CHIRALITY IN INCIDENCE GEOMETRY

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Abstract. Guided by the ideas of chirality in the abstract polytope theory, the present paper aims to extend the concept to a more general setting of incidence geometries. The purpose of this paper is to explore the more general framework of thin residually connected chiral geometries and also to take this opportunity to look at the regular case in a more detailed way. We give characterisations of automorphism groups of regular and chiral thin residually connected geometries in the same spirit as Coxeter groups.

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1. Introduction

A concept of a homogeneous honeycomb in euclidean space was introduced by Sommerville in 1929 in his book “An Introduction to the Geometry of n Dimensions” [29] as an object consisting of polyhedral cells, all alike, such that each rotation that is the symmetry operation of a cell is also a symmetry operation of the whole configuration. This definition inspired Coxeter to name a map regular whenever its group of automorphisms contains for each of its face elements that cyclically permute edges of that face and the automorphisms that for each of its vertices cyclically permute the edges meeting at the vertex. Coxeter distinguished two kinds of regular maps: reflexible and irreflexible [15]; nowadays commonly referred to respectively as regular and chiral maps (he called the map reflexible when the group of automorphisms of a map contains an element that fixes an edge by interchanging its two vertices while fixing the two faces that contain the edge, otherwise he called the map irreflexible).

Earliest known examples of chiral maps were produced by Heffter in 1898 [21] as a family of maps of Schlafli type (in which the first number in the symbol describes faces and the second number describes vertices of the map) \( \{2^k - 1, 2^k - 1\} \) for \( k > 2 \). In the 1940's Coxeter classified regular and chiral maps on the torus [13]. In 1966, Sherk [28], a PhD student of Coxeter, looked for chiral maps of small genus and constructed an infinite family of chiral maps of type \( \{6, 6\} \) (with the smallest number on a surface of genus 7). About the same time Edmonds re-discovered, but never published his findings, Heffter’s map of type \( \{7, 7\} \) (also on a surface of genus 7). In 1969, Garbe [18] proved that there are no chiral maps on surfaces of genus 2, 3, 4, 5 or 6. A number of papers appeared thereafter dealing with chiral maps. A first systematic search for chiral maps was conducted by Conder and Dobcsányi [9] and resulted in the complete list of chiral maps on surfaces of genus 2 to 15 in the orientable case and genus 3 to 30 in the non-orientable case. Consequently Conder extended this list several times. It now contains maps up to genus 301 in the orientable case and up to genus 602 in the non-orientable case [11].
In 1970 [14], Coxeter extended the above notions introducing the concept of a twisted honeycomb, a finite abstract object or rank 4 derived from a honeycomb, which is chiral in a sense that it inherits all the rotations of its cells but not its reflections. Two similar examples of such structures, both with only one polyhedral cell, were described earlier by Weber and Seifert in 1933 [32]. Coxeter produced a number of non-trivial examples, which he constructed from 3-dimensional euclidean and hyperbolic tessellations with spherical facets and vertex-figures, by looking at their Petrie polygons which naturally occur, in left- and right-handed varieties (Each such polygon has three, but not four, consecutive edges belonging to a cell.) Identifying vertices of each such, say left-handed Petrie polygon, that are separated by a fixed number of edges he observed that the resulting object may have the vertices on its right-handed Petrie polygons separated by a different number of edges. Coxeter called such a rank 4 object twisted as it is not symmetrical by a “reflection” in a sense that its automorphism group contains no involution that fixes, for example, a rank 2 face of the object but interchanges the two cells sharing this rank 2 face. The concept of twisted honeycomb inspired the modern definition of a chiral polytope, an abstract object of any rank that is maximally symmetric by abstract rotations but never by an abstract reflection (see Section 7).

The twisted honeycombs are finite structures resembling classical polytopes combinatorially as well as in the sense that their facets and vertex-figures are spherical. In 1977 Coxeter suggested to the last author to derive finite twisted honeycombs from 3-dimensional hyperbolic tessellations with horospherical facets and/or vertex-figures producing therefore rank 4 objects with toroidal facets and/or vertex-figures. About the same time Grunbaum [19] suggested to study abstract objects, which he called polystroma, whose faces and vertex-figures are not necessarily spherical. Inspired by the ideas of Coxeter and Grunbaum, in 1982 Schulte and Danzer [16] formalized and developed the theory of regular abstract polytopes (which they named incidence polytopes). In 1984, Colbourn and Weiss [8] unaware of the work of Schulte and Danzer, published a census of regular and chiral finite rank 4 polystroma derived from hyperbolic tessellations by applying the “twisted honeycomb” method of Coxeter. Not all such objects satisfied a more restrictive condition of abstract polytopality of [16].

By late 1980s a number of sporadic examples of chiral abstract polytopes in rank 3 and 4 were found. In 1991 Schulte and the last author of this paper developed the basic structure theory of abstract chiral polytopes [20] and in particular characterized their automorphism groups. These objects are now quite well understood and have been studied extensively over the past 30 years. Schulte, Monson and Weiss developed various methods of constructing such polytopes in rank 4. However, the classical approach to constructing higher rank polytopes inductively from the lower ones proved to be impossible for chiral polytopes. Although in 1995 [27] there was a universal extension method found leading to rank 5 chiral polytopes with regular facets, no chiral finite polytopes were known to exist in rank 5 or higher. It was only in 2008 that Conder, Hubard and Pisanski produced first examples of finite higher rank chiral polytopes [10] and in 2010 Pellicer [25] gave a construction for arbitrary rank.

In [20], Hartley, Hubard and Leemans constructed two atlases of chiral polytopes. Firstly they sought them as quotients of regular polytopes arising from the Atlas of Small Regular Polytopes (http://www.abstract-polytopes.com/atlas/); Secondly,
for each almost simple group $\Gamma$ such that $S \leq \Gamma \leq Aut(S)$ where $S$ is a simple group of order less than 900,000 listed in the Atlas of Finite Groups, they gave, up to isomorphism, the abstract chiral polytopes on which $\Gamma$ acts regularly. Such an atlas existed already in the regular case [22].

Guided by the ideas of chirality in the polytope theory, the present paper aims to extend the concept to a more general setting of incidence geometries. Indeed, when an incidence geometry is thin, it is possible to define chirality, as in the case of polytopes. It is then interesting to study how the group-theoretic counterpart of chiral polytopes extends in this more general framework, a chiral polytope being a thin incidence geometry with a linear diagram. The purpose of this paper is to explore this more general framework for chiral geometries and also to take this opportunity to look at the regular case in a more detailed way.

2. Thin residually connected regular geometries

In [30], Jacques Tits introduced the concept of geometry as an object generalizing the notion of incidence and established its close relation with groups (see also [4, Chapter 3]). Following [5], we begin by defining an incidence system $\Gamma$ (also called pre geometry or incidence structure in [4, 24]).

An incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- $X$ is a set whose elements are called the elements of $\Gamma$;
- $I$ is a set whose elements are called the types of $\Gamma$;
- $t : X \rightarrow I$ is a type function, associating to each element $x \in X$ of $\Gamma$ a type $t(x) \in I$;
- $*$ is a binary relation on $X$ called incidence, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The incidence graph of $\Gamma$ is the graph whose vertex set is $X$ and where two vertices are joined provided the corresponding elements of $\Gamma$ are incident. A flag is a set of pairwise incident elements of $\Gamma$, i.e. a clique of its incidence graph. The type of a flag $F$ is $\{t(x) : x \in F\}$. A chamber is a flag of type $I$. An element $x$ is incident to a flag $F$ and we write $x * F$ for that, provided $x$ is incident to all elements of $F$. An incidence system $\Gamma$ is a geometry or incidence geometry provided that every flag of $\Gamma$ is contained in a chamber (or in other words, every maximal clique of the incidence graph is a chamber). The rank of $\Gamma$ is the number of types of $\Gamma$, namely the cardinality of $I$.

We now define the notion of residue which is central in incidence geometry. Let $\Gamma := (X, *, t, I)$ be an incidence system and $F$ a flag of $\Gamma$. Given a flag $F$ of $\Gamma$, the residue of $F$ in $\Gamma$ is the incidence system $\Gamma_F := (X_F, *, t_F, I_F)$ where

- $X_F := \{x \in X : x * F, x \not\in F\}$;
- $I_F := I \setminus t(F)$;
- $t_F$ and $*_F$ are the restrictions of $t$ and $*$ to $X_F$ and $I_F$.

An incidence system $\Gamma$ is residually connected when each residue of rank at least two of $\Gamma$ has a connected incidence graph. It is called thin (resp. firm) when every residue of rank one of $\Gamma$ contains exactly two (resp. at least two) elements.

An incidence system $\Gamma := (X, *, t, I)$ is chamber-connected when for each pair of chambers $C$ and $C'$, there exists a sequence of successive chambers $C =: C_0, C_1, \ldots, C_n := C'$ such that $|C_i \cap C_{i+1}| = |I| - 1$. An incidence system $\Gamma := (X, *, t, I)$ is strongly chamber-connected when all its residues of rank at least 2 are chamber-connected.
We now state the following proposition which is a generalisation of Proposition 2A1 of [23].

**Proposition 2.1.** Let $\Gamma$ be a firm incidence geometry. Then $\Gamma$ is residually connected if and only if $\Gamma$ is strongly chamber-connected.

**Proof.** As in [23], the proof proceeds by induction on $n := \text{rank}(\Gamma)$, there being nothing to prove if $n \leq 1$. Let $n \geq 2$.

Suppose first that $\Gamma$ is residually connected and let us show that $\Gamma$ must be strongly chamber-connected. Let $C$ and $C'$ be two chambers of $\Gamma$. If $C \cap C' \neq \emptyset$, say $C \cap C' = \{x_i, \ldots, x_{jk}\}$ with $k \geq 1$, then the residue $\Gamma_{x_i}$ contains $C \setminus \{x_1\}$ and $C' \setminus \{x_1\}$ as chambers and $\Gamma_{x_i}$ is a residually connected firm geometry. Hence, by induction, we can find a sequence of successive chambers $C \setminus \{x_1\} =: C_0, C_1, \ldots, C_n := C' \setminus \{x_1\}$ in $\Gamma_{x_1}$ such that $|C_i \cap C_{i+1}| = n - 2$. The sequence $C =: C_0 \cup \{x_1\}, C_1 \cup \{x_1\}, \ldots, C_n \cup \{x_1\} := C'$ is then such that $|C_i \cap C_{i+1}| = n - 1$ as needed. Let $C \cap C' = \emptyset$. Since $\Gamma$ is residually connected, we can find a sequence of successively incident elements $x_0, \ldots, x_k$ such that $x_0 \in C$ and $x_k \in C'$. For each $i = 1, \ldots, k$, there is a chamber $C_i \supseteq \{x_{i-1}, x_i\}$. Set $C_0 := C$ and $C_k := C'$. We now appeal to the first part of the proof to move $C_{i-1}$ to $C_i$ by a sequence of adjacent chambers containing $x_i$ for each $i = 1, \ldots, k$; concatenation then gives the required sequence from $C$ to $C'$.

Suppose now that $\Gamma$ is strongly chamber-connected. In order to prove that $\Gamma$ is residually connected, it is enough to prove that it is connected as both properties are residual. Connectedness is obvious as every flag is contained in a chamber. Indeed, let $x_1$ and $x_2$ be two elements of $\Gamma$. Each of them is contained in at least one chamber. Let $x_1 \in C_1$ and $x_2 \in C_2$ where $C_1$ and $C_2$ are chambers of $\Gamma$. Using the fact that $\Gamma$ is strongly connected, we easily get a path from $x_1$ to $x_2$ using the sequence of successive chambers connecting $C_1$ to $C_2$. Hence $\Gamma$ is connected and therefore residually connected. \qed

A *weak hypertope* is a thin incidence geometry. A *hypertope* is a weak hypertope which is strongly chamber connected or equivalently residually connected.

Let $\Gamma := (X, *, t, I)$ be an incidence system. An *automorphism* of $\Gamma$ is a mapping $\alpha : (X, I) \to (X, I) : (x, t(x)) \mapsto (\alpha(x), t(\alpha(x)))$ where

- $\alpha$ is a bijection on $X$ inducing a bijection on $I$;
- for each $x, y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x, y \in X$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism $\alpha$ of $\Gamma$ is called *type preserving* when for each $x \in X$, $t(\alpha(x)) = t(x)$ (i.e. $\alpha$ maps each element on an element of the same type).

The set of type-preserving automorphisms of $\Gamma$ is a group denoted by $\text{Aut}_I(\Gamma)$. The set of automorphisms of $\Gamma$ is a group denoted by $\text{Aut}(\Gamma)$.

An incidence system $\Gamma$ is *flag-transitive* if $\text{Aut}_I(\Gamma)$ is transitive on all flags of a given type $J$. An incidence system $\Gamma$ is *chamber-transitive* if $\text{Aut}_I(\Gamma)$ is transitive on all chambers of $\Gamma$. Finally, an incidence system $\Gamma$ is *regular* if $\text{Aut}_I(\Gamma)$ acts regularly on the chambers (i.e. the action is semi-regular and transitive). The following proposition is folklore in incidence geometry. We recall it here as our paper makes bridges between different subjects.

**Proposition 2.2.** Let $\Gamma$ be an incidence geometry. $\Gamma$ is chamber-transitive if and only if $\Gamma$ is flag-transitive.
Proof. It is obvious that if $\Gamma$ is flag-transitive, then $\Gamma$ is chamber-transitive. Suppose $\Gamma$ is chamber-transitive. Let $F_1$ and $F_2$ be two flags of the same type. Each of them is contained in at least one chamber as $\Gamma$ is a geometry. Let $C_i$ be a chamber containing $F_i$ ($i = 1, 2$). Since $\Gamma$ is chamber-transitive, there exists an element $g \in Aut_\ell(\Gamma)$ such that $g(C_1) = C_2$. In particular, as $g$ preserves the types of the elements of $\Gamma$, we have $g(F_1) = F_2$. □

A regular weak hypertope is a flag-transitive weak hypertope. A regular hypertope is a flag-transitive hypertope. We use the adjective "regular" here as, if $\Gamma$ is a weak hypertope, it is thin and hence the action of $Aut_\ell(\Gamma)$ is necessarily faithful. Observe that every abstract regular polytope is a regular hypertope. Observe also that there are examples of regular geometries that are not thin (see for instance [4] geometry 2 for Sym(3)).

