Unified treatment for classical waves in two-dimensional media

Pi-Gang Luan\textsuperscript{1} and Tzong-Jer Yang\textsuperscript{2}

\textsuperscript{1}Institute of Optical Sciences, National Central University, Chung-Li 32054, Taiwan, Republic of China
\textsuperscript{2}Department of Electrophysics, National Chiao Tung University, Hsinchu 30050, Taiwan, Republic of China

(Dated: March 23, 2022)

A unified treatment for the propagation of classical waves in inhomogeneous media is proposed. We deal with four kinds of waves, they are the acoustic wave in fluid, the elastic shear wave in two dimensional solid, and the E- and H-polarized electromagnetic waves in two dimensional lossless medium. We first show that a universal wave equation governing the wave motion of all these kinds of waves can be derived. We then introduce an auxiliary field, and give the universal expressions for the energy densities and energy flows of these waves. This unified treatment provides intuitive insights, which may be helpful in understanding the essential physics of various wave phenomena, or useful for designing new photonic and sonic devices.

PACS numbers: 41.20.-q, 41.20.Jb, 43.20.+g

I. INTRODUCTION

Periodic media such as the photonic crystals \textsuperscript{1}234567 and phononic crystals \textsuperscript{3}89101112131415161718 have received considerable attention in the past decades because of their interesting physical properties and application potential in designing various new devices. Waves propagating inside these media will be modulated by the periodic structures. Therefore, the behavior of these waves will no longer be the same as they do in a free space, and the so called frequency band structures appear \textsuperscript{19}. Besides, if we destroy the periodicity to make these structures disordered, then under certain conditions there might appear another interesting phenomenon called Anderson localization \textsuperscript{20}21222324, which is caused by the multiple scattering of waves in the media. Recently, the fascinating and unusual negative refraction (NR) phenomenon \textsuperscript{25}26 in a kind of artificial media called left-handed materials (LHM) \textsuperscript{27}28 received even more attention and has already stimulated a large amount of investigations \textsuperscript{27}293031323334353637383940. Even more interestingly, the newest research results have shown that the NR-like phenomena can also happen in acoustic systems \textsuperscript{41}42.

All of these developments are about classical waves \textsuperscript{43}44, including the electromagnetic (EM) waves, the acoustic (AC) waves, and the elastic (EL) waves. Clearly there are similarities between the EM and AC systems (hereafter both the AC and EL systems will be denoted as the AC systems), and in the past decade these similarities indeed stimulated some parallel developments about the studies on EM and AC systems, such like the photonic and sonic crystals. Although these similarities have been noticed since as early as the 19th century \textsuperscript{45}, however, up to now it seems that a systematic study and a complete understanding on them are lacking. This situation leads to some inconveniences. For example, there have been proposed useful guiding rules for designing photonic crystals \textsuperscript{1}467 and phononic crystals \textsuperscript{17}161718 with large band gaps. However, it is not easy to understand the explicit meanings of these rules. Besides, since in most studies the researchers put restrictions on the values of the material parameters (for example, in a photonic crystal one usually assumes the permeability $\mu = 1$), the usefulness of these rules might be very restrictive. By extending the value of material parameters to the unexplored regions, some interesting or unexpected phenomena might be found \textsuperscript{22}2324. An abstract and unified treatment are thus useful and can avoid unnecessary repeated works. It can also help us to find the essential physics that governs various wave phenomena. Furthermore, since this treatment unifies the AC and EM systems, anyone who is familiar with only one side, say, the AC systems, can “translate” his knowledge to the other side, and thus can help him to understand the apparently different systems more easily.

In this paper, we shall show that for AC wave in fluid, the EL shear (ELSH) wave in two-dimensional solid, and the E- and H-polarized EM waves in two-dimensional nonabsorptive media, a unified theory can be constructed, which gives us a universal description of the dynamical behaviors of the above mentioned classical waves. We believe that a further development of this theory can help us much in clarifying the essential physics of the classical wave systems.

This paper is organized as follows. The explicit wave equations for the four kinds of waves mentioned before will be given in the next section. A unified treatment for these waves is developed in section III, in which we introduce an auxiliary field. Using this auxiliary field, not only a universal wave equation, but also all physical quantities such as energy densities and energy flows can be expressed in a unified manner. We then discuss in section IV the applications of this unified theory. Section V is the summary of this paper. Finally in Appendix A we derive the conservation laws for elastic waves.
II. THE WAVE EQUATIONS

In this section we first review the wave equations of the EL, AC and EM waves, respectively. We then demonstrate that for 2D propagation, the wave equations for the above mentioned four kinds of classical waves have the same form and thus can be treated in a unified manner.

A. Elastic waves

Let $\mathbf{u}$ be the deformation of the medium at certain point from its equilibrium position due to the stress. In terms of Cartesian coordinate system, the propagation of EL waves in an inhomogeneous medium is governed by

$$
\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_i} \left( \lambda_e \frac{\partial u_j}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left[ \mu_e \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]
$$

$$
= \partial T_{ij} \partial x_j,
$$

(1)

where $\rho$ is the mass density, $\lambda_e$ and $\mu_e$ are the Lamé coefficients, and

$$
T_{ij} = \lambda_e \left( \partial_i u_j \right) + \mu_e \left( \partial_i u_j + \partial_j u_i \right),
$$

(2)

is the $ij$th component of stress tensor, all quantities are position dependent. In terms of vector notations, the above equation can also be written as

$$
\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \left( \lambda_e \nabla \mathbf{u} \right) + \nabla \cdot \left( \mu_e \nabla \mathbf{u} + \mu_e \frac{\partial \mathbf{u}}{\partial x_i} \right).
$$

(3)

In a homogeneous medium $\mathbf{u}$ can always be written as the sum of a gradient and a curl, i.e.,

$$
\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t = \nabla \Phi + \nabla \times \mathbf{\Psi},
$$

