GOLDIE RANKS OF PRIMITIVE IDEALS AND INDEXES OF
EQUIVARIANT AZUMAYA ALGEBRAS

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Abstract. Let \( \mathfrak{g} \) be a semisimple Lie algebra. We establish a new relation between the Goldie rank of a primitive ideal \( J \subset U(\mathfrak{g}) \) and the dimension of the corresponding irreducible representation \( V \) of an appropriate finite W-algebra. Namely, we show that \( \text{Grk}(J) \leq \dim V/d_V \), where \( d_V \) is the index of a suitable equivariant Azumaya algebra on a homogeneous space. We also compute \( d_V \) in representation theoretic terms.

1. Introduction

In this paper we find a new relation between the Goldie ranks of primitive ideals and the dimensions of finite dimensional irreducible modules over finite W-algebras.

Let \( \mathfrak{g} \) be a semisimple Lie algebra over \( \mathbb{C} \), and let \( U \) denote its universal enveloping algebra. Recall that by a primitive ideal in \( U \) one means the annihilator of an irreducible module. Let \( J \) be a primitive ideal. Then \( U/J \) is a prime Noetherian algebra and so, by the Goldie theorem (see, e.g., [MR, Chapter 2]), it has the full fraction ring, \( \text{Frac}(U/J) \) that is a matrix algebra over a skew-field. The rank of this matrix algebra is called the \textit{Goldie rank} of \( J \) and is denoted by \( \text{Grk}(J) \). For example, when \( J \) is the annihilator of a finite dimensional irreducible representation, then the Goldie rank is the dimension of that representation. Finding a formula for \( \text{Grk}(J) \) in general is a well-known open problem in Lie representation theory.

Recall that to the primitive ideal \( J \) one can assign a nilpotent orbit in \( \mathfrak{g} \): the unique dense orbit in the subvariety \( V(J) \subset \mathfrak{g} \) defined by \( \text{gr} \ J \). Let \( \mathcal{O} \) denote this orbit. From \( (\mathfrak{g}, \mathcal{O}) \) one can construct an associative algebra known as the finite W-algebra, see [Pr1, L1]. We denote this algebra by \( \mathcal{W} \). According to [L2, Section 1.2], to \( J \) one can assign an irreducible representation of \( \mathcal{W} \) defined up to twisting with an outer automorphism. More precisely, pick \( e \in \mathcal{O} \) and include it into an \( \mathfrak{sl}_2 \)-triple \( (e, h, f) \). Let \( G \) be the simply connected group with Lie algebra \( \mathfrak{g} \). Set \( Q := Z_G(e, h, f) \), it is the reductive part of \( Z_G(e) \). The group \( Q \) acts on \( \mathcal{W} \) by algebra automorphisms, moreover, the action is Hamiltonian meaning that there is a compatible Lie algebra homomorphism (in fact, an inclusion) \( \mathfrak{q} \hookrightarrow \mathcal{W} \). So the component group \( A := Q/Q^o \) acts on the set \( \text{Irr}_{\text{fin}}(\mathcal{W}) \) of isomorphism classes of irreducible representations. The main result of [L2] is that the orbit set \( \text{Irr}_{\text{fin}}(\mathcal{W})/A \) is naturally identified with the set \( \text{Prim}_G(U) \) of primitive ideals in \( U \) corresponding to the orbit \( \mathcal{O} \). The papers [LO, BL] explain how to compute the \( A \)-orbits corresponding to the primitive ideals.

1.1. Known results. In particular, to \( J \) we can assign another numerical invariant, the dimension of the corresponding finite dimensional irreducible representation of \( \mathcal{W} \), denote this representation by \( V \). There is a lot of evidence that \( \dim V \) and \( \text{Grk}(J) \) are closely related. For example, it was shown in [L1] that \( \text{Grk}(J) \leq \dim V \). Premet improved this
result in [Pr2], where he proved that \( \dim V \) is divisible by \( \text{Grk}(\mathcal{J}) \). On the other hand, in \([L5, L6]\) the first named author proved that \( \text{Grk}(\mathcal{J}) = \dim V \) provided \( \mathcal{J} \) has integral central character for all orbits but one in type \( E_8 \). Another main result of \([L5]\) is a Kazhdan-Lusztig type formula for \( \dim V \) (for \( V \) with integral central character). So the equality \( \text{Grk}(\mathcal{J}) = \dim V \) yields a formula for \( \text{Grk}(\mathcal{J}) \).

On the other hand, Premet in [Pr2] found a series of examples of primitive ideals \( \mathcal{J} \), where the Goldie rank is always 1, while the dimension can be arbitrarily large. We will revisit that example in our paper.

1.2. Main result. The main result of this paper is the inequality \( \text{Grk}(\mathcal{J}) \leq \dim V/d_V \), where \( d_V \) is a positive integer determined as follows. Let \( Q_V \) denote the stabilizer of (the isomorphism class of) \( V \) in \( Q \). Since \( Q \) acts on \( \mathcal{W} \) by automorphisms and \( V \) is an irreducible \( \mathcal{W} \)-module, we see that \( V \) is a projective representation of \( Q_V \), let \( \psi \) denote the Schur multiplier. By definition, \( d_V \) is the GCD of the dimensions of the projective representations of \( Q_V \) with Schur multiplier \( \psi \).

Theorem 1.1. We have \( \text{Grk}(\mathcal{J}) \leq \dim V/d_V \).

Conjecture 1.2. We have \( \text{Grk}(\mathcal{J}) = \dim V/d_V \), at least for classical Lie algebras.

A current work in progress of the first named author and Bezrukavnikov should produce Kazhdan-Lusztig type formulas for \( \dim V \) (for \( V \) with an arbitrary central character). Together with Conjecture 1.2 this should give Kazhdan-Lusztig type formulas for Goldie ranks.

1.3. Ideas of proof and structure of the paper. Let us explain how Theorem 1.1 is proved.

The first step is as follows. Let \( H \) denote the preimage of \( Q_V \) under the natural epimorphism \( Z_G(e) \twoheadrightarrow Q \). For a projective \( H \)-module \( V' \) with Schur multiplier \( \psi \), we can form the equivariant Azumaya algebra \( A = G \times^H \text{End}(V') \) on \( G/H \). Our first important result is to show that the index \( \text{ind}(A) \) of \( A \) (we recall the definition of the index in 2.1) coincides with \( d_V \).

The second step is as follows. We can view \( V \) as a projective \( H \)-representation with Schur multiplier \( \psi \), where the action of \( H \) is inflated from \( Q_V \). Our second step is to produce a \( G \)-equivariant sheaf of \( \mathbb{C}[[\hbar]] \)-algebras \( A_\hbar \) on \( G/H \) (that is, in a suitable sense, the microlocalization of the Rees algebra of \( \mathcal{U}/\mathcal{J} \)) that modulo \( \hbar \) reduces to the equivariant Azumaya algebra \( A := G \times^H \text{End}(V) \).

In the third and final step we use the sheaf \( A_\hbar \) to show that \( \text{Grk}(\mathcal{J}) \) cannot exceed the Goldie rank of \( A \) that equals \( \dim V/d_V \).

The paper is organized as follows. In Section 2 we complete step 1 above computing the index of an equivariant Azumaya algebra on a homogeneous space. In Section 3 we complete the proof of Theorem 1.1. Then in Section 4 we present an example revisiting Premet’s example mentioned above.

