Breaking of general rotational symmetries by multi-dimensional classical ratchets

A. W. Ghosh\textsuperscript{1} and S. V. Khare\textsuperscript{2}

\textsuperscript{1} School of Electrical and Computer Engineering, Purdue University, W. Lafayette, IN 47907 and
\textsuperscript{2} Dept. of Materials Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801

(Dated: April 3, 2018)

We demonstrate that a particle driven by a set of spatially uncorrelated, independent colored noise forces in a bounded, multidimensional potential exhibits rotations that are independent of the initial conditions. We calculate the particle currents in terms of the noise statistics and the potential asymmetries by deriving an n-dimensional Fokker-Planck equation in the small correlation time limit. We analyze a variety of flow patterns for various potential structures, generating various combinations of laminar and rotational flows.

PACS numbers: PACS numbers: 05.10.Gg, 05.40.-a, 87.10.+e

\section{I. INTRODUCTION}

One of the corner-stones of equilibrium statistical mechanics is the second law of thermodynamics, which precludes extraction of pure work out of a heat source (e.g. thermal noise) without an accompanying change of state \cite{1}. Fluctuation-dissipation theorem makes it quite impossible for a system to extract work out of a noise source in equilibrium with it, even if the system is in a potential with a built-in directionality. The impossibility of such a thermal noise-induced rectification was lucidly explained by Feynman \cite{2} using the concept of a ‘ratchet’, a device whose static potential is periodic, but with an asymmetry within each period (such as a sawtooth potential). Although motion in a ratchet is easier in one direction than the other, it is impossible to exploit this asymmetry to drive a particle in the potential using a simple thermal environment. This is because the probability of a noise-induced jump over a barrier in a ratchet depends only on the barrier height, and is therefore the same to the right and the left, irrespective of the different slopes in the two directions.

Directed motion in a ratchet potential requires an external source of energy that is \textit{out of equilibrium} with the system, thus negating the necessity to obey the fluctuation-dissipation theorem \cite{3,4}. A trivial example is a fully correlated noise source such as a deterministic unidirectional force exerted to an axle attached to a ratchet-wheel. Remarkably, the rectification persists even for a non-equilibrium noise with zero time-average. The minimal condition for a ratchet to operate is \textit{broken detailed balance} (such as a “colored” or temporally correlated external noise) which in conjunction with the \textit{spatial asymmetry} within each ratchet period, and \textit{broken time-reversal symmetry} (dissipation) leads to directed motion.

The successful extraction of work out of a non-equilibrium source of energy has far-reaching implications. Thermal ratchets are not limited by energetic restrictions associated with equilibrium statistical mechanical principles \cite{4,5,6,7}. Massive (underdamped) ratchets exhibit a parametric current reversal that could be useful for continuum mass separation \cite{11} and designing ‘molecular shuttles’ \cite{11}. Furthermore, ratchet motion is considered to be a possible explanation for the long-range cellular transport of motor proteins \cite{12}. On the experimental front, Brownian ratchets have been demonstrated in the rectified motion of polystyrene spheres and a drop of mercury \cite{13}, and in current rectification in a DC SQUID \cite{14}.

A ratchet-induced rectification using a two or three dimensional apparatus has been investigated recently, primarily with the aim of separating particles of different diffusion constants or sizes \cite{10,15,16}. Though the particle motion in such studies \cite{10,15,16} is in multi-dimensions the ratchet effect is usually along a single direction. Relatively little investigation has been done of systems where the ratchet effect exists in multiple spatial directions \cite{11,14}. Extending ratchet motion to higher dimensions involves more than just a simple extension of one-dimensional arguments. One can envisage breaking of higher symmetries such as rotational symmetry, by the broken detailed balance, leading to a rich structure of loops and vortices. Generalizing the concept of a ‘coordinate’, one can map the motion of a stochastic particle in phase-space (semiclassical evolution in a multi-quantum well system for example), or in chemical coordinates, into a n-dimensional motion in real space. The interplay of rectification and dimensionality can lead to very interesting flow patterns. For instance, a judicious combination of one-dimensional ratchets can lead to steady-state rotations, even if the noise sources themselves are restricted only to apply along the orthogonal directions and are uncorrelated with each other. A simple realization in two dimensions (2D) is as follows: Consider Fig. \textbf{1} where a two-dimensional rotation due to ratchet motion is shown schematically. In the presence of spatial asymmetry in one-dimension, a time-correlated noise is known to produce a drift \cite{3,21}, the direction of the drift is determined by the sense of the potential asymmetry. For a potential in multiple dimensions, the sense of the potential asymmetry along one coordinate can be reversed by varying the other coordinates, leading to a change in sign (or reversal of asymmetry) of the potential. As indicated in Fig. \textbf{1} the coordinate-dependent reversal of...
the one-dimensional drifts could then conspire together to generate a steady-state rotation. Contrary to rotation generated by purely potential forces, determined by the initial conditions of the particle, the sense of our rotation is independent of initial conditions and is given by the combination of the potential asymmetry and the noise statistics. Furthermore, removing either the potential asymmetry or the correlation in the noise destroys the rotation. In effect, we have thus produced a rotational motion along a cyclical track using combinations of one-dimensional ratchets, although the x and y noise forces are totally uncorrelated. A generalization of this process can generate rotations for any asymmetric potential \[ V(x, y) \] that is built into the dynamical process of a particle in a potential \[ U(\vec{r}) \] driven by colored noise. In section V we present a systematic expansion in the correlation time for a Gaussian distributed noise. The work is presented as follows. Appendix A gives some algebraic details of the calculations. Appendix B discusses questions about convergence.

