The absolute continuity of convolutions of orbital measures in $SO(2n + 1)$

Kathryn E. Hare

Received: 25 October 2021 / Accepted: 11 March 2022 / Published online: 11 April 2022
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Abstract
Let $G$ be a compact Lie group of Lie type $B_n$, such as $SO(2n + 1)$. We characterize the tuples $(x_1, \ldots, x_L)$ of the elements $x_j \in G$ which have the property that the product of their conjugacy classes has non-empty interior. Equivalently, the convolution product of the orbital measures supported on their conjugacy classes is absolutely continuous with respect to Haar measure. The characterization depends on the dimensions of the largest eigenspaces of each $x_j$. Such a characterization was previously only known for the compact Lie groups of type $A_n$.

Keywords Orbital measure · Absolutely continuous measure · Compact Lie group

Mathematics Subject Classification Primary 43A80 · Secondary 58C35, 17B22

1 Introduction
Let $G$ be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Given $x \in G$ (or $X \in \mathfrak{g}$) we let $\mu_x$ (or $\nu_X$) denote the $G$-invariant measure on $G$ (or $\mathfrak{g}$) supported on the conjugacy class containing $x$ (or adjoint orbit containing $X$). These are known as orbital measures. It is a classical result due to Ragozin [11] that if the dimension of $G$ non-trivial orbital measures are convolved together, the resulting measure is absolutely continuous with respect to Haar measure on $G$ in the first case and with respect to Lebesgue measure on $\mathfrak{g}$ in the second. Ragozin proved this by using geometric
properties to deduce that the product of dimension of $G$ non-trivial conjugacy classes (or the sum of dim $G$ non-trivial adjoint orbits) had non-empty interior, an equivalent property.

In a series of papers (see [4–6, 8]) the author with various coauthors used tools from harmonic analysis and representation theory to improve upon Ragozin’s result, determining for each $x \in G$ (or $X \in g$) the minimal integer $L$ such that the convolution of $\mu_x$ (or $\nu_X$) with itself $L$ times was absolutely continuous. This number $L$, which depends on $x$ or $X$, never exceeds $2\text{rank}G$. In [13], Wright used geometric arguments to extend these results in the special case of the classical Lie groups and algebras of type $A_n$ and characterized the absolute continuity of $\mu_{x_1} \cdots \mu_{x_L}$ or $\nu_{X_1} \cdots \nu_{X_L}$ in terms of the dimensions of the largest eigenspaces of the elements $x_j$ and $X_j$.

Inspired by the algebraic methods that Gracyzk and Sawyer used to study related problems in the symmetric space setting (see [1–3]), the author with Gupta in [7] characterized the $L$-tuples $(X_1, \ldots, X_L)$ such that $\nu_{X_1} \cdots \nu_{X_L}$ is absolutely continuous for each of the compact classical Lie algebras, with the exception of one pair in the Lie algebra of type $D_n$ for each $n \geq 6$. The characterization can again be expressed in terms of the dimensions of the largest eigenspaces. We were also able to show that in the special case that each $x_j \in G$ had the property that $x_j = \exp X_j$ for some $X_j \in g$ with $\dim C_{x_j} = \dim O_{X_j}$, then $\mu_{x_1} \cdots \mu_{x_L}$ is absolutely continuous if and only if $\nu_{X_1} \cdots \nu_{X_L}$ is absolutely continuous. However, in all the compact Lie groups, except those of type $A_n$, there are many elements $x_j$ which do not have this special property. For instance, any element $x \in SO(2n + 1)$ with $-1$ an eigenvalue of multiplicity at least two, fails to have this special property.

In this note, we modify the strategy of [7] to characterize the absolute continuity of all the convolution products of orbital measures on the compact Lie groups of type $B_n$, the classical model being the group $SO(2n + 1)$. The characterization is more complicated than the Lie algebra case, but again can be expressed in terms of the dimensions of the largest eigenspace of each $x_j$. We refer the reader to Theorem 5.1 and Definition 3.7 for the precise statement. Our proof depends heavily upon the Lie theory of roots and root vectors, and particularly the structure of subsystems of annihilating roots of the elements $x_j$. Type $B_n$ is unusual in this regard as its root system has irreducible subsystems of type $D_j$, in addition to $B_j$ and $A_j$, in contrast to the situation in the Lie groups of types $C_n$, $D_n$, or even the Lie algebra of type $B_n$. It is this feature which seems to make the characterization so complex.

We conclude with a conjecture for the characterization of absolute continuity for the classical Lie algebras of type $C_n$ and $D_n$ and briefly discuss an approach to its proof.

\section{2 Notation and terminology}

\subsection{2.1 Basic notation}

For the remainder of this paper, $G_n$ will be a compact, connected simple Lie group of Lie type $B_n$, with Lie algebra $g_n$ and centre $Z(G_n)$. At times, we may suppress the subscript $n$, which is its rank. We denote by $T_n$ a maximal torus of $G_n$ and let $t_n$
denote the Lie algebra of $T_n$, a maximal torus of $g_n$. The Haar measure on $G$ will be denoted by $m_G$. Our model of these groups will be the matrix group $SO(2n+1)$, the group of $(2n+1) \times (2n+1)$ real, unitary matrices of determinant 1.

We take as its torus the block diagonal matrices, with $n$ $2 \times 2$ block matrices of the form
\[
\begin{bmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{bmatrix},
\]
and a 1 in the final $1 \times 1$ block, and identify such a torus element with the $n$-vector $(e^{i\theta_1}, \ldots, e^{i\theta_n})$.

The Lie algebra of this Lie group is $so(2n+1)$, the real $(2n+1) \times (2n+1)$ skew-Hermitian matrices and its torus can similarly be identified with vectors in $\mathbb{R}^n$.

We write $[\cdot, \cdot]$ for the Lie bracket action and $ad$ for the map: $g_n \to g_n$ given by $ad(\cdot) = [X, Y]$. The group acts on its Lie algebra by the adjoint action, denoted $Ad(\cdot)$, and the exponential map, $exp$, is a surjection from $g_n$ onto $G_n$. We recall that
\[
Ad(exp M) = exp(ad(M)) = Id + \sum_{k=1}^{\infty} \frac{ad^k(M)}{k!},
\]
where $ad^k(M)$ is the $k$-fold composition of $ad(M)$.

### 2.2 Orbital measures

Every element in $G_n$ is conjugate to some element $x \in T_n$ that has the vector form
\[
x = (1, \ldots, 1, -1, \ldots, -1, e^{i\alpha_1}, \ldots, e^{i\alpha_1}, \ldots, e^{i\alpha_m}, \ldots, e^{i\alpha_m}) \quad (2.1)
\]
where the $0 < \alpha_j < \pi$ are distinct and $u + v + s_1 + \cdots + s_m = n$. Thus $x$ has 1 as an eigenvalue with multiplicity $2u + 1$, $-1$ as an eigenvalue with multiplicity $2v$ and each $e^{\pm i\alpha_j}$ as an eigenvalue with multiplicity $s_j$. We denote the conjugacy class containing $x$ by
\[
C_x = \{gxg^{-1} : g \in G_n\}.
\]

Similarly, every element of $g_n$ is in the adjoint orbit $O_X = \{Ad(g)X : g \in G_n\}$ of some $X \in t_n$. Note that the element $x$ of (2.1) is equal to $\exp X_x$ where
\[
X_x = (0, \ldots, 0, \pi, \ldots, \pi, a_1, \ldots, a_1, \ldots, a_m, \ldots, a_m) \in t_n
\]
and that $C_x \subseteq \exp(O_{X_x})$. This inclusion can be proper.

**Definition 2.1** (i) For any $x \in T$, the orbital measure $\mu_x$ is the $G$-invariant probability measure supported on $C_x$, defined by
\[
\int_G f d\mu_x = \int_G f(gxg^{-1}) d m_G(g) \text{ for } f \text{ continuous on } G.
\]
(ii) For any \( X \in t \), the orbital measure \( v_X \) is the \( \text{Ad}(G) \)-invariant probability measure supported on \( O_X \), defined by

\[
\int_{\mathfrak{g}} f \, dv_X = \int_{G} f(\text{Ad}(g)X) \, dm_G(g) \quad \text{for } f \text{ bounded and continuous on } \mathfrak{g}.
\]

**Definition 2.2** We will say that the tuple \((x_1, \ldots, x_L) \in T^L \) (or \((X_1, \ldots, X_L) \in t^L \)) is **absolutely continuous** if the orbital measure \( \mu_{x_1} \ast \cdots \ast \mu_{x_L} \) on \( G \) (resp., \( v_{X_1} \ast \cdots \ast v_{X_L} \) on \( \mathfrak{g} \)) is absolutely continuous with respect to Haar measure on \( G \) (resp., Lebesgue measure on \( \mathfrak{g} \)). This is the same as saying the orbital measure has an \( L^1 \) density function.

### 2.3 Roots, root vectors and type

We denote by \( \Phi_n \) the set of roots of \( \mathfrak{g}_n \):

\[
\Phi_n = \{ \pm e_k, \pm e_i \pm e_j : 1 \leq i < j \leq n, 1 \leq k \leq n \}
\]

and let \( \Phi_n^+ \) denote the positive roots. Again, we may suppress the subscript. For each root \( \alpha \in \Phi_n \), we let \( E_{\alpha} \) denote a root vector corresponding to \( \alpha \), meaning that if \( H \in t_n \), then \( [H, E_{\alpha}] = i\alpha(H)E_{\alpha} \). (We make the convention that roots are real valued.) We define a function \( \alpha_G \) acting on \( x \in T \) by the rule

\[
\alpha_G(x) := \exp i\alpha(X_x).
\]

We will take a collection of root vectors \( \{E_{\alpha} : \alpha \in \Phi^+ \} \) that form a Weyl basis for \( \mathfrak{g} \) (see [9, p. 421] or [12, p. 290]) and set

\[
RE_{\alpha} = \frac{E_{\alpha} + E_{-\alpha}}{2}, \quad IE_{\alpha} = \frac{E_{\alpha} - E_{-\alpha}}{2i}.
\]

We also call these root vectors. These are vectors in \( \mathfrak{g} \) such that \( E_{\alpha} = RE_{\alpha} + iIE_{\alpha} \). Being a Weyl basis, we have

\[
\begin{align*}
[RE_{\alpha}, RE_{\beta}] &= cRE_{\alpha+\beta} + dRE_{\beta-a} \\
-[IE_{\beta}, RE_{\alpha}] &= [RE_{\alpha}, IE_{\beta}] = cIE_{\alpha+\beta} + dIE_{\beta-a} \\
[IE_{\alpha}, IE_{\beta}] &= -cRE_{\alpha+\beta} + dRE_{\beta-a}
\end{align*}
\]

where \( c, d \neq 0 \).