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2. INDEXES OF EQUIVARIANT AZUMAYA ALGEBRAS

2.1. Main result on indexes. Let \( G \) be a simply connected algebraic group over \( \mathbb{C} \) and \( H \subset G \) an algebraic subgroup. Let \( A \) be a \( G \)-equivariant Azumaya algebra on \( X := G/H \).
We are interested in computing the index \( \text{ind}(A) \). Recall that, for an Azumaya algebra \( A \) on an integral scheme \( X \), by the index of \( A \) we mean \( \sqrt{\dim D} \), where \( D \) is a skew-field such that the specialization \( A_{\mathbb{C}(X)} \) of \( A \) to the generic point of \( X \) is a matrix algebra over \( D \).

Since \( A \) is \( G \)-equivariant, it is the homogeneous vector bundle with fiber \( \text{End}(V) \), where \( V \) is a vector space such that \( H \) acts on \( \text{End}(V) \) by algebra automorphisms. In other words, \( V \) is a projective representation with Schur multiplier, say, \( -\psi \in H^2(G, \mathbb{C}^\times) \). Let \( \text{Rep}^\psi(H) \) denote the category of projective representations of \( H \) with Schur multiplier \( \psi \) and let \( d(\psi) \) denote the GCD of the dimensions of the representations in \( \text{Rep}^\psi(H) \). We also write \( d_V \) for \( d(\psi) \).

The main result of this section is as follows.

**Theorem 2.1.** In the notation above, we have \( \text{ind}(A) = d(\psi) \).

The proof occupies Sections 2.2-2.5. In Section 2.6 we provide some examples of computation that will become relevant for us in Section 4.

### 2.2. \( K \)-theoretic definition of the index.

Let us recall an alternative definition of the index. Let \( A' \) be a central simple algebra over a field \( K \). Let \( \tilde{K} \) be an extension of \( K \) such that the base change \( \tilde{A}' := \tilde{K} \otimes_K A' \) splits (note that we do not require that \( \tilde{K} \) is an algebraic extension). Then we have the base change map \( K_0(A'\text{-mod}) \to K_0(\tilde{A}'\text{-mod}) \). Note that both \( K_0 \) groups are isomorphic to \( \mathbb{Z} \) (if \( A' = \text{Mat}_n(D) \), where \( D \) is a skew-field, then \( A'\text{-mod} \) is equivalent to the category of right vector spaces over \( D \)).

The following lemma is straightforward.

**Lemma 2.2.** The image of \( K_0(A'\text{-mod}) \) in \( K_0(\tilde{A}'\text{-mod}) \cong \mathbb{Z} \) is \( \text{ind}(A')\mathbb{Z} \).

Let us return to the situation when we have an equivariant Azumaya algebra \( A \) over \( G/H \), \( \mathbb{K} = \mathbb{C}(G/H) \) and \( A' = A_{\mathbb{C}(G/H)} \). Let us explain our choice of \( \tilde{K} \). Let \( \pi \) denote the projection \( G \twoheadrightarrow G/H \).

**Lemma 2.3.** The pull-back \( \pi^*A \) (an Azumaya algebra on \( G \)) splits. In particular, for \( \tilde{\mathbb{K}} := \mathbb{C}(G) \), the algebra \( A' \) splits.

**Proof.** The pullback \( \pi^*A \) is the equivariant Azumaya algebra over \( G \) with fiber \( \text{End}(V) \) over 1 in \( G \). So it is the trivial Azumaya algebra. \( \square \)

Consider the categories \( \text{Coh}(G/H, A), \text{Coh}(G, \pi^*A) \) of coherent sheaves of modules over the corresponding Azumaya algebras. We have the pull-back map \( \pi^* : K_0(\text{Coh}(G/H, A)) \to K_0(\text{Coh}(G, \pi^*A)) \). The following statement, that should be thought as a global analog of Lemma 2.2, is an important part of our proof of Theorem 2.1.

**Proposition 2.4.** We have \( K_0(\text{Coh}(G, \pi^*A)) \cong \mathbb{Z} \) and \( \text{im} \pi^* = \text{ind}(A)\mathbb{Z} \).

**Proof.** We have the following commutative diagram, where the horizontal maps are pullbacks and the vertical maps are the specializations to the generic points. As before, we write \( A' := A_{\mathbb{K}} \) and \( \tilde{A}' := \tilde{\mathbb{K}} \otimes_{\mathbb{K}} A' \).

\[
\begin{array}{ccc}
K_0(\text{Coh}(G/H, A)) & \to & K_0(\text{Coh}(G, \pi^*A)) \\
\downarrow & & \downarrow \\
K_0(A'\text{-mod}) & \to & K_0(\tilde{A}'\text{-mod})
\end{array}
\]
The claim of the proposition will immediately follow from Lemma 2.2 once we know that
(1) the map \( K_0(\text{Coh}(G/H, A)) \to K_0(A'\text{-mod}) \) corresponding to the localization to the
generic point is surjective,
(2) and the map \( K_0(\text{Coh}(G, \pi^*A)) \to K_0(A_{\mathbb{K}}\text{-mod}) \) is an isomorphism.

(1) is straightforward. Let us explain why (2) holds. Since the Azumaya algebra \( \pi^*A \) splits,
(2) boils down to showing the map \( K_0(\text{Coh}(G)) \to \mathbb{Z} \) (sending the class of a any vector
bundle to its rank) is an isomorphism. This follows because \( G \) is simply connected as will
be explained after Proposition 2.6.

2.3. Equivariant \( K_0 \)-groups. We can also consider the categories \( \text{Coh}^G(G/H, A) \) and
\( \text{Coh}^G(G, \pi^*A) \) of \( G \)-equivariant sheaves of \( A \)- and \( \pi^*A \)-modules. We still have the pull-back functor \( \pi^* : \text{Coh}^G(G/H, A) \to \text{Coh}^G(G, \pi^*A) \). This gives rise to the pull-back map
on the level of \( K_0 \)-groups that will be denoted by \( \pi^* \) as well. We will need to describe the
image.

**Proposition 2.5.** We have identifications
\[
K_0(\text{Coh}^G(G/H, A)) \xrightarrow{\sim} K_0(\text{Rep}^\psi(H)), K_0(\text{Coh}^G(G, \pi^*A)) \xrightarrow{\sim} \mathbb{Z}
\]
so that the map \( \pi^* \) sends the class of \( \mathbf{U} \in \text{Rep}^\psi(H) \) to \( \dim \mathbf{U} \).

**Proof.** For any algebraic subgroup \( H^1 \subset G \) and any \( G \)-equivariant Azumaya algebra \( A^1 \) on
\( G/H^1 \), we have a category equivalence \( \text{Coh}^G(G/H^1, A^1) \to A^1_{\mathbb{K}}\text{-mod}^{H^1} \), where \( x = H^1 \subset \mathbb{G}^1, A^1_x \) stands for the fiber of \( A^1 \) in \( x \), and the notation \( \text{mod}^{H^1} \) means the category of
\( H^1 \)-equivariant modules. This equivalence is given by taking the fiber at \( x \).

Applying this to \( H^1 := \{1\}, A^1 := \pi^*A \), we get an equivalence
\( \text{Coh}^G(G, \pi^*A) \cong (\pi^*A)_x \text{-mod} \). Recall that \( (\pi^*A)_x = \text{End}(V) \), where \( V \) is a projective representation of \( H \) with Schur multiplier \( -\psi \).

Then \( \text{Vect} \cong \text{End}(V)\text{-mod} \) via \( U \mapsto V \otimes U \).

Similarly, for \( H^1 := H, A^1 := A \), we get \( \text{Coh}^G(G, \pi^*A) \cong A_x\text{-mod}^H \).

