GRADINGS AND SYMMETRIES ON HEISENBERG TYPE ALGEBRAS

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ABSTRACT. We describe the fine (group) gradings on the Heisenberg algebras, on the Heisenberg superalgebras and on the twisted Heisenberg algebras. We compute the Weyl groups of these gradings. Also the results obtained respect to Heisenberg superalgebras are applied to the study of Heisenberg Lie color algebras.

2010 MSC: 17B70, 17B75, 17B40, 16W50.
Key words and phrases: Heisenberg algebra, graded algebra, Weyl group.

1. INTRODUCTION

In the last years there has been an increasing interest in the study of the group gradings on Lie theoretic structures. In the case of Lie algebras, this study has been focused on the simple ones. A recent exhaustive survey on the matter is [23], here we can briefly mention that the (complex) finite-dimensional simple case has been studied, among other authors, by Bahturin, Elduque, Havlček, Kochetov, Patera, Pelantová, Shestakov, Zaicev and Zassenhaus [5, 6, 8, 19, 25, 31] in the classical case ([19] encloses $\mathfrak{d}_4$), while the exceptional cases $\mathfrak{g}_2, \mathfrak{f}_4$ and $\mathfrak{d}_4$ have been studied by Bahturin, Draper, Elduque, Kochetov, Martin, Tvalavadze and Viruel [7, 15, 16, 17, 20]. The fine group gradings on the real forms of $\mathfrak{g}_2$ and $\mathfrak{f}_4$, also simple algebras, have been classified by the three first authors [11]. Gradings have also been considered in certain $\mathbb{Z}_2$-graded structures (for instance superalgebras), so the three first authors have studied the case of the Jordan superalgebra $K_{10}$ (see [12]), and the second and third authors together with Elduque have classified the fine gradings on exceptional Lie superalgebras in [18]. In relation with other Lie structures, the Lie triple systems of exceptional type have also been considered from the viewpoint of gradings (see [13]).

There is not much work done in the field of gradings on non-simple Lie algebras. Some relevant references are [3] and [4]. Our work is one step further in this direction. We are interested in studying the gradings on a family of non-simple Lie algebras, superalgebras, color algebras and twisted algebras, namely, the Heisenberg algebras (resp. superalgebras, Lie color algebras and twisted ones). Note that Heisenberg (super) algebras are nilpotent and twisted Heisenberg algebras are solvable.

The Heisenberg family of algebraic structures was introduced by A. Kaplain in [28]. In the simplest case, that of a Heisenberg algebra, the notion is related to Quantum Mechanics. As it is well known, the Heisenberg Principle of Uncertainty implies the noncommutativity of position and momentum observables acting on fermions. This noncommutativity reduces to noncommutativity of the corresponding operators. If we represent by $x$ the operator associated to position and by $\frac{\partial}{\partial x}$ the one associated to momentum (acting for instance on

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The authors are partially supported by the MCYT grant MTM2010-15223 and by the Junta de Andalucía grants FQM-336, FQM-2467, FQM-3737. The first and fourth authors are also supported by the PCI of the UCA ‘Teoría de Lie y Teoría de Espacios de Banach’ and by the PAI with project number FQM-298.
a space \( V \) of differentiable functions of a single variable), then \( \left[ \frac{\partial}{\partial x}, x \right] = 1_V \) which is nonzero. Thus we can identify the subalgebra generated by \( 1, x \) and \( \frac{\partial}{\partial x} \) with the three-dimensional Heisenberg Lie algebra whose multiplication table in the basis \( \{1, x, \frac{\partial}{\partial x} \} \) has as unique nonzero product: \( \left[ \frac{\partial}{\partial x}, x \right] = 1_V \).

The literature about Heisenberg structures is ample. First, they have played an important role in Quantum Mechanics, where for instance twisted Heisenberg algebras appear by a quantizing process from the classical Heisenberg algebra \( H(4) \). Thus, in \( \text{Ref.}[1] \), coherent states for power-law potentials are constructed by using generalized Heisenberg algebras, being also shown that these coherent states are useful for describing the states of real and ideal lasers \( \text{Ref.}[9] \) or where a deformation of a Heisenberg algebra it is used to describe the solutions of the \( N \)-particle rational Calogero model and to solve the problem of proving the existence of supertraces \( \text{Ref.}[30] \).

Second, these structures are also important in Differential Geometry. The quotient Lie group \( H_3(\mathbb{R})/H_3(\mathbb{Z}) \), which is a compact smooth manifold without boundary of dimension 3, is, as mentioned by \( \text{Ref.[37]} \), one of the basic building blocks for 3-manifolds studied in \( \text{Ref.}[39] \). Our definition of twisted Heisenberg algebra is precisely motivated by this connection with Differential Geometry. In that context, the twisted Heisenberg algebra is the tangent algebra of a twisted Heisenberg group (certain semidirect product of a Heisenberg group with the real numbers). Its importance is due to a series of results (which can be consulted in \( \text{Ref.}[2] \)), that describe the groups which can act by isometries in a Lorentzian manifold. For instance:

**Theorem.** \( \text{Ref.[2, Theorem 11.7.3]} \) Let \( M \) be a compact connected Lorentzian manifold and \( G \) a connected Lie group acting isometrically and locally faithfully on \( M \). Then its Lie algebra \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{s} \) is a direct sum of a compact semisimple Lie algebra \( \mathfrak{k} \), an abelian algebra \( \mathfrak{a} \) and a Lie algebra \( \mathfrak{s} \), which is either trivial, or isomorphic to \( \text{diff}(\mathbb{R}) \), to a Heisenberg algebra \( H_n \), to a twisted Heisenberg algebra \( H_n^\lambda \) with \( \lambda \in \mathbb{Q}((n-1)/2), \) or to \( \mathfrak{sl}_2(\mathbb{R}) \).

Moreover, according to \( \text{Ref.[2, Chapter 8]} \), the converse of this result is also true: if \( G \) is a connected simply connected Lie group whose Lie algebra \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{s} \) is as above, then there is a locally faithful isometric action of \( G \) on a compact connected Lorentz manifold. The physical meaning of this result claims that if you want a Lie group to act on a compact connected Lorentzian manifold, then you must be ready to admit that the group may have a Heisenberg section. More results about the relationship between twisted Heisenberg algebras and Lorentzian manifolds can be found in \( \text{Ref.[24]} \).

Third, there are a variety of algebraic works about these structures. For instance the set of superderivations of a Heisenberg superalgebra is applied to the theory of cohomology, \( \text{Ref.[14]} \), the derivation algebra of a Heisenberg Lie color algebra is computed, being a simple complete Lie algebra \( \text{Ref.[27]} \). Other works in this line are \( \text{Ref.[29, 43]} \).

The study of Weyl groups of Lie gradings was inaugurated by Patera and Zassenhaus in \( \text{Ref.[31]} \). Some concrete examples were developed, for instance, in \( \text{Ref.[26]} \). Recently, Elduque and Kochetov have determined the Weyl groups of the fine gradings on matrix, octonions, Albert and simple Lie algebras of types \( A, B, C \) and \( D \) (see \( \text{Ref.[21, 22]} \)). The (extended) Weyl group of a simple Lie algebra is the Weyl group of the Cartan grading on that algebra (which is of course fine). Thus the notion of Weyl group of a grading encompasses that of the usual Weyl group with its countless applications. This is one of the reasons motivating the study of Weyl groups on fine Lie gradings. On the other hand, if we consider the category of graded Lie algebras (not in the sense of Lie superalgebras), then the automorphism group \( G \) of such an object is defined as the group of automorphisms of the algebra which
preserve the grading, and the Weyl group of the grading is an epimorphic image of \( G \). Thus the symmetries of a graded Lie algebra are present in the Weyl group of the grading. Our work includes the description of the Weyl group of the fine gradings on Heisenberg algebras, on Heisenberg superalgebras and on twisted Heisenberg algebras.

Over algebraically closed fields of characteristic zero, the study of gradings is strongly related to that of automorphisms. Although we get our classifications of gradings in Heisenberg algebras and on Heisenberg superalgebras in a more general context, we also compute the corresponding automorphism groups. We would like to mention the work [36] which contains a detailed study of the automorphisms on Heisenberg-type algebras. So it has been illuminating though our study of gradings goes a step further.

Finally, we devote some words to the distribution of results in our work. After a background on gradings in Section 2, we study the group gradings on Heisenberg algebras in Section 3, by showing that all of them are toral and by computing the Weyl group of the only (up to equivalence) fine one. In Section 4 we study the fine group gradings on Heisenberg superalgebras for \( F \) an algebraically closed field of characteristic different from 2, and calculate their Weyl groups. In Section 5 we discuss on the concept of Heisenberg Lie color algebra, give a description of the same and show how the results in the previous section can be applied to classify a certain family of Heisenberg Lie color algebras. Finally, in Section 6, we devote some attention to the concept of twisted Heisenberg algebras and also compute their group gradings for \( F \) algebraically closed of characteristic zero. We classify the fine gradings up to equivalence and find their symmetries, which turn out to be very abundant.

2. Preliminaries

Throughout this work the base field will be denoted by \( \mathbb{F} \). Let \( A \) be an algebra over \( \mathbb{F} \). A grading on \( A \) is a decomposition 
\[
\Gamma : A = \bigoplus_{s \in S} A_s
\]
of \( A \) into direct sum of nonzero subspaces such that for any \( s_1, s_2 \in S \) there exists \( s_3 \in S \) such that \( A_{s_1} A_{s_2} \subset A_{s_3} \). The grading \( \Gamma \) is said to be a group grading if there is a group \( G \) containing \( S \) such that \( A_{s_1} A_{s_2} \subset A_{s_1 s_2} \) (multiplication of indices in the group \( G \)) for any \( s_1, s_2 \in S \). Then we can write
\[
\Gamma : A = \bigoplus_{g \in G} A_g,
\]
by setting \( A_g = 0 \) if \( g \in G \setminus S \). In this paper all the gradings we consider will be group gradings where \( G \) is a finitely generated abelian group and \( G \) is generated by the set of all the elements \( g \in G \) such that \( A_g \neq 0 \), usually called the support of the grading (the above \( S \)).

Given two gradings \( A = \bigoplus_{g \in G} U_g \) and \( A = \bigoplus_{h \in H} V_h \), we shall say that they are isomorphic if there is a group isomorphism \( \sigma : G \rightarrow H \) and an (algebra) automorphism \( \varphi : A \rightarrow A \) such that \( \varphi(U_g) = V_{\sigma(g)} \) for all \( g \in G \). The above two gradings are said to be equivalent if there are a bijection \( \sigma : S \rightarrow S' \) between the supports of the first and second gradings respectively and an algebra automorphism \( \varphi \) of \( A \) such that \( \varphi(U_g) = V_{\sigma(g)} \) for any \( g \in S \).

Let \( \Gamma \) and \( \Gamma' \) be two gradings on \( A \). The grading \( \Gamma \) is said to be a refinement of \( \Gamma' \) (or \( \Gamma' \) a coarsening of \( \Gamma \)) if each homogeneous component of \( \Gamma' \) is a (direct) sum of some homogeneous components of \( \Gamma \). A grading is called fine if it admits no proper refinements.
A fundamental concept to obtain the coarsenings of a given grading is the one of universal grading group. Given a grading $\Gamma : A = \oplus_{g \in G} A_g$, one may consider the abelian group $\tilde{G}$ generated by the support of $\Gamma$ subject only to the relations $g_1g_2 = g_3$ if $0 \neq [A_{g_1}, A_{g_2}] \subset A_{g_3}$. Then $A$ is graded over $\tilde{G}$; that is $\tilde{\Gamma} : A = \oplus_{\tilde{g} \in \tilde{G}} A_{\tilde{g}}$, where $A_{\tilde{g}}$ is the sum of the homogeneous components $A_g$ of $\Gamma$ such that the class of $g$ in $\tilde{G}$ is $\tilde{g}$. Note that there is at most one such homogeneous component and that this $\tilde{G}$-grading $\tilde{\Gamma}$ is equivalent to $\Gamma$, since $G \to \tilde{G}$, $g \to \tilde{g}$ is an injective map (not homomorphism). This group $\tilde{G}$ has the following universal property: given any coarsening $A = \oplus_{h \in H} A'_h$ of $\tilde{\Gamma}$, there exists a unique group epimorphism $\alpha : G \to H$ such that

$$A'_h = \bigoplus_{g \in \alpha^{-1}(h)} A_g.$$  

The group $\tilde{G}$ is called the universal grading group of $\Gamma$. Throughout this paper, the gradings will be considered over their universal grading groups.

For a grading $\Gamma : A = \oplus_{g \in G} A_g$, the automorphism group of $\Gamma$, denoted $\text{Aut}(\Gamma)$, consists of all self-equivalences of $\Gamma$, i.e., automorphisms of $A$ that permute the components of $\Gamma$. The stabilizer of $\Gamma$, denoted $\text{Stab}(\Gamma)$, consists of all automorphisms of the graded algebra $A$, i.e., automorphisms of $A$ that leave each component of $\Gamma$ invariant. The quotient group $\text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ will be called the Weyl group of $\Gamma$ and denoted by $W(\Gamma)$.

It is well-known that any finitely generated abelian subgroup of diagonalizable automorphisms of $\text{Aut}(A)$ induces by simultaneous diagonalization a grading on $A$. When the field $F$ is algebraically closed of characteristic zero, the converse is true: any grading on $A$ (finite dimensional algebra) is induced by a finitely generated abelian subgroup of diagonalizable automorphisms of $\text{Aut}(A)$. The set of automorphisms inducing the grading (as simultaneous diagonalization) is contained in the normalizer of some maximal torus of the automorphism group (by [SS Theorem 6 and Theorem 3.15, p. 92]). A special kind of gradings arises when we consider the inducing automorphisms not only in the normalizer of a maximal torus, but in a torus. Indeed, a grading of an algebra $A$ is said to be toral if it is produced by automorphisms within a torus of the automorphism group of the algebra. In the case of a fine grading, the grading is toral if and only if its universal grading group is torsion-free, since the universal grading group is always isomorphic to the group of characters of the group of automorphisms inducing the grading.

Next, let us recall the situation for superalgebras. If $L = L_0 \oplus L_1$ is a Lie superalgebra over $F$ and $G$ a finitely generated abelian group, a $G$-grading on $L$ is a decomposition $\Gamma : L = \bigoplus_{g \in G} ((L_0)_g \oplus (L_1)_g)$ where any $(L_i)_g$ is a linear subspace of $L_i$ and where $[(L_0)_{g_1}, (L_1)_{g_2}] \subset (L_{i+1})_{g_1 g_2}$ holds for any $g_1, g_2 \in G$ and any $i, j \in \{0, 1\}$ (sum modulo 2). Here the support of $\Gamma$ is $\{g \in G : (L_i)_g \neq 0 \text{ for some } i\}$ and everything works analogously to the case of a Lie algebra with a grading. Note only a subtle difference: assuming that $L$ is non-abelian, the trivial grading, $L = L_0 \oplus L_1$, has as universal grading group $\mathbb{Z}_2$, while the trivial grading on $L$ as a Lie algebra has the trivial group as the universal grading group.

Finally we give two fundamental lemmas of purely geometrical nature that will be applied in future sections. Recall that a symplectic space $V$ is a linear space provided with an alternate nondegenerate bilinear form $\langle \cdot, \cdot \rangle$, and that in the finite-dimensional case a standard result states the existence of a “symplectic basis”, that is, a basis: $\{u_1, u'_1, \ldots, u_n, u'_n\}$ such that $\langle u_i, u'_i \rangle = 1$ while any other inner product is zero.
Lemma 1. Let \( (V, \langle \cdot, \cdot \rangle) \) be a finite-dimensional symplectic space and assume that \( V \) is the direct sum of linear subspaces \( V = \bigoplus_{i \in I} V_i \) where for any \( i \in I \) there is a unique \( j \in I \) such that \( \langle V_i, V_j \rangle \neq 0 \). Then there is a basis \( \{ u_1, u'_1, \ldots, u_n, u'_n \} \) of \( V \) such that:

- The basis is contained in \( \bigcup_i V_i \).
- For any \( i, j \) we have \( \langle u_i, u_j \rangle = \langle u'_i, u'_j \rangle = 0 \).
- For each \( i \) and \( j \) we have \( \langle u_i, u'_j \rangle = \delta_{i,j} \) (Kronecker’s delta).

Proof. First we split \( I \) into a disjoint union \( I = I_1 \cup I_2 \) such that \( I_1 \) is the set of all \( i \in I \) such that \( \langle V_i, V_i \rangle \neq 0 \) and in \( I_2 \) we have all the indices \( i \) such that there is \( j \neq i \) with \( \langle V_i, V_j \rangle \neq 0 \). Now for each \( i \in I_1 \) the space \( V_i \) is symplectic with relation to the restriction of \( \langle \cdot, \cdot \rangle \) to \( V_i \). So we fix in such \( V_i \) a symplectic basis. Take now \( i \in I_2 \) and let \( j \in I \) be the unique index such that \( \langle V_i, V_j \rangle \neq 0 \) (necessarily \( j \in I_2 \)). Consider now the restriction \( \langle \cdot, \cdot \rangle : V_i \times V_j \to \mathbb{F} \). This map is nondegenerate in the obvious sense (which implies \( \dim(V_i) = \dim(V_j) \)). If we fix a basis \( \{ e_1, \ldots, e_q \} \) of \( V_i \), then by standard linear algebra arguments we get that there is basis \( \{ f_1, \ldots, f_q \} \) of \( V_j \) such that \( \langle e_i, f_j \rangle = 1 \) being the remaining inner products among basic elements zero. Thus, putting together these bases suitable reordered we get the symplectic basis whose existence is claimed in the Lemma. \( \square \)

Since all the elements in the basis constructed above are in some component \( V_i \) we will refer to this basis as a “homogeneous basis” of \( V \).

