Comment on “Parameter-free scaling for nonequilibrium growth processes”

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In the paper [Phys. Rev. E 79, 051605 (2009)] by Chou and Pleimling a claim is made that a parameter-free scaling that gives data collapse for some simulation models would replace universal Family-Vicsek (FV) scaling. Here, by giving the explicit form of this scaling for competitive growth models, it is shown that data collapse in this procedure is obtained by a shift-and-scale operator that gives no information about stochastic dynamics and has no relation with FV function.

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Nonequilibrium suface-growth processes of Ref. [1] are SOS models in (1 + 1) dimensions with periodic boundary and time t is a number of deposited monolayers on a substrate of L sites. At t = 0 the substrate is flat. The growth rule is a competitive growth model: “either RD (active with probability q) or X (active with probability p = 1 − q),” where RD is random deposition and X is a process that builds correlations. For these models, the initial time evolution of the surface width w(t) has two growth regimes before cross-over time to saturation [1, 2].

In processes “RD or X” for any L and p, w(t) has three evolution regimes, as shown in Fig.1 of Ref. [1]:

\[ t \in [0; t_1] \cup (t_1; t_2) \cup [t_2; +\infty) = [0; +\infty). \]  (1)

First, w(t) obeys the RD universal power law:

\[ \forall t \in [0; t_1] : w(t) \propto t^{\beta_1}, \quad \beta_1 = 1/2. \]  (2)

Note, \( \beta_1 \) is not a scaling exponent [3]. Later, w(t) obeys:

\[ \forall t \in (t_1; t_2) : w(t) \propto t^{\beta_2}, \]  (3)

\[ \forall t \in [t_2; +\infty) : w(t) \propto L^{\alpha_2}, \]  (4)

where \( \beta_2 \) and \( \alpha_2 \) are scaling exponents (growth and roughness, respectively) of the universality class of X.

For fixed L and p, values of w(t) are in the interval [0; w_2], where \( w_2 = w(t_2) \propto L^{\alpha_2} \) by Eq. (4). After Ref. [1], this interval is [0; w_2] = [0; w_1] \cup (w_1; w_2], where \( w_1 = w(t_1) \propto \sqrt{t_1} \) by Eq. (2). In the limit of large but finite L, w_2 is large but finite; hence, in general:

\[ w(t) \in [0; w_1] \cup (w_1; w_2] = [0; w_2]. \]  (5)

In Eqs. (1) and (5), t_2 and w_2 are functions of L. But, w(t) in Eq. (2) does not depend on L [3]. In Ref. [1] this is seen in Fig.2a, where w(t) is for the model “RD or X” and X is “random deposition with surface relaxation,” i.e., in Edwards-Wilkinson (EW) universality class. For the model “RD or EW,” the data can be summarized as the family of curves parameterized by p and L:

\[ w(t; L; p) = \begin{cases} 
  c_1 \sqrt{t}, & t \in [0; t_1(p)] \\
  c_2 t^{\beta_2}, & t \in (t_1(p); t_2(L, p)) \\
  c_3 L^{\alpha_2}, & t \in [t_2(L, p); +\infty], 
\end{cases} \]  (6)

where \( c_1, c_2, \) and \( c_3 \) are constants.

In the language of ‘data collapse,’ Eq. (5) says that for \( t \in [0; t_1(p)] \) all the data for all L and for all p are already in one curve \( w = c_1 \sqrt{t} \). In order to collapse the data for \( t \in (t_1(p); +\infty) \) it is required to simultaneously multiply t and \( w(t) \) by different scale factors \( a(p, L) \) and \( b(p, L) \):

\[ t \rightarrow t/a(p, L), \quad w(t) \rightarrow w(t)/b(p, L). \]  (7)

But, if affine transformations in Eq. (7) give data collapse for \( t \in (t_1(p); +\infty) \), when they are applied to \( w(t) = c_1 \sqrt{t} \) for \( t \in [0; t_1(p)] \) they will produce ‘data scatter.’ This is because \( a(p, L) \neq b(p, L) \) and the data for \( t \in [0; t_1(p)] \) already follow one curve for any L and p. That is, Family-Vicsek (FV) scaling produces data collapse in EW-scaling regime for \( t \in (t_1; +\infty) \) and destroys data coalescence in RD-growth regime for \( t \in [0; t_1] \).

Because of this, the questions are about the rationale for the data collapse shown in Ref. [1] and about a relevance of the proposed parameter-free scaling to the universal dynamic scaling. The answers are given below.

In Ref. [1], data are collapsed in two-step transformations performed individually for each curve of Eq. (4), i.e., for each set of points \( (t, w) \) representing one curve indexed by p and L. In the first step, t is divided by t_1, and \( w(t) \) is divided by w_1, and the log is taken of all numbers. In this way intervals in Eq. (1) are mapped onto \( t \rightarrow t'' \in (-\infty; 0] \cup (0; \tau'') \cup [\tau''; +\infty) = (-\infty; +\infty), \)

\[ \tau'' = \log (t_2/t_1). \]  (8)

The intervals in Eq. (5) are mapped onto \( w \rightarrow w'' \in (-\infty; 0] \cup (0; w_2''] = (-\infty; w_2''], \)

\[ \text{where } w_2'' = \log (w_2/w_1). \]  (9)

In the second step, a number λ is selected such that \( \lambda w_2'' = 1 \), and \( t'' \) and \( w'' \) are multiplied by λ. This gives

\[ w'' \rightarrow w'' \in (-\infty; 0] \cup (0; 1] = (-\infty; 1), \]  (10)

\[ t'' \rightarrow t' \in (-\infty; 0] \cup (0; \tau) \cup [\tau; +\infty) \times \{1\}. \]  (11)

