Geometric Relationships Between Gaussian and Modulo-Lattice Error Exponents

Charles H. Swannack, Uri Erez, Member, IEEE, Gregory W. Wornell, Fellow, IEEE

Abstract

Lattice coding and decoding have been shown to achieve the capacity of the additive white Gaussian noise (AWGN) channel. This was accomplished using a minimum mean-square error scaling and randomization to transform the AWGN channel into a modulo-lattice additive noise channel of the same capacity. It has been further shown that when operating at rates below capacity but above the critical rate of the channel, there exists a rate-dependent scaling such that the associated modulo-lattice channel attains the error exponent of the AWGN channel. A geometric explanation for this result is developed. In particular, it is shown how the geometry of typical error events for the modulo-lattice channel coincides with that of a spherical code for the AWGN channel.

Index Terms
error exponents, Gaussian channels, lattice codes, lattice decoding, modulo-lattice channels

I. INTRODUCTION

THE capacity of the additive white Gaussian noise (AWGN) channel was analyzed by Shannon in 1948 in his foundational work [1]. In 1959 Shannon subsequently studied lower and upper bounds on the error exponent achieved by codes for this channel [2]. These bounds, while quite tedious to derive, relied on simple geometric arguments.

An alternative derivation of these results, which uses methods originally developed for general discrete memoryless channels (DMC), was later provided by Gallager in 1965 [3]. This derivation, while much simpler from an analytic standpoint lacked much of the geometry that was contained in Shannon’s original work. Further work by Shannon, Gallager and Berlekamp in 1967 [4], [5] provided a tighter upper bound on the reliability function for low rates, which was recently improved upon by Ashikhmin et al. [6].

The lower and upper bounds coincide for rates greater than the critical rate of the channel and therefore the error exponent is known for rates . These works further show that with (optimal) maximum likelihood (ML) decoding, the sphere-packing exponent can be achieved for by random spherical ensembles, i.e., by a code whose codewords are drawn uniformly over the surface of a sphere.

A different line of work aimed at developing structured codes for the AWGN channel using lattice codes was initiated by de Buda [2]. It was shown in [8] that the use of lattice codes in conjunction with lattice decoding can indeed achieve capacity on the AWGN channel. One of the key elements in the transmission scheme involves transforming the AWGN channel into an unconstrained modulo-lattice additive noise channel (as we describe in Section IV), having (asymptotically in the dimension of the lattice) the same capacity as the original channel. For the resulting channel, if one uses a lattice code such that the use of a lattice code that attains the error exponent of the AWGN channel. A geometric explanation for this result is developed. In particular, it is shown how the geometry of typical error events for the modulo-lattice channel coincides with that of a spherical code for the AWGN channel. Thus, MMSE scaling is a natural choice and indeed is unique if one aims for capacity [9].

We note that using a lattice code for transmission over the mod- channel does not incur any penalty in terms of error exponent of the mod- channel [3], i.e., the error probability (as measured by the best known lower bound on the error exponent) of a good (possibly randomly generated) lattice code is no greater than that of a random code [8] Thus, it suffices to study the error exponent of the mod- channel, which may be done by standard random coding arguments. Indeed, the error exponent of the mod- channel is interesting in its own right, as it plays a key role in other problems as well. For instance the error exponent of the mod- channel provides a lower bound on the error exponent of the dirty-paper channel [11] for arbitrarily strong interference [8].

This work was supported in part by the National Science Foundation under Grant No. CCF-0635191. This work was presented in part at the Allerton Conference on Communication, Control, and Computing, Monticello, IL, Sep. 2005.

U. Erez is with the Department of Electrical Engineering - Systems, Tel Aviv University, Ramat Aviv, 69978, Israel (Email: uri@eng.tau.ac.il).
C. H. Swannack and G. W. Wornell are with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 (Email: {swannack,gww}@mit.edu).

1The associated modulo-lattice channel is typically referred to as the mod- channel.
2This property is a counterpart to the sufficiency of linear codes for achieving the best known lower bounds on the error exponent of the binary symmetric channel [10].
It was further conjectured by the authors of [8] that the mod-$\Lambda$ transformation, while not incurring a loss in mutual information, does incur a loss in error exponent. Recently, however, Liu et al. [12] have shown that while MMSE scaling is not sufficient to obtain the error exponent for the mod-$\Lambda$ channel, a different scaling is nonetheless sufficient to obtain the random coding exponent. Through some quite rigorous computation, [12] shows that using a rate-dependent scaling, the sphere packing (i.e., optimal) error exponent can be achieved for rates exceeding the critical rate of the AWGN channel.

The goal of the present work is to provide a unifying geometrical framework for the derivation of the error exponents for both AWGN and mod-$\Lambda$ channels. We obtain a simple explanation for the results of [12], and in particular to the scaling that maximizes the error exponent of the mod-$\Lambda$ channel at high rates. We use geometric arguments in order to study the typical error events in both the AWGN and mod-$\Lambda$ channels. We start by analyzing random spherical codes and observe that the optimal mod-$\Lambda$ scaling occurs naturally in this context as well. We develop a simple geometrical picture of the relationship between the typical error events in the mod-$\Lambda$ and AWGN channels via identification of transmitted codewords.

At low rates the best known bounds for the error exponent are based on minimum distance arguments. The identification of transmitted codewords plays a key role in our development of the error exponent for the mod-$\Lambda$ in this regime, a region not explicitly characterized in [12]. We show that in this region there is a rate dependent scaling using which the error exponent of an ensemble of lattice codes matches the error exponent of an ensemble of spherical codes, provided that both code ensembles have been expurgated to meet the same minimum distance criterion. More precisely, the error exponent for an ensemble of (expurgated) spherical codes is equal to the error exponent for an ensemble of lattice codes provided that every code in each of these two ensembles have the same minimum distance. However, as the best known bound for the minimum distance of a spherical code exceeds that of a lattice, the resulting bound for the error exponent of the mod-$\Lambda$ channel is shown to be less than that in the AWGN channel. Therefore, at low rates, the lower bound for the error exponent of the mod-$\Lambda$ channel is less that that of the AWGN channel.

Another contribution of the present work is the derivation of exponentially tight bounds for the probability of a mixture of a spherical noise and AWGN noise leaving a sphere, which we require in our analysis. Beyond their use in this work, we believe they may also be useful in the analysis of other communication problems. In particular, it is known that quantization noise arising from a “good” high-dimensional quantizer behaves in much the same way as spherical noise [8]. Thus, the bounds derived in this work may be useful for the study of error probabilities in communication systems where both quantization and AWGN noise are present; see, e.g., [13] for a recent such application.

II. THE AWGN CHANNEL: CAPACITY AND ERROR EXponent

In the AWGN channel of interest, the received signal is

$$Y_i = X_i + Z_i, \quad i = 1, 2, \ldots, n,$$

where

$$\mathbf{x} = (X_1, X_2, \ldots, X_n)$$

is the transmitted signal of length $n$, which satisfies the power constraint $(1/n)||\mathbf{x}||^2 \leq P$, and where the noise $Z_1, \ldots, Z_n$ are independent, identically distributed (i.i.d.) Gaussian random variables with zero mean and variance $\sigma^2$. For $(n, R)$ codes, i.e., codebooks $\mathcal{C}$ of $2^{nR}$ codewords, each of length $n$, the largest rate $R$ such that vanishing error probability can be achieved is the channel’s capacity, which is given by

$$C = \frac{1}{2} \log(1 + SNR),$$

where $SNR = P/\sigma^2$ is the signal-to-noise (SNR) ratio on the channel.

There are many ways to generate random codebook ensembles in $n$-dimensional space that asymptotically achieve the capacity of the power constrained AWGN channel. Possible choices are: an i.i.d. Gaussian codebook, a codebook drawn uniformly over the interior of an $n$-dimensional sphere, a codebook drawn uniformly over the surface of that sphere, as well as a codebook drawn uniformly over the Voronoi region of a lattice that is “good for quantization” [8]. In essence, the codebook distribution should approach Gaussianity in a (Shannon) entropy sense, i.e., its entropy (for a given power) should be close to maximal.

A second-order figure-of-merit for a channel is the error exponent (or reliability function) of the channel, defined as

$$E(R) = \limsup_{n \to \infty} -\frac{\log P_e(n, R)}{n},$$

where $P_e(n, R)$ is the minimal value of the probability of error $P_e(\mathcal{C})$ over all $(n, R)$ codes $\mathcal{C}$, and where, in turn, $P_e(\mathcal{C})$ is the error probability of a given code $\mathcal{C}$ averaged over all codewords. The error exponent is more sensitive to the particular choice of codebook input distribution than the channel capacity.

The error exponent for the AWGN channel is still not known for all rates. For rates greater than the critical rate

$$R_{crit} = 1/2 \log \left( \frac{1}{2} + \frac{SNR}{4} + \frac{1}{2} \sqrt{1 + \frac{SNR^2}{4}} \right),$$
the error exponent for the AWGN channel is the sphere-packing error exponent $E_{sp}(R; \text{SNR})$. For rates less than $R_{crit}$, there are several known lower bounds, the best known of which is the maximum of the random coding error exponent and the expurgated error exponent.

Geometrically, the sphere-packing error exponent is the exponent of the probability that the received vector falls outside a cone with solid angle equal to the average solid angle of an ML decoding region (i.e., $\exp(-nR)$ times the surface area of the unit sphere). Recall that the sphere-packing error exponent of the channel (1) is

$$E_{sp}(R; \text{SNR}) = E_G(\beta_G, \rho_G; \text{SNR}),$$

(3)

where

$$E_G(\beta, \rho; \text{SNR}) \triangleq \frac{1}{2} \left[ (1 - \beta)(1 + \rho) + \text{SNR} + \rho \log \frac{\beta}{1 + \rho} \right] + \log \left( \beta - \frac{\text{SNR}}{1 + \rho} \right) - 2\rho R],$$

(4)

and with $\beta_G = e^{2R}$ and

$$\rho_G = \frac{\text{SNR}}{2\beta_G} \left( 1 + \sqrt{1 + \frac{4\beta_G}{\text{SNR}(\beta_G - 1)}} \right) - 1.$$  

(5)

As is well known, a Gaussian codebook does not achieve the error exponent of the channel, due to the impact of atypical codewords. As shown in [2], [3] the error exponent $E_{sp}(R; \text{SNR})$ is achieved by a codebook drawn uniformly over the surface of a sphere. We now re-derive the sphere-packing error exponent using simple geometric arguments in a way that highlights the relationship between the typical error events in the AWGN and mod-$\Lambda$ channels.

### III. Geometric Derivation of Sphere-Packing Exponent

In the sequel, we use random coding arguments to bound the probability of error. We use $\Omega_0^{(n)}$ to denote the ensemble of codes for which codewords are drawn independently and uniformly from the surface of a sphere, and use $\bar{c}$ to denote any codebook drawn from $\Omega_0^{(n)}$. We use $c$ to denote the transmitted codeword, and denote any other codeword by $c_e$, so that $\bar{c} = \{c\} \cup \{\cup_e c_e\}$. With $y = c + z$ denoting the received vector, we have than an error occurs under ML decoding when

$$\|y - c_e\| \leq \|y - c\|$$  

(6)

for some codeword $c_e \in \bar{c} \setminus c$. The error probability $P_e(c)$ given that the message $c$ is transmitted is then

$$P_e(c) = \mathbb{P} \{\|y - c_e\| \leq \|y - c\|, \text{ for some } c_e \in \bar{c} \setminus c\}$$

$$= \mathbb{P} \{y \notin \mathcal{R}_{ML}(c)\},$$

In [3] Gallager starts with a Gaussian distribution but applies expurgation to the same effect.
where
\[ \mathcal{R}_{\text{ML}}(c) = \{ x : \| x - c \| < \| x - c_e \| \text{ for all } c_e \neq c \} \]
is the ML decoding region of the codeword.

We denote by \( \overline{P}_e \) the average of \( P_e(\Omega) \) over the codebook ensemble \( \Omega^{(n)} \).

The sphere-packing (or in the present context “cone-packing”) lower bound on the probability of error is straightforward to derive. Indeed let \( \mathcal{R}_c(c) \) denote a cone with apex at the origin and axis passing through \( c \) such that its volume is equal to that of \( \mathcal{R}_{\text{ML}}(c) \). Then it is easy to show that
\[
P_e(c) = P \{ y \notin \mathcal{R}_{\text{ML}}(c) \} \\
\geq P \{ y \notin \mathcal{R}_c(c) \}.
\]
In effect, the ML decoding region cannot be better than a cone with equal volume. It further follows by convexity that, for any codebook,
\[
P_e(\Omega) \geq P \{ y \notin \mathcal{R}_c(c) \},
\]
where the cone \( \mathcal{R}_c(c) \) (with apex at the origin and axis running through \( c \)) has volume equal to the average of the volumes of the ML decoding regions.

The bound \( (7) \) is the well known sphere-packing bound and when evaluated explicitly (as will be done below) yields the expression \( (3) \). Because the sphere-packing bound is tight at sufficiently high rates, the ML decoding region may be well approximated (as far as error probability goes) by a cone. To establish the tightness of the bound we next turn to upper bounding the probability of error.

A. Gallager’s Bounding Technique

In the following sections we use the method due to Gallager \[15\] to bound the probability of decoding error. Recall that in general the probability of error can be upper bounded by considering only pairwise errors for all codewords. More precisely, the union bound yields
\[
P_e(c) \leq \sum_{c_e \neq c} \mathbb{P} ( \| y - c_e \| \leq \| y - c \| ) \triangleq P_{\text{union}}(c).
\]

While the union bound is tight for low rates (i.e., when the rate is kept fixed while the SNR and hence the capacity approach infinity) we require a more general bound in the sequel. Toward this end, let \( \mathcal{R}(c) \) be a region in \( \mathbb{R}^n \). Then, more generally, one can bound the probability of decoding error for a transmitted codeword \( c \) considering separately the probability of error when the received vector is and is not in \( \mathcal{R}(c) \). When \( y \in \mathcal{R}(c) \), we upper bound the probability of error by using a refined union bound over all codewords in the codebook, where the noise is bounded to lie within the region \( \mathcal{R}(c) - c \). When \( y \notin \mathcal{R}(c) \), we upper bound the probability of error by 1. Specifically, we have, in general,
\[
P_e(c) = \mathbb{P} ( \text{error}, c + z \in \mathcal{R}(c)) \\
+ \mathbb{P} ( \text{error}, c + z \notin \mathcal{R}(c)) \\
\leq \mathbb{P} ( \text{error}, c + z \in \mathcal{R}(c)) \\
+ \mathbb{P} ( c + z \notin \mathcal{R}(c)) \\
\leq P_{\text{region}}(c) + P_{\text{region}}(c)
\]
where
\[
P_{\text{region}}(c) \triangleq \mathbb{P} ( c + z \notin \mathcal{R}(c))
\]
and
\[
P_{\text{union}}^r(c) \triangleq \sum_{c_e \neq c} \mathbb{P} ( \| y - c_e \| \leq \| y - c \|, c + z \in \mathcal{R}(c)).
\]

As shown in the following sections, with a proper choice of the region \( \mathcal{R}(c) \), bound \eqref{eq:union-bound} is tight enough to obtain the sphere-packing error exponent of the AWGN channel. In fact, we will see that this is possible by taking \( \mathcal{R}(c) = \mathcal{R}_c(c) \), i.e., the same region used to derive the lower bound on the probability of error.

On the other hand, it is known that the standard union bound \eqref{eq:union-bound} is not sufficient to obtain the sphere packing error exponent. However, as shown in later sections, the standard union bound is tight enough to obtain the expurgated error exponent bound (the best known lower bound on the error exponent at low rates).

It is important to emphasize that \( \mathcal{R}(c) \) is not a decoding region in general. The region \( \mathcal{R}(c) \) can, in fact, be arbitrary for the above bound to hold. However, for a random spherical ensemble there is no loss in restricting \( \mathcal{R}(c) \) to be rotationally symmetric about the axis that passes through the origin and the codeword.

\[ \text{Note that the standard union bound corresponds to taking the region } \mathcal{R}(c) \text{ to be very large.} \]
For the remainder of this paper we assume that \( \mathcal{R}(c) \) is rotationally symmetric about the axis that passes through the origin and the codeword and that \( \mathcal{R}(c) \) is congruent for all codewords, and thus, for convenience, use \( \mathcal{R} \) for \( \mathcal{R}(c) \). Thus, \( P_{\text{region}}(c) \) is the same for all codewords \( c \in \mathcal{C} \) and we simply write this as \( P_{\text{region}} \). By averaging over the code and an ensemble of codes we have

\[
P_e \leq P_{\text{union}}^r(c) + P_{\text{region}} = P_{\text{union}}^r + P_{\text{region}}
\]  

(12)
since by averaging over the ensemble of codes the probability of error is independent of the codeword.

