Relativistic electrons in a rotating spherical magnetic dipole: localized three-dimensional states

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Paralleling a previous paper, we examine single- and many-body states of relativistic electrons in an intense, rotating magnetic dipole field. Single-body orbitals are derived semiclassically and then applied to the many-body case via the Thomas-Fermi approximation. The many-body case is reminiscent of the quantum Hall state. Electrons in a realistic neutron star crust are considered with both fixed density profiles and constant Fermi energy. In the first case, applicable to young neutron star crusts, the varying magnetic field and relativistic Coriolis correction lead to a varying Fermi energy and macroscopic currents. In the second, relevant to older crusts, the electron density is redistributed by the magnetic field.

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1. INTRODUCTION

In a previous paper [1] (hereafter paper I), we examined the relativistic semiclassical relativistic orbitals of a particle of mass $m$ and charge $q$ in an intense magnetic field, idealized as a dipole, in a rotating reference frame. The particle was confined to the spherical surface. In this paper, we present the treatment of three-dimensional orbitals, the local cyclotron or Landau states. These results are applied to many-body electron states defined in a semiclassical (Thomas-Fermi) approximation and calculated in a simplified neutron star crust model with electrons and nuclei. We consider quantum dynamics only for the electrons, not the hadrons [2], but effects of relativity and rotation are included.

We ignore gravity (derivatives of the metric) here, as this is negligible for charged particles compared to the magnetic field. Where numerical values are needed, the tilt angle $\theta_0$ between the dipole and rotation axes is assumed to be maximal, $\sin \theta_0 = 1$, and the rotational velocity $\Omega$ to be $\tilde{\omega} = \Omega R/c = 0.01$, a realistic value for high-field neutron stars with radius $R$ [3,4].

The treatment is based on expanding the particle motion in inverse powers of the field strength. Although electrons are stripped from the neutron star surface by the rotation-induced electric field, the bulk of electrons remain in the crust to preserve local charge neutrality, with the surface sheathed by a thin space charge. The space charge is stabilized by the Coulomb force (with the positive crystal) opposing the induced electric field. Only a small fraction of electrons are accelerated into the stellar wind [5]. Our final applications here are to quantum single- and many-body states where radiation emission is neglected. This is exact for charged particles in their ground states or in excited states unable to decay by Pauli exclusion blocking in the presence of other fermions. We also seek a general classification of possible orbitals based on the relevant kinematic parameters. The semiclassical quantization is based on the Wilson-Sommerfeld or Bohr-Sommerfeld rule, a result of the WKB approximation [6].

2. GENERAL RELATIONS

2.1 Coordinates, metric, and field

As the magnetic field dominates rotational effects (unless $r$ is quite large), we place the rotation axis at an angle $\theta_0$ with respect to the magnetic dipole in the $\phi = 0, \pi$ plane (Fig. 1). The metric in a spherical polar coordinate system $(r, \theta, \phi)$ rotating with the sphere is given by the line element

FIG. 1. Geometry of the magnetic dipole $M$ and sphere rotating at angular velocity $\Omega$, with relative tilt $\theta_0$. 

1. INTRODUCTION

In a previous paper [1] (hereafter paper I), we examined...
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]
\[ = c^2(1 - \omega^2)dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2r\omega_\phi\sin\theta\ dt\ d\phi - 2r\omega_\theta\ dt\ d\theta . \]  

The vector \( \omega \) is defined from the rotational angular velocity vector \( \Omega \) by \( \omega = \Omega \times r/c. \) We also use \( \bar{\omega} \equiv \Omega R/c, \) where \( r = R \) is a reference sphere. The components of \( \omega \) are
\[
\omega_\phi = (\bar{\omega}r/R)[\cos\theta_0\sin\theta - \sin\theta_0\cos\theta\cos\phi] ,
\omega_\theta = -(\bar{\omega}r/R)\sin\theta_0\sin\phi ,
\omega^2 = \omega_\phi^2 + \omega_\theta^2 .
\]

These are the appropriate generalizations of paper I and the treatment of Landau & Lifshitz to the case \( \sin\theta_0 \neq 0 \) and \( r \neq \text{constant}. \) The lightsphere is the surface \( \omega^2 = 1, \) or
\[
\omega^2(r/R)^2[\cos^2\theta_0\sin^2\theta + \sin^2\theta_0\sin^2\phi\sin^2\theta\cos^2\phi - (1/2)\sin\theta_0\sin2\theta_0\cos\phi] = 1 .
\]

This is the surface upon which \( g_{00} = 0 \) (Fig. 2).

