Some trapezoid and midpoint type inequalities via fractional \((p, q)\)-calculus

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Abstract
Fractional calculus is the field of mathematical analysis that investigates and applies integrals and derivatives of arbitrary order. Fractional \(q\)-calculus has been investigated and applied in a variety of research subjects including the fractional \(q\)-trapezoid and \(q\)-midpoint type inequalities. Fractional \((p, q)\)-calculus on finite intervals, particularly the fractional \((p, q)\)-integral inequalities, has been studied. In this paper, we study two identities for continuous functions in the form of fractional \((p, q)\)-integral on finite intervals. Then, the obtained results are used to derive some fractional \((p, q)\)-trapezoid and \((p, q)\)-midpoint type inequalities.

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1 Introduction
The ordinary calculus of Newton and Leibniz is well known to be investigated extensively and intensively to produce a large number of related formulas and properties as well as applications in a variety of fields ranging from natural sciences to social sciences. In the early eighteenth century, the well-known mathematician Leonhard Euler (1707–1783) established quantum calculus or \(q\)-calculus, which is the study of calculus without limits, in the way of Newton’s work for infinite series. Later, F. H. Jackson initiated a study of \(q\)-calculus in a symmetrical manner in 1910 and introduced \(q\)-derivative and \(q\)-integral in [1], see [2] for more details.

Many physical and mathematical problems have led to the necessity of studying \(q\)-calculus; for instance, Fock [3] studied the symmetry of hydrogen atoms using the \(q\)-difference equation. In addition, in modern mathematical analysis, \(q\)-calculus has lots of applications such as combinatorics, orthogonal polynomials, basic hypergeometric functions, number theory, quantum theory, mechanics, and theory of relativity, see also [4–24] and the references cited therein. The book by Kac and Cheung [25] covers the basic theoretical concepts of \(q\)-calculus.
As one of the major driving forces behind the modern approach of real analysis, inequalities have played a vital role in almost all branches of mathematics along with other fields of science. In 2015, Noor et al. [26] established $q$-analogue of classical integral identity to obtain $q$-trapezoid type inequalities for convex functions. Moreover, in 2016, Necmettin, Mehmet, and İmdat [27] proved the correctness of left part of $q$-Hermite–Hadamard and gave some $q$-midpoint type integral inequalities through $q$-differentiable convex function and $q$-differentiable quasi-convex functions. With these results, many researchers have extended some important topics of $q$-calculus together with applications in many fields, such as $q$-integral inequalities, see [28–37] for more details.

Since the exploration has been continued to generalize the existing results through creative thoughts and novel techniques of fractional calculus, in 2015, Tariboon, Ntouyas, and Agarwal [38] proposed a new $q$-shifting operator $\Phi_q(m) = qm + (1 - q)a$ for studying new concepts of fractional $q$-calculus. In 2016, Sudsutad, Ntouyas, and Tariboon [39] studied some fractional $q$-integral inequalities. In 2020, Kunt and Aljasem [40] proved Riemann–Liouville fractional $q$-trapezoid and $q$-midpoint type inequalities for convex functions. Furthermore, in 2021, Neang et al. [41] introduced fractional $(p, q)$-calculus on finite intervals and proved some well-known integral inequalities.

In 2018, as one of the most attractive areas, Kunt et al. [42] proved $(p, q)$-Hermite–Hadamard inequalities and gave some $(p, q)$-midpoint type integral inequalities via $(p, q)$-differentiable convex and $(p, q)$-differentiable quasi-convex functions. In 2019, Latif et al. [43] proved some $(p, q)$-trapezoid integral inequalities for convex and quasi-convex functions. Based on these results, many authors have generalized and developed $(p, q)$-calculus, which is used efficiently in many fields, and some results on the study of $(p, q)$-calculus can be found in [44–71].

Motivated by some of the above studies and applications, in this paper, we study two identities for continuous functions in the form of fractional $(p, q)$-integral on finite intervals. Then, the obtained results are used to derive some fractional $(p, q)$-trapezoid and $(p, q)$-midpoint type inequalities.

## 2 Preliminaries

In this section, we recall some well-known facts on fractional $(p, q)$-calculus, which can be found in [10, 11, 38, 53, 55]. Throughout this paper, let $[a, b] \subset \mathbb{R}$ be an interval with $a < b$, and $0 < q < p \leq 1$ be constants,

\[
[k]_{p,q} = \begin{cases} \frac{k^p - q^p}{p-q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}
\]

\[
[k]_{p,q}! = \prod_{i=1}^{k} \frac{p^i - q^i}{p - q}, \quad k \in \mathbb{N}, \quad k = 0.
\]

**Property 2.1** ([38]) Let $\Phi_q(m) = qm + (1 - q)a$. For any $m, n \in \mathbb{R}$ and for all positive integers $j, k$, we have

(i) $\Phi_q^j(m) = \Phi_q(m)$;

(ii) $\Phi_q^j(a \Phi_q^j(m)) = a \Phi_q^j(\Phi_q^j(m)) = a \Phi_q^{j+k}(m)$;

(iii) $\Phi_q(a) = a$

(iv) $\Phi_q^j(m) - a = q^j(m - a)$;

(v) $m - \Phi_q^j(m) = (1 - q^j)(m - a)$;
(vi) \( a \Phi_q^{k}(m) = m a_{im} \Phi_q^{k}(1) \) for \( m \neq 0 \);
(vii) \( a \Phi_q^{k}(m) - a \Phi_q^{k}(n) = q(m - a \Phi_q^{k-1}(n)) \).

**Property 2.2** ([38]) For any \( \gamma, n, m \in \mathbb{R} \) with \( n \neq a \) and \( k \in \mathbb{N} \cup \{0\} \), we have
(i) \( (n - m)^{(k)}_{n-a} = (n - a)^{(k)}_{n-a} q^k \);
(ii) \( (n - m)^{(k)}_{a} = (n - a)^{(k)}_{a} q^k \prod_{i=0}^{k-1} \frac{1 - (n - a)\gamma}{1 - q^{-1}\gamma} \);\( (n - a)^{(k)}_{a} q^k \);
(iii) \( (n - a \Phi_q^{k}(n))_a = (n - a)^{(k)}_{a} q^k \prod_{i=0}^{k-1} \frac{1 - (n - a)\gamma}{1 - q^{-1}\gamma} \).

For \( m, n \in \mathbb{R} \), the \((p, q)\)-analogue of the power function \( a(m - n)^{k}_{p_{q}} \) with \( k \in \mathbb{N} \cup \{0\} \) is defined follows:
\[
a(m - n)^{(0)}_{p_{q}} := 1, \quad a(m - n)^{(k)}_{p_{q}} := \prod_{i=0}^{k-1} \left( a \Phi_p^{i}(m) - a \Phi_q^{i}(n) \right), \tag{2.2}
\]
\[
a(m - n)^{(k)}_{p_{q}} = (m - a)^{k} q^k \prod_{i=0}^{k-1} \frac{1 - (n - a)\gamma}{1 - q^{-1}\gamma}, \tag{2.3}
\]

More generally, if \( \alpha \in \mathbb{R} \), then
\[
a(m - n)^{(\alpha)}_{p_{q}} = (m - a)^{\alpha} q^\alpha \prod_{i=0}^{\infty} \frac{1 - (n - a)\gamma}{1 - q^{-1}\gamma}, \tag{2.4}
\]
or
\[
a(m - n)^{(\alpha)}_{p_{q}} = (m - a)^{\alpha} q^\alpha \prod_{i=0}^{\infty} \frac{1 - (n - a)\gamma}{1 - q^{-1}\gamma}, \tag{2.5}
\]

**Property 2.3** ([41]) For \( \alpha > 0 \), the following formulas hold:
(i) \( a \Phi_{q}^{k}(m) = a = (\frac{q}{p})^{k}(m - a) \);
(ii) \( a(m - a \Phi_{q}^{k}(m))_{p} = (m - a)^{\alpha} \prod_{i=0}^{\infty} \frac{1 - (n - a)\gamma}{1 - q^{-1}\gamma} = (m - a)^{\alpha} \gamma^{(\frac{q}{p})^{k}(\frac{q}{p})} \).

