Group averaging in the \((p, q)\) oscillator representation of \(\text{SL}(2, \mathbb{R})\)

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Abstract

We investigate refined algebraic quantisation with group averaging in a finite-dimensional constrained Hamiltonian system that provides a simplified model of general relativity. The classical theory has gauge group \(\text{SL}(2, \mathbb{R})\) and a distinguished \(\mathfrak{o}(p, q)\) observable algebra. The gauge group of the quantum theory is the double cover of \(\text{SL}(2, \mathbb{R})\), and its representation on the auxiliary Hilbert space is isomorphic to the \((p, q)\) oscillator representation. When \(p \geq 2, q \geq 2\) and \(p + q \equiv 0 \pmod{2}\), we obtain a physical Hilbert space with a nontrivial representation of the \(\mathfrak{o}(p, q)\) quantum observable algebra. For \(p = q = 1\), the system provides the first example known to us where group averaging converges to an indefinite sesquilinear form.

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1 Introduction

In quantisation of constrained systems, an elegant proposal to obtain a physical inner product is to average unconstrained quantum states in an auxiliary Hilbert space over the gauge group \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]\). When the averaging is formulated within refined algebraic quantisation \([4, 8, 12]\) and converges in a sufficiently strong sense, it provides either the unique rigging map, and hence the unique inner product on the states that satisfy the constraints, or a proof that the system does not admit a rigging map \([8]\). Given the equivalence of refined algebraic quantisation to a wide class of methods of choosing the physical inner product \([13, 14]\), group averaging thus provides considerable control over the quantisation.

When the gauge group is compact, the averaging necessarily converges. For a non-compact gauge group the averaging need not converge on all of the auxiliary Hilbert space \(H_{\text{aux}}\) but may still converge on a suitable dense linear subspace \(\Phi\), and this is sufficient for recovering the physical Hilbert space \(H_{\text{RAQ}}\). The choice of the test space \(\Phi\) thus has a mathematical role in ensuring convergence, but it also has a deep physical role in that \(\Phi\) determines the algebra of operators represented on \(H_{\text{RAQ}}\) \([8, 14]\). While quantisation with group averaging can be carried out without the explicit construction of any physical observables, in concrete examples one may wish to choose \(\Phi\) so that certain explicitly-known physical observables of interest are contained in the algebra represented on \(H_{\text{RAQ}}\).

In this paper we study a quantum mechanical system whose constraints mimic the Hamiltonian structure of general relativity \([15]\). The constraint set consists of two “Hamiltonian”-type constraints, quadratic in the momenta, and one “momentum”-type constraint, linear in the momenta, and the classical gauge group generated by these constraints is \(\text{SL}(2, \mathbb{R})\). The unreduced phase space is \(T^*\mathbb{R}^{p+q} \simeq \mathbb{R}^{2(p+q)}\), where \(p \geq 1\) and \(q \geq 1\). The system was introduced by Montesinos, Rovelli and Thiemann with \(p = q = 2\) \([16]\), and its quantisation with \(p = q = 2\) was studied in \([10, 16, 17, 18]\) within Ashtekar’s algebraic quantisation \([19, 20]\), in \([21]\) within algebraic constraint quantisation \([22, 23]\), in \([10, 21, 24, 25, 26]\) within group theoretic quantisation \([27, 28]\), and in \([10]\) within refined algebraic quantisation with group averaging \([4, 8]\). All these quantisations relied in one way or another on a distinguished classical \(\mathfrak{o}(2, 2)\) observable algebra, constructing a quantum theory in which these observables are promoted into quantum operators. Within group averaging \([10]\), it was in particular found that a judicious choice for the test space is necessary to achieve both convergence of the averaging and the inclusion of the \(\mathfrak{o}(2, 2)\) observables in the physical operator algebra.

For \(p > 2\) and \(q = 2\), the system has been studied in the context of a “two-time” physical interpretation in \([29, 30, 31, 32]\). The case \(p = q = 2\) is currently being studied \([33]\) within the master constraint programme \([34]\). Related systems with \(\text{SL}(2, \mathbb{R})\) gauge invariance have been studied in \([35, 36, 37]\).

