Some properties of binomial coefficients and their application to growth modelling

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ABSTRACT

Some properties of diagonal binomial coefficients were studied in respect to frequency of their units’ digits. An approach was formulated that led to the use of difference tables to predict if certain units’ digits can appear in the values of binomial coefficients at quadratic terms of the binomial theorem. Frequency distributions of units’ digits of binomial coefficients contain gaps (zero frequency) under most common numbering systems with supposed exclusion to systems with $2^k$ bases. In the work, an application of binomial coefficient arithmetic to model cell population dynamics was suggested. For a multicellular organism, the growth of number of cells was presented as a succession of binomial coefficients. It was inferred that the number of cells in a multicellular organism may obey power function laws.

1. Introduction

Since long ago binomial coefficients are known to appear in the binomial theorem of power decomposition of:

$$(1 + x)^n = 1 + nx + C_n^2 \cdot x^2 + C_n^3 \cdot x^3 + \ldots + C_n^k \cdot x^k + \ldots,$$  

(1)

where, $x$ being a real variable, $C_n^2$, $C_n^3$, $C_n^k$, etc., are binomial coefficients. Because the first $k + 1$ terms of the power decomposition is a polynomial of $x$ at degree $k$ the expressions $C_n^2$, $C_n^3$ are coefficients at quadratic, cubic terms, correspondingly. One of the most vivid representations of the coefficients is Pascal triangle (Figure 1) which is a triangular table where every $n$th row is the coefficients of the decomposition.

Arithmetical properties of binomial coefficients are well known and relate mostly to the divisibility by prime numbers and their degrees (Guo, 2013; Guo & Krattenthaler, 2014; Winberg, 2008) as well as to sums of the coefficients in one row, i.e. where $n$ is a constant (Figure 1(a)). Besides subsets of the coefficients in horizontal rows, a theory has been developed which covers divisibility properties of central (or middle) binomial coefficients (Chen, 2016; Pomerance, 2015).

It has been shown (Gavrikov, 2017) that some subsets of the coefficients from diagonals of the Pascal triangle (Figure 1(b)) can possess other properties as well. Particularly, the frequency distributions of units’ digits of binomial coefficients at quadratic terms (which are minor totals of natural sequence) may or may not have gaps dependently on numbering system considered. This property was strictly proven on the example of base-three and base-four numbering systems. In terms of modular arithmetic, the properties look as follows, $l$ being a non-negative integer and $S_l$ being the minor total of the first $l$ terms of the natural sequence (the minor totals are numerically equivalent to binomial coefficients $C_n^k$ at quadratic terms). So, $S_{3l+0} \equiv 0 \pmod{3}$, $S_{3l+1} \equiv 1 \pmod{3}$, $S_{3l+2} \equiv 0 \pmod{3}$ while $S_n \neq 2 \pmod{3}$ at any $n$. Then, $S_{4l+0} \equiv 0 \pmod{4}$ at even $l$, $S_{4l+1} \equiv 2 \pmod{4}$ at odd $l$, $S_{4l+1} \equiv 1 \pmod{4}$ at even $l$, $S_{4l+3} \equiv 3 \pmod{4}$ at odd $l$.

Gavrikov (2017) has also suggested how to prove presence or absence of gaps (omissions) in frequency distributions of units’ digits of minor totals of natural sequence under numbering system with other bases. Empirically, the gaps are found in numbering systems with bases 5, 6, 7, 9 and 10 while there are no gaps in systems with bases 4, 8 and 16. For example, empirical frequency distributions of units’ digits for base-seven and base-eight numbering systems are given in Figure 2.

The aim of the work was to explore some properties of diagonal binomial coefficients which are coefficients at quadratic and cubic terms of the binomial theorem. Units’ digits of the diagonal binomial coefficients may have definite omissions (gaps in frequency distributions) depending on numbering system used. It would be therefore interesting to...
develop a tool which allows both proving the omission existence and predicting which omissions may be expected under a numbering system used. Another aim was to find if binomial coefficient arithmetic may be applied to modelling of cell population growth.

2. Generalization

The binomial coefficients at the quadratic term are minor totals of natural sequence (Figure 1(b)) and have been known since long ago. The minor totals themselves are a particular case of Faulhaber polynomials with unity exponent and are given by Faulhaber’s formula (also called Bernoulli formula). The nth minor total of the sequence $S_n$ is:

$$S_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{2}$$

The basic equation of the analysis looks like:

$$S_{Lk+i} = L \cdot m + j, \tag{3}$$

where, $L$ is the base of a numbering system, $i$ and $j$ are units’ digits in different representations of minor totals $S_n$, obviously $0 \leq i, j \leq (L-1)$. Values of $k$ and $m$ are non-negative integers, which is of core importance.

