Scattering solutions of Bethe-Salpeter equation in Minkowski and Euclidean spaces

Abstract. We shortly review different methods to obtain the scattering solutions of the Bethe-Salpeter equation in Minkowski space. We emphasize the possibility to obtain the zero energy observables in terms of the Euclidean scattering amplitude.

Keywords. Bethe-Salpeter equation · Euclidean scattering

1 Introduction

The Bethe-Salpeter (BS) equation \[^1\] deals with a – preexisting – Quantum Field Theory (QFT) object defined as a vacuum expectation value of the T-ordered product of Heisenberg field operators \[^2\]. For the two-body case in a scalar theory \(\phi\), this object has the expression

\[
\Phi(x_1, x_2; p) = \langle 0 | \, T \{ \phi(x_1) \phi(x_2) \} \, | p \rangle,
\]

where \( | p \rangle \) is the state vector with total momentum \( p \).

After removing some trivial dependence related to translational invariance,

\[
\Phi(x_1, x_2; p) = \frac{1}{(2\pi)^{3/2}} \Phi(x; p) \, e^{-ip(x_1 + x_2)/2},
\]

its Fourier transform

\[
\Phi(x; p) = \int d^4 x \, \Phi(k; p) \, e^{-ikx} \quad x = x_1 - x_2
\]
defines the momentum space BS amplitude. Bethe and Salpeter wrote a covariant 4D integral equation for the amplitude \(\Phi(k; p)\), which for the two-body bound state reads

\[
\Phi(k; p) = S_1(k, p) \, S_2(k, p) \, \int \frac{d^4 k'}{(2\pi)^4} \, iK(k, k'; P) \, \Phi(k'; p)
\]

where

\[
S_1(k, p) = \frac{i}{\left(\frac{\not{p}}{2} + k\right)^2 - m^2 + i\epsilon} \quad S_2(k, p) = \frac{i}{\left(\frac{\not{p}}{2} - k\right)^2 - m^2 + i\epsilon}
\]
are the free constituent propagators and $iK$ the interaction kernel. If the propagators were dressed and the kernel was a sum of all the irreducible Feynman graphs of the theory, solving (2) would be equivalent to solve the full QFT. This is however a wishful thinking. In practice only very poor representations of the interaction are used: most of nuclear physics calculations use ladder approximation and free propagators, though dressed propagators are systematically used in the Schwinger-Dyson approach [5].

There could be some misunderstanding on what BS amplitude denotes: either the quantity $\Phi$ computed with the full QFT solution or the quantity $\Phi$ obtained by solving (2) with a necessarily truncated interaction kernel. These quantities can strongly differ from each other, but we live with it.

Besides the bound states equation (2), we have also considered the corresponding scattering equation for the process $k_{1s} + k_{2s} \rightarrow k_1 + k_2$. The two-body scattering amplitude $F(k; p)$ satisfies an inhomogeneous BS equation which, for spinless particles in Minkowski space, reads:

$$F(k; p) = K(k, k_s) - i \int \frac{d^4k'}{(2\pi)^4} \frac{K(k, k') F(k'; p)}{\left((\frac{k}{2} + k')^2 - m^2 + i\epsilon\right) \left((\frac{k}{2} - k')^2 - m^2 + i\epsilon\right)}$$

with $2k = k_1 - k_2$ and $p = k_1 + k_2$.

This equation is plagued with singularities: (i) in the two-body propagators, (ii) in most of the interaction kernels even the simplest ones, (iii) in the (inhomogeneous) Born term and (iv) in the amplitude $F$ itself, being difficult to represent it in terms of smooth functions.

To avoid these troubles – and get some bound state solutions – Wick proposed [3] to introduce a new variable $k_0 = ik_4$ in terms of which the 4D Minkowski metric is changed into an Euclidian one:

$$k^2 = k_0^2 - k^2 = -(k_1^2 + k_2^2) \equiv -k_E^2$$

This change of metric leads to a smooth integral equation for the Euclidian amplitude, related to the Minkowski one by

$$F_E(k_4, k) = F_M(ik_4, k)$$

(5)

which is easily solvable by standard methods. Until very recently most of the existing solutions were found in this way. The are however some problems in performing the so called Wick rotation:

1. It is not a simple change of variable but underlies an application of the Cauchy theorem with two different integration contours. The equivalence of both theories – Minkowski and Euclidean – requires a careful analysis of the singularities crossed when moving from one to other. For the bound state case, the singularities come only in the interaction kernel and the validity has been proved only for simple cases. For the scattering case, they come also in the propagators and the corresponding residues must be taken into account.

