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Product Type Operators Involving Radial Derivative Operator Acting between Some Analytic Function Spaces

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Abstract: Let N denote the set of all positive integers and N0 = N \cup \{0\}. For m \in N, let Bm = \{z \in \mathbb{C}^m : |z| < 1\} be the open unit ball in the m-dimensional Euclidean space \mathbb{C}^m. Let H(Bm) be the space of all analytic functions on Bm. For an analytic self map \xi = (\xi_1, \xi_2, \ldots, \xi_m) on Bm and \phi_1, \phi_2, \phi_3 \in H(B^m), we have a product type operator T_{\phi_1,\phi_2,\phi_3,\xi} which is basically a combination of three other operators namely composition operator C_\xi, multiplication operator M_\phi and radial derivative operator \partial_t. We study the boundedness and compactness of this operator mapping from weighted Bergman–Orlicz space \mathbb{A}_p^B into weighted type spaces \mathcal{H}_\omega^\alpha and \mathcal{H}_\omega^\alpha,0.

Keywords: weighted Bergman–Orlicz spaces; composition operator; multiplication operator; radial derivative; weighted-type spaces; little weighted type spaces

MSC: 47B33; 30D55; 30H05; 30E05

1. Introduction

Let z = (z_1, z_2, \ldots, z_m), w = (w_1, w_2, \ldots, w_m) be points in \mathbb{C}^m and w = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m), where \bar{w}_k is the complex conjugate of w_k (k = 1, 2, \ldots, m). We write

|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_m|^2}

and

\langle z, w \rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_m\bar{w}_m.

Denote the open unit ball in \mathbb{C}^m by B^m = \{z \in \mathbb{C}^m : |z| < 1\}. Let H(B^m) be the space of all analytic functions on B^m, S^m the boundary of B^m called as the sphere in \mathbb{C}^m. Let dV be the Lebesgue measure on B^m and \sigma the normalized measure on S^m. For \sigma > -1, we write dV_\sigma(z) = C_\sigma(1 - |z|^2)^\sigma dV(z), where C_\sigma is such that V_\sigma(B^m) = 1. For the result in settings of unit ball, refer to Ref. [1] and the references therein.

For 0 < p < \infty and \sigma > -1, the weighted Bergman space A_p^\sigma(B^m) = A_p^\sigma consists of all those functions f \in H(B^m) for which we have the following norm

\|f\|_{A_p^\sigma} = \int_{B^m} |f(z)|^p dV_\sigma(z) < \infty.

A non-zero function \Psi : [0, \infty) \rightarrow [0, \infty) is said to be a growth function if it is continuous and non-decreasing. Clearly, every growth function fixes origin, that is \Psi(0) = 0. We say that the function \Psi is of positive upper type(respectively, negative lower type) for every s > 0 and t \geq 1, if there exist C > 0 and q > 0 (respectively, q < 0) such that

\frac{\Psi(\sigma z)}{\Psi(z)} \leq C z^s \quad \text{for} \quad 0 < \sigma \leq t.

Let \Psi : [0, \infty) \rightarrow [0, \infty) be a growth function and denote the weighted Bergman space A_p^\sigma(B^m) by A_p^\Psi(B^m).

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The class of all growth functions $\Psi$ of positive upper type $q$, for some $q \geq 1$, for which the function $t \rightarrow \Psi(t)/t$ is non-decreasing on $(0, \infty)$ is denoted by $\Sigma^q$. Similarly, for every $s > 0$ and $0 < t \leq 1$, a function $\Psi$ is said to be of positive lower type (respectively, negative upper type) if there are $C > 0$ and $p > 0$ (respectively, $p < 0$) such that $\Psi(st) \leq Ct^p \Psi(s)$. The set of all growth functions $\Psi$ of positive lower type $r$, for some $0 < r \leq 1$ such that the function $t \rightarrow \Psi(t)/t$ is non-increasing on $(0, \infty)$ is denoted by $\Sigma^r$.

For a growth function $\Psi$, the weighted Bergman-Orlicz space $A^\Psi_f(\mathbb{B}^m) = A^\Psi_f$ is the class of all functions $f$ in $H(\mathbb{B}^m)$ such that

$$\|f\|_{A^\Psi_f} = \int_{\mathbb{B}^m} \Psi(|f(z)|)dV_\omega(z) < \infty.$$ 

The quasi-norm on $A^\Psi_f$ is defined as follows:

$$\|f\|_{A^\Psi_f} = \inf\left\{ \lambda > 0 : \int_{\mathbb{B}^m} \Psi\left(\frac{|f(z)|}{\lambda}\right)dV_\omega(z) \leq 1 \right\}.$$ 

If $\Psi \in \Sigma^q$ or $\Psi \in \Sigma^r$, then the quasi-norm on $A^\Psi_f$ is finite and called the Luxembourg norm. A quasi-norm on a linear space $X$ is similar to a norm in that it satisfies the norm axioms, except that the triangle inequality is replaced by the quasi-triangle inequality, that is, $\|x + y\| \leq C(\|x\| + \|y\|)$, for some $C > 0$ and $x, y \in X$. The smallest $C$ for which quasi-triangle inequality holds will be called the quasi-norm constant of $(X, \| \cdot \|)$. The Luxembourg space equipped with the Luxembourg quasi-norm is really a quasi-normed function space with the same quasi-triangle constant as the one of the quasi-norm. If $\Psi(t) = t^p$, for $p > 0$, then we get the weighted Bergman space $A^\Psi_f$. If $p \geq 1$, then $A^\Psi_f$ will be a Banach space which for $0 < p < 1$ is a translation-invariant metric space such that $d(f, g) = \|f - g\|^p_{A^\Psi_f}$. To know more about these spaces one may refer [2,3] and the references therein.

A positive continuous function $\omega$ on $\mathbb{B}^m$ is called as a weight. The weight $\omega$ is called to be a standard weight, if for $z \in \mathbb{B}^m$, we have $\omega(z) \to 0$ as $|z| \to 1$. Further, for $z \in \mathbb{B}^m$, we call a weight $\omega$ to be radial, if $\omega(z) = \omega(|z|)$. For a weight $\omega$ the weighted-type space $H^\infty_\omega(\mathbb{B}^m) = H^\infty_\omega$ is the space of all functions $f \in H(\mathbb{B}^m)$ for which

$$\|f\|_{H^\infty_\omega} = \sup_{z \in \mathbb{B}^m} \omega(z)|f(z)| < \infty.$$ 

The little weighted-type space $H^\infty_\omega(0,\mathbb{B}^m) = H^\infty_\omega$ is the subspace of the space $H^\infty_\omega$ and contains all those $f \in H(\mathbb{B}^m)$ for which

$$\lim_{|z| \to 1} \omega(z)|f(z)| = 0.$$ 

Clearly, $H^\infty_\omega$ is closed in $H^\infty_\omega$. In particular, for $\omega(z) = (1 - |z|^2)^\sigma$; $\sigma > 0$, the space $H^\infty_\omega(0,\mathbb{B}^m)$ will be the classical weighted-type spaces $H^\infty$ (respectively, classical little weighted-type spaces $H^\infty$). For $\omega \equiv 1$, the space $H^\infty$ get reduced to the the space $H^\infty$ of bounded analytic function on $\mathbb{B}^m$. The weighted-type spaces have been studied by various authors see e.g., [4–6] and the references therein.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ be a holomorphic self-map of $\mathbb{B}^m$ and $\phi \in H(\mathbb{B}^m)$. Then, the composition, multiplication, differential and weighted composition operator on $H(\mathbb{B}^m)$ are respectively defined as

$$C_\xi f(z) = (f \circ \xi)(z) = f(\xi(z))$$
$$M_\phi f(z) = \phi(z)f(z)$$
$$D f(z) = f'(z)$$
$$W_\phi f(z) = (M_\phi C_\xi) f(z) = \phi(z)f(\xi(z)), \quad z \in \mathbb{B}^m; f \in H(\mathbb{B}^m).$$
Lemma 1. Let $W_{\phi,\xi} = M_\phi C_\xi$. More results on weighted composition operators on class of holomorphic functions can be found in [7,8] and the references therein. The product-type operators $W_{\phi,\xi}D$ and $DW_{\phi,\xi}$ were respectively, considered in [2] and [3]. To characterize the product-type operators in a unified way, new product-type operator $T_{\phi,\xi}^{\phi,\xi}$ was introduced which can be found in [9,10] and the references therein.