We choose our axes so that the magnetic dipole is along the \( \theta = 0 \) direction. The dipole magnetic field has polar strength \( B_0 \) at \( r = R: \)
\[
A_\phi = 0 \quad (4)
\]
\[
A_\phi = 0 \quad (5)
\]
\[
A_\phi = B_0 R^3/2r \quad (6)
\]

where \( A_\phi \) is covariant in the rotating spherical coordinates. For convenience, we rescale \( B_0 \) into dimensionless form as
\[
\beta_0 = |q|B_0/(2mc^2) .
\]

The magnetic moment of this dipole field is
\[
|\mathbf{M}| = B_0 R^3/2 .
\]

In a high-field neutron star, \( \beta_0 \sim 10^{15}. \)

The Lagrangian of the charged particle is expressed in terms of the proper time \( \tau, \) with \( x^\mu \equiv dx^\mu/d\tau. \) There are four equations of motion including one for each momentum component and one energy-momentum constraint. Since the Lagrangian does not explicitly depend on \( t, \) we have
\[
\frac{\partial L}{\partial t} = 0 .
\]

which implies
\[
\frac{dP_0}{d\tau} = 0 .
\]

Because
\[
d\tau = \frac{dt}{\sqrt{g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)}},
\]

we have also
\[
\frac{dP_0}{dt} = 0 ,
\]

where \( P_0 = E, \) the energy. The equations of motion for \( r, \phi, \) and \( \theta \) are non-trivial:
\[
\frac{\partial L}{\partial r} - \frac{dP_r}{d\tau} = \frac{\partial L}{\partial \phi} - \frac{dP_\phi}{d\tau} = \frac{\partial L}{\partial \theta} - \frac{dP_\theta}{d\tau} = 0 .
\]

The energy-momentum-mass constraint is
\[
g^{00}P_0^2 + 2g^{0i}P_0(P_i - \frac{q}{c}A_i) + g^{ij}(P_i - \frac{q}{c}A_i)(P_j - \frac{q}{c}A_j)
\]
\[
= (mc^2) ,
\]

including radial and angular terms, and the mixed rotational \( g^{0\phi}, g^{\phi\phi} \) terms. The contravariant metric components are
\[
g^{00} = 1/c^2 ,
\]
\[
g^{rr} = -1/r^2 ,
\]
\[
g^{\theta\theta} = -(1 - \omega_\theta^2)/r^2 ,
\]
\[
g^{\phi\phi} = -(1 - \omega_\phi^2)/(r^2\sin^2\theta) ,
\]
\[
g^{ij} = -\omega_\phi/(cr\sin\theta) .
\]

Along the actual worldpath in spacetime the energy-momentum constraint is equivalent to \( g_{\mu\nu}x^\mu x^\nu = c^2. \) This condition is valid after varying the action and simplifies the equations of motion.

### 2.2 Asymptotic orbits

As a check of this dynamical system, we consider briefly the \( r \rightarrow \infty \) orbits. The constraint (14) alone is sufficient to give essential information about the trajectory. Note first that \( \omega^2 < 1 \) for physical trajectories. Since \( \omega^2 = \omega_\phi^2 + \omega_\theta^2, \) each component must also have magnitude less than one. Then \( g^{00} \) and \( g^{\phi\phi} \) are never exactly zero.

In paper I the natural energy scale for unconfined charged particles at the surface \( r = R \) was set by \( c \sim 3, \) or \( E \sim |q|B_0R/2, \) independent of \( m \) (see also Longair [8]).