**Definition 2.1** ([72]) If \( f: [a, b] \to \mathbb{R} \) is a continuous function, then the \((p, q)\)-derivative of \( f \) on \([a, \frac{1}{p}(b - a) + a]\) at \( x \) is defined by
\[
aD_{p,q}^f(x) = \frac{f(px + (1 - p)a) - f(qx + (1 - q)a)}{(p - q)(x - a)}, \quad x \neq a, \tag{2.6}
\]
\[
aD_{p,q}^f(a) = \lim_{x \to a} aD_{p,q}^f(x).
\]

Obviously, a function \( f \) is \((p, q)\)-differentiable on \([a, \frac{1}{p}(b - a) + a]\) if \( aD_{p,q}^f(x) \) exists for all \( x \in [a, \frac{1}{p}(b - a) + a] \). In Definition 2.1, if \( a = 0 \), then \( 0D_{p,q}^f = D_{p,q}^f \), where \( D_{p,q}^f \) is defined by
\[
D_{p,q}^f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0. \tag{2.7}
\]

Furthermore, if \( p = 1 \) in (2.7), then it reduces to \( D_{q}^f \), which is \( q \)-derivative of the function \( f \), see [25, 73] for more details.
Definition 2.2 ([72]) If \( f : [a, b] \to \mathbb{R} \) is a continuous function, then the \((p, q)\)-integral is defined by

\[
\int_a^x f(t) \,a^{tq}t = (p - q)(x - a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} x^n \left( 1 - \frac{q^n}{p^{n+1}} \right) a
\]  

for \( x \in [a, \frac{1}{p}(b - a) + a] \). If \( a = 0 \) and \( p = 1 \) in (2.8), then we have the classical \( q \)-integral, see [25] for more details.

Theorem 2.1 ([72]) The following formulas hold for \( t \in [a, b] \):

(i) \( a^{tq}f(t) + b^{tq}g(t) = f(t) + g(t) \);  
(ii) \( a^{tq}f(t) + b^{tq}g(t) = f(t) - f(a) \);  
(iii) \( a^{tq}f(t) + b^{tq}g(t) = f(t) - f(c) \) for \( c \in (a, t) \).

Theorem 2.2 ([72]) If \( f, g : [a, b] \to \mathbb{R} \) are continuous functions and \( \lambda \in \mathbb{R} \), then the following formulas hold:

(i) \( \lambda \left(a^{tq} + b^{tq}\right) = a^{tq} + b^{tq} \);  
(ii) \( a^{tq}f(t) + b^{tq}g(t) = \lambda \int_a^t f(t) \,a^{tq}d t \);  
(iii) \( a^{tq}f(t) + b^{tq}g(t) = \int_a^t g(t) \,(1 - q)a^{tq} \).

For \( t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \), the \((p, q)\)-gamma function is defined by

\[
\Gamma_{pq}(t) = \frac{(p - q)^{t-1}}{(p - q)^{t-1}}, \quad (2.9)
\]

and an equivalent definition of (2.9) is given in [56] as

\[
\Gamma_{pq}(t) = p^{\frac{t-1}{p}} \int_0^\infty x^{t-1} E_{pq}^{-qx} \,d x, \quad (2.10)
\]

where

\[
E_{pq}^{-qx} = \sum_{n=0}^{\infty} \frac{q^n}{[n]_{pq}} (-qx)^n.
\]

Obviously, \( \Gamma_{pq}(t + 1) = [t]_{pq} \Gamma_{pq}(t) \). For \( s, t > 0 \), the definition of the \((p, q)\)-beta function is defined by

\[
B_{pq}(s, t) = \int_0^1 u^{q-1} (1 - q\Phi_{pq}(u))^{(t-1)} \,d u, \quad (2.11)
\]

and (2.11) can also be written as

\[
B_{pq}(s, t) = p^{(t-1)(2s+1)/2} \frac{\Gamma_{pq}(s) \Gamma_{pq}(t)}{\Gamma_{pq}(s + t)}, \quad (2.12)
\]

see [74, 75] for more details.
\textbf{Definition 2.3 ([41])} Let \( f \) be a function defined on \([a, b]\), and let \( \alpha > 0 \). The Riemann–Liouville fractional \((p, q)\)-integral is defined by

\[
(a^{p,q}_t f)(t) = \frac{1}{p^{(q)}(a)} \int_a^t (t-s)^{(\alpha-1)} f(s) \frac{1}{p^{(a)}(s)} a ds
\]

for \( t \in [a, p^q (b-a) + a] \).

\textbf{Theorem 2.3 ([41])} If \( f : [a, b] \rightarrow \mathbb{R} \) is a convex differentiable function and \( \alpha > 0 \), then we have

\[
f \left( \left( [\alpha + 1]_{p,q} - p^q \right) a + p^q b \right) \leq \frac{\Gamma_{p,q}(\alpha + 1)}{p^q (b-a)^a} \frac{\left( [\alpha + 1]_{p,q} - p^q \right) f(a) + p^q f(b)}{[\alpha + 1]_{p,q}} \leq \frac{\left( [\alpha + 1]_{p,q} - p^q \right) f(a) + p^q f(b)}{[\alpha + 1]_{p,q}}
\]

\( (2.13) \)

\textbf{3 Main results}

In this section, we give two identities for continuous functions in the form of fractional Riemann–Liouville \((p, q)\)-integral type which will be used to prove the fractional Riemann–Liouville \((p, q)\)-trapezoid and \((p, q)\)-midpoint type inequalities.

\textbf{Lemma 3.1} Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function and \( \alpha > 0 \). If \( a_{p,q} f \) is \((p, q)\)-integrable on \([a, \frac{1}{p}(b-a) + a]\), then the following equality holds:

\[
\frac{\Gamma_{p,q}(\alpha + 1)}{p^q (b-a)^a} (a^{p,q}_t f)(t) - \frac{\left( [\alpha + 1]_{p,q} - p^q \right) f(a) + p^q f(b)}{[\alpha + 1]_{p,q}} \leq \frac{\left( [\alpha + 1]_{p,q} - p^q \right) f(a) + p^q f(b)}{[\alpha + 1]_{p,q}} \leq \frac{\left( [\alpha + 1]_{p,q} - p^q \right) f(a) + p^q f(b)}{[\alpha + 1]_{p,q}}
\]

\( (3.1) \)

\textbf{Proof} By simple computation and using Definition 2.3, we have

\[
A_1 = \frac{b-a}{p^{(q)}} \int_0^1 (1 - q \Phi_{q}(t))^{(a)}_{p,q} a_{p,q} f((1-t)a + tb) a dt
\]

\[
= \frac{b-a}{p^{(q)}} \int_0^1 (1 - q \Phi_{q}(t))^{(a)}_{p,q} f((1-t)a + tb) - f((1-qt)a + qt(b) \frac{(p-q)(b-a)}{p^q (b-a)^a} a dt
\]

\[
= \frac{1}{p^{(q)}(p-q)} \int_0^1 (1 - q \Phi_{q}(t))^{(a)}_{p,q} f((1-t)a + tb) \frac{t}{t} a dt
\]

\[
= \frac{1}{p^{(q)}(p-q)} \int_0^1 (1 - q \Phi_{q}(t))^{(a)}_{p,q} f((1-qt)a + qt(b) \frac{t}{t} a dt
\]

\[
= \frac{1}{p^{(q)}(p-q)} \sum_{n=0}^{\infty} q^n \int_0^1 (1 - q \Phi_{q}^{n+1}(1))^{(a)}_{p,q} f((1-qt)a + qt(b) \frac{t}{t} a dt
\]

