Comment on a classical limit of Grover’s algorithm

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Abstract

A classical limit of Grover’s algorithm is discussed by assuming a very rapid decoherence (or dephasing) between consecutive Grover’s unitary operations, which leads pure quantum states to completely decohered mixed states. One can identify a specific element among \(N\) unsorted elements by a probability of the order of unity after \(k \sim N/4\) steps of classical amplification defined by the decohered mixed states, in contrast to Grover’s \(k \sim \pi\sqrt{N}/4\) steps in quantum mechanical amplification. This difference is caused by the loss of quantum coherence with or without the loss of entanglement depending on each case.

1 Introduction

Grover’s algorithm is fundamental in the study of the basic mechanism of speedups of the quantum computer relative to classical digital computers. The algorithm is defined for a search problem of a specific element among the unsorted \(N\)-number of elements \(\{1, 2, ..., N\}\) with \(N = 2^n\). The classical search problem is first transcribed into a quantum search problem; a specification of the quantum problem is important and we mainly work on a generic qubit model such as an Ising spin-type model [1].

Following the general idea of measurement theory in quantum mechanics [2], one may start with an initial state that is a superposition of \(N\)-number of quantum states with equal probability together with an ancilla \(|0\rangle\)

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |0\rangle.
\]
The unitary operator that searches the specific element $a$, which is represented by a quantum state $|a\rangle$, is given by

$$\tilde{U} = \sum_{i \neq a} |i\rangle \langle i| \otimes I + |a\rangle \langle a| \otimes X$$  \hspace{1cm} (1.2)

where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (1.3)

with $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$. The fact that we can identify the state $|a\rangle$ implies that we have some means to distinguish $|a\rangle$ from the rest of states. This identification is also required in the classical search and thus not specific to the present quantum search. It is also important that we assume the existence of such a state $|a\rangle$ when we search the entire given Hilbert space and we are not giving the existence (or absence) proof of such a state $|a\rangle$. After the application of the unitary measurement operator $\tilde{U}$ in (1.2), the state (1.1) becomes

$$\frac{1}{\sqrt{N}} \sum_{i \neq a} |i\rangle \otimes |0\rangle + \frac{1}{\sqrt{N}} |a\rangle \otimes |1\rangle.$$  \hspace{1cm} (1.4)

When one measures the ancilla states, one finds the state $|1\rangle$ with probability $1/N$ and one identifies the state $|a\rangle$, analogously to the use of the particle path (which corresponds to $|1\rangle$) to identify the spin direction (which corresponds to $|a\rangle$) in the Stern-Gerlach experiment. This probability is the same as in the classical search problem without a merit of dealing with the superposition of states

$$|+\rangle \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle.$$  \hspace{1cm} (1.5)

In this paper, we first recapitulate the essence of Grover’s algorithm which speeds up the data search and then study a classical limit of Grover’s algorithm using the idea of decohering a pure state to a completely mixed state.

## 2 Grover’s algorithm

The efficient algorithm of Grover [3, 4] is based on the measurement operator

$$U = \sum_{i \neq a} |i\rangle \langle i| - |a\rangle \langle a|$$  \hspace{1cm} (2.1)
which corresponds to the choice of the ancilla state as \( \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle] \) in (1.1), or

\[
U = \sum_{all \ i} (-1)^{f(i)}|i\rangle\langle i|
\]

(2.2)

with the oracle \( f(i) = 0 \) for \( i \neq a \) and \( f(a) = 1 \). How this identification of a specific state is done by the oracle (subroutine) is not the issue at this moment, but it could be time consuming. The appearance of the state \( |a\rangle \), which we are looking for, in the unitary operator (2.1) shows that we have a means to identify the state \( |a\rangle \) among the states in the unsorted data but it does not imply that we already know the answer to the search problem. We also introduce another unitary operator

\[
V = 2|+\rangle\langle +| - I,
\]

(2.3)

with \( I = \sum_i |i\rangle\langle i| \). The basic procedure is to amplify the target state \( |a\rangle \) in the initial state (1.5) that is written as

\[
|+\rangle = \sqrt{\frac{N-1}{N}} \sum_{i \neq a} |i\rangle + \frac{1}{\sqrt{N}} |a\rangle
\]

(2.4)

where

\[
|b\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \neq a} |i\rangle
\]

(2.5)

with \( \langle b|b\rangle = 1 \) and \( \langle a|b\rangle = 0 \), and then we measure the final state by the basis \( \{|i\rangle\}_i^N \). We also define

\[
|-\rangle = \sqrt{\frac{N-1}{N}} |a\rangle - \frac{1}{\sqrt{N}} |b\rangle
\]

(2.6)

which is normalized and orthogonal to \( |+\rangle \).

