Steiner Wiener index of block graphs

Matjaž Kovše, Rasila V A, Ambat Vijayakumar

Abstract
For a graph $G$ with vertex set $V(G)$, the Steiner distance $d(S) \subseteq V(G)$ of $S \subseteq V(G)$ is the smallest number of edges in a connected subgraph of $G$ that contains $S$. Such a subgraph is necessarily a tree called a Steiner tree for $S$. The Steiner distance is a type of multi-way metric measuring the size of a Steiner tree between vertices of a graph and it generalizes the geodetic distance. The Steiner Wiener index is the sum of all Steiner distances in a graph and it generalizes the Wiener index. A simple method for calculating the Steiner Wiener index of block graphs is presented.

1 Introduction
All graphs in this paper are simple, finite and undirected. If $G$ is a connected graph and $u, v \in V(G)$, then the (geodetic) distance $d(u, v)$ between $u$ and $v$ is the length of a shortest path connecting $u$ and $v$, see also [6]. The Wiener index $W(G)$ of a connected graph $G$ is defined by

$$W(G) = \sum_{u,v \in V(G)} d(u, v).$$

The first investigations of this distance-based graph invariant were done by Harold Wiener in 1947, who realized in [21] that there exist correlations between the boiling points of paraffins and their molecular structure and noted that in the case of a tree it can be easily calculated from the edge contributions by the following formula:

$$W(T) = \sum_{e \in E(T)} n(T_1)n(T_2)$$

where $n(T_1)$ and $n(T_2)$ denote the number of vertices in connected components $T_1$ and $T_2$ formed by removing an edge $e$ from the tree $T$.

The Steiner distance of a graph, introduced in [3] by Chartrand et al., is a natural generalization of the concept of the geodetic graph distance. For a graph
implies SW are on shortest paths joining u. K is a graph in which no induced subgraph is a claw, i.e. a complete bipartite graph n by joining terms of distance conditions [2].

characterized in various ways, for example, as certain intersection graphs [11], or in metric graph theory [1], molecular graphs [2] and phylogenetics [8]. They have been Block graphs are a natural generalization of trees, and they arise in areas such as graph in which every maximal 2-connected subgraph or block is a clique [1, 6].

connected vertex induced subgraph that has no cut vertices. A block graph is a maximal subgraph of G such that S ⊆ V(H) and |E(H)| = d(S), then H is a tree. Clearly, d(S) = min{|E(T)| : S ⊆ V(T)}, where T is a subtree of G. Furthermore, if S = {u, v}, then d(S) = d(u, v) coincides with the geodetic distance between u and v. Clearly, if |S| = k, then d(S) ≥ k − 1.

In [4, 5] Dankelmann et al. followed by studying the average k-Steiner distance µk(G), which is related to the k-Steiner Wiener index via the equality µk(G) = SWk(G)/n. In [10], Li, Mao and Gutman introduced a generalization of the Wiener index, by using Steiner distance. Thus, the k-th Steiner Wiener index SWk(G) of a connected graph G is defined by

\[ SW_k(G) = \sum_{S \subseteq V(G) \atop |S| = k} d(S). \]

For k = 2, the Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider SWk(G) for 2 ≤ k ≤ n − 1, but the above definition also implies SW1(G) = 0 and SWn(G) = n − 1 for a connected graph G of order n. They obtained the exact values of the Steiner Wiener k-index of the path, star, complete graph, and complete bipartite graph and sharp lower and upper bounds for SWk(G) for connected graphs and for trees. In [10] the application of k-Steiner Wiener index in mathematical chemistry is reported, and it is shown that the term W(G) + ASWk(G) provides a better approximation for the boiling points of alkanes than W(G) itself, and that the best such approximation is obtained for k = 7. See [15, 17, 18, 20] for recent results related to Steiner Wiener index and a survey on Steiner distance in [19].

In a graph G, a vertex u is a cut-vertex if deleting u and all edges incident to it increases the number of connected components. A block of a graph is a maximal connected vertex induced subgraph that has no cut vertices. A block graph is a graph in which every maximal 2-connected subgraph or block is a clique [1, 6]. Block graphs are a natural generalization of trees, and they arise in areas such as metric graph theory [1], molecular graphs [2] and phylogenetics [8]. They have been characterized in various ways, for example, as certain intersection graphs [11], or in terms of distance conditions [2].

