ON THE ADDITIVE AND MULTIPLICATIVE STRUCTURES OF THE EXCEPTIONAL UNITS IN FINITE COMMUTATIVE RINGS

SU HU AND MIN SHA

Abstract. Let \( R \) be a commutative ring with identity. A unit \( u \) of \( R \) is called exceptional if \( 1 - u \) is also a unit. When \( R \) is a finite commutative ring, we determine the additive and multiplicative structures of its exceptional units.

1. Introduction

1.1. Background. Let \( R \) be a commutative ring with \( 1 \in R \) and \( R^\ast \) its group of units. A unit \( u \in R \) is called exceptional if \( 1 - u \in R^\ast \). We denote by \( R^{\ast\ast} \) the set of exceptional units of \( R \). This concept was introduced by Nagell [9] in 1969 in order to solve certain cubic Diophantine equations. The key idea is that the solution of many Diophantine equations can be reduced to the solution of a finite number of unit equations of type

\[
ax + by = 1,
\]

where \( x \) and \( y \) are restricted to units in the ring of integers of some number field (see [2] for a treatise on unit equations). By choosing \( a = b = 1 \), we obtain the concept of exceptional unit.

Let \( \mathbb{Z}_n \) be the residue class ring of the integers \( \mathbb{Z} \) modulo a positive integer \( n \). Sander [11, Theorem 1.1] has determined the number of representations of an element in \( \mathbb{Z}_n \) as the sum of two exceptional units. Recently, Zhang and Ji [13, Theorem 1.5] has extended Sander’s result to the case when \( R \) is a residue class ring of a number field. Using a different approach with the aid of exponential sums, for any integer \( k \geq 2 \) Yang and Zhao [12, Theorem 1] has obtained an exact formula for the number of ways to represent an element of \( \mathbb{Z}_n \) as a sum of \( k \) exceptional units. Most recently, Miguel [8, Theorem 1] has generalized the result of Yang and Zhao to the case of finite commutative rings.

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1.2. **Our situation.** In this paper, we consider the additive and multiplicative structures of the exceptional units of a finite commutative ring. We show that as in \([8, 12]\), there is an exact formula for the number of ways to represent a unit as a product of \(k\) exceptional units. As an application and combining with Miguel's result, we completely determine the additive and multiplicative structures of such exceptional units in our situation.

From now on, \(R\) is a finite commutative ring with identity. It is well-known that \(R\) can be uniquely expressed as a direct sum of local rings; see [7, page 95]. So, in the following we assume that

\[
R = R_1 \oplus \cdots \oplus R_n,
\]

where each \(R_i\), for \(i = 1, \ldots, n\), is a local ring. Then, each element \(c \in R\) can be represented as \((c_1, \ldots, c_n)\) with \(c_i \in R_i\), \(i = 1, \ldots, n\). For each \(i = 1, \ldots, n\), suppose that \(M_i\) is the unique maximal ideal of \(R_i\), and put

\[
m_i = |M_i|, \quad q_i = |R_i/M_i|.
\]

Here, each residue field \(R_i/M_i\) is a finite field, and thus \(q_i\) is a power of a prime.

We first state our results in Sections 1.3, 1.4, 1.5 and 1.6, and then we give proofs in Section 2.

1.3. **Additive structure of units.** Before further work, we first determine the additive structure of \(R^*\). For \(c \in R\), we define the set

\[
\Psi_{k,R}(c) = \{(x_1, \ldots, x_k) \in (R^*)^k : x_1 + \cdots + x_k = c\},
\]

and put

\[
\psi_{k,R}(c) = |\Psi_{k,R}(c)|.
\]

That is, \(\psi_{k,R}(c)\) is the number of ways to represent \(c\) as a sum of \(k\) units. Kiani and Mollahajigahaei has obtained the following formula for \(\psi_{k,R}(c)\) (see [5, Theorem 2.5]), which generalizes the result in [10].