According to relation 2:

$$S_{Lk+i} = \frac{L \cdot L \cdot k^2 + L \cdot k(2i+1) + i(i+1)}{2}. \tag{4}$$

Then the generalizing relation linking $m, k, i$ and $j$ can be received from Equations (3) and (4) by expressing of $m$ through all other parameters:

$$m = \frac{k(L \cdot k + (2i+1))}{2} + \frac{(i+1) - j}{L}. \tag{5}$$

The wholeness of $m$ depends on evenness/oddness of term $k(L \cdot k + (2i+1))$ and on whether term $\frac{(i+1) - j}{L}$ is divisible by $L$ with or without a remainder. At given $k$ and $L$, those combinations of $i$ and $j$ that ensure wholeness of $m$ determine $S_{Lk+i} \equiv j \pmod{L}$.

Thus the generalizing relation 5–6 can be used to predict the units’ digits in values of binomial coefficients at quadratic terms ($S_n$) at a given base of numbering system $L$.

Properties of term 5 depend on evenness/oddness of $k$ and $L$. If $L$ is odd then term 5 is always an integer ($k(L \cdot k + (2i+1)) \equiv 0 \pmod{2}$). If $L$ is even then at even $k$ term 5 is an integer but at odd $k$ the term 5 is a fraction: $(k(L \cdot k + (2i+1)) \equiv \frac{1}{2} \pmod{2})$.

It is easy to see that $\frac{(i+1) - j}{L}$ brings about a table (Table 1) that may called a “difference table” which contains all the possible differences between $\frac{(i+1)}{2}$ and $j$. This table can be used to find the sought quantities of $j$.

Remark 2.1. The difference table may be used to predict not only units’ digits in $S_n$ but other digits as well. For example, considering $S_{Lk+i} = L^2 \cdot m + j$ one can study appearance of tens-and-units’ digits of.

Figure 1. Pascal triangle. a – A horizontal row with constant $n$ ($n=5$), as an example, is highlighted in bold face; b – diagonals of the triangle are highlighted in which $n$ is a variable. Minor totals of natural sequence (binomial coefficients at the quadratic term) are given in bold face. Binomial coefficients at the cubic term are given in bold italic face.

Figure 2. Empirical frequency distributions of units’ digits in binomial coefficients at the quadratic term (Figure 1(b)). a – Base-seven numbering system; b – base-eight numbering system. Numbers at abscissa axis are digits of the corresponding numbering system. Frequency is the number of the cases among first hundred of binomial coefficients values.
In the base-eight numbering system, values of differences divisible by 8 without a remainder or with a remainder 1/2 or –1/2 are given on grey background. Values of j satisfying the condition “m is a non-negative integer” are given in bold face.

### Table 1. A fragment of difference table.

| j   | 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  |
| 1   | –1  | 0   | 2   | 5   | 9   | 14  | 20  | 27  |
| 2   | –2  | –1  | 1   | 4   | 8   | 13  | 19  | 26  |
| 3   | –3  | –2  | 0   | 3   | 7   | 12  | 18  | 25  |
| 4   | –4  | –3  | –1  | 2   | 6   | 11  | 17  | 24  |
| 5   | –5  | –4  | –2  | 1   | 5   | 10  | 16  | 23  |
| 6   | –6  | –5  | –3  | 0   | 4   | 9   | 15  | 22  |
| 7   | –7  | –6  | –4  | –1  | 3   | 8   | 14  | 21  |

First eight values of f(i + 1)/2 and j are shown.

### Table 2. A 7 × 7 difference table.

| j   | 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  | 36  |
| 1   | –1  | 2   | 5   | 9   | 13  | 19  | 25  | 31  |
| 2   | –2  | 1   | 4   | 8   | 12  | 16  | 20  | 24  |
| 3   | –3  | –2  | 3   | 6   | 9   | 12  | 15  | 18  |
| 4   | –4  | –3  | –1  | 2   | 5   | 8   | 11  | 14  |
| 5   | –5  | –4  | –2  | 1   | 4   | 7   | 10  | 13  |
| 6   | –6  | –5  | –3  | 0   | 3   | 6   | 9   | 12  |

Values of differences divisible by 7 without a remainder are given on grey background. Values of j satisfying the condition “m is a non-negative integer” are given in bold face.

