Random matrices are used in fields as different as the study of multi-orthogonal polynomials or the enumeration of discrete surfaces. Both of them are based on the study of a matrix integral. However, this term can be confusing since the definition of a matrix integral in these two applications is not the same. These two definitions, perturbative and non-perturbative, are discussed in this chapter as well as their relation. The so-called loop equations satisfied by integrals over random matrices coupled in chain is discussed as well as their recursive solution in the perturbative case when the matrices are Hermitean.

1 Introduction: what is a matrix integral?

The diversity of aspects of mathematics and physics exposed in the present volume witnesses how rich the theory of random matrices can be. This large spectrum of applications of random matrices does not only come from the numerous possible ways to solve it but it is also intrinsically due to the existence, and use, of different definitions of the matrix integral giving rise to the partition function of the theory under study.

Back to the original work of Dyson [Dys62], the study of random matrices is aimed at computing integrals over some given set of matrices with respect to some probability measure on this set of matrices. In order to be computed, these integrals are obviously expected to be convergent. Nevertheless, one of the main applications of random matrices in modern physics follows from a
slightly different definition. Following the work of [Bre78], the matrix integral can be considered, through its expansion around a saddle point of the integrand, as a formal power series seen as the generating function of random maps, i.e. random surfaces composed of polygons glued by their sides\footnote{See chapter 26 for an introduction this topic and [Eyn06] and reference therein for the generalization to multi-matrix integrals.}. Whether this formal series has a non-vanishing radius of convergency or not does not make any difference: only its coefficients, which take finite values, are meaningful.

The issue whether these two definitions do coincide or not was not addressed for a long time and led to confusions. In particular it led to a puzzling non-coincidence of some result in the literature [Ake96b, Bre99, Kan98]. Their computations of the same quantity, even if proved to be right, did not match. This puzzled was solved by Bonnet, David and Eynard [Bon00] who were able to show that the mismatch between the two solutions is a consequence of the discrepancy between the definitions of the matrix integrals taken as partition functions.

Since some of the topics discussed in the present chapter do depend on the definition one considers for the partition function whereas some other issues do not, section 2 is devoted to the precise definition of these different matrix integrals. In section 3 we present the loop equations which can be used to compute the partition function and correlation functions of a large family of matrix models. Section 4 is devoted to a review of one the solution of the one Hermitean matrix model through the use of the so-called loop equations. Section 5 generalizes this method to an arbitrary number of Hermitean matrices coupled in chain. Finally, section 6 gives a short overview of generalizations and applications of this very universal method.

2 Convergent vs formal matrix integral

One of the most interesting features in the study of random matrices is the behavior of the statistic of eigenvalues, or correlation functions, as the random matrices become arbitrary large. This limit is not only very interesting for its applications in physics (study of heavy nuclei, condensed matter...) but also in mathematics: the knowledge of the large size limit allows to access the
asymptotics of a large set of multi-orthogonal polynomials.

Most of the usual technics used in random matrix theory fail in the study of the large matrix limit. However, one possible way to address this problem is to try to use naively some saddle point analysis. Let us consider the example of a Hermitean one matrix with polynomial potential to illustrate this procedure. The partition function is given by the matrix integral:

\[ Z(V) = \int_{\mathcal{H}_N} dM e^{-\frac{N}{2} \text{Tr} V(M)} \]

where one integrates over the group \( \mathcal{H}_N \) of Hermitean matrices of size \( N \) with respect to the measure

\[ dM := \frac{\prod_{i=1}^{N} k!}{\pi^{\frac{N(N-1)}{2}}} \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\Re(M_{ij}) d\Im(M_{ij}) \]

defined as the product of the Lebesgue measures of the real components of the matrix \( M \) divided by the volume of the unitary group of size \( N \). For the sake of simplicity, one assumes that the potential \( V(x) = \sum_{k=0}^{d} \frac{t_k}{k+1} x^{k+1} \) is a polynomial.

Notice that the direct saddle point analysis of this integral does not make sense in general.

In order to fix this, let us consider a more general problem. Instead of considering Hermitean matrices, we consider normal matrices of size \( N \) whose eigenvalues lie on some arbitrary path \( \gamma \) in the complex plane: \( H_N(\gamma) \) is the set of matrices \( M \) of size \( N \times N \) such that there exists \( U \in U(N) \) and \( X = \text{diag}(x_1,\ldots,x_N) \) with \( x_i \in \gamma \) satisfying \( M = UXU^\dagger \).

With this notation, the set of Hermitean matrices is \( H_N(\mathbb{R}) \). Given a fixed potential \( V(x) \), one considers the family of matrix integrals over formal matrices on arbitrary contours \( \gamma \):

\[ Z(V, \gamma) = \int_{\mathcal{H}_N(\gamma)} dM e^{-\frac{N}{2} \text{Tr} V(M)} \]

As in the Hermitean case, one can integrate out the unitary group to turn this

\[^{2}\text{See [Meh04] for a nice review of these application and all the other chapters of the present volume.}\]
partition function into an integral over the eigenvalues of the random matrix:

\[ Z(V, \gamma) = \int \gamma \cdots \gamma \int dx_i \prod_{i<j} (x_i - x_j)^2 e^{-Nt \sum_{i=1}^N V(x_i)} \]

However, given a polynomial potential of degree \( d + 1 \), not every path \( \gamma \) is admissible. Indeed, there are only \( d + 1 \) directions going to infinity where \( \Re[V(x)] > 0 \) as \( x \to \infty \) and where the integrand decreases rapidly enough for the integral to converge. Thus there exists \( d \) homologically independent paths on which the integral \( \int dx e^{-NtV(x)} \) is convergent. Let us choose a basis \( \{ \gamma_i \}_{i=1}^d \) of such paths. Every admissible path \( \gamma \) for the eigenvalues of the random matrix can thus be decomposed in this basis: \( \gamma = \sum_{i=1}^d \kappa_i \gamma_i \).

Using this decomposition, for any admissible path \( \gamma \), the partition function reduces to

\[
Z(V, \{ \gamma_i \} \mid \{ \kappa_i \}) = N! \sum_{\{n_i\}} \prod_{i=1}^d \frac{\kappa_i^{n_i}}{n_i!} \int_{\gamma_1 \times \cdots \times \gamma_d} dx_1 \cdots dx_d \prod_{i<j} (x_i - x_j)^2 e^{-Nt \sum_{i=1}^N V(x_i)}
\]

where one sums over all integer \( d \)-partitions \( (n_1, \ldots, n_d) \) of \( N \), i.e. the sets of \( d \) integers \( \{n_i\}_{i=1}^d \) satisfying \( n_1 + \cdots + n_d = N \).

The requirement of convergence of the integral only fixes the asymptotic directions of the paths \( \gamma_i \)’s. We still have the freedom to choose their behavior away from their asymptotic directions. Does there exist one choice better than the others? One is interested in performing a saddle point analysis of the matrix integral. One thus has to look for the singular points of the action, i.e. the solutions of \( V'(x) = 0 \). There exist \( d \) such solutions \( \xi_i, i = 1, \ldots, d \), i.e. as many as the number of paths \( \gamma_i \) in one basis. In the case of the one matrix model with polynomial potential exposed in the present section, it was proved following [Ber07] that there exists a good basis in the sense that every path \( \gamma_i \) is a steepest descent contour. More precisely, along any path \( \gamma_i \), the effective

---

\[ ^3 \] This procedure can be generalized to multi-matrix models using the HCIZ formula [Itz80, Har57] presented in chapter 17.

\[ ^4 \] The existence of a good path is conjectured to hold for all other matrix models discussed in this chapter. However, the proof is known, at the time these lines are written, only in the one matrix model case.
potential felt by an eigenvalue \( x \), \( V_{\mathrm{eff}}(x) = V(x) - \frac{1}{N} \langle \ln (\det x - M) \rangle \), behaves as follows: its real part decreases then stays constant on some interval and then increases, whereas its imaginary part is constant then increasing and finally constant.

