Compact gradient $\rho$-Einstein soliton is isometric to the Euclidean sphere

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Abstract In this paper, we have investigated some aspects of gradient $\rho$-Einstein Ricci soliton in a complete Riemannian manifold. First, we have proved that the compact gradient $\rho$-Einstein soliton satisfying some curvature conditions is isometric to the Euclidean sphere by showing that the scalar curvature becomes constant. Second, we have shown that in a non-compact gradient $\rho$-Einstein soliton satisfying an integral condition, the scalar curvature vanishes.

Keywords Gradient $\rho$-Einstein Ricci soliton · Scalar curvature · Riemannian manifold

Mathematics Subject Classification 53C20 · 53C21 · 53C44

1 Introduction and preliminaries

A 1-parameter family of metrics $\{g(t)\}$ on a Riemannian manifold $M$, defined on some time interval $I \subset \mathbb{R}$ is said to satisfy Ricci flow if it satisfies

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

where $R_{ij}$ is the Ricci curvature with respect to the metric $g_{ij}$. Hamilton [8] proved that for any smooth initial metric $g(0) = g_0$ on a closed manifold, there exists a unique solution $g(t)$, $t \in [0, \epsilon)$, to the Ricci flow equation for some $\epsilon > 0$. A solution $g(t)$ of the Ricci flow of the form
\[ g(t) = \sigma(t) \varphi(t)^* g(0), \]

where \( \sigma : \mathbb{R} \to \mathbb{R} \) is a positive function and \( \varphi(t) : M \to M \) is a 1-parameter family of diffeomorphisms, is called a Ricci soliton. It is known that if the initial metric \( g_0 \) satisfies the equation

\[
\text{Ric}(g_0) + \frac{1}{2} L_X g_0 = \lambda g_0,
\]

(1)

where \( \lambda \) is a constant and \( X \) is a smooth vector field on \( M \), then the manifold \( M \) admits Ricci soliton. Therefore, the equation (1), in general, is known as the Ricci soliton. If \( X \) is the gradient of some smooth function, then it is called gradient Ricci soliton. For more results of Ricci soliton, see [2, 6, 7]. In 1979, Bourguignon [1] introduced the notion of Ricci-Bourguignon flow, where the metrics \( g(t) \) is evolving according to the flow equation:

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + 2\rho R g_{ij},
\]

where \( \rho \) is a non-zero scalar constant and \( R \) is the scalar curvature of the metric \( g(t) \). Following the Ricci soliton, Catino and Mazzier [3] gave the definition of gradient \( \rho \)-Einstein soliton, which is the self-similar solution of Ricci-Bourguignon flow. This soliton is also called gradient Ricci-Bourguignon soliton by some authors.

**Definition 1.1** [3] Let \( (M, g) \) be a Riemannian manifold of dimension \( n \), \( n \geq 3 \), and let \( \rho \in \mathbb{R} \), \( \rho \neq 0 \). Then \( (M, g) \) is called gradient \( \rho \)-Einstein soliton, denoted by \( (M, g, f, \rho) \), if there is a smooth function \( f : M \to \mathbb{R} \) such that

\[
\text{Ric} + \nabla^2 f = \lambda g + \rho R g,
\]

(2)

for some constant \( \lambda \).

The soliton is trivial if \( \nabla f \) is a parallel vector field. The function \( f \) is known as \( \rho \)-Einstein potential function. If \( \lambda > 0 \) (resp. \( \lambda < 0 \)), then the gradient \( \rho \)-Einstein soliton \( (M, g, f, \rho) \) is said to be shrinking (resp. steady or expanding) . On the other hand, the \( \rho \)-Einstein soliton is called gradient Einstein soliton, gradient traceless Ricci soliton, or gradient Schouten soliton if \( \rho = 1/2 \), \( 1/n \) or \( 1/2(n-1) \) respectively. Later, this notion has been generalized in various directions such as \( m \)-quasi Einstein manifold [9], \( (m, \rho) \)-quasi Einstein manifold [11], Ricci-Bourguignon almost soliton [12].

