GEOMETRY, TOPOLOGY AND DYNAMICS OF
GEODESIC FLOWS ON NONCOMPACT POLYGONAL
SURFACES

EUGENE GUTKIN

Abstract. We establish the background for the study of geodesics on noncompact polygonal surfaces. For illustration, we study the recurrence of geodesics on Z-periodic polygonal surfaces. We prove, in particular, that almost all geodesics on a topologically typical Z-periodic surface with a boundary are recurrent.

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Introduction

The billiard ball problem on compact, planar polygons, $P \subset \mathbb{R}^2$, offers a variety of basic open questions. Some of them have to do with the statistical aspect of dynamics, e. g., the ergodicity of polygonal...
billiards; some others concern the topological aspect, e.g., the questions about periodic billiard orbits in $P$. We refer the reader to [10] for more detailed information on the subject.

However, the recurrence of billiard orbits on any compact polygon immediately follows from the Poincaré recurrence theorem, because the Liouville measure, which is the natural invariant measure, is finite. The situation changes, once we pass from compact to noncompact polygons. Although the billiard on certain noncompact polygons has been in the physics literature some hundred years [6] the subject has not been systematically studied.

All of the open questions for compact polygons are, of course, open for noncompact polygons, as well. In addition, because the Liouville measure is infinite, a noncompact polygon, $P \subset \mathbb{R}^2$, may yield a dissipative component in the billiard dynamics. Thus, we need to decompose the phase space for the billiard on $P$ into the conservative and dissipative components.

Before we begin the study of the billiard on noncompact polygons, the question arises: What do we mean by noncompact polygons? Indeed, this concept is so large and inhomogeneous that we do not expect nontrivial results pertaining to the billiard on every noncompact polygon. In order to obtain them, we should first identify a class of noncompact polygons which is i) sufficiently large to deserve a study; ii) sufficiently homogeneous to (hopefully) possess nontrivial properties that hold for all polygons in this class. Here we study such a class of noncompact polygons: The periodic polygons.

The recurrence, transience and ergodicity of the billiard on periodic polygons is also studied in [4]; the present paper is a supplement to [4] in a certain sense. We will now explain in what sense. The study of billiard on noncompact polygons leads to the concepts of noncompact polygonal surfaces and their coverings. It also leads to the notion of rationality for such surfaces, and hence to noncompact translation surfaces. Their counterparts in the compact case are well known; the basic properties and relationships between these notions are well established. They form the background for the study of billiard on compact polygons. See [8, 9] for more on this.

Here we do the same kind of ground work for noncompact polygonal surfaces. We introduce the basic notions in this subject; in particular,
we introduce and study the coverings of noncompact polygonal surfaces. We establish basic connections between the coverings and the holonomy of polygonal surfaces. We define the rationality for noncompact polygonal surfaces, and introduce noncompact translation surfaces. This material constitutes the geometry and topology part of our paper. See section 1.

In section 2 we do the ground work regarding the geodesic flows on noncompact polygonal surfaces. Let $P$ be one. If $P$ is rational, the geodesic flow on $P$ decomposes into a one-parameter family of directional flows. These can be realized as the linear flows on the noncompact translation surface $S = S(P)$. In the subject of compact, rational polygonal surfaces, arithmetic surfaces play a special role [7, 11]. In section 2.2 we introduce and study noncompact arithmetic polygonal surfaces.

The material of section 1 and section 2 is used in [4] to study the recurrence and ergodicity for geodesic dynamics on periodic surfaces. In particular, [4] contains results on ergodicity for certain $\mathbb{Z}^2$-periodic surfaces. Here, we restrict ourselves to the simpler class of $\mathbb{Z}$-periodic polygonal surfaces. We study in detail the recurrence of geodesics on these surfaces. Concentrating on $\mathbb{Z}$-periodic surfaces with boundaries, we establish a few results on recurrence of the geodesic flow and the directional flows. See Theorem 2 and Theorems 3, 4. Section 3 represents the dynamics part of our paper.

As we already mentioned, the goal in the present study is to establish the background for the subject of noncompact polygonal surfaces. This determined the style of the paper, which differs somewhat from that of a typical research paper. Our emphasis is on basic principles and the motivations, rather than on elaborate mathematical results. Thus, we present relatively few theorems, and offer many examples. The examples serve to illustrate the basic notions and to motivate the definitions.

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1. THE SETTING AND PRELIMINARIES

In this section we establish the subject of our study: Noncompact polygonal surfaces. For readers’ convenience, we will first recall the basic material on compact polygonal surfaces.

1.1. Compact polygonal surfaces.

We begin with the basic example of a compact polygonal surface: A planar polygon. By this we mean a compact region, $P \subset \mathbb{R}^2$, whose boundary consists of a finite number of linear intervals. We view $P$ as a riemannian manifold with boundary and corners. Then the geodesics are the billiard orbits in $P$; the geodesic flow is the billiard flow. Billiard orbits in polygons are of great interest to physicists [8, 9]; they have done extensive numerical studies of polygonal billiards. Many mathematical results in this subject concern rational polygons [10]. A polygon is rational if the angles between its sides are rational multiples of $\pi$. Denote by $\text{Hol}_r(P) \subset O(2)$ the linear part of the group generated by the orthogonal reflections with respect to the sides of $P$. Then $P$ is rational iff $\text{Hol}_r(P) = D_N$, the dihedral group of order $2N$, where $N = N(P)$ is determined by the angles of $P$. For irrational polygons $|\text{Hol}_r(P)| = \infty$.

By a surface we will always mean a connected, orientable surface. This requirement is not essential: We impose it to simplify the exposition. Any compact polygonal surface, say $S$, can be decomposed into a finite number of polygons, say $P_1 \cup \cdots \cup P_t$. To recover $S$ from $P_1, \ldots, P_t$, we glue them along (some of) their sides by isometries. This is the defining mapping, $f : \bigcup_{1 \leq i \leq t} P_i \rightarrow S$, of the disjoint union of comprising polygons onto the surface. As a space, $S$ is a two-dimensional manifold, in general, with a boundary. The boundary, $\partial S$ is the union of the sides of $P_1, \ldots, P_t$ that are left unglued. The euclidean structure of $P_1, \ldots, P_t$ endows $S$ with a locally euclidean metric. The metric may be singular only at the points of $S$ which come from the corners of $P_1, \ldots, P_t$. Thus, the singular set, $\Sigma \subset S$, is finite. Near a point $c \in \Sigma \cap \text{interior}(S)$ (resp. $c \in \Sigma \cap \partial S$) the surface is isometric to a euclidean cone (resp. euclidean wedge). The apex angle of this cone (resp. wedge) is the cone angle (resp. wedge angle) of the cone point (resp. wedge point). Cone (resp. wedge) angles of singular points may take any values except $2\pi$ (resp. $\pi$). Compact polygonal surfaces without singular points are either flat tori or flat cylinders.

**Example 1.** Let $P', P''$ be two copies of a simple $k$-gon $P = A_1 \ldots A_k$; let $\alpha_1, \ldots, \alpha_k$ be the respective angles. Let $S$ be the polygonal surface

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The meaning of this notation will become clear later on.
formed by gluing the corresponding sides of \(P'\) and \(P''\) reversing the orientation. This surface is the doubling of \(P\). We will denote it by \(DP\). Topologically, \(DP \sim S^2\). Its singular set consists of \(k\) cone points, with cone angles \(2\alpha_1, \ldots, 2\alpha_k\).

Let \(\text{Iso}(\mathbb{R}^2)\) (resp. \(\text{Iso}_0(\mathbb{R}^2)\)) be the group of (resp. orientation-preserving) isometries. We recall the notions of \((G, X)\)-manifolds and \((G, X)\)-orbifolds of Ehresmann and Thurston [20]. Heuristically, compact polygonal surfaces are \((\text{Iso}(\mathbb{R}^2), \mathbb{R}^2)\)-manifolds with singularities. In particular, we associate with a compact polygonal surface \(S\) its holonomy group \(\text{Hol}(S) \subset \text{Iso}(\mathbb{R}^2)\). We define \(\text{Hol}(S)\) the same way it is defined for \((\text{Iso}(\mathbb{R}^2), \mathbb{R}^2)\)-orbifolds. Namely, we choose a base point \(s_0 \in S \setminus \Sigma\), and consider regular (i.e., avoiding the singularities) loops based at \(s_0\). If \(\gamma\) is such a loop, we develop \(S\) along \(\gamma\) yielding \(h(\gamma) \in \text{Iso}(\mathbb{R}^2)\) which depends only on the regular homotopy class of \(\gamma\). The preceding construction is the holonomy homomorphism \(\text{hol} : \pi_1(S \setminus \Sigma) \to \text{Iso}(\mathbb{R}^2)\); its range is the holonomy group \(\text{Hol}(S)\).

The semi-direct product decomposition \(\text{Iso}(\mathbb{R}^2) = O(2) \times \mathbb{R}^2\) yields \(\text{Hol}(S) = \text{Hol}_r(S) \times \text{Hol}_t(S)\), where \(\text{Hol}_r(S)\) (resp. \(\text{Hol}_t(S)\)) is the rotational (resp. translational) holonomy. We say that \(S\) is a rational polygonal surface if \(|\text{Hol}_r(S)| < \infty\). If \(P\) is a polygon, \(\text{Hol}_r(P)\) is generated by linear orthogonal reflections about its sides. Thus, for polygons the present definition of “rational” agrees with the earlier one. We have \(\text{Hol}(DP) = \text{Hol}(P) \cap \text{Iso}_0(\mathbb{R}^2)\)\(^3\) Hence, the doubling surface \(DP\) in Example 1 is rational iff \(P\) is a rational polygon.

**Definition 1.** Let \(R, S\) be compact polygonal surfaces. A covering of polygonal surfaces is a continuous, surjective mapping \(\varphi : R \to S\) compatible with the respective \((\text{Iso}(\mathbb{R}^2), \mathbb{R}^2)\)-structures.

**Example 2.** Let \(S\) be any compact polygonal surface with a boundary. Gluing up the two copies \(S', S''\) along the boundaries, we obtain a closed polygonal surface \(DS\), the doubling of \(S\). When \(S\) is a polygon, we recover Example 1. The natural projection \(\varphi : DS \to S\) is a covering of polygonal surfaces.

**Remark 1.** The trick of doubling allows us to restrict our considerations, if need be, to boundaryless polygonal surfaces.

