PRYM-TYURIN VARIETIES USING SELF-PRODUCTS OF GROUPS

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Abstract. Given Prym-Tyurin varieties of exponent \( q \) with respect to a finite group \( G \), a subgroup \( H \) and a set of rational irreducible representations of \( G \) satisfying some additional properties, we construct a Prym-Tyurin variety of exponent \( |G : H|q \) in a natural way. We study an example of this result, starting from the dihedral group \( D_p \) for any odd prime \( p \). This generalizes the construction of [4] for \( p = 3 \). Finally, we compute the isogeny decomposition of the Jacobian of the curve underlying the above mentioned example.

1. Introduction

Consider a cartesian diagram of smooth projective curves

\[
\begin{array}{ccc}
X & & X \\
q_1 & & q_2 \\
\downarrow & & \downarrow \\
X_1 & & X_2 \\
\downarrow h_1 & & \downarrow h_2 \\
\mathbb{P}^1 & & \\
\end{array}
\]

with \( \text{deg } h_i = d_i \geq 2 \). Then the Jacobian \( JX \) cannot be equal to \( JX_1 \times JX_2 \), since the genus \( g_X \) of \( X \) is always bigger than the sum \( g_{X_1} + g_{X_2} \). However, if the \( h_i \) do not factorize via a cyclic étale covering for \( i = 1 \) and \( 2 \), then \( q_i^*: JX_i \to JX \) is injective for \( i = 1 \) and \( 2 \) and it is easy to see that

\[
q_1^* + q_2^*: JX_1 \times JX_2 \to JX
\]

is an isogeny onto its image. One might ask whether \( q_1^* + q_2^* \) is an embedding. This is certainly not the case if \( d_1 \) is not a divisor of \( d_2 \) (we assume that \( d_1 \leq d_2 \)), since then the type of the polarization on \( JX_1 \times JX_2 \) induced by the canonical polarization would be \((d_1, \ldots, d_1, d_2, \ldots, d_2)\). However, if we assume \( d_1 = d_2 =: d \), then the polarization would be the \( d \)-fold of a principal polarization and \( JX_1 \times JX_2 \) would be a Prym-Tyurin variety of exponent \( d \) in \( JX \). Recall that a Prym-Tyurin variety of exponent \( q \) in a Jacobian \( J \) is by definition an abelian subvariety of \( J \) on which the canonical polarization of \( J \) induces the \( q \)-fold of a principal polarization.

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An example for this was given by Mumford in [5, p. 346], where he showed that in the case of the above diagram with $\deg h_i = 2$ for $i = 1$ and 2, if $Y$ denotes the hyperelliptic curve ramified over the union of the branch loci of $h_1$ and $h_2$, then there is an étale double covering $X \to Y$ whose Prym variety in $JX$ is isomorphic (as principally polarized abelian varieties) to the product of Jacobians $JX_1 \times JX_2$.

Our main result is a generalization of Mumford’s theorem. In order to state it, we need to recall some facts of [2]. In that paper we started with a smooth projective curve $Z$ with action by a finite group $G$ such that $Z/G \cong \mathbb{P}^1$. Then we associated to a set of pairwise non-isomorphic irreducible rational representations and a subgroup $H$ of $G$ satisfying some additional properties, a Prym-Tyurin variety $P$ in $JX$ with $X = Z/H$. We say that this construction of the Prym-Tyurin variety is a presentation of $P$ with respect to the action of the group $G$, of the subgroup $H$ and the set of representations. For a more precise statement see Theorem 2.2 below and the subsequent definition. To state our result, we also need the following definition.

Let $\pi : Z \to Y$ be a Galois covering with Galois group $G$ and let $C_1, \ldots, C_t$ be pairwise different conjugacy classes of cyclic subgroups of $G$. We define the geometric signature of the covering $\pi$ to be the tuple $[\gamma, (C_1, s_1), \ldots, (C_t, s_t)]$, where $\gamma$ is the genus of the quotient curve $Y$, the covering has a total of $\sum_{j=1}^t s_j$ branch points in $Y$ and exactly $s_j$ of them are of type $C_j$ for $j = 1, \ldots, t$; that is, the corresponding points in its fiber have stabilizer belonging to $C_j$. Using this terminology, our first result can roughly be stated as follows.

**Theorem 1.1.** Suppose that for $i = 1$ and 2 the action of a group $G$ on a curve $Z_i$ has geometric signature $[0; (C_1^i, s_1^i), \ldots, (C_t^i, s_t^i)]$ and the branch loci of the corresponding coverings are disjoint in $\mathbb{P}^1$. Assume that we are given a presentation of a Prym-Tyurin variety $P_i$ (with the same exponent $q$) with respect to the action of $G$ on $Z_i$, a subgroup $H$ and a set of non-trivial rational representations. Then

(a): the induced action of the group $G^2 := G \times G$ on the fibre product $Z := Z_1 \times_{\mathbb{P}^1} Z_2$ defines a Prym-Tyurin variety $P$ in the Jacobian $J(Z/H^2)$ of exponent $\tilde{q} = |G : H|q$;

(b): under mild additional assumptions, $P \cong P_1 \times P_2$ (isomorphic as principally polarized abelian varieties).

For a more precise version of this theorem see Theorems 3.3 and 3.6. In [2] we showed that almost all known Prym-Tyurin varieties admit presentations in the above sense. Moreover, it is easy to find presentations of Prym-Tyurin varieties of exponent 1 (which are Jacobians according to the Theorem of Matsusaka-Ran) and exponent 2 (which essentially are classical Prym varieties according to a Theorem of Welters). In particular, this gives the above mentioned sufficient criterion for a product of Jacobians to be a subvariety of a Jacobian. Hence Theorem 1.1 can be considered as a generalisation of Mumford’s Theorem mentioned above. It should be noted that the special case of our proof in Mumford’s situation is different from Mumford’s original proof (see Example 3.7 and Corollary 3.9 below).
In the second part of the paper we work out an explicit example, which was, in fact, the starting point of this paper. In [4] a Prym-Tyurin variety of exponent 3 was associated to every étale degree 3 covering $\tilde{Y}$ of a hyperelliptic curve $Y$, such that the Galois group of its Galois closure over $\mathbb{P}^1$ is $S_3 \times S_3$, where $S_3$ denotes the symmetric group of degree 3. Solomon generalized in [6] this construction to define a Prym-Tyurin variety for every group $S_n \times S_n$. Since $S_3$ coincides with the dihedral group $D_3$ of order 6, one might ask whether there is also a generalization of the construction of [4] to $D_n \times D_n$. We show in the second part of the paper that this is in fact the case, at least for $D_p$ with $p$ an odd prime. The construction certainly also generalizes to the group $D_n$, however we restrict ourselves to a prime number, since the group theory of an arbitrary $D_n$ is more complicated.

Let $p$ be an odd prime number, and consider an étale $p$-fold covering $\tilde{Y} \to Y$ of a hyperelliptic curve $Y$ of genus $g$, such that the Galois closure of the composed map $\tilde{Y} \to Y \to \mathbb{P}^1$ has Galois group $D_p \times D_p$. Then the fibre product

$$X = \mathbb{P}^1 \times_{Y(2)} \tilde{Y}^{(2)}$$

is a smooth projective curve. Here $Y^{(2)}$ and $\tilde{Y}^{(2)}$ denote the second symmetric product of $Y$ and $\tilde{Y}$ and $\mathbb{P}^1 \to Y^{(2)}$ is the canonical embedding of the $g^1_2$ of $Y$. We define an effective symmetric fixed-point free $(p-1)^2$-correspondence $D$ on $X$ whose associated endomorphism $\gamma_D \in \text{End}(JX)$ satisfies the equation

$$\gamma_D^2 + (p-2)\gamma_D - (p-1) = 0.$$

This gives part (a) of the following theorem (see Corollary 6.2).

**Theorem 1.2.** (a): $D$ defines a Prym-Tyurin variety $P$ of exponent $p$ in the Jacobian $JX$.
(b): There exist smooth projective curves $X_1$ and $X_2$ whose fibre product over $\mathbb{P}^1$ is $X$, and such that

$$P \simeq JX_1 \times JX_2,$$

isomorphic as principally polarized abelian varieties.

This can be proven directly, which was our first approach. Then we realized that this is a special case of Theorem [1.1]. In fact, one can associate to the data the curves $X_1$ and $X_2$ in a canonical way, such that $X = X_1 \times_{\mathbb{P}^1} X_2$. Moreover it turns out that the Kanev correspondence, associated to the correspondence of Theorem [1.1], coincides with the correspondence $D$ (see Proposition 6.1). So Theorem [1.1] implies Theorem 1.2.

In the last part of the paper we compute the isogeny decomposition of the Jacobian of the curve $X$ of Theorem 1.2. Let $X$ be as above. The Galois group of the Galois closure $Z$ over $\mathbb{P}^1$ is the group $D_p^2$. Then, if $X_1$ and $X_2$ denote the curves of Theorem 1.2 we show in Section 7.
Theorem 1.3. There are étale \( p \)-fold coverings \( \tilde{Y}_j \to Y \), \( j = 1, \ldots, \frac{p-1}{2} \), of a hyperelliptic curve \( Y \), all of them subcovers of \( Z \) by explicitly given subgroups of \( \tilde{D}_p^2 \), such that 

\[
JX \sim JX_1 \times JX_2 \times P(\tilde{Y}_1/Y) \times \cdots \times P(\tilde{Y}_{(p-1)/2}/Y),
\]

where \( \sim \) denotes isogeny and \( P(\tilde{Y}_j/Y) \) denotes the (generalized) Prym variety of the covering \( \tilde{Y}_j \to Y \).

