On the dynamics of some vector fields
tangent to non-integrable plane fields

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Let $E^3 \subset TM^n$ be a smooth 3-distribution on a smooth $n$-manifold, and $W \subset E$ a line field such that $[W, E] \subset E$. We give a condition for the existence of a plane field $D^2$ such that $W \subset D$ and $[D, D] = E$ near a closed orbit of $W$. If $W$ has a non-singular Morse-Smale section, we get a condition for the global existence of $D$. As a corollary we obtain conditions for a non-singular vector field $W$ on a 3-manifold to be Legendrian, and for an even contact structure $E \subset TM^4$ to be induced by an Engel structure $D$.

1. Introduction

The only topologically stable families of smooth distributions on smooth manifolds are line fields, contact structures, even contact structures, and Engel structures [2, 6, 8, 17]. An even contact structure is a maximally non-integrable hyperplane field on an even dimensional manifold. An Engel structure is a 2-plane field $D$ on a 4-manifold $M$ such that $E = [D, D]$ is an even contact structure. Engel structures were discovered more than a century ago [2, 6] and they have sparked big interest throughout the years [3, 14, 16, 19, 20].

We want to understand which even contact structures $(M^4, E)$ are induced by Engel structures $D$, i.e. $[D, D] = E$. There are some obvious topological obstructions: $M$ admits an even contact structure if (up to a 2-cover) its Euler characteristic vanishes (see [12]), whereas it admits an Engel structure only if it is parallelizable (up to a 4-cover, see [20]). For this reason we only consider even contact structures $E$ which admit a framing $E = \langle W, A, B \rangle$ where $W$ spans the characteristic foliation, i.e. the unique

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line field $W \subset E$ satisfying $[W, E] \subset E$. In this case an orientable Engel structure compatible with $E$ takes the form $D_L = \langle W, L \rangle$, where $L \in \Gamma(A, B)$ and $[D_L, D_L] = E$.

The same framework can be used to describe different contexts. For example if $M$ is an orientable manifold of dimension 3 and $E := TM = \langle W, A, B \rangle$, then a plane field of the form $D_L = \langle W, L \rangle$, where $L \in \Gamma(A, B)$ and $[D_L, D_L] = E$, is an orientable contact structure for which $W$ is Legendrian. We introduce a more general family of distributions which permits to treat the above cases at once. For a given 3-distribution $E \subset TM$ on a manifold $M$, we say that $W = \langle W \rangle$ generates $E$ up to homotopy if there is a family of plane fields $D_L_s = \langle W, L_s \rangle \subset E$ continuous in $s \in [0, 1]$ and such that $D_L_0 = D_L$ and $D_L_1$ generates $E$.

**Definition 1.1.** If $E = \langle W, A, B \rangle$ and $W = \langle W \rangle$ satisfy $[W, E] \subset E$, we say that $D_L = \langle W, L \rangle \subset E$ generates $E$ up to homotopy if there is a family of plane fields $D_{L_s} = \langle W, L_s \rangle \subset E$ continuous in $s \in [0, 1]$ and such that $D_{L_0} = D_L$ and $D_{L_1}$ generates $E$.

If $\gamma$ is an orbit of $W = \langle W \rangle$, $p \in \gamma$ and $\phi_t$ denotes the flow of $W$ at time $t$, we introduce a rotation angle function $\theta(p; t)$ associated with $L$, whose derivative is non-vanishing if and only if $D_L = \langle W, L \rangle$ generates $E$ in a neighbourhood of $\gamma$. If $\gamma$ is closed of period $T$, we consider the quantity $\text{rot}_{\gamma, p}(L) = \theta(p; T) - \theta(p; 0)$, which we call the rotation number of $L$ along $\gamma$ at $p$. The maximal rotation number $\text{maxrot}_{\gamma}(L)$ of $L$ along $\gamma$ is the maximum of the rotation number under homotopies of $L$ and $\langle A, B \rangle$. This quantity gives an obstruction to the existence of $D$ generating $E$ in a neighbourhood of $\gamma$. If the dynamics of $W$ are particularly simple, we can give a necessary and sufficient condition for the global existence of $D$ generating $E$.

**Theorem A.** Let $E = \langle W, A, B \rangle$ be a rank 3 distribution on a manifold $M$, and denote by $W = \langle W \rangle$. Suppose that $[W, E] \subset E$, and let $W$ be a non-singular Morse-Smale vector field. There exists $D \subset E$ such that $W \subset D$ and that positively generates $E$ on $M$ if and only if there exists $L \in \Gamma(A, B)$ such that $\text{maxrot}_{\gamma}(L) > 0$ for all closed orbits $\gamma$ of $W$.