By the annihilating roots of \( x \in T \) or \( X \in t \) we mean the sets

\[
\Phi_x = \{ \alpha \in \Phi : \alpha_G(x) = 1 \} \quad \text{or} \quad \Phi^g_X = \{ \alpha \in \Phi : \alpha(X) = 0 \}.
\]

Equivalently,

\[
\Phi_x = \{ \alpha \in \Phi : \alpha(X_x) = 0 \mod 2\pi \}.
\]

These are root subsystems.
We note that $\Phi_x = \Phi$ if and only if $x \in Z(G)$. If $\Phi_x$ is empty, $x$ is called regular.

We point out that $\Phi^\rho_{X \pi} \subseteq \Phi_x$, and this inclusion can be proper. For example, if $x = (-1, \ldots, -1) \in SO(2n + 1)$, then we may take $X = (\pi, \ldots, \pi)$, so $\Phi_x = \{\pm e_i \pm e_j : i \neq j\}$, while $\Phi^\rho_{X \pi} = \{\pm (e_i - e_j) : i \neq j\}$. The centre of $SO(2n + 1)$ is trivial.

We also let

$$ N_x = \{RE_{\alpha}, IE_{\alpha} : \alpha \in \Phi^+, \alpha \notin \Phi_x\} \quad \text{and} \quad N^\rho_{X \pi} = \{RE_{\alpha}, IE_{\alpha} : \alpha \in \Phi^+, \alpha \notin \Phi^\rho_{X \pi}\}, $$

the sets of non-annihilating root vectors. Of course, $N_x \subseteq N^\rho_{X \pi}$. It is known that $\dim C_x = \vert N_x \vert$ and $\dim O_x = \vert N^\rho_{X \pi} \vert$. \cite{10, VI.4].

**Example 2.3** For the $x$ of (2.1), $\Phi_x = \Psi_0 \cup \Psi_\pi \cup \bigcup_{\ell=1}^m \Psi_\ell$ where

$$ \Psi_0 = \{\pm e_k, e_i \pm e_j : 1 \leq i, j, k \leq u, i \neq j\} $$
$$ \Psi_\pi = \{e_i \pm e_j : u < i \neq j \leq u + v\} \quad \text{and} \quad \Psi_\ell = \{e_i - e_j : u + v + s_1 + \cdots + s_{\ell-1} < i \neq j \leq u + v + s_1 + \cdots + s_\ell\} $$

for $\ell = 1, \ldots, m$. The root vectors $RE_{e_i \pm e_j}$ and $RE_{e_j}$ with $i \leq u$ and $j > u$ are some examples of non-annihilating root vectors of $x$.

There is an important geometric relationship between the absolute continuity of a convolution product of orbital measures and the product of the associated conjugacy classes or sum of orbits, as outlined in the next result. See \cite{7, Lemma 7.6}, \cite{11} and \cite{13, Prop. 2.2} for proofs.

**Proposition 2.4** The orbital measure $\mu_{x_1} \ast \cdots \ast \mu_{x_L}$ on $G$ (or $\nu_{X_1} \ast \cdots \ast \nu_{X_L}$ on $g$) is absolutely continuous with respect to Haar measure on $G$ (or Lebesgue measure on $g$) if and only if any of the following hold:

(i) The set $\prod_{i=1}^L C_{x_i} \subseteq G$ has non-empty interior (resp., the set $\sum_{i=1}^L O_{X_i} \subseteq g$ has non-empty interior);

(ii) The set $\prod_{i=1}^L C_{x_i} \subseteq G$ has positive measure (resp., the set $\sum_{i=1}^L O_{X_i} \subseteq g$ has positive measure);

(iii) There exist $g_i \in G$ with $g_1 = I_d$, such that

$$ sp(Ad(g_i)N_{x_i} : i = 1, \ldots, L) = g $$
$$ (resp., sp(Ad(g_i)N^\rho_{X_i} : i = 1, \ldots, L) = g). $$

We observe that this result implies that if $\mu_{x_1} \ast \cdots \ast \mu_{x_L}$ (or $\nu_{X_1} \ast \cdots \ast \nu_{X_L}$) is not absolutely continuous, then it is purely singular with respect to Haar measure (or Lebesgue measure).
3 Dominant type and eligibility

**Definition 3.1** Given \( x \in T \) as in (2.1), we will say that \( x \) is of type

\[
B_u \times D_v \times SU(s_1) \times \cdots \times SU(s_m)
\]

as this is the Lie type of \( \Phi_x \). Here by \( B_1 \) we mean the root system \( \{ \pm e_1 \} \), by \( D_2 \) we mean \( \{ \pm e_1 \pm e_2 \} \), and \( B_0, D_0, D_1, SU(0) \) and \( SU(1) \) are empty (and may be omitted if this does not cause confusion).

**Remark 3.2** It seems to be more helpful to say Lie type \( SU(s_j) \), rather than type \( A_{s_j-1} \), as the values \( s_j \) arise naturally.

There is a similar definition of type for elements of \( t \), (see [7]), but there is no \( D_v \) component in that case.

**Example 3.3** If \( x \) is type \( D_1 \) or \( SU(1) \), then \( x \) is regular. The only element in \( T_n \) of type \( B_n \) is the Identity.

**Example 3.4** When \( x \) is type \( B_u \times SU(s_1) \times \cdots \times SU(s_m) \), then \( x \) and \( X_x \) have the same type. If \( x \) is type \( B_u \times D_v \times SU(s_1) \times \cdots \times SU(s_m) \) with \( v \geq 2 \), then there is no choice of \( X \) of the same type as \( x \) with \( x = \exp X \). When \( v = 1 \), then \( x \) is also type \( B_u \times SU(1) \times SU(s_1) \times \cdots \times SU(s_m) \) and \( X_x \) is again of that type. When \( x = \exp X \), but \( x \) and \( X \) do not have the same type, then \( N_x \subsetneq N_x^B \), so \( \dim C_x < \dim O_x \). For instance, when \( x = (-1, \ldots, -1) \in T_n \), then \( x \) is type \( D_n \), whereas \( X_x \) is of type \( SU(n) \). Thus \( \dim C_x = 2n \), while \( \dim O_{X_x} = n(n+1) \).

**Remark 3.5** It was shown in [7, Prop. 7.7] that if \( x_j = \exp X_j \) and \( x_j, X_j \) are of the same type, then \( \mu_{x_1} \ast \cdots \ast \mu_{x_L} \) is absolutely continuous with respect to Haar measure on \( G \) if and only if \( \nu_{X_1} \ast \cdots \ast \nu_{X_L} \) is absolutely continuous with respect to Lebesgue measure on \( g \).

In [7] the notion of dominant type for \( X \in t \) was also defined: \( X \) of type \( B_u \times SU(s_1) \times \cdots \times SU(s_m) \) is of dominant type \( B \) if \( 2u > \max s_j \) and otherwise \( X \) is of dominant type \( S \). Set \( \mathcal{S}_X^B = 2u \) if \( X \) is of dominant type \( B \) and \( \mathcal{S}_X^B = \max s_j \) otherwise.

The analogous notions for elements of \( T \) are more complicated, as outlined below.

**Definition 3.6** Suppose \( x \in T \) is of type \( B_u \times D_v \times SU(s_1) \times \cdots \times SU(s_m) \). Let \( s = \max s_j \). We will say

\[
x \text{ is of dominant type } \begin{cases} B & \text{if } u > v, 2u + 1 > s \\ D & \text{if } u > v, 2v > s \\ BD & \text{if } v = u, 2v \geq s \\ S & \text{if } s \geq 2u + 1, 2v \end{cases}
\]

Put

\[
S_x = \begin{cases} 2u + 1 & \text{if } x \text{ is dominant type } B \\ 2v & \text{if } x \text{ is dominant type } D \\ s & \text{if } x \text{ is dominant type } S \\ (2u + 1, 2v) & \text{if } x \text{ is dominant type } BD \end{cases}
\]
In the case that \( x \) is of dominant type \( BD \), we will let \( S_x^{(1)} = 2u + 1 \) and \( S_x^{(2)} = 2v \).

It can be verified that each \( x \) satisfies precisely one dominant type criterion.

Note that if \( x \) is of dominant type \( B, D \) or \( S \), then \( Sx \) is the dimension of its largest eigenspace. When \( x \) is type \( BD \), \( S_x^{(1)} \) is the dimension of the eigenspace corresponding to the eigenvalue 1 and \( S_x^{(2)} \) is the dimension of the eigenspace of the eigenvalue \(-1\), the two largest eigenspaces. We will see that essentially we can treat \( x \) as either of dominant type \( B \) or \( D \), depending on which is more helpful in the circumstance.

In [7], the tuple \( (X_1, \ldots, X_L) \in T_n^L \) was said to be eligible if

\[
\sum_{i=1}^{L} S_{X_i}^{p_n} \leq 2n(L - 1).
\]  

In defining eligibility in the group setting, it is helpful to first say that a tuple \( (x_1, \ldots, x_L) \) has parity \( p = 1 \) if there are an odd number of \( x_j \) that are of dominant type \( D \) and parity \( p = 2 \) if there are an even number.