**Theorem 1.1** ([5]). For any integer \(k \geq 2\) and any \(c = (c_1, \ldots, c_n) \in R\), we have

\[
\psi_{k,R}(c) = \prod_{i=1}^n m_i^{k-1} q_i^{-1} \mu_{k,R_i}(c_i),
\]

where

\[
\mu_{k,R_i}(c_i) = \begin{cases} (q_i - 1)^k + (-1)^k (q_i - 1) & \text{if } c_i \in M_i, \\ (q_i - 1)^k + (-1)^{k+1} & \text{if } c_i \in R_i \setminus M_i. \end{cases}
\]

Theorem 1.1 can be directly used to establish the additive structure of the units of \(R\).
Theorem 1.2. The following hold:

(i) If $q_i > 2$ for each $i = 1, \ldots, n$, then we have
$$R^* + R^* = R.$$ 

(ii) If $q_1 = \cdots = q_s = 2$ ($s \geq 1$) and $q_j > 2$ for each $j > s$, then for any integer $k \geq 2$ we have
$$\sum_{i=1}^{k} R^* = \begin{cases} 
(\oplus_{i=1}^{s} M_i) \oplus (\oplus_{j>s} R_j) & \text{if } k \text{ is even}, \\
(\oplus_{i=1}^{s} R_i \setminus M_i) \oplus (\oplus_{j>s} R_j) & \text{otherwise}.
\end{cases}$$

In particular, each element of $R$ is a sum of units if and only if $s = 1$.

In Theorem 1.2 (ii), if $s = n$, then the part $\oplus_{j>s} R_j$ does not exist.

Corollary 1.3. The ring $R$ is generated by its units if and only if there is at most one $q_i$ equal to 2.

1.4. Additive structure of exceptional units. We now turn our attention to exceptional units. In Theorem 1.1, if we choose $k = 2$ and $c = 1$, then we directly get the size of $R^{**}$.

Theorem 1.4. We have
$$|R^{**}| = \prod_{i=1}^{n} m_i(q_i - 2).$$

We directly have:

Corollary 1.5. $R^{**} \neq \emptyset$ if and only if each $q_i$ is greater than 2, $i = 1, \ldots, n$.

For $c \in R$, define the set
$$\Phi_{k,R}(c) = \{(x_1, \ldots, x_k) \in (R^{**})^k : x_1 + \cdots + x_k = c\},$$
and denote
$$\varphi_{k,R}(c) = |\Phi_{k,R}(c)|.$$ 

That is, $\varphi_{k,R}(c)$ is the number of ways to represent $c$ as a sum of $k$ exceptional units. Miguel has given an exact formula for $\varphi_{k,R}(c)$; see [8, Theorem 1].

Theorem 1.6 ([8]). For any integer $k \geq 2$ and any $c = (c_1, \ldots, c_n) \in R$, we have
$$\varphi_{k,R}(c) = \prod_{i=1}^{n} (-1)^{k} m_i^{k-1} q_i^{-1} \rho_{k,R}(c_i),$$
where
$$\rho_{k,R_i}(c_i) = q_i \sum_{j=0}^{k} \binom{k}{j} + (2 - q_i)^k - 2^k.$$
Using Theorem 1.6, we completely determine the additive structure of $R^{**}$ in the following theorem. Note that by Corollary 1.5 we need to exclude the case when there is some $q_i$ equal to 2.