2. The total mass $M$ of the system is invariant under such transformation but the amplitude not. In practice it is very difficult – if possible at all – to recover $\Phi_M$ from $\Phi_E$ (see contribution [4]).

3. Other observable – scattering amplitudes, form factors – are defined and computable in terms of Minkowski amplitudes. It is worth noticing, however, that the distinction between Euclidian and Minkowski amplitudes is meaningful only when keeping for both of them real arguments ($k_0, k$) and ($k_4, k$) and the later statement must be understood in this conventional sense. Euclidian amplitudes with imaginary arguments [5] are equivalent to Minkowski ones and can lead to correct results.

All these reasons motivated several authors to solve the BS equation in Minkowski space, first for the bound state [6, 7, 8, 9, 10] and later for the scattering problem [11, 12, 13, 14]. Obtaining these numerical solution is indeed quite difficult and was not tried until very recently.

2 Minkowski space solutions: Light-Front projection and Nakanishi representation

Some years ago we proposed [7] a method based on an integral transform of the original BS equation which make all things regular. It was inspired by the relation between the Light-Front (LF) wave function $\Psi$ and the BS amplitude $\Phi$:

$$\Psi(k_1, k_2, p, \omega) = \frac{(\omega \cdot k_1)(\omega \cdot k_2)}{\pi(\omega \cdot p)} \int_{-\infty}^{+\infty} d\beta \Phi(k + \beta \omega)$$
This integral transform, denoted symbolically by \( \Psi = L\Phi \), maps the singular BS amplitude into a smooth wave function \( \Psi \) and was applied to the full BS equation (2) according to

\[
\Phi = G_0 K\Phi \quad \Longrightarrow \quad L\Phi = L G_0 K\Phi \quad \Longrightarrow \quad \Psi = L G_0 K\Phi
\]

In order to use the same unknown function on both sides of (6) and avoid the inversion of \( L \), we have used the Nakanishi weight function \( g \). Indeed, if \( g \) is known, the BS amplitude – either in Minkowski or in Euclidean space – can be computed by means of (7). A similar, one-dimensional, expression exists for the Light-Front wave function \( \Psi \) in terms of \( g \)

\[
\Psi(\gamma, z) = 1 - z^2 \int_0^\infty d\gamma' \frac{g(\gamma', z)}{[\gamma + m^2 - M^2 + (1 - z^2)\gamma']^2}
\]

These two possibilities make the Nakanishi weight function \( g \) especially well adapted for establishing a relation between Euclidean and Minkowski BS solutions. Indeed, if \( g \) could be determined either from the LF wave function – inverting (9) – or from an Euclidean BS solution – inverting (7) – it could be used to compute the Minkowski amplitude and related observables. Care must be taken however with the ill-defined character in the numerical inversion of (7) and (9).

Further applications of this LF/Nakanishi approach have been developed in the recent years.

- The solutions of (5) were used to compute the electromagnetic form factors \( E, M \) in terms of \( g \).
- Equation (5) was generalized to the two-fermion system. The usual one-boson exchange kernels have been derived and the bound state solution for the \( J^z = 0^+ \) states with the Yukawa coupling have been obtained in [4, 21].
- In [14], equation (5) was written in the form \( g = K' g \). This non-trivial analytic inversion of the left-hand-side kernel of (5), allows to compute unambiguously the function \( g \) and study its analytic properties. The bound state solutions found by solving this equation were in perfect agreement with the preceding results. The equation was extended to the scattering states and the scattering length could be accurately computed [14].

### 3 Minkowski space solutions: direct approach

A second method was developed in [11, 12] aiming to directly solve the original equation without any transform and taking into account all the singularities of equation (4). This direct approach has proved its efficiency both for the bound and, especially, for the scattering states. It provided for the first time the full off-shell scattering amplitude at the price of a careful analytical work to transform all the singular terms, from the kernels as well as from the propagators, into a smooth integrands.

