Local BRST Cohomology and Covariance

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Abstract

The paper provides a framework for a systematic analysis of the local BRST cohomology in a large class of gauge theories. The approach is based on the cohomology of $s + d$ in the jet space of fields and antifields, $s$ and $d$ being the BRST operator and exterior derivative respectively. It relates the BRST cohomology to an underlying gauge covariant algebra and reduces its computation to a compactly formulated problem involving only suitably defined generalized connections and tensor fields. The latter are shown to provide the building blocks of physically relevant quantities such as gauge invariant actions, Noether currents and gauge anomalies, as well as of the equations of motion.

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1 Introduction

1.1 Motivation

Gauge invariance underlies as a basic principle our present models of fundamental interactions and is widely used when one looks for extensions of these models. The BRST-BV formalism provides a general framework to deal with many aspects of gauge symmetry, both in classical and quantum field theory. It was first established by Becchi, Rouet and Stora \cite{1} in the context of renormalization of abelian Higgs–Kibble and Yang–Mills gauge theories, later extended by Kallosh to supergravity with open gauge algebra \cite{2} (see also \cite{3}) and by de Wit and van Holten to general gauge theories \cite{4}, resulting finally in the universal field-antifield formalism of Batalin and Vilkovisky \cite{5} which allows to treat all kinds of gauge theories within an elegant unified framework. The usefulness of this formalism is mainly based on the fact that it encodes the gauge symmetry and all its properties in a single antiderivation which is strictly nilpotent on all the fields and antifields. Throughout this paper, this antiderivation is called the BRST operator and denoted by $s$.

The nilpotency of $s$ establishes in particular the local BRST cohomology, i.e. the cohomology of $s$ in the space of local functionals (= integrated local volume forms) of the fields and antifields. This cohomology has many physically relevant applications. It determines for instance gauge invariant actions and their consistent deformations \cite{6}, the dynamical local conservation laws \cite{7} and the possible gauge anomalies (see e.g. \cite{8, 1, 9, 10, 11, 12}) of a gauge theory and is a useful tool in the renormalization of quantum field theories even when a theory is not renormalizable in the usual sense \cite{13}.

Since the BRST cohomology can be defined for any gauge theory and since the correspondence of its cohomology classes to the mentioned physical quantities is universal too, it is worthwhile to look for a suitable general framework within which this cohomology can be computed efficiently and which has a large range of applicability.

The purpose of this paper is to propose such a framework. It applies to a large class of gauge theories and relates the BRST cohomology to an underlying gauge covariant algebra. This includes a definition of tensor fields on which this algebra is realized and of generalized connections associated with it, and reduces the computation of the BRST cohomology locally to a problem involving only these quantities.

The reduced problem is formulated very compactly in terms of identities analogous to the “Russian formula” in Yang–Mills theory \cite{9, 14, 15}.

$$F = (s + d)(C + A) + (C + A)^2.$$ (1.1)

Here $C$, $A$ and $F$ are the familiar Lie algebra valued Yang–Mills ghost fields, connection and curvature forms respectively, $s$ is the Yang–Mills BRST operator, and $d$ is the spacetime exterior derivative. The usefulness of (1.1) is based, among others, on its remarkable property to compress the familiar BRST transformations of the Yang–Mills ghost and gauge fields, as well as the construction of the field strength in terms of the gauge field into a single identity. The combination $C + A$ occurring in (1.1) is an example of what will be called a generalized connection here.

\footnote{Originally the term “Russian formula” was introduced by Stora in the second ref. in \cite{15} for a different but related identity. Here it is used as in the last ref. in \cite{9}.}
1.2 Relations and differences to other approaches

The proposed approach generalizes a concept outlined in [16] (see also [10]) for the study of the “restricted” (= antifield independent) BRST cohomology in a special class of gauge theories characterized among others by (a) the presence of (spacetime) diffeomorphisms among the gauge symmetries, (b) the closure and irreducibility of the gauge algebra, (c) the presence of ‘enough’ independent gauge fields ensuring that all the derivatives of the ghost fields can be eliminated from the BRST cohomology. In such theories, the extension of the concept of [16] to the full cohomological problem, including the antifields, is (more or less) straightforward and was used already in [17, 18] within a complete computation of the BRST cohomology in Einstein gravity and Einstein–Yang–Mills theories.

Here these ideas are extended to general gauge theories. In particular none of the conditions (a)–(c) is needed as a prerequisite for the methods outlined in this paper. This is possible thanks to suitable generalizations of the concept [16] which at the same time modify and unite various techniques that have been developed over the last 20 years, thereby revealing relations between them which are less apparent in other approaches. Such techniques, to be described later in detail, are the so-called descent equation technique, the use of contracting homotopies in jet spaces, compact formulations of the BRST algebra analogous to the “Russian formula” (1.1), and spectral sequence techniques along the lines of homological perturbation theory [19, 20, 21]. Let me now briefly comment on the use of these techniques in this paper, as compared to other approaches.

Descent equations and the “Russian formula” were first used within the celebrated differential geometric construction of (representatives of) chiral anomalies in $D = 2n$ dimensions from characteristic classes in $D + 2$ dimensions [9], and also within the classification of such anomalies in [22]. Later it became clear that the descent equations are useful not only in connection with chiral anomalies, but to analyse the complete BRST cohomology, cf. e.g. [23, 24, 15, 11, 16]. The reason is that they allow to deal efficiently with the total derivatives into which the integrands of BRST invariant functionals transform in general.

In this paper we will compress the descent equations into a compact form. To this end the BRST operator $s$ and the exterior derivative $d$ will be united to the single operator

$$\tilde{s} = s + d$$

defined on local “total forms” (see section 3). This idea is not new; in fact it is familiar from the construction and classification of chiral anomalies mentioned above. However, somewhat surprisingly, it was not utilized systematically in a general approach to the BRST cohomology on local functionals later.

The systematic use of $\tilde{s}$ is fundamental to the method proposed here and has several advantages. In particular it allows us to extend the concept of [16] to theories which do not satisfy the assumptions (a)–(c) mentioned above, such as Yang–Mills theory whose BRST cohomology has been calculated by different means in [25, 24, 26, 27]. The use of $\tilde{s}$ is particularly well adapted to the analysis of the BRST cohomology on local functionals because the latter is in fact isomorphic to the cohomology of $\tilde{s}$.
on local total forms, at least locally, c.f. \([16]\) and section \([3]\).

Contracting homotopies similar to the ones used here were constructed and applied to BRST cohomological problems e.g. already in \([25, 23, 24]\). However, these contracting homotopies were designed for the cohomology of \(s\) \([24]\) and its linearized version \([25, 23]\) respectively. The method proposed here extends them to the \(\tilde{s}\)-cohomology. This has the important consequence that it leads directly to the mentioned compact formulation of the cohomological problem in terms of identities analogous to the “Russian formula” \((1.1)\). For instance, when applied to Yang–Mills theory, the contracting homotopy for \(\tilde{s}\) singles out the special combination (generalized connection) \(A + C\) occurring in \((1.1)\). As a result, \((1.1)\) itself arises naturally in this approach, cf. section \(7.1\). In contrast, the corresponding contracting homotopy \([24]\) for \(s\) gives instead of \(A + C\) just \(C\) and makes no contact with the “Russian formula” (it does however provide the same tensor fields).

The proposed approach also extends the methods developed in \([21]\) to use and deal with the antifields along the lines of homological perturbation theory \([19, 20]\). This extension is straightforward and, again, related to the use of \(\tilde{s}\) instead of \(s\). Among others it will allow us to trace the BRST cohomology at all ghost numbers (including negative ones) back to a weak (= on-shell) cohomological problem involving the tensor fields and generalized connections only. This has been utilized recently in \([28]\) in order to compute the BRST cohomology in four dimensional \(N = 1\) supergravity.

Finally, the approach provides a ‘cohomological’ perspective on tensor fields and connections. The latter are usually characterized through specific transformation properties under the respective symmetries. However, in a general gauge theory it is not always clear from the outset which transformation laws should be imposed for this purpose. An advantage of the approach proposed here is that such transformation laws need not be specified from the start. Rather, they emerge from the approach itself. Such a characterization of tensor fields, connections and the corresponding transformation laws has two major advantages: (i) it is purely algebraic and does not invoke any concepts in addition to the BRST cohomology itself; (ii) it is physically meaningful because the resulting tensor fields and generalized connections turn out to provide among others the building blocks of gauge invariant actions, Noether currents, anomalies and of the equations of motions.

### 1.3 Outline of the paper

The paper has been organized as follows. Section \(2\) sketches the basic algebraic approach to the BRST cohomology used in this paper and introduces some terminology and notation. Sections \(3\) and \(4\) relate the local BRST cohomology to the cohomology of \(\tilde{s}\) and its weak (= “on-shell”) counterpart. Section \(5\) introduces the concept of contracting homotopies for \(\tilde{s}\) in jet spaces, and section \(6\) shows that this concept is intimately related to the existence of a gauge covariant algebra and a compact formulation of the BRST algebra on tensor fields and generalized connections. Section \(7\) illustrates the method for various examples which do not satisfy the aforementioned

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3The isomorphism applies only to the BRST cohomology on local functionals, i.e. to the relative cohomology \(H(s|d)\) on local volume forms. It does not extend to \(H(s|d)\) at lower form degrees in general.
assumptions (a)–(c) of [15] (the examples are Yang–Mills theory, Einstein gravity in the metric formulation, supergravity with open gauge algebra and two-dimensional Weyl invariant sigma models). Sections 8–10 spell out implications for the structure of gauge invariant actions, Noether currents, gauge anomalies, etc., as well as for the classical equations of motion. In section 11 a special aspect of the cohomological problem is discussed, concerning the explicit dependence of the solutions on the coordinates of the base manifold which will be called “spacetime” henceforth, for no reason at all. The paper is ended by some concluding remarks in section 12 and two appendices containing details concerning the algebraic approach and conventions used in the paper.

2 Algebraic setting, definitions and notation

In order to define the local BRST cohomology in a particular theory one has to specify the BRST operator \( s \) and the space in which its cohomology is to be computed. The BRST operator is defined on a set of fields \( \Phi^A \) and corresponding antifields \( \Phi^* \) according to standard rules of the field-antifield formalism summarized in appendix B. In particular these rules include that the BRST operator is nilpotent and commutes with the spacetime derivatives \( \partial_\mu \),

\[
s^2 = s \partial_\mu s = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0.
\] (2.1)

The basic concept underlying these fundamental relations and the whole paper is the jet bundle approach [29] sketched in the appendix A. Essentially this means simply that the fields, antifields and all their derivatives are understood as local coordinates of an infinite jet space. For this set of jet coordinates the collective notation \([\Phi, \Phi^*]\) is used. The local jet coordinates are completed by the spacetime coordinates \( x_\mu \) and the differentials \( dx_\mu \). The differentials are counted among the jet coordinates by pure convention and convenience. The derivatives \( \partial_\mu \) are defined as total derivative operators in the jet space, cf. eq. (A.6), and become usual partial derivatives on the local sections of the jet bundle.

The concrete BRST transformations of the fields and antifields depend on the particular theory and its gauge symmetry, whereas the spacetime coordinates \( x_\mu \) and differentials \( dx_\mu \) are always BRST invariant in accordance with the second relation (2.1),

\[
s x_\mu = 0, \quad s dx_\mu = 0.
\] (2.2)

The use of the differentials is in principle not necessary but turns out to be very useful in order to analyse the local BRST cohomology. In particular it allows to define \( d = dx_\mu \partial_\mu \) and \( \tilde{s} = s + d \) in the jet space. The relations (2.1) are equivalent to the nilpotency of \( \tilde{s} \),

\[
\tilde{s}^2 = 0 \quad \iff \quad s^2 = sd + ds = d^2 = 0.
\] (2.3)

The usefulness of \( \tilde{s} \) in the context of the local BRST cohomology stems from the fact that it allows to write and analyse the descent equations in a compact form (cf.
The descent equations involve local $p$-forms

$$\omega_p = \frac{1}{p!} dx^{\mu_1} \ldots dx^{\mu_p} \omega_{\mu_1 \ldots \mu_p}(x, [\Phi, \Phi^*]).$$

(2.4)

These forms are required to be local in the sense that they are formal series in the antifields, ghosts and their derivatives such that each piece with definite antighost number (cf. [21] and section 4) depends polynomially on the derivatives of all the fields and antifields. From the outset no additional requirements are imposed on local forms here. In particular they are not restricted by power counting, it is not assumed that the indices $\mu_i$ of the functions $\omega_{\mu_1 \ldots \mu_p}$ occurring in (2.4) indicate their actual transformation properties under Lorentz or general coordinate transformations, and local forms are not required to be globally well-defined in whatever sense.

A local functional is by definition an integrated local volume form $\int \omega_D$ (throughout this paper $D$ denotes the spacetime dimension). It is called BRST invariant if $s\omega_D$ is $d$-exact in the space of local forms, i.e. if $s\omega_D + d\omega_{D-1} = 0$ holds for some local form $\omega_{D-1}$. Translated to the local sections of the jet bundle, in general this requires local functionals to be BRST invariant only up to surface integrals. Analogously a local functional $\int \omega_D$ is called BRST-exact (or trivial) if $\omega_D = sn_D + d\eta_{D-1}$ holds for some local forms $\eta_D$ and $\eta_{D-1}$. The BRST cohomology on local functionals considered here is thus actually the relative cohomology $H(s|d)$ of $s$ and $d$ on local volume forms. This cohomology is well-defined due to (2.3) and represented by solutions $\omega_D$ of

$$s\omega_D + d\omega_{D-1} = 0, \quad \omega_D \neq s\eta_D + d\eta_{D-1}.$$

(2.5)

In the next section $H(s|d)$ will be related to the cohomology of $\tilde{s}$ on local total forms $\tilde{\omega}$. The latter are by definition formal sums of local forms with different form degrees,

$$\tilde{\omega} = \sum_p \omega_p.$$

(2.6)

The $\tilde{s}$-cohomology on local total forms is then defined through the condition $\tilde{s}\tilde{\omega} = 0$ modulo trivial solutions of the form $\tilde{s}\tilde{\eta} + \text{constant}$ where $\tilde{\eta}$ is a local total form and the constant is included for convenience. The representatives of this cohomology are thus local total forms $\tilde{\omega}$ solving

$$\tilde{s}\tilde{\omega} = 0, \quad \tilde{\omega} \neq \tilde{s}\tilde{\eta} + \text{constant}.$$

(2.7)

The natural degree in the space of local total forms is the sum of the ghost number (gh) and the form degree (formdeg), called the total degree (totdeg),

$$\text{totdeg} = \text{gh} + \text{formdeg}.$$

(2.8)

A local total form with definite total degree $G$ is thus a sum of local $p$-forms with ghost number $g = G - p$ ($p = 0, \ldots, D$). $\tilde{s}$ has total degree 1, i.e. it maps a local total form with total degree $G$ to another one with total degree $G + 1$. 

