Classical $N = 1$ and $N = 2$ super W-algebras from a zero-curvature condition

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Abstract
Starting from superdifferential operators in an $N = 1$ superfield formulation, we present a systematic prescription for the derivation of classical $N = 1$ and $N = 2$ super W-algebras by imposing a zero-curvature condition on the connection of the corresponding first order system. We illustrate the procedure on the first non-trivial example (beyond the $N = 1$ superconformal algebra) and also comment on the relation with the Gelfand-Dickey construction of $W$-algebras.

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1. Introduction

It is well known how to derive classical W-algebras from (pseudo) differential operators by virtue of the second Gelfand-Dickey Hamiltonian structure [1]. This procedure has recently been generalized to the supersymmetric case [2] (see also [3]). Here we will generalize another construction which starts from the same differential operators. This method amounts to describing a $n$-th order operator in terms of a system of first order operators and imposing a zero-curvature condition on the associated connection (and its anti-analytic partner). This has previously been done for the case of $sl(n)$-valued connections [4]. Here, we will provide the necessary algebraic tools for the $N = 2$ super W-algebras (SWA’s) corresponding to the super Lie algebras $sl(n+1|n)$ and for the $N = 1$ SWA’s corresponding to $osp(2m \pm 1|2m)$. The connection between the super W- and the super Lie algebras [5][6][7] will become transparent in the discussion. The case of $osp(3|2)$ will be presented in detail in section 6 where we also exhibit the relation between the zero-curvature construction of (super) W-algebras and the Gelfand-Dickey approach. In Appendix A we develop the superspace formulation of gauge theories based on graded Lie algebras in order to give a precise geometric meaning to the zero-curvature condition which represents the starting point of our analysis.

2. Super differential operators

To start with, we consider the most general superconformally covariant differential operator of order $2n + 1$ acting on superconformal (primary) fields. As shown in reference [3], it may be cast into the form

$$L^{(n)} = D^{2n+1} + a_2 D^{2n-1} + \ldots + a_{2n+1},$$

(1)

where $D = \partial_\theta + \theta \partial$ and $\partial = \partial_z$. Under a superconformal change of the local coordinates $(z, \theta)$, the coefficient functions $a_k(z, \theta)$ transform in such a way that

$$L^{(n)} : \mathcal{F}_{-n} \rightarrow \mathcal{F}_{n+1};$$

where $\mathcal{F}_k$ represents the space of primary fields of weight $\frac{k}{2}$. The $a_k$ are even (odd) for $k$ even (odd). More specifically, $a_2$ belongs to $\mathcal{F}_2$ and $a_3 - \frac{1}{2} D a_2$ transforms like a superprojective connection (i.e. like the energy momentum tensor). The remaining coefficients $a_k$ ($k \geq 4$) transform in a more complicated way. However, one can express the operator $L^{(n)}$ entirely in terms of fields $V_2, V_3, \ldots, V_{2n+1}$ where $V_3$ transforms like the energy momentum tensor and where all the other $V_k$ are primary fields of weight $\frac{k}{2}$. The
relation between the $a_k$ and the $V_k$ is most easily obtained by rewriting the superdifferential equation

$$L^{(n)} f = 0 \quad (f \in F_{-n}),$$

which is of order $2n + 1$, as a system of first order differential equations. For the form (1) of the operator $L^{(n)}$, equation (2) is equivalent to

$$\nabla' F' = 0,$$

where

$$\nabla' \equiv D - A'(a_k) \equiv \begin{pmatrix} D & a_2 & a_3 & \ldots & a_{2n+1} \\ -1 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & -1 & D \end{pmatrix}, \quad F' = (D^{2n} f, D^{2n-1} f, \ldots, f)^T.$$

(Here and below the non-displayed matrix elements are assumed to be zero.) On the other hand, if $L^{(n)}$ is parametrized in terms of the fields $V_k$, equation (2) is equivalent to

$$\nabla F = 0,$$

where

$$\nabla \equiv D - A(V_k) \equiv \begin{pmatrix} D \\ -1 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & -1 \\ -1 & D \end{pmatrix}, \quad F = (f_1, \ldots, f_{2n+1})^T, \quad f_{2n+1} \equiv f.$$

The upper triangular part in $A(V_k)$ denoted by ‘$V'$s’ will be made explicit in a concrete example in section 6. The superfield $A$ takes its values in the super Lie algebra $sl(n+1|n)$ equipped with the diagonal grading, which means the following [3]. The graded algebra $sl(n+1|n)$ consists of $(2n+1) \times (2n+1)$ supermatrices with vanishing supertrace (e.g. see [8]). The standard grading is defined in terms of even and odd blocks: $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A_{(n+1) \times (n+1)}$ and $D_{n \times n}$ even and $B$ and $C$ odd. Supertracelessness means $0 = sTr M = Tr A - Tr D$. However, from the expression for $A$ we see that the natural grading for our discussion is not the standard one, but rather a grading by diagonals, i.e. we have
even and odd diagonals. The supertrace is then given by $\text{sTr} M = \sum_i (-)^i M_{ii}$ and the graded commutator by

$$[M, N]_{ik} = \sum_j \left( M_{ij} N_{jk} - (-)^{(i+j)(j+k)} N_{ij} M_{jk} \right).$$

It is straightforward (at least in principle) to express the $V_k$ in terms of the $a_k$ and vice versa. In fact, by virtue of the equation $\nabla \vec{F} = 0$ we can express $\vec{F}$ in terms of $\vec{F}'$ as $\vec{F} = X \vec{F}'$ where $X$ belongs to the supergroup $SL(n+1|n)$. Then $\nabla' = \hat{X}^{-1} \nabla X$ where the hat means that the signs of all odd generators in $X$ have been reversed. From now on, we will always work with $\nabla$ rather than $\nabla'$. Moreover, we will always assume that the coefficient functions $V_k$ are supersmooth rather than superholomorphic. (This assumption does not affect the transformation properties, since the operator $\mathcal{L}^{(n)}$ only contains derivatives with respect to $z$ and $\theta$.)

