PERMUTATION MODULES AND CHOW MOTIVES OF GEOMETRICALLY RATIONAL SURFACES

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Abstract. We prove that the Chow motive with integral coefficient of a geometrically rational surfaces $S$ over a perfect field $k$ is zero dimensional if and only if the Picard group of $\bar{k} \times_k S$, where $\bar{k}$ is an algebraic closure of $k$, is a direct summand of a $\text{Gal}(\bar{k}/k)$-permutation module, and $S$ possesses a zero cycle of degree one. As shown by Colliot-Thélène in a letter to the author (which we have reproduced in the appendix) this is in turn equivalent to $S$ having a zero cycle of degree 1 and $\text{CH}_0(k(S) \times_k S)$ being torsion free.

0. Introduction

Let $k$ be a perfect field with algebraic closure $\bar{k}$, and $S$ a geometrically rational $k$-surface. Set $S := \bar{k} \times_k S$. The Picard group $\text{Pic} \bar{S}$ is then a free $\mathbb{Z}$-module of finite rank and also a discrete $G_k := \text{Gal}(\bar{k}/k)$-module. It is called a $G_k$-permutation module if it has a $\mathbb{Z}$-basis which is permuted by the action of $G_k$, and $G_k$-invertible if it is a direct summand of a $G_k$-permutation module.

The aim of this short note is to prove the following result.

**Theorem A.** Let $S$ be a geometrically rational $k$-surface. Then the motive of $S$ in the category of effective Chow motives $\text{Chow}(k)$ with integral coefficients is 0-dimensional if and only if $\text{Pic} \bar{S}$ is an invertible $G_k$-module and $S$ has a 0-cycle of degree 1.

Recall that a motive $M$ in $\text{Chow}(k)$ is called 0-dimensional if it is a direct summand of a finite direct sum of Tate twists of motives of $k$-étale algebras.

Our proof of this theorem is constructive, and can be used to get motivic decompositions of motives of such surfaces. We illustrate this in the (rather easy) case of Del-Pezzo surfaces of degree 5. As another byproduct of our proof we get the (well known?) fact that a motive in the category of Chow motives with rational coefficients is zero dimensional if and only if it is geometrically split.

Being a 0-dimensional motive is stable under field extensions, and so if $S$ is a geometrically rational surface whose motive is 0-dimensional in $\text{Chow}(k)$ then the abelian group $\text{CH}_0(l \times_k S)$ is torsion free for all field extensions $l \supseteq k$. Colliot-Thélène has shown in a letter [5] to the author, which we have reproduced in the Appendix, that if $S$ is a geometrically rational surface with 0-cycle of degree 1 and no torsion in $\text{CH}_0(k(S) \times_k S)$, where $k(S)$ denotes the function field of $S$, then $\text{Pic} \bar{S}$

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is a invertible $G_k$-module. This together with Theorem A proves the converse of the assertion above.

In Sections 3.6 and 3.7 we extend these equivalences further:

**Theorem B.** Let $S$ be a geometrically rational surface over the perfect field $k$, and $T^S$ the $k$-torus with character group $\text{Pic} \bar{S}$. Then the following four assertions are equivalent:

(i) The motive of $S$ in $\text{Chow}(k)$ is 0-dimensional.

(ii) The $G_k$-module $\text{Pic} \bar{S}$ is invertible and $S$ has a 0-cycle of degree 1.

(iii) The group $\text{CH}_0(k(S) \times_k S)$ is torsion free and $S$ has a zero cycle of degree 1. (Note that for a geometrically rational surface $S$ the group $\text{CH}_0(S)$ is torsion free if and only if the degree map $\text{CH}_0(S) \to \mathbb{Z}$ is injective.)

(iv) The degree map $\text{CH}_0(l \times_k Y^c) \to \mathbb{Z}$ is an isomorphism for all smooth compactifications $Y^c$ of $T^S$-torsors $Y$ over $S$ and all field extensions $l$ of $k$.

One may wonder here, whether there exists an example of a geometrically rational $k$-surface $S$, whose motive is zero dimensional, but which has no $k$-rational point.

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1. Preliminaries and notations

1.1. Notations and conventions. Throughout this work $k$ denotes a perfect field with algebraic (and so also separable) closure $\bar{k}$. We denote the absolute Galois group of the field $k$ by $G_k$. If $M$ is a continuous $G_k$-module we denote by $H^i(k, M)$, or by $H^i(G_k, M)$, the Galois cohomology of $M$.

If $l/k$ is a field extension and $X$ a $k$-scheme we set $X_l := l \times_k X$, and $\bar{X} := X_{\bar{k}}$. Similarly we set $A_l := l \otimes_k A$ and $\bar{A} := A_{\bar{k}}$ if $A$ is a commutative $k$-algebra.

If $X$ is an integral $k$-scheme we denote the function field by $k(X)$.

1.2. Chow groups. Let $K$ be a field and $X$ a $K$-scheme of finite type. We denote by $\text{CH}_i(X)$ the Chow group of dimension $i$-cycles modulo rational equivalence as defined in Fulton’s book [11]. We denote the class of an $i$-dimensional subvariety $Z$ in $\text{CH}_i(X)$ by $[Z]$.

If $R$ is a commutative ring (with unit 1) we set $\text{CH}_i(X)_R := R \otimes_\mathbb{Z} \text{CH}_i(X)$, i.e. $\text{CH}_i(X)_R$ is the Chow group of dimension $i$-cycles with coefficients in $R$. In case $X = \text{Spec} A$ is an affine scheme we also use the notation $\text{CH}_i(A)_R$ instead of $\text{CH}_i(X)_R$.

For later use we state the following well known lemma, which e.g. follows from Karpenko and Merkurjev [14 Cor. RC.12].
Lemma. Let $K$ be an arbitrary field, and $X$ and $Y$ smooth and projective $K$-schemes which are $K$-birational to each other. Then $\deg : \text{CH}_0(X) \to \mathbb{Z}$ is an isomorphism if and only if $\deg : \text{CH}_0(Y) \to \mathbb{Z}$ is an isomorphism.

1.3. A commutative diagram. Let $X$ be a $k$-scheme and $\tau \in G_k$. Then $\tau$ induces a morphism of schemes $\tilde{\tau} : \text{Spec } \bar{k} \to \text{Spec } k$, which in turn induces by base change the automorphism $\tilde{\tau} \times \text{id}_X$ of $\bar{X} = \text{Spec } \bar{k} \times_k X$. The pull-backs of these morphisms define then the action of $G_k$ on $\text{CH}_0(\bar{X})$ by the rule $\tau.\alpha := (\tilde{\tau} \times \text{id}_X)^*(\alpha)$. Note that $(\tilde{\tau} \times \text{id}_X)^* = (\tilde{\tau}^{-1} \times \text{id}_X)_*$. As $\text{CH}_0(\bar{X})$ is the direct limit of all $\text{CH}_0(X_i)_R$, where $l$ runs through all finite subextensions $k \subseteq l \subseteq \bar{k}$, this makes $\text{CH}_0(X)_R$ a continuous $G_k$-module.

Assume now that $X$ is geometrically integral and smooth over $k$. Let $k \subseteq l \subseteq \bar{k}$ be a finite field extension of $k$. We identify the absolute Galois group $G_l$ of $l$ with a closed subgroup of $G_k$, and denote by $\Gamma$ the set of all $k$-embeddings $l \to \bar{k}$. Every $\gamma \in \Gamma$ induces a morphism of $k$-schemes $\gamma : \bar{X} \to \bar{k} \times_k X_l = \bar{k} \times_k l \times_k X$, and

$$
\bigcup_{\gamma \in \Gamma} \bar{X} \to \bar{k} \times_k X_l
$$

is an isomorphism of $k$-schemes, where $\bigcup_{\gamma \in \Gamma} \bar{X}$ is the disjoint union of $|\Gamma| = [l:k]$-copies of $\bar{X}$. The induced isomorphism

$$
(\gamma^*)_{\gamma \in \Gamma} : \text{CH}_0(\bar{k} \times_k X_l)_R \xrightarrow{\cong} \bigoplus_{\gamma \in \Gamma} \text{CH}_0(\bar{X})_R
$$

is $G_k$-linear for the following (by this isomorphism induced) $G_k$-action on the direct sum $\bigoplus_{\gamma \in \Gamma} \text{CH}_0(\bar{X})_R$: For $\tau \in G_k$ the $\gamma_0$-component of $\tau.(\alpha_\gamma)_{\gamma \in \Gamma}$ is given by $\tau.\alpha_{\tau^{-1} \gamma_0}$.