5
3 Descent equations

It is easy to see that the BRST cohomology on local functionals is locally isomorphic to the cohomology of \( \tilde{s} \) on local total forms\(^3\). To show this, one only needs (2.3) and a theorem on the cohomology of \( d \) on local forms, sometimes called the algebraic Poincaré lemma. The latter states that locally any \( d \)-closed local \( p \)-form is \( d \)-exact for \( 0 < p < D \) and constant for \( p = 0 \), while local volume forms (\( p = D \)) are locally \( d \)-exact if and only if they have vanishing Euler–Lagrange derivative with respect to all the fields and antifields \([30, 25, 16]\).

The local isomorphism of the cohomological problems associated with (2.5) and (2.7) can be derived by standard arguments which are therefore only sketched. Let me therefore sketch its derivation. The arguments are standard and therefore not given in detail. Suppose that \( \omega_D \) solves \( s\omega_D + d\omega_{D-1} = 0 \). Applying \( s \) to this equation results in \( d(s\omega_{D-1}) = 0 \) due to (2.3). Hence, \( s\omega_{D-1} \) is \( d \)-closed. Since it is not a volume form, it is thus also \( d \)-exact in the space of local forms according to the algebraic Poincaré lemma. Hence, there is a (possibly vanishing) local \((D-2)\)-form \( \omega_{D-2} \) satisfying \( s\omega_{D-1} + d\omega_{D-2} = 0 \). Iterating the arguments one concludes the existence of a set of local forms \( \omega_p, \, p = p_0, \ldots, D \) satisfying

\[
 s\omega_p + d\omega_{p-1} = 0 \quad \text{for } D \geq p > p_0; \quad s\omega_{p_0} = 0
\]

for some \( p_0 \). These equations are called the descent equations\(^4\). They can be compactly written in the form

\[
 \tilde{s} \tilde{\omega} = 0, \quad \tilde{\omega} = \sum_{p=p_0}^D \omega_p.
\]

Hence, any solution of \( s\omega_D + d\omega_{D-1} = 0 \) corresponds to an \( \tilde{s} \)-closed local total form and the reverse is evidently also true. Using again the algebraic Poincaré lemma and (2.3), it is easy to see that \( \omega_D \) is a trivial solution of the form \( s\eta_D + d\eta_{D-1} \) if and only if \( \tilde{\omega} \) is trivial too, i.e. if and only if \( \tilde{\omega} = \tilde{s}\eta + \text{constant} \). Since \( \tilde{\omega} \) has total degree \((g+D)\) if \( \omega_D \) has ghost number \( g \) we conclude

**Lemma 3.1:** The BRST-cohomology on local functionals with ghost number \( g \) and the \( \tilde{s} \)-cohomology on local total forms of total degree \( G = g+D \) are locally isomorphic.

That is to say, locally the solutions of (2.7) with ghost number \( g \) correspond one-to-one (modulo trivial solutions) to the solutions of (2.7) with total degree \( G = g+D \).

4 Equivalence to the weak cohomology of \( \tilde{\gamma} = \gamma + d \)

A simple and useful concept in the study of the BRST cohomology is a suitable expansion of local functionals and forms in powers of the antifields. Following the

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\(^3\)Here and in the following local equalities or isomorphisms refer to sufficiently small patches of the jet space. Global properties of the jet bundle are not taken into account.

\(^4\)For \( p_0 = 0 \) the algebraic Poincaré lemma alone actually implies only \( s\omega_0 = \text{const.} \); however, in meaningful gauge theories a BRST-exact constant vanishes necessarily, as one easily verifies (note that a constant can occur only if \( \omega_0 \) has ghost number \(-1\)). Notice that this might not hold anymore if one extends the BRST–BV formalism by including constant ghosts corresponding e.g. to global symmetries \([13, 16]\). Such an extension is always possible \([31]\) but not considered here.
lines of [21] it will now be used to show that the ˜

s-cohomology on local total forms

of the fields and antifields reduces to a weak (= on-shell) cohomology on antifield independent local total forms.

The most useful expansion in the antifields takes their respective ghost numbers into account. This is achieved through the so-called antighost number (antigh) defined according to

\[
\text{antigh}(\Phi^A) = -gh(\Phi^A), \quad \text{antigh}(\Phi^A) = \text{antigh}(dx^\mu) = \text{antigh}(x^\mu) = 0. \tag{4.1}
\]

In particular the BRST operator can be decomposed into pieces with definite antighost number (one says a piece has antighost number \(k\) if it raises the antighost number by \(k\) units). The decomposition of \(s\) starts always with a piece of antighost number \(-1,\)

\[
s = \delta + \gamma + \sum_{k \geq 1} s_k, \quad \text{antigh}(\delta) = -1, \quad \text{antigh}(\gamma) = 0, \quad \text{antigh}(s_k) = k. \tag{4.2}
\]

The most important pieces in this decomposition are \(\delta\) and \(\gamma;\) the other pieces have positive antighost number and play only a secondary role in the cohomological analysis. \(\delta\) is the so-called Koszul–Tate differential and is nonvanishing only on the antifields,

\[
\delta \Phi^A = 0, \quad \delta \phi^*_i = \frac{\partial^R L_{cl}}{\partial \phi^i}, \quad \ldots \tag{4.3}
\]

where \(\partial^R L_{cl}/\partial \phi^i\) denotes the Euler–Lagrange right-derivative of the classical Lagrangian \(L_{cl}\) w.r.t. \(\phi^i). In particular \(\delta\) thus implements the classical equations of motion in the cohomology. \(\gamma\) encodes the gauge transformations because \(\gamma \phi^i\) equals a gauge transformation of \(\phi^i\) with parameters replaced by ghosts,

\[
\gamma \phi^i = R^\alpha_\alpha C^\alpha, \tag{4.4}
\]

where the notation of appendix B is used.

(4.2) extends straightforwardly to the analogous decomposition of \(\tilde{s} = s + d\) into pieces with definite antighost numbers. Since \(d\) has vanishing antighost number, one simply gets

\[
\tilde{s} = \delta + \tilde{\gamma} + \sum_{k \geq 1} s_k \tag{4.5}
\]

with

\[
\tilde{\gamma} = \gamma + d. \tag{4.6}
\]

Note that \(\tilde{s}^2 = 0\) decomposes into

\[
\delta^2 = 0, \quad \delta \tilde{\gamma} + \tilde{\gamma} \delta = 0, \quad \tilde{\gamma}^2 = -\left(\delta s_1 + s_1 \delta\right), \quad \ldots \tag{4.7}
\]

The usefulness of the decomposition (4.3) is due to the acyclicity of the Koszul–Tate differential \(\delta\) on local functions at positive antighost number [24, 21, 32]. This means that the cohomology of \(\delta\) on local total forms is trivial at positive antighost number.

\[
\delta \tilde{\omega}_k = 0, \quad \text{antigh}(\tilde{\omega}_k) = k > 0 \quad \Rightarrow \quad \tilde{\omega}_k = \delta \tilde{\eta}_{k+1}. \tag{4.8}
\]

\footnote{An analogous statement does not hold for the relative cohomology of \(\delta\) and \(d\). Indeed there are in general solutions of (2.5) which contain no antifield independent part. Such solutions correspond to local conservation laws [3].}
Using standard arguments of spectral sequence techniques which are not repeated here, one concludes from (4.8) immediately that a nontrivial solution of $\tilde{s}\tilde{\omega} = 0$ contains necessarily an antifield independent part $\tilde{\omega}_0$ solving

$$\tilde{\gamma}\tilde{\omega}_0 \approx 0, \quad \tilde{\omega}_0 \not\approx \tilde{\gamma}\tilde{\eta}_0 + \text{constant}, \quad \text{antigh}(\tilde{\omega}_0) = 0$$

(4.9)

where $\approx$ denotes weak equality defined through

$$A_0 \approx 0 :\Leftrightarrow \exists A_1 : \quad A_0 = \delta A_1 \quad (\text{antigh}(A_k) = k). \quad (4.10)$$

Note that the weak equality is an “on-shell equality” since, due to (1.3), $A_0 \approx 0$ implies that $A_0$ vanishes for solutions of the classical equations of motion.

Furthermore (1.7) and (1.8) imply that each solution $\tilde{\omega}_0$ of (1.3) can be completed to a nontrivial solution $\tilde{\omega} = \tilde{\omega}_0 + \ldots$ of (2.7) and that two different completions with the same antifield independent part are equivalent in the cohomology of $\tilde{s}$ (the latter follows immediately from the fact that the difference of two such completions has no antifield independent part). This establishes the following result:

**Lemma 4.1:** The cohomology of $\tilde{s}$ on local total forms is isomorphic to the weak cohomology of $\tilde{\gamma}$ on antifield independent local total forms. That is to say, any solution $\tilde{\omega}$ of (2.7) contains an antifield independent part $\tilde{\omega}_0$ solving (4.9), and any solution $\tilde{\omega}_0$ of (4.9) can be completed to a solution of (2.7) which (for fixed $\tilde{\omega}_0$) is unique up to $\tilde{s}$-exact contributions.

**Remark:**

The weak cohomology of $\tilde{\gamma}$ on antifield independent local total forms is well-defined since $\tilde{\gamma}$ is weakly nilpotent on these forms,

$$\text{antigh}(A_0) = 0 \quad \Rightarrow \quad \tilde{\gamma}^2 A_0 \approx 0. \quad (4.11)$$

This follows immediately from the third identity (1.7) due to $\delta A_0 = 0$.

## 5 Elimination of trivial pairs

A well-known technique in the study of cohomologies is the use of contracting homotopies. I will now describe how one can apply it within the computations of the $\tilde{s}$-cohomology and of the weak $\tilde{\gamma}$-cohomology introduced in the previous sections. The idea is to construct contracting homotopy operators which allow to eliminate certain local jet coordinates, called trivial pairs, from the cohomological analysis. This reduces the cohomological problem to an analogous one involving only the remaining jet coordinates. For that purpose one needs to construct suitable sets of jet coordinates replacing the fields, antifields and their derivatives and satisfying appropriate requirements. In this section I will specify such requirements and show that they allow to eliminate trivial pairs. In section 6 various explicit examples will be discussed to illustrate how one constructs these special jet coordinates in practice.

The contracting homotopies and the trivial pairs for the $\tilde{s}$- and the weak $\tilde{\gamma}$-cohomology are usually closely related. Nevertheless, in practical computations the use of one or the other may be more convenient. Moreover it is often advantageous
to combine them. For instance one may first use a contracting homotopy for the \( \tilde{s} \)-cohomology that eliminates some fields or antifields completely, such as the antighosts and the corresponding Nakanishi–Lautrup auxiliary fields used for gauge fixing, and then analyse the remaining problem by investigating the weak \( \tilde{\gamma} \)-cohomology. The arguments will be worked out in detail only for the weak \( \tilde{\gamma} \)-cohomology which is more subtle due to the occurrence of weak instead of strict equalities. In contrast, the \( \tilde{s} \)-cohomology can be treated using standard arguments which imply:

**Lemma 5.1:** Suppose there is a set of local jet coordinates \( B = \{ U^\ell, V^\ell, W^i \} \) such that the change of local jet coordinates from \( \{ [\Phi^A, \Phi^{*A}], x^\mu, dx^\mu \} \) to \( B \) is local and locally invertible\(^6\) and

\[
\begin{align*}
\tilde{s} U^\ell &= V^\ell \quad \forall \ell , \\
\tilde{s} W^i &= R^i(W) \quad \forall i .
\end{align*}
\] (5.1) (5.2)

Then locally the \( U \)'s and \( V \)'s can be eliminated from the \( \tilde{s} \)-cohomology, i.e. the latter reduces locally to the \( \tilde{s} \)-cohomology on local total forms depending only on the \( W \)'s.

The \((U^\ell, V^\ell)\) are called trivial pairs. As already mentioned, lemma 5.1 can be used in particular to eliminate the antighosts, Nakanishi–Lautrup fields and their antifields completely from the cohomological analysis because they (and all their derivatives) form trivial pairs, cf. e.g. [25] and [4], section 14. In the following these fields will be therefore neglected without loss of generality.

Let me now turn to the derivation of an analogous result for the weak \( \tilde{\gamma} \)-cohomology on antifield independent local total forms. Let us assume that there is a local and locally invertible change of jet coordinates from the antifield independent set \( \{ [\Phi^A], x^\mu, dx^\mu \} \) to \( \{ U^\ell, V^\ell, W^i \} \) such that\(^7\)

\[
\begin{align*}
\tilde{\gamma} U^\ell &= V^\ell \quad \forall \ell , \\
\tilde{\gamma} W^i &= R^i(W) \quad \forall i .
\end{align*}
\] (5.3) (5.4)

Furthermore one can assume (without loss of generality) that each of the \( U \)'s, \( V \)'s and \( W \)'s has a definite total degree. Note that all these degrees are nonnegative because the \( U \)'s, \( V \)'s and \( W \)'s do not involve antifields and because it is assumed that antighosts and Nakanishi–Lautrup fields have been eliminated already.

Again, the \((U^\ell, V^\ell)\) are called trivial pairs. In order to deal with weak equalities the following lemma will be useful later on:

**Lemma 5.2:** Any weakly vanishing local total form \( f(U, V, W) \) is a combination of weakly vanishing functions \( L_K(W) \) in the sense that

\[
f(U, V, W) \approx 0 \iff f(U, V, W) = a^K(U, V, W)L_K(W), \quad L_K(W) \approx 0
\] (5.5)

for some local total forms \( a^K \).

\(^6\)I.e. locally any local total form \( f([\Phi, \Phi^*], dx, x) \) can be uniquely expressed as a local total form \( g(U, V, W) \) and vice versa.

