Massive higher spin supermultiplets unfolded

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Abstract

In this paper we construct an unfolded formulation for the massive higher spin $N = 1$ supermultiplets in four dimensional $\text{AdS}$ space. We use the same frame-like gauge invariant multispinor formalism that was used previously for their Lagrangian formulation. We also consider an infinite spin limit of such supermultiplets.

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Introduction

Lagrangian formulation for the massless higher spin supermultiplets (both on-shell and off-shell, both in flat space and in \(AdS\)) has been known for a long time [1–5]. However, any attempts to deform massless supermultiplets into the massive ones lead to the introduction of very complicated higher derivative corrections to the supertransformations without evident patterns. Moreover, the higher superspin of the supermultiplets is, the higher the number of derivatives one has to consider. Even the usage of the powerful superfield formalism allowed to construct only a couple of examples with relatively low superspins [6,7].

For the first time massive arbitrary superspin \(N = 1\) supermultiplets in flat four dimensional space were constructed in [8] using the gauge invariant formulation for the massive bosonic [9] and fermionic [10] fields. Initial idea was that the massive supermultiplet can be constructed out of the appropriately chosen set of the massless ones in the same way as the gauge invariant description for the massive fields can be constructed out of the appropriate set of the massless ones. The real picture (in a sense of the massless limit) appeared to be slightly more complicated, but anyway the construction was successful.

Later on, the Lagrangian formulation for the higher spin massive supermultiplets in flat three dimensional space has also been constructed [11], again using the gauge invariant formulation for massive bosonic and fermionic fields adopted for \(d = 3\) [12,13]. The correct procedure to deform such supermultiplets into \(AdS_3\) space was not evident form the very beginning. It so happened that firstly the unfolded formulation has been constructed [14] based on the results in [15]. After that the Lagrangian formulation for these supermultiplets in \(AdS_3\) has also been completed [16,17].

Recently, we have managed to construct the Lagrangian formulation for massive higher spin \(N = 1\) supermultiplets in \(AdS_4\) [18] using the frame-like gauge invariant formalism [19] in its multispinor version adopted for \(d = 4\). Note that though the traditional classification of the supermultiplets describes only massless and massive ones, recently it was shown [20] that in \(AdS_4\) space there exist the non-unitary higher spin supermultiplets containing partially massless fields. The explicit Lagrangian formulation for such supermultiplets has been constructed in [21]. Note also that the first examples of the infinite spin supermultiplets in flat space were constructed recently [22,23] (see also recent paper [24]). Here again it was crucial that the gauge invariant formalism used for the description of massive finite spin fields nicely works for the infinite spin limit as well [22,25–29].

The main aim of this paper is to construct unfolded formulation for the massive higher spin (including infinite spin limit) \(N = 1\) supermultiplets in \(AdS_4\). Recall that the unfolded formulation for massive higher spin bosons in arbitrary \(d \geq 4\) has been constructed in [30], while such formulation both for bosons as well as fermions in \(AdS_4\) appeared recently in our work [29]. Note here that, as far as we know, till now only unfolded formulation for the scalar supermultiplet was considered [31,32].

The paper is organized as follows. In Section 1 as a simple illustration of our formalism we provide an unfolded formulation for the massless \(N = 1\) supermultiplets. Also, in Section 2 we give a pair of simple examples for the lower spin massive supermultiplets, namely the scalar and the vector ones. Section 3 is devoted to the main task — construction of the unfolded formulation for the massive arbitrary superspin \(N = 1\) supermultiplets. We follow the same strategy as in the construction of their Lagrangian formulation in [18]. Namely, first of all we provide the unfolded equations for the massive bosons and fermions. Then we consider a pair of boson and fermion and construct supertransformations leaving their unfolded equations invariant. At last we consider complete supermultiplets containing two bosons and two fermions and adjust their parameters so that the algebra of the supertransformations is closed. Section 4 is devoted to the infinite spin supermultiplets.
1 Massless higher spin supermultiplets

In this section we provide an unfolded formulation for the massless higher spin supermultiplets [15] in the frame-like multispinor formalism we use later on for the construction of the massive supermultiplets.

1.1 Unfolded equations

Let us briefly recall the unfolded description of massless higher spin fields (see e.g. [33]). To build a system of unfolded equations for spin-$s$ boson, one needs a set of gauge one-forms $\Omega^\alpha(s-1)\dot{\alpha}(s-1)$, $0 \leq m < s$ and a set of gauge invariant zero-forms $W^\alpha(k+s)(k-s)$, $k \geq s$ with their conjugates. The field $\Omega^\alpha(s-1)\dot{\alpha}(s-1)$ is the physical one. The gauge transformations for the one-forms are:

$$
\delta\Omega^\alpha(s-1)\dot{\alpha}(s-1) = D\eta^\alpha(s-1)\dot{\alpha}(s-1) + e^\alpha_\beta \eta^\alpha(s-2)\dot{\alpha}(s-1)\dot{\beta}(s-2) + e^\alpha_\beta \eta^\alpha(s-1)\dot{\alpha}(s-1)\dot{\beta}(s-2)
$$

$$
\delta\Omega^\alpha(s-1+m)\dot{\alpha}(s-1-m) = D\eta^\alpha(s-1+m)\dot{\alpha}(s-1-m) + \lambda^2 e^\alpha_\beta \eta^\alpha(s-2+m)\dot{\alpha}(s-1-m)\dot{\beta}(s-2-m)
$$

$$
\delta\Omega^\alpha(2s-2) = D\eta^\alpha(2s-2) + \lambda^2 e^\alpha_\delta \eta^\alpha(2s-3)\dot{\alpha}(s-1)
$$

A set of gauge invariant two-forms - "curvatures" - can be build from these one-forms:

$$
\mathcal{R}^\alpha(s-1)\dot{\alpha}(s-1) = D\Omega^\alpha(s-1)\dot{\alpha}(s-1) + e^\alpha_\beta \Omega^\alpha(s-2)\dot{\alpha}(s-1)\dot{\beta}(s-2) + e^\alpha_\beta \Omega^\alpha(s-1)\dot{\alpha}(s-1)\dot{\beta}(s-2)
$$

$$
\mathcal{R}^\alpha(s-1+m)\dot{\alpha}(s-1-m) = D\Omega^\alpha(s-1+m)\dot{\alpha}(s-1-m) + \lambda^2 e^\alpha_\beta \Omega^\alpha(s-2+m)\dot{\alpha}(s-1-m)\dot{\beta}(s-2-m)
$$

$$
\mathcal{R}^\alpha(2s-2) = D\Omega^\alpha(2s-2) + \lambda^2 e^\alpha_\delta \Omega^\alpha(2s-3)\dot{\alpha}(s-1)
$$

The system of unfolded equations then can be split into the three parts. The first one is the zero-curvature conditions (analogue of the zero torsion condition in gravity):

$$
\mathcal{R}^\alpha(s-1+m)\dot{\alpha}(s-1-m) = 0, \quad m < s - 1
$$

while the second one connects the one-forms and zero-forms sectors:

$$
\mathcal{R}^\alpha(2s-2) = -2E_\beta(2)W^\alpha(2s-2)\beta(2),
$$

and the third one contains gauge invariant zero-forms only:

$$
0 = DW^\alpha(2s) + e^\alpha_\beta W^\alpha(2s)\beta
$$

$$
0 = DW^\alpha(2s+m)\dot{\alpha}(m) + e^\alpha_\beta W^\alpha(2s+m)\beta\dot{\alpha}(m)\dot{\beta} + \lambda^2 e^\alpha_\delta W^\alpha(2s-m-1)\dot{\alpha}(m-1), \quad m > 0
$$

The unfolded equations can be regarded as a chain of equations of the form $DA_i = cA_{i+1} + O(\lambda^2)$. This means that the field $A_{i+1}$ is a parametrization of the derivatives of $A_i$, which do not vanish on-shell, up to the gauge transformations.

In a similar fashion, the description of the massless fermion with spin $\tilde{s} = s + \frac{1}{2}$ is built. One needs a set of gauge one-forms $\Psi^\alpha(\tilde{s}-1+m)\dot{\alpha}(\tilde{s}-1-m)$, $\frac{1}{2} \leq m < \tilde{s}$ with their conjugates (where indices $k, m$ are half-integer). Here, the pair of fields $\Psi^\alpha(\tilde{s}-1+\frac{1}{2})\dot{\alpha}(\tilde{s}-1-\frac{1}{2})$ play the role of the physical ones. Similarly, a set of curvatures $F^\alpha(\tilde{s}-1+m)\dot{\alpha}(\tilde{s}-1-m)$ can be constructed. The expressions for
the gauge transformations and curvatures are similar to the bosonic case (up to the change $s \to \tilde{s}$, $\Omega \to \Psi$ and half-integer $m$), with the only exception being the case $m = \frac{1}{2}$:

$$
\delta \Psi^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2}) = D\eta^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2}) + \epsilon \lambda e_{\beta} \eta^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2})
$$

$$
+\epsilon \lambda e_{\gamma} \eta^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2})
$$

$$
F^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2}) = D\Psi^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2}) + \epsilon \lambda e_{\beta} \Psi^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2})
$$

$$
+\epsilon \lambda e_{\gamma} \Psi^\alpha (q-\frac{1}{2})\bar{\alpha}(q-\frac{1}{2})
$$

In case of AdS space, the parameter $\epsilon = \pm 1$ here corresponds to the choice of the sign of mass-like terms. In the flat space, however, this parameter is arbitrary for the massless particle. With the gauge forms encapsulated in curvatures, the unfolded equations reproduce the exact form of the bosonic ones:

$$
0 = F^\alpha (q-1+m)\bar{\alpha}(q-1-m), \quad m < q - 1
$$

$$
0 = F^\alpha (q-2) - 2\epsilon_{\beta} Y_{\alpha}(q-2)\bar{\beta}(2),
$$

$$
0 = D\Psi^\alpha (q-2) + \epsilon_{\beta} Y_{\alpha}(q)\bar{\alpha}(m)\bar{\beta}(m)\beta
$$

$$
+\lambda^2 \epsilon e_{\alpha} Y_{\alpha}(q+1)\bar{\beta}(m+1)\bar{\alpha}(m-1), \quad m \geq \frac{1}{2}
$$

Note once again that numbers $k, m$ are half-integers here.

Now we construct the massless supermultiplets. First, we introduce a supertransformation parameter $\zeta^\alpha$ with its hermitian conjugate $\zeta^\alpha$ which obeys $D\zeta^\alpha = -\epsilon \lambda e_{\alpha} \zeta^\alpha$ (similarly for the hermitian conjugate). In the supermultiplet, the spins of boson and fermion are connected by the relation $s - s = \pm \frac{1}{2}$, so there are two possibilities.

### 1.2 Half-integer superspin

Our task here to construct supertransformations transforming bosonic equations into the fermionic ones and vice versa. It is natural to begin with the gauge invariant zero-forms because they form a closed sector. The most general ansatz for their supertransformations is rather simple:

$$
\delta W^{\alpha(k+s)}\bar{\alpha}(k-s) = \delta_{k}^{-0} Y^{\alpha(k+s-1)}\bar{\alpha}(k-s)\zeta^\alpha + \delta_{k}^{0+} Y^{\alpha(k+s)}\bar{\alpha}(k-s)\beta
$$

$$
\delta Y^{\alpha(k+s-1)}\bar{\alpha}(k-s) = \tilde{\delta}_{k}^{0-} W^{\alpha(k+s-1)}\bar{\alpha}(k-s)\zeta^\alpha + \tilde{\delta}_{k}^{0+} W^{\alpha(k+s)}\bar{\alpha}(k-s-1)\beta
$$

where all the coefficients are in general complex. The solution for these coefficients turns out to be also simple:

$$
\tilde{\delta}_{k}^{0+} = C_{b}, \quad \tilde{\delta}_{k}^{0-} = \lambda C_{b}, \quad \tilde{\delta}_{k}^{0+} = C_{f}, \quad \tilde{\delta}_{k}^{0-} = \lambda C_{f}
$$

where $C_{b}$ and $C_{f}$ are two independent parameters (see below).

Similarly, the supertransformations for the gauge one-forms (except a pair of the highest ones) look like:

$$
\delta \Omega^{\alpha(s-1+m)}\bar{\alpha}(s-1-m) = \gamma_{m}^{-0} \Psi^{\alpha(s-2+m)}\bar{\alpha}(s-1-m)\zeta^\alpha + \gamma_{m}^{0-} \Psi^{\alpha(s-1+m)}\bar{\alpha}(s-2-m)\zeta^\alpha
$$

$$
\delta \Psi^{\alpha(s+2+m)}\bar{\alpha}(s-1-m) = \gamma_{m}^{0+} \Psi^{\alpha(s+2+m)}\bar{\alpha}(s-1-m)\zeta^\alpha + \gamma_{m}^{0+} \Psi^{\alpha(s-2+m)}\bar{\alpha}(s-1-m)\zeta^\alpha
$$

This gives the following solution for the coefficients with $m > 0$:

$$
\gamma_{m}^{-0} = C, \quad \gamma_{m}^{0-} = \lambda C, \quad \gamma_{m}^{0+} = \tilde{C}, \quad \gamma_{m}^{0+} = \lambda \tilde{C}
$$
where \( C \) and \( \tilde{C} \) are also independent. For \( m = 0 \), we obtain \( \gamma_0^0 = -C \), while for \( m < 0 \), we have:
\[
\gamma_0^{-m} = \epsilon \gamma_0^{-m}, \quad \gamma_m^0 = \epsilon \gamma_m^0, \quad \tilde{\gamma}_m^0 = \epsilon \tilde{\gamma}_m^0, \quad \tilde{\gamma}_m^0 = \epsilon \tilde{\gamma}_m^0. \tag{13}
\]
At last, we have to consider two highest one-forms \( \Omega^{a(2s-2)} \) and \( \Psi^{a(2s-3)} \) (with their conjugates) because their equations connect the two sectors. The ansatz for the supertransformations is now:
\[
\delta \Omega^{a(2s-2)} = \nu \epsilon e_{\alpha \beta} Y^{a(2s-1)} \zeta^\alpha + \gamma_{s-1}^{a(2s-3)} \zeta^\alpha, \quad \delta \Psi^{a(2s-3)} = \tilde{\gamma}_{s-2}^{a(2s-3)} \xi^\alpha + \gamma_{s-2}^{a(2s-3)} \zeta^\alpha \tag{14}
\]
and this provides the relations on the parameters from the two sections and fixes the only remaining coefficient:
\[
C_b = C, \quad C_f = \tilde{C}, \quad \nu = \frac{C}{2} \tag{15}
\]
The hermiticity requires that \( C = -\epsilon C^*, \tilde{C} = \epsilon \tilde{C}^* \). Then, either \( C \) is imaginary and \( \epsilon = 1 \) or \( C \) is real and \( \epsilon = -1 \). The sign of \( C^2 \) determines the parity of the boson: it is even if \( C^2 > 0 \) and odd if \( C^2 < 0 \). Thus, bosonic parity and fermionic mass terms sign are connected. It is impossible to link \( C \) and \( \tilde{C} \) by considering unfolded equations only. However, these constants can be connected if one requires that the sum of their Lagrangians is invariant under the supertransformations. If one chooses the normalization of the Lagrangians as in [29], it turns out that:
\[
C = 4i \epsilon (s - 1) \tilde{C} \tag{16}
\]
Finally, we evaluate a commutator of two supertransformations to show that the superalgebra is indeed closed. Consider, for instance, the field \( \Omega^{a(s-1+m)\dot{\alpha}(s-1-m)} \) for \( m > 0 \). We obtain:
\[
[\delta_1, \delta_2] \Omega^{a(s-1+m)\dot{\alpha}(s-1-m)} = C \tilde{C} \lambda \Omega^{a(s-2+m)\dot{\beta}(s-2-m)} \eta^\beta \dot{\alpha} + \lambda \Omega^{a(s-1+m)\dot{\beta}(s-2-m)} \eta^\beta \dot{\alpha} + \lambda^2 \Omega^{a(s-1+m)\dot{\beta}(s-2-m)} \eta^\beta \dot{\alpha} + \Omega^{a(s-2+m)\dot{\beta}(s-1-m)} \xi^\beta \dot{\alpha} \tag{17}
\]
where
\[
\xi^{\dot{\alpha} \dot{\beta}} = \xi_1^{\alpha \dot{\beta}}, \quad \eta^{\alpha(2)} = 2 \xi_2^{\alpha \dot{\alpha}}, \quad \eta^{\dot{\alpha}(2)} = 2 \xi_2^{\dot{\alpha} \dot{\alpha}}. \tag{18}
\]
and it is indeed a combination of pseudotranslations and Lorentz transformations. The expressions for other fields are similar.