(4)

where $\mathbf{u}_l = \nabla \Phi$ and $\mathbf{u}_t = \nabla \times \mathbf{\Psi}$ represent the longitudinal and transverse components of the waves, respectively. The fields $\Phi$ and $\Psi$ satisfy

$$
\nabla^2 \Phi - \frac{1}{c_l^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,
$$

$$
\nabla^2 \mathbf{\Psi} - \frac{1}{c_t^2} \frac{\partial^2 \mathbf{\Psi}}{\partial t^2} = 0,
$$

(5)

where

$$
c_l = \sqrt{\frac{\lambda_e + 2\mu_e}{\rho}}, \quad c_t = \sqrt{\frac{\mu_e}{\rho}}
$$

(6)

are the phase velocities of the longitudinal and transverse waves, respectively. From these results the Lamé coefficients can be expressed as

$$
\lambda_e = \rho (c_l^2 - 2c_t^2), \quad \mu_e = \rho c_t^2.
$$

(7)

1. Acoustic waves

One special case of Eq. (1) is the wave equation for AC waves in fluid. In fluid, we assume $\mu_e = 0$ (no shear force) and thus $c_t = 0$. Now we denote the phase velocity of the wave as $c$, where $c = c_l$. The pressure $p$ and vibration velocity $\mathbf{v}$ can be derived via the relations

$$
p = -\lambda_e \nabla \cdot \mathbf{u}, \quad \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}.
$$

(8)

Substitute (5) into (1) and one finds the wave equation

$$
\nabla \cdot \left( \frac{\nabla p}{\rho} \right) = \frac{1}{\rho \rho^2} \frac{\partial^2 p}{\partial t^2},
$$

(9)

and the equation relating $\mathbf{v}$ to $p$:

$$
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p.
$$

(10)

2. Elastic SH waves

Another special case of Eq. (1) is the wave equation for ELSH waves. Suppose the medium considered is homogeneous along the direction of $\mathbf{z}$, and wave is propagating along a path lying on the $xy$ plane, then the $\mathbf{u}$ vector can always be written as a sum of two decoupled vectors: a $\mathbf{u}_{xy}$ vector that lies on the $xy$ plane, and a $\mathbf{u}_z$ vector that parallels the $z$-axis. Now if $\mathbf{u}_{xy} = 0$, i.e., SH wave, then $\mathbf{u} = u \mathbf{z}$ and the wave equation becomes

$$
\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot (\mu_e \nabla \mathbf{u}) = \nabla \cdot (\rho c_t^2 \nabla \mathbf{u}).
$$

(11)

For the convenience of the following discussions, we define

$$
\eta \equiv \frac{1}{\mu_e} = \frac{1}{\rho c_t^2},
$$

(12)

and thus Eq. (11) can be rewritten as

$$
\nabla \cdot \left( \frac{\nabla \mathbf{u}}{\eta} \right) = \frac{1}{\eta \rho c_t^2} \frac{\partial^2 \mathbf{u}}{\partial t^2}.
$$

(13)

B. Electromagnetic waves

Now we turn to the discussion of EM waves. Adopting the Gaussian unit, and denoting the speed of light in vacuum as $c_0$, Maxwell's equations in a source free and locally isotropic medium are written as

$$
\nabla \times \mathbf{E} = -\frac{1}{c_0} \frac{\partial \mathbf{B}}{\partial t} = -\frac{\mu_e}{c_0} \frac{\partial \mathbf{H}}{\partial t}.
$$

(14)

$$
\nabla \times \mathbf{H} = \frac{1}{c_0} \frac{\partial \mathbf{D}}{\partial t} = \frac{\epsilon_e}{c_0} \frac{\partial \mathbf{E}}{\partial t}.
$$

(15)
Here $\epsilon = \epsilon(r)$ and $\mu = \mu(r)$ are position dependent permittivity and permeability. From (14) and (15), we can easily derive the wave equations in terms of $E$ and $H$ fields:

\[
\nabla \times \left( \frac{\nabla \times E}{\mu} \right) = -\frac{\epsilon}{c_0^2} \frac{\partial^2 E}{\partial t^2}
\]

\[
\nabla \times \left( \frac{\nabla \times H}{\epsilon} \right) = -\frac{\mu}{c_0^2} \frac{\partial^2 H}{\partial t^2}.
\]

1. E-polarized and H-polarized EM waves

Like that in the discussion of the elastic waves, we now assume that the medium is translational invariant along $\hat{z}$. It is well known that in this case if an EM wave propagating along a path on XY plane, then electromagnetic fields can always be decoupled into the E-polarized and H-polarized waves. For E-polarized waves, we mean $E = E\hat{z}$. Remember that in this system $\partial E/\partial z = 0$, thus we have

\[
\nabla \times \left( \frac{\nabla \times E}{\mu} \right) = -\hat{z} \nabla \cdot \left( \frac{\nabla E}{\mu} \right).
\]

So, Eq. (17) becomes

\[
\nabla \cdot \left( \frac{\nabla E}{\mu} \right) = \frac{\epsilon}{c_0^2} \frac{\partial^2 E}{\partial t^2} = \frac{1}{\mu c^2} \frac{\partial^2 E}{\partial t^2}.
\]

Similarly, for H-polarized waves, we have

\[
\nabla \cdot \left( \frac{\nabla H}{\epsilon} \right) = \frac{\mu}{c_0^2} \frac{\partial^2 H}{\partial t^2} = \frac{1}{\epsilon c^2} \frac{\partial^2 H}{\partial t^2}.
\]

Here $c = c_0/\sqrt{\epsilon\mu}$ represents the speed of light in the medium.

