TAKE-AWAY IMPARTIAL COMBINATORIAL GAMES ON HYPERGRAPHS AND OTHER RELATED GEOMETRIC AND DISCRETE STRUCTURES

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ABSTRACT

In a Take-Away Game on hypergraphs, two players take turns to remove the vertices and the hyperedges of the hypergraphs. In each turn, a player must remove either a single vertex or a hyperedge. When a player chooses to remove one vertex, all of the hyperedges that contain the chosen vertex are also removed. When a player chooses to remove one hyperedge, only that chosen hyperedge is removed. Whoever removes the last vertex wins the game. Following from the winning strategy for the Take-Away Impartial Combinatorial Games on only Oddly Uniform or only Evenly Uniform Hypergraphs, this paper is about the new winning strategy for Take-Away Games on neither Oddly nor Evenly Uniform Hypergraphs. These neither Oddly nor Evenly Uniform Hypergraphs, however, have to satisfy the specific given requirements.

Keywords Take-Away · combinatorial games · hypergraphs · geometric structures · discrete structures

1 The special requirements.

In this paper, only those hypergraphs satisfying these requirements are considered.

- In each hypergraph, these following special requirements must be satisfied:
  - Each vertex must appear in at least one hyperedge of at least 2 vertices.
  - All hyperedges must be under exactly one of two categories:
    - Category X contains exactly one hyperedge of an even number of vertices, and all vertices in that hyperedge.
    - Category Y contains any positive number of hyperedges of exactly 3 vertices, and all vertices in those hyperedges.

Notation:
- $E(CatX)$ denotes the set of hyperedges in category X, and $|E(CatX)|$ is the cardinality of $E(CatX)$.
- $V(CatX)$ denotes the set of vertices in category X, and $|V(CatX)|$ is the cardinality of $V(CatX)$. Note that $V(CatX)$ is also the set of all vertices of an only hyperedge in $E(CatX)$.
- $E(CatY)$ denotes the set of hyperedges in category Y, and $|E(CatY)|$ is the cardinality of $E(CatY)$.
- $V(CatY)$ denotes the set of vertices in category Y, and $|V(CatY)|$ is the cardinality of $V(CatY)$. Note that $V(CatY)$ is also the set of vertices of all hyperedges in $E(CatY)$.

With the previous special requirements:
- $|V(CatX)|$ is even.
- $|E(CatX)| = 1$.
- $|V(CatY)| > 0$.
- $|E(CatY)| > 0$. 
Figure 1: Several examples of category X:
Category X contains hyperedge \{A, B\}, vertex A, and vertex B.
Category X contains hyperedge \{C, D, E, F\}, vertex C, vertex D, vertex E, and vertex F.
Category X contains hyperedge \{G, H, I, J, K, L\}, vertex G, vertex H, vertex I, vertex J, vertex K, and vertex L.
Category X contains hyperedge \{M, N, O, P, Q, R, S, T\}, vertex M, vertex N, vertex O, vertex P, vertex Q, vertex R, vertex S, and vertex T.

Figure 2: Several examples of category Y:
Category Y contains hyperedge \{A, B, C\}, vertex A, vertex B, and vertex C.
Category Y contains hyperedge \{D, E, F\}, hyperedge \{G, H, I\}, vertex D, vertex E, vertex F, vertex G, vertex H, and vertex I.
Category Y contains hyperedge \{J, K, L\}, hyperedge \{M, N, O\}, and hyperedge \{P, Q, R\}, vertex J, vertex K, vertex L, vertex M, vertex N, vertex O, vertex P, vertex Q, and vertex R.

• Consider \(|E(CatX)|\) and \(|E(CatY)|\):
  – Any hypergraph with \(|E(CatX)| = 1\) and \(|E(CatY)| > 0\), by definition, is neither an evenly uniform hypergraph with a marked coloring nor an oddly uniform hypergraph. Therefore, any hypergraph with \(|E(CatX)| = 1\) and \(|E(CatY)| > 0\) must also satisfy these additional special requirements:
    * Consider E(CatX):
      Each vertex of an only hyperedge in E(CatX) must appear in 0 hyperedge, or 1 hyperedge, or 2 hyperedges in E(CatY).
    * Consider E(CatY):
      · Exactly 2 out of 3 vertices in each hyperedge in E(CatY) must appear in an only hyperedge in E(CatX).
      · The remaining 1 out of 3 vertices in each hyperedge in E(CatY) must appear in all hyperedges in E(CatY) and 0 hyperedge in E(CatX). That exactly 1 special vertex in V(CatY) is called vertex S. In summary, vertex S must appear in all hyperedges in E(CatY) and 0 hyperedge in E(CatX).

Following from the previous additional special requirements:
    * Consider V(CatX):
      Each vertex in V(CatX) must be under exactly 1 of 3 subcategories:
      · Subcategory A contains all vertices that appear in a hyperedge in E(CatX) and 1 hyperedge in E(CatY).
      · Subcategory B contains all vertices that appear in a hyperedge in E(CatX) and 2 hyperedges in E(CatY).
Subcategory C contains all vertices that appear in a hyperedge in E(CatX) and 0 hyperedge in E(CatY).

* Consider V(CatY):

Following from the previous parts, V(CatY) only contains a special vertex (vertex S), all vertices in subcategory A (if any), and all vertices in subcategory B (if any).

Figure 3: Several hypergraphs with \(|E(CatX)| = 1\) and \(|E(CatY)| > 0\):

Category X contains a pink hyperedge, and all black vertices in each hypergraph.
Category Y contains all hyperedges not in category X (not in pink), all vertices in those hyperedges, and a red vertex S in each hypergraph.