And we get the equivalence \( \text{Rep}^\psi(H) \cong A_x\text{-mod}^H \) by \( U \mapsto V \otimes U \) (since the Schur multipliers of \( U \) and \( V \) are opposite, \( V \otimes U \) is a genuine linear representation of \( H \)).

Under the resulting identifications \( \text{Coh}^G(G, \pi^*A) \xrightarrow{\sim} \text{Vect} \) and \( \text{Coh}^G(G/H, A) \xrightarrow{\sim} \text{Rep}^\psi(H) \)
the pull-back functor \( \pi^* : \text{Coh}^G(G/H, A) \to \text{Coh}^G(G, \pi^*A) \) becomes the forgetful functor
\( \text{Rep}^\psi(H) \to \text{Vect} \). The claim of the proposition follows.

2.4. Forgetful map. The goal of this section is to prove the following proposition.

**Proposition 2.6.** Let \( X \) be a smooth algebraic variety and \( G \) be a simply connected algebraic
group acting on \( X \). Let \( A \) be a \( G \)-equivariant Azumaya algebra on \( X \). Then the forgetful
map \( K_0(\text{Coh}^G(X, A)) \to K_0(\text{Coh}(X, A)) \) is surjective.

We remark that for \( A = \mathcal{O}_X \), this is a theorem of Merkurjev, \[Me\] Thm.40). This, in
particular, implies that \( K_0(\text{Coh}(G)) \xrightarrow{\sim} \mathbb{Z} \) via taking the rank (this has been already used
in the proof of Proposition 2.2): indeed, the inverse map is given by the forgetful map
\( \mathbb{Z} \xrightarrow{\sim} K_0(\text{Coh}^G(G)) \to K_0(\text{Coh}(G)) \).

We will reduce the proof of Proposition 2.6 to the case of \( A = \mathcal{O}_X \) using the Severi-Brauer
variety of \( A \). Recall that this is a variety \( \mathbb{SB}_X(A) \) whose \( \mathbb{C} \)-points are pairs \((x, J)\), where
\( x \in X, J \subset A_x \) is a minimal left ideal of \( A_x \). So \( \mathbb{SB}_X(A) \) comes with a natural projection
\( p : \mathbb{SB}_X(A) \to X \) and with the tautological vector bundle \( J \) whose fiber over a point \((x, J)\) is
\( J \). By the very definition, the Azumaya algebra \( p^*A \) splits: \( p^*A = \text{End}_\mathcal{O}(J) \), here we write
for the structure sheaf of $SB_X(A)$. Also, $p : SB_X(A) \to X$ is a projective bundle (that is locally trivial in the étale topology). In particular, $SB_X(A)$ is smooth.

Now if $G$ is an algebraic group acting on $X$ and $A$ is $G$-equivariant, then we have a natural action of $G$ on $SB_X(A)$, $p$ is $G$-equivariant and $J$ is a $G$-equivariant vector bundle on $SB_X(A)$.

**Proof of Proposition 2.6.** To simplify the notation, let us write $S$ for $SB_X(A)$. It is well-known that $K_0(\text{Coh}(X, A))$ splits as a direct summand of $K_0(\text{Coh}(S))$, [Pa1] Remark 3.3, Examples 3.6(c), [Pa2].

Let us recall how this works. Let us produce maps $\alpha : K_0(\text{Coh}(X, A)) \cong K_0(\text{Coh}(S)) : \beta$ with $\beta \circ \alpha = \text{id}$. Namely, $\alpha$ is induced by the (exact) functor $F \mapsto J^* \otimes_{p^*A}^L p^* F : \text{Coh}(X, A) \to \text{Coh}(S)$, while $\beta$ is induced by $G \mapsto Rp_*(J \otimes_{O_S} G) : D^b(\text{Coh}(S)) \to D^b(\text{Coh}(X, A))$. Now observe that the composition $D^b(\text{Coh}(X, A)) \to D^b(\text{Coh}(S)) \to D^b(\text{Coh}(X, A))$ is isomorphic to the identity functor. Indeed, since $p^*A = J \otimes_{O_S} J^*$, the composition is

$$Rp_*(J \otimes_{O_S} J^* \otimes_{p^*A} p^*(\bullet)) = Rp_*(p^*(\bullet)) = Rp_*(p^*O_X) \otimes_{O_X} \bullet,$$

where the last equality is the projection formula. Of course, $p^*O_X = O_S$. Since $p : S \to X$ is a projective bundle, we see that $Rp_*(O_S) = O_X$. This proves that $\beta \circ \alpha = \text{id}$.

Similarly, we have maps $\alpha_G : K_0(\text{Coh}^G(X, A)) \cong K_0(\text{Coh}^G(S)) : \beta_G$ with $\beta_G \circ \alpha_G = \text{id}$. Note that the forgetful maps $K_0(\text{Coh}^G(X, A)) \to K_0(\text{Coh}(X, A))$ and $K_0(\text{Coh}^G(S)) \to K_0(\text{Coh}(S))$ intertwine $\alpha_G$ with $\alpha$ and $\beta_G$ with $\beta$. The forgetful map $K_0(\text{Coh}^G(S)) \to K_0(\text{Coh}(S))$ is surjective as was explained after the statement of the proposition. Being a retraction of a surjective map, the forgetful map $K_0(\text{Coh}^G(X, A)) \to K_0(\text{Coh}(X, A))$ is surjective as well. □

### 2.5. Completion of the proof.

Let us finish the proof.

**Proof of Theorem 2.7.** We have the following commutative diagram, where the vertical maps $F_{G/H}, F_G$ are the forgetful ones.

$$\begin{array}{cccc}
K_0(\text{Rep}^{-\psi}(H)) & \cong & K_0(\text{Coh}^G(G/H, A)) & \pi^* & K_0(\text{Coh}^G(G, \pi^*A)) \\
\downarrow F_{G/H} & & \downarrow F_G & \cong & \downarrow \text{id} \\
K_0(\text{Coh}(G/H, A)) & & \pi^* & K_0(\text{Coh}(G, \pi^*A)) & \cong & \mathbb{Z}
\end{array}$$

First, let us show that $F_G : K_0(\text{Coh}^G(G, \pi^*A)) \to K_0(\text{Coh}(G, \pi^*A))$ is an isomorphism. Indeed, since $\pi^*A$ splits in a $G$-equivariant way, we reduce to showing that the forgetful map $K_0(\text{Coh}^G(G)) \to K_0(\text{Coh}(G))$ is an isomorphism. This was established after Proposition 2.6. The isomorphism sends the class of any vector bundle to its rank. So $F_G$ is an isomorphism as well.

By Proposition 2.6, $F_{G/H}$ is surjective. So the image of $K_0(\text{Coh}(G/H, A))$ in $K_0(\text{Coh}(G, A)) = \mathbb{Z}$ coincides with the image of $K_0(\text{Rep}^{-\psi}(H))$ in $K_0(\text{Coh}^G(G, A)) = \mathbb{Z}$. The latter is $d(\psi)\mathbb{Z}$ by Proposition 2.5. The former is $\text{ind}(A)\mathbb{Z}$ by Proposition 2.4. The equality $\text{ind}(A) = d(\psi)$ follows. □

### 2.6. Examples of computations of $d_V$.

Let us provide two examples that will be relevant for what follows.
Example 2.7. Let $H = \text{SO}_{2n+1}$ and $V$ be the spinor representation (of dimension $2^n$), which is the irreducible $\mathfrak{h}$-module with highest weight $\omega_n = \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_n)$ (in the standard notation). We claim that the dimension of any $H$-module with the same Schur multiplier is divisible by $2^n$ (and so $d_V = 2^n$). This is equivalent to saying that the dimension of any irreducible $\mathfrak{h}$-module $V(\lambda)$ such that $\lambda - \omega_n$ is in the root lattice is divisible by $2^n$. This will follow if we check that the cardinalities of the Weyl group orbits of dominant weights $\lambda$ with $\lambda - \omega_n$ in the root lattice are divisible by $2^n$. We have $\lambda = \sum_{i=1}^{n} (m_i + \frac{1}{2}) \epsilon_i$, where $m_1 \geq m_2 \geq \ldots \geq m_n \geq 0$ are integers. The stabilizer $W_{\lambda'}$ is included into $S_n \subset \hat{W}$ so $|W_{\lambda'}|$ is indeed divisible by $|W|/|S_n| = 2^n$.