II. FUNCTIONAL CALCULUS FOR THE PROBABILITY IN MULTIPLE-DIMENSIONS

Consider an overdamped classical particle in a multi-dimensional potential \[ U(\vec{r}) \] driven by an external time-correlated noise with force \[ \gamma \vec{f}(t) \], where \[ \gamma \] is the damping constant. The motion of the particle is governed by the Langevin equation, which is essentially Newton’s law without the inertial term:

\[
\dot{\vec{r}}(t) = \vec{W}(\vec{r}) + \vec{f}(t)
\]

(1)

where \( \vec{r} \) is the position vector of the particle, \( \gamma \vec{W} = -\nabla U(\vec{r}) \) is the force exerted by the potential on the particle, and the dot denotes a derivative with respect to time. The stochastic process described above depends crucially on the statistics of the noise \( \vec{f} \). We assume that the noise has a Gaussian probability distribution with a correlation time \( \tau_c \) and strength \( D_i \) along the \( i \)th coordinate direction. Furthermore, we assume that the noise along any two orthogonal coordinate directions are uncorrelated. The probability distribution \( P(\vec{r}, t) \) can be described in terms of a functional integral over different realizations of \( \vec{f} \) as follows:

\[
P(\vec{r}, t) = \int D\vec{f} P[\vec{f}] \delta(\vec{r} - \vec{r}(t)),
\]

\[
P[\vec{f}] = N \exp \left[ -\frac{1}{2} \int dsds' \sum_{ij} K_{ij}(s-s') f_i(s)f_j(s') \right],
\]

\[
\langle f_i \rangle = 0; \quad \langle f_i(t)f_j(t') \rangle = D_i \tau_c C_{ij} \left( \frac{|t-t'|}{\tau_c} \right) \delta_{ij}.
\]

(2)

The Dirac delta function in Eq. (2) equates the arbitrary position variable \( \vec{r} \) with the functional form \( \vec{r}(t) \) stipulated by Eq. (1). The dimensionless correlation function \( C_{ij} \) can in principle have multiple time-scales built into it. The formalism we develop can be easily generalized to take such extensions into account.

We generalize the functional calculus outlined by Fox [18] to develop a bona-fide Fokker-Planck equation for the probability distribution \( P(\vec{r}, t) \). Using Eqs. (1) and (2) we get

\[
\frac{\partial P(\vec{r}, t)}{\partial t} = -\int D\vec{f} P[\vec{f}] \sum_i \dot{r}_i \frac{\partial}{\partial r_i} \delta(\vec{r} - \vec{r}(t))
\]

\[
= -\sum_i \frac{\partial}{\partial r_i} [W_i(\vec{r}(t))P(\vec{r}, t) + Q_i(\vec{r})]
\]

(3)

where \( Q_i(\vec{r}) = \int D\vec{f} P[\vec{f}] \delta(\vec{r} - \vec{r}(t)) f_i(t) \). Using results derived in Appendix A, we can write the product \( P[\vec{f}]f_i(t) \) as

\[
\sum_i \frac{\partial}{\partial r_i} [W_i(\vec{r}(t))P(\vec{r}, t) + Q_i(\vec{r})]
\]
as follows:

\[ P[\tilde{f}_i(t)] = \sum_l \int ds P[\tilde{f}_i(s)] \delta_{il} \delta(t - s) \]

\[ = \sum_l \frac{D_l}{\tau_i^c} \int ds P[\tilde{f}_i(s)] \int ds' K_{li}(s - s') C_i \left( \frac{t - s'}{\tau_i^c} \right) \]

\[ = -\frac{D_l}{\tau_i^c} \int ds' C_i \left( \frac{t - s'}{\tau_i^c} \right) \delta P[\tilde{f}] \frac{\delta}{\delta f_i(s')} \]  

Substituting in Eq. (3)

\[ \frac{\partial P(\vec{r}, t)}{\partial t} = - \vec{\nabla} \cdot (\vec{W} P) - \sum_{ij} \frac{\partial^2 \hat{Q}_{ij}}{\partial r_i \partial r_j} \]  

where

\[ \hat{Q}_{ij}(t) = \frac{D_{ij}}{\tau_i^c} \int ds' C_i \left( \frac{t - s'}{\tau_i^c} \right) \times \int D \tilde{f} P[\tilde{f}] \delta(\vec{r} - \tilde{r}(t)) \frac{\delta r_j(t)}{\delta f_i(s')} \]  

At this stage, we see that in the presence of color, the stochastic process is non-Markovian, which means that the value of \( P \) at time \( t \) depends on earlier instants of time \( s' \) through the kernel \( K(t - s') \), or equivalently, through the correlation function \( C \) which represents the inverse of \( K \) (Appendix A). To evaluate the role of color, we now need to evaluate the functional derivative in Eq. (4). This is best done with the help of the Langevin Eq. (5) for \( \dot{r} \):

\[ \frac{d}{dt} \left[ \frac{\delta r_j(t)}{\delta f_i(s')} \right] = \frac{\delta r_j(t)}{\delta f_i(s')} \]

\[ = \frac{\delta}{\delta f_i(s')} \left[ W_j f_j(t) \right] \]

\[ \approx \sum_k M_{jk} \frac{\delta r_k(t)}{\delta f_i(s')} + \delta_{ij} \delta(t - s') \]  

(7)

where we define \( M_{ij} \equiv \partial W_i(t)/\partial r_j \). The solution to the above equation is:

\[ \frac{\delta r_j(t)}{\delta f_i(s')} = H(t - s') \delta_{ij} \mathbf{T} \left( \exp \left[ \int_{s'}^t ds M(\vec{r}(s)) \right] \right)_{ji} \]  

(8)

as can be verified by direct substitution. In the above, \( \mathbf{T} \) is the time-ordering operator and \( H(x) \) is the Heaviside step-function (defined to be unity for positive \( x \) and zero otherwise). Substituting this functional derivative in Eq. (8), we manage to reduce the equation for \( P(\vec{r}, t) \) into the form of a continuity equation:

\[ \dot{P} = -\vec{\nabla} \cdot \vec{J}, \quad \text{with} \]

\[ J_i = W_i P - \frac{\partial}{\partial r_i} \Theta_i P, \]

\[ \Theta_i = \frac{D_i}{\tau_i^c} \int_0^\infty dt' C_i \left( \frac{t'}{\tau_i^c} \right) \mathbf{T} \left( \exp \left[ \int_{t'}^t ds M(\vec{r}(s)) \right] \right)_{ji} \]  

(9)

**III. GETTING A BONA-FIDE FOKKER-PLANCK EQUATION**

To get the steady state \( (t \to \infty) \) limit of the above equation, we expand the integral inside the exponential:

\[ \int_{t - t'}^t ds M_{ij}(\vec{r}(s)) \approx t' M_{ij}(\vec{r}(t)) - \frac{t'^2}{2} \sum_k \frac{\partial^2 W_j}{\partial r_k \partial r_j} [W_k + f_k] + O(t'^2) \]

(10)

