Hidden Symmetries for Ellipsoid–Solitonic Deformations of Kerr–Sen Black Holes and Quantum Anomalies

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Abstract

We prove the existence of hidden symmetries in the general relativity theory defined by exact solutions with generic off–diagonal metrics, nonholonomic (non–integrable) constraints, and deformations of the frame and linear connection structure. A special role in characterization of such spacetimes is played by the corresponding nonholonomic generalizations of Stackel–Killing and Killing–Yano tensors. There are constructed new classes of black hole solutions and studied hidden symmetries for ellipsoidal and/or solitonic deformations of ”prime” Kerr–Sen black holes into ”target” off–diagonal metrics. In general, the classical conserved quantities (integrable and not–integrable) do not transfer to the quantized systems and produce quantum gravitational anomalies. We prove that such anomalies can be eliminated via corresponding nonholonomic deformations of fundamental geometric objects (connections and corresponding Riemannian and Ricci tensors) and by frame transforms.

Keywords: Hidden symmetries in general relativity, gravitational anomalies, nonholonomic deformations, generalized Stackel–Killing and Killing–Yano tensors.

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1 Introduction

A very important task in the study of classical and quantum gravitational and matter field interactions is to determine the corresponding symmetries of dynamical systems and to identify the constants of motion and related conservation laws. In general, the evolution of a dynamical system is described in the phase-space when nonholonomic (equivalently, anholonomic and/or non–integrable) constraints are imposed on dynamical variables. Various methods of nonholonomic geometry are considered in classical Lagrange/Hamilton mechanics, in quantum field theory of gauge fields and, for instance, in the Dirac approach to perturbative quantum gravity.

It is natural to search of conserved quantities corresponding to nonholonomically deformed and/or hidden symmetries of the complete phase–space, not just for the configuration one. For instance, a fundamental hidden symmetry for rotating black hole spacetimes with spherical horizon topology and Taub–NUT solutions is the Killing–Yano (KY) one \(^1\), see details in references in [2, 3, 4]. There were considered various types of generalized and associated hidden symmetries. A large class of symmetries is characterized by higher rank symmetric Stackel–Killing (SK) tensors which generalize the Killing vectors. \(^1\) There were considered also antisymmetric KY tensors, corresponding conformal extensions (CSK and CKY) etc, see reviews of results and applications in [5, 6, 7].

Quite generally, this paper is connected with three recent directions in modern string gravity, supergravity and applications of Ricci flow theory in physics. The first one is with extensions of the KY symmetry in the presence of skew–symmetric torsion [8, 9, 10, 11]. The second one was proposed in two papers on anholonomic frames, generalized Killing equations and Dirac operators considered on anisotropic Taub NUT spinning spaces [12, 13]. The third direction is related to some geometric methods in the geometry of nonholonomic Ricci flows and constructing wave type solutions for Einstein and/or Finsler spaces [14, 15].

In our recent works, there were proved two important results: 1) Using the anholonomic deformation method [14, 15], it is possible to decouple (equivalently, separate) the Einstein equations and generate exact solutions in very general forms. 2) Working with nonholonomic deformations of the Levi–Civita connection to certain auxiliary connections (also uniquely de-

\(^1\)Such higher order symmetries are called ”hidden symmetries” and the corresponding holonomic (un–constrained) values are quadratic, or (in general) polynomial in momenta. In this work, we study more general gravitational nonlinear systems with non–integrable constraints and ”nonholonomic hidden symmetries”. 

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fined by the metric and/or corresponding almost symplectic structure), the
Einstein gravity and generalizations were quantized in certain equivalent
forms using deformation quantization, A–brane formalism and two connec-
tion renormalization [16, 17]. Such constructions are characterized by special
types of symmetries derived for the corresponding class of admissible non-
holonomic deformations of geometric and physical objects in classical and
quantum gravity. This motivates our study of associated hidden symmetries
determined by nonholonomic deformations of KY and SK tensors.

The nonholonomic geometries with induced (by metric structure) tors-
sions have some similarities, for instance, with possible extensions (black
hole spacetimes, various supergravity theories etc) of the KY symmetry in
the presence of skew–symmetric torsion [8, 9, 10, 20, 21, 22, 23]. They
also admit generalized KY tensors with corrections which became trivial if
torsion fields vanish. In order to study the existing common features and
differences of spacetimes with “generic” torsions and induced (nonholonomi-
cally) torsions (which are equivalent to the field equations in GR), we provide
explicit constructions and analyze the basic properties of nonholonomic KY
tensors and theirs CKY extensions.

The results on exact off–diagonal solutions and quantum gravitational
field theories with nontrivial nonholonomic constraints give rise to two nat-
ural questions: 1) wether the nonholonomic KY and/or SK symmetries are
relied, or not, to certain fundamental properties of vacuum and non–vacuum
gravitational, and gravitational–matter fields interactions and 2) if there
are some interesting and physically important spacetimes characterized by
such nonholonomic symmetries? One of the aims of this work is to show
that nonholonomic hidden symmetries exist naturally for any generic off–
diagonal spacetime and associated nonholonomic frame structures. We shall
provide explicit examples of exact solutions (for black ellipsoids, nonholo-
nomic deformations and gravitational and solitons) and construct/study the
corresponding nonholonomic generalization of KY and SK tensors.

The above mentioned types of nonholonomic hidden symmetries are fun-
damental ones characterizing generic off–diagonal and non–integrable classical
gravitational interactions. Passing to quantized systems it is necessary to
investigate if such symmetries “survive”, or not, and to perform a rigorous
study of the corresponding possible conserved quantities and separability
of the field/motion equations can be preserved. It is well known that in
the cases of hidden symmetries for the Levi–Civita connection there are
anomalies representing discrepancies between the conservation laws the the
classical level and the corresponding ones at the quantum level [21, 25].
One of main goals of this work is to prove that certain classes of anomalies
can be eliminated via corresponding nonholonomic deformations and frame transforms. Such nonholonomic hidden symmetries exist for adapted distinguished connections, they are not violated under certain quantization schemes, being determined completely by the components of the metric tensor. The constructions can be redefined equivalently in terms of the ”standard” Levi–Civita connection but the corresponding formulas are quite sophisticate and less obvious then the nonholonomic ones if there are considered general nonlinear classical and quantum gravitational systems.

The plan of the paper is as follows: In section 2 we outline the anholonomic deformation method of constructing generic off–diagonal solutions in gravity theories. We show how the Einstein equations can be reformulated equivalently in nonholonomic variables which allows a ”magic splitting” into subsystems which can be integrated in very general forms. In section 3 the hidden symmetries are studied for explicit examples of exact solutions described by generic off–diagonal metrics. There are considered the nonholonomic symmetries for Kerr–Sen black ellipsoid/hole configurations and their solitonic deformations.

In section 4 we prove that nonholonomic hidden symmetries for the Einstein gravity and certain metric compatible generalizations survive under transitions from classical to quantum spacetimes, which is very different for the case of holonomic analogs with the Levi–Civita connection. We conclude that the gravitational anomalies can be canceled via nonholonomic deformations for very general nonlinear gravitational systems. Finally, section 5 is devoted to conclusions. Some necessary coefficient formulas are presented in Appendix.

## 2 Nonholonomic Variables and Exact Solutions

In general relativity theory (in brief, GR) the geometry nonlinear and ”non–integrable” interactions of particles and fields on a (pseudo) Riemannian spacetime \( V, \dim V = 4 \), (endowed with a metric tensor \( g \) of signature \((-1, 1, 1, 1)\)), is encoded into certain structures of moving frames (equivalently, tetrads/ vierbeins) \( e_\alpha = \gamma_\alpha(u) \partial/\partial u^\alpha \) and nonholonomic constraints \( e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma \). The anholonomy coefficients \( W^\gamma_{\alpha\beta}(u) \) vanish for holonomic, i. e. integrable, configurations. We shall follow such conventions: the small Greek indices \( \alpha, \beta, .. \) can be abstract or coordinate ones (we shall use also primed/underlined etc indices for distinguishing different types of (non)holonomic frame decompositions). Indices may be underlined, \( \underline{\alpha}, \underline{\beta}, .. \) in order to emphasize that they are coordinate ones, running values
\( \alpha, \beta \ldots = 1, 2, 3, 4 \); we shall omit, for simplicity, any priming/underlying etc if that will not result in ambiguities. On convenience, we shall consider also "permutations" of signatures with metrics locally parametrized in the form \((1, -1, 1, 1), (1, 1, -1, 1), \text{or} (1, 1, 1, -1)\). The local coordinates of a point \( u \in V \) are labeled \( u^\beta \), partial derivatives are \( \partial_\beta := \partial / \partial u^\beta \) and functions are written as \( f(u) \equiv f(u^\alpha) \).

Under frame transforms, the coefficients of a metric \( g = g_{\alpha\beta} e^\alpha \otimes e^\beta \), for a dual \( e^\alpha \), for \( e^\alpha \mid e^\beta = \delta^\alpha_\beta \) (where the 'hook' operator \( \mid \) corresponds to the inner derivative and \( \delta^\alpha_\beta \) is the Kronecker symbol) are re-defined following the rule

\[
g_{\alpha\beta} = e^\alpha_\alpha e^\beta_\beta g_{\alpha\beta}. \tag{1} \]

On a spacetime manifold \( V \), we can consider various types of nonholonomic structures parametrized by some sets of coefficients \( \{ e^{\alpha}_{\alpha} : \partial \alpha \rightarrow e_{\alpha} \} \).

### 2.1 Manifolds with associated N–connections

In a general context, we can use the definition of nonholonomic manifold \( V = (V, N) \), where \( V \) is a pseudo–Riemannian spacetime and \( N \) is a nonholonomic distribution.\(^2\) In this paper, we shall consider distributions defining a 2 + 2 splitting, i.e. a non–integrable fibration, with associated nonlinear connection (N–connection) structure \( N = N \) satisfying the properties:

1. A N–connection is introduced as a Whitney sum \( N : TV = hV \oplus vV \) defining a conventional horizontal (h) and vertical (v) splitting.