When one defines a new complete orthonormal basis set by \( \{|a\rangle, |b\rangle, \ldots\} \), we have

\[
U = |b\rangle\langle b| - |a\rangle\langle a|, \quad V = |+\rangle\langle +| - |\rangle\langle -|,
\]

(2.7)

which are relevant to the present problem. It is important that these two unitary operators are defined in the limited 2-dimesional space spanned by \( \{|a\rangle, |b\rangle\} \), which is equivalent to the space spanned by \( \{|+\rangle, |-\rangle\} \). By defining at \( t = 0 \), for example,

\[
\sin \theta = \langle a|+\rangle = \sqrt{1/N}, \quad \cos \theta = \langle b|+\rangle = \sqrt{(N-1)/N},
\]

(2.8)
we have $|+\rangle = \cos \theta |b\rangle + \sin \theta |a\rangle$, $|-\rangle = \cos \theta |a\rangle - \sin \theta |b\rangle$ and

$$VU|+\rangle = \cos 2\theta |+\rangle + \sin 2\theta |-\rangle,$$

$$= \cos 3\theta |b\rangle + \sin 3\theta |a\rangle$$  \hspace{1cm} (2.9)

and by noting

$$(VU)^k|a\rangle = \cos 2k\theta |a\rangle - \sin 2k\theta |b\rangle,$$

$$(VU)^k|b\rangle = \cos 2k\theta |b\rangle + \sin 2k\theta |a\rangle,$$  \hspace{1cm} (2.10)

we have for general $k$ assuming $\theta \simeq 1/\sqrt{N}$,

$$(VU)^k|+\rangle = \cos (2k + 1)\theta |b\rangle + \sin (2k + 1)\theta |a\rangle$$

$$\simeq [1 - \frac{1}{2}(2k + 1)\frac{1}{\sqrt{N}}]^2 |b\rangle + (2k + 1)\frac{1}{\sqrt{N}}|a\rangle.$$  \hspace{1cm} (2.11)

The optimal amplification of the state $|a\rangle$ is achieved for

$$k \sim \frac{\pi}{4}\sqrt{N},$$

namely, one can identify the specific state $|a\rangle$ with a probability of the order of unity after the amplification of $k$-number of steps. The unsorted quantum data is transformed to the data with a probability distribution

$$p_i = \langle +|(VU)^k|i\rangle\langle i|(VU)^k|+\rangle$$  \hspace{1cm} (2.13)

that is peaked at $i = a$, which is the location of the element we are looking for, when measured by the basis $\{|i\rangle\}_{i=1}^N$. The prediction of the formation of a peak at a specific point $i = a$ is analogous to the formation of peaks on the screen after the accumulation of data in the double-slit interference experiment. We know the characteristics of the state $|a\rangle$ beforehand, and thus it is essential to locate the position of the state $|a\rangle$ in the set $\{|i\rangle\}_{i=1}^N$. This is the essence of Grover’s algorithm \[3, 4, 5, 6, 7, 8\].

It is significant that the above operation is performed essentially on a superposition of only two states $\{|a\rangle, |b\rangle\}$; if more states are involved, one would need more parameters in addition to $\theta$ to point the state to the direction of $|a\rangle$. It will be crucial to construct a single well-defined state $|b\rangle$ as a superposition of all the rest of original states, which is strictly orthogonal to other $N - 2$ states formed of linear combinations of $\{|i\rangle\}_{i=1}^{N-1}$, in a realistic quantum computer.

It appears that the entanglement does not play any explicit role in the above analysis. But if one studies the simplest model of two Ising spins $H = -J\sigma_{1z}\sigma_{2z}$ -
\[(\sigma_z + \sigma_z) h \text{ with positive constants } J \text{ and } h, \text{ for example, one may start with a product state that has no entanglement}
\]

\[|+\rangle = \frac{1}{2} (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = \frac{1}{2} (|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \quad (2.14)\]

which becomes after marking the lowest energy state \(|a\rangle \equiv |0\rangle \otimes |0\rangle\), for example, by the unitary operator (2.1)

\[U|+\rangle = \frac{1}{2} [|-0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle]. \quad (2.15)\]

