Non-Hermitian Hamiltonians with real and complex eigenvalues: An sl(2,C) approach

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Abstract
Potential algebras are extended from Hermitian to non-Hermitian Hamiltonians and shown to provide an elegant method for studying the transition from real to complex eigenvalues for a class of non-Hermitian Hamiltonians associated with the complex Lie algebra $A_1$.

1 Introduction
In recent years there has been much interest in non-Hermitian Hamiltonians with real, bound-state eigenvalues. In particular, PT-symmetric Hamiltonians (such that $(PT)H(PT)^{-1} = H$, where $P$ is the parity and $T$ the time reversal) have been conjectured to have a real bound-state spectrum except when the symmetry is spontaneously broken, in which case their complex eigenvalues should come in conjugate pairs \cite{1}. It is also known that PT symmetry is not a necessary condition for the occurrence of real or complex-conjugate pairs of eigenvalues. A more general condition, namely pseudo-Hermiticity of the Hamiltonian (i.e., the existence of a Hermitian linear automorphism $\eta$ such that $\eta H \eta ^\dagger = H^\dagger$) has been identified as an explanation for the existence of this phenomenon for some non-PT-symmetric Hamiltonians \cite{2,3}. It should be noted that all such non-Hermitian Hamiltonians require a generalization of the normalization condition corresponding to an indefinite scalar product (see \cite{4} and references quoted therein).

Very recently there has been a growing interest in determining the critical strengths of the interaction, if any, at which PT symmetry (or some generalization thereof) becomes spontaneously broken. Among the various techniques that have been employed to construct and study non-Hermitian Hamiltonians with real or complex spectra, algebraic methods provide powerful approaches. In the present communication, we show how potential algebras can be used for such a purpose.
2 Potential algebras for Hermitian and non-Hermitian Hamiltonians

Potential algebras refer to Lie algebras whose generators connect eigenfunctions \( \psi_n^{(m)}(x) \) corresponding to the same eigenvalue (i.e., \( E_n^{(m)} \) is constant), but to different potentials \( V_m(x) \) of a given family. Here \( m \) is some parameter (generally related to the potential strength), which may change by one unit under the action of the generators.

Potential algebras were introduced for Hermitian Hamiltonians as real Lie algebras, the simplest example being that of \( \text{sl}(2,\mathbb{R}) \approx \text{so}(2, 1) \) [3, 4]. The latter is generated by \( J_0, J_+, J_- \), satisfying the commutation relations

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0, \tag{1}
\]

and the Hermiticity properties \( J_0^\dagger = J_0, J_\pm^\dagger = J_\mp \). Such operators are realized as differential operators

\[
J_0 = -i \frac{\partial}{\partial \phi}, \quad J_\pm = e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial x} + \left( i \frac{\partial}{\partial \phi} \mp \frac{1}{2} \right) F(x) + G(x) \right], \tag{2}
\]

depending upon a real variable \( x \) and an auxiliary variable \( \phi \in [0, 2\pi] \), provided the two real-valued functions \( F(x) \) and \( G(x) \) in (2) satisfy coupled differential equations

\[
F' = 1 - F^2, \quad G' = -FG. \tag{3}
\]

The \( \text{sl}(2,\mathbb{R}) \) Casimir operator, \( J^2 = J_0^2 \mp J_0 J_\mp J_\pm \), then becomes a second-order differential operator.

For bound states, to which we restrict ourselves here, one considers unitary irreducible representations of \( \text{sl}(2,\mathbb{R}) \) of the type \( D_n^\pm \), spanned by states \( |km\rangle \), \( k \in \mathbb{R}^+, m = k + n, n \in \mathbb{N} \), such that \( J_0|km\rangle = m|km\rangle \) and \( J^2|km\rangle = k(k-1)|km\rangle \). In the realization (4), these states are given by \( |km\rangle = \Psi_{km}(x, \phi) = \psi_{km}(x) e^{im\phi}/\sqrt{2\pi} \), where \( \psi_{km}(x) = \psi_n^{(m)}(x) \) satisfies the Schrödinger equation

\[
-\psi_n^{(m)''} + V_m \psi_n^{(m)} = E_n^{(m)} \psi_n^{(m)}. \tag{4}
\]

In (4), the potential \( V_m(x) \) is defined in terms of \( F(x) \) and \( G(x) \) by

\[
V_m = \left( \frac{3}{4} - m^2 \right) F' + 2mG' + G^2, \tag{5}
\]

and the energy eigenvalues are given by \( E_n^{(m)} = -\left( m - n - \frac{1}{2} \right)^2 \). The eigenfunction \( \psi_0^{(m)}(x) \) can be easily constructed by solving the first-order differential equation \( J_- \psi_0^{(m)}(x) = 0 \), while the remaining eigenfunctions \( \psi_n^{(m)}(x) \) can be obtained from the action of \( J_+ \) on \( \psi_{n-1}^{(m-1)}(x) \). Imposing the regularity condition for bound states \( \psi_n^{(m)}(\pm \infty) \to 0 \) restricts the allowed values of \( n \) to \( n = 0, 1, \ldots, n_{\text{max}} < m - \frac{1}{2} \).