For $f \in H(\mathbb{B}^m)$, the radial derivative is defined by

$$Wf(z) = \sum_{j=1}^{m} z_j \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), z \rangle,$$

where $\nabla f(z)$ denotes the gradient of $f$ which is defined by

$$\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \ldots, \frac{\partial f}{\partial z_m}(z) \right).$$

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ be a holomorphic self map of $\mathbb{B}^m$, then

$$|\xi| = \sqrt{\sum_{j=1}^{m} |\xi_j(z)|^2},$$

$$\nabla \xi(z) = (\nabla \xi_1(z), \nabla \xi_2(z), \ldots, \nabla \xi_m(z)),$$

$$|\nabla \xi(z)| = \sqrt{\sum_{j=1}^{m} |\nabla \xi_j(z)|^2}.$$

For more information about the radial derivative operator we refer to [11,12] and the references therein. Using radial derivative operator and $\phi_1, \phi_2, \phi_3 \in H(\mathbb{B}^m)$, Liu and Yu in [13] studied the extension of the operator $T_{\phi_1,\phi_2,\phi_3}$ which is denoted by $T_{\phi_1,\phi_2,\phi_3}$ and defined as

$$T_{\phi_1,\phi_2,\phi_3}f(z) = \phi_1(z)f(\xi(z)) + \phi_2(z)\nabla f(\xi(z)) + \phi_3(z)\mathcal{R}(f \circ \xi)(z),$$

where $f \in H(\mathbb{B}^m)$ and $z \in \mathbb{B}^m$. Clearly, by fixing $\phi_1, \phi_2, \phi_3$ in $T_{\phi_1,\phi_2,\phi_3}$, all possible products of above defined operators can be obtained. In particular, by setting $\phi_2(z) \equiv \phi_3(z) \equiv 0$, the operator $T_{\phi_1,\phi_2,\phi_3}$ get reduced to $W_{\phi_1,\xi}$. Similarly, for $\phi_1(z) \equiv \phi_3(z) \equiv 0$, the operator $T_{\phi_1,\phi_2,\phi_3}$ get reduced to $W_{\phi_2,\xi}\mathcal{R}$. The above defined operator can be found in [13] and the references therein. Various product-type operators have been studied for spaces of analytic functions on the unit disk. For this one can refer [14–19]. Product-type operators for the unit ball have also been considered by various experts which can be found in [5,13,20–23].

Operators involving radial derivative have been considered in many papers some of which are [6,24,25] and the references therein.

In this paper, we investigate the boundedness as well as the compactness of the operators $T_{\phi_1,\phi_2,\phi_3}$. This paper is represented in a systematic manner. Introduction and literature part is kept in Sections 1 and 2 consists of some auxiliary results which are used to derive the main results. In Section 3, we characterize the boundedness of operators $T_{\phi_1,\phi_2,\phi_3} : A^q_p \rightarrow \mathcal{H}_0^\infty$ and $T_{\phi_1,\phi_2,\phi_3} : A^\infty_p \rightarrow \mathcal{H}_0^\infty$. Finally, in Section 4, the compactness of operators $T_{\phi_1,\phi_2,\phi_3} : A^q_p \rightarrow \mathcal{H}_0^\infty$ and $T_{\phi_1,\phi_2,\phi_3} : A^\infty_p \rightarrow \mathcal{H}_0^\infty$ is given. Throughout the paper, for any two positive quantities $a$ and $b$, the notation $a \lesssim b$ means that $a \leq Cb$, for some constant $C > 0$. The constant $C$ may differ at each occurrence. Further, if both $a \gtrsim b$ and $b \gtrsim a$ hold, then we simply write $a \asymp b$.

2. Auxiliary Results

To obtain the desired results, we have used the following auxiliary results:

**Lemma 1.** Let $\Psi \in \mathcal{L}_p \cup 1^\infty$ and $\sigma > -1$. There is a constant $C > 1$ such that for any $f \in A^\Psi_p$,
\[ |f(z)| \leq C \Psi^{-1} \left( \frac{1}{(1 - |z|^2)^{m+\sigma+1}} \right) |f|_{\mathcal{B}^p}^{1\psi}. \]

**Lemma 2.** Let \(-1 < \sigma < \infty, w \in \mathbb{B}^m\). Let \(K > 1\). Suppose that \(\Psi \in \mathcal{L}_p \cup \Omega^q\). Then, the following function is in \(\mathcal{A}_\Psi^q\)
\[
f_w(z) = \Psi^{-1} \left( \frac{1}{(1 - |w|^2)^{m+\sigma+1}} \right) \left( \frac{1 - |z|^2}{1 - \langle z, w \rangle} \right)^K.\]
Moreover, \(\|f_w\|_{\mathcal{A}_\Psi^q} \ll 1\).

For the proof of Lemmas 1 and 2, we refer to [26,27].

**Lemma 3.** [28] Let \(\Psi \in \mathcal{L}_p \cup \Omega^q\) and \(\sigma > -1\). Then, there are two positive constants \(C_1\) and \(C_2\) such that for any \(f \in \mathcal{A}_\Psi^q\),
\[
|\nabla f(z)| \leq \frac{C_1}{1 - |z|^2} \Psi^{-1} \left( \frac{C_2}{(1 - |z|^2)^{m+\sigma+1}} \right) |f|_{\mathcal{B}^p}^{1\psi}, \quad \text{for any } z \in \mathbb{B}^m.
\]

**Lemma 4.** [20] Let \(\phi_1, \phi_2, \phi_3 \in H(\mathbb{B}^m), \xi = (\xi_1, \xi_2, \ldots, \xi_m)\) denote an analytic self-map of \(\mathbb{B}^m\). Then, \(T_{\phi_1,\phi_2,\phi_3} : \mathcal{A}_\Psi^q \rightarrow \mathcal{H}_\omega^\infty\) is compact if and only if \(T_{\phi_1,\phi_2,\phi_3} : \mathcal{A}_\Psi^q \rightarrow \mathcal{H}_\omega^\infty\) is bounded and for every sequence \(\{f_m\}\) in \(\mathcal{A}_\Psi^q\) which is bounded and on compact subsets of \(\mathbb{B}^m\) uniformly converges to zero as \(m \rightarrow \infty\), we have \(\|T_{\phi_1,\phi_2,\phi_3} f_m\|_{\mathcal{H}_\omega^\infty} \rightarrow 0\) as \(m \rightarrow \infty\). The compactness of \(T_{\phi_1,\phi_2,\phi_3}\) from a holomorphic space to \(\mathcal{H}_\omega^\infty\) can be obtained by using the following lemma which is similar to Lemma 1 in [21]. So the proof is omitted.

