Tests of $\Lambda$CDM and Conformal Gravity using GRB and Quasars as Standard Candles out to $z \sim 8$

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ABSTRACT

We compare the cosmology of conformal gravity (CG), (Mannheim 2006), to $\Lambda$CDM. CG cosmology has repulsive matter and radiation on cosmological scales, while retaining attractive gravity at local scales. Mannheim (2003) finds that CG agrees with $\Lambda$CDM for supernova data at redshifts $z < 1$. We use GRBs and quasars as standard candles to contrast these models in the redshift range $0 < z < 8$. We find CG deviates significantly from $\Lambda$CDM at high redshift and that $\Lambda$CDM is favoured by the data with $\Delta \chi^2 = 48$. Mannheim’s model has a bounded dark energy contribution, but we identify a $\lambda$ fine-tuning problem and a cosmic coincidence problem.

Key words: cosmology: theory – cosmology: observations – dark energy

1 INTRODUCTION

Currently, $\Lambda$CDM is a highly successful theory, providing satisfactory agreement with supernovae data (Riess et al. 1998), the cosmic microwave background (Planck Collaboration et al. 2016) and galaxy clustering (Alam et al. 2017), amongst others. However, dark matter and dark energy, required for these results, have their own issues (Bullock & Boylan-Kolchin 2017). There have been many indirect observations of dark matter, such as the Bullet Cluster (Clowe et al. 2006), but no confirmed direct detection. The cosmological constant has no firm theoretical basis, and the cosmological constant problem (Weinberg 1989) crops up when the amount of dark energy is derived according to the standard model of particle physics.

These problems motivate work on alternative gravity theories aiming to explain the observations without dark matter or dark energy, or to provide a new understanding for the cosmological constant. One such theory is conformal gravity (CG). CG is an alternative gravity theory proposed by Weyl (1918), and more recently developed by Mannheim & Kazanas (1989). For a review of CG, see Mannheim (2006). CG has features that are attractive for a quantum gravity theory: renormalization, conformal symmetry, unitarity and no ghosts (Bender & Mannheim 2008).

In addition to the Newtonian potential, CG has linear and quadratic terms, with respect to radius, which are small compared to the Newtonian potential at solar system scales. Consequently, the solar system predictions are close to those of general relativity (GR). The linear and quadratic terms can be chosen to become important at galaxy scales, which allowed Mannheim & O’Brien (2012) to fit the rotation curves of a sample of 111 spiral galaxies. However, the galaxy fits neglected the non-constant, non-zero Higgs field, whose conformal coupling is required to produce a nontrivial CG cosmology. The galaxy rotation curves of Mannheim & O’Brien (2012) have a Higgs field that varies with radius, implying that particle masses change with radius. When Horne (2016) used a conformal transformation to make the Higgs field constant, there is an additional term from the Higgs field in the galaxy rotation curves, which nearly cancels out the linear term of the potential, leaving CG unable to reproduce the typical, flat rotation curves of galaxies.

Because CG is a fourth-order theory, the scale of the quadratic potential (which is important for understanding galaxy rotation curves and lensing in CG) may be determined by the cosmology. Thus, it is important to understand the cosmological implications of CG, in addition to the galaxy scale effects. We look at the CG cosmology model, (Mannheim 1990), who found no significant difference between CG and $\Lambda$CDM for supernova data for redshifts $z < 1$. More recently, Diaferio et al. (2011) extended the Hubble diagram beyond supernovae (SNIa) by adding gamma-ray burst (GRB) standard candles to the supernova data to test the CG and $\Lambda$CDM cosmologies, and found that $\Lambda$CDM was slightly favoured. But when Diaferio et al. (2011) considered GRBs only, which probe higher redshifts than SNIa, $\Lambda$CDM was greatly preferred over CG.

In this paper, we analyse the Mannheim (2006) cosmological model of CG, and compare its predictions to $\Lambda$CDM’s using a Hubble diagram extended to redshift $z = 8$ by including updated GRB and quasar standard candles. The outline
With these sign conventions, the Einstein-Hilbert action is:

\[ S_{EH} = -\frac{\kappa^2}{8\pi G} \int d^4x \sqrt{-g} R \]

where \( \kappa^2 = \frac{\hbar c}{8\pi G} \), \( G_{\mu\nu} \) is the Einstein tensor and \( T_{\mu\nu} \) is the stress-energy tensor. The stress-energy tensor for a perfect fluid is:

\[ T_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} + p g_{\mu\nu}, \]

where \( \rho \) is the energy density, \( p \) is the pressure and \( U_{\mu} \) is the perfect fluid 4-velocity, with \( U_{\mu}U^{\mu} = -1 \). Hereafter, we shall use natural units, \( c = \hbar = 1 \).

