THE SPHERICAL METRIC AND UNIVALENT HARMONIC MAPPINGS

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Abstract. Let \( f = h + g \) be a harmonic univalent map in the unit disk \( D \), where \( h \) and \( g \) are analytic. We obtain an improved estimate for the second coefficient of \( h \). This indeed is the first qualitative improvement after the appearance of the papers by Clunie and Sheil-Small in 1984, and by Sheil-Small in 1990. Also, when the sup-norm of the dilatation is less than 1, it is shown that the spherical area of the covering surface of \( h \) is dominated by the spherical area of the covering surface of \( f \).

1. Introduction and Preliminaries

The famous Bieberbach conjecture of 1916 relates to the class \( S \) of normalized univalent analytic functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) defined on the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). The conjecture asserts that \( |a_n| \leq n \) for every \( f \in S \) and every \( n \geq 2 \). In 1984, de Branges proved this conjecture as well as some other stronger conjectures. Bieberbach’s coefficient conjecture was instrumental in the development of the theory of univalent functions. Numerous methods evolved and applied to investigate a number of extremal problems in geometric function theory. Yet there still exist many open problems and conjectures involving both univalent and non-univalent mappings. The Keobe function \( k(z) = 1/(1 - z)^2 \) and its rotations \( e^{-i\theta} k(e^{-i\theta} z) \) provide solutions to many extremal problems in the class \( S \) and related geometric subclasses. These include the class of functions that are close-to-convex, starlike, or convex in some direction (see [11, 16, 17]).

Another active topic studied in recent years is on planar harmonic mappings (see for instance [9, 10, 12] and the mini survey [18]). The present paper investigates such mappings. Specifically, we treat the family \( S_H \) of normalized univalent, sense-preserving harmonic mappings \( f = h + g \) in \( \mathbb{D} \), where

\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n.
\]

Here a mapping \( f = h + g \) is sense-preserving if the Jacobian \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \) of \( f \) is positive in \( \mathbb{D} \). Set

\[
S_H^0 = \{ f = h + g \in S_H : b_1 = g'(0) = 0 \}
\]
so that each $f \in S_H^0$ has the form (1.1) with $b_1 = g'(0) = 0$. An important member of this family is the so-called harmonic Koebe mapping $K$ given by

$$K(z) = H(z) + \frac{\alpha}{2} z^3 + i\alpha Im\left(\frac{z}{(1-z)^2}\right).$$

The classes $S_H$ and $S_H^0$ are known to be normal [12] with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. However only $S_H^0$ is compact. In 1984, as a generalization of Bieberbach conjecture, Clunie and Sheil-Small [10] investigated the class $S_H^0$ and conjectured that if $f = h + g \in S_H^0$ is given by (1.1), then for all $n \geq 2$,

$$|a_n| \leq \frac{(n+1)(2n+1)}{6}, \quad |b_n| \leq \frac{(n-1)(2n-1)}{6} \quad \text{and} \quad ||a_n| - |b_n|| \leq n$$

with equality occurring for $f(z) = K(z)$ given by (1.2). This conjecture has been verified for a few subclasses of $S_H^0$, namely the class of all functions starlike, close-to-convex, typically real and convex in one direction, in which $K$ plays the role of extremal function in these subfamilies. It is surprising that the sharp bound for $|a_2|$, $f = h + g \in S_H^0$, remains unsolved.

In [10], it was shown that $|a_2| < 12172$, which later in [22] was improved to $|a_2| < 57$. In [12], the estimate $|a_2| < 49$ was established, which is far from the conjectured bound $|a_2| \leq 5/2$. The field has not seen any further improvements on this problem. The right tools have not been found to deal with this problem and hence, the above coefficient conjecture in the case of harmonic mappings remains elusive, even in the case of $|a_2|$. One of our aims is to consider this problem and prove the following result as a consequence of our new approach.

**Theorem 1.** If $f = h + g \in S_H^0$, then $|a_2| \leq 20.9197$.

The proof of Theorem 1 is presented in Section 3. It requires several other basic results which will be discussed in Section 2.