Such a steepest descent path can thus be seen as a possible vacuum for one eigenvalue. Each \( d \)-partition of \( N \) hence corresponds to one vacuum for the theory, or one saddle configuration for the random matrix. The formula eq\[2.1\] can be understood as a sum over all possible vacua of the theory:

\[
Z(V, \gamma) = \sum_{n_1 + \ldots + n_d = N} N! \prod_{i=1}^{d} \frac{n_i!}{n_i!} Z(V, \{\gamma_i\} | n_1, \ldots, n_d)
\]

where the partition function with fixed filling fractions \( \epsilon_i = \frac{n_i}{N} \)

\[
Z(V, \{\gamma_i\} | n_1, \ldots, n_d) := \int_{\gamma_1^{n_1} \times \ldots \times \gamma_d^{n_d}} \prod_{i=1}^{N} dx_i \prod_{i<j} (x_i - x_j)^2 e^{-\frac{N}{t} \sum_{i=1}^{N} V(x_i)}
\]

is the weight of a fixed configuration of eigenvalues, or the partition function of the theory with a fixed vacuum labeled by a partition \((n_1, \ldots, n_d)\).

Assuming that the paths \( \gamma_i \) are good steepest descent paths, the partition functions with fixed filling fractions can be computed by saddle point approximation, i.e. perturbative expansion of the integral around a saddle as \( t \to 0 \). Further assuming that one can commute the integral and the power series expansion, the result is a formal power series in \( t \) whose coefficients are gaussian matrix integrals:

\[
Z(V, \{\gamma_i\} | n_1, \ldots, n_d) \sim Z_{\text{formal}}(V, \{\gamma_i\} | n_1, \ldots, n_d) \quad \text{when} \quad t \to 0
\]

with

\[
Z_{\text{formal}} := e^{-\frac{N}{t} \sum_i n_i V(\xi_i)} \sum_{k=0}^{\infty} \frac{(-1)^k N^k}{k!} \prod_{i=1}^{d} \left( \int_{H_{n_i}(\gamma)} dM_i \right)^k \left( \sum_{i=1}^{d} \text{Tr} \delta V_i(M_i) \right)^k \\
\times e^{-\frac{N}{2t} \sum_i V''(\xi_i) \text{Tr} (M_i - \xi_i 1_{n_i})^2} \prod_{j<i} \det (M_i \otimes 1_{n_j} - 1_{n_i} \otimes M_j)^2
\]

(2.2)

where \( \{\xi_i\}_{i=1}^{d} \) denote the \( d \) solutions of the saddle point equation \( V'(\xi_i) = 0 \), \( \delta V_i(x) := V(x) - V(\xi_i) - \frac{V''(\xi_i)}{2}(x - \xi_i)^2 \) denotes the non-gaussian part of
the Taylor expansion of the potential around the saddle $\xi_i$ and the notation
\[
\prod_{i=1}^{d} \left( \int_{H_{n_i}(\gamma_i)} dM_i \right)
\]
stands for the multiple integral $\int_{H_{n_1}(\gamma_1)} dM_1 \cdots \int_{H_{n_d}(\gamma_d)} dM_d$.
This formal series in $t$ is referred to as a formal matrix integral even though it is not a matrix integral but a formal power series in $t$.

This construction can be thought of as a perturbation theory: the matrix integral $Z(V)$ is the non-perturbative partition function of the theory whereas the formal matrix integral $Z_{\text{formal}}(V, R | n_1, \ldots, n_d)$ is a perturbative partition function corresponding to the expansion around a fixed vacuum $(n_1, \ldots, n_d)$ in the basis $(\gamma_1, \ldots, \gamma_d)$.

Since these two possible definitions of the partition function might be confused, let us emphasize their main differences, concerning their properties as well as their applications:

- The convergent matrix integral is fixed by a choice of potential $V$ together with an admissible path $\gamma$. The formal matrix integral depends on a potential $V$ of degree $d$, a basis of admissible paths $\{\gamma_i\}_{i=1}^{d}$ and a partition of $N$, $\{n_i\}_{i=1}^{d}$.

- By definition, the non-perturbative partition function is a convergent matrix integral for arbitrary potential, provided the paths $\gamma_i$ are chosen consistently. The perturbative integral is a power series defined for arbitrary potentials, integration paths and filling fractions. It might be non-convergent, and will be for most of combinatorial applications;

- The logarithm of the perturbative partition function always has a $\frac{1}{N^2}$ expansion, whereas the non-perturbative one does not have one most of the time (see section 3.3).

- The formal matrix integral is typically used to solve problems of enumerative geometry such as enumeration of maps or topological string theory. The convergent matrix integral is related for example to the study of multi-orthogonal polynomials.

3 Loop equations

Even if the perturbative and non-perturbative partition functions do not coincide in general, they share some common properties. One of the most useful
is the existence of a set of equations linking the correlation functions of the theory: the loop equations. These equations, introduced by Migdal [Mig83], are simply the Schwinger-Dyson equations applied to the matrix model setup. They proved to be an efficient tool for the computation of formal matrix integrals as the explicit computation of one class of one Hermitean formal matrix integral by Ambjorn and al [Amb93] proves.

3.1 Free energy and correlation functions

One of the main quantities used in the study of matrix integrals is the free energy which is defined as the logarithm of the partition function:

$$F := - \frac{1}{N^2} \mathcal{Z}.$$ 

In the formal case, where $\mathcal{Z}$ is the generating function of closed discrete surfaces, the free energy enumerates only connected such surfaces.

In order to be able to compute the free energy, but also for their own interpretation in combinatorics of maps or string theory, it is convenient to introduce the following correlation functions:

$$W_k(x_1, \ldots, x_k) := \left\langle \prod_{i=1}^k \frac{1}{x_i - M} \right\rangle_c$$

where the index $c$ denotes the connected part and

$$\frac{1}{x - M} = \sum_{i=1}^d \sum_{k=0}^\infty \frac{(M_i - \xi_i \mathbb{1}^j_i)^k}{(x - \xi_i)^{k+1}}$$ (3.1)

has to be understood as a formal power series. It is also useful to introduce the polynomial of degree $d - 1$ in $x$

$$P_k(x, x_1, \ldots, x_k) := \left\langle \prod_{i=1}^k \frac{1}{x_i - M} \right\rangle_c.$$

3.2 Loop equations

The non-perturbative partition function is given by a convergent matrix integral. It should thus be invariant under change of the integration variable $M$.

