Reply on
“Comment on neutrino–mixing interpretation of the GSI time anomaly”
by C. Giunti, nucl–th/0801.4639

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Abstract

Here we reply critically to the comments by Giunti (nucl–th/0801.4639) and justify our explanation of the experimentally observed periodic interference term in the rate of the K–shell electron capture decay of the H–like ions $^{140}$Pr$^{58+}$ and $^{142}$Pm$^{60+}$ as a neutrino–flavour mixing.

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1 Introduction

According to recent experimental data at GSI [1] on the K–shell electron capture (EC) decays of the H–like ions $^{140}\text{Pr}^{58+}$ and $^{142}\text{Pm}^{60+}$

$$
^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+} + \nu,
^{142}\text{Pm}^{60+} \rightarrow ^{142}\text{Nd}^{60+} + \nu,
$$

(1)

the rate of the number $N_{d}^{EC}(t)$ of daughter ions $^{140}\text{Ce}^{58+}$ or $^{142}\text{Nd}^{60+}$

$$
\frac{dN_{d}^{EC}(t)}{dt} = \lambda_{EC}^{(H)}(t) N_{m}(t),
$$

(I.1)

where $N_{m}(t)$ is the number of mother ions $^{140}\text{Pr}^{58+}$ or $^{142}\text{Pm}^{60+}$ [1] and $\lambda_{EC}^{(H)}(t)$ is the EC–decay rate, is a periodic function, caused by a periodic time–dependence of the EC–decay rate

$$
\lambda_{EC}^{(H)}(t) = \lambda_{EC}^{(H)}(1 + a_{EC}\cos \frac{2\pi t}{T_{d}})
$$

(I.2)

with a period $T_{d} \simeq 7$ sec and an amplitude $a_{EC} = 0.20(2)$ [1].

In our paper [2] such a periodic time–dependence of the EC–decay rate we have proposed to explain as an interference of two–neutrino flavours. We have related the period $T_{d}$ to the difference $\Delta m_{21}^{2} = m_{2}^{2} - m_{1}^{2}$ of the squared neutrino masses $m_{2}$ and $m_{1}$

$$
\frac{2\pi}{T_{d}} = \frac{\Delta m_{21}^{2}}{4\gamma M_{d}},
$$

(I.3)

where $M_{d}$ is the mass of the daughter ion and $\gamma = 1.43$ is a Lorentz factor [1].

The amplitude $A(I_{m} \rightarrow I_{d} + \nu)$ of the EC–decay $I_{m} \rightarrow I_{d} + \nu$, where $I_{m}$ and $I_{d}$ are the mother and daughter ions and $\nu$ is a neutrino, which is not detected experimentally [1], we define in the form of the coherent sum of the amplitudes $A(I_{m} \rightarrow I_{d} + \nu_{j})$

$$
A(I_{m} \rightarrow I_{d} + \nu) = \sum_{j=1,2,3} A(I_{m} \rightarrow I_{d} + \nu_{j}),
$$

(I.4)

where $\nu_{j}$ is a neutrino state with mass $m_{j}$ [3, 4].

This explanation has been recently criticised by Giunti [3]. Below we reply on this critique.

The paper is organised as follows. In section 2 we cite the paper by Giunti [5] in order to simplify the communication. In section 3 we give a detailed critical reply on Giunti’s critique. We show that the Eq.(9) and Eq.(10) (see Section 2 and [5]), proposed by Giunti for the definition of the wave function of a neutrino in the final state of the EC–decay and the amplitude of the EC–decay, cannot be used, since they contradict the main principles of time–dependent perturbation theory and quantum field theory. In section 4 we give arguments for the description of the amplitude of the EC–decay in the form Eq.(I.4). In Section 5 we apply the procedure, which we used for the analysis of the EC–decay, to the calculation of the time–dependent decay rate of the $\pi^{+} \rightarrow \mu^{+} + \nu$ decay by defining the amplitude of the decay as a coherent sum of the amplitudes $\pi^{+} \rightarrow \mu^{+} + \nu_{j}$. The obtained result agrees well with the experimental data on the measurement of the lifetime of the $\pi^{+}$–meson [4]. In the Conclusion we summarise our replies.
2 Comment on neutrino–mixing interpretation of the
GSI time anomaly by C. Giunti, nucl–th/0801.4639

The authors of Ref.[2] calculated the electron capture process using time–dependent perturbation theory with the effective time–dependent weak interactions Hamiltonian

$$H_W(t) = \frac{G_F}{\sqrt{2}} V_{ud} \int d^3x [\bar{\psi}_n(x)\gamma^\mu(1 - g_A\gamma^5)\psi_p(x)] \sum_j [U_{ej}^* \bar{\psi}_j(x)\gamma_\mu(1 - \gamma^5)\psi_e^-(x)]$$ (2)

with standard notations. They interpreted (see Eq.(3) of Ref.[2])

$$A(t) = \sum_k A_k(t)$$ (3)

as the time–dependent amplitude of the decay

$$I_i \rightarrow I_f + \nu_e,$$ (4)

where

$$A_k(t) = \int_0^t d\tau \langle I_f, \nu_k | H_W(\tau) | I_i \rangle$$ (5)

is the time–dependent amplitude of

$$I_i \rightarrow I_f + \nu_k,$$ (6)

transitions. Here $I_i$ is the initial ion ($^{140}\text{Pr}^{58+}$ or $^{142}\text{Pr}^{60+}$), $I_f$ is the final ion ($^{140}\text{Ce}^{58+}$ or $^{142}\text{Nd}^{60+}$, respectively), and $\nu_k$ are the massive neutrinos ($k = 1, 2, 3$).