For $f = h + g \in S_H^0$ given by (1.1), it was shown in [3] that the analytic part $h$ of $f$ lies in Hardy spaces $H^p$ for some small $p > 0$, namely, $0 < p < (2\alpha_0 + 2)^{-2}$ with $\alpha_0 = sup_{S_H^0} |a_2|$. Thus, determining sharp estimate for $|a_2|$ is an important problem. On the other hand, while the bound in Theorem 1 may not be sharp, it is indeed a better estimate than the known upper bound of 48.4 (see [12, p. 95–97]). We also note that as an attempt to solve the above conjecture, the following new conjecture was proposed in [19].

**Conjecture 1.** If $S_H^0(S) = \{h + g \in S_H^0 : \Phi_2 = h + e^{i\theta}g \in S \text{ for some } \theta \in \mathbb{R}\}$, then $S_H^0 = S_H^0(S)$. That is, for every function $f = h + g \in S_H^0$, there exists a $\theta \in \mathbb{R}$ such that $\Phi_2 = h + e^{i\theta}g \in S$.

The present article is organized as follows. Section 2 is devoted to establishing key ideas which lead to the proof of Theorem 1. In Section 3, normalized conformal maps are studied in relation to the elliptic modular function $Q$ on $\mathbb{D}$. We show in Theorem 2 and Corollary 2 a new estimate on the second coefficient is obtained for functions belonging to a certain class of conformal mappings. In Section 4, specifically in Theorem 4, we show that for a $K$–quasiconformal univalent harmonic map $f = h + g$ (that is, $|f_z| \leq |h_z|$ a.e. on $\mathbb{D}$, where $\alpha = (K - 1)/(K + 1)$ with $K \geq 1$), the spherical area $A_\alpha(h)$ is dominated by
the spherical area $A_s(f)$, and that it is finite. Finally, distortion estimates are obtained in Section 5 for univalent harmonic mappings.

2. Background for the proof of Theorem 1

A domain in the complex plane $\mathbb{C}$ is said to be hyperbolic if its complement contains at least two points. Let $\Omega$ be a hyperbolic domain in $\mathbb{C}$. Then the planar uniformization theorem assures the existence of a unique conformal universal covering $f : \mathbb{D} \to \Omega$ with a prescribed value $f(0) \in \Omega$, and $f'(0) > 0$. In the sequel, $f$ is conformal provided $f'(z) \neq 0$ in $\mathbb{D}$. When $\Omega$ is not a simply connected hyperbolic domain, then the universal covering $f$ cannot be univalent (see, for example, [5, p. 41]). For a hyperbolic domain $\Omega$ in $\mathbb{C}$, let $d(w, \partial \Omega)$ denote the Euclidean distance between $w \in \Omega$ and the boundary $\partial \Omega$ of $\Omega$.

The well-known principle of subordination defined by Littlewood in [14] will be referred to in this article. For two analytic functions $f$ and $g$ in the unit disk $\mathbb{D}$, the function $f$ is subordinate to $g$, written as $f(z) \prec g(z)$ or $f \prec g$, if there exists an analytic self-map $\varphi$ of $\mathbb{D}$ with $\varphi(0) = 0$ satisfying $f = g \circ \varphi$ (see also [11, 16]). When $g$ is univalent in $\mathbb{D}$, $f \prec g$ if and only if $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$.

When $F : D \to \Omega$ is a universal covering satisfying $f(0) = F(0)$, then $\varphi(z) = F^{-1}(f(z))$ has a branch at 0, which by the Monodromy theorem can be continued to all of $\mathbb{D}$. Hence $f$ is subordinate to $F$.

For our purpose, we shall consider the modular function $Q$

\begin{equation}
Q(z) = 16z \prod_{n=1}^{\infty} \left( \frac{1 + z^{2n}}{1 - z^{2n}} \right)^8 = \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D}.
\end{equation}

From the product expansion of $Q$, we see that the Taylor coefficients $A_n$ of $Q$ are all non-negative for $n \geq 1$ with $A_1 = 16$.

Properties of $Q$ have been comprehensively studied by Nehari in [15] (see also [16]) in which the author used the notation $J(z) : = -Q(-z)$. Moreover, the fact that the coefficients $\{A_n\}$ of $Q$ are nonnegative and form a non-decreasing convex sequence leads to the following useful result.

