A nonstandard proof of the Jordan curve theorem

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Abstract

We give a nonstandard variant of Jordan’s proof of the Jordan curve theorem which is free of the defects his contemporaries criticized and avoids the epsilontic burden of the classical proof. The proof is self-contained, except for the Jordan theorem for polygons taken for granted.

Introduction

The Jordan curve theorem was one of the starting points in the modern development of topology (originally called Analysis Situs). This result is considered difficult to prove, at least compared to its intuitive evidence.

C. Jordan [5] considered the assertion to be evident for simple polygons and reduced the case of a simple closed continuous curve to that of a polygon by approximating the curve by a sequence of suitable simple polygons.

Although the idea appears natural to an analyst it is not so easy to carry through. Jordan’s proof did not satisfy mathematicians of his time. On one hand it was felt that the case of polygons also needed a proof based on clearly stated geometrical principles, on the other hand his proof was considered incomplete (see the criticisms formulated in [11] and in [8]).

If one is willing to assume slightly more than mere continuity of the curve than much simpler proofs (including the case of polygons) are available (see Ames [1] and Bliss [3] under restrictive hypotheses).

O. Veblen [11] is considered the first to have given a rigorous proof which, in fact, makes no use of metrical properties, or, in the words of Veblen: We accordingly assume nothing about analytic geometry, the parallel axiom, congruence relations, nor the existence of points outside a plane.

His proof is based on the incidence and order axioms for the plane and the natural topology defined by the basis consisting of nondegenerate triangles. He also defines simple curves intrinsically as specific sets without parametrizations by intervals of the real line. He finally discusses how the introduction of one
additional axiom, existence of a point outside the plane, allows to reduce his result to the context JORDAN was working in.

VEBLEN also gave a specific proof for polygons based on the incidence and order axioms exclusively (see [10]) which was later criticized as inconclusive by H. HAHN [4] who published his own version of a proof based on VEBLEN’s incidence and order axioms of the plane (which, by the way, are equivalent to the incidence and order axioms of HILBERT’s system).

JORDAN’s proof in his Cours d’analyse of 1893 is elementary as to the tools employed. Nevertheless the proof extends over nine pages and, as mentioned above, cannot be considered complete. We are interested here in this proof. It depends on some facts for polygons and an approximation argument. It is, therefore, a natural idea to use nonstandard arguments to eliminate the epsilontic burden of the approximation.

There is an article by L. NARENS [6] in which this point of view is adopted. Unfortunately, beside some problems discovered in his proof by those who read it carefully, the reasoning involves more complicated topological ideas and anyway is not essentially shorter than or comparably elementary as the JORDAN’s proof.

It is certainly true that not all classical arguments can be replaced in some useful or reasonable way by simpler nonstandard arguments. But as we shall show it is possible to simplify the approximation argument specific to JORDAN’s proof. We shall follow the proof quite closely but take a somewhat different approach when proving path-connectedness.

That nonstandard analysis can even give some additional insight into the geometric problem is manifest from the proof by N. BERTOGLIO and R. CHUAQUI [2] which avoids polygons and approximations entirely by looking at a nonstandard discretization of the plane and reducing the problem to a combinatorial version of the JCT proved by L. N. STOUT [9]. This reduction of the problem to a (formally) discrete one is interesting and leads to a proof which establishes a link to a context totally different from JORDAN’s.

As a curiosity we note in passing that JORDAN speaks of infinitesimals in his proof but it is only a figure of speech for a number which may be chosen as small as one wishes or for a function which tends to zero.

For reference we state:

**The Jordan curve theorem** (abbreviated as JCT)

A simple closed continuous curve $K$ in the plane separates its complement into two open sets of which it is the common boundary: one of them is called the outer region (or exterior domain) $K_{\text{ext}}$ which is an open, unbounded, path-connected set and another set called the inner region (or interior domain) $K_{\text{int}}$ which is an open, simply path-connected, bounded set.