We now take the region \( \mathcal{R} \) to be the cone of half angle \( \theta \) with apex at the origin and whose axis passes through \( c \). We denote this region as \( \mathcal{R}_c(\theta) \). Let \( \theta_{\text{ML}}(R) \) be the angle such that the cone \( \mathcal{R}_c(\theta_{\text{ML}}(R)) \) has a solid angle equal to the average solid angle of an ML decoding region for a rate \( R \) code. Thus \( \mathcal{R}_c(\theta_{\text{ML}}(R)) = \mathcal{R}_c(c) \) as used in the lower bound (7). It is well known [2] that

\[
\theta_{\text{ML}}(R) = \theta(R)
\]

where \( \theta(R) \) satisfies \( \sin \theta(R) = \exp(-R) \) and \( \equiv \) denotes exponential equality.\(^5\) Thus, for a given half angle \( \theta \) we let \( R(\theta) = -\log \sin \theta \).

With this notation, the sphere-packing lower bound is

\[
P_e(\mathcal{C}) \geq \mathbb{P}(c + z \notin \mathcal{R}_c(\theta_{\text{ML}}(R)))
\]

(13)
\[
\equiv \mathbb{P}(c + z \notin \mathcal{R}_c(\theta(R)))
\]

(14)

Examining (12), it is clear that in order to show that the sphere-packing bound is tight it is sufficient to show that for \( R \geq R_{\text{crit}} \) the following properties hold

\[
\mathbb{P}(c + z \notin \mathcal{R}_c(\theta(R))) \equiv e^{-nE_{\text{sp}}(R)}
\]

(15)
\[
P_{\text{union}}^r \leq e^{-nE_{\text{sp}}(R)} \text{ for } R > R_{\text{crit}},
\]

(16)

with \( E_{\text{sp}}(R) \) as defined in (3), and where the choice of \( \mathcal{R}_c(\theta_{\text{ML}}(R)) \) is left implicit in (16).

We begin by examining \( \mathbb{P}(c + z \notin \mathcal{R}_c(\theta(R))) \) to derive (15). To do this we follow Berlekamp [17] and decompose the noise into a radial component \( z_y \) normal to the surface of the sphere at \( c \), and its orthogonal complement \( z_y^\perp \), as depicted in Fig. 2.

Then

\[
z = z_y \cdot e_y + z_y^\perp
\]

where \( e_y \) is the unit-vector normal to the sphere at \( c \). Let \( \beta \sqrt{nP} \) be the radial component of the noise \( z_y \), i.e., \( \beta = z_y / \sqrt{nP} \), and let \( r(\beta) \sqrt{nP} \) be the radius of the corresponding spherical cross section of the cone \( \mathcal{R}_c(\theta) \), as shown in Fig. 3. By simple geometry, we have \( r(\beta) = (1 + \beta) \tan \theta \). Thus, since the components of the noise are independent, we may condition on the radial component \( z_y \) and integrate over the distribution of that component, i.e.,

\[
\mathbb{P}(c + z \notin \mathcal{R}_c) = \mathbb{P}(z_y < -\sqrt{nP})
\]

\[
+ \int_{-\infty}^{\infty} p_{z_y}(\beta \sqrt{nP}) \cdot \mathbb{P}(\|z_y^\perp\| \geq r(\beta) \sqrt{nP}) \, d\beta.
\]

(17)

We now apply the following bounds on the norm of a Gaussian vector.

**Proposition 1:** [13] Let \( z = (z_1, z_2, \ldots, z_n) \) be i.i.d. Gaussian random variables with zero mean and variance \( \sigma^2 \). Then

\[
\mathbb{P}\left(\|z\| \geq r\sqrt{nP}\right) \equiv \exp(-nE_h(r^2P/\sigma^2))
\]

where

\[
E_h(\mu) \equiv \begin{cases} \frac{1}{2}(\mu - 1 - \log \mu) & \text{if } \mu \geq 1 \\ 0 & \text{otherwise} \end{cases}
\]

In the sequel we examine the exponential behavior of (17). First, recall that if \( z_1 \) is a zero-mean Gaussian random variable with variance \( \sigma^2 \), then

\[
p_{z_1}(\beta \sqrt{nP}) = K_\sigma \exp(-nE_v(\beta^2P/\sigma^2))
\]

where

\[
E_v(\mu) \equiv \mu/2
\]

\(^5\)This form of Gallager’s bound corresponds to Poltyrev’s tangential sphere bound (19).

\(^6\)Two functions \( f(n) \) and \( g(n) \) are said to be exponentially equal if \( \lim_{n \to \infty} \log(f(n)/g(n)) = 0 \) provided the two limits exist. The notation \( \geq \) and \( \leq \) are defined analogously.
and $K_{\sigma}$ is the normalizing constant. Now, bounding the second term on the right-hand side of (17) by the largest term of the integral (as it can be shown the first term on the right-hand side of (17) is never the dominating term) we have that the probability that the received vector is outside the cone satisfies

$$P(c + z \not\in R_c) \leq \exp\left(-n\min_{\beta} \left[ E_v(\beta^2 \text{SNR}) + E_h(r(\beta)^2 \text{SNR})\right]\right).$$

(18)

It is not hard to show that this inequality is in fact exponentially tight [2]. One can show that in the case that for $0 \leq R(\theta) \leq C$,

$$\beta^*(\theta; \text{SNR}) = \frac{\cos^2 \theta}{2} + \frac{\cos^2 \theta}{2} \sqrt{1 + \frac{4}{\text{SNR} \cos^2 \theta}} - 1.$$

(19)

Note that by letting $\theta = \theta(R)$ and with some additional algebra shown in Appendix I we see that indeed

$$E_v((\beta^*)^2 \text{SNR}) + E_h(r(\beta^*)^2 \text{SNR}) = E_{sp}(R; \text{SNR})$$

(20)

for $R > R_{\text{crit}}$. Thus,

$$P(c + z \not\in R_c(\theta(R))) = \exp(-nE_{sp}(R; \text{SNR})),$$

so that (15) holds.

We verify that (16) holds in Section VI and thus for $R > R_{\text{crit}}$ the sphere-packing error exponent is the proper exponent. To verify (16) we employ similar geometric arguments to those used in this section. In particular we examine the typical error events for $P_{\text{union}}$. We show that for all rates greater than the critical rate the error event is dominated by the same event of leaving a cone at a height of $\beta^*(\theta(R); \text{SNR}) \sqrt{nP}$.

Note that the probability of leaving the cone at any other height is in general smaller than at $\beta^*$. Thus, there is some slack in the choice of the region $R_c$. This leads to the question of what other regions one can use in order to derive the sphere-packing bound using Gallager’s bounding technique [9].

### B. Valid Regions for Geometric Derivation

The tightness of the bound we obtain using the cone $R_c(\theta(R))$ means that for rates greater than the critical rate, the error probability of leaving an ML decoding region is exponentially the same as that of leaving the cone $R_c(\theta(R))$. Thus, the cone approximates the ML decoding region in the error probability analysis for $R > R_{\text{crit}}$. We now investigate how much freedom we have in choosing a region that has this property.

It is important to note that the tightness of the sphere-packing bound does not imply the existence of good cone packings, and in fact these are known not to exist [19].
We refer to any region $\mathcal{R}$ that yields the sphere-packing bound as “valid”. More specifically, we refer as such to any region for which the probability that the received vector falls outside this region is exponentially equal to the sphere-packing error exponent and is no larger than the cone $\mathcal{R}_c(\theta(R))$. More precisely, any region satisfying the following is valid:

$$
A. \quad \mathbb{P}(c + z \notin \mathcal{R}) = \mathbb{P}(c + z \notin \mathcal{R}_c(\theta(R))) \\
B. \quad \mathcal{R} \subset \mathcal{R}_c(\theta(R))
$$

Condition A guarantees that $P_{\text{region}}$ remains exponentially the same for $\mathcal{R}$ as it was for $\mathcal{R}_c(\theta(R))$. That is, Condition A implies (15). Condition B ensures that $P_{\text{union}}$ is no greater for $\mathcal{R}$ than for $\mathcal{R}_c(\theta(R))$ and thus if $\mathcal{R}_c(\theta(R))$ satisfies (16) then by Condition B so does $\mathcal{R}$.

We seek the “smallest” valid region. Since, as we noted, the cross section of the cone that dominates the error event is that corresponding to $\beta^*$, for all other $\beta$ we should be able to choose a “narrower” cross section. Nonetheless, at a height $\beta^*$ the radius of $\mathcal{R}$ has to coincide with that of $\mathcal{R}_c(\theta(R))$ since we cannot hope to improve on the sphere-packing bound. Thus, $\mathcal{R}$ must be tangent to $\mathcal{R}_c(\theta(R))$ at $\beta^*$. Further, in order to ensure that $\mathbb{P}(c + z \notin \mathcal{R}) = \mathbb{P}(c + z \notin \mathcal{R}_c(\theta(R)))$, it is necessary that a valid region satisfy, for any $\beta$,

$$
p_{z_y}(\beta)\mathbb{P}(c + z \notin \mathcal{R} | z_y = \beta\sqrt{nP}) \\
\leq p_{z_y}(\beta^*)\mathbb{P}(c + z \notin \mathcal{R}_c | z_y = \beta^*\sqrt{nP}).
$$

It follows that the region that exactly meets (22) with equality for every $\beta$ is the smallest valid region, since the probability of leaving any other valid region is exponentially larger. This region is parametrized by

$$
E_p(\beta^2\text{SNR}) + E_h(r(\beta)^2 \text{ SNR}) = E_{sp}(R;\text{SNR}).
$$

As Fig. 3(a) depicts, the smallest region is contained in a general region $\mathcal{R}$; all are tangent to the cone at $\beta^*$.

C. Geometric Derivation Using Spherical Regions

We now consider the possibility of taking $\mathcal{R}$ to be a sphere. This provides the link to the mod-A channel. From the previous section we know that a valid sphere must be tangent to the cone of half angle $\theta$ at $\beta^*(\theta;\text{SNR})$ and must also contain the smallest region. As depicted in Fig. 3 in order to make the sphere tangent at $\beta^*(\theta;\text{SNR})$ we must draw a line perpendicular to the cone at $\beta^*(\theta;\text{SNR})$ and find the point where this line intersects the line passing through the origin and the transmitted codeword. We denote this point (the scaled codeword) as $c/\alpha_s(\theta)$. Using basic trigonometry we find that

$$
\alpha^*_s(\theta) = \alpha_s(\beta^*(\theta;\text{SNR}),\theta),
$$

where

$$
\alpha_s(\beta, \theta) = \frac{\cos^2 \theta}{1 + \beta}.
$$
Thus, the radius of this sphere is $\sqrt{nE}/\alpha^*_s(\theta)\sin \theta$ and
\[
\frac{1}{\alpha^*_s(\theta)} = \frac{1 + \beta^*(\theta; \text{SNR})}{\cos^2 \theta} = \frac{1}{2} \left( 1 + \sqrt{\frac{4}{\text{SNR} \cos^2 \theta}} \right),
\]
i.e.,
\[
\mathcal{R}_s(\theta) = \frac{\sqrt{nE}}{\alpha^*_s(\theta)} \cdot \mathbf{e}_y + \mathcal{B} \left( \frac{\sqrt{nE}}{\alpha^*_s(\theta)} \sin \theta \right),
\]
where $\mathcal{B}(r)$ is the ball of radius $r$ and where $\mathbf{e}_y$ is a vector of unit norm. We note that $\alpha^*_s(\theta(R))$ is the optimal scaling found in [12] for the mod-$\Lambda$ channel. In fact, we show in the sequel that the region $\mathcal{R}_s(\theta(R))$ is the natural counterpart of the cone when using Gallager’s bounding technique in the case of the mod-$\Lambda$ channel.

Before further explaining the connection to the mod-$\Lambda$ channel we give a second interpretation of our results thus far. We first rewrite the received vector as
\[
y = c + z = \frac{c}{\alpha} + \left( 1 - \frac{1}{\alpha} \right) c + z = \frac{c}{\alpha} + w
\]
where $w$ is a Gaussian vector with mean $(1 - 1/\alpha)c$. Thus, we can think of transmission as follows: $c/\alpha$ is the chosen codeword, a deterministic vector of magnitude $1 - 1/\alpha$ is added to enable us to meet the power constraint and then the result is transmitted through the channel where Gaussian noise is added. Thus, the probability of leaving the sphere satisfies
\[
P(c + z \notin \mathcal{R}_s(\theta)) = P\left( \frac{c}{\alpha} + w \notin \mathcal{R}_s(\theta) \right).
\]
Next, let $b$ denote a random vector that is uniform over the surface of a sphere of radius $\sqrt{nE}$ and define the effective noise as
\[
z_{\text{eff}} = \frac{1 - \alpha^*_s(\theta)}{\alpha^*_s(\theta)} b + z.
\]
From spherical symmetry it now follows that
\[
P\left( \frac{c}{\alpha} + w \notin \mathcal{R}_s(\theta) \right) = P\left( \frac{c}{\alpha} + z_{\text{eff}} \notin \mathcal{R}_s(\theta) \right).
\]
Thus, the effect of the deterministic vector $(1 - 1/\alpha)c$ is equivalent to that of a random spherical noise. This “noise” is the counterpart of the “self-noise” arising in the mod-$\Lambda$ channel as recounted below. Therefore, the probability of leaving the cone when transmitting the codeword $c$ through the channel (26) is the same as when transmitting it through the channel
\[
y_{\text{equiv}} = \frac{c}{\alpha} + \left( 1 - \frac{1}{\alpha} \right) b + z.
\]
The channel (31) is depicted in Figure 4.

This leads to the following lemma, establishing that the sphere-packing error exponent is exponentially equal to the probability that a random variable that is uniform over the surface of the ball of radius $(1 - \alpha^*_s(\theta(R)))/\alpha^*_s(\theta(R))\sqrt{nE}$ plus a Gaussian vector with independent identically distributed components of variance of $P/$SNR remains in a sphere of radius $\sqrt{nE}/\alpha^*_s(\theta(R))\exp(-R)$ about the scaled codeword.

**Lemma 1:** For $z_{\text{eff}}$ as defined in (29) with $\alpha^*_s$ the optimum scaling for the mod-$\Lambda$ channel, and $\theta(R)$ the half-angle of the cone to which the sphere is tangent, we have
\[
\exp(-nE_{\text{sp}}(R; \text{SNR})) = P\left( z_{\text{eff}} \notin \mathcal{B} \left( \frac{\sqrt{nE}}{\alpha^*_s(\theta(R)) \sin \theta(R)} \right) \right)
\]
\[
\text{Proof: This is simple to see using the following exponential equalities}
\]
\[
\exp(-nE_{\text{sp}}(R; \text{SNR}))
\]
\[
\equiv P\left( \frac{1}{\alpha^*_s(\theta(R))} c + z \notin \mathcal{B} \left( \frac{\sqrt{nE}}{\alpha^*_s(\theta(R)) \sin \theta(R)} \right) \right)
\]
\[
= P\left( \left[ \frac{1 - \alpha^*_s(\theta(R))}{\alpha^*_s(\theta(R))} b + z \right] \notin \mathcal{B} \left( \frac{\sqrt{nE}}{\alpha^*_s(\theta(R)) \sin \theta(R)} \right) \right)
\]

Fig. 4. A depiction of Lemma 1. A codeword is chosen at random from the codebook \( C \) and scaled by \( \frac{1}{\alpha} \). A random dither \( (1 - \frac{1}{\alpha}) b \) is added and the result is transmitted through the additive noise channel.

Fig. 5. The derivation of the sphere-packing error exponent using a spherical region. (a) The probability of a Gaussian leaving a sphere whose center is located at \( \frac{1}{\alpha} \) and tangent to a cone of half angle \( \theta(R) \). (b) The equivalence between this and the probability that a spherical noise plus a Gaussian leaves a larger sphere.

where the last equality follows from (30).

The equivalence of Lemma 1 is depicted in Fig. 5. We make the final connection to the error probability in the mod-\( \Lambda \) channel after briefly summarizing those aspects of the channel we’ll need.

IV. MODULO LATTICE ADDITIVE NOISE CHANNEL

In [8], a lattice-based transmission scheme was proposed for the power-constrained AWGN channel. The scheme transforms the AWGN channel into a mod-\( \Lambda \) channel. In this section, we relate the latter to the geometrical derivation of the AWGN error exponent we developed earlier. We briefly review the lattice transmission approach proposed in [8]. We first recall a few definitions pertaining to lattices.