**Lemma 1.** Suppose that $Q(z) = \sum_{n=1}^{\infty} A_n z^n$ is given by (2.1) and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ analytic in $\mathbb{D}$ satisfy $f(z) \prec Q(z)$ for $z \in \mathbb{D}$. Then $|a_n| \leq A_n$ for $n \geq 1$.

**Proof.** The assertion follows from [14, 15, 21]. Indeed, it is known from [15] p. 82 that $\{A_n\}$ is a convex non-decreasing sequence, that is, $B_n = A_n - A_{n-1}$ and $C_n = B_n - B_{n-1}$ are non-negative, where $A_0 = 0 = A_1$. Since $f \prec Q$, it readily follows from a theorem of Rogosinski [21] that $|a_n| \leq A_n$ for $n \geq 1$ (see also Littlewood [14] p. 169). □

Let $\mathcal{F}$ denote the class of all analytic functions in $\mathbb{D}$ of the form $f(z) = \sum_{n=1}^{\infty} a_n z^n$ that assumed the value 0 only at 0. Clearly the elliptic modular function $J$ and $Q(z) : = -J(-z)$ belong to $\mathcal{F}$.

**Lemma 2.** [16] If $f \in \mathcal{F}$, $D = f(\mathbb{D})$ and $a = d(0, \partial D)$, then $f(z) \prec aQ(z)$.

If $D$ is a region in the complex plane, denote by $D^c$ its complement $\overline{\mathbb{C}} \setminus D$. As an immediate consequence of Lemma 2, here is a result which reveals an important geometric fact.
Corollary 1. (Compare with [13, Theorem II]) Suppose that \( h \in \mathcal{F} \), \( h(z) = \sum_{n=1}^{\infty} a_n z^n \), \( h(\mathbb{D}) = D \), \( d(0, \partial D) = |a| \), \( a \in \partial D \), and \( Q(z) = \sum_{n=1}^{\infty} A_n z^n \) is given by (2.1). Then \(|a_n| \leq |a|A_n\) for \( n \geq 1 \).

An important question to ask is whether the coefficient estimate is sharp. At least in Theorem 2 in the next section, we present better estimates for \( a_2 \) and \( a_3 \).

3. Coefficient estimates for hyperbolic conformal maps

Here is our first basic result which gives better estimates for \( a_2 \) and \( a_3 \) than the estimates given by Corollary 1.

Theorem 2. Suppose that \( h \in \mathcal{F} \), \( h(z) = z + \sum_{n=2}^{\infty} a_n z^n \), \( h(\mathbb{D}) = D \), and \( d(0, \partial D) = |a| \) for some \( a \in \partial D \). Then \( a_2 \) and \( a_3 \) satisfy the following inequalities:

\[
\begin{align*}
16 & \leq |a|, \\
|a_2| & \leq 16|a| + \frac{1}{2|a|}, \\
|a_3| & \leq 704|a|.
\end{align*}
\]

If in addition \( D \) is hyperbolic, then \( |a| < 1 \).

Proof. For \( \rho \in (0, 1) \), let

\[
h_\rho(z) = (1/\rho)h(\rho z) = z + \sum_{n=2}^{\infty} a_n \rho^{n-1} z^n.
\]

We now apply Corollary 1 for \( h_\rho \) with \( a_\rho \in \partial D_\rho \) as the nearest point to the origin, where \( D_\rho = h_\rho(\mathbb{D}) \). Then it follows from (2.1), Corollary 1 and [16, p. 327] (see also Lemma 2) that \( h(z) \preceq aQ(z) \) and therefore,

\[
h_\rho(z) = -\frac{a_\rho}{\rho}Q(\varphi(\rho z)) = -\frac{16a_\rho}{\rho} [\varphi(\rho z) + 8\varphi^2(\rho z) + 44\varphi^3(\rho z) + \cdots],
\]

where \( \varphi \) is analytic in \( \mathbb{D} \) with \( \varphi(0) = 0, |\varphi(z)| < 1 \) in \( \mathbb{D} \) and \( a_\rho \to a \) as \( \rho \to 1^- \). With \( \varphi(z) = \beta_1 z + \beta_2 z^2 + \cdots \), and comparing the coefficients of \( z^n \) for \( n = 1, 2, 3 \) in the last expression of \( h_\rho \), gives the following three relations:

\[
\begin{align*}
a_1 &= -16a_\rho \beta_1 = 1, \\
a_2 &= -16a_\rho (\beta_2 + 8\beta_1^2), \quad \text{and} \\
a_3 &= -16a_\rho (\beta_3 + 16\beta_1\beta_2 + 44\beta_1^3).
\end{align*}
\]