We wish to quantise this system with group averaging for general \(p\) and \(q\), using test states built from eigenstates of the harmonic oscillator Hamiltonians that arise in the oscillator representation of \(\text{SL}(2, \mathbb{R})\) \([38]\). When \(p \geq 2\), \(q \geq 2\) and \(p + q \equiv 0 \pmod{2}\),
we obtain a quantum theory in which the classical $\mathfrak{o}(p, q)$ observables are promoted into a nontrivially-represented operator algebra. When $(p, q) = (1, 3)$ or $(3, 1)$, we obtain a quantum theory with a one-dimensional physical Hilbert space that is annihilated by all the $\mathfrak{o}(p, q)$ observables. For other values of $p$ and $q$ we recover no physical Hilbert space. In particular, for $p = q = 1$ the group averaging converges to an indefinite sesquilinear form, in a sense strong enough for the uniqueness theorem of [8] to imply that the system admits no rigging maps. This is the first example known to us in which group averaging fails to produce a Hilbert space owing to indefiniteness of the would-be inner product.

We show further that all our group averaging quantum theories can be obtained within Ashtekar’s algebraic quantisation [19, 20], using the $\mathfrak{o}(p, q)$ observables to determine the physical inner product, and we display explicitly the correspondence between the two schemes. We have not gained sufficient control over the $\mathfrak{o}(p, q)$ algebra to ascertain whether algebraic quantisation might for some $p$ and $q$ yield also quantum theories not recovered by the group averaging, but we show that this does not happen for $p+q \equiv 1 \pmod{2}$, nor does it happen for $p + q \equiv 0 \pmod{2}$ if $p \leq 3$ and $q \leq 3$.

We also give a detailed description of the classical reduced phase space. The reduced phase space contains a symplectic manifold if and only if $p \geq 2$ and $q \geq 2$. This manifold is separated by the $\mathfrak{o}(p, q)$ observables, and it is connected if and only if $p \geq 3$ and $q \geq 3$. This suggests that interesting quantum theories should exist only when $p \geq 2$ and $q \geq 2$, possibly with some subtleties when $\min(p, q) = 2$. As outlined above, this agrees with our findings.

The rest of the paper is as follows. Section 2 introduces and analyses the classical system. Section 3 discusses algebraic quantisation, laying out the task for general $p$ and $q$ and completing it for $\max(p, q) \leq 3$. Refined algebraic quantisation with group averaging is carried out in section 4 for $\min(p, q) \geq 3$ and in section 5 for other values of $p$ and $q$.

Section 6 presents a summary and concluding remarks. Appendix A collects some basic properties of $\text{SL}(2, \mathbb{R})$, and appendices B–E contain the proofs of several technical results stated in the main text.

## 2 Classical system

In this section we analyse a classical constrained system with the unreduced phase space $T^\ast \mathbb{R}^{p+q}$, where $p \geq 1$ and $q \geq 1$. The system was introduced for $p = q = 2$ in [16], and our discussion of the gauge transformations and the distinguished $\mathfrak{o}(p, q)$ observables generalises the observations of [16] in a straightforward manner. We shall however show that the structure of the reduced phase space depends sensitively on $p$ and $q$. 
2.1 The system

The system is defined by the action

\[
S = \int dt \left( p \cdot \dot{u} + \pi \cdot \dot{v} - NH_1 - MH_2 - \lambda D \right),
\]

where \( u \) and \( p \) are real vectors of dimension \( p \geq 1 \), \( v \) and \( \pi \) are real vectors of dimension \( q \geq 1 \), and the overdot denotes differentiation with respect to \( t \). \( p \) and \( \pi \) are respectively the momenta conjugate to \( u \) and \( v \), the symplectic structure is

\[
\Omega = \sum_{i=1}^{p} dp_i \wedge du_i + \sum_{j=1}^{q} d\pi_j \wedge dv_j
\]

and the phase space is \( \Gamma := T^*\mathbb{R}^{p+q} \cong \mathbb{R}^{2(p+q)} \). \( N \), \( M \) and \( \lambda \) are Lagrange multipliers associated with the constraints

\[
H_1 := \frac{1}{2} (p^2 - v^2)
\]

\[
H_2 := \frac{1}{2} (\pi^2 - u^2)
\]

\[
D := u \cdot p - v \cdot \pi
\]

The Poisson algebra of the constraints is the \( \mathfrak{sl}(2,\mathbb{R}) \) Lie algebra (see appendix A),

\[
\{ H_1, H_2 \} = D,
\]

\[
\{ H_1, D \} = -2H_1,
\]

\[
\{ H_2, D \} = 2H_2
\]

and the system is a first class constrained system [39, 40]. The finite gauge transformations on \( \Gamma \) generated by the constraints are

\[
\begin{pmatrix} u \\ p \end{pmatrix} \mapsto g \begin{pmatrix} u \\ p \end{pmatrix}, \quad \begin{pmatrix} \pi \\ v \end{pmatrix} \mapsto g \begin{pmatrix} \pi \\ v \end{pmatrix},
\]

where \( g \) is an SL(2,\( \mathbb{R} \)) matrix. The gauge group is thus SL(2,\( \mathbb{R} \)). As the Hamiltonian is a sum of the constraints, the constraints entirely determine the dynamics.

