AVERAGES ALONG CUBES FOR NOT NECESSARILY COMMUTING MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We study the pointwise convergence of some weighted averages linked to averages along cubes. We show that if \((X, \mathcal{B}, \mu, T_i)\) are not necessarily commuting measure preserving systems on the same finite measure space and if \(f_i, 1 \leq i \leq 6\) are bounded functions then the averages
\[
\frac{1}{N^3} \sum_{n,m,p=1}^{N} f_1(T_n^1 x) f_2(T_m^2 x) f_3(T_p^3 x) f_4(T_{n+m}^4 x) f_5(T_{n+p}^5 x) f_6(T_{m+p}^6 x)
\]
converge almost everywhere.

1. Introduction

Let \((X, \mathcal{B}, \mu, T_i), 1 \leq i \leq 3\), be three measure preserving systems on the same finite measure space. In [1] we proved that if \(f_i, 1 \leq i \leq 3\) are three bounded functions then the averages
\[
\frac{1}{N^2} \sum_{n=1}^{N} f_1(T_n^1 x) f_2(T_n^2 x) f_3(T_{n+m}^3 x)
\]
converge almost everywhere. This is a bit surprising as it is known [3] that the averages along diagonal terms such as \(\frac{1}{N} \sum_{n=1}^{N} f_1(T_n^1 x) f_2(T_n^2 x)\) do not converge even in norm when the transformations \(T_1\) and \(T_2\) do not necessarily commute. In the first section of this paper we will extend this result by proving the following theorem.

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Theorem 1. Let \((X, \mathcal{B}, \mu, T_i), 1 \leq i \leq 7\), be six measure preserving systems on the same finite measure space and consider \(f_i, 1 \leq i \leq 6\) bounded functions. Then the averages

\[
\frac{1}{N^3} \sum_{n,m,p=1}^{N} f_1(T_{n_1}^nx) f_2(T_{n_2}^mx) f_3(T_{n_3}^p_x) f_4(T_{n_4}^{m+p}x) f_5(T_{n_5}^{m+p}x) f_6(T_{n_6}^{m+p}x)
\]

converge almost everywhere and in norm.

The method used to prove this theorem is a combination of the following key estimates obtained in \([1]\) and the ergodic decomposition.

Lemma 1. Let \(a_n, b_n\) and \(c_n, n \in \mathbb{N}\) be three sequences of scalars that we assume for simplicity bounded by one. Then for each \(N\) positive integer we have

\[
\left| \frac{1}{N^2} \sum_{m,n=0}^{N-1} a_n b_m c_{n+m} \right|^2 \leq \min \left[ \sup_t \left| \frac{1}{N} \sum_{m'=1}^{2(N-1)} c_{m'} e^{2\pi i m't} \right|^2, \sup_t \left| \frac{1}{N} \sum_{n'=1}^{N} a_{n'} e^{2\pi i n't} \right|^2, \sup_t \left| \frac{1}{N} \sum_{n'=1}^{N} b_{n'} e^{2\pi i n''t} \right|^2 \right]
\]

Lemma 2. Let

\[
M_N(A_1, A_2, ..., A_7) = \frac{1}{N^3} \sum_{p,n,m=0}^{N-1} a_{1,p} a_{2,n} a_{3,p+n} a_{4,m} a_{5,n+m} a_{6,p+m} a_{7,n+m+p}
\]

the averages of seven bounded (by one) sequences \(A_i = (a_{i,n}), 1 \leq i \leq 7\). Let us denote by \(G\) the set of couples of integers between 1 and 7, \((i, j)\), which are connected by one of the indices \(n, m\) or \(p\). Then for each \(N\) positive integer we have

\[
\left| M_N(A_1, A_2, ..., A_7) \right|^2 \leq C \min_{(i,j) \in G} \left[ \max \left| \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m=0}^{N-1} a_{i,n} a_{j,n+m} e^{2\pi i m't} \right|^2, \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m=0}^{N-1} a_{i,m} a_{j,n+m} e^{2\pi i n'm} \right|^2 \right] \right].
\]
With these lemmas we will derive the pointwise convergence of Wiener-Wintner types of averages that will lead to the conclusion stated in Theorem 1. These pointwise results extend Wiener-Wintner classical ergodic theorem. (see [2], for instance for several proofs of this Wiener Wintner result).

This is done in a first subsection. In a second subsection we will study the problem of recurrence to a single set in the case of three transformations. We will be able to extend Khintchine's recurrence result by studying for any measurable set $A$ with positive measure the positivity of the limit

$$\lim N \frac{1}{N^2} \sum_{n,m=0}^{N-1} \mu \{ A \cap T_1^n A \cap T_2^m A \cap T_3^{n+m} A \} > 0$$

when the transformations are not necessarily commuting.

In the second section of the paper we will look at the convergence of weighted averages. For a measure preserving transformation $T$ we denote by $\mathcal{K}$ the $\sigma$-algebra spanned by the eigenfunctions of $T$. The method used in [1] to prove the pointwise convergence of averages along the cubes for the powers of the same measure preserving transformation led to the following results.

**Lemma 3.** Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and let $f \in \mathcal{K}^\perp$. Then for $\mu$ a.e. $x$ for all bounded sequences $a_n$, $b_n$, $c_n$,

1. $\lim N \frac{1}{N^2} \sum_{n,m=0}^{N-1} a_n b_m f(T^{n+m} x) = 0$,

2. $\lim N \frac{1}{N^2} \sum_{n,m=0}^{N-1} f(T^n x) b_m c_{n+m} = 0$ and

3. $\lim N \frac{1}{N^2} \sum_{n,m=0}^{N-1} a_n f(T^m x) c_{n+m} = 0$. 
and

**Proposition 2.** Let \((X, \mathcal{B}, \mu, T)\) be an ergodic dynamical system and let \(f \in L^2(\mu)\). Then for \(\mu\) a.e. \(x\) for all bounded sequences \(a_n, b_n\) such that \(\frac{1}{N} \sum_{n=0}^{N-1} a_n e^{2\pi int}\) and \(\frac{1}{N} \sum_{n=0}^{N-1} b_n e^{2\pi int}\) converge for each \(t\), the sequence

\[
\frac{1}{N^2} \sum_{n=0}^{N-1} a_n b_m f(T^{n+m}x)
\]

converges. A similar statement holds if one replaces \(a_n\) with \(f(T^n x)\) and uses instead \(b_m\) and \(c_{n+m}\) or if one chooses \(b_m = f(T^m x)\) and uses \(a_n\) and \(c_{n+m}\).