\( (3.1) \)
\[-\frac{1}{p(q)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} G_{p, q}^n \left( (1 - a) Q^{n+1}(1) + (1 - b) Q^n(1) \right) \]

\[
\sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
- \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
= \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
- \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
= \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
- \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
= \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
- \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
= \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
- \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
= \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]

\[
- \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^{n+1} \right] \sum_{n=0}^{\infty} \left[ (\frac{q}{p})^n \right] f \left( \left( 1 - \frac{a}{p} \right)^n + \left( \frac{a}{p} \right)^n \right) \]
\[
\times f\left(\frac{t}{p^{n-1}} + \left(1 - \frac{1}{p^{n-1}}\right)a\right)_{a\Delta_{p,q}t}
\]
\[
= -f(a) + \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha}(b - a)^{\alpha}} (a_{p,q}^{\alpha} f (p^\alpha b + (1 - p^\alpha) a),
\]
(3.2)

and
\[
A_2 = \frac{p^\alpha (b - a)}{[\alpha + 1]_{p,q}} \int_0^1 aD_{p,q} f ((1 - t)a + tb)_{a\Delta_{p,q}t}
\]
\[
= \frac{p^\alpha (b - a)}{[\alpha + 1]_{p,q}} \int_0^1 \frac{f ((1 - pt)a + ptb) - f ((1 - qt)a + qtb)}{(p - q)(b - a)t} \, dp_{p,q}t
\]
\[
= \left[ \frac{p^\alpha}{(p - q)[\alpha + 1]_{p,q}} \int_0^1 \frac{f ((1 - pt)a + ptb)}{t} \, dp_{p,q}t
\right.
\]
\[
- \left. \frac{p^\alpha}{(p - q)[\alpha + 1]_{p,q}} \int_0^1 \frac{f ((1 - qt)a + qtb)}{t} \, dp_{p,q}t \right]
\]
\[
= \frac{p^\alpha}{[\alpha + 1]_{p,q}} \left[ \sum_{n=0}^{\infty} f \left( \left(1 - \left(\frac{q}{p}\right)^n\right) a + \left(\frac{q}{p}\right)^{n+1} b \right)
\]
\[
- \sum_{n=0}^{\infty} f \left( \left(1 - \left(\frac{q}{p}\right)^{n+1}\right) a + \left(\frac{q}{p}\right)^n b \right) \right]
\]
\[
= \frac{p^\alpha f (b) - p^\alpha f (a)}{[\alpha + 1]_{p,q}}.
\] (3.3)

From (3.2) and (3.3), we obtain
\[
\frac{(b - a)}{[\alpha + 1]_{p,q}} \int_0^1 \left( \frac{[\alpha + 1]_{p,q}}{p^{\alpha}(b - a)^{\alpha}} (1 - \hat{\Phi}_q (s))_{p,q}^{(\alpha)} - p^\alpha \right) aD_{p,q} f ((1 - t)a + tb)_{a\Delta_{p,q}t}
\]
\[
= A_1 - A_2
\]
\[
= \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha}(b - a)^{\alpha}} (a_{p,q}^{\alpha} f (p^\alpha b + (1 - p^\alpha) a) - \frac{((\alpha + 1)_{p,q} - p^\alpha) f (a) + p^\alpha f (b)}{[\alpha + 1]_{p,q}}).
\] (3.4)

Thus the proof is completed. \(\Box\)

**Remark 3.1** If \(\alpha = 1\), then (3.1) reduces to Lemma 3.2 in [43] as
\[
\frac{1}{p(b - a)} \int_a^{p^{b+1-p}a} f(x)_{a\Delta_{p,q}x} - \frac{pf (a) + qf (a)}{p + q}
\]
\[
= \frac{q(b - a)}{p + q} \int_0^1 (1 - (p + q)t)_{a\Delta_{p,q}t}.
\] (3.5)

If \(p = 1\), then (3.1) reduces to Lemma 5.2 in [40] as
\[
\frac{\Gamma_q(\alpha + 1)}{(b - a)^{\alpha}} (a_{q}\hat{\Phi}_q^{(\alpha)} f (b) - \frac{((\alpha + 1)_{q} - 1)f (a) + f (b)}{[\alpha + 1]_{q}}
\]
\[
= \frac{(b - a)}{[\alpha + 1]_{q}} \int_0^1 ((\alpha + 1)_{q}(1 - \hat{\Phi}_q(t))_{q}^{(\alpha)} - 1) aD_{p,q} f ((1 - t)a + tb)_{a\Delta_{p,q}t}.
\] (3.6)
Moreover, if \( q \to 1 \) and \( \alpha = 1 \), then (3.6) reduces to

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(t(a + (1-t)b) \, dt,
\]

which can be found in [76].

**Theorem 3.1** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function, \( \alpha > 0 \), and \( \mathcal{D}_{p,q} f \) be \((p, q)\)-integrable on \((a, \frac{1}{p}(b - a) + a)\). If \( \mathcal{D}_{p,q} f \) is convex on

\[
\left( a, \frac{1}{p}(b - a) + a \right),
\]

then the following Riemann–Liouville fractional \((p, q)\)-trapezoid type inequality holds:

\[
\left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha (b - a)^\alpha} \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\right| \\
\leq \frac{(b-a)}{[\alpha + 1]_{p,q}} \left( |\mathcal{D}_{p,q} f(a)| B_1 + |\mathcal{D}_{p,q} f(b)| B_2 \right),
\]

where

\[
B_1 = \int_0^1 \left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha (b - a)^\alpha} \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\right| (1-t) \, d_{p,q} t
\]

and

\[
B_2 = \int_0^1 \left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha (b - a)^\alpha} \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\right| (1-t) \, d_{p,q} t
\]

**Proof** Using Lemma 3.1 and the convexity of \( |\mathcal{D}_{p,q} f| \), we have

\[
\frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha (b - a)^\alpha} \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\left| \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\right|
\]

\[
\leq \frac{(b-a)}{[\alpha + 1]_{p,q}} \left( |\mathcal{D}_{p,q} f(a)| \int_0^1 \left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha (b - a)^\alpha} \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\right| (1-t) \, d_{p,q} t
\]

\[
+ \frac{(b-a)}{[\alpha + 1]_{p,q}} \left( |\mathcal{D}_{p,q} f(b)| \int_0^1 \left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha (b - a)^\alpha} \mathcal{D}_{p,q} f \left( \frac{b^\alpha b}{p^\alpha} + (1 - p^\alpha) a \right) - \frac{\left[ \Gamma_{p,q}(\alpha + 1)p^\alpha - p^\alpha \right]f(a) + p^\alpha f(b)}{\alpha + 1}\right| (1-t) \, d_{p,q} t.
\]

This completes the proof. \(\square\)
Remark 3.2 If $p = 1$, then (3.8) reduces to

$$
\left| \frac{\Gamma_q(\alpha + 1)}{(b - a)^\alpha} \left( a D_{p,q}^\alpha f \right)(b) - \frac{((\alpha + 1)q - 1) f(a) + f(b)}{\alpha + 1} \right| \\
\leq \frac{(b - a)}{\alpha + 1} \left( |a D_{p,q} f(a)| \delta_1 + |a D_{p,q} f(b)| \delta_2 \right),
$$

(3.9)

where

$$
\delta_1 = \int_0^1 \left| (\alpha + 1)q(1 - \Phi_q(t))^{(\alpha)} - 1 \right| (1 - t) \, d_q t
$$

and

$$
\delta_2 = \int_0^1 \left| (\alpha + 1)q(1 - \Phi_q(t))^{(\alpha)} - 1 \right| t \, d_q t,
$$

which appeared in [40].