III. UNIFIED TREATMENT FOR CLASSICAL WAVES IN 2D SYSTEMS

All the wave equations of the four cases discussed before have the same form

\[
\nabla \cdot \left( \frac{\nabla U}{\alpha} \right) = \frac{1}{\alpha c^2} \frac{\partial^2 U}{\partial t^2}.
\]

thus their wave equations can be treated in a unified manner. However, this is not enough if we want to construct a complete unified description. Such a theory must also deal with the quantities like energy density and energy flow in a unified manner. To achieve this goal, we now introduce a field which we call the auxiliary field. We will show in the later discussion that the complete unified description can be constructed using the wave equation of the auxiliary field.

Consider an auxiliary field satisfying the wave equation

\[
\nabla \cdot \left( \frac{\nabla \Phi}{\alpha} \right) - \frac{1}{\alpha c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,
\]

which has the same form as in Eq. (22) with $\alpha = \alpha(r)$ and $c = c(r)$ being position-dependent parameters. Throughout the paper we assume that $\partial \Phi/\partial z = 0$, i.e., $\Phi = \Phi(x, y, t)$. Under this assumption the system becomes effectively two dimensional. We define the dynamical variables as $\varphi$ and $Q$:

\[
\varphi = \frac{\partial \Phi}{\partial t}, \quad Q = -\frac{1}{\alpha} \nabla \Phi.
\]

We will see in the later discussion that $\Phi$ plays the role of the vector potential in the electromagnetics. Using the definition (24) the wave equation (23) can be replaced by

\[
\frac{\partial Q}{\partial t} = -\frac{1}{\alpha} \nabla \varphi, \quad \frac{\partial \varphi}{\partial t} = -\alpha^2 \nabla \cdot Q.
\]

Note that if we choose the Lagrangian density

\[
\mathcal{L} = \frac{1}{2\alpha c^2} \left( \frac{\partial \Phi}{\partial t} \right)^2 - \frac{1}{2\alpha} \left( \nabla \Phi \right)^2 = \frac{1}{2\alpha c^2} \varphi^2 - \frac{\alpha}{2} Q^2,
\]

then Eq. (24) and (25) can be deduced from the Euler equation of motion

\[
\nabla \cdot \left[ \frac{\partial \mathcal{L}}{\partial (\nabla \Phi)} \right] = 0.
\]

Here we have employed the notation $\dot{\Phi} \equiv \partial \Phi/\partial t$. With this lagrangian, the canonical momentum $\Pi$ and the Hamiltonian $\mathcal{H}$ are given by

\[
\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{\alpha c^2} \dot{\Phi} = \frac{1}{\alpha c^2} \varphi.
\]

and

\[
\mathcal{H} = \Pi \dot{\Phi} - \mathcal{L} = \frac{\alpha c^2}{2} \Pi^2 + \frac{1}{2\alpha} (\nabla \Phi)^2 = \frac{\alpha}{2} Q^2 + \frac{1}{2\alpha c^2} \varphi^2.
\]

A. The Energy conservation law

In terms of Eq. (28), we now derive the energy conservation law. Multiplying $-\dot{\Phi}$ on Eq. (23), we have

\[
-\nabla \cdot \left( \frac{\Phi \nabla \Phi}{\alpha^2} \right) + \frac{\nabla \Phi \cdot \nabla \Phi}{\alpha} + \frac{\Phi}{\alpha c^2} \frac{\partial \Phi}{\partial t} = 0
\]
or
\[ \nabla \cdot \left( -\phi \frac{\nabla \Phi}{\alpha} \right) + \frac{\partial}{\partial t} \left[ \frac{1}{2\alpha c^2} \phi^2 + \frac{1}{2\alpha} (\nabla \Phi)^2 \right] = 0. \quad (30) \]

Define the energy flux \( J \) and energy density \( U \) as
\[ J \equiv -\frac{1}{\alpha} \phi \nabla \Phi = \varphi Q \quad (31) \]
and
\[ U \equiv \frac{1}{2\alpha} (\nabla \Phi)^2 + \frac{1}{2\alpha c^2} \varphi^2 = \frac{\alpha}{2} Q^2 + \frac{1}{2\alpha c^2} \varphi^2, \quad (32) \]
then Eq. (30) becomes
\[ \nabla \cdot J + \frac{\partial U}{\partial t} = 0, \quad (33) \]
which is the energy conservation law (the “continuity equation” for the energy). Comparing Eq. (32) with (29) confirms that \( U \) is indeed the energy density.

B. Identification of \( J \) and \( U \) with known physical quantities

1. \( J \) and \( U \) for acoustic waves in fluid

For the acoustic wave propagation in 2D we define
\[ \Phi = \int^t p dt, \quad \alpha \equiv \rho, \quad (34) \]
then
\[ \varphi = p, \quad Q = v. \quad (35) \]
The energy flux \( J \) and energy density \( U \) are given by
\[ J = p v, \quad U = \frac{1}{2} \rho v^2 + \frac{1}{2\rho c^2} p^2, \quad (36) \]
which are consistent with the results from the 3D formula (See appendix A).

2. \( J \) and \( U \) for elastic SH waves in solid

For the SH waves \( u = u(x, y, t) \hat{z} \). We define
\[ \Phi \equiv u, \quad \alpha \equiv \frac{1}{\mu c} = \frac{1}{\rho c^2}, \quad (37) \]
then
\[ \varphi = \dot{u} = v, \quad Q = -\mu v \nabla u = -\mathbf{T} \cdot \dot{\mathbf{z}}, \quad (38) \]
where \( v = v \hat{z} \) represents the vibration velocity of the media and \( \mathbf{T} \) stands for the stress tensor.

