), which together imply equiprobable divisions—which assumptions model the a prioristic abstraction from voter preferences.

In particular, PB is defined by specifying a voter’s division efficacy score as

$$\alpha^PB_i(S) = \begin{cases} 1 & \text{if } i \text{ is decisive in } S \\ 0 & \text{otherwise} \end{cases}$$

A division’s weight $\gamma(S)$, in turn, is typically interpreted in the Penrose–Banzhaf model as $S$’s ex ante probability. In the general, a posteriori case, a division’s probability $P(S)$ would be a function of the actual distribution of voter preferences; but a division’s probability $\mathbb{P}(S)$ in the a priori case (which we represent again using the lower case) assumes equiprobable divisions. For binary voting games $(z = 2)$, this yields:

$$\gamma^{PB}(S) = \mathbb{P}(S) = \frac{1}{|D|} = \frac{1}{2^n}$$

Notice that, because PB calculates a voter’s division efficacy score strictly on the basis of whether the voter is decisive in that division, its efficacy scores are what we shall call strategy symmetric, that is, a player’s efficacy score in a given division is equal to its efficacy score in any other division that is identical to it but for the player’s own vote. In the case of SVGs, where $z = 2$, this is because for every division in which the voter is yes-decisive there is precisely one corresponding division in which the voter is no-decisive (involving the two divisions that are identical except for the vote of the player in question). It follows that in SVGs the number of divisions in which the player plays a given strategy and is decisive equals the number
of divisions in which it plays any other strategy and is decisive. Strategy symmetry implies that $PB_i^+ = PB_i^-$. It follows that PB can be calculated via a shortcut $PB^*$, on the basis of solely yes-decisiveness (indeed, PB is typically defined in this way in the literature), setting a player’s division efficacy score as:

$$a_i^{PB^*}(S) = a_i^{PB}(S) = \begin{cases} 1 & \text{if } i \text{ is YES-decisive in } S \\ 0 & \text{otherwise} \end{cases}$$

and each division’s weight (again, for a priori power) as:

$$\gamma^{PB^*}(S) = \frac{1}{|D|/z} = \frac{1}{z^{n-1}} = \frac{1}{2^{n-1}}.$$}

Precisely because PB is strategy symmetric, we could also construct a corresponding shortcut based on no-decisiveness.

### 2.4 The Shapley–Shubik index

When Shapley and Shubik (1954) initially introduced their index, they characterized a player’s a priori voting power as equal to the proportion of permutations (ordered sequences) of voters in which a voter would be pivotal, i.e., the probability that the player would be pivotal if all permutations of voters are equiprobable. A pivotal voter is one who, in an ordered sequence of voters who sequentially vote in favour of an alternative, is the first whose vote secures it regardless of how subsequent voters vote. Subsequent analysis has shown that being pivotal is analytically reducible to the notion of decisiveness (Turnovec et al. 2008), which is why we can define the Shapley–Shubik (SS) index using our general formula for $\Psi$ above. We begin by specifying a player’s division efficacy score, as with PB, as follows:

$$a_i^{SS}(S) = \begin{cases} 1 & \text{if } i \text{ is decisive in } S \\ 0 & \text{otherwise} \end{cases}$$

and then set each division’s weight for SVGs, where $k$ equals the number of voters who vote as $i$ does in division $S$ (including player $i$), as:

$$\gamma^{SS}(S) = \frac{(k-1)! \cdot (n-k)!}{2n!}.$$}

Because SS, like PB, is strategy symmetric, it too can be calculated via a shortcut $SS^*$, on the basis solely of yes-decisiveness; indeed, this is how SS is typically defined in the literature (e.g. Felsenthal and Machover 1998):

$$a_i^{SS^*}(S) = a_i^{SS}(S) = \begin{cases} 1 & \text{if } i \text{ is YES-decisive in } S \\ 0 & \text{otherwise} \end{cases}$$

and:
As with PB, we could also construct the corresponding shortcut via no-decisiveness.

### 2.5 The recursive measure

The recursive measure (RM) specifies division efficacy scores in terms of not just the voter’s decisiveness in the division, but recursively in terms of its degree of efficacy in effecting the outcome, i.e., allowing for partial efficacy. We take being fully (causally) efficacious in a division to be equivalent to being decisive for its outcome, i.e., full decisiveness satisfies the “simple counterfactual dependence test” of causation (Lewis 1973). Concomitantly, we take a voter to be partially (causally) efficacious for an outcome when the player plays a causal role but is not fully decisive. This partial efficacy typically occurs, for example, when an outcome is causally overdetermined. Consider majority-rule voting with three voters. If all three voters vote yes, then no individual is decisive (fully efficacious), even though each player plays some causal role in effecting the yes outcome: each player is partially efficacious.\(^1\)

We formalize the notion of partial efficacy in voting outcomes, and a player’s degree of efficacy, via the concept of a division’s loyal children. Call a division \(\mathcal{S}\) winning if its outcome is yes and losing if its outcome is no. For winning divisions, we say that a division \(\mathcal{T}\) is a loyal child of \(\mathcal{S}\) (and \(\mathcal{S}\) is a loyal parent of \(\mathcal{T}\)) if and only if \(\mathcal{S} = \mathcal{T} \cup \{j\}\). That is, \(\mathcal{T}\) is identical to \(\mathcal{S}\) except that exactly one less player votes yes in \(\mathcal{T}\) than in \(\mathcal{S}\). The nomenclature loyal refers to the fact that \(\mathcal{S}\) and \(\mathcal{T}\) have the same outcome. Symmetrically, for losing divisions, we say that \(\mathcal{T}\) is a loyal child of \(\mathcal{S}\) (and \(\mathcal{S}\) is a loyal parent of \(\mathcal{T}\)) if and only if \(\mathcal{S} = \mathcal{T} \setminus \{j\}\). That is, \(\mathcal{T}\) is identical to \(\mathcal{S}\) except that exactly one less player votes no in \(\mathcal{T}\) than in \(\mathcal{S}\). Moreover, we call a division’s loyal descendants those divisions that are its loyal children, their loyal children, and so on. A player’s degree of causal efficacy can then be defined recursively: if the player is unsuccessful, then it has zero efficacy; if the player is successful and decisive, then it has full efficacy; otherwise, if the division has loyal children that in turn have no more loyal children, then the player’s efficacy is equal to the proportion of loyal children in which it would have been decisive, whereas if the division’s loyal children themselves have loyal children, then we continue in the same manner recursively, until there are no more loyal children.

We therefore define RM by specifying the division efficacy score recursively, for SVGs, as:

\[
\gamma^{\text{SSV}}(\mathcal{S}) = \frac{(|\mathcal{S}| - 1)! \cdot (n - |\mathcal{S}|)!}{n!}
\]

As with PB, we could also construct the corresponding shortcut via no-decisiveness.

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1. On causal overdetermination and defence of the notion of degrees of causal efficacy, see Wright 1985, 1988; Schaffer 2003; McDermott 1995; Ramachandran 1997; Hitchcock 2001; Halpern and Pearl 2005; Hall 2007; Braham and van Hees 2009. For a defence of the conceptual foundations of RM as grounded in partial efficacy, see Abizadeh (2022). For the motivation for the specific design of RM, see Abizadeh and Vetta (2021).
The blocker postulates for measures of voting power

\[ \alpha_i^{RM}(\mathbb{S}) = \begin{cases} 
1 & \text{if } i \text{ is decisive in } \mathbb{S} \\
0 & \text{if } i \text{ is not successful in } \mathbb{S} \\
\frac{1}{|LC(\mathbb{S})|} \cdot \sum_{\mathbb{S} \in LC(\mathbb{S})} \alpha_i^{RM}(\mathbb{S}) & \text{otherwise}
\end{cases} \]

where \( LC(\mathbb{S}) \) denotes a division \( \mathbb{S} \)’s set of loyal children in \( D \).

The division weight is interpreted (as with PB) as the division’s probability \( \mathbb{P}(\mathbb{S}) \). Since RM itself is a generalized (not specifically a priori) measure, we mark its a priori version by labelling it as \( \text{RM}' \). A priori voting power under RM again assumes equiprobable divisions; the division weight is therefore equal to:

\[ \gamma^{RM'}(\mathbb{S}) = \mathbb{P}(\mathbb{S}) = \frac{1}{|D|} = \frac{1}{2^n} \]

Whereas PB represents a probability (the player’s probability of being decisive), RM represents an expected value, namely, the player’s expected efficacy (which is a function of the player’s degree of efficacy in each division weighted by the division’s probability). Note that because RM’s division efficacy score tracks partial efficacy, the measure is not strategy symmetric; hence the familiar shortcut is unavailable, and no-efficacy must be accounted for separately from yes-efficacy. Since RM is the less familiar measure, we provide an illustrative calculation in the Appendix A.1.