**Proposition 1.** 1. Let \(\varphi : R \to S\) be a covering of compact polygonal surfaces. It induces an inclusion \(\text{Hol}(R) \subset \text{Hol}(S)\), compatible with the injection \(\varphi_* : \pi_1(R \setminus \Sigma(R)) \to \pi_1(S \setminus \Sigma(S))\) and with the holonomy

\[\text{hol} : \pi_1(S \setminus \Sigma) \to \text{Iso}(\mathbb{R}^2)\]

\[\text{hol}_r : \pi_1(S \setminus \Sigma) \to O(2)\]

\[\text{hol}_t : \pi_1(S \setminus \Sigma) \to \mathbb{R}^2\]

This is a special case of a general statement about the holonomy of coverings, see below.
homomorphisms $\text{hol}_R : \pi_1(R \setminus \Sigma(R)) \to \text{Iso}({\mathbb{R}}^2)$, $\text{hol}_S : \pi_1(S \setminus \Sigma(S)) \to \text{Iso}({\mathbb{R}}^2)$.

2. Let $S$ be a compact polygonal surface. Let $H \subset \text{Hol}(S)$ be a subgroup of finite index. Then there is a unique covering of compact polygonal surfaces $\varphi_H : \tilde{S}_H \to S$ such that $\text{Hol}(\tilde{S}_H) = H$.

Proof. Choose regular base points $r_0 \in R, s_0 \in S$ so that $\varphi(r_0) = s_0$. Let $\alpha$ be a regular loop in $(R, r_0)$. Then $\beta = \varphi(\alpha)$ is a regular loop in $(S, s_0)$. We denote by $[\alpha], [\beta]$ their homotopy classes. Simultaneously developing $R$ along $\alpha$ and $\hat{S}$ along $\beta$, we obtain $\text{hol}_R(\alpha) = \text{hol}_S(\beta)$. Replacing loops by their homotopy classes and using that $[\beta] = \varphi_*([\alpha])$, we obtain the first claim.

For the proof of the second claim we make use of Remark 1 and assume that $S$ is a closed surface. Set $G = \text{hol}_S^{-1}(H) \subset \pi_1(S \setminus \Sigma(S))$. The index $[G : \pi_1(S \setminus \Sigma(S))]$ is equal to the index of $H$ in $\text{Hol}(S)$. By assumption, $[G : \pi_1(S \setminus \Sigma(S))] = d \in \mathbb{N}$. Let $\varphi_0 : R_0 \to S \setminus \Sigma(S)$ be the corresponding topological covering of degree $d$. The $(\text{Iso}({\mathbb{R}}^2), {\mathbb{R}}^2)$-structure pulls back from $S \setminus \Sigma(S)$ to $R_0$. The covering $\varphi_0 : R_0 \to S \setminus \Sigma(S)$ uniquely extends to a projection $\varphi : R \to S$ of their completions; the mapping $\varphi : R \to S$ is a branched covering of degree $d$. The branching locus is $\Sigma(S)$. Thus, $R$ is a compact polygonal surface. Set $\tilde{S}_H = R$. By construction, $\varphi_H : \tilde{S}_H \to S$ is a covering of polygonal surfaces. It is straightforward to check that it has the required properties.

Corollary 1. Let $\varphi : R \to S$ be a covering of compact polygonal surfaces. Then one of the surfaces is rational iff both are rational.

Proof. By Proposition 1, $\text{Hol}_r(R) \subset \text{Hol}_r(S)$. Thus, if $S$ is a rational surface, then $|\text{Hol}_r(R)| < \infty$, i.e., $R$ is a rational surface. Suppose now that $S$ is irrational, i.e., $|\text{Hol}_r(S)| = \infty$. By Proposition 1, $[\text{Hol}_r(R) : \text{Hol}_r(S)] < \infty$, implying $|\text{Hol}_r(R)| = \infty$.

A compact translation surface is a compact polygonal surface whose rotational holonomy is trivial. Equivalently, a compact translation surface is a compact polygonal surface which carries a $({\mathbb{R}}^2, {\mathbb{R}}^2)$-structure [11, 12]. Also equivalently, a compact polygonal surface $S$ is a translation surface iff $\text{Hol}(S) = \text{Hol}_r(S)$. Compact translation surfaces arise in several contexts, e.g., in complex analysis. They are instrumental in the analysis of billiard in rational polygons [19]. A covering of translation surfaces is a covering of polygonal surfaces $\varphi : R \to S$, where $R$ and $S$ are translation surfaces.

Corollary 2. Let $S$ be a compact, rational polygonal surface. Then it has a unique minimal covering $\varphi_t : R \to S$ by a compact translation
surface. The minimality means that if \( \varphi : R' \to S \) is any covering by a compact translation surface then there is a covering of translation surfaces \( \psi : R' \to R \) such that \( \varphi = \varphi_t \circ \psi \).

**Proof.** We have \([\text{Hol}_t(S) : \text{Hol}(S)] = |\text{Hol}_t(S)| < \infty\). Set \( H = \text{Hol}_t(S) \); let \( R = \tilde{S}_H \) and let \( \varphi_t : R \to S \) be the covering in Proposition 1. Then \( \text{Hol}_t(R) = \text{Hol}(R) = \text{Hol}_t(S) \), i.e., \( R \) is a translation surface. We will now prove the minimality of \( \varphi_t : R \to S \). If \( \varphi : R' \to S \) is any covering by a translation surface, then \( \text{Hol}(R') \subset \text{Hol}_t(S) = \text{Hol}(R) \). Let \( \psi : R' \to R \) be the covering constructed in Proposition 1. Then \( \psi : R' \to R \) is a covering of polygonal surfaces. By construction, \( \varphi = \varphi_t \circ \psi \).

Let \( P \) be any compact, rational polygonal surface. Let \( \varphi_t : S(P) \to P \) be the minimal covering ensured by Corollary 2. We will refer to \( S(P) \) as the (canonical) translation surface of \( P \) and to \( \varphi_t : S(P) \to P \) as the canonical translation covering. Note that the degree of \( \varphi_t : S(P) \to P \) is \(|\text{Hol}_t(P)|\). If \( P \) is a rational polygon, \( S(P) \) is often called the Katok-Zemlyakov surface. The term acknowledges the work [21] which derived nontrivial dynamical consequences from \( \varphi_t : S(P) \to P \). We point out that these coverings, for any compact rational polygonal surface, are in the literature since 1907. See the references in [7].

### 1.2. Noncompact polygons: Definitions and examples.

Our goal is to extend the above material to noncompact polygonal surfaces. As in section 1.1, the basic example of such surface is a noncompact polygon. By this we mean a closed, noncompact region, \( \Omega \subset \mathbb{R}^2 \) such that \( \partial \Omega \) consists of (maximal) line segments: The sides of \( \Omega \). They may have finite or infinite length; there may be infinitely many sides. We will introduce some requirements on our noncompact polygons.

A. Any set \( \{ |x|, |y| \leq a \} \) intersects at most a finite number of sides.

B. The side lengths are bounded away from zero.

**Remark 2.** In view of condition A, we exclude from consideration fractal polygons. We feel that the billiard on fractal polygons and fractal polygonal surfaces is a separate subject. Condition B is not as paramount as condition A; some natural noncompact polygons do not satisfy it. See Example 7 below. We point out that regions, such as the whole plane, half-planes, wedges, infinite bands, etc are noncompact polygons.

We will now discuss a few examples.
Example 3. Let \( P_1, P_2, \ldots, P_t \subset \mathbb{R}^2 \) be simple polygons, whose interiors are pairwise disjoint. Set \( \Omega = \mathbb{R}^2 \setminus \text{interior}(P_1 \cup P_2 \cup \cdots \cup P_t) \). Then \( \Omega \) is a noncompact polygon. We will refer to it as the plane with (a finite number of) polygonal obstacles.

Example 4. By an infinite band we will mean the planar region bounded by two parallel lines. The standard band \( B_0 \) is bounded by \( \{y = 0\} \) and \( \{y = 1\} \). By a standard \( a \times b \) rectangle we mean any rectangle \( R(a, b; \xi, \eta) = \{(x, y) : \xi \leq x \leq \xi + a, \eta \leq y \leq \eta + b\} \). Setting \( \xi = \eta = 0, a = b = 1 \), we obtain the standard unit rectangle \( R \). We set \( R_{(m,n)} = R + (m, n) \).

Let \( P \subset \text{interior}(R) \) be a simple polygon. The region \( \Omega = B_0 \setminus \{\cup_{k \in \mathbb{Z}} (P + (k, 0))\} \) is a noncompact polygon. This is an infinite band with a \( \mathbb{Z} \)-periodic configuration of polygonal obstacles. Figure 1 shows a special case.

Example 5. Let \( P \) be a simple polygon, as in Example 4. Set \( \Omega = \mathbb{R}^2 \setminus \{\cup_{(m,n) \in \mathbb{Z}^2} (P + (m, n))\} \). This noncompact polygon is the plane with a \( \mathbb{Z}^2 \)-periodic configuration of polygonal obstacles. The special case, when \( P \) is a rectangle, has been in the literature for some hundred years [6]. Following [6] and [14], this polygonal surface is often called the wind-tree model. Figure 2 shows a \( \mathbb{Z}^2 \)-periodic configuration of rectangular obstacles.

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**Figure 1.** An infinite band with a \( \mathbb{Z} \)-periodic configuration of rectangular obstacles.

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Example 6. We will denote by \( \rho_\theta \) the rotation of \( \mathbb{R}^2 \) by the angle \( \theta \) around any center point. Let \( P \subset \text{interior}(R) \) be a simple polygon with a distinguished point, \( o \in P \), which will be our center point. We assume that \( \rho_\theta(P) \subset \text{interior}(R) \) for any \( \theta \).

Let \( 0 < \alpha, \beta < 2\pi \). The region \( \Omega = B_0 \setminus \{\cup_{k \in \mathbb{Z}} (\rho_k \alpha(P) + (k, 0))\} \) is a noncompact polygon. If \( \alpha/\pi \) is rational (resp. irrational), \( \Omega \) is the standard band with a \( \mathbb{Z} \)-periodic (resp. \( \mathbb{Z} \)-quasi-periodic) configuration.
of polygonal obstacles. We will say that $\Omega$ is a periodic or a quasi-periodic band, for brevity. Figure 3 shows an example.

Let $\Omega = \mathbb{R}^2 \setminus \{ \cup_{(m,n) \in \mathbb{Z}^2} (\rho_{m\alpha+n\beta}(P) + (m,n)) \}$. This noncompact polygon is the plane with a $\mathbb{Z}^2$-periodic or $\mathbb{Z}^2$-quasi-periodic configuration of polygonal obstacles, depending on $\alpha, \beta$. See figure 4 for an example.

Figure 2. The euclidean plane with a $\mathbb{Z}^2$-periodic configuration of rectangular obstacles.

Figure 3. A band with a $\mathbb{Z}$-quasi-periodic (or $\mathbb{Z}$-periodic) configuration of rectangular obstacles.
Figure 4. The euclidean plane with a \( \mathbb{Z}^2 \)-quasi-periodic (or \( \mathbb{Z}^2 \)-periodic) configuration of rectangular obstacles.