Let $\rho_\gamma \in G_k$ be an extension of $\gamma \in \Gamma$ to $\bar{k}$. We have then a commutative diagram

$$
\begin{array}{ccc}
\text{CH}_0(X_l)_R & \xrightarrow{\text{res}_{k/l}} & \text{CH}_0(\bar{k} \times_k X_l)^{G_k} \\
\text{res}_{k/l} & \downarrow \cong & \uparrow (\gamma^*)_{\gamma \in \Gamma} \\
\text{CH}_0(\bar{X})^{G_l}_R & \cong & \left[ \bigoplus_{\gamma \in \Gamma} \text{CH}_0(\bar{X})_R \right]^{G_k}_R
\end{array}
$$

(1)

where the bottom arrow maps $\alpha \in \text{CH}_0(\bar{X})_R^{G_k}$ to $(\rho_\gamma \alpha)_{\gamma \in \Gamma}$. Note that this map does not depend on the choice of the extensions $\rho_\gamma$. As indicated in the diagram the bottom arrow as well as the arrow on the right hand side are isomorphisms.

1.4. Chow motives. Let $X, Y$ be smooth and projective $k$-schemes. A correspondence of degree $0$ with coefficients in $R$ from $X$ to $Y$ is an element $\alpha$ of $\bigoplus_{i=1}^r \text{CH}_{\dim X_i}(X_i \times_k Y)_R$, where $X_1, \ldots, X_r$ are the irreducible components of $X$. We indicate this by $\alpha : X \dashrightarrow Y$. The composition of two correspondences is denoted by $\beta \circ \alpha$.

Smooth projective varieties with correspondences of degree $0$ (with coefficients in $R$) as morphisms and disjoint unions as direct sums are an additive category, the category of correspondences of degree $0$ with $R$ as coefficient ring. The idempotent completion of this category is the category of (effective) Chow motives over $k$ with
coefficient ring \( R \). We denote this category \( \text{Chow}(k, R) \) or \( \text{Chow}(k) \) if \( R = \mathbb{Z} \). The objects of \( \text{Chow}(k, R) \) are pairs \( (X, \pi) \), where \( X \) is a smooth projective \( k \)-scheme and \( \pi : X \rightarrow X \) is an idempotent correspondence of degree 0, i.e. \( \pi \circ \pi = \pi \). By some abuse of notation we denote the motive of a smooth projective scheme \( X \) by the same letter \( X \).

If \( M, N \) are motives, i.e. objects of \( \text{Chow}(k, R) \), we denote the group of morphisms between them by \( \text{hom}_k(M, N) \), respectively by \( \text{hom}_k(M, N) \) if \( R = \mathbb{Z} \). Similarly \( \text{end}_k(M) \) and \( \text{end}_k(M) \) denote the endomorphism group of the motive \( M \) in \( \text{Chow}(k, R) \) and \( \text{Chow}(k) \), respectively.

If \( l/k \) is a field extension we denote the base change functor \( \text{Chow}(k, R) \rightarrow \text{Chow}(l, R) \) by \( \text{res}_{l/k} \), and set \( M_l := \text{res}_{l/k}(M) \) and \( \alpha_l := \text{res}_{l/k}(\alpha) \) for a motive \( M \) and a morphism \( \alpha \) in \( \text{Chow}(k, R) \). If \( l = \bar{k} \) is the algebraic closure we use then also \( M \) and \( \bar{\alpha} \) for \( M_{\bar{k}} \) and \( \alpha_{\bar{k}} \), respectively.

1.5. Split motives. We denote the Tate motive in \( \text{Chow}(k, R) \) by \( R(1) \), and set \( R(i) := R(1)^{\otimes i} \) and \( M(i) := M \otimes R(i) \) for a motive \( M \) and an integer \( i \geq 0 \). In particular, \( R = R(0) = \text{Spec } k \) is the motive of the point. A motive \( M \) is called split if \( M \) is isomorphic in \( \text{Chow}(k, R) \) to a finite direct sum of twisted Tate motives, and geometrically split if \( M = M_{\bar{k}} \) is split. Recall, see e.g. Merkurjev [17, Prop. 1.5], that the motive of a smooth projective and integral scheme \( X \) is split in \( \text{Chow}(k, R) \) if and only if

\[
\text{CH}_i(X)_R \otimes_R \text{CH}_{\dim X-i}(X)_R \rightarrow R, \quad \alpha \otimes \beta \mapsto \deg(\alpha \otimes \beta)
\]

is a perfect duality for all \( 0 \leq i \leq \dim X \). In particular, there are bases \( u_1^1, \ldots, u_n^i \) and \( v_1^i, \ldots, v_m^i \) of the free \( R \)-modules \( \text{CH}_i(X)_R \) and \( \text{CH}_{\dim X-i}(X)_R \), respectively, such that

\[
\deg(u_r^i \cap v_j^i) = \delta_{rs},
\]

where \( \delta_{rs} \) is the Kronecker delta. Moreover, in this case we have

\[
\text{id}_X = \sum_{i=0}^{\dim X} \sum_{j=1}^{n_i} u_j^i \times u_j^i \in \text{end}_k(X)_R,
\]

and the product of cycles induces an isomorphism

\[
\bigoplus_{i=0}^{\dim X} \text{CH}_i(X)_R \otimes_R \text{CH}_{\dim X-i}(X)_R \xrightarrow{\sim} \text{CH}_{\dim X}(X \times_k \bar{X})_R = \text{end}_k(X)_R.
\]

Example. Let \( S \) be a geometrically rational \( k \)-surface, i.e. a smooth projective geometrically integral \( k \)-scheme of dimension 2, such that \( \bar{S} = \bar{k} \times_k S \) is rational. The motive of such a surface \( S \) is geometrically split in \( \text{Chow}(k) \) and so also in \( \text{Chow}(k, R) \) for all coefficient rings \( R \), see e.g. [13, Thm. 2.6].

1.6. Motives of étale algebras and permutation modules. We denote by \( RG_k \) the group ring of \( G_k \) over \( R \).

A continuous \( RG_k \)-module \( M \) is called an \( RG_k \)-permutation module if \( M \) is a free and finitely generated \( R \)-module which has a basis which is permuted by the action of \( G_k \). If the coefficient ring is \( \mathbb{Z} \) we call a \( ZG_k \)-permutation module a \( G_k \)-permutation module. A direct summand of a \( RG_k \)-permutation module is called an invertible \( RG_k \)-module, respectively an invertible \( G_k \)-module if \( R = \mathbb{Z} \).
An example of a $RG_k$-permutation module is $CH_0(\bar{E})_R$ for $E$ a $k$-étale algebra, and in fact every $RG_k$-permutation module is isomorphic to $CH_0(\bar{E})_R$ for some $k$-étale algebra $E$.

If $\alpha : \text{Spec } E \to \text{Spec } F$ is a correspondence of degree 0 between the spectra of two $k$-étale algebras then

$$\alpha_* : CH_0(\bar{E})_R \to CH_0(\bar{F})_R, \beta \mapsto \alpha \circ \beta$$

is a $RG_k$-morphism. This defines a functor from the subcategory of $\text{Chow}(k)$ generated by étale $k$-algebras into the category of $RG_k$-permutation modules which is full and faithful, cf. [3, Sect. 7].

1.7. 0-dimensional motives. A motive $M$ in $\text{Chow}(k)$ is called 0-dimensional if there exist $k$-étale algebras $E_0, E_1, \ldots, E_n$, such that $M$ is a direct summand of the motive

$$\bigoplus_{i=0}^n (\text{Spec } E_i)(i).$$

Note that if the motive of a smooth projective scheme $X$ is 0-dimensional in $\text{Chow}(k) = \text{Chow}(k, \mathbb{Z})$ then it is also 0-dimensional in $\text{Chow}(k, R)$ for all coefficient rings $R$.