The expansion does not introduce any singularities, as discussed in the Appendix B. After changing variables \( t' \to x = t'/\tau_i^c \), we can now rewrite \( \Theta_i \) as:

\[ \Theta_i = D_i \int_0^\infty dx C_i(x) \mathbf{T} \left( \exp \left[ x \tau_i^c M - x^2 (\tau_i^c)^2 R/2 \right] + O(x^3(\tau_i^c)^3) \right) \]  

(11)

where the matrix \( R \) has components \( R_{ij} \) given by \( R_{ij} = \sum_k W_k \partial^2 W_i/\partial r_j \partial r_k \). The components of \( M \) and \( R \) have dimensions of \( 1/\tau_i^c \cong U_0/\gamma L_i^2 \) \( (U_0) \) is the maximum height of the potential and \( L_i \) is the length scale of variation of the potential in the \( i \)-th direction, which equals its period for a periodic potential) while \( x \) is dimensionless. For small correlation times \( \tau_i^c \ll \tau_i^c \) (time scales described in detail in the next section), we can further Taylor expand the exponent to give us the effective diffusion constant \( \Theta_i \) in terms of the noise statistics defined by \( \{ \mu_n^i \} \), where \( \mu_n^i \equiv \int_0^\infty dx C_i(x) x^n \) is the \( n \)th moment of the noise correlation function. For well-behaved functions with rapidly vanishing higher moments, terms like \( x^n(\tau_i^c)^n \) can be ignored for small \( \tau_i^c \) and large \( n \) (cf. Appendix B). Then we can truncate the equation of motion for \( P(\vec{r}, t) \) to second order, leading thereby to a bona-fide Fokker-Planck equation. For small correlation time \( \tau_i^c \), the equation reads:

\[ \frac{\partial P(\vec{r}, t)}{\partial t} = - \sum_i J_i = - \sum_i \frac{\partial}{\partial r_i} \left( W_i P - \frac{\partial}{\partial r_i} \right) \Theta_i P \]

\[ \Theta_i = D_i \left[ 1 + \mu_1^i \tau_i^c M_{ii} - \frac{1}{2} \mu_2^i (R - M^2)_{ii} \right] \]

\[ M_{ij} = \frac{\partial W_i(t)}{\partial r_j} \]

\[ R_{ij} = \sum_k W_k \frac{\partial^2 W_i}{\partial r_j \partial r_k} \]  

(12)

The above set of equations are the central equations for all our analyses. **The effect of the noise correlation shows up in the effective diffusion constant \( \Theta_i \), which picks up a position dependence \( \{ \mu_n^i \} \) in a well-defined manner through the potential gradient terms \( M \) and \( R \). The statistics of the noise shows up through the moments \( \{ \mu_n^i \} \) of the temporal correlation function.**
IV. APPLICATION: 1-D RATCHET

Having established an approximate though bona-fide Fokker-Planck equation for arbitrary correlation functions \( C_t \) and arbitrary dimensions at small correlation times, we now use our results to obtain the dynamics of a classical particle in various kinds of ratchet potentials. As a first step, we calculate the steady-state current density in a one-dimensional periodic potential using our Fokker-Planck formalism for exponential correlation \((\mu_1 = 1, \mu_2 = 2)\). In one-dimension, \( \Theta \) can be written as:

\[
\Theta = D \left[ 1 + \tau^c W' - (\tau^c)^2 \{ WW'' - W'^2 \} \right] \tag{13}
\]

to second order in \( \tau^c \) and where prime denotes a derivative. The expression for \( \Theta \) to first order in \( \tau^c \) is well known in the literature. However, we need to retain the second order corrections in \( \tau^c \) to get any nontrivial current density out of the noise, as we shall shortly see.

At steady state \( (\dot{P} = 0) \), the Fokker-Planck equation reads:

\[
\frac{dJ}{dx} = 0 \quad J = WP - \frac{d}{dx}[\Theta P] \tag{14}
\]

The continuity equation dictates a constant \((x\text{-independent})\) current. Using an integrating factor \( e^{-\phi(x)} \) where \( \phi(x) = \int_0^x dy W(y)/\Theta(y) \), \('0'\) being an arbitrary point on the \( x \) axis, we solve the above first order ordinary differential equation for \( P(x) \):

\[
P(x) e^{-\phi(x)} = P(0) e^{-\phi(0)} - J \int_0^x dxe^{-\phi(y)} \tag{15}
\]

Imposing periodic boundary conditions \( P(0) = P(L) \), \( \Theta(0) = \Theta(L) \), one then gets the following equation for the drift current \( J \):

\[
J = P(0) \Theta(0) \left[ 1 - e^{-\phi(L)} \right] \int_0^L dxe^{-\phi(x)} \tag{16}
\]

We can solve for \( P(0) / \Theta(0) \) by normalizing \( P(x) \) within a period with a normalization constant \( N \) representing the number of particles within a period at steady-state. This yields finally:

\[
J = \frac{N \int_0^L dxe^{\phi(x)} [ \int_0^L dxe^{-\phi(y)} - \left[ 1 - e^{-\phi(L)} \right] \int_0^L dxe^{-\phi(y)} ]}{\Theta(x) \int_0^L dxe^{\phi(x)}} \tag{17}
\]

For small correlation times \( \tau^c \), using the expression for \( \Theta(x) \) from Eq. (12), one can expand \( \phi(L) \) as follows:

\[
\phi(L) = \int_0^L \left[ W(x)/\Theta(x) \right] dx
\]

\[
\approx \int_0^L dx \frac{W(x)}{D} \left[ 1 - \tau^c W'(x) + (\tau^c)^2 W(x) W''(x) + O \left( (\tau^c)^3 \right) \right]
\]

The first and second terms in the integrand are proportional to exact derivatives of \( U \) and \( W^2 / 2 \) respectively, and vanish due to periodic boundary conditions, so that \( J \propto (\tau^c)^2 \). This allows us to replace \( 1 - \exp(-\phi(L)) \) in Eq. (10) by the small quantity \( \phi(L) \). The current density is then given, to leading order in \( \tau^c \), by:

\[
J = (\tau^c)^2 \frac{N \int_0^L dx W^2(x) W''(x)}{\int_0^L dx e^{U(x)/\gamma D} \int_0^L dx e^{U(x)/\gamma D}} \tag{19}
\]

A non-zero drift current density in a ratchet is generated because the following two quantities don’t vanish: (a) correlation time \((\tau^c \neq 0)\), which signifies a nonequilibrium noise that breaks detailed balance, and (b) potential asymmetry, meaning that there exists no \( \Delta x \) such that \( U(x + \Delta x) = -U(x) \). This makes the integral in the numerator of Eq. (19) non-zero although the integrand itself is periodic.