Theorem 3.2 Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function, $\alpha > 0$, and $a D_{p,q} f$ be $(p, q)$-integrable on $(a, \frac{b}{2}, (b - a) + a)$. If $|a D_{p,q} f|^r$ is convex on $(a, \frac{b}{2}, (b - a) + a)$ for $r \geq 0$, then the following Riemann–Liouville fractional $(p, q)$-trapezoid type inequality holds:

$$
\left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha+1}} \left( a D_{p,q}^{\alpha} f \right)(p^\alpha b + (1 - p^\alpha)a) - \frac{[(\alpha + 1)_{p,q} - p^\alpha] f(a) + p^\alpha f(b)}{\alpha + 1} \right| \\
\leq \frac{(b - a)}{\alpha + 1} B_3^{1-1/r} \left( |a D_{p,q} f(a)| B_1 + |a D_{p,q} f(b)| B_2 \right)^{1/r},
$$

(3.10)

where $B_1$ and $B_2$ are given in Theorem 3.1 and

$$
B_3 = \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha+1}} (1 - \Phi_q(t))^{(\alpha)} - p^\alpha \right| \, d_{p,q} t.
$$

Proof Using Lemma 3.1, the convexity of $|a D_{p,q} f|^r$, and the power mean inequality, we have

$$
\left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha+1}} \left( a D_{p,q}^{\alpha} f \right)(p^\alpha b + (1 - p^\alpha)a) - \frac{[(\alpha + 1)_{p,q} - p^\alpha] f(a) + p^\alpha f(b)}{\alpha + 1} \right| \\
\leq \frac{(b - a)}{\alpha + 1} \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha+1}} (1 - \Phi_q(t))^{(\alpha)} - p^\alpha \right| \, d_{p,q} t \\
\leq \frac{(b - a)}{\alpha + 1} \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha+1}} (1 - \Phi_q(t))^{(\alpha)} - p^\alpha \right| \, d_{p,q} t \right)^{1-1/r} \\
\times \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha+1}} (1 - \Phi_q(t))^{(\alpha)} - p^\alpha \right| \, d_{p,q} t \right)^{1/r}.
$$

$$
\leq \frac{(b - a)}{\alpha + 1} \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha+1}} (1 - \Phi_q(t))^{(\alpha)} - p^\alpha \right| \, d_{p,q} t \right)^{1-1/r} \\
\times \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha+1}} (1 - \Phi_q(t))^{(\alpha)} - p^\alpha \right| \, d_{p,q} t \right)^{1/r}.
$$
\[ + \left| a_{D_{pq}f(b)}\right|^r \right|_{\alpha}^{\frac{1}{r}} \]
\[ \leq \frac{(b-a)}{[\alpha + 1]_{pq}} \left( \int_{a}^{b} \left| \left[ \frac{\alpha + 1}{p} \right]_{pq} (1 - a_{\Phi_{pq}}(t))^{(\alpha)}_{pq} - p^\alpha \right| da_{pq} t \right) \]
\[ \times \left[ \left| a_{D_{pq}f(a)}\right|^r \right|_{\alpha}^{\frac{1}{r}} \times \left[ \left| \left[ \frac{\alpha + 1}{p} \right]_{pq} (1 - a_{\Phi_{pq}}(t))^{(\alpha)}_{pq} - p^\alpha \right| (1 - t) da_{pq} t \right] \]
\[ + \left| a_{D_{pq}f(b)}\right|^r \right|_{\alpha}^{\frac{1}{r}} \times \left[ \left| \left[ \frac{\alpha + 1}{p} \right]_{pq} (1 - a_{\Phi_{pq}}(t))^{(\alpha)}_{pq} - p^\alpha \right| (1 - t) da_{pq} t \right] \]

Therefore, the proof is completed. \[ \square \]

**Remark 3.3** If \( \alpha = 1 \), then (3.10) reduces to
\[ \left| \frac{1}{p(b-a)} \int_{a}^{b} \left| f(x) \right|_{pq} da_{pq} x - \frac{pf(a) + qf(a)}{p + q} \right| \]
\[ = \frac{q(b-a)}{p + q} \left[ \frac{2(p + q - 1)}{(p + q)^2} \right]^{1-1/r} \left[ \lambda_1(p,q) \left| a_{D_{pq}f(b)}\right|^r + \lambda_2(p,q) \left| a_{D_{pq}f(a)}\right|^r \right]^{1/r}, \] (3.11)

where
\[ \lambda_1(p,q) = \frac{q(5p^3 - 2 + 2p) + (2p^2 + 2)q + pq^2 + 2p - 2p}{(p + q)^2(p^2 + pq + q^2)} \]

and
\[ \lambda_2(p,q) = \frac{1}{(p + q)^2(p^2 + pq + q^2)} \left[ q \left( (5p^3 - 4p^2 - 2p + 2) + (6p^2 - 4p - 2)q \right) \right. \]
\[ + (5p - 2)q + 2q^3 \left. + (2p^4 - 2p^3 - 2p^2 + 2p) \right] \]

which appeared in [43].

Moreover, if \( p = 1 \), then (3.10) reduces to
\[ \Gamma_q(\alpha + 1) \left[ \frac{1}{(b-a)^{\alpha(q)}} \left[ a_{D_{pq}f(b)}\right]^{(\alpha)}_{pq} - \frac{[(\alpha + 1)]_{q} - 1)f(a) + f(b)}{[\alpha + 1]_q} \right] \]
\[ \leq \frac{(b-a)}{[\alpha + 1]_q} M_1^{1-1/r} \left[ \left| a_{D_{pq}f(a)}\right|^r \right] M_1 + \left| a_{D_{pq}f(b)}\right|^r M_2^{1/r}, \] (3.12)

where \( \delta_1 \) and \( \delta_2 \) are given in Remark 3.2 and
\[ \delta_3 = \int_{0}^{1} \left| \left[ \frac{\alpha + 1}{q} (1 - a_{\Phi_{pq}}(t))^{(\alpha)}_q - 1 \right| da_{q} t, \]

which appeared in [40].

