Coded Caching with Polynomial Subpacketization

Wentu Song, Kui Cai, and Long Shi
Science and Math Cluster, Singapore University of Technology and Design, Singapore
Email: {wentu_song, cai_kui, shi_long}@sutd.edu.sg

Abstract—Consider a centralized caching network with a single server and $K$ users. The server has a database of $N$ files with each file being divided into $F$ packets ($F$ is known as subpacketization), and each user owns a local cache that can store $\frac{M}{N}$ fraction of the $N$ files. We construct a family of centralized coded caching schemes with polynomial subpacketization. Specifically, given $M$, $N$ and an integer $n \geq 0$, we construct a family of coded caching schemes for any $(K, M, N)$ caching system with $F = O(K^{n+1})$. More generally, for any $t \in \{1, 2, \ldots, K-2\}$ and any integer $n$ such that $0 \leq n \leq t$, we construct a coded caching scheme with $\frac{M}{N} = \frac{1}{K}$ and $F \leq K\left(1 - \frac{M}{N}\right)^{K+n}$.

I. INTRODUCTION

A $(K, M, N)$ caching system consists of one server and $K$ users, where all users connect to the server through a shared, error-free link. The server has a database of $N$ files and each user may request a specific file from the server at certain time in the future. The user requests are random and not known to the server in advance. Each user has a cache that can store $\frac{M}{N}$ fraction of the $N$ files of the server. A centralized coded caching scheme operates in two separated phases: the placement phase and the delivery phase. In the placement phase, the server allocates certain packets of the data files into the cache of the users, while in the delivery phase, the server, upon receiving the specific demands of all users, broadcasts coded packets through the shared link to all users so that each user can extract its requested file from the received packets and its cache content. The rate $R$ of the scheme is defined as the maximal transmission amount in the delivery phase among all possible combinations of the user demands, and the primary goal is to design coded caching scheme with as small rate as possible.

Coded caching problem was first investigated by Maddah-Ali and Niesen in their award-winning paper [1]. The coded caching scheme proposed in [1] attains the rate

$$R^* = \frac{K(1 - \frac{M}{N})}{1 + K\frac{M}{N}},$$

where $1 - \frac{M}{N}$ is called the local caching gain and $1 + K\frac{M}{N}$ is called the global caching gain, and $R^*$ was proved to be optimal among schemes with uncoded placement [2], [3].

A major limitation of the Maddah-Ali-Niesen scheme is the exponential subpacketization problem: by this caching scheme, each file is divided into $F = (\frac{K}{K\frac{M}{N}})$ packets ($F$ is referred to as the file size or subpacketization), which grows exponentially with $K$ [4]. Since high subpacketization may result in transmission delay in practical implementations, coded caching with low subpacketization, especially polynomial subpacketization, is of great interest.

Many works have been engaged to reduce the subpacketization, with the sacrifice of increasing the rate. A user-grouping method was adopted in [4] to reduce the subpacketization level, and a more general concatenating construction method was used in [5]. A framework of constructing centralized coded caching scheme, named placement delivery array design (or PDA design for simplicity), was introduced in [6], based on which some new classes of coded caching schemes were obtained in [9] and [7]. Caching schemes constructed using other techniques, such as hypergraphs, bipartite graphs combinatorial designs, and projective geometries over finite fields, are reported in [3]-[10]. Most of these schemes have exponential or subexponential subpacketization. More interestingly, a family of coded caching schemes with linear subpacketization (i.e., $F = K$), were constructed in [17], using the Ruzsa-Szemerédi graphs. However, this construction is valid only for sufficiently large $K$. Another family of linear-subpacketization schemes were constructed in [10] using balanced incomplete block designs (BIBD), which exists only for some special parameters.

