A Note on Rectilinearity and Angular Resolution

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Abstract

We connect two aspects of graph drawing, namely angular resolution, and the possibility to draw with all angles an integer multiple of $2\pi/d$. A planar graph with angular resolution at least $\pi/2c$ can be drawn with all angles an integer multiple of $\pi/2$ (rectilinear). For $d \neq 4$, $d > 2$, an angular resolution of $2\pi/d$ does not imply that the graph can be drawn with all angles an integer multiple of $2\pi/d$. We argue that the exceptional situation for $d = 4$ is due to the absence of triangles in the rectangular grid.

Keywords: Rectilinear drawing, plane graph, angular resolution, integer flow.

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1 Introduction

Angular resolution and rectilinearity are well-studied aspects of plane graphs. The angular resolution of a (plane) graph is the minimum angle made by line segments at a vertex. A graph is rectilinear if it can be drawn with all angles a multiple of $\pi/2$ radians. In this note we give an easy proof of the “folk conjecture” that graphs with an angular resolution at least $\pi/2$ are rectilinear.

We generalise rectilinearity and call a graph $d$-linear if it allows a drawing with all edges a multiple of $2\pi/d$ (thus rectilinearity is 4-linearity). Unlike the case $d = 4$, for $d > 4$ it is not the case that an angular resolution of $2\pi/d$ implies $d$-linearity.

This is the organization of the paper. The remainder of this section introduces some preliminaries, including Tamassia’s flow model for the angles in a drawing, on which our first result is based. Section 2 proves our positive result (for $d = 4$). Section 3 contains the negative result (for $d > 4$). Section 4 lists conclusions.

1.1 Preliminaries

We assume familiarity of the reader with basic graph notions. A plane graph is a planar graph given together with an embedding; this embedding should be respected in a drawing. Given a drawing, its angular resolution is the minimum angle made by line segments at any vertex, and the angular resolution of a plane graph is the maximum angular resolution of any drawing. A drawing is called $d$-linear if all angles are an integer multiple of $2\pi/d$ radians.

A vertex of degree $\delta$ has $\delta$ angles: one between each two successive incident edges. For vertex $v$ and face $f$, let $d(v, f)$ be the number of angles of $v$ that belong to face $f$ (only for a cutvertex $v$ there is an $f$ for which $d(v, f) \geq 2$). For every face $f$, we let $a(f)$ denote the number of angles that belong to $f$. In a biconnected graph, $a(f)$ also equals the number of edges at the border of $f$ and it equals the number of vertices on the border of $f$.

1.2 The Flow Model for Angles

The embedding contained in a plane graph defines the position of the nodes in a qualitative manner, but to convert the embedding into a drawing, in addition two more things need be specified: the angles between the edges at each node, and the lengths of all edges. Tamassia [3] has shown that the angle values in a drawing satisfy the constraints of a suitably chosen multi-source multi-sink flow network. In any drawing, the angles around a node sum up to $2\pi$ and if an internal face is drawn as an $a$-gon its angles sum up to $\pi(a - 2)$ radians. (The angles of the outer face with $a$ edges sum up to $\pi(a + 2)$.)

Because we want rectangular angles to correspond to integers, we shall now express angles in units of $\pi/2$ radians. Thus, with $\alpha_{v,f}$ the angle at node $v$ in face $f$, the collection of angles in a drawing with angular resolution 1 is a
solution for this set of linear equations:

\[
\begin{align*}
\sum_f \alpha_{v,f} &= 4 & \text{for all nodes } v \\
\sum_v \alpha_{v,f} &= 2(\alpha(f) - 2) & \text{for all internal faces } f \\
\sum_v \alpha_{v,f} &= 2(\alpha(f) + 2) & \text{for outer faces } f \\
\alpha_{v,f} &\geq 1 & \text{for all incident } v \text{ and } f
\end{align*}
\]

We refer to these equations as the network model for \( G \); observe that all constraints are integer numbers. The description of this set of equations as a flow network can be found in \([1, 3]\).

**Relations between flows and drawings.** The following two results are known.

**Theorem 1** If plane graph \( G \) has a drawing with angular resolution at least 1 unit (\( \pi/2 \) radians), then the associated network model has a solution.

**Theorem 2** (Tamassia \([3]\)) If the network model associated to plane graph \( G \) has an integer solution, then \( G \) has a rectilinear drawing.

2 Angular Resolution \( \pi/2 \) Implies Rectilinear

Our main result is obtained by combining Theorems 1 and 2 with a result from standard flow theory.

**Theorem 3** If a graph has angular resolution at least \( \pi/2 \), then it is rectilinear.

**Proof.** Assume \( G \) has angular resolution at least \( \pi/2 \) radians. By definition, it has a drawing with angular resolution at least 1 unit, hence by Theorem 1 the associated network model has a solution. It is known from flow theory (see, e.g., \([2, \text{Chapters 10, 11}]\)) that if a flow network with integer constraints admits a flow, then it admits an integer flow. By Theorem 2, we have that \( G \) has a rectilinear drawing.

