Curci-Ferrari–type condition in Hamiltonian formalism: A free spinning relativistic particle

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Abstract – The Curci-Ferrari (CF)–type restriction emerges in the description of a free spinning relativistic particle within the framework of the Becchi-Rouet-Stora-Tyutin (BRST) formalism when the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for this system are derived from the application of the horizontality condition (HC) and its supersymmetric generalization (SUSY-HC) within the framework of the superfield formalism. We show that the above CF condition, which turns out to be the secondary constraint of our present theory, remains time-evolution invariant within the framework of Hamiltonian formalism. This time-evolution invariance i) physically justifies the imposition of the (anti-)BRST invariant CF-type condition on this system, and ii) mathematically implies the linear independence of BRST and anti-BRST symmetries of our present theory.

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Introduction. – The standard model of high-energy physics is one of the most successful theories of modern times which is based on the four (3 + 1)-dimensional (4D) non-Abelian 1-form \((A^{(1)}=dx^\mu A_\mu; \mu=0,1,2,3)\) gauge theory. Such theories are covariantly quantized by using Becchi-Rouet-Stora-Tyutin (BRST) formalism where the unitarity and “quantum” gauge (i.e., BRST) invariance are respected together at any arbitrary order of perturbative computations for a given physical process (that is allowed by the interacting gauge theory).

One of the decisive features of BRST approach to any arbitrary \(p\)-form \((p=1,2,3\ldots)\) non-supersymmetric gauge theories is the existence of Curci-Ferrari (CF)–type restriction\(^{1}\) (see, e.g., [1]) which has been recently shown to be deeply connected with the mathematical concept of gerbes (see, e.g., [2,3]). This condition ensures the absolute anticommutativity of BRST and anti-BRST symmetries thereby making the BRST and anti-BRST symmetries have completely independent identity and meaning.

The geometrical superfield formalism\(^{4,5}\) provides a sound mathematical basis for the derivation of CF-type restriction for any arbitrary \(p\)-form non-supersymmetric gauge theory where the celebrated horizontality condition (HC) plays a very decisive role. In our recent publication\(^{6}\), we have applied the superfield formulation to a toy model of supersymmetric (SUSY) gauge theory (i.e., a free spinning relativistic particle) where we have generalized the HC to its supersymmetric counterpart (SUSY-HC). The application of the latter leads to the emergence of a CF-type restriction (cf. eq. (6) below) and derivation of proper (anti-)BRST symmetries (cf. eqs. (2) and (3) below). The Noether conserved charges, corresponding to the latter symmetries, have been derived and some novel features in the physicality criteria (with the conserved and nilpotent (anti-)BRST charges) have been pointed out in [7].

In the above context, it is important to stress that neither the Lagrangian nor the superfield formalism answers the question as to why the CF-type restriction should be imposed on the theory within the framework of BRST formalism. The central purpose of our present

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\(^{1}\)For the Abelian 1-form gauge theory, the CF-type restriction turns out to be trivial (as it is the limiting case of CF condition present in its non-Abelian counterpart).
is a modest step towards our main goal of applying the CF-type restriction in one stroke (cf. (24), (28) and (30) below).

The motivating factors behind our present investigations are as follows. First, the Lagrangian and superfield formulations [4,5] do not shed any light on the time-evolution invariance of the CF-type condition. Thus, it is very important for us to prove it within the framework of Hamiltonian formulation. Second, for aesthetic reasons, it is essential to obtain the CF-type condition from a single Lagrangian and/or Hamiltonian. We have accomplished this goal in our present endeavor. Finally, our present work is a modest step towards our main goal of applying the superfield and BRST formulations to the SUSY p-form gauge theories of phenomenological importance.

