The inverse spatial Laplacian of spherically symmetric spacetimes

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Abstract
We derive the inverse spatial Laplacian for static, spherically symmetric backgrounds by solving Poisson’s equation for a point source. This is different from the electrostatic Green function, which is defined on the four dimensional static spacetime, while the equation we consider is defined on the spatial hypersurface of such spacetimes. This Green function is relevant in the Hamiltonian dynamics of theories defined on spherically symmetric backgrounds, and closed form expressions for the solutions we find are absent in the literature. We derive an expression in terms of elementary functions for the Schwarzschild spacetime, and comment on the relation of this solution with the known Green function of the spacetime Laplacian operator. We also find an expression for the Green function on the static pure de-Sitter space in terms of hypergeometric functions. We conclude with a discussion of the constraints of the electromagnetic field.

Keywords: Green function, spatial Laplacian, Schwarzschild spacetime, de-Sitter

1. Introduction
Let us consider a four dimensional, static, spherically symmetric background with metric
\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\lambda^2 dr^2 + h_{\alpha\beta} dx^\alpha dx^\beta. \] (1.1)

By static, we mean that the spacetime admits a hypersurface orthogonal timelike Killing vector \( \xi^\mu \), such that \( \xi^\mu \xi_\mu = -\lambda^2 \). The induced metric on the spacelike hypersurface \( \Sigma \) is \( h_{\alpha\beta} = g_{\alpha\beta} + \lambda^{-2} \xi_\alpha \xi_\beta \), and since the hypersurface is assumed to be spherically symmetric, we can write \( \lambda = \lambda(r) \). Any four-dimensional object can be projected to the three-dimensional hypersurface \( \Sigma \) by use of \( h_{\alpha\beta} \). We will use the same indices in three dimensions as in
four, but different symbols for the four-dimensional object and its three-dimensional counterpart, e.g. $V^\mu$ and $v^\mu = h^\mu_\nu V^\nu$ for some vector $V$ and its $\Sigma$-projection $v$. This is a particularly useful general notation for setting up a Hamiltonian formalism, which lies at the root of our motivation for the problem considered in this paper.

It is a straightforward exercise to work out the constraints for the electromagnetic gauge field $A_\mu$ on this background. Let us assume that there is also a horizon at $r = r_H$, i.e. $\lambda(r_H) = 0$, as these are spaces that interest us in this paper. Then there are two first-class constraints [36],

$$\pi^\phi \approx 0,$$

$$D_\mu \pi^\mu |_{r=r_H} \approx 0,$$  \hspace{1cm} (1.2)

where $\pi^\phi$ and $\pi^\mu$ are respectively the momenta canonically conjugate to $\phi = \xi^\mu A_\mu$ and $a_\mu = h^\mu_\nu A^\nu$. $n^\mu$ is the $\Sigma$-ward normal on the horizon at $r = r_H$, and $D_\mu$ is the induced covariant derivative compatible with the induced metric,

$$D_\mu h_{\alpha\beta} = 0.$$  \hspace{1cm} (1.3)

The $\approx$ symbols stands for weak equality, i.e. equality only on the constrained subspace of the phase space. The second term on the left hand side of equation (1.3) is non-zero.

The reduced phase space may be found by using gauge-fixing conditions, which convert the above first-class constraints into second-class ones. The matrix of Poisson brackets is then inverted and used to define Dirac brackets. If the gauge-fixing functions are chosen to be

$$\phi \approx 0,$$  \hspace{1cm} (1.4)

$$D^\nu a_\nu |_{r=r_H} \approx 0,$$  \hspace{1cm} (1.5)

the inverse of the Poisson bracket matrix involves the Green function $\tilde{G}(\vec{x}, \vec{y})$ for the induced spatial Laplacian operator, which formally satisfies the equation

$$D_\mu D^\mu \tilde{G}(\vec{x}, \vec{y}) = -4\pi \delta (\vec{x}, \vec{y}),$$  \hspace{1cm} (1.6)

where $D_\mu$ is the induced covariant derivative compatible with the induced metric,

$$D_\mu h_{\alpha\beta} = 0.$$  \hspace{1cm} (1.7)

and the 3-dimensional covariant delta function $\delta (\vec{x}, \vec{y})$ is defined by

$$\int d^3x \sqrt{\det h(\vec{x})} f(\vec{x}) \delta (\vec{x}, \vec{y}) = f(\vec{y}),$$  \hspace{1cm} (1.8)

for all well-behaved functions $f(\vec{x})$ if $\sigma \subseteq \Sigma$ includes the point $\vec{y}$, and zero otherwise. It is this Green function that we will be primarily concerned with in this paper.

On the other hand, a different Green function appears in solving for the Coulomb potential in static spherically symmetric spacetimes, and a closed form expression for it is well known. For Maxwell’s equation

$$\nabla_\alpha F^{\alpha\mu} = -4\pi J^\mu,$$  \hspace{1cm} (1.9)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic field strength tensor. In what follows, $a_\mu = h^\mu_\nu A^\nu$ and $\phi = \xi^\alpha A_\alpha$ represent the spatial and temporal components of the spacetime field $A_\mu$, respectively. By defining the electric field as
\( e^\mu := \lambda^{-1} \xi_\alpha F^{\mu \alpha} \),

we find that the contraction of equation (1.10) with \( \lambda^{-1} \xi_\mu \), equivalent to setting \( \mu = 0 \), leads to

\[
D_\mu e^\mu = D_\mu \left( \lambda^{-1} D^\mu \phi - \lambda^{-1} \mathcal{L}_\xi a^\mu \right) = -4\pi J^0 ,
\]

where \( J^0 = \lambda^{-1} \xi_\mu J^\mu \) and \( \mathcal{L}_\xi \) is the Lie derivative with respect to \( \xi^\alpha \). If we also set \( \mathcal{L}_\xi \phi = 0 = \mathcal{L}_\xi a_\mu \), and take a point charge by setting \( J^0 = \delta (\vec{x}, \vec{y}) \), equation (1.12) reduces to

\[
\mathcal{D}_\mu (\lambda^{-1} (\vec{x}) D^\mu G(\vec{x}, \vec{y})) = -4\pi \delta (\vec{x}, \vec{y}) .
\]

The left hand side of equation (1.13) is nothing but the action of the d’Alembertian on time-independent functions, for which the expansion

\[
\nabla_\mu \nabla^\mu \phi = \lambda D_\mu \left( \lambda^{-1} D^\mu \phi \right) = D_\mu D^\mu \phi + \lambda D_\mu \left( \lambda^{-1} \right) D^\mu \phi
\]

reveals that while equations (1.7) and (1.13) are the same in flat space, they differ on curved backgrounds where \( \lambda \) is not a constant. We will call the Green function corresponding to equation (1.13) the 4d static scalar Green function, and that of equation (1.7) the inverse spatial Laplacian.

For the Schwarzschild background, the 4d Green function \( G(\vec{x}, \vec{y}) \) is known in closed form. It can be derived by direct construction of the Hadamard elementary solution [1] and also using the method of multipole expansion [2, 3]. A closed form expression was given in [4], which included an additional term missed in [1]. This term accounts for the induced charge behind the horizon of the black hole on the Schwarzschild background, and corresponds to the zero mode contribution in the multipole expansion result. The closed form expression for the static, scalar Green function for the spacetime Laplacian on curved backgrounds has found numerous applications [5–9] predominantly in its use in determining the self force acting on the particle placed on such backgrounds [10–16]. Such closed form expressions have additionally been determined for the Reissner–Nördstrom [17], and more recently for Kerr backgrounds [18].

In contrast, the Green function of equation (1.7) arises in various contexts which involve fields on static foliations of spacetime. These include the gravitational initial value problem [19–21], metric fluctuations around solutions of the Einstein field equations [22–24], classical radiation of free-falling charges [25], and more recently, renormalization group equations on curved backgrounds [26–28], to name a few. This Green function is particularly relevant in the context of Hamiltonian dynamics of fields. The specific context we have in mind is the constrained dynamics of gauge field theories, where this function appears for gravitational [29–31] and electromagnetic [32–35] fields. For example, the Maxwell field has the first class Gauss law constraint \( D_\mu \Pi^\mu \approx 0 \), which implies the existence of redundant or gauge degrees of freedom, which can be eliminated by fixing the gauge and then applying Dirac’s procedure. The resultant Dirac brackets of the fields and their momenta in the radiation gauge on curved backgrounds with horizons involves this Green function [36]. However, while well motivated in the literature, we found no closed form expressions for them on curved backgrounds. In this work, we consider these functions for spherically symmetric backgrounds, and derive their expressions for the Schwarzschild and pure de-Sitter cases. For these backgrounds, the metric of equation (1.15) takes the form
\[ ds^2 = -\lambda(r)^2dr^2 + \frac{1}{\lambda(r)^4}dr^2 + r^2d\Omega^2. \]  

Both backgrounds possess a horizon, defined by \( \lambda = 0 \).