Example. The motive of a $k$-rational surface $S$ is 0-dimensional in $\text{Chow}(k, R)$ for every (commutative) coefficient ring $R$. In fact, since $k$ is perfect there exists a smooth projective surface $S'$, and morphisms $S' \to S$ and $S' \to \mathbb{P}^2_k$ which are compositions of blow-ups in closed points, see [10, Chap. II, Cor. 21.4.2]. Therefore by the blow-up formula (cf. [15] or [11, Chap. 6]) the motive of $S$ is a direct summand of the motive of $S'$ which in turn by the blow-up formula and the projective bundle theorem is isomorphic to

$$R \oplus \bigoplus_{i=1}^m (\text{Spec } l_i)(1) \oplus R(2)$$

for some finite field extensions $l_i \supseteq k$.

For later use we mention the following easy lemma.

Lemma. Let $X$ be a smooth projective geometrically integral $k$-scheme whose motive is 0-dimensional in $\text{Chow}(k, R)$. Then $\deg : CH_0(X)_R \to R$ is surjective. In particular, if moreover $R$ is torsion free and the motive of $X$ in $\text{Chow}(k, R)$ is geometrically split then the degree map is an isomorphism.

Proof. Since $X$ is geometrically integral the field $k$ is algebraically closed in its function field $k(X)$, and therefore

$$\text{res}_{k(X)/k} : \text{hom}_k(F, L)_R \to \text{hom}_{k(X)}(k(X) \otimes_k F, k(X) \otimes_k L)_R$$

is an isomorphism for all finite field extensions $F, L$ of $k$. This implies since the motive of $X$ in $\text{Chow}(k, R)$ is a direct summand of motives of Tate twists of separable field extensions of $k$ that the restriction map $CH_0(X)_R \to CH_0(X_{k(X)})_R$ is an isomorphism. The first claim follows.

The last assertion is a consequence of the fact that the kernel of the degree map is torsion if the motive of $X$ in $\text{Chow}(k, R)$ is geometrically split. □
1.8. **Z-trivial varieties.** There is a closely related property of schemes. A geometrically integral $k$-scheme $X$ is called $Z$-trivial if $\deg : CH_0(X_l) \to \mathbb{Z}$ is an isomorphism for all field extensions $l \supseteq k$ with $X(l) \neq \emptyset$, and strongly $Z$-trivial if $\deg : CH_0(X_l) \to \mathbb{Z}$ is an isomorphism for all field extensions $l \supseteq k$.

**Remark.** The notion $Z$-trivial, or more general $R$-trivial for an arbitrary coefficient ring $R$, has been introduced by Karpenko and Merkurjev [14, Def. 2.3].

As shown by Merkurjev [18, Thm. 2.11] a smooth projective $k$-scheme $X$ which is geometrically integral is strongly $Z$-trivial if and only if the class of the generic point in $CH_0(X_{k(X)})$ is defined over $k$, and this is in turn equivalent to the assertion that for any cycle module $M_*$ in the sense of Rost [19] the cycle cohomology group $H^n(X, M_*)$ is equal $M_*(k)$ via the canonical map $M_*(k) \to H^n(X, M_*) \subseteq M_*(k(X))$.

**Example.** Let $X$ be a geometrically rational smooth and projective $k$-scheme whose motive is 0-dimensional in $\text{Chow}(k)$. Then by the lemma in [17] above the degree map $CH_0(X_l) \to \mathbb{Z}$ is an isomorphism for all $l \supseteq k$, i.e. $X$ is strongly $Z$-trivial. This property has the following consequence.

1.9. **Lemma.** Let $X$ be a geometrically integral smooth and projective $k$-scheme, which is strongly $Z$-trivial. Then $\text{Br}(k) = H^2(k, \mathbb{G}_m) \to H^2_{et}(X, \mathbb{G}_m)$ is injective, and so the natural homomorphism $\text{Pic} X \to (\text{Pic} X)^{G_k}$ is an isomorphism.

**Proof.** The latter statement follows from the first by the Hochschild-Serre spectral sequence.

For the first assertion it is enough to show that the canonical map $\text{Br}(k) \to \text{Br}(k(X))$ is injective. Since $k$ is by assumption perfect there are no elements of exponent $p = \text{char} k > 0$ in $\text{Br}(k)$ if $k$ has positive characteristic $p$ by a theorem of Albert [11, p. 109]. Hence it is enough to show that for $n$ prime to $\text{char} k$ the natural homomorphism $H^2(k, \mu_n) \to H^2(k(X), \mu_n)$ is injective, where $\mu_n \subseteq \mathbb{G}_m$ denotes the group of $n$th roots of unity. For such $n$ the assignment $l \mapsto H^2(l, \mu_n^{\otimes 1})$ is a cycle module in the sense of Rost, see [19], and so by the above mentioned result of Merkurjev [18, Thm. 2.11] the natural homomorphism $H^2(k, \mu_n) \to H^2(k(X), \mu_n)$ is injective. \hfill $\square$

2. **Maps between étale algebras and smooth projective schemes in the category of Chow motives**

2.1. **Morphisms from motives of Tate twists of étale algebras to motives of smooth projective schemes.** We fix throughout this section a coefficient ring $R$. Let $X$ be a smooth projective and geometrically integral $k$-scheme and $E$ a $k$-étale algebra, i.e. $E = l_1 \times \ldots \times l_d$ for some finite separable extensions $l_i \supseteq k$.

Let $0 \leq i \leq n := \dim X$ be an integer. We have a homomorphism

$$\psi_{E,X} : \text{hom}_k((\text{Spec} E)(i), X)_R \to \text{Hom}_{RG_k}(CH_0(\bar{E})_R, CH_i(\bar{X})_R), \alpha \mapsto \bar{\alpha}_*,$$

where $\bar{\alpha}_* : CH_0(\bar{E})_R \to CH_i(\bar{X})_R$ maps the cycle $\beta$ to $\bar{\alpha} \circ \beta$. Note that since $\alpha$ is defined over $k$ the homomorphism $\bar{\alpha}_*$ is $RG_k$-linear.
Let $\bar{k} \supseteq l \supseteq k$ be an algebraic extension of $k$. The absolute Galois group $G_l$ of $l$ can be identified with a closed subgroup of $G_k$ and so any $RG_l$-homomorphism is also a $RG_l$-homomorphism. The following diagram commutes:

$$\begin{align*}
\hom_l((\Spec E_l)(i), X_l)_R & \xrightarrow{\phi_{E_l}(i)} \Hom_{RG_l}(CH_0(E_l), CH_i(\bar{X})_R) \\
\res_{l/k} & \downarrow \quad \downarrow
\hom_k((\Spec E)(i), X)_R & \xrightarrow{\psi_{E,X}} \Hom_{RG_k}(CH_0(E), CH_i(\bar{X})_R).
\end{align*}$$

(2)

**Lemma.** Let $X$ and $E = l_1 \times \ldots \times l_d$ be as above. Assume that the natural homomorphism

$$\res_{\bar{k}/l_i} : CH_i(X_{l_i})_R \rightarrow CH_i(\bar{X})_R^{G_{l_i}}$$

is an isomorphism for all $i = 1, \ldots, d$, where $G_{l_i} \subseteq G_k$ is the absolute Galois group of $l_i$. Then $\psi_{E,X}$ is an isomorphism.

**Proof.** This is clear if $k = \bar{k}$ is an algebraically closed field. For the general case we can assume that $E = l$ is a finite separable field extension of $k$. Using the commutative diagram (2) we see then that it is enough to show that the restriction map

$$\res_{l/k} : CH_i(X_l)_R = \hom_k(l, X)_R \rightarrow \hom_k(\bar{l}, \bar{X})_R = CH_i(l \times_k \bar{X})_R$$

is an isomorphism onto the $G_k$-invariant elements in $\hom_l(\bar{l}, \bar{X})_R$. But this follows from the commutative diagram (1) in §1.3 and the assumption that $\res_{\bar{k}/l} : CH_i(X_{\bar{l}})_R \rightarrow CH_i(\bar{X})_R^{G_l}$ is an isomorphism. \qed

2.2. Morphisms from motives of smooth projective schemes to motives of Tate twists of étale algebras.

We continue with the notation of the last subsection.