In this paper, we propose a family of centralized coded caching schemes with polynomial subpacketization. Specifically, for any $t \in \{1, 2, \ldots, K-2\}$ and any integer $n$ such that $0 \leq n \leq t$, we construct a coded caching scheme for any $(K, M, N)$ caching systems with $\frac{M}{N} = \frac{1}{K}$.

$$R = \frac{m}{m-1} \sum_{i=1}^{m-1} (-1)^{i-1} \binom{m-1}{i} \binom{K-1-i(\ell-1)}{m-2}$$

and

$$F = \frac{K}{m} \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \binom{K-1-i(\ell-1)}{m-1},$$

where $m = K-t$ and $\ell = K - m - n + 1$, and we can prove that

$$F \leq K\left(1 - \frac{M}{N}\right)^{K+n}.$$

In particular, given $M$, $N$ and integer $n \geq 0$, for any positive integer $K$ such that $m = K(1 - \frac{M}{N}) \geq 2$ is an integer and $2 \leq m \leq K - n$, our construction gives a coded caching scheme for any $(K, M, N)$ caching system with $F \leq K\left(1 - \frac{M}{N}\right)^{K+n} = O(K^{n+1})$. Our construction is based on a family of subsets of $\mathbb{Z}_K = \{0, 1, \ldots, K-1\}$, called $(m)_t$-bounded subsets of $\mathbb{Z}_K$, and can be viewed as a generalization of the construction in [1].
The rest of this paper is organized as follows. We give a formal formulation of the centralized coded caching problem in Section II. We introduce the bounded subsets of \( Z_K \) and discuss their properties in Section III. Our construction of coded caching scheme is presented in Section IV. Finally, the paper is concluded in Section V.

II. Preliminaries

For any positive integer \( n \), denote \([n] := \{1, 2, \ldots, n\}\). For any set \( X \), \(|X|\) is the size (cardinality) of \( X \). If \( Y \subseteq X \) and \(|Y| = m \), where \( 0 \leq m \leq |X| \), we call \( Y \) an \( m \)-subset of \( X \). We use \( \binom{N}{m} \) to denote the collection of all \( m \)-subsets of \( X \).

We consider a \((K, M, N)\) caching system, where one server is connected by \( K \) users through a shared, error-free link. The server has \( N \) files, denoted by \( W_1, \ldots, W_N \), such that each file \( W_i \in \mathbb{F}^F \) for some fixed real number \( F \), where \( F \) is referred to as the subpacketization.

The caching system operates in two phases: the placement phase and the delivery phase. In the placement phase, the vector \( Z_k \) is computed and allocated into the cache memory of each user \( k \). In the delivery phase, each user \( k \) demands a file \( W_{d_k} \) for some fixed real number \( R \) and transmits it to the users, where \( d = (d_0, d_1, \ldots, d_{K-1}) \in [N]^K \) is a demand vector. An \( F \)-division coded caching scheme with a rate \( R \) is specified by three sets of functions:

(i) (Placement Scheme) a set of caching functions
\[
\{\phi_k: \mathbb{F}^N \rightarrow \mathbb{F}^{MF}\} \quad k \in \mathbb{Z}_K, 
\]
(ii) (Delivery Scheme) a set of encoding functions
\[
\{\Phi_d: \mathbb{F}^N \rightarrow \mathbb{F}^{RF}\} \quad d \in [N]^K, 
\]
(iii) (Decoding Scheme) a set of decoding functions
\[
\{\mu_{k,d}: \mathbb{F}^{MF} \times \mathbb{F}^{RF} \rightarrow \mathbb{F}^N\} \quad k \in \mathbb{Z}_K, d \in [N]^K, 
\]
such that for all \( k \in \mathbb{Z}_K \) and \( d = (d_0, d_1, \ldots, d_{K-1}) \in [N]^K \),
\[
W_{d_k} = \mu_{k,d}(Z_k, X_d),
\]
where \( Z_k = \phi_k(W_1, \ldots, W_N) \) and \( X_d = \Phi_d(W_1, \ldots, W_N) \).

Clearly, the decoding scheme is completely determined by the placement scheme and the delivery scheme. A caching scheme is said to have uncoded placement if \( Z_k \) consists of an exact copy of some subpackets of \( W_1, \ldots, W_N \). Otherwise, it is said to have coded placement.