**Definition 3.7** We will say that the tuple \( (x_1, \ldots, x_L) \in T_n^L \) of parity \( p \) is **eligible** if one of the following (exclusive) conditions hold.

(i) Any two or more of \( x_1, \ldots, x_L \) are of dominant type \( BD \) or \( S \).
(ii) Only one \( x_j \) is of dominant type \( S \), none are of dominant type \( BD \) and

\[
\sum_{i=1}^{L} S_{x_i} \leq (2n + 1)(L - 1).
\]

(iii) Only one \( x_j \), say \( x_1 \), is of dominant type \( BD \), none are of dominant type \( S \) and

\[
S_{x_1}^{(p)} + \sum_{i=2}^{L} S_{x_i} \leq (2n + 1)(L - 1).
\]

(iv) No \( x_j \) is of dominant type \( BD \) or \( S \) and

\[
\sum_{i=1}^{L} S_{x_i} \leq (2n + 1)(L - 1) + p - 1.
\]

**Example 3.8** A pair \( (x, y) \) of dominant type \( (B, B) \) (meaning both \( x \) and \( y \) are of dominant type \( B \)) is eligible if \( 2u_1 + 2u_2 \leq 2n \), a pair of dominant type \( (D, D) \) is eligible if \( 2v_1 + 2v_2 \leq 2n + 2 \), and pairs of dominant type \( (B, D), (B, BD) \) or \( (BD, D) \) are eligible if \( 2u_1 + 2v_2 \leq 2n \). For instance, if \( x \) is type \( B_1 \) and \( y \) is type \( D_2 \) in the Lie group \( B_2 \), then \( (x, y) \) is not eligible. One can similarly determine the requirements for types \((S, B)\) and \((S, D)\). All pair types \((S, S), (S, BD)\) or \((BD, BD)\) are eligible.
Example 3.9 A central element in $G_n$ is type $B_n$. Thus if $x_1$ is central, then $(x_1, \ldots, x_L)$ is eligible if and only if $(x_2, \ldots, x_L)$ is eligible.

Our goal is to prove that eligibility characterizes absolute continuity. The necessity of eligibility is easy to prove:

**Proposition 3.10** Any absolutely continuous tuple in $T_n^L$ is eligible.

**Proof** Assume $(x_1, \ldots, x_L)$ is absolutely continuous, but not eligible. We will use the fact that the absolute continuity of $\mu_{x_1} \ast \ldots \ast \mu_{x_L}$ implies $\prod_{i=1}^L Cx_i$ has non-empty interior, Proposition 2.4.

First, consider the case that no $x_j$ is of dominant type $BD$ or $S$. Let $y_j = g_j^{-1}x_jg_j$ be an arbitrary element of $Cx_j$ and assume $V_j$ is the eigenspace of $y_j$ corresponding to the eigenvalue $\alpha_j = 1$ if $x_j$ is of dominant type $B$ and eigenvalue $-1$ if $x_j$ is of dominant type $D$. In either case, $V_j$ is the eigenspace of $y_j$ of dimension $S_{x_j}$.

Let $y = y_1 \cdots y_L$ and note that if $v \in \bigcap_{j=1}^L V_j$, $v \neq 0$, then $y(v) = \left(\prod_{j=1}^L \alpha_j\right)v$, so $v$ is an eigenvector of matrix $y$ with eigenvalue $\alpha = \prod_{j=1}^L \alpha_j$. Of course, $\alpha = 1$ if there are an even number of $x_j$ of dominant type $D$ and $\alpha = -1$ otherwise.

Now

$$\dim \bigcap_{j=1}^L V_j = \sum_{j=1}^L \dim V_j - \left(\dim(V_1 + V_2) + \dim \left(\left(V_1 \cap V_2\right) + V_3\right) + \ldots + \dim \left(\bigcap_{j=1}^{L-1} V_j + V_L\right)\right) \geq \sum_{j=1}^L S_{x_j} - (L - 1)(2n + 1).$$

If there are an odd number of $x_j$ of dominant type $D$, then the failure of eligibility implies $\dim \bigcap_{j=1}^L V_j \geq 1$. This shows that any $y \in \prod_{i=1}^L Cx_i$ has a non-zero eigenvector with eigenvalue $\alpha = -1$ and that is impossible for a set with non-empty interior.

If there are an even number of $x_j$ of dominant type $D$, then the failure of eligibility implies $\dim \bigcap_{j=1}^L V_j \geq 2$, so any such $y$ has two linearly independent eigenvectors with eigenvalue 1. Every element of $SO(2n+1)$ has one eigenvector with eigenvalue 1, but a subset of $SO(2n+1)$ with non-empty interior cannot have the property that each element of the set has two linearly independent eigenvectors with eigenvalue 1, so again we obtain a contradiction.

Next, suppose there is one $x_j$, say $x_1$, of dominant type $BD$ and none of dominant type $S$. Assume there are an odd number of $x_j$ of dominant type $D$. As before, let $y_j$ be an arbitrary element of $Cx_j$ and $y = y_1 \cdots y_L$. Let $V_1$ be the eigenspace of $y_1$...
corresponding to the eigenvalue \(1\) having dimension \(S_{x_1}^{(1)}\), and let \(V_j\) be eigenspace of \(y_j\) of dimension \(S_{x_j}\) for \(j \neq 1\). The calculations as above and failure of eligibility implies

\[
\dim \bigcap_{j=1}^{L} V_j \geq S_{x_1}^{(1)} + \sum_{j=2}^{L} S_{x_j} - (L - 1)(2n + 1) \geq 1.
\]

Moreover, any non-zero \(v \in \bigcap_{j=1}^{L} V_j\) is an eigenvector of \(y\) corresponding to the eigenvalue \((-1)^{\#x_{j}}\) dominant \(D = -1\) as we have assumed there are an odd number of \(x_j\) of dominant type \(D\). Thus every element of \(\prod_{i=1}^{L} C_{x_i}\) has an eigenvector with eigenvalue \(\alpha = -1\) and that is impossible for a set with non-empty interior.

If there are an even number of \(x_j\) of dominant type \(D\) the arguments are similar, but we begin with \(V_1\) the eigenspace of \(y_1\) with eigenvalue \(-1\). This space has dimension \(S_{x_1}^{(2)}\). In this case, any non-zero \(v \in \bigcap_{j=1}^{L} V_j\) is an eigenvector of \(Y\) corresponding to the eigenvalue \((-1)(-1)^{\#x_{j}}\) dominant \(D = -1\) and again the failure of eligibility implies that \(\bigcap_{j=1}^{L} V_j\) has dimension at least one which is a contradiction.

Finally, if \(x_1\) is of dominant type \(S\), with eigenvalue \(\alpha_1 \neq \pm 1\) having eigenspace of dimension \(S_{x_1}\), and no other \(x_j\) is type \(S\) or \(BD\), then we begin with \(V_1\) the eigenspace of \(y_1\) corresponding to \(\alpha_1\). Arguing as above, we deduce that non-eligibility implies every element of \(\prod_{i=1}^{L} C_{x_i}\) has an eigenvector with eigenvalue equal to either \(\pm \alpha_1\), again impossible for a set with non-empty interior.

As we assumed \((x_1, \ldots, x_L)\) was not eligible, these are the only cases to consider.

\(\square\)

## 4 Preliminary results towards proving absolute continuity

Our proof that eligible tuples are absolutely continuous will proceed by induction on the rank of the Lie group. This will require associating each \(x\) in the torus of \(G_n\) with some \(x'\) in the torus of \(G_{n-1}\), as was done for elements of the torus of the Lie algebra in [7].

**Notation 4.1** Assume \(x \in T_n\) is as in (2.1). We call \(x' \in T_{n-1}\) the reduction of \(x\) when

\[
x' = \begin{cases} 
(1, \ldots, 1, -1, \ldots, -1, e^{ia_1}, \ldots, e^{ia_1}, \ldots, e^{ia_m}, \ldots, e^{ia_m}) & \text{if } x \text{ is dominant } B \text{ or } BD \\
1, \ldots, 1, -1, \ldots, -1, e^{ia_1}, \ldots, e^{ia_1}, \ldots, e^{ia_m}, \ldots, e^{ia_m} \quad & \text{if } x \text{ is dominant } D \\
1, \ldots, 1, -1, \ldots, -1, e^{ia_1}, \ldots, e^{ia_1}, \ldots, e^{ia_m}, \ldots, e^{ia_m} \quad & \text{if } x \text{ is dominant } S \text{ and } s_1 = \max s_j 
\end{cases}
\]

We can embed \(t_{n-1}\) into \(t_n\) by taking the standard basis vectors \(e_1, \ldots, e_n\) for \(t_n\) and omitting the first for \(t_{n-1}\). This gives a natural embedding of \(\Phi_{n-1}\) into \(\Phi_n\), an embedding of \(\mathfrak{g}_{n-1}\) into \(\mathfrak{g}_n\), \(G_{n-1}\) into \(G_n\) and \(T_{n-1}\) into \(T_n\). It is in this way that we view \(x'\) as an element of \(T_{n-1}\).
Observe that if \( x \) is of dominant type \( BD \), then \( x' \) is either of dominant type \( D \) or \( S \). Also, \( x \) is central if and only if \( x' \) is central. We record here some other simple facts.

**Lemma 4.2** (a) If \( x \) is either of dominant type \( BD \) or \( S \), then \( \mu_x^2 \in L^2 \).

(b) If \( x \) is of dominant type \( BD \) or \( S \), then \( \mu_{x'}^2 \in L^2 \).

**Proof** (a) This follows directly from Theorem 9.1(B) of [6] with the observation that if \( x \) is of dominant type \( BD \), then \( x \) is either \( B_n/2 \times D_n/2 \) or \( B_u \times D_u \times SU(s_1) \times \cdots \times SU(s_m) \) where \( u < n/2 \).