Now let us consider the flat space case. Contrary to the \( \text{AdS} \) case, the equations for the coefficients with positive and negative \( m \) fall into two independent subsystems so that we loose the hermiticity conditions on the parameters \( C \) and \( \tilde{C} \). The non-zero coefficients now are:
\[
\delta_k^{0+} = C, \quad \tilde{\delta}_{k-\frac{1}{2}}^{0+} = \tilde{C}, \quad \nu = \frac{C}{2}, \quad m \geq 0, \tag{19}
\]
\[
\gamma_k^{-m} = C, \quad \tilde{\gamma}_{k-\frac{1}{2}}^{-m} = \tilde{C}, \quad m \leq 0.
\]
To fix the phases of the coefficients \( C \) and \( \tilde{C} \), one has to consider a commutator of two supertransformations. Consider, for instance, field \( \Omega^{a(s-1+m)\dot{\alpha}(s-1-m)} \), \( m > 0 \). The commutator of the supertransformations parametrized by \( \zeta_1^\alpha, \zeta_2^\alpha \) is:
\[
[\delta_1, \delta_2] \Omega^{a(s-1+m)\dot{\alpha}(s-1-m)} = C \tilde{C} \Omega^{a(s+1+m)(s-2-m)} \zeta_1^\alpha \zeta_2^\alpha. \tag{20}
\]
The hermiticity requires that \( C \tilde{C} \) is imaginary. With the requirement that the sum of the Lagrangians is invariant, a stronger condition can be obtained:
\[
C = 4i(s - 1) \tilde{C}. \tag{21}
\]
1.3 Integer superspin

Again, we consider the AdS case first. As in the previous case we begin with the sector of the gauge invariant zero-forms. In this case the most general ansatz for the supertransformations is:

\[
\begin{align*}
\delta W^{\alpha(k+s)\dot{\alpha}(k-s)} &= \zeta_k^{00} W^{\alpha(k+s)\beta\dot{\alpha}(k-s)} \zeta_{\beta} + \delta_k^{0-} W^{\alpha(k+s)\dot{\alpha}(k-s-1)} \zeta_{\alpha}, \\
\delta Y^{\alpha(k+s+1)\dot{\alpha}(k-s)} &= \tilde{\zeta}_{k+1/2}^{00} W^{\alpha(k+s)\dot{\alpha}(k-s)} \zeta_{\alpha} + \delta_{k+1/2}^{0+} W^{\alpha(k+s+1)\dot{\alpha}(k-s)} \zeta_{\beta},
\end{align*}
\]

where all coefficients are in general complex. The invariance of the unfolded equations under these supertransformations leads to:

\[
\begin{align*}
\delta_k^{0+} &= C_b, & \delta_k^{0-} &= \lambda C_b, & \delta_{k+1/2}^{0+} &= C_f, & \delta_{k+1/2}^{0-} &= \lambda C_f.
\end{align*}
\]

where \(C_b\) and \(C_f\) are two independent parameters.

Now let us consider a sector of gauge one-forms (except two highest ones \(\Omega^{\alpha(2s-2)}\) and \(\Psi^{\alpha(2s-1)}\) with their conjugates). Here the ansatz for the supertransformations looks like:

\[
\begin{align*}
\delta \Omega^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} &= \gamma_m^{+0} \Psi^{\alpha(s-1+m)\beta\dot{\alpha}(s-1-m)} \zeta_{\beta} + \gamma_m^{0+} \Psi^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} \zeta_{\alpha}, \\
\delta \Psi^{\alpha(s+1+m)\dot{\alpha}(s-1-m)} &= \tilde{\gamma}_{m+1/2}^{0-} \Omega^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} \zeta_{\alpha} + \tilde{\gamma}_{m+1/2}^{0+} \Omega^{\alpha(s+1+m)\dot{\alpha}(s-2-m)} \zeta_{\alpha},
\end{align*}
\]

and the solution gives us two additional independent parameters:

\[
\begin{align*}
\gamma_m^{+0} &= C, & \gamma_m^{0+} &= \lambda C, & \tilde{\gamma}_{m+1/2}^{0-} &= \tilde{C}, & \tilde{\gamma}_{m+1/2}^{0+} &= \lambda \tilde{C} & m > 0.
\end{align*}
\]

For \(m = 0\), we obtain \(\gamma_0^{0+} = -C\), while for \(m < 0\), we have:

\[
\begin{align*}
\gamma_0^{-0} &= e\gamma_0^{-0}, & \gamma_0^{0+} &= e\gamma_0^{0+}, & \tilde{\gamma}_0^{0-} &= e\tilde{\gamma}_0^{0-}, & \tilde{\gamma}_0^{0+} &= e\tilde{\gamma}_0^{0+}.
\end{align*}
\]

At last, we consider supertransformations for the remaining one-forms:

\[
\begin{align*}
\delta \Omega^{\alpha(2s-2)} &= \gamma_{s-1}^{+0} \Psi^{\alpha(2s-2)\beta\dot{\alpha}} \zeta_{\beta} + \gamma_{s-1}^{0+} \Psi^{\alpha(2s-2)\dot{\alpha}} \zeta_{\alpha}, \\
\delta \Psi^{\alpha(2s-1)} &= \tilde{\nu} \Omega^{\alpha(2s-2)\dot{\alpha}} \zeta_{\alpha} + \tilde{\nu}_{s+1/2} \Omega^{\alpha(2s-2)\dot{\alpha}} \zeta_{\alpha},
\end{align*}
\]

which gives us the relations between the parameters of the two sectors and determines the only remaining one:

\[
C_b = C, \quad C_f = \tilde{C}, \quad \tilde{\nu} = \frac{\tilde{C}}{2}.
\]

Again, this gives \(C = -eC^*,\ \tilde{C} = e\tilde{C}^* \) together with the hermiticity requirement. Hence, the boson has the parity opposite to \(e\), similarly to the half-integer superspin case. By considering the unfolded equations only, the only thing one can establish is that the product of the parameters \(C\) and \(\tilde{C}\) must be imaginary. The constants \(C\) and \(\tilde{C}\) can be linked by requirement that the sum of bosonic and fermionic Lagrangians is invariant under the supertransformations:

\[
(s - 1)C = 4ie\tilde{C}.
\]

The expression for the commutator of two supertransformations parametrized by \(\zeta_1^{\alpha}\) and \(\zeta_2^{\alpha}\) is the same as in the previous case:

\[
[\delta_1, \delta_2] \Omega^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} = C\tilde{C}[\lambda \Omega^{\alpha(s-2+m)\beta\dot{\alpha}(s-1-m)} \eta_{\beta} \alpha + \lambda \Omega^{\alpha(s-1+m)\beta\dot{\alpha}(s-2-m)} \eta_{\beta} \dot{\alpha} + \lambda^2 \Omega^{\alpha(s-1+m)\beta\dot{\alpha}(s-2-m)} \zeta_{\beta} \dot{\alpha} + \Omega^{\alpha(s-2+m)\beta\dot{\alpha}(s-1-m)} \zeta_{\beta} \alpha],
\]

(30)
In the flat space, the invariance of the unfolded equations does not fix the phases of $C$ and $\tilde{C}$, so that the solution for the coefficients is:

$$
\delta^+_{k} = C, \quad \tilde{\delta}^+_{k+\frac{1}{2}} = \tilde{C}, \quad \tilde{\nu} = \frac{\tilde{C}}{2}.
$$

$$
\gamma^+_{m} = C, \quad \tilde{\gamma}^+_{m+\frac{1}{2}} = \tilde{C}, \quad m \geq 0,
$$

$$
\gamma^0_{m} = C^*, \quad \tilde{\gamma}^0_{m+\frac{1}{2}} = \tilde{C}^*, \quad m < 0.
$$

In this case the requirement that the $C\tilde{C}$ is imaginary follows only from the commutator of two supertransformations. A stronger relation

$$(s - 1)C = 4i\tilde{C}^*$$

can still be obtained from the invariance of the sum of the two Lagrangians.

## 2 Low spins examples

In this section we present two simplest examples of the massive $N = 1$ supermultiplets: a scalar and a vector ones.

### 2.1 Unfolded equations

First of all we need the unfolded equations for massive spin 1, spin $\frac{1}{2}$ and spin 0 fields. **Massive vector** In this case the unfolded formulations require three infinite chains of the zero-forms: $W^{\alpha(k+m)\bar{\alpha}(k-m)}$, $k \geq 1$, $m = \pm 1, 0$ corresponding to the three physical helicities $\pm 1, 0$. The most general (up to the normalization) ansatz has the form:

$$
0 = D W^{\alpha(k+1)\bar{\alpha}(k-1)} + e_{\beta\bar{\gamma}} W^{\alpha(k+1)\beta\bar{\alpha}(k-1)\bar{\gamma}} + \beta_{k,1} e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k-2)} + \beta_{k,0}^+ e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k-1)} + \beta_{k,0}^- e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k-1)} + \beta_{k,0}^- e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k-1)}
$$

$$
0 = D W^{\alpha(k+1)\bar{\alpha}(k+1)} + e_{\beta\bar{\gamma}} W^{\alpha(k+1)\beta\bar{\alpha}(k+1)} + \beta_{k,1} e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k+2)} + \beta_{k,0}^- e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k+1)} + \beta_{k,0}^- e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k+1)} + \beta_{k,0}^- e^\alpha_\beta W^{\alpha(k+1)\bar{\gamma}^{\alpha}(k+1)}
$$

The self-consistency of these equations leads to the following solutions for the coefficients:

$$
\beta_{k,0}^+ = \beta_{k,0}^- = \frac{1}{k(k+1)}
$$

$$
\beta_{k,1}^+ = \beta_{k,1}^- = \frac{2m^2}{(k+1)(k+2)}
$$

$$
\beta_{k,1}^- = -\frac{1}{k(k+1)} [m^2 - k(k+1)\lambda^2]
$$

(34)

$$
\beta_{k,0}^- = -\frac{(k-1)(k+2)}{k^2(k+1)^2} [m^2 - k(k+1)\lambda^2]
$$

(35)

As is well known, in the flat Minkowski space all the members of the supermultiplet must have equal masses. But in AdS space, as it has been shown in [13], there must be a small splitting between the bosonic and fermionic masses of the order of the cosmological constant. For the lower spins we consider in this section, the bosonic mass $m$ and the fermionic one $\tilde{m}$ must satisfy:

$$
m^2 = \tilde{m}(\tilde{m} \pm \lambda)
$$
In this case the $\beta$-functions take the form:

\[
\begin{align*}
\beta_{k,0}^- &= \beta_{k,0}^+ = \frac{1}{k(k+1)} \\
\beta_{k,1}^- &= \beta_{k,1}^+ = \frac{2\tilde{m}(\tilde{m} \pm \lambda)}{(k+1)(k+2)} \\
\beta_{k,1}^- &= -\frac{1}{k(k+1)}[\tilde{m} \pm (k+1)\lambda][\tilde{m} \mp k\lambda] \\
\beta_{k,0}^- &= -\frac{1}{k^2(k+1)^2}[\tilde{m} \pm (k+1)\lambda][\tilde{m} \mp k\lambda]
\end{align*}
\] (36)

It is this factorization of the $\beta^{--}$-functions that appears to be crucial for the construction of the supermultiplets in what follows.

**Massive spinor** In this case there are two physical helicities $\pm 1/2$ and we need a pair of (conjugated) chains of the zero-forms $Y^{\alpha(k+1)\dot{\alpha}(k)}$, $Y^{\alpha(k)\dot{\alpha}(k+1)}$, $k \geq 0$. We choose the following ansatz for the unfolded equations:

\[
\begin{align*}
0 &= DY^{\alpha(k+1)\dot{\alpha}(k)} + e_{\beta\dot{\gamma}}Y^{\alpha(k+1)\beta\dot{\alpha}\dot{\gamma}} + \tilde{\beta}^{++} e_{\dot{\alpha}}Y^{\alpha(k)\dot{\alpha}\dot{\beta}} + \tilde{\beta}^{-+} e_{\dot{\alpha}}Y^{\alpha(k)\dot{\alpha}(k-1)} \\
0 &= DY^{\alpha(k)\dot{\alpha}(k+1)} + e_{\beta\dot{\gamma}}Y^{\alpha(k)\beta\dot{\alpha}\dot{\gamma}} + \tilde{\beta}^{++} e_{\dot{\alpha}}Y^{\alpha(k)\dot{\alpha}\dot{\beta}} + \tilde{\beta}^{-+} e_{\dot{\alpha}}Y^{\alpha(k-1)\dot{\alpha}(k)}
\end{align*}
\] (37)

The self-consistency of these equations requires:

\[
\begin{align*}
\tilde{\beta}^{++} &= \tilde{\beta}^{-+} = \frac{\epsilon\tilde{m}}{(k+1)(k+2)}, \quad \epsilon = \pm 1 \\
\tilde{\beta}^{+-} &= -\frac{1}{(k+1)^2}[\tilde{m}^2 - (k+1)^2\lambda^2]
\end{align*}
\] (38)

Note that in what follows we always assume that the fermionic masses are positive and take into account the two possible signs of the $\tilde{\beta}^{+-}$ (which also play an important role in our construction) using the parameter $\epsilon = \pm 1$.

**Massive scalar** In this case we have one chain of the zero-forms only with the unfolded equations:

\[
0 = DW^{\alpha(k)\dot{\alpha}(k)} + e_{\beta\dot{\gamma}}W^{\alpha(k)\beta\dot{\alpha}\dot{\gamma}} + \beta^{--} e_{\dot{\alpha}}W^{\alpha(k-1)\dot{\alpha}(k-1)}
\] (39)

where

\[
\beta^{--} = -\frac{1}{k(k+1)}[m_0^2 - k(k+1)\lambda^2]
\]

As in the spin 1 case, the factorization of the $\beta^{--}$ function is achieved at $m_0^2 = \tilde{m}(\tilde{m} \pm \lambda)$:

\[
\beta^{--} = -\frac{1}{k(k+1)}[\tilde{m} \pm (k+1)\lambda][\tilde{m} \mp k\lambda]
\] (40)

### 2.2 Scalar supermultiplet

In the flat case such supermultiplet was considered in [31,32]. We begin with a one pair of spinor and scalar fields. Our first task is to find supertransformations such that the variations of the fermionic equations be proportional to the bosonic ones and vice versa.