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Comment

We shall use the IST language of nonstandard analysis (see Nelson [7]) to present the proof. However, as the reasoning involves only some very basic nonstandard notions, the exposition will be equally well understood by the followers of the “asterisk” version of nonstandard analysis (although nonstandard polygons should be called hyperpolygons or *-polygons, K should be sometimes replaced by *K etc.)

Plan of the proof

Starting the proof of the Jordan theorem, we consider a simple closed curve $K = \{ K(t) : 0 \leq t < 1 \}$ where $K : \mathbb{R} \rightarrow \mathbb{R}^2$ is a standard continuous 1-periodic function which is injective modulo 1 (i.e. $K(t) = K(t')$ implies $t - t' \equiv 0 \mod 1$). From $K(t) \approx K(t')$ it follows then that $t \approx t' \mod 1$.

Section 1. We infinitesimally approximate $K$ by a simple polygon $\Pi$, using a construction, due to Jordan, of consecutive cutting loops in an originally self-intersecting approximation.

Section 2. We define the interior $K_{\text{int}}$ as the open set of all (standard) points which belong to $\Pi_{\text{int}}$ but does not belong to the monad of $\Pi$. (The Jordan theorem for polygons is taken for granted; this attaches definite meaning to $K_{\text{int}}$ and $K_{\text{ext}}$.) $K_{\text{ext}}$ is defined accordingly.

Section 3. To prove that $K_{\text{int}}$ is nonempty, we take a longest diameter $AB$ of $\Pi$, draw two straight lines, $\alpha$ and $\beta$, through resp. $A$ and $B$ orthogonally to $AB$, and a parallel line $\gamma$ between $\alpha$ and $\beta$ at equal distance from them. Now $\Pi$ is divided by $A$ and $B$ on two simple disjoint broken lines, $\mathcal{L}$ (the left part) and $\mathcal{R}$ (the right part). The straight segment $CD$ of $\gamma$ bounded by the rightmost intersection $C$ of $\gamma$ with $\mathcal{L}$ and the next to the right intersection $D$ of $\gamma$ with $\mathcal{R}$ is included in $\Pi_{\text{int}}$ and contains a point which does not belong to the monad of $\Pi$.

Section 4. To prove that $K_{\text{int}}$ is path-connected we define a simple polygon $\Pi'$ which lies entirely within $\Pi_{\text{int}}$, does not intersect $K$, and contains all points of $K_{\text{int}}$. This easily implies the path-connectedness.

1 Approximation by a simple polygon

We say that an (internal) polygon $\Pi = P_1P_2\ldots P_nP_1$ (n may be infinitely large) approximates $K$ if there is an internal sequence of reals $0 \leq t_1 < t_2 < \ldots < t_n < 1$ such that

(†) $P_i = K(t_i)$ for $1 \leq i < n$

(‡) $t_n - t_1 > \frac{1}{2}$ and $t_{i+1} - t_i < \frac{1}{2}$ for all $1 \leq i < n$.

We say that $\Pi$ approximates $K$ infinitesimally if in addition $\Delta(\Pi) \approx 0$, where $\Delta(\Pi) = \max_{1 \leq k \leq n} |P_kP_{k+1}|$ (it is understood that $P_{n+1} = P_1$).
Lemma 1 Let $\Pi = P_1 \ldots P_n P_1$ approximate $K$ infinitesimally. Then

(i) $n$ is infinitely large, $t_{i+1} \approx t_i$ for all $1 \leq i < n$, $t_1 \approx 0$, and $t_n \approx 1$;

(ii) $\text{monad} K = \text{monad} \Pi$.

(iii) If $k < l$ and $P_k \approx P_l$ then: either $t_k \approx t_l$ and the arcs $P_k P_l$ of both $K$ and $\Pi$ are contained in $\text{monad} P_k = \text{monad} P_l$; or $t_k \approx 0$, $t_l \approx 1$, and the arcs $P_1 P_k$ of both $K$ and $\Pi$ are contained in $\text{monad} P_k = \text{monad} P_l$.