An \( n \)-dimensional lattice \( \Lambda \) is a discrete subgroup of the Euclidean space \( \mathbb{R}^n \). The fundamental Voronoi region of a lattice \( V = V(\Lambda) \) can be taken as any set such that the following is satisfied:

- If \( x \in \mathcal{V} \) then \( \|x - 0\| \leq \|x - \lambda\| \) for any \( \lambda \in \Lambda \setminus \{0\} \).
- Any point \( y \in \mathbb{R}^n \) can be uniquely written as \( y = \lambda + x \), where \( \lambda \in \Lambda \) and \( x \in \mathcal{V} \). Thus, \( x \) is the remainder when reducing \( y \) modulo \( \Lambda \).

Clearly all fundamental regions have the same volume. Thus, we let \( V = V(\Lambda) \) be the volume of any (every) fundamental region. To each lattice we may associate an “effective radius,” \( r^{\text{eff}}_{\Lambda} \), which is the radius of the sphere having the same volume of \( V \), i.e.,

\[
r^{\text{eff}}_{\Lambda} = \left( \frac{V(\Lambda)}{\text{Vol}(B(1))} \right)^{1/n}.
\]

We next recall two important figures of merit for any lattice that are required in the sequel; see, e.g., [20] for a further discussion of these figures of merit.

The normalized second-moment of a lattice \( G(\Lambda) \) is

\[
G(\Lambda) \triangleq \frac{\sigma^2(\Lambda)}{|V|^{2/n}}.
\]

Note that this is simply a scaled version of \( 1 + \rho_G \).
where, in turn, $\sigma^2(\Lambda)$ is the second-moment of the lattice
\[
\sigma^2(\Lambda) \triangleq \frac{1}{n} \iint_{V} \|x\|^2 \, dx
\]
It is known that $G(\Lambda)$ is always greater than $1/(2\pi e)$, the normalized second-moment of a sphere. Lattices such that $G(\Lambda) \approx 1/(2\pi e)$ are useful in quantization theory and are said to be “good for quantization.” A second important figure of merit of any lattice is its covering radius, $r_{\text{cov}}^{\Lambda}$. To be precise, recall that the set $\Lambda + B(r)$ is a covering of Euclidean space if
\[
\mathbb{R}^n \subseteq \Lambda + B(r).
\]
The covering radius is the smallest radius $r$ such that $\Lambda + B(r)$ is a covering, i.e.,
\[
r_{\text{cov}}^{\Lambda} = \min \{ r : \Lambda + B(r) \text{ is a covering} \}.
\]
A sequence of lattices $\{\Lambda_n\}$ is said to be “good for covering” if
\[
\liminf_{n \to \infty} \frac{r_{\text{cov}}^{\Lambda_n}}{r_{\text{eff}}^{\Lambda_n}} = 1.
\]
We now summarize the coding scheme of \cite{8}. In particular, with $u$ denoting a random variable (dither) that is uniformly distributed over $V$, i.e., $u \sim \text{Unif}(V)$, we have:

- **Transmitter:** The input alphabet is restricted to $V$. For any $v \in V$, the encoder sends
  \[
  x = \lfloor v - u \rfloor \; \text{mod} \; \Lambda.
  \]  \hfill (35)

- **Receiver:** The receiver computes
  \[
  y' = \left\lfloor y + \frac{1}{\alpha} \cdot u \right\rfloor \; \text{mod} \; \Lambda / \alpha \quad \text{where} \quad 0 < \alpha \leq 1.
  \]  \hfill (36)

The resulting channel is described by the following lemma \cite{21}.

**Lemma 2:** The channel from $v$ to $y'$ defined by (11), (35) and (36) is equivalent in distribution to the mod-$\Lambda$ channel
\[
y' = \left\lfloor \frac{1}{\alpha} \cdot v + z'_{\text{eff}} \right\rfloor \; \text{mod} \; \Lambda / \alpha
\]  \hfill (37)
with
\[
z'_{\text{eff}} = \frac{1 - \alpha}{\alpha} \cdot u + z
\]  \hfill (38)

Note the similarity of the mod-$\Lambda$ channel, depicted in Fig. 6, to the equivalent channel representation (31) for transmission of a codeword from a spherical code as depicted in Fig. 4.

The capacity of the mod-$\Lambda$ channel (37) is characterized by the following theorem:

**Proposition 2** (\cite{8}, \cite{9}): The capacity $C(\Lambda, \alpha)$ of the mod-$\Lambda$ transmission system is lower bounded by
\[
C(\Lambda, \alpha) \geq C - \frac{1}{2} \log 2\pi e G(\Lambda) - \frac{1}{2} \log \frac{\sigma_{\alpha}}{\sigma_{\text{MMSE}}}
\]
where $\sigma_{\alpha}$ and $\sigma_{\text{MMSE}}$ are the expected estimation error per dimension using the linear estimator $\hat{X} = \alpha \cdot X$ and the MMSE estimate $\hat{X}_{\text{MMSE}} = \alpha_{\text{MMSE}} \cdot X$, respectively, where $\alpha_{\text{MMSE}} = \text{SNR}/(1 + \text{SNR})$.

Thus, the gap to capacity may be made arbitrarily small by taking a lattice $\Lambda$ such that $G(\Lambda)$ is sufficiently close to $1/(2\pi e)$ and $\alpha = \alpha_{\text{MMSE}}$. That is, to achieve capacity it is sufficient for $\Lambda$ to be good for quantization and for $\alpha = \alpha_{\text{MMSE}}$. However,

\footnote{This particular form of the theorem is due to Forney \cite{9}.}
as previously noted, the error exponent is more sensitive to the input distribution. In particular, it is no longer sufficient for \( \Lambda \) to be good for quantization (as was sufficient to achieve capacity). We require the additional condition that \( \Lambda \) is good for covering, i.e., that \( r_{\Lambda}^{\text{cov}} / r_{\Lambda}^{\text{eff}} \rightarrow 1 \) as \( n \rightarrow \infty \). Furthermore, as shown in [12], a scaling that is strictly less than \( \alpha_{\text{MMSE}} \) in order to achieve the error exponent. In this direction we define the error exponent for the mod-\( \Lambda \) channel using a scaling \( \alpha \) as

\[
E_{\Lambda}(R, \alpha) = \lim_{n \rightarrow \infty} \sup \left[ -\log P_{e,\Lambda}(n, R, \alpha) / n \right],
\]

where \( P_{e,\Lambda}(n, R, \alpha) \) is the minimal value of the average probability of error, \( P_{e,\Lambda}(\mathcal{C}) \), over all \( (n, R) \) codes using a scaling \( \alpha \), and, in turn, where \( P_{e,\Lambda}(\mathcal{C}) \) is the average error probability of a given \( (n, R) \) lattice code averaged over all codewords.

The error exponent for the mod-\( \Lambda \) (just as for the AWGN channel) is only known for a range of rates. However, it is shown in [12] that the error exponent for the mod-\( \Lambda \) channel achieves the random coding exponent, and the scaling \( \alpha \) that achieves this exponent was explicitly found in [12]. Indeed, as previously noted it is precisely \( \alpha_{\text{MMSE}}(R) \) [cf. (25)] which we have shown corresponds to valid spherical regions in Gallager’s bound [9]. We now provide an intuitive explanation for this result, which was a question left open by [12].

We begin by noting that the noise \( z'_{\text{eff}} \) in (38) looks very much like the effective noise appearing in (33). Additionally note that the random vectors \( b \) and \( u \) have the same second-moment but while \( b \) is spherical, \( u \) is uniform over \( \mathcal{V} \). The following proposition makes this notion precise.

**Proposition 3 ([8]):** Let \( \Lambda \) be any \( n \)-dimensional lattice that is good for quantization and covering such that \( \sigma^2(\Lambda) = nP \). Now, consider the random variable \( u \) that is chosen uniformly from the fundamental region \( \mathcal{V}(\Lambda) \) and the random variable \( b \) chosen uniformly from the surface of the ball \( \mathcal{B}(\sqrt{nP}) \). Then

\[
\log p_u(x) = \log p_b(x) + o(1)
\]

where \( p_u(x) \) and \( p_b(x) \) are the probability density functions of \( u \) and \( b \) respectively.

Define the following modified channel that replaces the self-noise in (38) with a spherical noise:

\[
y'' = \left[ \frac{1}{\alpha} \cdot v + z'_{\text{eff}} \right] \mod \Lambda / \alpha \quad \text{with} \quad z''_{\text{eff}} = \frac{1 - \alpha}{\alpha} \cdot b + z.
\]

(39)

Then it follows from Proposition 3 that the error exponent of the original channel (37) is no worse (in an exponential sense) than that of the modified channel (39). We now bound the error exponent of the channel (39) by using Gallager’s technique [9] as before.

It is conceptually much easier if we first consider how one may remove the \( \mod \Lambda / \alpha \) operation appearing in (39). In this direction let

\[
\tilde{y} = \frac{1}{\alpha} \cdot v + z''_{\text{eff}}.
\]

Then, following in the footsteps of [12], we may upper bound the probability of error by using a suboptimal decoder which first performs Euclidean distance decoding in the extended codebook

\[
\mathcal{E}^{\Lambda} \triangleq \mathcal{C} + \Lambda = \bigcup_{c \in \mathcal{C}} \{ c + \Lambda \},
\]

and then reduces the result mod-\( \Lambda \) to obtain the coset leader. We call this sub-optimal decoder the closest coset decoder. In other words, the decoder searches for the coset with minimum Euclidean distance to the received vector and an error occurs whenever the closest codeword to the received vector does not belong to the coset of the transmitted codeword (we also take equality as an error). That is, using the closest coset decoder an error occurs when the event

\[
E_{\alpha}(\tilde{y}, c_e) \triangleq \left\| \frac{c_e}{\alpha} - \tilde{y} \right\| \leq \left\| \frac{c}{\alpha} - \tilde{y} \right\|
\]

occurs for some \( c_e \). Thus, the closest coset decoder satisfies

decoding rule: \( \hat{c} = \arg \min_{c \in \mathcal{E}^\Lambda} \left\| \frac{c}{\alpha} - \tilde{y} \right\| \mod \Lambda / \alpha \)

error event: \( E_{\alpha}(\tilde{y}, c_e) \) some \( c_e \in \mathcal{E}^{\Lambda} \setminus \{ c + \Lambda \} \)

We may further upper bound the probability of error by using a still simpler decoder that does not perform coset decoding but rather searches for the codeword with the minimum Euclidean distance in the extended codebook. Thus, with this decoding rule, an error results even if the closest (to the received vector) codeword in the extended code belongs to the same coset as
the transmitted codeword. We call this sub-optimal decoder the Euclidean distance decoder. The Euclidean distance decoder satisfies

\[
\hat{c} = \arg \min_{c \in \mathcal{C}} \| \frac{c}{\alpha} - \hat{y} \|,
\]

error event: \( \mathcal{E}_\alpha(\hat{y}, c) \) some \( c \in \mathcal{C} \setminus \{ c \} \)

Now, by replacing the self-noise by spherical noise and using a Euclidean distance decoder, we have the following upper bound on the probability of error in a mod-\( \Lambda \) channel:

\[
P^\lambda_e(c) \leq P \left( \text{error, } \frac{c}{\alpha} + z''_{\text{eff}} \in \mathcal{R}(c) \right) + P \left( \text{error, } \frac{c}{\alpha} + z''_{\text{eff}} \notin \mathcal{R}(c) \right) \\
\leq P \left( \text{error, } \frac{c}{\alpha} + z''_{\text{eff}} \in \mathcal{R}(c) \right) + P \left( \frac{c}{\alpha} + z''_{\text{eff}} \notin \mathcal{R}(c) \right) \\
\leq \sum_{c \in \mathcal{C} \setminus \{ c \}} P \left( \mathcal{E}_\alpha(\hat{y}, c), \frac{c}{\alpha} + z''_{\text{eff}} \in \mathcal{R}(c) \right) + P \left( \frac{c}{\alpha} + z''_{\text{eff}} \notin \mathcal{R}(c) \right) \\
= P_{\text{union}}^\lambda(c) + P_{\text{region}}^\lambda(c) \tag{40}
\]

where \( P_{\text{region}}^\lambda(c) \) is the probability that the received vector is not in the region \( \mathcal{R}(c) \) and \( P_{\text{union}}^\lambda(c) \) is the sum appearing in (40). As before, we consider congruent regions for all \( c \) and thus simply write \( \mathcal{R}(c) = \mathcal{R} \) and \( P_{\text{region}}^\lambda(c) = P_{\text{region}}^\lambda \). As done for the AWGN channel, we use random coding arguments to bound the probability of error. In this direction, we denote the ensemble of codes for which the codewords are drawn uniformly from the Voronoi of a lattice \( V \subset \mathbb{R}^n \) as \( \Omega_{\Lambda}^{0,\Lambda,n} \) and write any codebook drawn from \( \Omega_{\Lambda}^{0,\Lambda,n} \) as \( \mathcal{C} \). Averaging over all the codewords in the code and the ensemble \( \Omega_{\Lambda}^{0,\Lambda,n} \) yields

\[
P_e^\lambda \leq P_{\text{union}}^\lambda + P_{\text{region}}^\lambda, \tag{41}
\]

As before, begin by considering \( P_{\text{region}}^\lambda \). We have a choice over which region we choose to use in (41). Note that by taking \( \alpha = \alpha^*_s(R) \), the effective noise \( z''_{\text{eff}} \) is precisely the same as that found in (29) in Section III-C. That is the effective noise \( z''_{\text{eff}} \) is the sum of a spherical noise and a Gaussian. Thus, taking the region \( \mathcal{R} = \mathcal{R}_s(\theta(R)) \) one has that

\[
P_{\text{region}}^\lambda = P_{\text{region}} \tag{42}
\]

where \( P_{\text{region}} \) was defined via (9) and the choice of region is left implicit. Further, as a consequence of Proposition 3 we can replace \( z''_{\text{eff}} \) with \( z_{\text{eff}} \) and obtain an asymptotic equality.

This yields the following lemma, establishing that the sphere-packing error exponent is exponentially equal to the probability that a random vector that is uniform over the fundamental region \( \mathcal{V}(\Lambda_n) \) plus a Gaussian vector with independent identically distributed components of variance \( \sqrt{nP/\alpha^*_s(R)} \) remains in a sphere of radius \( \sqrt{nP/\alpha^*_s(R)} \) about the scaled codeword.

Lemma 3: Let \( \{ \Lambda_n \} \) be a sequence of lattices that is good for coding and quantization such that \( \sigma^2(\Lambda_n) = nP \). Then

\[
\exp(-nE_{\text{sp}}(R;\text{SNR})) = \mathbb{P} \left( \frac{z'_{\text{eff}}}{\alpha^*_s(R)} \notin \mathcal{B} \left( \sqrt{nP/\alpha^*_s(R)} \sin \theta(R) \right) \right),
\]

where

\[
z'_{\text{eff}} = \frac{1 - \alpha^*_s(R)}{\alpha^*_s(R)} u + z,
\]

and in turn, where \( u \) is the random variable that is uniform over \( \mathcal{V} \).

Thus, (13) holds for the mod-\( \Lambda \) channel using the region and scaling that corresponds to a valid sphere in the AWGN channel. We note that using the distance distribution of a random ensemble of lattice codes it was shown in [12] that (16) holds for the mod-\( \Lambda \) channel. This leads to the following theorem of [12].

Proposition 4 ([12]): If \( R_{\text{crit}} \leq R \leq C \), then there exists an \( \alpha(R) \) such that \( 0 \leq \alpha(R) \leq \alpha_{\text{MMSE}} \) and for mod-\( \Lambda \) transmission

\[
\exp(-nE\Lambda(R, \alpha(R))) = \exp(-nE_{\text{sp}}(R;\text{SNR})).
\]

Furthermore, the exponent \( E_{\text{sp}}(R;\text{SNR}) \) can be achieved with Euclidean decoding.

Reexamining Fig. 5 of Section III-C we can see why both spherical codes and the mod-\( \Lambda \) transmission scheme may achieve the sphere-packing error exponent. By identifying the transmitted codewords \( x \), as depicted in Fig. 7 we can interpret the
center of the spherical region, \( e/\alpha \), as the selected codeword in the mod-\( \Lambda \) channel and \( \mathcal{B}_e(\theta(R)) \) as a spherical approximation to the Voronoi region of the codeword.

We can extend our analysis to rates less than the critical rate. For completeness we present a separate proof of this in Section \[V\] which shows that not only are the error exponents in the AWGN and mod-\( \Lambda \) channels equal, but the typical error events coincide. In the following section we summarize our characterization of the AWGN error exponent for rates above and below the critical rate.

