A Novel Transformation Approach of Shared-link Coded Caching Schemes for Multiaccess Networks

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Abstract

Coded caching is a promising technique to smooth the network traffic by storing parts of a content library at the users’ caches. The original coded caching scheme was proposed by Maddah-Ali and Niesen (MN) for shared-link caching systems, which is referred to as MN scheme. With combinatorial cache placement and delivery phases, the MN scheme provides an additional coded caching gain compared to the conventional uncoded caching scheme. In this paper, we consider the multiaccess caching systems formulated by Hachem et al., including a central server containing \( N \) files connected to \( K \) cache-less users through an error-free shared link, and \( K \) cache-nodes, each equipped with a cache memory size of \( M \) files. Each user has access to \( L \) neighbouring cache-nodes with a cyclic wrap-around topology. The coded caching scheme proposed by Hachem et al. suffers from the case that \( L \) does not divide \( K \), where the needed number of transmissions (a.k.a. load) is at most four times the load expression for the case where \( L \) divides \( K \). Our main contribution is to propose a novel transformation approach to smartly extend the MN scheme to the multiaccess caching systems, such that the load expression of the scheme by Hachem et al. for the case where \( L \) divides \( K \), remains achievable in full generality. The resulting scheme has the maximum local caching gain (i.e., the cached contents stored at any \( L \) neighbouring cache-nodes are different such that each user can totally retrieve \( LM \) files from the connected cache-nodes) and the same coded caching gain as the related MN scheme. Moreover, our transformation approach can also be used to extend other coded caching schemes (satisfying some constraints) for the original MN caching systems to the multiaccess systems, such that the resulting scheme achieves the maximum local caching gain and the same coded caching gain as the considered caching scheme. Finally we also show The resulting scheme is also able to reduce the loads of all the existing caching schemes for the multiaccess caching systems. Finally for some PDAs, the transmission load of our new multiaccess coded caching schemes in delivery phase can be reduced by further compressing the transmitted multicast messages.

Index Terms

Coded caching, multiaccess networks, placement delivery array.

I. INTRODUCTION

WIRELESS networks are increasingly under stress because of the ever-increasing large number of wireless devices consuming high-quality multimedia content (e.g., on-demand video streaming). The high temporal variability of network traffic results in congestions during the peak traffic times and underutilization of the network during off-peak traffic times. Caching can effectively shift traffic from peak to off-peak times [1], by storing fractions of popular content at users’ local memories during the peak traffic times, such that users can be partly served from their local caches, thereby reducing the network traffic.

Coded caching was originally proposed by Maddah-Ali and Niesen (MN) in [2] for the shared-link broadcast network, where a single server with access to a library containing \( N \) equal-length files is connected to \( K \) users through a shared link. Each of the \( K \) users has a cache memory which can store up to \( M \) files. A coded caching scheme operates in two phases. In the placement phase, the server populates the users’ caches without knowledge of future demands. In the delivery phase, each user demands one file. According to users’ demands and caches, the server broadcasts coded messages through the shared link to all users such that each user’s demand is satisfied. The goal is minimize the worst-case number of transmissions normalized by the file size (referred to as worst-case load or just load) in the delivery phase among all possible demands. The MN coded caching scheme uses a combinatorial design in the placement phase, such that in the delivery phase each message broadcasted by the server can simultaneously satisfy multiple users’ demands. The achieved load of the MN scheme is

\[
R_{MN} = \frac{K(1 - M/N)}{KM/N + 1}, \quad \forall M = \frac{Nt}{K} : t \in \{0, 1, \ldots, [K]\}.
\]

When \( N \geq K \), the MN scheme was showed to be optimal under uncoded placement (i.e., each user directly copies some bits of files in its cache) [3] and to be generally order optimal within a factor of 2 [4].

A main drawback of the MN scheme is its high subpacketization level. The first coded caching scheme with reduced subpacketization level was proposed in [5] with a grouping strategy. The authors in [6] proposed a class of caching schemes based on the concept of placement delivery array (PDA), and showed that the MN scheme is a special PDA, referred to

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as MN PDA. Based on the concept of PDA, various caching schemes were proposed, e.g., [6]–[13], in order to reduce the subpacketization level of the MN scheme.

A. Multiaccess caching

Caching at the wireless edge nodes is a promising way to boost the spatial and spectral efficiency, for the sake of reducing communication cost of wireless systems [14], [15]. Edge caches enable storing Internet-based content, including web objects, videos and software updates. Compared to the end-user-caches which are heavily limited by the storage size of each user’s device (e.g., a library size is usually much larger than the storage size of a mobile device) and only useful to one user, the caches at edge devices such as local helpers, mobile edge computing (MEC) servers and hotspots, usually have much larger storage sizes and can be accessed by a number of users. In this paper, we consider an ideal coded edge caching model, referred to as multiaccess caching model (illustrated in Fig. 1), which was originally introduced in [16]. Different from the original MN caching model, in the multiaccess caching system, there are \( K \) cache-nodes each of storage \( M \) files, and \( K \) cache-less users each of which can access \( L \) cache-nodes in a cyclic wrap-around fashion, while also receiving the broadcast transmission directly from the server. As assumed in [16], the cache-nodes in the considered model are accessed at no load cost; that is, we count only the broadcasted load of the server. This can be justified by the fact that the access to the cache-nodes can be offloaded on a different network (e.g., a wifi offloading, for hotspots).

A multiaccess coded caching scheme was proposed in [16], which deals with the following two cases separately.

- When \( L \) divides \( K \) (i.e., \( L|K \)), the authors in [16] proposed a scheme that divides the \( K \) users into \( L \) groups and divides each file into \( L \) subfiles. The MN scheme is then used in each user group with \( N \) subfiles. This results in a total load equal to

\[
R_{HKD} = \frac{K(1 - LM/N)}{KM/N + 1}, \quad \forall M = \frac{Nt}{K} : t \in \left\{ 0, 1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor \right\}.
\]

(2)

In this case, the multiplicative gap between the resulting scheme and the MN scheme (with memory ratio \( \frac{LM}{N} \)) is at most \( L \).

- When \( L \) does not divide \( K \), the authors showed that a load which is three times larger than the one in (2) can be achieved. The authors in [17] proposed an new scheme based on erasure-correcting code which has lower load than the scheme in [16] when \( K < \frac{KLM}{N} \). In addition, when \( L \geq K^2 \), by extending the converse bound in [8], a converse bound under uncoded cache placement was also proposed in [17]. Under the constraint of \( N \geq K, L \geq K^2 \), and uncoded cache placement, the scheme in [17] was shown to be order optimal within a factor of 2. For some special parameters of the multiaccess coded caching problem, some improved schemes with reduced load compared to the scheme in [17] were proposed in [18]–[20]. The authors in [21] extended the the linear topology multiaccess coded caching scheme [16] to two-dimensional cellular networks, where the mobile users move on a two-dimensional grid and can access the nearest cache-nodes located at the grid.

Notice that if each user has an individual cache of size \( LM \), then the MN scheme would achieve the load \( \frac{K(1 - LM/N)}{KLM/N + 1} \). Instead, if each user has an individual cache of size \( M \), then the MN scheme would achieve the load in (1). Therefore, the multiaccess case model here where each user accesses to \( L \) neighbouring cache-nodes, yields an intermediate performance in between the two extremes.

Fig. 1: The \((K, L, M, N)\) multiaccess coded caching system.
B. Contribution and paper organization

This paper focuses on the multiaccess caching system in [16]. Interestingly, we show that the MN PDA can be used to construct a multiaccess coded caching scheme by a novel three-step transformation. The load of the resulting scheme achieves the same load as in (2) for any system parameters, i.e., without constraint that \( L \) divides \( K \) in [16]. Our transformation approach can also extend other PDAs (e.g., the ones in [6], [9], [11]) to the multiaccess caching system, as long as the considered caching schemes satisfy some constraints (given in Proposition 1). Finally for some PDAs, we can reduce the transmission load of our new schemes by further compressing the transmitted multicast messages in delivery phase.

The rest of this paper is organized as follows. The multiaccess coded caching model and some preliminary results are introduced in Section [II]. The main results of this paper are listed in Section [III]. Section [IV] provides the detailed construction of the proposed caching scheme. Section [V] describes that the load of the constructed caching scheme can be further reduced. Finally, we conclude the paper in Section [VI] and some proofs are provided in the Appendices.

Notations

In this paper, we use the following notations unless otherwise stated.

- Bold capital letter, bold lower case letter and curlicue font will be used to denote array, vector and set respectively. We assume that all the sets are increasing ordered and \( | \cdot | \) is used to represent the cardinality of a set or the length of a vector;
- For any positive integers \( a, b, t \) with \( a < b \) and \( t \leq b \), and any nonnegative set \( \mathcal{V} \),
  - \( \{a, b\} = \{a, a+1, \ldots, b\} \), especially \( [1, b] \) be shorten by \( [b] \), and \( \binom{b}{t} = \{\mathcal{V} | \mathcal{V} \subseteq [b], |\mathcal{V}| = t\} \), i.e., \( \binom{b}{t} \) is the collection of all \( t \)-sized subsets of \([b]\);
  - \( \text{Mod}(b, a) \) represents the modulo operation on positive integer \( b \) with positive integer divisor \( a \). In this paper we let \( \text{Mod}(b, a) = \{1, \ldots, a\} \) (i.e., we let \( \text{Mod}(b, a) = a \) if \( a \) divides \( b \)). \( \mathcal{V}[h] \) represent the \( h \)-th smallest element of \( \mathcal{V} \), where \( h \in [||\mathcal{V}||] \);
  - \( \text{Mod}(\mathcal{V}, a) = \{\text{Mod}(\mathcal{V}[h], a) : h \in [||\mathcal{V}||]\} \);
  - \( \mathcal{V} + a = \{\mathcal{V}[h] + a : h \in [||\mathcal{V}||]\} \).

II. System Model and Related Works

In this section, we first introduce the original MN caching model in [2], and the MN caching scheme from the viewpoint of placement delivery array (PDA) [6]. Then the multiaccess caching model and previously known results are introduced.

A. The original caching model

In the shared-link coded caching system [2], a server containing \( N \) files with equal length in \( \mathcal{W} = \{W_1, W_2, \ldots, W_N\} \) connects through an error-free shared link to \( K \) users in \( \{U_1, U_2, \ldots, U_K\} \) with \( K \leq N \), and every user has a cache which can store up to \( M \) files for \( 0 \leq M \leq N \). An \( F \)-division \((K, M, N)\) coded caching scheme contains two phases.

