AN EFFECTIVE BOUND ON GENERALIZED DIOPHANTINE \(m\)-TUPLES

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Abstract. For non-zero integers \(n\) and \(k \geq 2\), a generalized Diophantine \(m\)-tuple with property \(D_k(n)\) is a set of \(m\) positive integers \(S = \{a_1, a_2, \ldots, a_m\}\) such that \(a_i a_j + n\) is a \(k\)-th power for \(1 \leq i < j \leq m\). Define \(M_k(n) := \sup\{|S| : S \text{ has property } D_k(n)\}\). In a recent work, the second author, S. Kim and M. R. Murty proved that \(M_k(n)\) is \(O(\log n)\), for a fixed \(k\), as we vary \(n\). In this paper, we obtain effective upper bounds on \(M_k(n)\). In particular, we show that for \(k \geq 2\), \(M_k(n) \leq 3 \phi(k) \log n\), if \(n\) is sufficiently larger than \(k\).

1. Introduction

Given a non-zero integer \(n\), we say a set of natural numbers \(S = \{a_1, a_2, \ldots, a_m\}\) is a Diophantine \(m\)-tuple with property \(D(n)\) if \(a_i a_j + n\) is a perfect square for all \(1 \leq i < j \leq m\). Diophantus first studied such sets of numbers and found the quadruple \(\{1, 33, 68, 105\}\) with property \(D(256)\). The first \(D(1)\)-quadruple \(\{1, 3, 8, 120\}\) was discovered by Fermat, and this was later generalized by Euler, who found the following infinite family of quadruples with property \(D(1)\), namely
\[
\{a, b, a + b + 2r, 4r(r + a)(r + b)\},
\]
where \(ab + 1 = r^2\). In fact, it is known that any \(D(1)\)-triple can be extended to a Diophantine quadruple [1]. In 1969, using Baker’s theory of linear forms in logarithm of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [2] proved that Fermat’s example is the only extension of \(\{1, 3, 8\}\) with property \(D(1)\). In 2004, Dujella [10], using similar methods, proved that there are no \(D(1)\)-sextuples and there are only finitely many \(D(1)\)-quintuples, if any. The conjecture on the non-existence of \(D(1)\)-quintuples was finally settled in 2019 by He, Togbé, and Ziegler in [15].

However, in general, there are \(D(n)\)-quintuples for \(n \neq 1\). For instance,
\[
\{1, 33, 105, 320, 18240\} \quad \text{and} \quad \{5, 21, 64, 285, 6720\}
\]
are Diophantine quintuples satisfying property \(D(256)\). Also, we note that there are known examples of \(D(n)\)-sextuples, but no \(D(n)\)-septuple is known. So, it is natural to study the size of the largest tuple with property \(D(n)\). Define
\[
M_n := \sup\{|S| : S \text{ satisfies property } D(n)\}.
\]
In 2004, Dujella [9] showed that
\[
M_n \leq C \log |n|,
\]
where $C$ is an absolute constant. He also showed that for $n > 10^{100}$, one can choose $C = 8.37$. This constant was improved by Becker and Murty [3], who showed that for any $n$, 

$$M_n \leq 2.6071 \log |n| + O \left( \frac{\log |n|}{(\log \log |n|)^2} \right).$$

(1)

Our goal is to study this problem when squares are replaced by higher powers.

**Definition 1** (generalized Diophantine $m$-tuples). Fix a natural number $k \geq 2$. A set of natural numbers $S = \{a_1, a_2, \ldots, a_m\}$ is said to satisfy property $D_k(n)$ if $a_i a_j + n$ is a $k$-th power for all $1 \leq i < j \leq m$.

We analogously define the following quantity for each non-zero integer $n$,

$$M_k(n) := \sup \{|S| : S \text{ satisfies property } D_k(n)\}.$$ 

For $k \geq 3$ and $m \geq 3$, we can apply the celebrated theorem of Faltings [12] to deduce that a superelliptic curve of the form

$$y^k = f(x) = (a_1 x + n)(a_2 x + n)(a_3 x + n)(a_4 x + n) \cdots (a_m x + n)$$

has only finitely many rational points and a fortiori, finitely many integral points. Therefore, a set $S$ satisfying property $D_k(n)$ must be finite. When $n = 1$, Bugeaud and Dujella [6] showed that

$$M_3(1) \leq 7, \quad M_4(1) \leq 5, \quad M_5(1) \leq 4 \text{ for } 5 \leq k \leq 176, \quad \text{and } M_k(1) \leq 3 \text{ for } k \geq 177.$$ 