### V. The AWGN Error Exponents for Low Rates

For rates less than the critical rate the best known lower bound is the maximum of the random coding error exponent, \( E^r_{\text{AWGN}}(R; \text{SNR}) \), and the expurgated error exponent, \( E^e_{\text{AWGN}}(R; \text{SNR}) \). As Fig. 1 reflects, the random coding error exponent is the larger error exponent for all rates greater than \( R^*_{x} = \frac{1}{2} \log \left( \frac{1}{2} \left( 1 + \sqrt{1 + \frac{\text{SNR}}{4}} \right) \right) \)

and is equal to the sphere-packing error exponent for rates greater than the critical rate. Hence, the sphere-packing error exponent is tight for all rates greater than the critical rate. The error exponent of the AWGN channel, however, is still not known for all rates. Recent progress has been made to show that the random coding error exponent is indeed the correct error exponent for a range of rates less than the critical rate; see [22] and references therein. For all rates less than the critical rate the random coding error exponent is linear. More precisely, the random coding error exponent is

\[
E^r_{\text{AWGN}}(R; \text{SNR}) \triangleq \begin{cases} E^e_{\text{AWGN}}(R; \text{SNR}) & \text{if } 0 \leq R \leq R^*_{x} \\ E_{sp}(R; \text{SNR}) & \text{if } R^*_{x} < R \leq C \end{cases}
\]

(43)

where \( E^e_{\text{AWGN}}(R; \text{SNR}) = E_G(\beta'_{G}, 1; \text{SNR}) \)

and, in turn,

\[
\beta'_{G} = \frac{1}{2} \left( 1 + \frac{\text{SNR}}{2} + \sqrt{1 + \frac{\text{SNR}^2}{4}} \right).
\]

(44)

Recall that \( E_G(\beta, \rho; \text{SNR}) \) was defined in [4] and note that \( \beta'_{G} \) is independent of the rate.

For all rates less than \( R_x \) the expurgated error exponent is greater than the random coding error exponent. In order to precisely define the expurgated error exponent, recall that the minimum distance of a code is the smallest distance between any two codewords in a code. The expurgated error exponent geometrically corresponds to errors occurring between the closest two codewords of a code that achieves the best known minimum distance (as this is the dominating error event at low rates; see, e.g., [10]). Conversely, by using an upper bound on the minimum distance one can arrive at the minimum distance upper
bound on the error exponent. These error exponents are apparent in Fig. I. The expurgated error exponent is
\[ E_{\text{AWGN}}^\tau(R; \text{SNR}) \triangleq \frac{\text{SNR}}{4} \left( 1 - \sqrt{1 - \exp(-2R)} \right). \]
Thus, the best known lower bound on the error exponent of the AWGN channel is
\[ E_{\text{AWGN}}(R; \text{SNR}) \triangleq \begin{cases} E_{\text{AWGN}}^\tau(R; \text{SNR}) & \text{if } 0 \leq R \leq R_c \\ E_{\text{AWGN}}^\tau(R; \text{SNR}) & \text{if } R_c < R \leq C \end{cases} \]

In the preceding sections we provided a simple proof that, for a variety of regions \( \mathcal{R} \), one may achieve the sphere-packing bound using Gallager’s bounding technique for \( R > R_{\text{crit}} \) under the assumption that (16) holds and used this to provide a simple explanation for the error exponent of the mod-A channel. In the following section we prove that (16) does indeed hold, provide exponential bounds for \( E_{\text{union}}^\tau \) and show that this bound is exponentially equal to \( E_{\text{AWGN}}(R; \text{SNR}) \).

VI. GEOMETRIC DERIVATION OF THE RANDOM CODING AND EXPURGATED ERROR EXPONENTS

Begin by recalling from (12) that \( E_{\text{union}}^\tau \) is a union bound over pairwise errors averaged over the random ensembles of spherical codes \( \Omega_0^{(n)} \). While this ensemble is sufficient to achieve the random coding error exponent we require a more general ensemble of codes to derive the best known bound on the error exponent for the AWGN and mod-A channels as well as provide the final geometric link between the error exponents of these two channels. In this direction let \( \Omega_{d,0}^{(n)}(R) = (\Omega_0^{(n)}, d_{\Omega_0}(R)) \) be the ensemble of rate \( R \) random spherical codes where expurgation has been applied such that the minimum distance is \( d_{\Omega_0}(R) \). Note, with this notation \( \Omega_{d,0}^{(n)}(R) \) is the ensemble with minimum distance 0 or the random spherical ensemble \( \Omega_0^{(n)} \) that was introduced previously in Section III. Moreover, it is known from (2) that no rate loss in incurred from the expurgation process for any \( d_{\Omega_0}(R) \) such that
\[ d_{\Omega_0}(R) \leq d_{\text{min}}(R), \]
where
\[ d_{\text{min}}(R) \triangleq \sqrt{2 - 2\sqrt{1 - \exp(-2R)}}. \] (45)
Henceforth we consider ensembles of codes such that \( d_{\Omega_0}(R) \leq d_{\text{min}}(R) \). In particular, we consider the ensembles
\[ \Omega_{1,0}^{(n)}(R) = (\Omega_0^{(n)}, 0) \] (46)
\[ \Omega_{1,1}^{(n)}(R) = (\Omega_0^{(n)}, e^{-R}) \] (47)
\[ \Omega_{1,1}^{(n)}(R) = (\Omega_0^{(n)}, d_{\text{min}}(R)) \] (48)
and denote exponent of the average probability of error for these ensembles as \( E_1(R; \text{SNR}) \), \( E_{\Omega_1}(R; \text{SNR}) \) and \( E_{\Omega_{1,1}}(R; \text{SNR}) \) respectively. We show in the sequel that
\[ \exp(-nE_1(R; \text{SNR})) \leq \exp(-nE_{\text{AWGN}}^\tau(R; \text{SNR})) \]
and
\[ \exp(-nE_{\Omega_{1,1}}(R; \text{SNR})) \leq \exp(-nE_{\text{AWGN}}(R; \text{SNR})). \]
Moreover, in Section VII we show that the mod-A channel can obtain an average probability of error that is exponentially equal to that obtained by the ensemble \( \Omega_{1,1}^{(n)}(R) \). We begin by examining the exponential behavior of the probability of having a pairwise error while remaining inside the cone \( \mathcal{R}_c(\theta) \) for the ensemble \( \Omega_{1,1}^{(n)}(R) \). That is, the exponent of \( E_{\text{union}}^\tau \) for the code ensemble with no minimum distance constraint.

Recall that conditioned on the event that the sum of the codeword and the noise remains inside \( \mathcal{R}_c(\theta) \) an error occurs between a codeword, say \( c_r \), at a distance \( d \) if the codeword plus the noise crosses the ML plane between \( c \) and \( c_r \). We let \( \mathcal{D}_c(d, \theta) \) be the region corresponding to this event. That is, we let \( \mathcal{D}_c(d, \theta) \) be the intersection of the cone \( \mathcal{R}_c(\theta) \) with the half space that orthogonally bisects a cord of length \( d \) that has one end point at the transmitted codeword and the other end at \( c_r \). This can be seen as the shaded region in Fig. 8. We write \( \mathcal{D}_c(d) \) for simplicity.

Similar to the previous sections we find the typical error events or the distance, \( d \), and \( \beta \) that maximize \( E_{\text{union}}^\tau \). More precisely for each \( d \) we find the typical \( \beta \) and then find the typical \( d \). It is often simpler to consider the angle made between the transmitted codeword and any codeword at a distance \( d \) instead of the distance itself and denote this angle by \( \Theta(d) \), i.e.,
\[ \Theta(d) \triangleq 2\arcsin(d/2) = \arccos\left(1 - \frac{d^2}{2}\right). \] (49)

10 This error exponent may be achieved by drawing a uniform code over the sphere and then expurgating all codewords that fall within a distance equal to the best known minimum distance of any other codeword.
In Appendix V it is shown that
\[ P_{\text{union}} \leq 2K \max_{0 \leq d \leq 2} \mathbb{P}\left( \sqrt{nP} \cdot e_y + z \in D_c(d) \right) \times \exp\left( nR + (n-1) \log \left( d \sqrt{1 - \frac{d^2}{4}} \right) \right) \]

where \( K \) is a normalizing constant. In order to form an exponential bound for (50) we begin by examining the exponential behavior of \( \mathbb{P}\left( \sqrt{nP} \cdot e_y + z \in D_c(d) \right) \), i.e., the probability of having a pairwise error with a codeword at a distance \( d \) while remaining inside the cone \( R_c(\theta) \). We again use the tangential sphere bound and thus we let \( D_c(d, \theta, \beta) \) be the intersection of \( D_c(d, \theta) \) with the hyperplane, say \( \mathcal{H} \), at a distance of \( \beta \sqrt{nP} \) from the transmitted codeword. More precisely, let \( \mathcal{H} \) be the hyperplane such that \( e_y' x = \beta \sqrt{nP} \) for all \( x \in \mathcal{H} \). The \( n-1 \) dimensional region \( D_c(d, \theta, \beta) = D_c(d, \theta) \cap \mathcal{H} \) and can be seen in Fig. 8.

Integrating along the radial component of the noise we have
\[ P\left( \sqrt{nP} \cdot e_y + z \in D_c(d) \right) = \int_{-\infty}^{\infty} e^{-\frac{2nR_\theta z^2}{2}} \mathbb{P}\left( \sqrt{nP} \cdot e_y + z \in D_c(d, \theta, \beta) \right) \, d\beta_c \]
\[ = \int_{-\infty}^{\infty} e^{-\frac{2nR_\theta z^2}{2}} \mathbb{P}\left( z_2 \geq x_c; \sum_{i=2}^{\infty} z_i^2 \leq y_c^2 \right) \, d\beta_c \]
\[ = \int_{-\infty}^{\infty} e^{-\frac{2nR_\theta z^2}{2}} \mathbb{P}\left( \beta_c \geq x_c; \sum_{i=2}^{\infty} z_i^2 \leq y_c^2 \frac{1}{\beta_c} \right) \, d\beta_c \]

where
\[ x_c = x_c(\beta_c, \Theta(d)) = \sqrt{nP}(1 + \beta_c) \tan \frac{\Theta(d)}{2} \]
and
\[ y_c = y_c(\beta_c) = \sqrt{nP}(1 + \beta_c) \tan \theta(R) \]
can be derived through the geometry in Fig. 8. That is, when the radial component of the noise has magnitude \( \beta_c \), the probability \( \mathbb{P}\left( \sqrt{nP} \cdot e_y + z \in D_c(d) \right) \) is, geometrically speaking, simply the probability that the second component of the noise is greater than \( x_c \) (so that the codeword plus the noise is in the decoding region for a different codeword) while the magnitude of the second thorough \( n \)th component of the noise is less than \( y_c \) (so that the codeword plus the noise is in \( R_c(\theta) \)).

Before proceeding, we recall the following bound on Gaussian vectors.

**Proposition 5 (178):** Let \( z_1, z_2, \ldots, z_n \) be i.i.d zero-mean Gaussian random variables with variance \( \sigma^2 \). Let \( z = (z_1, \ldots, z_n) \). Then, if \( n \geq 2 \),
\[ \mathbb{P}( |z| \geq \sqrt{nP} x, \|z\| \leq \sqrt{nP} y ) \leq e^{-n \tilde{E}_d(x,y;\sigma^2)} \]
where the exponent \( \tilde{E}_d \) is defined via
\[ 2 \tilde{E}_d(x,y;\tau) = \begin{cases} \frac{\tau x^2}{\tau y^2 - \log(\epsilon \tau y^2 - x^2))} & \text{if } y^2 - x^2 \geq \frac{1}{\tau} \\ \tau y^2 - \log(\epsilon \tau y^2 - x^2) & \text{otherwise} \end{cases} \]
Thus, (50) becomes
\[
\mathbb{P} \left( \sqrt{nP} \cdot e_y + z \in \mathcal{D}_e(d) \right) \\
\leq \max_{\beta_c} \exp \left( -nE_d(\beta_c, x_c(\beta_c, \Theta(d)), y_c(\beta_c); \text{SNR}) \right)
\]  
(53)

where
\[
2E_d(\beta, x, y; \tau) \triangleq \tau \beta^2 + E_d(x, y; \tau)
\]  
(54)

Thus, (50) becomes
\[
\mathcal{P}_{\text{union}} \leq \max_{0 \leq d \leq 2} \max_{-1} \exp \left( -nE_{\text{bnd}}(\theta, d, \beta, R; \text{SNR}) \right)
\]  
(55)

where
\[
E_{\text{bnd}}(\theta, d, \beta, R; \text{SNR}) \\
= E_d(\beta, x_c, y_c; \text{SNR}) - \frac{1}{2} \log \left[ d^2 \left( 1 - \frac{d^2}{4} \right) \right] - R.
\]  
(56)

It is a simple, yet lengthy, process to find the value of $\beta$ that maximizes $\mathcal{P}_{\text{union}}$. We provide a full derivation of the optimal $\beta$ in Appendix II but for now note that it is equal to
\[
\beta_c^*(\theta, \Theta(d); \text{SNR}) = \begin{cases} 
\beta^*(\theta; \text{SNR}) & \text{if } R(\theta) > R_{\text{crit}}(\Theta(d)) \\
\cos^2 \left( \frac{\Theta(d)}{2} \right) & \text{otherwise}
\end{cases}
\]  
(57)

where $\beta^*(\theta; \text{SNR})$ was defined in (19) and
\[
R_{\text{crit}}(\Theta(d)) \triangleq -\log \left[ 1 - \frac{2 \text{SNR} \cos^4 \frac{\Theta(d)}{2}}{2 + \text{SNR} (1 + \cos \Theta(d))} \right].
\]

Note that for a given $\Theta(d)$ if $R(\theta) > R_{\text{crit}}(\Theta(d))$ then $\beta^*_c$ is independent of $d$ and exactly equal to the optimal $\beta$ in the derivation of the upper bound for $P_{\text{region}}$ [cf. (19)]. That is, the typical error event of $\mathcal{P}_{\text{union}}$ corresponds to the typical error event for $P_{\text{region}}$. In fact, if this were not the case for $\theta = \Theta(R)$ then we would either be able to improve the sphere-packing error exponent by making the region $\mathcal{R}_c(\theta(R))$ larger (if $P_{\text{region}} < \mathcal{P}_{\text{union}}$) or be unable to show that the sphere-packing error exponent is tight (if $P_{\text{region}} < \mathcal{P}_{\text{union}}$). Thus, the minimizing distance should be the point that is tangent to $\mathcal{R}_c(\theta)$ at $\beta^*(\theta(R); \text{SNR})$. By the law of cosines this would imply that $d^*_c(\theta) = \sqrt{2} \sin \theta$ for $R(\theta) > R_{\text{crit}}(\Theta(d))$. We show in Appendix II that this is indeed the case and the value of $d$ that maximizes $\mathcal{P}_{\text{union}}$ is
\[
d^*_c(\theta) = \begin{cases} 
\sqrt{2} \sin \theta & \text{if } R(\theta) > R_{\text{crit}} \\
d_{\text{crit}} & \text{otherwise}
\end{cases}
\]  
(58)

where
\[
d_{\text{crit}} = \sqrt{\frac{2}{\text{SNR}} + \frac{4}{\text{SNR}^2}} - 2 \sqrt{1 + \frac{4}{\text{SNR}^2}}
\]

Combining (57) and (58) in to one equation yields the following definition and lemma. Let
\[
f_{\text{bnd}}(d, \theta, R; \text{SNR}) = \exp \left( -nP_{\text{bnd}}(d, \theta, R; \text{SNR}) \right)
\]  
(59)

where $P_{\text{bnd}}(d, \theta; \text{SNR})$ is defined at the bottom of the page in (60).

This yields the following lemma.

**Lemma 4:** Consider the sequence of ensembles of random spherical codes $\{\Omega_1^{(n)}(R)\}$. Then, if $R = \mathcal{R}_c(\theta(R))$ and $R_{\text{crit}} < R < C$,
\[
\mathcal{P}_{\text{union}} \leq f_{\text{bnd}}(d^*_c(\theta(R), \theta(R); \text{SNR})
\]

\[1\] Note this is equal to $\sqrt{2/\beta^*_c}$. 

\[
E_{\text{bnd}}(d, \theta; \text{SNR}) = \begin{cases} 
E_{\text{ap}}(R(\theta); \text{SNR}) - \log (\sin \theta) - R & \text{if } R(\theta) > R_{\text{crit}}(\Theta(d)) \\
\text{SNR}/8 \cdot d^2 - \log \left( d\sqrt{1 - d^2/4} \right) - R & \text{otherwise}
\end{cases}
\]  
(60)
From the above discussion, it is clear that in the case that \( \theta = \theta(R) \) we have that the typical error events for \( \overline{P} \) and \( P_{\text{region}} \) are equivalent and \( \overline{P}_{\text{union}} = P_{\text{region}} \) for all \( R \geq R_{\text{crit}} \). Thus (16) holds and we have shown that the sphere-packing error exponent is indeed a valid lower bound on the error exponent of the AWGN channel. We state this in the following lemma.