Proof (i) The requirement (†) does not allow the reals $t_k$ to collapse into a sort of infinitesimal “cluster” or into a pair of them grouped around 0 and 1; both the cases are compatible with the conjunction of (†) and $\Delta(\Pi) \approx 0$.

(ii) $\delta_i = \max_{t_i \leq t \leq t_{i+1}} |K(t) - K(t_i)|$ is infinitesimal for each $1 \leq i \leq n$ and therefore $\varepsilon = 2 \max_{1 \leq i \leq n} \delta_i$ is infinitesimal and proves the assertion. □

Lemma 2 There exists a simple polygon which infinitesimally approximates $K$.

Proof Taking $t_i = \frac{i}{n}$ for some infinitely large $n$ results in a polygon which infinitesimally approximates $K$. But it may have self-intersections.

First of all two adjacent sides, $P_k P_{k+1}$ and $P_{k+1} P_{k+2}$ may have an “illegal” self-intersection other than the common vertex $P_{k+1}$, i.e. either $P_k$ is an inner point of $P_{k+1} P_{k+2}$ or $P_{k+2}$ is an inner point of $P_k P_{k+1}$. In this case we simply eliminate the vertex $P_{k+1}$, so that the polygon $P_1 \ldots P_k P_{k+1} P_{k+2} P_n P_1$ reduces to $P_1 \ldots P_k P_{k+2} \ldots P_n P_1$.

Assume two non-adjacent sides intersect, i.e. $P_i P_{i+1}$ intersects $P_j P_{j+1}$ for some $1 \leq i < j - 1 < n$. By the triangle inequality the shorter of the segments $P_i P_j$ and $P_{i+1} P_{j+1}$ is not longer than the longer of the segments $P_i P_{i+1}$ and $P_j P_{j+1}$ which is bounded in length by $\Delta(\Pi)$.

If $|P_i P_j| \leq |P_{i+1} P_{j+1}|$ then we consider that one of the polygons

$$P_1 \ldots P_j P_{j+1} \ldots P_n P_1$$

and

$$P_1 P_{i+1} \ldots P_j P_i$$

which is parametrically longer, i.e. the first one if $t_j - t_i \leq \frac{1}{2}$ or the second one if $t_j - t_i > \frac{1}{2}$. In the case when $|P_{i+1} P_{j+1}| \leq |P_i P_j|$ we take that one of the polygons $P_1 \ldots P_{i+1} P_{j+1} \ldots P_n P_1$ and $P_{i+1} P_{i+2} \ldots P_{j+1} P_{j+1}$ which is parametrically longer. (... means that all indices in between are involved.)

In all the cases the resulting polygon $\Pi_{\text{new}}$ still approximates $K$ (Lemma 1(iii)) easily implies (†) for $\Pi_{\text{new}}$ and satisfies $\Delta(\Pi_{\text{new}}) \leq \Delta(\Pi)$ because the only new side is not longer than a certain side of $\Pi$.

This (internal) procedure does not necessarily reduce the number of self-intersections because for the one which is removed there may be others appearing on the newly introduced side of the reduced polygon $\Pi_{\text{new}}$. But the number of vertices of $\Pi_{\text{new}}$ is strictly less than that of $\Pi$.

Therefore the internal series of polygons arising from $\Pi$ by iterated applications of this reduction procedure eventually ends by a simple polygon $\Pi'$. This
polygon approximates \( K \) and satisfies \( \Delta(\Pi') \leq \Delta(\Pi) \) by the construction, so \( \Delta(\Pi') \approx 0 \). Finally, as the (internal) requirement (\( \dagger \)) is preserved at each step of the procedure, we conclude, by internal induction, that \( \Pi' \) still satisfies (\( \dagger \)), hence approximates \( K \) infinitesimally.

2 Definition of the inner and outer region

Let us fix for the remainder a polygon \( \Pi = P_1P_2 \ldots P_nP_1 \) which approximates \( K \) infinitesimally.