3 The subadditivity blocker postulates

In this section we will study the subadditivity blocker postulates; we consider the minimum-power blocker postulates and the added-blocker postulate in Sects. 4 and 5, respectively. In each section, we shall conclude by stating which measures satisfy the postulates; proofs for all technical theorems are provided in the Appendix A.2. However, since the subadditivity blocker postulates that we shall consider concern blockers (vetoers) that are members of a bloc of voters, we set up and motivate our analysis of these postulates by first considering (in Sect. 3.1) three bloc postulates not involving blockers; with these preliminaries in place, we then motivate two subadditivity blocker postulates (in Sect. 3.2). Readers beware: despite the similar spelling in English, a bloc (without the k) is entirely distinct from a blocker. A bloc is a group of amalgamated voters that acts as a single unit; a blocker is a voter who can veto outcomes on its own.

3.1 A preliminary consideration of some bloc (not blocker) postulates

The “conventional wisdom that the whole is greater than—or at least equal to—the sum of its parts” might suggest that it would be paradoxical if the a priori voting power of a bloc of voters turned out to be less than the sum of the a priori voting power of each individual bloc member prior to forming the bloc (Brams 1975: 178). We can formalize this conventional wisdom via a superadditivity postulate, concerning the lower bounds of a bloc’s voting power, as follows. Let \( \mathcal{G} \) be the voting game
derived from $G$ when a subset of players $I \subseteq N$ form a voting bloc. We model the formation of a bloc in two stages: first, all members of $I$ fully donate their votes to a single lead member, effectively rendering the donating players dummies; second, the dummies are then deleted from the game and the lead member is relabeled as the bloc $I$. (Recall that, by one of the dummy postulates, every player’s a priori voting power remains the same if a dummy is deleted.) Consider the case of $I = \{i, j\}$, $|I| = 2$. The full donation from $j$ to $i$ induces a new (monotonic) game $\tilde{G}$ given (for all $S$ containing neither $i$ nor $j$) by:

$$S \cup \{i, j\} \in \tilde{W} \iff S \cup \{i\} \in W$$

$$S \cup \{i\} \in \tilde{W} \iff S \cup \{i, j\} \in W$$

$$S \cup \{j\} \in \tilde{W} \iff S \in W$$

$$S \in \tilde{W} \iff \bar{S} \in W$$

This completes the first stage, where we have a voting game $\tilde{G}$ in which $j$ has fully donated to $i$. Next, since $j$ is now a dummy, we can derive $\hat{G}$ from $\tilde{G}$ by eliminating $j$, and relabelling $i$ as the bloc $I$. For the case $|I| \geq 3$ we generate $\hat{G}$ simply by iterating this transformation.

A measure of voting power $\Psi$ (where $\hat{\Psi}_i$ is $i$’s voting power in $\hat{G}$) satisfies the superadditivity bloc postulate if, for any bloc $I \subseteq N$:

$$\hat{\Psi}_I \geq \sum_{i \in I} \Psi_i$$  

As Felsenthal and Machover (1998: 224–231) have argued, however, such a postulate would be poorly motivated for measures of voting power, and its violation not truly paradoxical. (Indeed, all three of our candidate measures would violate such a postulate.\(^2\)) There are two basic reasons for this. The first is that players who form a bloc lose their ability to act as separate individuals, thus foreclosing possible strategies that otherwise might have been available to them. It is therefore unreasonable to expect a bloc’s power always to equal or exceed the sum of its members’ power individually. Call this argument for the unreasonability of a superadditivity postulate the loss-of-freedom rationale.

The second argument stems from a decreasing-marginal-returns dynamic. Decreasing marginal returns, a discrete analogue of concavity, is a central concept in economics. Individually, it applies when each additional unit of effort yields less incremental benefit than the previous unit. Collectively, it applies when the effort of an additional individual yields less incremental benefit when added to a larger group than a smaller group. For voting games, this effect is widespread. For example, a voter may be decisive in a small bloc but not decisive in a larger bloc. More generally, increasing the number of voters who vote with a voter above the minimum sufficient to ensure success may decrease the efficacy of the voter.

\(^2\) However, it is well-known that the Penrose–Banzhaf measure satisfies a weaker variant of the postulate, restricted to when the bloc is formed from only two players; indeed, the superadditivity two-player bloc postulate has been used as an axiom to provide axiomatizations of the Banzhaf value (Lehrer 1988; Casajus 2012).
Neither the loss-of-freedom nor the decreasing-marginal-returns rationale, however, rules out all expectations concerning lower bounds on a bloc’s voting power. For example, it is reasonable to expect a bloc to be just as powerful as any member would have been individually on its own: on the one hand, the bloc as a whole has just as much freedom as any of its individual members would have had on their own; on the other, there is no reason why adding or donating one voter’s power to another would diminish the latter’s individual power (as noted, forming a bloc can be represented as all members transferring their voting power to a lead member). We can formalize this expectation as follows. Again, let \( \mathcal{G} \) be the voting game derived from \( G \) by forming a voting bloc \( I \subseteq N \). A measure of voting power \( \Psi \) satisfies the minimum-power bloc postulate if, for any bloc \( I \subseteq N \):

\[
\Psi_I \geq \max_{i \in I} \Psi_i.
\]

Felsenthal and Machover (1998: 255–56) have already shown that PB and SS satisfy the postulate,\(^3\) while Abizadeh and Vetta (2021) show that RM satisfies it.

The superadditivity bloc postulate, which we rejected, and the minimum-power bloc postulate, which we accept, both concern the lower bounds of a bloc’s power. Are there reasonable expectations about upper bounds, motivated by the loss-of-freedom and decreasing-marginal-returns rationales? The most stringent expectation would be that a bloc’s voting power never be greater than the sum of its members’ voting power prior to forming the bloc. A measure of voting power \( \Psi \) would meet this expectation if it satisfied the subadditivity bloc postulate, i.e., if, for any bloc \( I \subseteq N \):

\[
\Psi_I \leq \sum_{i \in I} \Psi_i.
\]

It is not reasonable, however, to expect measures of voting power to satisfy the subadditivity bloc postulate. The reason is because the loss-of-freedom and decreasing-marginal-returns dynamics may in some circumstances be undercut or neutralized by two corresponding countervailing dynamics. First, as Felsenthal and Machover (1998: 229) have argued, when a bloc of at least three members is formed, whether the bloc’s voting power is less or greater than the sum of its members’ original, pre-bloc voting power will often depend on whether, in the original voting structure, the divisions in which any given individual would-be bloc member was efficacious tend to be ones in which other would-be bloc members vote against or with each other.\(^4\) From amongst divisions in which would-be bloc members are

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\(^3\) Although they only show this for blocs of two players, the result follows for blocs of any size by applying their result sequentially.

\(^4\) One could weaken the super- and subadditivity bloc postulates by restricting them to two-player blocs in order to sidestep this dynamic. For example, proxy neutrality (Haller 1994; Casajus 2014) is the property according to which the sum of two voters’ voting power would remain the same if one were to fully donate its voting power to the other. Once we eliminate the donating player (now a dummy), proxy neu-
efficacious, the higher the proportion of divisions in which some members vote against each other, the more we should expect the sum of their individual voting powers to be higher relative to the bloc’s voting power (because, once the bloc is formed, these “high-efficacy” divisions no longer exist); by contrast, the higher the proportion of divisions, from amongst those in which would-be members are efficacious, in which would-be members vote together, the more we should expect the sum of individual voting powers to be less relative to the bloc’s voting power—and hence the more we should expect the voting structure to induce a violation of the subadditivity bloc postulate. In essence, this latter, inefficacious-dissension dynamic neutralizes the loss-of-freedom dynamic.

Second, decreasing marginal returns may sometimes (for example with small voting blocs) be countervailed by increasing marginal returns. The latter dynamic may arise because increasing the proportion of divisions in which more voters vote with a given voter often increases (and, for monotonic games, never reduces) the proportion of divisions in which the voter is successful and, hence, potentially efficacious or decisive.

Indeed, none of our candidate measures satisfies the subadditivity bloc postulate. For example, consider three-voter unanimity rule in which the three voters subsequently form a bloc. This is clearly a case displaying both inefficacious dissension (each voter is decisive in only two divisions, in both of which other would-be bloc members all vote together) and increasing marginal returns (the bloc is successful in all divisions). Prior to forming the bloc, for each voter $PB_i = \frac{1}{3}$ and, hence, $\sum PB_i = \frac{3}{4}$. But once the bloc is formed, $PB_i = 1$, in violation of subadditivity. For SS and RM, consider the weighted voting game $G = \{2 : 1, 1, 2, 2, 2\}$ where there are five voters with weights $\{1, 1, 2, 2, 2\}$ and a quota of 2. The reader may verify that for both SS and RM, $\hat{\psi}_{(1,2)} > \psi_1 + \psi_2$. (In particular, $\hat{SS}_{(1,2)} = \frac{1}{4} > SS_1 + SS_2 = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$; and $\hat{RM}_{(1,2)} = \frac{19}{64} > RM_1 + RM_2 = \frac{41}{320} + \frac{41}{320} = \frac{41}{160}$.)