**Lemma 5:** Consider the sequence of ensembles of random spherical codes \( \{1^{(n)}(R)\} \). Then, if \( R = R_{\text{c}}(\theta(R)) \), the average probability of error for \( R_{\text{crit}} < R < C \) is upper bounded by

\[
\overline{P}_e \leq \exp(-nE_{\text{AWGN}}(R;\text{SNR})).
\]

Note that if \( R(\theta) < R_{\text{crit}} \) both \( d_\text{c}^c(\theta) \) and \( \beta_\text{c}^c \) are independent of \( \theta \). Thus, if \( \theta = \theta(R) \) then for all rates \( R < R_{\text{crit}} \) it is clear that one may fix \( \theta = \pi/2 \) and obtain the same result for \( \overline{P}_{\text{union}} \) as if one had used \( \theta = \theta(R) \). In this direction, let

\[
\theta_{\text{AWGN}}(R) \triangleq \begin{cases}
\pi/2 & \text{if } 0 \leq R \leq R_{\text{crit}} \\
\theta(R) & \text{if } R_{\text{crit}} < R \leq C
\end{cases}
\]

Note that this implies that \( \overline{P}_{\text{union}} \) must be equal to the union bound over all codewords, \( \overline{P}_{\text{union}} \). Now we have the following theorem.

**Theorem 1:** Consider the sequence of ensembles of random spherical codes \( \{1^{(n)}(R)\} \). If \( R = R_{\text{c}}(\theta_{\text{AWGN}}(R)) \), then Gallager’s bounding technique (9) for the average probability of error satisfies the following two properties for \( 0 \leq R \leq C \):

1'. \( \overline{P}_{\text{union}} \leq \exp(-nE_{\text{AWGN}}^r(R;\text{SNR})) \)

2'. \( \overline{P}_{\text{union}} \geq P_{\text{region}} \)

where \( E_{\text{AWGN}}^r(R;\text{SNR}) \) was defined in (43).

A proof is provided in Appendix II.

Note that the typical error events of Theorem 1 happen with codewords at a distance \( d_{\text{crit}} \) for rates less than \( R_{\text{crit}} \). Hence, by using a code ensemble such that \( d_1(R) > d_{\text{crit}} \) one expects to be able to improve upon our current bound for rates such that \( d_1(R) > d_{\text{crit}} \). For example, we consider the ensemble of codes \( \{1^{(n)}(R)\} \). It is straightforward to check that for the ensemble \( \{1^{(n)}(R)\} \) one has \( d_{\text{min}}(R) > d_{\text{crit}} \) for \( R < R_{\text{III}} \), where

\[
R_{\text{III}} = R_x = \frac{1}{2} \log \left( \frac{1}{2} \left( 1 + \sqrt{1 + \frac{\text{SNR}}{4} \right)} \right).
\]

Additionally, it is easy to see that for \( \theta = \theta_{\text{AWGN}}(R) \) the ensemble \( \{1^{(n)}(R)\} \) has typical error events that occur at a distance \( d_\text{c}^c(\theta) = d_{\text{min}}(R) \) if \( R < R_{\text{III}} \). Hence, the typical error events occur with codewords at a distance

\[
d_{\text{typ}}(R) \triangleq \begin{cases}
d_{\text{min}}(R) & \text{if } 0 \leq R \leq R_x \\
d_{\text{crit}} & \text{if } R_x < R \leq R_{\text{crit}} \\
2\exp(-R) & \text{if } R_{\text{crit}} < R \leq C
\end{cases}
\]

(61)

This yields the following characterization of the error exponent of the AWGN channel.

**Theorem 2:** Consider the sequence of ensembles of random spherical codes \( \{1^{(n)}(R)\} \). If \( R = R_{\text{c}}(\theta_{\text{AWGN}}(R)) \), then the typical error events occur with codewords at a distance \( d_{\text{typ}}(R) \) and for \( 0 \leq R \leq C \):

\[
A. \quad \overline{P}_{\text{union}} \leq f_{\text{bnd}}(d_{\text{typ}}(R), \theta_{\text{AWGN}};\text{SNR})
\]

\[
B. \quad P_{\text{region}} \leq f_{\text{bnd}}(d_{\text{typ}}(R), \theta_{\text{AWGN}};\text{SNR})
\]

(62)

(63)

Moreover,

\[
f_{\text{bnd}}(d_{\text{typ}}(R), \theta_{\text{AWGN}};\text{SNR}) \triangleq e^{-nE_{\text{AWGN}}(R;\text{SNR})}.
\]

Note, in Theorem 2 \( P_{\text{region}} \) is less than our bound on \( \overline{P}_{\text{union}} \), \( f_{\text{bnd}} \). In order to improve the bound on the error exponent it is natural to ask whether the choice of \( \theta = \theta_{\text{AWGN}} \) in Theorem 2 is the best choice for all rates \( R < R_{\text{crit}} \) since there is some slack in the choice \( \theta = \theta_{\text{AWGN}} \) due to the fact that \( \overline{P}_{\text{union}} \geq P_{\text{region}} \). We have shown that for the ensemble \( \{1^{(n)}(R)\} \) one can not do better in terms of the average probability of error for the ensemble using Gallager’s technique. We next show that so long as \( \theta \) is within reason the choice of \( \theta \) has no effect on the resulting bound on the average probability of error.

In order to precisely characterize the freedom one has in choosing \( \theta \), we require the following definitions. Let, for \( K \geq 1/\text{SNR} \),

\[
\theta_{\text{c}}(K;\text{SNR}) = \arcsin \left( \sqrt{1 - \frac{1}{K(1 + K)\text{SNR}}} \right).
\]
Note that with this parametrization of $\theta_{c}$ one has a simple parametrization of the sphere-packing exponent (or alternatively a simple parametrization of the upper bound $P_{\text{region}}$ for the region $R = R_{c}(\theta_{c}(K; \text{SNR}))$ in terms of $K$. More precisely, one has

$$E_{\text{sp}}(R(\theta_{c}(K; \text{SNR})); \text{SNR}) = -1 + K \text{SNR} - K \log \left(1 + \frac{1}{R} - \frac{1}{K^2 \text{SNR}}\right).$$

and

$$P_{\text{region}} \leq \exp \left(-n \left(-1 + K \text{SNR} - K \log \left(1 + \frac{1}{R} - \frac{1}{K^2 \text{SNR}}\right)\right) \right).$$

Using this parametrization we now precisely characterize the freedom one has in choosing $\theta$. We let, for $R(\theta_{c}(K; \text{SNR})) < R_{\text{crit}}$, $z(K; d, R, \text{SNR})$ be $2K$ times the difference of $E_{\text{sp}}$ and $E_{\text{bnd}}$ for the region $R = R_{c}(\theta_{c}(K; \text{SNR}))$ and an ensemble with typical error events that occur with codewords at a distance $d_{c}^{*} = d$. More precisely we let

$$z(K; d, R, \text{SNR}) = -1 + K \left(2R + \left(1 - \frac{d^{2}}{4}\right) \frac{\text{SNR}}{\text{SNR}}\right) + K \log \left[\frac{d^{2} \left(1 - \frac{d^{2}}{4}\right)}{K(1 + K) \text{SNR} - 1}\right].$$

We can now use the characterization of the typical error events (61) to help find the minimal $\theta$ such that one may obtain the proper error exponent. First, we state the following property of the function $z(K; d, R, \text{SNR})$, as a function of $K$, has a unique zero on the interval $[1/\text{SNR}, \infty)$ for $d \geq 0$, $R \geq 0$ and $\text{SNR} > 0$.

A proof is provided in Appendix VII.

We let $K_{c}(d; R, \text{SNR})$ be the unique root of $z(K; d, R, \text{SNR})$ on the interval $[1/\text{SNR}, \infty)$. Hence, for rates $R \leq R_{\text{crit}}$, if one chooses $\theta \geq \theta_{c}(K_{c}(d; R, \text{SNR}); \text{SNR})$, then

$$P_{\text{region}} \leq f_{\text{bnd}}(d_{\text{typ}}(\theta(R)), \theta(R); \text{SNR})$$

Thus, we let $\theta_{\text{AWGN}}(R; \text{SNR})$ be the smallest $\theta$ such that (65) holds for the ensemble $\Omega_{\text{III}}^{(R)}$ using a region $R = R_{c}(\theta)$. That is, $\theta_{\text{AWGN}}(R; \text{SNR})$ satisfies (66) at the bottom of the page. Then, if we use any region $R = R_{c}(\phi(R))$ such that

$$\theta_{\text{AWGN}}(R; \text{SNR}) \leq \phi(R) \leq \arcsin \exp(-R),$$

the results of Theorem 2 still hold. We state this in the following theorem.

**Theorem 3:** Consider the sequence of ensembles of random spherical codes $\{\Omega_{\text{III}}^{(R)}(R)\}$ and let $\phi(R)$ be given such that

$$\theta_{\text{AWGN}}(R; \text{SNR}) \leq \phi(R) \leq \arcsin \exp(-R).$$

If $R = R_{c}(\phi(R))$, then the typical error events occur with codewords at a distance $d_{\text{typ}}(R)$ and for $0 \leq R \leq C$:

**A.** $P_{\text{union}}^{r} \leq f_{\text{bnd}}(d_{\text{typ}}(R), \theta_{\text{AWGN}}^{*}(R); \text{SNR})$

**B.** $P_{\text{region}}^{r} \leq f_{\text{bnd}}(d_{\text{typ}}(R), \theta_{\text{AWGN}}^{*}(R); \text{SNR})$

Moreover,

$$f_{\text{bnd}}(d_{\text{typ}}(R), \theta_{\text{AWGN}}^{*}(R); \text{SNR}) \equiv e^{-n E_{\text{AWGN}}(R; \text{SNR})}.$$

Taking $\phi(R) = \theta_{\text{AWGN}}(R; \text{SNR})$ in the preceding theorem yields our final characterization of the error exponent of the AWGN channel.

**Corollary 1:** Consider the sequence of ensembles of random spherical codes $\{\Omega_{\text{III}}^{(R)}(R)\}$. If $R = R_{c}(\theta_{\text{AWGN}}(R))$, then Gallager’s bounding technique (9) for the average probability of error satisfies the following two properties for $0 \leq R \leq C$:

1. $P_{\text{region}}^{r} \leq \exp(-n E_{\text{AWGN}}(R; \text{SNR}))$

2. $P_{\text{union}}^{r} \leq P_{\text{region}}^{r}$

$$\theta_{\text{AWGN}}(R; \text{SNR}) = \begin{cases} \theta_{c}(K_{c}(d_{\text{typ}}(R); R, \text{SNR}); \text{SNR}) & \text{if } 0 \leq R < R_{\text{crit}} \\ \arcsin \exp(-R) & \text{if } R_{\text{crit}} \leq R \leq C \end{cases}$$

(66)
Further, if \( \mathcal{R} = \mathcal{R}_c(\theta_{\text{AWGN}}(R)) \) then the typical error events occur with codewords at a distance \( d_{\text{typ}}(R) \).

As in the case of the sphere-packing bound, one may ask whether there are smaller regions than the cone \( \mathcal{R}_c(\theta_{\text{AWGN}}) \) that may be used to derive the same exponential upper bound on \( P_{\text{union}} \). We extend our previous definition of “valid” regions [cf. (21)] to be the regions \( \mathcal{R} \) such that:

1. \( \mathbb{P}(\mathbf{c} + \mathbf{z} \notin \mathcal{R}) \equiv \mathbb{P}(\mathbf{c} + \mathbf{z} \notin \mathcal{R}_c(\theta_{\text{AWGN}})) \) \hspace{1cm} (67)
2. \( \mathcal{R} \subset \mathcal{R}_c(\theta_{\text{AWGN}}) \) \hspace{1cm} (68)

Recall that in order to show that the mod-\( \Lambda \) channel can achieve the sphere-packing error exponent for the AWGN channel we took a scaling \( \alpha \) that corresponded to a valid sphere. More precisely, in Section III-C we showed that one may use the sphere tangent to the cone \( \mathcal{R}_c(\theta(R)) \) and achieve the same exponential upper bound on \( P_{\text{region}} \). Replacing \( \theta(R) \) with \( \theta_{\text{AWGN}}(R) \) in that context one may do the same. However, in the sequel we show that the spherical region \( \mathcal{R}_c(\theta_{\text{AWGN}}(R)) \) does not correspond to the optimal scaling in the mod-\( \Lambda \) channel. Indeed, we have shown for rates less than \( R_{\text{crit}} \) the upper bound on the union bound, \( f_{\text{typ}} \), dominates our bound on the error exponent. Hence, one would expect that the optimal scaling would relate to the half angle \( \Theta(d)/2 \) and not to the half angle of the cone \( \mathcal{R}_c(\theta) \). We show in the sequel that the optimal scaling does indeed relate to the half angle \( \Theta(d)/2 \) and, for the ensemble \( \Omega_1^{(n)}(R) \), is equal to:

\[
\alpha_{\text{AWGN}}^*(R) = \alpha_e^*(\max\{R, R_{\text{crit}}\})
\]

and, for the ensemble \( \Omega_{\text{HI}}^{(n)}(R) \), is equal to \( \alpha_{\text{AWGN}}(R; \text{SNR}) \) defined in (69) at the bottom of the page\(^1\)

We show in Appendix IV that the ensemble \( \Omega_{\text{HI}}^{(n)}(R) \) achieves the full AWGN error exponent \( E_{\text{AWGN}}(R; \text{SNR}) \) using the scaling \( \alpha_{\text{AWGN}}(R; \text{SNR}) \). We now provide our final characterization of the error exponent of the mod-\( \Lambda \) channel and show that in general a scaling different than \( \alpha_{\text{AWGN}}(R; \text{SNR}) \) is needed to achieve our best bound on the error exponent of the mod-\( \Lambda \) channel.

**VIII. Error Exponents in the Mod-\( \Lambda \) Channel**

In Section IV we provided a simple proof that the mod-\( \Lambda \) channel achieves the sphere-packing error exponent for rates greater than the critical rate under the assumption that \( P_{\text{union}}^{\lambda,r} \leq P_{\text{region}}^{\lambda} \). Here we show that this is indeed true and provide an exponential bound for \( P_{\text{union}}^{\lambda,r} \) that is exponentially equal to the random coding exponent \( E_{\text{AWGN}}(R; \text{SNR}) \). As in Section IV we provide a simple relation to the derivation of the AWGN error exponent using spherical regions. For this reason we take the region \( \mathcal{R} \) to be a sphere of radius \( \sqrt{nP} \cdot r \) centered at the codeword. We denote this region \( \mathcal{R}_\lambda(r) \).

Note that \( \mathcal{R}_\lambda(r) \) is not an actual decoding region. However, in order to choose the radius of the sphere \( \mathcal{R}_\lambda(r) \) we may use the same intuition that led to our choice of the cone \( \mathcal{R}_c(\theta(R)) \) in our derivation for the AWGN channel. That is, we can choose \( r \) such that \( \mathcal{R}_\lambda(r) \) has a volume equal to the average volume of the Voronoi under ML decoding. Thus, for rates greater than the critical rate we consider \( r = r^*(R) = \sin(\theta(R))/\alpha = \exp(-R)/\alpha \). Analogous to our definition of \( R(\theta) \) we let

\[
R^*(\alpha) = -\log(\alpha \cdot r)
\]

so that \( R^*(\alpha(R)) = R \).

We consider the ensemble of random coset codes that are drawn i.i.d from a uniform distribution over the Voronoi region of a lattice \( \Lambda \) that is good for quantization and good for covering. Recall from (22) that for any code \( \mathcal{C} \) and any given codeword \( \mathbf{c} \in \mathcal{C} \) we have

\[
P^\lambda_{\mathcal{C}} \leq P_{\text{region}}^\lambda + P_{\text{union}}^{\lambda,r},
\]

where

\[
P_{\text{union}}^{\lambda,r} = \mathbb{E} \left[ \frac{1}{|\mathcal{C}|} \right] \sum_{\mathbf{c} \in \mathcal{C}} \sum_{\mathbf{c}', \mathbf{z} \in \mathcal{C}} \mathbb{P} \left( \mathcal{E}_\alpha(\mathbf{y}, \mathbf{c}), \mathbf{c}' + \mathbf{z} \in \mathcal{R}_\lambda(r) \right)
\]

and the expectation is taken over the ensemble \( \hat{\Omega}_1^{(n)}(\Lambda, \alpha) \) of random coset codes. It is important to note that for a fixed code each codeword, say \( \mathbf{c} \), and its translates \( \mathbf{c} + \lambda \) for \( \lambda \in \Lambda \) are dependent. However, by averaging over the ensemble of codes the distribution of the codewords \( \mathbf{c}_e \) are uniform over \( \mathbb{R}^n \) with a density\(^1\) of \( e^{nR}/\sqrt{n}P \).

