CONVERGENCE RATES OF SOLUTIONS FOR A TWO-SPECIES CHEMOTAXIS-NAVIER-STOKES SYSTEM WITH COMPETITIVE KINETICS

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Abstract. In this paper, we study the convergence rates of solutions to the two-species chemotaxis-Navier-Stokes system with Lotka-Volterra competitive kinetics:

\[
\begin{aligned}
  (n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1 (1 - n_1 - a_1 n_2), \quad x \in \Omega, \quad t > 0, \\
  (n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2), \quad x \in \Omega, \quad t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - (\alpha n_1 + \beta n_2)c, \quad x \in \Omega, \quad t > 0, \\
  u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla p + (\gamma n_1 + \delta n_2) \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0
\end{aligned}
\]

under homogeneous Neumann boundary conditions for \( n_1, n_2, c \) and no-slip boundary condition for \( u \) in a bounded domain \( \Omega \subset \mathbb{R}^d (d \in \{2, 3\}) \) with smooth boundary. The global existence, boundedness and stabilization of solutions have been obtained in 2-D \[6\] and 3-D for \( \kappa = 0 \) and \( \chi_i (i = 1, 2) \) being sufficiently large \[4\]. Here, we examine convergence and derive the explicit rates of convergence for any supposedly given global bounded classical solution; more specifically, we show that

1. when \( a_1, a_2 \in (0, 1) \), the global-in-time bounded classical solution components \((n_1, n_2, u)\) converge at least exponentially to \((1, 1, 0)\) as \( t \to \infty \);
2. when \( a_1 \geq 1 > a_2 \), the global-in-time and bounded classical solution components \((n_1, n_2, u)\) converge at least algebraically to \((1, 0, 0)\) as \( t \to \infty \);
3. when \( a_2 \geq 1 > a_1 \), the global-in-time and bounded classical solution components \((n_1, n_2, u)\) converge at least algebraically to \((1, 0, 0)\) as \( t \to \infty \);
4. in either one of the three cases above, the global-in-time and bounded classical classical solution component \( c \) converges at least exponentially to 0 as \( t \to \infty \).

Moreover, it is shown that the rate of convergence for \( u \) in the first case is expressed in terms of the model parameters and the first eigenvalue of \(-\Delta\) in \( \Omega \) under homogeneous Dirichlet boundary conditions, and all other rates of convergence are explicitly expressed only in terms of the model parameters \( a_i, \mu_i, \alpha \) and \( \beta \) and the space dimension \( d \).

1. Introduction

We consider the following two-species chemotaxis-fluid system with competitive terms:

\[
\begin{aligned}
  (n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1 (1 - n_1 - a_1 n_2), \quad x \in \Omega, \quad t > 0, \\
  (n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2), \quad x \in \Omega, \quad t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - (\alpha n_1 + \beta n_2)c, \quad x \in \Omega, \quad t > 0, \\
  u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla p + (\gamma n_1 + \delta n_2) \nabla \phi, \quad \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0, \\
  \partial_{\nu} n_1 &= \partial_{\nu} n_2 = \partial_{\nu} c = 0, \quad u = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  n_i(x, 0) &= n_{i,0}(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad i = 1, 2,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^d (d \geq 2) \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( \partial_{\nu} \) denotes differentiation with respect to the outward normal of \( \partial \Omega \); \( \kappa \in \{0, 1\} \) \( \chi_1, \chi_2, a_1, a_2 \geq 0 \) and

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\( \mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0 \) are constants; \( n_{1,0}, n_{2,0}, c_0, u_0, \phi \) are known functions satisfying
\[
0 < n_{1,0}, n_{2,0} \in C(\bar{\Omega}), \quad 0 < c_0 \in W^{1,q}(\Omega), \quad u_0 \in D(A^\vartheta), \quad \phi \in C^{1+\eta}(\bar{\Omega})
\] (1.2) (1.3)
for some \( q > d, \vartheta \in (\frac{2}{3}, 1), \eta > 0 \) and \( A \) is the Stokes operator.

The system (1.1), an extension of the chemotaxis-fluid system introduced by Tuval et al. [15], depicts the evolution of two competing species which react on a single chemoattractant in a liquid surrounding environment. Here, \( n_1 \) and \( n_2 \) denote densities of species, \( c \) means the chemical concentration, and finally, \( u \) and \( P \) represent the fluid velocity field and its associated pressure. So, it is the mixed combination of the complex interaction between chemotaxis, the Lotka-Volterra kinetics and fluid.

The model (1.1) and its variants have been received considerable attention in 2- and 3-dimensional settings. In one-species context \((n_2 \equiv 0)\), global existence of weak (and/or eventual smoothness of solutions) and classical solutions and asymptotic behavior have been investigated, e.g., in [20, 21, 22] without logistic source \((\mu_1 = 0)\) and also the convergence rate has been explored [23] and in [8, 15, 17] with logistic source.

In two-species context, related studies first begin with fluid-free systems with signal production (in which the asymptotic stability usually depends on some smallness condition on the chemo-sensitivities) to understand the influence of chemotaxis and the Lotka-Volterra kinetics [1, 2, 10, 9, 11, 12, 14]. For the two-species chemotaxis-fluid system with competition (in which the asymptotic stability usually depends on some smallness condition on the chemo-sensitivities), the global existence, boundedness of classical solutions and stabilization to equilibria were very recently studied by Hirata et al. [6] in the 2-D setting and by Cao et al. [4] in the 3-D setting for \( \kappa = 0 \), as precisely stated as follows:

(B2) **(Boundedness in 2-D)** [6] In the case that \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary, let \( \chi_1, \chi_2, a_1, a_2 \geq 0, \mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0 \) and let (1.2) and (1.3) hold. The the IBVP (1.1) possesses a unique classical solution \((n_1, n_2, c, u, P)\), up to addition of constants to \( P \), such that
\[
n_1, n_2 \in C(\bar{\Omega} \times [0, \infty)) \cap C^2(\bar{\Omega} \times (0, \infty)), \quad c \in C(\bar{\Omega} \times [0, \infty)) \cap C^2(\bar{\Omega} \times (0, \infty)) \cap L^\infty_{\text{loc}}([0, \infty); W^{1,q}(\Omega)), \quad u \in C(\bar{\Omega} \times [0, \infty)) \cap C^2(\bar{\Omega} \times (0, \infty)) \cap L^\infty_{\text{loc}}([0, \infty); D(A^\vartheta)), \quad P \in C^1(\bar{\Omega} \times (0, \infty)).
\]
Moreover, there exists a constant \( C > 0 \) such that for all \( t > 0 \)
\[
\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,q}(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C.
\] (1.4)

(B3) **(Boundedness in 3-D)** [4] In the case that \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary, besides the assumptions in (B2), let \( \kappa = 0 \). Then there exists a constant \( \xi_0 > 0 \) such that whenever \( \frac{\max\{|\chi_1, \chi_2|}{\min\{\mu_1, \mu_2\}} < \xi_0 \), the statements of (B2) hold.

(UC) **(Uniform Convergence)** [4] Let \((n_1, n_2, c, u, P)\) be the solution of (1.1) obtained from (B2) or (B3). Then it fulfills the following convergence properties:

(i) Assume that \( a_1, a_2 \in (0, 1) \). Then
\[
n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \text{ in } L^\infty(\Omega) \text{ as } t \to \infty,
\]
where
\[
N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}.
\] (1.5)

(ii) Assume that \( a_1 \geq 1 > a_2 \). Then
\[
n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \text{ in } L^\infty(\Omega) \text{ as } t \to \infty.
\]
In this paper, we study dynamical properties for any supposed global-in-time and bounded classical solution to (1.1), with particular focus on the model of convergence as
well as their explicit rates of convergence. Before proceeding to our main results, let us observe that, the \( n_1 \) - and \( n_2 \)-equations in (1.1) are symmetric. Thus, we should have a result about stabilization to \((0,1,0,0)\). This was not mentioned in related works, cf. [1, 4, 6, 9]. Now, let \( \lambda_P \) denote the Poincaré constant, cf. [4, 10], and, finally, let

\[
\kappa = \frac{1}{2} \min \left\{ \left( 1 - a_1 a_2 \right) \mu_1 \max \left\{ \frac{\mu_1}{\lambda N_1}, \frac{a_1}{a_2} \right\}, \left( \alpha N_1 + \beta N_2 \right) \right\}.
\]

Then we are at the position to state our main results on exponential and algebraic convergence of bounded solutions to (1.1).

