On the High-Level Error Bound for Gaussian Interpolation

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Abstract

It’s well-known that there is a very powerful error bound for Gaussians put forward by Madych and Nelson in 1992. It’s of the form $|f(x) - s(x)| \leq (Cd)^{\frac{n}{2}} \|f\|_h$ where $C, c$ are constants, $h$ is the Gaussian function, $s$ is the interpolating function, and $d$ is called fill distance which, roughly speaking, measures the spacing of the points at which interpolation occurs. This error bound gets small very fast as $d \to 0$. The constants $C$ and $c$ are very sensitive. A slight change of them will result in a huge change of the error bound. The number $c$ can be calculated as shown in [9]. However, $C$ cannot be calculated, or even approximated. This is a famous question in the theory of radial basis functions. The purpose of this paper is to answer this question.

Keyword: radial basis function, interpolation, error bound, Gaussian

1.INTRODUCTION

Let $h$ be a continuous function on $R^n$ which is conditionally positive definite of order $m$. Given data $(x_j, f_j), j = 1, ..., N$, where $X = \{x_1, ..., x_N\}$ is a subset of points in $R^n$ and the $f_j's$ are real or complex numbers, the so-called $h$ spline interpolant of these data is the function $s$ defined by

$$s(x) = p(x) + \sum_{j=1}^{N} c_j h(x - x_j),$$  \hspace{1cm} (1)

where $p(x)$ is a Polynomial in $\mathcal{P}_{m-1}$ and $c_j's$ are chosen so that

$$\sum_{j=1}^{N} c_j q(x_j) = 0$$ \hspace{1cm} (2)
for all polynomials $q$ in $\mathcal{P}_{m-1}$ and

$$p(x_i) + \sum_{j=1}^{N} c_j h(x_i - x_j) = f_i, \quad i = 1, \ldots, N. \quad (3)$$

Here $\mathcal{P}_{m-1}$ denotes the class of those polynomials of $\mathbb{R}^n$ of degree $\leq m - 1$.

It is well known that the system of equations (2) and (3) has a unique solution when $X$ is a determining set for $\mathcal{P}_{m-1}$ and $h$ is strictly conditionally positive definite. For more details see [7]. Thus, in this case, the interpolant $s(x)$ is well defined.

We remind the reader that $X$ is said to be a determining set for $\mathcal{P}_{m-1}$ if $p$ is in $\mathcal{P}_{m-1}$ and $p$ vanishes on $X$ implies that $p$ is identically zero.

1.1 A Bound for Multivariate Polynomials

A key ingredient in the development of our estimates is the following lemma which gives a bound on the size of a polynomial on a cube in $\mathbb{R}^n$ in terms of its values on a discrete subset which is scattered in a sufficiently uniform manner. For its proof, please see [9].

**LEMMA1.** For $n = 1, 2, \ldots$, define $\gamma_n$ by the formulas $\gamma_1 = 2$ and, if $n > 1$, $\gamma_n = 2n(1 + \gamma_{n-1})$. Let $Q$ be a cube in $\mathbb{R}^n$ that is subdivided into $q^n$ identical subcubes. Let $Y$ be a set of $q^n$ points obtained by selecting a point from each of those subcubes. If $q \geq \gamma_n(k+1)$, then for all $p$ in $\mathcal{P}_k$

$$\sup_{x \in Q} |p(x)| \leq e^{2n\gamma_n(k+1)} \sup_{y \in Y} |p(y)|. \quad (4)$$

1.2 A Variational Framework for Interpolation

The precise statement of our estimate concerning $h$ splines requires a certain amount of technical notation and terminology which is identical to that used in [8]. For the convenience of the reader we recall several basic notions.
The space of complex valued functions on $\mathbb{R}^n$ that are compactly supported and infinitely differentiable is denoted by $\mathcal{D}$. The Fourier transform of a function $\phi$ in $\mathcal{D}$ is

$$\hat{\phi}(\xi) = \int e^{-i<x,\xi>} \phi(x) dx.$$ 

In what follows $h$ will always denote a continuous conditionally positive definite function of order $m$. The Fourier transform of such functions uniquely determines a positive Borel measure $\mu$ on $\mathbb{R}^n \sim \{0\}$ and constants $a_\gamma, |\gamma| = 2m$ as follows: For all $\psi \in \mathcal{D}$

