ON A CONNECTION BETWEEN NAIMARK’S DILATION THEOREM, SPECTRAL REPRESENTATIONS, AND CHARACTERISTIC FUNCTIONS

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ABSTRACT. We give a Herglotz-type representation of an arbitrary generalized spectral measure. As an application, a new proof of the classical Naimark’s dilation theorem is given. The same approach is used to describe the spectrum of all unitary rank-one perturbations of a given partial isometry.

1. Introduction

Let \( \mathcal{B} \) be a family of all Borel sets on the unit circle \( \mathbb{T} \). By a generalized spectral measure on \( \mathbb{T} \) we mean a function \( B : \mathcal{B} \to B(\mathcal{H}) \) whose values are positive bounded self-adjoint operators on \( \mathcal{H} \) such that \( B(\emptyset) = 0, B(\mathbb{T}) = I \) and for every sequence \( \Delta_1, \Delta_2, ... \) of mutually disjoint Borel sets, we have

\[
B(\Delta_1 \cup \Delta_2 \cup ...) = \sum_{i=1}^{\infty} B(\Delta_i)
\]

in the strong operator topology. If we require all the values to be orthogonal projections then we have an (ordinary) spectral measure. A classical theorem of Naimark [3] says that any generalized spectral measure can be represented as a projection of an ordinary spectral measure. This theorem is considered by many as the beginning of Dilation Theory. Since then many different proofs and generalization have appeared (e.g. [4, 6, 8, 7]). We propose yet another approach, which involves in a natural way characteristic functions and spectral representations of unitary operators; it also relates Naimark’s theorem for the first time to the subject of rank-one perturbations of a given operator. The latter is another classical subject with a rich literature behind (see [5] and the references therein). The key idea in our approach is to obtain a representation for a generalized spectral measure which is reminiscent to the well known one:

\[
\langle (U + zI)(U - zI)^{-1}h_1|h_2 \rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d \langle (E(\xi)h_1|h_2) \rangle
\]

relating a unitary operator \( U \) to its spectral measure \( E \). We will show that if \( K \) is a closed subspace of \( \mathcal{H} \) and the generalized spectral measure \( B : \mathcal{B} \to B(K) \) is obtained from a spectral measure \( E : \mathcal{B} \to B(\mathcal{H}) \) by \( B(\Delta) = P_K E(\Delta) \) then the above mentioned representation is given by:

\[
\langle (\tilde{U} + \Theta_S(z))(\tilde{U} - \Theta_S(z))^{-1}k_1|k_2 \rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d \langle (B(\xi)k_1|k_2) \rangle
\]

where \( \tilde{U} : U^*(K) \to K \) is the restriction of \( P_K U \) on \( U^*(K) \) and \( \Theta_S(z) \) is the characteristic function corresponding to the operator \( S := (I - P_K)U \). In the case
when $\mathcal{K}$ is one-dimensional this relation reduces to a relation that D. Clark [2] used in his treatment of the rank-one perturbations of a restricted shift. This is the point where the connection between these seemingly unrelated concepts is made.

The paper is organized as follows. The next section presents some preliminary material. Section 3 contains the main theorem which contains the above mentioned representation. As an application, a new proof of Naimark’s dilation theorem is given. In the last section an application to the rank-one perturbations of a partial isometry is given.

2. Preliminaries

We recall the basic facts from the the model theory of completely nonunitary contractions developed by Sz. Nagy and Foias [6]. Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction, $\|T\| \leq 1$. The self adjoint operators $D_T := (I - T^*T)^{1/2}$ and $D_{T^*} := (I - TT^*)^{1/2}$ are called the defect operators of $T$ and the spaces $D_T := \overline{D_T \mathcal{H}}$, $D_{T^*} := \overline{D_{T^*} \mathcal{H}}$ are called the defect spaces for $T$. The defect indices are defined by $\partial_T := \dim D_T$ and $\partial_{T^*} := \dim D_{T^*}$. These indices measure, in a certain sense, how much a contraction differs from a unitary operator. If $T$ is a partial isometry then the defect operators $D_T$ and $D_{T^*}$ are orthogonal projections onto the initial and the final space of $T$ respectively.