The dynamics of the particle has three time-scales built into it:

(i) Correlation time, \( \tau^c \). This is the time governing the rate of loss of ‘memory’ in the noise. A convenient way to view the correlation time (as we establish later in this section) is an effective partitioning of the dynamics such that for times less than the correlation time the motion is ballistic, while for times greater than the correlation time the motion is diffusive.

(ii) Diffusion time, \( \tau^D = L^2 / \gamma D \). \( L \) is the typical length scale built into the potential. For a periodic potential, for example, \( L \) denotes the period. \( \tau^D \) describes the time taken to diffuse over one unit length scale \( L \) of the potential.

(iii) Drift time, \( \tau^\gamma = \gamma L^2 / U_0 \). \( U_0 \) is the maximum height of the potential, which serves as a typical energy scale in the problem. The drift time describes the amount of time for an overdamped particle to drift from the maximum to the minimum of the potential over a distance \( \sim L \), and serves as the time-scale for noise-free motion in the system.

Let us calculate \( J \) for a 1-D periodic asymmetric potential with period \( L \) and height \( U_0 \), and impose periodic boundary conditions \( P(0) = P(L) \). Using Eq. (19) for \( J \) we get the 1-D drift current as:

\[
J = \frac{(\tau^c \tau^D)^2}{(\tau^\gamma)^5} g \left( \frac{\tau^D}{\tau^\gamma} \right) \tag{20}
\]

where \( g(x) \) is a factor that depends on the geometry of the potential \( V(x) \). Fig. 4 shows the steady-state current density for a specific example of a periodic potential for varying correlation times \( \tau^c \). The current density is positive, tending to drive the particle out of the potential well.
in the direction where the restoring force $-dU(x)/dx$ is less.

A heuristic argument can illustrate the origin of the directionality of the current. For a correlation function $\langle f(t)f(0) \rangle = (D/τ^c)C(|t|/τ^c)$, one can use Langevin’s equation to get:

$$\langle r^2(t) \rangle = 2Dt \int_0^{t/τ^c} dY C(Y) - 2Dτ^c \int_0^{t/τ^c} dYY C(Y)$$
$$\quad = 2DC(0)t^2/τ \quad (t \ll τ^c),$$
$$\quad = 2D[t - μ/τ] \quad (t \gg τ^c)$$

where $μ$ is the first moment of the noise. The above scaling of the particle coordinate with time means that the particle motion can be looked upon as *ballistic* for $t \ll τ^c$ and *diffusive* for $t \gg τ^c$. In absence of correlations ($τ^c = 0$), the particle jumps over the barriers purely due to diffusive transport, and this has equal likelihood in either direction, since the barrier height is the same to the right and the left. However, for a correlated noise, there is a net drift over a time $τ^c$ that pushes the particle ballistically more to the right than the left, since the restoring force is less to the right (see Fig. 2). At the end of the drift process therefore, the particle has reached a higher elevation to the right than the left. Thereafter, the barrier height is smaller to the right, so there is a higher probability of crossing it to the right. Thus the net drift in the positive $x$ direction is a consequence of *drift assisted thermal activation, produced by the presence of correlation in the noise*.

Fig. 2 shows the steady-state probability distribution function for varying values of the correlation time $τ^c$. For white noise ($τ^c = 0$), the probability distribution is given by the Maxwell-Boltzmann result $P \propto \exp[-U(x)/γD]$. As the correlation time is cranked up, however, the probability distribution within each well of the potential progressively shifts in weight to the left. This corresponds to a non-vanishing steady-state current density $J$ in the positive direction. The direction of the current density can be obtained easily from Eq. (14), whereby the peak ($dp/dx = 0$) of the probability density is given for small correlation times ($τ^c \approx D$) by the point where $J = WP$. Since $P > 0$, therefore for positive current $J > 0$, we have $-W = dU/dx < 0$, meaning the peak shifts to the left. Note that the directionality of the ratchet motion depends on the specific nature of the correlation, and can be different depending on whether the time-correlation is incorporated into the driving force, or into a fluctuating version of the periodic asymmetric potential we have been dealing with.

The current density plotted vs. the barrier height shows a maximum, as is seen in Fig. 3. For small barrier heights, there is an equal chance of jumping over the barrier to the right and the left, so the current is small. Similarly at large barrier heights, the chance is equally small, so the current is small once again. Thus the peak in the current arises out of a competition between ratchet motion to the right, and current back-flow to the left.

One dimensional ratchet motion has been observed for
polystyrene beads in an aqueous solution driven by a sinusoidal current between lithographically patterned electrodes. In addition, ratchet motion has also been observed for beads in a “flashing potential” generated by a laser system modulated suitably by a chopper. In the flashing geometry, the beads are trapped in deep wells on the attractive potential with period noise passed through an RC low-pass filter. Imagine a colored noise generator (the simplest example is a white noise). Such drifts of turning on, switching off and turning on again leads to a slow drift of the fluorescent beads under colored noise (Fig. 3). Our set-ups can be imagined to be the same, except the driving force is neither an AC signal nor a switching potential. Instead, we imagine a static asymmetric potential generated as above, but driven by a colored noise generator (the simplest example is a white noise passed through an RC low-pass filter). Imagine a potential with period \( \sim 1 \mu m \) (\( a = 0.7 \mu m, b = 0.3 \mu m \)) and fluorescent charged polystyrene beads \( (\sim 0.07 - 1 \mu m) \) in an aqueous solution at room temperature. The energy of the potential \( U_0 \) is set to \( \approx 75 \) meV. For this set of parameters, \( \tau^D \approx 3 \) s and \( \tau^\gamma \approx 1 \) s. For a piecewise linear potential, \( f(\tau^D/\tau^\gamma) \approx \tau^D/\tau^\gamma \). Assume a bandwidth of the colored noise of \( \sim 40 \) Hz (these parameters satisfy \( \tau^c \ll \tau^\gamma \leq \tau^D \)). Then \( \tau \approx 10 \) hours. Such a slow drift of the fluorescent beads under colored noise should be readily observable with a microscope. Subsequently, measuring the current density for varying values of \( U_0 \) should generate a graph similar to Fig. 3, the position of the peak depending on the shape of the potential, but the tail varying as \( \exp(-U_0/\gamma D) \).