$$\int h(x)\psi(x) dx = \int \left\{ \hat{\psi}(\xi) - \hat{\chi}(\xi) \sum_{|\gamma|<2m} D^\gamma \hat{\psi}(0) \frac{\xi^\gamma}{\gamma!} \right\} d\mu(\xi)$$

$$+ \sum_{|\gamma|\leq 2m} D^\gamma \hat{\psi}(0) a_\gamma \frac{\xi^\gamma}{\gamma!},$$

where for every choice of complex numbers $c_\alpha, |\alpha| = m$,

$$\sum_{|\alpha|=m} \sum_{|\beta|=m} a_{\alpha+\beta} c_\alpha \overline{c_\beta} \geq 0.$$ 

Here $\chi$ is a function in $\mathcal{D}$ such that $1 - \hat{\chi}(\xi)$ has a zero of order $2m + 1$ at $\xi = 0$; both of the integrals

$$\int_{0<|\xi|<1} |\xi|^{2m} d\mu(\xi), \int_{|\xi|\geq 1} d\mu(\xi)$$

are finite. The choice of $\chi$ affects the value of the coefficients $a_\gamma$ for $|\gamma| < 2m$.

Our variational framework for interpolation is supplied by a space we denote by $\mathcal{C}_{h,m}$. If

$$\mathcal{D}_m = \{ \phi \in \mathcal{D} : \int x^\alpha \phi(x) dx = 0 \text{ for all } |\alpha| < m \}$$
then $C_{h,m}$ is the class of those continuous functions $f$ which satisfy
\[
\left| \int f(x)\phi(x)dx \right| \leq c(f) \left\{ \int h(x-y)\phi(x)\phi(y)dxdy \right\}^{\frac{1}{2}}
\]

for some constant $c(f)$ and all $\phi$ in $D_m$. If $f \in C_{h,m}$ let $\|f\|_{h}$ denote the smallest constant $c(f)$ for which (4) is true. Recall that $\|f\|_{h}$ is a semi-norm and $C_{h,m}$ is a semi-Hilbert space; in the case $m = 0$ it is a norm and a Hilbert space respectively.

Given a function $f$ in $C_{h,m}$ and a subset $X$ of $R^n$, there is an element $s$ of minimal $C_{h,m}$ norm which is equal to $f$ on $X$.

The function space $C_{h,m}$ is not very easy to understand. However Luh made a lucid characterization for this space in [3] and [4]. For an easier understanding of this space we suggest that reader read [3] and [4] first.

2. MAIN RESULTS

Before showing our main results, we need some lemmas. The following lemma is cited directly from [9].

**LEMMA 2.** Let $Q, Y, \gamma_n$ be as in LEMMA 1. Then, given a point $x$ in $Q$, there is a measure $\sigma$ supported on $Y$ such that
\[
\int p(y)d\sigma(y) = p(x)
\]
for all $p$ in $P_k$, and
\[
\int d|\sigma|(y) \leq e^{2n\gamma_n(k+1)}.
\]

Now we need a famous formula.

**Stirling’s Formula:** $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$. 
Remark The approximation is very reliable even for small $n$. For example, when $n = 10$, the relative error is only 0.83%. The larger $n$ is, the better the approximation is. For further details see [1] and [2].

LEMMA 3. Let $\rho_1 = \frac{1}{e}$ and $\rho_2 = \frac{3^k}{e} - \frac{1}{2}$. Then

$$\sqrt{2\pi} \rho_1^k k^k \leq k! \leq \sqrt{2\pi} \rho_2^k k^k$$

for all positive integer $k$.

Proof. Note that $\frac{1}{e}, \frac{2\pi}{e}, \frac{3\pi}{e}, \frac{4\pi}{e}, \ldots$ can be expressed by $\frac{1}{e}, \left(\frac{2\pi}{e}\right)^2, \left(\frac{3\pi}{e}\right)^3, \left(\frac{4\pi}{e}\right)^4, \left(\frac{5\pi}{e}\right)^5, \ldots$