\(^7\)One may replace the equalities in (5.3) and (5.4) by weak equalities without essential changes in the following arguments.
Proof: Since the classical equations of motion have vanishing total degree and do not involve antifields, they are expressible solely in terms of the $U$’s and $W$’s because the $V$’s have positive total degrees as a direct consequence of (5.3) (in fact only those $U$’s and $W$’s with vanishing total degrees can occur in the equations of motion). To prove (5.5) it is therefore sufficient to consider functions depending only on the $U$’s and $W$’s. Now, if a function $f(U, W)$ vanishes weakly then the same holds for its $\tilde{\gamma}$-transformation due to the second identity in (4.7), for the latter implies $f = \delta g \Rightarrow \tilde{\gamma} f = -\delta(\tilde{\gamma} g) \approx 0$. Using (5.3) and (5.4) one concludes

$$f(U, W) \approx 0 \Rightarrow \tilde{\gamma} f(U, W) \approx 0.$$  \hspace{1cm} (5.6)

Since the $U$’s, $V$’s and $W$’s are by assumption independent local jet coordinates, and since the $V$’s do not occur in the equations of motion, one concludes from (5.6) (for instance by differentiating $\tilde{\gamma} f(U, W)$ w.r.t. to $V^\ell$) that $f(U, W) \approx 0$ implies $\partial f(U, W)/\partial U^\ell \approx 0$. Iteration of the argument yields

$$f(U, W) \approx 0 \Rightarrow \frac{\partial^k f(U, W)}{\partial U^\ell_1 \ldots \partial U^\ell_k} \approx 0 \quad \forall k.$$ \hspace{1cm} (5.7)

Thus a weakly vanishing function $f(U, W)$ must be a combination of weakly vanishing functions of the $W$’s which proves (5.5). \hfill $\Box$

I remark that lemma 5.2 implies in particular that the equations of motion themselves are equivalent to a set of equations involving only those $W$’s with vanishing total degree. This result will be interpreted in section 10 as the covariance of the equations of motion. We are now prepared to prove that the $U$’s and $V$’s can be eliminated from the weak $\tilde{\gamma}$-cohomology:

Lemma 5.3: Suppose there is a local and locally invertible change of jet coordinates replacing $\{[\Phi^A], x^\mu, dx^\mu\}$ by a set $\{U^\ell, V^\ell, W^i\}$ satisfying (5.3) and (5.4). Then locally the $U$’s and $V$’s can be eliminated from the weak $\tilde{\gamma}$-cohomology on antifield independent local total forms,

$$\tilde{\gamma} \tilde{\omega}_0(U, V, W) \approx 0 \quad \Rightarrow \quad \tilde{\omega}_0(U, V, W) \approx f(W) + \tilde{\gamma}\tilde{\eta}_0(U, V, W),$$ \hspace{1cm} (5.8)

i.e. locally this cohomology is represented by solutions of

$$\tilde{\gamma} f(W) \approx 0, \quad f(W) \not\approx \tilde{\gamma} g(W) + \text{constant}.$$ \hspace{1cm} (5.9)

Proof: By assumption, locally any antifield independent local total form can be written in terms of the $U$’s, $V$’s and $W$’s. To construct a contracting homotopy a parameter $t$ is introduced scaling the $U$’s and $V$’s according to

$$U^\ell_t := tU^\ell, \quad V^\ell_t := tV^\ell.$$ \hspace{1cm} (5.10)

On total forms $\tilde{\omega}_0(U_t, V_t, W)$ one then defines an operator $b$ through

$$b = U^\ell \frac{\partial}{\partial V^\ell_t} = \frac{1}{t} U^\ell \frac{\partial}{\partial V^\ell}.$$ \hspace{1cm} (5.11)
\( \tilde{\gamma} U^\ell_t \) and \( \tilde{\gamma} V^\ell_t \) are defined by replacing in \( \tilde{\gamma} U^\ell \) and \( \tilde{\gamma} V^\ell \) all \( U \)'s and \( V \)'s by the corresponding \( U_t \)'s and \( V_t \)'s. Now, (5.3) implies \( \tilde{\gamma} V^\ell = \tilde{\gamma}^2 U^\ell \approx 0 \). Using lemma 5.2 one thus concludes

\[
\tilde{\gamma} V^\ell = a^{\ell,K}(U, V, W) L_K(W), \quad L_K(W) \approx 0
\]

for some \( a^{\ell,K} \) and \( L_K \). Hence one defines

\[
\tilde{\gamma} U^\ell_t = V^\ell_t, \quad \tilde{\gamma} V^\ell_t = a^{\ell,K}(U_t, V_t, W) L_K(W).
\]

This shows in particular \( \tilde{\gamma} V^\ell_t \approx 0 \) and one now easily verifies

\[
(\tilde{\gamma} b + b\tilde{\gamma}) \tilde{\omega}_0(U_t, V_t, W) \approx \frac{\partial \tilde{\omega}_0(U_t, V_t, W)}{\partial t}
\]

which implies

\[
\tilde{\omega}_0(U, V, W) - \tilde{\omega}_0(0, 0, W) \approx \int_0^1 dt (\tilde{\gamma} b + b\tilde{\gamma}) \tilde{\omega}_0(U_t, V_t, W).
\] (5.12)

Applying again lemma 5.2 one concludes that \( \tilde{\gamma} \tilde{\omega}_0(U, V, W) \approx 0 \) implies \( \tilde{\gamma} \tilde{\omega}_0(U_t, V_t, W) \approx 0 \). Using this in (5.13) we finally get

\[
\tilde{\gamma} \tilde{\omega}_0(U, V, W) \approx 0 \Rightarrow \tilde{\omega}_0(U, V, W) \approx \tilde{\omega}_0(0, 0, W) + \tilde{\gamma} \int_0^1 dt b \tilde{\omega}_0(U_t, V_t, W) \quad (5.14)
\]

where we used \( \int_0^1 dt \tilde{\gamma} b \tilde{\omega}_0(\ldots) \approx \int_0^1 dt b \tilde{\omega}_0(\ldots) \) (the latter holds since \( \tilde{\gamma} \) does not change the \( t \)-dependence up to weakly vanishing terms). This proves the lemma. \( \square \)

Remarks:

a) It is very important to realize that both (5.1) and (5.2) must hold in order to eliminate \( U \)'s and \( V \)'s from the cohomology, and that the existence of a pair of jet coordinates satisfying (5.1) does in general not guarantee the existence of complementary \( W \)'s fulfilling (5.2). A simple and important counterexample is given by \( x^\mu \) and \( dx^\mu \) which always satisfy \( s x^\mu = dx^\mu \) but usually do not form a trivial pair except in diffeomorphism invariant theories, cf. section 11. Analogous remarks apply of course to (5.3) and (5.4). The reader may check that the contracting homotopies for \( s \) used in \[24, 15, 16\] are in fact also based on the construction of variables satisfying requirements analogous to (5.3) and (5.4).

b) Clearly the aim is the construction of a set of local jet coordinates containing as many trivial pairs as possible. The difficulty of this construction is in general not the finding of pairs of local jet coordinates satisfying (5.1) resp. (5.3) but the construction of complementary \( W \)'s resp. \( W \)'s satisfying (5.2) resp. (5.4).

c) Typically the \( U \)'s are components of gauge fields and their derivatives and the \( V \)'s contain the corresponding derivatives of the ghosts, cf. section 7. The \( W \)'s will be interpreted as tensor fields and generalized connections, cf. section 6.

d) Lemmas 5.1 and 5.3 are not always devoid of global subtleties, i.e. they can fail to be globally valid. E.g. if the manifold of the \( U \)'s has a nontrivial de Rham cohomology, one cannot always eliminate all the \( U \)'s and \( V \)'s globally (important counterexamples are the vielbein fields in gravitational theories, cf. \[18\], section 5). In such cases the proof of lemma 5.3 breaks down globally because some of the functions of the \( U \)'s, \( V \)'s and \( W \)'s occurring in the proof have no globally well-defined extensions. This problem can be dealt with along the lines of \[18\].
6 Gauge covariant algebra, tensor fields and generalized connections

It will now be shown that the existence of a set of local jet coordinates \( \{U^\ell, V^\ell, W^i\} \) (with nonempty subset \( \{U^\ell, V^\ell\} \)) satisfying (5.3) and (5.4) has a deep origin. Namely it is intimately related to an algebraic structure encoded in (5.4) which will be interpreted as a gauge covariant algebra and leads to the identification of tensor fields and generalized connections mentioned in the introduction.

Recall that each local jet coordinate \( W^i \) has a definite nonnegative total degree since it neither involves antifields nor antighosts. Those \( W^i \)’s with vanishing total degree are called tensor fields and are denoted by \( T_ı \); the other \( W^i \)’s are called generalized connections for reasons which will become clear soon. Those generalized connections with total degree 1 are denoted by \( \tilde{C}_N \); the other generalized connections are denoted by \( \tilde{Q}^{NG} \) where \( G \) indicates their total degree,

\[
\{ T^i \} = \{ W^i : \text{totdeg}(W^i) = 0 \}, \quad \{ \tilde{C}^N \} = \{ W^i : \text{totdeg}(W^i) = 1 \}, \quad \{ \tilde{Q}^{NG} \} = \{ W^i : \text{totdeg}(W^i) = G \geq 2 \}. \tag{6.1}
\]

Note that the tensor fields have necessarily vanishing ghost number and form degree, whereas a generalized connection decomposes in general into a sum of local forms with different ghost numbers and corresponding form degrees,

\[
\tilde{C}^N = \tilde{C}^N + A^N, \quad \text{gh}(\tilde{C}^N) = 1, \quad \text{gh}(A^N) = 0, \tag{6.2}
\]

\[
\tilde{Q}^{NG} = \sum_{p=0}^{G} \hat{Q}_{p}^{NG}, \quad \text{gh}(\hat{Q}_{p}^{NG}) = G - p. \tag{6.3}
\]

The \( \tilde{C}^N \) are called covariant ghosts, the \( A^N \) connection 1-forms and the \( \hat{Q}^{NG} \) connection \( G \)-forms.

Since \( \tilde{\gamma} \) raises the total degree by one unit, (5.4) and (6.1) imply in particular

\[
\tilde{\gamma} T^i = \tilde{C}^N R_N^i(\mathcal{T}), \tag{6.4}
\]

\[
\tilde{\gamma} \tilde{C}^N = \frac{1}{2}(-)^{\varepsilon + 1} \tilde{C}^L \tilde{C}^K F_{KL} N(\mathcal{T}) + \hat{Q}^{M2} Z_{M2} N^2(\mathcal{T}), \tag{6.5}
\]

\[
\tilde{\gamma} \tilde{Q}^{N2} = \frac{1}{2}(-)^{\varepsilon + 1} \tilde{C}^L \tilde{C}^M \tilde{C}^L Z_{MLK}^{N2}(\mathcal{T}) + \hat{Q}^{M3} Z_{M3}^{N2}(\mathcal{T}) + \hat{Q}^{M2} \tilde{C}^K Z_{KM2}^{N2}(\mathcal{T}) \tag{6.6}
\]

\[\vdots\]

for some functions \( \mathcal{R} \), \( \mathcal{F} \) and \( \mathcal{Z} \) of the tensor fields. Here \( (\varepsilon_M + 1) \) denotes the Grassmann parity of \( \tilde{C}^M \),

\[
\varepsilon(\tilde{C}^M) = \varepsilon_M + 1. \tag{6.7}
\]

From \( \tilde{\gamma}^2 T^i \approx 0 \) one concludes, using (6.4) and (6.5),

\[
\mathcal{R}_{M}^{N} \frac{\partial \mathcal{R}_{N}^{i}}{\partial T} - (-)^{\varepsilon_M \varepsilon_N} \mathcal{R}_{N}^{i} \frac{\partial \mathcal{R}_{M}^{i}}{\partial T} \approx - \mathcal{F}_{MN}^{K} \mathcal{R}_{K}^{i}, \tag{6.8}
\]

\[
\mathcal{Z}_{M2}^{N} \mathcal{R}_{N}^{i} \approx 0. \tag{6.9}
\]
(6.8) can be written in the compact form
\[ [\Delta_M, \Delta_N] \approx -\mathcal{F}_{MN}^K(T)\Delta_K \] (6.10)
where \([\cdot, \cdot]\) denotes the graded commutator,
\[ [\Delta_M, \Delta_N] = \Delta_M \Delta_N - (-)^{\varepsilon_M \varepsilon_N} \Delta_N \Delta_M , \] (6.11)
and \(\Delta_N\) is the operator
\[ \Delta_N = \mathcal{R}_N \frac{\partial}{\partial T} . \] (6.12)
Analogously \(\tilde{\gamma} \tilde{\gamma}^T \tilde{C}^N \approx 0\) implies in particular
\[ \sum_{MNP} \left( \Delta_M \mathcal{F}_{NP}^K + \mathcal{F}_{MN}^R \mathcal{F}_{RP}^K + Z_{MNP}^M Z_{M_2}^K \right) \approx 0 \] (6.13)
where the graded cyclic sum was used defined by
\[ \sum_{MNP} X_{MNP} = (-)^{\varepsilon_M \varepsilon_P} X_{MNP} + (-)^{\varepsilon_N \varepsilon_M} X_{NPM} + (-)^{\varepsilon_P \varepsilon_N} X_{PMN} . \] (6.14)
(6.13) are nothing but the Jacobi identities for the algebra (6.10) in presence of possible reducibility relations (6.9). Note that the Grassmann parities of \(\tilde{\gamma}\) and of the \(\tilde{C}\)'s imply the following Grassmann parities and symmetries of the \(\Delta\)'s and \(\mathcal{F}\)'s
\[ \varepsilon(\Delta_N) = \varepsilon_N , \quad \varepsilon(\mathcal{F}_{MN}^K) = \varepsilon_M + \varepsilon_N + \varepsilon_K \pmod{2} , \]
\[ \mathcal{F}_{MN}^K = -(-)^{\varepsilon_M \varepsilon_N} \mathcal{F}_{NM}^K . \] (6.15)
In order to reveal the geometric content of this algebra it is useful to decompose (6.4) and (6.5) into parts with definite ghost numbers. Note that (6.4) reads
\[ \tilde{\gamma} T^i = \tilde{C}^N \Delta_N T^i \] (6.16)
and thus decomposes due to \(\tilde{\gamma} = \gamma + d\) and (6.2) into
\[ \gamma T^i = \tilde{C}^N \Delta_N T^i , \]
\[ d T^i = A^N \Delta_N T^i . \] (6.17) (6.18)
(6.17) can be interpreted as a characterization of tensor fields as gauge covariant quantities. Indeed, recall that tensor fields are constructed solely out of the ‘classical fields’ \(\phi\), their derivatives and the spacetime coordinates due to (6.1). Therefore \(\gamma T\) equals just a gauge transformation of \(T\) with parameters replaced by ghosts. (6.17) requires thus that the gauge transformation of a tensor field involves only special combinations of the parameters and their derivatives (which may involve the classical fields too), corresponding to the covariant ghosts \(\tilde{C}\). Hence, (6.17) characterizes tensor fields indeed through a specific transformation law.
Now, the derivatives \(\partial_\mu T\) of a tensor field are in general not tensor fields since \(\gamma(\partial_\mu T)\) contains \(\partial_\mu \tilde{C}^N\). The question arises how to relate \(\partial_\mu T\) to gauge covariant quantities. The answer is encoded in (6.18). Indeed, recall that the \(A^N\) are 1-forms,
\[ A^N = dx^\mu A^N_\mu . \] (6.19)
(6.18) is therefore equivalent to
\[ \partial_\mu T^i = A_\mu^N \Delta_N T^i. \] (6.20)

By assumption, (6.20) holds \textit{identically} in the fields and their derivatives, with the same set \( \{ A_\mu^N \} \) for all \( i \). In general this requires that \( \{ A_\mu^N \} \) contains a locally invertible subset \( \{ v_\mu^m \} \). Then (6.20) just defines those \( \Delta \)'s corresponding to \( \{ v_\mu^m \} \) in terms of the \( \partial_\mu \) and the other \( \Delta \)'s, and can be regarded as a definition of \textit{covariant derivatives}. To put this in concrete terms I introduce the notation
\[ \{ A_\mu^N \} = \{ v_\mu^m, A_\mu \}, \quad \{ \Delta_N \} = \{ D_m, \Delta_\hat{r} \}, \quad m = 1, \ldots, D \] (6.21)

where the matrix \( (v_\mu^m) \) is assumed to be invertible. The \( D_m \) are called covariant derivatives and according to (6.20) they are given by
\[ D_m = V_m^\mu (\partial_\mu - A_\mu \hat{r} \Delta_\hat{r}) \] (6.22)

where \( V_m^\mu \) denotes the inverse of \( v_\mu^m \),
\[ v_\mu^m V_m^\nu = \delta^\nu_\mu, \quad v_\mu^m V_n^\mu = \delta^m_n. \] (6.23)

I note that neither the \( v_\mu^m \) nor the \( A_\mu \) are necessarily elementary fields. In particular, some of them may be constant or even zero.