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Footnote:

$^5$They are generating functions of open surfaces, as opposed to the free energy which generates surfaces without boundaries. The interested reader can refer to chapter 31 of the present book or the review [Eyn06] for details of this interpretation.
(or its entries). The name loop equation refers to any equation obtained from the invariance to first order in $\epsilon \to 0$ of the partition function under a change of variable of the form $M \to M + \epsilon \delta(M)$.

$$\int_{\mathcal{H}_N(\gamma)} dMe^{-\frac{N}{t} \text{Tr} V(M)} = \int_{\mathcal{H}_N(\gamma)} d(M + \epsilon \delta(M)) e^{-\frac{N}{t} \text{Tr} V(M + \epsilon \delta(M))}.$$  

To first order in $\epsilon$, this means that the variation of the action should be compensated by the Jacobian of the change of variables:

$$\frac{N}{t} \langle \text{Tr} V'(M) \delta(M) \rangle = \langle J(M) \rangle.$$

Actually, the form of the changes of variable considered is limited to two main families of $\delta(M)$. This allows to give a recipe to compute the Jacobian rather easily as follows.

- Leibnitz rule:

$$J[A(M)B(M)] = \{J[A(M)B(m)]\}_{m \to M} + \{J[A(m)B(M)]\}_{m \to M};$$

- Split rule: $J[A(m)M^i B(m)] = \sum_{j=0}^{l-1} \text{Tr} \left[A(m)M^j\right] \text{Tr} \left[M^{l-j-1}B(m)\right];$

- Merge rule: $J[A(m) \text{Tr} \left(M^i B(m)\right)] = \sum_{j=0}^{l-1} \text{Tr} \left[A(m)M^j B(m)M^{l-j-1}\right];$

- if there is no $M$: $J[A(m)] = 0.$

The formal matrix integral is obtained from Gaussian convergent integrals by algebraic computations which commute with the loop equations thus

**Theorem 3.1** The formal matrix integrals satisfy the same loop equations as the convergent matrix integrals.

### 3.3 Topological expansion

The loop equations are a wonderful tool for the study of formal matrix integrals. From now on, we restrict our study to these formal power series, leaving aside the convergent matrix integrals.

---

\footnote{It can be equivalently seen as integration by parts.}
Following an observation originally made by t’Hooft in the study of Feynman graphs of QCD \cite{thooft1974}, one can see that the exponent of $N$ in the free energy $F$ is the Euler characteristic of the surface enumerated by this partition function. Thus, $F$ admits a $\frac{1}{N^2}$ expansion

$$F = \sum_{g=0}^{\infty} N^{-2g} F^{(g)}$$

commonly called *topological expansion* since the terms $F^{(g)}$ of this expansion are generating functions of connected closed surfaces of fixed genus $g$.

As for the free energy, one can collect together coefficients with the same power of $N$ in the correlation function and get

$$W_k(x_1, \ldots, x_k) = \sum_{h=0}^{\infty} \left( \frac{N}{t} \right)^{2h-k} W_k^{(h)}(x_1, \ldots, x_k)$$

as well as

$$P_k(x, x_1, \ldots, x_k) = \sum_{h=0}^{\infty} \left( \frac{N}{t} \right)^{1-2h-k} P_k^{(h)}(x, x_1, \ldots, x_k)$$

where the coefficient are formal power series in $t$ independent of $N$.

Both the convergent (non-perturbative) partition function and the formal (perturbative) matrix integral are solution to the loop equations. Nevertheless they do not coincide in general considered that the loop equations have not a unique solution. Indeed, in order to make the solution of these equations unique, one has to fix some "initial conditions" satisfied by the sought for solution and the convergent and formal matrix integrals are not constrained by the same kind of conditions.

On the one hand, the formal matrix integral has well defined constraints: it has a $\frac{1}{N^2}$ expansion and the small $t$, large $x$, limit of any correlation function is fixed by the choice of filling fractions. In other words, by fixing the filling fractions, one prevents the eigenvalues of the random matrix from tunneling from one saddle to another, i.e. from one steepest descent path to another. There is no instanton contribution.

On the other hand, the convergent matrix integral does not admit, in general, any $\frac{1}{N^2}$ expansion. Moreover, its resolvent is not normalized by any arbitrarily fixed choice of filling fraction: it is rather normalized by some equilibrium
conditions on the configuration of the eigenvalues, which, thanks to tunneling, gives instanton corrections to the classical partition function around the true vacuum of the theory. This means that the eigenvalues of the matrix distribute on the different paths of the basis in such a way that they are in equilibrium under the action of the potential and their mutual logarithmic repulsion.

In the formal case, one of the main properties of the correlations functions is the existence of a topological expansion. Let us plug these topological expansions into one set of equations obtained by considering the change of variable of type \( \delta M = \frac{1}{x-M} \prod_{i=1}^{k} \text{Tr} \frac{1}{x_i-M} \). They read, order by order in \( N^{-2} \):

\[
V'(x)W_{n+1}^{(h)}(x, J) = W_{n+1}^{(h-1)}(x, x, J) + \sum_{m=0}^{h} \sum_{I \subset J} W^{(m)}_{1+|I|}(x, I)W_{1+n-|I|}^{(h-m)}(x, J \setminus I)
+ P_n^{(h)}(x, J) + \sum_{i=1}^{n} \frac{\partial}{\partial x_j} W_n^{(h)}(x, J \setminus \{x_j\}) - W_n^{(h)}(J)
\]

(3.2)

where \( J \) stands for \( \{x_1, \ldots, x_n\} \). This is the hierarchy of equations which is solved in the following section.

**Remark 3.1** Remember that the correlation functions can be seen as the generating functions of discrete surfaces of given topology. In this picture, the loop equations get a combinatorial interpretation: they summarize all the possible ways of erasing one edge from surfaces of a given topology. This gives a recursive relation among generating functions of surfaces with different Euler characteristics. This inductive method was introduced in the case of triangulated surface by Tutte [Tut62] without any matrix model’s representation of the considered generating functions.

## 4 Solution of the loop equations in the 1MM

The solution of the loop equations in their topological expansion has been under intensive study since their introduction by Migdal [Mig83]. In particular, [Amb93] proposed a general solution of these equations in the one matrix model case for the so-called one cut case, i.e. the case where only one of the filling fractions \( \epsilon_i \) doesn’t vanish. The first steps in the study of the 2-cut case were then performed by Akemann in [Ake96a].

Later, in 2004, Eynard [Eyn04] solved the loop equations eq (3.2) for the formal integral for an arbitrary number of cuts, i.e. compute all the terms in the topological expansion of any correlation function as well as the free energy’s
\( \frac{1}{N^2} \)-expansion for arbitrary filling fractions. This solution relies heavily on the existence of an algebraic curve encoding all the properties of the considered matrix model: the spectral curve. Let us first remind how the latter can be derived.

### 4.1 Spectral curve

Consider eq.3.2 for \((h, n) = (0, 0)\): it is a quadratic equation satisfied by the genus 0 one point function:

\[
W_1^{(0)}(x)^2 - V'(x)W_1^{(0)}(x) = -P^{(0)}(x)
\]

called the master loop equation. This can be written as an algebraic equation

\[
H_{1MM}(x, W_1^{(0)}) = 0
\]

where \(H_{1MM}(x, y)\) is a polynomial of degree \(d\) in \(x\) and 2 in \(y\). The algebraic equation \(H_{1MM}(x, y) = 0\) is the basis of the solution presented in this section and will be referred to as the spectral curve of the considered matrix model.