The organization of our paper is as follows. In section 2, we review the derivation of the static, scalar Green function for the spacetime Laplacian defined on the Schwarzschild background. In section 3, we derive the solutions of equation (1.7) for the Schwarzschild and static pure de-Sitter backgrounds. While we were able to determine the closed form expression for the Schwarzschild case in terms of elementary functions, we were unable to find a similar expression for the pure de-Sitter background. Finally, in section 6, we discuss the relevance of our result in the constrained quantization of the Maxwell field on spherically symmetric backgrounds.

2. Derivation of the 4d static, scalar Green function

The Green function corresponding to equation (1.13) is relevant for Coulomb’s law, as we have seen. Let us briefly review its derivation on the Schwarzschild background following [2, 3], as we will follow a similar procedure for deriving the Green function for equation (1.7).

We take \( \lambda^2 = g_{rr} = 1 - \frac{2m}{r} \), and place a unit charge at \((r', \theta', \phi')\). With this choice equation (1.13) becomes, in explicit coordinates,

\[
\sin \theta \partial_r \left( r^2 \partial_r G \right) + \frac{1}{(1 - \frac{2m}{r})} \partial_{\theta} \left( \sin \theta \partial_{\theta} G \right) + \frac{1}{(1 - \frac{2m}{r}) \sin \theta} \partial_{\phi}^2 G = -4\pi \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi'),
\]

(2.1)

where the delta functions are normalized according to

\[
\int_{2m}^{\infty} rdr \delta(r - r') = 1, \quad \int_0^\pi d\theta \delta(\theta - \theta') = 1, \quad \int_0^{2\pi} d\phi \delta(\phi - \phi') = 1.
\]

(2.2)

Away from the point charge, the right hand side of equation (2.1) vanishes, and we can expand \( G \) as

\[
G(r, r') = \sum_{l=0}^{\infty} R_l(r, r') P_l(\cos \gamma),
\]

(2.3)

where \( \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \). While we could have used the azimuthal symmetry to reduce this to a problem in plane polar coordinates \((r, \theta)\), the calculations are no more complicated for \((r, \theta, \phi)\), so we have chosen to display all coordinates. We note that since \( P_l(\cos \gamma) \) is related to the spherical harmonics \( Y_{lm}(\theta, \phi) \) via the Legendre addition theorem (see equations (14.30.8), (14.30.9), (14.30.11) of [37])

\[
\frac{2l + 1}{4\pi} P_l(\cos \gamma) = \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) Y_{l'm}(\theta', \phi'),
\]

(2.4)

it further satisfies

\[
\frac{1}{\sin \theta} \partial_{\theta} \left( \sin \theta \partial_{\theta} P_l(\cos \gamma) \right) + \frac{1}{\sin^2 \theta} \partial_{\phi}^2 P_l(\cos \gamma) = -l(l + 1) P_l(\cos \gamma),
\]

(2.5)
Substituting equations (2.3) in (2.1) away from the source, we find that \( R_l(r) \) is a linear combination of two independent solutions,

\[
R_l(r, r') = A_l(r')g_l(r) + B_l(r')f_l(r),
\]

where \( g_l(r) \) and \( f_l(r) \) are given by [2, 38]

\[
g_l(r) = \begin{cases} 
\frac{1}{\sqrt{\pi}} \frac{2^{l+1} l^m}{(2m)!} (r - 2m) \frac{d}{dr} P_l \left( \frac{r}{m} \right) & \text{for } l = 0, \\
\frac{1}{\sqrt{\pi}} \frac{2^{l+1} l^m}{(2m)!} (r - 2m) \frac{d}{dr} P_l \left( \frac{r}{m} \right) & \text{for } l \neq 0,
\end{cases}
\]

\[
f_l(r) = -\frac{(2l + 1)!}{2^l (l + 1)! l^{m+1}} (r - 2m) \frac{d}{dr} Q_l \left( \frac{r}{m} \right).
\]

Here \( P_l \) and \( Q_l \) are the Legendre functions of the first and second kind, respectively. With the exception of \( g_0(r) = 1 \), the leading term of \( g_l(r) \) is proportional to \( r^l \) and diverges as \( r \to \infty \). Thus this solution is ruled out for large values of \( r \). Both \( g_l(r) \) and \( f_l(r) \) are well behaved at the horizon \( r = 2m \). However, \( \frac{d}{dr} f_l(r) \) diverges logarithmically as \( r \to 2m \), except when \( l = 0 \). On the other hand, the leading behaviour of \( f_l(r) \) for large \( r \) is proportional to \( r^{-l-1} \).

We can thus write equation (2.3) as

\[
G(\vec{r}, \vec{r}') = \begin{cases} 
\sum_{l=0}^{\infty} A_l(r') f_l(r) P_l(\cos \gamma) & r > r', \\
\sum_{l=0}^{\infty} B_l(r') g_l(r) P_l(\cos \gamma) & r < r'.
\end{cases}
\]

The continuity of \( G \) and discontinuity of \( \nabla G \) at \( \vec{r} = \vec{r}' \) tells us that by defining \( r_\gamma = \min(r, r') \) and \( r_\gamma = \max(r, r') \), we can write \( G(\vec{r}, \vec{r}') \) as

\[
G(\vec{r}_\gamma, \vec{r}_\gamma) = \sum_{l=0}^{\infty} g_l(r_\gamma) f_l(r_\gamma) P_l(\cos \gamma).
\]

A bit of algebra now shows that these solutions can be rewritten in the form

\[
G(\vec{r}, \vec{r}') = \frac{1}{r r'} \left[ \frac{(r - m)(r' - m) - m^2 \cos \gamma}{\sqrt{(r - m)^2 + (r' - m)^2 - 2(r - m)(r' - m) \cos \gamma - m^2 \sin^2 \gamma}} + m \right].
\]

This expression, found in [4], differs from a solution provided many years earlier [1] because of the term \( \frac{1}{r r'} \), which accounts for the zero-mode contribution in equation (2.11). The result in equation (2.12) has been derived recently using the heat kernel method and bi-conformal symmetry in [40].

3. Inverse spatial Laplacian of the Schwarzschild background

Now let us get back to the solution of equation (1.7) in the Schwarzschild background. With the source at \( (r', \theta', \phi') \) as before, equation (1.7) takes the form
\[
\sin \theta \partial_r \left( r^2 \sqrt{1 - \frac{2m}{r} \partial_r \tilde{G}} \right) + \frac{1}{\sqrt{1 - \frac{2m}{r}}} \sin \theta \partial_\theta \left( \sin \theta \partial_\theta \tilde{G} \right) + \frac{1}{\sqrt{1 - \frac{2m}{r}} \sin \theta} \partial_\phi^2 \tilde{G} \\
= -4\pi \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi').
\] (3.1)

It will be convenient to make a change of variables from \( r \) to \( y = \frac{r}{m} - 1 \). After we find the solution, we can change variables again to express the Green function in terms of the original coordinates.

In terms of \( y \), equation (3.1) takes the form

\[
\sin \theta \left[ \partial_y \left( (y + 1)^2 \sqrt{\frac{y - 1}{y + 1} \partial_y \tilde{G}} \right) + \frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta \tilde{G} \right) + \frac{1}{\sin^2 \theta} \partial_\phi^2 \tilde{G} \right] \\
= -4\pi \frac{\delta(y - y')}{m} \delta(\theta - \theta') \delta(\phi - \phi'),
\] (3.2)

with the point source located at \((y', \theta', \phi')\) in the new coordinates.