III. Bounded Subsets of \( \mathbb{Z}_K \)

In this section, we always assume that \( K, m, \ell \) are positive integers such that \( K \geq 2, m \leq K \) and \( \ell \leq K - m + 1 \). Denote \( \mathbb{Z}_K = \{0, 1, \ldots, K - 1\} \). A family of subsets of \( \mathbb{Z}_K \), referred to as \((m)\)-bounded subsets of \( \mathbb{Z}_K \), is introduced, which will be used, in the next section, to construct coded caching schemes with polynomial subpacketization.

We first give a different representation of the \( m \)-subsets of \( \mathbb{Z}_K \). Denote
\[
V_K(m) = \left\{(k, a_1, \cdots, a_m) \in \mathbb{Z}^{m+1}: 0 \leq k \leq K - 1, a_i \geq 1 \right\}
\]
for all \( i \in [m] \), and 
\[
|V_K(m)| = m.
\]

Clearly, \( f(v) \) is an \( m \)-subset of \( \mathbb{Z}_K \), and from (3), we obtain a mapping \( f: V_K(m) \rightarrow \mathbb{Z}_K^m \). Hence, each \( v \in V_K(m) \) can be used to represent an \( m \)-subset of \( \mathbb{Z}_K \).

As an example, consider \( K = 20 \) and \( m = 5 \). Suppose \( v = (12, 3, 2, 6, 7, 2) \). Then we have \( v \in V_{20}(5) \). By (3), we can obtain \( f(v) = (12, 15, 17, 3, 10) \in \mathbb{Z}_5^5 \).

Lemma I: Let \( f \) be the mapping defined according to (3).
1) \( f \) is surjective.
2) If \( A \) is an \( m \)-subset of \( \mathbb{Z}_K \), then \(|f^{-1}(A)| = m \) and \(|f(A)| = m \).

It is a mechanical way to verify that \( v_{A,k} \in V_K(m) \) and \( f(v_{A,k}) = A \), so \( f \) is surjective and \( \{v_{A,k}: k \in A\} \subseteq f^{-1}(A) \).

2) According to (3), \( (a_1^{(A,k)}, a_2^{(A,k)}, \cdots, a_m^{(A,k)}) \) is uniquely determined by \( A \) and \( k \). Moreover, if \( i \) and \( i' \) are two distinct elements of \( A \), then \( (a_1^{(A,k)}, a_2^{(A,k)}, \cdots, a_m^{(A,k)}) \) is a circular shift of \( (a_1^{(A,k')}, a_2^{(A,k')}, \cdots, a_m^{(A,k')}) \).

Proof: 1) Suppose \( A = \{k_1, k_2, \ldots, k_m\} \) such that \( k_1 < k_2 < \cdots < k_m \). For each \( k = k_{i_0} = A_i \), \( i_0 \in [m] \), let
\[
a_{i_0}^{(A,k)} = \begin{cases} k_{i_0+i} - k_{i_0+i-1}, & \text{for } 1 \leq i \leq m - i_0, \\ K + k_{i_0+i} - k_{i_0+i-1}, & \text{for } i = m - i_0 + 1, \\ k_{i_0+i} - m - k_{i_0+i} - m - 1, & \text{for } m - i_0 + 1 < i \leq m, \end{cases}
\]
and let
\[
v_{A,k} = (k_{i_0}^{(A,k)}, a_2^{(A,k)}, \cdots, a_m^{(A,k)}).
\]
can prove this by contradiction. Suppose \(|f^{-1}(A)| > m\) for some \(A \in \binom{Z}{m}\). Since by 1), \(f\) is surjective, then we have

\[
|V_K(m)| = \left| \bigcup_{A \subseteq (\mathbb{Z}_+/m)} f^{-1}(A) \right| > m \left( \frac{K-1}{m-1} \right). \tag{6}
\]

On the other hand, the number of integer solutions to the equation \(a_1 + \cdots + a_m = K\) under the condition that \(a_i \geq 1\) for all \(i = 1, \ldots, n\), is \(\binom{K-1}{m-1}\) (e.g., see Chapter 1 of [19]). So by (2), we have

\[
|V_K(m)| = K \left( \frac{K-1}{m-1} \right) = m \left( \frac{K}{m} \right).
\]

which contradicts to (4), so it must be the case that \(|f^{-1}(A)| = m\) for all \(A \in \binom{Z}{m}\), and hence, we have \(f^{-1}(A) = \{v_{A,k} : k \in A\}\) for all \(A \in \binom{Z}{m}\). ■

