Hydrodynamic theory of motion of quantized vortex rings in trapped superfluid gases.

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I study vortex ring oscillations in a superfluid, trapped in an elongated trap, under the conditions of the Local Density Approximation. On the basis of the Hamiltonian formalism I develop a hydrodynamic theory, which is valid for an arbitrary superfluid and depends only on the equation of state. The problem is reduced to an ordinary differential equation for the ring radius. The cases of the dilute BEC and the Fermi gas at unitarity are investigated in detail. Simple analytical equations for the periods of small oscillations are obtained and the equations of non-linear dynamics are solved in quadratures. The results agree with available numerical calculations. Experimental possibilities to check the predictions are discussed.

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Introduction. The quantized vortex ring in a superfluid is one of the most unusual objects of modern physics. It can be quite macroscopic in size but still keeps its quantum nature, carrying one quanta of circulation. Rings of large size have small phase velocity and play a crucial role in the phenomenon of critical velocity. The existence of such rings was predicted by Feynman [1]. They were discovered, in an indirect way, in experiments on ion motion in liquid $^4$He [2]. Vortex rings were also observed in gaseous Bose-Einstein Condensates (BEC) in traps [3]. There is every reason to believe that the development of experimental technique will permit the detailed investigation of rings, both in trapped BEC and Fermi gases near unitarity. The dynamics of rings under such conditions should be quite peculiar. In a uniform fluid a ring always moves with an anomalous energy-velocity relation - its velocity decreases with increasing energy. Under the non uniform conditions in a trap the situation is different. It was discovered in [4] that in a spherical trap a configuration of maximum energy exists, where a circular ring is at rest. It was checked in [4] that the same situation takes place also in an elongated trap. An initial deviation from this equilibrium configuration will result in oscillations of the ring. These oscillations in a spherical trap were investigated numerically in [4].

The problem of oscillation of a ring along a symmetry axis of a superfluid sample in an elongated harmonic trap is one of the most natural subjects of experimental investigation. Numerical simulations have been recently performed in [6] for a Fermi gas at unitarity using a time-dependent density functional theory and in [7] for a BEC using the Gross-Pitaevskii (GP) equation. The main goal of these papers is the interpretation of experimental results [8]. An approximate theory for ring motion in a cylindrical trap in the presence of dissipation was developed in [5].

General theory. In this letter I will present the Hamiltonian theory of the motion of a ring of radius $R$ along the axis of a superfluid gas in a harmonic trap. I assume that the trap is elongated, $\omega_\perp \gg \omega_z$. I also assume the Local Density Approximation (LDA) conditions, that is that the transverse radius of the gas is much larger than the healing length, $R_\perp \gg \xi$. Moreover I will assume that these conditions are satisfied strongly in the sense that also the logarithmic factor $L \equiv \log(R/\xi)$, which enters in the theory, is large and can be considered as a constant in all calculations. We will see that in this “logarithmic” approximation one can solve the problem analytically in a simple way for a superfluid of any nature. I will finally assume that $R \sim R_\perp$, excluding very small rings.

The LDA conditions permit us to use hydrodynamics. The energy of a ring, i. e. the Hamiltonian, can be written directly. The key point is that the energy can be obtained by integrating the kinetic energy of the flow $\rho v^2/2$ over the volume ($\rho$ is the density of the fluid).

The logarithmic approximation the main contribution to the integral is due to a small region near the vortex line. Thus one can use the expression for the energy in a uniform fluid, taking the density $\rho(r, z)$ to be its value near the ring:

$$E_R(R, Z) = \frac{2m^2\hbar^2}{M^2} R \rho(R, Z) \log \left( \frac{R}{\xi} \right), \quad (1)$$

where $M$ is the atomic mass $m$ for the Bose superfluid and the pair mass $2m$ for the Fermi one, and $Z$ is the $z$-coordinate of the ring. Notice that the coordinate dependence of the density in the LDA regime can be written in the form

$$\rho(r, z) = \rho \left[ \mu (1 - r^2/R_\perp^2 - z^2/R_z^2) \right], \quad (2)$$

where $\mu$ is the chemical potential in the center of the trap and $R_\perp = (2\mu/m\omega_\perp^2)^{1/2}$, $R_z = (2\mu/m\omega_z^2)^{1/2}$ are the Thomas-Fermi (TF) radii of the fluid.