A first corollary of this equation is the multi-valuedness of \(W_1^{(0)}(x)\) as a function of \(x\). Indeed, considered \(P(x)\) known, one can solve this equation and get:

\[
W_1^{(0)}(x) = \frac{V'(x) \pm \sqrt{V'(x)^2 - 4P^{(0)}(x)}}{2}.
\]

A priori, for any value of the complex variable \(x\), there exist two values of \(W_1^{(0)}(x)\). Since, from the definition 4.2, its large \(x\) behavior is known to be

\[
W_1^{(0)}(x) \sim \frac{\sum_{i=1}^{d} n_i}{x N} = \frac{t}{x} \quad \text{as} \quad x \to \infty,
\]

one has to select the \(-\) sign in order to get the physically meaningful correlation function.

If one can relieve this ambiguity by hand for the genus zero one point function, the computation of the complete topological expansion of all the correlation functions would imply such a choice at each step.

On the other hand, one can totally get rid of this problem by understanding where it originates from. Any correlation function is defined as a formal power series both in \(t \to 0\) and in \(x \to \infty\). The coefficients of the \(\frac{1}{N^2}\)-expansion of
$W_1(x)$ are thus well defined only around $x \to \infty$, as this series might have a finite radius of convergency: it is not an analytic function of $x$. In order to get a monovalued function, one has to extend this series further than this radius. The master loop equation tells us how one can proceed: instead of considering the correlation function $W_1(x)$ as a function of the complex variable $x$, one should consider it as a function defined on the spectral curve. That is to say that one should not consider $W_1^{(h)}(x)$ as functions of a complex variable $x$ but rather as functions of a complex variable $x$ together with a $+\text{ or } -$ sign corresponding to the choice of one branch of solution of the master loop equation. The tools of algebraic geometry are built to be able to deal with such situations and we present it in the next section.

4.2 Algebraic geometry

Consider an algebraic equation $E(x, y) = 0$ in $y$ and $x$ of respective degrees $d_y + 1$ and $d_x + 1$.

A classical result of algebraic geometry states that there exists a compact Riemann surface $\mathcal{L}$ and two meromorphic functions $x(p)$ and $y(p)$ on it such that:

\[ \forall p \in \mathcal{L}, \ E(x(p), y(p)) = 0. \]

By abuse of language, one shall use the term spectral curve to denote the Riemann surface $\mathcal{L}$, the triple $(\mathcal{L}, x, y)$ and the equation $E(x, y) = 0$ in the following, when no ambiguity can occur.

Let us detail some general properties of this spectral curve useful for the resolution of the matrix model\footnote{Most of the properties needed for the study of matrix models can be found in \cite{Far92,Fay73} as well chapter 29 of this volume.}.

4.2.1 Sheeted structure and branch points

For a generic fixed value of $x$, there exists $d_y + 1$ functions of $x$, $y^i(x)$, $i = 0, \ldots, d_y$ solution of the equation $E(x, y^i(x)) = 0$. In other words, a given complex number $x(p)$ has $d_y + 1$ preimages $p^i$, $i = 0, \ldots, d_y$ on the surface $\mathcal{L}$ corresponding to different values of $y(p^i)$: one can thus see the Riemann surface $\mathcal{L}$ as $d_y + 1$ copies of the Riemann sphere, denoted as $x$-sheets, glued together,
the function $x$ being injective on each copy.

How are these sheets glued together to form the Riemann surface $\mathcal{E}$? Two sheets merge when two branches of solution in $y$ coincide: $y(p^i) = y(p^j)$ for $i \neq j$. These critical points $a_i$, called branch points, are characterized by the vanishing of the differential $dx$, i.e. the branch points are solutions of the equation $dx(a_i) = 0$. From now on, we suppose that all the branch points are simple zeroes of the one form $dx$. This means that only two sheets merge at these points.

This last assumption implies that, around a branch point $a_i$, one has

$$y(p) \sim y(a) + \sqrt{x(p) - x(a)}.$$ 

This assumption also implies that, for any branch point $a_i$ and any point $z$ close to $a_i$, there exists a unique point $\overline{z}$ such that $x(\overline{z}) = x(z)$ and $\overline{z} \to a_i$ as $z \to a_i$. We call $\overline{z}$ the point conjugated to $z$ around $a_i$.

The spectral curve $L$ is thus a $d_y + 1$ covering of the Riemann sphere with simple ramification points solutions of $dx(a_i) = 0$.

**Example: hyperelliptic curve**

Let us consider an hyperelliptic spectral curve, i.e. a curve given by a quadratic equation $d_y = 1$, as in the one Hermitean matrix model case:

$$\mathcal{E}(x, y) = y^2 - \prod_{i=1}^{d} (x - x(a_i))(x - x(b_i)) = y^2 - \sigma(x)$$

where $d_x := 2d$ to match the notations of the previous section. The corresponding Riemann surface can be seen as a two sheeted cover of the Riemann sphere: one sheet corresponding to the branch $y(x) = \sqrt{\sigma(x)}$, and the other one to the other branch $y(x) = -\sqrt{\sigma(x)}$. These two sheets merge when $y(x)$ takes the same value on both sheets, i.e. when $y(x)$ vanishes. The branch points are thus the preimages of the points $x(a_i)$ and $x(b_i)$ on the spectral curve. The latter can thus be described as two copies of $\mathbb{CP}^1$ glued by $d$ cuts $[a_i, b_i]$.

---

8 Each copy of the Riemann sphere corresponds to a branch of solution in $y$ of the equation $\mathcal{E}(x, y) = 0$.

9 The application $z \to \overline{z}$ is defined only locally around the branch points and depends on the branch point considered and the notation $\overline{z}$ is abusive. Nevertheless, this application will always be used in the vicinity of a branch point in such a way that no ambiguity will occur.
4.2.2 Genus and cycles

Generically, the compact Riemann surface $\mathcal{L}$ associated to an algebraic equation may have non-vanishing genus $g$, and it will be the case in most of the applications of the present chapter.

The Riemann-Hurwitz theorem allows us to get this genus out of the branched covering picture of the spectral curve. For example, if there are only simple ramification points, it states that

$$ g = -d_y + \frac{\text{number of branch points}}{2} $$

where $d_y + 1$ is the number of $x$-sheets.

If the Riemann surface has non-vanishing genus, i.e. it is not conformally equivalent to $\mathbb{CP}^1$, there exist non-contractible cycles on it. In order to deal with them, it will be useful to choose a canonical homology basis of cycles $(\mathcal{A}_1, \ldots, \mathcal{A}_g, \mathcal{B}_1, \ldots, \mathcal{B}_g)$ satisfying the intersection conditions

$$ \forall i, j = 1, \ldots, g, \ A_i \cap A_j = B_i \cap B_j = 0, \ A_i \cap B_j = \delta_{i,j}. $$

Example: hyperelliptic curve

Let us keep on considering the example of a hyperelliptic curve with $2d$ branch points. From the Riemann-Hurwitz theorem, it has genus $g = d - 1$. This follows the intuitive picture of two Riemann sphere glued by $d$ segments giving rise to a genus $d - 1$ surface.

One can also explicit a canonical homology basis as follows. First choose one cut, for example $[a_1, b_1]$ and one sheet, e.g. the sheet corresponding to the minus sign of the square root. Then define the $\mathcal{A}_i$-cycle as the cycles on the chosen sheet around the cut $[a_{i+1}, b_{i+1}]$. Finally, define the $\mathcal{B}_i$-cycle as the composition of the segment $[a_1, a_{i+1}]$ in the chosen sheet and $[a_{i+1}, a_1]$ followed in the opposite direction in the other sheet (see fig.1 for the simplest example of a genus 1 surface).