**Lemma 5.** A set \(A\) closed in \(\mathcal{H}_\omega^\infty\) is compact if and only if it is bounded and satisfy the following condition
\[
\lim_{|z| \rightarrow 1} \sup_{h \in A} \omega(z)|h(z)| = 0.
\]

### 3. Boundedness of Operator \(T_{\phi_1,\phi_2,\phi_3}\) from Weighted Bergman–Orlicz Space to Weighted and Little Weighted Type Spaces

**Theorem 1.** Let \(\xi = (\xi_1, \xi_2, \ldots, \xi_m)\) denote an analytic self-map of \(\mathbb{B}^m\) and \(\phi_1, \phi_2, \phi_3 \in H(\mathbb{B}^m)\). Then, the following statements hold true:

(a) If
\[
\sup_{z \in \mathbb{B}^m} \omega(z)|\phi_1(z)|\Psi^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) < \infty
\]
and
\[
\sup_{z \in \mathbb{B}^m} \frac{\omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Re\xi(z)|}{1 - |\xi(z)|^2} \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) < \infty,
\]
then \(T_{\phi_1,\phi_2,\phi_3} : \mathcal{A}_\Psi^q \rightarrow \mathcal{H}_\omega^\infty\) is bounded.

(b) If \(T_{\phi_1,\phi_2,\phi_3}\) is bounded then the condition (1) hold and
\[
\sup_{z \in \mathbb{B}^m} \frac{\omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Re\xi(z), \xi(z)|}{1 - |\xi(z)|^2} \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) < \infty.
\]
Proof. (a) First, suppose that conditions (1) and (2) are true. Then, for \( f \in \mathcal{A}_\xi^\beta \) and any \( z \in \mathbb{B}^m \), we have

\[
\Re(f \circ \xi)(z) = \sum_{j=1}^m z_j \frac{\partial (f \circ \xi)}{\partial z_j}(z)
\]

\[
= \sum_{j=1}^m z_j \left( \sum_{k=1}^m \frac{\partial f}{\partial z_k}(\xi(z)) \frac{\partial \xi_k}{\partial z_j}(z) \right)
\]

\[
= \sum_{k=1}^m \left( \frac{\partial f}{\partial z_k}(\xi(z)) \sum_{j=1}^m z_j \frac{\partial \xi_k}{\partial z_j}(z) \right)
\]

\[
= \sum_{j=1}^m \left( \frac{\partial f}{\partial z_j}(\xi(z)) \Re \xi_j(z) \right)
\]

\[
= (\nabla f(\xi(z)), \Re \xi(z)).
\]

Thus, we have

\[
\omega(z)|T_{\phi_1, \phi_2, \phi_3, \xi} f(z)|
\]

\[
= \omega(z)|\phi_1(z)f(\xi(z)) + \phi_2(z)\Re f(\xi(z)) + \phi_3(z)\Re(f \circ \xi)(z)|
\]

\[
\leq \omega(z)|\phi_1(z)f(\xi(z))| + \omega(z)|\phi_2(z)\Re f(\xi(z))| + \omega(z)|\phi_3(z)\Re(f \circ \xi)(z)|
\]

\[
= \omega(z)|\phi_1(z)f(\xi(z))| + \omega(z)|\phi_2(z)(\nabla f(\xi(z)), \xi(z)) + \phi_3(z)(\nabla f(\xi(z)), \Re \xi(z))| + \omega(z)|\phi_3(z)\Re(f \circ \xi)(z)|
\]

\[
\leq \omega(z)|\phi_1(z)f(\xi(z))| + \omega(z)|\nabla f(\xi(z))| + \omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Re \xi(z)|\nabla f(\xi(z))|
\]

\[
\leq C \omega(z)|\phi_1(z)|\Psi^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) \|f\|_{\mathcal{A}^\beta z}^\Psi
\]

\[
+ C \frac{|\phi_2(z)\xi(z) + \phi_3(z)\Re \xi(z)|}{1 - |\xi(z)|^2} \Psi^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) \|f\|_{\mathcal{A}^\beta z}^\Psi
\]

\[
= C \omega(z)|\phi_1(z)|\Psi^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right)
\]

\[
+ \frac{|\phi_2(z)\xi(z) + \phi_3(z)\Re \xi(z)|}{1 - |\xi(z)|^2} \Psi^{-1} \left( \frac{C_2}{1 - |\xi(z)|^2} \right) \|f\|_{\mathcal{A}^\beta z}^\Psi. \quad (4)
\]

From (1), (2) and (4), it follows that \( T_{\phi_1, \phi_2, \phi_3, \xi} : \mathcal{A}_\beta^\xi \rightarrow \mathcal{H}^{\alpha}_{\omega} \) is bounded.

(b) Conversely, suppose that \( T_{\phi_1, \phi_2, \phi_3, \xi} : \mathcal{A}_\beta^\xi \rightarrow \mathcal{H}^{\alpha}_{\omega} \) is bounded. Thus, there exist an independent constant \( C > 0 \) such that

\[
\|T_{\phi_1, \phi_2, \phi_3, \xi} f\|_{\mathcal{H}^{\alpha}_{\omega}} \leq C \|f\|_{\mathcal{A}^\beta z}.
\]

For \( w \in \mathbb{B}^m \), let

\[
f_w(z) = a_0 \left( \frac{1 - |\xi(z)|^2}{1 - \langle z, \xi(z) \rangle} \right)^{K(m+\sigma+1)} + a_1 \left( \frac{1 - |\xi(w)|^2}{1 - \langle z, \xi(w) \rangle} \right)^{K(m+\sigma+1)+1},
\]

where \( a_0 = K(m + \sigma + 1) + 1 \) and \( a_1 = -K(m + \sigma + 1) \). This implies

\[
\Re f_w(z) = a_0 (1 - |\xi(w)|^2)^{K(m+\sigma+1)} \sum_{j=1}^m z_j \frac{\partial}{\partial z_j} \left( \frac{1}{1 - \langle z, \xi(w) \rangle} \right)^{K(m+\sigma+1)}
\]

\[
+ a_1 (1 - |\xi(w)|^2)^{K(m+\sigma+1)+1} \sum_{j=1}^m z_j \frac{\partial}{\partial z_j} \left( \frac{1}{1 - \langle z, \xi(w) \rangle} \right)^{K(m+\sigma+1)+1}
\]
\[ a_0 [K(m + \sigma + 1)] \left( \frac{\langle z, \xi(w) \rangle}{1 - \langle z, \xi(w) \rangle} \right) \left( \frac{1 - |\xi(w)|^2}{1 - \langle z, \xi(w) \rangle} \right)^{K(m+\sigma+1)} \]

\[ + a_1 [K(m + \sigma + 1) + 1] \left( \frac{\langle z, \xi(w) \rangle}{1 - \langle z, \xi(w) \rangle} \right) \left( \frac{1 - |\xi(w)|^2}{1 - \langle z, \xi(w) \rangle} \right)^{K(m+\sigma+1)+1}. \quad (6) \]

By Lemma 2, we have \[ \sup_{w \in \mathcal{B}^m} \| f_w \|_{\mathcal{A}^F} \lesssim 1. \] Using (5) and (6), we get

\[ f_w(\xi(w)) = 1 \quad \text{and} \quad \Re f_w(\xi(w)) = 0. \quad (7) \]