In the limit \( r \rightarrow \infty, \) Eq. (14) simplifies to
\[
P_r^2 = (E/c^2) - (mc^2) ,
\]

where \( E \) is naturally \( \gtrsim O(|q|B_0R). \) In this regime, \( B_0 \) is much smaller than at the surface of the star, as \( r \) is much larger. The magnetic potential term vanishes as \( 1/r^2, \) while the angular momentum terms vanish as \( 1/r. \) Particles can escape as \( r \rightarrow \infty \) if \( P_r^2 > 0 \) and if they remain within the lightsphere surface \( \omega^2 = 1 \) (Fig. 2). Both graphically and analytically, it is seen that \( (\phi, \theta) \rightarrow (0, \theta_0) \) or \( (\pi, \pi - \theta_0), \) as \( r \rightarrow \infty. \) That is, an escaping particle is forced into the common plane of \( \mathbf{M} \) and \( \Omega \) and leaves along the rotation axis.
3. LOCAL SINGLE-BODY STATES

The local Landau states can be obtained by expanding the energy-momentum constraint in a Taylor series around $\mathbf{r} = \bar{\mathbf{r}}$, i.e., let $\mathbf{r}$ go to $\bar{\mathbf{r}} + \Delta \mathbf{r}$. The local point is defined by its spherical coordinates $(\bar{r}, \bar{\theta}, \bar{\phi})$. We retain terms up to second order in $\Delta \mathbf{r}$ and momentum $\mathbf{P}$, but leave the local metric $g_{\mu\nu}(\bar{r})$ constant.

3.1 Local coordinates and field

We define a local Cartesian coordinate system so that the $z$ coordinate is along the local magnetic field vector, the $x$ coordinate is orthogonal to $z$ and pointing outward, and the $y$ coordinate is azimuthal.

The dipole magnetic field is

$$
\mathbf{B} = \left(\frac{B_0}{2}\right) \left(\frac{R}{r}\right)^3 \left[2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\mathbf{\theta}}\right] .
$$

In terms of the local displacement $\Delta \mathbf{r}$, the new coordinates are given by

$$
x = \Delta r \sin \theta - 2 r \Delta \theta \cos \theta \sqrt{1 + 3 \cos^2 \theta} ,
$$

$$
z = \frac{2 \Delta r \cos \theta + r \Delta \theta \sin \theta}{\sqrt{1 + 3 \cos^2 \theta}} ,
$$

$$
y = -r \sin \theta \Delta \phi ,
$$

while the local Cartesian basis is

$$
\hat{x} = \sin \theta \hat{\mathbf{r}} - 2 \cos \theta \hat{\mathbf{\theta}} \sqrt{1 + 3 \cos^2 \theta} ,
$$

$$
\hat{z} = \frac{2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\mathbf{\theta}}}{\sqrt{1 + 3 \cos^2 \theta}} ,
$$

$$
\hat{y} = -\hat{\phi} .
$$

The bar on $\bar{\mathbf{r}}$ is now omitted unless needed.

3.2 Local Landau states

The constraint can be rewritten in the local coordinates as

$$
\epsilon^2 - \Pi_x^2 - \Pi_z^2 - (\Pi_y - \beta x/r)^2 - 1 +
2 \omega_0' \epsilon (\Pi_y - \beta x/r) - 2 \omega_0' \epsilon \frac{\sin \theta \Pi_z - 2 \cos \theta \Pi_x}{\sqrt{1 + 3 \cos^2 \theta}} +
2 \omega_0' \epsilon \left(\frac{\beta \cos \theta}{r^2 \sin \theta (1 + 3 \cos^2 \theta)^{3/2}}\right) [4 \cos \theta x^2 - 3 \cos \theta \sin^2 \theta z^2 + 2 \sin \theta (3 \cos^2 \theta - 1) x z] +
(\Pi_y' - \beta x/r)^2 \omega_0'' +
\frac{(\sin \theta \Pi_z - 2 \cos \theta \Pi_x)^2}{1 + 3 \cos^2 \theta} \omega_0 = 0 .
$$

Here we use dimensionless local magnetic field strength, energy, and momenta given by

$$
\beta = \frac{|q| B \bar{r}}{2 m c^2} = \beta_0 \left(\frac{R}{r}\right)^2 \sqrt{1 + 3 \cos^2 \theta} ,
$$

$$
\epsilon = E/mc^2 ,
$$

$$
\Pi_{x,z} = P_{x,z}/mc .
$$

$\Pi_y'$ is the new canonical momentum given by the transformation

$$
\Pi_y' = -\frac{P_\phi}{mc \sin \theta} + \frac{q B_0 R^3 \sin \theta}{2 mc^2 r^2} .
$$

The second term is a shift which is constant in the local coordinate system. The first line of the constraint (20) gives the classical version of the Landau system in terms of the local magnetic field $\mathbf{B}(\bar{\mathbf{r}})$, which also defines the local $z$ axis. The other terms are linear and quadratic corrections (Coriolis and centrifugal terms) due to the local $\omega$ components $\omega_x$ and $\omega_\phi$.