**Example 1:** Let \(K = 20, m = 5\) and \(A = \{2, 3, 11, 15, 19\}\).

By (4), we have \(a_1^{(A,2)} = 3 - 2 = 1, a_2^{(A,2)} = 11 - 3 = 8, a_3^{(A,2)} = 15 - 11 = 4, a_4^{(A,2)} = 19 - 15 = 4,\) and \(a_5^{(A,2)} = 2 + 20 - 19 = 3\). So by (5), \(v_{A,2} = (2, 1, 8, 4, 3)\).

Similarly, \(v_{A,3} = (3, 8, 4, 4, 3, 1)\), \(v_{A,11} = (11, 4, 4, 3, 1, 8)\), \(v_{A,15} = (15, 4, 3, 1, 8, 4)\) and \(v_{A,19} = (3, 3, 1, 8, 4, 4)\). By Lemma I we obtain \(f^{-1}(A) = \{v_{A,2}, v_{A,3}, v_{A,11}, v_{A,15}, v_{A,19}\}\).

**Lemma 2:** Suppose \(2 \leq m \leq K\) and \(A\) is an \((m)\)-bounded subset of \(Z_K\). Then any \((m-1)\)-subset of \(A\) is an \((m-1)\)-bounded subset of \(Z_K\).

**Proof:** Suppose \(A = \{k_1, k_2, \ldots, k_m\}\) such that \(0 \leq k_1 < k_2 < \cdots < k_m \leq K - 1\), and \(B = A \setminus \{k_i\}\), where \(i \in [m]\). Let \(i = i_0 + 1 \pmod{m}\). By (4) and (5), we can verify that \(v_{B,k_{i_0}} = (k_1, a_1^{(B,k_1)}, \ldots, a_{m-1}^{(B,k_{m-1})})\).

By 2) of Lemma I \(v_{A,k_i} \in f^{-1}(A)\). Since \(A\) is an \((m)\)-bounded subset of \(Z_K\), we have \(a_i^{(A,k_i)} \geq \ell\) for some \(i \in [m]\), and so \(a_j^{(B,k_i)} \geq \ell\) for some \(i' \in [m-1]\). By (7), we have \(v_{B,k_{i}} \in V_{K,\ell}(m-1)\). Moreover, by 2) of Lemma I \(v \in f^{-1}(B)\), and so \(f^{-1}(B) \cap V_{K,\ell}(m-1) \neq \emptyset\). Hence, \(B\) is an \((m-1)\)-bounded subset of \(Z_K\). ■

The following lemma counts the number of \((m)\)-bounded subsets of \(Z_K\).

**Lemma 3:** Suppose \(K, m, \ell\) are positive integers such that \(K \geq 2, m \leq K\) and \(\ell \leq K - m + 1\). We have

1) For each \(k \in Z_K\), the number of \((m)\)-bounded subsets of \(Z_K\) containing \(k\), denoted by \(C(K, m, \ell)\), is independent of \(k\), and we have

\[
C(K, m, \ell) = \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \left( \frac{K - 1 - i(\ell - 1)}{m - 1} \right). \tag{7}
\]

2) The number of \((m)\)-bounded subsets of \(Z_K\) is

\[
|B_{K,\ell}(m)| = \frac{K}{m} \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \left( \frac{K - 1 - i(\ell - 1)}{m - 1} \right). \tag{8}
\]

3) The number of \((m)\)-bounded subsets of \(Z_K\) satisfies

\[
|B_{K,\ell}(m)| \leq K \left( \frac{K - \ell + 1}{m} \right). \tag{9}
\]

**Proof:** 1) For \(k \in Z_K\), let \(S_K(k)\) denote the collection of all \(m\)-subsets of \(Z_K\) containing \(k\). Clearly, \(|S_K(k)| = \binom{K-1}{m-1}\).

