Integrable nonlinear coupled waves with an exact asymptotic singular solution in the context of laser–plasma interaction

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Abstract
A special case of coupled integrable nonlinear equations with a singular dispersion law is derived in the context of the small amplitude limit of general wave equations in a fluid-type warm electrons/cold ions plasma irradiated by a continuous laser beam. This model accounts for a nonlinear mode coupling of the electrostatic wave with the ion sound wave and is shown to be highly unstable. Its instability is understood as a continuous secular transfer of energy from the electrostatic wave to the ion sound wave through the ponderomotive force. The exact asymptotic solution of the system is constructed and shows that the dynamics of the energy transfer results in a singular asymptotic behavior of the ion sound wave, which explains the low penetration of the incident laser beam.

Keywords: inverse scattering transform, singular dispersion relation, nonlinear coupled waves, laser–plasma interaction

Mathematics Subject Classification numbers: 35A20, 35B30, 45D05, 45Q05

1. Introduction
Recent progress in laser generation has aroused great interest in the problem of laser–plasma interaction, which is the source of various nonlinear effects, such as self-focusing, self-modulation, filamentation and several parametric instabilities [1–5]. These phenomena have applications in many advanced physical events, such as inertial confinement fusion, higher order harmonic generation, x-ray sources and particle acceleration [6–9].
This paper concerns the general theory of nonlinear coupled waves in plasmas, in the context of a two-component non-magnetized fluid-type plasma. In a plasma produced and irradiated by a laser beam, the electrostatic wave (EW) induces scattered waves by means of different types of stimulated emission [10, 11]. Here we are interested in the process of stimulated Brillouin (back) scattering (SBS) for longitudinal EW. The resulting incident and reflected high-frequency EW can drive some low-frequency waves of acoustic type, such as the ion sound wave (ISW), by means of the ponderomotive effect on the local charge density. In this paper, it is shown that this nonlinear effect is at the origin of plasma instability.

It is well known that the model of the nonlinear propagation of the ISW, where the ponderomotive effects are neglected (no mode coupling), is the famous Korteweg–de Vries (KdV) equation [10, 11]. This equation has the advantage of being integrable, which allows us to describe the solution on the basis of its ‘nonlinear Fourier spectrum’ (see, for instance, [12] and [13]). When the ISW is decomposed into its harmonic modes, the KdV equation maps into the nonlinear Schrödinger equation (NLS), another integrable model [10–13]. Both models propagate soliton solutions which provide general fluid equations with stable small-amplitude localized coherent structures. But these models ignore the possibility of energy transfer from the long wavelength domain to the dissipative short wavelength range.

This energy transfer is considered by including the ponderomotive force (the low-frequency effect of the high-frequency electrostatic field) that results in the nonlinear mode coupling in the momentum transfer equation for the electrons. Using this approach, Karpman [14] derived an integrable model, but this does not account for dispersion or nonlinear dynamical effects (ISW nonlinearity). However, this model has proved to be very useful and it was slightly expanded in [15] and [16]. All these models and their stability properties are discussed in [17].

Later, Casanova et al [18] showed that the wave dynamics in plasma is well described by a model where the ISW is nonlinear and is coupled to the electrostatic field through the ponderomotive term. In particular, the ISW nonlinearity was shown to play a crucial role in the Brillouin reflexivity.

In this paper, starting from the most general one-dimensional fluid equation for a plasma with a Maxwellian velocity distribution [10], a model for the nonlinear propagation of the ISW coupled to the EW (the pump wave and the wave reflected through SBS) is established. This model is obtained as an exact asymptotic limit in the regime of ‘warm electrons–cold ions’ and a small amplitude of the pump wave. The technique used is the well-known multi-scale expansion [19, 21], and the model expounded here then appears to be universal in the sense given in [20]. This model is an integrable nonlinear evolution in the context of laser–plasma interaction with unexpected properties. The integrability permits us to obtain general theorems about the nature and behavior of the solution. In particular, it is shown that the EW transfers its energy to the ISW through a highly nonlinear and unstable process: any infinitely small initial disturbance of the ISW asymptotically evolves towards a singular (soliton-like) solution.

2. Derivation

Let us consider a linearly polarized electrostatic field vector \( \vec{E} = \xi E(z, t) \) with a frequency close to the plasma frequency \( \omega_p = 4\pi e^2 n_0/m_e \), with some broadening represented by the following wave packet:

\[
E(z, t) = \int_{-\infty}^{+\infty} d\omega \tilde{E}(\omega, z, t)e^{-i\omega t}.\tag{2.1}
\]
Note that \( m_0 \) and \(-e\) are respectively the mass and the charge of the electrons, and \( n_0 \) is the ambient ion density. \( E(\omega, z, \tau) \) is the slowly varying envelope of \( E(z, t) \), where \( \tau \) is the slow-scale time defined below (2.13).

The low-frequency effect on the electrons due to the high-frequency field \( E \) results in the ponderomotive force obtained by considering the real part of \( \hat{\delta z}(\partial E/\partial z) \) around the average position \( \langle z \rangle \), where \( \delta z \) is the complex valued displacement and solution of

\[
m_0(\delta z)_m = eE.
\]  

(2.2)

The ponderomotive force then reads

\[
f_p = -\frac{e^2}{2m_e} \partial z \int_{-\infty}^{+\infty} d\omega \, \omega^2 |\tilde{E}(\omega, z, \tau)|^2.
\]  

(2.3)

Using the ratio of the electron and ion masses \( \epsilon = (m_e/m_i)^2 \) and the electron Debye wavelength \( \lambda_D^2 = K_BT_e/4\pi n_0 e^2 \) (with \( T_e \) being the electron temperature and \( K_B \) the Boltzmann constant), a dimensionless set of variables can be set as follows:

\[
z' = \lambda_D^{-1} z, \quad t' = e\omega_D t.
\]  

(2.4)

In these variables, we have the usual system of fluid-type equations [10], with the additional term \( f_p \) in the momentum transfer equation for the electrons [22].