2. N–adapted parametrizations of local coordinates and frames and respective tensor indices on \( V \) are considered for \( u^a = (x^i, y^a) \), where h–indices take values \( i, j, ... = 1, 2 \) and v–indices take values \( a, b, ... = 3, 4 \), for \( a = i + 1 \) and \( b = j + 2 \) respectively contracted with \( i \) and \( j \). This introduces a local fibred structure when the coefficients of N–connection, \( N^a_i \), for \( N = N^a_i (u) dx^i \otimes \partial / \partial y^a \), define N–adapted /–elongated local bases (partial derivatives), \( e_\nu = (e_i, e_a) \), and cobases

\(^2\)Such a distribution can be stated by an arbitrary function (or a set of functions) on \( V \) prescribing a vierbein structure \( e^a_\alpha \) following certain geometric principles. In modern gravity, it is largely used the so–called ADM (Arnowit–Deser–Misner) splitting, 3+1, see details in \[19\]. For our purposes, it is convenient to work with an alternative non–integrable 2+2 splitting, which allows us to decouple the Einstein equations and integrate them in "very" general forms \[14\] \[15\]. Such a technique of generating exact solutions can not be elaborated working only with 3+1 decompositions.
(differentials), \(e^\mu = (e^t, e^a)\), when
\[
e_t = \frac{\partial}{\partial x^t} - N^a_i \frac{\partial}{\partial y^a}, \quad e_a = \frac{\partial}{\partial y^a},
\]
and
\[
e^i = dx^i, \quad e^a = dy^a + N^a_i dx^i.
\]
We shall use boldface symbols for spaces enabled with N–connection structure.

3. In general a metric structure
\[
g = g_{\alpha\beta} (u) \, du^\alpha \otimes du^\beta
\]
on \(V\) is generic off–diagonal, i.e. it can not be diagonalized via coordinate transforms, and can be always parametrized (up to certain classes of frame/coordinate transforms of type (1)) as
\[
g_{\alpha\beta} (u) = \begin{bmatrix}
g_{ij} + h_{ab} N^a_i N^b_j & h_{ac} N^c_i \\
h_{bc} N^c_i & h_{ab}
\end{bmatrix}.
\]
Equivalently, such a metric can be represented in N–adapted form (as a distinguished metric/tensor, d–metric, d–tensor)
\[
g = g_{ij} (x, y) \, e^i \otimes e^j + g_{ab} (x, y) e^a \otimes e^b.
\]
A parametrization of type (5) should be not confused with similar ones in the Kaluza–Klein theory when \(y^a\) are considered as extra dimension coordinates on which, for instance, a "cylindrical" compactification is performed and \(N^c_j (x, y) \sim A^e_{a j} (x^k) y^a\) are certain (non) Abelian gauge fields. In our approach, \(N^c_i\) are with arbitrary nonlinear dependence on \(y^a \rightarrow y^a\) and state naturally certain frame coefficients \(e^\alpha_\alpha\) with 2+2 splitting and linear elongations on \(N^a_i\) in \(e\) (2) and \(e^\mu\) (3). This subclass of anholonomic frames is subjected to conditions
\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma,
\]
when the (antisymmetric) nontrivial anholonomy coefficients \(W^b_{ia} = \partial_a N^b_i\) and \(W^a_{ji} = \Omega^a_{ij}\) are determined respectively by partial \(v\)–derivatives of \(N^b_i\) and the coefficients of curvature of N–connection \(\Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j)\). Having prescribed a N–connection 2+2 splitting, we can say that our (pseudo) Riemannian spacetime is modelled as N–anholonomic manifold (a similar situation exists for the ADM formalism when the role of N–coefficients is played by the so–called "shift" and "lapse" functions).
2.2 Nonholonomic deformations of linear connections

From the class of linear connections which can be defined on a manifold $V$, we can select a subclass which is adapted to the $N$–connection structure.

Definition 2.1 A distinguished connection, $d$–connection, $D = (hD, vD)$ on a $N$–anholonomic $V$ is defined as a linear connection preserving under parallelism the $N$–connection structure.

The $N$–adapted components $\Gamma^\alpha_{\beta\gamma}$ of a $d$–connection $D$ are computed following equations $D^\alpha e^\beta = \Gamma^\gamma_{\alpha\beta} e^\gamma$ and parametrized in the form $\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bc})$, where $D^\alpha = (D_i, D_a)$, with $hD = (L^i_{jk}, L^a_{bc})$ and $vD = (C^i_{je}, C^a_{bc})$ determining covariant, respectively, $h$– and $v$–derivatives. We can associate a differential 1–form $\Gamma^\alpha_{\beta\gamma} e^\gamma$ and perform a $N$–adapted differential form calculus.

The torsion $T^\alpha = \{T^\alpha_{\beta\gamma}\}$ and curvature $R^\alpha_{\beta} = \{R^\alpha_{\beta\gamma\delta}\}$ of $D$ are computed respectively

$$D^r T^\alpha := De^\alpha = de^\alpha + \Gamma^\alpha_{\beta\gamma} e^\gamma$$
and
$$D^r R^\alpha_{\beta} := D\Gamma^\alpha_{\beta} = d\Gamma^\alpha_{\beta} - \Gamma^\gamma_{\beta\delta} \wedge \Gamma^\alpha_{\gamma} = R^\alpha_{\beta\gamma\delta} e^\gamma \wedge e^\delta,$$

see Refs. [15] for explicit calculi of coefficients (we provide some necessary component formulas in Appendix).

For any metric $g$ (equivalently, $d$–metric $\bar{g}$), we can introduce the Levi–Civita (LC) connection $\nabla = \{\bar{\Gamma}^\alpha_{\beta\gamma}\}$ as the unique one satisfying the metric compatibility condition, $\nabla g = 0$, and zero torsion condition, $\nabla T^\alpha := 0$. Such a linear connection is not a $d$–connection because it does not preserve under frame/coordinate/parallel transforms a prescribed $N$–connection splitting. Nevertheless, it is possible to construct decompositions of type $g \nabla = gD + gZ$, for $gD = 0$, when both the linear connection $g \nabla$ and metric compatible $d$–connection $gD$ and the distortion tensor $gZ$ are defined in a unique form, following well defined geometric principles, by a metric tensor $g$ when a $N$–connection structure $N$ is prescribed on a $N$–anholonomic spacetime $V$.

Theorem 2.1 –Definition. There is a canonical $d$–connection $\hat{D}$ completely and uniquely defined by a (pseudo) Riemannian metric $g$ [15] for a

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3We shall use certain left "up" or "low" labels in order to emphasize that certain geometric objects are determined by another fundamental geometric object, for instance, that the torsion $D^r T^\alpha$ is determined by $d$–connection $D$. We shall omit such labels if that will not result in ambiguities.
chosen $N = \{N^a_i\}$ if and only if $\hat{D}g = 0$ and the horizontal and vertical torsions are zero, i.e. $h\hat{T} = \{T^i_{jk}\} = 0$ and $v\hat{T} = \{T^a_{bc}\} = 0$.

**Proof.** It follows from explicit constructions provided in Appendix, see formulas (A.2). □

We have the canonical distortion relation
\[ \nabla = \hat{D} + \hat{Z}, \]  
(10)
when both linear connections $\nabla = \{\hat{\Gamma}^\alpha_{\beta\gamma}\}$ and $\hat{D} = \{\hat{\Gamma}^\gamma_{\alpha\beta}\}$ and the distorting tensor $\hat{Z} = \{\hat{Z}^\gamma_{\alpha\beta}\}$ are uniquely defined by the same metric tensor $g$ (see coefficient formulas (A.1) and (A.4); for simplicity, we omitted the left label ”$g$”). The connection $\hat{D}$ is with nontrivial torsion (in general, the coefficients $\hat{T}^i_{ja}, \hat{T}^a_{ji}$ and $\hat{T}^a_{bi}$ are not zero, see (A.3)).

**Definition 2.2** The torsion $\hat{T} = \{\hat{T}^\gamma_{\alpha\beta}\}$ of $\hat{D}$, completely defined by data $(g, N)$ is called the canonical $d$–torsion of a $N$–anholonomic (pseudo) Riemannian manifold $V$.

Such a torsion is nonholonomically induced by $N$–connection coefficients and completely determined by certain off–diagonal $N$–terms in (5). The GR theory can be formulated equivalently using the connection $\nabla$ and/or $\hat{D}$ if the distorting relation (10) is used.

### 2.3 The Einstein equations in $N$–adapted variables

The Ricci $d$–tensor $Ric = \{R_{\alpha\beta}\}$ of a $d$–connection $D$ is constructed using a respective contracting of coefficients of the curvature tensor (9), $R_{\alpha\beta} \doteq R^\gamma_{\alpha\beta\gamma}$. The $h$–/ $v$–components of this $d$–tensor
\[ R_{\alpha\beta} = \{R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ai} \doteq R^b_{ai}b, \quad R_{ab} \doteq R^c_{abc}\}, \]  
(11)
see explicit coefficients formulas in Refs. [15]. The scalar curvature of $D$ is constructed by using the inverse $d$–metric to $g$ (6), $R \doteq g^{\alpha\beta}R_{\alpha\beta} = g^{ij}R_{ij} + h^{ab}R_{ab}$, with $R = g^{ij}R_{ij}$ and $S = h^{ab}R_{ab}$ being respectively the $h$– and $v$–components of scalar curvature.

The Einstein equations for a metric $g_{\beta\delta}$ are written in standard form using the LC–connection $\nabla$ (and various tetradic, spinor, 3+1 splitting etc representations). Using the distortion relations of (10), we can compute the respective distortions of the Ricci tensor and of the scalar curvature.
and consider such values as "effective" sources additionally to the energy–momentum tensor for matter fields, i.e. to $\mathcal{z}T_{\beta\delta}$, where $\mathcal{z}$ is defined by the gravitational constant. If $\mathcal{g}\nabla = \mathcal{g}\mathbf{D} + \mathcal{g}\mathbf{Z}$, we do not need additional field equations (for torsion fields) like in the Einstein–Cartan, gauge or string gravity theories.

**Theorem 2.2** The Einstein equations in GR can be rewritten equivalently for a nonholonomic 2+2 splitting and respective canonical d–connection $\hat{\mathbf{D}}$,

\[
\hat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathcal{g}_{\beta\delta} \mathcal{g}R = \Upsilon_{\beta\delta},
\]

\[
\hat{L}_{\alpha}^{\gamma} \equiv e_{\alpha}(N_{\gamma}), \quad \hat{C}_{ij}^{\gamma} = 0, \quad \Omega_{ji} = 0,
\]

where the source $\Upsilon_{\beta\delta}$ is such way constructed that $\Upsilon_{\beta\delta} \rightarrow \kappa T_{\beta\delta}$ for $\hat{\mathbf{D}} \rightarrow \nabla$.

**Proof.** If the constraints (13) are satisfied the tensors $\hat{\Gamma}_{\alpha\beta}^{\gamma}$ (A.3) and $Z_{\alpha\beta}^{\gamma}$ (A.4) are zero. This states that $\hat{\Gamma}_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$, with respect to N–adapted frames, see (A.1), even, in general, $\hat{\mathbf{D}} \neq \nabla$ (the transformation laws under frame/coordinate transforms of a d–connection and of the LC–connection are different). In such a case, the equations (12) are completely equivalent to the Einstein equations for $\nabla$. □

For various purposes, it is more convenient to work with a d–connection $\mathcal{g}\mathbf{D}$ instead of $\mathcal{g}\nabla$. For instance, We can prescribe such a nonholonomic 2+2 splitting via N–coefficients $\mathcal{N} = \{N_{a}^{i}\}$ when $\mathcal{g}\mathbf{D}$ became a canonical almost symplectic connection, which is important for deformation quantization and/or A–brane quantization, or two–connection renormalization of GR, see [16, 17, 18]. In section 5, we shall show that cancelations of anomalies are possible for $\hat{\mathbf{D}}$ being a solution (12) even, in general, quantum anomalies exist for $\nabla$.