The angular delta functions satisfy the expressions in equation (2.2), while the \( y \) delta function now satisfies

\[
\int_1^\infty dy \delta(y - y') = 1.
\] (3.3)

The first step in our derivation is to consider equation (3.2) far removed from the source. Thus we need to solve the following equation

\[
0 = \sqrt{\frac{y - 1}{y + 1}} \partial_y \left( (y + 1)^2 \sqrt{\frac{y - 1}{y + 1} \partial_y \tilde{G}} \right) + \frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta \tilde{G} \right) + \frac{1}{\sin^2 \theta} \partial_\phi^2 \tilde{G}.
\] (3.4)

Writing

\[
\tilde{G}(\vec{y}, \vec{y}') = \sum_{l=0}^\infty R_l(y, y') P_l(\cos \gamma),
\] (3.5)

and substituting equations (3.5) in (3.4), we get the differential equation

\[
\sqrt{\frac{y - 1}{y + 1}} \frac{d}{dy} \left( (y + 1)^2 \sqrt{\frac{y - 1}{y + 1}} \frac{d}{dy} R_l(y, y') \right) - l(l + 1) R_l(y, y') = 0.
\] (3.6)

We have described the solution of equation (3.6) in appendix A. The general solution is given in equation (A.11), and it is of the form

\[
R_l(y, y') = A_l(y') g_l(y) + B_l(y') f_l(y),
\] (3.7)

where the functions \( g_l(y) \) and \( f_l(y) \) involve Legendre polynomials of fractional degree, with the argument \( y > 1 \). The functions \( A_l \) and \( B_l \) can be found from continuity conditions, and will be found below for the Schwarzschild spacetime. They will also turn out to be related to the same functions. Legendre polynomials of fractional degree can be described in terms of hypergeometric functions, for which there exist several representations. A particular representation which we will use is (see pp 153–63, table entry 10 and 28, of [39]).
\[ P_{\nu}^{\mu}(y) = \frac{\Gamma(-\nu - \frac{1}{2})}{2^{\nu+1}\sqrt{\pi}\Gamma(-\nu - \mu)} y^{-\nu+\mu-1}(y^2 - 1)^{-\frac{3}{4}} \]
\[ \times 2F_1 \left( \frac{1 + \nu - \mu}{2}, \frac{2 + \nu - \mu}{2}; \nu + \frac{3}{2}; \frac{1}{y^2} \right) \]
\[ + 2^\nu \Gamma(\nu + \frac{1}{2})^{-\nu+\mu}(y^2 - 1)^{-\frac{3}{2}} \]
\[ \times 2F_1 \left( \frac{-\nu - \mu}{2}, \frac{1 - \nu - \mu}{2}; -\nu + \frac{1}{2}; \frac{1}{y^2} \right), \]
\[ e^{-i\pi \mu} Q_{\nu}^{\mu}(y) = \frac{\sqrt{\pi}\Gamma(1 + \nu + \mu)}{2^{\nu+1}(\frac{3}{2} + \nu)} y^{-\nu-\mu-1}(y^2 - 1)^{-\frac{3}{2}} \]
\[ \times 2F_1 \left( \frac{\nu + 2}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{y^2} \right). \] (3.8)

The solutions \( g_i(y) \) and \( f_i(y) \) make use of these solutions for the case of \( \mu = \frac{1}{2} \) and \( \nu = 1 \) as shown in equation (A.11) of appendix A, and can be written as

\[ g_1(y) = \frac{1}{\sqrt{y} + 1} \left[ \frac{1}{\frac{1}{2}\sqrt{\pi}} y^{-l+\frac{1}{2}} 2F_1 \left( \frac{l + \frac{1}{2}, \frac{l + 3}{2}, l + \frac{3}{2}}{l + \frac{1}{2}; \frac{1}{y^2}} \right) \right. \]
\[ + 2^l y^{l+\frac{1}{2}} 2F_1 \left( \frac{-l - \frac{1}{2}, -l + \frac{1}{2}, -l + \frac{1}{2}; \frac{1}{y^2}}{l + \frac{1}{2}; \frac{1}{y^2}} \right) \right], \]
\[ f_1(y) = \sqrt{y - 1} \left[ \frac{1}{\frac{1}{2}\sqrt{\pi}} y^{-l-\frac{1}{2}} 2F_1 \left( \frac{l + \frac{1}{2}, \frac{l + 3}{2}, l + \frac{3}{2}}{l + \frac{1}{2}; \frac{1}{y^2}} \right) \right. \] (3.9)

It turns out that the functions given in equation (3.9) admit expressions in terms of more elementary functions, which we will now describe. These expressions will be relevant in determining the final form of the Green function for the spatial Laplacian. The hypergeometric functions contained in \( g_1(y) \) in equation (3.9) are both of the following generic form, with the known representation

\[ 2F_1 \left( a, a + \frac{1}{2}, 2a + 1, \frac{1}{y^2} \right) = 2^{2a} \left( \frac{y + \sqrt{y^2 - 1}}{y} \right)^{-2a}. \] (3.10)

where \( a \) stands for both \( \frac{l+\frac{1}{2}}{2} \) and \( \frac{l-\frac{1}{2}}{2} \) in the above expression. We can thus write the expression for \( g_1(y) \) as

\[ g_1(y) = \frac{1}{\sqrt{2\sqrt{\pi} + 1}} \left[ \left( y + \sqrt{y^2 - 1} \right)^{-l+\frac{1}{2}} + \left( y + \sqrt{y^2 - 1} \right)^{l+\frac{1}{2}} \right]. \] (3.11)

Likewise, the hypergeometric function given in \( f_1(y) \) has the following expression in terms of elementary functions,

\[ 2F_1 \left( b, b + \frac{1}{2}, 2b, \frac{1}{y^2} \right) = \frac{2^{2b-1} \sqrt{y^2 - 1}}{\sqrt{3 - y^2}} \left( y + \sqrt{y^2 - 1} \right)^{-2b+1}. \] (3.12)
where \( b = \frac{l + \frac{1}{2}}{2} \). We can thus write \( f_l(y) \) as

\[
  f_l(y) = \sqrt{2} \left( \frac{y + \sqrt{y^2 - 1}}{\sqrt{y^2 - 1}} \right)^{-l - \frac{1}{2}}.
\]  

(3.13)

The calculation below will require the Wronskian of the solutions given in equation (3.9). Using the above expressions, we readily find that the Wronskian \( W(g_l(y), f_l(y), y) \) is given by

\[
  W(g_l(y), f_l(y), y) = -\frac{(2l + 1)}{(1 + y)^2 \sqrt{y - 1}}.
\]  

(3.14)

There are two limits to consider of the solutions given in equations (3.11) and (3.13), and their derivatives. These are the \( y \to 1 \) and \( y \to \infty \) limits, which correspond to \( r \to 2m \) and \( r \to \infty \) respectively. Before describing these, we note that \( g_0(y) \) is a special case in that it is a constant, \( g_0(y) = 1 \) for all values of \( y \).

For all the other terms we find the following. As \( y \to 1 \), both \( g_l(y) \to 1 \) and \( f_l(y) \to 1 \) for all values of \( l \), i.e. they are both finite. However, all derivatives of \( f_l(y) \) diverge as \( y \to 1 \), while \( \frac{\delta}{\delta y} g_l(y) \to l(l + 1) \) as \( y \to 1 \). Thus the near horizon solution must only contain \( g_l(y) \), and we must set \( B_l(y') = 0 \) in equation (3.7) in the region between \( (\gamma', \theta', \phi') \) and the event horizon of the black hole.

On the other hand, as \( y \to \infty \), we find that \( f_l(y) \to 0 \) for all values of \( l \), and the derivatives of \( f_l(y) \) are also well behaved, but \( g_l(y) \) diverges for \( l \neq 0 \). We must thus set \( A_l(y') = 0 \) in equation (3.7) to describe the region from \( (\gamma', \theta', \phi') \) to \( \infty \).