### 3. Zero-curvature condition

In order to impose a zero-curvature condition on the connection, we need to pair the connection component $A_\theta \equiv A$ with components $A_\bar{\theta} \equiv \bar{A}$ and $A_z, A_{\bar{z}}$. These smooth superfields all take their values in $sl(n+1|n)$ or a subalgebra thereof. The component $A$ will always be assumed to be in the so-called highest weight gauge form. We will see that choosing $\bar{A} \in sl(n+1|n)$ will lead to $N = 2$ SWA’s and choosing $\bar{A} \in osp(2m\pm 1|2m) \subset sl(n+1|n)$ with $4m\pm 1 = 2n+1$ will lead to $N = 1$ SWA’s. As usual in supergauge theories, the spatial components $A_z$ and $A_{\bar{z}}$ of the connection depend on the other components, $A$ and $\bar{A}$, by virtue of redefinition constraints (see Appendix A) and therefore we will not consider them in the sequel.

If we simultaneously consider the two first-order systems $\nabla \vec{F} = 0$ and $\nabla' \vec{F} = 0$, i.e.

$$D \vec{F} = A \vec{F}, \quad D' \vec{F} = \bar{A} \vec{F},$$

we have to require their compatibility, i.e. $\{D, D'\} \vec{F} = 0$. In order to investigate the implications of this relation, we decompose $A \equiv A_\theta$ and $\bar{A} \equiv A_{\bar{\theta}}$ along the even and odd basis elements ($t_a$ and $t_\alpha$) of the super Lie algebra under consideration:

$$A = A^a t_a + A^\alpha t_\alpha, \quad \bar{A} = \bar{A}^a t_a + \bar{A}^\alpha t_\alpha.$$  

* If $X = \exp x$ with $x \in sl(n+1|n)$, then $\hat{X} = \exp \hat{x}$. 

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The superfields $A^a$ and $\bar{A}^a$ have odd parity while $A^\alpha$ and $\bar{A}^\alpha$ have even parity. Since $\bar{D}$ is odd, we get

$$\bar{D}(D\bar{F}) = (\bar{D}A)\bar{F} + (-A^a t_a + A^\alpha t_\alpha)\bar{D}\bar{F}$$

$$= (D\bar{A})\bar{F} - \bar{D}F$$

$$= (D\bar{A} - \hat{A}\bar{A})\bar{F}. \quad (9)$$

Thus the integrability condition $\{D, \bar{D}\}\bar{F} = 0$ is equivalent to the relation

$$D\bar{A} + D\bar{A} - \hat{A}\bar{A} - \hat{A}\bar{A} = 0. \quad (10)$$

This equation represents a zero-curvature condition for the connection since it corresponds to the vanishing of the curvature component $\mathcal{F}_{\theta\theta}$ (see Appendix A).

4. The superalgebra $sl(n+1|n)$

In this section we briefly summarize some results which are relevant for our subsequent discussions. The Cartan matrix $(a_{ij})$ of $sl(n+1|n)$ has the following non-vanishing elements $[5][7]$:

$$a_{i,i+1} = a_{i+1,i} = (-)^{i+1} \quad (i = 1, \ldots, 2n). \quad (11)$$

A basis for the Chevalley generators with diagonal grading and with all simple roots fermionic is given by $[3]$

$$h_i = (-)^{i+1}(E_{i,i} + E_{i+1,i+1})$$

$$e_i = (-)^{i+1}E_{i,i+1}$$

$$f_i = E_{i+1,i}, \quad (12)$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. A general element of $sl(n+1|n)$ with diagonal grading is then simply a $(2n+1) \times (2n+1)$ matrix of numbers together with an even (odd) grading for the even (odd) diagonals, the only restriction being the vanishing of the supertrace.

The principal embedding $osp(1|2)_{pal} \subset sl(n+1|n)$ is given by $[7][3]$

$$J_- = \sum_i f_i = \text{diag}_{-1}(1, \ldots, 1)$$

$$J_+ = \sum_{ij} a^{ij}e_i = \text{diag}_1(n, -1, n-1, -2, \ldots, -n) \quad (13)$$

$$H = \{J_+, J_-\} = \text{diag}_0(n, n-1, \ldots, -n),$$

where we count the main diagonal as the 0-th one. We have the commutation relations

$$[H, e_i] = e_i, \quad [H, f_i] = -f_i.$$
The highest weight generators of $osp(1|2)_{pal}$ are defined as those $M_k \in sl(n+1|n)$ which satisfy

$$[H, M_k] = kM_k \quad \text{and} \quad [J_+, M_k] = 0 .$$

For the solutions of these conditions one finds $M_k = M_1^k$ with

$$M_1 = \text{diag}_{+1}(n, 1, n-1, 2, \ldots, n) \quad (M_1^{2n+1} = 0 ) .$$

Henceforth \[3\], the super Lie algebra $sl(n+1|n)$ is associated with a connection (in highest weight gauge)

$$-A_{N=2} = -J_+ + \sum_{k=1}^{2n} V_{k+1} M_k ,$$

where $V_3$ is a superprojective connection and $V_k \in F_k$ for $k = 2, 4, \ldots, 2n + 1$. The first order system $D - A_{N=2}$ is equivalent to a covariant superdifferential operator of order $2n + 1$ of the form \[1\] without any restrictions on its coefficients.

5. The superalgebras $osp(2m \pm 1|2m)$

For the discussion of $N = 1$ SWA’s, the graded algebras $osp(2m \pm 1|2m) \subset sl(n+1|n)$ with $4m \pm 1 = 2n + 1$ are relevant \[3\]. In order to specify the zero-curvature condition to this case, we need a general characterization of the elements of these algebras in diagonal grading. This will be provided in the following. Some more details concerning the diagonal grading representation of $osp(2m \pm 1|2m)$ are presented in Appendix B.

