Derivatives of normal functions and $\omega$-models

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Abstract In this note the well-ordering principle for the derivative $g'$ of normal functions $g$ on ordinals is shown to be equivalent to the existence of arbitrarily large countable coded $\omega$-models of the well-ordering principle for the function $g$.

Keywords Derivative of normal functions · $\omega$-models · Well-ordering principle

Mathematics Subject Classification 03F15 · 03F35

1 Well-ordering principles

In this note we are concerned with a proof-theoretic strength of a $\Pi^1_2$-statement $\text{WOP}(g)$ saying that ‘for any well-ordering $X$, $g(X)$ is a well-ordering’, where $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a computable functional on sets $X$ of natural numbers. $\langle n, m \rangle$ denotes an elementary recursive pairing function on $\mathbb{N}$.

Definition 1 $X \subset \mathbb{N}$ defines a binary relation $<_X := \{(n, m) : \langle n, m \rangle \in X\}$.

\[
\begin{align*}
\text{LO}(X) :& \iff [\forall n (n \not<_X n) \land \forall n, m, k (n <_X m <_X k \to n <_X k) \\
& \land \forall n, m (n <_X m \lor n = m \lor m <_X n)] \\
\text{Prg}[<_X, Y] :& \iff \forall m (\forall n <_X m Y(n) \to Y(m)) \\
\text{TI}(<_X, Y) :& \iff \text{Prg}[<_X, Y] \to \forall n Y(n) \\
\text{WO}(X) :& \iff \text{LO}(X) \land \forall Y \text{TI}(<_X, Y)
\end{align*}
\]

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For a functional $g : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$,

$$\text{WOP}(g) :\Leftrightarrow \forall X \, (\text{WO}(X) \rightarrow \text{WO}(g(X)))$$

The theorem due to J.-Y. Girard is a base for further results on the strengths of the well-ordering principles WOP($g$). For second order arithmetics RCA$_0$, ACA$_0$, etc. see [10]. For a set $X \subset \mathbb{N}$, $\omega^X$ denotes an ordering on $\mathbb{N}$ canonically defined such that its order type is $\omega^\alpha$ when $\prec_X$ is a well ordering of type $\alpha$.

**Theorem 1** (Girard [4], also cf. [5])

*Over RCA$_0$, ACA$_0$ is equivalent to WOP($\lambda X.\omega^X$).*

The following theorem summarizes some known results on the strengths of WOP($g$) for $g$ growing faster than the exponential function. ACA$_0^+$ is an extension of ACA$_0$ by the axiom of the existence of the $\omega$-th jump of a given set. $\varphi_\alpha \beta = \varphi_\alpha(\beta)$ denotes the binary Veblen function starting with $\omega^\alpha$. For example $\varphi_1(\beta) = \varepsilon_\beta$ with the $\beta$-th epsilon number $\varepsilon_\beta$.

**Theorem 2**

1. (Marcone and Montalbán [6])

*Over RCA$_0$, ACA$_0^+$ is equivalent to WOP($\lambda X.\varepsilon_X$).*

2. (H. Friedman)

*Over RCA$_0$, ATR$_0$ is equivalent to WOP($\lambda X.\varphi_0 X$).*

Theorem 2 is proved in [6] computability theoretically. M. Rathjen noticed that the principle WOP($g$) is tied to the existence of countable coded $\omega$-models.

**Definition 2**

A countable coded $\omega$-model of a second-order arithmetic $T$ is a set $Q \subset \mathbb{N}$ such that $M(Q) \models T$, where $M(Q) = \langle \mathbb{N}, \{(Q)_n\}_{n \in \mathbb{N}}, +, \cdot, 0, 1, \prec \rangle$ with $(Q)_n = \{m \in \mathbb{N} : \langle n, m \rangle \in Q\}$.

Let $X \in_\omega Y :\Leftrightarrow (\exists n[X = (Y)_n])$ and $X =_\omega Y :\Leftrightarrow (\forall Z(Z \in_\omega X \leftrightarrow Z \in_\omega Y))$.

It is not hard to see that over ACA$_0$, ACA$_0^+$ is equivalent to the fact that there exists an arbitrarily large countable coded $\omega$-model of ACA$_0$, cf. [1] and Lemma 1 below. The fact means that there is a countable coded $\omega$-model $Q$ of ACA$_0$ containing a given set $X$, i.e., $X = (Q)_0$. From this characterization, Afshari and Rathjen [1] gives a purely proof-theoretic proof of Theorem 2.1. Their proof is based on Schütte's method of complete proof search in $\omega$-logic. The proof is extended by Rathjen and Weiermann [8] to give an alternative proof of Theorem 2.2. Furthermore Rathjen [7] lifts Theorem 2.1 up to $\Gamma$-function and ATR$_0$ as follows.

**Definition 3**

A continuous and strictly increasing function on ordinals is said to be a normal function.

For a normal function $f$, its derivative $f'$ is a normal function enumerating the fixed points of the function $f$.

The $(\alpha + 1)$-th branch $\varphi_{\alpha+1} : \beta \mapsto \varphi_{\alpha+1}(\beta)$ of the Veblen function is the derivative $(\varphi_\alpha)'$ of the previous one $\varphi_\alpha$, and for limit $\lambda$, $\varphi_\lambda$ enumerates the common fixed points of the functions $\varphi_\alpha (\alpha < \lambda)$. The $\Gamma$-function $\alpha \mapsto \Gamma_\alpha$ is the derivative of the normal function $\alpha \mapsto \varphi_0$.
Theorem 3 (Rathjen [7])
Over RCA₀, WOP(\(\lambda X.\Gamma_X\)) is equivalent to the existence of arbitrarily large countable coded \(\omega\)-models of ATR₀.

In view of Theorems 1 and 2.1 is equivalently stated: over RCA₀, WOP(\(\lambda X.\varepsilon_X\)) is equivalent to the existence of arbitrarily large countable coded \(\omega\)-models of WOP(\(\lambda X.\omega^X\)). Moreover relying on Theorem 2.2, Theorem 3 states that over RCA₀, WOP(\(\lambda X.\Gamma_X\)) is equivalent to the existence of arbitrarily large countable coded \(\omega\)-models of WOP(\(\lambda X.\varphi_X0\)). Here is a striking similarity: \(\lambda\alpha.\varepsilon\alpha\) is the derivative of the function \(\lambda\alpha.\omega\alpha\), and \(\lambda\alpha.\Gamma\alpha\) is the one of \(\lambda\alpha.\varphi\alpha0\).

Definition 4
\(T^+\) denotes the extension of a second-order arithmetic \(T\) by the axiom stating that

\[ '\text{there exists an arbitrarily large countable coded } \omega\text{-model of } T' \quad (1) \]

Note that when \(T\) is axiomatized by a \(\Pi^1_2\)-sentence over RCA₀, \(T^+\) is axiomatized by the \(\Pi^1_2\)-sentence (1) over RCA₀.

These results suggest us a general fact:

\[ \text{WOP}(g^') \text{ is equivalent to } \text{WOP}(g)^+ \text{ over ACA}_0. \quad (2) \]

In this note we confirm it for a variety of normal functions \(g\). Theorem 2.1 follows from (2) for \(g(\alpha) = \omega\alpha\), and Theorem 3 from Theorem 2.2 and (2) for \(g(\alpha) = \varphi_\alpha(0)\).

