LARGE N LIMIT OF THE O(N) LINEAR SIGMA MODEL VIA STOCHASTIC QUANTIZATION

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Abstract. This article studies large N limits of a coupled system of N interacting $\Phi^4$ equations posed over $\mathbb{T}^d$ for $d = 1, 2$, known as the $O(N)$ linear sigma model. Uniform in N bounds on the dynamics are established, allowing us to show convergence to a mean-field singular SPDE, also proved to be globally well-posed. Moreover, we show tightness of the invariant measures in the large N limit. For large enough mass, they converge to the (massive) Gaussian free field, the unique invariant measure of the mean-field dynamics, at a rate of order $1/\sqrt{N}$ with respect to the Wasserstein distance. We also consider fluctuations and obtain tightness results for certain $O(N)$ invariant observables, along with an exact description of the limiting correlations in $d = 1$.

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1. Introduction

In this paper, we consider the following system of equations on the $d$-dimensional torus $\mathbb{T}^d$ for $d = 1, 2$

$$\mathcal{L} \Phi_i = -\frac{1}{N} \sum_{j=1}^{N} \Phi_j^2 \Phi_i + \xi_i, \quad \Phi_i(0) = \phi_i,$$

(1.1)

where $\mathcal{L} = \partial_t - \Delta + m$ with $m \geq 0$, $N \in \mathbb{N}$, and $i \in \{1, \cdots, N\}$. The collection $(\xi_i)_{i=1}^N$ consists of $N$ independent space-time white noises on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$. In $d = 2$, the system (1.1) requires renormalization, and the formal product $\Phi_j^2 \Phi_i$ will be interpreted as the Wick product $\Phi_j^2 \Phi_i$: whose definition is postponed to Section 3.

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This system arises as the stochastic quantization of the following $N$-component generalization of the $\Phi^4_3$ model, given by the (formal) measure
\[
d\nu_N^N(\Phi) \overset{\text{def}}{=} \frac{1}{C_N} \exp \left( - \int_{\mathbb{R}^d} \sum_{j=1}^N |\nabla \Phi_j|^2 + m \sum_{j=1}^N \Phi_j^2 + \frac{1}{2N} \left( \sum_{j=1}^N \Phi_j^2 \right)^2 \, dx \right) D\Phi \tag{1.2}
\]
onumber
over $\mathbb{R}^N$ valued fields $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N)$ and $C_N$ is a normalization constant. In $d = 2$, the interaction should be Wick renormalized : $\left( \sum_{j=1}^N \Phi_j^2 \right)^2$: for the measure to make sense. This is also referred to as the $O(N)$ linear sigma model, since this formal measure is invariant under a rotation of the $N$ components of $\Phi$. ¹ This symmetry will play an important role throughout the paper.

Our focus in this article is on the asymptotic behavior of the system (1.1) and its invariant measures $\nu_N$ as $N \to \infty$. Note that a factor $1/N$ has been introduced in front of the nonlinearity (resp. the quartic term in the measure), and heuristically, this compensates the sum of $N$ terms so that one could hope to obtain an interesting limit as $N \to \infty$. The study of physically meaningful quantities associated with a quantum field theory model (1.2) as $N \to \infty$ is generally referred to as a large $N$ problem; see Section 1.1 where we introduce more background, references in physics and mathematics, and different approaches to this problem. To the best of our knowledge, the present article provides the first rigorous results on large $N$ problems in the formulation of stochastic quantization.

In Theorem 1.1 below, we study the $N \to \infty$ limit of each component in the Wick renormalized version of (1.1) in $d = 2$, c.f. (3.1) below, and show that a suitable mean-field singular SPDE governs the limiting dynamics. Before giving the statement, let us first comment on the notion of solution used. Recall that the well-posedness of (1.1) in the case $N = 1$ and $d = 2$ (i.e. the dynamical $\Phi^4_3$ model) is now well developed: two classical works being [AR91] where martingale solutions are constructed and [DPD03] where strong solutions are addressed, as well as the more recent approach to global well-posedness in [MW17b]. These results can be generalized to the vector case (with fixed $N > 1$) without much extra effort. As in [DPD03] and [MW17b] the solutions are defined by the decomposition $\Phi_i = Z_i + Y_i$, where
\[
\mathcal{L}Z_i = \xi_i, \tag{1.3}
\]
\[
\mathcal{L}Y_i = -\frac{1}{N} \sum_{j=1}^N (Y_j^2 Y_i + Y_j^2 Z_i + 2Y_j Y_i Z_j + 2Y_j :Z_i Z_j; + :Z_j^2; Y_i + :Z_i Z_j^2;) \tag{1.4}
\]
and $:Z_i Z_j; , :Z_i Z_j^2$: are Wick renormalized products (see Section 3). For the uninitiated reader, note that (1.4) arises by inserting the decomposition of $\Phi_i$ into (1.1) and re-interpreting the ill-defined products $Z_i Z_j$ and $Z_i Z_j^2$ that appear.

The mean-field SPDE formally associated to (1.1) takes the form
\[
\mathcal{L}\Psi_i = -E[\Psi_i^2]|\Psi_i + \xi_i, \quad \Psi_i(0) = \psi_i. \tag{1.5}
\]
This equation arises naturally as follows: assuming the initial conditions $\{\phi_i\}_{i=1}^N$ are exchangeable, the components $\{\Phi_i\}_{i=1}^N$ will have identical laws, so that replacing the empirical average $\frac{1}{N} \sum_{j=1}^N \Phi_j^2$ in (1.1) by its mean and re-labelling $\Phi$ as $\Psi$ leads us to (1.5), at least formally. In two space dimensions, (1.5) is a singular SPDE.

¹The word “linear” here only means that the target space $\mathbb{R}^N$ is a linear space. “Nonlinear” sigma model on the other hand refers to similar models where $\Phi$ takes value in a sphere in $\mathbb{R}^N$, or more generally in a manifold.
where the ill-defined non-linearity depends on the law of the solution and similar
to (1.1) it also requires a renormalization. Postponing for the moment a more
complete discussion of this point, we now state our first main result.

**Theorem 1.1** (Large $N$ limit of the dynamics for $d = 2$). Let $\{(\phi_i^N, \psi_i)\}_{i=1}^N$ be
exchangeable random initial datum with components in $C^{-\kappa}$ for some small $\kappa > 0$
and all moments finite. Assume that for each $i \in \mathbb{N}$, $\phi_i^N$ converges to $\psi_i$ in
$L^p(\Omega; C^{-\kappa})$ for all $p > 1$, and $(\psi_i)$ are independent.

Then for each component $i$, the solution $\Phi_i^N$ defined by (1.5) with initial datum
$\phi_i^N$ satisfies for all $t > 0$
\[
\lim_{N \to \infty} E\|\Phi_i^N(t) - \Phi_i(t)\|^2_{L^2(\Omega)} = 0,
\]
where $\Phi_i$ is the unique solution to the mean-field SPDE formally described by
\[
\mathcal{L}\Phi_i = -E[\Psi_i^2 - Z_i^2]\Phi_i + \xi_i, \quad \Phi_i(0) = \psi_i,
\]
and $Z_i$ is the stationary solution to (1.3). Furthermore, for any $T > 0$ it holds that
$\Phi_i^N$ converges in probability to $\Phi_i$ in $C([0, T], C^{-1}(T^2))$ as $N \to \infty$.

Along the way to Theorem 1.1, we prove new uniform in $N$ bounds on the
Wick renormalized dynamics through suitable energy estimates on the remainder
equation (1.4). We are inspired in part by the approach in [MW17b], but subtleties
arise as we track carefully the dependence of the bounds on $N$. Indeed, the natural
approach (e.g. [MW17b] for dynamical $\Phi^4_3$ model) to obtain global in time bounds
for fixed $N$ is to exploit the damping effect from $Y_i^2Y_i$. However, the extra factor
$1/N$ before the nonlinear terms makes this effect weaker as $N$ becomes large. In
fact, the moral is that we cannot exploit the strong damping effect at the level
of a fixed component $Y_i$, rather we’re forced to consider aggregate quantities, and
ultimately we focus on the empirical average of the $L^2$-norm (squared), c.f. Lemma
3.3. This is natural on one hand due to the coupling of the components, but also
for the slightly more subtle point that we ought to respect the structure of the
mean-field SPDE (1.7), for which the damping effect seems to hold only in the
mean square sense, not at the path-by-path level.

In this direction, we now discuss a bit more the solution theory for the mean-field
SPDE (1.7). While the notion of solution we use is again via the Da-Prato/Debussche
trick, the well-posedness theory for (1.5) requires more care than for $\Phi^4_3$ since we
cannot proceed by pathwise arguments alone. In fact, similar to (1.3) – (1.4), we
understand (1.7) via the decomposition $\Psi_i = Z_i + X_i$ with $X_i$ satisfying
\[
\mathcal{L}X_i = -E[X_i^2]X_i + E[X_i^2]Z_i + 2E[X_iZ_i]X_i + 2E[X_iZ_i]Z_i.
\]
Here we actually introduce an independent copy $(X_j, Z_j)$ of $(X_i, Z_i)$, which turns
out to be useful for both the local and global well-posedness of (1.7). Indeed,
one point is that the term $E[X_jZ_j]Z_i$ in (1.8) cannot be understood in a classical
sense; however we can view it as a conditional expectation $E[X_jZ_jZ_i|Z_i]$ and use
properties of the Wick product $Z_jZ_j$ to give a meaning to this, c.f. Lemma 4.1.
Furthermore, to obtain global bounds, using this independent copy allows us to
approach the a priori estimates for (1.8) much like the uniform in $N$ bounds for
1.4. Indeed, after taking expectation, $E[X_j^2]X_i$ in (1.8) also plays the role of the
damping mechanism, which helps us to obtain uniform bounds on the mean-squared
$L^2$-norm of $X_i$ c.f. Lemma 4.3.

Theorem 1.1 can be viewed as a mean field limit result in the context of singular
SPDE systems. Our proof is indeed inspired by certain mean field limit techniques,
and we combine them with a priori estimates that are specific to our model. We
will provide more background discussion on mean field limits below in Section 1.2. By a classical coupling argument, this result also yields a propagation of chaos type statement: if the initial condition is asymptotically chaotic (i.e. independent components as $N \to \infty$) then, although the $\Phi$-system is interacting, as $N \to \infty$ the limiting system becomes decoupled ([Jab14, Def.3,Def.5]).

The second part of this paper (Section 6) is concerned with equilibrium theories, namely stationary solutions, invariant measures, and large $N$ convergence. For $N = 1$, the long-time behavior of the solutions was investigated in [RZZ17] and [TW18]. In the vector valued setting, by lattice approximation and [HM18b, HS19] it can be shown that $\nu^N$ is the unique invariant measure to (1.1) and the law of $\Phi_t(t)$ converges to $\nu^N$ as $t \to \infty$. Our goal then is to study the large $N$ limit of the $O(N)$ linear sigma model $\nu^N$. Our second main result yields the convergence of the unique invariant measure $\nu^N$ of (1.1) to the invariant measure of (1.7), provided the mass is sufficiently large.

To state the result, consider the projection onto the $i$th component,

$$\Pi_i : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d), \quad \Pi_i(\Phi) \overset{\text{def}}{=} \Phi_i.$$  

Noting that $\nu^N$ is a measure on $\mathcal{S}'(\mathbb{T}^d)^N$, we define the marginal law $\nu^{N,i} \overset{\text{def}}{=} \nu^N \circ \Pi_i^{-1}$. Furthermore, consider

$$\Pi^{(k)} : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d)^k, \quad \Pi^{(k)}(\Phi) = (\Phi_i)_{1 \leq i \leq k}$$

and define the marginal law of the first $k$ components by $\nu^N_k \overset{\text{def}}{=} \nu^N \circ (\Pi^{(k)})^{-1}$.

**Theorem 1.2** (Large $N$ limit of the invariant measures). There exists $m_0 > 0$ such that the following results hold:

- The Gaussian free field $\nu \overset{\text{def}}{=} \mathcal{N}(0, (m - \Delta)^{-1})$ is an invariant measure for (1.7).
- The sequence of probability measures $(\nu^{N,i})_{N \geq 1}$ are tight on $\mathcal{C}^{-\kappa}$ for $\kappa > 0$.
- For $m \geq m_0$, the Gaussian free field $\mathcal{N}(0, (m - \Delta)^{-1})$ is the unique invariant measure to equation (1.7).
- For $m \geq m_0$, $\nu^{N,i}$ converges to $\nu$ and $\nu^N_k$ converges to $\nu \times \cdots \times \nu$, as $N \to \infty$. Furthermore, $\mathcal{W}_2(\nu^{N,i}, \nu) \lesssim N^{-\frac{1}{2}}$.

These statements will follow from Theorem 6.9, Theorem 6.4 and Theorem 6.11. Here $\mathcal{W}_2(\nu_1, \nu_2)$ is the $\mathcal{C}^{-\kappa}$-Wasserstein distance defined before Theorem 6.11. The Gaussian free field limit is expected (at a heuristic level) by physicists e.g. [Wil73] and also in mathematical physics [Kup80b]. Our result Theorem 1.2 provides a precise justification provided $m \geq m_0$, with the convergence rate (which is expected to be optimal, see [JW18, Remark 4]) in terms of Wasserstein distance.

Note that the study of ergodicity properties of the dynamic (1.7) is nontrivial. In fact, the dynamic for $\Psi$ depends on the law of $\Psi$ itself, so the associated semigroup is generally nonlinear (see Section 6.1). As a result, the general ergodic theory for Markov process (see e.g. [DPZ96], [HMS11], [HM18b]) could not be directly applied here. Instead, we prove the solutions to (1.7) converge to the limit directly as time goes to infinity, which requires $m \geq m_0$.

We now comment on our approach to the fourth part of Theorem 1.2. It would be natural to try and use Theorem 1.1 together with the tightness result from the

\footnote{In [Wil73] it was written that “If one now looks at vacuum expectation values of individual $\Phi$ fields, all diagrams vanish like $1/N$ (at least), except for the free-field terms.” In the Introduction of [Kup80b] it was mentioned that “the $1/N$ expansion predicts that the theory is close to Gaussian as $N$ becomes large enough”, but this reference did not intend to prove this statement (see Section 1.1 below).}
second part of Theorem 1.2 to derive the convergence of $\nu^{N,i}$ to $\nu$ directly (see e.g. [HM18a]). However, it is not clear to the authors how to implement this strategy in the present setting. Indeed, to apply Theorem 1.1, it is important that each component $\psi_i$ of the initial data is independent of each other. However, we are not able to deduce that an arbitrary limit point $\nu^*$ has this property. If we use $P_t^\nu \nu^*$ to denote the solution to (1.7) starting from the initial distribution $\nu^*$, we cannot write $P_t^\nu \nu^*$ as $\int (P_t^\nu \delta_\psi) \nu^*(d\psi)$ due to the lack of linearity, which makes it difficult to overcome the assumption of independence. Alternatively, we follow the idea in [GH18] and construct a jointly stationary process $((\Phi, \Psi))$ whose components satisfy (1.1) and (1.7), respectively. In this case $\Psi = Z$, since the Gaussian free field gives the unique invariant measure to (1.7). We then establish the convergence of $\nu^{N,i}$ to $\nu$ by deriving suitable uniform estimates on the stationary process.

Our next result is concerned with observables, in the stationary setting. In QFT models with continuous symmetries, physically interesting quantities involve more than just a component of the field itself but also quantities composed by the fields which preserve the symmetries, called invariant observables. These acquire the same interest in SPDE (a natural example being the gauge invariant observables, e.g. [She18, Section 2.4]). In the present setting of (1.1), we consider $O(N)$ invariant observables: for $\Phi \sim \nu^N$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :\Phi_i^2: , \quad \frac{1}{N} \left( \sum_{i=1}^{N} \Phi_i^2 \right)^2 :.$$  \hspace{1cm} (1.9)

Here the Wick product is with respect to the Gaussian free field measure; the precise definition is given in Section 6.3. One could consider more general renormalized polynomials of $\sum_i \Phi_i^2$ but we choose to focus on the above two in this article. We establish the large $N$ tightness of these observables as random fields in suitable Besov spaces.

**Theorem 1.3.** Suppose that $\Phi \sim \nu^N$. For $m$ large enough, the following result holds for any $\kappa > 0$:

- $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :\Phi_i^2:$ is tight in $B_{2,2}^{-2\kappa}$.
- $\frac{1}{N} \left( \sum_{i=1}^{N} \Phi_i^2 \right)^2 :$ is tight in $B_{1,1}^{-3\kappa}$.

Note that the physics literature usually considers integrated quantities, i.e. partition function of correlations of these observables. Our SPDE approach allows us to study these observables as random fields with precise regularity as $N \to \infty$ which is new. The above result can also be shown in dimension one, and in fact we can obtain more refined information about the limit in this case. This is the focus of the final part of our paper. In $d = 1$ the equations are less singular and uniform estimates are simpler; we provide in Section 7 the arguments in $d = 1$ that simplify in comparison to $d = 2$ (while skipping details that follow essentially the same way as $d = 2$). A reader might also use Section 7 to grasp the main ideas of our analysis before embarking on the more challenging case of $d = 2$.

Moreover, in $d = 1$, we investigate the nontrivial statistics of the large $N$ limit of the $O(N)$ invariant observables. To this end, we suitably recenter the system (1.1), that is, the Wick ordered system defined by subtracting the finite variance of the stationary solution $Z_i$ (see (7.31) for the precise definition), and also Wick order the observables (see (7.32)). We show that although for large enough $m$ the invariant measure of $\Phi_i$ converges as $N \to \infty$ to the invariant measure of $Z_i$, i.e. Gaussian free field, the limits of the observables (1.9) have different laws than those
if $\Phi_i$ in (1.9) were replaced by $Z_i$:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :Z_i^2:, \quad \frac{1}{N} \left( \sum_{i=1}^{N} Z_i^2 \right)^2. \tag{1.10}$$

**Theorem 1.4** (Large N limit in $d = 1$). Under the same distributional assumption on the initial conditions as in Theorem 1.4 together with the additional hypothesis that for some $q > 1$, $E\|\phi_i^N\|_{L^q}^q + \|\psi_i\|_{L^q}^q + \|\psi_i\|_{L^\infty} \lesssim 1$, and $E\|\phi_i^N - \psi_i\|_{L^2}^2 \to 0$ as $N \to \infty$, for every $i \in \mathbb{N}$, $t > 0$, the solution $\Phi_i^N(t)$ to (1.1) starting from $\phi_i^N$ converge in $L^2(\Omega; C([0, T]; L^2(\mathbb{T})))$ to the solution of

$$\mathcal{L}\Psi_i = -E[\Psi_i^2]\Psi_i + \xi_i, \tag{1.11}$$

with $\Psi_i(0) = \psi_i$, where $\xi_i$ are the space-time white noises as in (1.1).

Moreover, there is a unique constant $\mu > 0$ such that the Gaussian free field $\nu \overset{\text{def}}{=} N(0, (m - \Delta + \mu)^{-1})$ is an invariant measure to (1.11). The sequence of measures $(\nu^{N,i})_{N \geq 1}$ are tight, and for large enough $m$, the above Gaussian free field is the unique invariant measure to equation (1.11), $\nu^{N,i}$ converges to $\nu$ and $\nu_N^{(i)}$ converges to $\nu \times \cdots \times \nu$, as $N \to \infty$.

Finally, when the system (1.1) is Wick ordered (see (7.31)), for $m$ large enough, the stationary solution $\Phi_i$ converges to the stationary solution $Z_i$ as $N \to \infty$; the observables (1.9) are tight in $L^2$ and $L^1$ respectively. However, the limiting laws of the observables (1.9) are different from those of (1.10); in fact, we have an explicit formula for the Fourier transform of the two point correlation function of $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :\Phi_i^2:$ as $N \to \infty$ given by $2\hat{C}^2/(1 + \hat{C}^2)$, where $C = (m - \Delta)^{-1}$ and $\hat{C}$ is the Fourier transform, and the formula for $E\frac{1}{N^2} \langle \Phi^2 \rangle^2(x)$ as $N \to \infty$ given by $-2 \sum_{k \in \mathbb{Z}} \hat{C}^2(k)^2/(1 + \hat{C}^2(k))$.

These results are proved in Theorem 7.6, Theorem 7.7, Theorem 7.8, Theorem 7.9 and Theorem 7.10. The last statement on correlation formulas of the observables are known – first heuristically by physicists who expressed these formulas in terms of the sum of “bubble” diagrams, and then derived in [Kup80b, Eq. (15)] using constructive field theory techniques such as “chessboard estimates”. Our new proofs of these correlation formulas using PDE methods are quite simple and straightforward once all the a priori estimates are available. We expect that these methods can be applied to study more $O(N)$ invariant observables and higher order correlations; we will pursue these in future work.

Let’s also mention the three dimensional construction of local solutions [Hai14, CC18], global solutions [MW17a, GH19, AK17], as well as a priori bounds in fractional dimension $d < 4$ by [CMW19], though we only focus on $d = 1, 2$ in this paper. It would also be interesting to see if our methodology could be used to study limits of other mathematical objects as dimensionality of the target space tends to infinity, such as coupled dynamical $\Phi^4_3$, coupled KPZ systems [FH17], random loops in $N$ dimensional manifolds [BGHZ19, Hai16] (and [RWZZ20, CWZZ18]) and Yang-Mills model [CCHS] with $N$ dimensional Lie groups (or abelian case [She18] with Higgs field generalized to value in $\mathbb{C}^N$). These are of course left to further work.

### 1.1. Large N problem in quantum field theory

Large $N$ methods (or “$1/N$ expansions”) in theoretical physics are ubiquitous and are generally applied to models where dimensionality of the target space is large. It was first used in [Sta68]

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3In fact, we have obtained some partial results for coupled dynamical $\Phi^4_3$, such as convergence of invariant measures to the Gaussian free field.
for spin models, and then developed in quantum field theories (QFT) which was pioneered by [Wit73] (Φ⁴ type and Fermionic models), [GN74] (Fermionic models), [%H74] (Yang-Mills model), and the idea was soon popularized and extended to many other systems - see [BW93] for an edited comprehensive collection of articles on large N as applied to a wide spectrum of problems in quantum field theory and statistical mechanics; see also the review articles [Wit80], [Col88, Chapter 8] and [MZJ03] for summaries of the progress. Loosely speaking, the large N expansion is an expansion in the parameter 1/N, as opposed to the ordinary perturbation theory in a coupling constant in QFT. Typically, each term in the large N expansion is a (formal) sum of infinitely many orders of the ordinary perturbation theory (so that even the zeroth order reveals non-trivial structure as N → ∞). Besides directly examining the perturbation theory, alternative methodologies of analyzing the large N expansion were discovered in physics, for instance a method via “dual” field [%CJP74], [MZJ03, Section 2], via Schwinger-Dyson equations [%Sym77], or via stochastic quantization (with references below).

Rigorous study of large N in mathematical physics was initiated by Kupiainen [Kup80c, Kup80b, Kup80a]. The literature most related to the present article is [Kup80b], which studied the QFT in continuum in d = 2 given by (1.2), and proved that the 1/N expansion of the pressure (i.e. vacuum energy, or log of partition per area) is asymptotic, and each order in this expansion can be described by sums of infinitely many Feynman diagrams of certain types. Borel summability of 1/N expansion of Schwinger functions for this model was discussed in [BR82].

In [Kup80c] Kupiainen also proved that on the lattice with fixed lattice spacing, the large N expansion of correlation functions of the N-component nonlinear sigma model (which simplifies to “spherical model” as N → ∞) is asymptotic above the spherical model criticality; asymptoticity was later extended to Borel summability by [FMR82]. Large N limit and expansion for Yang-Mills model has also been rigorously studied: see [Lév11] (also [AS12]) for convergence of Wilson loop observables to master field in the continuum plane, and [Cha19] (resp. [CJ16]) for computation of correlations of the Wilson loops in the large N limit (resp. 1/N expansion) which relates to string theory.

Large N problems in the stochastic quantization formalism have also been discussed in the physics literature, for instance [AS83], [DH87, Section 8]. [MZJ03, Section 5.1] is close to our setting; it makes an “ansatz” that \( \frac{1}{N} \sum_{j=1}^{N} \Phi_j^2 \) in (1.1) would self-average in the large N limit to a constant; our present paper justifies this ansatz and in the non-equilibrium setting generalizes it.

1.2. Mean field limits. As mentioned above the proof of our main theorems borrows some ingredients from mean field limit theory. To the best of our knowledge, the study of mean field problems originated from McKean [McK67]. Typically, a mean field problem is concerned with a system of N particles interacting with each other, which is often modeled by a system of stochastic ordinary differential equations, for instance, driven by independent Brownian motions. A prototype of such systems has the form \( dX_i = \frac{1}{N} \sum_j f(X_i, X_j) dt + dB_i \), see for instance the classical reference by Sznitman [Szn91, Sec 1(1)], and in the N → ∞ limit one could obtain decoupled SDEs each interacting with the law of itself: \( dY_i = \int f(Y_i, y) \mu(dy) dt + dB_i \), where \( \mu(dy) \) is the law of \( Y_i \). In simple situations the interaction \( f \) is assumed to be “nice”, for instance globally Lipschitz ([McK67]); much of the literature aims to prove such limits under more general assumptions.
on the interaction, see [Szn91] for a survey. Our Theorem 1.1 can be viewed as a result of this flavor, in an SPDE setting, and in fact the starting point of our proof is indeed close in spirit to [Szn91, Sec I(1)] where one subtracts $X_i$ from $Y_i$ to cancel the noise and then bound a suitable norm of the difference.

We note that mean field limits are studied under much broader frameworks or scopes of applications, such as mean field limit in the context of rough paths (e.g. [CL15, BCD18, CDFM18]), mean field games (e.g. survey [LL07]), quantum dynamics (e.g. [ESY10] and references therein). We do not intend to have a comprehensive list, but rather refer to survey articles [Gol16, Jab14] and the book [Spo91, Chapter 8] besides [Szn91].

The study of mean field limit for SPDE systems also has precursors, see for instance the book [KX95, Chapter 9] or [ES13]. However these results make strong assumptions on the interactions of the SPDE systems such as linear growth and globally Lipschitz drift, and certainly do not cover the singular regime where renormalization is required as in our case.

1.3. Structure of the paper. This paper is organized as follows. In Section 2, we collect the notations and useful lemmas used throughout the paper. Sections 3-5 are devoted to proof of Theorem 1.1. First in Section 3.1 we recall the definition of the renormalization for $Z_i$, which satisfies the linear equation (1.3). Then a uniform in $N$ estimate for the average of the $L^2$-norm of $Y_i$, the solutions to equation (1.4), is derived in Section 3.2. Local well-posedness to equation (1.7) is proved in Section 4.1. Global well-posedness to equation (1.7) is proved in Section 4.2 by combining a uniform $L^p$-estimate with Schauder theory. The difference estimate for $\Phi_i - \Psi_i$ is given in Section 5, which gives the proof of Theorem 1.1.

Section 6 is concerned with the proof of Theorem 1.2 and Theorem 1.3. In Section 6.1, uniqueness of invariant measures to (1.7) for large $m$ is proved. The convergence of invariant measures from $\nu^{N,i}$ to the Gaussian free field $\nu$ is shown in Section 6.2 by comparing the stationary solutions $(\Phi_i, Z_i)$. Section 6.3 is devoted to the study of the observables and the proof of Theorem 1.3.

Finally in Section 7 all the results have been derived in the one dimensional case and the proof of Theorem 1.4 is given. In Section 7.1 the convergence of the dynamics is shown with uniform in $N$ estimates. Section 7.2 is the corresponding part of the uniqueness of the invariant measure and the convergence of the invariant measures. Section 7.3 mainly concentrates on the proof of the nontriviality of the statistics of the observables. In Appendix A we give the proof of global well-posedness of equation (1.4). In Appendix B the application of Dyson-Schwinger equations has been derived, which is useful in studying the limiting law of the observables.

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\footnote{In the context of SDE systems, one also considers the empirical measures of the particle configurations, and aims to show their convergence as $N \to \infty$ to the McKean-Vlasov PDEs, which are typically deterministic. Note that in this paper we do not consider the "analogue" of McKean-Vlasov PDE (which would be infinite dimensional) in the context of our model.}
applications and the support by key Lab of Random Complex Structures and Data Science, Chinese Academy of Science.

2. Preliminary

Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Given a Banach space $E$ with a norm $\| \cdot \|_E$ and $T > 0$, we write $C_T E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to $E$, equipped with the supremum norm $\| f \|_{C_T E} = \sup_{t \in [0, T]} \| f(t) \|_E$. For $p \in [1, \infty]$ we write $L^p_T E = L^p([0, T]; E)$ for the space of $L^p$-integrable functions from $[0, T]$ to $E$, equipped with the usual $L^p$-norm. Let $S'$ be the space of distributions on $\mathbb{T}^d$. We use $(\Delta_1)_{i \geq -1}$ to denote the Littlewood–Paley blocks for a dyadic partition of unity. Besov spaces on the torus with general indices $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ are defined as the completion of $C^\infty$ with respect to the norm

$$\| u \|_{B^\alpha_{p,q}} : = \left( \sum_{j \geq -1} (2^j)^\alpha \| \Delta_j u \|_{L^p_T}^q \right)^{1/q},$$

and the Hölder-Besov space $C^\alpha$ is given by $C^\alpha = B^\alpha_{\infty, \infty}$. We will often write $\| \cdot \|_{C^\alpha}$ instead of $\| \cdot \|_{B^\alpha_{\infty, \infty}}$ in the following.

Set $\Lambda = (1 - \Delta)^{\frac{1}{2}}$. For $s \geq 0$, $p \in [1, \infty]$ we use $H^p_s$ to denote the subspace of $L^p$, consisting of all $f$ which can be written in the form $f = \Lambda^{-s} g$, $g \in L^p$ and the $H^p_s$ norm of $f$ is defined to be the $L^p$ norm of $g$, i.e. $\| f \|_{H^p_s} : = \| \Lambda^s f \|_{L^p}$. For $s < 0$, $p \in (1, \infty)$, $H^p_s$ is the dual space of $H^p_{-s}$. Let $\Phi : = H^2_{-1}$.

The following embedding results will be frequently used.

**Lemma 2.1.** (i) Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B^\alpha_{p_1,q_1} \subset B^\alpha_{p_2,q_2}$. (cf. [GIP15, Lemma A.2])

(ii) Let $s \in \mathbb{R}$, $1 < p < \infty$, $\epsilon > 0$. Then $H^s_{\epsilon} \subset B^s_{2,2}$, and $B^s_{p,1} \subset H^s_{\epsilon} \subset B^s_{p,\infty}$. (cf. [Tri78, Theorem 4.6.1])

(iii) Let $1 \leq p_1 \leq p_2 < \infty$ and let $\alpha \in \mathbb{R}$. Then $H^\alpha_{p_1} \subset H^{\alpha-d(1/p_1-1/p_2)}_{p_2}$. Here $\subset$ means continuous and dense embedding.

A good reference for these results is [Tri78].

We recall the following interpolation inequality and multiplicative inequality for the elements in $H^p_{\alpha}$.

**Lemma 2.2.** (i) Suppose that $s \in (0, 1)$ and $p \in (1, \infty)$. Then for $f \in H^1_p$

$$\| f \|_{H^\alpha_p} \lesssim \| f \|_{L^p_T} \| f \|_{H^s_p}.$$

(cf. [Tri78, Theorem 4.3.1])

(ii) Suppose that $s > 0$ and $p \in [1, \infty)$. It holds that

$$\| \Lambda^s (fg) \|_{L^p} \lesssim \| f \|_{L^{p_1}} \| \Lambda^s g \|_{L^{p_2}} + \| g \|_{L^{p_3}} \| \Lambda^s f \|_{L^{p_4}},$$

with $p_i \in (1, \infty)$, $i = 1, \ldots, 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

(cf. see [GO14, Theorem 1])

(iii) (Gagliardo-Nirenberg inequality) For $s \in [0, 1)$, $\alpha \in (0, 1)$, $r \geq 1$,

$$\| u \|_{H^\alpha_q} \lesssim \| u \|_{H^s_p} \| u \|_{L^r_{\nu}}^{1-\alpha}.$$
with $\frac{1}{q} = \frac{1}{d} + \frac{1 - \alpha}{p}$.

**Lemma 2.3.** (i) Let $\alpha, \beta \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\]
The bilinear map $(u, v) \mapsto uv$ extends to a continuous map from $B^\alpha_{p_1, q} \times B^\beta_{p_2, q}$ to $B^{\alpha + \beta}_{p, q}$ if $\alpha + \beta > 0$. (cf. [MW17b, Corollary 2])

(ii) (Duality.) Let $\alpha \in (0, 1)$, $p, q \in [1, \infty]$, $p'$ and $q'$ be their conjugate exponents, respectively. Then the mapping $(u, v) \mapsto \int uv dx$ extends to a continuous bilinear form on $B^\alpha_{p, q} \times B^{-\alpha}_{p', q'}$, and one has $|\langle u, v \rangle| \lesssim \|u\|_{B^\alpha_{p, q}} \|v\|_{B^{-\alpha}_{p', q'}}$ (cf. [MW17b, Proposition 7]).

We recall the following smoothing effect of the heat flow $S_t = e^{t(\Delta - m)}$, $m \geq 0$ (e.g. [GIP15, Lemma A.7], [MW17b, Proposition 5]).

**Lemma 2.4.** Let $u \in B^\alpha_{p, q}$ for some $\alpha \in \mathbb{R}, p, q \in [1, \infty]$. Then for every $\delta \geq 0$
\[
\|S_t u\|_{B^{\alpha + \delta}_{p, q}} \lesssim t^{-\delta/2} \|u\|_{B^\alpha_{p, q}},
\]
where the proportionality constant is independent of $t$.