Preceding examples satisfied conditions A and B. Our next example is a class of noncompact polygons satisfying A but not necessarily B.

**Example 7.** Let \( (h) = h_0, h_1, \ldots \) be an infinite sequence of (strictly) positive numbers. The noncompact polygons \( P(h) = \bigcup_{k \geq 0} R(1, h_k; k, 0) \) are *infinite stairway polygons*, or *stairways* for brevity. The work [5] studied the billiard on \( P(h) \) when the sequence \( (h) = h_0, h_1, \ldots \) is strictly decreasing. We call the polygons \( P(h) \) with \( (h) \) strictly decreasing (resp. increasing) the *descending* (resp. *ascending*) *stairways*. If the sequence \( (h) \) is not monotone, we will informally say that \( P(h) \) is an *up and down stairway*. See figures 5 and 6 for an illustration.

If \( \Omega \) is a noncompact polygon, we denote by \( \text{Hol}(\Omega) \subset \text{Iso}(\mathbb{R}^2) \) the group generated by orthogonal reflections about the lines extending the sides of \( \Omega \); it is a semi-direct product \( \text{Hol}(\Omega) = \text{Hol}_r(\Omega) \times \text{Hol}_l(\Omega) \). The polygon \( \Omega \) is rational if \( |\text{Hol}_r(\Omega)| < \infty \). Then \( \text{Hol}_r(\Omega) = D_N \); compare with section 1.1. If \( \Omega \) is irrational, the group \( \text{Hol}_r(\Omega) \) may be infinitely generated.
In Example 3, $\Omega$ is rational iff for any pair $1 \leq i, j \leq t$ the angles between the sides of $P_i$ and $P_j$ are $\pi$-rational. In Example 4, $\Omega$ is rational iff the angles of $P$ and the angles between the sides of $P$ and the
horizontal axis are $\pi$-rational. In particular, the noncompact polygons in figures 1 and 2 are rational, and $N = 2$ for both. The infinite band with rectangular obstacles in figure 3 is rational iff $\alpha \in \pi \mathbb{Q}$, iff $\Omega$ is $\mathbb{Z}$-periodic. The polygon $\Omega$ in figure 4 is rational iff $\alpha, \beta \in \pi \mathbb{Q}$, iff $\Omega$ is $\mathbb{Z}^2$-periodic. The stairway polygons in Example 7 are rational, and $N = 2$.

1.3. Noncompact polygonal surfaces: definitions and examples.

A noncompact polygonal surface is assembled from a disjoint union of a countable (at most) collection of compact and noncompact polygons. All boundary identifications are via isometries on full sides of these polygons. Let $S$ be a polygonal surface, let $\bigcup_{i \in I} P_i$ be the disjoint union of polygons comprising $S$, and let $f : \bigcup_{i \in I} P_i \to S$ be the defining mapping. We impose the following restriction.

C. For any $c \in S$ we have $|f^{-1}(c)| < \infty$.

The boundary $\partial S$ is the union of the sides of polygons $P_i, i \in I$, that are left unglued. The euclidean structure of polygons $P_i, i \in I$, endows $S$ with a locally euclidean metric; it may be singular only at the points which come from the corners of polygons $P_i, i \in I$. Let $\Sigma' \subset S$ be the $f$-range of the set of corners. Since, by condition A, the set $\Sigma'$ is discrete, the singular set $\Sigma \subset \Sigma'$ is discrete, as well. In view of condition C, near a point $c \in \Sigma' \cap \text{interior}(S)$ (resp. $c \in \Sigma' \cap \partial S$) the surface is isometric to a euclidean cone (resp. euclidean wedge). The singular set $\Sigma$ consists of those points $c \in \Sigma' \cap \text{interior}(S), c \in \Sigma' \cap \partial S$ whose cone angle (resp. wedge angle) is not $2\pi$ (resp. not $\pi$). If condition B is not satisfied, then $\Sigma$ may contain infinite sequences $a_k \neq b_k$ such that the distance $d(a_k, b_k)$ goes to zero. We finish this section with a few examples.

Example 8. Let $\Omega$ be a noncompact polygon. Its doubling $D\Omega$ is the noncompact polygonal surface without boundary obtained by identifying the respective sides of two copies of $\Omega$. This is the noncompact version of Example 1.

If $\Omega$ is the half-plane, then $D\Omega = \mathbb{R}^2$. If $\Omega$ is the wedge with wedge angle $\alpha$, then $D\Omega$ is the euclidean cone with the cone angle $2\alpha$. Let $\Omega = \mathbb{R}^2 \setminus \text{interior}(P_1 \cup P_2 \cup \cdots \cup P_t)$ be as in Example 3. The topology of the surface $D\Omega$ depends only on $t$. If $t = 1$ then $D\Omega \sim \mathbb{R} \times S^1$ is a topological cylinder. The geometry of $D\Omega$ does depend on the polygons $P_1, P_2, \ldots, P_t$. Let $t = 1, P_1 = P$, and let $\alpha_1, \ldots, \alpha_p$ be the interior angles of $P$. Then $D\Omega$ has $p$ cone points, with cone angles
4π − 2α₁, . . . , 4π − 2αₚ respectively. We leave the analysis of $DΩ$ for $t > 1$ to the reader.

Our next example is a variation on the theme of Example 8.

**Example 9.** Let $Ω$ be a noncompact polygon, and let $L$ be the collection of its sides, i.e., $∂Ω = \cup_{l \in L} l$. Let $L = G \cup O$ be a partition, with $G \neq \emptyset$. Let $Ω', Ω''$ be two copies of $Ω$, and let $∂Ω', ∂Ω''$ be the respective boundaries. For $l \in L$ let $l', l''$ be the corresponding pair of sides. Let $DΩ$ be the union $Ω' \cup Ω''$, where $l'$ and $l''$ are identified iff $l \in G$. Then $DΩ$ is a noncompact, connected polygonal surface. We have $∂(DΩ) = \∪_{l \in O}(l' \cup l'')$. In particular, $D∅Ω = DΩ$ is the only case when $∂(DΩ) = \emptyset$.

Let $Ω$ be the band in figure 1. Let $G$ (resp. $O$) be the pair of infinite segments (resp. the collection of finite segments) in $∂Ω$. Then $DΩ$ is the infinite flat cylinder with $\mathbb{Z}$-periodic collections of rectangular obstacles on its front and its back. This surface resembles a building in the shape of a tower with infinitely many floors; each floor of this tower has two identical rectangular windows, one in the front and the other in the back. See figure 7. Let $P_0, P_1, \ldots, P_t$ be simple, disjoint polygons. Let $Ω = \mathbb{R}^2 \ \text{interior}(P_0 \cup P_1 \cup \cdots \cup P_t)$ be the polygon in Example 3. Let $G$ be the collection of sides of $P_0$. Thus, $O$ consists of the sides of $P_1, \ldots, P_t$. The polygonal surface $DΩ$ is the flat cylinder $D(\mathbb{R}^2 \ \text{interior}(P_0))$ with the polygonal obstacles $P'_1 \cup P''_1 \cup \cdots \cup P'_t \cup P''_t$.

1.4. **Holonomy and coverings of noncompact polygonal surfaces.** Definitions of the holonomy group $\text{Hol}(S) \subset \text{Iso}(\mathbb{R}^2)$ and the holonomy homomorphism $\text{hol} : π_1(S \ \text{interior}(S)) \rightarrow \text{Iso}(\mathbb{R}^2)$ of section 1.1 directly extend to noncompact polygonal surfaces. As in section 1.1, we have a semidirect product $\text{Hol}(S) = \text{Hol}_r(S) \times \text{Hol}_l(S)$, where the group $\text{Hol}_r(S) \subset O(2)$ (resp. $\text{Hol}_l(S) \subset \mathbb{R}^2$) is the rotational (resp. translational) holonomy. These groups may be infinitely generated. Thus, let $P$ be a noncompact polygon with infinitely many sides. The group $\text{Hol}(P)$ (resp. $\text{Hol}_r(P)$) is generated by (resp. linear parts of) orthogonal reflections about the lines extending the sides of $P$. Obvious modifications of Example 4 and Example 5 yield polygons with infinitely generated groups $\text{Hol}_r(P)$, and hence $\text{Hol}(P)$. Since $\text{Hol}(DΩ) = \text{Hol}(P) \cap \text{Iso}_0(\mathbb{R}^2)$, their doublings are examples of boundaryless noncompact polygonal surfaces with infinitely generated holonomy groups.

The concept of covering of polygonal surfaces, as stated in Definition 1 for compact surfaces, applies to arbitrary polygonal surfaces.
The following lemma summarizes the basic properties of coverings. See Proposition 1 for a proof.

**Lemma 1.** Let \( \varphi : R \rightarrow S \) be a covering of polygonal surfaces. It induces an inclusion \( \text{Hol}(R) \subset \text{Hol}(S) \), compatible with the injection \( \varphi_* : \pi_1(R \setminus \Sigma(R)) \rightarrow \pi_1(S \setminus \Sigma(S)) \) and with the holonomy homomorphisms \( \text{hol}_R : \pi_1(R \setminus \Sigma(R)) \rightarrow \text{Iso}(\mathbb{R}^2) \), \( \text{hol}_S : \pi_1(S \setminus \Sigma(S)) \rightarrow \text{Iso}(\mathbb{R}^2) \).
Definition 2. A covering of polygonal surfaces $\varphi : R \to S$ is tame if the index $[\text{Hol}_r(R) : \text{Hol}_r(S)] < \infty$.

Proposition 2. Let $S$ be a polygonal surface. Let $H \subset \text{Hol}(S)$ be a subgroup. Let $H = H_r \times H_t$ be the decomposition corresponding to $\text{Hol}(S) = \text{Hol}_r(S) \times \text{Hol}_t(S)$.

Let $[H_r : \text{Hol}_r(S)] < \infty$. Then there is a unique covering of polygonal surfaces $\varphi_H : \tilde{S}_H \to S$ such that $\text{Hol}(\tilde{S}_H) = H$. Its degree is equal to the index of $H$ in $\text{Hol}(S)$.

Proof. In view of the doubling construction of Example 8, we restrict the discussion to surfaces without boundary. Set $G = \text{hol}^{-1}_S(H) \subset \pi_1(S \setminus \Sigma(S))$. Let $\varphi_0 : R_0 \to S \setminus \Sigma(S)$ be the unique topological covering corresponding to $G$. Pulling back by $\varphi_0$ the $(\text{Iso}(\mathbb{R}^2), \mathbb{R}^2)$-structure on $S \setminus \Sigma(S)$, we endow $R_0$ with a $(\text{Iso}(\mathbb{R}^2), \mathbb{R}^2)$-structure.