This last expression is no more a product state and in fact entangled, as is confirmed by considering the partial trace of the density matrix \(\rho' = U|+\rangle \langle +|U^\dagger\) that is reduced to a mixed state [9, 10]. In this sense, the entanglement plays a role in Grover’s algorithm in the process of identifying the states \(|a\rangle\) and \(|b\rangle\). In fact, any state (2.11) is entangled except for the cases with \(\sin 2k\theta = 0\) for which \((VU)^k|+\rangle = \cos \theta |b\rangle + \sin \theta |a\rangle = |+\rangle\) or \(\cos (2k+1)\theta = 0\) for which \((VU)^k|+\rangle = |a\rangle\), if one chooses the initial state as a generalization of (2.14). See also [11] and references therein.

Incidentally, the initial state such as (2.14) and the final state \(|a\rangle\), which are in principle described by both the classical and quantum formulations, are generally chosen to be product states and not entangled.

It is also important to recall that the quantum states \(|a(t)\rangle\) and \(|b(t)\rangle\) in (2.11) are time dependent in general and contain complex phases, although the orthonormality, \(\langle a(t)|a(t)\rangle = 1\), \(\langle b(t)|b(t)\rangle = 1\) and \(\langle a(t)|b(t)\rangle = 0\), are assumed to be preserved in the present paper.

### 3 Classical probability amplification

To analyze the notion of classical amplification it is natural to start with a density matrix corresponding to the initial state (1.5) with no quantum coherence

\[\rho = \sum_{i=1}^{N} \frac{1}{N} |i\rangle \langle i| \quad (3.1)\]

with \(\text{Tr} \rho = 1\), which gives an equal probability

\[p_i = \text{Tr} |i\rangle \langle i| \rho = \frac{1}{N} \quad (3.2)\]

for any state $|i\rangle$. The amplification of the appearance probability of the specific state $|a\rangle$ implies that we realize a density matrix $\rho = \sum_i w_i |i\rangle\langle i|$. 

We discuss how a classical amplification in the above sense is realized in Grover’s algorithm when one assumes that the very rapid decoherence takes place between consecutive Grover’s unitary operations.

In quantum mechanics which keeps $\langle \psi | \psi \rangle$ invariant under the unitary transformation, we have

$$|\psi_0\rangle = \cos \theta |b\rangle + \sin \theta |a\rangle \rightarrow |\psi_1\rangle = VU |\psi_0\rangle = \cos 3\theta |b\rangle + \sin 3\theta |a\rangle$$

after an operation of amplification as in (2.9). The incoherent mixed state after the complete decoherence (or dephasing), $|\psi_1\rangle \equiv \rho_0 = |\psi_0\rangle\langle \psi_0|$, is defined by

$$\rho_0 = |\psi_0\rangle\langle \psi_0| \Rightarrow Tr_\rho_0 = \sin^2 \theta |a\rangle\langle a| + \cos^2 \theta |b\rangle\langle b|$$

with

$$Tr_\rho_0 = Tr\{\sin^2 \theta |a\rangle\langle a| + \cos^2 \theta |b\rangle\langle b|\} = 1. \quad (3.5)$$

After the quantum amplification operation (2.10) of $\rho_0$, we have

$$\rho_1 = VU \rho_0 (VU)\dagger = \sin^2 \theta VU |a\rangle\langle a| (VU)\dagger + \cos^2 \theta VU |b\rangle\langle b| (VU)\dagger$$

which becomes after the complete decoherence (or dephasing),

$$\bar{\rho}_1 = VU \rho_0 (VU)\dagger = \sin^2 \theta VU |a\rangle\langle a| (VU)\dagger + \cos^2 \theta VU |b\rangle\langle b| (VU)\dagger$$

where we used $VU |a\rangle = \cos 2\theta |a\rangle - \sin 2\theta |b\rangle$ and $VU |b\rangle = \cos 2\theta |b\rangle + \sin 2\theta |a\rangle$ in (2.10), and thus after the complete decoherence (or dephasing)

$$\bar{VU} |a\rangle (\bar{VU} |a\rangle)\dagger = (\cos 2\theta |a\rangle - \sin 2\theta |b\rangle)(\cos 2\theta |a\rangle - \sin 2\theta |b\rangle)\dagger$$

$$= \cos^2 2\theta |a\rangle\langle a| + \sin^2 2\theta |b\rangle\langle b|,$$

$$\bar{VU} |b\rangle (\bar{VU} |b\rangle)\dagger = (\cos 2\theta |b\rangle + \sin 2\theta |a\rangle)(\cos 2\theta |b\rangle + \sin 2\theta |a\rangle)\dagger$$