We assume that the normal function \(g\) enjoys the following conditions. The computability of the functional \(g\) and the linearity of \(g(X)\) for linear orderings \(X\) are assumed. Moreover \(g(X)\) is assumed to be a term structure over constants \(g(c) (c \in X)\) and some function symbols \(f\). We need the term structures \(G(X) = \langle g(X), <_g(X); f, \ldots \rangle\) to enjoy the following two, cf. Definition 5 of extendible structures: First if \((X, <_X)\) is a substructure of \((Y, <_Y)\), then \(G(X)\) is a substructure of \(G(Y)\). Second \(\langle g(c) : c \in X \rangle\) is an indiscernible sequence for \(G(X)\). These two postulates allow us to extend an order preserving map \(f\) between linear orderings \(X, Y\) to an order preserving map \(F\) between \(g(X)\) and \(g(Y)\), cf. Proposition 1:

\[ \begin{array}{c}
g(X) \xrightarrow{f} g(Y) \\
\downarrow \quad \downarrow \\
X \xrightarrow{f} Y \\
\end{array} \]

Moreover we assume that \((g^'(X); 0, +, \lambda\alpha.\omega^\alpha)\) is a substructure of the term structure \(G'(X)\) for the derivative \(g^'\), cf. Definition 6 of exponential structures. Then (2) is shown in Theorem 4.

Next (2) suggests us a result on common fixed points. Let \(\varphi[g]_\alpha(\beta)\) denote the \(\alpha\)-th Veblen function starting with \(\varphi[g]_0(\beta) = g(\beta)\). For \(\alpha > 0\)

\[ \text{WOP}(\varphi[g]_\alpha) \text{ is equivalent to } (\forall \beta < \alpha \text{WOP}(\varphi[g]_\beta))^+ \text{ over ACA}_0 + \text{LO}(\alpha). \quad (3) \]
Under a mild condition on the Veblen hierarchy \( \{ \varphi[g]_\alpha \}_\alpha \), we confirm (3) in Theorem 6.

Next consider WOP(\( \varphi_\alpha \)) with \( \varphi_\alpha = \varphi[g]_\alpha \) for the most familiar \( g(\beta) = \omega^\beta \).

Let TJ(\( X \)) denote the Turing jump of sets \( X \). Hier\( _\alpha(X, Y) \) designates that \( \{(Y)_\beta\}_{\beta<\alpha} \) is the Turing jump hierarchy starting with \( X = (Y)_0 \) for the least element 0 in the ordering \( <:\) for any non-zero \( \beta < \alpha \), if \( \beta = \gamma + 1 \), then \( (Y)_\beta = TJ((Y)_\gamma) \), and when \( \beta \) is limit, \( (Y)_\beta = \sum_{\gamma<\beta}(Y)_\gamma \).

The following Lemma 1 is shown in Theorem 1.9 of [6], and it yields Theorem 2.2.

**Lemma 1 ([6]).**

Over ACA\(_0 + \) WO(\( \alpha \)), WOP(\( \varphi_\alpha \)) is equivalent to \( \forall X \exists Y \) Hier\( _{\omega^\alpha}(X, Y) \):

\[
\text{ACA}_0 \vdash \forall \alpha[\text{WO}(\alpha) \rightarrow (\text{WOP}(\varphi_\alpha) \leftrightarrow \forall X \exists Y \text{ Hier}_{\omega^\alpha}(X, Y))].
\]

**Proof** It is well known that WOP(\( \varphi_\alpha \)) follows from \( \forall X \exists Y \text{ Hier}_{\omega^\alpha}(X, Y) + \) WO(\( \alpha \)), cf. [9].

Conversely let \( A(\alpha) :\leftrightarrow [\text{WOP}(\varphi_\alpha) \rightarrow \forall X \exists Y \text{ Hier}_{\omega^\alpha}(X, Y)] \). We show first that

\[
\text{ACA}_0 \vdash \forall Z[Z \models (\text{ACA}_0 \rightarrow \forall \beta < \alpha A(\beta))] \rightarrow A(\alpha) \quad (4)
\]

Assume that \( A(\beta) \) holds for any \( \beta < \alpha \) in any countable coded \( \omega \)-models of ACA\(_0 \). Suppose WOP(\( \varphi_\alpha \)) for \( \alpha > 0 \). Then by (3) we have \( (\forall \beta < \alpha \text{WOP}(\varphi_\beta))^+ \).

Given a set \( X \), pick a countable coded \( \omega \)-model \( Z \) of \( \forall \beta < \alpha \text{WOP}(\varphi_\beta) \) such that \( X \in_\omega Z \). \( Z \) is an \( \omega \)-model of ACA\(_0 \) by Theorem 1. By the assumption we obtain

\[\forall \beta < \alpha \forall X \exists Y \text{ Hier}_{\omega^\beta}(X, Y) \in Z.\]

For the given set \( X \) let \( W = \{ (\gamma, m) : \gamma < \omega^\beta, Z \models \exists Y[\text{Hier}_\gamma(X, Y) \land m \in (Y)_\gamma] \} \). \( W \) is a set by ACA\(_0 \). If \( \alpha \) is a limit number, then \( \forall \beta < \alpha \text{ Hier}_{\omega^\beta}(X, W) \) yields \( \text{Hier}_{\omega^\beta}(X, W) \). When \( \alpha = \beta + 1 \), we see by induction on \( k < \omega \) that \( \forall k < \omega \text{ Hier}_{\omega^k}(X, W) \), and hence \( \text{Hier}_{\omega^\alpha}(X, W) \). This shows (4).

For sequences \( s \) of natural numbers, let \( Z_s \) be defined recursively by \( Z_{\emptyset} = Z \) for the empty sequence \( \emptyset \), and \( Z_{\langle n \rangle} = (Z_s)_n \) with \( (n_0, \ldots, n_k) \ast (n) = (n_0, \ldots, n_k, n) \).

Let \( B(\alpha) :\leftrightarrow \forall s[Z_s \models (\text{ACA}_0 \rightarrow \forall \beta < \alpha A(\beta))] \). From (4) we see that \( \text{ACA}_0 \vdash \forall \alpha[\forall \beta < \alpha B(\beta) \rightarrow B(\alpha)] \) for the arithmetical formula \( B \). We obtain \( \text{ACA}_0 \vdash \) WO(\( \alpha \)) \( \rightarrow B(\alpha) \). Hence \( \text{ACA}_0 \vdash \) WO(\( \alpha \)) \( \rightarrow \forall Z[Z \models (\text{ACA}_0 \rightarrow \forall \beta < \alpha A(\beta))] \).