Neglecting rotational effects yields the classical cyclotron motion $\bar{\mathbf{r}}$. In the semiclassical form, these are the Landau orbitals, with energy eigenvalues

$$
\epsilon_0^2 = 1 + \Pi_z^2 + \left(\frac{\hbar}{mc}\right) \left(\frac{2 \beta}{r}\right) [2 n_L + 1 - \text{sgn}(q) \sigma] ,
$$

where the principal Landau quantum number is $n_L = 0, 1, 2, ...$ and the spin $\sigma$ has been included, $\sigma = \pm 1$. For sgn$(q) < 0$, the state $(n_L, \sigma = 1)$ is degenerate with $(n_L + 1, \sigma = -1)$. The Landau orbitals are characterized by length and momentum scales $L = \sqrt{\hbar c/(2 q |B|)}$ and $P_L = \sqrt{\hbar q |B|/(2 e)}$ in the $(x, y)$ plane transverse to the field. The longitudinal momentum $P_z = mc \Pi_z$ is an eigenvalue continuous over the range $-\infty < P_z < +\infty$, as the motion in that direction is force-free. Note that the magnetic factor proportional to $\beta$ is suppressed by the ratio of the Compton wavelength $\hbar/(mc)$ to the spherical distance $r$. The effect of the magnetic field in the energy is thus negligible compared to $mc^2$ unless $\beta$ is very large, as seen in paper I. Single-body Landau states of fermions are unstable against radiation, unless the lower energy states are already filled, as they are in Sect. IV below.

We now consider the effect of rotation as a first-order perturbation in $\bar{\omega}$, approaching the problem semiclassically. (The terms quadratic in $\bar{\omega}$ are much smaller.)
The unperturbed orbitals are Landau states in the \((x, y)\) transverse plane. We keep the terms linear in \(\omega\) (Cori
colis effect) and average them over one Landau orbit, as
in paper I, Sect. V. But unlike that case, this averaging is
over a local microscopic orbit, not a macroscopic or-
bit over the whole sphere. The terms linear in \(\bar{\omega}\) thus
do not average to zero in general. The modified classical
energy-momentum constraint is then

\[
e^2 - \Pi_x^2 - \Pi_y^2 = (\Pi_y - \beta x/r)^2 - 1 - \frac{2\omega_0\epsilon_0}{\sin \theta \Pi_z \sqrt{1 + 3 \cos^2 \theta}} + \frac{2\omega_0\epsilon_0}{r^2 \sin \theta (1 + 3 \cos^2 \theta)^{3/2}} \times [4 \cos \theta x^2 - 3 \cos \theta \sin^2 \theta z^2] = 0 ,
\]

noting that \(\langle x \rangle_0 = \langle y \rangle_0 = \langle \Pi_x \rangle_0 = \langle \Pi_y \rangle_0 = 0\), averaged
over an unperturbed cyclotron orbit. These orbits have
semiclassical radius \(a(n_L) = 2L\sqrt{n_L} + 1/2\). The semi-
classical result for the Landau energies corrected through
\(O(\bar{\omega})\) is then

\[
\epsilon^2_1 = \epsilon_0^2 + 2\omega_0\epsilon_0 \sin \theta \Pi_z \sqrt{1 + 3 \cos^2 \theta} - \frac{2\omega_0\epsilon_0}{r^2 \sin \theta (1 + 3 \cos^2 \theta)^{3/2}} \times [4(n_L + 1) \cos \theta L^2 - 3 \cos \theta \sin^2 \theta \bar{z}_C^2] ,
\]

where we have used \(\langle \Pi_x \rangle_0 = \Pi_x, \langle x^2 \rangle_0 = a^2(n_L)/2\), and
a distance cutoff \(z_C\) for motion in the \(z\) direction. (This
cutoff is discussed further in Sect. IV below.) The last
term can be neglected if the Landau orbits are much
smaller than the spherical distance \(r : L, z_C << r\), as
it is another factor of \(L/r\) smaller than the zeroth-order
term. The energies depend on \(\Pi_x^2\) at zeroth order but
receive a correction proportional to \(\Pi_x\) at first order in
\(\bar{\omega}\), breaking the symmetry \(\Pi_x \rightarrow -\Pi_x\). The Landau pole
states of paper I can be recovered if \(\sin \theta \rightarrow 0, r = R\),
identifying \(n_L\) with the old \(n_0\) and setting \(\Pi_x = 0\).

