Abstract

The alpha complex efficiently computes persistent homology of a point cloud $X$ in Euclidean space when the dimension $d$ is low. Given a subset $A$ of $X$, relative persistent homology can be computed as the persistent homology of the relative Čech complex $\tilde{C}(X, A)$. But this is not computationally feasible for larger point clouds $X$. The aim of this note is to present a method for efficient computation of relative persistent homology in low dimensional Euclidean space. We introduce the relative Delaunay Čech complex $\text{Del}\tilde{C}(X, A)$ whose homology is the relative persistent homology. It can be constructed from the Delaunay complex of an embedding of the point clouds in $(d+1)$-dimensional Euclidean space.

Keywords—topological data analysis, relative homology, Delaunay-Čech complex, alpha complex

1 Introduction

Persistent homology is receiving growing attention in the machine learning community. In that light, the scalability of persistent homology computations is of increasing importance. To date, the alpha complex is the most widely used method to compute persistent homology for large low-dimensional data sets.

Relative persistent homology has been considered several times in recent years. For example Edelsbrunner and Harrer [1] have presented an application of relative persistent homology to estimate the dimension of an embedded manifold. Relative persistent homology is also a way to introduce the concept of extended persistence [2]. De Silva and others have shown that the relative persistent homology $H_\ast(X, A_t)$ with an increasing family of sets $A_t$ and a constant $X = \cup_t A_t$, and the corresponding relative persistent cohomology have the same barcode [3]. They also show that absolute persistent homology of $A_t$ can be computed from this particular type of relative persistent homology. More recently, Pokorny and others [4] have used relative persistent homology to cluster two-dimensional trajectories. Some software, such as PHAT [5], even allows for the direct computation of relative persistent homology. For an example see the PHAT github repository.

Despite the fact that relative persistent homology has been considered in many different situations, we are not aware of a relative version of the alpha- or Delaunay Čech complexes being used.

Our contributions are as follows.

1. We give a new elementary proof that the Delaunay Čech complex is homotopy equivalent to the Čech complex. This has previously been shown using discrete Morse theory [6].

2. We extend this proof to the relative versions of the Delaunay Čech complex and the Čech complex.

3. We explain how the relative Delaunay Čech complex can be computed through embedding in a higher dimension.

Together, these contributions result in theorem[14] which shows how the relative persistent homology of Čech persistence modules $\tilde{C}_\ast(X; k)/\tilde{C}_\ast(A; k)$ of low-dimensional spaces can be efficiently computed using a relative Delaunay-Čech complex.
Theorem 1.1. Let $A \subseteq X \subseteq \mathbb{R}^d$ be finite. The relative Delaunay-Čech complex $\text{DelČ}(X, A)$ defined in section 2 is homotopy equivalent to the relative Čech complex $\check{C}(X, A)$.

Moreover, given the cardinalities $n_X$ of $X$ and $n_A$ of $A$, the relative Delaunay-Čech complex contains at most $O((n_X + n_A)(d + 1)/2)$ simplices.

This manuscript is structured as follows. In section 2 we introduce relative persistent homology. Section 3 introduces Dowker Nerves, the theoretical foundation we use to prove that the relative Delaunay Čech complex is homotopy equivalent to the relative Čech complex. In section 4 we introduce the alpha- and Delaunay-Čech complexes using the Dowker Nerve notation and show that they are homotopy equivalent to the Čech complex. Section 5 introduces the relative alpha- and Delaunay-Čech complexes, and proves that they are homotopy equivalent to the relative Čech complex. Finally, in section 6 we show how the relative Delaunay-Čech complex can actually be computed.

2 Relative Persistent Homology

Let $X$ be a finite subset of Euclidean space $\mathbb{R}^d$. Given $t > 0$, the Čech complex $\check{C}_t(X)$ of $X$ is the abstract simplicial complex with vertex set $X$ and with $\sigma \subseteq X$ a simplex of $\check{C}_t(X)$ if and only if there exists a point $p \in \mathbb{R}^d$ with distance less than $t$ to every point in $\sigma$. Varying $t$ we obtain the filtered Čech complex $\check{C}(X)$.

Given a subset $A$ of $X$ we obtain an inclusion $\check{C}(A) \subseteq \check{C}(X)$ of filtered simplicial complexes and an induced inclusion $\check{C}_*(A;k) \subseteq \check{C}_*(X;k)$ of associated chain complexes of persistence modules over the field $k$. The relative persistent homology of the pair $(X, A)$ is defined as the homology of the factor chain complex of persistence modules $\check{C}_*(X;k)/\check{C}_*(A;k)$.