Let me finally discuss (6.24) which generalizes the Russian formula (6.13). Its decomposition into pieces with definite ghost number (resp. form degree) reads
\[ \gamma \hat{C}^N = \frac{1}{2} (-)^{\hat{c}+1} \hat{C}^L \hat{C}^K F_{KL}^N (T) + \hat{Q} M^2 Z_{M2}^N (T), \] (6.24)
\[ \gamma A_\mu^N = \partial_\mu \hat{C}^N - \hat{C}^K A_\mu^K F_{KL}^N (T) - \hat{C}^L A_\mu^L F_{KL}^N (T), \] (6.25)
\[ \partial_\mu A_\mu^N = -A_\mu^L A_\nu^K F_{KL}^N (T) + B_{\mu \nu} M^2 Z_{M2}^N (T) \] (6.26)

where the following notation was used:
\[ \hat{Q} N_2 = \frac{1}{2} dx^\mu dx^\nu B_{\mu \nu} N_2 + dx^\mu \hat{C}_\mu N_2 + \hat{Q} N_2. \] (6.27)

(6.24) and (6.25) give the \( \gamma \)-transformations of the covariant ghosts and of the \( A_\mu^N \) respectively. (6.26) determines the \textit{curvatures} (field strengths) corresponding to the “gauge fields” \( A_\mu^N \). They are given by
\[ F_{mn}^N = V_m^\mu V_n^\nu \left( 2 \partial_\mu A_\nu \right)^N + 2 v_{[\mu}^k A_{\nu]} \hat{r} F_{k \hat{r}}^N (T) \]
\[ + A_\mu \hat{r} A_\nu \hat{r} F_{\hat{r} \hat{r}}^N (T) - B_{\mu \nu} M^2 Z_{M2}^N (T) \] (6.28)

where the invertibility of the \( v_\mu^m \) was used again in order to solve (6.26) for the \( F_{mn}^N \). That the latter should indeed be identified with curvatures follows from the fact that they occur in the commutator of the covariant derivatives,
\[ [D_m, D_n] \approx -F_{mn}^N \Delta_N. \] (6.29)

Note however that some (or all) of these curvatures may be constant or even zero. The Bianchi identities arising from (6.29) are a subset of the identities (6.13),
\[ D_{[m} F_{nk]}^N = F_{[mn}^M F_{k]M}^N + Z_{mnk} M^2 Z_{M2}^N \approx 0. \] (6.30)
Remarks:

a) (6.10) can be regarded as a covariant version of the gauge algebra. However it is important to realize that the number of $\Delta$’s exceeds in general the number of gauge symmetries, cf. section 7.

b) $\tilde{Q}$’s occur only in reducible gauge theories because otherwise there are no local jet variables which can correspond to them.

c) Considerations similar to those performed here for the $W$’s can be of course also applied to the $\mathcal{W}$’s satisfying (5.2). That leads in particular to an extension of the concept to antifield dependent tensor fields. Examples can be found in [17, 18, 33].

7 Examples

The concept outlined in the previous sections will now be illustrated for four examples, exhibiting different facets of the general formalism. First the concept is shown to reproduce the standard tensor calculus in the familiar cases of Yang–Mills theory and of gravity in the metric formulation. Then pure four dimensional $N=1$ supergravity without auxiliary fields is discussed. This illustrates the case of an open gauge algebra and is the only example where the number of $\Delta$’s and gauge symmetries coincide. Finally Weyl and diffeomorphism invariant sigma models in two spacetime dimensions are considered. In this example one gets an infinite set of generalized connections and corresponding $\Delta$-transformations, but no (nonvanishing) curvatures (6.28). I remark that the approach of [15] does not apply to any of these examples (not even to gravity in the metric formulation!) because each of them violates one of the assumptions (a)–(c) mentioned in section 1.2. Hence, one really needs the extended concept outlined in the previous sections to perform the following analysis.

As the gauge algebra is closed in the first, second and last example, the formulae of section 6 are in these cases promoted to strict instead of weak equalities, with $\tilde{\gamma}$ replaced by $\tilde{s}$ and without making reference to a particular gauge invariant action.

7.1 Yang–Mills theories

For simplicity I consider pure Yang–Mills theories (no matter fields). The standard BRST transformations of the Yang–Mills gauge fields $A_{\mu}^i$ and the corresponding ghosts $C^i$ read

$$sA_{\mu}^i = \partial_{\mu}C^i + C^k A_{\mu}^j f_{jk}^i, \quad sC^i = \frac{1}{2} C^k C^j f_{jk}^i$$ (7.1)

where $i$ labels the elements of the Lie algebra of the gauge group with structure constants $f_{ij}^k$. The trivial pairs are in this case given by

$$\{U^\ell\} = \{\partial_{(\mu_1...\mu_k} A_{\mu_{k+1})}^i : k = 0, 1, \ldots\},$$ (7.2)

$$\{V^\ell\} = \{sU^\ell\} = \{\partial_{(\mu_1...\mu_k+1} C^i + \ldots : k = 0, 1, \ldots\}.$$ (7.3)

Hence, in the new set of local jet coordinates the $V$’s replace one by one all the derivatives of the ghosts. The undifferentiated ghosts themselves are replaced by the generalized connections

$$\tilde{C}^i = C^i + A^i, \quad A^i = dx^\mu A_{\mu}^i.$$ (7.4)
The complete set of generalized connections contains in addition the differentials,

\[ \{ \tilde{C}^N \} = \{ dx^\mu, \tilde{C}^i \}. \]  (7.5)

The \( v_\mu^m \) are thus in this case just the entries of the constant unit matrix, \( v_\mu^m = \delta_\mu^m \).

Hence, indices \( m \) and \( \mu \) need not be distinguished in this case. The \( \Delta \)-operations corresponding to (7.5) are

\[ \{ \Delta_N \} = \{ D_\mu, \delta_i \}, \quad D_\mu = \partial_\mu - A_\mu^i \delta_i \]  (7.6)

where the \( \delta_i \) are the Lie algebra elements. (6.5) reproduces for \( N = i \) the “Russian formula” (1.1) in the form

\[ \tilde{s} \tilde{C}^i = \frac{1}{2} \tilde{C}^k \tilde{C}^j f_{jk}^i + \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}^i. \]  (7.7)

The algebra (6.10) of the \( \Delta \)'s reads in this case

\[ [D_\mu, D_\nu] = -F_{\mu\nu}^i \delta_i, \quad [D_\mu, \delta_i] = 0, \quad [\delta_i, \delta_j] = f_{ij}^k \delta_k \]  (7.8)

with the standard Yang–Mills field strengths arising from (6.28) and transforming under the \( \delta_i \) according to the adjoint representation,

\[ F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + A_\mu^j A_\nu^k f_{jk}^i, \quad \delta_i F_{\mu\nu}^j = -f_{ik}^j F_{\mu\nu}^k. \]  (7.9)

A complete set of tensor fields is in this case given by the \( x^\mu \) and a choice of algebraically independent components of the field strengths and their covariant derivatives,

\[ \{ \mathcal{T}^i \} \subset \{ x^\mu, D_\mu \ldots D_\mu_k F_{\nu\rho}^i : k = 0, 1, \ldots \}. \]  (7.10)

Remark:

Notice that the above choice of variables is very similar to the one in [24]. In fact the tensor fields coincide in both approaches (except that here also the \( x^\mu \) are counted among them). The difference is that the present approach singles out the \( \tilde{C}^i \) and \( dx^\mu \) as generalized connections, rather than just the \( C^i \). Note that, as a direct consequence of the presence of \( d \) in \( \tilde{s} \), one cannot simply choose \( \tilde{C}^i = C^i \) here because that choice would not fulfill requirement (5.4).

### 7.2 Gravity in the metric formulation

I consider now pure gravity with the metric fields \( g_{\mu\nu} = g_{\nu\mu} \) as the only classical fields and diffeomorphisms as the only gauge symmetries. The BRST transformations of the metric and the diffeomorphism ghosts \( \xi^\mu \) read

\[ \tilde{s} g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + (\partial_\mu \xi^\rho) g_{\rho\nu} + (\partial_\nu \xi^\rho) g_{\mu\rho}, \quad \tilde{s} \xi^\mu = \xi^\nu \partial_\nu \xi^\mu. \]  (7.11)

The trivial pairs can be chosen as

\[ \{ U^\ell \} = \{ x^\mu, \partial_\mu \ldots \partial_\mu_k \Gamma_{\mu_{k+1}\mu_{k+2}}^\nu : k = 0, 1, \ldots \} \]  (7.12)

\[ \{ V^\ell \} = \{ \tilde{s} U^\ell \} = \{ dx^\mu, \partial_\mu \ldots \partial_\mu_k \xi^\nu + \ldots : k = 0, 1, \ldots \} \]  (7.13)
where
\[ \Gamma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \] (7.14)

Note that the \( V \)'s replace all derivatives of the ghosts of order \( > 1 \). The undifferentiated ghosts and their first order derivatives give rise to the generalized connections
\[ \{ \tilde{C}^N \} = \{ \tilde{\xi}^\mu, \tilde{\xi}^\nu \}, \quad \tilde{\xi}^\mu = \xi^\mu + dx^\mu, \quad \tilde{\xi}^\nu = \partial_\mu \xi^\nu + \Gamma_{\mu\rho}^\nu \xi^\rho. \] (7.15)

The generalized Russian formulae (6.13) read in this case
\[ \tilde{\eta} \tilde{\xi}^\mu = \xi^\nu \tilde{C}_{\nu}^\mu, \quad \tilde{\xi} \tilde{\xi}^\nu = \tilde{\xi}^\rho \tilde{C}_{\rho}^\nu + \frac{1}{2} \tilde{\xi}^\rho \tilde{\xi}^\sigma R_{\rho\sigma}^\mu \] (7.16)

where \( R_{\mu\nu} \) is the standard Riemann tensor constructed of the \( \Gamma \)'s. The \( v_{\mu m} \) are, as in the case of the Yang–Mills theory, just the entries of the constant unit matrix. Hence, indices \( \mu \) and \( m \) are not distinguished. One gets
\[ \{ \Delta \} = \{ D_\mu, \Delta_\mu^\nu \}, \quad D_\mu = \partial_\mu - \Gamma_{\mu\rho}^\nu \Delta_\nu^\rho \] (7.17)
where the \( \Delta_\mu^\nu \) generate \( GL(D) \)-transformations of world indices according to
\[ \Delta_\mu^\nu T_\mu = \delta_\nu^\nu T_\mu, \quad \Delta_\mu^\nu T_\rho = -\delta_\mu^\rho T_\nu. \] (7.18)

The algebra (6.14) reads now
\[ [D_\mu, D_\nu] = -R_{\mu\nu}^\rho \Delta_\rho^\mu, \quad [\Delta_\mu^\nu, D_\rho] = \delta_\rho^\nu D_\mu, \quad [\Delta_\mu^\nu, \Delta_\rho^\sigma] = \delta_\rho^\sigma \Delta_\mu^\nu - \delta_\mu^\sigma \Delta_\nu^\rho. \] (7.19)

The set of tensor fields contains the \( g_{\mu\nu}, \mu \geq \nu \) and a maximal set of algebraically independent components of \( R_{\mu\nu}^\rho \) and their covariant derivatives,
\[ \{ T^i \} \subset \{ g_{\mu\nu}, D_{\mu_1} \ldots D_{\mu_k} R_{\lambda\nu}^\rho \} : \quad k = 0, 1, \ldots. \] (7.20)

Remark:
Recall that tensor fields are characterized by the transformation law (6.17). One might wonder whether this transformation law agrees in this case with the standard transformation law for tensor fields under diffeomorphisms which is in BRST language the Lie derivative along the diffeomorphism ghosts. The answer is affirmative because (6.17) yields in this case, e.g. for a tensor field \( T_\mu \),
\[ \gamma T_\mu = \xi^\nu D_\nu T_\mu + (\partial_\mu \xi^\nu + \Gamma_{\mu\rho}^\nu \xi^\rho) T_\nu = \xi^\nu \partial_\nu T_\mu + (\partial_\mu \xi^\nu) T_\nu. \] (7.21)

7.3 D=4, N=1 minimal supergravity

The classical field content of the D=4, N=1 minimal pure supergravity theory without auxiliary fields is given by the vielbein fields and the gravitinos, denoted by \( e_\mu^a \) and \( \psi_{\mu}^\alpha \), \( \bar{\psi}^\dot{\alpha}_\mu \) respectively (\( \alpha, \dot{\alpha} \) denote indices of two-component complex Weyl spinors with conventions as in [28]). The gauge symmetries are diffeomorphism invariance, local supersymmetry and local Lorentz invariance. The corresponding ghosts are
denoted by $\xi^\mu$, $\xi^a$, $\tilde{\xi}^a$ and $C^{ab} = -C^{ba}$ respectively. For simplicity the analysis is restricted to the action

$$S_{cl} = \int d^4x \left[ \frac{1}{2} e R - 2 e^{\mu\nu\rho\sigma}(\psi^\mu \sigma^\nu \nabla^\rho \bar{\psi}_\sigma - \bar{\psi}_\sigma \sigma^\rho \nabla^\nu \psi^\sigma) \right]$$

(7.22)

with $e = \det(e^\mu_\alpha)$, $e^{0123} = 1$ and

$$R = R_{ab}^{\quad ba}, \quad R_{ab}^{\quad cd} = 2E_{[a}^{\mu} E_{b]}^{\nu}(\partial_\mu \omega^c_{\nu} + \omega^c_\mu \omega_\nu^d),$$

(7.23)

$$\nabla_\mu \psi^\alpha = \partial_\mu \psi^\alpha - \frac{1}{2} \omega^\mu_\alpha \epsilon_{\alpha\beta\gamma} \psi^\beta \psi^\gamma,$$

(7.24)

$$\nabla_\mu \bar{\psi}_\alpha = \partial_\mu \bar{\psi}_\alpha + \frac{1}{2} \omega^\mu_\alpha \bar{\sigma}^\beta_\mu \bar{\psi}_\beta \psi^\gamma,$$

(7.25)

where the $E^\mu_a$ are the entries of the inverse vielbein and $\omega^ab_\mu$ denotes the gravitino dependent spin connection

$$\omega^ab_\mu = E^{au} E^{b\rho}(\omega^a_{[\mu\nu]}\rho - \omega^a_{[\nu\rho]}\mu + \omega^a_{[\rho\mu]}\nu),$$

$$\omega^a_{[\mu\nu]}\rho = e_{\rho a} \partial^a_\mu e^b_\nu - i \bar{\psi}_\rho \sigma_\mu \psi_\nu + i \psi_\nu \sigma_\rho \bar{\psi}_\mu.$$  

(7.26)

(Lorentz indices $a,b,\ldots$ are lowered and raised with the Minkowski metric $\eta_{ab} = \text{diag}(1,-1,-1,-1)$.)