Let $R$ be the completion of $R_0$ with respect to the induced metric. The assumption $[H_r : \text{Hol}_r(S)] < \infty$ ensures that $R \setminus R_0$ consists of cone points. The covering $\varphi_0 : R_0 \to S \setminus \Sigma(S)$ uniquely extends to a branched covering $\varphi : R \to S$ whose ramification locus is contained in $R \setminus R_0$. The ramification number $r(\tilde{c})$ at a point $\tilde{c} \in R \setminus R_0$ is less than or equal to the index $[H_r : \text{Hol}_r(S)]$. Let $c = \varphi(\tilde{c})$ and let $\alpha(c), \alpha(\tilde{c})$ be the respective cone angles. Then

\[ \alpha(c) \leq \alpha(\tilde{c}) = r(\tilde{c}) \alpha(c) \leq [H_r : \text{Hol}_r(S)] \alpha(c). \]

Thus, $R$ is a polygonal surface. Set $R = \tilde{S}_H, \varphi = \varphi_H$. The equality $\text{Hol}(\tilde{S}_H) = H$ holds by construction. The reader will easily check the remaining claims.

Definition 3. A polygonal surface $S$ is rational (resp. translation surface) if $|\text{Hol}_r(S)| < \infty$ (resp. $|\text{Hol}_r(S)| = 1$).

A polygon is rational if it is a rational polygonal surface. Note that a compact polygon is rational in the sense of Definition 3 if it is rational in the sense of section 1.1. A noncompact polygon, say $P \subset \mathbb{R}^2$, is rational iff there exists $N \in \mathbb{N}$ such that all angles between the sides of $P$ belong to the set $\{k\pi/N : 0 \leq k \leq 2N-1\}$. The following is immediate from Lemma 1.

Corollary 3. Let $\varphi : R \to S$ be a tame covering of polygonal surfaces. Then one of them is rational iff the other is as well.

A (tame) covering of translation surfaces is a (tame) covering of polygonal surfaces $\varphi : R \to S$, where $R$ and $S$ are translation surfaces.

Corollary 4. A rational polygonal surface $P$ has a unique minimal tame covering $\varphi_t : S(P) \to P$ by a translation surface. The minimality
means that if $\varphi : R' \to P$ is any tame covering by a translation surface then there is a tame covering of translation surfaces $\psi : R' \to S(P)$ such that $\varphi = \varphi_t \circ \psi$.

Proof. Follow the proof of Corollary 2 and use Proposition 2.

The surface $S(P)$ in Corollary 4 is the (canonical) translation surface of $P$ and $\varphi_t : S(P) \to P$ is the canonical translation covering. Its degree is equal to $|\text{Hol}_r(P)|$.

Example 10. Let $P$ be a noncompact, rational polygonal surface with boundary, and let $DP$ be its doubling. Then $\text{Hol}_r(DP) = \text{Hol}_r(P) \cap \text{Iso}_0(\mathbb{R}^2)$ has index 2 in $\text{Hol}_r(P)$. We have $S(DP) = S(P)$. The canonical translation covering $\varphi_t : S(P) \to DP$ is the composition of the canonical translation covering $\varphi_t : S(P) \to P$ and the 2-to-1 projection $\psi : DP \to P$.

2. Geodesic flow and directional flows

Let $P$ be a polygonal surface, let $\Sigma \subset P$ be the singular set, and let $\partial P$ be the boundary. The tangent plane $T_\xi P$ is defined for $\xi \in \text{interior}(P) \setminus \Sigma$. For $\xi \in \partial P \setminus \Sigma$ we denote by $T_\xi P$ the tangent half-plane, i. e., the quotient of $\mathbb{R}^2$ by the orthogonal reflection about the side of $P$ containing $\xi$. For $\xi \in \Sigma \cap \text{interior}(P)$ (resp. $\xi \in \Sigma \cap \partial P$) we denote by $T_\xi P$ the tangent cone (resp. tangent wedge). The space $TP = \bigcup_{\xi \in P} T_\xi P$ is the tangent bundle. The euclidean norm on $T_\xi P$ defines the set $U_\xi P \subset T_\xi P$ of unit vectors; $UP = \bigcup_{\xi \in P} U_\xi P$ is the unit tangent bundle.

Geodesic curves in $P$ are straight in local coordinates. By geodesics we will mean these curves, parameterized by arclength. We will use the notation $\{\xi(t) : t \in \mathbb{R}\}$; it means that the point mass is freely moving on $P$ with the unit speed. Thus, $\dot{\xi}(t) \in U_{\xi(t)} P$ is the velocity vector. We continue geodesics through points in $\partial P \setminus \Sigma$ using the usual reflection law. However, continuation through $\Sigma$ is not defined. Because of this, not all geodesics are parameterized by $\mathbb{R}$; those that are parameterized by half-lines and finite intervals are singular geodesics. A geodesic curve of finite length is a geodesic segment. Singular geodesic segments are the saddle connections; their endpoints belong to $\Sigma$.

In view of this, the geodesic flow $G^t : UP \to UP$ is not defined for all $t \in \mathbb{R}$ on the union of all singular geodesics. Let $\xi \in P \setminus \Sigma$. The set of $v \in U_\xi P$ that yield singular geodesics is countable, thus the singular

\footnote{It can be defined through wedge points with wedge angles $\pi/n$ and cone points with cone angles $2\pi/n$. Since such points rarely occur, we do not treat them here differently from others.}
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set \((UP)_{\text{sing}} \subset UP\) has positive codimension. Although the flow \(G^t\) is defined for all \(t\) on the regular set \(UP \setminus (UP)_{\text{sing}}\), we will use the notation \(G^t : UP \to UP\). Thus, we will ignore that \(G^t(v)\) is defined for all \(t\) only if \(v \in UP \setminus (UP)_{\text{sing}}\). The reader should keep this in mind, and make obvious adjustments, whenever necessary.

The reason we can ignore the singular set is that its volume with respect to the invariant Liouville measure on \(UP\) is zero. Indeed, locally \(UP \sim U \times \mathbb{R}^2\); the Liouville measure is the product of respective Lebesgue measures. Thus, sets of smaller dimension have measure zero. We will denote by \(d\lambda_P, d\lambda_U\) the Lebesgue measures on \(P, U\) respectively. For the Liouville measure we have \(d\mu = d\lambda_P d\lambda_U\). Note that \(d\lambda_P, d\lambda_U\) and hence \(d\mu\) are determined up to a factor. If \(\text{vol}(P) < \infty\), then we can normalize so that all relevant measures have volume one.

2.1. Rational polygonal surfaces and directional geodesic flows.

If \(O \subset P\) is a coordinate neighborhood, we denote by \(UO \subset UP\) the set of vectors with base points in \(O\). The \((\text{Iso}(\mathbb{R}^2), \mathbb{R}^2)\)-structure of \(P\) induces representations \(UO = O \times U\), and hence projections \(p_O : UO \to U\). If \(\theta = p_O(\xi)\), then \(\theta\) is the “direction” of the vector \(\xi \in UP\). The maps \(p_O\) are defined up to the action of \(O(2)\) on \(U\). If \(g \in \text{Iso}(\mathbb{R}^2)\), let \(\bar{g} \in O(2)\) be its linear part. Choosing a particular \(p_O\) for some \(O \subset P\), continuing the projection along a closed path, say \(\gamma\), and returning to \(O\), we obtain the projection \(\bar{p}_O = \text{hol}(\gamma) \circ p_O\). Thus, the direction of a tangent vector is defined modulo the action of \(\text{Hol}_r(P)\) on the unit circle.

If \(P\) be rational, then the quotient space \(U/\text{Hol}_r(P)\) exists. Thus, the local projections \(p_O\) define the mapping \(p : UP \to U/\text{Hol}_r(P)\). Let \(\bar{\theta} \in U/\text{Hol}_r(P)\). Since the flow \(G^t\) commutes with the local projections, the set \(UP_{\bar{\theta}} = \{\xi \in UP : \bar{\theta}(\xi) = \bar{\theta}\}\) is \(G^t\)-invariant. Varying \(\theta \in U/\text{Hol}_r(P)\), we obtain a measurable decomposition \(UP = \bigcup_{\bar{\theta} \in U/\text{Hol}_r(P)} UP_{\bar{\theta}}\) into \(G^t\)-invariant sets. We will use the following convention. Let \(\theta \in U\) and let \(\bar{\theta} \in U/\text{Hol}_r(P)\) be its projection. We set \(UP_\theta = UP_{\bar{\theta}}\). We denote by \(G^t_\theta\) the restriction of the geodesic flow to \(UP_\theta\). The Liouville measure on \(UP\) decomposes into directional Liouville measures: \(d\mu = \int_{\theta \in U/\text{Hol}_r(P)} d\mu_\theta\). We summarize the preceding discussion as follows.

**Proposition 3.** Let \(P\) be a rational polygonal surface. Then the geodesic flow \((UP, G^t, d\mu)\) decomposes:

\[
(2) \quad (UP, G^t, d\mu) = \int_{\theta \in U/\text{Hol}_r(P)} (UP_\theta, G^t_\theta, d\mu_\theta) d\theta.
\]
The decomposition is compatible with tame coverings of polygonal surfaces.

We will call \((UP_\theta, G_{\theta}, d\mu_\theta)\) the directional (geodesic) flows on the rational polygonal surface \(P\). Let \(\theta \in U\). Then \(UP_\theta = \{v \in UP : p(v) \in \text{Hol}_r(P) \cdot \theta\}\). Let \(q : UP \to P\) be the projection. Then \(q : UP_\theta \to P\) is onto; for any point \(\xi \in P\) the fibre \(q^{-1}(\xi)\) consists of tangent vectors \(v \in U_\xi P\) whose directions belong to the finite orbit \(\text{Hol}_r(P) \cdot \theta\).

We specialize this discussion to a translation surface, say \(S\). Then \(US_\theta\) consists of unit tangent vectors \(v \in US\) with direction \(\theta\). Identifying this set with \(S\), we obtain the linear flow \(L^t_\theta\) on \(S\) in direction \(\theta\). The orbits of \(L^t_\theta\) are the geodesics \(\xi(t)\) on \(S\) with direction \(\theta\). The flows \(L^t_\theta\) preserve the lebesgue measure \(\lambda_S\).

**Corollary 5.** Let \(P\) be a rational polygonal surface. Let \(S = S(P)\) be the corresponding translation surface, and let \(\varphi_t : S \to P\) be the minimal tame covering. Let \(\theta \in U\) be a direction whose isotropy subgroup in \(\text{Hol}_r(P)\) is trivial. The projection \(\varphi_t : S \to P\) induces the isomorphism of flows \((S, L^t_\theta, \lambda_S) \simeq (UP_\theta, G_{\theta}, d\mu_\theta)\).