Let us calculate the efficiency of the ratchet process. Essentially, the energy for the directed motion is extracted from the correlated pieces of the noise force. The input power, given by the product of the force of the noise and the velocity obtained in the absence of the potential, is given by:

\[
P_{in} = \langle \gamma f \times f \rangle = \frac{\tau^\gamma U_0}{\tau^D/\tau^\gamma}.
\]

To calculate the output power, we use the generated drift velocity, and the force \(-dU/dx(x)\) required to overcome a potential barrier and get:

\[
P_{out} \approx J \times \frac{U_0}{L}.
\]

This yields an efficiency

\[
\eta = \frac{P_{out}}{P_{in}} = \frac{(\tau^\gamma)^3(\tau^D)^2}{(\tau^\gamma)^3} \exp(-\tau^D/\tau^\gamma)
\]

For the typical values cited above for our proposed experiment, this corresponds to \( \eta \approx 10^{-5} \). The transduction is not very efficient, although there have been proposals for ratchet mechanisms that perform almost with 100 percent Carnot efficiency.

V. TIME-DEPENDENT PROBLEM

The steady-state probability distribution obtained in Eq. 12 is in effect the long-term \((t \to \infty)\) solution to the full time-dependent Langevin problem. To solve for the transient response \( P(\vec{r}, t) \), let us expand the spatially periodic variables \( \vec{W} \) and \( \Theta_i \) in their Fourier modes:

\[
\vec{W} = \sum_{\vec{k}} \vec{W}_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} \text{ and } \Theta_i = \sum_{\vec{k}} \Theta_{i, \vec{k}} e^{i\vec{k} \cdot \vec{r}}.
\]

Expanding

\[
P(\vec{r}, t) = \sum_{\vec{k}} P_{\vec{k}} e^{i\vec{k} \cdot \vec{r} - i\omega_{\vec{k}} t}
\]

we get from Eq. 12:

\[
\sum_{\vec{k}} -i\omega_{\vec{k}} P_{\vec{k}} e^{i\vec{k} \cdot \vec{r} - i\omega_{\vec{k}} t} = \sum_{\vec{k}, \vec{k}'} \left[-i(\vec{k} + \vec{k}') \cdot \vec{W}_{\vec{k}} - (\vec{k} + \vec{k}')^2 \Theta_{\vec{k}'} \right] e^{i(\vec{k} + \vec{k}') \cdot \vec{r} - i\omega_{\vec{k}'} t}
\]

Changing summation variables on the right from \( \vec{k}, \vec{k}' \) to \( \vec{k}, \vec{k} + \vec{k}' \), we have:

\[
\sum_{\vec{k}} P_{\vec{k}} e^{-i\omega_{\vec{k}} t} \times \left[i\omega_{\vec{k}} e^{-i\vec{k} \cdot \vec{r}} - \sum_{\vec{k}'} i\vec{k}' \cdot \vec{W}_{\vec{k}'} e^{-i\vec{k}' \cdot \vec{r}} - \sum_{\vec{k}'} (\vec{k})^2 \Theta_{\vec{k}'} e^{-i\vec{k} \cdot \vec{r}} \right]
\]

Setting the terms in square brackets equal to zero, and performing an inverse Fourier transform, we get the dispersion relation:

\[
-i\omega_{\vec{k}} = -k^2(\Theta) - i\vec{k} \cdot (\vec{W}),
\]

where \( \langle \ldots \rangle \) denotes a spatial average over the period \( L \). Substituting back into the definition of \( P \), and including the steady-state solution as the \( t \to \infty \), i.e., the long wavelength \( (k = 0) \) limit, we get:

\[
P(\vec{r}, t) = \sum_{\vec{k}} P_{\vec{k}} \exp \left\{i\vec{k} \cdot \vec{r} - \langle \vec{W} \rangle t - k^2(\Theta) t\right\} + P_{st}(\vec{r}),
\]

as one expects for a drift-diffusion equation. Since the average effective diffusion constant \( \langle \Theta \rangle \) is positive, the probability distribution decays with time to the stable steady-state solution.

The “initial conditions” for the Fourier coefficients \( P_{\vec{k}} \) in Eq. 23 are set by the value of \( P(\vec{r}, t_0) \), where \( t_0 \gg \tau^c \). We get \( P_{\vec{k}} = \int d^3r P(\vec{r}, t_0) \exp \left[-\vec{k} \cdot \vec{r} + k^2(\Theta) t_0\right]/L^3 \). At times less than \( \tau^c \), memory effects become important,
where the temporal evolution is non-Markovian and the corresponding equation for the probability distribution $P(x,t)$ cannot be truncated to second derivatives in $x$ to construct the bona-fide FPE. This means that for small times the transient equation is not of a drift–diffusion form, but depends in fact on higher correlations. This disallows the use of the FPE structure to start from an initial condition at $t = 0$ and propagate to steady-state. However using an intermediate distribution at $t_0$ with $t_0 \gg \tau^c$ is allowed. In order to obtain this intermediate distribution from an initial condition at $t = 0$, one actually needs to work with Eq. 23, which has a more complicated time-dependence and associated memory effects built into it. However, for times much larger than the lifetime of the memory effects ($\sim \tau^c$), Eq. 28 should work well. For times larger than the correlation time $\tau^c$, the approach to equilibrium is governed by the diffusion constant averaged over a potential period, $\langle \Theta \rangle$, while the associated drift is governed by the corresponding average potential force, $\langle W \rangle$, which is zero for a periodic potential.