\(^1\)Recall \( \alpha_e^*(R) \) was defined in (23), \( \beta_e^*(\theta; \Theta; \text{SNR}) \) was defined in (57), \( \Theta(d) \) was defined in (49) and \( \alpha_e(\beta, \theta) \) was defined in (23).

\(^{13}\)Note that in the scaled lattice \( \Lambda/\alpha \) the codewords are uniform over \( \mathbb{R}^n \) with a density of \( \alpha e^{nR}/\sqrt{n}P \).

\[
\alpha_{\text{AWGN}}(R; \text{SNR}) = \begin{cases} 
\alpha_e(\beta_e^*(\theta(R), \Theta(d_{\text{typ}}(R))/2; \text{SNR}), \Theta(R)) & \text{if } R_{\text{crit}} < R < C, \\
\alpha_e(\text{SNR} \cdot \beta_e^*(\theta(R), \Theta(d_{\text{typ}}(R))/2; \text{SNR}), \Theta(d_{\text{typ}}(R))/2) & \text{if } 0 < R \leq R_{\text{crit}}.
\end{cases}
\]

(69)
Although the distribution of codewords are rotationally symmetric since they are uniformly distributed over $\mathbb{R}^n$, the dither introduces an asymmetry for error events between codewords at a given distance from a selected codeword. It is necessary to consider the orientation of any codeword to the dither when examining the error events. As depicted in Fig. 9, let $\Theta_p$ be the angle made between the direction of the dither and the line from the original codeword to an arbitrary codeword at distance $d$. Note that in the case of a spherical code using the relation shown in Fig. 7, $\Theta_p = \pi/2 - \Theta(d)$ was always such that $\sin \Theta_p = 1 - d^2/2$ since the codewords were constrained to lay on the surface of a sphere [cf. (49)]. However, in the case of the mod-$\Lambda$ channel $\Theta_p$ varies independently of $d$.

We use the same method as used previously to further bound (70). That is, we use the tangential sphere bound and integrate with respect to the radial component of the noise. As before, we consider the region that is the intersection of $\mathcal{R}_\lambda(r)$ and the half space that orthogonally bisects the line connecting the original codeword to any codeword at a distance $d$ and angle $\Theta_p$. As Fig. 9 depicts, it is much simpler to consider this region when parametrized by $l = d^2 \cos \Theta_p$, and thus we denote this region by $D_\lambda(r, d, l)$. Note that in order to find the dominating event it is sufficient to optimize over the distance $d$ and $l$ since this pair uniquely specifies $\Theta_p$.

It is shown in Appendix VI that

$$
\mathbb{P}_{\text{union}}^{c/\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l) | \mathbf{u}} \leq \max_{0 \leq d \leq 2r, l \geq K_\alpha} \frac{c}{\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l)}
$$

\begin{equation}
\times \exp \left( nR + (n-1) \log \left( 2^{d^2/4l^2} \right) \right).
\end{equation}

In order to arrive at an exponential bound to the right-hand side of (71) we begin by providing an exponential bound for $\mathbb{P}(c/\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l) | \mathbf{u})$. We use the tangential sphere bound as in Section VI by defining the radial direction to be the direction of the dither $\mathbf{u}$. It is clear due to the rotational symmetry that

$$
\mathbb{P}(c/\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l) | \mathbf{u}) = \mathbb{P}(c/\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l))
$$

and henceforth we write $\mathbb{P}(c/\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l))$. From Fig. 9 we see that conditioned on the event that the radial component of the noise has a magnitude of $\sqrt{nP} \cdot \beta$, we have that $c/\alpha + z'_{\text{eff}} \in D_\lambda(r, d, l)$ if the second component is greater than $x_\lambda$ while the sum of the second through $n$th component is less than $y_\lambda$ where

$$
x_\lambda = x_\lambda(\beta, l, \alpha) = \frac{l - (\beta + K_\alpha)}{\sqrt{\frac{d^2}{4l^2} - 1}}
$$

and

$$
y_\lambda^2 = y_\lambda^2(\beta, r, \alpha) = r^2 - (\beta + K_\alpha)^2.
$$
Thus, applying Proposition 5 and Proposition 3 we have as in (55)
\[
\mathbb{P}\left(\frac{c}{\alpha} + z_{\text{eff}} \in \mathcal{D}_\lambda(r, d, l)\right) = e^{o(1)} \mathbb{P}\left(\frac{c}{\alpha} + z_{\text{eff}}'' \in \mathcal{D}_\lambda(r, d, l)\right) \leq \max_{\beta_\lambda} \exp\left(-n E_{\text{bad}}(\beta_\lambda, x_\lambda, y_\lambda; \text{SNR})\right)
\]
(72)

Thus, (71) becomes
\[
P_{\text{union}}^{\lambda,r} \leq \max_{(d,l)} \max_{\beta_\lambda} \exp\left(-n E_{\text{bad}}(r, K_\alpha, l, d, \beta, R; \text{SNR})\right).
\]
(74)

where \(E_{\text{ind}}^\lambda\) is defined in (75) at the bottom of the page. Then, it is simple to check that the \(\beta_\lambda\) that maximizes (74) for a fixed \(d\) and \(l\) is

\[
\beta_\lambda^*(r, d, l; \text{SNR}) = \begin{cases} 
\beta_\lambda^*(r, d, l; \text{SNR}) & \text{if } R^\alpha(r) > R_{\text{cr}}^\alpha(d, l) \\
\beta_\lambda^*(r, d, l; \text{SNR}) & \text{otherwise}
\end{cases}
\]
(76)

where \(\beta_\lambda^*(r, d, l; \text{SNR})\) satisfies (77) at the bottom of the page and in turn where

\[
R_{\text{cr}}^\alpha(d, l) \triangleq -\frac{1}{2} \log\left[\frac{1}{1 + K_\alpha} \left(\frac{d^2}{4} + K_\alpha^2 \left(1 - \frac{d^2}{4l^2}\right) + \frac{1}{\text{SNR}}\right)\right].
\]

In our derivation of an exponential bound for \(P_{\text{region}}\) we were able to show that the typical error events in the AWGN channel and the mod-\(\Lambda\) channel coincide if we use the scaling that is equivalent to a valid sphere. We now show that this is again the case for \(P_{\text{union}}^{\lambda,r}\) and thus the mod-\(\Lambda\) channel can achieve the random coding error exponent, \(E_{\text{AWGN}}^\lambda(R; \text{SNR})\). We first consider what parameters one must choose in order for the geometry of Fig. 9 to equal that in Fig. 7. Thus, applying Proposition 5 and Proposition 3 we have as in (53)

\[
P_{\text{union}}^{\lambda,r} \leq \exp\left(-n E_{\text{bad}}(r, K_\alpha, l, d, \beta, R; \text{SNR})\right).
\]

Examining these figures it is clear that for this to be true we must have that

\[
l - K_\alpha = \beta_\lambda^*(r, d, l; \text{SNR}) = 1 + \beta_\lambda^*(\theta; \text{SNR})
\]
(78)

where \(d_\lambda^* = d_\lambda^*(r, l, \alpha; \text{SNR})\) and \(l_\lambda^* = l_\lambda^*(r, \alpha; \text{SNR})\) are the distance and value of \(l\) that maximize \(P_{\text{union}}^{\lambda,r}\) respectively. Further, it is clear that if \(r = (\sin \theta)/\alpha\) then one must have \(\Theta_\lambda = \frac{\pi}{2} - \frac{2\pi\lambda d}{\alpha^2}\) and \(l = 1/\alpha\) for the geometry to agree. Thus, to show that mod-\(\Lambda\) channel is able to achieve the random coding error exponent using our relation we would require that

\[
l_\lambda^* \left(\frac{\sin \theta_{\text{AWGN}}^r(R)}{\alpha_{\text{AWGN}}^r(R)}, r, \alpha_{\text{AWGN}}^r(R); \text{SNR}\right) = \left[1 + \frac{1}{\alpha_{\text{AWGN}}^r(R)}\right]^{\alpha_{\text{AWGN}}^r(R)} - R
\]

and \(d_\lambda^*\) to satisfy (79) at the bottom of the page

and further that these values satisfy (78) for \(\theta = \theta_{\text{AWGN}}^r\). It is easy to check that these values do indeed satisfy (78) through direct substitution. We provide the general derivation of the parameters \(l_\lambda^*\) and \(d_\lambda^*\) for general \(\alpha\) in Appendix III. The case when \(\alpha = \alpha_{\text{AWGN}}^r(R)\) is easy to verify and leads to the following theorem.

**Theorem 4:** Consider the sequence of random random coset ensembles \(\{\Omega_\alpha^i(n)\}\) where \(\{\Lambda_n\}\) is a sequence of lattices that are good for covering and quantization. If \(R = R_{\text{cr}}^\alpha(R)\) and \(\alpha = \alpha_{\text{AWGN}}^r(R)\), then Gallager’s bounding technique [9] for the average probability of error satisfies the following two properties for \(0 \leq R \leq C\):

1'. \(P_{\text{union}}^{\lambda,r} \leq \exp\left(-n E_{\text{AWGN}}^r(R; \text{SNR})\right)\)

2'. \(P_{\text{union}}^{\lambda,r} \geq P_{\text{region}}\)

where \(E_{\text{AWGN}}^r(R)\) was defined in (43).

A proof is provided in Appendix III.

Recall that when we considered the AWGN error exponent, the random coding error exponent \(E_{\text{AWGN}}^r(R; \text{SNR})\) could be improved for some rates by considering an ensemble of codes with minimum distance, \(d_{\text{min}}(R)\). It is natural to expect that the same can be done here and indeed it can. That is, as done in Section VI, we can attempt to improve on the random coding error exponent by considering the ensembles of rate \(R\) coset codes that meet a constraint on the minimum distance, which

\[
E_{\text{bad}}^\lambda(r, K_\alpha, l, d, \beta; \text{SNR}) = E_{\text{bad}}(\beta, x_\lambda, y_\lambda; \text{SNR}) - \frac{1}{2} \log\left[\frac{d^2}{4l^2} \left(1 - \frac{d^2}{4l^2}\right)\right] - \frac{1}{2} \log\left[\frac{l^2}{(1 + K_\alpha)^2}\right] - R.
\]
(75)

\[
l - K_\alpha - \beta_\lambda^*(r, d, l; \text{SNR}) = l \left(1 - \frac{d^2}{4l^2}\right) + \frac{1}{2K_\alpha \text{SNR}} - \sqrt{\frac{1}{4K_\alpha^2 \text{SNR}^2} + \left(r^2 - \frac{d^2}{4l^2}\right) \left(1 - \frac{d^2}{4l^2}\right)}.
\]
(77)
we denote as \( r_\Omega(R) \). That is, the ensembles \( \hat{\Omega}_{\Lambda}^{(A,n)}(R) = (\hat{\Omega}_{\Lambda}^{(A,n)}, r_\Omega(R)) \) (provided such an ensemble exists). Note, for the mod-\( \Lambda \) channel we must consider the minimum distance for codes distributed over \( \mathbb{R}^N \) rather than over the unit sphere.

In order to determine which \( r_\Omega(R) \) led to valid ensembles it is often easier to study a normalized version of the minimum distance. Define, \( r_{\Lambda} \), as

\[
\rho_{\Lambda} = \frac{r_{\Omega}(R)}{r_{\Lambda}^\text{eff}},
\]

where \( r_{\Lambda}^\text{eff} \) was defined in (34). It is a classic problem to determine the largest possible value for \( \rho_{\Lambda} \). In this direction, let

\[
\rho = \lim_{n \to \infty} \sup_{\Lambda} \rho_{\Lambda}.
\]

Then, we can improve upon the random coding exponent for rates such that \( \alpha d_\star^r < 2\rho r_{\Lambda}^\text{eff} / \sqrt{nF} \). From (79) if \( \alpha = \alpha_{\text{AWGN}}^r(R) \) this is equivalent to

\[
d_{\text{crit}} \leq 2\rho r_{\Lambda}^\text{eff} / \sqrt{nF} \leq 2\rho \exp(-R) \tag{80}
\]

if \( \rho < 1/\sqrt{7} \) and where the last inequality is satisfied with equality if \( \Lambda \) is good for covering. To date the best known bounds on \( \rho \) are [20]

\[
\frac{1}{2} \leq \rho \leq 0.660211...
\]

Henceforth we consider ensembles of codes such that \( r_\Omega(R) \leq \exp(-R) \). In particular, we consider the ensemble of coset codes

\[
\hat{\Omega}_{\Omega\Lambda}^{(A,n)}(R) = (\hat{\Omega}_{\Omega\Lambda}^{(A,n)}, e^{-R})
\]

where \( \Lambda \) is good for covering and quantization.

We now return to our original problem of improving upon the random coding error exponent. Begin by noting that for the ensemble \( \hat{\Omega}_{\Omega\Lambda}^{(A,n)}(R) \) if \( \Lambda \) is good for covering we can improve upon the random coding exponent for rates \( R < R_{\Omega\Lambda} \) where

\[
R_{\Omega\Lambda} = \max \{ 0, - \log(d_{\text{crit}}) \}.
\]

Further, note the definition of \( \hat{\Omega}_{\Omega\Lambda}^{(A,n)}(R) \) is similar to the definition of \( \Omega_{\Omega\Lambda}^{(n)}(R) \) in (47) in Section VI. In fact, one can use similar arguments to those leading to Theorem 4 to show that the typical error events of \( \hat{\Omega}_{\Omega\Lambda}^{(A,n)}(R) \) and \( \Omega_{\Omega\Lambda}^{(n)}(R) \) coincide given the appropriate scaling for the lattice. In this direction we provide the following definition. Let, for any \( r_\Omega(R) \leq \exp(-R) \),

\[
K_{\alpha}^* = K_{\alpha}^*(r_\Omega(R); \text{SNR})
\]

where \( K_{\alpha}^*(r_\Omega(R); \text{SNR}) \) is defined in (81) at the bottom of the page, and let

\[
\alpha_{\Lambda}^*(r_\Omega(R); \text{SNR}) = \frac{1}{1 + K_{\alpha}^*(r_\Omega(R); \text{SNR})}.
\]

In Appendix III we show that choosing \( \alpha = \alpha_{\Lambda}^*(r_\Omega(R); \text{SNR}) \) one has

\[
l_{\alpha}^* (r_{\alpha}(\Lambda, n); \Omega_{\Lambda}^*, n) = 1 + K_{\alpha}^*(r_\Omega(R); \text{SNR}) \tag{82}
\]

yielding

\[
\min_{d \geq r_{\Omega}(R) \beta_{\Lambda}^{-1}} \min_{d \geq r_{\Omega}(R) \beta_{\Lambda}^{-1}} E_{\text{bnd}}((1 + K_{\alpha}^*) \cdot \sin \theta, K_{\alpha}^*, d, \beta, R; \text{SNR})
\]

\[
= \min_{d \geq r_{\Omega}(R) \beta_{\Lambda}^{-1}} E_{\text{bnd}}(\theta, d, \beta, R; \text{SNR}) \tag{83}
\]

\[
= \tilde{E}_{\text{bnd}}(r_{\Omega}(R), \theta; \text{SNR}) \tag{84}
\]

\[
\alpha_{\text{AWGN}}^r(R) \cdot d_{\Lambda}^r \left( \frac{\sin \theta_{\text{AWGN}}^r(R)}{\alpha_{\text{AWGN}}^r(R)} \cdot \frac{1}{\alpha_{\text{AWGN}}^r(R)} \cdot \alpha_{\text{AWGN}}^r(R); \text{SNR} \right) = \left\{ \begin{array}{ll}
\sqrt{2} \sin \theta_{\text{AWGN}}^r & \text{if } R > R_{\text{crit}} \\
\frac{1}{d_{\text{crit}}} & \text{otherwise}
\end{array} \right. \tag{79}
\]

\[
K_{\alpha}^*(r_\Omega(R); \text{SNR}) = \left\{ \begin{array}{ll}
(1 - \alpha_{\Lambda}^*(\theta(R))) / \alpha_{\Lambda}^*(\theta(R)) & \text{if } R_{\text{crit}}(\theta) < R < C \\
(1 - d_{\text{crit}}/4)^{-1} \text{SNR}^{-1} & \text{if } R \leq R_{\text{crit}} \text{ and } d_{\text{crit}} > d_{\Omega}(R) \\
(1 - r_{\Omega}(R)^2/4) \cdot \text{SNR}^{-1} & \text{if } R \leq R_{\text{crit}} \text{ and } d_{\text{crit}} \leq d_{\Omega}(R) \tag{81}
\end{array} \right.
\]
We note that \( \mathbf{82} - \mathbf{84} \) is the formal statement of our geometric equivalence depicted in Fig. I. That is, choosing the appropriate scaling, \( \alpha^*_\Lambda (r_\Omega (R); \text{SNR}) \), the typical error events in the mod-\( \Lambda \) and AWGN channels coincide. This yields the following theorem.