Thus, the known estimates \(|\beta_n| \leq 1\) for \( n \geq 1 \) readily establish

\[
|a_\rho| \geq \frac{1}{16|\beta_1|} \geq \frac{1}{16},
\]

\[
|\beta_1| = 1/(16|a_\rho|) \quad \text{and} \quad |a_2| \leq 16|a_\rho|(1 + 8|\beta_1|^2) = 16|a_\rho| + \frac{1}{2|a_\rho|}.
\]
Note that $|a_\rho| \to |a|$ as $\rho \to 1^-$. Clearly, the right side of (3.1) follows from the analog of Koebe one-quarter theorem for hyperbolic domains ([6] and [23, p. 894]). See also the inequality recalled in (4.2).

Finally, we present a proof of (3.2). To do this, we recall the sharp upper bounds for the functionals $|\beta_3 + \mu \beta_1 \beta_2 + \nu \beta_3^3|$ when $\mu$ and $\nu$ are real. In [20], Prokhorov and Szynal proved among other results that

$$ |\beta_3 + \mu \beta_1 \beta_2 + \nu \beta_3^3| \leq |\nu| $$

if $|\mu| \geq 4$ and $\nu \geq (2/3)(|\mu| - 1)$. From the third relation above for $a_3$, this condition is fulfilled (since $\mu = 16$ and $\nu = 44$) and thus,

$$ |a_3| = 16 |a_\rho| |\beta_3 + 16 \beta_1 \beta_2 + 44 \beta_3^3| \leq 16 \times 44 |a_\rho| = 704 |a_\rho| $$

which proves the desired inequality (3.2).

As $A(x) = 16x + 1/(2x)$ is increasing on $[1/(4\sqrt{2}), 1]$, it follows that $A(x) \leq A(1) = 16.5$. This observation leads to

**Corollary 2.** If $h \in \mathcal{F}$ and $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_2| \leq 16.5$ and $|a_3| \leq 704$.

Now, we are in a position to formulate an important general result. First note that if $h(z) = \sum_{n=1}^{\infty} a_n z^n$ is analytic on $\overline{D}$, $h(\overline{D}) = D$ and $d(0, \partial D) = |a|$ for some $a \in \partial D$, then the function $z(h(z) - a)/(a)$ belongs to $\mathcal{F}$. Furthermore, we remark that $z(h(z) - a)$ is zero only at $0$.

**Theorem 3.** Suppose that $h$ is conformal in $D$, $h(\overline{D}) = D$ is hyperbolic and $d(0, \partial D) = |a|$, where $a \in \partial D$ and $h(0) = h'(0) - 1 = 0$. Then

$$ \frac{1}{16.5} \leq |a| < 1.$$

**Proof.** Let

$$ g(z) = z \frac{a - h(z)}{a} = z - \frac{1}{a} z^2 + \cdots, $$

so that $g$ belongs to $\mathcal{F}$. Consequently, by Theorem [2] it follows that $|1/a| \leq 16.5$. Since $D$ has a hyperbolic metric $\lambda(z)$ and that

$$ \lambda(0)d(0, \partial D) = \frac{1}{h'(0)} d(0, \partial D) = |a| < 1, $$

the result follows. At this place it is worth recalling that $\lambda(z)d(z, \partial D) \leq 1$ holds always. \(\Box\)

For the proof of our next lemma, we need to establish some preliminaries. It is known from the work of Abu Muhanna and Hallenbeck [2] that if $\alpha > 0$, and

$$ E_{\alpha} = \{ f \in \mathcal{A} : f(z) < \exp(\alpha z/(1 - z)), f(0) = 1 \}, $$

then

$$ E_{\alpha} = \left\{ \int_{\partial D} \exp \left( \alpha \frac{xz}{1 - xz} \right) d\mu(x) : \mu \text{ is a probability measure on } \partial D \right\}. $$
Each function $f \in E_\alpha$ maps $\mathbb{D}$ into $|w| > r = \exp(-\alpha/2)$ and $f(0) = 1$. Clearly, the inclusion $E_\alpha \subset E_\beta$ holds for $\alpha > \beta$ and thus, the coefficients of $f \in E_\alpha$ are dominated by the corresponding coefficients of $F(z) = \exp(\alpha z/(1 - z))$, where

$$F(z) = \exp(\alpha z/(1 - z)) = 1 + \alpha z + \frac{\alpha(\alpha + 2)}{2} z^2 + \cdots = 1 + A_1 z + A_2 z^2 + \cdots.$$ 