3 Angular Resolution and \( d \)-Linearity

This section answers the question for what values of \( d \), any plane graph with angular resolution \( 2\pi/d \) radians is \( d \)-linear. The cases \( d = 1 \) and \( d = 2 \) are somewhat trivial, as the classes of drawable graphs are collections of isolated edges, or paths, respectively.

The case of odd \( d \) is somewhat degenerate as, while drawings are supposed to be built up of straight lines, a straight angle at a vertex is not allowed. The cycle with \( 2d + 1 \) points can be drawn as a regular \( (2d + 1) \)-gon, witnessing that its angular resolution is at least \( 2\pi/d \). But it does not have a \( d \)-linear drawing, as its angle sum, \((2d - 1) \cdot \pi\), is not an integer multiple of \( 2\pi/d \).
In the remainder of this section \( d = 2d' \) is even and larger than 4. Measuring angles in units of \( \pi/d' \) radians, we observe that the angles of any triangle add to exactly \( d' \) units. We introduce rigid triangles as gadgets with a fixed shape. The graph \( T_{d'} \) has a top node \( t \) and base nodes \( b_1 \) and \( b_2 \), forming a cycle. For even \( d' \), \( t \) has a third neighbor \( i \), inserted between \( b_2 \) and \( b_1 \) in the planar embedding. Each base node has \( \left\lfloor \frac{d' - 3}{2} \right\rfloor \) neighbors, also located inside the triangle; see Figure 1. The graph \( T_{d'} \) has an angular resolution of \( 2\pi/d \) radians, and in every drawing with that resolution, each segment of each angle measures exactly \( \pi/d' \) radians, that is, the proportions of \( T_{d'} \) are fixed in every drawing. The ratio between height and base length of the rigid triangle in such a drawing, \( b_{d'} \), can be computed as

\[
b_{d'} = \frac{1}{2} \tan\left(\frac{\left\lfloor \frac{d' - 1}{2} \right\rfloor \cdot \left(\frac{\pi}{d'}\right)}{2}\right).
\]

![Figure 1: Rigid Triangles and the Crane \( C_{d',6,2} \)](image)

The crane graph \( C_{d',k,l} \) contains \( 1 + 2k + 4l \) copies of \( T_{d'} \) joined together. A central triangle is extended with a leg consisting of \( k \) pairs of triangles on one side, and an arm consisting of \( l \) quartets of triangles on the other side. In any drawing where all internal angles of the triangles satisfy the constraint for angular resolution \( \pi/d' \), the angle \( \alpha \) at the bottom of the drawing satisfies \( \tan \alpha = \frac{2l}{k} b_{d'} \). By choosing \( k \) and \( l \), any angle between \( \pi/d' \) and \( \pi/2 \) radians can be approximated arbitrarily closely, contradicting the possibility to draw any crane with all angles a multiple of \( \pi/d' \) radians.

**Theorem 4** For each \( d' > 2, \beta \geq \pi/d', \epsilon > 0 \), there exists a graph \( G = C_{d',k,l} \) such that

1. \( G \) has angular resolution \( \pi/d' \) radians;
2. each drawing of \( G \) with angular resolution \( \pi/d' \) contains an angle \( \alpha \) such that \( |\alpha - \beta| < \epsilon \).

### 4 Conclusions

Our note compares two types of drawings, namely (1) those where all angles are at least \( 2\pi/d \) (angular resolution) and (2) those where all angles are an integer
multiple of $2\pi/d$. The flow model introduced by Tamassia [3] implies that if angles can be assigned satisfying (1), then it is also possible to assign all angles satisfying (2). However, only in the special case $d = 4$ it is also possible to assign edge lengths in such a way that a drawing results.

When drawing graphs in a rectilinear way, the drawing is built up from rectangular elements and in such drawings it is possible to shift parts of the drawing without disrupting angles in other parts; see Figure 2. Indeed, a rectangle can be stretched in one direction while preserving the orthogonality of its angles. In a drawing containing triangles, this is not possible, because stretching a triangle in one direction changes its angles. We therefore conclude that the special position of $d = 4$ in the studied problem is due to the absence of triangles in a rectilinear grid.

![Figure 2: Orthogonal and non-orthogonal movements](image)

The observations in this note can be extended to the situation where the embedding is free; that is, only a (planar) graph is given and the question is, what type of drawings does it admit. For the positive result (for $d = 4$), if the graph has some drawing with angular resolution $\pi/2$, it can be drawn rectilinearly with the same embedding. Our triangles $T_d'$ loose their rigidity if the embedding is free, because one can draw the internal nodes on the outside and then modify the triangle shape. By connecting the internal nodes in a cycle this can be prevented; in fact the triangles are modified to enforce the same embedding.

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