The intriguing aspect of these results is the fact that the set of convergence for \(x\) is independent of the bounded sequences \(a_n, b_n\) and \(c_n\). An illustration of such property can be given by taking \(a_n = (f_1(T^n_1 x))\), \(b_m = f_2(T^m_2 x)\) with \(f_1, f_2 \in L^\infty\). One obtains immediately the almost everywhere convergence of the averages

\[
\frac{1}{N^2} \sum_{n,m=1}^{N} f_1(T^n_1 x) f_2(T^m_2 x) f(T^{n+m}x)
\]

if the transformation \(T\) is ergodic. Other choices for the sequences \(a_n\) and \(b_n\) are also possible. For instance one could easily take \(a_n = f(T^{p(n)}(x))\) where \(p(x)\) is a real polynomial with positive integer coefficients. Such observation seemed to indicate that the almost everywhere convergence of the averages along the cubes of a single transformation, namely

\[
\frac{1}{N^2} \sum_{n,m=1}^{N} f(T^n x) g(T^m x) h(T^{n+m}x),
\]

relies more on the underlying arithmetic structure than on its dynamical structure. This is one of the reasons why we asked in [1] if the assumption of ergodicity made in Lemma 3 and the Proposition 2 above was necessary. In this paper we will answer in part this question by showing that the ergodicity assumption
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is indeed necessary in Lemma 3. At the present time we do not know if Proposition 2 is true without ergodicity assumption. With the method used in [1] we have the following

**Proposition 3.** Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(\mu)$. Define the set $WW_1$ as

$$WW_1 = \{ a \in l^\infty; \lim_N \frac{1}{N} \sum_{n=0}^{N-1} a_n e^{2\pi i nt} \text{ exists for all } t \}$$

If the set

$$D = \{ x : \frac{1}{N^2} \sum_{n,m=0}^{N-1} a_n b_m f(T^{n+m} x), \text{ converge for all bounded sequences } (a_n) \in WW_1, (b_m) \in WW_1 \}$$

is measurable then for $\mu$ a.e. $x$ for all bounded sequences $(a_n) \in WW_1, (b_m) \in WW_1$ the averages

$$\frac{1}{N^2} \sum_{n,m=0}^{N-1} a_n b_m f(T^{n+m} x)$$

converge.

The currently open question is the measurability of $D$ that we will not address in this paper.

It is worth pointing out that if one looks only at the norm convergence Proposition 2 is true without ergodicity.

We will also look at the higher order averages. We denote by $A_i = (a_{n,i}) 1 \leq i \leq 6$, six bounded sequences of scalars. We consider the averages

$$M_N(A_1, A_2, \ldots A_6, f)(x) = \frac{1}{N^3} \sum_{n,m,p=0}^{N-1} a_{1,p} a_{2,n} a_{3,p+n} a_{4,m} a_{5,n+m} a_{6,p+m} f(T^{n+m+p} x).$$
In \[1\] we proved that if \(f \in \mathcal{CL}^\perp\) then we have a similar result to Lemma 1. More precisely we have;

**Proposition 4.** Let \((X, \mathcal{B}, \mu, T)\) be an ergodic dynamical system and let \(f \in \mathcal{CL}^\perp\). Then for \(\mu\) a.e. \(x\) for all bounded sequences \(A_i = (a_{i,n})\), \(1 \leq i \leq 6\) the sequence

\[
M_N(A_1, A_2, ... A_6, f)(x) = \frac{1}{N^3} \sum_{n,m,p=0}^{N-1} a_{1,p} a_{2,n} a_{3,p+n} a_{4,n} a_{5,n+m} a_{6,p+m} f(T^{n+m+p}x)
\]

converge to zero.

A natural question is to find the precise condition on the sequences \(A_i\) that will give the almost everywhere convergence of the averages \(M_N(A_1, A_2, ... A_6, f)(x)\) when \(f \in \mathcal{CL}\). We will show that a condition such as \(\lim_N \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi int}\) exists for each \(t \in \mathbb{R}\) which is actually necessary and sufficient for the convergence of the weighted averages

\[
\frac{1}{N^2} \sum_{n=0}^{N-1} a_n b_m f(T^{n+m}x)
\]

in the universal sense described by Proposition 2 is no longer sufficient for the convergence of the averages \(M_N(A_1, A_2, ... A_6, f)(x)\). At the present time sufficient conditions on the sequences that would guarantee the almost convergence are not yet clear to us.

2. **Almost everywhere convergence and recurrence for not necessarily commuting measure preserving transformations**

2.1. **Proof of Theorem 1.** We recall that if \((X, \mathcal{B}, \mu, T)\) is a measure preserving dynamical system then the measure \(\mu\) can be disintegrated in a product so that \(d\mu = d\mu_c dc\) and \((X, \mathcal{B}, \mu_c)\) becomes an ergodic dynamical system. This disintegration allows to lift several results from the ergodic case to the not necessarily ergodic one.
The proof of Theorem 1 will be completed after several steps. First we will need a Wiener Wintner strengthening of Theorem 10 in [1].

**Lemma 4.** Let \((X, \mathcal{B}, \mu, T_i)\) be three measure preserving transformations on the same finite measure space. Consider three bounded functions \(f_i, 1 \leq i \leq 3\) then for \(\mu\) a.e. \(x\) for all \(\varepsilon_1, \varepsilon_2 \in \mathbb{R}\) the averages

\[
\frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T_1^nx)f_2(T_2^mx)f_3(T_3^{n+m}x)e^{2\pi i \varepsilon_1}e^{2\pi i \varepsilon_2}
\]

converge.

**Proof.** Without loss of generality we can assume that the functions \(f_i\) are bounded by one. We use the ergodic decomposition with respect to \(T_3\) to obtain a disintegration of \(\mu, d\mu = d\mu_{c,3} dc\) into ergodic components. By the same disintegration and because of the Wiener Wintner theorem for measure preserving transformations for \(c\) a.e., for \(\mu_{c,3}\) a.e. \(y\), the averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T_1^ny)e^{2\pi i \varepsilon_1}
\]

and

\[
\frac{1}{N} \sum_{m=0}^{N-1} f_2(T_2^my)e^{2\pi i \varepsilon_2}
\]

converge for all \(\varepsilon_1, \varepsilon_2 \in \mathbb{R}\). It is clear that the transformations \(T_1\) and \(T_2\) may no longer be measure preserving with respect to \(\mu_{c,3}\) but we are only using here the disintegration of measurable sets of full measure given by Wiener Wintner ergodic theorem. Let us consider the Kronecker factor \(K_{3,c}\) of \(T_3\) with respect to \((X, \mathcal{B}, \mu_{c,3})\) and let us decompose the function
\( f_3 \) into the sum \( f_{3,K_e} + f_{3,K_e^\perp} \) where \( f_{3,K_e} \) is its projection onto \( K_{3,e} \). By Bourgain’s uniform Wiener Wintner ergodic theorem (see [2] for instance for a proof) we have for \( \mu_{3,e} \) a.e. \( y 
abla \)

\[
\lim_N \sup_t \left| \frac{1}{N} \sum_{n=0}^{N-1} f_{3,K_e}(T_3^n y) e^{2\pi int} \right| = 0.
\]