Now we denote \( \mathbf{n} \) (\( ||\nabla u|| \)) as the direction of \( Q \), i.e.,
\[ Q = Q \mathbf{n}, \quad Q = |Q|. \quad (39) \]

Suppose there is a region enclosed by a surface with normal vector \( \mathbf{n} \) (here we have \( \mathbf{n} \perp \hat{z} \)). The surface exerts a shear force \( \mathbf{f}_s \) (along \( \hat{z} \)) on its surrounding medium, which is given by
\[ \mathbf{f}_s = -\mathbf{T} \cdot \mathbf{n} = Q \hat{z}. \quad (40) \]

From this observation we find that \( Q \) in this case is a vector that has the magnitude of the shear force and the direction of \( \nabla u \). The energy flux \( J \) and energy density \( U \) are given by
\[ J = -\mu c \nabla u, \quad U = \frac{\mu}{2} v^2 + \frac{\mu c^2}{2} (\nabla u)^2, \quad (41) \]
which are consistent with the results from the 3D formula (See appendix A).

3. \( J \) and \( U \) for E-polarized wave

For EM waves, we start with the consideration of the E-polarized wave. In this case the dynamical variables are \( E \) and \( H \) fields. Since \( \nabla \cdot \mathbf{B} = 0 \), in terms of the vector potential \( \mathbf{A} \) we have \( \mathbf{B} = \nabla \times \mathbf{A} \). Now we choose
\[ \mathbf{A} = -\sqrt{\frac{4\pi}{c_0}} \Phi \hat{z} \quad (42) \]
and
\[ \alpha = \mu/c_0, \quad \alpha c^2 = c_0/\epsilon, \quad (43) \]
then we have
\[ \mathbf{E} = -\frac{1}{c_0} \frac{\partial \mathbf{A}}{\partial t} = \sqrt{\frac{4\pi}{c_0}} \varphi \hat{z}, \quad (44) \]
\[ \mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \sqrt{\frac{4\pi}{c_0}} Q \times \hat{z}. \quad (45) \]

Using these relations then Eq. (25) indeed gives the correct dynamical equations for E-polarized waves (i.e., Eq. (13) and (15)). In addition, the textbook formulas for the Poynting vector
\[ \mathbf{S} = \frac{c_0}{4\pi} \mathbf{E} \times \mathbf{H} = \varphi \hat{z} \times (Q \times \hat{z}) = \varphi Q = J, \quad (46) \]
and the energy density
\[ \mathcal{W} = \frac{1}{8\pi} (\epsilon E^2 + \mu H^2) = \frac{\alpha}{2} Q^2 + \frac{1}{2\alpha c^2} \varphi^2 = U. \quad (47) \]
give exactly the same results as those obtained from our formulas for energy flux (Eq. (31)) and for energy density (Eq. (32)).

4. \( J \) and \( U \) for H-polarized waves

In this case the dynamical variables are also \( E \) and \( H \) fields. Since we consider only the source free case, we have \( \nabla \cdot \mathbf{D} = 0 \). Thus we can introduce a vector \( \mathbf{A}_m \), such
that $\mathbf{D} = \nabla \times \mathbf{A}_m$. Here $\mathbf{A}_m$ represents another kind of vector potential, used in deriving the $\mathbf{D}$ field. Now we relate $\mathbf{A}_m$ to $\Phi$ via

$$
\mathbf{A}_m = \sqrt{4\pi c_0} \Phi \hat{z}
$$

(48)

and choose

$$
\alpha = \epsilon/c_0, \quad \alpha c^2 = c_0/\mu,
$$

(49)

then we have

$$
\mathbf{H} = \frac{1}{c_0} \frac{\partial \mathbf{A}_m}{\partial t} = \sqrt{\frac{4\pi}{c_0}} \varphi \hat{z},
$$

(50)

$$
\mathbf{E} = \frac{1}{\epsilon} \nabla \times \mathbf{A}_m = -\sqrt{\frac{4\pi}{c_0}} \mathbf{Q} \times \hat{z}.
$$

(51)

Using these relations then Eq. (26) gives the correct dynamical equations for H-polarized waves (i.e., Eq. (14) and (15)). Furthermore, the Poynting vector

$$
\mathbf{S} = \frac{-\alpha}{4\pi} \mathbf{H} \times \mathbf{E} = -\varphi \hat{z} \times (-\mathbf{Q} \times \hat{z}) = \varphi \mathbf{Q},
$$

(52)

together with the electromagnetic energy density

$$
\mathcal{W} = \frac{1}{8\pi} (\epsilon \mathbf{E}^2 + \mu \mathbf{H}^2) = \frac{\alpha}{2} \mathbf{Q}^2 + \frac{1}{2\alpha c^2} \varphi^2 = \mathcal{U}.
$$

(53)

are the same as those obtained from Eq. (31) and Eq. (52).

C. Monochromatic waves

Now we consider the monochromatic wave. The auxiliary field $\Phi$ can be written as

$$
\Phi = \frac{1}{2} (\hat{\varphi} + \hat{\varphi}^*).
$$

(54)

Here $\hat{\varphi} \propto e^{-i\omega t}$ satisfies $\hat{\varphi} = -i\omega \hat{\varphi}$. $\hat{\varphi}^*$ is the complex conjugate of $\hat{\varphi}$.

The $\varphi$ and $\mathbf{Q}$ now become

$$
\varphi = \frac{1}{2} (\hat{\varphi} + \hat{\varphi}^*) , \quad \mathbf{Q} = \frac{1}{2} (\hat{\mathbf{Q}} + \hat{\mathbf{Q}}^*),
$$

(55)

with $\hat{\varphi}$ and $\hat{\mathbf{Q}}$ given by

$$
\hat{\varphi} = -i\omega \hat{\Phi} , \quad \hat{\mathbf{Q}} = -\frac{1}{\alpha} \nabla \hat{\Phi}.
$$

(56)

Since both $\hat{\varphi}$ and $\hat{\mathbf{Q}}$ contain a factor $e^{-i\omega t}$, they can be factorized as

$$
\hat{\varphi}(r,t) = \psi(r) e^{-i\omega t} , \quad \hat{\mathbf{Q}}(r,t) = \left( \frac{\nabla \psi(r)}{i\omega \alpha(r)} \right) e^{-i\omega t},
$$

(57)

where $\psi$ is a time independent field.