Detailed inspection of the system of differential equations (3) shows that it admits three classes of solutions corresponding to the (nonsingular) Scarf II, the (singular) generalized Pöschl-Teller, and the Morse potentials, respectively [4, 8].
The transition from Hermitian to non-Hermitian Hamiltonians can now be performed by replacing real Lie algebras by complex ones \[9, 10\] (see also \[11, 12\] for a related approach). In the case of sl(2,R), we find the algebra known as A\(_1\) in Cartan’s classification of complex Lie algebras (which we shall refer to as sl(2,C), considered here as a complex algebra, not a real one as is usually the case). Its generators still satisfy the commutation relations \(4\), but their Hermiticity properties remain undefined. This means that the realization \(2\) is still applicable with \(F(x)\) and \(G(x)\) now some complex-valued functions.

For bound states, we restrict ourselves to irreducible representations spanned by states \(|km\rangle\), for which both \(k = k_R + ik_I\) and \(m = m_R + im_I\) may be complex with \(m_R = k_R + n \in \mathbb{R}\), \(m_I = k_I \in \mathbb{R}\), and \(n \in \mathbb{N}\). Equations \((4)\) and \((5)\) remain valid and we get some complexified forms of the Scarf II, generalized Pöschl-Teller, and Morse potentials, given by

\[
\begin{align*}
\text{I:} & \quad V_m = \left(b^2 - m^2 + \frac{1}{4}\right) \text{sech}^2 \tau - 2mb \text{sech} \tau \tanh \tau, \\
\text{II:} & \quad V_m = \left(b^2 + m^2 - \frac{1}{4}\right) \text{cosech}^2 \tau - 2mb \text{cosech} \tau \coth \tau, \\
\text{III:} & \quad V_m = b^2 e^{\pm 2x} \mp 2mbe^{\mp x},
\end{align*}
\]

where \(b = b_R + ib_I\) and \(\tau = x - c - i\gamma\) with \(b_R, b_I, c \in \mathbb{R}\), \(-\frac{\pi}{4} \leq \gamma < \frac{\pi}{4}\). For generic values of the parameters, such potentials are neither PT-symmetric nor pseudo-Hermitian. The corresponding energy eigenvalues become \(E_n^{(m)} = -\left(m_R + im_I - n - \frac{1}{2}\right)^2\) and are therefore complex if \(m_I \neq 0\). From the explicit form of \(\psi_0^{(m)}(x)\), it can be shown that the potentials \((3) - (8)\) have at least one regular eigenfunction (namely that corresponding to \(n = 0\)) provided \(m_R > 1/2\) and \(b_R > 0\), where the second condition applies only to class III.

It is worth noting that the same potentials \((3) - (8)\) can alternatively be derived in the framework of supersymmetric quantum mechanics by considering a complex superpotential \(W(x) = \left(m - \frac{1}{2}\right) F(x) - G(x)\) \[13\]. They turn out to be shape-invariant as their real counterparts.

### 3 Some examples

As a first example, let us consider a special case of class I potentials,

\[
V(x) = -V_1 \text{sech}^2 x - iV_2 \text{sech} x \tanh x, \quad V_1 > 0, \quad V_2 \neq 0,
\]

which is both PT-symmetric and P-pseudo-Hermitian. It corresponds to \(c = \gamma = 0\) in \(3\), while the other four parameters \(m_R, m_I, b_R, b_I\) are related to \(V_1\) and \(V_2\) through four quadratic equations,

\[
\begin{align*}
b_R^2 - b_I^2 - m_R^2 + m_I^2 + \frac{1}{4} = -V_1, \\
b_R b_I - m_R m_I = 0, \\
m_R b_R - m_I b_I = 0, \\
2(m_R b_I + m_I b_R) = V_2.
\end{align*}
\]
On solving the latter to express $m_R$, $m_I$, $b_R$, $b_I$ in terms of $V_1$, $V_2$, taking the regularity condition $m_R > 1/2$ into account, and inserting the $m_R$ and $m_I$ values into $E_n^{(m)}$, we find one critical strength (for a given sign of $V_2$) corresponding to $|V_2| = V_1 + \frac{1}{4}$. For $|V_2| < V_1 + \frac{1}{4}$, there are (in general) two series of real eigenvalues (instead of one for the real Scarf II potential), given by

$$E_{n,\pm} = -\left[\frac{1}{2}\left(\sqrt{|V_1 + \frac{1}{4} + |V_2|} \pm \sqrt{|V_1 + \frac{1}{4} - |V_2|}\right) - n - \frac{1}{2}\right]^2,$$

where $n = 0, 1, 2, \ldots < \frac{1}{2}\left(\sqrt{|V_1 + \frac{1}{4} + |V_2|} \pm \sqrt{|V_1 + \frac{1}{4} - |V_2| - 1}\right)$. When $|V_2|$ reaches the value $V_1 + \frac{1}{4}$, the two series of real energy eigenvalues merge and for higher $|V_2|$ values they move into the complex plane so that we get a series of complex-conjugate pairs of eigenvalues,

$$E_{n,\pm} = -\left[\frac{1}{2}\left(\sqrt{|V_2| + V_1 + \frac{1}{4} \pm i\sqrt{|V_2| - V_1 - \frac{1}{4}}\right) - n - \frac{1}{2}\right]^2,$$

where $n = 0, 1, 2, \ldots < \frac{1}{2}\left(\sqrt{|V_2| + V_1 + \frac{1}{4} - 1}\right)$. This agrees with results obtained elsewhere by another method \[14\].