For $X$ of small cardinality, the relative persistent homology can be calculated as the reduced persistent homology of the relative Čech complex $\check{C}(X, A)$, where $\sigma \subseteq X$ is a simplex of $\check{C}(X, A)$, if either $\sigma \subseteq A$ or $\sigma \in \check{C}_t(X)$. However, as the cardinality of $X$ grows, this quickly becomes computationally infeasible.

3 Dowker Nerves

A dissimilarity is a continuous function of the form $\Lambda: X \times Y \to [0, \infty]$, for topological spaces $X$ and $Y$, where $[0, \infty]$ is given the order topology. A morphism $f: \Lambda \to \Lambda'$ of dissimilarities $\Lambda: X \times Y \to [0, \infty]$ and $\Lambda': X' \times Y' \to [0, \infty]$ consists of a pair $(f_1, f_2)$ of continuous functions $f_1: X \to X'$ and $f_2: Y \to Y'$ so that for all $(x, y) \in X \times Y$ the following inequality holds:

$$\Lambda'(f_1(x), f_2(y)) \leq \Lambda(x, y).$$

This notion of morphism is less general than the definition in for example [1, Definition 2.10], but it is simpler and suffices for our purposes.

The Dowker Nerve $\Lambda N$ of $\Lambda$ is the filtered simplicial complex described as follows: For $t > 0$, the simplicial complex $\Lambda N_t$ consists of the finite subsets $\sigma$ of $X$ for which there exists $y \in Y$ so that $\Lambda(x, y) < t$ for every $x \in \sigma$.

Let $f: \Lambda \to \Lambda'$ be a morphism of dissimilarities as above and let $\sigma \in \Lambda N_t$. Given $y \in Y$ with $\Lambda(x, y) < t$ for every $x \in \sigma$ we see that

$$\Lambda'(f_1(x), f_2(y)) \leq \Lambda(x, y) < t.$$
1. The subset $\alpha^{-1}(0, 1]$ of $V$ consisting of elements where $\alpha$ is strictly positive is a simplex in $K$. In particular it is finite.

2. The sum of the values of $\alpha$ is one, that is $\sum_{v \in V} \alpha(v) = 1$.

The subspace topology on $|K|$ is called the strong topology on the geometric realization. It is convenient for construction of functions into $|K|$. The weak topology on $|K|$, which we are not going to use here, is convenient for construction of functions out of $|K|$. The homotopy type of $|K|$ is the same for these two topologies [S p. 355, Corollary A.2.9]. Given a simplex $\sigma \in K$, the simplex $|\sigma|$ of $|K|$ is the closure of $\{\alpha : V \to [0, 1] | \alpha(v) > 0 \text{ for all } v \in \sigma\}$.

The simplices of $|K|$ are the sets of this form.

A partition of unity subordinate to the dissimilarity $\Lambda : X \times Y \to [0, \infty]$ consists of continuous maps $\varphi^t : \Lambda^t \to |N\Lambda_t|$ such that given $x \in X$, the closure of the set $\{y \in Y | \varphi^t(y)(x) > 0\}$ is contained in $B_{\Lambda}(x, t)$. We say that $\Lambda$ is numerable if a partition of unity subordinate to $\Lambda$ exists. If $Y$ is paracompact, then every dissimilarity of the form $\Lambda : X \times Y \to [0, \infty]$ is numerable [S p. 355, paragraph after Definition A.2.10].

Let $y \in \Lambda^t$ and let $\{\varphi^t : \Lambda^t \to |N\Lambda_t|\}$ be a partition of unity subordinate to $\Lambda$. If $x \in X$ with $\varphi^t(y)(x) > 0$, then $\Lambda(x, y) < t$. Therefore $\varphi^t(y)$ is contained in a simplex $|\sigma|$ in $|N\Lambda_t|$ with $\sigma$ contained in $\{x \in X | \Lambda(x, y) < t\}$. Every finite subset of this set is an element of $N\Lambda_t$. This implies that for $s \leq t$ there is a simplex of $|N\Lambda_s|$ containing both $\varphi^s(y)$ and $\varphi^t(y)$. It also implies that given another partition of unity $\{\psi^t : \Lambda^t \to |N\Lambda_t|\}$ subordinate to $\Lambda$ there is a simplex of $|N\Lambda_t|$ containing both $\varphi^t(y)$ and $\psi^t(y)$. This is exactly the definition of contiguous maps, so $\varphi^t$ and $\psi^t$ are contiguous, and thus homotopic maps [S Remak 2.22, p. 350].

Similarly, the diagram

$$
\Lambda^t \xrightarrow{\varphi^t} |N\Lambda_t| \\
\downarrow \downarrow \\
\Lambda^t \xrightarrow{\psi^t} |N\Lambda_t|
$$

commutes up to homotopy [S paragraph on the nerve starting on page 355 and ending on page 356].

Recall that a cover $U$ of $Y$ is good if all non-empty finite intersections of members of $U$ are contractible. We now state the Nerve Lemma in the context of dissimilarities.