**Theorem 1.7.** Assume that each $q_i$ is greater than 2, $i = 1, \ldots, n$. Then, the following hold:

(i) If each $q_i$ is greater than 4, $i = 1, \ldots, n$, we have

$$R^{**} + R^{**} = R.$$ 

(ii) If $q_1 = \cdots = q_s = 3$ ($s \geq 1$) and $q_j > 4$ for each $j > s$, then for any $k \geq 2$ we have

$$\sum_{i=1}^{k} R^{**} = \begin{cases} 
(\oplus_{i=1}^{s} M_i) \oplus (\oplus_{j>s} R_j) & \text{if } k \equiv 0 \pmod{3}, \\
(\oplus_{i=1}^{s} 2 + M_i) \oplus (\oplus_{j>s} R_j) & \text{if } k \equiv 1 \pmod{3}, \\
(\oplus_{i=1}^{s} 1 + M_i) \oplus (\oplus_{j>s} R_j) & \text{if } k \equiv 2 \pmod{3}.
\end{cases}$$ 

In particular, every element of $R$ is a sum of its exceptional units if and only if $s = 1$.

(iii) If $q_1 = \cdots = q_t = 4$ ($t \geq 1$) and $q_j > 4$ for each $j > t$, then for any $k \geq 2$ we have

$$\sum_{i=1}^{k} R^{**} = \begin{cases} 
(\oplus_{i=1}^{t} (M_i \cup 1 + M_i)) \oplus (\oplus_{j>t} R_j) & \text{if } k \text{ is even}, \\
(\oplus_{i=1}^{t} (R_i \setminus (M_i \cup 1 + M_i))) \oplus (\oplus_{j>t} R_j) & \text{otherwise}.
\end{cases}$$ 

In particular, every element of $R$ is a sum of its exceptional units if and only if $t = 1$.

(iv) Assume that $q_1 = \cdots = q_s = 3$ ($s \geq 1$), $q_{s+1} = \cdots = q_{s+t} = 4$ ($t \geq 1$), and $q_j > 4$ for each $j > s+t$. Then, for any $k \geq 2$ we have

$$\sum_{i=1}^{k} R^{**} = \begin{cases} 
(\oplus_{i=1}^{s} M_i) \oplus (\oplus_{i=s+1}^{s+t} (M_i \cup 1 + M_i)) \oplus (\oplus_{j>s+t} R_j) & \text{if } k \equiv 0 \pmod{6}, \\
(\oplus_{i=1}^{s} 2 + M_i) \oplus (\oplus_{i=s+1}^{s+t} (R_i \setminus (M_i \cup 1 + M_i))) \oplus (\oplus_{j>s+t} R_j) & \text{if } k \equiv 1 \pmod{6}, \\
(\oplus_{i=1}^{s} 1 + M_i) \oplus (\oplus_{i=s+1}^{s+t} (M_i \cup 1 + M_i)) \oplus (\oplus_{j>s+t} R_j) & \text{if } k \equiv 2 \pmod{6}, \\
(\oplus_{i=1}^{s} 2 + M_i) \oplus (\oplus_{i=s+1}^{s+t} (R_i \setminus (M_i \cup 1 + M_i))) \oplus (\oplus_{j>s+t} R_j) & \text{if } k \equiv 3 \pmod{6}, \\
(\oplus_{i=1}^{s} 1 + M_i) \oplus (\oplus_{i=s+1}^{s+t} (M_i \cup 1 + M_i)) \oplus (\oplus_{j>s+t} R_j) & \text{if } k \equiv 4 \pmod{6}, \\
(\oplus_{i=1}^{s} 2 + M_i) \oplus (\oplus_{i=s+1}^{s+t} (R_i \setminus (M_i \cup 1 + M_i))) \oplus (\oplus_{j>s+t} R_j) & \text{if } k \equiv 5 \pmod{6}.
\end{cases}$$
In particular, every element of $R$ is a sum of its exceptional units if and only if $s = t = 1$.

1.5. Multiplicative structure of exceptional units. Here, we want to determine the multiplicative structure of exceptional units of $R$.

For a unit $u \in R^*$, define the set

$$\Theta_{k,R}(u) = \{(x_1, \ldots, x_k) \in (R^{**})^k : x_1x_2 \cdots x_k = u\},$$

and denote

$$\theta_{k,R}(u) = |\Theta_{k,R}(u)|.$$

That is, $\theta_{k,R}(u)$ is the number of ways to represent $u$ as a product of $k$ exceptional units.