2.4 On generic off–diagonal Einstein spaces

It is possible a "magic" separation (i.e. decoupling; in our works, such words are used as equivalent ones stating some properties of a system of nonlinear partial differential equations) of the Einstein equations (12) for $\hat{\mathbf{D}}$ if certain types of parametrizations for the coefficients of N–connection $\mathcal{N} = \{N_{a}^{i}\}$ of the metric $\mathbf{g}$ (4) (equivalently, d–metric (6)) are considered, see details, proofs and examples in Refs. [14, 15, 12, 13]. This allows us to solve the Einstein equations in very general forms for any general sources.

\footnote{such sources should be defined in explicit form from certain additional suppositions on interactions of gravitational and matter fields; we omit such considerations in this work}
parametrized in the form
\[ \Upsilon_\delta = \text{diag}(\Upsilon_2(x^k, y^3), \Upsilon_2(x^k, y^3), \Upsilon_4(x^k), \Upsilon_4(x^k)), \]
for coordinates \(u^\alpha = (x^k, y^3 = v, y^4)\). In a particular case, we can consider Einstein spaces with \(\Upsilon_2 = \Upsilon_4 = \lambda = \text{const}\) which define off–diagonal Einstein spaces with, in general, nontrivial cosmological constant \(\lambda\).

**Lemma 2.1** Any metric \(g\) with coefficients of necessary smooth class on a (pseudo) Riemannian \(V\) can be represented in \(N\)-adapted form as
\[ g = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^k, v, y^4)h_a(x^k, v)e^a \otimes e^a, \]
\[ e^3 = dy^3 + w_j(x^k, v)dx^j, \quad e^4 = dy^4 + n_j(x^k, v)dx^j. \]

**Proof.** A d–metric of type (15) is of type (6) with \(N^3 = w_j\) and \(N^4 = n_j\). With respect to a chosen coordinate base, we express (15) in the form (4),
\[ g_{\alpha\beta} = \begin{bmatrix} g_1 + \omega^2(w_1^2h_3 + n_1^2h_4) & \omega^2(w_1w_2h_3 + n_1n_2h_4) & \omega^2w_1h_3 & \omega^2n_1h_4 \\
\omega^2(w_1w_2h_3 + n_1n_2h_4) & g_2 + \omega^2(w_2^2h_3 + n_2^2h_4) & \omega^2w_2h_3 & \omega^2n_2h_4 \\
\omega^2w_1h_3 & \omega^2w_2h_3 & \omega^2h_3 & 0 \\
\omega^2n_1h_4 & \omega^2n_2h_4 & 0 & \omega^2h_4 \end{bmatrix}. \]

A general metric \(g_{\alpha\beta}(x^k, y^\alpha)\) on \(V\) can be always parametrized in the form (16) via certain frame transform of type \(g_{\alpha\beta} = \epsilon^\alpha_\alpha \epsilon^\beta_\beta g_{\alpha\beta}\). For certain given values \(g_{\alpha\beta}\) and \(g_{\alpha\beta}\) (in GR, there are 6 + 6 independent components), we have to solve a quadratic algebraic equation in order to determine 16 coefficients \(\epsilon^\alpha_\alpha\), up to a fixed coordinate system. We have to fix such nonholonomic 2+2 splitting when the algebraic equations have real nondegenerate solutions.

Let us introduce brief denotations for partial derivatives: \(a^* = \partial a/\partial x^1\), \(a' = \partial a/\partial x^2\), \(a^* = \partial a/\partial y^3\) and \(a^* = \partial a/\partial y^4\) and consider such parametrizations of (15) when \(h^*_a \neq 0\) (such conditions can be satisfied for some correspondingly chosen systems of coordinates and frame transforms \(\epsilon^\alpha_\alpha\)). Using Lemma 2.1 the solutions of the Einstein equations can be also constructed in very general forms, see details and proofs in [14] [15] (see also Appendix B); we note that in different works there are used some different parametrizations and/or systems of reference/coordinates).

**Theorem 2.3** A general class of solutions of the gravitational field equations (12) for the canonical d–connection \(\widehat{D}\) with a general \(N\)-adapted diagonal source \(\Upsilon_\delta\), with \(\Upsilon_2 \neq 0\), is determined by d–metrics of type
(equivalently, by generic off–diagonal metrics of type (16)) with the coefficients computed in the form

\[ g_i = \epsilon_i e^{\psi(x^k)}, \]

\[ h_4 = \mathcal{H}_4(x^k) \pm \int \left[ \mathcal{Y}_2(x^k, v) \right]^{-1} \left( \exp[2\phi(x^k, v)] \right)^* dv, \quad (17) \]

\[ h_3 = \left[ \left( \sqrt{|h_4(x^k, v)|} \right)^2 \exp[-2\phi(x^k, v)] \right], \]

\[ w_1 = \phi^*/\phi^*, w_2 = \phi'/\phi^*, \]

\[ n_k = 1n_k(x^i) + 2n_k(x^i) \int \left[ h_3/\left( \sqrt{|h_4|} \right)^3 \right] dv \]

determined by an arbitrary generating function \( \phi(x^k, v), \phi^* \neq 0 \), any functions \( \psi(x^k) \) and \( \omega(x^k, y^a) \) satisfying the conditions

\[ \epsilon_1 \psi^{**} + \epsilon_2 \psi''' = \mathcal{Y}_4, \]

\[ e_k \omega = \partial_k \omega + w_k \omega^* + n_k \omega^* = 0, \quad (18) \]

and integration functions \( \mathcal{H}_4(x^k), \ 1n_k(x^i), \ 2n_k(x^i) \) and \( \mathcal{H}_4(x^k) \) to be determined from certain boundary conditions; in the above formulas \( \epsilon_i = \pm 1 \) and other signs \( \pm \) must be fixed such way to have a fixed necessary signature for the chosen class of solutions.

The solutions defined by data (17) and (18) are very general ones, with generic off–diagonal metrics when theirs coefficients depend on all coordinates \( u^\alpha = (x^k, y^a) \). They may be of arbitrary smooth class and with possible singularities, nontrivial topological configurations which depends on the type of prescribed symmetries (singularities etc) for generating and integration functions. For such generic nonlinear systems, we can not state in general form any uniqueness black hole theorems etc. Certain physical interpretation for some classes of solutions can be provided only for special classes of parametrization, see examples in section 4.

We can prescribe such nonholonomic distributions with conventional 2+2 splitting when the general solutions from Theorem 2.3 are restricted to generate generic off–diagonal solutions in GR.

**Corollary 2.1** A d–metric (15) with coefficients (17) and (18) define exact solutions of the Einstein equations for the LC–connection \( \nabla \) if the generating and integration functions are constrained to satisfy the conditions

\[ w_i^* = e_i \ln |h_4|, e_k w_i = e_i w_k, n_i^* = 0, \partial_i n_k = \partial_k n_i. \quad (19) \]
Proof. By straightforward computations we can verify that if the constraints (19) are solved the conditions (13) are satisfied, i.e. the torsion (A.3) of $D$ is zero. □

We note that if $\omega = \text{const}$ the generating solutions are with Killing symmetry because the ansatz for metrics does not depend on variable $y^4$. Such subclasses of off–diagonal solutions are also very general ones.

Finally, it should be emphasized that a similar theorem can be formulated for vacuum solutions with $\Upsilon^\alpha_\delta = 0$, see details in Refs. [14, 15]. A limit $\Upsilon^\alpha_\delta \to 0$ may be a not smooth one on $\lambda$ for such generic nonlinear solutions. For simplicity, in this work we shall consider only Einstein spaces with nontrivial cosmological constant $\lambda$, $\Upsilon^\alpha_\delta \to \lambda \delta^\alpha_\delta$. The conditions (19) have nontrivial solutions, see Corollary B.2.

3 Hidden Symmetries for Generic Off–Diagonal Solutions

Following the conditions of Theorem 2.3 the Einstein equations can be solved in very general forms if the coefficients of a generic off–diagonal metric are defined by some data (17), (18) and (19). In this section, we construct in explicit form some classes of solutions describing nonholonomic deformations of the metrics for Kerr–Sen black holes, analyze possible solitonic deformations and study the associated hidden nonholonomic symmetries. Finally we shall study the hidden symmetries of nonholonomic deformations of Kerr–Sen metrics.

3.1 Nonholonomic analogous Kerr–Sen black holes in GR

Let us consider an ansatz for a "primary" metric

$$^0g = e^\Phi (b^2) \left( S^{-1}dr \otimes dr + d\theta \otimes d\theta \right) + e^\Phi (b^2) \left( a^2 \sin^2 \theta e^3 \otimes e^3 - S e^4 \otimes e^4 \right),$$

where $e^3 = \delta t - \frac{r^2 - 2(M - b)r + a^2}{a^2} \delta \varphi, e^4 = \delta t - a \sin^2 \theta \delta \varphi$, $\delta \varphi = d\varphi + w_i(u)dx^i$, $\delta t = dt + n_i(u)dx^i$. 

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with local coordinates $u^\alpha = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$, for some given functions and constants

\[ \Phi = \Phi(x^i), w_k = w_k(u^\beta), n_k = n_k(u^\beta), \rho^2 = r^2 + a^2 \cos^2 \theta, \]

\[ b = \rho^2(x^i) + 2br, S(r) = r^2 - 2(M - b)r + a^2, \]

\[ M = \text{const}, a = \text{const}, b = \text{const}. \]

In general, such a metric (20) is not a solution of the Einstein equations. If we chose a respective class of coefficients for the metric ansatz and matter fields,

\[ w_i = n_i = 0, \Phi = 2 \ln(\rho/b \rho), A = -rQ(b \rho)^{-2}(dt - a \sin^2 \theta d\varphi), \]

\[ H = 2ab(b \rho)^{-4}d\varphi \wedge dt \wedge [(r^2 - a^2 \cos^2 \theta) \sin^2 \theta dr - r \circ S \sin 2\theta dt], \]

we generate the Kerr–Sen black hole solutions in the low–energy string theory given by the effective action

\[ I = \int \sqrt{\circ g} d^4u e^\Phi \left( \circ R - \frac{1}{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} + \circ g^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi - \frac{1}{8} F_{\alpha \beta} F^{\alpha \beta} \right), \]

where $\circ R$ is the Ricci scalar determined by the LC–connection $\circ \nabla$ of $\circ g_{\alpha \beta}$, i.e. the metric in the string frame; $\Phi$ is the dilaton field; $F = dA$ is the Maxwell field and $H = dB - \frac{1}{4} A \wedge dA$ is the 3–form for the skew–symmetric torsion field $H_{\alpha \beta \gamma}$ determined by an antisymmetric tensor field $B_{\alpha \beta}$ in string gravity (see details in [31, 32, 33, 11]). The data (21) describe a black hole with mass $M$ and angular momentum $J = Ma$, when a nontrivial charge $Q$ induces a magnetic dipole momentum $\mu = QA$. When the twist parameter $b = Q^2/2M$ is zero, we get the Kerr black hole known in GR.