Thus, while the loss-of-freedom and decreasing-marginal-returns rationales render unreasonable the expectation that the bloc’s power always be greater than or equal to the sum of its members (the superadditivity bloc postulate), they do not rule out the possibility that, on some occasions, a bloc’s voting power may indeed be greater.

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Footnote 4 (continued)

tality becomes equivalent to the combination of the superadditivity and subadditivity bloc properties when restricted to two-player blocs. Although some of have used proxy neutrality (and two other, closely related “collusion” properties, namely association neutrality and distrust neutrality (Haller 1994; Malawski 2002; Casajus 2014)) to provide axiomatizations of the Penrose–Banzhaf measure, none of these authors has provided any independent normative justification for why it would be reasonable to require measures of voting power in general to satisfy a postulate corresponding to such properties. Because our interest in this paper is with blocker postulates and not bloc postulates per se, we do not here investigate these alternative, weaker postulates any further.

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3.2 Two subadditivity blocker postulates

We are nevertheless able to specify reasonable, weaker expectations about a bloc’s upper bounds, grounded in the loss-of-freedom and decreasing-marginal-returns rationales, under conditions in which the countervailing inefficacious-dissension and/or increasing-marginal-returns rationales are neutralized or overwhelmed. We shall specify two such weaker postulates concerning a bloc’s upper bounds under the condition that the bloc contains at least one blocker (or vetoer). We say that a player \( i \) is a **yes-blocker** if for every \( S \in W \) we have \( i \in S \). Consequently, if \( i \) votes **no** then the outcome is **no**. That is, \( i \) can veto a **yes**-outcome. Similarly, a player \( j \) is a **no-blocker** if for every \( S \notin W \) we have \( j \notin S \). That is, if \( j \) votes **yes** then the outcome is **yes** and \( j \) can block or veto a **no**-outcome.

Let \( \hat{G} \) be the voting game derived from \( G \) by forming a voting bloc \( I \subseteq N \). Then a measure of voting power \( \Psi \) satisfies the **strong subadditivity blocker postulate** if:

\[
\text{(sbk-1)} \quad \hat{\psi}_I \leq \sum_{i \in I} \psi_i \text{ for any } I \text{ containing a yes-blocker } b.
\]

\[
\text{(sbk-2)} \quad \hat{\psi}_I \leq \sum_{i \in I} \psi_i \text{ for any } I \text{ containing a no-blocker } b.
\]

That is, the a priori voting power of \( I \) in \( \hat{G} \) is no greater than the sum of the a priori voting power of each of its members separately in \( G \), provided \( I \) contains a yes-blocker (or no-blocker).

Why would the strong subadditivity blocker postulate be reasonable? On the one hand, the loss-of-freedom dynamic, which favours subadditivity, is still in place: since a bloc member \( j \in I \) is forced to coordinate its vote with the bloc, any contributions it could have made in divisions in which members would have voted against each other are excluded; only the player’s contribution when the bloc votes together counts. But the presence of a blocker \( b \) also puts into play the countervailing inefficacious-dissension dynamic: if the blocker is a yes-blocker, for example, other would-be bloc members could not have been efficacious on their own in any division in which the blocker voted **no** and they voted against it. So the loss-of-freedom dynamic on its own is insufficient to secure the subadditivity blocker postulate. On the other hand, the presence of a blocker \( b \)—let us say a yes-blocker—*strengthens* the decreasing-marginal-returns dynamic. When \( j \) transfers its voting power to a bloc that contains a yes-blocker, its presence cannot help increase the bloc’s division efficacy score on the no-side any further than the score would have been without \( j \): when the bloc votes **no**, the outcome is **no** regardless of whether or not \( j \) joins the bloc. That is, for (sbk-1), the presence of player \( j \) can help increase the bloc’s voting power only on the basis of divisions in which the bloc votes together and when the yes-blocker \( b \) votes yes. Thus \( j \)'s marginal contributions will be even less. A symmetric argument motivates (sbk-2).

We can further weaken the subadditivity blocker postulate by considering only the case in which *every* member of the bloc is a blocker. A measure of voting power \( \Psi \) satisfies the **weak subadditivity blocker postulate** if:
(wbk-1) \( \hat{\psi}_I \leq \sum_{i \in I} \psi_i \) for any \( I \) containing only YES-blockers.

(wbk-2) \( \hat{\psi}_I \leq \sum_{i \in I} \psi_i \) for any \( I \) containing only NO-blockers.

That is, the a priori voting power of \( I \) in \( \mathcal{G} \) is no greater than the sum of the a priori voting power of each of its members separately in \( \mathcal{G} \), provided every member of \( I \) is a YES-blocker (or every member of \( I \) is a NO-blocker). Observe that (wbk-1) is indeed weaker than (sbk-1) because it only need hold in the restricted case where the bloc \( I \) consists entirely of YES-blockers. For example, the decreasing-marginal-returns rationale may hold only when the added agent \( j \) is itself a YES-blocker (as well as the other members of the bloc) and need not hold when \( j \) is not a blocker.

We now determine whether or not PB, SS and RM satisfy the subadditivity blocker postulates. (See Appendix A.2 for proofs.)

**Theorem 3.1** PB fails to satisfy the weak subadditivity blocker postulate (and, thus, fails to satisfy the strong subadditivity blocker postulate).

**Theorem 3.2** SS satisfies the weak subadditivity blocker postulate but does not satisfy the strong subadditivity blocker postulate.

**Theorem 3.3** RM satisfies the strong subadditivity blocker postulate (and, thus, satisfies the weak subadditivity blocker postulate).

## 4 The minimum-power blocker postulates

Felsenthal and Machover (1998: 264) introduce what they call the **blocker’s share postulate**, which is satisfied by a measure \( \Psi \) if the share of any YES-blocker’s a priori voting power, out of the sum total of all players’ a priori voting power, is at least as great as the reciprocal of the number of players in any YES-successful set of voters (and similarly for NO-blockers):

(bsp-1) If \( b \in S \) is a YES-blocker, then \( \frac{\psi_b}{\sum_{i \in S} \psi_i} \geq \frac{1}{|S|} \), for any \( S \in \mathcal{W} \).

(bsp-2) If \( b \in \bar{T} \) is a NO-blocker, then \( \frac{\psi_b}{\sum_{i \in \bar{T}} \psi_i} \geq \frac{1}{|\bar{T}|} \), for any \( T \not\in \mathcal{W} \).

These lower bounds of a blocker’s voting power are of course most stringent when \( S \) is the smallest possible set of successful YES-voters \( S^* \), and \( \bar{T} \) is the smallest possible set of successful NO-voters \( \bar{T}^* \). In effect, the postulate requires that for any YES-blocker, its share of a priori voting power, out of the sum of all players’ a priori voting power, be no less than \( \frac{1}{|S^*|} \). Felsenthal and Machover then prove that PB violates the postulate, i.e., that when PB is normalized such that all players’ scores sum to 1,
which yields each player’s relative voting power according to what they call the Banzhaf index, a yes-blocker’s relative a priori voting power according to the Banzhaf index may be less than $\frac{1}{|S^*|}$, whereas SS (which is itself already a relative index, since $\sum_{i \in N} SS_i = 1$) satisfies it. It can be shown that RM, like PB, also violates the blocker’s share postulate.

Does this speak against PB and RM? It does not: the postulate is unmotivated for voting power. The intuition behind the postulate does not concern voting power as such but, rather, its expected value (as is implicit to Felsenthal and Machover’s justification for the postulate). Consider a non-voter for whom the value of a yes-outcome is equal to $f > 0$, and who would therefore be willing to spend up to $f$ to buy the votes of a set of voters capable of ensuring a yes-outcome. This set must include any yes-blocker, if there is one. Assume the voting structure has at least one yes-blocker, the non-voter knows the voting structure, but has no information about the distribution of player preferences (modeling the a priori case). Without information about voter inclinations, the most efficient strategy is to bribe the smallest possible set of successful yes-voters $S^*$, to minimize the total bribe necessary to secure the desired outcome. What is the (subjective) expected value, to the non-voter, of the yes-blocker’s vote, i.e., the value of bribing the yes-blocker rather other players to realize the desired yes-outcome? If every member of $S^*$ is a yes-blocker, then the expected value of any yes-blocker’s vote to the non-voter will be equal to that of any other yes-blocker, which implies the non-voter would be willing to offer each yes-blocker a bribe of up to $\frac{1}{|S^*|} \cdot f$. If, by contrast, not all members of $S^*$ are yes-blockers, then the smallest possible set of successful yes-voters need not be unique, i.e., there may be more than one such minimal set of voters. And since only a yes-blocker will be a member of every such minimal set, the expected value of its vote to the non-voter will be at least as great as that of any other potential member of these minimal sets. Therefore, $\frac{1}{|S^*|} \cdot f$ is the minimum value of a yes-blocker’s vote to the non-voter, and $\frac{1}{|S^*|}$ its minimum relative value. And this is precisely what the blocker’s share postulate says: that the relative value of a yes-blocker’s a priori voting power should be at least $\frac{1}{|S^*|}$. We therefore conclude that the blocker’s share postulate is appropriate not for measures of a priori voting power, but, rather, for measures of the expected value of a player’s a priori voting power. (The fact that SS satisfies the blocker’s share postulate provides support, in other words, for the view that, considered as an a priori index, it is best interpreted, not as an index of relative a priori voting power, but, rather, as an index of a player’s expected payoff assuming a cooperative game with transferable utility).