2.2 Classical observables

Recall that an observable is a function on \( \Gamma \) whose Poisson brackets with the first class constraints vanish when the first class constraints hold [40]. Consider on \( \Gamma \) the functions \( O_{kj} := x_k \times x_j \), where \( x_k = (u_k, p_k)^T \) for \( 1 \leq k \leq p \), \( x_{p+k} = (\pi_k, v_k)^T \) for \( 1 \leq k \leq q \), and the cross stands for the scalar-valued cross product on \( \mathbb{R}^2 \). As the SL(2,\( \mathbb{R} \)) action on \( \mathbb{R}^2 \) preserves areas, [25] shows that \( O_{kj} \) are invariant under the gauge transformations.
Hence $O_{k,j}$ are observables. The Poisson algebra of these observables is the $\mathfrak{o}(p,q)$ Lie algebra,

$$\{O_{ij}, O_{kl}\} = g_{ik}O_{jl} - g_{il}O_{jk} + g_{jl}O_{ik} - g_{jk}O_{il} ,$$

where

$$g_{ik} = \text{diag}(1,\ldots,1,-1,\ldots,-1)_{ik} .$$

The algebra generated by $\{O_{ij}\}$ is denoted by $\mathcal{A}_{\text{class}}$. The finite transformations that $\mathcal{A}_{\text{class}}$ generates on $\Gamma$ are

$$\begin{pmatrix} u \\ \pi \end{pmatrix} \mapsto R \begin{pmatrix} u \\ \pi \end{pmatrix}, \quad \begin{pmatrix} p \\ v \end{pmatrix} \mapsto R \begin{pmatrix} p \\ v \end{pmatrix},$$

where $R$ is an $O(p,q)$ matrix, in the connected component $O_c(p,q)$. Note that as none of the above relies on the constraints being satisfied, the $\text{SL}(2,\mathbb{R})$ action (2.5) and the $O_c(p,q)$-action (2.8) commute on all of $\Gamma$, not just on the subset where the constraints hold.

It will be useful to decompose the basis $\{O_{ij}\}$ of $\mathcal{A}_{\text{class}}$ as $\mathfrak{o}(p,q) = \mathfrak{o}(p) \oplus \mathfrak{o}(q) \oplus \mathfrak{p}$, where $\mathfrak{p}$ is spanned by the observables transverse to those in the Lie algebra of the maximal compact subgroup $O_c(p) \times O_c(q)$ [41]. Explicitly, we write

$$A_{ij} := O_{ij} = u_ip_j - u_jp_i , \quad 1 \leq i \leq p, \quad 1 \leq j \leq p ;$$  
$$B_{ij} := O_{p+i,p+j} = v_i\pi_j - v_j\pi_i , \quad 1 \leq i \leq q, \quad 1 \leq j \leq q ;$$  
$$C_{ij} := O_{i,p+j} = u_iv_j - p_i\pi_j , \quad 1 \leq i \leq p, \quad 1 \leq j \leq q ,$$

where $A_{ij} \in \mathfrak{o}(p), B_{ij} \in \mathfrak{o}(q)$ and $C_{ij} \in \mathfrak{p}$.

Other observables of interest in $\mathcal{A}_{\text{class}}$ are the Casimir elements of the universal enveloping algebra of $\mathfrak{o}(p,q)$ [42]. We consider only the quadratic Casimir observable,

$$C := \frac{1}{2} \sum_{ijkl} g_{ij}g_{kl}O_{ik}O_{jl}$$  
$$= \sum_{i<j} (A_{ij})^2 + \sum_{i<j} (B_{ij})^2 - \sum_{i,j} (C_{ij})^2$$  
$$= -4H_1H_2 - D^2 ,$$

where the last equality follows by direct computation. When the constraints hold, $C$ thus vanishes.

### 2.3 Reduced phase space

Let $\Gamma$ be the subset of $\Gamma$ where the constraints hold. The reduced phase space, denoted by $\mathcal{M}$, is the quotient of $\Gamma$ under the gauge action (2.5). As the Hamiltonian is a linear
combination of the constraints, there is no dynamics on \( M \), and \( M \) can be identified with the space of classical solutions. As the functions in \( A_{\text{class}} \) are gauge invariant, they project to functions on \( M \): We use for these functions the same symbols.