3. Thin regular geometries as coset geometries

Given an incidence system $\Gamma$ and a chamber $C$ of $\Gamma$, we may associate to the pair $(\Gamma, C)$ a pair consisting of a group $G$ and a set $\{G_i : i \in I\}$ of subgroups of $G$ where $G := Aut_\ell(\Gamma)$ and $G_i$ is the stabilizer in $G$ of the element of type $i$ in $C$. The following proposition shows how reverse this construction, that is starting from a group and some of its subgroups, construct an incidence system.

Proposition 3.1. (Tits, 1961) Let $n$ be a positive integer and $I := \{1, \ldots, n\}$ a finite set. Let $G$ be a group together with a family of subgroups $(G_i)_{i \in I}$, $X$ the set consisting of all cosets $Gg$, $g \in G$, $i \in I$ and $t : X \to I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by:

\[
G_i g_1 * G_j g_2 \text{ iff } G_i g_1 \cap G_j g_2 \text{ is non-empty in } G.
\]

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having a chamber. Moreover, the group $G$ acts by right multiplication as an automorphism group on $\Gamma$. Finally, the group $G$ is transitive on the flags of rank less than 3.

Observe that, in the proposition above, $G \leq Aut_\ell(\Gamma)$. When a geometry $\Gamma$ is constructed using the proposition above, we denote it by $\Gamma(G; (G_i)_{i \in I})$ and call it a coset geometry. The subgroups $(G_i)_{i \in I}$ are called the maximal parabolic subgroups. The Borel subgroup of the incidence system is the subgroup $B = \cap_{i \in I} G_i$. The action of $G$ on $\Gamma$ involves a kernel $K$ which is the largest normal subgroup of $G$ contained in every $G_i$, $i \in I$. If the kernel is the identity, we say that $G$ acts faithfully on $\Gamma$. If $G$ acts transitively on all chambers of $\Gamma$, hence also on all flags of any type $J$, where $J$ is a subset of $I$, we say that $G$ is flag-transitive on $\Gamma$ or that $\Gamma$ is flag-transitive (under the action of $G$). In that case, any chamber of $\Gamma$ is obtained by multiplying the cosets of the base chamber $\{G_0, \ldots, G_{r-1}\}$ on the right by an element $g \in G$.

If $G$ acts faithfully and flag-transitively on $\Gamma$, we say that $G$ acts regularly on $\Gamma$ or that $\Gamma$ is regular (under the action of $G$).

If $\Gamma(G; (G_i)_{i \in I})$ is a flag-transitive geometry and $F$ is a flag of $\Gamma$, the residue of $F$ is isomorphic to the incidence system

\[
\Gamma_F = \Gamma((\cap_{j \in \ell(F)} G_j), (G_i \cap (\cap_{j \in \ell(F)} G_j))_{i \in I \setminus \ell(F)})
\]

and we readily see that $\Gamma_F$ is also a flag-transitive geometry.

If $\Gamma$ is a geometry of rank 2 with $I = \{0, 1\}$ such that each of its 0-elements is incident with each of its 1-elements, then we call $\Gamma$ a generalized digon.
We refer to [2] and [3] for the notion of the Buekenhout diagram of a geometry and simplify it here to deal only with thin geometries. For a thin, residually connected, flag-transitive coset geometry $\Gamma(G; (G_i)_{i \in I})$, the Buekenhout diagram $B(\Gamma)$ is a graph whose vertices are the elements of $I$. Elements $i, j$ of $I$ are not joined by an edge of the diagram provided that a residue $\Gamma_F$ of type $\{i, j\}$ is a generalized digon. Otherwise, $i$ and $j$ are joined by an edge endowed with a number $g_{ij}$ that is equal to half the girth of the incidence graph of a residue $\Gamma_F$ of type $\{i, j\}$. On a picture of the diagram, this structure can locally be depicted as follows.

The following theorem, due to Jacques Tits, shows that if a geometry $\Gamma(G; (G_i)_{i \in I})$ has a non-trivial kernel $K$ then this geometry can be constructed from a smaller group, namely $G/K$.

**Theorem 3.2.** [30] Let $\Gamma(G; (G_i)_{i \in I})$ be a geometry. If $K$ is the kernel of the action of $G$ on $\Gamma$, then $\Gamma(G; (G_i)_{i \in I}) \cong \Gamma(G/K; (G_i/K)_{i \in I})$.

For this reason it is natural to assume that a thin flag-transitive geometry is regular. The following lemma shows that we then may assume that the Borel subgroup of a thin geometry is the identity subgroup.

**Lemma 3.3.** Let $\Gamma(G; (G_i)_{i \in I})$ be a thin regular geometry. Then $B = \cap_{i \in I} G_i = 1$.

**Proof:** Since $\Gamma$ is thin and regular, the groups $G_J = \cap_{j \in J} G_j$ where $J$ is any subset of $I$ of cardinality $|I| - 1$ contain $B$ as a subgroup of index 2. Hence $B$ is a normal subgroup of all these groups. Now, the subgroups $G_J$ generate $G$ and thus $B$ must also be a normal subgroup of $G$. This means $B$ is a kernel. Then in order to have a faithful action of $G$ on $\Gamma$, we must have $B = 1$. □

Observe that the subgroups $G_J$ appearing in the proof above are cyclic groups of order 2. This will enable us to make the connection between thin regular geometries and groups generated by involutions.

The following Theorem translate the residual connectedness (or strong flag-connectedness) condition in a group-theoretic condition.

**Theorem 3.4.** [17] Let $\Gamma(G; (G_i)_{i \in I})$ be a flag-transitive geometry. $\Gamma$ is residually connected if and only if for every subset $J \subseteq I$ of cardinality at most $|I| - 2$,

$$\cap_{j \in J} G_j = \langle \cap_{j \in J \cup \{k\}} G_j : k \in I - J \rangle.$$  

For $\Gamma(G; (G_i)_{i \in I})$ a thin regular residually connected geometry, we define $\rho_i$ as the generator of the minimal parabolic subgroup $\cap_{j \in I \setminus \{i\}} G_j$. Observe that all the $\rho_i$'s are involutions. We call the set $\{\rho_i : i \in I\}$ the distinguished generators of $\Gamma(G; (G_i)_{i \in I})$.

The following result gives a way to check whether $\Gamma$ is a flag-transitive geometry. See also Dehon [17].