Another example of occurrence of just one critical strength is provided by a PT-symmetric and P-pseudo-Hermitian special case of class II potentials,

$$V(x) = V_1 \coth^2 \tau - V_2 \coth \tau \coth \tau, \quad V_1 > -\frac{1}{4}, \quad V_2 \neq 0,$$

which is defined on the entire real line (contrary to its real counterpart).

A rather different situation is depicted by the potential

$$V(x) = (V_{1R} + iV_{1I})e^{-2x} - (V_{2R} + iV_{2I})e^{-x}, \quad V_{1R}, V_{1I}, V_{2R}, V_{2I} \in \mathbb{R},$$

which is the most general class III potential (for the upper sign choice in \(8\)) and has no special property for generic values of the parameters.

By proceeding as in the first example, we find that the regularity conditions $m_R > 1/2$ and $b_R > 0$ impose that $V_{1I}$ be non-vanishing and

$$(V_{1R} + \Delta)^{1/2}V_{2R} + \nu(-V_{1R} + \Delta)^{1/2}V_{2I} > \sqrt{2}\Delta,$$

where $\nu$ denotes the sign of $V_{1I}$ and $\Delta \equiv \sqrt{V_{1R}^2 + V_{1I}^2}$.

The results for the bound-state energy eigenvalues strongly contrast with those obtained hereabove. Indeed real eigenvalues belonging to a single series,

$$E_n = -\left[\frac{V_{2R}}{\sqrt{2}|V_{1I}|}(-V_{1R} + \Delta)^{1/2} - n - \frac{1}{2}\right]^2,$$

where $n = 0, 1, 2, \ldots < (V_{2R}/\sqrt{2}|V_{1I}|)(-V_{1R} + \Delta)^{1/2} - \frac{1}{2}$, only occur for a special value of $V_{2I}$, namely $V_{2I} = \nu(-V_{1R} + \Delta)^{1/2}(V_{1R} + \Delta)^{-1/2}V_{2R}$, while complex eigenvalues,

$$E_n = -\left\{\frac{1}{2\sqrt{2}\Delta}\left[(V_{1R} + \Delta)^{1/2} - i\nu(-V_{1R} + \Delta)^{1/2}\right](V_{2R} + iV_{2I}) - n - \frac{1}{2}\right\}^2,$$

\begin{equation}
\text{for } n = 0, 1, 2, \ldots < (V_{2R}/\sqrt{2}|V_{1I}|)(-V_{1R} + \Delta)^{1/2} - \frac{1}{2},
\end{equation}
where $n = 0, 1, 2, \ldots < \frac{1}{2\sqrt{2\Delta}} \left[(V_{1R} + \Delta)^{1/2}V_{2R} + \nu(-V_{1R} + \Delta)^{1/2}V_{2I}\right] - \frac{1}{2}$ and which do not form complex-conjugate pairs, occur for all the remaining values of $V_{2I}$.

Such results can be interpreted by choosing the parametrization $V_{1R} = A^2 - B^2$, $V_{1I} = 2AB$, $V_{2R} = \gamma A$, $V_{2I} = \delta B$, where $A$, $B$, $\gamma$, $\delta$ are real, $A > 0$, and $B \neq 0$. The complexified Morse potential (17) then becomes $V(x) = (A + iB)^2e^{-2x} - (2C + 1)(A + iB)e^{-x}$, where $C = [(\gamma - 1)A + i(\delta - 1)B]/[2(A + iB)]$. Its (real or complex) eigenvalues can be written in a unified way as $E_n = -(C - n)^2$, while the regularity condition (18) amounts to $\gamma > 1$, and therefore $C = \frac{1}{2}(\gamma - 1) \in \mathbb{R}^+$, $V(x)$ is pseudo-Hermitian under imaginary shift of the coordinate [3]. It has only real eigenvalues corresponding to $n = 0, 1, 2, \ldots < C$, thus exhibiting no symmetry breaking over the whole parameter range. For the values of $\delta$ different from $\gamma$, the potential indeed fails to be pseudo-Hermitian. In such a case, $C$ is complex as well as the eigenvalues. Nevertheless, the eigenfunctions associated with $n = 0, 1, 2, \ldots < \text{Re}C$ remain regular. Note that the existence of regular eigenfunctions with complex energies for general complex potentials is a well-known phenomenon (see e.g. [17]).

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