In other words, the size of $D_k(1)$-tuples is bounded by 3 for large enough $k$. In the general case, for any $n \neq 0$ and $k \geq 3$, Bérczes, Dujella, Hajdu and Luca [5] obtained upper bounds for $M_k(n)$. In particular, they showed that for $k \geq 5$

$$M_k(n) \leq 2|n|^5 + 3.$$ 

In [8], the second author, S. Kim and M. R. Murty improved the above bounds on $M_k(n)$ for large $n$ and a fixed $k$. Define

$$M_k(n; L) := \sup \{|S \cap [n^L, \infty)| : S \text{ satisfies property } D_k(n)\}.$$ 

They proved that for $k \geq 3$, as $n \to \infty$,

$$M_k(n, L) \ll_{k, L} 1, \text{ for } L \geq 3 \quad \text{and} \quad M_k(n) \ll_{k} \log n.$$ 

(2)

The purpose of this paper is to explicitly obtain the implied constants in (2). In [8], the bounds for $M_k(n)$ were proven under the further assumption that $n > 0$. However, this is not necessary, as was remarked in [8], but an argument was not provided. We first prove the bounds (2) for all non-zero integers $n$.

**Theorem 1.1.** Let $k \geq 3$ be an integer. Then the following holds as $|n| \to \infty$:

1. For $L \geq 3$,

$$M_k(n, L) \ll 1,$$

where the implied constant depends on $k$ and $L$, but is independent of $n$.

2. Moreover,

$$M_k(n) \ll \log |n|$$

where the implied constant depends on $k$.

We now state our main theorem, which is the effective version of Theorem 1.1.
Theorem 1.2. Let \( k \) be a positive integer \( \geq 3 \). Then the following holds.

(a) For \( L \geq 3 \),
\[
M_k(n, L) \leq 2^{28} \log(2k) \log(2 \log(2k)) + 14. 
\]

(b) Suppose \( n \) and \( k \) vary such that as \( |n| \to \infty \) and
\[
k = o(\log \log |n|).
\]

Then
\[
M_k(n) \leq 3 \phi(k) \log |n| + O \left( \frac{(\phi(k))^2 \log |n|}{\log \log |n|} \right),
\]
where \( \phi(n) \) denotes the Euler totient function.

Remarks.

(1) It is possible to replace 14 on the right hand side of \((3)\) with a smaller positive integer for large values of \( k \).

(2) For a fixed \( k > 2 \), from Theorem 1.2(b), we obtain that as \( |n| \to \infty \)
\[
M_k(n) \leq 3 \phi(k) \log |n| + O \left( \frac{\log |n|}{\log \log |n|} \right).
\]

This upper bound is very close to the case \( k = 2 \), where the best known upper bound due to Becker and Ram Murty is given by \((1)\).

2. Preliminaries

In this section, we recall and develop the necessary tools to prove our main theorems.

2.1. Gallagher’s larger sieve. In 1971, Gallagher [13] discovered an elementary sieve inequality which he called the larger sieve. We refer the reader to [7] for the general discussion but record the result in a form applicable to our context.

Theorem 2.1. Let \( N \) be a natural number and \( S \) a subset of \( \{1, 2, \ldots, N\} \). Let \( \mathcal{P} \) be a set of primes. For each prime \( p \in \mathcal{P} \), let \( S_p = S \pmod{p} \). For any \( 1 < Q \leq N \), we have
\[
|S| \leq \sum_{\substack{p \leq Q, p \in \mathcal{P} \\backslash \{p\}}} \log p - \log N \sum_{p \leq Q, p \in \mathcal{P}} \frac{\log p}{|S_p|} - \log N, \tag{4}
\]
where the summations are over primes \( p \leq Q, p \in \mathcal{P} \) and the inequality holds provided the denominator is positive.
2.2. A quantitative Roth’s theorem. Quantitative results related to counting exceptions in Roth’s celebrated theorem on Diophantine approximations were established by a variety of authors. We will use the following result due to Evertse \cite{Evertse1985}. For an algebraic number $\xi$ of degree $r$, we define the (absolute) height by

$$H(\xi) := \left( a \prod_{i=1}^{r} \max(1, |\xi(i)|) \right)^{1/r},$$

where $\xi^{(i)}$ for $1 \leq i \leq r$ are the conjugates (over $\mathbb{Q}$) and $a$ is the positive integer such that

$$a \prod_{i=1}^{r} (x - \xi^{(i)})$$

has rational integer coefficients with gcd 1.