**Theorem 3.5.** (Buekenhout, Hermand [2]) Let $P(I)$ be the set of all the subsets of $I$ and let $\alpha : P(I) - \{\emptyset\} \to I$ be a function such that $\alpha(J) \in J$ for every $J \subset I$, $J \neq \emptyset$. the geometry $\Gamma$ is flag-transitive if and only if, for every $J \subset I$ such that $|J| \geq 3$, we have

$$\cap_{j \in J - \alpha(J)} (G_j G_{\alpha(J)}) = \left( \cap_{j \in J - \alpha(J)} G_j \right) G_{\alpha(J)}.$$
A proof of this result is also available in the book by Buekenhout and Cohen (see [3 Theorem 1.8.10]).

4. C-groups

A C-group of rank \( r \) is a pair \((G, S)\) such that \( G \) is a group and \( S := \{\rho_0, \ldots, \rho_{r-1}\} \) is a generating set of involutions of \( G \) that satisfy the following property.

\[
\forall I, J \subseteq \{0, \ldots, r - 1\}, \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle \tag{4.1}
\]

This property is called the intersection property and denoted by (IP). We call any subgroup of \( G \) generated by a subset of \( S \) a parabolic subgroup of the C-group \((G, S)\). In particular, a parabolic subgroup generated by exactly one involution of \( S \) is called minimal and a parabolic subgroup generated by all but one involutions of \( S \) is called maximal. We write \( G_J := \langle \rho_j \mid j \in J \rangle \) for \( J \subseteq \{0, \ldots, r - 1\} \) and \( G_i := G_{\{i\}} \). Obviously, \( G_0 = \{1_G\} \) and \( G_{\{0, \ldots, r-1\}} = G \).

A C-group is a string C-group provided its generating involutions can be reordered in such a way that \((\rho_i, \rho_j)^2 = 1_G\) for all \( i, j \) with \(|i - j|^2 > 1\).

We say that two C-groups \((G, S)\) and \((G', S')\) are isomorphic if there is an isomorphism \( \alpha : G \to G' \) such that \( \alpha(S) = S' \).

The Coxeter diagram \( \mathcal{C}(G, S) \) of a C-group \((G, S)\) is a graph whose vertex set is \( S \). Two vertices \( \rho_i \) and \( \rho_j \) are joined by an edge labelled by \( \alpha(\rho_i, \rho_j) \). As a consequence, the Coxeter diagram is a complete graph. We take the convention of dropping an edge if its label is 2 and of not writing the label if it is 3. The Coxeter diagram of a string C-group has a string shape.

**Proposition 4.1.** Let \( I := \{0, \ldots, r - 1\} \) and let \( \Gamma := \Gamma(G; (G_i)_{i \in I}) \) be a regular hypertope of rank \( r \). The pair \((G, S)\) where \( S \) is the set of distinguished generators of \( \Gamma \) is a C-group of rank \( r \).

**Proof.** The proposition is obviously true for \( r = 2 \). Suppose the proposition is true for \( r - 1 \) and let us show it is then true for \( r \) by way of contradiction. Let us denote by \( G_{\overline{I}} \) the subgroup \(<\rho_k \mid k \in I \setminus K>\).

If \((G, S)\) does not satisfy (4.1), then there is a pair of subgroups, \( G_{\overline{K}} \) and \( G_{\overline{J}} \) with \( K, J \subseteq I \) such that \( G_{\overline{K}} \cap G_{\overline{J}} \neq G_{\overline{K} \cup J} \). Hence \( G_{\overline{K}} \cap G_{\overline{I}} \neq G_{\overline{K} \cup J} \). Take \( g \in (G_{\overline{K}} \cap G_{\overline{J}}) \setminus G_{\overline{K} \cup J} \). This \( g \) fixes a flag of type \( K \cup J \) in the base flag \( \{G_0, \ldots, G_{r-1}\} \). But the action of \( G_{\overline{K} \cup J} \) must be regular on the residue \( \Gamma_F \) of the flag \( F := \{G_i \mid i \in K \cup J\} \). Indeed, that residue is also a thin regular residually connected geometry and its distinguished generators are exactly those of \( G_{\overline{K} \cup J} \) and satisfy (4.1) by induction. Any element of \( G_{\overline{K} \cup J} \) will fix all elements of \( \{G_i \mid j \in K \cup J\} \). Since \( G_{\overline{K} \cup J} \) is regular on \( \Gamma_F \), there must exist an element \( h \in G_{\overline{K} \cup J} \) that sends the flag \( \{G_k \mid k \in I \setminus (K \cup J)\} \) onto \( \{G_k \ast g \mid k \in I \setminus (K \cup J)\} \). But then \( g \ast h^{-1} \neq 1_G \) fixes the base chamber \( \{G_i \mid i \in I\} \), a contradiction with the regularity of the action of \( G \) on the chambers of \( \Gamma \).

Observe that we can construct a coset geometry \( \Gamma(G; (G_i)_{i \in I}) \) in a natural way from a C-group \((G, S)\) of rank \( r \) by letting \( G_i := \langle \rho_j \mid \rho_j \in S, j \in I \setminus \{i\} \rangle \) for all \( i \in I := \{0, \ldots, r - 1\} \). This construction always gives a thin, residually connected, regular coset geometry when the rank is at most 2. We next show that, this construction will not give, for rank three (and therefore also not for higher ranks), a thin, residually connected, regular coset geometry. To this end, we recall a group-theoretical result of Tits.
Lemma 4.2. (Tits [31]) Let $G_0, G_1, G_2$ be three subgroups of a group $G$. Then the following conditions are equivalent.

1. $G_0 G_1 \cap G_0 G_2 = G_0 (G_1 \cap G_2)$
2. $(G_0 \cap G_1) : (G_0 \cap G_2) = (G_1 G_2) \cap G_0$
3. If the three cosets $G_0 x, G_1 y$ and $G_2 z$ have pairwise nonempty intersection, then $G_0 x \cap G_1 y \cap G_2 z \neq \emptyset$.

Proposition 4.3. Let $(G, \{\rho_0, \rho_1, \rho_2\})$ be a C-group of rank three and let $\Gamma := \Gamma(G; \{\langle \rho_1, \rho_2 \rangle, \langle \rho_0, \rho_2 \rangle, \langle \rho_0, \rho_1 \rangle\})$. Then $\Gamma$ is thin if and only if $\Gamma$ is regular. Moreover, if $\Gamma$ is thin (or regular), it is a regular hypertope.

Proof. First as $G_0 \cap G_1 \cap G_2 = \{1_G\}$, $G$ acts faithfully on $\Gamma$. Now suppose that $\Gamma$ is flag transitive. Take a flag of type $\{i, j\}$, say $\{G_i, G_j\}$. As $G_i \cap G_j \cong C_2$ there are exactly two elements of type $k$ incident to $G_i$ and $G_j$. Hence $\Gamma$ is thin.

Conversely, suppose that $\Gamma$ is thin. If $\Gamma$ is not flag transitive, there exists a chamber $\{G_0, G_1, G_2z\}$ such that $G_0 \cap G_1 \cap G_2z = \emptyset$. But $G_2$ and $G_2 \rho_2$ as well as $G_2z$ are incident to both $G_0$ and $G_1$. Hence the residue of the flag $\{G_0, G_1\}$ has at least three elements and $\Gamma$ is not thin.