**Supertransformations for spinor** We choose the following ansatz for the supertransformations where the coefficients are in general complex:

\[
\begin{align*}
\delta Y^{\alpha(k+1)\dot{\alpha}(k)} &= \delta^{--}_{k} W^{\alpha(k)\dot{\alpha}(k)} \zeta^{\alpha} + \delta^{0+}_{k} W^{\alpha(k+1)\dot{\alpha}(k)} \zeta^{\dot{\alpha}} \\
\delta Y^{\alpha(k)\dot{\alpha}(k+1)} &= \delta^{++}_{k} W^{\alpha(k)\dot{\alpha}(k+1)} \zeta^{\dot{\alpha}} + \delta^{0-}_{k} W^{\alpha(k)\dot{\alpha}(k)} \zeta^{\alpha}
\end{align*}
\] (41)
where

\[ \delta_{k+}^0 = (\delta_{k+}^0)^*, \quad \delta_{k-}^0 = (\delta_{k-}^0)^* \]

The solution appears to be:

\[ \delta_{k+}^0 = \epsilon \tilde{C}, \quad \delta_{k-}^0 = \frac{1}{(k+1)} [\tilde{m} \pm (k+1)\lambda] \tilde{C}, \quad \tilde{C}^* = \mp \epsilon \tilde{C} \quad (42) \]

where the \( \pm \)-sign corresponds to that of the relation \( m_0^2 = \tilde{m}(\tilde{m} \pm \lambda) \) and \( \epsilon \) comes from the \( \beta^{+-} \) function.

**Supertransformations for scalar** Similarly, for the spin-0 field we take the following supertransformations (also with the complex coefficients):

\[ \delta W^{\alpha(k)\dot{\alpha}(k)} = \delta_{k+}^0 \delta^{\alpha(k)\beta(\dot{k})} \zeta_\beta + \delta_{k-}^0 \delta^{\alpha(k-1)\dot{\alpha}(k)} \zeta^\alpha + \delta_{k+}^0 \delta^{\alpha(k)\dot{\alpha}(k)} \zeta^\alpha + \delta_{k-}^0 \delta^{\alpha(k)\dot{\alpha}(k-1)} \zeta^\dot{\alpha} \quad (43) \]

where

\[ \delta_{k+}^0 = - (\delta_{k+}^0)^*, \quad \delta_{k-}^0 = - (\delta_{k-}^0)^* \]

with the solution:

\[ \delta_{k+}^0 = C, \quad \delta_{k-}^0 = - \frac{\epsilon}{(k+1)} [\tilde{m} \pm (k+1)\lambda] C, \quad C^* = \pm \epsilon C \quad (44) \]

Now having the explicit form of the supertransformations at our disposal, it is easy to calculate their commutators and find that their superalgebra is not closed. The reason is clear: we must have an equal number of bosonic and fermionic degrees of freedom in each supermultiplet. As is well known the scalar supermultiplet contains two scalar fields, moreover, it is important that they must be scalar and pseudo-scalar. So we consider the supermultiplet \( (1/2, 0, 0') \). For concreteness we take \( \epsilon = +1 \), then to have opposite parities for two scalars we choose:

\[ m_1^2 = \tilde{m}(\tilde{m} + \lambda), \quad m_2^2 = \tilde{m}(\tilde{m} - \lambda) \quad (45) \]

The complete set of the supertransformations for the spinor now has the form:

\[ \delta Y^{\alpha(k+1)\dot{\alpha}(k)} = i \tilde{\delta}_{1,k} Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta - i \tilde{C}_1 W^{\alpha(k+1)\dot{\alpha}(k)} \zeta_\beta \]

\[ + \tilde{\delta}_{2,k} W^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta + \tilde{C}_2 W^{\alpha(k)\dot{\alpha}(k+1)} \zeta_\beta + \tilde{\delta}_{1,k} W^{\alpha(k+1)\dot{\alpha}(k)} \zeta_\beta \quad (46) \]

where

\[ \tilde{\delta}_{1,k} = \frac{1}{(k+1)} [\tilde{m} - (k+1)\lambda] \tilde{C}_1 \]

\[ \tilde{\delta}_{2,k} = \frac{1}{(k+1)} [\tilde{m} + (k+1)\lambda] \tilde{C}_2 \quad (47) \]

For the supertransformations of the two scalars we have:

\[ \delta W^{\alpha(k)\dot{\alpha}(k)} = C_1 Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta + \delta_{1,k} Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta - C_1 Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta - \delta_{1,k} Y^{\alpha(k-1)\dot{\alpha}(k)} \zeta^\alpha \]

\[ \delta W^{\alpha(k)\dot{\alpha}(k+1)} = i C_2 Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta + i \delta_{2,k} Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta + i C_2 Y^{\alpha(k)\dot{\alpha}(k)} \zeta_\beta + i \delta_{2,k} Y^{\alpha(k-1)\dot{\alpha}(k)} \zeta^\alpha \quad (48) \]
where
\[
\delta_{1,k}^- = -\frac{1}{(k+1)}[\tilde{m} + (k+1)\lambda]C_1
\]
\[
\delta_{2,k}^- = -\frac{1}{(k+1)}[\tilde{m} - (k+1)\lambda]C_2
\]

So we have four (real) arbitrary parameters \(C_{1,2}\) and \(\tilde{C}_{1,2}\). We proceed with calculations of the commutators. For the first scalar field we find:
\[
[\delta_1, \delta_2]W_1^{\alpha(k)\dot{\alpha}(k)} = -2iC_1\tilde{C}_1[\xi_{\beta}\beta W_1^{\alpha(k)\dot{\beta}(k)} + \beta_k^- \xi^{\dot{\alpha}\alpha} W_1^{\alpha(k-1)\dot{\alpha}(k-1)} \\
+ \lambda(\eta^{\alpha}_{\beta} W_1^{\alpha(k)\dot{\beta}(k)} + \eta^{\dot{\alpha}}_{\beta} W_1^{\alpha(k)\dot{\beta}(k-1)})]
\]
where
\[
\xi^{\dot{\alpha}\alpha} = \zeta_1^\alpha \zeta_2^\dot{\alpha} - (1 \leftrightarrow 2), \quad \eta^{\alpha(2)} = \zeta_1^{\alpha} \zeta_2^{\dot{\alpha}} - (1 \leftrightarrow 2)
\]
The results for the second scalar \(W_2\) are the same provided
\[
C_2\tilde{C}_2 = -C_1\tilde{C}_1
\]
At last, for the spinor we obtain:
\[
[\delta_1, \delta_2]Y^{\alpha(k+1)\dot{\alpha}(k)} = -2iC_1\tilde{C}_1[\xi_{\beta}\beta Y^{\alpha(k+1)\dot{\beta}(k)} + \beta_k^- \xi^{\dot{\alpha}\alpha} Y^{\alpha(k)\dot{\alpha}(k)} \\
+ \lambda(\eta^{\alpha}_{\beta} Y^{\alpha(k)\dot{\beta}(k)} + \eta^{\dot{\alpha}}_{\beta} Y^{\alpha(k+1)\dot{\beta}(k-1)})]
\]
Comparison with the initial unfolded equations shows that the supertransformations close on-shell and give AdS4 superalgebra:
\[
\{Q^\alpha, Q^{\dot{\alpha}}\} \sim P^{\alpha\dot{\alpha}}, \quad \{Q^\alpha, Q^\beta\} \sim \lambda M^{\alpha\beta}, \quad \{Q^{\dot{\alpha}}, Q^{\dot{\beta}}\} \sim \lambda M^{\dot{\alpha}\dot{\beta}}
\]

### 2.3 Vector supermultiplet

Let us turn to our second example — vector supermultiplet. We begin with the pair vector-spinor.

**Supertransformations for vector** The most general ansatz (taking into account the hermicity conditions) has the form:
\[
\delta W^{\alpha(k+1)\dot{\alpha}(k-1)} = \delta^{0}_{k,1} Y^{\alpha(k)\dot{\alpha}(k-1)} \zeta^\alpha - (\delta^{+0}_{k,1})^* Y^{\alpha(k+1)\dot{\alpha}(k-1)} \dot{\zeta}^\beta \\
\delta W^{\alpha(k)\dot{\alpha}(k)} = \delta^{+0}_{k,0} Y^{\alpha(k)\dot{\alpha}(k)} \zeta^\beta + \delta^{-0}_{k,0} Y^{\alpha(k-1)\dot{\alpha}(k)} \zeta^\alpha \\
\delta W^{\alpha(k-1)\dot{\alpha}(k+1)} = \delta^{-0}_{k,1} Y^{\alpha(k-1)\dot{\alpha}(k+1)} \dot{\zeta}^\beta - (\delta^{-0}_{k,1})^* Y^{\alpha(k)\dot{\alpha}(k-1)} \zeta^\alpha
\]
where all the coefficients are in general complex. The invariance of the unfolded equations gives:
\[
\delta^{0+}_{k,1} = 2\epsilon(\tilde{m} \pm \lambda)C, \quad \delta^{+0}_{k,0} = C, \quad C^* = \mp \epsilon C
\]
\[
\delta^{-0}_{k,1} = \frac{2}{(k+1)}[\tilde{m} \pm \lambda][\tilde{m} \pm (k+1)\lambda]C
\]
\[
\delta^{-0}_{k,0} = \epsilon \frac{(k+2)}{k(k+1)}[\tilde{m} \pm (k+1)\lambda]C
\]
Supertransformations for spinor 

Similarly, we introduce:

\[
\delta Y^{\alpha(k+1)\dot{\alpha}(k)} = \tilde{\delta}_{k,1}^{+0} W^{\alpha(k+1)\beta(k)} \zeta_{\beta} + \tilde{\delta}_{k,1}^{-0} W^{\alpha(k)\dot{\alpha}(k)} \zeta^{\dot{\alpha}} \\
+ (\tilde{\delta}_{k,0}^{+0})^* W^{\alpha(k+1)\beta(k)} \zeta_{\beta} + (\tilde{\delta}_{k,0}^{-0})^* W^{\alpha(k)\dot{\alpha}(k-1)} \zeta^{\dot{\alpha}} \\
(56)
\]

and obtain:

\[
\tilde{\delta}_{k,1}^{+0} = \tilde{C}, \quad \tilde{\delta}_{k,0}^{+0} = em_1 \tilde{C}, \quad \tilde{C}^* = \pm \epsilon \tilde{C} \\
\tilde{\delta}_{k,1}^{-0} = -\frac{k}{(k+1)(k+2)} \tilde{m} \equiv (k+1)\lambda \tilde{C} \\
\tilde{\delta}_{k,0}^{-0} = -\epsilon \frac{1}{(k+1)} \tilde{m} \equiv (k+1)\lambda \tilde{C} \tag{57}
\]

It is straightforward to check that these supertransformations do not close and the reason is again that we have three physical degrees of freedom for the massive vector and only two — for spinor. So we turn to the complete vector supermultiplet \((1,1/2,1/2,0')\). In this case it is also important that the spin 1 and spin 0 have opposite parities. We assume that the coefficients for the vector field supertransformations are real and choose:

\[m_v^2 = m_1(m_1 + \lambda) = m_2(m_2 - \lambda) = m_s^2, \quad \epsilon_1 = -1, \quad \epsilon_2 = +1\] (58)

where \(m_1, m_2\) are masses of the two spinors. This leads to the following expressions for the four possible boson-fermion pairs. For the vector and first spinor we have formulas (51), (53) with the parameter \(C_1\) and (56), (57) with the parameter \(i\tilde{C}_1\) (all with upper signs), while for the second spinor — the same formulas but with the parameters \(C_2, i\tilde{C}_2\) (with lower signs). Similarly, for the first spinor and the pseudo-scalar we have formulas (51), (52) with the parameter \(i\tilde{C}_3\) and (56), (57) with the parameter \(C_3\) (with upper signs), while for the second spinor — the same with the parameters \(iC_4, \tilde{C}_4\) (with lower signs).

So we have eight (real) parameters \(C_1, C_4, \tilde{C}_1, \tilde{C}_4\). Let us consider the commutators for these supertransformations. Note that all subsequent formulas are given up to the common multiplier \(-2i(m_1 + m_2)C_1\tilde{C}_1\).

The closure of the superalgebra on the vector field requires:

\[C_1\tilde{C}_1 + C_2\tilde{C}_2 = 0, \quad m_2C_1\tilde{C}_3 + m_1C_2\tilde{C}_4 = 0\]

In this case we obtain:

\[
[\delta_1, \delta_2]W^{\alpha(k+1)\dot{\alpha}(k-1)} \sim \xi_{\beta} W^{\alpha(k+1)\beta(k-1)} \delta_{k,1}^{+0} + \beta_{k,1}^{+0} \xi_{\beta} W^{\alpha(k)\dot{\alpha}(k-1)} \delta_{k,1}^{-0} \\
+ \lambda[\eta^+_{\alpha} W^{\alpha(k)\beta(k-1)} + \eta^-_{\alpha} W^{\alpha(k+1)\beta(k-2)}] \tag{59}
\]

\[
[\delta_1, \delta_2]W^{\alpha(k)\dot{\alpha}(k)} \sim \xi_{\beta} W^{\alpha(k)\beta(k)} \delta_{k,1}^{+0} + \beta_{k,1}^{+0} \xi_{\beta} W^{\alpha(k-1)\dot{\alpha}(k)} \delta_{k,1}^{-0} \\
+ \beta_{k,0}^{+0} \xi_{\beta} W^{\alpha(k)\beta(k-1)} + \beta_{k,0}^{-0} \xi_{\beta} W^{\alpha(k-1)\dot{\alpha}(k-1)} \\
+ \lambda[\eta^+_{\alpha} W^{\alpha(k-1)\beta(k-1)} + \eta^-_{\alpha} W^{\alpha(k)\beta(k-1)}] \tag{60}
\]

For the first spinor we obtain the conditions:

\[m_1C_1\tilde{C}_1 + C_3\tilde{C}_3 = 0, \quad m_1C_2\tilde{C}_1 + C_4\tilde{C}_3 = 0\]
leading to

\[ [\delta_1, \delta_2] Y^{a(k+1)\dot{a}(k)} \sim \xi_{\dot{\beta}\dot{\gamma}} Y^{a(k+1)\dot{\dot{a}}(k)\dot{\dot{\beta}}} + \gamma_{k}^{a} + \xi_{\dot{\beta}} Y^{a(k)\dot{a}(k)} + \gamma_{k}^{a} \xi_{\dot{\dot{a}}} Y^{a(k)\dot{a}(k-1)} + \lambda[\eta^{a}_{\dot{\beta}} Y^{a(k)\dot{a}(k)} + \eta^{a}_{\dot{\dot{\beta}}} Y^{a(k+1)\dot{a}(k-1)}\dot{\beta}] \] (61)

The results for the second spinor are the same provided

\[ m_2 C_2 \tilde{C}_2 + C_4 \tilde{C}_4 = 0, \quad m_2 C_1 \tilde{C}_2 + C_3 \tilde{C}_4 = 0 \]

At last the commutator on the pseudo-scalar closes if

\[ C_3 \tilde{C}_1 + C_4 \tilde{C}_2 = 0 \]

and gives

\[ [\delta_1, \delta_2] \tilde{W}^{a(k)\dot{a}(k)} \sim \xi_{\dot{\beta}\dot{\gamma}} \tilde{W}^{a(k)\dot{\dot{a}}(k)\dot{\dot{\beta}}} + \beta_{k}^{a} \xi_{\dot{\dot{a}}} \tilde{W}^{a(k)\dot{a}(k-1)} + \lambda[\eta^{a}_{\dot{\beta}} \tilde{W}^{a(k)\dot{a}(k)} + \eta^{a}_{\dot{\dot{\beta}}} \tilde{W}^{a(k+1)\dot{a}(k-1)}\dot{\beta}] \] (62)

Thus we indeed obtain the correct on-shell superalgebra provided a number of relations on the parameters hold. It is easy to check that these relations are consistent, one of the possible simple solutions being

\[ C_2 = C_3 = C_4 = C_1, \quad \tilde{C}_2 = -\tilde{C}_1, \quad \tilde{C}_3 = -m_1 \tilde{C}_1, \quad \tilde{C}_4 = m_2 \tilde{C}_1 \]

### 3 Massive higher spin supermultiplets

Lagrangian formulation for the massive higher spin \( N = 1 \) supermultiplets in \( AdS_4 \) has been developed in [18]. In this section we consider an unfolded formulation for these supermultiplets. First of all we recall the unfolded equations for massive bosonic and fermionic fields developed in [29]. Then we consider the pairs of bosonic and fermionic fields which differ in spin by \( \frac{1}{2} \) (we call them superblock) and construct the supertransformations transforming bosonic equations into fermionic ones and vice versa. At last we consider two types of massive supermultiplets (with integer and half-integer superspins) and adjust the parameters of their four superblocks so that the superalgebra is closed.