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^d (d \in \{2,3\}) \) be a bounded and smooth domain and let (1.2) and (1.3) be in force, and, finally, let \((n_1, n_2, c, u, P)\) be a global classical solution of (1.1) with uniform-in-time bound. Then this solution enjoys the following decay properties.

(I) When \( a_1, a_2 \in (0,1) \), the solution components \((n_1, n_2, u)\) converge at least exponentially to \((N_1, N_2, 0)\) in the following way:

\[
\begin{align*}
\| n_1(\cdot, t) - N_1 \|_{L^\infty(\Omega)} &\leq m_1 e^{-\frac{\kappa}{\mu_1} t}, \quad \forall t \geq 0, \\
\| n_2(\cdot, t) - N_2 \|_{L^\infty(\Omega)} &\leq m_2 e^{-\frac{\kappa}{\mu_2} t}, \quad \forall t \geq 0, \\
\| u(\cdot, t) \|_{L^\infty(\Omega)} &\leq m_3 e^{-\frac{\kappa}{\mu_2} \min(\lambda_P, \frac{\kappa}{\mu_1}) t}, \quad \forall t \geq 0.
\end{align*}
\]

(II) When \( a_1 \geq 1 > a_2 \), the solution components \((n_1, n_2, u)\) converge at least algebraically to \((0,1,0)\) in the following way:

\[
\begin{align*}
\| n_1(\cdot, t) \|_{L^\infty(\Omega)} &\leq m_4(t + 1)^{-\frac{1}{2+\epsilon}}, \quad \forall t \geq 0, \\
\| n_2(\cdot, t) - 1 \|_{L^\infty(\Omega)} &\leq m_5(t + 1)^{-\frac{1}{2+\epsilon}}, \quad \forall t \geq 0, \\
\| u(\cdot, t) \|_{L^\infty(\Omega)} &\leq m_6(t + 1)^{-\frac{1}{2+\epsilon}}, \quad \forall t \geq 0.
\end{align*}
\]

(III) When \( a_2 \geq 1 > a_1 \), the solution components \((n_1, n_2, u)\) converge at least algebraically to \((1,0,0)\) in the following way:

\[
\begin{align*}
\| n_1(\cdot, t) - 1 \|_{L^\infty(\Omega)} &\leq m_7(t + 1)^{-\frac{1}{2+\epsilon}}, \quad \forall t \geq 0, \\
\| n_2(\cdot, t) \|_{L^\infty(\Omega)} &\leq m_8(t + 1)^{-\frac{1}{2+\epsilon}}, \quad \forall t \geq 0, \\
\| u(\cdot, t) \|_{L^\infty(\Omega)} &\leq m_9(t + 1)^{-\frac{1}{2+\epsilon}}, \quad \forall t \geq 0.
\end{align*}
\]

(IV) In either one of the three cases above, the solution component \(c\) converges at least exponentially to 0 in the following way:

\[
\| c(\cdot, t) \|_{L^\infty(\Omega)} \leq m_{10} e^{-\frac{\alpha N_1}{\epsilon} t}, \quad \forall t \geq 0,
\]

where \((\hat{N}_1, \hat{N}_2) = (N_1, N_2)\) in Case (I), \((\hat{N}_1, \hat{N}_2) = (0,1)\) in Case (II), and \((\hat{N}_1, \hat{N}_2) = (1,0)\) in Case (III).

Here, \( \epsilon \in (0,1) \) is arbitrarily given, only \( m_3, m_6 \) and \( m_9 \) depend on \( \epsilon \); all \( m_i (i = 1, 2, 3, \cdots, 9) \) are suitably large constants depending on the initial data \( n_{1,0}, n_{2,0}, c_0, u_0, \phi \) and Sobolev embedding constants but not on time \( t \), see Section 4. Moreover,

\[
m_4 \geq O(1) \left( 1 + (1 - a_2)^{-\frac{1}{2+\epsilon}} \right), \quad m_5 \geq O(1) \left( 1 + (1 - a_2)^{-\frac{1}{2+\epsilon}} \right), \quad m_6 \geq O(1) \left( 1 + (1 - a_2)^{-\frac{1}{2+\epsilon}} \right)
\]

and

\[
m_7 \geq O(1) \left( 1 + (1 - a_1)^{-\frac{1}{2+\epsilon}} \right), \quad m_8 \geq O(1) \left( 1 + (1 - a_1)^{-\frac{1}{2+\epsilon}} \right), \quad m_9 \geq O(1) \left( 1 + (1 - a_1)^{-\frac{1}{2+\epsilon}} \right).
\]

From these estimates, we see the facts that \( 1 - a_1 a_2 > 0, 1 - a_2 > 0 \) and \( 1 - a_1 > 0 \) are very important in Case (I), (II) and (III), respectively.
Remark 1.2. In our argument, we don’t need any restriction on the space dimension $d$. Thus, Theorem 1.1 works equally well in any dimension as long as the solution is global-in-time and bounded. While, by the well-known difficulty about the Navier-Stokes system, we restrict ourselves to the physically relevant cases $d = 2$ and $d = 3$.

With certain regularity and dissipation properties of global bounded solutions, the argument for the proof of convergence is quite known and developed, cf. [1] [4] [9] [15] [16] [17] [21] for example. The strategy for obtaining the explicit rates of convergence as described in Theorem 1.1 consists mainly of four steps. In the first step, we present more strong regularity properties, e.g., $W^{1,\infty}$-regularity for $n_i$, $W^{1,p}$-regularity for $u$ with any finite $p$, and $W^{2,\infty}$-regularity for $c$, for any bounded solution of (1.1) than those shown in [6] [4]; this are done in Section 2. In the crucial second step done in Section 3, we use refined computations to make those widely known Lyapunov functionals (cf. eg. [1] [4] [6] [9]) explicit, which will enable us to derive the explicit rates of convergence. Armed with the information provided by Step two, we then move on to calculate precisely the rates of convergence in $L^1$- and $L^2$-norm for the considered bounded solution, and related necessary estimates are also studied in great details. These constitute our Step three and are conducted in Section 4. Finally, thanks to the improved regularities, we apply the well-known Gagliardo-Nirenberg interpolation inequality to pass the obtained $L^1$- and $L^2$-convergence to the $L^\infty$-convergence; these are our Step four and are also done in Section 4.

2. Regularities of bounded solutions

Let $(n_1, n_2, c, u, P)$ be a supposedly given global-in-time and bounded classical solution to (1.1) in the sense of (1.4). In this section, we provide more strong regularity properties for any such bounded solution than those shown in [6] [4], which are needed to achieve our desired rates of convergence in $L^\infty$-norm. We start with the regularity of $u$ and $c$.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded and smooth domain. For $d < p < \infty$, there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{W^{1,p}} \leq C, \quad \forall t > 1$$

and

$$\|c(\cdot, t)\|_{W^{1,\infty}} + \|\Delta c(\cdot, t)\|_{L^\infty} \leq C, \quad \forall t > 1.$$  \hspace{2cm} (2.1)

Proof. Using the essentially same argument as in [21] Lemma 6.3, we obtain (2.1). The $W^{1,\infty}$-boundedness of $c$ can be seen in [6] Lemma 3.9 and [14] Lemma 3.12. With these and the $L^\infty$-boundedness of $n_1, n_2$ and $u$, an direct application of the standard parabolic schauder theory (cf. [7] [13] to the third equation in (1.1) yields (2.2).  \hfill \Box

With the regularity properties in Lemma 2.1 at hand, we now utilize the quite commonly used arguments (cf. [16] [17]) to show the following $W^{1,\infty}$-regularity of $n_1$ and $n_2$.