**Lemma 2.5.** For $s \in (0, 1)$
\[
\langle g, f \rangle \lesssim (\|\nabla g\|_{L^1} \|g\|_{L^s}^{1-s} + \|g\|_{L^1}) \|f\|_{C^{-s}}.
\]

**Proof.** This follows from Lemma 2.3 which states that $\langle g, f \rangle$ is a continuous bilinear form on $B^1_{1,1} \times C^{-s}$, together with [MW17b, Proposition 8] which states that
\[
\|g\|_{B^1_{1,1}} \lesssim \|\nabla g\|_{L^1} \|g\|_{L^s}^{1-s} + \|g\|_{L^1}.
\]

We also recall the following comparison test result, which has been proved in [TW18, Lemma 3.8].

**Lemma 2.6.** Let $f : [0, T] \to [0, \infty)$ differentiable such that for every $t \in [0, T]$
\[
\frac{d f}{d t} + c_1 f^2 \leq c_2.
\]
Then for $t > 0$
\[
f(t) \leq \left( t^{-\frac{2}{c_1}} \right) \vee \left( \frac{2c_2}{c_1} \right)^{\frac{1}{2}}.
\]

### 3. Uniform in $N$ bounds on the dynamical linear sigma model

In this section, we obtain new estimates on the Wick renormalized version of (1.1), given by
\[
\mathcal{L} \Phi_i = -\frac{1}{N} \sum_{j=1}^{N} \phi_j^2 \Phi_i + \xi_i, \quad \Phi_i(0) = \phi_i.
\]
The notion of solution to (3.1) is the same as in [DPD03] and [MW17b], where the case $N = 1$ is treated. For a fixed $N$, these well-posedness results are easy to generalize to the present setting, so we only give the statement here and refer the reader to Appendix A for the proof. Our primary goal in this section is rather to obtain bounds which are stable with respect to the number of components $N$, which we will send to infinity in Section 5.
As is well known, it’s natural to consider initial datum to (3.1) belonging to a negative Hölder space with exponent just below zero. We will be slightly more general and consider random initial datum of the form \( \phi(t) = z_i + y_i \) defined on the same stochastic basis \((\Omega, \mathcal{F}, \mathbb{P})\) and satisfying \( \mathbb{E}\|z_i\|^{p}_{C^{-\kappa}} < 1 \) for \( \kappa > 0 \) small enough and every \( p > 1 \), and \( \mathbb{E}\|y_i\|_2 < 1 \), where the implicit constants are independent of \( i, N \). Furthermore, we impose the qualitative hypothesis that \( (\phi_i)_i \) is independent of the space-time white noises \( (\xi_i)_i \).

The notion of solution to (3.1) is based on the now classical trick of Da-Prato and Debussche, c.f. [DPD03]. Namely, we say that \( \Phi \) is a solution to (3.1) provided the decomposition \( \Phi = Z_i + Y_i \) holds, where \( Z_i \) is a solution to the linear SPDE

\[
\mathcal{L}Z_i = \xi_i, \quad Z_i(0) = z_i, \tag{3.2}
\]

and \( Y_i \) is a weak solution to the remainder equation

\[
\mathcal{L}Y_i = -\frac{1}{N} \sum_{j=1}^{N} (Y_j Y_i + Y_i Y_j + 2 Y_j Z_i + 2 Y_j Z_j + Z_i Z_j + Z_j Z_i); \quad Y_i(0) = y_i. \tag{3.3}
\]

The notation \( :Z_i Z_j; \), \( :Z_i^2; \) and \( :Z_i Z_j^2; \) denotes a renormalized product of Wick type which will be defined in Section 3.1 below.

### 3.1. Renormalization

To define the renormalized products appearing in (3.3), it is convenient to make a further splitting of \( Z_i \) relative to the corresponding stationary solution to (3.2), which we will denote by \( \tilde{Z}_i \). For \( \tilde{Z}_i \), these products have a canonical definition that we now recall. Namely, let \( \xi_{i, \varepsilon} \) be a space-time white noise defined on \( \mathbb{R} \times \mathbb{T}^2 \) and let \( \tilde{Z}_{i, \varepsilon} \) be the stationary solution to

\[
\mathcal{L}\tilde{Z}_{i, \varepsilon} = \xi_{i, \varepsilon}. \tag{3.3}
\]

For convenience, we assume that all the noises are mollified with a common bump function. In particular, \( \tilde{Z}_{i, \varepsilon} \) are i.i.d. mean zero Gaussian. For \( k \geq 1 \) and \( i_1, \ldots, i_k \in \{1, \ldots, N\} \) we then write \( \tilde{Z}_{i_1, \varepsilon} \cdots \tilde{Z}_{i_k, \varepsilon}^{;} \) as the limit of \( \tilde{Z}_{i_1, \varepsilon}^{;} \cdots \tilde{Z}_{i_k, \varepsilon}^{;} \) as \( \varepsilon \to 0 \). Here \( \tilde{Z}_{i_1, \varepsilon}^{;} \cdots \tilde{Z}_{i_k, \varepsilon}^{;} \) is the canonical Wick product, which in particular is mean zero. More precisely,

\[
\tilde{Z}_{i_1}^{;} \tilde{Z}_{j_1}^{;} = \begin{cases} 
\lim_{\varepsilon \to 0} (\tilde{Z}_{i_1, \varepsilon}^{3} - a_{i_1, \varepsilon} \tilde{Z}_{i_1, \varepsilon}^{;}) & (i = j), \\
\lim_{\varepsilon \to 0} \tilde{Z}_{i_1, \varepsilon}^{;} \tilde{Z}_{j_1, \varepsilon}^{;} & (i \neq j) 
\end{cases},
\]

\[
\tilde{Z}_{i_1}^{;} \tilde{Z}_{j_1}^{;} = \begin{cases} 
\lim_{\varepsilon \to 0} (\tilde{Z}_{i_1, \varepsilon}^{3} - 3a_{i_1, \varepsilon} \tilde{Z}_{i_1, \varepsilon}^{;}) & (i = j), \\
\lim_{\varepsilon \to 0} \tilde{Z}_{i_1, \varepsilon}^{;} \tilde{Z}_{j_1, \varepsilon}^{;} - a_{i_1, \varepsilon} \tilde{Z}_{i_1, \varepsilon}^{;} & (i \neq j) 
\end{cases}
\]

where \( a_{i_1, \varepsilon} = \mathbb{E} [\tilde{Z}_{i_1, \varepsilon}^{3}(0, 0)] \) is a diverging constant independent of \( i \) and the limits are understood in \( C^{\kappa}_T \mathbb{C}^{-\kappa} \) for \( \kappa > 0 \). (see [MW17b] for more details).

We now define the Wick products for \( Z_i \) by combining the above with the smoothing properties of the heat semi-group \( S_t \) associated with \( \mathcal{L} \). Defining \( \tilde{z}_i = z_i - Z_i(0) \), we have the decomposition

\[
Z_i = \tilde{Z}_i + S_t \tilde{z}_i.
\]

We then overload notation and define the Wick products of \( Z_i \) by the binomial formula \(^5\) namely

\[
:Z_i^2; = :\tilde{Z}_i^2; + 2S_t \tilde{z}_i \tilde{Z}_j + (S_t \tilde{z}_j)^2, \\
:Z_i^3; = :\tilde{Z}_i^3; + 3S_t \tilde{z}_i \tilde{Z}_j^2; + 3(S_t \tilde{z}_j)^2 \tilde{Z}_j + (S_t \tilde{z}_j)^3;
\]

\(^5\)This definition is in line with [MW17b, (5.42)], which first considers a linear solution with 0 initial condition rather than a stationary solution as here.
and for \( i \neq j \)
\[
:Z_i Z_j: = \hat{Z}_i \hat{Z}_j: + S_i \hat{Z}_j \hat{Z}_i + S_i \hat{Z}_i \hat{Z}_j + S_i \hat{Z}_i S_i \hat{Z}_j,
\]
\[
:Z_i Z_j^2: = \hat{Z}_i \hat{Z}_j^2: + S_i \hat{Z}_j : \hat{Z}_i : + 2 S_i \hat{Z}_j : \hat{Z}_j:
+ 2 S_i \hat{Z}_i S_i \hat{Z}_j + (S_i \hat{Z}_j)^2 \hat{Z}_i + S_i \hat{Z}_i (S_i \hat{Z}_j)^2.
\]

We caution the reader that this definition is non-canonical, in the sense that these renormalized products are not necessarily mean zero. By the calculation in [MW17b, Corollary 3] (see also [RZZ17, Lemma 3.5]) we have the following estimate:

**Lemma 3.1.** For each \( \kappa' > \kappa > 0 \) and all \( p \geq 1 \), we have the following bounds
\[
\mathbb{E}\|\hat{Z}_i\|_{C_T\mathcal{C}^{\kappa}}^p + \mathbb{E}\|Z_i\|_{C_T\mathcal{C}^{-\kappa}}^p \lesssim 1.
\]
\[
\mathbb{E}\|\hat{Z}_i \hat{Z}_j: \|_{C_T\mathcal{C}^{-\kappa}}^p + \mathbb{E}\|\hat{Z}_i \hat{Z}_j^2: \|_{C_T\mathcal{C}^{-\kappa}}^p \lesssim 1.
\]
\[
\mathbb{E}( \sup_{t \in [0,T]} t^\kappa ' \|Z_i Z_j: \|_{\mathcal{C}^{-\kappa}}^p + \mathbb{E}( \sup_{t \in [0,T]} t^{2\kappa '} \|Z_i Z_j^2: \|_{\mathcal{C}^{-\kappa}}^p \lesssim 1.
\]

Furthermore, the proportional constants in the inequalities are independent of \( i, j, N \).

By Lemma 3.1, there exists a measurable set \( \Omega_0 \subset \Omega \) with \( \mathbb{P}(\Omega_0) = 1 \) such that for \( \omega \in \Omega_0 \) and every \( i, j \)
\[
\|Z_i\|_{C_T\mathcal{C}^{-\kappa}} + \sup_{t \in [0,T]} t^\kappa ' \|Z_i Z_j: \|_{\mathcal{C}^{-\kappa}} + \sup_{t \in [0,T]} t^{2\kappa '} \|Z_i Z_j^2: \|_{\mathcal{C}^{-\kappa}} < \infty.
\]

In the following we always consider \( \omega \in \Omega_0 \). With the above choice of renormalization, classical arguments from [DPD03] can be used to obtain local existence and uniqueness to equation (3.3) by a pathwise fixed point argument. This solution can also be shown to be global, as a simple consequence of a much stronger result, Lemma 3.3, which will be established in detail below. Since the well-posedness arguments for solving equation (3.3) with a fixed number of components is essentially known, we relegate the proof to Appendix A and only state the result here.

**Lemma 3.2.** For each \( N \), there exist unique global solutions \( (Y_i) \) to equation (3.3) such that for \( 1 \leq i \leq N, Y_i \in C_T L^2 \cap L^4_T L^4 \cap L^4_T H^1 \).

3.2. Uniform in \( N \) estimate. We now turn to our uniform in \( N \) bounds on equation (3.3) and note that \( Y_i \) itself depends on \( N \), but we omit this throughout. In the following lemma, we show that the empirical averages of the \( L^2 \) norms of \( Y_i \) can be controlled pathwise in terms of averages of the \( C_T \mathcal{C}^{-\kappa} \) norms of \( Z_i, :Z_i Z_j: \) and \( :Z_i^2 Z_j: \); discussed in Lemma 3.1.

**Lemma 3.3.** Let \( s \in [2\kappa, \frac{1}{2}] \). There exists a universal constant \( C \) such that
\[
\frac{1}{N} \sup_{t \in [0,T]} \sum_{j=1}^N \|Y_j\|_{L^2}^2 + \frac{1}{N} \sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^N Y_i^2\|_{L^2_T L^2}^2 \leq C \int_0^T R_N dt + \frac{1}{N} \sum_{j=1}^N \|y_j\|_{L^2}^2,
\]
where
\[
R_N := 1 + \left( \frac{1}{N} \sum_{j=1}^N \|Z_j\|_{\mathcal{C}^{-\kappa}}^2 \right)^{\frac{1}{2\kappa}} + \left( \frac{1}{N} \sum_{j=1}^N \|Z_j^2: \|_{\mathcal{C}^{-\kappa}} \right)^{\frac{1}{2\kappa}}.
\]
\[ + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \left< :Z_j Z_i^2 : \right>_{C^{-s}} \right)^{\frac{2}{2s}} + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \left< :Z_j^2 Z_i : \right>_{C^{-s}} \right). \]

**Proof.** The proof is based on an energy estimate. In Step 1 we establish the energy identity (3.6) which identifies the coercive quantities and involves three types of terms on the RHS. These are labelled \( I_N^1 \), \( I_N^2 \), and \( I_N^3 \), which are respectively linear, quadratic, and cubic in \( Y \). In Steps 2-4, we estimate each of these quantities, proceeding in order of difficulty, in terms of the coercive terms and the quantities \( R_N^i \) for \( i = 1, 2, 3 \) defined below. The main ingredient is Lemma 2.5, restated here: for \( s \in (0, 1) \)

\[ \left< g, f \right> \lesssim \left( \| \nabla g \|_{L^2} \| g \|_{L^2}^{-s} + \| g \|_{L^2} \right) \| f \|_{C^{-s}}. \]  

(3.5)

The final output of Steps 1-4 is that for some universal constant \( C \) it holds

\[ \frac{1}{N} \sum_{i=1}^{N} \frac{d}{dt} \| Y_i \|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^{N} \| \nabla Y_i \|_{L^2}^2 + \frac{1}{N^2} \sum_{i=1}^{N} Y_i^2 \leq \frac{m}{N} \sum_{j=1}^{N} \| Y_j \|_{L^2}^2 \]

\[ \leq C \frac{R_N^1}{N} + C(R_N^2 + R_N^3) \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2. \]

Noting that by Hölder’s inequality

\[ \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 = \frac{1}{N} \left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^1} \leq \frac{1}{N} \left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2}, \]

the estimate (3.4) follows from Young’s inequality with exponents \((2, 2)\) and an integration over \([0, T]\). The condition \( s \in [2\kappa, \frac{1}{2})\) ensures that \( R_N \) is integrable on near the origin, c.f. Lemma 3.1.

**Step 1 (Energy balance)**

In this step, we justify the energy identity

\[ \frac{1}{2} \sum_{i=1}^{N} \frac{d}{dt} \| Y_i \|_{L^2}^2 + \sum_{i=1}^{N} \| \nabla Y_i \|_{L^2}^2 + m \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 + \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i^2 \right\|_{L^2}^2 \]

\[ = I_N^1 + I_N^2 + I_N^3, \]  

(3.6)

where the quantities \( I_N^i \) for \( i = 1, 2, 3 \) are defined by

\[ I_N^1 = \frac{1}{N} \sum_{i,j=1}^{N} \left< Y_i, :Z_j^2 Z_i : \right> \]

\[ I_N^2 = -\frac{1}{N} \sum_{i,j=1}^{N} \left< Y_i Y_j, :Z_i Z_j : \right> + \left< Y_i^2, :Z_j^2 : \right> \]

\[ I_N^3 = \frac{1}{N} \sum_{i,j=1}^{N} \left< Y_i^2 Y_j, Z_j \right> \]  

(3.7)

Notice that \( I_N^1, I_N^2, \) and \( I_N^3 \) are linear, quadratic, and cubic in \( Y \), respectively. Formally, the identity (3.6) follows from testing (3.3) by \( Y_i \), integrating by parts, summing over \( i = 1, \ldots, N \), and using symmetry with respect \( i \) and \( j \). Since \( Y_i \) is not sufficiently smooth in the time variable, some care is required to make this fully rigorous, and we direct the reader to [MW17b, Proposition 6.8] for more details.

**Step 2 (Estimates for \( I_N^1 \))**
In this step, we show there is a universal constant $C$ such that
\[
I_N^1 \leq \frac{1}{4} \left( \sum_{i=1}^{N} \|\nabla Y_i\|_2^2 + m \sum_{i=1}^{N} \|Y_i\|_2^2 + 1_{m=0} \sum_{i=1}^{N} \|Y_i\|_2^2 + C R_N^1, \right)
\] (3.8)
where
\[
R_N^1 \stackrel{\text{def}}{=} \sum_{i=1}^{N} \left\| \frac{1}{N} \sum_{j=1}^{N} :Z_j^2 Z_i: \right\|^2_{C^{-s}}.
\]
To establish (3.8), we apply (3.5) with $Y_i$ playing the role of $g$ and $\frac{1}{N} \sum_{j=1}^{N} :Z_j^2 Z_i:$ playing the role of $f$ to find
\[
I_N^1 \lesssim \sum_{i=1}^{N} \left( \|Y_i\|_{L^1} \|\nabla Y_i\|_{L^1} + \|Y_i\|_{L^1} \right) \left( \|Y_i\|_{L^2} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \frac{1}{N} \sum_{j=1}^{N} :Z_j^2 Z_i: \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{N} \|Y_i\|_{L^2} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \frac{1}{N} \sum_{j=1}^{N} :Z_j^2 Z_i: \right)^{\frac{1}{2}}, \tag{3.9}
\]
where we applied Hölder’s inequality for the summation in $i$ using exponents $(\frac{2}{1-s}, \frac{2}{2})$ for the first term and $(2, 2)$ for the second term. We now use Young’s inequality with the same exponents and then the embedding of $L^2$ into $L^1$ to obtain (3.8), considering separately the cases $m = 0$ and $m > 0.$

STEP 3 (Estimates for $I_N^2$)
In this step, we show there is a universal constant $C$ such that
\[
I_N^2 \leq \frac{1}{4} \sum_{i=1}^{N} \|\nabla Y_i\|_2^2 + C(1 + R_N^2) \left( \sum_{i=1}^{N} \|Y_i\|_2^2 \right), \tag{3.10}
\]
where
\[
R_N^2 \stackrel{\text{def}}{=} \left( \sum_{i,j=1}^{N} \| :Z_j Z_i: \|_{C^{-s}}^2 \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{N} \| :Z_j Z_i: \|_{C^{-s}}^2 \right)^{\frac{1}{2}}.
\]
Applying (3.5) with $Y_i Y_j$ playing the role of $g$ and $:Z_j Z_i:$ playing the role of $f$ followed by Hölder’s inequality in $L^2$, the product rule and symmetry with respect to $i, j$ we find
\[
\frac{1}{N} \sum_{i,j=1}^{N} \langle Y_i Y_j, :Z_j Z_i: \rangle \lesssim \frac{1}{N} \sum_{i,j=1}^{N} \left( \|Y_i Y_j\|_{L^1}^{1-s} \|\nabla (Y_i Y_j)\|_{L^1}^s + \|Y_i Y_j\|_{L^1} \right) \| :Z_j Z_i: \|_{C^{-s}} \lesssim \frac{1}{N} \sum_{i,j=1}^{N} \left( \|Y_i\|_{L^2} \|Y_j\|_{L^2} \|\nabla Y_i\|_{L^2} + \|Y_i\|_{L^2} \|Y_j\|_{L^2} \right) \| :Z_j Z_i: \|_{C^{-s}} \lesssim \left( \sum_{j=1}^{N} \|Y_j\|_{L^2} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^{2(1-s)} \|\nabla Y_i\|_{L^2}^s \right)^{\frac{1}{2}} \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| :Z_j Z_i: \|_{C^{-s}}^2 \right)^{\frac{1}{2}}.
\]
where we used Hölder’s inequality for the summation in $i, j$ with exponents $(2, 2)$ followed by Hölder’s inequality for the summation in $i$ with exponents $(\frac{1}{1-s}, \frac{2}{s})$. Finally, applying (3.5) with $Y_i^2$ playing the role of $g$ and $\frac{1}{N} \sum_{j=1}^N :Z_j^2:$ playing the role of $f$ we find

\[
\frac{1}{N} \sum_{i,j=1}^N \langle Y_i^2, :Z_j^2: \rangle \\
\lesssim \sum_{i=1}^N (\|Y_i^2\|_L^2) \|Y_i \nabla Y_i\|_{L^1} + \|Y_i^2\|_{L^1}) \left( \frac{1}{N} \sum_{j=1}^N :Z_j^2: \right) \|_{C^{-s}} \\
\lesssim \sum_{i=1}^N (\|Y_i^2\|_L^2 + \|Y_i\|_{L^2}^2) \left( \frac{1}{N} \sum_{j=1}^N :Z_j^2: \right) \|_{C^{-s}} \\
\lesssim \left( \sum_{i=1}^N \|Y_i^2\|_L^2 \right)^{1-\frac{2}{s}} \left( \sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^{\frac{2}{s}} + \sum_{i=1}^N \|Y_i^2\|_L^2 \left( \frac{1}{N} \sum_{j=1}^N :Z_j^2: \right) \|_{C^{-s}},
\]

where we used Hölder’s inequality for the summation in $i$ with exponents $(\frac{2}{2-s}, \frac{2}{s})$. The inequality (3.10) now follows from (3.11)-(3.12) by Young’s inequality with exponents $(\frac{2}{2-s}, \frac{2}{s})$.

STEP 4 (Estimates for $I_N^3$: cubic terms in $Y$)

In this step, we show there exists a universal constant $C$ such that

\[
I_N^3 \leq \frac{1}{4} \left( \sum_{i=1}^N \|\nabla Y_i\|_L^2 + \left( \frac{1}{\sqrt{N}} \sum_{i=1}^N \|Y_i^2\|_{L^2} \right)^2 \right) + C(1 + R_N^3) \left( \sum_{i=1}^N \|Y_i^2\|_{L^2} \right),
\]

where

\[
R_N^3 \overset{\text{def}}{=} \left( \frac{1}{N} \sum_{j=1}^N \|Z_j\|_{C^{-s}} \right)^{1-s}.
\]

Appealing again to (3.5), we find

\[
I_N^3 \lesssim \frac{1}{N} \sum_{j=1}^N \left( \left\| \sum_{i=1}^N Y_i^2 Y_j \right\|_{L^1} \left\| \nabla \left( \sum_{i=1}^N Y_i^2 Y_j \right) \right\|_{L^1} + \left\| \sum_{i=1}^N Y_i^2 Y_j \right\|_{L^1} \right) \|Z_j\|_{C^{-s}} \\
\lesssim \frac{1}{N} \left( \sum_{j=1}^N \left\| \sum_{i=1}^N Y_i^2 Y_j \right\|_{L^1} \left\| \nabla \left( \sum_{i=1}^N Y_i^2 Y_j \right) \right\|_{L^1} \right)^{2(1-s)} \left( \sum_{j=1}^N \|Z_j\|_{C^{-s}} \right)^{1/2} + \frac{1}{N} \left( \sum_{j=1}^N \left\| \sum_{i=1}^N Y_i^2 Y_j \right\|_{L^1} \right)^{1/2} \left( \sum_{j=1}^N \|Z_j\|_{C^{-s}} \right)^{1/2}.
\]
By Hölder’s inequality, it holds that
\[ \left\| \sum_{i=1}^{N} Y_i^2 Y_j \right\|_{L^1} \lesssim \left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2} \left\| Y_j \right\|_{L^2}. \] (3.15)

Furthermore, we find that
\[
\left\| \nabla \left( \sum_{i=1}^{N} Y_i^2 Y_j \right) \right\|_{L^1} \lesssim \left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2} \left\| \nabla Y_j \right\|_{L^2} + \left( \sum_{i=1}^{N} \left\| \nabla Y_i \right\|_{L^2} \right)^{1/2} \left( \sum_{i=1}^{N} Y_i^2 Y_j \right)^{1/2} \] (3.16)

Hence, we find that
\[
\sum_{j=1}^{N} \left\| \sum_{i=1}^{N} Y_i^2 Y_j \right\|_{L^1}^{2(1-s)} \left\| \nabla \left( \sum_{i=1}^{N} Y_i^2 Y_j \right) \right\|_{L^1}^{2s} \lesssim \left( \sum_{i=1}^{N} Y_i^2 \right)^{2} \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2}^{2(1-s)} \right)^{1-s} \left( \sum_{j=1}^{N} \left\| \nabla Y_j \right\|_{L^2}^{2} \right)^{s} \] 
\[
+ \left( \sum_{i=1}^{N} Y_i^2 \right)^{2(1-s)} \left( \sum_{j=1}^{N} \left\| \nabla Y_i \right\|_{L^2}^{2} \right)^{s} \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2}^{2} \right)^{1-s} \left( \sum_{j=1}^{N} \left\| \nabla Y_j \right\|_{L^2}^{2} \right)^{s} . \]

Inserting this into (3.14), taking the square root, and using (3.15) we find
\[
I_N^1 \lesssim \frac{1}{N} \left\| \sum_{i=1}^{N} Y_i \right\|_{L^2} \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2} \right)^{1-s} \left( \sum_{j=1}^{N} \left\| \nabla Y_j \right\|_{L^2} \right)^{s} \left( \sum_{j=1}^{N} \left\| Z_j \right\|_{C^{2-s}} \right)^{1/2} 
+ \frac{1}{N} \left\| \sum_{i=1}^{N} Y_i \right\|_{L^2} \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2} \right)^{1-s} \left( \sum_{j=1}^{N} \left\| \nabla Y_j \right\|_{L^2} \right)^{s} \left( \sum_{j=1}^{N} \left\| Z_j \right\|_{C^{2-s}} \right)^{1/2} . \] (3.17)

Applying Young’s inequality with exponent \((2, \frac{2}{1-s}, \frac{2}{s})\) we arrive at (3.13).

\[ \square \]

**Corollary 3.4.** Let \( q \geq 1, s \in [2\kappa, \frac{2}{q+1}) \). There exists a universal constant \( C \) such that
\[
\sup_{t \in [0,T]} \left( \frac{1}{N} \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2}^{2} \right)^{q} + \int_{0}^{T} \left( \frac{1}{N} \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2}^{2} \right)^{q-1} \left[ \frac{1}{N} \sum_{j=1}^{N} \left\| \nabla Y_j \right\|_{L^2}^{2} + \left\| \frac{1}{N} \sum_{i=1}^{N} Y_i^{2} \right\|_{L^2}^{2} \right] dt 
\leq C \int_{0}^{T} R_N^{\frac{2}{2+s}} dt + \left( \frac{1}{N} \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2}^{2} \right)^{q} ,
\]

with \( R_N \) introduced in Lemma 3.3.
Proof. Set \( V = \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \) and \( G = \frac{1}{N} \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2}^2 + \| \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \|_{L^2}^2 \). By Lemma 3.3 we deduce for \( q \geq 1 \)

\[
\frac{d}{dt} V^q + G V^{q-1} \leq C R_N V^{q-1} \leq C R_N^{q+1} + \frac{1}{2} V^{q+1}.
\]

Note that \( G \geq V^2 \) since \( \sum_{i=1}^{N} Y_i^2 \|_{L^1} = \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \), which implies the result. \( \square \)

Lemma 3.5. Let \( s \in [2\kappa, 1/4) \). There exists a universal constant \( C \) such that

\[
\sup_{t \in [0,T]} \sum_{j=1}^{N} \| Y_j \|_{L^2}^2 + \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \|_{L^2}^2 \leq C \left( \sum_{i=1}^{N} \| A_i \|_{L^p} \right) \exp \left\{ \int_{0}^{T} \left( 1 + R_N^2 + R_N^3 \right) dt \right\},
\]

where \( R_N^2, R_N^3 \) given in the proof of Lemma 3.3 and

\[
R_N^0 = \frac{1}{N^2} \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} A_j^{\infty} \left( : Z_j^2 Z_i : \right) \right\|_{L^2}^2.
\]

Proof. The proof is almost the same as Lemma 3.3. We appeal to Steps 1, 3, and 4 of Lemma 3.3 and only modify Step 2. To estimate \( I_N^{2,0} \) we write

\[
I_N^{2,0} \leq \frac{1}{N} \sum_{i=1}^{N} \| \Lambda^s Y_i \|_{L^2} \left\| \sum_{j=1}^{N} \Lambda^{-s} \left( : Z_j^2 Z_i : \right) \right\|_{L^2} \leq \frac{1}{8} \sum_{i=1}^{N} \| Y_i \|_{L^2} + \frac{1}{8} \sum_{i=1}^{N} \| \nabla Y_i \|_{L^2}^2 + \frac{C}{N^2} \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} \Lambda^{-s} \left( : Z_j^2 Z_i : \right) \right\|_{L^2}^2,
\]

where, in the last step, we applied Young’s inequality for products, and then interpolation. Combining (3.10), (3.11) (3.12) with (3.13) and inserting these inequalities into the energy identity (3.6), we obtain

\[
\sum_{j=1}^{N} \frac{d}{dt} \| Y_j \|_{L^2}^2 + \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \|_{L^2}^2 \leq C R_N^0 + \sum_{j=1}^{N} \| Y_j \|_{L^2} C (1 + R_N^2 + R_N^3).
\]

The estimate (3.18) now follows from Gronwall’s inequality. \( \square \)

Remark 3.6. For the estimate of \( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \) in Lemma 3.3, the dissipation term \( \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \) could be used to avoid Gronwall’s Lemma. However, for \( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \) or \( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^p}^p \) for \( p > 2 \) the corresponding dissipation term is less clear how to exploit and we need to use Gronwall’s inequality to derive a uniform estimate. Since the \( R_N^2, R_N^3 \) appear in the exponential, this makes it unclear how to obtain moment estimates directly.
4. Global solvability of the mean-field SPDE

In this section, we develop a solution theory for the mean-field SPDE (1.7), the renormalized version of the formal equation

$$\mathcal{L}\Psi_i = -E[\Psi_i^2]\Psi_i + \xi_i.$$

We work here in the two dimensional setting, so (1.5) is a singular SPDE where the ill-defined non-linearity depends on the law of the solution. As a result, we cannot proceed via path-by arguments alone as in [DPD03] and [MW17b] and we need to develop a few new tricks for both the local and global well-posedness.

We begin by explaining our assumptions on the initial data and our notion of solution to (1.7). The initial datum $\psi_i$ decompose as $\psi_i = z_i + \eta_i$, where $E[z_i]^{p}_{C^{-\kappa}} < \infty$ for $\kappa > 0$ and every $p > 1$ and $E[\eta_i]^{1}_{L^4} \lesssim 1$. Both parts of the initial condition are defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$ and independent of the space-time white noises $(\xi_i)_{i \in \mathbb{N}}$. We define $\Psi_i$ to be a solution to the renormalized, mean-field SPDE (1.7) starting from $\psi_i$ provided that $\Psi_i = Z_i + X_i$ holds, where $Z_i$ is the solution to (3.2) with $Z_i(0) = z_i$ as in Section 3 and $X_i$ is a random process satisfying

$$\mathcal{L}X_i = -\mu(X_i + Z_i), \quad X_i(0) = \eta_i.$$  

Here, $\mu$ depends on the law of $X_i$ and is defined as

$$\mu \overset{\text{def}}{=} E[X_i^2] + 2E[X_i Z_i] + E[\mid Z_i^2].$$

We now comment on the meaning of the non-linearity in equation (4.1). Since $Z_i \in C^{-\kappa}$ by Lemma 3.1, while $E[\mid Z_i^2] = E[(S_i \hat{z}_i)^2]$ with $\hat{z}_i = z_i - Z_i(0)$, and hence is smooth, c.f. Section 3.1, by Schauder theory we expect that $X_i$ is Hölder continuous. Hence, we anticipate that $E[X_i^2]$ is a well-defined function, while $E[X_i Z_i]$ is a distribution satisfying for $t > 0$ and $\beta > \kappa$

$$\|E[X_i Z_i(t)]\|_{C^{-\kappa}} \lesssim \|E[X_i(t)]\|_{C^{s}}\|Z_i(t)\|_{C^{-\kappa}}.$$

We immediately find that all terms in $\mu(X_i + Z_i)$ are classically defined in the sense of distributions except for $E[X_i Z_i]Z_i$, which requires more care and a suitable probabilistic argument. The idea used to overcome this difficulty, which is repeated in different ways throughout the section, is to view the expectation $\mu$ as coming from a suitable independent copy of $(X_i, Z_i)$. To avoid notational confusion, we now comment further on our convention throughout this section. We consider equation (1.7) for a fixed $i$ and we write $(\eta_j, z_j)$ for $j \neq i$ we mean an independent copy of $(\eta_i, z_i)$, and when we write $Z_j$ (resp. $X_j$) it means an independent copy of $Z_i$ (resp. $X_i$).

4.1. Local well-posedness.

Lemma 4.1. For $p \in [1, \infty]$ and $0 < \kappa < s$ it holds

$$\|Z_i E[Z_i X_i]\|_{B^\kappa_{p, \infty}} \lesssim (E[\|X_i\|_{B^\kappa_{p, \infty}}^2])^{\frac{1}{2}} (E[\|Z_i Z_j\|_{C^{-\kappa}}]^{2} \mid Z_i])^{\frac{1}{2}}.$$  

Here the conditional expectation is on the $\sigma$-algebra generated by the stochastic process $Z_i$.

Proof. Letting $(X_j, Z_j)$ be an independent copy of $(X_i, Z_i)$ we have

$$Z_i E[Z_i X_i] = Z_i E[Z_j X_j] = E[\mid Z_i Z_j\mid X_j \mid Z_i].$$
We then use Jensen’s inequality to find
\[ \| Z_t \mathbb{E}[ Z_j X_j ] \|_{B^\beta_{r, \infty}} \leq \mathbb{E} \left[ \| Z_t Z_j X_j \|_{B^\beta_{r, \infty}} \mid Z_i \right] \leq \mathbb{E} \left[ \| Z_i Z_j \|_{C^{-\gamma}} \| X_j \|_{B^\beta_{r, \infty}} \mid Z_i \right], \]
where we used Lemma 2.3 in the last line. The claim now follows from conditional Hölder’s inequality and the independence of \( X_j \) from \( Z_i \).

\[ \square \]

We now apply the above result to obtain a local well-posedness result for (4.1), which yields in turn a local well-posedness result for (1.7).