(b) If \( x \) as in (2.1) is of dominant type \( BD \), then \( x' \) is \( B_{u-1} \times D_u \times SU(s_1) \times \cdots \times SU(s_m) \). Unless \( \sum s_j = 0,1 \), we are in the ‘else’ case in the notation of [6, Thm. 9.1(B)] and consequently \( \mu_{x'}^2 \in L^2 \). If \( \sum s_j = 0 \) (or 1), then \( u = n/2 \) (resp., \( u = (n - 1)/2 \)) and one can check from [6, Thm. 9.1(B)] that both situations imply \( \mu_{x'}^2 \in L^2 \).

The reasoning is similar if \( x \) is of dominant type \( S \) and is left to the reader. \( \square \)

**Corollary 4.3** (a) If \( x \) and \( y \) are both of dominant types \( BD \) or \( S \), then \( \mu_x \ast \mu_y \) and \( \mu_{x'} \ast \mu_{y'} \) \( \in L^2 \).

(b) If \( x \) is of dominant type \( BD \) or \( S \) and changes type upon reduction, then \( \mu_x^2 \in L^2 \).

**Proof** (a) This follows immediately from parts (a) and (b) of the previous Lemma upon applying Plancherel’s theorem and Holder’s inequality, as follows. From Plancherel’s theorem, \( \mu_x \ast \mu_y \in L^2 \) if and only if

\[
\| \mu_x \ast \mu_y \|_2 = \left( \sum_{\sigma \in \hat{G}} d_{\sigma}^2 \left( \frac{Tr\sigma(x)}{d_{\sigma}} \right)^2 \left( \frac{Tr\sigma(y)}{d_{\sigma}} \right)^2 \right) < \infty.
\]

By Holder’s inequality, another application of Plancherel’s theorem and part (a) of the Lemma, we see that

\[
\| \mu_x \ast \mu_y \|_2 \leq \left( \sum_{\sigma \in \hat{G}} d_{\sigma}^2 \left( \frac{Tr\sigma(x)}{d_{\sigma}} \right)^4 \right)^{1/2} \left( \sum_{\sigma \in \hat{G}} d_{\sigma}^2 \left( \frac{Tr\sigma(y)}{d_{\sigma}} \right)^4 \right)^{1/2} = \left\| \mu_x^2 \right\|_2 \left\| \mu_y^2 \right\|_2 < \infty.
\]

(b) For this we merely need to observe that the change of dominant type of \( x \) upon reduction implies \( x' \) is either of dominant type \( BD \) or \( S \). \( \square \)

An important property for our induction argument is that eligibility is preserved under reduction.

**Proposition 4.4** If \((x_1, \ldots, x_L)\) in \(G_n\) is eligible, then so is the reduced tuple \((x'_1, \ldots, x'_L)\) in \(G_{n-1}\).

**Proof** Note that Lemma 4.2 and its Corollary imply that if \( x \) is of dominant type \( BD \) or \( S \), or \( x \) and \( x' \) are different dominant types, then \( \mu_{x'}^2 \in L^2 \). Consequently,
$\mu x_1' \ast \cdots \ast \mu x_L' \in L^2 \subseteq L^1$, and hence $(x_1', \ldots, x_L')$ is eligible, if any of the following situations occur:

(i) two or more $x_j$ are of dominant type $BD$ or $S$;
(ii) two or more $x_j$ switch dominant type upon reduction;
(iii) one $x_j$ is of dominant type $BD$ or $S$ and another switches dominant type upon reduction.

Thus there are two cases that we need to analyze.

Case 1: All $x_j$ are of dominant type $B$ or $D$ and at most one switches dominant type upon reduction.

Case 2: One $x_j$ is of dominant type $BD$ or $S$ and no other $x_i$ switches dominant type upon reduction.

We remark that it is a routine exercise to check that if $x$ is of dominant type $B$ or $D$ and $x$ does not switch dominant type upon reduction, then $Sx' = Sx - 2$, while if $x$ switches dominant type, then $Sx' \leq Sx - 1$ (where if $x'$ is of dominant type $BD$ by ‘$Sx' \leq Sx - 1$’ we mean both coordinates of $Sx'$ are dominated by $Sx - 1$). If $x$ is of dominant type $S$, then $Sx' \leq Sx$ (again, meaning both coordinates if $x'$ is of dominant type $BD$) and if $x$ is of dominant type $BD$, then $Sx' = S^{(2)}_x = \min(S^{(1)}_x, S^{(2)}_x)$. (In this final situation, $x'$ cannot be type $BD$.)

Proof of Case 1: Regardless of the parity of the tuple, eligibility certainly implies

$$\sum_{i=1}^{L} S_{x_i} \leq (2n + 1)(L - 1) + 1.$$ 

Assume $x_1$ is the one that switches dominant type (if any do). Then

$$\sum_{i=1}^{L} S_{x_i'} \leq S_{x_1} - 1 + \sum_{i=2}^{L} (S_{x_i} - 2) = \sum_{i=1}^{L} S_{x_i} - 2(L - 1) - 1 \leq (2(n - 1) + 1)(L - 1),$$

so $(x_1', \ldots, x_L')$ is eligible in $G_{n - 1}$.

Proof of Case 2: If $x_1$ is of dominant type $BD$, then eligibility implies

$$S^{(p)}_{x_1} + \sum_{i=2}^{L} S_{x_i} \leq (2n + 1)(L - 1)$$

where $p$ is the parity of the tuple. As $Sx_i' = Sx_i - 2$ for $i \neq 1$ and $Sx_1' \leq S^{(p)}_{x_1}$, we have

$$\sum_{i=1}^{L} S_{x_i'} \leq \sum_{i=1}^{L} S_{x_i} - 2(L - 1) \leq (2(n - 1) + 1)(L - 1),$$

and hence the tuple is eligible.

The arguments are similar if $x_1$ is of dominant type $S$. \qed
Our proof of the sufficiency of eligibility for absolute continuity will make heavy use of the following Proposition, which we refer to as the general strategy. Its proof relies on Proposition 2.4 and is essentially the same as given in Proposition 5.6 in [7] in the Lie algebra setting and is omitted.

**Notation 4.5** For \( x \in G \) or \( X \in \mathfrak{g} \) put

\[
\Omega_x = N_x \setminus N_{x'} \quad \text{and} \quad \Omega^B_x = N^B_x \setminus N^B_{x'},
\]

where we use the natural embedding described above.

**Proposition 4.6** (General strategy) Let \( L \geq 2 \). Let \( x_i \in T_n \) for \( i = 1, \ldots, L \) and assume \((x'_1, \ldots, x'_L)\) is an absolutely continuous tuple in \( G_{n-1} \). Suppose \( \Omega \subseteq \{ RE_\alpha, IE_\alpha : \alpha \in \Phi_n^+ \setminus \Phi^+_{n-1} \} \), \( \Omega \) contains each \( \Omega_{x_i} \) and \( \Omega \) has the property that \( \text{ad}(H)(\Omega) \subseteq \text{sp}\Omega \) whenever \( H \in \mathfrak{g}_{n-1} \). Fix \( \Omega_0 \subseteq \Omega_{x_L} \) and assume there exist \( g_1, \ldots, g_{L-1} \in G_{n-1} \) and \( M \in \mathfrak{g}_n \) such that

(i) \( \text{sp}(\text{Ad}(g_i)(\Omega_{x_i}), \Omega_{x_L} \setminus \Omega_0 : i = 1, \ldots, L - 1) = \text{sp}\Omega; \)

(ii) \( \text{ad}^k(M) : N_{x_L} \setminus \Omega_0 \to \text{sp}(\Omega, \mathfrak{g}_{n-1}) \) for all positive integers \( k \); and

(iii) The span of the projection of \( \text{Ad}(\exp tM)(\Omega_0) \) onto the orthogonal complement of \( \text{sp}(\mathfrak{g}_{n-1}, \Omega) \) in \( \mathfrak{g}_n \) is a surjection for all small \( t \neq 0 \).

Then \((x_1, \ldots, x_L)\) is an absolutely continuous tuple in \( G_n \).

The following linear algebra result can sometimes be helpful in verifying the hypotheses of this proposition. Its proof is essentially the same as that of Lemma 5.7 of [7]. It is based on the fact that if \( \{v_1, \ldots, v_n\} \) are a linearly independent set of vectors in vector space \( V \) and \( w_1, \ldots, w_n \in V \), then the collection \( \{v_1 + tw_1, \ldots, v_n + tw_n\} \) is also linearly independent for sufficiently small \( t > 0 \), and the observation that

\[
\|\text{Id} - \text{Ad}(\exp tH_0)\| \quad \text{and} \quad \left\| \text{ad}(H_0) - \frac{1}{t}(\text{Ad}(\exp(tH_0) - \text{Id}) \right\|
\]

tend to 0 as \( t \to 0 \).

**Lemma 4.7** Let \( \Xi_i, i = 1, \ldots, L \), \( L \geq 2 \), be given and suppose \( \Omega \) is as (4.1). Assume \( \Omega \supseteq \bigcup_{i=1}^L \Xi_i \) and has the property that \( \text{ad}(H)(\Omega) \subseteq \text{sp}\Omega \) whenever \( H \in \mathfrak{g}_{n-1} \). Fix \( \Omega_0 \subseteq \Xi_L \) and \( \Omega_\ast \subseteq \left( \Xi_L \cap \bigcup_{i=1}^{L-1} \Xi_i \right) \setminus \Omega_0 \), and assume that for some \( H_0 \in \mathfrak{g}_{n-1} \),

\[
\text{sp}\left( \text{ad}(H_0)\Omega_\ast, \Xi_L \setminus \Omega_0, \bigcup_{i=1}^{L-1} \Xi_i \setminus \Omega_\ast \right) = \text{sp}\Omega.
\]

Then, for small \( t \neq 0 \),

\[
\text{sp}\left( \text{Ad}(\exp tH_0) \bigcup_{i=1}^{L-1} \Xi_i, \Xi_L \setminus \Omega_0 \right) = \text{sp}\Omega.
\]
5 Characterizing absolute continuity

5.1 Proof of the characterization theorem

**Theorem 5.1** Assume $x_j$, $j = 1, \ldots, L$ are non-central, torus elements of $G_n$. The tuple $(x_1, \ldots, x_L)$ is eligible if and only if $\mu_{x_1} \ast \cdots \ast \mu_{x_L}$ is absolutely continuous.