The first figure (from left to right):
Category X contains hyperedge \{A, B\}, and all vertices in that hyperedge.
Category Y contains hyperedge \{S, A, B\}, and all vertices in that hyperedge.

The second figure (from left to right):
Category X contains hyperedge \{A, B, C, D\}, and all vertices in that hyperedge.
Category Y contains hyperedge \{S, A, B\}, hyperedge \{S, B, C\}, and all vertices in those hyperedges.

The third figure (from left to right):
Category X contains hyperedge \{A, B, C, D, E, F\}, and all vertices in that hyperedge.
Category Y contains hyperedge \{S, A, B\}, hyperedge \{S, B, C\}, hyperedge \{S, C, D\}, and all vertices in those hyperedges.

The fourth figure (from left to right):
Category X contains hyperedge \{A, B, C, D, E, F, G, H\}, and all vertices in that hyperedge.
Category Y contains hyperedge \{S, A, B\}, hyperedge \{S, C, D\}, hyperedge \{S, E, F\}, hyperedge \{S, G, H\}, and all vertices in those hyperedges.

For example: In a third hypergraph (from left to right) in Figure 3:
Consider category X:

- E(CatX) contains hyperedge \{A, B, C, D, E, F\}
- V(CatX) contains vertex A, vertex B, vertex C, vertex D, vertex E, and vertex F.

Consider category Y:

- E(CatY) contains hyperedge \{S, A, B\}, hyperedge \{S, B, C\}, hyperedge \{S, C, D\}
- V(CatY) contains all vertices in the hyperedges in E(CatY).

Since vertex A and vertex D appear in an only hyperedge in E(CatX) and 1 hyperedge in E(CatY), vertex A and vertex D are in subcategory A.
Since vertex B and vertex C appear in an only hyperedge in E(CatX) and 2 hyperedges in E(CatY), vertex B and vertex C are in subcategory B.
Since vertex E and vertex F appear in an only hyperedge in E(CatX) and 0 hyperedge in E(CatY), vertex E and vertex F are in subcategory C.

For hypergraphs with

- Either an only hyperedge (with all vertices in that hyperedge) in category X, and 0 hyperedge in category Y.
- Or only hyperedges (with all vertices in those hyperedges) in category Y, and 0 hyperedge in category X.

the previous additional special requirements do not apply. Therefore, in those cases, there does not exist a special vertex S nor any subcategory. Instead, the winning strategy was already found in those cases. By definition:
• Any hypergraph with an only hyperedge (with all vertices in that hyperedge) in category X, and 0 hyperedge in category Y, is an evenly uniform hypergraph with a marked coloring.
• Any hypergraph with only hyperedges (with all vertices in those hyperedges) in category Y, and 0 hyperedge in category X is an oddly uniform hypergraph.

Take-Away impartial combinatorial games on both evenly uniform hypergraphs with a marked coloring, and oddly uniform hypergraphs were already solved [1]. Therefore, in this article, the focus is only on the Take-Away impartial combinatorial games on any hypergraphs with $|E(Cat_X)| = 1$ and $|E(Cat_Y)| > 0$ that satisfies the special requirements in Part 1.

2 Lemmas, theorems, and corollary.

2.1 Lemma 1

There does not exist a hypergraph where all vertices in $V(Cat_X)$ are in subcategory C.

Proof
To the contrary, assume that there exists a hypergraph where all vertices in $V(Cat_X)$ are in subcategory C. Vertices in subcategory C appear in a hyperedge in $E(Cat_X)$ and 0 hyperedge in $E(Cat_Y)$. Furthermore, from the assumption and the additional special requirements, $V(Cat_Y)$ only contains a vertex S. It means that when all vertices in $V(Cat_X)$ are in subcategory C, a hypergraph contains a hyperedge (and all vertices in that hyperedge) in category X. Consequently, the previous additional special requirements do not apply. Therefore, in those cases, there does not exist a special vertex S nor any subcategory. Thus, by contradiction, there does not exist a hypergraph where all vertices in $V(Cat_X)$ are in subcategory C.

2.2 Lemma 2

Prove that $|E(Cat_Y)| = |V(Cat_X)|$ for all hypergraphs where all vertices in $V(Cat_X)$ are in subcategory B.

Proof
Consider all hypergraphs where all vertices in $V(Cat_X)$ are in subcategory B. Vertices in subcategory B appear in a hyperedge in $E(Cat_X)$ and 2 hyperedges in $E(Cat_Y)$. Furthermore, from the hypothesis and the additional special requirements, $V(Cat_Y)$ only contains a vertex S, and all vertices in subcategory B. Suppose that 2 out of 3 vertices in each hyperedge in $E(Cat_Y)$ can be the same vertices, it leads to $|E(Cat_Y)| = |V(Cat_X)|$. With the fact that 2 out of 3 vertices in each hyperedge in $E(Cat_Y)$ must be distinct, with a vertex S and the same number of vertices in subcategory B, there are a lot of different combinations to make valid hyperedges in $E(Cat_Y)$, so $|E(Cat_Y)|$ and $|V(Cat_X)|$ both remain unchanged, and $|E(Cat_Y)| = |V(Cat_X)|$. Thus, $|E(Cat_Y)| = |V(Cat_X)|$ for all hypergraphs where all vertices in $V(Cat_X)$ are in subcategory B. There is an additional remark that with a given $|V(Cat_X)|$, as exactly 2 out of 3 vertices in each hyperedge in $E(Cat_Y)$ must appear in a hyperedge in $E(Cat_X)$, the maximum possible number that $|E(Cat_Y)|$ can reach is the number which is equal to $|V(Cat_X)|$. Consequently, for all hypergraphs where all vertices in $V(Cat_X)$ are in subcategory B, $|E(Cat_Y)|$ reaches the maximum possible number with the given $|V(Cat_X)|$, which is equal to $|V(Cat_X)|$.