We have a homomorphism

$$\phi_{X,E} :$$

$$\hom_k(X, (\Spec E)(i))_R \rightarrow \Hom_{RG_k}(CH_i(\bar{X})_R, CH_0(E))_R, \alpha \mapsto \bar{\alpha}.$$  

This homomorphism commutes also with base change, i.e. if $l \supseteq k$ is a field extension then the diagram

$$\begin{align*}
\hom_l((\Spec E_l)(i))_R & \xrightarrow{\phi_{E_l}(i)} \Hom_{RG_l}(CH_i(\bar{X})_R, CH_0(E)_R) \\
\res_{l/k} & \downarrow \quad \downarrow
\hom_k((\Spec E)(i))_R & \xrightarrow{\phi_{E,X}} \Hom_{RG_k}(CH_i(\bar{X})_R, CH_0(E)_R)
\end{align*}$$

commutes.

**Lemma.** Let $E = l_1 \times \ldots \times l_d$ be as above. Assume that the motive of $X$ in $\text{Chow}(k, R)$ is geometrically split and that

$$\res_{\bar{k}/l_i} : CH_{n-i}(X_{l_i})_R \rightarrow CH_{n-i}(\bar{X})_R^{G_{l_i}}$$

is an isomorphism for the finite field extensions $\bar{k} \supseteq l_i \supseteq k$ with absolute Galois group $G_{l_i} \subseteq G_k$, $t = 1, \ldots, d$. Then $\phi_{E,X}$ is an isomorphism.
Proof. If $k = \kbar$ is algebraically closed this follows since the motive of $X$ is split and so (cf. [1,5])
\[
\text{CH}_i(\bar{X})_R \times \text{CH}_{n-i}(\bar{X})_R \rightarrow R, \ (\alpha, \beta) \mapsto \deg(\alpha \cap \beta)
\]
is a perfect pairing. As in the proof of the lemma in [2.1] this implies the general case using now the commutative diagram (3) instead of (2). \qed

2.3. Theorem. Let $X$ be a smooth and projective $k$-scheme of dimension $n$, whose motive is geometrically split in $\mathcal{CH}_0(k, R)$. Assume that there is an $k$-étale algebra $E$ and correspondences of degree $0$
\[
\alpha : (\text{Spec } E)(i) \xrightarrow{\cdot} X \quad \text{and} \quad \beta : X \xrightarrow{\cdot} (\text{Spec } E)(i)
\]
with corresponding $RG_k$-linear maps
\[
f = \psi^E_{\alpha, X} : \text{CH}_0(E)_R \rightarrow \text{CH}_i(\bar{X})_R
\]
and
\[
g = \phi^E_{\beta, X} : \text{CH}_i(\bar{X})_R \rightarrow \text{CH}_0(E)_R
\]
for some integer $0 \leq i \leq n = \dim X$.

Then we have $f \cdot g = \text{id}_{\text{CH}_i(\bar{X})_R}$ if and only if
\[
\bar{\alpha} \circ \bar{\beta} = \sum_{j=1}^{m} v_j \times u_j,
\]
where $u_1, \ldots, u_m \in \text{CH}_i(\bar{X})_R$ and $v_1, \ldots, v_m \in \text{CH}_{n-i}(\bar{X})_R$ are dual basis with respect to the perfect pairing $\text{CH}_i(\bar{X})_R \times \text{CH}_{n-i}(\bar{X})_R \rightarrow R, \ (\gamma, \rho) \mapsto \deg(\gamma \cap \rho)$, see [1,5].

Proof. We have a $k$-algebra isomorphisms $\bar{E} \xrightarrow{\sim} \kbar^r$ with corresponding open immersions $p_j : \text{Spec } k \rightarrow \text{Spec } \bar{E}$. We can then write $\bar{\alpha} = \sum_{j=1}^{r} (p_j \times \text{id}_{\bar{X}})^*(\alpha_j)$ and $\bar{\beta} = \sum_{j=1}^{r} (\text{id}_{\bar{X}} \times p_j)^*(\beta_j)$ for appropriate $\alpha_j$’s in $\text{CH}_i(\bar{X})_R$ and $\beta_j$’s in $\text{CH}_{n-i}(\bar{X})_R$, respectively. In terms of the dual basis we have $\alpha_j = \sum_{s=1}^{m} a_{sj} \cdot u_s$ and $\beta_j = \sum_{t=1}^{r} b_{jt} \cdot v_t$ for some $a_{sj}, b_{jt} \in R$. Since $\bar{\alpha} \circ \bar{\beta}$ is in the image of the injective homomorphism $\text{CH}_i(\bar{X})_R \otimes_R \text{CH}_{n-i}(\bar{X})_R \rightarrow \text{CH}_n(\bar{X} \times \bar{X})_R, \ \gamma \otimes \rho \mapsto \gamma \times \rho$, and this image is a free $R$-module with basis $u_i \times v_j, \ 1 \leq i, j \leq m$, by the Künneth isomorphism, see [1,5] the equation (4) is equivalent to
\[
\sum_{j=1}^{r} a_{sj} \cdot b_{jt} = \delta_{st}
\]
for all $1 \leq s, t \leq m$, where $\delta_{st}$ denotes the Kronecker delta.

On the other hand the cycles $\alpha_j$ and $\beta_j$ correspond by [2.1] and [2.2] to maps $f_j$ in $\text{Hom}_R(\text{CH}_0(k)_R, \text{CH}_i(\bar{X})_R)$ and $g_j$ in $\text{Hom}_R(\text{CH}_i(\bar{X})_R, \text{CH}_0(k)_R)$, respectively, such that
\[
f = \sum_{j=1}^{r} f_j \quad \text{and} \quad g = \sum_{j=1}^{r} g_j
\]
via the isomorphism $\text{CH}_0(\bar{E})_R \simeq \text{CH}_0(\kbar)_R$ induced by the pull-backs along the open immersions $p_1, \ldots, p_r$. 

We identify now $\text{CH}_i(\bar{X})_R$ with $\text{Hom}_R(\text{CH}_0(\bar{k})_R, \text{CH}_i(\bar{X})_R)$ naturally, and the group $\text{CH}_{n-i}(\bar{X})_R$ with $\text{Hom}_R(\text{CH}_i(\bar{X})_R, \text{CH}_{n-i}(\bar{X})_R)$ via the intersection product using that the motive of $\bar{X}$ is split. Then we have $f_j = \sum_{s=1}^{m} a_{sj} \cdot u_s$ and $g_j = \sum_{t=1}^{n} b_{jt} \cdot v_t$, and therefore $f(g(u)) = \sum_{j=1}^{r} f_j(g_j(u)) = \sum_{s=1}^{m} \left( \sum_{j=1}^{r} a_{sj} \cdot b_{jt} \right) \cdot u_s$ for all $1 \leq l \leq m$.

Hence since $u_1, \ldots, u_m$ is an $R$-basis of $\text{CH}_i(\bar{X})_R$ the equation $f \cdot g = \text{id}_{\text{CH}_i(\bar{X})_R}$ is equivalent to $\mathfrak{H}$.

This has the following consequence.

**Corollary.** Let $X$ be a smooth and projective $k$-scheme of dimension $n$, whose motive is geometrically split in $\text{Chow}(k,R)$. Assume there are $k$-étale algebras $E_0, \ldots, E_n$, and correspondences of degree $0$

$$\alpha = \sum_{i=0}^{n} \alpha_i : \bigoplus_{i=0}^{n} (\text{Spec } E_i)(i) \to X$$

and

$$\beta = (\beta_i)_{i=0}^{n} : X \to \bigoplus_{i=0}^{n} (\text{Spec } E_i)(i)$$

with corresponding $R G_k$-linear maps $f_i = \psi_{E_i,X}(\alpha_i) : \text{CH}_0(\bar{E}_i)_R \to \text{CH}_i(\bar{X})_R$ and $g_i = \phi_{X,E_i}(\beta_i) : \text{CH}_i(\bar{X})_R \to \text{CH}_0(\bar{E}_i)_R$ for $i = 0, \ldots, n$.