4.2.3 Differentials

The meromorphic differentials on the Riemann surface $\mathcal{L}$ and their properties will play a crucial role in the following. In particular, let us remind a fundamental result concerning meromorphic differentials: a meromorphic differential $df$ on Riemann surface $\mathcal{L}$ of genus $g$ equipped with a basis of cycles $\{\mathcal{A}_i, \mathcal{B}_i\}_{i=1}^g$, is
Figure 1: Genus 1 hyperelliptic curve: it is built as two copies of the Riemann sphere glued by two cuts $[a_1, a_2]$ and $[a_3, a_4]$. The unique $A$-cycle encircles $[a_3, a_4]$ while the $B$-cycles goes through both cuts.

defined uniquely by its $A$-cycles $\int_{A_i} df$ on the one hand and its singular behavior, i.e. the position of its poles and the divergent part of its Laurent expansion around the latter.

For example, one introduce one of the main character of the resolution of loop equations as follows:

**Definition 4.1** Let the Bergman kernel $B(p, q)$ be the unique bi-differential in $p$ and $q$ on $\mathcal{L}$ defined by the constraints as a differential in $p$:

- it has a unique pole located at $p \to q$ which is double without residue. In local coordinates, it reads

\[
B(p, q) \sim \frac{dp \, dq}{(p - q)^2} + \text{regular when } p \to q;
\]

- it has vanishing $A$-cycle integrals:

\[
\forall i = 1, \ldots, g, \int_{A_i} B(p, q) = 0.
\]

It is also useful to define the primitive of the Bergman kernel:

\[
dS_{p_1, p_2}(q) = \int_{z=p_1}^{p_2} B(z, q)
\]

which is a one form in $q$ with simple poles in $q \to p_1$ and $q \to p_2$ with respective residues $-1$ and $+1$. 

4.3 The one point function and the spectral curve

With these few elements of algebraic geometry in hand, let us complete our study of the spectral curve of the Hermitean one matrix model. Up to now, one has obtained that $W_1^{(0)}(x)$ is solution of a quadratic equation which depends on a polynomial $P^{(0)}(x)$ of degree $d - 1$, i.e. $d$ variables remain to be fixed.

From the definition eq.3.1 of the correlation functions, considering the $A_i$-cycles as circle, independent of $t$ around $\xi_i$ one gets $d$ constraints

$$\forall i = 1, \ldots, d, \quad \frac{1}{2i\pi} \oint_{A_i} W_1^{(0)}(x) dx = \frac{n_i t}{N}$$

allowing to fix the coefficients of the polynomial $P^{(0)}(x)$, since the contour integral $\oint_{A_i}$ picks only one residue at $\xi_i$. One thus gets all the parameters of the spectral curve as well as the one point function $W_1^{(0)}(x)$.

Properties of the one matrix model’s spectral curve

The polynomial $H_{1MM}(x, y)$ has degree 2 in $y$. This means that the embedding of $L_{1MM}$ is composed by 2 copies of the Riemann sphere glued by $g + 1$ cuts so that the resulting Riemann surface $L_{1MM}$ has genus $g$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{1MM}(x, y) = 0$. Since there are only two sheets in involution, this spectral curve is said to be hyperelliptic. It also means that the application $z \rightarrow \overline{z}$ is globally defined since it is the map which exchanges both sheets, i.e. which exchange the two branches of the square root in 4.2.

The Riemann surface $L_{1MM}$ has genus $g$ lower than $d - 1$.

The function $x(z)$ on the Riemann surface $L_{1MM}$ has two simple poles (call them $\alpha_+$ and $\alpha_-$), one in each sheet. Near $\alpha_\pm$, $y(z)$ behaves like:

$$y(z) \sim_{z \rightarrow \alpha_+} \frac{t}{x(z)} + O(1/x(z)^2)$$

and

$$y(z) \sim_{z \rightarrow \alpha_-} t_d x^d(z) + O(x^{d-1}(z)).$$

\[\text{[10]}\]Indeed, when $t \rightarrow 0$, the cuts are reduced to double points at $\xi_i$’s. As $t$ grows, these double points give rise to cuts of length of order $\frac{n_i t}{N}$.

\[\text{[11]}\]Notice that $d - 1$ is an upper bound. It might happen that two branch points coincide resulting in the closing of one cut and decreasing of the genus by one. For some special value of the coefficients of the polynomial $H_{1MM}$ one can even get a genus zero spectral curve. For application of matrix models to enumeration of surfaces, this very non-generic constraint is almost always satisfied (see [Eyn06] for further considerations on this point).
4.4 Two point function

Let us go one step further and consider the loop equation (3.2) for \( k = 2 \) and \( h = 0 \). It allows to obtain a formula for \( W_2^{(0)}(x, x_1) \):

\[
W_2^{(0)}(x, x_1) = \frac{\partial x_1}{\partial x} W_1^{(0)}(x) - \frac{W_1^{(0)}(x_1)}{2(V'(x) - W_1^{(0)}(x))} + P_2^{(0)}(x_1).
\]

A first look at this expression allows to see that this function is multivalued in term of the complex variable \( x \) and \( x_1 \). However, one can lift it to a monovalued function, actually a 2-form, on the spectral curve by defining

\[
\hat{\omega}_2^{(0)}(z, z_1) := W_2^{(0)}(x(z), x(z_1))dx(z)dz(z_1).
\]

\( \hat{\omega}_2^{(0)} \) is thus a meromorphic bi-differential on \( \mathcal{L} \). One can then study all possible singularities of this formula and see that \( \hat{\omega}_2^{(0)}(z, z_1) \) has poles only at \( z \to \bar{z}_1 \). On the other hand, the normalization of the two point function around the \( \mathcal{A} \)-cycles reads \( \oint_{A_i} \hat{\omega}_2^{(0)}(z, z_1) = 0 \) for \( i = 1, \ldots, d \). These two conditions imply that \( \hat{\omega}_2^{(0)}(z, z_1) \) is given by the Bergman kernel (see for example section 5.2.3 of \[Eyn09\])

\[
\hat{\omega}_2^{(0)}(z, z_1) = -B(z, \bar{z}_1) = B(z, z_1) - \frac{dx(z)dx(z_1)}{(x(z) - x(z_1))^2}.
\]

4.5 Correlation functions

We have now everything in hand to compute any correlation function by solving the loop equations. First of all, the study of the one and two point functions proved that it is more convenient to promote the multivalued functions \( W_n^{(h)} \) on the complex plane to monovalued meromorphic forms on \( \mathcal{L} \):

\[
\omega_n^{(h)}(z_1, \ldots, z_n) := W_n^{(h)}(z_1, \ldots, z_n) \prod_{i=1}^n dz(z_i) + \delta_{n,2} \delta_{h,0} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}
\]

and

\[
y(z)dx(z) := W_1^{(0)}(z)dz.
\]

It is important to remember that the physical quantities encoded in the correlation functions are obtained as the terms of the expansion of the latter when their variables approach the physical pole \( \alpha_+ \) of the spectral curve.

\[\text{The monovaluedness of the differential form } \omega_n^{(h)} \text{ on the spectral curve is obtained by induction on the Euler characteristic } 2h + n - 2 \text{ through the use of the loop equations } (3.2).\]
From the loop equations (3.2), one can prove by induction that $\omega_n^h(z_1, \ldots, z_n)$ with $2h + n \geq 3$ can have pole neither at coinciding points $x(z_i) = x(z_j)$, nor at the poles of $x$ nor at the double points. It may have poles only at the branch points.