**Theorem 2.2.** Let $\alpha$ be a real algebraic number of degree $r$ over $\mathbb{Q}$, and $0 < \kappa \leq 1$. The number of rational numbers $p/q$ satisfying $\max(|p|, |q|) \geq \max(H(\alpha), 2)$,

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{\max(|p|, |q|)^{2+\kappa}}$$

is at most

$$2^{25}\kappa^{-3} \log(2r) \log(\kappa^{-1} \log(2r)).$$

2.3. Vinogradov’s theorem. The following bound on character sums was proved by Vinogradov (see \cite{Vinogradov1937}).

**Lemma 2.3.** Let $\chi \pmod{q}$ be a non-trivial Dirichlet character and $n$ be an integer such that $(n, q) = 1$. If $A \subseteq (\mathbb{Z}/q\mathbb{Z})^*$ and $B \subseteq (\mathbb{Z}/q\mathbb{Z})^* \cup \{0\}$, then

$$\sum_{a \in A} \sum_{b \in B} \chi(ab + n) \leq \sqrt{q|A||B|}.$$

The original method of Vinogradov does not produce the bound above and instead gives the right hand side as $\sqrt{2q|A||B|}$. However, the above bound holds and a short proof of this can be found in \cite[Proposition 2.5]{Bennet2015}.

2.4. Prime bounds in arithmetic progression. Let $Q, k, a$ be positive integers with $(a, k) = 1$. Denote by $\theta(Q; k, a)$ the sum of the logarithms of the primes $p \equiv a \pmod{k}$ with $p \leq Q$, i.e.,

$$\theta(Q; k, a) := \sum_{\substack{p \equiv a \pmod{k} \\ p \text{ prime} \leq Q}} \log p.$$

We will need the following bound on $\theta(Q; k, a)$ obtained by Bennet, Martin, O’Bryant, Rechnitzer in \cite[Theorem 1.2]{Bennet2015}:

**Theorem 2.4.** For $k \geq 3$ and $(a, k) = 1$,

$$\left| \theta(Q; k, a) - \frac{Q}{\phi(k)} \right| < \frac{1}{160} \frac{Q}{\log Q}$$

for all $Q \geq Q_0(k)$ where

$$Q_0(k) = \begin{cases} 8 \cdot 10^9 & \text{if } 3 \leq k \leq 10^5 \\ \exp(0.03 \sqrt{k} \log^3 k) & \text{if } k > 10^5. \end{cases}$$
2.5. **Gap principle.** The next two lemmas are variations of a gap principle of Gyarmathi [14]. The following lemma was proved in [8].

**Lemma 2.5** ([8], Lemma 2.4). Let \( k \geq 2 \). Suppose that \( a, b, c, d \) are positive integers such that \( a < b \) and \( c < d \). Suppose further that
\[
ac + n, \quad bc + n, \quad ad + n, \quad bd + n
\]
are perfect \( k \)-th powers. Then,
\[
bd \geq k^k n^{-k} (ac)^{k-1}.
\]

An immediate Corollary of this lemma shows that “large” elements of any set with property \( D_k(n) \) have “super-exponential growth.”

**Corollary 2.1** ([8] Corollary 2). Let \( k \geq 3 \) and \( m \geq 5 \). Suppose that \( n^3 \leq a_1 < a_2 < \ldots < a_m \) and the set \( \{a_1, a_2, \ldots, a_m\} \) has property \( D_k(n) \). Then \( a_{2+3j} \geq a_2^{(k-1)^j} \) provided \( 1 \leq j \leq (m-2)/3 \).

A modification of the proof of the above Lemma 2.5 yields a gap principle for negative values of \( n \).