For \( p = q = 2 \), the generic sectors of \( M \) were found in [16, 21] and the global properties of \( M \) were exhibited in [10]. We now analyze \( M \) for general \( p \geq 1 \) and \( q \geq 1 \). \( \Gamma \) is clearly connected. Hence also \( M \) is connected.

To proceed, we decompose \( \Gamma \) into three subsets. Let \( \Gamma_0 = \{ q_0 \} \), where \( q_0 \) is the origin of \( \Gamma \), \( u = p = 0 = v = \pi \). Let \( \Gamma_{\text{ex}} \) contain all other points of \( \Gamma \) at which at least one of the pairs \((u, p)\) and \((v, \pi)\) is linearly dependent. Finally, let \( \Gamma_{\text{reg}} \) contain the rest of \( \Gamma \). We refer to \( \Gamma_{\text{ex}} \) and \( \Gamma_{\text{reg}} \) as respectively the “exceptional” and “regular” parts of \( \Gamma \).

We show in appendix B that the gradients of the constraints are all vanishing on \( \Gamma_0 \), linearly dependent but not all vanishing on \( \Gamma_{\text{ex}} \), and linearly independent on \( \Gamma_{\text{reg}} \). \( \Gamma_0 \) and \( \Gamma_{\text{ex}} \) are nonempty for all \( p \) and \( q \), while \( \Gamma_{\text{reg}} \) is nonempty if and only if \( p \geq 2 \) and \( q \geq 2 \).

As \( \Gamma_0, \Gamma_{\text{ex}} \) and \( \Gamma_{\text{reg}} \) are preserved by the gauge transformations, they project onto disjoint subsets of \( M \). We denote these sets respectively by \( M_0, M_{\text{ex}} \) and \( M_{\text{reg}} \) and analyze each in turn.

### 2.3.1 \( M_0 \)

\( M_0 \) contains only one point, the projection of \( q_0 \). All observables in \( A_{\text{class}} \) vanish on \( M_0 \).

### 2.3.2 \( M_{\text{ex}} \)

As \( q_0 \notin \Gamma_{\text{ex}} \), the constraints \( H_1 = 0 = H_2 \) show that all points in \( \Gamma_{\text{ex}} \) have \((u, p) \neq (0, 0)\) and \((v, \pi) \neq (0, 0)\). Given a point at which the pair \((u, p)\) is linearly dependent, there thus exists a gauge-equivalent point with \( u = 0 \) and \( p^2 = 1 \), at which the constraints imply \( \pi = 0 \) and \( v^2 = 1 \). Given a point at which the pair \((v, \pi)\) is linearly dependent, a similar argument shows that there exists a gauge-equivalent point at which \( \pi = 0 \), \( v^2 = 1 \), \( u = 0 \) and \( p^2 = 1 \). Thus, each point in \( \Gamma_{\text{ex}} \) is gauge-equivalent to a point that satisfies

\[
v^2 = p^2 = 1 \quad , \quad u = 0 = \pi .
\]

It follows that both the pair \((u, p)\) and the pair \((v, \pi)\) are linearly dependent on \( \Gamma \).

The gauge transformations that preserve the set (2.11) act on it either trivially or by

\[
(v, p) \mapsto (-v, -p) .
\]

\( M_{\text{ex}} \) can therefore be represented as the quotient of the set (2.11), with topology \( S^{p-1} \times S^{q-1} \), under the \( \mathbb{Z}_2 \) action generated by (2.12). If in particular \( p = 1 \) (respectively \( q = 1 \)), \( M_{\text{ex}} \) has topology \( S^{p-1} (S^{q-1}) \). If \( p = q = 1 \), \( M_{\text{ex}} \) contains just two points.

Other representations of \( M_{\text{ex}} \) are obtained by replacing in (2.11) the first equations by \( v^2 = p^2 = r \), where \( r \) is an arbitrary prescribed positive number. This shows that in the topology of \( M \) induced from \( \Gamma \), every open set that includes \( M_0 \) includes also \( M_{\text{ex}} \).
Equations (2.9) and (2.11) show that all observables in \( \mathcal{A}_{\text{class}} \) vanish on \( \mathcal{M}_{\text{ex}} \). Equations (2.2) and (2.11) show that the projection of the symplectic form \( \Omega \) vanishes on \( \mathcal{M}_{\text{ex}} \). We refer to \( \mathcal{M}_{\text{ex}} \) as the “exceptional” part of \( \mathcal{M} \).