In particular,

$$|f''(0)| \leq A_2 = \frac{\alpha(\alpha + 2)}{2}.$$ 

Clearly, as $\alpha \to \infty$ (circle $|w| = \rho = r = \exp(-\alpha/2)$ shrinks) $r \to 0$ and $A_1, A_2 \to \infty$.

**Lemma 3.** Suppose that $h(z) = z + a_2 z^2 + \cdots$ is analytic in $\overline{\mathbb{D}}$ and misses the disk $D(c, r) := \{z : |z-c| < r\}$ which touches the boundary $\partial h(\mathbb{D})$. Then the function $\Psi$ defined by $\Psi(z) = \frac{c-h(z)}{c}$ misses the disk $D(0, \rho)$, where $\rho = r/|c| > 1/16$, and $|a_2| < 20.9197|c|$.

**Proof.** By assumption, the function

$$\Psi(z) = \frac{c-h(z)}{c} = 1 - \frac{1}{c} z - \frac{a_2}{c} z^2 + \cdots$$

misses the disk $D(0, r/|c|)$ and its boundary touches the circle $|w| = r/|c|$. Then $g$ defined by $g(z) = z \Psi(z)$ belongs to the family $\mathcal{F}$ and thus, if the nearest point to 0 is $a = g(e^{\theta})$, then $|a| = r/|c| \geq 1/16$. Note that $z(c - h(z))$ is zero only at 0 and thus, $\rho > 1/16$ is indeed a consequence of Nehari’s result.

Consequently,

$$|a_2| \leq \frac{\alpha(\alpha + 2)}{2},$$

where

$$\frac{r}{|c|} = \exp(-\alpha/2).$$

Thus

$$\alpha = \log (|c|/r)^2 < 2 \log (16) = 8 \log 2 \approx 5.54518.$$ 

This gives the estimate

$$|a_2| \leq |c| \frac{\alpha(\alpha + 2)}{2} < |c| [8 \log(2)(4 \log 2 + 1)] \approx 20.9197|c|$$

and completes the proof. \qed

The proof of Theorem 1 below depends on the following remark.

**Remark 1.** Suppose that $h(z) = z + a_2 z^2 + \cdots$ is conformal on $\overline{\mathbb{D}}$ and $d(0, \partial h(\mathbb{D})) = |a| < 1$, where $a \in \partial h(\mathbb{D})$. It is worth pointing out that $D(a, 1 - |a|) \cap (h(\mathbb{D}))^c$ is non-empty and open, where $D(a, r) := \{z : |z - a| < r\}$. Thus, there is a complex number $c$ in the complement $(h(\mathbb{D}))^c$ with $|c| < 1 - |a|$ and a positive number $r$ so that

$$D(c, r) \subset (h(\mathbb{D}))^c$$

and touches the boundary $\partial h(\mathbb{D})$. With this $c$, $|a_2| < 20.9197$. 

In this remark, it suffices to assume that \( h(z) = z + a_2 z^2 + \cdots \) is conformal on \( \mathbb{D} \); otherwise, consider \( h_\rho(z) = (1/\rho)h(\rho z), \rho \in (0,1) \) in the proof, and then let \( \rho \to 1^- \).

Although \( h'(z) \neq 0 \) for \( f = h + \mathcal{H} \in \mathcal{S}_0^H \), the function \( h \) can however vanish several times. This fact is illustrated by the first author in [1] by the function
\[
h(z) = -\coth z + z + (\log \sinh z - \log \sinh 1) + \coth 1 - 1/2, \quad z \in \mathbb{R}.
\]

Here \( \mathbb{R} \) is the open right half-plane and the dilatation \( \varphi(z) = (\coth z - 1)/(\coth z + 1) \) maps \( \mathbb{R} \) onto the punctured disk \( \mathbb{D} \setminus \{0\} \). In [1], the function \( h \) was shown to have infinite valence and finitely many zeros.