Figure 4: Several hypergraphs with $|E(Cat_Y)| = |V(Cat_X)|$. 
2.3 Lemma 3

Prove that $|E(CatY)| \geq 4$ for all hypergraphs where all vertices in $V(CatX)$ are in subcategory B.

Proof

To the contrary, assume that $0 \leq |E(CatY)| < 4$ in hypergraphs where all vertices in $V(CatX)$ are in subcategory B. From Lemma 2, $|E(CatY)| = |V(CatX)|$ for all hypergraphs where all vertices in $V(CatX)$ are in subcategory B. From the special requirement, since $|V(CatX)|$ is even, $|E(CatY)|$ is even. Since $|E(CatY)|$ is even and $0 \leq |E(CatY)| < 4$, there are only 2 cases to consider: either $|E(CatY)| = 0$ or $|E(CatY)| = 2$.

- Case 1: $|E(CatY)| = 0$.
  
  For hyperedges with only a hyperedge (and all vertices in that hyperedge) in category X, the previous additional special requirements do not apply. Therefore, in those cases, there does not exist a special vertex S nor any subcategory. Consequently, this case reaches a contradiction.

- Case 2: $|E(CatY)| = 2$.
  
  From the same argument in Lemma 2, for all hypergraphs where all vertices in $V(CatX)$ are in subcategory B, $V(CatY)$ only contains a vertex S, and all vertices in subcategory B. Vertices in subcategory B appears in a hyperedge in $E(CatX)$ and 2 hyperedges in $E(CatY)$. Consequently, in case 2, those 2 hyperedges of 3 distinct vertices in $E(CatY)$ must be exactly the same. Therefore, there is only 1 valid hyperedge of 3 vertices in $E(CatY)$, which reaches a contradiction.

In conclusion, a contradiction is found in either case. Thus, by contradiction, $|E(CatY)| \geq 4$ for all hypergraphs where all vertices in $V(CatX)$ are in subcategory B.

Figure 5: Several hypergraphs with all vertices in $|V(CatX)|$ are in subcategory B.

2.4 Lemma 4

There does not exist a hypergraph where all vertices in $V(CatX)$ are in subcategory B and subcategory C.

Proof

To the contrary, assume that there exists a hypergraph where all vertices in $V(CatX)$ are in subcategory B and subcategory C. Vertices in subcategory C appear in a hyperedge in $E(CatX)$ and 0 hyperedge in $E(CatY)$. Vertices in subcategory B appear in a hyperedge in $E(CatX)$ and 2 hyperedges in $E(CatY)$. Furthermore, from the hypothesis and the additional special requirements, $V(CatY)$ only contains a special vertex (vertex S), and all vertices in subcategory B. From Lemma 2, $|E(CatY)| = |V(CatX)|$ for all hypergraphs where $V(CatY)$ only contains a vertex S, and all vertices in subcategory B. As there is 0 vertex that appears in a hyperedge in $E(CatX)$ and 0 hyperedge in $E(CatY)$, there is 0 vertex in subcategory C. Thus, by contradiction, there does not exist a hypergraph where all vertices in $V(CatX)$ are in subcategory B and subcategory C.
2.5 Lemma 5

Prove that $|E(CatY)| \geq 3$ for all hypergraphs where all vertices in $V(CatX)$ are in subcategory A and subcategory B.

Proof
To the contrary, assume that $0 \leq |E(CatY)| < 3$ in hypergraphs where all vertices in $V(CatX)$ are in subcategory A and subcategory B.

- Case 1: $|E(CatY)| = 0$.
  For hypergraphs with only a hyperedge (and all vertices in that hyperedge) in category X, the previous additional special requirements do not apply. Therefore, in those cases, there does not exist a special vertex S nor any subcategory.
  Consequently, this case reaches a contradiction.

- Case 2: $|E(CatY)| = 1$.
  From the hypothesis, and the additional special requirements, $V(CatY)$ must contain only a vertex S and 2 vertices in subcategory A.
  Therefore, $V(CatX)$ can only contain all vertices in subcategory A and all vertices in category C (if any).
  Consequently, this case reaches a contradiction.

- Case 3: $|E(CatY)| = 2$.
  From the hypothesis, and the additional special requirements, $V(CatY)$ can only contain a vertex S, 1 vertex in subcategory B and 2 vertices in subcategory A. However, this leads to $|V(CatX)| = 3$ (2 vertices in subcategory A and 1 vertex in subcategory B), which is not an even number of vertices.
  Consequently, this case reaches a contradiction.

In conclusion, a contradiction is found in all 3 cases.
Thus, by contradiction, $|E(CatY)| \geq 3$ for all hypergraphs where all vertices in $V(CatX)$ are in subcategory A and subcategory B.

2.6 Lemma 6

From Lemma 1 and Lemma 4, there is a corollary:
With 3 subcategories (subcategory A, subcategory B, and subcategory C), there are exactly 5 distinct groups (counted in Roman numerals for easier use later) for all hypergraphs of all vertices in $V(CatX)$:

- Group I: Hypergraphs where all vertices in $V(CatX)$ are in subcategory A.
- Group II: Hypergraphs where all vertices in $V(CatX)$ are in subcategory B. For all hypergraphs in this case, from Lemma 2 and Lemma 3, $|E(CatY)| = |V(CatX)|$, and $|E(CatY)|$ is even (as $|V(CatX)|$ is even).
- Group III: Hypergraphs where all vertices in $V(CatX)$ are in subcategory A and subcategory B.
- Group IV: Hypergraphs where all vertices in $V(CatX)$ are in subcategory A and subcategory C.
- Group V: Hypergraphs where all vertices in $V(CatX)$ are in subcategory A, subcategory B, and subcategory C.