**Theorem 3.3** Let \( f : [a,b] \rightarrow \mathbb{R} \) be a continuous function, \( \alpha > 0 \) and \( a_{D_{pq}f} \) be \((p,q)\)-integrable on \((a, \frac{1}{p}(b-a) + a)\). If \( \left| a_{D_{pq}f} \right|^r \) is convex on \([a, \frac{1}{p}(b-a) + a]\) for \( r > 1 \) and
$1/r + 1/p = 1$, then the following Riemann–Liouville fractional $(p,q)$-trapezoid type inequality holds:

$$\left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha}b^{\alpha}}(a^{p\alpha}f(b) + (1 - p\alpha)a) - \frac{[(\alpha + 1)p,q - p\alpha)f(a) + p\alpha f(b)]}{\alpha + 1} \right| \leq \frac{(b - a)}{[\alpha + 1]p,q} B_4 \left( \frac{(p + q - 1)aD_{p,q}f(a) + |aD_{p,q}f(b)|}{p + q} \right)^{1/r},$$

(3.13)

where

$$B_4 = \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha}} \left( 1 - 0 \Phi_q(t) \right)^{\alpha - p\alpha} - p\alpha \right|_{\alpha}^{1/r}.$$

**Proof** Using Lemma 3.1, the convexity of $|aD_{p,q}f|^r$, and Hölder’s inequality, we have

$$\left| \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha}b^{\alpha}}(a^{p\alpha}f(b) + (1 - p\alpha)a) - \frac{[(\alpha + 1)p,q - p\alpha)f(a) + p\alpha f(b)]}{\alpha + 1} \right| \leq \frac{(b - a)}{[\alpha + 1]p,q} \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha}} \left( 1 - 0 \Phi_q(t) \right)^{\alpha - p\alpha} - p\alpha \right|_{\alpha}^{1/r} \right)^{1/r},$$

$$\times \left( \int_0^1 |aD_{p,q}f((1 - t)a + tb)|_{\alpha}^{1/r} \right)^{1/r} \leq \frac{(b - a)}{[\alpha + 1]p,q} \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha}} \left( 1 - 0 \Phi_q(t) \right)^{\alpha - p\alpha} - p\alpha \right|_{\alpha}^{1/r} \right)^{1/r},$$

$$\times \left( \int_0^1 [aD_{p,q}f(a)]^{r}(1 - t) + |aD_{p,q}f(b)|_{\alpha}^{r} \right)_{\alpha}^{1/r} \leq \frac{(b - a)}{[\alpha + 1]p,q} \left( \int_0^1 \left| \frac{\alpha + 1}{p^{\alpha}} \left( 1 - 0 \Phi_q(t) \right)^{\alpha - p\alpha} - p\alpha \right|_{\alpha}^{1/r} \right)^{1/r},$$

$$\times \left( \frac{(p + q - 1)aD_{p,q}f(a) + |aD_{p,q}f(b)|}{p + q} \right)^{1/r}. $$

This completes the proof.

**Remark 3.4** If $\alpha = 1$, then (3.13) reduces to

$$\left| \frac{1}{p(b - a)} \int_a^{b+(1-p)a} f(x)_{a}d_{p,q} - \frac{pf(a) + qf(a)}{p + q} \right| \leq \frac{q(b - a)}{p + q} \left[ \lambda_3 \right]^{1/s} \left( \frac{|aD_{p,q}f(b)|^{r} + (p + q - 1)|aD_{p,q}f(a)|^{r}}{p + q} \right)^{1/r},$$

(3.14)

where

$$\lambda_3 = \int_0^1 |1 -(p + q)t|_{\alpha}^{s}.$$
Moreover, if \( p = 1 \), then (3.13) reduces to

\[
\Gamma_q(\alpha + 1) \left( \frac{\sigma_p^a f(b)}{(a + 1)_p^a} \right) = \frac{([\alpha + 1]_q - f(a) + f(b))}{[\alpha + 1]_q} \left| \delta_4 \right|^\alpha \\
\leq \frac{(b - a)_q}{[\alpha + 1]_q} \delta_4^{1/\alpha} \left( \frac{q|aD_qf(a)|^r + |aD_qf(b)|^r}{1 + q} \right)^{1/r} ,
\]

where

\[
\delta_4 = \int_0^1 \left| [\alpha + 1]_q (1 - \Phi_q(t))^{(\alpha)} - 1 \right|^r \, dt , \tag{3.15}
\]

which appeared in [40].

Now we will prove the following lemma to obtain the Riemann–Liouville fractional 
\((p, q)\)-midpoint type inequalities.

**Lemma 3.2** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function and \( \alpha > 0 \). If \( aD_{p,q}f \) is \((p, q)\)-integrable on \((a, b)\), then the following equality holds:

\[
f \left( \frac{([\alpha + 1]_p - p^\alpha a + p^\alpha b)}{[\alpha + 1]_p} \right) - \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha(b - a)} \left( aD_{p,q}f \right) (p^\alpha b + (1 - p^\alpha)a)
\]

\[
= (b - a) \left[ \int_0^1 \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha(b - a)} \left( 1 - \Phi_q(t) \right)^{(\alpha)} p(t) \right] aD_{p,q}f((1 - t)a + tb) \, dt
\]

\[
+ \int_0^1 \frac{\Gamma_{p,q}(\alpha + 1)}{p(t)} \left( 1 - \Phi_q(t) \right)^{(\alpha)} aD_{p,q}f((1 - t)a + tb) \, dt . \tag{3.16}
\]

**Proof** By direct computation and using Definitions 2.1 and 2.2, we have

\[
A_3 = \int_0^1 \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha(b - a)} \left( 1 - \Phi_q(t) \right)^{(\alpha)} aD_{p,q}f((1 - t)a + tb) \, dt
\]

\[
= \int_0^1 \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha(b - a)} \left( 1 - \Phi_q(t) \right)^{(\alpha)} f((1 - pt)a + ptb - f((1 - qt)a + qt) \, dt
\]

\[
= \frac{1}{(p - q)(b - a)} \int_0^1 \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha(b - a)} \left( 1 - \Phi_q(t) \right)^{(\alpha)} f((1 - pt)a + ptb) \, dt
\]

\[
- \frac{1}{(p - q)(b - a)} \int_0^1 \frac{\Gamma_{p,q}(\alpha + 1)}{p^\alpha(b - a)} \left( 1 - \Phi_q(t) \right)^{(\alpha)} f((1 - qt)a + qt) \, dt
\]

\[
= \frac{p^\alpha}{(b - a)[\alpha + 1]_p} \sum_{n=0}^\infty \frac{q^n}{\Gamma_{p,q}^{(\alpha + 1)\, p^\alpha}} \left( a + \frac{q^n p^\alpha}{\Gamma_{p,q}^{(\alpha + 1)\, p^\alpha}} b \right)
\]

\[
- \frac{p^\alpha}{(b - a)[\alpha + 1]_p} \sum_{n=0}^\infty \frac{q^n}{\Gamma_{p,q}^{(\alpha + 1)\, p^\alpha}} \left( a + \frac{q^n p^\alpha}{\Gamma_{p,q}^{(\alpha + 1)\, p^\alpha}} b \right)
\]

where...
Remark
Therefore, the proof is completed. □

On the other hand, in Lemma 3.1, the following integral was given:

\[
A_1 = \frac{b-a}{p^{(\alpha)}} \int_{0}^{1} (1 - \Phi_p(t)_{p,q}^{(a)}) a_{D_p,q} f((1-t)a + tb) \ dt \quad (3.17)
\]

\[
= -f(a) + \frac{\Gamma_{p,q}(\alpha + 1)}{p^{\alpha}(b-a)^{\alpha}} (a_{p,q}^{(a)} \varphi_{p,q}^{(a)})(p^{\alpha}b + (1-p^{\alpha})a). 
\quad (3.18)
\]

Consequently, from (3.17) and (3.18), we have

\[
A_3 + A_1
\]

\[
= (b-a) \left[ \int_{0}^{\alpha} a_{D_p,q} f((1-t)a + tb) \ dt \right] + \int_{0}^{1} \left( \frac{1 - \Phi_p(t)_{p,q}^{(a)}}{p^{(\alpha)}} \right) a_{D_p,q} f((1-t)a + tb) \ dt
\]

\[
= (b-a) \left[ \int_{0}^{\alpha} a_{D_p,q} f((1-t)a + tb) \ dt \right] - \int_{0}^{1} \left( \frac{1 - \Phi_p(t)_{p,q}^{(a)}}{p^{(\alpha)}} \right) a_{D_p,q} f((1-t)a + tb) \ dt
\]

\[
= f \left( (\alpha + 1)_{p,q} - p^{\alpha}a + p^{\alpha}b \right) \alpha \frac{(\alpha + 1)}{p^{\alpha}(b-a)^{\alpha}} \left( a_{p,q}^{(a)} \varphi_{p,q}^{(a)} \right)(p^{\alpha}b + (1-p^{\alpha})a).
\]