The momentum of the ring can be calculated as $P_R(R, Z) = (\hbar/M) \int \rho(r, z) \partial_z \phi d^3r$, where $\phi$ is the phase of the order parameter (condensate wave function in BEC case). However, for an elongated trap with $R_z \gg R_\perp$ one can substitute $z \approx Z$. Then the integration will be reduced to an integration on the surface, stretched on the
ring aperture, where the phase \( \phi \) undergoes a 2\( \pi \) jump:

\[
P_R \approx \frac{\hbar}{M} \int \rho(r, Z) \partial_z \phi \rho dr = \frac{2\pi \hbar}{M} \int_0^R \rho(r, Z) 2\pi rdr.
\]

The velocity of the ring can be calculated with the Hamilton equation as

\[
V \equiv V_z = \left( \frac{\partial E_R}{\partial P_R} \right)_Z = \left( \frac{\partial E_R/\partial R}{\partial P_R/\partial R} \right)_Z.
\]

\[
V(R, Z) = \frac{\hbar}{2M} \frac{L}{R_\rho} \left( \frac{\partial (R\rho)}{\partial R} \right)_Z.
\]

Notice that the equation (4) admits a transparent interpretation. The quantity \( P_z = \frac{2\pi \hbar}{(2\pi \hbar \rho)} \) is the force, acting on a unit of length of the ring. According to the Magnus equation, the ring drifts when the velocity \( FM/ (2\pi \hbar \rho) \) in agreement with (3).

Equations (11) and (12) give a full description of the motion of the ring in a trap. Notice that the theory is completely hydrodynamic in its nature. Properties of the fluid enter only through the equation of state \( \rho(\mu) \). It is convenient to introduce the dimensionless variables \( X = Z/R_\rho \) and \( Y = R/R_\perp \). Energy can be presented as \( E_R(R, Z) = (2\pi \hbar^2 R_\perp \rho_0 L/M^2) f(Y, X) \), where \( \rho_0 \) is the density in the center of the trap and \( f \) is a dimensionless function. The trajectory of the ring on the \( X, Y \) plane is given by the energy conservation equation

\[
f(Y, X) = f_0,
\]

where \( f_0 \) fixes the energy of the ring. The plane where the ring can be at rest is always at \( X = 0 \). The equilibrium radius \( R_{\text{EQ}} = R_\perp Y_{\text{EQ}} \) is defined by the equation \( \partial f(Y, 0)/\partial Y = 0 \). The energy \( E_R \) has a maximum at this point. The equation \( f(Y, 0) = f_0 \) has two positive solutions, \( Y_0 \) and \( Y_1 \), where \( 0 < Y_0 < Y_1 \). Then \( R_{\text{min}} = R_\perp Y_0 \) is the minimal radius of the ring on the given trajectory and \( R_{\text{max}} = R_\perp Y_1 \) is the maximal one. Below I will consider \( Y_0 \) as an "initial point" of the trajectory. The equation \( \partial f/\partial Y \big|_X = 0 \) defines the line of "turning points" on \( Y, X \) plane, where the velocity changes sign. Together with (6) it gives the turning point \( Y_A, X_A \) for a trajectory of given energy. It follows that a ring with \( Y < Y_A \) moves in the same direction as in an uniform fluid and a ring with \( Y > Y_A \) in the opposite direction. The amplitude of the oscillations in \( z \)-direction is \( Z_A = R_\perp |X_A| \). It is worth noting, that the ring at rest has zero velocity, but finite momentum, in analogy with rotons in superfluid \(^4\)He. The quantity, which can be easily measured, is a period of oscillation of the ring. One can calculate the period of small oscillations \( T_0 \) in a general form by writing the energy near the point \( R = R_{\text{EQ}} \) and \( Z = 0 \) in the oscillator form

\[
E_R(R, Z) - E_R(R_{\text{EQ}}, 0) \approx - \left[ \frac{\partial f}{\partial Y} \right]^2 V^2 + Z^2.
\]