Regrettably, the amplitude in Eq.(3) does not describe the decay (4), but a decay in which the final neutrino state is

$$|\nu\rangle = \sum_k |\nu_k\rangle,$$ (7)

which is clearly different from an electron neutrino state. Indeed, in the standard theory of neutrino oscillations (see references of Ref.[5]) electron neutrinos are described by the state

$$|\nu_e\rangle = \sum_k U_{ek}^* |\nu_k\rangle,$$ (8)

where $U$ is the unitary mixing matrix of the neutrino fields in Eq.(2). More accurately, if the neutrino mass effects in the interaction processes are taken into account (see references in [5]), in the time–dependent perturbation theory used in Ref.[2] the final electron neutrino in the process (4) is described by the normalised state

$$|\nu_e(t)\rangle = \frac{\sum_k A_k(t)|\nu_k\rangle}{\sqrt{\sum_j |A_j(t)|^2}}.$$ (9)
The time dependence of this electron neutrino state takes into account the fact that in time–dependent perturbation theory the final state of a process is studied during formation.

Using the correct electron neutrino state in Eq.(9), the decay amplitude is not given by Eq.(3), but by

\[ A(t) = \sqrt{\sum_j |A_j(t)|^2} \]

Then, it is clear that the electron capture probability is given by the incoherent sum over the different channels of massive neutrino emission. In other words, there is no interference term between different massive neutrinos contributing to the rates of the electron capture processes $^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+} + \nu_e$ and $^{142}\text{Pm}^{60+} \rightarrow ^{142}\text{Ce}^{60+} + \nu_e$, as well as all decays and cross sections.

In conclusion, I have shown that neutrino mixing cannot explain the GSI time anomaly, refuting the claims presented in Ref.[2].

3 Reply on “Comment on neutrino–mixing interpretation of the GSI time anomaly” by C. Giunti, nucl–th/0801.4639

According to Giunti’s assertion [5], the wave function of the neutrino in the final state of the EC–decay $I_m \rightarrow I_d + \nu$ should be taken in form Eq.(9) as $|\nu_e(t)\rangle$ and it should have a non–trivial dependence on time. But “regrettably” such an assertion contradicts the main principles of time–dependent perturbation theory [7]–[9] and quantum field theory [10]. In order to show this in detail we make an excursion to time–dependent perturbation theory [7]–[9].

Analysis of the wave function Eq.(9) in time–dependent perturbation theory

According to [7]–[9], time–dependent perturbation theory describes transitions $i \rightarrow f$ from the initial stationary state $|i\rangle$ with the wave function $\psi_i^{(0)}(0)$ to the final stationary state $|f\rangle$ with the wave function $\psi_f^{(0)}(0)$, caused by a time–dependent perturbation $H_W(t)$. The wave functions $\psi_i^{(0)}(0)$ and $\psi_f^{(0)}(0)$ are eigenfunctions of an unperturbed Hamilton operator $H_0$ with eigenvalues $E_i^{(0)}$ and $E_f^{(0)}$, respectively. These wave functions have no information about a perturbation Hamilton operator $H_W(t)$ [7]–[9].

The amplitude $a_{mk}(t)$ of the transition of the stationary state $|k\rangle$ with the wave function $\psi_k^{(0)}(0)$ to the stationary state $|m\rangle$ with the wave function $\psi_m^{(0)}(0)$ obeys the following differential equation [7]–[9]

\[ i \frac{\partial}{\partial t} a_{mk}(t) = \sum_n e^{i(E_m^{(0)} - E_n^{(0)})t} \langle m|H_W(t)|n\rangle a_{nk}(t), \]
where the matrix element \( \langle m | \hat{H}_W(t) | n \rangle \) is defined by

\[
\langle m | \hat{H}_W(t) | n \rangle = \int dv \psi_m^{(0)*}(0) H_W(t) \psi_n^{(0)}(0)
\]  

(12)

and \( dv \) is an element of a configuration space. Since the interaction is weak, Eq.(11) can be solved perturbatively. Keeping the contributions up to the first order in the Fermi coupling constant \( O(G_F) \), for the coefficients \( a_{nk}(t) \) we get the following expression

\[
a_{nk}(t) = a_{nk}^{(0)} + a_{nk}^{(1)}(t) = \delta_{nk} + a_{nk}^{(1)}(t)
\]

(13)

where \( a_{nk}^{(0)} = \delta_{nk} \) means that a quantum system at \( H_W(t) = 0 \) does not change the state.