From Corollary 6, there is a remark:

- Except all hypergraphs in group II, for any hypergraph in 1 of 4 remaining groups, there is always at least a vertex in $V(CatX)$ that is in subcategory A.
- Consider all hypergraphs in group III, group IV, and group V. For any hypergraph in 1 of those 3 groups, there is always at least a vertex in $V(CatX)$ that is in subcategory B or there is always at least a vertex in $V(CatX)$ that is in subcategory C.

Theorem 7
If $T$ is a hypergraph where $|E(CatX)| = 1$ and $|E(CatY)|$ is odd, then $g(T) = 1$.

Proof
Proof by induction.
Base case: Considering a hypergraph $T$ where $|E(CatX)| = 1$ and $|E(CatY)| = 1$.

- Since $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.
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• Since $|E(CatY)| = 1$, and by the additional special requirements, category Y contains a vertex S and 2 vertices in subcategory A.

Consequently, all hypergraphs in group I and group IV are valid with this hypothesis.

Therefore, there is always at least a vertex in subcategory A in all valid hypergraphs.

By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

• Case 1: Remove a vertex S.

After a vertex S is removed:
- $|V(CatX)|$ is even.
- $|E(CatX)| = 1$, and $|E(CatY)| = 0$.

By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
- The resulting vertex set $|V(T_1)|$ is even.
- The resulting hyperedge set $|E(T_1)|$ is odd.

By Theorem 2.8, $g(T_1) = 2$.

• Case 2: Remove a vertex in subcategory A.

After a vertex in subcategory A is removed:
- $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
- $|E(CatX)| = 0$, and $|E(CatY)| = 0$.

By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_2)|$ is even.
- The resulting hyperedge set $E(T_2)$ is even.

By Theorem 2.4, $g(T_2) = 0$.

• Case 3: Remove a vertex in subcategory C (if any).

After a vertex in subcategory C is removed:
- $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
- $|E(CatX)| = 0$, and $|E(CatY)| = 1$.

By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_3)|$ is even.
- The resulting hyperedge set $|E(T_3)|$ is odd.

By Theorem 2.4, $g(T_3) = 3$.

• Case 4: Remove a hyperedge in E(CatX).

After a hyperedge in E(CatX) is removed,
- $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y).
- $|E(CatX)| = 0$, and $|E(CatY)| = 1$.

By definition, the resulting hypergraph $T_4$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_4)|$ is odd.
- The resulting hyperedge $|E(T_4)|$ is odd.

By Theorem 2.4, $g(T_4) = 2$.

• Case 5: Remove a hyperedge in E(CatY). After a hyperedge in E(CatY) is removed:
- $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y).
- $|E(CatX)| = 1$, and $|E(CatY)| = 0$.

By definition, the resulting hypergraph $T_5$ is an evenly uniform hypergraph with a marked coloring with:
- The resulting vertex set $|V(T_5)|$ is odd.
- The resulting hyperedge set $|E(T_5)|$ is odd.

By Theorem 2.8, $g(T_5) = 3$.

• Consider all hypergraphs $T$ in group I. By Sprague–Grundy Theorem, $g(T) = \text{mex}\{2,0,2,3\} = 1$. 

• Consider all hypergraphs $T$ in group IV. By Sprague–Grundy Theorem, $g(T) = \text{mex}\{2,0,3,2,3\} = 1$.

Therefore, for all hypergraphs $T$ in either group I or group IV, $g(T) = 1$.

Inductive hypothesis: assume that if $T$ is a hypergraph where $|E(CatX)| = 1$ and $|E(CatY)| = n$ (n is odd), then $g(T) = 1$.

Consider a hypergraph $T$ where $|E(CatX)| = 1$ and $|E(CatY)| = n + 1$ (n + 1 is odd).

Prove that $g(T) = 1$.

• Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.

• The hyperedge set $E(T)$ contains $|E(CatX)| = 1$ and $|E(CatY)|$ is odd.

By Corollary 6, all hypergraphs in group II are not valid in this Theorem (as the inductive hypothesis states that $|E(CatY)|$ is odd). Only all hypergraphs in group I, group III, group IV, and group V are valid.

By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

• Case 1: Remove a vertex $S$.
After a vertex $S$ is removed:
  – $|V(CatX)|$ is even.
  – $|E(CatX)| = 1$, and $|E(CatY)| = 0$.

By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
  – The resulting vertex set $|V(T_1)|$ is even.
  – The resulting hyperedge set $|E(T_1)|$ is odd.

By Theorem 2.8, $g(T_1) = 2$.

• Case 2: Remove a vertex in subcategory A.
After a vertex in subcategory A is removed:
  – $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
  – $|E(CatX)| = 0$, and $|E(CatY)|$ is even.

By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
  – The resulting vertex set $|V(T_2)|$ is even.
  – The resulting hyperedge set $|E(T_2)|$ is even.

By Theorem 2.4, $g(T_2) = 0$.

• Case 3: Remove a vertex in subcategory B (if any).
After a vertex in subcategory B is removed:
  – $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
  – $|E(CatX)| = 0$, and $|E(CatY)|$ is odd.

By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
  – The resulting vertex set $|V(T_3)|$ is even.
  – The resulting hyperedge set $|E(T_3)|$ is odd.

By Theorem 2.4, $g(T_3) = 3$.

• Case 4: Remove a vertex in subcategory C (if any).
After a vertex in subcategory C is removed:
  – $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
  – $|E(CatX)| = 0$, and $|E(CatY)|$ is odd.

