Logarithmic Potential Model of Quigg and Rosner as a Generalization of Naive Quark Model

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ABSTRACT

Exploiting the explicit mass formulae for logarithmic potential model of Quigg and Rosner it is shown that at least on the level of mass-relations this model reproduces the naive quark model relations and generalizes the last one in case of highly non-trivial potential. Generalization includes the relations for higher values of orbital quantum numbers. In particular, predictions for recently discovered atom-like P-states are no worse than for any other potential models. The advantage consists in simplicity of approach.
Examining the approximate equality of the spacing between the ground states and first radial excitations in the Upsilon and J/Ψ systems Quigg and Rosner [1] demonstrated that the logarithmic potential is the only nonrelativistic potential in quantum mechanics that predicts independence of the radial excitations over the constituent masses.

They also calculated the numeric values for all observable physical quantities for this potential. It turned out that from the phenomenological point of view this potential is no worse then the other ones. So, though the linear potential has more sound theoretical basis, the logarithmic one also deserves attention.

In this note I will concentrate on some of the mass relations (sum rules) which can be derived explicitly only within the logarithmic potential model. In the literature it was never noticed that the logarithmic potential, beeing nonrelativistic, generalizes naive quark model. So its predictions are simple and can be formulated analytically. In the last years Martin’s phenomenological potential [2] $V(r) = -0.8064 + 6.87r^{0.1}$ has gained much interest. Coupled with the simplest spin-spin Hamiltonian this model leads to satisfactory results for lower levels of quarkonia.

Due to small exponent Martin’s potential is quite close to the logarithmic one. So it’s natural to expect that numerical results will be quite similar. And we will see below that in the latter case simple analytic results can be derived.

So, for each quark-antiquark pair we will use the following potential [1]:

$$V_{ij}(r) = g_{ij} \ln \frac{r}{R_{ij}}$$  \hspace{1cm} (1)

Then it is easy to show that after introduction of the dimensionless parameter $\rho = (2\mu_{ij}g_{ij})^{1/2}r$ (where $\mu_{ij} = m_i m_j / (m_i + m_j)$ is a reduced mass) into the Schrödinger equation, masses of the states with some orbital momentum $l$ and the radial quantum number $n$ are given by [1][3]:

$$M_{nl}^{ij}(q_i, q_{\bar{j}}) = m_i + m_j - \frac{g_{ij}}{2} \ln(2\mu_{ij}R_{ij}^2g_{ij}) + g_{ij}E_{nl},$$  \hspace{1cm} (2)

where the eigenvalues $E_{nl}$ depend only on $n$ and $l$ and not on the other parameters — namely, the masses of constituents. Their numeric [1] and also approximate analytical expressions [4] are known but we will not need them now.

Of course one can take $g_{ij}$ and $R_{ij}$ flavour-dependent in order to fine-tune phenomenological predictions but hypothesis of flavour-independence, as in the Martin potential case, is more attractive. It was flavour independence that served as an argument for initial consideration of the Logarithmic Potential [3] because of approximate equality of $\psi' - \psi$ and $\Upsilon' - \Upsilon$ splittings. It is clear from the very start that absolute flavour independence is not the case here, but existing difference can be attributed to the relativistic corrections.

In the case of flavour independence there arise number of mass relations. The one that interests us is the following:

$$M_{n_2}^{l_2}(Q\bar{q}) - M_{n_1}^{l_1}(Q\bar{q}) = M_{n_2}^{l_2}(Q\bar{Q}) - M_{n_1}^{l_1}(Q\bar{Q}).$$  \hspace{1cm} (3)

This relation is symmetric under interchange of $Q$ and $q$ i.e. together with (3) we have:

$$M_{n_2}^{l_2}(q\bar{q}) - M_{n_1}^{l_1}(q\bar{q}) = M_{n_2}^{l_2}(Q\bar{Q}) - M_{n_1}^{l_1}(Q\bar{Q}).$$  \hspace{1cm} (4)

This relation is critical for estimation of flavour independence — it gets less precise the lighter the quarks are. It is not surprising because the relativistic effects are more important for light quarks.

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2 As long as we neglect particle spins, and the mass relations are written in the center of mass of multiplets, we deal only with the spin-independent relativistic corrections.
As for the relation (3), one can expect that the relativistic corrections are of the same order for both sides: though in the left-hand-side one of the quarks is light, but the system is atom-like and so the light quark is predominantly located at large ”distances” [4] and it can lead to effective damping of the matrix elements of the perturbation Hamiltonian.

The above considerations sound quite speculative until they are not checked in actual calculations, but the underlying physical picture seems quite realistic.

Formula (2) leads to the rule for the level shifts in the different families without making assumption of universality of constants [3]:

\[
\frac{M_{n_1}^{l_1}(Q\bar{Q}) - M_{n_2}^{l_2}(Q\bar{Q})}{M_{n_1}^{l_1}(Q\bar{Q}) - M_{n_2}^{l_2}(Q\bar{Q})} = \frac{M_{n_1}^{l_1}(q\bar{q}) - M_{n_2}^{l_2}(q\bar{q})}{M_{n_1}^{l_1}(q\bar{q}) - M_{n_2}^{l_2}(q\bar{q})}
\] (5)

These rules can be used to relate mass differences in the light and heavy families.

Below we will consider some consequences for the examples where more or less experimental data is available for the states with \( n = 1 \) and \( l = 0, 1 \).