Lemma 2.2. There exists a constant $C > 0$ such that

$$\|n_1(\cdot, t)\|_{W^{1,\infty}} + \|n_2(\cdot, t)\|_{W^{1,\infty}} \leq C, \quad \forall t > 1.$$  \hspace{2cm} (2.3)

Proof. First, we show there exists a $c_1 > 0$ such that

$$\|n_1(\cdot, t)\|_{W^{1,\infty}} \leq c_1, \quad \forall t > 1.$$  \hspace{2cm} (2.4)

To this end, for any $T > 2$, we let

$$M(T) := \sup_{t \in (2, T)} \|\nabla n_1(\cdot, t)\|_{L^\infty}.$$  Since clearly $\nabla n_1$ is continuous on $\overline{\Omega} \times [0, T]$, it follows that $M(T)$ is finite. Moreover, since by our universal assumption $n_1$ is bounded in $L^\infty(\Omega \times (0, \infty))$, to prove (2.4), it is sufficient to derive the existence of $c_2 > 0$ satisfying

$$M(T) \leq c_2, \quad \forall T > 2.$$  \hspace{2cm} (2.5)
To achieve (2.6), for any given \( t \in (2, T) \), using the variation-of-constants formula to the first equation in (1.1), we get

\[
n_1(\cdot, t) = e^{\Delta t} n_1(\cdot, t - 1) - \chi \int_{t-1}^{t} e^{(t-s)\Delta} \nabla \cdot \left( n_1(\cdot, s) \nabla c(\cdot, s) \right) ds \\
- \int_{t-1}^{t} e^{(t-s)\Delta} u(\cdot, s) \cdot \nabla n_1(\cdot, s) ds \\
+ \mu_1 \int_{t-1}^{t} e^{(t-s)\Delta} n_1(\cdot, s) \left( 1 - n_1(\cdot, s) - a_1 n_2(\cdot, s) \right) ds,
\]

which implies

\[
\| \nabla n_1(\cdot, t) \|_{L^\infty} \leq \| \nabla e^{\Delta t} n_1(\cdot, t - 1) \|_{L^\infty} + \chi \int_{t-1}^{t} \| \nabla e^{(t-s)\Delta} \nabla \cdot \left( n_1(\cdot, s) \nabla c(\cdot, s) \right) \|_{L^\infty} ds \\
+ \int_{t-1}^{t} \| \nabla e^{(t-s)\Delta} u(\cdot, s) \cdot \nabla n_1(\cdot, s) \|_{L^\infty} ds \\
+ \mu_1 \int_{t-1}^{t} \| \nabla e^{(t-s)\Delta} n_1(\cdot, s) \left( 1 - n_1(\cdot, s) - a_1 n_2(\cdot, s) \right) \|_{L^\infty} ds \tag{2.6}
\]

Next, we shall employ the widely known smoothing \( L^p-L^q \) properties of the Neumann heat semigroup \( \{e^{t\Delta}\}_{t \geq 0} \) in \( \Omega \) (see [19, 33, 45] for instance) to estimate \( I_i, i = 1, 2, 3, 4 \).

Thanks to the boundedness of \( n_1, n_2, u \) in \( \Omega \times (1, \infty) \), (2.1) and (2.2), we employ those smoothing Neumann heat semigroup estimates to obtained that

\[
I_1 = \| \nabla e^{\Delta t} n_1(\cdot, t - 1) \|_{L^\infty} \leq c_3 \| n_1(\cdot, t - 1) \|_{L^\infty} \leq c_4 \tag{2.7}
\]

and that

\[
I_2 = \chi \int_{t-1}^{t} \| \nabla e^{(t-s)\Delta} \nabla \cdot \left( n_1(\cdot, s) \nabla c(\cdot, s) \right) \|_{L^\infty} ds \\
\leq c_5 \int_{t-1}^{t} \left[ 1 + (t-s)^{-\frac{q}{2} - \frac{n}{pq}} \right] e^{-\lambda_1(t-s)} \| \nabla \cdot \left( n_1(\cdot, s) \nabla c(\cdot, s) \right) \|_{L^p} ds \\
\leq c_5 \int_{t-1}^{t} \left[ 1 + (t-s)^{-\frac{q}{2} - \frac{n}{pq}} \right] e^{-\lambda_1(t-s)} \| \nabla n_1(\cdot, s) \cdot \nabla c(\cdot, s) \|_{L^p} ds \tag{2.8}
\]

\[
+ c_5 \int_{t-1}^{t} \left[ 1 + (t-s)^{-\frac{q}{2} - \frac{n}{pq}} \right] e^{-\lambda_1(t-s)} \| n_1(\cdot, s) \Delta c(\cdot, s) \|_{L^p} ds \\
\leq c_6 \int_{t-1}^{t} \left[ 1 + (t-s)^{-\frac{q}{2} - \frac{n}{pq}} \right] e^{-\lambda_1(t-s)} \| \nabla n_1(\cdot, s) \|_{L^p} ds + c_7,
\]

where \( \lambda_1(> 0) \) is the first nonzero eigenvalue of \( -\Delta \) under homogeneous boundary condition and we have used the choice of \( p > n \) to ensure the finiteness of the Gamma integral.

Similarly, we can estimate \( I_3 \) as follows:

\[
I_3 = \int_{t-1}^{t} \| \nabla e^{(t-s)\Delta} u(\cdot, s) \cdot \nabla n_1(\cdot, s) \|_{L^\infty} ds \tag{2.9}
\]

\[
\leq c_8 \int_{t-1}^{t} \left[ 1 + (t-s)^{-\frac{q}{2} - \frac{n}{pq}} \right] e^{-\lambda_1(t-s)} \| \nabla n_1(\cdot, s) \|_{L^p} ds + c_9.
\]

At last, using the boundedness of \( n_1 \) and \( n_2 \) again, one has

\[
I_4 = \mu_1 \int_{t-1}^{t} \| \nabla e^{(t-s)\Delta} n_1(\cdot, s) \left( 1 - n_1(\cdot, s) - a_1 n_2(\cdot, s) \right) \|_{L^\infty} ds \\
\leq c_{10} \int_{t-1}^{t} \left[ 1 + (t-s)^{-\frac{q}{2}} \right] e^{-\lambda_1(t-s)} ds \leq c_{10} \int_{0}^{1} (1 + \tau^{-\frac{q}{2}}) e^{-\lambda_1 \tau} ds \leq \left( \frac{1}{\lambda_1} + 2 \right) c_{10}. \tag{2.10}
\]
Substituting (2.7), (2.8), (2.9) and (2.10) into (2.6), we infer that
\[
\|\nabla n_1(\cdot,t)\|_{L^p} \leq c_{11} \int_{t_1}^t \left[ 1 + (t-s)^{-\frac{\theta}{p}} \right] e^{-\lambda_1(t-s)} \|\nabla n_1(\cdot,s)\|_{L^p} ds + c_{12}. \tag{2.11}
\]
Then invoking the Gagliardo-Nirenberg inequality, the smoothness and hence boundedness of \(\nabla n_1\) on \(\Omega \times [1,2]\), and the definition of \(M(T)\), we estimate
\[
\|\nabla n_1(\cdot,s)\|_{L^p} \leq c_{13}\|\nabla n_1(\cdot,s)\|_{L^\infty}^{\theta} \|n_1(\cdot,s)\|_{L^\infty}^{1-\theta} + c_{13}\|n_1(\cdot,s)\|_{L^\infty}
\]
\[
\leq c_{14}(M^\theta(T) + 1), \quad \forall s \in (1, T), \tag{2.12}
\]
where \(\theta = \frac{p-n}{p} \in (0,1)\) due to \(p > n\).

Finally, since \(\frac{p}{2} + \frac{p}{2p} < 1\), then a substitution of (2.12) into (2.11) entails
\[
M(T) \leq c_{15}M^\theta(T) + c_{16}, \quad \forall T > 2,
\]
which upon a use of elementary inequality gives
\[
M(T) \leq \max\{2c_{16}, (2c_{15})^{1-\theta}\}, \quad \forall T > 2,
\]
and hence (2.4) follows.