Moreover, the Maxwell equation for the EW is obtained on the basis of the dispersion relation [10]

\[
\omega^2 = \omega_p^2 + 3V_T^2 k^2
\]  

(2.5)

where \( V_T = \lambda_D \omega_D \) is the thermal electron velocity, \( k \) is the wave number and \( \omega_p \) stands for the plasma frequency

\[
\omega_p^2 = \omega_D^2 (1 + q_e).
\]  

(2.6)

\( q_e \) is the fractional change in the electron density of average value \( n_0 \), namely

\[
n_e = n_0 [1 + q_e(z, t)].
\]  

(2.7)

Now, redefining the fields as

\[
\tilde{E}' = \tilde{E} \left( \frac{e^2}{2m_e \omega_D^2 K_B T_e} \right), \quad \phi' = \frac{e}{K_B T_e} \phi, \quad v'_i = v_i \left( \frac{m_i}{K_B T_e} \right)^{1/2},
\]  

(2.8)

where \( \phi \) is the electrostatic potential and \( v_i \) is the ion velocity, the momentum transfer equation for electrons reads [10, 22]

\[
\frac{\partial \phi'}{\partial z'} - \frac{1}{(1 + q_e)} \frac{\partial q_e}{\partial z'} - \frac{\partial}{\partial z'} \int_{-\infty}^{+\infty} d\omega |\tilde{E}'(\omega, z', \tau)|^2 = 0,
\]  

(2.9)

and for the ions it reads

\[
\frac{\partial v'_i}{\partial t'} + v'_i \frac{\partial v'_i}{\partial z'} = -\frac{\partial \phi'}{\partial z'}
\]  

(10.10)

The above system has to be completed with the continuity equation

\[
\frac{\partial q_i}{\partial t'} + \frac{\partial}{\partial z'} [(1 + q_e) v'_i] = 0,
\]  

(2.11)
and the Poisson equation
\[
\frac{\partial^2 \phi}{\partial z'^2} = q_e - q_i, \tag{2.12}
\]
where \(q_i\) in (2.11) and (2.12) is the fractional change in the ion density \(n_i = n_e (1 + q_i)\).

Let us define a slowly varying set of variables in the co-moving frame of the ISW at the sound speed \(C_s = (K_B T_e/m_i)^{1/2} = \epsilon \omega_0 \lambda_B\):
\[
\xi = \epsilon (z' - \tau), \quad \tau = (\epsilon \tau)^{3/2}.
\tag{2.13}
\]
Now, by expanding the electrostatic field \(\phi\), the density variations \(q_e\) and \(q_i\) and the scaled ion velocity \(v_i'\) in powers of \(\epsilon\) as
\[
v_i' = \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + O(\epsilon^3), \tag{2.14}
\]
and by defining \(E'\) as follows
\[
E' = (\epsilon \omega_0^{3/2}) [E + O(\epsilon)] \tag{2.15}
\]
the set of equations (2.9)–(2.12), after the integration of (2.9), gives at the first order
\[
r \div q_e^{(1)} = q_i^{(1)} = \phi^{(1)} = v_i^{(1)}, \tag{2.16}
\]
and at the second order, by eliminating \(q_e^{(2)}, q_i^{(2)}, v_i^{(2)}\) and \(\phi^{(2)}\), one obtains the following evolution for \(r\) (defined in (2.16)):
\[
2r + 2rr_t + r_{\xi\xi} = -\partial_\xi \int_{-\infty}^{+\infty} 3\omega_0 \nu d\nu |E|^2. \tag{2.17}
\]
On the other hand, the Maxwell equation from (2.5) reads
\[
\left[ \frac{\partial^2}{\partial t^2} - 3 (V_T e)^2 \frac{\partial^2}{\partial z^2} \right] E(z, t) = -\omega_0^2 [1 + q_e(z, t)] E(z, t). \tag{2.18}
\]
Using the change of variable (2.4), the scalings (2.13), (2.15), and the expansion (2.14), equation (2.18) gives at first order
\[
\partial_\xi E + \left( \nu^2 - \frac{r}{3} \right) E = 0, \tag{2.19}
\]
where the parameter \(\nu\) is defined through the relative difference between the squared frequency \(\omega\) of the applied laser beam and the squared plasma frequency \(\omega_0^2\):
\[
\frac{\omega^2 - \omega_0^2}{\omega_0^2} = 3 \epsilon \nu^2. \tag{2.20}
\]
The system (2.17)–(2.19) representing the nonlinear propagation of the EW coupled to the ISW does not account for SBS and has been shown to not be integrable [22]. To take SBS into account, one has to allow the electric field envelope \(E\) to contain simultaneously at first order a left-going wave \(\exp [i(\omega_1 \tau + k_1 \xi)]\) and a right-going wave \(\exp [i(\omega_2 \tau + k_2 \xi)]\) with small,
slowly varying amplitudes \( \varepsilon_2(\xi, \tau) \) and \( \varepsilon_2a_2(\xi, \tau) \), respectively, \( \varepsilon_2 \) being a small parameter submitted to the constraint (2.29). The wave number moduli are close to the same average value \( k_e \), namely:

\[
k_1 = k_e + O(\varepsilon_2), \quad k_2 = -k_e + O(\varepsilon_2),
\]

which means

\[
E(\xi, t) = \varepsilon_2 [a_1 e^{i \omega_1 t + ik_1 \xi} + a_2 e^{i \omega_2 t + ik_2 \xi}],
\]

and the frequencies are related through (resonant scattering conditions)

\[
\omega_1 = \omega_2 + \omega_s, \quad k_1 = k_2 + k_s,
\]

where \( \omega_s \) and \( k_s \) stand for the ISW parameters. In the frame \((\xi, \tau)\), the dispersion relation reads

\[
\omega = k_s^2.
\]