#### 3.1 Unfolded equations

Let us recall the unfolded equations developed in [29].

##### 3.1.1 Bosonic case

To describe a massive spin-\( s \) boson, one needs gauge one-forms (physical, auxiliary and extra) \( \Omega^{a(k+m)\dot{a}(k-m)}, |m| \leq k \leq s-1 \), Stueckelberg zero-forms \( W^{a(k+m)\dot{a}(k-m)}, |m| \leq k \leq s-1 \), and gauge invariant zero-forms \( \tilde{W}^{a(k+m)\dot{a}(k-m)}, |m| \leq s \leq k \). We use a convenient normalization of the Stueckelberg zero-forms where their transformations are just shifts:

\[ \delta W^{a(k+m)\dot{a}(k-m)} = \eta^{a(k+m)\dot{a}(k-m)} \] (63)

As for the gauge one-forms, their gauge transformations:

\[ \delta \Omega^{a(k+m)\dot{a}(k-m)} = D \eta^{a(k+m)\dot{a}(k-m)} + \alpha_{k,m} \eta^{a(k)\dot{a}(k-1)} \eta^{a(k-1)\dot{a}(k-2)} + \alpha_{k,m}^{+} e_{\dot{\beta}\dot{\gamma}} \eta^{a(k)\dot{a}(k-1)\dot{\dot{\beta}}} + \alpha_{k,m}^{+} e_{\dot{\dot{\beta}}} \eta^{a(k)\dot{a}(k-1)\dot{\dot{\gamma}}} + \alpha_{k,m}^{+} e_{\dot{\gamma}} \eta^{a(k)\dot{a}(k-1)\dot{\dot{\beta}}} \] (64)

\[ \delta \Omega^{a(2k)} = D \eta^{a(2k)} + \alpha_{k,k}^{+} e_{\dot{\beta}\dot{\gamma}} \eta^{a(2k)\dot{a}(2k)} + \alpha_{k,k}^{+} e_{\dot{\dot{\beta}}} \eta^{a(2k-1)\dot{a}(2k-2)} + \alpha_{k,k}^{+} e_{\dot{\gamma}} \eta^{a(2k-1)\dot{a}(2k-2)} \]
are the modification of the massless ones by the cross terms with the coefficients $\alpha^+, \alpha^-$. In what follows we assume that all functions $\alpha$ are real and satisfy the hermiticity conditions:

$$\alpha^+_k = \alpha^{--}_k, \quad \alpha^{++}_k = \alpha^{++}_k, \quad \alpha^{-+}_k = \alpha^{-+}_k.$$

All these functions can be expressed in terms of the main one $\alpha^+_m$:

$$\begin{align*}
\alpha^{++}_{k,m} &= \frac{\alpha^+_m}{(k-m+1)(k-m+2)(k+m)(k+m+1)}, \quad m > 0, \\
\alpha^{+-}_{k,m} &= \frac{\alpha^{++}_k}{(k-m+1)(k-m+2)}, \quad m \geq 0, \\
\alpha^{+-}_{-k,m} &= \frac{\alpha^{+-}_m}{(k+m)(k+m+1)}, \quad m \geq 0, \\
\alpha^{++}_{k,m} &= k(k+1)\alpha^{++}_{k+2,m}, \quad k \geq 2, \quad \alpha^{++}_0 = 2\alpha^{--}_2, \quad \alpha^{--}_k = \frac{\alpha^{++}_{k+1}}{k(k-1)}.
\end{align*}$$

For the massive spin-$s$ boson we consider in this subsection the function $\alpha^{++}_m$ is:

$$\alpha^{++}_m = (s-m+1)(s+m)[M^2 - m(m-1)\lambda^2]$$

As is well known, in the flat space masses of the all members of the same supermultiplet must be equal. As it was shown in [18], in $AdS_4$ case bosonic $M$ and fermionic $\tilde{M}$ mass parameters must satisfy the relation $M^2 = \tilde{M}[\tilde{M} \pm \lambda]$. In this case the function $\alpha^{++}_m$ takes the form:

$$\alpha^{++}_m = (s-m+1)(s+m)[\tilde{M} \pm m\lambda][\tilde{M} \mp (m-1)\lambda]$$

and this factorization appears to be crucial for the construction of the superblocks and hence the supermultiplets.

The explicit expressions for all the $\alpha$-functions given above were found [29] in the construction of the gauge invariant self-consistent two-forms (curvatures) for each gauge one-form $(0 \leq m < k)$:

$$\begin{align*}
\mathcal{R}^{(k+m)\alpha(k-m)} &= D\Omega^{(k+m)\alpha(k-m)} + \alpha^{--}_{k,m}e^{\alpha\alpha}\Omega^{(k+m-1)\alpha(k-m-1)}
\quad + \alpha^{++}_{k,m}e^{\alpha\beta}\Omega^{(k+m)\beta\alpha(k-m)\beta} + \alpha^{+-}_{k,m}e^{\alpha\beta}\Omega^{(k+m-1)\alpha(k-m)\beta}
\quad + \alpha^{+-}_{k,m}e^{\beta\alpha}\Omega^{(k+m)\beta\alpha(k-m)}, \\
\mathcal{R}^{(2k)} &= D\Omega^{(2k)} + \alpha^{++}_{k,k}e^{\alpha\beta}\Omega^{(2k)\beta\alpha} + \alpha^{+-}_{k,k}e^{\alpha\beta}\Omega^{(2k-1)\beta}
\quad - 2\alpha^{++}_{k,k}e^{\alpha\beta}\Omega^{(2k-2)} - 2\alpha^{++}_{k,k}e^{\alpha\beta}\Omega^{(2k-2)\beta} + \frac{\alpha^{++}_{k+1,k}}{k + 1}E^{(2k)\beta}W^{(2k-2)\beta}, \\
\mathcal{C}^{(k+m)\alpha(k-m)} &= DW^{(k+m)\alpha(k-m)} - \Omega^{(k+m)\alpha(k-m)} + \alpha^{--}_{k,m}e^{\alpha\alpha}W^{(k+m-1)\alpha(k-m-1)}
\quad + \alpha^{++}_{k,m}e^{\alpha\beta}W^{(k+m)\beta\alpha(k-m)\beta} + \alpha^{+-}_{k,m}e^{\beta\alpha}\tilde{W}^{(k+m+1)\alpha(k-m)\beta}
\quad + \alpha^{+-}_{k,m}e^{\beta\alpha}\Omega^{(k+m)\beta\alpha(k-m)}.
\end{align*}$$
Now we are ready to present a set of unfolded equations. The whole system can be subdivided into three subsystems. The first subsystem is just the zero curvature conditions for most of the gauge invariant two- and one-forms (except some highest ones, see below):

\[
\begin{align*}
0 &= R^{\alpha(k+m)\hat{\alpha}(k-m)}, & k < s - 1 \\
0 &= R^{\alpha(s-1+m)\hat{\alpha}(s-1-m)}, & |m| < s - 1 \\
0 &= C^{\alpha(k+m)\hat{\alpha}(k-m)}, & k < s - 1
\end{align*}
\] (70)

The second one contains these remaining gauge invariant curvatures and gives a connection with the sector of the gauge invariant zero-forms:

\[
\begin{align*}
0 &= R^{\alpha(2s-2)} + 2E_{\beta(2)}W^{\alpha(2s-2)\beta(2)} \\
0 &= C^{\alpha(s-1+m)\hat{\alpha}(s-1-m)} - e_{\beta\hat{\beta}}W^{\alpha(s-1+m)\beta\hat{\alpha}(s-1-m)\hat{\beta}}
\end{align*}
\] (71)

Finally, the third one contains the gauge invariant zero-forms only. Its structure reproduces the structure of the unfolded equations for massless components with added cross terms \((m < k)\):

\[
\begin{align*}
0 &= DW^{\rho(k+m)\hat{\rho}(k-m)} + \frac{\beta^{++}_{k,m}}{e^{\hat{\alpha}\hat{\rho}}}W^{\rho(k+m-1)\hat{\rho}(k-m-1)} \\
&\quad + \frac{\beta^{+-}_{k,m}}{e_{\beta\hat{\rho}}}W^{\rho(k+m)\beta\hat{\rho}(k-m)\hat{\beta}} + \frac{\beta^{-+}_{k,m}}{e_{\hat{\beta}\beta}}W^{\rho(k+m)\hat{\beta}\beta(k-m)\hat{\beta}} \\
&\quad + \frac{\beta^{+\beta}_{k,m}}{e_{\rho\rho}}W^{\rho(k+m)\hat{\rho}(k-m)\hat{\rho}} \\
0 &= DW^{\rho(2k)} + \frac{\beta^{++}_{k,k}}{e_{\beta\hat{\alpha}}}W^{\rho(2k)\beta\hat{\alpha}} + \frac{\beta^{+-}_{k,k}}{e_{\hat{\beta}\beta}}W^{\rho(2k)\hat{\beta}\beta} \\
&\quad + \frac{\beta^{-+}_{k,k}}{e_{\rho\rho}}W^{\rho(2k)\hat{\rho}\rho}
\end{align*}
\] (72)

Here we also assume that all the functions \(\beta\) are real and satisfy the hermiticity conditions:

\[
\beta^{++}_{k,m} = \beta^{-+}_{k,-m}, \quad \beta^{+\beta}_{k,m} = \beta^{+\beta}_{k,-m}, \quad \beta^{-\beta}_{k,m} = \beta^{-\beta}_{k,-m}.
\]

The coefficients \(\beta_{k,m}^{ij}\) are determined by the self-consistency of these equations (taking into account their connection with the gauge sector). It appears that all of them can be expressed via the very same function \(\alpha_{m}^{+}\):

\[
\begin{align*}
\beta^{++}_{k,m} &= \frac{\beta^{++}_{m}}{(k+m)(k+m+1)}, & m \geq 0, \\
\beta^{+\beta}_{k,m} &= \frac{\beta^{+\beta}_{m}}{(k-m)(k-m+1)}, & m \geq 0, \\
\beta^{-\beta}_{k,m} &= \frac{\alpha_{k+1}^{+}}{(k+m)(k+m+1)(k-m)(k-m+1)}, & k > s, \quad \beta^{-\beta}_{s,m} = 0, \\
\beta^{+\beta}_{m} &= \frac{\alpha_{m}^{+}}{(s-m)(s-m+1)}, & 1 \leq m < s, \quad \beta^{+\beta}_{s-1} = \frac{\alpha_{s-1}^{+}}{2}, \\
\beta^{+\beta}_{m} &= \frac{\alpha_{m}^{+}}{(s-m)(s-m+1)}, & 0 \leq m < s - 1, \quad \beta^{+\beta}_{s-1} = 2.
\end{align*}
\]

### 3.1.2 Fermionic case

Similarly to the massive boson, to describe a massive spin-\(\tilde{s}\) = \(s + \frac{1}{2}\) fermion, one needs one-forms (physical and extra ones) \(\Psi^{\alpha(k+m)\hat{\alpha}(k-m)}, |m| \leq k \leq \tilde{s} - 1\), Stueckelberg zero-forms \(Y^{\alpha(k+m)\hat{\alpha}(k-m)}, |m| \leq k \leq \tilde{s} - 1\), and gauge invariant zero-forms \(Y^{\alpha(k+m)\hat{\alpha}(k-m)}, |m| \leq \tilde{s} \leq k\); the indices \(k, m\) are half-integers now. The ansatz for gauge transformations and gauge invariant curvatures for the fermions has the same form as the corresponding expressions
for bosons; but the coefficients \( \tilde{\alpha}_{ij}^{k,m} \) are different from the corresponding bosonic ones. The gauge transformations are:

\[
\delta \Psi^{\alpha(k+m)\dot{\alpha}(k-m)} = D\tilde{\eta}^{\alpha(k+m)\dot{\alpha}(k-m)} + \tilde{\alpha}_{k,m}^{++} e^{\alpha \dot{\alpha}} \Psi^{\alpha(k+m-1)\dot{\alpha}(k-m-1)}
+ \tilde{\alpha}_{k,m}^{+\dot{\alpha}} e^{\alpha \dot{\alpha}} \Psi^{\alpha(k+m)\dot{\alpha}(k-m)\dot{\beta}} + \tilde{\alpha}_{k,m}^{-\dot{\alpha}} e^{\alpha \dot{\alpha}} \Psi^{\alpha(k+m-1)\dot{\alpha}(k-m)\dot{\beta}}
\]

\[
\delta Y^{\alpha(k+m)\dot{\alpha}(k-m)} = \tilde{\eta}^{\alpha(k+m)\dot{\alpha}(k-m)},
\]

where all the functions \( \tilde{\alpha} \) are assumed to be real and satisfying the hermiticity conditions:

\[
\tilde{\alpha}_{k,m}^{++} = \tilde{\alpha}_{k,m}^{-\dot{\alpha}}, \quad \tilde{\alpha}_{k,m}^{+\dot{\alpha}} = \tilde{\alpha}_{k,m}^{++}, \quad \tilde{\alpha}_{k,m}^{-\dot{\alpha}} = \tilde{\alpha}_{k,m}^{--},
\]

All of them also can be expressed in terms of one main function \( \tilde{\alpha}_{m}^{--} \):

\[
\tilde{\alpha}_{k,m}^{++} = \frac{\tilde{\alpha}_{m}^{++}}{(k - m + 1)(k - m + 2)(k + m)(k + m + 1)}, \quad m > \frac{1}{2},
\]

\[
\tilde{\alpha}_{k,\frac{1}{2}}^{++} = \frac{\sqrt{\tilde{\alpha}_{\frac{1}{2}}^{++}}}{(k + \frac{1}{2}) (k + \frac{3}{2})},
\]

\[
\tilde{\alpha}_{k,m}^{++} = 1, \quad m \geq \frac{1}{2},
\]

\[
\tilde{\alpha}_{k,m}^{--} = \frac{\tilde{\alpha}_{k}^{--}}{(k - m + 1)(k - m + 2)}, \quad m \geq \frac{1}{2},
\]

\[
\tilde{\alpha}_{k,m}^{-\dot{\alpha}} = \frac{\tilde{\alpha}_{k}^{--}}{(k + m)(k + m + 1)}, \quad m \geq \frac{1}{2},
\]

\[
\tilde{\alpha}_{k,m}^{++} = (k + \frac{1}{2})^{2} \tilde{\alpha}_{k+1,m}^{--}, \quad \tilde{\alpha}_{k,m}^{--} = \frac{\tilde{\alpha}_{k+1,m}^{++}}{(k - \frac{1}{2})^{2}}.
\]

Here, the function \( \tilde{\alpha}_{m}^{--} \) is:

\[
\tilde{\alpha}_{m}^{--} = (\tilde{s} - m + 1)(\tilde{s} + m) \left( \tilde{M}^{2} - (m - \frac{1}{2})^{2} \lambda^{2} \right)
\]

In particular, \( \sqrt{\tilde{\alpha}_{\frac{1}{2}}^{--}} = (\tilde{s} + \frac{\lambda}{2}) \tilde{M} \). One of the essential differences between bosons and fermions is that bosons have the mass-like terms proportional to \( M^{2} \), while fermions — to \( \tilde{M} \). And as it was shown in \[18\], the sign of the fermionic mass term plays an important role in the construction of the supermultiplets. Namely, the signs for the two fermions entering the supermultiplet must be opposite. Thus in the expressions given above we introduced the parameter \( \epsilon = \pm 1 \) corresponding to the choice of mass-like terms sign, while we always assume that the parameters \( M \) and \( \tilde{M} \) are positive.