**Lemma 2.6.** For \( n > 0 \) and natural numbers \( a, b, c, d \) such that \( n^3 \leq a < b < c < d \), we have
\[
(ac - n)(bd - n) \geq \frac{abcd}{2}.
\]

**Proof.** Since
\[
(ac - n)(bd - n) = abcd - n(ac + bd) + n^2,
\]
it is enough to prove that
\[
\frac{abcd}{2} \geq n(ac + bd) - n^2.
\]
Also, since \( a \geq n^3 \) and \( c > n^3 \), we get for all cases other than \( n = 1, a = 1, b = 2, c = 3 \),
\[
abcd \geq 4nbd
\]
\[
\geq 2nbd + 2nac
\]
\[
\geq 2nbd + 2nac - 2n^2
\]
where the first inequality is obvious as \( a \geq n^3 \) and \( c \geq n^3 + 2 \). Thus, we get the desired result. For the case \( n = 1, a = 1, b = 2, c = 3, \) clearly
\[
2n(ac + bd) - 2n^2 = 4 + 4d < 6d = abcd
\]
as \( d > c \).

We are now ready to prove the following analog of Lemma 2.5.

**Lemma 2.7.** Let \( n > 0 \) and \( k \geq 2 \). Suppose that \( a, b, c, d \) are positive integers such that \( n^3 \leq a < b < c < d \). Suppose further that \( ac - n, bc - n, ad - n, bd - n \) are perfect \( k \)-th powers. Then,
\[
bd \geq k^k 2^{-k} n^{-k} (ac)^{k-1}.
\]

**Proof.** Since \((b - a)(d - c) > 0\), we have
\[
bd + ac > ad + bc,
\]
and it is easily seen that
\[
(ad - n)(bc - n) > (ac - n)(bd - n).
\]
As \((ac - n)(bd - n)\) and \((ad - n)(bc - n)\) are both perfect \(k\)-th powers, we have
\[
(ad - n)(bc - n) \geq [(ac - n)(bd - n)]^{1/k} + 1]^k
\]
\[
\geq (ac - n)(bd - n) + k((ac - n)(bd - n))^{k-1/k}
\]
\[
\geq (ac - n)(bd - n) + k \left( \frac{abcd}{2} \right)^{k-1/k}
\]
where the last inequality follows from Lemma 2.6. Thus,
\[
-n(ad + bc) \geq -n(ac + bd) + k \left( \frac{abcd}{2} \right)^{k-1/k}.
\]
Since \(bd > ad + bc - ac > 0\), we have \(bd + ac - ad - bc < bd\) and hence we obtain
\[
nbd > k \left( \frac{abcd}{2} \right)^{k-1/k}.
\]
Therefore,
\[
bd \geq k^{2^{1-k}2^{-k}(ac)^{k-1}} \geq k^{2^{-k}n^{-k}(ac)^{k-1}},
\]
which proves the lemma. \(\square\)

This enables us to prove super-exponential growth for large elements of a set with \(D_k(n)\), when \(n < 0\).

**Corollary 2.2.** Let \(k \geq 3\). If \(n^3 \leq a < b < c < d < e\) are natural numbers such that the set \(\{a, b, c, d, e\}\) has property \(D_k(-n)\), then \(e \geq b^{k-1}\).

**Proof.** From Lemma 2.7
\[
ce \geq k^{2^{1-k}2^{-k}(bd)^{k-1}} \geq k^{2^{-k}n^{-k}(bc)^{k-1}}.
\]
Therefore,
\[
e \geq b^{k-1}e^{k-2}2^{-k}n^{2k-6}2^{-k} \geq b^{k-1}n^{2k-6}2^{-k} \geq b^{k-1}.
\]
\(\square\)

Using induction on the previous corollary, we deduce

**Corollary 2.3.** Let \(k \geq 3\) and \(m \geq 5\). Suppose that \(n^3 \leq a_1 < a_2 < \ldots < a_m\) and the set \(\{a_1, a_2, \ldots, a_m\}\) has property \(D_k(-n)\). Then \(a_{2+3j} \geq a_2^{(k-1)^j}\) provided \(1 \leq j \leq (m-2)/3\).

3. **Proof of the main theorems**

3.1. **Proof of Theorem 1.1.** We first prove Theorem 1.1 as the proof follows a similar method as in [8].

Let \(n\) be a positive integer and \(m = M_k(-n)\) and \(S = \{a_1, a_2, a_3, \ldots, a_m\}\) be a generalized \(m\)-tuple with the property \(D_k(-n)\). Suppose \(n^L < a_1 < a_2 < \ldots < a_m\) for some \(L \geq 3\). Consider the system of equations
\[
a_1x - n = u^k
\]
\[
a_2x - n = v^k.
\]
(6)
Clearly, \( x = a_i \) for \( i \geq 3 \) are solutions to this system. Also,

\[
\left| a_2 u^k - a_1 v^k \right| = n(a_2 - a_1).
\]

Let \( \alpha := (a_1/a_2)^{1/k} \) and \( \zeta_k := e^{2\pi i/k} \). Then, we have the following two lemmas analogous to the ones proved in \[8\].