**Proof.** We have already proved the claim in the special case of translation surfaces. The general case follows from the canonical covering \(\varphi_t : S \to P\), by Proposition 4.

**Remark 3.** Let \(\theta \in U\) be such that its isotropy subgroup in \(\text{Hol}_r(P)\) is nontrivial. We will say that \(\theta\) is a singular direction. A rational boundaryless polygonal surface has no singular directions. A rational polygonal surface with a boundary has a finite number of them. Let \(P\) be a rational polygonal surface with a boundary; let \(\theta \in U\) be a singular direction. Then the relationship between \((S, L^t_\theta, \lambda_S)\) and \((UP_\theta, G_{\theta}, d\mu_\theta)\) is not an isomorphism, as in Corollary 5. Instead, it is a 2-to-1 covering of flows \(q : (S, L^t_\theta, \lambda_S) \to (UP_\theta, G_{\theta}, d\mu_\theta)\) This is contained in [7] for compact polygonal surfaces. The discussion in [7] extends mutatis mutandis to all rational polygonal surfaces.

### 2.2. Arithmetic polygonal surfaces.

The simplest compact boundaryless polygonal surfaces are flat tori. These are translation surfaces \(T = \mathbb{R}^2/L\), where \(L \subset \mathbb{R}^2\) is a lattice, i.e., a closed cocompact subgroup. The integer lattice \(\mathbb{Z}^2 \subset \mathbb{R}^2\) yields the standard torus \(T_0 = \mathbb{R}^2/\mathbb{Z}^2\). A compact translation surface (of positive genus) is arithmetic if it admits a translation covering \(\varphi : S \to T\) whose branching locus is a point. Modifying the translation structure by \(\text{GL}_+(2, \mathbb{R})\), if need be, we can assume that our translation covering is
\( \varphi : S \rightarrow T_0 \) and the branching locus is \( \{0\}/\mathbb{Z}^2 \). Representing \( T_0 \) by the unit square, we obtain a unit square tiling of \( S \). This explains the term \textit{square tiled} for arithmetic translation surfaces. Another popular name for these surfaces is \textit{origami} \cite{18}. There are several characterizations of (compact) arithmetic translation surfaces \cite{7,11,12}. A \textit{compact, rational polygonal surface} \( P \) is \textit{arithmetic} if its translation surface \( S(P) \) is arithmetic.\footnote{This definition is already in the literature, if only implicitly.} We will extend these notions to noncompact polygonal surfaces.

**Definition 4.** Let \( P \) be a rational polygonal surface. Then \( P \) is \textit{arithmetic} if its translation surface \( S(P) \) admits a tame translation covering of a flat torus whose branching locus is (at most) a point.

Since we view polygons as polygonal surfaces, Definition 4 applies to them as well. We will give a few examples of noncompact arithmetic polygons.

**Example 11.** 1. We will use the notation of Example 4. Let \( \Omega = B_0 \setminus \{ \cup_{k \in \mathbb{Z}}(R(a,b;\xi,\eta) + (k,0)) \} \) be the standard infinite band with a periodic sequence of rectangular obstacles. See figure 1. Then \( \Omega \) is arithmetic iff \( \eta,a,b \in \mathbb{Q} \). 2. We will use the notation of Example 5. Let \( \Omega = \mathbb{R}^2 \setminus \{ \cup_{(m,n) \in \mathbb{Z}^2} (R(a,b;\xi,\eta) + (m,n)) \} \) be the plane with a \( \mathbb{Z}^2 \)-periodic configuration of rectangular obstacles. Then \( \Omega \) is arithmetic iff \( a,b \in \mathbb{Q} \).

**Example 12.** Let \( P \subset \mathbb{R}^2 \) be a polygon. We say that \( P \) is \textit{drawn on the integer lattice} if there is a set \( I \subset \mathbb{Z}^2 \) such that \( P = \cup_{(m,n) \in I} R(m,n) \). In particular, \( P \) is an arithmetic polygon; it is compact iff \( |I| < \infty \). Each side of \( P \) is a horizontal or a vertical segment with integer endpoints. Gluing some of the sides by integer translations, we obtain an arithmetic polygonal surface. If no sides of \( P \) remain unglued, we obtain an arithmetic translation surface.

We may call these polygonal surfaces \textit{square tiled}. Any arithmetic translation surface is \( \text{GL}(2,\mathbb{R}) \)-equivalent to a square tiled surface. A square tiled polygon \( P \) is naturally partitioned by horizontal rows (resp. vertical columns). A row (resp. column) is the union of a maximal set of horizontally (resp. vertically) adjacent squares. This suggests a particular translation surface \( S \) represented by \( P \). To obtain \( S \), we glue the left (resp. lower) side of every row (resp. column) with its right (resp. upper) side. Following \cite{18}, we call \( S \) the \textit{origami translation surface} corresponding to \( P \).

Figure 8 shows a particular noncompact origami surface \( S \). Note that each row and each column of \( S \) consists of 3 unit squares. It
is invariant under the translation by (1, 1) and has the shape of an infinite stairway going from South-East to North-West. Recurrence of geodesics on such stairways is studied in [15]. See also section 3.

![Figure 8. A noncompact origami translation surface.](image)

Directional flows $G^t_\theta$ on a compact, arithmetic polygonal surface $P$ satisfy a dichotomy, depending on whether $\theta$ is a rational direction or an irrational direction (for $P$). In order to define this notion, we assume first that $S(P)$ is square tiled. We use the cartesian coordinates to identify the circle $U$ of directions with $\mathbb{R}/2\pi\mathbb{Z}$; the slope of a direction is $\tan \theta$. Then $\theta$ is rational (resp. irrational) if $\tan \theta \in \mathbb{Q} \cup \{\infty\}$ (resp. $\tan \theta \notin \mathbb{Q} \cup \{\infty\}$). Let now $P$ be any compact, arithmetic polygonal surface, and let $S = S(P)$. Let $g \in \text{GL}_+(2, \mathbb{R})$ be such that $S_1 = g \cdot S$ is square tiled. A direction $\theta \in U$ is rational for $P$ if $g \cdot \theta$ has a rational slope. The set of slopes of rational directions for $P$ does not depend on the choices involved. It has the form $g^{-1} \cdot (\mathbb{Q} \cup \{\infty\})$, where $g^{-1} \in \text{GL}_+(2, \mathbb{R})$ is a fractional linear transformation, hence countable.
Theorem 1. (See [7].) Let $P$ be a compact, arithmetic polygonal surface. If $\theta$ is irrational, then $G^t_\theta$ is uniquely ergodic. If $\theta$ is rational, then every geodesic in direction $\theta$ is either periodic or a saddle connection.

Note that the above definition of rational and irrational directions applies to noncompact, arithmetic polygonal surfaces as well. We conjecture that the dichotomy of Theorem 1 extends to arbitrary arithmetic polygonal surfaces. We point out that the noncompact version of Theorem 1 should take into account the possibility of transient geodesics. We discuss this in the next section.

3. Conservativeness and dissipation for noncompact polygonal surfaces

First, we will recall basic notions pertaining to recurrence and transience in dynamical systems [11, 17]. For simplicity of exposition, we gear the discussion to the dynamics with time $\mathbb{Z}$. We will use the shorthand $\nu$-a.e. to mean almost every with respect to (the measure) $\nu$. The dynamical system $(X, \tau, \nu)$ is conservative if for every measurable set $B \subset X$ and for $\nu$-a.e. point $x \in B$ there is $n = n(x) > 0$ such that $\tau^n x \in B$. It is dissipative if there is a measurable subset $A \subset X$ such that $X = \bigcup_{k \in \mathbb{Z}} \tau^k A$ and the sets $\tau^k A \cap \tau^l A = \emptyset$ for $k \neq l$. Every dynamical system uniquely decomposes as a disjoint union of its conservative part and the dissipative part $X = X_{\text{cnsv}} \cup X_{\text{dspt}}$. Points $x \in X_{\text{cnsv}}$ (resp. $x \in X_{\text{dspt}}$) are the recurrent (resp. transient) points in $X$. If $\nu(X) < \infty$, then, by the poincare recurrence theorem, $\nu$-a.e. point is recurrent; equivalently, $X = X_{\text{cnsv}}$. If $\nu(X) = \infty$, which is our case, the dissipative part may be nontrivial. Note that $\nu(X_{\text{dspt}}) > 0$ iff $\nu(X_{\text{dspt}}) = \infty$.

It is straightforward to reformulate the above notions for the dynamics with time $\mathbb{R}$, i.e., for measure preserving flows. We will study the recurrence of geodesic flows on certain noncompact polygonal surfaces. The following observation will allow us to replace the dynamics with time $\mathbb{R}$ by the dynamics with time $\mathbb{Z}$.

Proposition 4. Let $(Y, T^t, \mu)$ be a flow, let $X \subset Y$ be a cross-section, and let $(X, \tau, \nu)$ be the induced return transformation. Then the time $\mathbb{R}$ dynamical system $(Y, T^t, \mu)$ is conservative iff the time $\mathbb{Z}$ dynamical system $(X, \tau, \nu)$ is conservative.

3.1. Geodesic flows on $\mathbb{Z}$-periodic polygonal surfaces.
We begin by defining the notion.
Definition 5. Let $\tilde{P}$ be a noncompact polygonal surface, and let $\Gamma$ be a countably infinite group acting by isometries on $\tilde{P}$. Suppose that $\Gamma$ acts freely and cocompactly. Then $\tilde{P}$ is a $\Gamma$-periodic polygonal surface.

In our examples, $\Gamma = \mathbb{Z}$ or $\Gamma = \mathbb{Z}^2$. For instance, the surfaces in Example 4 and Figure 7 are $\mathbb{Z}$-periodic; the surface in Example 5 is $\mathbb{Z}^2$-periodic. If $\Gamma$ is not specified, we will speak of periodic polygonal surfaces.

Let $\tilde{P}$ be a $\Gamma$-periodic polygonal surface. Then $P = \tilde{P}/\Gamma$ is a compact polygonal surface; let $p : \tilde{P} \to P$ be the projection. Let $U\tilde{P}, UP$ be the unit tangent bundles for $\tilde{P}, P$; let $\tilde{G}^t, G^t$ be the respective geodesic flows; let $\tilde{\mu}, \mu$ be the liouville measures for $U\tilde{P}, UP$ respectively. The action of $\Gamma$ on $\tilde{P}$ uniquely extends to a free, cocompact action on $U\tilde{P}$.

