Singular states of relativistic fermions in the field of a circularly polarized electromagnetic wave and constant magnetic field

B. V. Gisin

IPO, Ha-Tannaim St. 9, Tel-Aviv 69209, Israel. E-mail: borisg2011@bezeqint.net

(Dated:)

Dirac’s equation in the field of a circularly polarized electromagnetic wave and constant magnetic field has exact localized non-stationary solutions. The solutions corresponds relativistic fermions only. Among them singular solutions with energy eigenvalues close to each other are found. The solutions are most practicable and can be separated by means of the phase matching between the momentum of the electromagnetic wave and spinor. Characteristic parameters of the singular states are defined.

PACS numbers: 03.65.Ge, 71.70.Di, 13.49.Em;

1. INTRODUCTION

Recently a new class of exact localized non-stationary solutions to Dirac’s equation in the field of a traveling circularly polarized electromagnetic wave and constant magnetic field was presented [1]. These solutions correspond to stationary states (Landau levels) in a rotating and co-moving frame of references. In contrast to classical case, the wave function of such solutions never can be presented as large and small two-component spinor. They are relevant only to relativistic fermions.

In [1], as an example, ”ground state” of these solutions was considered (quotation marks are used here because states are non-stationary). The problem of the magnetic resonance was studied in a general manner in the framework of the classical field theory [2] and an exact expression for oscillations of the magnetic moment found. It was shown that in the condition of the magnetic resonance, i.e., at the amplitude maximum in respect to the constant magnetic field, this amplitude still remains dependent on the momentum, frequency and amplitude of the electromagnetic wave. Some evaluations of parameters for practical realization of such solutions were performed.

In the given paper the consideration is expanded to ”excited states”. Moreover, singular solutions are found. These solutions have some interesting features applicable to practice. In particular, the phase matching between the momentum of the electromagnetic wave and spinor may enhance the precision technic [3] for measurements of the fermion mass and magnetic moment.

For simplicity, the consideration is restricted by ”pure quantum mechanics” without reference to quantum field theory [4], [5].

2. STATES OF DIRAC’S EQUATION IN ROTATING ELECTROMAGNETIC FIELD

We consider Dirac’s equation

\[ i\hbar \frac{\partial}{\partial t} \Psi = c\alpha(p - \frac{e}{c} A)\Psi + \beta mc^2 \Psi = 0 \]  

in the electromagnetic field with potential \( A_x = -\frac{1}{2}Hz_y + \frac{1}{2} H \cos(\Omega t - kz) \), \( A_y = \frac{1}{2}Hz_x + \frac{1}{2} H \sin(\Omega t - kz) \), where \( k = \varepsilon \Omega/c \) is the propagation constant, \( \Omega \) is the frequency, the sign change of \( \Omega \) corresponds to the opposite polarization, values \( \varepsilon = 1 \) and \( \varepsilon = -1 \) are used when the wave propagates along the \( z \)-axis and opposite direction respectively, \( c \) is the speed of light, \( \alpha_k, \beta \) are Dirac’s matrices, \( H \) is the amplitude of this wave. This potential corresponds a plane circularly polarized wave and constant magnetic field.

Stationary solutions of Eq. (1) exist in a rotating and co-moving frame of reference. Coordinates of the frame are

\[
\tilde{x} = r \cos \tilde{\varphi}, \quad \tilde{y} = r \sin \tilde{\varphi}, \\
\tilde{\varphi} = \varphi - \Omega t + k z, \quad \tilde{t} = t, \quad \tilde{z} = z,
\]

where the tilde is used for these coordinates. The transformation \( \tilde{\Psi} = \exp \left[ \frac{i}{2} \alpha_1 \alpha_2 (\Omega t - k z) \right] \Psi \) describes the translation of a spinor into this rotating reference frame. Obviously the operator

\[ \hbar \left( \frac{\partial}{\partial \tilde{t}} + \varepsilon \frac{\partial}{\partial \tilde{z}} \right) \]

arXiv:1203.2600v2 [physics.gen-ph] 24 Mar 2012
is invariant under the transformation (2).