The previous theorem is new already in the special case of Legendrian vector fields. This question has already been studied in the case of Morse-Smale gradient vector fields in [7].

\[1\] We do not consider all possible homotopies of the plane field $D_L \subset E$, only those tangent to $W$. 

1. Structure of the paper

In Section 2 we introduce the rotation number, and we study its behaviour under homotopies in Section 3. In Section 4 we apply the theory to Morse-Smale vector fields. In Sections 5 we study the case of Legendrian vector fields and even contact structures.

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2. Rotation number

Suppose that \( \mathcal{E} \subset T M \) is a rank 3 distribution which admits a global framing \( \mathcal{E} = \langle W, A, B \rangle \), such that the flow of \( W \) preserves \( \mathcal{E} \), and denote \( W = \langle W \rangle \). Given \( L \in \Gamma \mathcal{E} \) nowhere tangent to \( W \), we want to determine when the distribution \( D_L := \langle W, L \rangle \) is homotopic within \( \mathcal{E} \) to a maximally non-integrable plane field within \( \mathcal{E} \). Since we have fixed a framing, \( \mathcal{E} \) is oriented. Moreover every \( D_L \) that generates \( \mathcal{E} \) uniquely defines an orientation of \( \mathcal{E} \) given by \( \{W, L, [W, L]\} \). We say that \( D_L \) positively (resp. negatively) generates \( \mathcal{E} \) if these orientations coincide (resp. they are opposite).

In analogy with [1, 14], for a given orbit of \( W \) parametrized by the immersion \( \gamma : [a, b] \to M \) we define the developing map

\[
\delta_\gamma : [a, b] \to \mathbb{RP}^1 \equiv \mathbb{P}(\mathcal{E}/\mathcal{W}|_{\gamma})
\]

via

\[
\delta_\gamma(t) = \left[D_L|_{\gamma(t)}\right] \in \mathbb{P}(\mathcal{E}/\mathcal{W}|_{\gamma(t)}) \equiv \mathbb{P}(\mathcal{E}/\mathcal{W}|_{\gamma(0)}),
\]

where the identification \( \mathcal{E}/\mathcal{W}|_{\gamma(t)} \equiv \mathcal{E}/\mathcal{W}|_{\gamma(0)} \) is given by \( \gamma^{-1} \). A framing \( \{A, B\} \) fixes a trivialization of \( \mathcal{E}/\mathcal{W} \) and permits to lift \( \delta_\gamma \) to an angle function \( \theta \) at \( p = \gamma(a) \), up to choosing a lift \( \theta(p; 0) \). If we fix \( W \) such that \( \mathcal{W} = \langle W \rangle \), this furnishes a parametrization \( \gamma \) via the flow \( \phi_t \) of \( W \), so that the angle function \( \theta \) satisfies

\[
\delta_\gamma(t) = [\phi_{-t_a} L(p)] = [\cos \theta(p; t) A(p) + \sin \theta(p; t) B(p)].
\]

With techniques similar to the ones used in [1, 4] we can prove the following
Proposition 2.1. The distribution $\mathcal{D}_L = \langle W, L \rangle \subset \mathcal{E}$ generates $\mathcal{E}$ in a neighbourhood of an orbit parametrized by an immersion $\gamma$ if and only if $\delta_\gamma$ is an immersion.

Definition 2.2. Let $\mathcal{E} = \langle W, A, B \rangle$ be as above and let $\phi_t : [0, T] \to M$ parametrize a closed orbit of $W$ of period $T$. We call rotation number of $L$ around $\gamma$ at $p = \gamma(0)$ the quantity

$$\text{rot}_{\gamma, p}(L) = \theta(p; T) - \theta(p; 0).$$

The rotation number is not an integer. Moreover it depends on the choice of $\{A, B\}$, and is not invariant under homotopies of $L$, as the following example shows.

Example 2.3. The Lie algebra $\mathfrak{g}$ of the Lie group $\text{Sol}^1_4$ is generated by $\{W, X, Y, Z\}$ satisfying $[W, X] = -X, [W, Y] = Y, [X, Y] = Z$, and all other brackets are zero. We have a left-invariant Engel structure $\mathcal{D} = \langle W, X + Y \rangle$ (see [19] for more details). The left-invariant even contact structure $\langle W, X, Y \rangle$ has characteristic foliation spanned by $W$, whose flow preserves $\langle X \rangle$ and $\langle Y \rangle$. Hence for each compact quotient $\text{Sol}^1_4/\Gamma$ such that $W$ admits a closed orbit $\gamma$, we have $\text{rot}_{\gamma, p}(X) = 0$. Notice that $L_s = X + sY$ gives a homotopy $\mathcal{D}_L_s = \langle W, L_s \rangle$ between $\mathcal{D}_L_0 = \langle W, X \rangle$ and $\mathcal{D}_L_1 = \langle W, X + Y \rangle$, which is an Engel structure. In particular $\text{rot}_{\gamma, p}(X + Y) \neq 0$ by Proposition 2.1.