The error arising from neglecting the cubic terms in
the Taylor expansion can be estimated and varies as
\(~ |\Delta r|/r\) times the quadratic terms. That is, the
cubic and higher terms in the expansion are suppressed by
additional powers of the Landau length \(L\) over the sphere
size \(r\).

4. LOCAL MANY-BODY STATES

4.1 Density of states

The full density of states is a product of four factors:
the density of Landau states, the degeneracy factor \(D_\perp\)
in the \((x, y)\) transverse plane for each Landau state, the
density of longitudinal states \(D_\parallel\) for \(z\) motion, and the
spin factor (one for the ground state \(n_L = 0\), two other-
wise).

The degeneracy of a given Landau state \(n_L\) in the
transverse plane is

\[
D_\perp = \frac{|qB|}{2\pi \hbar} (26)
\]

per unit planar area, a result valid in both non-
relativistic \([11]\) and relativistic regimes, while the lon-
gitudinal motion contributes a factor

\[
D_\parallel = \frac{mc \, d\Pi_z}{2\pi \hbar} (27)
\]

per unit longitudinal length. Thus the number of states,
including the spin factor, is

\[
\frac{d^2N}{dS \, dz} = (2 - \delta_{n_L, 0}) \cdot D_\perp \cdot D_\parallel , (28)
\]

per unit transverse surface area \(dS\) and unit longitudinal
length \(dz\).

In the semiclassical limit, where \(n_L\) is quasi-
continuous, the density of Landau states per unit energy
is given in dimensionless form by

\[
\frac{dn_L}{de} = \left(\frac{mc \, e}{4\hbar^2}\right) \left[1 + (2\omega_0/\sqrt{1 + 3 \cos^2 \theta})\right]^{-1} \times [1 + \Pi_z^2 + 2\hbar \beta/(mc) |2n_L + 1 - \text{sgn}(q)\sigma| + 2\omega_0\epsilon_0 \sin \theta \Pi_z / \sqrt{1 + 3 \cos^2 \theta}] . (29)
\]

4.2 Thomas-Fermi approximation

The Thomas-Fermi method approximates quantum
many-body fermion states in a varying potential with
local states defined by a locally constant field \([6]\). In our
case, the local electron states are filled up to some highest
and partially-filled Landau level \(n_L^*\) by the charge carri-
ers, assumed here to be electrons. That part \(\zeta^2\) of the
squared energy \(\epsilon^2\) arising from the \(\Pi_z\) terms alone,

\[
\zeta^2(\Pi_z) = \Pi_z^2 + 2\omega_0\epsilon_0 \sin \theta \Pi_z / \sqrt{1 + 3 \cos^2 \theta} ,
\]

must be cut off at some maximum, as must the corre-
sponding \(z\) motion. In a real system, the cutoffs are
provided naturally by the presence of lattice ions: \(|z_C| \gtrsim
\hbar/(Ze_{\text{eff}}mc)\) (Bohr length) and \(|\Pi_z| \gtrsim Z_{\text{eff}}\alpha\), where \(Z_{\text{eff}}\)
is an effective (screened) positive ionic charge \([6]\). In the
realistic case the \(z_C\) term is unnecessary; assume that
\(\zeta^2 < \zeta^2_0 = (Ze_{\text{eff}}\alpha)^2\). Then the longitudinal momentum
\(\Pi_z\) is cut off asymmetrically at \(\Pi_z = \Pi_z^0 > 0\) and \(\Pi_z < 0\),
with \(\Pi_z^0 \neq -\Pi_z^0\) if \(\omega \neq 0\) (Fig. 3).
FIG. 3. Momentum-dependent part $\zeta^2$ of the squared energy $\epsilon^2$ as a function of longitudinal momentum $\Pi_z$, including $\mathcal{O}(\omega)$ correction, showing the solutions $\Pi^{\pm}_z$ of $\zeta^2 = \zeta^2$.