2.3.3 \( \mathcal{M}_{\text{reg}} \)

When \( p = 1 \) or \( q = 1 \) (or both), \( \mathcal{T}_{\text{reg}} \) and hence also \( \mathcal{M}_{\text{reg}} \) are empty. We now assume \( p \geq 2 \) and \( q \geq 2 \).

We show in appendix B that the gradients of the constraints are linearly independent on \( \mathcal{T}_{\text{reg}} \). It follows ([10], Section 1.1.2 and Appendix 2A) that \( \mathcal{M}_{\text{reg}} \) is a manifold of dimension \( 2p + 2q - 6 \) with a symplectic form induced from \( \Gamma \). We refer to \( \mathcal{M}_{\text{reg}} \) as the “regular” part of \( \mathcal{M} \).

Given a point in \( \mathcal{T}_{\text{reg}} \), the linear independence of the pair \((u, p)\) implies that there exists a gauge-equivalent point at which \( u \cdot p = 0 \) and \( u^2 = p^2 > 0 \). The constraints imply that at this point \( v \cdot \pi = 0 \), \( v^2 = p^2 \) and \( \pi^2 = u^2 \). Hence each point in \( \mathcal{T}_{\text{reg}} \) is gauge-equivalent to a point that satisfies
\[
 u^2 = p^2 = v^2 = \pi^2 > 0, \quad u \cdot p = v \cdot \pi = 0.
\] (2.13)

The gauge transformations that preserve the set (2.13) are (2.5) with
\[
g = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\] (2.14)

where \( 0 \leq \theta < 2\pi \). It follows that \( \mathcal{M}_{\text{reg}} \) can be represented as the quotient of the set (2.13) under the \( U(1) \) action given by (2.5) and (2.14).

We show in appendix C that \( \mathcal{A}_{\text{class}} \) separates \( \mathcal{M}_{\text{reg}} \): Given two distinct points in \( \mathcal{M}_{\text{reg}} \), there exist functions in \( \mathcal{A}_{\text{class}} \) that take distinct values at the two points.

For \( p = q = 2 \), \( \mathcal{M}_{\text{reg}} \) consists of four connected components [10] [16] [24], which can be pairwise joined into two connected symplectic manifolds by adding certain points from \( \mathcal{M}_{\text{ex}} \) [10].

Suppose \( p = 2 \) and \( q > 2 \). Within each gauge equivalence class in (2.13), there is a unique representative at which \( p_1 = 0 \) and \( u_1 > 0 \). It follows that at this point \( p_2 \neq 0 \) and \( u_2 = 0 \). A gauge transformation by \( g = \text{diag}(\vert p_2 \vert, \vert p_2 \vert^{-1}) \) brings this point to
\[
v^2 = 1, \quad v \cdot \pi = 0, \quad \pi^2 > 0, \quad p = (0, \epsilon), \quad u = (\vert \pi \vert, 0),
\] (2.15)

where \( \epsilon = \pm 1 \). For each \( \epsilon \), the set (2.15) is recognised as the cotangent bundle over \( S^{q-1} \), with the zero fibres omitted. Hence \( \mathcal{M}_{\text{reg}} \) consists of two connected components, given by (2.15) with the respective values of \( \epsilon \). Equations (2.2) and (2.15) show that the symplectic structure of this cotangent bundle description is precisely the symplectic structure induced from \( \Gamma \). For each \( \epsilon \), it is possible to include the zero fibres by allowing \( \pi^2 = 0 \) in (2.15); this means adding from \( \mathcal{M}_{\text{ex}} \) the subset represented uniquely by (2.11) with \( p = (0, \epsilon) \) and \( \pi = 0 \). Note that because of the identification (2.12) in (2.11), this subset of \( \mathcal{M}_{\text{ex}} \) is the same for both signs of \( \epsilon \). The mechanism of pairwise smoothly
joining the disconnected sectors for \( q = 2 \) [10] is not available now because the fibres without origin are disconnected for \( q = 2 \) but connected for \( q > 2 \).

The case \( q = 2 \) and \( p > 2 \) is isomorphic to \( p = 2 \) and \( q > 2 \).

When \( p > 2 \) and \( q > 2 \), \( \mathcal{M}_{\text{reg}} \) is connected. We have not found a simpler description of the global properties in this case. Convenient local gauge fixings are introduced in appendix C.