Lemma 2. Let \( (V, \langle \cdot, \cdot \rangle) \) be a finite-dimensional linear space \( V \) with a symmetric non-degenerate bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{F} \) where \( \mathbb{F} \) is of characteristic other than 2. Assume that \( V = \bigoplus_{i \in I} V_i \) is the direct sum of linear subspaces in such a way that for each \( i \in I \) there is a unique \( j \in I \) such that \( \langle V_i, V_j \rangle \neq 0 \). Then there is a basis \( B = \{ u_1, v_1, \ldots, u_r, v_r, z_1, \ldots, z_q \} \) of \( V \) such that:

- \( B \subset \bigcup_i V_i \).
- \( \langle z_i, z_i \rangle \neq 0, \langle u_i, u_i \rangle = 1 \).
- Any other inner product of elements in \( B \) is zero.

Proof. Let \( I_1 \) be the subset of \( I \) such that for any \( i \in I_1 \) we have \( \langle V_i, V_i \rangle \neq 0 \) and let \( I_2 \) be the complementary \( I_2 := I \setminus I_1 \). On one hand, each \( V_i \) with \( i \in I_1 \) has an orthogonal basis (char \( \mathbb{F} \neq 2 \)). On the other hand, for any \( i \in I_2 \) consider the unique \( j \neq i \) such that \( \langle V_i, V_j \rangle \neq 0 \). The couple \( (V_i, V_j) \) gives a dual pair \( \langle \cdot, \cdot \rangle : V_i \times V_j \to \mathbb{F} \) (implying \( \dim(V_i) = \dim(V_j) \)) and for any basis \( \{ e_k \} \) of \( V_i \), there is a dual basis \( \{ f_k \} \) in \( V_j \) such that \( \langle e_k, f_k \rangle = \delta_{k,h} \) (Kronecker’s delta). So putting together all these bases (suitably ordered) the required base on \( V \) is got. \( \square \)

Observe that if \( \mathbb{F} \) is algebraically closed the inner products \( \langle z_i, z_i \rangle \) in Lemma 2 may be chosen to be 1. The basis constructed in Lemma 2 will be also termed a “homogeneous basis”.

We will have the occasion to use basic terminology of finite groups: \( \mathbb{Z}_n \) for the cyclic group of order \( n \), \( S_n \) for the permutation group of \( n \) elements and \( D_n \) for the dihedral group of order \( 2n \).

3. GRADINGS ON HEISENBERG ALGEBRAS

A Lie algebra \( H \) is called a Heisenberg algebra if it is nilpotent in two steps (that is, not abelian and \( [[H, H], H] = 0 \)) with one-dimensional center \( \mathcal{Z}(H) := \{ x \in H : [x, H] = 0 \} \). If \( z \neq 0 \) is a fixed element in \( \mathcal{Z}(H) \) and we take \( P \) any complementary subspace of \( \mathbb{F}z \),
then the map \( \langle \cdot, \cdot \rangle : P \times P \to \mathbb{F} \) given by \( \langle v, v' \rangle z = [v, v'] \) is a nondegenerate skewsymmetric bilinear form, or, in other words, \((P, \langle \cdot, \cdot \rangle)\) is a symplectic space. Of course, any Lie algebra constructed from a symplectic space \((P, \langle \cdot, \cdot \rangle)\) as \( H = P \oplus \mathbb{F} z \) with \( z \in Z(H) \) and \([v, v'] = \langle v, v' \rangle z\) for all \( v, v' \in P \), is a Heisenberg algebra. Recall that the dimension of \( P \) is necessarily even.

In particular, there is one Heisenberg algebra up to isomorphism for each odd dimension \( n = 2k + 1 \), which we will denote \( H_n \), characterized by the existence of a basis
\[
B = \{e_1, \hat{e}_1, \ldots, e_k, \hat{e}_k, z\}
\]
in which the nonzero products are \([e_i, \hat{e}_i] = -[\hat{e}_i, e_i] = z\) for \( 1 \leq i \leq k \).

A fine grading on \( H_n \) is obviously provided by this basis as:
\[
H_n = \langle e_1 \rangle \oplus \cdots \oplus \langle e_k \rangle \oplus \langle \hat{e}_1 \rangle \oplus \cdots \oplus \langle \hat{e}_k \rangle \oplus \langle z \rangle,
\]
where each summand is a homogeneous component. This grading is also a group grading.

For instance, it can be considered as a \( \mathbb{Z} \)-grading by letting \( H_n = \bigoplus_{i=-k}^{k} (H_n)_i \) for
\[
(H_n)_{-i} = \langle \hat{e}_i \rangle, \quad (H_n)_0 = \langle z \rangle, \quad (H_n)_i = \langle e_i \rangle, \quad i = 1, \ldots, k.
\]
Moreover, this grading on \( H_n \) is toral. It is enough to observe that the group of automorphisms \( \mathcal{T} \) which are represented by scalar matrices relative to the basis \( B \) is a torus. In fact, it is a maximal torus of dimension \( k + 1 \). Indeed, an element \( f \in \mathcal{T} \) will be determined by the nonzero scalars \((\lambda_1, \ldots, \lambda_k, \lambda_i) \in \mathbb{F}^{k+1}\) such that \( f(z) = \lambda z \) and \( f(e_i) = \lambda_i e_i \), being then \( f(\hat{e}_i) = \frac{\lambda_i}{\lambda} e_i \). If we denote such an automorphism by \( t_{(\lambda_1, \ldots, \lambda_k, \lambda)} \), it is straightforward that \( t_{(\lambda_1, \ldots, \lambda_k, \lambda)} t_{(\lambda'_1, \ldots, \lambda'_k, \lambda')} = t_{(\lambda_1 \lambda'_1, \ldots, \lambda_k \lambda'_k, \lambda \lambda')} \) and that any automorphism commuting with every element in \( \mathcal{T} \) preserves the common diagonalization produced by \( \mathcal{T} \), which is precisely the one given by \( \mathcal{T} \). All this in particular implies that this fine toral grading can be naturally considered as a grading over the group \( \mathbb{Z}^{k+1} \), which is its universal grading group:
\[
\Gamma : \quad (H_n)_{(0, \ldots, 0; 2)} = \langle z \rangle, \\
(H_n)_{(0, \ldots, 1; 0; 1)} = \langle e_i \rangle, \quad \text{(1 in the } i \text{-th slot)} \\
(H_n)_{(0, \ldots, -1; 0; 1)} = \langle \hat{e}_i \rangle,
\]
if \( i = 1, \ldots, k \).

Our first aim is to prove that, essentially, this is the unique fine grading.

**Theorem 1.** For any (group) grading on \( H_n \), there is a basis \( B = \{z, u_1, u'_1, \ldots, u_n, u'_n\} \) of homogeneous elements of \( H_n \) such that \( [u_i, u'_i] = z \) and the remaining possible brackets among elements of \( B \) are zero.

In particular, there is only one fine grading up to equivalence on \( H_n \), the \( \mathbb{Z}^{k+1} \)-grading given by \( \Gamma \). Thus, any grading on \( H_n \) is toral.

**Proof.** As before we will consider the bilinear alternate form \( \langle \cdot, \cdot \rangle : H_n \times H_n \to \mathbb{F} \). In any (group) graded Lie algebra the center admits a basis of homogeneous elements, so, if \( H_n \) is graded by a group \( G \), then there is some \( g_0 \in G \) such that \( z \in \langle H_n \rangle_{g_0} \). Thus, denoting by \( Z(H_n) = \mathbb{F} z \) the center of \( H_n \), the (abelian) quotient Lie algebra \( P := H_n/ Z(H_n) \) is a symplectic space relative to \( (x + Z(H_n), y + Z(H_n)) := \langle x, y \rangle \). Denote by \( \pi : H_n \to P \) the canonical projection. By defining for each \( g \in G \) the subspace \( P_g := \pi(\langle H_n \rangle_g) \), it is easy to check that \( P = \bigoplus_{g \in G} P_g \) and that for any \( g \in G \) there is a unique \( h, (h = -g + g_0) \), such that \( \langle P_g, P_h \rangle \neq 0 \). Then Lemma 1 provides a symplectic basis of \( P \) of homogeneous elements. Let \( \{u_1 + Z(H_n), u'_1 + Z(H_n), \ldots, u_n + Z(H_n), u'_n + Z(H_n)\} \) be such basis.
(observe that each $u_i$ and $u'_i$ may be chosen in some homogeneous component of $H_n$ being $u_i, u'_i \notin \mathbb{F}z$). Then $B := \{ z, u_1, u'_1, \ldots, u_n, u'_n \}$ is a basis of homogeneous elements of $H_n$ such that $[u_i, u'_i] = z$ and the remaining possible brackets among basis elements are zero.

To finish the proof we can consider the maximal torus of $\text{Aut}(H_n)$ formed by the automorphisms which are diagonal relative to the basis $B$. Up to conjugations, this torus is formed by all the automorphisms $\lambda_{(\lambda_1, \ldots, \lambda_k, \lambda)}$ constructed above. Since $B$ are eigenvectors for any of these elements, the initial grading is toral and in fact it is a coarsening of the fine grading described in Equation 3.

In order to work on the group of symmetries of this grading, the Weyl group, we compute next the automorphism group of the Heisenberg algebra, $\text{Aut}(H_n)$. For any $f \in \text{Aut}(H_n)$ we have $f(z) \in Z(H_n)$ and so $f(z) = \lambda_f z$ for some $\lambda_f \in \mathbb{F}^\times$. If we denote by $i: P \rightarrow H_n$ the inclusion map, by $\pi: H_n \rightarrow P$ the projection map and define $\bar{f} := \pi \circ f \circ i$, we easily get that $\bar{f}$ is a linear automorphism of $P$ satisfying $\langle \bar{f}(x_p), \bar{f}(y_p) \rangle = \lambda_f \langle x_p, y_p \rangle$ for any $x_p, y_p \in P$. Indeed, given any $x = x_p + \lambda z$, and $y = y_p + \mu z$ in $H_n$, $\lambda, \mu \in \mathbb{F}$, we have, taking into account $[z, H_n] = 0$, that

$$f([x, y]) = f([x_p, y_p]) = f(\langle x_p, y_p \rangle z) = \langle x_p, y_p \rangle f(z) = \langle x_p, y_p \rangle \lambda_f z$$

and

$$[f(x), f(y)] = [\bar{f}(x_p), \bar{f}(y_p)] = \langle \bar{f}(x_p), \bar{f}(y_p) \rangle z.$$ 

Hence $\bar{f}$ belongs to $\text{GSp}(P) := \{ g \in \text{End}(P) : \text{there is } \lambda_g \in \mathbb{F}^\times \text{ with } \langle g(x), g(y) \rangle = \lambda_g \langle x, y \rangle \ \forall x, y \in P \}$, the similitude group of $(P, \langle \cdot, \cdot \rangle)$.

As a consequence, an arbitrary $f \in \text{Aut}(H_n)$ has as associated matrix relative to the basis $B$ in Equation 4,

$$M_B(f) = \left( \begin{array}{cc} \text{diag}(\alpha) & 0 \\ 0 & \lambda_f \end{array} \right)$$

for $B_P = B \setminus \{ z \}$ a basis of $P$, $\bar{f} = \pi \circ f \circ i \in \text{GSp}(P)$ and for the vector $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{F}^{2k}$ given by $\lambda_{2i-1} z = (f - \bar{f})(e_i)$ and $\lambda_{2i} z = (f - \bar{f})(\bar{e}_i)$ if $i \leq k$.

Moreover, if we define in the set $\text{GSp}(P) \times \mathbb{F}^{2k}$ the (semidirect) product

$$(\bar{f}, \lambda)(\bar{g}, \eta) := (\bar{f} \bar{g}, \lambda M_{B_P}(\bar{g}) + \lambda_f \eta),$$

it is straightforward to verify that the mapping

$$\Omega: \text{Aut}(H_n) \rightarrow \text{GSp}(P) \ltimes \mathbb{F}^{2k}$$

given by $\Omega(f) = (\bar{f}, \lambda)$ is a group isomorphism.

Take, for each permutation $\sigma \in S_k$, the map $\tilde{\sigma}: H_n \rightarrow H_n$ given by $\tilde{\sigma}(e_i) = e_{\sigma(i)}$, $\tilde{\sigma}(\bar{e}_i) = \bar{e}_{\sigma(i)}$ and $\tilde{\sigma}(z) = z$. It is clear that $\tilde{\sigma}$ is an automorphism permuting the homogeneous components of $\Gamma$, the grading in 3, that is, $\tilde{\sigma} \in \text{Aut}(\Gamma)$.

Other remarkable elements in the automorphism group of the grading are the following ones: for each index $i \leq k$, take $\mu_i: H_n \rightarrow H_n$ given by $\mu_i(e_i) = \bar{e}_i, \mu_i(\bar{e}_i) = -e_i, \mu_i(e_j) = e_j, \mu_i(\bar{e}_j) = \bar{e}_j$ (for $j \neq i$) and $\mu_i(z) = z$.

Denote by $[f]$ the class of an automorphism $f \in \text{Aut}(\Gamma)$ in the quotient $\mathcal{W}(\Gamma) = \text{Aut}(\Gamma) / \text{Stab}(\Gamma)$.

**Proposition 1.** The Weyl group $\mathcal{W}(\Gamma)$ is generated by $\{ [\tilde{\sigma}] : \sigma \in S_k \}$ and $[\mu_1]$. 

Proof. Let $f$ be an arbitrary element in $\text{Aut}(\Gamma)$. The elements in $\text{Aut}(\Gamma)$ permute the homogeneous components of the grading $\Gamma$, but $\mathbb{F}z$ remains always invariant. Thus $f(e_i)$ belongs to some homogeneous component different from $\mathbb{F}z$, and there is an index $i \leq k$ such that either $f(e_i) \in \mathbb{F}^*e_i$ or $f(e_i) \in \mathbb{F}^*\tilde{e}_i$. We can assume that $f(e_i) \in \mathbb{F}^*e_i$ by replacing, if necessary, $f$ with $\mu_i f$. Now, take the permutation $\sigma = (1, i)$ which interchanges $1$ and $i$, so that $f' = \sigma f$ maps $e_1$ into $\alpha e_1$ for some $\alpha \in \mathbb{F}^*$. Note that $\alpha [e_1, f'(e_i)] = [f'(e_1), f'(e_i)] = f'([e_1, e_i]) = f'(z)$ is a nonzero multiple of $z$, hence $f'(e_i) \not\in \{x \in H_n : [x, e_1] = 0\} = \langle z, e_1, e_i \rangle$. But $f' \in \text{Aut}(\Gamma)$, so $f'(e_i) \in \langle \tilde{e}_1 \rangle$. In a similar manner the automorphism $f'$ sends $e_2$ to some $\langle e_j \rangle$ or some $\langle \tilde{e}_j \rangle$ for $j \not= 1$, hence we can replace $f'$ by $f'' \in \{ (2, j) f', (\tilde{2}, j) \mu_j f' \}$ such that $f''$ preserves the homogeneous components $\langle e_1 \rangle, \langle \tilde{e}_1 \rangle, \langle e_2 \rangle, \langle \tilde{e}_2 \rangle$ and $\langle z \rangle$. By arguing as above, we can multiply $f$ by an element in the subgroup generated by $\mu_j$ and $\tilde{\sigma}$, for $1 \leq j \leq k$ and $\sigma \in S_k$, such that the product stabilizes all the components, so that it belongs to $\text{Stab}(\Gamma)$. The proof finishes if we observe that $\tilde{\sigma} \mu_i = \mu_{\sigma(i)} \sigma$ for all $i \leq k$, so that all elements belong to the noncommutative group generated by $\{ [\tilde{\sigma}] : \sigma \in S_k \}$ and $\{ \mu_i \}$. \hfill \Box

Hence $\mathcal{W}(\Gamma) = \{ \mu_1, \ldots, \mu_k, [\tilde{\sigma}] : \sigma \in S_k, 1 \leq i_1 \leq \cdots \leq i_n \leq k \}$ has $2^k k!$ elements. Observe that, although any $\mu_i$ has order 4, its class $[\mu_i]$ has order 2. Besides $\mu_i$ and $\mu_j$ commute, so we can identify $\mathcal{W}(\Gamma)$ with the group $\mathcal{P}(K) \times S_k$, where the product is given by $(A, \sigma)(B, \eta) = (A \triangle \sigma(B), \sigma \eta)$ if $A, B \subset K = \{1, \ldots, k\}$, $\sigma, \eta \in S_k$, and where the elements in $\mathcal{P}(K)$ are the subsets of $\{1, \ldots, k\}$ and $\triangle$ denotes the symmetric difference. Thus

$$\mathcal{W}(\Gamma) \cong \mathbb{Z}_2^k \times S_k.$$ 

4. GRADINGS ON HEISENBERG SUPERALGEBRAS

In this section we will assume the ground field $\mathbb{F}$ to be algebraically closed and of characteristic other than 2. A Heisenberg superalgebra $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a nilpotent in two steps Lie superalgebra with one-dimensional center such that $[\mathcal{H}_0, \mathcal{H}_1] = 0$. In particular this implies that the even part is a Heisenberg algebra, so that it is determined up to isomorphism by its dimension. Note that, if $x, y \in \mathcal{H}_1$, then $[x, y] = [y, x] \in \mathbb{F}z = \mathcal{Z}(\mathcal{H})$, so there is a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{F}$ such that $[x, y] = \langle x, y \rangle z$ for all $x, y \in \mathcal{H}_1$. Hence there is a basis of $\mathcal{H}_1$ in which the matrix of $\langle \cdot, \cdot \rangle$ is the identity matrix. If $B_1 = \{ w_1, \ldots, w_m \}$ denotes such a basis, and $B_0 = \{ z, e_1, e_1, \ldots, e_k, \tilde{e}_k \}$ denotes the basis of $\mathcal{H}_0$ as in Section 3 then the product is given by

$$[e_i, \tilde{e}_i] = -[\tilde{e}_i, e_i] = z, \quad 1 \leq i \leq k,$n

$[w_j, w_j] = z, \quad 1 \leq j \leq m,$

with all other products being zero. As this algebra is completely determined by $n = 2k + 1$ and $m$, the dimensions of the even and odd part respectively, we denote it by $H_{n,m}$.