The characteristic function of a contraction $T$ is an operator-valued function $\Theta_T(z) : D_T \to D_{T^*}$ defined by:

$$\Theta_T(z) := -T + zD_{T^*}(I - zT^*)^{-1}D_T|D_T.$$ 

It is always an analytic contraction-valued function. A contraction-valued function $\Theta : \mathbb{D} \to \mathcal{B}(\mathcal{L}, \mathcal{L}^*)$ is called pure if $\|\Theta(l)\| < \|l\|$ for $l \in \mathcal{L}$, $l \neq 0$. Two contraction-valued analytic functions $\Theta_1 : \mathbb{D} \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_1^*)$, $\Theta_2 : \mathbb{D} \to \mathcal{B}(\mathcal{L}_2, \mathcal{L}_2^*)$ are said to coincide if there exist unitaries $\omega : \mathcal{L}_1 \to \mathcal{L}_2$, $\omega_1 : \mathcal{L}_1^* \to \mathcal{L}_2^*$ such that $\Theta_1(z) = \omega^{-1}_1\Theta_2(z)\omega$. There is an easy characterization of partial isometries in terms of their characteristic functions. Namely, a contraction $T$ is a partial isometry if and only if $\Theta_T(0) = 0$.

A contraction $T$ is said to be completely nonunitary (c.n.u.) if it is not unitary on any of its invariant subspaces. There always exists a unique decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ into subspaces reducing $T$, such that $T|\mathcal{H}_0$ is unitary and $T|\mathcal{H}_1$ is c.n.u..

One of the main theorems of Sz. Nagy and Foias theory is the fact that every c.n.u. contraction $T$ on a separable Hilbert space is unitarily equivalent to an operator $T$ acting on the space

$$\mathcal{H} = [H^2(D_{T^*}) \oplus \Delta_T L^2(D_T)] \ominus \{\Theta_T u + \Delta_T u : u \in H^2(D_T)\},$$

by

$$T(u \oplus v) = P_\mathcal{H}(zu \oplus zv),$$

where $P_\mathcal{H}$ is the orthogonal projection of $H^2(D_{T^*}) \oplus \Delta_T L^2(D_T)$ onto $\mathcal{H}$ and $\Delta_T(t) := (I - \Theta_T(e^{it})\Theta_T(e^{it}))^{1/2}$. Here, as usual, $L^2(D_T)$ is the space of $D_T$-valued square integrable functions on $T$ and $H^2(D_T)$ is the corresponding Hardy space.

The following fact will be important for us. For any given contraction-valued analytic function $\Theta : \mathbb{D} \to \mathcal{B}(\mathcal{L}, \mathcal{L}^*)$ which is pure, there exists a contraction $T$ whose characteristic function coincides with $\Theta(z)$. 

In the case when $\Theta_T(z)$ is an inner function (i.e. $\Theta_T(\zeta)$ is an isometry for a.e. $\zeta \in \mathbb{T}$) then $H$ has a simpler form $H = H^2(D_T) \ominus \{\Theta_T u : u \in H^2(D_T)\}$. This happens if and only if $T^{*n} \to O$ as $n \to \infty$. If, in addition, the defect indices of $T$ are both equal to 1 then $H = H^2(\mathbb{D}) \ominus \Theta_T H^2(\mathbb{D})$ is a space of analytic scalar-valued functions. In this case $\Theta_T(z)$ can be viewed as a usual (scalar-valued) inner function. In [2], all the unitary rank-one perturbations of this type of operators $T$ are examined. It is shown there that all the rank-one unitary perturbations can be parametrized by points $\alpha \in \mathbb{T}$ and that the spectral measures $\sigma_\alpha$ for each of these operators can be easily obtained from the characteristic function $\Theta_T(z)$. Namely, they are determined by:

$$\frac{\alpha + \Theta_T(z)}{\alpha + \Theta_T(z)} = \int_T \frac{\xi + z}{\xi - z} d\sigma_\alpha(\xi).$$

The collection of measures that are associated to a given inner function (or more generally to an analytic self-map of the disc) in this way are called Clark (or Aleksandrov-Clark) measures [5].