VI. MULTIDIMENSIONAL ANALYSIS; ROTATIONS AND PATTERNS

Let us generalize our results of the previous sections to higher dimensions. We consider a periodic potential so that $U(\vec{r})|_{r_i=0} = U(\vec{r})|_{r_i=L_i}$, where $L_i$ is the period of the potential along the $i$th direction. Imposing periodic boundary conditions along the $i$th direction and integrating Eq. (12) leads to an integral equation for $J_i$:

$$\int_0^{L_i} dr_i J_i(\vec{r}) e^{-\phi_i(\vec{r})} = \left[ P(\vec{r}) \Theta_i(\vec{r}) \right]_{r_i=0} \times \left[ 1 - e^{-\phi_i(\vec{r})} \right]_{r_i=L_i}$$

where

$$\phi_i(\vec{r}) \equiv \int_0^{r_i} dz_i [W_i(\vec{z})/\Theta_i(\vec{z})],$$

In one-dimensional, steady-state implies constant current density, which allows us to pull $J$ out of the integrals (as shown in Eq. 13) and solve for it, with the boundary value of $P(\vec{r})$ at $r_i = 0$ and $L_i$ being fixed by normalization. The situation is quite different in multi-dimensions. At steady-state, the current density is not a constant in general. However, we can still make a few observations that lead to nontrivial conclusions: (i) the right hand side of Eq. 28 is in general not identically zero, (ii) the integrand on the left hand side is a product of $J_i(\vec{r})$ and a positive definite quantity, and (iii) definition of steady state ($\dot{P}(\vec{r},t) \equiv 0$) implies $\nabla \cdot \vec{J}(\vec{r}) \equiv 0$. The first two observations imply that $\vec{J}$ cannot be identically zero everywhere. Combining this with the third observation leads to the unavoidable conclusion that $\nabla \times \vec{J}$ is not identically zero over one period of the potential (excluding the trivial case where $\vec{J}$ is a constant vector). In other words, there necessarily are local rotational patterns.

The necessity of color and potential asymmetry in our arguments is now easily seen. Analogous to Eq. 13 of the one-dimensional case, for small correlation times $\tau^c$, we get:

$$\phi_i(L_i) = \int_0^{L_i} dz_i \frac{W_i(\vec{z})}{D_i} \left[ 1 - \mu_i^1 \tau^c M_i(\vec{z}) + \frac{(\tau^c)^2}{2} \left[ (\mu_i^1 R(\vec{z}) - \mu_i^2 M(\vec{z}))_{ii} + 2(\mu_i^1 M_{ij}(\vec{z}))^2 \right] + O((\tau^c)^3) \right].$$

Terms of zeroth and first order in $\tau^c$ in Eq. 31 are zero since the integrands in them are proportional to exact derivatives (of $U$ and $W^2$ respectively), and vanish due to periodic boundary conditions. Hence, $\phi_i(L_i) \propto (\tau^c)^2$. From Eq. 31 we see that $\phi_i(L_i) = 0$ either for white noise (i.e. when $\tau^c = 0$) or when the potential is symmetric within a single period (i.e. the net integral multiplying $(\tau^c)^2$ vanishes). If either of this holds then our observation (i) is invalidated leaving open the possibility that $J_i(\vec{r}) = 0$ everywhere.

Having established the existence of rotations, we need to solve Eq. 29 numerically in $n$-dimensions with given boundary conditions to get specific flow patterns for $\vec{J}(\vec{r})$. For the purpose of illustrations, we specifically adopt the following simplifications: (a) we concentrate on two-dimensions ($i = x,y$), where Eq. 29 becomes:

$$\phi_x(L_x,y) = \int_0^{L_x} dx W(x,y) \left[ \frac{\mu_x^2}{2} \left( \frac{\partial W_x}{\partial y} \right)^2 - \frac{3\mu_x^2}{4} \frac{\partial^2 W_x}{\partial x^2} - \frac{\mu_x^2 W_y^2}{2} \frac{\partial^2 W_x}{\partial x \partial y} \right]$$

(b) Next we will consider the specific case of exponential correlation function ($\mu_x^1 = 1$, $\mu_x^2 = 2$); (c) finally, we consider a restricted class of potentials and noise such that $U(x,y) = U(y,x)$, $D_x = D_y$ and $\tau^2 = \tau_y^2$.

At steady-state, this implies that $J_x(x,y) = J_x(y,x)$ and $J_y(x,y) = J_y(x,y)$, i.e. the $x$ component of vector $\vec{J}(x,y)$ is only a function of $y$ and its $y$ component is only a function of $x$. Before proceeding we emphasize that our arguments (following Eq. 29) showing the necessary existence of rotations do not depend on this restricted class of noises and potentials; the class of potentials is adopted just to simplify the algebra for illustrative purposes. The calculation of $J_x(y)$ and $J_y(x)$ can now proceed smoothly by observing that $J_x$ (or $J_y$) may be pulled out of the integral in Eq. 29. The total current density $\vec{J}(\vec{r})$ is then obtained by solving the set of Eqs. 29 - 32. The current density $J_x(y)$ depends on the Dirichlet boundary conditions $P(0,y)$, which we will set to constant ($P(0,y) = P(x,0) = constant$), since it allows us to get simple flow patterns.
Flow pattern Coupling mechanism
Rotation Ratchets coupled in x and y
Laminar flow Decoupled ratchets in x and y
Rotation + net drift Coupled ratchets asymmetric under $x \leftrightarrow -x$, $y \leftrightarrow -y$

Table I shows the three general cases of multidimensional flows which may be created by appropriate choice of the potential. Figures 4-7 show contour plots of different potentials and their corresponding current densities $\vec{J}$. The first case is shown in Fig. 4 which shows that for non-separable potentials one gets in general rotational vortices. This demonstrates the breaking of rotational symmetry of the system by construction of an appropriate ratchet potential. Notice that the total circulation along the boundary is zero, which means that although there are local circulation patterns, the global average is zero. This is a consequence of periodic boundary conditions that we use. Global rotations will need a net rotation along the boundary, which is a different set of boundary conditions (similar to that in Fig. 4) than we are analyzing here. The absence of any global circulation causes the current circulation patterns to come in vortex-antivortex pairs.

Fig. 5 shows how in a separable potential one gets laminar flow. By adjusting the relative magnitude of the potential terms involving the x and y coordinates one can change the angle the laminar flow makes with the X axis. Thus we can obtain the breaking of reflection symmetry about the two axes x and y, with this flow pattern. Notice that although there are local swings in the current density, there are no vortex-antivortex pairs, since only the relative magnitudes (but not the signs) of $J_x(y)$ and $J_y(x)$ change.