We have $UP = U\tilde{P}/\Gamma$; let $q : U\tilde{P} \to UP$ be the projection. The polygonal surfaces $\tilde{P}$ and $P$ are rational or not rational simultaneously. Suppose they are rational; let $(U\tilde{P}_\theta, \tilde{G}^t_\theta, \tilde{\mu}_\theta)$ and $(UP_\theta, G^t_\theta, \mu_\theta)$ be the respective directional geodesic flows. By Proposition 3, the projection $q : U\tilde{P} \to UP$ is compatible with the decomposition equation (2), yielding the directional projections $q_\theta : (U\tilde{P}_\theta, \tilde{G}^t_\theta, \tilde{\mu}_\theta) \to (UP_\theta, G^t_\theta, \mu_\theta)$. Note that $q$ and all $q_\theta$ are coverings of flows.

The geodesic flow $(U\tilde{P}, \tilde{G}^t, \tilde{\mu})$ (resp. directional flow $(U\tilde{P}_\theta, \tilde{G}^t_\theta, \tilde{\mu}_\theta)$) is a skew product over the flow $(UP, G^t, \mu)$ (resp. $(UP_\theta, G^t_\theta, \mu_\theta)$) with the fibre $\Gamma$. To simplify the exposition, we recall the concept of skew products for time $\mathbb{Z}$ dynamical systems. The time $\mathbb{R}$ case is similar. See [3] for more information.

Let $(X, \tau, \nu)$ be a measure preserving automorphism. Let $\Gamma$ be a countably infinite abelian group, and let $\nu_\Gamma$ be the counting measure. Let $\varphi : X \to \Gamma$ be a measurable mapping. Set $\tilde{X} = X \times \Gamma$, $\tilde{\nu} = \nu \times \nu_\Gamma$, and

$$ (3) \quad \tilde{\tau}(x, g) = (\tau x, g + \varphi(x)). $$

The dynamical system $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ is the skew product over $(X, \tau, \nu)$ with the fibre $\Gamma$ and the displacement function $\varphi$.

Let $(\tilde{Y}, \tilde{T}^t, \tilde{\mu})$ be a measure preserving flow which is a skew product with fibre $\Gamma$ over the flow $(Y, T^t, \mu)$. Let $q : (\tilde{Y}, \tilde{T}^t, \tilde{\mu}) \to (Y, T^t, \mu)$ be the projection. Let $X \subset Y$ be a cross-section for $(Y, T^t, \mu)$. Let $\nu$ be the induced measure on $X$, and let $(X, \tau, \nu)$ be the poincare map. Set $\tilde{X} = q^{-1}(X) \subset \tilde{Y}$. Then $\tilde{X}$ is a cross-section for $(\tilde{Y}, \tilde{T}^t, \tilde{\mu})$. Let $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$.

\footnote{The notion of skew products makes sense for any locally compact group. We restrict the exposition to this class of groups because in our applications, $\Gamma = \mathbb{Z}$ or $\Gamma = \mathbb{Z}^2$.}
be the corresponding poincare map. Then \((\tilde{Y}, \tilde{\mu}) = (Y \times \Gamma, \mu \times \nu \tau)\) as measure spaces. This induces the isomorphism \((\tilde{X}, \tilde{\nu}) = (X \times \Gamma, \nu \times \nu \tau)\) and determines a mapping \(\varphi: X \to \Gamma\) such that \(\tilde{\tau}: \tilde{X} \to \tilde{X}\) is given by equation (3). Thus, \((\tilde{X}, \tilde{\tau}, \tilde{\nu})\) is the skew product over \((X, \tau, \nu)\); the displacement function \(\varphi: X \to \Gamma\) is uniquely determined by \(q: (\tilde{Y}, \tilde{T}^\mu, \tilde{\mu}) \to (Y, T^\mu, \mu)\) and the choice of cross-section \(X \subset Y\). The following example illustrates this material in the context of periodic polygonal surfaces.

**Example 13.** We will use two subgroups \(Z \subset \mathbb{R}^2\): The **horizontal** \(Z_h = \{(k, 0)\}\) and the **vertical** \(Z_v = \{(0, k)\}\). The quotient \(\tilde{S} = \mathbb{R}^2/Z_h\) is a flat vertical cylinder; \(Z_v\) acts on \(\tilde{S}\) by vertical translations. The action is free and cocompact. The quotient \(S = \tilde{S}/Z_v\) is the flat unit torus. We view \(S\) as the unit square \(R = \{(x, y) : 0 \leq x, y \leq 1\}\) with its opposite sides glued in the usual way. Let \(US, US\) be the unit tangent bundles of \(\tilde{S}, S\); let \((US, \tilde{G}^t, \tilde{\mu})\) and \((US, G^t, \mu)\) be the respective geodesic flows. Identifying the **circle of directions** with \([0, 2\pi)\), we have \(US = \{(x, y, \theta) : (x, y) \in R, \theta \in [0, 2\pi)\}\). The set \(X = \{(x, 0, \theta)\}\) is a cross-section for \((US, G^t, \mu)\). With the notation \((x, \theta) \in X\), the poincare map is given by \(\tau(x, \theta) = ((x + \cot \theta) \mod 1, \theta)\). Let \(p: \tilde{S} \to S\) and \(q: US \to US\) be the projections. We identify \(US\) with \(\{(x, y, \theta) : (x, \theta) \in X, y \in \mathbb{R}\}\). Then \(X = q^{-1}(X) = \{(x, k, \theta) : (x, \theta) \in X, k \in \mathbb{Z}\}\). Thus, \(\tilde{X} = X \times Z\) up to a set of measure zero. Set \(\varphi(x, \theta) = 1\) if \(0 < \theta < \pi\), \(\varphi(x, \theta) = -1\) if \(\pi < \theta < 2\pi\). The poincare map for the cross-section \(\tilde{X}\) is given by \(\tilde{\tau}((x, \theta), k) = (\tau(x, \theta), k + \varphi(x, \theta))\). Thus, \(\varphi: X \to \mathbb{Z}\) is the displacement function.

In this example \(\tilde{S}, S\) are translation surfaces; \(\theta\) corresponds to the direction of a tangent vector. Fixing \(\theta\), we obtain the subset \(X_\theta \subset X\) which is a cross-section for the directional flow \((US_\theta, G^t_\theta, \mu_\theta)\). We have \(X_\theta = \mathbb{R}/Z\). The directional poincare map is given by \(\tau_\theta \cdot x = (x + \cot \theta) \mod 1\). Set \(q^{-1}(X_\theta) = \tilde{X}_\theta = (\mathbb{R}/Z) \times \mathbb{Z}\). The directional poincare map for \(\tilde{S}\) is \(\tilde{\tau}_\theta \cdot (x, k) = ((x + \cot \theta) \mod 1, k + \varphi_\theta(x))\). The **directional displacement functions** \(\varphi_\theta: \mathbb{R}/Z \to \mathbb{Z}\) are \(\varphi_\theta = 1\) for \(0 < \theta < \pi\), and \(\varphi_\theta = -1\) for \(\pi < \theta < 2\pi\).

Example 13 shows the importance of choice of cross-sections for (directional) geodesic flows. For polygonal surfaces with boundaries, there is a canonical choice: The set of tangent vectors with base points on the boundary. We will now give the relevant definitions.

Let \(P\) be an arbitrary polygonal surface. Let \(A \subset P\) be a closed subset. We denote by \(U_A P \subset UP\) the set of tangent vectors \(v \in UP\) whose base points belong to \(A\). Suppose that \(P\) has a boundary, \(\partial P\).
Let $BUP \subset UP$ be the smallest $G'$-invariant set containing $U_{\partial P}P$. Let $\mu_B = \mu|_{BUP}$. By definition, $U_{\partial P}P$ is a cross-section for the flow $(BUP, G', \mu_B)$. Let $\nu$ be the induced measure on $U_{\partial P}P$. We will use the following terminology: The flow $(BUP, G', \mu_B)$ is the billiard flow for the polygonal surface $P$; the set $U_{\partial P}P \subset BUP$ is the standard cross-section for the billiard flow; the induced transformation $(U_{\partial P}P, \tau, \nu)$ is the billiard map for $P$.

For $N \in \mathbb{N}$ denote by $D_N \subset O(2)$ the group generated by orthogonal reflections about two axes forming the angle $\pi/N$. It is the dihedral group of order $2N$. Suppose now that $P$ is a rational polygonal surface. Then $\text{Hol}_r(P) = D_N$ where $N = N(P)$. If $P$ is a rational polygon, then $N$ is the least common denominator of its angles. We identify $U/D_N$ and $[0, \pi/N]$. For $\theta \in [0, \pi/N]$ set $U_{\partial P}P_{\theta} = U_{\partial P}P \cap U_{\partial P}P$, $BUP_{\theta} = BUP \cap U_{\partial P}P$.

**Definition 6.** Let $P$ be a rational polygonal surface with a boundary. A direction $\theta \in [0, \pi/N]$ is transversal to the boundary of $P$ if $\mu_{\theta}(BUP_{\theta}) > 0$.

If $\theta$ is transversal to $\partial P$, we denote by $b\mu_{\theta}$ the restriction of $\mu_{\theta}$ to $BUP_{\theta}$; then $(BUP_{\theta}, G'_{\theta}, b\mu_{\theta})$ is the billiard flow in direction $\theta$. The set $U_{\partial P}P_{\theta} \subset BUP_{\theta}$ is the standard directional cross-section. Let $\nu_{\theta}$ be the induced measure on $U_{\partial P}P_{\theta}$. The induced dynamical system $(U_{\partial P}P_{\theta}, \tau_{\theta}, \nu_{\theta})$ is the directional billiard map. When $P \subset \mathbb{R}^2$ is a compact rational polygon, this is the standard terminology [10]. When the surface $P$ is clear from the context, we will simply speak of transversal directions. We will now illustrate the above material with examples.

Let $S$ be a translation surface. Let $O \subset S$ be a polygon, in general disconnected, such that $S \setminus O$ is connected. The polygonal surface $P = S \setminus \text{interior}(O)$ is a translation surface with polygonal obstacles. A barrier $L \subset S$ is a connected polygon without interior, e. g., a linear segment. Making the billiard ball bounce off of each side of $L$, we obtain the polygonal surface which is a translation surface with a barrier. We will denote it by $P = S \setminus L$. Combining the two notions, we come to the concept of translation surfaces with polygonal obstacles and/or barriers. Note that these are polygonal surfaces with boundaries.