If we denote by $\circ : (H_{n,m})_1 \times (H_{n,m})_1 \to (H_{n,m})_0$ the bilinear mapping $x_1 \circ y_1 := [x_1, y_1]$, we get that the product in $H_{n,m}$ can be expressed by

$$[(x_0, x_1), (y_0, y_1)] = [x_0, y_0] + x_1 \circ y_1,$$

for all $x_i, y_i \in (H_{n,m})_i$.

Assume now that $H_{n,m}$ is a graded superalgebra and $G$ is the grading group. Then the even part $(H_{n,m})_0$ admits a basis $\{ z, e_1, \tilde{e}_1, \ldots, e_k, \tilde{e}_k \}$ of homogeneous elements as has been proved in the previous section. On the other hand the product in $(H_{n,m})_1$ is of the
form $x \circ y = \langle x, y \rangle z$ for any $x, y \in (H_{n,m})_1$ and where $\langle \cdot, \cdot \rangle: (H_{n,m})_1 \times (H_{n,m})_1 \to \mathbb{F}$ is a symmetric nondegenerate bilinear form. Furthermore, for any $g \in G$, denote by $L_g$ the subspace $L_g := (H_{n,m})_1 \cap (H_{n,m})_1$ (that is, the odd part of the homogeneous component of degree $g$ of $H_{n,m}$). Then $(H_{n,m})_1 = \bigoplus_{g \in G} L_g$ is a decomposition on linear subspaces and for any $g \in G$ there is a unique $h \in G$ such that $\langle L_g, L_h \rangle \neq 0$: indeed, assume $\langle L_g, L_h \rangle = 0$. Then $0 \neq L_g \circ L_h \subset \mathbb{F}z$ and this implies that $g + h = g_0$ where $g_0$ is the degree of $z$. Thus $h = g_0 - g$ is unique. Next we apply Lemma 2 to get a basis $\{z, e_1, \ldots, e_k, e_1, u_1, \ldots, u_r, v_r, z_1, \ldots, z_q\}$ of $H_{n,m}$ (of homogeneous elements) such that $z, e_i, \hat{e}_i$ generate the even part of $H_{n,m}$ while $u_1, v_1, \ldots, u_r, v_r, z_1, \ldots, z_q$ generate the odd part, and the nonzero products are:

\[
[\hat{e}_i, \hat{e}_j] = [u_j, v_j] = [z_l, z_l] = z,
\]

for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, r\}$ and $l \in \{1, \ldots, q\}$.

This basis provides a $\mathbb{Z}^{1+k+r} \times \mathbb{Z}^{m-2r}$-grading on $H_{n,m}$ given by

\[
\Gamma^r: \quad (H_{n,m})_1^{(2,0,\ldots,0;0,\ldots,0,0,\ldots,0)} = \langle z \rangle,
\]

\[
(H_{n,m})_1^{(1,0,\ldots,1,0;0,\ldots,0,0,\ldots,0)} = \langle e_i \rangle,
\]

\[
(H_{n,m})_1^{(1,0,\ldots,1,0,0,\ldots,0,0,\ldots,0)} = \langle \hat{e}_i \rangle,
\]

\[
(H_{n,m})_1^{(1,0,\ldots,0,0,\ldots,1,0,\ldots,0)} = \langle u_j \rangle,
\]

\[
(H_{n,m})_1^{(1,0,\ldots,0,0,\ldots,1,0,\ldots,0)} = \langle v_j \rangle,
\]

\[
(H_{n,m})_1^{(1,0,\ldots,0,0,\ldots,0,1,\ldots,0)} = \langle z_l \rangle,
\]

if $i \leq k, j \leq r, l \leq q$. This grading is a refinement of the original $G$-grading of the algebra.

Observe that for each $r$ such that $0 \leq 2r \leq m$, there exists a basis of $(H_{n,m})_1$ satisfying the relations (4) by taking for $j \leq r, l \leq m - 2r$,

\[
u_j := \frac{1}{\sqrt{2}} (w_{2j-1} + iw_{2j}),
\]

\[
u_j := \frac{1}{\sqrt{2}} (w_{2j-1} - iw_{2j}),
\]

\[
u_l := u_{l+2r},
\]

if $i \in \mathbb{F}$ is a primitive square root of the unit. If the starting $G$-grading is fine, then it is equivalent to the $\mathbb{Z}^{1+k+r} \times \mathbb{Z}^{m-2r}$-grading $\Gamma^r$ provided by the above basis. Therefore we have proved the following result.

**Theorem 2.** Up to equivalence, there are $m^2 + 1$ fine gradings on $H_{n,m}$ if $m$ is even and $m + 1$ if in case $m$ is odd, namely, $\{\Gamma^r : 2r \leq m\}$. All of these are inequivalent, and only one is toral, $\Gamma^m/\mathbb{Z}^m$ when $m$ is even.

The inequivalence follows easily from the fact that the universal grading groups are not isomorphic. Note that Theorem 1 is a particular case of Theorem 2 for $m = 0$.

Next, let us compute the group $\text{Aut}(H_{n,m})$ of automorphisms of $H_{n,m}$. Recall that $\text{Aut}(H_{n,m})$ is formed by the linear automorphisms $f: H_{n,m} \to H_{n,m}$ such that $f([x, y]) = [f(x), f(y)]$ for all $x, y \in H_{n,m}$ and $f((H_{n,m})_1) = (H_{n,m})_1$ for $i \in \{0, 1\}$.

If we identify $(H_{n,m})_1$ with the underlying vector space endowed with the inner product $\langle \cdot, \cdot \rangle$, then we can consider the group $\text{GO}((H_{n,m})_1) = \{g \in \text{End}((H_{n,m})_1) : \langle g(x), g(y) \rangle = \lambda \langle x, y \rangle \forall x, y \in (H_{n,m})_1 \}$ for some $\lambda \in \mathbb{F}^\times$, and the group homomorphism $\pi: \text{Aut}(H_{n,m}) \to \text{GO}((H_{n,m})_1)$ such that $\pi(f) = f((H_{n,m})_1)$. We prove next that $\pi$ is an epimorphism. Given $f \in \text{GO}((H_{n,m})_1)$, we know that there is $\lambda f \in \mathbb{F}^\times$ such that $\langle f(x), f(y) \rangle = \lambda \langle x, y \rangle$. Then we can define the linear map $\hat{f}: H_{n,m} \to H_{n,m}$ such that $\hat{f}$ restricted to $(H_{n,m})_1 = f$ and $\hat{f}(z) := \lambda_f z$, $\hat{f}(e_i) := \lambda_f e_i$ and $\hat{f}(\hat{e}_i) := \hat{e}_i$ for any $i$. 

Then \( f \in \text{aut}(H_{n,m}) \) and \( \pi(f) = f \). Furthermore \( \ker(\pi) \cong \text{Sp}(P) \times F^{2k} \), taking into account the results in Section[3] and that any \( f \in \text{aut}(H_{n,m}) \) such that \( \pi(f) = f|_{\text{aut}(H_{n,m})} = 1 \) satisfies \( f(z) = z \). Therefore we have a short exact sequence

\[
1 \rightarrow \text{Sp}(P) \times F^{2k} \rightarrow \text{Aut}(H_{n,m}) \rightarrow \text{GO}(\text{aut}(H_{n,m})_1) \rightarrow 1
\]

which is split since \( j : \text{GO}(\text{aut}(H_{n,m})_1) \rightarrow \text{Aut}(H_{n,m}) \) defined by \( j(f) := \hat{f} \) satisfies \( \pi j = 1 \). Then \( \text{Aut}(H_{n,m}) \) is the semidirect product

\[
\text{Aut}(H_{n,m}) \cong (\text{Sp}(P) \times F^{2k}) \rtimes \text{GO}(\text{aut}(H_{n,m})_1).
\]

Finally, we compute the Weyl groups of the fine gradings on \( H_{n,m} \) described in Theorem[2]. Consider, as in Section[3] the maps \( \hat{\sigma}, \mu_i \in \text{Aut}(H_{n,m})_0 \) if \( \sigma \in S_k, i \leq k \), and extend to automorphisms of \( H_{n,m} \) by setting \( \hat{\sigma}|_{H_{n,m}} = \mu_i|_{H_{n,m}} \). Thus, \( \hat{\sigma}, \mu_i \in \text{Aut}(\Gamma^r) \), for \( \Gamma^r \) the grading in [5]. Take too, for each permutation \( \sigma \in S_r \), the map \( \hat{\sigma} : H_{n,m} \rightarrow H_{n,m} \) given by \( \hat{\sigma}|_{H_{n,m}} = \text{id} \), \( \hat{\sigma}(u_i) = \sigma(u_i) \), \( \hat{\sigma}(v_i) = \sigma(v_i) \) and \( \hat{\sigma}(\bar{\rho}) = \bar{\rho} \). Also consider for each permutation \( \rho \in S_q \), the map \( \mu_i : H_{n,m} \rightarrow H_{n,m} \) given by \( \mu_i|_{H_{n,m}} = \text{id} \), \( \mu_i(u_i) = v_i, \mu_i(v_i) = u_i \) and \( \mu_i(\bar{\rho}) = \bar{\rho} \). Finally consider for each index \( i \leq r \), the map \( \mu_i : H_{n,m} \rightarrow H_{n,m} \) given by \( \mu_i|_{H_{n,m}} = \text{id} \), \( \mu_i(u_i) = v_i, \mu_i(v_i) = u_i \) and \( \mu_i(\bar{\rho}) = \bar{\rho} \). It is clear that \( \hat{\sigma}, \mu_i \in \text{Aut}(\Gamma^r) \) in any case.

**Proposition 2.** The Weyl group \( \mathcal{W}(\Gamma^r) \) is generated by \( [\mu_1], [\mu_1'], [\hat{\sigma}] : \sigma \in S_k \), \( [\hat{\sigma}] : \sigma \in S_r \) and \( [\hat{\sigma}] : \sigma \in S_q \), with \( k = (n-1)/2, q = m - 2r \).

**Proof.** We know by Section[5] that the subgroup \( W \) of \( \mathcal{W}(\Gamma^r) \) generated by the classes of the elements fixing all the homogeneous components of \( (H_{n,m})_1 \) is \( \{[\mu_1, \ldots, \mu_i, \hat{\sigma}] : \sigma \in S_k, 1 \leq i_1 \leq \cdots \leq i_s \leq k \} \).

Let \( f \) be an arbitrary element in \( \text{Aut}(\Gamma^r) \). As \( f|_{H_{n,m}} \) preserves the grading \( \Gamma \) in Equation[4], then we can compose \( f \) with an element in \( W \) to assume that \( f \) preserves all the homogeneous components of \( (H_{n,m})_0 \).

Then the element \( f(u_1) \) belongs to some homogeneous component of \( (H_{n,m})_1 \), but it does not happen that there is \( j \leq q \) such that \( f(u_1) \in Fz_j \), since then \( 0 = f(u_1, u_1) = z_j \circ z_j = z \). So there is an index \( i \leq r \) such that either \( f(u_1) \in Fu_i \) or \( f(u_1) \in Fv_i \). The same arguments as in the proof of Proposition[1] show that we can replace \( f \) by \( \mu_j \circ \cdots \mu_1 \circ f \) for \( 1 \leq j_i \leq \cdots \leq j_s \leq r \) and \( \sigma \in S_r \) to get that \( f(u_1) \in Fv_i \) and \( f(v_i) \in Fu_i \) for all \( i \leq r \).

Now it is clear that \( f(z_1) \in Fz_l \) for some \( 1 \leq l \leq q \). If \( l \neq 1 \), we can replace \( f \) with \( \hat{\rho} f \), for \( \rho = (1, l) \), so that we can assume \( f(z_1) = \bar{\rho} \). And, in the same way, we can assume that \( f(z_1) = Fz_1 \) for \( 1 \leq l \leq q \). Our new \( f \) belongs to \( \text{Stab}(\Gamma^r) \).

The proof finishes if we observe that \( \sigma \mu_j = \mu_j' \sigma \) for all \( i \leq r \) and \( \sigma \in S_r \), and that \( \hat{\sigma} \) as well as \( \mu_j' \) commute with \( \bar{\rho} \) for all \( \rho \in S_q \). \( \square \)

Hence, an arbitrary element in \( \mathcal{W}(\Gamma^r) \) is

\[
[\mu_1, \ldots, \mu_i, \hat{\sigma}, \mu_j', \ldots, \mu_j' \hat{\rho}]
\]

for \( 1 \leq i_1 \leq \cdots \leq i_k, 1 \leq j_1 \leq \cdots \leq j_l \leq r, \hat{\sigma} \in S_k, \hat{\rho} \in S_q, \) so that \( \mathcal{W}(\Gamma^r) \) is isomorphic to

\[
(P(1, \ldots, k)) \rtimes S_k \times (P(1, \ldots, r)) \rtimes S_r \times S_q
\]
with the product as in Section [3] and in a more concise form,
\[ \mathcal{W}(\Gamma^r) \cong \mathbb{Z}_{2}^{r+k} \times (S_k \times S_r \times S_q). \]

5. An Application to Heisenberg Lie Color Algebras

The base field \( \mathbb{F} \) will be supposed throughout this section algebraically closed and of characteristic other than 2, as in Section 4. Lie color algebras were introduced in [35] as a generalization of Lie superalgebras and hence of Lie algebras. This kind of algebras has attracted the interest of several authors in the last years, (see [10, 32, 34, 41, 42]), being also remarkable the important role they play in theoretical physics, specially in conformal field theory and supersymmetries (38, 40).

**Definition 1.** Let \( G \) be an abelian group. A skew-symmetric bicharacter of \( G \) is a map \( \epsilon: G \times G \rightarrow \mathbb{F}^\times \) satisfying
\[
\epsilon(g_1, g_2) = \epsilon(g_2, g_1)^{-1}, \\
\epsilon(g_1, g_2 + g_3) = \epsilon(g_1, g_2)\epsilon(g_1, g_3),
\]
for any \( g_1, g_2, g_3 \in G \).

Observe that \( \epsilon(g, 0) = 1 \) for any \( g \in G \), where 0 denotes the identity element of \( G \).

**Definition 2.** Let \( L = \bigoplus_{g \in G} L_g \) be a \( G \)-graded \( \mathbb{F} \)-vector space. For a nonzero homogeneous element \( v \in L \), denote by \( \deg v \) the unique element in \( G \) such that \( v \in L_{\deg v} \). We shall say that \( L \) is a Lie color algebra if it is endowed with an \( \mathbb{F} \)-bilinear map (the Lie color bracket)
\[ [\cdot, \cdot]: L \times L \rightarrow L \]
satisfying \([L_g, L_h] \subset L_{g+h}\) for all \( g, h \in G \) and
\[
[v, w] = -\epsilon(\deg v, \deg w)[w, v], \quad \text{ (color skew-symmetry)} \\
[v, [w, t]] = [[v, w], t] + \epsilon(\deg v, \deg w)[w, [v, t]], \quad \text{ (Jacobi color identity)}
\]
for all homogeneous elements \( v, w, t \in L \) and for some skew-symmetric bicharacter \( \epsilon \).

Two Lie color algebras are isomorphic if they are isomorphic as graded algebras.

Clearly any Lie algebra is a Lie color algebra and also Lie superalgebras are examples of Lie color algebras (take \( G = \mathbb{Z}_2 \) and \( \epsilon(i, j) = (-1)^{ij} \), for any \( i, j \in \mathbb{Z}_2 \)).

Heisenberg Lie color algebras have been previously considered in the literature (see [27]). Fixed a skew-symmetric bicharacter \( \epsilon: G \times G \rightarrow \mathbb{F}^\times \), a Heisenberg Lie color algebra \( H \) is defined in [27] as a \( G \)-graded vector space, where \( G \) is a torsion-free abelian group with a basis \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \), in the form
\[ H = \bigoplus_{i=1}^{n} \mathbb{F}p_i \oplus \bigoplus_{j=1}^{n} \mathbb{F}q_j \oplus \mathbb{F}c \]
where \( p_i \in H_{\epsilon_i}, q_j \in H_{-\epsilon_j}, \) and \( c \in H_0; \) and where the Lie color bracket is given by
\[ [p_i, q_j] = \delta_{ij}c \quad \text{and} \quad [p_i, p_j] = [q_i, q_j] = [p_i, c] = [q_i, c] = [c, p_j] = [c, q_j] = 0. \]

Observe that, following this definition, the class of Heisenberg superalgebras is not contained in the one of Heisenberg Lie color algebras. Hence, this definition seems to us very restrictive. So let us briefly discuss about the concept of Heisenberg Lie color algebras. Any Heisenberg algebra (respectively Heisenberg superalgebra) \( H \) is characterized among the Lie algebras (respectively among the Lie superalgebras) for satisfying \([H, H] = Z(H)\) and \( \dim(Z(H)) = 1 \). Hence, it is natural to introduce the following definition.
Definition 3. A Heisenberg Lie color algebra is a Lie color algebra \( L \) such that \( [L, L] = \mathcal{Z}(L) \) and \( \dim(\mathcal{Z}(L)) = 1 \).

Examples.
1. The above so called Heisenberg Lie color algebras, given in [27], satisfy these conditions and so can be seen as a particular case of the ones given by Definition 3. As examples of Heisenberg Lie color algebras we also have the Heisenberg algebras \( (G = \{0\}) \) and the Heisenberg superalgebras \( (G = \{Z_2\}) \). We can also consider any Heisenberg \( G \)-graded algebra as a Heisenberg Lie color algebra for the group \( G \) and the trivial bicharacter given by \( \epsilon(g, h) = 1 \) for all \( g, h \in G \).