In the laboratory frame \((X, T)\) with the scaled variables [19] defined as follows

\[
X = \varepsilon_2 \left[ \xi + \frac{3}{2} (k_s)^2 \tau \right], \quad T = 3(\varepsilon_2)^2 k_s \tau,
\]

we consider \( r(\xi, \tau) \) (defined in (2.16)) to be a superposition of its harmonic modes expanded in powers of \( \varepsilon_2 \) as the formal series

\[
r = (\varepsilon_2)^2 \tilde{q}_0 + \sum_{n=1}^{N} (\varepsilon_2)^n \tilde{q}_n e^{i n \omega \tau + ik n \xi} + \text{C.C.}
\]

with

\[
\tilde{q}_n = \sum_{j=0}^{M} q_{n,j}(X, T)(\varepsilon_2)^j + O(\varepsilon^{M+1}), \quad n = 0, ..., N.
\]

For \( N = 2 \) and \( M \leq 3 \), by means of the standard multiscale techniques [19, 20, 22]—i.e. by inserting (2.22) and (2.25) into (2.17) and (2.19), and then examining the relevant orders in the expansions in powers of \( \varepsilon_2 \) and averaging over the fast oscillation \( \exp(i \omega \tau) \) (which means only looking at the coefficients of \( \exp(i \omega \tau) \), which is known as the ‘rotating wave approximation’)—one obtains for

\[
q = \frac{1}{3i k_e} q_0(X, T), \quad \varepsilon_2 \zeta = \frac{k_s^2 - \nu^2}{2k_e}
\]

the following system of coupled equations:

\[
\begin{cases}
-iq_T + \frac{1}{2}q_{XX} - q |q|^2 &= i \frac{\omega_0}{12} \int_{-\infty}^{\infty} dq_0 \bar{q}_2 \\
\alpha_{1,X} + i\zeta a_1 &= qa_2 \\
\alpha_{2,X} - i\zeta a_2 &= \bar{q} a_1
\end{cases}
\]

The left-going wave of amplitude \( a_1(X, T) \) results from the incident (pump) laser beam with wave number \( \zeta \) and an initial given amplitude of, say, \( A(\zeta) \). The right-going wave of amplitude \( a_2(X, T) \) is the wave reflected by SBS.
Note that the two multiscale analyses with the small parameters $\epsilon_1$ and $\epsilon_2$ applied in this section have to be consistent, which is guaranteed by the ordering

$$
\epsilon_1 \ll \epsilon_2 \ll \epsilon_2 \ll 1.
$$

To the system (2.28), we adjoin the following initial/boundary values:

$$
\begin{aligned}
q(X, 0) &\in L^1(\mathbb{R}) \\
\lim_{X \to +\infty} a_1(\zeta, X, T) &= A(\zeta)e^{-i\zeta X} \\
\lim_{X \to -\infty} a_2(\zeta, X, T) &= 0
\end{aligned}
$$

Thus, the system of coupled equations (2.28) completed by the initial/boundary conditions (2.30) is the central point of discussion in the following sections.

3. Method of solution

The system (2.28) with the initial/boundary conditions (2.30) belongs to a general class of integrable nonlinear evolution [23] with a singular dispersion law [23, 24, 28, 29]. This system is a special case of the following system for the $2 \times 2$ matrices $\mu(k, x, t)$ and $Q(x, t)$:

$$
\begin{aligned}
&Q - \frac{i}{2} \sigma_3 Q_{xx} + i\sigma_3 Q^2 = i[\sigma_3, \langle \mu \sigma \mu^{-1} \rangle], \\
&\mu_x + ik[\sigma_3, \mu] = Q\mu
\end{aligned}
$$

where

$$
Q = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

and

$$
\langle \mu \sigma \mu^{-1} \rangle \doteq \frac{1}{2\pi i} \int \int_{\mathbb{C}} d\lambda \wedge d\bar{\lambda} g(\lambda, t)\mu(\lambda, x, t)\sigma_3 \mu^{-1}(\lambda, x, t),
$$

for any given distribution $g(k, t)$ in the complex plane. We also have

$$
d\lambda \wedge d\bar{\lambda} = -2i d\lambda d\bar{\lambda} \quad \text{for} \quad \lambda = \lambda_k + i\lambda_t.
$$

The above system has been shown to be integrable [24, 28, 29]. It is related to the general theory of the spectral transform [12, 13] in the context of the singular dispersion relation [22]. A complete proof and details are given in the appendix.

One can summarize the solution method using the following set of linear steps.

**Step 1.** Given $q(x, 0)$ on $L^1(\mathbb{R})$, solve the following Volterra equations for the vector $\mu^+(\zeta, X, 0)$ and $\mu^-(\zeta, X, 0)$ with components $\mu_1$ and $\mu_2$

$$
\begin{aligned}
\mu_1^+ &= 1 - \int_X^{+\infty} dX' q\mu_2^+ \\
\nu_2^+ &= \int_X^{-\infty} dX' q\nu_1^+ \\
\mu_1^- &= 1 - \int_X^{+\infty} dX' q\mu_2^- \\
\nu_2^- &= -\int_X^{+\infty} dX' q\nu_1^-
\end{aligned}
$$

with $\nu = \mu \exp(2i\zeta X)$. 

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The vector $\mu^+$ (respectively $\mu^-$) is homomorphic in $\mathcal{H}(\zeta) > 0$ (respectively in $\mathcal{H}(\zeta) < 0$) and the discontinuity $\mu^+ - \mu^-$ as $\zeta$ crosses the real axis leads to the following Riemann–Hilbert problem for $\zeta \in \mathbb{R}$:

$$\mu^+(\zeta + i0) - \mu^-(\zeta - i0) = R(\zeta + i0)e^{2i\lambda \sigma_1}(\zeta - i0).$$  \hspace{1cm} (3.6)

The function $R(\zeta + i0)$ is called the nonlinear Fourier transform of $q(X, 0)$ and $\sigma_1$ is the usual Pauli matrix. $R$ is computed out of the Volterra equation (3.5) for given $q(X, 0)$ by

$$R(\zeta, 0) = \int_{-\infty}^{+\infty} dX' q(X', 0)\mu_1^+(\zeta, X', 0)e^{-2i\lambda X'}. \hspace{1cm} (3.7)$$

This first step solves the direct problem, namely the determination of $R(\zeta)$ from $q(X, 0)$.