**Theorem 3.1.** If $Y$ is paracompact, then there exists a partition of unity $\{\varphi^t : \Lambda^t \to |N\Lambda_t|\}$ subordinate to every dissimilarity $\Lambda : X \times Y \to [0, \infty]$. Moreover, if the cover of $\Lambda^t$ by $\Lambda$-balls of radius $t$ is a good cover, then $\varphi^t$ is a homotopy equivalence.

**Proof.** By the above discussion, we only need to note that the last statement about good covers is [S Theorem 4.3].

A functorial version of the Nerve Lemma can be stated as follows:

**Proposition 3.2.** Let $\Lambda : X \times Y \to [0, \infty]$ and $\Lambda' : X' \times Y' \to [0, \infty]$ be dissimilarities and let $f = f_1 \times f_2 : X \times Y \to X' \times Y'$ be a morphism $f : \Lambda \to \Lambda'$ of dissimilarities. If $\{\varphi^t : \Lambda^t \to |N\Lambda_t|\}$ is a partition of unity subordinate to $\Lambda$ and $\{\psi^t : (\Lambda')^t \to |N\Lambda'_{t}|\}$ is a partition of unity subordinate to $\Lambda'$, then for every $t \geq 0$ the diagram

$$
\Lambda^t \xrightarrow{\varphi^t} |N\Lambda_t| \\
\downarrow f_2 \downarrow |f_1| \\
(\Lambda')^t \xrightarrow{\psi^t} |N\Lambda'_{t}|
$$

commutes up to homotopy.

**Proof.** We show that the two compositions are contiguous. Recall that $|f_1|$ takes a point $\alpha : X \to [0, 1]$ of $|N\Lambda_t|$ to the point $|f_1|(|\alpha|) = \sum_{x \in X} f_1(x) \alpha(x)$. Recall further that $\varphi^t(y)$ is contained in a simplex $|\sigma|$ in $|N\Lambda_t|$, where $\sigma$ is contained in $\{x \in X | \Lambda(x, y) < t\}$. Then we have that for $y \in \Lambda^t$, the elements $|f_1|(|\varphi^t(y)|)$ and $|\psi^t(f_2(y))|$ of $|N\Lambda'_{t}|$ are contained in simplices $|\sigma|$ and $|\tau|$ respectively. Both $\sigma$ and $\tau$ are subsets of the set $\{x' \in X' | \Lambda'(x', f_2(y)) < t\}$. However every finite subset of this set is a simplex in $N\Lambda'_{t}$. In particular, so is the union $\sigma \cup \tau$. 

\[ \square \]
4 The Alpha- and Delaunay Čech Complexes

Given a finite subset $X$ of $\mathbb{R}^d$ we define the Voronoi cell of $x \in X$ as

$$\text{Vor}(X, x) = \{ p \in \mathbb{R}^d \mid d(x, p) \leq d(y, p) \text{ for all } y \in X \}.$$ 

Let $\mathbb{R}^d$ be Euclidean space with the discrete topology. The discrete Delaunay dissimilarity of $X$ is defined as

$$d^X: X \times \mathbb{R}^d \to [0, \infty], \quad d^X(x, p) = \begin{cases} 0 & \text{if } p \in V(X, x) \\ \infty & \text{if } p \notin V(X, x). \end{cases}$$

The Delaunay complex $\text{Del}(X)$ is the simplicial complex with vertex set $X$ and with $\sigma \subseteq X$ a simplex of $\text{Del}(X)$ if and only if there exists a point in $\mathbb{R}^d$ belonging to $\text{Vor}(X, x)$ for every $x \in \sigma$. That is, $\text{Del}(X) = N d^X_i$ for $t > 0$.

Note that with respect to Euclidean topology, the discrete Delaunay dissimilarity is not continuous, and hence $d^X: X \times \mathbb{R}^d \to [0, \infty]$ is not a dissimilarity. One way to deal with this is to use the Nerve Lemma for absolute neighbourhood retracts [10, Theorem 8.2.1]. In order to use theorem 3.1 and proposition 3.2 from above, instead we construct a continuous version of the Delaunay dissimilarity.

Given a subset $\sigma$ of $X$ and $p \in \mathbb{R}^d$, let

$$d_{\text{Vor}}(p, \sigma) = \max\{d(p, \text{Vor}(X, x)) \mid x \in \sigma\},$$

where for any $A \subseteq \mathbb{R}^d$, we define $d(p, A) = \inf_{a \in A}\{d(p, a)\}$.