It is possible, however, to reformulate the postulate in a way that would be appropriate for measures of voting power, by focussing on a player’s a priori power itself, rather than its share of overall power. The key is to compare its a priori power against the voting power that a measure of voting power would assign to a dictator in a dictator-rule SVG. (A dictator is a player $d$ such that if $d \in S$ then $S \in \mathcal{W}$ and

---

5 Felsenthal and Machover put it in terms of their distinction between I-power and P-power.

6 On SS as a bribe index, see Morriss (2002); on the equivalent expected payoff interpretation, see Felsenthal and Machover (1998).
if \( d \not\in S \) then \( S \not\in \mathcal{W} \), i.e., whose vote always unilaterally determines the outcome.)

To formalize the comparison, let \( S \) be a yes-successful set in \( \mathcal{G} \) (that is, \( S \in \mathcal{W} \)), \( \bar{T} = N \setminus T \) be a no-successful set in \( \mathcal{G} \) (that is, \( T \not\in \mathcal{W} \)), and \( \psi_d \) be the voting power of player \( d \) in \( \mathcal{G}' \), where \( \mathcal{G}' \) is any dictator-rule SVG where \( d \) is the dictator.

A measure of voting power \( \Psi \) satisfies the **strong minimum-power blocker postulate** if:

\[
\text{(SMP-1)} \quad \text{If } b \in S \text{ is a yes-blocker, then } \psi_b \geq \frac{\psi_d}{|S|}, \text{ for any } S \in \mathcal{W}.
\]

\[
\text{(SMP-2)} \quad \text{If } b \in \bar{T} \text{ is a no-blocker, then } \psi_b \geq \frac{\psi_d}{|\bar{T}|}, \text{ for any } T \not\in \mathcal{W}.
\]

A measure of voting power \( \Psi \) satisfies the **weak minimum-power blocker postulate** if:

\[
\text{(WMP-1)} \quad \text{If every voter in } S \text{ is a yes-blocker, then } \psi_b \geq \frac{\psi_d}{|S|}, \text{ for all } b \in S, \text{ for any } S \in \mathcal{W}.
\]

\[
\text{(WMP-2)} \quad \text{If every voter in } \bar{T} \text{ is a no-blocker, then } \psi_b \geq \frac{\psi_d}{|\bar{T}|}, \text{ for all } b \in \bar{T}, \text{ for any } T \not\in \mathcal{W}.
\]

The strong minimum-power blocker postulate obviously implies the weak minimum-power blocker postulate. We remark that \( \psi_d = 1 \) for a dictator \( d \) for each of the three voting measures studied in this paper. In fact, multiplying \( \psi_i \) by a fixed scalar for each voter \( i \) has no effect on the structure of the voting game nor on the satisfaction of any of the postulates. The fact that \( \psi_d = 1 \) for each of three voting measures essentially means their scaling factors are all identical.

To understand the minimum-power blocker postulates, let’s begin with a sanity check. Suppose \( S = \{b\} \) with cardinality 1. If \( \{b\} \) is a yes-successful set and \( b \) is a yes-blocker, then a division \( \mathbb{R} = (R, \bar{R}) \) is winning if and only if \( b \in R \). That is, \( b \) is a dictator! It follows that \( \psi_b = \psi_d \geq \frac{\psi_d}{1} \) and the weak minimum-power blocker postulate (WMP-1) holds for \( S = \{b\} \). A similar argument applies for (WMP-2).

Suppose, by contrast, that \( S \) is a yes-successful set containing more than one player, but each is a yes-blocker. Then \( \mathbb{R} = (R, \bar{R}) \) is winning if and only if \( S \subseteq R \). Thus, collectively \( S \) is a dictatorship. But, in addition, each voter \( b \in S \) has the individual power to veto any yes-outcome. Thus the agents in \( S \) have the option of choosing to act collectively as dictator, and individually each must be in any yes-successful set. It is thus reasonable to expect the voting power of any member of \( S \) to satisfy \( \psi_b \geq \frac{\psi_d}{|S|} \) and the weak minimum-power blocker postulate to hold.

A similar argument applies for the strong minimum-power blocker postulate. The agents in \( S \) can choose to act collectively as a dictator. But since \( S \) may contain non-blockers, this is only because a yes-blocker \( b \) has the power to veto any yes-outcome. Thus \( b \) should have power at least as large as any other member of \( S \) and, specifically, at least as the large as the average, and hence \( \psi_b \geq \frac{\psi_d}{|S|} \).
We next determine whether PB, SS and RM satisfy the minimum-power blocker postulates. (See Appendix A.2 for proofs.)

**Theorem 4.1**  PB does not satisfy the weak minimum-power blocker postulate (and, thus, does not satisfy the strong minimum-power blocker postulate).

**Theorem 4.2**  SS satisfies the strong minimum-power blocker postulate (and, thus, satisfies the weak minimum-power blocker postulate).

**Theorem 4.3**  RM satisfies the strong minimum-power blocker postulate (and, thus, satisfies the weak minimum-power blocker postulate).

5  The added-blocker postulate

We conclude with the added-blocker postulate, which concerns changes to other players’ a priori voting power when a blocker is added to a game.

The first step is to formulate an added-blocker postulate that is appropriate for a priori voting power in general. Given a game \( G = (N, W) \), let \( G^{YB} = (N \cup \{0\}, W^{YB}) \) be the game resulting from adding an added yes-blocker, i.e., a new player 0 that is a yes-blocker but who otherwise does not affect the original voting structure. Specifically, \( W^{YB} = \{S \cup \{0\} : \forall S \in W\} \). Similarly, let \( G^{NB} = (N \cup \{0\}, W^{NB}) \) be the game resulting from adding an added no-blocker 0. Specifically, \( W^{NB} = \{S \cup \{0\} : \forall S \in W\} \cup \{S : \forall S \in W\} \).

Felsenthal and Machover (1998: 266–275) argue that any reasonable measure of voting power \( \Psi \) must satisfy the following postulate for a priori voting power. For any pair of players \( i \) and \( j \),

\[
\frac{\psi_i(G)}{\psi_j(G)} = \frac{\psi_i(G^{YB})}{\psi_j(G^{YB})}.
\]

That is, the relative measures of a priori voting power for \( i \) and \( j \) should be unaffected by an added yes-blocker. They argue “there is nothing at all to imply that the addition of the new” yes-blocker “is of greater relative advantage to some of the voters” of the original game than to others, because there is “no reasonable mechanism that would create a differential effect” (Felsenthal and Machover 1998: 267). They then show that PB satisfies this postulate, but SS violates it, and, on this basis, conclude that the latter cannot be considered a valid index of a priori voting power.7

But there is a problem: their specification is asymmetric between yes-voting and no-voting power.8 Contrary to their assertion, in general we do have good

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7 Where voting power is understood as the capacity to influence voting outcomes (Felsenthal and Machover 1998: 267–275).

8 A similar issue arises with the bicameral postulate (Felsenthal et al. 1988).
reasons to expect an added yes-blocker sometimes to have differential relative impact on players’ a priori voting power as a whole, depending on the relative importance, to each player’s total voting power, of its yes- as opposed to no-voting power. This is because an added yes-blocker may diminish the relative significance or share of yes-voting power within a player’s total voting power.

We should expect this potential asymmetry between yes- and no-voting power to be neutralized only for measures of voting power that, like PB and SS, give a positive efficacy score only in cases of (full) decisiveness. Such measures, by ignoring partial efficacy, effectively render a player’s a priori yes- and no-voting power perfectly symmetrical, that is, $\psi_i^+ = \psi_i^-$: any player that is yes-decisive in a winning division will also be no-decisive in the corresponding losing division in which the only difference is that player’s vote. By contrast, this symmetry between yes- and no-voting power will not hold for measures that, like RM, take degrees of efficacy into account. A player that is only partially efficacious in a winning division will not be efficacious at all in the corresponding division in which all other players’ votes are held constant, because, not being (fully) decisive, the player’s switch from yes to no will not change the outcome—which switches the player from successful in one division to unsuccessful in the other. Recall: RM is not strategy symmetric. The implication is that, if a player’s total voting power relies more heavily on partial efficacy in winning divisions than does that of another player, then an added yes-blocker may have a disproportionately negative impact on the former than on the latter. We therefore have no reason in general to expect an added yes-blocker never to result in some players’ relative advantage.