Throughout the paper we work over the field of complex numbers.

2. Preliminaries

In this section we recall the main result of [2]. Let \( G \) be a finite group. In order to fix the notation, we recall some basic properties of representations of \( G \). For any complex irreducible representation \( V \) of \( G \), we denote by \( \chi_V \) its character, by \( L_V \) its field of definition and by \( K_V = \mathbb{Q}(\chi_V(g) \mid g \in G) \). Then \( L_V \) and \( K_V \) are finite abelian extensions of \( \mathbb{Q} \); we denote by \( m_V = [L_V : K_V] \) the Schur index of \( V \). For any automorphism \( \varphi \) of \( L_V/\mathbb{Q} \) we denote by \( V^{\varphi} \) the representation conjugate to \( V \) by \( \varphi \).

If \( \mathcal{W} \) is a rational irreducible representation of \( G \), then there exists a complex irreducible representation \( V \) of \( G \), uniquely determined up to conjugacy in \( \text{Gal}(L_V/\mathbb{Q}) \), such that 

\[
\mathcal{W} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{\varphi \in \text{Gal}(L_V/\mathbb{Q})} V^{\varphi} \simeq m_V \bigoplus_{\tau \in \text{Gal}(K_V/\mathbb{Q})} V^\tau.
\]

We call \( V \) a complex irreducible representation associated to \( \mathcal{W} \).

Let \( \mathcal{W}_1, \ldots, \mathcal{W}_r \) denote nontrivial pairwise non-isomorphic rational irreducible representations of the group \( G \) with associated complex irreducible representations \( V_1, \ldots, V_r \). In our applications \( r \) will be either 1 or 2. We make the following hypothesis on the \( \mathcal{W}_i \) and a subgroup \( H \) of \( G \).

Hypothesis 2.1. For all \( k, l = 1, \ldots, r \) we assume

a) \( \dim V_k = \dim V_l =: n \),  
b) \( K_{V_k} = K_{V_l} =: L \),  
c) \( \dim V_k^H = 1 \),  
d) \( H \) is maximal with property c); that is, for every subgroup \( N \) of \( G \) with \( H \subseteq N \) there is an index \( k \) such that \( \dim V_k^N = 0 \).

Choose a set of representatives 

\[
\{ g_{ij} \in G \mid i = 1, \ldots, d \text{ and } j = 1, \ldots, n_i \}
\]

for both the left cosets and right cosets of \( H \) in \( G \), such that 

\[
G = \bigsqcup_{i=1}^d Hg_{i1}H \quad \text{and} \quad Hg_{i1}H = \bigsqcup_{j=1}^{n_i} g_{ij}H = \bigsqcup_{j=1}^{n_i} Hg_{ij}
\]

are the decompositions of \( G \) into double cosets, and of the double cosets into right and left cosets of \( H \) in \( G \). Moreover, we assume \( g_{11} = 1_G \).
Now let $Z$ be a (smooth projective) curve with $G$-action and $\pi_H : Z \to X := Z/H$. In [2] we defined a correspondence on $X$, which is given by

$$\mathcal{D}(x) = \sum_{i=1}^{d} b_i \sum_{j=1}^{n_i} \pi_H g_{ij}(z).$$

for all $x \in X$ and $z \in Z$ with $\pi_H(z) = x$, where

$$b_i := \sum_{k=1}^{r} \sum_{h \in H} \text{tr}_{L/Q}(\chi_{V_k}(hg_i^{-1}))$$

is an integer for $i = 1, \ldots, d$.

Let $\delta_{\mathcal{D}}$ denote the endomorphism of the Jacobian $J_X$ associated to the correspondence $\mathcal{D}$. We denote by

$$P_{\mathcal{D}} := \text{Im}(\delta_{\mathcal{D}})$$

the image of the endomorphism $\delta_{\mathcal{D}}$ in the Jacobian $J_X$ and call it the (generalized) Prym variety associated to the correspondence $\mathcal{D}$.

Setting

$$b := \gcd\{b_1 - b_i \mid 2 \leq i \leq d\},$$

[2] Theorem 4.8] can be stated as follows:

**Theorem 2.2.** Let $\mathcal{W}_1, \ldots, \mathcal{W}_r$ denote nontrivial pairwise non-isomorphic rational irreducible representations of the group $G$ with associated complex irreducible representations $V_1, \ldots, V_r$ satisfying Hypothesis [2.1] for a subgroup $H$ of $G$. Suppose that the action of the finite group $G$ has geometric signature $[0; (C_1, s_1), \ldots, (C_t, s_t)]$ satisfying

$$\sum_{j=1}^{t} s_j \left[ q[L : Q] \left( \sum_{k=1}^{r} (\dim V_k - \dim V_k^{G_j}) \right) - ([G : H] - |H\backslash G/G_j|) \right] = 0,$$

where $G_j$ is a subgroup of $G$ of class $C_j$ and

$$q = \frac{|G|}{b \cdot n}.$$  

Then $P_{\mathcal{D}}$ is a Prym-Tyurin variety of exponent $q$ in $J_X$.

Furthermore, we showed in [2] Section 4.4,

$$\dim P_{\mathcal{D}} = [L : Q] \sum_{i=1}^{r} \left[ -n + \frac{1}{2} s_j (\dim V_i - \dim V_i^{G_j}) \right]$$

and

$$g_X = 1 - [G : H] + \frac{1}{2} \sum_{j=1}^{t} s_j ([G : H] - |H\backslash G/G_j|)$$
In the sequel we will use the following definition: We say that the construction of the Prym-Tyurin variety $P = P_D$ of Theorem 2.2 is a presentation of $P$ with respect to the action of the group $G$, the subgroup $H$ and the set of representations $\{W_1, \ldots, W_r\}$.

3. The construction

In this section we show how to construct new Prym-Tyurin varieties out of given ones. As above let $G$ be a finite group. For $i = 1, 2$, let $Z_i$ denote a smooth projective curve, on which $G$ acts with geometric signature $[0; (C_1^i, s_1^i), \ldots, (C_t^i, s_t^i)]$ on $Z_1$ and $[0; (C'_{1j}^i, s_{1j}^i), \ldots, (C'_{t_j}^i, s_{t_j}^i)]$ on $Z_2$. Let $h_i : Z_i \rightarrow \mathbb{P}^1$ denote the corresponding covering maps. Then we consider the product group

$$G^2 := G \times G.$$  

For any subgroup $H$ of $G$, we denote $H_1 = H \times \{1\}_G$, $H_2 = \{1\}_G \times H$ and write $C_j^1 = C_j \times \{1\}_G$ and similarly $C_j^2 = \{1\}_G \times C_j'$. So the $C_j^\ell$ are considered as conjugacy classes of cyclic subgroups of $G^2$ for $\ell = 1$ and 2.

**Lemma 3.1.** Suppose the branch loci of $h_i : Z_i \rightarrow \mathbb{P}^1$ are disjoint in $\mathbb{P}^1$. Then the fibre product

$$Z := Z_1 \times_{\mathbb{P}^1} Z_2$$

is a smooth projective curve, Galois over $\mathbb{P}^1$ with Galois group $G^2$ and geometric signature $[0; (C_1^1, s_1^1), \ldots, (C_{t_1}^1, s_{t_1}^1), (C_1^2, s_1^2), \ldots, (C_{t_2}^2, s_{t_2}^2)]$.

**Proof.** This is elementary. The branch loci being disjoint implies that $Z$ is smooth and classical Galois theory implies that $Z$ is Galois over $\mathbb{P}^1$ with Galois group $G^2$. The last statement is clear from the definitions. $\square$

Now let $H$ be a subgroup of $G$ and consider $X_1 := Z_1/H$ and $X_2 := Z_2/H$. Clearly the curve $X := Z/(H \times H)$ is the fibre product of $X_1$ and $X_2$ over $\mathbb{P}^1$,

$$X = Z/H^2 = X_1 \times_{\mathbb{P}^1} X_2$$

and we have the following diagram, where the maps are the obvious ones.

(3.1)

Here

$$\deg h_1 = \deg h_2 = \deg p_1 = \deg p_2 = |G|,$$

$$\deg \varphi_1 = \deg \varphi_2 = \deg q_1 = \deg q_2 = [G : H],$$

$$\deg \pi = |H|^2.$$
In the rest of this section we change slightly the notation for the geometric signature, for the sake of simplicity. Let \( C_1, \ldots, C_t \) denote all conjugacy classes of nontrivial cyclic subgroups of \( G \). Then in the tuple \([0; (C_1, s_1), \ldots, (C_t, s_t)]\) a number \( s_j \) will be 0 precisely if there is no branch point of type \( C_j \). Before constructing a Prym-Tyurin variety using the action of \( G^2 \) on \( Z \), we need the following Lemma for which use the following notation. For \( i = 1 \) and 2 let \( \mathcal{D}_i \) denote the correspondence on the curve \( X_i \) defined in (2.1) for the group \( G \) with respect to the subgroup \( H \) and the representations \( \mathcal{W}_1, \ldots, \mathcal{W}_r \). Similarly let \( \mathcal{D} \) denote the correspondence on the curve \( X \) defined in (2.1) for the Group \( G^2 \) with respect to the subgroup \( H^2 \) and the representations \( \mathcal{W}_i^1 = \mathcal{W}_i \otimes V_0 \) and \( \mathcal{W}_i^2 = V_0 \otimes \mathcal{W}_i \) for \( i = 1, \ldots, r \), where \( V_0 \) denotes the trivial representation of \( G \). Then we have

**Lemma 3.2.**

\[ \mathcal{D} = \left| H \right| \cdot (q_1^* \mathcal{D}_1 + q_2^* \mathcal{D}_2) \]

**Proof.** For simplicity we assume \( r = 1 \) and write \( \mathcal{W} = \mathcal{W}_1 \) etc. The proof for the general case is the same, only notationally more complicated.