We use constants $E$ and $p$ as "energy" and "momentum along the z-axis" for stationary states in the rotating frame. Once these states are found, the wave function as well as coordinates are translated back into the initial (non-rotating) frame. In this frame the operator (4), in contrast to operators of energy and momentum, commutes with the Hamiltonian of Eq. (1).

The Dirac equation has exact localized non-stationary solutions $\Psi = \exp[-iEt/\hbar + ipz/\hbar - \alpha_1\alpha_2(\Omega t - kz)/2 + D] \psi$, $D = -d(x^2 + y^2)/2 + d_1\tilde{x} + d_2\tilde{y}$, were $\psi$ is a spinor. States may be classified in accordance with the form of this spinor $\psi$. A constant spinor describes "ground state". A spinor polynomial in $\tilde{x}, \tilde{y}$ corresponds to "excited states".

Localized solutions exist if the parameter $d$ is positive and defined by the equality

$$d = \frac{|eH_z|}{2\hbar c}. \tag{5}$$

In accordance with Eq. (5), two types of solutions are possible. We denote them as $\psi_-$ for $eH_z < 0$ and $\psi_+$ for $eH_z > 0$. These normalized spinors of the ground state have the form

$$\psi_0 = N_- \begin{pmatrix} \frac{hE}{\varepsilon} \\ -\varepsilon(\varepsilon + 1)(\varepsilon - \varepsilon_0) \\ -(\varepsilon - 1)(\varepsilon - \varepsilon_0) \end{pmatrix}, \quad \psi_{+0} = N_+ \begin{pmatrix} (\varepsilon + 1)(\varepsilon + \varepsilon_0) \\ \varepsilon\varepsilon h \\ -\varepsilon(\varepsilon - 1)(\varepsilon + \varepsilon_0) \end{pmatrix}, \tag{6}$$

where $N_\mp$ is defined by the normalization condition $\int \Psi^*\Psi dx dy = 1$.

$$N_\mp = \sqrt{d/2\pi}\exp(-d_2^2/2d)/\sqrt{(\varepsilon_0^2 + 1)(\varepsilon_0^2 + 2\hbar^2\varepsilon^2)}, \tag{7}$$

$$d_1 = \mp id_2, \quad d_2 = \frac{\varepsilon_0mc}{2\hbar(\varepsilon_0^2 + \varepsilon^2)}. \tag{8}$$

The upper and lower sign before a parameter corresponds to solutions with negative and positive $eH_z$ respectively.

The normalized eigenvalue of the operator (4) is $E = (E - \varepsilon pc)/mc^2$. Eigenvalues of the "ground state" obey the characteristic equation

$$-\varepsilon(\varepsilon + \Lambda_\mp) + 1 + \frac{\varepsilon}{\varepsilon + \varepsilon_0}\hbar^2 = 0, \tag{9}$$

where

$$\varepsilon_0 = \frac{2\hbar d}{\Omega m}, \quad \Lambda_\mp = \frac{2\varepsilon pc \mp\hbar\Omega}{mc^2}, \quad h = \frac{e}{kmc^2}H. \tag{10}$$

Obviously, wave functions (6) cannot be presented as a small and large two-component spinor. It means that the difference $E^2 - m^2c^2$ cannot be small and these solutions correspond only to the relativistic case.

For the first "excite state" $\psi = \psi_0 + \tilde{x}\psi_x + \tilde{y}\psi_y$, where $\psi_0, \psi_x, \psi_y$ are constant spinors. There exist non-degenerate solutions with $\psi = \psi_{+0}(1 - id\tilde{x}/d_2 \mp id\tilde{y}/d_2)$, where $\psi_{+0}$ is defined by Eq. (6) and the normalization coefficient

$$N_{+1} = \exp\left[-\frac{d_2^2}{2d}\right] \frac{d_2\sqrt{d}}{\sqrt{2\pi(d + d_2^2)}} \frac{1}{\sqrt{\hbar^2\varepsilon^2 + (\varepsilon_0^2 + 1)(\varepsilon_0^2 + 2\hbar^2\varepsilon^2)}} \tag{11}$$