(4) yields \( \text{ACA}_0 \vdash \) WO(\( \alpha \)) \( \rightarrow A(\alpha) \). \( \square \)

Let us mention the contents of the paper. In the next Sect. 2 \( g(X) \) is defined as a term structure. Extendible structures and exponential ones are defined. The easy half of Theorem 4 is shown. In Sect. 3 an infinitary sequent calculus is introduced. We search canonically a derivation of the empty sequent (contradiction) from the assumption WOP(\( g \)) in the calculus. This yields a tree \( T \) such that if \( T \) is not well-founded, then we can find a countable coded \( \omega \)-model of WOP(\( g \)) from an infinite path through \( T \). In Sect. 4 assuming that the tree \( T \) is well-founded, (hidden) cut inferences are eliminated from the derivation tree \( T \) so that the depth of the resulting tree is bounded by \( g'(\Lambda) \) for the order type \( \Lambda \) of the well-founded tree \( T \) in the Kleene-Brouwer ordering. We see that this is not the case by transfinite induction up to the ordinal \( g'(\Lambda) \), and the

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harder half of Theorem 4, i.e., (2) is concluded. The elimination is based on a result due to Takeuti \[11,12\]: suppose a cut-free derivation \(\pi\) of the transfinite induction on an ordering \(<\) is given in depth \(\alpha\). Then there exists an embedding from \(<\) to the ordinal \(\omega^\alpha\) such that the embedding is computable from the derivation \(\pi\), the ordering \(<\) and an ordering of type \(\omega^\alpha\), cf. Theorem 5. In Sect. 5 we show Theorem 6, i.e., (3) on common fixed points.

2 Term structures

Let us compare the proof-theoretic strength WOP(\(g'\)) with WOP(\(g\)) for normal function \(g\). First of all, both \(g'\) and \(g\) are assumed to be definable in such a way that WOP(\(g'\)) and WOP(\(g\)) are \(\Pi^1_2\)-formulas. Also the fact that \(g\) sends linear orderings \(X\) to linear orderings \(g(X)\) should be provable in an elementary way. \(g\) sends a binary relation \(<_X\) on a set \(X\) to a binary relation \(<_{g(X)} = g(<_X)\) on a set \(g(X)\). We further assume that \(g(X)\) is a Skolem hull, i.e., a term structure over constants \(0\) and \(\omega\) such that the embedding is computable from the derivation \(\pi\), the ordering \(<\) and an ordering of type \(\omega^\alpha\), cf. Theorem 5. In Sect. 5 we show Theorem 6, i.e., (3) on common fixed points.

**Definition 5** 1. \(g(X)\) is said to be a computably linear term structure if there are three \(\Sigma^0_1(X)\)-formulas \(g(X), <_{g(X)} = g(<_X)\) for which all of the following facts are provable in RCA\(_0\): let \(\alpha, \beta, \gamma, \ldots\) range over terms.

(a) (Computability) Each of \(g(X), <_{g(X)} = g(<_X)\) and \(\alpha, \beta, \gamma, \ldots\) range over terms.

(b) (Congruence)

\[=\] is a congruence relation on the structure \((g(X); <_{g(X)}, f, \ldots)\).

Let us denote \(g(X)/=\) the quotient set.

In what follows assume that \(<_X\) is a linear ordering on \(X\).

(c) (Linearity) \(<_{g(X)}\) is a linear ordering on \(g(X)/=\) with the least element \(0\).

(d) (Increasing) \(g\) is strictly increasing: \(c <_X d \Rightarrow g(c) <_{g(X)} g(d)\).

(e) (Continuity) \(g\) is continuous: Let \(\alpha <_{g(X)} g(c)\) for a limit \(c \in X\) and \(\alpha \in g(X)\).

Then there exists a \(d <_X c\) such that \(\alpha <_{g(X)} g(d)\).

2. A computably linear term structure \(g(X)\) is said to be extendible if it enjoys the following two conditions.

(a) (Suborder) If \((X, <_X)\) is a substructure of \((Y, <_Y)\), then \((g(X); =, <_{g(X)}, f, \ldots)\) is a substructure of \((g(Y); =, <_{g(Y)}, f, \ldots)\).

(b) (Indiscernible)

\((g(c) : c \in [0] \cup X)\) is an indiscernible sequence for linear orderings \((g(X), <_{g(X)}): \) Let \(\alpha[0, g(c_1), \ldots, g(c_n)], \beta[0, g(c_1), \ldots, g(c_n)] \in g(X)\) be terms such that constants occurring in them are among the list \(0, g(c_1), \ldots, g(c_n)\). Then for any increasing sequences \(c_1 <_X \ldots <_X c_n\) and \(d_1 <_X \ldots <_X d_n\), the following holds.

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\[ \alpha[0, g(c_1), \ldots, g(c_n)] <_{g(X)} \beta[0, g(c_1), \ldots, g(c_n)] \]
\[ \Leftrightarrow \alpha[0, g(d_1), \ldots, g(d_n)] <_{g(X)} \beta[0, g(d_1), \ldots, g(d_n)] \]

**Remark 1** As pointed out by the referee, the fact that a term structure \( g(X) \) is extendible is closely related to the fact that \( g(X) \) is effectively relatively categorical in the sense of Feferman [3]. A similar fact as the following proposition is proved in Theorem 2.7 (or better in the footnote 4) of [3] under the assumption of the relative categoricity.

**Proposition 1** Suppose \( g(X) \) is an extendible term structure. Then the following is provable in RCA\(_0\): Let both \( X \) and \( Y \) be linear orderings.

Let \( f : \{0\} \cup X \to \{0\} \cup Y \) be an order preserving map, \( n <_X m \Rightarrow f(n) <_Y f(m) \). Then there is an order preserving map \( F : g(X) \to g(Y) \), \( n <_{g(X)} m \Rightarrow F(n) <_{g(Y)} F(m) \), which extends \( f \) in the sense that \( F(g(n)) = g(f(n)) \).

**Proof** Let \( F(0) = 0 \). Let \( \alpha[0, g(c_1), \ldots, g(c_n)] \in g(X) \) be a term such that constants occurring in it are among the list \( 0, g(c_1), \ldots, g(c_n) \) for \( c_i \in \{0\} \cup X \). Define \( F(\alpha[0, g(c_1), \ldots, g(c_n)]) = \alpha[0, g(f(c_1)), \ldots, g(f(c_n))] \). From (5) on \( g(X) + Y \), we see that \( F \) is an order preserving map from \( g(X) \) to \( g(Y) \).

Moreover we see that if \( \alpha[0, g(c_1), \ldots, g(c_n)] = \beta[0, g(c_1), \ldots, g(c_n)] \), then \( F(\alpha[0, g(c_1), \ldots, g(c_n)]) = F(\beta[0, g(c_1), \ldots, g(c_n)]) \).

**Definition 6** Suppose that function symbols \( +, \lambda \xi. \omega^\xi \) are in the list \( F \) of function symbols for a computably linear term structure \( g(X) \). Let \( 1 := \omega^0 \), and \( 2 := 1 + 1 \), etc.

\( g(X) \) is said to be an exponential term structure (with respect to function symbols \( +, \lambda \xi. \omega^\xi \)) if all of the followings are provable in RCA\(_0\).