Applying Lemma 1 with \( a_n = f_1(T_1^n y) e^{2\pi in_1} \), \( b_m = f_2(T_2^m y) e^{2\pi im_2} \), and \( c_k = f_3(T_3^k y) \) we obtain the estimate

\[
\sup_{\varepsilon_1, \varepsilon_2} \left| \frac{1}{N_2} \sum_{n,m=0}^{N-1} f_1(T_1^n y) f_2(T_2^m y) f_{3,K_e^\perp}(T_3^{n+m} y) e^{2\pi in_1} e^{2\pi im_2} \right| \leq \sup_t \left| \frac{1}{N} \sum_{k=0}^{N-1} f_{3,K_e^\perp}(T_3^k y) e^{2\pi ikt} \right|.
\]

As a consequence of the uniform Wiener Wintner theorem we have

\[
\lim_N \sup_{\varepsilon_1, \varepsilon_2} \left| \frac{1}{N_2} \sum_{n,m=0}^{N-1} f_1(T_1^n y) f_2(T_2^m y) f_{3,K_e^\perp}(T_3^{n+m} y) e^{2\pi in_1} e^{2\pi im_2} \right| = 0.
\]

The function \( f_{3,K_e} \) projects onto the eigenfunctions of \( T_3 \) with respect to \( \mu_{e,3} \). If \( e_{j,3} \) is one of these eigenfunctions with corresponding eigenvalue \( e^{2\pi i \theta_j} \) then we have

\[
f_{3,K_e} = \left[ \sum_{j=0}^{\infty} \int f_{3,K_e}(y) e_{j,3}(y) d\mu_{e,3}(y) e_{j,3} \right].
\]

Hence by linearity and approximation it is enough to consider the case where \( f_{3,K_e} \) is one of the eigenfunctions \( e_{j,3} \). In this case \( f_{3,K_e}(T_3^{n+m} y) = e^{2\pi i (n+m) \theta_j} e_{j,3} \) and the averages become

\[
e_{j,3} \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T_1^n y) f_2(T_2^m y) e^{2\pi i (n+m) \theta_j} e^{2\pi in_1} e^{2\pi im_2} = e_{j,3} \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T_1^n y) e^{2\pi i (\theta_j + \varepsilon_1)} f_2(T_2^m y) e^{2\pi i (\theta_j + \varepsilon_2)}
\]

The convergence can be derived now by the disintegration, done at the beginning of the proof, of the sets where the Wiener Wintner ergodic theorem applied to the functions \( f_1 \) and \( f_2 \).
As the set of $x$ for which for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ the averages

$$\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T_1^n x) f_2(T_2^m x) f_3(T_3^{n+m} x) e^{2\pi i \varepsilon_1} e^{2\pi i \varepsilon_2}$$

converge is $\mathcal{B}$ measurable we can integrate with respect to $\mu_{c,3}$ and $dc$ to show that this set has full measure.

□

Lemma 5. Let $(X, \mathcal{B}, \mu, T_i)$ be four measure preserving transformations on the same finite measure space. Then for all bounded functions $f_i, 1 \leq i \leq 4$, the averages

$$\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T_1^n x) f_2(T_2^m x) f_3(T_3^{n+m} x) f_4(T_4^p x)$$

converge $\mu$ a.e. and in norm.

Proof. We can write these averages as

$$\left[ \frac{1}{N} \sum_{p=0}^{N-1} f_4(T_4^p x) \right] \left[ \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T_1^n x) f_2(T_2^m x) f_3(T_3^{n+m} x) \right].$$

The conclusion now follows from Birkhoff’s pointwise ergodic theorem and Theorem 10 in [1]. The convergence in norm is an easy consequence of Lebesgue dominated convergence theorem.

□

We need now a Wiener Wintner version of Lemma 5.

Lemma 6. Let $(X, \mathcal{B}, \mu, T_i)$ be four measure preserving transformations on the same finite measure space. Then for all bounded functions $f_i, 1 \leq i \leq 4$, for $\mu$ a.e. $x$, for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ the averages

$$\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T_1^n x) f_2(T_2^m x) f_3(T_3^{n+m} x) f_4(T_4^p x) e^{2\pi i (n+p) \varepsilon_1} e^{2\pi i (m+p) \varepsilon_2}$$
converge.

Proof. We can rewrite the averages as
\[
\left[ \frac{1}{N} \sum_{p=0}^{N-1} f_4(T^p_4 x) e^{2\pi i p (\epsilon_1 + \epsilon_2)} \right] \left[ \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T^n_1 x) f_2(T^m_2 x) f_3(T^{n+m}_3 x) e^{2\pi i n \epsilon_1} e^{2\pi i m \epsilon_2} \right].
\]
The a.e. convergence is a consequence of the Wiener Wintner ergodic theorem for measure preserving transformations and Lemma 4.

Lemma 7. Let \((X, \mathcal{B}, \mu, T_i)\) be five measure preserving transformations on the same finite measure space. Then for all bounded functions \(f_i, 1 \leq i \leq 5\), for \(\mu\) a.e. \(x\) the averages
\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T^n_1 x) f_2(T^m_2 x) f_3(T^{n+m}_3 x) f_4(T^p_4 x) f_5(T^{n+p}_5 x)
\]
converge.

Proof. We follow the path of the proof of Lemma 4. The set where the averages converge is \(\mathcal{B}\) measurable. We use the ergodic decomposition of \((X, \mathcal{B}, \mu, T_5)\) into ergodic components on \((X, \mathcal{B}, \mu_{c,5})\). We disintegrate the set where the averages converge for each \(\epsilon \in \mathbb{R}\). We decompose the function \(f_5\) into its projection onto the corresponding Kronecker factor \(f_{5,K_c}\) and \(f_{5,K_c}^\perp\). By considering first the case of one eigenfunction then by approximation and linearity we obtain for \(\mu_{c,5}\) a.e. \(y\) the convergence of the averages
\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T^n_1 y) f_2(T^m_2 y) f_3(T^{n+m}_3 y) f_4(T^p_4 y) f_{5,K_c}(T^{n+p}_5 y).
\]
We can dominate the averages with the function $f_{5,K^c}$ by their absolute value
\[
\left| \frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T^n_1 y) f_2(T^m_2 y) f_3(T^{n+m}_3 y) f_4(T^p_4 y) f_{5,K^c}(T^{n+p}_5 y) \right|
\]
which in turn are bounded by
\[
\left| \frac{1}{N^3} \sum_{m=0}^{N-1} |f_2(T^m_2 y)| \sum_{n=0}^{N-1} |f_1(T^n_1 y)||f_3(T^{n+m}_3 y)| \sum_{p=0}^{N-1} f_4(T^p_4 y) f_{5,K^c}(T^{n+p}_5 y) \right|.
\]
Using the fact that the functions are uniformly bounded (by one without loss of generality) we get the upper bound
\[
\frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{p=0}^{N-1} f_4(T^p_4 y) f_{5,K^c}(T^{n+p}_5 y) \right|.
\]
We can apply the remark made after the proof of Lemma 5 in [1] to obtain the bound
\[
\sup_t \left| \frac{1}{N} \sum_{k=0}^{N-1} f_{5,K^c}(T^k y) e^{2\pi i k t} \right|
\]
which converges to zero by the uniform Wiener-Wintner ergodic theorem. By combining the convergence obtained for functions $f_{5,K^c}$ and $f_{5,K^c}$ we can reach the $\mu_{5,c}$ a.e. $y$ convergence of the averages
\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T^n_1 y) f_2(T^m_2 y) f_3(T^{n+m}_3 y) f_4(T^p_4 y) f_{5,K^c}(T^{n+p}_5 y).
\]
The convergence $\mu$ a.e. $x$ can be obtained by integration with respect to $d\mu_{5,c}dc$.