- Placement phase: Each file is divided into \( F \) packets with equal size\(^2\) and then each user \( U_k \) where \( k \in [K] \) caches some packets of each file, which is limited by its cache size \( M \). Let \( Z_{U_k} \) denote the cache contents at user \( U_k \), which is assumed to be known to the server. Notice that the placement phase is done without knowledge of later requests.

- Delivery phase: Each user randomly requests one file from the server. The requested file by user \( U_k \) is represented by \( W_{d_{U_k}} \), and the request vector by all users is denoted by \( \mathbf{d} = (d_{U_1}, d_{U_2}, \ldots, d_{U_K}) \). According to the cached contents and requests of all the users, the server transmits a broadcast message including \( S_d \) packets to all users, such that each user’s request can be satisfied.

In such system, the number of worst-case transmitted files (a.k.a. load) for all possible requests is expected to be as small as possible, which is defined as

\[
R = \max_{\mathbf{d} \in [N]^K} \frac{S_d}{F}.
\]

For the shared-link coded caching problem, the authors in [6] proposed a class of solutions based on the concept of placement delivery array (PDA) which is defined as follows.

Definition 1. ([6]) For positive integers \( K, F, Z \) and \( S \), an \( F \times K \) array \( \mathbf{P} = (p_{j,k})_{j \in [F], k \in [K]} \), composed of a specific symbol “*” and \( S \) positive integers from \([S]\), is called a \((K, F, Z, S)\) PDA if it satisfies the following conditions:

C1. The symbol “*” appears \( Z \) times in each column;

C2. Each integer occurs at least once in the array;

C3. For any two distinct entries \( p_{j_1, k_1} \) and \( p_{j_2, k_2} \), \( p_{j_1, k_1} = p_{j_2, k_2} = s \) is an integer only if

a. \( j_1 \neq j_2, k_1 \neq k_2 \), i.e., they lie in distinct rows and distinct columns; and

\(^2\)In this paper, we only consider the uncoded cache placement.
b. \( p_{j_1,k_2} = p_{j_2,k_1} = \ast \), i.e., the corresponding \( 2 \times 2 \) subarray formed by rows \( j_1, j_2 \) and columns \( k_1, k_2 \) must be of the following form
\[
\begin{pmatrix}
  s & \ast \\
  \ast & s
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
  \ast & s \\
  s & \ast
\end{pmatrix}.
\]

In a \((K, F, Z, S)\) PDA \( \mathcal{P} \), each column represents one user’s cached contents, i.e., if \( p_{j,k} = \ast \), then user \( U_k \) has cached the \( j^{th} \) packet of all the files. If \( p_{j,k} = s \) is an integer, it means that the \( j^{th} \) packet of all the files is not stored by user \( U_k \). Then the XOR of the requested packets indicated by \( s \) is broadcasted by the server at time slot \( s \). The property C2 of Definition 1 implies that the number of signals transmitted by the server is exactly \( S \). So the load is \( R = \frac{S}{T} \). Finally the property C3 of Definition 1 guarantees that each user can get the requested packet, since it has cached all the other packets in the signal except its requested one. Hence, the following lemma was proved by Yan et al. in [6].

Lemma 1. (6) Using Algorithm 1 an \( F \)-division caching scheme for a \((K, M, N)\) caching system can be realized by a \((K, F, Z, S)\) PDA with \( \frac{M}{N} = \frac{2}{4} \). Each user can decode his requested file correctly for any request \( d \) at the rate \( R = \frac{S}{T} \). □

Algorithm 1: PDA Coded caching scheme for original MN caching model based on PDA in [4]

1. procedure PLACEMENT(\( \mathcal{P}, \mathcal{W} \))
2. Split each file \( W_n \in \mathcal{W} \) into \( F \) packets, i.e., \( W_n = (W_n,j : j \in [F]) \).
3. for \( k \in [K] \) do
4. \( Z_{U_k} \leftarrow \{ W_n,j : p_{j,k} = \ast, \forall n \in [N] \} \)
5. end for
6. end procedure
7. procedure DELIVERY(\( \mathcal{P}, \mathcal{W}, d \))
8. for \( s = 1, 2, \ldots, S \) do
9. Server sends \( \bigoplus_{p_{j,k} = s, j \in [F], k \in [K]} W_{dU_k,j} \).
10. end for
11. end procedure

Let us briefly introduce the MN coded caching scheme in [2] from the viewpoint of MN PDA, where the resulting PDA is referred to as MN PDA. For any integer \( t \in [K] \), we let \( F = \left( \begin{smallmatrix} K \\ t \end{smallmatrix} \right) \). We arrange all the subsets with size \( t + 1 \) of \([K]\) in the lexicographic order and define \( \phi(S) \) to be its order for any subset \( S \) of size \( t + 1 \). Clearly, \( \phi \) is a bijection from \( \left( \begin{smallmatrix} K \\ t+1 \end{smallmatrix} \right) \) to \( \left( \begin{smallmatrix} K \\ t \end{smallmatrix} \right) \). Then, an MN PDA is defined as a \( \left( \begin{smallmatrix} K \\ t \end{smallmatrix} \right) \times K \) array \( \mathcal{P} = (p_{T,k})_{T \in [K], k \in [K]} \) by
\[
p_{T,k} = \begin{cases} 
\phi(T \cup \{k\}), & \text{if } k \notin T \\
\ast, & \text{otherwise}
\end{cases}
\]
where the rows are labelled by all the subsets \( T \in [K] \). Thus, by the definition of PDA, the achieved load of the MN PDA is as follows.

Lemma 2. (MN PDA [2]) For any positive integers \( K \) and \( t \) with \( t < K \), there exists a \( \left( \begin{smallmatrix} K \\ t \end{smallmatrix} \right), \left( \begin{smallmatrix} K-t \end{smallmatrix} \right), \left( \begin{smallmatrix} K-t+1 \end{smallmatrix} \right) \) PDA, which gives a \( \left( \begin{smallmatrix} K \\ t \end{smallmatrix} \right) \)-division \((K, M, N)\) coded caching scheme for original MN caching model with memory ratio \( \frac{M}{N} = \frac{K-t+1}{t+1} \) and load \( R = \frac{K-t+1}{t+1} \).

Example 1. We then illustrate the MN PDA by the following example where \( N = K = 4 \) and \( M = 2 \). By the construction of the MN PDA, we have the following \((4, 6, 3, 4)\) PDA.

\[
P = \begin{pmatrix}
  \ast & 1 & 2 \\
  1 & \ast & 3 \\
  2 & 3 & \ast \\
  1 & \ast & 4 \\
  2 & 4 & \ast \\
  3 & 4 & \ast
\end{pmatrix}.
\]

Using Algorithm 1 the detailed caching scheme is as follows.

- **Placement Phase:** From Line 2 of the algorithm we have \( W_n = (W_{n,1}, W_{n,2}, W_{n,3}, W_{n,4}, W_{n,5}, W_{n,6}) \) where \( n \in [4] \). Then by Lines 3-5 in Algorithm 1 the users’ caches are

\[
Z_{U_1} = \{ W_{n,1}, W_{n,2}, W_{n,3} : n \in [4] \}; \quad Z_{U_2} = \{ W_{n,1}, W_{n,4}, W_{n,5} : n \in [4] \}; \\
Z_{U_3} = \{ W_{n,2}, W_{n,4}, W_{n,6} : n \in [4] \}; \quad Z_{U_4} = \{ W_{n,3}, W_{n,5}, W_{n,6} : n \in [4] \}.
\]
Transmitted Signal

| Time Slot | W_{1,4} \oplus W_{2,2} \oplus W_{3,1} |
|-----------|----------------------------------|
| 2         | W_{1,5} \oplus W_{2,3} \oplus W_{4,1} |
| 3         | W_{1,6} \oplus W_{3,3} \oplus W_{4,2} |
| 4         | W_{2,6} \oplus W_{3,5} \oplus W_{4,4} |

TABLE I: Delivery steps in Example 1

- **Delivery Phase**: Assume that the request vector is $d = (1, 2, 3, 4)$. By the transmitting process by Lines 8-10 in Algorithm 1, the server transmits the multicast messages in Table I with total load $R = \frac{4}{3} = \frac{4}{3}$.

\[ \square \]

**B. Multiaccess coded caching model**

We then introduce the $(K, L, M, N)$ multiaccess coded caching problem in [16] (as illustrated in Fig. 1), containing a server with a set of $N$ equal-length files (denoted by $\mathcal{W} = \{W_1, \ldots, W_N\}$), $K$ cache-nodes (denoted by $C_1, \ldots, C_K$), and $K \leq N$ users (denoted by $U_1, \ldots, U_K$). Each cache-node has a memory size of $M$ files where $0 \leq M \leq \frac{N}{K}$. Each user is connected to $L$ neighbouring cache-nodes in a cyclic wrap-around fashion. Each user is also connected via an error-free shared link to the server. In this paper, we assume that the communication bottleneck is on the shared link from the server to the users; thus we assume that each user can retrieve the cached contents from its connected cache-nodes without any cost.

An $F$-division $(K, L, M, N)$ multiaccess coded caching scheme runs in two phases,

- **Placement phase**: each file is divided into $F$ packets of equal size, and then each cache-node $C_k$ where $k \in [K]$, directly caches some packets of each file, which is limited by its cache size $M$. Let $\mathcal{Z}_{C_k}$ denote the cache contents at cache-node $C_k$. The placement phase is also done without knowledge of later requests. Each user $U_k$ where $k \in [K]$ can retrieve the contents cached at the $L$ neighbouring cache-nodes in a cyclic wrap-around fashion. Let $\mathcal{Z}_{U_k}$ denote the retrievable cache contents by user $U_k$.
- **Delivery phase**: each user randomly requests one file. According to the request vector $d = (d_1, d_2, \ldots, d_K)$ and the cached contents in the cache-nodes, the server transmits $S_d$ multicast messages to all users, such that each user’s request can be satisfied.

We aim to design a multiaccess coded caching scheme with minimum worst-case load as defined in (3). The first multiaccess coded caching scheme was proposed in [16], where the following result was proved.