For instance, the BS equation for the S-wave can be written in the following form

\[
F_0(k_0, k, k') = F_0'(k_0, k) + \frac{i\pi k_0}{8\epsilon_{k_0}} W_0^S(k_0, k, 0, a_+ k_0) F_0(0, a_+ k_0)
\]

\[
+ \frac{\pi}{2M} \int_0^\infty \frac{dk'}{\epsilon_{k'}(2\epsilon_{k'} - M)} \left[ k'^2 W_0^S(k_0, k, a_-, k') F_0(|a_-|, k') - \frac{2k'_2\epsilon_{k'}}{\epsilon_{k'} + \epsilon_{k_0}} W_0^S(k_0, k, 0, k_0) F_0(0, k_0) \right]
\]

\[
- \frac{\pi}{2M} \int_0^\infty \frac{k'^2 dk'}{\epsilon_{k'}(2\epsilon_{k'} + M)} W_0^S(k_0, k, a_+, k') F_0(a_+, k')
\]
where $W^S_0$ is the S-wave kernels and $F^B_0$ the corresponding Born term. Notice, apart from the inhomogeneous term, the existence of zero-, one- and two-dimensional integral terms. The details of the derivation and some illustrative results can be found in [12]. Having computed the scattering solutions, the corresponding "bound $\rightarrow$ scattering state" transition electromagnetic form factor was found in [13].

4 Deriving an Euclidean equation

In contrast to the bound state case, the scattering BS equation cannot be in general transformed into an equation for the Euclidean amplitude $F^E_E$ alone. Indeed, let us consider equation (4) and assume we want to obtain from it an equation for the Euclidean amplitude $F^E_E$ defined in (5). In order to properly apply the Wick rotation, we have to take into account the pole singularities associated with the propagators in (4) which are given by

\[
\begin{align*}
    k_0^{(1)}(k, k_s) &= \varepsilon_{k_s} + \varepsilon_{k'} - i\epsilon = +a_+ - i\epsilon \\
    k_0^{(2)}(k, k_s) &= \varepsilon_{k_s} - \varepsilon_{k'} + i\epsilon = -a_- + i\epsilon \\
    k_0^{(3)}(k, k_s) &= -\varepsilon_{k_s} + \varepsilon_{k'} - i\epsilon = +a_- - i\epsilon \\
    k_0^{(4)}(k, k_s) &= -\varepsilon_{k_s} - \varepsilon_{k'} + i\epsilon = -a_+ + i\epsilon
\end{align*}
\]

with $a_\pm(k', k_s) = \varepsilon_{k'} \pm \varepsilon_{k_s}$

In the case $k' < k_s$, their positions in the complex $k_0$-plane are shown in Fig.1. When the integration contour is rotated, the singularities $k_0^{(2)}$ and $k_0^{(3)}$ are crossed and the corresponding residues of the integrand at these poles will contribute to the integral term.

\[\text{Fig. 1 Singularities of propagators for scattering state and integration contour in complex } k_0 \text{ plane (}k' < k_s).\]

Equation (4) is then transformed into:

\[
F^E_E(k_4; k; k_s) = V^B(k_4; k; k_s) + \int \frac{d^4k'}{(2\pi)^4} \frac{V(k_4, k'; k_4', k')F^E_E(k_4', k'; k_s)}{(k_4'^2 + a_+^2)(k_4'^2 + a_-^2)} + S(k_4, k, k_s)
\]

where $V$ and $V^B$ are respectively the Wick-rotated interaction kernel and Born term.
Of particular interest is the zero energy scattering. It can be shown \[20\] that in the limit and with those of \[22\].

The phase shifts. The results provided by our direct method \[10\] are in agreement with the solution of (13)

This system of equations – whose Minkowski part remains singular – was first derived in \[22\], re-derived and solved in \[12\] to check our direct Minkowski solution \[F_M\]. Howev er, on mass-shell \(F_M\) does not provide the full solution of the scattering problem. However, on mass-shell \(k_0 = k_3 = 0\) and since \(F_M(0, k) = F_E(0, k)\), the Euclidean amplitude computed this way can be also used to obtain the phase shifts. The results provided by our direct method \[10\] are in agreement with the solution of \[12\] and with those of \[22\].