For a combination of separable and non-separable po-
FIG. 7: 3-D flow patterns in the potential $U(x, y, z) = A [\sin x \sin y \sin z - \alpha \sin (2x) \sin (2y) \sin (2z)]$ for $A = 1$, $\alpha = 1$. The 3-D potential leads to a rich structure of loops and vortices (left). In the panels to the right, components $J_x$ and $J_y$ of the current are plotted at various $z$ slices, corresponding to heights $z/2\pi = (a) 0.2$ (top left), (b) 0.4 (top right), (c) 0.6 (bottom left) and (d) 0.8 (bottom right). Note that the sizes of the arrows show the relative magnitudes of the $J_x, J_y$ currents within each panel, and should not be compared for varying heights (such a comparison is shown in the 3-D plot on the left).

tentials as shown in Fig. 6, one gets equivalent combinations of patterns i.e. one can generate net drifts along with local rotations. Fig. 7 shows a flow pattern for a 3-dimensional potential, shown as a 3-D plot, and as 2-D plots at different $z$ slices. Infinitely many combinations of such potentials can be generated to break different kinds of symmetries in higher dimensions as shown in Figs. (4-7). As an example, in 3-D one could have a 1-D ratchet along the $z$-direction completely decoupled from a rotating 2-D ratchet in the $x$-$y$ plane. This would give rise to a helix with its axis along the $z$-direction.

Finally, we note that we have used Dirichlet boundary conditions with specified $P(x, 0)$ and $P(0, y)$ functions (assumed to be constant), in order to get our flow patterns. We could use Neumann boundary conditions as well, by rewriting $P(0, y)$ in $J_x(y)$ in terms of the normal derivative $\partial P(x, y)/\partial x|_{x=0}$. This is done using the definition of $\bar{J}$ from Eq. 4 and in Eq. 29 setting $r_i = x$:

$$P(0, y) = \frac{1}{\chi(y)} \Theta_x(0, y) \frac{\partial P(x, y)}{\partial x} \bigg|_{x=0}$$

where

$$\chi(y) = \left( W_x(x, y) - \frac{\partial \Theta_x(x, y)}{\partial x} \right)_{x=0} - \Theta_x(0, y) \left[ 1 - e^{-\psi_x(x, y)} \right] \int_0^{L_x} dxe^{-\phi_x(x, y)}$$

and an analogous equation for $P(x, 0)$ in terms of $\partial P(x, y)/\partial y|_{y=0}$.

VII. CONCLUSIONS

We have shown that a Fokker-Planck Equation may be derived for weak colored noise for a bounded potential in multiple dimensions. Using this we have demonstrated that an asymmetric periodic potential with a colored noise in $n$ dimensions will necessarily lead to breaking of many types of symmetries of the particle motion. Specific examples have been solved for laminar flow and laminar flow combined with rotations in two and three dimensions. Such symmetry-breaking in higher dimensions should also be readily generalizable to other types of noise statistics, such as a non-Gaussian probability distribution, and other types of ratchet potentials. This could include time-dependent (“flashing”) ratchet potentials 25, as well as discrete versions such as multidimensional Parrondo’s games 26.

VIII. ACKNOWLEDGMENTS

We would like to thank O. Pierre-Louis for suggesting the problem, and C. Jayaprakash, F. Julicher, J. W. Wilkins, S. Datta, D. Basu and S. S. Khare for useful discussions.
IX. APPENDIX (A) : MATHEMATICAL DETAILS

(i) Lemma: \( \delta N/\delta f_k(t) = 0 \)
Proof:

\[
\frac{\delta N}{\delta f_k(t)} = -N^2 \frac{\delta (1/N)}{\delta f_k(t)}
\]

(35)

From the normalization condition \( \int P[\vec{f}] \mathcal{D}\vec{f} = 1 \) and Eq. [2] we get

\[
\frac{1}{N} = \int \mathcal{D}\vec{f} \exp \left[ -\frac{1}{2} \int \int dsds' \sum_{ij} K_{ij}(s-s') f_i(s) f_j(s') \right].
\]

(36)

Hence we get

\[
\frac{\delta N}{\delta f_k(t)} = -N^2 \int \mathcal{D}\vec{f} \exp (\ldots) \left[ -\frac{1}{2} \int \int dsds' \sum_{ij} K_{ij}(s-s') \right. \\
\times \left. \left\{ \delta_{ik} \delta(t-s) f_j(s') + \delta_{jk} \delta(t-s') f_i(s) \right\} \right]
\]

\[
= N^2 \int \mathcal{D}\vec{f} \exp (\ldots) \sum_i \int ds (f_i(s)/2) \\
\times [K_{kk}(t-s) + K_{kk}(s-t)]
\]

\[
= (N/2) \sum_i \int ds [K_{kk}(t-s) + K_{kk}(s-t)]
\]

\[
\times \int \mathcal{D}\vec{f} P[\vec{f}] f_i(s)
\]

(37)

where the second integral on the last line can be rewritten as \( \langle f_i \rangle \), which is zero since \( f_i \) is a noise. This completes the proof.

As a corollary to the above, we obtain the following two equations:

\[
\frac{\delta P[\vec{f}]}{\delta f_k(t)} = -\sum_i \int ds K_{ik}(t-s) f_i(s) P[\vec{f}]
\]

(38)

\[
\frac{\delta^2 P[\vec{f}]}{\delta f_k(t) \delta f_l(t')} = -K_{kl}(t-t') P[\vec{f}]
\]

(39)

\[
\sum_{ij} \int \int dsds' K_{ik}(t-s) f_i(s) K_{jl}(t'-s') f_j(s') P[\vec{f}]
\]

\[
= \int ds' K_{kl}(t'-s') C_i([s-s']/\tau_i^0) = \tau_i^0 \delta_{i0} \delta(t-s')/D_i + \int ds' K_{kl}(s-s') C_i([t'-s']/\tau_i^0).
\]

This equation establishes that \( K \) is a diagonal matrix, whose inverse gives \( C \). Differentiating the normalization equation \( 1 = \int \mathcal{D}\vec{f} P[\vec{f}] \), we get:

\[
0 = \frac{\delta^2}{\delta f_k(t) \delta f_l(t') \delta f_l(t')} \int \mathcal{D}\vec{f} P[\vec{f}]
\]

\[
= \int \mathcal{D}\vec{f} \frac{\delta^2 P[\vec{f}]}{\delta f_k(t) \delta f_l(t') \delta f_l(t')}
\]

(40)

Rewriting \( K_{kl}(t-t') \) as \( \sum_i \int ds K_{ik}(t-s) \delta_{ij}(s-t') \), and using \( \langle f_i(s)f_j(s') \rangle = \delta_{ij} (D_i/\tau_i^0) C_i([s-s']/\tau_i^0) \), leads immediately to the above proof.