is used instead of $N_\mp$. The characteristic equation coincides with Eq. (9) if

$$\Lambda_\mp = \frac{2\varepsilon pc \mp 3\hbar\Omega}{mc^2}. \tag{12}$$

Two types of degenerate solutions are

$$\psi_{-11} = N_{-11} \begin{pmatrix} \varepsilon(\varepsilon + 1) \\ h - i\frac{2\hbar d}{mc}\tilde{x} + \frac{2\hbar d}{mc}\tilde{y} \\ -\varepsilon(\varepsilon - 1) \\ -\varepsilon[h - i\frac{2\hbar d}{mc}\tilde{x} + \frac{2\hbar d}{mc}\tilde{y}] \end{pmatrix}, \tag{13}$$

$$N_{-11} = \exp\left[-\frac{d_2^2}{2d}\right] \frac{\sqrt{d}(\varepsilon - \varepsilon_0)/\sqrt{2\pi}}{\sqrt{\hbar^2\varepsilon^2 + (\varepsilon - \varepsilon_0)^2(\varepsilon_0^2 + 1 + 4\hbar^2d/m^2c^2)}}. \tag{14}$$
Hamiltonian, therefore, average values of these operators have to be used. The average value of any operator functions. Except for the operator (4), operators of energy, angular momentum, spin, etcetera don’t commute with $E$

For simplicity we consider below "minus solutions" for $eH$

For the non-degenerate solutions of the first excited state the form of roots coincides with (20), but in the condition $H/H_z \ll 1$. Therefore, expansions of roots in terms of this parameter is more convenient. The characteristic equations of the singular states have a pair of roots in the form of the expansion in a vicinity of $±E_0$. Every such a root is expanded in power series in $h$

The necessary condition for existence of singular states is

$$\Lambda_± = ± 1/\varepsilon_0 + \varepsilon_0.$$  

For simplicity we consider below "minus solutions" for $eH_z < 0$. For "plus solutions" $eH_z > 0$ the parameter $\varepsilon_0$ changes the sign.

Three roots of the ground state are

$$\varepsilon'_{1,2} = \varepsilon_0 ± \frac{\varepsilon_0 h}{\sqrt{\varepsilon_0^2 + 1}} + \frac{\varepsilon_0 h^2}{2(\varepsilon_0^2 + 1)^2} + \ldots, \quad \varepsilon_3 = \frac{1}{\varepsilon_0} - \frac{\varepsilon_0 h^2}{(\varepsilon_0^2 + 1)^2} + \ldots.$$  

For the non-degenerate solutions of the first excited state the form of roots coincides with (20), but in the condition the term $\Lambda_±$ must be replaced by the expression (12).

For degenerate solutions of the first "excited state" roots are

$$\varepsilon''_{1,2} = \varepsilon_0 ± \frac{\varepsilon_0 h}{\sqrt{\varepsilon_0^2 + 1 + \varepsilon}} + \frac{\varepsilon_0(1 + \varepsilon) h^2}{2(\varepsilon_0^2 + 1 + \varepsilon)^2} + \ldots, \quad \varepsilon_3 = \frac{1 + \varepsilon}{\varepsilon_0} - \frac{\varepsilon_0(1 + \varepsilon) h^2}{(\varepsilon_0^2 + 1 + \varepsilon)^2} + \ldots,$$

and the same condition (19) is used.

### 4. PHASE MATCHING CONDITION

The condition $\Lambda_± = 0$ for singular states is the phase matching between momentum of the wave and spinor

$$\hbar \Omega = 2\varepsilon pc.$$  

In accordance with Eq. (19), this condition corresponds to equality $\varepsilon_0 = \pm 1$. Using expressions for $d, \varepsilon_0$ in non-normalized parameters (10), (5), it is easily shown that the equality is none other than the classical condition of the magnetic resonance

$$h\Omega = \pm \frac{eH}{2mc}$$

(23)

for the “normal magnetic moment” of electron.