**Lemma 4.2.** There exists \( T^* > 0 \) small enough such that (4.1) has a unique mild solution \( X_i \in C_T L^4 \cap C((0, T^*]; C^\beta) \) such that for \( \beta > 3\kappa \) small enough
\[ \mathbb{E} \left[ \sup_{t \in [0, T^*]} t^\gamma \| X_i \|_{C^\beta}^2 \right] \leq 1, \]
for \( \gamma = \beta + \frac{1}{2} \).

**Proof.** For \( T > 0 \) define the ball
\[ B_T \overset{\text{def}}{=} \left\{ X_i \in C((0, T]; C^\beta) \mid \mathbb{E} \left[ \sup_{t \in [0, T]} t^\gamma \| X_i \|_{C^\beta}^2 \right] \leq 1, \quad X(0) = \eta_i \right\}. \]

For \( X_i \in B_T \), define \( \mathcal{M}_T X_i : (0, T] \to C^\beta \) via
\[ \mathcal{M}_T X_i(t) := \int_0^t S_{t-s} \mathbb{E}[ X_i^2 + 2 X_i Z_i + 2 Z_i^2 ] (X_i + Z_i) ds + S_t \eta_i. \]

Using Lemma 2.4 and Lemma 2.3 noting \( \beta > \kappa \), we find that
\[ \| \mathcal{M}_T X_i(t) - S_t \eta_i \|_{C^\beta} \leq \int_0^t (t-s)^{-\frac{\alpha+\kappa}{2}} \mathbb{E}[ X_i Z_i \| Z_i \|_{C^{-\gamma}} ds \]
\[ + \int_0^t \left( \| \mathbb{E} X_i^2 \|_{C^\beta} + \| \mathbb{E} Z_i^2 \|_{C^\beta} + (t-s)^{-\frac{\alpha+\kappa}{2}} \mathbb{E}[ X_i Z_i \|_{C^{-\gamma}} X_i \|_{C^\beta} ds \]
\[ + \int_0^t (t-s)^{-\frac{\alpha+\kappa}{2}} \mathbb{E}[ X_i^2 \|_{C^\beta} + \| \mathbb{E} Z_i^2 \|_{C^\beta} \| Z_i \|_{C^{-\gamma}} ds \]
\[ = \sum_{i=1}^3 J_i(t). \]

We start by applying Lemma 4.1 to obtain the pathwise bound
\[ J_i(t) \leq \int_0^t (t-s)^{-\frac{\alpha+\kappa}{2}} \left( \mathbb{E}[ X_i \|_{C^\beta}^2 ] \right)^{\frac{1}{2}} \left( \mathbb{E}[ Z_i Z_j \|_{C^{-\gamma}} Z_i \|_{C^\beta} ] \right)^{\frac{1}{2}} ds \]
\[ \leq \left( \mathbb{E} \left[ \sup_{r \in [t, T]} (r^\nu \| Z_i Z_j \|_{C^{-\gamma}} Z_i \|_{C^\beta} )^2 \right] \right)^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{\alpha+\kappa}{2}} s^{-\frac{\gamma+2\kappa}{2}} ds, \]
for \( \kappa' > \kappa > 0 \), provided \( \beta < 2 - \kappa \) and \( \gamma < 2 - 2\kappa' \). We may now apply Lemma 3.1 to find that
\[ \mathbb{E}[J_i(t)]^2 \leq \mathbb{E} \left[ \sup_{r \in [t, T]} (r^\nu \| Z_i Z_j \|_{C^{-\gamma}} Z_i \|_{C^\beta} )^2 \right] t^{2-(\beta+\kappa+2\kappa'+\gamma)} \leq t^{2-(\beta+\kappa+2\kappa'+\gamma)}. \]

Before estimating \( J_2(t) \) and \( J_3(t) \), we make three observations. First note that by Jensen’s inequality and Lemma 2.3 it holds
\[ \| \mathbb{E} Z_i^2 \|_{C^\beta} \leq \mathbb{E}[ \| X_i \|_{C^\beta}^2 ] \leq s^{-\gamma}. \]

Furthermore, using again Lemma 2.4 we find
\[ \| \mathbb{E} Z_i^2 \|_{C^\beta} = \mathbb{E}[ (S_\xi Z_i)^2 ] \leq \mathbb{E}[ S_\xi Z_i \|_{C^\beta}^2 ] \leq s^{-(\beta+\kappa)}. \]
Finally, note that
\[ \|E[X_t Z_t]\|_{C^{-\gamma}} \lesssim E[\|X\|_{C^\beta}\|Z\|_{C^{-\gamma}}] \lesssim (E[\|X\|_{C^\beta}^2])^{1/2} (E[\|Z\|_{C^{-\gamma}}^2])^{1/2} \lesssim s^{-\tilde{\gamma}}. \]
Inserting these three bounds, we find the inequalities
\[ J_2(t) \lesssim \int_0^t (s^{-\gamma} + s^{-(\beta+\gamma)} + (t-s)^{-\frac{\beta+\gamma}{2}}) \|X_t\|_{C^\beta} \, ds. \]
\[ J_3(t) \lesssim \int_0^t (t-s)^{-\frac{\beta+\gamma}{2}} (s^{-\gamma} + s^{-(\beta+\gamma)}) \|Z_t\|_{C^{-\gamma}} \, ds. \]

Squaring and taking expectation, we find
\[ E[J_2(t)]^2 \lesssim t^{-\gamma} (t^{2-2\gamma} + t^{2-2(\beta+\gamma)} + t^{2-(\beta+\gamma+\gamma)}). \]
\[ E[J_3(t)]^2 \lesssim (t^{2-(\beta+\gamma+\gamma)} + t^{2-3(\beta+\gamma)}). \]

Finally, note that by Lemma 4.3 and the embedding \( L^4 \subset C^{-\frac{1}{2}} \), we obtain
\[ \|S_t \eta\|_{C^\beta} \lesssim t^{-\frac{1+2\beta}{4}} \|\eta\|_{L^4}. \]
Combining the above estimates we could find \( T^* \) small enough to have
\[ E\left[ \sup_{t \in [0,T]} t^{-\gamma} \|M_T X(t)\|_{C^\beta}^2 \right] \lesssim 1, \]
which implies that for \( T^* \) small enough \( M_T \) maps \( B_T \) into itself. The contraction property follows similarly. Now the local existence and uniqueness in \( C((0,T), C^\beta) \) follows. Furthermore, we know \( \int_0^T S_{t-s} \mu(X(s) + Z(s)) \, ds \) is continuous in \( C^\beta \) and \( S_t \eta \in C_T L^4 \). The result follows. \( \square \)

4.2. Global well-posedness. We now extend our local solution to a global solution through a series of a priori bounds, starting with a uniform in time on the \( L^2(\Omega; L^2) \) norm of \( X_t \) together with an \( L^2(\Omega; L^2 H^1) \) bound.

**Lemma 4.3.** There exists a universal constant \( C \) such that
\[ \sup_{t \in [0,T]} E[\|X_t\|_{L^2}^2 + E[\|\nabla X_t\|_{L^2}^2 + \|E X^2\|_{L^2}^2 + mE[\|X_t\|_{L^2}^2] \leq C \int_0^T R \, dt + E[\|\eta\|_{L^2}^2], \quad (4.3) \]
where, for \( i \neq j \) we define
\[ R \overset{\text{def}}{=} 1 + \left( E[\|Z_i\|_{C^{-\gamma}}] \right)^{\frac{1}{1-\gamma}} + E[\|Z_i^2 Z_j\|_{C^{-\gamma}}] + C(E[\|Z_j Z_i\|_{C^{-\gamma}}]^2 + C(E[\|Z_i^2\|_{C^{-\gamma}}] + C(E[\|Z_j^2\|_{C^{-\gamma}}]^4). \]

**Proof.** The proof is similar in spirit to the proof of Lemma 3.3, proceeding by energy estimates.

**Step 1 (Expected energy balance)**
In this step, we establish the following identity
\[ \frac{1}{2} \frac{d}{dt} E[\|X_t\|_{L^2}^2 + E[\|\nabla X_t\|_{L^2}^2 + \|E X^2\|_{L^2}^2 + mE[\|X_t\|_{L^2}^2] = I^1 + I^2 + I^3, \quad (4.4) \]
where
\[ I^1 \overset{\text{def}}{=} E(X_i, Z_i : Z_j^2), \]
\[ I^2 \overset{\text{def}}{=} E(X_i^2, Z_j^2) + 2E(X_i Z_j, Z_j Z_j), \]
\[ I^3 \overset{\text{def}}{=} 3E(X_i^2 X_j, Z_j). \]
Formally, testing (4.1) with $X_i$, integrating by parts and using that $X_i, X_j Z_i$, and $Z_i^2$ are respectively equal in law to $X_j, X_j Z_j$, and $Z_j^2$: we find

$$
\frac{1}{2} \frac{d}{dt} \|X_i\|_{L^2}^2 + \|\nabla X_i\|_{L^1}^2 + m \|X_i\|_{L^2}^2 + \|X_i^2 E Z_i\|_{L^1}^2
$$

Taking expectation on both sides, using independence, and the fact that $X_i Z_i$ with exponents

$$
= - \langle X_i, Z_i E (Z_i^2) \rangle - \langle X_i^2, E (Z_i^2) \rangle - 2 \langle X_i, Z_i E (X_j Z_j) \rangle
$$

To prove the claim, we apply (4.5) with Young’s inequality followed by two applications of Hölder’s inequality to obtain

$$
I \lesssim E \left( \|X_i\|_{L^2}^2 + \|\nabla X_i\|_{L^1} \right) \|Z_i^2 Z_i: \|_{C^{-r}}
$$

Hence, (4.5) follows from the inequality $E \|X_i\|_{L^2}^2 \leq \|E X_i^2\|_{L^2}$ and Young’s inequality with exponents $(\frac{2}{2}, \frac{2}{2})$ and $(2, 2)$.

**Step 2 (Estimates for $I^1$)**

In this step, we show there is a universal constant $C$ such that

$$
I^1 \lesssim \frac{1}{4} \left( \|E X_i^2\|_{L^2}^2 + \|E \nabla X_i\|_{L^2}^2 \right) + C(1 + E \|Z_i^2 Z_i: \|_{C^{-r}}).
$$

Using again (3.5), Hölder’s inequality, and the independence of $X_i$ and $X_j$ we obtain $E (X_i X_j, Z_j Z_i: )$

$$
\lesssim E \left( \|X_i X_j\|_{L^2} + \|\nabla X_i X_j\|_{L^1} + \|X_i X_j \nabla X_j\|_{L^1} \right) \|Z_i Z_i: \|_{C^{-r}}
$$

Similarly, using this time independence of $X_j^2$ and $Z_j^2$, we obtain $E (X_j^2, Z_j^2: )$

$$
\lesssim E \left( \|X_j^2\|_{L^1} + \|\nabla X_j\|_{L^2} \right) \|Z_j^2 Z_j: \|_{C^{-r}}.
$$

To obtain (4.7) we use Young’s inequality with exponents $(2, 2)$ and $(2, 4)$ for both (4.8) (4.9).
Step 4 (Estimates for $I^3$) 
In this step, we show there is a universal constant $C$ such that

$$I^3 \leq \frac{1}{4}(E\|\nabla X_i\|_{L^2}^2 + E\|X_i\|_{L^2}^2) + C\left((E\|Z_j\|_{C^{-s}}^2)^{\frac{1}{1-s}} + 1\right).$$

(4.10)

To this end, we write

$$I^3 \leq E\left(\|X_i^2 X_j\|_{L^1} + \|\nabla(X_i^2 X_j)\|_{L^1} + \|X_i^2 X_j\|_{L^1}\right)\|Z_j\|_{C^{-s}}$$

$$\leq (E\|X_i^2 X_j\|_{L^1}\|Z_j\|_{C^{-s}}) + (E\|\nabla(X_i^2 X_j)\|_{L^1}\|Z_j\|_{C^{-s}})^{1-s}$$

$$+ E\|X_i^2 X_j\|_{L^1}\|Z_j\|_{C^{-s}}.$$

By independence and Hölder’s inequality, it holds that

$$E\|X_i^2 X_j\|_{L^1}\|Z_j\|_{C^{-s}} \leq \|E X_i^2 E(X_j\|Z_j\|_{C^{-s}})\|_{L^1},$$

$$\leq \|E X_i^2\|_{L^2}\left(\|E X_j\|^2\|Z_j\|_{C^{-s}}\right)^{1/2}$$

$$\leq \|E X_i^2\|_{L^2}\left(\|E \nabla X_j\|_{L^2}\right)^2\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2},$$

(4.11)

where we used that $\|(E X_j^2)^{1/2}\|_{L^2} = \|E X_j^2\|_{L^1}^{1/2}$.

Furthermore,

$$E\|X_i^2 \nabla X_j\|_{L^1}\|Z_j\|_{C^{-s}}$$

$$\leq \left\|E X_i^2\right\|_{L^2}\left\|E \left(\|\nabla X_j\|_{L^2}\|Z_j\|_{C^{-s}}\right)\right\|_{L^2}$$

$$\leq \left\|E X_i^2\right\|_{L^2}\left(\|E \nabla X_j\|_{L^2}\right)^{1/2}\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2}$$

(4.12)

Similarly, note that

$$E\|X_i X_j \nabla X_i\|_{L^1}\|Z_j\|_{C^{-s}}$$

$$\leq E\|X_i X_j\|_{L^2}\left(\|\nabla X_i\|_{L^2}\|Z_j\|_{C^{-s}}\right)$$

$$\leq \left(\|E X_i\|_{L^2}^2\right)^{1/2}\left(\|E \nabla X_i\|_{L^2}\right)^{1/2}\|E\|Z_j\|_{C^{-s}}\}^{1/2}$$

$$\leq \left\|E X_i^2\right\|_{L^2}\left(\|E \nabla X_i\|_{L^2}\right)^{1/2}\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2}.$$ (4.13)

Combining the above estimate we arrive at

$$I^3 \leq \left\|E X_i^2\right\|_{L^2}\left(\|E \nabla X_i\|_{L^2}^{1-s}\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2}\right)$$

$$+ \left\|E X_i^2\right\|_{L^2}\left(\|E X_i\|_{L^2}^{1/2}\right)^{1/2}\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2}$$

(4.14)

$$\leq \left\|E X_i^2\right\|_{L^2}^{1/2}\left(\|E \nabla X_i\|_{L^2}^{1-s}\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2}\right) + \left\|E X_i\right\|_{L^2}^{1/2}\left(\|E\|Z_j\|_{C^{-s}}\right)^{1/2}.$$ 

Applying Young’s inequality with exponents $(\frac{1}{1-s}, \frac{1}{2}, \frac{1}{1-s})$ we arrive at (4.10).

In Section 5 we will study the large N limit of (3.1) by comparing the dynamics of each component to the corresponding mean-field evolution. To control the equation for the difference, we will need a stronger control on $X_i$ than the $L^2$ type bound obtained above. In the following lemma, we show that $L^p$ bounds can be propagated in time, which will turn out to be a necessary ingredient in Section 5.

**Lemma 4.4.** Let $p > 2$ and assume that $E\|\eta_i\|_{L^p}^p \lesssim 1$. Then, we have

$$\sup_{t \in [0,T]} \left\|E\|X_i\|_{L^p}^p + E\|X_i\|_{L^2}^{1-\frac{2}{s}} \nabla X_i\|_{L^2}^{2}\|E\|X_i\|_{L^2}^p \left\|E X_i^2\right\|_{L^1,L^1} \lesssim 1,$$
where the implicit constant is independent of $i$.

Proof. Given $p > 2$, we fix $s > 0$ sufficiently small such that $sp < \frac{1}{2}$ and $\frac{2}{p} + s < 1$. We will perform an $L^p$ estimate: integrating (4.1) against $|X_i|^{p-2}X_i$ we get

$$
\frac{1}{p} \frac{d}{dt} E \|X_i\|^p_{L^p} + (p-1)E \|X_i\|^{p-2} \|\nabla X_i\|^2_{L^1} + E \|X_i\|^{p+2} \|X_i\|_{L^p} + mE \|X_i\|_{L^p}^2
$$

$$
= -2E \left( E[X_j Z_i] \right) |X_i|^p - E \left( E[X_j] \right) |X_i|^{p-2}X_i, Z_i) - 2E \left( E[X_j Z_i] Z_i |X_i|^{p-2}X_i \right) + E \left( E[ : Z_j^2 : ] \right) |X_i|^p + E \left( E[ : Z_j^2 : ] |X_i|X_i|^{p-2}, Z_i \right) =: \sum_{k=1}^5 I_k. \quad (4.15)
$$

Set

$$
D \overset{\text{def}}{=} \|X_i^{p-2}\|\|\nabla X_i\|^2_{L^1}, \quad A \overset{\text{def}}{=} \|X_i^p X_i^2\|_{L^1}.
$$

Below we denote by $\varepsilon \in (0, \frac{1}{10})$ a fixed constant, and $C > 0$ a constant which differs from line to line, and we will frequently exploit the independence of $(X_i, Z_i)$ with $(X_j, Z_j)$.

Step 1 (Estimate of $I_1$)

By Lemma 2.5, we have

$$
I_1 \lesssim E \left[ \|X_j X_i^p 1^{\frac{1}{p_1}} \|_{L^p} \|\nabla \langle X_j, X_i \rangle \|_{C^{-\alpha}} \right] + E \left[ \|X_j X_i^p 1^{\frac{1}{p_1}} \|_{L^p} \|Z_i\|_{C^{-\alpha}} \right] =: I_1^{(1)} + I_1^{(2)}.
$$

In order to obtain reasonable bounds in terms of $A$ and $D$, it will be helpful to view $X_j X_i^p = X_j (X_i^2)^{\frac{1}{2}}$. By Cauchy-Schwarz and independence

$$
I_1^{(2)} \leq E \left[ \|X_i^p X_i^2 1^{\frac{1}{p_1}} \|_{L^p} \|X_i^p X_i^2 1^{\frac{1}{p_1}} \|_{L^p} \|Z_i\|_{C^{-\alpha}} \right] \lesssim \varepsilon EA + CE \|X_i^p \|_{L^p} E \|Z_i\|_{C^{-\alpha}}.
$$

Regarding $I_1^{(1)}$, one has

$$
\|X_j X_i^p\|_{L^1} \lesssim A^{1/2} \|X_i^{\frac{p}{2}}\|_{L^1}.
$$

Moreover, using Hölder inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ where $p_1 < 2$ and $p_2 > 2$, and then with $\frac{1}{p_3} + \frac{1}{p_2} = \frac{1}{p}$, we have

$$
\|\nabla \langle X_j, X_i \rangle \|_{L^1} \lesssim \|\nabla \langle X_j, X_i \rangle \|_{L^p} \|X_j\|_{L^{p_3}} \|X_i\|_{L^{p_2}} \lesssim \|X_j\|_{L^{p_3}} \|X_i\|_{L^{p_2}} \|\nabla X_i^{\frac{1}{2}}\|_{L^2}
$$

$$
\lesssim \|X_j\|_{L^{p_3}} \|X_i\|_{L^{p_2}} + \|X_j\|_{L^{p_3}} \|\nabla X_i^{\frac{1}{2}}\|_{L^2} + \|X_i\|_{L^{p_3}} \|\nabla X_j^{\frac{1}{2}}\|_{L^2} + A^{1/2} D^{1/2}
$$

$$
\lesssim D^{1+\alpha/2} \|X_j\|_{H^1} \|X_i\|_{L^{p_2}}^{1-\alpha} + D^{1/2} \|X_j\|_{H^1} \|X_i\|_{L^{p_2}}^{1/2} + D^{1/2} \|X_j\|_{H^1} \|X_i\|_{L^{p_2}}^{1/2},
$$

for $\alpha \overset{\text{def}}{=} 1 - \frac{2}{p_2} < 1$. Here in the last inequality, we used the 2d embedding

$$
H^1 \subset L^p \quad (\forall p > 1), \quad H^1 \subset L^{p_2}, \quad H^{1-\alpha} \subset L^{p_3}
$$

together with interpolation; for instance in the first term we have

$$
\|X_j\|_{L^{p_2}}^{1-\alpha} \lesssim \|X_j\|_{H^1}^{1-\alpha} \lesssim \|X_j\|_{L^{p_2}}^{1-\alpha} \|X_j\|_{H^1}^{1-\alpha} \lesssim \|X_j\|_{L^{p_2}}^{1-\alpha} D^{\alpha/2} + \|X_j\|_{L^2},
$$

By the above two bounds, we obtain

$$
I_1^{(1)} \lesssim \sum_{\beta \in (0, \alpha)} E \left[ A^{1-\alpha} \|X_i^p\|_{L^1}^{1-\alpha} \|\nabla X_j^{\frac{1}{2}}\|_{L^2} \|X_j\|_{H^1} \|Z_j\|_{C^{-\alpha}} \right]
$$

$$
+ E \left[ A^{1-\alpha} \|X_i^p\|_{L^1}^{1-\alpha} \|X_j\|_{H^1} \|Z_j\|_{C^{-\alpha}} \right] + E \left[ A^{1-\alpha} \|X_i^p\|_{L^1}^{1-\alpha} \|X_j\|_{H^1} \|Z_j\|_{C^{-\alpha}} \right]
$$

$$
\lesssim \varepsilon EA + \varepsilon ED + C E \|X_i^p\|_{L^1} \times \left( \sum_{\beta \in (0, \alpha)} E \|X_j\|_{L^{p_2}}^{2/(1-s \beta)} \|Z_j\|_{C^{-\alpha}}^{2/(1-s \beta)} + E \|Z_j\|_{C^{-\alpha}}^{2/1-s} + E \|X_j\|_{H^1}^{2/(1-s)} \|Z_j\|_{C^{-\alpha}}^{2/1-s} \right).
$$


where in the last inequality we used independence and Young’s inequality for products with exponents \((\frac{2}{1-s}, \frac{2}{1-s})\) for the first term and \((2, \frac{2}{1-s}, \frac{2}{s})\) for the second term.

Noting that \(s/(1-s\beta) < 1\) by our choice of small \(s\), by Lemma 4.3 and Lemma 3.1, the integral over \([0, T]\) of the quantity in the parenthesis above is bounded by a constant. Therefore we have obtained a bound of the form

\[
I_1 \leq 2\varepsilon E A + \varepsilon E D + E[\|X_i^p\|_{L^1}] R_N^1, \quad \text{with} \quad \int_0^T R_N^1 \lesssim 1. \tag{4.16}
\]

**Step 2. (Estimates for \(I_2\))**

For the second term on the right hand side of (4.15) we use Lemma 2.2 to have

\[
I_2 = E(\Lambda^s(X_i^p - X_i)_{X_i}^{p-2}, \Lambda^{-s} Z_i)
\]

\[
\lesssim E[\|\Lambda^s(X_i^p - X_i)\|_{L^p} \|X_i^{p-1}\|_{L^{p-1}} \|\Lambda^{-s} Z_i\|_{L^\infty}] + E[\|X_i^2\|_{L^p} \|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{p-1}} \|\Lambda^{-s} Z_i\|_{L^\infty}]
\]

\[
=: I_2^{(1)} + I_2^{(2)}.
\]

For \(I_2^{(1)}\), by independence one only needs to bound the following two expectations

\[
E[\|X_i\|_{L^p}^{p-1} \|\Lambda^{-s} Z_i\|_{L^\infty}] \lesssim E[\|X_i\|_{L^p}^p] + 1
\]

which follows from Young’s inequality for products and Lemma 3.1; and

\[
E[\|\Lambda^s(X_i^p)^2\|_{L^p}] \lesssim E[\|\Lambda^s X_i\|_{L^{2p}} \|X_i\|_{L^{2p}}] \lesssim E[\|X_i\|_{H^1}^2]
\]

which follows from (2.1) and embedding (Lemma 2.1), provided that \(sp < 1\).

Regarding \(I_2^{(2)}\), by the interpolation Lemma 2.2 followed by H"older’s inequality,

\[
\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{p-1}} \lesssim \|\nabla(X_i|X_i|^{p-2})\|_{L^{p-1}} \|X_i^{p-1}\|_{L^{p-1}} + \|X_i^{p-1}\|_{L^{p-1}}
\]

\[
\lesssim D^{s/2} \|X_i^p\|_{L^1}^{\frac{1-s}{2}} \|\Lambda^{-s} Z_i\|_{L^\infty} + \|X_i^p\|_{L^1}^{\frac{1-s}{2}}. \tag{4.17}
\]

Also, by Sobolev embedding \(H^\beta \subset L^{2p}, \beta = 1 - \frac{1}{p} < 1\) and interpolation Lemma 2.2, followed by H"older inequality with exponents \((\frac{1}{p}, \frac{1}{2})\), we have

\[
E[\|X_i^2\|_{L^p}] \lesssim E[\|X_i\|_{H^1}^2] \|X_i\|_{L^p}^{2(1-\beta)} \lesssim (E[\|X_i\|_{H^1}^2])^\beta (E[\|X_i\|_{L^p}^2])^{1-\beta}. \tag{4.18}
\]

Recall from Lemma 4.3 that \(E[\|X_i\|_{L^p}^2] \lesssim 1\).

With the above two estimates (4.17)-(4.18), using again independence, and H"older’s inequality with exponents \((\frac{2}{p}, \frac{2}{2^\beta})\), together with Lemma 3.1, we obtain

\[
I_2^{(2)} \lesssim \left(E[\|X_i\|_{H^1}^2]\right)^\beta (E D)^{\eta/2} E[\|X_i^p\|_{L^1} \|\Lambda^{-s} Z_i\|_{L^\infty}^{2\eta}] \left[\|X_i\|_{H^1}^2 \|X_i\|_{L^p}^2\right]^{1-\frac{\eta}{2}}
\]

\[
+ E[\|X_i\|_{H^1}^2]^{\beta} (E[\|X_i^p\|_{L^1}] + 1)
\]

where \(\eta \eqdef (1 - \frac{1}{p} - \frac{s}{2}) \frac{2}{2^\beta}\) and clearly \(\eta < 1\). The first line on the RHS can be bounded by, using Young’s inequality with exponents \((\frac{2}{s}, \frac{2}{2^\beta})\) and then with exponents \((\frac{1}{\eta}, \frac{1}{1-\eta})\),

\[
\varepsilon E[D] + CE[\|X_i^p\|_{L^1} \|\Lambda^{-s} Z_i\|_{L^\infty}^{2\eta}] \left[\|X_i\|_{H^1}^2 \|X_i\|_{L^p}^2\right]^{\frac{\eta}{2}}
\]

\[
\leq \varepsilon E[D] + C\left(E[\|X_i^p\|_{L^1} + E[\|\Lambda^{-s} Z_i\|_{L^\infty}^{2\eta}] \left[\|X_i\|_{H^1}^2 \|X_i\|_{L^p}^2\right]^{\frac{\eta}{2}}
\]

\[
\leq \varepsilon E[D] + C(E[\|X_i^p\|_{L^1} + 1])E[\|X_i\|_{H^1}^2]^{\frac{\beta}{2}}
\]
where Lemma 3.1 is applied in the last step, and \(2\beta/(2-s) < 1\) by our smallness assumption on \(s > 0\).

Therefore, in view of Lemma 4.3, we have obtained a bound of the form (4.16) for \(I_2\).

**STEP 3** (Estimate of \(I_3-I_5\))

Using Lemma 2.2 we obtain

\[
I_3 = E\langle \Lambda^s(X_jX_i|X_j|^{p-2}), \Lambda^{-s}(Z_iZ_j) \rangle \\
\lesssim E \left[ \| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty} \| \Lambda^s X_j \|_{L^p} \| X_j^p \|_{L^{\frac{p}{p-1}}} \right] \\
+ E \left[ \| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty} \| X_j \|_{L^p} \| \Lambda^s (X_i|X_i|^{p-2}) \|_{L^{\frac{p}{p-1}}} \right] \\
=: I_3^{(1)} + I_3^{(2)}.
\]

For \(I_3^{(1)}\) we use Sobolev embedding \(H^1 \subset H^s_p\) and Young’s inequality and independence to have

\[
I_3^{(1)} \lesssim E[\| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty}^p] + E[\| X_j \|_{H^s_p}^p] E[\| X_i^p \|_{L^1}] 
\]

For \(I_3^{(2)}\) we use (4.17) to have

\[
I_3^{(2)} \lesssim E \left[ \left( D^{s/2} \| X_j^p \|_{L^1}^{(1-s)} \frac{d}{dp} + \frac{d(1-s)}{dp} \right) \| X_j \|_{L^p} \| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty} \right] \\
\lesssim E[D]^{s/2} E[\| X_j^p \|_{L^1}] E[\| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty}^{1-s/2}] \\
+ E[\| X_j \|_{H^s_p}^p] E[\| X_i^p \|_{L^1}] + E[\| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty}^p],
\]

where \(\eta = \left( (1-s)\frac{1}{2} + \frac{d(1-s)}{2p} \right) < 1\). The last step follows from Sobolev embedding and Hölder and Young’s inequality. Using Young’s inequality, the first term could controlled by

\[
\varepsilon E[D] + CE[\| X_j^p \|_{L^1} E[\| X_j \|_{H^s_p}^{2s/(2-\eta)}] + CE[\| \Lambda^{-s}(Z_iZ_j) \|_{L^\infty}^{2s/(2-\eta)}],
\]

where \(q = \frac{1}{\eta d/2}\), and for \(s > 0\) small enough \(\frac{2}{p} + s < 1\).

By (4.17) we have for \(\varepsilon > 0\) small enough

\[
I_4 + I_5 \\
\lesssim \| E[Z_j^2] \|_{L^\infty} E[\| X_j \|_{L^p}] + \| E[Z_j^2] \|_{C^{++}} E[\| \Lambda^s(X_jX_i|X_i|^{p-2}) \|_{L^1}] \| \Lambda^{-s} Z_i \|_{L^\infty} \\
\lesssim \| E[Z_j^2] \|_{L^\infty} E[\| X_j \|_{L^p}] + \| E[Z_j^2] \|_{C^{++}} E[D^{s/2} \| X_j^p \|_{L^1}^{(1-s)} \frac{d}{dp} + \frac{d(1-s)}{dp} \| \Lambda^{-s} Z_i \|_{L^\infty}] \\
+ \| E[Z_j^2] \|_{C^{++}} E[X_j^p \|_{L^1}] \| \Lambda^{-s} Z_i \|_{L^\infty} \\
\lesssim E[D]^{s/2} E[\| X_j^p \|_{L^1} \| \Lambda^{-s} Z_i \|_{L^\infty}^{1-s/2}] E[Z_j^2] + \| E[Z_j^2] \|_{C^{++}} (E[\| X_j \|_{L^p}] + 1) \\
\lesssim \varepsilon E[D] + E[\| X_j^p \|_{L^1}] + \| E[Z_j^2] \|_{C^{++}} E[\| \Lambda^{-s} Z_i \|_{L^\infty}] + \| E[Z_j^2] \|_{C^{++}} (E[\| X_j \|_{L^p}] + 1)
\]

Combining all the above estimates and using Gronwall’s inequality, we obtain the claimed bound.

\[ \square \]

We now conclude this section by combining our energy estimates with Schauder theory to obtain a global Hölder bound on \(X_i\).
Lemma 4.5. Assume that \( E \| \eta_i \|^4_{L^4} \lesssim 1 \). For \( \beta > \kappa \) sufficiently small it holds
\[
E \left[ \sup_{t \in [0,T]} t^\gamma \| X_i \|^2_{C^4} \right] \lesssim 1,
\]
where \( \gamma = \beta + \frac{1}{2} \).