In the so called Einstein frame metric when $E g = e^{-\Phi} \circ g$, the solutions present certain fundamental string modifications of the Kerr geometry and possess some inherit properties and hidden symmetries.

Our goal is to construct a new class of solutions in GR which will generalize the Kerr metric as certain nonholonomic Kerr–Sen spacetimes when some nontrivial N–connection coefficients $w_k = w_k(u^\beta)$ and $n_k = n_k(u^\beta)$ approximate off–diagonal gravitational contributions of metrics which are similar to $H$ and $\Phi$ fields. Namely, we shall demonstrate that such solutions possess hidden nonholonomic symmetries with canonical d–torsion occurring in the theory for more general classes of frame deformations but vanishing if the nonholonomic structure is correspondingly re–defined. For the reason to understand the physical properties of nonholonomic modifications of Kerr–Sen solutions for GR, compare with their analogs in the string frame and
show how geometric method can be applied for constructing exact solutions in gravity theories, our study will be mainly concentrated on ellipsoidal and solitonic deformation.

The ansatz (20) can be expressed equivalently in a diagonal form

\[ \circ g = \circ g_1 \, dr \otimes dr + \circ g_3 \, d\theta \otimes d\theta + \circ h_3 \, d\varphi \otimes d\varphi + \circ h_4 \, dt \otimes dt, \]  

(23)

for a new system of coordinates

\[ \tilde{u}^{\beta} = (\tilde{x}^1 = \int \sqrt{|0S|}^{-1}(r) \, dr, \tilde{x}^2 = \theta, \tilde{y}^3 = \varphi, \tilde{y}^4 = t), \]  

(24)

where

\[ \circ g_1(r, \theta) = e^\Phi(b \rho)^2, \circ g_2(r, \theta) = 0, \circ h_4(r, \theta) = 1K + 2K, \]  

(25)

\[ \circ h_3(r, \theta) = \beta^2(2K)(1 + 2K/1Kq^2)^2 \]  

for \( \beta = -a \sin^2 \theta, q = -(r^2 + 2br + a^2)/a, \)

\[ 1K = -0^2(\rho)^2, 2K = a^2 \sin^2 \theta(\rho)^2 - e^\Phi. \]

At the next step, we consider a nonholonomic deformation of the above metric

\[ \eta g = \eta g_i = 1 + \chi_i(x^k), \eta h_a = 1 + \chi_a(x^k, \varphi) \]  

(27)

and off–diagonal (N–connection) coefficients \( \tilde{w}_i(x^k, \varphi) \) and \( \tilde{n}_i(x^k, \varphi) \), when

\[ d\varphi \rightarrow e^3 = d\varphi + \tilde{w}_i dx^i, \quad dt \rightarrow e^4 = dt + \tilde{n}_i dx^i. \]

Such a ”target” d–metric \( \eta g \) is supposed to be generic off–diagonal of type (15) (equivalently (16)), with respective coefficients chosen in the forms (17) and (18), and must define exact solutions of the Einstein equations for the canonical d–connection and, for the corresponding restrictions, for the LC–connection, when the source \( \Upsilon^\alpha_\delta \rightarrow \lambda \delta^\alpha_\delta \), see formula (14) and the conditions stated in Lemma 2.1, Theorem 2.3 and Corollary 2.1.

For some classes of nonholonomic deformations, we can consider that some \( \chi_\alpha \) in polarizations (27) are small values, \( |\chi_\alpha| < 1 \), which allows us to provide certain physical interpretation of such classes of solutions to be very similar to that for the Kerr–Sen metrics but (in our case) in GR, with some off–diagonal modifications, let say, with ellipsoid and/or solitonic symmetries.

Using the results provided in Corollary B.2 and Remark B.1, we prove
Theorem 3.1 The set of stationary solutions for (26) defining nonholonomic deformations of a originating from string gravity d–metric (20) into generic off–diagonal metrics for Einstein spacetimes in GR are parametrized by gravitational polarizations

\[ \eta_i = 1 + \chi_i = e^{\psi(\tilde{x}^i)}, \eta_i = 1 + \chi_a \]

\[ w_i = (\phi^*)^{-1} \frac{\partial \phi}{\partial \tilde{x}^i}, n_k = 1 n_k(\tilde{x}^i), \]

generated by a function \( \chi_4(\tilde{x}^i, \varphi) \), for \( \chi_4^4 \neq 0 \), where \( \phi = \ln \sqrt{|\lambda \circ h_4 \chi_4|} \) and

\[ \chi_3 = -1 + (\lambda \circ h_3)^{-1} \left[ (\ln \sqrt{|1 - \chi_4|})^2 \right] \]

with \( \psi(\tilde{x}^i) \) being a solution of

\[ \left[ \frac{\partial^2}{\partial \tilde{x}_i^2} + \frac{\partial^2}{\partial \tilde{x}_j^2} \right] [\psi(\tilde{\Phi} + 2 \ln \ |b \rho|)] = \lambda \] and \( \partial(1 n_k)/\partial \tilde{x}^i = \partial(1 n_i)/\partial \tilde{x}^k \).

The set of solutions (28) is defined for any \( \lambda \neq 0 \) (in a similar form, it is possible to generate vacuum solutions, see details in [14, 15]; the lengths of this paper does not allow us to consider such metrics).

3.2 Ellipsoidal deformations and gravitational solitons

3.2.1 Rotoid deformations

We can chose a class of nonholonomic deformations when data (28) define a black ellipsoid solution in GR (such solutions were studied also in (non) commutative and/or string/brane models of gravity; they seem to be stable and, for Einstein configurations, do not violate the conditions of black hole uniqueness theorems, see details in [34 14 15] and references therein).

The generating function (gravitational polarization) is chosen \( \chi_4 = -1 + (\circ h_4)^{-1} (\tilde{q} + \varepsilon\tilde{q}) \), where

\[ q(\tilde{x}^1, \theta, \varphi) = 1 - \frac{2^{1 \mu(r, \theta, \varphi)}}{r}, \quad \tilde{q}(\tilde{x}^1, \theta, \varphi) = \frac{q_0(r)}{4\mu_0^2} \sin(\omega_0 \varphi + \varphi_0). \]

Following Theorem 3.1 there is a class of exact solutions (for any fixed parameter \( \varepsilon \))

\[ r_{\lambda} \tilde{g} = e^{\tilde{\psi}(\tilde{x}^1, \theta)} \circ g_1 (d\tilde{x}^1 \otimes d\tilde{x}^1 + d\theta \otimes d\theta) + \]

\[ \lambda^{-1} \left[ (\ln \sqrt{|2 - q - \varepsilon\tilde{q}|})^2 \right] \delta \varphi \otimes \delta \varphi - (q + \varepsilon\tilde{q}) \delta t \otimes \delta \tilde{t}, \]

\[ \delta \varphi = d\varphi + w_1 d\tilde{x}^1 + w_2 d\theta, \delta t = dt + n_1 d\tilde{x}^1 + n_2 d\theta.\]
Such generic off–diagonal (anisotropic) stationary metrics posses rotoid symmetry with ”ellipsoidal horizon” when the condition of vanishing of the metric coefficient before $\delta t \otimes \delta t$, i.e. $h_4 = 0$, states a parametric elliptic configuration of type $r_+ \simeq 2^{1/2} \mu \left( 1 + \varepsilon \frac{g_4(r)}{\mu^2} \sin(\omega_0 \varphi + \varphi_0) \right)$, for a corresponding $g_4(r)$. The parameter $\varepsilon$ is just the eccentricity for a rotating ellipsoid.

For small values of $\varepsilon \geq 0$, a metric (30) describes nonholonomic black ellipsoid – de Sitter configurations, when the coefficients of metrics contain dependencies on ”primary” values like $^0 g_1$ and $^0 h_4$ in (20) (equivalently, (23)). We argue that such solutions define in GR certain rotoid black hole objects mimicking ”nonholonomically deformed” black holes. The physical properties of such solutions are determined by three sets of data: the ”primary” data for the Kerr–Sen metrics, the ellipsoidal type of nonholonomic deformations stated by values (29) for $\chi_4$, and a nontrivial cosmological constant $\lambda$.

In the limit $\varepsilon \to 0$, we get a subclass of solutions with spherical symmetry but with generic off–diagonal coefficients induced by the N–connection coefficients, i.e. by the corresponding nonholonomic deformations. This class of spacetimes depend on cosmological constants which, in general, are polarized nonholonomically by nonlinear gravitational interactions. We can extract from such configurations the Schwarzschild solution if we select a set of functions with the properties $\phi \to const, w_i \to 0, n_i \to 0$ and $h_4 \to \varpi^2$, where $\varpi^2$ is chosen to determine the horizon of a static black hole. Finally, we emphasize that the parametric dependence on cosmological constants, for such nonholonomic configurations, is not smooth.

### 3.2.2 Kerr–Sen rotoids and solitonic distributions

The solutions for the Kerr–Sen rotoid configurations in GR $^{rot} g$ (30) can be generalized by introducing additional stationary deformations induced by a static three dimensional solitonic distribution $\eta(x^1, \theta, \varphi)$ as a solution of the solitonic equation $^5$

$$\eta^{**} + c(\eta' + 6\eta \eta^* + \eta^{***})^* = 0, \quad \epsilon = \pm 1.$$  \hspace{1cm} (31)

We construct and analyze two different types of solitonic nonholonomic deformations of Kerr–Sen black holes.

$^5$ $\eta$ can be a solution of any three dimensional gravitational solitonic stationary solitonic distribution and/ or other nonlinear wave equations if we construct exact solutions with running in time solitons when are functions of type $\eta(t, \theta, \varphi)$, or $\eta(\xi, \theta, t)$, see [34].
De Sitter type solutions with 3–d solitonic polarizations of masses:
The geometric "target" data are generated by a "vertical" distribution \( \eta(\tilde{x}^1, \theta, \varphi) \) when the coefficients of \( v \)-metric are computed in the form
\[
h_4 = \eta^{**} = (q + \varepsilon \tilde{q}), \quad h_3 = \left[ (\sqrt{|h_4|})^* \right] e^{-2\phi},
\]
where
\[
\phi(\tilde{x}^1, \theta, \varphi) = \frac{1}{2} \ln |\lambda \left[ 0 h_4 + \varepsilon \left( \eta' + 6\eta \eta^* + \eta^{**} \right) \right] |. \tag{32}
\]
By straightforward computations, we can verify that the conditions of Theorem 3.1 with
\[
h_4 = \gamma h_4 \pm \lambda^{-1} e^{2\phi}
\]
are satisfied if and only if \( \eta \) is a solution of (31).