By contrast, we have every reason to expect that an added yes-blocker will be of no relative advantage to players’ a priori yes-voting power and that an added no-blocker will be of no relative advantage to players’ a priori no-voting power in particular. This induces an easy fix to Felsenthal and Machover’s proposal, so as to yield a postulate appropriate for all efficacy measures in general (and not just decisiveness measures). We simply reformulate their postulate to distinguish between yes-voting power and no-voting power. Accordingly, we say that a measure of voting power $\Psi$ satisfies the added-blocker postulate if, for any pair of players $i$ and $j$, the following conditions hold for a priori voting power:

\[(\text{ADD-1}) \quad \frac{\psi_i^+(G)}{\psi_j^+(G)} = \frac{\psi_i^+(G^{\text{yes}})}{\psi_j^+(G^{\text{yes}})}, \text{ and} \]

\[(\text{ADD-2}) \quad \frac{\psi_i^-(G)}{\psi_j^-(G)} = \frac{\psi_i^-(G^{\text{no}})}{\psi_j^-(G^{\text{no}})}. \]

(ADD-1) says that the relative measures of a priori yes-voting power are unaffected by an added yes-blocker. Hence, we call this element of the postulate the added-yes-blocker postulate for any pair of players $i$ and $j$. (ADD-2) says the relative measures of a priori no-voting power are unaffected by an added no-blocker. We similarly call this element the added-no-blocker postulate.
To finish we determine whether or not PB, SS and RM satisfy the added-blocker postulate. (See Appendix A.2 for proofs).

**Theorem 5.1** PB satisfies the added-blocker postulate.

**Theorem 5.2** SS does not satisfy the added-blocker postulate.

**Theorem 5.3** RM satisfies the added-blocker postulate.

## 6 Conclusion

We have specified and motivated five reasonable postulates involving blockers, and concerning a priori voting power, that a measure of voting power should satisfy: the strong and weak subadditivity blocker postulates, the strong and weak minimum-power blocker postulates, and the added-blocker postulate. We further showed that the classic Penrose–Banzhaf measure violates the two subadditivity blocker postulates and the two minimum-power blocker postulates, while the classic Shapley–Shubik index violates the strong subadditivity blocker postulate and the added-blocker postulate. These violations weaken the plausibility of PB and SS as measures of voting power. By contrast, the recursive measure, alone amongst the three measures studied here, withstands full scrutiny: it satisfies all five postulates. We take this finding considerably to buttress the plausibility of RM as a measure of voting power.

## Appendices

### A.1 An illustrative calculation of RM

Consider the weighted voting game $\{4 : 3, 2, 1\}$. Here players 1, 2 and 3 have voting weights 3, 2 and 1, respectively, and a voting threshold of 4 votes is required for a YES-outcome. Consequently, the winning divisions are those whose set of YES-successful voters are $\{1, 2\}$, $\{1, 3\}$, and $\{1, 2, 3\}$, while the losing divisions are those whose set of (unsuccessful) YES-voters are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{2, 3\}$. For example, let’s calculate the RM division efficacy scores for player 2. We begin with the winning divisions and its two base cases. First, for any winning division in which player 2 is YES-decisive, the player’s division efficacy score is 1. Thus $a_{2}^{RM}(\{(1, 2), \{3\}\}) = a_{2}^{RM}(\{(1, 2), \{3\}\}) = 1$. Second, for any winning division in which player 2 votes NO, its efficacy score is 0. Thus $a_{2}^{RM}(\{(1, 3), \{2\}\}) = 0$. Player 2’s efficacy scores in the remaining winning divisions are calculated recursively and equal its average scores in the division’s loyal children. There is only one such division, namely $\{(1, 2, 3), \emptyset\}$. It has two loyal children, namely $\{(1, 2), \{3\}\}$.
and $\{1, 3\}, \{2\}$, in which player 2’s efficacy scores are 1 and 0, respectively. Thus $\alpha_2^{RM}(\{1, 2, 3\}, \emptyset) = \frac{1}{2}$.

Next, consider the losing divisions. First, player 2’s efficacy score is 1 in any losing division in which it is no-decisive. Thus $\alpha_2^{RM}(\{1\}, \{2, 3\}) = \alpha_2^{RM}(\{1\}, \{2, 3\}) = 1$. Second, for any losing division in which player 2 votes yes, its efficacy score is 0. Thus $\alpha_2^{RM}(\{2, 3\}, \{1\}) = \alpha_2^{RM}(\{2, 1\}, \{3\}) = 0$. The efficacy scores of the remaining losing divisions are calculated recursively and equal the average scores of their loyal children. There are two such divisions, namely $\{3\}, \{1, 2\}$ and $\emptyset, \{1, 2, 3\}$. The former has only one loyal child, namely $\{2, 3\}, \{1\}$, in which player 2’s efficacy score is 0. Thus $\alpha_2^{RM}(\{3\}, \{1, 2\}) = 0$. The division $\emptyset, \{1, 2, 3\}$, by contrast, has three loyal children, namely $\{1\}, \{1, 2, 3\}, \{2\}, \{1, 3\}$, and $\{3\}, \{1, 2\}$, in which player 2’s efficacy scores are 1, 0, and 0, respectively. Thus $\alpha_2^{RM}(\emptyset, \{1, 2, 3\}) = \frac{1}{3}$.

Therefore, the a priori voting power of player 2 under RM is

$$RM'_2 = \frac{1}{8} \cdot \left(4 \cdot 0 + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{2} + 2 \cdot 1\right) = \frac{17}{48}.$$ 

While not necessary for the results in this paper, there is an elucidative way to calculate RM using a division lattice. (See Abizadeh and Vetta (2021) for motivations for the division lattice approach and for formal definitions.) For the
weighted voting game \( \{4 : 3, 2, 1\} \), the division lattice is shown in Fig. 1, where each division is represented by its set of \(\text{yes}\)-voters and winning divisions are shaded. The winning and losing divisions induce two posets called the \(\text{yes}\)-poset and the \(\text{no}\)-poset, respectively. These posets, illustrated in Figs. 2 and 3, where the \(\text{no}\)-poset is shown inverted from the division lattice, naturally show the loyal children of each division. These posets can be used to recursively compute the RM division efficacy scores. Player 2’s efficacy scores are shown to the left of each division node, with the base case efficacy scores in bold.

A.2 Proofs for formal theorems

Proofs for Sect. 3

**Theorem 3.1** PB fails to satisfy the weak subadditivity blocker postulate (and, thus, fails to satisfy the strong subadditivity blocker postulate).

**Proof** Let \( G \) be a three-player unanimity-rule voting game, and let \( \hat{G} \) be the game derived from \( G \) when all three players form a unanimous bloc \( I \). By the unanimity rule, all three players are \(\text{yes}\)-blockers in \( G \). Now PB gives a value to each player of \( \frac{1}{4} \) and thus \( \sum_{i=1}^{3} PB_i = \frac{3}{4} \). But \( I \) is a dictator in \( \hat{G} \), so \( \hat{PB}_I = 1 > \frac{3}{4} \), in violation of each subadditivity blocker postulate.

**Theorem 3.2** SS satisfies the weak subadditivity blocker postulate but does not satisfy the strong subadditivity blocker postulate.

**Proof** First we show SS does not satisfy the strong subadditivity blocker postulate via a counterexample. Consider the weighted voting game \( \hat{G} = \{3 : 2, 1, 1\} \). Observe that voter 1 is a \(\text{yes}\)-blocker since voters 2 and 3 have voting weight 1 + 1 which is smaller than the quota of 3. Here \( SS_1 = \frac{2}{3} \) and \( SS_2 = SS_3 = \frac{1}{6} \). Now let \( \hat{G} \) be the game derived from \( G \) in which the first two players form a bloc \( I = \{1, 2\} \). It follows that \( I \) is a dictator in \( \hat{G} \), so \( \hat{SS}_I = 1 > \frac{2}{3} + \frac{1}{6} = \frac{5}{6} \), in violation of the postulate.

Second, we prove SS does satisfy the weak subadditivity blocker postulate. It suffices to show (wbk-1) holds. Assume \( I = \{i,j\} \). Let \(\text{yes}\)-blocker \( j \) donate to
YES-blocker $i$. Then $j$ becomes a dummy. Take an ordering $\sigma$ of the agents. We have three cases.

(i) Let $j$ be the pivotal voter in the ordering $\sigma$ for $G$. Let $S$ be the set of agents before $j$ in the ordering. Since $j$ is decisive in $S \cup \{j\}$ in $G$ but is a dummy in $\hat{G}$, it must be the case that $S$ is a YES-successful set in $G$. In particular, $i$ must appear before $j$ in $\sigma$ since $i$ is a YES-blocker. Thus $i \in S$. Since $S$ is YES-successful in $\hat{G}$ there must be some agent in $S$ that is now decisive. This may or may not be agent $S$ must be some agent in $\{i, j\}$.

(ii) Let $i$ be the pivotal voter in the ordering $\sigma$ for $G$. Let $S$ be the set of agents before $i$ in the ordering. So $S$ is YES-unsuccesful and $S \cup \{i\}$ is YES-successful in $G$. This must still be the case after the donation from $j$ to $i$. Thus $i$ remains the pivotal voter in the ordering $\sigma$ for $\hat{G}$.