Since

\[ f_w(\xi(z)) = a_0 \left( \frac{1 - |\xi(w)|^2}{1 - \langle \xi(z), \xi(w) \rangle} \right)^{K(m+\sigma+1)} + a_1 \left( \frac{1 - |\xi(w)|^2}{1 - \langle \xi(z), \xi(w) \rangle} \right)^{K(m+\sigma+1)+1}. \]

Therefore,

\[ \Re (f_w \circ \xi)(z) = a_0 [K(m + \sigma + 1)] \left( \frac{\Re \xi(z), \xi(w)}{1 - \langle \xi(z), \xi(w) \rangle} \right) \left( \frac{1 - |\xi(w)|^2}{1 - \langle \xi(z), \xi(w) \rangle} \right)^{K(m+\sigma+1)} \]

\[ + a_1 [K(m + \sigma + 1) + 1] \left( \frac{\Re \xi(z), \xi(w)}{1 - \langle \xi(z), \xi(w) \rangle} \right) \left( \frac{1 - |\xi(w)|^2}{1 - \langle \xi(z), \xi(w) \rangle} \right)^{K(m+\sigma+1)+1}, \]

which implies

\[ \Re (f_w \circ \xi)(w) = 0. \quad (8) \]

Using function \( f_w \), define a function as

\[ f(z) = \Psi^{-1} \left( \frac{1}{1 - |\xi(w)|^2} \right)^{m+\sigma+1} f_w(z). \quad (9) \]

From conditions (7), (8) and (9), we obtain

\[ \Re (f \circ \xi)(w) = \Re (f \circ \xi)(w) = 0, f(\xi(w)) = \Psi^{-1} \left( \frac{1}{1 - |\xi(w)|^2} \right)^{m+\sigma+1}. \quad (10) \]

Using Lemma 2, we have \( f \in \mathcal{A}^F \) and \( \| f \|_{\mathcal{A}^F} \lesssim 1 \). By Equation (10) and boundedness of the operator \( T_{\phi_1, \phi_2, \phi_3} : \mathcal{A}^F \rightarrow \mathcal{H}^0_{\mathcal{A}^F} \), we have

\[ \omega(w) |\phi_1(w)| \left( \frac{1}{1 - |\xi(w)|^2} \right)^{m+\sigma+1} \leq C \| T_{\phi_1, \phi_2, \phi_3} \|. \]

Thus,

\[ \sup_{w \in \mathcal{B}^m} \omega(w) |\phi_1(w)| \left( \frac{1}{1 - |\xi(w)|^2} \right)^{m+\sigma+1} < \infty. \]

This proves (1). For \( f(z) \equiv 1 \in \mathcal{A}^F \), we have

\[ K_1 := \sup_{z \in \mathcal{B}^m} \omega(z) |\phi_1(z)| < \infty. \quad (11) \]

Taking the function \( f_j(z) = z_j \in \mathcal{A}^F \), we obtain
\[ \sup_{z \in B^m} \omega(z) |\phi_1(z)\tilde{c}_j(z) + \phi_2(z)\tilde{c}_j(z) + \phi_3(z)\Re\tilde{c}_j(z)| < \infty. \tag{12} \]

Using (11) and (12) with the fact that \(|\tilde{c}(z)| \leq 1\), we get
\[
\sup_{z \in B^m} \omega(z) |\phi_2(z)\tilde{c}(z) + \phi_3(z)\Re\tilde{c}(z)| \\
= \sup_{z \in B^m} \left[ \sum_{j=1}^{m} |\phi_2(z)\tilde{c}_j(z) + \phi_3(z)\Re\tilde{c}_j(z)|^2 \right]^{1/2} \\
\leq C + \sup_{z \in B^m} \left[ \sum_{j=1}^{m} |\tilde{c}_j(z)|^2 \right]^{1/2} \\
\leq C + \sup_{z \in B^m} |\phi_1(z)||\tilde{c}(z)| \\
\leq C + K_1.
\]

Thus,
\[ K_2 := \sup_{z \in B^m} \omega(z) |\phi_2(z)\tilde{c}(z) + \phi_3(z)\Re\tilde{c}(z)| < \infty. \tag{13} \]

For a fixed \( w \in B^m \), define
\[
{g}_w(z) = \left( \frac{1 - |\tilde{c}(w)|^2}{1 - \langle z, \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1)} - \left( \frac{1 - |\tilde{c}(w)|^2}{1 - \langle z, \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1) + 1}. \tag{14} \]

This implies
\[
\Re{g}_w(z) = \left[ K(m + \sigma + 1) \right] \left( \frac{\langle z, \tilde{c}(w) \rangle}{1 - \langle z, \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1)} \left( \frac{1 - |\tilde{c}(w)|^2}{1 - \langle z, \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1)} \\
- \left[ K(m + \sigma + 1) + 1 \right] \left( \frac{\langle z, \tilde{c}(w) \rangle}{1 - \langle z, \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1) + 1}. \tag{15} \]

By Lemma 2, we have \( g_w \in \mathcal{A}^\Psi(B^m) \) and \( \sup_{w \in B^m} \|g_w\|_{\mathcal{A}^\Psi} \leq 1 \). Conditions (14) and (15) implies that
\[ g_w(\tilde{c}(w)) = 0, \quad \Re{g}_w(\tilde{c}(w)) = \frac{-|\tilde{c}(w)|^2}{1 - |\tilde{c}(w)|^2}. \tag{16} \]

Now, we have
\[
\Re(g_w \circ \tilde{c})(z) \\
= \left[ K(m + \sigma + 1) \right] \left( \frac{\Re\tilde{c}(z), \tilde{c}(w)}{1 - \langle \tilde{c}(z), \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1)} \left( \frac{1 - |\tilde{c}(w)|^2}{1 - \langle z, \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1)} \\
- \left[ K(m + \sigma + 1) + 1 \right] \left( \frac{\Re\tilde{c}(z), \tilde{c}(w)}{1 - \langle \tilde{c}(z), \tilde{c}(w) \rangle} \right)^{K(m + \sigma + 1) + 1},
\]

which implies
\[ \Re(g_w \circ \tilde{c})(w) = \frac{-\Re\tilde{c}(w), \tilde{c}(w)}{1 - |\tilde{c}(w)|^2}. \tag{17} \]

Using function \( g_w \), define another function
\[ g(z) = \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) g_w(z). \]  

(18)

Therefore, by conditions (16), (17) and (18), we get

\[ g(\xi(w)) = 0, \]
\[ \mathfrak{R} g(\xi(w)) = \left( \frac{-|\xi(w)|^2}{1 - |\xi(w)|^2} \right) \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right), \]
\[ \mathfrak{R}(g \circ \xi)(w) = \left( \frac{-\mathfrak{R} \xi(w), \xi(w)}{1 - |\xi(w)|^2} \right) \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right). \]

For \( w \in \mathbb{B}^m \), we have

\[ C \geq \| T_{\Phi_1, \Phi_2, \Phi_3} \|_{\mathcal{H}_w^\infty} \]
\[ \geq \omega(w) |\Phi_1(w) g(\xi(w)) + \Phi_2(w) \mathfrak{R} g(\xi(w)) + \Phi_3(z) \mathfrak{R}(g \circ \xi)(w)| \]
\[ = \omega(w) \left| \frac{\Phi_2(w)|\xi(w)|^2 + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w)}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) \]
\[ = \omega(w) \left| \frac{\langle \Phi_2(w) \xi(w) + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w) \rangle}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right). \]

(19)

From (19), it follows that

\[ \sup_{w \in \mathbb{B}^m} \omega(w) \left| \frac{\langle \Phi_2(w) \xi(w) + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w) \rangle}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) < \infty, \]

which proves condition (3). This completes the theorem. \( \square \)

**Corollary 1.** Let \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \) be an analytic self map of \( \mathbb{B}^m \), \( \Phi_1, \Phi_2, \Phi_3 \in H(\mathbb{B}^m) \) and suppose the functions \( \zeta \) and \( \Phi_2 \zeta + \Phi_3 \mathfrak{R} \zeta \) are linearly dependent. Then, \( T_{\Phi_1, \Phi_2, \Phi_3} : A^\infty \rightarrow \mathcal{H}_w^\infty \) is bounded if and only if it satisfies (1) and (2).