Then we have $\alpha \circ \beta = \text{id}_X$ in $\text{Chow}(k,R)$ if and only if $f_i \circ g_i = \text{id}_{\text{CH}_i(\bar{X})_R}$ for all $i = 0, \ldots, n$.

**2.4. Zero dimensional motives.** Recall that a motive $M$ in $\text{Chow}(k,R)$ is called zero dimensional if it is a direct summand of $\bigoplus_{i=0}^{n} (\text{Spec } E_i)(i)$ for some $k$-étale algebras $E_i$.

If the coefficient ring $R$ has the property that all projective $R$-modules are free this implies that $M$ is also geometrically split. Hence one implication of the corollary above can be formulated as follows:

**Corollary.** Let $X$ be a smooth projective $k$-scheme whose motive in $\text{Chow}(k,R)$ is zero dimensional. Assume that

(i) the motive of $X$ is geometrically split, or that

(ii) every projective $R$-module is free.

Then the $R G_k$-module $\text{CH}_i(\bar{X})_R$ is invertible for all $0 \leq i \leq \text{dim } X$.

**2.5. A splitting criterion.** The converse of the corollary in 2.4 above seems to be only true under some further assumptions on $X$. Let for this $X$ be a smooth and projective $k$-schemes whose motive is geometrically split in $\text{Chow}(k,R)$. We assume further that

(GC) If $k \supset l \supset k$ is an algebraic field extension then the restriction homomorphism $\text{CH}_i(X_l)_R \to \text{CH}_i(X_k)_R$ is an isomorphism onto the $G_l$-invariant elements of $\text{CH}_i(X_k)_R$ for all $0 \leq i \leq n := \text{dim } X$ (where as above we identify $G_l$ with a subgroup of $G_k$), and

(RN) Rost nilpotence is true for $X$ in $\text{Chow}(k,R)$, i.e. given a field extension $l \supset k$, and a correspondence $\alpha$ in $\text{end}_k(X)_R$, such that $\alpha_l = 0$ then $\alpha$ is nilpotent: $\alpha^{\circ N} = 0$ for some integer $N \geq 1$.  

We choose for all $0 \leq i \leq n$ a surjective $RG_k$-linear morphism $P_i \xrightarrow{f_i} \text{CH}_i(X)_R$ with $P_i$ a $RG_k$-permutation module. We can assume, see [1.6] that there is a $k$-étale algebra $E_i$, such that $P_i = \text{CH}_0((E_i)^\bar{k})_R$ for all $0 \leq i \leq n$. We have then the following result.

**Theorem.** If the morphisms $f_i$ are split, i.e. if there are $RG_k$-linear maps $g_i : \text{CH}_i(X)^\bar{k})_R \to \text{CH}_0((E_i)^\bar{k})_R$, such that $f_i \cdot g_i = \text{id}_{\text{CH}_i(X)}$ for all $0 \leq i \leq n$, then the motive of the scheme $X$ is a direct summand of $\bigoplus_{i=0}^n \text{Spec}(E_i)(i)$ in the category $\text{Chow}(k, R)$.

**Proof.** By assumption (GC) the lemmas in [2.1] and [2.2] imply the existence of cycles $\alpha_i \in \text{CH}_i(E_i \times_k X)_R = \text{hom}_k((\text{Spec } E_i)(i), X)_R$ and $\beta_i \in \text{CH}_{n-i}(X \times_k E_i)_R = \text{hom}_k(X, (\text{Spec } E_i)(i))_R$, such that

$$f_i = \psi_{E_i,X}^i(\alpha_i) \quad \text{and} \quad g_i = \phi_{X,E_i}^i(\beta_i)$$

for $i = 0, \ldots, n = \dim X$. By the corollary to Theorem [2.3] together with [1.3] we have then

$$\sum_{i=0}^n \alpha_i \circ \beta_i = \text{id}_X.$$

Since Rost nilpotence is true for $X$ in $\text{Chow}(k, R)$ by assumption (RN) we get therefore $\sum_{i=0}^n \alpha_i \circ \beta_i = \text{id}_X + \gamma$, where $\gamma$ is a nilpotent correspondence of degree 0 on $X$. Hence the claim.

**2.6. Example.** Let $R = F$ be a field of characteristic 0. Then for any field extension $l/k$ and any finite type $k$-scheme $Y$ the base change homomorphism $\text{CH}_i(Y)_F \to \text{CH}_i(Y)_l$ is injective for all $i \in \mathbb{N}$ and so Rost nilpotence is true for all motives in $\text{Chow}(k, F)$. Moreover, if $k \supseteq l \supseteq k$ is an algebraic field extension and $Y$ is geometrically integral then the base change map $\text{CH}_i(Y)_l \to \text{CH}_i(Y)_k$ is an isomorphism for all $0 \leq i \leq \dim Y$, where $G_l \subseteq G_k$ denotes the absolute Galois group of $l$.

Since by Maschke’s theorem every surjective $G_k$-linear map between continuous and $F$-finite dimensional $G_k$-modules splits we conclude from the corollary in [2.3] and the theorem in [2.5] the following (well known?) fact.

**Corollary.** Let $F$ be a field of characteristic 0, and $X$ a smooth and projective $k$-scheme. Then the motive of $X$ in $\text{Chow}(k, F)$ is zero dimensional if and only if it is geometrically split.

3. Geometrically rational surfaces with zero dimensional Chow motive

3.1. Geometrically rational surfaces. Let $S$ be a geometrically rational $k$-surface. Recall that by [12, 13] Rost nilpotence is true for $S$ in $\text{Chow}(k)$.

**Theorem.** Let $S$ be a geometrically rational $k$-surface. Then the motive of $S$ in $\text{Chow}(k)$ is zero dimensional if and only if Pic $\bar{S}$ is an invertible $G_k$-module and $S$ has a zero cycle of degree 1.
Proof. One direction is a consequence of the corollary to Theorem 2.3 and the lemma in [17].

To prove the converse, we show first that if the motive of $S$ is 0-dimensional then $S$ is strongly $\mathbb{Z}$-trivial. By Merkurjev [18, Thm. 2.11], see [18, 5] it is enough to show that the class of the generic point in $\text{CH}_0(S_{k(S)})$ is defined over $k$. Since $S$ has by assumption a 0-cycle of degree one this follows if we show that $\text{CH}_0(S_{k(S)})$ is torsion free.

Let $T^S$ be the $k$-torus with character group $\text{Pic} \bar{S}$, i.e. $T^S = \text{Spec}(\bar{k}[\text{Pic} \bar{S}]^{G_k})$, and set $K := \bar{k}(S)$. Since $S_{k(S)}$ is rational we have by [2, Thm. 0.1] an exact sequence (identifying $\text{Gal}(K/k(S)) = G_k$)

$$H^1(G_k, K^M_2(K(S))/K^M_2(k(S))) \rightarrow A_0(S_{k(S)}) \rightarrow H^1(G_k, T^S_{k(S)}),$$

where $A_0(S_{k(S)}) = \text{Ker}(\deg : \text{CH}_2(S_{k(S)}) \rightarrow \mathbb{Z})$ is the torsion part of $\text{CH}_0(S_{k(S)})$.

Since $\text{Pic} \bar{S}$ is $G_k$-invertible the torus $T^S$ is a direct factor of a quasi-trivial torus and thus by Hilbert’s Theorem 90 we have $H^1(G_k, T^S_{k(S)}) = 0$. The group on the left hand side vanishes by the main result of Colliot-Thélène [4]. Hence $A_0(S_{k(S)}) = 0$, and therefore $S$ is strongly $\mathbb{Z}$-trivial.

It follows then that $\text{res}_{k/l} : \text{CH}_0(S_l) \rightarrow \text{CH}_0(S_{\bar{k}}) = \text{CH}_0(S)^{G_l}$ is an isomorphism, and by Lemma 1.9 that $S_l \rightarrow (\text{Pic} \bar{S})^{G_l}$ is an isomorphism for all intermediate fields extensions $k \subseteq l \subseteq \bar{k}$, where $G_l \subseteq G_k$ denotes the absolute Galois group of $l$.