Direct calculation gives

\[
T_0 = 2\sqrt{2T_z} \frac{M \mu}{mL \omega_\perp} \left( \frac{f^2}{\partial X^2 \partial Y} \right)^{1/2}, \quad (7)
\]

where \( T_z = 2\pi/\omega_z \) is the trap period and quantities in the parenthesis should be taken at \( X = 0, Y = Y_{\text{EQ}} \). In LDA regime \( (\mu/L \omega_\perp) \gg 1 \) and \( T_0 \gg T_z \), as revealed in numerical calculations in (R, R). Notice that the prefactor, fixed by the parameters of the system, can be presented as \( (T_z \mu/mL \omega_\perp) = (\pi MR_\perp \rho_0 \hbar) \), demonstrating a simple dependence on pure geometric factors.

One can find the time dependence of \( Z \) and \( R \) from the equation \( t = \int \frac{dZ}{\rho} \), where the integral should be taken along the trajectory. It is more convenient to go to variable \( R \). Using Eq. (6) we get

\[
t(Y) = -T_z \frac{M \mu}{\pi^2 mL \omega_\perp} 2\pi f_0 \int_0^Y \left( \frac{\partial X}{\partial f} \right)_Y dY.
\]

The period for oscillations for an arbitrary amplitude can be found as \( T = 2t(Y_1) \). This quantity is interesting, because it reflects the peculiarity of the dynamics of the rings, and also important, because it is difficult to observe oscillations of a small amplitude in an experiment.

Vortex ring in a trapped dilute BEC. In a dilute BEC the chemical potential is \( \mu = gn \), where \( g \) is the coupling constant and \( n \) is the atom density. This means that the energy function \( f \) is

\[
f(Y, X) = Y (1 - Y^2 - X^2).
\]

\[
\int_{0}^{R_{\text{max}}} \rho(r, Z) 2\pi rdr.
\]

\[
\left( \frac{\partial E_R/\partial R}{\partial P_R/\partial R} \right)_Z.
\]
Then the equilibrium radius is \( R_{EQ} = R_1 Y_{EQ} = R_1 / \sqrt{3} \). This value coincides with one obtained in [14] in the logarithmic approximation for a spherical trap. The line of the turning points is \( 3Y_A + X_A^2 = 1 \). The minimal radius of the ring is related to the energy as \( f_0 = Y_0 (1 - Y_0^2) \) and the maximal radius \( Y_1 = \frac{1}{2} \sqrt{4 - 3Y_0^2} - \frac{1}{2} Y_0 \). The equation of the trajectory with initial radius \( Y_0 \) can be written as \( Y (1 - Y^2 - X^2) = Y_0 (1 - Y_0^2) \) and the amplitude of oscillation can be expressed through energy as \( X_A = \sqrt{1 - \frac{3}{2Y_0^2} f_0^{2/3}} \). Trajectories for different values of \( Y_0 \) are shown in the left panel in Fig. 1. (One can see values of \( Y_0 \) as initial values of \( Y \) on the \( y \)-axis.)