Our present paper is organized as follows. In section two, we concisely recapitulate the bare essentials of the proper (i.e., off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetries, coupled Lagrangians and CF-type restriction. Our section three is devoted to the derivation of alternative sets of Lagrangians (and Hamiltonians) that are useful in the description of spinning relativistic particle and which yield the CF-type restriction in one step. Section four deals with the time-evolution invariance of the CF-type restriction within the framework of Hamiltonian formalism. Finally, in section five, we make some concluding remarks.

Preliminaries: off-shell nilpotent symmetries in the Lagrangian formulation. – We begin with the one (0 + 1)-dimensional (1D) SUSY system of a massless spinning particle whose Lagrangian is (see, e.g., [8])

\[ L_0 = p_\mu \dot{x}_\mu - \frac{c}{2} \dot{\varphi}^2 + \frac{L}{2} \dot{\chi}_\mu \dot{\psi}_\mu + i \chi p_\mu \psi^\mu, \]

where the motion of the 1D SUSY system is parameterized by the monotonically increasing parameter \( \tau \) and \( \dot{x}_\mu = (dx^\mu/d\tau) \) and \( \dot{\psi}_\mu = (d\psi^\mu/d\tau) \) are the generalized “velocities” corresponding to the D-dimensional target space position variable \( x^\mu(\tau) (\mu = 0, 1, 2 \ldots D-1) \) and its supersymmetric partner \( \psi^\mu(\tau) \). The latter turns out to be the spin variable of this particle. The constraints \( \dot{p}_\mu = 0 \) and \( p_\mu \dot{\psi}_\mu = 0 \) of our present theory have been incorporated through the Lagrange multipliers \( c(\tau) \) and \( \chi(\tau) \) which are bosonic and fermionic in nature, respectively. These latter variables are analogues of the vierbein and Rarita-Schwinger fields of the 4D supergravity theory. In addition, these variables transform exactly like the gauge potentials of the 4D supersymmetric gauge theories. The fermionic variables \( (\psi_\mu, \chi) \) anticommute (i.e., \( \chi^2 = 0, \chi \psi^\mu = 0, \) etc.) amongst themselves and commute (i.e., \( \psi_\mu \psi_\nu - \psi_\nu \psi_\mu = 0, \chi \psi_\mu - x^\mu \chi = 0, e \chi - \chi e = 0, \psi_\mu e - e \psi_\mu = 0, \) etc.) with the bosonic variables \( (x^\mu, c, p_\mu, \dot{p}_\mu, \) etc.) of the theory. The basic fermionic \( (\psi_\mu, \chi) \) and bosonic \( (x^\mu, c) \) degrees of freedom match which is the indication of the presence of supersymmetry in the theory.

The well-known gauge and supergauge symmetry transformations [8] of the above Lagrangian can be generalized to the proper (i.e., off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations \( (s_{(a)b}) \) at the quantum level of the theory, as [6]

\[ s_{ab} x_\mu = c p_\mu + \beta \psi_\mu, \quad s_{ab} c = \dot{c} + 2 \dot{\beta} \chi, \quad s_{ab} p_\mu = 0, \]

\[ s_{ab} \dot{c} = -i \beta^2, \quad s_{ab} c = i \dot{\beta}, \quad s_{ab} \dot{\beta} = 0, \quad s_{ab} \chi = i \dot{\beta}, \quad s_{ab} b = 2 \dot{\beta} \gamma, \]

\[ s_{ab} \psi_\mu = i \dot{\beta} p_\mu, \] (2)

\[ s_{a} x_\mu = c p_\mu + \beta \psi_\mu, \quad s_{a} c = \dot{c} + 2 \dot{\beta} \chi, \quad s_{a} p_\mu = 0, \]

\[ s_{a} \dot{c} = -i \beta^2, \quad s_{a} c = i \dot{\beta}, \quad s_{a} \dot{\beta} = 0, \quad s_{a} \chi = i \dot{\beta}, \quad s_{a} b = -2 \dot{\beta} \gamma, \]

\[ s_{a} \psi_\mu = i \dot{\beta} p_\mu, \] (3)