Using formula (29), we can relate the solitonic function \( \eta \) to certain gravitational polarizations of mass, when (for simplicity we can consider \( \varepsilon = 0 \))
\[
\eta^{**}(\tilde{x}^1, \theta, \varphi) = 1 - \frac{2}{r} \frac{\mu(r, \theta, \varphi)}{r}. \tag{33}
\]
Putting together the coefficients, we get the metric
\[
^{rot}_{1sd} g_{\text{rot}} = e^{\psi(\tilde{x}^1, \theta)} \circ g_1 (d\tilde{x}^1 \otimes d\tilde{x}^1 + d\theta \otimes d\theta)
\]
\[
+ \left[ (\sqrt{|\eta^{**}|})^* \right] e^{-2\phi} \delta \phi \otimes \delta \varphi - \eta^{**} \delta t \otimes \delta t,
\]
\[
\delta \varphi = d\varphi + w_1(\tilde{x}^1, \theta, \varphi)d\tilde{x}^1 + w_2(\tilde{x}^1, \theta, \varphi)d\theta,
\]
\[
\delta t = dt + n_1(\tilde{x}^1, \theta)d\tilde{x}^1 + n_2(\tilde{x}^1, \theta)d\theta,
\]
where the N–connection coefficients \( w_i = (\phi^*)^{-1} \frac{\partial \phi}{\partial \tilde{x}^i} \) and \( n_k = n_k(\tilde{x}^i) \) are respectively computed using the generating function \( \phi \) (32) and subjected to the LC–conditions (19).

The class of generic off–diagonal metrics (34) are similar to the rotoid Kerr–Sen configurations in GR (31). For small \( \varepsilon \), such solutions define black ellipsoids with solitonically polarized mass \( \mu(r, \theta, \varphi) = \mu_0 + \varepsilon \mu_1(r, \theta, \varphi) \), following (33). We suppose that it is possible to detect such black rotoid objects via anisotropic polarizations of their masses when certain gravitational solitonic waves act on such generic off–diagonal black ellipsoid solutions.

**Arbitrary solitonic deformations of the \( v \)-components of \( d \)-metrics:**
We construct another class of nonholonomic solitonic transforms from \( ^{rot} g \) (30) to a stationary \( ^{rot}_{2sd} g \) determining stationary metrics for a rotoid in
solitonic backgrounds,

\[ \text{rot}_{\text{2d}} \mathbf{g} = e^{\psi(x^1, \theta)} \circ g_1 (d\bar{x}^1 \otimes d\bar{x}^1 + d\theta \otimes d\theta) + \]

\[ + \left[ \frac{1}{\sqrt{|\eta_1(q + \varepsilon g)|}} \right]^2 \delta \varphi \otimes \delta \varphi - \eta (q + \varepsilon g) \delta t \otimes \delta t, \]

\[ \delta \varphi = d\varphi + w_1(\bar{x}^1, \varphi) d\bar{x}^1 + w_2(\bar{x}^1, \varphi) d\theta, \]

\[ \delta t = dt + n_1(\bar{x}^1, \theta) d\bar{x}^1 + n_2(\bar{x}^1, \theta) d\theta. \]

The conditions of Theorem 3.1 and Remark 4.1 are satisfied for any nontrivial solution \( \eta(\bar{x}^1, \theta, \varphi) \) of the solitonic equation (31) if we chose the generating function

\[ \phi = \frac{1}{2} \ln \lambda \left[ \begin{pmatrix} h_1 & \pm \eta(q + \varepsilon g) \end{pmatrix} \right] \]

for computing the values \( w_i = (\phi^*)^{-1} \frac{\partial \phi}{\partial x^i} \) and \( n_k = 1_{n_k}(\bar{x}^i) \) subjected to the LC-conditions (19).

There is a substantial difference between two classes of solitonic gravitational solutions (31) and (35). In the first case, the 3-d distribution \( \eta(\bar{x}^1, \theta, \varphi) \) is a solitonic one if the Einstein equations are satisfied. In the second class, the distribution may be a solitonic type, or another one, and such off–diagonal metric ansatz defining Einstein spacetime manifolds do not transform the field equations into certain solitonic ones. The generating function \( \phi(\bar{x}^1, \theta, \varphi) \) (35) is such way chosen that \( \text{rot}_{\text{2d}} \mathbf{g} \) (35) describes a rotoid Kerr–Sen configuration \( \text{rot} \mathbf{g} \) (31) imbedded self–consistently into a solitonic background \( \eta(\bar{x}^1, \theta, \varphi) \) (31) in such forms that, for instance, black ellipsoid properties are preserved for various types of nonlinear waves.

### 3.3 Hidden symmetries for nonholonomic Kerr–Sen metrics

The quadratic elements for all classes of solitonic Kerr–Sen rotoid configurations constructed in this section can be represented as nonholonomic deformations (20) of the metric (23) via gravitational polarizations (27),

\[ \delta s^2 = ^0 g_1(r, \theta) (\sqrt{|\eta_1(r, \theta)|} d\bar{x}^1)^2 + ^0 h_a(r, \theta) (\sqrt{|\eta_2(r, \theta)|} e^a)^2, \]

\[ \bar{e}^3 = d\varphi + \bar{w}_1(r, \theta, \varphi) d\bar{x}^1, \quad \bar{e}^4 = d\varphi + \bar{w}_2(r, \theta, \varphi) d\bar{x}^1, \]

for \( \bar{x}^1 = (\bar{x}^1(r, \theta), \bar{y}^a = \bar{y}^a(\varphi, t). \) The coefficients/multiples in this formula are considered in a form when an exact solution for the Kerr–Sen and/or ellipsoidal/solitonic configuration is defined.

Let us introduce the following basis of 1–forms:

\[ \bar{e}^a = [\bar{x}^1 = \rho \sqrt{|\eta_1|} d\bar{x}^1, e^2 = \sqrt{|\eta_2|} \rho d\theta, \quad \bar{e}^3 = (b, \rho)^{-2} \rho a \sin \theta (\sqrt{|\eta_4|} e^4 + \frac{\sqrt{|\eta_3|}}{\sqrt{|\eta_5|}} e^3), \]

\[ \bar{e}^4 = (b, \rho)^{-2} \rho \sqrt{|\eta_5|} (\sqrt{|\eta_4|} e^4 + \beta \sqrt{|\eta_3|} e^3) \]

(37)
when the target metric is written
\[ \eta = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 - e_4 \otimes e_4. \]  
(38)

In general, the nonholonomically induced canonical d–torsion \( \hat{T} \) (A.3) is not completely antisymmetric. Nevertheless, alternatively to this d–torsion, we can define another torsion field (which is also completely determined nonholonomically by the metric structure and parametrized by two functions before absolute anti–symmetric tensors, \( e^{234} \) and \( e^{124} \), with respect to a dual base (37)),

\[ \triangleright T_\pm = -2a \sin \theta \left[ (-b)^{-2}(r+b) \pm r \rho^{-2} \right], \]
\[ \triangleleft T_\pm = -2 \rho^{-1} a (\cos \theta) \sqrt{S} \left[ (-b)^{-2} \pm \rho^{-2} \right], \]

when
\[ H T_\pm := \triangleright T_\pm e^{234} + \triangleleft T_\pm e^{124}. \]  
(39)

The torsions \( H T_\pm \) are similar to the torsions associated to \( H \) studied in Ref. \[11\]. Nevertheless, constructing our solutions we have not involved additional field equations for the matter fields and torsion (we shall write in brief \( H T \) if it will not be not necessary to state exactly with what sign +, or −, we are working for some constructions). There is not a general smooth transform \( H T \rightarrow H \) because our gravitational field equations for target metrics are not derived from an action of type (22). For some classes of nonholonomic constraints on the off–diagonal gravitational field dynamics we can mimic certain types of matter field interactions.

There are two obvious isometries \( \partial_t \) and \( \partial_\varphi \) for a general (non) holonomic Kerr–Sen metric. Such a geometry also admits an irreducible Killing tensor\[6\]
\[ \eta K = a^2 \cos^2 \theta (e^4 e^4 - e^1 e^1) + r^2 (e^2 \otimes e^2 + e^3 \otimes e^3). \]

The first one can be related to the canonical d–torsion \( \hat{T} \) (A.3) and/or auxiliary torsion \( H T \) (39) in such a form when a nonholonomically induced torsion is included naturally into a generalized closed CKY 2–form. We change \( T \rightarrow H T^+ \) and consider \( H D_X \Psi := \nabla_X \Psi + \frac{1}{2} (X \cdot H T^+) \wedge \Psi \), when

\[ H d\Psi = d\Psi - H T^+ \wedge_1 \Psi = e^v \wedge H D_a \Psi = e^i \wedge H D_i \Psi + e^v \wedge H D_a \Psi, \]
\[ H d^* \Psi = d^* \Psi - H T^+ \wedge_2 \Psi = -e_a \wedge H D_a \Psi = -e_i \wedge H D_i \Psi - e_a \wedge H D_a \Psi. \]

\[6\] such a geometric object is responsible for separability of (charged) Hamilton–Jacobi equation in the complete integrability of the motion of particles; with respect to anholonomic frames, the constructions can be generalized to include nonholonomic variables and non–integrable dynamics
We can verify explicitly that:

**Corollary 3.1** On \((n+m)\)-dimensional \(N\)-anholonomic (pseudo) Riemannian space, the value \(+h = a \cos \theta e^3 \wedge e^2 + r e^4 \wedge e^1\) is a closed nonholonomic CKY 2-form obeying the conditions

\[
H D_X( +h) = X^3 \wedge H \xi, \quad H \xi = (1 - n - m)^{-1} H d^*( +h).
\]

If both conditions \(\hat{Z} = 0\) \((13)\) and \(H T^+ = 0\), i.e. for holonomic configurations, the existence of \(+h\) determines explicitly a tower of hidden symmetries and states uniquely (up to \((n + m)/2\) functions of one variable) the canonical form of metric. Such holonomic principal CKY tensors and related symmetries are studied in Refs. \([35, 36]\). For generic off–diagonal metrics, hidden nonholonomic symmetries exist naturally and they are associated with induced torsions. The constructions are similar to those presented in sections 2 and 3 of Ref. \([11]\) for generalized closed CKY 2-forms with nontrivial torsion and generated towers of hidden symmetries with that difference that in our work we consider different types of torsions.