By Corollary 1.5, if there is some $q_i = 2$, then $R^{**} = \emptyset$. Certainly we need to exclude this case. Applying the same arguments as in [8], we obtain an exact formula for $\theta_{k,R}(u)$.

**Theorem 1.8.** Assume that each $q_i > 2$, $i = 1, \ldots, n$. Then, for any integer $k \geq 2$ and any $u = (u_1, \ldots, u_n) \in R^*$, we have

$$\theta_{k,R}(u) = \prod_{i=1}^{n} m_i^{k-1}(q_i - 1)^{-1}\sigma_{k,R_i}(u_i),$$

where

$$\sigma_{k,R_i}(u_i) = \begin{cases} (q_i - 2)^k + (-1)^k(q_i - 2) & \text{if } u_i \in 1 + M_i, \\ (q_i - 2)^k + (-1)^{k+1} & \text{if } u_i \not\in 1 + M_i. \end{cases}$$

Then, we can determine the multiplicative structure of $R^{**}$.

**Theorem 1.9.** The following hold:

(i) If each $q_i > 3$, $i = 1, \ldots, n$, then we have

$$R^{**} \cdot R^{**} = R^*.$$

(ii) If $q_1 = \cdots = q_s = 3$ ($s \geq 1$) and $q_j > 3$ for each $j > s$, then for any integer $k \geq 2$ we have

$$\prod_{i=1}^{k} R^{**} = \begin{cases} (\oplus_{i=1}^{s} 1 + M_i) \oplus (\oplus_{j>s} R_j^*) & \text{if } k \text{ is even}, \\ (\oplus_{i=1}^{s} R_i^* \setminus 1 + M_i) \oplus (\oplus_{j>s} R_j^*) & \text{otherwise}. \end{cases}$$

In particular, every unit of $R$ is a product of its exceptional units if and only if $s = 1$.

Finally, we determine under which condition the ring $R$ can be generated by its exceptional units.

**Corollary 1.10.** Assume that each $q_i \geq 3$, $i = 1, \ldots, n$. Then, the ring $R$ is generated by its exceptional units if and only if there is at most one $q_i$ equal to 3.
1.6. Applications. Here, we apply the above results to the case of number fields. Let $F$ be a number field, and let $\mathcal{O}_F$ be the ring of integers of $F$.

For the additive structure of the units in $\mathcal{O}_F$, Jarden and Narkiewicz [4] has showed that there is no integer $k$ such that every element of $\mathcal{O}_F$ is a sum of at most $k$ units. Several questions have been posed in that paper, and one of them has been answered affirmatively by Frei [3], which asserts that there exists a number field $K$ containing $F$ such that $\mathcal{O}_K$ is generated by its units. By Theorem 1.2 (ii), we can get a simple sufficient condition to decide whether $\mathcal{O}_F$ is generated by its units. Note that the statement that $\mathcal{O}_F$ is generated by its units is equivalent to that every element of $\mathcal{O}_F$ is a sum of its units.

**Corollary 1.11.** If there are at least two prime ideals of $\mathcal{O}_F$ lying above the prime 2 and of norm 2, then $\mathcal{O}_F$ can not be generated by its units.

However, the condition in Corollary 1.11 is sufficient but not necessary.

**Example 1.12.** Choose $F = \mathbb{Q}(\sqrt{6})$. Then, the prime 2 is ramified in $\mathcal{O}_F$. By Theorem 1.2, for any non-zero ideal $I$ every element of $\mathcal{O}_F/I$ is a sum of its units. However, by Belcher’s result [1, Lemma 1], not every element of $\mathcal{O}_F$ is a sum of its units.