The homogeneous, isotropic spacetime is described by the Robertson-Walker (RW) metric:

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \]

where \( a(t) \) is the scale factor, \( k \) is the curvature and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). ACDM uses a 3-component perfect fluid, each component having \( p = \rho W \) with the equation of state parameter, \( w = 0, \frac{1}{3}, -1 \) for matter, radiation and dark energy respectively. The corresponding Friedmann equations (dropping the \( t \) from \( a(t) \) here on) of ACDM are:

\[ \left( \frac{H}{H_0} \right)^2 = \left( \frac{\dot{a}}{a} \right)^2 H_0^{-2} = \Omega_{\Lambda} + \Omega_K a^{-2} + \Omega_M a^{-3} + \Omega_R a^{-4}, \]

\[ \Omega_K = 1 - (\Omega_{\Lambda} + \Omega_M + \Omega_R), \]

where \( \dot{a} \) denotes derivative with respect to time, \( H_0 \) is the current value of the Hubble parameter, \( \Omega_w = \frac{\rho_w(t)}{\rho_c} \) with \( \rho_c = \frac{3H_0^2}{8\pi G} \) as the critical density at present time and \( \Omega_w(t) = \frac{\rho_w(t)}{\rho_c} \).

Figure 1 serves as a useful comparison to later figures, showing how the scale factor evolves, beginning with a radiation-dominated phase \( a \propto t^{1/2} \) that extends back to the singularity, through to a matter-dominated era \( a \propto t^{2/3} \), to finally a dark energy-dominated era \( a \propto e^{Ht} \). Throughout the rest of the paper, we shall refer to \( \Omega_M = 0.3, \Omega_{\Lambda} = 0.7, \Omega_R = 0 \) as the ACDM model.

### 3 MANNHEIM’S CONFORMAL GRAVITY COSMOLOGY

Mannheim’s CG cosmology is derived from the Weyl action instead of the Einstein-Hilbert action:

\[ S_{Weyl} = -\alpha g \int d^4x \sqrt{-g} C_{\mu\nu\lambda\kappa} C^{\mu\nu\lambda\kappa}, \]

where \( \alpha_g \) is a dimensionless constant and \( C_{\mu\nu\lambda\kappa} \) is the Weyl or conformal tensor. CG is conformally invariant, whereas Einstein’s GR is not.

CG cosmology differs from ACDM by its matter action, which includes a conformal coupling of gravity to the Higgs field:

\[ S_M = -\int d^4x \sqrt{-g} \left( \frac{1}{2} S [\nabla^\mu S_{\mu} - \frac{R}{6} S^2] + \lambda S^4 + \bar{\psi}(\not{D} - m)\psi \right), \]

where \( S \) is the Higgs field, \( X^\mu \) represents the covariant derivative, \( \lambda \) is the dimensionless Higgs self-coupling constant, \( \psi \) is the fermion’s wave function, \( \not{D} \) is the slashed Dirac operator and \( m = h\bar{s} \) is the fermion mass induced by coupling to the Higgs field. In CG, the field equations analogous to the Einstein equations are the Bach equations:

\[ 4\alpha g W_{\mu\nu} = T_{\mu\nu}, \]

where \( W_{\mu\nu} \) is the Bach tensor, created by taking the variation

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**Figure 1.** Top: Scale factor vs time for ACDM, with \( \Omega_{\Lambda} = 0.7 \) and \( \Omega_M = 0.3 \). The majority of the past expansion occurs during the matter dominated Universe. Bottom: The matter, radiation and dark energy contributions vs \( \log(1+z) \) for the ACDM model. \( \Omega_M(t) \) is the black dot-dashed line, \( \Omega_{\Lambda}(t) \) is the red dashed line and \( \Omega_R(t) \) is the blue dotted line. This model is transitioning from matter dominated to dark energy dominated at present time.
of the Weyl action with respect to the metric, and $T_{\mu\nu}$ is the conformal stress-energy tensor.