1. \((s\bar{q})\) system \((q \equiv u \text{ or } d)\):

From (3) we have:

\[
M_1^{l_1}(s\bar{q}; 1^+) = M_0^{l_1}(s\bar{q}; 1^-) + [M_1^{l_1}(s\bar{s}; 1^{++}) - M_1^{l_1}(s\bar{s}; 1^{--})].
\] (6)

Using identification from [6] we get:

\[
M_1^{l_1}(s\bar{q}; 1^+) = K^+(892) + [f_1(1530) - \Phi(1020)] \approx 1402\text{MeV}.
\] (7)

Experimental data for the corresponding multiplet is [6]: \(2^+ K^*_2(1430), 1^+ K^*_1(1400), 0^+ K^*_0(1350)\).

Hence \(M_{cm} = (5K^*_2 + 3K^*_1 + K^*_0)/9 \approx 1411\text{MeV}\).

2. \((c\bar{s})\) system:

In this case in correspondence with (3) we have:

\[
M_1^{l_1}(c\bar{s}; 1^+) = D^*_s(2110) + [\chi_c(3522) - \psi(3097)] \approx 2535\text{MeV}.
\] (8)

It is in good agreement with experimental value 2536MeV [8], and also with the value calculated by Martin [2], 2532MeV.

3. \((c\bar{q})\) system:

The prediction is:

\[
M_1^{l_1}(c\bar{q}; 1^+) = D^*(2010) + [\chi_c(3522) - \psi(3097)] \approx 2435\text{MeV}.
\] (9)

Candidates for the \(1^+\) and \(2^+\) states were found at 2424 and 2459MeV [9], respectively. Our result is in good agreement with them.

4. \((b\bar{q})\) and \((b\bar{s})\) systems:

Corresponding formulae have the form:

\[
M_1^{l_1}(b\bar{q}; 1^+) = B^*(5324.6?) + [\chi_b(9900) - \Upsilon(9460)] \approx 5764\text{MeV}.
\]

\[
M_1^{l_1}(b\bar{s}; 1^+) = B^*_s(5416) + [\chi_b(9900) - \Upsilon(9460)] \approx 5856\text{MeV}.
\] (10)

Unfortunately there is no valid experimental data yet to check these results. Readers well acquainted with the spectroscopy of particles can find other examples of employing (3) too.

What can be said in conclusion about the mass relation (3)? It is designed in the way to comply with the naive quark counting, being at the same time valid for far from trivial type of
potential (1). It is worth noting here that similar mass relations emerge also for the multiquark bound states and lead to fair predictions for the Baryon masses \([10], [11]\).

For multiquark systems it is very convenient to use the method of hyperspheric harmonics \([10]\). In the so-called diagonal approximation this formalism for the logarithmic potential between pairs leads to the following mass formula for N-particle bound state:

\[
m(1, 2, \ldots, N) \equiv m_n^{\lambda(L)} = \sum_{i=1}^{N} m_i + C_{[L][L]} - \frac{1}{2} \sum_{i<j} g_{ij} \ln \left[ \frac{2m_i m_j}{m_i + m_j} R_{ij}^2 \right]
\]

\[
+ (\sum_{k<l} g_{kl}) E_{n\lambda} .
\]

Here \(\lambda = L + 3(N - 2)/2\) and \(L\) is the grand momentum. The usual QCD relation between the quarks’ and quark-antiquark coupling constants \(g_{ij} \rightarrow \frac{1}{2} g_{ij}\) was also exploited in (11).

The eigenvalues \(E_{n\lambda}\) are also mass and coupling \(g_{ij}\) independent. If the constituents have different masses then the coefficients \(C_{[L][L]}\) depend on the masses and so the mass-dependence arises in the orbital (but not radial) excitations.

We do not have mass-dependence in the \(L = 0\) case when:

\[
C_{[L=0][L=0]} = \frac{1}{2} \sum_{k<l} g_{kl} \left\{ \Psi\left(\frac{3}{2}\right) - \Psi\left(\frac{3(N - 1)}{2}\right) \right\}
\]

where \(\Psi(z)\) is the logarithmic derivative of the Euler’s \(\Gamma\)-function.

Using these formulae we can obtain relations between spin averaged masses. They have the naive quark counting form. E.g. \([11]\):

\[
M(\bar{s}s) - M(\bar{c}c) = 2[M(ssc) - M(ccs)] = \frac{2}{3} [M(sss) - M(ccc)] ,
\]

\[
M(\bar{c}s) - M(\bar{s}s) = M(ssc) - M(sss) ,
\]

\[
M(\bar{c}s) - M(\bar{c}c) = M(ssc) - M(ccc) etc.
\]

These are the meson-baryon mass formulae.

One more relation for light quarks is the following \([11]\):

\[
M(\bar{q}c) - M(\bar{q}s) = M(qqc) - M(qqs) .
\]

If we take into account also the spin interactions we’ll get sum rules which fit with experiments quite well:

\[
M_{\Lambda_c} = M_{\Lambda} + \frac{1}{4} [M_D - M_K + 3(M_{D^*} - M_{K^*})] \sim 2295\text{MeV}(exp. 2285\text{MeV})
\]

\[
M_{\Sigma_c} = M_{\Sigma} + \frac{1}{12} [7(M_D - M_K) + 5(M_{D^*} - M_{K^*})] \sim 2448\text{MeV}(exp. 2455\text{MeV})
\]

So we see that the logarithmic potential model is a good generalization of the naive quark model and is quite simple in use.
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