**Theorem 5:** Consider the sequence of ensembles of random lattice codes \( (\hat{\Omega}^{(\Lambda,n)}, r_\Omega (R)) \) where \( \{ \Lambda_n \} \) is a sequence of lattices that are good for covering and quantization. If \( \mathcal{R} = \mathcal{R}_\Lambda (r^* (R)), 0 \leq R < C, \) and \( \alpha = \alpha^*_\Lambda (r_\Omega (R); \text{SNR}) \), then Gallager’s bounding technique \( \mathbf{9} \) for the average probability of error is exponentially equal to that of the ensemble \( (\hat{\Omega}^{(\Lambda,n)}, r_\Omega (R)) \). That is,

\[
\mathcal{P}^{\Lambda,r}_{\text{union}} \leq f_{\text{bound}}(r_\Omega (R), \theta (R); \text{SNR})
\]

with \( f_{\text{bound}} (d, \theta; \text{SNR}) \) as defined in \( \mathbf{59} \). Moreover, with \( \alpha = \alpha^*_\Lambda (r_\Omega (R); \text{SNR}) \), \( l_{\Lambda}^* (r, \alpha; \text{SNR}) = 1/\alpha \) and the typical error events for the ensemble \( (\hat{\Omega}^{(\Lambda,n)}, r_\Omega (R)) \) and \( (\hat{\Omega}^{(\Lambda)}, r_\Omega (R)) \) coincide. A proof is provided in Appendix \( \mathbf{III} \).

In order to provide the desired relation to the ensemble \( \Omega^{(\Lambda)}_\Omega (R) \) we let

\[
\alpha^*_{\Lambda} (R; \text{SNR}) = \frac{1}{1 + K_{\alpha}^* (\exp(-R); \text{SNR})}.
\]

**Corollary 2:** Consider the sequence of random coset ensembles \( \{ \hat{\Omega}^{(\Lambda,n)} \} \) where \( \{ \Lambda_n \} \) is a sequence of lattices that are good for covering and quantization. If \( \mathcal{R} = \mathcal{R}_\Lambda (r^* (R)) \) and \( \alpha = \alpha^*_{\Lambda} (R; \text{SNR}) \), then Gallager’s bounding technique \( \mathbf{9} \) for the average probability of error of the ensemble \( \hat{\Omega}^{(\Lambda,n)}_\Omega (R) \) is exponentially equal to that of the ensemble \( \Omega^{(\Lambda)}_\Omega (R) \). Moreover,

\[
\exp(-n E^*_{\text{AWGN}} (R; \text{SNR})) = \exp(-n E_\Omega (R; \text{SNR}))
\]

\[
\leq \exp(-n E_{\Omega} (R; \text{SNR})) = \exp(-n E^*_{\text{AWGN}} (R; \text{SNR}))
\]

As done in Section \( \mathbf{VI} \) it is natural to ask whether one can improve upon the error exponent for the mod-\( \Lambda \) with a different choice of region than that taken in Corollary \( \mathbf{2} \) as \( \mathcal{P}^{\Lambda,r}_{\text{union}} \geq P_{\text{region}} \). We now characterize the freedom one has in this choice. Let

\[
d^\Omega_{\text{typ}} (R) \triangleq \begin{cases} \exp(-R) & \text{if } 0 \leq R \leq R_\Omega \\ d_{\text{crit}} & \text{if } R_\Omega < R \leq R_{\text{crit}} \\ \sqrt{2} \exp(-R) & \text{if } R_{\text{crit}} < R \leq C \end{cases}
\]

and let \( \theta^*_{\Lambda} (R; \text{SNR}) \) be defined as in \( \mathbf{85} \) at the bottom of the page. In turn, let

\[
r^*_{\Lambda} (R; \text{SNR}) = \frac{\exp(-\sin \theta^*_{\Lambda} (R; \text{SNR}))}{\alpha^*_{\Lambda} (R; \text{SNR})}
\]

That is, \( r^*_{\Lambda} (R; \text{SNR}) \) is the smallest radius such that \( f_{\text{bound}} (d^\Omega_{\text{typ}} (R); R, \text{SNR}) \geq P_{\text{region}} \). This is characterized in the following theorem.

**Theorem 6:** Consider the sequence of random random coset ensembles \( \{ \hat{\Omega}^{(\Lambda,n)}_\Omega \} \) where \( \{ \Lambda_n \} \) is a sequence of lattices that are good for covering and quantization. If \( \mathcal{R} = \mathcal{R}_\Lambda (r^*_\Lambda (R)) \) and \( \alpha = \alpha^*_{\Lambda} (R; \text{SNR}) \), then Gallager’s bounding technique \( \mathbf{9} \) for the average probability of error satisfies the following two properties for \( 0 \leq R \leq C \):

\[
1'. \quad \mathcal{P}^{\Lambda,r}_{\text{union}} \leq \exp (-n E_{\Omega} (R; \text{SNR}))
\]

\[
2'. \quad \mathcal{P}^{\Lambda,r}_{\text{union}} \leq P_{\text{region}}
\]

It is easy to see by examining \( \mathbf{80} \) and \( \mathbf{45} \) that using our derivation, the error exponent of the mod-\( \Lambda \) channel would be to equal that of the AWGN channel had the minimum distance of the coset code been equal to that of the spherical code. However, the best known lower bound on the minimum distance in a constellation in \( \mathbb{R}^N \) with a given density is less than that of a spherical code \( \mathbf{13} \). Thus, the error exponent for the mod-\( \Lambda \) channel cannot be shown to be equivalent to that of the AWGN channel using this approach for rates less than \( R_x \). In fact, by examining the exponents at low rates one can see that even the best known upper bound for \( \rho \) is not sufficient to achieve the AWGN error exponent for all rates less than \( R_x \).

\[
\theta^*_{\Lambda} (R; \text{SNR}) = \begin{cases} \theta^*_{\cal C} (d^\Omega_{\text{typ}} (R); R, \text{SNR}; \text{SNR}) & \text{if } 0 \leq R < R_{\text{crit}} \\ \arcsin \exp (-R) & \text{if } R_{\text{crit}} \leq R \leq C \end{cases}
\]
VIII. Conclusion

It remains an open problem to show whether the mod-$\Lambda$ channel can achieve the expurgated error exponent for all rates. Note, that in our derivation of the error exponent we used the sub-optimal Euclidean distance decoder. One may ask whether the closest coset decoder or a true ML decoder could achieve the expurgated error exponent. It is our conjecture that this in fact cannot be done. This conjecture is motivated by the fact that it was shown in (83)–(84) that using a sub-optimal Euclidean distance decoder a linear scaling existed such that the mod-$\Lambda$ channel meets the best known lower bound on the reliability of the AWGN channel if the mod-$\Lambda$ channel and AWGN channel codes have the same minimum distance. However, as it is known that the minimum distance of a lattice is less than that of a spherical code at low rates it is unlikely that the mod-$\Lambda$ channel can achieve the expurgated error exponent for all rates using a ML decoder. However, the mod-$\Lambda$ channel is itself suboptimal in the fact that it uses a linear estimator at the receiver. It may be possible to show that lattice encoding and decoding could achieve the expurgated error exponent by using a non-linear receiver.

Appendix I

Proof of (20)

Begin by noting that

$$(1 + \beta^*) = \frac{1 + \rho_G}{\text{SNR}} (\beta_G - 1)$$

and

$$\left( \frac{(1 + \beta^*)^2}{\cos^2 \theta(R)} - (1 + \beta^*) \right) = \frac{1}{\text{SNR}}.$$ 

Then, we have

$$1 + \rho_G = \text{SNR} e^{-2R} \frac{1 + \beta^*}{1 - e^{-2R}}$$

$$= \text{SNR} \frac{1 - e^{-2R}}{1 - e^{-2R}} (1 + \beta^*)$$

$$= \frac{r^2(\beta^*) \text{SNR}}{(1 + \beta^*)}$$

so that

$$\beta_G - \frac{\text{SNR}}{1 + \rho_G} = e^{2R} \frac{1}{r^2(\beta^*)} \left( (1 + \beta^*) - \frac{(1 + \beta^*)^2}{\cos^2 \theta(R)} - (1 + \beta^*) \right)$$

$$= \frac{1}{r^2(\beta^*)} \left( (1 + \beta^*)^2 - (1 + \beta^*) \right)$$

$$= \frac{1}{r^2(\beta^*)} \left( (1 + \beta^*) - (1 + \beta^*) \right)$$

$$= \frac{1}{r^2(\beta^*)} \left( (1 + \beta^*) \right)$$

Now,

$$2E_v(\beta^* \text{SNR})$$

$$= \text{SNR} (\beta^*)^2$$

$$= \text{SNR} \left( -1 + (1 + \beta^*)^2 \right)$$

$$= \text{SNR} \left( 1 - 2(1 + \beta^*) + (1 + \beta^*)^2 \right)$$

$$= \text{SNR} \left( 1 - 2(1 + \beta^*) + (1 + \beta^*)^2 \left( \frac{1}{\cos^2 \theta(R)} - \frac{1}{\text{SNR}} \tan^2 \theta(R) \right) \right)$$

$$= \text{SNR} \left( 1 - (1 + \beta^*) + \frac{1}{\text{SNR}} - r^2(\beta^*) \right)$$

$$= \text{SNR} - \text{SNR} (1 + \beta^*) - 2E_h(r^2(\beta^*) \text{SNR}) - \log(r^2(\beta^*) \text{SNR})$$

Thus,

$$2E_v(\beta^* \text{SNR}) + 2E_h(r^2(\beta^*) \text{SNR})$$

$$= \text{SNR} - \text{SNR} (1 + \beta^*) - \log(r^2(\beta^*) \text{SNR})$$

$$= \text{SNR} - (1 - \beta_G)(1 + \rho_G) + \log \left( \frac{\beta_G - \frac{\text{SNR}}{1 + \rho_G}}{\beta_G} \right)$$

$$= 2E_{sp}(R; \text{SNR})$$
APPENDIX II
DERIVATION OF MAXIMIZING PARAMETERS FOR RANDOM CODING EXPONENT

We now provide a derivation of the minimizing parameters for the random coding exponent. Begin by examining (53) and note that the exponent \( E_d(\beta, x, y;\text{SNR}) \) has two cases based on the values of \( x \) and \( y \) relative to \( \text{SNR} \). Note that by using the inequality \( \log \frac{x}{x - 1} \) for \( x > 1 \) we have that

\[
\text{SNR} \cdot (\beta^2 + y^2) - \log(e\text{SNR}(y^2 - x^2)) \geq \text{SNR} \cdot (x^2 + y^2).
\]

Thus, since \( E_d \) is continuous and increasing in \( \beta \) we can always considering minimizing \( E_d \) under the assumption \( y^2 - x^2 \leq 1/\text{SNR} \) and provide an improvement if this is not the case. However, since we have freedom in our choice of \( x \) it should be clear that for any \( \beta_c \) it is always advantageous to pick \( \theta \) such that \( y_c(\beta_c, \Theta(d))^2 - x_c(\beta_c, \Theta(d))^2 \geq 1/\text{SNR} \). We examine how this may be done after first considering the minimization under the assumption \( y^2 - x^2 \leq 1/\text{SNR} \).

Consider the minimization of the exponent \( E_d(R) \) under the assumption \( y_c^2 - x_c^2 \leq 1/\text{SNR} \). In this case the exponent \( E_d(R) \) is greatly simplified and in fact simply reduces to the exponent of a one dimensional Gaussian. That is, in the case that \( y_c^2 - x_c^2 \leq 1/\text{SNR} \) the pairwise error probability between two codewords at a distance \( d \), say \( c \) and \( c_c \), is dominated by the probability that a one dimensional component of the noise crosses the ML decoding plane and stays in the cone \( \mathcal{R}_c(\theta) \).

Now, using the tangential sphere bound we have that the minimal \( \beta_c \) is

\[
1 + \beta_c^* = 1 - \frac{\tan^2 \frac{\Theta(d)}{2}}{1 + \tan^2 \frac{\Theta(d)}{2}} = 1 - \frac{\sin^2 \Theta(d)}{2} = \cos^2 \frac{\Theta(d)}{2}
\]

and (50) becomes (86) at the bottom of the page, which is what one would have if one considered the noise along the line from \( c \) to \( c_c \). Thus, minimizing (86) we find that

\[
d_c^* = d_{\text{crit}} = \sqrt{2 + \frac{4}{\text{SNR}} - 2\sqrt{1 + \frac{4}{\text{SNR}^2}}}
\]

Note that \( d_c^* \) is independent of the choice of the half angle of the cone \( \mathcal{R}_c(\theta) \). Thus, by choosing \( \theta \) properly we can guarantee that

\[
y_c(\beta_c^*, \Theta(d))^2 - x_c(\beta_c^*, \Theta(d))^2 \geq 1/\text{SNR}.
\]

However, in order to optimize the overall probability of error we may not take \( \theta \) arbitrarily since we must have \( P_{\text{union}} = P_{\text{region}} \). Thus, for some rates (87) may not hold. Indeed, closely examining (87) for \( \theta = \Theta(R) \) we have that if (87) is true then

\[
R > 1/2 \log \left( \frac{1}{2} + \frac{\text{SNR}}{4} + \frac{1}{2} \sqrt{1 + \frac{\text{SNR}^2}{4}} \right) = R_{\text{crit}}.
\]

Thus, if \( R > R_{\text{crit}} \) then \( d_{\text{crit}} \) is not the dominating error event. In this case we must consider optimizing \( E_d(R) \) with \( y_c^2 - x_c^2 \geq 1/\text{SNR} \).

It is a simple computation to show that the minimizing \( \beta_c \) in the exponent of (53) in the case that \( y_c^2 - x_c^2 \geq 1/\text{SNR} \) is

\[
1 + \beta_c^*(\theta, \Theta(d); \text{SNR}) = \frac{\cos^2 \theta}{2} + \frac{\cos \theta}{2} \sqrt{\cos^2 \theta + \frac{4}{\text{SNR}^2}}.
\]

Substituting \( \beta_c^* \) in to (50) we have (89) at the bottom of the page. Hence, in the case that \( y_c^2 - x_c^2 \geq 1/\text{SNR} \), we have the minimizing \( d \) as

\[
d_c^* = \frac{\sqrt{2} y_c}{\sqrt{(1 + \beta_c^*)^2 + y_c^2}} = \frac{\sqrt{2} \tan \Theta(d)}{\sqrt{1 + \tan^2 \Theta(d)}} = \sqrt{2} \sin \theta
\]

\[
\mathcal{P}_{\text{union}}^c \leq \exp \left( -n \min_{0 \leq d \leq 2} \left( \frac{\text{SNR}}{2} \cdot \sin^2 \frac{\Theta(d)}{2} - \log (\sin (\Theta(d)) - R) \right) \right)
\]

and

\[
\mathcal{P}_{\text{union}}^c \leq \exp \left( -n \min_{0 \leq d \leq 2} E_d(\beta_c^*, x_c, y_c; \text{SNR}) - \log \left( d \sqrt{1 - d} - R \right) \right)
\]

(89)
APPENDIX III
DERIVATION OF PARAMETERS MINIMIZING THE MOD-Λ ERROR EXPONENT

As done in Appendix III we consider the two cases of $\beta_\lambda^*$ separately. First, we consider the case where $\beta_\lambda^* = \beta_\lambda^*(r, d, l; \text{SNR})$ and begin by examining the partial derivative of $E_{\text{bnd}}^\lambda(r, K_\alpha, l, d, \beta_\lambda^*; \text{SNR})$ with respect to $l$. One can check that this partial is zero if

$$l = l_o^*(r, K_\alpha; \text{SNR}) \text{ or } l = -l_o^*(r, K_\alpha; \text{SNR}) + \frac{2}{2K_\alpha \text{SNR}}$$

where

$$l_o^*(r, K_\alpha; \text{SNR}) = \frac{1 + \sqrt{1 + 4K_\alpha^2r^2\text{SNR}^2}}{2K_\alpha \text{SNR}}$$

which is independent of $d$. Hence, taking the partial derivative of $E_{\text{bnd}}^\lambda(r, K_\alpha, l, d, \beta_\lambda^*; \text{SNR})$ with respect to $d$ and substituting $l$ with $l_o^*(r, K_\alpha; \text{SNR})$ one can show that this partial is zero if

$$d^*_\lambda(r) = \sqrt{2r}.$$

Hence, if $\beta_\lambda^* = \beta_\lambda^*(r, d, l; \text{SNR})$ then one can show

$$l_o^*(r, \alpha; \text{SNR}) = l_o^*(r, K_\alpha; \text{SNR}) \text{ and } d^*_\lambda(r, l, \alpha; \text{SNR}) = d^*_\lambda(r).$$

The case where $\beta_\lambda^* = d^2/4l^2 \cdot (l - K_\alpha)$ is a bit more tedious. In the sequel we provide a general derivation for the $l$ and $d$ that maximize the union bound for a code that has been expurgated so that the minimum distance is at least $d_\Omega(R)$. In this direction we let

$$\tilde{E}_{\text{bnd}}^\lambda(r, K_\alpha, l, d; \text{SNR}) = E_{\text{bnd}}^\lambda(r, K_\alpha, l, d, \frac{d^2}{4l^2}(l - K_\alpha); \text{SNR}).$$

It is simple to check via direct substitution that

$$\tilde{E}_{\text{bnd}}^\lambda(r, K_\alpha, l, d; \text{SNR}) = \frac{d^2}{8l^2}(K_\alpha - l)^2 \cdot \text{SNR} - \log \left[ \frac{d}{1 + K_\alpha} \sqrt{1 - \frac{d^2}{4l^2}} \right] - R. \quad (91)$$

Examining (91) it is easy to see that if the $l$ and $d$ that maximize the union bound are equal to $1 + K_\alpha$ and $d_\Omega(R)(1 + K_\alpha)$ respectively then (91) is exactly the union bound for a spherical code with minimum distance $d_\Omega(R)$ in the expurgated regime. We now show that there exists a scaling $\alpha$ such that this is true. That is, we solve for the $\alpha$ such that the $l$ and $d$ that maximize the union bound are equal to $1 + K_\alpha$ and $d_\Omega(R)(1 + K_\alpha)$ respectively.