The wave equation Eq. (23) now becomes

$$
\nabla \cdot \left( \frac{\nabla \psi(r)}{\alpha(r)} \right) + \frac{\omega^2 \psi(r)}{\alpha(r)c^2(r)} = 0.
$$

(58)

The time averaged energy flux and energy density are defined as

$$
\mathbf{j} = \frac{1}{T} \int_0^T \mathbf{J}(t) dt = \frac{1}{2} \text{Re} \left( \hat{\varphi}^* \hat{\mathbf{Q}} \right),
$$

$$
\mathcal{U} = \frac{1}{2} \text{Re} \left( \frac{\psi^* \nabla \psi}{ik} \right).
$$

(59)

and

$$
\mathcal{U} = \frac{1}{2} \text{Re} \left( \frac{\psi^* \nabla \psi}{ik} \right) = \frac{1}{2} \text{Re} \left( \frac{\psi^* \nabla \psi}{ik} \right).
$$

(60)

Here $T = 2\pi/\omega$ is the time period, and $k = k(r) = \omega/c(r)$ is the wave number in the media.

IV. APPLICATIONS

Now we turn to the applications of this unified theory. In the following subsections we will discuss the use of the theory, only briefly, on two topics: the wave scattering in a two dimensional system, and the band gap engineering problem.

A. Scattering in a two dimensional system

We discuss only a simple case. Suppose a (monochromatic) incident wave from a source propagates to and be scattered by a circular cylinder located at the origin.

Now we assume the radius of the cylinder is $a$. The material parameter $\alpha(r)$ inside and outside of the cylinder are given by constants $\alpha_1$ and $\alpha$, respectively. Similarly, the wave speed are given by $c_1$ and $c$. We also define $k_1 = \omega/c_1$ and $k = \omega/c$ as the corresponding wave numbers, and denote the wave inside and outside of the cylinder as $\psi_1(r)$ and $\psi(r)$. According to Eq. (26), they satisfy the Helmholtz equations,

$$
(\nabla^2 + k_1^2)\psi_1 = 0, \quad (\nabla^2 + k^2)\psi = 0.
$$

(61)

Since the Bessel and Hankel functions of all orders $(J_n, H_n^{(1)})|n = -\infty \rightarrow \infty|$ forms a complete set of eigenfunctions, the solutions of $\psi$ and $\psi_1$ can be written as the sums of them:

$$
\psi(r, \theta) = \sum_{n=-\infty}^{\infty} [A_n H_n^{(1)}(kr) + B_n J_n(kr)] e^{in\theta},
$$

$$
\psi_1(r, \theta) = \sum_{n=-\infty}^{\infty} [C_n H_n^{(1)}(k_1 r) + D_n J_n(k_1 r)] e^{in\theta}.
$$

(62)
To determine the coefficients, we have to find the boundary conditions. According to Eq. (5), and by using the divergence theorem, we get the boundary conditions at \( r = a \):

\[
\psi_1|_{r=a} = \psi|_{r=a}, \quad \frac{1}{\alpha_1} \frac{\partial \psi_1}{\partial r} \bigg|_{r=a} = \frac{1}{\alpha} \frac{\partial \psi}{\partial r} \bigg|_{r=a}.
\]  

(63)

Furthermore, since the wave source is outside of the cylinder, the wave amplitude inside the cylinder cannot go to infinity, we therefore have \( C_n = 0 \). Substitute Eq. (62) and Eq. (66), we have used the identity

\[
\frac{k_1}{\alpha}[A_n H_n^{(1)'}(ka) + B_n J_n'(ka)] = \frac{k_1}{\alpha} D_n J_n'(k_1a).
\]  

(64)

These two equations lead to

\[
\frac{A_n}{B_n} = \frac{2gh/(i\pi ka)}{H_n^{(1)}(ka)J_n'(ka/h) - gh H_n^{(1)'}(ka)J_n(ka/h)}
\]  

and

\[
\frac{D_n}{B_n} = \frac{2gh/(i\pi ka)}{H_n^{(1)}(ka)J_n'(ka/h) - gh H_n^{(1)'}(ka)J_n(ka/h)}.
\]  

(65)

(66)

Here we have defined \( g \equiv \alpha_1/\alpha \) and \( h \equiv c_1/c \). In deriving Eq. (66), we have used the identity

\[
J_n(x) H_n^{(1)'}(x) - J_n'(x) H_n^{(1)}(x) = \frac{2i}{\pi x}.
\]

Suppose the incident wave is a plane wave:

\[
\psi_0(r) = e^{ikr} = e^{ikr \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{in\theta},
\]  

(67)

then we have

\[
B_n = i^n.
\]  

(68)

Now the coefficients \( A_n \) and \( D_n \) can be obtained from Eq. (65) and (66). The scattered wave is then given by

\[
\psi_{\text{scat}}(r) \equiv \psi(r) - \psi_0(r) = \sum_{n=-\infty}^{\infty} A_n H_n^{(1)'}(kr)e^{in\theta}.
\]  

(69)

If the incident wave is not a plane wave, the only difference will be to give a different set of \( B_n \)'s, and calculate its corresponding \( A_n \) and \( D_n \)'s from Eq. (65) and (66).

**B. Band gap engineering**

Now we turn to the discussion about the photonic and phononic crystals. In these wave crystals \( \alpha(r) \) and \( c(r) \) are periodic functions of \( r \), i.e.,

\[
\alpha(r + \mathbf{R}) = \alpha(r), \quad c(r) = c(r + \mathbf{R}).
\]  

(70)

Here

\[
\mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2
\]  

is an arbitrary translation vector, \( n_1 \) and \( n_2 \) are two integers, and \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) are the fundamental translation vectors \[11\]. According to Bloch’s Theorem, the eigenwave in wave crystal has the form

\[
\psi_{\mathbf{k}}(r) = e^{i\mathbf{K} \cdot r} \xi(r),
\]  

(72)

where \( \mathbf{K} \) is the Bloch wave vector, and \( \xi(r) \) is a periodic function, satisfying

\[
\xi(r) = \xi(r + \mathbf{R}).
\]  

(73)

The band structure of the wave crystal can be calculated in a unified manner from Eq. (58) by using the method that used in the second paper of Ref. \[8\]. We will not go through the details of the band structure calculations but instead give a brief discussion about the band gap engineering.

