Current-carrying ground states in mesoscopic and macroscopic systems

Michael R. Geller

Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
and Department of Physics, University of Missouri, Columbia, Missouri 65211

(March 23, 2022)

We extend a theorem of Bloch, which concerns the net orbital current carried by an interacting electron system in equilibrium, to include mesoscopic effects. We obtain a rigorous upper bound to the allowed ground-state current in a ring or disc, for an interacting electron system in the presence of static but otherwise arbitrary electric and magnetic fields. We also investigate the effects of spin-orbit and current-current interactions on the upper bound. Current-current interactions, caused by the magnetic field produced at a point \( r \) by a moving electron at \( r' \), are found to reduce the upper bound by an amount that is determined by the self-inductance of the system. A solvable model of an electron system that includes current-current interactions is shown to realize our upper bound, and the upper bound is compared with measurements of the persistent current in a single ring.

PACS numbers: 71.27.+a, 71.70.Ej, 72.15.-v, 75.20.En

I. INTRODUCTION

A theorem due to Bloch holds that an interacting electron system in equilibrium carries no net orbital current \( J = 0 \). This question originally had been motivated by early attempts, before the Bardeen-Cooper-Schrieffer theory, to explain superconductivity by proposing that electron-electron interactions lead to special current-carrying states of lower energy than the current-free states. However, it is now understood that supercurrent-carrying states are in fact metastable nonequilibrium states, which, because of their off-diagonal long-range order or wave function rigidity, have an extremely long lifetime.

The great interest over the past several years in the physics of mesoscopic systems again has made the question of allowed equilibrium currents an important one. More than ten years ago, Büttiker et al. \[2\] predicted the existence of equilibrium currents in mesoscopic normal-metal rings threaded by a magnetic flux. Recent experimental evidence \[3\] in support of this conjecture has stimulated considerable interest in these so-called persistent currents. Although a satisfactory explanation of the experiments is still lacking, the present consensus is that both electron-electron interaction and disorder effects are important \[3\]. A related phenomena is that of spontaneous orbital currents occurring in the absence of any applied magnetic field or twisted boundary conditions. Although there is no experimental evidence for this symmetry-breaking state, spontaneous orbital currents have been predicted to occur by several authors \[3\].

Given the diverse situations in which equilibrium current-carrying states may occur, it is worthwhile to revise Bloch’s theorem to incorporate these mesoscopic effects. To this end, Vignale \[4\] has recently derived a rigorous upper bound to the persistent current in a single ring, the results being valid for both noninteracting and interacting electrons in the presence of arbitrary magnetic fields and impurity potentials. One surprising result of Vignale’s analysis is that although the upper bound to the persistent current in a thin ring of uniform density and radius \( R \) vanishes as \( 1/R \) for large \( R \), it does not vanish for a thick ring or punctured disc with the ratio \( R_{\text{in}}/R_{\text{out}} \) of the inner radius and outer radius fixed as these radii become infinite. Although Vignale’s result does not preclude the existence of a more stringent upper bound that always vanishes in the macroscopic limit, the upper bound is actually realized in calculations of the persistent current (the integrated azimuthal current density) in a two-dimensional noninteracting electron gas in a large quantum dot \[5\].

One motivation for this work is to extend the analysis of Ref. \[1\] to include the effects of spin-orbit interaction, which has received considerable attention in connection with persistent currents and spontaneous currents. Spin-orbit interaction is known to lead to a topological interference effect, called the Aharonov-Casher effect \[2\], which is an electromagnetic dual of the Aharonov-Bohm effect. Meir et al. \[13\] have shown that spin-orbit scattering in one-dimensional disordered rings induces an effective magnetic flux which reduces the persistent current in a universal manner. The effect of spin-orbit interaction on mesoscopic persistent currents has been studied by several other authors, who also find reduced currents \[14\] bound by an amount that is determined by the self-inductance of the system. We shall show here that this is not the case.