1. 0 is the least element in \( <_{g(X)} \), and \( \alpha + 1 \) is the successor of \( \alpha \).
2. \( + \) and \( \lambda \xi. \omega^\xi \) enjoy the following familiar conditions.
   (a) \( \alpha <_{g(X)} \beta \rightarrow \omega^\alpha + \omega^\beta = \omega^\beta \).
   (b) \( \gamma + \lambda = \sup\{\gamma + \beta : \beta < \lambda\} \) when \( \lambda \) is a limit number, i.e., \( \lambda \neq 0 \) and \( \forall \beta <_{g(X)} \lambda(\beta + 1 <_{g(X)} \lambda) \).
   (c) \( \beta_1 <_{g(X)} \beta_2 \rightarrow \alpha + \beta_1 <_{g(X)} \alpha + \beta_2 \), and \( \alpha_1 <_{g(X)} \alpha_2 \rightarrow \alpha_1 + \beta <_{g(X)} \alpha_2 + \beta \).
   (d) \( (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \).
   (e) \( \alpha <_{g(X)} \beta \rightarrow \exists \gamma <_{g(X)} \beta(\alpha + \gamma = \beta) \).
   (f) Let \( \alpha_n <_{g(X)} \cdots \leq_{g(X)} \alpha_0 \) and \( \beta_m <_{g(X)} \cdots <_{g(X)} \beta_0 \). Then \( \omega^{\alpha_0} + \cdots + \omega^{\alpha_n} <_{g(X)} \omega^{\beta_0} + \cdots + \omega^{\beta_m} \) iff either \( n < m \) and \( \forall i \leq n(\alpha_i = \beta_i) \), or \( \exists j \leq \min\{n, m\}[\alpha_j <_{g(X)} \beta_j \land \forall i < j(\alpha_i = \beta_i)] \).
3. Each \( f(\beta_1, \ldots, \beta_n) \in g(X) \) (\( + \neq f \in F \)) as well as \( g(c) (c \in \{0\} \cup X) \) is closed under \( + \). In other words the terms \( f(\beta_1, \ldots, \beta_n) \) and \( g(c) \) denote additively closed ordinals (additive principal numbers) when \( <_{g(X)} \) is a well ordering.

It is clear that each of \( \varepsilon_X \) in [1], \( \varphi \chi'0 \) in [8] and \( \Gamma \chi' \) in [7] is an extendible and exponential term structure.

In what follows we assume that \( g(X) \) is an extendible term structure, and \( g'(X) \) is an exponential term structure. Constants in the term structure \( g'(X) \) are 0 and \( g'(c) \) for
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\(c \in \{0\} \cup X\), and function symbols in \(\mathcal{F} \cup \{0, +\} \cup \{g\}\) with a unary function symbol \(g\). When \(\mathcal{F} = \emptyset\), let \(\omega^\alpha := g(\alpha)\). Otherwise we assume that \(\lambda \xi. \omega^\xi\) is in the list \(\mathcal{F}\). Furthermore assume that RCA\(_0\) proves that

\[
\beta_1, \ldots, \beta_n < g'(X) g'(c) \rightarrow f(\beta_1, \ldots, \beta_n) < g'(X) g'(c) (f \in \mathcal{F} \cup \{+, g\})
\]

\[
\omega g' \beta = g(g'(\beta)) = g'(\beta)
\]

\[
g'(0) = \sup_n g^n(0)
\]

\[
g'(c + 1) = \sup_n g^n(g'(c) + 1) (c \in \{0\} \cup X)
\]

(6)

where \(g^n\) denotes the \(n\)-th iterate of the function \(g\), and we are assuming in the last that the successor element \(c + 1\) of \(c\) in \(X\) exists. Note that the last two in (6) hold for normal functions \(g\) when \(g(0) > 0\).

Assume that \(<_X\) is a linear ordering. Each non-zero term \(\beta \in g'(X)\) is written as a Cantor normal form \(\beta = \beta_1 + \cdots + \beta_n\) where \(\beta_n \leq g'(X) \ldots \leq g'(X) \beta_1\) and each \(\beta_i\) is an \(f\)-term \(f(\gamma_1, \ldots, \gamma_m)\) with \(f \in \mathcal{F}\) or \(g'(c)\). Using the Cantor normal form, we can define the natural (commutative) sum \(\alpha \# \beta\) of terms \(\alpha, \beta \in g'(X)\) which enjoys \(\alpha \# \beta = \beta \# \alpha\) and \(\alpha_1 < g'(X) \alpha_2 \Rightarrow \alpha_1 \# \beta < g'(X) \alpha_2 \# \beta\).

**Theorem 4** Let \(g(X)\) be an extendible term structure, and \(g'(X)\) an exponential term structure for which (6) holds.

Then the following two are mutually equivalent over ACA\(_0\):

1. WOP\((g')\).
2. \((\text{WOP}(g))^+ : \Leftrightarrow \forall X \exists Y [X \in Y \land M_Y \models \text{WOP}(g)]\). Namely there exists an arbitrarily large countable coded \(\omega\)-model of WOP\((g)\).

First let us show the easy half \(2 \Rightarrow 1\). Let sets \(X, U\) be given such that WO\((\leq_0)\) for \(\leq_0 = \leq_X\). We have LOI\((\leq_X)\). Pick a countable coded \(\omega\)-model \(M\) of WOP\((g)\) such that \(X, U \in M\). Then \(g(X), g'(X) \in M\). Let \(<_1\) be obtained from \(<_0\) by adding the largest element \(\alpha\). This means that \(a <_1 \alpha\) for any \(a\) in the field of \(<_0\). We have WO\((<_1)\) by WO\((<_0)\). We show \(\text{Pr}g[<_1, (a)]\) for an arithmetical formula

\[
C(a) : \Leftrightarrow M \models \forall Y \left( \text{Pr}g[<_2, Y] \rightarrow \forall x <_2 g'(a) Y(x) \right)
\]

for \(<_2 = \leq g'(<_1)\). This yields \(C(a)\). Since by (6), \(x <_2 g'(a)\) for any \(x\) in the field of \(\leq g'(<_0)\), we obtain \(M \models \forall Y \left( \text{Pr}g[\leq g'(X), Y] \rightarrow \forall x \in \text{fld}(g'(X)) Y(x) \right)\). Hence we obtain \(M \models \left( \text{Pr}g[\leq g'(X), U] \rightarrow \forall x \in \text{fld}(g'(X)) U(x) \right)\), i.e., TI \((\leq g'(X), U)\). Since \(U\) is an arbitrary set, we conclude WO\((\leq g'(X))\).

It remains to show that \(\text{Pr}g[<_1, (a)]\). When \(a\) is a limit element, this follows from the continuity of the function \(g'(a)\). Assuming \(C(a)\), let us show \(C(a + 1)\). Argue in the model \(M\). Suppose \(\text{Pr}g[<_2, Y]\) and \(x <_2 g'(a + 1) = \sup_n g^n(g'(a) + 1)\) by (6). By induction on \(n < \omega\) we see that \(\forall x <_2 g^n(g'(a) + 1) Y(x)\) using WOP\((g)\) and \(C(a)\), i.e., WO\((<_2[ g'(a) + 1])\). Hence we obtain \(C(a + 1)\). \(C(0)\) is seen similarly.
3 Proof search

Conversely assume $\text{WOP}(g')$. We need to find a countable coded $\omega$-model of $\text{WOP}(g)$. The idea in [1,7,8] is to search a derivation of the negation of $\text{WOP}(g)$ in $\omega$-logic. Construct a locally correct $\omega$-branching tree in a canonical way. If the search results in a fail, i.e., if the constructed tree is not well-founded, then we can believe in the consistency of $\text{WOP}(g)$ in $\omega$-logic. In fact we can find a countable coded $\omega$-model of $\text{WOP}(g)$ from an infinite path through the tree. Otherwise the tree is well-founded, i.e., a derivation in a depth $\alpha$. It turns out that the derivation can be converted to a cut-free deduction with the empty sequent at its root, and in depth bounded by $g'(\alpha)$. Then by our assumption $\text{WOP}(g')$, the deduction is well-founded, i.e., a derivation of the empty sequent. We see that this is not the case by transfinite induction up to $g'(\alpha)$. This shows the consistency of $\text{WOP}(g)$ in $\omega$-logic based on $\text{WOP}(g')$. Now details follow.