It remains to add one more transformation and function, namely $T_6$ and $f_6$. The path is quite clear. We start with a Wiener Wintner version of the Lemma 7.
Lemma 8. Let \((X, \mathcal{B}, \mu, T_i)\) be five measure preserving transformations on the same finite measure space. Then for all bounded functions \(f_i, 1 \leq i \leq 5\), for \(\mu\) a.e. \(x\) for all \(t \in \mathbb{R}\) the averages

\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T_1^n x) f_2(T_2^m x) f_3(T_3^{n+m} x) f_4(T_4^p x) f_5(T_5^{n+p} x) e^{2\pi i (m+p)t}
\]

converge.

Proof. We will use several tools in the proof of the previous Lemmas 7 and 8. We reconsider the disintegration of the measure \(\mu\) into ergodic components with respect to \((X, \mathcal{B}, \mu_5, c)\).

We disintegrate the measurable set where the averages converge for all \(\varepsilon_1, \varepsilon_2 \in \mathbb{R}\). We disintegrate also the measurable set where the averages converge for all \(t \in \mathbb{R}\). We decompose the function \(f_5\) into its projection onto the corresponding Kronecker factor \(f_{5,K_c}\) and \(f_{5,K_c^\perp}\). Again by approximation and linearity it is enough to look at the case of an eigenfunction \(e_j, 5\) with eigenvalue \(e^{2\pi i \theta_j}\). The averages in this case are equal to

\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T_1^n y) f_2(T_2^m y) f_3(T_3^{n+m} y) f_4(T_4^p y) f_5(T_5^{n+p} y) e^{2\pi i (m+p)t}
\]

and converge \(\mu_{c,5}\) a.e. \(y\) for all \(t\). We are left with the averages related to the function \(f_{5,K_c^\perp}\). By observations similar to those made in Lemma 7 we obtain for each \(t\) the upper
bound
\[
\frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{N} \sum_{p=0}^{N-1} f_4(T_4^n y)e^{2\pi ipt} f_5,_{K_c}(T_5^{n+p} y)\]

This last term is dominated by
\[
\sup_s \left| \frac{1}{N} \sum_{k=0}^{N-1} f_5,_{K_c}(T_5^k y)e^{2\pi iks} \right|
\]
and the convergence follows by the uniform Wiener Wintner ergodic theorem.

\[\square\]

**End of the proof of Theorem 1**

We consider \((X, \mathcal{B}, \mu, T_i)\) six measure preserving transformations on the same finite measure space and six bounded functions \(f_i, 1 \leq i \leq 6\). We want to prove that for \(\mu\) a.e. \(x\) the averages

\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T_1^m x)f_2(T_2^m x)f_3(T_3^{n+m} x)f_4(T_4^p x)f_5(T_5^{n+p} x)f_6(T_6^{m+p} x)
\]

converge.

We use an ergodic decomposition of the measure \(\mu\) into ergodic components for \(T_6\). As in the previous lemmas this reduces the study of the convergence on these components.

The function \(f_6\) is decomposed into the sum \(f_{6, K_c}\) and \(f_{6, K_c^\perp}\). The convergence \(\mu_{6,c}\) a.e. \(y\) is obtained by linearity, approximation and the use of Lemma 8. It remains to prove the convergence for the averages related to \(f_{6, K_c^\perp}\). We can use Lemma 2 with the sequence
\(a_{7,k} = 1\) (see also the proof of Lemma 6 in [1] to obtain the following inequalities:

\[
\left| \frac{1}{N^3} \sum_{n,m,p=0}^{N-1} f_1(T^{n}x) f_2(T^{m}x) f_3(T_3^{n+m}x) f_4(T_4^{p}x) f_5(T_5^{n+p}x) f_{6,K_6^c}(T_6^{m+p}x) \right|
\]
\[
\leq \frac{1}{N^3} \sum_{n,m=0}^{N-1} \left| \sum_{p=0}^{N-1} f_4(T_4^p y) f_5(T_5^{n+p}y) f_{6,K_6^c}(T_6^{m+p}y) \right|
\]
\[
\leq C \sup_s \left| \frac{1}{N} \sum_{k=0}^{N-1} f_{6,K_6^c}(T_6^k y) e^{2\pi iks} \right|
\]

The conclusion of the theorem follows after using the uniform Wiener Wintner ergodic theorem and integration.

### 2.2. An extension of Khintchine recurrence theorem.

Khintchine classical recurrence theorem says that if \(A\) is a set of positive measure, \(T\) an invertible measure preserving system and \(\varepsilon > 0\) the set

\[
\{n \in \mathbb{Z} : \int 1_A 1_A \circ T^n d\mu \geq |\int 1_A d\mu|^2 - \varepsilon\}
\]

has bounded gaps. This recurrence result states that for any measurable set \(A\) with positive measure its images under the iterates of \(T\) come back and overlap the set with bounded gaps. This is a consequence of von Neumann mean ergodic theorem as

\[
\lim_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} 1_A 1_A \circ T^n d\mu \geq \mu(A)^2.
\]

In this section we study similar recurrence properties with two and three measure preserving transformations that do not necessarily commute. We first give a two dimensional extension of Khintchine’s theorem. We can remark that an example given in [5] shows that the averages

\[
\frac{1}{N^2} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n}A \cap T_2^{-m}A \cap T_1^{-n}T_2^{-m}A)
\]
may diverge if $T_1$ and $T_2$ do not necessarily commute.