**Lemma 3.1.** Let \( k \geq 3 \) be odd. Suppose \( u, v \) satisfy the system of equations (6). Let

\[
c(k) := \prod_{j=1}^{(k-1)/2} \left( \frac{\sin \frac{2\pi j}{k}}{k} \right)^2.
\]

Then, for \( n > 2^{1/(L-1)} c(k)^{-1/(L-1)} \),

\[
\left| \frac{u}{v} - \alpha \right| \leq \frac{a_2}{2v^k}.
\]

We omit the proof of this lemma here as it is identical to the proof of \[8, Lemma 3.1\].

**Lemma 3.2.** Let \((u_i, v_i)\) denote distinct pairs that satisfy the system of equations (6) with \( v_{i+1} > v_i \). For \( n > 2^{1/(L-1)} c(k)^{-1/(L-1)} \) and \( i \geq 14 \),

\[
\left| \frac{u}{v} - \alpha \right| < \frac{1}{v_i^{k-1/2}},
\]

and \( v_i > a_2^4 \).

**Proof.** From Lemma 3.1 we have \( \left| \frac{u}{v_i} - \alpha \right| < \frac{a_2}{2v_i^k} \). Thus, we need to show \( a_2 < 2v_i^{1/2} \) for \( i > 14 \). Since \( v_i^k = a_2 a_i - n \), we have \( v_i \geq a_i^{1/k} \). By Corollary 2.3,

\[
a_{2+3j} \geq a_2^{(k-1)^j}
\]

so that \( v_{2+3j} \geq a_2^{(k-1)^j} \). We choose a positive integer \( j_0 \) such that \((k-1)^{j_0} > 4k \). Since \( k \geq 3 \), \( j_0 = 4 \) satisfies the condition. As \( 2 + 3j_0 = 14 \), we have \( v_i \geq v_{14} > a_2^4 \) for all \( i \geq 14 \). This completes the proof. \( \square \)

For larger values of \( k \), the number 14 in the above Lemma can be improved to \( 2 + 3j_0 \), where \( j_0 \) satisfies the condition \((k-1)^{j_0} > 4k \).

Now, assume that \((u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m)\) satisfy the system of equations (6) with

\[
v_i > \max(a_2^{1/k}, 2) \geq \max(H(\alpha), 2).
\]

By Lemma 3.2, for \( 14 \leq i \leq m \),

\[
\left| \frac{u}{v} - \alpha \right| < \frac{1}{v_i^{k-1/2}} \leq \frac{1}{v_i^{2.5}},
\]

as \( k \geq 3 \). Since \( \alpha = (a_1/a_2)^{1/k} < 1 \) and \( \max(u_i, v_i) = v_i \), from Theorem 2.2 the number of such \( i \)'s is \( O(\log k \log \log k) \). This proves Theorem 1.1.
3.2. Proof of Theorem 1.2. Let \( m = M_k(n) \) and \( S = \{a_1, a_2, a_3, \ldots, a_m\} \) be a generalized \( m \)-tuple with the property \( D_k(n) \). Suppose \( |n|^L < a_1 < a_2 < \ldots < a_m \) for some \( L \geq 3 \). We consider the system of equations

\[
\begin{align*}
  a_1 x + n &= u^k, \\
  a_2 x + n &= v^k.
\end{align*}
\]

(7)

As before, \( x = a_i \) for \( i \geq 3 \) are solutions to this system. The statements of Lemma 3.1 and 3.2 hold for all non-zero integers \( n \). For \( n > 0 \) this was proved in [8].