(iii) Let $\ell \neq i, j$ be the pivotal voter in the ordering $\sigma$ for $G$. Let $S$ be the set of agents before $\ell$ in the ordering. Since $S \cup \{\ell\}$ is YES-successful in $G$, it must be the case that $i$ and $j$ are in $S$ since they are both YES-blockers. (Note, this is where the distinction between the weak and strong postulates is important.) But then, by definition, $S$ and $S \cup \{\ell\}$ have the same outcomes in $G$ and $\hat{G}$. Thus $\ell$ remains pivotal in the ordering $\sigma$ for $\hat{G}$.

It immediately follows that $\psi_i \leq \psi_i + \psi_j$. Iterating this argument, we have that $\psi_I \leq \sum_{i \in I} \psi_i$ for any $I$ consisting only of three or more blockers.

**Theorem 3.3** $RM$ satisfies the strong subadditivity blocker postulate (and, thus, satisfies the weak subadditivity blocker postulate).

The crux to proving Theorem 3.3 is the following lemma.

**Lemma 7.1** Let $j$ fully donate to $i$. If $j$ is a YES-blocker (or a NO-blocker), then $\hat{RM}^I_j \leq RM^I_i + RM^I_j$.

**Proof** Observe that

$$RM^I_i = \frac{1}{2^n} \cdot \sum_{S \in D : i \notin S} (\alpha_i(S) + \alpha_i(S \cup j) + \alpha_i(S \cup i) + \alpha_i(S \cup \{i, j\}))$$

$$= \frac{1}{2^n} \cdot \sum_{S \in D : i \notin S} (\alpha^-_i(S) + \alpha^-_i(S \cup j) + \alpha^+_i(S \cup i) + \alpha^+_i(S \cup \{i, j\}))$$

Thus to prove $\hat{RM}^I_i \leq RM^I_i + RM^I_j$ it suffices to show that

$$\hat{\alpha}^-_i(S) + \hat{\alpha}^-_i(S \cup j) + \hat{\alpha}^+_i(S \cup i) + \hat{\alpha}^+_i(S \cup \{i, j\})$$

$$\leq \left(\alpha^-_i(S) + \alpha^-_j(S)\right) + \left(\alpha^+_i(S \cup j) + \alpha^+_j(S \cup i)\right) + \left(\alpha^+_i(S \cup i) + \alpha^-_j(S \cup i)\right) + \left(\alpha^+_i(S \cup \{i, j\}) + \alpha^-_j(S \cup \{i, j\})\right)$$

Take any $S$ containing neither $i$ nor $j$. Since $j$ is a YES-blocker, $S$ and $S \cup \{i, j\}$ both win, or
Similarly, in Case (ii) observe that both

\[ i \text{ and } j \]

are decisive at both \( \mathbb{S} \cup j \) and \( \mathbb{S} \cup \{i, j\} \) both lose.

Thus there are three cases. In the modified game these three cases imply:

(i) \( \mathbb{S} \cup i \) and \( \mathbb{S} \cup \{i, j\} \) both win but \( \mathbb{S} \) and \( \mathbb{S} \cup j \) both lose.

(ii) \( \mathbb{S} \cup i \) and \( \mathbb{S} \cup \{i, j\} \) both win but \( \mathbb{S} \) and \( \mathbb{S} \cup j \) both lose.

(iii) \( \mathbb{S}, \mathbb{S} \cup j, \mathbb{S} \cup i \) and \( \mathbb{S} \cup \{i, j\} \) all lose.

The first two cases are easier to deal with. In Case (i) observe that \( j \) is yes-decisive at both \( \mathbb{S} \cup j \) and \( \mathbb{S} \cup \{i, j\} \) and no-decisive at both \( \mathbb{S} \) and \( \mathbb{S} \cup i \). Thus

\[
\left( \alpha_i^- (\mathbb{S}) + \alpha_j^- (\mathbb{S}) \right) + \left( \alpha_i^- (\mathbb{S} \cup j) + \alpha_j^+ (\mathbb{S} \cup j) \right) + \left( \alpha_i^+ (\mathbb{S} \cup i) + \alpha_j^- (\mathbb{S} \cup i) \right)
\]

\[
+ \left( \alpha_i^+ (\mathbb{S} \cup \{i, j\}) + \alpha_j^+ (\mathbb{S} \cup \{i, j\}) \right)
\]

\[
\geq (0 + 1) + (0 + 1) + (0 + 1) + (0 + 1)
\]

\[
\geq 4
\]

\[
\geq \hat{\alpha}_i^- (\mathbb{S}) + \hat{\alpha}_j^- (\mathbb{S} \cup j) + \hat{\alpha}_i^+ (\mathbb{S} \cup i) + \hat{\alpha}_j^+ (\mathbb{S} \cup \{i, j\})
\]

Similarly, in Case (ii) observe that both \( i \) and \( j \) are yes-decisive at \( \mathbb{S} \cup \{i, j\} \). Furthermore, \( i \) is no-decisive at \( \mathbb{S} \cup j \) and \( j \) is no-decisive at \( \mathbb{S} \cup i \). Thus

\[
\left( \alpha_i^- (\mathbb{S}) + \alpha_j^- (\mathbb{S}) \right) + \left( \alpha_i^- (\mathbb{S} \cup j) + \alpha_j^+ (\mathbb{S} \cup j) \right) + \left( \alpha_i^+ (\mathbb{S} \cup i) + \alpha_j^- (\mathbb{S} \cup i) \right)
\]

\[
+ \left( \alpha_i^+ (\mathbb{S} \cup \{i, j\}) + \alpha_j^+ (\mathbb{S} \cup \{i, j\}) \right)
\]

\[
\geq (0 + 0) + (1 + 0) + (0 + 1) + (1 + 1)
\]

\[
\geq 4
\]

\[
\geq \hat{\alpha}_i^- (\mathbb{S}) + \hat{\alpha}_j^- (\mathbb{S} \cup j) + \hat{\alpha}_i^+ (\mathbb{S} \cup i) + \hat{\alpha}_j^+ (\mathbb{S} \cup \{i, j\})
\]

Case (iii) is more complex. Recall that in this case \( \mathbb{S}, \mathbb{S} \cup j, \mathbb{S} \cup i \) and \( \mathbb{S} \cup \{i, j\} \) all lose in both the original game and the modified game. This implies that the yes-efficacy score of each voter is zero at these divisions. Thus it suffices to prove that

\[
\hat{\alpha}_i^- (\mathbb{S}) + \hat{\alpha}_j^- (\mathbb{S} \cup j) \leq \alpha_i^- (\mathbb{S}) + \alpha_j^- (\mathbb{S} \cup j) + \alpha_i^+ (\mathbb{S} \cup i) + \alpha_j^- (\mathbb{S} \cup i)
\]

(1)

In fact, we will prove something stronger. The following two inequalities hold.

\[
\hat{\alpha}_i^- (\mathbb{S}) \leq \alpha_i^- (\mathbb{S}) + \alpha_j^- (\mathbb{S})
\]

(2)

\[
\hat{\alpha}_i^- (\mathbb{S} \cup j) \leq \alpha_i^- (\mathbb{S} \cup j) + \alpha_j^- (\mathbb{S} \cup i)
\]

(3)

To show this we need the following important fact from Abizadeh and Vetta (2021). A no-efficacy score \( \alpha_i^- (\mathbb{S}) \) for RM can be calculated by considering paths in the division lattice from \( \mathbb{S} \) to \( \mathbb{N} = (\mathbb{N}, \emptyset) \). The division lattice contains a node for each division \( \mathbb{S} \). There is an arc in the lattice from \( \mathbb{S} \) to \( \mathbb{S} \cup j \), for each \( j \in \mathbb{N} \setminus \mathbb{S} \). Then the no-efficacy score \( \alpha_i^- (\mathbb{S}) \) is the fraction of paths from \( \mathbb{S} \) to \( \mathbb{N} \) that contains a division at which \( i \) is no-decisive.

Using this fact we proceed to prove (2).
Claim 7.1 \( \hat{\alpha}_i^-(S) \leq \alpha_i^-(S) + \alpha_j^-(S) \)

**Proof** Take any path \( P \) from \( S \) to \( N \) that contains a division at which \( i \) is \textit{no}-decisive in the modified game \( \hat{G} \). Thus \( P \) contributes to \( \hat{\alpha}_i^-(S) \). We wish to find a matching contribution to \( \alpha_i^-(S) + \alpha_j^-(S) \). Again we break the analysis into cases.

1. \( i \) is \textit{no}-decisive at \( T \) on path \( P \) in \( \hat{G} \)
   
   By monotonicity, \( i \) is \textit{no}-decisive at the highest \textit{no}-division on \( P \) in \( \hat{G} \). Thus, we may assume \( T \cup k \) on path \( P \) is winning \( \hat{G} \). But \( i \) is a \textit{yes}-blocker in \( \hat{G} \) because \( j \) is a \textit{yes}-blocker in \( G \). Hence, it must be the case that \( k = i \).