Therefore, the proof is completed. □

Remark 3.5 If \( \alpha = 1 \), then (3.16) reduces to

\[
\begin{align*}
\left| \int_{a}^{x} \left( \frac{qa + pb}{p + q} \right) - \frac{1}{p} \int_{a}^{\frac{x-a}{p}} f(x) \ dt \right| \\
= q(b-a) \left[ \int_{0}^{\frac{x-a}{p}} t_{a} D_p,q f((1-t)a + tb) \ dt \right] + \int_{0}^{1} \left( t \frac{1}{q} \right) a_{D_p,q} f((1-t)a + tb) \ dt.
\end{align*}
\quad (3.19)
\]

which appeared in [42].
Moreover, if \( p = 1 \), then (3.16) reduces to
\[
\begin{align*}
    f \left( \frac{[\alpha + 1]_{p,q} - 1}{\alpha + 1} \right) + \frac{\Gamma(\alpha + 1)}{(b - a)\alpha} \left( aD_{p,q} f \right)(b) \\
    = (b - a) \left[ \int_{0}^{\frac{1}{\alpha + 1}} (1 - \Phi_{q}(t))^{(\alpha)}_{q} aD_{p,q} \left( (1 - t)a + tb \right) \, \theta dq_{t} \\
    + \int_{\frac{1}{\alpha + 1}}^{1} (1 - \Phi_{q}(t))^{(\alpha)}_{q} aD_{p,q} \left( (1 - t)a + tb \right) \, \theta dq_{t} \right] 
\end{align*}
\]
which appeared in [40].

**Theorem 3.4** Let \( f : [a,b] \to \mathbb{R} \) be a continuous function, \( \alpha > 0 \), and \( aD_{p,q}f \) be \((p,q)\)-integrable on \((a,\frac{1}{p}(b-a) + a)\). If \(|aD_{p,q}f|\) is convex on \((a,\frac{1}{p}(b-a) + a)\), then the following Riemann–Liouville fractional \((p,q)\)-midpoint type inequality holds:
\[
\begin{align*}
    \left| f \left( \frac{[\alpha + 1]_{p,q} - p^{\alpha}a + p^{\alpha}b}{\alpha + 1} \right) - \frac{\Gamma(\alpha + 1)}{p^{\alpha}(b - a)\alpha} \left( aD_{p,q} f \right)(p^{\alpha}b + (1 - p^{\alpha})a) \right| \\
    \leq (b - a) \left[ B_{5} \left| aD_{p,q}f(a) \right| + B_{6} \left| aD_{p,q}f(b) \right| \right], 
\end{align*}
\]
where
\[
\begin{align*}
    B_{5} &= \int_{0}^{\frac{1}{p\alpha + 1}} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| (1-t) \, \theta dq_{t}, \\
    B_{6} &= \int_{0}^{\frac{1}{p\alpha + 1}} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| t \, \theta dq_{t}, \\
    B_{7} &= \int_{\frac{1}{p\alpha + 1}}^{1} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| (1-t) \, \theta dq_{t}, \\
    B_{8} &= \int_{\frac{1}{p\alpha + 1}}^{1} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| t \, \theta dq_{t}. 
\end{align*}
\]

**Proof** Using Lemma 3.2 and the convexity of \(|aD_{p,q}f|\), we have
\[
\begin{align*}
    \left| f \left( \frac{[\alpha + 1]_{p,q} - p^{\alpha}a + p^{\alpha}b}{\alpha + 1} \right) - \frac{\Gamma(\alpha + 1)}{p^{\alpha}(b - a)\alpha} \left( aD_{p,q} f \right)(p^{\alpha}b + (1 - p^{\alpha})a) \right| \\
    \leq (b - a) \left[ \int_{0}^{\frac{1}{p\alpha + 1}} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| aD_{p,q} \left( (1 - t)a + tb \right) \, \theta dq_{t} \\
    + \int_{\frac{1}{p\alpha + 1}}^{1} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| aD_{p,q} \left( (1 - t)a + tb \right) \, \theta dq_{t} \right] \\
    \leq (b - a) \left[ \int_{0}^{\frac{1}{p\alpha + 1}} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| \left| aD_{p,q}f(a) \right| (1-t) \, \theta dq_{t} \\
    + \int_{\frac{1}{p\alpha + 1}}^{1} \left| 1 - \frac{1 - \Phi_{q}(t)^{(\alpha)}_{q}}{p^{(\alpha)}} \right| \left| aD_{p,q}f(b) \right| t \, \theta dq_{t} \right] \\
    \leq (b - a) \left[ B_{5} \left| aD_{p,q}f(a) \right| + B_{6} \left| aD_{p,q}f(b) \right| \right]. 
\end{align*}
\]
which appeared in [42].

Moreover, if \( p = 1 \), then (3.21) reduces to

\[
\left| f\left( qa + \frac{pb}{p + q} \right) - \frac{1}{b - a} \int_a^{b(1-p)a} f(x) \, d\nu_{D_qf}(x) \right| 
\]

\[
\leq q(b - a) \left[ \lambda_4(p, q) \left| aD_{p,q}f(a) \right| + \lambda_5(p, q) \left| aD_{p,q}f(b) \right| + \lambda_6(p, q) \left| aD_{p,q}f(a) \right| + \lambda_7(p, q) \left| aD_{p,q}f(b) \right| \right], \tag{3.22}
\]

where

\[
\lambda_4(p, q) = \frac{p^3}{(p + q)^3(p^2 + pq + q^2)}, \quad \lambda_5(p, q) = \frac{p^2(p + pq + q^2) - p^3}{(p + q)^3(p^2 + pq + q^2)},
\]

\[
\lambda_6(p, q) = \frac{2p^3}{(p + q)^3(p^2 + pq + q^2)}, \quad \lambda_7(p, q) = \frac{p^4 + p^3q + p^2q^2 - 2p^3}{(p + q)^3(p^2 + pq + q^2)},
\]

which appeared in [42].

Moreover, if \( \alpha = 1 \), then (3.21) reduces to

\[
\left| f\left( b - a \right) \frac{[\alpha + 1]a + b}{[\alpha + 1]q} - \frac{\Gamma_q(\alpha + 1)}{(b - a)\nu_{aD_f}(b)} \right| 
\]

\[
\leq (b - a) \left[ \delta_5 \left| aD_{\nu_f}(a) \right| + \delta_6 \left| aD_{\nu_f}(b) \right| + \delta_7 \left| aD_{\nu_f}(a) \right| + \delta_8 \left| aD_{\nu_f}(b) \right| \right], \tag{3.23}
\]

where

\[
\delta_5 = \left[ \int_0^{[\alpha + 1]q} |1 - (1 - \Phi_q(t))_{q}^{(a)}| (1 - t) \, d\nu_q(t) \right],
\]

\[
\delta_6 = \left[ \int_0^{[\alpha + 1]q} |1 - \Phi_q(t)_{q}^{(a)}| t \, d\nu_q(t) \right],
\]

\[
\delta_7 = \left[ \int_0^{[\alpha + 1]q} |1 - \Phi_q(t)_{q}^{(a)}| (1 - t) \, d\nu_q(t) \right],
\]