**Proof.** First suppose that \( T_{\Phi_1, \Phi_2, \Phi_3} : A^\infty \rightarrow \mathcal{H}_w^\infty \) is bounded. Then, by condition (b) of Theorem 1, we see that (1) holds. So, we only need to prove (2). Using Cauchy-Schwartz inequality and (19), for \( w \in \mathbb{B}^m \) we get

\[ C \geq \omega(w) \left| \frac{\Phi_2(w)|\xi(w)|^2 + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w)}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) \]
\[ = \omega(w) \left| \frac{\langle \Phi_2(w) \xi(w) + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w) \rangle}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) \]
\[ = \omega(w) \left| \frac{\Phi_2(w)|\xi(w)|^2 + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w)}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right). \]

(20)

Thus, for \( \delta \in (0, 1) \), condition (20) implies that

\[ \sup_{\delta < |\xi(w)| < 1} \omega(w) \left| \frac{\Phi_2(w)|\xi(w)|^2 + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w)}{\delta(1 - |\xi(w)|^2)} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) \]
\[ \leq \sup_{\delta < |\xi(w)| < 1} \omega(w) \left| \frac{\Phi_2(w)|\xi(w)|^2 + \Phi_3(w) \mathfrak{R} \xi(w), \xi(w)}{\delta(1 - |\xi(w)|^2)} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) \]
\[ \leq \frac{C}{\delta}. \]

(21)
Therefore, by Equation (13), we have
\[
\sup_{\xi(w) \leq \delta} \omega(w) \left| \frac{\phi_2(w)\xi(w) + \phi_3(w)\Re \xi(w)}{1 - |\xi(w)|^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - |\xi(w)|^2)^{m+\sigma+1}} \right) \leq \sup_{\xi(w) \leq \delta} \omega(w) \left| \frac{\phi_2(w)\xi(w) + \phi_3(w)\Re \xi(w)}{1 - \delta^2} \right| \Psi^{-1} \left( \frac{C_2}{(1 - \delta^2)^{m+\sigma+1}} \right) \leq \frac{K_2}{1 - \delta^2} \Psi^{-1} \left( \frac{C_2}{(1 - \delta^2)^{m+\sigma+1}} \right) < \infty.
\]

Hence, (2) follows from the combination of (21) and (22). This completes the proof.

Conversely suppose that (1) and (2) hold. Then, from condition (a) in Theorem 1 we get that \( T_{\phi_1, \phi_2, \phi_3} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \) is bounded.

In Corollary 1, if we take \( \phi_1(z) = \phi(z) \) and \( \phi_2(z) \equiv \phi_3(z) \equiv 0 \), then the operator gets reduced to the weighted composition operator \( W_{\phi, \xi} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \). Thus we get the following corollary for the boundedness of \( W_{\phi, \xi} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \) as:

**Corollary 2.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) be an analytic self map of \( \mathbb{B}^m \) and \( \phi \in H(\mathbb{B}^m) \). Then the operator \( W_{\phi, \xi} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \) is bounded if and only if
\[
\sup_{z \in \mathbb{B}^m} \omega(z)|\phi(z)|\Psi^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) < \infty.
\]

**Theorem 2.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) be an analytic self map of \( \mathbb{B}^m \) and \( \phi_1, \phi_2, \phi_3 \in H(\mathbb{B}^m) \). Then, the operator \( T_{\phi_1, \phi_2, \phi_3} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \) is bounded if and only if \( T_{\phi_1, \phi_2, \phi_3} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \) is bounded and satisfy the conditions
\[
\lim_{|z| \to 1} \omega(z)|\phi_1(z)| = 0 \quad (23)
\]
and
\[
\lim_{|z| \to 1} \omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Re \xi(z)| = 0 \quad (24)
\]

**Proof.** First suppose that \( T_{\phi_1, \phi_2, \phi_3} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \) is bounded. This implies the boundedness of \( T_{\phi_1, \phi_2, \phi_3} : A_\psi^\psi \to \mathcal{H}_{\phi_1}^\phi \). Thus for each \( f \in A_{\psi^\phi} \), we have \( T_{\phi_1, \phi_2, \phi_3}f \in \mathcal{H}_{\phi_1}^\phi \). On setting \( f(z) \equiv 1 \in A_{\psi^\phi} \), we get that
\[
\lim_{|z| \to 1} \omega(z)|\phi_1(z)| = 0,
\]
which proves (23). Now, for \( k = 1, 2, \ldots, m \) take \( f_k(z) = z_k \in A_{\psi^\phi} \). So, we get
\[
\lim_{|z| \to 1} \omega(z)|\phi_1(z)\xi_k(z) + \phi_2(z)\xi_k(z) + \phi_3(z)\Re \xi_k(z)| = 0.
\]

By using (23) and (25) and triangle inequality with condition \( |\xi_k(z)| \leq |\xi(z)| \leq 1 \), we get
\[
\lim_{|z| \to 1} \omega(z)|\phi_2(z)\xi_k(z) + \phi_3(z)\Re \xi_k(z)| = 0, \quad k = 1, 2, \ldots, m.
\]

Thus,
\[
\lim_{|z| \to 1} \omega(z)|\phi_2(z)\xi_k(z) + \phi_3(z)\Re \xi_k(z)| = 0,
\]
which proves (24).

Conversely, suppose that (23), (24) hold and \( T_{\phi_1, \phi_2, \phi_3} : A^\gamma \to H^\omega_{\omega_0} \) is bounded. Then, for each polynomial \( q \) and \( z \in \mathbb{B}^m \), we obtain

\[
|q(z)| \leq \|q\|_{\infty}, \quad \text{and} \quad |\nabla q(z)| \leq C < \infty.
\]

Therefore,

\[
\omega(z) |T_{\phi_1, \phi_2, \phi_3} q(z)|
\]

\[
= \omega(z) |\phi_1(z)q(\xi(z)) + \phi_2(z)\nabla q(\xi(z)) + \phi_3(z)\nabla(\phi \circ \xi)(z)|
\]

\[
\leq \omega(z) |\phi_1(z)q(\xi(z))| + \omega(z) |\phi_2(z)\nabla q(\xi(z))| + \omega(z) |\phi_3(z)\nabla(\phi \circ \xi)(z)|
\]

\[
= \omega(z) |\phi_1(z)| q(\xi(z))| + \omega(z) |\nabla q(\xi(z))| \phi_2(z) \xi(z) + \phi_3(z) \nabla \xi(z)|
\]

\[
\leq \omega(z) |\phi_1(z)| q(\xi(z))| + \omega(z) |\phi_2(z) \xi(z) + \phi_3(z) \nabla \xi(z)||\nabla q(\xi(z))|
\]

\[
\leq \omega(z) |\phi_1(z)||q||_{\infty} + C \omega(z) |\phi_2(z) \xi(z) + \phi_3(z) \nabla \xi(z)|
\]

\[
\rightarrow 0 \quad \text{as} \quad |z| \to 1.
\]