Example 2.8. Let $H = \text{SO}_{2n}$ and $V$ be one of the half-spinor representations (of dimension $2^{n-1}$). Similarly to the previous example, we see that $d_V = 2^{n-1}$.

Note that, in general, it is not true that $d_V = \dim V$ for a minuscule representation of $\mathfrak{h}$. For example, for $\mathfrak{h} = \mathfrak{sl}_n$, the dimension of $S^2(\mathbb{C}^n)$ is not divisible by that of the corresponding minuscule representation $\Lambda^2(\mathbb{C}^n)$.

3. Inequality on Goldie ranks

3.1. W-algebras. Let us recall some results about W-algebras, see [Pr1] [L2].

Let $G$ be a simply connected semisimple algebraic group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{O} \subset \mathfrak{g}$ a nilpotent orbit, $e \in \mathfrak{O}$. We will write $\hat{H}$ for $Z_G(e)$ and $Q$ for $Z_G(e, h, f)$.

Pick an $\mathfrak{sl}_2$-triple $(e, h, f)$. Set $S = e + \mathfrak{z}_\mathfrak{g}(f)$, this is a transverse slice to $\mathfrak{O}$. The group $Q$ naturally acts on $S$. Also we have a $\mathbb{C}^\times$-action on $S$. Namely, we introduce a grading on $\mathfrak{g}$, $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ by eigenvalues of $[h, \cdot]$. We define a $\mathbb{C}^\times$-action on $\mathfrak{g}$ by $t.x := t^{-i/2}x$ for $x \in \mathfrak{g}(i)$. Clearly, the action fixes $S$.

A finite W-algebra $W$ as constructed by Premet in [Pr1] (see also [L1] for an equivalent alternative definition) is a filtered quantization of the graded Poisson algebra $\mathbb{C}[S]$, i.e., $W$ is a $\mathbb{Z}_{>0}$-filtered associative algebra and we have an isomorphism $\text{gr } W \cong \mathbb{C}[S]$ of graded Poisson algebras. The group $Q$ acts on $W$ by filtered algebra automorphisms and this action is Hamiltonian: we have a $Q$-equivariant inclusion $\mathfrak{q} \hookrightarrow W$ such that the adjoint action of $\mathfrak{q}$ on $W$ coincides with the differential of the $Q$-action.

In what follows we will need an isomorphism of completions from $[L_1, L_2]$ that connects $\mathcal{U}$ and $W$. Namely, consider the Rees algebra $\mathcal{U}_h$ of $\mathcal{U}$ with $\deg \mathfrak{g} = 2$. Let $\chi \in \mathfrak{g}^*$ be the image of $e$ under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ coming from the Killing form. We can view $\chi$ as a homomorphism $\mathcal{U}_h \to \mathbb{C}$ and consider the completion $\mathcal{U}_h^{\chi} := \varprojlim_{n \to \infty} \mathcal{U}_h/(\ker \chi)^n$. This is a complete and separated topological $\mathbb{C}[[h]]$-algebra that is flat over $\mathbb{C}[[h]]$. Note that it carries an action of $H$ by algebra automorphisms and this action is Hamiltonian in the sense that the differential of the $H$-action coincides with the map $x \mapsto h^{-2}[x, \cdot] : \mathfrak{h} \to \text{Der}(\mathcal{U}_h^{\chi})$. There is also a $\mathbb{C}^\times$-action on $\mathcal{U}_h^{\chi}$ coming from the grading on $\mathfrak{g}$ with $\deg \mathfrak{g}(i) = i + 2$.

Then there is a $Q \times \mathbb{C}^\times$-equivariant decomposition

$$U_h^{\chi} \cong A_h^{\Lambda_0} \otimes_{\mathbb{C}[I]} W_h^{\chi},$$

where the notation is as follows. We consider the vector space $[\mathfrak{g}, f]$. This space is symplectic with form $\omega(v_1, v_2) := \langle \chi, [v_1, v_2] \rangle$ so we can form its Weyl algebra $A$. Let $A_h$ denote the Rees algebra of $A$, and let $A_h^{\Lambda_0}$ be the completion of $A_h$ at 0. Similarly, $W_h$ is the Rees algebra of $W$ and $W_h^{\chi}$ is the completion of the former. The action of $Q \times \mathbb{C}^\times$ on the right hand side of (1) is diagonal, we will not need a precise description of the action on $A_h^{\Lambda_0}$.
This construction was used in [L2] to establish the bijection \( \text{Prim}_G(U) \cong \text{Irr}_{\text{fin}}(W)/A \). Namely, let \( J \) be a two-sided ideal in \( U \). We can consider the corresponding Rees ideal \( J_h \) in \( U_h \) and its closure \( J_h^{\wedge} \) in \( U_h^{\wedge} \). Then there is a unique two-sided ideal \( J^+_1 \subset W \) such that
\[
J_h^{\wedge} \cong \mathcal{A}_h^{\wedge} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} J_{1,h}^{\wedge}
\]
If \( \overline{O} \) is an irreducible component of the associated variety of \( J \), then \( J^+_1 \) has finite codimension. If \( J \) is, in addition, primitive, then \( J^+_1 \) is a maximal \( Q \)-stable ideal of finite codimension in \( W \). Such ideals are in a natural one-to-one correspondence with the \( A \)-orbits in \( \text{Irr}_{\text{fin}}(W) \). This gives rise to a bijection \( \text{Prim}_G(U) \sim \sim \text{Irr}_{\text{fin}}(W)/A \) mentioned in the introduction.

Conversely, to a two-sided ideal \( I \subset W \) we can assign a two-sided ideal \( I^+_1 \subset W \): the maximal two-sided ideal in \( U \) with the property that \( (I^+_1)_I \subset I \), see [L1, Section 3.4].

3.2. Jet bundles. Here we will explain various constructions of jet bundles to be used in the construction of the sheaf of algebras \( \mathcal{A}_h \) on \( G/H \) in the next section. Here the meaning of \( H \) is the same as in the introduction: \( H \) is a finite index subgroup of \( \overline{H} \).

Let us start with the usual jet bundle \( J^\infty O_X \) of a smooth algebraic variety \( X \). Let \( I_\Delta \) be the sheaf of ideals of the diagonal in \( X \times X \). Then, by definition, \( J^\infty O_X \) is the formal completion of \( O_{X \times X} \) with respect to \( I_\Delta \). We view \( J^\infty O_X \) as a pro-coherent sheaf on \( X \) with respect to the projection \( p_1 : X \times X \to X \) to the first copy. Note that the fiber of \( J^\infty O_X \) over \( x \in X \) is the formal completion of the stalk \( O_{X,x} \). The sheaf \( J^\infty O_X \) comes with a natural flat connection \( \nabla \), whose flat sections are again identified with \( O_X \). This construction trivially extends to the jet bundle \( J^\infty A \) of an Azumaya algebra \( A \) on \( X \). We note that we have a homomorphism of sheaves of algebras with flat connection \( J^\infty O_X \to J^\infty A \).