3. Main Results

Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Fix $h \in \mathcal{H}$. Then the function $\langle (U + zI)(U - zI)^{-1}h|h \rangle$ is holomorphic in $z \in \mathbb{D}$ with positive real part. Thus, there exists a unique measure $\mu_h$ on the unit circle $\mathbb{T}$ such that

$$\langle (U + zI)(U - zI)^{-1}h|h \rangle = \int_\mathbb{T} \frac{\xi + z}{\xi - z} d\mu_h(\xi).$$

Using polarization, for any $h_1, h_2 \in \mathcal{H}$ there exists a measure $\mu_{h_1, h_2}$ such that:

$$\langle (U + zI)(U - zI)^{-1}h_1|h_2 \rangle = \int_\mathbb{T} \frac{\xi + z}{\xi - z} d\mu_{h_1, h_2}(\xi).$$

Let $\Delta$ be any Borel set on the unit circle $\mathbb{T}$. Then $\mu_{h_1, h_2}(\Delta)$ is a skew-symmetric function of $h_1$ and $h_2$, linear in $h_1$, and bounded by $\|h_1\|\|h_2\|$. Therefore, it can be represented as

$$\mu_{h_1, h_2}(\Delta) = \langle E(\Delta)h_1|h_2 \rangle,$$

for some positive bounded operator $E(\Delta)$. It is well known that $E : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is an ordinary spectral measure.

Let $\mathcal{K} \subset \mathcal{H}$ be a closed subspace and let $P_\mathcal{K} : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$. Define $T, S : \mathcal{H} \to \mathcal{H}$ by $T = P_\mathcal{K}U$ and $S = (I - P_\mathcal{K})U$, respectively. Then $T$ and $S$ are partial isometries and the characteristic function $\Theta_S(z) : U^*(\mathcal{K}) \to \mathcal{K}$ satisfies $\Theta_S(0) = O$ (and hence is pure). It will be useful to denote by $\hat{U}$ the restriction of $U$ on the closed subspace $U^*(\mathcal{K})$, and view it as an operator from $U^*(\mathcal{K})$ to $\mathcal{K}$. Obviously, $\hat{U}u = Tu = Uu$ for any vector $u \in U^*(\mathcal{K})$ and $\hat{U}^*k = T^*k = U^*k$ for any $k \in \mathcal{K}$. Observe also that $U^*P_\mathcal{K} = T^*P_\mathcal{K} = U^*P_\mathcal{K}^2 = T^*$.

For fixed $k \in \mathcal{K}$, again $\langle (\hat{U} + \Theta_S(z))(\hat{U} - \Theta_S(z))^{-1}k|k \rangle$ is a holomorphic function in $z \in \mathbb{D}$ with positive real part and consequently there exists a unique measure $\sigma_k$ satisfying

$$\langle (\hat{U} + \Theta_S(z))(\hat{U} - \Theta_S(z))^{-1}k|k \rangle = \int_\mathbb{T} \frac{\xi + z}{\xi - z} d\sigma_k(\xi).$$
By polarization again, for any $k_1, k_2 \in \mathcal{K}$ there exists a measure $\sigma_{k_1, k_2}$ such that:

$$\left\langle (\hat{U} + \Theta_S(z))\hat{U} - \Theta_S(z) \right\rangle^{-1} k_1|k_2\right\rangle = \int_{\xi} \frac{\xi + z}{\xi - z} d\sigma_{k_1, k_2}(\xi).$$

For any Borel set $\Delta$ on the unit circle let $B(\Delta)$ be the positive self-adjoint operator satisfying

$$\sigma_{k_1, k_2}(\Delta) = \left\langle B(\Delta)k_1|k_2\right\rangle,$$

for all $k_1, k_2 \in \mathcal{K}$. It can be shown that $B : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ is a generalized spectral measure.