Note that if $\sigma \notin \text{Del}(X)$, then the infimum $\varepsilon_{\sigma}$ of the continuous function $d_{\text{Vor}}(-, \sigma): \mathbb{R}^d \to \mathbb{R}$ is strictly positive. Choose $\varepsilon > 0$ so that $2\varepsilon < \varepsilon_{\sigma}$ for every subset $\sigma$ of $X$ that is not in $\text{Del}(X)$. Given $x \in X$ we define the $\varepsilon$-thickened Voronoi cell $\text{Vor}(X, x)^\varepsilon$ by

$$\text{Vor}(X, x)^\varepsilon = \{ p \in \mathbb{R}^d \mid d(p, \text{Vor}(X, x)) < \varepsilon \}.$$ 

By construction the nerve of the open cover $(\text{Vor}(X, x)^\varepsilon)_{x \in X}$ of $\mathbb{R}^d$ is equal to $\text{Del}(X)$.

Let $h: [0, \infty] \to [0, \infty]$ be the order preserving map

$$h(t) = \begin{cases} -\ln(1 - t/\varepsilon) & \text{if } t < \varepsilon \\ \infty & \text{if } t \geq \varepsilon. \end{cases}$$

For each $x \in X$ we let $\text{Del}^t_x: \mathbb{R}^d \to [0, \infty]$ be the function defined by $\text{Del}^t_x(p) = h(d(p, \text{Vor}(X, x)))$ so that $\text{Del}^t_x(\text{Vor}(X, x)) = 0$ and $\text{Del}^t_x(\mathbb{R}^d \setminus \text{Vor}(X, x)^\varepsilon) = \infty$.

The Delaunay dissimilarity of $X$ is defined as

$$\text{Del}^X: X \times \mathbb{R}^d \to [0, \infty], \quad \text{Del}^X(x, p) = \text{Del}^t_x(p).$$

By the above discussion we know that $N \text{Del}^1_X = N \text{Del}^X_i = \text{Del}(X)$ whenever $t > 0$.

The Čech dissimilarity of $X$ is defined as

$$d^X: X \times \mathbb{R}^d \to [0, \infty],$$

where $d^X(x, p)$ is the Euclidean distance between $x \in X$ and $p \in \mathbb{R}^d$.

The alpha dissimilarity of $X$ is defined as

$$A^X = \max(\text{Del}^X, d^X): X \times \mathbb{R}^d \to [0, \infty].$$

The Delaunay Čech dissimilarity is defined as

$$\text{Del}^C^X: X \times (\mathbb{R}^d \times \mathbb{R}^d) \to [0, \infty], \quad \text{Del}^C^X(x, (p, q)) = \max(d^X(x, p), \text{Del}^X(x, q)).$$

Note the nerve of the dissimilarity

$$\text{del}^C^X: X \times (\mathbb{R}^d \times \mathbb{R}^d) \to [0, \infty], \quad \text{del}^C^X(x, (p, q)) = \max(d^X(x, p), \text{del}^X(x, q))$$

is identical to the nerve of $\text{Del}^C^X$. Moreover, the Dowker nerves of the Delaunay-, Čech-, alpha- and Delaunay Čech dissimilarities are the Delaunay-, Čech-, alpha- and Delaunay Čech complexes respectively. For all these dissimilarities, the corresponding balls are convex, so the geometric realizations are homotopy
equivalent to the corresponding thickenings. In order to see that the morphism \( A^X \rightarrow d^X \) of dissimilarities induces homotopy equivalences \(|NA^X| \xrightarrow{\simeq} |Nd^X|\) it suffices to note that the corresponding map \((A^X)^t \rightarrow (d^X)^t\) is the identity map. This holds because \(B_{A^X}(x,t) = B_{d^X}(x,t)\) and \(y \leq \gamma(y)\) that the point \((x',t)\) for some \(x' \in X\) with \(d^X(y,x')\) minimal, so \(d^X(y,x') < t\) and \(y \in B_{d^X}(x',t) \cap B_{A^X}(x',t)\).

In order to see that the morphism \(\text{Del}C^X \rightarrow d^X\) of dissimilarities induces homotopy equivalences \(|N\text{Del}C^X| \xrightarrow{\simeq} |Nd^X|\) we use the following lemma:

**Lemma 4.1.** For every \((p,q) \in (\text{Del}C^X)^t\), the entire line segment between \((p,p)\) and \((p,q)\) is contained in \((\text{Del}C)^t\).

**Proof.** In order not to clutter notation we omit superscript script \(X\) on dissimilarities. Let \(\gamma: [0,1] \rightarrow \mathbb{R}^d\) be the function \(\gamma(s) = (p,(1-s)p + sq)\). We claim that given \((p,q) \in \text{Del}C^t\) and \(s \in [0,1]\) the point \((p,\gamma(s)) = (p,(1-s)p + sq)\) is in \(\text{Del}C^t\).