**Lemma 3 ([16])**. For the $(K, L, M, N)$ multiaccess coded caching problem,

- when $L|K$, the lower convex envelope of the memory-load tradeoff points

  \[ (M, R_{\text{HKD}}) = \left( \frac{Nt}{K}, \frac{K - tL}{t + 1} \right), \forall t \in \left\{ 0, 1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor \right\} \quad (6) \]

  and $(M, R_{\text{HKD}}) = \left( \frac{N}{K}, 0 \right)$, is achievable.

- when $L \nmid K$, the lower convex envelope of the memory-load tradeoff points

  \[ (M, R_{\text{HKD}}) = \left( \frac{Nt}{K}, \frac{K - tL}{t + 1} \right), \forall t \in \left\{ 0, 1, \ldots, \left\lfloor \frac{K}{2L} \right\rfloor \right\} \quad (7) \]

  and $(M, R_{\text{HKD}}) = \left( \frac{N}{K}, 0 \right)$, is achievable.

\[ \square \]

When $L|K$, the authors in [16] showed that the multiplicative gap between the $R_{\text{HKD}}$ and the MN scheme with memory ratio $\frac{LN}{K}$ is at most $L$. Based on Minimum Distance Separable (MDS) codes, the authors in [17] proposed the following improved scheme.

**Lemma 4 ([17])**. For the $(K, L, M, N)$ multiaccess coded caching problem, the lower convex envelope of the memory-load tradeoff points

\[ (M, R_{\text{RK}}) = \left( \frac{Nt}{K}, \frac{(K - tL)^2}{K} \right), \forall t \in \left\{ 0, 1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor \right\} \quad (8) \]

is achievable.

\[ \square \]

The load in (8) was shown in [17] to be optimal under uncoded cache placement when $L = K - 1; L = K - 2; L = K - 3$ for $K$ is even. The authors in [18] proposed a scheme with subachetization level $F = K(K - 1)$ achieving the following load.
Lemma 5 ([18]). For the \((K, L, M, N)\) multiaccess coded caching problem, the lower convex envelope of the memory-load tradeoff points \(\left(\frac{Nt}{K}, R_{SR}\right)\) for all \(t \in \{0, 1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor\}\) and \(\left(\frac{N}{L}, 0\right)\), is achievable, where

- \(R_{SR} = \frac{1}{K} R_{SR}\) if \(K - 1 = tL\);
- \(R_{SR} = \sum_{h=0}^{K-tL-1} \frac{tL}{1+\left(\frac{K}{L}\right)}\) if \(K - tL\) is even;
- \(R_{SR} = \frac{1}{K}\sum_{h=0}^{K-tL-1} \frac{tL}{1+\left(\frac{K}{L}\right)}\) if \(K - tL > 1\) is odd.

\[\square\]

When \(M = \frac{2N}{K}\) (i.e., \(t = \frac{KM}{N} = 2\)), the authors in [19] proposed a caching scheme with load \(\frac{K - tL}{g}\) where \(g > t + 1 = 2\), which is strictly lower than the load in [6]. When \(M = \frac{N}{K}\), the authors in [20] proposed a caching scheme with linear subpacketization level, but achieving a higher load than the above schemes when \(L < \frac{K}{2}\).

III. MAIN RESULTS

In this section, by a non-trivial transformation from the MN PDA, we propose a novel multiaccess coded caching scheme, which achieves the same load as the scheme in [16] but for any parameters \(K\) and \(L\).

A. Main results and performance analyses

The performance of the new proposed scheme is given the following, whose detailed description could be found in Section IV.

Theorem 1. For the \((K, L, M, N)\) multiaccess coded caching problem, the lower convex envelope of the following memory-load tradeoff corner points are achievable,

\[\left(M_t, R_t\right) = \left(\frac{Nt}{K}, \frac{K - tL}{t + 1}\right), \quad \forall t \in \left\{0, 1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor\right\},\]

and \(\left(M, R_1\right) = \left(\frac{N}{L}, 0\right)\).

\[\square\]

Notice that as the existing schemes in Lemmas 3-5 the non-trivial corner points of the proposed scheme are at the memory sizes \(M = \frac{Nt}{K}\) where \(t \in \{1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor\}\).

We then compare the achieved load by the proposed scheme in Theorem 1 with the existing schemes in Lemmas 3-5.

- Comparison to Lemma 3
  1) When \(L\) divides \(K\), the proposed scheme in Theorem 1 achieves the same load as Lemma 3, i.e., \(R_1 = R_{HKD}\).
  2) When \(L\) does not divide \(K\), we have \(R_{HKD} = \frac{K - \left\lfloor \frac{K}{L} \right\rfloor}{K - L} R_1\) for \(M = \frac{Nt}{K}\) where \(t \in \{0, 1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor\}\). In addition, for \(\frac{N}{K} \left\lfloor \frac{K}{2L} \right\rfloor \leq M < \frac{N}{K}\), we have

\[\frac{R_{HKD}}{R_1} > \frac{K - \left\lfloor \frac{K}{2L} \right\rfloor}{K - L\left\lfloor \frac{K}{2L} \right\rfloor}\]

which will be proved in Appendix A-A.

- Comparison to Lemma 4. For the non-trivial corner points with \(t = \frac{KM}{N} \in \{1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor\}\), it will be proved in Appendix A-B that

\[R_1 < R_{RK}, \quad \text{if} \quad K > (t + 1)L.\]

- Comparison to Lemma 5. Due to the high complexity of the closed-form of the load in Lemma 5, we cannot provide the exact comparison between the proposed scheme and the scheme in Lemma 5. Instead, in Appendix A-C we will show that for the non-trivial corner points with \(t = \frac{KM}{N} \in \{1, \ldots, \left\lfloor \frac{K}{L} \right\rfloor\}\), we have

\[R_1 < R_{SR}, \quad \text{if} \quad K \gg L.\]

\[\Box\]

Remark 1. It will be explained in Section III-B and Section IV, the main novelty of the proposed scheme in Theorem 1 is a smart transformation of the MN PDA for the multiaccess caching problem. By a similar transformation approach, some other existing PDAs for the original MN problem (e.g., the PDAs in [6], [9], [11]) can also be extended to the multiaccess caching problem, as long as the conditions introduced in Proposition [1] are satisfied. Please refer to Remark 3 for a detailed example.

We conclude this subsection with some numerical evaluations to compare the proposed scheme in Theorem 1 and the existing schemes in Lemmas 3-5. In Fig. 2, we consider the multiaccess caching problem with \(K = N = 20, L = 3\), and \(0 \leq \frac{M}{N} \leq \frac{3}{5}\). It can be seen that the proposed scheme performs the best when \(0 \leq \frac{M}{N} \leq \frac{K - L - 1}{K} = \frac{20/3 - 1}{20} \approx 0.28\) by the condition \(K > (t + 1)L\) in [11]. In Fig. 3, we consider the multiaccess caching problem with \(K = N = 30, L = 4, \) and \(0 \leq \frac{M}{N} \leq \frac{1}{4}\). It can be seen that the proposed scheme performs the best when \(0 \leq \frac{M}{N} \leq 0.217\).
B. Sketch of the proposed scheme in Theorem [7]

Let us consider the \((K, L, M, N) = (8, 3, 2, 8)\) multiaccess caching problem. We aim to propose a multiaccess coded caching scheme based on the \((K', F, Z, S) = (4, 6, 3, 4)\) MN PDA, i.e., \(P\) in (5). It will be clear in Section IV that we choose the MN PDA where \(K' = K - \frac{KM}{N}(L - 1)\) and \(K' \frac{Z}{F} = K \frac{M}{N}\). Generally speaking, the main ingredients of the proposed scheme contain two key points:

- According to the multiaccess caching model, in order to fully use the cache-nodes, we design a cache placement such that any two cache-nodes connected to some common user(s) should not cache the same packets. Since each user can access \(L\) cache-nodes, the local caching gain of the proposed scheme is

\[
g_{\text{local}} = 1 - \frac{LM}{N} = \frac{1}{4}.
\]

- The structure of the proposed placement and delivery phases is based on the MN PDA, such that the coded caching gain is the same as the \((4, 6, 3, 4)\) MN PDA, which is equal to

\[
g_{\text{coded}} = \frac{K'Z}{F} + 1 = \frac{KM}{N} + 1 = 3.
\]

Thus the load of the proposed scheme is

\[
K \frac{g_{\text{local}}}{g_{\text{coded}}} = \frac{2}{3}.
\]

which coincides with (9). Then we introduce the more details on the construction. We divide each file into \(K = 8\) subfiles with equal length, \(W_n = \left(W^{(g)}_n : g \in [8]\right)\), and divide the caching procedure into 8 rounds, where in the \(g^{\text{th}}\) round we only consider the \(g^{\text{th}}\) subfile of each file.
Definition 2. For each $g \in [K]$, 

- A $F \times K$ node-placement array $C^{(g)}$ consists of star and null, where $F$ and $K$ represent the subpacketization of each subfile and the number of cache-nodes, respectively. The entry located at the position $(j,k)$ in $C^{(g)}$ is star if and only if the $k^{th}$ cache-node caches the $j^{th}$ packet of each $W^{(g)}_n$ where $n \in [N]$. 
- A $F \times K$ user-retrieve array $U^{(g)}$ consists of star and null, where $F$ and $K$ represent the subpacketization of each subfile and the number of users, respectively. The entry at the position $(j,k)$ in $U^{(g)}$ is star if and only if the $k^{th}$ user can retrieve the $j^{th}$ packet of each $W^{(g)}_n$ where $n \in [N]$. 
- A $F \times K$ user-delivery array $Q^{(g)}$ consists of $\{\ast\} \cup [S]$, where $F$, $K$ and the stars in $Q^{(g)}$ have the same meaning as $F$, $K$ of $U^{(g)}$ and the stars in $U^{(g)}$, respectively. Each integer represents a multicast message, and $S$ represents the total number of multicast messages transmitted in the first round of the delivery phase.