Proof. Recall that \( X_i \) satisfies the mild formulation of (4.1), which we write using our independent copy \((X_j, Z_j)\) as
\[
X_i(t) = S_t \eta_i + \int_0^t S_{t-s} E[X_j t^2 + 2X_j Z_j + :Z_j^2:] (X_i + Z_i) ds.
\]
We start by applying the Schauder estimate, Lemma 2.4, with \( \delta \) playing the role of \( \beta + 1, \beta + \kappa, \) and \( \beta + \kappa + \frac{2}{3} \) respectively to find
\[
\| X_i(t) - S_t \eta_i \|_{C^3} \lesssim \int_0^t (t-s)^{-\frac{\beta+1}{2}} \| E[X_j^2] X_i \|_{C^{\frac{1}{2}}} ds
\]
\[
+ \int_0^t (t-s)^{-\frac{\beta+3}{2}} \| E[ :Z_j^2:] (X_i + Z_i) \|_{C^{-\kappa}} ds
\]
\[
+ \int_0^t (t-s)^{-\frac{\beta+2}{3}+\frac{1}{3}} \| E[X_j Z_j] Z_i \|_{C^{-\kappa-\frac{2}{3}}} ds
\]
\[
+ \int_0^t (t-s)^{-\frac{\beta+2}{3}+\frac{1}{3}} \| E[X_j Z_j] X_i \|_{C^{-\kappa-\frac{2}{3}}} + \| E[X_j^2] Z_i \|_{C^{-\kappa-\frac{2}{3}}} \| := \sum_{i=1}^4 J_i.
\]
To estimate \( J_1 \), first recall Lemma 4.4 implies that
\[
\sup_{t \in [0,T]} \| E[X_j^2] \|_{L^2} = \sup_{t \in [0,T]} \| E[X_j^2] \|_{L^2} \lesssim \sup_{t \in [0,T]} \| E[X_i] \|_{L^4} \| \lesssim 1, \quad (4.19)
\]
which can be combined with the Sobolev embedding \( L^2 \hookrightarrow C^{-1} \) in \( d = 2 \) corresponding to Lemma 2.1 with \( \alpha = 0, p_1 = q_1 = 2 \) and \( p_2 = q_2 = \infty \) to find
\[
J_1 \lesssim \int_0^t (t-s)^{-\frac{\beta+1}{2}} \| E[X_j^2] \|_{L^2} \| X_i \|_{C^{\frac{1}{2}}} ds \lesssim \int_0^t (t-s)^{-\frac{\beta+1}{2}} \| X_i \|_{C^{\frac{1}{2}}} ds.
\]
We now turn to \( J_2 \) and apply Lemma 2.3 to find for \( \beta > \kappa \) and \( \kappa' > \kappa \)
\[
J_2 \lesssim \int_0^t (t-s)^{-\frac{\beta+3}{2}} \| E[ :Z_j^2:] \|_{C^{-\kappa}} \| X_i \|_{C^{\frac{1}{2}}} \| + \| E[ :Z_j^2:] \|_{C^{\frac{1}{2}}} \| Z_i \|_{C^{-\kappa}} ds
\]
\[
\lesssim \int_0^t (t-s)^{-\frac{\beta+3}{2}} s^{-\frac{\kappa}{2'}} \| X_i \|_{C^{\frac{1}{2}}} ds + \| Z_i \|_{C^{\frac{1}{2}}}.
\]
We now turn to \( J_3 \) and \( J_4 \) and use the Sobolev embedding \( B^\kappa_{3,\infty} \hookrightarrow C^{-\kappa-\frac{\beta}{2}} \) in \( d = 2 \) corresponding to to Lemma 2.1 with \( \alpha = -\kappa, p_1 = 3 \) and \( q_1 = p_2 = q_2 = \infty \). Let’s begin with \( J_3 \) which is simpler. Using Lemma 4.4 and Lemma 2.1 and Lemma 2.2 we have
\[
\int_0^T \| E[X_j^2] \|_{B^\kappa_{3,\infty}}^2 ds \lesssim \int_0^T \| E[X_j^2] \|_{B^\kappa_{3,\infty}}^2 ds \lesssim \int_0^T \| E[ X_j^2] \|_{L^3}^2 ds
\]
\[
\lesssim \int_0^T \| E[X_i] \|_{H^1} \| X_i \|_{L^3}^2 ds \lesssim \int_0^T \| E[X_i] \|_{H^1} \| X_i \|_{L^4}^2 ds \lesssim 1, \quad (4.20)
\]
where we used (2.1) in the third inequality and Hölder inequality in the fourth inequality. Note that by Hölder’s inequality in time which exponents \( (\frac{2}{3}, 3) \) and
taking into account that \( \frac{\beta}{4} + \frac{\gamma}{2} + \kappa < 1 \) for \( \beta \) small enough we find

\[
J_3 \lesssim \int_0^T \| E[X_j Z_j] Z_i \|_{B_{2 \infty}^\beta}^3 \, ds \lesssim \int_0^T (E[\|X_j\|_{B_{2 \infty}^\beta}^2])^{\frac{3}{2}} (E[\|Z_i\|_{C^{-\kappa}}^2 | Z_i])^{\frac{3}{2}} \, ds
\]

\[
\lesssim 1 + \int_0^T (E[\|Z_i Z_j\|_{C^{-\kappa}}^2 | Z_i])^6 \, ds.
\] (4.21)

Here we used Lemma 4.1 and (4.20). Finally, we turn to \( J_4 \). By Lemma 2.3 and Lemma 2.2 we deduce

\[
\| X_i \|_{B_{2 \infty}^\beta} \lesssim \| X_i \|_{B_2^\beta} \lesssim \| X_i \|_{L^2} \lesssim \| X_i \|_{B_0^\beta} \lesssim \| X_i \|_{H^1} \| X_i \|_{L^4}^{1-2\kappa},
\]

which implies that

\[
\int_0^T \| E(X_i Z_i) \|_{B_{2 \infty}^\beta}^3 \, ds \lesssim \int_0^T E[\|X_i\|_{B_{2 \infty}^\beta}^3] \| Z_i \|_{C^{-\kappa}}^3 \lesssim \int_0^T E[\|X_i\|_{H^1}^6] \| X_i \|_{L^4}^{3(1-2\kappa)} \| Z_i \|_{C^{-\kappa}}^3 \]

\[
\lesssim \int_0^T E[\|X_i\|_{H^1}^2] + \int_0^T E[\|X_i\|_{L^4}^2] + \int_0^T E[\|Z_i\|_{C^{-\kappa}}] \lesssim 1,
\] (4.22)

for some \( l > 1 \). This combined with (4.20) implies that

\[
J_4 \lesssim \int_0^t (t - s)^{-\frac{2+2/3+\kappa}{3}} \| E[X_j Z_j] \|_{B_{2 \infty}^\beta} \| X_i \|_{C^\beta} + \| E[X_j^2] \|_{B_{2 \infty}^\beta} \| Z_i \|_{C^{-\kappa}} \, ds
\]

\[
\lesssim \int_0^t (t - s)^{-\frac{2+2/3+\kappa}{4}} \| E[X_j Z_j] \|_{B_{2 \infty}^\beta} \| X_i \|_{C^\beta} \| Z_i \|_{C^{-\kappa}} \, ds + \| Z_i \|_{C^\beta} \| Z_i \|_{C^{-\kappa}} \]

Combining the above estimates, using (4.22), Hölder’s inequality and Gronwall’s inequality the result follows.

Combining the local well-posedness result and the uniform estimate Lemma 4.5 we conclude the following result:

**Theorem 4.6.** For given \( Z_i \) as the solution to (3.2) and \( E[\|\eta_i\|_{L^4}^4] \lesssim 1 \), there exists a unique solution \( X_i \in C([0, T]; C^\beta) \cap C_T L^4 \) to (4.1) such that

\[
E[\sup_{t \in [0, T]} t^\gamma \| X_i \|_{L^4}^2] + \sup_{t \in [0, T]} E[\| X_i \|_{L^2}^2] + E[\| X_i \|_{L^4}^2] \lesssim 1.
\]

In particular, for every \( \psi_i \in C^{-\kappa} \) with \( E[\|\psi_i\|_{C^{-\kappa}}^p \lesssim 1, p > 1 \), there exists a unique solution \( \Psi_i \in C_T C^{-\kappa} \) to (1.7) such that

\[
E[\sup_{t \in [0, T]} t^\gamma \| \Psi_i - Z_i \|_{C^\beta}^2] \lesssim 1,
\]

for \( 0 < 3\kappa < \beta \) small enough and \( \gamma = \frac{1}{2} + \beta \).

5. LARGE N LIMIT OF THE DYNAMICS

In this section we study the large \( N \) behavior of a fixed component \( \Phi_N \) satisfying (3.1) with initial condition \( \phi_N^0 = \eta_i + z_i^N \). We will show that under suitable assumptions on the initial conditions, as \( N \to \infty \), the component converges to the corresponding solution \( \Psi_i \) (1.7) with initial condition \( \psi_i = \eta_i + z_i \).

Set

\[
Z_i^N = \tilde{Z}_i + S_i(z_i^N - \tilde{Z}_i(0)), \quad Z_i = \tilde{Z}_i + S_i(z_i - \tilde{Z}_i(0)).
\]
We know that $\Phi^N_i - \Psi_i = Y_i^N - X_i + Z_i^N - Z_i := v_i + e^{i\Delta}(z_i^N - z_i) := v_i^N + u_i$, where we define $v_i^N \overset{\text{def}}{=} Y_i^N - X_i$ which satisfies

$$\mathcal{L}v_i = -\frac{1}{N} \sum_{j=1}^N \Phi_j^2 \Phi_i + \mu \Psi_i,$$

where we omit the superscript $N$ for notation’s simplicity.

**Assumption 5.1.** Suppose the following assumptions:

- The random variables $\{(z_i^N, y_i^N, z_i, \eta_i)\}_{i=1}^N$ are exchangeable and $\{(z_i, \eta_i)\}_{i=1}^N$ are independent.
- For every $p > 1$, and every $i$,

$$\mathbf{E}[\|z_i^N - z_i\|_{C^{-\kappa}}^p] \to 0, \quad \mathbf{E}[\|y_i^N - \eta_i\|_{L^2}^2] \to 0,$$

as $N \to \infty$.
- For some $q > 1$, $p_0 > 4/(1-4\kappa)$, and every $p > 1$

$$\mathbf{E}[\|z_i^N\|_{C^{-\kappa}}^p + \|z_i\|_{C^{-\kappa}}^p] \lesssim 1, \quad \mathbf{E}[\|\eta_i\|_{L^2}^{p_0}] \lesssim 1, \quad \mathbf{E}[\frac{1}{N} \sum_{i=1}^N \|y_i^N\|_{L^2}^q] \lesssim 1,$$

where the implicit constant is independent of $i, N$.

**Theorem 5.1.** If the initial datum $(z_i^N, y_i^N, z_i, \eta_i)_{i=1}^N$ satisfy Assumption 5.1, then for all $t > 0$ and for every $i$ it holds

$$\lim_{N \to \infty} \mathbf{E}[\|\Phi_i^N(t) - \Phi_i(t)\|_{L^2}^2] = 0. \quad (5.1)$$

Furthermore, for every $i$ and every $T > 0$, $\|v_i^N\|_{C^T L^2}$ converges to zero in probability, as $N \to \infty$.

**Proof.** The proof has a similar flavor to the Lemma 3.3, and in fact we will continue to use the notation $R_i^N$ for $i = 1, 2, 3$ for the same quantities. One additional ingredient required is the following instance of the Gagliardo-Nirenberg inequality (a special case of Lemma 2.2),

$$\|g\|_{L^q} \lesssim C\|g\|_{H^1}^{1/2}\|g\|_{L^2}^{1/2}. \quad (5.2)$$

In the proof we omit the superscript $N$ for notation’s simplicity.

**Step 1** (Energy balance)

In this step, we suppose that $u_i = 0$, i.e. $Z_i^N = Z_i$ and justify the following energy identity

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \|v_i\|_{L^2}^2 + \sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + m \sum_{i=1}^N \|v_i\|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^N \|Y_j v_i\|_{L^2}^2 + \frac{1}{N} \sum_{j=1}^N \|X_j v_j\|_{L^2}^2 = I_1^N + I_2^N + I_3^N, \quad (5.3)$$

where

$$I_1^N \overset{\text{def}}{=} -\frac{1}{N} \sum_{i,j=1}^N \left( 2\langle v_i v_j, :Z_i Z_j : \rangle + \langle v_i^2, :Z_i^2 : \rangle + 2\langle v_i^2 Y_j, Z_j \rangle \right),$$

$$I_2^N \overset{\text{def}}{=} -\frac{1}{N} \sum_{i,j=1}^N \left( v_i v_j, (X_j Y_j + (3X_j + Y_j)Z_i) \right),$$

and

$$I_3^N \overset{\text{def}}{=} -\frac{1}{N} \sum_{i,j=1}^N \left( v_i v_j, (X_j Y_j + (3X_j + Y_j)Z_i) \right).$$
Similarly, we find
\[ \frac{I_3^N}{N} = \frac{1}{N} \sum_{i,j=1}^{N} \left( [ :Z_i^2 : - \mathbf{E} :X_i^2 : + X_j (X_j + 2Z_j) - \mathbf{E}X_j (X_j + 2Z_j) ] (X_i + Z_i, v_i) \right), \]  
(5.4)

In the definition of \( I_3^N \), to have a compact formula, we slightly abuse notation for the contribution of the diagonal part \( i = j \), where we understand \( Z_i Z_j \) to be :\( Z_i^2 : \) and :\( Z_i : Z_i^2 : \).

We now turn the justification of this identity, and for the convenience of the reader, we write the equations for \( Y_i \) and \( X_i \) side by side as

\[
\mathcal{L} Y_i 
= -\frac{1}{N} \sum_{j=1}^{N} (Y_j^3 Y_i + Y_j^2 Z_i + 2Y_j Z_j Y_i + 2Y_j :Z_i Z_j: + Y_i :Z_i^2 : + :Z_i Z_i^2 :) \quad (5.5)
\]

\[
\mathcal{L} X_i 
= -\frac{1}{N} \sum_{j=1}^{N} \left( \mathbf{E}(X_j^2)X_i + \mathbf{E}(X_j^2)Z_i + 2\mathbf{E}(X_j Z_j)X_i + 2\mathbf{E}(X_j Z_j)Z_i \right. \\
+ X_i \mathbf{E}( :Z_i^2 :) + Z_i \mathbf{E}( :Z_i^2 :) \right),
\]
(5.6)

where we used that \( X_j \) and \( X_i \) are equal in law. We now compare each of the first 4 terms in (5.5) to the corresponding terms in (5.6). Note first that

\[
Y_j^3 Y_i - \mathbf{E}(X_j^2)X_i = Y_j^3 Y_i - X_j^2 X_i + (X_j^2 - \mathbf{E}(X_j^2)) X_i \\
= Y_j^3 v_i + v_j (Y_j + X_j) X_i + (X_j^2 - \mathbf{E}(X_j^2)) X_i.
\]

Similarly, we find

\[
(Y_j^2 - \mathbf{E}(X_j^2)) Z_i = v_j (Y_j + X_j) Z_i + (X_j^2 - \mathbf{E}(X_j^2)) Z_i. \\
2Y_j Z_j Y_i - 2\mathbf{E}(X_j Z_j)X_i = 2 (v_j Y_j + v_j X_i) Z_j + 2 (X_j Z_j - \mathbf{E}(X_j Z_j)) X_i. \\
2Y_j :Z_i Z_j: - 2\mathbf{E}(X_j Z_j)Z_i = 2 v_j :Z_i Z_j: + 2 (X_j :Z_i Z_j: - \mathbf{E}(X_j Z_j)Z_i). \\
Y_i :Z_i^2 : = v_i :Z_i^2 : + X_i :Z_i^2 :.
\]

Taking the difference of (5.5) and (5.6), using the identities above, multiplying by \( v_i \), integrating by parts, and summing over \( i \) leads to (5.3). Indeed, notice that each equality is a sum of two pieces, one with a factor of \( v \) and one without any factor of \( v \), but with a re-centering. The terms which have a factor of \( v \) lead to \( I_1^N \) and \( I_2^N \), except for \( Y_j^2 v_i \) and \( v_j X_j X_i \), which lead to the two coercive quantities on the LHS of (5.3). The terms which have been re-centered lead to \( I_3^N \).

**Step 2 (Estimates for \( I_1^N \))**

In this step, we show there is a universal constant \( C \) such that

\[
I_1^N \leq \frac{1}{8} \left( \sum_{i=1}^{N} \| \nabla v_i \|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^{N} \| Y_j v_i \|_{L^2}^2 \right. \\
+ C (1 + R_1^2 + R_3^2 + R_5) \sum_{i=1}^{N} \| v_i \|_{L^2}^2, \]
(5.7)

where \( R_2 \) and \( R_3 \) are defined in terms of \( Z \) in the same way as in Lemma 3.3 and

\[
R_3^2 \overset{\text{def}}{=} \left( 1 + \frac{1}{N} \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2}^2 \right)^s \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{-2}} \right).
\]
Furthermore, we find that

$$\frac{1}{N} \sum_{i,j=1}^{N} (2 \langle v_i v_j, : Z_j Z_i \rangle + (v_i^2, : Z_j^2)) \leq \frac{1}{16} \left( \sum_{i=1}^{N} \| \nabla v_i \|_{L^2}^2 \right)$$

$$+ C(1 + R_N^2) \sum_{i=1}^{N} \| v_i \|_{L^2}^2. \quad (5.8)$$

Finally, for the contribution of $v_i^2 Y_j Z_j$, the argument is similar as for (3.13). This leads to the estimate

$$- \frac{1}{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} v_i^2 Y_j, Z_j \right)$$

$$\lesssim \frac{1}{N} \sum_{j=1}^{N} \left( \| \sum_{i=1}^{N} v_i^2 Y_j \|_{L^1} \right)^{1-s} \| \nabla \left( \sum_{i=1}^{N} v_i^2 Y_j \right) \|_{L^1} + \left( \sum_{j=1}^{N} \| v_i^2 Y_j \|_{L^1}^2 \right)^{1/2} \left( \sum_{j=1}^{N} \| Z_j \|_{C^{-s}}^2 \right)^{1/2} \quad (5.9)$$

$$+ \frac{1}{N} \left( \sum_{j=1}^{N} \left( \sum_{i=1}^{N} v_i^2 Y_j \right) \right)^{1/2} \left( \sum_{j=1}^{N} \| Z_j \|_{C^{-s}} \right)^{1/2}.$$

By Hölder’s inequality, it holds that

$$\left\| \sum_{i=1}^{N} v_i^2 Y_j \right\|_{L^1} \lesssim \left( \sum_{i=1}^{N} \| v_i Y_j \|_{L^2} \right)^{1/2} \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{1/2}. \quad (5.10)$$

Furthermore, we find that

$$\left\| \nabla \left( \sum_{i=1}^{N} v_i^2 Y_j \right) \right\|_{L^1} \lesssim \left( \sum_{i=1}^{N} \| v_i^2 \|_{H^s} \right)^{1/2} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{1/2} \left( \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2} \right)^{1/2} \left( \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2} \right)^{1/2}$$

$$\lesssim \left( \sum_{i=1}^{N} \| v_i \|_{H^s} \right)^{1/2} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{1/2} \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{1/2} \left( \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2} \right)^{1/2}$$

where we used (5.2) in the last step. Hence, we find that

$$\sum_{j=1}^{N} \left\| \sum_{i=1}^{N} v_i^2 Y_j \right\|_{L^1} \left\| \nabla \left( \sum_{i=1}^{N} v_i^2 Y_j \right) \right\|_{L^1}^{2s} \lesssim \left( \sum_{i,j=1}^{N} \| v_i Y_j \|_{L^2} \right)^{1-s} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{s} \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{s} \left( \sum_{j=1}^{N} \| \nabla Y_j \|_{L^2} \right)^s$$

$$+ \left( \sum_{i=1}^{N} \| \nabla v_i \|_{L^2} \right)^{1-s} \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{1-s} \left( \sum_{j=1}^{N} \| v_i Y_j \|_{L^2} \right).$$
Inserting this into (5.9), taking the square root, and using (5.10) and Young’s inequality with exponent \((\frac{2}{1-s}, \frac{2}{s}, 2)\) we arrive at
\[
-\frac{1}{N} \sum_{i,j=1}^{N} (2v_i^2 Y_j, Z_j) \leq \frac{1}{16} \left( \sum_{i=1}^{N} \|\nabla v_i\|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^{N} \|Y_j v_j\|_{L^2}^2 \right) + C(1 + R_N^3 + R_N^5) \sum_{i=1}^{N} \|v_i\|_{L^2}^2.
\] (5.11)

Combining (5.8) and (5.11) and recalling the definition of \(I_1^N\) we obtain (5.7).

**STEP 3 (Estimates for \(I_2^N\))**

In this step, we show there is a universal constant \(C\) such that
\[
I_2^N \leq \frac{1}{4} \left( \sum_{i=1}^{N} \|\nabla v_i\|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^{N} \|v_i Y_j\|_{L^2}^2 + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} v_j X_j \right)^{\frac{1}{2}} + C \left( 1 + R_N^3 + R_N^5 + R_N^6 + \left( \frac{1}{N} \sum_{i=1}^{N} \|X_i\|_{L^1}^2 \right)^2 \left( \sum_{i=1}^{N} \|v_i\|_{L^2}^2 \right) \right),
\] (5.12)
where \(R_N^3\) is defined as in Lemma 3.3 and \(R_N^4\) and \(R_N^6\) are defined by
\[
R_N^4 \overset{\text{def}}{=} \left( \frac{1}{N} \sum_{j=1}^{N} \|X_j\|_{L^1}^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{i=1}^{N} \|Z_i\|_{C^{-s}}^2 \right) \quad \text{and} \quad \text{R}_N^6 \overset{\text{def}}{=} \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|_{C^{-s}}^2 \right).
\]

We break \(I_2^N\) into the separate contributions where \(v_i v_j\) multiplies \(X_i X_j, X_j Z_i,\) and \(Y_j Z_i\) respectively. For the first contribution, Cauchy’s inequality yields
\[
\frac{1}{N} \sum_{i,j=1}^{N} \int v_i v_j X_j Y_j dx \leq \frac{1}{8} \sum_{i,j=1}^{N} \int v_i^2 Y_j^2 dx + C \frac{1}{N} \sum_{i,j=1}^{N} \int v_j^2 X_i^2 dx.
\]
\[
\leq \frac{1}{8} \frac{1}{N} \sum_{i,j=1}^{N} \int v_i^2 Y_j^2 dx + C \left( \sum_{i=1}^{N} \|v_i\|_{L^2}^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} \|X_i\|_{L^1} \right),
\]
\[
\leq \frac{1}{8} \frac{1}{N} \sum_{i,j=1}^{N} \|v_i Y_j\|_{L^2}^2 + C \left( \sum_{i=1}^{N} \|v_i\|_{L^2}^2 \right)^{1/2} \left( \sum_{i=1}^{N} \|v_j\|_{L^2}^2 \right)^{1/2} \left( \frac{1}{N} \sum_{i=1}^{N} \|X_i\|_{L^1}^2 \right),
\]
where we used (5.2). Using Young’s inequality with exponents \((2, 2)\) leads to the last contribution to (5.12). The remaining contributions to \(I_2^N\) are more involved to estimate. Our next claim is that
\[
\frac{3}{N} \sum_{i=1}^{N} \left( v_i \sum_{j=1}^{N} v_j X_j, Z_i \right) \leq \frac{1}{8} \left( \sum_{i=1}^{N} \|\nabla v_i\|_{L^2}^2 + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} v_j X_j \right) + C(1 + R_N^3 + R_N^4) \sum_{i=1}^{N} \|v_i\|_{L^2}^2.
\] (5.13)
The basic setup is the same as the bound leading to (3.14) via the inequality (3.5) with \( v_i \sum_{j=1}^{N} v_j X_j \) playing the role of \( g \) and \( Z_i \) playing the role of \( f \), followed by an application of the Cauchy-Schwarz inequality for the summation in \( i \).

\[
\frac{1}{N} \sum_{i=1}^{N} \left( \frac{N}{i=1} \left( \frac{N}{j=1} v_i \sum_{j=1}^{N} v_j X_j, Z_i \right) \right) ^{2(1-s)} \leq \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{-s}} \right) ^{1/2} + \frac{1}{N} \sum_{i=1}^{N} \left( \frac{N}{j=1} \right) ^{1/2} \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{-s}} \right) ^{1/2}.
\]

(5.14)

Using Hölder’s inequality in the form \( \| v_i \sum_{j=1}^{N} v_j X_j \|_{L^1} \leq \| v_i \|_{L^2} \| \sum_{j=1}^{N} v_j X_j \|_{L^2} \) together with

\[
\left\| \nabla \left( \frac{N}{i=1} \sum_{j=1}^{N} v_j X_j \right) \right\|_{L^1} \leq \left\| \nabla v_i \right\|_{L^2} \left\| \sum_{j=1}^{N} v_j X_j \right\|_{L^2}
\]

(5.15)

where we used (5.2), and inserting this into (5.14) and applying Hölder’s inequality for the summation in \( i \) together with

\[
\left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{2(1-s)} \leq \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{2-\frac{s}{2}} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{\frac{s}{2}},
\]

we obtain a majorization by

\[
\frac{1}{\sqrt{N}} \sum_{j=1}^{N} v_j X_j \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{2-\frac{s}{2}} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{\frac{s}{2}} \times \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{-s}} \right) ^{1/2}
\]

\[
+ \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} v_j X_j \right)^{1-\frac{s}{2}} \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{\frac{2-s}{4}} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{\frac{2+s}{4}} \times \left( \frac{1}{N} \sum_{i=1}^{N} \| X_i \|_{L^4} \right)^{\frac{s}{2}} \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{-s}} \right) ^{1/2}
\]

\[
+ \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} v_j X_j \right)^{1-\frac{s}{2}} \left( \sum_{i=1}^{N} \| v_i \|_{L^2} \right)^{\frac{2-s}{4}} \left( \sum_{i=1}^{N} \| v_i \|_{H^1} \right)^{\frac{2+s}{4}} \times \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{-s}} \right) ^{1/2}
\]
\[
\times \left( \frac{1}{N} \sum_{j=1}^{N} \| \nabla X_j \|_{L^2}^2 \right)^{\frac{\alpha}{2}} \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{\alpha_{-}}}^2 \right)^{1/2}.
\]

Finally, we apply Young’s inequality with exponents \((2, \frac{2}{1-s}, \frac{2}{s})\) for the first term, \((\frac{2}{1-s}, \frac{4}{1-s}, \frac{2}{s})\) for the second term, and \((\frac{2}{1-s}, 2, \frac{2}{s})\) for the third term which leads to (5.13).

Similar to (5.13), we now claim that
\[
\frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} v_i v_j Y_j, Z_i \right) \leq \frac{1}{8} \left( \sum_{i=1}^{N} \| \nabla v_i \|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^{N} \| v_i Y_j \|_{L^2}^2 \right)
+ C(1 + R_N^3 + R_N^6) \sum_{i=1}^{N} \| v_i \|_{L^2}^2. \tag{5.16}
\]

The basic setup is again similar to the bound leading to (3.14) via the inequality (3.5) with \(\sum_{j=1}^{N} v_i v_j Y_j\) playing the role of \(g\) and \(Z_i\) playing the role of \(f\), followed by an application of the Cauchy-Schwartz inequality for the summation in \(i\).

\[
\frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} v_i v_j Y_j, Z_i \right) \leq \frac{1}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} v_i v_j Y_j \right)^2 \right)^{1/2} \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{B_{s_{\infty}}^\infty}^2 \right)^{1/2}
+ \frac{1}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} v_i v_j Y_j \right)^2 \right)^{1/2} \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{\alpha_{-}}}^2 \right)^{1/2}. \tag{5.17}
\]

By Hölder’s inequality and the Cauchy-Schwartz inequality we find
\[
\left( \sum_{j=1}^{N} v_i v_j Y_j \right)^2 \leq \left( \sum_{j=1}^{N} \| v_j Y_j \|_{L^2}^2 \right)^{1/2} \left( \sum_{j=1}^{N} \| v_j \|_{L^2}^2 \right)^{1/2},
\]

together with (5.2) to have
\[
\left( \sum_{j=1}^{N} \nabla (v_i v_j Y_j) \right)^2 \leq \left( \sum_{j=1}^{N} \| v_j Y_j \|_{L^2}^2 \right)^{1/2} \left( \sum_{j=1}^{N} \| \nabla v_j \|_{L^2}^2 \right)^{1/2}
+ \| \nabla v_i \|_{L^2} \left( \sum_{j=1}^{N} \| v_j \|_{L^4}^2 \right)^{1/2} \left( \sum_{j=1}^{N} \| Y_j \|_{L^4}^2 \right)^{1/2}
+ \| v_i \|_{L^4} \left( \sum_{j=1}^{N} \| v_j \|_{L^4}^2 \right)^{1/2} \left( \sum_{j=1}^{N} \| \nabla Y_j \|_{L^4}^2 \right)^{1/2}.
\]

Inserting this into (5.17) and applying Hölder’s inequality for the summation in \(i\) with exponents \((\frac{1}{1-s}, \frac{1}{s})\) leads to a majorization by
\[
\left( \frac{1}{N} \sum_{j=1}^{N} \| v_i Y_j \|_{L^2}^2 \right)^{\frac{\alpha}{2}} \left( \sum_{i=1}^{N} \| v_i \|_{L^2}^2 \right)^{\frac{\alpha}{2}} \left( \sum_{i=1}^{N} \| \nabla v_i \|_{L^2}^2 \right)^{\frac{\alpha}{2}}
\times \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^{\alpha_{-}}}^2 \right)^{1/2}.
\]
We show there is a universal constant $C$ with 
\[ Note that for the third term, we also took advantage of (5.2). We now apply Young’s inequality with exponents $(2, \frac{2}{1+s}, \frac{2}{s})$ for the first term, $(\frac{2}{1+s}, \frac{2}{s}, \frac{2}{s}, \frac{2}{s})$ for the second term, and $(\frac{2}{1+s}, \frac{2}{s}, 2)$ for the third term, which leads to (5.16). Finally, combining (5.13) and (5.16) we obtain (5.12).

**Step 4** (Law of large numbers type bounds: estimates for $I_3^N$)

For $I_3^N$ we obtain a bound in expectation in the spirit of the law of large numbers in a Hilbert space to generate cancellations. To this end, we define 
\[ G_j \overset{\text{def}}{=} (X_j^2 - \mathbf{E}X_j^2) + 2(X_jZ_j - \mathbf{E}X_jZ_j) + (\mathbf{E} : Z_j^2) - \mathbf{E} : Z_j^2) := G_j^1 + G_j^2 + G_j^3. \]

We show there is a universal constant $C$ such that 
\[ I_3^N \leq C(R_N + \bar{R}_N) + \frac{1}{8N} \left( \sum_{i=1}^N X_i v_i \right)^2_{L^2} + \frac{1}{4} \sum_{i=1}^N \| v_i \|_{H^s}^2, \quad (5.18) \]

with 
\[ R_N \overset{\text{def}}{=} \frac{1}{N} \left( \sum_{i=1}^N \| X_i \|_{L^4}^4 (1 - 2s) \right), \]
\[ \bar{R}_N^1 \overset{\text{def}}{=} \frac{1}{N^2} \left( \sum_{i=1}^N \sum_{j=1}^N \| G_j^1 Z_i \|_{H^s}^2 \right), \]
\[ \bar{R}_N^2 \overset{\text{def}}{=} \frac{1}{N^2} \left( \sum_{i=1}^N \sum_{j=1}^N \| G_j^2 Z_i \|_{H^s}^2 \right), \]
\[ R_N^7 \overset{\text{def}}{=} \left( \frac{1}{N} \sum_{i=1}^N \| Z_i \|_{L^\infty}^2 \right)^{\frac{1}{1-s}}. \]

We write $I_3^N = \sum_{k=1}^3 (I_{3,k}^N + J_{3,k}^N)$ with 
\[ I_{3,k}^N \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \left( \sum_{j=1}^N G_j^k X_i v_i \right), \quad J_{3,k}^N \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \left( \sum_{j=1}^N G_j^k Z_i v_i \right). \]

We consider each term separately: For $I_{3,1}^N$ we have the following 
\[ I_{3,1}^N \leq \frac{1}{N} \left( \sum_{j=1}^N \| G_j \|_{L^2} \right) \left( \sum_{i=1}^N X_i v_i \right)_{L^2} \leq C \frac{1}{N} \left( \sum_{j=1}^N \| G_j \|_{L^2} \right)^2 + \frac{1}{8N} \left( \sum_{i=1}^N X_i v_i \right)_{L^2}. \]
For $J_{3,1}^N$ we use (2.1), the interpolation Lemma 2.2 and Young’s inequality to obtain

$$J_{3,1}^N = \frac{1}{N} \sum_{i=1}^{N} \left\langle \Lambda^{s} \left( \sum_{j=1}^{N} G_j^1 v_i \right), \Lambda^{-s} Z_i \right\rangle$$

$$\lesssim \frac{1}{N} \sum_{i=1}^{N} \left( \left\| \Lambda^{s} \left( \sum_{j=1}^{N} G_j^1 \right) \right\|_{L^2} \left\| v_i \right\|_{L^2} + \left\| \sum_{j=1}^{N} G_j^1 \right\|_{L^2} \left\| \Lambda^{s} Z_i \right\|_{L^\infty} \right)$$

$$\lesssim \frac{1}{N} \sum_{i=1}^{N} \left\| G_j^1 \right\|_{H^s}^2 + \left( \sum_{i=1}^{N} \left\| v_i \right\|_{L^2}^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \Lambda^{s} Z_i \right\|_{L^\infty} \right)$$

$$+ \left( \sum_{i=1}^{N} \left\| v_i \right\|_{L^2}^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \Lambda^{-s} Z_i \right\|_{L^\infty} \right)^{\frac{1}{2-s}}.$$

For $I_{3,2}^N$ we have

$$I_{3,2}^N = \frac{1}{N} \sum_{i=1}^{N} \left\langle \Lambda^{-s} \sum_{j=1}^{N} G_j^2, \Lambda^{s} (X_i v_i) \right\rangle$$

$$\lesssim \frac{1}{N} \left\| \sum_{j=1}^{N} G_j^2 \right\|_{H^s}^2 + \frac{1}{N} \left( \sum_{i=1}^{N} \left\| \Lambda^{s} (X_i v_i) \right\|_{L^2} \right)^2$$

$$\lesssim \frac{1}{N} \sum_{j=1}^{N} \left\| G_j^2 \right\|_{H^s}^2 + \frac{1}{N} \sum_{i=1}^{N} \left\| v_i \right\|_{H^1}^2$$

$$+ \left( \sum_{i=1}^{N} \left\| v_i \right\|_{L^2}^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} \left( \left\| X_i \right\|_{L^4}^{4/(1-2s)} + \left\| \Lambda^{s} X_i \right\|_{L^4} \right) \right),$$

where we used (2.1), (5.2) to have

$$\frac{1}{N} \left( \sum_{i=1}^{N} \left\| \Lambda^{s} (X_i v_i) \right\|_{L^2} \right)^2 \lesssim \frac{1}{N} \left( \sum_{i=1}^{N} \left\| \Lambda^{s} X_i \right\|_{L^4} \left\| v_i \right\|_{L^4} + \left\| \Lambda^{s} v_i \right\|_{L^4} \left\| X_i \right\|_{L^4} \right)^2$$

$$\lesssim \frac{1}{N} \sum_{j=1}^{N} \left\| v_i \right\|_{H^s}^2 \left\| v_i \right\|_{L^2}^2 \left\| \Lambda^{s} X_i \right\|_{L^4} + \sum_{i=1}^{N} \left\| v_i \right\|_{H^s}^2 \left\| v_i \right\|_{L^2}^2 \left\| X_i \right\|_{L^4}^2$$

$$\lesssim \frac{1}{N} \left( \sum_{i=1}^{N} \left\| v_i \right\|_{H^1}^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} \left( \left\| X_i \right\|_{L^4}^{4/(1-2s)} + \left\| \Lambda^{s} X_i \right\|_{L^4} \right) \right),$$

in the last inequality.