2. Any \( G \)-graded \( L = \bigoplus_{g \in G} M_g \) Heisenberg superalgebra \( L = L_0 \oplus L_1 \) gives rise to a Heisenberg Lie color algebra relative to the group \( G \times \mathbb{Z}_2 \) and an adequate skew-symmetric bicharacter \( \epsilon \). In fact, we just have to \((G \times \mathbb{Z}_2)\)-graduate \( L \) as \( L = \bigoplus_{(g, i) \in G \times \mathbb{Z}_2} M_{(g, i)} \) where \( M_{(g, i)} = M_g \) with \( M_{(g, i)} \subseteq L_i \), and define \( \epsilon : (G \times \mathbb{Z}_2) \times (G \times \mathbb{Z}_2) \to \mathbb{F}^x \) as \( \epsilon((g, i), (h, j)) := (-1)^{ij} \). Observe that both gradings on \( L \) are equivalent.

3. Consider some \( g_0 \in G \), a graded vector space \( V = \bigoplus_{g \in G} V_g \) such that \( \dim(V_g) = \dim(V_{-g+g_0}) \) for any \( g \neq \{g_0, 0\} \) and \( \dim(V_{g_0}) = \dim(V_0) + 1 \) in case \( g_0 \neq 0 \); and any skew-symmetric bicharacter \( \epsilon : G \times G \to \mathbb{F}^x \) satisfying \( \epsilon(g, g) = -1 \) for any \( g \in G \) such that \( 2g = g_0 \) and \( V_0 \neq 0 \). Fix bases \( \{z, u_{g_0,1}, ..., u_{g_0,n_{g_0}}\} \) and \( \{\hat{u}_{g_0,1}, ..., \hat{u}_{g_0,n_{g_0}}\} \) of \( V_{g_0} \) and \( V_0 \) respectively when \( g_0 \neq 0 \) or \( \{z, u_{g_0,1}, ..., u_{g_0,n_{g_0}}, \hat{u}_{g_0,1}, ..., \hat{u}_{g_0,n_{g_0}}\} \) when \( g_0 = 0 \). For any subset \( \{g, -g + g_0\} \neq \{g_0, 0\} \) of \( G \), fix also basis \( \{u_{g,1}, ..., u_{g,n_g}\} \) and \( \{\hat{u}_{g,1}, ..., \hat{u}_{g,n_g}\} \) of \( V_g \) and \( V_{-g+g_0} \) in case \( 2g \neq g_0 \) and \( \{u_{g_1,1}, ..., u_{g,n_{g_1}}\} \) basis of \( V_{g_0} \) in case \( 2g = g_0 \). Then by defining a product on \( V \) given by \( [u_{g,i}, \hat{u}_{g,i}] = z \), \( [u_{g,i}, u_{g,i}] = -\epsilon(-g + g_0, g) z \) in the cases \( g = 0 \) or \( 2g \neq g_0 \), \( [u_{g,i}, \hat{u}_{g,i}] = z \) in the cases \( 2g = g_0 \) with \( g \neq 0 \), and the remaining brackets zero, for any subset \( \{g, -g + g_0\} \) of \( G \), we get that \( V \) becomes a Heisenberg Lie color algebra that we call of type \((G, g_0, \epsilon)\). We note that for an easier notation we allow empty basis in the above definition which correspond to trivial subspaces \( \{0\} \).

Lemma 3. A Heisenberg Lie color algebra \( L \) of type \((G, g_0, \epsilon)\) is a graded Heisenberg superalgebra if and only if \( \epsilon(g, -g + g_0) \in \{1\} \) for any \( g \in G \) such that \( L_g \neq 0 \).

Proof. Suppose \( L \) is a grading of a Heisenberg superalgebra \( L = L_0 \oplus L_1 \) and there exists \( g \in G \) with \( L_g \neq 0 \) and such that \( \epsilon(g, -g + g_0) \notin \{1\} \). Since \( [L_0, L_1] = 0 \), \( L_g + L_{-g+g_0} \subset L_i \) for some \( i \in \mathbb{Z}_2 \). By taking either \( u_g \in L_g \) and \( \hat{u}_g \in L_{-g+g_0} \) in the cases \( g = 0 \) or \( 2g \neq g_0 \); or \( u_g \in L_g \) in the case \( 2g = g_0 \) with \( g \neq 0 \), elements of the standard basis of \((G, g_0, \epsilon)\) described in Example 3, we have either \( 0 \neq [u_g, \hat{u}_g] = -\epsilon(g, -g + g_0)[u_g, u_g] \) if \( 2g \neq g_0 \) or \( 0 \neq [u_g, u_g] = -\epsilon(g, -g + g_0)[u_g, u_g] \) if \( 2g = g_0 \), being \( \epsilon(g, -g + g_0) \neq \pm 1 \), which contradicts the identities of a Lie superalgebra.

Conversely, if \( \epsilon(g, -g + g_0) \in \{1\} \) for any \( g \in G \) with \( L_g \neq 0 \), we can \( \mathbb{Z}_2 \)-graduate \( L \) as \( L = \bigoplus_{g \in \text{supp}(G) : \epsilon(g, -g + g_0) = 1} L_g \bigoplus_{h \in \text{supp}(G) : \epsilon(h, -h + g_0) = -1} L_h \), this one becoming a Heisenberg Lie superalgebra, graded by the group generated by the support \( \text{supp}(G) := \{g \in G : L_g \neq 0\} \). 

Proposition 3. Any Heisenberg Lie color algebra is isomorphic to a Heisenberg Lie color algebra of type \((G, g_0, \epsilon)\).
Proof. Consider a Heisenberg Lie color algebra \( L = \bigoplus_{g \in G} L_g \). We rule out the trivial case \( G = \{0\} \) which gives the Heisenberg Lie color algebra of type \( \{0\} \), that is, a Heisenberg algebra with trivial grading. Since \( \dim(Z(L)) = 1 \), we can write \( Z(L) = \langle z \rangle \), being \( z = \sum_{i=1}^n x_{g_i} \), with any \( 0 \neq x_{g_i} \in L_{g_i} \), and \( g_i \neq g_j \) if \( i \neq j \). If \( n \neq 1 \), then \( [x_{g_1}, L_h] \subset L_{g_1+h} \cap \langle z \rangle = 0 \) for any \( h \in G \) and \( i \in \{1, \ldots, n\} \). Hence any \( x_{g_i} \in Z(L) = \langle z \rangle \), a contradiction. Thus \( z \in L_{g_0} \) for some \( g_0 \in G \). Now the fact \([L, L] \subset Z(L)\) gives us that for any \( g \in G \), \([L_g, L_h] = 0 \) if \( g \neq g_0 \), and consequently \([L_g, L_{g+g_0}] \neq 0 \) if \( g \neq g_0 \).

Since for any skew-symmetric bicharacter \( \epsilon : G \times G \rightarrow \mathbb{R}^\times \) and \( g \in G \) we have \( \epsilon(g, g) \in \{\pm 1\} \), we can \( \mathbb{Z}_2 \)-graduate \( L \) as

\[
L = (\bigoplus_{\{g \in G : \epsilon(g, g) = 1\}} L_g) \oplus (\bigoplus_{\{h \in G : \epsilon(h, h) = -1\}} L_h).
\]

This is a grading on the algebra \( L \) since \( \epsilon(g \pm h, g + h) = \epsilon(g, g) \epsilon(h, g) \epsilon(h, g) = \epsilon(g, g) \epsilon(h, h) = \epsilon(g, g) \epsilon(h, h) \).

Let us distinguish two cases, according to the dichotomy \( g_0 = 0 \) or \( g_0 \neq 0 \).

First, assume \( g_0 = 0 \). For any \( g \in G \) it is easy to check that \( \epsilon(g, g) = \epsilon(-g, -g) = \epsilon(g, -g) = \epsilon(-g, g) \in \{\pm 1\} \). This fact together with the observations in the previous paragraph tell us that \( L = (\bigoplus_{\{g \in G : \epsilon(g, g) = 1\}} L_g) \oplus (\bigoplus_{\{h \in G : \epsilon(h, h) = -1\}} L_h) \) is actually a Lie superalgebra, satisfying \([L, L] \subset Z(L)\) and \( \dim(Z(L)) = 1 \), and being the initial Lie color grading a refinement of the \( \mathbb{Z}_2 \)-grading as superalgebra. By arguing as in [14] we easily get that \( L \) is of type \((G, 0, \epsilon)\), being also a grading of a Heisenberg superalgebra.

Second, assume that \( g_0 \neq 0 \). Since \([L, L] \subset \langle z \rangle \subset L_{g_0} \), then \([L_{g_0}, L_{g_0}] \subset L_{2g_0} \cap L_{g_0} = 0 \), so that \( L' := L_{g_0} \oplus L_0 \) is a Lie algebra (take into consideration \( \epsilon(g_0, 0) = \epsilon(0, 0) = 1 \)). If \( L_0 = 0 \), then \( L' = \langle z \rangle \) and otherwise \([L', L'] = Z(L') \) with \( \dim(Z(L')) = 1 \). In the second situation, we have that \( L' = L_{g_0} \oplus L_0 \) is a Heisenberg algebra, so that taking into account Section 3 the grading is toral and there exist basis \( \{z, u_{g_0,1}, \ldots, u_{g_0,n_{g_0}}\} \) and \( \{\tilde{u}_{g_0,1}, \ldots, \tilde{u}_{g_0,n_{g_0}}\} \) of \( L_{g_0} \) and \( L_0 \) respectively such that \([u_{g_0,i}, u_{g_0,i}] = z, [\tilde{u}_{g_0,i}, u_{g_0,i}] = -z \) and such that the remaining products in \( L' \) are zero. Consider now any subset \( \{g, -g + g_0\} \neq \{g_0, 0\} \) of \( G \). In case \( L_g \neq 0 \), then necessarily \( L_{-g+g_0} \neq 0 \) and we cannot distinguish two possibilities. First, if \( 2g \neq g_0 \), the facts \([L_g, L_h] = 0 \) if \( g \neq g_0 \) and \([L_g, L_{g+g_0}] = \langle z \rangle \) allow us to apply standard linear algebra arguments to obtain basis \( \{u_{g,1}, \ldots, u_{g,n_g}\} \) and \( \{\tilde{u}_{g,1}, \ldots, \tilde{u}_{g,n_g}\} \) of \( L_g \) and \( L_{-g+g_0} \) such that \([u_{g,i}, \tilde{u}_{g,i}] = z, [u_{g,i}, u_{g,i}] = -\epsilon(-g + g_0, g)z \) and being null the rest of the products among the elements of the basis. Second, in the case \( 2g = g_0 \), a similar argument gives us \( \{u_{g,1}, \ldots, u_{g,n_g}\} \) a basis of \( L_g \) such that \([u_{g,i}, u_{g,i}] = z \) with the remaining brackets zero. Also observe that necessarily \( \epsilon(g, g) = -1 \) for any \( g \in G \) such that \( 2g = g_0 \) and \( L_g \neq 0 \) because in the opposite case \( \epsilon(g, g) = 1 \) and then \( 0 \neq L_g \subset Z(L) \subset L_0 \), a contradiction. Summarizing, we have showed that \( L \) is isomorphic to a Heisenberg Lie color algebra of type \((G, g_0, \epsilon)\) with \( g_0 \neq 0 \). \( \square \)

We finish this section by showing how the results in Section 3 can be applied to the classification of Heisenberg Lie color algebras. Following Lemma 3 and the arguments in the proof of Proposition 3 any Heisenberg Lie color algebra \( L = \bigoplus_{g \in G} L_g \) is isomorphic to a grading of a Heisenberg superalgebra if and only if either \( Z(L) \subset L_0 \) or \( L \) is of the type \((G, g_0, \epsilon)\) with \( \epsilon(g, -g + g_0) \in \{\pm 1\} \) for any \( g \in G \) such that \( L_g \neq 0 \). In particular, this is the case of the Heisenberg Lie color algebras considered in [27]. Hence, \( L \) is isomorphic to a coarsening of a fine grading \( H_{n,m} = \bigoplus_{k \in K} (H_{n,m})_k \) of a Heisenberg
superalgebra. Since it is known the procedure to compute all of the coarsenings of a given grading when this is given by its universal group grading (see Section 2 and [17]), we can apply Theorem 2 to get the list of all of these Heisenberg Lie color algebras $L$, in the moment $G$ and $K$ are generated by their supports and $K$ is the universal grading group. Of course this procedure does not hold for a Heisenberg Lie color algebra which is not a grading of a Heisenberg superalgebra.

6. Gradings on twisted Heisenberg algebras

In this final section we consider the so called twisted Heisenberg algebras. As mentioned in the introduction, these algebras appear naturally as some of the direct summands of the Lie algebras of connected Lie groups acting isometrically and locally faithfully on compact connected Lorentzian manifolds.

The ground field $F$ will be assumed to be algebraically closed of characteristic zero.

6.1. Geometric definition of twisted Heisenberg algebra. We follow the approach of [2, Chapter 8], adapted to our context. Take $λ := (λ_1, \ldots, λ_k) \in (\mathbb{R}^\times)^k$. For each $t \in \mathbb{R}$, take $θ_t : H_n \to H_n$ the automorphism of the Lie algebra given in terms of the basis in Equation (1) by

$$
θ_t(e_j) = \cos(λ_j t)e_j + \sin(λ_j t)\hat{e}_j,
$$

$$
θ_t(\hat{e}_j) = -\sin(λ_j t)e_j + \cos(λ_j t)\hat{e}_j,
$$

$$
θ_t(z) = z.
$$

Thus we have $\{θ_t : t \in \mathbb{R}\}$ a uniparametric subgroup of $Aut(H_n)$. This induces, by monodromy, an action of $\mathbb{R}$ by Lie group automorphisms of $\mathfrak{h}_n$, the Lie group of $H_n$. The group $\mathfrak{h}_n^λ := \mathbb{R} \ltimes \mathfrak{h}_n$ is then called a twisted Heisenberg group, and its tangent Lie algebra is called a twisted Heisenberg Lie algebra, and it is denoted by $H_n^λ$. According to [2, Lemma 8.2.1], $H_n^λ$ has a basis $\{X_1, \ldots, X_k, Y_1, \ldots, Y_k, W, Z\}$ such that $[X_i, Y_i] = Z$ (a central element), $[W, X_i] = λ_i Y_i$ and $[W, Y_i] = -λ_i X_i$.

6.2. Algebraic definition of twisted Heisenberg algebra. Consider the Heisenberg algebra $H_n$ over our algebraically closed field $F$ and take $d$ to be any derivation of $H_n$. Then one can define in $F \times H_n$ the product

$$
[(\lambda, a), (\mu, b)] := (0, \lambda d(b) - \mu d(a) + [a, b]),
$$

for any $\lambda, \mu \in \mathbb{R}$, $a, b \in H_n$. This defines a Lie algebra structure in $F \times H_n$ and we will denote this Lie algebra by $H_n^d$. If we define $u = (1, 0)$, then $H_n^u = Fu \oplus H_n$ and its product can be rewritten as $[\lambda u + a, \mu u + b] = \lambda[u, b] - \mu[u, a] + [a, b]$. Thus, $d = ad(u)|H_n$. If $d_1$ and $d_2$ are derivations of $H_n$ such that $H_n = Im(d_1) + Fz$, it is easy to see that $H_n^{d_1} \cong H_n^{d_2}$ if and only if there is some nonzero scalar $λ_0$, an element $x_0 \in H_n$, an automorphism $θ \in Aut(H_n)$ such that $θ(λ_0 d_1 + ad(x_0))θ^{-1} = d_2$. In particular if the two derivations $d_1$ and $d_2$ are in the same orbit on the action of $Aut(H_n)$ on $Der(H_n)$ by conjugation, then both algebras $H_n^{d_1}$ and $H_n^{d_2}$ are isomorphic.

There is a more intrinsic definition of this kind of algebras which is equivalent to that of $H_n^d$. On one hand, the Lie algebra $H_n^d$ fits in a split exact sequence

$$
0 \to H_n \overset{i}{\to} H_n^d \overset{p}{\to} F \to 0
$$

where $i$ is the inclusion map and $p(λ, a) = λ$ for any $a \in H_n$. The sequence is split because we can define $j : F \to H_n^d$ by $j(1) = (1, 0)$ and then $pj = 1_F$. On the other hand if we consider any algebra $A$ which is a split extension of the type

$$
0 \to H_n \to A \overset{p}{\to} F \to 0,
$$

we can class $A$ by its universal group grading $L$, in the moment $G$ and $K$ are generated by their supports and $K$ is the universal grading group.
then $A$ is isomorphic to some $H^d_n$ for a suitable derivation $d$ of $H_n$.