Now, $\sup \left\{ \frac{1}{e}, \frac{2\pi}{e}, \frac{3\pi}{e}, \frac{4\pi}{e}, \frac{5\pi}{e}, \ldots \right\} = \frac{3\pi}{e}$ implies that $\frac{\sqrt{2\pi}}{e} \leq \rho_2$ for all $k$. Thus $k! \sim \sqrt{2\pi} \frac{\sqrt{2\pi}}{e} \cdot k^k \leq \sqrt{2\pi} \rho_2^k \cdot k^k$. The remaining part $\sqrt{2\pi} \rho_1^k k^k \leq k!$ follows by observing that $\sqrt{2\pi} \left(\frac{1}{e}\right)^k k^k \leq \sqrt{2\pi} \cdot k^k \sim k!$

LEMMA 4. Let $\rho = \frac{\sqrt{2\pi}}{e}$. Then $k! \leq 2\pi \rho^k k^{-1}$ for all $k \geq 1$.

Proof. $k! \sim \sqrt{2\pi} \left(\frac{1}{e}\right)^k \cdot \sqrt{k} \cdot k^k = \sqrt{2\pi} \cdot \frac{1}{e} \cdot k^{-1}$. Note that $\left\{ \frac{1}{e}, \left(\frac{2\pi}{e}\right)^2, \left(\frac{3\pi}{e}\right)^3, \left(\frac{4\pi}{e}\right)^4, \ldots \right\}$ can be expressed by $\left\{ \frac{1}{e}, \left(\frac{2\pi}{e}\right)^2, \left(\frac{3\pi}{e}\right)^3, \left(\frac{4\pi}{e}\right)^4, \ldots \right\}$. Our lemma follows by noting that $\sup \left\{ \frac{1}{e}, \frac{2\pi}{e}, \frac{3\pi}{e}, \frac{4\pi}{e}, \ldots \right\} = \frac{\sqrt{2\pi}}{e}$.

LEMMA 5. Let $h(x) = e^{-\beta|x|^2}$, $\beta > 0$, be the Gaussian function in $R^n$, and $\mu$ be the measure defined in (4). For any positive even integer $k$,

$$\int_{R^n} |\xi|^k d\mu(\xi) \leq \pi^{n+\frac{1}{2}} \cdot n \cdot \alpha_n \cdot \frac{2^{k+n+2}}{2} \cdot \rho^\frac{k+n-1}{2} \cdot \beta^\frac{k}{2} \cdot (k+n-1)^\frac{k+n-3}{2} \cdot (2 + \frac{1}{e})$$

for add $n$, and

$$\int_{R^n} |\xi|^k d\mu(\xi) \leq \pi^{n+\frac{1}{2}} \cdot n \cdot \alpha_n \cdot \frac{2^{k+n+3}}{2} \cdot \rho^\frac{k+n-2}{2} \cdot \beta^\frac{k}{2} \cdot (k+n-2)^\frac{k+n-4}{2}$$
for even $n$, where $\rho = \frac{\sqrt{3}}{e}$ and $\alpha_n$ is the volume of the unit ball in $R^n$.

Proof. $\int_{R^n} |\xi|^k d\mu(\xi)$

$$= \left(\frac{\sqrt{3}}{\beta}\right)^{\frac{k}{2}} \int_{R^n} \frac{|\xi|^k}{e^{\frac{3k}{2}}} d\xi$$

$$= \left(\frac{\sqrt{3}}{\beta}\right)^{\frac{k}{2}} \cdot n \cdot \alpha_n \cdot \int_{0}^{\infty} \frac{e^{k+n-1}}{e^{\frac{3k}{2}}} d\gamma$$

$$= \left(\frac{\sqrt{3}}{\beta}\right)^{\frac{k}{2}} \cdot n \cdot \alpha_n \cdot (2\sqrt{\beta})^{k+n} \cdot \int_{0}^{\infty} \frac{e^{k+n-1}}{e^{ir^2}} dr$$

$$= \left\{ \begin{array}{ll}
\frac{\pi^{\frac{k}{2}}}{n} \cdot n \cdot \alpha_n \cdot 2^{n-1+k} \cdot \beta^{\frac{k}{2}} \cdot \frac{(k+n-2)}{2} (k+n-2) - 1 \cdot \left(\frac{1}{2}\right) \int_{0}^{\infty} \frac{1}{\sqrt{ue}} du & \text{if } n \text{ is odd}, \\
\frac{\pi^{\frac{k}{2}}}{n} \cdot n \cdot \alpha_n \cdot 2^{n-1+k} \cdot \beta^{\frac{k}{2}} & \text{if } n \text{ is even}. 
\end{array} \right.$$
\[ \int_{\mathbb{R}^n} |\xi|^k \, d\mu(\xi) \leq \pi^{\frac{n+1}{2}} \cdot n \cdot \alpha_n \cdot 2^{\frac{k+n+1}{2}} \cdot \beta^{\frac{k}{2}} \cdot \rho^{\frac{k+n-2}{2}} \cdot (k+n-2)^{\frac{k+n-4}{2}}. \]

\[ \square \]