For $J_{3,2}^N$ we have

$$J_{3,2}^N \lesssim \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} G_j^2 Z_i \left\| Z_i \right\|_{H^s}^2 + \frac{1}{N} \sum_{i=1}^{N} \left\| v_i \right\|_{H^1}^2.$$

For $I_{3,3}^N$ we have

$$I_{3,3}^N \lesssim \frac{1}{N} \left\| \sum_{j=1}^{N} G_j^3 \right\|_{H^s}^2 + \frac{1}{N} \left( \sum_{i=1}^{N} \left\| \Lambda^{s} (X_i v_i) \right\|_{L^2} \right)^2$$

$$\lesssim \frac{1}{N} \left\| \sum_{j=1}^{N} G_j^3 \right\|_{H^s}^2 + \frac{1}{N} \sum_{i=1}^{N} \left\| v_i \right\|_{H^1}^2.$$
where we used (5.19) in the last inequality.

For the last term we have

\[ J_{N,k}^N \lesssim \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \| G_j^3 Z_i \|_{H^{-\epsilon}}^2 + \frac{1}{20} \sum_{i=1}^N \| v_i \|_{H^\epsilon}^2. \]

Combining all the estimates for \( I_{N,k}^N \) and \( J_{N,k}^N \) we arrive at (5.18). In the following we calculate \( E\| \tilde{R}_N \|_{L^1_T} + E\| \check{R}_N \|_{L^1_T} \). To this end, we recall the following general fact: for mean-zero independent random variables \( U_1, \ldots, U_N \) taking values in a Hilbert space \( H \), it holds that

\[ E \left\| \sum_{i=1}^N U_i \right\|_H^2 = E \sum_{i=1}^N \| U_i \|_H^2. \]

Applying this we obtain

\[ E\| \tilde{R}_N \|_{L^1_T} \lesssim E\| G_1 \|_{L^2_T H^\epsilon}^2 + E\| G_2 \|_{L^2_T H^{-\epsilon}}^2 + E\| G_3 \|_{L^2_T H^{-\epsilon}}^2. \]

It is obvious that \( E\| G_1 \|_{L^2_T H^\epsilon}^2 \lesssim 1 \). By Lemma 4.4 we know

\[ E\| G_2 \|_{L^2_T H^{-\epsilon}}^2 \lesssim \int_0^T E \| \nabla X_1 \|_{L^2_T}^2 + \| X_1 \|_{L^4_T}^2 \right) dt \lesssim 1, \]

\[ E\| G_3 \|_{L^2_T H^{-\epsilon}}^2 \lesssim \int_0^T E \| Z_1 \|_{L^2_T}^2 \right) + \| X_1 \|_{L^4_T}^2 \right) + 1 \right) dt \lesssim 1, \]

where we used Lemma 2.2 and Lemma 2.3. Then we deduce \( E\| \tilde{R}_N \|_{L^1_T} \lesssim 1 \). For \( \check{R}_N \) we have

\[ E \frac{1}{N^2} \sum_{i=1}^N \left\| \sum_{j=1}^N G_j Z_i \right\|_{H^{-\epsilon}}^2 = \frac{1}{N^2} \sum_{i,j}^N \left\| G_j Z_i \right\|_{H^{-\epsilon}}^2 \]

\[ = \frac{1}{N^2} \left( \sum_{i=j}^N + 2 \sum_{i=j}^N + \sum_{\ell \neq i} \right) \lesssim E \| Z_1 : Z_1 \|_{H^{-\epsilon}}^2 + E \| Z_1 : Z_1 \|_{H^{-\epsilon}}^2, \]

where we used independence to have \( \sum_{i \neq j \neq \ell} = 0 \). Similarly, we have

\[ E \frac{1}{N^2} \sum_{i=1}^N \left\| \sum_{j=1}^N G_j Z_i \right\|_{H^{-\epsilon}}^2 \lesssim E \| Z_1 \|_{H^{-\epsilon}}^2 + E \| Z_2 \|_{H^{-\epsilon}}^2. \]

Combining the above two estimates and using Lemma 3.1 and the same argument as in (5.20) with \( Z_1 \) replaced by \( Z_1 Z_2 \) and \( Z_2^2 \), we obtain \( E\| \check{R}_N \|_{L^1_T} \lesssim 1 \).

STEP 5 (Convergence of \( v_t \) to zero in \( L^2(\Omega) \))

We now combine our estimates and conclude the proof. Namely, we insert the estimates (5.7) and (5.12) into (5.3) and also appeal to our bounds from Step 4 to obtain

\[ \frac{d}{dt} \sum_{i=1}^N \| v_i \|_{L^2}^2 \leq C(\tilde{R}_N + \check{R}_N) \]
\[ + C \left( 1 + \sum_{i=2}^{N} R_N^i + \left( \frac{1}{N} \sum_{i=1}^{N} \left( \|X_i\|_{L^4}^{1/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4 \right) \right) \sum_{i=1}^{N} \|v_i\|_{L^2}^2, \right) \tag{5.21} \]

where \( R_N + \tilde{R}_N^i \) is uniformly bounded in \( L^1(\Omega \times [0, T]) \). Furthermore, by using Lemma 3.1, Lemma 3.3, Lemma 4.4 and (5.2) we deduce

\[ \sum_{i=2}^{N} \mathbb{E} \int_{0}^{T} R_N^i \, dt < \infty. \]

By Galliardo-Nirenberg inequality in Lemma 2.2 we have for \( s \geq 2\kappa, r > 4, \frac{1}{r} = \frac{1}{2} + \frac{1-2s}{r} \), \( \|\Lambda^s X_i\|_{L^4}^4 \lesssim \|X_i\|_{H^1}^4 \|X_i\|_{L^4}^{4(1-s)} \), which combined with Lemma 4.4 implies that

\[ \mathbb{E} \int_{0}^{T} \left( \frac{1}{N} \sum_{i=1}^{N} \left( \|X_i\|_{L^4}^{1/r} + \|\Lambda^s X_i\|_{L^4}^4 \right) \right) \, dt \]

\[ \leq \mathbb{E} \int_{0}^{T} \left( \frac{1}{N} \sum_{i=1}^{N} \left( \|X_i\|_{H^1}^4 + \|X_i\|_{L^4}^{1/r} + 1 \right) \right) \, dt < \infty. \]

By Gronwall’s inequality, it follows that the sequence of random variables \( N \mapsto \sup_{t \in [0, T]} \sum_{i=1}^{N} \|v_i\|_{L^2}^2 \) is uniformly bounded in probability if the initial data satisfies \( \sum_{i=1}^{N} \|v_i(0)\|_{L^2}^2 < \infty \). In particular, it follows that

\[ \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^{N} \|v_i\|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^{N} \|v_i\|_{H^1}^4 + \frac{1}{N} \sum_{i,j=1}^{N} \|Y_{ij}\|_{L^4}^4 \|X_j v_j\|_{L^2}^2 \]

\[ \leq \frac{1}{N} \sum_{i=1}^{N} \|Y_{ij}\|_{L^4}^4 \|X_j v_j\|_{L^2}^2 \] \tag{5.22}

converges to zero in probability under the assumption by using Lemma 5.2 below. Furthermore, note that \( \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^{N} \|v_i\|_{L^2}^2 \) is uniformly bounded in \( L^q(\Omega) \) for \( q \) in Assumption 5.1 by Lemma 3.3. Additionally, by Lemma 4.4, Jensen’s inequality, and the fact that \( X_i \) and \( X_j \) are identically distributed it holds

\[ \sup_{N \geq 1} \sup_{t \in [0, T]} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|X_i(t)\|_{L^2}^2 \right)^2 < \infty. \]

Hence, by the triangle inequality we find that

\[ \sup_{N \geq 1} \sup_{t \in [0, T]} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|v_i(t)\|_{L^2}^2 \right)^q < \infty, \]

which implies the following convergence upgrade: \( \frac{1}{N} \sum_{i=1}^{N} \|v_i(t)\|_{L^2}^2 \) converges to zero in \( L^2(\Omega) \) for each \( t \in [0, T] \). Since \( v_i \) and \( v_j \) have the same law, it follows that

\[ \mathbb{E} \|v_i(t)\|_{L^2}^2 = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \|v_i(t)\|_{L^2}^2 \to 0, \]

completing the proof of the first result.

**Step 6** (Convergence as a stochastic process) Furthermore, by a similar argument as before we deduce that

\[ \frac{d}{dt} \|v_i\|_{L^2}^2 \leq C \left( \frac{\tilde{R}_N}{N} + \tilde{R}_N^i \right) + \frac{1}{N} \sum_{i=1}^{N} \|v_i\|_{H^1}^4 + \frac{1}{N^2} \sum_j \|X_j v_j\|_{L^2}^2 \] \tag{5.23}
For the first term we have
\[ + C \left( 1 + \tilde{R}_N + \frac{1}{N} \sum_{j=1}^{N} \|X_j\|_{L^4}^4 + \|X_i\|_{L^4}^{4(1-2s)} + \|X^s_iX_i\|_{L^4}^4 \right) \|v_i\|_{L^2}^2 \]

\[ + C \left( 1 + \tilde{R}_{2N} + \tilde{R}_N^3 + \tilde{R}_N^5 + \tilde{R}_N^6 + \|X_i\|_{L^4}^4 \right) \frac{1}{N} \sum_{j=1}^{N} \|v_j\|_{L^2}^2, \quad (5.24) \]

where
\[ \tilde{R}_N^1 \overset{\text{def}}{=} \frac{1}{N^2} \left\| \sum_{j=1}^{N} G_j^2 Z_i \right\|_{L^\infty}^2 + \frac{1}{N^2} \left\| \sum_{j=1}^{N} G_j^2 Z_i \right\|_{L^\infty}^2, \]

\[ \tilde{R}_N^2 \overset{\text{def}}{=} \frac{1}{N} \sum_j \|Z_j^2 : \|_{C_{(2-s)}^1}, \quad \tilde{R}_N^3 \overset{\text{def}}{=} \frac{1}{N} \sum_j \|Z_iZ_j : \|_{C_{(2-s)}}, \]

\[ \tilde{R}_N^4 \overset{\text{def}}{=} \left( 1 + \frac{1}{N} \sum_{j=1}^{N} \|X_j\|_{L^4}^2 \right)^{\frac{\nu+2}{2}} \|Z_i\|_{L^\infty}^{\frac{\nu+2}{2}} + \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|_{L^\infty}^2 \right)^{\frac{\nu+2}{2}} \|X_i\|_{H^1}^{\frac{\nu+2}{2}}, \]

\[ \tilde{R}_N^5 \overset{\text{def}}{=} \left( \frac{1}{N} \sum_{j=1}^{N} \|X_j\|_{L^4}^2 \right) \|Z_i\|_{L^\infty}^{\frac{\nu+2}{2}}, \quad \tilde{R}_N^6 \overset{\text{def}}{=} \|X^s_iX_i\|_{L^\infty}^{2(1-s)}. \]

In fact all the terms is similar as above except the following two terms:
\[ - \frac{1}{N} \sum_{j=1}^{N} \int X_j v_j v_i dx - \frac{2}{N} \sum_{j=1}^{N} \langle X_j v_j v_i, Z \rangle : = J_1 + J_2. \]

For the first term we have
\[ J_1 \leq \frac{1}{N} \sum_{j=1}^{N} \int v_j^2 X_j^2 dx + \frac{1}{N} \sum_{j=1}^{N} \int v_i^2 X_j^2 dx \]

\[ \leq C \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j\|_{L^4}^2 \right) \|X_i\|_{L^4}^2 + C \|v_i\|_{L^4}^2 \frac{1}{N} \sum_{j=1}^{N} \|X_j\|_{L^4}^2 \]

\[ \leq C \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j\|_{L^4}^2 \right)^{1/2} \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j\|_{L^4}^2 \right)^{1/2} \|X_i\|_{L^4}^2 + \frac{1}{8} \|v_i\|_{L^4}^2, \]

\[ + C \|v_i\|_{L^2}^2 \left( \frac{1}{N} \sum_{j=1}^{N} \|X_j\|_{L^4}^2 \right)^2, \]

which by using Young’s inequality deduce one contribution to (5.24). For the second term we have
\[ J_2 \leq \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j v_i X_j\|_{L^4}^{2(1-s)} \|\nabla (v_j v_i X_j)\|_{L^4}^{2s} \right)^{1/2} \times \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|_{C_{(2-s)}}^2 \right)^{1/2} \]
\[ + \left( \frac{1}{N} \sum_{j=1}^{N} \|v_i v_j X_i\|_{L^2}^2 \right)^{1/2} \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|^2_{C^{\kappa}} \right)^{1/2} \]. \tag{5.25}

Using Hölder’s inequality in the form \( \|v_i v_j X_i\|_{L^1} \leq \|v_i\|_{L^2} \|v_j X_i\|_{L^2} \) together with the bound for the first term in \( J_1 \) we obtain the estimate for the second term in \( J_2 \), which corresponds to the last term in \( \tilde{R}_N^1 \). Moreover, we have

\[ \| \nabla (v_i v_j X_i) \|_{L^1} \leq \| \nabla v_i \|_{L^2} \| v_j X_i \|_{L^2} + \| v_i \|_{L^2}^{1/2} \| \nabla v_j \|_{L^2} \| X_i \|_{L^4} + \| v_i \|_{L^2}^{1/2} \| \nabla v_j \|_{L^2} \| v_j \|_{L^2}^{1/2} \| \nabla X_i \|_{L^2}, \] \tag{5.26}

and inserting this into the first term in (5.25) and applying Hölder’s inequality for the summation in \( j \) leads to the following

\[ \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j X_i\|_{L^2}^2 \right)^{1/2} \|v_i\|_{L^2}^{1-\kappa} \|v_i\|_{H^1}^{\kappa} \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|^2_{C^{\kappa}} \right)^{1/2} \]  
\[ + \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j X_i\|_{L^2}^2 \right)^{1/2} \|v_i\|_{L^2}^{1-\kappa} \|v_i\|_{H^1}^{\kappa} \|X_i\|_{L^4}^{\kappa} \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|^2_{C^{\kappa}} \right)^{1/2} \]
\[ + \left( \frac{1}{N} \sum_{j=1}^{N} \|v_j X_i\|_{L^2}^2 \right)^{1/2} \|\nabla X_i\|_{L^2}^{\kappa} \|v_i\|_{L^2}^{1-\kappa} \|v_i\|_{H^1}^{\kappa} \left( \frac{1}{N} \sum_{j=1}^{N} \|Z_j\|^2_{C^{\kappa}} \right)^{1/2}. \]

Finally, we apply Young’s inequality and obtain the contribution of \( \tilde{R}_N^1 \) in the estimate (5.24).

Using the fact that (5.22) converges to zero in probability, we deduce the \( L^1(0,T) \) norm of (5.24) and the right hand side of (5.23) converges to zero in probability. Then by Gronwall’s inequality and Lemma 5.2 imply \( \sup_{t \in [0,T]} \|v_i(t)\|_{L^2}^2 \rightarrow 0 \) in probability, as \( N \rightarrow \infty \).

**Step 7** (General initial datum)

In this final and independent step, we sketch how to modify the arguments above to treat the case that \( u_i \neq 0 \). In this case we have the following extra terms:

\[ \tilde{I}_N^u := -\frac{1}{N^3} \sum_{i,j,k} \left[ \langle Y_j, v_i u_j \rangle + 2 \langle Y_j Y_i u_j, v_i \rangle + 2 \langle Y_j v_i, :Z_i^N Z_j^N: - :Z_i Z_j: \rangle 
+ \langle Y_i v_i, :Z_i^N Z_j^N: - :Z_i Z_j^N: \rangle + \langle v_i, :Z_i^N Z_j^N: - :Z_i Z_j^N: \rangle \right]. \]

These terms could also be estimated similarly as that for \( I_1^N \) and \( I_2^N \) by using \( \|u_i(t)\|_{C^{\kappa}} \lesssim t^{-(\kappa+\kappa)/2} \|z_i^N - z_i\|_{C^{\kappa}}. \)

\[ \square \]

We recall an elementary lemma from classical probability theory.

**Lemma 5.2.** Let \( \{U_N\}_{N=1}^{\infty} \) be a non-negative sequence of 1d random variables converging to zero in probability. Let \( \{V_N\}_{N=1}^{\infty} \) be a non-negative sequence of random variables with tight laws. Then the sequence \( \{U_N V_N\}_{N=1}^{\infty} \) converges to zero in probability.
Proof. For each $M, \overline{M} > 0$ it holds
\[ P(U_N V_N > M) \leq P(U_N > \frac{M}{\overline{M}}) + P(V_N > \overline{M}). \]
Given $\epsilon > 0$, we may first choose $\overline{M}$ to make the second term small uniformly in $N$ and then send $N \to \infty$ to make the first term small.

\[ \square \]

6. Invariant measure and observables

We now study the invariant measure for the equation
\[ \mathcal{L} \Psi = -E[\Psi^2 - Z^2] \Psi + \xi, \quad (6.1) \]
with $E[\Psi^2 - Z^2] = E[X^2] + 2E[XZ]$ for $X = \Psi - Z$ and $\xi$ space-time white noise. Here, since we are only interested in the stationary setting in this section, we overload the notation in the previous sections and simply write $Z$ for the stationary solution to the linear equation
\[ \mathcal{L} Z = \xi, \quad (6.2) \]
and we consider the decomposition (slightly different from Section 4) $X \overset{\text{def}}{=} \Psi - Z$, so that
\[ \mathcal{L} X = -E[X^2 + 2XZ](X + Z), \quad X(0) = \Psi(0) - Z(0). \quad (6.3) \]
For the case that $m = 0$ we restrict the solutions $\Psi$ and $Z$ satisfying $\langle \Psi, 1 \rangle = \langle Z, 1 \rangle = 0$. By Theorem 4.6 for every initial data
\[ \Psi(0) = \psi \in C^{-\kappa} \]
with $E[\|\psi\|_{C_T}^p] \lesssim 1$ there exists a unique global solution $\Psi$ to (6.1). We immediately find that $Z$ is a stationary solution to (6.1). This follows since the unique solution to (6.3) starting from zero is identically zero. Furthermore, we define a semigroup $P^* t \nu$ to denote the law of $\Psi(t)$ with the initial condition distributed according to a measure $\nu$. By uniqueness of the solutions to (6.1), we know
\[ P^* t = P^* t - s P^* s, \quad t \geq s \geq 0. \]
By direct probabilistic calculation we could easily obtain the following result, which implies that the implicit constant in Lemma 3.1 is independent of $m$.

Lemma 6.1. For $\kappa' > \kappa > 0$ and $p \geq 1$, it holds
\[ \sup_{m \geq 0} E[\|Z_i\|_{C_T C^{-\kappa}}^p] + \sup_{m \geq 0} E[\|Z_i Z_j\|_{C_T C^{-\kappa}}^p] + \sup_{m \geq 0} E[\|Z_i Z_j^2\|_{C_T C^{-\kappa}}^p] \lesssim 1, \]
where the proportional constants are independent of $i, j, N$.

Proof. By a standard technique (c.f. [GP17]), it is sufficient to calculate
\[ E[\Delta_q Z_i(t)]^2 \lesssim \sum_{k \in \mathbb{Z}^2} \int_\mathbb{R} \theta(2^{-q}k)^2 e^{(t-s)(|k|^2 + m)} ds \lesssim \sum_{k \in \mathbb{Z}^2} 2^{qk} \frac{1}{|k|^2(|k|^2 + m)}, \]
where $\Delta_q$ is a Littlewood-Paley block and $\theta$ is the Fourier multiplier associated with $\Delta_q$. From here we see the bound is independent of $m$. Other terms could be calculated similarly.

We also prove the following lemma. For $R^3 N$ defined in Lemma 3.5 with $Z_i$ stationary, we have the following result.
Lemma 6.2. For every $q \geq 1$ it holds
\[ \mathbb{E}[(R_N^0)^q] \lesssim 1. \] (6.4)

Proof. Since we will have several similar calculations in the sequel, we first demonstrate such calculation in the case $q = 1$. We have
\[ \mathbb{E}\frac{1}{N^2} \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} \Lambda^{-s}(\cdot; Z_i^2 Z_i^2) \right\|_{L^2}^2 = \frac{1}{N^2} \sum_{i,j_1,j_2=1}^{N} \mathbb{E}\langle \Lambda^{-s} : Z_{j_1}^2 Z_i^2; \Lambda^{-s} : Z_{j_2}^2 Z_i^2 \rangle. \]

We have 3 summation indices and a factor $1/N^2$. The contribution to the sum from the cases $j_1 = i$ or $j_2 = i$ or $j_1 = j_2$ is bounded by a constant in light of Lemma 6.1. If $i, j_1, j_2$ are all different, by independence and the fact that Wick products are mean zero, the terms are zero.

For general $q \geq 1$, by Gaussian hypercontractivity and the fact that $R_N^0$ is a random variable with finite Wiener chaos decomposition, we have
\[ \mathbb{E}[(R_N^0)^q] \lesssim \mathbb{E}[(R_N^0)^{q/2}]. \]

For the case that $q = 2$ we write it as
\[ \frac{1}{N^4} \sum_{i_1,i_2,j_k=1}^{N} \mathbb{E}\langle \Lambda^{-s} : Z_{j_1}^2 Z_{i_1}; \Lambda^{-s} : Z_{j_2}^2 Z_{i_2}; \rangle \langle \Lambda^{-s} : Z_{j_3}^2 Z_{i_2}; \Lambda^{-s} : Z_{j_4}^2 Z_{i_2}; \rangle. \]

We have 6 indices $i_1, i_2, j_k, k = 1, \ldots, 4$ summing from 1 to $N$ and an overall factor $1/N^4$. Using again Lemma 6.1, we reduce the problem to the cases where five or six of the indices are different. However, in these two cases, by independence the expectation is zero, so (6.4) follows. \hfill \square

6.1. Uniqueness of invariant measure. We now turn to the question of uniqueness for the invariant measure of (6.1). Since the non-linearity in the SPDE (6.1) involves the law of the solution, the associated semigroup $P_t^*$ is generally nonlinear i.e.
\[ P_t^* \nu \neq \int (P_t^* \delta_\psi) \nu(d\psi), \]

for a non-trivial distribution $\nu$ (see e.g. [Wan18]). As a result, its unclear if the general ergodic theory for Markov processes (see e.g. [DPZ96], [HMS11]) can be applied directly in our setting. Fortunately, (6.3) has a strong damping property in the mean-square sense, which comes to our rescue and allows us to proceed directly by a priori estimates.

Lemma 6.3. There exists $C_0 > 0$ such that for all
\[ m \geq 2 C_0 (\mathbb{E}\|Z_2 Z_1\|_{C^{-s}}^2 + (\mathbb{E}\|Z_1\|_{C^{-s}}^2)^{\frac{1}{r-s}} + 1) := m_0, \]
there exists a universal $C$ with the following property: for every solution $\Psi$ to (6.1) with $\Psi(0) \in C^{-s},$
\[ \sup_{t \geq 1} e^{\frac{m}{2}} \mathbb{E}\|\Psi(t) - Z(t)\|_{L^2}^2 \leq C. \] (6.5)

Proof. The proof relies heavily on several computations performed in Lemma 4.3 where we used slightly different notation, so we will write $X_i$ instead of $X$ and $Z_i$ instead of $Z$ for the remainder of this proof. Revisiting the first step of Lemma 4.3
where we established (4.4), we find that $I_1$ and the first contribution to $I_2$ vanishes in light of $E(\mathbb{L}_t^2) = 0$. It follows that
\[
\frac{1}{2} \frac{d}{dt} E\|X_t\|_{L^2}^2 + E\|\nabla X_t\|_{L^2}^2 + mE\|X_t\|_{L^2}^2 + \|EX_t\|_{L^2}^2 \\
= -2E\langle X_t, X_j, Z_i \rangle - 3E\langle X, X_j^2, Z_i \rangle.
\]
Furthermore, in light of (4.8) and (4.14), we obtain
\[
E\langle X_t, X_j, Z, Z_i \rangle \lesssim (\|EX_t\|_{L^2}^2 + E\|\nabla X_t\|_{L^2}^2)_{1/2}^2 (E : Z_j Z_i : \mathbb{C}^{-\varepsilon})^{1/2} + \|EX_t\|_{L^2}^2 \lesssim 1.
\]
As a consequence, noting that $\|EX_t\|_{L^2}^2 \geq (E\|X_t\|_{L^2}^2)^2$, applying Lemma 2.6 it holds
\[
\sup_{t \geq 0} (t + 1) E\|X_t(t)\|_{L^2}^2 \lesssim 1,
\]
where the implicit constant is independent of the initial data. On the other hand, Young’s inequality also yields
\[
\frac{d}{dt} E\|X_t\|_{L^2}^2 + mE\|X_t\|_{L^2}^2 \leq C_0(E : Z_2Z_1 : \mathbb{C}^{-\varepsilon} + (E\|Z_t\|_{\mathbb{C}^{-\varepsilon}})^{1/2} + 1) E\|X_t\|_{L^2}^2.
\]
Applying Gronwall’s inequality over $[1, t]$ leads to
\[
e^{(m - m_0)t} E\|X_t(t)\|_{L^2}^2 \lesssim E\|X_t(1)\|_{L^2}^2,
\]
so choosing $m \geq m_0$, using (6.7), and taking the supremum over $t \geq 1$, we arrive at (6.5).

We now apply the above result to show that for sufficiently large mass, the unique invariant measure to (6.1) is Gaussian. To this end, define the $C^{-1}$-Wasserstein distance
\[
\mathbb{W}_p(\nu_1, \nu_2) := \inf_{\pi \in \mathcal{P}(\nu_1, \nu_2)} \left( \int \|\phi - \psi\|^p_{C^{-1}} \pi(d\phi, d\psi) \right)^{1/p},
\]
where $\mathcal{P}(\nu_1, \nu_2)$ denotes the collection of all couplings of $\nu_1, \nu_2$ satisfying $\int \|\phi\|^p_{C^{-1}} \nu_i(d\phi) < \infty$ for $i = 1, 2$.

**Theorem 6.4.** For $m_0$ as in Lemma 6.3 and $m \geq m_0$ the unique invariant measure to (6.1) supported on $C^{-\kappa}$ is $\mathcal{N}(0, (-\Delta + m)^{-1})$, the law of the Gaussian free field.

**Proof.** Recall that $Z$ is a stationary solution to (6.1). Indeed, by definition, $\Psi = X + Z$, where $X$ solves (6.3). However, since $X(0) = 0$, the identically zero process is the unique solution to (6.3). Hence, the law of $Z$, which we now denote by $\nu$, is invariant under $P_t^\nu$. We now claim that for $m \geq m_0$, this is the only invariant measure supported on $C^{-\kappa}$. Indeed, let $\nu_1$ be another such measure, then modifying the stochastic basis if needed, we may assume there exists $\psi \in C^{-\kappa}$ on it such that $\psi \sim \nu_1$. By similar arguments as in Theorem 4.6 we may construct a solution $\Psi$ to
(6.1) with \( \Psi(0) = \psi \). By invariance of \( \nu_1 \) and \( \nu \) and the embedding \( L^2 \hookrightarrow C^{-1} \), c.f. Lemma 2.1, it follows that

\[
\mathbb{W}_2'(\nu, \nu_1) = \mathbb{W}_2'(P_t^\nu \nu, P_t^\nu \nu_1)^2 \leq E\|\Psi(t) - Z(t)\|_{C^{-1}}^2 \lesssim e^{-\frac{mt}{2}},
\]

for \( t \geq 1 \) by Lemma 6.3. Letting \( t \to \infty \) we obtain \( \nu = \mu \). \( \square \)

**Remark 6.5.** Note that for the limiting equation \( \mathcal{L}\Psi = -\mu \Psi + \xi \), if we assume that \( \mu \) is simply a constant (which is the case in the spatially translation invariant setting), it has a Gaussian invariant measure \( \mathcal{N}(0, (-\Delta + m + \mu)^{-1}) \). Assuming \( \Psi \sim \mathcal{N}(0, (-\Delta + m + \mu)^{-1}) \) and \( Z \sim N(0, (-\Delta + m)^{-1}) \), the self-consistent requirement \( E[\Psi^2 - Z^2] = \mu \) then yields

\[
\sum_{k \in \mathbb{Z}^2} \left( \frac{1}{|k|^2 + m + \mu} - \frac{1}{|k|^2 + m} \right) = \mu \quad (6.9)
\]

and for \( \mu + m \geq 0 \) we only have one solution \( \mu = 0 \), since the LHS is monotonically decreasing in \( \mu \). Unfortunately, we don’t know how to deduce the translation invariance directly unless the mass is sufficiently large.

**Remark 6.6.** If we change the renormalization constant in (6.1) to \( EZ^2_N(0,0) \) with \( Z_i \) the stationary solution to the equation \( (\partial_t - (\Delta - 1))Z_i = \xi_i \), one invariant measure to (6.1) is given by the Gaussian free field \( \mathcal{N}(0, (-\Delta + m + \mu_0)^{-1}) \) with \( \mu_0 \) satisfying

\[
\sum_{k \in \mathbb{Z}^2} \left( \frac{1}{|k|^2 + m + \mu_0} - \frac{1}{|k|^2 + 1} \right) = \mu_0. \quad (6.10)
\]

For \( m + \mu_0 \) large enough, \( \mathcal{N}(0, (-\Delta + m + \mu_0)^{-1}) \) is the unique invariant measure to (6.1). In fact, we separate \( \Psi = \bar{X} + \bar{Z} \) with \( \bar{Z} \) the stationary solution to \( \mathcal{L}\bar{Z} = -\mu_0 \bar{Z} + \xi \). Then \( \bar{X} \) satisfies the following equation:

\[
\mathcal{L}\bar{X} = -\mu_0 \bar{X} - E[\bar{X}^2 + 2\bar{X} \bar{Z}](\bar{X} + \bar{Z}),
\]

which is the same case as (4.1) with \( m \) replaced by \( m + \mu_0 \). By the same proof of Lemma 6.3 and Theorem 6.4 we know for \( m + \mu_0 \) large enough, the invariant measure is unique.

### 6.2. Convergence of the Invariant Measures

As a consequence of Lemma 3.2, the solutions \( (\Phi_i)_{1 \leq i \leq N} \) to (3.1) form a Markov process on \( (C^{-\infty})^N \) which, by general results of [HM18b], will turn out to admit a unique invariant measure, henceforth denoted by \( \nu^N \). Our goal in this section is to study the large \( N \) behavior of \( \nu^N \) and show that for sufficiently large mass, as \( N \to \infty \), its marginals are simply products of the Gaussian invariant measure for \( \Psi \) identified in Theorem 6.4. For this we rely heavily on the computations from Section 3.2 for the remainder \( Y \), but we leverage these estimates with consequences of stationarity. To this end, it will be convenient to have a stationary coupling of the linear and non-linear dynamics (3.2) and (3.1) respectively, which is the focus of the following lemma.

**Lemma 6.7.** There exists a unique invariant measure \( \nu^N \) on \( (C^{-\infty})^N \) to (3.1). Furthermore, there exists a stationary process \( (\Phi_i, Z_i)_{1 \leq i \leq N} \) such that the components \( \Phi_i, Z_i \) are stationary solutions to (3.1) and (3.2), respectively. Moreover, \( E\|\Phi_i(0) - Z_i(0)\|_{L^2}^q \approx 1 \) and for every \( q > 1 \)

\[
E\left( \frac{1}{N} \sum_{i=1}^N \|\Phi_i(0) - Z_i(0)\|_{L^2}^2 \right)^q \lesssim 1 \quad (6.11)
\]
Proof. Let $\Phi_i$ and $Z_i$ be solutions to (3.1) and (3.2), respectively. By the general results of [HM18b], it follows that $(\Phi_i, Z_i)_{1 \leq i \leq N}$ is a Markov process on $(\mathbb{C}^{-\kappa})^{2N}$, and we denote by $(P^N_t)_{t \geq 0}$ the associated Markov semigroup. To derive the desired structural properties about the limiting measure, we will follow the Krylov-Bogoliubov construction with a specific choice of initial condition that allows to exploit Lemma 3.3. Namely, we denote by $\Phi_i$ the solution to (3.1) starting from the stationary solution $\tilde{Z}_i(0)$, so that the process $Y_i = \Phi_i - Z_i$ starts from the origin. Using Lemma 3.3 and Corollary 3.4 with $y_i = 0$ together with Lemma 6.1 to obtain a uniform bound on $E R_N$, we find for every $T \geq 1$

\[
\int_0^T \mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^N \| Y_i(t) \|_{H^1}^2 \right] dt \lesssim T, \tag{6.12}
\]

and

\[
\mathbb{E}\left[ \int_0^T \left( \frac{1}{N} \sum_{i=1}^N \| Y_i(t) \|_{L^2}^2 \right)^q dt \right] \lesssim T, \tag{6.13}
\]

where the implicit constant is independent of $T$ and for $m = 0$ we used the Poincaré inequality. By (6.12) we obtain

\[
\int_0^T \mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^N \| \Phi_i(t) \|_{C^{-\kappa/2}}^2 \right] + \int_0^T \mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^N \| Z_i(t) \|_{C^{-\kappa/2}}^2 \right] \lesssim T.
\]

Defining $R^N_\lambda := \frac{1}{T} \int_0^T \int_r^{P^N_t} dr$, the above estimates and the compactness of the embedding $C^{-\kappa/2} \hookrightarrow C^{-\kappa}$ imply the induced laws of $\{R^N_\lambda\}_{t \geq 0}$ started from $(\tilde{Z}(0), \tilde{Z}(0))$ are tight on $(\mathbb{C}^{-\kappa})^{2N}$. Furthermore, by the continuity with respect to initial data, it is easy to check that $(P^N_\lambda)_{t \geq 0}$ is Feller on $(\mathbb{C}^{-\kappa})^{2N}$. By the Krylov-Bogoliubov existence theorem (see [DPZ96, Corollary 3.1.2]), these laws converge weakly in $(\mathbb{C}^{-\kappa})^{2N}$ along a subsequence $t_k \rightarrow \infty$ to an invariant measure $\pi_N$ for $(P^N_\lambda)_{t \geq 0}$. The desired stationary process $(\Phi_i, Z_i)_{1 \leq i \leq N}$ is defined to be the unique solution to (3.1) and (3.2) obtained by sampling the initial datum $(\phi_i, z_i)_i$ from $\pi_N$. By (6.12) we deduce

\[
E^\pi_N \| \Phi_i(0) - Z_i(0) \|_{H^1}^2 = E^\pi_N \sup_{\varphi} \langle \Phi_i(0) - Z_i(0), \varphi \rangle^2
\]

\[
= E \sup_{\varphi} \lim_{T \to \infty} \left[ \frac{1}{T} \int_0^T \langle Y_i(t), \varphi \rangle dt \right]^2 \leq \lim_{T \to \infty} \frac{1}{T} \int_0^T E \| Y_i(t) \|_{H^1}^2 dt \lesssim 1,
\]

where $\sup_{\varphi}$ is over smooth functions $\varphi$ with $\| \varphi \|_{H^{-1}} \leq 1$. Similarly using (6.13), (6.11) follows. Finally, we project onto the first component and consider the map $\Pi_1 : S'(T^2)^{2N} \rightarrow S'(T^2)^N$ defined through $\Pi_1(\Phi, Z) = \Phi$. Letting $\nu_N = \pi_N \circ \Pi_1^{-1}$ yields an invariant measure to (3.1), and uniqueness follows from the general results of [HM18b] and [HS19].