The same is obviously true for $\text{CH}_2$, i.e. $\text{CH}_2(S_l) \xrightarrow{\sim} \text{CH}_2(S)^{G_l}$, and so we can apply the theorem in [2,5] which shows that the motive of $S$ in $\mathbf{Chow}(k)$ is 0-dimensional as claimed. We are done. \hfill $\square$

3.2. A computational remark. Let $S$ be a geometrically rational $k$-surface. Assume that the motive of $S$ in $\mathbf{Chow}(k)$ is 0-dimensional. Then by the lemma in [17] the $G_k$-module $\text{CH}_0(\bar{S})$ is isomorphic to the trivial $G_k$-module $\mathbb{Z}$, and the same is true for $\text{CH}_2(\bar{S})$. Since the trivial $G_k$-module $\mathbb{Z}$ corresponds to the motive $\mathbb{Z}$ this implies by the corollary to Theorem 2.3 and the theorem in [2,5] that the motive of $S$ is a direct summand of

$$\mathbb{Z} \oplus (\text{Spec } E)(1) \oplus \mathbb{Z}(2),$$

(6)

for every $k$-étale algebra $E$, such that $\text{Pic} \bar{S}$ is a direct $G_k$-summand of $\text{CH}_0(\bar{E})$.

Note that the summand $\mathbb{Z}$ corresponds to the idempotent $\eta \times [S] \in \text{CH}_2(S \times_k S)$, and the summand $\mathbb{Z}(2)$ to the idempotent $[S] \times \eta \in \text{CH}_2(S \times_k S)$, where $\eta \in \text{CH}_0(S)$ has degree 1. In particular the “middle part” $(S, \rho)$ of the motive of $S$, where $\rho = \text{id}_S - (\eta \times [S] + [S] \times \eta)$, is a direct summand of $(\text{Spec } E)(1)$.

In some cases the complement of the summand $S$ in $\mathbf{Chow}(k)$ can be computed explicit. Recall for this that a $G_k$-module $C$ is called coflabby if $H^1(H, C) = 0$ for all open subgroups $H \subseteq G_k$. A coflabby resolution of $\text{Pic} \bar{S}$ is an exact sequence

$$0 \rightarrow C \rightarrow P \rightarrow \text{Pic} \bar{S} \rightarrow 0$$

(7)

with $P$ a $G_k$-permutation module and $C$ a coflabby $G_k$-module. Such a resolution always exists, and splits if and only if $\text{Pic} \bar{S}$ is an invertible $G_k$-module, i.e. a direct summand of a $G_k$-permutation module, see e.g. [7, Lem. 1]. Hence the motive of $S$ is zero dimensional in $\mathbf{Chow}(k)$ if and only if one (and so all) coflabby resolutions of $\text{Pic} \bar{S}$ split.
If there is such a split coflabby resolution (7) with C also a permutation module then
\[(S, \rho) \oplus (\text{Spec } F)(1) \simeq (\text{Spec } E)(1)\]
in Chow(k), where E and F are a k-étale algebras, such that \(P \simeq \text{CH}_0(E)\) and \(C \simeq \text{CH}_0(F)\) as \(G_k\)-modules. We leave the details to the reader.

3.3. Del Pezzo surfaces of degree 5. Let \(S\) be a Del Pezzo surface of degree \(d = 5\). Recall, see Manin’s book [16, Sect. 25], that Pic \(\bar{S}\) is a free \(\mathbb{Z}\)-module which has a basis \(\ell_0, \ell_1, \ell_2, \ell_3, \ell_4\), such that \((\ell_i, \ell_j) = 0\) for \(i \neq j\), \((\ell_0, \ell_0) = 1\), and \((\ell_i, \ell_i) = -1\) for \(1 \leq i \leq 4\), where \((-,-)\) is the intersection pairing. The class of the canonical bundle is then
\[\varpi_S = -3\ell_0 + \sum_{i=1}^{4} \ell_i \in \text{Pic } \bar{S},\]
and
\[\{ \ell \in \text{Pic } \bar{S} | (\varpi_S, \ell) = 0, (\ell, \ell) = -2 \}\]
is a root system of type \(A_4\). A set of simple roots is given by \(s_1 = \ell_1 - \ell_2, s_2 = \ell_2 - \ell_3, s_3 = \ell_3 - \ell_4, \) and \(s_4 = \ell_0 - \ell_1 - \ell_2 - \ell_3\).

Identifying \(\mathbb{Z}^{1+4} \xrightarrow{\cong} \text{Pic } \bar{S}, (a, b_1, b_2, b_3, b_4) \mapsto a\ell_0 - \sum_{i=1}^{4} b_i \ell_i\), the to the simple roots \(s_i\) corresponding reflections \(\sigma_i\) act on Pic \(\bar{S}\) as follows: The map \(\sigma_i\) interchanges the coordinates \(b_i\) and \(b_{i+1}\) and keeps the other coordinates for \(i = 1, 2, 3,\) and
\[\sigma_4(a, b_1, b_2, b_3, b_4) = (2a - b_1 - b_2 - b_3, a - b_2 - b_3, a - b_1 - b_3, a - b_1 - b_2, b_4).\]

As the action of the Galois group \(G_k\) on Pic \(\bar{S}\) fixes \(\varpi_S\) and preserves the intersection pairing this action factors through the above described action of the Weyl group W(\(A_4\)) of type \(A_4\), see [16, Thm. 23.9].

Using these data we construct now a coflabby resolution for the \(G_k\)-module Pic \(\bar{S}\). Consider the elements \(h_i := \ell_0 - \ell_i \in \text{Pic } \bar{S}, i = 1, 2, 3, 4, \) and \(h_5 = 2\ell_0 - \ell_1 - \ell_2 - \ell_3 - \ell_4\). Together with \(\varpi_S\) these elements generate the abelian group Pic \(\bar{S}\). The reflection \(\sigma_i\) interchanges \(h_i\) and \(h_{i+1}\) and leaves the other \(h_j\)'s fixed for \(i = 1, 2, 3, 4\). Let
\[P := \mathbb{Z} \cdot e \oplus \bigoplus_{i=1}^{5} \mathbb{Z} \cdot e_i\]
a free \(\mathbb{Z}\)-module of rank 6. We let W(\(A_4\)) act on this module as follows: The simple reflection \(\sigma_i\) interchanges \(e_i\) and \(e_{i+1}\) and leaves the other \(e_j\)'s and \(e\) fixed. Hence it is a W(\(A_4\))-permutation module, and therefore also a \(G_k\)-permutation module. The map
\[f : P \longrightarrow \text{Pic } \bar{S},\]
which maps \(e_i\) to \(h_i\) and \(e\) to \(\varpi_S = -3\ell_0 + \sum_{i=1}^{4} \ell_i\) is surjective, and W(\(A_4\))- and so also \(G_k\)-linear. Its kernel is generated by the W(\(A_4\))- and so also \(G_k\)-invariant
element \( x := (e_1 + e_2 + e_3 + e_4 + e_5) + 2e \) and so is isomorphic to the trivial \( G_k \)-module \( \mathbb{Z} \). Hence the exact sequence

\[
0 \rightarrow \mathbb{Z} \xrightarrow{\iota} P = \mathbb{Z} \cdot e \oplus \bigoplus_{i=1}^{5} \mathbb{Z} \cdot e_i \xrightarrow{f} \text{Pic} \bar{S} \rightarrow 0,
\]

where \( \iota \) maps 1 to \( x \), is a coflabby resolution of the \( G_k \)-module \( \text{Pic} \bar{S} \). It is split, as for instance \( e \mapsto -2 \) and \( e_i \mapsto 1 \) for \( i = 1, 2, 3, 4, 5 \) splits \( \iota \).

Since a Del Pezzo surface of degree 5 always has a rational point (in fact is rational) by a classical result of Enriques, see for instance Skorobogatov [20, Sect. 3.1] for a “modern” proof, the considerations above imply the following computation of motives.