As in the bosonic case, for each gauge one-form one can construct a gauge invariant two-form (0 ≤ m < k):

\[
F^{\alpha(k+m)\dot{\alpha}(k-m)} = D\tilde{\eta}^{\alpha(k+m)\dot{\alpha}(k-m)} + \tilde{\alpha}_{k,m}^{++} e^{\alpha \dot{\alpha}} F^{\alpha(k+m-1)\dot{\alpha}(k-m-1)}
+ \tilde{\alpha}_{k,m}^{+\dot{\alpha}} e^{\alpha \dot{\alpha}} F^{\alpha(k+m)\dot{\alpha}(k-m)\dot{\beta}} + \tilde{\alpha}_{k,m}^{-\dot{\alpha}} e^{\alpha \dot{\alpha}} F^{\alpha(k+m-1)\dot{\alpha}(k-m)\dot{\beta}}
\]

\[
F^{\alpha(2k)} = D\tilde{\eta}^{\alpha(2k)} + \tilde{\alpha}_{k,k}^{++} e^{\alpha \dot{\alpha}} F^{\alpha(2k)\dot{\alpha} \dot{\beta} \dot{\gamma}} + \tilde{\alpha}_{k,k}^{+\dot{\alpha}} e^{\alpha \dot{\alpha}} F^{\alpha(2k-1)\dot{\alpha}}
- 2\tilde{\alpha}_{k,k}^{++} \tilde{\alpha}_{k,k}^{-\dot{\alpha}} E^{\alpha(2)} Y^{\alpha(2k-2)} - 2\tilde{\alpha}_{k,k}^{++} E^{\dot{\beta}(2)} Y^{\alpha(2k)\dot{\beta}(2)}
\]

\[
- \tilde{\alpha}_{k,k}^{++} E^{\dot{\alpha}} Y^{\alpha(2k-1)\dot{\beta}}.
\]
as well as a gauge invariant one-form for each Stueckelberg zero-form:

\[ \mathcal{D}^{a(k+m)}\dot{\alpha}(k-m) = DY^{a(k+m)}\dot{\alpha}(k-m) - \Phi^{a(k+m)}\dot{\alpha}(k-m) + \tilde{\alpha}_m^{-+} e^{a\dot{\alpha}} Y^{a(k+1)}\dot{\alpha}(k-m) \]

\[ + \tilde{\alpha}_m^{+-} e^{a\dot{\alpha}} Y^{a(k+1)}\dot{\alpha}(k-m) \]

\[ + \tilde{\alpha}_m^{++} e^{a\dot{\alpha}} Y^{a(k+1)}\dot{\alpha}(k-m) \]

\[ + \tilde{\alpha}_m'^{+} e^{a\dot{\alpha}} Y^{a(k+1)}\dot{\alpha}(k-m) \]  \hspace{1cm} (77)

Now let us consider a set of the unfolded equation. Here the whole system also can be subdivided into three subsystems. The first subsystem is just the zero curvature conditions for most of the gauge invariant two- and one-forms:

\[ 0 = \mathcal{F}^{a(k+m)}\dot{\alpha}(k-m), \quad k < s - 1 \]
\[ 0 = \mathcal{F}^{a(s-1+m)}\dot{\alpha}(s-1-m), \quad |m| < s - 1 \]  \hspace{1cm} (78)

The second one contains the remaining gauge invariant curvatures and gives a connection with the sector of the gauge invariant zero-forms:

\[ 0 = \mathcal{F}^{a(2s-2)} + E_{(2)} Y^{a(2s-2)}\beta(2) \]
\[ 0 = \mathcal{D}^{a(s-1+m)}\dot{\alpha}(s-1-m) - e_{\beta\beta} Y^{a(s-1+m)}\beta(a-1-m) \]  \hspace{1cm} (79)

Finally, the third one contains the gauge invariant zero-forms only. Its structure reproduces the structure of the unfolded equations for massless components with added cross terms \((m < k)\):

\[ 0 = DY^{a(k+m)}\dot{\alpha}(k-m) + \tilde{\beta}_{m}^{+-} e^{a\dot{\alpha}} Y^{a(k+1)}\dot{\alpha}(k-m) \]
\[ + e_{\beta\beta} Y^{a(k+m)}\beta(a-1-m) \]
\[ + \tilde{\beta}_{m}^{++} e^{a\dot{\alpha}} Y^{a(k+1)}\dot{\alpha}(k-m) \]
\[ 0 = DY^{a(2k)} + e_{\beta\beta} Y^{a(2k)}\beta(a) + \tilde{\beta}_{m}^{+-} e^{a\dot{\alpha}} Y^{a(2k)} \]  \hspace{1cm} (80)

The coefficients \( \tilde{\beta}_{m}^{\pm} \) (which assumed to be real and satisfying the hermiticity conditions similar to that of \( \tilde{\alpha}_{m}^{\pm} \)) are determined by the self-consistency of these equations (taking into account the connection with the gauge sector). They resemble the corresponding bosonic coefficients, the most significant difference being the behavior of some of the coefficients at \( m = \pm \frac{1}{2} \). As in the bosonic case, they all can be expressed via the same main function \( \tilde{\alpha}_{m}^{+} \):

\[ \tilde{\beta}_{m}^{+} = \frac{\tilde{\alpha}_{m}^{+}}{\bar{k}(m)(m+1)}, \quad m \geq \frac{1}{2}, \]
\[ \tilde{\beta}_{m}^{-} = \frac{\tilde{\alpha}_{m}^{-}}{\bar{k}(m)(m+1)}, \quad m \geq \frac{1}{2}, \]
\[ \tilde{\beta}_{m}^{+} = \frac{\tilde{\alpha}_{m+1}}{\bar{k}(m)(m+1)}, \quad k > s, \quad \tilde{\beta}_{m}^{-} = 0, \]  \hspace{1cm} (81)
\[ \tilde{\beta}_{m}^{+} = \frac{\tilde{\alpha}_{m+1}}{\bar{k}(s-1)(s-0)}, \quad \frac{1}{2} \leq m < \bar{s}, \quad \tilde{\beta}_{s}^{+} = \frac{\tilde{\alpha}_{s+1}}{2}, \quad \tilde{\beta}_{s-1}^{+} = \frac{\tilde{\alpha}_{s+1}}{2}, \]
\[ \tilde{\beta}_{m}^{+} = \frac{\tilde{\alpha}_{m+1}}{\bar{k}(s-1)(s-0)}, \quad \frac{1}{2} \leq m < \bar{s}, \quad \tilde{\beta}_{s}^{+} = \frac{\tilde{\alpha}_{s+1}}{2}, \quad \tilde{\beta}_{s-1}^{+} = \frac{\tilde{\alpha}_{s+1}}{2}, \]

### 3.2 Superblocks

Similarly to the massless supermultiplets, it is possible to construct a system of massive higher spin boson and fermion which is invariant under the supertransformations, which we
call a superblock. However, in contrast to the massless case, the algebra of such supertransformations is not closed. To make it closed, one needs four particles — two bosons and two fermions \[18,21,23,\] Each pair of one boson and one fermion forms a superblock with its own transformations, so that each particle enters two such superblocks. Moreover, it is possible to adjust the parameters of these superblocks so that the superalgebra is closed.

We begin with the construction of the superblocks. Naturally, supersymmetry requires that the parameters of the particles are connected. First, a well-known relation \(\tilde{s} = s \pm \frac{1}{2}\) holds for the spins of fermion and boson. Secondly, as it was shown in [18,21,23], the mass parameters of the particles are also must be connected: \(M^2 = \tilde{M}(\tilde{M} \pm \lambda)\). At first, we consider the general properties of these superblocks and then provide the explicit solutions for the two possible types with \(\tilde{s} = s \pm \frac{1}{2}\).

As we have seen, the whole set of unfolded equations both for the bosons as well for the fermions can be subdivided into the three sub-sectors. It is natural to begin with the subsector of the gauge invariant zero-forms since they must form a closed subsystem under the supertransformations as well. The most general ansatz is thus:

\[
\begin{align*}
\delta W^{\alpha(k+m)\dot{\alpha}(k-m)} &= \delta_{k,m}^{0+} Y^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \delta_{k,m}^{0-} Y^{\alpha(k+m)\dot{\alpha}(k-m-1)} \zeta_{\alpha} \\
\delta Y^{\alpha(k+m)\dot{\alpha}(k-m)} &= \tilde{\delta}_{k,m}^{0+} W^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \tilde{\delta}_{k,m}^{0-} W^{\alpha(k+m)\dot{\alpha}(k-m-1)} \zeta_{\alpha}
\end{align*}
\]

(82)

Here \(k, m\) are integers in the first equation and half-integers in the second one. All these functions \(\delta, \tilde{\delta}\) are in general complex and satisfy the hermiticity conditions:

\[
\begin{align*}
\delta_{k,m}^{0+} &= -(\delta_{k,m}^{0+})^* \quad \delta_{k,m}^{0-} = -(\delta_{k,m}^{0-})^* \\
\tilde{\delta}_{k,m}^{0+} &= (\tilde{\delta}_{k,m}^{0+})^* \quad \tilde{\delta}_{k,m}^{0-} = (\tilde{\delta}_{k,m}^{0-})^* 
\end{align*}
\]

(83)

For lower \(k\), some of the fields \(W\) or \(Y\) on the right-hand side may turn out to be the Stueckelberg ones. Such terms are forbidden by gauge invariance, so we must impose the following boundary conditions depending on the type of the superblock:

\[
\begin{align*}
\delta_{s, m}^{0+} &= 0 \quad \tilde{s} = s - \frac{1}{2} \\
\delta_{s, m}^{0-} &= 0 \quad \tilde{s} = s + \frac{1}{2}
\end{align*}
\]

(84)

The requirement that the gauge invariant subsector of the unfolded equations is preserved by these supertransformations leads to the number of equations on the functions \(\delta, \tilde{\delta}\) given in Appendix. These equations completely determine these functions up to the two arbitrary constants. Their explicit solutions given in the two subsequent subsections. Note, that the relation \(M^2 = \tilde{M}(\tilde{M} \pm \lambda)\) appears already at this level.

Then, we consider the supertransformations for the gauge sector. The most general ansatz for the Stueckelberg zero-forms is:

\[
\begin{align*}
\delta W^{\alpha(k+m)\dot{\alpha}(k-m)} &= \gamma_{k,m}^{0+} Y^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \gamma_{k,m}^{0-} Y^{\alpha(k+m)\dot{\alpha}(k-m-1)} \zeta_{\alpha} \\
\delta Y^{\alpha(k+m)\dot{\alpha}(k-m)} &= \tilde{\gamma}_{k,m}^{0+} W^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \tilde{\gamma}_{k,m}^{0-} W^{\alpha(k+m)\dot{\alpha}(k-m-1)} \zeta_{\alpha}
\end{align*}
\]

(85)

where all functions \(\gamma, \tilde{\gamma}\) are in general complex and satisfy the hermiticity conditions similar to that for the \(\delta, \tilde{\delta}\):

\[
\begin{align*}
\gamma_{k,m}^{0+} &= -(\gamma_{k,m}^{0+})^* \\
\gamma_{k,m}^{0-} &= -(\gamma_{k,m}^{0-})^* \\
\tilde{\gamma}_{k,m}^{0+} &= (\tilde{\gamma}_{k,m}^{0+})^* \\
\tilde{\gamma}_{k,m}^{0-} &= (\tilde{\gamma}_{k,m}^{0-})^*$
\]

(86)
Most of the unfolded equations for the Stueckelberg zero-forms are just the zero-curvature conditions. Thus the invariance of these equations under the supertransformations is equivalent to the following transformations for these curvatures:

\[
\delta \mathcal{C}^{\alpha(k+m)\dot{\alpha}(k-m)} = \gamma^{0+}_{k,m} \mathcal{D}^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \gamma^{0-}_{k,m} \mathcal{D}^{\alpha(k+m)\dot{\alpha}(k-m-1)} \zeta_{\alpha} \\
\delta \mathcal{D}^{\alpha(k+m)\dot{\alpha}(k-m)} = \gamma^{+0}_{k,m} \mathcal{C}^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \gamma^{-0}_{k,m} \mathcal{C}^{\alpha(k+m-1)\dot{\alpha}(k-m)} \zeta_{\alpha}
\]

This leads to the number of equations on the functions \(\gamma, \dot{\gamma}\) also given in Appendix. Their solutions also determine all the functions \(\gamma, \dot{\gamma}\) up to the two arbitrary constants. Note, that the supertransformations for the Stueckelberg zero-forms can (and have to) contain gauge invariant zero forms for highest \(k = \max\{s, \tilde{s}\}\) possible. The ansatz (85) has to be modified in a different way for the two types of the superblocks. We will present the modified ansatz in the following subsubsections.

At last let us turn to the gauge one-forms. Recall that the general form for the Stueckelberg field curvatures are \(\mathcal{C} = DW + \Omega + \ldots, \mathcal{D} = DY + \Psi + \ldots\). This fix the supertransformations for the gauge one-forms entirely. Except for the \(|m| = k\), the structure and coefficients for the supertransformations of one-forms are the same:

\[
\delta \Omega^{\alpha(k+m)\dot{\alpha}(k-m)} = \gamma^{0+}_{k,m} \Psi^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \gamma^{0-}_{k,m} \Psi^{\alpha(k+m)\dot{\alpha}(k-m-1)} \zeta_{\alpha} \\
\delta \Psi^{\alpha(k+m)\dot{\alpha}(k-m)} = \gamma^{+0}_{k,m} \Omega^{\alpha(k+m)\dot{\alpha}(k-m)} \zeta_{\beta} + \gamma^{-0}_{k,m} \Omega^{\alpha(k+m-1)\dot{\alpha}(k-m)} \zeta_{\alpha}
\]

The supertransformations for one-forms \(\Omega^{\alpha(2k)}, \Psi^{\alpha(2k+1)}\) must contain terms with zero-forms (both Stueckelberg and the gauge invariant ones). Now we consider the two cases \(\tilde{s} = s \pm \frac{1}{2}\).