The constructing flow diagram from $P$ to $Q^{(1)}$ (listed in Fig. 4) contains the following three steps:

- **Step 1.** In the first step, we construct the node-placement array $C^{(1)}$ from $P$. More precisely, let us focus on each row $j \in [6]$ of $P$. It can be seen that row $j$ contains two stars, which are assumed to be located at columns $k_1$ and $k_2$, respectively. We let row $j$ of $C^{(1)}$ also contain two stars, where the first star is located at column $k_1 + (L - 1) = k_1 + 2$ and the second star is located at column $k_2 + 2(L - 1) = k_2 + 4$. For example, as the blue line showed in Step 1, the $*$ at position $(1,1)$ of $P$ is put at position $(1,1+2) = (1,3)$ of $C^{(1)}$, and the $*$ at position $(1,2)$ of $P$ is put at position $(1,2+4) = (1,6)$ of $C^{(1)}$. By this construction, any $L$ neighbouring cache-nodes do not cache any common packets.

- **Step 2.** In the second step, we construct the user-retrieve array $U^{(1)}$ from $C^{(1)}$. More precisely, since each user can access $L$ neighboring cache-nodes in a cyclic wrap-around fashion, we put the stars in $U^{(1)}$ according to position of the stars in $C^{(1)}$. In other words, if the entry at the position $(j,k)$ of $C^{(1)}$ is star, then the entry at the position $(j,k)$ of $U^{(1)}$ is set to be star where $0 \leq k - k_1 \leq L - 1 = 2$.

- **Step 3.** In the third step, we construct the user-delivery array $Q^{(1)}$ from $U^{(1)}$. First, we let $Q^{(1)} = U^{(1)}$. Recall that the null entries in the $k^{th}$ column represent the required packets in $W^{(g)}_{d_{U_k}}$ which can not be retrieved by user $k$ from its connected cache-nodes. For example, the entry at the position $(1,7)$ is null, because user $U_7$ requires the first packet of $W^{(g)}_{d_{U_7}}$ which can not be retrieved from its connected cache-nodes $C_7$, $C_8$ and $C_1$. We then fill the null entries in $Q^{(1)}$ by the integers in $P$ following the delivery strategy of the MN PDA. Let us focus on the row $j \in [6]$ of $P$ and $Q^{(1)}$. Notice that row $j$ of $P$ contains two integers, and that row $j$ of $Q^{(1)}$ also contains two nulls. Thus we set the first null to be the first integer and the second null to be the second integer.

**After determining** $Q^{(1)}$, **it can be seen that** $Q^{(1)}$ **satisfies the condition C3 of Definition 1.** Hence, we use the delivery strategy in Line 9 of Algorithm 1. For example, assume that the request vector is $d = (1,2,\ldots,8)$. In the first transmission which corresponds to the integer $1$ in $Q^{(1)}$, the server sends the XOR of the fourth packet of $W^{(1)}_4$ (denoted by $W^{(1)}_{43}$), the second packet of $W^{(1)}_4$ (denoted by $W^{(1)}_{42}$), and the first packet of $W^{(1)}_7$ (denoted by $W^{(1)}_{71}$), i.e., $W^{(1)}_{43} \oplus W^{(1)}_{42} \oplus W^{(1)}_{71}$.

Finally for each $g \in [8]$, the arrays $C^{(g)}$ and $Q^{(g)}$ can be obtained by cyclically right-shifting $C^{(1)}$ and $Q^{(1)}$ by $g - 1$ positions, respectively. After obtaining $C^{(g)}$ and $Q^{(g)}$, the placement and delivery phases in the $g^{th}$ round can be done as above.

Let $C = [C^{(1)};C^{(2)};\cdots;C^{(8)}]$ be the total placement array of the cache-nodes. By the construction, each column of $C$ is a concatenation of all the columns of $C^{(1)}$; thus each column of $C$ has exactly 12 stars. In addition, the array $C$ contains

![Fig. 4: The flow diagram of constructing $C^{(1)}$, $U^{(1)}$ and $Q^{(1)}$ via (4,6,3,4) MN PDA](image-url)
6 \times 8 = 48$ rows. Hence, the needed memory size is $M = \frac{12N}{8} = 2$, satisfying the memory size constraint. The total subpacketization level of the proposed scheme is $48$, which is equal to the number of rows in $C$.

It can be seen from the above example that, if we want to extend some existing PDA (represented by the array $P$) for the original MN caching problem to the multiaccess model by the proposed transformation approach, the PDA $P$ should satisfy that (i) the number of stars in each row of $P$ is the same; (ii) the resulting array $Q^{(1)}$ from $P$ satisfies Condition C3 in Definition 3.

Besides the MN PDA, the PDAs in [6], [9], [11] also satisfy the above constraints. For the above $(K, L, M, N) = (8, 3, 2, 8)$ multiaccess caching problem, based on the $(K', F, Z, S) = (4, 2, 1, 2)$ PDA which can be obtained by either [6] Theorem 4 ($m = 1$ and $q = 2$) or [9] Theorem 3 ($m = 1$ and $q = 2$, $z = 1$), we can obtain a multiaccess coded caching scheme with total subpacketization level $F \times K = 2 \times 8 = 16$ and load $\frac{8}{2} = 4 = 1$. Based on the $(K', F, Z, S) = (4, 4, 2, 4)$ PDA in [11] Theorem 18 ($m = 2$ and $q = 2$, $t = 1$), a multiaccess coded caching scheme with total subpacketization level $F = 4 \times 8 = 32$ and load $\frac{32}{8} = 4 = 1$ can be obtained.

IV. THE PROOF OF THEOREM 1

Let us consider the $(K, L, M, N)$ multiaccess coded caching problem, where $t = \frac{KM}{F}$ is in the set of integers $\{0, 1, \ldots, \lfloor \frac{K}{Z} \rfloor \}$. In other words, the whole library is totally cached $t$ times in the system. Define that $K' = K - t(L - 1)$. For this multiaccess caching problem, we search a $(K', F, Z, S)$ PDA $P = (p_{j,k})_{j \in [F], k \in [K']}^p$ for the original MN caching model where the whole library is also totally cached $t$ times, i.e., $\frac{K'F}{Z} = \frac{KM}{N} = t$. We define

$$A_j = \{k : p_{j,k} = *, k \in [K']\}, \forall j \in [F]$$

as the column label set of $P$ where the entries in $j$-th row are stars. We have $|A_j| = t$ because each packet is cached exactly $t$ times.

For the $(K, L, M, N)$ multiaccess coded caching problem, we divide each file $W_n$ where $n \in [N]$ into $K$ subfiles with equal length, $W_n = \{W_{n,1}^{(1)}, \ldots, W_{n,K}^{(1)}\}$. Denote the set of the $g^a$ subfiles by $W^{(g)} = \{W_{1,1}^{(g)}, \ldots, W_{N,1}^{(g)}\}$, for each $g \in [K]$. As shown in the sketch of the proof in Section II-B we divide the whole caching procedure into $K$ separate rounds, where in the $g^a$ round we only deal with $W^{(g)}$. In the $g^a$ round where $g \in [K]$, our construction consists of three steps: the generations for the node-placement array $C^{(g)}$, the user-retrieve array $U^{(g)}$, and the user-delivery array $Q^{(g)}$, where the definitions of these three arrays are given in Definition 2.

A. Caching strategy for cache-nodes: Generation of $C^{(g)}$ for $g \in [K]$

The main objective in this step is that each L neighbouring cache-nodes do not cache any common packet. Let us first consider the case where $g = 1$. The detailed placement is as follows, which is based on a $(K', F, Z, S)$ PDA $P = (p_{j,k})_{j \in [F], k \in [K']}^p$ for the original MN caching model with $K' = K - t(L - 1)$ and $\frac{K'F}{Z} = \frac{KM}{N}$.

We divide each subfile $W_{n,1}^{(1)}$ where $n \in [N]$ into $F$ equal-length packets, $W_{n,1}^{(1)} = \{W_{n,1,1}^{(1)}, \ldots, W_{n,1,F}^{(1)}\}$. Then each cache-node $C_k$ where $k \in [K]$ caches the following set of packets (recall that $\text{Mod}(b, a) \in \{1, \ldots, a\}$),

$$C_{j}^{(1)} = \{W_{n,j}^{(1)} : k \in C_j^{(1)}, j \in [F], n \in [N]\},$$

where $C_j^{(1)} = \text{Mod}(|A_j| + h(L - 1) : h \in [|t|], K)$. \hfill (14)

Notice that $c_j^{(1)}$ represents the set of cache-nodes which cache the $j$-th packet of each subfile in $W^{(1)}$. As the meaning of stars in PDA, we use the following $F \times K$ node-placement array $C^{(1)} = (c_{j,k})_{j \in [F], k \in [K]}$ where

$$C_{j,k}^{(1)} = \begin{cases} * & \text{if } k \in C_j^{(1)} \\ \text{null} & \text{otherwise} \end{cases},$$

and each entry at the position $(j, k)$ of $C^{(1)}$ is star if and only if the packets $W_{n,j}^{(1)}$ for all $n \in [N]$ are cached by cache-node $C_k$. By the above construction, for any two distinct integers $h, h' \in [|t|]$ and for any $j \in [F], g \in [K]$, the following inequality

$$|C_j^{(1)}[h] - C_j^{(1)}[h']| = |A_j[h] - A_j[h'] + (h - h')(L - 1)| \geq L \hfill (17)$$

holds. Thus any L neighbouring cache-nodes do not cache any common packet.

After determining $C^{(1)}$, we can obtain $C^{(g)}$ where $g \in [K]$ by simply cyclically right-shifting $C^{(1)}$ by $g - 1$ positions. Then we let

$$C = \left[ C^{(1)}; C^{(2)}; \ldots; C^{(K)} \right]$$

3 From Lemmas 4 and 5 when $K \leq tL + 1$ the scheme with minimum load is obtained. So we only need to consider the case $K > tL + 1$. Then the assumption $K' > 0$ always holds when $K > tL + 1$. 

to represent the cached contents of the cache-nodes. Each column of $C$ is the concatenation of all the columns of $C^{(1)}$. In addition, by the constraint, the total number of stars in $(K', F, Z, S)$ PDA equals the total number of stars in $C^{(1)}$. Hence, the number of stars in each column of $C$ is $ZK'$. The total number of packets of each file equals the number of rows of $C$, i.e., $KF$. So the needed memory size of each cache-node is

$$\frac{ZK'}{KF}N = \frac{Z}{F} \cdot \frac{K'}{K}N = M,$$

satisfying the memory size constraint.

**Example 2.** Let us return to the example in Section III-B with $K = N = 8$ and $L = 3$, where the caching procedure is divided into 8 rounds.