From the above fact, we get that \( T_{\phi_1, \phi_2, \phi_3} q \in H^\omega_{\omega_0} \). As we know that the class of all the polynomial is dense in \( A^\gamma \), so for each \( f \in A^\gamma \), there will be a sequence \( \{q_k\}_{k \in \mathbb{N}} \) such that

\[
\lim_{k \to \infty} \|q_k - f\|_{A^\gamma} = 0.
\]

By the boundedness of the operator \( T_{\phi_1, \phi_2, \phi_3} : A^\gamma \to H^\omega_{\omega_0} \), we get that

\[
\|T_{\phi_1, \phi_2, \phi_3} q_k - T_{\phi_1, \phi_2, \phi_3} f\|_{H^\omega_{\omega_0}} \leq \|T_{\phi_1, \phi_2, \phi_3} q_k - f\|_{A^\gamma} \to 0 \text{ as } k \to \infty.
\]

Thus, \( T_{\phi_1, \phi_2, \phi_3} f \in H^\omega_{\omega_0} \). As \( H^\omega_{\omega_0} \subset H^\omega_{\omega} \) is closed, this implies \( T_{\phi_1, \phi_2, \phi_3} (A^\gamma) \subset H^\omega_{\omega_0} \). Hence, the boundedness of \( T_{\phi_1, \phi_2, \phi_3} : A^\gamma \to H^\omega_{\omega_0} \) follows from the boundedness of \( T_{\phi_1, \phi_2, \phi_3} : A^\gamma \to H^\omega_{\omega} \). This completes the proof. \( \Box \)

By taking \( \phi_1(z) = \phi(z) \) and \( \phi_2(z) \equiv \phi_3(z) \equiv 0 \) in Theorem 2, we get \( \text{W}_{\phi_3} : A^\gamma \to H^\omega_{\omega_0} \). Thus, we get the following corollary for the boundedness of \( \text{W}_{\phi_3} : A^\gamma \to H^\omega_{\omega_0} \) as:

**Corollary 3.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) be an analytic self map of \( \mathbb{B}^m \) and \( \phi \in H(\mathbb{B}^m) \). Then, the operator \( \text{W}_{\phi_3} : A^\gamma \to H^\omega_{\omega_0} \) is bounded if and only if \( \text{W}_{\phi_3} : A^\gamma \to H^\omega_{\omega_0} \) is bounded and \( \phi \in H^\omega_{\omega_0} \).

**Theorem 3.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) denotes an analytic self map of \( \mathbb{B}^m \) and \( \phi_1, \phi_2, \phi_3 \in H(\mathbb{B}^m) \). Then, the operator \( T_{\phi_1, \phi_2, \phi_3} : A^\gamma \to H^\omega_{\omega_0} \) is bounded if it satisfy the conditions

\[
\lim_{|z| \to 1} \omega(z) |\phi_1(z)| \gamma^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) = 0 \quad (26)
\]

and

\[
\lim_{|z| \to 1} \frac{\omega(z) |\phi_2(z) \xi(z) + \phi_3(z) \nabla(\phi \circ \xi)(z)|}{1 - |\xi(z)|^2} \gamma^{-1} \left( \frac{C_2}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) = 0 . \quad (27)
\]

**Proof.** Suppose that (26) and (27) hold. It clearly implies that (1) and (2) hold. By using condition (a) in 1, we obtain that \( T_{\phi_1, \phi_2, \phi_3} : A^\gamma \to H^\omega_{\omega_0} \) is bounded. So, by taking \( f \in A^\gamma \) and using (4), we get
(b) If \( T \) when \( \delta \) on compact subset of \( B \) follows from the boundedness of \( T \) and \( \phi \). Then, the following statements hold true:

First suppose that \( |z| \to 0 \) as \( \delta \to 0 \). Then \( T \) is bounded and satisfy the condition

\[
\lim_{|z| \to 1} \omega(z)|\phi_1(z)|^{\Psi^{-1}} \left( \frac{1}{(1 - |z|^2)^{m+\nu+1}} \right) = 0
\]

and

\[
\lim_{|z| \to 0} \frac{\omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Omega(z)|^{\Psi^{-1}}}{1 - |z|^2} \left( \frac{C_2}{(1 - |z|^2)^{m+\nu+1}} \right) = 0,
\]

then, \( T \) is compact then, it satisfy (28) along with the condition

\[
\lim_{|z| \to 1} \frac{\omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Omega(z)|^{\Psi^{-1}}}{1 - |z|^2} \left( \frac{C_2}{(1 - |z|^2)^{m+\nu+1}} \right) = 0.
\]

Proof. (a) First suppose that \( T \) is bounded and satisfy the conditions (28) and (29). Let \( \{f_k\} \in \mathcal{A}_0^\Psi \) be a bounded sequence converging to zero uniformly on compact subset of \( B \) as \( k \to \infty \). Let \( \|f_k\|_{\mathcal{A}_0^\Psi} \leq 1 \).

Now, by conditions (28) and (29), we have that for any \( \epsilon > 0 \), there will be a \( \delta \in (0,1) \) for which

\[
\omega(z)|\phi_1(z)|^{\Psi^{-1}} \left( \frac{1}{(1 - |z|^2)^{m+\nu+1}} \right) < \epsilon
\]

and

\[
\frac{\omega(z)|\phi_2(z)\xi(z) + \phi_3(z)\Omega(z)|^{\Psi^{-1}}}{1 - |z|^2} \left( \frac{C_2}{(1 - |z|^2)^{m+\nu+1}} \right) < \epsilon,
\]

when \( \delta < |\xi(z)| < 1 \). Since \( T \) is bounded, so condition (1) and (3) hold. In addition, on compact subset of \( B \), the sequence \( \{f_k\} \in \mathcal{A}_0^\Psi \) converges to zero. So, using the Cauchy’s estimate we get that on compact subsets of \( B \), the sequence \( \nabla f_k \) uniformly converges to zero as \( k \to \infty \). This implies that \( \lim_{k \to \infty} \sup_{|W| \leq \delta} |\nabla f_k(W)| = 0 \). Thus, there will be a \( K_0 \in \mathbb{N} \) such that
\[
    \sup_{|\xi(z)| \leq \delta} \omega(z)|T_{\phi_1, \phi_2, \phi_3, \xi} f_k(z)| \\
    \leq \sup_{|\xi(z)| \leq \delta} \omega(z)|f_k(\xi(z))| + \sup_{|\xi(z)| \leq \delta} \omega(z)|\phi_2(\xi(z))| + \phi_3(\xi(z))|f_k(z)| \\
    \leq K_1 \sup_{|\xi(z)| \leq \delta} |f_k(\xi(z))| + \sup_{|\xi(z)| \leq \delta} \omega(z)|\phi_2(\xi(z))| + \phi_3(\xi(z))|f_k(z)| \\
    \leq K_1 \sup_{|W| \leq \delta} |f_k(W)| + K_2 \sup_{|W| \leq \delta} |\nabla f_k(W)| \\
    \leq C \epsilon.
\]