A second motivation for this work is to examine the influence of current-current interactions, an order \( v^2/c^2 \) relativistic effect caused by the magnetic field produced at a point \( r \) by a moving electron at \( r' \). The possibility of these magnetic interactions leading to a spontaneous current-carrying state in mesoscopic metal rings has been discussed in a remarkable paper by Wohlleben et al. \[1\], where it is shown that, in zero field, a small ring exhibits
a transition to a state with persistent current. The combined effects of spin-orbit coupling and current-current interactions has been studied recently by Choi [8]. We shall show below that current-current interactions reduce the upper bound on the ground-state current by an amount which is determined by the self-inductance of the system.

In this paper, we derive an upper bound to the ground-state current in an arbitrary many-electron system, including spin-orbit coupling and current-current interaction effects. To best demonstrate the modifications to Bloch’s theorem from the effects of finite sample size, we restrict our analysis to zero temperature. However, our final results are also valid at finite temperature, as may be shown by following the method of Ref. [3].

II. RIGOROUS UPPER BOUND

We begin by obtaining a many-electron Hamiltonian that includes current-current interactions to order \( v^2/c^2 \). In the transverse gauge, the vector potential seen by an electron at \( \mathbf{r}_n \) in the presence of the other moving electrons (of charge \(-e\)) is

\[
A^i(\mathbf{r}_n) = -\frac{e}{2c} \sum_{n' \neq n} \frac{T^{ij}(\mathbf{r}_n - \mathbf{r}_{n'}) v^j_{n'}}{|\mathbf{r}_n - \mathbf{r}_{n'}|},
\]

where \( v_n \) is the velocity of the \( n \)th electron, and where

\[
T^{ij}(\mathbf{r}) \equiv \delta^{ij} + \frac{\mathbf{r}^i \mathbf{r}^j}{|\mathbf{r}|^2}.
\]

This vector potential leads to a current-current interaction term in the Lagrangian of the form

\[
L_{\text{int}} = \frac{e^2}{2c^2} \sum_{n<n'} \frac{v^i_n T^{ij}(\mathbf{r}_n - \mathbf{r}_{n'}) v^j_{n'}}{|\mathbf{r}_n - \mathbf{r}_{n'}|},
\]

which in turn leads to a current-current interaction term in the Hamiltonian of the form

\[
H_{\text{int}} = -\frac{e^2}{2m^2c^2} \sum_{n<n'} p^i_n T^{ij}(\mathbf{r}_n - \mathbf{r}_{n'}) p^j_{n'},
\]

to leading order in \( v^2/c^2 \). The complete Hamiltonian, including spin-orbit coupling, Coulomb and current-current interactions, and coupling to additional electric and magnetic fields, may be written as

\[
H = \sum_n \left( \frac{\Pi^2_n}{2m} - \frac{\Pi^4_n}{8m^3c^2} \right) + V(\mathbf{r}_n) + \frac{1}{2m^2c^2} \mathbf{S}_n \cdot \left[ \nabla V(\mathbf{r}_n) \times \Pi_n \right] + \sum_{n<n'} \frac{e^2}{|\mathbf{r}_n - \mathbf{r}_{n'}|} - \frac{e^2}{2m^2c^2} \sum_{n<n'} \frac{\Pi^{ij}_n T^{ij}(\mathbf{r}_n - \mathbf{r}_{n'})}{|\mathbf{r}_n - \mathbf{r}_{n'}|} \Pi^{ij}_{n'},
\]

where \( \Pi_n \equiv \frac{\mathbf{p}_n + \frac{e}{c} \mathbf{A}(\mathbf{r}_n)}{m} \), the \( S^i \) are spin operators, and where the potentials \( \mathbf{A} \) and \( V \) are time-independent but otherwise arbitrary. Spin-spin interactions and the coupling of the spin degrees of freedom to the external magnetic field are not important here and shall be ignored. The velocity operator \( v^i_n \equiv [\mathbf{r}_n, H]/i\hbar \) is given by

\[
v^i_n = \frac{\Pi^i_n}{m} \left( 1 - \frac{\Pi^2_n}{2m^2c^2} \right) + \frac{1}{2m^2c^2} [\mathbf{S}_n \times \nabla V(\mathbf{r}_n)]^i - \frac{e^2}{4m^2c^2} \sum_{n' \neq n} \left( \frac{\Pi^{ij}_n T^{ij}(\mathbf{r}_n - \mathbf{r}_{n'})}{|\mathbf{r}_n - \mathbf{r}_{n'}|} + \frac{T^{ij}(\mathbf{r}_n - \mathbf{r}_{n'})}{|\mathbf{r}_n - \mathbf{r}_{n'}|} \Pi^{ij}_{n'} \right).
\]