   Now \( T \cup i \) is losing in \( G \) and \( j \) is a \textit{yes}-blocker in \( G \). But \( T \cup i \) is winning in \( \hat{G} \).

   So, by definition, \( T \cup \{i, j\} \) is winning in \( G \). This implies \( j \) is \textit{no}-decisive at \( T \cup i \) on path \( P \) in \( G \). Consequently, \( P \) contributes to \( \alpha_j^-(S) \).

2. \( i \) is \textit{no}-decisive at \( T \cup j \) on path \( P \) in \( \hat{G} \)

   (a) \( T \cup j \) is losing in \( G \): note that \( T \cup \{i, j\} \) is winning in \( G \) as it is winning in \( \hat{G} \). Therefore, \( i \) is also \textit{no}-decisive at \( T \cup j \) on path \( P \) in \( G \). Thus \( P \) contributes to \( \alpha_i^-(S) \).

   (b) \( T \cup j \) is winning in \( G \): Now consider the \textit{mirror path} \( P^M \) from \( S \) to \( N \) which is identical to \( P \) except the roles of \( i \) and \( j \) are switched (that is \( i \) and \( j \) swap their positions in \( P \)). In particular, \( P^M \) passes through the division \( T \cup i \).

   But as \( j \) is a \textit{yes}-blocker in \( G \) it must be the case that \( T \cup i \) is losing in \( G \). As \( T \cup \{i, j\} \) is winning in both \( G \) and \( \hat{G} \), it follows that \( j \) is \textit{no}-decisive at \( T \cup i \) on the path \( P^M \) in \( G \). Consequently, \( P^M \) contributes to \( \alpha_j^-(S) \). On the other hand, suppose \( P^M \) also contributes to \( \hat{\alpha}_i^-(S) \). This can only happen if \( i \) is \textit{no}-decisive at \( \mathbb{R} \) in \( \hat{G} \), where \( \mathbb{R} \) is the division where \( P \) and \( P^M \) diverge. By definition, this implies \( \mathbb{R} \cup \{i, j\} \) is winning in \( G \). In particular, \( i \) is \textit{no}-decisive at \( \mathbb{R} \cup j \) on \( P \) in \( \hat{G} \). So in this case \( P \) also contributes to \( \alpha_i^-(S) \).

   Ergo, the combined contribution of \( P \) and \( P^M \) to \( \alpha_j^-(S) \) is at most their combined contribution to \( \alpha_i^-(S) + \alpha_j^-(S) \). The claim follows.

\( \square \)

Now we prove (3).

Claim 7.2 \( \hat{\alpha}_i^-(S \cup j) \leq \alpha_i^-(S \cup j) + \alpha_j^-(S \cup i) \)

**Proof** Take any path \( P \) from \( S \cup j \) to \( N \) that contains a division at which \( i \) is \textit{no}-decisive in the modified game \( \hat{G} \). Thus \( P \) contributes to \( \hat{\alpha}_i^-(S \cup j) \). We wish to find a matching contribution to \( \alpha_i^-(S \cup j) + \alpha_j^-(S \cup i) \). Let \( i \) be \textit{no}-decisive at \( T \cup j \) on path \( P \) in \( \hat{G} \). We now have two cases.

1. \( T \cup j \) is losing in \( G \): in this case, \( i \) is also \textit{no}-decisive at \( T \cup j \) on path \( P \) in \( G \). Thus \( P \) contributes to \( \alpha_i^-(S \cup j) \).

\( \square \)
2. \( T \cup j \) is winning in \( G \): Now consider the twin path \( P^T \) from \( S \cup i \) to \( N \) which is identical to \( P \) except the roles of \( i \) and \( j \) are switched (note that unlike for the mirror path \( P^M \) the twin path starts at a different division than \( P \)). So \( P^T \) passes through \( T \cup i \). But since \( j \) is a yes-blocker in \( G \) it must be the case that \( T \cup i \) is losing in \( G \). It follows that \( j \) is no-decisive at \( T \cup i \) on the path \( P^T \) in \( G \). Thus \( P^M \) contributes to \( \alpha_j^{-}(S \cup i) \). Furthermore, observe that since the path \( P^T \) originates at \( S \cup i \), the voter \( i \) cannot be no-decisive on the path. 

Ergo, the combined contribution of \( P \) and \( P^T \) to \( \hat{\alpha}_i^{-}(S \cup j) \) is at most their combined contribution to \( \alpha_i^{-}(S \cup j) + \alpha_j^{-}(S \cup i) \). The claim follows. □

Together Claim 7.1 and Claim 7.2 imply (2). Thus the proof of Lemma 7.1 is complete. □

We are now ready to prove Theorem 3.3.

**Proof of Theorem 3.3** Lemma 7.1 and the dummy postulate imply that the subadditivity blocker postulate holds for \( I = \{i, j\} \) where \( j \) is a yes-blocker. But if \( j \) is a yes-blocker in \( G \) then \( I \) is a yes-blocker in \( \hat{G} \). Thus for \( |I| > 2 \) the result then follows by iteratively adding the voters of \( I \) to the set \( \{i, j\} \). □

**Proofs for Sect. 4**

**Theorem 4.1** \( PB \) does not satisfy the weak minimum-power blocker postulate (and, thus, does not satisfy the strong minimum-power blocker postulate).

**Proof** Take a unanimity game. Then each player \( b \) is a yes-blocker and has voting power \( PB_b = \sum_{S \in D} a^PB_j(S) \cdot \gamma^{PB}(S) = \frac{2}{2^n} = \frac{1}{2^{n-1}} \). Furthermore the smallest possible yes-successful set is \( S^* = N \), which has cardinality \( n \). A dictator \( d \) has voting power \( PB_d = 1 \). But then, for large \( n \), we have

\[
PB_b = \frac{1}{2^{n-1}} \ll \frac{\psi_d}{|S^*|} = \frac{1}{n}
\]

Ergo, the weak minimum-power blocker postulate does not hold. □

**Theorem 4.2** \( SS \) satisfies the strong minimum-power blocker postulate (and, thus, satisfies the weak minimum-power blocker postulate).

**Proof** Since \( \sum_{i \in N} SS_i = 1 = SS_d \), this follows immediately from the fact that \( SS \) satisfies the blocker’s share postulate (Felsenthal and Machover 1998). □

**Theorem 4.3** \( RM \) satisfies the strong minimum-power blocker postulate (and, thus, satisfies the weak minimum-power blocker postulate).
**Proof** Let $|S^*| = k$. By unanimity, we may assume $k \geq 1$. Let $b$ be a yes-blocker; it immediately follows that $b \in S^*$. Now any set of players $S$ can be written as $S = (S \cap S^*) \cup (S \cap (N \setminus S^*))$. Thus we have

$$RM_b' = \frac{1}{2^n} \cdot \sum_{S \in D} \alpha_b(S) = \frac{1}{2^n} \cdot \sum_{S \subseteq N\setminus S^*} \sum_{T \subseteq S^*} \alpha_b(S \cup T)$$

(4)

Now $|T| \leq k$ for any subset $T$ or $S^*$. Hence

$$\sum_{T \subseteq S^*: |T| = \ell} \alpha_b(S \cup T) = \sum_{\ell=0}^{k} \sum_{T \subseteq S^*: |T| = \ell} \alpha_b(S \cup T) = \sum_{\ell=0}^{k} \sum_{T \subseteq S^*: |T| = \ell} (\alpha_b^+(S \cup T) + \alpha_b^-(S \cup T))$$

(5)

Now if $S \cup T$ is winning then $\alpha_b^+(S \cup T) = 1$ because $b$ is a yes-blocker and, hence, is yes-decisive in $S \cup T$. It follows that to lower bound (5) we may assume that $S \cup T$ is losing for any $T \subseteq S^*$. Note the strict subset is necessary here since, by monotonicity, $S \cup S^*$ must be winning. Thus for any $l < k$ we obtain a lower bound of

$$\sum_{T \subseteq S^*: |T| = \ell} \alpha_b(S \cup T) = \sum_{T \subseteq S^*: |T| = \ell} \alpha_b^-(S \cup T)$$

$$= \sum_{T \subseteq S^*: |T| = \ell, b \in T} \alpha_b^-(S \cup T) + \sum_{T \subseteq S^*: |T| = \ell, b \notin T} \alpha_b^-(S \cup T)$$

$$= 0 + \sum_{T \subseteq S^*: |T| = \ell, b \notin T} \alpha_b^-(S \cup T)$$

$$= \sum_{T \subseteq S^*: |T| = \ell, b \notin T} \alpha_b^-(S \cup T)$$

(6)

