Solitary waves propagation in three-level atomic media

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Abstract. In this paper analytical solutions of a non-linear differential equations describing solitary wave-pair propagation are derived as well as the existence conditions in absorbing optical dense media.

1. Introduction

Solitons are non-dispersive wave packets which travel at constant velocity [1-7]. Solitons are formed via non-linear interaction effects with media. A soliton is considered as solitary, traveling wave pulse solution of nonlinear partial differential equation. The beginning of soliton physics dated back to the 1834 when John Scott Russell observed the "great wave of translation". His report [8] on waves represents the first scientific account of solitons in history. Later, in 1877 a mathematical model, KdV equation, of waves on shallow water surfaces was introduced by Boussinesq which was rediscovered by Diederik Korteweg and Gustav de Vries in 1895 [9]. It was notable that this non-linear partial differential equation has solutions that can be exactly specified. On 1965 Zabusky and Kruskal [10] solved numerically the KdV equation. It was on 1967, Gardner, Greene, Kruskal and Miura [11] discovered an inverse scattering transform enabling analytical solution of the KdV equation. In 1973, Robin Bullough (see [12] for details) showed that solitons could exist in optical fibers while presenting the first mathematical report of the existence of optical solitons. The most important practical use for solitons has been in fiber-optic communications. In 1988 Mollenauer and Smith [13] have proven by experiments at the Bell Laboratory that optical solitons could travel without significant deformation. Recently researchers from Karlsruhe and Lausanne [14] have shown a record-high speed optical communication via soliton. Most of the previous analytical studies were limited to the lossless systems. The previous models usually neglect the dissipation of the system. By including the dissipation, existence conditions of the soliton propagation appear. This paper aims to study the soliton-pair propagation in an atomic dissipative medium. The soliton and the soliton-pair propagation in absorbing atomic three-level system has been explored in a previous works [15,16,17,18], however the study was limited to the case for equal coupling constants between the two coherent fields and the two allowed atomic transitions. In this paper we will explore the analytical soliton-pair solutions for unbalanced coupling between the two coherent lights and the atomic transitions. Limiting condition of the soliton-pair propagation will be derived. The speed of the soliton-pair will be determined.

2. Model
In this paper we study the solitary wave propagation in a three-level system in lambda configuration. The two allowed atomic transitions are interacting with two laser fields applied to the stocks and pump transitions. The atomic system has 3 levels, one is for the excited state \(|0\rangle\) and two ground states \(|1\rangle\) and \(|2\rangle\). Due to the selection rule, the possible transitions of the atomic system from one quantum state to another are contraint. The two allowed atomic transitions are only between the excited state and the ground states. These atomic transitions are excited resonantly by two optical fields with amplitudes \(E_1\) and \(E_2\) and frequencies \(\omega_1\) and \(\omega_2\). The optical driving field is then given by:

\[
E(x, t) = E_1(x, t)e^{k_1x-\omega_1t} + E_2(x, t)e^{k_2x-\omega_2t}
\]  

where \(k_j\) designs the wavenumbers defined by \(k_{1,2} = \frac{\omega_{1,2}}{c}\), \(c\) is the vacuum light speed. Here we consider that the amplitudes of the classical light amplitudes are slowly varying in space and time verifying \([13, 14, 15]\):

\[
\left| \frac{\partial E_j}{\partial t} \right| \ll \omega_j |E_j| \text{ and } \left| \frac{\partial E_j}{\partial x} \right| \ll \frac{\omega_j}{c} |E_j|
\]

This approximation known as Slowly Varying Envelope Approximation (SVEA) is often used when the variation of the envelope of the optical is slowly varying in time and space compared to a period or wavelength. This restraints the spectrum of the pulse to be narrow-banded.

This is not the case if the pulses have a wide band spectrum or they are very short such as atto-second pulses \([19,20,21]\) the study then needs careful consideration and generates much more complicated expressions which are analytically unsolvable.