### Table 3. A 8 × 8 difference table.

| j   | 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  | 36  |
| 1   | –1  | 2   | 5   | 9   | 13  | 19  | 25  | 31  |
| 2   | –2  | 1   | 4   | 8   | 12  | 16  | 20  | 24  |
| 3   | –3  | –2  | 3   | 6   | 9   | 12  | 15  | 18  |
| 4   | –4  | –3  | –1  | 2   | 5   | 8   | 11  | 14  |
| 5   | –5  | –4  | –2  | 1   | 4   | 7   | 10  | 13  |
| 6   | –6  | –5  | –3  | 0   | 3   | 6   | 9   | 12  |
| 7   | –7  | –6  | –4  | –1  | 3   | 8   | 14  | 21  |

Values of differences divisible by 8 without a remainder or with a remainder 1/2 or –1/2 are given on grey background. Values of j satisfying the condition “m is a non-negative integer” are given in bold face.

### Table 4. A 9 × 9 difference table.

| j   | 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  | 36  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  | 36  | 44  |
| 1   | –1  | 2   | 5   | 9   | 13  | 19  | 25  | 31  | 37  |
| 2   | –2  | 1   | 4   | 8   | 12  | 16  | 20  | 24  | 29  |
| 3   | –3  | –2  | 3   | 6   | 9   | 12  | 15  | 18  | 13  |
| 4   | –4  | –3  | –1  | 2   | 5   | 8   | 11  | 14  | 9   |
| 5   | –5  | –4  | –2  | 1   | 4   | 7   | 10  | 13  | 5   |
| 6   | –6  | –5  | –3  | 0   | 3   | 6   | 9   | 12  | 1   |
| 7   | –7  | –6  | –4  | –1  | 3   | 8   | 14  | 21  | 0   |
| 8   | –8  | –7  | –5  | –2  | 1   | 7   | 13  | 19  | 6   |

Values of differences divisible by 9 without a remainder are given on grey background. Tens-and-units values of j from that do not satisfy the condition “m is a non-negative integer” are given in bold italic face.

In this section, examples of difference table usage are considered. These are cases of base-seven, base-eight and ternary numbering systems.

#### Proposition 3.1.

In base-seven numbering system, $S_7 \equiv j \ (mod \ 7)$ where $j \in \{0,1,3,6\}$ while $S_7 \not\equiv \alpha \ (mod \ 7)$ where $\alpha \in \{2,4,5\}$ for any $n$.

**Proof.** Because 7 is an odd number, term 5 is an integer. Therefore combinations of $i$ and $j$ are sought that ensure the wholeness of term 6. In other words, the values of the difference table must be divisible by 7 without a remainder.

For base-seven numbering system, one should consider a $7 \times 7$ difference table (Table 2). As follows from the table, only values of $j \in \{0,1,3,6\}$ satisfy to the condition “$m$ is a non-negative integer” and therefore $S_7 \equiv j \ (mod \ 7)$ only from this set (Figure 2(a)).

#### Proposition 3.2.

In base-eight numbering system, $S_8 \equiv j \ (mod \ 8)$ where $j \in \{0,1,2,3,4,5,6,7\}$.

**Proof.** Because 8 is an even number, the term 5 may be non-negative integer (for even $k$) and a fractional number (for odd $k$). Namely, for odd $k$ $\frac{k(k+1)(2k+1)}{2} \equiv \frac{1}{2} \ (mod \ 2)$. Therefore, the differences are sought that are divisible by 8 either without a remainder or with the remainder 1/2 or -1/2.

A $8 \times 8$ difference table should be considered (Table 3). As follows from the table, all the digits of the base-eight numbering system satisfy the condition “$m$ is a non-negative integer”, i.e. $S_8 \equiv j \ (mod \ 8)$ for all the $j \in \{0,1,2,3,4,5,6,7\}$ (Figure 2(b)).

#### Remark 3.3.

Not all even $L$ will bring about that $j$ covers all the digits of the numbering system. For example, for $L=6$, $L=10$ and others there will be gaps in the sets of units’ digits. Empirically, only the cases $L=2^c$ (c being a non-negative integer) lead to that $j$ covers all the system digits.

#### Proposition 3.4.

In ternary system, $S_9 \not\equiv 11 \ (mod \ 100)$ and $S_9 \not\equiv 21 \ (mod \ 100)$ for any $n$.

**Proof.** Converting the relations to decimal system one gets $S_9 \not\equiv 4 \ (mod \ 9)$ and $S_9 \not\equiv 7 \ (mod \ 9)$.

A $9 \times 9$ difference table should be considered in which difference values are sought that contain 9 (Table 4). As follows from the table, neither ternary 11 nor ternary 21 are present among values appearing as tens-and-units’ digits in ternary representation of $S_9$.