Set \( d = \left| H \setminus G/H \right| \) and \( \{ g_{ij} : i = 1, \ldots, d; j = 1, \ldots, n_i \} \) as in Section 2. Therefore \( \left| H^2 \setminus G^2/H^2 \right| = d^2 \) and \( \{ (g_{ij}, g_{kl}) : i, k = 1, \ldots, d; j = 1, \ldots, n_i; l = 1, \ldots, n_k \} \) are representatives of both left and right cosets of \( H^2 \) in \( G^2 \). According to (2.1) we have

\[ \mathcal{D}_\nu(x_\nu) = \sum_{i=1}^{d} a_i \sum_{j=1}^{n_i} \pi_H g_{ij}(z_\nu) \]

for all \( x_\nu \in X_\nu \), where \( z_\nu \in Z_\nu \) is a preimage of \( x_\nu, 1 \leq \nu \leq 2 \), and where

\[ a_i := \sum_{h \in H} \text{tr}_{L/Q}(\chi_V(h g_{i1})) \]

is the same integer for both \( \mathcal{D}_\nu \) and \( i = 1, \ldots, d \).

By definition we have \( (q_\nu^* \mathcal{D}_\nu)(x_1, x_2) = q_\nu^{-1} \mathcal{D}_\nu q_\nu(x_1, x_2) = q_\nu^{-1} \mathcal{D}_\nu(x_\nu) \) for \( \nu = 1, 2 \). Therefore

\begin{equation}
(q_\nu^* \mathcal{D}_\nu)(x_1, x_2) = q_\nu^{-1} \left( \sum_{i=1}^{d} a_i \sum_{j=1}^{n_i} \pi_H g_{ij}(z_\nu) \right) = \sum_{i=1}^{d} a_i \sum_{j=1}^{n_i} \sum_{l=1}^{n_k} (\pi_H g_{ij}(z_1), \pi_H g_{kl}(z_2)).
\end{equation}

On the other hand, according to (2.1) we have for \( \mathcal{D} \),

\begin{equation}
\mathcal{D}(x_1, x_2) = \sum_{i=1}^{d} \sum_{k=1}^{d} (b_{ik} + b_{\nu k}^*) \sum_{j=1}^{n_i} \sum_{l=1}^{n_k} (\pi_H g_{ij}(z_1), \pi_H g_{kl}(z_2))
\end{equation}

with \( z_1 \) and \( z_2 \) as above,

\begin{equation}
b_{ik} = \sum_{(h_1, h_2) \in H^2} \text{tr}_{L/Q}(\chi_V(h_1 g_{i1}, h_2 g_{k1})) = \sum_{(h_1, h_2) \in H^2} \text{tr}_{L/Q}(\chi_V(h_1 g_{i1}) \chi_V(h_2 g_{k1})) = |H| \sum_{h \in H} \text{tr}_{L/Q}(\chi_V(h g_{i1})) = |H| \cdot a_i
\end{equation}
and similarly
\begin{equation}
(3.5)\quad b'_{ik} = \sum_{(h_1,h_2)\in H^2} \text{tr}_{L/Q}(\chi_{(V_0\otimes V)}(h_1g_i^{-1}, h_2g_i^{-1})) = |H| \cdot a_k.
\end{equation}

Inserting the last equations into (3.3) and comparing with (3.2) we deduce
\[
D(x_1,x_2) = |H| \cdot (q_1^*D_1 + q_2^*D_2)(x_1,x_2)
\]

\[
\square
\]

**Theorem 3.3.** Let \(W_1, \ldots, W_r\) denote nontrivial pairwise non-isomorphic rational irreducible representations of the group \(G\) with associated complex irreducible representations \(V_1, \ldots, V_r\) satisfying Hypothesis \(\mathbb{H}\) with respect to a subgroup \(H\) of \(G\).

Suppose that the action of \(G\) on \(Z_i\) has geometric signature \([0; (C_1, s_1^i), \ldots, (C_t, s_t^i)]\) on \(Z_i\) for \(i = 1, 2, \ldots\), and satisfies
\begin{equation}
(3.6)\quad \sum_{j=1}^{t} s_j^i \left[ q[L : Q] \sum_{k=1}^{r} (\dim V_k - \dim V_k^{G_j}) - ([G : H] - |H\setminus G/G_j|) \right] = 0.
\end{equation}

for \(i = 1, 2, \) where \(G_j\) is a subgroup of type \(C_j\) and \(q = \frac{|G|}{b\cdot n}\). Furthermore assume that the branch loci of \(Z_i \rightarrow \mathbb{P}^1\) are disjoint in \(\mathbb{P}^1\).

Then the action of the group \(G^2\) on the curve \(Z\) defines a Prym-Tyurin variety \(P\) in the Jacobian \(JX\) of exponent \(\tilde{q} = [G : H]q\) and dimension
\[
\dim P = [L : Q] \sum_{i=1}^{r} [-2n + \frac{1}{2} \sum_{j=1}^{t} (s_j^i + s_j^i)(\dim V_i - \dim V_i^{G_j})].
\]

**Proof.** First note that Hypothesis \(\mathbb{H}\) is satisfied for the subgroup \(H^2\) of \(G^2\) and the representations \(W_i \otimes V_0, V_0 \otimes W_j\), with associated complex representations \(V_j^1 := V_j \otimes V_0\) and \(V_j^2 := V_0 \otimes V_j\), where \(V_0\) denotes the trivial representation of \(G\).

To see this, notice that
\[
\dim(V_j^i)_{H \times H} = \langle V_j^i, \rho_{H \times H}^{G \times G} \rangle_{G \times G} = \langle V_j^i, \rho_H^{G} \rangle = 1
\]
for all \(i\) and \(j\). The maximality of \(H \times H\) with respect to this property is a consequence of the fact that for every \(W_j\) both representations \(W_j \otimes V_0\) and \(V_0 \otimes W_j\) occur in \(\rho_{H \times H}^{G \times G}\).

First, we need to compute the exponent \(\tilde{b}_i\) this data determines. In order to do that, we need to compute the greatest common divisor \(\tilde{b}\) of the differences between the first coefficient \(\tilde{b}_i\) of \(D\) with the others. Using Equations (3.3), (3.4) and (3.5) of Lemma 3.2 we obtain that
\[
\tilde{b}_1 = 2|H|a_1.
\]

The rest of the coefficients of \(D\) are of the following types
\[
|H|(a_1 + a_j) \quad \text{or} \quad 2|H|a_j
\]
therefore the differences are of the following types
\[
|H|(a_1 - a_j) \quad \text{or} \quad 2|H|(a_1 - a_j)
\]
hence the corresponding greatest common divisor \( \tilde{b} \) is \( |H| \) times the corresponding one \( b \) for one copy of \( G \). The new exponent is then computed as

\[
\tilde{q} = \frac{|G^2|}{b \cdot \dim V_j^1} = \frac{|G| \cdot |G|}{|H|b \cdot \dim V_j} = [G : H]q
\]

Now, the assertion follows from Theorem 2.2 as soon as we show that

\[
(3.7) \quad 2 \sum_{\ell=1}^t \sum_{j=1}^r \left[ \tilde{q}[L : \mathbb{Q}] \sum_{i=1}^2 \sum_{k=1}^r (\dim V_k^i - \dim(V_k^{G_j^i})) - ([G^2 : H^2] - |H^2 \backslash G^2/G_j^i|) \right] = 0.
\]

where \( G_j^\ell \) is of class \( C_j^\ell \) with \( C_j^\ell \) as in Lemma 3.1.

To see this, observe that

\[
(3.8) \quad [G^2 : H^2] = [G : H]^2, \quad \text{and} \quad |H^2 \backslash G^2/G_j^i| = [G : H]|H \backslash G/G_j|.
\]

Moreover, we have

\[
(3.9) \quad \dim(V_k^{G_j^i}) = \langle V_k^i, \rho_{G_j^i}^{G_j} \rangle = \begin{cases} 
\langle V_k, \rho_{G_j}^{G_j} \rangle = \dim V_k^{G_j}, & \text{if } \ell = i; \\
\dim V_k^i = \dim V_k, & \text{otherwise}.
\end{cases}
\]

It now follows from (3.8) and (3.9) that the left hand side of (3.7) equals \( [G : H] \) times the sum with \( i = 1 \) and \( i = 2 \) of the left hand side of (3.6). But these are zero by assumption, which implies the assertion. Finally, the computation of the dimension is a consequence of equation (2.4). \( \square \)

For our main applications we need only the following special case of the above theorem.