If \((p,q) \in \text{Del}C^t\), there exists a point \(x \in X\), such that \(p \in B_d(x,t)\) and \(q \in B_{\text{Del}}(x,t)\), that is, \(d(q,\text{Vor}(X,x)) < h^{-1}(t)\), where \(h^{-1}\) is the generalized inverse of \(h\). Pick \(q' \in \text{Vor}(X,x)\) so that \(d(q,q') < h^{-1}(t)\). Let \(\gamma': [0,1] \rightarrow \mathbb{R}^d\) be the function \(\gamma'(s) = (p,(1-s)p + sq')\). Given \(s \in [0,1]\), suppose that the point \((p,\gamma'(s)) = (p,(1-s)p + sq')\) is in \(\text{Del}C^t\). Then \(\gamma(s)\) is in \(\text{Del}C^t\) since the distance between \((1-s)p + sq\) and \((1-s)p + sq'\) is less than \(h^{-1}(t)\).

We are left to show that, given \(s \in [0,1]\), the point \((p,\gamma'(s)) = (p,(1-s)p + sq')\) is in \(\text{del}C^t\). Suppose \(\gamma'(s) \in \text{Vor}(X,y)\) for some \(s \in [0,1]\) and some \(y \in X\). We claim that then \(p \in B_d(y,t)\). To see this, we may without loss of generality assume that \(y \neq x\). Let \(H\) be the hyperplane in between \(x\) and \(y\), i.e.

\[
H = \{z \in X \mid d(x,z) = d(y,z)\}.
\]

Let

\[
H_+ = \{z \in X \mid d(x,z) \geq d(y,z)\}
\]

and

\[
H_- = \{z \in X \mid d(x,z) \leq d(y,z)\}.
\]

Since \(\gamma'(s) \in \text{Vor}(X,y)\) we have \(\gamma'(s) \in H_+\). Since \(q \in \text{Vor}(X,x)\) we have \(q \in H_-\). Since the line segment between \(p\) and \(q\) either is contained in \(H\) or intersects \(H\) at most once we must have \(p \in H_+\). That is, \(d(y,p) \leq d(x,p) < t\), so \(p \in B_d(y,t)\) as claimed. \(\square\)

By lemma 41 the inclusion

\[
(d^X)^t = \cup_{x \in X} B_{d^X}(x,t) \rightarrow \cup_{x \in X} B_{\text{Del}C^X}(x,t) = (\text{Del}C^X)^t, \quad p \mapsto (p,p)
\]

is a deformation retract. In particular it is a homotopy equivalence.

## 5 The Relative Delaunay Čech Complex

In this section we consider two subsets \(X_1\) and \(X_2\) of \(d\)-dimensional Euclidean space \(\mathbb{R}^d\).

The **Voronoi diagram** of a finite subset \(X\) of \(\mathbb{R}^d\) is the set of pairs of the form \((x,\text{Vor}(X,x))\) for \(x \in X\), that is,

\[
\text{Vor}(X) = \{(x,\text{Vor}(X,x)) \mid x \in X\}.
\]

This may seem overly formal since the projection on the first factor gives a bijection \(\text{Vor}(X) \rightarrow X\). However, when we work with Voronoi cells with respect to different subsets \(X_1\) and \(X_2\) of \(\mathbb{R}^d\) it may happen that \(\text{Vor}(X_1,x_1) = \text{Vor}(X_2,x_2)\) even when \(x_1 \neq x_2\). The **Voronoi diagram** of the pair of subsets \(X_1\) and \(X_2\) of \(\mathbb{R}^d\) is the set

\[
\text{Vor}(X_1,X_2) = \text{Vor}(X_1) \cup \text{Vor}(X_2).
\]

The **discrete Delaunay dissimilarity** of \(X_1\) and \(X_2\) is defined as

\[
\text{del}^{X_1,X_2}: \text{Vor}(X_1,X_2) \times \mathbb{R}^d \rightarrow [0,\infty], \quad \text{del}^{X_1,X_2}((x,V),p) = \begin{cases} 0 & \text{if } p \in V, \\ \infty & \text{if } p \notin V. \end{cases}
\]
The simplicial complex $N \Delta_t^{X_1, X_2}$ is independent of $t > 0$. It is the Delaunay complex $\text{Del}(X_1, X_2)$ on $X_1$ and $X_2$. In order to describe the homotopy type of this simplicial complex we thicken the Voronoi cells like we did in the previous section:

Given a subset $\sigma$ of $\text{Vor}(X_1, X_2)$ and $p \in \mathbb{R}^d$, let

$$d_{\text{Vor}}(p, \sigma) = \max \{ d(p, V) \mid (x, V) \in \sigma \}.$$ 

Note that if $\sigma \notin \text{Del}(X_1, X_2)$, then the infimum $\epsilon_{\sigma}$ of the continuous function $d_{\text{Vor}}(\cdot, \sigma) : \mathbb{R}^d \to \mathbb{R}$ is strictly positive. Choose $\epsilon > 0$ so that $2\epsilon < \epsilon_{\sigma}$ for every subset $\sigma$ of $\text{Vor}(X_1, X_2)$ that is not in $\text{Del}(X_1, X_2)$. Given $(x, V) \in \text{Vor}(X_1, X_2)$ we define the $\epsilon$-thickening $V^\epsilon$ of $V$ by

$$V^\epsilon = \{ p \in \mathbb{R}^d \mid d(p, V) < \epsilon \}.$$ 

By construction, the nerve of the open cover $((x, V^\epsilon))_{(x, V) \in \text{Vor}(X_1, X_2)}$ is equal to $\text{Del}(X_1, X_2)$. The Delaunay dissimilarity $\text{Del}^{X_1, X_2}$ of $X_1$ and $X_2$ is defined as

$$\text{Vor}(X_1, X_2) \times \mathbb{R}^d \xrightarrow{\text{Del}^{X_1, X_2}} [0, \infty], \quad (x, V, p) \mapsto h(d(p, V))$$

for $h : [0, \infty) \to [0, \infty]$ the order preserving map defined in the previous section.

The inclusion $X_1 \to \text{Vor}(X_1, X_2)$ taking $x \in X_1$ to $(x, \text{Vor}(x, X_1))$ induces a morphism of dissimilarities $\text{Del}^{X_1} \to \text{Del}^{X_1, X_2}$ and an inclusion of nerves $N \text{Del}^{X_1} \subseteq N \text{Del}^{X_1, X_2}$ for $t > 0$.

Next, we construct the dissimilarity $A^{X_1, X_2}$ as

$$\text{Vor}(X_1, X_2) \times \mathbb{R}^d \xrightarrow{A^{X_1, X_2}} [0, \infty], \quad (x, V, p) \mapsto \max(d(x, p), \text{Del}^{X_1, X_2}((x, V), p)).$$

Also here we have an obvious inclusion $N A_t^{X_1} \to N A_t^{X_1, X_2}$, and the $A^{X_1, X_2}$-balls are convex so the nerve lemma yields a homotopy equivalence

$$|N A_t^{X_1, X_2}| \simeq \bigcup_{(x, V) \in \text{Vor}(X_1, X_2)} B_{A^{X_1, X_2}}((x, V), t) = \bigcup_{x \in X_1 \cup X_2} B_{A^{X_1, X_2}}(x, t) = (X_1 \cup X_2)^t.$$

Finally, we construct the dissimilarity $\text{Del}^{X_1, X_2}$

$$\text{Vor}(X_1, X_2) \times (\mathbb{R}^d \times \mathbb{R}^d) \xrightarrow{\text{Del}^{X_1, X_2}} [0, \infty], \quad ((x, V), (p, q)) \mapsto \max(d(x, p), \text{Del}^{X_1, X_2}((x, V), q)).$$

Here again we have an obvious inclusion $N \text{Del}^{X_1} \to N \text{Del}^{X_1, X_2}$, and the $\text{Del}^{X_1, X_2}$-balls are convex so the nerve lemma yields a homotopy equivalence

$$|N \text{Del}^{X_1, X_2}_t| \simeq (\text{Del}^{X_1, X_2}_t)^t.$$

The following variant of lemma implies that $(\text{Del}^{X_1, X_2}_t)^t$ is a deformation retract of $(X_1 \cup X_2)^t$.

**Lemma 5.1.** For every $(p, q) \in (\text{Del}^{X_1, X_2}_t)^t$, the entire line segment between $(p, p)$ and $(p, q)$ is contained in $(\text{Del}^{X_1, X_2}_t)^t$.

**Proof.** Given $(p, q) \in (\text{Del}^{X_1, X_2}_t)^t = (\text{Del}^{X_1}_t)^t \cup (\text{Del}^{X_2}_t)^t$, we have $(p, q) \in (\text{Del}^{X_1}_t)^t$ for some $i \in \{1, 2\}$. Then also $(p, p)$ lies in $(\text{Del}^{X_1}_t)^t$, and lemma proves the claim.

## 6 Implementation Of The Relative Delaunay Čech Complex

In this section we explain how the relative Delaunay complex can be realized as a standard Delaunay complex by embedding in one dimension higher.

We fix some notation used in this section: $X_1 \subseteq \mathbb{R}^d$ and $X_2 \subseteq \mathbb{R}^d$ are finite subsets. We let $s$ be a positive real number, we let $Z = X_1 \times \{s\} \cup X_2 \times \{-s\}$ and we let $\text{pr} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection omitting the last coordinate.
Lemma 6.1. The projection $pr: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ induces a surjection

$$Vor(Z) \xrightarrow{pr} Vor(X_1, X_2), \quad ((x, s), V) \mapsto (x, V(x_1, x)),$$

$$((x, -s), V) \mapsto (x, V(x(X_2), x)),$$

with $pr(V) \subseteq V(X_1, x)$ for $x \in X_i$. Given $(x, V) \in Vor(X_1, X_2)$ the fiber $g^{-1}((x, V))$ consists of all elements of $Vor(Z)$ of the form $((x, a), V)$ for $a \in \{\pm s\}$.