Let \(T_K(k)\) denote the collection of all \(m\)-subsets of \(Z_K\) that contain \(k\) but are not an \((m)\)-bounded subsets of \(Z_K\). We now compute \(|T_K(k)|\). If \(A \in T_K(k)\), by 1) of Remark II \((k, a_1^{(A,k)}, \ldots, a_m^{(A,k)}) \notin V_{K,\ell}(m)\), so we obtain an \(m\)-tuple
\( a_1^{(A,k)}, \ldots, a_m^{(A,k)} \in \mathbb{Z}_m^K \) satisfying \( \sum_{i=1}^m a_i^{(A,k)} = K \) and \( 1 \leq a_i^{(A,k)} \leq \ell - 1 \) for all \( i \in [m] \). Conversely, for any \( m \)-tuple \((a_1, \ldots, a_m) \in \mathbb{Z}_m^K \) satisfying \( \sum_{i=1}^m a_i = K \) and \( 1 \leq a_i \leq \ell - 1 \) for all \( i \in [m] \), by \( [3] \), we have \( f(v) \in \mathcal{T}_K(k) \), where \( v = (k, a_1, \ldots, a_m) \in V_K(m) \). Hence, \( |\mathcal{T}_K(k)| \) equals to the number of \( m \)-tuples \((a_1, \ldots, a_m) \in \mathbb{Z}_m^K \) satisfying \( \sum_{i=1}^m a_i = K \) and \( 1 \leq a_i \leq \ell - 1 \) for all \( i \in [m] \), by \( [18] \), we can obtain \( v_{A,1} = (1, 3, 9, 1, 4, 3) \), so \( A \in \mathcal{B}_{20,9}(5) \). Note that by \( \{4\} \), \( a_5^{(A,2)} = 9 = k_2 - k_2 \), and we can verify that \( A \subseteq X_{(k_3, k_4)} = \{k_3, k_3 + K = 1, \ldots, k_3 + K (K - \ell) = 13, 14, \ldots, 19, 1, 2, 3, 4 \} \).

**IV. Coded Caching With Polynomial Subpacketization**

In this section, we construct a family of coded caching schemes using the \((m)_{i,\ell}\)-bounded subsets of \( \mathbb{Z}_K \).

Suppose \( K, m, \ell \) are positive integers such that \( 2 \leq m \leq K - 1 \) and \( \ell \leq K - m - 1 \). We use \( \mathcal{B}_{K,\ell}(m) \) to denote the set of \( K \) users, and each file \( W_n \) is divided into \( F = |\mathcal{B}_{K,\ell}(m)| \) packets. (Note that \( \mathcal{B}_{K,\ell}(m) \) is the collection of all \((m)_{i,\ell}\)-bounded subsets of \( \mathbb{Z}_K \).) Then we can denote

\[ W_n = \{W_{n,S} \in \mathcal{F}_2 : S \in \mathcal{B}_{K,\ell}(m)\}. \]

Moreover, for each \( T \in \mathcal{B}_{K,\ell}(m - 1) \), denote

\[ U(T) = \{k \in \mathbb{Z}_K : (T \cup \{k\}) \in \mathcal{B}_{K,\ell}(m)\}, \]

and for each \( k \in \mathbb{Z}_K \), denote

\[ \mathcal{V}(k) = \{T \in \mathcal{B}_{K,\ell}(m - 1) : (T \cup \{k\}) \in \mathcal{B}_{K,\ell}(m)\}. \]

Now, we have the following construction.

**Construction 1:** A coded caching scheme is as follows.

(i) (Placement Scheme) For each \( k \in \mathbb{Z}_K \), the user \( k \) caches

\[ Z_k = \{W_{n,S} : n \in [N], S \in \mathcal{B}_{K,\ell}(m) \text{ and } k \notin S\}. \]

(ii) (Delivery Scheme) Given any \( d = (d_1, d_2, \ldots, d_{K-1}) \in [N]^{K-1} \), for each \( T \in \mathcal{B}_{K,\ell}(m - 1) \), the server transmits

\[ X_T = \bigoplus_{k \in U(T)} W_{d_k,T \cup \{k\}} \]

where \( \oplus \) denotes the bitwise XOR.