The coefficient \( a_{nk}^{(1)}(t) \) is of order \( O(G_F) \). It defines the amplitude of the \( k \rightarrow n \) transition, caused by a weak interaction \( H_W(t) \). Substituting Eq.(13) into Eq.(11) we obtain the amplitude \( a_{mk}^{(1)}(t) \) of the \( k \rightarrow m \) transition, caused by a weak interaction \( H_W(t) \), equal to

\[
a_{mk}^{(1)}(t) = -i \int_0^t d\tau e^{i(E_m^{(0)} - E_k^{(0)})\tau} \langle m | H_W(\tau) | k \rangle.
\]

(14)

For the transition \( i \rightarrow f \) we set \( k = i \) and \( m = f \) and get

\[
a_{i-f}^{(1)}(t) = -i \int_0^t d\tau e^{i(E_f^{(0)} - E_i^{(0)})\tau} \langle f | H_W(\tau) | i \rangle,
\]

(15)

where \( |i\rangle \) and \( |f\rangle \) are stationary states with the wave functions \( \psi_i^{(0)}(0) \) and \( \psi_f^{(0)}(0) \), respectively.

Thus, according to standard time–dependent perturbation theory \([7]–[9]\), wave functions of the initial and final states of the \( i \rightarrow f \) transition are independent of time. Moreover wave functions of the initial and final states are eigenfunctions of a non–perturbed Hamilton \( H_0 \) and have no information about a perturbation interaction \( H_W(t) \).

Since the time derivative of the wave function \( |\nu_e(t)\rangle \) in Eq.(9) is not equal to zero

\[
\frac{i}{\hbar} \frac{\partial |\nu_e(t)\rangle}{\partial t} \neq 0,
\]

(16)

the wave function Eq.(9) does not describe a stationary state and, correspondingly, cannot be used for the calculation of the amplitude of the \( EC \)–decay within standard time–dependent perturbation theory \([7]–[9]\). A strong dependence of the wave function Eq.(9) on a structure of a perturbation Hamilton operator \( H_W(t) \) confirms also the impossibility to use this wave function for the analysis of the \( EC \)–decays within standard time–dependent perturbation theory.

**Does the wave function Eq.(9) define the asymptotic neutrino state at \( t \rightarrow \infty \)?**

Another confirmation of the falseness of the wave function Eq.(9) as a true wave function for a neutrino in the final state of the \( EC \)–decay is the fact that such a wave function does not describe an “asymptotic neutrino state” at \( t \rightarrow \infty \). In order to illustrate this assertion we propose to investigate in detail the application of the wave function Eq.(9).
to the description of the EC-decay of the H–like $^{140}$Pr$^{58+}$ ion. For simplicity we can use plane waves for the wave functions of neutrinos $\nu_j$ with masses $m_j$. This gives [2]

$$A_j(t) = \sqrt{3} \sqrt{2M_mE_d} \mathcal{M}_{GT} \langle \psi_{1s}^{(Z)} \rangle (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{q}) U_{ej} \sqrt{E_j} \sin \left( \frac{\Delta E_j}{2} t \right) e + i \frac{\Delta E_j(k)}{2} t,$$

$$|A_j(t)|^2 = \left[ \sqrt{3} \sqrt{2M_mE_d} \mathcal{M}_{GT} \langle \psi_{1s}^{(Z)} \rangle (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{q}) \right]^2 |U_{ej}|^2 E_j \frac{\sin^2 \left( \frac{\Delta E_j}{2} t \right)}{\Delta E_j^2}.$$

where $\vec{k}$ and $\vec{q}$ are momenta of the neutrino $\nu_j$ and the daughter ion, respectively, and

$$\Delta E_j = \sqrt{k^2 + m_j^2 + E_d(k) - M_m},$$

where $E_d(k)$ and $M_m$ are an energy and mass of the daughter and mother ions, respectively. The wave function of Eq.(9) with $A_j(t)$, defined by Eq.(17), takes the form

$$|\nu_e(t)\rangle = \sum_j U_{ej} \sqrt{E_j} \sin \left( \frac{\Delta E_j}{2} t \right) e + i \frac{\Delta E_j}{2} t |\nu_j\rangle \sqrt{\sum_j |U_{ej}|^2 \delta(\Delta E_j)}.$$

Since, according to Giunti [5] (see also Section 2), the wave function Eq.(19) describes a neutrino state at any finite time $t$, it should also define the asymptotic neutrino state, calculated at $t \rightarrow \infty$, related to an observable detectable neutrino state [10]. Using the relations

$$\frac{\sin \left( \frac{\Delta E_j}{2} t \right)}{\frac{\Delta E_j}{2}} \xrightarrow{t \rightarrow \infty} 2\pi \delta(\Delta E_j), \quad \frac{\sin^2 \left( \frac{\Delta E_j}{2} t \right)}{\Delta E_j^2} \xrightarrow{t \rightarrow \infty} 2\pi t \delta(\Delta E_j).$$

and energy conservation we get

$$|\nu_e(t)\rangle \xrightarrow{t \rightarrow \infty} \sqrt{\frac{2\pi}{t}} \sum_j U_{ej} \delta(\Delta E_j) |\nu_j\rangle = O\left( \frac{1}{\sqrt{t}} \right).$$