**Theorem 1** Let \( h(x) = e^{-\beta|x|^2}, \beta > 0, \) be the Gaussian function in \( \mathbb{R}^n, \) and \( \mu \) be the measure defined in (4). Then, given a positive number \( b_0, \) there are positive constants \( \delta_0, c, \) and \( C \) for which the following is true: If \( f \in C_{h,m} \) and \( s \) is the \( h \) spline that interpolates \( f \) on a subset \( X \) of \( \mathbb{R}^n, \) then

\[ |f(x) - s(x)| \leq \Delta''(C_\delta)^{\frac{t}{t_0}} \cdot \|f\|_h \]

where \( \Delta'' = \pi^{\frac{n-1}{4}} \cdot (n \cdot \alpha_n)^{\frac{1}{2}} \cdot 2^{\frac{n+1}{4}} \left(\frac{\sqrt{3}}{e}\right)^{\frac{n-1}{2}} \) for even \( n \) and \( \Delta'' = \pi^{\frac{n-1}{4}} \cdot (n \cdot \alpha_n)^{\frac{1}{2}} \cdot (2 + \frac{1}{e})^{\frac{n-1}{2}} \cdot \left(\frac{\sqrt{3}}{e}\right)^{\frac{n-1}{2}} \) for odd \( n, \) holds for all \( x \) in a cube \( E \) provided that (i) \( E \) has side \( b \) and \( b \geq b_0, (ii) 0 < \delta \leq \delta_0, \) and (iii) every subcube of \( E \) of side \( \delta \) contains a point of \( X. \) Here, \( \alpha_n \) denotes the volume of the unit ball in \( \mathbb{R}^n. \)

The number \( c \) is equal to \( \frac{b_0}{8 \gamma_n} \) where \( \gamma_n \) is defined in LEMMA1. The number \( C \) is equal to \( \left(3^{\frac{3}{4}} \cdot e \cdot \sqrt{2 \rho \beta} \cdot \sqrt{n} \cdot e^{2n \gamma_n}\right)^4 \cdot b_0^3 \cdot \gamma_n, \) where \( \rho = \frac{\sqrt{3}}{e}. \) Moreover, \( \delta_0 \) can be defined by

\[ \delta_0 = \min \left\{ \frac{1}{(3^{\frac{3}{4}} \cdot e \cdot \sqrt{2 \rho \beta} \cdot \sqrt{n} \cdot e^{2n \gamma_n})^4 \cdot b_0^3 \cdot \gamma_n}, \delta_n \right\} \]

where

\[ \delta_n = \begin{cases} \frac{b_0}{2 \gamma_n} & \text{if } n = 1, \\ \frac{b_0}{2^{\gamma_n(n-1)}} & \text{if } n \text{ is odd, and } n > 1, \\ \frac{b_0}{2^{\gamma_n(n-2)}} & \text{if } n \text{ is even, and } n > 2, \\ \frac{b_0}{2 \gamma_n} & \text{if } n = 2. \end{cases} \]

**Proof.** Let \( \delta_0 \) be as in the theorem. If \( \delta \leq \delta_0, \) then \( \delta \leq \frac{b_0}{2 \gamma_n} \) and \( \frac{\gamma_n \delta}{b_0} \leq 1. \) There is an integer \( k \geq 1 \) such that \( 1 \leq (\frac{\gamma_n \delta}{b_0})k \leq 2. \) This implies

\[ \frac{b_0}{2} \leq \gamma_n \delta k \leq b_0. \]
Now, let \( x \) be any point of the cube \( E \) and recall that Theorem 4.2 of [8] implies that

\[
|f(x) - s(x)| \leq c_k \cdot \|f\|_h \cdot \int_{\mathbb{R}^n} |y-x|^k \, d|\sigma|(y)
\]

whenever \( k > 0 \), where \( \sigma \) is any measure supported on \( X \) such that

\[
\int p(y) \, d\sigma(y) = p(x)
\]

for all polynomials \( p \) in \( \mathcal{P}_{k-1} \). Here

\[
c_k = \left\{ \int_{\mathbb{R}^n} |\xi|^{2k} \, d\mu(\xi) \right\}^{\frac{1}{2}}
\]

whenever \( k > 0 \).