**Proposition 5.** Let $(X, \mathcal{B}, \mu)$ be a probability measure space and $T_1$, $T_2$ two measure preserving transformations on this measure space. We denote by $\mathcal{I}_1$ and $\mathcal{I}_2$ the $\sigma$ algebras of the invariant sets for $T_1$ and $T_2$. Consider $A$ a set of positive measure. Then

$$\lim_{N} \frac{1}{N^2} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n}A \cap T_2^{-n-m}A) = \int_A \mathbb{E}(1_A, \mathcal{I}_1)(x) \mathbb{E}(1_A, \mathcal{I}_2)(x) d\mu.$$  

In particular

$$\lim_{N} \frac{1}{N^2} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n}A \cap T_2^{-n-m}A) \geq \mu(A)^4.$$  

**Proof.** The averages

$$\frac{1}{N^2} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n}A \cap T_2^{-n-m}A)$$

are the integrals of the functions

$$\frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(x) 1_A(T_1^n x) 1_A(T_2^{n+m} x)$$

with respect to the measure $\mu$. As a particular case of Theorem 1 we have the pointwise convergence of these averages. Thus

$$\lim_{N} \frac{1}{N^2} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n}A \cap T_2^{-n-m}A)$$

exists after integration. So we just have to prove that

$$\lim_{N} \frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T_1^n x) 1_A(T_2^{n+m} x) = \mathbb{E}(1_A, \mathcal{I}_1)(x) \mathbb{E}(1_A, \mathcal{I}_2)(x)$$
in $L^2$ norm to conclude. For each $N$ we have

$$\frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T^n_1 x) 1_A(T^{n+m}_2 x)$$

$$= \frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T^n_1 x)E(1_A, I_2)(x) + \frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T^n_1 x)[1_A(T^{n+m}_2 x) - E(1_A, I_2)(x)]$$

The first term of the last equation converges by Birkhoff’s pointwise ergodic theorem to $E(1_A, I_1)(x)E(1_A, I_2)(x)$. Noticing that the function $E(1_A, I_2)(x)$ is $T_2$ invariant we can bound the $L^2$ norm of the second term by

$$\| \frac{1}{N} \sum_{n=1}^{N} \left[ \frac{1}{N} \sum_{m=1}^{N} [1_A \circ T^m_2 - E(1_A, I_2)] \circ T^n_2 \right] \|_2.$$ 

This term is less than

$$\frac{1}{N} \sum_{n=1}^{N} \left\| \sum_{n=1}^{N} \left[ \frac{1}{N} \sum_{m=1}^{N} [1_A \circ T^m_2 - E(1_A, I_2)] \right] \right\|_2$$

which is equal to

$$\| \frac{1}{N} \sum_{m=1}^{N} [1_A \circ T^m_2 - E(1_A, I_2)] \|_2$$

This last term tends to zero by the mean ergodic theorem applied to $T_2$. This proves that

$$\lim_{N} \| \frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T^n_1 x) 1_A(T^{n+m}_2 x) - E(1_A, I_1)(x)E(1_A, I_2)(x) \|_2 = 0.$$ 

It remains to show that

$$\int_A E(1_A, I_1)(x)E(1_A, I_2)(x)d\mu \geq \mu(A)^4.$$
We have
\[
\int_A E(1_A, I_1)(x).E(1_A, I_2)(x)d\mu = \int E(1_A E(1_A, I_1), I_2))1_A d\mu
\]
\[
= \int E(1_A E(1_A, I_1), I_2))E(1_A, I_2)d\mu
\]
(and as \(E(1_A, I_1)(x) \leq 1\) we have \(E(1_A, I_2) \geq E(1_A E(1_A, I_1), I_2)\)
\[
\geq \int \left( E(1_A E(1_A, I_1), I_2) \right)^2 d\mu
\]
\[
\geq \left( \int 1_A E(1_A, I_1)d\mu \right)^2 \geq \left( \int (E(1_A, I_1))^2 d\mu \right)^2
\]
\[
\geq (1_A d\mu)^4 = \mu(A)^4
\]

The study of the case of three measure preserving transformations seems much more complex.

**Lemma 9.** Let \((X, \mathcal{B}, \mu, T)\) be an invertible measure preserving system on a finite measure space, \(\mathcal{K}\) the \(\sigma\) algebra spanned by the eigenfunctions of \(T\) and \(f\) a bounded function. Let us denote by \(X_f\) the set of full measure given by the Wiener Wintner ergodic theorem such that for each \(x \in X_f\) the averages
\[
\frac{1}{N} \sum_{n=1}^N f(T^nx)e^{-2\pi int}
\]
converge for each \(t\). For each \(t \in \mathbb{R}\) let us denote by \(E_t(f)\) the limit function of these averages.

1. \(E_t(f)\) is the projection of the function \(f\) onto the eigenspace of \(T\) corresponding to the eigenvalue \(e^{2\pi it}\). In particular \(E_0(f)\) is equal to \(E(f, \mathcal{I})\) the conditional expectation with respect to the \(\sigma\) algebra of invariant sets for \(T\).
(2) If $t \neq s$ we have $\int E_t(f)\overline{E_s(f)}d\mu = 0$.

(3) If $e^{2\pi it_k}$ is the countable sequence of eigenvalues for $T$ and $e^{2\pi it_k}$ any countable set of distinct complex numbers then

$$\sum_{k=0}^{\infty} \|E_{t_k}(f)\|_2^2 \leq \sum_{k=0}^{\infty} \|E_{\theta_k}(f)\|_2^2 \leq \|E(f,K)\|_2^2$$

Proof. This is a simple consequence of the spectral theorem. If we denote by $P_t(f)$ the projection onto the eigenspace corresponding to the eigenvalue $e^{2\pi it}$ and by $\sigma_{f-P_t(f)}$ the spectral measure of the function $f - P_t(f)$ then we have

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x)e^{-2\pi int} = \frac{1}{N} \sum_{n=1}^{N} P_t(f)(T^n x)e^{-2\pi int} + \frac{1}{N} \sum_{n=1}^{N} [f - P_t(f)](T^n x)e^{-2\pi int}$$

$$= P_t(f)(x) + \frac{1}{N} \sum_{n=1}^{N} [f - P_t(f)](T^n x)e^{-2\pi int}$$

As

$$\lim_{N} \frac{1}{N} \sum_{n=1}^{N} [f - P_t(f)](T^n x)e^{-2\pi int}^2 = \int \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n(\theta - t)} \right|^2 d\sigma_{f-P_t(f)}(\theta) = \sigma_{f-P_t(f)}(\{0\}) = 0$$

we can conclude that $P_t(f) = E_t(f)$.

From this identification the remaining parts of the lemma follow without difficulty. For the last part of the lemma we just need to observe that $E_t(f) = 0$ if $e^{2\pi it}$ is not an eigenvalue of $T$. □

Remark It is worth noticing that there is a key difference at the pointwise level between $E_t(f)(x)$ and $P_t(f)(x)$. This difference highlights the difficulty one faces when dealing with ergodic versus not necessarily ergodic transformations. The function $E_t(f)$ is defined off a single set of measure zero for ALL $t \in \mathbb{R}$. For each $t \in \mathbb{R}$ it is almost everywhere equal
to the function $P_t(f)(x)$ and so for each $t$ the $L^2$ functions $P_t(f)$ and $E_t(f)$ are equal. However we can not claim that there is a universal null set off which one could write that $E_t(f)(x) = P_t(f)(x)$ for all $t \in \mathbb{R}$. One can look at the example given in Proposition 7 below.

**Proposition 6.** Let $(X, B, \mu, T_i), \ 1 \leq i \leq 3$, be three measure preserving systems on the same finite measure space. There exists a constant $0 < \delta < 1$ (independent from the $T_i$) such that for all measurable set $A$ with measure $\mu(A) > 1 - \delta$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n} A \cap T_2^{-m} A \cap T_3^{n-m} A) \geq \frac{1}{2} \mu(A)^8$$

**Proof.** We consider three measure preserving transformations $T_j, \ 1 \leq j \leq 3$ and a measurable set $A$ with positive measure. We list the following notations and properties.