We have invariance of the rotation number under a smaller family of homotopies of $L$.

Lemma 2.4. Let $L_\tau$ for $\tau \in [0, 1]$ be a smooth family of vector fields tangent to $\mathcal{E} = \langle W, A, B \rangle$ and nowhere tangent to $W$. If $L_\tau(p) = L_0(p)$ for all $\tau \in [0, 1]$ then $\text{rot}_{\gamma, p}(L_1) = \text{rot}_{\gamma, p}(L_0)$.

Proof. Parametrize $\gamma$ via $\phi_t$, the flow of $W$, and denote by $\theta_\tau$ the angle function associated with $L_\tau$. Now the angle functions $\theta_0$ and $\theta_1$ are homotopic relative to the end points through the family of angle functions $\theta_\tau$, which concludes the proof. \qed

If $L(p)$ and $W$ are fixed, but the homotopy class of $L$ is allowed to vary, the rotation number may vary by an integer multiple of $2\pi$. By definition, $\text{rot}_{\gamma, p}(L)$ is invariant under homotopies of $\{A, B\}$ relative to $p$. One can show that changing representative in the homotopy class of $\{A, B\}$ changes
the rotation number at most \( \pi \) (see the proof of Proposition 2.4). This suggests to take into account all possible “initial phases” of \( L \) and choices of \( \mathcal{B}(p) = \{ A(p), B(p) \} \). More precisely, identifying \( L \) with a map \( L : M \to S^1 \) via the framing \{A, B\}, and denoting by \( R(\eta) \) the rotation of \( S^1 \) of angle \( \eta \in \mathbb{R} \), we define

\[
(2.1) \quad \Phi_{\gamma, B(p)}^L : \mathbb{R} \to \mathbb{R} \text{ s.t. } \eta \mapsto \text{rot}_{\gamma, p}(R(\eta) \circ L).
\]

Taking the maximum with respect to all possible initial phases and choices of \( \mathcal{B}(p) \) we get

**Definition 2.5.** The maximal rotation number of \( L \) along \( \gamma \) is

\[
\maxrot_{\gamma}(L) = \max \left\{ \Phi_{\gamma, B(p)}^L(\eta) \mid \eta \in \mathbb{R}, \mathcal{B}(p) \right\}.
\]

**Lemma 2.6.** The maximal rotation number of \( L \) along \( \gamma \) does not depend on \( p \).

**Proof.** Applying the linearised flow of \( W \), we see that \( r = \Phi_{\gamma, B(p)}^L(\eta) \) coincides with the rotation number of \( \phi_t R(\eta) \circ L \) calculated with respect to \( \phi_t \mathcal{B}(p) \). There is an angle \( \eta' \in \mathbb{R} \) such that \( R(\eta') \circ L \) and \( \phi_t R(\eta) \circ L \) coincide at \( \phi_t(p) \), up to a positive rescaling. Since both \( R(\eta') \circ L \) and \( \phi_t R(\eta) \circ L \) are homotopic to \( L \), they must be homotopic to each other relative to \( \{\phi_t(p)\} \). By Lemma 2.4 \( r = \Phi_{\gamma, \phi_t \mathcal{B}(p)}^L(\eta') \leq \maxrot_{\gamma}(L) \) calculated in \( \phi_t(p) \). Now using transitivity of \( \phi_t \) we conclude the proof. \( \square \)

**Theorem 2.7.** Let \( \mathcal{E} = \langle W, A, B \rangle \) be a distribution of rank 3 such that \([W, \mathcal{E}] \subset \mathcal{E}\), and let \( \gamma \) be a closed orbit for \( W \). Then \( \mathcal{D}_L = \langle W, L \rangle \) generates \( \mathcal{E} \) in a neighbourhood of \( \gamma \) up to homotopy if and only if \( |\maxrot_{\gamma}(L)| > 0 \).