The argument done for \(n_1\) can also be similarly applied to \(n_2\) to find that
\[
\|n_2(\cdot,t)\|_{W^{1,\infty}} \leq c_{17}, \quad \forall t > 1.
\]
This along with (2.3) yields simply (2.4), finishing the proof of the lemma. \(\square\)

3. Existence of explicit Lyapunov functionals

From boundedness to convergence, besides enough information on regularity, we still need some decaying estimates of bounded solutions under investigation. For the latter, the availability of a Lyapunov functional is crucial, see [1, 4, 6, 9] for instance. In this section, for our purpose, we particularize those known Lyapunov functionals used in those papers to obtain the explicit rates of convergence as stated in Theorem 1.1. Let us start with the case of \(a_1, a_2 \in (0,1)\). In this case, the explicit Lyapunov functional that we obtain for the chemotaxis-fluid system (1.1) reads as follows:

**Lemma 3.1.** Define
\[
E_1 := \int_{\Omega} \left( n_1 - N_1 - N_1 \log \frac{n_1}{N_1} \right) + \frac{a_1\mu_1}{a_2\mu_2} \int_{\Omega} \left( n_2 - N_2 - N_2 \log \frac{n_2}{N_2} \right)
\]
\[
+ \frac{1}{2} \left( \frac{N_1 \chi_1^2}{4} + \frac{a_1\mu_1 N_2 \chi_2^2}{4a_2\mu_2} + 1 \right) \int_{\Omega} e^2 \tag{3.1}
\]
and
\[
F_1 := \int_{\Omega} (n_1 - N_1)^2 + \int_{\Omega} (n_2 - N_2)^2.
\]

Then, in the case of \(a_1, a_2 \in (0,1)\), the nonnegative functions \(E_1\) and \(F_1\) satisfy
\[
\frac{d}{dt}E_1(t) \leq -(1 - a_1a_2)\mu_1 \min\left\{ \frac{1}{2}, \frac{a_1}{(1 + a_1a_2)\mu_2} \right\} F_1(t) := -\tau F_1(t), \quad \forall t > 0. \tag{3.2}
\]

**Proof.** By honest differentiation of \(E_1\) in (3.1) and using elementary Cauchy-Schwarz inequality, one can easily derive the dissipation estimate (3.2); or alternatively, in the proof of [6] Lemma 4.1, by taking
\[
k_1 = \frac{a_1\mu_1}{a_2\mu_2}, \quad l_1 = \left( \frac{N_1 \chi_1^2}{4} + \frac{a_1\mu_1 N_2 \chi_2^2}{4a_2\mu_2} + 1 \right), \quad \epsilon = \frac{\mu_1}{2}(1 - a_1a_2),
\]
upon honest calculations, one can easily arrive at (3.2). \(\square\)
For the purpose of driving our explicit Lyapunov functionals in Cases (II) and (III), we wish to perform honest computations here. We illustrate it for Case (II), i.e., $a_1 > a_2$. In this case, from \([11], \text{the fact that } a_1 \geq 1 \text{ and the positivity of } n_1, n_2, \text{ we calculate that}\)

$$\frac{d}{dt} \int_{\Omega} n_1 = \mu_1 \int_{\Omega} (1 - n_1 - a_1 n_2) n_1 \leq -\mu_1 \int_{\Omega} n_1^2 - \mu_1 \int_{\Omega} n_1 (n_2 - 1),$$

$$\int_{\Omega} (n_2 - 1 - \log n_2) = -\int_{\Omega} \frac{\nabla n_2}{n_2^2} + \chi_2 \int_{\Omega} \frac{\nabla n_2}{n_2^2} \nabla c - \mu_2 \int_{\Omega} (n_2 - 1)^2 - a_2 \mu_2 \int_{\Omega} n_1 (n_2 - 1)$$

as well as

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} c^2 = -\int_{\Omega} |\nabla c|^2 - \int_{\Omega} (a n_1 + b n_2) c^2.$$

Therefore, for any positive constants $\sigma_1, \sigma_2$ and $\eta \in (0, 1)$, in view of the passivity of $n_i, \alpha$ and $\beta$, a clear linear combination of the three estimates above shows

$$-\frac{d}{dt} \int_{\Omega} \left[ n_1 + \sigma_1 (n_2 - 1 - \log n_2) + \frac{\sigma_2}{2} c^2 \right]$$

$$\geq \int_{\Omega} \left[ \mu_1 n_1^2 + (\mu_1 + a_2 \mu_2 \sigma_1) n_1 (n_2 - 1) + \mu_2 \sigma_1 (n_2 - 1)^2 \right]$$

$$+ \int_{\Omega} \left[ \sigma_1 \frac{\nabla n_2}{n_2^2} - \chi_2 \frac{n_2}{n_2} \nabla c + \sigma_2 |\nabla c|^2 \right]$$

$$= \int_{\Omega} \left\{ \left[ \sigma_1 \frac{\nabla n_2}{n_2^2} - \chi_2 \frac{n_2}{n_2} \nabla c + \sigma_2 |\nabla c|^2 \right] \right\}$$

$$+ \mu_2 \sigma_1 (1 - \eta) \int_{\Omega} (n_2 - 1)^2 + \int_{\Omega} \left[ \left( \frac{\sqrt{\mu_2 \sigma_1 \eta}}{n_2} - \frac{(\mu_1 + a_2 \mu_2 \sigma_1)}{2 \sqrt{\mu_2 \sigma_1 \eta}} n_1 \right) \right]$$

$$\geq \mu_2 \sigma_1 (1 - \eta) \int_{\Omega} (n_2 - 1)^2 + \left[ \mu_1 - \frac{(\mu_1 + a_2 \mu_2 \sigma_1)}{4 \mu_2 \sigma_1 \eta} \right] \int_{\Omega} n_1^2 + \int_{\Omega} (c_2 - \frac{\lambda_2^2 \sigma_1}{4}) \int_{\Omega} |\nabla c|^2.$$

With these calculations above, we obtain the next explicit decay property, which is a specification of \([9, \text{Lemma } 4.3], \text{ see also } [11, \text{Section } 4.2].\)

**Lemma 3.2.** Define

$$E_2 := \int_{\Omega} n_1 + \frac{\mu_1}{2 a_2 \mu_2} \int_{\Omega} (n_2 - 1 - \log n_2) + \frac{\mu_1 \lambda_2^2}{8 a_2 \mu_2} \int_{\Omega} c^2$$

and

$$F_2 := \int_{\Omega} n_1^2 + \int_{\Omega} (n_2 - 1)^2.$$

Then, in the case of $a_1 > 1 > a_2$, the nonnegative functions $E_2$ and $F_2$ satisfy

$$\frac{d}{dt} E_2(t) \leq -(1 - a_2) \mu_1 \min\left\{ \frac{1}{2 a_2}, \frac{1}{1 + a_2} \right\} F_2(t) := -\sigma F_2(t), \quad \forall t > 0. \quad (3.4)$$

**Proof.** The fact that $a_2 < 1$ allows us to select

$$\sigma_1 = \frac{\mu_1}{a_2 \mu_2}, \quad \sigma_2 = \frac{\mu_1 \lambda_2^2}{4 a_2 \mu_2}, \quad \eta = \frac{1 + a_2}{2} \in (0, 1),$$

then, upon a plain calculation from \([33, \text{Lemma } 4.3], \text{ we obtain the dissipation inequality } (3.3). \square$$

By the symmetry of the $n_1$-and $n_2$-equations in \([11], \text{ when } a_2 \geq 1 > a_1, \text{ using similar arguments leading to Lemma } 3.2 \text{ we have a dissipation inequality as follows:}\)

**Lemma 3.3.** Define

$$E_3 := \int_{\Omega} (n_1 - 1 - \log n_1) + \frac{a_1 \mu_1}{\mu_2} \int_{\Omega} n_2 + \frac{\lambda_1^2}{8} \int_{\Omega} c^2$$

and

$$F_3 := \int_{\Omega} n_1^2 + \int_{\Omega} (n_2 - 1)^2.$$
and

\[ F_3 := \int_{\Omega} (n_1 - 1)^2 + \int_{\Omega} n_2^2. \]

Then, in the case of \( a_2 \geq 1 > a_1 \), the nonnegative functions \( F_3 \) and \( F_3 \) satisfy

\[ \frac{d}{dt} E_3(t) \leq - (1 - a_1) \mu_1 \min \left\{ \frac{1}{2}, \frac{a_1}{1 + a_1} \right\} F_3(t) = p F_3(t), \quad \forall t > 0. \]  

(3.5)

**Proof.** For any constants \( l_1, l_2 > 0 \) and \( \varepsilon \in (0, 1) \), using the fact that \( a_2 \geq 1 \) and similar computations to the ones leading to (3.4), we infer that

\[ \frac{d}{dt} \int_{\Omega} \left[ (n_1 - 1 - \log n_1) + l_1 n_2 + \frac{l_2}{2} n_2^2 \right] \geq \mu_1 (1 - \varepsilon) \int_{\Omega} (n_1 - 1)^2 + l_2 \mu_2 - \frac{(a_1 \mu_1 + \mu_2 l_1)}{4 \mu_1 \varepsilon} \int_{\Omega} n_2^2 + (l_2 - \frac{\lambda_1^2}{4}) \int_{\Omega} |\nabla c|^2. \]  