In a basis of the root space with only fermionic simple roots, the algebras $osp(2m \pm 1|2m)$ are characterized by Cartan matrices $a_{ij}$ whose only non-zero elements are \[3\]

$$a_{11} = 1 , \quad a_{i,i+1} = a_{i+1,i} = (-)^i ,$$

where $i = 1, \ldots, 2m$ for $osp(2m + 1|2m)$ and $i = 1, \ldots, 2m - 1$ for $osp(2m - 1|2m)$. The Chevalley generators can be chosen as follows:

$osp(2m + 1|2m)$:

$$h_i = (-)^{i+1} \left( E_{2m+1-i,2m+1-i} - E_{2m+1+i,2m+1+i} + E_{2m+2-i,2m+2-i} - E_{2m+i,2m+i} \right)$$

$$e_i = (-)^i \left( E_{2m+i,2m+1+i} - E_{2m+1-i,2m+2-i} \right)$$

$$f_i = E_{2m+1+i,2m+i} + E_{2m+2-i,2m+1-i}$$

(18)
\(osp(2m - 1|2m)\):

\[
h_i = (-)^{i+1} \left( E_{2m-i,2m-i} - E_{2m+i,2m+i} + E_{2m+1-i,2m+1-i} - E_{2m-1+i,2m-1+i} \right)
\]
\[
e_i = (-)^i \left( E_{2m-1+i,2m+i} - E_{2m-i,2m+1-i} \right)
\]
\[
f_i = E_{2m+i,2m-1+i} + E_{2m+1-i,2m-i}
\]

They satisfy the algebra

\[
[h_i, e_j] = a_{ij} e_j
\]
\[
[h_i, f_j] = -a_{ij} f_j
\]
\[
\{e_i, f_j\} = \delta_{ij} h_j.
\]

A general \(osp(2m \pm 1|2m)\) element in diagonal grading is then a \((4m \pm 1) \times (4m \pm 1)\) matrix satisfying the following conditions:

\[
M_{i,i+2k} = (-)^{k+1} M_{p+1-i-2k,p+1-i}
\]
\[
M_{i,i+2k+1} = (-)^{k+1} M_{p-i-2k,p+1-i}
\]

\((p = 4m \pm 1; \ i = 1, \ldots, p; \ k = 0, \pm 1, \ldots)\).

The principal embedding of \(osp(1|2)_{\text{pal}} \subset osp(2m \pm 1|2m)\) is again given by

\[
J_- = \sum_i f_i \quad J_+ = \sum_{ij} a_{ij} e_i \quad H = \{J_+, J_-\}.
\]

The bases (18) and (19) have been chosen such as to lead to the same explicit expressions for \(J_\pm\) and \(H\) as in the \(sl(n+1|n)\) case, see eq. (13). The highest weight generators are also the same, except that we have to exclude those \(M_k\) which do not belong to \(osp(2m \pm 1|2m)\). This leaves us with \(M_{2,3(\text{mod} \ 4)}\). The \(\theta\)-component of an \(osp(2m \pm 1|2m)\)-valued connection in highest weight gauge is thus given by

\[
-A_{N=1} = -J_- + \sum_{k=2,3(\text{mod} \ 4)} V_{k+1} M_k.
\]

The number of fields occurring in \(A_{N=1}\) equals the rank of the corresponding algebra, namely \(2m\) for \(osp(2m + 1|2m)\) and \(2m - 1\) for \(osp(2m - 1|2m)\). Note that in contrast to the \(sl(n+1|n)\) case there is no superfield \(V_2\) of weight one. We are thus dealing with higher spin extensions of the \(N = 1\) superconformal algebra. The \((2n+1)\)-th order operator associated to the corresponding first order system is symmetric in the sense of ref. [6] (thus allowing for the factorization considered in eq. (37) below).
6. An example: \(osp(3|2)\)

**General set-up:** The simplest \(N = 1\) SWA occurs for the algebra \(osp(1|2)\); in this case the zero-curvature condition \((10)\) yields the \(N = 1\) superconformal Ward identity which is easily integrated to give the \(N = 1\) superconformal algebra \([11]\). Instead of this very simple example, we will rather consider the algebra \(osp(3|2) \subset sl(3|2)\) to illustrate the procedure outlined above. For \(sl(3|2)\), the operator \((3)\) takes the form

\[
\nabla = \begin{pmatrix}
D & 2V_2 & 2V_3 & V_4 & V_5 \\
-1 & D & V_2 & V_3 & V_4 \\
-1 & 2V_3 & D & V_2 & V_4 \\
-1 & V_2 & 2V_3 & D & V_2 \\
-1 & D & V_2 & V_2 & D
\end{pmatrix},
\]

(23)

where \(V_k\) is even (odd) for \(k\) even (odd) \([3]\). For the matrix which generates the transformation between \(\nabla\) and \(\nabla'\) one then finds

\[
X = \begin{pmatrix}
1 & 0 & 4V_2 & D V_2 - V_3 & 2\partial V_2 + 2V_2^2 + 2D V_3 + V_4 \\
1 & 0 & 3V_2 & 2D V_2 + 2V_3 \\
1 & 0 & 2V_3 & 2V_2 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

(24)

and for the relations between the \(a_k\) and the \(V_k\) one gets the expressions

\[
a_2 = 6V_2 \\
a_3 = 3V_3 + 3D V_2 \\
a_4 = 2V_4 + D V_3 + 3\partial V_2 + 8V_2^2 \\
a_5 = V_5 + D V_4 + 2\partial V_3 + 2D^3 V_2 + 8V_2 D V_2 + 8V_2 V_3.
\]

(25)

So far we have still looked at the graded algebra \(sl(3|2)\) which would lead to the \(N = 2\) \(W_3\)-algebra of ref.\([2]\). To go over to the \(osp(3|2)\) case, we now set \(V_2 = V_5 \equiv 0\); then we have

\[
\nabla = D - J_+ + M_2 V_3 + M_3 V_4.
\]