Since the r.h.s. of Eq.(21) vanishes in the limit $t \rightarrow \infty$, the wave function Eq.(9) describes no “asymptotic neutrino” state, which can be detected [10].
Analysis of the amplitude Eq. (10) and the wave function Eq. (9) for the EC–decay with the neutrino \( \nu_e \), treated as an elementary particle

The incorrectness of the relation Eq. (10) becomes obvious if one treats the neutrino \( \nu_e \) in the final state of the \( I_m \rightarrow I_d + \nu_e \) decay as an elementary particle. According to Giunti [5], the wave function of the neutrino \( \nu_e \) should be taken in the form

\[
|\nu_e(t)\rangle = \frac{A(t)|\nu_e\rangle}{\sqrt{|A(t)|^2}},
\]

where \( A(t) \) is

\[
A(t) = \int_0^t d\tau \langle I_d, \nu_e | H_W(\tau) | I_m \rangle
\]

In accordance with Eq. (10) (see Section 2 and [5]), the amplitude \( A(t) \) of the \( I_m \rightarrow I_d + \nu_e \) decay should be defined by

\[
A(t) = \sqrt{|A(t)|^2} = |A(t)|.
\]

This means that the amplitude \( A(t) \) is positive and has no imaginary part. Unlike the result, obtained following Giunti’s prescription Eq. (10), the direct calculation of the amplitude \( A(t) \) of the \( I_m \rightarrow I_d + \nu_e \) decay gives the expression [2]

\[
A(t) = i \sqrt{3} \sqrt{2 M_m 2 E_d E_\nu} M_{GT} \langle \psi^{(Z)}_1 | (2\pi)^3 \delta^{(3)}(\vec{q} + \vec{k}_\nu) \frac{\sin \left( \frac{\Delta E_\nu}{2} t \right)}{\left( \frac{\Delta E_\nu}{2} \right)} e^{i \frac{\Delta E_\nu}{2} t} \nu_e \rangle,
\]

where \( \Delta E_\nu = E_\nu(k_\nu) + E_d(k_\nu) - M_m \). It is seen that unlike Eq. (24), imposed by Giunti’s prescription Eq. (10), the amplitude Eq. (25) has a nontrivial imaginary phase.

Substituting Eq. (25) into Eq. (22) we get the wave function

\[
|\nu_e(t)\rangle = i \text{sign} \left[ \sin \left( \frac{\Delta E_\nu}{2} t \right) \right] e^{i \frac{\Delta E_\nu}{2} t} |\nu_e\rangle,
\]

where \( \text{sign}[f(t)] = \pm 1 \) for \( f(t) \geq 0 \).

Thus, the wave function Eq. (26), constructed in accordance with Giunti’s prescription Eq. (9), does not describe a stationary neutrino state, therefore it cannot be used for the calculation of the amplitude of the \( I_m \rightarrow I_d + \nu_e \) decay in standard time–dependent perturbation theory [7]–[9] with the neutrino \( \nu_e \), treated as an elementary particle.

4 Amplitudes of decays with undetected neutrinos

The problem, which we discuss in this section, concerns the definition of the amplitudes of two–body weak decays \( I_m \rightarrow I_d + \nu \) with undetected neutrinos, where \( I_m \) and \( I_d \) are the initial and final nuclear states. The experimental analysis of the reaction \( I_m \rightarrow I_d + \nu \)
contains the preparation of the initial nuclear state $I_m$, which should be the H–like ion, and the detection of the final nuclear state $I_d$, which is a bare nucleus. The neutrino is not detected.

According to a modern theory of neutrino physics [3], neutrinos can be detected only in the states with definite leptonic flavours $|\nu_e\rangle$, $|\nu_\mu\rangle$ or $|\nu_\tau\rangle$, which are superpositions of the neutrino states $|\nu_j\rangle$ with masses $m_j$

$$|\nu_\alpha\rangle = \sum_{j=1,2,3} U_{\alpha j}^* |\nu_j\rangle, \quad (27)$$

where $\alpha = e, \mu$ and $\tau$ for the electron $e^-$, the muon $\mu^-$ and $\tau$–lepton $\tau^-$, respectively, $U_{\alpha j}$ are elements of the $3 \times 3$ unitary matrix $U$ [3] [4]. The neutrinos $\nu_j$ are not detectable directly, since they have no definite leptonic flavour [3].