For the Robertson-Walker metric, the Bach tensor vanishes. Thus, all cosmological terms come from the stress-energy tensor, which is comprised of ordinary matter and radiation, treated as a perfect fluid, plus the scalar Higgs field. Working in the Higgs frame, where the radiation are repulsive and the cosmological constant is derived from the Higgs vacuum energy, $\rho_{\Lambda} = \Delta_{\bar{S}}^0$. By defining $\bar{\Omega}_{\Lambda} = -\epsilon \Omega_{\Lambda}$, we recover the same form as Eqn (5) in $\Lambda$CDM but with $\bar{\Omega}_{\Lambda}$, $\bar{\Omega}_{R} < 0$ and, if we choose $\epsilon < 0$, $\bar{\Omega}_{\Lambda} > 0$. Here on, we use this form of the equations and drop the bar notation.

Figures 1 and 2 show the evolution of the scale factor against time for both $\Lambda$CDM and Mannheim’s model respectively. Mannheim (2006) fits supernova data up to redshift $z \sim 1$ using the model $\Omega_{\ Lambda} = 0.37$, $\Omega_{K} = 0.63$, $\Omega_{R} \approx -10^{-60}$, $\Omega_{M} = 0$. We refer to these values as the Mannheim model. Mannheim (2006) requires $\Omega_{R} \approx -10^{-60}$ to have $T_{\text{max}} > 10^{15}$K. One difference between the models is that $\Lambda$CDM has a singularity, whereas Mannheim’s cosmology has a minimum scale factor, $a_{\text{min}} = \Delta_{\bar{S}}^0$. Thus, all cosmological terms come from the stress-energy tensor, which is comprised of ordinary matter and radiation, treated as a perfect fluid, plus the scalar Higgs field. Working in the Higgs frame, where the radiation are repulsive and the cosmological constant is derived from the Higgs vacuum energy, $\rho_{\Lambda} = \Delta_{\bar{S}}^0$. By defining $\bar{\Omega}_{\Lambda} = -\epsilon \Omega_{\Lambda}$, we recover the same form as Eqn (5) in $\Lambda$CDM but with $\bar{\Omega}_{\Lambda}$, $\bar{\Omega}_{R} < 0$ and, if we choose $\epsilon < 0$, $\bar{\Omega}_{\Lambda} > 0$. Here on, we use this form of the equations and drop the bar notation.

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Figure 2. Top: Scale factor against time for Mannheim’s model universe. The present scale factor is $a = 1$. There is a minimum scale factor, $a_{\text{min}} = 5.7 \times 10^{-30}$, so this model has no singularity at $t = 0$. Thus, the maximum redshift is $z = 1.75 \times 10^{37}$. The vertical dashed lines denote the time where the Universe transitions to curvature dominated and dark energy dominated respectively. The horizontal dashed lines are for $a = a_{\text{min}}$ and $a = 1$. Bottom: The dark energy, curvature and radiation contributions plotted against $\log(1+z)$. $\Omega_k(t)$ is the blue line, $\Omega_f(t)$ is the red dashed line and $\Omega_R(t)$ is the blue dotted line. There are three phases in this model: expansion from the minimum radius in the early Universe driven by radiation, a long period of linear growth via curvature, and an exponential era due to dark energy.

Figure 3. The dark energy contribution vs the sum of the radiation and matter contributions for $\Lambda$CDM in black and Mannheim in red. The dots denote values at $z = 0$, the squares denote values at $z = 1$, and the diamonds denote values at $z = 10$. 

\[ -\epsilon G, \text{ where } \epsilon = \frac{3}{4\pi G \bar{S}_{0}^2}. \]

\[
\left( \frac{H}{H_0} \right)^2 = -\epsilon \left( \Omega_{\Lambda} + \Omega_{M} a^{-3} + \Omega_{R} a^{-4} + \Omega_{K} a^{-2} \right), \quad (11)
\]

\[
\Omega_{K} = 1 + \epsilon \left( \Omega_{\Lambda} + \Omega_{M} + \Omega_{R} \right). \quad (12)
\]

A full derivation from the action may be found in Mannheim (2006) and Appendix A. Therefore, in Mannheim’s model, homogeneous, isotropic matter and radiation are repulsive and the cosmological constant is derived from the Higgs vacuum energy, $\rho_{\Lambda} = \Delta_{\bar{S}}^0$. By defining $\bar{\Omega}_{\Lambda} = -\epsilon \Omega_{\Lambda}$, we recover the same form as Eqn (5) in $\Lambda$CDM but with $\bar{\Omega}_{\Lambda}$, $\bar{\Omega}_{R} < 0$ and, if we choose $\epsilon < 0$, $\bar{\Omega}_{\Lambda} > 0$. Here on, we use this form of the equations and drop the bar notation.