which are the symmetry transformations for the following coupled (but equivalent) Lagrangians of our present physical system [6]

\[ L_\delta = L_0 + b \dot{\beta} + b(\dot{b} + 2 \dot{\beta} \beta) - i \dot{\beta}(\dot{c} + 2 \dot{\beta} \chi) + 2 i \dot{\beta} \dot{c} \]

\[ -2 \dot{c}(\dot{\gamma} + \beta \dot{\beta}) + 2 \beta \dot{c} \dot{\gamma} + \beta^2 \dot{\beta}^2 + 2 \beta \dot{\beta} \dot{\gamma}, \] (4)

\[ L_\delta = L_0 - b \dot{\beta} + b(\dot{b} + 2 \dot{\beta} \beta) - i \dot{\beta}(\dot{c} + 2 \dot{\beta} \chi) + 2 i \dot{\beta} \dot{c} \]

\[ -2 \dot{c}(\dot{\gamma} - \beta \dot{\beta}) + 2 \beta \dot{c} \dot{\gamma} + \beta^2 \dot{\beta}^2 + 2 \beta \dot{\beta} \dot{\gamma}, \] (5)

where \( (b, \dot{b}) \) are the Nakanishi-Lautrup-type auxiliary variables, \( (\dot{c}, \dot{\beta}) \) are the fermionic (anti-)ghost variables (i.e., \( c^2 = c = 0, \dot{c} + c = 0 \) and \( \beta \bar{\beta} \) are the bosonic (anti-)ghost variables. The fermionic (i.e., \( \gamma \chi + \chi \gamma = 0, \gamma \dot{\psi}_\mu + \psi_\mu \dot{\gamma} = 0 \) variable \( \gamma \) is also included in the theory for its complete and consistence description within the framework of BRST formalism. The above (anti-)ghost variables are required for the validity of unitarity in the theory. One of the key observations, from the coupled Lagrangians (4) and (5), is the following equations of motion:

\[ b = \frac{\dot{c}}{2} - \beta \dot{\beta}, \quad \dot{b} = \frac{\dot{c}}{2} - \beta \dot{\beta} \quad \Rightarrow \quad b + \dot{b} + 2 \beta \dot{\beta} = 0. \] (6)

The above final expression \( b + \dot{b} + 2 \beta \dot{\beta} = 0 \) is nothing but the Curci-Ferrari (CF)–type restriction that is responsible for the absolute anticommutativity of the (anti-)BRST
symmetry transformations. For instance, the basic anticommutators \( \{s_b, s_a\} = 0 \) and \( \{s_b, s_{ab}\} x_a = 0 \) are true if and only if \( b + b + 2\beta \beta = 0 \). The existence of the CF-type restriction, in the context of spinning relativistic particle, is a completely novel observation which has emerged out from the description of our present SUSY system within the framework of superfield formalism (see, e.g., [6]).

We also note that the (anti-)BRST invariant (i.e., \( s_{(a)} [b + b + 2\beta \beta = 0] \)) CF-type restriction has been obtained in two steps because first we have derived the expressions for \( b \) and \( \bar{b} \) from Lagrangians (4) and (5) and, then, we have added them together to obtain (6). Further, it is not clear as to why one should impose the CF-type condition for the absolute anticommutativity property (within the framework of Lagrangian formalism). Thus, in the forthcoming sections, we shall discuss the time-evolution invariance of CF-type restriction within the framework of Hamiltonian formulation so that we could physically justify its imposition (during the complete time-evolution) of our present SUSY system.