**Theorem 3.1.** For any $k \in \mathcal{K}$ the following equality holds:

$$\left\langle (U + z I)(U - z I)^{-1} k|k\right\rangle = \left\langle (\hat{U} + \Theta_S(z))\hat{U} - \Theta_S(z) \right\rangle^{-1} k|k\right\rangle.$$

**Proof:** The left- and right-hand side of the equality we are aiming to prove are equal to $2 \left\langle (I - \Theta_S(z)\hat{U}^*)^{-1} k|k\right\rangle - \|k\|^2$ and $2 \left\langle (I - z \hat{U}^*)^{-1} k|k\right\rangle - \|k\|^2$, respectively. Hence, it is equivalent to show that:

$$\left\langle (I - \Theta_S(z)\hat{U}^*)^{-1} k|k\right\rangle = \left\langle (I - z \hat{U}^*)^{-1} k|k\right\rangle.$$ 

To prove this equality, we first notice that

$$\left\langle (I - \Theta_S(z)\hat{U}^*)^{-1} k|k\right\rangle = \left\langle (I - z P_{\mathcal{K}}(I - z S^*)^{-1} \hat{U}^*)^{-1} k|k\right\rangle$$

$$= \left\langle (\hat{U}\hat{U}^* - z P_{\mathcal{K}}(I - z S^*)^{-1} \hat{U}^*)^{-1} k|k\right\rangle$$

$$= \left\langle \hat{U}(\hat{U} - z P_{\mathcal{K}}(I - z S^*)^{-1})^{-1} k|k\right\rangle$$

$$= \left\langle \hat{U}(\hat{U}(I - z S^*) - z P_{\mathcal{K}})^{-1} k|k\right\rangle$$

$$= \left\langle \hat{U}(I - z S^*)(\hat{U}(I - z S^*) - z \hat{U}^* P_{\mathcal{K}})^{-1} k|k\right\rangle$$

$$= \left\langle \hat{U}(I - z S^*)(I - z S^* - z \hat{U}^* P_{\mathcal{K}})^{-1} \hat{U}^* k|k\right\rangle.$$ 

Now, recalling that $\hat{U}^* P_{\mathcal{K}} = T^*$, $T^* k = \hat{U}^* k$ we obtain

$$\left\langle (I - \Theta_S(z)\hat{U}^*)^{-1} k|k\right\rangle = \left\langle (I - z S^*)(I - z S^* - z T^*)^{-1} \hat{U}^* k|\hat{U}^* k\right\rangle$$

$$= \left\langle (I - z S^* - z T^*)^{-1} T^* k|T^* k\right\rangle - \left\langle z(I - z S^* - z T^*)^{-1} T^* k|ST^* k\right\rangle$$

$$= \left\langle (I - z S^* - z T^*)^{-1} T^* k|T^* k\right\rangle.$$ 

Since $ST^* k = 0$, we finally have

$$\left\langle (I - \Theta_S(z)\hat{U}^*)^{-1} k|k\right\rangle = \left\langle (I - z(T + S)^*)^{-1} T^* k|T^* k\right\rangle$$

$$= \left\langle (I - z U^*)^{-1} U^* k|U^* k\right\rangle$$

$$= \left\langle U^*(I - z U^*)^{-1} k|U^* k\right\rangle$$

$$= \left\langle (I - z U^*)^{-1} k|k\right\rangle.$$ 

\qed
Remark 1. If \( K = H \) then \( S = O \) on \( H \) and hence \( \Theta_S(z) = zI \). Also, clearly \( \tilde{U} = U \). Thus, we can view \((U + \Theta_S(z))(U - \Theta_S(z))^{-1}\) as a substitution for \((U + zI)(U - zI)^{-1}\) in the case when \( K \) is a true subspace.