Observe that $H^d_n$ is isomorphic to $H^t_n$ if one takes the particular derivation of $H_n$ given in terms of the basis $\{z, e_1, \hat{e}_1, \ldots, e_k, \hat{e}_k\}$ of $H_n$ ($n = 2k + 1$) by $d(z) = 0$, $d(e_i) = \lambda_i \hat{e}_i$ and $d(\hat{e}_i) = -\lambda_i e_i$ for a fixed $k$-tuple $\lambda := (\lambda_1, \ldots, \lambda_k) \in (\mathbb{Q}^\times)^k \subset (\mathbb{R}^\times)^k$, since $[u, e_i] = \lambda_i \hat{e}_i$ while $[u, \hat{e}_i] = -\lambda_i e_i$. This motivates that, for any choice of $\lambda \in (\mathbb{R}^\times)^k$, we denote such an algebra by $H^\lambda_n$ and call it also a twisted Heisenberg Lie algebra, although algebraically it is certain extension of $H_n$ more than a twist. A definition depending on a basis is convenient because of our necessity of making explicit computations when dealing with fine gradings. Besides note that under a suitable change of coordinates, the basis $\{z, u, e_1, \hat{e}_1, \ldots, e_k, \hat{e}_k\}$ can be chosen to satisfy

\begin{equation}
\begin{aligned}
[e_i, \hat{e}_i] &= \lambda_i z, \quad [u, e_i] = \lambda_i \hat{e}_i \quad \text{and} \quad [u, \hat{e}_i] = \lambda_i e_i,
\end{aligned}
\end{equation}

where we have used $i = \sqrt{-1} \in \mathbb{F}$. Thus, to fix the ideas we give the following:

**Definition 4.** Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbb{R}^\times)^k$, $k > 0$. The corresponding twisted Heisenberg algebra $H^\lambda_n$ of dimension $n = 2k + 2$ is the Lie algebra spanned by the elements

\begin{equation}
\{z, u, e_1, \hat{e}_1, \ldots, e_k, \hat{e}_k\},
\end{equation}

and the nonvanishing Lie brackets are given by Equation (6) for any $i = 1, \ldots, k$.

As before, $H^\lambda_n = \mathbb{F} u \oplus H_n$ where $H_n$ can be identified with the subalgebra spanned by $\{z, e_1, \hat{e}_1, \ldots, e_k, \hat{e}_k\}$. This algebra is not nilpotent, but it is solvable, since $[H^\lambda_n, H^\lambda_n] = H_n$.

The isomorphism condition given above in terms of derivations can be applied to $H^\lambda_n$ and so a direct argument proves that given $\lambda = (\lambda_i) \in (\mathbb{R}^\times)^n$ and $\mu = (\mu_i) \in (\mathbb{R}^\times)^n$, one has $H^\lambda_n \cong H^\mu_n$ if and only if there is a permutation $\sigma \in S_n$ and a scalar $k \in \mathbb{R}^\times$ such that $\mu_i = k\lambda_{\sigma(i)}$ for all $i \leq k$.

### 6.3. Torality and basic examples

We are now dealing with two fine gradings on $H^\lambda_n$ which will be relevant to our work. One of them is toral while the other is not.

A fine grading on $H^\lambda_n$ is obviously provided by our basis in Equation (7):

\begin{equation}
H^\lambda_n = \langle z \rangle \oplus \langle u \rangle \oplus (\oplus_{i=1}^k \langle e_i \rangle) \oplus (\oplus_{i=1}^k \langle \hat{e}_i \rangle).
\end{equation}

Again it is also a group grading. In order to find $G_0$ the universal grading group, note that necessarily (we denote $\deg x = g$ when $x \in (H^\lambda_n)_g$) the following assertions about the degrees are verified:

\begin{align*}
\deg e_i + \deg \hat{e}_i & = \deg z, \\
\deg e_i + \deg u & = \deg \hat{e}_i, \\
\deg \hat{e}_i + \deg u & = \deg e_i.
\end{align*}

Hence $\deg u \in G_0$ has order 1 or 2 and $2(\deg e_i - \deg \hat{e}_i) = 0$ (providing $k - 1$ order two elements). The universal grading group $G_0$ is the abelian group with generators $\deg u, \deg z, \deg e_i, \deg \hat{e}_i$ and relations above. It can be computed to be $G_0 = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2^{k-1}$ and the grading is given as follows:

\begin{equation}
\Gamma_1 : \quad (H^\lambda_n)_{(2;1;0;\ldots;0;\ldots;0)} = \langle z \rangle, \\
(H^\lambda_n)_{(0;1;0;\ldots;0;\ldots;0)} = \langle u \rangle, \\
(H^\lambda_n)_{(1;1;0;\ldots;0;\ldots;0)} = \langle e_i \rangle \quad (i \text{ in the } i\text{-th slot when } i \neq k), \\
(H^\lambda_n)_{(1;0;0;\ldots;0;\ldots;0)} = \langle \hat{e}_i \rangle, \\
(H^\lambda_n)_{(1;1;0;\ldots;0;\ldots;0)} = \langle e_k \rangle, \\
(H^\lambda_n)_{(1;0;0;\ldots;0;\ldots;0)} = \langle \hat{e}_k \rangle.
\end{equation}
Lemma 4. For each group grading on $H_n^\lambda$, there is a basis $\{z, u', u'_1, v'_1, \ldots, u'_k, v'_k\}$ of $H_n^\lambda$ with $u'$ a homogeneous element of the grading, such that the only nonzero brackets are
given by
\[
[u', u'_i] = \lambda_i u'_i, \\
[u', v'_i] = -\lambda_i v'_i, \\
[u'_i, v'_i] = -2\lambda_i z.
\]

Proof. Let $\Gamma = \oplus_{g \in G} L_g$ be a group grading on $L = H^\lambda$, as any automorphism leaves invariant $[L, L] = H_n$ and $\mathcal{Z}(L) = \langle z \rangle$, this implies that $z$ is homogeneous and $H_n$ is a graded subspace (the homogeneous components of any element in $H_n$ are again elements in $H_n$).

Since not every homogeneous element is contained in $H_n$ there must be someone of the form $a = \lambda u + h$ with $\lambda$ a nonzero scalar and $h \in H_n$. Then $u' := \lambda^{-1} a$ is a homogeneous element and $u' - u \in H_n$. Hence $u' = u + \alpha z + \sum_{i=1}^k \alpha_i u_i + \sum_{j=1}^h \beta_j v_j$ for some choice of scalars $\alpha, \alpha_i, \beta_j \in F$. If we take $u'_i = u_i + 2\beta_i z$ and $v'_i = v_i + 2\alpha_i z$, then the basis $\{z, u', u'_1, u'_2, \ldots, u'_k, v'_0\}$ trivially satisfies the required conditions.

Consequently, fixed $\Gamma = \oplus_{g \in G} L_g$ a group grading on $L = H^\lambda$, there is not loss of generality in supposing that $u$ is homogeneous. Let us denote by $h \in G$ the degree of $u$ in $\Gamma$. Our next aim is to show that $h$ is necessarily of finite order. From now on we are going to denote by $\varphi$ be the inner derivation
\[
\varphi := \text{ad}(u) : H^\lambda \rightarrow H^\lambda, \quad x \mapsto [u, x],
\]
which is going to be a key tool in the study of the group gradings on $H^\lambda$. If $0 \neq x \in [u, L]$ is a homogeneous element, then there is $g \in G$ such that $x = \sum_{i=1}^k (c_i u_i + d_i v_i) \in L_g$ for some $c_i, d_i \in F$, so that $\varphi^t(x) = \sum_i (c_i u_i + (-1)^t d_i v_i) \lambda^{t}_i \in L_{g+x}$ is not zero for all $t \in \mathbb{N}$. Taking into account that at most there are $2k$ independent elements in the left set
\[
\{ \sum_{i=1}^k (c_i u_i + (-1)^t d_i v_i) \lambda^{t}_i : t = 0, 1, 2, \ldots \} \subset \langle \{u_1, v_1, \ldots, u_k, v_k\} \rangle,
\]
hence there is a positive integer $r \leq 2k$ with $\varphi^r(x) \in \langle \{\varphi^t(x) : 0 \leq t < r\} \rangle$. Since $\varphi^r(x) \in L_{g+r} \cap (\sum_{t < r} L_{g+th})$, necessarily there exists $t < r$ such that $g + rh = g + th$, so that $(r - t)h = 0$, as we wanted to show.

Let us denote by $l$ the order of $h$ in $G$. Recall that the set of eigenvalues of $\varphi$ is $\{\lambda_1, -\lambda_1, \ldots, \lambda_k, -\lambda_k, 0, 0\}$ with respective eigenvectors $\{u_1, v_1, \ldots, u_k, v_k, u, z\}$ (recall Equation (8)), so that the set of eigenvalues of $\varphi|_{[u, L]}$ is
\[
\{\lambda_1, -\lambda_1, \ldots, \lambda_k, -\lambda_k\} = \text{Spec}(\text{ad}(u)|_{[u, L]}) =: \text{Spec}(u).
\]
Fix some $\lambda_i \in \text{Spec}(u)$, consider the eigenspace of $\varphi$ given by $V_{\lambda_i} := \{x \in L : \varphi(x) = \lambda_i x\}$ and denote by
\[
V^{l}_{\lambda_i} := \{x \in L : \varphi^l(x) = \lambda_i^l x\}.
\]
It is obviously nonzero, because $u_i \in V_{\lambda_i}$, $\subset V^l_{\lambda_i}$. Moreover, as $V^{l}_{\lambda_i}$ is invariant under $\varphi$, we have that $\varphi|_{V^{l}_{\lambda_i}}$ is diagonalizable and

\begin{align}
V^{l}_{\lambda_i} &= \bigoplus_{j=0}^{l-1} V_{\xi^j \lambda_i}.
\end{align}

for $\xi$ a fixed primitive $l$th root of the unit. Note that if $x \in V^{l}_{\lambda_i}$, then $\sum_{q=0}^{l-1} (\xi^{-j} \lambda_i^{-1})^q \varphi^q(x) \in V_{\xi^j \lambda_i}$ for any $j = 0, \ldots, l - 1$.

Recall that if $f \in \text{End}(L)$ satisfies $f(L_g) \subset L_g$ for all $g \in G$, then for each $\alpha \in F$, the eigenspace $\{x \in L : f(x) = \alpha x\}$ is graded. This can be applied to the endomorphism
since \( \varphi^l \), since \( \varphi^l(L_g) \subset L_{g+h} = L_g \), so that the eigenspace \( V_{\lambda_i} \) is a graded subspace of \( L \). Thus we can take \( 0 \neq x \in V_{\lambda_i} \cap L_g \) for some \( g \in G \). For each \( j = 0, \ldots, l - 1 \), the element \( \sum_{l=0}^{l-1}(\xi^j-\xi^{-1})^{l-1} \xi^{j} (x) \in \sum_{l=0}^{l-1} L_{g+h} \) must be nonzero, because the involved homogeneous pieces are different \( (g+h) = g+ph \) implies \((q-p)h=0\) and the projection in the component \( L_g \) is \( x \neq 0 \). Consequently \( V_{\xi^i \lambda_i} \neq 0 \) for all \( j \), so that

\[
\{ \xi^j \lambda_i : j = 0, \ldots, l - 1 \} \subset \text{Spec}(u)
\]

for any \( \lambda_i \in \text{Spec}(u) \), and hence \( \text{Spec}(u) = \{ \pm \xi^j \lambda_i : 0 \leq j < l, 1 \leq i \leq k \} \).

**Proposition 5.** Assume that \( \lambda_i/\lambda_j \) is not a root of unit if \( i \neq j \). Then the unique fine (group) gradings on \( H_n^\lambda \) (up to equivalence) are \( \Gamma_1 \) and \( \Gamma_2 \). Moreover, the Weyl groups of these fine gradings are \( \mathcal{W}(\Gamma_1) \cong \mathbb{Z}_2^k \) and \( \mathcal{W}(\Gamma_2) \cong \mathbb{Z}_2 \).

**Proof.** Let \( \Gamma : L = \oplus_{g \in G} L_g \) be a grading on \( L = H_n^\lambda \). By the above discussion we can suppose that \( u \) is homogeneous with degree \( h \in G \) of finite order \( l \). Let us show that either \( l = 1 \) or \( l = 2 \). Otherwise, take \( \xi \) a primitive \( l \)th root of unit. If \( \xi \neq \lambda_1 \) \( \text{Spec}(u) \) according to Equation (11), then there is \( i \neq k \) such that \( \xi_{\lambda_1} \in \{ \pm \lambda_i \} \) and hence either \( \left( \frac{\xi}{\xi_1} \right)^i = 1 \) or \( \left( \frac{\xi}{\xi_1} \right)^{2l} = 1 \), what is a contradiction. Hence, we can distinguish two cases.

First consider \( h = 0 \in G \). Thus \( \varphi(L_g) \subset L_g \) for any \( g \). Restrict \( \varphi : [u, L] \to [u, L] \), we can take a basis of homogeneous elements which are eigenvectors for \( \varphi \). Recall that the spectrum of \( \varphi |_{[u, L]} \) consists of \( \{ \pm \lambda_1, \ldots, \pm \lambda_k \} \). Take \( x_1 \neq 0 \) some homogeneous element in \( V_{\lambda_1} \). As \( [x_1, V_{-\lambda_1}] \neq 0 \), there is some element \( y_1 \in V_{-\lambda_1} \) in the above basis such that \( [x_1, y_1] = -2\lambda_1 z \) (by scaling \( y_1 \) if necessary). Now \( [u, L] = W \oplus Z_{[u, L]}(W) \) for \( W \) and \( \text{centralizer} Z_{[u, L]}(W) \) are graded and \( \varphi \)-invariant. We continue by induction until finding a basis of homogeneous elements \( \{ x_1, y_1, \ldots, x_k, y_k \} \) of \( [u, L] \) such that \( [x_1, y_1] = -2\lambda_1 z, [u, x_1] = \varphi(x_1) = \lambda_1 x_1 \) and \( [u, y_i] = \varphi(y_i) = -\lambda_1 y_i \). Since \( L = \langle z \rangle \oplus \langle u \rangle \oplus \langle u, L \rangle \), we have that the map \( u \mapsto u, z \mapsto z, x_i \mapsto x_i \) and \( y_i \mapsto y_i \) is a Lie algebra isomorphism which applies \( \Gamma \) into a coarsening of \( \Gamma_2 \).

Second consider the case when \( 2h = 0 \) but \( h \neq 0 \). Thus \( \varphi^2 \) preserves the grading \( \Gamma \) and it is diagonalizable with eigenvalues \( \{ \lambda_1^2, \lambda_2^2, \ldots, \lambda_k^2, 0, 0 \} \) (counting each one with multiplicity 1). Observe that \( \varphi \) applies \( \{ x \in L : \varphi^2(x) = \lambda_1^2 x \} = V_{\lambda_1^2} \) onto itself. Moreover this set is graded for each \( i \), because \( \varphi^2 \) preserves the grading. For any \( x_1 \neq 0 \) \( V_{\lambda_1^2} \cap L_g \) a homogeneous element of the grading, \( \varphi(x_1) \) is independent with \( x_1 \) (otherwise \( \varphi(x_1) \in L_g \cap L_{g+h} \) but \( h \neq 0 \) and \( \varphi(x_1) \neq 0 \)). Take \( y_1 = \frac{\lambda_1}{\lambda_1^2} \varphi(x_1) \), which verifies \( \varphi(y_1) = \lambda_1 x_1 \). Since our ground field is algebraically closed, it contains the square roots of all its elements, so that if \( [x_1, y_1] \neq 0 \), we can scale to get \( [x_1, y_1] = \lambda_1 z \), and then we can continue because, as before, \( [u, L] = W \oplus Z_{[u, L]}(W) \) for \( W = \langle x_1, y_1 \rangle \), where both \( W \) and its centralizer is graded and \( \varphi \)-invariant. The case \( [x_1, y_1] = 0 \) does not occur under the hypothesis of the theorem, since \( \lambda_1^2 \neq \lambda_2^2 \) if \( i \neq j \), so that \( \dim V_{\lambda_1^2} = 2 \) for all \( i \). As \( V_{\alpha}, V_{\beta} = 0 \) if \( \alpha + \beta \neq 0 \), this implies that there is \( y \in V_{\lambda_1} \) with \( [x_1, y] \neq 0 \) but \( V_{\lambda_1} = \langle x_1, y_1 \rangle \). To summarize, if \( h \) has order 2 and \( \lambda_1/\lambda_2 \notin \{ \pm 1 \} \), we find a basis of homogeneous elements \( \{ x_1, y_1, \ldots, x_k, y_k \} \) of \([u, L]\) such that \( [x_1, y_1] = \lambda_1 z, [u, x_1] = \lambda_1 x_1 \) and \( [u, y_i] = \lambda_i x_i \), so that the map \( u \mapsto u, z \mapsto z, x_i \mapsto e_i \) and \( y_i \mapsto e_i \) is a Lie algebra isomorphism which applies \( \Gamma \) into a coarsening of \( \Gamma_1 \).
In order to compute the Weyl groups of these fine gradings, recall that any $f \in \text{Aut}(L)$ verifies $0 \neq f(z) \in \langle z \rangle$. If besides $f \in \text{Aut}(\Gamma_1)$, then $f(u) \in \langle u \rangle$. Otherwise, there would exist some $i \leq k$ such that either $f(e_i) \in \langle u \rangle$ or $f(\hat{e}_i) \in \langle u \rangle$, so that $0 \neq f(\lambda z) = [f(e_i), f(\hat{e}_i)] \in [u, L] \cap \langle z \rangle = 0$. Consider for each index $i \leq k$, the element in $\text{Aut}(\Gamma_1)$ defined by $\mu_i(e_i) = i\hat{e}_i$, $\mu_i(\hat{e}_i) = i\hat{e}_i$, $\mu_i(e_j) = e_j$, $\mu_i(\hat{e}_j) = \hat{e}_j$ for each $j \neq i$, $\mu_i(z) = z$ and $\mu_i(u) = u$. Note that if $r = 1, \ldots, k$, there are not any $i, j \leq k$ such that $f(e_i) \in \langle e_r \rangle$ and $f(e_j) \in \langle \hat{e}_r \rangle$. Hence we can compose $f$ with some $\mu_i$’s if necessary to obtain that $f' := \mu_1 \cdots \mu_i f$ satisfies

$$f'(e_i) \in \langle e_1 \rangle \cup \langle e_2 \rangle \cup \cdots \cup \langle e_k \rangle$$

for each $i = 1, \ldots, k$. Thus, there is $\sigma \in S_k$ such that $f'(z) = \mu z$, $f'(u) = \beta u$, $f'(e_i) = \gamma_i e_{\sigma(i)}$ and $f'((\hat{e}_i)) = \gamma_i' \hat{e}_{\sigma(i)}$, for any $i = 1, \ldots, k$, with $\mu, \beta, \gamma_i, \gamma_i' \in \mathbb{F}^\times$. From here, the equality $f'([u, e_i]) = [f'(u), f'(e_i)]$ implies

$$\gamma_i' \lambda_i = \beta \gamma_i \lambda_{\sigma(i)},$$

and finally the condition $f'([u, \hat{e}_i]) = [f'(u), f'((\hat{e}_i))]$ allows us to assert

$$\gamma_i \lambda_i = \beta \gamma_i' \lambda_{\sigma(i)}.$$ (13)

From Equations (12) and (13) we easily get $\lambda_{\sigma(i)} \in \pm \beta^{-1} \lambda_i$ for any $i = 1, \ldots, k$. By multiplying, $\Pi_{i=1}^k \lambda_{\sigma(i)} \in \pm \beta^{-k} \Pi_{i=1}^k \lambda_i$ so that $\beta^{2k} = 1$. As $\lambda_{\sigma(i)}/\lambda_i$ is not a root of unit if $\sigma(i) \neq i$, we conclude that $\sigma = \text{id}$, so that $f' \in \text{Stab}(\Gamma_1)$. In other words, since $\mu_i \mu_j = \mu_j \mu_i$,

$$W(\Gamma_1) = \{[\mu_{i_1} \cdots \mu_{i_s}] : 1 \leq i_1 \leq \cdots \leq i_s \leq k \} \cong \mathbb{Z}_2^k,$$

where $[\cdot]$ is used for denoting the class of an element of $\text{Aut}(\Gamma_1)$ modulo $\text{Stab}(\Gamma_1)$.