When $\Gamma$ is flag transitive we use Theorem 3.4 to conclude that $\Gamma$ residually connected. By Proposition 2.1 $\Gamma$ is therefore a hypertope. □

Example 4.4. Take the hypermap of type $(3,3,3)$, whose automorphism group $G \cong E_6 : C_2$ is of order 18. Defining relations for the automorphism group are

\[ \rho_0^3 = \rho_1^3 = \rho_2^3 = (\rho_0 \rho_1 \rho_2)^2 = (\rho_0 \rho_1)^3 = (\rho_1 \rho_2)^3 = 1. \]

The pair $(G, \{\rho_0, \rho_1, \rho_2\})$ is a C-group (satisfying IP). However, the coset geometry $\Gamma := \Gamma(G; \{\langle \rho_1, \rho_2 \rangle, \langle \rho_0, \rho_2 \rangle, \langle \rho_0, \rho_1 \rangle\})$ is a $K_{3,3,3}$ and hence $G$ cannot be flag-transitive on $\Gamma$ as $\Gamma$ has $3^3 = 27$ chambers. Moreover, $\Gamma$ is not thin, but it is residually connected.

We give a second example to show that even residual connectedness may be lost in higher ranks.

Example 4.5. Let $G \cong A_6$ and define $S := \{\rho_0 = (1, 2)(3, 4), \rho_1 = (2, 6)(3, 5), \rho_2 = (1, 4)(2, 3), \rho_3 = (1, 4)(3, 5)\}$. It can be checked by hand or using MAGMA that $(G, S)$ is a C-group. This C-group was mentioned in [12]. It has the following Coxeter diagram and sublattice.
However it is not giving a thin residually connected regular geometry with the construction above. Indeed, the subgroups \(G_1 := \langle (1, 2)(3, 4), (1, 4)(2, 3), (1, 4)(3, 5) \rangle\) and \(G_2 := \langle (1, 2)(3, 4), (1, 4)(2, 3), (1, 4)(3, 5) \rangle\) are both isomorphic to \(A_5\) and their intersection is a dihedral group of order 10. This means that the corresponding coset geometry will have 6 elements of type 1 and 6 elements of type 2, and that each element of type 1 is incident to each element of type 2. But then, the residue of an element of type \(\{0, 3\}\), which is supposed to be a 4-gon by the Coxeter diagram, consists of 4 elements of type 1 and four elements of type 2, forming a complete bipartite graph \(K_{4,4}\). Therefore, the coset geometry cannot be thin. In addition, it can be checked that the coset geometry is also not residually connected nor flag-transitive.

It follows that, given a thin, residually connected, regular coset geometry \(\Gamma := \Gamma(G; (G_i)_{i \in I})\) and the pair \((G, S)\) defined as above, we have \(C(G, S) \cong B(\Gamma)\).

**Proposition 4.6.** Let \((G, \{\rho_0, \ldots, \rho_{r-1}\})\) be a C-group of rank \(r\) and let \(\Gamma := \Gamma(G; (G_i)_{i \in I})\) with \(G_i := \langle \rho_j | \rho_j \in S, j \in I \setminus \{i\} \rangle\) for all \(i \in I := \{0, \ldots, r - 1\}\). If \(\Gamma\) is flag-transitive, then \(\Gamma\) is a regular hypertope.

**Proof.** Residual connectedness follows from Theorem 3.4 the fact that \(\Gamma\) is flag-transitive and the definition of \(\Gamma\). The minimal parabolic subgroups of \(\Gamma\) are cyclic groups of order 2 by the intersection property of the C-group, hence \(\Gamma\) is thin. Regularity also follows from the intersection property as the intersection of all maximal parabolic subgroups of \(\Gamma\) must be reduced to the identity. \(\square\)

### 5. Applications to Polytopes

An abstract polytope \(P\) is a ranked partially ordered set whose elements are called *faces*. A polytope \(P\) of rank \(n\) has faces of ranks \(-1, 0, \ldots, n\); \(P\) has a smallest and a largest face, of ranks \(-1\) and \(n\), respectively. Each maximal chain (or chamber) of \(P\) contains \(n + 2\) faces, one for each rank. \(P\) is strongly chamber-connected. \(P\) is *thin*, that is, for every flag and every \(j = 0, \ldots, n - 1\), there is precisely one other \((j\text{-adjacent})\) flag with the same faces except the \(j\)-face. This condition is also called the diamond condition.

Abstract regular polytopes can be identified with labelled string C-groups as shown in [23, Theorem 2E11]. In this case, the involutions of a string C-group \((G, S)\) can be ordered in such a way that \((\rho_i \rho_j)^2 = 1_G\) for all \(i, j\) with \(|i - j| > 1\).

The right cosets of the maximal parabolic subgroups of \((G, S)\) correspond to faces of the polytope. The rank of faces of the polytope is induced by the labeling of the generators of \((G, S)\). By reversing the order of the generators of a string C-group, one obtains the dual of the corresponding polytope.

The main theorem of [1] can be rephrased in the framework of string C-groups as follows.

**Theorem 5.1** (Aschbacher, 1983). Let \((G, \{\rho_0, \ldots, \rho_{r-1}\})\) be a string C-group of rank \(r\) and let \(\Gamma := \Gamma(G; (G_i)_{i \in I})\) with \(G_i := \langle \rho_j | \rho_j \in S, j \in I \setminus \{i\} \rangle\) for all \(i \in I := \{0, \ldots, r - 1\}\). Then \(\Gamma\) is thin, residually connected and regular. Moreover, \(\Gamma\) has a string diagram.

**Proof.** The intersection property of \((G, S)\) implies assumption (i) of Aschbacher’s theorem while the string condition implies assumption (ii). Therefore we can apply his result to string C-groups to show that \(\Gamma\) is indeed thin, residually connected and regular in all ranks. \(\square\)
Therefore, abstract regular polytopes, being also string C-groups, are thin regular residually connected coset geometries (or hypertopes). The diamond condition in polytopes corresponds to thinness in geometries. A chain (respectively flag) of a polytope is a flag (respectively chamber) in the corresponding geometry. Strong flag-connectedness in polytopes corresponds to residual connectedness in geometries. The commuting property of non-consecutive generators in string C-groups corresponds to the linearity of the Buekenhout diagram in geometries. The concept of adjacent flags in polytopes is the same as that of adjacent chambers. Two chambers of an incidence geometry $\Gamma$ of rank $r$ are adjacent provided their intersection is a flag of cardinality $r - 1$.

**Theorem 5.2.** Let $\Gamma \coloneqq \Gamma(G; (G_i)_{i \in I})$ be a thin, residually connected, regular coset geometry with a string diagram. Let $C \coloneqq (G, \{\rho_0, \ldots, \rho_{r-1}\})$ where $\{\rho_0, \ldots, \rho_{r-1}\}$ is the set of distinguished generators of $\Gamma$. Then $C$ is a string C-group.