Notice that every exceptional unit of $\mathcal{O}_F$ automatically yields an exceptional unit in $\mathcal{O}_F/I$ for any ideal $I$. By Corollary 1.5, we immediately have:

**Corollary 1.13.** If $\mathcal{O}_F$ has a prime ideal of norm 2, then $\mathcal{O}_F^{**} = \emptyset$.

In view of quadratic number fields, one can see that the condition in Corollary 1.13 is sufficient but not necessary. In fact, by definition it is easy to see that the only exceptional units in quadratic fields are the roots of the four polynomials:

$$X^2 - X + 1, \quad X^2 - 3X + 1, \quad X^2 - X - 1, \quad X^2 + X - 1.$$

They correspond to the quadratic fields $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{5})$. For each of the two fields, the ring of integers is generated by its exceptional units.

Using Corollary 1.13 and Theorem 1.7, we directly obtain:

**Corollary 1.14.** Not every element of $\mathcal{O}_F$ is a sum of its exceptional units if one of the following conditions holds:

- $\mathcal{O}_F$ has a prime ideal of norm 2;
- $\mathcal{O}_F$ has at least two prime ideals of norm 3;
- $\mathcal{O}_F$ has at least two prime ideals of norm 4.
The following is a direct consequence of Corollary 1.10.

**Corollary 1.15.** \( \mathcal{O}_F \) can not be generated by its exceptional units if \( \mathcal{O}_F \) has at least two prime ideals of norm 3.

The condition in Corollary 1.15 is also only sufficient.

**Example 1.16.** Choose \( F = \mathbb{Q}(\sqrt{21}) \). Then, the prime 2 is inert and the prime 3 is ramified in \( \mathcal{O}_F \). By Corollary 1.10, for any non-zero ideal \( I \) the quotient ring \( \mathcal{O}_F/I \) is generated by its exceptional units. However, \( \mathcal{O}_F \) has no exceptional unit.

2. **Proofs**

2.1. **Proof of Theorem 1.2.** (i) Notice that for any \( c = (c_1, \ldots, c_n) \in R \), if \( q_i > 2 \), then

\[
\mu_{2,R_i}(c_i) > 0.
\]

This together with Theorem 1.1 implies the result in (i).

(ii) Now, assume that \( q_i = 2 \), then by Theorem 1.1, we have

\[
\mu_{k,R_i}(c_i) = \begin{cases} 
1 + (-1)^k & \text{if } c_i \in M_i, \\
1 + (-1)^{k+1} & \text{if } c_i \in R_i \setminus M_i.
\end{cases}
\]

Then, letting \( k \) be even and applying (2.1), we obtain

\[
\sum_{i=1}^{k} R_i^* = (\oplus_{i=1}^{s} M_i) \oplus (\oplus_{j>s} R_j).
\]

Similarly, letting \( k \) be odd and using (2.1) we obtain the second identity.

2.2. **Proof of Theorem 1.7.** (i) Notice that given \( c = (c_1, \ldots, c_n) \in R \), for each \( i = 1, \ldots, n \),

\[
\rho_{2,R_i}(c_i) > 0 \quad \text{if } q_i > 4.
\]

This together with Theorem 1.6 implies the result in (i).

(ii) If \( q_i = 3 \), we have

\[
\rho_{2,R_i}(c_i) = 3 \left( \sum_{j=0}^{2} \left( \binom{2}{j} - 1 \right) \right), \quad \text{for any } c_i \in R_i.
\]

Notice that \( q_i = 3 \), then the residue classes modulo \( M_i \) can be represented by 0, 1, 2 respectively. So, \( \rho_{2,R_i}(c_i) \neq 0 \) if and only if \( c_i \in 1 + M_i \). Then, using Theorem 1.6 we get

\[
R_i^{**} + R_i^{**} = (\oplus_{i=1}^{s} 1 + M_i) \oplus (\oplus_{j>s} R_j).
\]
We again have
\[ \rho_{3,R_i}(c_i) = 3 \left( \sum_{j=0}^{3} \binom{3}{j} - 3 \right), \text{ for any } c_i \in R_i. \]

Thus, \( \rho_{3,R_i}(c_i) \neq 0 \) if and only if \( c_i \in M_i \). So, by Theorem 1.6 we obtain
\[ R^{**} + R^{**} + R^{**} = (\oplus_{i=1}^{s} M_i) \oplus (\oplus_{j=s}^{s} R_j). \]

Then, combining (2.2) with (2.3) we get the identities in (ii).