**Example 14.** Let $S = \mathbb{R}^2/\mathbb{Z}^2$ be the flat unit torus. A linear barrier $L \subset S$ is determined by one of its end points, say $A \in S$, its direction $\eta$, and its length $l$. We assume without loss of generality, that $A = (0, 0)$ and that $0 \leq \eta < \pi/2$. The other endpoint of the barrier is $B = (l \cos \eta \mod 1, l \sin \eta \mod 1)$. We require that $B \neq A$; otherwise $P = S \setminus L$ is disconnected. If $\eta$ is $\pi$-rational, this yields an upper bound on $l$; otherwise $l$ is arbitrary.
As in Example 13, we represent the flat unit torus by the unit square \( R \). Let \( l < \frac{1}{\cos \eta} \). Then \( P = S \setminus L \) is represented by \( R \) with the linear segment \([A = (0, 0), B = (l \cos \eta, l \sin \eta)]\). Figure 9 shows a billiard orbit in \( P \). Denote by \( P(l, \eta) \) the above surface. We will now illustrate Definition 6. The group \( \text{Hol}_r(P) \) is generated by a single reflection, thus \( N = 1 \). Let \( 0 \leq \theta \leq \pi \). The set \( BUP_\theta \) is formed by the billiard orbits crossing the barrier in direction \( \theta \). Up to a universal normalizing factor, \( \mu_\theta(BUP_\theta) = l|\sin(\theta - \eta)| \). Thus, the only direction which is not transversal to the boundary of \( P(l, \eta) \) is \( \eta \).

![Figure 9. A billiard orbit in the standard torus with a linear barrier.](image)

Lemma 2. Let \( P \) be a compact, rational polygonal surface with a boundary. Then all but a finite number of directions are transversal.

Proof. Let \( \eta \) be the direction of a side in \( \partial P \). The argument in Example 14 shows that every direction \( \theta \neq \eta, \eta + \pi \) is transversal. \( \square \)

Remark 4. In fact, a stronger statement holds. But for a very special class of surfaces, every direction is transversal to \( \partial P \). For \( P \) in that class, all but two directions are transversal. We will not use these facts in what follows.
Theorem 2. Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface such that the quotient $P = \tilde{P}/\mathbb{Z}$ is a translation surface with obstacles and/or barriers. Suppose that $P$ is a rational polygonal surface and that $N = N(P)$ is even. Then for almost every $\theta \in [0, \pi/N]$ the directional flow $(U\tilde{P}_\theta, \tilde{G}^t_{\theta}, \tilde{\mu}_{\theta})$ is recurrent.

Proof. Let $\theta$ be an ergodic direction for $P$ which is transversal to $\partial P$. Then $(U_{P\theta}, G^t_{\theta}, \mu_{\theta}) = (BUP_{\theta}, G^t_{\theta}, b\mu_{\theta})$. In other words, $\partial P$ provides a cross-section, say $X = X(\theta)$, for the directional flow $(U_{P\theta}, G^t_{\theta}, \mu_{\theta})$. Let $(X, \tau_{\theta}, \nu_{\theta})$ be the poincare map. Let $\tilde{X}$ be the corresponding cross-section for the directional flow $(U\tilde{P}_\theta, \tilde{G}^t_{\theta}, \tilde{\mu}_{\theta})$ on the noncompact surface. Let $\varphi_{\theta}$ be the directional displacement function. Since $N$ is even, we have satisfies

$$\int_X \varphi_{\theta} d\nu_{\theta} = 0.$$  

See Lemma 5 in [4]. By [2], the poicare map $(\tilde{X}, \tilde{\tau}_{\theta}, \tilde{\nu})$ is recurrent. By Proposition 4, the flow $(U\tilde{P}_\theta, \tilde{G}^t_{\theta}, \tilde{\mu}_{\theta})$ is recurrent as well.

By [16], the set of ergodic directions for a compact, rational polygonal surface has full measure. In view of Lemma 2, almost every direction is ergodic and transversal. 

Corollary 6. Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface satisfying the assumptions of Theorem 2. Then the geodesic flow on $\tilde{P}$ is conservative.

Proof. Immediate from Theorem 2 and equation (2). 

Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface. Let $P = \tilde{P}/\mathbb{Z}$ be the quotient. Suppose that i) $P$ is a rational polygonal surface; ii) $P$ is a translation surface with a boundary and/or obstacles; and iii) $N(P)$ is even. Then, by Corollary 6, the geodesic flow on $\tilde{P}$ is conservative. In view of Example 13 condition i) alone does not ensure the conservativeness of the geodesic flow on $\tilde{P}$. The quotient surface in Example 13 is rational but has no boundary. What happens if the quotient surface has a boundary but not rational? We will partially answer this question.

Compact euclidean polygons form a topological space. Fixing the type of a polygon, we obtain closed subspaces in this space. Thus, we can ask questions about the billiard in topologically typical polygons (of a particular type). See [21, 16, 13].

We fix a compact translation surface $S_0$. Let $O \subset S_0$ be a collection of polygons/barriers of a particular type such that $P = S_0 \setminus O$ is

\[ \text{centered.} \]
connected. For instance, $O$ can be an arbitrary triangle, or a quadrilateral, or a right triangle, or a linear segment, or a disjoint union of two triangles, etc. We denote the type of $O$ by $\kappa$. Let $\mathcal{P} = \mathcal{P}(S_0, \kappa)$ be the space of polygonal surfaces $P = S_0 \setminus O$, endowed with the natural topology. The space $\mathcal{P}$ is homeomorphic to a bounded domain in some euclidean space. Let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(S_0, \kappa)$ be the space of $\mathbb{Z}$-periodic polygonal surfaces $\tilde{P}$ such that $\tilde{P}/\mathbb{Z} \in \mathcal{P}(S_0, \kappa)$. The topology of $\mathcal{P}(S_0, \kappa)$ induces a topology on $\tilde{\mathcal{P}}(S_0, \kappa)$.

Let $\mathcal{P}_1 \subset \mathcal{P}$ be the subset of rational polygonal surfaces $P = S_0 \setminus O$ such that the number $N(P)$ is even.

**Definition 7.** We say that $\kappa$ is an amenable type if $\mathcal{P}_1$ is dense in $\mathcal{P}$.

Saying that a noncompact polygonal surface $\tilde{P}$ is conservative, we will mean that the geodesic flow $(U\tilde{P}, \tilde{G}^t, \tilde{\mu})$ is recurrent.

**Theorem 3.** Let $\kappa$ be a type of polygons/barriers in $S_0$. Let $\mathcal{P} = \mathcal{P}(S_0, \kappa)$ be the topological space of polygonal surfaces $P = S_0 \setminus O$, $O \in \kappa$. Let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(S_0, \kappa)$ be the corresponding space of $\mathbb{Z}$-periodic polygonal surfaces.

If $\kappa$ is an amenable type then the set of conservative surfaces in $\tilde{\mathcal{P}}$ is a dense $G_\delta$.

**Proof.** Let $\kappa$ be arbitrary. It is standard to check that the set $\mathcal{P}_{\text{cnsv}}(S_0, \kappa)$ of conservative surfaces is a countable intersection of open subsets in $\mathcal{P}(S_0, \kappa)$, i. e., it is a $G_\delta$ set. Since $\kappa$ is amenable, $\mathcal{P}_{\text{cnsv}}(S_0, \kappa)$ contains a dense subset of surfaces $\tilde{P}$ such that $P = \tilde{P}/\mathbb{Z}$ is rational and $N(P)$ is even. The claim now follows from Corollary 6.

Let $P_0$ be a particular compact polygonal surface. As always, we assume that $P_0$ is connected. In our applications, $P_0$ will be an “elementary” polygonal surface. Choose a type $\kappa$ of obstacles/barriers $O \subset P_0$. Analogously to the preceding discussion, we define the space $\mathcal{P} = \mathcal{P}(P_0, \kappa)$ of polygonal surfaces $P = P_0 \setminus O$, $O \in \kappa$, and endow it with the natural topology. Let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(P_0, \kappa)$ be the space of $\mathbb{Z}$-periodic polygonal surfaces $\tilde{P}$ such that $\tilde{P}/\mathbb{Z} \in \mathcal{P}(P_0, \kappa)$. The topology of $\mathcal{P}(P_0, \kappa)$ induces a topology on $\tilde{\mathcal{P}}(P_0, \kappa)$.

By analogy with Definition 7, we say that the pair $(P_0, \kappa)$ is amenable if i) $P_0$ is a rational polygonal surface; ii) $\mathcal{P}(P_0, \kappa)$ contains a dense subspace of rational polygonal surfaces $P = P_0 \setminus O$ such that $N(P_0 \setminus O)$ is even.

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8Typical examples: i) $P_0$ is the standard torus; ii) $P_0$ is the standard cylinder.
**Theorem 4.** Let \((P_0, \kappa)\) be an amenable pair. Then the set of conservative \(\mathbb{Z}\)-periodic polygonal surfaces \(\tilde{P} \in \mathbb{P}(P_0, \kappa)\) is a dense \(G_\delta\).

**Proof.** Let \(S_0 = S(P_0)\) be the minimal translation covering of \(P_0\); let \(\varphi_0 : S_0 \to P_0\) be the canonical projection. See Corollary 2. Let \(\Gamma \subset O(2)\) be the corresponding subgroup, so that \(P_0 = S_0/\Gamma\). Then \(\kappa\) defines a \(\Gamma\)-invariant type \(\lambda\) of obstacles/barriers \(O\) in \(S_0\). The action of \(\Gamma\) on \(S_0\) uniquely extends to the actions of \(\Gamma\) on associated spaces. In particular, \(\Gamma\) acts on the space \(\mathbb{P}(S_0, \lambda)\). Pulling back by \(\varphi_0\) yields an isomorphism of \(\mathbb{P}(P_0, \kappa)\) and \(\mathbb{P}(S_0, \lambda)^\Gamma\). Now the proof of Theorem 3 applies and yields the claim. 

We will now apply Theorem 3 and Theorem 4 to particular families of \(\mathbb{Z}\)-periodic polygonal surfaces.

**3.2. Examples and concluding remarks.**

Let \(0 \leq a, b \leq 1\), let \(0 \leq \theta < 2\pi\). Let \(R\) be the standard unit square, \(R = \{(x, y) : 0 \leq x, y \leq 1\}\). Let \(l \geq 0\) be such that \((a + l \cos \theta, b + l \sin \theta) \in R\). We denote by \(O(a, b, l, \theta) \subset R\) the linear segment with endpoints \((a, b)\) and \((a + l \cos \theta, b + l \sin \theta)\).

Let \(C_0\) be the polygonal surface obtained by identifying the vertical sides of \(R\) by the parallel translation \((\xi, \eta) \mapsto (\xi + 1, \eta)\). It is a compact flat cylinder; we will refer to \(C_0\) as the standard cylinder. The surface \(C_0(a, b, l, \theta) = C_0 \setminus O(a, b, l, \theta)\) is the standard cylinder with a linear barrier. Let \(\mathcal{O} \subset \mathbb{R}^4\) be the bounded domain formed by \((a, b, l, \theta)\) satisfying the above conditions. We denote by \(\mathcal{O}(a, b) \subset \mathcal{O}\) the subset obtained by fixing the values of the first two variables. The sets \(\mathcal{O}(l), \mathcal{O}(a, b, l)\), etc are similarly defined.