**Lagrangians and corresponding canonical Hamiltonians: alternative forms.** As we discuss in this section, the Lagrangian and Hamiltonian formulations of our present theory in an explicit fashion. In this context, first of all, we re-express the coupled Lagrangians (cf. (4), (5)) in the following forms:

\[
L_b^{(1)} = L_0 + b \dot{c} + b(b + 2\beta \beta) - i \dot{e} \dot{c} + \beta^{2} \beta^{2} + 2i \dot{\bar{e}} \dot{c} \chi + 2i(e \chi + i e \beta) \beta \\
+ 2(e \chi - \beta \dot{e} + \beta c) \gamma, \tag{7}
\]

\[
L_b^{(2)} = L_0 - b \dot{c} - b(b + 2\beta \beta) - i \dot{e} \dot{c} + \beta^{2} \beta^{2} - 2i \dot{\bar{e}} \dot{c} \chi + 2i(e \chi + i e \beta) \beta \\
+ 2(e \chi - \beta \dot{e} + \beta c) \gamma, \tag{8}
\]

which differ from (4) and (5) by the total derivative, namely,

\[
L_{(b,b)} = L_{(b,b)}^{(1)} + \frac{d}{d\tau} \left[ 2i \beta \dot{c} \chi - 2i \beta \dot{e} c \right]. \tag{9}
\]

As a consequence, the dynamical equations of motion for the theory remain intact (because the action remains the same for physical variables which vanish off at infinity). In our further discussions, the forms (7) and (8) would be preferred over the coupled Lagrangians (4) and (5) because the former coupled set supports the maximum number of variables having their non-vanishing conjugate momenta.

Let us first concentrate on the Lagrangian (7) which is perfectly BRST invariant [6]. The Euler-Lagrange equations of motion, from (7), are

\[
\dot{x}_\mu - e p_\mu + i \dot{c} \chi = 0, \quad \dot{p}_\mu - \chi p_\mu = 0, \\
\dot{c} + 2(b + 2\beta \beta) = 0, \quad \dot{b} + \frac{p^2}{2} + 2\gamma \chi + 2\beta \beta = 0,
\]

\[
p_\mu \psi^\mu + 2 \dot{e} \beta \beta - 2 \dot{\bar{e}} \beta \beta - 2 i \epsilon \gamma = 0, \quad \dot{p}_\mu = 0,
\]

\[
\dot{\bar{e}} + 2 \dot{\beta} \chi + 2 \dot{\beta} \chi + 2i \beta \gamma = 0, \quad e \chi - \beta \dot{e} + \beta c = 0, \\
\dot{\beta} + 2 \dot{\beta} \chi + 2 \dot{\beta} \chi + 2i \beta \gamma = 0, \\
\beta \dot{\beta} - e \dot{c} \beta + \beta \beta^{2} + c \gamma = 0, \\
\dot{\beta} + i \dot{\beta} - i \dot{c} \chi + \gamma \gamma + \beta \beta^{2} = 0, \tag{10}
\]

where we have used the convention of the “left-derivative” in the operation of derivatives with respect to the fermionic variables. Exactly the same kind of equations of motion emerge out from the Lagrangian (8) except the following:

\[
\dot{e} - 2(\dot{b} + \beta \beta) = 0, \quad \dot{b} - \frac{p^2}{2} - 2\gamma \chi + 2 \beta \beta = 0, \\
\dot{\beta} + e \dot{c} - b \beta - i e \chi + \gamma c - \beta \beta^{2} = 0, \tag{11}
\]

\[
\dot{\beta} - i \dot{e} \chi - \dot{\gamma} + \bar{b} \beta = 0.
\]

It is to be noted that the equations of motion, corresponding to \( b \) and \( \bar{b} \) in (10) and (11), lead to the derivation of CF condition (6). In addition, the variation of the action with respect to the variable \( e(\tau) \) also yields the CF-type condition (6). At this juncture, it is crystal clear that the CF-type condition (6) is derived in two steps by adding appropriate equations of motion from (10) and (11) which are true for the Lagrangians (7) and (8). However, this sort of derivation is not preferable. It would be nice to obtain the CF-type condition from a single Lagrangian.