Let $Q \subset \mathbb{N}$ be a given set, which is viewed as a family $\{(Q)_i : i \in \mathbb{N}\}$ of sets of natural numbers. The language $\mathcal{L}_\omega$ here consists of function symbols for elementary recursive functions including 0 and the successor $S$, predicate symbols $=, \neq$ and unary predicate variables $\{X_i, E_i : i < \omega\}$ and their complements $\bar{X}_i, \bar{E}_i$. Let us write $n \prec_i m$ for $n <_{X_i} m$, i.e., for $X_i(\langle n, m \rangle)$, and $n \prec g_i m$ for $n <_{g(X_i)} m$. Each $E_i$ is a fresh variable used to express the well-foundedness $\text{TI}(\prec_i, E_i)$ of the relation $\prec_i$. Each closed term $t$ is identified with its value $t^\mathbb{N}$, a numeral.

$$
D_Q(i, n) = \begin{cases} X_i(n) & \text{if } n \in (Q)_i \\
\bar{X}_i(n) & \text{if } n \notin (Q)_i
\end{cases}
$$

and $\text{Diag}(Q) = \{D_Q(i, n) : i, n \in \mathbb{N}\}$.

A true literal is one of the form $t_0 = t_1 (t_0^\mathbb{N} = t_1^\mathbb{N})$, $s_0 \neq s_1 (s_0^\mathbb{N} \neq s_1^\mathbb{N})$, and $D_Q(i, n)$ for $i, n \in \mathbb{N}$.

**Axioms** in $G(Q) + (prg) + (W)$ are

$$\Gamma, \bar{E}_i(n), E_i(n)$$

and for true literals $L$

$$\Gamma, L$$

**Inference rules** in $G(Q) + (prg) + (W)$.

$$
\frac{\Gamma, A_0 \lor A_1, A_i}{\Gamma, A_0 \lor A_1} \quad (\lor) \\
\frac{\Gamma, A_0}{\Gamma, A_0 \land A_1} \quad (\land) \\
\frac{\Gamma, \exists x A(x), A(n)}{\Gamma, \exists x A(x)} \quad (\exists) \\
\frac{\Gamma, A(n) : n \in \mathbb{N}}{\Gamma, \forall x A(x)} \quad (\forall) \\
$$

($\exists^2$) for $i \in \mathbb{N}$, ($\forall^2$) with an eigenvariable $Z$ and the repetition rule ($Rep$)

$$
\frac{\Gamma, \exists Y A(Y), A(X_i)}{\Gamma, \exists Y A(Y)} \quad (\exists^2) \\
\frac{\Gamma, A(Z)}{\Gamma, \forall Y A(Y)} \quad (\forall^2) \\
\frac{\Gamma}{\Gamma} \quad (Rep)
$$

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and the following two for $i, m \in \mathbb{N}$:

$$\frac{[\Gamma, E_i(n) : n < i, m \text{ is true}]}{\Gamma, E_i(m)} (\text{prg})_i$$

where by saying that $n < i, m$ is true we mean $\langle n, m \rangle \in (Q)_i$.

$$\frac{\Gamma, \text{LO}(\langle i \rangle) \Gamma, \forall x E_i(x) \exists y \text{TI}(\langle g_i, y \rangle), \Gamma}{\Gamma} (W)_i$$

Let us construct a tree $T \subset <^\omega \omega$ recursively as follows. For $a \in T$, $\text{Seq}(a)$ is a label attached with the node $a$, which is a sequent at $a$. First put the empty sequent at the root $\emptyset$. **Leaf condition** on the tree runs as follows: If $\text{Seq}(a)$ is an axiom in $G(Q)$, then $a$ is a leaf in $T$. The construction is divided into three cases. Suppose that the tree $T$ has been constructed up to a node $a \in <^\omega \omega$.

**Case 0.** $lh(a) = 3i$ for an $i \geq 0$: Apply the inference $(W)_i$ backwards.

**Case 1.** $lh(a) = 3i + 1$: Apply one of inferences ($\vee$), ($\wedge$), ($\exists$), ($\forall \omega$), ($\exists^3$) if it is possible. Otherwise repeat, i.e., apply the inference $(\text{Rep})$ backwards.

Note that the inference ($\forall^2$) is not applicable since no second-order universal quantifier occurs in the upper sequents of the inferences $(\text{prg})_i$, $(W)_i$.

**Case 2.** $lh(a) = 3\langle j, \langle i, m \rangle \rangle + 2$: Apply the inference $(\text{prg})_i$ with the main formula $E_i(m)$ backwards if it is possible. Otherwise repeat.

If the tree $T$ is not well-founded, then let $P$ be an infinite path through $T$. Let $(M)_i \subset \mathbb{N}$ be a set such that for any $n \in \mathbb{N}$, $(X_i(n)) \in P \Rightarrow n \notin (M)_i$ and $(\bar{X}_i(n)) \in P \Rightarrow n \in (M)_i$. Then for any $n$ for which one of $X_i(n)$, $\bar{X}_i(n)$ is in $P$, we obtain $n \in (Q)_i \iff n \in (M)_i$. For other $n$, $n \in (M)_i$ is arbitrarily determined for $i \neq 0$: set $(M)_0 := (Q)_0$.

$M$ is shown to be a countable coded $\omega$-model of $\text{WOP}(g)$ as follows. The search procedure is fair, i.e., each formula is eventually analyzed on every path as in [1,7,8]. To ensure fairness, formulas in sequents $\Gamma$ are assumed to stand in a queue. The head of the queue is analyzed in **Case 1**, and the analyzed formula moves to the end of the queue in the next stage. We see from the fairness that $M \not\models A$ by induction on the number of occurrences of logical connectives in formulas $A$ on the path $P$, where by definition $M \not\models E_i(m)$ for any $i, m \in \mathbb{N}$. $M \models \text{WOP}(g)$ since the inference rules $(\text{prg})_i$, $(W)$, are analyzed.

### 4 Cut elimination

In what follows assume that $T$ is well founded. Since we are working in $\text{ACA}_0$, we know that the Kleene-Brouwer ordering $<_\text{KB}$ on $T$ is a well-ordering, cf. [10]. Let $\Lambda = \text{otp}(<_\text{KB})$ denote the order type of the well-ordering $<_\text{KB}$. We have $\text{WO}(g'(\Lambda))$ by $\text{WO}(g')$ and $\text{WO}(\Lambda)$.

For $b < \Lambda$ let us write $\vdash^b \Gamma$ when there exists a derivation of $\Gamma$ in $G(Q) + (\text{prg}) + (W)$ whose depth is bounded by $b$. On the other side for $\alpha < g'(\Lambda)$, $\vdash^\alpha \Gamma$ designates
that there exists a derivation of $\Gamma$ in $G(\mathcal{Q}) + (prg)$ of depth $\alpha$. In the derivation no inference $(W)_i$ occurs. Specifically for a function $\pi$ on $<^\omega \omega$, $\pi \vdash_0^\alpha \Gamma$ designates that there exists a derivation of the sequent $\Gamma$ in $G(\mathcal{Q}) + (prg)$ with the repetition rule $(Rep)$ in depth $\alpha$, and this fact is witnessed by the function $\pi$. The last means the following. For each $a \in <^\omega \omega$, either $\pi(a) = *$ designating that $a$ is not in the naked tree for the derivation, or $\pi(a) = (\text{Seq}(a), \text{Rule}(a), \text{Mfml}(a), \text{Sfml}(a), \text{ord}(a))$, where $\text{Seq}(a)$ denotes the sequent at the node $a$, $\text{Rule}(a)$ the inference rule whose lower sequent is $\text{Seq}(a)$, $\text{Mfml}(a)$ is the main (principal) formula of $\text{Rule}(a)$, $\text{Sfml}(a)$ the minor (auxiliary or side) formulas of $\text{Rule}(a)$ and $\text{ord}(a) < \Lambda$.