Now suppose that \( X \) is a symplectic variety that comes with a Hamiltonian action of \( G \). We are going to define a pro-coherent sheaf \( J^\infty U_{h,X} \) on \( X \) whose fiber at \( x \in X \) is the completion of \( U_h \) at \( \mu(x) \), where \( \mu : X \to g^* \) is the moment map. Namely, note that we have a \( G \)-equivariant homomorphism of sheaves \( O_X \otimes g \to O_X \otimes \mathbb{C}[X], f \otimes \xi \mapsto f \otimes \mu^*(\xi) \).

It extends to a homomorphism of sheaves of algebras \( O_X \otimes U_h \to O_X \otimes \mathbb{C}[X] \) (that sends \( h \) to \( 0 \)). Let \( I_{\mu,\Delta} \subset O_X \otimes U_h \) denote the pre-image of the ideal sheaf of the diagonal. We set \( J^\infty U_{h,X} := \varprojlim_{n \to \infty} O_X \otimes \mathcal{U}_h/I_{\mu,\Delta} \). This sheaf again carries a natural flat connection \( \nabla \) extended by continuity from the trivial connection on \( O_X \otimes U_h \). Note that \( \nabla \) satisfies the following identity, where \( \xi_X \) stands for the velocity vector field induced by \( \xi \in g \):

\[
\nabla_{\xi_X} = \xi_{J^\infty U_h} - \frac{1}{\hbar^2}[\xi, \bullet].
\]

On the right hand side, the notation is as follows. We write \( \xi_{J^\infty U_h} \) for the derivative of the \( G \)-equivariant sheaf \( J^\infty U_h \) induced by \( \xi \). In the bracket \( \xi \) is viewed as an element of \( U_h \hookrightarrow O_X \otimes U_h \). The reason why (2) holds is that it holds on \( O_X \otimes U_h \) for trivial reasons and then extends to \( J^\infty U_h \) by continuity.

Furthermore, we have a homomorphism of sheaves of algebras with flat connections \( J^\infty U_{h,X} \to J^\infty O_X \) (induced from \( O_X \otimes U_h \to O_X \otimes \mathbb{C}[X] \)). Note that \( J^\infty U_{h,X} \to J^\infty O_X \) when \( X \) is a homogeneous \( G \)-space. We also observe that the fiber of \( J^\infty U_h \) at \( x \in X \) is the completion \( U_h^{\wedge,x} \).

We will apply these constructions in the case when \( X = G/H \), where \( H \) is a finite index subgroup of \( \overline{H} = Z_G(e) \). Here \( J^\infty U_h \) is the \( G \)-homogeneous pro-vector bundle whose fiber at \( 1H \) is \( U_h^{\wedge,x} \).
3.3. Sheaves $\mathfrak{A}_h, \mathcal{A}_h$. Let us make a remark regarding finite dimensional representations of $\mathcal{W}$. Let $V$ be a finite dimensional $\mathcal{W}$-module that also comes with a projective representation of a finite index subgroup $Q_V \subset Q$ whose differential is the linear representation of $\mathfrak{q}$ that comes from restricting the $\mathcal{W}$-action to $\mathfrak{q}$. Let $H$ be the preimage of $Q_V$ in $\tilde{H}$.

Take the trivial filtration on $V$ and form the Rees $\mathbb{C}[h]$-module $V_h$ and its completion $V_h^{\wedge \alpha}$. Then $\mathcal{A}_h^{\wedge \alpha} := \text{End}_{\mathbb{C}[h]}(V_h^{\wedge \alpha})$ is an algebra that comes with a $Q_V \times \mathbb{C}^\times$-action by automorphisms and with a $Q_V \times \mathbb{C}^\times$-equivariant $\mathbb{C}[h]$-algebra homomorphism $\mathcal{W}_h^{\wedge \alpha} \to \mathcal{A}_h^{\wedge \alpha}$.

Set $\mathcal{A}_h^{\wedge \times} := \mathcal{A}_h^{\wedge 0} \otimes_{\mathbb{C}[h]} \mathcal{A}_h^{\wedge 0}$. This algebra carries a $Q_V \times \mathbb{C}^\times$-action by automorphisms and, thanks to $\mathfrak{q}$, comes with a $Q_V \times \mathbb{C}^\times$-equivariant $\mathbb{C}[h]$-algebra homomorphism $\mathcal{U}_h^{\wedge \times} \to \mathcal{A}_h^{\wedge \times}$. This homomorphism allows to extend a $Q_V$-action on $\mathcal{A}_h^{\wedge \times}$ to an $H$-action because the action of $H$ on $\mathcal{U}_h^{\wedge \times}$ is Hamiltonian.

So we can form a $G$-homogeneous pro-vector bundle $\mathfrak{A}_h$ on $G/H$ with fiber $\mathcal{A}_h^{\wedge \times}$. It comes with a flat connection $\nabla$ given by a formula similar to (2), namely:

$$\nabla_{\xi} = \xi_{\mathfrak{A}_h} - \frac{1}{\hbar^2} [\xi, \bullet].$$

So we get a $G \times \mathbb{C}^\times$-equivariant homomorphism of $\mathbb{C}[[h]]$-algebras $\mathcal{J}^{\infty} \mathcal{U}_h \to \mathfrak{A}_h$ that intertwines the flat connections.

Now let us describe the bundle $\mathfrak{A}_h/(h)$ with a flat connection. Consider the equivariant Azumaya algebra $A := G \times^H \mathfrak{A}_h$, where $\mathfrak{A}_h := \mathfrak{A}_h/(h) (= \text{End}(V))$.

**Lemma 3.1.** The sheaf of algebras $\mathfrak{A} := \mathfrak{A}_h/(h)$ with a flat connection is isomorphic to $\mathcal{J}^{\infty} A$.

**Proof.** The connection on $\mathfrak{A}$ is induced from $\mathcal{J}^{\infty} S(\mathfrak{g}) = \mathcal{J}^{\infty} \mathcal{U}_h/(h)$. On $\mathcal{J}^{\infty} S(\mathfrak{g})$ the connection is given by

$$\nabla_{\xi} = \xi_{\mathcal{J}^{\infty} S(\mathfrak{g})} - \{\xi, \bullet\}.$$

Note that $\mathbb{C}[S]^\times \to \mathfrak{A}_h$ factors through the residue field of $\mathbb{C}[S]^\times$. Also note that we have a Poisson bracket map $Z(\mathfrak{A}) \otimes \mathfrak{A} \to \mathfrak{A}$ thanks to the presence of the deformation $\mathfrak{A}_h$. The image of $\mathcal{J}^{\infty} S(\mathfrak{g}) \to \mathfrak{A}$ lies in the center of $\mathfrak{A}$. So the connection on $\mathfrak{A}$ is also given by

$$\nabla_{\xi} = \xi_{\mathfrak{A}} - \{\xi, \bullet\}.$$

Set $T := [\mathfrak{g}, f]$, recall that this is a symplectic vector space. Now we observe that $\mathfrak{A} = G \times^H (\mathbb{C}[[T]] \otimes \mathfrak{A}_h)$, where the action of $H$ on $\mathbb{C}[[T]] \otimes \mathfrak{A}_h$ is as follows: the action of $Q_V$ is diagonal, while the action of the Lie algebra $\mathfrak{h}$ comes from $\mathbb{C}[\mathfrak{g}]^\times \to \mathbb{C}[[T]]$. The connection is given by (3). But the sheaf of algebras with a flat connection $\mathcal{J}^A H$ has the same description. This gives a required isomorphism $\mathfrak{A} \cong \mathcal{J}^{\infty} A$. \(\square\)

So we see that $\mathfrak{A}_h$ is a formal deformation of $\mathcal{J}^A H$. Set $\mathcal{A}_h = \mathfrak{A}_h^{\wedge}$, the sheaf of flat sections. This is a $G$-equivariant sheaf of $\mathbb{C}[[h]]$-algebras on $G/H$ (but not a sheaf of $\mathcal{O}_{G/H}$-modules).