Proof of Theorem 1.2 (a). Let \((u_1, v_1), \ldots, (u_m, v_m)\) satisfy the system of equations (7) with \( v_i > \max(a_2^{1/k}, 2) \geq \max(H(\alpha), 2) \). By Lemma 3.2, we get for \( 14 \leq i \leq m \),

\[
\left| \frac{u_i}{v_i} - \alpha \right| \leq \frac{1}{v_i^{k-1/2}} \leq \frac{1}{v_i^2}.
\]

as \( k \geq 3 \). Since \( \alpha = (a_1/a_2)^{1/k} < 1 \) and \( \max(u_i, v_i) = v_i \), applying Theorem 2.2 with \( \kappa = 0.5 \), we get that the number of \( i \)'s satisfying the above inequality is

\[
2^{25}(0.5)^{-3} \log(2k) \log((0.5)^{-1} \log(2k))
\]

which is

\[
2^{28} \log(2k) \log(2 \log(2k)).
\]

So, the total number of solutions is at most

\[
2^{28} \log(2k) \log(2 \log(2k)) + 14
\]

for \( k \geq 3 \).

\[\square\]

Proof of Theorem 1.2 (b). Let \( S = \{a_1, a_2, \ldots, a_m\} \) be a generalized Diophantine \( m \)-tuple with property \( D_k(n) \) such that each \( a_i \leq |n|^3 \). We shall apply Gallagher’s larger sieve with primes \( p \leq Q \) satisfying \( p \equiv 1 \mod k \). Let \( \mathcal{P} \) be the set of all primes \( p \equiv 1 \mod k \). For all such primes \( p \in \mathcal{P} \), there exists a Dirichlet character \( \chi \mod p \) of order \( k \).

Denote by \( S_p \) the image of \( S \mod p \) for a given prime \( p \). For \( p \in \mathcal{P} \), applying Lemma 2.3 with \( \mathcal{A} = \mathcal{B} = S_p \) and \( \chi \mod p \) a character of order \( k \), we obtain

\[
|S_p| (|S_p| - 1) \leq \sum_{a \in S_p - \{0\}} \sum_{b \in S_p} \chi(ab + n) + |S_p| \leq \sqrt{p} |S_p| + |S_p|.
\]

Thus,

\[
|S_p| \leq \sqrt{p} + 2.
\]

Take \( N = |n|^3 \). Since \( a_i \leq |n|^3 \), applying Theorem 2.1, we obtain

\[
|S| \leq \sum_{p \in \mathcal{P}, p \leq Q} \log p - \log N
\]

\[
\sum_{p \in \mathcal{P}, p \leq Q} \frac{\log p}{|S_p|} - \log N.
\]

By Theorem 2.4

\[
\sum_{p \leq Q} \log p = \frac{Q}{\phi(k)} + O \left( \frac{Q}{\log Q} \right)
\]
when \( Q > Q_0(k) \). Using partial summation,

\[
\sum_{\substack{p \leq Q \\ p \equiv 1 \mod k}} \frac{\log p}{\sqrt{p} + 2} = \frac{2\sqrt{Q}}{\phi(k)} + O\left(\frac{\sqrt{Q}}{\log Q}\right).
\]

Thus,

\[
|S| \leq \frac{\frac{Q}{\phi(k)} + O\left(\frac{Q}{\log Q}\right) - \log N}{2\sqrt{Q} \phi(k) + O\left(\frac{\sqrt{Q}}{\log Q}\right) - \log N}.
\]

Choose \( Q = (\phi(k) \log N)^2 \). Note that the condition \( Q > Q_0(k) \) is same as

\[
\log N > \exp\left(0.015 \sqrt{k (\log k)^3}\right).
\]

Since \( k = o(\log \log |n|) \), (9) holds for \( N \) large enough. Now, for both the numerator and the denominator, multiply by \( \phi(k) \) and divide by \( \log N \), to get

\[
|S| \leq \frac{\phi(k) \log N - 1 + O\left(\frac{Q}{\log N \log Q}\right)}{1 + O\left(\frac{\sqrt{Q}}{\log N \log Q}\right)}.
\]

Because \( k = o(\log \log N) \), it is easy to see that

\[
\frac{\sqrt{Q}}{\log N \log Q} = o(1).
\]

Hence, the denominator in (10) is \( 1 + o(1) \) and we obtain

\[
|S| \leq \phi(k) \log N + O\left(\frac{(\phi(k))^2 \log N}{\log \log N}\right).
\]

Since \( N = |n|^3 \),

\[
M_k(n) \leq 3 \phi(k) \log |n| + O\left(\frac{(\phi(k))^2 \log |n|}{\log \log |n|}\right)
\]

as required. \(\square\)

**Acknowledgements**

We thank Prof. Alain Togbé for his question regarding effective versions of the bounds in [8] during the second author’s talk in the Leuca 2022 conference, which motivated this paper.

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