The electron number density $n_e$ at any point is given by

$$n_e = \frac{|qB_{\Pi}|}{2\pi^2\hbar c} \left[ 1 + 2(n_L^* - 1) + 2\nu \right] \cdot \Delta\Pi_z (mc/2\pi\hbar) \ ,$$

(30)

where $\nu$ is the partial filling factor of the highest Landau level $n_L^*$, $0 \leq \nu \leq 1$, and $n_L^*$ is assumed $\geq 1$. (If $n_L^* = 0$, the entire term in brackets is replaced simply by $\nu$.) Because of the $z$ degree of freedom, each Landau level is actually a band, with lowest energy level at some $\Pi_z \neq 0$, whose sign depends on that of $\omega_0$. In addition, there is the planar degeneracy, modified by the partial filling factor $\nu$ in the highest Landau level. If $n_e$ is specified along with $\Delta\Pi_z$, then $n_L^*$, $\nu$, and $\epsilon_F$ can be determined. Procedural details are found in the Appendix.

At zero temperature (the case examined in this paper), the electrons cannot radiate into already-filled lower-energy single-body states. Radiation occurs only if fermions are externally excited above the Fermi energy. If $n_e$ is held constant over a sphere of radius $r$, then $n_L^*$, $\nu$, and $\epsilon_F$ change over that surface as $|B|$ changes. The variation of $n_L^*$ and $\nu$ over the surface is given by

$$n_L^* = \text{Int} \left[ \frac{2\pi\hbar}{mc} \left( \frac{n_e}{\Delta\Pi_z} \right) \left( \frac{B_c}{B_0\sqrt{\alpha}} \right) \left( \frac{r}{R} \right)^3 \times \right.$$

$$\frac{1}{4\sqrt{1 + 3\cos^2 \theta}} + \frac{1}{2} \bigg] ,$$

$$\nu = \left( \frac{2\pi\hbar}{mc} \right)^3 \left( \frac{n_e}{\Delta\Pi_z} \right) \left( \frac{B_c}{B_0\sqrt{\alpha}} \right) \times$$

$$\frac{1}{4\sqrt{1 + 3\cos^2 \theta}} + \frac{1}{2} - n_L^* \ ,$$

(31)

where $\text{Int}[f]$ in (31) denotes the integer value of the function $f$. Typical variations of $n_L^*$ and $\nu$ over the sphere are shown in Figs. 4 and 5. The Fermi energy at any point on the surface is

$$\epsilon_F^2 = 1 + \zeta_F^2 + \left( \frac{B_0\sqrt{\pi}}{B_c, \pi} \right) \left( \frac{R}{r} \right)^3 \left( 2n_L^* + 1 + \sigma \right) \times$$

$$\sqrt{1 + 3\cos^2 \theta} \ ,$$

(32)

where $\text{sgn}(q) = -1$, and $\sigma_F = -1$ for the $n_L^*$ level; the critical field strength is $B_c = (n_e^2/\pi)\sqrt{e^2/h^3} = 1.3 \times 10^{12}$ G, defined in paper I. Matching variations of $\epsilon_F$ are shown in Fig. 6. As the Fermi energy $\epsilon_F$ varies over a sphere, electron currents flow as the electrons seek the lowest energy. There are also radial currents (see Sect. IV.C below).

Quantitative details of the subsequent evolution [13] are beyond the scope of this paper, but certain qualitative features are clear. The magnitude and evolution of the currents depend on the electrical conductivity, which in turn depends on the nuclear crystal and the magnetic field. (Relativistic electrons travel at essentially the speed of light, leading to saturated current densities $\mathcal{O}(n_e c)$, apart from magnetic effects on the density of states [1].) As the magnetic field and rotation affect the electron Fermi energy, the beta equilibria of neutrons [13] and muons are also affected above their respective thresholds. This last effect cannot be included without revising the hadronic equation of state [3]. The Fermi energy will be unequal over the sphere at first, a limit expected to apply to young neutron star crusts. With time, $\epsilon_F$ will equilibrate to the same value everywhere, and the $B$ and $n_e$, $n_p$ profiles will change in parallel.