Corollary 3.2. For any Borel set \( \Delta \subset \mathbb{T} \), \( B(\Delta) = P_K E(\Delta) \).

Proof: To show the claim it is equivalent to show

\[ \langle P_K E(\Delta)k_1 | k_2 \rangle = \langle B(\Delta)k_1 | k_2 \rangle \]

for all \( k_1, k_2 \in K \). Since \( P_K \) is an orthogonal projection, this is equivalent to:

\[ \langle E(\Delta)k_1 | k_2 \rangle = \langle B(\Delta)k_1 | k_2 \rangle . \]

Therefore, using polarization it suffices to prove

\[ \langle E(\Delta)k | k \rangle = \langle B(\Delta)k | k \rangle \]

for all \( k \in K \). But, this easily follows from Theorem 3.1.

Next we give a new proof of Naimark’s dilation theorem.

Theorem 3.3 (Naimark [3]). Let \( B : \mathcal{B} \to \mathcal{B}(K) \) be a generalized spectral measure. Then there exist \( H \supset K \) and an ordinary spectral family \( E : \mathcal{B} \to \mathcal{B}(H) \) such that for any Borel set \( \Delta \subset \mathbb{T} \), \( B(\Delta) = P_K E(\Delta) \).

Proof: Let \( k_1, k_2 \in K \). Define \( \sigma_{k_1,k_2}(\Delta) = \langle B(\Delta)k_1 | k_2 \rangle \) for any Borel set \( \Delta \subset \mathbb{T} \). Clearly, \( \sigma_{k_1,k_2} \) is a Borel measure on \( \mathbb{T} \). For any \( z \in \mathbb{D} \) define \( F(z) : K \to K \) such that

\[ \langle F(z)k_1 | k_2 \rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\sigma_{k_1,k_2}(\xi) \]

for every \( k_1, k_2 \in K \). It is easy to see that \( F(z) \) is a linear operator with a positive real part and \( F(0) = I \). It is also analytic as an operator-valued function in \( z \in \mathbb{D} \). The inverse \((I + F(z))^{-1}\) is defined on a dense subset of \( K \) since \( F(z) \) has a positive real part. Set \( \Theta(z) = (F(z) - I)(F(z) + I)^{-1} \) on that dense set. By continuity, \( \Theta(z) \) can be extended on the whole \( K \) with \( ||\Theta(z)|| \leq 1 \). Moreover, \( \Theta(z) \) is an analytic contraction-valued function with \( \Theta(0) = 0 \) (and hence is pure). Thus, there exists a completely non-unitary contraction \( S : H \to H \) whose characteristic function \( \Theta_S(z) \) coincides with \( \Theta(z) \). More precisely, there exist unitary operators \( \omega : D_\omega \to K \) and \( \omega_s : D_{\omega_s} \to K \) such that \( \Theta_S(z) = \omega_s^{-1}\Theta(z)\omega \). Define \( \tilde{U} = \omega_s^{-1}\omega : D_{\omega_s} \to D_\omega \) and set \( T : H_1 \to H_1 \) to be

\[ Th = \begin{cases} \tilde{U}h, & h \in D_\omega \\ 0, & h \in \text{Ker}D_\omega. \end{cases} \]

Finally, define \( U_1 = T + S : H_1 \to H_1 \). Clearly, \( U_1 \) is unitary. Let \( E_1(\Delta) \) be the spectral measure corresponding to \( U_1 \) and let \( B_1(\Delta) \) be the generalized spectral measure such that

\[ \langle (\tilde{U} + \Theta_S(z))(\tilde{U} - \Theta_S(z))^{-1}s_1 | s_2 \rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\langle B_1(\xi)s_1 | s_2 \rangle \]

for every \( s_1, s_2 \in D_{\omega_s} \). By the previous Corollary, for any Borel set \( \Delta \) we have

\[ B_1(\Delta) = P_{D_{\omega_s}} E_1(\Delta). \]