Remark 6.8. Using a lattice approximation (see e.g. [GH18], [HM18a, ZZ18]) one can show that the measure $\nu^N(d\Phi)$ indeed has the form (1.2) (with Wick renormalization).

The next step is to study tightness of the marginal laws of $\nu^N$ over $S'(T^2)^N$. To this end, consider the projection $\Pi_i : S'(T^2)^N \rightarrow S'(T^2)$ defined by $\Pi_i(\Phi) = \Phi_i$ and let $\nu^N_i := \nu^N \circ \Pi_i^{-1}$ be the marginal law of the $i^{th}$. Furthermore, for $k \leq N$, define the map $\Pi^{(k)} : S'(T^2)^N \rightarrow S'(T^2)^k$ via $\Pi^{(k)}(\Phi) = (\Phi_i)_{1 \leq i \leq k}$ and the let
\(\nu^N_k \overset{\text{def}}{=} \nu^N \circ (\Pi^{(k)})^{-1}\) be the marginal law of the first \(k\) components. We have the following result:

**Theorem 6.9.** \(\{\nu^{N,i}\}_{N \geq 1}\) form a tight set of probability measures on \(\mathcal{C}^{-\kappa}\) for \(\kappa > 0\).

**Proof.** Let \((\Phi^N_j, Z_j)_{1 \leq j \leq N}\) be the jointly stationary solutions to (3.1) and (3.2) constructed in Lemma 6.7. To prove the result, in light of the compact embedding of \(\mathcal{C}^{-\kappa/2} \hookrightarrow \mathcal{C}^{-\kappa}\) and the stationarity of \(\Phi^N_i\), it suffices to show that the second moment of \(\|\Phi^N_i(0)\|_{\mathcal{C}^{-\kappa/2}}\) is bounded uniformly in \(N\). To this end, let \(Y_i^N = \Phi^N_i - Z_i\), which is also stationary and note that

\[
E\|\Phi^N_i(0)\|^2_{\mathcal{C}^{-\kappa/2}} = \frac{2}{T} \int_{T/2}^T E\|\Phi^N_i(s)\|^2_{\mathcal{C}^{-\kappa/2}} ds
\]

\[
\leq \frac{4}{T} \int_{T/2}^T E\|Z_i(s)\|^2_{\mathcal{C}^{-\kappa/2}} ds + \frac{4}{T} \int_{T/2}^T E\|Y_i^N(s)\|^2_{\mathcal{H}_1} ds.
\]

The first term is controlled by Lemma 6.1. For the second term, symmetry yields \(Y_i^N\) and \(Y_j^N\) are identical in law, which combined with Lemma 3.3 implies that

\[
\frac{2}{T} \int_{T/2}^T E\|Y_i^N(s)\|^2_{\mathcal{H}_1} ds = \frac{2}{T} \int_{T/2}^T \frac{1}{N} \sum_{i=1}^N E\|Y_i^N(s)\|^2_{\mathcal{H}_1} ds
\]

\[
\leq \frac{C}{T} E\int_0^T R_N dt \leq C,
\]

where we used that by stationarity \(\sum_{i=1}^N E\|Y_i^N(T)\|^2_{L_2} = \sum_{i=1}^N E\|Y_i^N(T/2)\|^2_{L_2}\), with both being finite in view of Lemma 6.7. For \(m = 0\) we also used the Poincaré inequality. \(\square\)

**Remark 6.10.** It is reasonable to expect that any limiting measure obtained in Theorem 6.9 is an invariant measure for (1.7) assuming only \(m \geq 0\). However, this cannot be directly deduced from our main result in Section 5 because we do not know a-priori that any limiting measure of \(\nu^N\) is a product measure. This is problematic because the initial conditions for each component of \(\Psi_i\) are assumed to be independent in Section 5. Nevertheless, we can prove below that this is indeed true if \(m\) is large.

In the following we prove the convergence of the measure to the unique invariant measure by using the estimate in Lemma 3.5, which requires \(m\) large enough.

Define the \(\mathcal{C}^{-\kappa}\)-Wasserstein distance

\[
\mathcal{W}_2(\nu_1, \nu_2) := \inf_{\pi \in \mathcal{C}(\nu_1, \nu_2)} \left( \int \|\phi - \psi\|_{\mathcal{C}^{-\kappa}}^2 \pi(d\phi, d\psi) \right)^{1/2},
\]

where \(\mathcal{C}(\nu_1, \nu_2)\) denotes the set of all couplings of \(\nu_1, \nu_2\) satisfying \(\int \|\phi\|_{\mathcal{C}^{-\kappa}}^2 \nu_i(d\phi) < \infty\) for \(i = 1, 2\).

**Theorem 6.11.** Let \(\nu = \mathcal{N}(0, (m - \Delta)^{-1})\). There exist \(C_0 > 0\) such that for all \(m \geq m_1\) where

\[
m_1 \overset{\text{def}}{=} C_0(\mathbb{E}\|Z_1\|_{\mathcal{C}^{-\kappa}}^2 + \mathbb{E}\|Z_1\|_{\mathcal{C}^{-\kappa}}^2 + \mathbb{E}\|Z_2Z_1\|_{\mathcal{C}^{-\kappa}}^2 + 1)
\]

one has

\[
\mathcal{W}_2(\nu^{N,i}, \nu) \leq CN^{-\frac{1}{2}}.
\]

Furthermore, \(\nu^N_k\) converges to \(\nu \times \ldots \times \nu\). (6.14)
Proof. By Lemma 6.7 we may construct a stationary coupling \((\Phi^N, Z_i)\) of \(\nu_N\) and \(\nu\) whose components satisfy (1.1) and (3.2), respectively. The stationarity of the joint law of \((\Phi^N, Z_i)\) implies that also \(Y^N_i = \Phi^N_i - Z_i\) is stationary. In the following we freely omit the time argument of expectations of stationary quantities. We now claim that

\[
E\|Y^N_i\|_{H^1}^2 \leq CN^{-1},
\]

which implies (6.14) by definition of the Wasserstein metric and the embedding \(H^1 \hookrightarrow C^{-n}\) in \(d = 2\), c.f. Lemma 2.1. To ease notation, we write \(Y_i = Y^N_i\) in the following. By (3.20) combined with the stationarity of \((Y_j)_j\) and \((Z_j)_j\), we find

\[
\sum_{j=1}^{N} E\|\nabla Y_j\|_{L^2}^2 + m \sum_{j=1}^{N} E\|Y_j\|_{L^2}^2 + \frac{1}{N} E\left\|\sum_{i=1}^{N} Y_i^2\right\|_{L^2}^2 \\
\leq C E R^0_N + E\left(\sum_{j=1}^{N} \|Y_j\|_{L^2}^2 (D_N + D^1_N)\right),
\]

where \(R^0_N\) is defined in Lemma 3.5 and

\[
D_N = C\left(\frac{1}{N} \sum_{j=1}^{N} \|Z_j\|_{C^{-1}}^2 + \frac{1}{N} \sum_{j=1}^{N} \|:Z_j^2\|_{C^{-1}}^2 + 1\right),
\]

\[
D^1_N = C\left(\frac{1}{N^2} \sum_{i,j=1}^{N} \|:Z_i Z_j\|_{C^{-1}}^2\right).
\]

Setting \(A \overset{\text{def}}{=} E D_N\) and \(A_1 \overset{\text{def}}{=} E\|:Z_2 Z_1\|_{C^{-1}}^2\), we may re-center \(D_N\) and \(D^1_N\) above and divide by \(N\) to obtain

\[
\frac{1}{N} \sum_{j=1}^{N} E\|\nabla Y_j(t)\|_{L^2}^2 + (m - A - A_1) \frac{1}{N} \sum_{j=1}^{N} E\|Y_j(t)\|_{L^2}^2 + \frac{1}{N^2} E\left\|\sum_{i=1}^{N} Y_i^2(t)\right\|_{L^2}^2 \\
\leq C \frac{1}{N} E R^0_N + \frac{1}{N} E\left(\sum_{j=1}^{N} \|Y_j(t)\|_{L^2}^2 (|D_N - A| + |D^1_N - A_1|)\right) \\
\leq C \frac{1}{N} E R^0_N + \frac{1}{2} \frac{1}{N^2} E\left(\sum_{j=1}^{N} \|Y_j\|_{L^2}^2\right)^2 + E|D_N - A|^2 + E|D^1_N - A_1|^2.
\]

For \(m \geq A + A_1 + 1\), using that \(Y_i\) and \(Y_j\) are equal in law, we obtain

\[
E\|Y_i\|_{H^1}^2 \leq \frac{1}{N} \sum_{j=1}^{N} E\|\nabla Y_j(t)\|_{L^2}^2 + (m - A - A_1) \frac{1}{N} \sum_{j=1}^{N} E\|Y_j\|_{L^2}^2 \\
\leq C \frac{1}{N} E R^0_N + E|D_N - A|^2 + E|D^1_N - A_1|^2.
\]

Using independence, we find

\[
E|D_N(t) - A|^2 \leq \frac{1}{N} \text{Var}\left(\|Z_i\|_{B_{2,\infty}^{2,-2}}^2 + \|:Z_i^2\|_{B_{2,\infty}^{2,-2}}\right) \leq \frac{C}{N}.
\]

To estimate \(D^1_N\), we write \(M_{i,j} = \|:Z_i Z_j\|_{C^{-1}} - A_1\) for \(i \neq j\) and \(M_{i,i} = \|:Z_i^2\|_{C^{-1}} - A_1\) and have

\[
E\left(\frac{1}{N^2} \sum_{i,j=1}^{N} M_{i,j}\right)^2 \leq E\left(\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i} M_{i,j} + \frac{1}{N^2} \sum_{i=1}^{N} M_{i,i}\right)^2.
\]


\[
\leq \frac{2}{N^4} \sum_{i \neq j, i \neq j} E(M_{i,j}M_{i_1,j_1}) + \frac{2}{N^2} E(M_{i,1}^2)
\]

\[
\leq \frac{1}{N} + \frac{2}{N^2} E(M_{i,1}^2) \lesssim \frac{1}{N},
\]

where we used that for the case that \((i, j, i_1, j_1)\) are different, 

\[E(M_{i,j}M_{i_1,j_1}) = EM_{i,j}EM_{i_1,j_1} = 0.\]

Then we have

\[E|D_1^N - A_1|^2 \leq \frac{1}{N}.\]  

(6.18)

Inserting the estimates (6.17), and (6.18) into (6.16) and using (6.4), we obtain (6.15), completing the proof. \(\square\)

**Remark 6.12.** If we change the renormalization constant as in Remark 6.12, For \(m + \mu_0\) large enough, \(\nu \overset{\text{def}}{=} N(0, (-\Delta + m + \mu_0)^{-1})\) is the unique invariant measure to (6.1). In this case, we separate \(\Phi_i = Y_i + Z_i\) with \(Z_i\) the stationary solution to \(\mathcal{L}Z_i = -\mu_0 Z_i + \xi_i\). Then \(Y_i\) satisfies the following equation:

\[
\mathcal{L}Y_i = -\mu_0 Y_i - \frac{1}{N} \sum_{j=1}^N (Y_j^2 Y_i + 2Y_j Y_i Z_i + 2Y_j :Z_i: Y_i :Z_j: + :Z_i :Z_j:) + \frac{2\mu_0}{N} (Y_i + Z_i),
\]

which is the same case as (3.3) with \(m\) replaced by \(m + \mu_0\) and an extra term \(\frac{2\mu_0}{N} (Y_i + Z_i)\). Here the Wick product of \(Z_j\) could be defined similarly as in section 3.1. By the same proof of Theorem 6.11 we know for \(m + \mu_0\) large enough, \(\nu^{N,i}\), with renormalization constant given as in Remark 6.12, converges to \(\tilde{\nu}\) and the other results in Theorem 6.11 also hold in this case.

### 6.3. Observables

In quantum field theories with symmetries, quantities that are invariant under action of the symmetry group are of particular interest; examples of such quantities in the SDE setting include gauge invariant observables e.g. [She18, Section 2.4]. The model we study here exhibits \(O(N)\) rotation symmetry and formally, functions of the squared “norm” \(\sum_i \Phi_i^2\) are quantities that are \(O(N)\) invariant. Of course, such observables need to be suitably renormalized to be well-defined and suitably scaled by factors of \(N\) to have nontrivial limit as \(N \to \infty\).

In this section we study the following two observables:

\[
\frac{1}{N^{1/2}} \sum_{i=1}^N :\Phi_i^2: = \frac{1}{N} :\left( \sum_{i=1}^N \Phi_i^2 \right)^2:, \quad \tag{6.19}
\]

with \(\Phi = (\Phi_i)_{1 \leq i \leq N} \sim \nu^N\). These are defined as follows. By Lemma 6.7 we decompose \(\Phi_i = Y_i + Z_i\) with \((Y_i, Z_i)\) stationary. With this we define

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^N :\Phi_i^2: \overset{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{i=1}^N (Y_i^2 + 2Y_i Z_i + :Z_i^2:), \tag{6.20}
\]

\[
\frac{1}{N} \left( \sum_{i=1}^N \Phi_i^2 \right)^2 \overset{\text{def}}{=} \frac{1}{N} \sum_{i,j=1}^N \left( Y_i^2 Y_j^2 + 4Y_i^2 Y_j Z_j + 2Y_i^2 :Z_j^2: + :Z_i^2 :Z_j^2: + 4Y_i Y_j :Z_i :Z_j: + 4Y_i Y_j :Z_i Z_j : \right). \tag{6.21}
\]
Here the Wick products for Gaussians are canonically defined as in Section 3.1, for instance
\[ :Z_i^2 Z_j^2 : = \begin{cases} 
\lim_{\epsilon \to 0} (Z_{i,\epsilon}^4 - 6a_\epsilon Z_{i,\epsilon}^2 + 3a^2) & (i = j) \\
\lim_{\epsilon \to 0} (Z_{j,\epsilon}^2 - a_\epsilon)(Z_{i,\epsilon}^2 - a_\epsilon) & (i \neq j)
\end{cases} \]
with \( Z_{i,\epsilon} \) as mollification of \( Z_i \) and \( a_\epsilon = E[Z_{i,\epsilon}^2 (0,0)] \).

**Remark 6.13.** One could also define (6.19) in \( L^p(\nu^N) \) directly without using the decomposition \( \Phi_i = Y_i + Z_i \). In fact, by similar argument as in [GJ87, Section 8.6] or [Sim74], one can show that \( \nu^N \) is absolutely continuous with respect to the corresponding Gaussian free field \( \tilde{\nu} \) with a density in \( L^p(\tilde{\nu}) \) for \( p \in (1, \infty) \). Since (6.19) with each \( \Phi_i \) replaced by \( Z_i \) can be defined via \( L^p(\tilde{\nu}) \) limit of mollification, using argument along the line of [RZZ17, Lemma 3.6] we know that (6.19) can be also defined as \( L^p(\nu^N) \) limit of mollification (essentially Hölder inequality), and they have the same law as the right hand side of (6.20) and (6.21).

In this section we also consider \( Y_i, Z_i \) as stationary process with \( Z_i \) as the stationary solution of (6.2) and \( Y_i \) as the solution of (3.3).

**Lemma 6.14.** There exists an \( m_0 \) such that for \( m \geq m_0 \) and \( q \geq 1 \)
\[
E \left[ \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 \right)^q \right] + E \left[ \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 + 1 \right)^q \left( \sum_{i=1}^{N} \|\nabla Y_i\|_{L^2}^2 \right) \right] \lesssim 1, \tag{6.22}
\]
\[
E \left[ \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 + 1 \right)^q \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 \right)^{2} \right] \lesssim 1, \tag{6.23}
\]
where the implicit constant is independent of \( N \).

**Proof.** First we observe that (6.23) may be quickly deduced from (6.22) with the help of the inequality
\[
\left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2}^2 \lesssim \left( \sum_{i=1}^{N} \|Y_i\|_{H^1}^2 \right) \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 \right). \tag{6.24}
\]
To obtain (6.24), note first that
\[
\left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2}^2 = \sum_{i,j=1}^{N} \|Y_i Y_j\|_{L^2}^2.
\]
Furthermore, by Hölder’s inequality, (5.2), and Young’s inequality
\[
\|Y_i Y_j\|_{L^2}^2 \leq \|Y_i\|_{L^2}^2 \|Y_j\|_{L^2}^2 \lesssim \|Y_i\|_{H^1} \|Y_j\|_{H^1} \|Y_i\|_{L^2} \|Y_j\|_{L^2} \lesssim \|Y_i\|_{H^1}^2 \|Y_j\|_{L^2}^2 + \|Y_i\|_{L^2}^2 \|Y_j\|_{H^1}^2.
\]
Summing both sides over \( i, j \) and using symmetry with respect to the roles of \( i \) and \( j \), we obtain (6.24). The remainder of the proof is devoted to (6.22).

To shorten the expressions that follow, we introduce the quantities
\[
F \overset{\text{def}}{=} \sum_{i=1}^{N} \|\nabla Y_i\|_{L^2}^2 + \frac{1}{N} \left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2}^2 \quad \text{and} \quad U \overset{\text{def}}{=} \sum_{i=1}^{N} \|Y_i\|_{L^2}^2.
\]
Note that \( F \) and \( U \) are stationary, so we will freely omit the time argument below. Our starting point is the key inequality (3.20), which may be recast in terms of \( U \) and \( F \) as
\[
\frac{d}{dt} U + F + mU \leq CR_N^0 + C(D_N + D_N^1)U.
\]
Multiplying the above by $U_q^{-1}$ we find that for $q \geq 1$ it holds

$$
\frac{1}{q} \frac{d}{dt} U_q + U_q^{-1} F + mU_q \leq CR_N^q U_q^{-1} + C(D_N + D_N^1)U_q.
$$

As in the proof of Theorem 6.11, we now define $A = \mathbf{E}(D_N)$ and $A_1 = \mathbf{E} \| :Z_1 Z_2: \|_{C_{-\infty}}^2$. Subtract the mean from $D_N + D_N^1$ and take expectation on both sides to find

$$
\mathbf{E}[U_q^{-1} F] + (m - A - A_1) \mathbf{E}[U_q] \leq C \mathbf{E}[R_N^q U_q^{-1}] + C \mathbf{E}[(D_N + D_N^1 - A - A_1)U_q].
$$

Substituting into (6.24) and using the hypercontractivity upgrade of (6.17) and (6.18) in the last line. Using Young’s inequality, we may absorb the first term to the left and obtain

$$
\mathbf{E}[U_q^{-1} F] + (m - A - A_1 - 1) \mathbf{E}[U_q] \leq C + CN^{-\frac{q}{2q+1}} \mathbf{E}[U_{q+1}].
$$

(6.25)

The strategy now is to first use the dissipative quantity on the LHS of (6.25) to obtain $\mathbf{E}(U_q) \leq CN^{\frac{q}{q+1}}$, and then use the massive term on the LHS of (6.25) to iteratively decrease the power of $N$ and eventually arrive at $\mathbf{E}(U_q) \leq C$. Once this is established, plugging the bound back into (6.25) completes the proof.

Indeed, first observe that $F \geq N^{-1} U^2$ so that $\mathbf{E}(U_q^{-1} F) \geq N^{-1} \mathbf{E}(U_q)$. Hence, Young’s inequality with exponents $(q + 1, \frac{q+1}{q})$ leads to $\mathbf{E}(U_q) \leq CN^{\frac{q}{q+1}}$.

Defining $A_q = \mathbf{E} U_q$ and discarding the dissipative term, (6.25) implies

$$
A_q \lesssim A_{q+1}^\frac{q}{q+1} N^{-1/2} + 1.
$$

(6.26)

We have $A_q \lesssim N^{\frac{q}{q+1}}$, which gives

$$
A_q \lesssim N^{\frac{q^2}{2(q+1)} - \frac{q}{q+1} + 1}.
$$

Substituting into (6.26) and use induction we have for $m \geq 1$

$$
A_q \lesssim N^{a_{m,q}} + 1,
$$

(6.27)

with $a_{m,q} = \frac{q(q+1-m)}{2(q+1)} - \frac{q}{2}(\sum_{k=1}^{m-1} \frac{1}{q+k} - \frac{1}{2})$. Here $\sum_{k=1}^{0} = 0$. In fact, we could check (6.27) by

$$
A_q \lesssim (N^{a_{m,q}} + 1)^{\frac{q}{q+1}} N^{-\frac{q}{q+1}} + 1 \lesssim N^{a_{m+1,q}} + 1.
$$

For fixed $q \geq 1$ we could always find $m$ large enough such that $a_{m,q} < 0$, which implies that $A_q \lesssim 1$ and the result follows.

□

**Theorem 6.15.** Let $\Phi = (\Phi_i)_{1 \leq i \leq N} \sim \nu_N$ and $m$ be given as in Lemma 6.14, then the following hold:

1. The laws of $N^{-\frac{1}{2}} \sum_{i=1}^{N} :\Phi_i^2:$ are tight on $B_{2,2}^{-2\nu}$ and the subsequential limits can be identified with those of

$$
N^{-\frac{1}{2}} \sum_{i=1}^{N} ( :Z_i^2: + 2Y_iZ_i).
$$

(6.28)
(2) The laws of $\frac{1}{N} : \left( \sum_{i=1}^{N} \Phi_i^2 \right)^2$: are tight on $B_{1,1}^{-\kappa}$ and the subsequential limits can be identified with those of

$$N^{-1} \sum_{i,j=1}^{N} \left( :Z_i^2 Z_j^2: + Y_i :Z_i Z_j^2: + Y_i Y_j :Z_i Z_j: \right). \quad (6.29)$$

**Proof.**

Recall the definition:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Phi_i^2 := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i^2 + \frac{2}{\sqrt{N}} \sum_{i=1}^{N} Y_i Z_i + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_i^2 : . \quad (6.30)$$

Note that the first sum on RHS of (6.30) converges to zero in $L^2(\Omega; L^2)$ as an immediate consequence of (6.23). We will now show that the other two quantities induce tight laws on $B_{2,2}^{-\kappa}$, which implies the first part of the theorem. The second sum in (6.30) can be estimated using Lemma 2.3 and Lemma 2.1 to find for $s \in (\kappa, 2\kappa)$

$$\left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i Z_i \right\|_{B_{2,2}^{-\kappa}} \lesssim \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \|Y_i\|_{B_{2,2}^{2}} \|Z_i\|_{C^{-\kappa}}$$

$$\leq \sum_{i=1}^{N} \|Y_i\|_{B_{2,2}^{2}}^2 + \frac{1}{N} \sum_{i=1}^{N} \|Z_i\|_{C^{-\kappa}}^2 \lesssim \sum_{i=1}^{N} \|Y_i\|_{B_{2,2}^{2}}^2 \|Y_i\|_{L^2}^{2(1-s)} + \frac{1}{N} \sum_{i=1}^{N} \|Z_i\|_{C^{-\kappa}}^2$$

which combined with Lemma 6.14 for $q = 1$ and Lemma 6.1 implies that

$$E \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i Z_i \right\|_{B_{2,2}^{-\kappa}} \lesssim 1.$$

For the third sum in (6.30) we use independence to find for $s \in (\kappa, 2\kappa)$

$$E \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_i^2 : \right\|_{B_{2,2}^{-\kappa}}^2 = E \left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} Z_i^2 \right\|_{B_{2,2}^{-\kappa}}^2 \lesssim 1.$$

By the triangle inequality and the embedding $L^2 \hookrightarrow B_{2,2}^{-\kappa}$ we find that $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} :\Phi_i^2:$ is bounded in $L^1(\Omega; B_{2,2}^{-\kappa})$. In light of the compact embedding $B_{2,2}^{-\kappa} \subset B_{2,2}^{-2\kappa}$, the tightness claim follows. Furthermore, since the first contribution to (6.30) converges strongly to zero, only contributions from (6.28) can contribute to the limiting law.

For the second part of the theorem, by definition,

$$\frac{1}{N} \sum_{i,j=1}^{N} :\Phi_i^2 \Phi_j^2: = \frac{1}{N} \sum_{i,j=1}^{N} (Y_i^2 Y_j^2 + 4Y_i^2 Y_j Z_j + 2Y_i^2 :Z_j^2:) \quad (6.31)$$

$$+ \frac{1}{N} \sum_{i,j=1}^{N} ( :Z_i^2 Z_j^2: + 4Y_i :Z_i Z_j^2: + 4Y_i Y_j :Z_i Z_j: ). \quad (6.32)$$
We start with the terms in (6.31), which will be shown to converge to zero in $L^1(\Omega; B^{-2s}_{1,1})$ for $s > \kappa$. For the first term of (6.31) we use Lemma 6.14 to obtain

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i,j=1}^N Y_i^2 Y_j^2 \right\|_{L^1} = \frac{1}{N} \mathbb{E} \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^2}^2 \lesssim \frac{1}{N},$$

so this term converges to zero in $L^1(\Omega; L^1)$. For the second term of (6.31), using (3.17) of Lemma 3.3 with $\varphi Z_j$ in place of $Z_j$ we obtain

$$\sup_{\|\varphi\|_{C^{2s}} \leq 1} \left\| \frac{1}{N} \sum_{i,j=1}^N Y_i^2 Y_j Z_j, \varphi \right\| \lesssim \frac{1}{N} \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^2} \left( \sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^N \|Z_j\|_{C^{\infty}}^2 \right)^{1/2} + \frac{1}{N} \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^2} \left( \sum_{j=1}^N \|Y_j\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^N \|Z_j\|_{C^{\infty}}^2 \right)^{1/2}.$$ 

Hence, using Young’s inequality we find for $\delta > 0$ small enough

$$\left\| \frac{1}{N} \sum_{i,j=1}^N Y_i^2 Y_j Z_j \right\|_{B^{-2s}_{1,1}} \leq N^{-\delta} \left\| \frac{1}{N} \sum_{i=1}^N Y_i^2 \right\|_{L^2}^2 + N^{-\delta} \left( \sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2 \right) \left( \sum_{j=1}^N \|Z_j\|_{C^{\infty}}^2 + 1 \right)^{1/2}.$$ 

Both terms above converge to zero in $L^1(\Omega)$ as a consequence of (6.22), (6.23) and Lemma 6.1. For the third term in (6.31), a calculation similar to (3.12) using Lemma 2.3 with $Y_i$ replaced by $\varphi Y_i$ with $\varphi \in C^{2s}$ yields

$$\sup_{\|\varphi\|_{C^{2s}} \leq 1} \left\| \frac{1}{N} \sum_{i,j=1}^N (Y_i^2 : Z_j^2), \varphi \right\| \lesssim \frac{1}{N} \sum_{i=1}^N \|\Lambda^s (Y_i^2)\|_{L^2} \left\| \sum_{j=1}^N \Lambda^{-s} ( : Z_j^2) \right\|_{L^2} \lesssim \left( \sum_{i=1}^N \|\nabla Y_i\|_{L^2}^{1+s} \|Y_i\|_{L^2}^{1-s} + \|Y_i^2\|_{L^2} \right) \left\| \frac{1}{N} \sum_{j=1}^N \Lambda^{-s} ( : Z_j^2) \right\|_{L^2} \lesssim \left( \sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \|Y_i\|_{L^2}^2 \right) \left\| \frac{1}{N} \sum_{j=1}^N \Lambda^{-s} ( : Z_j^2) \right\|_{L^2},$$

(6.33)

where we used (2.1) and Lemma 2.2 to have

$$\|\Lambda^s (Y_i^2)\|_{L^2} \lesssim \|\Lambda^s Y_i\|_{L^1} \|Y_i\|_{L^4} \lesssim \|\nabla Y_i\|_{L^2}^{1+s} \|Y_i\|_{L^2}^{1-s} + \|Y_i^2\|_{L^2}.$$ 

The first part of the product in (6.33) is bounded in $L^1(\Omega)$ by (6.22). For the second part of the product, we use independence to obtain

$$\mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N \Lambda^{-s} ( : Z_j^2) \right\|_{L^2}^2 \lesssim \frac{1}{N^2} \sum_{j=1}^N \mathbb{E} \|\Lambda^{-s} ( : Z_j^2)\|_{L^2}^2 \lesssim \frac{1}{N},$$
so together and using Lemma 5.2 we find
\[
\left\| \frac{1}{N} \sum_{i,j=1}^{N} Y_i^2 : Z_j^2 \right\|_{B_{2,1}^{-2\varepsilon}} \to 0 \quad \text{in probability.}
\]

We now turn to the quantities in (6.32) and derive suitable moment bounds. For the first term in (6.32) we have
\[
E \left\| \frac{1}{N} \sum_{i,j=1}^{N} : Z_i^2 Z_j^2 : \right\|_{B_{2,2}^{-\varepsilon}}^2 = E \left( \frac{1}{N^2} \sum_{i,j=1}^{N} : Z_i^2 Z_j^2 : , \Lambda^{-\kappa} \sum_{i,j=1}^{N} : Z_i^2 Z_j^2 : \right) \lesssim E \left\| : Z_i^2 Z_j^2 : \right\|_{B_{2,2}^{-\varepsilon}}^2 + \frac{1}{N} E \left\| : Z_i^4 : \right\|_{B_{2,2}^{-\varepsilon}}^2 \lesssim 1.
\]

For the second term in (6.32), using (3.19) with \( \varphi Y_i \) in place of \( Y_i \) we find
\[
\sup_{\| \varphi \|_{\ell^2} \leq 1} \left| \frac{1}{N} \sum_{i,j=1}^{N} \langle Y_i : Z_i Z_j^2 : , \varphi \rangle \right| \lesssim \left( \sum_{i=1}^{N} \| \Lambda^s Y_i \|_{L^2} \right)^{1/2} \left( \sum_{i=1}^{N} \| Z_j^2 Z_i^2 : \right)^{1/2} \left( \sum_{i=1}^{N} \| \Lambda^{-s} ( : Z_i Z_j : ) \right)^{1/2} \left( \sum_{i=1}^{N} \| \Lambda^{-s} ( : Z_i Z_j : ) \right)^{1/2}.
\]

Using (6.4) and Lemma 6.14 we deduce for some \( p \) satisfying \( sp < 2 \)
\[
E \left\| \frac{1}{N} \sum_{i,j=1}^{N} Y_i : Z_i Z_j^2 : \right\|_{B_{2,1}^{-2\varepsilon}}^p \lesssim 1.
\]

Finally, for the third term in (6.32), we argue similarly to (3.10) but with \( Y \) replace by \( \varphi Y \) for \( \varphi \in C^* \) to deduce boundedness in \( L^1(\Omega; B_{1,1}^{-2\varepsilon}) \).

Combining the above observations with the triangle inequality, we find that the observable \( N^{-1} \sum_{i,j=1}^{N} : \Phi_1^2 \Phi_j^2 : \) is uniformly bounded in probability as a \( B_{1,1}^{-2\varepsilon} \) valued random variable. By compactness of the embedding of \( B_{1,1}^{-2\varepsilon} \) into \( B_{1,1}^{-2\varepsilon-\delta} \) for \( \delta > 0 \), we obtain the result. Since the contributions from (6.31) converge strongly to zero, only the contributions of (6.29) contribute in the limit, completing the proof. \( \square \)

### 7. One Space Dimension

In this section we turn to one space dimension \( d = 1 \) and provide proofs to the analogous results obtained in the previous sections. Moreover we prove a result (Theorem 7.10) which states that in the large \( N \) limit, \( O(N) \) invariant observables have nontrivial law. We will omit some details which follow the same way as 2d, but meanwhile give proofs for results such as uniform estimates and convergence of the dynamics which are simpler in 1d, and we believe that this also serves pedagogical purpose which makes the understanding of the 2d proofs a bit easier.