Finally, we conclude the section with the proof of Theorem 1.

**Proof of Theorem 1.** Let \( \rho_n \) be a sequence of radii increasing to 1. Now, consider the analytic part of \( f_{\rho_n}(z) = (1/\rho_n)f(\rho_n z) \), namely, the function
\[
h_{\rho_n}(z) = (1/\rho_n)h(\rho_n z) = z + \sum_{k=2}^\infty a_k \rho_n^{k-1} z^k.
\]

Since \( D_{\rho_n} = h_{\rho_n}(\mathbb{D}) \) is hyperbolic, by Lemma 3, the second coefficient of \( h_{\rho_n} \) gives the estimate \( |a_2 \rho_n| \leq 20.9197 \) for each \( n \). The desired conclusion follows when \( n \to \infty \).

\[\square\]

4. **Spherical area of the covering surface of \( h \) over \( D \).**

The spherical metric on \( \overline{\mathbb{C}} \) (the Riemann sphere) is defined by
\[
\sigma(z)|dz| = \frac{|dz|}{1 + |z|^2},
\]
and the spherical area of (a surface of \( f \) above) \( \mathbb{D} \) given by a harmonic map \( f \) is
\[
A_s(f) = \int_\mathbb{D} \frac{J_f(z) \, dA}{(1 + |f(z)|^2)^2},
\]
where \( J_f \) denotes the Jacobian of \( f \). When \( f \) is analytic, then the spherical area becomes
\[
A_s(f) = \int_\mathbb{D} \frac{|f'(z)|^2 \, dA}{(1 + |f(z)|^2)^2}.
\]

Clearly, if the surface covers the plane exactly once then \( A_s(f) = 4\pi \).

The hyperbolic (or Poincaré) metric in \( \mathbb{D} \) [5, 4, 13, 23] is the Riemannian metric defined by \( \lambda_\mathbb{D}(z)|dz| \), where
\[
\lambda_\mathbb{D}(z) = \frac{1}{1 - |z|^2}.
\]

Note that metrics \( \sigma \) and \( \lambda_\mathbb{D} \) have constant curvatures 4 and \( -4 \), respectively. Using analytic maps, hyperbolic metrics can be transferred from one domain to another. Indeed, for a given hyperbolic domain \( \Omega \), and a (conformal) universal covering map \( f : \mathbb{D} \to \Omega \), the hyperbolic metric of \( \Omega \) is given by
\[
\lambda_\Omega(f(z))|f'(z)||dz| = \lambda_\mathbb{D}(z)|dz|.
\]
In particular,
\[ \lambda_{\Omega}(f(0)) = \frac{1}{|f'(0)|}, \]
a fact which is already used in the proof of Theorem 3. It is well-known that the metric \( \lambda_{\Omega} \) is independent of the choice of the conformal map \( f \) used.

When \( \Omega \) is simply connected, the Koebe one-quarter theorem [17, p. 22] gives the sharp estimates
\[ \frac{1}{4} \leq d(w, \partial \Omega) \lambda_{\Omega}(w) \leq 1. \]

When \( \Omega \) is a hyperbolic domain, the estimates established in [6] and [23, p. 894] are
\[ \frac{1}{2(\beta_{\Omega}(w) + C_0)} \leq d(w, \partial \Omega) \lambda_{\Omega}(w) \leq \min \left\{ 1, \frac{2C_0 + \pi/2}{2(\beta_{\Omega}(w) + C_0)} \right\}, \]
where
\[ \beta_{\Omega}(w) = \inf_{b \in \partial \Omega} \left| \frac{w - a}{b - a} \right|, \]
with \( a \in \partial \Omega, |w - a| = d(w, \partial \Omega) \), and \( C_0 \approx 4.37688 \). The lower bound in (4.2) is known to be sharp but not the upper bound. Indeed it is difficult to estimate \( \beta_{\Omega}(w) \), which is the modulus of the largest annulus inside \( \Omega \) separating the boundary \( \partial \Omega \). However if \( \partial \Omega \) is a connected set, then \( \beta_{\Omega}(w) = 0 \) and a lower bound is 1/8, which is the right estimate that one could get from (4.2).