1. We denote by $E_t^j(1_A)(x)$ the limit of

$$\frac{1}{N} \sum_{n=1}^{N} 1_A(T_j^nx)e^{-2\pi int}$$

for all $t \in \mathbb{R}$ off a single set of measure zero.

2. For each $1 \leq j \leq 3$ we consider the universal sets $X_{1A}^j$ such that $E_t^j(1_A)(x)$ exists for all $t \in \mathbb{R}$.

3. We consider an ergodic decomposition of $(X, B, \mu, T_3)$ with the measures $\mu_c$ where $d\mu = d\mu_c dc$.

4. We call $K_c$ the Kronecker factor of $T_3$ relative to the measure space $(X, B, \mu_c)$. The basis of eigenfunctions of $T_3$ relative to $\mu_c$ is denoted by $e_{k,c}$. The constant function $1$ corresponds to $e_{0,c}$.

5. The eigenvalue corresponding to the eigenfunction $e_{k,c}$ is $e^{-2\pi i \theta_{k,c}}$. 

(6) By Birkhoff pointwise ergodic theorem combined with the disintegration of \( \mu \) we have for a.e. \( c \) for \( \mu_c \) a.e. \( y \) \[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(T^n_3 y) = E(1_A, \mathcal{I})(y) = E_3^0(1_A)(y), \] where \( \mathcal{I} \) denotes the \( \sigma \) algebra of invariant sets with respect to \( \mu \).

As a consequence of the ergodicity of \( T_3 \) with respect to \( \mu \) we have
\[ E(1_A, K_c) = \sum_{k=0}^{\infty} \left( \int 1_A e^{k,c} d\mu_c \right) e^{k,c}. \]

We disintegrate the measurable sets \( \{X_j^j\}_{1_A} \) with respect to the measure \( d\mu_c \). We obtain for \( \mu_c \) a.e. \( y \) for all \( t \in \mathbb{R} \) the pointwise convergence of the averages
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(T^n_3 y)e^{-2\pi i nt}. \]

This is crucial for our method as with respect to the measure \( \mu_c \) the transformations \( T_1 \) and \( T_2 \) are not necessarily measure preserving.

For each eigenfunction \( e_{k,c} \) we have
\[ \lim_{N \to \infty} \frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T^n_1 y)1_A(T^m_2 y)e_{k,c}(T^{n+m}_3 y) \]
\[ = e_{k,c}(y) \lim_{N \to \infty} \frac{1}{N^2} \sum_{n,m=1}^{N} 1_A(T^n_1 y)e^{-2\pi i \theta_{k,c}}1_A(T^m_2 y)e^{-2\pi i \theta_{k,c}} = e_{k,c}(y)E_{\theta_{k,c}}^1(1_A)(y)E_{\theta_{k,c}}^2(1_A)(y) \]

As a consequence we have
\[ \lim_{N \to \infty} \frac{1}{N^2} \sum_{n,m=1}^{N} \mu_c(A \cap T^n_1 A \cap T^m_2 A \cap T^{n+m}_3 A) \]
\[ = \mu_c(A)(\int 1_A E_{\theta_{k,c}}^1(1_A) E_{\theta_{k,c}}^2(1_A) d\mu_c) + \sum_{k=1}^{\infty} (\int 1_A e_{k,c} d\mu_c)(\int 1_A E_{\theta_{k,c}}^1(1_A) E_{\theta_{k,c}}^2(1_A) d\mu_c) \]
The first term of the previous line is for a.e. $c$ equal to

$$\mathbb{E}(1_A, I_3) \int 1_A(y)\mathbb{E}(1_A, I_1)\mathbb{E}(1_A, I_2)d\mu_c.$$ 

In view of the constance of $\mathbb{E}(1_A, I_3)$ with respect to $\mu_c$, this last term can be written as

$$\int \mathbb{E}(1_A, I_3)1_A(y)\mathbb{E}(1_A, I_1)\mathbb{E}(1_A, I_2)d\mu_c.$$ 

Integrating with respect to $dc$ and using properties of the conditional expectation we get

$$\int \mathbb{E}(1_A, I_3)1_A(y)\mathbb{E}(1_A, I_1)\mathbb{E}(1_A, I_2)d\mu_c,dc = \int \mathbb{E}(1_A, I_3)1_A\mathbb{E}(1_A, I_1)\mathbb{E}(1_A, I_2)d\mu(x)$$

$$\geq \int \mathbb{E}(1_A\mathbb{E}(1_A, I_3)\mathbb{E}(1_A, I_2), I_1)1_A d\mu = \int \mathbb{E}(1_A\mathbb{E}(1_A, I_3)\mathbb{E}(1_A, I_2), I_1)\mathbb{E}(1_A, I_1)d\mu$$

$$\geq (\int (\mathbb{E}(1_A\mathbb{E}(1_A, I_3)\mathbb{E}(1_A, I_2), I_1))^2d\mu)^2 \geq \left(\int 1_A\mathbb{E}(1_A, I_3)\mathbb{E}(1_A, I_2)d\mu\right)^2$$

$$\geq \left(\int \mathbb{E}(1_A, I_3, I_2)\mathbb{E}(1_A, I_2)d\mu\right)^2$$

$$= \left(\int (\mathbb{E}(1_A, I_3, I_2))^2d\mu\right)^2 \geq \left(\int 1_A\mathbb{E}(1_A, I_3)d\mu\right)^4 \geq \mu(A)^8.$$ 

This shows that after integration with respect to $dc \mu_c(A)(\int 1_A E_0^1(1_A)E_0^2(1_A)d\mu_c)$ is bounded below by $\mu(A)^8$. Our proof will be complete if one can show that if $\mu(A) > 1 - \delta$ for some universal $0 < \delta < 1$ then

$$\int |I_c|d\mu = \int \int \sum_{k=1}^{\infty} |\int 1_A e_{k,c} d\mu_c| |\int 1_A e_{k,c} E_{\theta_{k,c}}^1(1_A)E_{\theta_{k,c}}^2(1_A) d\mu_c| d\mu_c \leq \frac{1}{2} \mu(A)^8.$$ 

By Cauchy-Schwartz’s inequality we have

$$|I_c| \leq \left(\sum_{k=1}^{\infty} |\int 1_A e_{k,c} d\mu_c|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\int 1_A e_{k,c} E_{\theta_{k,c}}^1(1_A)E_{\theta_{k,c}}^2(1_A) d\mu_c|^2\right)^{1/2}.$$
The vectors $e_{k,c}$ form an orthonormal basis of $L^2(X, \mathcal{K}_c, \mu_c)$ because $T_3$ on this space is ergodic. Thus we have

$$\left( \sum_{k=1}^{\infty} \left| \int 1_A e_{k,c} d\mu_c \right|^2 \right)^{1/2} = \left( \int |\mathbb{E}(1_A, \mathcal{K}_c)|^2 d\mu_c - \mu_c(A)^2 \right)^{1/2} \leq (\mu_c(A) - \mu_c(A)^2)^{1/2}$$