We shall consider a system of \( N \) electrons confined to a ring or disc, oriented with its axis along the \( z \) direction, and we write (5) in cylindrical coordinates \( \mathbf{r} = (r, \theta, z) \). The thickness and cross-sectional shape of the system is arbitrary. The many-body ground state \( \psi(\mathbf{r}_1, s_1, \cdots, \mathbf{r}_N, s_N) \) satisfies

\[
H \psi = E \psi,
\]

where \( E \) is the ground-state energy.

Suppose that the ground state \( \psi \) carries an orbital persistent current

\[
I = -\frac{e}{4\pi} \left\langle \psi \left( \sum_n \left( \frac{\mathbf{e}_\theta(\mathbf{r}_n)}{r_n} \cdot \mathbf{v}_n + \mathbf{v}_n \cdot \frac{\mathbf{e}_\theta(\mathbf{r}_n)}{r_n} \right) \right) \right| \psi \rangle,
\]

where \( \mathbf{e}_\theta(\mathbf{r}_n) = (0, 0, 1) \).
where $e_\theta(\mathbf{r})$ is an azimuthal unit vector at $\mathbf{r}$. We may construct a rotating state $\psi' = U \psi$, where

$$U \equiv \prod_n e^{i\delta L_n / \hbar},$$

which is not necessarily an eigenstate of $H$, and which has a mean energy $E' \equiv \langle \psi' | H | \psi' \rangle$ given by

$$E' = E - \frac{2\pi I}{e} \delta L + \frac{1}{2m} \left( \sum_n \frac{1}{r_n^2} \right) \delta L^2 - \frac{1}{8m^2 c^2} \left( \sum_n \left( \Pi_n^2 \frac{1}{r_n^2} + \frac{1}{r_n^2} \Pi_n^2 + 4 \langle \Pi_n^0 \rangle^2 \right) \right) \delta L^2$$

$$- \frac{e^2}{4m^2 c^2} \sum_{n \neq n'} \frac{\epsilon^i_\theta(r_n) T^{ij}(r_n - r_{n'}) \epsilon^j_\theta(r_{n'})}{r_n | r_n - r_{n'} | r_{n'}} \delta L^2 - \frac{1}{2m^2 c^2} \left( \sum_n \frac{\Pi^0_n}{r_n^2} \right) \delta L^3 - \frac{1}{8m^3 c^2} \left( \sum_n \frac{1}{r_n^4} \right) \delta L^4.$$ (10)

Here $\langle \cdots \rangle$ denotes an expectation value in the original ground state $\psi$, $\Pi_n^0 \equiv \Pi_n \cdot e_\theta(r_n)$, and we have used

$$U^\dagger \Pi_n U = \Pi_n + \frac{1}{r_n} e_\theta(r_n) \delta L.$$ (11)

The energy difference, $\delta E \equiv E' - E$, plotted as a function of the parameter $\delta L$, is shown in Fig. 1. For values of $\delta L$ such that

$$0 < \delta L < \delta L^*,$$ (12)

where $\delta L^*$ is the zero of $\delta E$ defined in Fig. 1, the rotating state $\psi'$ has a lower mean energy than $\psi$, so $\psi$ cannot be the ground state.

This is the essential content of Bloch’s theorem. It applies whenever there is a nonzero $\delta L$ satisfying (12). However, the smallest nonzero $\delta L$ permitted by the condition that the wave function $\psi'$ be single-valued is $\delta L = \hbar$. When the relativistic corrections in (3) are neglected, $\delta L^* = \delta L^0$, where

$$\delta L^0 = \frac{4\pi m I}{N e} \left( \frac{1}{r^2} \right)^{-1}.$$ (13)

Here

$$\left\langle \frac{1}{r^\gamma} \right\rangle = \frac{1}{N} \left( \sum_n \frac{1}{r_n^{\gamma}} \right) = \frac{1}{N} \int d^3 r \frac{n(r)}{r^{\gamma}},$$ (14)

where $n(r)$ is the ground-state number density. Therefore, Bloch’s theorem applies only when $\delta L^* > \hbar$, or whenever $|I| > I_{\text{max}}^0$, where

$$I_{\text{max}}^0 = \frac{N e \hbar}{4 \pi m} \left( \frac{1}{r^2} \right).$$ (15)