$$= \cos^2 2\theta |b\rangle\langle b| + \sin^2 2\theta |a\rangle\langle a|.$$
and thus

$$\text{Tr}_1 = 1. \quad (3.10)$$

Similarly, we have

$$\rho_2 = VU\rho_1(VU)^\dagger = [\sin^2 2\theta \cos^2 2\theta + \cos^2 2\theta \sin^2 2\theta][\cos^2 2\theta |a\rangle\langle a| + \sin^2 2\theta |b\rangle\langle b|]$$

$$+ [\sin^2 2\theta \sin^2 2\theta + \cos^2 2\theta \cos^2 2\theta][\cos^2 2\theta |b\rangle\langle b| + \sin^2 2\theta |a\rangle\langle a|]$$

$$= [\sin^2\theta(\cos^4 2\theta + \sin^4 2\theta) + \cos^2\theta(2 \sin^2 2\theta \cos^2 2\theta)]|a\rangle\langle a|$$

$$+ [\cos^2\theta(\cos^4 2\theta + \sin^4 2\theta) + \sin^2\theta(2 \sin^2 2\theta \cos^2 2\theta)]|b\rangle\langle b| \quad (3.11)$$

with

$$\text{Tr}_2 = 1. \quad (3.12)$$

One thus has an iteration formula

$$\rho_{k-1} = c_{k-1}|a\rangle\langle a| + d_{k-1}|b\rangle\langle b|,$$

$$\rho_k = (c_{k-1} \cos^2 2\theta + d_{k-1} \sin^2 2\theta)|a\rangle\langle a|$$

$$+ (c_{k-1} \sin^2 2\theta + d_{k-1} \cos^2 2\theta)|b\rangle\langle b|, \quad (3.13)$$

with $\rho_0 = \sin^2\theta|a\rangle\langle a| + \cos^2\theta|b\rangle\langle b|$. This relation is exactly solved as,

$$\rho_k = c_k|a\rangle\langle a| + d_k|b\rangle\langle b|,$$

$$c_k = \cos^4 2\theta \sin^2 2\theta + \frac{1 - \cos^k 4\theta}{1 - \cos 4\theta} \sin^2 2\theta$$

$$= \frac{1}{2} - \left(\frac{1}{2} - \sin^2\theta\right) \cos^k 4\theta$$

$$= \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{N}\right)[1 - \frac{8}{N}(1 - \frac{1}{N})]^k$$

$$\simeq \frac{1}{2} \left(1 - \exp[-\frac{8}{N}\frac{k}{N}]\right) \quad (3.14)$$

for $N \to$ large with fixed $k/N$ \footnote{\[1 - \frac{8}{N}(1 - \frac{1}{N})]^k = \exp[k\ln(1 - \frac{8}{N}(1 - \frac{1}{N}))] \simeq \exp[-8\frac{k}{N}]\] for $N \to$ large with fixed $k/N.$}, and $d_k = 1 - c_k$. The maximum amplification $c_k \simeq 1/2$ is achieved at $k/N = 1$ if one considers $k \leq N$; in the classical amplification, (3.13) shows that $d_k$ is amplified if $c_k > 1/2$.

One can then confirm that, after a $k$-number of operations,

$$\rho_k \simeq [\theta^2 + k(2\theta)^2]|a\rangle\langle a| + [1 - (\theta^2 + k(2\theta)^2)]|b\rangle\langle b| \quad (3.15)$$
to the accuracy of $O(\theta^2) = O(\frac{1}{N})$. This $\rho_k$, in both the exact form and approximate form, satisfies the basic condition of the density matrix

$$\text{Tr} \rho_k = 1,$$

and thus the present way of achieving decoherence (or dephasing) is consistent as a model of the very rapid decoherence of the density matrix between consecutive unitary operations.