\[
\delta_8 = \left[ \int_0^{[\alpha + 1]q} |1 - \Phi_q(t)_{q}^{(a)}| t \, d\nu_q(t) \right],
\]

which appeared in [40].
Theorem 3.5 Let \( f : [a, b] \to \mathbb{R} \) be a continuous function, \( \alpha > 0 \) and \( aD_{p,q}f \) be \((p, q)\)-integrable on \((a, \frac{b}{2}(b-a) + a)\). If \( aD_{p,q}f \)' is convex on \((a, \frac{b}{2}(b-a) + a)\) for \( r \geq 0 \), then the following Riemann–Liouville fractional \((p, q)\)-midpoint type inequality holds:

\[
\left| \frac{1}{\binom{n+1}{p,q}} \Gamma_{p,q}(\alpha + 1) \right| (p^{\alpha}b + (1 - p^{\alpha})a) \right|
\leq (b-a) \left[ B_{5}^{1-1/r}(B_{5} |aD_{p,q}f(a)|^{r} + B_{6} |aD_{p,q}f(b)|^{r}) \right]^{1/r}
\]

where \( B_{5}, B_{6}, B_{7}, \) and \( B_{8} \) are given in Theorem 3.4 and

\[
B_{9} = \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t}
\]

and

\[
B_{10} = \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t}.
\]

Proof Using Lemma 3.2, the power mean inequality and the convexity of \( aD_{p,q}f \)', we have

\[
\left| \frac{1}{\binom{n+1}{p,q}} \Gamma_{p,q}(\alpha + 1) \right| (p^{\alpha}b + (1 - p^{\alpha})a) \right|
\leq (b-a) \left[ \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{1/r} \right]
\]

\[
\times \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{11/r}
\]

\[
+ \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{11/r}
\]

\[
\times \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{11/r}
\]

\[
\leq (b-a) \left[ \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{1/r} \right]
\]

\[
\times \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{11/r}
\]

\[
+ \left( \int_{0}^{1} \frac{1 - \frac{(1 - 0\Phi_{q}(t))^{a}_{pb}}{p^{(\frac{2}{3})}}} {0d_{p,q}t} \right)^{11/r}
\]
\[
\times \left( \int_{\mu}^{\nu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \left[ aD_p q f(a)^{\alpha} (1 - t) + aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/r} \right]
\]

\[
\leq (b-a) \left[ \left( \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \right) \left[ aD_p q f(a)^{\alpha} (1 - t) + aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1-1/r} 
\times \left( \left[ aD_p q f(a)^{\alpha} \right] \left[ \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \right) \left[ aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/1/r} 
\times \left( b-a \right) \left( \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \right) \left[ aD_p q f(a)^{\alpha} (1 - t) + aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1-1/r} 
\times \left( \left[ aD_p q f(a)^{\alpha} \right] \left[ \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \right) \left[ aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/1/r} 
\]

This completes the proof. 

\[\square\]

 Remark 3.7 If $\alpha = 1$, then (3.24) reduces to

\[
\left| \int f \left( \frac{qa + pb}{p + q} \right) - \frac{1}{p(b-a)} \int f(x) aD_p q x \right|
\leq q(b-a) \left( \frac{p^2}{(p + q)^2} \right)^{1-1/r} \left[ \left( \lambda_4(p,q) \right) aD_p q f(a)^{\alpha} (1 - t) + \lambda_5(p,q) aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/1/r} 
\times \left( \left[ aD_p q f(a)^{\alpha} \right] \left[ \int f(x) aD_p q x \right] \right) \left[ \int f(x) aD_p q x \right] \right)^{1-1/r} 
\times \left( \left[ aD_p q f(b)^{\alpha} \right] \left[ \int f(x) aD_p q x \right] \right) \left[ \int f(x) aD_p q x \right] \right)^{1/1/r} 
\]

where $\lambda_4(p,q), \lambda_5(p,q), \lambda_6(p,q)$, and $\lambda_7(p,q)$ are given in Remark (3.6), which appeared in [42].

Moreover, if $p = 1$, then (3.24) reduces to

\[
\left| \int f \left( \frac{qa + 1}{\alpha + q} \right) - \frac{1}{b-a} \int f(x) aD_q f(x) \right|
\leq q(b-a) \left( \frac{1}{(b-a)^2} \right)^{1-1/r} \left[ \left( \delta_5 q^2 \right) aD_q f(a)^{\alpha} (1 - t) + \delta_6 q^2 aD_q f(b)^{\alpha} |t| \cdot dD_q t \right] \right)^{1/1/r} 
\times \left( \left[ aD_q f(a)^{\alpha} \right] \left[ \int f(x) aD_q f(x) \right] \right) \left[ \int f(x) aD_q f(x) \right] \right)^{1-1/r} 
\times \left( \left[ aD_q f(b)^{\alpha} \right] \left[ \int f(x) aD_q f(x) \right] \right) \left[ \int f(x) aD_q f(x) \right] \right)^{1/1/r} 
\]

where $\delta_5, \delta_6, \delta_7, \delta_8$ are given in Remark (3.6) and

\[
\delta_5 = \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \left[ aD_p q f(a)^{\alpha} (1 - t) + aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/r} 
\delta_6 = \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \left[ aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/r} 
\delta_7 = \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \left[ aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/r} 
\delta_8 = \int_{\nu}^{\mu} \left[ \frac{1 - \Phi_q(t)(\alpha)}{p(t)^2} \right] \left[ aD_p q f(b)^{\alpha} |t| \cdot dD_p q t \right] \right)^{1/r} 
\]

which appeared in [40].
\textbf{Theorem 3.6} Let \( f : [a, b] \to \mathbb{R} \) be a continuous function, \( \alpha > 0 \), and \( \alpha_{D_p,q}f \) be \((p,q)\)-integrable on \( (a, \frac{1}{r}(b-a) + a) \). If \( |\alpha_{D_p,q}f|^r \) is convex on \([a, \frac{1}{r}(b-a) + a]\) for \( r > 1 \) and \( 1/r + 1/s = 1 \), then the following Riemann–Liouville fractional \((p,q)\)-midpoint type inequality holds:

\[
\left| \int \left( \frac{[(\alpha + 1)_{p,q} - p^\alpha]a + p^\alpha b}{[\alpha + 1]_{p,q}} \right) - \Gamma_{p,q}(\alpha + 1) p^\alpha (b - a)^\alpha \left( \alpha_{p,q}f \right) (1 - p^\alpha) \right| \\
\leq (b-a) \left[ B_{11} \left( |\alpha_{D_p,q}(a)|^r \left( \frac{p^\alpha (p + q) [\alpha + 1]_{p,q} - p^\alpha}{(p + q) [\alpha + 1]_{p,q}^2} \right) \right) \right]^{1/r} \\
+ |\alpha_{D_p,q}(b)|^r \left( \frac{p^\alpha}{(p + q) [\alpha + 1]_{p,q}^2} \right)^{1/r} \\
+ (b-a) \left[ B_{12} \left( |\alpha_{D_p,q}(a)|^r \left( \frac{p^\alpha (p + q) [\alpha + 1]_{p,q} - p^{2\alpha}}{(p + q) [\alpha + 1]_{p,q}^2} \right) \right) \right]^{1/r},
\]

(3.27)

where

\[
B_{11} = \int_0^{\rho_{p,q}} \left| 1 - \frac{(1 - t \Phi_q(t)_{p,q})^{[\alpha]}}{\rho(t)} \right| \, d\rho_{p,q} t
\]

and

\[
B_{12} = \int_0^{\rho_{p,q}} \left| - \frac{(1 - t \Phi_q(t)_{p,q})^{[\alpha]}}{\rho(t)} \right| \, d\rho_{p,q} t.
\]