By definition, the resulting hypergraph $T_4$ is an oddly uniform hypergraph with:
  – The resulting vertex set $|V(T_4)|$ is even.
  – The resulting hyperedge set $|E(T_4)|$ is odd.

By Theorem 2.4, $g(T_4) = 3$. 
• Case 5: Remove a hyperedge in \( \text{E(CatX)} \).
  After a hyperedge in \( \text{E(CatX)} \) is removed:
  
  - \( |V(\text{CatX})| \) is even, and there is a vertex \( S \) (not in category \( X \) but in category \( Y \)).
  - \( |\text{E(CatX)}| = 0 \), and \( |\text{E(CatY)}| \) is odd.
  
  By definition, the resulting hypergraph \( T_5 \) is an oddly uniform hypergraph with:

  - The resulting vertex set \( |V(T_5)| \) is odd.
  - The resulting hyperedge set \( |E(T_5)| \) is odd.

  By Theorem 2.4, \( g(T_5) = 2 \).

• Case 6: Remove a hyperedge in \( \text{E(CatY)} \).
  After a hyperedge in \( \text{E(CatY)} \) is removed:

  - Since \( |V(\text{CatX})| \) is even, and there is a vertex \( S \) (not in category \( X \) but in category \( Y \)), \( |V(T_6)| \) is odd.
  - \( |\text{E(CatX)}| = 1 \), and \( |\text{E(CatY)}| = 0 \).
  
  By definition, the resulting hypergraph \( T_6 \) is neither an oddly uniform hypergraph nor an evenly uniform hypergraph with a marked coloring.

  Starting from here, \( |\text{E(CatY)}| \) is even, but from the starting note above, all hypergraphs in group II are already not valid. Therefore, only all hypergraphs in group I, group III, group IV, and group V are valid. Therefore, by Corollary 6, there is always at least a vertex in subcategory A in all valid hypergraphs.

  Continue with the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

  * Subcase 1: Remove a vertex \( S \).
    After a vertex \( S \) is removed:
    
    - \( |V(\text{CatX})| \) is even.
    - \( |\text{E(CatX)}| = 1 \), and \( |\text{E(CatY)}| = 0 \).
    
    By definition, the resulting hypergraph \( T_{61} \) is an evenly uniform hypergraph with a marked coloring with:

    - The resulting vertex set \( |V(T_{61})| \) is even.
    - The resulting hyperedge set \( |E(T_{61})| \) is odd.

    By Theorem 2.8, \( g(T_{61}) = 2 \).

  * Subcase 2: Remove a vertex in subcategory A.
    After a vertex in subcategory A is removed:
    
    - \( |V(\text{CatX})| \) is odd, and there is a vertex \( S \) (not in category \( X \) but in category \( Y \)).
    - \( |\text{E(CatX)}| = 0 \), and \( |\text{E(CatY)}| \) is odd.
    
    By definition, the resulting hypergraph \( T_{62} \) is an oddly uniform hypergraph with:

    - The resulting vertex set \( |V(T_{62})| \) is even.
    - The resulting hyperedge set \( |E(T_{62})| \) is odd.

    By Theorem 2.4, \( g(T_{62}) = 3 \).

  * Subcase 3: Remove a vertex in subcategory B (if any).
    After a vertex in subcategory B is removed:
    
    - \( |V(\text{CatX})| \) is odd, and there is a vertex \( S \) (not in category \( X \) but in category \( Y \)).
    - \( |\text{E(CatX)}| = 0 \), and \( |\text{E(CatY)}| \) is even.
    
    By definition, the resulting hypergraph \( T_{63} \) is an oddly uniform hypergraph with:

    - The resulting vertex set \( |V(T_{63})| \) is even.
    - The resulting hyperedge set \( |E(T_{63})| \) is even.

    By Theorem 2.4, \( g(T_{63}) = 0 \).

  * Subcase 4: Remove a vertex in subcategory C (if any).
    After a vertex in subcategory C is removed:
    
    - \( |V(\text{CatX})| \) is odd, and there is a vertex \( S \) (not in category \( X \) but in category \( Y \)).
    - \( |\text{E(CatX)}| = 0 \), and \( |\text{E(CatY)}| \) is even.
    
    By definition, the resulting hypergraph \( T_{64} \) is an oddly uniform hypergraph with:

    - The resulting vertex set \( |V(T_{64})| \) is even.
    - The resulting hyperedge set \( |E(T_{64})| \) is even.

    By Theorem 2.4, \( g(T_{64}) = 0 \).

  * Subcase 5: Remove a hyperedge in \( \text{E(CatX)} \).
    After a hyperedge in \( \text{E(CatX)} \) is removed:
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- $|V(CatX)|$ is even, and there is a vertex $S$ (not in category $X$ but in category $Y$).
- $|E(CatX)| = 0$, and $|E(CatY)|$ is odd.

By definition, the resulting hypergraph $T_{6_5}$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_{6_5})|$ is odd.
- The resulting hyperedge set $|E(T_{6_5})|$ is even.

By Theorem 2.4, $g(T_{6_5}) = 1$.

* Subcase 6: Remove a hyperedge in $E(CatY)$. After a hyperedge in $E(CatY)$ is removed:
- Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category $X$ but in category $Y$), $|V(T_{6_5})|$ is odd.
- $|E(CatX)| = 1$, and $|E(CatY)|$ is odd.

By the Inductive Hypothesis, $g(T_{6_5}) = 1$.