In the first round, we consider $W^{(1)} = \{W^{(1)}_1, \ldots, W^{(1)}_8\}$. Each subfile $W^{(1)}_n$ where $n \in [8]$ is divided into 6 packets, i.e., $W^{(1)}_n = (W^{(1)}_{n,1}, W^{(1)}_{n,2}, \ldots, W^{(1)}_{n,6})$. By (13) we have

$$A_1 = \{1, 2\}, \ A_2 = \{1, 3\}, \ A_3 = \{1, 4\}, \ A_4 = \{2, 3\}, \ A_5 = \{2, 4\}, \ A_6 = \{3, 4\}.$$

By (15) we have

$$C^{(1)}_1 = \{3, 6\}, \ C^{(1)}_2 = \{3, 7\}, \ C^{(1)}_3 = \{3, 8\}, \ C^{(1)}_4 = \{4, 7\}, \ C^{(1)}_5 = \{4, 8\}, \ C^{(1)}_6 = \{5, 8\}.$$

Then each cache-node $C_k, k \in [8]$, caches the packets from $W^{(1)}$ as follows by (14).

$$Z^{(1)}_{C_1} = Z^{(1)}_{C_2} = 0;$$
$$Z^{(1)}_{C_3} = \{W^{(1)}_{n,1}, W^{(1)}_{n,2}, W^{(1)}_{n,3} : n \in [8]\};$$
$$Z^{(1)}_{C_4} = \{W^{(1)}_{n,4}, W^{(1)}_{n,5} : n \in [8]\};$$
$$Z^{(1)}_{C_5} = \{W^{(1)}_{n,6} : n \in [8]\};$$
$$Z^{(1)}_{C_6} = \{W^{(1)}_{n,1} : n \in [8]\};$$
$$Z^{(1)}_{C_7} = \{W^{(1)}_{n,2}, W^{(1)}_{n,3} : n \in [8]\};$$
$$Z^{(1)}_{C_8} = \{W^{(1)}_{n,4}, W^{(1)}_{n,5}, W^{(1)}_{n,6} : n \in [8]\}.$$

By (16) the above packets cached by cache-nodes can be represented by the first array $C^{(1)}$ in Table II which is exactly the array $C^{(1)}$ in Fig. 4. To get the array $C^{(2)}$, we right-shift $C^{(1)}$ by one position. The total number of stars in $C^{(1)}$ is 12, which

**TABLE II: Node-placement arrays $C^{(1)}$ and $C^{(2)}$.**

| $n \in [8]$ | Node-placement array $C^{(1)}$ for $W^{(1)}$ | Node-placement array $C^{(2)}$ for $W^{(2)}$ |
|-------------|-----------------------------------------------|-----------------------------------------------|
| $W^{(1)}_1$  | $\ast$                                        | $W^{(2)}_1$                                  |
| $W^{(1)}_2$  | $\ast$                                        | $W^{(2)}_2$                                  |
| $W^{(1)}_3$  | $\ast$                                        | $W^{(2)}_3$                                  |
| $W^{(1)}_4$  | $\ast$                                        | $W^{(2)}_4$                                  |
| $W^{(1)}_5$  | $\ast$                                        | $W^{(2)}_5$                                  |
| $W^{(1)}_6$  | $\ast$                                        | $W^{(2)}_6$                                  |
| $W^{(1)}_7$  | $#1$                                         | $W^{(2)}_7$                                  |
| $W^{(1)}_8$  | $#1$                                         | $W^{(2)}_8$                                  |

is the total number of stars in the $(4, 6, 3, 4)$ PDA $P$ in [5]. Then the number of stars in each column of $C$ is 12. Since $C$ has $6 \times 8 = 48$ rows, the memory ratio of each cache-node is $M/N = 12/48 = 1/4$.

B. The packets retrievable to users: Generation of $U^{(g)}$ for $g \in[K]$

Let us also start with $g = 1$. Since each user is connected to $L$ neighbouring cache-nodes in a cyclic wrap-around fashion, and can retrieve the cached contents in those cache-nodes. Hence, each user $U_k$ where $k \in[K]$ can retrieve the packet $W^{(1)}_{n,j}$ for all $n \in[N]$ if and only if $k$ is an element of the following set

$$U^{(1)}_j = \bigcup_{h \in [t]} \text{Mod}(\{C^{(1)}_j[h] - (L - 1), \ldots, C^{(1)}_j[h]\}, K)$$

$$= \bigcup_{h \in [t]} \text{Mod}(\{A_j[h] + (h - 1)(L - 1), \ldots, A_j[h] + h(L - 1)\}, K).$$  

(18)
Notice that $U_j^{(1)}$ is the set of users who can retrieve the $j$th packet of each subfile in $W^{(1)}$. Then user $U_k$ can retrieve the following packets of $W^{(1)}$,

$$Z_{U_k}^{(1)} = \{ W_{n,j}^{(1)} : k \in U_j^{(1)}, n \in [N] \}. \quad (19)$$

From (17), we showed that any neighbouring cache-nodes do not cache any common packet. Recall that each packet is stored by $t$ cache-nodes and each cache-node is connected to $L$ users. Hence, we have $|U_j^{(1)}| = tL$ for each $j \in [F]$, which means that each packet is retrievable by $tL$ users.

From (19), we can generate the $F \times K$ user-retrieve array $U^{(1)} = (u_{j,k}^{(1)})$ for each $j \in U_k$.

$$u_{j,k}^{(1)} = \begin{cases} * & \text{if } k \in U_j^{(1)} \\ \text{null} & \text{otherwise} \end{cases}. \quad (20)$$

In other words, each entry at position $(j,k)$ of $U^{(1)}$ is star if and only if the packets $W_{n,j}^{(1)}$ for all $n \in [N]$ can be retrieved by user $U_k$.

**Remark 2.** The number of null entries in $j$th row of $U^{(1)}$ equals the number of integer entries in the $j$th row of PDA $P$ since the number of null entries in each $j$th row is $K - nL = K - t(L - 1) - t = K' - t$.

After determining $U^{(1)}$, we can obtain $U^{(g)}$ where $g \in [K]$ by simply cyclically right-shifting $U^{(1)}$ by $g - 1$ positions.

**Example 3.** Let us return to the example in Section III-B. From (18), we have

- $U_1^{(1)} = \{1, 2, 3, 4, 5, 6\}$, $U_2^{(1)} = \{1, 2, 3, 5, 6, 7\}$, $U_3^{(1)} = \{1, 2, 3, 6, 7, 8\}$,
- $U_4^{(1)} = \{2, 3, 4, 6, 7\}$, $U_5^{(1)} = \{2, 3, 4, 6, 7, 8\}$, $U_6^{(1)} = \{3, 4, 5, 6, 7\}$.

From (19), the users can retrieve the packets from $W^{(1)}$ as follows,

- $Z_{U_1}^{(1)} = \{ W_{n,1}^{(1)}, W_{n,2}^{(1)}, W_{n,3}^{(1)} : n \in [8] \}$,
- $Z_{U_2}^{(1)} = \{ W_{n,1}^{(1)}, W_{n,2}^{(1)}, W_{n,3}^{(1)}, W_{n,4}^{(1)}, W_{n,5}^{(1)} : n \in [8] \}$,
- $Z_{U_3}^{(1)} = \{ W_{n,1}^{(1)}, W_{n,2}^{(1)}, W_{n,3}^{(1)}, W_{n,4}^{(1)}, W_{n,5}^{(1)}, W_{n,6}^{(1)} : n \in [8] \}$,
- $Z_{U_4}^{(1)} = \{ W_{n,1}^{(1)}, W_{n,3}^{(1)}, W_{n,4}^{(1)}, W_{n,5}^{(1)}, W_{n,6}^{(1)} : n \in [8] \}$,
- $Z_{U_5}^{(1)} = \{ W_{n,1}^{(1)}, W_{n,2}^{(1)}, W_{n,4}^{(1)}, W_{n,5}^{(1)}, W_{n,6}^{(1)} : n \in [8] \}$,
- $Z_{U_6}^{(1)} = \{ W_{n,1}^{(1)}, W_{n,2}^{(1)}, W_{n,3}^{(1)}, W_{n,4}^{(1)}, W_{n,5}^{(1)}, W_{n,6}^{(1)} : n \in [8] \}$.

Hence, we can generate the user-retrieve array $U^{(1)}$ in Table III which is exactly the array $U^{(1)}$ in Fig. 4. To get the array $U^{(2)}$, we right-shift $U^{(1)}$ by one position. It can be also seen that the number of null entries in each row of $U^{(1)}$ equals the number of integer entries in each row of the $(4, 6, 3, 4)$ PDA $P$.

**TABLE III: User-retrieve arrays $U^{(1)}$ and $U^{(2)}$.**

| $n \in [8]$ | User-retrieve array $U^{(1)}$ for $W^{(1)}$ | User-retrieve array $U^{(2)}$ for $W^{(2)}$ |
|----------------|--------------------------------|--------------------------------|
| $W_{n,1}^{(1)}$ | * * * * * * | $W_{n,1}^{(2)}$ |
| $W_{n,2}^{(1)}$ | * * * * * * | $W_{n,2}^{(2)}$ |
| $W_{n,3}^{(1)}$ | * * * * * * | $W_{n,3}^{(2)}$ |
| $W_{n,4}^{(1)}$ | * * * * * * | $W_{n,4}^{(2)}$ |
| $W_{n,5}^{(1)}$ | * * * * * * | $W_{n,5}^{(2)}$ |
| $W_{n,6}^{(1)}$ | * * * * * * | $W_{n,6}^{(2)}$ |

**C. Delivery strategy: Generation of $Q^{(g)}$ for $g \in [K]$**

We also start with $g = 1$. By first letting $Q^{(1)} = U^{(1)}$, the next step is to fill the nulls in $Q^{(1)}$ by some integers. From Remark 2, for each $j \in [F]$ we can get a new array $Q^{(1)}$ as follows. We replace the $h$th (where $h \in [K' - t]$) null entry in the $j$th row of $U^{(1)}$ by the $h$th integer entry in the $j$th row of $P$. Specifically, for each $j \in [F]$, define

$$\overline{A}_j = \{ k : p_{j,k} \in [S], k \in [K'] \}, \quad (21)$$
i.e., the column label set of $\mathcal{P}$ where the entries in $j^{th}$ row are integers, and

$$\overline{\mathcal{U}}_j^{(1)} = \{ k : p_{j,k} = \text{null}, k \in [K] \},$$

(22)

i.e., the column label set of $\mathcal{U}^{(1)}$ where the entries in $j^{th}$ row are nulls. From Remark 2 we have

$$|\overline{\mathcal{A}}_j| = |\mathcal{U}_j| = K' - t.$$

We then define a one-to-one mapping $\psi_j$ from $\overline{\mathcal{U}}_j^{(1)}$ to $\overline{\mathcal{A}}_j$:

$$\psi_j(\overline{\mathcal{U}}_j^{(1)}[h]) = \overline{\mathcal{A}}_j[h], \forall h \in [K' - t], j \in [F].$$

Then the entry of the new array $Q^{(1)} = (q^{(1)}_{j,k})_{j \in [F], k \in [K]}$ can be written as follows.

$$q^{(1)}_{j,k} = \begin{cases} s & \text{if } k \in \overline{\mathcal{U}}_j^{(1)} \text{ and } p_{j,\psi_j(k)} = s; \\ * & \text{otherwise.} \end{cases}$$

(23)

Hence, the alphabet set of the resulting $Q^{(1)}$ consists of $|S|$ and symbol star.