where \( k > K_0 \). On combining conditions (31), (32) and (33) with Lemma 1, we get

\[
    \|T_{\phi_1, \phi_2, \phi_3, \xi} f_k\|_{H_\omega^0} \\
    = \sup_{z \in \mathbb{B}^m} \omega(z)|T_{\phi_1, \phi_2, \phi_3, \xi} f_k(z)| \\
    \leq \sup_{|\xi(z)| \leq \delta} \omega(z)|T_{\phi_1, \phi_2, \phi_3, \xi} f_k(z)| + \sup_{|\xi(z)| \leq \delta} \omega(z)|\phi_1(\xi(z))| \\
    \leq C \epsilon + C \sup_{\delta < |\xi(z)| < 1} \omega(z)|\phi_1(\xi(z))| \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) \|f_k\|_{L^2(\mathbb{B}^m)} \\
    + C \sup_{\delta < |\xi(z)| < 1} \frac{\omega(z)|\phi_2(\xi(z))| + \phi_3(\xi(z))|\nabla f_k(z)|}{1 - |\xi(z)|^2} \left( \frac{C_2}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) \|f_k\|_{L^2(\mathbb{B}^m)} \\
    \leq 3C \epsilon,
\]

when \( k > K_0 \), which approaches to zero as \( k \to \infty \). Hence the operator \( T_{\phi_1, \phi_2, \phi_3, \xi} : A_\sigma^Y \to H_\omega^0 \) is compact.

(b) Conversely, suppose that \( T_{\phi_1, \phi_2, \phi_3, \xi} : A_\sigma^Y \to H_\omega^0 \) is compact. This implies that \( T_{\phi_1, \phi_2, \phi_3, \xi} \) is bounded. If \( \|\xi\|_\infty < 1 \), then conditions (28) and (30) hold. Let \( \|\xi\|_\infty = 1 \) and \( \{z_k\} \in \mathbb{B}^m \) be a sequence such that \( |\xi(z_k)| \to 1 \) as \( k \to \infty \). Using sequence \( |\xi(z_k)| \) define a function

\[
f_k(z) = \Psi^{-1} \left( \frac{1}{(1 - |\xi(z_k)|^2)^{m+\sigma+1}} \right) f_{\xi(z_k)}(z),
\]

where

\[
f_{\xi(z_k)}(z) = [K(m + \sigma + 1) + 1] \left( \frac{1 - |\xi(z_k)|^2}{1 - \langle z, \xi(z_k) \rangle} \right)^{(m+\sigma+1)} \\
- [K(m + \sigma + 1)] \left( \frac{1 - |\xi(z_k)|^2}{1 - \langle z, \xi(z_k) \rangle} \right)^{(m+\sigma+1)+1}.
\]

Clearly, \( f_{\xi(z_k)} \in A_\sigma^Y \) and \( \sup_{k \in \mathbb{N}} \|f_{\xi(z_k)}\|_{A_\sigma^Y} \leq C \). Therefore,
where \(|z| \leq r\), we get that \(f_k(z_k)\) converges to zero uniformly on compact subsets of \(B^m\). Thus, \(f_k\) converges to zero uniformly on compact subsets of \(\mathbb{B}^m\). Therefore, by Lemma 4, it follows that the sequence \(\{f_k\}\) uniformly converges to zero on any compact subsets of \(\mathbb{B}^m\) as \(k \to \infty\) such that

\[
\lim_{k \to \infty} \|T_{\Phi_1, \Phi_2, \Phi_3} f_k\|_{\mathcal{H}^m} = 0.
\]

In addition, we have

\[
f_k(\xi(z_k)) = \Psi^{-1}\left(\frac{1}{1 - (|\xi(z_k)|^2)^{m+\sigma+1}}\right)\text{ and } \Re f_k(\xi(z_k)) = \Re (f_k \circ \xi)(z_k) = 0.
\]

Thus, we obtain

\[
\omega(z_k)|\Phi_1(z_k)|\Psi^{-1}\left(\frac{1}{1 - (|\xi(z_k)|^2)^{m+\sigma+1}}\right) \leq \|T_{\Phi_1, \Phi_2, \Phi_3} f_k\|_{\mathcal{H}^m} \to 0 \text{ as } k \to \infty,
\]

which implies that (28) holds. In order to prove condition (30), we define another sequence of functions

\[
g_k(z) = \Psi^{-1}\left(\frac{C_2}{(1 - |\xi(z_k)|^2)^{m+\sigma+1}}\right)g_k(\xi(z_k))(z),
\]

where

\[
g_k(\xi(z_k))(z) = \left(\frac{1 - |\xi(z_k)|^2}{1 - \langle \xi(z), \xi(z_k) \rangle}\right)^{k(m+\sigma+1)} - \left(\frac{1 - |\xi(z_k)|^2}{1 - \langle z, \xi(z_k) \rangle}\right)^{k(m+\sigma+1)+1}.
\]

Therefore,

\[
(g_k(\xi(z_k)) \circ \xi)(z) = \left(\frac{1 - |\xi(z_k)|^2}{1 - \langle \xi(z_k), \xi(z) \rangle}\right)^{k(m+\sigma+1)} - \left(\frac{1 - |\xi(z_k)|^2}{1 - \langle \xi(z), \xi(z_k) \rangle}\right)^{k(m+\sigma+1)+1},
\]

which implies

\[
\Re g_k(\xi(z_k))(z) = \left[\Re f_k(\xi(z_k))\right] \left(\frac{1 - |\xi(z_k)|^2}{1 - \langle z, \xi(z_k) \rangle}\right)^{k(m+\sigma+1)}
\]

\[
+ \left[\Re f_k(\xi(z_k)) \circ \xi\right] \left(\frac{1 - |\xi(z_k)|^2}{1 - \langle \xi(z_k), \xi(z) \rangle}\right)^{k(m+\sigma+1)+1}.
\]
and
\[ R(g_k(z_k) \circ \xi)(z) \]
\[ = [K(m + \sigma + 1)] \left( \frac{\langle R_k(z_k), \xi(z_k) \rangle}{1 - \langle \xi(z), \xi(z_k) \rangle} \right) \left( \frac{1 - |\xi(z_k)|^2}{1 - |\xi(z)|^2} \right)^{k(m + \sigma + 1)} \]
\[ + [K(m + \sigma + 1)] \left( \frac{\langle R_k(z), \xi(z_k) \rangle}{1 - \langle \xi(z), \xi(z_k) \rangle} \right) \left( \frac{1 - |\xi(z_k)|^2}{1 - |\xi(z)|^2} \right)^{k(m + \sigma + 1) + 1}. \]

Similar to the sequence \( f_k \), the sequence \( g_k(z_k) \) and hence the sequence \( g_k \) converges to zero uniformly on compact subsets of \( \mathbb{B}^m \), \( g_k \in A^\infty_r \) and \( \sup_{k \in \mathbb{N}} \|g_k\|_{A^\infty_r} \leq C \). By Lemma 4, we have
\[
\lim_{k \to \infty} \|T_{\phi_1, \phi_2, \phi_3, \xi} g_k\|_{H^\infty_w} = 0. \tag{36}
\]

In addition, we have
\[ g_k(\xi(z_k)) = 0, \quad Rg_k(\xi(z_k)) = \frac{-|\xi(z_k)|^2}{1 - |\xi(z_k)|^2}, \quad R(g_k \circ \xi)(z_k) = \frac{-\langle R_k(z_k), \xi(z_k) \rangle}{1 - |\xi(z_k)|^2}. \]