The $\gamma$-transformations read in this case

$$\gamma_\mu e^a_\mu = \xi^\nu \partial_\nu e^a_\mu + (\partial_\mu \xi^\nu) e^a_\nu + C^a_\mu e^b_\mu + 2 i C^{a\mu_\alpha}_\alpha \psi^\alpha + \xi^\alpha \psi^\alpha,$$

(7.27)

$$\gamma_\mu \bar{\psi}_\alpha = \nabla_\mu \xi^\alpha + \xi^\nu \partial_\nu \bar{\psi}_\alpha + (\partial_\mu \xi^\nu) \bar{\psi}_\nu + \frac{1}{2} C^{ab}_\mu \sigma_{ab} \bar{\psi}_\alpha \psi^\beta,$$

(7.28)

$$\gamma_\mu \xi^\alpha = \xi^\nu \partial_\nu \xi^\alpha + 2 i C^{a\mu}_\alpha \bar{\psi}_\alpha \xi^\beta,$$

(7.29)

$$\gamma_\mu \xi^\alpha = \xi^\mu \partial \xi^\alpha + 2 i C^{ab}_\mu \sigma_{ab} \xi^\beta - 2 i C^{a\mu}_\beta \bar{\psi}_\alpha \xi^\beta,$$

(7.30)

$$\gamma_\mu C^{ab} = \xi^\mu \partial C^{ab} - C^{a\mu} C^{c\mu} - C^{a\mu} C^{c\mu} - 2 i C^{a\mu\beta} \bar{\psi}_\alpha \xi^\beta.$$  

(7.31)

(and analogous expressions for $\gamma \bar{\psi}_\mu$ and $\gamma \bar{\xi}$) where

$$\nabla_\mu \xi^\alpha = \partial_\mu \xi^\alpha - \frac{1}{2} \omega^\mu_\alpha \sigma_{ab} \bar{\psi}_\alpha \psi^\beta.$$  

The gauge algebra is open (it closes modulo the equations of motion for the gravitinos). Hence $\gamma$ is nilpotent only on-shell and does not agree with $s$ on all the fields.

One can choose the $U^a$’s in this case as

$$\{ U^a \} = \{ x^\mu, \partial_{(\mu_1 \ldots \mu_k)} e^a_{\mu_{k+1}}, \partial_{(\mu_1 \ldots \mu_k)} \omega^a_{\mu_{k+1}}, \partial_{(\mu_1 \ldots \mu_k)} \bar{\psi}_{(\mu_{k+1})}, \partial_{(\mu_1 \ldots \mu_k \bar{\psi}_{(\mu_{k+1})})} : c > d; \ k = 0, 1, \ldots \}.$$  

(7.32)

Note that the $\omega^a_{\mu\nu}$ correspond one by one to the antisymmetrized first order derivatives $\partial_{(\mu_1 \ldots \mu_k)} e^a_{\nu}$ of the vielbein fields due to (7.26). Hence, all the $U^a$ are indeed algebraically independent new local jet coordinates. The corresponding $V^a$ replace one by one the $dx^\mu$ and all the derivatives of the ghosts due to $\gamma e^a_\mu = \partial_\mu \xi^a + \ldots$ $\gamma \omega^a_{\mu\nu} = \partial_\mu C^{ab} + \ldots$, $\gamma \psi^a_\alpha = \partial_\mu \xi^a + \ldots$ and $\bar{\gamma} \bar{\psi}_\alpha = -\partial_\mu \bar{\xi}^a + \ldots$.

The undifferentiated ghosts give rise to the generalized connections

$$\{ \bar{\xi}^a : a > b \}, \  \xi^a = \xi^a, \ \tilde{\xi}^a = \xi^a, \ \tilde{\xi}^a = \tilde{\xi}^a, \ \bar{\xi}^a = \bar{\xi}^a,$$

(7.33)
with $\tilde{\xi}^\mu$ as in (7.13). The corresponding $\Delta$’s are denoted by

$$\{\Delta_N\} = \{\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}_\alpha, l_{ab} : a > b\},$$

$$\mathcal{D}_a = E_a^\mu \left( \partial_\mu - \frac{1}{2} \omega^a_{\beta\gamma} l^{\beta\gamma} - \psi_\alpha^a \mathcal{D}_\alpha + \bar{\psi}_\bar{\alpha} \bar{\mathcal{D}}_{\bar{\alpha}} \right)$$ (7.34)

where $l_{ab} = -l_{ba}$ denote the elements of the Lorentz algebra, and $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\bar{\alpha}}$ are supersymmetry transformations represented on the tensor fields given below (these tensor fields are ordinary fields, not superfields; accordingly $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\bar{\alpha}}$ are not ‘superspace operators’). The Grassmann parities of the $\Delta$’s are $\varepsilon_a = \varepsilon_{[ab]} = 0$, $\varepsilon_\alpha = \varepsilon_{\alpha\bar{\beta}} = 1$ (the supersymmetry ghosts commute). (7.34) indicates that in this case the vielbein fields are identified with the $v_\mu^m$, i.e. the indices $m$ coincide here with Lorentz vector indices,

$$v_\mu^m \equiv e_\mu^a, \quad V_m^\mu \equiv E_a^\mu.$$ (7.35)

Using the shorthand notation

$$\{\tilde{\xi}^A\} = \{\tilde{\xi}^a, \tilde{\xi}^\alpha, \tilde{\xi}^\bar{\alpha}\}, \quad \{\mathcal{D}_A\} = \{\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\bar{\alpha}}\}$$

the algebra of the $\mathcal{D}_A$ reads

$$[\mathcal{D}_A, \mathcal{D}_B] \approx -T_{AB}^C \mathcal{D}_C - \frac{1}{2} F_{AB}^{\ cd} l_{cd}$$ (7.36)

where the nonvanishing $T_{AB}^C$ and $F_{AB}^{\ cd}$ are

$$T_{a\dot{a}}^a = T_{\dot{a}a}^a = 2i \sigma_{a\dot{a}}^a,$$ (7.37)

$$T_{ab}^\alpha = E_a^\mu E_b^\nu (\nabla_\mu \psi_\nu^\alpha - \nabla_\nu \psi_\mu^\alpha),$$ (7.38)

$$F_{ab}^{\ cd} = i (T_{c[ab]} \sigma_{b\dot{a}\dot{a}} - 2 \sigma_{[c\dot{a}]} T_{b\dot{a}]}^{\dot{a}}),$$ (7.39)

$$F_{ab}^{\ cd} = R_{ab}^{\ cd} + 2 (\bar{\psi}_[a|\dot{a} F_{b]}^{\ cd} - \bar{\psi}_[a|\dot{a} F_{b]}^{\ cd})$$ (7.40)

and analogous expressions for $T_{ab}^{\ \dot{a}}$ and $F_{ab}^{\ cd}$. The remaining commutators of the $\Delta$’s are

$$[l_{ab}, \mathcal{D}_A] = -g_{[ab]} A^B \mathcal{D}_B, \quad [l_{ab}, l_{cd}] = 2 \eta_{[a|c|d]} \eta_{b]} - 2 \eta_{[a|c|d]} \eta_{b]}$$ (7.41)

where

$$g_{[ab]} c^d = 2 \eta_{[a|\delta]} c^d, \quad g_{[ab]} c^\beta = \sigma_{ab} c^\beta, \quad g_{[ab]} c^\dot{a} = -\sigma_{ab} c^\dot{a}. \quad (7.42)$$

Accordingly the generalized Russian formulae (6.5) read in this case

$$\tilde{\gamma} \tilde{\xi}^A = \frac{1}{2} \tilde{C}^{ab} [g_{[ab]} B^{\dot{A}} \tilde{\xi}^B - \frac{1}{2} (-)^{\varepsilon_B} \tilde{C}^{CD} \tilde{\xi}^C T_{CB}^A],$$ (7.43)

$$\tilde{\gamma} \tilde{C}^{ab} = -\tilde{C}^{ac} \tilde{C}_c^{\ dot{b}} - \frac{1}{2} (-)^{\varepsilon_D} \tilde{\xi}^{DE} \tilde{\xi}^C F_{CD}^{\ ab}. \quad (7.44)$$

Note that these identities encode all the equations (7.26)–(7.31), (7.38) and (7.40). The set of independent tensor fields consists in this case of a subset of $F_{ab}^{\ cd}$, $T_{ab}^\alpha$, $T_{ab}^{\ \dot{a}}$ and their covariant derivatives,

$$\{T^i\} \subset \{\mathcal{D}_{a1} \ldots \mathcal{D}_{ak} F_{bc}^{\ de}, \mathcal{D}_{a1} \ldots \mathcal{D}_{ak} T_{bc}^\alpha, \mathcal{D}_{a1} \ldots \mathcal{D}_{ak} T_{bc}^{\ \dot{a}} : k = 0, 1, \ldots \}. \quad (7.45)$$
Remark:

Notice that the formalism provides ‘super-covariant’ tensor fields and, in particular, ‘super-covariant’ derivatives \((\ref{eq:7.34})\) containing the gravitino and the supersymmetry transformations. Note also that these tensor fields do not carry “world indices” \(\mu, \nu, \ldots\), in contrast to the example discussed in the previous subsection. The reason is that the undifferentiated vielbein fields count among the \(U\)’s. Indeed, the corresponding \(V\)’s replace all the first order derivatives of the diffeomorphism ghosts \(\xi^\mu\) and therefore the BRST transformation of a tensor field must not involve \(\partial_\nu \xi^\mu\). Hence, tensor fields are indeed ‘world scalars’ in this case. One could of course instead count the undifferentiated vielbein fields also among the tensor fields and promote the \(\partial_\nu \xi^\mu\) to generalized connections. Then tensor fields could also carry world indices and one would get additional \(\Delta\)’s generating \(GL(4)\) transformations of world indices, as in the metric formulation of gravity discussed in the previous subsection. However, such a choice would not correspond to a maximal set of trivial pairs and would thus complicate unnecessarily the analysis of the BRST cohomology!

### 7.4 Two dimensional sigma models

Consider two dimensional sigma models whose set of classical fields consists of scalar fields \(\phi^i\) and the two dimensional metric fields \(g_{\mu\nu}\) and whose gauge symmetries are given by two dimensional diffeomorphism and Weyl invariance, with corresponding ghosts \(\xi^\mu\) and \(C\) respectively. The BRST transformations of the fields read

\[
\begin{align*}
sg_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + (\partial_\mu \xi^\rho) g_{\rho\nu} + (\partial_\nu \xi^\rho) g_{\mu\rho} + C g_{\mu\nu}, \\
Y &= \xi^\nu \partial_\nu Y \text{ for } Y \in \{\phi^i, \xi^\mu, C\}.
\end{align*}
\]  

(7.46)

Following closely the lines (but not the notation) of \cite{33} I first introduce new local jet coordinates \(h, \bar{h}, e, \eta, \bar{\eta}\) replacing the undifferentiated metric components and diffeomorphism ghosts \((h, \bar{h})\) are “Beltrami variables”\(^8\)]

\[
\begin{align*}
h &= \frac{g_{11}}{g_{12} + \sqrt{g}}, \quad \bar{h} = \frac{g_{22}}{g_{12} + \sqrt{g}}, \quad e = \sqrt{g}, \\
\eta &= (\xi^1 + dx^1) + \bar{h}(\xi^2 + dx^2), \quad \bar{\eta} = (\xi^2 + dx^2) + h(\xi^1 + dx^1)
\end{align*}
\]  

(7.47)

with \(g = - \det(g_{\mu\nu}) > 0\). The \(U\)’s are

\[
\{U^\xi\} = \{x^\mu, \partial^p \bar{\partial}^q h, \partial^p \bar{\partial}^q \bar{h}, \partial^p \bar{\partial}^q e : p, q = 0, 1, \ldots\}
\]  

(7.49)

where

\[
\partial \equiv \partial_1, \quad \bar{\partial} \equiv \partial_2.
\]  

(7.50)

Hence, in this case all the metric components and all their derivatives occur in trivial pairs. The corresponding \(V\)’s replace one by one the \(C, \partial \bar{\eta}, \partial \eta\), all their derivatives, and the \(dx^\mu\). Therefore one gets in this example an infinite set of generalized connections, given by \(\eta, \bar{\eta}\) and their remaining derivatives,

\[
\begin{align*}
\{\bar{\xi}^N\} &= \{\eta^p, \bar{\eta}^\bar{p} : p, \bar{p} = -1, 0, 1, \ldots\}, \\
\eta^p &= \frac{1}{(p+1)!} \partial^{p+1} \eta, \quad \bar{\eta}^\bar{p} = \frac{1}{(p+1)!} \bar{\partial}^{\bar{p}+1} \bar{\eta}.
\end{align*}
\]  

(7.51)

(7.52)

\(^8\)This change of jet coordinates is not globally well-defined in general.
(7.46)–(7.48) imply $\tilde{s}\eta = \eta \partial \eta$ and $\tilde{s}\bar{\eta} = \bar{\eta} \partial \bar{\eta}$. Therefore (6.5) reads in this case

$$\tilde{s}\eta^p = \frac{1}{2} \sum_{r=-(p+1)}^{p+1} (p-2r)\eta^r \eta^{p-r}$$

(7.53)

and an analogous formula for $\tilde{s}\bar{\eta}^\bar{p}$. The infinite set of $\Delta$’s corresponding to the $\tilde{C}$’s is denoted by

$$\{\Delta_N\} = \{L_p, \tilde{L}_\beta : p, \bar{\beta} = -1, 0, 1, \ldots\}.$$

(7.54)

Recall that the r.h.s. of (7.53) contains the structure functions occurring in the algebra of the $\Delta$’s. In this case all of these functions are constant and the algebra of the $L$’s and $\tilde{L}$’s is isomorphic to two copies of the algebra of regular vector fields $(-z^{p+1})\partial/\partial z$,

$$[L_p, L_q] = (p-q)L_{p+q}, \quad [L_{\bar{\beta}}, \tilde{L}_q] = (\bar{\beta}-\bar{q})\tilde{L}_{\bar{\beta}+\bar{q}}, \quad [L_p, \tilde{L}_\beta] = 0.$$

(7.55)

The set of tensor fields on which this algebra is realized is given by

$$\{T^1\} = \{T^i_{p,\bar{\beta}} : p, \bar{\beta} = 0, 1, \ldots\}, \quad T^i_{p,\bar{\beta}} = (L_{-1})^p (\tilde{L}_{-1})^{\bar{p}} \varphi^i.$$

(7.56)

The explicit form of the $T^i_{p,\bar{\beta}}$ in terms of the fields and their derivatives was discussed in [33] and will be rederived below for the first few $T$’s. The algebraic representation of the $L$’s and $\tilde{L}$’s on the tensor fields can be derived from the algebra (7.56) using $L_p T^i_{0,0} = \tilde{L}_\beta T^i_{0,0} = 0 \forall p, \bar{\beta} \geq 0$. The latter follows from the identification $\tilde{s}T^i_{0,0} = \tilde{C}^N \Delta_N T^i_{0,0}$; cf. (6.16). This yields

$$q < p : \quad L_q T^i_{p,\bar{\beta}} = \frac{p!}{(p-q)!} T^i_{p-q,\bar{\beta}}; \quad q \geq p : \quad L_q T^i_{p,\bar{\beta}} = 0$$

(7.57)

and analogous formulae for $\tilde{L}_q T^i_{p,\bar{\beta}}$.