X. APPENDIX (B) : CONVERGENCE ISSUES

It is important to establish the validity of various expansions that we do in the correlation time at various stages of the derivation of Eq. [12]. In particular, the current distribution may not necessarily be an analytic function of \( \tau_c \) for an arbitrary noise-source, in which case any perturbative expansion in \( \tau_c \) yields results that are wrong. The non-analyticity can arise from three possible sources: (i) the potential itself may be non-analytic, (ii) the noise statistics has a finite support, (iii) the perturbative expansion may be of a non-analytic function of \( \tau_c \).

Case(i) A piecewise linear sawtooth potential has infinite derivatives at the kinks, leading to potential divergent terms in the current. This can be avoided in principle by restricting our arguments to a smoothened function.

Case(ii) The derivation of the Fokker-Planck structure itself depends crucially on the statistics of the noise. If the noise distribution has a finite support so that arbitrarily large noise amplitudes are excluded from consideration, then any computation of the current density along the lines we prescribed would be totally wrong. For example, if the height of the potential barrier is larger than the maximum allowed noise amplitude, then there will be no current, contrary to what an injudicious application of the formalism will yield. In our analyses, we have assumed a Gaussian distribution function for the noise (Eq. [2]). This has an infinite support and thereby avoids such non-analyticities [24]. However, for a discrete noise process such as dichotomous noise, care must be exercised in obtaining the Fokker-Planck description. Often, an additional white noise source is included with the explicit purpose of handling such non-analyticities. We do not need such sources since our probability distribution is Gaussian.

Case(iii) The functional calculus as outlined by Fox and extended by us to several dimensions is non-perturbative in the correlation time. As argued by Fox in Ref. [13], the prescription leads to currents that are
uniformly convergent for $\tau^c = 0$. The non-perturbative description leads us to Eq. [1] that involves the exponential of integrals of matrix elements of $M$. We finally performed an explicit evaluation of the matrix elements in Eq. [11] for small $\tau^c$. This does involve a perturbative expansion in $\tau^c$, but of an exponential function, analytic in $\tau^c$. The $n$th term of the expansion is proportional to $\mu_n (\tau^c / \tau^\gamma)^n / n!$. For well-behaved correlation functions such as an exponential or a Gaussian, this term tends to zero rapidly as $n$ increases to infinity, provided $\tau^c < \tau^\gamma$, as we have assumed.

[1] K. Huang, Statistical Mechanics (Wiley, New York, 1963).
[2] R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures on Physics, Vol. 1, chapter 46, Addison Wesley, Reading MA, 1963.
[3] For an overview and references, see P. Reimann, Phys. Rep. 361, 57 (2002) and also D. Astumian and P. Hänggi, Phys. Today 55, 33 (November 2002).
[4] F. Jülicher and J. Prost, Phys. Rev. Lett. 75, 2618 (1995).
[5] F. Jülicher, A. Ajdari and J. Prost, Rev. Mod. Phys. 69, 1269 (1997); A. B. Kolomeisky and B. Widom, J. Stat. Phys. 93, 633 (1998); M. E. Fisher and A. B. Kolomeisky, Proc. Natl. Acad. Sci. USA 98, 7748 (2001).
[6] J. Maddox, Nature 365, 203 (1993).
[7] M. O. Magnasco, in Fluctuations and Order: The New Synthesis Ed. M. Millonas, Springer-Verlag, 1994.
[8] R. D. Astumian and M. Bier, Biophys. J. 70, 637 (1990); C. S. Peskin, G. B. Ermentrout and G. F. Oster in Cell Mechanics and Cellular Engineering, Ed. V. C. Mov, F. Guilak, R. Tran-Son-Tay and R. M. Hochmuth, Springer, New York, 1994.
[9] I. Derémy and T. Vicsek, Proc. Natl. Acad. Sci. USA 93, 6775 (1996).
[10] A. van Oudenaarden and S. G. Boxer, Science 285, 1046 (1999); I. Derényi and R. D. Astumian, Phys. Rev. E 58, 7781 (1998).
[11] F. Marchesoni, Phys. Lett. A 237, 126 (1998).
[12] I. Derényi and T. Vicsek Physica A 249, 397 (1998).
[13] J. Rousselet et al., Nature (London), 370, 446 (1994); L. P. Faucheux et al., Phys. Rev. Lett. 74, 1504 (1995).
[14] I. Zapata, R. Bartussek, F. Sols and P. Hänggi, Phys. Rev. Lett. 77, 2292 (1996).
[15] G. W. Slater, H. L. Guo, and G. I. Nixon, Phys. Rev. Lett. 78, 1170 (1997); D. Ertas, ibid 80, 1548 (1998); T. A. J. Duke and R. H. Austin, ibid 80, 1552 (1998); J. F. Wambaugh, C. Reichhardt, C. J. Olson, F. Marchesoni, and F. Nori, ibid 83, 5106 (1999); A. Lorke, S. Wimmer, B. Jager, J. P. Kotthaus, W. Wegscheider and M. Bichler, Physica B 249, 312 (1998); C. Keller, F. Marquardt, and C. Bruder, Phys. Rev. E 65, 041927 (2002); C. S. Lee, B. Janko, I. Derényi, and A. L. Barabasi, Nature 400, 337 (1999).
[16] J. K. Gimzewski, C. Joachim, R. R. Schüttler, V. Langlais V, H. Tang H, and I. Johannsen Science 281, 531 (1998); T. R. Kelly, H. DeSilva, and R. A. Silva, Nature 401, 150 (1999).
[17] H. Qian, Phys. Rev. Lett. 81, 3063 (1998).
[18] A. W. Ghosh and S. V. Khare, Phys. Rev. Lett. 84, 5243 (2000).
[19] R. F. Fox, Phys. Rev. A 33, 467 (1986), R. F. Fox, Phys. Rev. A 34, 4525 (1986).
[20] I. M. Sokolov, cond-mat/0002251.
[21] M. O. Magnasco, Phys. Rev. Lett. 71, 1477 (1993).
[22] As an alternative to calculating the probability distribution $P[\vec{f}]$ as in Eq. (2), one could write a dynamic equation for $\vec{f}(t)$ having the form: $d\vec{f}/dt = -A\vec{f} + B\vec{\zeta}(t)$, where $\vec{\zeta}(t)$ is a white noise. The solution $\vec{f}(t)$ represents a diffusion process, although it is singular in terms of the diffusion tensor. The rigorous mathematics for the singular diffusion has been recently studied by J. -P. Eckmann and M. Hairer, Comm. Math. Phys. 219, 523 (2001).