We can therefore write the solution in the following way in the two regions,

\[
  \tilde{G}(y', y) = \begin{cases} 
    \sum_{l=0}^{\infty} A_l(y') g_l(y) P_l(\cos \gamma), & (y < y') \\
    \sum_{l=0}^{\infty} B_l(y') f_l(y) P_l(\cos \gamma). & (y > y')
  \end{cases}
\]  

(3.15)

Continuity of \( \tilde{G} \) at \( y = y' \) implies that \( A_l(y') g_l(y') = B_l(y') f_l(y') \). Then we can define a constant \( C_l \) such that

\[
  C_l = \frac{A_l(y')}{f_l(y')} = \frac{B_l(y')}{g_l(y')},
\]  

(3.16)

using which we can write the solution in the form

\[
  \tilde{G}(y', y) = \begin{cases} 
    \sum_{l=0}^{\infty} C_l f_l(y') g_l(y) P_l(\cos \gamma), & (y < y') \\
    \sum_{l=0}^{\infty} C_l g_l(y') f_l(y) P_l(\cos \gamma). & (y > y')
  \end{cases}
\]  

(3.17)

We can now determine the constants \( C_l \) by appropriately integrating equation (3.2). To begin with, we insert equations (3.5) into (3.2), multiply both sides with \( P_l(\cos \gamma) \) and integrate with respect to \( \theta \) and \( \phi \) to find

\[
  \frac{1}{2l + 1} \left[ \frac{d}{dy} \left( (y + 1)^2 \sqrt{\frac{y - 1}{y + 1}} \frac{d}{dy} R_l(y) \right) - l(l + 1) \sqrt{\frac{y + 1}{y - 1}} R_l(y) \right] = -\frac{\delta(y - y')}{m}.
\]  

(3.18)
Integrating equation (3.18) over an infinitesimal region from \( y' - \epsilon \) to \( y' + \epsilon \), we get

\[
- \frac{1}{m} = \frac{1}{2l + 1} C_l(y' + 1)^2 \left( \sqrt{y' - 1} \right)^{l+1/2} \left[ \frac{g_l(y') \, df_l(y)}{dy} \bigg|_{y' + \epsilon} - f_l(y') \, dg_l(y) \bigg|_{y' - \epsilon} \right]
\]

\[
= \frac{1}{2l + 1} C_l(y' + 1)^{3/2} \sqrt{y' - 1} W(g_l(y'), f_l(y') , y')
\]

\[
= -C_l,
\]

(3.19)

where in going from the second to the third equality in equation (3.19), we made use of equation (3.14). Thus we have determined that \( C_l \) is independent of \( l \),

\[
C_l = \frac{1}{m},
\]

(3.20)

and we can write the solution of equation (3.2) as

\[
\tilde{G}(\vec{y}_<, \vec{y}_>) = \frac{1}{m} \sum_{l=0}^{\infty} g_l(y_<) f_l(y_>) P_l(\cos \gamma),
\]

(3.21)

where \( y_\leq = \min(y, y') \) and \( y_\geq = \max(y, y') \). Using equations (3.13) and (3.11), we find that the product \( g_l(y_<) f_l(y_>) \) is given by

\[
g_l(y_<) f_l(y_>) = \frac{1}{\sqrt{y_< + 1} \sqrt{y_>} + 1} \left[ \left( y_< + \sqrt{y_<^2 - 1} \right)^{l+1/2} \right.
\]

\[
+ \left. \left( y_> + \sqrt{y_>^2 - 1} \right)^{l+1/2} \right].
\]

(3.22)

For the sake of notational convenience, let us define

\[
A = y_\geq + \sqrt{y_\geq^2 - 1} \quad \text{and} \quad B = y_< + \sqrt{y_<^2 - 1}.
\]

(3.23)

Using equation (3.22), and the standard expression for the generating function for Legendre polynomials

\[
\sum_{l=0}^{\infty} t^l P_l(x) = \frac{1}{\sqrt{1 - 2xt + t^2}},
\]

(3.24)

we find that equation (3.21) takes the form

\[
\tilde{G}(\vec{y}_<, \vec{y}_>) = \frac{1}{m} \frac{1}{\sqrt{y_< + 1} \sqrt{y_>} + 1}
\]

\[
\times \left[ \frac{\sqrt{AB}}{\sqrt{A^2 + B^2 - 2AB \cos \gamma}} + \frac{\sqrt{AB}}{\sqrt{A^2 B^2 + 1 - 2AB \cos \gamma}} \right].
\]

(3.25)

To write the solution in terms of Schwarzschild coordinates, we simply make the substitution for \( y \), and write
\[ \tilde{G}(\vec{r}, \vec{r}') = \frac{1}{\sqrt{rr'}} \left[ \frac{\sqrt{\kappa(\vec{r})r - m} \sqrt{\kappa(\vec{r}')r' - m}}{\sqrt{(\kappa(\vec{r})r - m)^2 + (\kappa(\vec{r}')r' - m)^2} - 2 (\kappa(\vec{r})r - m)(\kappa(\vec{r}')r' - m) \cos \gamma} \right. \\
\left. + \frac{m \sqrt{\kappa(\vec{r})r - m} \sqrt{\kappa(\vec{r}')r' - m}}{\sqrt{(\kappa(\vec{r})r - m)^2 (\kappa(\vec{r}')r' - m)^2} + m^2 - 2 m^2 (\kappa(\vec{r})r - m)(\kappa(\vec{r}')r' - m) \cos \gamma} \right], \]

where we have defined \( \kappa(\vec{r}) = 1 + \lambda(\vec{r}) = 1 + \frac{1}{1 - \frac{2m}{r}} \), and \( \kappa(\vec{r}') \) similarly. As noted earlier, we see that as we take the flat space limit \((m \to 0)\), this solution as well as that of equation (2.12) reduce to the Green function of flat space. We also note that just as in the Green function result given in the previous section, this solution is regular at the horizon.

4. Inverse spatial Laplacian of the de-Sitter background

We now turn our attention to writing a closed form expression for the Green function on a de-Sitter background. The scalar de-Sitter–Green function for cosmological de-Sitter spacetimes has been derived in [41–43]. In static coordinates, the thermal Green function for the massless scalar field equation [44], as well as the Green function for the massive scalar field equation [45, 46] are known in the literature. These Green functions correspond to the de-Sitter generalization of equation (1.14), whereas we will be concerned with the derivation of the solution of the inverse spatial Laplacian, i.e. of equation (1.7). The procedure described in this section can be used for finding the solution of equation (1.13) as well.

For pure de-Sitter space with cosmological constant \( \Lambda \), we have \( \lambda(\vec{r})^2 = 1 - \frac{3}{L^2} \), working in the quadrant of de-Sitter space where the time coordinate increases into the future. We again make a change of coordinates and write \( y = \frac{r}{L} \). For this choice, equation (1.7) takes the form

\[ \sin \theta \left[ \frac{\partial_y \left( y^2 \sqrt{1 - y^2} \partial_y \tilde{G} \right)}{\sqrt{1 - y^2}} + \frac{1}{\sin \theta} \left( \frac{1}{\sin \theta} \partial_{\theta} \left( \sin \theta \partial_{\theta} \tilde{G} \right) + \frac{1}{\sin^2 \theta} \partial^2_{\phi} \tilde{G} \right) \right] \]

\[ = -4\pi \frac{\delta(y - y')}{L} \delta(\theta - \theta') \delta(\phi - \phi'). \]  

(4.1)

The delta functions for the angular variables satisfy equation (2.2), but the \( y \) delta function now satisfies

\[ \int_0^1 dy \delta(y - y') = 1. \]

As in the Schwarzschild case, we begin by solving the above equation far away from the source

\[ \sqrt{1 - y^2} \partial_y \left( y^2 \sqrt{1 - y^2} \partial_y \tilde{G} \right) + \frac{1}{\sin \theta} \partial_{\theta} \left( \sin \theta \partial_{\theta} \tilde{G} \right) + \frac{1}{\sin^2 \theta} \partial^2_{\phi} \tilde{G} = 0, \]  

(4.2)
with
\[ \tilde{G}(y, y') = \sum_{l=0}^{\infty} R_l(y, y') P_l(\cos \gamma). \] (4.3)

Substituting equations (4.3) in (4.2), and using (2.5), we get the equation
\[ \sqrt{1 - y^2} \frac{d}{dy} \left( y^2 \sqrt{1 - y^2} \frac{d}{dy} R_l(y, y') \right) - l(l + 1)R_l(y, y') = 0. \] (4.4)

To find the general solution in this case, it will be convenient to express equation (4.4) in terms of \( t = \sqrt{1 - y^2} \), which results in
\[ \sqrt{1 - t^2} \frac{d}{dt} \left( (1 - t^2)^{\frac{1}{2}} \frac{d}{dt} R_l(t, t') \right) - l(l + 1)R_l(t, t') = 0. \] (4.5)