(3.6)

Now, thanks to \( a_1 < 1 \), we set

\[ l_1 = \frac{a_1 \mu_1}{\mu_2}, \quad l_2 = \frac{\lambda_1^2}{4}, \quad \varepsilon = \frac{1 + a_1}{2} \in (0, 1), \]

and then we easily conclude (3.5) upon trivial computations from (3.6). \( \Box \)

4. **Convergence rates**

Aided by those dissipation estimates as provided in Lemmas 3.1, 3.2 and 3.3, even weaker regularity properties than those in Section 2, using the quite known arguments, cf. [H1, H2, H3, H4, H5, H6, H7, H8] for example, we know that any global-in-time and bounded classical solution of (1.1) satisfies the convergence properties as follows:

\[ \left\| (n_1(\cdot, t), n_2(\cdot, t), c(\cdot, t), u(\cdot, t)) - (\hat{N}_1, \hat{N}_2, 0, 0) \right\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \]  

(4.1)

Here, \((\hat{N}_1, \hat{N}_2) = (N_1, N_2)\) when \( a_1, a_2 \in (0, 1) \), \((\hat{N}_1, \hat{N}_2) = (0, 1)\) when \( a_1 \geq 1 > a_2 \), and \((\hat{N}_1, \hat{N}_2) = (1, 0)\) when \( a_2 \geq 1 > a_1 \), where \( N_1 \) and \( N_2 \) are defined by (1.5). In this section, we derive the explicit rates of convergence as described in Theorem 1.1 for any supposedly bounded and global-in-time solution to (1.1). Firstly, it follows from (1.1), as \( t \to \infty \), that \( n_1(\cdot, t) \to \hat{N}_1 \) and \( n_2(\cdot, t) \to \hat{N}_2 \) uniformly in \( \Omega \). Henceforth, we fix a \( t_0 > 1 \) such that

\[ \frac{\hat{N}_1}{2} \leq n_1 \leq \frac{3 \hat{N}_1}{2} \quad \text{and} \quad \frac{\hat{N}_2}{2} \leq n_2 \leq \frac{3 \hat{N}_2}{2} \quad \text{on} \quad \Omega \times [t_0, \infty). \]  

(4.2)

4.1. **Convergence rate of \( c \).** We first take up the convergence rate of \( c \), which is based on a parabolic comparison argument.

**Lemma 4.1.** The \( c \)-solution component of any bounded solution of (1.1) stabilizes to zero exponentially:

\[ \|c(\cdot, t)\|_{L^\infty} \leq \|c_0\|_{L^\infty} e^{-\frac{\alpha N_1}{2}(t-t_0)}, \quad \forall t \geq t_0. \]

**Proof.** We show the proof only for the case that \( a_1, a_2 \in (0, 1) \). Then (1.2) shows

\[ \alpha n_1 + \beta n_2 \geq \frac{\alpha N_1 + \beta N_2}{2} \quad \text{on} \quad \Omega \times [t_0, \infty). \]

This along with the third equation in (1.1) and the positive of \( c \) gives

\[ c_t \leq \Delta c - u \cdot \nabla c - \frac{\alpha N_1 + \beta N_2}{2} c \quad \text{on} \quad \Omega \times [t_0, \infty). \]  

(4.3)

Let \( z(t) \) be the solution of the following associated ODE problem:

\[ \begin{cases} z'(t) + \frac{\alpha N_1 + \beta N_2}{2} z(t) = 0, \quad t \geq t_0, \\ z(t_0) = \|c(\cdot, t_0)\|_{L^\infty}. \end{cases} \]
It is clear that \( z(t) \) satisfies (4.3) together with \( \partial_t z = 0 \), and hence an application of the comparison principle and Hopf boundary point lemma immediately yields

\[
c(x, t) \leq z(t) = \|c(\cdot, t_0)\|_{L^\infty} e^{-\frac{\alpha N_1 + \beta N_2}{2}(t-t_0)} \quad \text{for all } x \in \Omega, t \geq t_0.
\]

Using the basic fact that \( t \to \|c(\cdot, t)\|_{L^\infty} \) is non-increasing again by comparison principle, cf. \cite[Lemma 2.1]{21}, one has

\[
c(x, t) \leq \|c_0\|_{L^\infty} e^{-\frac{\alpha N_1 + \beta N_2}{2}(t-t_0)} \quad \text{for all } x \in \Omega, t \geq t_0.
\]

This completes the proof of Lemma 4.1 by noting the positivity of \( c \). \( \Box \)

4.2. **Convergence rates in Case I**: \( a_1, a_2 \in (0, 1) \). In this case, we will show that the solution components \((n_1, n_2, u)\) converge at least exponentially to \((N_1, N_2, 0)\).

4.2.1. **Convergence rates of \( n_1 \) and \( n_2 \) in Case I.** In this subsection, we shall establish the convergence rate of \( n_1 \) and \( n_2 \) on the basis of the convergence rate of \( c \) in Lemma 4.1 and the regularity of \( n_1 \) and \( n_2 \) provided by Lemma 2.2. We first use Lemmas 4.1 and 3.1 to obtain the exponential convergence rate of \( \|n_1 - N_1\|_{L^2} \) and \( \|n_2 - N_2\|_{L^2} \).

**Lemma 4.2.** The \( n_1 \)- and \( n_2 \)-solution components of bounded solution of (1.1) verify

\[
\|n_1(\cdot, t) - N_1\|_{L^2}^2 + \|n_2(\cdot, t) - N_2\|_{L^2}^2 \leq K_1 e^{-\kappa(t-t_0)}, \quad \forall t \geq t_0,
\]

where \( K_1 = K_1(t_0) \) is defined by

\[
K_1 = \frac{\left( 1 - a_1 a_2 \right) \mu_1 |\Omega| \|c_0\|_{L^\infty}^2}{2 \min \left\{ a_1 \mu_1, \frac{a_1 \mu_2}{a_2 \mu_2 N_2} \right\}} \max \left\{ \frac{1}{4}, \frac{1}{a_1 N_1 + \beta N_2} \right\} \left( \frac{N_1 \chi_2^2 + a_2 \mu_2 N_2 \chi_2^2}{a_1 \chi_2^2 (a_2 \mu_2 N_2 + 1)} \right)
\]

and the exponential decay rate \( \kappa \) is defined by (4.6).

**Proof.** Applying Taylor’s formula to the function \( \psi(z) = z - N_1 \ln z \) at \( z = N_1 \), we obtain

\[
n_1 - N_1 - N_1 \log \frac{n_1}{N_1} = \psi(n_1) - \psi(N_1) = \frac{\psi''(\xi)}{2} (n_1 - N_1)^2 = \frac{N_1}{2\xi^2} (n_1 - N_1)^2 \quad (4.6)
\]

for some \( \xi > 0 \) between \( n_1 \) and \( N_1 \). Then the combination of (4.2) and (4.6) gives

\[
\frac{2}{g N_1} (n_1 - N_1)^2 \leq n_1 - N_1 - N_1 \log \frac{n_1}{N_1} \leq \frac{2}{N_1} (n_1 - N_1)^2, \quad \forall t \geq t_0,
\]

and hence

\[
\frac{2}{g N_1} \int_\Omega (n_1 - N_1)^2 \leq \int_\Omega \left( n_1 - N_1 - N_1 \log \frac{n_1}{N_1} \right) \leq \frac{2}{N_1} \int_\Omega (n_1 - N_1)^2, \quad \forall t \geq t_0. \quad (4.7)
\]

Similarly, one gets

\[
\frac{2}{g N_2} \int_\Omega (n_2 - N_2)^2 \leq \int_\Omega \left( n_2 - N_2 - N_2 \log \frac{n_2}{N_2} \right) \leq \frac{2}{N_2} \int_\Omega (n_2 - N_2)^2, \quad \forall t \geq t_0. \quad (4.8)
\]