Let us now write down the Cauchy formula on the spectral curve:

$$\omega_{n+1}^h(z, z_1, \ldots, z_n) = \text{Res}_{z' \to z} dS_{z', o}(z) \omega_{n+1}^h(z', z_1, \ldots, z_n)$$

where $o$ is an arbitrary point of $L$. Since $\omega_{n+1}^h(z', z_1, \ldots, z_n)$ has poles only at the branch point $a_i$, moving the integration contours on $L$ (and not $C$!), one gets contributions from the latter and the boundaries of the fundamental domain of $L$ according to Riemann bilinear formula [Far92]

$$\omega_{n+1}^h(z, z_1, \ldots, z_n) = -\sum_i \text{Res}_{z' \to a_i} dS_{z', o}(z) \omega_{n+1}^h(z', z_1, \ldots, z_n)$$

$$+ \sum_{i=1}^g \left[ \oint_{z' \in A_i} B(z, z') \oint_{z' \in B_i} \omega_{n+1}^h(z', z_1, \ldots, z_n) \right].$$

Since the correlation functions and the Bergmann kernel have vanishing $A$-cycle integrals, the second and third line vanish. One can then plug the expression for $\omega_{n+1}^h(z', z_1, \ldots, z_n)$ coming from the loop equations (3.2) into this formula. Since the polynomial $P_{n+1}^{(g)}(x(z'), z_1, \ldots, z_n)$ is regular at the branch points, it does not give any contribution and one gets the recursion formula

$$\omega_{n+1}^h(z, z_1, \ldots, z_n) = \sum_i \text{Res}_{z' \to a_i} K(z, z') \omega_{n+1}^h(z', z_1, \ldots, z_n)$$

$$+ \sum_{j=0}^h \sum_{I \subset \{z_1, \ldots, z_n\}} \omega_{|I|+1}^{(j)}(z', I) \omega_{n-|I|+1}^{(h-j)}(\overline{z'}, \{z_1, \ldots, z_n\} \setminus I)$$

where the sign $\sum'$ means that the sum does not involve the terms with $(j, |I|) = (0, 0)$ or $(j, |I|) = (h, n)$ and the recursion kernel is

$$K(z, z') := \frac{dS_{z, \overline{z}}(z)}{2(y(z') - y(\overline{z}'))dx(z')}.$$

It is easy to see that this recursive relation on $2h + n - 2$, i.e. provided that $\omega_2^{(0)}(z, z_1) = B(z, z_1)$ is known, it determines all the other correlation functions through their topological expansion.
Remark 4.1 This recursion can be graphically represented in such a way that it becomes very easy to remember and allows to recover some of the properties of the correlation functions using only diagrammatic proofs. Details on this diagrammatic representation can be found in [Eyn09].

4.6 Free energies

In the preceding section, we have been able to compute the topological expansion of any correlation function $W_n$ for $n > 0$. Let us now address the case $n = 0$, that is to say the computation of the topological expansion of the free energy.

For this purpose, one can build an operator acting from the space of $n + 1$-differentials on $\mathcal{L}$ into the space of $n$-differentials mapping the $n + 1$-point function to the $n$-point function.

**Theorem 4.1** For any $h$ and $n$ satisfying $2 - 2h - n < 0$ and any primitive $\Phi$ of $ydx$, one has

$$\omega^{(h)}_n(z_1, \ldots, z_n) = \frac{1}{2 - 2h - n} \sum_i \text{Res}_{z \to a_i} \Phi(z) \omega^{(h)}_{n+1}(z, z_1, \ldots, z_n).$$

One can guess that this definition can be extended to $n = 0$ in order to get the topological expansion of the free energies as follows:

**Theorem 4.2** The terms of the topological expansion of the free energy of the Hermitean one matrix model are given by:

$$F^{(h)} = \frac{1}{2 - 2h} \sum_i \text{Res}_{z \to a_i} \Phi(z) \omega^{(h)}_1(z)$$

for $h \geq 2$.

This guess can be proved to be right by looking at the derivative of the result with respect to all the moduli of the formal integral, i.e. the coefficient of the potential and the filling fractions. Indeed, they match the expected variations of the free energies when varying these moduli [Che06a].

5 Matrices coupled in a chain plus external field

It is remarkable that the recursive formula giving the topological expansion of the free energy and the correlation functions depends on the moduli of the model only through the spectral curve. One can thus wonder whether the same
procedure can be applied to solve other matrix models which are known to be related to a spectral curve. This is indeed the case for the model of two matrices coupled in chain [Che06b] but also for the arbitrary long chain of matrices in an external field [Eyn08].

In order to deal with a large family of Hermitean matrix models at once, let us consider an arbitrarily long sequence of matrices coupled in chain and submitted to the action of an external field.

The partition function is given by the chain of matrices formal matrix integral:

\[
Z_{\text{chain}} = \int_{\text{formal}} e^{-\frac{N}{T} \text{Tr} \left( \sum_{k=1}^{m} V_k(M_k) - \sum_{k=1}^{m} c_{k,k+1} M_k M_{k+1} \right)} \, dM_1 \ldots dM_m
\]

where the integral is a formal integral in the sense of the preceding section\(^ {13}\).\(^ {13}\)

\(M_{m+1}\) is a constant given diagonal matrix \(M_{m+1} = \Lambda\) with \(s\) distinct eigenvalues \(\lambda_i\) with multiplicities \(l_i\):

\[
M_{m+1} = \Lambda = \text{diag} \left( \lambda_1, \ldots, \lambda_1, \ldots, \lambda_s, \ldots, \lambda_s \right)
\]

with \(\sum l_i = N\) and one considers the \(m\) polynomial potentials:

\[
V_k(x) = -\sum_{j=2}^{d_k+1} \frac{k_j}{j} x^j.
\]

As in the one matrix model case, the definition of the formal integral requires to choose around which saddle point one expands. Saddle points are solutions of the set of equations

\[
\forall k = 1, \ldots, m, \quad V_k'(\xi_k) = c_{k-1,k} \xi_{k-1} + c_{k,k+1} \xi_{k+1}, \quad \exists j, \xi_{m+1} = \lambda_j
\]

which can be reduced to an algebraic equation with \(D = s d_1 d_2 \ldots d_m\) solutions.

This choice is thus equivalent to the choice of a \(D\)-partition \((n_1, \ldots, n_D)\) of \(N\) giving rise to the filling fractions:

\[
\epsilon_i = \frac{n_i}{N}.
\]

\(^{13}\) The formal integral is a power series in \(t\) whose coefficients are Gaussian integrals. See [Eyn06] for a review on this topic.

\(^{14}\) It is possible to generalize all this section to potentials whose derivative are arbitrary rational functions without any significant modification of the present procedure.
for \( i = 1, \ldots, D \) with \( D = d_1 d_2 \ldots d_m s \) and \( n_i \) arbitrary integers satisfying
\[
\sum_i n_i = N.
\]

### 5.0.1 Definition of the correlation functions

The loop equations of the chain of matrices were derived in [Eyn03, Eyn08], and they require the definition of several quantities.