Here the third equality holds since, by definition, the no-efficacy of $b$ is zero for any division in which $b$ votes yes. Now recall that to calculate $\alpha_b^-(S \cup T)$ we may perform a random walk in the no-poset. We need to find the probability that, starting the walk at the node for division $S \cup T$ we reach a node where $b$ is no-decisive. In particular, $b$ is no-decisive at the node for $S \cup S^* \setminus \{b\}$. Moreover, since $b$ is a yes-blocker, it is no-decisive at the node for $S \cup X \cup S^* \setminus \{b\}$ for any $X \subseteq N \setminus (S \cup S^*)$, by monotonicity. This implies that if we randomly add players in order to $S \cup T$ we will reach a node where $b$ is no-decisive if $b$ appears after every other node of $S^*$. Since $|T| = \ell$, this occurs with probability $\frac{1}{k-\ell}$. Thus $\alpha_b^-(S \cup T) \geq \frac{1}{k-\ell}$. Note this is an inequality not equality. (We have a lower bound on no-efficacy since there may be other losing divisions, reachable from $S \cup T$, that do not contain $S^* \setminus \{b\}$ where $b$ is no-decisive.) Simple counting arguments then give

$$\sum_{T \subseteq S^*: |T| = \ell, b \notin T} \alpha_b^-(S \cup T) \geq \left(\frac{k-1}{\ell}\right) \cdot \frac{1}{k-\ell} = \left(\frac{k}{\ell}\right) \cdot \frac{1}{k}$$

(7)

Observe that $\alpha_b^+(S \cup S^*) = 1$. So, plugging (6) and (7) into (4) gives
The blocker postulates for measures of voting power

We thus obtain

\[ RM'_b \geq \frac{1}{k} \cdot \frac{2^k}{2^n} \sum_{S \subseteq N \setminus S^*} 1 = \frac{1}{k} \cdot \frac{2^k}{2^n} \cdot 2^{n-k} = \frac{1}{k} = \psi_d \]

Therefore, (smp-1) is satisfied. A symmetrical argument applies to (smp-2) for a no-blocker. Ergo, the strong minimum-power blocker postulate is satisfied. \( \square \)

We remark that an alternative proof of Theorem 4.3 is via the fact that RM satisfies the strong subadditivity blocker postulate (Theorem 3.3). In contrast, such an approach cannot be used to prove Theorem 4.2 since SS does not satisfy the strong subadditivity blocker postulate (Theorem 3.2).

Proofs for Sect. 5

Theorem 5.1 PB satisfies the added-blocker postulate.

Proof Recall \( \psi_j^+ = \psi_j^- \) for any decisiveness measure. Thus, a decisiveness measure that satisfies Felsenthal and Machover’s (1998: 266–75) specification of the added-blocker postulate ipso facto satisfies our specification. It follows that PB satisfies our added-blocker postulate, since, as they show, it satisfies theirs. \( \square \)

Theorem 5.2 SS does not satisfy the added-blocker postulate.

Proof Consider the weighted voting games \( G = \{3;2,1,1\} \) and \( G^{\text{VB}} = \{8;2,1,1,5\} \) (Felsenthal and Machover 1998). Observe that the new player in \( G^{\text{VB}} \) is a yes-blocker. It can be verified that \( SS^+_1(G) = \frac{2}{6} \) and \( SS^+_2(G) = \frac{1}{12} \) whereas \( SS^+_1(G^{\text{VB}}) = \frac{5}{24} \) and
Thus \( SS_1^+(G^{\text{YB}}) = \frac{1}{24} \). Thus \( \frac{SS_1^+(G)}{SS_1^+(G^{\text{YB}})} = 4 < \frac{SS_2^+(G^{\text{YB}})}{SS_1^+(G^{\text{YB}})} = 5 \), in violation of (ADD-1) and hence our postulate.

\( \square \)

**Theorem 5.3** \( PB \) satisfies the added-blocker postulate.

To prove that \( RM \) satisfies the added-blocker postulate, we show it satisfies (ADD-1) and (ADD-2). We begin with a useful lemma.

**Lemma 7.2** For any player \( i \), the efficacy scores \( \alpha^+_{RM} \) and \( \alpha^{-RM} \) in \( G \) and \( G^{\text{YB}} \) satisfy

\[
\alpha^+_{i,G^{\text{YB}}}(S \cup \{0\}) = \alpha^+_{i,G}(S) \quad \forall S \tag{B1}
\]

\[
\alpha^+_{i,G^{\text{YB}}}(S, \bar{S} \cup 0)) = 0 \quad \forall S \tag{B2}
\]

**Proof** Consider the games \( G \) and \( G^{\text{YB}} \). Recall the winning divisions in \( G^{\text{YB}} \) are of the form \( S \cup \{0\} \) where \( S \in \mathcal{W} \) in the original voting game. The key facts are then the following. Let \( S \subseteq N \) contain player \( i \). Then in \( G^{\text{YB}} \) player \( i \) is never \( \text{YES}-\)decisive at \( (S, \bar{S} \cup 0) \) because the division is a losing division (given that \( S \) does not contain the blocker \( 0 \)). Furthermore, player \( i \) is \( \text{YES}-\)decisive at \( S \cup \{0\} \) in \( G^{\text{YB}} \) if and only if it is originally \( \text{YES}-\)decisive at \( S \) in \( G \).

Next take \( S \subseteq N \) where \( S \) does not contain player \( i \). Then in \( G^{\text{YB}} \) player \( i \) is never \( \text{NO}-\)decisive at \( (S, \bar{S} \cup 0) \) because \( S \cup \{i\} \notin \mathcal{W} \) (given that it does not contain the blocker \( 0 \)). Furthermore, player \( i \) is \( \text{NO}-\)decisive at \( S \cup \{0\} \) in \( G^{\text{YB}} \) if and only if \( i \) is originally \( \text{NO}-\)decisive at \( S \) in \( G \).

These key facts imply that the \( \text{YES}-\)outcomes for \( G^{\text{YB}} \) are identical to the \( \text{YES}-\)outcomes for \( G \), except that for each winning coalition the set of \( \text{YES}-\)voters now also contains the blocker \( 0 \) (equivalently, the set of \( \text{NO}-\)voters in their corresponding divisions are identical). This immediately implies that we recursively calculate the \( \text{YES}-\)efficacy scores, they are identical for the corresponding winning coalitions in \( G \) and \( G^{\text{YB}} \). That is, \( \alpha^+_{i,G^{\text{YB}}}(S \cup \{0\}) = \alpha^+_{i,G}(S) \) and (B1) holds.

Furthermore, given that \( 0 \) is a \( \text{YES}-\)blocker, any \( S \subseteq N \) is a losing division. Thus \( \alpha^+_{i,G^{\text{YB}}}(S, \bar{S} \cup 0) = 0 \) and (B2) holds. \( \square \)

**Lemma 7.3** \( RM \) satisfies (ADD-1).

**Proof** For any player \( i \) in the original game
The blocker postulates for measures of voting power

Here the third and fourth equalities hold by (B1) and (B2) of Lemma 7.2, respectively. Similarly, for any player \( j \) in the original game, we have

\[
RM_{i}^{'+}(G^{YB}) = \sum_{S \in D} \alpha^{+}_{i,G^{YB}}(S) \cdot \frac{1}{2^{n+1}} = \sum_{S \in D} \alpha^{+}_{i,G^{YB}}((S, \bar{S} \cup \{0\})) \cdot \frac{1}{2^{n+1}} + \sum_{S \in D} \alpha^{+}_{i,G^{YB}}(\bar{S} \cup \{0\}) \cdot \frac{1}{2^{n+1}}
\]

\[
= 0 + \sum_{S \in D} \alpha^{+}_{i,G}(S \cup \{0\}) \cdot \frac{1}{2^{n+1}}
\]

\[
= \sum_{S \in D} \alpha^{+}_{i,G}(S) \cdot \frac{1}{2^{n+1}}
\]

\[
= \frac{1}{2} \sum_{S \in D} \alpha^{+}_{i,G}(S) \cdot \frac{1}{2^{n}}
\]

\[
= \frac{1}{2} \cdot RM_{i}^{'+}(G)
\]

Ergo, the added-YES-blocker postulate (ADD-1) is satisfied.

\[\square\]

**Lemma 7.4** \( RM \) satisfies (ADD-2).

**Proof** Applying a symmetric argument to that used in the proof of Lemma 7.3 we have the following. For any player \( i \) in the original game, the efficacy scores \( \alpha^{+} \) and \( \alpha^{-} \) in \( G \) and \( G^{NB} \) satisfy

\[
\alpha^{-}_{i,G^{NB}}(S \cup \{0\}) = 0 \quad \forall S
\]

\[
\alpha^{-}_{i,G^{NB}}((S, \bar{S} \cup \{0\})) = \alpha^{-}_{i,G}(S) \quad \forall S
\]

Then, applying a symmetric argument to that used in the proof of Lemma 7.3 completes the proof.

\[\square\]

It follows by Lemmas 7.3 and 7.4 that RM satisfies the added-blocker postulate and Theorem 5.3 holds.

**Funding Information** This work was supported by Social Sciences and Humanities Research Council of Canada (Insight Grant) 435-2018-0386 (Abizadeh) and Natural Sciences and Engineering Research Council of Canada RGPIN-2017-06107 (Vetta).
Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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