**Corollary 3.4.** Let \( W \) be a nontrivial rational irreducible representation of \( G \), with associated complex irreducible representation \( V \), such that the subgroup \( H \) of \( G \) is maximal with the property

\[
\dim V^H = 1.
\]

Suppose that the action of \( G \) on \( Z_i \) has geometric signature \([0; (C_1, s_1^i), \ldots, (C_t, s_t^i)]\) for \( i = 1, 2 \), and satisfies equation (3.6) with \( r = 1 \) for \( i = 1 \) and \( 2 \). Furthermore assume that the branch loci of \( Z_i \rightarrow \mathbb{P}^1 \) are disjoint in \( \mathbb{P}^1 \).

Then the action of the group \( G^2 \) on the curve \( Z \) defines a Prym-Tyurin variety \( P \) in the Jacobian \( J_X \) of exponent \( \tilde{q} = [G : H]q \) and dimension

\[
\dim P = [K_V : \mathbb{Q}][-2 \dim V + \frac{1}{2} \sum_{j=1}^t (s_j^1 + s_j^2)(\dim V - \dim V^{G_j})].
\]

**Remark 3.5.** One could generalize Theorem 3.3 to an \( n \)-fold self-product \( G^n \) of \( G \), a subgroup \( H \) of \( G \) and \( n \) Galois coverings \( h_i : Z \rightarrow \mathbb{P}^1 \) with Galois group \( G \) and pairwise disjoint branch loci in \( \mathbb{P}^1 \). We omit the details.
The fact that the Prym-Tyurin variety $P$ is constructed via a product of groups suggests that it is a product of Prym-Tyurin varieties, and in fact, this is the case, as we will show in the next theorem. Let the notation be as in Theorem 3.3. According to Theorem 2.2 the action of the group $G$ and its subgroup $H$ induce a Prym-Tyurin variety $P_i$ of exponent $q$ in $JX_i$ for $i = 1$ and 2. Let $P$ denote the Prym-Tyurin variety in $JX$ of Theorem 3.3. Then we have:

**Theorem 3.6.** Suppose the coverings $q_i : X \to X_i$ do not factorize via an étale cyclic covering for $i = 1, 2$. Then we have

$$P \simeq P_1 \times P_2.$$ 

*Proof.* The map $q_1^* + q_2^* : JX_1 \times JX_2 \to JX$ is an isogeny onto its image. According to Lemma 3.2 it maps $P_1 \times P_2$ into $P$. From equation (2.4) and Theorem 3.3 we get

$$\dim(P_1 \times P_2) = \dim P.$$ 

Hence $q_1^* + q_2^*$ induces an isogeny $P_1 \times P_2 \to P$. Now, since the maps $q_i : X \to X_i$ do not factorize via a cyclic étale covering, the canonical polarization of $JX$ induces a polarization of the same type on $P$ and $P_1 \times P_2$, namely the $([G : H] \cdot q)$-fold of a principal polarization. This implies that $q_1^* + q_2^* : P_1 \times P_2 \to P$ is an isomorphism. \hfill \qed

**Example 3.7.** As a first example consider the cyclic group $G = \mathbb{Z}_2$ of order 2. It certainly satisfies Hypothesis 2.1 for the trivial subgroup $H = \{0\}$. For $i = 1$ and 2 let $X_i$ be a hyperelliptic curve of genus $g_{X_i}$. We assume that the hyperelliptic coverings $f_i : X_i \to \mathbb{P}^1$ have disjoint branch loci in $\mathbb{P}^1$, so that we are in the situation of diagram (1.1). The fibre product $X = X_1 \times_{\mathbb{P}^1} X_2$ is Galois over $\mathbb{P}^1$ with Galois group the Klein group $G^2$, so that in diagram (3.1) the curves $Z = X$ and $Z_i = X_i$ coincide. Moreover the group $G$ acts on $X_i$ with geometric signature $[0; (G, 2g_{X_i} + 2)]$ and satisfies equation (2.3) with $q = 1$. So the Prym-Tyurin variety of the action coincides with the Jacobian $JX_i$. Moreover the assumptions of Theorem 3.3 are fulfilled and we obtain a Prym-Tyurin variety $P$ of exponent 2 for the group $G^2$ in the Jacobian $JX$.

Let $K$ denote the third subgroup of $G^2$, i.e. $K = \{(0, 0), (1, 1)\}$ and denote

$$Y := X/K.$$ 

**Proposition 3.8.** (a) The curve $Y$ is hyperelliptic of genus $g_Y = g_{X_1} + g_{X_2} + 1$ and the map $X \to Y$ is an étale double covering.

(b) The Prym-Tyurin variety $P$ of the action of $G^2$ coincides with the classical Prym variety of the étale covering $X \to Y$.

*Proof.* (a): Set $H_1 = G \times \{0\} = \{(0, 0), (1, 0)\}$ and $H_2 = \{0\} \times G = \{(0, 0), (0, 1)\}$. Then $G^2$ acts on $X$ with geometric signature $[0; (H_1, 2g_{X_2} + 2), (H_2, 2g_{X_1} + 2), (K, 0)]$. From this we conclude that the double covering $X \to Y$ is étale. In fact, if $x \in X$ is a branch point of the covering $X \to \mathbb{P}^1$, it cannot be one of $X \to Y$, since $\text{Stab}_{G^2}(x) \cap K = \{(0, 0)\}$. It follows that $Y \to \mathbb{P}^1$ is a double covering ramified in all $2(g_{X_1} + g_{X_2}) + 4$ points, which means that $Y$ is hyperelliptic of genus $g_{X_1} + g_{X_2} + 1$. 


(b): First note that the Prym variety of the covering $X \to Y$ is given by the correspondence $1 - \iota$, where $\iota$ is the involution on $X$ with quotient $Y$. Hence $\iota$ is given by the element $(1,1)$ of $G^2$.

On the other hand, the Prym-Tyurin variety $P$ for the action of $G^2$ is defined by the correspondence $\mathcal{D}$ (see (2.1)) constructed using the sum of the representations $\mathcal{W}_1 = \mathcal{W} \otimes V_0$ and $\mathcal{W}_2 = V_0 \otimes \mathcal{W}$ of $G^2$, where $V_0$ and $\mathcal{W}$ are the trivial and the alternating representation of $G$ respectively. We have then $H^2 = \{(0,0)\}$, the trivial subgroup of $G^2$, and the double coset representatives are just the elements of $G^2$, i.e. $g_{11} = (0,0), g_{21} = (1,0), g_{31} = (0,1), g_{41} = (1,1)$. They are also the left and right coset representatives of $H^2$ in $G^2$.

By definition, we have for $i = 1, \ldots, 4$,

$$b_i := \sum_{k=1}^{2} \sum_{h \in H^2} (\chi_{\mathcal{W}_k}(h g_{i1}^{-1})) = \chi_{\mathcal{W}_1}(g_{i1}) + \chi_{\mathcal{W}_2}(g_{i1}),$$

which gives $b_1 = 2, b_2 = b_3 = 0$ and $b_4 = -2$. Hence $b := \gcd\{b_1 - b_i \mid 2 \leq i \leq 4\} = 2$. Therefore

$$\mathcal{D} = 1 \cdot (0,0) - 1 \cdot (1,1).$$

Hence the correspondences coincide, which implies the assertion. \hfill \square

As an immediate consequence of Theorem 3.3 and Proposition 3.8 we obtain:

**Corollary 3.9.** (Mumford)
Let $X_i$ be a hyperelliptic curve of genus $g_i$ for $i = 1$ and 2, whose hyperelliptic coverings have disjoint branch loci in $\mathbb{P}^1$. Let $X = X_1 \times_{\mathbb{P}^1} X_2$ and $Y$ the hyperelliptic curve ramified over all branch points of $X_1$ and $X_2$. Then the natural map $X \to Y$ is an étale double covering whose Prym variety is isomorphic to $JX_1 \times JX_2$ as principally polarized abelian varieties.

4. The Main Example

Given two Prym-Tyurin varieties of exponent $q$ presented with respect to the same group $G$, subgroup $H$ and rational irreducible representations, but with group actions with disjoint branch loci, Theorem 3.3 gives a new Prym-Tyurin variety of exponent $[G : H] \cdot q$. Hence every Prym-Tyurin constructed with a presentation with respect to a group action (and these are almost all such varieties known up to now, see [2]) gives a new one. We will only give one example, which we think is interesting, since it arises also from a completely different geometric construction, as we shall see in the next section.

Let $p$ be an odd prime and

$$G = D_p = \langle \sigma, \tau : \sigma^p, \tau^2, (\sigma \tau)^2 \rangle$$

the dihedral group of order $2p$. Any complex irreducible representation $V$ of degree two of $D_p$ is defined over the field $K_V = \mathbb{Q}(\zeta + \zeta^{-1})$, where $\zeta$ denotes a $p$-th root of unity. The
Galois group $\text{Gal}(K_V/Q)$ is cyclic of order $\frac{p-1}{2}$, and the associated rational irreducible representation $\mathcal{W}$ is of degree $p-1$ and given by

$$\mathcal{W}: \sigma \to \begin{bmatrix} 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \cdot \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}, \quad \tau \to \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdot \\ -1 & 0 & \cdots & 0 & \cdot \end{bmatrix}$$

Apart from the trivial subgroup, $D_p$ admits exactly 2 conjugacy classes of cyclic subgroups, namely the classes $C_1$ of subgroups of order 2 and $C_2$ of subgroups of order $p$. Since $D_p$ can be generated by any even number $s \geq 4$ of involutions with product equal to 1, there exist curves $Z$ with $D_p$-action and geometric signature $[0; (C_1, s)]$.