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In contrast to $\Lambda$CDM, which has a radiation era, then matter dominated era followed by the dark energy era, Mannheim’s model has three phases: a quadratic expansion from the minimum radius in the early universe driven by
radiation, a long period of linear growth via curvature, and an exponential era due to negative Higgs vacuum energy. Mannheim’s cosmology always has positive acceleration, unlike ΛCDM which has a decelerating phase when matter or radiation are dominant. Figure 3 plots $Ω_M(t) + Ω_R(t)$ against $Ω_{M}(t)$, ΛCDM begins close to (0,0) and progresses along the flat-geometry diagonal line to (0,1). However, this requires fine-tuning of the initial conditions, because a small deviation from (0,1) leads to large excursions from flat geometry, before arriving at (0,1). The Mannheim cosmology begins at (−∞,0) and evolves rapidly towards the origin, where it stays for 32 Gyr in the curvature-dominated era before moving up the y-axis towards (0,1). CG predicts that the age of the Universe is approximately 32 Gyr, which is roughly 2.5 times greater than the ΛCDM age. For both the flat ΛCDM model and the Mannheim model, the point (0,1) is an attractor.

4 LUMINOSITY DISTANCES

Mannheim (2006) showed that ΛCDM and CG provide equally good fits to SNIa distances out to $z \approx 1$. Here in Fig 4, we extend the Hubble diagram to $z \approx 8$ by including GRB and quasar distance moduli. The distance modulus, $μ$, is:

$$μ = 25 + 5\log_{10}(D_L/\text{Mpc}), \quad (13)$$

where $D_L$, the luminosity distance, is given by:

$$D_L = (1+z) \frac{c}{H_0} \frac{1}{\sqrt{|Ω_{K}|}} S_0 \left( \int_0^z \frac{H_0}{H(z')} dz' \right), \quad (14)$$

with $S_0(x) = (\sin x, x, \sinh x)$ for $(k > 0, k = 0, k < 0)$ respectively.

We use the binned dataset from Figure 5 of Risaliti & Lusso (2015), comprising 808 quasars, in 24 redshift bins that are 0.1 dex in width, as shown in Table 1. The quasars’ distance moduli are calibrated using an empirical relationship between the UV and X-ray fluxes of quasars.

For the GRBs, we utilise the Mayflower sample of 79 GRBs from Liu & Wei (2015), shown in Table 2, in which they calibrate the Amati relation via the Padé approximant method for 138 GRBs, with $z < 1.4$, with the Union 2.1 SNIa data. We used 0.1 dex bins in log($1+z$) to produce the averages in Table 2.

With these distance moduli in hand, we produce the Hubble diagram for both these cosmologies in Figure 4 using Eqs (13) and the cosmological solutions for each theory: ΛCDM in Eqs (5) and (6) and Mannheim in Eqs (11) and (12). The quasar data points are blue and the GRB points are 0.1 dex wide, as shown in Table 1. The quasars’ distance moduli are calibrated using an empirical relationship between the UV and X-ray fluxes of quasars.

![Figure 4](image_url)
(Weinberg 1989), or a Λ fine-tuning problem, where the
predicted value from standard particle physics results in an
extremely high dark energy component, ΩΛ ∼ 10^{120} if it is asso-
ciated to the Planck scale, ΩΛ ∼ 10^{90} if the electroweak scale.
CG cosmology has a self-quenching mechanism (Mannheim 2006),
so that for any vacuum energy ΔS_0^4 < 0 (where ΔS is the value of the
Higgs field in the Higgs frame), ΩΛ(t) must lie within the range 0 ≤ ΩΛ(t) ≤ 1; see Figure 3.