Using the ansatz \( R_l(t, t') = B_l(t') P_{\mu}^l(t) A(t) \) as before (see appendix A), we find the following general solution
\[ R_l(t, t') = A_l'(t') (1 - t^2)^{-\frac{1}{2}} P_{\mu}^{l+\frac{1}{2}}(t) + B_l'(t') (1 - t^2)^{-\frac{1}{2}} P_{\mu}^{l-\frac{1}{2}}(t). \] (4.6)

The Legendre polynomials described in equation (4.6) can be described in terms of hypergeometric functions. For Legendre polynomials defined in the region between \(-1\) and \(+1\), we have (see p 166 of [39])
\[ \Gamma(1 - \mu)P_{\nu}^l(x) = 2^{\mu}(1 - x^2)^{-\frac{\nu}{2}} \frac{\Gamma\left(\frac{l + \nu}{2}\right)}{\Gamma\left(\frac{l + 1}{2}\right)} F_1\left(\frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2}, -\frac{\nu}{2} - \frac{\mu}{2}; 1 - \mu; 1 - x^2\right). \] (4.7)

By using the expressions in equation (4.7), and writing the results in terms of the variables \( y \) by substituting \( 1 - t^2 = y^2 \), one can find the following general solution
\[ R_l(y, y') = A_l(y') g_l(y) + B_l(y') f_l(y), \] (4.8)
where \( g_l(y) \) and \( f_l(y) \) are now given by
\[ g_l(y) = y' F_1\left(\frac{l}{2}, 1; \frac{3}{2} + l; y^2\right) \] and
\[ f_l(y) = \frac{1}{y' + 1} F_1\left(-l - 1, -l + 1; 1; y^2\right). \] (4.9)

Here, \( A_l(y') \) and \( B_l(y') \) are real coefficients, and the solutions themselves are positive and real in the region between \( 0 \) and \(+1\). The Wronskian of the two solutions given in equation (4.9) satisfies the following relation
\[ W(g_l(y), f_l(y), y) = -\frac{2l + 1}{y^2 \sqrt{1 - y^2}}. \] (4.10)

Unlike in the Schwarzschild case, we were unable to determine a closed form expression of the solutions in terms of elementary functions for arbitrary \( l \). The solutions for specific choices of \( l \) however can be easily determined. Using the derivative relations satisfied by the hypergeometric functions, we have derived in appendix B the following general form of the \( f_l(y) \) solutions
\[ f_l(y) = \begin{cases} \sum_{n=0}^{\frac{l-1}{2}} \frac{c_l}{n!} y^n & (l \text{ odd}), \\ \frac{1}{y} & (l = 0), \\ \sqrt{1 - y^2} \sum_{n=1}^{\frac{l}{2}} \frac{c_l}{n!} y^n & (l \text{ even}; l \neq 0), \end{cases} \]  
where \( c_{l\pm 1} = 1 \) for the odd \( l \) case, and \( c_{l} = 1 \) for the even \( l \) case.

To proceed further, we need to determine the behaviour of the solution in the limit \( y \to 0 \) and \( y \to 1 \). As before, \( g_0(y) = 1 \), which follows from \( \mathcal{F}_1 \left( 0, 1; \frac{3}{2}; y^2 \right) = 1 \), and will not be considered in the following. As \( y \to 0 \), we can make use of the following derivative relation satisfied by the hypergeometric functions

\[ \frac{d}{dy} \mathcal{F}_1(a, b, c, x) = \frac{a b}{c} \mathcal{F}_1(a + 1, b + 1, c + 1, x), \tag{4.12} \]

as well as \( \mathcal{F}_1(a, b, c, 0) = 1 \), to determine the behaviour of the solutions. We see that \( g_l(y) \) and its first derivative vanish, while \( f_l(y) \) and its first derivative diverge for all values of \( l \), as \( y \to 0 \). We must thus set \( B_0 = 0 \) in equation (4.8) in the region where \( y \) can vanish, in order to have regular solutions.

As \( y \to 1 \), we need to consider the integral representation of the hypergeometric function to demonstrate that \( g_l(y) \) is finite while its first derivative diverges, for \( l \neq 0 \). This is shown in appendix B. The solutions provided in equation (4.11) tell us the following about the behaviours of the first derivatives differ for even and odd \( l \). The first derivative of \( f_l(y) \) diverges when \( l \) is even, and is finite when \( l \) is odd. Regularity of the solutions requires that in the region where \( y \to 1 \), we not only set \( A_0 = 0 \) for all \( l \neq 0 \), but also set \( B_0 = 0 \) for even \( l \).

We can now determine the general solution \( \mathcal{G}(\vec{y}, \vec{y'}) \) for the point source located at \((y', \theta', \phi')\). Away from the source the solution is given by equation (4.3). As explained above, in the region \( y < y' \) we simply set \( B_l(y') = 0 \) and sum over all \( l \). In the region \( y > y' \) we set \( A_l(y') = 0 \) for all \( l \neq 0 \) and sum over all odd \( l \), but we in addition have the \( g_0(y) = 1 \) term which contributes a constant term. Thus, we can write

\[ \mathcal{G}(\vec{y}, \vec{y'}) = \begin{cases} \sum_{l=0}^{\infty} A_l(y') g_l(y) P_l(\cos \gamma) & (y < y'), \\ A'_0 + \sum_{l=0}^{\infty} B_{2l+1}(y') f_{2l+1}(y) P_{2l+1}(\cos \gamma) & (y > y'). \end{cases} \tag{4.13} \]

Finally, we need to match these solutions at \( y = y' \). This sets \( A_0 = A'_0 \), and leads us to define the constant \( C_{2l+1} = \frac{A_{2l+1}(y)}{f_{2l+1}(\sqrt{y')})} = \frac{B_{2l+1}(y)}{g_{2l+1}(\sqrt{y')})}, \) and we also find that \( A_k \) vanishes for even \( k(\neq 0) \). Then we can write

\[ \mathcal{G}(\vec{y}, \vec{y'}) = C_0 + R_{2l+1}(y, y') P_{2l+1}(\cos \gamma), \tag{4.14} \]

where \( C_0 \equiv A_0 \) is the constant zero-mode contribution, and

\[ R_{2l+1}(y, y') = \begin{cases} \sum_{l=0}^{\infty} C_{2l+1} g_{2l+1}(y) f_{2l+1}(y') & (y < y'), \\ \sum_{l=0}^{\infty} C_{2l+1} f_{2l+1}(y) g_{2l+1}(y') & (y > y'). \end{cases} \tag{4.15} \]
Multiplying both sides of equation (4.1) with $P_{2l+1}(\cos \gamma)$ and integrating with respect to $\theta$ and $\phi$, we get

$$-rac{\delta(y - y')}{L} = \frac{1}{4l + 3} \left[ \frac{d}{dy} \left( y^2 \sqrt{1 - y^2} \frac{d}{dy} R_{2l+1}(y, y') \right) - \frac{(2l + 1)(2l + 3)}{\sqrt{1 - y^2}} R_{2l+1}(y, y') \right],$$

where we have used equation (2.6). We next integrate over $y$ from $y' - \epsilon$ to $y' + \epsilon$, i.e. over an infinitesimal region about the point source, for which we find

$$-\frac{1}{L} = \frac{1}{4l + 3} C_{2l+1} y'^2 \sqrt{1 - y'^2}$$

$$\times \left[ g_{2l+1}(y') \left( \frac{d}{dy} f_{2l+1}(y) \right) \right]_{y' = y'} \left. - f_{2l+1}(y') \left( \frac{d}{dy} g_{2l+1}(y) \right) \right|_{y' = y}$$

$$= \frac{1}{4l + 3} C_{2l+1} y'^2 \sqrt{1 - y'^2} W(g_{2l+1}(y'), f_{2l+1}(y'), y')$$

$$= -C_{2l+1},$$

where we have made use of equation (4.10) in going from the second to the third equality in equation (4.17). Using this, we can write the Green function in the de-Sitter case as

$$\tilde{G}(\vec{r}_<, \vec{r}_>) = \frac{1}{L} \sum_{l=0}^{\infty} g_{2l+1}(y_<) f_{2l+1}(y_) P_{2l+1}(\cos \gamma),$$

where $y_< = \min(y, y')$ and $y_>$ = $\max(y, y')$ as before. Unlike in the Schwarzschild case, we have not been able to write this in a simpler form. We can nonetheless substitute for $y$ in equations (4.9) and use this in (4.18), by writing $y_< = \frac{r_<}{\Lambda}$ and $y_ман = \frac{r_мен}{\Lambda}$, to find the solution in terms of $r$,

$$\tilde{G}(\vec{r}_<, \vec{r}_>) = \frac{1}{r_>^2} \sum_{l=0}^{\infty} \left( \frac{r_<}{r_>} \right)^{2l+1} F_1 \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3r_<^2}{\Lambda} \right)$$

$$\times F_1 \left( -l - 1, -l, -2l - \frac{3r_<^2}{\Lambda} \right) P_{2l+1}(\cos \gamma).$$

5. Discussion

As we mentioned earlier, one of the places where the spatial Laplacian appears is in the constrained quantization of Maxwell fields on static spherical symmetric spacetimes with horizons. Let us now briefly revisit the role of the Green function in that problem; for more details we refer the reader to [36].