### 3.2.1 Superblock \(\tilde{s} = s - \frac{1}{2}\)

We begin with the ansatz (82) for the gauge invariant zero-forms. The gauge invariant sector of the unfolded equations system is preserved under the conditions given in Appendix (125). Those conditions require that \(M^2 = \tilde{M}(\tilde{M} \pm \lambda)\); the explicit expressions for the coefficients \(\delta^{ij}_{k,m}\) are \((m \geq 0)\):

\[
\delta^{+0}_{k,m} = (s - m)(s - m - 1)C_b, \\
\delta^{0-}_{k,m} = \pm \frac{(k + s + 1)(\tilde{M} \pm (k + 1)\lambda)}{(k - m)(k - m + 1)} \delta^{+0}_{k,m}, \\
\delta^{0+}_{k,m} = \pm (s + m)(\tilde{M} \pm m\lambda)C_b, \quad m > 0, \\
\delta^{+0}_{k,0} = \pm \epsilon s(s - 1)C_b, \\
\delta^{-0}_{k,m} = \pm \frac{(k + s + 1)(\tilde{M} \pm (k + 1)\lambda)}{(k + m)(k + m + 1)} \delta^{+0}_{k,m}, \\
\delta^{0+}_{k,0} = \pm \frac{(k + s + 1)(\tilde{M} \pm (k + 1)\lambda)}{(k - m)(k - m + 1)} \delta^{+0}_{k,0},
\]

while those for the functions \(\tilde{\delta}^{ij}_{k,m}\) are \((m \geq \frac{1}{2})\):

\[
\tilde{\delta}^{+0}_{k+\frac{1}{2},m+\frac{1}{2}} = C_f, \\
\tilde{\delta}^{-0}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(k - s + 1)(\tilde{M} \pm (k + 1)\lambda)}{(k - m)(k - m + 1)} C_f.
\]
\[ \delta_{k+\frac{1}{2}, m+\frac{1}{2}}^{0+} = \pm \frac{\hat{M} \mp m\lambda}{(s - m - 1)} C_f, \quad (90) \]

\[ \delta_{k+\frac{1}{2}, m+\frac{1}{2}}^{-0} = \pm \frac{(k - s + 1)(\hat{M} \mp (k + 1)\lambda)}{(k + m + 1)(k + m + 2)} \delta_{k+\frac{1}{2}, m+\frac{1}{2}}^{0+}. \]

The sign choice corresponds to the sign in the relation \( M^2 = \hat{M}(\hat{M} \pm \lambda). \) Note that \( \delta_{s,m}^{0-} = 0 \) as it should be. Thus all the functions \( \delta, \tilde{\delta} \) are determined up to the two arbitrary complex parameters \( C_b \) and \( C_f. \) Moreover, in \( AdS \) case, i.e. when \( \lambda \neq 0, \) we obtain a pair of additional relations on these constants:

\[ C_b^* = \mp \epsilon C_b, \quad C_f^* = \pm \epsilon C_f \quad (91) \]

Now let us turn to the gauge sector. The invariance of the corresponding set of the unfolded equations under the supertransformations \( \mathbb{S}_5 \) leads to a number of equations \( (146) \) given in Appendix. These equations determine all the functions \( \gamma \) and \( \tilde{\gamma} \) also up to the two arbitrary complex constants \( C \) and \( \tilde{C}. \) Explicit expressions for the functions \( \gamma \) look like \( (m \geq 0): \)

\[ \gamma_{k,m}^{0+} = \mp \sqrt{k(s - k - 1)(\hat{M} \mp (k + 1)\lambda)} C, \quad k > 0, \]

\[ \gamma_{0,0}^{0+} = \mp \sqrt{2(s - 1)(\hat{M} \pm \lambda)} C, \]

\[ \gamma_{k,m}^{0-} = -\sqrt{\frac{(s + k + 1)(\hat{M} \pm (k + 1)\lambda)}{k}} C, \quad (92) \]

\[ \gamma_{k,m}^{0+} = \pm \frac{(s + m)(\hat{M} \pm m\lambda)}{(k - m + 1)(k + m + 2)} \gamma_{k,m}^{0+}, \quad m > 0, \]

\[ \gamma_{k,m}^{0-} = \pm \frac{(s + m)(\hat{M} \pm m\lambda)}{(k + m)(k + m + 1)} \gamma_{k,m}^{0-}, \quad m > 0, \]

while those for the \( \tilde{\gamma} \) \( (m \geq \frac{1}{2}):\)

\[ \tilde{\gamma}_{k+\frac{1}{2}, m+\frac{1}{2}}^{0+} = \mp \sqrt{(k + 1)(s + k + 2)(\hat{M} \mp (k + 2)\lambda)} \tilde{C}, \]

\[ \tilde{\gamma}_{k+\frac{1}{2}, m+\frac{1}{2}}^{0-} = -\sqrt{\frac{(s - k - 1)(\hat{M} \mp (k + 1)\lambda)}{k}} \tilde{C}, \quad k > \frac{1}{2}, \]

\[ \tilde{\gamma}_{\frac{1}{2}, \frac{1}{2}}^{0-} = -\sqrt{\frac{(s - 1)(\hat{M} \mp \lambda)}{2}} \tilde{C}, \quad (93) \]

\[ \tilde{\gamma}_{k+\frac{1}{2}, m+\frac{1}{2}}^{0+} = \pm \frac{(s - m)(\hat{M} \mp m\lambda)}{(k - m + 1)(k + m + 2)} \tilde{\gamma}_{k+\frac{1}{2}, m+\frac{1}{2}}^{0+}, \]

\[ \tilde{\gamma}_{k+\frac{1}{2}, m+\frac{1}{2}}^{0-} = \pm \frac{(s - m)(\hat{M} \mp m\lambda)}{(k + m + 1)(k + m + 2)} \tilde{\gamma}_{k+\frac{1}{2}, m+\frac{1}{2}}^{0-}. \]

Similarly to the previous case, for \( \lambda \neq 0 \) we obtain a pair of additional relations on these constants:

\[ C^* = \mp \epsilon C, \quad \tilde{C}^* = \pm \epsilon \tilde{C} \quad (94) \]

Similarly to the case with the gauge invariant two-forms, the supertransformations for one-forms at \( m = \pm k \) differ from the general case and have to contain zero-forms:

\[ \delta \Omega^{(2k)} = \gamma_{k,k}^{0+} \Omega^{(2k)} \beta_{\beta} + \gamma_{k,k}^{0+} \Omega^{(2k)} \beta_{\beta} + \gamma_{k,k}^{-0} \Omega^{(2k-1)} \hat{c}^{\alpha} \]

\[ + \gamma_{k,k}^{-0} \alpha_{k+\frac{1}{2}}^{0+} e^{\alpha} \Omega^{(2k-1)} \beta_{\beta} + \gamma_{k,k}^{-0} \alpha_{k+\frac{1}{2}}^{0+} e^{\beta} \Omega^{(2k)} \beta_{\beta}, \quad k > 0, \]

19
\[
\delta \Omega = \gamma_{0,0}^+ \Phi^\beta \zeta_\beta + a_0 e_{\alpha\alpha} Y^\alpha \zeta_\alpha + h.c., \quad \tag{95}
\]
\[
\delta \Psi^{(2k)} = \gamma_{k,k}^+ \Omega^{(2k)} \beta \zeta_\beta + \gamma_{k,k}^+ \Omega^{(2k-1)} \zeta_\alpha + \gamma_{k,k}^+ \zeta_\beta + \gamma_{k,k}^+ \alpha_{k+\frac{1}{2}} \epsilon_\beta \alpha \Psi^{(2k)} \zeta_\alpha.
\]

where the coefficient \(a_0\) stands for:

\[
a_0 = -(s + 1)(\tilde{M} \pm \lambda) \sqrt{2(s - 1)(\tilde{M} \pm \lambda)} C \tag{96}
\]

At last, we have to consider remaining unfolded equations which connect gauge sector with the sector of the gauge invariant zero-forms. The corresponding supertransformations have the form:

\[
\delta G^{(2s-2)} = \gamma_{s-1,s-1}^+ \Psi^{(2s-2)} \beta \zeta_\beta + \gamma_{s-1,s-1}^+ \Psi^{(2s-2)} \zeta_\beta + \gamma_{s-1,s-1}^+ \Psi^{(2s-3)} \zeta_\alpha
\]
\[
+ \gamma_{s-1,s}^+ \Psi^{(2s-2)} \beta \zeta_\beta + \gamma_{s-1,s-1}^+ \Psi^{(2s-2)} \zeta_\alpha \tag{97}
\]
\[
\delta W^{(2s-2)} = \gamma_{s-1,m}^+ \Psi^{(2s-2)} \beta \zeta_\beta + \gamma_{s-1,m}^+ \Psi^{(2s-2)} \zeta_\beta + \gamma_{s-1,m}^+ \Psi^{(2s-3)} \zeta_\alpha
\]
\[
+ \gamma_{s-1,m}^+ \Psi^{(2s-2)} \beta \zeta_\beta + \gamma_{s-1,m}^+ \Psi^{(2s-2)} \zeta_\alpha
\]

In particular, this gives us the relations between the constants \(C, \tilde{C}\) and \(C_b, C_f\):

\[
C_b = \mp \frac{C}{\sqrt{2s(s - 1)(\tilde{M} \pm s)}}, \quad C_f = \mp \tilde{C} \sqrt{2s(s - 1)(\tilde{M} \pm s)}. \tag{98}
\]

The parameters \(C, \tilde{C}\) are restricted by the hermiticity conditions only. Similarly to the massless case, their product \(CC\) is always imaginary. It is possible to restrict them further by requiring the invariance of the sum of the bosonic and fermionic Lagrangians. If one takes the normalization of the Lagrangians as in [29], the connection between the parameters is:

\[
\tilde{C} = 4i e C \tag{99}
\]

One can see that this relation is in agreement with the hermiticity conditions.

### 3.2.2 Superblock \(\hat{s} = s + \frac{1}{2}\)

Now we repeat the same steps. The ansatz for the supertransformations for the sector of gauge invariant zero-forms as well as the ansatz for the gauge sector are the same as before — [82] and [85], [88] correspondingly. Hence, the equations on the parameters of the supertransformations are also the same [145], [146]. But the fermionic functions \(\beta, \tilde{\beta}\) are different now and this leads to the essentially different solution. For the sector of the gauge invariant zero-forms we obtain for the bosonic functions \(\delta (m \geq 0)\):

\[
\delta_{k,m}^{\pm 0} = C_b,
\]
\[
\delta_{k,m}^{0} = \pm \frac{(k - s)(\tilde{M} \pm (k + 1)\lambda)}{(k - m)(k - m + 1)} C_b, \quad \delta_{k,0}^{0} = \mp \epsilon C_b, \quad m > 0, \quad \delta_{k,0}^{0} = \mp \epsilon C_b, \tag{100}
\]
\[
\delta_{k,m}^{0+} = \pm \frac{(M \pm m \lambda)}{(s - m)} C_b, \quad m > 0, \quad \delta_{k,0}^{0+} = \mp \epsilon C_b.
\]
and for the fermionic functions $\tilde{\delta}$ ($m \geq \frac{1}{2}$):

$$
\tilde{\delta}^{+0}_{k+\frac{1}{2},m+\frac{1}{2}} = (s - m)(s - m - 1)C_f,
$$

$$
\tilde{\delta}^{-0+}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(k + s + 2)(\tilde{M} \mp (k + 1)\lambda)}{(k - m)(k - m + 1)} \tilde{\delta}^{+0}_{k+\frac{1}{2},m+\frac{1}{2}},
$$

$$
\tilde{\delta}^{+0+}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp (s + m + 1)(\tilde{M} \mp m\lambda)C_f,
$$

$$
\tilde{\delta}^{-0-}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(k + s + 2)(\tilde{M} \mp (k + 1)\lambda)}{(k + m + 1)(k + m + 2)} \tilde{\delta}^{-0+}_{k+\frac{1}{2},m+\frac{1}{2}}.
$$

(101)

Note that in this case $\delta_{s,m}^0 = 0$ as it should be. As in the previous case, for $\lambda \neq 0$ we obtain a pair of additional relations on the two arbitrary constants:

$$
C_b^* = \pm \epsilon C_b, \quad C_f^* = \mp \epsilon C_f
$$

(102)

For the gauge sector supertransformation parameters $\gamma$ we obtain ($m \geq 0$):

$$
\gamma^{+0}_{k,m} = \pm \sqrt{k(s + k + 2)(\tilde{M} \mp (k + 1)\lambda)}C, \quad k > 0,
$$

$$
\gamma^{-0+}_{0,0} = \pm \sqrt{2(s + 2)(\tilde{M} \mp \lambda)}C,
$$

$$
\gamma^{-0-}_{k,m} = \mp \frac{(s - k)(\tilde{M} \mp (k + 1)\lambda)}{k} \gamma^{+0}_{k,m}, \quad m > 0,
$$

$$
\gamma^{0+}_{k,m} = \mp \frac{(s - m + 1)(\tilde{M} \pm m\lambda)}{(k - m + 1)(k - m + 2)} \gamma^{-0+}_{k,m}, \quad m > 0,
$$

(103)

while for the parameters $\tilde{\gamma}$, correspondingly ($m \geq \frac{1}{2}$):

$$
\tilde{\gamma}^{+0}_{k+\frac{1}{2},m+\frac{1}{2}} = \pm \sqrt{(k + 1)(s - k - 1)(\tilde{M} \pm (k + 2)\lambda)}C, \quad k > \frac{1}{2},
$$

$$
\tilde{\gamma}^{-0+}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(s + k + 2)(\tilde{M} \mp (k + 1)\lambda)}{k} \tilde{\gamma}^{+0}_{k+\frac{1}{2},m+\frac{1}{2}}, \quad k > \frac{1}{2},
$$

$$
\tilde{\gamma}^{-0-}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(s + 2)(\tilde{M} \mp \lambda)}{2} \tilde{\gamma}^{-0+}_{k+\frac{1}{2},m+\frac{1}{2}},
$$

$$
\tilde{\gamma}^{0+}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(s + m + 1)(\tilde{M} \mp m\lambda)}{(k - m + 1)(k - m + 2)} \tilde{\gamma}^{0+}_{k+\frac{1}{2},m+\frac{1}{2}},
$$

$$
\tilde{\gamma}^{-0-}_{k+\frac{1}{2},m+\frac{1}{2}} = \mp \frac{(s + m + 1)(\tilde{M} \mp m\lambda)}{(k + m + 1)(k + m + 2)} \tilde{\gamma}^{-0-}_{k+\frac{1}{2},m+\frac{1}{2}}.
$$

(104)

In the flat space $C$ and $\tilde{C}$ are the two arbitrary complex constants while in $AdS$ ($\lambda \neq 0$) they must satisfy the relations similar to that of $C_b$ and $C_f$:

$$
C^* = \pm \epsilon C, \quad \tilde{C}^* = \mp \epsilon \tilde{C}
$$

(105)

The supertransformations for the one-forms with $m = \pm k$ have to contain zero-forms as well. The expressions for their supertransformations are still given by (105), but the expression for the coefficient $a_0$ is now:

$$
a_0 = -s(\tilde{M} \pm \lambda)\sqrt{2(s + 2)(\tilde{M} \mp \lambda)}C
$$

(106)
At last let us turn to the remaining unfolded equations connecting two sectors. In this case, it is fermionic fields supertransformations which have to be modified:

\[
\delta \Psi^{(2s-2)} = \gamma^{+}_{s-1,\bar{s}-1} (\gamma^{(2s-2)} \delta \zeta_{\beta}^{\alpha} + \gamma^{0}_{s-1,\bar{s}-1} (\gamma^{(2s-2)} \delta \zeta_{\beta}^{\alpha} ) + \gamma^{0}_{s-1,\bar{s}-1} (\gamma^{(2s-2)} \delta \zeta_{\beta}^{\alpha} )
\]

\[
\delta Y^{(2s-2)} = \gamma^{+}_{s-1,\bar{s}-1} (\gamma^{(2s-2)} \delta \zeta_{\beta}^{\alpha} ) + \gamma^{0}_{s-1,\bar{s}-1} (\gamma^{(2s-2)} \delta \zeta_{\beta}^{\alpha} )
\]

For the consistency the constants \( C, \tilde{C} \) have to be connected with the constants \( C_{b}, C_{f} \) as follows:

\[
C_{b} = \pm C \sqrt{(s-1)(2s+1)} , \quad C_{f} = \pm \frac{\tilde{C}}{\sqrt{(s-1)(2s+1)}}
\]

Apart from the hermiticity conditions, the constants \( C \) and \( \tilde{C} \) are arbitrary. If the sum of the Lagrangians is required to be invariant, these constants turn out to be connected:

\[
\tilde{C} = 4iC
\]

Again, this relation is in agreement with the hermiticity conditions.