However, we have identified a similar issue in
Mannheim’s cosmology: the Λ fine-tuning problem. Similar to the Λ fine-tuning problem in ΛCDM, we find that to fit Mannheim’s preferred parameters the Higgs self-coupling constant must be tiny, on the order of ∼10^{-117}. Recalling the definition of ΩΛ, Λ, and ϵ, we may rewrite ΩΛ as:

$$\Omega_\Lambda = -\frac{24\lambda^6}{\ell^2_0^4}. \quad (15)$$

Now by rearranging the above equation for Λ by and using
S_\psi = 1 \frac{c}{\hbar} r_{CMB} \Rightarrow \frac{24\lambda^6}{\ell^2_0^4} yields:

$$\Lambda = -\frac{\epsilon \Omega_\Lambda H_0^4}{4\rho_C}. \quad (16)$$

By defining the radiation energy-density as ρR = 4\pi r_{CMB}^4, where T_{CMB} is the present temperature of the CMB, we can rearrange the definition of ΩR to get:

$$\epsilon = -\frac{\Delta R_{\mu} \rho_C}{4\pi r_{CMB}^4}. \quad (17)$$

Then by substituting Eqn (17) into Eqn (16) and re-
turning to SI units, we find that:

$$\Lambda = \frac{\pi \Omega_\Lambda \Omega_R H_0^4}{16\ell^2_0^4}. \quad (18)$$

The supernova distances give ΩΛ = 0.37, and Mannheim adopts ΩR = −10^{−60} so that T_max = T_{CMB}/c/\Omega_\mu exceeds the electroweak scale. Using the observed values of ΩH and T_{CMB}, we find that Mannheim’s model predicts a value for Λ that is of the order Λ ≈ −10^{−117}. This is very small, why is it not zero? One could appeal to some of the same proposed solutions to the fine-tuning problem in ΛCDM here, such as the existence of a multiverse.

ΛCDM also has the cosmic coincidence problem, where a high degree of fine tuning is required so that ΩΛ ∼ Ω_M at current time. Mannheim (2006) claims that his model solves the cosmic coincidence problem, because Ω_M ≈ Ω_Λ. However, there remains a similar cosmic coincidence problem; that the current era just so happens to be when the Universe transitions from curvature to dark energy driven expansion, or that Ω_Λ ∼ Ω_M which also requires fine tuning. Thus CG and ΛCDM have similar fine-tuning problems.

6 CONCLUSIONS

We have compared Mannheim’s CG cosmology to the stan-
dard ΛCDM cosmology. Mannheim (2006) indicated close
agreement with ΛCDM for the predicted distance moduli for redshift z < 1. By collating GRB (Liu & Wei 2015) and quasar (Risaliti & Lusso 2015) data, we have extended the Hubble diagram out to z = 8 and reanalysed the predictions
of the CG cosmology. We find that the CG cosmology deviates significantly from ΛCDM and the data for redshift z > 2. We performed the χ^2 test and found that ΛCDM is the favoured model with Δχ^2 = 48.

We have discussed theoretical problems with these cos-
mological models. Mannheim (2006) claims that his model
solves the cosmic coincidence problem. However, in this paper we found an analogue to the cosmic coincidence problem, involving Ω_R and Ω_Λ, for Mannheim’s model. Additionally, we identified a Λ fine-tuning problem analogous to that of ΛCDM. We determined that using Mannheim’s model, the Higgs self-coupling constant is of the order Λ ≈ −10^{−117}. In summary, ΛCDM and CG have similar fine-tuning issues, but ΛCDM fits the data far better than CG.

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APPENDIX A: DERIVATION OF MANNHEIM’S COSMOLOGY

This treatment follows that of Mannheim (2006). The combined
case is S_\text{total} = S_\text{Weyl} + S_M, where the matter action is:

$$S_M = -\int d^4x \sqrt{-\mathcal{g}} \left( \frac{1}{2} \mathcal{S}^\mu_\nu \mathcal{S}_\mu^\nu - \frac{1}{3} \mathcal{S}^2 \mathcal{R} + \mathcal{A} \mathcal{S}^4 + \psi(\mathcal{D} - h\mathcal{S})\psi \right). \quad (A1)$$

where \mathcal{S}(x) is the Higgs scalar field; \mathcal{R} is the Ricci scalar;
\mathcal{A} is the Higgs self-coupling constant; \psi(x) is the fermion’s
wave function; \( h \) is the Yukawa coupling constant between the Higgs field and the fermions; \( D_\mu \equiv i\gamma_\mu D_\mu \) is the slashed Dirac operator; and \( D_\mu \equiv \partial_\mu + \Gamma_\mu \) is the Dirac operator, where \( \Gamma(x) \) is the fermion spin connection and \( \gamma(x) \) are the gamma matrices.