For the Maxwell field, an important distinction between its treatment on spacetimes with or without horizons is that in the presence of a horizon, the Gauss law constraint picks up an additional horizon term,

$$\Omega \equiv -n_\mu \pi^\mu \bigg|_{r=\hat{r}_h} + D_\mu \pi^\mu \approx 0. \tag{5.1}$$

This is a first class constraint, and one way of handling it is to fix a gauge and find the corresponding Dirac brackets. An interesting choice of gauge fixing function is one that includes a surface term,
For this choice, the relevant Dirac bracket becomes

\[ [a_\mu(\vec{x}), \pi^\nu(\vec{y})]_D = \delta(\vec{x}, \vec{y})\delta_\mu^\nu - D_\mu^r D_\nu^s \tilde{G}(\vec{x}, \vec{y}), \]

(5.3)

where \( \delta(\vec{r}, \vec{y}) \) is defined in equation (1.9).

A different gauge choice would produce a different set of brackets, for example the gauge choice \( \Omega_{gf} = D_\mu^r(\lambda a^\mu) \) produces Dirac brackets in which \( \tilde{G} \) is replaced by \( G \) of equation (1.13) in the second term on the right hand side, and that term also picks up a factor of \( \lambda \).

Consider the Dirac bracket given in equation (5.3) in the Schwarzschild background. When one of the arguments is at the horizon, e.g. in the limit \( y \to r_H \), we find for the \( r - r \) component that

\[ [a_\mu(\vec{x}), \pi^\nu(\vec{y})]_D \bigg|_{y \to r_H} = \delta(r, r_H) + \kappa_H \frac{2r - m(1 + \cos \gamma)}{2(r^2 - mr(1 + \cos \gamma))^{3/2}}. \]

(5.4)

For the other gauge choice mentioned above, only the \( \delta(r, r_H) \) remains on the right hand side of the above equation in the limit \( y \to r_H \). The Dirac brackets comprise one aspect that enters into the quantization of theories. Since the Gauss’ law constraint must be respected by physical states of the theory, the surface term contained in the constraint on these backgrounds will be relevant to states at the horizon. A complete treatment of the quantization of the Maxwell field on static, spherically symmetric backgrounds with horizons lies outside the scope of the present work. In light of the preceding discussion, we can nevertheless expect that the inverse spatial Laplacian will affect the quantization of gauge fields near the horizon.

Since both the de-Sitter and Schwarzschild cases admit a mode expansion, where the functions depending on \( r \) are ultimately associated Legendre polynomials, it seems plausible to presume that a similar result would hold for the Schwarzschild–de-Sitter background. Unfortunately, we have been unable to find a simple transformation for this case since the cubic dependence on \( r \) in the lapse function \( \lambda \) poses a significant obstacle to the procedure. From the nature of the equation to solve for the Schwarzschild–de-Sitter background, it appears that the solution for the corresponding Green function will require a different approach from what was considered here.

### 6. Conclusion

In this paper, we have discussed a new class of static, scalar Green functions on spherically symmetric spacetimes, those corresponding to the inverse spatial Laplacian defined exclusively on the spatial hypersurface of the spherically symmetric spacetime. Specifically, we have derived the inverse spatial Laplacian in the form of mode solutions for the Schwarzschild and pure de-Sitter backgrounds. We have determined the closed form expression for Green function on Schwarzschild spacetime in terms of elementary functions, and on the pure de-Sitter space in terms of hypergeometric functions.

### Appendix A. Derivation of the general solution of the homogeneous equations

We seek to solve equations (3.6) and (4.5), which take the general form

\[ (1 - y^2) \frac{d^2}{dy^2} R_l(y, y') + f(y) \frac{d}{dy} R_l(y, y') + g(y) R_l(y, y') - l(l + 1) R_l(y, y') = 0. \]

(A.1)
We will solve this equation, for the cases of equations (3.6) and (4.5) by making use of the ansatz \( R_l(y, y') = B_l(y') P_{\nu}^{\mu}(y) A(y) \). We first recall that the Legendre polynomial is a solution of the following differential equation

\[
(1 - y^2) \frac{d^2}{dy^2} P_{\nu}^{\mu}(y) - 2y \frac{dy}{dy} P_{\nu}^{\mu}(y) + \left[ \nu(\nu + 1) - \frac{\mu^2}{1 - y^2} \right] P_{\nu}^{\mu}(y) = 0. \tag{A.2}
\]

Expanding equation (3.6), we find

\[
(1 - y^2) \frac{d^2}{dy^2} R_l(y, y') - (2y - 1) \frac{dy}{dy} R_l(y, y') + l(l + 1)R_l(y, y') = 0. \tag{A.3}
\]

Substituting the ansatz and making use of equation (A.2), we get

\[
A(y) \left[ - \left( \nu(\nu + 1) - \frac{\mu^2}{1 - y^2} \right) P_{\nu}^{\mu}(y) + \frac{d}{dy} P_{\nu}^{\mu}(y) \right] + P_{\nu}^{\mu}(y) \left[ (1 - y^2) \frac{d^2}{dy^2} A(y) - (2y - 1) \frac{dy}{dy} A(y) \right] + 2(1 - y^2) \frac{d}{dy} P_{\nu}^{\mu}(y) \frac{d}{dy} A(y) + l(l + 1)P_{\nu}^{\mu}(y)A(y) = 0. \tag{A.4}
\]

Collecting terms, we have

\[
\frac{d}{dy} P_{\nu}^{\mu}(y) \left[ 2(1 - y^2) \frac{d}{dy} A(y) + A(y) \right] + P_{\nu}^{\mu}(y) \left[ (1 - y^2) \frac{d^2}{dy^2} A(y) - (2y - 1) \frac{dy}{dy} A(y) \right] - \left( \nu(\nu + 1) - \frac{\mu^2}{1 - y^2} - l(l + 1) \right) A(y) = 0. \tag{A.5}
\]

It follows that the coefficients of \( \frac{d}{dy} P_{\nu}^{\mu}(y) \) and \( P_{\nu}^{\mu}(y) \) must vanish separately. For, suppose they did not. Then we can write

\[
\frac{d}{dy} P_{\nu}^{\mu}(y) = \tilde{A}(y) P_{\nu}^{\mu}(y), \tag{A.6}
\]

where we have written

\[
\tilde{A}(y) = - \left[ \frac{(1 - y^2) \frac{d}{dy} A(y) - (2y - 1) \frac{dy}{dy} A(y) - \left( \nu(\nu + 1) - \frac{\mu^2}{1 - y^2} - l(l + 1) \right) A(y)}{2(1 - y^2) \frac{d}{dy} A(y) + A(y)} \right]. \tag{A.7}
\]

However, the associated Legendre functions also satisfy the general identity (see 14.10 of [37])

\[
(1 - y^2) \frac{d}{dy} P_{\nu}^{\mu}(y) = (\nu + \mu) P_{\nu-1}^{\mu}(y) - \nu y P_{\nu}^{\mu}(y), \tag{A.8}
\]

which is incompatible with equation (A.6) for all values of \( \mu, \nu \), as can be checked by a simple calculation.