3 Algebraic quantisation

In this section we apply the algebraic quantisation framework of [19], adopting \( \mathcal{A}_{\text{class}} \) as the classical observable algebra whose complex conjugation relations are promoted into adjointness relations. Seeking solutions to the quantum constraints by separation of variables, we show in subsection 3.1 that necessary conditions for obtaining a quantum theory with a nontrivially-represented observable algebra are \( p \geq 2 \), \( q \geq 2 \) and \( p + q \equiv 0 \) (mod 2). The case \( p = q = 2 \) was analysed in [10, 16]. In subsection 3.2 we complete the quantisation for \( p = q = 3 \).

Detailed expositions of algebraic quantisation can be found in [19, 20].

3.1 Setup for \( p \geq 1 \) and \( q \geq 1 \)

We take the elementary ‘position’ and ‘momentum’ operators to act on smooth functions \( \Psi(u, v) \) as

\[
\hat{u} \Psi(u, v) = u \Psi(u, v), \quad \hat{p} \Psi(u, v) = -i \nabla_u \Psi(u, v),
\hat{v} \Psi(u, v) = v \Psi(u, v), \quad \hat{\pi} \Psi(u, v) = -i \nabla_v \Psi(u, v),
\]

so that \([\hat{u}_k, \hat{p}_j] = i \delta_{kj}\) and \([\hat{v}_k, \hat{\pi}_j] = i \delta_{kj}\). Inserting these operators into the classical constraints (2.3) and making a judicious ordering choice, we obtain the quantum constraints

\[
\hat{H}_1 := -\frac{1}{2} (\Delta_u + v^2), \quad \hat{H}_2 := -\frac{1}{2} (\Delta_v + u^2), \quad \hat{D} := -i \left( u \cdot \nabla_u - v \cdot \nabla_v + \frac{p - q}{2} \right),
\]

where \( \Delta_u \) (respectively \( \Delta_v \)) stands for the Laplacian in \( u \) (\( v \)). The non-derivative term in \( \hat{D} \) is needed to make the commutators close as the \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebra,

\[
[\hat{H}_1, \hat{H}_2] = i \hat{D}, \quad [\hat{H}_1, \hat{D}] = -2i \hat{H}_1, \quad [\hat{H}_2, \hat{D}] = +2i \hat{H}_2.
\]
We define the quantum observables $\hat{O}_{ij}$ by substituting the elementary quantum operators (3.1) in the expressions of the classical observables $O_{ij}$. These quantum observables commute with the quantum constraints (3.2), and their commutators form the $\mathfrak{o}(p, q)$ Lie algebra, obtained by hatting (2.9) and multiplying the right-hand side by $i$. As $O_{ij}$ are real, we introduce on the algebra generated by $\{\hat{O}_{ij}\}$ an antilinear involution by $\hat{O}_{ij}^* = \hat{O}_{ij}$. We denote the resulting star-algebra of quantum observables by $A^{(s)}_{\text{phys}}$.

Following (2.9), we decompose the basis of $A^{(s)}_{\text{phys}}$ as

$$
\hat{A}_{ij} := \hat{O}_{ij} = -i (u_i \partial_{u_j} - u_j \partial_{u_i}) \ , \quad 1 \leq i \leq p, \quad 1 \leq j \leq p ;
$$
$$
\hat{B}_{ij} := \hat{O}_{p+i,p+j} = -i (v_i \partial_{v_j} - v_j \partial_{v_i}) \ , \quad 1 \leq i \leq q, \quad 1 \leq j \leq q ;
$$
$$
\hat{C}_{ij} := \hat{O}_{i,p+j} = u_i v_j + \partial_{u_i} \partial_{v_j} \ , \quad 1 \leq i \leq p, \quad 1 \leq j \leq q .
$$

The quantum quadratic Casimir observable is

$$
\hat{C} := \frac{1}{2} \sum_{ijkl} g_{ij} g_{kl} \hat{O}_{ik} \hat{O}_{jl}
$$

$$
= \sum_{i<j} (\hat{A}_{ij})^2 + \sum_{i<j} (\hat{B}_{ij})^2 - \sum_{i,j} (\hat{C}_{ij})^2
$$

$$
= -2(\hat{H}_1 \hat{H}_2 + \hat{H}_2 \hat{H}_1) - \hat{D}^2 - \frac{1}{4}(p+q)(p+q-4) \ ,
$$

(3.5)

where the last equality follows by direct computation. In contrast to the classical Casimir (2.10), $\hat{C}$ vanishes on states annihilated by the constraints only for $p + q = 4$.