Let \( I \) be the open set of all (standard) points \( A \in \Pi_{\text{int}} \) which have a non-infinitesimal distance from \( \Pi \). We put \( K_{\text{int}} = I \) and call this the \textit{inner region} of the curve \( K \). In the same way we define the open set \( E \) of all (standard) points from \( \Pi_{\text{ext}} \) which have non-infinitesimal distance from \( \Pi \), put \( K_{\text{ext}} = E \) and call this the \textit{outer region} of the curve \( K \).

Omitting rather elementary proofs that \( K_{\text{int}} \) is bounded, \( K_{\text{ext}} \) is unbounded, and the complement of the union of both sets equals the curve \( K \), let us prove that for \( A \in K_{\text{int}} \) and \( B \in K_{\text{ext}} \) any (standard) continuous arc from \( A \) to \( B \) intersects \( K \). Indeed the arc must intersect \( \Pi \) in some point \( P \) because it starts in \( \Pi_{\text{int}} \) and ends in \( \Pi_{\text{ext}} \). (The Jordan theorem for polygons, transferred to the nonstandard domain, is applied.) By Lemma \( \ref{lemma1} \), there is a (standard) point \( P' \in K \) infinitesimally close to \( P \in \Pi \). As \( K \) and the arc are standard and closed, \( P' \) is in \( K \) and the arc.

3 The inner region is non-empty

We prove that the inner region is not empty (that the outer region is not empty is trivial). By definition it suffices to prove the existence of a point in \( \Pi_{\text{int}} \) having non-infinitesimal distance from \( \Pi \).

Arguing in the nonstandard domain, we let \( A \) and \( B \) be the two vertices of \( \Pi \) with maximal distance between them; the distance by necessity must be non-infinitesimal. Let \( \alpha \) and \( \beta \) be two straight lines through \( A \) and \( B \) respectively and orthogonal to the segment \( AB \). The polygon \( \Pi \) now consists of two broken lines \( \mathcal{L} \) and \( \mathcal{R} \) joining \( A \) and \( B \), having no common points and not intersecting \( \alpha \) and \( \beta \) except for the points \( A \) and \( B \).

Let \( \gamma \) be a straight line parallel to \( \alpha \) and \( \beta \) and drawn between them at equal distance from both. Suppose that \( \mathcal{L} \) is the arc (among \( \mathcal{L} \) and \( \mathcal{R} \)) first encountered when coming from left along \( \gamma \). There must be a rightmost intersection point \( C \) for \( \gamma \) and \( \mathcal{L} \). Continuing from there further to the right there must be a first intersection \( D \) with \( \mathcal{R} \). One easily sees that the points between these two must be contained in the interior region \( \Pi_{\text{int}} \).

Consider a point \( E \) in the segment \( CD \) which has the equal distance \( d = d(E, \mathcal{L}) - d(E, \mathcal{R}) \) from both \( \mathcal{L} \) and \( \mathcal{R} \). Note that \( d \) is not infinitesimal. Indeed otherwise there are vertices \( L \in \mathcal{L} \) and \( R \in \mathcal{R} \) such that \( L \approx E \approx R \), which is impossible by Lemma \( \ref{lemma1} \) as \( \gamma \) has non-infinitesimal distance from \( \alpha \) and \( \beta \).
4 Path-connectedness

Let $A$ and $B$ be two (standard) points in $\mathcal{K}_{\text{int}}$. We have to prove that there is a broken line joining $A$ with $B$ and not intersecting $\mathcal{K}$. This is based on the following lemma.

**Lemma 3** There exists a simple polygon $\Pi'$ lying entirely within $\Pi'_{\text{int}}$, containing no point of $\,*\mathcal{K}*$ in $\Pi'_{\text{int}}$, and containing both $A$ and $B$ in $\Pi'_{\text{int}}$.

The lemma clearly implies the path-connectedness: indeed, $A$ can be connected to $B$ by a broken line which lies within $\Pi'_{\text{int}}$ therefore does not intersect $\,*\mathcal{K}$.* By Transfer we get a standard broken line which connects $A$ and $B$ and does not intersect $\mathcal{K}$, as required.