**LEMMA 5.** now applies.

For odd \( n \),

\[
c_k = \frac{1}{k!} \left\{ \int_{\mathbb{R}^n} |\xi|^{2k} \, d\mu(\xi) \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{k!} \left\{ \pi^{\frac{n+1}{2}} \cdot n \cdot \alpha_n \cdot 2^{\frac{2+k+n-1}{4}} \cdot \rho^{\frac{2k+n-1}{4}} \cdot \beta^k \cdot (2k + n - 1)^{\frac{2k+n-1}{4}} \cdot (2 + \frac{1}{\varepsilon}) \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{k!} \left\{ \pi^{\frac{n+1}{4}} \cdot (n \cdot \alpha_n)^{\frac{1}{2}} \cdot 2^{\frac{n+2}{4}} \cdot \rho^{\frac{n-1}{4}} \cdot (\sqrt{2\rho \beta})^k \cdot (2k + n - 1)^{\frac{2k+n-3}{4}} \cdot \frac{2k+n-1}{4} \cdot \sqrt{2 + \frac{1}{\varepsilon}} \right\}
\]

\[
\leq \frac{\pi^{\frac{n+1}{2}} \cdot (n \cdot \alpha_n)^{\frac{1}{2}} \cdot 2^{\frac{n+2}{4}} \cdot \rho^{\frac{n-1}{4}} \cdot (\sqrt{2\rho \beta})^k \cdot (2k + n - 1)^{\frac{2k+n-3}{4}}}{\sqrt{2\pi \cdot \rho^k \cdot k^k}} \cdot \sqrt{2 + \frac{1}{\varepsilon}} \quad \text{by LEMMA 3}
\]
\[
= \pi^{\frac{n-1}{4}} \cdot (n \cdot \alpha_n)^\frac{1}{2} \cdot 2^{\frac{n}{4}} \cdot \rho_3^{\frac{n-1}{4}} \cdot \rho_3^k \cdot (\frac{1}{k^k}) \cdot (2k + n - 1)^{\frac{2k+n-3}{4}} \cdot (\sqrt{2 + \frac{1}{e}})
\]

where \( \rho_3 = \frac{\sqrt{2} \rho_3}{\rho_1} \)

\[
= \Delta'' \cdot \rho_3^k \cdot \frac{1}{k^k} \cdot (2k + n - 1)^{\frac{2k+n-3}{4}}
\]

where \( \Delta'' = \sqrt{2 + \frac{1}{e}} \cdot \pi^{\frac{n-1}{4}} \cdot (n \cdot \alpha_n)^\frac{1}{2} \cdot 2^{\frac{n}{4}} \cdot \rho^{\frac{n-1}{4}}. \)

To obtain the desired bound on \(|f(x) - s(x)|\), it suffices to find a suitable bound for

\[
I = c_k \int_{R^n} |y - x|^k d|\sigma|(y).
\]

Let \( Q \) be any cube which contains \( x \), has side \( \gamma_n k \delta \), and is contained in \( E \). Subdivide \( Q \) into \( (\gamma_n k)^n \) congruent subcubes of side \( \delta \). Since each of these subcubes must contain a point of \( X \), select a point of \( X \) from each subcube and call the resulting discrete set \( Y \). By Lemma 1 we may conclude that there is a measure \( \sigma \) supported on \( Y \) which satisfies (6) and enjoys the estimate

\[
\int_{R^n} d|\sigma|(y) \leq e^{2n\gamma_n k}.
\]

We use this measure in (6) to obtain an estimate on \( I \).