A nontrivial nonholonomic structure does not allow a simple recovering of symmetries from the divergence of \(+h\) (contrary to the "holonomic" Kerr solution). Following the conditions of above Corollary, we can introduce such nonholonomic values

\[
H \xi_+ = -\frac{1}{3} H d^*( +h) = \rho^{-1}(-\sqrt{|S|} e^4 + a \sin \theta e^3),
\]

\[
H \xi_- = (-\sqrt{|\eta_4|} h_4)^{-1} \partial_t.
\]

For holonomic configurations, there is a smooth limit \(\eta_4 \to 0\) to data \((21)\) with \(b = 0\) in \((20)\), when both torsions \(\hat{T}\) and \(H T\) vanish and we recover the standard Kerr geometry.

There is another closed nonholonomic CKY 2–form associated to nonholonomic and holonomic Kerr–Sen type geometries

\[
-h = a \cos \theta e^2 \wedge e^3 - r e^1 \wedge e^4,
\]

\[
H \xi_- = -\frac{1}{3} H d^*( -h) = -\rho^{-1} \left(\sqrt{|S|} e^4 + a \sin \theta e^3\right),
\]

with respect to a different torsion \(H T^- := \xi T^2 e^{234} + \eta T^1 e^{124}\). In Ref. \([11]\), see formulas \((3.15) - (3.17)\), it is considered that such a torsion is "rather peculiar" because it remains non–trivial even in the holonomic Kerr geometry. In terms of the geometry of nonholonomic manifolds, this is not surprising.
because rotating Kerr black holes can be described naturally with respect to rotating system of coordinates. Such geometric/physical objects have already a specific anisotropy determined by rotations and the associated torsion $H^T$ are induced by rotating frames of reference. In a more general context, the constructions can be extended to arbitrary nonholonomic frames including those related to N–connections and certain off–diagonal terms in the metrics.

Finally, in this section, we note the nontrivial nonholonomic structures associated to off–diagonal solutions are characterized additionally by various types of hidden nonholonomic symmetries with induced torsions which play an important role in stating criteria of separability of Hamilton–Jacobi, Klein–Gordon, Dirac equations etc, which nonholonomic variables. Such constructions were provided, for instance, in Refs. [11] for holonomic Kerr–Sen and Taub NUT backgrounds, and in Refs. [12, 13], for nonholonomic Taub NUT solutions and Dirac operators. The length of this paper does not allow analyze certain constructions related to nonholonomic Kerr–Sen configurations.

4 Nonholonomic Symmetries and Quantum Gravitational Anomalies

In previous section, we proved that there are natural nonholonomic hidden symmetries determined by non–diagonal components of metrics which characterize different classes of generic off–diagonal solutions of Einstein equations. Usual hidden symmetries, for holonomic configurations, with the LC–connection, result in gravitational anomalies (studied in various models of quantum gravity). The surprising thing is that we can find such nonholonomic hidden symmetries when certain classes of anomalies can be canceled (for some auxiliary, but not less fundamental, connections); at the end, all constructions can be re–defined equivalently in terms of the LC–configurations. The aim of this section is to show how some types of quantum gravitational anomalies derived for nonholonomic hidden symmetries can be eliminated.

We consider a physical system when possible non–integrable constraints are encoded into the frame structure of a nonholonomic manifold $V$, $\dim V = n + m$. To find the necessary conditions for the existence of constants of, in general, constrained motion in a first-quantized system we replace momenta by derivatives and look for operators commuting with the Hamiltonian $\hat{H}$. 

In our approach, it is defined using the canonical d–connection,

$$\hat{\mathcal{H}} = \Box = \hat{D}_\beta g^{\beta\gamma} \hat{D}_\gamma = \hat{D}_\beta \hat{D}_\beta.$$  (40)

For holonomic configurations, $\hat{\mathcal{H}}$ transforms into the Hamiltonian $H$ defined by $\nabla$ and corresponds to a free scalar particle when the covariant Laplacian (d’Alambert operator $\Box = \nabla_\beta \nabla^\beta$, constructed for the same metric) is acting on scalars.

In general, the classical conserved quantities associated with SK tensors do not transfer to the quantized systems which results in quantum anomalies. In what follows we shall analyze the quantum anomalies in the case of nonholonomic SK and CSK d–tensors and, to make things more specific, we confine ourselves to the case of tensors of rank 1 and 2.

Let us consider a conserved operator corresponding to a conformal Killing d–vector $K^\alpha$ in the quantized system $V$ $\hat{\mathcal{Q}} = K^\alpha \hat{D}_\alpha$. A quantum gravitational anomaly can be identified by evaluating the commutator $[\hat{\Box}, \hat{\mathcal{Q}}] \Phi$ for the solutions of the Klein-Gordon equation, $\Phi \in C^\infty(V)$, with the Klein-Gordon d–operator (40). A straightforward calculation with respect to $N$–adapted frames is similar to that in usual (pseudo) Riemannian spaces but using $\hat{\mathcal{D}}$ instead of $\nabla$. It gives

$$[\hat{\mathcal{H}}, \hat{\mathcal{Q}}] = 2 - \frac{n - m}{n + m} \left( \hat{D}_\alpha \hat{D}_\gamma K_\gamma \right) \hat{D}_\alpha + \frac{2}{n + m} \hat{D}_\alpha K^\alpha \hat{\Box}.$$  (41)

For a Killing d–vector $K^\alpha$, this commutator vanishes and there are no quantum gravitational anomalies (this property holds both for the holonomic and/or nonholonomic systems). But for conformal Killing d–vectors the situation is very different. Even for solutions of the massless nonholonomic Klein–Gordon equation, $\hat{\Box} \Phi(x) = 0$ (we can also consider $\Box \Phi(x) = 0$, when the zero torsion conditions are satisfied), the term $\left( \hat{D}_\alpha \hat{D}_\gamma K_\gamma \right) \hat{D}_\alpha$ from (41) survives and, in general, the system is affected by quantum gravitational anomalies.

Using a tedious evaluation for $N$–adapted canonical operators (which is similar to that for the LC configurations, provided in Refs. [25, 26]), we prove

Lemma 4.1 a) For a quantity $\tau \hat{\mathcal{Q}} := \hat{D}_\alpha K^{\alpha\beta} \hat{D}_\beta$, the commutator

$$[\hat{\Box}, \tau \hat{\mathcal{Q}}] = 2(\hat{D}(\gamma K^{\alpha\beta}) \hat{D}_\gamma \hat{D}_\alpha \hat{D}_\beta + 3 \hat{D}_\mu (\hat{D}(\gamma K^{\mu\beta}) \hat{D}_\beta \hat{D}_\gamma + \left\{ \frac{4}{3} \hat{D}_\gamma (\hat{R}_\nu K^{\beta\nu}) + \hat{D}_\gamma \frac{1}{2} g_{\mu\tau} (\hat{D}^\gamma \hat{D}(\mu K^{\tau\beta}) \hat{D}_\beta \hat{D}_\gamma + \hat{D}_\alpha \hat{D}(\gamma K^{\alpha\beta}) \right\}) \hat{D}_\beta.$$  

arranging the right side into groups with three, two and just one derivatives and consequently it is impossible to have compensations between them
b) In the case of SK d–tensors all the symmetrized derivatives vanish and
\[
[\hat{\Box}, T\hat{Q}] = -\frac{4}{3}\hat{\mathcal{D}}^\gamma \left( \hat{\mathcal{R}}_\nu [\gamma \mathbf{K}^{3}]^{\nu} \right) \hat{\mathcal{D}}_3.
\] (42)

Similar formulas for \( \hat{\mathcal{D}} \to \nabla \) and SK tensors positively exhibit quantum anomalies, i.e. the classical conservation law does not transfer to the quantum level. Even if we evaluate the commutator for CSK tensors associated with CKY tensors and the LC connection we do not obtain a cancelation of anomalies [26]. Therefore we are not able to identify any favorable circumstances on the CSK tensors in order to achieve a conserved quantum operator. This problem seems to exist for all hidden symmetries of gravitational models with, or not, torsion.

Nevertheless, there are possibilities to consider such nonholonomic distributions on a spacetime \( V \), when the associated nonholonomic commutator \( \hat{\mathcal{D}} \) vanishes.

**Theorem 4.1** For any given data \((g, \nabla)\) and prescribed \( N \)-connection structure \( N \), we can construct nonholonomic deformations to some chosen canonical data \((\eta g, \eta N, \eta \hat{D})\) when \([\eta \hat{\square}, \eta T\hat{Q}] = 0\). If \( N \) is fixed to solve the LC conditions (13) (for a zero nonholonomically induced canonical torsion), the nonholonomic gravitational anomaly vanishes for the Einstein spaces, \([\eta \hat{\square}, \eta T\hat{Q}] = 0\).

**Proof.** In general, a metric \( g \) is not a solution of Einstein equations for \( \nabla \). We fix a nonholonomic splitting with a "primary" \( N \) consider nonholonomic deformations of type (similarly to (26) and (27))

\[
\eta g = [\eta g_i, \eta h_a, \eta N^a_i] \rightarrow \eta g = [\eta^i g_i, \eta^a h_a = \eta^a \eta^i w_i, \eta^3 N^i_3 = \eta^4 N^i_4 = \eta^4 n_i],
\] (43)

where \( \eta_\alpha = 1 + \chi_\alpha = (\eta_i = 1 + \chi_1 (u^3), \eta_\alpha = 1 + \chi_\alpha (u^3), \eta^a = \delta^a_i + \chi^a_i (u^3) \).

We do not consider summation on repeating indices in the above formulas. The gravitational polarizations \( \eta_\alpha \) and \( \eta^a \) are uniquely defined if we chose any \( \eta g \) defining an exact solution of

\[
\eta \hat{\mathcal{R}}_{\beta \delta} = \lambda \eta g_{\beta \delta},
\] (44)

see equations (12) for source (14) defined by a nontrivial cosmological constant \( \lambda \). The solutions \( \eta g \) can be always constructed following the conditions of Lemma 2.1 and Theorem 2.3 and (for the LC conditions) Corollary 2.1.
For different classes of solutions \( \eta g \), we have different types of hidden nonholonomic symmetries determined by the primary data \((g, \nabla)\) and target data \((\eta g, \eta N, \eta \hat{D})\). If the LC conditions (19) are satisfied and \( g \) is also a solution of the Einstein equations, we can constrain the class of nonholonomic transforms, i.e. the gravitational polarizations \( \eta_\alpha \) and \( \eta_a \) in such a form that \( \eta g \) is generated via a frame transform of \( g \). In both cases, for a chosen class of target exact solutions, the \( \eta \)-coefficients characterize the "flexibility", i.e. symmetry of with respect to possible nonholonomic deforms (13). For explicit geometric constructions, we can fix the target metrics to be certain classes of nonholonomic Kerr–Sen black hole configurations in GR (for instance, with small ellipsoidal and/or solitonic deformations).