The windmill graph WD(p, n) is a block graph constructed for p ≥ 2 and n ≥ 2 by joining n copies of the complete graph Kp at a shared vertex. A claw-free graph is a graph in which no induced subgraph is a claw, i.e. a complete bipartite graph K1,3. Claw-free block graphs are block graphs which are claw-free.

The interval I(u, v) between two vertices u and v consists of all vertices that are on shortest paths joining u and v. More generally for a subset A ⊂ V(G) the
\(k\)-interval of \(A\), denoted by \(I_k(A)\), consist of all vertices of \(G\) that are on some \(k\)-Steiner tree joining vertices of \(A\). A graph \(G\) is modular \([12]\) if for every three vertices \(x, y, z\) there exists a vertex \(w\) that lies on a shortest path between every two vertices of \(x, y, z\), i.e., \(|I(x, y) \cap I(x, z) \cap I(y, z)| \geq 1\). It is easy to see that a modular graph is a bipartite graph. Examples of modular graphs are trees, hypercubes, grids, complete bipartite graphs, etc. The simplest examples of non-modular graphs are cycles on \(n\) vertices, for \(n \neq 4\), and complete graphs.

In this paper we obtain simple methods for calculating the \(k\)-Steiner Wiener index of block graphs. We obtain exact values for \(k\)-Steiner Wiener index of windmill graphs and claw free block graphs. We generalise the relation between 3-Steiner Wiener index and Wiener index of a tree from \([16]\) to modular graphs and obtain the corresponding similar relation for block graphs. We conclude with the Steiner Wiener decomposition formula for the special family of block graphs - trees via their subtrees.

2 Decomposition formula of \(k\)-Steiner Wiener index of block graphs

For a graph \(G\), let \(n(G)\) denote the number of its vertices. For a forest \(F\) with \(p, p > 1\), connected components \(T_1, T_2, \ldots, T_p\) denote by \(N_k(F)\) the sum over all partitions of \(k\) into at least two nonzero parts of products of combinations distributed among the \(p\) components of \(F\):

\[
N_k(F) = \sum_{l_1 + l_2 + \ldots + l_p = k}^{0 \leq l_1, l_2, \ldots, l_p < k} \left( \binom{n(T_1)}{l_1} \binom{n(T_2)}{l_2} \cdots \binom{n(T_p)}{l_p} \right)
\]

For a tree \(T\) and \(e \in E(T)\), let \(T - e\) denote a graph obtained by removing edge \(e\) from \(T\). Then the following formula has been shown in \([15]\)

\[
SW_k(T) = \sum_{e \in E(T)} N_k(T - e).
\]

For a given partition \(l_1 + l_2 + \ldots + l_p = k\), let \(\alpha(l_1, l_2, \ldots, l_p)\) denote the number of nonzero summands minus 1. For a graph \(G\) with \(p, p > 1\), connected components \(G_1, G_2, \ldots, G_p\), we define \(N'_k(G)\) to be the sum over all partitions of \(k\) into at least two nonzero parts of products of combinations distributed among the \(p\) components of \(G\) multiplied by \(\alpha(l_1, l_2, \ldots, l_p)\):

\[
N'_k(G) = \sum_{l_1 + l_2 + \ldots + l_p = k}^{0 \leq l_1, l_2, \ldots, l_p < k} \left( \binom{n(G_1)}{l_1} \binom{n(G_2)}{l_2} \cdots \binom{n(G_p)}{l_p} \right) \cdot \alpha(l_1, l_2, \ldots, l_p).
\]

For a connected graph \(G\), we define \(N'_k(G) = 0\). Note that by the definition \(\binom{n}{0} = 1\), and \(\binom{n}{k} = 0\) whenever \(n < k\).
Theorem 2.1. Let $G$ be a block graph with blocks $B_1, B_2, \ldots, B_m$, and let $G\setminus B_i$ denote a graph obtained from $G$ by deleting all edges from block $B_i$. Then

$$SW_k(G) = \sum_{i=1}^{m} N'_k(G\setminus B_i).$$

Proof. $N'_k(G\setminus B_i)$ counts the contribution of $B_i$ to the $k$-Steiner Wiener index of vertices from $G\setminus B_i$. Let $G_1, G_2, \ldots, G_p$ be the connected components of $G\setminus B_i$. For a given partition $l_1 + l_2 + \ldots + l_p = k$, let $\alpha(l_1, l_2, \ldots, l_p)$ denote the number of nonzero summands minus one. A block $B_i$ of $G$ contributes $\alpha(l_1, l_2, \ldots, l_p)$ to the Steiner distance of $k$ vertices from $G\setminus B_i$. Then,

$$N'_k(G\setminus B_i) = \sum_{\substack{l_1 + l_2 + \ldots + l_p = k \\ 0 \leq l_1, l_2, \ldots, l_p < k}} \binom{n(G_1)}{l_1} \binom{n(G_2)}{l_2} \cdots \binom{n(G_p)}{l_p} \cdot \alpha(l_1, l_2, \ldots, l_p).$$

And the formula follows. \hfill \Box

Illustration 2.1. Consider the block graph $G$ of the Figure 1.