The main aim of the band gap engineering in the wave crystal study is to create a large band gap. Recently, Zhang \emph{et.al} have developed a systematic method for enlarging a photonic \[9\] or phononic \[17\] band gap. Their formulas tell them how and where to vary the material parameters can enlarge a band gap that already exists. Here we will derive the unified version of their formulas (See Eq. (5) and Eq. (7) of Ref. \[9\] and Eq. (2) of Ref. \[17\]).

Suppose \( \psi_{\mathbf{k}}(r) \equiv \psi(r) \) is an eigenfunction of the operator

\[
-\alpha(r)c^2(r)\nabla \cdot \frac{1}{\alpha(r)} \nabla,
\]

and \( \omega_K^2 \equiv \omega^2 \) is the corresponding eigenvalue. That is, \( \psi(r) \) satisfies Eq. (58). Multiplying Eq. (58) by \( \psi^* \), we find

\[
\nabla \cdot \left( \frac{\psi^* \nabla \psi}{\alpha} \right) - \frac{\nabla |\psi|^2}{\alpha} + \frac{\omega^2 |\psi|^2}{\alpha c^2} = 0.
\]  

(74)

According to Eq. (72), the quantity

\[
\psi^* \nabla \psi = i|\xi|^2 \mathbf{K} + \xi^* \nabla \xi
\]

is a periodic vector field. Integrating Eq. (74) over a unit cell, we get

\[
\int_{\text{cell}} \frac{\nabla |\psi(r)|^2}{\alpha(r)} d^2r = \omega^2 \int_{\text{cell}} \beta(r)|\psi(r)|^2 d^2r.
\]  

(75)

Here we have defined

\[
\beta(r) = \frac{1}{\alpha(r)c^2(r)}
\]  

(76)

Rewriting Eq. (58) as

\[
\nabla \cdot \left( \frac{\nabla \psi}{\alpha} \right) = -\omega^2 \beta \psi,
\]  

(77)
we can derive
\[
\nabla \cdot \left[ -\frac{1}{\alpha} \nabla \psi + \frac{1}{\alpha} \nabla \delta \psi \right] = -\delta \omega^2 \beta \psi - \omega^2 (\delta \beta \psi + \beta \delta \psi). \tag{78}
\]

Note that in any case we do not vary the Bloch wave vector \( \mathbf{K} \).

Multiplying Eq. \eqref{78} by \( \psi^* \) and integrating the resulting product over one unit cell, we find
\[
\int_{\text{cell}} \nabla \psi^* \cdot \left[ -\frac{1}{\alpha} \nabla \psi + \frac{1}{\alpha} \nabla \delta \psi \right] d^2 r = \delta \omega^2 \int_{\text{cell}} \beta |\psi|^2 d^2 r + \omega^2 \int_{\text{cell}} (\delta \beta |\psi|^2 + \beta |\psi|^2 \delta \psi) d^2 r. \tag{79}
\]

Multiplying the complex conjugation of Eq. \eqref{77} by \( \delta \psi \), we can also derive
\[
\int_{\text{cell}} \nabla \delta \psi \cdot \left[ -\frac{1}{\alpha} \nabla \psi^* \right] d^2 r = \omega^2 \int_{\text{cell}} \beta \psi^* \delta \psi d^2 r. \tag{80}
\]

By substituting Eq. \eqref{80} into Eq. \eqref{79}, and using Eq. \eqref{77}, we finally obtain
\[
\frac{\delta \omega}{\omega} = \frac{1}{2} \left[ \frac{\int_{\text{cell}} \frac{1}{\alpha} |\nabla \psi|^2 d^2 r}{\int_{\text{cell}} |\nabla \psi|^2 d^2 r} \right] - \frac{\int_{\text{cell}} \delta \beta |\psi|^2 d^2 r}{\int_{\text{cell}} \beta |\psi|^2 d^2 r}. \tag{81}
\]

This is the unified version of the band gap engineering formula derived by Zhang et al. \[5, 6, 16\].

For acoustic wave we have \( \psi = \rho \), \( \alpha = \rho \) and \( \beta = 1/\alpha c^2 \), and hence
\[
\frac{\delta (\omega^2)}{\omega^2} = 2 \frac{\delta \omega}{\omega}, \quad \delta \left( \frac{1}{\rho} \right) = \frac{\delta \rho}{\rho^2}, \quad \delta \beta = - \frac{\delta (\alpha c^2)}{(\alpha c^2)^2}. \tag{82}
\]

We therefore have from Eq. \eqref{81} and Eq. \eqref{77} the result
\[
\frac{\delta (\omega^2)}{\omega^2} = \frac{\int_{\text{cell}} |\nabla \psi|^2 \delta (\rho c^2 |\psi|^2)}{\int_{\text{cell}} |\nabla \psi|^2} d^2 r - \frac{\int_{\text{cell}} \nabla (\rho c^2 |\psi|^2) / \rho c^2 d\psi d^2 r}{\omega^2 \int_{\text{cell}} |\nabla (\rho c^2 |\psi|^2) / \rho c^2| d^2 r}. \tag{83}
\]

which is just the Eq. \eqref{2} of Ref. \[17\]. Results for the EM waves can also be obtained in a similar way.

We now give several remarks about Eq. \eqref{75}.