To begin we introduce a Lagrange multiplier $\mu$ and consider minimizing

$$\tilde{E}_{\text{bnd}}^\lambda(r, K_\alpha, l, d; \text{SNR}) + \mu (d_\Omega(R) - d/(1 + K_\alpha)).$$

It can be shown that in the regime of interest $\mu > 0$, $d^*_\lambda = d_\Omega(R)(1 + K_\alpha)$ and $l_o^*(r, \alpha; \text{SNR})$ satisfies

$$\frac{d_\Omega(R)^2}{4} = \frac{(l_o^*)^2 + (K_\alpha^2 \cdot (l_o^*)^2 - K_\alpha^2 \cdot (l_o^*)^3) \text{SNR}}{K_\alpha^2 (l_o^* - l_o^* \text{SNR})}. \quad (92)$$

In order to minimize the probability of error we are left to maximize

$$2\tilde{E}_{\text{bnd}}^\lambda(r, K_\alpha, l_o^*(r, K_\alpha; \text{SNR}), d_\Omega(R)(1 + K_\alpha); \text{SNR})$$

as a function of $K_\alpha$. It is straightforward\(^{14}\) to check that $\tilde{E}_{\text{bnd}}^\lambda(r, K_\alpha, l_o^*(r, K_\alpha; \text{SNR}), d_\Omega(R)(1 + K_\alpha); \text{SNR})$ is maximized when

$$K_\alpha = K_\alpha^*(d_\Omega(R); \text{SNR}) = \frac{4}{(4 - d_\Omega(R)^2) \text{SNR}},$$

yielding

$$l_o^*(r, 1/(1 + K_\alpha^*(d_\Omega(R); \text{SNR})); \text{SNR}) = 1 + K_\alpha^*(d_\Omega(R); \text{SNR}); \text{SNR}).$$

\(^{14}\)Recall that $l_o^*$ is a function of $K$. 

Substituting this into (75) we have

$$\bar{E}_{\text{bnd}}^\Lambda(r, K^*, l^*, R_3(R)(1 + K^*); \text{SNR}) = \frac{\text{SNR}}{8} \cdot d_1(R)^2 - \log \left(d_1(R)\sqrt{1 - d_1(R)^2/4}\right) - R$$

which is precisely the error exponent for the union bound using a spherical code with minimum distance $d_1(R)$ in the expurgated regime.

**APPENDIX IV**

**DERIVATION OF (16) WITH A SPHERICAL REGION**

We have now provided a geometric characterization of the typical error events for the AWGN error exponent. For this derivation we used cone $R_c(\theta_{\text{AWGN}})$ for the region $R$ in (9). Recall that in order to show that the mod-$\Lambda$ channel can achieve the sphere-packing error exponent for the AWGN channel we took a scaling $\alpha$ that corresponded to a valid sphere. We extend our previous definition of “valid” regions [cf. (21)] to be the regions $R$ such that:

1. $P(c + \mathbf{z} \notin R) = P(c + \mathbf{z} \notin R_c(\theta_{\text{AWGN}}))$ (92)
2. $R \subset R_c(\theta_{\text{AWGN}})$ (93)

We now show that by using the valid sphere $R_c(\theta_{\text{AWGN}})$ one may obtain the best known lower bounds on the error exponent. We then use this derivation to show that the mod-$\Lambda$ channel also can achieve the random coding error exponent $E_{\text{AWGN}}'(R)$.

Again using the tangential sphere bound we consider the intersection of the region $R_c(\theta)$ with the half space that orthogonally bisects a cord of length $d$ that has one end point at transmitted codeword and the other at $c$. This is the shaded region in Fig. 10. We let $D_s(d, \alpha)$ be this intersection. We also let $D_s(d, \alpha, \beta_s)$ be the intersection of $D_s(d, \alpha)$ with the hyperplane, say $\mathbf{H}$, such that $\mathbf{e}_s \cdot \mathbf{x} = \beta_s \sqrt{nP}$ for all $\mathbf{x} \in \mathbf{H}$. The $n - 1$ dimensional region $D_s(d, \alpha, \beta_s)$ can be seen in Fig. 10. We may bound $P_{\text{union}}$ as in (50) yielding

$$P_{\text{union}} \leq \max_{0 \leq d \leq 2} \left[ \mathbb{P}\left(\sqrt{nP} \cdot \mathbf{e}_y + \mathbf{z} \in D_s(d, \alpha)\right)\right] \times \exp\left(nR + (n - 1)\log \left(d\sqrt{1 - \frac{d^2}{4}}\right)\right)$$ (94)

In order to obtain an exponential bound to $\mathbb{P}\left(\sqrt{nP} \cdot \mathbf{e}_y + \mathbf{z} \in D_s(d, \alpha)\right)$ we again integrate along the radial component of the noise. This can be seen in Fig. 10. When the radial component of the noise has magnitude $\beta_s$, the probability $\mathbb{P}\left(\sqrt{nP} \cdot \mathbf{e}_y + \mathbf{z} \in D_s(d, \alpha)\right)$ is simply the probability that the second component is greater than $x_s$, while the magnitude of the second thorough $n$th component is less than $y_s$, where

$$x_s = x_s(\beta_s, \Theta(d)) = x_c(\beta_s, \Theta(d))$$ (95)

and

$$y_s^2 = y_s^2(\alpha, \beta_s) = \sqrt{nP} \left(\frac{\sin^2 \theta}{\alpha^2} - \left(\frac{1}{\alpha} - (1 + \beta_s)^2\right)\right)$$ (96)

and can be derived through the the geometry in Fig. 10.

Thus, applying Proposition 5 we have, similar to (53),

$$\mathbb{P}\left(\sqrt{nP} \cdot \mathbf{e}_y + \mathbf{z} \in D_s(d, \alpha)\right) \leq \max_{\beta_s \geq -1} \exp\left(-n E_d(\beta_s, x_s, y_s; \text{SNR})\right)$$ (97)

where $E_d$ was defined in (54).

It is clear from the geometry in Fig. 9 and Fig. 10 that the error exponent using spherical regions is exactly that of the mod-$\Lambda$ channel if $l = 1 + K_\alpha$. Hence, if $\alpha = \alpha_\Lambda^* \alpha$, the geometry of the error events coincide. That is, $\beta_s^* = \beta_\Lambda^*$ and $d_s^* = d_\Lambda^*$.

Clearly this analysis extends to the analysis using a cone so that typical error events for the AWGN channel using $R_c(\theta_{\text{AWGN}})$ and the typical events using $R_c(\theta_{\text{AWGN}})$ coincide.
Fig. 10. The parameters for the derivation of the AWGN error exponent using spherical regions. (a) A 2D representation of the bounding technique. The region $D_s(d)$ corresponding to an error with a codeword at distance $d$ condition on the event that the noise remains in $\mathbb{R}$ can be seen shaded in gray. (b) A three dimensional representation of the region corresponding to an error with a codeword at distance $d$ condition on the event that the noise remains in $\mathbb{R}$ and the radial component of the noise.

APPENDIX V
DERIVATION OF (50)

For a fixed codebook $C$ and fixed codeword $c \in C$ we have from (9)

$$P_{\text{union}}^r(c) = \sum_{c_e \neq c} \mathbb{P}(\|y - c_e\| \leq \|y - c\|, c + z \in \mathbb{R})$$

$$= \sum_{c_e \neq c} \mathbb{P}(c + z \in D_c(\|c - c_e\|)).$$

(98)

Note that (98) is a function of the distance between the transmitted codeword and all other code words. Recall that for a fixed code $C$ the distance distribution or spectrum of the code is defined as follows. Let $b_c(s, t)$ be the number of codewords of $C$ that are at least a distance $s$ from $c$ but not further than $t$. That is,

$$b_c(s, t) = |\{c_e \in C : s \leq \|c - c_e\| < t\}|$$

We further let $b(s, t)$ be the average of $b_c(s, t)$ over the code. That is,

$$b(s, t) = \frac{1}{|C|} \sum_{c \in C} b_c(s, t).$$

For a spherical code $0 < \|c - c_e\| < 2\sqrt{nP}$. Thus, by discretizing the interval $[0, 2\sqrt{nP}]$ into intervals of length $\Delta$, we may upper bound (98) as

$$P_{\text{union}}^r(c) \leq \mathbb{E} \sum_{i=0}^{k_\Delta} b(i\Delta, (i+1)\Delta) \mathbb{P}(c + z \in D_c(i\Delta) \cup D_c((i+1)\Delta))$$

(99)

where $k_\Delta = [2\sqrt{nP}/\Delta] - 1$. By spherical symmetry and linearity of expectation we have

$$\overline{P}_{\text{union}} \leq \mathbb{E} \sum_{i=0}^{k_\Delta} b(i\Delta, (i+1)\Delta) \mathbb{P}\left(\sqrt{nP} \cdot e_y + z \in D_c(i\Delta) \cup D_c((i+1)\Delta)\right)$$

$$= \sum_{i=0}^{k_\Delta} \mathbb{P}\left(\sqrt{nP} \cdot e_y + z \in D_c(i\Delta) \cup D_c((i+1)\Delta)\right) \mathbb{E} b(i\Delta, (i+1)\Delta).$$

(100)

In the limit $\Delta \to 0$ we have that $\mathbb{E} b(i\Delta, (i+1)\Delta)$ is proportional to the radius of the spherical cross section at a height $d = i\Delta$, i.e.,

$$\mathbb{E} b(i\Delta, (i+1)\Delta) \to \sin \Theta(d) = d \sqrt{1 - \frac{d^2}{4}}$$
since the codewords are chosen uniformly over the surface of the sphere. Thus,

\[ P_{\text{union}}^{\alpha} \leq K \int_0^2 e^{nR} \left( s \sqrt{1 - \frac{s^2}{4}} \right)^{n-1} \mathbb{P} (e_y + z \in D_c(s) \cup D_c ((i + 1)\Delta)) \, ds \]

\[ \leq 2K \max_{0 \leq d \leq 2} e^{nR} \left( d \sqrt{1 - \frac{d^2}{4}} \right)^{n-1} \mathbb{P} (e_y + z \in D_c(d)) \]  

(101)

where \( K \) is a normalizing constant.

**APPENDIX VI**

**DERIVATION OF (71)**

Recall that in order to derive bounds for \( P_{\text{union}}^{\alpha} \) we used the spectrum of the ensemble of codes. Now, we use a random coding argument to derive the error probability for the mod-\( \Lambda \) channel. We consider the ensemble of random codes that are i.i.d. and uniform over the Voronoi of the lattice. Recall that in order to derive bounds for \( P_{\text{union}}^{\alpha} \) we used the spectrum of the ensemble of codes. In this direction, let \( b_c(s, t, \theta_{d_1}, \theta_{d_2}) \) be the number of codewords of \( \mathcal{C} \) that are at least a distance \( s \) from the transmitted codeword \( c \) but not further than \( t \) and form an angle with the dither between \( \theta_{d_1} \) and \( \theta_{d_2} \). That is,

\[ b_c(s, t, \theta_{d_1}, \theta_{d_2}) \triangleq \left| \left\{ c_c \in \mathcal{C} : s \leq \| c - c_c \| < t \text{ and } \cos \theta_{d_2} \leq \frac{\| u \| \| c_c \|}{\| u \|} \leq \cos \theta_{d_1} \right\} \right| \]

We further let \( b(s, t, \theta_{d_1}, \theta_{d_2}) \) be the average of \( b_c(s, t, \theta_{d_1}, \theta_{d_2}) \) over the code. By appropriately discretizing by \( \Delta \) and taking the limit it can be shown that

\[ \lim_{\Delta \to 0} b_c(d, d + \Delta, \theta_p, \theta_p + \Delta) = \alpha e^{nR} (d \sin \theta_p)^{n-1} \]

where the \( \alpha \) appears due to the scaling of the lattice. Thus, (70) becomes

\[ P_{\text{union}}^{\alpha, r} \leq \max_{(d, l)} e^{nR} \left( \alpha d \sqrt{1 - \frac{d^2}{4l^2}} \right)^{n-1} \mathbb{P} \left( \frac{y}{\alpha} + z_{\text{eff}} \in D_{\alpha}(r, d, l) \right) \]  

(102)

**APPENDIX VII**

**PROOF OF LEMMA [6]**

It is simple to check that

\[ z(K; d, R, \text{SNR}) = 2K \cdot (c(d; R, \text{SNR}) - E_{\text{ap}}(R(\theta_\zeta (K; \text{SNR}); \text{SNR})) \]

where \( c(d; R, \text{SNR}) \geq 0 \) if \( d \geq 0, R \geq 0 \) and \( \text{SNR} > 0 \), and is independent of \( K \). Furthermore, \( 2K \cdot E_{\text{ap}}(R(\theta_\zeta (K; \text{SNR}); \text{SNR})) \geq 0 \) and is monotonically increasing on the interval \([1/\text{SNR}, \infty)\) as a function of \( K \). Hence, since \( E_{\text{ap}}(R(\theta_\zeta (1/\text{SNR}; \text{SNR}); \text{SNR}) = 0 \) the equation \( E_{\text{ap}}(R(\theta_\zeta (K; \text{SNR}); \text{SNR})) = c(d; R, \text{SNR}) \) has a unique solution for \( K \) and thus \( z(K; d, R, \text{SNR}) \) has one root on the interval \([1/\text{SNR}, \infty)\).

**REFERENCES**

[1] C. Shannon, “A mathematical theory of communication,” BSTJ, pp. 379–423, 623–656, 1948.
[2] C. E. Shannon, “Probability of error for optimal codes in a Gaussian channel,” BSTJ, vol. 3, pp. 1–17, April 1959.
[3] R. Gallager, “A simple derivation of the coding theorem and some applications,” IEEE Trans. Inform. Theory, vol. 15, no. 1, pp. 23–33, January 1965.
[4] C. Shannon, R. Gallager, and E. Berlekamp, “Lower bounds to error probability for coding on discrete memoryless channels,” Inform. Contr., vol. 10, pp. 65–103, January 1967.
[5] ——, “Low density parity check codes,” Ph.D. dissertation, MIT, Cambridge, Massachusetts, 1962.
[16] G. Poltyrev, “Bounds on the decoding error probability of binary linear codes via their spectra,” *IEEE Trans. Inform. Theory*, vol. 40, no. 4, pp. 1284–1292, July 1994.

[17] E. R. Berlekamp, “The technology of error-correcting codes,” *Proc. IEEE*, vol. 68, no. 5, pp. 564–592, May 1980.

[18] G. Poltyrev, “On coding without restrictions for the AWGN channel,” *IEEE Trans. Inform. Theory*, vol. 40, no. 2, pp. 409–417, March 1994.

[19] G. Kabatiansky and V. I. Levenshtein, “Bounds for packings on the sphere and in the space,” *Probl. Pered. Inform.*, vol. 14, no. 1, pp. 3–25, February 1978.

[20] U. Erez, S. Litsyn, and R. Zamir, “Lattices which are good for (almost) everything,” *IEEE Trans. Inform. Theory*, vol. 51, no. 10, October 2005.

[21] U. Erez, S. Shamai (Shitz), and R. Zamir, “Capacity and lattice-strategies for cancelling known interference,” in *Proc. ISITA 2000*, November 2000, pp. 681–684.

[22] M. V. Burnashev, “Code spectrum and the reliability function: Gaussian channel,” *Problems of Information Transmission*, vol. 43, no. 2, March 2007.