![Figure 1: A block graph with three different blocks.](image)

We illustrate Theorem 2.1 for $k = 3$. By the theorem, $SW_k(G) = \sum_{i=1}^{m} N'_k(G\setminus B_i)$ Here,

$$N'_3(G\setminus B_1) = \binom{1}{1} \binom{3}{1} \frac{1}{2} + \binom{1}{1} \binom{3}{2} = 12$$

$$N'_3(G\setminus B_2) = \binom{3}{1} \binom{2}{1} + \binom{3}{2} \frac{1}{2} = 9$$

$$N'_3(G\setminus B_3) = \binom{4}{2} \frac{1}{1} = 6.$$ 

Therefore $SW_3(G) = 12 + 9 + 6 = 27$. 

4
3 Steiner Wiener index of Windmill graphs

Theorem 3.1. Let $Wd(p, n)$ be windmill graph. Then

$$SW_k(Wd(p, n)) = \binom{n(p-1)}{k-1}(k-1) + n\binom{p-1}{k}(k-1) + \sum_{l_1+l_2+...+l_p=k} k\binom{p-1}{l_1} \cdots \binom{p-1}{l_n}$$

Proof. We prove the theorem by considering two cases.

Case 1. Set of $k$ terminals includes the central vertex of the windmill graph.

Since the central vertex is adjacent to any other vertex, we get a star graph as Steiner tree of $k$ vertices which has size $(k-1)$. Then, the contribution of this case to $SW_k(G)$ is $(k-1)^{\binom{n(p-1)}{k-1}}$.

Case 2. Set of $k$ terminals does not contain the central vertex of the windmill graph.

Sub case 2.1: Set of $k$ terminals are from the same clique.

We get a path of length $k-1$ as Steiner tree consisting of the set of $k$ vertices and there are $n\binom{p-1}{k}$ possible ways of choosing them. Therefore, the contribution of this case to $SW_k(G)$ is $n(k-1)^{\binom{p-1}{k}}$.

Sub case 2.2: Set of terminals are from at least two different cliques.

Let $l_1 + l_2 + \ldots + l_n = k$ be a partition. Since any path from $K_{p_i}$ to $K_{p_j}$, $i \neq j$ must pass through central vertex and the set of $l_i$ vertices in $K_{p_i}$ form $l_i-1$ path, the Steiner distance is $l_1 + l_2 + \ldots + l_n = k$. Then, the contribution of this case to $SW_k(G)$ is

$$\sum_{l_1+l_2+...+l_n=k, 0 \leq l_1, l_2, \ldots, l_n < k} k\binom{p-1}{l_1} \cdots \binom{p-1}{l_n}$$

Thus, the equation holds.

4 Vertex decomposition of Steiner Wiener index of block graphs

For a tree $T$ and $v \in V(T)$, let $T \setminus v$ denote a graph obtained by removing $v$ from $T$. Note that $T \setminus v$ may consists of several components and that their number equals the degree of $v$. In [15], the vertex version of Steiner Wiener index of a tree is given by

$$SW_k(T) = \sum_{v \in V(T)} N_k(T \setminus v) + (k-1)^{\binom{n}{k}}$$
Theorem 4.1. For a block graph $G$ with set of cut vertices $V_c(G)$,
\[
SW_k(G) = \sum_{v \in V_c(G)} N_k(G \setminus v) + (k - 1) \binom{n}{k}
\]

Proof. $N_k(G \setminus v)$ counts number of times a cut vertex $v$ is a non terminal vertex of Steiner tree. Since each such vertex adds 1 to Steiner distance of $k$ vertex set, Steiner distance between $k$ vertices is by $k - 1$ greater than the number of non-terminal vertices in the corresponding Steiner tree, adding $k - 1$ for each set of $k$ vertices, we get the sum of Steiner distances between all $k$ sets of vertices, and the equality in formula holds.

The line graphs of trees are exactly the block graphs in which every cut vertex is incident to at most two blocks, or equivalently these are the claw-free block graphs.