After determining $Q^{(1)}$, the delivery procedure in the first round is as follows. From Line 9 of Algorithm 1 for each integer $s \in [S]$ and $g = 1$, the server sends the following multicast message

$$\bigoplus_{q^{(1)}_{j,k}=s,j\in[F],k\in[K]} W^{(1)}_{d_{\psi_j},j},$$

(24)

if the following requirement is satisfied, such that each user can directly decode its required packet from each multicast message transmitted by the server.

**Requirement:** $Q^{(1)}$ defined in (23) satisfies the condition C3 in Definition 1.

For each $g \in [K]$, $\mathcal{U}^{(g)}$ can be obtained by cyclically right-shifting $\mathcal{U}^{(1)}$ by $g - 1$ positions, and then the delivery procedure in the $g^{th}$ round is based on $\mathcal{U}^{(g)}$ as above. It is not difficult to check that if **Requirement** holds for $Q^{(1)}$, we also have that $Q^{(g)}$ satisfies the condition C3 in Definition 1.

Since the server totally sends $SK$ multicast messages of packets in all rounds and each file is divided into $KF$ packets, the achieved load is $R = \frac{SK}{KF} = \frac{S}{K}$. So the following result can be obtained.

**Proposition 1.** For a $(K', F, Z, S)$ PDA $P$ where $K' = K - t(L - 1)$ and $\frac{K'L'}{Z} = \frac{KM}{N}$, if the number of stars in each row of $P$ is $t$ and $Q^{(1)}$ generated in (23) from $P$ satisfies **Requirement**, then there exists a multiaccess coded caching scheme with load $R_1 = \frac{S}{K}$.

Let us then consider the MN PDA, and we have the following lemma whose proof can be found in Appendix B.

**Lemma 6.** If $P$ is the MN PDA, then the $F \times K$ array $Q^{(1)}$ defined in (23) satisfies the condition C3 of Definition 1. \(\square\)

From Lemma 6 when $P$ is a $\left(K', (\frac{K'}{t}), (\frac{K' - 1}{t - 1}), (\frac{K'}{t + 1}) \right)$ MN PDA, from Proposition 1 we can get a multiaccess coded caching scheme for the $(K, L, M, N)$ multiaccess caching problem, with subpacketization level $K(K'/t) = K(\frac{K' - t(L - 1)}{t})$ and load $R_1 = (\frac{K'L'}{K'}) = \frac{K' - t}{K' + t}$. Hence, we proved Theorem 1.

**Example 4.** Let us return to the example in Section III-B again. From (21) and (22) we have

$$\overline{\mathcal{A}}_1 = \{3, 4\}, \quad \overline{\mathcal{A}}_2 = \{2, 4\}, \quad \overline{\mathcal{A}}_3 = \{2, 3\}, \quad \overline{\mathcal{A}}_4 = \{1, 4\}, \quad \overline{\mathcal{A}}_5 = \{1, 3\}, \quad \overline{\mathcal{A}}_6 = \{1, 2\},$$

and

$$\overline{\mathcal{U}}_1^{(1)} = \{7, 8\}, \quad \overline{\mathcal{U}}_2^{(1)} = \{4, 8\}, \quad \overline{\mathcal{U}}_3^{(1)} = \{4, 5\}, \quad \overline{\mathcal{U}}_4^{(1)} = \{1, 8\}, \quad \overline{\mathcal{U}}_5^{(1)} = \{1, 5\}, \quad \overline{\mathcal{U}}_6^{(1)} = \{1, 2\},$$

respectively. Then the following mappings can be obtained.

$$\psi_1(7) = 3, \quad \psi_1(8) = 4, \quad \psi_2(4) = 2, \quad \psi_2(8) = 4, \quad \psi_3(4) = 2, \quad \psi_3(5) = 3, \quad \psi_4(1) = 1, \quad \psi_4(8) = 4, \quad \psi_5(1) = 1, \quad \psi_5(5) = 3, \quad \psi_6(1) = 1, \quad \psi_6(2) = 2.$$ 

From (23), the user-delivery array $Q^{(1)}$ can be obtained in Table V which is exactly the array $Q^{(1)}$ in Fig. 4. To get the array $Q^{(2)}$, we right-shift $Q^{(1)}$ by one position. It can be checked that both of $Q^{(1)}$ and $Q^{(2)}$ satisfy the condition C3 of Definition 1.

Then the server sends the following multicast messages in the first two rounds, listed in Table V. We can easily check that each user can decode its required packets from the multicast messages in Table V. Then the load is $R_1 = \frac{4 \times 8}{2 \times 8} = \frac{2}{3}$. \(\square\)
TABLE IV: User-delivery arrays $Q^{(1)}$ and $Q^{(2)}$.  

| $n \in \mathbb{N}$ | User-delivery array $Q^{(1)}$ for $W^{(1)}$ | User-delivery array $Q^{(2)}$ for $W^{(2)}$ |
|---|---|---|
| 1 | $W_{n,1}^{(1)}$ | $W_{n,1}^{(2)}$ |
| 2 | $W_{n,2}^{(1)}$ | $W_{n,2}^{(2)}$ |
| 3 | $W_{n,3}^{(1)}$ | $W_{n,3}^{(2)}$ |
| 4 | $W_{n,4}^{(1)}$ | $W_{n,4}^{(2)}$ |
| 5 | $W_{n,5}^{(1)}$ | $W_{n,5}^{(2)}$ |
| 6 | $W_{n,6}^{(1)}$ | $W_{n,6}^{(2)}$ |

Remark 3 (Application of other PDAs). From Proposition [1] for any PDA $P$ for the original MN caching model, if $P$ satisfies the following conditions:

- each row of $P$ has the same number of stars;
- the array $Q^{(1)}$ generated in [23] from $P$, satisfies the condition C3 of Definition [1];

then we can use the proposed transformation approach to extend this PDA to the multiaccess caching problem.

Let us consider the $(K = 16, L = 3, M = 4, N = 16)$ multiaccess caching problem and construct the coded caching scheme from the $(K', F, Z, S)$ PDA in [6] Theorem 4 ($m = 3$ and $q = 2$) for the original MN caching problem, where $K' = K - t(L - 1)$ and $K'Z/F = KM/N = t = 4$. In other words, we obtain the following $(8, 8, 4, 8)$ PDA in [6] for the original MN caching problem.

$$P = \begin{pmatrix}
* & 2 & * & 3 & 5 & * & 1 \\
1 & * & 4 & * & 6 & 2 & *
\end{pmatrix}

\begin{pmatrix}
* & 4 & 1 & * & 7 & 3 & *
3 & 2 & * & 8 & 4 & \\
* & 6 & 7 & 1 & 5 & *
5 & 8 & 2 & * & 6 & \\
* & 8 & 5 & 3 & * & 7 & \\
7 & 6 & 4 & * & 8 & *
\end{pmatrix}

$$

It can be seen that each row of $P$ has $t = 4$ stars. In addition, by the definition in [13], we have $|A_1| = \ldots = |A_8| = 4$. From [13], [15], and [18], we have

- $A_1 = \{1, 3, 5, 7\}$, $A_2 = \{2, 3, 5, 8\}$, $A_3 = \{1, 4, 5, 8\}$, $A_4 = \{2, 4, 5, 7\}$, $A_5 = \{1, 3, 6, 8\}$, $A_6 = \{2, 3, 6, 7\}$, $A_7 = \{1, 4, 6, 7\}$, $A_8 = \{2, 4, 6, 8\}$

and

- $C_1^{(1)} = \{3, 7, 11, 15\}$, $C_2^{(1)} = \{4, 7, 11, 16\}$, $C_3^{(1)} = \{3, 8, 11, 16\}$, $C_4^{(1)} = \{4, 8, 11, 15\}$, $C_5^{(1)} = \{3, 7, 12, 16\}$, $C_6^{(1)} = \{4, 7, 12, 15\}$, $C_7^{(1)} = \{3, 8, 12, 15\}$, $C_8^{(1)} = \{4, 8, 12, 16\}$

respectively. From [21] and [22], we have

- $U_1^{(1)} = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15\}$, $U_2^{(1)} = \{2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 15, 16\}$, $U_3^{(1)} = \{1, 2, 3, 6, 7, 8, 9, 10, 11, 14, 15, 16\}$, $U_4^{(1)} = \{2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15\}$, $U_5^{(1)} = \{1, 2, 3, 5, 6, 7, 10, 11, 12, 14, 15, 16\}$, $U_6^{(1)} = \{2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15\}$, $U_7^{(1)} = \{1, 2, 3, 6, 7, 8, 10, 11, 12, 13, 14, 15\}$, $U_8^{(1)} = \{2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16\}$

and

- $\mathcal{A}_1 = \{2, 4, 6, 8\}$, $\mathcal{A}_2 = \{1, 4, 6, 7\}$, $\mathcal{A}_3 = \{2, 3, 6, 7\}$, $\mathcal{A}_4 = \{1, 3, 6, 8\}$, $\mathcal{A}_5 = \{2, 4, 5, 7\}$, $\mathcal{A}_6 = \{1, 4, 5, 8\}$, $\mathcal{A}_7 = \{2, 3, 5, 8\}$, $\mathcal{A}_8 = \{1, 3, 5, 7\}$

and

- $\mathcal{U}_1^{(1)} = \{4, 8, 12, 16\}$, $\mathcal{U}_2^{(1)} = \{1, 8, 12, 13\}$, $\mathcal{U}_3^{(1)} = \{4, 5, 12, 13\}$, $\mathcal{U}_4^{(1)} = \{1, 5, 12, 16\}$, $\mathcal{U}_5^{(1)} = \{4, 8, 9, 13\}$, $\mathcal{U}_6^{(1)} = \{1, 8, 9, 16\}$, $\mathcal{U}_7^{(1)} = \{4, 5, 9, 16\}$, $\mathcal{U}_8^{(1)} = \{1, 5, 9, 13\}$.  