The weak interactions of neutrinos $\nu_j$ with leptons and hadrons (or nuclei) are defined by the current $\times$ current interactions

$$H_W^{(h)}(t) = \frac{G_F}{\sqrt{2}} V_{\text{CKM}}^{(h)} \sum_\alpha \int d^3x J_\rho^{(h)}(x) J_\rho^{(\alpha)}(x) =$$

$$= \frac{G_F}{\sqrt{2}} V_{\text{CKM}}^{(h)} \sum_{\alpha,j} U_{\alpha j} \int d^3x J_\rho^{(h)}(x) [\bar{\psi}_{\nu_j}(x) \gamma^\rho (1 - \gamma^5) \psi_\alpha(x)], \quad (28)$$

cauised by the W–boson exchange, where $G_F$ is the Fermi weak constant, $J_\rho^{(h)}(x)$ is a charged hadronic current, $V_{\text{CKM}}^{(h)}$ is a matrix element of the Cabibbo–Kobayashi–Maskawa (CKM) matrix dependent on the structure of the hadronic current [4] and $J_\rho^{(\alpha)}(x)$ is the charged leptonoc current, defined by [3]

$$J_\rho^{(\alpha)}(x) = \sum_{j=1,2,3} U_{\alpha j} \bar{\psi}_{\nu_j}(x) \gamma^\rho (1 - \gamma^5) \psi_\alpha(x), \quad (29)$$

where $\bar{\psi}_{\nu_j}(x)$ and $\psi_\alpha(x)$ are operators of the neutrino fields $\nu_j$ with masses $m_j$ and lepton fields – the electron $e^-$, the muon $\mu^-$ and the $\tau$–lepton $\tau^-$ for $\alpha = e, \mu$ and $\tau$, respectively.

Using the Hamilton operator of the weak interactions Eq.(28) we can solve the following problem [11 [12]. Let a neutrino $\nu$, emitted in the reaction $I_m \rightarrow I_d + \nu$, be used for the subsequent reaction $\nu + X \rightarrow Y + e^-$, where $X$ and $Y$ are two hadronic or nuclear states. In this case, according to [11 [12], the amplitude of the transition $I_m \rightarrow I_d + \nu \rightarrow \nu + X \rightarrow Y + e^-$ can be defined to the second order of time–dependent perturbation theory as

$$A_{I_m \rightarrow Y + e^-}(t) = -i \sum_{j=1,2,3} \int_0^t dt'' \langle e^- Y | H_W | X \nu_j \rangle e^{i(E_e + E_Y - E_X - E_j)t''} A(I_m \rightarrow I_d + \nu_j)(t) =$$

$$= - \sum_{j=1,2,3} \int_0^t dt'' \langle e^- Y | H_W | X \nu_j \rangle e^{i(E_e + E_Y - E_X - E_j)t''} \sum_{j=1,2,3} \int_0^{t''} dt' \langle \nu_j I_d | H_W | I_m \rangle e^{i(E_j + E_d - E_m)t'}. \quad (30)$$

where $E_e$, $E_Y$, $E_X$ and $E_j$ are the energies of the electron, the hadronic (or nuclear) states $Y$ and $X$ and the neutrino, respectively. The amplitudes of the $I_m \rightarrow I_d + \nu_j$ transitions
are defined to the first order of time–dependent perturbation theory as \[11, 12\]

\[
A(I_m \rightarrow I_d + \nu_j)(t) = -i \int_0^t dt' \langle \nu_j, I_d | H_W | I_m \rangle \, e^{i(E_j + E_d - E_m)t'},
\]  

(31)

where \(E_m\) and \(E_d\) are energies of the initial and final hadronic (nuclear) states and \(E_j\) is a neutrino energy.

Thus, the amplitude of the transition \(I_m \rightarrow I_d + \nu \rightleftharpoons \nu + X \rightarrow Y + e^-\) is a coherent sum of the amplitudes of the transitions \(I_m \rightarrow I_d + \nu_j \rightleftharpoons \nu_j + X \rightarrow Y + e^-\). As a result the rate of the transition \(I_m \rightarrow I_d + \nu \rightleftharpoons \nu + X \rightarrow Y + e^-\), defined by \(|A_{I_m \rightarrow Y + e^-}(t)|^2\), should contain both the squared absolute values of the amplitudes of the \(I_m \rightarrow I_d + \nu_j \rightarrow \nu_j + X \rightarrow Y + e^-\) transitions and the interference terms \[11, 12\].