This is the upper bound derived in Ref. [3]. When the relativistic corrections are included to leading order, $\delta L^*$ is given by

$$\delta L^* = \delta L^0 + \frac{\pi I}{e m c^2 N^2 (1/r^2)^2} \left( \sum_n \left( \Pi_n^2 \frac{1}{r_n^2} + \frac{1}{r_n^2} \Pi_n^2 + 4 \langle \Pi_n^0 \rangle^2 \right) \right)$$

$$+ \frac{2\pi I}{e c^2 N^2 (1/r^2)^2} \sum_{n \neq n'} \frac{\epsilon^i_\theta(r_n) T^{ij}(r_n - r_{n'}) \epsilon^j_\theta(r_{n'})}{r_n | r_n - r_{n'} | r_{n'}} + \frac{16 \pi^2 I^2}{e^2 c^2 N^3 (1/r^2)^3} \left( \sum_n \frac{\Pi^0_n}{r_n^2} \right) + \frac{16 \pi^3 m I^3 (1/r^4)}{e^3 c^2 N^3 (1/r^2)^4}.$$ (16)

Bloch’s theorem therefore applies whenever $|I| > I_{\text{max}}$, where

$$I_{\text{max}} = (1 - \Lambda) I_{\text{max}}^0,$$ (17)

and

3
\[ \Lambda \equiv \frac{1}{4m^2c^2 N(1/r^2)} \left\{ \sum_n \left( \Pi_n^2 \frac{1}{r_n^2} + \frac{c_n^2}{r_n^2} + 4 \left( \Pi_n^2 \right)^2 \right) \right\} + \frac{e^2}{2mc^2 N(1/r^2)} \left\{ \sum_{n \neq n'} \frac{e^d_n(r_n)}{r_n|r_n - r_{n'}|} \right\} \]
\[ + \frac{\hbar}{m^2 c^2 N(1/r^2)} \left\{ \sum_n \Pi_n^2 \right\} + \frac{\hbar^2 (1/r^2)}{4m^2 c^2 (1/r^2)} \]

is a dimensionless reduction factor. States carrying orbital currents larger than \( I_{\text{max}} \) cannot be ground states of (3).

The upper bound (17) applies to interacting electron systems in the presence of static but otherwise arbitrary electric and magnetic fields, and includes the effects of spin-orbit coupling and current-current interaction. The upper bound (18) applies to noninteracting systems and also to electrons with Coulomb interaction. In particular, (15) applies to noninteracting electrons in a periodic potential, and this fact leads to a general constraint on the band structure of any one-dimensional crystal (18).

### III. UPPER BOUND FOR A THIN RING

Now consider the case of a thin ring with cross-sectional dimensions much less than the radius \( R \) of the ring. In this case,

\[ I_{\text{max}}^0 = \frac{Ne \hbar}{4\pi m R^2} = \frac{2e v_F}{L}, \]

where \( v_F \) is the Fermi velocity, \( L \equiv 2\pi R \) is the circumference of the ring, and

\[ \Lambda \approx \frac{3}{2m^2 c^2 N} \left\{ \sum_n \left( \Pi_n^2 \right)^2 \right\} + \frac{e^2}{2mc^2 N} \left\{ \sum_{n \neq n'} \frac{e^d_n(r_n) T^{ij}(r_n - r_{n'}) e^d_{n'}(r_{n'})}{|r_n - r_{n'}|} \right\} + \frac{\hbar}{m c^2 N R} \left\{ \sum_n \Pi_n^2 \right\} + \frac{\hbar^2}{4m^2 c^2 R^2}. \]

The first term in (20) is approximately equal to \( E_F/mc^2 \), where \( E_F \) is the Fermi energy, and hence this term is entirely negligible here. The magnitude of the third term may be estimated by using the approximation \( \left\{ \sum_n \Pi_n^2 \right\} \approx 4\pi m R I_{\text{max}}^0/e \), which shows that the third term and fourth term in (20) are both of order \( \lambda_c^2 / R^2 \), where \( \lambda_c \equiv \hbar / mc \) is the Compton wavelength of the electron. These terms are therefore negligible here as well.