For the other case, define the automorphism $\mu \in \text{Aut}(\Gamma_2)$ by means of $\mu(u_i) = iv_i$ and $\mu(v_i) = iv_i$ for all $i$, $\mu(z) = z$ and $\mu(u) = -u$. Consider $f \in \text{Aut}(\Gamma_2)$, and note that again there is $\beta \in \mathbb{F}^\times$ such that $f(u) = \beta u$. If $f(u_i)$ is a multiple of either $u_j$ or $v_j$ for some $j$, this clearly implies that $f(v_i)$ also is, so that there is $\sigma \in S_k$ such that $f(u_i) \in \langle u_{\sigma(i)} \rangle \cup \langle v_{\sigma(i)} \rangle$ for all $i \leq k$. As $\beta [u, f(u_i)] = \lambda_i f(u_i)$, then $\lambda_i \in \pm \beta \lambda_{\sigma(i)}$. Multiplying as in the above case, we get that $\beta$ is a root of unit, and again we conclude that $\sigma = \text{id}$. By composing with $\mu$ if necessary, we can assume that $f(u_i) \in \langle u_i \rangle$, which implies $\beta = 1$. If $f(u_i) \in \langle v_i \rangle$ for some $i$, then $\beta = -1$, which is a contradiction, so that $f(u_i) \in \langle u_i \rangle$ for all $i$ and $f$ belongs to $\text{Stab}(\Gamma_2)$. We have then proved that

$$W(\Gamma_2) = \langle [\mu] \rangle \cong \mathbb{Z}_2.$$

\hfill \Box

6.4. Fine gradings on twisted Heisenberg algebras. In the general case (possible roots of the unit among the fractions of $\lambda_i$’s), the situation is much more involved. On one hand, a lot of different fine gradings arise, and on the other hand even the Weyl groups of $\Gamma_1$ and $\Gamma_2$ change. There is a lot of symmetry in the related twisted Lie algebra, and its fine gradings are also symmetric. In order to figure out what is happening, we previously need to show a couple of key examples.

First, for $\xi$ a primitive $l$th root of the unit and $\alpha$ a nonzero scalar, we consider the twisted Heisenberg algebra $H^\xi_{2l+2}$ corresponding to

$$\lambda = (\lambda_1, \ldots, \lambda_l) = (\xi \alpha, \xi^2 \alpha, \ldots, \xi^{l-1} \alpha, \alpha).$$
Thus \([u, u_i] = \xi^i \alpha u_i, [u, v_i] = -\xi^i \alpha v_i\) and \([u_i, v_j] = -2\xi^i \alpha z\) for \(i = 1, \ldots, l\), with the
definition of \(u_i\)’s and \(v_i\)’s as in Equation (15). Take now
\[
\begin{align*}
x_j &= \sum_{i=1}^{l} \xi^{j} u_i, \\
y_j &= -\frac{1}{2} \sum_{i=1}^{l} (-1)^{j} \xi^{(j-1)} i v_i,
\end{align*}
\]
if \(j = 1, \ldots, l\). These elements verify \([u, x_j] = \alpha x_{j+1}\) and \([u, y_j] = \alpha y_{j+1}\) for all \(j \leq l - 1\).

Besides \([x_i, y_j] = \sum_{i=1}^{l} (-1)^{i} \xi^{(i+j)} z\) is not zero if and only if \(i + j = l, 2l\), and in such a case \([x_i, y_i] = \sum_{i=1}^{l} \gamma_i \alpha z\) and \([x_i, y_{l-i}] = \sum_{i=1}^{l} \gamma_i l \alpha z\) for \(i = 1, \ldots, l - 1\). Note that obviously \([x_1, y_1, \ldots, x_l, y_l]\) is a family of independent vectors such that \([x_i, x_j] = 0 = [y_i, y_j]\) for all \(i, j\).

Therefore we have a fine grading on \(L = H^2_{2l+2}\) over the group
\[G = \mathbb{Z}^2 \times \mathbb{Z}_l,\]
given by
\[
\begin{align*}
L_{(0,0,1)} &= \langle u \rangle, \\
L_{(1,1,0)} &= \langle z \rangle, \\
L_{(1,0,l)} &= \langle x_l \rangle, \\
L_{(0,1,7)} &= \langle y_l \rangle,
\end{align*}
\]
for all \(i = 1, \ldots, l\).

Take \(\gamma \in \mathbb{F}\) such that \(\gamma^l = (-1)^l\), and consider the automorphisms \(\theta, \vartheta \in \text{Aut}(L)\)
defined by
\[
\begin{align*}
\theta(x_i) &= x_{i+1}, & \theta(y_i) &= y_{i-1}, & \theta(z) &= -z, & \theta(u) &= u; \\
\vartheta(x_i) &= \gamma y_i, & \vartheta(y_i) &= -\gamma y_i, & \vartheta(z) &= z, & \vartheta(u) &= \gamma u;
\end{align*}
\]
where the indices are taken modulo \(2l\). It is not difficult to check that the Weyl group of the grading described in Equation (15) is generated by the classes \([\theta]\) and \([\vartheta]\), elements of order \(l\) and \(2\) respectively which do not commute, so that the Weyl group is isomorphic to the Dihedral group \(D_l\).

This example motivates the following definition.

**Definition 5.** Let \(L\) be any Lie algebra, \(z \in L\) a fixed element, \(u\) an arbitrary element and \(\alpha \in \mathbb{F}^\times\). A set \(B^l(u, \alpha)\), which will be referred as a block of type \(l\), is given by a family of \(2l\) independent elements in \(L\),
\[
B^l(u, \alpha) = \{x_1, y_1, \ldots, x_l, y_l\},
\]
satisfying that the only non-vanishing products among them are the following:
\[
\begin{align*}
[u, x_i] &= \alpha x_{i+1}, & \forall i &= 1, \ldots, l - 1, \\
[u, x_1] &= \alpha y_1, \\
[u, y_i] &= \alpha y_{i+1}, & \forall i &= 1, \ldots, l - 1, \\
[u, y_1] &= (-1)^l \alpha y_1, \\
x_i, y_{l-i} &= (-1)^{l-i} \alpha z, & \forall i &= 1, \ldots, l - 1, \\
x_1, y_l &= (-1)^l \alpha z.
\end{align*}
\]

As a second example, fix \(\zeta\) a primitive \(2l\)th root of the unit and \(\alpha\) a nonzero scalar. Consider now the twisted Heisenberg algebra \(H^\lambda_{2l+2}\) corresponding to
\[
\lambda = (\lambda_1, \ldots, \lambda_l) = (\zeta \alpha, \zeta^2 \alpha, \ldots, \zeta^{l-1} \alpha, -\alpha).
\]
Again \([u, u_i] = \zeta^i \alpha u_i, [u, v_i] = -\zeta^i \alpha v_i\) and \([u_i, v_i] = -2\zeta^i \alpha z\) for \(i = 1, \ldots, l\). Take now

\[
x_j = \frac{1}{2\sqrt{l}} \sum_{i=1}^{l} (u_i + (-1)^{j-1} v_i) \zeta^{(j-1)i}
\]

for each integer \(j\). Observe that \(\{x_1, \ldots, x_{2l}\}\) is a family of linearly independent elements satisfying \([u, x_j] = \alpha x_{j+1}\) for any \(j = 1, \ldots, 2l - 1\) and \([u, x_{2l}] = \alpha x_1\). A direct computation gives

\[
[x_i, x_j] = \frac{1}{2l} \alpha((-1)^i + (-1)^{j-1})(\sum_{k=1}^{l} \zeta^{(i+j-1)k}) z
\]

for any \(i\) and \(j\). If \(i + j - 1 = 2l\), then \(i\) and \(j - 1\) are either both odd or both even and \([x_1, x_{2l+1-i}] = (-1)^i \alpha z \neq 0\). Hence,

\[
[x_1, x_{2l}] = -[x_2, x_{2l-1}] = \cdots = (-1)^{l-1}[x_l, x_{l+1}].
\]

Again Equation \((18)\) tells us that the remaining brackets are zero: if \(r = i + j - 1\) is odd, then \((-1)^i + (-1)^{j-1} = 0\), and, if \(r = i + j - 1 \neq 2l\) is even, then \(\sum_{k=1}^{l} \zeta^{rk} = 0\) since \(\zeta^2\) is a primitive \(l\)th root of unit \((\zeta^r = 1\) with \(\zeta^r \neq 1\)).

We note that this provides a fine grading on \(H^{2l}_{\lambda+2}\) over the group

\[G = \mathbb{Z} \times \mathbb{Z}_{2l},\]

given by

\[
L_{(0, \overline{r})} = \langle u \rangle, \quad L_{(2, 1)} = \langle z \rangle, \quad L_{(1, \overline{r})} = \langle x_i \rangle,
\]

for \(i = 1, \ldots, 2l\).

Take \(\rho \in \text{Aut}(L)\) defined by

\[
\rho(x_i) = x_{i+1}, \quad \rho(z) = (-1)^i z, \quad \rho(u) = u,
\]

for all \(i = 1, \ldots, 2l\) (mod \(2l\)). This time the Weyl group of the grading is isomorphic to \(\mathbb{Z}_{2}\), since it is easily proved to be generated by the class \([\rho]\).

This example gives rise to the next concept.

**Definition 6.** Let \(L\) be any Lie algebra, \(z \in L\) a fixed element, \(u \in L\) an arbitrary element and \(\alpha \in \mathbb{F}^\times\). A set \(B^{\alpha}_{\lambda}(u, \alpha)\), which will be referred as a block of type II, is given by a family of \(2l\) independent elements in \(L\),

\[B^{\alpha}_{\lambda}(u, \alpha) = \{x_1, \ldots, x_{2l}\},\]

satisfying that the only non-vanishing products among them are the following:

\[
[u, x_i] = \alpha x_{i+1} \quad \forall i = 1, \ldots, 2l \text{ (mod } 2l\text{)},
\]

\[
[x_i, x_{2l-i+1}] = (-1)^i \alpha z \quad \forall i = 1, \ldots, 2l.
\]

In fact, all of the gradings on a twisted Heisenberg algebra can be described with blocks of types I and II, according to the following theorem.
Theorem 3. Let $\Gamma$ be a $G$-grading on $L = H^*_x$. Let $z \in \mathcal{Z}(L)$. Then there exist $u \in L$, positive integers $l, s, r$ such that $l(r + 2s) = 2k = n - 2$ ($r = 0$ when $l$ is odd) and scalars $\beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r \in \{\pm \lambda_1, \ldots, \pm \lambda_k\}$ such that

\begin{equation}
\{z, u\} = \left( \bigcup_{j=1}^{s} B^l_j(u, \beta_j) \right) \cup \left( \bigcup_{i=1}^{r} B^h_i(u, \alpha_i) \right)
\end{equation}

is a basis of homogeneous elements of $\Gamma$, being zero the bracket of any two elements belonging to different blocks.

Proof. Recall that $z$ is always a homogeneous element. By Lemma 4 and the arguments below, we can assume that $u$ is also homogeneous of degree $h \in G$, necessarily of finite order. Let $l \in \mathbb{Z}_{\geq 0}$ be the order of $h$. Take $\varphi = \text{ad}(u)$ and consider again the subspaces $V_{\lambda_1}$ and $V_{\lambda_1}^l$. Recall that $V_{\lambda_1}$ is a $\varphi$-invariant graded subspace for all $\lambda_1 \in \text{Spec}(u)$.

Let us discuss first the case that $l$ is odd. Fix any $0 \neq x \in V_{\lambda_1}^l \cap L_g$ for some $g \in G$. Since each $\varphi^l(x) \in L_{g+ih}$, we have that

$$\{x, \varphi(x), \ldots, \varphi^{l-1}(x)\}$$

is a family of linearly independent elements of $L$. Now observe that Equation 10, together with the fact that $l$ is odd, says that $[V_{\lambda_1}^l, V_{\lambda_1}^l] = 0$ and $[V_{\lambda_1}^l, V_{-\lambda_1}^l] \neq 0$. From here,

\begin{equation}
[\varphi^l(x), \varphi^l(y)] = 0 \text{ for any } i, j = 0, 1, \ldots, l - 1,
\end{equation}

and we can take a nonzero homogeneous element $0 \neq y \in V_{-\lambda_1}^l \cap L_g$ such that $[x, y] \neq 0$. By scaling if necessary, we can suppose $[x, y] = \lambda_1 z$, being then $\deg z = g + p$. As above, we also have that

$$\{y, \varphi(y), \ldots, \varphi^{l-1}(y)\}$$

is a family of linearly independent elements of $L$ satisfying

\begin{equation}
[\varphi^i(y), \varphi^j(y)] = 0 \text{ for any } i, j = 0, 1, \ldots, l - 1.
\end{equation}

Taking into account that $\varphi^l(x) \in L_{g+ih}$ and $\varphi^l(y) \in L_{p+ih}$, we get that if $[\varphi^l(x), \varphi^l(y)] \neq 0$, then $g + p + (i + j)h = \deg z = g + p$, which is only possible if $i + j$ is a multiple of $l$. That is, for each $0 \leq i, j < l$,

\begin{equation}
[\varphi^i(x), \varphi^j(y)] = 0 \text{ if } i + j \neq l.
\end{equation}

Also note that

\begin{equation}
[\varphi^i(x), \varphi^{l-i}(y)] = (-1)^{l-i} x_{i,l} [x, y] \neq 0.
\end{equation}

Indeed, take $\xi$ a primitive $l$th root of the unit and write $x = \sum_{j=0}^{l-1} a_j$ and $y = \sum_{j=0}^{l-1} b_j$ for $a_j \in V_{\xi^j \lambda_1}$ and $b_j \in V_{-\xi^j \lambda_1}$, taking into consideration Equation 10. Then

$$[\varphi^i(x), \varphi^{l-i}(y)] = \sum_{j,k} [\xi^{ji} \lambda_1 \lambda_1^l a_j, (-\xi^k)^{l-i} \lambda_1^{l-i} b_k] = \lambda_1^l (-1)^{l-i} \sum_j [\xi^{ji} \lambda_1 \lambda_1^l a_j, b_j] = (-1)^{l-i} x_{i,l} [x, y].$$

Finally note that the family $\{x, \varphi(x), \ldots, \varphi^{l-1}(x), y, \varphi(y), \ldots, \varphi^{l-1}(y)\}$ is linearly independent. Indeed, in the opposite case some $\varphi^i(y) = \beta \varphi^j(x)$, $\beta \in F^x$, because we are dealing with a family of homogeneous elements, and then $[\varphi^i(y), \varphi^{l-i}(y)] = \beta [\varphi^i(x), \varphi^{l-i}(y)] \neq 0$, what contradicts Equation 22.

Taking into account Equations 21, 22, 23 and 24, we have that

$$\begin{pmatrix}
\varphi(x) / \lambda_1, & \varphi(y) / \lambda_1, & \varphi^i(x) / \lambda_1, & \varphi^j(y) / \lambda_1, \ldots, & \varphi^{l-i}(x) / \lambda_1, & \varphi^{l-j}(y) / \lambda_1
\end{pmatrix} = \begin{pmatrix} x, y \end{pmatrix} = (-1)^l [x, y]$$

is a block $B^l_j(u, \lambda_1)$ of type I.
Now \([u, L] = W \oplus Z_{[u,L]}(W)\) for \(W := \langle B_l^1(u, \lambda_1) \rangle\), where \(W\) as well as its centralizer are graded and \(\varphi\)-invariant. We continue by iterating this process on \(Z_{[u,L]}(W)\) until finding a basis of \([u, L]\) formed by \(s = \frac{l}{2}\) blocks of type I of homogeneous elements.

Now consider the case with \(l\) even. If we think as above of the linear subspace \(0 \neq V_l^i\), we have two different cases to distinguish.

Assume first that for any \(g \in G\) and any \(x \in V_l^1 \cap L_g\) we have \([x, \varphi(x)] = 0\). Fix \(0 \neq x \in V_l^1 \cap L_g\) for some \(g \in G\), being then \(\{x, \varphi(x), \ldots, \varphi^{l-1}(x)\}\) a family of linearly independent elements of \(L\). By induction on \(n\) it is easy to verify, taking into account that \(\varphi\) is a derivation, that for any \(i = 0, \ldots, l - 1\), we have \([\varphi^i(x), \varphi^{i+n}(x)] = 0\) for any \(n = 1, \ldots, l\). That is, \([\varphi^i(x), \varphi^j(x)] = 0\) for any \(i, j = 0, \ldots, l - 1\).