(26)

For \(\bar{A}\) we make the most general Ansatz compatible with the \(osp(3|2)\) structure:

\[
\bar{A} = \begin{pmatrix}
\alpha_1 & a_1 & \beta_1 & b & 0 \\
c_1 & \alpha_2 & a_2 & \beta_2 & b \\
\gamma_1 & c_2 & 0 & -a_2 & \beta_1 \\
d & \gamma_2 & c_2 & -a_2 & -a_1 \\
0 & -d & \gamma_1 & c_1 & -\alpha_1
\end{pmatrix}.
\]

(27)
Here, small Latin (Greek) letters denote Grassmann even (odd) superfields (cf. eq.\((8)\)). In eq.\((27)\) we have suppressed the indices of all matrix elements. These can be determined by going back to eq.\((4)\) and noting that \(f \equiv f^z \in F_{-2}\) in the present case. This fixes the indices of all elements of \(\vec{F}'\). By virtue of the relation \(\vec{F} = X\vec{F}'\) the components of \(\vec{F}\) then have the same index structure as those of \(\vec{F}'\). In the system of equations \(0 = \vec{\nabla}\vec{F} = (\bar{D} - \bar{A})\vec{F}\), we encounter the derivatives \(\bar{D} - \alpha_1\) and \(\bar{D} - \alpha_2\) so that \(\alpha_1 = (\alpha_1)_{\tilde{\bar{\theta}}}\) and \(\alpha_2 = (\alpha_2)_{\tilde{\bar{\theta}}}\). This determines the index structure of all other elements of \(\bar{A}\):

\[
\begin{align*}
&b_{\theta z}, \quad (\beta_i)_{\tilde{\bar{\theta}}z}, \quad (a_i)_{\tilde{\bar{\theta}}\theta}, \quad (c_i)_{\tilde{\bar{\theta}}}, \quad (\alpha_i)_{\tilde{\bar{\theta}}}, \quad (\gamma_i)_{\tilde{\bar{\theta}}}^z, \quad d_{\theta z}^{\theta z} \quad (i = 1, 2) \quad .
\end{align*}
\]

The transformation laws of these variables under a superconformal change of coordinates, \((z, \tilde{z}, \theta, \tilde{\theta}) \to (\tilde{z}, \tilde{\bar{z}}, \tilde{\theta}, \tilde{\bar{\theta}})\), are derived along the same lines from the one of \(f \in F_{-2}\), i.e. from \(f \to e^{-2w}f\) where \(e^{-w} \equiv D\tilde{\theta}\).

From this transformation law we first deduce the one of \(\vec{F}'\) which fixes the one of \(\vec{F}\) by virtue of the relation \(\vec{F} = X\vec{F}'\) and of the transformation laws of the \(V_k\). One finds that \(\vec{F} \to S\vec{F}\) with

\[
S = \begin{pmatrix}
    e^{2w} & 2e^{2w}Dw & -2e^{2w}\partial w & -2e^{2w}(Dw)\partial w & 2e^{2w}(\partial w)^2 \\
    0 & e^w & -e^wDw & -e^w\partial w & 2e^w(Dw)\partial w \\
    0 & 0 & 1 & Dw & -2\partial w \\
    0 & 0 & 0 & e^{-w} & -2e^{-w}Dw \\
    0 & 0 & 0 & 0 & e^{-2w}
\end{pmatrix}.
\]

Since \(\bar{D}w = 0\), we have \((\bar{D}\vec{F}) \to e^{\bar{w}}\bar{S}(\bar{D}\vec{F})\) and therefore the condition \(\vec{\nabla}\vec{F} = 0\) is superconformally covariant if and only if the matrix \(\bar{A}\) transforms according to

\[
\bar{A} \to e^{\bar{w}}\bar{S}\bar{A}\bar{S}^{-1} \quad .
\]

The elements of \(\bar{A}\) which have the simplest transformation laws are easily seen to be \(d\) and \(\gamma_1\):

\[
d \to e^{\bar{w}}e^{-3w}d \\
\gamma_1 \to e^{\bar{w}}e^{-2w}[\gamma_1 - (Dw)d] \quad .
\]

Henceforth,

\[
\left[\gamma_1 - \frac{1}{3}Dd\right] \to e^{\bar{w}}e^{-2w}\left[\gamma_1 - \frac{1}{3}Dd\right]
\]

and we conclude that both \(d\) and \(\gamma_1 - \frac{1}{3}Dd\) transform like superconformal fields.
Ward identities: Substituting $A$ and $\bar{A}$ (as given by eqs. (26) and (27), respectively) into the zero-curvature condition (10), we obtain a set of coupled equations for $V_3$, $V_4$ and the coefficients of $\bar{A}$. Almost all of these can be solved algebraically for the coefficients of $\bar{A}$ in terms of $V_3$, $V_4$ and $\gamma_1$, $d$; one is only left with two differential equations which are in fact the Ward identities for the underlying SWA. In terms of the superconformal fields $H_3 \equiv \gamma_1 - \frac{1}{3}Dd$ and $H_4 \equiv \frac{4}{3}d$, these identities take the form

\[
\begin{align*}
\{\bar{D} - H_3 \partial + \frac{1}{2}(DH_3)D - \frac{3}{2}(\partial H_3)\}V_3 &= -\frac{1}{2}D^5H_3 + \left\{\frac{3}{2}H_4 \partial + \frac{1}{2}(DH_4)D + 2(\partial H_4)\right\}V_4 \\
\{\bar{D} - H_3 \partial + \frac{1}{2}(DH_3)D - 2(\partial H_3) - \frac{3}{4}H_4 D^3 - \frac{5}{4}(DH_4)\partial - \frac{5}{4}(\partial H_4)D - \frac{5}{2}(D^3H_4)\}V_4 &= \frac{1}{8}D^7H_4 + \left\{\frac{3}{8}H_4 \partial^2 + \frac{1}{4}(DH_4)D^3 + (\partial H_4)\partial + \frac{1}{2}(D^3H_4)D + \frac{3}{4}(\partial^2 H_4)\right\}V_3 \\
\end{align*}
\]