Let us now make contact with section 4. Using (7.48) one easily reads off from (7.52) the connection forms $\mathcal{A}^p$ and $\tilde{\mathcal{A}}^{\bar{p}}$ contained in $\eta^p$ and $\bar{\eta}^{\bar{p}}$:

$$\mathcal{A}^p = \delta^p_{-1} dx^1 + H^p dx^2, \quad \tilde{\mathcal{A}}^{\bar{p}} = \delta^{\bar{p}}_{-1} dx^2 + \bar{H}^{\bar{p}} dx^1,$$

$$H^p = \frac{1}{(p+1)!} \partial^{p+1} h, \quad \bar{H}^{\bar{p}} = \frac{1}{(\bar{p}+1)!} \bar{\partial}^{\bar{p}+1} \bar{h}.$$

(7.58)

The components of $\mathcal{A}^{-1}$ and $\tilde{\mathcal{A}}^{-1}$ are identified with the $v^m_\mu$ according to

$$\{v^m_\mu\} \equiv \{\mathcal{A}^{-1}_\mu, \tilde{\mathcal{A}}^{-1}_\mu\}.$$

(7.60)

Explicitly one thus gets in matrix form

$$(v^m_\mu) = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}, \quad (V^m_\mu) = \frac{1}{1-hh} \begin{pmatrix} 1 & -\bar{h} \\ -\bar{h} & 1 \end{pmatrix}.$$

(7.61)

Due to (7.60) the covariant derivatives $\{\mathcal{D}_m\} \equiv \{\mathcal{D}, \tilde{\mathcal{D}}\}$ are identified with $L_{-1}$ and $\tilde{L}_{-1}$. (6.24) yields now

$$\mathcal{D} = \frac{1}{1-hh} \left( \partial - h \bar{\partial} - \sum_{\bar{p} \geq 0} \bar{H}^{\bar{p}} \tilde{L}_{\bar{\beta}} + h \sum_{p \geq 0} H^p L_p \right).$$

(7.62)
and an analogous expression for $\bar{D}$, with $H$'s as in (7.59). Thanks to (7.57), the occurrence of infinitely many $L$'s and $L$'s in $D$ and $\bar{D}$ does not result in nonlocal expressions (tensor fields). Note that $D \equiv L_{-1}$ and $\bar{D} \equiv L_{-1}$ commute according to (7.55), i.e. in this case all the (infinitely many) curvatures (5.28) vanish! Due to (7.56) the set of tensor fields is given by the $\varphi^i$ and all their covariant derivatives. The latter may now be constructed recursively using (7.57). With $\varphi$ to the solution of $\tilde{\varphi}$ gauge invariant actions, conserved currents and anomalies.

In this section some conclusions are drawn about the geometric structure of the solutions of the cohomological problem and the related physical quantities such as gauge invariant actions, conserved currents and anomalies.

According to section 4 the cohomological problem in question can be reduced to the solution of $\tilde{\varphi}_0 \approx 0$ where $\tilde{\varphi}_0$ does not depend on antifields. This cocycle condition decomposes into the “weak descent equations”

$$d\alpha_M \approx 0, \quad \gamma\alpha_p + d\alpha_{p-1} \approx 0 \quad \text{for } p_0 < p \leq M, \quad \gamma\alpha_{p_0} \approx 0 \quad (8.1)$$

where $\alpha_p$ denotes the $p$-form contained in $\tilde{\varphi}_0$,

$$\tilde{\varphi}_0 = \sum_{p=p_0}^M \alpha_p, \quad M = \min\{D, G\}, \quad G = \text{totdeg}(\tilde{\varphi}_0). \quad (8.2)$$

Here $M = \min\{D, G\}$ holds because $\tilde{\varphi}_0$ does not contain antifields \footnote{I also assume, without loss of generality, that $\tilde{\varphi}_0$ does not depend on antighosts or Nakanishi–Lautrup fields (cf. section 5) and that it has a definite total degree.} and thus involves only $p$-forms with $p \leq G$. Note that the first equation in (8.1) is trivially satisfied if $M = D$, i.e. if $G \geq D$.

Now, the only local forms which are weakly $d$-closed but not necessarily (locally) weakly $d$-exact are volume forms and forms which do not depend on the ghosts, i.e. all other weakly $d$-closed local forms are also weakly $d$-exact,

$$d\alpha_q \approx 0, \quad q < D, \quad gh(\alpha_q) > 0 \quad \Rightarrow \quad \alpha_q \approx d\eta_{q-1} \quad (8.3)$$

This follows by means of the algebraic Poincaré lemma (cf. section 3) immediately from a general result on the relative cohomology of $\delta$ and $d$ derived in [35]. Since all the local forms $\alpha_p$ with $p < M$ occurring in (8.2) have positive ghost number, one can analyse the weak descent equations (8.1) by means of (8.3) like the usual descent equations by means of the algebraic Poincaré lemma in section 3. This leads to the conclusion that $\tilde{\varphi}_0$ is a nontrivial solution of $\tilde{\varphi}_0 \approx 0$ if and only if its part $\alpha_M$ fulfills

$$G > D: \quad \gamma\alpha_D + d\alpha_{D-1} \approx 0, \quad \alpha_D \neq \gamma\eta_D + d\eta_{D-1}; \quad (8.4)$$

$$G = D: \quad \gamma\alpha_D + d\alpha_{D-1} \approx 0, \quad \alpha_D \neq d\eta_{D-1}; \quad (8.5)$$

$$G < D: \quad d\alpha_G \approx 0, \quad \alpha_G \neq d\eta_{G-1} + \text{constant.} \quad (8.6)$$

8 Structure of the solutions

In this section some conclusions are drawn about the geometric structure of the solutions of the cohomological problem and the related physical quantities such as gauge invariant actions, conserved currents and anomalies.

According to section 4 the cohomological problem in question can be reduced to the solution of $\tilde{\varphi}_0 \approx 0$ where $\tilde{\varphi}_0$ does not depend on antifields. This cocycle condition decomposes into the “weak descent equations”

$$d\alpha_M \approx 0, \quad \gamma\alpha_p + d\alpha_{p-1} \approx 0 \quad \text{for } p_0 < p \leq M, \quad \gamma\alpha_{p_0} \approx 0 \quad (8.1)$$

where $\alpha_p$ denotes the $p$-form contained in $\tilde{\varphi}_0$,

$$\tilde{\varphi}_0 = \sum_{p=p_0}^M \alpha_p, \quad M = \min\{D, G\}, \quad G = \text{totdeg}(\tilde{\varphi}_0). \quad (8.2)$$

Here $M = \min\{D, G\}$ holds because $\tilde{\varphi}_0$ does not contain antifields \footnote{I also assume, without loss of generality, that $\tilde{\varphi}_0$ does not depend on antighosts or Nakanishi–Lautrup fields (cf. section 5) and that it has a definite total degree.} and thus involves only $p$-forms with $p \leq G$. Note that the first equation in (8.1) is trivially satisfied if $M = D$, i.e. if $G \geq D$.

Now, the only local forms which are weakly $d$-closed but not necessarily (locally) weakly $d$-exact are volume forms and forms which do not depend on the ghosts, i.e. all other weakly $d$-closed local forms are also weakly $d$-exact,

$$d\alpha_q \approx 0, \quad q < D, \quad gh(\alpha_q) > 0 \quad \Rightarrow \quad \alpha_q \approx d\eta_{q-1} \quad (8.3)$$

This follows by means of the algebraic Poincaré lemma (cf. section 3) immediately from a general result on the relative cohomology of $\delta$ and $d$ derived in [35]. Since all the local forms $\alpha_p$ with $p < M$ occurring in (8.2) have positive ghost number, one can analyse the weak descent equations (8.1) by means of (8.3) like the usual descent equations by means of the algebraic Poincaré lemma in section 3. This leads to the conclusion that $\tilde{\varphi}_0$ is a nontrivial solution of $\tilde{\varphi}_0 \approx 0$ if and only if its part $\alpha_M$ fulfills

$$G > D: \quad \gamma\alpha_D + d\alpha_{D-1} \approx 0, \quad \alpha_D \neq \gamma\eta_D + d\eta_{D-1}; \quad (8.4)$$

$$G = D: \quad \gamma\alpha_D + d\alpha_{D-1} \approx 0, \quad \alpha_D \neq d\eta_{D-1}; \quad (8.5)$$

$$G < D: \quad d\alpha_G \approx 0, \quad \alpha_G \neq d\eta_{G-1} + \text{constant.} \quad (8.6)$$
Furthermore one concludes, using (8.3) again, that all solutions of (8.4)–(8.6) can be completed to nontrivial solutions of \( \tilde{\omega}_0 \approx 0 \). Hence, the complete local BRST cohomology is in fact (locally) isomorphic to the cohomological problems established by (8.4)–(8.7).

Let me now discuss the implications of sections 5 and 6 for the structure of the solutions of (8.4)–(8.6) and briefly comment on their physical interpretation. For notational convenience I will restrict this discussion to the case of an irreducible gauge algebra. Using the notation of section 6, one can then assume \( \tilde{\omega}_0 \) to be of the form

\[
\tilde{\omega}_0 = \tilde{C}^{N_1} \cdots \tilde{C}^{N_G} a_{NG \ldots N_1}(T), \quad \tilde{C}^N = \tilde{C}^N + A^N.
\]

(8.7)

This implies that, in irreducible gauge theories, the general solutions of (8.4)–(8.6) are of the form

\[
G > D : \quad \alpha_D = A^{N_1} \cdots A^{ND} \tilde{C}^{ND+1} \cdots \tilde{C}^{NG} a_{NG \ldots N_1}(T) ;
\]

(8.8)

\[
G = D : \quad \alpha_D = A^{N_1} \cdots A^{ND} a_{NG \ldots N_1}(T) ;
\]

(8.9)

\[
G < D : \quad \alpha_G = A^{N_1} \cdots A^{NG} a_{NG \ldots N_1}(T) ,
\]

(8.10)

up to trivial and “topological” (= locally but not globally trivial) solutions, of course. (8.8) applies for \( G = D + 1 \) to the antifield independent part of integrands of candidate gauge anomalies. Well-known examples are representatives of chiral anomalies in Yang–Mills theory \( \ddagger \). Their integrands have indeed the form (8.8) (recall that in Yang–Mills theory the differentials count among the connection forms, cf. subsection 7.1).

Solutions of (8.5) give rise to BRST invariant functionals with ghost number 0 and thus (8.9) applies to integrands of gauge invariant actions and their continuous first order deformations \( \ddagger \). However, concerning these solutions a few more remarks are in order which I postpone to the next section.

The solutions of (8.6) provide the local conservation laws of the theory. They correspond for \( G = D - 1 \) one-to-one to the nontrivial conserved currents\( ^{10} \) and generalize for smaller \( G \) the concept of nontrivial conserved currents to form degrees \( G < D - 1 \) \( \ddagger \).

We conclude that all nontrivial “dynamical” conserved local \( G \)-forms can be written in the form (8.10) if the gauge algebra is irreducible, and in a similar form, involving possibly connection forms of higher form degree, if the gauge algebra is reducible (“topological” conserved local forms cannot always be cast in this form). In fact, in “normal” theories, dynamical solutions of (8.6) exist at most at form degrees \( G \geq D - (2 + r) \) where \( r \) denotes the reducibility order of the theory, see \( \ddagger \) \( (r = -1 \) for theories without gauge invariance, \( r = 0 \) for irreducible gauge theories, \( \ldots \) ). The weak \( d \)-cohomology established by (8.6) goes sometimes under the name “characteristic cohomology” \( \ddagger \).

To illustrate the result on the conservation laws I consider the Noether current corresponding to the invariance of the supergravity action (7.22) under global \( U(1) \)-transformations of the gravitino. One finds for this Noether current \( j^\mu \) and the

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\( ^{10} \)A conserved current \( j^\mu \) (\( \partial_\nu j^\mu \approx 0 \)) is called trivial in this context if \( j^\mu \approx \partial_\nu S^{\nu \mu} \) holds for some local \( S^{\nu \mu} = -S^{\mu \nu} \).
corresponding solution $\alpha_3$ of (8.6)

\[ j^\mu = -2ie^{\mu\nu\rho\lambda}\bar{\psi}_\nu \sigma_\rho \psi_\lambda , \quad (8.11) \]

\[ \alpha_3 = \frac{1}{6} dx^\mu dx^\nu dx^\rho \epsilon^{\mu\nu\rho\lambda} \bar{\psi}_\lambda = 2ie^a \psi_\mu \bar{\psi}_\mu , \quad (8.12) \]

where $\epsilon^{0123} = -\epsilon_{0123} = 1$ and

\[ e^a = dx^\mu e^a_\mu , \quad \psi^a = dx^\mu \psi^a_\mu , \quad \bar{\psi}^\alpha = -dx^\mu \bar{\psi}^\alpha_\mu . \quad (8.13) \]

$\alpha_3$ is indeed of the form (8.10) because the 1-forms (8.13) are among the connection forms $A^N$ of the supergravity theory, cf. subsection 7.3. Note that this solution of (8.6) is constructed solely out of connection forms, i.e. it does not involve tensor fields at all!