If the conditions (44) are satisfied for the commutator (42), we get \([\eta \Box, \eta \hat{T}Q] = 0\), even (in general), computing for \( g \) we have \([\Box, \hat{T}Q] \neq 0\). We conclude that we can always eliminate such gravitational anomalies for certain classical symmetries of a (pseudo) Riemannian metric \( g \) if we suppose that such a metric can be nonholonomically deformed to a solution of the Einstein equations for a canonical d–connection \( \eta \hat{D} \) uniquely defined by the same metric structure. Such nonholonomic hidden symmetries characterize both the "off–diagonal" nonlinear properties of \( g \) and its "flexibility" to be transformed in Einstein spaces. □

It is not possible to provide proofs of the above Lemma and Theorem on nonholonomic cancelation of anomalies (or other results related to nonholonomic hidden symmetries) for spaces endowed with general torsion structure even there are various attempts and examples related to (super) string theory [11, 9, 22, 23, 6]. For generic off–diagonal configurations in GR, such constructions can be performed in a simplified form if it is known how a result can be derived, for instance, for a (non) vacuum solution \( \circ g \) and corresponding LC–connection \( \circ \nabla \) using certain vanishing commutators. Via nonholonomic deformations, \( \circ g \to \eta g \), we can derive necessary type canonical decomposition \( \circ \nabla \to \eta \hat{D} = \eta \nabla - \eta \hat{Z} \) generalizing the "anomaly" formulas for \( \eta \hat{D} \) which (in general) contain non-trivial torsion components.

**Remark 4.1** If we select for \( \eta g \) such nonholonomic parametrizations that the conditions (44) and (19) are satisfied, we get \([\eta \Box, \eta \hat{T}Q] = 0\), i.e. canceling of certain anomalies for \( \eta \hat{D} \) and, for zero torsions, for \( \eta \nabla \). We can always define such \( \eta \hat{D} \) with nonholonomic hidden symmetries even we begin with a primary metric \( \circ g \) for which \([\circ \Box, \circ \hat{T}Q] \neq 0\). Computations for nontrivial \( \eta \hat{T} \) are much simple and similar to those for \( \circ \nabla \), with \( \circ T = 0 \), because the distortion tensor \( \eta \hat{Z} \) from (13) is completely determined by the
metric tensor and finally constrained to be zero.

We emphasize this important consequence of the above Theorem: We can always cancel anomalies of a Klein–Gordon operator in GR if we chose a corresponding class of nonholonomic deformations of the frame and linear connection structures uniquely defined by the metric structure. Finally, we recover the conditions for the LC configurations for some specials subclasses of nonholonomic deformations.

5 Conclusions and Discussion

In this work we have studied a new class of hidden nonholonomic symmetries of generic off–diagonal solutions in the general relativity theory. Such Einstein spacetimes can be generated via nonholonomic frame transforms and deformations of fundamental geometric objects (for instance, of the linear connection structure and related differential operators, Riemann and Ricci tensors etc) to certain gravitational configurations with explicit decoupling of some generalized Einstein gravitational field equations. This allows us to find solutions in very general forms when the coefficients of off–diagonal metrics depend on some classes of generating and integration functions. Imposing nonholonomic constraints on such integral varieties, we select the torsionless Levi–Civita, LC, configurations and generate solutions in the Einstein gravity theory (such methods and various examples are provided in Refs. [14, 15, 12, 13, 34]).

Via nonholonomic deformations, various types of torsion fields can be induced on a (pseudo) Riemannian manifold. Such torsions are very different from those, for instance, in the Einstein–Cartan and/or string gravity theories because, in our approach, they are completely defined by the off–diagonal coefficients of the metric tensor following certain well defined geometric principles. We do not need additional field equations for such nonholonomically induced torsion fields, as it is considered, for instance, in [11], and all constructions can be performed equivalently to those for the LC connection.

The gravitational and matter field interactions in the Einstein gravity theory and modifications have a generic nonlinear character. They are characterized by various types of nonlinear symmetries and different classes of nonholonomic constraints. An effective tool for study such classical and quantum interactions is the formalism of Killing–Yano and Stackel–Killing tensors and their higher order generalizations and extensions to nontrivial torsion fields, off–diagonal metrics, non–integrable constraints etc. Their
generalized conformal and anholonomic analogs are related to a multitude of theoretical and mathematical physics issues such as classical integrability of systems together with their quantization, supergravity, string theories, hidden symmetries in higher dimensional black–holes spacetimes, etc.

The considerations in this article are totally classical. Nevertheless, certain constructions emphasize the existence of important quantum effects because of possible elimination of certain types of quantum anomalies by classical nonholonomic transforms/deformations. Let us briefly discuss such issues: We found new classes of generic off–diagonal exact solutions in general relativity and modifications possessing hidden symmetries generated by corresponding Killing–Yano and Stackel–Killing tensors. The classical conserved quantities are not generally preserved when we pass to the quantum mechanical level, i.e. certain anomalies may occurs. In this work, there are analyzed gravitational anomalies determined by surviving (in general) of the second term in commutator (41) by evaluations for the solutions of the Klein–Gordon equation with nonholonomically deformed Hamiltonian (40).

It should be emphasized here that we consider the concept of ’quantum gravitational anomaly’ in a very restricted sense, i.e. for certain quantum operators on classical curved spacetimes when we do not have a well defined and generally accepted model of quantum gravity. See, for instance, [26] and, for a review of main concepts on such quantum anomalies related to the wave, Klein–Gordon and Dirac operators, the section 6 of [40].

The physical importance of Theorem 4.1 is that it concludes a procedure of nonholonomic deformations resulting in vanishing of certain classes of (nonholonomic) gravitational anomalies (even we are not able to identify any general favorable circumstances of CSK tensors in GR, and modifications, in order to construct conserved quantum operators). This allows us to transfer certain classical conserved quantities on off–diagonal spacetimes to the quantum mechanical level if necessary types of nonholonomic constraints are imposed on a corresponding classical nonlinear gravitational dynamics.

Let us speculate on possible quantum effects of ”nonholonomic elimination” of anomalies. For the Kerr and Reisner–Nordström curved–space backgrounds, there is a relationship between the quantum conservation laws and the corresponding quantum numbers associated to operators that commute with the wave operator in a first–quantized field theory. If the Ricci curvature vanishes for any such type exact solution, this implies the existence of a Killing vector/tensor in the first-/ second–order cases and a corresponding constant for theories in the classical limit. Surprisingly, such a property can be preserved under certain classes of off–diagonal nonholonomic deformations of solutions (as we proved in our work). Perhaps, this

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is explained in some sense as straightforward consequences of something more fundamental. In [41], we proved that such nonholonomic deformations can be constructed in a general form depending on finite sets of parameters inducing corresponding hierarchies of conservation laws which, in their turn, result in bi–Hamilton and associated solitonic hierarchies [42]. Such terms contribute efficiently in the Hamiltonian (40) determined by nonholonomic deformations of the Klein-Gordon d–operator and may result in observable quantum effects and new types of conservation laws of solitonic nature. In section 3.2 we provided explicit examples of solitonic deformations of exact solutions which via nonlinear and linear connection coefficients distort the d’Alambert operator.

Another application is similar to that for the Runge–Lenz constants in the nonrelativistic hydrogen atom problem. The hidden symmetries of generic off–diagonal solutions involve certain generalized types of conserved quantities. If in the "hydrogen atom problem" the Runge–Lenz constants and conservation law are additional to the well known for the axial angular momentum, energy, and proper mass, their nonholonomic deformation generalizations arising with hidden ellipsoid, solitonic, parametric or other symmetries also may give rise to the degeneracy of energy levels. We can prescribe a kind of Geroch multi-parametric group for anholonomic deformations and associated solitonic parameters [41] for certain classes of solutions but, in general, only the existence of certain hidden symmetries can hardly provide a satisfactory explanation in itself. It connects certain mysteries to some possible prescribed symmetries. Perhaps, this provides a generalization of the "Schiff conjecture" concerning the relationship between classical mechanics, with nonholonomic constraints, generalizations to nonholonomic frames in gravity and the equivalence principle, see details in [37].

It is an interesting question how the fact of canceling gravitational anomalies for certain hidden nonholonomic symmetries demonstrated in this work can be applied in modern quantum gravity. There are certain quantization schemes based on introducing alternative connections in the Einstein gravity theory and generalizations (for instance, the almost Kähler Cartan one, which is also completely defined by the metric structure in a compatible form), see Refs. [16, 17, 18]. How such various bi-connection formulations of a classical and quantum gravitational model can be related to new insights on nonholonomic canceling of anomalies and renormalization problem of gravitational interactions is a matter of further research and elaborating new geometric methods.

Finally, it should be emphasized that an obvious extension of the results of this paper can be performed for study hidden symmetries for nonholo-
nomic generalizations of particle like gravitational solutions and Einstein–Yang–Mills–Higgs systems considered in Refs. [38, 39, 32]. We should apply the nonlinear connection formalism and the results on generic off–diagonal Taub NUT configurations, running black holes and generalized solutions for the Einstein–Dirac solitonic and pp–waves [34, 12, 13]. We plan to address such issues in our further works.

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A N–adapted Coefficients for d–Connections

For convenience, we present some necessary formulas from the geometry of N–anholonomic (pseudo) Riemannian spaces, see details in [15].