Catino and Mazzier [3] showed that compact gradient Einstein, Schouten, and traceless Ricci soliton are trivial. They classified three-dimensional gradient shrinking Schouten soliton and proved that it is isometric to a finite quotient of either \( \mathbb{S}^3 \) or \( \mathbb{R}^3 \) or \( \mathbb{R} \times \mathbb{S}^2 \). Huang [10] deduced a sufficient condition for the compact gradient shrinking \( \rho \)-Einstein soliton to be isometric to a quotient of the round sphere \( \mathbb{S}^n \).

**Theorem 1.1** [10] Let \( (M, g, f, \rho) \) be an \( n \)-dimensional \( (4 \leq n \leq 5) \) compact gradient shrinking \( \rho \)-Einstein soliton with \( \rho < 0 \). If the following condition holds

\[
\left( \int_M \left| W + \frac{\sqrt{2}}{\sqrt{n} (n-2)} Z \otimes g \right|^2 \right)^{\frac{2}{n}} + \sqrt{\frac{(n-4)(n-1)}{8(n-2)}} \lambda \text{vol}(M)^{\frac{2}{n}} \leq \sqrt{\frac{n-2}{32(n-1)}} Y(M, [g]),
\]

where \( Z = \text{Ric} - \frac{\rho}{n} g \) is the trace-less Ricci tensor, \( W \) is the Weyl tensor and \( Y(M, [g]) \) is the Yamabe invariant associated to \( (M, g) \), then \( M \) is isometric to a quotient of the round sphere \( \mathbb{S}^n \).

In 2019, Mondal and Shaikh [13] proved the isometry theorem for gradient \( \rho \)-Einstein soliton in case of conformal vector field. In particular, they proved the following result:

**Theorem 1.2** [13] Let \( (M, g, f, \rho) \) be a compact gradient \( \rho \)-Einstein soliton. If \( \nabla f \) is a non-trivial conformal vector field, then \( M \) is isometric to the Euclidean sphere \( \mathbb{S}^n \).
Dwivedi [12] proved an isometry theorem for gradient Ricci-Bourguignon soliton.

**Theorem 1.3** [12] A non-trivial compact gradient Ricci-Bourguignon soliton is isometric to an Euclidean sphere if any one of the following holds

1. \( M \) has constant scalar curvature.
2. \( \int_M g(\nabla R, \nabla f) \leq 0 \).
3. \( M \) is a homogeneous manifold.

We note that Catino et al. [4] proved many results for gradient \( q \)-Einstein soliton in non-compact manifold.

**Theorem 1.4** [4] Let \( (M, g, f, \rho) \) be a complete non-compact gradient shrinking \( q \)-Einstein soliton with \( 0 < \rho < 1/2(n-1) \) bounded curvature, non-negative radial sectional curvature, and non-negative Ricci curvature. Then the scalar curvature is constant.

In this paper, we have shown that a non-trivial compact gradient \( q \)-Einstein soliton with a curvature condition is isometric to the Euclidean sphere. The main results of this paper are as follows:

**Theorem 1.5** A nontrivial compact gradient \( q \)-Einstein soliton \( (M, g, f, \rho) \) of dimension \( n \neq 2 \) with \( |Ric|^2 = \frac{K_n}{n} \) has constant scalar curvature and therefore \( M \) is isometric to an Euclidean sphere.

We have also showed that in a non-compact gradient \( q \)-Einstein soliton satisfying some conditions the scalar curvature vanishes.

**Theorem 1.6** Suppose \( (M, g, f, \rho) \) is a non-compact gradient non-expanding \( q \)-Einstein soliton with non-negative scalar curvature. If \( \rho > 1/n \) and the \( q \)-Einstein potential function satisfies

\[
\int_{M-B(p,r)} d(x,p)^{-q} f < \infty,
\]

then the scalar curvature vanishes in \( M \).