The second term $(II)_c = \left( \sum_{k=1}^{\infty} \left| \int 1_A e_{k,c} E_{\theta_{k,c}}^1(1_A) E_{\theta_{k,c}}^2(1_A) d\mu_c \right|^2 \right)^{1/2}$ can also be bounded above by

$$\left( \sum_{k=1}^{\infty} \left( \int 1_A |E_{\theta_{k,c}}^1(1_A)|^2 d\mu_c \right)^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} \left( \int 1_A |E_{\theta_{k,c}}^2(1_A)|^2 d\mu_c \right)^2 \right)^{1/2}.$$

Using Lemma 9 part (3), this last term is bounded above by

$$\left( \int |\mathbb{E}(1_A, \mathcal{K}_c)|^2 d\mu_c - \mu_c(A)^2 \right)^{1/2} \left( \int |\mathbb{E}(1_A, \mathcal{K}_c)|^2 d\mu_c - \mu_c(A)^2 \right)^{1/2}$$

which is equal to

$$\left( \int |\mathbb{E}(1_A, \mathcal{K}_c)|^2 d\mu_c - \mu_c(A)^2 \right) \leq (\mu_c(A) - \mu_c(A)^2)$$

Combining the bounds found in (2) and in (3) we get

$$|I_c| \leq (\mu_c(A) - \mu_c(A)^2)^{3/2}.$$ As $(\mu_c(A) - \mu_c(A)^2)^{3/2} \leq (\mu_c(A) - \mu_c(A)^2)$ and $\int \mu_c(A)^2 dc \geq (\int \mu_c(A) dc)^2$, integrating with respect to $c$ we obtain

$$\int |I_c| dc \leq \int (\mu_c(A) - \mu_c(A)^2)^{3/2} dc \leq \int (\mu_c(A) - \mu_c(A)^2) dc \leq \mu(A) - \mu(A)^2.$$

Going back to (1) we will reach our conclusion if we can find $0 < \delta < 1$ such that

$$\mu(A) - \mu(A)^2 \leq \frac{1}{2} \mu(A)^8,$$

for all measurable set $A$ with measure greater or equal to $1 - \delta$. This is an easy consequence of the uniqueness of the root for the polynomial $1/2x^7 + x - 1$ on $(0, 1)$. 
Remark The constant $\frac{1}{2}$ for the lower bound $\frac{1}{2}\mu(A)^8$ is certainly not optimal. Following the same path one can show that 

$$
\lim_{N \to \infty} \frac{1}{N^2} \sum_{n,m=1}^{N} \mu(A \cap T_1^{-n} A \cap T_2^{-m} A \cap T_3^{-n-m} A) > 0
$$

for all measurable set $A$ when $\mu(A) > \beta$ where $\beta$ is the root of $x^7 + x - 1$ on $(0,1)$.

\[\square\]

3. On the almost everywhere convergence of weighted averages

3.1. The averages $\frac{1}{N^2} \sum_{n,m=1}^{N} a_n b_m f(T^{n+m}x)$. Our goal is to prove first that the ergodicity assumptions are necessary in Lemma 3. We recall that we denote by $K$ the $\sigma$ algebra generated by the eigenfunctions of a measure preserving transformation. Even without the ergodicity assumption this $\sigma$-algebra is well defined.

**Proposition 7.** There exists a non ergodic measure preserving system $(Y, B, \nu, S)$, a function $f \in L^\infty(\nu) \cap K^\perp$ such that for $\nu$ a.e. $y$ we can find bounded sequences $a_n$ and $b_n$ such that the averages

$$
\frac{1}{N^2} \sum_{n,m=1}^{N} a_n b_m f(S^{n+m}y)
$$

do not converge when $N$ tends to $\infty$. In other words Lemma 3 is false if we remove the ergodicity assumption.

**Proof.** Let $S(x, y) = (x + \alpha, x + y)$ be the ergodic measure preserving transformation defined on the two Torus where $\alpha$ is an irrational number. We consider the measure preserving transformation $T = S \times S$ on $\mathbb{T}^4$ defined as

$$
T(x_1, x_2, x_3, x_4) = (x_1 + \alpha, x_1 + x_2, x_3 + \alpha, x_3 + x_4).
$$
The transformation $T$ is not ergodic and the Kronecker factor ($\sigma$ algebra spanned by the eigenfunctions of $T$) corresponds to the functions depending on the first and third coordinates $x_1$ and $x_3$. This is because the eigenfunctions of $S$ depend on their first coordinates (see also Lemma 4.18 in [4] on the way in general the eigenfunctions of $T$ are created from those of $S$).

Consider the function $f(x_1, x_2, x_3, x_4) = e^{-2\piix_2}e^{2\piix_4}$. This function belongs to $K_\perp$. We have $f(T^{n+m}(x_1, x_2, x_3, x_4)) = e^{2\pii(x_4-x_2+(n+m)(x_3-x_1)}$. Let us assume that Lemma 3 was true without ergodicity assumption then we could find a set of full measure such that for a.e. $x_1, x_2, x_3, x_4$ in this set and for all bounded sequences $a_n$ and $b_n$ we would have

$$\lim_{N} \frac{1}{N^2} \sum_{n,m=0}^{N-1} a_nb_m f(T^{n+m}(x_1, x_2, x_3, x_4)) = 0.$$

To disprove this we can take a bounded sequence $v_n$ such that the averages $\frac{1}{N} \sum_{n=0}^{N-1} v_n$ diverge. Then we can take $a_n = v_ne^{-2\piinx_3-x_1}$, and $b_m = e^{-2\piimx_3-x_1}$. As

$$\frac{1}{N^2} \sum_{n,m=0}^{N-1} a_nb_m f(T^{n+m}(x_1, x_2, x_3, x_4)) = \frac{1}{N} \sum_{n=0}^{N-1} v_ne^{2\pii(x_4-x_2)},$$

this shows that Lemma 3 is false once we remove the ergodicity assumption. This ends the proof of Proposition 7.

□

Remarks 1

(1) Proposition 7 shows that Lemma 3 as stated is quite sharp as one can not even expect to have the convergence of the averages $\frac{1}{N^2} \sum_{n,m=0}^{N-1} a_nb_m f(T^{n+m}x)$ as in this example they are equal to $\frac{1}{N} \sum_{n=1}^{N} v_ne^{2\pii(x_4-x_2)}$

(2) The same measure preserving system can be used to show that the uniform Wiener-Wintner ergodic theorem is no longer valid if $T$ is not ergodic. By this we mean
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that if we denote by $\mathcal{K}$ the $\sigma$ algebra spanned by the eigenfunctions of $T$ then we
do not have in general for functions $f \in \mathcal{K}^\perp$,$$
\limsup_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)e^{2\pi i nt} = 0.