Now let us consider the \(I_m \rightarrow I_d + \nu_e\) decay. Using the definition of the wave function of the electronic neutrino Eq.(27), the weak interaction Hamilton operator Eq.(28) and standard time–dependent perturbation theory \[7, 9\], for the amplitude of the \(I_m \rightarrow I_d + \nu_e\) decay, dependent on time \(t\), we obtain the following expression

\[
A(I_m \rightarrow I_d + \nu_e)(t) = \sum_j \langle \nu_e | \nu_j \rangle A(I_m \rightarrow I_d + \nu_j)(t) = \sum_j U_{ej} A(I_m \rightarrow I_d + \nu_j)(t) =
\]

\[
= -i \sum_j \int_0^t dt' U_{ej} \langle \nu_j, I_d | H_W | I_m \rangle \, e^{i(E_j + E_d - E_m)t'}.
\]  

(32)

Thus, the amplitude of the \(I_m \rightarrow I_d + \nu_e\) decay is a coherent sum of the amplitudes \(I_m \rightarrow I_d + \nu_j\) with a weight \(|U_{ej}|^2\). The decay rate \(\lambda(t)\) is defined by

\[
\lambda(t) \propto \int |A(I_m \rightarrow I_d + \nu_e)(t)|^2 d\rho,
\]  

(33)

where \(d\rho\) is an element of a phase volume of the final state. The decay rate contains both the contributions of the squared absolute values of the amplitudes of the transitions \(I_m \rightarrow I_d + \nu_j\) and the interference terms.

For the \(EC\)–decay of the H–like \(^{140}Pr^{58+}\) ion \(^{140}Pr^{58+} \rightarrow ^{140}Ce^{58+} + \nu_e\), calculated for the mixing angle \(\theta_{13} = 0\) \[4\], the decay rate is equal to

\[
\lambda_{EC}^{(H)}(t) = \left(1 - \frac{1}{2} \sin^2(2\theta_{12})\right) \lambda_{EC}^{(H)} \left\{1 + a_{EC} \cos \left(\frac{\Delta m^2_{21} t}{4M_d}\right)\right\}.
\]  

(34)

where \(\lambda_{EC}^{(H)}\) has been calculated in \[6\] (see also \[2\]) and reads

\[
\lambda_{EC}^{(H)} = \frac{1}{2F + 1} \frac{3}{2} |M_{GT}|^2 |\langle \psi^{(Z)}_{1s} |^2 \frac{Q_H^2}{\pi}.
\]  

(35)

The amplitude \(a_{EC}\) of a periodic dependence of the decay rate is

\[
a_{EC} = \frac{\sin^2(2\theta_{12})}{1 + \cos^2(2\theta_{12})} e^{-\delta^2 \Delta^2_{21}}.
\]  

(36)

For the averaged over time decay rate \(\langle \lambda_{EC}^{(H)}(t) \rangle\) we obtain the following expression

\[
\langle \lambda_{EC}^{(H)}(t) \rangle = \left(1 - \frac{1}{2} \sin^2(2\theta_{12})\right) \lambda_{EC}^{(H)}.
\]  

(37)
According to [6], the appearance of the factor $1 - \frac{1}{2} \sin^2(2\theta_{12}) = 0.57$, calculated for the experimental value of the mixing angle $\theta_{12} = 33.9$ degrees [4], contradicts the experimental data by GSI on the ratios of the EC and $\beta^+$ decays of the H–like $^{140}$Pr$^{58+}$ and He–like $^{140}$Pr$^{57+}$ ions [13].

This shows that unlike Giunti’s assertion (see a discussion above Eq.(8) in Section 2 and Ref.[5]) one cannot use the wave function $|\nu_e\rangle = \sum_j U_{ej}^* |\nu_j\rangle$ for the analysis of the EC–decay of the H–like $^{140}$Pr$^{58+}$ ion.

Thus, the amplitude of the two–body weak decay $I_m \to I_d + \nu$ with neutrinos in the final state should be taken in the form of a coherent sum of the amplitudes $I_m \to I_d + \nu_j$. For the undetected neutrino and unfixed leptonic flavour of the neutrino state the amplitude of the $I_m \to I_d + \nu$ is equal to

$$A(I_m \to I_d + \nu) = \sum_j A(I_m \to I_d + \nu_j).$$

As has been shown in [2], the decay rate of the EC–decay of the H–like $^{140}$Pr$^{58+}$ ion $^{140}$Pr$^{58+} \to ^{140}$Ce$^{58+} + \nu_e$, calculated for the mixing angle $\theta_{13} = 0$ [4], is

$$\lambda^{(H)}_{EC}(t) = \lambda^{(H)}_{EC} \left\{ 1 + a_{EC} \cos \left( \frac{\Delta m^2_{21}}{4M_d} t \right) \right\},$$

where $\lambda^{(H)}_{EC}$ is given by Eq.(35). For the averaged over time decay rate $\langle \lambda^{(H)}_{EC}(t) \rangle$ we obtain the following expression [6]

$$\langle \lambda^{(H)}_{EC}(t) \rangle = \lambda^{(H)}_{EC},$$

which describes well the experimental data by GSI on the ratios of the EC and $\beta^+$ decays of the H–like $^{140}$Pr$^{58+}$ and He–like $^{140}$Pr$^{57+}$ ions [13].