### 3.3 Supermultiplets

We build the supermultiplets now. A massive supermultiplet contains two bosons and two fermions; each pair of one boson and one fermion forms a superblock. It was shown in [13] that the bosons have the opposite parity and the fermions have opposite mass terms sign. This leaves four possible structures of the supermultiplet, as shown in the Figure 1. Each pair of fields connected by a pair of arrows forms a superblock. One can see that the commutator of two supertransformations transforms a field into a combination of two fields and one of these fields corresponds to another particle. The coefficients \( C_{i} \) and \( \tilde{C}_{i} \) have to be tuned to get rid of such terms. This gives certain equalities for the products \( C_{i}\tilde{C}_{i} \). The rest of the terms must form the transformations of the \( AdS \) algebra. Again, we consider integer and half-integer superspin (i.e. average spin of the supermultiplet \( \langle s \rangle \)) cases separately.

#### 3.3.1 Integer superspin case

In case of integer superspin, the coefficients \( C_{i} \) and \( \tilde{C}_{i} \) must satisfy:

\[
C_{1}\tilde{C}_{1} = -C_{2}\tilde{C}_{2} = C_{3}\tilde{C}_{3} = -C_{4}\tilde{C}_{4} = iC^{2} , \quad C_{1}C_{3} = C_{2}C_{4} , \quad \tilde{C}_{1}\tilde{C}_{3} = \tilde{C}_{2}\tilde{C}_{4}
\]

If one also requires the invariance of the sum of the Lagrangians for all four members, the coefficients become fixed up to a single scale factor \( C \). If the highest-spin fermion has \( \epsilon = 1 \), the constants are:

\[
C_{1} = \frac{C}{2} , \quad C_{2} = \frac{C}{2} , \quad C_{3} = \frac{iC}{2} , \quad C_{4} = \frac{iC}{2} , \quad \tilde{C}_{1} = 2iC , \quad \tilde{C}_{2} = -2iC , \quad \tilde{C}_{3} = 2C , \quad \tilde{C}_{4} = -2C.
\]
In case of half-integer superspin, the products of the coefficients

3.3.2 Half-integer integer superspin case

Recall that

We give the resulting expression for the commutator for the bosonic field $\Omega^{\alpha(k+m)\hat{\alpha}(k-m)}$ as an example:

$$[\delta_1, \delta_2]\Omega^{\alpha(k+m)\hat{\alpha}(k-m)} = 4iC^2 \tilde{M}(\langle s \rangle + \frac{1}{2})$$

$$= \left[ \lambda \Omega^{\alpha(k+m)\hat{\alpha}(k-m-1)\hat{\beta}} \eta_{\beta}^{\hat{\alpha}} + \lambda \Omega^{\alpha(k+m-1)\hat{\beta}(k-m)} \eta_{\beta}^{\alpha} + \alpha_{k,m}^{+} \Omega^{\alpha(k+m-1)\hat{\beta}(k-m-1)} \xi_{\beta}^{\hat{\alpha}} + \alpha_{k,m}^{+ \dagger} \Omega^{\alpha(k+m)\hat{\beta}(k-m)} \xi_{\beta}^{\hat{\alpha}} \right] (113)$$

Recall that

$$\eta^{\alpha(2)} = 2\zeta_1^{\alpha} \zeta_2^{\alpha}, \quad \eta^{\hat{\alpha}(2)} = 2\zeta_1^{\hat{\alpha}} \zeta_2^{\hat{\alpha}}, \quad \xi^{\alpha\hat{\alpha}} = \zeta_1^{\alpha} \zeta_2^{\hat{\alpha}} - \zeta_1^{\hat{\alpha}} \zeta_2^{\alpha} \quad (114)$$

The factor $4iC^2 \tilde{M}(\langle s \rangle + \frac{1}{2})$ is the same for all fields. The coefficients $\alpha_{k,m}^{+}$ correspond to the same particle as the field $\Omega^{\alpha(k+m)\hat{\alpha}(k-m)}$. By comparing the expression with then unfolded equations, one can see that it is indeed a combination of pseudotranslations and Lorentz transformations.

3.3.2 Half-integer integer superspin case

In case of half-integer superspin, the products of the coefficients $C_i$ and $\tilde{C}_i$ are fixed by the same relations $[110]$. The requirement of the invariance for the sum of the Lagrangians fixes the coefficients up to the single scale factor. In case of even-parity highest-spin boson, the coefficients are:

$$C_1 = \frac{C}{2}, \quad C_2 = \frac{C}{2}, \quad C_3 = \frac{C}{2}, \quad C_4 = \frac{i C}{2},$$

$$\tilde{C}_1 = 2C, \quad \tilde{C}_2 = -2i C, \quad \tilde{C}_3 = 2i C, \quad \tilde{C}_4 = -2C. \quad (115)$$

Figure 1: Structure of massive HS supermultiplets. Sharp boxes represent bosons, while the skew ones represent fermions. The letter $P$ is boson parity. Each arrow from $A$ to $B$ corresponds to the terms with $B$ fields in the variation of $A$ fields under the supertransformation. All such terms are proportional to their own constant $C_i$ (resp. $\tilde{C}_i$). The parameters $M_+, M_-$ (resp. $\tilde{M}_+, \tilde{M}_-$) are the roots of $M^2 = \tilde{M}(\tilde{M} \pm \lambda)$ with a sign chosen respectively; note that $M_- - M_+ = \lambda$.

If the highest-spin fermion has $\epsilon = -1$, the constants are:

$$C_1 = -\frac{i C}{2}, \quad C_2 = -\frac{C}{2}, \quad C_3 = \frac{C}{2}, \quad C_4 = \frac{C}{2},$$

$$\tilde{C}_1 = -2C, \quad \tilde{C}_2 = 2C, \quad \tilde{C}_3 = 2i C, \quad \tilde{C}_4 = -2i C. \quad (112)$$
If the highest-spin boson is parity-odd, the constants are:

\[
C_1 = \frac{C}{2}, \quad C_2 = -i\frac{C}{2}, \quad C_3 = -i\frac{C}{2}, \quad C_4 = \frac{C}{2}, \\
\tilde{C}_1 = 2iC, \quad \tilde{C}_2 = 2C, \quad \tilde{C}_3 = -2C, \quad \tilde{C}_4 = -2iC.
\] (116)

Again, we present a commutator of the supertransformations for the field \(\Omega^{\alpha(k+m)\hat{\alpha}(k-m)}\) as an example:

\[
[\delta_1, \delta_2]\Omega^{\alpha(k+m)\hat{\alpha}(k-m)} = 2iC^2(\hat{M}_+ + \hat{M}_-)((s) + \frac{1}{2})
\]

\[= \left[ \lambda\Omega^{\alpha(k+m)\hat{\alpha}(k-m-1)}\eta_\beta^{\hat{\alpha}} + \lambda\Omega^{\alpha(k+m-1)\hat{\beta}(k-m)}\eta_\alpha^{\hat{\beta}} + \alpha_{k+m}^{\alpha \hat{\alpha}}\Omega^{\alpha(k+m-1)\hat{\beta}(k-m)}\xi_\beta^{\hat{\beta}} + \alpha_{k+m}^{\alpha \hat{\alpha}}\Omega^{\alpha(k+m-1)\hat{\beta}(k-m)}\xi_\beta^{\hat{\beta}} \right] (117)

One can see that the structure of the commutator is the same as in the previous case. The factor \(2iC^2(\hat{M}_+ + \hat{M}_-)((s) + \frac{1}{2})\) is slightly different now. Again, it is the same for all fields. The coefficients \(\alpha_{k,m}^{ij}\) correspond to the same particle as the field \(\Omega^{\alpha(k+m)\hat{\alpha}(k-m)}\).

4 Infinite spin supermultiplets

Recently it became clear that the gauge invariant formalism we use for the description of massive higher spin fields nicely works for the infinite spin limit as well \[22\,25\,29\]. Moreover, the first examples of the infinite spin supermultiplets in the flat space were constructed \[22\,23\] (see also recent paper \[24\]). In this section we consider unfolded formulation of the infinite spin supermultiplets both in the flat and \(AdS_4\) spaces. These two cases turns out to be rather different, so we consider them separately in the two subsequent subsections.

Let us begin with the general considerations. In the infinite spin limit the gauge invariant formulation does not contain any gauge invariant zero-forms so we have the gauge one-forms \(\Omega, \Psi\) and Stueckelberg zero-forms \(W, Y\) only. In this, the unfolded equations is just the infinite set of the zero-curvature conditions:

\[
\mathcal{R}^{\alpha(k+m)\hat{\alpha}(k-m)} = 0, \quad \mathcal{C}^{\alpha(k+m)\hat{\alpha}(k-m)} = 0 \\
\mathcal{F}^{\alpha(k+m)\hat{\alpha}(k-m)} = 0, \quad \mathcal{D}^{\alpha(k+m)\hat{\alpha}(k-m)} = 0
\] (118)

The expressions for the bosonic curvatures \(\mathcal{R}\) and \(\mathcal{C}\) are still given by \[68\], \[69\], while the fermionic ones are still defined by \[73\], \[77\] but with different functions \(\alpha, \hat{\alpha}\) (see below). Similarly, the general ansatz for the supertransformations for the Stueckelberg zero-forms is still \[83\] and for the one-forms is still \[88\] and \[95\].

4.1 Flat space

In the infinite spin limit the gauge invariant formalism leads to the massless and tachyonic solutions for bosons and only massless ones for fermions (because the tachyonic ones are non unitary) \[25\,26\,29\]. This leaves us the only possibility — a massless infinite spin supermultiplet in agreement with the classification in \[34\].

For the massless infinite spin boson the functions \(\alpha\) have a rather simple form:

\[
\alpha_{k,m}^{++} = \sqrt{k(k+1)\mu} / (k-m+1)(k-m+2)
\]
\[
\alpha_{k,m}^+ = \frac{\mu^2}{(k+m)(k+m+1)(k-m+1)(k-m+2)}
\]
\[
\alpha_{k,0}^- = 1
\]
\[
\alpha_{k,m}^- = \frac{\mu}{(k+m)(k+m+1)\sqrt{k(k-1)}}
\]

where \(\mu\) is a dimensionful parameter related with the eigenvalue of the second Casimir operator of Poincare group. Similarly, for the massless infinite spin fermions we have:

\[
\tilde{\alpha}_{k,m}^+ = \frac{(k+1)\tilde{\mu}}{(k-m+1)(k-m+2)}
\]
\[
\tilde{\alpha}_{k,0}^- = \frac{\tilde{\mu}^2}{(k-m+1)(k-m+2)(k+m+1)(k+m+2)}
\]
\[
\tilde{\alpha}_{k,m}^- = \epsilon \frac{\tilde{\mu}}{(k+1)(k+2)}, \quad \epsilon = \pm 1
\]
\[
\tilde{\alpha}_{k,0}^+ = \frac{\mu}{(k+1)(k+2)}
\]

**Superblock** Let us consider a superblock containing one such boson and one fermion. First of all, supersymmetry requires that their dimensionful parameters must be equal \(\mu = \tilde{\mu}\). Then we obtain the following expressions for the parameters of the supertransformations for the boson:

\[
\gamma_{k,m}^{+0} = \sqrt{k}C
\]
\[
\gamma_{k,m}^{-0} = \frac{\mu}{(k+m)(k+m+1)\sqrt{k}}
\]
\[
\gamma_{k,m}^{0+} = -\epsilon \frac{\sqrt{k}\mu}{(k-m+1)(k-m+2)}C^*
\]
\[
\gamma_{k,m}^{0-} = -\epsilon \frac{1}{\sqrt{k}}C^*
\]

and for the fermion:

\[
\tilde{\gamma}_{k,m}^{+0} = \sqrt{(k+1)}\tilde{C}
\]
\[
\tilde{\gamma}_{k,m}^{-0} = \frac{\mu}{(k+m+1)(k+m+2)\sqrt{k}}\tilde{C}
\]
\[
\tilde{\gamma}_{k,m}^{0+} = \epsilon \frac{\sqrt{(k+1)}\mu}{(k-m+1)(k-m+2)}\tilde{C}^*
\]
\[
\tilde{\gamma}_{k,m}^{0-} = \epsilon \frac{1}{\sqrt{k}}\tilde{C}^*
\]

Here \(C\) and \(\tilde{C}\) are two arbitrary complex constants. It is easy to check that the algebra of these supertransformations is not closed so to construct a supermultiplet we have to consider a pair of bosons and a pair of fermions.

**Supermultiplet** In the flat space, there exists only one infinite spin supermultiplet, with its structure shown in the Figure 2. As in the Lagrangian formulation [23], we have found that the two bosons must have opposite parity, while the two fermions must have opposite signs of the mass-like terms \(\epsilon_2 = -\epsilon_1\). Moreover, all the products \(C_i\tilde{C}_i, \ i = 1, 2, 3, 4\) must be imaginary and satisfy the following relations:

\[
C_1\tilde{C}_1 = -C_2\tilde{C}_2 = C_3\tilde{C}_3 = -C_4\tilde{C}_4
\]
\[
C_2\tilde{C}_3 = -C_1\tilde{C}_4, \quad C_3\tilde{C}_4 = -C_2\tilde{C}_1.
\]
Figure 2: Structure of the infinite spin supermultiplet. The parameters $\mu$ are equal for each particle and are omitted. Again, bosons are represented by sharp boxes, while the fermions - by rounded ones. The sign choice for each superblock is now indicated in the corresponding corner of the picture.

For definiteness, we assume that the first boson is parity-even, and the first fermion has $\epsilon_1 = 1$. If we also require that not only unfolded equations but also the sum of the four Lagrangians is invariant under the supertransformations we obtain

$$C_1 = \frac{C}{2}, \quad C_2 = \frac{C}{2}, \quad C_3 = \frac{C}{2}, \quad C_4 = \frac{C}{2},$$

$$\tilde{C}_1 = 2iC, \quad \tilde{C}_2 = -2iC, \quad \tilde{C}_3 = 2C, \quad \tilde{C}_4 = -2C.$$ (124)

Once again, we provided as an example the explicit expressions for the commutator of the two supertransformations on the one-form $\Omega$:

$$[\delta_1, \delta_2] \Omega^\alpha(k+m)\check{a}(k-m) = 2iC^2 \left[ \alpha_{k,m}^+ \Omega^\alpha(k+m-1)\check{a}(k-m-1)\check{a} + \Omega^\alpha(k+m+1)\check{a}(k-m-1)\check{a} \right] + \alpha_{k,m}^- \Omega^\alpha(k+m-1)\check{a}(k-m-1)\check{a} + \alpha_{k,m}^+ \Omega^\alpha(k+m+1)\check{a}(k-m-1)\check{a} \right] (125)$$

4.2 AdS$_4$ space

In this case for the infinite spin limit the gauge invariant formalism provides a whole range of the unitary solutions both for the bosons as well as for the fermions $[25, 26, 29]$. But as we have already noted for the construction of the supermultiplets it is crucial to have a factorization of the main functions $\alpha^{-+}$ and $\check{a}^{-+}$. The only such possibility we have found — so called ”partially massless” infinite spin particles when the spectrum of helicities is $s \leq |h| < \infty$, where integer or half-integer $s$ denotes the lowest helicity. In this case the main functions look very similar to the massive finite spin case:

$$\alpha_m^{-+} = (m - s - 1)(m + s)[m(m - 1)\lambda^2 - M^2]$$

$$\check{a}_m^{-+} = (m - \check{s} - 1)(m + \check{s})[m - \check{s})^2\lambda^2 - \check{M}^2]$$ (126)

Moreover, it appears that the bosonic and fermionic mass parameters must still satisfy the same relation $M^2 = \tilde{M}[\tilde{M} \pm \lambda]$. As a result, we obtain:

$$\alpha_m^{-+} = (m - s - 1)(m + s)[m\lambda \pm \tilde{M}][(m - 1)\lambda \mp \tilde{M}]$$ (127)
As in the massive case, we begin with the construction of two possible superblocks with \( \hat{s} = s \pm \frac{1}{2} \).