The following Theorem 5 is due to Takeuti [11, 12].

**Theorem 5** The following is provable in $\text{RCA}_0 + \text{WO}(\alpha)$:

Suppose that $<$ is a linear ordering with the least element 0, and $<$ denotes the well-ordering up to $\omega^\alpha$. $(prg)_<$ denotes the sequent calculus with inference rules $(prg)_<$ and the repetition rule $(Rep)$.

$$\begin{align*}
\{ \Gamma, E(n) : n < m \text{ is true} \} & \quad (prg)_< \\
\Gamma, E(m) & \quad (prg)_<
\end{align*}$$

Suppose $\pi \vdash_0^\alpha \forall x E(x)$. Then there exists an embedding $f$ such that $n < m \implies f(n) < f(m)$, $f(m) < \omega^\alpha$ for any $n, m$, and $f$ is $\alpha$-recursive in the function $\pi$ and the relations $<, <$.

**Proof** Let us write $\Gamma : \alpha$ for $\vdash_0^\alpha \Gamma$, and $<_\omega$ for the usual $\omega$-ordering in the proof. First search the $\omega$-rule $(\forall \omega)$ nearest to the root in the derivation $\pi$:

$$\begin{align*}
\vdots & \\
\pi_m & \\
\{ E(m) : \alpha_m \} & \quad (\forall \omega) \\
\forall x E(x) : \alpha' & \\
\vdots & \\
\forall x E(x) : \alpha
\end{align*}$$

where $\alpha_m < \alpha' \leq \alpha$ and there are some (possibly none) $(Rep)$’s below the $(\forall \omega)$. Such an $(\forall \omega)$ exists by $\text{WO}(\alpha)$. By induction on $m$, we define a derivation $\rho_m$ of $\Gamma_m : \beta_m$ for a finite set $\Gamma_m \subset \{ E(n) : n \in \mathbb{N} \}$ such that $E(m) \in \Gamma_m$ and $\forall n [E(n) \in \Gamma_m \implies m \leq n]$ as follows. If $\forall n <_\omega m(n < m)$, then $\rho_m = \pi_m$ and $\beta_m = \alpha_m$. Otherwise let

$$n_0 < \cdots < n_{j-1} < n_j (= m) < n_{j+1} < \cdots < n_m \quad (7)$$

with $\{ n_i : i \leq m \} = \{ 0, \ldots, m \}$ and $j <_\omega m$.

---

1 Actually Takeuti proved a similar result when we have in hand a finite proof figure of transfinite induction in $\text{PA}$. Under the assumption we can take an order preserving map $f$ elementarily recursive in the ordering, cf.[2].
Search the nearest inference \((prg)_{<}\) in \(\rho_{n_{j+1}}\):

\[
\frac{\{\Gamma_{n_{j+1}}, E(n) : \beta\}_{n < n'} \hspace{1cm} (prg)_{<}}{\Gamma_{n_{j+1}} : \beta'_{n_{j+1}} \hspace{1cm} \rho_{n_{j+1}} \hspace{1cm} \Gamma_{n_{j+1}} : \beta_{n_{j+1}}}
\]

where \(\beta < \beta'_{n_{j+1}} \leq \beta_{n_{j+1}}, E(n') \in \Gamma_{n_{j+1}}\) is the main formula of the inference \((prg)_{<}\). We have \(m < n_{j+1} \leq n'\). Let \(\rho_m\) be the following

\[
\vdots \hspace{1cm} \Gamma_{n_{j+1}}, E(m) : \beta
\]

with \(\beta_m = \beta < \beta_{n_{j+1}}\).

Define a function \(f(m)\) by induction on \(m\) as follows. \(f(0) = \omega^{\beta_0} = \omega^{\beta_{n_0}}\) for the least element 0 with respect to \(<\). For \(m \neq 0\), \(f(m) = f(n_{j-1}) + \omega^{\beta_m}\) with the largest element \(n_{j-1} <\omega m\) with respect to \(<\) in (7). Let us show that \(f\) is a desired embedding. In (7), it suffices to show by induction on \(m\) that

\[
\forall i <\omega \ m[f(n_{i+1}) = f(n_i) + \omega^{\beta_{n_{i+1}}}]
\]

First by the definition of \(f\) we have \(f(m) = f(n_{j-1}) + \omega^{\beta_m}\) with \(m = n_j\). On the other hand we have \(f(m) + \omega^{\beta_{n_{j+1}}} = f(n_{j-1}) + \omega^{\beta_m} + \omega^{\beta_{n_{j+1}}} = f(n_{j-1}) + \omega^{\beta_{n_{j+1}}} = f(n_{j+1})\) by \(\beta_m < \beta_{n_{j+1}}\) and IH. This shows (8), and our proof is completed.

Let us call a sequent \(\Delta\) an E-sequent if \(\Delta \subseteq \{\forall x E_i(x), E_i(n) : i, n < \omega\}\). An E-free formula is a formula in which no \(E_i\) occurs.

**Lemma 2** For an E-sequent \(\Delta\) and an E-free sequent \(\Gamma\), if \(\vdash^b \Delta, \Gamma\) for \(b < \Lambda\), then \(\vdash^{g'(b)} \Delta, \Gamma\).

**Proof** by induction on \(b < \Lambda\).

**Case 1.** \(\Delta, \Gamma\) is an axiom: There is nothing to prove.

**Case 2.** \(\Delta, \Gamma\) is a lower sequent of an inference such that its principal formula is in \(\Delta \cup \Gamma\):

\[
\vdots \hspace{1cm} \vdash^c n \Delta_n, \Gamma_n \hspace{1cm} \vdots
\]

\[
\vdash^b \Delta, \Gamma
\]

By IH we have \(\vdash^{g'_0} \Delta_n, \Gamma_n\). From \(g'(c_n) < g'(b)\) we obtain

\[
\vdots \hspace{1cm} \vdash^{g'_0} \Delta_n, \Gamma_n \hspace{1cm} \vdots
\]

\[
\vdash^{g'_0} \Delta, \Gamma
\]

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When there are no upper sequents, i.e., when \((E_i(m)) \in \Delta\) with the minimal \(m\) with respect to \(<_i\), we have \(\vdash^0 \Delta, \Gamma\).

**Case 3.** \(\Delta, \Gamma\) is a lower sequent of an inference \((W)_i\).

\[
\begin{align*}
\vdash^{c'} \Delta, \Gamma, \text{LO}(<_i) & \quad \vdash^c \Delta, \Gamma, \forall x E_i(x) \quad \vdash^d \Delta, \Gamma, \exists Y \neg \text{TI}(<g_i, Y) \\
\vdash^b \Delta, \Gamma
\end{align*}
\]

where \(c', c, d < b\).