**Proposition 4.1.** Every $D_p$-action on a curve $Z$ of geometric signature $[0; (C_1, s)]$, any subgroup $H$ of order 2 of $D_p$, and the representation $\mathcal{W}$, give a presentation of the Jacobian $JX$ (with $X = Z/H$) as a Prym-Tyurin variety of exponent 1.

**Proof.** Without loss of generality we may assume that $H = \{1_G, \tau\}$. It suffices to show that the assumptions of Theorem 22 are satisfied with $r = 1$ and $q = 1$. Let the notation be as in Section 2. Set $\pi_H: Z \to X$ and let $V$ denote a complex irreducible representation associated to $W$ (deg $V = 2$).

First we claim that $b = p$. To see this, note that any double coset $HgH$ contains exactly $[H : H \cap g^{-1}Hg]$ right cosets of $H$. Thus $H1_GH$ consists of one right coset, and for any $g$ not in $H$ we have $[H : H \cap g^{-1}Hg] = |H| = 2$. Hence the number $d = |H\backslash G/H|$ of double cosets is given by $p = |G : H| = 1 + 2(d - 1)$, from where

$$d = \frac{p + 1}{2}.$$

Moreover, $H\sigma^iH = H\sigma^jH$ if and only if $\sigma^i \in H\sigma^jH = \{\sigma^i, \tau\sigma^i, \sigma^i\tau, \tau\sigma^i\tau = \sigma^{-j}\}$. We conclude that $H\sigma^iH = H\sigma^jH$ if and only if $i = j$ or $i = p - j$. This implies that a set of double coset representatives for $H$ in $G$ is given by

$$\left\{ g_{i1} = 1_G, g_{i1} = \sigma^{i-1} \mid i = 2, \ldots, \frac{p + 1}{2} \right\}.$$

Therefore,

$$b_1 = \sum_{h \in H} \chi_{W}(hg_{i1}^{-1}) = \chi_{W}(1_G) + \chi_{W}(\tau) = p - 1$$

and for $i = 2, \ldots, \frac{p+1}{2}$

$$b_i = \sum_{h \in H} \chi_{W}(hg_{i1}^{-1}) = \chi_{W}(1_G \cdot \sigma^{p-i+1}) + \chi_{W}(\tau \cdot \sigma^{p-i+1}) = -1,$$

since $\chi_{W}(\tau \cdot \sigma^i) = 0$ and $\chi_{W}(\sigma^i) = -1$, for all $i = 1, \ldots, p - 1$. This implies

$$b = \gcd \left\{ b_1 - b_i \mid 2 \leq i \leq \frac{p + 1}{2} \right\} = p.$$
We conclude
\[ q = \frac{|D_p|}{b \cdot \dim V} = 1. \]

Consider \( G_1 = H \) as a representative of the conjugacy class \( C_1 \), then we have that \( |H \backslash G/G_1| = \frac{p+1}{2} \).

Due to Frobenius reciprocity,
\[ \dim V^{G_1} = \langle \Res_{G_1} V, 1_{G_1} \rangle = \frac{1}{\dim V} \sum_{h \in G_1} \chi_V(h) = \frac{1}{2} (2 + 0) = 1, \]
and \( G_1 \) is a maximal subgroup with this property. It remains to show that equation (2.3) is satisfied for \( q = 1 \). In fact,
\[ q[L_V : \mathbb{Q}] (\dim V - \dim V^{G_1}) - ([G : H] - |H \backslash G/G_1|) = \frac{p-1}{2} (2 - 1) - \left( p - \frac{p+1}{2} \right) = 0. \]
The assertion now follows from Theorem 3.3. \( \Box \)

As an immediate consequence of Proposition 4.1, Corollary 3.4 and Theorem 3.6 we obtain

**Proposition 4.2.** Set \( G = D_p \). For \( i = 1 \) and \( 2 \) let \( Z_i \) be a curve with \( G \)-action of geometric signature \( [0, (C_1, s_i)] \), with \( s_i \) even integers \( \geq 4 \). Let \( X_i \) be as in Proposition 4.1. Assume moreover that the coverings \( Z_i \to \mathbb{P}^1 \) have disjoint branch loci. Denote \( \tilde{Z} = Z_1 \times_{\mathbb{P}^1} Z_2 \) and \( X = X_1 \times_{\mathbb{P}^1} X_2 \).

Then the action of the group \( G^2 \) on the curve \( Z \) defines a Prym-Tyurin variety \( P \) in the Jacobian \( JX \) of exponent \( p \) and dimension \( \frac{p-1}{4} (s_1 + s_2 - 8) \).

Moreover,
\[ P \simeq JX_1 \times JX_2. \]

In the sequel we use the following presentation of \( D_p^2 \):
\[ D_p^2 = \langle \sigma_1, \tau_1, \sigma_2, \tau_2 \mid \sigma_i^p, \tau_i^2, (\sigma_i \tau_i)^{2}, [\sigma_1, \sigma_2], [\tau_1, \tau_2], [\sigma_i, \tau_j] \forall 1 \leq i \neq j \leq 2 \rangle. \]
So \( \sigma_1, \tau_1 \) generate the first factor and \( \sigma_2, \tau_2 \) generate the second factor of \( D_p^2 \). Then, with \( H^2 = \langle \tau_1, \tau_2 \rangle \), we have
\[ Z_1 = Z/\langle \sigma_2, \tau_2 \rangle, \quad Z_2 = Z/\langle \sigma_1, \tau_1 \rangle, \]
\[ X_1 = Z/\langle \tau_1, \sigma_2, \tau_2 \rangle, \quad X_2 = Z/\langle \sigma_1, \tau_1, \tau_2 \rangle \quad \text{and} \quad X = Z/H^2. \]
Moreover, consider the following quotient curves of \( Z \):

- \( \tilde{X} := Z/\langle \tau_1 \tau_2 \rangle \);
- \( \tilde{Y}_j := Z/\langle (\sigma_1 \sigma_2, \tau_1 \tau_2) \rangle \) for \( j = 1, \ldots, \frac{p-1}{2} \);
- \( \tilde{Y} := Z/\langle \sigma_1, \sigma_2, \tau_1 \tau_2 \rangle \).
Then we have the following diagram with the degrees of the maps as indicated:

\[
\begin{align*}
\begin{array}{c}
\text{Z} \\
\downarrow^{2p:1} \\
\text{X} \\
\downarrow^{p:1} \\
\text{Z}_1 \\
\end{array} & \quad \begin{array}{c}
\text{2:1} \\
\uparrow^{g_1} \\
\text{p:1} \\
\uparrow^{p:1} \\
\text{X}_1 \\
\end{array} & \quad \begin{array}{c}
\text{2:1} \\
\downarrow^{g_2} \\
\text{p:1} \\
\uparrow^{p:1} \\
\text{X}_2 \\
\end{array} & \quad \begin{array}{c}
\text{2:1} \\
\downarrow^{f_1} \\
\text{p:1} \\
\uparrow^{p:1} \\
\text{Y} \\
\end{array} & \quad \begin{array}{c}
\text{2:1} \\
\downarrow^{f_{z+1}} \\
\text{p:1} \\
\uparrow^{p:1} \\
\text{\text{Y}_p:1} \\
\end{array}
\end{align*}
\]

The next lemma gives the genus of the curves and the ramification of the maps of this diagram, which we need in the sequel.

**Lemma 4.3.** (a): The covering \(Z \to Y\) is étale, and hence so are \(Z \to \tilde{X}\), \(\tilde{X} \to \tilde{Y}_j\) and \(\tilde{Y}_j \to Y\) for all \(j\). \(Y\) is hyperelliptic, ramified over all \(s_1 + s_2\) branch points of \(Z \to \mathbb{P}^1\). Hence

\[
g_Y = \frac{s_1 + s_2}{2} - 1 \quad \text{and} \quad g_{\tilde{Y}_j} = \frac{p}{2}(s_1 + s_2) - 2p + 1 \quad \text{for all } j;
\]

(b): The covering \(X_i \to \mathbb{P}^1\), for \(i = 1\) and \(2\), is ramified exactly over the \(s_i\) branch points of \(Z_i \to \mathbb{P}^1\), each one with fibre consisting of \(\frac{p-1}{2}\) points of order 2 and one unramified point. Hence

\[
g_{X_i} = \frac{s_i(p-1)}{4} - p + 1 \quad \text{for } i = 1, 2 \quad \text{and} \quad g_X = \frac{(s_1 + s_2)(p^2 - p)}{4} - p^2 + 1.
\]

**Proof.** The stabilizers of branch points of \(Z \to \mathbb{P}^1\) are conjugate subgroups of either \(\langle \tau_1 \rangle\) or \(\langle \tau_2 \rangle\), hence are of the form \(\langle \sigma_1^i \tau_1 \rangle\) or \(\langle \sigma_2^i \tau_2 \rangle\) for \(i = 0, \ldots, p-1\). For any subgroup \(U\) of \(G\), the stabilizers for the covering \(Z \to \mathbb{P}^1\) are given by the intersection of \(U\) with the stabilizers for the action of \(G\). In our case the stabilizers for the action of \(D_p^2\) are either trivial or are of order 2, hence one has to check whether the subgroup \(U\) contains subgroups of the form \(\langle \sigma_1^i \tau_1 \rangle\) or \(\langle \sigma_2^i \tau_2 \rangle\). Doing this for the subgroups in question, together with the Riemann-Hurwitz formula, gives the assertions. \(\square\)

**Lemma 4.4.** Let \(Z \to \mathbb{P}^1\) be a Galois covering with Galois group \(D_p^2\) and geometric signature \([0; (C_1^1, s_1), (C_1^2, s_2)]\) as above. Then

(a): \(Y\) is the only intermediate hyperelliptic curve of \(Z \to \mathbb{P}^1\) of genus \(\frac{s_1 + s_2}{2} - 1\);

(b): Up to isomorphism, the curves \(\tilde{Y}_j\), \(j = 1, \ldots, \frac{p-1}{2}\) are the only intermediate curves of \(Z \to \mathbb{P}^1\) which are étale \(p\)-fold coverings of \(Y\). They are pairwise non-conjugate in \(Z \to \mathbb{P}^1\);

(c): Up to isomorphism, \(X\) is the only intermediate curve of \(Z \to \mathbb{P}^1\) which is a \(4:1\)-quotient of \(Z\).
Proof. This is a consequence of the subgroup structure of the group $D_p^2$.