FIG. 4. The last, partially-filled Landau level $n_L^*$ on a spherical surface, at constant density, defined by Eq. (31); $B_0 = 10B_e$. (a) $\rho = 10^3$ g/cm$^3$; (b) $\rho = 10^5$ g/cm$^3$; (c) $\rho = 10^{13}$ g/cm$^3$. $\theta$ and $\phi$ in radians.
FIG. 5. The partial filling factor $\nu$ of the last Landau level $n^*_L$ on a spherical surface at constant density, defined by Eq. (31); $B_0 = 10B_c$. (a) $\rho = 10$ g/cm$^3$; (b) $\rho = 10^7$ g/cm$^3$; (c) $\rho = 10^{13}$ g/cm$^3$. $\theta$ and $\phi$ in radians.

FIG. 6. The dimensionless Fermi energy $\epsilon_F$ on spherical a surface at constant density, defined by Eq. (32); $B_0 = 10B_c$. (a) $\rho = 10$ g/cm$^3$; (b) $\rho = 10^7$ g/cm$^3$; (c) $\rho = 10^{13}$ g/cm$^3$. $\theta$ and $\phi$ in radians.

4.3 Radial structure

For a simple neutron star, we can assume a given profile of positive ions, and thus electrons, with radius $r$. With the field $|B(r)|$, in addition, the radial dependence of $\epsilon_F$ can be found.

In Fig. 8 the radial profile is shown for $\epsilon_F$ from $r/R = 0.7$ to 1.0, with an expanded subfigure of $r/R = 0.9998–1.0$. Even at this detail, the thin surface non-relativistic layer cannot be seen, but the quantum regime of discretized $\epsilon_F$ steps is clearly visible. Fig. 9 shows $\epsilon_F$ as a function of $r/R$ and $\theta$. The rotational correction drops out in certain cases. If $\sin \phi = 0$ (in the $\mathbf{M} - \Omega$ plane) or $\sin \theta_0 = 0$ (no tilt), $\omega_\theta$ vanishes. Also, if $\sin \theta = 0$ (along the dipole axis), $P_\theta \sim r \sin \theta \cdot P_z \to 0$, and the Coriolis effect disappears. Otherwise the $\epsilon_F$ radial profile is affected dramatically by the rotational correction for relativistic $\epsilon_F$.

FIG. 7. The electron number density $n_e$ on spherical surface at constant Fermi energy, defined by Eq. (30); $B_0 = B_c$. Units are inverse Compton volume. (a) $\epsilon_F = 1.06$; (b) $\epsilon_F = 5.0$; (c) $\epsilon_F = 1800$. $\theta$ and $\phi$ in radians.

FIG. 8. Dimensionless Fermi energy $\epsilon_F$ as function of $r/R$ for simplified neutron star crust model of text; $\sin \theta = \sin \phi = 1$, and $B_0 = 10B_c$. (a) $r/R = 0.7–1.00$; (b) $r/R = 0.9998–1.00$. Terraced steps indicate quantum regime at the surface.

The opposite limit is the case where the Fermi energy is constant in space and no currents flow, a situation probably holding for crusts at later times. Since the field $|\mathbf{B}|$ varies, the electron density $n_e$, and thus the positive ion density $n_p$, must also vary. This is the case where the magnetic field is strong enough to dominate the mechanical structure of dense matter. We show three examples of how $n_e$ is redistributed over a sphere of constant $r$ for fixed $\epsilon_F$ (Fig. 7). The proton density is $n_p = (Z/Z_{\text{eff}})n_e$, and $\rho$ can be inferred from Eq. (A2).
FIG. 9. Dimensionless Fermi energy $\epsilon_F$ as a function of $r/R$ and $\theta$ for simplified neutron star crust model of text. (a) $\sin \phi = 1$, $\tau/R = 0.7-1.00$, $B_0 = 10B_c$, $\sin \theta > 0$; (b) detail showing $\sin \theta > 0.2$. $\theta$ and $\phi$ in radians.

This concludes the treatment of semiclassical orbitals begun in paper I. In this paper, we have found the local one- and many-body states of relativistic charged particles confined to a sphere with an intense, rotating magnetic dipole field.