Corollary 4.2. For a claw-free block graph $G$, Steiner Wiener index is given by
\[
SW_k(G) = \sum_{v \in V_c(G)} \sum_{0 \leq l < k} \binom{n(T_1)}{l} \binom{n(T_2)}{k - l} + (k - 1) \binom{n}{k}.
\]

Proof. Claw-free block graphs are graphs for which a cut vertex is adjacent to at most two blocks. Let the components of $T \setminus v$ be $T_1$ and $T_2$. Then formula follows by applying Theorem 4.1.

Illustration 4.1. Consider the block graph $G$ given by Figure 1. We illustrate Theorem 4.1 for $k = 3$. By the theorem, $SW_3(G) = \sum_{v \in V_c(G)} N_3(G \setminus v) + (k - 1) \binom{n}{k}$. Here, $V_c(G) = \{3, 4\}$
\[
\sum_{v \in V_c(G)} N_3(G \setminus v) = \binom{2}{1} \binom{2}{2} + \binom{2}{2} \binom{2}{1} + \binom{3}{2} \binom{1}{1} = 7
\]
\[
(k - 1) \binom{n}{k} = 2 \binom{5}{3} = 20
\]
Therefore $SW_3(G) = 20 + 7 = 27$.

5 3-Steiner Wiener index of modular and block graphs

For $S \subseteq V(G)$, the 2-intersection interval of $S$ is the intersection of all intervals between pairs of vertices from $S$: $I_2(S) = \bigcap_{a, b \in S, a \neq b} I(a, b)$. Hence modular graphs are those graphs for which the 2-intersection interval of every triple of vertices is non-empty. The following result is from [13].
Theorem 5.1. Let \( S = \{u_1, u_2, \ldots, u_n\} \) be a set of \( n > 2 \) vertices of a graph \( G \). If the 2-intersection interval of \( S \) is nonempty and \( x \in I_2(S) \), then \( d(S) = \sum_{i=1}^{n} d(u_i, x) \).

Next we provide the connection between 3-Steiner Wiener index and Wiener index.

Theorem 5.2. Let \( G \) be a graph on \( n \) vertices. Then, \( SW_3(G) \geq n - \frac{2}{2} W(G) \), with the equality if and only if \( G \) is a modular graph.

Proof. Let \( G \) be a connected graph. A triplet of vertices \( x, y, z \in V(G) \) is called a modular triplet if \( I(x, y) \cap I(x, z) \cap I(y, z) \neq \emptyset \).

Let \( S = \{a, b, c\} \subseteq V(G) \), \( |S| = 3 \), and let \( G \) be a modular graph. Then there exist \( x \in I_2(S) \). By Theorem 5.1 it follows \( d(S) = d(a, x) + d(b, x) + d(c, x) \). There are two possibilities: \( x \in S \) or \( x \notin S \).

Case 1 \( x \in S \)
Without loss of generality, let \( x = b \). Hence \( d(a, c) = d(a, b) + d(b, c) \) and therefore \( d(S) = d(a, c) = \frac{1}{2}(d(a, b) + d(b, c) + d(a, c)) \).

Case 2 \( x \notin S \)
It follows that \( d(a, x) + d(x, b) = d(a, b) \) and \( d(b, x) + d(x, c) = d(b, c) \) and \( d(a, x) + d(x, c) = d(a, c) \). Therefore \( d(S) = d(a, x) + d(b, x) + d(c, x) = \frac{1}{2}(d(a, b) + d(b, c) + d(a, c)) \). Each pair of vertices in a graph on \( n \) vertices belongs to \( n - 2 \) different triples of vertices, hence it follows.

\[
SW_3(G) = \sum_{S \subseteq V(G)} d(S) \\
= \sum_{a,b,c \in V(G), |\{a,b,c\}|=3} \frac{1}{2}(d(a, b) + d(b, c) + d(a, c)) \\
= \frac{1}{2} \sum_{a,b \in V(G)} d(a, b) (n - 2) \\
= \frac{n - 2}{2} W(G).
\]

For a non modular triplet \( x, y \) and \( z \) we always have \( d(x, y, z) > \frac{1}{2}(d(x, y) + d(x, z) + d(y, z)) \), hence we get the strict inequality, for any non modular graph. \( \Box \)