(a) follows from the fact that the subgroup $\langle \sigma_1, \sigma_2, \tau_1 \tau_2 \rangle$ is the only subgroup of index 2 of $D_p^2$ not containing any stabilizer for the action of $D_p^2$ on $Z$.

(b) First note that the subgroup $\langle \sigma_1 \sigma_2, \tau_1 \tau_2 \rangle$ corresponding to $\tilde{Y}_j$ is a subgroup of index $p$ in the subgroup corresponding to $Y$. According to Lemma 4.3 (a), $\tilde{Y}_j \to Y$ is étale of degree $p$. Moreover, note that whenever we have a curve $Z$ with group action by a group $G$, the covering $Z \to Z/G$ is the Galois closure of an intermediate covering $Z/U \to Z/G$ if and only if the core $\text{Core}_G U := \bigcap_{g \in G} U^g$ is trivial. In our situation, the only subgroups, up to conjugacy, of $D_p^2$ of index $p$ with trivial core in $D_p^2$ are the ones determining the $\tilde{Y}_j$. It is easy to see that they are pairwise non-conjugate.

(c): In $D_p^2$ there is just one conjugacy class of subgroups of order 4. All of them are non-cyclic. Therefore the subgroup defining $X$ is uniquely determined up to conjugacy in $D_p^2$. \qed

As an immediate consequence we get

**Corollary 4.5.** Suppose $\tilde{Y} \to Y$ is an étale $p$-fold covering of a hyperelliptic curve $Y$ such that the Galois closure $Z$ of the composed map $\tilde{Y} \to Y \to \mathbb{P}^1$ has Galois group $D_p^2$. Then $\tilde{Y}$ is (isomorphic to) one of the curves $\tilde{Y}_j$ of diagram $\text{[4.4]}$. In particular it determines the curves $Z_1$ and $Z_2$ of that diagram uniquely and $X, X_1$ and $X_2$ uniquely up to conjugacy.

5. Étale $p$-fold coverings of hyperelliptic curves

In this section we give a geometric construction of the $p$-fold coverings $X_1$ and $X_2$ of diagram $\text{[4.4]}$ in terms of the covering $f : \tilde{Y} \to Y$.

5.1. The curves. So let $f : \tilde{Y} \to Y$ denote an étale covering of degree $p$ of a hyperelliptic curve $Y$. Suppose the (2:1)-covering $h : Y \to \mathbb{P}^1$ is ramified over $B_h = \{a_1, \ldots, a_{2g+2}\} \subset \mathbb{P}^1$.

Define the curve $X$ as the following fibre product,

\begin{equation}
X := (f^{(2)})^{-1}(g_2^{(2)})^{-1} \tilde{Y}^{(2)} \overset{\pi = f^{(2)}|_X}{\rightarrow} \mathbb{P}^1 \cong g_2^{(2)} \overset{f^{(2)}}{\rightarrow} Y^{(2)}.
\end{equation}

Here $Y^{(2)}$ and $\tilde{Y}^{(2)}$ denote the second symmetric product of $Y$ and $\tilde{Y}$. In order to work out conditions for $X$ to be smooth and irreducible, we need some notation.

Fix a point $z_0 \in \mathbb{P}^1 \setminus B_h$ and let $\gamma_i$ denote the class of a path at $z_0$ going around $a_i$ clockwise as usual, then the fundamental group of $\mathbb{P}^1 \setminus B_h$ is

\begin{equation}
\pi_1(\mathbb{P}^1 \setminus B_h, z_0) = \langle \gamma_1, \ldots, \gamma_{2g+2} : \prod_{1}^{2g+2} \gamma_i = 1 \rangle
\end{equation}
If $\iota$ is the hyperelliptic involution of $Y$, we denote
\[ h^{-1}(z_0) = \{x, \iota x\}, \]
\[ f^{-1}(x) = \{x_1, \ldots, x_p\} \subset \tilde{Y} \quad \text{and} \quad f^{-1}(\iota x) = \{y_1, \ldots, y_p\} \subset \tilde{Y}. \]
If $\mu: \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to S_{2p}$ is a classifying homomorphism of $h \circ f : \tilde{Y} \to Y \to \mathbb{P}^1$, we know that
\[ \mu(\gamma_i) = t_1 t_2 \cdots t_p \]
where $t_1, t_2, \ldots, t_p$ are disjoint transpositions. But not all such products can occur. In fact, if we identify for $i = 1, \ldots, p$,
\[ x_i = i \quad \text{and} \quad y_i = p + i, \]
then we have the following lemma, the proof of which is obvious.

**Lemma 5.1.** A homomorphism $\mu: \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to S_{2p}$ is a classifying homomorphism for $h \circ f : \tilde{Y} \to Y \to \mathbb{P}^1$ if and only if
\begin{enumerate}
  \item $G := \text{Im}(\mu)$ is an imprimitive transitive subgroup of imprimitivity degree $p$ of $S_{2p}$ and
  \item $\mu(\gamma_i) = t_1 t_2 \cdots t_p$ with disjoint transpositions $t_i$ of the form $(j \ p + k)$ with $1 \leq j, k \leq p$.
\end{enumerate}

Finally, if we denote for $i, j = 1, \ldots, p$,
\[ P_{ij} := x_i + y_j \in X \subset \tilde{Y}^{(2)}, \]
then $\pi^{-1}(z_0) = \{P_{ij} \mid i, j = 1, \ldots, p\}$, with $\pi: X \to \mathbb{P}^1$ from Diagram 5.1.

Now consider the group $G \subset S_{2p}$ generated by
\begin{itemize}
  \item $\varphi_1 = \prod_{i=1}^p (i \ i + p)$
  \item $\varphi_2 = [\prod_{i=1}^{p-1} (i \ i + p + 1)](p \ p + 1)$
  \item $\varphi_3 = (1 \ p + 1)\prod_{i=2}^p (i \ 2p + 2 - i)$ and
  \item $\varphi_4 = (1 \ p + 2)(2 \ p + 1)\prod_{i=3}^p (i \ 2p + 3 - i)$
\end{itemize}

**Lemma 5.2.**
\[ G = D_p \times D_p \subset S_{2p}, \]
where $\varphi_1, \varphi_2$ generate the first factor $D_p$ and $\varphi_3, \varphi_4$ the second.

**Proof.** Obviously all $\varphi_i$’s are of order 2. Moreover we have the following relations:
\[ \varphi_1 \cdot \varphi_2 = (1 \ p \ p - 1 \ p - 2 \ldots \ 2)(p + 1 \ p + 2 \ldots \ 2p), \]
\[ \varphi_3 \cdot \varphi_4 = (1 \ 2 \ 3 \ 4 \ldots \ p)(p + 1 \ p + 2 \ldots \ 2p). \]
Hence $\langle \varphi_1, \varphi_2 \rangle$ and $\langle \varphi_3, \varphi_4 \rangle$ are both groups isomorphic to $D_p$. Moreover, one easily checks that $\varphi_1$ and $\varphi_2$ commute with $\varphi_3$ and $\varphi_4$, which completes the proof. \[\square\]
Proposition 5.3. If \( f : \tilde{Y} \to Y \) is an étale covering of degree \( p \) of a hyperelliptic curve \( Y \) such that the image of a classifying homomorphism \( \mu : \pi_1(\mathbb{P}^1 \setminus B_h, z_0) \to S_{2p} \) is the group \( \mathcal{G} \), then the curve \( X \) of diagram (5.1) is smooth and irreducible.

Proof. The stabilizer of the element \( P_{11} = x_1 + y_1 \) of the fibre \( \pi^{-1}(z_0) \) of the map \( \pi : X \to \mathbb{P}^1 \) is the group

\[
\mathcal{G}_{P_{11}} = \langle \varphi_1, \varphi_3 \rangle,
\]

which is Klein’s group of 4 elements. Since \( \mathcal{G} \) is of order \( 4p^2 \), this means that \( \mathcal{G} \) acts transitively on the set \( \{ P_{ij} \mid i, j = 1, \ldots, p \} \) implying that \( X \) is irreducible. The proof of the fact that \( X \) is smooth is a slight generalization of the proof of [1], Lemma 12.8.1. (see also [4], Lemma 3.1, where the special case \( p = 3 \) is proved).

Proposition 5.4. The curve \( X \) of diagram (5.1) coincides with the curve \( X \) in diagram (4.1).

Proof. According the the proof of Proposition 5.3, the curve \( X \) corresponds to the Klein subgroup \( \langle \varphi_1, \varphi_3 \rangle \) of \( D_p^2 \). In Lemma 4.4 we show that, up to conjugacy, there is only one such a subgroup.