Finally from [23] we have

\[ Q^{(1)} = \begin{pmatrix} * & * & 2 & * & * & 3 & * & * & 5 & * & * & 1 \\ 1 & * & * & * & * & 4 & * & * & 6 & 2 & * & * \\ * & * & 4 & 1 & * & * & * & * & 7 & 3 & * & * \\ 3 & * & * & 2 & * & * & * & * & 8 & * & * & 4 \\ * & * & 6 & * & * & 7 & 1 & * & * & 5 & * & * \\ 5 & * & * & * & * & 8 & 2 & * & * & * & 6 & * \\ 7 & * & * & 6 & * & * & 4 & * & * & 8 & * & * \end{pmatrix}. \]

We can check that the above \( Q^{(1)} \) satisfies the condition C3 of Definition 1. Then from Proposition 1 we obtain a multiaccess coded caching scheme with subpacketization level \( 8 \times 16 = 128 \) and load \( \frac{S}{8} = 1 \).

\[ \square \]

V. FURTHER IMPROVED RESULT

In this section, we will show that based on the placement strategy proposed in Subsection IV-A, we can further reduce the transmission load of the scheme proposed in Subsection IV-C if some constraints in the following Proposition 2 are satisfied. This method can also work for some other existing PDAs.

Given a \((K', F, Z, S)\) PDA, assume \( |A_1| = \ldots = |A_F| \). For each \( h \in [t] \), let \( a_h = \min\{A_j[h] : j \in [F]\} \) and \( b_h = \max\{A_j[h] : j \in [F]\} \).

**Proposition 2.** Given a \((K', F, Z, S)\) PDA \( P \), if \( |A_1| = \ldots = |A_F| = t \) and \( \lambda_h = a_h + L - b_h > 0 \) holds for each \( h \in [t] \) and the \( Q^{(1)} \) generated by \( P \) in (23) satisfies the **Requirement** in Subsection IV-C, then for any positive integers \( L, M, N \), there exists a \((K = K' + t(L - 1), L, M, N)\) multiaccess coded caching scheme with memory ratio \( \frac{M}{N} = \frac{K^2}{K'F} \) and transmission load \( R_2 = \frac{S}{F} \cdot \frac{K - \sum_{h=1}^{t} \lambda_h}{K'F} \).

**Proof.** For any integer \( j \in [F] \), we know that

\[ U_j^{(1)} = \bigcup_{h \in [t]} \{A_j[h] + (h - 1)(L - 1), \ldots, A_j[h] + h(L - 1)\} \]

is the set of users in \([K]\) who can retrieve the packets \( W_{n,j}^{(1)} \) for all \( n \in [N] \). So for any two different positive integers \( j_1, j_2 \in [F] \), a user \( U_{j_1} \cap U_{j_2} \) can retrieve the packets \( W_{n,j_1}^{(1)} \) and \( W_{n,j_2}^{(1)} \) for all \( n \in [N] \) if and only if \( k \in U_{j_1} \cap U_{j_2} \). For each integer \( h \in [t] \),

\[ \bigcap_{j \in [F]} \{A_j[h] + (h - 1)(L - 1), \ldots, A_j[h] + h(L - 1)\} = \{a_h + (h - 1)(L - 1), \ldots, a_h + h(L - 1)\} \]

(25a)

\[ = \{b_h + (h - 1)(L - 1), \ldots, b_h + h(L - 1)\} \]

(25b)

\[ = \{b_h + (h - 1)(L - 1), \ldots, a_h + h(L - 1)\}, \]

(25c)

always holds where (25b) comes from that \( a_h = \min\{A_j[h] : j \in [F]\} \) and \( b_h = \max\{A_j[h] : j \in [F]\} \). From (25a) and (25c), each user from \( \{b_h + (h - 1)(L - 1), \ldots, a_h + h(L - 1)\} \) can retrieve all the packets of the first subfile. From (25c), it can be seen that there are exactly

\[ \sum_{h=1}^{t} \{b_h + (h - 1)(L - 1), \ldots, a_h + h(L - 1)\} = \sum_{h=1}^{t} \lambda_h \]

users who can re-construct all the \( S \) multicast messages for \( Q^{(1)} \) from their retrieved cache-nodes. By the symmetry, considering all \( Q^{(g)} \) where \( g \in [K] \), among all the \( KS \) multicast messages in the delivery phase, each user can re-construct \( \sum_{h=1}^{t} \lambda_h S \) multicast messages. Hence, we can transmit \((K - \sum_{h=1}^{t} \lambda_h)S\) random linear combinations of the \( KS \) multicast messages. Then the transmission load is

\[ R_2 = \frac{(K - \sum_{h=1}^{t} \lambda_h)S}{KF} = \frac{S}{F} \cdot \frac{K - \sum_{h=1}^{t} \lambda_h}{K'F}. \]

Then the proof is completed.

\[ \square \]

Instead of random linear combinations, we can also use the parity check matrix of Minimum Distance Separable (MDS) code or Cauchy matrix as in [23], to encode the \( KS \) multicast messages. In each of these matrices whose dimension is dimension \( m_1 \times m_2 \) where \( m_1 \leq m_2 \), every \( m_1 \) columns are linearly independent.
By \[4\], in MN PDA we have \(|A_1| = \ldots = A_{(K')} = t\) and \(a_h = h, b_h = K' - (t - h)\) for each \(h \in [t]\). For any positive integer \(L\), if
\[t + 1 < K' < tL\]
which implies \(tL + 1 < K < tL + L\) in Theorem \[1\] we have
\[\lambda_h = a_h + L - b_h = h + L - (K' - (t - h)) = t + L - K' > 0.\]
From Proposition \[2\] we can directly obtain the following result.

**Corollary 1.** For the \((K, L, M, N)\) centralized multiaccess coded caching problem, if \(M = \frac{N}{K}\), where \(t \in \{0, 1, \ldots, \lfloor \frac{K}{L} \rfloor\}\) and \(tL + 1 < K < tL + L\), the following load is achievable,
\[R_2 = \frac{K - tL (t + 1)(K - tL)}{K t + 1} = \frac{(K - tL)^2}{K}.\]  

We can check that the transmission load in Corollary \[1\] is exactly the transmission load in Lemma \[4\]. This implies that when \(tL + 1 < K < tL + L\), the scheme in \[17\] has smaller transmission load. As shown in \((11)\), when \((t + 1)L < K\) the proposed scheme in Theorem \[1\] has strictly lower transmission load than the scheme in \[17\].

Finally let us see the example in Remark \[3\] to show that when \((t + 1)L \geq K\), we can also reduce the transmission loads of our new schemes based on some other PDAs. Firstly it is easy to check that \((t + 1)L = 16 \geq 16 = K\) holds in the example of Remark \[3\]. Since \((a_1, b_1) = (1, 2)\), \((a_2, b_2) = (3, 4)\), \((a_3, b_3) = (5, 6)\) and \((a_4, b_4) = (7, 8)\), i.e., \(\lambda_h = a_h + L - b_h = 3 - 1 = 2\) holds for each \(h \in [4]\). From Proposition \[2\] the improved transmission load \(R_2 = \frac{8}{8} \cdot \frac{16 - 8}{16} = 0.5\). Notice that the schemes in Lemma \[4\] and Lemma \[5\] achieve the loads equal to 1 and 0.9, with subpacketization levels equal to 140 and 240, respectively.

**VI. Conclusion**

In this paper, we considered the multiaccess coded caching problem and proposed a novel transformation approach, which extends the MN caching scheme for the shared-link caching systems to the considered problem. We show that the resulting scheme can achieve a strictly lower load than all the exiting schemes for the multiaccess coded caching problem. In addition, the proposed transformation approach can also extend some other PDA caching schemes for the shared-link caching systems to the considered problem, such that we can further reduce the subpacketization level of the scheme based on the MN caching scheme. Further works include characterizing the fundamental limits of the considered problem and extending the proposed approach to more general multiaccess caching networks.

**APPENDIX A**

**Proof of (10), (11) and (12)**

**A. Proof of (10)**

We focus on the case where \(L\) does not divide \(K\). When \(M_1 = \frac{N}{K} \lfloor \frac{K}{2L} \rfloor\), we have
\[\frac{R_{HKD}}{R_1} = \frac{K - \lfloor \frac{K}{2L} \rfloor}{\frac{K}{t + 1}} > \frac{K - \lfloor \frac{K}{2L} \rfloor}{\frac{K}{t + 1}} \cdot \frac{t + 1}{t + \frac{K}{2L}} = \frac{K - \lfloor \frac{K}{2L} \rfloor}{K} \cdot \frac{t + 1}{\frac{K}{2L}} = \frac{K - \lfloor \frac{K}{2L} \rfloor}{K - L \lfloor \frac{K}{2L} \rfloor}.\]  

Let us then focus on a memory size \(M_1 \leq M\). The achieved load by the scheme in Lemma \[5\] is obtained by memory-sharing between the memory-load tradeoff points \((\frac{N}{K} \lfloor \frac{K}{2L} \rfloor, R_{HKD})\) and \((\frac{N}{K}, 0)\). In addition, the achieved load by the proposed scheme is no worse than the load obtained by memory-sharing between the memory-load tradeoff points \((\frac{N}{K} \lfloor \frac{K}{2L} \rfloor, R_1)\) and \((\frac{N}{K}, 0)\). Additionally with \(27\), we can prove that with \(M\), the multiplicative gap between the achieved loads by Lemma \[5\] and by the proposed scheme is larger than \(\frac{K - \lfloor \frac{K}{2L} \rfloor}{K - L \lfloor \frac{K}{2L} \rfloor}\), which coincides with \(10\).