Proof. We show that $pr(V) \subseteq V(X_1, x_1)$ for $((x_1, s), V) \in Vor(Z)$ with $x_1 \in X_1$. Given $(p, r) \in V$ we have for all points of the form $(x_1', s)$ for $x_1' \in X_1$ that $d((p, r), (x_1, s)) \leq d((p, r), (x_1', s))$. This implies that $d(p, x_1) \leq d(p, x_1')$, and thus $p \in V(X_1, x_1)$. We conclude that $pr(V) \subseteq V(X_1, x_1)$. An analogous argument applies for elements of the form $((x_2, -s), V) \in Vor(Z)$.

Let $s_1$ be larger than the largest filtration value of the alpha complex of $X_1$. Then the function $j_1: Vor(X_1) \rightarrow Vor(Z)$ defined by $j_1(x_1, V) = ((x_1, s), V(Z, (x_1, s)))$ induces a simplicial map of nerves $del(X_1) \rightarrow del(Z)$ for all $s > s_1$. Similarly, there is a simplicial map $del(X_2) \rightarrow del(Z)$ for all $s > s_2$ when $s_2$ is larger than all filtration values of the alpha complex of $X_2$. Let $s(X_1, X_2) = \max(s_1, s_2)$.

Choose $\varepsilon > 0$ satisfying the following two criteria:

1. $2\varepsilon < \varepsilon_0$ for every subset $\sigma$ of $Vor(X_1, X_2)$ that is not in $Del(X_1, X_2)$.
2. $2\varepsilon < \varepsilon_0$ for every subset $\sigma$ of $Vor(Z)$ that is not in $Del(Z)$.

Let $h: [0, \infty] \rightarrow [0, \infty]$ be the order preserving map defined in eq. (1), and let $Del^Z$ and $Del^{X_1, X_2}$ be constructed using $h$. We define a new dissimilarity

$$D: Vor(Z) \times (\mathbb{R}^d \times \mathbb{R}^{d+1}) \rightarrow [0, \infty], \quad D((z, V), (p, q)) = \max(d(pr(z), p), Del^Z((z, V), q)).$$

Note that the underlying simplicial complex $\bigcup_{t>0} ND_t$ of the nerve of $D$ is the Delaunay complex $del(Z)$. The filtration value of $\sigma \in del(Z)$ in the neve of $D$ is the filtration value of $g(\sigma)$ in the nerve of $Del^{X_1, X_2}$.

Proposition 6.2. Let $X_1 \subseteq \mathbb{R}^d$ and $X_2 \subseteq \mathbb{R}^d$ be finite. Choose $s > s(X_1, X_2)$. Then $Vor(Z) \xrightarrow{pr} Vor(X_1, X_2)$ and $id \times pr: \mathbb{R}^d \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ form a morphism

$$f = (g, id \times pr): D \rightarrow Del^{X_1, X_2}$$

of dissimilarities inducing a homotopy equivalence

$$g: ND_t \rightarrow NDel^{X_1, X_2}_t$$

for every $t > 0$.

Proof. For $i = 1, 2$ the inclusion $pr(V) \subseteq V(X_i, x)$ for $((x, (-1)^{i-1}s), V) \in Vor(Z)$ implies that

$$Del^{X_1, X_2}(g(z, V), pr(q)) \subseteq Del^Z((z, V), q)$$

for all $((z, V), q) \in Vor(Z)$. So we have a morphism $f = (g, id \times pr): D \rightarrow Del^{X_1, X_2}$.

In order to show that $g$ induces a homotopy equivalence of geometric realizations, by the Nerve Lemma, it suffices to show that given a simplex $\sigma$ of $N Del^{X_1, X_2}_t$, the inverse image $g^{-1}(\sigma)$ is a simplex of $ND_t$. Let $p$ be a point in the intersection of the Voronoi cells in $\sigma$. Write $g^{-1}(\sigma) = \tau_1 \cup \tau_2$, where $\tau_1$ consists of Voronoi cells with centers at height $s$ and $\tau_2$ consists of Voronoi cells with centers at height $-s$. Let $\sigma_1 = \{(x_1, s) \mid (x_1, V(x_1, x_1)) \in \sigma\}$ and $\sigma_2 = \{(x_2, -s) \mid (x_2, V(x_2, x_2)) \in \sigma\}$.