If \(\text{LO}(<_i)\) is false, i.e., \(<_Q>_i\) is not a linear ordering, then we see that \(\vdash^{c'} \Delta, \Gamma\) with \(c' < b\). IH yields the assertion.

In what follows assume that \(<_Q>_i\) is a linear ordering. By IH we have for the \(E\)-sequent \(\Delta \cup \{\forall x E_i(x)\}\)

\[
\vdash^{g'(c)}_0\Delta, \forall x E_i(x), \Gamma
\]

If \(\vdash^{g'(c)}_0 \Delta, \Gamma\), then we obtain the assertion. Assume that this is not the case. Then we claim that

\[
\vdash^{g'(c)}_0\forall x E_i(x) \tag{9}
\]

This is seen by induction on \(g'(c) < g'(\Lambda)\) as follows. If \(\Delta, \forall x E_i(x), \Gamma\) is an axiom, then so is \(\Gamma\), i.e., either a true literal is in \(\Gamma\) or \(\{L, \bar{L}\} \subset \Gamma\) for a literal \(L\). Then \(\vdash^{g'(c)}_0 \Delta, \Gamma\). Next assume that \(\Delta, \forall x E_i(x), \Gamma\) is derived by an inference whose principal formula is in \(\Delta \cup \Gamma\).

\[
\{\vdash^{b_n}_0 \Delta_n, \forall x E_i(x), \Gamma_n\}_n
\]

We can assume that there exists an \(n\) for which \(\vdash^{b_n}_0 \Delta_n, \Gamma_n\) does not hold. By IH we obtain \(\vdash^{b_n}_0 \forall x E_i(x)\). Finally let

\[
\vdash^{g'(c)}_0 \Delta, \forall x E_i(x), \Gamma
\]

We can assume that \(\vdash^{b_n}_0 \Delta, \Gamma\) does not hold for any \(n\). Then we show that \(\vdash^{b_n}_0 E_i(n)\) holds for any \(n\) by induction on \(b_n\). Consider the case

\[
\vdash^{a_n}_0 \Delta, E_i(n), \Gamma : n <_i m \tag{prg}_i
\]

By IH we see that \(\vdash^{a_n}_0 E_i(n)\) for any \(n <_i m\). Thus (9) is shown.

Let \(\beta_0 = g'(c)\). By Theorem 5 there is an embedding \(f\) such that \(n <_i m \Rightarrow f(n) < f(m), f(m) < \omega^{\beta_0}\) for any \(n, m\), and \(f\) is \(\beta_0\)-recursive in the computable function \(\pi\) for the derivation witnessing the fact (9) and the relations \(<_i, <\).
By Proposition 1 let $F$ be an order preserving map from $g(<_i)$ to $g(<): n < g_i m \Rightarrow F(n) < F(m)$, $F(m) < g(\omega^{\beta_0})$, and $F$ is computable from $f$.

The following shows that $\vdash_{0} G(m) + 3 \rightarrow Prg[<_g, Z], Z(m)$ with a fresh variable $Z$ and $G(m) = \omega + 1 + 4F(m)$ by induction on $F(m)$:

\[
\frac{\vdash_{0} G(n) + 3 \rightarrow Prg[<_g, Z], Z(n) : n < g_i m]}{\vdash_{0} G(n) \rightarrow Prg[<_g, Z], \forall Z : n \in \omega} \quad (\forall \omega)
\]

\[
\frac{\vdash_{0} G(n) \rightarrow Prg[<_g, Z], \forall y < g_i m Z(y)}{\vdash_{0} G(n) + 1 \rightarrow Prg[<_g, Z], \forall y < g_i m Z(y)} \quad (\forall \omega)
\]

\[
\frac{\vdash_{0} G(n) \rightarrow Prg[<_g, Z], \forall y < g_i m Z(y), Z(m)}{\vdash_{0} G(n) + 2 \rightarrow Prg[<_g, Z], \forall y < g_i m Z(y) \land Z(m), Z(m)} \quad (\exists)
\]

where $n < g_i m$ denotes the formula $\neg(g(X))(\langle n, m \rangle)$, which is a $\Delta^0_1$-formula in $X_i$. Thus for $G(m) + 3 < g(\omega^{\beta_0})$ we obtain

\[
\frac{\vdash_{0} G(n) + 3 \rightarrow Prg[<_g, Z], Z(m) : m < \omega)}{\vdash_{0} G(\omega^{\beta_0}) \rightarrow Prg[<_g, Z], \forall x Z(x)} \quad (\forall \omega)
\]

\[
\frac{\vdash_{0} G(\omega^{\beta_0}) \rightarrow Prg[<_g, Z], \forall x Z(x)}{\vdash_{0} G(\omega^{\beta_0}) + 2 \rightarrow TI[<_g, Z]} \quad (\forall 2)
\]

\[
\frac{\vdash_{0} G(\omega^{\beta_0}) + 3 \rightarrow \forall Y TI[<_g, Y]}{\vdash_{0} G(\omega^{\beta_0}) + 3} \quad (\forall Y)
\]

On the other hand we have by IH $\vdash_{0} g^{(d)} \Delta, \Gamma, \exists Y \rightarrow TI[<_g, Y]$ for the $E$-free sequent $\Gamma \cup \{\exists Y \rightarrow TI[<_g, Y]\}$. By cut-elimination we obtain $\vdash_{0} \beta_1 \Delta, \Gamma$ for $\beta_1 = \omega_k (g(\omega^{\beta_0})#3g^{(d)})$ for a $k < \omega$ depending only on the formula $\forall Y TI[<_g, Y]$. Now $\beta_1 = \omega_k (g(\omega^{\beta_0}(c)#3g^{(d)})) < g^{(b)}$ since $c, d < b$ and $g^{(b)}$ is closed under $+, \lambda \xi, \omega^x$ and $g$ by (6).

Let us finish our proof of the harder direction in Theorem 4. By our assumption we have $\vdash_b \emptyset$ for a $b < \Lambda$ and the empty sequent $\emptyset$. Lemma 2 yields $\vdash_{0} g^{(b)} \emptyset$. We see that this is not the case by induction on $g^{(b)} < g^{(\Lambda)}$. Therefore the tree $T$ is not well founded.

Finally let us spend a few words on a formalization of the above proof in ACA0. In the proof one can agree that each infinite derivation is a computable function $\pi$ on the set of finite sequences $a$ of natural numbers. $\pi(a)$ is a bunch of data as described before Theorem 5. $\vdash_{0} \alpha$ $\Gamma$ denotes the fact that there exists a computable function $\pi$ such that $Seq(\emptyset) = \Gamma$ and $ord(\emptyset) = \alpha$ for the empty sequence $\emptyset$, i.e., the root of the derivation tree, $\vdash_{0} \alpha$ $\Gamma$ is arithmetically definable, defined by a $\Sigma^0_3$-formula. The above proof of Lemma 2 is formalizable in ACA0 with the assumption $WO(g^{(\Lambda)})$.