Remark:

The usual construction of Noether currents does not always provide the corresponding solutions of (8.6) directly in the form (8.10). The statement here is that one can always redefine the Noether currents by subtracting trivial currents (if necessary) such that the corresponding $(D-1)$-forms take the geometric form (8.10).

A famous example for such a redefined current is the “improved” energy momentum tensor in Yang–Mills theory.

9 Structure of gauge invariant actions

The field-antifield formalism is usually constructed starting from a given gauge invariant classical action. One may then ask whether it is possible to deform this action without destroying the gauge invariance. This question is relevant for instance in the quantum theory where deformations of the action can be caused by quantum corrections, or for the deformation of free gauge theories to interacting ones. The BRST cohomology provides a powerful tool to tackle these problems [6].

One may distinguish two kinds of deformations of a given action: those which do not change the gauge transformations up to local field redefinitions, and those which modify simultaneously the action and the gauge transformations in a nontrivial way.

The integrands (volume forms) of actions which are invariant under given gauge transformations have to satisfy

\[ \gamma \alpha_D + d\alpha_{D-1} = 0 , \quad \alpha_D \neq d\eta_{D-1} , \quad \alpha_D = d^D x a(x, [\phi]) \quad (9.1) \]

where $\gamma$ encodes the gauge transformations under study. Note that (9.1) is a stronger condition than (8.5) and replaces the latter for two reasons: the integrands of gauge invariant actions are (i) required to be strictly $\gamma$-invariant up to a total derivative and (ii) not necessarily to be considered as trivial if they are weakly zero up to a total derivative – for instance one would not call the Einstein–Hilbert action $\int d^4x \sqrt{-g}R$ trivial even though its integrand is weakly zero.

Now, if the gauge algebra is (off-shell) closed, $\gamma$ is strictly nilpotent on all the fields (but not necessarily on the antifields). Therefore (9.1) implies descent equations for $\gamma$ and $d$ which do not involve antifields and read in a compact form $\bar{\gamma} \bar{\omega} = 0 , \bar{\omega} \gamma = \bar{\gamma} \eta$.

This problem can be analysed like the weak $\gamma$-cohomology in sections 5 and 6 – all...
the arguments go through also for strict instead of weak equalities since $\gamma$ and $\tilde{\gamma}$ are strictly nilpotent on all the fields. In particular we conclude that the general solution of (9.1) has again the form (8.9) (up to $d$-exact contributions) if the gauge algebra is closed and irreducible. The general solution of (9.1) provides the most general action which is invariant under a given set of gauge transformations encoded in $\gamma$. It has been determined by means of the BRST cohomology for Yang–Mills theory [25], gravity [23], minimal $N=1, D=4$ supergravity [37, 28] (both in the old minimal formulation [38] and in the new minimal one [39]) and for the sigma models considered in subsection 7.4 [33]. One can check that in all these cases the integrand of the most general gauge invariant action can indeed be expressed in the geometric form (8.9) even though this is not completely obvious in all cases. For instance, written in this form the integrand of the supergravity action (7.22) reads

$$\alpha_4 = -\epsilon_{abcd} e^a e^b e^c \left( \frac{1}{48} e^d \mathcal{R} + \frac{i}{3} S \sigma^d \bar{\psi} - \frac{i}{3} \psi \sigma^d \bar{S} \right),$$

(9.2)

$$\mathcal{R} = F_{ab}{}^{ba}, \quad S_{a} = T_{ab}{}^\beta \sigma_{ab}{}^{\alpha \beta},$$

(9.3)

with $e^a$, $\psi$ and $\bar{\psi}$ as in (8.13), $T_{ab}{}^\alpha$ and $F_{ab}{}^{cd}$ as in (7.38) and (7.40), and $\epsilon_{0123} = -1$.

The determination of deformations of a given action which modify nontrivially the gauge transformations is more subtle. A method which allows to attack this problem systematically and is based on the BRST cohomology was outlined in [1]. The idea is to deform the solution of the master equation instead of the classical action itself. This has many advantages. In particular it shows that to first order in the deformation parameter the deformed action is required to be weakly invariant under the original (undeformed) $\gamma$. The integrand of this first order deformation thus has the form (8.9) up to weakly vanishing terms. However, in general an analogous statement does not apply to the terms of higher orders in the deformation parameter because these terms are not necessarily weakly $\gamma$-invariant.

10 Gauge covariance of the equations of motion

A direct corollary of lemma 5.2 is the gauge covariance of the equations of motion (cf. remark after the proof of that lemma). Indeed, lemma 5.2 implies that the classical equations of motion are equivalent to a set of weakly vanishing functions of those $W$’s with vanishing total degree. Since the latter are just the tensor fields (cf. section 6), we conclude:

**Lemma 10.1:** *The classical equations of motion in a gauge theory are gauge covariant in the sense that they are equivalent to a set of weakly vanishing functions of the tensor fields.*

This is of course well-known for standard gauge theories such as Yang–Mills theory and Einstein gravity where the Euler–Lagrange equations themselves turn out to be expressible solely in terms of the tensor fields. A less trivial check of lemma 10.1 can be performed for the supergravity action (7.22). Indeed one can verify that the corresponding equations of motion are equivalent to the following equations involving only the tensor fields (7.43):

$$\mathcal{R}_{ab} \approx 0, \quad S^{\alpha} \approx 0, \quad U_{\tilde{\alpha} \tilde{\beta}}{}^{\alpha} \approx 0$$

(10.1)
with $S^\alpha$ as in (9.3) and
\[ \mathcal{R}_{ab} = F_{acb}, \quad U_{\dot{\alpha}\dot{\beta}}^\alpha = T_{ab}^\alpha \sigma_{\dot{\alpha}\dot{\beta}}^ab. \] 

(10.2)

11 Discussion of $x$-dependence

This section is devoted to the discussion of a special aspect of the cohomological problem concerning the explicit dependence of the solution $s$ on the spacetime coordinates. In particular it is emphasized that the result of the cohomological analysis depends on whether it is carried out in the space of $x$-dependent or $x$-independent local forms. It is therefore important to make clear in every analysis of the BRST cohomology in which space one works and to be aware of the consequences of the chosen approach.

11.1 General remarks

In general the results of the cohomological analysis will depend in two respects on whether or not one considers the problem in the space of $x$-dependent local forms:

(a) some nontrivial representatives of the cohomology might be overlooked if one performs the cohomological analysis in the space of $x$-independent local forms; (b) solutions which are nontrivial in the space of $x$-independent local forms can become trivial in the space of $x$-dependent local forms.

It is important to realize that (b) applies also to theories which do not admit solutions of the cohomological problem depending nontrivially on the $x^\mu$ at all. An important subclass of theories with this property are those which are invariant under spacetime diffeomorphism. They will be discussed in the next subsection in this context.

Let me first illustrate (a) and (b) for the simple example of the free $D$-dimensional Maxwell action
\[ S_{\text{maxwell}} = \int d^Dx F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Important examples for $x$-dependent solutions are the Noether currents associated with the Lorentz invariance of $S_{\text{maxwell}}$. In “improved” (= gauge invariant) form these currents read
\[ j_{\text{lorentz}}^\mu = \lambda_{\rho\nu} x^\rho T^{\nu\mu}, \quad \lambda_{\rho\nu} = -\lambda_{\nu\rho} = \text{constant} \]
where $T^{\mu\nu} = F^{\rho\mu} F_\rho^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$ is the “improved” energy momentum tensor ($\eta^{\mu\nu}$ is the Minkowski metric). The $x$-dependence of $j_{\text{lorentz}}^\mu$ cannot be removed by subtracting trivial currents from it. This illustrates (a). To demonstrate (b) I consider the $x$-independent current
\[ j_{\text{triv}}^\mu = \lambda_\nu F^{\nu\mu}, \quad \lambda_\nu = \text{constant}. \]
It is clearly conserved too, $\partial_\mu j_{\text{triv}}^\mu \approx 0$, and the corresponding $(D - 1)$-form is nontrivial in the space of $x$-independent forms. However it is trivial in the space of $x$-dependent forms, as one has
\[ j_{\text{triv}}^\mu \approx \partial_\nu (\lambda_\rho x^\rho F^{\nu\mu}). \]
This reflects that the global symmetry of $S_{\text{maxwell}}$ which corresponds via Noether’s theorem to $j_{\mu}^{\text{triv}}$ in $\mathbb{R}^{3+1}$ is trivial too [7]: it is the shift symmetry $A_\mu \rightarrow A_\mu + \lambda_\mu$ and is trivial because it is just a gauge transformation with parameter $\lambda_\mu x^\mu$.

### 11.2 Implications of diffeomorphism invariance

As mentioned already, in diffeomorphism invariant theories (of the standard type) one can remove any explicit $x$-dependence locally from all the solutions of the cohomological problem by subtracting trivial solutions. This was observed and used first in [23] for the antifield independent BRST cohomology in standard gravity. It is instructive to see how this result arises naturally within the framework of section 5. Namely it follows simply from the fact that all the $x^\mu$ and $dx^\mu$ form trivial pairs $(\bar{U}_\ell, \bar{V}_\ell)$ as a direct consequence of diffeomorphism invariance. To see this, note first that $x^\mu$ and $dx^\mu$ indeed satisfy requirement (5.1),

$$\tilde{s} x^\mu = dx^\mu.$$  

Now, this is valid for any theory but does not imply in general that $x^\mu$ and $dx^\mu$ form a trivial pair because to that end (5.2) must hold as well. However, in contrast to other theories (such as Yang–Mills theory), one can usually fulfill this additional requirement in diffeomorphism invariant theories through a simple change of variables (jet coordinates): one just replaces the diffeomorphism ghosts $\xi^\mu$ with the combinations

$$\tilde{\xi}^\mu = \xi^\mu + dx^\mu.$$  

Indeed, in standard diffeomorphism invariant theories the $\tilde{s}$-transformation of all the fields and antifields depends on the $\xi^\mu$ and $dx^\mu$ only via $\tilde{\xi}^\mu$ for one has

$$sZ = \xi^\mu \partial_\mu Z + \ldots,$$

$$dZ = dx^\mu \partial_\mu Z$$

and thus $\tilde{s}Z = \tilde{\xi}^\mu \partial_\mu Z + \ldots$ for any field or antifield $Z$ (the nonwritten terms in $sZ$ do not contain undifferentiated $\xi$’s in standard diffeomorphism invariant theories). This reflects that the diffeomorphisms are encoded in the BRST operator through the Lie derivative along $\xi$ and implies that (i) $x^\mu$ and $dx^\mu$ indeed form a trivial pair and can thus be eliminated locally from the cohomology, i.e. the nontrivial solutions of $\tilde{s}\omega = 0$ can be chosen so as not to depend explicitly on the $x^\mu$ and to depend on $\xi^\mu$ and $dx^\mu$ only via $\tilde{\xi}^\mu$, (ii) on $x$-independent functions and local total forms respectively, $s$ and $\tilde{s}$ arise from each other through the replacements $\xi^\mu \leftrightarrow \tilde{\xi}^\mu$, i.e.

$$\tilde{s} = \rho \circ s \circ \rho^{-1}, \quad s = \rho^{-1} \circ \tilde{s} \circ \rho$$

where

$$\rho = \exp \left( dx^\mu \frac{\partial}{\partial \tilde{\xi}^\mu} \right), \quad \rho^{-1} = \exp \left( -dx^\mu \frac{\partial}{\partial \xi^\mu} \right).$$

In particular this implies the now well-known result, first derived in [23], that the descent equations go in standard diffeomorphism invariant theories always down to a BRST-invariant $x$-independent 0-form $\omega_0$, and that the “integration” of the descent

---

\[11\] Assuming again that antighosts and Nakanishi–Lautrup fields have been eliminated from the cohomological problem already, cf. section 5.
equations starting from such a 0-form is not obstructed and results in a solution
\[ \tilde{\omega} = \rho \omega_0 \] of \( \tilde{s} \tilde{\omega} = 0 \).

I stress however that this result is valid only in the space of \( x \)-dependent forms. Indeed, in the space of \( x \)-independent forms there are additional solutions “\( \tilde{\omega} \) times monomial of the \( dx^\mu \)” where \( \tilde{\omega} \) is an \( x \)-independent solution because in that space \( dx^\mu \) is an “\( \tilde{s} \)-singlet”. In particular it is not true that the descent equations go always down to a 0-form if one restricts the cohomological analysis to \( x \)-independent forms.

12 Conclusion

The framework proposed in this paper to analyse the local BRST cohomology is based on a few very simple ideas: (i) the formulation of the local BRST cohomology in the jet bundle approach, (ii) the mapping of the BRST cohomology to the cohomology of \( \tilde{s} = s + d \) and to its on-shell counterpart, the antifield independent weak cohomology of \( \tilde{\gamma} = \gamma + d \), (iii) the construction of contracting homotopies to eliminate certain jet coordinates, called trivial pairs, from the \( \tilde{s} \)-cohomology.

In spite of its conceptual simplicity, (iii) is not straightforward because it requires the construction of an appropriate set of local jet coordinates splitting into two subsets one of which contains the trivial pairs whereas the other one consists of complementary jet coordinates which are required to generate an \( \tilde{s} \)- resp. \( \tilde{\gamma} \)-invariant subalgebra and are interpreted as tensor fields and generalized connections. The existence (and finding) of such complementary jet coordinates is a crucial prerequisite for the elimination of trivial pairs and was shown to be intimately related to a gauge covariant algebra. The construction of such jet coordinates has been illustrated for various examples to demonstrate the proposed method and its large range of applicability.

The outlined method simplifies the computation of the BRST cohomology considerably by reducing it locally to a cohomological problem involving only the tensor fields and generalized connections. The simplification does not only consist in the fact that some jet coordinates, the trivial pairs, are eliminated. On top of that, and equally important, one obtains a very compact and useful formulation of the remaining cohomological problem on tensor fields and generalized connections through equations such as (6.4)–(6.6). For specific models the compact formulation of the BRST algebra obtained in this way is in fact well-known in the literature. For instance (6.5) reproduces in the Yang–Mills case the celebrated “Russian formula” (1.1) which was used especially within the algebraic construction and classification of chiral anomalies [9, 22].

It should be remarked that this simplifies the computation of the BRST cohomology, but of course does not solve it. Nevertheless it allows remarkable conclusions about the ‘geometric’ structure and covariance properties of the solutions of the cohomological problem and the related physical quantities (Noether currents, gauge invariant actions, candidate gauge anomalies, etc.), as well as of the classical equations of motion, cf. sections 8–10.