Setting the coefficient of \( \frac{d}{dy} P_{\nu}^{\mu}(y) \) to zero, we get an equation which can be trivially solved to give
\[ A(y) = \left( \frac{y - 1}{y + 1} \right)^\frac{1}{2}. \]  
(A.9)

Substituting this solution back in the other coefficient in equation \((A.5)\) and setting that to zero gives us
\[ \left( \frac{1}{4} - \mu^2 \right) (y - 1)^{-\frac{1}{2}} (y + 1)^{-\frac{1}{2}} - (\nu(\nu + 1) - l(l + 1)) (y - 1)^{\frac{1}{2}} (y + 1)^{-\frac{1}{2}} = 0. \]  
(A.10)

Equation \((A.10)\) makes sense provided \(\mu = \frac{1}{2}\) and \(\nu = l\).

One solution of equation \((A.4)\) is thus \(\left( \frac{y - 1}{y + 1} \right)^\frac{1}{2} P_l^\nu(y)\). Since our procedure made use of the Legendre polynomials, we would get another solution by simply using \(R_l(y, y') = B_l(y') Q_{\nu'}^\mu(y) A(y)\), with the same solution for \(A(y)\). The general solution is thus found to be
\[ R_l(y, y') = A_l(y') \left( \frac{y - 1}{y + 1} \right)^\frac{1}{2} P_l^\nu(y) + B_l(y') \left( \frac{y - 1}{y + 1} \right)^\frac{1}{2} \left( i Q_{\nu'}^\mu(y) \right). \]  
(A.11)

Equation \((3.6)\) is written as it is since \(i Q_{\nu'}^\mu(y)\) is a real solution.

This procedure can similarly be used in equation \((4.5)\), which can be written as
\[ (1 - r^2) \frac{d^2}{dt^2} R_l(t, t') - 3t \frac{d}{dt} R_l(t, t') - \frac{l(l + 1)}{(1 - r^2)} R_l(t, t') = 0. \]  
(A.12)

Substitution of the ansatz \(R_l(t, t') = B_l(t') P_{\nu'}^\mu(t) A(t)\) now leads to the following equation
\[ \frac{d}{dt} P_{\nu'}^\mu(t) \left[ 2 \frac{d}{dt} A(t)(1 - r^2) - A(t) \right] + P_{\nu'}^\mu(t) \left[ (1 - r^2) \frac{d^2}{dt^2} A(t) - 3t \frac{d}{dt} A(t) \right. 
\[ \left. - \left( \nu(\nu + 1) - \frac{\mu^2}{1 - r^2} + \frac{l(l + 1)}{1 - r^2} \right) A(t) \right] = 0. \]  
(A.13)

As before, we assume the possibility that the coefficients of the \(P_{\nu'}^\mu(t)\) and \(\frac{d}{dt} P_{\nu'}^\mu(t)\) separately vanish. The coefficient of the latter term vanishing leads to the following simple result for \(A(t)\)
\[ A(t) = (1 - r^2)^{-\frac{1}{2}}. \]  
(A.14)

Substituting this equation back into equation \((A.13)\) leads to the following result
\[ - \left[ \frac{3}{4} - \nu(\nu + 1) \right] r^2 - \left[ \nu(\nu + 1) - \frac{1}{2} + l(l + 1) - \frac{\mu^2}{2} \right] = 0, \]  
(A.15)

which is satisfied for the choice of \(\nu = \frac{l}{2}\) and \(\mu = l + \frac{1}{2}\). Since in this case \(\nu \pm \mu\) is an integer but \(\mu\) is not, the other independent solution is not \(Q_{\nu'}^\mu\), but rather \(P_{\nu'}^\mu\). Thus the general solution can be written as
\[ R_l(t, t') = A_l(t') (1 - r^2)^{-\frac{1}{2}} P_{\frac{l}{2}}^{\nu + \frac{1}{2}}(t) + B_l(t') (1 - r^2)^{-\frac{1}{2}} P_{\frac{l}{2}}^{\nu - \frac{1}{2}}(t), \]  
(A.16)

which is equation \((4.6)\).
Appendix B. Limits of the de-Sitter solutions as $y \to 1$

Let us first note that the hypergeometric functions given in equation (4.9) are of the form $\binom{2}{1}(a, a + 1; 2a + \frac{3}{2}; y^2)$, where $a = \frac{2}{3}$ and $a = \frac{-1}{2}$ correspond to the two hypergeometric functions contained in $g_t(y)$ and $f_t(y)$ respectively. There exists a known formula for evaluating the hypergeometric functions at the point $y^2 = 1$. This formula is given by (see equation (15.4.20) of [37])

$$\binom{2}{1}(a, b, c, 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad \Re(a + b - c) < 0; \quad c \neq 0, -1, -2, \ldots \quad (B.1)$$

This formula applies to the hypergeometric functions included in $f_t$ and $g_t$, but not to their derivatives. Let us consider the functions separately to find their derivatives at $y^2 = 1$.

B.1. $f_t(y)$ solutions and hypergeometric functions

For the $f_t$ solutions, we need to find the expressions explicitly in order to determine the nature of the derivatives at the point $y = 1$. For the values of $l = 0, 1, 2$ and 3, the corresponding hypergeometric functions are, respectively,

$$\binom{2}{1}(- \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; y^2) \quad (l = 0);$$

$$\binom{2}{1}(-1, 0; - \frac{1}{2}; y^2) \quad (l = 1);$$

$$\binom{2}{1}(- \frac{3}{2}, - \frac{1}{2}; - \frac{3}{2}; y^2) \quad (l = 2);$$

$$\binom{2}{1}(-2, -1; - \frac{5}{2}; y^2) \quad (l = 3). \quad (B.2)$$

Here we see that $\binom{2}{1}(a, b; c; y^2)$ and $\binom{2}{1}(a - 1, b - 1; c - 2; y^2)$ represent two successive even (odd) solutions when $(a, b; c) = \left(\frac{-l - 1}{2}, \frac{-l + 1}{2}; -l + \frac{1}{2}\right)$.

Two hypergeometric functions which are contiguous are related to one another through certain differentiation formulas (see equations (15.5.4) and (15.5.9) of [37]). Let us look at the ones relevant to the $f_t$ functions. These are

$$\binom{2}{1}(a - n, b - n; c - n; z) = \frac{1}{(c - n)_n} (1 - z)^{n-a-b} z^{1+n-c} \frac{d^n}{dz^n} \left[ (1 - z)^{n+b-c} z^{c-1} \binom{2}{1}(a, b; c; z) \right],$$

$$\binom{2}{1}(a, b; c; n; z) = \frac{1}{(c - n)_n} z^{1+n-c} \frac{d^n}{dz^n} \left[ z^{c-1} \binom{2}{1}(a, b; c; z) \right], \quad (B.3)$$

where $(k)_n = \frac{\Gamma(k+n)}{\Gamma(k)}$ is Pochhammer’s symbol. Using the two relations in equation (B.3), we can write

$$\binom{2}{1}(a - n, b - n; c - 2n; z)
= \frac{z^{1+2n-c}}{(c - n)_n (c - 2n)_n} \frac{d^n}{dz^n} \left[ (1 - z)^{c+n-a-b} \right.
\times \frac{d^n}{dz^n} \left[ z^{c-1}(1 - z)^{n+b-c} \binom{2}{1}(a, b; c; z) \right] \quad (B.4)$$
Using $\, _2F_1\left(a, b; c; z\right) = \, _2F_1\left(a, a + 1; 2a + \frac{1}{2}; z\right)$ in equation (B.4) provides the relevant recursive relation for the $f_l$ hypergeometric functions. To simplify the notation in what follows, let us define

$$\, _2F_1\left(-l - \frac{1}{2}, -l + \frac{1}{2}; -l + \frac{1}{2}; z\right) = F_l(z). \quad (B.5)$$

With this definition, the solutions we seek are given by $f_l(y) = \frac{F_l(z)}{D_l(z)}$. Combining equations (B.5) with (B.4), we can write

$$F_{l+2}(z) = \frac{z^{2n+\frac{1}{2}+l}}{(l + n - \frac{1}{2})_n (l + (2n - \frac{1}{2})_n)} \times \frac{d^n}{dz^n} \left[(1 - z)^{n+\frac{1}{2}} \frac{d^n}{dz^n} \left[z^{-l-\frac{1}{2}} (1 - z)^{-\frac{1}{2}} F_l(z)\right]\right]. \quad (B.6)$$