**Theorem.** Let \( S \) be a Del-Pezzo surface of degree 5 over \( k \). Then we have an isomorphism

\[
\mathbb{Z}(1) \oplus S \cong \mathbb{Z} \oplus \mathbb{Z}(1) \oplus (\text{Spec} E)(1) \oplus \mathbb{Z}(2)
\]

in \( \mathcal{C}how(k) \), where \( E \) is a \( k \)-étale algebra of degree 5.

**Remarks.**

(i) A small alteration of the construction above gives a coflabby resolution for the Picard group \( \text{Pic} \bar{S} \) where \( S \) is a Del Pezzo surface of degree 6 (we have essentially only to remove \( \ell_4 \)).

(ii) The summand \( \mathbb{Z}(1) \) on both sides of the isomorphism (9) does not cancel if \( \text{Pic} \bar{S} \) is not a \( G_k \)-permutation module.

### 3.4. Torsion in the group of zero cycles of a geometrically rational surface.

Let \( S \) be a geometrically rational \( k \)-surface. If \( S \) is zero dimensional in \( \mathcal{C}how(k) \) then \( S \) is strongly \( \mathbb{Z} \)-trivial, see the example in [1,8].

The following theorem of Colliot-Thélène [5], see the Appendix, implies that the converse is also true.

**Theorem (Colliot-Thélène).** Let \( K \) be a field with separable closure \( K_s \) and \( X \) a smooth projective and geometrically integral \( K \)-scheme with \( K \)-rational point, such that

(i) \( \text{Pic} X_K \) is a free and finitely generated \( \mathbb{Z} \)-module, and

(ii) \( \deg : \text{CH}_0(X_K(X)) \rightarrow \mathbb{Z} \) is an isomorphism.

Then \( \text{Pic} X_K \) is an invertible \( G_K \)-module, where \( G_K = \text{Gal}(K_s/K) \) is the absolute Galois group of \( K \).

(Note that if \( K \) is a perfect field the assumption \( X(K) \neq \emptyset \) can be replaced by \( X \) has a 0-cycle of degree 1 as under the assumption \( K \) is perfect the existence of such a cycle assures already that \( X \) possesses a universal torsor, see [9] (2.0.2), and Props. 2.2.2 (b) and 2.2.5].)

Let \( S \) be a geometrically rational \( k \)-surface with function field \( K = k(S) \), which is strongly \( \mathbb{Z} \)-trivial. Then also \( S_K \) is strongly \( \mathbb{Z} \)-trivial and so by the theorem of Colliot-Thélène the \( G_K \)-module \( \text{Pic} S_K \) is invertible. But \( \text{Gal}(K_s/k(S)) \) acts trivially on \( \text{Pic} S_K \) since the restriction maps \( \text{Pic} \bar{S} \rightarrow \text{Pic} S_{k(S)} \rightarrow \text{Pic} S_K \) are both isomorphisms. Identifying \( G_k \) with \( \text{Gal}(k(S)/K) \) it follows that also the \( G_k \)-module \( \text{Pic} \bar{S} \) is invertible. Therefore:
3.5. Theorem. Let $S$ be a geometrically rational surface over $k$. Then the following statements are equivalent:

(i) The degree map $\text{CH}_0(S_{k(S)}) \to \mathbb{Z}$ is an isomorphism and $S$ has a 0-cycle of degree 1.

(ii) The surface $S$ is strongly $\mathbb{Z}$-trivial.

(iii) The $G_k$-module $\text{Pic} \bar{S}$ is invertible and $S$ has a 0-cycle of degree 1.

(iv) The motive of $S$ in $\text{Chow}(k)$ is zero-dimensional.

3.6. Strong $\mathbb{Z}$-triviality of $T^S$-torsors over $S$. The following is a corollary of the arguments in the letter [5] by Colliot-Thélène.

Let $S$ be a geometrically rational surface over $k$, whose motive in $\text{Chow}(k)$ is 0-dimensional, and $T^S$ the torus with character group $\text{Pic} \bar{S}$.

Let $Y$ be a $T^S$-torsor over $S$, and $Y^c$ and $(T^S)^c$ smooth compactifications of $Y$ and $T^S$, respectively. Such compactifications exist, see Colliot-Thélène, Harari, and Skorobogatov [3], and Colliot-Thélène and Sansuc [9, Rem. 2.1.4]. We aim to show that under the assumption that the Chow motive of $S$ is 0-dimensional the $k$-scheme $Y^c$ is strongly $\mathbb{Z}$-trivial, i.e.

$$\text{deg} : \text{CH}_0(Y^c_K) \to \mathbb{Z}$$

is an isomorphism for all field extensions $K \supseteq k$ (cf. 1.8).

For this we observe first that since the motive of $S$ is 0-dimensional in $\text{Chow}(k)$ it is a direct summand of $\mathbb{Z} \oplus (\text{Spec} E)(1) \oplus \mathbb{Z}(2)$ for some $k$-étale algebra $E$, see [5]. Hence $S_K$ is a direct summand of $\mathbb{Z} \oplus (\text{Spec} E_K)(1) \oplus \mathbb{Z}(2)$ in $\text{Chow}(K)$ and so it is also 0-dimensional for all fields $K \supseteq k$.

By Theorem 3.5 above we know that then $\text{Pic} \bar{S}$ is an invertible $G_k$-module and therefore $T^S$ is a direct summand of a quasi trivial torus, which in turn by Hilbert 90 implies that $H^1(K, T^S_K) = 0$ for all $K \supseteq k$. Hence every $T^S$-torsor over a field extensions of $k$ is trivial and so in particular $(S \times_k T^S)_K$ is trivial. It follows that $Y$ is birational to $S \times_k T^S$. Then also $Y^c_K$ is birational isomorphic to $(S \times_k (T^S)^c)_K$ for all $K \supseteq k$, and so we have by the lemma in [1.2] that $\text{deg} : \text{CH}_0(Y^c_K) \to \mathbb{Z}$ is an isomorphism if and only if $\text{deg} : \text{CH}_0(S_K \times_K (T^S)^c_K) \to \mathbb{Z}$ is an isomorphism. But since the Chow motive of $S_K$ is a direct summand of $\mathbb{Z} \oplus (\text{Spec} E_K)(1) \oplus \mathbb{Z}(2)$ the latter group is isomorphic to $\text{CH}_0((T^S)^c_K)$ via the push-forward along the projection $S_K \times_K (T^S)^c_K \to (T^S)^c_K$ for all field extensions $K$ of $k$. Hence it is enough to show that the degree map

$$\text{deg} : \text{CH}_0((T^S)^c_K) \to \mathbb{Z}$$

is an isomorphism for all field extensions $K \supseteq k$.

Let $K$ be such an extension with separable closure $K_s$ and absolute Galois group $G_K = \text{Gal}(K_s/K)$. The projections from $S_K \times_K (T^S)^c_K$ to $S_K$ and to $(T^S)^c_K$ induce a $G_K$-linear map

$$\text{Pic} S_K \oplus \text{Pic}(T^S)^c_K \to \text{Pic} \left(S_K \times_K (T^S)^c_K \right)$$

which is an isomorphism by Colliot-Thélène and Sansuc [7, Lem. 11] since $S_K$ is rational as shown by Coombes [10]. Therefore $\text{Pic}(T^S)^c_K$ is a direct $G_K$-summand of $\text{Pic}(S_K \times_K (T^S)^c_K)$. 


On the other hand by [9] Prop. 2.A.1 there are $G_K$-permutation modules $P$ and $Q$, such that we have an isomorphism of $G_K$-modules

$$\text{Pic}(S_K \times_K (T^S)_K^c) \oplus P \simeq \text{Pic} Y_K^c \oplus Q$$

(10)
since $Y_K^c$ and $S_K \times_K (T^S)_K^c$ are $K$-birational to each other.

This is in particular true for an universal $T^S$-torsor $Z$ over $S$. For such a $T^S$-torsor the $G_K$-module $\text{Pic} Z_K^c$ is invertible by [9, Thm. 2.1.2] and therefore by the isomorphism (10) for $Y = Z$ also $\text{Pic}(T^S)_K^c$ is an invertible $G_K$-module. But this implies by Colliot-Thélène and Sansuc [7, Prop. 6] (cf. the remark below) that $T^S_K$ and so also its compactification $(T^S)_K^c$ is stably rational, and so the degree map $\text{deg} : \text{CH}_0((T^S)_K^c) \rightarrow \mathbb{Z}$ is an isomorphism as claimed. We have proven one direction of the following result.