We already saw the necessity of eligibility in Proposition 3.10, thus we only need to prove sufficiency. This will be a proof by induction on $n$, the rank of $G_n$, and we begin with the base case, $n = 2$.

**Lemma 5.2** All eligible tuples of non-central torus elements in $G_2$ are absolutely continuous.

**Proof** This will use the generalized Holder’s inequality which states that

$$\sum_{k} \prod_{i=1}^{L} |a_i(k)| < \infty$$

if each $(a_i) \in \ell^{p_i}$ where $\sum_{i=1}^{L} 1/p_i = 1$.

It is shown in Theorem 9.1 of [6] that if $x \in G_2 \setminus Z(G_2)$, then $\mu_x^2 \in L^2$, except if $x$ is type $D_2$ when $\mu_x^4 \in L^2$. Thus if $L \geq 4$, it follows from Plancherel’s theorem and the generalized Holder’s inequality that $(x_1, \ldots, x_L)$ is absolutely continuous since even $(x_1, \ldots, x_4)$ is absolutely continuous (applying Holder’s inequality with $p_i = 4$).

Similarly, if $L = 3$ and only two $x_j$ are type $D_2$, then applying Holder’s inequality with $p_1 = 2$ and $p_2 = p_3 = 4$, we see the triple is absolutely continuous. If all three $x_j$ are type $D_2$, the triple is not eligible. That proves all eligible triples are absolutely continuous.

Likewise, any pair with neither $x_1$ nor $x_2$ of type $D_2$ is absolutely continuous. The only eligible pairs for which $x_1$ is type $D_2$ are the pairs $(x_1, x_2)$ where $x_2$ is regular, and Theorem 1.3 of [13] implies that such pairs are absolutely continuous. □

The major work in proving the theorem is done in the next lemma.

**Lemma 5.3** Suppose $n \geq 3$. Assume that all eligible tuples of non-central elements in $G_{n-1}$ are absolutely continuous and that $(x_1, \ldots, x_L)$ is eligible in $G_n$.

(a) If all $x_j$ are either of dominant type B or D, and the number of $x_j$ of dominant type D is even, then $(x_1, \ldots, x_L)$ is absolutely continuous.

(b) If one $x_j$ is of dominant type S and none are of dominant type BD, then $(x_1, \ldots, x_L)$ is absolutely continuous.

**Proof** For both (a) and (b) we will use the general strategy, Proposition 4.6, taking

$$\Omega := \{ R \varepsilon_\alpha, I \varepsilon_\alpha : \alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+ \} = \{ F E_{e_1}, F E_{e_1 \pm e_j} : j = 2, \ldots, n, F = R, I \}.$$
Note that the orthogonal complement of \( sp(\mathfrak{g}_{n-1}, \Omega) \) in \( \mathfrak{g}_n \) is the one dimensional space spanned, for example, by the torus element \([RE_{e_1}, IE_{e_1}]\). It is also spanned by the projection onto the orthogonal complement of the non-zero torus element \([RE_{e_1+e_n}, IE_{e_1+e_n}]\), and many other choices.

Since \((x_1, \ldots, x_L)\) is eligible in \( G_n \), Proposition 4.4 shows that the reduced tuple, \((x'_1, \ldots, x'_L)\) in \( G_{n-1} \), is eligible. By the hypothesis of the lemma, it is an absolutely continuous tuple.

(a) Without loss of generality, we can assume \( x_j \) are of dominant type \( D \) for \( j = 1, \ldots, m \) and \( x_j \) are of dominant type \( B \) for \( j = m + 1, \ldots, L \), with \( m \) even. In this situation, eligibility simplifies to

\[
\sum_{i=1}^{m} 2v_i + \sum_{i=m+1}^{L} 2u_i \leq 2n(L - 1) + m. \tag{5.1}
\]

We begin with the case \( m \neq 0 \), so \( m \geq 2 \). We will put \( \Omega_0 = \{RE_{e_1}, IE_{e_1}\} \) and we have

\[
\Omega_{x_i} = \begin{cases} 
\{FE_{e_1}, FE_{e_1} \pm e_j : j \in J_i, F = R, I\} & \text{for } i = 1, \ldots, m \smallsetminus 1 \\
\{FE_{e_1, e_1} \pm e_j : j \in J_i, F = R, I\} & \text{for } i = m + 1, \ldots, L
\end{cases}
\]

where the sets \( J_i \subseteq \{2, \ldots, n\} \), \( |J_i| = n - v_i \) for \( i = 1, \ldots, m \), and \( |J_i| = n - u_i \) for \( i = m + 1, \ldots, L \). For notational ease, we will write \( \Omega_i = \Omega_{x_i} \).

Clearly, \( \Omega_i \subseteq \Omega \), \( \Omega_0 \subseteq \Omega_1 \cap \Omega_2 \) and \( ad(H)(\Omega) \subseteq sp\Omega \) for all \( H \in \mathfrak{g}_{n-1} \).

By replacing \( x_i \) by Weyl conjugates that permute the appropriate indices, if necessary, (or, equivalently, replacing \( \Omega_i \) by \( Ad(g_i)\Omega_i \) where \( g_i \in G_{n-1} \) is a suitable Weyl conjugation), there is no loss of generality in assuming that if \( \sum_{i=1}^{L} |J_i| \geq n - 1 \), then \( \bigcup_{i=1}^{L} J_i = \{2, \ldots, n\} \) and otherwise the sets \( J_i \) are disjoint.

In particular, if \( \sum_{i=1}^{L} |J_i| \geq n - 1 \), then since \( RE_{e_1}, IE_{e_1} \in \Omega_2 \),

\[
(\Omega_1 \setminus \Omega_0) \bigcup_{i=2}^{L} \Omega_i = \Omega,
\]

so property (i) of the general strategy, Proposition 4.6, holds.

Put \( M = RE_{e_1} \). Property (iii) of the general strategy holds since

\[
Ad(\exp tM)IE_{e_1} = a_t IE_{e_1} + tb_t[RE_{e_1}, IE_{e_1}]
\]

where \( b_t \rightarrow b \neq 0 \) as \( t \rightarrow 0 \). Moreover, \( ad(M)(FE_{e_1} \pm e_j) \in spFE_{e_1} \) for \( j \neq 1 \), \( ad(M)(FE_{e_1} \pm e_j) = 0 \) if neither \( i, j = 1 \) and \( ad(M)(FE_{e_1} \pm e_j) \in sp(FE_{e_1} \pm e_j) \) if \( j \neq 1 \). It follows that

\[
ad^k(M) : \mathcal{N}_{x_1} \setminus \Omega_0 \subseteq sp(\Omega, \mathfrak{g}_{n-1}),
\]

so (ii) of the general strategy holds. Consequently, Proposition 4.6 implies \((x_1, \ldots, x_L)\) is an absolutely continuous tuple.
So assume \( \sum |J_i| < n - 1 \). The number of indices from \( \{2, \ldots, n\} \) that are missing from \( \bigcup_{i=1}^{L} J_i \) is equal to
\[
n - 1 - \left( \sum_{i=1}^{m} (n - v_i) + \sum_{i=m+1}^{L} (n - u_i) \right) = n(1 - L) - 1 + \sum_{i=1}^{m} v_i + \sum_{i=m+1}^{L} u_i
\]
and the eligibility assumption (5.1) ensures this is bounded above by
\[
n(1 - L) - 1 + n(L - 1) + \frac{m}{2} = \frac{m}{2} - 1.
\]
Thus \( (\Omega_1 \setminus \Omega_0) \bigcup \bigcup_{i=2}^{L} \Omega_i \) contains all of \( \Omega \), except possibly as many as \( m/2 - 1 \) of the pairs \( FE_{e_1 \pm e_j} \). Moreover, we must have \( m \geq 4 \) since we are assuming that some indices are missing. There is no loss of generality in assuming that it is the indices \( j \in \{2, \ldots, K\} \) with \( K \leq m/2 \) which have the property that \( FE_{e_1 \pm e_j} \) does not belong to \( (\Omega_1 \setminus \Omega_0) \bigcup \bigcup_{i=2}^{L} \Omega_i \).

The idea is now to use the fact that \( FE_{e_1} \) occurs in each of \( \Omega_1, \ldots, \Omega_m \). One of these copies will be needed to ‘cover’ \( \Omega \), one will be used to obtain the torus element as above, and we will see that the other \( m - 2 \) copies can be used to obtain the pairs \( FE_{e_1 \pm e_j} \in \Omega \) that are not present in \( (\Omega_1 \setminus \Omega_0) \bigcup \bigcup_{i=2}^{L} \Omega_i \).

An important observation towards this is that since \( ad(R e_j)(FE_{e_1}) = c FE_{e_1 + e_j} + d FE_{e_1 - e_j} \) and \( ad(I e_j)(FE_{e_1}) = -c F' e_1 + e_j + d F' e_1 - e_j \) where \( F' = I \) if \( F = R \) and vice versa (see 2.2), we have
\[
Ad(\exp t R E_{e_j})(FE_{e_1}) = a_t FE_{e_1} + t(c_t FE_{e_1 + e_j} + d_t FE_{e_1 - e_j}) \quad \text{and}
Ad(\exp t I E_{e_j})(FE_{e_1}) = a'_t F' e_1 + t(-c'_t F' e_1 + e_j + d'_t F' e_1 - e_j)
\]
where \( c_t, d_t \) converge to \( c, d \neq 0 \) respectively as \( t \to 0 \). Since the pairs
\[
c_t FE_{e_1 + e_j} + d_t FE_{e_1 - e_j}, -c_t FE_{e_1 + e_j} + d_t FE_{e_1 - e_j}
\]
are linearly independent for small \( t \neq 0 \), it follows that
\[
sp(FE_{e_1}, Ad(\exp t R E_{e_j})(FE_{e_1}), Ad(\exp t I E_{e_j})(FE_{e_1}) : F = R, I) = sp(FE_{e_1}, FE_{e_1 \pm e_j} : F = R, I).
\]
Moreover, \( Ad(\exp t F' E_{e_j})(FE_{e_1 \pm e_k}) = FE_{e_1 \pm e_j} \) if \( F' = R, I \) and \( j \neq k \).