On the other hand, Lemma 4.1 quickly gives rise to

\[
\int c^2 \leq |\Omega| \|c(\cdot, t)\|_{L^\infty}^2 \leq |\Omega| \|c_0\|_{L^\infty}^2 e^{-\alpha N_1 + \beta N_2} (t-t_0), \quad \forall t \geq t_0. \quad (4.9)
\]

A substitution of (4.7), (4.8) and (4.9) into the definition of \( E_1 \) in (3.1) gives

\[
E_1(t) \leq \frac{2}{N_1} \int_\Omega (n_1 - N_1)^2 + \frac{2 a_1 \mu_1}{a_2 \mu_2 N_2} \int_\Omega (n_2 - N_2)^2 + m e^{-\alpha N_1 + \beta N_2} (t-t_0) \leq \theta F_1 + m e^{-\alpha N_1 + \beta N_2} (t-t_0), \quad \forall t \geq t_0.
\]

(4.10)
where

$$\theta = 2 \max \left\{ \frac{1}{N_1} \frac{a_1 \mu_1}{a_2 \mu_2 N_2}, \frac{1}{a_2 \mu_2 N_2^2}, m_c = \|\Omega\|_{L^\infty}^2 \left( \frac{N_1 \chi_1^2}{4} + \frac{a_2 \mu_2 N_2^2}{4a_1 \mu_1} + 1 \right) \right\}.$$  \hspace{1cm} (4.11)

Hence, from (4.10) and the dissipation estimate (3.2), we derive that

$$\frac{d}{dt} E_1 + \frac{\tau}{\theta} E_1 \leq \frac{\tau m_c}{\theta} e^{-c N_1 + \beta N_2 (t-t_0)}, \quad \forall t \geq t_0,$$

and then, solving this Gronwall differential inequality, we readily get

$$E_1(t) \leq E_1(t_0) e^{-\frac{\tau}{\theta} (t-t_0)} + \frac{2 \tau m_c}{(\alpha N_1 + \beta N_2) \theta} e^{-c N_1 + \beta N_2 (t-t_0)} - \frac{2}{(\alpha N_1 + \beta N_2) \theta} e^{-c N_1 + \beta N_2 (t-t_0)}, \quad \forall t \geq t_0.$$  \hspace{1cm} (4.12)

where we have used the following algebraic calculations:

$$e^{(\alpha N_1 + \beta N_2) t} e^{-\frac{\tau}{\theta} (t-t_0)} = \begin{cases} \frac{1}{\theta} e^{-\frac{\tau}{\theta} (t-t_0)}, & \text{if } \frac{\tau}{\theta} = (\alpha N_1 + \beta N_2) \\ \frac{1}{\theta} e^{-\frac{\tau}{\theta} (t-t_0)} - e^{-\frac{\tau}{\theta} (t-t_0)}, & \text{if } \frac{\tau}{\theta} \neq (\alpha N_1 + \beta N_2) \end{cases} e^{-c N_1 + \beta N_2 (t-t_0)}, \quad \forall t \geq t_0.$$  \hspace{1cm} (4.13)

By the definition of $E_1$ in (4.11) and the estimates (4.7), (4.8), we see that

$$E_1(t) \geq \frac{2}{\theta} \min \left\{ \frac{1}{N_1}, \frac{a_1 \mu_1}{a_2 \mu_2 N_2} \right\} \left[ \int_{\Omega} (n_1 - N_1)^2 + \int_{\Omega} (n_2 - N_2)^2 \right].$$  \hspace{1cm} (4.14)

Joining (4.11) and (4.12) and substituting the definitions of $\tau$, $\theta$ and $m_c$ in (3.2) and (4.11), we finally arrive at (4.1) with $K_1$ and $\kappa$ given by (4.5) and (1.10), respectively. □

Thanks to the regularity in Lemma 2.2, we employ the well-known Gagliardo-Nirenberg interpolation inequality pass the $L^2$-convergence of $n_1$ and $n_2$ in (4.11) to the $L^\infty$-convergence.

**Lemma 4.3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded and smooth domain. Then the $n_1$- and $n_2$- solution components of bounded solution of (1.1) decay exponentially to $(N_1, N_2)$:

$$\|n_1(\cdot, t) - N_1\|_{L^\infty} + \|n_2(\cdot, t) - N_2\|_{L^\infty} \leq C e^{-\frac{\tau}{\theta} (t-t_0)}, \quad \forall t \geq t_0.$$  \hspace{1cm} (4.15)

for some $C > 0$ independent of $t$. Here, the exponential decay rate $\kappa$ is defined by (1.6).

**Proof.** Due to the $L^2$-convergence of $n_1, n_2$ in (4.11) and the uniform $W^{1,\infty}$-boundedness of $n_1, n_2$ in (2.3), the Gagliardo-Nirenberg inequality enables us to conclude that

$$\|n_1(\cdot, t) - N_1\|_{L^\infty} + \|n_2(\cdot, t) - N_2\|_{L^\infty} \leq c_1 \left( \|n_1(\cdot, t)\|_{W^{1,\infty}}^2 + \|n_2(\cdot, t)\|_{W^{1,\infty}}^2 \right) \leq c_2 \left( \|n_1(\cdot, t)\|_{L^2}^2 + \|n_2(\cdot, t)\|_{L^2}^2 \right) \leq c_3 e^{-\frac{\tau}{\theta} (t-t_0)}, \quad \forall t \geq t_0.$$  \hspace{1cm} (4.15)

This is nothing but the exponential decaying estimate (4.15). □
where we also used the fact for the Poincaré constant $\lambda$

Solving this ODI and performing similar computations to (4.13), we readily obtain

Recalling that $\mathbf{u}(1.1)$ by

Proof. Recalling that $\nabla \cdot \mathbf{u} = 0$ in $\Omega$ and $\mathbf{u}|_{\partial \Omega} = 0$, we multiply the fourth equation in (1.1) by $\mathbf{u}$ and integrate it over $\Omega$ to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} |\nabla \mathbf{u}|^2 = \int_{\Omega} (\gamma n_1 + \delta n_2) \nabla \phi \cdot \mathbf{u}
\]

where we also used the fact $\int_{\Omega} \nabla \phi \cdot \mathbf{u} = 0$. Therefore, we apply the Poincaré inequality:

\[
\lambda_P \int_{\Omega} |\mathbf{u}|^2 \leq \int_{\Omega} |\nabla \mathbf{u}|^2
\]

for the Poincaré constant $\lambda_P$ to (4.17) deduce that

\[
\frac{d}{dt} \int_{\Omega} |\mathbf{u}|^2 + 2\lambda_P \int_{\Omega} |\mathbf{u}|^2
\]

As a result, for

\[
\tilde{K}_1 = 2 \max\left\{ \frac{\gamma^2}{\lambda_P}, \frac{\delta^2}{\lambda_P} \right\} ||\nabla \phi||^2_{L_2^\infty},
\]

it follows that

\[
\frac{d}{dt} \int_{\Omega} |\mathbf{u}|^2 + \lambda_P \int_{\Omega} |\mathbf{u}|^2 \leq \tilde{K}_1 \left( \int_{\Omega} |n_1 - N_1|^2 + \int_{\Omega} |n_2 - N_2|^2 \right).
\]

Substituting (4.4) into (4.20), we derive that

\[
\frac{d}{dt} \int_{\Omega} |\mathbf{u}|^2 + \lambda_P \int_{\Omega} |\mathbf{u}|^2 \leq \tilde{K}_1 e^{-\kappa(t-t_0)}, \quad \forall t \geq t_0.
\]

Solving this ODI and performing similar computations to (4.13), we readily obtain

\[
||\mathbf{u}(\cdot, t)||^2_{L_2^\infty} \leq ||\mathbf{u}(\cdot, t_0)||^2_{L_2^\infty} e^{-\lambda_P(t-t_0)} + K_1 \tilde{K}_1 e^{\kappa t_0} e^{-\lambda_P t} \int_{t_0}^t e^{\lambda_P s} ds
\]

which is precisely the desired exponential decay estimate (4.16). 