We seek states annihilated by the constraints,

$$
\hat{H}_1 \Psi(u, v) = 0 \ , \quad \hat{H}_2 \Psi(u, v) = 0 \ , \quad D \Psi(u, v) = 0 \ ,
$$

(3.6)

by separation of variables. If $p \geq 2$ and $q \geq 2$, we make the ansatz

$$
\Psi(u, v) = \psi(u, v) Y_{l_{ku}} (\theta(u)) Y_{j_{kv}} (\theta(v)) \ ,
$$

(3.7)

where $u := |u|$, $v := |v|$ and $Y_{l_{ku}} (\theta(u))$ (respectively $Y_{j_{kv}} (\theta(v))$) are the spherical harmonics on unit $S^{p-1}$ in $u$ ($S^{q-1}$ in $v$) \textsuperscript{13} \textsuperscript{44}. Here $\theta(u)$ denotes the coordinates on $S^{p-1}$, the index $l$ ranges over non-negative integers, the eigenvalue of the scalar Laplacian on $S^{p-1}$ is $-l(l + p - 2)$, the index $k_u$ labels the degeneracy for each $l$, and similarly for the quantities appearing in $Y_{j_{kv}} (\theta(v))$. We extend the ansatz (3.7) to $p = 1$, in which case $\theta(u) := u_1 / u \in \{1, -1\}$, $l \in \{0, 1\}$, the index $k_u$ takes only a single value and can be dropped, and the spherical harmonics are $Y_l (\theta(u)) := (\theta(u))^l / \sqrt{2}$, and similarly for $q = 1$. For all $p \geq 1$ and $q \geq 1$, equations (3.6) then reduce to

$$
\left[ \frac{1}{u^{p-3}} \frac{\partial}{\partial u} \left( u^{p-1} \frac{\partial}{\partial u} \right) - l(l + p - 2) + u^2 v^2 \right] \psi(u, v) = 0 \ ,
$$

(3.8a)

$$
\left[ \frac{1}{v^{q-3}} \frac{\partial}{\partial v} \left( v^{q-1} \frac{\partial}{\partial v} \right) - j(j + q - 2) + u^2 v^2 \right] \psi(u, v) = 0 \ ,
$$

(3.8b)

$$
\left( u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + \frac{p-q}{2} \right) \psi(u, v) = 0 .
$$

(3.8c)
The general solution to \((3.8c)\) is \(\psi(u, v) = u^{(2-p)/2}v^{(2-q)/2} \chi(\zeta)\), where \(\zeta := uv\). Substituting this in \((3.8a)\) and \((3.8b)\), we find that the indices satisfy
\[
2l + p = 2j + q
\]
and \(\chi(\zeta)\) satisfies the Bessel equation of order \(l + (p - 2)/2\) \([43]\).

Equation \((3.9)\) shows that solutions exist only when \(p + q \equiv 0 \pmod{2}\). If \(p = 1\) or \(q = 1\), inspection of \((3.9)\) further shows that solutions exist only when \((p, q) = (1, 1), (1, 3)\) or \((3, 1)\). Let us consider these exceptional cases first.

When \(p = q = 1\), the linearly independent solutions are \(\Psi_\pm := \exp(\pm iu_1v_1)\), \(A_{\text{phys}}^{(s)}\) is generated by the single observable \(\hat{C}_{11}\), and \(\hat{C}_{11} \Psi_\pm = \pm i \Psi_\pm\). The representation of \(A_{\text{phys}}^{(s)}\) on \(V_{\text{phys}} := \text{span}\{\Psi_\pm\}\) is irreducible, but the only sesquilinear forms in which \(\hat{C}_{11}\) is symmetric have indefinite signature.

When \(p = 1\) and \(q = 3\), the only (smooth) solution is \(\Psi_0 := v^{-1} \sin(u_1v)\), which is annihilated by all operators in \(A_{\text{phys}}^{(s)}\). Promoting \(\text{span}\{\Psi_0\}\) into a one-dimensional Hilbert space gives thus a quantum theory in which \(A_{\text{phys}}^{(s)}\) is represented trivially. The situation for \(p = 3\) and \(q = 1\) is similar.

We therefore see that necessary conditions for obtaining a quantum theory with a nontrivial representation of \(A_{\text{phys}}^{(s)}\) are \(p \geq 2\), \(q \geq 2\) and \(p + q \equiv 0 \pmod{2}\). When these conditions hold, we have found for the quantum constraints the linearly independent solutions
\[
\Psi_{ljk_{k,u}} := \delta_{2l+p,2j+q} u^{(2-p)/2}v^{(2-q)/2} J_{l+(p-2)/2}(uv) Y_{lk_u}(\theta^{(u)}) Y_{jk_v}(\theta^{(v)})
\]
where \(J_{l+(p-2)/2}\) is the Bessel function of the first kind \([43]\). The Bessel function of the second kind has been excluded to make \(\Psi_{ljk_{k,u}}\) smooth at \(uv = 0\). The motivation for this exclusion may be debatable within algebraic quantisation, but we shall see that it is precisely the smooth solutions \((3.10)\) that will emerge from group averaging in sections \([44]\) and \([5]\).