In the remainder of this section, we will explore these equations and the information they contain. First, we note that the superfields involved in eq. (31) have the index structure $(H_3)_\theta^z$, $(H_4)_{\bar{\theta}}^z$ and $(V_3)_\theta z$, $(V_4)_{zz}$, respectively. More specifically, $(H_3)_\theta^z$ can be identified with the super Beltrami coefficient $H_\theta^z$ parametrizing superconformal classes of metrics and $(V_3)_\theta z$ is a multiple of the superstress tensor; this correspondence is confirmed by the fact that the previous set of identities reduces to the $N = 1$ superconformal Ward identity for $H_4 = 0 = V_4$. (The latter is the anomalous Ward identity expressing the superdiffeomorphism invariance of the generating functional $Z_c[H_3]$ in superconformal field theory.)

Operator product expansions: To derive the singular parts of the operator products from the Ward identities (31), we follow reference [11] and we consider them at $Z_1 = (z_1, \bar{z}_1, \theta_1, \bar{\theta}_1)$, then multiply by $\theta_{12}/Z_{12}$ (with $\theta_{12} = \theta_1 - \theta_2$ and $Z_{12} = z_1 - z_2 - \theta_1 \theta_2$) and subsequently integrate with respect to $Z_1$. Thereafter, we replace $V_3$ by $\delta Z_c/\delta H_3$ and $V_4$ by $\delta Z_c/\delta H_4$ where $Z_c[H_3, H_4]$ denotes a generating functional. Application of the functional derivatives $\delta/\delta H_3, \delta/\delta H_4$ to the so-obtained equations at the point $H_3 = 0 = H_4$ then leads to the operator products. If we define the correspondence $\delta/\delta H_3 \sim T$, where $T$ is
the superstress tensor and $\delta/\delta H_4 \sim W_4$, where $W_4$ is a superconformal field of weight 2, we find the following expressions for the singular parts of the operator products:

\begin{align*}
T(Z_1)T(Z_2) &= \frac{1}{Z_{12}^3} + \left[ \frac{3}{2} \frac{\theta_{12}}{Z_{12}^2} + \frac{1}{Z_{12}} D_2 + \frac{\theta_{12}}{Z_{12}} \partial_2 \right] T(Z_2) \\
T(Z_1)W_4(Z_2) &= \left[ \frac{2}{2} \frac{\theta_{12}}{Z_{12}^2} + \frac{1}{2} \frac{\theta_{12}}{Z_{12}} D_2 + \frac{\theta_{12}}{Z_{12}} \partial_2 \right] W_4(Z_2) \\
-W_4(Z_1)W_4(Z_2) &= \frac{3}{4} \frac{1}{Z_{12}^4} + \left[ \frac{5}{2} \frac{\theta_{12}}{Z_{12}^2} + \frac{5}{4} \frac{\theta_{12}}{Z_{12}^2} D_2 + \frac{5}{4} \frac{1}{Z_{12}} \partial_2 + \frac{3}{4} \frac{\theta_{12}}{Z_{12}} D_2^3 \right] W_4(Z_2) \\
&\quad + \left[ \frac{3}{2} \frac{\theta_{12}}{Z_{12}^2} + \frac{1}{2} \frac{\theta_{12}}{Z_{12}} D_2 + \frac{\theta_{12}}{Z_{12}} \partial_2 + \frac{1}{4} \frac{1}{Z_{12}} D_2^3 + \frac{3}{8} \frac{\theta_{12}}{Z_{12}} \partial_2^2 \right] T(Z_2) \\
&\quad + \frac{9}{8} \frac{\theta_{12}}{Z_{12}} T(Z_2) \left[ D_2 T(Z_2) + 4W_4(Z_2) \right].
\end{align*}

The product of $T$ with $W_4$ expresses the fact that $W_4$ is a primary field of weight 2. The terms in the operator products which involve products of fields have to be regularized in the quantum version of the algebra *.

**Relation with Poisson Algebra:** Let us now go back to the identities (31) and rewrite them in the form

\[ \bar{D} \partial_i = \sum_{j=3,4} (-)^i L^{ij} \partial_j \quad (i = 3, 4) \quad (33) \]

with

\begin{align*}
L^{33} &= \frac{1}{2} \left[ D^5 + 3V_3 \partial + (DV_3)D + 2(\partial V_3) \right] \\
L^{34} &= - \left[ 2V_4 \partial - \frac{1}{2} (DV_4)D + \frac{3}{2} (\partial V_4) \right], \quad L^{43} = 2V_4 \partial - \frac{1}{2} (DV_4)D + (\partial V_4) \\
L^{44} &= \frac{1}{8} \left[ D^7 + 6V_3 \partial^2 + 4(DV_3)D^3 + 8(\partial V_3) \partial + 2(D^3 V_3)D + 3(\partial^2 V_3) + 9V_3(D^3 V_3) \right] \\
&\quad + \frac{5}{2} \left[ V_4 D^3 + \frac{1}{2} (DV_4) \partial + \frac{1}{2} (\partial V_4)D + \frac{3}{10} (D^3 V_4) + \frac{9}{5} V_3 V_4 \right].
\end{align*}

Then, the different operators $L^{ij}$ are just the operators defining the Poisson brackets between the fields $V_i$,

\[ \{V_i(Z_1), V_j(Z_2)\} = L^{ij}_{Z_1} \delta(Z_1 - Z_2), \quad (35) \]

where $\delta(Z_1 - Z_2) = (\theta_1 - \theta_2)\delta(z_1 - z_2)$. The equivalence between the Poisson algebra (34) and the operator product algebra (32) follows easily from the correspondence

\[ \frac{\theta_{12}}{Z_{12}^{r+1}} \leftrightarrow (-)^r \frac{1}{r!} \partial_1^r \delta(Z_1 - Z_2), \quad \frac{1}{Z_{12}^{r+1}} \leftrightarrow (-)^r \frac{1}{r!} D_1^{2r+1} \delta(Z_1 - Z_2). \quad (36) \]