Since the fact that \(l\) is even implies \(V_l^1 = V_l^2\), we can choose a homogeneous element \(0 \neq y \in V_l^1 \cap L_p\), for some \(p \in G\), such that \(0 \neq [x, y] = \lambda_1 z\). The same arguments that in the odd case say that again

\[
\{\varphi(x), \varphi(y), \varphi^i(x), \varphi^i(y), \varphi^i(x), \varphi^i(y), \varphi^i(x), \varphi^i(y)\}
\]

is a block \(B_l^1(u, \lambda_1)\) of type I. Now we can write

\[
[u, L] = W \oplus Z_{[u,L]}(W)
\]

for \(W := \langle B_l^1(u, \lambda_1) \rangle\), where \(W\) as well as its centralizer are graded and \(\varphi\)-invariant.

Second, assume that there exist \(g \in G\) and \(0 \neq x \in V_l^1 \cap L_g\) such that \([x, \varphi(x)] \neq 0\). By scaling if necessary, we can assume \([x, \varphi(x)] = \lambda_1^2 z\). We have as above that \(\{x, \varphi(x), \ldots, \varphi^{l-1}(x)\}\) is a family of homogeneous linearly independent elements of \(L\) satisfying \([\varphi^i(x), \varphi^j(x)] = 0\) if \(i + j \neq 1, l + 1\) (take into account that \(\deg z = 2g + h\)). Besides Equation (10) allows us to get in a direct way that

\[
[\varphi^i(x), \varphi^{l-i+1}(x)] = (-1)^i \lambda_1^2 [x, \varphi(x)] \neq 0
\]

for any \(i = 1, \ldots, l\). Thus the set

\[
\{\varphi(x), \varphi(x), \varphi(x), \varphi(x), \varphi(x), \varphi(x), \varphi(x), \varphi(x)\}
\]

is a block \(B_l^2(u, \lambda_1)\) of type II. Now we can write

\[
[u, L] = W \oplus Z_{[u,L]}(W)
\]

for \(W\) the vector space spanned by the above block \(B_l^2(u, \lambda_1)\), where \(W\) as well as its centralizer \(Z_{[u,L]}(W)\) are graded and \(\varphi\)-invariant.

Taking into account Equations (25) and (26), we can iterate this process on \(Z_{[u,L]}(W)\) until finding the required basis of \([u, L]\) formed by \(s\) blocks of type I and \(r\) blocks of type II.

Theorem 8 provides, in particular, a description of all the fine gradings on \(H_n^\lambda\). It is clear that each basis as the one in Equation (20) determines a fine grading, which we will denote by

\[
\Gamma(l, s, r; \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r).
\]

For instance, our well-known fine gradings of Subsection 6.3 are

\[
\Gamma_1 = \Gamma(2, 0, k; \lambda_1, \ldots, \lambda_k),
\]

\[
\Gamma_2 = \Gamma(1, k, 0; \lambda_1, \ldots, \lambda_k),
\]
taking into consideration that \( \{ u_i, \frac{1}{2} v_i \} = B_l(u, \lambda_i) \) and that \( \{ \bar{e}_i, e_i \} = B_{\pm}^l(u, \lambda_i) \).

The point is that the grading in Equation (27) does not necessarily exist for all choice of integers \( l, s, r \) such that \( l(r + 2s) = n - 2 \) and all nonzero scalars \( \beta_i, \alpha_j \in \text{Spec}(u) \). If we are in the situation of Theorem 3 and take \( \xi \) a primitive \( l \)th root of the unit, then

\[
\text{Spec}(u) = \{ \pm \lambda_i : i = 1, \ldots, k \} = \{ \xi^i \beta_j, -\xi^l \beta_j : j = 1, \ldots, s, t = 0, \ldots, l - 1 \} \cup \{ \xi^i \alpha_z : i = 1, \ldots, l, r = 0, \ldots, l - 1 \}.
\]

This condition is not only necessary, but sufficient.

**Theorem 4.** A grading \( \Gamma \) on \( H^\lambda_n \) is fine if and only if \( \Gamma \) is isomorphic to

\[
\Gamma(l, s, r; \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r)
\]

for some \( l, s, r \in \mathbb{Z}_{\geq 0} \) such that \( l(r + 2s) = 2k = n - 2 \) and some scalars \( \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r \in \{ \pm \lambda_1, \ldots, \pm \lambda_k \} \) such that (28) holds, with \( l \) even if \( r \neq 0 \). The universal grading group in this case is

\[
\begin{align*}
\mathbb{Z}_l \times \mathbb{Z}^{s+1} \times \mathbb{Z}^{r-1} & \quad \text{if } r \neq 0, \\
\mathbb{Z}_l \times \mathbb{Z}^{s+1} & \quad \text{if } r = 0.
\end{align*}
\]

**Proof.** The fact that any fine grading is like the above has been proved in Theorem 5. For the converse, reorder

\[
\lambda = (\xi \beta_1, \ldots, \xi^l \beta_1, \ldots, \xi \beta_s, \ldots, \xi^l \beta_s, \xi \alpha_1, \ldots, \xi^l \alpha_1, \ldots, \xi \alpha_r, \ldots, \xi^l \alpha_r),
\]

(with \( \xi^2 = \xi \)), what is possible because of (25). Take \( \{ z, u, u_1, \ldots, u_k, v_1, \ldots, v_k \} \) a basis as in Equation (3). Take \( \{ x_1^j, y_1^j, \ldots, x_s^j, y_s^j \} \) a block \( B_l^j(u, \beta_j) \) for each \( j \leq s \), chosen as in Equation (14). Take \( \{ a_1^t, \ldots, a_1^r \} \) a block \( B_{\pm}^l(u, \alpha_t) \) for each \( t \leq r \), chosen as in Equation (17). Now the union of these blocks as in Equation (20) provides a basis which determines a fine grading (since the product of two elements in the basis is multiple of another one). This grading has universal group \( \mathbb{Z}_l \times \mathbb{Z}^{s} \times \mathbb{Z} \times \mathbb{Z}^{r-1} \) and it is given by

\[
\begin{align*}
\deg(x_1^j) &= (\bar{t} + 1; 0, \ldots, 1, 0; 0, \ldots, 0) \quad \text{for each } j, \\
\deg(y_1^j) &= (\bar{t}; 0, \ldots, 1, 0; 0, \ldots, 0), \\
\deg(z) &= (\bar{T}; 0, \ldots, 0; 1, \bar{0}, \ldots, 0), \\
\deg(u) &= (\bar{T}; 0, \ldots, 0; 0, \bar{0}, \ldots, 0), \\
\deg(u_1) &= (\bar{t}; 0, \ldots, 0; 1, \bar{0}, \ldots, 0), \\
\deg(u_2) &= (\bar{t}; 0, \ldots, 0; 1, \bar{0}, \ldots, 0).
\end{align*}
\]

\[\square\]

In practice, when one wants to know how many gradings are in a particular twisted Heisenberg algebra \( H^\lambda_n \), it is enough to see how many ways are of splitting \( \{ \pm \lambda_1, \ldots, \pm \lambda_k \} \) in the way described in Equation (28).

**Example 1.** Let us compute how many fine gradings are there in \( L = H^{(1,1,i,i)}_{10} \). As \( l \) must divide 8, the possibilities are \( l = 1, l = 2 \) with \( (r, s) = (4, 0), (2, 1), (0, 2) \) and \( l = 4 \) with \( (r, s) = (2, 0), (0, 1) \). But we have seven (not six) fine gradings:

- the \( \mathbb{Z}^5 \)-grading \( \Gamma(1, 4, 0; 1, 1, i, i) \) (the only toral fine grading, \( \Gamma_2 \));
- the \( \mathbb{Z} \times \mathbb{Z}^4 \)-grading \( \Gamma(2, 0, 4; 1, 1, i, i) \) (again \( \Gamma_1 \));
- two \( \mathbb{Z}^2 \times \mathbb{Z}^2 \)-gradings: \( \Gamma(2, 1, 2; 1, 1, i) \) and \( \Gamma(2, 1, 2; 1, i, i) \);
- the \( \mathbb{Z}^3 \times \mathbb{Z}^2 \)-grading \( \Gamma(2, 2, 0; 1, i) \);
- the \( \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2 \)-grading \( \Gamma(4, 0, 2; 1, 1) \);
- the \( \mathbb{Z}^2 \times \mathbb{Z}_4 \)-grading \( \Gamma(4, 1, 0; 1) \).
The possibility \( l = 8 \) does not happen taking into account that \( \pm \frac{\lambda_i}{\lambda_j} \) is never a primitive eighth root of the unit when \( \lambda_i, \lambda_j \in \text{Spec}(u) \).

For boarding the problem of the classification of the fine gradings up to equivalence, we need to make some considerations.

**Lemma 5.** Let \( L \) be a Lie algebra, \( z \in Z(L) \), \( u \in L, \alpha, \beta \in F^\times \) such that \( L \) contains blocks of types \( B_1^l(u, \alpha) \) and \( B_2^l(u, \beta) \) for some \( \nu \in \{ I, II \} \). Then

i) \( \langle B_1^l(u, \alpha) \rangle = \langle B_1^l(u, \beta) \rangle \) if and only if \( \left( \frac{\alpha}{\beta} \right)^l = 1 \) if \( l \) is even and \( \left( \frac{\alpha}{\beta} \right)^{2l} = 1 \) if \( l \) is odd.

ii) \( \langle B_2^l(u, \alpha) \rangle = \langle B_2^l(u, \beta) \rangle \) if and only if \( \left( \frac{\alpha}{\beta} \right)^l = 1 \).

**Proof.** As usual, denote \( \varphi = \text{ad}(u) \) and \( \xi \) a primitive \( l \)th root of the unit. For i), take \( B_1^l(u, \alpha) = \{ x_1, y_1, \ldots, x_l, y_l \} \) and \( V \) the vector space spanned by these elements. Note that \( \varphi^l \) diagonalizes \( V \) with only eigenvalues \( \alpha^l \) and \( (1/l)\alpha^l \) respectively. Thus either \( \alpha^l = \beta^l \) or \( \alpha^l = -(1/l)\beta^l \).

Conversely, let us see that there is a block of type \( B_2^l(u, \xi \alpha) \) contained in \( \langle B_2^l(u, \alpha) \rangle \). Indeed, take \( \zeta \) such that \( \zeta^2 = \xi \). The elements \( x'_i := \xi^{1-2l} x_i \) and \( y'_i := \xi^{1-2l} y_i \) constitute a block of type \( B_2^l(u, \zeta \alpha) \). Moreover, if we take \( \delta \) such that \( \delta^2 = \zeta \), then the elements \( x'_i := (-1)^l \delta^{1-2l} x_i \) and \( y'_i := (-1)^l \delta^{1-2l} y_i \) constitute a block of type \( B_2^l(u, \zeta \alpha) \) if \( l \) is odd.

The case ii) is proved with similar arguments. \( \square \)

These arguments make convenient to consider the following equivalence relation in \( F^\times \).

Two nonzero scalars \( \alpha \) and \( \beta \) are \( m \)-related if and only if \( \left( \frac{\alpha}{\beta} \right)^m = 1 \). The equivalence classes of the element \( \alpha \) for the \( l \)-relation and the \( 2l \)-relation will be denoted respectively by

\[
\tilde{\alpha} := \{ \alpha \xi^t : t = 0, \ldots, l-1 \},
\tilde{\alpha} := \{ \alpha \xi^t : t = 0, \ldots, 2l-1 \},
\]

where \( \xi, \zeta \in F \) will denote from now on some fixed \( l \)th and \( 2l \)th primitive roots of the unit, respectively.

**Theorem 5.** Two fine gradings on \( H_n^\lambda \), \( \Gamma = \Gamma(l, s, r; \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r) \) and \( \Gamma' = \Gamma'(l', s', r'; \beta'_1, \ldots, \beta'_s, \alpha'_1, \ldots, \alpha'_r) \), are equivalent if and only if \( l = l', s = s', r = r' \) and there are \( \varepsilon \in F^\times, \eta \in S_n \) and \( \sigma \in S_r \) such that for all \( j = 1, \ldots, s \), \( \varepsilon \tilde{\beta}_j = \tilde{\beta}_{\eta(j)} \) if \( l \) is odd and \( \varepsilon \tilde{\beta}_j = \tilde{\beta}_{\eta(j)} \) if \( l \) is even, and for all \( i = 1, \ldots, r \), \( \tilde{\alpha}_i = \tilde{\alpha}_{\sigma(i)} \).

**Proof.** The grading \( \Gamma \) is given by blocks \( B_1^l(u, \beta_j) = \{ x_1^l, y_1^l, \ldots, x_l^l, y_l^l \} \) if \( j \leq s \), and blocks \( B_2^l(u, \alpha_i) = \{ a_1^l, \ldots, a_l^l \} \) if \( i \leq r \). Also the grading \( \Gamma' \) is given by blocks \( B_1^l(u', \beta'_j) \) if \( j \leq s' \), and blocks \( B_2^l(u', \alpha'_i) \) if \( i \leq r' \), for some homogeneous element \( u'' \in L \) (\( z \) is fixed).

Let \( f : H_n^\lambda \to H_n^\lambda = L \) be an automorphism applying any homogeneous component of \( \Gamma \) into a homogeneous component of \( \Gamma' \) (take into account that all of them are one-dimensional). Note that \( f \) applies \( Z(L) = \langle z \rangle \) into itself. We can assume that \( f(z) = z \), by replacing \( f \) with the composition of \( f \) with the automorphism given by

\[
z \mapsto \alpha^2 z, \ u \mapsto u, \ x_i^l \mapsto \alpha x_i^l, \ y_i^l \mapsto \alpha y_i^l, \ a_i^l \mapsto \alpha a_i^l \]

for a convenient \( \alpha \in F^\times \).
As $f$ leaves $H_n = [L, L]$ invariant, the homogeneous element $f(u) \notin f([L, L])$, so that there is $\varepsilon \in \mathbb{F}^\times$ such that $f(u) = \varepsilon u'$. Fixed $j \leq s$, we have that
\[
f(x_1^j) \in f(B^1_j(u, \beta_j)) = B^1_j(f(u, \beta_j)) = B^1_j(u', \beta_j / \varepsilon),
\]
but the homogeneous element $f(x_1^j)$ must also belong either to some $\langle B^1_j(u', \beta_j') \rangle$ or to some $\langle B^1_j(u', \alpha_q') \rangle$, and so it is multiple of one of the independent elements forming such blocks (by homogeneity). Observe that the second possibility can be ruled out, because any element $x$ in a block of type I (related to $u'$) verifies that $[x, \varphi^m(x)] = 0$ for all $m \in \mathbb{N}$ (being $\varphi = \text{ad}(u')$), while any element $x$ in a block of type II (related to $u'$) verifies that there is $m \in \mathbb{N}$ such that $[x, \varphi^m(x)] = 0$.

Necessarily $l' = l$, since the size $m$ of a concrete block of type I can be computed by taking $x$ any of its elements (obviously no problem if we take a nonzero multiple) and then $m \geq 1$ is the minimum integer such that $\varphi^m(x) \in \langle x \rangle$. As $\varepsilon(\text{ad } u') f = f(\text{ad } u)$, the result follows.

The also homogeneous element $f(y_1^j)$ must belong to the same $\langle B^1_j(u', \beta_j') \rangle$, since $[f(x_1^j), f(y_1^j)] \neq 0$ while the bracket of two different blocks is zero. And $f(x_{i+1}) = \frac{1}{\varepsilon} \varphi(f(x_i))$ belongs to $\langle B^1_j(u', \beta_j') \rangle$ too for induction on $i$, since the space spanned by the block is $\varphi$-invariant. We have proved that $\langle B^1_j(u', \beta_j / \varepsilon) \rangle = \langle \{f(x_i'), f(y_j') : i = 1, \ldots, l\} \rangle \subset \langle B^1_j(u', \beta_j') \rangle$, but both spaces have the same dimension $2l$, so they coincide.

Now we apply Lemma (4) to get that $\varepsilon_{\beta_j} = \beta_j$ if $l$ is odd and $\varepsilon_{\beta_j} = \beta_j$ if $l$ is even.

We proceed analogously with the blocks of type II.

\[ \square \]

6.5. Weyl groups. Finally, we would like to compute the Weyl groups of the fine gradings on the twisted Lie algebra $L = H_n^\lambda$.

Let $\Gamma$ be a fine grading on $L$. Then, taking into account Theorem (9) there are $l, s, r \in \mathbb{Z}_{\geq 0}$ with $n - 2 = l(2s + r)$ such that $\Gamma = \Gamma(l, s, r; \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r)$, where the scalars $\beta_j, \alpha_i \in \mathbb{F}^\times$ can be chosen such that
\[
\begin{align*}
\{\beta_1, \ldots, \beta_s\} &= \{\delta_1, \ldots, \delta_1, \ldots, \delta_{s'}, \ldots, \delta_{s'}\} \\
\{\alpha_1, \ldots, \alpha_r\} &= \{\gamma_1, \ldots, \gamma_1, \ldots, \gamma_{r'}, \ldots, \gamma_{r'}\}
\end{align*}
\]
(reordering the blocks) and such that
\[
\begin{align*}
\tilde{\delta}_i &\neq \tilde{\delta}_j, \tilde{\gamma}_i &\neq \tilde{\gamma}_j &\forall i \neq j &\text{if } l \text{ is even}, \\
\tilde{\alpha}_i &\neq \tilde{\alpha}_j &\forall i \neq j &\text{if } l \text{ is odd (so } r = 0).'
\end{align*}
\]

(Obviously $m_1 + \cdots + m_{s'} = s$ and $n_1 + \cdots + n_{r'} = r$).