5.2. The correspondence. In the last subsection we saw how to describe the curve \( X \) of diagram (4.1) in terms of the covering \( f : \tilde{Y} \to Y \). We can use this to define a fixed-point free symmetric effective \((p - 1)\)-correspondence \( D \) on \( X \).

For this fix a point \( z_0 \in U := \mathbb{P}^1 \setminus B_h \) and denote the fibre \( \pi^{-1}(z_0) = \{ P_{ij} \mid i, j = 1, \ldots, p \} \) as in equation (5.4). Moreover we use the notation

\[
I_{ij} := \{(k, l) \in \{1, \ldots, p\}^2 \mid k + l \equiv i + j \pmod p \quad \text{and} \quad k - l \equiv i - j \pmod p \}.
\]

Then in the fibre \( \pi^{-1}(z_0) \) the correspondence \( D \) is defined by

\[
D_{z_0}(P_{ij}) := \sum_{(k, l) \in I_{ij}} P_{kl}
\]

We extend \( D_{z_0} \) to a correspondence \( D_U \) on \( \pi^{-1}(U) \) in the usual way (see e.g. [4, Section 3]) as follows: We enumerate the \( x_i \) and \( y_j \) in such a way that the stabilizer of \( P_{11} \) is the group \( H^2 = \langle \varphi_1, \varphi_3 \rangle \). If \( \{(g_{ij}, g_{kl}) \mid 1 \leq i, k \leq d, 1 \leq j, l \leq n_i \} \) denotes the set of representatives of the right and left cosets of \( H^2 \) in \( \mathcal{G} \) as in the proof of Lemma 3.2 with \( d = \frac{p^2 - 1}{2}, n_1 = 1 \) and \( n_i = 2 \) for \( i \geq 2 \) (see proof of Proposition 4.1), this induces a \( \mathcal{G} \)-equivariant bijection

\[
\{(g_{11}, g_{11}), \ldots, (g_{dn_d}, g_{dn_d})\} \longrightarrow \pi^{-1}(z_0) = \{ P_{ij} \mid i, j = 1, \ldots, p \}
\]

to be described in the proof of Proposition 6.1.

Then for every point \( z \in U \) choose a path \( \gamma_z \) in \( U \) connecting \( z \) and \( z_0 \). The path defines a bijection

\[
\mu : \pi^{-1}(z) \to \pi^{-1}(z_0) = \{ P_{11}, \ldots, P_{pp} \}
\]

in the following way: For any \( x \in \pi^{-1}(z) \) denote by \( \tilde{\gamma}_x \) the lift of \( \gamma_z \) starting at \( x \). If \( P_{ij} \in \pi^{-1}(z_0) \) denotes the end point of \( \tilde{\gamma}_x \), define

\[
\mu(x) = P_{ij}.
\]
Define
\[ D_U := \{(x, x') \in \pi^{-1}(U) \times_{\mathbb{P}^1} \pi^{-1}(U) \mid \mu(x') \in D_{z_0}(\mu(x))\}. \]

Finally, define \( D \) to be the closure of \( D_U \) in \( X \times X \).

**Proposition 5.5.** \( D \) is a correspondence on \( X \).

In the next section we will see that \( D \) coincides with the Kanev correspondence associated by the correspondence \( D \) defining the Prym-Tyurin variety \( P \) of Proposition 4.2. In particular \( D \) is an effective symmetric fixed-point free correspondence whose associated endomorphism \( \gamma_D \) on the Jacobian \( J_X \) satisfies the equation
\[ \gamma_D^2 + (p-2)\gamma_D - (p-1) = 0. \]

Of course it is easy to see this also directly.

**Proof.** We have to show that the definition of \( D_U \) is independent of the choice of the paths \( \gamma_z \). For this it suffices to show that \( D_U \) is invariant under the action of \( G = \mathbb{D}_p \times \mathbb{D}_p; \) that is, the diagram

\[
\begin{array}{ccc}
P_{ij} & \xrightarrow{\varphi_k} & \varphi_k(P_{ij}) \\
D \downarrow & & \downarrow D \\
D(P_{ij}) & \xrightarrow{\varphi_k} & D(\varphi_k(P_{ij})) = \varphi_k(D(P_{ij}))
\end{array}
\]

commutes for \( 1 \leq k \leq 4 \). In fact, for \( \varphi_1 \) we have
\[ \varphi_1^{-1}(D(\varphi_1(P_{ij}))) = \varphi_1(D(P_{ji})) = \varphi_1 \left( \sum_{(l,k) \in I_{ji}} P_{lk} \right) = \sum_{(k,l) \in I_{ij}} P_{kl} = D(P_{ij}). \]

For the other \( \varphi_k \) the proof is similar, using
\[ \varphi_2(P_{ij}) = P_{kl} \text{ with } k \equiv j - 1 \mod p, \quad l \equiv i + 1 \mod p, \]
\[ \varphi_3(P_{ij}) = P_{kl} \text{ with } k \equiv p - j + 2 \mod p, \quad l \equiv p - i + 2 \mod p, \]
\[ \varphi_4(P_{ij}) = P_{kl} \text{ with } k \equiv p - j + 3 \mod p, \quad l \equiv p - i + 3 \mod p. \]

\[ \square \]

**Example 5.6.** In the special case \( p = 3 \) the curve \( X \) coincides with the corresponding curve in [4]. Moreover we have
\[ D(P_{ij}) = \sum_{(k,l) \in I_{ij}} P_{kl} = \sum_{k=1,k \neq i}^3 P_{kj} + \sum_{l=1,l \neq i}^3 P_{il}. \]

This is just the correspondence of [4]. Hence our construction of Prym-Tyurin varieties is a generalisation to arbitrary odd prime \( p \) of the Prym-Tyurin varieties of [4] for \( p = 3 \).
6. Comparison of the correspondences $\mathcal{D}$ and $D$

According to Proposition 5.4 the curve $X$ of diagram (5.1) coincides with the curve $X$ of diagram (4.1). Hence we have two correspondences on $X$, namely $\mathcal{D}$ defined by equation (3.3) (in the special case $G = D_p$) and $D$ defined in the previous section. In this section we compare both and show that they induce the same Prym-Tyurin variety in $JX$.

Recall from [2] that for a general $D$ as in equation (2.1), we constructed a correspondence $K_D$, called the associated Kanev correspondence, which is effective, symmetric, fixed-point free and whose associated endomorphism $\gamma_{K_D}$ satisfies the equation $\gamma_{K_D}^2 + (q - 2)\gamma_{K_D} - (q - 1) = 0$. In fact, in the proof of [2, Lemma 3.8] we saw that $K_D(x)$ is given by

$$K_D(x) = \sum_{i=2}^{d} \left( \frac{b_i}{b_{i-1}} - 1 \right) \sum_{j=1}^{n_i} \pi_H g_{ij}(z)$$

for all $x \in X$ and $z \in Z$ with $\pi_H(z) = x$, where

$$b_i := \sum_{k=1}^{r} \sum_{h \in H} \text{tr}_{L/Q}(\chi_{V_k}(h g_{i1}^{-1}))$$

Moreover, in terms of $K_D$ the associated Prym-Tyurin variety $P$ is given by

$$P = \text{Im}(1 - \gamma_{K_D})$$

Now let the notation be as in Section 5. In particular $Z$ is a Galois covering of $\mathbb{P}^1$ with Galois group $D_p^2$ and $\pi_{H^2}: Z \to X$ is the covering of diagram (5.1). Let $\mathcal{D}$ denote the correspondence defined in equation (3.3) in our special situation. Then we have

**Proposition 6.1.** The Kanev correspondence $K_D$ associated to $\mathcal{D}$ coincides with the correspondence $D$ of Proposition 5.5.

**Proof.** For the proof we show that the right hand side of (6.1) in this special situation is equal to the right hand side of (5.6) under the identifications given below.

For this we recall the notations. The underlying group is $G^2 = D_p \times D_p$ with generators of the two factors $\sigma_i$ of order $p$ and $\tau_i$ of order 2 for $i = 1$ and 2. We are considering the representations $\mathcal{W} \otimes V_0$ and $V_0 \otimes \mathcal{W}$, with $\mathcal{W}$ as in Proposition 4.1. Therefore here $r = 2$.

We need to collect some of the previous results for $G = D_p$ and its subgroup $H = \langle \tau \rangle$ from Proposition 4.3

- the double cosets representatives for $H \backslash G \backslash H$ we consider are $\sigma^i$ for $i = 1, \ldots, \frac{p+1}{2}$;
- the coefficients of the correspondence for $D_p$ and $H$ on $X_i$ are $a_1 = p - 1$ and $a_i = -1$ for $i = 2, \ldots, \frac{p+1}{2}$;
- the double coset represented by $1_G$ has 2 elements, hence just one right coset. The double coset represented by $\sigma^i$ for $i > 1$ has 4 elements, hence two right cosets. The representatives for the right and left cosets in this double coset are $\sigma^i$ and $\sigma^{p-i}$.
We now work out the induced representatives and coefficients for the correspondence for $G^2 = D_p \times D_p$ and its subgroup $H^2 = \langle \tau_1 \rangle \times \langle \tau_2 \rangle$ on the curve $X$.