For curves in Eq. (6) (shown here in the left Fig.1) the procedure of Ref. [1] is described by the operator \( \hat{G}_{(p, L)} \),

\[ \hat{G}_{(p, L)} : (x, y) \rightarrow (x', y'), \]  (12)
where $x = \log t$, $y = \log w$, $x' = \log t'$, and $y' = \log w'$. Denoting $x_i = \log t_i$ and $y_i = \log w_i$ for $i = 1, 2$, standard algebra methods give the explicit form of $\hat{G}(p, L)$:

$$x \rightarrow x' = \frac{x - x_1}{x_2 - x_1} \beta_2, \quad y \rightarrow y' = \frac{y - y_1}{y_2 - y_1}.$$  \hfill (11)

Explicitly, $\hat{G}(p, L) = \hat{S}(p, L) \circ \hat{T}(p)$ is the composition of the translation operator, $\hat{T}(p) : (x, y) \rightarrow (x', y') = (x - x_1, y - y_1)$, and the scale operator, $\hat{S}(p, L) : (x'', y'') \rightarrow (x', y') = (x''/\beta_2, y''/\beta_2)$, in $(x, y)$-plane. Thus, $\hat{G}(p, L)$ is a shift-and-scale operator that translates the curves to one position and adjusts the length of the correlation-growth phase to $\sqrt{t} + 1$, as shown here in Fig. 1. Such shift-and-scale operation is possible because each curve in this family carries one universal footprint: the initial RD transient — where each curve has the same one slope of 1/2 but its length depends on $p$ — that is followed by a specific universal correlation phase, where each curve has the same one slope of $\beta_2$ and ends at saturation phase. However, a similar picture may or may not occur in other competitive growth phenomena, so $G(p, L)$ is not general. Moreover, as this explicit derivation of $\hat{G}(p, L)$ shows, the shift-and-scale operator in $(x, y)$-plane is not a dynamic scaling.

Data collapse by shift-and-scale operation is a nice illustration of the known fact that all systems in one universality class follow one universal curve. But it has no connection with finite-size dynamic scaling and with FV function. Affine scaling of interfaces such as the one in Eq. (7) reflects universal dynamics of correlations. FV function summarizes a relation between scaling properties of growing surfaces and symmetry properties of equations that describe growth dynamics. In FV function, argument and prefactor contain explicit information about the way the dynamics is affected by growth parameters. Physics wise, FV function provides explanation not only for data collapse but first of all for the universal shape of $w(t)$, in contrast to the scaling of Ref. [1].

Proof that parameter-free scaling is a dynamic scaling calls for showing that it connects with stochastic dynamics, i.e., it must be shown that Eqs. (11) give $t_1$ and $t_2$ as functions of $L$ and $p$. But Eqs. (11) and (6) are equivalent, hence, Eqs. (11) do not contain any new information above that in Eq. (6): Parameter-free scaling does not explain the universal shape of $w(t)$. In order to find $t_1$ and $t_2$ one must analyze the scale invariance of stochastic growth equation, as universal scaling functions express this invariance: Parameter-free scaling does not express dynamical scale invariance.

Even for models with $p = 1$ parameter-free scaling does not connect with dynamics. When in Eq. (7) of Ref. [1] a heuristic parameter $\lambda$ is set $\lambda = 1$, in order to find $w_2(L)$ and $t_2(L)$ one must perform finite-size scaling analysis. On the other hand, when $w_2(L)$ and $t_2(L)$ are known beforehand, it is possible to verify if Eq. (7) of Ref. [1] represents dynamic scaling, which shows that there are some examples of dynamics where the scaling of Ref. [1] may map directly on FV function when heuristically $\lambda = 1$. However, there is no physical reason to set $\lambda = 1$. When $\lambda \neq 1$, then Eq. (7) of Ref. [1] does not give FV scaling function even if $w_2(L)$ and $t_2(L)$ are known.

Perhaps Eq. (7) in Ref. [1] should have been supplemented by an explanation that we must always have $\lambda = 1$ and explicitly know $w_1$, $w_2$, $t_1$, and $t_2$ as functions of $L$ and $p$ for a model, in order to be able to identify the parameter-free scaling of Ref. [1]. Of course then $t_1$ and $t_2$ must come from dynamical scale-invariance analysis and from FV function, because there is no other way to obtain crossover times. Therefore the claim made in Ref. [1] that parameter-free scaling is ‘something more’ than FV scaling does not hold: To the contrary, parameter-free scaling is ‘something less.’ This is because when we know FV function we can construct parameter-free scaling in such a way that it reflects the scale invariance of stochastic dynamics. But if we do not know the symmetries of the stochastic growth equation the data collapse via parameter-free scaling is meaningless.

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[1] Y.-L. Chou and M. Pleimling, Phys. Rev. E 79, 051605 (2009).
[2] A. Kolakowska, M. A. Novotny, and P. Verma, Phys. Rev. E 73, 011603 (2006); A. Kolakowska and M. A. Novotny, arXiv:cond-mat/0511688v3, (2006); C. M. Horowitz and E. V. Albano, J. Phys. A: Math. Gen. 34, 357 (2001); C. M. Horowitz et al, Phys. Rev. E 63, 066132 (2001).
[3] In RD universality class, all curves $w(t)$ for any $L$ follow only one curve $w(t) = c \sqrt{t}$, where $c$ is nonuniversal, i.e., $c$ is model-dependent and does not depend on $L$. 

FIG. 1: (Color on line) Domain (left) and image (right) of the shift-and-scale operator $\hat{G}(p, L)$. In the image, cross-over to saturation is at $(\tau = 1/\beta, 1)$. The curves are for “RD or EW” competitive growth process in Eq. (6). Here, $\log(\cdot) = \log_{10}(\cdot)$. 

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