**Proof** of the lemma. Let an infinitesimal $\varepsilon > 0$ be defined as in the proof of Lemma 1(iii) so that $\,*\mathcal{K}*$ is included in the $\varepsilon$-nbhd of $\Pi$.

Note that each side of $\Pi$ is infinitesimal by definition. For any side $PQ$ of $\Pi$ we draw a rectangle of the size $\,(|PQ| + 4\varepsilon) \times (4\varepsilon)$ so that the side $PQ$ lies within the rectangle at equal distance $2\varepsilon$ from each of the four sides of the rectangle.

Let us say that a point $E$ is the inner intersection of two straight segments $\sigma$ and $\sigma'$ iff $E$ is an inner point of both $\sigma$ and $\sigma'$, and $\sigma \cap \sigma' = \{E\}$. For any point $C \in \Pi_{\text{int}}$ which is either a vertex of some of the rectangles above, or an inner intersection of sides of two different rectangles in this family — let us call points of this type special points — let $CC'$ be a shortest straight segment which connects $C$ with a point $C'$ on $\Pi$; obviously $CC'$ is infinitesimal.

The parts of the rectangles lying within $\Pi$ and the segments $CC'$ for special points $C$ decompose the interior $\Pi_{\text{int}}$ in a number of polygonal domains. Let $\Pi'$ be the polygon among them such that $\Pi'_{\text{int}}$ contains $A$. (Note that all the lines involved lie in the monad of $\Pi$, hence none of them contains $A$ or $B$.) It remains to prove that $\Pi'_{\text{int}}$ also contains $B$, the other point.

It is clear that each vertex of $\Pi'$ is a special point (in the sense above). It follows that each side of $\Pi'$ is a part of either a side of one of the rectangles covering $\Pi$ or a segment of the form $CC'$ — therefore it is infinitesimal.

Let $\Pi' = C_1C_2 \ldots C_n$. We observe that, for any $k = 1, ..., n$, the shortest segment $\sigma_k = C_kC'_k$, connecting $C_k$ with a point $C'_k$ in $\Pi$, does not intersect $\Pi'_{\text{int}}$ by construction. Moreover, by the triangle equality, the segments $\sigma_k$ have no inner intersections. Therefore we may suppose that any two of them intersect each other only in such a manner that either the only intersection point is the common endpoint $C'_k = C'_l$ or one of them is an end-part of the other one. Then the segments $\sigma_k$ decompose the ring-like polygonal region $\mathcal{R}$ between $\Pi$ and $\Pi'$ into $n$ open domains $D_k$ ($k = 1, ..., n$) defined as follows.

If $\sigma_k$ and $\sigma_{k+1}$ are disjoint ($\sigma_{n+1}$ equals $\sigma_1$) then the border of $D_k$ consists of $\sigma_k$, $\sigma_{k+1}$, the side $C_kC_{k+1}$ of $\Pi'$, and that part $\overline{C'_kC'_{k+1}}$ of $\Pi$ which does not contain any of $C'_k$ as an inner point. If $\sigma_k$ and $\sigma_{k+1}$ have the common endpoint $C_k = C_{k+1}$ and no more common points then the border shrinks to
σ_k, σ_{k+1}, and C_kC_{k+1}. If, finally, one of the segments is an end-part of the other one then \( \mathcal{D}_k \) is empty.

If now \( B \in \Pi'_{\text{ext}} \) then \( B \) belongs to one of the domains \( \mathcal{D}_k \). If this is a domain of the first type then the infinitesimal simple arc \( C'_kC_kC'_{k+1}C'_{k+1} \) separates \( A \) from \( B \) within \( \Pi \), which easily implies, by Lemma 1(iii), that either \( A \) or \( B \) belongs to \( \text{monad}\Pi \), which is a contradiction with the choice of the points. If \( \mathcal{D}_k \) is a domain of second type then the barrier accordingly shrinks, with the same contradiction. \( \square \)

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