4.1 The case of zero energy scattering

Of particular interest is the zero energy scattering. It can be shown \[20\] that in the limit \(k_5 \to 0\) the term \(S\) in Eq. \[12\] vanishes and the four-dimensional BS equation for the Euclidean amplitude in the zero energy limit takes the form

\[
F_E(k_4, k) = \mathcal{I}_1[F_E(k'_4, k'), F_M(\varepsilon_{k_5} - \varepsilon_{k'_3}, k')] + \mathcal{I}_2[F_E(k'_4, k'), F_M(\varepsilon_{k_5} - \varepsilon_{k'_3}, k')]
\]

(13)

This equation determines \(F_E(k_4, k; k_5 = 0)\) without any coupling to the Minkowski amplitude \(F_M\). This decoupling is valid only in the limit \(k_5 \to 0\).

In the S-wave case, Eq. \[14\] reduces to the following two-dimensional integral equation:

\[
F_0(k_4, k) = V_0^E(k_4, k) + \int_0^\infty k'^2 dk' \int_0^\infty \frac{V_0(k_4, k'; k'_3, k'_4)}{(k'^2 + a'^2_3)(k'_4 + a'_2)} F_0(k'_4, k')
\]

(15)

with \(a'_3 = \sqrt{m^2 + k'^2} \pm m\) and where we have omitted the superscript on \(F_E\). This equation determines the S-wave zero energy half-mass shell scattering amplitude \(F_0\). The scattering length is given by

\[
a_0 = -\frac{1}{m} F_0(k_4 = 0, k = 0)
\]

(16)

A more detailed derivation can be found in \[20\].

The interest in deriving equation \[15\] is not only a significant simplification in computing the BS scattering length but in the fact that this fundamental quantity can be obtained from a purely Euclidean solution. Although considered here in the approximate framework of BS equation, the "exact" Euclidean amplitude \[14\] – solution of the full QFT problem – is nowadays accessible in the Lattice approach. It is the basic ingredient of the HAL-QCD collaboration to obtain the \(ab\ initio\) Nucleon-Nucleon potential from QCD \[22\] but it has never been used to extract the scattering observables.

The possibility to compute a scattering amplitude from Euclidean correlators in infinite space is forbidden by the Maiani-Testa theorem \[24\]. This result is in agreement with the impossibility discussed above to obtain BS scattering observable without a coupling to the Minkowski amplitudes, but does not apply to zero scattering energy where the phase shifts vanish.

Luscher et al. \[25\] circumvented this problem and proposed a method to compute the phaseshifts based on the \(V\)-dependence of the 2-particle energies confined in a box with periodic boundary conditions. This approach was successfully applied to several hadronic systems in \(ab\ initio\) QCD \[24\].
The use of Eq. (16) constitutes an alternative method to compute the scattering length in the Lattice calculations, directly from the Fourier transform of the Euclidean BS amplitude defined in (1). In the framework of BS equation, such a possibility is justified by the existence of Eq. (15) allowing to compute the Euclidean BS amplitude in a purely Euclidean formalism as in the Lattice approach.

The numerical solutions of Eq. (15) with one-boson exchange kernel were obtained in [20]. The scattering length $a_0$ was extracted from the Euclidean amplitude (16) at the origin. The value is in full agreement with the results of our previous work [11] and from those of Ref. [14], obtained by independent methods.

Figure 2 displays the amplitude $F_0(k_4, k)$ as a function of $k$ at fixed values of $k_4$ (left panel) and as a function of $k_4$ at fixed values of $k$ (right panel) for the parameters $m = 1$, $\mu = 0.15$ and $\alpha = 2.5$. The scattering length $a_0 = 12.3$ is directly readable in both panels. For these parameters the two-body system has two bound states, which are responsible for the structure in the amplitude.

**Fig. 2** Euclidean scattering amplitude $F_0(k_4, k)$ as a function of $k$ for different values of $k_4$ (left) and as a function of $k_4$ for different values of $k$ (right). It corresponds to $m = 1$, $\mu = 0.15$ and $\alpha = 2.5$. The scattering length value is given by $a_0 = -F(0, 0) = 12.3$.

### 5 Conclusion

We reviewed different methods to obtain the scattering solutions of the Bethe-Salpeter equation in Minkowski space.

We showed that the Bethe-Salpeter scattering amplitude at zero energy obeys a purely Euclidean equation, as it is the case for the bound states. This decoupling between Euclidean and Minkowski BS amplitudes is only possible for zero energy scattering and allows determining the scattering length from the Euclidean BS amplitude.

Such a possibility suggests to extract the scattering length in Lattice calculations from a direct computation of the Euclidean Bethe-Salpeter amplitude $F_0$ in momentum space and provides an alternative to the Luscher method.

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