Thus,
\[
\|T_{\phi_1, \phi_2, \phi_3, \xi} g_k\|_{H^\infty_w}
\geq \omega(z_k)\|\phi_1(z)g_k(\xi(z_k)) + \phi_2(z_k)Rg_k(\xi(z_k)) + \phi_3(z_k)R(g_k \circ \xi)(z_k)\|
\]
\[ = \omega(z_k)\|\phi_2(z_k)g_k(\xi(z_k)) + \phi_3(z_k)R(g_k \circ \xi)(z_k)\|
\]
\[ = \omega(z_k) \left| \frac{\langle \phi_2(z) \xi(z_k) + \phi_3(z_k)R \xi(z_k), \xi(z_k) \rangle}{1 - |\xi(z_k)|^2} \right|. \]

By condition (36)
\[
\frac{\omega(z_k) \left| \langle \phi_2(z) \xi(z_k) + \phi_3(z_k)R \xi(z_k), \xi(z_k) \rangle \right|}{1 - |\xi(z_k)|^2} \to 0 \text{ as } k \to \infty,
\]
from which condition (30) follows. □

**Corollary 4.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) be an analytic self map of \( \mathbb{B}^m \), \( \phi_1, \phi_2, \phi_3 \in H(\mathbb{B}^m) \) and suppose the functions \( \xi \) and \( \phi_2 \xi + \phi_3 R \xi \) are linearly dependent. Then, we have the following equivalent statements:
(a) \( T_{\phi_1, \phi_2, \phi_3, \xi} : A^\infty_r \to H^\infty_w \) is compact;
(b) \( T_{\phi_1, \phi_2, \phi_3, \xi} : A^\infty_r \to H^\infty_w \) is bounded and satisfy (28) and (29).

**Proof.** We omit the proof as it is easy to prove. □

In Corollary 4 if we take \( \phi_1(z) = \phi(z) \) and \( \phi_2(z) \equiv \phi_3(z) \equiv 0 \), then we get the operator \( W_{\phi, \xi} : A^\infty_r \to H^\infty_w \) whose compactness can be given by the following corollary:

**Corollary 5.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) be an analytic self map of \( \mathbb{B}^m \) and \( \phi \in H(\mathbb{B}^m) \). Then, the operator \( W_{\phi, \xi} : A^\infty_r \to H^\infty_w \) is compact if and only if \( W_{\phi, \xi} : A^\infty_r \to H^\infty_w \) is bounded and
\[
\lim_{|\xi(z)| \to 1} \omega(z)|\phi(z)|^{-1} \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) = 0.
\]
Theorem 5. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ denote a holomorphic self-map of $B^m$ and $\phi_1, \phi_2, \phi_3 \in H(B^m)$. Then, the following statements hold true

(i) If (26) and (27) hold, then the operator $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ is compact;

(ii) If $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ is compact, then (26) hold and

\[
\lim_{|z| \to 1} \omega(z) \left| \left( \phi_2(z) \xi(z) + \phi_3(z) \rho(\xi(z), \xi(z)) \right) \right|^{-1} \left( \frac{C_2}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) = 0.
\]  

Proof. (i) Initially, suppose that conditions (26) and (27) hold. Then, from Theorem 3 we get that $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ is bounded. Let $|z| \to 1$ in (4) and take supremum on the unit ball $\|f\|_{A^\infty_\omega} \leq 1$, we get that

\[
\limsup_{|z| \to 1} \sup_{\|f\|_{A^\infty_\omega} \leq 1} \omega(z) |T_{\phi_1, \phi_2, \phi_3} f(z)| = 0.
\]

Thus, on applying lemma 5, we get that the operator $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ is compact.

(ii) Assume that $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ is compact. This implies the compactness of $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$. Condition (b) in Theorem 4 implies that (28) and (30) hold. Thus, for $\varepsilon > 0$, there exists a $\delta \in (0, 1)$ for which (31) hold along with the condition

\[
\frac{\omega(z) \left| \left( \phi_2(z) \xi(z) + \phi_3(z) \rho(\xi(z), \xi(z)) \right) \right|^{-1} \left( \frac{C_2}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right)}{1 - |\xi(z)|^2} < \varepsilon,
\]

where $\delta < |\xi(z)| < 1$. Further, the compactness of $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ implies that $T_{\phi_1, \phi_2, \phi_3} : A^\infty_\omega \rightarrow H\omega$ is bounded. Using Theorem 2, we see that (23) and (24) hold. Thus, for any $\varepsilon > 0$, there exists a $\eta \in (0, 1)$ such that

\[
\omega(z) |\phi_1(z)| < \varepsilon \left( 1 \left( \frac{1}{(1 - \delta^2)^{m+\sigma+1}} \right) \right)
\]

and

\[
\omega(z) |\phi_2(z) \xi(z) + \phi_3(z) (R) \xi(z)| < \varepsilon \left( \frac{C_2}{(1 - \delta^2)^{m+\sigma+1}} \right),
\]

when $\eta < |z| < 1$. Therefore, with $\delta < |\xi(z)| < 1$ and $\eta < |z| < 1$ condition (31) implies that

\[
\omega(z) |\phi_1(z)| < \varepsilon \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) < \varepsilon.
\]

Again, for $|\xi(z)| < \delta$ and $\eta < |z| < 1$ condition (39) implies that

\[
\omega(z) |\phi_1(z)| < \varepsilon \left( \frac{1}{(1 - |\xi(z)|^2)^{m+\sigma+1}} \right) < \varepsilon.
\]

Thus, (26) follows from (41) and (42). Similarly, by using Cauchy-Schwartz inequality, (38) and (40) we can obtain (37). This completes the proof.

Corollary 6. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ be an analytic self map of $B^m$ and $\phi_1, \phi_2, \phi_3 \in H(B^m)$ and suppose the functions $\xi$ and $\phi_2 \xi + \phi_3 \rho \xi$ are linearly dependent. Then, we have the following equivalent statements:
(a) \( T_{\phi_1, \phi_2, \phi_3, \xi} : A_{\sigma}^\Psi \to H_{\omega, 0}^\infty \) is compact;
(b) (26) and (27) hold.

**Proof.** We omit the proof as it is easy to prove.  

In Corollary 6 if we take \( \phi_1(z) = \phi(z) \) and \( \phi_2(z) \equiv \phi_3(z) \equiv 0 \), then we get the operator \( W_{\phi, \xi} : A_{\sigma}^\Psi \to H_{\omega, 0}^\infty \) whose compactness can be given by the following corollary:

**Corollary 7.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \) be an analytic self map of \( B^m \) and \( \phi \in H(B^m) \). Then, the operator \( W_{\phi, \xi} : A_{\sigma}^\Psi \to H_{\omega, 0}^\infty \) is compact if and only if

\[
\lim_{|z| \to 1} \frac{|z|\omega(z)|\phi(z)|^\Psi^{-1}}{(1 - |\xi(z)|^2)^{m+\sigma+t}} = 0.
\]

5. Discussion and Conclusions

In this paper, we have considered the product type operators formed by the combination of composition, multiplication, differentiation and radial derivative operators acting between weighted Bergman–Orlicz spaces and weighted type spaces taken over the unit ball. We analysed these operators for basic properties including boundedness and compactness. The basic aim of this paper is to give the operator-theoretic characterization of these operators in terms of function–theoretic characterization of their including functions.

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