The classical limit of Grover’s algorithm thus shows

$$p_i = \text{Tr}\{|i\rangle\langle i|\rho_k\} \simeq (1 + 4k)/N$$

for $i = a$ after $k$-iteration, where we used $\theta^2 = 1/N$ and $p_i$ is the probability of finding the state $|i\rangle$. Thus the probability of finding the rest of states becomes

$$p_i = \text{Tr}\{|i\rangle\langle i|\rho_k\} \simeq 1 - (1 + 4k)/N$$

for $i \neq a$ if one uses the original states $\{|i\rangle\}_{i=1}^{N}$. After the iteration of the order $k \sim N/4$ times, we thus have the amplified probability of the order of unity $|3.17\rangle$ (in fact $p_a \simeq 1/2$ in the more accurate estimate using (3.14)) to find the state $a$ by a single trial. The classical amplification does not change the total $k \sim N$ steps required to identify a specific state $a$ although the numerical factor in front of $N$ is generally modified, in contrast to the quantum Grover’s algorithm where $k \sim \sqrt{N}$. The appearance of $k \sim \sqrt{N}$ in Grover’s algorithm is due to the fact that we work on the complex probability amplitude in quantum mechanics instead of the real classical probability, without any irreversible non-unitary effects such as the external thermal agitation.

The reduction of the problem to an effective two-dimensional space spanned by $\{|a\rangle, |b\rangle\}$, i.e., the target state and a superposition of all the rest of states, using the superposition principle plays an important role in the present classical amplification; otherwise, the natural incoherent density matrix would follow the equal a priori probability rule for all the involved states (see (3.14)) and consequently a smaller amplification of a specific state. An illuminating analogy is to imagine a system consisting of two almost degenerate states with very unbalanced occupation numbers $n_a \ll n_b$ with $n_a + n_b = N$; one would then obtain an enormous amplification of $n_a, n_a \sim n_b \sim N/2$, by simply touching the system to a heat bath (a kind of inverse purification).

4 Discussion and conclusion

Grover once posed the following question [12]: Why is it not possible to search in fewer than $O(\sqrt{N})$ steps? What is lacking is a simple and convincing two line argument that shows why one would expect this to be the case.
The present analysis was motivated by this remark, and we studied a classical limit of Grover’s algorithm. Our analysis may not answer the basic question, but it is natural to assume that the classical amplification does not reduce the total search steps to fewer than $O(N)$-steps when formulated in terms of a classical Ising spin, for example. If this is the case, the quantum search process, which deals with the complex probability amplitude rather than the real classical probability itself, can accomplish the search in $O(\sqrt{N})$ steps but not in fewer steps, since otherwise the classical search could be achieved in fewer than $O(N)$ steps. To be more explicit, the coherent treatment of the probability amplitude (2.11) gives after the $k$-steps of amplification

$$\sim k(2\theta)|a\rangle,$$  \hspace{1cm} (4.1)

while the treatment of the incoherent density matrix (3.15) gives

$$\sim k(2\theta)^2|a\rangle\langle a|.$$  \hspace{1cm} (4.2)

To achieve the coefficient of the state $|a\rangle$ of the order of unity, one thus needs $k \sim 1/\theta \sim \sqrt{N}$ and $k \sim 1/\theta^2 \sim N$, respectively.

An analog model of Farhi and Gutmann [13], which does not use entanglement in an obvious way, is similarly understood. The time evolution operator is defined by [13]

$$U(t) \equiv e^{-iHt} = e^{-iEt}\left\{ \cos(xEt) - i\sin(xEt) \left( \frac{x}{\sqrt{1-x^2}} \right) \frac{\sqrt{1-x^2}}{-x} \right\},$$  \hspace{1cm} (4.3)

with the initial condition

$$\psi(0) = \left( \begin{array}{c} x \\ \sqrt{1-x^2} \end{array} \right) = x|w\rangle + \sqrt{1-x^2}|r\rangle,$$

$$\rho(0) = \frac{\psi(0)\psi^\dagger(0)}{x^2|w\rangle\langle w| + (1-x^2)|r\rangle\langle r|}.$$  \hspace{1cm} (4.4)

where $x$ and $E$ are constant parameters and $|w\rangle$ and $|r\rangle$ correspond to our $|a\rangle$ and $|b\rangle$, respectively. The solution of the Schroedinger equation $\psi(t) = U(t)\psi(0)$ is given by

$$\psi(t) = e^{-iEt}\left\{ (x\cos(xEt) - i\sin(xEt))|w\rangle + \sqrt{1-x^2}\cos(xEt)|r\rangle \right\}.$$  \hspace{1cm} (4.5)