**Step 2.** When $q(X, T)$ obeys the nonlinear evolution equation of (2.28), the time dependence of $R$ is given by

$$R(\zeta, T) = R(\zeta)e^{-\theta T}. \hspace{1cm} (3.8)$$

with

$$\theta = \zeta^2 + \frac{\omega_0}{12} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - \zeta}|A(\lambda)|^2, \hspace{1cm} (3.9)$$

where the slashed integral denotes the Cauchy principal value. It is worth remarking that $\theta$ has a positive imaginary part $\theta_1 = (\pi/12)\omega_0|A(\lambda)|^2$, which implies the exponential growth of $R(\zeta + i0, T)$ in time. We shall see that this fact is the reason for the instability of the system (2.28).

**Step 3.** Solve the Riemann–Hilbert problem (3.6) in which $R(\zeta, T)$ is given in terms of $R(\zeta, 0)$ by means of (3.7). The solution is obtained from the following Cauchy–Green integral equation (omitting the $(X, T)$-dependence):

$$\mu(\zeta) = \left(\frac{1}{1} + \frac{1}{2\pi i} \oint_C \frac{d\lambda}{\lambda - \zeta} \sigma_1 R(\lambda)e^{2i\lambda X}, \hspace{1cm} (3.10)$$

where $C$ represents any contour in the upper half plane ($\mathcal{H}(\zeta) > 0$) equivalent to the real axis. Equation (3.10) gives the solution $\mu^+(\zeta, X, T)$, while the solution $\mu^-(\zeta, X, T)$ is obtained from (3.6).

**Step 4.** Finally, the solution $(q, a_1, a_2)$ of the system (2.28) is obtained from the function $\mu$ through

$$\begin{cases}
q(X, T) = 2i\mu^{(1)}(X, T) \\
a_i(X, T) = A_i(\zeta, X, T)e^{-i\lambda X}, \quad i = 1, 2
\end{cases} \hspace{1cm} (3.11)$$

where $\mu^{(1)}(X, T)$ denotes the coefficient of $\zeta^{-1}$ in the Laurent expansion of $\mu_2(\zeta, X, T)$.

This last step achieves the resolution of the initial/boundary value problem for the system (2.28)–(2.30).

Note that equations (3.5) and (3.6) imply

$$\lim_{X \to +\infty} \mu_2^+ = Re^{2i\lambda X}. \hspace{1cm} (3.12)$$
Hence, \( R(\zeta + i0) \) is in fact the reflection coefficient of the potential \( q \) at any fixed time \( t \). It is easy to verify that the transmission coefficient \( T \), defined by
\[
\lim_{\chi \to -\infty} \mu^+ = T,
\] (3.13)
and the reflection coefficient \( R \) obey the unitarity relation
\[
R(\zeta + i0)R(\zeta - i0) + T(\zeta + i0)T(\zeta - i0) = 1.
\] (3.14)

4. Conservation laws

The first usual consequence of the integrability of a system is the existence of an infinite sequence of local conservation laws.

The explicit calculation of the conservation laws requires the Lax pair \( \Omega(k) \) and \( W(k) \) [12]. Here, \( \Omega \) is the (singular) dispersion relation (this choice is justified in the appendix; see (A.35) and (A.36))
\[
\Omega = ik^2\sigma^2 + \frac{1}{2i\pi} \int_C \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} g(\lambda)\sigma^2
\] (4.1)
and \( W \) can be computed using the method expounded in [22]
\[
W(k) = -i\sigma k^2 + Qk + \frac{i}{2}\sigma_3(Q_k - Q^2) - \frac{1}{2i\pi} \int_C \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} g(\lambda)\psi\sigma_3\psi^{-1},
\] (4.2)
where \( Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \) and \( \psi \) is defined in (A.2) and (A.3).

Using the compatibility condition
\[
U_T - W_X + [U, W] = 0,
\] (4.3)
the infinite sequence of conservation laws can be obtained by equating the powers of \((2ik)^{-1}\).

The first law of this sequence reads
\[
\partial_T |q|^2 = \partial_X \left[ \frac{i}{2}(q_X\bar{q} - \bar{q}_Xq) + \frac{\omega_0}{24}\mathcal{J} \right]
\] (4.4)
with
\[
\mathcal{J} = \int_{-\infty}^{+\infty} d\zeta \left[ a_1(\zeta + i0)\bar{a}_1(\zeta - i0) + a_2(\zeta + i0)\bar{a}_2(\zeta - i0) \right].
\] (4.5)

Now, by integrating equation (4.4) with respect to \( X \) using (2.30), (3.6) and (3.14), one obtains
\[
\frac{\partial}{\partial T} \int_{-\infty}^{+\infty} dX' |q|^2 = \int_{-\infty}^{+\infty} d\zeta |A|^2 R(\zeta + i0)\bar{R}(\zeta - i0),
\] (4.6)
where \([A] = (\omega_0/12)|\lambda|\).

Due to (3.8), the product \( R(\zeta + i0)R(\zeta - i0) \) is time-independent and therefore the energy \( \int_{-\infty}^{+\infty} dX' |q|^2 \) transferred from the EW to the ISW grows linearly in time. This causes the instability of the ISW that is described in the next section by evaluation of the time-asymptotic solution of \( q(X, T) \).
5. The time-asymptotic solution

As shown in equation (3.8), the reflection coefficient $R(\xi + i0, T)$ shows an exponential growth that is characteristic of integral evolutions with a singular dispersion law [23–25]. Following [25], the asymptotic behavior $T \to +\infty$ of the solution of (3.10) is evaluated using the saddle-point method.