Using (8) and the fact that support of \( \sigma \) is contained in \( Q \) whose diameter is \( \sqrt{n}\gamma_n k \delta \) we may write

\[
I \leq \Delta'' \cdot \rho_3^k \cdot \frac{1}{k^k} \cdot (2k + n - 1)^{\frac{2k+n-3}{4}} \cdot (\sqrt{n}\gamma_n k \delta)^k \cdot e^{2n\gamma_n k}
\]

\[
= \Delta'' \cdot (B'\gamma_n \delta)^k \cdot (2k + n - 1)^{\frac{2k+n-3}{4}} \text{ where } B' = \rho_3 \sqrt{ne^{2n\gamma_n}}
\]

\[
\leq \Delta'' \cdot (B'\gamma_n \delta)^k \cdot (3k)^\frac{3k}{4} \text{ (if } k \geq n - 1)\]
\[ = \Delta'' \cdot (B'\gamma_n \delta \frac{3}{2} k \frac{1}{k})^k \]

\[ \leq \Delta'' \cdot \left[ 3^\frac{3}{2} B'\gamma_n \delta \left( \frac{b_0}{\gamma_n \delta} \right)^\frac{1}{2} \right]^k \quad \text{(since } k \leq \frac{b_0}{\gamma_n \delta}) \]

\[ = \Delta'' \cdot \left[ B'' \gamma_n \frac{1}{2} b_0 \frac{3}{2} \delta \frac{1}{2} \right]^k \quad \text{where } B'' = 3^\frac{3}{2} B' \]

\[ = \Delta'' \cdot \left[ B'' b_0 \gamma_n \frac{1}{2} \delta \frac{1}{2} \right]^\frac{b_0}{2\gamma_n} \quad \text{(since } \frac{b_0}{2\gamma_n} \leq k) \]

where \( B'' b_0^\frac{3}{2} \gamma_n \frac{1}{2} \delta \frac{1}{2} \leq 1 \) if and only if \( B'' b_0^\frac{3}{2} \gamma_n \delta \leq 1 \) and is guaranteed by \( \delta \leq \delta_0 \)

\[ = \Delta'' \cdot \left[ C \frac{b_0}{2\gamma_n} \right]^\frac{b_0}{2\gamma_n} \quad \text{where } C = B'' b_0^\frac{3}{2} \gamma_n \text{ and } c = \frac{b_0}{2\gamma_n}. \]

Note that \( k \geq n - 1 \) is guaranteed by \( n - 1 \leq \frac{b_0}{2\gamma_n} \leq k \).

For even \( n \),

\[ c_k = \frac{1}{k!} \left\{ \int_{R^n} |\xi|^{2k} \ d\mu(\xi) \right\} \frac{1}{2} \]

\[ \leq \frac{1}{k!} \left\{ \pi^{-\frac{n+1}{2}} \cdot n \cdot \alpha_n \cdot 2^{\frac{2k+n+1}{2}} \cdot \rho^{\frac{2k+n-2}{2}} \cdot \beta^k \cdot (2k + n - 2)^{\frac{2k+n-4}{2}} \right\} \frac{1}{2} \]

\[ = \frac{1}{k!} \left\{ \pi^{-\frac{n+1}{2}} \cdot (n\alpha_n)^{\frac{1}{2}} \cdot 2^{\frac{2k+3}{2}} \cdot \rho^{\frac{n-2}{4}} \cdot (\sqrt{2}\rho \beta)^k \cdot (2k + n - 2)^{\frac{2k+n-4}{4}} \right\} \]
\[ \leq \pi^{\frac{n+1}{2}} (n\alpha_n)^{\frac{n}{2}} \rho^{\frac{n-2}{2}} (\sqrt{2}\rho^3)^{k} \cdot (2k+n-2) \frac{2k+n-4}{4} \]

\[ = \Delta'' \cdot \rho_3^k \cdot (2k+n-2) \frac{2k+n-4}{4} \]

where \( \Delta'' = \pi^{\frac{n-1}{2}} \cdot (n\alpha_n)^{\frac{1}{2}} \cdot 2^{\frac{n+1}{4}} \cdot \rho^{\frac{n-2}{2}} \)

and \( \rho_3 = \frac{\sqrt{2}\rho^3}{\rho^1} \)

