ON THE SIZE OF NIKODYM SETS IN FINITE FIELDS

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ABSTRACT. Let $\mathbb{F}_q$ denote a finite field of $q$ elements. Define a set $B \subset \mathbb{F}^n_q$ to be Nikodym if for each $x \in B^c$, there exists a line $L$ such that $L \cap B^c = \{x\}$. The main purpose of this note is to show that the size of every Nikodym set is at least $C_n \cdot q^n$, where $C_n$ depends only on $n$.

1. INTRODUCTION

The finite field Kakeya problem, posed by Wolff in his influential survey [13], asks for the smallest subset of $\mathbb{F}^n_q$ that contains a line in each direction, where $\mathbb{F}_q$ denotes a finite field of $q$ elements. A subset containing a line in each direction is called a Kakeya set. In analogy with the Euclidean Kakeya problem, Wolff conjectured that $\sharp K \geq C_n q^n$ holds for any Kakeya set $K \subset \mathbb{F}^n_q$, where $C_n$ depends only on the dimension $n$. For $n = 2$ Wolff immediately proved the bound $\sharp K \geq q(q + 1)/2$, and it is best possible when $q$ is even. To the author’s knowledge, Blokhuis and Mazzocca [1] studied the finite field Kakeya problem in two dimensions and proved the sharp bound $\sharp K \geq q(q + 1)/2 + (q - 1)/2$ when $q$ is odd, as conjectured by Faber in [6]. The higher dimensional finite field Kakeya problem has been extensively investigated in [2, 8, 10, 12, 13] such as proving the bound $\sharp K \geq C_n q^{(n+2)/2}$ or $\sharp K \geq C_n q^{(4n+3)/7}$. Recently, using the polynomial method in algebraic extremal combinatorics, Dvir [5] completely confirmed this conjecture by proving

$$\sharp K \geq \binom{n+q-1}{n}.$$

On the other hand, Nikodym [9] proved that there exists a null set in the unit square such that every point of the complement is “linearly accessible through the set”, which means it lies on a line that is otherwise included in the set. Falconer [7] extended Nikodym’s result to higher dimensions proving there exists a set $N \subset \mathbb{R}^n$ of zero Lebesgue measure such that for each $x \in N^c$, there is a hyperplane $P$ satisfying $P \cap N^c = \{x\}$. In the Euclidean spaces Nikodym sets are closely related to Kakeya sets through Carbery’s transformation [4, 11].

Motivated by the above works, we shall define a set $B$ in $\mathbb{F}^n_q$ to be Nikodym if for each $x \in B^c$ there exists a line $L$ such that $L \cap B^c = \{x\}$. The main purpose of this note is to prove the lower bound

$$\sharp B \geq \binom{n+q-2}{n}.$$

Slightly different with the two dimensional finite field Kakeya problem, this bound is not best possible in two dimensions.
2. General dimensions

**Theorem 2.1.** Any Nikodym set \( B \subset \mathbb{F}_q^n \) satisfies
\[
|B| \geq \binom{n + q - 2}{n},
\]
where \( \mathbb{F}_q \) denotes a finite field of \( q \) elements.

**Proof.** We argue by contradiction and suppose
\[
|B| < \binom{n + q - 2}{n}.
\]
A basic result in combinatorics [3] says that the number of monomials in \( \mathbb{F}[x_1, \ldots, x_n] \) of degree at most \( d \) is
\[
\binom{n + d}{n},
\]
hence there exists a nonzero polynomial \( g \in \mathbb{F}[x_1, \ldots, x_n] \) of degree at most \( q - 2 \) such that
\[
g(y) = 0 \quad (\forall y \in B).
\]
For each \( x \in B^c \), there exists a line \( L \) such that
\[
L \cap B^c = \{x\}.
\]
The restriction of \( g \) to this line is a univariate polynomial of degree at most \( q - 2 \), and since it has at least \( q - 1 \) zeros, it must be zero on the entire line \( L \). Considering \( x \) belongs to this line, it follows that
\[
g(x) = 0.
\]
This would mean \( g \) is the zero polynomial, a contradiction.

\[\square\]

3. Two dimensions

**Theorem 3.1.** Any Nikodym set \( B \subset \mathbb{F}_q^2 \) satisfies
\[
\sharp B \geq \frac{2q^2}{3} + O(q) \quad (q \to \infty),
\]
where \( \mathbb{F}_q \) denotes a finite field of \( q \) elements.

**Proof.** Write \( s = \lfloor \frac{q}{3} \rfloor \). First, assume that
\[
\sharp B^c \leq s(q - 1) + 2q,
\]
then
\[
(3.1) \quad \sharp B \geq q^2 - s(q - 1) - 2q \geq q^2 - \frac{q}{3}(q - 1) - 2q = \frac{2q^2}{3} - \frac{5q}{3}.
\]
Else suppose that
\[
\sharp B^c \geq s(q - 1) + 2q.
\]
Since \( B \) is a Nikodym set, for each \( x \in B^c \) there exists a line \( L_x \) such that
\[
L_x \cap B^c = \{x\}.
\]
Obviously, all of these lines are distinct from each other. Noting that there are in total $q+1$ directions in $\mathbb{F}_q^2$, we partition \( \{L_x\}_{x \in \mathbb{F}_q^2} \) into classes \( \{G_i\}_{i=0}^q \) according to their directions. Without loss of generality we may assume that

\[ \sharp G_0 \geq \sharp G_1 \geq \sharp G_2 \geq \cdots \geq \sharp G_q. \]

Thus

\[ q + q + \sharp G_2 \cdot (q - 1) \geq \sum_{i=0}^q \sharp G_i = \sharp B^c \geq s(q - 1) + 2q, \]

from which yields

\[ \sharp G_2 \geq s. \]

Choose \( s \) parallel lines \( \{X_l\}_{l=1}^s \) from \( G_0 \), \( s \) parallel lines \( \{Y_m\}_{m=1}^s \) from \( G_1 \) and \( s \) parallel lines \( \{Z_n\}_{n=1}^s \) from \( G_2 \), then it follows that

\[
\begin{align*}
\sharp B & \geq \sum_{l=1}^s (\sharp X_l - 1) + \sum_{m=1}^s (\sharp Y_m - 1 - s) + \sum_{n=1}^s (\sharp Z_n - 1 - 2s) \\
& = s(q - 1) + s(q - 1 - s) + s(q - 1 - 2s) = 3s(q - 1 - s) \\
& \geq 3 \left( \frac{q - 2}{3} \right) (q - 1 - \frac{q}{3}) = \frac{2q^2}{3} - \frac{7q}{3} + 2.
\end{align*}
\]

Combining (3.1) and (3.2) yields the desired result.

\[ \square \]

**Question:** How small can the Nikodym sets really be in two dimensions?

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