It follows from these observations that for small \( t \neq 0 \) and \( j \in \{2, \ldots, K\} \),
\[
sp(FE_{e_1}, Ad(\exp t R E_{e_j})(\Omega_{2j-1}), Ad(\exp t I E_{e_j})(\Omega_{2j}) : F = R, I) = sp(FE_{e_1}, FE_{e_1 \pm e_j}, FE_{e_1 \pm e_i} : i \in J_{2j-1} \cup J_{2j}, F = R, I).
\]
Since \( FE_{e_1} \in \Omega_2 \), we conclude that \( sp(\Omega) \) is equal to
\[ sp(\Omega_1 \setminus \Omega_0, \Omega_2, \Omega_{2K+1}, \ldots, \Omega_L, Ad(\exp tRE_{e_j}))(\Omega_{2j-1}), \]
\[ Ad(\exp tIE_{e_j})(\Omega_{2j}) : j = 2, \ldots, K). \]

That establishes property (i) of the general strategy. We again take \( M = RE_{e_1} \) to complete the argument as we did in the case when there were no missing indices.

Now assume \( m = 0 \). This case can be handled by arguments similar to those used in the Lie algebra setting of [7]. We outline the key ideas here for completeness. We have

\[ \Omega_{x_i} = \{ FE_{e_1 \pm e_j} : j \in J_i, F = R, I\}, i = 1, \ldots, L \]

and we set

\[ \Omega_0 = \{ FE_{e_1 + e_n} : F = R, I\}. \]

With our usual notation, \( |J_i| = n - u_i \) and eligibility states \( \sum_{i=1}^{L} u_i \leq n(L - 1) \), so \( \sum_{i=1}^{L} |J_i| \geq n \). By taking Weyl conjugates if necessary, there is no loss of generality in assuming

\[ \bigcup_{i=2}^{L} \Omega_{x_i} = \left\{ FE_{e_1 \pm e_j} : j = n - \sum_{i=2}^{L} |J_i| + 1, \ldots, n, F = R, I \right\} \]

(where we replace \( n - \sum_{i=2}^{L} |J_i| + 1 \) by the index 2 if \( \sum_{i=2}^{L} |J_i| \geq n \)) and

\[ \Omega_{x_1} = \{ FE_{e_1 \pm e_j} : j \in [2, \ldots, |J_1|, n], F = R, I\}. \]

It is important to note that the eligibility criterion ensures \( \Omega_{x_1} \bigcup \bigcup_{i=2}^{L} \Omega_{x_i} \) contains all the vectors \( FE_{e_1 \pm e_j} \) for \( j = 2, \ldots, n \) and that \( RE_{e_1 \pm e_n}, IE_{e_1 \pm e_n} \) belong to both \( \Omega_{x_1} \) and \( \bigcup_{i=2}^{L} \Omega_{x_i} \).

Put \( g_t = \exp tRE_{e_n} \) for \( t > 0 \) and notice that

\[ Ad(g_t)(FE_{e_1 \pm e_n}) = a(t)FE_{e_1 \pm e_n} + tb(t)FE_{e_1} + t^2 c(t)FE_{e_1 + e_n} \]

where \( a(t) \to 1 \) as \( t \to 0 \) and \( b(t), c(t) \) converge to non-zero constants. Thus

\[ sp\{FE_{e_1 - e_n}, Ad(g_t)FE_{e_1 \pm e_n} : F = R, I\} \]
\[ = sp\{FE_{e_1 - e_n}, FE_{e_1 \pm e_n} + tb'(t)FE_{e_1}, FE_{e_1} + t c'(t)FE_{e_1 + e_n} : F = R, I\} \]

where \( b'(t), c'(t) \) converge to non-zero constants as \( t \to 0 \). As the collection of six vectors \( \{ FE_{e_1}, FE_{e_1 \pm e_n} \} \) are linearly independent, so are the vectors

\[ \{ FE_{e_1 - e_n}, FE_{e_1 \pm e_n} + t b'(t)FE_{e_1}, FE_{e_1} + t c'(t)FE_{e_1 + e_n} : F = R, I\} \]

for small enough \( t > 0 \), and hence the latter set spans the same set as \( \{ FE_{e_1}, FE_{e_1 \pm e_n} : F = R, I\} \).
Also, $Ad(g_t)(FE_{e_1} \pm e_j) = FE_{e_1} \pm e_j$ for $j \neq n$. From these comments it follows that for small enough $t > 0$,

$$sp \left\{ Ad(g_t) \left( \bigcup_{i=2}^{L} \Omega_{x_i} \right), \Omega_{x_1} \setminus \Omega_0 \right\} = sp \Omega,$$

so property (i) of the general strategy is satisfied.

Now take $M = RE_{e_1 + e_n}$. One can easily verify that properties (ii) and (iii) of the general strategy are also satisfied and hence $(x_1, \ldots, x_L)$ is an absolutely continuous tuple.

That completes the proof of (a).

(b) In this case, we can assume $x_j$ are of dominant type $D$ for $j = 1, \ldots, m$, $x_j$ are of dominant type $B$ for $j = m + 1, \ldots, L - 1$, and $x_L$ is of dominant type $S$.

Eligibility tells us that

$$\sum_{i=1}^{m} 2v_i + \sum_{i=m+1}^{L-1} 2u_i + s_L \leq 2n(L - 1) + m. \quad (5.2)$$

First, assume $m \geq 1$. As before, we have

$$\Omega_{x_i} = \begin{cases} \{ FE_{e_1}, FE_{e_1} \pm e_j : j \in J_i, F = R, I \} & \text{for } i = 1, \ldots, m \\ \{ FE_{e_1} \pm e_j : j \in J_i, F = R, I \} & \text{for } i = m + 1, \ldots, L - 1 \end{cases},$$

where the sets $J_i \subseteq \{2, \ldots, n\}$, $|J_i| = n - v_i$ for $i = 1, \ldots, m$, and $|J_i| = n - u_i$ for $i = m + 1, \ldots, L - 1$. But, in this case

$$\Omega_{x_L} = \{ FE_{e_1}, FE_{e_1+e_j}, FE_{e_1+e_k}, F = R, I : j \in J_L, k \geq 2 \}$$

where $|J_L| = n - S_L$. Again, by replacing $x_i$ by Weyl conjugates that permute the appropriate indices there is no loss of generality in assuming that if $\sum_{i=1}^{L} |J_i| \geq n - 1$, then $\bigcup_{i=1}^{L} J_i = \{2, \ldots, n\}$ and otherwise the sets $J_i$ are disjoint. Thus we may assume the cardinality of $\bigcup_{i=1}^{L} J_i$ is equal to

$$\min \left( n - 1, \sum_{i=1}^{m} (n - v_i) + \sum_{i=m+1}^{L-1} (n - u_i) + n - s_L \right).$$

If $\left| \bigcup_{i=1}^{L} J_i \right| = n - 1$, then already

$$\Omega = \bigcup_{i=1}^{L} \Omega_{x_i} \bigcup (\Omega_{x_L} \setminus \{FE_{e_1}\})$$

and we can directly apply the general strategy with $M = RE_{e_1}$, as in the first case.
So assume otherwise and choose \( m - 1 \) indices from \( \{2, \ldots, n\} \setminus \bigcup_{i=1}^{L} J_i \) (or all these indices if there are less than \( m - 1 \)), say the indices \( k_i \) for \( i = 2, \ldots, m' \).

For \( j \neq 1 \),

\[
Ad(\exp t RE_{e_j})(FE_{e_1}) = a_1 FE_{e_1} + tb_1 FE_{e_1+e_j} + tc_1 FE_{e_1-e_j}
\]

where \( a_1, b_1, c_1 \) converge to non-zero constants as \( t \to 0 \). For \( \ell \neq j \),

\[
Ad(\exp t RE_{e_j})(FE_{e_1} \pm e_\ell) = FE_{e_1} \pm e_\ell.
\]

Put \( g_i = \exp t RE_{e_{k_i}} \in G_{n-1} \) for \( i = 2, \ldots, m' \). Let \( J_0 \subseteq \{2, \ldots, n\} \) consist of the union of the sets \( J_j, j = 1, \ldots, L, \) together with the additional indices \( k_i, i = 2, \ldots, m' \). Since \( \Omega_{xL} \) contains all the root vectors of the form \( FE_{e_1+e_k} \) and \( \Omega_{xL} \) contains \( FE_{e_1} \), it follows that for small \( t \neq 0 \),

\[
sp \left( \bigcup_{j=M+1}^{L-1} \Omega_{x,j}, \Omega_{xL} \setminus \{FE_{e_1}\}, Ad(g)(\Omega_{xL}) : i = 2, \ldots, M, F = R, I \right)
\]

\[
= \left( FE_{e_1}, FE_{e_1+e_k} : k \geq 2, FE_{e_1-e_j}, j \in J_0, F = R, I \right).
\]

If \( m' < m \), then \( J_0 = \{2, \ldots, n\} \) and we complete the proof by taking \( M = RE_{e_1} \) and appealing to the general strategy.