The operator in the second term of (20) may be written in second-quantized form as

\[ \sum_{n \neq n'} \frac{e^d_n(r_n) T^{ij}(r_n - r_{n'}) e^d_{n'}(r_{n'})}{|r_n - r_{n'}|} = \int d^3 r d^3 r' F(r, r') \psi^\dagger(r) \psi^\dagger(r') \psi(r), \]

where \( \psi(r) \) and \( \psi^\dagger(r) \) are electron field operators, and where

\[ F(r, r') \equiv \frac{e^d_n(r) T^{ij}(r - r') e^d_{n'}(r')}{|r - r'|}. \]

This term is a consequence of the current-current interactions. In a mesoscopic or macroscopic system, the largest contribution to the expectation value of (21) comes from the direct term

\[ \int d^3 r d^3 r' F(r, r') n(r) n(r'), \]

which is normally absent in the case of Coulomb interactions in a uniform system. For a thin wire of approximately uniform density and current density, we may write the latter as \( I \) divided by the cross-sectional area \( N/nL \),

\[ \mathbf{j}(r) \approx \frac{n I L}{N} \mathbf{e}_\theta(r). \]

Then we have

\[ \left\{ \sum_{n \neq n'} \frac{e^d_n(r_n) T^{ij}(r_n - r_{n'}) e^d_{n'}(r_{n'})}{|r_n - r_{n'}|} \right\} = \frac{N^2}{4\pi^2 R^2 T^2} \int d^3 r d^3 r' \frac{\mathbf{j}(r) \cdot \mathbf{j}(r')}{|r - r'|} \]
\[ = \frac{N^2}{2\pi^2 R^2 T^2} \int d^3 r d^3 r' \frac{\mathbf{j}(r) \cdot \mathbf{j}(r')}{|r - r'|}. \]
where $j_t$ is the transverse current density, defined as

$$
j_t(r) = \frac{1}{4\pi} \nabla \times \nabla \times \int d^3r' \frac{j(r')}{|r-r'|},$$

$$= j(r) + \frac{1}{4\pi} \int d^3r' \frac{\nabla' \cdot j(r')}{|r-r'|}.
$$

(26)

Because the equilibrium current density satisfies $\nabla \cdot j = 0$, the second term in (26) vanishes and $j$ is purely transverse. The reduction factor (20) for a thin ring may therefore be written approximately as $\Lambda = 2I_{\text{max}}^0 \mathcal{L}/c\Phi_0$, where

$$\mathcal{L} \equiv \frac{1}{\ell^2} \int d^3r d^3r' \frac{j(r) \cdot j(r')}{|r-r'|}$$

(27)

is the classical self-inductance of the ring, and where $\Phi_0 \equiv hc/e$ is the quantum of flux. It is also useful to rewrite the reduction factor as

$$\Lambda = \frac{2I_{\text{max}}^0}{I_c},$$

(28)

where $I_c \equiv c\Phi_0/\mathcal{L}$ is the magnitude of the current needed to produce one quantum of flux. This latter form makes explicit the relative importance of the inductive effects. Therefore, the upper bound in a thin ring may be written as

$$I = \left(1 - \frac{2I_{\text{max}}^0}{I_c}\right)I_{\text{max}}^0.$$

(29)

We see that the current-current interactions always reduce the allowed ground-state current by an amount that depends on the self-inductance of the ring. This reduction occurs because the energy required to sustain a persistent current now includes the magnetic field energy. As is clear from our derivation, which treated the current-current interaction as a small perturbation, (29) is valid only when $\Lambda \ll 1$, or when $I_{\text{max}}^0 \ll I_c$. We shall evaluate (29) for realistic thin-ring geometries in the final section of this paper.