The contribution \( S^\mu S^\mu - \frac{1}{6} S^2 R \) is a conformally invariant addition to the matter action. Now, we take the variation with respect to \( \psi \). \( S \) and the metric to obtain the following equations of motion.

The Dirac equation:

\[
(\not\partial - hS)\psi = 0 .
\]

The Higgs equation:

\[
S^\mu_\mu + \frac{R}{6} S - 4\lambda S^3 + h\psi \psi = 0 .
\]

The conformal stress-energy tensor:

\[
T^\mu_\nu = i\bar{\psi} \gamma_\mu \partial_\nu \psi + \frac{2}{3} \bar{\psi} \gamma_\mu S_\nu - \frac{1}{6} \gamma_\mu \gamma^\sigma S^\sigma_\nu
\]

\[
- \frac{1}{3} (\bar{\psi} \gamma_\mu \partial_\nu \psi + \frac{2}{3} \bar{\psi} \gamma_\mu S_\nu) - \frac{1}{6} S^2 \left( R^\mu_\nu - \frac{1}{2} \gamma^\sigma \gamma_\sigma R \right)
\]

\[
- \gamma^\rho \left( \lambda S^4 + \bar{\psi}(\not\partial - hS)\psi \right) .
\]

Inserting (A2) into (A4) yields:

\[
T^\mu_\nu = i\bar{\psi} \gamma_\mu \partial_\nu \psi + \frac{2}{3} \bar{\psi} \gamma_\mu S_\nu - \frac{1}{6} \gamma_\mu \gamma^\sigma S^\sigma_\nu
\]

\[
- \frac{1}{3} (\bar{\psi} \gamma_\mu \partial_\nu \psi + \frac{2}{3} \bar{\psi} \gamma_\mu S_\nu) - \frac{1}{6} S^2 \left( R^\mu_\nu - \frac{1}{2} \gamma^\sigma \gamma_\sigma R \right)
\]

\[
- \lambda S^4 \gamma^\rho .
\]

Mannheim (2006) uses a conformal transformation to the Higgs frame, where \( S = S_0 \), so that fermion mass is a space-time constant, \( hS \to hS_0 \). (A5) becomes:

\[
T^\mu_\nu = i\bar{\psi} \gamma_\mu \partial_\nu \psi - \frac{1}{6} S^2 G^\mu_\nu - \gamma^\rho \left( \lambda S^4 \right) .
\]

The incoherent averaging of \( i\bar{\psi} \gamma_\mu \partial_\nu \psi \) (Mannheim 1990) allows a perfect fluid representation for the ordinary matter and radiation: \((\rho + p)U_\mu U_\nu + p g^\mu_\nu\) with pressure \( p \), energy density \( \rho \) and \( U_\alpha \) is the fluid’s 4-velocity. Hence, (A6) is now:

\[
T^\mu_\nu = (\rho + p)U_\mu U_\nu \quad (p - \rho_\Lambda) g^\mu_\nu + \frac{1}{6} S^2 G^\mu_\nu .
\]

where \( \rho_\Lambda = \lambda S^4_0 \) is the Higgs vacuum energy density. The equivalent equation to the Einstein equation of CG is the Bach equation:

\[
4\alpha_g W^\mu_\nu = T^\mu_\nu .
\]

where \( \alpha_g \) is a dimensionless constant and \( W^\mu_\nu \) is the Bach tensor, created by varying the Weyl action with respect to the metric. The FRW metric is a conformally flat metric, thus the Weyl tensor, and hence the Bach tensor, vanishes. Thus (A8) becomes:

\[
0 = T^\mu_\nu .
\]

Thus, to obtain Mannheim’s cosmology, rearrange Equation (A7) to find:

\[
G^\mu_\nu = 8\pi\epsilon G \left( (\rho + p)U_\mu U_\nu + (p - \rho_\Lambda) g^\mu_\nu \right) .
\]

where the effective gravitational constant is \( G_{eff} = -\epsilon G \) with \( \epsilon = \frac{3}{4\pi\lambda S^4_0 G} \).

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