For the recursion relation, we need only consider equation (B.6) with $n = 1$,

$$F_{l+2}(z) = \frac{z^{\frac{3}{2}+l}}{(l + \frac{1}{2}) (l + \frac{3}{2})} \frac{d}{dz} \left[(1 - z)^{\frac{1}{2}} \frac{d}{dz} \left[z^{-l-\frac{1}{2}} (1 - z)^{-\frac{1}{2}} F_l(z)\right]\right], \quad (B.7)$$

which upon evaluating the derivatives can be written as

$$F_{l+2}(z) = \left(1 - \frac{(2l + 1)(2l + 2)z}{3 + 8l + 4l^2}\right) F_l(z) - \frac{(8l + 4)z - (8l + 2)z^2}{3 + 8l + 4l^2} F_l'(z) - \frac{4(z^3 - z^2)}{3 + 8l + 4l^2} F_l''(z), \quad (B.8)$$

where primes denote differentiation with respect to $z$. As we will see below, for even $l$ we can extract a factor of $\sqrt{1 - z}$ to write the functions $F_l(z)$ in the form $F_l(z) = \sqrt{1 - z} D_l(z)$. Substituting this in equation (B.7), we find a recursion relation for the functions $D_l(z)$,

$$D_{l+2}(z) = \left[1 - \frac{2(2l + 1)z}{3 + 8l + 4l^2}\right] D_l(z) - \frac{(8l + 4)z - (8l - 2)z^2}{3 + 8l + 4l^2} D_l'(z) - \frac{4(z^3 - z^2)}{3 + 8l + 4l^2} D_l''(z). \quad (B.9)$$

We will now need the lowest order solutions to proceed further. The lowest order solution for even $l$ is $F_0(z)$, which corresponds to $\, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; z\right)$, as shown in equation (B.2). The hypergeometric function $\, _2F_1\left(\frac{\alpha}{2}, -\frac{\alpha}{2}; \frac{1}{2}; z\right)$ has the following known representation

$$\, _2F_1\left(\frac{\alpha}{2}, -\frac{\alpha}{2}; \frac{1}{2}; z\right) = cos \left(\alpha \sin^{-1}\left(\sqrt{z}\right)\right), \quad (B.10)$$

and the case where $\alpha = -1$ is the one we require. The lowest order solution for odd $l$ is $f_l(z)$, which corresponds to $F_1(z) = \, _2F_1\left(-1, 0; -\frac{1}{2}; z\right)$. From the definition of the hypergeometric function $\, _2F_1\left(a, b; c; z\right)$

$$\, _2F_1\left(a, b; c; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (B.11)$$

Using $\, _2F_1\left(a, b; c; z\right) = \, _2F_1\left(a, a + 1; 2a + \frac{1}{2}; z\right)$ in equation (B.4) provides the relevant recursive relation for the $f_l$ hypergeometric functions. To simplify the notation in what follows, let us define
we know that \( _2F_1(0, b; c; z) = _2F_1(a, 0; c; z) = _2F_1(a, b; c; 0) = 1 \). This tells us that the lowest order solutions are simply

\[
_F^0(z) = _2F_1 \left( -\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; z \right) = \sqrt{1 - z},
_F^1(z) = _2F_1 \left( -1, 0; -\frac{1}{2}; z \right) = 1.
\] (B.12)

We will use these to derive the expressions for the \( f_l(y) \) functions given in equation (4.11). We begin with the even \( l \) solutions. It can be seen that all even \( l \) solutions are of the form \( F_l(z) = \sqrt{1 - z}D_l(z) \). This follows directly from the fact that \( F_0(z) = \sqrt{1 - zD_0(z)} \), where \( D_0(z) = 1 \), and equation (B.9). The operator in equation (B.9) takes a polynomial and produces another polynomial of one order higher. The only exception is \( D_0(z) = 1 \) which, when inserted into equation (B.9), produces \( D_2(z) = 1 \). We can also calculate directly that \( D_2(z) = \sqrt{1 - z} = \sqrt{1 - zD_2(z)} \). It follows from equation (B.9) that \( D_{2k}(z) \) is a polynomial of order \( k \) in \( z \).

For \( F_l(z) \) corresponding to odd \( l \), we can use the recursion relation of equation (B.8) directly. For example, substitution of \( F_l(z) = 1 \) in equation (B.8) leads to \( F_{l+1}(z) = 1 - \frac{1}{2}z \), substituting this result back in equation (B.8) results in \( F_{l+2}(z) = 1 - \frac{2}{2}z + \frac{1}{3!}z^2 \), etc. Thus \( F_{2k+1}(z) \) is a polynomial of order \( k \) in \( z \).

We can now make a change of variable to \( z = y^2 \) in all the \( F_l(z) \) solutions, and consider \( f_l(y) = \frac{F_l(y^2)}{y^l} \) to find the \( f_l(y) \) solutions shown in equation (4.11).

**B.2. Limits of the \( g_l(y) \) solutions**

The limit of the \( g_l \) hypergeometric functions and their derivatives are most easily determined by making use of the following integral representation for hypergeometric functions (see equation (15.6.1) of [37])

\[
_2F_1(a, b, c, x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - xt)^{-a}dt \tag{\text{B.13}}
\]

\( \Re(c) > \Re(b) > 0 \).

As we have seen, the \( f_l \) hypergeometric functions have \( c \leq b \) for all choices of \( l \), with the equality holding true for the \( l = 0 \) case. Hence, we could not use equation (B.13) for those functions, and had to make use of the treatment described earlier. For the \( g_l \) hypergeometric functions, \( a = \frac{1}{2} \), which guarantees that \( _2F_1(a, a + 1, 2a + \frac{3}{2}, y^2) \) always has \( c > b > 0 \). This also holds true for the derivative of this function on account of equation (4.12). We can thus consider equation (B.13) in terms of the \( g_l \) hypergeometric functions we are dealing with, in which case we have

\[
_2F_1 \left( \frac{l}{2}, \frac{1}{2}; 1, l + \frac{3}{2}; y^2 \right) = \frac{\Gamma \left( \frac{l}{2} + 1 \right)}{\Gamma \left( \frac{l}{2} + 1 \right) \Gamma \left( \frac{l}{2} + 1 \right)} \int_0^1 \left( \frac{t}{1 - ty^2} \right)^{\frac{l}{2}} (1 - t)^{\frac{l}{2}} \frac{1}{2} dt,
\] (B.14)

while the derivative of this function takes the form

\[
\frac{\partial_j}{\partial_j} \left( _2F_1 \left( \frac{l}{2}, \frac{1}{2}; 1, l + \frac{3}{2}; y^2 \right) \right) = \frac{\Gamma \left( \frac{l}{2} + 1 \right)}{\Gamma \left( \frac{l}{2} + 1 \right) \Gamma \left( \frac{l}{2} + 1 \right)} \int_0^1 \left( \frac{t}{1 - ty^2} \right)^{\frac{l}{2} + 1} (1 - t)^{\frac{l}{2}} \frac{1}{2} dt,
\] (B.15)
The explicit representation of the $g_l$ hypergeometric functions and their derivatives, in terms of elementary functions, can now be derived using these equations for specific choices of $l$. Since we are interested in the nature of the limit of these functions as $y \rightarrow 1$ for any choice of $l$, we can simply take this limit in the above expressions, and then evaluate the integrals. This amounts to the evaluation of standard integrals. We find that equation (B.13) gives us the following finite result

$$2F_1\left(\frac{l}{2}, \frac{l}{2} + 1, l + \frac{3}{2}, 1\right) = \frac{2\sqrt{\pi}}{(l+1)} \Gamma\left(l + \frac{3}{2}\right) \left(\Gamma\left(\frac{l+1}{2}\right)\right)^2,$$  \hspace{1cm} (B.16)

while equation (B.15) diverges for all choices of $l$.

Similarly, the $y \rightarrow 0$ limits can also be determined very simply by using substituting $y = 0$ in equations (B.14) and (B.15). One can easily find that $2F_1\left(\frac{l}{4}, \frac{l}{4} + 1, l + \frac{3}{4}, 0\right) = 1$, by working out the integral. The derivative of this function at $y = 0$ vanishes on account of an overall factor of $y$, and the fact that the integral is finite.

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