Now, since

\[ \langle (\tilde{U} + \Theta_S(z))(\tilde{U} - \Theta_S(z))^{-1}s_1 | s_2 \rangle = \langle (I + \Theta(z))(I - \Theta(z))^{-1}\omega_s s_1 | \omega_s s_2 \rangle \]
and $F(z) = (I + \Theta(z))(I - \Theta(z))^{-1}$, it follows that $B(\Delta) = \omega_s B_1(\Delta)\omega_s^{-1}$.

Define $\mathcal{H} = \mathcal{K} \oplus \ker D\sigma$ and $W_\omega = \omega_s \oplus I : \mathcal{H}_1 \rightarrow \mathcal{H}$. Clearly, $W_\omega$ is unitary. Then $B(\Delta) = P_\mathcal{K} E(\Delta)$ where $P_\mathcal{K} = W_\omega D\omega W_\omega^{-1}$ and $E(\Delta) = W_\omega E_1(\Delta) W_\omega^{-1}$.

One readily sees that $P_\mathcal{K} : \mathcal{H} \rightarrow \mathcal{H}$ defined this way is the orthogonal projection onto $\mathcal{K}$ and $E(\Delta)$ is a spectral measure on $\mathcal{H}$.

\[ \square \]

4. Rank-one perturbations of partial isometries

Let $S : \mathcal{H} \rightarrow \mathcal{H}$ be a c.n.u. partial isometry with both defect indices equal to 1, i.e., $\partial S = \partial S^* = 1$. Denote by $\mathcal{K}$ and $\bar{\mathcal{K}}$ the orthogonal complements of the final and the initial space of $S$, respectively. Fix two unit vectors $k \in \mathcal{K}$ and $\bar{k} \in \bar{\mathcal{K}}$. For any complex number $\alpha$ of modulus 1, define the linear operator $U_\alpha$ by

$$U_\alpha h = \begin{cases} S h, & \text{if } h \perp \bar{\mathcal{K}}; \\ \alpha k, & \text{if } h = \bar{k}. \end{cases}$$

Clearly, $U_\alpha$ are unitary operators; these are the only unitary rank-one perturbations of $S$. Consider $\mathcal{K}_\alpha := \overline{\text{span}}\{U_\alpha^n k : n \in \mathbb{Z}\}$. One can show by induction that for any $h \in \mathcal{K}_\alpha^\perp$ and $\beta \in T$ we have that $U_\alpha^\beta h = S^\alpha h$ and $U_\alpha h = S^{-\alpha} h$. Therefore, $\mathcal{K}_\alpha^\perp \subset \mathcal{K}_\beta^\perp$ and, by symmetry, $\mathcal{K}_\alpha^\perp = \mathcal{K}_\beta^\perp$. This space $\mathcal{K}_\alpha^\perp$ is reducing for $S$ and $S$ is unitary there. Since $S$ is c.n.u., we have that $\mathcal{K}_\alpha = \mathcal{H}$ for all $\alpha \in T$.

Next we will describe the spectrum of $U_\alpha$. Let $\Theta_S(z)$ be the characteristic function of $S$. For each $z \in \mathbb{D}$, $\Theta_S(z)$ is an operator between two one-dimensional spaces $\bar{\mathcal{K}}$ and $\mathcal{K}$. Denote by $\phi(z)$ the scalar valued analytic function such that $\Theta_S(z)\bar{k} = \phi(z)k$ for each $z \in \mathbb{C}$. Clearly, $|\phi(z)| \leq 1$. Define also $\bar{U}_\alpha : \bar{\mathcal{K}} \rightarrow \bar{\mathcal{K}}$ to be the operator sending $\bar{k}$ to $\alpha k$. Let $\sigma_\alpha$ be the measure on $T$ for which

$$\langle (\bar{U}_\alpha + \Theta_S(z)){\bar{U}_\alpha} - \Theta_S(z)^{-1} k|k \rangle = \int_T \frac{\xi + z}{\xi - z} \, d\sigma_\alpha(\xi).$$