Since trees are modular graphs, Theorem 5.2 generalises Corollary 4.5 from [16], where a special case for trees is proved.
Theorem 5.3. Let $G$ be a block graph with blocks $B_1, B_2, \ldots, B_m$. Then

$$SW_3(G) = \frac{n-2}{2} W(G) + \frac{1}{2} \sum_{i=1}^{m} \left( \frac{|B_i|}{3} \right).$$

Proof. In a block graph $G$, any nonmodular triplet $x, y, z$ must belong to the same clique. In this case $d(x, y, z) = 2 = \frac{1}{2}(d(x, y) + d(x, z) + d(y, z)) + \frac{1}{2}$. Let $M(G)$ denote the set of all modular triplets of $G$. Then it follows

$$SW_3(G) = \sum_{x, y, z \in M(G)} d(x, y, z) + \sum_{x, y, z \in V(G) \setminus M(G)} d(x, y, z)$$

$$= \sum_{x, y, z \in M(G)} \left( \frac{1}{2}(d(x, y) + d(x, z) + d(y, z)) \right)$$

$$+ \sum_{x, y, z \in V(G) \setminus M(G)} \left( \frac{1}{2}(d(x, y) + d(x, z) + d(y, z)) + \frac{1}{2} \right)$$

$$= \frac{n-2}{2} W(G) + \frac{1}{2} \sum_{i=1}^{m} \left( \frac{|B_i|}{3} \right).$$

Here the last sum counts the number of nonmodular triplets in a block graph. □

6 The k-intersection intervals and k-Steiner Wiener index of trees

In this section we extend result on Wiener index of trees from Doyle and Graver [7], by generalizing the original notions and proof technique to obtain a formula for $k$-Steiner Wiener index of a tree, see also [9, 14] for alternative proofs in the case when $k = 2$. For a subset of vertices $A$, let $S(A)$ denote the Steiner tree connecting them. A subset of $k$ distinct vertices $\{v_1, v_2, \ldots, v_k\} = A \subseteq V(G)$ is said to be $k$-collinear if there exist $i, 1 \leq i \leq k$, such that $a_i \in S(A \setminus \{a_i\})$. Let $\tau_k(G)$ denote the number of non-$k$-collinear subsets of $G$.

Theorem 6.1. Let $G$ be a graph on $n$ vertices, such that every subset of $k$ vertices is connected by a unique $k$-Steiner tree. Then

$$SW_k(T) = \binom{n}{k+1} + (k-1) \binom{n}{k} - \tau_k(T). \quad (1)$$

Proof. Let $C$ be the collection of $(k+1)$-collinear subsets of $V(G)$ and let $D$ be the collection of all $k$-subsets of $V(G)$. Define $\phi : C \rightarrow D$ by letting $\phi(A)$, $A \subseteq C$, be the $k$-subset of vertices whose Steiner tree includes all vertices from $A$.

None that for a $B \subseteq D$, $\phi^{-1}(B)$ is the collection of all $(k+1)$-subsets of $V(G)$ which contain vertices from $B$ and a vertex on the unique Steiner tree between them. Therefore $|\phi^{-1}(B)| = d(B) - (k-1)$, which is precisely the number of all
inner vertices on a unique Steiner tree with $k$-terminal vertices, forming a set $B$. Hence

$$|C| = \sum_{A \subset V(G) \atop |A| = k} d(A) - (k - 1) = SW_k(G) - (k - 1) \binom{n}{k}.$$  

Since $|C| + \tau_k(G) = \binom{n}{k+1}$ the theorem is proved. \hfill \Box

We can extend the definition of the 2-intersection interval as follows. For $S \subseteq V(G)$, the $k$-intersection interval of $S$ is the intersection of all $k$-intervals between $k$-subsets of vertices from $S$: $I_k(S) = \bigcap_{A \subseteq S \atop |A| = k} I_k(A)$.

Let $G$ be a graph with $p$ connected components $G_1, G_2, \ldots, G_p$. Then we define

$$M_k(G) = \sum_{\{i_1, i_2, \ldots, i_k\} \subseteq \{1, \ldots, p\} \atop \{|i_1, i_2, \ldots, i_k\| = k}} n(G_{i_1}) \cdot n(G_{i_2}) \cdot \ldots \cdot n(G_{i_k}).$$

When $p < k$ then we define $M_k(G) = 0$. Let $T' \subseteq T$ denote that $T'$ is a subtree of tree $T$.