3.7. Corollary. Let $S$ be a geometrically rational $k$-surface. Denote by $T^S$ the $k$-torus with character group $\text{Pic} \bar{S}$. Then the following is equivalent:

(i) The motive of $S$ in $\text{Chow}(k)$ is 0-dimensional.
(ii) Every compactification of every $T^S$-torsor over $S$ is strongly $\mathbb{Z}$-trivial.

Proof. We are left to show (ii) $\implies$ (i). Assuming (ii) the scheme $S \times_k (T^S)_K^c$ is strongly $\mathbb{Z}$-trivial, where $(T^S)_K^c$ is a compactification of $T^S$, and so the degree map $\text{CH}_0((S \times_k (T^S)_K^c)_K^c) \rightarrow \mathbb{Z}$ is an isomorphism for all field extensions $K \supseteq k$. Since this degree map factors through $\text{deg} : \text{CH}_0(S_K^c) \rightarrow \mathbb{Z}$ also the latter is surjective. On the other hand $T^S_K$ and so also $(T^S)_K^c$ has a $K$-rational point and therefore $\text{CH}_0(S_K^c)$ is a direct summand of $\text{CH}_0((S \times_k (T^S)_K^c)_K^c)$ and consequently torsion free. It follows that $\text{deg} : \text{CH}_0(S_K^c) \rightarrow \mathbb{Z}$ is an isomorphism as the kernel of this map is the torsion subgroup of $\text{CH}_0(S_K^c)$ since $S$ is geometrically rational. Hence $S$ is strongly $\mathbb{Z}$-trivial and so the motive of $S$ in $\text{Chow}(k)$ is 0-dimensional by Theorem 3.5 above. We are done.

Remark. The assertion of [7, Prop. 6] we use assumes that the base field has characteristic 0 to assure the existence of a smooth compactification for every torus over this field. Only later it has been shown by Brylinski and Kümemann that such compactifications exist also in positive characteristic, cf. [6].

APPENDIX A. A LETTER OF COLLIO-T-THÉLÈNE TO THE AUTHOR

The following is a reproduction of the letter [5] by Colliot-Thélène except that (i) the list of references has been deleted, (ii) all citations refer now to the bibliography of the article, and (iii) a remark at the end has been added.

Toronto, den 8. Mai 2013

Brief von J.-L. Colliot-Thélène an Stefan Gille

Hier ist ein allgemeiner Satz.

Satz. Sei $F$ ein Körper, $X$ eine glatte, projektive, geometrisch irreduzible Varietät mit einem rationalen Punkt. Sei $G = \text{Gal}(F_s/F)$ die absolute Galoisgruppe von $F$.

Nehmen wir an :

(a) Für jede Erweiterung von Körpern $L/F$ ist die Gradabbildung

$$\text{CH}_0(X_L) \rightarrow \mathbb{Z}$$
bijektiv.

(b) \( M = \text{Pic}(X \times_F F_s) \) ist ein Gitter.

Dann gibt es ein \( G \)-Gitter \( N \), einen Permutationsmodul \( P \) und einen Isomorphismus \( M \odot N \simeq P \) von Galois Gittern.

Beweis – auch im Falle \( \text{Char}(F) > 0 \).

Sei \( p_0 \in X(F) \), \( T \) der \( F \)-Torus mit Charaktergruppe \( \hat{T} = M \), und \( Y \longrightarrow X \) der universelle Torsor über \( X \) mit trivaler Faser im Punkt \( p_0 \). Dies ist ein Torsor unter der Gruppe \( T \). Sei \( L/F \) eine Erweiterung von Körpern. So ein Torsor definiert Abbildungen

\[ X(L) \longrightarrow H^1(L,T) \]

und allgemeiner

\[ CH_0(X_L) \longrightarrow H^1(L,T) \]

(letztere eine Gruppenabbildung) die \( p_0 \) nach 0 schicken. (Siehe [7, Prop. 12 Seite 198]).

Sei \( \eta \) der generische Punkt von \( X \) und \( L = F(X) \) der Funktionenkörper von \( X \). Unter der Annahme \( \text{Grad} : CH_0(X_L) \simeq \mathbb{Z} \) folgt, daß das Bild von \( \eta - p_0 \) unter

\[ CH_0(X_L) \longrightarrow H^1(L,T) \]

null ist, also ist das Bild von \( \eta \) unter

\[ X(L) \longrightarrow H^1(L,T) \]

auch null. Das besagt aber, daß der Torsor \( Y \longrightarrow X \) unter \( T \) einen rationalen Schnitt hat, und somit ist die \( F \)-Varietät \( Y \) \( F \)-birational zum Produkt \( X \times_F T \).

Sei \( T^c/F \) eine äquivariante glatte Kompaktifizierung vom \( F \)-torus \( T \) ([8]).

Dann ist \( Y^c = Y \times^T T^c \) (contracted product) eine vollständige glatte Kompaktifizierung von \( Y \).

Die glatten projektiven Varietäten \( Y^c \) und \( X \times_F T^c \) sind \( F \)-birational zueinander. In [3 Théorème 1] und [9 Thm. 2.12 Seite 411] wird gezeigt : Da \( Y \longrightarrow X \) ein universeller Torsor ist, folgt \( F_s \simeq F_s[Y]^c \) und \( \text{Pic}(Y \times_F F_s) = 0 \). Daraus folgt leicht : \( \text{Pic}(Y^c) \) ist ein Gitter und als \( G \)-Modul ein Permutationsmodul \( P_0 \).

Weil \( T^c \) geometrisch rational ist, ist die natürliche Abbildung

\[ \text{Pic}(X_{F_s}) \oplus \text{Pic}(T^c_{F_s}) \longrightarrow \text{Pic}((X \times_F T^c)_{F_s}) \]

ein Isomorphismus ([9, Lemme 11 Seite 188]).

In [9 Prop. 2.A.1 Seite 461] findet man einen Beweis von Moret-Bailly für das folgende Lemma.

**Lemma.** Seien \( V \) und \( W \) zwei glatte vollständige geometrisch irreduzible Varietäten die \( F \)-birational zueinander sind. Dann gibt es Permutationsmodule \( P_1 \) und \( P_2 \), und einen Isomorphismus von Galoismodulen

\[ P_1 \oplus \text{Pic}(V_{F_s}) \simeq P_2 \oplus \text{Pic}(W_{F_s}) \].
Wenn man alles zusammenfasst, erhält man einen Isomorphismus von Galoismodulen
\[ P_1 \oplus P_0 \simeq P_2 \oplus \text{Pic}(X_{F_s}) \oplus \text{Pic}(T_{F_s}), \]
also ist Pic(X_{F_s}) invertierbar.

Was zu beweisen war.

Bemerkungen.

1. Wenigstens wenn Char(F) = 0 ist, sollte man zeigen können, daß Annahme (b) aus Annahme (a) folgt. Man kann schon sehen, daß Pic(X \times_F F_s) von endlichem Typ über \( \mathbb{Z} \) ist.

   (Hinzugefügt, 10. April 2014) Sei Char(F) = 0, und sei \( n > 0 \) eine ganze Zahl.

   Aus Annahme (a) und der Kummerschen Sequenz folgt:
   \[ 0 = H^1_{et}(X_{F_s}/F_s, \mu_n) = \text{Pic}(X_{F_s})[n]. \]

   Wegen der bekannten Struktur der Picard Gruppe folgt, daß die Picard Varietät von X null ist, und daß die Néron-Severi Gruppe torsionsfrei ist, also ist Pic(X \times_F F_s) ein Gitter.

2. Es ist nicht klar, ob die Arbeit von Merkurjev [18] einen alternativen Beweis des Satzes geben könnte. Aus dieser Arbeit kann man wenigstens folgern, daß Pic(X \times_F F_s) co-welk ist.

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