In the following result, \( A_s(D) := A_s(D, h) \) denotes spherical area of the covering surface of \( h \) over \( D \). Moreover, if \( f = h + \overline{g} \) is a sense-preserving harmonic mapping in \( \mathbb{D} \), then \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \) in \( \mathbb{D} \) and thus, the analytic dilatation \( \varphi(z) := \varphi_f(z) = g'(z)/h'(z) \) of \( f \) satisfies \( |\varphi(z)| < 1 \) for all \( z \in \mathbb{D} \). Define \( ||\varphi||_\infty := \sup_{z \in \mathbb{D}} |\varphi(z)| \).

**Theorem 4.** Let \( f = h + \overline{g} \) be a sense-preserving univalent harmonic mapping in \( \mathbb{D} \) given by (1.1) with the dilatation \( \varphi \). Suppose that \( \alpha = ||\varphi||_\infty < 1 \) and \( D = h(\mathbb{D}) \) is hyperbolic. Then the spherical area of the covering surface of \( D \) satisfies
\[ \frac{1 - \alpha^2}{4} A_s(D) \leq A_s(\Omega) := A_s(\Omega, f) \leq 4\pi, \]
where \( f(\mathbb{D}) = \Omega \) and
\[ A_s(\Omega) = \iint_{\mathbb{D}} \frac{|h'(z)|^2 \, dA}{(1 + |h(z)|^2)^2}. \]

**Proof.** Let \( f = h + \overline{g} \) and \( N(a, r) \) be a disk in \( D \) so that the components of \( h^{-1}(N(a, r)) \), namely, \( \{U_n^a\} \), are disjoint, open, connected and \( h \) maps each component univalently onto the disk \( N(a, r) \). In other words, \( \{U_n^a\} \) is the covering above \( N(a, r) \). Let \( U_n^a \) be an arbitrary component with \( h(b) = a \) and \( b \in U_n^a \). We now introduce
\[ F_n(w) = f(h^{-1}(w)) \quad \text{for} \quad w \in N(a, r). \]
Then \( F_n(w) = w + g(h^{-1}(w)) \) and therefore, with \( D = h(\mathbb{D}) \) and \( z = h^{-1}(w) \), it follows easily that
\[ (F_n)_w(w) = 1 \quad \text{and} \quad (F_n)^w(w) = \frac{g'(z)}{h'(z)} = \varphi(z) = \varphi(h^{-1}(w)). \]
Since \( \{U^n_a\} \) are disjoint and \( f \) is univalent, \( F_n(N(a,r)) = f(U^n_a) \) are also disjoint. Consequently, the spherical area of the image of \( F_n \) above \( U^n_a \) is given by

\[
A_s(F_n) = \int\int_{N(a,r)} \frac{J_{F_n}(z) \, dA_w}{(1 + |F_n(w)|^2)^2} = \int\int_{N(a,r)} \frac{(1 - |\varphi(h^{-1}(w))|^2) \, dA_w}{(1 + |F_n(w)|^2)^2}.
\]

Note that

\[
|F_n(w) - w| \leq |\varphi(h^{-1}(w))| \leq \int_a^w |\varphi(h^{-1}(w))| \, |dw| \leq |w - a|
\]

and therefore, \( |F_n(w)| \leq |w| + |w - a| \). When \( a = 0 \), we get the estimate \( |F_n(w)| \leq 2|w| \). On the other hand, when \( a \neq 0 \) we may add the condition \( r < |a|/2 \) so that \( |w - a| \leq |a|/2 \leq |w| \), which again yields \( |F_n(w)| \leq 2|w| \). In both cases, (4.3) gives

\[
A_s(F_n) \geq \frac{1 - \alpha^2}{4} \int\int_{N(a,r)} \frac{4 \, dA_w}{(1 + 4|w|^2)^2} \geq \frac{1 - \alpha^2}{4} A_s(2N(a,r)),
\]

where \( A_s(2N(a,r)) \) is the spherical area of \( 2N(a,r) \). Thus, the spherical area of the part of the covering surface of \( D \) above \( N(a,r) \) is dominated by the spherical area of the union of \( \{f(U^n_a)\} \). In other words,

\[
A_s(N(a,r)) \leq \sum_n A_s(f(U^n_a)) \leq \frac{4}{1 - \alpha^2} A_s(\Omega).
\]