**B. Proof of (11)**

Let us focus on the case where \(K > (t + 1)L\). From \(8\) and \(9\), we have
\[\frac{R_1}{R_{BK}} = \frac{K - tL}{(t + 1)^2} > \frac{1}{(t + 1)(1 - \frac{K}{L})} < \frac{1}{(t + 1)(1 - \frac{K}{L})} = 1.\]
C. Proof of (12)

From Lemma 5 it can be seen that

\[
R_{SR} \geq \sum_{h = \frac{K-tL}{2} \pm 2}^{K-tL} \frac{2}{1+\left\lfloor \frac{L}{N} \right\rfloor},
\]

We focus on the non-trivial corner points at the memory sizes \(M = \frac{Nt}{K}\) where \(t \in \{1, \ldots, \left\lceil \frac{K}{L} \right\rceil\}\). We first show that when \(\frac{KM}{2N}(1 - \frac{ML}{N}) \geq 1\), we have \(R_1 < R_{SR}\). More precisely, we have

\[
\sum_{h = \frac{K-tL}{2} \pm 2}^{K-tL} \frac{2}{1+\left\lfloor \frac{L}{N} \right\rfloor} \geq \sum_{h = \frac{K-tL}{2} \pm 2}^{K-tL} \frac{2}{2+\frac{L}{N}} = \frac{1}{2+\frac{L}{N}} + \frac{1}{2+\frac{L}{N}} + \cdots + \frac{1}{2+\frac{L}{N}} \geq (K-tL) \frac{1}{2+\frac{L}{N}}. \tag{28}
\]

From (28), it can be seen that

\[
\frac{R_{SR}}{R_1} > \frac{(K-tL) \frac{1}{2+\frac{L}{N}}}{K-tL} = \frac{1}{2+\frac{L}{N}} + \frac{L}{N} + 2 = \frac{KML}{N} + 1 - \frac{KM}{N} + 2 = \frac{KML}{N} + 1 \left(1 - \frac{KM}{N} - \frac{2}{N+1} \right) > \frac{KML}{N} + 1 \left(1 - \frac{KM}{N} - \frac{2}{N+1} \right) > \frac{KML}{N} + 1 \left(1 - \frac{KM}{N} - \frac{2}{N+1} \right) \geq 1.
\]

Hence, we showed that when \(\frac{KM}{2N}(1 - \frac{ML}{N}) \geq 1\), it holds that \(R_1 < R_{SR}\).

Let us then consider the regime where \(\frac{KM}{2N}(1 - \frac{ML}{N}) < 1\). In this case, we have either \(M < \frac{L}{N} + \sqrt{\frac{L}{4\pi} - \frac{2}{KL}}\) or \(M > \frac{L}{N} - \sqrt{\frac{L}{4\pi} - \frac{2}{KL}}\). When \(K \gg L\), we have

\[
\frac{1}{2L} + \sqrt{\frac{1}{4L^2} - \frac{2}{KL}} \to \frac{1}{L};
\]

and \(\frac{1}{2L} - \sqrt{\frac{1}{4L^2} - \frac{2}{KL}} \to 0\).

Recall that \(M \leq \frac{N}{L}\). Hence, we can prove that when \(K \gg L\), the regime \(\frac{KM}{2N}(1 - \frac{ML}{N}) < 1\) does not exist.

In conclusion, if \(K \gg L\), we can prove that \(R_1 < R_{SR}\) for \(M = \frac{Nt}{K}\) where \(t \in \{1, \ldots, \left\lceil \frac{K}{L} \right\rceil\}\).

APPENDIX B

PROOF OF LEMMA 6

For any integer \(s \in [S]\), assume that there exist two different entries, say, \(q_{j_1,k_1}^{(1)} = q_{j_2,k_2}^{(1)} = s\). Since the integers in each row of the \((K', F, Z, S)\) PDA \(P\) are different, then all the integers in each row of \(Q^{(1)}\) must be different by (23). Then we have \(j_1 \neq j_2\).

Now let us see the other requirements of Condition C3 of Definition 1. From (22) there exist two unique integers \(h_1 \in [K'-t]\) and \(h_2 \in [K'-t]\) satisfying \(k_1 = \Pi_{j_1}^{(1)}[h_1]\) and \(k_2 = \Pi_{j_2}^{(1)}[h_2]\). So we have \(\psi_{j_1}(k_1) = \alpha_{j_1}[h_1]\) and \(\psi_{j_2}(k_2) = \alpha_{j_2}[h_2]\), and then
Then we have
\[ S = \overline{A}_{j_1} \cup \{ A_{j_2}[h_1] \} = A_{j_2} \cup \{ \overline{A}_{j_2}[h_2] \}. \]

Then we have
\[ \overline{A}_{j_1}[h_1] \in A_{j_2} \quad \overline{A}_{j_2}[h_2] \in A_{j_1} \]

Let \( A_{j_2}[h_2'] = \overline{A}_{j_1}[h_1] \) and \( A_{j_1}[h_1'] = \overline{A}_{j_2}[h_2] \). When \( A_{j_1}[h_1'] < A_{j_2}[h_2'] \), we have
\[ S[h_1] = A_{j_1}[h_1'] = \overline{A}_{j_2}[h_2] \quad \text{and} \quad S[h_2] = A_{j_2}[h_2'] = \overline{A}_{j_1}[h_1]. \]

These imply that
\begin{itemize}
  \item in \( A_{j_2} \) there are \( h_1' - 1 \) integers which are smaller than \( \overline{A}_{j_2}[h_2] \) and
  \item in \( A_{j_1} \) there are \( h_2' \) integers which are smaller than \( \overline{A}_{j_1}[h_1] \).
\end{itemize}

Furthermore,
\begin{itemize}
  \item \( h_1 \) is the number of columns in \( P \), which contain the null entries and are less than or equal to \( \overline{A}_{j_1}[h_1] \) in \( j_1^{th} \) row, and
  \item \( h_2 \) is the number of columns in \( P \), which contain the null entries and are less than or equal to \( \overline{A}_{j_2}[h_2] \) in \( j_2^{th} \) row.
\end{itemize}

So we have
\[ \overline{A}_{j_1}[h_1] = h_2' + h_1 \quad \overline{A}_{j_2}[h_2] = (h_1' - 1) + h_2. \]

We then introduce the following lemma, whose proof could be found in Appendix C.

**Lemma 7.** For any integer \( k \in [K] \) and \( j \in [F] \), if \( k = \overline{U}_{j}(1)[h] \) for some positive integer \( h \in [K - tL] \), then \( \psi_j(k) = x + h \) and \( k = xL + h \) where \( x \) is the number of stars before \( \psi_j(k) \) in \( j^{th} \) row of \( P \).

From Lemma 7 we have \( k_1 = h_2' L + h_1 \) and \( k_2 = (h_1' - 1)L + h_2 \). Since
\begin{align*}
  k_1 &= h_2' L + h_1 \in \{ h_2' + h_1 + (h_2' - 1)(L - 1), \ldots, h_1 + h_2' L \} \\
  &= \{ \overline{A}_{j_1}[h_1] + (h_2' - 1)(L - 1), \ldots, h_1 + h_2' L \} \\
  &= \{ \overline{A}_{j_2}[h_2'] + (h_2' - 1)(L - 1), \ldots, \overline{A}_{j_1}[h_1] + h_2' L \} \\
  &= \{ A_{j_2}[h_2'] + (h_2 - 1)(L - 1), \ldots, A_{j_2}[h_2'] + h_2 L \} \\
  &= \overline{U}_{j_2}(1); \\
\end{align*}
and
\begin{align*}
  k_2 &= (h_1' - 1) L + h_2 \in \{ (h_1' - 1)L + h_2, \ldots, (h_1' - 1)L + h_2 + L + 1 \} \\
  &= \{ (h_1' - 1) + h_2 + (h_1' - 1)(L - 1), \ldots, (h_1' - 1) + h_2 + h_1' L \} \\
  &= \{ A_{j_2}[h_2] + (h_1' - 1)(L - 1), \ldots, A_{j_2}[h_2] + h_1' L \} \\
  &= \{ A_{j_1}[h_1'] + (h_1' - 1)(L - 1), \ldots, A_{j_1}[h_1'] + h_1' L \} \\
  &= \overline{U}_{j_1}(1),
\end{align*}

we have \( q_{j_1,k_1}^{(1)} = q_{j_2,k_2}^{(1)} = * \). Similarly we can show that \( q_{j_1,k_2}^{(1)} = q_{j_2,k_1}^{(1)} = * \) also holds when \( A_{j_1}[h_1'] > A_{j_2}[h_2'] \). Consequently \( k_1 \neq k_2 \) always holds. Then the proof is completed.

**APPENDIX C**

**PROOF OF LEMMA 7**

Since \( k = \overline{U}_{j}(1)[h] \), \( k \) is the \( h^{th} \) non-star entry in \( j^{th} \) row of \( Q^{(1)} \). Then \( \psi_j(k) \) is the \( h^{th} \) integer entry in \( j^{th} \) row of \( P \) by the definition of mapping \( \psi_j(k) = \psi_j(\overline{U}_{j}(1)[h]) = \overline{A}_{j}[h] \). Assume that there are \( x \) stars before \( \psi_j(k) \) in the \( j^{th} \) row of \( P \). So we have \( \psi_j(k) = x + h \). Let \( A_j = \{ a_1, a_2, \ldots, a_x, a_{x+1}, \ldots, a_t \} \). Then \( a_x < \psi_j(k) < a_{x+1} \). By (18) we have
\[ a_x + x(L - 1) < \psi_j(k) < x(L - 1) < a_{x+1} + x(L - 1) \in \overline{U}_{j}(1). \]

So \( \psi_j(k) + x(L - 1) \in \overline{U}_{j}(1) \). In addition since \( \psi_j(k) \) is the \( h^{th} \) integer entry in \( j^{th} \) row of \( P \), then there are exactly \( h - 1 \) integers before \( \psi_j(k) + x(L - 1) \) in \( j^{th} \) row of \( U^{(1)} \). This implies that \( \psi_j(k) + x(L - 1) \) is the \( h^{th} \) element of \( \overline{U}_{j}(1) \), i.e., \( \overline{U}_{j}(1)[h] = \psi_j(k) + x(L - 1) \). So we have \( k = \psi_j(k) + x(L - 1) = x + h + x(L - 1) = xL + h \). Then the proof is completed.
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