Otherwise, the number of indices in \( \{2, \ldots, n\} \setminus J_0 \) (we call these the ‘missing’ indices) is

\[
N := n - 1 - \left( \sum_{i=1}^{m} (n - v_i) + \sum_{i=m+1}^{L-1} (n - u_i) + n - s_L + m - 1 \right).
\]

The eligibility assumption (5.2) implies

\[
N \leq n(1 - L) + \sum_{i=1}^{m} 2v_i + \sum_{i=m+1}^{L-1} 2u_i + s_L - \sum_{i=1}^{m} v_i - \sum_{i=m+1}^{L-1} u_i - m
\]

\[
\leq n(L - 1) - \sum_{i=1}^{m} v_i - \sum_{i=m+1}^{L-1} u_i,
\]

which coincides with the cardinality of \( \bigcup_{i=1}^{L-1} J_i \). As \( \bigcup_{i=1}^{L} J_i \subseteq J_0 \), we deduce that \( N \leq (n - 1)/2 \). There is no loss of generality in assuming the missing indices are \( \{n - N + 1, \ldots, n\} \) (in other words, \( J_0 = \{2, \ldots, n - N\} \)), and the indices in \( \bigcup_{i=1}^{L-1} J_i \) include \( \{2, \ldots, N + 1\} \). Note that \( N + 1 < n - N + 1 \).

These observations imply that

\[
H_0 := \sum_{k=2}^{N+1} RE_{e_{k}+e_{n-N+1+k}} \in g_{n-1}
\]
is well defined. Furthermore, whenever \( j \in \{2, \ldots, N + 1\} \), then
\[
ad(H_0)(F E_{e_1+e_j}) = a_F F E_{e_1-e_{n-N-1+j}}
\]
where \( a_F \neq 0 \).

We will now appeal to Lemma 4.7, where we take \( \Xi_i = Ad(g_i)\Omega_{x_i} \) for \( 2 \leq i \leq m \), \( \Xi_j = \Omega_{x_j} \) for \( j = 1 \) and \( m < j \leq L \), \( \Omega_0 = \{RE_{e_1}, IE_{e_1}\} \) and \( \Omega_* = \{FE_{e_1+e_k} : k = 2, \ldots, N + 1, F = R, I\} \).

As the root vectors \( FE_{e_1+e_j} \), for each choice of \( j \in \bigcup_{i=1}^{L-1} J_i \), occur in both the set
\[
\Omega_{x_1} \bigcup_{k=m+1}^{L-1} \bigcup_{k=2}^{m} Ad(g_k)\Omega_{x_k}
\]
and the set \( \Omega_{x_L} \), we see that
\[
\Omega_* \subseteq \left( \bigcap_{i=1}^{L-1} \Xi_i \right) \setminus \Omega_0.
\]
Since \( J_0 = \{2, \ldots, n - N\} \) and
\[
ad(H_0)(\Omega_*) = \{FE_{e_1-e_k} : k \geq n - N + 1, F = R, I\},
\]
follows that
\[
sp\left( ad(H_0)\Omega_*, \Xi_L \setminus \Omega_0, \bigcup_{i=1}^{L-1} \Xi_i \setminus \Omega_* \right) = sp\left( FE_{e_1}, FE_{e_1+e_i}, FE_{e_1-e_j}, FE_{e_1-e_k} : i \geq 2, j \in J_0, k \geq n - N + 1, F = R, I\right)
\]
\[
= sp\Omega
\]
Thus Lemma 4.7 implies property (i) of the general strategy is satisfied with the \( g_i \) there replaced by \( \exp tH_0 \cdot g_i \) for \( i = 2, \ldots, m \). We now complete the proof in the usual way with \( M = RE_{e_1} \).

Lastly, suppose \( m = 0 \). The argument is similar, but with a few important differences. To begin, put
\[
J = n - \sum_{i=1}^{L-1} (n - u_i) \text{ and } \Omega_0 = \{FE_{e_1+e_{n-J+1}} : F = R, I\}.
\]
Without loss of generality, we can assume
\[
\bigcup_{i=1}^{L-1} \Omega_{x_i} = \{ FE_{e_1 \pm e_j} : j = 2, \ldots, n - J + 1, F = R, I \} \quad \text{and} \quad \Omega_{x_L} = \{ FE_{e_1}, FE_{e_1 + e_k}, FE_{e_1 - e_j} : k \geq 2, j > s_L, F = R, I \}.
\]

Note that \( \Omega_0 \subseteq \bigcup_{i=1}^{L-1} \Omega_{x_i} \cap \Omega_{x_L} \).

If \( n - J + 1 \geq s_L \), then \( \bigcup_{i=1}^{L-1} \Omega_{x_i} \cup (\Omega_{x_L} \setminus \Omega_0) = \Omega \) and we complete the argument in the usual fashion with \( M = RE_{e_1 + e_{n-J+1}} \).

Otherwise, let
\[
H_0 = \sum_{j=2}^{s_L+J-n} RE_{e_j} + e_{n-J+j}
\]
and
\[
\Omega_\ast = \{ FE_{e_1} + e_k : 2 \leq k \leq n - J \} \subseteq \left( \bigcup_{i=1}^{L-1} \Omega_{x_i} \cap \Omega_{x_L} \right) \setminus \Omega_0.
\]

Using arguments similar to the previous case, the eligibility condition allows us to deduce that
\[
sp \left\{ ad(H_0)(\Omega_\ast), \bigcup_{i=1}^{L-1} \Omega_{x_i} \setminus \Omega_\ast, \Omega_{x_L} \setminus \Omega_0 \right\} = sp \Omega,
\]
and again we conclude that for a suitable \( g \in G_{n-1} \) we have
\[
sp \left\{ Ad(g) \left( \bigcup_{i=1}^{L-1} \Omega_{x_i} \right), \Omega_{x_L} \setminus \Omega_0 \right\} = sp \Omega.
\]

The proof is completed by taking \( M = RE_{e_1 + e_{n-J+1}} \) and applying the general strategy. We refer the reader to the proof of Prop. 5.8 Case 3 in [7] where all the details of a similar argument can be found.

**Proof of Theorem 5.1** The necessity of eligibility was shown in Proposition 3.10, thus we only need prove its sufficiency. We proceed by induction on the rank \( n \) of \( G_n \). Lemma 5.2 establishes the base case, thus we may assume that \( n \geq 3 \) and that all eligible, non-central tuples in \( G_{n-1} \) are absolutely continuous.

Let \((x_1, \ldots, x_L)\) be an eligible tuple with each \( x_j \in T_n \). If two or more \( x_i \) are of dominant type \( S \) or \( BD \), then according to Lemma 4.2 we even have \( \mu_{x_1} \ast \cdots \ast \mu_{x_L} \in L^2 \), so the convolution is clearly absolutely continuous.

Thus we may assume at most one \( x_j \) is of dominant type either \( S \) or \( BD \).

First, assume there are none. Suppose \( x_1, \ldots, x_m \) are each of dominant type \( D \) and \( x_{m+1}, \ldots, x_L \) are each of dominant type \( B \), with \( x_i \) of type \( B_{u_i} \times D_{v_i} \times SU(s_{i,1}) \times \)
\( \cdots \times SU(s_{i_1}, t_{i_1}) \). Simple calculation shows that if \( m \) is odd, then eligibility means

\[
\sum_{i=1}^{m} 2v_i + \sum_{i=m+1}^{L} 2u_i \leq 2n(L - 1) + m - 1. \tag{5.3}
\]

Consider \( y_1 \) of type \( B_{v_1} \times D_{u_1} \times SU(s_{1,1}) \times \cdots \times SU(s_{1, t_1}) \). We have \( v_1 > u_1 \) and \( 2v_1 > s_1 \), hence \( y_1 \) is of dominant type \( B \). Furthermore, by taking a Weyl conjugate, if necessary, we can assume \( y_1 \) has the same annihilating roots of the form \( e_i \pm e_j \) as \( x_1 \), and more annihilating roots of the form \( e_k \), so \( N_{x_1} \supseteq N_{y_1} \). Consequently, Proposition 2.4 implies that if \((y_1, x_2, \ldots, x_L)\) is absolutely continuous, so is \((x_1, x_2, \ldots, x_L)\). This shows there is no loss of generality in assuming that if the tuple admits only terms of dominant type either \( B \) or \( D \), then we can assume the number of terms of dominant type \( D \) is even.

Next, suppose \( x_1, \ldots, x_m \) is of dominant type \( D \), \( x_{m+1}, \ldots, x_{L-1} \) is of dominant type \( B \), and \( x_L \) is of dominant type \( BD \). We can choose to define \( \Omega_{x_L} \) either as \( \Omega_y \) where \( y \) is dominant \( B \) type with \( S_y = S^{(1)}_{x_L} \), or as \( \Omega_z \) where \( z \) is dominant type \( D \) with \( S_z = S^{(2)}_{x_L} \), depending on which is more helpful.

First, suppose \( m \) is even. Then eligibility requires

\[
\sum_{i=1}^{m} 2v_i + 2v_L + \sum_{i=m+1}^{L-1} 2u_i \leq 2n(L - 1) + m.
\]

This is the eligibility requirement for the case of a tuple with \( m + 1 \) terms of dominant type \( D \) and \( L - m - 1 \) terms of dominant type \( B \), thus if we define \( \Omega_{x_L} \) as we would for a dominant \( D \) type, we can prove the absolute continuity of \( x_1, \ldots, x_L \) using the same arguments as for the case of \( m + 1 \) terms of dominant type \( D \) and \( L - m - 1 \) terms of dominant type \( B \).

Similarly, if \( m \) is odd, eligibility requires

\[
\sum_{i=1}^{m} 2v_i + \sum_{i=m+1}^{L} 2u_i \leq 2n(L - 1) + m - 1.
\]

In this situation, we define \( \Omega_{x_L} \) as we would for a dominant \( B \) type and note that the inequality above is precisely the eligibility requirement for a tuple with \( m \) terms of dominant type \( D \) and \( L - m \) of dominant type \( B \).

This reasoning allows us to reduce the case of one \( x_j \) of dominant type \( BD \) to the case where all \( x_j \) are of dominant type \( B \) or \( D \).

These observations reduce the problem to establishing the absolute continuity of eligible tuples satisfying either the hypothesis of Lemma 5.3(a) or (b), and hence that Lemma completes the proof. \( \square \)
5.2 Applications

Here are some consequences of our characterization theorem.