The total Hamiltonian of the system is formed by the proper energies of the atom and the optical fields as well as by the interaction of the two optical fields with the medium. The expression of the Hamiltonian is then given by

\[
H = \sum_{j=0,1,2} \varepsilon_j |j\rangle\langle j| + \sum_{i=1,2} g_i' (E_i |0\rangle\langle l| + c.c)
\]

The first term describes the proper energy of the atomic three-level system. The second term includes the interaction between the two classical fields and the two allowed atomic transitions \(|1\rangle\rangle 0\rangle\) and \(|2\rangle\rangle 0\rangle\). Where \(g_1'\) and \(g_2'\) are the coupling constants between the atomic transitions \(|1\rangle\rangle 0\rangle\) and \(|2\rangle\rangle 0\rangle\) and respectively the classical light fields \(E_1\) and \(E_2\). \(\varepsilon_j\) represents the energy of the atomic level \(|j\rangle\) verifying in the resonant case: \(\omega_1 = \frac{\varepsilon_0 - \varepsilon_1}{\hbar}\) and \(\omega_2 = \frac{\varepsilon_0 - \varepsilon_2}{\hbar}\).

We consider here that the atomic system is initially prepared in the way that the excited state \(|0\rangle\) is almost empty and the population are distributed equally in the ground levels \(|1\rangle\) and \(|2\rangle\). In addition, the coherence between the ground levels at \(t=0\) is negligible. We are exploring the soliton-pair propagation in this three-level system with the same velocity, we can write then

\[
E_j(x, t) = E_j(x - vt)
\]

where \(v\) represents the speed of the soliton-pair.

Using the master equation and allowing the same procedure of calculations in \([13]\), we get in the moving frame \(z = x - vt\) the following coupled non-linear differential equations

\[
\begin{align*}
1. & \frac{d\alpha_1}{dz} = \alpha_1^3 - \frac{g_1}{2} \alpha_1 + \alpha_2^2 \alpha_1 \\
2. & \frac{d\alpha_2}{dz} = \alpha_2^3 - \frac{g_2}{2} \alpha_2 + \alpha_1^2 \alpha_2
\end{align*}
\]
where $\Gamma = \frac{Z}{y}$ is the normalized dissipation rate, here we suppose that the spontaneous emission rates from the excited state to the both ground states are the same. $g_j = \frac{g_{ej} \beta_j}{\hbar (c - \nu)}$ are the effective normalized coupling constant with $g_{ej}$ represent the propagation constants of the optical fields inside the atomic media.

$\alpha_j$ are the normalized solitons amplitudes defined by $\alpha_j = \frac{g_j}{\hbar y} E_j$

3. Soliton-pair solutions

In order to solve the coupled differential equations for the soliton-pair propagation we divide the two differential equations in (5) and we obtain

$$\frac{d\alpha_2}{d\alpha_1} = \frac{\alpha_3^2 - \frac{\alpha_2^2}{2} \alpha_2 + \alpha_1^2 \alpha_2}{\alpha_1^2 - \frac{\alpha_2^2}{2} \alpha_1 + \alpha_2^2 \alpha_1}$$

(6)

The above differential equation has an implicit solution in the form

$$\frac{1}{2} \frac{(g_1 - g_2)}{g_2} \ln \left( -\frac{g_1}{2} \frac{g_2}{2} \alpha_1^2 + \frac{g_1}{2} \alpha_2^2 \right) + \ln(\alpha_1) - \frac{g_1}{g_2} \ln(\alpha_2) = c_1$$

(7)

where $c_1$ is a free constant.