- Consider all hypergraphs in group I. By Sprague-Grundy Theorem, $g(T_6) = \text{mex} \{2, 3, 1, 1\} = 0$.
- Consider all hypergraphs in group III, group IV, and group V. By Sprague-Grundy Theorem, $g(T_6) = \text{mex} \{2, 3, 0, 1, 1\} = 4$.

Therefore, for all hypergraphs in all groups, either $g(T_6) = 0$ or $g(T_6) = 4$.

- Consider all hypergraphs in group I. By Sprague-Grundy Theorem, $g(T_6) = \text{mex} \{2, 0, 2, 0\} = 1$, or by Sprague-Grundy Theorem, $g(T_6) = \text{mex} \{2, 0, 2, 4\} = 1$.
- Consider all hypergraphs in group III, group IV, and group V. By Sprague-Grundy Theorem, $g(T_6) = \text{mex} \{2, 0, 3, 2, 0\} = 1$ or by Sprague-Grundy Theorem, $g(T_6) = \text{mex} \{2, 0, 3, 2, 4\} = 1$.

Therefore, for all hypergraphs in all groups, $g(T) = 1$.

Thus, by induction, if $T$ is a hypergraph where $|E(CatX)| = 1$ and $|E(CatY)|$ is odd, then $g(T) = 1$.

**Theorem 8**

For all hypergraphs $T$ in group III, group IV, and group V where $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number, then $g(T) = 4$.

**Proof**

Proof by induction.

Base case: Consider a hypergraph $T$ in group III, or group IV, or group V where $|E(CatX)| = 1$ and $|E(CatY)| = 2$.

- Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category $X$ but in category $Y$). Consequently, the vertex set $|V(T)|$ is odd.
- By Corollary 6, since a hypergraph $T$ is in group III, or group IV, or group V, there is always at least a vertex in subcategory $A$ in all valid hypergraphs.
- Furthermore, by Corollary 6, since a hypergraph $T$ is in group III, or group IV, or group V, there is always at least a vertex in subcategory $B$, or there is always at least a vertex in subcategory $C$ in all valid hypergraphs.

By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

- Case 1: Remove a vertex $S$.
  After a vertex $S$ is removed:
  - $|V(CatX)|$ is even.
  - $|E(CatX)| = 1$, and $|E(CatY)| = 0$.

By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
  - The resulting vertex set $|V(T_1)|$ is even.
  - The resulting hyperedge set $|E(T_1)|$ is odd.

By Theorem 2.8, $g(T_1) = 2$.

- Case 2: Remove a vertex in subcategory $A$.
  After a vertex in subcategory $A$ is removed:
  - $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category $X$ but in category $Y$).
By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_2)|$ is even.
- The resulting hyperedge set $|E(T_2)|$ is odd.

By Theorem 2.4, $g(T_2) = 3$.

• Case 3: Remove a vertex in subcategory B (if any).
  After a vertex in subcategory B is removed:
  - $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)| = 0$.
  
  By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_3)|$ is even.
  - The resulting hyperedge set $|E(T_3)|$ is even.

By Theorem 2.4, $g(T_3) = 0$.

• Case 4: Remove a vertex in subcategory C (if any).
  After a vertex in subcategory C is removed:
  - $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)| = 0$.

By definition, the resulting hypergraph $T_4$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_4)|$ is even.
- The resulting hyperedge set $|E(T_4)|$ is even.

By Theorem 2.4, $g(T_4) = 0$.

• Case 5: Remove a hyperedge in $E(CatX)$.
  After a hyperedge in $E(CatX)$ is removed:
  - $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)| = 2$.

By definition, the resulting hypergraph $T_5$ is an oddly uniform hypergraph with:
- The resulting vertex set $|V(T_5)|$ is odd.
- The resulting hyperedge set $|E(T_5)|$ is even.

By Theorem 2.4, $g(T_5) = 1$.

• Case 6: Remove a hyperedge in $E(CatY)$.
  After a hyperedge in $E(CatY)$ is removed:
  - $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 1$, and $|E(CatY)| = 1$.

By Theorem 7, $g(T_6) = 1$.

Consider all hypergraphs $T$ in group III, group IV, and group V. By Sprague-Grundy Theorem, $g(T) = \text{mex } \{2, 3, 0, 1, 1\} = 4$.

Inductive hypothesis: Consider all hypergraphs $T$ in group III, group IV, and group V where $|E(CatX)| = 1$ and $|E(CatY)| = n$ (n is a positive even number), then $g(T) = 4$.

Consider all hypergraphs $T$ in group III, group IV, and group V where $|E(CatX)| = 1$ and $|E(CatY)| = n+1$ (n+1 is a positive even number). Prove that $g(T) = 4$.

- Since $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.
- The hyperedge set $E(T)$ contains $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number.
By Corollary 6, since a hypergraph T is group III, or group IV, or group V, there is always at least a vertex in subcategory A in all valid hypergraphs.
Furthermore, by corollary 6, since a hypergraph T is group III, or group IV, or group V, there is always at least a vertex in subcategory B, or there is always at least a vertex in subcategory C in all valid hypergraphs.
By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

- Case 1: Remove a vertex S.
  After a vertex S is removed:
  - $|V(CatX)|$ is even.
  - $|E(CatX)| = 1$, and $|E(CatY)| = 0$.
  By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
  - The resulting vertex set $|V(T_1)|$ is even.
  - The resulting hyperedge set $|E(T_1)|$ is odd.
  By Theorem 2.8, $g(T_1) = 2$.

- Case 2: Remove a vertex in subcategory A.
  After a vertex in subcategory A is removed:
  - $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is odd.
  By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_2)|$ is even.
  - The resulting hyperedge set $|E(T_2)|$ is odd.
  By Theorem 2.4, $g(T_2) = 3$.