**IV. PERSISTENT CURRENT IN A SOLVABLE MODEL WITH CURRENT-CURRENT INTERACTIONS**

In this section we shall calculate the ground-state current in an electron gas with current-current interactions, which is confined to a thin wire loop of circumference $L \equiv 2\pi R$. We shall assume that only a single transverse mode in the wire is occupied, so that the system is effectively one-dimensional, with a width $a \ll L$. For modest applied perpendicular magnetic fields, such that the magnetic length $\ell \equiv (hc/eB)^{\frac{1}{2}}$ satisfies $\ell >> a$, the primary effect of the magnetic field on the electron gas is to induce an Aharonov-Bohm phase shift around the ring. The effect of the magnetic field in this regime may therefore be accounted for by imposing the twisted boundary conditions

$$\psi_n(x + L) = e^{2\pi \phi} \psi_n(x),
$$

(30)

on the single-particle states. Here $x \equiv R\theta$ is an arc-length coordinate going around the circumference of the ring, and $\phi \equiv \Phi/\Phi_0$ is the total enclosed flux in units of the flux quantum $\Phi_0 \equiv hc/e$.

We shall calculate the ground-state current in a self-consistent mean field approximation in which the effect of the current-current interaction is to increase the flux enclosed by the ring by an amount that is determined by the persistent current. The persistent current is, in turn, determined by the total flux. Coulomb interactions, which are not expected to affect the persistent current in a disorder-free system, are ignored altogether. The single-particle states are therefore of the form

$$\psi_n = \frac{1}{\sqrt{L}} e^{ik_n x},
$$

(31)

where $n$ is an integer and, according to (30), the allowed wave numbers are $k_n \equiv 2\pi(n + \phi)/L$. The energies of the states (31) are $\epsilon_n = \hbar^2 k_n^2/2m$, and

$$I_n = \frac{2\pi e\hbar}{mL^2}(n + \phi)
$$

(32)
are their contributions to the total persistent current.

Let \( N \) be the total number of electrons, which we shall take to be twice an odd integer. (A similar analysis applies to the case of \( N/2 \) even.) The total persistent current at zero temperature is given by

\[
I = 2 \sum_n \theta(E_F - \epsilon_n) I_n, \tag{33}
\]

where \( E_F \) is the Fermi energy, \( \theta(x) \) is the unit step function, and where the factor of 2 accounts for the spin degeneracy. For a small flux, the set of occupied states in (33) is not changed by the flux. Then we find

\[
I = -2 I_{\text{max}}^0 \phi_1, \tag{34}
\]

which is the well-known result for the persistent current in a one-dimensional ring in the presence of an external dimensionless flux \( \phi \), when disorder and electron-electron interactions are ignored. The expression (34) is valid until the set of occupied states is changed by the flux. In the usual case where \( \phi \) is independent of \( I \), the range of allowed flux is \(-\frac{1}{2} < \phi < \frac{1}{2}\), and the upper bound (19) is realized when \( \phi \to \pm \frac{1}{2} \).

In our case of interest, however, there are two contributions to \( \phi \),

\[
\phi = \phi_1 + \phi_2. \tag{35}
\]

The first, \( \phi_1 \), is the flux from the external magnetic field, and the second, \( \phi_2 \), is the flux that originates from the current-current interactions. The latter is the flux through the ring induced classically by the current \( I \),

\[
\Phi_2 = \frac{1}{c} \int_{\text{ring}} \mathbf{a} \cdot \nabla \times \int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{36}
\]

In a thin ring, this leads to \( \phi_2 = IL/c\Phi_0 \), where \( L \) is the self-inductance. Therefore, we obtain from (34),

\[
I = -2 I_{\text{max}}^0 \phi_1 \left( 1 - 2 \frac{I_{\text{max}}^0 L}{c\Phi_0} \right), \quad \left( -\frac{1}{2} < \phi_1 < \frac{1}{2} \right) \tag{37}
\]

In this simple model, the current is zero when the external flux is zero. The maximum current occurs when \( \phi_1 \to \pm \frac{1}{2} \). The magnitude of the persistent current in this optimal case is then found to be

\[
|I| = I_{\text{max}}^0 \left( 1 - 2 \frac{I_{\text{max}}^0 L}{c\Phi_0} \right), \tag{38}
\]

which realizes our thin-ring upper bound (29) for electrons with current-current interactions.