**Corollary 5.4** (a) The product of conjugacy classes, \( \prod_{i=1}^{L} C_{x_i} \), has positive measure (or non-empty interior) if and only if \((x_1, \ldots, x_L)\) is eligible.

(b) If \(x_j\) and \(y_j\) are of the same dominant type and \(S \cdot x_j = S \cdot y_j\) for all \(j\), then \(\mu_{x_1} \cdots \mu_{x_L}\) is absolutely continuous if and only if \(\mu_{y_1} \cdots \mu_{y_L}\) is absolutely continuous.

**Proof** These statements follow from our Theorem 5.1, together with Proposition 2.4 in the case of (a) and the observation that \((x_1, \ldots, x_L)\) is eligible if and only if \((y_1, \ldots, y_L)\) is eligible in (b).

**Example 5.5** Suppose each \(x_j\) is of dominant type \(D\) and \(S \cdot x_j = 2 \cdot y_j\). Take \(y_j\) of type \(D_{y_j}\). Then \((x_1, \ldots, x_L)\) is absolutely continuous if and only if \((y_1, \ldots, y_L)\) is absolutely continuous. This is despite the fact that \(\Phi_{x_j}\) is a proper subset of \(\Phi_{x_j}\) (unless \(x_j\) is conjugate to \(y_j\)), so \(N_{s_{x_j}} \not\subseteq N_{y_j}\) and \(\dim C_{x_j} < \dim C_{y_j}\).

Since it is always the case that \(\Phi_{x_j}^N \subseteq \Phi_{x_j}\) if \((x_1, \ldots, x_j)\) is absolutely continuous in \(G\), then \((X_{x_1}, \ldots, X_{x_L})\) is absolutely continuous in \(g\). The converse is true when the Lie group \(G\) is Lie type \(A_n\), but not when \(G\) is Lie type \(B_n\). For example, if \(x\) is type \(D_n\) in a Lie group of type \(B_n\), then \(X_x\) is type \(SU(n)\). Hence \((X_{x}, X_x)\) is absolutely continuous for all choices of \(n\). But \((x, x)\) is never eligible, not even when \(n = 2\). Indeed, it was shown in [4] that if \(x\) is type \(D_n\) in \(B_n\), then \(\mu_{x}^{2n}\) is absolutely continuous and \(\mu_{x}^{2n-1}\) is purely singular. In fact, more can be said.

**Corollary 5.6** Suppose \(x_j \in G_n \setminus Z(G_n)\) for \(j = 1, \ldots, L\).

(a) Suppose \(L = 2n - 1\). The convolution \(\mu_{x_1} \cdots \mu_{x_L}\) is not absolutely continuous with respect to Haar measure on \(G_n\) if and only if all \(x_j\) are type \(D_n\).

(b) If \(L = 2n\), then \(\mu_{x_1} \cdots \mu_{x_L}\) is absolutely continuous and \(\prod_{i=1}^{2n} C_{x_i}\) has non-empty interior.

**Proof** (a) One can easily check that if \(x \notin Z(G_n)\) is not type \(D_n\), then \(S_x \leq 2n - 1\), with equality only if \(x\) is type \(B_{n-1}\), and that for any \(x \notin Z(G_n), S_x \leq 2n\), with equality only if \(x\) is type \(D_n\).

Suppose \(x_1\) is not type \(D_n\). If either \(S_{x_1} < 2n - 1\) or \(S_{x_j} < 2n\) for some \(j > 1\), then

\[
\sum_{i=1}^{2n-1} S_{x_i} \leq (2n - 1) + (2n - 2)(2n) - 1 = (2n + 1)(2n - 2).
\]

Thus \((x_1, \ldots, x_{2n-1})\) is eligible and therefore \(\mu_{x_1} \cdots \mu_{x_{2n-1}}\) is absolutely continuous.

So assume \(S_{x_1} = 2n - 1\) and all \(S_{x_j} = 2n\) for \(j > 1\). That means \(x_1\) is of dominant type \(B\) and \(x_2, \ldots, x_{2n-1}\) are of dominant type \(D\). Consequently, there are an even number of dominant type \(D\). That means \((x_1, \ldots, x_{2n-1})\) is eligible provided \(\sum_{i=1}^{2n-1} S_{x_i} \leq (2n + 1)(2n - 2) + 1\), and that is obviously true.

(b) One can similarly check that any \(2n\)-tuple of elements \(x_j \notin Z(G_n)\) in \(G_n\) is eligible. \(\square\)
Remark 5.7 We note that (b) also follows from the fact that for all \( x \notin Z(G_n), \mu_x^{2n} \in L^2 \), as shown in [4].

Corollary 5.8 If \((x_1, \ldots, x_L)\) is eligible, then there are matrices \( g_i \) such that \( \prod_{i=1}^{L} g_i x_i g_i^{-1} \) has distinct eigenvalues.

Proof This is due to the fact that a set in \( G \) with non-empty interior must contain elements with distinct eigenvalues. \( \square \)

It is an open problem to characterize the tuples \((x_1, \ldots, x_L)\) such that \( \prod_{i=1}^{2n} C_{x_i} \) admits an element with distinct eigenvalues.

5.3 Types \(C_n\) and \(D_n\)

We conclude with some comments on extending this characterization to the other classical Lie algebras, types \(C_n\) and \(D_n\). Every element of the torus of one of these Lie groups is conjugate to an element of the form

\[
\begin{align*}
  x &= (1, \ldots, 1, -1, \ldots, -1, e^{ia_1}, \ldots, e^{ia_1}, \ldots, e^{(\pm)ia_m}, \ldots, e^{(\pm)ia_m}) \\
  &= (u, v, s_1, s_m)
\end{align*}
\]

where the \( 0 < a_j < \pi \) are distinct, \( u \geq v, s_1 = \max s_j \) and \( u + v + s_1 + \cdots + s_m = n \).

The minus sign is only needed for \( D_n \). Such an \( x \) will be said to be of type \( C_u \times C_v \times SU(s_1) \times \cdots \times SU(s_m) \) or \( D_u \times D_v \times SU(s_1) \times \cdots \times SU(s_m) \)

as these are the Lie types of their sets of annihilating roots.

We will say \( x \) is of dominant type \( C \) (or \( D \)) if \( 2u \geq s_1 \) and then set \( S_x = 2u \), and say \( x \) is of dominant type \( S \) otherwise and set \( S_x = s_1 \). We will say that \((x_1, \ldots, x_L)\) is eligible if

\[
\sum_{i=1}^{L} S_{x_i} \leq 2n(L - 1).
\]

One can again show that eligibility is a necessary condition for absolute continuity.

We will say the pair \((x, y)\) in \(C_n\) or \(D_n\) is exceptional if:

(i) \( x \) is type \( C_{n/2} \times C_{n/2} \) (or \( D_{n/2} \times D_{n/2} \)) \((n \text{ even})\) and \( y \) is the same type as \( x \) or type \( SU(n) \);

(ii) \( x \) is type \( C_{n/2} \times C_{n/2} \) and \( y \) is type \( C_{n/2} \times C_{n/2-1} \) \((n \text{ even})\) or \( x \) is type \( C_{(n+1)/2} \times C_{(n-1)/2} \) and \( y \) is type \( C_{(n-1)/2} \times C_{(n-1)/2} \) \((n \text{ odd})\);

(iii) (in \( D_n \) only) \( x \) is type \( SU(n) \) and \( y \) is type \( SU(n) \) or \( SU(n - 1) \).

The pairs of (i) are eligible, but their reductions (defined in a similar manner as in \( B_n \)) are not. In fact, it can be seen from [6, Thm. 9.1] that the pair of type \((C_{n/2} \times C_{n/2}, C_{n/2} \times C_{n/2})\) is not absolutely continuous (although it is eligible). The pairs of (ii) are entwined under the reduction process and it is unclear whether their base case, \((C_2 \times C_1, C_1 \times C_1)\) in \( C_3 \), is an absolutely continuous pair. The pairs in (iii) coincide...
with exceptional pairs from the Lie algebra problem, whose absolute continuity was undetermined in [7].

All non-exceptional, eligible tuples reduce to non-exceptional, eligible tuples.

**Conjecture 1** Suppose the non-central elements $x_j$ belong to the torus of $C_n$ or $D_n$ for $n \geq 5$ and that $(x_1, \ldots, x_L)$ is not exceptional. Then $(x_1, \ldots, x_L)$ is absolutely continuous if and only if it is eligible.

It is natural to appeal to an induction argument on the rank of the Lie group, as we did for type $B_n$. But for the Lie groups of types $C_n$ and $D_n$, the induction step will be much easier than it was for $B_n$ as it will follow from the arguments given for the proof in the Lie algebras setting in [7]. This is essentially because the eligibility condition is the same for $(x_1, \ldots, x_L)$ and $(X_{x_1}, \ldots, X_{x_L})$, and even though it need not be true that $\Phi_{X_{x}}^{g_n} = \Phi_x$ for $g_n$ the Lie algebra of type $C_n$ or $D_n$, it is true that $N_{X_{x}}^{g_n} \setminus N_{X_{x'}}^{g_n-1} = N_x \setminus N_{x'}$.

However, establishing the base cases will be onerous with our current techniques. For $C_n$, this task is complicated by the fact that when $x$ is type $C_2 \times C_1$ in the group $C_3$, then $\mu^3_x$ is not absolutely continuous. In $D_4$ we have the same issue with $x$ of type $SU(4)$. In fact, for all $D_n$ and $x$ of type $SU(n)$, $\mu^2_x$ is not absolutely continuous and that means we cannot immediately reduce to the case of at most one $x_j$ of type $S$, as we did for $B_n$. This latter problem arose in the Lie algebra setting, as well, and a modification of the approach taken there will work here.

For $C_3$, $C_4$ and $D_4$, it appears there will be other exceptional tuples and that new methods will be needed to determine which of these, as well as which of the exceptional tuples listed above, are absolutely continuous.

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