**Proof.** Let \( L_t \) for \( t \in [0, 1] \) be a homotopy such that \( L = L_0 \) and \( \langle W, L_1 \rangle \) is maximally non-integrable within \( \mathcal{E} \) in a neighbourhood of \( \gamma \). We need to show that for some \( p \in \gamma \), there is a homotopy relative to \( L_1(p) \) between \( L_1 \) and \( R(\eta) \circ L \) for some \( \eta \in \mathbb{R} \). This is done by taking \( \eta \) such that \( R(\eta) \circ L(p) = L_1(p) \), exactly as in the proof of Lemma 2.6. This implies the claim thanks to Lemma 2.4 and Proposition 2.1.

Conversely let \( |\maxrot_{\gamma}(L)| > 0 \). Without loss of generality we can suppose that \( \text{rot}_{\gamma, p}(L) > 0 \). First homotope \( L \) relative to \( \{p\} \) and to the boundary of \( \mathcal{O}(p) \) to a maximally non-integrable distribution within \( \mathcal{E} \) near \( p \). The rotation number does not change by Lemma 2.4. Fix \( \mathcal{W} = \langle W \rangle \) and let \( \phi_t \) denote its flow. For \( \epsilon > 0 \) small, take a disc \( D^3 \to M \) centered at \( \phi_t(p) \) and
everywhere transverse to \(W\). Up to shrinking the disc \(D^3\), we can suppose that the map \(F : D^3 \times [\epsilon, T - \epsilon] \to M\) given by the flow \((q, t) \mapsto \phi_t(q)\) is an embedding and hence a flow box for \(W\). In this chart for \(q \in D^3\) we can express

\[
F^* L(\phi_t(q)) = \rho(q; t) \left( \cos \theta(q; t) F^* A(\phi_t(q)) + \sin \theta(q; t) F^* B(\phi_t(q)) \right),
\]

for some functions \(\rho > 0\) and \(\theta\). Since \(\text{rot}_{W_p}(L) > 0\), up to choosing \(\epsilon > 0\) small enough, we can suppose that \(\theta(0; T - \epsilon) - \theta(0; \epsilon) > 0\). Hence there exists a homotopy \(\theta_\tau : D^3 \times [\epsilon, T - \epsilon] \to \mathbb{R}\) such that \(\theta_0 = \theta\), the restriction of \(\theta_\tau\) to the boundary \(\partial(D^3 \times [\epsilon, T - \epsilon])\) is \(\theta\), and

\[
\theta_1(0; t) = h(t) \left( \theta(0; T - \epsilon) - \theta(0; \epsilon) \right) + \theta(0; \epsilon)
\]

for a smooth step function \(h\) (see Figure 2.1). This defines a family of vector fields \(L_t\) such that \(D_{L_1}\) generates \(E\) on a (possibly smaller) neighbourhood of \(\gamma\).

\[\Box\]

3. Character of closed orbits of \(W\)

We now consider the action \([\phi_t] : \mathbb{P}(E_p/W_p) \to \mathbb{P}(E_{\phi_t(p)}/W_{\phi_t(p)})\) of the flow \(\phi_t\) of a section \(W\) of \(W\) on \(\mathbb{P}(E/W)\). This is discussed in detail in [13] for the case of Engel structures. If \(p \in M\) is contained in a closed orbit of \(W\) of period \(T\), then \(P := [\phi_T] \in \text{PSL}(2, \mathbb{R})\), where we identify \(\mathbb{R}P^1 = \mathbb{P}(E_p/W_p)\).

We say that a closed orbit \(\gamma\) is:

- **Elliptic** if \(|\text{tr} P| < 2\) or \(P = \pm \text{id}\), in which case we can represent \(P\) by a rotation \(P \equiv R(\delta)\) with \(\delta \in \mathbb{R}\).
Proposition 3.1. Let $W$, $A$, $B$ be a distribution of rank 3 such that $\langle W, E \rangle \subset \mathcal{E}$, $\mathcal{D}_L$ a distribution of rank 2 such that $\mathcal{D}_L = \langle W, L \rangle \subset \mathcal{E}$, $\gamma$ a closed orbit for $W$, and $p$ a point on $\gamma$.

1) If $\gamma$ is hyperbolic, then for every $\eta \in \mathbb{R}$ we have $|\Phi^L_{\gamma,p}(\eta) - \text{rot}_{\gamma,p}(L)| < \pi$. Moreover there exists a constant $c \in (0, \pi)$ such that $\mathcal{D}_L$ positively generates $\mathcal{E}$ in a neighbourhood of $\gamma$ up to homotopy if and only if $\text{rot}_{\gamma,p}(L) > -c$.