In general case, the normalized parameter $\varepsilon_0$ defines the $g$-factor $\varepsilon_0 = 2/g$. Quantum field theory produces a coefficient at the first term in the right part of Eq. (19). As a result, $\varepsilon_0$ is changed and the magnetic moment becomes anomalous one. But for simplicity, as it is noticed in Introduction, the consideration here is restricted by “pure quantum mechanics”.

Every term in (18) has multiplier $\exp\left[-d^2/(2d^2)\right]$, where $d^2, d_2^0$ correspond to two eigenvalues of $E$. In particular, for singular pair of the ground state this factor in the first approximation equals $\exp\left[-(d^2_0 + 1)/2\lambda^2d\right]$, where $\lambda$ is the Compton wavelength. For electron $\lambda^2d = 1.13 \cdot 10^{-14} HZ/G$, therefore, this factor equals zero with huge accuracy for all magnetic fields attained in laboratory conditions. The same is valid for singular pair of excited states. In contrast to that, for a sum of the ground and first excited degenerative state this factor is $\exp[-2d\lambda^2]$. It equals one with huge accuracy. Therefore, for an illustration of the temporal behavior of the average value, it is suffice to consider this combination of wave functions.

Consider the average value of spin $\mathbf{s} = -i\frac{1}{2} \int \Psi^* \alpha_1 \alpha_2 \Psi dx dy$ for such a mixed state with the wave function $\Psi = (C_0 \Psi_{-0} + C_1 \Psi_{-1})$. Without loss generality, constants $C_0, C_1$ are assumed to be real: $C_0 = \sin \tau, C_1 = \cos \tau$. It may be straightforwardly shown that the constant part of $\mathbf{s}$ is negligible, because it is a sum of two small terms of the order of $\hbar$ or $c$. The oscillation term at $\varepsilon^2_0 = 1$ is

$$\mathbf{s}(t) = \frac{1}{\sqrt{8}} \sin 2\tau \cos F t, \quad F = (E' - E'')mc^2/h \approx \frac{eH}{mc\sqrt{8}}$$

(24)

where $F$ is the oscillation frequency.

Basic parameters applicable to measurements are determined as follows. The allowable value of $H_z$ is defined from Eq. (5) by the appropriate transverse localization length $\sim 1/\sqrt{2d}$; some examples are presented in [1]. $\Omega$ is defined by the condition (23), in particular, it may be a optical frequency [1]. The value of the "momentum in the rotating and co-moving frame of references" $|\mathbf{p}| = |h\Omega/2c|$ is small in contrast to the average momentum $\bar{p} \equiv \int \Psi^* (-i\hbar \partial / \partial z) \Psi dx dy$. This momentum with the accuracy of the order of $\hbar$ equals $\varepsilon mc^2$ in this expression $\mathbf{p}$ is neglected. An average velocity $\bar{v}$ associated with the average momentum $\bar{p}$ is $\bar{v} = c/\sqrt{2}$. This value falls in the diapason of relativistic velocities.

5. CONCLUSION

Singular states pertaining to the new class of exact solutions of Dirac’s equation in the rotating electromagnetic field are most practicable. They may be separated from totality of states with help of the phase matching condition. The use both the magnetic resonance and the resonance velocity gives a chance for an improvement of the precision measurements of the fermion mass and magnetic moment in the relativistic range.

[1] B. V. Gisin, arXiv: 2011.3832v5 [math-ph] 29 Sep 2011. In Eq. (28) must be $A_+ = A_- (-\varepsilon_0)$.
[2] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Pergamon Press, 1962).
[3] B. Odom, D. Hanneke, B. D’Urso, and G. Gabrielse, Phys. Rev. Lett. 97, 030801 (2006).
[4] S. S. Schweber, *QED and Men Who Made It: Dyson, Feyman Schwinger, and Tomonaga*, (Princeton University Press, Princeton, New Jersey, 1994).
[5] N. N. Bogoliubov, D. V. Shirkov, *The Theory of Quantized Fields*, (New York, Interscience, 1959).