Let \(B_0\) be the standard band. See Example 4. Set \(B_0(a, b, l, \theta) = B_0 \setminus \bigcup_{n \in \mathbb{Z}} (O(a, b, l, \theta) + (n, 0))\) be the standard band with a \(\mathbb{Z}\)-periodic collection of linear barriers. See figure 10. Then \(B_0(a, b, l, \theta)\) is a \(\mathbb{Z}\)-periodic polygonal surface, and \(B_0(a, b, l, \theta)/\mathbb{Z} = C_0(a, b, l, \theta)\).

**Corollary 7.** 1. The set of parameters in \(\mathcal{O}\) such that the noncompact polygonal surface \(B_0(a, b, l, \theta)\) is conservative is a dense \(G_\delta\). 2. Let \((a, b) \in R\) be arbitrary. Let \(\mathcal{O}(a, b) \subset \mathcal{O}\) be the corresponding subset. Then set of parameters \((l, \theta)\) such that the noncompact polygonal surface \(B_0(a, b, l, \theta)\) is conservative is a dense \(G_\delta\) in \(\mathcal{O}(a, b)\). 3. Let \(l\) be such that the set \(\mathcal{O}(l) \subset \mathcal{O}\) has nonempty interior. Then set of parameters \((a, b, \theta)\) such that the noncompact polygonal surface \(B_0(a, b, l, \theta)\) is conservative is a dense \(G_\delta\) in \(\mathcal{O}(l)\). 4. Let \((a, b, l)\) be such that the set \(\mathcal{O}(a, b, l) \subset \mathcal{O}\) has nonempty interior. Then set of directions \(\theta\) such that the noncompact polygonal surface \(B_0(a, b, l, \theta)\) is conservative is a dense \(G_\delta\) in \(\mathcal{O}(a, b, l)\).
Proof. Since all claims fit into the framework of Theorem \[\text{4}\], we only need to prove that the relevant types \((C_0, \kappa)\) are amenable. In each case the angle \(\theta\) is free to vary in certain intervals. 9 The group in question is generated by the reflection about the horizontal axis and about the \(\theta\)-axis. It is finite iff \(\frac{\theta}{\pi} = \frac{m}{n}\), and then \(N(P) = n\). All claims now follow from the observation that rational numbers with even denominators are dense in any interval.

Let \(\tilde{C}_0\) be the translation surface obtained by gluing the boundary components of \(B_0\) via the parallel translation \((x, 0) \mapsto (x, 1)\). The infinite flat cylinder \(\tilde{C}_0\) satisfies \(\tilde{C}_0 = \mathbb{R}^2/\mathbb{Z}_v\). See Example \[\text{13}\]. Let \(T_0 = \mathbb{R}^2/\mathbb{Z}^2\) be the standard torus. Let \(0 \leq \theta_1, \theta_2 < 2\pi\) be arbitrary angles. Let \(a_i, b_i, l_i, i = 1, 2\) be parameters satisfying the conditions described prior to Corollary \[\text{7}\]. Denote by \(O_i = O(a_i, b_i, l_i, \theta_i) \subset R, i = 1, 2\) the corresponding linear segments. The compact polygonal surface \(P = P(a_i, b_i, l_i, \theta_i : i = 1, 2) = T_0 \setminus (O_1 \cup O_2)\) is the torus with two linear barriers. If \(\theta_2 - \theta_1 \neq k\pi\), the barriers are transversal. Otherwise, they may overlap, forming a single barrier. Denote by \(\tilde{P} = \tilde{P}(a_i, b_i, l_i, \theta_i : i = 1, 2) = C_0 \setminus \left((\mathbb{Z} \cdot O_1) \cup (\mathbb{Z} \cdot O_2)\right)\) the \(\mathbb{Z}\)-periodic covering surface. Then \(\tilde{P}\) is the infinite cylinder with two \(\mathbb{Z}\)-periodic collections of linear barriers. See figure \[\text{11}\].

Let \(O \subset \mathbb{R}^4\) be the bounded domain defined prior to Corollary \[\text{7}\]. For \(r_i \in O, i = 1, 2\) let \(\tilde{P}(r_1, r_2)\) be the corresponding infinite cylinder with two \(\mathbb{Z}\)-periodic collections of linear barriers. This is a family of \(\mathbb{Z}\)-periodic polygonal surfaces parameterized by points in \(O \times O \subset \mathbb{R}^8\).

9Note that the intervals may depend on relevant parameters.
Figure 11. Infinite cylinder with two $\mathbb{Z}$-periodic collections of linear barriers.

**Corollary 8.** The set of parameters $(r_1, r_2) \in \mathcal{O} \times \mathcal{O}$ such that the noncompact polygonal surface $\tilde{P}(r_1, r_2)$ is conservative is a dense $G_\delta$.

**Proof.** Set $P(r_1, r_2) = \tilde{P}(r_1, r_2)/\mathbb{Z} = T_0 \setminus (O_1 \cup O_2)$. The claim fits into the framework of Theorem 3. Hence, it suffices to prove that the type $(T_0, \kappa)$ of two linear barriers in the torus is amenable. If we fix non-angular parameters, the angles $\theta_1, \theta_2$ are free to vary in certain intervals which in general depend on these parameters. The surface $P(r_1, r_2)$ is rational iff $\frac{\theta_1 - \theta_2}{\pi} = \frac{m}{n} \text{ 10}$ and $N(P(r_1, r_2)) = n$. The claim

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$^{10}$Where $m$ and $n$ are relatively prime.
now follows from the observation that the condition $\frac{\theta_1 - \theta_2}{\pi} = \frac{m}{n}$ with $n$ even is dense in the product of any intervals.

**Remark 5.** Corollary 8 is the analogue of claim 1 in Corollary 7. Fixing some of the coordinates in $(r_1, r_2)$, we obtain analogs of the other claims in Corollary 7. The dense $G_\delta$ claim will hold as long as at least one of the angles $\theta_i$ is free to vary in an interval. We leave the details to the reader.

Let $R(a, b; \xi, \eta)$ be the $a \times b$ rectangle with the lower left corner $(\xi, \eta)$. See Example 4 and Example 6 for notation. Let $\rho_\theta R(a, b; \xi, \eta)$ be the rectangle $R(a, b; \xi, \eta)$ rotated by the angle $\theta$ about its center-point. Let $R$ be the unit rectangle. We assume that the parameters $a, b; \xi, \eta$ are such that $\rho_\theta R(a, b; \xi, \eta) \subset R$ for $0 \leq \theta \leq 2\pi$. We fix $a, b; \xi, \eta$ and set $O_\theta = \rho_\theta R(a, b; \xi, \eta) \subset R$. Representing the standard cylinder $C_0$ by $R$, we have $O_\theta \subset C_0$. Set $P_\theta = C_0 \setminus O_\theta$. Let $\kappa$ be the type of obstacles $O_\theta \subset C_0, 0 \leq \theta \leq 2\pi$. Thus, $\mathfrak{P} = \mathfrak{P}(C_0, \kappa) = \{P_\theta : 0 \leq \theta \leq 2\pi\}$ is a space of flat tori with rectangular obstacles. Note that topologically the space $\mathfrak{P}$ is the circle. Let $\mathfrak{P} = \mathfrak{P}(C_0, \kappa)$ be the corresponding space of $\mathbb{Z}$-periodic polygonal surfaces. Let $B_0$ be the standard infinite band. The surface $\tilde{P}_\theta \in \mathfrak{P}$ is $B_0$ with a $\mathbb{Z}$-periodic collection of tilted rectangular obstacles $O_\theta$. We have $\tilde{P}_\theta = B_0 \setminus (\bigcup_{k \in \mathbb{Z}} (O_\theta + (k, 0)))$. See figure 12.

![Figure 12](image)

**Figure 12.** Infinite band with a $\mathbb{Z}$-periodic collection of tilted rectangular obstacles.

**Corollary 9.** The set of parameters $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that the noncompact polygon $\tilde{P}_\theta$ is conservative is a dense $G_\delta$.

**Proof.** In view of Theorem 4 it suffices to check that $(C_0, \kappa)$ is an amenable pair. The surface $P_\theta$ is rational iff $\frac{\theta}{\pi} = \frac{m}{n}$, with $m$ and $n$ relatively prime. We then have $N(P_\theta) = n$ if $n$ is even, and $N(P_\theta) = 2n$.
if $n$ is odd. Thus, $N(P_\theta)$ is even for any $\pi$-rational $\theta$. The claim now follows from the density of rational numbers.

We expect that among $\mathbb{Z}$-periodic polygonal surfaces, conservativeness is generic in a stronger sense than that ensured by Theorem 3 and Theorem 4. For instance, let $\{\tilde{P}_\theta : \theta \in \mathbb{R}/2\pi \mathbb{Z}\}$ be the family in Corollary 9. We conjecture that the noncompact surfaces $\tilde{P}_\theta$ are conservative for all $\theta$. However, as the following example shows, not every $\mathbb{Z}$-periodic polygonal surface with a boundary is conservative.

**Example 15.** Let $0 < l < 1$. Let $\tilde{P}$ be the standard band with a $\mathbb{Z}$-periodic collection of horizontal barriers of length $l$. See figure 13 where the barriers are in the middle of the band. This surface belongs to the family of surfaces considered in Corollary 7. Also, $\tilde{P}$ is a degeneration of the surface $\tilde{P}_0$ in the family $\{\tilde{P}_\theta : \theta \in \mathbb{R}/2\pi \mathbb{Z}\}$ of Corollary 9. The quotient $P = \tilde{P}/\mathbb{Z}$ is the standard cylinder with a horizontal barrier. It is a rational polygonal surface; $N(P) = 1$. Thus, Theorem 2 does not apply.

The horizontal component of a tangent vector does not change under the geodesic flow of $\tilde{P}$. Hence, the geodesic flow is transient. The directional flows $(U\tilde{P}_\theta, \tilde{G}_t^\theta, \tilde{\mu}_\theta)$ are also transient for $\theta \neq 0$, while the flow $\tilde{G}_{t_0}^0$ is periodic.

The fact that our barriers are positioned at the height $1/2$ plays no role in this case. Any $\mathbb{Z}$-periodic collection of horizontal barriers in $B_0$ leads to a transitive polygonal surface. We leave further generalizations to the reader.

![Figure 13. Infinite band with a $\mathbb{Z}$-periodic collection of horizontal barriers.](image)
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Nicolaus Copernicus University (UMK), Chopina 12/18, Torun 87-100 and Mathematics Institute of the Polish Academy of Sciences (IMPAN), Sniadeckich 8, Warszawa 10, Poland

*E-mail address: gutkin@mat.umk.pl, gutkin@impan.pl*