* The quantum algebra has been computed for this case in ref. [12] by solving the Jacobi identities.
As an independent check of the algebra (33) we have explicitly derived these brackets by using the second Gelfand-Dickey Hamiltonian structure. For that purpose it is convenient to take the basic differential operator in the factorized form [6],

\[ \mathcal{L}^{(n)} = (D + \phi_1)(D + \phi_2)D(D - \phi_2)(D - \phi_1), \tag{37} \]

where the fields \( \phi_i \) are related to \( V_3, V_4 \) by the generalized Miura transformation

\[
V_3 = -\frac{2}{3} \partial \phi_1 - \frac{1}{3} \phi_1 D \phi_1 + \frac{1}{3} \phi_2 + \frac{1}{3} \phi_2 D \phi_2 \\
V_4 = \frac{1}{3} D^3 \phi_1 + \frac{1}{3} \phi_1 \partial \phi_1 + \frac{1}{6} (D \phi_1)^2 - \frac{2}{3} D^3 \phi_2 \\
+ \frac{2}{3} \phi_2 \partial \phi_2 - \frac{2}{3} (D \phi_2)^2 - \phi_1 \partial \phi_2 - \phi_1 \phi_2 D \phi_2. \tag{38} \]

By virtue of the supersymmetric version of the Kupershmidt-Wilson theorem, the second Gelfand-Dickey bracket for the fields \( V_i \) is then given by the first bracket for the fields \( \phi_i \), which reads [6]

\[
\{ \phi_i(Z_1), \phi_j(Z_2) \} = (-)^i \delta_{ij} DZ_1 \delta(Z_1 - Z_2). \tag{39} \]

After a considerable amount of work and rescaling the fields we have explicitly derived eq.(35). Concerning the derivation we remark that the following general result is helpful: if the fundamental Poisson bracket between the fields \( u_i \) with Grassmann parity \( |u_i| = (-)^i \) is given by

\[
\{ u_i(Z_1), u_j(Z_2) \} = L_{Z_1}^{ij} \delta(Z_1 - Z_2),
\]

then the bracket of two differential polynomials of \( u_i \) is given by

\[
\{ \mathcal{F}(Z_1), \mathcal{H}(Z_2) \} = \sum_{i,j} \sum_{p,q=0}^{\infty} (-)^{|\mathcal{H}|+|\mathcal{F}|+i+q}(i+q) + \delta_{p(p+1)} + j(p+1) \frac{\partial \mathcal{F}}{\partial u^p_{i(q)}}(Z_1) D^q_{Z_1} \circ L_{Z_1}^{ij} \circ D^p_{Z_1} \circ \frac{\partial \mathcal{H}}{\partial u^p_{j(q)}}(Z_1) \delta(Z_1 - Z_2),
\]

where \( u_i^{(p)} = D^p u_i \).

**Covariance:** If written in the form (33), the Ward identities are manifestly covariant since the operators \( L^{ij} \) represent linear combinations of superconformally covariant operators of the types constructed in refs.[8] and [3]. In fact, if we identify the superstress tensor \( V_3 \) with a superprojective connection \( \mathcal{R} \), eqs.(33) take the compact form

\[
\bar{D}V_3 = -\frac{1}{2} \mathcal{L}_2 H_3 + 6J_4^{(2)}(V_4, \cdot) H_4 \\
\bar{D}V_4 = \left[ \frac{1}{8} \mathcal{L}_3 + \frac{5}{2} M_{V_4}^{(3)} \right] H_4 + 4J_4^{(2)}(V_4, \cdot) H_3. \tag{40} \]

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Here, \( L_2 \) and \( L_3 \) are super Bol operators (depending only on the projective connection \( V_3 = \mathcal{R} \)) while \( M_{V_4}^{(3)} \) and \( J_{4,k}^{2}(V_4, \cdot) \) are linear operators depending on both \( V_3 \) and the conformal field \( V_4 \). These operators are superconformally covariant when acting on \( H_3 \in \mathcal{F}_{-2} \otimes \mathcal{F}_1 \) and \( H_4 \in \mathcal{F}_{-3} \otimes \mathcal{F}_1 \), respectively.

**Generalization to \( osp(2m \pm 1|2m) \):** Quite generally, for \( W_n \) to be a primary field of weight \( \frac{n}{2} \), the Ward identities for the fields \( V_3 \) and \( V_n \) need to have the covariant form

\[
\left\{ \bar{D} - H_3 \partial + \frac{1}{2} (DH_3) D - \frac{3}{2} (\partial H_3) \right\} V_3 = -\frac{1}{2} D^5 H_3 + \left\{ \theta \right. \frac{n - 1}{2} H_n \partial + \frac{1}{2} (-)^n (DH_n) D + \frac{n}{2} (\partial H_n) \left. \right\} V_n
\]

\[
\left\{ \bar{D} - H_3 \partial + \frac{1}{2} (DH_3) D - \frac{n}{2} (\partial H_3) \right\} V_n = \text{terms involving } H_{n \neq 3}. \tag{41}
\]

This corresponds to the operator products

\[
T(Z_1)T(Z_2) = \frac{1}{Z_{12}^3} + \left[ \frac{3}{2} \frac{\theta_{12}}{Z_{12}^2} + \frac{1}{2} \frac{D_2 + \theta_{12}}{Z_{12}} \partial_2 \right] T(Z_2)
\]

\[
T(Z_1)W_n(Z_2) = \left[ \frac{n \theta_{12}}{2} \frac{Z_{12}^2}{Z_{12}} + \frac{1}{2} \frac{D_2 + \theta_{12}}{Z_{12}} \partial_2 \right] W_n(Z_2), \tag{42}
\]

or, equivalently, to Poisson brackets involving

\[
L^{3n} = (-)^{n+1} \left[ \frac{n}{2} V_n \partial - \frac{1}{2} (-)^n (DV_n) D + \frac{n-1}{2} (\partial V_n) \right] \quad \text{for } n \neq 3 \tag{43}
\]

and \( L^{33} \) as in eq.(34).