5 On time–dependence of $\pi^+ \to \mu^+ + \nu$ decay rate

In this section we calculate the decay rate $\lambda_{\pi^+}(t)$ of the $\pi^+$–meson decay $\pi^+ \to \mu^+ + \nu$. Following standard theory of weak interactions, standard time–dependent perturbation theory and using a coherent contribution of the decay channels $\pi^+ \to \mu^+ + \nu_j$ [2], for the amplitude of the $\pi^+ \to \mu^+ + \nu$ decay we obtain the following expression

$$A(\pi^+ \to \mu^+ + \nu) = \sum_j A(\pi^+ \to \mu^+ + \nu_j) = G_F V_{ud} F_\pi m_\mu \sum_j U_{\mu j} i \int_0^t d\tau (2\pi \delta^2)^{3/2}$$

$$\times \int \frac{d^3k}{(2\pi)^3} e^{-\frac{i}{2} \delta^2 (\vec{k} - \vec{\tau})^2} u_\mu^\dagger (\vec{r}, \sigma) (1 + \gamma^5) v_\mu (\vec{p}_+, \sigma_+) e^{-i(\vec{k} + \vec{p}_+) \cdot \vec{r} - i(E_j(\vec{k}) + E_+ (\vec{p}_+) - m_\pi)\tau} =$$

$$= -2m_\mu \sqrt{2} m_\pi G_F V_{ud} F_\pi \sum_j U_{\mu j} \sqrt{E_j} e^{-\frac{i}{2} \delta^2 (\vec{p}_+ + \vec{\tau})^2} \frac{\sin \left( \frac{\Delta E_j t}{2} \right)}{\left( \frac{\Delta E_j}{2} \right)} e^{i\Delta E_j \frac{t}{2}},$$

where $F_\pi = 92.4$ MeV is the $\pi^+$–meson leptonic constant [4], $m_\pi = 139.57$ MeV and $m_\mu = 105.66$ MeV are the pion and muon masses, respectively, $\Delta E_j = E_j(\vec{p}_+) + E_+ (\vec{p}_+) - m_\pi$,
The absolute value of the amplitude is equal to
\[ |A(\pi^+ \rightarrow \mu^+ \nu)|^2 = 8m_{\mu}^2 m_{\pi} G_F^2 |V_{ud}|^2 F_\pi^2 (2\pi \delta^2)^3 \]

\[
\times \left\{ \frac{|U_{\mu j}|^2}{E_+(\vec{k}_j)} E_j(\vec{k}_j) \sin^2 \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right) \frac{\sin^2 \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right)}{\sin^2 \left( \frac{\Delta E_i(\vec{k}_j)}{2} t \right)} \right. \\
\times e^{-\delta^2 \Delta^2 \vec{k}_{ij}} \sin \left( \frac{\Delta E_i(\vec{k}_i)}{2} t \right) \frac{\sin \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right)}{\sin \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right)} \cos \left( \frac{E_i(\vec{k}_i) - E_j(\vec{k}_j)}{2} t \right) \right\}.
\]

Following [2], we obtain the neutrino spectrum. For this aim we integrate over the phase volume of the positron. This gives [2]

\[ N_\nu(t) = \frac{1}{2m_\pi} \int |A(\pi^+ \rightarrow \mu^+ \nu)|^2 \frac{d^3p_+}{(2\pi)^3 2E_+(\vec{p}_+)} = 2m_{\mu} G_F^2 |V_{ud}|^2 F_\pi^2 (\pi \delta^2)^{3/2} \]

\[
\times \left\{ \frac{|U_{\mu j}|^2}{E_+(\vec{k}_j)} E_j(\vec{k}_j) \sin^2 \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right) \frac{\sin^2 \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right)}{\sin^2 \left( \frac{\Delta E_i(\vec{k}_i)}{2} t \right)} \right. \\
\times e^{-\delta^2 \Delta^2 \vec{k}_{ij}} \sin \left( \frac{\Delta E_i(\vec{k}_i)}{2} t \right) \frac{\sin \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right)}{\sin \left( \frac{\Delta E_j(\vec{k}_j)}{2} t \right)} \cos \left( \frac{E_i(\vec{k}_i) - E_j(\vec{k}_j)}{2} t \right) \right\}.
\]

The \( \pi^+ \)-meson decay rate \( \lambda_{\pi^+}(t) \) is equal to [2]

\[ \lambda_{\pi^+}(t) = \int \frac{d^3k}{(2\pi)^3 2E_\nu} \frac{N_\nu(t)}{t(\pi \delta^2)^{3/2}} = \lambda_{\pi^+} \left( 1 + \sum_{i<j} 2U_{\mu i}^* U_{\mu j} e^{-\delta^2 \Delta^2 \vec{k}_{ij}} \cos \left( \frac{2\pi t}{T_{ij}} \right) \right), \]

where we have denoted

\[ T_{ij} = \frac{4\pi}{m_\pi} \frac{m_{\pi}^2 - m_\mu^2}{m_i^2 - m_j^2} \]

and

\[ \lambda_{\pi^+} = \frac{G_F^2 |V_{ud}|^2}{4\pi} F_\pi^2 m_\mu^2 m_\pi \left( 1 - \frac{m_i^2}{m_j^2} \right)^2 = 2.49 \times 10^{-14} \text{ MeV}. \]