(3) As indicated earlier the norm convergence holds without difficulty as the next proposition shows. We give the proof just for the sake of completeness and to show the
difference between the pointwise and norm convergence.

Definition 1. We will denote by $\mathcal{W}W_1$ the set of bounded sequences $a = (a_n)$ of scalars
such that $\lim_N \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi i nt}$ exists for each $t \in \mathbb{R}$.

Proposition 8. Let $T$ be a unitary operator and let $a = (a_n)$ and $b = (b_m)$ be bounded
sequences. Then the averages
$$\frac{1}{N^2} \sum_{n,m=1}^{N} a_n b_m T^{n+m}$$
converge in norm if $a$ and $b$ belong to $\mathcal{W}W_1$

Proof. It is a simple consequence of the spectral theorem. If we denote by $\sigma_f$ the spectral
measure of the function $f$ with respect to $T$ then we have
$$\| \frac{1}{N^2} \sum_{n,m=1}^{N} a_n b_m T^{n+m} - \frac{1}{M^2} \sum_{n,m=1}^{M} a_n b_m T^{n+m} \|^2$$
$$= \int | \frac{1}{N^2} \sum_{n,m=1}^{N} a_n b_m e^{2\pi i (n+m)t} - \frac{1}{M^2} \sum_{n,m=1}^{M} a_n b_m e^{2\pi i (n+m)t} |^2 d\sigma_f(t)$$
$$= \int | \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi i nt} \frac{1}{N} \sum_{m=1}^{N} b_m e^{2\pi i mt} - \frac{1}{N} \sum_{n=1}^{N} a_n e^{2\pi i nt} \frac{1}{N} \sum_{m=1}^{N} b_m e^{2\pi i mt} |^2 d\sigma_f(t)$$
which easily shows that the averages form a Cauchy sequence. □
3.2. Higher order averages. Proposition 2 shows that if the transformation $T$ is ergodic and the function $f \in L^2$ then for $\mu$ a.e. $x$ the averages
\[
\frac{1}{N^2} \sum_{n,m=0}^{N-1} a_n b_m f(T^{n+m}x)
\]
converge for all sequences $a = (a_n), b = (b_n)$ that belong to $W W_1$.

The next proposition shows that the class $W W_1$ does not characterize those bounded sequences for which the similar averages for seven terms converge a.e. even under the condition of ergodicity of the transformation

**Proposition 9.** There exists an ergodic dynamical system $(X, A, \mu, T)$ and a function $f \in L^\infty(\mu)$ such that for $\mu$ a.e. $x$ we can find bounded sequences $A_i = (a_{n,i}) \in W W_1$ for which the averages
\[
M_N(A_1, A_2, ... A_6, f)(x) = \frac{1}{N^3} \sum_{n,m,p=0}^{N-1} a_{1,p}a_{2,n}a_{3,p+n}a_{4,m}a_{5,n+m}a_{6,p+m}f(T^{n+m+p}x)
\]
do not converge.

**Proof.** We consider the sequence $v_n$ with values 1 or -1 such that the averages $\frac{1}{N} \sum_{n=1}^{N} v_n$ diverge. The sequence is built from longer and longer stretches of 1 and -1 so that the averages get close to 1 then close to -1 and so on. We extend $v_n$ to negative indices by putting $v_{-n} = v_n$. We can observe that this sequence has a correlation in the sense that for any $h \in \mathbb{Z}$ the averages $\frac{1}{N} \sum_{n=1}^{N} v_n v_{n+h}$ converge to a scalar $\gamma(h)$. Simple considerations show that the limit for all $h$ is equal to one. The quantity $\gamma(h)$ represents the $h$ Fourier coefficients of a positive measure $\sigma$ that is equal then to the Dirac measure at zero, $\delta_0$, a discrete measure.
We take now an irrational number \( \alpha \). We claim that the sequence \( a_n = v_n e^{2\pi i n^2} \alpha \) belongs to \( \mathcal{WW}_1 \). To see this first one can observe that the sequence \( e^{2\pi i n^2} e^{2\pi i nt} \) does have a correlation; for each \( h \in \mathbb{Z} \) the limit of \( \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i [(n^2 - (n+h)^2)\alpha e^{2\pi i nt} - (n+h)t]} \) is equal to zero for \( h \neq 0 \) and zero otherwise. Therefore the measure associated with these Fourier coefficients is Lebesgue measure, \( m \). As a consequence of the Affinity principle the measures \( m \) and \( \delta_0 \) being orthogonal we have for each \( t \in \mathbb{R} \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} v_n e^{2\pi i n^2} e^{2\pi i nt} = 0.
\]

Thus we have shown that the sequence \( a_n = v_n e^{2\pi i n^2} \alpha \) belongs to \( \mathcal{WW}_1 \).

We consider the ergodic measure preserving transformation \( S(x, y) = (x + \alpha, x + y) \) defined on the two Torus where \( \alpha \) is the irrational number used to define the sequence \( a_n \).

Our goal is to prove that for the function \( f(x, y) = e^{4\pi iy} \) it is impossible to find a set of full measure off which for all six bounded sequences \( A_i = (a_{i,n}), 1 \leq i \leq 6 \) the averages

\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} a_{1,p} a_{2,n} a_{3,p+n} a_{4,m} a_{5,n+m} a_{6,p+m} f(S^{n+m+p}(x, y))
\]

converge. To reach this conclusion we can use the simple equality

\[
(n + m + p)^2 = (n + m)^2 + (n + p)^2 + (m + p)^2 - n^2 - p^2 - m^2.
\]

We have \( f(T^{n+m+p}(x, y)) = e^{4\pi iy(n+m+p)(x-\alpha/2)} e^{2\pi i(n+m+p)^2} \alpha \). As a consequence if we take \( a_{1,p} = e^{2\pi ip^2} \alpha, a_{2,n} = v_n e^{2\pi in^2} \alpha, a_{3,p+n} = e^{-2\pi i(p+n)^2} e^{-2\pi i(p+n)(x-\alpha/2)}, a_{4,m} = e^{2\pi im^2} \alpha, a_{5,n+m} = e^{-2\pi i(n+m)^2} e^{-2\pi i(n+m)(x-\alpha/2)} \), and \( a_{6,p+m} = e^{-2\pi i(p+m)^2} e^{-2\pi i(p+m)(x-\alpha/2)} \), then

\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} a_{1,p} a_{2,n} a_{3,p+n} a_{4,m} a_{5,n+m} a_{6,p+m} f(S^{n+m+p}(x, y)) = \frac{1}{N} \sum_{n=0}^{N-1} v_n e^{4\pi iy}.
\]
Therefore the averages
\[
\frac{1}{N^3} \sum_{n,m,p=0}^{N-1} a_{1,n}a_{2,p+n}a_{3,n+m}a_{4,m+n+p+m}f(S^{n+m+p}(x,y))
\]
do not converge. (The arguments in the previous paragraphs can also be used to show that each sequence \( A_i \in \mathcal{W}_1 \).)

\[ \square \]

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