- Case 3: Remove a vertex in category B (if any).
  After a vertex in category B is removed:
  - $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is even.
  By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_3)|$ is even.
  - The resulting hyperedge set $|E(T_3)|$ is even.
  By Theorem 2.4, $g(T_3) = 0$.

- Case 4: Remove a vertex in category C (if any).
  After a vertex in category C is removed:
  - $|V(CatX)|$ is odd, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is even.
  By definition, the resulting hypergraph $T_4$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_4)|$ is even.
  - The resulting hyperedge set $|E(T_4)|$ is even.
  By Theorem 2.4, $g(T_4) = 0$.

- Case 5: Remove a hyperedge in E(CatX).
  After a hyperedge in E(CatX) is removed:
  - $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is even.
  By definition, the resulting hypergraph $T_5$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_5)|$ is odd.
  - The resulting hyperedge set $|E(T_5)|$ is even.
  By Theorem 2.4, $g(T_5) = 1$.

- Case 6: Remove a hyperedge in E(CatY).
  After a hyperedge in E(CatY) is removed:
Since $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y), $|V(T_6)|$ is odd.

By Theorem 7, $g(T_0) = 1$.

Consider all hypergraphs in group III, group IV, and group V. By Sprague-Grundy Theorem, $g(T) = \text{mex \{2, 3, 0, 1, 1\}} = 4$.

Thus, consider all hypergraphs in group III, group IV, and group V. By induction, if $T$ is one of the considered hypergraphs where $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number, then $g(T) = 4$.

**Theorem 9**
For all hypergraphs $T$ in group I where $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number, then $g(T) = 0$.

**Proof**
Proof by induction.
Base case: Consider a hypergraph $T$ in group I where $|E(CatX)| = 1$ and $|E(CatY)| = 2$.

- Since $|V(CatX)|$ is even, and there is a vertex S (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.
  - Since a hypergraph $T$ is in group I, before the game starts, there are only vertices in category A.

By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

- **Case 1:** Remove a vertex $S$.
  - After a vertex $S$ is removed:
    - $|V(CatX)|$ is even.
    - $|E(CatX)| = 1$, and $|E(CatY)| = 0$.
  - By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
    - The resulting vertex set $|V(T_1)|$ is even.
    - The resulting hyperedge set $|E(T_1)|$ is odd.
  - By Theorem 2.8, $g(T_1) = 2$.

- **Case 2:** Remove a vertex in category A.
  - After a vertex in category A is removed:
    - $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
    - $|E(CatX)| = 0$, and $|E(CatY)| = 1$.
  - By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
    - The resulting vertex set $|V(T_2)|$ is even.
    - The resulting hyperedge set $E(T_2)$ is odd.
  - By Theorem 2.4, $g(T_2) = 3$.

- **Case 3:** Remove a hyperedge in $E(CatX)$.
  - After a hyperedge in $E(CatX)$ is removed:
    - $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y).
    - $|E(CatX)| = 0$, and $|E(CatY)| = 2$.
  - By definition, the resulting hypergraph $T_3$ is an evenly uniform hypergraph with:
    - The resulting vertex set $|V(T_2)|$ is odd.
    - The resulting hyperedge set $E(T_3)$ is even.
  - By Theorem 2.4, $g(T_3) = 1$.

- **Case 4:** Remove a hyperedge in $E(CatY)$.
  - After a hyperedge in $E(CatY)$ is removed:
    - $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y).
    - $|E(CatX)| = 1$, and $|E(CatY)| = 1$. 

By Theorem 7, $g(T_4) = 1$.

By Sprague-Grundy Theorem, $g(T) = \text{mex} \{2, 3, 1, 1\} = 0$.

Inductive hypothesis: For all hypergraphs $T$ in group I where $|E(CatX)| = 1$ and $|E(CatY)| = n$ ($n$ is a positive even number), then $g(T) = 0$.
Consider all hypergraphs $T$ in group I where $|E(CatX)| = 1$ and $|E(CatY)| = n+1$ ($n+1$ is a positive even number). Prove that $g(T) = 4$.

- Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.
- Since a hypergraph $T$ is in group I, before the game starts, there are only vertices in category A.

By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

- Case 1: Remove a vertex $S$.
  After a vertex $S$ is removed:
  - $|V(CatX)|$ is even.
  - $|E(CatX)| = 1$, and $|E(CatY)| = 0$.
  By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
    - The resulting vertex set $|V(T_1)|$ is even.
    - The resulting hyperedge set $|E(T_1)|$ is odd.
  By Theorem 2.8, $g(T_1) = 2$.

- Case 2: Remove a vertex in category A.
  After a vertex in category A is removed:
  - $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is odd.
  By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
    - The resulting vertex set $|V(T_2)|$ is even.
    - The resulting hyperedge set $|E(T_2)|$ is odd.
  By Theorem 2.4, $g(T_2) = 3$.

- Case 3: Remove a hyperedge in $E(CatX)$.
  After a hyperedge in $E(CatX)$ is removed:
  - $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is even.
  By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
    - The resulting vertex set $|V(T_3)|$ is odd.
    - The resulting hyperedge set $|E(T_3)|$ is even.
  By Theorem 2.4, $g(T_3) = 1$.

- Case 4: Remove a hyperedge in $E(CatY)$.
  After a hyperedge in $E(CatY)$ is removed:
  - Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y), $|V(T_6)|$ is odd.
  - $|E(CatX)| = 1$, and $|E(CatY)|$ is odd.
  By Theorem 7, $g(T_6) = 1$.

Consider all hypergraphs $T$ in group I. By Sprague-Grundy Theorem, $g(T) = \text{mex} \{2, 3, 1, 1\} = 0$.
Thus, by induction, for all hypergraphs $T$ in group I where $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number, then $g(T) = 0$.