\[
\text{Lemma 4.5. Let } \Omega \subset \mathbb{R}^d \text{ be a bounded and smooth domain. Then, for any } \epsilon \in (0, 1), \text{ there exists a constant } C > 0 \text{ such that}
\]

\[
||\mathbf{u}(\cdot, t)||_{L_\infty} \leq C e^{-\frac{\epsilon}{\lambda_P} \min\{\lambda_P, \tilde{\lambda}\}(t-t_0)}, \quad \forall t \geq t_0,
\]

where the exponent rate $\kappa$ is defined by (1.1).
Proof. Thanks to the $L^2$-convergence of $u$ in \((4.16)\) and the uniform $W^{1,p}$-boundedness of $u$ in \((2.41)\), the Gagliardo-Nirenberg inequality allows us to infer
\[
\|u(\cdot,t)\|_{L^\infty} \leq c_1 \|u(\cdot,t)\|_{W^{1,p}}^{\frac{2p-2d}{mp}} \|u(\cdot,t)\|_{L^2}^{\frac{2p-2d}{mp}} \leq c_2 e^{-\frac{(p-2d)\min\{\lambda p, \frac{1}{2}\} (t-t_0)}{d+2(1-\epsilon)(d+2(1-\epsilon))}} \quad \forall t \geq t_0,
\]
which implies \((4.39)\) upon choosing $p = \lceil d + 2(1-\epsilon)d/[(d+2)(1-\epsilon)] \rceil > d$. \hfill \Box

4.3. Convergence rates in Case II: $a_1 \geq 1 > a_2$. In this section, we will show that the solution components $(n_1, n_2, u)$ converge at least algebraically to $(0, 1, 0)$.

4.3.1. Convergence rates of $n_1$ and $n_2$ in Case II. Again, we start with the $L^1$- and $L^2$-convergence rates of $n_1$ and $n_2$.

**Lemma 4.6.** There exists $t_1 \geq \max\{1, t_0\}$ such that
\[
\|n_1(\cdot,t)\|_{L^1} + \|n_2(\cdot,t) - 1\|_{L^2} \leq \frac{K_2}{t + t_1}, \quad \forall t \geq t_1, \tag{4.22}
\]
where
\[
K_2 = \max\left\{2t_1 E_2(t_1), \frac{k_1(k_1 + k_2 + 2\sigma)}{\sigma} \right\} \min\{1, \frac{2a_1}{a_2 + 2a_2} \} \geq O(1)(1 + \frac{1}{\sigma}) = O(1)(1 + (1 - a_2)^{-1}) \tag{4.23}
\]
with $\sigma, k_1, k_2, k_3$ and $\alpha$ defined in \((3.4)\), \((4.24)\), \((4.25)\) and \((4.26)\), respectively.

**Proof.** Our proof makes use of the explicit Lyapunov functional provided by Lemma 4.1. To proceed, we first apply the Hölder inequality to find
\[
\int_{\Omega} n_1 \leq |\Omega|^\frac{1}{2} \left( \int_{\Omega} n_1^2 \right)^{\frac{1}{2}}. \tag{4.24}
\]
Next, since $\|n_2(\cdot,t) - 1\|_{L^\infty} \to 0$ as $t \to \infty$ and
\[
\lim_{z \to 1} \frac{z - 1 - \log z}{z - 1} = 0,
\]
we can take $t_0 > 0$ such that $|n_2(x, t) - 1 - \log n_2(x, t)| \leq |n_2(x, t) - 1|$ for all $x \in \Omega$ and $t \geq t_0$. Accordingly, we have
\[
\int_{\Omega} (n_2 - 1 - \log n_2) \leq \int_{\Omega} |n_2 - 1| \leq |\Omega|^\frac{1}{2} \left( \int_{\Omega} (n_2 - 1)^2 \right)^{\frac{1}{2}}, \quad \forall t \geq t_0. \tag{4.25}
\]
In this case, $(\hat{N}_1, \hat{N}_2) = (0, 1)$, so the exponential decay of $c$ in Lemma 4.1 warrants that
\[
\int_{\Omega} c^2 \leq |\Omega||c||e||L^\infty \leq |\Omega||c_0||L^\infty e^{-\beta(t-t_0)}, \quad \forall t \geq t_0. \tag{4.26}
\]
From the definitions of $E_2$ and $F_2$ in Lemma 3.2, upon a combination of \((4.24)\), \((4.25)\) and \((4.26)\) and the fact that $\sqrt{A} + \sqrt{B} \leq \sqrt{2(A + B)}$ for $A, B \geq 0$, we find two constants
\[
k_1 = \max\{1, \frac{\mu_1}{a_2 + 2\mu_2}\} (2|\Omega|)^{\frac{1}{2}}, \quad k_2 = \frac{\mu_1 \lambda_2^2}{8\sigma a_2 \mu_2} |\Omega||c_0||L^\infty \tag{4.27}
\]
such that
\[
E_2(t) \leq k_1 F_2(t) + k_2 e^{-\beta(t-t_0)}, \quad \forall t \geq \max\{t_0, \hat{t}_0\},
\]
which further gives us
\[
E_2^2(t) \leq 2k_1^2 F_2(t) + 2k_2^2 e^{-2\beta(t-t_0)}, \quad \forall t \geq \max\{t_0, \hat{t}_0\}.
\]
A substitution of this into the dissipation inequality \((3.4)\) entails
\[
\frac{d}{dt} E_2(t) + \frac{\sigma}{2k_1^2} E_2(t) \leq \frac{k_2^2 \sigma}{k_1^2} e^{-2\beta(t-t_0)}, \quad t \geq \max\{t_0, \hat{t}_0\}. \tag{4.28}
\]
Now, to illustrate (4.22), we first take \( t_1 = \max \{ t_0, \hat{t}_0, 1, \beta^{-1} \} \) so that
\[
a = \frac{k_3^2 \sigma}{k_1^2} \max \{ (t + t_1)^2 e^{-2\beta(t-t_0)} : t \geq t_0 \} = \frac{4k_3^2 \sigma}{k_1} e^{-2\beta(t_1-t_0)};
\]
and then, for any
\[
b \geq \frac{k_1}{\sigma} \left( k_1 + \sqrt{k_1^2 + 2a\sigma} \right),
\]
we put
\[
y(t) = \frac{b}{t + t_1}, \quad t \geq 0.
\]
We use straightforward calculations from (4.31) and use (4.29) to see that
\[
y'(t) + \frac{\sigma}{2k_1^2} y^2(t) - \frac{k_3^2 \sigma}{k_1^2} e^{-2\beta(t-t_0)}
\]
\[= (t + t_1)^{-2} \left[ \frac{\sigma}{2k_1^2} b^2 - \frac{k_3^2 \sigma}{k_1^2} (t + t_1)^2 e^{-2\beta(t-t_0)} \right]
\]
\[\geq (t + t_1)^{-2} \left( \frac{\sigma}{2k_1^2} b^2 - b - a \right) \geq 0, \quad \forall t \geq t_1.
\]
This, upon a clear choice of \( b \) in (4.30), an ODE comparison argument to (4.28) shows
\[
E_2(t) \leq \frac{2t_1 E_2(t_1)}{t + t_1} \left( \frac{k_1}{\sigma} \left( k_1 + \sqrt{k_1^2 + 2a\sigma} \right) \right): = \frac{k_3}{t + t_1}, \quad \forall t \geq t_1.
\]
Then since \( t_1 \geq t_0 \), we infer from (4.32) and the definition of \( E_2(t) \) in Lemma 3.2 that
\[
\min \{ 1, \frac{2\mu_1}{9d^2 \mu_2} \} \left( \| n_1(\cdot, t) \|_{L^1} + \| n_2(\cdot, t) - 1 \|_{L^2}^2 \right) \leq \frac{k_3}{t + t_1}, \quad \forall t \geq t_1.
\]
This, upon a substitution of the respective definitions of \( k_1, k_2, k_3 \) and \( a \) in (4.27), (4.28) and (4.29), proves our desired algebraic decay estimate (4.22).