**Superblock \( \hat{s} = s - \frac{1}{2} \)** For the bosonic functions \( \gamma \) we obtain \( (k \geq s, m \geq 0): \)

\[
\begin{align*}
\gamma^{+0}_{k,m} & = \sqrt{k(k+1-s)((k+1)\lambda \mp \tilde{M})} C, \\
\gamma^{-0}_{k,m} & = \sqrt{\frac{(k-s)((k+1)\lambda \pm M)}{k}} C, \\
\gamma^{0+}_{k,m} & = \frac{(s+m)(\tilde{M} \pm m\lambda)}{(k-m+1)(k-m+2)} \gamma^{+0}_{k,m}, \quad m > 0, \\
\gamma^{0-}_{k,m} & = \frac{(s+m)(\tilde{M} \pm m\lambda)}{(k+m)(k+m+1)} \gamma^{-0}_{k,m}, \quad m > 0,
\end{align*}
\]

while for the fermionic functions \( \tilde{\gamma} \) \( (k \geq \hat{s}, m \geq \frac{1}{2} \): \)

\[
\begin{align*}
\gamma^{+0}_{k,\frac{1}{2},m+\frac{1}{2}} & = \sqrt{(k+1)(s+k+2)((k+2)\lambda \pm \tilde{M})} \tilde{C}, \\
\gamma^{-0}_{k,\frac{1}{2},m+\frac{1}{2}} & = \sqrt{\frac{(k-s-1)((k+1)\lambda \pm \tilde{M})}{k}} \tilde{C}, \\
\gamma^{0+}_{k,\frac{1}{2},m+\frac{1}{2}} & = \frac{(s-m)(\tilde{M} \mp m\lambda)}{(k-m+1)(k-m+2)} \gamma^{+0}_{k,\frac{1}{2},m+\frac{1}{2}}, \\
\gamma^{0-}_{k,\frac{1}{2},m+\frac{1}{2}} & = \frac{(s-m)(\tilde{M} \mp m\lambda)}{(k+m)(k+m+1)} \gamma^{-0}_{k,\frac{1}{2},m+\frac{1}{2}},
\end{align*}
\]

Since \( \lambda \neq 0 \), we obtain also a pair of relations on these two parameters \( C \) and \( \tilde{C} \):

\[
C^* = \mp \epsilon C, \quad \tilde{C}^* = \pm \epsilon \tilde{C}
\]

At the same time, the relation between \( c \) and \( \tilde{C} \) from the invariance for the sum of the two Lagrangians appears to be different form the massive case:

\[
\tilde{C} = \pm 4\epsilon C
\]

and this turns out to be important (see below).

**Superblock \( \hat{s} = s + \frac{1}{2} \)** In this case the bosonic functions \( \gamma^{ij}_{k,m} \) are \( (k \geq s, m \geq 0): \)

\[
\begin{align*}
\gamma^{+0}_{k,m} & = \sqrt{k(s+k+2)((k+1)\lambda \mp \tilde{M})} C, \\
\gamma^{-0}_{k,m} & = \sqrt{\frac{(k-s)((k+1)\lambda \pm M)}{k}} C, \\
\gamma^{0+}_{k,m} & = \frac{(s-m+1)(\tilde{M} \mp m\lambda)}{(k-m+1)(k-m+2)} \gamma^{+0}_{k,m}, \quad m > 0, \\
\gamma^{0-}_{k,m} & = \frac{(s-m+1)(\tilde{M} \mp m\lambda)}{(k+m)(k+m+1)} \gamma^{-0}_{k,m}, \quad m > 0,
\end{align*}
\]

and for the fermionic ones \( \tilde{\gamma} \) \( (k \geq \hat{s}, m \geq \frac{1}{2} ):\)

\[
\begin{align*}
\gamma^{+0}_{k,\frac{1}{2},m+\frac{1}{2}} & = \sqrt{(k+1)(k+1-s)((k+2)\lambda \pm \tilde{M})} \tilde{C}, \\
\gamma^{-0}_{k,\frac{1}{2},m+\frac{1}{2}} & = \pm \sqrt{\frac{(s+k+2)((k+1)\lambda \pm \tilde{M})}{k}} \tilde{C}, \\
\gamma^{0+}_{k,\frac{1}{2},m+\frac{1}{2}} & = \pm \frac{(s+m+1)(\tilde{M} \mp m\lambda)}{(k-m+1)(k-m+2)} \gamma^{+0}_{k,\frac{1}{2},m+\frac{1}{2}}, \\
\gamma^{0-}_{k,\frac{1}{2},m+\frac{1}{2}} & = \pm \frac{(s+m+1)(\tilde{M} \mp m\lambda)}{(k+m)(k+m+1)} \gamma^{-0}_{k,\frac{1}{2},m+\frac{1}{2}},
\end{align*}
\]

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In this case we also obtain
\[ C^* = \pm \epsilon C, \quad \tilde{C}^* = \mp i\epsilon \tilde{C} \] (134)
Again, the relation between \( C \) and \( \tilde{C} \) which follows from the Lagrangians invariance, is slightly different:
\[ \tilde{C} = \mp 4i\epsilon C \] (135)

**Supermultiplets** Similarly to the massive case, there exist two different solutions for the infinite spin supermultiplet in \( AdS_4 \), which resemble those with the integer superspin and half integer superspin. Their structure is the same as in the massive case (see Figure 1). The coefficients \( C_i, \tilde{C}_i \) are restricted by the same conditions as in (110):
\[ C_1 \tilde{C}_1 = -C_2 \tilde{C}_2 = C_3 \tilde{C}_3 = C_4 \tilde{C}_4 = iC^2, \quad C_1 C_3 = C_2 C_4, \quad \tilde{C}_1 \tilde{C}_3 = \tilde{C}_2 \tilde{C}_4 \] (136)
The expressions for the commutators are also the same as in the massive supermultiplet case. However, the restrictions following from the Lagrangian invariance cannot be satisfied, as they require, for instance, the bosons to have the same parity. A possible way to restore the invariance is to change the sign of one bosonic and one fermionic Lagrangians so that the connection between \( C_i \) and \( \tilde{C}_i \) becomes \( \tilde{C}_i = 4i\epsilon C \) as in in the massive case. But this spoils the unitarity of the theory and this resembles the situation with the non-unitary partially massless finite spin supermultiplets constructed in [21].

**Conclusion**

In this paper we have constructed the unfolded formulation for the massive higher spin \( N = 1 \) supermultiplets in \( AdS_4 \). Our results are in complete agreement with the results of [18] where the Lagrangian formulation of such supermultiplets were developed. We also consider an infinite spin limit for these supermultiplets with the results also consistent with that of [21].

**Acknowledgements**

Authors are grateful to the I. L. Buchbinder and T. V. Snegirev for collaboration. M.Kh. is grateful to Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS" for their support of the work.

**A Notations and conventions**

In the paper, we adopt the "condensed notation" of the indices. Namely, if an expression contains \( n \) consecutive indices, denoted by the same letter with different indices (e.g. \( \alpha_1, \alpha_2, \ldots, \alpha_n \)) and is symmetric on them, we simply write the letter, with the number \( n \) in parentheses if \( n > 1 \) (e.g. \( \alpha(n) \)). For example:
\[ \Phi^{\alpha_1, \alpha_2, \alpha_3} = \Phi^{\alpha(3)}, \quad \zeta^{\alpha_1} \Omega^{\alpha_2 \alpha_3} = \zeta^{\alpha} \Omega^{\alpha(2)} \] (137)
We define symmetrization over indices as the sum of the minimal number of terms necessary without normalization multiplier.

We use the multispinor formalism in four dimensions as in the paper [33]. Every vector index is transformed into a pair of spinor indices: \( V^\mu \sim V^{\alpha, \dot{\alpha}} \), where \( \alpha, \dot{\alpha} = 1, 2 \). Dotted and undotted indices are transformed into one another under the hermitian conjugation:
\[ (\Omega^{\alpha \dot{\alpha}(2)})^\dagger = \Omega^{\alpha(2)\dot{\alpha}} \] (138)
The spin-tensors, i.e. fields with odd number of indices, are Grassmannian. For example,

\[ A^{\alpha(2)\dot{\alpha}} \eta^\alpha = -\eta^\alpha A^{\alpha(2)\dot{\alpha}} \]  

(139)

Under the hermitian conjugation, the order of fields is reversed:

\[ (A^{\alpha(2)\dot{\alpha}} \eta^\alpha)^\dagger = \eta^\alpha A^{\alpha(2)\dot{\alpha}} = -A^{\alpha(2)\dot{\alpha}} \eta^\alpha \]  

(140)

The metrics for the spinor indices is an antisymmetric bispinor:

\[ \delta_{\alpha\beta} \xi^\beta = -\xi_\alpha, \quad \epsilon^{\alpha\beta} \xi_\beta = \xi^\alpha, \]  

(141)

similarly for dotted indices. Hence, symmetry over a set on indices implies tracelessness. This feature greatly simplifies the work with traceless mixed symmetry tensors and spin-tensors. The mixed symmetry tensor \( \Phi^{\mu(k),\nu(l)} \) which corresponds to the two-row Young tableaux \( Y(k, l) \) is described by a pair of multispinors \( \Phi^{\alpha(k+l)\dot{\alpha}(k+l)} \), \( \Phi^{\alpha(k-l)\dot{\alpha}(k+l)} \) in multispinor formalism; if the tensor \( \Phi^{\mu(k),\nu(l)} \) is real then:

\[ (\Phi^{\alpha(k+l)\dot{\alpha}(k+l)})^\dagger = \Phi^{\alpha(k-l)\dot{\alpha}(k+l)}. \]  

(142)

Similarly, the mixed symmetry spin-tensor \( \Psi^{\mu(k),\nu(l)} \) which corresponds to the Young tableaux \( Y(k + \frac{1}{2}, l + \frac{1}{2}) \) is described by a pair of multispinors \( \Psi^{\alpha(k+l+1)\dot{\alpha}(k+l)}, \Psi^{\alpha(k-l+1)\dot{\alpha}(k+l+1)} \); if the spin-tensor \( \Psi^{\mu(k),\nu(l)} \) is Majorana one then

\[ (\Psi^{\alpha(k+l+1)\dot{\alpha}(k+l)})^\dagger = \Psi^{\alpha(k-l+1)\dot{\alpha}(k+l+1)}. \]  

(143)

In the frame-like formalism, two bases, namely the world one and the local one are used. We denote the local basis vectors as \( e^{\alpha\dot{\alpha}} \); the world indices are omitted, and all the fields are assumed differential forms with respect to them. Similarly all the products are exterior with respect to the world indices. In the paper, we use basis forms, i.e. antisymmetrized products of basis vectors \( e^{\alpha\dot{\alpha}} \). The forms are 2-form \( E^{\alpha(2)} + h.c. \), 3-form \( E^{\alpha\dot{\alpha}} \) and 4-form \( E \).

The transformation law of the forms under the hermitian conjugation is:

\[ (e^{\alpha\dot{\alpha}})^\dagger = e^{\alpha\dot{\alpha}} \quad (E^{\alpha(2)})^\dagger = E^{\alpha(2)} \quad (E^{\alpha\dot{\alpha}})^\dagger = -E^{\alpha\dot{\alpha}} \quad (E)^\dagger = -E \]  

(144)

### B Equations on the parameters of superblock

Here we provide a complete set of equations which follows from the requirement that unfolded equations be invariant under the supertransformations. For the supertransformations of the bosonic sector of gauge invariant zero-forms we obtain:

\[
\begin{align*}
\delta^0_{k,m} & \beta^i_{k+\frac{1}{2},m-\frac{1}{2}} + \beta^i_{k,m} \delta^0_{k+\frac{1}{2}(1+i),m-\frac{1}{2}(1-i)} - \lambda \delta^0_{k,m} \\
\delta^0_{k,m} & \beta^{-i}_{k+\frac{1}{2},m+\frac{1}{2}} + \beta^{-i}_{k,m} \delta^0_{k+\frac{1}{2}(1+i),m+\frac{1}{2}(1-i)} - \lambda \delta^0_{k,m} \\
\beta^{ij}_{k,m} & \delta^0_{k+\frac{1}{2},j} + \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2}(1+i),m+\frac{1}{2}(1-j)} \\
\beta^{ij}_{k,m} & \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2},j} + \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2}(1+i),m+\frac{1}{2}(1-j)} \\
\beta^{ij}_{k,m} & \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2},j} + \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2}(1+i),m+\frac{1}{2}(1-j)} \\
\beta^{ij}_{k,m} & \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2},j} + \delta^0_{k,m} \beta^{ij}_{k+\frac{1}{2}(1+i),m+\frac{1}{2}(1-j)}
\end{align*}
\]  

(145)
and similar conditions with inverted tildes (i.e. tilde is added above the coefficients which do not possess one and removed from those which have one) for the fermionic sector with half-integer \( k, m \). Here \( i, j \) are numbers \( \pm 1 \); when written as upper indices of the coefficients, they stand for + and − respectively.

Similarly, for the gauge sector supertransformation parameters we get:

\[
\begin{align*}
\gamma_{k,m}^{0+} & \hat{\alpha}_{k+1/2,m-1/2}^{+i,j} + \alpha_{k,m}^{i,j} \gamma_{k+1/2,m+1/2}^{0-} - \lambda_{k,m}^{0i} = \alpha_{k,m}^{i,j} \gamma_{k-1/2,m+1/2}^{0-} - \gamma_{k,m}^{0+} \hat{\alpha}_{k+1/2,m-1/2}^{+i,j} \\
\gamma_{k,m}^{0-} & \hat{\alpha}_{k+1/2,m+1/2}^{-i,j} + \alpha_{k,m}^{i,j} \gamma_{k+1/2,m+1/2}^{0+} - \lambda_{k,m}^{0i} = \alpha_{k,m}^{i,j} \gamma_{k-1/2,m-1/2}^{-i,j} - \gamma_{k,m}^{0+} \hat{\alpha}_{k+1/2,m+1/2}^{+i,j} \\
\hat{\alpha}_{k,m}^{i,j} &= \alpha_{k,m}^{i,j} \gamma_{k+1/2,m+1/2}^{0+} \gamma_{k,m}^{0i} \gamma_{k+1/2,m+1/2}^{0-} \gamma_{k,m}^{0i}
\end{align*}
\]

The relations for \( \hat{\gamma}_{k,m}^{ij} \) are obtained by inverting tildes.

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