**Theorem 10**
For all hypergraphs $T$ in group II where $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number, then $g(T) = 3$. 
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Proof
Proof by induction.
Base case: Consider a hypergraph $T$ in group II where $|E(CatX)| = 1$ and $|E(CatY)| = 2$.

- Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.
- Since a hypergraph $T$ is in group II, before the game starts, there are only vertices in category B.

By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

- Case 1: Remove a vertex $S$.
  After a vertex $S$ is removed:
  - $|V(CatX)|$ is even.
  - $|E(CatX)| = 1$, and $|E(CatY)| = 0$.
  By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
    - The resulting vertex set $|V(T_1)|$ is even.
    - The resulting hyperedge set $|E(T_1)|$ is odd.
  By Theorem 2.8, $g(T_1) = 2$.

- Case 2: Remove a vertex in category B (if any).
  After a vertex in category B is removed:
  - $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)| = 0$.
  By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
    - The resulting vertex set $|V(T_2)|$ is even.
    - The resulting hyperedge set $|E(T_2)|$ is even.
  By Theorem 2.4, $g(T_2) = 0$.

- Case 3: Remove a hyperedge in $E(CatX)$.
  After a hyperedge in $E(CatX)$ is removed:
  - $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)| = 2$.
  By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
    - The resulting vertex set $|V(T_3)|$ is odd.
    - The resulting hyperedge set $|E(T_3)|$ is even.
  By Theorem 2.4, $g(T_3) = 1$.

- Case 4: Remove a hyperedge in $E(CatY)$.
  After a hyperedge in $E(CatY)$ is removed:
  - $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 1$, and $|E(CatY)| = 1$.
  By Theorem 7, $g(T_4) = 1$.

By Sprague-Grundy Theorem, $g(T) = \text{mex} \{2, 0, 1, 1\} = 3$.

Inductive hypothesis: For all hypergraphs $T$ in group II where $|E(CatX)| = 1$ and $|E(CatY)| = n$ ($n$ is a positive even number), then $g(T) = 4$.
Consider all hypergraphs $T$ in group II where $|E(CatX)| = 1$ and $|E(CatY)| = n+1$ ($n+1$ is a positive even number). Prove that $g(T) = 3$.

- Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y). Consequently, the vertex set $|V(T)|$ is odd.
- Since a hypergraph $T$ is in group II, before the game starts, there are only vertices in category B.
By the rule of the game, either 1 vertex or 1 hyperedge is removed at a time.

- Case 1: Remove a vertex $S$.
  After a vertex $S$ is removed:
  - $|V(CatX)|$ is even.
  - $|E(CatX)| = 1$, and $|E(CatY)| = 0$.
  By definition, the resulting hypergraph $T_1$ is an evenly uniform hypergraph with a marked coloring with:
  - The resulting vertex set $|V(T_1)|$ is even.
  - The resulting hyperedge set $|E(T_1)|$ is odd.
  By Theorem 2.8, $g(T_1) = 2$.

- Case 2: Remove a vertex in category B.
  After a vertex in category B is removed:
  - $|V(CatX)|$ is odd, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is even.
  By definition, the resulting hypergraph $T_2$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_2)|$ is even.
  - The resulting hyperedge set $|E(T_2)|$ is even.
  By Theorem 2.4, $g(T_2) = 0$.

- Case 3: Remove a hyperedge in $E(CatX)$.
  After a hyperedge in $E(CatX)$ is removed:
  - $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y).
  - $|E(CatX)| = 0$, and $|E(CatY)|$ is even.
  By definition, the resulting hypergraph $T_3$ is an oddly uniform hypergraph with:
  - The resulting vertex set $|V(T_3)|$ is odd.
  - The resulting hyperedge set $|E(T_3)|$ is even.
  By Theorem 2.4, $g(T_3) = 1$.

- Case 4: Remove a hyperedge in $E(CatY)$.
  After a hyperedge in $E(CatY)$ is removed:
  - Since $|V(CatX)|$ is even, and there is a vertex $S$ (not in category X but in category Y), $|V(T_4)|$ is odd.
  - $|E(CatX)| = 1$, and $|E(CatY)|$ is odd.
  By Theorem 7, $g(T_4) = 1$.

For all hypergraphs in group II, by Sprague-Grundy Theorem, $g(T) = \text{mex} \{2, 0, 1, 1\} = 3$.

Thus, by induction, for all hypergraphs in group II where $|E(CatX)| = 1$ and $|E(CatY)|$ is a positive even number, then $g(T) = 3$.

**Part 3: Applications.**

- A game on any geometric object whose Planar Graph satisfying all of the requirements in part 1 can be played by using the winning strategy in part 2.
  Example: Graphs.

- A game on any geometric object whose Schlegel diagram satisfying all of the requirements in part 1 can be played by using the winning strategy in part 2.
  Example: Pyramids.

- A game on any geometric object whose graphical projection satisfying all of the requirements in part 1 can be played by using the winning strategy in part 2.
  Example: Chocolate boxes.
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• A game on any geometric object whose n-polytope satisfying all of the requirements in part 1 can be played by using the winning strategy in part 2.
  Example: polygons that are self-intersecting with varying densities of different regions.
• A game on any complex polygons satisfying all of the requirements in part 1 can be played by using the winning strategy in part 2.
• A game on any polyhedrons which have the net of the polyhedrons satisfying all of the requirements in part 1 can be played by using the winning strategy in part 2.

Acknowledgments

This article was served as T. H. Molena’s Graduating Senior Capstone from Berea College, which is a liberal arts college in Berea, Kentucky, in August of 2021.

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