The approach of difference tables will naturally work for other subsets of binomial coefficients. Consider the case of binomial coefficients at cubic term of the binomial theorem (Figure 1(b)). The basic equation to analyze is:

$$T_{i,k+1} = Lm + j,$$

where $T_{i,k+1}$ is a representation of the binomial coefficients at cubic term. Obviously, the values of the
coefficients may be obtained with the help of
\[( \frac{Lk + i}{2} ) \frac{(Lk + i + 1)(Lk + i + 2)}{3} \]

Thus, the Equation (7) may be re-written as:
\[
\frac{(Lk + i)(Lk + (i + 1))(Lk + (i + 2))}{2 \cdot 3} = Lm + j, \tag{8}
\]
from which the value of coefficient \( m \) may be expressed as:
\[
m = \frac{k(i^2 \cdot k^2 + Lk(3i + 3) + (i + 1)(i + 2) + (2i(3i + 1)))}{2 \cdot 3} + \frac{ii(i+1)i+2)}{23} - \frac{j}{L}. \tag{9}
\]

As it has been shown above, the key point is to find such a combination of \( L, i \) and \( j \) that ensures that \( m \) is a non-negative integer. The term (9) produces residues which are multiples of 1/6: 0, 1/6, 2/6, 3/6, 4/6 and 5/6. Consequently, term (10) has to produce the same row of multiples of 1/6 to provide the wholeness of the sum. The expression \( \frac{ii(i+1)i+2)}{23} \) – \( j \) is a difference table and the values of the table have to be checked for divisibility by \( L \). Let’s consider an example case of base-ten numbering system, i.e. \( L = 10 \).

**Proposition 3.5.** In base-ten numbering system, \( T_n \neq 2 \pmod{10} \). \( T_n \neq 3 \pmod{10} \). \( T_n \neq 7 \pmod{10} \) and \( T_n \neq 8 \pmod{10} \) for any \( n \).

**Proof.** A 10 x 10 difference table should be considered in which difference values are sought that while divided by 10 produce residues from 0, 1/6, 2/6, 3/6, 4/6 or 5/6 (Table 5). Obviously, a division by 10 can produce only the residues of 0 and 3/6 (i.e. 1/2). As follows from the table, the rows corresponding to \( j \) equal to 2, 3, 7 and 8 contain no values that produce the residues 0 or 1/2. Therefore, the relations given in the proposition are true. Also, empirical distribution of units’ digits (Figure 3) supports the inference.

**Figure 3.** Empirical frequency distribution of units’ digits in binomial coefficients at the cubic term at base-ten numbering system. Numbers at abscissa axis are digits of the corresponding numbering system. Frequency is the number of the cases among first hundred of binomial coefficients values.

Both are non-negative integers; binomial coefficients are generated by a summation algorithm as well as new cells are added to the existing cells giving rise to growth. The binomial coefficients arithmetic could be therefore considered in terms of biological growth modelling.

A classical example of non-negative integer modelling is provided by a consideration of population growth of cells each of which doubles in equal time intervals producing the next cell generation. Such a dynamics gives rise to sequence of the cell numbers like 1, 2, 4, 8, …, \( b^{t} \), \( b \) being the number of the cell generation while value 2 in this particular case being the reproduction factor. In other words, this is an example of how local divisions of cell lead to an exponential law of cell population growth.

Within a multicellular organism, however, the cell population growth obeys much more complicated rules. On the example of higher plant organisms, the following rules (kinds of cells) may be identified. First, there are the so-called initial cells that preserve the ability to divide in the course of the entire life span of the organism. Second, the cells – immediate descendants of the initials – can divide for some time but sooner or later transform to the third kind of cells. The third kind are remote descendants that are differentiated cells that may be dead or alive but no longer divide. Because the amount of initial cells compared to other kinds is low, a modelling may “imply” their existence but not to take into account explicitly.

Let’s \( M \) be the number of differentiated cells and \( m \) be the number of dividing cells at the moment (generation number) 0. The next moment (generation) the number of dividing cells is multiplied by 2 – each of them divides in two. A share of the new cells goes to the pool of differentiated cells and another share remains to be dividing. In order to get a law of the whole population growth, it is necessary
to suppose how the number of dividing cells alters from generation to generation. Why their number can change is a question of a separate sort and not considered here in detail. One can note, however, that the number of newly differentiated cells may influence the number of the new generation of dividing cells.