The period of small oscillations can be calculated according to Eq. (10). A simple calculation gives

\[
\frac{T_0^{(B)}}{T_z} = \frac{4}{3\sqrt{3}} \frac{\mu}{\hbar \omega_\perp} \approx 0.77 \frac{\mu}{\hbar \omega_\perp} \quad \text{ (10)}
\]

Scaling of \( T \) as \( gn \log(gn) \) was predicted in [1] on the basis of qualitative considerations. For oscillations of an arbitrary amplitude let us present the period as \( T^{(B)} = T_z \left( \frac{\mu}{\pi^2 \hbar \omega_\perp} \right) \tau^{(B)} \). One obtains the expression for the dimensionless period \( \tau^{(B)} \)

\[
\tau^{(B)} = \int_{Y_0}^{Y_2} \frac{2\pi f_0 dY}{\sqrt{Y(Y - Y_0)(Y_1 - Y)(Y - Y_2)}} \quad \text{ (11)}
\]

where \( Y_2 = -\frac{1}{2} \sqrt{4 - 3Y_0^2} - \frac{1}{2} Y_0 \) is a negative root of the equation \( f(Y,0) = f_0 \). Equation (11) can be expressed in terms of the complete elliptic integral of the first order:

\[
\tau^{(B)} = \frac{4\pi f_0}{\sqrt{2} \sqrt{Y_1 (Y_0 - Y_2)}} K(k), \quad k = \sqrt{\frac{(Y_1 - Y_0)(-Y_2)}{Y_1 (Y_0 - Y_2)}} \quad \text{ (12)}
\]

For small oscillations \( Y_0 \rightarrow Y_1 \rightarrow 1/\sqrt{3}, f_0 = 2/(3\sqrt{3}), Y_2 = -2/\sqrt{3} \). Then \( \tau^{(B)} \rightarrow 4\pi^2/3\sqrt{3} \) in accordance with (11). At small \( Y_0 \) one gets formally \( \tau^{(B)} \approx 2\pi Y_0 \ln(16/Y_0) \). However, this limit violates the applicability of the approximation. The dependence of the period on the minimal radius of a ring \( Y_0 \) is shown in Fig. 2.
with the minimal radius \( y_0 = y_0^3/2 \) this equation can be presented as \( (y - y_0) Q(y) = 0 \), where
\[
Q(y) = y^3 + y^2 y_0 + y y_0^2 + y_0^3 - 1. \tag{15}
\]
The equation \( Q(y) = 0 \) has 3 roots. The root \( y_1 \) is real and defines the maximum value of the radius, \( Y_1 = y_1^{3/2} \). Roots \( y_2 \) and \( y_3 \) are complex conjugated. One can find the roots analytically or numerically.

Changing the variable of integration in \( \Theta \) from \( Y \) to \( y \), I present the time of motion as \( t^{(F)} = T_z (2 \mu / \pi^2 L \hbar \omega_\perp) \tau^{(F)} \), where
\[
\tau^{(F)}(y) = \int_{y_0}^y \frac{2 \pi f_0^{2/3} dy}{\sqrt{(y_1 - y)(y - y_0)(y^2 - 2Re(y_2)y + |y_2|^2)}}. \tag{16}
\]
The integral can be expressed in terms of an elliptic integral (see \( [12] \, \text{Eq. 3.145} \))
\[
\tau^{(F)} = \frac{4 \pi f_0^{2/3}}{\sqrt{pq}} F(\varphi, k), \tag{17}
\]
where the parameters are
\[
\varphi = 2 \text{arccot} \sqrt{\frac{q(y_1 - y)}{p(y - y_0)}}, \quad k = \sqrt{\frac{(y_1 - y_0)^2 - (p - q)^2}{4pq}} \tag{18}
\]
and
\[
p = \left[ |y_0|^2 + |y_2|^2 - 2y_0 \text{Re}(y_2) \right]^{1/2}, \\
q = \left[ |y_1|^2 + |y_2|^2 - 2y_1 \text{Re}(y_2) \right]^{1/2}. \tag{19}
\]