(iii) (Decoding Scheme) Given any \( d = (d_1, d_2, \ldots, d_{K-1}) \in [N]^{K-1} \), for each \( k \in \mathbb{Z}_K \) and each \( T \in \mathcal{V}(k) \),

\[ W_{d_k,T \cup \{k\}} = \bigoplus_{k' \in U(T) \setminus \{k\}} W_{d_{k'},T \cup \{k'\}} \oplus X_T. \]

Clearly, the decoding equality \( (14) \) can be derived directly from \( (13) \). We still have to prove that each user can recover its requested file by the decoding scheme.

**Lemma 4:** In Construction 1, for each \( k \in \mathbb{Z}_K \), the user \( k \) can successfully recover its requested file \( W_{d_k} \).

**Proof:** By \( (9) \) and \( (12) \), it suffices to prove that for each \( k \in \mathbb{Z}_K \), and \( S \in \mathcal{B}_{K,\ell}(m) \) such that \( k \in S \), the user \( k \) can recover \( W_{d_k} \) from its cached packets and received packets. Let \( T = S \setminus \{k\} \). By Lemma \( [4] \), we have \( T \in \mathcal{B}_{K,\ell}(m - 1) \), \( T \in \mathcal{V}(k) \) and \( k \in U(T) \), where \( \mathcal{V}(k) \) and \( U(T) \) are defined.
as in (11) and (10), respectively. For each $k' \in \mathcal{U}(T) \setminus \{k\}$, since $T = S \setminus \{k\}$, we have $k \not\in T \cup \{k'\}$. Moreover, by (10), we have $T \cup \{k'\} \in \mathcal{B}_{K, t}(m)$. Then by (12), the user $k$ caches $W_{d_k, T \cup \{k\}}$ for each $k' \in \mathcal{U}(T) \setminus \{k\}$, and hence it can recover $W_{d_k, T \cup \{k\}} = W_{d_k, s}$ by (14).

Theorem 1: Construction 1 gives a coded caching scheme for any $(K, M, N)$ caching system with $\frac{M}{N} = 1 - \frac{m}{K}$, for $F = \frac{K}{m} \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \left(K - 1 - i(\ell - 1)\right)$, and $\ell = \frac{m}{K} - 1$.

Moreover, denoting $n = K - m + 1 - \ell$, then $F \leq K \left(\frac{1 - \frac{m}{K}}{n} K + n\right)$.

Proof: By Lemma 4, Construction 1 is a coded caching scheme for any $(K, M, N)$ caching system with $K$ users and $N$ files, and we have seen that each file is divided into $F = \left|\mathcal{B}_{K, \ell}(m)\right| = \frac{K}{m} \sum_{i=1}^{m} (-1)^{i-1} \binom{m-1}{i} \left(K - 1 - i(\ell - 1)\right)$ packets.

For each $k \in \mathbb{Z}_K$, by (12), each user caches $\left|\mathcal{B}_{K, \ell}(m)\right| - C(K, m, \ell)$ packets of each file, where $C(K, m, \ell)$ is the number of $(m)$-bounded subsets of $Z_K$ containing $k$. In the proof of 2) of Lemma 3, we have seen that $C(K, m, \ell) = \left|\mathcal{B}_{K, \ell}(m)\right|$, so we can obtain

\[
\frac{M}{N} = \frac{\left|\mathcal{B}_{K, \ell}(m)\right| - C(K, m, \ell)}{F} = \frac{\left|\mathcal{B}_{K, \ell}(m)\right|}{\frac{K}{m} \sum_{i=1}^{m} (-1)^{i-1} \binom{m-1}{i} \left(K - 1 - i(\ell - 1)\right)} = 1 - \frac{m}{K}.
\]

By the delivery scheme of Construction 1, the total number of packets transmitted by the server is $RF = \left|\mathcal{B}_{K, \ell}(m)\right|$, so $R = \frac{\left|\mathcal{B}_{K, \ell}(m)\right|}{F} = \frac{K}{m} \sum_{i=1}^{m} (-1)^{i-1} \binom{m-1}{i} \left(K - 1 - i(\ell - 1)\right) = \frac{m}{m-1} \sum_{i=1}^{m} (-1)^{i-1} \binom{m-1}{i} \left(K - 1 - i(\ell - 1)\right)$.