The coefficients of the Levi–Civita (LC) connection $\nabla = \{\hat{\Gamma}_\alpha^\gamma_{\beta\delta}\}$ for the metric (6) (computed with respect to N–adapted basis (2) and (3)) can be written in the form

$$\hat{\Gamma}_\alpha^\gamma_{\beta\delta} = \hat{\Gamma}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta}. \quad (A.1)$$

The value $\hat{D} = \{\hat{\Gamma}_\alpha^\gamma_{\beta\delta}\} = \{\hat{L}_{jk}^a, \hat{L}_{ab}^a, \hat{\tilde{C}}_{jc}^a, \hat{\tilde{C}}_{bc}\}$ with coefficients

$$\hat{\tilde{C}}_{jc}^a = \frac{1}{2} g^{ik} e_c g_{jk}, \hat{L}_{ab}^a = e_b(N^a_k) + \frac{1}{2} h^{ac}(e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k),$$

$$\hat{L}_{jk}^a = \frac{1}{2} g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \hat{\tilde{C}}_{bc}^a = \frac{1}{2} h^{ad}(e_c h_{bd} + e_d h_{cd} - e_d h_{bc}) \quad (A.2)$$

defines the canonical distinguished connection (d–connection). By straightforward computations, we can check that it is metric compatible, $\hat{D}g = 0$, and its torsion $\mathcal{T} = \{\hat{T}_\alpha^\gamma_{\beta\delta} = \hat{\Gamma}_\alpha^\gamma_{\beta\delta} - \hat{\Gamma}_\alpha^\gamma_{\beta\delta} \hat{T}_{jk}^a, \hat{T}_{ja}^a, \hat{T}_{ji}^a, \hat{T}_{bi}^a, \hat{T}_{bc}\}$, is with zero horizontal and vertical coefficients, $\hat{T}_{jk}^a = 0$ and $\hat{T}_{bc}^a = 0$. There are also nontrivial $h$–v coefficients

$$\hat{T}_{jk}^i = \hat{L}_{jk}^i - \hat{L}_{kj}^i, \hat{T}_{ja}^i = \hat{\tilde{C}}_{jb}^i, \hat{T}_{ji}^a = -\Omega_{ji}^a, \quad (A.3)$$

$$\hat{T}_{aj}^c = \hat{L}_{aj}^c - e_a(N_{cj}^d), \hat{T}_{bc}^a = \hat{\tilde{C}}_{bc}^a - \hat{\tilde{C}}_{cb}^a.$$

The distortion tensor $Z^\gamma_{\alpha\beta}$ in (A.1) is also constructed in a unique form from the coefficients of metric N–connection,

$$Z_{jk}^i = 0, Z_{jk}^a = -\hat{C}_{jk}^a, Z_{bk}^i = \frac{1}{2} \Omega_{jk}^i h_{cb} g^{ji} - \hat{\Xi}_{jk}^i \hat{\tilde{C}}_{hb}^j,$$

$$Z_{bk}^a = -\frac{1}{2} \hat{\Xi}_{jk}^j \hat{\tilde{C}}_{bc}^j, Z_{bk}^i = \frac{1}{2} \Omega_{jk}^i h_{cb} g^{ji} + \hat{\Xi}_{jk}^i \hat{\tilde{C}}_{hb}^j, Z^a_{bc} = 0,$$

$$Z_{jb}^a = -\frac{1}{2} \hat{\Xi}_{jk}^j \hat{\tilde{C}}_{jd}^a, Z_{ab}^i = -\frac{1}{2} \left[ \hat{T}_{ja}^c h_{cb} + \hat{T}_{jb}^c h_{ca} \right] \quad (A.4)$$
for \( \Xi^h_{jk} = \frac{1}{2}(\delta^h_j \delta^h_k - g_{jk}g^{ih}) \) and \( \pm \Xi^a_{cd} = \frac{1}{2}(\delta^a_c \delta^a_d + h_{cd}h^{ab}) \).

Any geometric and physical formulas for the connection \( \nabla \) can be equivalently redefined for the canonical \( \hat{d} \)-connection \( \hat{\nabla} \), and inversely, using (A.1) because all involved geometric objects (two different connections and the distorting tensor) are uniquely defined by the same metric structure.

## B Decoupling and Integration of Einstein Eqs

We briefly summarize the results on generating off–diagonal solutions in gravity \([14, 15]\).

Using the conditions of Lemma 2.1 and computing in explicit form the Ricci and Einstein tensors, we prove:

**Theorem B.1 (decoupling of equations).** The Einstein equations for \( \hat{\nabla} \) (A.2) and ansatz for the metric \( g \) (15) with \( \omega = 1 \) and any general source \( \Upsilon^a_{\alpha} \) [14] are

\[
\begin{align*}
\hat{R}_1^1 &= \hat{R}_2^2 = -\frac{1}{2g_1g_2} g_{22} - \frac{g_1 g_2}{g_1} \frac{(g_2')^2}{g_2} + g_1'' - \frac{g_1 g_2'}{g_1} - \frac{(g_1')^2}{g_1} = -\Upsilon_2(x^k), \\
\hat{R}_3^3 &= \hat{R}_4^4 = -\frac{1}{2h_3h_4} [h_4^{**} - \frac{(h_4^*)^2}{2h_4} - h_3^* h_4^*] = -\Upsilon_4(x^k, y^a), \\
\hat{R}_{3k} &= \frac{w_k}{2h_4} [h_4^{**} - \frac{(h_4^*)^2}{2h_4} - h_3^* h_4^*] + \frac{h_3^*}{4h_4} (\partial_k h_3 + \partial h_4) - \frac{\partial_k h_4^*}{2h_4} = 0 \\
\hat{R}_{4k} &= \frac{h_4^*}{2h_3} n_k^{**} + \left( \frac{h_4^*}{h_3} \frac{3}{2} h_4^* \right) \frac{n_k}{2h_3} = 0.
\end{align*}
\]

The system of partial differential equations (B.1) is with decoupling of equations (the "splitting" of equations is used as an equivalent one; we should not confuse this with the property of separation of variables). For simplicity, we can consider a subclass of solutions when for the chosen N–system of reference \( h_4^* \neq 0 \)

**Corollary B.1** The system of equations (B.1) for \( y^3 = v, g_i = e^i_\varepsilon \varepsilon^{\psi}(x^k) \) and \( h_4^* \neq 0, \Psi_2 \neq 0, \Psi_4 \neq 0 \), can be written equivalently in the form

\[
\begin{align*}
\psi^{**} + \psi'' &= 2\Psi_4(x^k) \\
\frac{h_4^*}{h_3 h_4} &\Psi_2(x^k, v)/\phi^*, \beta \omega_1 + \alpha_i = 0, n_i^{**} + \gamma n_i^* = 0,
\end{align*}
\]

for \( \phi = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \gamma := (\ln \left| \frac{h_4^*}{|h_3|} \right|)^*, \alpha_i = h_4^* \partial_i \phi, \beta = h_4^* \phi^* \).

\[\text{If } h_4^* = 0, \text{ or } h_4^* = 0, \text{ the solutions can be constructed similarly (in certain cases, they can be transformed from one to another one via frame/coordinate transforms).} \]
The systems of equations (B.1) and (B.2) can be integrated in general forms following the results of Theorem [2.3]. We can generate solutions for the LC connection if the zero torsion conditions of Corollary 2.1 are satisfied.

Let us study the "vanishing torsion" conditions (19). For general sources, Υαδ, it is quite difficult to prove in an explicit analytic form that such equations have nontrivial solutions.

**Corollary B.2** We can adapt the nonholonomic distributions for generic off–diagonal Einstein spaces with Υαδ = λδαδ, when Ψ2 = Ψ4 = λ, and parametrize the data (17) and (18) for the coefficients of metric ansatz in such a form that (19) determine some classes of nontrivial solutions, when the d–torsions (A.3) for selecting from data (17) and (18) we generate solutions with separation of variables, 0wixi(xk) = 0 and 0nixi(xk). In such cases, we must prove that h4 = ±λ−1eφ and wi = ∂iφ/φ* have for some classes of functions φ(xi, xj), φ* ≠ 0, certain nontrivial solutions of (19), i.e.

\[ w_i^* = e_i \ln |h_4| \text{ and } \partial_i w_j = \partial_j w_i. \]  

Expressing h4 and wi explicitly via φ, we get from (B.3) that φ*∂iφ − φ*(∂iφ)* = 0. As a particular case, these equations can be written in the form (∂iφ/φ*)* = w_i^* = 0, when w_i = 0wi(xk). Choosing φ = 0φ(xk)φ(v), we generate solutions with separation of variables, 0wixi(xk) = i∂i 0φ and \( x_k \) for a nonzero constant i. In general, we can consider various types of nonholonomic constraints (19) for selecting from data (17) and (18) very different families of solutions in GR.

**Proof.** Let us consider a solution (17) and (18) when the coordinate system and boundary conditions are fixed in the form that \( h_4(x^k) = 0 \), \( 2nixi(xk) = 0 \) and \( \partial_i 1nixi(xk) = \partial_j 2nixi(xk) \). In such cases, we must prove that h4 = ±λ−1eφ and wi = ∂iφ/φ* have for some classes of functions φ(xi, xj), φ* ≠ 0, certain nontrivial solutions of (19), i.e.

\[ w_i^* = e_i \ln |h_4| \text{ and } \partial_i w_j = \partial_j w_i. \]  

Expressing h4 and wi explicitly via φ, we get from (B.3) that φ*∂iφ − φ*(∂iφ)* = 0. As a particular case, these equations can be written in the form (∂iφ/φ*)* = w_i^* = 0, when w_i = 0wi(xk). Choosing φ = 0φ(xk)φ(v), we generate solutions with separation of variables, 0wixi(xk) = i∂i 0φ and \( x_k \) for a nonzero constant i. In general, we can consider various types of nonholonomic constraints (19) for selecting from data (17) and (18) very different families of solutions in GR.

**Remark B.1** Integrating on variable \( y^3 = \varphi \) and taking \( b_{\nu} = 0h_4(\bar{x}^i) \), \( \epsilon_i = 1 \), \( g_i = \eta_i e^{\psi(\bar{x}^i)} 0g_1(\bar{x}^i) \), \( \eta_i = \epsilon_i e^{\psi(\bar{x}^i)} \), \( \omega = 1 \), in coordinates \( \bar{u}^3 \) [24] and for the "initial" data \( g_\alpha \) [23], the "one–Killing" off–diagonal solutions (17) and (18) for the Einstein spaces can be represented in the form

\[ \epsilon_i \frac{\partial^2}{\partial u^3} \left[ \psi(\bar{\Phi} + 2 \ln |b_{\rho}|) \right] = \lambda, \]  

\[ h_4 = 0h_4 \pm \lambda^{-1}e^{2\varphi}, \quad h_3 = \left[ \left( \sqrt{|h_4|} \right)^3 \right] e^{-2\varphi}, \]  

\[ w_i = -(\varphi^*)^{-1} \frac{\partial \varphi}{\partial u^i}, \quad n_k = 1n_k(\bar{x}^i), \quad 2n_k(\bar{x}^i) \]  

subjected to the conditions (B.3).
To consider possible small nonholonomic deformations $^\circ g_\alpha \to ^\eta g_\alpha$ is convenient to express the formulas (B.3) via polarization functions (27) when, for $\lambda \neq 0$, $h_a = ^\circ h_a \eta_\alpha = ^\circ h_a (1 + \chi_a)$. Using the second formula in (B.4), for $h_4$, we can express $\phi = \ln \sqrt{\lambda ^\circ h_4 \chi_4}$ and consider for such classes of solutions a "new" generating function $\chi_4 (\tilde{x}^i, \varphi)$. The third formula for $h_3$ from that system of solutions allows us to express

$$\chi_3 = -1 + (\lambda ^\circ h_3)^{-1} \left[ \left( \ln \sqrt{1 - \chi_4} \right)^* \right]^2.$$  \hspace{2cm} (B.5)

We conclude that all $v$–coefficients of off–diagonal metrics and N–connections, see ansatz (15) (equivalently (16)) for the Einstein spaces with Killing symmetry on $\partial/\partial \tilde{y}^4$, up to arbitrary frame transforms, are functionally determined by $\chi_4$, i.e. $\phi[\chi_4], h_a[\chi_4], w_i[\chi_4], n_k[\chi_4]$.

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