**Proof.** This is an immediate consequence of Proposition 4.1. □

In rank three, a string C-group induces an abstract regular polyhedron with vertices, edges and faces. When finite, such a polyhedron can be embedded on a closed surface (orientable or not) without boundary and is usually called reflexible map [15]. However, not all maps are abstract polytopes as some of them do not satisfy the diamond condition.

Thin regular geometries induced by rank three non-string C-groups, provide examples of reflexible hypermaps. For example, starting from the following Coxeter diagram

```

\( \rho_0 \quad \rho_1 \quad \rho_2 \)
```

we get a string C-group generated by three reflections $\rho_0$, $\rho_1$ and $\rho_2$, that is universal in this case, meaning it is the full Coxeter group $[6,3]$. This C-group gives the tessellation of a plane by hexagons which is an abstract regular polyhedron of Schl"afli type $\{6,3\}$. The regular maps with the same Schl"afli type $\{6,3\}$ will be embedded on the torus and induced by adding the relation

\[ (\rho_1 \rho_2 (\rho_1 \rho_0)^2)^b (\rho_2 \rho_1 (\rho_0 \rho_1)^2)^c = 1 \]

to the universal Coxeter group $[6,3]$. The resulting regular map is denoted by $\{6,3\}_{(b,c)}$.

Doubling the fundamental region of the C-group or, in other words, looking at the subgroup $H$ generated by $\rho_1^{\rho_0}, \rho_1, \rho_2$, we get the following non-string Coxeter diagram and $H$ is a non-string C-group.
Taking a quotient of \( H \) by adding the above relation, we get a finite incidence geometry which can be seen as a regular hypermap of type \((3, 3, 3)\) on the torus. Note that \((3, 3, 3)\) is regular if and only if \( b = c \). Observe that hypermaps are generalisations of maps obtained by dropping a string condition.

Extending \( H \) by an involution \( \rho_3 \) such that \( \rho_2 \rho_3 \) is of order \( p \geq 3 \), and in addition \( \rho_3 \) commutes with \( \rho_0 \) and \( \rho_1 \), we get a rank four C-group with the following diagram.

![Diagram](image)

Adding \( \rho_0 \) to this group, we get the Coxeter group \([6, 3, p] \) generated by \( \rho_0, \rho_1, \rho_2 \) and \( \rho_3 \), which for \( p = 3, 4, 5 \) and 6, is the symmetry group of a regular tessellation of the hyperbolic 3-space. We will give examples of regular hypertopos of this type in section 6.

6. \( C^+ \)-groups

We now consider groups that are not necessarily generated by involutions.

Consider a pair \((G^+, R)\) with \( G^+ \) being a group and \( R := \{\alpha_1, \ldots, \alpha_{r-1}\} \) an independent generating set for \( G^+ \), that is, \( \alpha_i \notin \langle \alpha_j : j \neq i \rangle \) and \( G^+ = \langle R \rangle \).

Define \( \alpha_0 := 1_{G^+} \) and \( \alpha_{ij} := \alpha_i^{-1} \alpha_j \) for all \( 0 \leq i, j \leq r-1 \). Let \( G^+_I := \langle \alpha_{ij} \mid i, j \in I \rangle \) for \( I \subseteq \{0, \ldots, r-1\} \).

If the pair \((G^+, R)\) satisfies condition (6.1) below called the intersection property \( IP^+ \) (obtained in analogy with the intersection property \( IP \) of C-groups keeping only those equalities that involve subsets \( I \) and \( J \) of cardinality at least two), we say that \((G^+, R)\) is a \( C^+ \)-group.

\[ (6.1) \quad \forall I, J \subseteq \{0, \ldots, r-1\}, \text{with } |I|, |J| \geq 2, G^+_I \cap G^+_J = G^+_I \cap J \]

Examples of \( C^+ \)-groups may be constructed from C-groups as follows. Given a C-group \((G, S)\) with \( S := \{\rho_0, \ldots, \rho_{r-1}\} \), we define the rotation subgroup \((G^+, R)\) where \( R := \{\alpha_j := \rho_0 \rho_j : j \in \{1, \ldots, r-1\}\} \) and \( G^+ := \langle R \rangle \). Let \( \alpha_0 := 1_{G^+} \).

Obviously, \( \alpha_{ij} := \rho_i \rho_j = \alpha_j^{-1} \alpha_i \in G^+ \) for any choice of \( i, j \in \{0, \ldots, r-1\} \). The subgroup \( G^+ \) is of index 1 or 2 in \( G \).

Lemma 6.1. Let \((G, S)\) be a C-group and \((G^+, R)\) its rotation subgroup as defined above. Then \((G^+, R)\) is a \( C^+ \)-group.

Proof. As \( S \) is an independent set it is straightforward that \( R \) is also independent. Let \( A = \langle \alpha_{ij} \mid i, j \in I \rangle \), \( B = \langle \alpha_{ij} \mid i, j \in J \rangle \) and \( C = \langle \alpha_{ij} \mid i, j \in I \cap J \rangle \). We have that \( A \cap B \supseteq C \). Let us prove the inclusion in the other direction. Suppose that \( \alpha_{k,l} \in A \cap B \) and that either \( k \notin I \cap J \) or \( l \notin I \cap J \). First suppose that \( k \notin J \). We have that \( \rho_k \rho_l \in B \). As \( B \subseteq \langle \rho_i \mid j \in J \rangle \) we have that \( \rho_k \in \langle \rho_i \cup \{\rho_i \mid i \in J \} \rangle \) with \( k \notin J \cup \{\} \) contradicting the independence of \( S \). Thus \( k \in J \). Using the same argument we prove that \( k \) is also in \( I \), and, similarly, \( l \in I \cap J \). Hence \( A \cap B = C \). \( \square \)
Let \((G^+, R)\) be a \(C^+\)-group. It is convenient to represent \((G^+, R)\) by the following complete graph with \(r\) vertices which we will call the \(B\)-diagram of \((G^+, R)\) and denote by \(B(G^+, R)\). The vertex set of \(B\) is the set \(\{\alpha_0, \ldots, \alpha_{r-1}\}\). The edges \(\{\alpha_i, \alpha_j\}\) of this graph are labelled by \(o(\alpha_i^{-1} \alpha_j) = o(\alpha_j^{-1} \alpha_i) = o(\alpha_i \alpha_j^{-1})\). We take the convention of dropping an edge if its label is 2 and of not writing the label if it is 3. Vertices of \(B\) are represented by small circles in order to distinguish from the vertices of a Coxeter diagram, which represent involutions. Observe that a \(C\)-group \((G, S)\) and its corresponding \(C^+\)-group \((G^+, R)\) will have isomorphic diagrams. The main difference is that the vertex set of the Coxeter diagram of \((G, S)\) is \(S\) while the vertex set of the \(B\)-diagram of a \((G^+, R)\) is \(R \cup \{1_{G^+}\}\).

For instance, the automorphism group of a chiral 4-polytope of Sch"afli type \(\{6, 3, 3\}\) with toroidal facets has the following \(B\)-diagram.

\[
\begin{array}{ccc}
\alpha_1 & \alpha_0 = 1_{G^+} & \alpha_2 \\
\alpha_3 & & \alpha_4 \\
\end{array}
\]

Observe that the two generators \(\alpha_1\) and \(\alpha_2\) represent rotations: \(\alpha_1\) corresponds to a rotation around a 2-face on a facet and \(\alpha_2\) corresponds to a rotation around a vertex of that 2-face on the same facet but they rotate in opposite directions.