#### 7.1. Convergence of the dynamics

Consider the following equation on \( \mathbb{R}^+ \times \mathbb{T} \):
\[
\mathcal{L} \Phi_i = -\frac{1}{N} \sum_{j=1}^{N} \Phi_j^2 \Phi_i + \xi_i, \quad \Phi_i(0) = \phi_i, \tag{7.1}
\]
with \( \mathcal{L} = \partial_t - \Delta + m \) for \( m \geq 0, \ 1 \leq i \leq N, \ i \in \mathbb{N} \). Here \((\xi_i)_i\) are independent space-time white noises on \( \mathbb{R} \times \mathbb{T} \) defined on a stochastic basis \((\Omega, \mathcal{F}, \mathbb{P})\). The initial values \( \phi_i \) defined on the same stochastic basis \((\Omega, \mathcal{F}, \mathbb{P})\) are independent of the space-time white noises \( \xi_i \); and \( \mathbb{E}\|\phi_i\|_{L^2}^2 \lesssim 1 \), where the implicit constants are independent of \( i, N \). Now we decompose (7.1) to the following two equations:

\[
\mathcal{L}Z_i = \xi_i, \tag{7.2}
\]

where \( Z_i \) is the stationary solution to (7.2), and

\[
\mathcal{L}Y_i = -\frac{1}{N} \sum_{j=1}^{N} (Y_j + Z_j)^2 (Y_i + Z_i), \quad Y_i(0) = y_i \overset{\text{def}}{=} \phi_i - Z_i(0). \tag{7.3}
\]

For \( d = 1 \) it is well-known that

\[
\sup_{m \geq 0} \mathbb{E}\|Z_i\|^p_{C_T^{\frac{1}{2},-\kappa}} \lesssim 1, \tag{7.4}
\]

for \( \kappa > 0, p > 1 \) and, of course, no renormalization is required to define products of \( Z_i, Z_j \).

By similar arguments as the two dimensional case (see Lemma 3.2), we can show existence and uniqueness of solutions \( Y_i \in C_T L^2 \cap L_T^4 L^4 \cap L_T^2 H^1 \) to (7.3) \( \mathbb{P} \)-a.s.. In the following we first give a uniform in \( N \) estimate for \( Y_i \).

**Lemma 7.1.** There exists a universal constant \( C \) such that

\[
\frac{1}{N} \left( \sum_{i=1}^{N} \|Z_i\|^2_{L^\infty} \right)^2 + \int \frac{1}{N^2} \sum_{i,j=1}^{N} Z_i^2 Z_j^2 \, dx \leq C \int_0^T R_N \, dt + \frac{1}{N} \sum_{j=1}^{N} \|y_j\|^2_{L^2}, \tag{7.5}
\]

where

\[
R_N := \left( \frac{1}{N} \sum_{i=1}^{N} \|Z_i\|^2_{L^\infty} \right)^2 + \int \frac{1}{N^2} \sum_{i,j=1}^{N} Z_i^2 Z_j^2 \, dx.
\]

**Proof.** Taking \( L^2 \)-inner product with \( Y_i \) in (7.3), one has

\[
\frac{1}{2} \frac{d}{dt} \|Y_i\|^2_{L^2} + \|\nabla Y_i\|^2_{L^2} + \|Y_i\|^2_{L^2} + \int \frac{1}{N} \sum_{j=1}^{N} (Y_j + Z_j)^2 Y_i^2 \, dx
\]

\[
= - \int \frac{1}{N} \sum_{j=1}^{N} (Y_j + Z_j)^2 (Y_i Z_i) \, dx \tag{7.6}
\]

\[
\leq \frac{1}{2} \int \frac{1}{N} \sum_{j=1}^{N} (Y_j + Z_j)^2 Y_i^2 \, dx + \frac{1}{2} \int \frac{1}{N} \sum_{j=1}^{N} (Y_j + Z_j)^2 Z_i^2 \, dx.
\]

Half of the third term on the LHS is canceled by the first term on the RHS, and for the rest half, since \( \frac{1}{2} Y_i^2 Y_j^2 + 2 Y_i^2 Y_j Z_j + 2 Y_i^2 Z_j^2 \geq 0 \) one has

\[
\int \frac{1}{2N} \sum_{j=1}^{N} (Y_j + Z_j)^2 Y_i^2 \, dx \geq \frac{1}{2N} \sum_{j=1}^{N} \int \frac{1}{2} Y_j^2 Y_i^2 - Z_j^2 Y_i^2 \, dx.
\]
Now we obtain
\[
\frac{1}{2} \frac{d}{dt} \| Y_i \|_{L^2}^2 + \| \nabla Y_i \|_{L^2}^2 + m \| Y_i \|_{L^2}^2 + \int \frac{1}{4N} \sum_{j=1}^N Y_j^2 Y_i^2 \, dx
\]
\[
\leq \int \frac{1}{2N} \sum_{j=1}^N Y_j^2 Z_j^2 \, dx + \frac{1}{2} \int \frac{1}{N} \sum_{j=1}^N (Y_j + Z_j)^2 Z_i^2 \, dx.
\]
Multiplying $1/N$ and taking sum over $i$ we obtain
\[
\frac{1}{2N} \frac{d}{dt} \sum_{j=1}^N \| Y_j \|_{L^2}^2 + \frac{1}{N} \sum_{j=1}^N \| \nabla Y_j \|_{L^2}^2 \, dx + \frac{m}{N} \sum_{j=1}^N \| Y_j \|_{L^2}^2 + \int \frac{1}{4N^2} \sum_{i,j=1}^N Y_j^2 Y_i^2 \, dx
\]
\[
\leq \int \frac{3}{2N^2} \sum_{i,j=1}^N Y_j^2 Z_i^2 \, dx + \int \frac{1}{N^2} \sum_{i,j=1}^N Z_j^2 Z_i^2 \, dx.
\]
Using $(\sum_{j=1}^N Y_j^2)^2 = \frac{1}{N^2} \sum_{i,j=1}^N Y_i^2 Y_j^2$, we obtain
\[
\frac{1}{2N} \frac{d}{dt} \frac{1}{N} \sum_{j=1}^N \| Y_j \|_{L^2}^2 + \frac{1}{2N} \sum_{j=1}^N \| \nabla Y_j \|_{L^2}^2 + \frac{1}{8N^2} \sum_{i,j=1}^N Y_j^2 Y_i^2
\]
\[
\leq \left( \frac{3C}{2N} \sum_{i=1}^N \| Z_i \|_{L^\infty}^2 \right)^2 + \int \frac{1}{N^2} \sum_{i,j=1}^N Z_j^2 Z_i^2, \quad (7.7)
\]
which implies (7.5). Here we use Young's inequality for products,
\[
\frac{3}{2} \int \left( \frac{1}{N} \sum_{j} Y_j^2 \right) \left( \frac{1}{N} \sum_{i} Z_i^2 \right) \, dx \leq \int \frac{1}{8} \left( \frac{1}{N} \sum_{j} Y_j^2 \right)^2 + C \left( \frac{1}{N} \sum_{i} Z_i^2 \right)^2 \, dx.
\]

\begin{corollary}
For every $p > 1$ it holds that
\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_{j=1}^N \| Y_j(t) \|_{L^2}^2 \right)^p \right] \lesssim 1 + \mathbb{E} \left[ \left( \frac{1}{N} \sum_{j=1}^N \| \phi_j \|_{L^2}^2 \right)^p \right], \quad (7.8)
\]
\[
\mathbb{E} \left[ \| Y_i(t) \|_{L^p}^{2p} \right] \lesssim 1 + \mathbb{E} \left[ \left( \frac{1}{N} \sum_{j=1}^N \| \phi_j \|_{L^2}^2 \right)^{p'} \right] + \mathbb{E} \left[ \| \phi_i \|_{L^2}^{2p} \right], \quad (7.9)
\]
where $p' > p$ and the proportionality constant is independent of $N$ and $i$. Furthermore,
\[
\sup_{t>0} (t \wedge 1) \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \| Y_j(t) \|_{L^2}^2 \right] \lesssim 1, \quad (7.10)
\]
where the proportionality constant is independent of initial data.
\end{corollary}

\begin{proof}
It is easy to deduce (7.8) by using (7.5) and (7.4). By (7.6) we know
\[
\frac{d}{dt} \| Y_i \|_{L^2}^2 \lesssim \int \frac{1}{N} \sum_{j=1}^N (Y_j^2 + Z_j^2) Z_i^2 \lesssim \| Z_i \|_{L^2}^2 + \frac{1}{N} \sum_{j=1}^N \| Y_j \|_{L^2}^2 + \frac{1}{N} \sum_{j=1}^N Z_j^2 Z_i^2. \quad (7.10)
\]
Using (7.8), (7.4) and Young’s inequality we know (7.9) follows. The last bound (7.10) follows from Lemma 2.6 and (7.7).
\end{proof}
In $d = 1$, we will show that the limiting equation as $N \to \infty$ will be given by
\begin{equation}
\mathcal{L} \Psi_i = -\mu \Psi_i + \xi_i, \quad \text{with} \quad \mu(t, x) = E[\Psi_i(t, x)^2]
\end{equation}
where $(\xi_i)_{i \in \mathbb{N}}$ are the same collection of noises as in (7.1). We start by establishing some a priori bounds for this equation. We assume that the initial conditions $\Psi_i(0) = \psi_i$ (generally random) are defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$ and are independent of the noise $(\xi_i)_{i \in \mathbb{N}}$, and $E[\|\psi_i\|_{L^2}^2]$ is bounded by a constant independent of $i$.

Again, let $Z_i$ be the stationary solution to the linear equation (7.2). Set $\Psi_i = Z_i + X_i$ so that
\begin{equation}
\mathcal{L} X_i = -\mu(X_i + Z_i), \quad X_i(0) = \psi_i - Z_i(0).
\end{equation}

In what follows, we will focus on the a priori estimate for the equation (7.11). Existence and uniqueness of solutions for (7.12) could be obtained by similar arguments as in the two dimensional case and one has $X_i \in C_T L^\infty \cap L^2_T H^1$, but we skip the details since it’s not the main purpose of this section.

**Lemma 7.3.** There exists a universal constant $C_T$ such that
\begin{equation}
\sup_{t \in [0, T]} E[\|X_i\|_{L^2_P L^2}^2] + E[\|\nabla X_i\|_{L^2_P L^2}^2] + E[\|X_i\|_{L^4_P L^2}^2] \leq C_T + E[\|\psi_i\|_{L^2}^2],
\end{equation}
\begin{equation}
\|X_i\|_{L^4_P L^2}^2 + \|\nabla X_i\|_{L^2_P L^2}^2 \leq C_T \|Z_i\|_{L^\infty_P L^\infty}^2 + \|\psi_i\|_{L^2}^2,
\end{equation}
\begin{equation}
\|X_i\|_{L^4_P L^2}^2 + \int_0^T \|\nabla X_i\|_{L^2_P L^2}^2 dt \leq C_T \|Z_i\|_{L^\infty_P L^\infty}^2 + \|\psi_i\|_{L^2}^4.
\end{equation}

**Proof.** Taking inner product with $X_i$ for (7.12) we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|X_i\|_{L^2}^2 + \|\nabla X_i\|_{L^2}^2 + m \|X_i\|_{L^2}^2 + \int \mu x^2 dx \geq -\int \mu X_i Z_i dx \leq \frac{1}{2} \int \mu x^2 dx + \frac{1}{2} \int \mu Z_i dx.
\end{equation}
The first term on the RHS can be absorbed to the LHS. Write $X_j, Z_j$ ($i \neq j$) for independent copies of $X_i, Z_i$ respectively. Note that
\begin{equation}
\frac{1}{2} E[X_j(t, x)^2] - E[Z_j(t, x)^2] \leq \mu(t, x) \leq 2E[X_j(t, x)^2] + 2E[Z_j(t, x)^2].
\end{equation}
Taking expectation, one then has
\begin{equation}
\frac{1}{2} \frac{d}{dt} E[\|X_i\|_{L^2}^2] + E[\|\nabla X_i\|_{L^2}^2] + \frac{1}{4} \int E[X_i^2 Z_i^2] dx \leq \int E[X_i^2 Z_i^2] dx + \int E[Z_i^2 Z_i^2] dx
\end{equation}
\begin{equation}
\leq E[\|X_i\|_{L^2}^2] E[\|Z_i\|_{L^\infty}^2] + E[\|Z_i\|_{L^\infty}^2]^2 \leq \frac{1}{8} E[\|X_i\|_{L^2}^2] + C E[\|Z_i\|_{L^\infty}^2]^2
\end{equation}
where we applied Young’s inequality in the last step. Now the claimed bound (7.13) follows by integrating time.

Furthermore, by (7.16) we have
\begin{equation}
\frac{1}{2} \|X_i\|_{L^2_P L^2}^2 + \|\nabla X_i\|_{L^2_P L^2}^2 \leq \|Z_i\|_{L^\infty_P L^\infty}^2 (\|\mu\|_{L^1_T L^1} + 1) + \|\psi_i\|_{L^2}^2.
\end{equation}
Now we apply the bound (7.13), which we have just proved; this yields
\begin{equation}
\|\mu\|_{L^1_T L^1} \lesssim E[\|X_i\|_{L^2_P L^2}^2] + E[\|Z_i\|_{L^2_P L^2}^2] \lesssim 1,
\end{equation}
which implies the second result \((7.14)\). Moreover, using \((7.16)\) we deduce
\[
\frac{1}{4} \frac{d}{dt} \|X_i\|_{L^2}^2 + \|\nabla X_i\|_{L^2}^2 \leq \frac{1}{2} \|X_i\|_{L^2}^2 \int \mu Z_i^2 \, dx.
\]
Taking integration with respect to time and using \((7.14)\) and \((7.18)\), \((7.15)\) follows. \(\square\)

**Corollary 7.4.** It holds that
\[
\sup_{t > 0} (t \wedge 1) E[\|X_i(t)\|_{L^2}^2] \lesssim 1, \tag{7.19}
\]
\[
\sup_{t \geq 1} \frac{1}{t} \int_1^t E[\|\nabla X_i(t)\|_{L^2}^2] \, dt \lesssim 1. \tag{7.20}
\]

Here the implicit constants are independent of the initial data and \(m\).

**Proof.** Combining \((7.17)\) and \((7.4)\) one has
\[
\frac{d}{dt} E[\|X_i\|_{L^2}^2] + \frac{1}{4} E[\|X_i\|_{L^2}^2]^2 \leq C E[\|Z_i\|_{L^\infty}^2]^2
\]
Using Lemma 2.6 we obtain \((7.19)\). Similarly, integrating \((7.17)\) over time interval \([1, t]\),
\[
\int_1^t E[\|\nabla X_i(t)\|_{L^2}^2] \, dt \lesssim E[\|X_i(1)\|_{L^2}^2] + t \lesssim t
\]
which proves \((7.20)\). \(\square\)

We also establish an \(L^\infty\) bound, as stated in the following Lemma.

**Lemma 7.5.** It holds that
\[
\|\Psi_i\|_{L^\infty L^\infty} \lesssim \|Z_i\|_{L^\infty L^\infty} + \|\psi_i\|_{L^\infty} + 1. \tag{7.21}
\]

**Proof.** Using \((7.13)\) we obtain for \(\delta > 0\)
\[
\|\mu\|_{L^\infty} \lesssim E[\|X_i\|_{L^\infty}^2] + E[\|Z_i\|_{L^\infty}^2] \lesssim E[\|X_i\|_{H^1}^{1+\delta}] \|X_i\|_{L^2}^{1-\delta} + E[\|Z_i\|_{L^\infty}^2]
\]
\[
\lesssim (E[\|X_i\|_{H^{1+\delta}}^2] E[\|X_i\|_{L^2}^{1-\delta}])^{1/2} + E[\|Z_i\|_{L^\infty}^2],
\]
where we used Sobolev embedding \(H^{1+\delta} \subset L^\infty\) and interpolation in the second inequality and we used Hölder’s inequality in the last inequality. This combined with \((7.13)\) and \((7.4)\) implies that
\[
\|\mu\|_{L^\infty L^\infty}^{1/2} \lesssim 1. \tag{7.22}
\]
By using mild formulation we have
\[
\Psi_i(t) = S_t \psi_i + Z_i(t) - \int_0^t S_{t-s} \mu(s) \Psi_i(s) \, ds,
\]
which implies that
\[
\|\Psi_i(t)\|_{L^\infty} \lesssim \|\psi_i\|_{L^\infty} + \|Z_i(t)\|_{L^\infty}
\]
\[
+ \int_0^T \|\mu(s)\|_{L^\infty} \|X_i(s)\|_{L^\infty} \, ds + \int_0^T \|\mu(s)\|_{L^\infty} \|Z_i(s)\|_{L^\infty} \, ds.
\]
For the third term on the right hand side we use Sobolev embedding and interpolation as above
\[
\int_0^T \|\mu(s)\|_{L^\infty} \|X_i(s)\|_{L^\infty} \, ds \lesssim \int_0^T \|\mu(s)\|_{L^\infty} \|X_i(s)\|_{H^{1+\delta}} \|X_i(s)\|_{L^\infty} \, ds
\]
\[
\lesssim \left( \int_0^T \| \mu(s) \|_{L^\infty} \frac{2}{N} \, ds \right)^{\frac{2+\delta}{2+\delta}} \left( \int_0^T \| X_i(s) \|_{L^2}^2 \, ds \right)^{\frac{1+\delta}{2}} \sup_{s \in [0,T]} \| X_i(s) \|_{L^2}^{\frac{1-\delta}{2}}
\lesssim \| Z_i \|_{L^\infty T L^\infty} + \| \psi_i \|_{L^\infty} + 1,
\]
where we used (7.22) and (7.14) in the last inequality. Similarly, the last term could be estimated by
\[
\int_0^T \| \mu(s) \|_{L^\infty} \| Z_i(s) \|_{L^\infty} \, ds \lesssim \| Z_i \|_{L^\infty T L^\infty}.
\]
Combining all the above estimates the result follows.

We now turn to the convergence of the dynamics. Recall that $\Phi_i^N$ is the solution to (7.1) with initial condition $\phi_i^N$. $\Psi_i$ is the solution to (1.7) with initial condition $\psi_i$. We set $v_i^N = \Phi_i^N - \Psi_i$ and have
\[
\mathcal{L} v_i = -\frac{1}{N} \sum_{j=1}^N \Phi_j^N \Phi_i + \mu \psi_i
\]
\[
= -\frac{1}{N} \sum_{j=1}^N \Phi_j^N (\Phi_i - \Psi_i) - \frac{1}{N} \sum_{j=1}^N (\Phi_j^N - \Psi_j^N) \Psi_i - \left( \frac{1}{N} \sum_{j=1}^N \Psi_j^N - \mu \right) \psi_i. \tag{7.23}
\]
Here we write $\Phi_i = \Phi_i^N$, $v_i = v_i^N$ for notation’s simplicity.

**Assumption 7.1.** Suppose the following assumptions:
- The random variables $\{(\phi_i, \psi_i)\}_{i=1}^N$ are exchangeable and $(\psi_i)_i$ are independent.
- For every $i$,
  \[
  \mathbb{E}[\| \phi_i^N - \psi_i \|_{L^2}] \to 0, \quad \text{as } N \to \infty.
  \]
- For some $q > 1$,
  \[
  \mathbb{E}[\| \psi_i \|_{L^2}] \lesssim 1, \quad \mathbb{E}[\| \phi_i^N \|_{L^2}^{2q}] \lesssim 1, \quad \tag{7.24}
\]
  where the implicit constant is independent of $N$. For $1 \leq i \leq N$, $\psi_i \in L^\infty$ $\mathbb{P}$-a.s..

**Theorem 7.6.** Under Assumption 7.1, for every $i$ and every $T > 0$ $\mathbb{E}[v_i^N]_{C_T L^2}$ converges to zero as $N \to \infty$.

**Proof.** In the proof we omit the superscript $N$ for simplicity. Taking $L^2$-inner product with $v_i$ on both side of (7.23) we obtain
\[
\frac{1}{2} \frac{d}{dt} \| v_i \|_{L^2}^2 + \| \nabla v_i \|_{L^2}^2 + m \| v_i \|_{L^2}^2 + \int \frac{1}{N} \sum_{j=1}^N \Phi_j^N v_i \, dx + \frac{1}{N} \sum_{j=1}^N \int \psi_j v_i \, dx
\]
\[
= -\frac{1}{N} \sum_{j=1}^N \int \Phi_j v_j \psi_i \, dx - \int \left( \frac{1}{N} \sum_{j=1}^N \Psi_j^N - \mu \right) \psi_i v_i \, dx. \tag{7.25}
\]
Summing over $i$, we obtain
\[
\frac{1}{2} \sum_{i=1}^N \frac{d}{dt} \| v_i \|_{L^2}^2 + \sum_{i=1}^N \| \nabla v_i \|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^N \| \Phi_j v_i \|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^N \| \Psi_i v_i \|_{L^2}^2
\]
\[ \leq \frac{1}{N} \sum_{i,j=1}^{N} \left( \frac{1}{2} \Phi^2_i v_i^2 + \frac{1}{2} L_2^2 \right) + \frac{1}{2N} \left\| \sum_{i=1}^{N} \Psi_i v_i \right\|_{L^2}^2 + \frac{1}{2N} \left\| \sum_{j=1}^{N} (\Psi^2_j - \mu) \right\|_{L^2}^2. \]

Therefore we have
\[ \frac{1}{2} \sum_{i=1}^{N} \frac{d}{dt} \|v_i\|_{L^2}^2 + \sum_{i=1}^{N} \|\nabla v_i\|_{L^2}^2 + \frac{1}{2N} \sum_{i,j=1}^{N} \|\Phi_{ij} v_i\|_{L^2}^2 + \frac{1}{2N} \sum_{i=1}^{N} \|\Psi_i v_i\|_{L^2}^2 \]
\[ \leq \left( \frac{1}{N} \sum_{i=1}^{N} \|\Psi_i\|_{L^\infty}^2 \right) \left( \sum_{j=1}^{N} \|v_j\|_{L^2}^2 \right) + \frac{1}{2N} \sum_{j=1}^{N} \|\Psi^2_j - \mu\|_{L^2}^2. \] (7.26)

Recall the general fact: for mean-zero independent random variables \( U_1, \ldots, U_N \) taking values in a Hilbert space \( H \), it holds that
\[ \mathbb{E} \left\| \sum_{i=1}^{N} U_i \right\|_H^2 = \mathbb{E} \sum_{i=1}^{N} \|U_i\|_H^2. \]

Applying this we obtain
\[ \mathbb{E} \frac{1}{N} \left\| \sum_{i=1}^{N} (\Psi^2_i - \mu) \right\|_{L^2}^2 \leq \mathbb{E} \|\Psi_1^2 - \mu\|_{L^2}^2 \lesssim \mathbb{E} \|\Psi_1\|_{L^4}^4 \lesssim \mathbb{E} \|X_1\|_{L^2}^8 \|X_1\|_{H^1} + \mathbb{E} \|Z_1\|_{L^4}. \]

Combining this, (7.26), (7.15) and Gronwall’s inequality, we obtain that \( \sup_{t \in T} \sum_i \|v_i\|_{L^2}^2 \) is uniformly bounded in probability. Hence,
\[ \lim_{N \to \infty} \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^{N} \|v_i\|_{L^2}^2 = 0 \quad \text{in probability.} \]

Furthermore, using the bound (7.8) on moments of \( \frac{1}{N} \sum_j \|Y_j(t)\|_{L^2}^2 \) and the bound (7.14) on \( X_i \) and triangular inequality, we obtain that the above convergence can be upgraded to convergence in \( L^1(\Omega) \), i.e.
\[ \mathbb{E} \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^{N} \|v_i\|_{L^2}^2 \text{ converges to zero}. \] (7.27)

On the other hand, we use (7.25) to have
\[ \frac{1}{2} \frac{d}{dt} \|v_i\|_{L^2}^2 + \|\nabla v_i\|_{L^2}^2 + \int \frac{1}{2N} \sum_{j=1}^{N} \Phi_{ij}^2 v_i^2 \, dx \]
\[ \leq \|\Psi_i\|_{L^\infty} \sum_{j=1}^{N} \int v_j^2 \, dx + \frac{1}{2N} \sum_{j=1}^{N} \|\Psi_j\|_{L^2}^2 + \int v_i^2 \, dx \]
\[ + \|v_i\|_{L^2} \|\Psi_i\|_{L^\infty} + \frac{1}{N^2} \sum_{j=1}^{N} (\Psi_j^2 - \mu) \right\|_{L^2}^2, \] (7.28)

which combined with Gronwall’s inequality and (7.27), (7.13) implies that \( \sup_{t \in [0, T]} \|v_i\|_{L^2}^2 \) converges to zero in probability. By using (7.9) and (7.14), and assumption (7.24), we deduce the result. \( \square \)
7.2. Invariant measure. Now we study the invariant measure to (7.11). We first note that (7.11) has a Gaussian invariant measure with mean zero and covariance 
\((-\Delta + m + \mu)^{-1}\) with \(\mu\) being a constant. Indeed if \(\mu\) is constant, the stationary solution is given by 
\[ \int_{-\infty}^{t} e^{(t-s)(\Delta - m - \mu)} \, dW_{s} \]
with \(W\) being an \(L^{2}(\mathbb{T})\)-cylindrical Wiener process, and by definition of \(\mu\) it should satisfy the following self-consistent equation
\[ \sum_{k=1}^{\infty} \frac{1}{k^2 + m + \mu} = \mu. \]
By monotonicity, there is a unique solution \(\mu > 0\).

We define the \(L^{2}\)-Wasserstein distance
\[ \mathbb{W}_{2}(\nu_{1}, \nu_{2}) := \inf_{\pi \in \mathcal{G}(\nu_{1}, \nu_{2})} \left( \int \| \phi - \psi \|_{L^{2}(\mathbb{T})}^{2} \pi(\mathrm{d}\phi, \mathrm{d}\psi) \right)^{1/2}, \]
where \(\mathcal{G}(\nu_{1}, \nu_{2})\) is the set of all couplings for \(\nu_{1}, \nu_{2} \in \mathcal{P}_{2}\) with
\[ \mathcal{P}_{2} = \{ \nu : \int \| \phi \|_{L^{2}(\mathbb{T})}^{2} \mathrm{d}\nu < \infty \}. \]
Similar as 2d case, we define a semigroup \(P^{*}_{t}\nu\) to denote the law of \(\Psi(t)\) with initial condition distributed by \(\nu\) which satisfies \(\int \| \psi \|_{L^{2}}^{2} \mathrm{d}\nu < \infty\). By uniqueness of the solutions to (7.11), we know \(P^{*}_{t} = P^{*}_{t-s}P^{*}_{s}\), \(t \geq s \geq 0\).

**Theorem 7.7.** There exists \(m_{0} \geq 0\) such that for \(m \geq m_{0}\), there exist a unique invariant measure \(\nu\) satisfying \(\int \| \psi \|_{L^{\infty}}^{2} \mathrm{d}\nu < \infty\) to (7.11).

**Proof.** Let \(\Psi_{1}\) and \(\Psi_{2}\) be two solutions to (7.11) with the initial distributions given by \(\nu_{1}\) and \(\nu_{2}\) satisfying \(\int \| \psi \|_{L^{2}}^{2} \mathrm{d}\nu_{i} < \infty\) for \(i = 1, 2\). Set \(u = \Psi_{2} - \Psi_{1}\). We have
\[ \partial_{t}u = (\Delta - m)u - \mathbf{E}[\Psi_{2}^{2}]u + \mathbf{E}[\Psi_{1}^{2} - \Psi_{2}^{2}]\Psi_{1}. \]
Now we take \(L^{2}\)-inner product with \(u\) and obtain
\[ \frac{1}{2} \frac{d}{dt} \| u \|_{L^{2}}^{2} + m \| u \|_{L^{2}}^{2} + \| \nabla u \|_{L^{2}}^{2} + \int \mathbf{E}[\Psi_{2}^{2}]u^{2} \, dx = \int \mathbf{E}[\Psi_{1}^{2} - \Psi_{2}^{2}]\Psi_{1} u \, dx \]
\[ \leq 2 \left( \mathbf{E}[u^{2}] \right)^{1/2} \left( \left( \mathbf{E}[\Psi_{2}^{2}] \right)^{1/2} + \left( \mathbf{E}[\Psi_{1}^{2}] \right)^{1/2} \right) \| \Psi_{1} u \| \, dx \]
\[ \leq \frac{1}{2} \int \mathbf{E}[\Psi_{2}^{2}]u^{2} \, dx + \int \mathbf{E}[u^{2}]\Psi_{1}^{2} \, dx + \int \mathbf{E}[\Psi_{1}^{2}]u^{2} \, dx. \]
Now we take expectation and obtain
\[ \frac{1}{2} \frac{d}{dt} \mathbf{E}[\| u \|_{L^{2}}^{2}] + m \mathbf{E}[\| u \|_{L^{2}}^{2}] + \mathbf{E}[\| \nabla u \|_{L^{2}}^{2}] + \frac{1}{2} \int \mathbf{E}[\Psi_{2}^{2}]\mathbf{E}[u^{2}] \, dx \]
\[ \leq \int \mathbf{E}[u^{2}]\mathbf{E}[\Psi_{1}^{2}] \, dx \lesssim \mathbf{E}[\| u \|_{L^{2}}^{2}] \mathbf{E}[\| \Psi_{1} \|_{L^{\infty}}^{2}]. \]  
(7.29)
By Gronwall’s inequality we have
\[ \mathbf{E}[\| u(t) \|_{L^{2}}^{2}] \lesssim \mathbf{E}[\| u(1) \|_{L^{2}}^{2}] \exp \left( -m(t-1) + C \int_{1}^{t} \mathbf{E}[\| \Psi_{1} \|_{L^{\infty}}^{2}] \, ds \right). \]
Choosing \(m\) large enough and invoking Corollary 7.4 we obtain
\[ \mathbf{E}[\| u(t) \|_{L^{2}}^{2}] \lesssim \exp (-mt/2). \]
Now we take two different invariant measures \(\nu_{1}, \nu_{2}\) supported on \(L^{\infty}\) to equation (7.11). We could construct a stochastic basis \((\Omega, \mathcal{F}, P)\) and two random variables
ψ_1, ψ_2 ∈ L^∞ on it such that ψ_i ∼ ν_i. For ψ_i we could construct two solutions Ψ_1, Ψ_2 to equation (7.11) with Ψ_i(0) = ψ_i. Since ν_i is invariant measure we have

\[ \|W^2(ν_1, ν_2)\|^2 = W^2(P_tν_1, P_tν_2)^2 \leq E[\|Ψ_1(t) − Ψ_2(t)\|_2^2] \lesssim e^{-mt/2}. \]

Letting t → ∞ we obtain

ψ_1 = ψ_2 \quad P - a.s.,

which implies that ν_1 = ν_2 and the result follows.

Now we turn to the question of convergence of invariant measures. Consider

\[ dv^N(Φ) \overset{\text{def}}{=} \frac{1}{C_N} \exp \left( -\int \sum_{j=1}^N |∇Φ_j|^2 + m \sum_{j=1}^N Φ_j^2 + \frac{1}{2N} \sum_{j=1}^N (Φ_j^2)^2 \right) dΦ, \quad (7.30) \]

Φ = (Φ_1, Φ_2, ..., Φ_N). The measure ν^N is defined on S'(𝕋)^N. Consider the map Π_i : S'(𝕋)^N → S'(𝕋) given by Π_i(Φ) = Φ_i. Let \( ν^{N,i} := ν^N \circ Π_i^{-1} \).

One can show that ν^N is the unique invariant measure to (7.1), by similar argument as the two dimensional case, and for any m ≥ 0, ν^{N,i} are tight (as in Theorem 6.9 for d = 2 case). Here we do not repeat these details.

We only sketch the proof for the convergence of measures as N → ∞.

ν^{N,i} should converge to the invariant measure of the linear equation. Furthermore, define the map Π^{(k)} : S'(𝕋)^N → S'(𝕋)^k, Π^{(k)}(Φ) = (Φ_i)_{1 ≤ i ≤ k} and the measure ν^k_N := ν^N \circ Π^{(k)}^{-1}. We have the following result:

**Theorem 7.8.** There exist \( m_0 > 0 \) such that for \( m \geq m_0 \), ν^{N,i} converges to the law ν of the unique invariant measure to (7.11). Furthermore, ν^k_N converges to ν × ... × ν.

**Proof.** Similar as in Lemma 6.7 we could find stationary process (Φ_i, Ψ_i)_{1 ≤ i ≤ N} such that Φ_i, Ψ_i are the stationary solution to (7.1) and (7.11), respectively. Set \( v_i = Φ_i − Ψ_i \). By using (7.26) for the difference \( v_i \), we obtain

\[ \frac{1}{2N} \sum_{i=1}^N \frac{d}{dt} \|v_i\|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^N (∇v_i)_{L^2}^2 + m \frac{1}{N} \sum_{i=1}^N \|v_i\|_{L^2}^2 \]

\[ \lesssim \frac{1}{N} \sum_{i=1}^N \|v_i\|_{L^2}^2 D_N + R^0_N, \]

with

\[ D_N = \frac{1}{N} \sum_{j=1}^N \|Ψ_j\|_{L^∞}^2 + 1, \]

\[ R^0_N = \frac{1}{N^2} \sum_{j=1}^N (ψ_j^2 − μ)^2. \]

Then the rest of the proof is the same as 2d case (see Theorem 6.11).
7.3. Observables. In this section we study $O(N)$ invariant observables of our SPDE system. We consider the following model, where $m$ is a sufficiently large constant

$$
\partial_t \Phi_i = (\Delta - m) \Phi_i - \frac{1}{N} \sum_{j=1}^{N} (\Phi_j^2 - C_w) \Phi_i + \xi_i, \quad (7.31)
$$

$$
\partial_t Z_i = (\Delta - m) Z_i + \xi_i, \quad C_w \overset{\text{def}}{=} \mathbb{E}(Z(t,x)^2)
$$

where $Z$ is the stationary solution. Here, $C_w$ is finite, and we incorporate this constant for convenience (namely it will be easier to compare with the two dimensional case, certain products below will be centered, and the limiting SPDE will have simpler form). Of course, (7.31) is equivalent with (7.1) via a finite shift of $C_w$.