V. DISCUSSION

We now evaluate the upper bound (29) for realistic thin-ring geometries. The upper bound (19) for a metal with a Fermi velocity of \( 2 \times 10^8 \text{ cm/s} \), a typical value, may be written as

\[
I_{\text{max}}^0 \approx 0.64 \mu\text{A}, \tag{39}
\]

where \( L(\mu\text{m}) \) is the circumference of the ring in microns. As discussed above, the relevance of inductive effects are characterized by the ratio of \( I_c \equiv c\Phi_0/L \), which is the current needed to produce one quantum of flux, to \( I_{\text{max}}^0 \). If we measure the self-inductance in microns, then

\[
I_c \approx \frac{41.2 \text{ mA}}{L(\mu\text{m})}. \tag{40}
\]

The reduction factor (28) for a metal ring may then be written as

\[
\Lambda \approx 3.0 \times 10^{-5} \frac{L}{L}. \tag{41}
\]
The self-inductance of a thin toroidal ring with major radius $R$ and minor radius $a$ (wire radius) is $\mathcal{L} = 4\pi R [\ln(8R/a) - \frac{7}{4}]$. Therefore, we see that $\Lambda$ depends only on the aspect ratio $R/a$ of the ring, and not on its circumference:

$$\Lambda \approx 6.0 \times 10^{-5} \left[ \ln \left( \frac{8R}{a} \right) - \frac{7}{4} \right].$$

(42)

For a thick ring, where $R/a \approx 1$, $\mathcal{L}$ is approximately equal to the size $L$ of the ring. For a thin ring with $R/a \approx 10$ or $R/a \approx 100$, $\mathcal{L}$ is substantially larger.

Consider, for example, a gold ring with $L \approx 12 \mu m$ and $R/a \approx 30$, characteristic of the rings studied by Chandresakhar et al. [4], where persistent currents of order $I_0^{\text{max}}$ where measured. Then $\mathcal{L}/L \approx 7.5$, and $\Lambda \approx 10^{-4}$, a negligible reduction that is consistent with the experiments.

**ACKNOWLEDGMENTS**

This work was supported by the National Science Foundation through Grant DMR-9403908. I would like to thank Giovanni Vignale for many useful discussions on this subject. I also acknowledge the hospitality of the Institute for Theoretical Physics, Santa Barbara, where part of this work was completed under NSF Grant PHY89-04035.

[1] See D. Bohm, Phys. Rev. 75, 502 (1949), and references therein.
[2] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. 96A, 365 (1983).
[3] L. P. Lévy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. 64, 2074 (1990).
[4] V. Chandresakhar, R. A. Webb, M. J. Brady, M. B. Ketchen, W. J. Gallagher, and A. Kleinsasser, Phys. Rev. Lett. 67, 3578 (1991).
[5] V. Ambegaokar and U. Eckern, Phys. Rev. Lett. 65, 3578 (1991).
[6] D. Wohlleben, M. Esser, P. Freche, E. Zipper, and M. Szopa, Phys. Rev. Lett. 66, 3191 (1991).
[7] M. Rasolt and F. Perrot, Phys. Rev. Lett. 69, 2563 (1992).
[8] M. Y. Choi, Phys. Rev. Lett. 71, 2987 (1993).
[9] G. Vignale, Phys. Rev. B 51, 2612 (1995).
[10] Y. Avishai and M. Kohmoto, Phys. Rev. Lett. 71, 279 (1993).
[11] M. R. Geller and G. Vignale, Phys. Rev. B 50, 11714 (1994).
[12] Y. Aharonov and A. Casher, Phys. Rev. Lett. 53, 319 (1984).
[13] Y. Meir, Y. Gefan, and O. Entin-Wohlman, Phys. Rev. Lett. 63, 798 (1989).
[14] D. Loss, P. Goldbart, and A. V. Balatsky, Phys. Rev. Lett. 65, 1655 (1990).
[15] A. V. Balatsky and B. L. Altshuler, Phys. Rev. Lett. 70, 1678 (1993).
[16] S. Fujimoto and N. Kawakami, Phys. Rev. B 48, 17406 (1993).
[17] T. Z. Qian and Z. B. Su, Phys. Rev. Lett. 72, 2311 (1994).
[18] M. R. Geller and G. Vignale, Phys. Rev. B 51, 2616 (1995).

**FIG. 1.** Energy difference between the ground state $\psi$ and the rotating state $\psi'$, as a function of the imparted angular momentum $\delta L$. 