Let consider an arbitrary initial local distribution $q(X, 0)$ of the charge density by considering some generic reflection coefficient $r_0$.

\[ R(\xi + i0, 0) = \frac{r_0}{(\xi + i0)^n}. \tag{5.1} \]

where $r_0$ is a real constant and $n > 1$. Let us also consider a very sharp (i.e. monochromatic) input laser beam, so that the evolution (3.8) is reduced to

\[ R(\lambda) \sim r_0\lambda^{-n}\exp \left[ iT \left( \frac{\gamma}{\lambda} - 2\lambda^2 \right) \right]. \tag{5.2} \]

for $\Im(\lambda) > 0$ and $\gamma = \int_{-\infty}^{\infty} d\zeta' |\mathcal{A}(|\zeta'|)|^2$. From (5.2), the saddle point $T \to +\infty$ is

\[ \lambda_e = \frac{1 + i\sqrt{3}}{2} \left( \frac{\gamma}{4} \right)^{(1/3)}. \tag{5.3} \]

For the contour $C$ in (3.10) one may choose any contour that is equivalent to the straight line parallel to the real $\lambda$-axis passing through this saddle point. This contour, which is equivalent to the real axis, is precisely the critical path. Then the asymptotic behavior of the solution of (3.10) satisfies the following algebraic system ($\Im(\xi) > 0$)

\[ \begin{pmatrix} \mu_1(\xi) \\ \mu_2(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{T}{\lambda_0 - \xi} B \exp \left[ 2i\lambda_e X + iT \left( \frac{\gamma}{\lambda} - 2\lambda_e^2 \right) \right] \begin{pmatrix} \bar{\mu}_2(\bar{\xi}) \\ \bar{\mu}_1(\bar{\xi}) \end{pmatrix}, \tag{5.4} \]

where the constant $B$ is

\[ B = \frac{\pi}{6} \frac{r_0}{(\lambda_0)^n} \frac{1}{2i\pi}. \tag{5.5} \]

The solution of the above algebraic system gives the behavior of $\mu^-(\xi, X, T)$ and $\mu^+(\xi, X, T)$ through (3.6) and hence $a_1$ and $a_2$ through (3.11). Namely,

\[ \begin{aligned}
    a_1(X, T) &= \left( 1 - \frac{\bar{\lambda}_e}{\lambda_e} \right) \frac{\Gamma(X, T)^2}{1 - \Gamma(X, T)^2} \\
    a_2(X, T) &= \left( 1 - \frac{\bar{\lambda}_e}{\lambda_e} \right) \frac{\bar{\Gamma}(X, T)}{1 - \bar{\Gamma}(X, T)}
\end{aligned} \tag{5.6} \]

with

\[ \Gamma(X, T) = \frac{\lambda_e}{\lambda_e - \bar{\lambda}_e} \frac{1}{2i\pi} \left[ \frac{-2\pi}{T\omega^0(\lambda_e)} \right]^{1/2} \alpha(\lambda_e)e^{i\phi}. \tag{5.7} \]
where
\[ \Phi = 2i\lambda X + 2i\left(\frac{\lambda^2 - \frac{\gamma}{2\lambda}}{T}\right) \] (5.8)

and \( \omega'' \) stands for the second derivative of \( \omega(\lambda) = 2i\left[\frac{\lambda^2}{T} + \lambda^2 - \frac{\gamma}{2\lambda}\right] \) with respect to \( \lambda \).

Finally, the time-asymptotic behavior of \( q \) is obtained through (3.11)
\[ \lim_{T \to +\infty} q(X, T) = -23(\lambda_\nu) - \frac{e^{i\phi}}{\sinh \rho}. \] (5.9)

with
\[ \phi = 2\Im(\lambda_\nu)[X + 6\Re(\lambda_\nu)T] + \phi_0 \] (5.10)

and
\[ \rho = 2\sqrt{3}\Re(\lambda_\nu)\left[X - \Re(\lambda_\nu)T + \ln\left(\frac{\beta\alpha}{4\Im(\lambda_\nu)\sqrt{6\pi}T}\right)\right]. \] (5.11)

where the constants \( \alpha \) and \( \phi_0 \) are related via
\[ \frac{r_0}{(\lambda_\nu)^n} = \alpha e^{i\phi_0}. \] (5.12)

The consistency of solutions (5.6) and (5.9) can easily be verified by replacing these solutions into the initial system (2.28), satisfying the initial/boundary condition (2.30).

6. Conclusions and discussion

In conclusion, we make the following comments.

(i) Although the asymptotic behavior of \( q(X, T) \) (5.9) is a singular expression, \( q(X, T) \) itself is not singular. Thus the direct consequence of (5.9) is the growing amplitude of \( q(X, T) \), which is the signature of the energy transfer.

(ii) The singular point in (5.9) travels with the asymptotic velocity \(-6\Im(\lambda_\nu)\). Hence, the solution \( q(X, T) \) accumulates energy in the region where the laser beam is applied, which accounts for the low penetration of the laser in the plasma.

(iii) The basic system (2.28) is obtained as the small amplitude limit of the fluid-type behavior of plasma and the Maxwell equations. Therefore, it is proved here that this system does not propagate stable small solutions when there is a balance between SBS emission and the ISW nonlinearity. It is worth remarking that if one neglects the ponderomotive effect (which means no right-hand side in the evolution equation in (2.28)), one will obtain an equation for the ISW where any initial disturbance will eventually disperse away as \( T \to \infty \). Therefore, while the nonlinearity in the left-hand side of the evolution equation in (2.28) is a mechanism for the saturation of the (linear) instability of the ISW, the nonlinear ponderomotive effect of the electric field is the mechanism responsible for the plasma instability.