**Lemma 4.7.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded and smooth domain. Then the \( n_1 \) and \( n_2 \)-solution components of bounded solution of (4.17) decay at least algebraically to \( (0,1) \): there exist two constants \( K_3 \) and \( K_4 \) independent of \( t \) fulfilling
\[
K_3 \geq O(1) \left( 1 + (1 - a_2)^{-\frac{1}{\beta(t_1)}} \right), \quad K_4 \geq O(1) \left( 1 + (1 - a_2)^{-\frac{1}{\beta(t_1)}} \right)
\]
such that
\[
\| n_1(\cdot, t) \|_{L^\infty} \leq \frac{K_3}{(t + t_1)^{\frac{1}{\beta(t_1)}}, \quad \forall t \geq t_1}
\]
as well as
\[
\| n_2(\cdot, t) - 1 \|_{L^\infty} \leq \frac{K_4}{(t + t_1)^{\frac{1}{\beta(t_1)}}, \quad \forall t \geq t_1.
\]
**Proof.** Equipped with the uniform \( W^{1,\infty} \)-bounds of \( n_1, n_2 \) in Lemma 2.2 as before, by means of the Gagliardo-Nirenberg inequality, we readily infer, for all \( t \geq t_1 \),
\[
\| n_1(\cdot, t) \|_{L^\infty} \leq c_1 \| n_1(\cdot, t) \|_{W^{1,\infty}} | n_1(\cdot, t) |_{L^1}^{\frac{1}{\beta(t_1)}} \leq c_2 \| n_1(\cdot, t) \|_{L^1}^{\frac{1}{\beta(t_1)}}
\]
and
\[
\| n_2(\cdot, t) - 1 \|_{L^\infty} \leq c_3 \| n_2(\cdot, t) \|_{W^{1,\infty}} | n_2(\cdot, t) - 1 |_{L^2}^{\frac{1}{\beta(t_1)}} \leq c_4 \| n_2(\cdot, t) - 1 \|_{L^2}^{\frac{1}{\beta(t_1)}}.
\]
These along with the \( L^1 \)- and \( L^2 \)-convergence of \( n_1, n_2 \) in (4.22) and the bound for \( K_2 \) in (4.23) yield immediately (4.33) and (4.34).
4.3.2. Convergence rate of \( u \) in Case II.

**Lemma 4.8.** The \( u \)-solution component of (1.1) fulfills

\[
\|u(\cdot,t)\|_{L^2}^2 \leq \frac{K_5}{t + t_1}, \quad \forall t \geq t_1
\]

for some positive constant \( K_5 \) independent of \( t \) satisfying \( K_5 \geq O(1)(1 + (1 - a_2)^{-1}) \).

**Proof.** Thanks to the \( L^1 \)-and \( L^2 \)-convergence of \( n_1, n_2 \) in (4.22), we can easily adapt the proof of Lemma 4.4 here. Indeed, we derive from (4.17), the Poincaré inequality (4.18) and the boundedness of \( u \) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 + \frac{\lambda_p}{2} \int_{\Omega} |u|^2 \leq \gamma \|\nabla \phi\|_{L^\infty} \int_{\Omega} n_1 + \frac{\delta^2 \|\nabla \phi\|_{L^\infty}^2}{2\lambda_p} \int_{\Omega} \|n_2 - 1\|^2.
\]

Thus, for

\[
\tilde{K}_5 = 2 \max\{\gamma \|u\|_{L^\infty(\Omega \times (0,\infty))}, \frac{\delta^2 \|\nabla \phi\|_{L^\infty}}{2\lambda_p}\} \|\nabla \phi\|_{L^\infty} < \infty,
\]

from (4.36) and (4.22), we obtain an ODI as follows:

\[
\frac{d}{dt} \int_{\Omega} |u|^2 + \lambda_p \int_{\Omega} |u|^2 \leq \tilde{K}_5 \left( \int_{\Omega} n_1 + \int_{\Omega} |n_2 - 1|^2 \right) \leq \frac{K_2 \tilde{K}_5}{t + t_1}, \quad \forall t \geq t_1.
\]

Solving the ODI (4.37), we end up with

\[
\|u(\cdot,t)\|_{L^2}^2 \leq \|u(\cdot,t_1)\|_{L^2}^2 e^{-\lambda_p(t-t_1)} + K_2 \tilde{K}_5 e^{-\lambda_p t_1} \int_{t_1}^{t} \frac{e^{\lambda_p s}}{s + t_1} ds
\]

\[
\leq \|u(\cdot,t_1)\|_{L^2}^2 e^{-\lambda_p(t-t_1)} + \frac{K_2 \tilde{K}_5 \tilde{K}_5}{t + t_1}
\]

\[
\leq \left( \|u(\cdot,t_1)\|_{L^2}^2 e^{-\lambda_p t_1} \tilde{K}_5 + K_2 \tilde{K}_5 \tilde{K}_5 \right)(t + t_1)^{-1}, \quad \forall t \geq t_1,
\]

from which (4.35) follows. Here, we have used the following facts

\[
\tilde{K}_5 = \max\{(t + t_1)e^{-\lambda_p t_1} : \; t \geq t_1\} < \infty
\]

and

\[
\tilde{K}_5 = \max\left\{(t + t_1)e^{-\lambda_p t_1} \int_{t_1}^{t} \frac{e^{\lambda_p s}}{s + t_1} ds : \; t \geq t_1\right\} < \infty.
\]

The latter is due to

\[
\lim_{t \to \infty} \left[ (t + t_1)e^{-\lambda_p t_1} \int_{t_1}^{t} \frac{e^{\lambda_p s}}{s + t_1} ds \right] = \lim_{t \to \infty} \frac{\int_{t_1}^{t} \frac{e^{\lambda_p s}}{s + t_1} ds}{(t + t_1)^{-1} e^{\lambda_p t_1}} = \frac{1}{\lambda_p} < \infty
\]

With the \( L^2 \)-convergence of \( u \) in Lemma 4.8 at hand, the same argument as done to Lemma 4.5 shows the following \( L^\infty \)-convergence of \( u \).

**Lemma 4.9.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded and smooth domain. Then, for any \( \epsilon \in (0,1) \), there exists a positive constant

\[
K_6 \geq O(1) \left( 1 + (1 - a_2)^{-\frac{\epsilon}{1 + \epsilon}} \right)
\]

such that

\[
\|u(\cdot,t)\|_{L^\infty} \leq \frac{K_6}{(t + t_1)^{\frac{\epsilon}{1 + \epsilon}}}, \quad \forall t \geq t_1.
\]
4.4. Convergence rates in Case III: \( a_2 \geq 1 > a_1 \). In this case, we shall show that the solution components \((n_1, n_2, u)\) converge at least algebraically to \((1, 0, 0)\). Comparing Lemmas 3.2 and 3.3 and the \(n_1\)-and \(n_2\)-equations in (1.1), we see that this subsection is fully parallel to Section 4.3, and so we simply write down their respective final outcomes.

**Lemma 4.10.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded and smooth domain. Then the \(n_1\) - and \(n_2\)-solution components of bounded solution of (1.1) decay at least algebraically to \((1, 0)\); there exist \( t_2 \geq \max\{t_1, t_0\} \) and positive constants \( K_7 \) and \( K_8 \) independent of \( t \) fulfilling

\[
K_7 \geq O(1)\left(1 + (1 - a_1)^{-\frac{1}{d+2}}\right), \quad K_8 \geq O(1)\left(1 + (1 - a_1)^{-\frac{1}{d+1}}\right)
\]

such that

\[
\|n_1(\cdot, t) - 1\|_{L^\infty} \leq \frac{K_7}{(t + t_2)^{\frac{1}{d+2}}}, \quad \forall t \geq t_2
\]

and

\[
\|n_2(\cdot, t)\|_{L^\infty} \leq \frac{K_8}{(t + t_2)^{\frac{1}{d+1}}}, \quad \forall t \geq t_2.
\]

**Lemma 4.11.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded and smooth domain. Then, for any \( \epsilon \in (0, 1) \), there exists a positive constant \( C_\epsilon \geq O(1)(1 + (1 - a_1)^{-\frac{1}{d+2}}) \) such that

\[
\|u(\cdot, t)\|_{L^\infty} \leq \frac{C_\epsilon}{(t + t_2)^{\frac{1}{d+2}}}, \quad \forall t \geq t_2.
\]

**Proof of Theorem 1.1.** Notice that \( t_0, t_1, t_2 \geq 1 \); the respective decay estimates asserted in Theorem 1.1 follow from some lemmas in this section with perhaps some large constants \( m_\epsilon \). More specifically, the exponential decay estimate (1.7) follows from Lemmas 4.3 and 4.8, the algebraical decay estimate (1.2) follows from Lemmas 4.4 and 4.5, the algebraical decay estimate (1.9) follows from Lemmas 4.10 and 4.11, and, finally, the exponential decay estimate (1.10) follows simply from Lemma 4.1. \( \square \)

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