**Proposition 3.2.** The sheaf $\mathcal{A}_h$ is a formal deformation on $A$.

**Proof.** It follows from the construction that $\mathcal{A}_h$ is flat over $\mathbb{C}[[h]]$ and complete and separated in the $h$-adic topology. So we need to check that $\mathcal{A}_h/(h) = A$. Note that, since taking the flat sections is a left exact functor, we have $\mathcal{A}_h/(h) \to A$.

Since $\mathcal{A}_h$ is $G$-equivariant, the isomorphism $\mathcal{A}_h/(h) \cong A$ will follow if we check that

$$\mathcal{A}_h^{\wedge H}/(h) \cong \mathbb{C}[[T]] \otimes \mathfrak{A}_h.$$
Note that the right hand side is the restriction of $A$ to the formal neighborhood of $1H$. Again, we have $\mathcal{A}^{\rho \lambda}_{h} / (\mathfrak{h}) \hookrightarrow \mathcal{C}[T] \otimes \mathcal{A}_{h}$. Now note that $\mathcal{U}_{h}$ naturally maps to $\Gamma(\mathcal{A}_{h})$ that gives rise to a homomorphism $\mathcal{U}_{h}^{\times} \to \mathcal{A}_{h}^{\times \lambda}$. In particular, we get a homomorphism $\mathcal{A}_{h}^{\rho \lambda} \to \mathcal{A}_{h}^{\rho \lambda}$. It is injective. This reduces the proof of (6) to checking that the rank of the centralizer of $\mathcal{A}_{h}^{\rho \lambda}$ in $\mathcal{A}_{h}^{\rho \lambda}$ equals $\dim \mathcal{A}$. Note that the kernel of $\mathcal{U}_{h} \to \Gamma(\mathcal{A}_{h})$ is $\mathcal{J}_{h}$. So the homomorphism $\mathcal{W}_{h}^{\times} \to \mathcal{A}_{h}$ factors through $\mathcal{W}_{h}^{\times} / \mathcal{T}_{h}^{\times}$. The latter is the direct sum of matrix algebras over $\mathbb{C}[\mathfrak{h}]$, each of rank $\dim \mathcal{A}$. This implies our claim about the rank of the centralizer and finishes the proof of the proposition. \(\square\)

3.4. Inequality for Goldie ranks. The following proposition together with Theorem 2.1 prove Theorem 1.1.

**Proposition 3.3.** We have $\text{Grk}(\mathcal{J}) \leq \dim V / \text{ind}(A)$.

**Proof.** The algebra $\mathcal{U}_{h} / \mathcal{J}_{h}$ is prime and Noetherian, because $U / \mathcal{J}$ is so. Note that the Goldie ranks of $\mathcal{U} / \mathcal{J}$ and $\mathcal{U}_{h} / \mathcal{J}_{h}$ coincide (because $(\mathcal{U}_{h} / \mathcal{J}_{h})[h^{-1}] = (U / \mathcal{J})[h^{\pm 1}]$). For any affine open subset $U \subset G / H$, we have an inclusion $\mathcal{U}_{h} / \mathcal{J}_{h} \subset \Gamma(U, \mathcal{A}_{h})$. Further, $\Gamma(U, \mathcal{A}_{h}) / (\mathfrak{h}) = \Gamma(U, A)$. The algebra $\Gamma(U, A)$ is an Azumaya $\mathbb{C}[U]$-algebra hence is Noetherian and prime. As a formal deformation of a prime Noetherian algebra, the algebra $\Gamma(U, \mathcal{A}_{h})$ is prime and Noetherian as well. By a result of Warfield, [W, Theorem 1], the inclusion $\mathcal{U}_{h} / \mathcal{J}_{h} \subset \Gamma(U, \mathcal{A}_{h})$ implies the inequality of Goldie ranks $\text{Grk}(\mathcal{U}_{h} / \mathcal{J}_{h}) \leq \text{Grk}(\Gamma(U, \mathcal{A}_{h}))$. So it remains to show that we can choose $U$ so that $\text{Grk}(\Gamma(U, \mathcal{A}_{h})) = \dim V / \text{ind}(A)$, which is the Goldie rank of $\Gamma(U, A)$.

We can pick $U$ in such a way that $\Gamma(U, A) = \text{Mat}_{k}(D)$, where $D$ is a domain, and $k = \dim V / \text{ind}(A)$. Since $\Gamma(U, \mathcal{A}_{h})$ is a formal deformation of $\text{Mat}_{k}(D)$, the inclusion $\text{Mat}_{k}(\mathbb{C}) \hookrightarrow \text{Mat}_{k}(\mathcal{A}_{h})$ lifts to $\text{Mat}_{k}(\mathbb{C}) \hookrightarrow \Gamma(U, \mathcal{A}_{h})$. The centralizer $D_{h}$ of $\text{Mat}_{k}(\mathbb{C})$ in $\Gamma(U, \mathcal{A}_{h})$ is the formal deformation of $D$. So $\Gamma(U, \mathcal{A}_{h}) = \text{Mat}_{k}(D_{h})$ and the Goldie rank equals $k$. This finishes the proof of $\text{Grk}(\mathcal{J}) \leq \dim V / \text{ind}(A)$. \(\square\)

4. Example of computation

Here we will use Theorem 1.1 to revisit an example of a completely prime primitive ideal in [Pr2] that corresponds to an irreducible representation of $\mathcal{W}$ with large dimension.

4.1. Main result. Let us start by recalling a classical theorem of Duflo.

Let us write $t$ for a Cartan subalgebra of $\mathfrak{g}$. For $\lambda \in t^{*}$ let $L(\lambda)$ denote the irreducible module in the usual BGG category $\mathcal{O}$ with highest weight $\lambda - \rho$, where $\rho$ is half the sum of the positive roots. Let $\mathcal{J}(\lambda)$ denote the annihilator of $L(\lambda)$ in $\mathcal{U}$. As Duflo proved, the ideals $\mathcal{J}(\lambda)$ exhaust the primitive ideals of $\mathcal{U}$.

Here is the main result of this section.

Let $\mathfrak{g} = \mathfrak{sp}^{\ast}_{2n}$. We consider the primitive ideal $\mathcal{J} := \mathcal{J}(\rho / 2)$ in $\mathcal{U}$. It follows from results of McGovern that $\text{Grk}(\mathcal{J}) = 1$, while the codimension of the corresponding ideal $\mathcal{I}$ in $\mathcal{W}$ equals $2^{n-1}$, see [McG]. We will independently show that the corresponding representation of $\mathcal{W}$ restricts to a spinor representation of $\mathfrak{g}$, hence the inequality in Theorem 1.1 becomes an equality. From here we will deduce the equality $\text{Grk}(\mathcal{J}) = 1$.

Here is a completed statement.