Now,

\[ I = c_k \int_{R^n} |y - x|^k \, d|\sigma|(y) \]

\[ \leq \Delta'' \cdot \rho_3^k \cdot (2k+n-2) \frac{2k+n-4}{4} \cdot (\sqrt{n} \cdot \gamma_n \cdot k \cdot \delta)^k \cdot e^{2n\gamma_n k} \]

\[ = \Delta'' \cdot (B'\gamma_n \delta)^k \cdot (2k+n-2) \frac{2k+n-4}{4} \]

where \( B' = \rho_3\sqrt{n} e^{2n\gamma_n} \)

\[ \leq \Delta'' \cdot (B'\gamma_n \delta)^k \cdot (3k)^{\frac{3k}{4}} \]

(whenever \( k \geq n - 2 \))

\[ = \Delta'' \cdot (B'\gamma_n \delta^3 \frac{3k}{4})^k \]

\[ \leq \Delta'' \cdot \left[ 3^{\frac{3}{4}} B' \gamma_n \delta (\frac{b_0}{\gamma_n \delta})^\frac{3}{4} \right]^k \]

\( (k \leq \frac{b_0}{\gamma_n \delta}) \)

\[ = \Delta'' \cdot \left[ B'' \gamma_n \frac{1}{4} b_0^\frac{3}{2} \delta^\frac{3}{4} \right]^k \]

where \( B'' = 3^{\frac{3}{4}} \cdot B' \)

\[ \leq \Delta'' \cdot \left[ B'' b_0^\frac{3}{4} \gamma_n^\frac{1}{4} b_0^\frac{1}{4} \delta^\frac{1}{4} \right]^k \]

\[ \leq \Delta'' \cdot \left[ B'' b_0^\frac{3}{4} \gamma_n^\frac{1}{4} \delta^\frac{1}{4} \frac{b_0}{2\gamma_n \delta} \right] \]

since \( \frac{b_0}{2\gamma_n \delta} \leq k \)

and \( B'' b_0^\frac{3}{4} \gamma_n^\frac{1}{4} \delta^\frac{1}{4} \leq 1 \) due to \( \delta \leq \delta_0 \).

This gives
\[ I \leq \triangle'' \left[ B^\beta b_0^\gamma \delta \right]^{b_0 \over \delta \gamma_n} \]

\[ = \triangle'' \cdot [C\delta]^{\frac{b_0}{\delta \gamma_n}} \] where \( C = B^\beta b_0^\gamma \) and \( c = {b_0 \over \delta \gamma_n} \).

Note that \( k \geq n-2 \) is guaranteed by \( \delta \leq \delta_0 \leq {b_0 \over \delta \gamma_n(n-2)} \) and \( n-2 \leq {b_0 \over \delta \gamma_n \delta} \leq k \).

We conclude that

\[ |f(x) - s(x)| \leq \triangle'' \cdot [C\delta]^{\frac{b_0}{\delta \gamma_n}} \|f\|_h \]

whenever \( 0 < \delta \leq \delta_0 \) as stated in the theorem.

Remark The high-level error bound for Gaussians was first put forward by Madych and Nelson in Theorem 3. of [9]. However their proof is incomplete. In their theorem the radial basis function \( h \) must satisfy the key condition

\[ \int_{\mathbb{R}^n} |\xi|^k \, d\mu(\xi) \leq \rho^k r^k \]  \hspace{1cm} (9)

for \( k > 2m \), where \( r \) is a real constant and \( \rho \) is a positive constant. As pointed out by them in page 102 of [9], \( r = \frac{1}{2} \) if \( h \) is a Gaussian function. Moreover, they only treated the case \( \beta = 1 \). However (9) holds only when \( n = 1 \). For \( n > 1 \), (9) should be replaced by Lemma 5 of this paper. Consequently their high-level error bound is essentially only suitable for \( \mathbb{R}^1 \).

Note that Theorem 1 can be improved if \( n \) is known in advance, especially when \( n = 1 \) or 2. For example, suppose \( n = 1 \). Let

\[ \delta_0 = \min \left\{ \delta \left| \delta \leq \frac{1}{(\sqrt{e} \sqrt{2\rho^\beta} \sqrt{n} e^{2\gamma_n}})^{2\gamma_n b_0} \right. \right\}. \]
Then for $\delta \leq \delta_0$,