2) If $\gamma$ is parabolic, then for every $\eta \in \mathbb{R}$ we have $|\Phi^L_{\gamma,p}(\eta) - \text{rot}_{\gamma,p}(L)| < 2\pi$. Moreover there exists a constant $c \in (0, 2\pi)$ such that $\mathcal{D}_L$ positively generates $\mathcal{E}$ in a neighbourhood of $\gamma$ up to homotopy if and only if $\text{rot}_{\gamma,p}(L) > -c$.

3) If $\gamma$ is elliptic, then $\Phi^L_{\gamma,p}(\eta)$ does not depend on $\eta$.

Proof. The developing map $\delta^L_\gamma$ of the rotated plane field $\mathcal{D}_{R(\eta) L}$ satisfies

$$\delta^L_\gamma(T) = [\phi_{-\tau \gamma}R(\eta)L(p)] = [\phi_{-\tau \gamma}R(\eta)\phi_{\tau \gamma}]\delta^L_\gamma(T) = P^{-1}R(\eta)P \circ \delta^L_\gamma(T).$$

We need to analyse the rotation induced by $M = P^{-1}R(\eta)P$ on $v = \phi_{-\tau \gamma}L(p)$. $P$ will rotate $v$ by an angle $r$, $R(\eta)$ will further rotate it by an angle $\eta$, and finally $P^{-1}$ by an angle $r'$. Now denoting by $\theta_{R(\eta)L}$ and $\theta_L$ the rotation angles associated with $R(\eta)L$ and $L$ we have

$$\text{rot}_{\gamma,p}(R(\eta)L) = \theta_{R(\eta)L}(p; T) - \theta_{R(\eta)L}(p; 0) = \theta_L(p; T) + r + \eta + r' - \theta_L(p; 0) - \eta,$$

hence it suffices to study the term $r + r'$. In the case of a hyperbolic orbit we have $|r|, |r'| < \pi/2$, whereas for a parabolic orbit we have $|r|, |r'| < \pi$. If $\gamma$ is elliptic, then $M = R(\delta)R(\eta)R(-\delta) = R(\eta)$, so that $r + r' = 0$. \qed
Remark 3.2. The cases where $\text{rot}_{\gamma, p}(L) \leq 0$ and nonetheless $D_L = \langle W, L \rangle$ positively generates $\mathcal{E}$ up to homotopy on $\gamma$ occur only when $\gamma$ is hyperbolic or parabolic. In these cases $\text{rot}_{\gamma, p}(L)$ is not allowed to be “too negative”.

Corollary 3.3. Let $\mathcal{E} = \langle W, A, B \rangle$ be a distribution of rank 3 such that $[W, \mathcal{E}] \subset \mathcal{E}$, and let $\gamma$ be an unknotted elliptic closed orbit for $W$. Then the rotation number $r = \text{rot}_{\gamma, p}(L)$ of $L \in \Gamma(A, B)$ does not depend on $L$. In particular there exists an oriented plane field $D$ such that $W \subset D \subset \mathcal{E}$ and which positively generates $\mathcal{E}$ on a neighbourhood of $\gamma$ if and only if $r > 0$.

Proof. Let $L$ and $L'$ be non-singular vector fields in $\langle A, B \rangle$; identify them with maps $L, L': M \to S^1$. Since $\gamma$ is unknotted, there is an embedded disc $D^2$ such that $\partial D^2 = \gamma$, hence there exists a homotopy between $L$ and $L'$. Since $\gamma$ is elliptic, by point (3) of Proposition 3.1 we have that $r = \text{rot}_{\gamma, p}(L) = \text{rot}_{\gamma, p}(L')$. The second claim now follows directly from Theorem 2.7.

Notice that the hypothesis that $\gamma$ is unknotted is equivalent to $\gamma$ being null-homotopic if the dimension of $M$ is greater than 3.

4. Morse-Smale vector fields

Since the dynamics of non-singular Morse-Smale (NMS) vector fields can be described once we understand neighbourhoods of the closed orbits, it is reasonable to expect that the rotation number will play a central role when $W$ is NMS. For the basic theory of Morse-Smale vector fields see [10].