**Proposition 4.1.** Let $\mathfrak{g} = \mathfrak{sp}^{\ast}_{2n}$ with $n > 2$. Set $\mathcal{J} = \mathcal{J}(\rho / 2)$. Then the following are true:

1. For $n = 2m$, the $A$-orbit in $\text{Irr}_{fin}(\mathcal{W})$ corresponding to $\mathcal{J}$ consists of 2 irreducible representations. Their restrictions to $\mathfrak{q} = \mathfrak{so}_{n}$ are the (non-isomorphic) half-spinor representations.
(2) For \( n = 2m + 1 \), the \( A \)-orbit in \( \text{Irr}_{\text{fin}}(W) \) corresponding to \( \mathcal{J} \) consists a single irreducible representation. Its restriction to \( q = \mathfrak{so}_n \) is the spinor representation.

(3) \( \text{Grk}(\mathcal{J}) = 1 \).

4.2. Category \( \mathcal{O} \) for \( W \). Let \( T_Q \subset Q \) denote the maximal torus and let \( \nu : \mathbb{C}^\times \to Q \) be a one-parameter subgroup that is generic in the sense that its centralizer in \( g \) coincides with the centralizer of \( T_Q \). The one-parameter subgroup \( \nu \) gives rise to the weight decomposition \( W = \bigoplus_{i \in \mathbb{Z}} W_i \). Set \( W_{\geq 0} := \bigoplus_{i \geq 0} W_i, C_{\nu}(W) := W_{\geq 0}/(W_{\geq 0} \cap W_{\geq 0}) \).

Following \( \text{BGK} \), define the category \( \mathcal{O}_\nu(W) \) as the full subcategory in the category of finitely generated \( W \)-modules consisting of all modules \( M \) such that

1. \( M \) admits a weight decomposition with respect to \( t \): \( M = \bigoplus_{\alpha \in \mathfrak{t}^*} M_\alpha \), where \( M_\alpha := \{ m \in M | x m = \langle \alpha, x \rangle m \} \).
2. \( W_{>0} \) acts on \( M \) locally nilpotently.

In particular, any finite dimensional irreducible \( W \)-module lies in \( \mathcal{O} \). We remark that, modulo (1), (2) can be shown to be equivalent to the condition that the real numbers \( \text{Re}(\alpha, \nu) \) are bounded from above.

Note that for \( M \in W \)-mod, the annihilator \( M^{W_{>0}} \) is \( W_{\geq 0} \)-stable and the action of \( W_{>0} \) on \( M^{W_{>0}} \) factors through \( C_{\nu}(W) \). It turns out that if \( M \) is simple in \( \mathcal{O}_\nu(W) \), then \( M^{C_{\nu}(W)} \) is a simple \( C_{\nu}(W) \)-module. The assignment \( M \mapsto M^{W_{>0}} \) is a bijection between the simples in \( \mathcal{O}_\nu(W) \) and the simple \( C_{\nu}(W) \)-modules. For an irreducible \( C_{\nu}(W) \)-module \( N \), we write \( L_\nu(N) \) for the corresponding simple object in \( \mathcal{O}_\nu(W) \).

The structure of the algebra \( C_{\nu}(W) \) was determined in \( \text{BGK} \), see also \( \text{[L3]} \). Let \( \mathfrak{g} \) denote the centralizer of \( \nu \) in \( g \), note that \( e \in \mathfrak{g} \). Let \( \mathcal{U} \) denote \( U(\mathfrak{g}) \) and \( \mathcal{W} \) denote the \( W \)-algebra for \( (\mathfrak{g}, e) \). Then there is an isomorphism \( C_{\nu}(W) \cong \mathcal{W} \), see \( \text{[BGK]} \) Section 4. This isomorphism is \( T_Q \)-equivariant but it does not intertwine the quantum comoment maps \( t \mapsto C_{\nu}(W), \mathcal{W} \). Rather it induces a shift by a character \( \delta \) that we define now. Pick a Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{g} \) that contains \( t \) and \( h \). Let \( \Delta \subset \mathfrak{t}^* \) be the root system. Set \( \Delta^- := \{ \beta \in \Delta | \langle \beta, \nu \rangle < 0 \} \).

We set

\[
\delta = \frac{1}{2} \sum_{\alpha \in \Delta^-; \langle \alpha, h \rangle = -1} \alpha + \sum_{\alpha \in \Delta^-; \langle \alpha, h \rangle \leq -2} \alpha.
\]

Then the isomorphism \( C_{\nu}(W) \cong \mathcal{W} \) restricts to \( x \mapsto x - \langle \delta, x \rangle t \) on \( \mathfrak{t} \).

Now pick a finite dimensional irreducible \( W \)-module \( V \). It lies in \( \mathcal{O}_\nu(W) \) and so there is a unique finite dimensional irreducible \( W \)-module \( W \)-module \( V \) with \( V = L_\nu(V) \). Let \( V \) correspond to the primitive ideal \( \mathcal{J}(\lambda) \subset \mathcal{U} \) (annihilating the irreducible \( \mathfrak{g} \)-module with highest weight \( \lambda - \rho \)). It was shown in \( \text{[L3]} \) Theorem 5.1.1 that the primitive ideal in \( W \) corresponding to \( V \) is \( \mathcal{J}(\lambda) \).

While we cannot read the Schur multiplier for the \( Q \)-action on \( V \), we can recover that for the \( \mathcal{O} \)-action. Namely, \( t_Q \) embeds into the center of \( \mathcal{W} \). The subalgebra \( t_Q \subset \mathcal{W} \) acts on \( V \) by \( (\lambda - \rho)|_{t_Q} \). Then the character of the action of \( t_Q \subset \mathcal{W} \) on \( V = V^{W_{>0}} \) is

\[
(\lambda - \rho - \delta)|_{t_Q}.
\]

It follows that the weights of \( t \) in \( V \) are congruent to \( (\lambda - \delta)|_{t_Q} \). This allows to recover the Schur multiplier for the projective action of \( Q \) on \( V \).

4.3. Proof of Proposition 4.1. Let \( \mathfrak{g} = \mathfrak{sp}_{2n} \). We consider the orbit \( \mathcal{O} \) corresponding to the partition \( (2^n) \) (where the superscript indicates the multiplicity). The group \( Q \) is \( O_n \). It is easy to see that the codimension of \( \mathcal{O} \) in \( \mathfrak{g} \) is \( n^2 \).
Proof. The proof is in several steps.

Step 1. It is easy to see that \( \dim V(J) = \dim \mathcal{O} \). Indeed, the integral Weyl group for \( \rho/2 \) has type \( B_{[n/2]} \times D_{[n/2]} \). For the integral root system, \( \rho/2 \) is integral and dominant. By [1] Corollary 3.5, this implies that \( \dim V(J) = \dim \mathfrak{g} - \dim \mathfrak{g}^\vee_{\text{int}} \), where \( \mathfrak{g}^\vee_{\text{int}} \) stands for the integral subalgebra for \( \rho/2 \) in \( \mathfrak{g}^\vee = \mathfrak{so}_{2n+1} \). Since \( \dim \mathfrak{g}^\vee_{\text{int}} = n^2 \), the claim in the beginning of the paragraph follows.

Step 2. The symplectic form we use to define \( \mathfrak{sp}_{2n} \) is \( \omega(u, v) = \sum_{i=1}^n (u_i v_{2n+1-i} - v_i u_{2n+1-i}) \). Identify \( t \) with \( \mathbb{C}^n \) in a standard way, \( t = \text{diag}(x_1, \ldots, x_n, -x_n, \ldots, -x_1) \). When \( n = 2m \), \( t_Q \) is embedded into \( t \) as

\[
\{(x_1, x_1, x_2, x_2, \ldots, x_m, x_m)\}.
\]

while for \( n = 2m+1 \), we get

\[
t_Q = \{(x_1, x_1, \ldots, x_m, x_m, 0)\}.
\]

We choose \( \nu \) corresponding to the vector \( (x_1, \ldots, x_m) \in \mathbb{Z}^m \) with \( x_1 > x_2 > \ldots > x_m > 0 \), it is dominant for \( \mathfrak{g} \).