A basis of homogeneous elements of this grading is formed by $z \in Z(L), u, \text{blocks } B^1_j(u, \beta_j) = \{x^1_i, y^1_i, \ldots, x^1_i, y^1_i\}$ if $j \leq s$, and blocks $B^1_j(u, \alpha_i) = \{a^1_i, a^1_i\}$ if $i \leq r$.

First observe that each $\sigma = (\sigma_1, \ldots, \sigma_{r'}) \in S_{n_1} \times \cdots \times S_{n_{r'}}$ (that is, $\sigma(n_1 + \cdots + n_j + t) = n_1 + \cdots + n_j + \sigma_{j+1}(t)$ if $1 \leq t \leq n_{j+1}$) and each $\eta \in S_{m_1} \times \cdots \times S_{m_{s'}}$ allow to define the automorphism $\Upsilon_{(\sigma, \eta)} \in \text{Aut}(\Gamma)$ by
\[
z \mapsto z, \ u \mapsto u, \ x^1_i \mapsto x^1_i \eta(j), \ y^1_i \mapsto y^1_i \eta(j), \ a^1_i \mapsto a^1_i \sigma(t),
\]
which obviously preserves the grading but interchanges the blocks.

We have besides some remarkable elements in the group of automorphisms of the grading which fix all the spaces spanned by the blocks (generalizing Equations (16) and (19)). For each $j \leq s$, consider $\theta_j \in \text{Aut}(\Gamma)$ leaving invariant $\langle \{z, u, x^1_i, y^1_i, a^1_i : i \leq t, p \leq r, t \leq
\]

\[ \Box \]
Lemma 6. If \( l \) is even and the automorphism \( f \in \text{Aut}(\Gamma) \) is such that \( f(z) = z \) and \( f(u) = u \), then \([f]\) belongs to the group generated by
\[
\{[\Gamma_{(\eta,\sigma)}], [\theta_j], [\varrho_t] : j \leq s, t \leq r \}
\]
with \( \eta \in S_{m_2} \cdots \times S_{m_r} \) and \( \sigma \in S_{m_2} \cdots \times S_{m_r} \), permutations as above.

Proof. Fixed \( j = 1, \ldots, r \), there is \( p = 1, \ldots, s \) such that \( \delta_p = \beta_j \) (\( j = m_1 + \cdots + m_{p-1} + b \) for some \( 1 \leq b \leq m_p \)). The image of this \( j \)th block is \( f(B^1_j(u, \delta_p)) = B^1_j(f(u), \delta_p) \), so it must generate the subspace spanned by the \( q \)th block for some \( q \) such that \( \delta_p = \beta_q \) (taking into account Lemma 5 and the conditions required after Equation (30)). In other words, there is \( \eta \in S_s \) such that \( \eta \) applies the \( j \)th block of type I into the span of the \( \eta(j) \)th block of type I, with this \( \eta \in S_{m_1} \times \cdots \times S_{m_r} \). In the same way, there is \( \sigma \in S_{n_1} \times \cdots \times S_{n_r} \), such that any \( n \)th block of type II is applied into the span of the \( \sigma(n) \)th block of type II. Hence, the automorphism \( \Gamma_{(\eta,\sigma)}^{-1}f \) applies each block into the subspace spanned by itself. Now take into account that inside each block, we had computed the Weyl groups in Equations (16) and (19).

Moreover, note that \( \langle [\theta_j], [\varrho_t] \rangle \cong D_1 \) for each fixed \( j \), and these elements commute with all \([\theta_j], [\varrho_t] \) if \( i \neq j \) (any \( t \)). The other elements multiply as follows:
\[
\Gamma_{(\eta,\sigma)}\theta_j\Gamma_{(\eta,\sigma)}^{-1} = \theta_{\eta(j)}, \quad \Gamma_{(\eta,\sigma)}\theta_j\Gamma_{(\eta,\sigma)}^{-1} = \theta_j, \quad \Gamma_{(\eta,\sigma)}\rho_t\Gamma_{(\eta,\sigma)}^{-1} = \rho_{\sigma(t)},
\]
so that their classes do not commute in general. Thus, the group \( \mathcal{W}' \) generated by the set in Equation (31) is isomorphic to
\[
\mathcal{W}' \cong (S_{n_1} \times \cdots \times S_{n_r} \times S_{m_1} \times \cdots \times S_{m_r}) \ltimes (D^*_1 \times \mathbb{Z}_2^r).
\]

The following technical lemma will be useful, too.

Lemma 7. Under the hypothesis above,

a) If \( f \in \text{Aut}(L) \) verifies that \( f(z) = z \) and \( f(u) = \varepsilon u \), then \( \varepsilon \) is a primitive \( p \)th root of the unit and the spectrum of \( \text{ad}(u) \) can be splitted as
\[
\{\hat{\alpha}_1, \ldots, \hat{\alpha}_r\} = \bigcup_{i=0}^{p-1} \varepsilon^i X, \quad \text{(disjoint union)} \quad \{\hat{\beta}_1, \ldots, \hat{\beta}_s\} = \bigcup_{i=0}^{p-1} \varepsilon^i Y,
\]
for some sets \( X \) and \( Y \) of classes such that \( X \cap \varepsilon^j X \) is either \( \emptyset \) or \( X \) for any \( j \), and in the same way \( Y \cap \varepsilon^j Y \) is either \( \emptyset \) or \( Y \).
b) If there is \( p \) a divisor of \( r \) and \( s \) such that \( \beta_j = \beta_{j+r} \) for all \( 1 \leq j \leq \frac{r}{p} \) and \( \alpha_i = \alpha_{i+\frac{r}{p}} \) for all \( 1 \leq i \leq \frac{s}{p} \), then the map \( g_p \) given by

\[
z \mapsto z, \quad u \mapsto e^{-1}u, \quad x_i \mapsto x_i^{j+\frac{r}{p}}, \quad y_i \mapsto y_i^{j+\frac{r}{p}}, \quad a_i \mapsto a_i^{j+\frac{r}{p}},
\]

if \( t \leq r, \ i \leq l, \ j \leq s, \) for \( \epsilon \) a primitive \( p \)th root of the unit, is an \( \epsilon \) order \( p \) automorphism.

**Proof.** As \( f(B_1'(u, \beta_j)) = B_1'(f(u), \beta_j) = B_1'(u, \beta_j/\epsilon) \) for all \( j \), by Lemma 5 there is \( k \leq r \) such that \( \beta_j/\epsilon = \beta_k \). Thus there are \( \sigma \in S_r \) and \( \eta \in S_s \) such that

\[
\epsilon \alpha_i = \alpha_{\sigma(i)}, \quad \epsilon \beta_j = \beta_{\eta(j)}
\]

for all \( i \leq r \) and \( j \leq s \). In particular (multiplying for all the indices), we get that \( \hat{\sigma} = 1 = \hat{\epsilon} \) and hence \( \epsilon^{rt} = 1 \). Take \( q, p \) the minimum integers such that \( \hat{\epsilon}^q = 1 = \hat{\epsilon}^p \) (so \( q \) divides \( p \)). Thus, for all \( j \), the set \( \{ \beta_j, \epsilon \beta_{j+1}, \ldots, \epsilon^{p-1} \beta_j \} \) has \( q \) different classes repeated \( p/q \) times. Now we take \( j_1 = 1 \), and by induction \( j_i \notin \{ j_d, \eta(j_d), \ldots, \eta^{p-1}(j_d) \} : d \leq i - 1 \) but such that \( \beta_{j_i} = \beta_{j_d} \) if there is \( d < i \) such that \( \beta_{j_i} \in \{ \epsilon^{\delta} \beta_{j_d} : g \leq q \} \). Hence there is a set \( Y = \{ \beta_{j_1}, \ldots, \beta_{j_q} \} \) of classes such that \( \{ \beta_1, \ldots, \beta_s \} = \bigcup_{j=1}^{p-1} \epsilon^{j}Y \) (disjoint union) which verifies that \( Y \cap \epsilon^{j}Y = \emptyset \) if \( j \) is not a multiple of \( q \) and \( Y \cap \epsilon^{j}Y = Y \) otherwise. We can deal with \( X \) in the same way.

The second item b) is clear. Also it is clear that given any decomposition as in Equation (32), we can reorder the blocks to are in the situation of item b) precisely for \( p \) and consider \( g_p \) correspondingly.

**Proposition 6.** Take \( p \in \mathbb{N} \) the greatest integer such that, for \( \epsilon \) a primitive \( p \)th root of the unit, there are \( X = W(\Gamma) \) and \( Y \) sets of classes such that Equation (32) holds. Let \( q \) be the minimum positive integer such that \( \hat{\epsilon}^q = 1 \).

The Weyl group of \( \Gamma \), if \( l \) is even, is isomorphic to

\[
\left( (S_n \times \cdots \times S_n) \times (S_m \times \cdots \times S_m) \times (D_1^{\star} \times \mathbb{Z}_p^\star) \right) \times \mathbb{Z}_p^{\mathbb{N}/q}
\]

**Proof.** Take \( f \in \text{Aut}(\Gamma) \), so that \( f(z) \in \langle z \rangle \) and \( f(u) \in \langle u \rangle \). Since we can compose \( f \) with an automorphism in the stabilizer as the given one in Equation (29), then we can assume \( f(z) = z \).

Consider \( g_p \) the automorphism described in Lemma 1b). Note that if we apply Lemma 7b) to \( f \), there is \( \epsilon \) a primitive \( p' \)th root of the unit such that \( f(u) = \epsilon u \). So \( g_p^{-1}(u) = \epsilon u \), being \( \epsilon \) a root \( lcm(p, p') \)-primitive. Again by Lemma 7b), we have the corresponding splitting, so that, by maximality of \( p \), we conclude that \( lcm(p, p') = p \) and \( \epsilon \) is a power of \( \epsilon \). Thus there is \( c \in \mathbb{N} \) such that \( g_p^c(u) = u \) and thus Lemma 6 is applied and \( [f] \in \langle (g_p), W' \rangle \). Moreover, \( W(\Gamma) = W' \rtimes \langle (g_p) \rangle \), since \( g_p f g_p^{-1} \) fixes \( u \) for any \( f \in W' \).

In order to make precise the composition of \( g_p \) with \( Y_{(\eta, \sigma)} \), we need to choose the order of the scalars \( \beta_j \)’s and \( \alpha_i \)’s adapted to both automorphisms, changing slightly the order taken in Equation (30). We mean: order the set \( Y = \{ \delta_1, \ldots, \delta_1, \ldots, \delta_a, \ldots, \delta_a \} \) where \( \delta_i \) is repeated \( l_i \) times and \( \delta_i \notin \{ \epsilon \delta_i \} \) if \( i \neq j \). So the set of different representatives for the blocks of type I are \( \{ \epsilon^i \delta_b : 1 \leq b \leq a, 0 \leq t \leq q - 1 \} \), where \( \epsilon^i \delta_b \) appears \( l_b p \) times in the positions

\[
\{ l_1 + \cdots + l_{b-1} + i + \frac{sq}{p} t + \frac{sq}{p} c : i = 1, \ldots, l_b; \ c = 0, \ldots, \frac{p}{q} - 1 \}.
\]
(With our notation \( \{m_1, \ldots, m_s\} = \cup_{l=0}^{q-1} \{l \mathbb{Z} : 1 \leq b \leq a\} \)) we denote by \( S'_\alpha \) the set of permutations which fix the sets in Equation (33) for all \( 1 \leq b \leq a, 0 \leq t \leq q - 1 \), which is of course isomorphic to \( S_{m_1} \times \cdots \times S_{m_s} \). Proceed in the same way with the set \( X \) and denote by \( \widetilde{S}_r \) the set of suitable permutations (interchanging the positions corresponding to the same scalars). Therefore,

\[
\mathcal{W}(\Gamma) = \langle \{ [g_p], [\Upsilon_{(\eta, \sigma)}], [\vartheta_j], [\vartheta_t], [\vartheta_i] \ : \ j \leq s, t \leq r, \eta \in \bar{S}'_s, \sigma \in \tilde{S}_r \rangle,
\]

where we can observe that if \( \theta'(j) := \eta(j + \frac{t}{p}) - \frac{j}{p} \) and \( \sigma(t) := \sigma(t + \frac{r}{p}) - \frac{t}{p} \), then \( \theta' \in S'_s, \sigma \in \tilde{S}_r \) and

\[
g_p^{-1}\Upsilon_{(\eta, \sigma)}g_p = \Upsilon_{(\theta', \sigma')}.
\]

The remaining elements multiply as follows:

\[
g_p^{-1}\theta_j g_p = \theta_{j-\frac{p}{r}}, \quad g_p^{-1}\vartheta_j g_p = \vartheta_{j-\frac{r}{s}}, \quad g_p^{-1}p g_p = p - \frac{z}{r}.
\]

This explains how the actions in the semidirect products are defined.

Finally, the new generator \( [g_p] \) has order \( p \). Besides \( g_p^n(u) = x^u \in \langle \xi \rangle u \), so its composition with some power of the following element in the stabilizer

\[
z \mapsto \xi z, \ u \mapsto \xi u, \ x^j_i \mapsto \xi^j_i x^j_i, \ y^j_i \mapsto \xi^{j+1} y^j_i, \ a^j_i \mapsto \xi^j_i a^j_i,
\]

(followed by a convenient element as the one in Equation (29)) fixes \( u \) (and \( z \)). Thus \( [g_p]^q \in \mathcal{W}' \). Moreover, \( q \) is the least number with this property (a power of less order does not stabilize the blocks). \( \square \)

We aboard now the case \( l \) odd. Recall that there are no blocks of type II. As above, if \( f \in \text{Aut}(\Gamma) \) verifies \( f(u) = \varepsilon u \), then \( f(B_l^j(u, \beta_j)) = B_l^j(u, \beta_j/\varepsilon) \). Thus there is \( \eta \in S_s \) such that

\[
\varepsilon \beta_j = \tilde{\beta}_\eta(j)
\]

for all \( j = 1, \ldots, s \), and \( \varepsilon \) turns out to be a \( p \)-th root of the unit for some \( p \). So we can divide \( \{\tilde{\beta}_1, \ldots, \tilde{\beta}_s\} = \cup_{l=0}^{p-1} e^j \Upsilon', \) with two different pieces disjoint and \( p \) maximum such that this decomposition exists. Take \( g_p \) as in the even case.

The maps \( \vartheta_j \) are not longer automorphisms, but we can consider \( \vartheta' \in \text{Aut}(\Gamma) \) given by

\[
\vartheta'(z) = z, \ \vartheta'(u) = -u, \ \vartheta'(x^j_i) = (-1)^j y^j_i, \ \vartheta'(y^j_i) = (-1)^{j+1} x^j_i.
\]

If \( f \in \text{Aut}(\Gamma) \) fixes the subspaces spanned by the blocks, it is not difficult to check that \( f \) belongs to the subgroup generated by

\[
\{[\vartheta_j] \ : \ j \leq s\} \cup \{[\vartheta']\},
\]

which is isomorphic to \( \mathbb{Z}_l^+ \times \mathbb{Z}_2, \) and hence

**Proposition 7.** The Weyl group of \( \Gamma = \Gamma(l, s, 0; \beta_1, \ldots, \beta_s) \), if \( l \) is odd, is isomorphic to

\[
\mathcal{W}(\Gamma) \cong \left( S_{m_1} \times \cdots \times S_{m_s} \times \mathbb{Z}_2 \times \mathbb{Z}_l^+ \right) \rtimes \mathbb{Z}_p,
\]

with \( m_1 + \cdots + m_s = s = \frac{n^2}{2p} \), for \( m_j \)'s defined as in Equation (30), \( p \in \mathbb{N} \) the greatest integer such that, for \( \varepsilon \) a primitive \( p \)-th root of the unit, there is \( Y \) a set of classes such that Equation (32) holds, and \( q \) minimum such that \( \varepsilon^q = 1 \).
Example 2. We now describe the Weyl groups of the fine gradings on $L = H_{11,11,11}^{(1,1,1,1)}$ computed in Example 1. Observe first that the results for $\mathcal{W}(\Gamma_1)$ and $\mathcal{W}(\Gamma_2)$ are quite different than those ones in Proposition 5 since

$$\mathcal{W}(\Gamma_2) \cong \frac{\mathbb{Z}_2^4 \times \mathbb{Z}_4}{\mathbb{Z}_2^2} \quad \mathcal{W}(\Gamma_1) \cong \frac{\mathbb{Z}_2^6 \times \mathbb{Z}_4}{\mathbb{Z}_2}.$$ 

Indeed, $\mathcal{W}(\Gamma_2)$ has 4 generators: $[g] \equiv [g_4]$ (the only element with order 4), $[\vartheta']$ and the classes of the two automorphisms $f_1$ and $f_2$ coming from permutations, such that the three latter ones commute, $[g]$ commute with $[\vartheta']$, $[g_1 f_1 g^{-1}] = [f_2]$ and $[g]^2 = [\vartheta' f_1 f_2]$. And for $\Gamma_1 ((l, r, s) = (2, 4, 0))$, the generators of the Weyl group are $\{ [\varphi_i] : i = 1, \ldots, 4 \}$, $[g] \equiv [g_4]$; the automorphism interchanging $e_1$ with $e_2$ and $\bar{e}_1$ with $\bar{e}_2$ and the one interchanging $e_3$ with $e_4$ and $\bar{e}_3$ with $\bar{e}_4$.

The remaining cases of Example 1 correspond, respectively, to Weyl groups isomorphic to $\mathbb{Z}_3^4 \times \mathbb{Z}_2$, $\mathbb{Z}_4^2$, $\mathbb{Z}_2^2$ and $D_4$.

Acknowledgment. The authors would like to thank the referee for his exhaustive review of the paper as well as for many suggestions which have helped to improve the work.

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