In [26], the set of generators of \(G^+\) is usually denoted by \(\sigma_1, \ldots, \sigma_{r-1}\). We note that in the example above, \(\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4\), and more generally, given an abstract chiral polytope of Sch"afli type \(\{p_1, p_2, \ldots, p_{r-1}\}\) with generators \(\sigma_1, \ldots, \sigma_{r-1}\), we have \(\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4\) for \(2 \leq i \leq r-1\). Given an abstract chiral polytope and a set of generators \(\sigma_1, \ldots, \sigma_{r-1}\) of its automorphism group \(G^+\), there is no automorphism \(g\) of \(G^+\) such that \(g(\sigma_1) = \sigma_1^{-1}\), \(g(\sigma_2) = \sigma_2^{-1}\), \(g(\sigma_3) = \sigma_3\), and \(g(\sigma_4) = \sigma_4\) for \(3 \leq i \leq r-1\) as, if such an automorphism exists, the polytope is not chiral. In this case the conjugation by \(\rho_0\) defines an automorphism of \(G^+\). In terms of the generators \(\{\sigma_i : i \in \{1, \ldots, r-1\}\}\), this condition is equivalent to having no \(g \in \text{Aut}(G^+)\) such that \(g(\sigma_i) = \sigma_i^{-1}\) for all \(1 \leq i \leq r-1\). The generators \(\sigma_i\)'s are a natural choice in the case of geometries with a linear diagram. We adopt a different set of generators, needed for the cases where the diagram is not linear. Our generators correspond in rank three to the ones usually chosen in the theory of maps and hypermaps.

7. Chiral geometries

We now extend the notion of chirality in abstract polytopes to the more general framework of incidence geometries. Although thinness was not necessary in order to define what is a regular geometry, for defining chiral geometry, thinness is necessary.

Two chambers \(\Phi\) and \(\Phi'\) of an incidence geometry of rank \(r\) are called \(i\)-adjacent if \(\Phi\) and \(\Phi'\) differ only in their \(i\)-elements. We then denote \(\Phi'\) by \(\Phi^i\). Let \(\Gamma(X, *, t, I)\) be a thin incidence geometry. We say that \(\Gamma\) is chiral if \(\text{Aut}_t(\Gamma)\) has two orbits on the chambers of \(\Gamma\) such that any two adjacent chambers lie in distinct orbits.

Given a chiral hypertope \(\Gamma(X, *, t, I)\) (with \(I := \{0, \ldots, r-1\}\)) and its automorphism group \(G^+ := \text{Aut}_t(\Gamma)\), pick a chamber \(\Phi\). For any pair \(i \neq j \in I\), there exists an automorphism \(\alpha_{ij} \in G^+\) that maps \(\Phi\) to \((\Phi^i)^j\). Also, \(\Phi_{\alpha_i} := (\Phi^i)^i = \Phi\) and \(\alpha_{ij} = \alpha_{ji}\). We define the distinguished generators of \(G^+\) with respect to \(\Phi\) as follows:

\[
\alpha_0 := 1_{G^+}, \alpha_1 := \alpha_{10}, \alpha_2 := \alpha_{12}, \alpha_3 := \alpha_{12} \cdots \alpha_{i-1, i} (i = 3, \ldots, r-1)
\]
Arguments similar to those used in the proof of Proposition 4.1 permit to show that the pair \((G^+, R)\) is a \(C^+\)-group, that is, the distinguished generators of \(G^+\) satisfy the relations implicit in the \(B\)-diagram of the \(C^+\)-group as well as the intersection property \(IP^+\). The following proposition is the chiral equivalent of Proposition 4.1.

**Proposition 7.1.** Let \(I := \{0, \ldots, r-1\}\) and let \(\Gamma\) be a chiral hypertope of rank \(r\). The pair \((G^+, R)\), where \(R\) is the set of distinguished generators of \(\Gamma\) associated to the chamber \(\Phi\), is a \(C^+\)-group.

**Proof.** The proposition is obviously true for \(|R| = 2\). Suppose the proposition is true for \(r-1\) and let us show it is then true for \(r\) by way of contradiction. Let us denote by \(G^+_{R/K}\) the subgroup \(\langle \alpha_{ij} | i, j \in I \setminus K \rangle\). Let \(G_i\) be the stabiliser in \(G^+\) of the element of type \(i\) in \(\Phi\) for \(i = 0, \ldots, r-1\).

If \((G^+, R)\) does not satisfy (6.1), then there is a pair of subgroups, \(G^+_{R/K} \text{ and } G^+_{R/J}\) with \(K, J \subseteq I\), both of size at most \(r-2\), such that \(G^+_{R/K} \cap G^+_{R/J} \neq G^+_{R/K \cup J}\). Hence \(G^+_{R/K} \cap G^+_{R/J} > G^+_{R/K \cup J}\). Take \(g \in (G^+_{R/K} \cap G^+_{R/J}) \setminus G^+_{R/K \cup J}\). This \(g\) fixes a flag of type \(K \cup J\) in the base flag \(\{G_0, \ldots, G_{r-1}\}\). But the action of \(G^+_{R/K \cup J}\) must be faithful on the residue \(\Gamma_F\) of the flag \(F := \{G_i | i \in K \cup J\}\). Indeed, that residue is also a thin residually connected geometry and its distinguished generators are exactly those of \(G^+_{R/K \cup J}\) and satisfy (6.1) by induction. Any element of \(G^+_{R/K \cup J}\) will fix all elements of \(F\). If \(|K \cup J| = r\), the element \(g\) fixes a chamber, a contradiction with faithfulness. If \(|K \cup J| = r-1\), there are exactly two chambers containing the flag \(F\). Since the action is chiral, \(g\) must also fix these chambers, again a contradiction with faithfulness. Finally, suppose that \(|K \cup J| < r-1\). Since \(G^+_{R/K \cup J}\) has at most two orbits on the flags of \(\Gamma_F\), there must exist an element \(h \in G^+_{R/K \cup J}\) that sends the flag \(\{G_k | k \in I \setminus (K \cup J)\}\) onto \(\{G_k \ast g | k \in I \setminus (K \cup J)\}\) or onto one of its adjacent flags. In the first case, \(g \ast h^{-1} \neq 1_G\) fixes the base chamber \(\{G_i | i \in I\}\), a contradiction with the faithfulness of the action of \(G\) on the chambers of \(\Gamma\). In the second case, \(g \ast h^{-1} \neq 1_G\) maps the base chamber onto one of its adjacent chambers, contradicting chirality of the hypertope.