\textbf{Proof} Applying Lemma 3.2, Hölder’s inequality, and the convexity of \( |\alpha_{D_p,q}f|^r \), we have

\[
\left| \int \left( \frac{[(\alpha + 1)_{p,q} - p^\alpha]a + p^\alpha b}{[\alpha + 1]_{p,q}} \right) - \Gamma_{p,q}(\alpha + 1) p^\alpha (b - a)^\alpha \left( \alpha_{p,q}f \right) (1 - p^\alpha) \right| \\
\leq (b-a) \left[ \int_0^{\rho_{p,q}} \left| 1 - \frac{(1 - t \Phi_q(t)_{p,q})^{[\alpha]}}{\rho(t)} \right| \, d\rho_{p,q} ((1-t)a + tb) \right] \, d\rho_{p,q} t \\
+ \int_0^{\rho_{p,q}} \left| - \frac{(1 - t \Phi_q(t)_{p,q})^{[\alpha]}}{\rho(t)} \right| \, d\rho_{p,q} ((1-t)a + tb) \right] \, d\rho_{p,q} t
\]

\[
\leq (b-a) \left[ \left( \int_0^{\rho_{p,q}} \left| 1 - \frac{(1 - t \Phi_q(t)_{p,q})^{[\alpha]}}{\rho(t)} \right|^{1/p} \, d\rho_{p,q} t \right)^{1/p} \\
\times \left( \int_0^{\rho_{p,q}} \left| \alpha_{D_p,q}f ((1-t)a + tb) \right|^{1/r} \, d\rho_{p,q} t \right)^{1/r} \\
+ \left( \int_0^{\rho_{p,q}} \left| - \frac{(1 - t \Phi_q(t)_{p,q})^{[\alpha]}}{\rho(t)} \right|^{1/p} \, d\rho_{p,q} t \right)^{1/p} \\
\times \left( \int_0^{\rho_{p,q}} \left| \alpha_{D_p,q}f ((1-t)a + tb) \right|^{1/r} \, d\rho_{p,q} t \right)^{1/r} \right].
\]
\[
\leq (b-a) \left[ \left( \int_{\alpha}^{\mu_{n+1}[p,q]} \left| 1 - \frac{(1 - \Phi_p(t))^{(a)}}{p^{(2)}} \right|^p \alpha_d p_q d \alpha \right)^{1/p} \right] \\
\times \left[ \left| a D_{p,q} f(a) \right| \int_{\alpha}^{\mu_{n+1}[p,q]} (1 - t) \alpha d p_q d t + \left| a D_{p,q} f(b) \right| \int_{\alpha}^{\mu_{n+1}[p,q]} t \alpha d p_q d t \right]^{1/r} \\
+ (b-a) \left[ \left( \int_{\alpha}^{\mu_{n+1}[p,q]} \left| 1 - \frac{(1 - \Phi_p(t))^{(a)}}{p^{(2)}} \right|^p \alpha d p_q d \alpha \right)^{1/p} \right] \\
\times \left[ \left| a D_{p,q} f(a) \right| \int_{\alpha}^{\mu_{n+1}[p,q]} (1 - t) \alpha d p_q d t \right. \\
+ \left| a D_{p,q} f(b) \right| \int_{\alpha}^{\mu_{n+1}[p,q]} t \alpha d p_q d t \right]^{1/r} \\
\leq (b-a) \left[ \left( \int_{\alpha}^{\mu_{n+1}[p,q]} \left| 1 - \frac{(1 - \Phi_p(t))^{(a)}}{p^{(2)}} \right|^p \alpha d p_q d \alpha \right)^{1/p} \right] \\
\times \left[ \left| a D_{p,q} f(a) \right| \left( \frac{p^2 (p + q) \left[ (\alpha + 1)_{p,q} - p^2 \right]}{(p + q) \left[ (\alpha + 1)_{p,q}^2 \right]} \right)^{1/r} \right] \\
+ (b-a) \left[ \left( \int_{\alpha}^{\mu_{n+1}[p,q]} \left| 1 - \frac{(1 - \Phi_p(t))^{(a)}}{p^{(2)}} \right|^p \alpha d p_q d \alpha \right)^{1/p} \right] \\
\times \left[ \left| a D_{p,q} f(a) \right| \left( \frac{p + q - 1}{p + q} - \frac{p^2 (p + q) \left[ (\alpha + 1)_{p,q} - p^{2\alpha} \right]}{(p + q) \left[ (\alpha + 1)_{p,q}^2 \right]} \right) \right] \\
+ \left| a D_{p,q} f(b) \right| \left( \frac{1}{p + q} - \frac{p^{2\alpha}}{(p + q) \left[ (\alpha + 1)_{p,q}^2 \right]} \right) \right]^{1/r}.
\]

This completes the proof. \qed

Remark 3.8 If \( \alpha = 1 \), then (3.27) reduces to

\[
\left| f \left( \frac{qa + pb}{p + q} \right) - \frac{1}{p(b-a)} \int_{\alpha}^{\mu_{n+1}(1-p)x} f(x) \alpha d p_q d \alpha \right| \\
\leq q(b-a) \left[ \left( \left( \frac{p}{p + q} \right)^{\alpha} \left( \frac{p - q}{p^{x+1} - q^{x+1}} \right) \right)^{1/s} \left| a D_{p,q} f(a) \right| \left( \frac{p^3 + 2p^2 q + p q^2 - p^2}{(p + q)^3} \right) \right] \\
+ \left| a D_{p,q} f(b) \right| \left( \frac{p^2}{(p + q)^3} \right) \right]^{1/r} \\
+ \left( \int_{\alpha}^{\mu_{n+1}[p,q]} \left( \frac{1 - t}{q} \right) \alpha d p_q d t \right)^{1/s} \left| a D_{p,q} f(b) \right| \left( \frac{2pq + q^2}{(p + q)^3} \right) \\
+ \left| a D_{p,q} f(a) \right| \left( \frac{p^2 q + 2pq^2 - pq - q^3}{(p + q)^3} \right) \right]^{1/r},
\]

which appeared in [42].
Moreover, if \( p = 1 \), then (3.27) reduces to

\[
\left| f \left( \frac{(\alpha + 1)q - 1)a + b}{\alpha + 1} \right) - \Gamma_q(\alpha + 1) \left( _qD^{\alpha}_qf(b) \right) \right| \\
\leq (b - a) \left[ (\delta_{11})^{\frac{1}{t}} \left( \left| _qD^\alpha_qf(a) \right| \left( \frac{1 + q}{1 + q((\alpha + 1)q - 1)} \right) \right) \right] \\
+ (b - a) \left[ (\delta_{12})^{\frac{1}{t}} \left( \left| _qD^\alpha_qf(b) \right| \left( \frac{1}{(1 + q)((\alpha + 1)q - 1)} \right) \right) \right],
\]

where

\[
\delta_{11} = \int_0^{1/q} \left| 1 - (1 - 0\Phi_q(t))^{(\alpha)s} \right| q d_q t
\]

and

\[
\delta_{12} = \int_{1/q}^1 \left| -(1 - 0\Phi_q(t))^{(\alpha)s} \right| q d_q t,
\]

which appeared in [40].

4 Conclusions

In this work, we studied two identities for continuous functions in the form of fractional Riemann–Liouville \((p, q)\)-integral. Based on these two identities, some fractional Riemann–Liouville \((p, q)\)-trapezoid and \((p, q)\)-midpoint type inequalities are given. From this idea, as well as the techniques of this paper, we hope that it will inspire interested readers working in this field.

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Authors’ contributions

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