(iv) It has been admitted that \( q(X, T) \in L^1(\mathbb{R}) \), which implies in particular that \( |q(X, T)| \to 0 \) as \( |X| \to \infty \). It is interesting to speculate whether different asymptotic behaviors might change the stability properties of the system. For instance, the large thermal conductivity of the electrons might act as a saturation mechanism.
(v) In many physical situations the length of the plasma is finite. Then the question of the properties of the system (2.28) on a finite X-interval arises. It is possible (as in [18]) to observe a chaotic behavior in this case.

(vi) It has been known since the pioneering work of Zakharov [26] that the route to Langmuir turbulence passes through the ‘collapsing of the EW’, i.e. the formation of local singularities of the wave amplitude (and hence also of the ISW amplitude, named cavities). This result was originally obtained in three dimensions with spherical symmetry, and it was obtained later in one dimension [27] but with higher order nonlinearity with no physical origin. Here we have a model equation, derived from the basic hydrodynamic Poisson–Maxwell system equations, whose solution generically evolves through a singular solution.

(vii) Finally, note that the monochromatic approximation led to evolution (5.2) and subsequently to only one saddle point and one singularity in the time-asymptotic behavior of the ISW. Releasing the monochromatic approximation will introduce more and more saddle points, and hence more and more singular points or asymptotically collapsing waves.

Appendix. Proof of the solution method

The differential equations
\[
\begin{align*}
\dot{a}_1 + i \omega a_1 &= qa_2 \\
\dot{a}_2 - i \omega a_2 &= qa_1
\end{align*}
\]
extracted from system (2.28) are the vectorial form of the Zakharov–Shabat spectral problem [30, 31], which can be written in the general matrix form [24, 28, 29]
\[
\psi_1 = U \psi_1, \quad U = -k \sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}.
\]

Case (A.1) is recovered by the reduction \( r = \bar{q}_1 \), which will be adopted from now on. Here, \( \psi \) is a \( 2 \times 2 \) matrix, built with two independent column vector solutions \( (\psi_1, \psi_2) \), and it is determined completely by giving its asymptotic behavior. Then the set of the differential equations \( \psi_1 = U \psi_1 \) can be equivalently written as a set of Volterra integral equations. For convenience, let us write these equations for the matrix \( \mu(k,x) \), defined by
\[
\mu = \psi \exp(ikx) \Rightarrow \mu_x = i k [\mu, \sigma_3] + Q \mu.
\]

Two dependent solutions \( \mu^+ \) and \( \mu^- \) can be defined through
\[
\begin{align*}
\mu_1^+ &= 1 - \int_x^{+\infty} dx' q \mu_2^+ \\
nu_2^+ &= \int_{-\infty}^x dx' q \mu_1^+ e^{2ik(x-x')} \\
nu_{12}^+ &= - \int_{-\infty}^{+\infty} dx' q \mu_{22}^+ e^{-2ik(x-x')} \\
nu_{22}^+ &= 1 - \int_{-\infty}^{+\infty} dx' q \mu_{12}^+ \\
\mu_1^- &= 1 - \int_x^{+\infty} dx' q \mu_2^- \\
nu_2^- &= \int_{-\infty}^x dx' q \mu_1^- e^{2ik(x-x')} \\
nu_{12}^- &= - \int_{-\infty}^{+\infty} dx' q \mu_{22}^- e^{-2ik(x-x')} \\
nu_{22}^- &= 1 - \int_{-\infty}^{+\infty} dx' q \mu_{12}^- 
\end{align*}
\]
One can easily verify (using the Leibniz formula) that \( \mu \) satisfies (A.3). In the reduction \( r = \tilde{q} \), one has

\[
Q = \sigma_1 \tilde{Q} \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]  

(A.5)

and it is easy to prove that

\[
\mu^+(k, x) = \sigma_1 \overline{\mu^-}(\bar{k}, x) \sigma_1.
\]  

(A.6)

The set of Volterra equations (A.4) can be mapped into a Riemann–Hilbert problem as follows. \( \mu^+ \) (respectively \( \mu^- \)) is holomorphic in the complex upper half plane \( \Im(k) > 0 \) (respectively in the complex lower half plane \( \Im(k) < 0 \)). Let us write

\[
\mu = \begin{cases} 
\mu^+ \text{ in } \Im(k) > 0 \\
\mu^- \text{ in } \Im(k) < 0.
\end{cases}
\]  

(A.7)

The matrix \( \mu \) is holomorphic all over the complex plane except on the real axis, where it stands as a discontinuity. Writing the Riemann–Hilbert problem consists of expressing this discontinuity in terms of \( \mu^+ \) (or \( \mu^- \)).

To do so, let us compute

\[
D(k, x) = [\mu^+(k + i0, x) - \mu^-(k - i0, x)] e^{2ikx}, \quad x \in \mathbb{R}
\]  

(A.8)

out of the integral equations (A.4) and obtain the following Volterra equations for \( D_1 \) and \( D_2 \), components of \( D \).

\[
\begin{aligned}
D_1 &= -\int_{-\infty}^{+\infty} dx' qD_2 e^{2ikx'} \\
D_2 &= \alpha^+(k) - \int_{-\infty}^{+\infty} dx' qD_1,
\end{aligned}
\]  

(A.9)

where

\[
\alpha^+(k) = \int_{-\infty}^{+\infty} dx' \tilde{q}(x') \mu^+_{11}(k, x') e^{-2ikx'}, \Im(k) = 0^+.
\]  

(A.10)

An integral equation with the same Green function as (A.9) can be obtained readily out of (A.4). Indeed, the vector

\[
L = \mu^+_{21}(k + i0, x) \alpha^+(k), \quad k \in \mathbb{R}
\]  