7. **Concluding remarks**

To summarize, we have provided a general algorithm for deriving the classical SWA’s associated to the super Lie algebras \( sl(n+1|n) \) and \( osp(2m \pm 1|2m) \). This was explicitly demonstrated with the \( osp(3|2) \) example. As mentioned above, a similar calculation for the simple case of \( osp(1|2) \) leads to the super Virasoro algebra or, equivalently, to the second Hamiltonian structure for the super KdV equation. The Poisson structure derived here for the \( osp(3|2) \) case should correspond to another supersymmetric integrable hierarchy, associated with the Lax operator \( D \mathcal{L}^{(2)} \).

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Appendix A: Gauge theories with supergroups

In this appendix we work out the superspace formulation of gauge theories based on supergroups \cite{7} and we discuss the integrability conditions for gauge covariant differential equations. Upon restriction to the even part of the supergroup, all formulae reduce to the standard ones for ordinary groups \cite{13}.

For a super Riemann surface, the canonical basis of the tangent space is given by the vector fields \((D_A) = (\partial, \bar{\partial}, D, \bar{D})\). The associated Lie brackets read

\[ [D_A, D_B] \equiv D_A D_B - (-)^{ab} D_B D_A = -T_{AB}^C D_C. \]

Here, the anholonomy coefficients \(T_{AB}^C\) correspond to the rigid superspace torsion and the only non-vanishing coefficients are given by \(T_{\theta\bar{\theta}}^z = -2 = T_{\bar{\theta}\theta}^z\). The graded commutator of the gauge covariant derivatives \(\nabla_A = D_A - A_A\) defines the components \(F_{AB}\) of the curvature associated to the connection \(A\):

\[ [\nabla_A, \nabla_B] = -F_{AB} - T_{AB}^C \nabla_C \]

\[ F_{AB} = D_A A_B - (-)^{ab} D_B A_A - [A_A, A_B] + T_{AB}^C A_C. \]

If the connection takes its values in a super Lie algebra, the graded commutators of the connection components are defined by the relations

\[ [A_z, A_{\bar{z}}] = A_z A_{\bar{z}} - A_{\bar{z}} A_z \]

\[ [A_\theta, A_{\bar{\theta}}] = \hat{A}_\theta A_{\bar{\theta}} + \hat{A}_{\bar{\theta}} A_\theta \]

\[ [A_z, A_{\bar{\theta}}] = \hat{A}_z A_{\bar{\theta}} - A_{\bar{\theta}} A_z = -[A_{\bar{\theta}}, A_z] \]

and analogously for \(A_z \leftrightarrow A_{\bar{z}}, A_\theta \leftrightarrow A_{\bar{\theta}}\) (as well as for the commutators of covariant derivatives). In order to obtain these graded commutators in a systematic way, it is convenient to multiply the odd superfields \(A_\theta, A_{\bar{\theta}}\) by Grassmann numbers \(\varepsilon, \varepsilon'\) so as to obtain even superfields: the relations \((45)\) then follow from the ordinary commutators

\[ [\varepsilon A_\theta, \varepsilon' A_{\bar{\theta}}] = (\varepsilon A_\theta)(\varepsilon' A_{\bar{\theta}}) - (\varepsilon' A_{\bar{\theta}})(\varepsilon A_\theta) \equiv -\varepsilon \varepsilon' [A_\theta, A_{\bar{\theta}}] \]

\[ [A_z, \varepsilon A_{\bar{\theta}}] = A_z (\varepsilon A_{\bar{\theta}}) - (\varepsilon A_{\bar{\theta}}) A_z \equiv \varepsilon [A_z, A_{\bar{\theta}}] \]

and the relations

\[ A_\theta \varepsilon = -\varepsilon \hat{A}_\theta \]

\[ A_z \varepsilon = \varepsilon \hat{A}_z. \]

If the vector \(\vec{F}\) satisfies the covariant differential equations

\[ \nabla_\theta \vec{F} = 0 \quad \text{and} \quad \nabla_{\bar{\theta}} \vec{F} = 0, \]

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then the relations (44) imply the integrability condition 0 = [∇_θ, ∇_{\bar{θ}}] \vec{F} = -F_{\theta \bar{θ}} \vec{F}, i.e.

\[ F_{\theta \bar{θ}} = 0. \] (49)

This is just our zero-curvature equation (10). Furthermore, eqs. (44) imply the integrability condition

0 = [∇_θ, ∇_θ] \vec{F} = -F_{θθ} \vec{F} + 2\nabla_z \vec{F}

and analogously for [∇_{\bar{θ}}, ∇_{\bar{θ}}] \vec{F}, i.e. one has to require that

\[ \nabla_z \vec{F} = \frac{1}{2} F_{θθ} \vec{F}, \quad \nabla_{\bar{θ}} \vec{F} = \frac{1}{2} F_{\bar{θ}\bar{θ}} \vec{F}. \] (50)

If we now impose the redefinition constraints \( F_{θθ} = 0 \) and \( F_{\bar{θ}\bar{θ}} = 0 \), we can use the definition (44) to express the spatial components A_z and A_{\bar{z}} in terms of A_θ and A_{\bar{θ}}, respectively:

\[ A_z = D A_θ - \tilde{A}_θ A_θ, \quad A_{\bar{z}} = \bar{D} A_{\bar{θ}} - \tilde{A}_{\bar{θ}} A_{\bar{θ}}. \] (51)