2 **Proof of the results**

**Proof of the Theorem 1.5** Since the gradient \( q \)-Einstein soliton is non-trivial, it follows that \( \rho \neq 1/n \), see [3]. Taking the trace of (2) we get

\[
R + \Delta f = \lambda n + \rho R n.
\]

From the commutative equation, we obtain

\[
\Delta \nabla f = \nabla_i \Delta f + R_{ij} \nabla f.
\]

By using contracted second Bianchi identity, we have

\[
\Delta \nabla f = \nabla_j \nabla_i f = \nabla_j (\lambda g_{ij} + \rho R g_{ij} - R_{ij})
\]

\[
= \nabla_i \left( \rho R - \frac{1}{2} R \right)
\]

and

\[
\nabla_i \Delta f = \nabla_i (\lambda n + \rho R n - R) = \nabla_i (\rho R n - R).
\]

Therefore, (5) yields

\[
(n-1) \rho \nabla_i R - \frac{1}{2} \nabla_i R + R_{ij} \nabla f = 0,
\]

Taking covariant derivative \( \nabla_i \), we get

\[
(n-1) \rho \nabla_i \nabla R - \frac{1}{2} \nabla_i R + \nabla_i R_{ij} \nabla f + R_{ij} \nabla_i \nabla f = 0.
\]

Taking trace in both sides, we obtain
\[
\left( \frac{n-1}{2} + \frac{1}{2} \right) \Delta R + \frac{1}{2} g(\nabla R, \nabla f) + \frac{R}{n} (\lambda n + \rho Rn - R) = 0. \tag{7}
\]

Now integrating using divergence theorem we get
\[
\int_M (\lambda n + \rho Rn - R) = - \int_M ((n-1) \rho + \frac{1}{2}) \Delta R - \frac{n}{2} \int_M g(\nabla R, \nabla f) = \frac{n}{2} \int_M R \nabla f = \frac{n}{2} \int_M (\lambda n + \rho Rn - R).
\]

The above equation is true only if
\[
\int_M (\lambda n + \rho Rn - R) = 0, \tag{8}
\]
which implies
\[
\int_M R \left( R + \frac{\lambda n}{\rho - 1} \right) = 0. \tag{9}
\]

Again integrating (4), we obtain
\[
\int_M \left( R + \frac{\lambda n}{\rho - 1} \right) = 0. \tag{10}
\]

Therefore, (9) and (10) together imply that
\[
\int_M \left( R + \frac{\lambda n}{\rho - 1} \right)^2 = 0.
\]

Hence, \( R = \lambda n/(1 - \rho n) \). Then from Theorem 1.3 we can conclude our result.

\textit{Proof of the Theorem 1.6} \quad From (4) we get
\[
(n \rho - 1) R = \Delta f - \lambda n.
\]

Since \( \lambda \geq 0 \), the above equation implies that
\[
(n \rho - 1) R \leq \Delta f. \tag{11}
\]

Now, we consider the cut-off function, introduced in [5], \( \varphi_r \in C^2_0(B(p, 2r)) \) for \( r > 0 \) such that
\[
\begin{aligned}
0 &\leq \varphi_r \leq 1 & \text{in } B(p, 2r) \\
\varphi_r &= 1 & \text{in } B(p, r) \\
|\nabla \varphi_r|^2 &\leq \frac{C}{r^2} & \text{in } B(p, 2r) \\
\Delta \varphi_r &\leq \frac{C}{r^2} & \text{in } B(p, 2r),
\end{aligned}
\]

where \( C > 0 \) is a constant. Then for \( r \to \infty \), we have \( \Delta \varphi_r^2 \to 0 \) as \( \Delta \varphi_r^2 \leq \frac{C}{r^2} \). Then we calculate
\[
(n \rho - 1) \int_M R \varphi_r^2 \leq \int_M \varphi_r^2 \Delta f = \int_{B(p, 2r) \setminus B(p, r)} f \Delta \varphi_r^2 \tag{12}
\]
\[
\leq \int_{B(p, 2r) \setminus B(p, r)} \frac{C}{r^2} \to 0, \tag{13}
\]
as \( r \to \infty \). Hence, we obtain
\[
(n \rho - 1) \lim_{r \to \infty} \int_{B(p, r)} R \leq 0. \tag{14}
\]

Since \( \rho > 1/n \), it follows that...
\[
\lim_{r \to \infty} \int_{B(r)} R \leq 0.
\]

But \( R \) is non-negative everywhere in \( M \). Therefore, \( R \equiv 0 \) in \( M \). \( \square \)

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