Using the Legendre’s transformation, one can calculate the Hamiltonian, corresponding to the Lagrangian (7) and (8), as given below

\[
H_b = i \Pi_{(c)} \Pi_{(c)} + \frac{e}{2} \dot{\beta}^2 - i \chi (p_\mu \psi^\mu - b(b + 2\beta \beta)) \\
- \beta \beta^{2} - 2e \chi \dot{c} + i \Pi_{(c)} \gamma, \tag{12}
\]

\[
H_b = i \Pi_{(c)} \Pi_{(c)} + \frac{e}{2} \dot{\beta}^2 - i \chi (p_\mu \psi^\mu - b(b + 2\beta \beta)) \\
- \beta \beta^{2} - 2e \chi \dot{c} + i \Pi_{(c)} \gamma, \tag{13}
\]

where we have used \( H_{(b,b)} = i \Pi_{(c)} \Pi_{(c)} - L_{(b,b)} \) for the generic variable \( \phi_i \) (i.e., \( \phi_i = x_\mu, \psi^\mu, e, \chi, \beta, c, \gamma \)) and corresponding momenta \( \Pi_{(c)}^{(c)} \). In fact, the explicit forms of the canonically conjugate momenta \( \Pi_{(c)}^{(c)} \) are

\[
\Pi_{(c)}^{(c)} = p^\mu, \quad \Pi_{(c)}^{(c)} = - \frac{i}{2} \psi^\mu, \quad \Pi_{(c)} = d, \\
\Pi_{(c)} = -2i \chi, \quad \Pi_{(c)} = i \dot{c}, \quad \Pi_{(c)} = -i \dot{c}, \tag{14}
\]

which have been derived from the Lagrangian (7). The conjugate momenta, from the Lagrangian (8), are also the same except the following:

\[
\Pi_{(c)} = -\bar{b}, \quad \Pi_{(c)} = 2i \dot{\bar{c}}, \quad \Pi_{(c)} = -2i(c \chi + i e \beta). \tag{15}
\]
It should be noted that $\Pi(\gamma) \approx 0$ is the primary constraint on the theory and $\Pi(\gamma) \approx 0$ (calculated from $H(\beta, \bar{\beta})$) leads to $e \chi = \beta \dot{c} - \beta c$ [cf. (10) as well]. In fact, one can add (i.e., $H_T = H(b, \bar{b}) + c \Pi_b$) the primary constraints $\Pi_b$ in the Hamiltonians $H(\beta, \bar{\beta})$ following Dirac’s prescription. However, the equation of motion from $H_T$ imply that the arbitrary coefficient $v = \tilde{\gamma}$ (so that $H_T = H(b, \bar{b}) + \tilde{\gamma} \Pi_b$). We have purposely ignored the presence of $\tilde{\gamma} \Pi(\gamma)$ in the above Hamiltonian mainly to avoid the explicit appearance of the “velocity” dependence.

We can use the following Heisenberg’s equations of motion (with $\hbar = 1$) for the generic variable $\phi_i$ (with generic entity $\phi_i$ corresponding to all the variables and corresponding momenta), namely

$$\dot{\phi}_i = \pm i[\phi_i, H(b, \bar{\beta})], \quad \dot{\bar{\phi}}_i = \frac{d\phi_i}{dt},$$

(16)

where $(\pm)$ signs, in front of the commutator, are chosen depending on the nature of the generic variable $\phi_i$ being (fermionic) bosonic in nature, to derive the dynamics of the theory. In fact, we obtain the following Heisenberg’s equations of motion from the Hamiltonian $H_b$, namely

$$e \chi = \beta \dot{c} - \beta c, \quad \bar{c} \neq -i \gamma, \quad \Pi(\chi) = i p_\mu \psi^\mu + 2 c \gamma, \quad \dot{\bar{c}} = \bar{c} = 0, \quad \Pi(\gamma) = 0, \quad H(\beta) = 2 \beta \dot{\beta} + 2 \beta \bar{\beta}, \quad \dot{\beta} = \dot{\bar{\beta}} = 0, \quad \gamma = \gamma, \quad \bar{b} + \frac{1}{2} p_2 + 2 \gamma \chi = 0, \quad \bar{\Pi}(\beta) = 2 \beta \dot{\beta} + 2 \beta \bar{\beta},$$