**Corollary 7.2.** The set \(R := \{\alpha_i | i \in I \setminus \{0\}\}\) is an independent generating set for \(G^+\).

The notion of chirality in incidence geometries was well explored in the case when the diagram of the geometry is linear, that is when the induced chiral geometry is an abstract chiral polytope [20]. The automorphism groups of chiral polytopes are characterized as groups with specific generators \(\sigma_1, \ldots, \sigma_{r-1}\) such that \(\sigma_i\sigma_{i+1}\ldots\sigma_j\) is of order 2 for all \(1 \leq i < j \leq r-1\). Examples can be found where the \(\sigma_i\)’s are not independent when \(r \geq 4\). For instance, all chiral polytopes of \(S_3\) given in [20] have their \(\sigma_i\)'s not independent.

Regular polytopes are often constructed inductively from regular polytopes of lower rank. Similar constructions can be applied to hypertopes. However such constructions of chiral polytopes is not possible as the \((n-2)\)-faces of a chiral polytope of rank \(n\) are necessarily regular (see [20] Proposition 9). The result for polytopes cannot be extended to thin geometries with a nonlinear diagram as these geometries are not necessary posets. We proceed to prove a similar result for chiral hypertopes. Let \(\Gamma(X, *, t, I)\) be an incidence geometry. For \(J \subseteq I\), we define the
If the chambers can be extended to chambers $C^+$, then $C^+$ can be mapped to an adjacent chamber of $|−|$ generating set of this group.

Indeed, every element $ρ$ of $G^+$ that sends each element of $H$ to its inverse, is involutory and we call it $ρ_0$. The group $G^+$ is then a subgroup of index at most 2 in $G := ⟨R, ρ_0⟩$. Moreover, the pair $(G, S) where $S := ⟨ρ_0⟩ ∩ {ρ_i := ρ_0^i α_i | α_i ∈ R}$ is a C-group. Indeed, every element $ρ_i$ is an involution as $ρ_i^2 = ρ_0 α_i ρ_0 ρ_0 = α_i^−1 α_i = 1_G$ and the intersection condition $IP$ follows from the fact that $(G^+, R)$ is a C-group.

Theorem 7.4. Let $G^+$ be a group and let $R := {α_1, . . . , α_r−1}$ be an independent generating set of $G^+$. Let $Γ := Γ(G^+, R)$ be the coset geometry associated to $(G^+, R)$ using Construction [7]. If $Γ$ is a hypertope, then $Γ$ is chiral if and only if there is no automorphism of $G^+$ that inverts all the elements of $R$.

Proof. It suffices to realise that the group $G^+$ has at most two orbits on the chambers of $Γ$. Indeed, $G^+$ acts faithfully on the chambers of $Γ$ as, if an element of $G^+$ fixes a chamber, it fixes all its adjacent chambers and their adjacent chambers and
so on. So it is the identity element of $G$. The existence of the automorphism then implies a unique orbit on the chambers of $\Gamma$. And the non-existence implies that there are exactly two orbits. □

If one looks at the rotation subgroup of the group $A_6$ of Example 4.5, it is clear that a $C^+$-group does not necessarily give a coset geometry $\Gamma$ that is thin and strongly chamber connected and hence does not give automatically a hypertope.

8. Examples of locally toroidal incidence geometries

Starting from tessellations of the hyperbolic space one can derive locally toroidal polytopes (LTP), that is, those polytopes whose facets or vertex-figures are either spherical or toroidal, but at least one of them toroidal. Partial classification of universal, regular, finite such objects exists [23, Chapters 10–12] but no classification for chiral finite LTP is known.

For the rank 4 polytopes (and their duals) with Schlafli symbol \(\{6, 3, p\}\) when \(p = 3, 4, 5, 6\) the classification of finite regular LTP is complete [23, Chapters 10–11]. We used Magma to search for the finite chiral LTP with the same Schlafli symbol. The results are compiled in Table 1 where we list each finite universal polytope of type $\{6, 3\}_s, \{3, p\}$ (with $s = (b, c)$) we obtained along with numbers $v$ of its vertices and $f$ of its facets, the order $g$ of the group. Observe that for type $(1, 1)$ with $p = 4$, the corresponding polytope has 6 vertices, not 12 as mentioned in [23]. In Table 2 we look at index 2 subgroups which yield regular and chiral locally toroidal hypertopes. We conjecture that these lists are complete.

9. Open problems

We conclude this paper with a series of open problems that we think are interesting to investigate in future work.

**Problem 9.1.** What is a minimal set of conditions for the IP$^+$ condition?

**Problem 9.2.** Classify all finite locally toroidal incidence geometries of type $(3, 3, 3; p)$.

**Problem 9.3.** Find an example of a $C$-group that gives a thin, residually connected geometry of rank $\geq 4$ that is not flag-transitive.

| $p$ | $s$ | $v$ | $f$ | $g$ | Group | Chiral/Regular |
|-----|-----|-----|-----|-----|-------|----------------|
| 3   | (2, 0) | 10  | 5   | 240 | $S_5 \times C_2$ | regular         |
| (3, 0) | 54 | 12  | 1296 | $[1 \ 1 \ 2]^3 \times C_2$ | regular |
| (4, 0) | 640 | 80  | 15360 | $[1 \ 1 \ 2]^4 \times C_2$ | regular |
| (1, 2) | 28  | 8   | 363 | $PGL_2(7)$ | chiral          |
| (1, 3) | 182 | 28  | 2184 | $PSL_2(13) \times C_2$ | chiral |
| (1, 4) | 672 | 64  | 8064 | $SL_2(7) \times A_4 \times C_2$ | chiral |
| (2, 2) | 120 | 20  | 2880 | $S_5 \times S_4$ | regular         |
| (2, 3) | 570 | 60  | 6840 | $PGL_2(19)$ | chiral          |

Table 1. Finite polytopes of type $\{6, 3\}_s, \{3, p\}$
Table 2. Finite thin geometries of type \{3, 3, 3; p\}

| p | s   | v | f  | g       | Group                        | Chiral/Regular |
|---|-----|---|----|---------|------------------------------|----------------|
| 3 | (2,0) | 5 | 5  | 120     | $S_5$                        | regular        |
|   | (3,0) | 27 | 12 | 648     | $[1,1,2]^3$                  | regular        |
|   | (4,0) | 320| 80 | 7680    | $[1,1,2]^4$                  | regular        |
|   | (1,2) | 14 | 8  | 168     | $PSL_2(7)$                   | chiral         |
|   | (1,3) | 91 | 28 | 1092    | $PSL_2(13)$                  | chiral         |
|   | (1,4) | 336| 64 | 4032    | $SL_2(7) \rtimes A_4$       | chiral         |
|   | (2,2) | 60 | 20 | 1440    | $A_5 \times S_4$            | regular        |
|   | (2,3) | 285| 60 | 3420    | $PSL_2(19)$                  | chiral         |
| 4 | (1,2) | 42 | 48 | 1008    | $PSL_2(7) \times S_5$       | chiral         |
|   | (2,0) | 8  | 16 | 384     | $[3,3,4]$                    | regular        |
| 5 | (2,0) | 120| 600 | 14400   | $[3,3,5]$                    | regular        |

**Problem 9.4.** Find an example of a C-group of rank 3 that gives a geometry that is not thin, not residually connected and not flag-transitive.

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