Moreover, noticing that $\frac{M}{N} = 1 - \frac{m}{K}$, we can obtain $m = K \left(1 - \frac{m}{K}\right)$. Since $n = K - m + 1 - \ell$, then $K - \ell + 1 = m + n = \left(1 - \frac{m}{K}\right) K + n$. So by 3) of Lemma 3, we have

\[
\left|\mathcal{B}_{K, \ell}(m)\right| \leq K \left(\frac{K - \ell + 1}{m}\right)m = K \left(\frac{K - \ell + 1}{m}\right) K - \ell + 1 - m = K \left(\frac{1 - \frac{m}{K}}{n}\right) K + n,
\]

which completes the proof.

We can compare our construction with the Maddah-Ali-Niesen scheme (1). For any $t \in [K - 2]$ and any integer $n$ such that $0 \leq n \leq t$, let $m = K - t$ and $\ell = K - m + 1 - n$. Then from Construction 1, we obtain a coded caching scheme for any $(K, M, N)$ caching system with $\frac{M}{N} = 1 - \frac{m}{K}$ and $F \leq K \left(\frac{1 - \frac{m}{K}}{n}\right) K + n$. Moreover, we have

1) For $n > t - \frac{K}{K-1}$, we have $\ell < \frac{m}{K} + 1$, and by 2) of Remark 1, $\mathcal{B}_{K, \ell}(m) = \left(\frac{2K}{m}\right)$ and $\mathcal{B}_{K, \ell}(m-1) = \left(\frac{K}{m-1}\right)$.

By Theorem 1, it can be verified that the caching scheme obtained from Construction 1 has $F = \left(\frac{K}{K+1}\right)$ and $R = \frac{K(1 - \frac{M}{N})}{1 + K}$, which are the same as the Maddah-Ali-Niesen scheme (1).

2) As $n$ decreases, $\ell$ increases and by Theorem 1, $F$ decreases while $R$ increases. As an example, the log($F$) versus $n + 1$ and the $R$ versus $n + 1$ for a system with $K = 50$ and $\frac{M}{N} = \frac{1}{2}$ are shown in Fig. 1.

![Fig. 1. The log($F$) versus $n + 1$ and the $R$ versus $n + 1$ figure for a caching system with $K = 50$ and $\frac{M}{N} = \frac{1}{2}$, where we can obtain $1 \leq n + 1 \leq 26$.](image)

Construction 1 gives a family of coded caching schemes with polynomial subpacketization, as stated by the following theorem.

Theorem 2: Given an integer $n \geq 0$, for any $K$ such that $m = K \left(1 - \frac{m}{K}\right)$ is an integer and $2 \leq m \leq K - n$, there exists a coded caching scheme for any $(K, M, N)$ caching system with $F \leq K \left(\frac{1 - \frac{m}{K}}{n}\right) K + n = O(K^{n+1})$ and $R = \frac{m}{m-1} \sum_{i=1}^{m-1} (-1)^{i-1} \binom{m-1}{i} \left(K - i(\ell - 1)\right)$, where $\ell = K - m + 1$.

Proof: By assumption, we have $2 \leq m \leq K - 1$ and $1 \leq \ell \leq K - m + 1$. Therefore, Construction 1 gives a coded caching scheme for any $(K, M, N)$ caching system with $F \leq K \left(\frac{1 - \frac{m}{K}}{n}\right) K + n$ and $R = \frac{m}{m-1} \sum_{i=1}^{m-1} (-1)^{i-1} \binom{m-1}{i} \left(K - i(\ell - 1)\right)$.