(17)

However, this is not the case. We may note that, with the inputs $\beta = \dot{\beta} = 0, \chi = -\gamma$, emerging from the Heisenberg’s equation of motion and using the definition of the canonical momenta in (14) and (15), one can prove that the Euler-Lagrange equations of motion (10) and (11) reduce to the Heisenberg’s equation of motion (17) and (19). Thus, we conclude that there is no inconsistency between the equations of motion that emerge from the Lagrangian and Hamiltonian formulations.

**Time-evolution invariance of CF-type condition: Hamiltonian approach.** - As is obvious from our discussions in the previous sections, the CF-type condition $(b \dot{b} + 2 \beta \dot{\beta} = 0)$ emerges in two steps from the coupled (but equivalent) Lagrangians (7) and (8) as well as from the Hamiltonians (12) and (13). However, this derivation is clumsy, as it is derived from, apparently different looking Lagrangians and Hamiltonians which are equivalent only on a super world-line, defined by the CF-type restriction, in the $D$-dimensional target spacetime manifold. To derive the CF condition in one stroke, we add the Lagrangians in (7) and (8) to redefine the following new Lagrangian:

$$L^{(2)} = \frac{1}{2} (L_b^{(1)} + L_b^{(1)}) = L_0 + L_g + L_{extra},$$

(20)

where $L_0$ is given in eq. (1) and the explicit expressions for $L_g$ and $L_{extra}$, in terms of the variables of the theory, are

$$L_g = -i \dot{\bar{c}} \dot{c} + 2 i (\beta \dot{c} - \beta c) \dot{\chi} + 2 (e \chi - \beta \dot{c} + \beta c) \gamma + \beta^2 \dot{\beta}^2 + (2 i e \chi - e \beta) \dot{\beta} - (2 i c \chi - c \beta) \dot{\bar{\beta}},$$

$$L_{extra} = \left( b - \frac{b}{2} \right) \dot{c} + \left( \frac{b^2 + \bar{b}^2}{2} \right) + (b + \bar{b}) \beta \beta.$$ 

(21)

We note, in passing, that the other linearly independent combination $L^{(3)} = \frac{1}{2} (L_0^{(1)} - L_0^{(1)})$ is not interesting for our discussions because “$L_0$” cancels out along with the other useful as well as dynamically important terms.

Even at this stage, the Lagrangian (20) does not lead to the derivation of CF-type restriction in one step. To corroborate the above statement, we observe that the following equations of motion emerge from (20), namely

$$b + \dot{b} \beta = \frac{\dot{\bar{b}}}{2}, \quad \bar{b} + \dot{\bar{b}} \beta = \frac{\dot{b}}{2},$$

(22)

which, ultimately, lead to two linearly independent relationships between the Nakanishi-Lautrup-type auxiliary variables “$b$” and “$\bar{b}$” as

$$b + \bar{b} = -2 \beta \beta, \quad b - \bar{b} = -\dot{c}.$$ 

(23)

Thus, it is clear that we obtain CF-type condition $b + \bar{b} + 2 \beta \dot{\beta} = 0$ in two steps from the Lagrangian (20), too. To demonstrate the time-evolution invariance of CF-type condition, it is essential that we obtain the above restriction, in one stroke, from the appropriate Lagrangian
and we should obtain the suitable Hamiltonian, corresponding to this Lagrangian, for our further theoretical analysis.