(iii) If \( q_i = 4 \), for even \( k \) we have
\[ \rho_{k,R_i}(c_i) = 4 \sum_{j=0}^{k} \binom{k}{j}, \text{ for any } c_i \in R_i. \]

So, \( \rho_{k,R_i}(c_i) \neq 0 \) if and only if \( c_i \in M_i \) or \( c_i \in 1 + M_i \), where one should note that the characteristic of the residue field \( R_i/M_i \) is 2 (because \( q_i = 4 \)). Thus, by using Theorem 1.6 we get the first identity in (iii).

For the second identity, we note that for odd \( k \), we have
\[ \rho_{k,R_i}(c_i) = 4 \left( \sum_{j=0}^{k} \binom{k}{j} - 2^{k-1} \right), \text{ for any } c_i \in R_i. \]

Since the characteristic of the residue field \( R_i/M_i \) is 2, if \( c_i \in M_i \), then \( j \equiv c_i \equiv 0 \pmod{M_i} \) for any even integer \( j \). Also, if \( c \in 1 + M_i \), then \( j \equiv c_i \equiv 1 \pmod{M_i} \) for any odd integer \( j \). Thus, \( \rho_{k,R_i}(c_i) \neq 0 \) if and only if \( c \not\in M_i \) and \( c \not\in 1 + M_i \). So, similarly we obtain the second identity.

(iv) The desired results in (iv) follow directly from (ii) and (iii).

2.3. Proof of Theorem 1.8. Applying the same arguments as in [8], we have the following two lemmas, for which we omit the proofs.

**Lemma 2.1.** For any \( u = (u_1, \ldots, u_n) \in R^* \), we have
\[ \theta_{k,R}(u) = \prod_{i=1}^{n} \theta_{k,R_i}(u_i). \]

**Lemma 2.2.** For each \( i = 1, \ldots, n \) and for any unit \( u_i \in R_i^* \) we have
\[ \theta_{k,R_i}(u_i) = m_i^{k-1} \theta_{k,R_i/M_i}(u_i). \]
Let \( F_q \) be a finite field of \( q \) elements. Recall that a multiplicative character \( \chi \) of \( F_q^* \) is a homomorphism from \( F_q^* \) to the complex roots of unity. The trivial character \( \chi_0 \) is the one sending every element of \( F_q^* \) to 1. Let \( G_q \) be the group of multiplicative characters of \( F_q^* \), and let \( G_q^* = G_q \setminus \{ \chi_0 \} \). Then, \(|G_q| = q - 1\). Furthermore, we have the following orthogonality relations (for instance, see \([6]\)):

\[
\sum_{a \in F_q^*} \chi(a) = \begin{cases} q - 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise}; \end{cases}
\]

and

\[
\sum_{\chi \in G_q} \chi(a) = \begin{cases} q - 1 & \text{if } a = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof of Theorem 1.8.** From the above lemmas, it is only need to calculate \( \theta_{k,F_q}(c) \) for \( c \in F_q^* \) and \( q > 2 \). Using the above formulas about multiplicative characters, we obtain

\[
\theta_{k,F_q}(c) = |\{(x_1, \ldots, x_k) \in (F_q^*)^k : x_1 x_2 \cdots x_k = c\}|
\]