(A.11)

is also a solution of (A.9). General theorems about integral equations [32] allow us to prove that (A.9) has only the trivial solution \( \alpha = 0 \). Then we have \( D = L \), which gives the following Riemann–Hilbert problem:

\[
\mu^+_{21}(k + i0, x) - \mu^-_{21}(k - i0, x) = \alpha^+(k) e^{2ikx} \mu^+_{21}(k + i0, x).
\]  

(A.12)

A similar calculation with

\[
D'(k, x) = [\mu^+_{21}(k + i0, x) - \mu^-_{21}(k - i0, x)] e^{-2ikx}, \quad x \in \mathbb{R}
\]  

(A.13)
leads to the second Riemann–Hilbert problem

\[ \mu_1^+(k + i0, x) - \mu_2^-(k - i0, x) = \alpha^{-}(k)e^{-2ikx}\mu_2^-(k - i0, x). \]  

(A.14)

Using (A.4), we have

\[ \alpha^{-}(k) = -\int_{-\infty}^{+\infty} dx' q(x')\mu_{22}^{-}(k, x')e^{2ikx'} \equiv \overline{\alpha^{+}(k)}, \quad \cal{I}(k) = 0^{+}. \]  

(A.15)

Then (A.12) and (A.14) can be written in the following matrix form:

\[ \mu^+ - \mu^- = (\mu_1^+, \mu_2^+)S \]  

(A.16)

with

\[ S(k, x) = e^{-ikx}\left( \begin{array}{cc} 0 & -\overline{\alpha^{+}(k)} \\ \alpha^{+}(k) & 0 \end{array} \right) e^{ikx}, \quad k \in \mathbb{R}. \]  

(A.17)

Note that, as with \( \mu \) in (A.6), the matrix \( S \) satisfies the reduction relation

\[ S(k, x) = -\sigma \overline{S}(k, x)\sigma. \]  

(A.18)

The function \( \alpha^{+}(k) \) is called the reflection coefficient. \( \alpha^{+}(k) \) and \( q(x) \) are equivalent in the sense that with \( q(x) \) being given, one solves the Volterra equations (A.4) and computes \( \alpha^{+}(k) \) through

\[ \alpha^{+}(k) = \lim_{s \to +\infty} \mu_{21}^{+}(k, x)e^{-2ikx}, \]  

(A.19)

which solves the direct problem.

The inverse problem consists of constructing \( Q(x) \) from a given \( S(k, x) \) and solving the Riemann–Hilbert problem (A.16) to obtain \( \mu(k, x) \). Once \( \mu \) is obtained, then \( q(x) \) is computed in the following way.

Write the Laurent expansion

\[ \mu_{11} = 1 + \frac{1}{k}\mu_{11}^{-(1)} + \frac{1}{k^2}\mu_{11}^{-(2)} + \ldots \]

\[ \mu_{21} = \frac{1}{k}\mu_{21}^{-(1)} + \frac{1}{k^2}\mu_{21}^{-(2)} + \ldots \]  

(A.20)

obtained from the Volterra equation (A.4) by integrating by parts. Then, insert (A.20) into (A.3) and use the Liouville theorem to obtain

\[ q = 2i\mu_{21}^{-(1)}, \]  

(A.21)

which gives \( q(x) \) out of \( \mu(k, x) \).

To solve the Riemann–Hilbert problem, it is convenient to use the ‘DBAR problem’ formulation:

\[ \frac{\partial \mu}{\partial k} = \frac{1}{2} [\mu(k)\delta^{+}(k) - \mu(k)\delta^{-}(k)]] \]  

(A.22)

where

\[ \frac{\partial}{\partial k} = \frac{i}{2} \left( \frac{\partial}{\partial k_R} + i \frac{\partial}{\partial k_I} \right) \]  

(A.23)
with
\[ k = k_R + i k_T, \quad \delta^+ = \delta(k_I - i0), \quad \delta^- = \delta(k_I + i0), \]
(24)
\( \delta \) being the usual Dirac distribution.

Thus the equation (A.16) becomes
\[ \frac{\partial \mu}{\partial k} = \mu R, \quad R = \frac{1}{2} \begin{pmatrix} \delta^+ & 0 \\ 0 & \delta^- \end{pmatrix} \]
(A.25)
and the generalized Cauchy formula reads
\[ \mu(k, x) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{d\lambda}{\lambda - k} \mu(\lambda, x) + \frac{1}{2\pi} \int_{\mathcal{D}} \frac{d\lambda}{\lambda - k} \mu(\lambda, k) R(\lambda, k) \]
(A.26)
with \( \mathcal{D} \) being the complex plane.

The Laurent expansion (A.20) allows the computation of the first integral in (A.26) and, taking into account the analytic properties of \( \mu \), the second integral of (A.26) is reduced to an integration over the real line. Consequently, for the first column vector \( \mu_1 \) with the reduction of (A.6), we have
\[ \mu_1^{-1}(k, x) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - k} \sigma_1 \mu_1^{-1}(\lambda, x) \alpha^+(\lambda) e^{2i\lambda x}. \]
(A.27)
A similar equation holds for the second column vector.

Hence the inverse problem (construction of \( q(x) \) from \( \alpha^+(k) \)) is resolved by solving the Cauchy–Green integral equation and calculating \( q(x) \) out of (A.21).

Note that the preceding formalism remains valid if \( q(x) \) is assumed to also depend on a real external parameter \( t \) (time). Then, the eigenfunction \( \mu(k, x) \) and the spectral transform \( R(k, x) \) will also depend on \( t \).

In the next step, we must construct integral evolutions out of a simple choice of a given time dependence of the spectral transform \( R(k, x, t) \).