Towards this goal in mind, we redefine the following linearly independent variables from the Nakanishi-Lautrup–type auxiliary variables, namely

\[ B = \frac{b + b}{2}, \quad \bar{B} = \frac{b - b}{2} \implies \frac{b^2 + \bar{b}^2}{2} = B^2 + \bar{B}^2. \] (24)

This leads us to obtain the explicit expression for \( L_{\text{extra}} \) as follows:

\[ L_{\text{extra}} = \dot{B}c + (B^2 + \bar{B}^2) + 2B\beta\bar{\beta} + e(\beta\bar{\beta} - \bar{\beta}\beta). \] (25)

The Euler-Lagrange equation of motion, with respect to \( B \), yields \( B + \beta\bar{\beta} = 0 \) which is nothing but the CF-type restriction \((b + \bar{b} + 2\beta\bar{\beta} = 0)\). The other equations of motion, emerging from the Lagrangian \( L^{(2)} \), are as follows:

\[ \ddot{c} + 2\beta\chi + 2i\beta\gamma + 2\dot{\beta}\bar{\beta} = 0, \quad \dot{\psi}^\mu = \chi p^\mu, \]
\[ \ddot{\bar{c}} + 2\beta\chi + 2i\beta\gamma + 2\dot{\beta}\bar{\beta} = 0, \quad \dot{\bar{\psi}} = i\chi \bar{c}, \]
\[ 2i(\dot{\bar{\beta}} - \dot{\beta}) = i\bar{p} \cdot \psi + 2e\gamma, \quad \dot{\bar{B}} = -\dot{\beta}, \]
\[ \dot{\bar{c}}\bar{\beta} + 2e\dot{\beta} = 2c\dot{\gamma} + 2\dot{\bar{c}}\chi - 2\beta\bar{\beta}^2 - 2B\bar{\beta}, \]
\[ \dot{\bar{B}} + \frac{1}{2}\bar{B}^2 + (\beta\bar{\beta} - \bar{\beta}\beta - 2\chi\gamma) = 0, \]
\[ e\dot{\beta} = 2c\gamma + 2i\dot{\bar{c}}\chi + 2\beta\bar{\beta}^2 - 2B\beta, \]
\[ \ddot{x}^\mu = ep^\mu - i\chi \bar{c}, \quad \bar{\epsilon} = \beta\bar{\epsilon} - \beta c. \] (26)

Thus, the appropriate Lagrangian for our present analysis, in terms of the auxiliary variables \( B \) and \( \bar{B} \), is

\[ L_B = p_\mu \dot{x}^\mu - \frac{e}{2}p^2 + \bar{c} + i\bar{c}p_\mu \bar{\psi}^\mu + i\bar{\psi}^\mu \bar{c} + \bar{B}c \]
\[ (B^2 + \bar{B}^2) + 2B\beta\bar{\beta} - i\bar{c}\bar{\beta} + 2i(\dot{\beta} - \dot{\beta})\bar{c} \]
\[ + 2(e\chi - \beta\bar{\beta} + \beta\gamma) + \beta^2\beta^2 + (2i\dot{\bar{c}}\chi - e\dot{\beta}) \]
\[ -(2i\chi - e\beta)\dot{\bar{\beta}}, \] (27)

which leads to the definition of non-vanishing canonical conjugate momenta as quoted in (14). The vanishing canonical momenta are \( \Pi_{(B)} \approx 0 \) and \( \Pi_{(\gamma)} \approx 0 \) which are the primary constraints\(^3\) on the theory and both of them are canonically conjugate to the auxiliary variables \( B \) and \( \gamma \), respectively.