Step 3. The subalgebra \( \mathfrak{g} = \mathfrak{z}_\mathfrak{g}(\nu) \) is \( \mathfrak{sl}_n^\mathfrak{g} \) and \( e \) is a principal nilpotent element in \( \mathfrak{g} \). The W-algebra \( \mathcal{W} \) is, therefore, the center of \( \mathcal{U} \). Consider the primitive ideal \( \mathcal{J}(\rho/2) \) for \( \mathfrak{g} \). It is minimal, so it gives rise to an irreducible representation of \( \mathcal{W} \), to be denoted by \( \overline{V} \). Set\n\[
V = L_{\nu}(\overline{V}).
\]
By [L3] Theorem 5.1.1, \( \mathcal{J}(\rho/2) = \text{Ann}_\mathcal{W}(\overline{V}) \). It follows that \( \mathcal{O} \subset V(J) \) and, since the codimensions of \( \mathcal{O} \) and \( V(J) \) in \( \mathfrak{g} \) coincide (both are equal to \( n^2 \)), we recover the equality \( V(J) = \mathcal{O} \). Equivalently, \( V \) is finite dimensional.

Step 4. Let us determine how \( t_Q \) acts on \( \overline{V} \). According to (8), the action is by the character \( (-\rho/2 - \delta)|_{t_Q} \). Below in this step, we will compute this character explicitly.

Note that \( e \) is an even nilpotent element so \( \delta = \sum_{\alpha \in \Delta_- \setminus \Delta_0} \epsilon_{\alpha} \leq -2 \alpha \). Note that \( \delta|_{t_Q} = \frac{1}{2}(\sum_{\alpha \in \Delta_- \setminus \Delta_0} \epsilon_{\alpha})|_{t_Q} \) because \( t_Q \) centralizes the \( \mathfrak{sl}_2 \)-triple \((e, h, f)\). So \( (-\rho/2 - \delta)|_{t_Q} = (\rho/2 - \rho_0)|_{t_Q} \), where \( \rho_0 = \frac{1}{2} \sum_{\alpha > 0} \epsilon_{\alpha} \). The element \( h \in t_Q \) equals \((1, -1, 1, -1, \ldots)\). The positive roots that vanish on \( h \) are of the form \( \epsilon_i - \epsilon_j \), where \( i - j \) is even, and \( \epsilon_i + \epsilon_j \), where \( i - j \) is odd.

It is easy to see that \( \rho = n\epsilon_1 + (n-1)\epsilon_2 + \ldots + \epsilon_n \). Let \( \eta_i \) denote the \( i \)th coordinate function on \( t_Q \subset \mathbb{C}^m \).

Consider the case of \( n = 2m \) first. Here \( \rho|_{t_Q} = (4m - 1)\eta_1 + (4m - 5)\eta_2 + \ldots + 3\eta_m \). We get \( 2\rho_0 = (m-1)(\epsilon_1 + \epsilon_2) + (m-3)(\epsilon_3 + \epsilon_4) + \ldots + (1-m)(\epsilon_{2m-1} + \epsilon_{2m}) + m(\epsilon_1 + \ldots + \epsilon_{2m}) = (2m-1)(\epsilon_1 + \epsilon_2) + (2m-3)(\epsilon_3 + \epsilon_4) + \ldots + (\epsilon_{2m-1} + \epsilon_{2m}) \). Hence \( 2\rho_0|_{t_Q} = (4m-2)\eta_1 + \ldots + 2\eta_m \). We conclude that \( (\rho/2 - \rho_0)|_{t_Q} = (\eta_1 + \ldots + \eta_m)/2 \). This is the highest weight of a half-spinor representation.

Now consider the case of \( n = 2m+1 \). Here \( \rho|_{t_Q} = (4m + 1)\eta_1 + (4m - 3)\eta_2 + \ldots + 5\eta_m \). We get

\[
2\rho_0 = m\epsilon_1 + (m-2)\epsilon_3 + \ldots - m\epsilon_{2m+1} + (m-1)\epsilon_2 + (m-3)\epsilon_4 + \ldots + (1-m)\epsilon_{m-1} + \epsilon_m + (m+1)(\epsilon_2 + \ldots + \epsilon_{2m}) = 2m(\epsilon_1 + \epsilon_2) + (2m-2)(\epsilon_3 + \epsilon_4) + \ldots + 2(\epsilon_{2m-1} + \epsilon_{2m}).
\]

So \( 2\rho_0|_{t_Q} = 4m\eta_1 + 4(m-1)\eta_2 + \ldots + 4\eta_m \). It follows that \( (\rho/2 - \rho_0)|_{t_Q} = (\eta_1 + \ldots + \eta_m)/2 \), the highest weight of the spinor representation.

Step 5. Now consider a more general situation. Let \( \nu \) be as in Section 4.2 let \( V \) be an irreducible \( \mathcal{W} \)-module such that \( V = L_{\nu}(\overline{V}) \) is finite dimensional. Assume that
(i) \( \dim V = 1 \) (which always holds when \( e \) is principal in \( \mathfrak{g} \), which holds in the case of interest for us),

(ii) The algebra \( \mathfrak{q} \) is semisimple.

(iii) The action of \( t_{Q} \) on \( V \) via \( t_{Q} \hookrightarrow C_{\nu}(\mathcal{W}) \) is by a minuscule weight, say \( \omega \).

We claim that in this case \( V \) is irreducible over \( \mathfrak{q} \) (with highest weight \( \omega \)). This partially generalizes [L3, Theorem 5.2.1]. The proof is similar but we provide it for readers convenience.

Note that all \( t_{Q} \)-weights \( \beta \) in \( V \) are congruent to \( \omega \) modulo the root lattice of \( \mathfrak{q} \) and also \( \langle \beta, \nu \rangle \leq \langle \omega, \nu \rangle \) with equality if and only if \( \beta \) is a weight of \( V \). If \( V \) is not irreducible over \( \mathfrak{q} \), then there is another highest weight, say \( \omega' \). Since \( \omega \) is minuscule, we have \( \omega' \geq \omega \) (meaning that \( \omega' - \omega \) is the sum of positive roots). This contradicts the inequality \( \langle \omega', \nu \rangle \leq \langle \omega, \nu \rangle \) and proves the irreducibility of \( V \) over \( \mathfrak{q} \).

**Step 6.** Let us now complete the proof. By Step 4, \( V \) satisfies condition (iii) of Step 5, while conditions (i),(ii) were established above in the proof. So \( V \) is the spinor representation of \( \mathfrak{so}_{2m+1} \) when \( n = 2m + 1 \) and one of the half-spinor representations of \( \mathfrak{so}_{2m} \) in the case when \( n = 2m \). In the former case, \( V \) is \( Q \)-stable because \( V \) is stable under \( N_{Q}(t_{Q}) \). In the latter case, \( V \) is not stable because of an outer automorphism in \( Q \) that permutes the half-spinor representations. This proves (1),(2) of the proposition. (3) now follows from Theorem [L4] and Examples 2.7 (the case of \( n = 2m + 1 \)) and 2.8 (the case of \( n = 2m \)).

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