Finally I remark that local jet coordinates with the mentioned properties are also useful when one needs to take global (topological) aspects into account which have been completely neglected in this paper. In particular, global obstructions to the
elimination of trivial pairs may be taken into account using K"unneth’s theorem à la

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A Fields, antifields and their jet space

The so-called minimal set of fields contains the ‘classical fields’ $\phi^i$ which occur in the gauge invariant classical action, the ghosts $C^\alpha$ corresponding one-to-one to the nontrivial gauge symmetries and the ghosts for ghosts $Q^\alpha_k$ of first and higher order $k = 1, \ldots, r$ where $r$ denotes the reducibility order of the theory ($r = 0$ for irreducible gauge theories),

$$\{\Phi^A\}_{\text{min}} = \{\phi^i, C^\alpha, Q^\alpha_k\}, \quad \{\Phi^*_A\}_{\text{min}} = \{\phi^*_i, C^*_\alpha, Q^*_\alpha_k\}. \quad (A.1)$$

In order to fix the gauge one usually extends the minimal set of fields to a nonminimal one by adding antighosts and Nakanishi–Lautrup auxiliary fields. Each field $\Phi^A$ has a definite Grassmann parity $\varepsilon(\Phi^A)$ and ghost number $\text{gh}(\Phi^A)$. The Grassmann parities and ghost numbers of the antifields are related to those of the fields according to

$$\text{gh}(\phi^i) = 0, \quad \text{gh}(C^\alpha) = 1, \quad \text{gh}(Q^\alpha_k) = k + 1,$$

$$\text{gh}(\Phi^*_A) = -\text{gh}(\Phi^A) - 1, \quad \varepsilon(\Phi^*_A) = \varepsilon(\Phi^A) + 1 \quad (\text{mod } 2). \quad (A.2)$$

The Grassmann parity of the classical fields is 0 for bosonic (commuting) fields and 1 for fermionic (anticommuting) fields, the Grassmann parity of the ghosts is opposite to the Grassmann parity of the corresponding gauge symmetry, and the Grassmann parity of the ghosts for ghosts is determined analogously. Fields and antifields commute or anticommute according to their Grassmann parities,

$$Z_1 Z_2 = (-)^{\varepsilon(Z_1)\varepsilon(Z_2)} Z_2 Z_1. \quad (A.3)$$

The fields and antifields and all their derivatives are considered as local coordinates of an infinite jet space. For this set of jet coordinates I use the collective notation

$$[\Phi, \Phi^*] \equiv \{\partial_{\mu_1\ldots\mu_k} \Phi^A, \partial_{\mu_1\ldots\mu_k} \Phi^*_A : k = 0, 1, \ldots\} \quad (A.4)$$

and these jet coordinates are regarded as independent apart from the identities

$$\partial_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_k} Z = \partial_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\mu_k} Z \quad \forall i, j. \quad (A.5)$$

The derivatives $\partial_\mu$ have vanishing Grassmann parity and ghost number. The set of local jet coordinates is completed by the spacetime coordinates $x^\mu$ and by the differentials $dx^\mu$ which are counted among the jet coordinates by convenience. The former have even, the latter odd Grassmann parity, both have vanishing ghost number.
The derivatives $\partial_\mu$ are defined as total derivative operators in the jet space according to
\[
\partial_\mu = \frac{\partial}{\partial x_\mu} + \sum_{k \geq 0; \nu_i \geq 0; \sum_{i+1} \geq \nu_i} (\partial_{\nu_1 \ldots \nu_k} Z_I) \frac{\partial}{\partial (\partial_{\nu_1 \ldots \nu_k} Z_I)} \tag{A.6}
\]
where $\{Z_I\} = \{\Phi^A, \Phi^*_A\}$. The sum in (A.6) runs only over those $\partial_{\nu_1 \ldots \nu_k} Z$ with $\nu_{i+1} \geq \nu_i$ because of the identities (A.5). It is further understood that
\[
\frac{\partial (\partial_{12} Z)}{\partial (\partial_{12} Z)} = \frac{\partial (\partial_{21} Z)}{\partial (\partial_{12} Z)} = 1 \text{ etc.} \tag{A.7}
\]

**B BRST operator**

The BRST operator is constructed from a solution $S$ of the master equation [5] of the form
\[
S = \int d^Dx \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{cl}(x, [\phi]) - (R^i_\alpha C^\alpha) \phi_i^* + \ldots \tag{B.1}
\]
where $\mathcal{L}_{cl}$ denotes the Lagrangian of the gauge invariant classical action and $R^i_\alpha C^\alpha$ is the gauge transformation of $\phi^i$ with gauge parameters replaced by the ghosts, i.e. $R^i_\alpha$ is an operator of the form
\[
R^i_\alpha (x, [\phi], \partial) = \sum_{k \geq 0} r^i_{\alpha_1 \ldots \alpha_k} (x, [\phi]) \partial_{\alpha_1} \ldots \partial_{\alpha_k}. \tag{B.2}
\]

If the gauge algebra is reducible with reducibility order $r$, then $\mathcal{L}$ is required to contain also a piece of the form
\[
C^*_\alpha Z_{\alpha_1}^{\alpha_1} Q^{\alpha_1} + \sum_{k=2}^r Q^*_\alpha Q_{\alpha k-1}^{\alpha k-1} Q^{\alpha_k}, \tag{B.3}
\]
where the $Z$’s are operators of the form (B.2) implementing the reducibility relations. For the purpose of gauge fixing one may also include pieces involving antighosts and Nakanishi–Lautrup auxiliary fields [5].

The BRST transformations of $\Phi^A$ and $\Phi^*_A$ are given by their antibrackets [5] with $S$ according to $s \cdot = (S, \cdot)$. This results in
\[
s \Phi^A = -\frac{\hat{\partial}^R \mathcal{L}}{\partial \Phi^*_A}, \quad s \Phi^*_A = \frac{\hat{\partial}^R \mathcal{L}}{\partial \Phi^A} \tag{B.4}
\]
where $\hat{\partial}^R \mathcal{L}/\hat{\partial} Z$ denotes the Euler–Lagrange right-derivative of $\mathcal{L}$ with respect to $Z$ (derivatives $\partial^R/\partial$ act from the right),
\[
\frac{\hat{\partial}^R \mathcal{L}}{\partial Z} = \sum_{k \geq 0; \mu_{i+1} \geq \mu_i} (-)^k \partial_{\mu_1} \ldots \partial_{\mu_k} \frac{\partial^R \mathcal{L}}{\partial (\partial_{\mu_1} \ldots \partial_{\mu_k} Z)}. \tag{B.5}
\]

The BRST transformations of derivatives of the fields and antifields are obtained from (B.4) simply by requiring $s \partial_\mu = \partial_\mu s$, i.e.
\[
s (\partial_\mu \Phi^A) = \partial_\mu (s \Phi^A) = -\partial_\mu \frac{\hat{\partial}^R \mathcal{L}}{\partial \Phi^*_A} \text{ etc.} \tag{B.6}
\]
References

[1] C. Becchi, A. Rouet and R. Stora, Phys. Lett. 52B (1974) 344; Commun. Math. Phys. 42 (1975) 127; Ann. Phys. 98 (1976) 287; I. V. Tyutin, Gauge invariance in field theory and statistical mechanics, Lebedev preprint FIAN, n°39 (1975) (unpublished).

[2] R. E. Kallosh, Zh. Eksp. Teor. Fiz. Pis’ma 26 (1977) 575; Nucl. Phys. B141 (1978) 141.

[3] G. Sterman, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D14 (1978) 1501.

[4] B. de Wit and J. W. van Holten, Phys. Lett. 79B (1978) 389.

[5] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. 102B (1981) 27; Phys. Rev. D28 (1983) 2567 (E: D30 (1984) 508).

[6] G. Barnich and M. Henneaux, Phys. Lett. 311B (1993) 123.

[7] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. 174 (1995) 57.

[8] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.

[9] R. Stora, Continuum Gauge Theories in: New Developments in Quantum Field Theory and Statistical Physics, eds. M. Levy and P. Mitter, NATO ASI Series B26 (Plenum, New York, 1977); Algebraic Structure and Topological Origin of Anomalies in: Progress in Gauge Field Theory, Cargese Lectures 1983, eds. G. ’t Hooft et al. (Plenum, New York, 1984); B. Zumino, Chiral Anomalies and Differential Geometry in: Relativity, Groups and Topology II, Les Houches Lectures 1983, eds. B. S. De Witt and R. Stora (North-Holland, Amsterdam, 1984); W. A. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421; B. Zumino, Y.S. Wu and A. Zee, Nucl. Phys. B239 (1984) 477; L. Baulieu, Nucl. Phys. B241 (1984) 557; Phys. Rep. 129 (1985) 1; J. Mañes, R. Stora and B. Zumino, Commun. Math. Phys. 102 (1985) 157.

[10] P.S. Howe, U. Lindström and P. White, Phys. Lett. 246B (1990) 430; W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, Nucl. Phys. B333 (1990) 727.

[11] O. Piguet and S.P. Sorella, Algebraic Renormalization, Lecture Notes in Physics Vol. m28 (Springer Verlag, Berlin, Heidelberg, 1995), and refs. therein.

[12] J. Gomis, J. París and S. Samuel, Phys. Rep. 259 (1995) 1, and refs. therein.

[13] J. Gomis and S. Weinberg, Nucl. Phys. B469 (1996) 473.

[14] L. Baulieu and J. Thierry-Mieg, Nucl. Phys. B197 (1982) 477.

[15] F. Brandt, Structure of BRS-invariant local functionals, preprint NIKHEF-H 93-21, hep-th/9310123 (unpublished).
[16] N. Dragon, *BRS symmetry and cohomology*, Saalburg Lectures, preprint ITP-UH–3/96, hep-th/9602163 (unpublished).

[17] G. Barnich, F. Brandt and M. Henneaux, *Phys. Rev. D51* (1995) 1435.

[18] G. Barnich, F. Brandt and M. Henneaux, *Nucl. Phys. B455* (1995) 357.

[19] G. Hirsch, *Bull. Soc. Math. Belg. B* 6 (1953) 79;
J. D. Stasheff, *Trans. Am. Math. Soc.* 108 (1963) 215, 293;
V. K. A. M. Gugenheim, *J. Pure Appl. Alg.* 25 (1982) 197;
V. K. A. M. Gugenheim and J. D. Stasheff, *Bull. Soc. Math. Belg. B* 38 (1986) 237.

[20] J. M. L. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, *Commun. Math. Phys.* 127 (1989) 379.

[21] J. M. L. Fisch and M. Henneaux, *Commun. Math. Phys.* 128 (1990) 627;
M. Henneaux, *Nucl. Phys. B (Proc. Suppl.)* 18A (1990) 47;
M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).

[22] M. Dubois-Violette, M. Talon and C. M. Viallet, *Phys. Lett. 158B* (1985) 231;
*Commun. Math. Phys. 102* (1985) 105.

[23] F. Brandt, N. Dragon and M. Kreuzer, *Nucl. Phys. B340* (1990) 187.

[24] M. Dubois-Violette, M. Henneaux, M. Talon and C. M. Viallet, *Phys. Lett. 289B* (1992) 361.

[25] F. Brandt, N. Dragon and M. Kreuzer, *Phys. Lett. 231B* (1989) 263; *Nucl. Phys. B332* (1990) 224.

[26] G. Barnich and M. Henneaux, *Phys. Rev. Lett.* 72 (1994) 1588.

[27] G. Barnich, F. Brandt and M. Henneaux, *Commun. Math. Phys.* 174 (1995) 93.

[28] F. Brandt, *Local BRST cohomology in minimal D=4, N=1 supergravity*, hep-th/9609192, to appear in *Ann. Phys. (N.Y.)*.

[29] I. M. Anderson, *The variational bicomplex* (Academic Press, Boston, 1994); *Contemp. Math.* 132 (1992) 51;
D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series 142 (Cambridge University Press, Cambridge, 1989).

[30] A. M. Vinogradov, *Sov. Math. Dokl.* 18 (1977) 1200, 19 (1978) 144, 19 (1978) 1220;
F. Takens, *J. Differential Geometry* 14 (1979) 543;
I. M. Anderson and T. Duchamp, *Amer. J. Math.* 102 (1980) 781;
M. De Wilde, *Lett. Math. Phys.* 5 (1981) 351;
W. M. Tulczyjew, *Lecture Notes in Math.* 836 (1980) 22;
P. Dedecker and W. M. Tulczyjew, *Lecture Notes in Math.* 836 (1980) 498;
T. Tsujishita, *Osaka J. of Math.* 19 (1982) 311;
L. Bonora and P. Cotta-Ramusino, *Commun. Math. Phys.* 87 (1983) 589;
P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, volume 107 (Springer Verlag, New York, 1986);
R. M. Wald, *J. Math. Phys.* 31 (1990) 2378;
M. Dubois-Violette, M. Henneaux, M. Talon and C. M. Viallet, *Phys. Lett.* 267B (1991) 81;
L. A. Dickey, *Contemp. Math.* 132 (1992) 307;
cf. also first ref. in [9].

[31] F. Brandt, M. Henneaux and A. Wilch, *Phys. Lett.* 387B (1996) 320; *Ward identities for rigid symmetries of higher order*, UB–ECM–PF 96/18, ULB–TH 96/17, hep-th/9611056 (unpublished).

[32] S. Vandoren and A. Van Proeyen, *Nucl. Phys.* B411 (1994) 257.

[33] F. Brandt, W. Troost and A. Van Proeyen, *Nucl. Phys.* B464 (1996) 353.

[34] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Phys. Rev.* D13 (1976) 3214;
D. Z. Freedman and P. van Nieuwenhuizen, *Phys. Rev.* D14 (1976) 912;
S. Deser and B. Zumino, *Phys. Lett.* 62B (1976) 335.

[35] M. Henneaux, *Commun. Math. Phys.* 140 (1991) 1.

[36] R. L. Bryant and P. A. Griffiths, *Characteristic Cohomology of Differential Systems (I): General Theory*, Duke University Mathematics Preprint Series, volume 1993 n01 (January 1993).

[37] F. Brandt, *Lagrangian densities and anomalies in four-dimensional supersymmetric theories*, doctoral thesis (in German), RX–1356 (unpublished, Hannover, 1991); *Class. Quantum Grav.* 11 (1994) 849.

[38] K. S. Stelle and P. C. West, *Phys. Lett.* 74B (1978) 330;
S. Ferrara and P. van Nieuwenhuizen, *Phys. Lett.* 74B (1978) 333.

[39] M. F. Sohnius and P. C. West, *Phys. Lett.* 105B (1981) 353.