\[
I \leq \triangle'' \cdot (B'\gamma_n \delta)^k \cdot (2k)^{2k-2}
= \triangle'' \cdot (B'\gamma_n \delta)^k \cdot (2k)^{\frac{k}{2}}
\leq \triangle'' \cdot \left[ B'\gamma_n \delta \cdot \sqrt{2} \sqrt{k} \right]^k
= \triangle'' \cdot \left[ \sqrt{2} B'\gamma_n \delta \left( \frac{b_0}{\gamma_n \delta} \right)^{\frac{1}{2}} \right]^k
= \triangle'' \cdot \left[ B'' \gamma_n \delta^\frac{1}{2} b_0^{\frac{1}{2}} \right]^\frac{b_0}{\gamma_n \delta}
= \triangle'' \cdot \left[ B'' \gamma_n \delta^\frac{1}{2} b_0^{\frac{1}{2}} \right]^\frac{b_0}{\gamma_n \delta}
\]

where $B'' \gamma_n \delta^\frac{1}{2} b_0^{\frac{1}{2}} \leq 1$ is guaranteed by $\delta \leq \delta_0$,

\[
= \triangle'' \cdot \left[ B'' \gamma_n b_0 \delta \right]^{\frac{b_0}{\gamma_n \delta}} = \triangle'' \cdot [C \delta]^{\frac{\gamma_n}{b_0}}
\]

where $C = B'' \gamma_n b_0$ and $c = \frac{b_0}{4 \gamma_n}$.

In this case, $\delta_0, C, c$ are the same as Madych and Nelson’s results without any sacrifice.

What’s noteworthy is that in Theorem 1 the parameter $\delta$ is not the generally used fill distance. For easy use we should transform Theorem 1 into a statement described by the fill distance.

Let

\[
d(\Omega, X) = \sup_{y \in \Omega} \inf_{x \in X} |y - x|
\]

be the fill distance. Observe that every cube of side $\delta$ contains a ball of radius $\frac{\delta}{2}$. Thus the subcube condition in Theorem 1 is satisfied when $\delta = 2d(E, X)$. More generally, we can easily conclude the following:
Corollary Suppose that \( h \) satisfies the hypotheses of the theorem, \( \Omega \) is a set which can be expressed as the union of rotations and translations of a fixed cube of side \( b_0 \), and \( X \) is a subset of \( \mathbb{R}^n \). Then there are positive constants \( d_0, c', C' \) for which the following is true:

If \( f \in C_{h,m} \) and \( s \) is the \( h \) spline that interpolates \( f \) on a subset \( X \) of \( \mathbb{R}^n \), then

\[
|f(x) - s(x)| \leq \Delta'' (C'd) \frac{C'}{2} \cdot \|f\|_h
\]

where \( \Delta'' \) is as in Theorem1 and \( C' = 2C, c' = 2 \) with \( C \) and \( c \) defined in Theorem1, holds for all \( x \) in a cube \( E \subseteq \Omega \) provided that (i) \( E \) has side \( b \) and \( b \geq b_0 \), (ii) \( 0 < d \leq d_0 \), and (iii) every subcube of \( E \) of side \( 2d \) contains a point of \( X \). Here \( d \) denotes \( d(\Omega, X) \) or \( d(E, X) \) and \( m = 0 \) since Gaussians are c.p.d of order 0.

Proof. Let \( d_0 = \frac{\delta_0}{2} \) and \( \delta = 2d \) where \( \delta_0 \) is as in the theorem. If \( 0 < d \leq d_0, 0 < \delta \leq 2d_0 = \delta_0 \), By the theorem,

\[
|f(x) - s(x)| \leq \Delta'' (C\delta) \frac{\delta}{2} \cdot \|f\|_h
\]

\[
= \Delta'' (C2d) \frac{2d}{2} \cdot \|f\|_h
\]

\[
= \Delta'' (C'd) \frac{c'}{2} \cdot \|f\|_h
\]

holds for all \( x \) in \( \Omega \).

Remark The space \( C_{h,m} \) probably is unfamiliar to most people. It's introduced by Madych and Nelson in [7] and [8]. Later Luh made characterizations for it in [3] and [4]. Many people think that it's defined by Gelfand and Shilov's definition of generalized Fourier transform, and is therefore difficult to deal with. This is wrong. In fact, it can be characterized by Schwartz's definition of generalized Fourier transform. The situation is not so bad. Moreover, many people mistake \( C_{h,m} \) to be the closure of Wu and Schaback's function space which is defined in [10]. This is also wrong. The two spaces have very little connection. Luh also made a clarification for this problem. For further details, please see [5] and [6].
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