We start with the following generic DBAR problem
\[ \begin{cases} \frac{\partial}{\partial k} \mu(k) = \mu(k) R(k) \\ \mu(k) = 1 + O\left( \frac{1}{k} \right), |k| \to \infty \end{cases} \]
(A.28)
and ask for \( R \) an \((x, t)\)-dependence through
\[ \begin{cases} \mathcal{R}_t = [R, \Lambda] \\ \mathcal{R}_e = [R, \Omega] \end{cases} \]
(A.29)
where \( \Lambda \) and \( \Omega \) are given distributions of \( k \in \mathbb{C} \), functions of \( x \) and \( t \) [24, 28, 29].

It is a simple task to check the following relations
\[ \begin{cases} \frac{\partial}{\partial k} (\mu \mu^{-1} - \mu \Lambda \mu^{-1}) = -\mu \frac{\partial \Lambda}{\partial k} \mu^{-1} \\ \frac{\partial}{\partial k} (\mu \mu^{-1} - \mu \Omega \mu^{-1}) = -\mu \frac{\partial \Omega}{\partial k} \mu^{-1} \end{cases} \]
(A.30)
By integrating the above equations, one obtains

1939
\[
\begin{align*}
\mu_\lambda(k,x,t) &= \left[ U - \frac{1}{2\pi i} \int_C \frac{d\lambda \wedge d\tilde{\lambda}}{\lambda - k} \mu(\lambda, x, t) \frac{\partial \mu(\lambda, x, t)}{\partial \tilde{\lambda}} \right] \\
\mu_\kappa(k,x,t) &= \left[ V - \frac{1}{2\pi i} \int_C \frac{d\lambda \wedge d\tilde{\lambda}}{\lambda - k} \mu(\lambda, x, t) \frac{\partial \Omega(\lambda, x, t)}{\partial \tilde{\lambda}} \right] \\
\end{align*}
\]

(A.31)
in which the matrices \( U \) and \( V \) (the ‘constants’ of integration of the \( \partial \) operator) are given by

\[
\begin{align*}
U(k,x,t) &= -\mathrm{Pol}_k [\mu(k,x,t)\Lambda(k,x,t)\mu^{-1}(k,x,t)] \\
V(k,x,t) &= -\mathrm{Pol}_k [\mu(k,x,t)\Omega(k,x,t)\mu^{-1}(k,x,t)] \\
\end{align*}
\]

(A.32)
where \( \mathrm{Pol}_k [...] \) means ‘the polynomial part in \( k \) of ...’.

To obtain equation (A.3), we choose and adopt from now on

\[
\Lambda = ik\sigma_3. 
\]

(A.33)
The compatibility condition \( \mu_\lambda = \mu_\kappa \) when \( \partial\Lambda/\partial k = 0 \) leads to

\[
U(k) - V(k) + [U(k), V(k)] = -\frac{1}{2\pi i} \int_C \frac{d\lambda \wedge d\tilde{\lambda}}{\lambda - k} \left[ U(\lambda) - U(k), \mu(\lambda) \frac{\partial \Omega(\lambda)}{\partial \tilde{\lambda}} \mu^{-1}(\lambda) \right].
\]

(A.34)

Now, with the choice

\[
\Omega = ik^2\sigma_3 + \frac{1}{2\pi i} \int_C \frac{d\lambda \wedge d\tilde{\lambda}}{\lambda - k} g(\lambda)\sigma_3
\]

(A.35)
the time evolution (A.34) is transformed into

\[
Q_t = \frac{i}{2} \sigma_3 Q_{xx} + i\sigma_3 Q^2 = i \left[ \sigma_3, \frac{1}{2\pi i} \int_C \frac{d\lambda \wedge d\tilde{\lambda}}{\lambda - k} g(\lambda) \mu(\lambda) \sigma_3 \mu^{-1}(\lambda) \right],
\]

(A.36)
which is the evolution equation in (A.31).

Therefore, it has been proved that when \( Q(x, t) \) evolves according to a nonlinear evolution, its spectral transform \( R(k, x, t) \) related to \( \alpha^+(k, t) \) by (A.17) evolves in time according to (A.30) with \( \Omega(k) \) given in (A.35).

In the reduced case \( r = \dot{q} \), we have the relation (A.18), and therefore the explicit time evolution of the reflected coefficient \( \alpha = \alpha^-(k) \) is given by

\[
\alpha(k_R, t) = 2i \left[ k_R^2 - \frac{i}{\pi} \int_C \frac{d\lambda \wedge d\lambda}{\lambda - (k_R - i0)} g(\lambda) \right] \alpha(k_R, t).
\]

(A.37)
The example of the physical model studied in section 2 justifies the most interesting sub-case:

\[
g(\lambda) = \frac{ik\pi}{6} \delta(\lambda) \delta(\lambda - k_0) |A(k_R)|.
\]

(A.38)
Indeed, case (A.37) gives after integration

\[
\alpha(k,t) = \alpha(k,0) \exp \left[ 2i \left( k^2 + \frac{1}{6} \int_{-\infty}^{+\infty} \frac{dkA(k)}{k_0 - k} \right) \right].
\]

(A.39)
Clearly, the function $\alpha(k)$ has an essential singularity at $k = k_0$. Hence $\alpha(k)$ is not defined for $k = k_0$ unless $a(k_0, 0) \equiv 0$, which is actually required by system (2.28). Indeed, for $q(x, 0)$ vanishing at both ends of the $x$-axis, consistency requires that $a(k_0, 0) \equiv 0$, which is actually required by system (2.28). Therefore, as a consequence of equations (A.14) and (3.11), $\alpha(k)$ must vanish for $k = k_0$. This condition actually determines the parameter $k_0$ from the initial datum $q(x, 0)$, which is the solution of the equation

$$\alpha(k_0, 0) = \int_{-\infty}^{+\infty} dx' \dot{q}(x') \mu_{22}(k_0, x', 0)e^{-ik_0x'} = 0. \quad (A.40)$$

Hence, it is a property of system (2.28) that the initial value $q(x, 0)$ of the ISW determines the small correction of the wave number of the scattered ESW.

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