**Theorem 6.2.** Let $T$ be a tree on $n$ vertices. Then

$$SW_k(T) = \left( \binom{n}{k+1} + (k - 1) \binom{n}{k} \right) - \sum_{T' \subseteq T} M_k(T - T').$$  \hfill (2)

**Proof.** In a tree, any subset of vertices is joined by a unique Steiner tree, hence we can use Theorem 6.1.

If $v_1, v_2, \ldots, v_k$ are $k$ distinct non-$(k - 1)$-collinear vertices of $T$, joined by Steiner tree $S$, then there exist a unique minimal subtree $S'$ such that $S \setminus S'$ has exactly $k$ components - this is exactly the $k$-intersection interval of $v_1, v_2, \ldots, v_k$.

The function $M_k(T - T')$, is precisely the number of non-collinear $k$-subsets of $V(T)$ with $T'$ as its $k$-intersection interval. \hfill \Box

**References**

[1] H.-J. Bandelt, H. M. Mulder, Distance-hereditary graphs, J. of Combinatorial Theory, Series B 41 (1986) 182–208.

[2] A. Behtoei, J. Mohsen, T. Bijan, A characterization of block graphs, Discrete Appl. Math. 158 (2010) 219–221.

[3] G. Chartrand, O.R. Oellermann, S. L. Tian, H. B. Zou, Steiner distance in graphs, Časopis pro pěstování matematiky 114 (1989) 399–410.

[4] P. Dankelmann, O. R. Oellermann, H. C. Swart, The average Steiner distance of a graph, J. Graph Theory 22 (1996) 15–22.
[5] P. Dankelmann, O. R. Oellermann, H. C. Swart, On the average Steiner distance of graphs with prescribed properties, Discrete Appl. Math. 79 (1997) 91–103.

[6] M. M. Deza, E. Deza, Encyclopedia of distances, Springer Berlin Heidelberg, 2009.

[7] J. K. Doyle, J. E. Graver, Mean distance in a graph, Discrete Math. 17 (1977) 147–154.

[8] A. Dress, K. T. Huber, J. Koolen, V. Moulton, A. Spillner, Characterizing block graphs in terms of their vertex-induced partitions, Australas. J. Comb. 66 (2016) 1–9.

[9] I. Gutman, A new method for the calculation of the Wiener number of acyclic molecules, Journal of Molecular Structure: THEOCHEM 285 (1993) 137–142.

[10] I. Gutman, B. Furtula, X. Li, Multicenter Wiener indices and their applications, J. Serb. Chem. Soc. 80 (2015) 1009–1017.

[11] F. Harary, A characterization of block-graphs, Canadian Mathematical Bulletin 6 (1963) 1–6.

[12] E. Howorka, Betweenness in graphs, Abstracts Amer. Math. Soc (1981), *783–06–5:

[13] Kubicka, Ewa and Kubicki, Grzegorz and Oellermann, Ortrud R, Steiner intervals in graphs, Discrete Appl. Math. 81 (1998) 181–190.

[14] M. Knor, R. Škrekovski, A. Tepeh, Mathematical aspects of Wiener index, Ars Mathematica Contemporanea 11 (2016) 327–352.

[15] M. Kovše, Vertex decomposition of Steiner Wiener index and Steiner betweenness centrality, arXiv:1605.00260 [math.CO] 2016.

[16] X. Li, Y. Mao, I. Gutman, The Steiner Wiener index of a graph, Discuss. Math. Graph Theory 36 (2016) 455–465.

[17] X. Li, Y. Mao, I. Gutman, Inverse problem on the Steiner Wiener index, Discuss. Math. Graph Theory 38 (2018) 83–95.

[18] L. Lu, Q. Huang, J. Hou, X. Chen, A sharp lower bound on Steiner Wiener index for trees with given diameter, Discrete Mathematics 341 (2018) 723–731.

[19] Y. Mao, Steiner Distance in Graphs–A Survey, preprint, arXiv:1708.05779 [math.CO] 2017.

[20] Y. Mao, Z. Wang, and I. Gutman, Steiner Wiener index of graph products, Trans. Combin. 5(3) (2016), 39–50.

[21] H. Wiener, Structural determination of paraffin boiling points, Journal of the American Chemical Society 69 (1947) 17–20.

Matjaž Kovše, School of Basic Sciences, IIT Bhubaneswar, Bhubaneswar, India
E-mail address: kovse@iitbbs.ac.in

10
Rasila V A, Department of Mathematics, Cochin University of Science and Technology, Cochin, India.

E-mail address: 17rasilai7@gmail.com

Ambat Vijayakumar, Department of Mathematics, Cochin University of Science and Technology, India.

E-mail address: vambat@gmail.com