Suppose, first, that the number of dividing cells grows from generation to generation linearly, namely, by one cell a generation. Then the whole dynamics of population of $N$ cells in generations from 0 to $q$ may look as in expressions 11–15. In Expression 12 (generation 1), for example, the term $2m - (m + 1)$ denotes a new portion of differentiated cells while $(m + 1)$ is the new amount of dividing cells which is by one cell bigger as in generation 0:

$$M + m$$

$$M + 2m - (m + 1) + (m + 1)$$

$$M + 2m - (m + 1) + 2(m + 1) - (m + 2) + (m + 2) - (m + 3) + (m + 3)$$

$$...$$

$$M + 2m + 2(m + 1) + 2(m + 2) + 2(m + q - 1) - [(m + 1) + (m + 2) + ... + (m + q - 1)]$$

In Expression 15, collecting terms and taking into account that $1 + 2 + 3 + ... + q - 1$ is $\frac{(q - 1)q}{2}$, give $N_q$ the size of the cell population at generation $q$ in the form:

$$N_q = M + m + m \cdot q + \frac{(q - 1)q}{2},$$

which is an equivalent to coefficients of the binomial theorem (1) truncated to the quadratic term. In other words, the idealized multicellular organism grows as minor totals of the natural sequence. A simple corollary of the dynamics is that the growth obeys a power law, in this particular case it is a quadratic power function. It is easy to show that if the number of dividing cells stays constant then the entire organism grows linearly. If the number of dividing cells falls linearly, then the entire organism undergoes a decay of growth which obeys a power law as well.

The number of dividing cells may alter in a non-linear manner. Suppose then that their number grows as the minor totals of the natural sequence, which is the same as binomial coefficients at the quadratic term. As seen from Pascal triangle (Figure 1(b)), one can expect that the total growth should obey the pattern of binomial coefficients at the cubic term. In fact, relations between binomial coefficients at the quadratic and cubic terms, $S_n$ and $T_m$, correspondingly, are rather transparent. The summation of $S_n$ naturally gives $T_n$:

$$T_n = \sum_{k=1}^{n} \binom{n}{k} = \frac{1}{2} \sum_{k=1}^{n} k^2 + \frac{1}{2} \sum_{k=1}^{n} k = \frac{n(n + 1)(2n + 1)}{12} + \frac{n(n + 1)}{4} = \frac{n(n + 1)(n + 2)}{2},$$

$$M + \frac{m(m + 1)}{2}$$

$$M + m(m + 1) - \frac{(m + 1)(m + 2)}{2} + (m + 1)(m + 2) - \frac{(m + 2)(m + 3)}{2} + (m + 2)(m + 3) - \frac{(m + 3)(m + 4)}{2} + (m + 3)(m + 4) - \frac{(m + q - 1)(m + q)}{2} + (m + q)(m + q)$$

Expressions 18–22 show the whole cell population dynamics in detail implying that the cell population consists of $M$ differentiated and $\frac{m(m + 1)}{2}$ dividing cells at the generation 0. Analogically to Expression 12, in 19, $m(m + 1) - \frac{(m + 1)(m + 2)}{2}$ is the new differentiated cell while $\frac{m(m + 1)(m + 2)}{2}$ is the new number of dividing cells.

Collecting terms in Expression 22 one gets:

$$N_q = M + m + m \cdot q + \frac{q(q - 1)q}{2}$$

$$N_q = M + m(m + 1) + \frac{(m + 1)(m + 2)}{2} + \frac{(m + 2)(m + 3)}{2} + ... + \frac{(m + q - 1)(m + q)}{2},$$

which is with the help of Equations (2) and (17) transferred to:

$$N_q = M + S_m + S_m + S_{m+1} + S_{m+2} + ... + S_{m+q-1} = M + S_m + T_{m+q-1} - T_{m-1}.$$
organism. Obviously, $T_{n+q-1}$ is a polynomial of the third order.

An important implication of the cell population behaviour modelled with diagonal binomial coefficient arithmetic is that it is a power law that governs the growth of a multicellular organism, naturally, in terms of cell number. In mathematical modelling of biological objects, the use of polynomial forms at approximating of various biological data has been called into question. While providing high approximation accuracy, the polynomial forms often lack profound justifications and interpretability of parameters. The approach presented in the above analysis may help to fill the gap in polynomial usage in biological growth modelling.

To give a summary, an approach of difference tables was suggested in the work which provides a tool to study some properties of diagonal binomial coefficients. The properties are presence or absence of certain units’ digits in the coefficients, that is, gaps in frequency distributions of the units’ digits. The difference tables help to formulate mathematical proves for any numbering system. Hypothetically, the gaps will be found under all numbering systems which do not have bases of $2^c$ ($c$ being a non-negative integer). It has been also suggested to use binomial coefficients arithmetic to model multicellular growth in term of number of cells. The use of the arithmetic implies application of power laws to model the growth.

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