For convenience, we introduce \( G_i(x_i) := \frac{1}{x_i - M_i} = \sum_{k=0}^{\infty} \frac{M^k}{x_i^k} \) as a formal power series in \( x_i \to \infty \) as well as a polynomial in \( x \), \( Q(x) = \frac{1}{c_{n,n+1}} S(x) - S(\Lambda) \), where \( S(x) \) is the minimal polynomial of \( \Lambda \), \( S(x) = \prod_{i=1}^{s} (x - \lambda_i) \). We also define the polynomials \( f_{i,j}(x_i, \ldots, x_j) \) by \( f_{i,j} = 0 \) if \( j < i - 1 \), \( f_{i,i-1} = 1 \), and
\[
f_{i,j}(x_i, \ldots, x_j) = \det \begin{pmatrix}
V_i'(x_i) & -c_{i,i+1} x_{i+1} & 0 \\
-c_{i,i+1} x_i & V_{i+1}'(x_{i+1}) & \ddots \\
& \ddots & \ddots & -c_{j-1,j} x_{j-1} \\
0 & \ddots & V_j'(x_j)
\end{pmatrix}
\]
if \( j \geq i \). The latter satisfy the recursion relations
\[
c_{i-1,i} f_{i,j}(x_i, \ldots, x_j) = V_i'(x_i) f_{i+1,j}(x_{i+1}, \ldots, x_j) - c_{i,i+1} x_i x_{i+1} f_{i+2}(x_{i+2}, \ldots, x_j).
\]

Let us finally define the correlation functions and some useful auxiliary functions. In the following \( \text{Pol}_x f(x) \) refers to the polynomial part of \( f(x) \) as \( x \to \infty \). For \( i = 2, \ldots, m \), we define
\[
W_i(x_1, x_i, \ldots, x_m, z) := \text{Pol}_{x_1,\ldots,x_m} f_{i,m}(x_i, \ldots, x_m) \langle \text{Tr} (G_1(x_1)G_i(x_i) \ldots G_m(x_m)Q(z)) \rangle,
\]
which is a polynomial in variables \( x_i, \ldots, x_m, z \), but not in \( x_1 \), for \( i = 1 \),
\[
W_1(x_1, x_2, \ldots, x_m, z) := \text{Pol}_{x_1,\ldots,x_m} f_{1,m}(x_1, \ldots, x_m) \langle \text{Tr} (G_1(x_1)G_2(x_2) \ldots G_m(x_m)Q(z)) \rangle
\]
which is a polynomial in all variables and, for \( i = 0 \), \( W_0(x) = \langle \text{Tr} G_1(x) \rangle \). We also define:
\[
W_{i;1}(x_1, x_i, \ldots, x_m, z; x'_i) := \text{Pol}_{x_1,\ldots,x_m} f_{i,m}(x_i, \ldots, x_m) \langle \text{Tr} (G_1(x_1)) \text{Tr} (G_1(x'_i)) \text{Tr} (G_1(x_1)G_i(x_i) \ldots G_m(x_m)Q(z)) \rangle.
\]
All these functions admit a topological expansion:
\[
W_i = \sum_g (N/t)^{1-2g} W_i^{(g)} \quad \text{and} \quad W_{i;1} = \sum_g (N/t)^{-2g} W_{i;1}^{(g)}.
\]
5.0.2 Loop equations and spectral curve

In this model, the master loop equation reads [Eyn03, Eyn08]:

\[
W_{2,1}(x_1, \ldots, x_{m+1}; x_1) + \frac{t}{N} W_1(x_1, \ldots, x_{m+1}) - (V_1'(x_1) - c_{1,2} x_2) S(x_{m+1}) \\
+ (c_{1,2} x_2 - V_1'(x_1) + \frac{t}{N} W_0(x_1)) \left( \frac{t}{N} W_2(x_1, \ldots, x_{m+1}) - S(x_{m+1}) \right)
\]

\[
= \frac{t}{N} \sum_{i=2}^{m} \left( V_i'(x_i) - c_{i-1,i} x_{i-1} - c_{i,i+1} x_{i+1} \right) W_{i+1}(x_1, x_i, \ldots, x_{m+1}).
\]

(5.1)

Let us consider specific values for the variables \(x_i\) in order to turn it into an equation involving only \(x_1\) and \(x_2\). One defines \(\{\hat{x}_i(x_1, x_2)\}_{i=3}^{m+1}\) as functions of the two first variables \(x_1\) and \(x_2\) by

\[
c_{i,i+1} \hat{x}_{i+1}(x_1, x_2) = V_i'(\hat{x}_i(x_1, x_2)) - c_{i-1,i} \hat{x}_{i-1}(x_1, x_2).
\]

(5.2)

for \(i = 2, \ldots, m\) with the initial conditions \(\hat{x}_1(x_1, x_2) = x_1\) and \(\hat{x}_2(x_1, x_2) = x_2\).

Choosing \(x_i = \hat{x}_i(x_1, x_2)\), reduces the master loop equation to an equation in \(x_1\) and \(x_2\):

\[
\hat{W}_{2,1}(x_1, x_2; x_1) + \frac{t}{N} (c_{1,2} x_2 - Y(x_1)) \hat{U}(x_1, x_2) = \hat{E}(x_1, x_2)
\]

where \(Y(x) = V_1'(x) - \frac{t}{N} W_0(x)\), the hat means that the functions are considered at the value \(x_i = \hat{x}_i(x_1, x_2)\), i.e. \(\hat{f}(x_1, x_2) := f(x_1, x_2, \hat{x}_3, \hat{x}_4, \ldots, \hat{x}_n)\) for an arbitrary function \(f\), and one has defined

\[
\hat{U}(x_1, x_2) = W_2(x_1, x_2, \hat{x}_3, \ldots, \hat{x}_{m+1}) - \frac{N}{t} S(\hat{x}_{m+1}),
\]

and

\[
\hat{E}(x_1, x_2) = -\frac{t}{N} \hat{W}_1(x_1, x_2) + (V_1'(x_1) - c_{1,2} x_2) \hat{S}(x_1, x_2).
\]

Finally, the leading order in the topological expansion gives

\[
\hat{E}^{(0)}(x_1, x_2) = \left( c_{1,2} x_2 - Y^{(0)}(x_1) \right) \hat{U}^{(0)}(x_1, x_2)
\]

(5.3)

where one should notice that \(\hat{W}_1(x_1, x_2)\), and thus \(\hat{E}(x_1, x_2)\), is a polynomial in both \(x_1\) and \(x_2\).

Again, this equation is valid for any \(x_1\) and \(x_2\), and, if we choose \(x_2\) such that \(c_{1,2} x_2 = Y^{(0)}(x_1)\), we get:

\[
H_{chain}(x_1, x_2) := \hat{E}^{(0)}(x_1, x_2) = 0.
\]

(5.4)
This algebraic equation is the spectral curve of our model.

**Study of the spectral curve**

The algebraic plane curve $H_{\text{chain}}(x_1, x_2) = 0$, can be parameterized by a variable $z$ living on a compact Riemann surface $L_{\text{chain}}$ of some genus $g$, and two meromorphic functions $x_1(z)$ and $x_2(z)$ on it. Let us study it in greater details.

The polynomial $H_{\text{chain}}(x_1, x_2)$ has degree $1 + d$ in $x_2$ (resp. $x_1$). This means that the embedding of $L_{\text{chain}}$ is composed of $1 + d$ copies of the Riemann sphere, called $x_1$-sheets (resp. $x_2$-sheets), glued by cuts so that the resulting Riemann surface $L_{\text{chain}}$ has genus $g$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{\text{chain}}(x_1, x_2) = 0$ in $x_2$ (resp. $x_1$).