Let \( D_1 = \bigcup_j N(a_j, r_j) \) be disjoint union of disks in \( D \), and \( U_1 = \bigcup_j \bigcup_n U^n_{a_j} \) the corresponding disjoint covers. Note that \( \{U^n_{a_j}\} \) are disjoint, for all \( a_j \) and \( n \). Then the spherical area of the covering surface of \( D \) above \( D_1 \) is

\[
A_s(h(U_1)) = \sum_{j=1}^\infty \sum_n A_s(h(U^n_{a_j})) \leq \frac{4}{1 - \alpha^2} \sum_{j=1}^\infty \sum_n A_s(f(U^n_{a_j})) \leq \frac{4}{1 - \alpha^2} A_s(\Omega) \leq \frac{16\pi}{1 - \alpha^2}.
\]

Consequently, the spherical area of the covering surface of \( D \) is less than or equal to \( 16\pi/(1 - \alpha^2) \), which completes the proof. \( \square \)

We end the section with the following.

**Conjecture 2.** The condition on the dilatation in Theorem 4 can be removed.
5. Distortion estimates for univalent harmonic mappings

For $f = h + g \in S_H$, it is known \cite{12} pp. 92, 98 that

\begin{equation}
\frac{1}{16} (1 - |\mu(z)|) \leq d(f(z), \partial \Omega) \lambda_D(h(z)) \leq c,
\end{equation}

where $\mu$ is the dilatation of $f$, $1 \leq c < 2$, $f(\mathbb{D}) = \Omega$, and $h(\mathbb{D}) = D$. If $D$ is hyperbolic, $h$ satisfies the general inequality in (4.2). Moreover, since $D$ is a simply connected domain, (4.1) gives

\begin{equation}
\frac{1}{4} \leq d(f(z), \partial \Omega) \lambda_\Omega(f(z)) \leq 1.
\end{equation}

It follows from (5.1) and (5.2) that

\[ \frac{1}{16} (1 - |\mu(z)|) \lambda_\Omega(f(z)) \leq \lambda_D(h(z)) \leq 4c \lambda_\Omega(f(z)). \]

Combining (4.2) and (5.1) yields

\begin{equation}
\frac{1}{16} (1 - |\mu(z)|) \leq \frac{d(f(z), \partial \Omega)}{d(h(z), \partial D)} \leq 2c(\beta_D(w) + C_0).
\end{equation}

The next result gives better estimates than (5.3).

Theorem 5. Let $f = h + g \in S_H^0$, $f(\mathbb{D}) = \Omega$, $h(\mathbb{D}) = D$ is hyperbolic, and $a \in \partial D$ be the nearest point to the origin 0. Then

\begin{equation}
\frac{1}{16} \leq d(0, \partial D) \leq 1
\end{equation}

and

\begin{equation}
\frac{1}{16} (1 - |\mu(z)|) \leq d(f(z), \partial \Omega) \leq 2d(h(z), \partial D).
\end{equation}

Proof. The inequalities (5.4) have already appeared in Theorem 2. To show the inequalities (5.5), we fix $z \in \mathbb{D}$ and let $b \in \partial D$ be nearest to $h(z)$ and $\gamma = [h(z), b] \subset D$ be the line segment from $h(z)$ to $b$. As $b$ is accessible from inside $D$, $b$ is a radial limit for $h$. The function $h$ being conformal assures a branch of $\Gamma = h^{-1}(\gamma)$ connecting $z$ to $\partial \mathbb{D}$. Now, we find that

\[ d(f(z), \partial \Omega) \leq \int_\Gamma \left| \frac{\partial f}{\partial \zeta} d\zeta + \frac{\partial f}{\partial \bar{\zeta}} d\bar{\zeta} \right| \]

\[ \leq \int_\Gamma (|h'(\zeta)| + |g'(\zeta)|) |d\zeta| \]

\[ \leq 2 \int_\Gamma |h'(\zeta)| |d\zeta| = 2d(h(z), \partial D) \]

which proves the right-hand inequality of (5.5). The left-hand inequality of (5.5) is a known estimate (see \cite{12}).
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