Suppose that $\tau_2$ is empty. Then actually $\sigma \in Del^{X_1}_t$, and since $s > s_1$ we know that $j_1(\sigma) \in del(Z)$. Since $g \circ j_1$ is the inclusion of $Vor(X_1)$ in $Vor(X_1, X_2) = Vor(X_1) \cup Vor(X_2)$ we know that $j_1(\sigma) \subseteq g^{-1}(\sigma) = \tau_1$ and that $j_1(\sigma) \in ND_t$. On the other hand, since $\tau_2$ is empty and $j_1$ is injective, we know that $g^{-1}(\sigma)$ has the same cardinality as $j_1(\sigma)$, so they must be equal. We conclude that $g^{-1}(\sigma)$ is a simplex of $ND_t$. A similar argument applies when $\tau_1$ is empty.

In the remaining case where both $\tau_1$ and $\tau_2$ are nonempty, the function

$$f: \mathbb{R}^{d+1} \rightarrow \mathbb{R}: \quad f(a) = d_{Vor}(a, \sigma_1) - d_{Vor}(a, \sigma_2)$$

has $f((p, -s)) > 0$ and $f((p, s)) < 0$. By the intermediate value theorem there exists $t \in [-s, s]$ with $f(p, t) = 0$. Since $(p, t)$ has the same distance to all elements of $\sigma_1$ and also has the same distance to all elements of $\sigma_2$ we conclude that $(p, t)$ is in the intersection of the Voronoi cells in $g^{-1}(\sigma) = \tau_1 \cup \tau_2$. Thus $Del^Z((z, V), p) = 0$ and $d(pr(z), p) < t$ for all $(z, V) \in g^{-1}(\sigma)$. In particular $g^{-1}(\sigma) \in ND_t$. 

\[7\]
We are now ready to compute persistent homology of \( X_1 \cup X_2 \) relative to \( X_1 \). The relative Delaunay-Čech complex \( \text{Del} \mathcal{C}(X_1 \cup X_2, X_1) \) is the filtered simplicial complex with \( \text{Del} \mathcal{C}(X_1 \cup X_2, X_1)_t = j_1(\text{Del}(X_1)) \cup ND_t \).

**Theorem 6.3.** Let \( X_1 \subseteq \mathbb{R}^d \) and \( X_2 \subseteq \mathbb{R}^d \) be finite. Choose \( s > s(X_1, X_2) \). Then there is an isomorphism

\[
(H_\ast(\text{Del} \mathcal{C}(X_1 \cup X_2, X_1)_t)_{t > 0} \cong (H_\ast((X_1 \cup X_2)^t, X_1^t))_{t > 0}
\]

of persistence modules.

**Proof.** Since \( j_1(\text{Del}(X_1)) \) is contractible, the geometric realization of \( \text{Del} \mathcal{C}(X_1 \cup X_2, X_1)_t \) is homotopy equivalent to the quotient space \( |\text{Del} \mathcal{C}(X_1 \cup X_2, X_1)_t|/|j_1(\text{Del}(X_1))| \). This quotient space is homeomorphic to \( |ND_t|/|ND_t \cap j_1(\text{Del}(X_1))| \). By proposition 6.2 the map \( g: ND_t \to N \text{Del} \mathcal{C}_t^{X_1} \cup X_2 \) induces a homotopy equivalence of geometric realizations. Moreover \( g \) induces an isomorphism \( ND_t \cap j_1(\text{Del}(X_1)) \to N \text{Del} \mathcal{C}_t^{X_1} \cup X_2 \). Combining these two statements, \( g \) induces a homotopy equivalence \( |ND_t|/|ND_t \cap j_1(\text{Del}(X_1))| \to |N \text{Del} \mathcal{C}_t^{X_1} \cup X_2|/|N \text{Del} \mathcal{C}_t^{X_1} \cup X_2| \). The space \( |N \text{Del} \mathcal{C}_t^{X_1} \cup X_2| \) is homotopy equivalent to the Euclidean \( t \)-thickening \( (X_1 \cup X_2)^t \) of \( X_1 \cup X_2 \) and \( |N \text{Del} \mathcal{C}_t^{X_1}| \) is homotopy equivalent to the Euclidean \( t \)-thickening \( X_1^t \) of \( X_1 \).

Finally, we note that the size of the relative Delaunay-Čech complex grows linearly with the sizes \( n_i \) of the finite subsets \( X_i \). The Delaunay triangulation of \( n \) points in \( d \) dimensions contains at most \( O(n(d/2)) \) simplices [11]. Since we use the Delaunay triangulation of \( n_1 + n_2 \) points in \( d + 1 \) dimensions to compute the relative Delaunay-Čech complex, it contains at most \( O((n_1 + n_2)((d + 1)/2)) \) simplices. This concludes the proof of theorem [11].

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