In particular, for $m$ large enough, by Theorem 7.8, the unique invariant measure $\nu^N$ to (7.31) converge to the unique invariant measure of (7.2), as $N \to \infty$. In fact, it is easy to see that the stationary solution $\Psi$ to the limiting equation in this case is simply given as the stationary solution $Z$.

We consider the stationary setting, namely, suppose $\Phi = (\Phi_i)_{1 \leq i \leq N} \sim \nu^N$ (where $\nu^N$ is as in (7.30) but with $m$ therein replaced by $m - C_w$). Similar as in Section 6.3 the stationary solution $\Phi$ to (7.31) has decomposition $\Phi_i = Y_i + Z_i$ with $Z_i$ stationary, and $Y_i$ solves (7.3) (with $m$ therein replaced by $m - C_w$). We then write for shorthand

$$
\Phi^2 \overset{\text{def}}{=} \sum_{i=1}^{N} \Phi_i^2, \quad :\Phi^2: \overset{\text{def}}{=} \sum_{i=1}^{N} (\Phi_i^2 - C_w), \quad Z^2 \overset{\text{def}}{=} \sum_{i=1}^{N} Z_i^2, \quad :Z^2: \overset{\text{def}}{=} \sum_{i=1}^{N} (Z_i^2 - C_w)
$$

Consider the observables

$$
\frac{1}{\sqrt{N}} :\Phi^2: \quad \frac{1}{N} :(:\Phi^2):^2;
$$

Here the precise definition of $(:\Phi^2):^2$: is given by

$$
(:\Phi^2):^2 = \sum_{i=1}^{N} (\Phi_i^4 - 6C_w \Phi_i^2 + 3C_w^2) + \sum_{i \neq j} (\Phi_i^2 - C_w)(\Phi_j^2 - C_w)
$$

We can decompose them as

$$
\frac{1}{\sqrt{N}} :\Phi^2: = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( Y_i^2 + 2Y_i Z_i + :Z_i^2: \right),
$$

$$
\frac{1}{N} (:\Phi^2):^2 = \frac{1}{N} \sum_{i,j=1}^{N} \left( Y_i^2 Y_j^2 + :Z_i^2 Z_j: + 4Y_i^2 Y_j Z_j + 2Y_i^2 :Z_j^2: + 4Y_i :Z_i Z_j: + 4Y_j :Z_i Z_j: \right).
$$

Here the notation is close to the two dimensional case, and in particular,

$$
:Z_i Z_j: = \begin{cases} 
Z_i^2 - C_w & (i = j) \\
Z_i Z_j & (i \neq j)
\end{cases}, \quad :Z_i^2 Z_j: = \begin{cases} 
Z_i^3 - 3C_w Z_i & (i = j) \\
Z_i Z_j^2 - C_w Z_i & (i \neq j).
\end{cases}
$$

Analogous results as in Lemma 6.14, Theorem 6.15 (tightness of these observables) could also be proved in the same way. More precisely, we have
Theorem 7.9. Suppose that $\Phi = (\Phi_i)_{1 \leq i \leq N} \sim \nu^N$. For $m$ large enough, the following result holds:

(1) for $q \geq 1$ the following bound holds uniformly in $N$
\[
\mathbb{E} \left[ \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 \right)^q \right] + \mathbb{E} \left[ \left( \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 + 1 \right)^q \left( \sum_{i \neq j} \|\nabla Y_i\|_{L^2}^2 + \|Y_i^2\|_{L^2}^2 \right) \right] \leq 1. \tag{7.33}
\]

(2) $\frac{1}{N} \cdot \mathbb{E}^{\frac{2}{q}}$: is tight in $L^2$.

(3) $\frac{1}{N} \cdot (\mathbb{E}^{2})$: is tight in $L^1$.

We will skip the proof details of the above statement, and turn to study the statistical property of the limiting observable, namely, we show that the limiting observables have nontrivial laws, in the sense that although $\Phi_i$ converges to the nontrivial stationary solution $Z_i$ (and $\Phi_i^2 - C_w \rightarrow Z_i^2 - C_w$ as $N \rightarrow \infty$ for each $i$), the observables do not converge to the ones with $\Phi_i$ replaced by $Z_i$.

To state such “nontriviality” results, we define
\[
G_N(x-z) \stackrel{\text{def}}{=} \mathbb{E} \left[ \frac{1}{\sqrt{N}} : \Phi^2 : (x) \frac{1}{\sqrt{N}} : \Phi^2 : (z) \right].
\]

Here we use the translation invariance of $\nu^N$. For comparison, we first note that
\[
\mathbb{E} \left[ \frac{1}{\sqrt{N}} : \Phi^2 : (x) \frac{1}{\sqrt{N}} : \Phi^2 : (z) \right] = 2C(x-z)^2
\]
for any $N$ and $x, z \in \mathbb{T}$, where $C = (m - \Delta)^{-1}$, which follows from definition of $: \Phi^2 :$ and Wick’s theorem. Also, $\mathbb{E} : (\Phi^2) = 0$ for any $N$.

We denote $\hat{f}$ the Fourier transform of a function $f$.

Theorem 7.10. Under the same setting as in Theorem 7.9, it holds that
\[
\lim_{N \rightarrow \infty} G_N = 2\hat{C^2} / (1 + \hat{C^2}), \tag{7.34}
\]
and
\[
\lim_{N \rightarrow \infty} \frac{1}{N} : (\Phi^2)^2 : (x) = -2 \sum_{k \in \mathbb{Z}} \hat{C^2}(k)^2 / (1 + \hat{C^2}(k)).
\]

In particular,
\[
\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} : \Phi^2 : \neq \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} : \Phi^2 : \neq \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} : \Phi^2 : \neq \lim_{N \rightarrow \infty} \frac{1}{N} : (\Phi^2)^2 :.
\]

Proof. Integration by parts formula gives us the following identities (derivation can be found in Appendix B)
\[
-\frac{1}{4N} \mathbb{E} : (\Phi^2)^2 : (x) = \frac{(N + 2)}{4N} \int C(x-z)^2 G_N(x-z) \, dz + R_N, \tag{7.35}
\]
\[
G_N(x-z) = 2C(x-z) \mathbb{E}[\Phi_1(x) \Phi_1(z)] - \frac{N + 2}{N} \int C(x-y)^2 G_N(y-z) \, dy + \frac{1}{N} Q_N(x-z), \tag{7.36}
\]
with
\[
R_N = -\frac{1}{4N^2} \int C(x-z_1)C(x-z_2) \mathbb{E} \left[ : \Phi_1(z_1) \Phi^2(z_1) : : \Phi_1(z_2) \Phi^2(z_2) : : \Phi^2(x) : \right] \, dz_1dz_2,
\]
\[
Q_N(x-z) = -2 \int C(x-y)C(x-z) \mathbb{E} \left[ : \Phi_1(y) \Phi^2(y) : : \Phi_1(z) : \right] \, dy
\]
\[ + \frac{1}{N} \int C(x - z_1)C(x - z_2)E \left[ :\Phi_1(z_1)\Phi^2(z_1): :\Phi_1(z_2)\Phi^2(z_2): :\Phi^2(z): \right] dz_1dz_2 \]

\[ \overset{\text{def}}{=} Q^N_1(x - z) + Q^N_2(x - z). \]

One can rewrite (7.36) as

\[ (1 + \frac{N + 2}{N} \hat{C}^2) \hat{G}_N = 2\hat{C}_N + \frac{Q_N}{N} \]

where \( C_N(x - z) = E[\Phi_1(x)\Phi_1(z)] \to C(x - z) \) as \( N \to \infty \). Here, since \( C \) is positive definite, \( \hat{C} \) is a positive function, and so is \( \hat{C}^2 \); this allows us to divide both sides by \( 1 + \hat{C}^2 \). In Lemmas 7.11 and 7.12 below we show that \( Q_N/N \) vanishes in \( L^1 \) as \( N \to \infty \), which implies that \( \hat{Q}_N/N \) vanishes in \( L^\infty \) in the limit. We therefore have (7.34). In particular, \( \lim_{N \to \infty} \hat{G}_N \neq 2\hat{C}^2 \), showing that the random fields \( \frac{1}{\sqrt{N}} :\Phi^2: \) and \( \frac{1}{\sqrt{N}} :Z^2: \) have different limiting laws.

Furthermore, note that \( R_N \) is independent of \( x \), by spatial translation invariance. By Lemma 7.12 we have

\[ \lim_{N \to \infty} E \left[ \frac{1}{N} \frac{1}{\sqrt{N}} :\Phi^2: \right] (x) = - \int C(x - z)^2 \lim_{N \to \infty} G_N(x - z) dz = - 2 \sum_{k \in \mathbb{Z}} \hat{C}^2(k) \cdot \hat{C}^2(k)/(1 + \hat{C}^2(k)) \]

where the sum is over integers (i.e. Fourier variables). This is non-zero, showing that the limiting law of \( \frac{1}{\sqrt{N}} :\Phi^2: \) is different from that of \( \frac{1}{\sqrt{N}} :Z^2: \). \( \square \)

**Lemma 7.11.** It holds that

\[ \|Q^N_1\|_{L^1} \lesssim N^{1/2}. \]

**Proof.** We first expand the term inside the expectation part in \( Q^N_1 \) as

\[ E \left[ :\Phi_1(y)\Phi^2(y): :\Phi_1(z) \right] \]

\[ = E \left[ (Y_1 + Z_1) \sum_{i=1}^N (Y_i^2 + 2Y_iZ_i + Z_i^2:) (y)(Y_1 + Z_1)(z) \right] \]

For the term involving only \( (Z_i)_i \) in \( Q^N_1 \) can be computed explicitly with Wick’s theorem

\[ E \left[ Z_1(z) :Z_1(y) \sum_{i=1}^N Z^2_i(y): \right] \]

\[ = E \left[ Z_1(z) \left( Z_1(y)^3 - 3C_w Z_1(y) \right) \right] + (N - 1) E \left[ Z_1(z)Z_1(y) \left( Z_2(y)^2 - C_w \right) \right] = 0. \]

In the following we only have consider other terms. Since \( C \) is a continuous kernel in \( d = 1 \), we deduce that

\[ E \left[ \int |C(x - y)C(x - z)f(y)g(z)| dxdy \right] \lesssim \|f\|_{L^1} \|g\|_{L^1}, \]

which implies that \( \|Q^N_1\|_{L^1} \) is bounded by

\[ E \left[ \left\| (Y_1 + Z_1) \sum_{i=1}^N (Y_i^2 + 2Y_iZ_i + Z_i^2:) \right\|_{L^1} \right] \|Y_1\|_{L^1} \]

\[ + \left( \left\| Y_1 \sum_{i=1}^N (Y_i^2 + 2Y_iZ_i + Z_i^2:) \right\|_{L^1} \right) + \left( \left\| Z_1 \sum_{i=1}^N (Y_i^2 + 2Y_iZ_i) \right\|_{L^1} \right) \|Z_1\|_{L^1}. \]
We will expand the above multiplication and bound each term separately. In what follows, we will frequently exploit symmetry to rewrite certain quantity of $Y_1$ in terms of an average, e.g. replacing $\|Y_1\|_{L^2}^2$ by $\frac{1}{N} \sum_i \|Y_i\|_{L^2}^2$ under expectation; this will give us nice factor $1/N$ since we have moment bounds on $\sum_i \|Y_i\|_{L^2}^2$ thanks to Theorem 7.9. We will also often invoke moment bounds (7.4) on $Z_i$ without explicitly mentioning.

For each of the terms below, we will bound them by an expression of the form as the LHS of (7.33) and then apply Theorem 7.9 to get bound by as required.

The first of these terms is bounded by

$$\mathbb{E}\left[\left\|Y_1 \sum_{i=1}^N Y_i^2 \right\|_{L^1} \cdot \left\|Y_1 \right\|_{L^1}\right] \lesssim \mathbb{E}\left[\left\|Y_1\right\|_{L^2}^2 \cdot \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2\right]$$

(7.37)

$$\lesssim \frac{1}{N} \mathbb{E}\left[\sum_{i=1}^N \left\|Y_i\right\|_{L^2}^2 \cdot \left(\left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}^2 + 1\right)\right] \lesssim \frac{1}{N} \lesssim 1.$$  

Next,

$$\mathbb{E}\left[\left\|Z_1 \sum_{i=1}^N Y_i^2 \right\|_{L^1} \cdot \left\|Z_1 \right\|_{L^1}\right] \lesssim \mathbb{E}\left[\left\|Z_1 \right\|_{L^\infty} \cdot \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2} \cdot \left\|Y_1 \right\|_{L^2}\right]$$

where we used Hölder’s inequality to separate the Y’s and Z’s. This can be bounded similarly as (7.37). Next,

$$\mathbb{E}\left[\left\|Z_1 \sum_{i=1}^N Y_i^2 \right\|_{L^1} \cdot \left\|Z_1\right\|_{L^1}\right] \lesssim \mathbb{E}\left[\left\|Z_1\right\|_{L^\infty} \cdot \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2} \cdot \left\|Y_1\right\|_{L^2}\right]$$

so it’s bounded as in the previous one. Next,

$$\mathbb{E}\left[\left\|Z_1 \sum_{i=1}^N Y_i^2 \right\|_{L^1} \cdot \left\|Z_1\right\|_{L^1}\right] \lesssim \mathbb{E}\left[\left\|Z_1\right\|_{L^\infty} \cdot \left\|\sum_{i=1}^N Y_i^2\right\|_{L^2}\right]$$

and it’s again bounded by Hölder’s inequality. Next,

$$\mathbb{E}\left[\left\|Y_1 \sum_{i=1}^N Y_i Z_i \right\|_{L^1} \cdot \left\|Y_1 \right\|_{L^1}\right] \lesssim \mathbb{E}\left[\left\|Y_1\right\|_{L^2}^2 \cdot \left\|\sum_{i=1}^N Y_i Z_i\right\|_{L^2}\right]$$

$$\lesssim \mathbb{E}\left[\left\|Y_1\right\|_{L^2}^2 \sum_{i=1}^N \left\|Y_i\right\|_{L^2} \cdot \left\|Z_i\right\|_{L^\infty}\right] \lesssim \mathbb{E}\left[\frac{1}{N} \sum_i \left\|Y_i\right\|_{L^2}^2 \left(\sum_i \left\|Y_i\right\|_{L^2}^2\right)^{\frac{1}{2}} \left(\sum_i \left\|Z_i\right\|_{L^\infty}^2\right)^{\frac{1}{2}}\right]$$

which can be bounded with Hölder’s inequality as before by throwing the $\frac{1}{N}$ to the Z-factor. Next, proceeding similarly as the last term, and then using Hölder’s inequality followed by symmetry

$$\mathbb{E}\left[\left\|Y_1 \sum_{i=1}^N Y_i Z_i \right\|_{L^1} \cdot \left\|Z_1\right\|_{L^1}\right] \lesssim \mathbb{E}\left[\left\|Y_1\right\|_{L^2} \cdot \left\|Z_1\right\|_{L^\infty} \cdot \left(\sum_i \left\|Y_i\right\|_{L^2}^2\right)^{\frac{1}{2}} \left(\sum_i \left\|Z_i\right\|_{L^\infty}^2\right)^{\frac{1}{2}}\right]$$

$$\lesssim \mathbb{E}\left[\frac{1}{N} \left(\sum_i \left\|Y_i\right|_{L^2}^2\right)^{\frac{1}{2}} \cdot \left\|Z_1\right\|_{L^\infty} \cdot \left(\sum_i \left\|Z_i\right|_{L^\infty}^2\right)^{\frac{1}{2}} \right] \lesssim 1.$$  

Next, by Young’s inequality,

$$\mathbb{E}\left[\left\|Z_1 \sum_{i=1}^N Y_i Z_i \right\|_{L^1} \left(\left\|Y_1\right\|_{L^1} + \left\|Z_1\right\|_{L^1}\right)\right]$$
Proof. \( \sum_{i=1}^{N} \| Y_i \|_{L^2} \) is bounded since Lemma 7.12. □ 

in the last inequality Summarizing all the estimate the result follows.

\[ \sum \| Y_i \|_{L^2}^{2} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1} \]

\[ \sum \| Y_i \|_{L^2}^{4} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1}^{4} \]

\[ \sum \| Y_i \|_{L^2}^{2} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1} \]

where we used (7.33). Next,

\[ \sum \| Y_i \|_{L^2}^{2} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1} \]

\[ \sum \| Y_i \|_{L^2}^{4} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1}^{4} \]

which is bounded since \( \sum \| Y_i \|_{L^2}^{2} \leq 1 \) by independence. Next,

\[ \sum \| Y_i \|_{L^2}^{2} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1} \]

\[ \sum \| Y_i \|_{L^2}^{4} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1}^{4} \]

which is bounded using symmetry. Here we also used independence to have

\[ \sum \| Y_i \|_{L^2}^{2} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1} \]

\[ \sum \| Y_i \|_{L^2}^{4} \leq \sum \| Y_i \|_{L^1} + \| Z_i \|_{L^1}^{4} \]

in the last inequality Summarizing all the estimate the result follows. □

**Lemma 7.12.** It holds that

\[ \| Q_N^N \|_{L^1} \leq N^{1/2}, \quad \| R_N \|_{L^1} \leq N^{-1/2}. \]

**Proof.** First the term involving only \( Z \)

\[ \frac{1}{N} \int C(x-z_1)C(x-z_2) \sum \| Z_i \|_{L^2}^2 dz_1 dz_2 \]

can be computed explicitly using Wick’s theorem. Indeed, the above expression equals (up to a constant independent of \( N \))

\[ \int C(x-z_1)C(x-z_2)C(z_1-z_2)C(z-z_2)dz_1 dz_2 \]

which as a function of \( x-z \) is bounded in \( L^\infty \) since \( C \) is continuous in \( d = 1 \).

In the following we only consider the other terms. Since \( C \) is continuous in \( d = 1 \), \( \| Q_N^N \|_{L^1} \) is bounded by a constant times the following

\[ \frac{1}{N} \sum \| (Y_i + Z_i) \sum \| (Y_i^2 + 2Y_i Z_i + : Z_i^2 : ) \|_{L^1}^2 \]

\[ + \frac{1}{N} \sum \| (Y_i + Z_i) \sum \| (Y_i Z_i) \|_{L^1}^2 + \| Y_i \sum \| : Z_i^2 : \|_{L^1}^2 + \| \sum \| : Z_i^2 : \|_{L^1} \|_{L^1} + 1, \]
and \( \|NR_N\|_{L^1} \) satisfies exactly the same bound. We expand the above expression. We then proceed as in the proof of Lemma 7.11, by repeatedly using symmetry and Theorem 7.9. As in the proof of Lemma 7.11 for the most terms we will actually have better bound than what’s claimed.

The first term is controlled by
\[
\frac{1}{N} E \left[ \left\| (Y_1 + Z_1) \sum_{i=1}^N Y_i^2 \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left[ \left( \|Y_1\|_{L^\infty}^2 + \|Z_1\|_{L^\infty}^2 \right) \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left( \|Y_1\|_{L^1}^2 + \|Z_1\|_{L^2}^2 \right) \left( \sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^3 \leq \frac{1}{N},
\]

The next term is bounded by
\[
\frac{1}{N} E \left[ \left\| (Y_1 + Z_1) \sum_{i=1}^N Y_i^2 \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left[ \left( \|Y_1\|_{L^\infty}^2 + \|Z_1\|_{L^\infty}^2 \right) \left\| \sum_{i=1}^N Y_i^2 \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left( \|Y_1\|_{H^1}^2 \|Y_i\|_{L^2}^{2(1-s)} + \|Z_1\|_{L^\infty}^2 \right) \left( \sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^{5/2} \left( \sum_{i=1}^N \|Z_i\|_{L^2}^2 \right)^{1/2} \lesssim 1,
\]

where \( s > 1/2 \) and we used Hölder’s inequality (and symmetry and Theorem 7.9 as usual) in the last step. Next,
\[
\frac{1}{N} E \left[ \left\| (Y_1 + Z_1) \sum_{i=1}^N Y_i^2 \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left[ \left( \|Y_1\|_{H^1}^2 \|Y_i\|_{L^2}^{2(1-s)} + \|Z_1\|_{L^\infty}^2 \right) \left( \sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^2 \left( \sum_{i=1}^N \|Z_i\|_{L^2}^2 \right) \lesssim 1,
\]

where \( s > 1/2 \) and we used Hölder inequality, symmetry and Theorem 7.9 in the last step. Next,
\[
\frac{1}{N} E \left[ \left\| (Y_1 + Z_1) \sum_{i=1}^N Y_iZ_i \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left[ \left( \|Y_1\|_{L^2}^2 + \|Z_1\|_{L^2}^2 \right) \left\| \sum_{i=1}^N Y_i \sum_{i=1}^N Z_i \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left( \|Y_1\|_{L^2}^2 + \|Z_1\|_{L^2}^2 \right) \left( \sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^2 \left( \sum_{i=1}^N \|Z_i\|_{L^\infty}^2 \right) \lesssim 1,
\]

where we used symmetry and Theorem 7.9 in the last step. Next,
\[
\frac{1}{N} E \left[ \left\| Y_1 \sum_{i=1}^N Y_iZ_i \right\|_{L^1} \right] 
\leq \frac{1}{N} E \left[ \left( \|Y_1\|_{L^2}^2 + \|Z_1\|_{L^2}^2 \right) \left( \sum_{i=1}^N \|Y_i\|_{L^2}^2 \right)^2 \left( \sum_{i=1}^N \|Z_i\|_{L^\infty}^2 \right) \lesssim 1.
\]
\[
\lesssim \frac{1}{N} E\left[ \left\| Y_1 \right\|_{L^2}^2 \left( \sum_{i=1}^N \left\| Y_i \right\|_{L^2}^{1/2} \left( \sum_{i=1}^N \left\| Z_i \right\|_{L^\infty}^{1/2} \right) \right) \right] \\
\lesssim \frac{1}{N} E\left[ \left\| Y_1 \right\|_{L^2}^2 \left( \sum_{i=1}^N \left\| Y_i \right\|_{L^2}^{3/2} \left( \sum_{i=1}^N \left\| Z_i \right\|_{L^\infty}^{3/2} \right) \right) \right] \lesssim 1,
\]
where we used symmetry and Theorem 7.9 in the last step. For the next term,
\[
\frac{1}{N} E\left[ \left\| Z_1 \right\|_{L^2}^2 \left( \sum_{i=1}^N \left\| Y_i \right\|_{L^2}^{1/2} \left( \sum_{i=1}^N \left\| Z_i \right\|_{L^\infty}^{1/2} \right) \right) \right] \\
\lesssim \frac{1}{N} E\left[ \left\| Z_1 \right\|_{L^2}^2 \left( \sum_{i=1}^N \left\| Y_i \right\|_{L^2}^{3/2} \left( \sum_{i=1}^N \left\| Z_i \right\|_{L^\infty}^{3/2} \right) \right) \right] \lesssim \sqrt{N}.
\]
Next,
\[
\frac{1}{N} E\left[ \left\| (Y_1 + Z_1) \right\|_{L^2}^2 \sum_{i=1}^N \left\| Y_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \\
\lesssim \frac{1}{N} E\left[ \left\| (Y_1 \right\|_{L^2} + \left\| Z_1 \right\|_{L^2} \right) \sum_{i=1}^N \left\| Y_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \lesssim 1,
\]
where we used Hölder inequality and \( E\left[ \frac{1}{N} \sum_{i=1}^N \left\| Z_i \right\|_{L^2}^2 \right] \lesssim 1 \) in the last step. The next term is controlled by
\[
\frac{1}{N} E\left[ \left\| (Y_1 + Z_1) \right\|_{L^2}^2 \sum_{i=1}^N \left\| Y_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \\
\lesssim \frac{1}{N} E\left[ \left\| (Y_1 \right\|_{L^2} + \left\| Z_1 \right\|_{L^2} \right) \sum_{i=1}^N \left\| Y_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \lesssim \frac{1}{N} E\left[ \left\| Y_1 \right\|_{L^2}^2 + \left\| Z_1 \right\|_{L^2}^2 \right] \lesssim 1,
\]
where we used \( E\left( \frac{1}{N} \right) \sum_{j=1}^N \left\| Z_j \right\|_{L^2}^q \lesssim 1 \) for \( q > 1 \). For the next term
\[
\frac{1}{N} E\left[ \left\| (Y_1 + Z_1) \right\|_{L^2}^2 \sum_{i=1}^N \left\| Y_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \\
\lesssim \frac{1}{N} E\left[ \left\| (Y_1 \right\|_{L^2} + \left\| Z_1 \right\|_{L^2} \right) \sum_{i=1}^N \left\| Y_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \lesssim \sqrt{N},
\]
where we used independence to have \( E\left( \frac{1}{N} \right) \sum_{j=1}^N \left\| Z_j \right\|_{L^2}^q \lesssim 1 \) for \( q > 1 \). For the next term,
\[
\frac{1}{N} E\left[ \left\| Y_1 \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \sum_{i=1}^N \left\| Z_i \right\|_{L^2} \right] \lesssim 1
\]
where we again used \( E\left( \frac{1}{N} \right) \sum_{i=1}^N \left\| Z_i \right\|_{L^2}^q \lesssim 1 \) for \( q > 1 \). Combining all the above estimate the result follows. \( \square \)
APPENDIX A. PROOF OF LEMMA 3.2

Proof. For initial value $y_i \in C^\beta(T^2)$, $\beta \in (1, 2)$ we could use similar argument as in [MW17b, Theorem 6.1] to obtain global solutions $(Y_i)$ to (3.3) with each $Y_i \in C_T C^\beta$. In fact, we use mild solutions and fixed point argument to obtain unique local solutions. Furthermore, for fixed $N$ we obtain a global in time $L^p$-estimate, $p > 1$, which gives the required global solutions.

Moreover, for general initial data $y_i \in L^2$, we consider smooth approximation $(y_i^\epsilon)$ to initial data $y_i$. For $(y_i^\epsilon)$ we construct solutions $Y_i^\epsilon \in C_T C^\beta$ by the above argument. For $Y_i^\epsilon$ we could do the uniform estimate as in Lemma 3.3 and obtain

$$
\frac{1}{N} \sup_{t \in [0,T]} \sum_{j=1}^{N} \|Y_j^\epsilon\|_{L^2}^2 + \frac{1}{N} \sum_{j=1}^{N} \|\nabla Y_j^\epsilon\|_{L^2(0,T;L^2)}^2 + \frac{1}{N} \sum_{i=1}^{N} \|Y_i^\epsilon\|_{L^2(0,T;L^2)}^2 \leq C,
$$

where $C$ is independent of $\epsilon$. By standard compactness argument we deduce that there exist a sequence $(\epsilon_k)$ and $Y_i \in L_\infty T^2 L^2 \cap L_\infty T^2 H^1 \cap L_\infty T^2 L^4$ such that $Y_i^\epsilon \rightarrow Y$ in $L_\infty T^2 H^1 \cap C_T H^{-1}$, $\delta < 1$. Furthermore, by similar argument as in the proof of [RYZ18, Theorem 4.3] we obtain $Y_i \in C_T L^2 \cap L_\infty T^2 L^4 \cap L_\infty T^2 H^1$. For the uniqueness part we could do similar estimate as $I_1^N$ and $I_2^N$ for the difference $v_i$ in Section 5. From the estimate in Section 5 the regularity for $Y_i$ is enough for the uniqueness.

\[\Box\]

APPENDIX B. CONSEQUENCES OF DYSON-SCHWINGER EQUATIONS

Dyson-Schwinger equations are relations between correlation functions of different orders. Here we derive the identities (7.35) and (7.36) using Dyson-Schwinger equations; these are essentially in [Kup80b](Eq (7)(8) therein), but since we’re in slightly different setting, we give some details here to be self-contained. Dyson-Schwinger equations are direct consequences of integration by parts formula, for instance, for the $\Phi^4$ model, we refer to [GH18, Theorem 6.7]. In the case of the $N$-component $\Phi^4$ model (i.e. linear sigma model), for a fixed $N$, it is easy to derive the following integration by parts formula

$$
E \left( \frac{\delta F(\Phi)}{\delta \Phi_1(x)} \right) = E \left( (m - \Delta_x) \Phi_1(x) F(\Phi) \right) + \frac{1}{N} E \left( F(\Phi) : \Phi_1(x) \Phi(x)^2 : \right)
$$

where $\Phi \sim \nu^N$ and $\Phi_i^2 \equiv \sum_{i=1}^{N} \Phi_i^2$ following the notation in Section 7.3; or in terms of Green’s function $C(x-y) = (m - \Delta_x)^{-1}(x-y)$

$$
\int C(x-z) E \left( \frac{\delta F(\Phi)}{\delta \Phi_1(z)} \right) dz = E \left( \Phi_1(x) F(\Phi) \right) + \frac{1}{N} \int C(x-z) E \left( F(\Phi) : \Phi_1(z) \Phi(z)^2 : \right) dz
$$

(B.1)

Here we will assume $d = 1$ and the Wick products are understood as Sec 7.

The proof of (B.1) is standard by using Gaussian integration by parts. Here we apply (B.1) to prove (7.35). Recall that $C_W = C(0)$. Taking $F(\Phi) = :\Phi_1(x)\Phi^2(x):$ one has

$$
\frac{1}{N} \int C(x-z) E \left( :\Phi_1(x)\Phi^2(x): :\Phi_1(z)\Phi^2(z): \right) dz = C_W E \left( \frac{\delta :\Phi_1(x)\Phi^2(x):}{\delta \Phi_1(x)} \right) - E[\Phi_1(x) :\Phi_1(x)\Phi^2(x):] = - \frac{1}{N} E \left( :\Phi^2 : \right)(x)
$$


where the last step can be checked straightforwardly 6 using the definition of Wick products and the symmetry (i.e. exchangeability of $\langle \Phi_i \rangle_z$).

Next, taking $F = \Phi^2(x) \cdot \Phi_1 \Phi^2(z)$; one has

$$\frac{1}{N} \int C(x - z_1) E\left( \Phi^2(x) : \Phi_1 \Phi^2(z) : \Phi_1 \Phi^2(z) : \Phi_1 \Phi^2(z) : \right) dz_1$$

$$= C(x - z) E\left( \Phi^2(x) : \Phi_1 \Phi^2(z) : \frac{\delta \cdot \Phi_1 \Phi^2(z)}{\delta \Phi_1(z)} \right) + C_w E\left( \frac{\delta \cdot \Phi_2(x)}{\delta \Phi_1(x)} : \Phi_1 \Phi^2(z) : \right)$$

$$- E\left( \Phi_1(x) : \Phi^2(x) : \Phi_1 \Phi^2(z) : \right)$$

$$= \frac{N + 2}{N} C(x - z) E[ \Phi^2(x) : \Phi^2(z) : ] - E\left( : \Phi_1(x) \Phi^2(x) : \Phi_1(z) \Phi^2(z) : \right)$$

where we again used symmetry, for instance we replaced $\frac{\delta \cdot \Phi_1 \Phi^2(z)}{\delta \Phi_1(z)}$ by $\frac{N + 2}{N} \Phi^2(z)$; under expectation, as well as a simple relation

$$2C_w \Phi_1(x) - \Phi_1(x) : \Phi_1^2(z) : = - : \Phi_1(x) \Phi^2(x) :$$

From the above two correlation identities, we cancel out the 6th order correlation term and then obtain the first equation in (7.35).

Now take $F(\Phi) = \int C(x - y) : \Phi_1 \Phi^2(y) : \Phi^2(z) : dy$. We have, by (B.1),

$$\frac{1}{N} \int \int C(x - y_1) C(x - y_2) E\left( : \Phi_1 \Phi^2(y_1) : \Phi_1 \Phi^2(y_2) : \Phi^2(z) : \right) dy_1 dy_2$$

$$= \int C(x - z) C(x - y) E\left( \frac{\delta \cdot \Phi_1 \Phi^2(z)}{\delta \Phi_1(z)} : \Phi_1 \Phi^2(y) : \right) dy$$

$$+ \int C(x - y)^2 E\left( : \Phi^2(z) : \frac{\delta \cdot \Phi_1 \Phi^2(y)}{\delta \Phi_1(y)} : \right) dy$$

$$- \int C(x - y) E\left( \Phi_1(x) : \Phi_1 \Phi^2(y) : \Phi^2(z) : \right) dy.$$

Note that the LHS is precisely $Q_N^2$ and the first term on the RHS is just $Q_1^N$. The second term on the RHS equals

$$\frac{N + 2}{N} \int C(x - y)^2 E\left( : \Phi^2(z) : \Phi^2(y) : \right) dy.$$

To deal with the last term above, taking $F(\Phi) = \Phi_1(x) : \Phi_1 \Phi^2(z) :$ and applying (B.1), one has

$$\int C(x - y) E\left( \Phi_1(x) : \Phi_1 \Phi^2(y) : \Phi^2(z) : \right) dy$$

$$= NC_w E \phi(z]^2 + NC(x - z) E\left( \Phi_1(x) \frac{\delta \cdot \Phi^2(z)}{\delta \Phi_1(z)} : \right) - N E\left[ \Phi_1(x)^2 \phi(z) : \right]$$

$$= - E[ : \Phi^2(x) : \phi(z)^2 : ] + 2 NC(x - z) E[ \Phi_1(x) \Phi_1(z) ]$$.

We then obtain (7.36).

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6 This could be viewed as an $N$ dimensional generalization of the well-known relation $H_{n+1}(x) = xH_n(x) - H_n(x)$ for Hermite polynomials $H_n$. 
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