- the double coset representatives for $H^2 \backslash G^2 / H^2$ we consider are $(\sigma_i^{j-1}, \sigma_j^{i-1})$ for $i, j = 1, \ldots, \frac{p+1}{2}$;
- the representatives for the right and left cosets inside the double coset are: for $i = j = 1$ there is only 1 right coset represented by $(1_G, 1_G)$; for $i = 1$ and $1 \leq k \leq \frac{p-1}{2}$ the right and left representatives are $\{(1_G, \sigma_k^2), (1_G, \sigma_2^{-p-k})\}$.

Hence we get for any $x$ corresponding to the subgroup $P_{x, 2}$ that maps to $X$ in $\pi_{H^2} : Z \to X$ is the covering corresponding to the subgroup $H^2$.

Now the identification of the two definitions of the curve $X$ is as follows. If $P_{11}$ corresponds to the point $x \in X$, then $P_{t,k}$ corresponds to the point $\pi_{H^2}(\sigma_1^t, \sigma_2^k)(z)$ where $z \in Z$ maps to $x$. This follows from the fact that the identification is $G^2$-equivariant. This means that for $P_{rs} \in X$ we have $
$
with $u$ and $v$ such that $u + v \equiv r + s + \ell \mod p$ and $u - v \equiv r - s + k \mod p$. Inserting this in (6.1), we obtain
\[
\mathcal{K}_D(P_{ij}) = \sum_{(k,l) \in I_{ij}} P_{kl},
\]
which completes the proof of the proposition. $\square$

As an immediate consequence of Propositions 4.2 and 6.1 we obtain

**Corollary 6.2.** Suppose $\tilde{Z} \to Y$ is an étale $p$-fold covering of a hyperelliptic curve $Y$ such that the Galois closure $Z$ of the composed map $\tilde{Z} \to Y \to \mathbb{P}^1$ has Galois group $G = D_p^2$.

(a): The correspondence $D$ on the curve $X$ of diagram (5.1) defines a Prym-Tyurin variety $P$ of exponent $p$ in the Jacobian $JX$.

(b): There exist curves $X_1$ and $X_2$ over $\mathbb{P}^1$, whose fibre product over $\mathbb{P}^1$ is $X$ such that $P \simeq JX_1 \times JX_2$.

as principally polarized abelian varieties.

**7. Decomposition for the Jacobian of $X$**

Let the notation be as in Section 4. So we are given the curve $Z = Z_1 \times_{\mathbb{P}^1} Z_2$ with $D_p^2$-action as in Proposition 4.2. Consider the curve $X$ of diagram (5.1) (or equivalently the curve $X$ of diagram (5.1)). If $X_i, Y$ and $\tilde{Y}_j$ are as in diagram (4.1), recall that $\tilde{Y}_j \to Y$ are étale $p$-fold coverings. If then $P(\tilde{Y}_j/Y)$ denotes the corresponding (generalized) Prym variety (i.e. the connected component of the origin of the kernel of the norm map $J\tilde{Y}_j \to JY$), the Jacobian $JX$ decomposes up to isogeny in the following way.

**Theorem 7.1.**

\[
JX \sim JX_1 \times JX_2 \times P(\tilde{Y}_1/Y) \times \cdots \times P(\tilde{Y}_{(p-1)/2}/Y).
\]

For the proof we use some results of [3] which we recall first: For a given finite group $G$, let $Z \to \mathbb{P}^1$ denote a Galois covering with Galois group $G$, $H$ a subgroup of $G$ and $Z_H := Z/H$. To every rational irreducible representation $W$ of $G$ one can associate an abelian subvariety $B_W$ of the Jacobian $JZ$ which is uniquely determined up to isogeny. Let $V_0$ denote the trivial representation and $W_1, \ldots, W_s$ the non-trivial rational irreducible representations of $G$ with associated complex irreducible representations $V_1, \ldots, V_s$. Moreover, $\rho_H$ denotes the character of $G$ induced by the trivial character of $H$. Then [3, Lemma 4.3] says that
\[
\rho_H = V_0 \oplus \bigoplus_{j=1}^s c_j W_j \quad \text{with} \quad c_j = \frac{\dim V_j^H}{m_j},
\]
where $m_j$ denotes the Schur index of the representation $V_j$. Moreover, [3, Proposition 5.2] says that the Jacobian $JZ_H$ decomposes up to isogeny as

\[
JZ_H \sim B_{W_1}^{c_1} \times \cdots \times B_{W_s}^{c_s}.
\]
Finally, if \( H \subset N \) are two subgroups of \( G \), then
\[
\rho_H = \rho_N \oplus \bigoplus_{j=1}^{s} d_j \mathcal{W}_j \quad \text{with} \quad d_j = \frac{\dim V_j^H - \dim V_j^N}{m_j} \quad (\geq 0)
\]
and, according to [3, Corollary 5.4], the Prym variety \( P(Z_H/Z_N) \) of the morphism \( Z_H \to Z_N \) decomposes up to isogeny as
\[
(7.2) \quad P(Z_H/Z_N) \sim \mathcal{B}_{\mathcal{W}_1}^{d_1} \times \cdots \times \mathcal{B}_{\mathcal{W}_s}^{d_s}.
\]
Returning to the dihedral group \( \mathcal{D}_p \), we denote by \( V_j, j = 1, \ldots, \frac{p-1}{2} \) its complex irreducible representations of degree 2, with corresponding characters given by
\[
\chi_{V_j}(\sigma^h) = \omega^{jh} + \omega^{-jh}, \quad \chi_{V_j}(\tau \sigma^h) = 0,
\]
where \( \omega \) denotes a fixed \( p \)-th root of unity. Then \( \mathcal{W} = \bigoplus_{j=1}^{\frac{p-1}{2}} V_j \) is the irreducible rational representation of degree \( p - 1 \).
Consider, as in Section 4, the group \( \mathcal{D}_p^2 = \langle \sigma_1, \tau_1, \sigma_2, \tau_2 \rangle \). The complex irreducible representation of \( \mathcal{D}_p^2 \) are the (outer) tensor products of the irreducible representations of the factors. We consider the following rational irreducible representations of \( \mathcal{D}_p^2 \); the trivial \( \mathcal{V}_0 = V_0 \otimes V_0 \), the alternating representation \( \mathcal{V}'_0 = V'_0 \otimes V'_0 \),
\[
\mathcal{W}_1 = \mathcal{W} \otimes V_0 \quad \text{and} \quad \mathcal{W}_2 = V_0 \otimes \mathcal{W},
\]
and, for \( j = 1, \ldots, \frac{p-1}{2} \),
\[
\mathcal{U}_j = \bigoplus_{i=1}^{\frac{p-1}{2}} (V_i \otimes V_k) = (V_1 \otimes V_j) \oplus (V_2 \otimes V_{j+1}) \oplus \cdots \oplus (V_{p-1/2} \otimes V_{j-1})
\]
where \( 1 \leq k \leq \frac{p-1}{2} \) is given by \( j + i - 1 \) if \( j + i - 1 \leq (p - 1)/2 \) and \( k = j + i - 1 - (p - 1)/2 \) otherwise.

Recall the subgroups defining the curves of diagram 4.3:
- \( X \) is defined by \( H^2 := \langle \tau_1, \tau_2 \rangle \),
- \( X_1 \) is defined by \( H_1 := \langle \sigma_2, \tau_1, \tau_2 \rangle \),
- \( X_2 \) is defined by \( H_2 := \langle \sigma_1, \tau_1, \tau_2 \rangle \),
- \( \tilde{Y}_j \) is defined by \( L_j := \langle \sigma_1^j, \sigma_2, \tau_1, \tau_2 \rangle \),
- \( Y \) is defined by \( M := \langle \sigma_1, \sigma_2, \tau_1, \tau_2 \rangle \).

**Lemma 7.2.** (a): \( \rho_{H^2} = \mathcal{V}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_{\frac{p-1}{2}} \),
(b): \( \rho_{H_i} = \mathcal{V}_0 \oplus \mathcal{W}_i \) for \( i = 1 \) and 2,
(c): \( \rho_{M} = \mathcal{V}_0 \oplus \mathcal{V}'_0 \),
(d): \( \rho_{L_j} = \rho_{M} \oplus \mathcal{U}_j \) for \( j = 1, \ldots, \frac{p-1}{2} \).

**Proof.** (a): As \( H^2 \) is a subgroup of \( \mathcal{D}_p^2 \) satisfying Hypothesis 2.1 we have for the character product
\[
\langle \rho_{H^2}, \mathcal{W}_1 \rangle = \langle \rho_{H^2}, \mathcal{W}_2 \rangle = 1.
\]
By direct computation of the character product using Frobenius reciprocity, one shows \( \langle \rho_H^2, \mathcal{U}_j \rangle = 1 \) for all \( j \). Finally, as the sum of the degrees of these representations is the degree of \( \rho_H^2 \), we get the result. The proofs of the other assertions are similar.

Proof of Theorem 7.1. Let \( B_{W_i} \) and \( B_{U_j} \) denote the abelian varieties associated to the representations \( W_i \) and \( U_j \). Then we have, according to Lemma 7.2 (a) and equation (7.1),

\[
J_X \sim B_{W_1} \times B_{W_2} \times B_{U_1} \times \cdots \times B_{U_{p-1}}.
\]

It remains to identify the \( B \)'s. According to Lemma 7.2 (b) and equation (7.1),

\[
B_{W_i} \sim JX_i \quad \text{for} \quad i = 1, 2
\]

and Lemma 7.2 (c) and (d) and equation (7.2) imply

\[
B_{U_j} \sim P(\tilde{Y}_j/Y) \quad \text{for all} \quad j.
\]

This completes the proof of the theorem.

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