The implicit solution of (7) in general does not have an explicit relation between the two amplitudes of the soliton-pair. However, if we suppose that $g_1 = 2g_2$ it is possible to derive an explicit relation. In this work we focus on such case. Therefore, from (7) we get

$$\alpha_2^2 = \frac{1}{g_1} \sqrt{R(\alpha_1^2 - A)^2 + B} + \frac{1}{2} g_1 - \alpha_1^2$$

(8)

Where $A$, $B$ and $R$ are positive constants satisfying

$$A = \frac{g_1^2}{2(g_1 + 2e^{-4c_1})}, \quad R = \frac{1}{g_1^2 + 2g_1 e^{-4c_1}} \quad \text{and} \quad B = \frac{g_1^2}{4(g_1 + 2e^{-4c_1})} \left( 1 - \frac{g_1}{(g_1 + 2e^{-4c_1})} \right)$$

We will suppose that $g_1$ is very small then $B<<1$. Therefore, it is possible to derive an explicit relation between the two solitary waves

$$\alpha_2^2 = \frac{R}{g_1} (\alpha_1^2 - A) + g_1^2 - \alpha_1^2$$

(9)

By substituting (9) in the first differential equation of (5) we get a separable non-linear first-order differential equation for the amplitude of the first soliton

$$\frac{d\alpha_1}{dz} = M \alpha_1^2 - N \alpha_1$$

where the new parameters are defined by

$$M = \frac{R}{\Gamma g_1} \quad \text{and} \quad N = \frac{RA}{\Gamma g_1}.$$

Therefore
Following the same procedure we will get the following differential equation for $\alpha_2$

$$\frac{d\alpha_2}{dz} = K\alpha_2^3 - H\alpha_2$$

which have two explicit solutions given by

$$\alpha_2 = \pm \sqrt[4]{\frac{H}{e^{2H(z+c_2)} + K}}$$

where $K$ and $H$ are positive constants defined by

$$K = \frac{R}{\Gamma(R-g_1)}, \quad H = \frac{g_1-2\Lambda}{2\Gamma(1-\frac{R}{R})}$$

and $c_2$ is a free constant.

### 4. Velocity and existence condition

In this section we are interested to derive the expression of the speed that the two soliton-pair are propagating with.

In order to determine the speed, let us first calculate the limits of the soliton amplitudes.

$$\lim_{z \to -\infty} \alpha_1 = \pm \sqrt[4]{\frac{N}{M}}, \quad \lim_{z \to 0} \alpha_1 = 0$$

$$\lim_{z \to +\infty} \alpha_2 = 0; \quad \lim_{z \to -\infty} \alpha_2 = \pm \sqrt[4]{\frac{H}{K}}$$

It is worth mentioning here that each soliton from the pair of the solitary waves its limit is independent of the atomic dissipation. It depends crucially on the coupling constant between the atomic transition levels and the optical light. The free parameters allow some freedom to choose the amplitudes of the solitons.

Applying the limit to (9) when $z$ approaches $\infty$ we get

$$g_1 = \sqrt{2RA}$$

By substituting $g_1 = \frac{a}{(c-v)\nu}$ we will get the quadratic equation

$$\nu^2 - cv + P = 0$$

Where $P = \frac{a}{\sqrt{2RA}}$

The above quadratic equation has only one possible acceptable solution of the soliton-pair velocity (the second solution is for the velocity higher than the speed of light which is not physical acceptable). The expression of the physical solution of the speed is given by
\[ v = \frac{c - \sqrt{c^2 - 4P}}{2} \]

Under the condition that

\[ c^2 > \frac{4a(g_1 + 2e^{-4c_1})}{\sqrt{g_1}} \]

This condition is important for the existence of the soliton-pair. The weak condition can be written as following

\[ 2\sqrt{\frac{a}{g_1}} < c \]

Since \( a \) and \( g_1 \) are parameter linked to the propagation of the soliton waves there are some media where the soliton pair are not allowed to propagate or do not exist.

![Figure 1](image)

**Figure 1.** The velocity of the soliton-pair versus constant \( g_1 \) the coupling and \( \frac{a}{c^2} \).

We observe from figure 1 that for small values of the coupling parameters \( a \) and \( g_1 \) the soliton-pair propagates with very slow speed compared to the light speed in the medium. This effect can be used for conceiving optical memory.

**5. Conclusion**
We investigated the analytical soliton-pair solutions for different coupling constants between the two coherent lights and the atomic transitions. We have derived analytical expressions describing the soliton-pair shapes. We have highlighted the condition of the soliton-pair propagation in such media and have determined the allowed soliton-pair velocity.

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