V. CONCLUSIONS

We construct a family of coded caching schemes, which includes the schemes with optimal rate as well as the schemes with polynomial subpacketization. Like all existing constructions, our method reduces the subpacketization at the cost of increasing the rate. It is still an open problem to characterize the tight bound on the rate for coded caching with polynomial subpacketization.
REFERENCES

[1] M. A. Maddah-Ali and U. Niesen, “Fundamental limits of caching.” IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2856-2867, May 2014.

[2] K. Wan, D. Tuninetti, and P. Piantanida, “On the optimality of uncoded cache placement,” in Proc. IEEE Inf. Theory Workshop (ITW), 2016, pp. 161-165.

[3] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “The exact rate-memory tradeoff for caching with uncoded prefetching,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2017, pp. 1613-1617.

[4] K. Shanmugam, M. Ji, A. M. Tulino, J. Llorca, and A. G. Dimakis, “Finite-length analysis of caching-aided coded multicasting.” IEEE Trans. Inf. Theory, vol. 62, no. 10, pp. 5524-5537, Oct 2016.

[5] M. Cheng, J. Jiang, Q. Wang, and Y. Yao, “A Generalized Grouping Scheme in Coded Caching.” IEEE Trans. Communications, vol. 67, no. 5, pp. 3422-3430, May 2019.

[6] Q. Yan, M. Cheng, X. Tang, and Q. Chen, “On the placement delivery array design for centralized coded caching scheme,” IEEE Trans. Inf. Theory, vol. 63, no. 9, pp. 5821-5833, Sep. 2017.

[7] M. Cheng, J. Jiang, Q. Yan, and X. Tang, “Constructions of Coded Caching Schemes With Flexible Memory Size.” IEEE Trans. Communications, vol. 67, no. 6, pp. 4166-4176, Jun. 2019.

[8] C. Shangguan, Y. Zhang, and G. Ge, “Centralized Coded Caching Schemes: A Hypergraph Theoretical Approach.” IEEE Trans. Inf. Theory, vol. 64, no. 8, pp. 5755-5766, Aug. 2018.

[9] Q. Yan, X. Tang, Q. Chen, and M. Cheng, “Placement delivery array design through strong edge coloring of bipartite graphs.” IEEE Communications Letters, vol. 22, no. 2, pp. 236-239, Feb 2018.

[10] J. Michel and Q. Wang, “Placement Delivery Arrays from Combinations of Strong Edge Colorings,” 2019, available online at https://arxiv.org/abs/1907.03177.

[11] L. Tang and A. Ramamoorthy, “Coded Caching Schemes With Reduced Subpacketization From Linear Block Codes.” IEEE Trans. Inf. Theory, vol. 64, no. 4, pp. 3099-3120, Apr 2018.

[12] P. Krishnan, “Coded caching via line graphs of bipartite graphs,” in Proc. IEEE Information Theory Workshop (ITW), 2018, pp. 1-5.

[13] C. Hari Hara Suthan, M. Bhavana, and P. Krishnan, “Coded caching via projective geometry: A new low subpacketization scheme,” in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2019, pp. 682-686.

[14] S. Agrawal, K. V. S. Sree, and P. Krishnan, “Coded Caching based on Combinatorial Designs,” in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2019, pp. 1227-1231.

[15] M. Cheng, J. Wang, and X. Zhong, “A Unified Framework for Constructing Centralized Coded Caching Schemes,” 2019, available online at https://arxiv.org/abs/1908.05865.

[16] W. Song, K. Cai, and L. Shi, “Some New Constructions of Coded Caching Schemes with Reduced Subpacketization,” 2019, available online at https://arxiv.org/abs/1908.06570.

[17] K. Shanmugam, A. M. Tulino, and A. G. Dimakis, “Coded caching with linear subpacketization is possible using Ruzsa-Szemerédi graphs,” in Proc. IEEE Int. Symp. Inform. Theory (ISIT), 2017, pp. 1237-1241.

[18] J. Ratsaby, “Estimate of the number of restricted integer-partitions,” Applicable Analysis and Discrete Mathematics, 2(2): 222-233, 2008.

[19] S. Jukna. Extremal Combinatorics: With Applications in Computer Science. EATCS Texts in Theoretical Computer Science. Springer-Verlag, 2001.