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For an infinitesimal $\Delta t$

$$U(\Delta t) \sim e^{-iE\Delta t}\{1 - \frac{1}{2}(xE\Delta t)^2 - i(xE\Delta t)\left(\frac{x}{\sqrt{1 - x^2}} \sqrt{1 - x^2} - \frac{1}{x}\right)\},$$

$$\psi(\Delta t) \sim e^{-iE\Delta t}\{(x - ixE\Delta t)|w\rangle + \sqrt{1 - x^2}(1 - \frac{1}{2}(xE\Delta t)^2)|r\rangle\},$$

$$\overline{\rho}(\Delta t) = \frac{U(\Delta t)\overline{\rho}(0)U^\dagger(\Delta t)}{\overline{U}(\Delta t)|w\rangle\langle w|U^\dagger(\Delta t)} \approx (x^2 + (xE\Delta t)^2)|w\rangle\langle w| + (1 - x^2)(1 - (xE\Delta t)^2)|r\rangle\langle r|, \quad \text{(4.6)}$$

where we used the complete decoherence (or dephasing)

$$\overline{U}(\Delta t)|w\rangle\langle w|U^\dagger(\Delta t) = (1 - (xE\Delta t)^2(1 - x^2))|w\rangle\langle w| + (xE\Delta t)^2(1 - x^2)|r\rangle\langle r|,$$

$$\overline{U}(\Delta t)|r\rangle\langle r|U^\dagger(\Delta t) = (xE\Delta t)^2(1 - x^2)|w\rangle\langle w| + (1 - (xE\Delta t)^2(1 - x^2))|r\rangle\langle r|. \quad \text{(4.7)}$$

After $k$-iteration for $xE\Delta t \ll 1$ and choosing $\Delta t$ to be the minimum uncertainty time $E\Delta t = 1$, we have

$$\psi(k\Delta t) \approx e^{-ikE\Delta t}\{(x - ik\theta E\Delta t)|w\rangle + \sqrt{1 - x^2}(1 - \frac{1}{2}(kxE\Delta t)^2)|r\rangle\}$$

$$\approx e^{-ik}\{x(1 - ik)|w\rangle + \sqrt{1 - x^2}(1 - \frac{1}{2}(kx)^2)|r\rangle\},$$

$$\overline{\rho}(k\Delta t) \approx (x^2 + k(xE\Delta t)^2)|w\rangle\langle w| + (1 - x^2)(1 - k(xE\Delta t)^2)|r\rangle\langle r|$$

$$\approx x^2(1 + k)|w\rangle\langle w| + ((1 - x^2) - kx^2)|r\rangle\langle r|. \quad \text{(4.8)}$$

When one sets $x = \theta = 1/\sqrt{N}$ in (4.8) following [13], the essence of Grover’s formula is recovered. The coefficients of the state $|w\rangle$ show that the transition from the quantum $k \sim \sqrt{N}$ to classical $k \sim N$ behavior is caused by the loss of quantum coherence but not by the loss of entanglement in this example.

The present analysis will help understand what is the probability amplification in the classical context, which will simultaneously deepen our understanding of quantum amplification. The analysis of the effect of decoherence on Grover’s algorithm is also crucial in the realistic study of the quantum search [14, 15, 16, 17, 18]. Our simple analytic formula (4.14) will be useful to understand the decoherence effect in an idealized model. We also mention that other approaches to the classical analogues of Grover’s algorithm have been discussed in the past [19, 20]. These approaches are, however, very different from the present analysis based on the rapid decoherence (or dephasing) of a pure quantum state to a decohered mixed density matrix.

In conclusion, the quantum computation is based on the processing of superposition states [21] and Grover’s algorithm shows speedups of the quantum search problem, for example, the reduction of $N = 10^6$ steps to $\sqrt{N} = 10^3$ steps. The practical
implementation of Grover’s algorithm in quantum search problems in comparison with the classical computation has been carefully examined [22], and the prospect of the superiority of quantum algorithm does not appear to be self-evident depending on each case. In the related optimization problems, which are often formulated in terms of Ising spin glass models [1], the idea of simulated annealing is known [23]. This is the standard approximation procedure used in classical digital computers. Recently, the possible speedup in the search of the ground state of the Ising model was suggested using the idea of quantum annealing [24]. A closely related idea of quantum computation is the scheme of quantum adiabatic evolution [25]. A naive application of Grover’s algorithm to a search of the ground state in the Ising model with 150 spins, for example, implies \( k \sim \sqrt{N} = 2^{75} \) which is a very large number. Practically, the quantum mechanical unitary operation does not speed up the search of the ground state enough, and a modified scheme which allows the non-unitary operation (and possibly phase transition) has been suggested [26]. In practical combinatorial optimization problems, which are expected to require about \( 10^5 \) qubits, one would need quantum heuristics and approximation schemes using some clever mixtures of quantum and classical computations.

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