4.1. Morse-Smale vector fields and round handle decompositions

Recall that a NMS vector field $W$ is a non-singular vector field which has finitely many non-degenerate closed orbits $\gamma_1, \ldots, \gamma_m$, whose union is the non-wandering set $\Omega = \gamma_1 \cup \cdots \cup \gamma_m$. Moreover for every $i, j \in \{1, \ldots, m\}$ the stable manifold $W^s(\gamma_i)$ and the unstable manifold $W^u(\gamma_j)$ intersect transversely. A round handle decomposition (RHD) of $M$ is a filtration $M_1 \subset M_2 \subset \cdots \subset M_m = M$ where $M_k$ is obtained from $M_{k-1}$ by attaching a round handle $R_h = D^h \times D^{n-h-1} \times S^1$. We call $h$ the index of the round handle, $\partial_+ R_h = D^h \times S^{n-h-2} \times S^1$ the enter region or the positive boundary and $\partial_- R_h = S^{h-1} \times D^{n-h-1} \times S^1$ the exit region or the negative boundary.
**Theorem 4.1** [15]. Let $W$ be a non-singular Morse-Smale vector field on $M$. Then $M$ admits a RHD $M_1 \subset M_2 \subset \cdots \subset M_m = M$ such that every round handle $R$ is a neighbourhood of a closed orbit $\gamma$ of $W$, and the index of $R$ (as a handle) is the index of $\gamma$ (as a closed orbit). The attaching procedure is performed via the flow of $W$, which is transverse to $\partial M_k$ pointing outwards for every $k$.

We include a sketch of the proof for completeness, since it will be relevant in the proof of Theorem 4.2.

**Sketch of proof.** The idea is to order the closed orbits of $M$ via $\gamma_i \leq \gamma_j$ if $W^u(\gamma_i) \cap W^s(\gamma_j) \neq \emptyset$, and reason by induction. In other words, $\gamma_i \leq \gamma_j$ if there is a orbit whose $\alpha$-limit is $\gamma_i$ and whose $\omega$-limit is $\gamma_j$. The no cycle condition (see [18]) ensures that this is compatible with a total ordering of $\{\gamma_1, \ldots, \gamma_m\}$.

The first orbits in the ordering are the source orbits, i.e. the ones for which $W^u = \emptyset$, hence we construct $M_1$ by taking a neighbourhood of $\gamma_1$. Suppose that we have constructed inductively $M_{k-1}$ such that $\gamma_1, \ldots, \gamma_{k-1} \subset M_{k-1}$, $\gamma_j \cap M_{k-1} = \emptyset$ for $j > k - 1$, and the flow is transverse to $\partial M_{k-1}$ pointing outwards. If $\gamma_k$ is a source orbit, then we take a neighbourhood $R_k$ disjoint from $M_{k-1}$ and define $M_k = M_{k-1} \cup R_k$.

If $\gamma_k$ is not a source orbit, this means that $M_{k-1}$ contains all of them, so a generic point in $M \setminus \{\gamma_1, \ldots, \gamma_{k-1}\}$ has to have one of the source orbits as $\alpha$-limit. We take a small tubular neighbourhood $R_k$ of $\gamma_k$ and we attach it using all flow lines of $W$ that have $\alpha$-limit in $M_{k-1}$. This might introduce corners and the boundary of $M_k$ will not be transverse to $W$. For these reasons we smoothen it as illustrated in Figure 4.1. For further details on the proof see [19].

4.2. Morse-Smale flows preserving a 3-distribution

We give a necessary and sufficient condition for the existence of $\mathcal{D} \subset \mathcal{E}$ that generates $\mathcal{E}$ when $W$ is NMS.

**Theorem 4.2.** Let $\mathcal{E} = \langle W, A, B \rangle$ be a rank 3 distribution on a manifold $M$, and denote by $\mathcal{W} = \{W\}$. Suppose that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, and let $W$ be a non-singular Morse-Smale vector field. There exists $\mathcal{D} \subset \mathcal{E}$ such that $\mathcal{W} \subset \mathcal{D}$ and that positively generates $\mathcal{E}$ on $M$ if and only if there exists $L \in \Gamma\langle A, B \rangle$ such that $\text{maxrot}_\gamma(L) > 0$ for all closed orbits $\gamma$ of $W$. 

Figure 4.1: Smoothen the corners.

Proof. If such a plane field $D$ exists, then we can take $L$ to be any vector field satisfying $D = (W, L)$ and the claim follows by Proposition 2.1. The idea for the converse is to construct $D$ inductively using the RHD of Theorem 4.1. First we construct $D$ in a neighbourhood of the source orbit $\gamma_1$ using Theorem 2.7. Suppose that we have attached $k-1$ handles to obtain $M_{k-1}$, and that we want to attach the $k$-th handle $R_k$. If $\gamma_k$ is a source orbit then we construct $D$ on $R_k = \mathcal{O}(\gamma_k)$ as above, and attach it by disjoint union $M_k = M_{k-1} \cup R_k$. This procedure yields a plane field $D$ homotopic to $D_L = (W, L)$ which generates $E$ along the core of each handle.