A. In fact, Eq. \eqref{75} is a consequence of the harmonic property of the wave field. In the second line of Eq. \eqref{60} we note that the energy density can be written as the sum of two terms: \( \alpha Q^2/4 = |\nabla \psi|^2/4\alpha \omega^2 \) and \( |\varphi|^2/4\alpha c^2 = |\psi|^2/4\alpha c^2 \). Let's call them the type I energy and the type II energy. Eq. \eqref{75} means that the time average of this two kinds of energies in one unit cell is equal. This is reasonable because to form a harmonic wave the type I energy has to transform itself to type II completely and then transform back in one time period \( T/\omega \).

B. From Eq. \eqref{74}, we have
\[
\omega^2 = \frac{\int_{\text{cell}} |\nabla \psi(r)|^2 d^2 r}{\int_{\text{cell}} |\nabla \psi(r)/\alpha(r)|^2 d^2 r}, \tag{84}
\]

which yields
\[
\ln \omega^2 = \ln \int_{\text{cell}} |\nabla \psi(r)/\alpha(r)|^2 d^2 r - \ln \int_{\text{cell}} |\nabla \psi(r)|^2 d^2 r. \tag{85}
\]

Taking the variation of Eq. \eqref{81} and keeping the \( |\psi| \) and \( |\nabla \psi| \) terms unchanged then gives us Eq. \eqref{81}. Why can we assume in this variational procedure that \( |\psi| \) is unchanged? The reason is that the change of \( |\psi| \) caused by the first order perturbation of \( 1/\alpha \) and \( \beta \) is second order.

C. Although in a unit cell the total type I energy is equal to the total type II energy, however, their distributions are different. Consider a binary system, i.e., the wave crystal that composed of two kinds of homogeneous materials. A unit cell of such a wave crystal can be divided into region \( a \) and region \( b \), with material parameters \( \alpha_a, \beta_a \) and \( \alpha_b, \beta_b \), respectively. According to Ref. \[3, 4, 17\], it is the impedance \( \beta/\alpha \) that determine the size of the band gap. More explicitly, a wave crystal with large \( \beta_a/\alpha_a \) and small \( \beta_b/\alpha_b \), or a small \( \beta_a/\alpha_a \) and large \( \beta_b/\alpha_b \), will most probably have a large band gap. This seems to imply that to have a large gap we have to design the \( \alpha \) and \( \beta \) parameters such that the type I and type II energies are efficiently separated in the unit cell.

V. SUMMARY

In this paper, we have shown that for four kinds of classical waves propagating in two-dimensional media, a unified treatment can be constructed. These waves are the AC wave in fluid, the EL shear (ELSH) wave in solid, and the E- and H-polarized EM waves in nonabsorptive media. This unified theory helps us to find the essentials of various wave phenomena and thus give us a more complete and comprehensive understanding of these phenomena.

ACKNOWLEDGMENTS

The authors would like to acknowledge helpful discussions with Dr. Y. M. Kao, D. H. Lin, J. T. Lin, and M. C. Wu. This work was supported by the National Science Council, Republic of China.
APPENDIX A: CONSERVATION LAWS FOR ELASTIC WAVES

The equation of motion for elastic wave is of the form
\[
\rho \ddot{u}_i = \partial_j T_{ji},
\]  
(A1)

where
\[
T_{ji} = \frac{1}{2} \left( \partial_j u_j + \partial_j u_i \right).
\]
(A3)

Multiplying both side by \( \partial_i \), we find
\[
\partial_i \left( \rho \ddot{u}_i / 2 \right) = \partial_j (T_{ji} \ddot{u}_j) - T_{ji} \partial_i (\partial_j u_i),
\]  
(A2)

Thus the energy flux \( J = \rho v \), and remembering that \( \lambda_c = \rho c^2 \) then one finds
\[
J = \rho v, \quad \mathcal{U} = \frac{\rho}{2} v^2 + \frac{1}{2 \rho c^2} p^2.
\]  
(A8)

Another special case is the elastic SH wave. In this case \( u = u_z \). Remember that \( \partial u / \partial z = 0 \), therefore \( \nabla \cdot u = 0 \) and we have
\[
J = -\mu_c \nabla u, \quad \mathcal{U} = \frac{\rho}{2} v^2 + \frac{\mu_c}{2} (\nabla u)^2.
\]  
(A11)

For the special case that acoustic waves propagating in fluid, we have \( \mu_c = 0 \). Using the relation \( p = -\lambda \nabla \cdot u \), we find
\[
\mathbf{T} = -\rho \mathbf{I},
\]  
(A7)

where \( \mathbf{I} \) is the unit matrix.

Denoting \( \dot{\mathbf{u}} \) as \( \mathbf{v} \) and remembering that \( \lambda_c = \rho c^2 \) then one finds
\[
\mathcal{J} = \rho v, \quad \mathcal{U} = \frac{\rho}{2} v^2 + \frac{1}{2 \rho c^2} \rho^2.
\]  
(A8)

The energy flow and energy density are thus given by
\[
\mathbf{J} = -\rho c \mathbf{v} \mathbf{u}, \quad \mathcal{J} = \frac{\rho}{2} v^2 + \frac{\mu_c}{2} (\nabla u)^2.
\]  
(A11)

Here \( \mathbf{v} = \dot{\mathbf{u}} = \dot{\mathbf{u}} \mathbf{z} \) is the vibration velocity of the media.