The canonical Hamiltonian \( (H_B) \), that is derived for the appropriate Lagrangian \( (L_B) \) using Legendre transformation, is as follows:

\[ H_B = i\Pi_{(c)}\Pi_{(\bar{c})} + \frac{e}{2}p^2 - i\bar{\psi}^\mu \bar{\psi}^\mu - \beta^2\beta^2 - 2B\beta\bar{\beta} \]
\[ -(B^2 + \bar{B}^2) - 2e\chi\gamma + i\Pi_{(\chi)}\gamma, \] (28)

where the canonical conjugate momenta, corresponding to the variables in (27), are same as (14) except the following:

\[ \Pi_{(c)} = \bar{B}, \quad \Pi_{(\bar{c})} = 2i\dot{\bar{c}}\chi - e\dot{\beta}, \quad \Pi_{(\beta)} = -2i\epsilon\chi + e\beta. \] (29)

\[^3\text{We discuss about these issues in our fifth section (in somewhat more detail).}\]

It is now straightforward to check that

\[ \dot{\Pi}_{(B)} = -i[\Pi_{(B)}, H_B] = 2(B + \beta\bar{\beta}) \approx 0, \]
\[ [B + \beta\bar{\beta}, H_B] = 0. \] (30)

The above equations establish the derivation of the CF condition from \( \dot{\Pi}_{(B)} \approx 0 \) as the secondary constraint \((i.e., B + \beta\bar{\beta} = 0)\) of the theory. Furthermore, we also obtain the time-evolution invariance of the CF-type condition as the above secondary constraint commutes with the Hamiltonian of the theory. We do not discuss here, in detail, the constraint analysis and classifications of the constraints because this is not our main objective here.

**Conclusions.** – In our present investigation, we have derived the appropriate Lagrangian that yields the CF-type restriction as an Euler-Lagrange equation of motion by redefining the auxiliary variables \((e.g., B \) and \( \bar{B} \)) in terms of the Nakanishi-Lautrup auxiliary variables \((e.g., b \) and \( \bar{b} \)). As it turns out, this Lagrangian is endowed with three primary constraints \( \Pi_{(\gamma)} \approx 0, \Pi_{(B)} \approx 0 \) and \( \Pi_{(\beta)} \approx 0 \) as there are no “velocity” terms corresponding to the variables \( \gamma, B \) and \( \bar{B} \). It is essential, however, to lay emphasis on the fact that variable \( \bar{B} \) is somewhat special because it is the canonical conjugate momentum for the variable \( e \) (cf. eqs. (29)). Thus, effectively, there are only two primary constraints on the theory.

In the Lagrangian formulation, the time-evolution invariance of the CF-type restriction does not appear in a straightforward manner even we use the CF-type restriction in proving the absolute anticommutativity of the (anti-)BRST symmetries of the theory. We have, in our present investigation, derived the appropriate Hamiltonian for the theory and demonstrated that the CF-type condition is nothing other than the secondary constraint on the theory that emerges when we demand the time-evolution invariance \( (\Pi_{(B)} \approx 0) \) of the primary constraint \( \Pi_{(B)} \approx 0 \). This secondary constraint, as it turns out, commutes with the Hamiltonian of the theory thereby proving its time-evolution invariance. This justifies the imposition of CF-type condition on the theory as it remains the same during the time-evolution of the whole system.

In the above context, it is important to point out that the time-evolution invariance of \( \Pi_{(\gamma)} \) \((\Pi_{(\gamma)} \approx 0)\) produces \( e\chi = \beta\bar{\epsilon} - \beta c \) which is one of the equations of motion that emerges out from the Lagrangian (27). However, in the language of constraint analysis, the equation \( e\chi = \beta\bar{\epsilon} - \beta c \) is also a secondary constraint. We do not perform here a detailed constraint analysis and their classifications.

We have established the time-evolution invariance of the CF-type restriction in the context of 4D Abelian 2-form gauge theory [9] and achieved the same goal for our present SUSY toy model of the 1D free spinning relativistic particle. It would be nice endeavor to carry out these kinds of exercise in the context of other physically interesting models so that we could lay our method of analysis on the firmer footings. We are intensively involved in this
endeavor at the moment and our result would be reported later in our future publication [10] .

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