If $\gamma_k$ is not a source orbit, the proof Theorem 4.1 ensures that $R_k$ is a neighbourhood of $\gamma_k$, and that the attaching procedure happens via the flow of $W$. We first construct $D$ on $R_k$ using Theorem 2.7. The existence of $L$ ensures that the $D$ extends to a plane field on $M_k$ which generates $E$ on a neighbourhood of $M_{k-1}$ and of $\gamma_k$.

In general we cannot homotope this plane field to a maximally non-integrable one on $M_k$. The problem is that the attaching region is of the form $\partial_+ R_k \times I$, where $\partial_+ R_k \times \{1\}$ is the subset of $R_k$ where $W$ points inwards, and $W$ is tangent to the $I$-factor on $\partial_+ R_k \times I$. This means that, on the universal cover $\partial_+ \tilde{R}_k \times I$, a lift of $L$ takes the form $\tilde{L} = \cos f_t \tilde{A} + \sin f_t \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are lifts of $A$ and $B$, and $f_t : \partial_+ \tilde{R}_k \to \mathbb{R}$ is a $I$-family of angle functions. Hence we can homotope $L$ transversely to $\partial_t$ so that $\langle \partial_t, L \rangle$ generates $E$ if and only if $f_1 > f_0$. There is no reason for this to happen in general.
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Let $K = \max \{f_1(p) - f_0(p) | p \in \partial_x \tilde{R}_k\}$. For any $p \in \partial M_{k-1} \times (-\epsilon, \epsilon)$ on a collared neighbourhood $\partial M_{k-1}$, the vector field $L$ can be described by an embedding $h_p : (-\epsilon, \epsilon) \to S^1$. We substitute it by $\tilde{h}_p : (-\epsilon, \epsilon) \to S^1$ which coincides with $h$ on $Op\{\{-\epsilon, \epsilon\}\}$, and such that it makes a number of turns around $S^1$ bigger than $K/2\pi$. In this way we obtain a new vector field $L'$, not homotopic to $L$ in general, and such that its associated family of angle functions $f'_t$ satisfies $f'_1 > f'_0$. We can now homotope $L'$ to a maximally non-integrable plane field within $E$ on the attaching region. We might now need to round the corners of $M_k$, and this can be done exactly as in the proof of Theorem 4.1 (see Figure 4.1).

$\square$

5. Morse-Smale Legendrians and even contact structures

Theorem 4.2 gives a necessary and sufficient condition for a NMS vector field to be Legendrian. An interesting example of 3-manifold admitting NMS vector fields is $S^3$. However only very few 3-manifolds admit such vector fields (see [15, Theorem A]). It is interesting to know when a given vector field $L$ is transverse to a contact structure. This question has already been studied in [9] for $L$ tangent to the fibres of a $S^1$-bundle over a surface, and in [11] for $L$ tangent to the fibres of a Seifert fibration.

If $L$ is Legendrian for some orientable contact structure $D$, then there is a contact structure $\tilde{D}$ transverse to $L$. Indeed choose $\tilde{D}$ such that $D = \langle L, \tilde{L} \rangle$ and consider $\tilde{D} = \phi_\epsilon^* D$, where $\phi_\epsilon$ denotes the flow of $\tilde{L}$ for small time $\epsilon$. The contact condition ensures that $\tilde{D}$ is transverse to $D$, moreover it contains $L$, so it is transverse to $L$.

Corollary 5.1. There exists a vector field on $S^3$ which is transverse to a contact structure but never Legendrian.

Proof. Consider the vector field $W$ normal to the canonical Reeb foliation on $S^3$. Using the theory of confoliations [5, Chapter 2] we can $C^0$-deform the tangent bundle of the Reeb foliation to get a contact structure, so that $L$ is transverse to a contact structure. $L$ has two unknotted elliptic closed orbits with trivial monodromy, which obstructs the existence of a contact structure for which $L$ is Legendrian.

Theorem 4.2 suggests to study even contact structures which admit a NMS section of $W$. It is not clear if every parallelizable 4-manifold admits such structures, and in fact many NMS flows on 4-manifolds cannot span the characteristic foliation of an even contact structure. On the other hand this property becomes true if we allow perturbations of $W$. 
Lemma 5.2. Near every closed orbit \( \gamma \) of a NMS vector field \( W \) on \( M \) there exists an even contact structure \( E \), whose characteristic line field is spanned by a \( C^0 \)-perturbation of \( W \) fixing \( \gamma \).