There remain full quantization with the Dirac equation and the inclusion of the positively-charged lattice structure to determine the local neutron star matter state. A full calculation requires a self-consistent treatment of gravity, nuclear matter, magnetic fields, and currents, with chemical equilibrium and Coulomb neutrality. Depending on the degree of lattice disorder and interelectron forces, various conducting, insulating, or quantum Hall-like many-body states can arise, affecting the macroscopic currents inferred in Sect. IV.

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APPENDIX: MANY-BODY THEORY

To determine the filling of momentum states labelled by $\Pi_z$, the cutoff equation $\zeta^2(\Pi_z) = \zeta_F^2$ must be solved for some given value of $\zeta_F^2$, taken here as $(Z_{eff})^2$. At zeroth order in $\tilde{\omega}$, this equation is trivial. Once the rotational corrections are included, the equation is not only quadratic and linear in $\Pi_z$, but has an implicit dependence on $\Pi_z$ through $\nu_0$. In the $O(\tilde{\omega})$ correction, our procedure is to take $\nu_0^2 = 1 + \epsilon_L^2$, where $\epsilon_L^2$ is the purely two-dimensional Landau term:

$$\epsilon_L^2 = \frac{\hbar |qB|}{m^2 c^2} [2n_L + 1 - \text{sgn}(q)\sigma]$$

and neglect the $\Pi_z^2$ term, as the latter is typically much smaller than one. In that case, the cutoff equation $\zeta^2(\Pi_z) = \zeta_F^2$ is a simple quadratic with the two roots $\Pi_z^\pm$ and

$$\Delta \Pi_z = \Pi_z^+ - \Pi_z^- =$$

$$2\sqrt{\zeta_F^2 + \omega_F^2(1 + \epsilon_L^2) \sin^2 \theta/(1 + 3 \cos^2 \theta)}$$  \hspace{1cm}(33)$$

as the allowed spread of $z$ momenta. Given a value of $n_e$, the values of $\epsilon_F$, $n_L^*$, and $\nu$ are found iteratively, starting with $\tilde{\omega} = 0$, then with this solution used in the $O(\tilde{\omega})$ corrections.

In the opposite case, of fixed $\epsilon_F$, $n_L^*$ is determined, while we set $\nu = 0.1$ and 0.9 as illustrative (Fig. 10). The density $n_e$ is then found. As long as $\nu \neq 0.5$, $n_e$ can fall or rise with $r$. A complete treatment of the bulk crust requires inclusion of nuclear matter and gravity, as well as an interior $B$ field profile, not necessarily a dipole.

A semi-realistic spatial profile of electron density requires the proton density $n_p = n_e$, usually determined in terms of mass density $\rho$. The “effective” electron density $n_e$, the density available for conduction, is
\[
(2\pi \hbar/mc)^3 n_{e,\text{cond}} = \frac{Z_{\text{eff}}}{A} \cdot \frac{1.1 \times 10^{-5} \rho}{\text{g cm}^{-3}},
\]
(34)

where \(Z_{\text{eff}}\) is the number of electrons per nucleus available for conduction (unbound electrons). The atomic number \(A = N + Z\), where the neutron number \(N\) per nucleus is abnormally large for nuclei in an electron Fermi sea. The number density is normalized to a Compton volume \((2\pi \hbar/mc)^3\). From \(\rho = 10\) to about \(3 \times 10^4\ \text{g/cm}^3\), the inner electrons of the atoms remain bound, not participating in conduction; in this case, \(Z_{\text{eff}} < Z\) and can be read off from standard atomic structure \([13]\). The nuclei are always iron \((Z = 26\) and \(Z_{\text{eff}} = 8, 16, 24\) at these densities \([3,18]\). For higher densities, the orbitals of different atoms merge, and \(Z_{\text{eff}} = Z\). (We neglect the formation of partially ionized atoms at ultrahigh densities, assuming that all electrons are stripped from their nuclei.) The results do not depend sensitively on \(Z_{\text{eff}}\).

For nuclear matter composition at densities below complete nuclear dissociation, but for \(Z > 26\), we use the results from \([16,17]\) (see also \([18]\)) at \(\rho < 2 \times 10^{13}\ \text{g/cm}^3\). Our method can be applied at higher densities with free nucleons, but it needs to incorporate the presence of muons and then of heavier strange and non-strange hadrons, and then possibly of a quark-gluon plasma \([3]\). In this paper, only the simplest case of electrons and positive ions is examined.

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