\[
= \sum_{x_1 \in F_q^{**}} \sum_{x_2 \in F_q^{**}} \cdots \sum_{x_k \in F_q^{**}} \frac{1}{q - 1} \sum_{\chi \in G_q} \chi(x_1 \cdots x_k / c)
\]

\[
= \frac{1}{q - 1} \sum_{\chi \in G_q} \left( \sum_{x_1 \in F_q^{**}} \chi(x_1) \right) \cdots \left( \sum_{x_k \in F_q^{**}} \chi(x_k) \right) \chi(c^{-1})
\]

\[
= \frac{1}{q - 1} \left( \sum_{\chi \in G_q^*} \left( \sum_{x_1 \in F_q^{**}} \chi(x_1) \right) \cdots \left( \sum_{x_k \in F_q^{**}} \chi(x_k) \right) \chi(c^{-1}) + (q - 2)^k \right)
\]

(since \( F_q^{**} = F_q^* \setminus \{1\} \)).

Notice that for any \( \chi \in G_q^* \), we have

\[
0 = \sum_{x_i \in F_q^*} \chi(x_i) = 1 + \sum_{x_i \in F_q^{**}} \chi(x_i).
\]

Then, we further have

\[
\theta_{k,F_q}(c) = \frac{1}{q - 1} \left( (q - 2)^k + (-1)^k \sum_{\chi \in G_q^*} \chi(c^{-1}) \right)
\]

\[
= \begin{cases} \frac{1}{q - 1} \left( (q - 2)^k + (-1)^k (q - 2) \right) & \text{if } c = 1, \\ \frac{1}{q - 1} \left( (q - 2)^k + (-1)^{k+1} \right) & \text{if } c \neq 1. \end{cases}
\]

This, together with Lemma 2.1 and Lemma 2.2, implies the desired result. \( \square \)
2.4. Proof of Theorem 1.9. (i) Notice that given \( u = (u_1, \ldots, u_n) \in R^* \), for each \( i = 1, \ldots, n \), we have

\[
\sigma_{2,R_i}(u_i) > 0 \quad \text{if} \quad q_i > 3.
\]

This together with Theorem 1.8 implies the result in (i).

(ii) If \( q_i = 3 \), we have

\[
\sigma_{k,R_i}(u_i) = \begin{cases} 
1 + (-1)^k & \text{if} \ u_i \in 1 + M_i, \\
1 + (-1)^{k+1} & \text{if} \ u_i \notin 1 + M_i.
\end{cases}
\]

Then, letting \( k \) be even and applying (2.4), we get

\[
\prod_{i=1}^{k} R^{**} = (\oplus_{i=1}^{s} 1 + M_i) \oplus (\oplus_{j>s} R^*_j).
\]

This completes the proof of the first identity.

Similarly, letting \( k \) be odd and using (2.4) we obtain the second identity.

2.5. Proof of Corollary 1.10. The sufficient part follows directly from Theorem 1.2 (i) and Theorem 1.9.

For the necessary part, we suppose that \( q_1 = q_2 = 3 \). By assumption, the ring \( R \) is generated by its exceptional units. Then, the ring \( R_1 \oplus R_2 \) is also generated by its exceptional units, and so is the ring \( R_1/M_1 \oplus R_2/M_2 \). On the other hand, since both finite fields \( R_1/M_1 \) and \( R_2/M_2 \) have only three elements, by Theorem 1.4 we have

\[
| (R_1/M_1 \oplus R_2/M_2)^{**} | = 1.
\]

In fact, we have \( (R_1/M_1 \oplus R_2/M_2)^{**} = \{(2, 2)\} \). So, the ring \( R_1/M_1 \oplus R_2/M_2 \) can not be generated by its unique exceptional unit. This leads to a contradiction.

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Department of Mathematics, South China University of Technology, Guangzhou 510640, China

E-mail address: mahusu@scut.edu.cn

Department of Computing, Macquarie University, Sydney, NSW 2109, Australia

E-mail address: shamin2010@gmail.com