Proof. Up to a perturbation, \( \gamma \) has a tubular neighbourhood \( \nu_\gamma = S^1 \times D^3 \) where

\[
W|_{\nu_\gamma} = \partial_\theta + 2\epsilon_1 x \partial_x + 2\epsilon_2 y \partial_y + 4\epsilon_3 z \partial_z,
\]

with \( \epsilon_i = \pm 1 \) depending on the index of \( \gamma \). This is proven using the linearised Poincaré map (see [10] for more details). If \( \epsilon_i \) are all equal, then \( W \) is Liouville for the symplectic form \( \omega = dx \wedge dy + dz \wedge d\theta \), so we have an even contact form \( \alpha = i_W \omega \). If the \( \epsilon_i \) are not all equal, then the vector field \( V = 2\epsilon_1 x \partial_x + 2\epsilon_2 y \partial_y + 4\epsilon_3 z \partial_z \) preserves the contact structure defined by \( \eta = dz - x dy + y dx \) on \( D^3 \), so that \( \nu_\gamma \) is the suspension of the time 1 flow of \( V \) (see [13] for more details on this construction).

The methods developed in this paper are well-suited for constructing examples of even contact structures which do not admit compatible Engel structures.

Proposition 5.3. Every even contact structure is \( C^0 \)-close to one which is not induced by an Engel structure.

Proof. On a manifold \( M \) consider an even contact structure \( E \) with characteristic foliation \( W \). On a small neighbourhood \( U \) construct an even contact structure \( E' \) with a contractible characteristic closed orbit \( \gamma \) having trivial monodromy. Make sure that \( E'|_{O(\gamma)} \) extends to a formal even contact structure on \( M \), which coincides with \( E \) on \( M \setminus U \). We conclude the proof using the (relative) complete h-principle for even contact structures (see [12]).

References

[1] R. L. Bryant and L. Hsu, Rigidity of integral curves of rank 2 distributions, Invent. Math. 114 (1993), no. 1, 435–461.

[2] E. Cartan, Sur quelques quadratures dont l’élément différentiel contient des fonctions arbitraires, Bull. Soc. Math. France 29 (1901), 118–130.

[3] R. Casals, J. L. Perez, A. del Pino, and F. Presas, Existence h-principle for Engel structures, Invent. Math. 210 (2017), 417–451.

[4] R. Casals, A. del Pino, and F. Presas, Loose Engel structures, Compos. Math. 156 (2020), 412–434.
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[5] Y. Eliashberg and W. P. Thurston, *Confoliations*, American Mathematical Soc., Vol. 13 (1998).

[6] F. Engel, *Zur Invariantentheorie der Systeme Pfaff’scher Gleichungen*, Leipz. Ber. Band 41 (1889), 157–176.

[7] J. Etnyre and R. Ghrist, *Gradient flows within plane fields*, Comment. Math. Helv. 74 (1999), no. 4, 507–529.

[8] V. Y. Gershkovich and A. M. Vershik, *Nonholonomic dynamical systems*, Geometry of Distributions and Variational Problems (Russian) (1987).

[9] E. Giroux, *Structures de contact sur les variétés fibrées en cercles au dessus d’une surface*, Comment. Math. Helv. 76 (2001), 218–262.

[10] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and Its Applications. Cambridge: Cambridge University Press (1995).

[11] P. Lisca and G. Matić, *Transverse contact structures on Seifert 3-manifolds*, Algebr. Geom. Topol. 4 (2004), no. 2, 1125–1144.

[12] D. McDuff, *Applications of convex integration to symplectic and contact geometry*, Ann. Inst. Fourier 37 (1987), 107–133.

[13] Y. Mitsumatsu, *Geometry and dynamics of Engel structures*, preprint (2018), arXiv:1804.09471.

[14] R. Montgomery, *Engel deformations and contact structures*, North. Calif. Symppl. Geom. Sem., AMS Transl. Ser. 2 196 (1999), 103–117.

[15] J. W. Morgan, *Non-singular Morse-Smale flows on 3-dimensional manifolds*, Topology 18 (1978), 41–53.

[16] A. del Pino and T. Vogel, *The Engel-Lutz twist and overtwisted Engel structures*, to appear in Geometry and Topology. Preprint (2018), arXiv:1712.09286.

[17] F. Presas, *Non-integrable distributions and the h-principle*, Eur. Math. Soc. Newsl. 99 (2016), 18–26.

[18] S. Smale, *Differentiable dynamical systems*, Bull. Am. Math. Soc. 73 (1967), 747–817.

[19] T. Vogel, *Maximally non-integrable plane fields on Thurston geometries*, International Mathematics Research Notes (2006), 1–30.
[20] T. Vogel, *Existence of Engel structures*, Ann. of Math. (2) **169** (2009), no. 1, 79–137.

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