This is equivalent to

$$\frac{\alpha + \phi(z)}{\alpha - \phi(z)} = \int_T \frac{\xi + z}{\xi - z} \, d\sigma_\alpha(\xi),$$

which implies that $\sigma_\alpha$ is a Clark measure for $\phi(z)$. It is also immediate that the corresponding generalized spectral measure $B_\alpha(\Delta)$ is simply given by $B_\alpha(\Delta) k = \sigma_\alpha(\Delta) k$. If $E_\alpha(\Delta)$ is the spectral measure corresponding to $U_\alpha$, then it follows from Corollary 3.2 that

$$\langle E_\alpha(\Delta) k|k \rangle = \sigma_\alpha(\Delta).$$

Since $\overline{\text{span}}\{E_\alpha(\Delta) k : \Delta \text{ Borel subset of } T\} = \mathcal{H}$ the last equality proves the following:

**Corollary 4.1.** The spectrum of $U_\alpha$ coincides with the support of $\sigma_\alpha$. Thus, it consists of the union of those points in $T$ at which $\phi(z)$ cannot be analytically continued and those $\zeta \in T$ at which $\phi(z)$ is analytically continuable with $\phi(\zeta) = \alpha$. The set of eigenvalues of $U_\alpha$ coincides with the set of all the atoms of $\sigma_\alpha$.

**Remark 2.** There are several well-known conditions describing the atoms of a Clark measure $\sigma_\alpha$. An important one (goes back to M. Riesz) is the following: $\sigma_\alpha$
has a point mass at $\zeta$ if and only if $\phi(z)$ has the nontangential limit $\alpha$ at $\zeta$ and for all (or one) $\beta$ in $T$ different from $\alpha$,
\[
\int_T \frac{d\sigma_{\beta}(\zeta)}{|\xi - \zeta|^2} < \infty.
\]

**Remark 3.** In [2], a similar description (although with different methods) of the spectra is obtained for unitary rank-one perturbations of a restricted shift. Notice that not every partial isometry can be represented as a restricted shift on the space $H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D})$ of scalar-valued functions. Thus, the proposition above is not implied by the results in [2].

Finally, we can also consider the more general case when $S$ is c.n.u. with equal (possibly infinite) defect indices. Let again $K$ and $\tilde{K}$ be the orthogonal complements of the final and the initial space of $S$, respectively. For a unitary operator $A : \tilde{K} \to K$, similarly as in [1], we can define a unitary perturbation $U_A$ of $S$ by
\[
U_A h = \begin{cases} Sh, & \text{if } h \perp \tilde{K} \\ Ah, & \text{if } h \in \tilde{K}. \end{cases}
\]

As in the case of the rank-one perturbations, one can show that $S$ c.n.u. implies $\text{spur}\{U_A^n k : k \in K, n \in \mathbb{Z}\} = \mathcal{H}$. To describe the spectrum of $U_A$ notice that by Theorem 3.1 we have
\[
\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1}k|k\rangle = \langle (\tilde{U}_A + \Theta_S(z))(\tilde{U}_A - \Theta_S(z))^{-1}k|k\rangle = \int_T \frac{\xi + z}{\xi - z} d\langle E_A(\xi)k|k\rangle.
\]

**Corollary 4.1.** The spectrum of $U_A$ consists of the union of those points in $T$ at which $\Theta_S(z)$ cannot be analytically continued and those $\zeta \in T$ at which $\Theta_S(z)$ is analytically continuable with $\Theta_S(\zeta) - A$ not invertible.

Further extensions of these results will be considered in a subsequent paper.

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