The Riemann surface $L_{\text{chain}}$ has genus $g$ lower than $D - s$ with $D = s d_1 \ldots d_m$.

One can consider all the variables $x_i(p) := \hat{x}_i(x_1(p), x_2(p))$ as meromorphic functions on $L_{\text{chain}}$ as opposed to only $x$ and $y$ in the one matrix model case. Their negative divisors are given by

$$[x_k(p)]_\infty = -r_k - s_k \sum_{i=1}^s \hat{\lambda}_i$$

where $\infty$ is the only point of $L_{\text{chain}}$ where $x_1$ has a simple pole, the $\hat{\lambda}_i$ are the preimages of $\lambda_i$ under the map $x_{m+1}(p), x_{m+1}(\hat{\lambda}_i) = \lambda_i$, and the degrees $r_k$ and $s_k$ are integers given by $r_1 := 1, r_k := d_1 d_2 \ldots d_{k-1}, s_{m+1} := 0, s_m := 1, s_k := d_k+1 d_{k+2} \ldots d_m s$.

Note that the presence of an external matrix creates as many poles as the number of distinct eigenvalues of this external matrix $M_{m+1} = \Lambda$.

**5.0.3 Solution of the loop equations**

The procedure used to solve the loop equations in the one matrix model cannot be generalized in this setup, mainly because the involution $z \to \bar{z}$ is not globally defined on the spectral curve. However, the loop equations can be solved using a detour [Eyn08]. This resolution proceeds in three steps. One first shows

---

\footnote{The cases of matrix models without external field correspond to a totally degenerate external matrix $\Lambda = c \text{Id}$ with only 1-eigenvalue. There are thus two poles as in the 1 matrix model studied earlier.}
that the loop equations eq\[5.1\] have a unique solution admitting a topological
expansion. One then finds a solution of these equations:

\textbf{Lemma 5.1}

\[ E(x(z), y) = -K \left( \prod_{i=0}^{d} (y - V_i'(x(z^i))) + \frac{t}{N} \frac{1}{\text{Tr}(x(z^i) - M_1)} \right) \]

(5.5)

where \( K \) is a constant and the inverted comas \( " \langle . \rangle " \) means that every time
one encounters a two point function \( \langle \frac{1}{\text{Tr}(x(z^i) - M_1)} \frac{1}{\text{Tr}(x(z^j) - M_1)} \rangle_c \), one replaces
it by \( W_{0,1}^{(0)}(z_i; z_j) := \langle \frac{1}{\text{Tr}(x(z^i) - M_1)} \frac{1}{\text{Tr}(x(z^j) - M_1)} \rangle_c + \frac{\delta_{n,2}\delta_{g,0}}{(x(z_i) - x(z_j))^2}. \)

The matching of the coefficients of the polynomials in \( y \) in the left- and right
hand sides of \ref{5.5} and a few algebro-geometrical computations allows to solve
the loop equations\[\text{[16]}\] to get

\textbf{Theorem 5.1} The correlation functions of a chain of matrices formal integral
are recursively obtained by computing residues on \( L_{\text{chain}} \):

\[ \omega_{n+1}^{(h)}(z, z_1, \ldots, z_n) = \sum_i \text{Res}_{z^i \to a_i} K(z, z') \left[ \omega_{n+2}^{(h-1)}(z', z, z_1, \ldots, z_n) \right. \\
+ \sum_{j=0}^{h} \sum_{I \subset \{z_1, \ldots, z_n\}} \omega_{|I|+1}^{(j)}(z', I) \omega_{n-|I|+1}^{(h-j)}(z, \{z_1, \ldots, z_n\} \setminus I) \right] \]

(5.6)

where, as in the preceding section,

\[ \omega_n^{(h)}(z_1, \ldots, z_n) = \text{Res}_{N \to \infty} N^{n+2h-3} \left( \prod_{i=1}^{n} \text{Tr} G_1(x_i) \right) \frac{dx(z_1) \ldots dx(z_n)}{c} + \delta_{n,2}\delta_{g,0} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}. \]

the two point function \( \omega_2^{(0)} \) is the Bergman kernel of the spectral curve \( L_{\text{chain}} \),
the recursion kernel is

\[ K(z, z') := \frac{dS_{z', z}(z)}{2(y(z') - y(z))dx(z')}, \]

\( x \) and \( y \) are two meromorphic functions on \( L_{\text{chain}} \) such that \( H_{\text{chain}}(x(z), y(z)) = 0 \) for any point \( z \in L_{\text{chain}} \) and \( a_i \) are the \( x \)-branch points, i.e. solutions to \( dx(a_i) = 0. \)

\[\text{[16]}\text{See } \text{Che06b, Eyn08} \text{ for the detailed proof.}\]
Remember that $H_{\text{chain}}$ was defined in 5.4 by $H_{\text{chain}}(x_1, c_{12}x_2) = 0$ for $c_{12}x_2 = Y^{(0)}(x_1)$. Thus the function $x$ and $y$ can be thought of as continuation to the whole spectral curve of $x_1$ and $c_{12}x_2$ respectively.

The free energy is also obtained by using the same formula as in the one matrix case:

**Theorem 5.2** For $h > 1$ and any primitive $\Phi$ of $y \, dx$, one has

$$F^{(h)} = \frac{1}{2 - 2h} \sum_i \text{Res}_{z \to a_i} \Phi(z) \omega_1^{(h)}(z).$$ (5.7)

Thus, the solution of any chain-matrix model with an external field is obtained by the exact same formula as the solution of the one matrix model: the only difference is the spectral curve used to apply this recursion.

### 6 Generalization: topological recursion

We have seen that the loop equation method gives a unique solution for a large family of formal matrix models. The only input of this solution is the spectral curve of the considered model. In [Eyn07], it has been proposed to use equations 5.6 and 5.7 to associate infinite sets of correlation functions and free energies to any spectral curve $(\mathcal{L}, x, y)$ where $\mathcal{L}$ is a compact Riemann surface and $x$ and $y$ two functions analytic in some open domain of $\mathcal{L}$.

The free energies and correlation functions built from this recursive procedure show many interesting properties such as invariance under a large set of transformations of the spectral curve, special geometry relations, modular invariance or integrable properties. In particular, it is a very convenient tool to study critical regimes and get the universal properties of the matrix integrals described in the chapter 6 of this volume. It is also very useful to compare different matrix integrals. Eventually, this procedure proved to be efficient in the resolution of many problems of enumerative geometry or statistical physics such as string theory, Gromov-Witten invariants theory, Hurwitz theory or exclusion processes such as TASEP or PASEP. Most of the results proposed by this approach are still conjectures up to now but the numerous checks passed so far tends to prove that this generalization of the loop equation method is a very promising field.\(^{17}\)

\(^{17}\)For a review on this subject, see [Eyn09] and references therein.
The inductive procedure presented in this chapter only allows to compute one particular set of observables of multi-matrix models. It does not compute correlation functions involving more than one type of matrix inside the same trace. These more complicated objects are very important for their application to quantum gravity or conformal field theories where they correspond to the insertion of boundary operators. In the two matrix model, the loop equation method allowed to compute the topological expansion of any of these operators [Eyn08]. In the chain of matrices case, only a few of them were computed in their large $N$ limit only [Eyn03]. The computation of any observable of the chain of matrices is still an open problem which is very likely to be solved by the use of loop equations.

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