\[\begin{align*}
1 & \text{E. Yablonovitch, Phys. Rev. Lett. 58, 2059 (1987); Sci. Am. Vol. 285, N6, 34 (2001).} \\
2 & \text{S. John, Phys. Rev. Lett. 58, 2486 (1987).} \\
3 & \text{K. Sakoda, Optical Properties of Photonic Crystals (Springer-Verlag, 2001).} \\
4 & \text{J. D. Joannopoulos, R. D. Meade, and J. N. Winn, Photonic Crystals—Molding the Flow of Light (Princeton University Press, 1995).} \\
5 & \text{M. M. Sigalas, C. M. Soukoulis, R. Biswas, and K. M. Ho, Phys. Rev. B 56, 959 (1997).} \\
6 & \text{Chul-Sik Kee, Jae-Eun Kim, Hae Yong Park, S. J. Kim, H. C. Song, Y. S. Kwon, N. H. Myung, S. Y. Shin, and H. Lim, Phys. Rev. E 59, 4695 (1999).} \\
7 & \text{Xiangdong Zhang, Zhao-Qing Zhang, Lie-Ming Li, C. Jin, D. Zhang, B. Man, and B. Cheng, Phys. Rev. B 61, 1892 (2000).} \\
8 & \text{M. S. Kushwaha, P. Halevi, L. Dovrzyński, and B. Djafari-Rouhani, Phys. Rev. Lett. 71, 2022 (1993); M. S. Kushwaha and P. Halevi, Appl. Phys. Lett. 69, 31 (1996); M. S. Kushwaha, Appl. Phys. Lett. 70, 3218 (1997); J. O. Vasseur, P. A. Deymier, B. Djafari-Rouhani, L. Dobrzynski, and D. Prevost, Phys. Rev. Lett. 86, 3012 (2001).} \\
9 & \text{J. V. Sánchez-Pérez, D. Caballero, R. Martínez-Sala, C. Rubio, J. Sánchez-Dehesa, F. Meseguer, J. Linares, and F. Galván, Phys. Rev. Lett. 80, 5325 (1998).} \\
10 & \text{M. Torres, F. R. Montero de Espinosa, D. García-Pablos, and N. García, Phys. Rev. Lett. 82, 3054 (1999).} \\
11 & \text{M. Kafesaki, R. S. Penciu, and E. N. Economou, Phys. Rev. Lett. 84, 6050 (2000).} \\
12 & \text{I. E. Psarobas and N. Stefanou, Phys. Rev. B 62, 278 (2000).} \\
13 & \text{Z. Liu, X. Zhang, Y. Mao, Y. Y. Zhu, Z. Yang, C. T. Chan, and P. Sheng, Science 289, 1734 (2000).} \\
14 & \text{Ph. Lambin and A. Khelif, Phys. Rev. B 63, 066605 (2001).} \\
15 & \text{C. Goffaux and J. P. Vigneron, Phys. Rev. B 64, 075118 (2001).} \\
16 & \text{Chul-Sik Kee, Jae-Eun Kim, Hae Yong Park, K. J. Chang, and H. Lim, J. Appl. Phys. 87, 1593 (2000).} \\
17 & \text{Yun Lai, Xiangdong Zhang, and Zhao-Qing Zhang, Appl. Phys. Lett. 79, 3224 (2001).} \\
18 & \text{Yun Lai and Zhao-Qing Zhang, Appl. Phys. Lett. 83,}
\end{align*}\]
[19] C. Kittel, Introduction to Solid State Physics, 7th ed., (John Wiley & Sons, Inc., 1996)
[20] A. Ishimaru, Wave Propagation and Scattering in Random Media (Academic, New York, 1978).
[21] Scattering and Localization of Classical Waves in Random Media, edited by P. Sheng (World Scientific, 1990).
[22] E. Hoskinson and Zhen Ye, Phys. Rev. Lett. 83, 2734 (1999).
[23] Pi-Gang Luan and Zhen Ye, Phys. Rev. E 63, 066611 (2001); 64, 066609 (2001). Zhen Ye and Pi-Gang Luan, J. Appl. Phys. 91, 4761 (2002).
[24] F. A. Pinheiro, A. S. Martinez, and L. C. Sampaio, Phys. Rev. Lett. 85, 5563 (2000).
[25] V.G. Veselago, Sov. Phys. Usp. 10, 509 (1968).
[26] J. B. Pendry, Phys. Rev. Lett. 85, 3966 (2000).
[27] D. R. Smith, W. J. Padilla, D.C. Vier, S. C. Nematic-Nasser, and S. Schultz, Phys. Rev. Lett. 84, 4184 (2000). R. A. Shelby, D. R. Smith, and S. Schultz, Science 292, 77 (2001).
[28] R. W. Ziolkowski and E. Heyman, Phys. Rev. E 64, 056625 (2001).
[29] M. Notomi, Phys. Rev. B 62, 10696 (2000).
[30] G. W. tHooft, Phys. Rev. Lett. 87, 249701 (2001).
[31] J. M. Williams, Phys. Rev. Lett. 87, 249703 (2001).
[32] R.M. Walser, A.P. Valanju, and P.M. Valanju, Phys. Rev. Lett 87, 119701 (2001)
[33] N. Garcia, and M. Nieto-Vesperinas, Phys. Rev. Lett. 88, 207403 (2002).
[34] Z. Ye, Phys. Rev. B 67, 193106 (2003).
[35] D. R. Smith, and D. Schurig, and J. B. Pendry, Appl. Phys. Lett. 81, 2713 (2002).
[36] J. Pacheco, Jr., T. M. Grzegorczyk, B.-I. Wu, Y. Zhang, and J. A. Kong, Phys. Rev. Lett. 89, 257401 (2002).
[37] S. Foteinopoulou, E. N. Economou, and C.M. Soukoulis, Phys. Rev. Lett. 90, 107402 (2003).
[38] S. Foteinopoulou and C. M. Soukoulis, Phys. Rev. B 67, 235107 (2003).
[39] C. Luo, S. G. Johnson, J. D. Joaannopoulos, and J. B. Pendry, Phys. Rev. B 65, 201104 (R) (2002).
[40] Liang-Shan Chen, Chao-Hsien Kuo, and Zhen Ye, Appl. Phys. Lett. 85, 1072 (2004).
[41] L. Brillouin, Wave Propagation in Periodic Structures – Electric Filters and Crystal Lattices, 2nd ed., (Dover Publications, Inc., 1946)