Inverse dependence \( y(\tau^{(F)}) \) can be expressed in terms of the Jacobi cn\((t, k)\) function. The period of oscillations is \( 2t^{(F)}(Y_1) \). For small amplitudes the result coincides with \( [11] \). For a ring of small radius, \( y_0 \rightarrow 0 \), one gets \( \tau^{(F)} \rightarrow 15.3 y_0^{2/3} \). Notice that the authors of \( [6] \) used a slightly different dependence of period on radius for relatively small rings: \( T^{(F)} \propto R \) as opposite to \( T^{(F)} \propto R^{2/3} \) here. In Fig. 3 I show the time dependence of the radius of a ring, oscillating in the Fermi gas at unitarity. One can see that the oscillations are almost harmonic. Only for the small initial radius \( y_0 = 0.1 \) the difference between growing and shrinking motion, which was revealed in \( [7] \), can be noticed. In contrast, the dependence of the period on the amplitude of oscillations is strong. This dependence is shown in Fig. 4. It is interesting, that the analogous curve for a ring in BEC is practically indistinguishable from Fig. 4.

The results of the present analytical theory are in qualitative agreement with numerical calculations \( [6, 7, 18] \): the period of oscillations is much longer than \( T_z \) and increases when the radius of the ring decreases or the interaction increases. I tried to compare quantitatively my results with numerical calculations \( [7] \). These calculations were produced for a ring in a BEC with \( R_\perp / \xi = 2 \mu / \hbar \omega_\perp \approx 27 \), giving \( L \approx 3.3 \) and \( \omega_\perp / \omega_z = 4 \), that, of course, does not ensure applicability of my asymptotic theory. However, formal use of the theory gives \( T^{(B)} / T_z \approx 3 \) for \( R_{\text{min}} / \omega_z \approx 1.2 \). The data presented in Fig. 5 of \( [7] \) give \( T^{(B)} / T_z \approx 2.6 \) in surprisingly good agreement. Quantitative comparison with calculations of Ref. \( [6] \) for the Fermi gas at unitarity is unreasonable, because the LDA conditions are not satisfied there.

It is worth noting that the present theoretical scheme can be generalized to relax the strong inequality, which I used. For example, one can calculate the energy and momentum numerically, as done in \( [6] \), and still use equation \( [4] \). Of course, there is no problem to use this theory for a non-harmonic trap. The theory also can be generalized for more complicated ”solitonic vortices” excitations, observed in numerical calculations \( [12, 13] \). An interesting direction of applications of the theory is the dynamics of topologically nontrivial excitations in two interpenetrating superfluids. (See, for example, \( [10, 12] \).)

Experimental confirmations of the present results demand strong LDA conditions. However, I believe that in the Fermi gas they can be satisfied in a natural way. For example, in experiments \( [18] \) the value \( R_\perp / \xi \approx 100 \) of the LDA parameter at \( \omega_\perp / \omega_z = 6.2 \) was reached, which is sufficient for the theory, and nothing prevents using even larger values. The problem of the creation of a ring in a controlled way is not an easy one. It can be solved as a result of the snake instability of a soliton, as it already observed in \( [3] \). It was suggested in \( [11] \) to create a ring by a moving bright spot of laser light, in analogy with rings creation by moving impurities in experiments \( [2] \). In experiments \( [10] \) and \( [20] \) gravitational accelerations of impurity atoms in a trapped superfluid was observed. This effect can also be used to create rings. Recently, an ingenious method of creating vortex rings in BEC by a fast increase interaction near the Feshbach resonance was suggested in \( [21] \). The same method can be used in a Fermi superfluid.

To conclude, I have developed a Hamiltonian theory for the radius and center of a quantized vortex ring, oscillating along an axis of a superfluid, trapped in an elongated cylindrical trap. The theory is valid in the strong Local Density Approximation, when \( L = \log (R_\perp / \xi) \gg 1 \). The theory is pure hydrodynamical and demands only knowledge of the equation of state of the fluid. It occurs that the equations can be solved in quadratures and the period of oscillations is scaled as \( T \sim T_z (\mu / L \hbar) \). For the cases of dilute BEC and Fermi gas at unitarity simple expressions for the periods are obtained and solutions can be expressed in terms of elliptic integrals. Possible generalizations are mentioned. Ways of verifying of the predictions in experiments are discussed in short.

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