**Remark 2** We can show one of equivalences due to Girard [4] in the spirit of Rathjen [1, 7, 8]: ACA0 is equivalent to WFP($\lambda X.2^X$) over RCA0, where WFP($g$) : $\iff \forall X (WF(X) \rightarrow WF(g(X)))$ with WF($X$) : $\iff \forall Y TI[<_X, Y]$. The direction ACA0 $\rightarrow$ WFP($\lambda X.2^X$) is well known. The reverse direction is seen as follows. Consider the proof search of the contradiction in a sequent calculus $G(Q) + (Jcut) + (J)$. Pick a fresh unary predicate symbol $J$. Let $\exists x B(x, y)$ be a
fixed $\Sigma^0_1$-formula. $G(\mathcal{Q}) + (Jcut) + (J)$ is obtained from the sequent calculus $G(\mathcal{Q})$ by adding the following three inference rules $(Jcut), (J), (\bar{J})$.

\[
\frac{\Gamma, J(n) \quad \bar{J}(n) \quad \Gamma}{\Gamma} \quad (Jcut) \quad \frac{\Gamma, J(n) \quad \exists x B(x, n) \quad \Gamma, J(n)}{\Gamma} \quad (J) \quad \frac{\Gamma, \bar{J}(n) \quad \forall x \neg B(x, n) \quad \Gamma, \bar{J}(n)}{\Gamma} \quad (\bar{J})
\]

$J(n)$ is intended to denote $\exists x B(x, n)$. If the tree in the proof search is not well founded, then an infinite path through the tree yields a set $\mathcal{J}$ such that $\forall n [n \in \mathcal{J} \iff \exists x B(x, n)]$. Thus ACA$_0$ follows. Suppose contrarily that the tree is well founded, and let $\Lambda$ be the depth of the well founded tree. Then a cut elimination yields a cut-free derivation of the empty sequent in $G(\mathcal{Q})$ in depth $2^c(\Lambda)$ for a constant $c$ depending only on the $\Delta^0_0$-formula $B$. From WFP($\lambda X. 2^X$) we see that the cut-free derivation is well founded, and this is not the case.

### 5 Common fixed points

Let $\alpha$ be the order type of a computable well ordering on $\mathbb{N}$. $\varphi[g]_{\alpha}(\beta)$ denotes the $\alpha$-th Veblen function starting with $\varphi[g]_0 \beta = g(\beta)$.

We assume that $\varphi[g]_{\alpha}(X)$ is a term structure over constants $\{\varphi[g]_{\alpha}(c) : c \in X \cup \{0\}\}$ and unary function symbols $\varphi[g]_{\beta}(\beta < \alpha)$ and the addition $+$. Also a function symbol for the exponential $\omega^\beta$ is included when $\varphi[g]_0 = g$ is not the exponential.

In what follows we assume that each term structure $\varphi[g]_{\beta}(X) (\beta < \alpha)$ is extendible, and $\varphi[g]_{\alpha}$ is exponential. Moreover we assume that the following properties are provable in RCA$_0$, cf. (6).

\[
\begin{align*}
\beta_1, \ldots, \beta_n \varphi[g]_{\alpha}(X) & \varphi[g]_{\alpha}(c) \rightarrow f(\beta_1, \ldots, \beta_n) <_{\varphi[g]_{\alpha}(X)} \varphi[g]_{\alpha}(c) \\
(f \in \{\varphi[g]_{\beta} : \beta < \alpha\} \cup \{+\}) \\
\omega^\varphi[g]_{\alpha}(c) &= \varphi[g]_{\beta}(\varphi[g]_{\alpha}(c)) = \varphi[g]_{\alpha}(c) (\beta < \alpha) \\
\varphi[g]_{\alpha}(0) &= \sup\{(\varphi[g]_{\beta})^n(0) : \beta < \alpha, n \in \omega\} \\
\varphi[g]_{\alpha}(c + 1) &= \sup\{(\varphi[g]_{\beta})^n(\varphi[g]_{\alpha}(c) + 1) : \beta < \alpha, n \in \omega\}
\end{align*}
\]

(10)

We see the following as in Proposition 1.

**Proposition 2** Suppose $\varphi[g]_{\beta}(X)$ is an extendible term structure. Then the following is provable in RCA$_0$; let both $X$ and $Y$ be linear orderings.

Let $f : \{0\} \cup X \rightarrow \{0\} \cup Y$ be an order preserving map, $n <_X m \Rightarrow f(n) <_Y f(m), n, m \in \{0\} \cup X$. Then there is an order preserving map $F : \varphi[g]_{\beta}(X) \rightarrow \varphi[g]_{\beta}(Y), n <_{\varphi[g]_{\beta}(X)} m \Rightarrow F(n) <_{\varphi[g]_{\beta}(Y)} F(m)$.

**Theorem 6** Let each term structure $\varphi[g]_{\beta}(X) (\beta < \alpha)$ be extendible, and $\varphi[g]_{\alpha}$ is exponential for which (10) holds. Then the following two are mutually equivalent over ACA$_0 +$ LO($\alpha$) for $\alpha > 0$. 

\[\square\ Springer\]
1. WOP($\varphi[g]_\alpha$).
2. $(\forall \beta < \alpha \text{WOP}(\varphi[g]_\beta))^+$.

The easier direction states that WOP($\varphi[g]_\alpha$) follows from $(\forall \beta < \alpha \text{WOP}(\varphi[g]_\beta))^+$, and it is seen as for the easy half of Theorem 4 in Sect. 2 using the fact (10).

The harder direction is seen as in Theorem 4 by slight modifications. Suppose WOP($\varphi[g]_\alpha$) and LO($\alpha$) for $\alpha > 0$. Replace the inference rule $(W)_i$ by

$$\Gamma, \text{LO}(\prec_i) \vdash \forall x. E_i(x) \exists Y. \neg \text{TI}(\prec_i, \beta, Y), \Gamma \vdash (W)_i, \beta,$$

where $\beta < \alpha$ and $n \prec_i, \beta m \iff n < \varphi[g]_\beta(\prec_i) m$.

Construct fairly a tree in the sequent calculus $G(\varnothing) + (prg) + \{(W)_i, \beta\}_{i \in \omega, \beta < \alpha}$ ending with the empty sequent. When the tree is not well founded, an infinite path through the tree yields a countable coded $\omega$-model of $(\forall \beta < \alpha \text{WOP}(\varphi[g]_\beta))$. Suppose that the search tree $T$ is well founded with its order type $\Lambda$ in the Kleene-Brouwer ordering. We obtain WO($\varphi[g]_\alpha(\Lambda)$) by WOP($\varphi[g]_\alpha$). As in Lemma 2 we see the following Lemma 3 from Proposition 2 and (10):

$c, d < b \& \beta < \alpha \Rightarrow \omega_k(\varphi[g]_\beta(\omega_{\varphi[g]_\alpha(c)}(d)) < \varphi[g]_\alpha(b)$

**Lemma 3** For an $E$-sequent $\Delta$ and an $E$-free sequent $\Gamma$, if $\vdash^b \Delta, \Gamma$ for $b < \Lambda$, then $\vdash^0 \varphi[g]_\alpha(b) \Delta, \Gamma$.

The harder direction in Theorem 6 is concluded as follows. By our assumption we have $\vdash^b \emptyset$ for a $b < \Lambda$ and the empty sequent $\emptyset$. Lemma 3 yields $\vdash^0 \varphi[g]_\alpha(b) \emptyset$. We see that this is not the case by induction on $\varphi[g]_\alpha(b) < \varphi[g]_\alpha(\Lambda)$. Therefore the tree $T$ is not well founded.

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