In this case, eqs. (50) reduce to \( \nabla_z \vec{F} = 0 \) and \( \nabla_{\bar{θ}} \vec{F} = 0 \) and compatibility of these equations with each other and with eqs. (48) requires that all remaining components of the curvature vanish:

\[ 0 = F_{z\bar{z}} = F_{zθ} = F_{z\bar{θ}} = F_{\bar{z}θ} = F_{\bar{z}\bar{θ}}. \] (52)

**Appendix B: osp(2m ± 1|2m) in diagonal grading**

One usually defines [8] the superalgebra \( osp(2m \pm 1|2m) \) as the set of supermatrices \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) which satisfy \( M^{sT} G + GM = 0 \) where the supertranspose of \( M \) is defined by \( M^{sT} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix} \) and an invariant metric by \( G = G_{so(2m \pm 1)} \oplus G_{sp(2m)} \). For the invariant \( so(2m \pm 1) \) and \( sp(2m) \) metrics we choose, respectively,

\[ G_{so(2m \pm 1)} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad G_{sp(2m)} = \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix}. \]

Let us now discuss the two cases \( osp(2m - 1|2m) \) and \( osp(2m + 1|2m) \) in turn.
\textit{osp}(2m-1|2m): There are $4m^2 - 2m + 1$ even and $2m(2m - 1)$ odd generators. The Chevalley basis in block grading is represented by the elements

\begin{align*}
e_{2i-1} &= E_{m+1-i,4m-i} + E_{3m-i,m+1-i} \quad (i = 1, \ldots, m) \\
e_{2i} &= E_{m+i,3m-i} - E_{4m-i,m-i} \quad (i = 1, \ldots, m-1)
\end{align*}

\begin{align*}
f_{2i-1} &= E_{m-1+i,3m-i} - E_{4m-i,m+1-i} \\
f_{2i} &= -E_{m-i,4m-i} - E_{3m-i,m+i}
\end{align*}

\begin{align*}
h_{2i-1} &= E_{3m-i,3m-i} - E_{4m-i,4m-i} - E_{m+1-i,m+1-i} + E_{m-1+i,m-1+i} \\
h_{2i} &= E_{m-i,m-i} - E_{m+i,m+i} - E_{3m-i,3m-i} + E_{4m-i,4m-i}.
\end{align*}

The transition between matrices in block and diagonal grading is mediated by a similarity transformation, $M_{\text{diag.}} = L^{-1}M_{\text{block}}L$. The choice of $L$ that leads to the expression for the generators of $\text{osp}(1|2)_{\text{pal}}$ given by eq.\,(13) is

\begin{equation}
L = \sum_{i=0}^{m-1} (-)^{i+1} E_{2m+i,2i+1} + \sum_{i=1}^{m} (-)^{i} E_{2m-i,2i} + (-1)^{m+1} \sum_{i=1}^{n} E_{4m-i,2m+2i-1} + (-)^{m} \sum_{i=1}^{m-1} E_{m-i,2m+2i}.
\end{equation}

The invariant metric then becomes

\begin{equation}
G = \sum_{i=1}^{4m-1} (-)^{m+[\frac{i+1}{2}]} E_{i,4m-i}.
\end{equation}

where $[i]$ denotes the integer part. For the elements of $M^{sT}$ we find

\begin{align*}
(M^{sT})_{i,i+2k} &= M_{i+2k,i} \quad (i = 1, \ldots, 4m-1 \quad ; \quad k = 0, \pm 1, \ldots) \\
(M^{sT})_{i,i+2k+1} &= (-)^{i} M_{i+2k+1,i}.
\end{align*}

The Chevalley generators and the general matrix element in diagonal grading are also obtained easily and are as given in eqs.\,(19) and (21).

\textit{osp}(2m+1|2m): There are $2m(2m+1)$ even and odd generators. The Chevalley basis in
block grading is \((i = 1, \ldots, m)\)
\[
e_{2i-1} = E_{m+2-i,4m+2-i} + E_{3m+2-i,m+i}
\]
\[
e_{2i} = E_{m+1+i,3m+2-i} - E_{4m+2-i,m+1-i}
\]
\[
f_{2i-1} = E_{m+i,3m+2-i} - E_{4m+2-i,m+2-i}
\]
\[
f_{2i} = -E_{m+1-i,4m+2-i} - E_{3m+2-i,m+1+i}
\]
\[
h_{2i-1} = E_{3m+2-i,3m+2-i} - E_{4m+2-i,4m+2-i} - E_{m+2-i,m+2-i} + E_{m+i,m+i}
\]
\[
h_{2i} = E_{m+1-i,m+1-i} - E_{m+1+i,m+1+i} - E_{3m+2-i,3m+2-i} + E_{4m+2-i,4m+2-i}.
\]

For the matrix \(L\) we find
\[
L = \sum_{i=0}^{m} (-)^{i} E_{2m+1-i,2i+1} + \sum_{i=1}^{m} (-)^{i} E_{2m+1+i,2i}
\]
\[
+ (-1)^{n+1} \sum_{i=1}^{m} E_{4m+2-i,2m+2i} + (-)^{m} \sum_{i=1}^{m} E_{m+1-i,2m+1+2i}.
\]

The invariant metric then becomes
\[
G = \sum_{i=1}^{4m+1} (-)^{m+\left[\frac{i}{2}\right]} E_{i,4m+2-i}
\]

and the elements of \(M^{sT}\) take the form
\[
(M^{sT})_{i,i+2k} = M_{i+2k,i} \quad (i = 1, \ldots, 4m+1 ; \; k = 0, \pm 1, \ldots)
\]
\[
(M^{sT})_{i,i+2k+1} = (-)^{i+1} M_{i+2k+1,i}.
\]

For the Chevalley generators and the general matrix element in diagonal grading we refer again to the main body of the text (eqs. (18) and (21)).
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