The theoretical value \( \lambda_{\pi^+} = 2.49 \times 10^{-14} \text{ MeV} \) agrees well with the experimental one \( \lambda_{\pi^+}^{\text{exp}} = 2.53 \times 10^{-14} \text{ MeV} \) [4].
The time–dependence of the decay rate $\lambda_\pi^+(t)$ is defined by

$$
\lambda_\pi^+(t) = \lambda_\pi^+ \left( 1 - \sin 2\theta_{12} \cos^2 \theta_{23} e^{-\delta^2 \Delta^2 k_{21}} \cos \left( \frac{2\pi t}{T_{21}} \right) 
- \sin \theta_{12} \sin 2\theta_{23} e^{-\delta^2 \Delta^2 k_{31}} \cos \left( \frac{2\pi t}{T_{31}} \right) 
+ \cos \theta_{12} \sin 2\theta_{23} e^{-\delta^2 \Delta^2 k_{32}} \cos \left( \frac{2\pi t}{T_{32}} \right) \right),
$$

(47)

where we have used matrix elements $U_{\mu j}$ [4] and set $\theta_{13} = 0$ [4, 14]. The periods $T_{ij}$ are equal to

$$
T_{21} = \frac{4\pi}{m_\pi} \frac{m_\pi^2 - m_\mu^2}{\Delta m_{21}^2} = 1.11 \times 10^{-3} \text{ sec},
$$

$$
T_{31} = \frac{4\pi}{m_\pi} \frac{m_\pi^2 - m_\mu^2}{\Delta m_{31}^2} = 2.04 \times 10^{-4} \text{ sec},
$$

$$
T_{32} = \frac{4\pi}{m_\pi} \frac{m_\pi^2 - m_\mu^2}{\Delta m_{32}^2} = 2.04 \times 10^{-4} \text{ sec},
$$

(48)

were we have set $\Delta m_{31}^2 \simeq \Delta m_{32}^2 = 2.40 \times 10^{-3} \text{ eV}^2$ [4, 14], $m_\pi = 139.57 \text{ MeV}$ and $m_\mu = 105.66 \text{ MeV}$ [4]. The periods of oscillations are much greater than the lifetime $\tau_{\pi^+} = 2.60 \times 10^{-8} \text{ sec}$ [4].

### 6 Conclusion

Concluding our replies on Giunti’s critique we argue that

- The amplitude of the $EC$–decay $I_m \rightarrow I_d + \nu$ with an undetected neutrino can be written in the form of the coherent sum of the amplitudes $I_m \rightarrow I_d + \nu_j$ of the $EC$–decays:

$$
A(I_m \rightarrow I_d + \nu) = \sum_j A(I_m \rightarrow I_d + \nu_j).
$$

- The wave function of the neutrino in the final state of the $EC$–decay $I_m \rightarrow I_d + \nu$ cannot be taken in the form of Eq. (8), as it is proposed by Giunti [5]. This leads to the contradiction with the experimental data on the ratios of the $EC$ and $\beta^+$ decays of the H–like and He–like ions at GSI [13].

- Since the wave function Eq. (9), proposed by Giunti as a wave function of a neutrino in the final state of the $EC$–decay of the H–like ion [5], does not describe a stationary neutrino state, required by time–dependent perturbation theory, it cannot be used as a wave function of a neutrino in the final state of the $EC$–decay $I_m \rightarrow I_d + \nu$ of the H–like ion.
• The falseness of the wave function Eq.(9), proposed by Giunti as a wave function of a neutrino in the final state of the $EC$–decay of the H–like ion [5], is confirmed also by the failure of this wave function to describe an asymptotic neutrino state, related to an observable detectable neutrino state [10].

• The relation Eq.(10), proposed by Giunti for the definition of the amplitude of the $EC$–decay of the H–like ions, is incorrect, since it is based on the use of the incorrect wave function Eq.(9).

• The direct calculation shows also that Eq.(10) cannot be used for the definition of the amplitude of the $I_m \rightarrow I_d + \nu_e$ decay even if the neutrino $\nu_e$ is an elementary particle.

• The calculation of the $\pi^+ \rightarrow \mu^+ + \nu$ decay rate shows that the definition of the amplitude of the $\pi^+ \rightarrow \mu^+ + \nu$ decay in the form of the coherent sum of the amplitudes of the decay channels $\pi^+ \rightarrow \mu^+ + \nu_j$

$$A(\pi^+ \rightarrow \mu^+ + \nu) = \sum_j A(\pi^+ \rightarrow \mu^+ + \nu_j)$$

leads to the correct description of the decay rate $\lambda_{\pi^+}(t)$, agreeing well with the experimental data on the lifetime of the $\pi^+$–meson [4], and with a periodic time–dependence [2]. Since the periods of variation of the time–dependent terms are much greater than the lifetime of the $\pi^+$–meson, such a time–dependence cannot be measured.

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