To proceed, we would need to examine the representation of \(A_{\text{phys}}^{(s)}\) on \(\text{span}\{\Psi_{ljk_{k,u}}\}\). The representation of the \(\mathfrak{o}(p) \oplus \mathfrak{o}(q)\) subalgebra is given directly by its representation on the spherical harmonics \([43]\) \([44]\), but the observables \(\hat{C}_{ij}\) mix the states in a more complicated way. The special case \(p = q = 2\) was analysed in \([10]\) \([16]\). In subsection \(3.2\) we address the special case \(p = q = 3\).

### 3.2 Completion for \(p = q = 3\)

When \(p = q = 3\), the states \(\Psi_{lmn}\) can be written as
\[
\Psi_{lmn} = j_l(uv) Y_{lm}(\theta^{(u)}) Y_{ln}(\theta^{(v)})
\]
where \(l\) ranges over nonnegative integers, \(j_l(uv)\) is the spherical Bessel function of the first kind of order \(l\) \([43]\), \(m\) and \(n\) are integers satisfying \(|m| \leq l\) and \(|n| \leq l\) and the \(Y\)'s are the usual spherical harmonics on \(S^2\) \([43]\). We write \(V_{\text{phys}} := \text{span}\{\Psi_{lmn}\}\).
We introduce for $A_{\text{phys}}^{(\ast)}$ the basis

\[
\begin{align*}
\hat{L}_3 & := \hat{A}_{12} , \\
\hat{L}_\pm & := \hat{A}_{23} \pm i\hat{A}_{31} , \\
\hat{J}_3 & := \hat{B}_{12} , \\
\hat{J}_\pm & := \hat{B}_{23} \pm i\hat{B}_{31} , \\
\hat{C}_0 & := \hat{C}_{33} , \\
\hat{C}_\pm & := \hat{C}_{31} \pm i\hat{C}_{32} , \\
\hat{C}_{22} & := \hat{C}_{13} \pm i\hat{C}_{23} , \\
\hat{C}_{33} & := \left( \hat{C}_{11} + \hat{C}_{22} \right) \pm i(\hat{C}_{21} - \hat{C}_{12}) , \\
\hat{C}_{44} & := \left( \hat{C}_{11} - \hat{C}_{22} \right) \pm i(\hat{C}_{21} + \hat{C}_{12}).
\end{align*}
\tag{3.12}
\]

Note that the $\hat{L}$'s (respectively $\hat{J}$'s) are a standard raising and lowering operator basis for the $\mathfrak{o}(3)$ algebra in $\mathfrak{u}(v)$ \[43\]. The action of the basis (3.12) on $V_{\text{phys}}$ can be computed from standard properties of the spherical harmonics and spherical Bessel functions \[43, 45\] and is displayed in Table \[1\]. It follows that $V_{\text{phys}}$ is invariant under $A_{\text{phys}}^{(\ast)}$. We show in appendix \[D\] that the representation of $A_{\text{phys}}^{(\ast)}$ on $V_{\text{phys}}$ is irreducible.

The star-relations of the basis (3.12) read

\[
\begin{align*}
(\hat{L}_3)^* & = \hat{L}_3 , \\
(\hat{L}_\pm)^* & = \hat{L}_\mp , \\
(\hat{J}_3)^* & = \hat{J}_3 , \\
(\hat{J}_\pm)^* & = \hat{J}_\mp , \\
(\hat{C}_0)^* & = \hat{C}_0 , \\
(\hat{C}_k)^* & = \hat{C}_k^\mp , & 1 \leq k \leq 4.
\end{align*}
\tag{3.13}
\]

From Table \[1\] it follows by direct computation that these star-relations coincide with the adjoint relations in the inner product

\[
(\Psi_{lmn}, \Psi_{l'm'n'})_{AQ} := (2l + 1)\delta_{ll'}\delta_{mm'}\delta_{nn'}. \tag{3.14}
\]

We show in appendix \[D\] that the only inner products on $V_{\text{phys}}$ with this property are multiples of (3.14).

The physical Hilbert space is the Cauchy completion of $V_{\text{phys}}$ in the inner product (3.14). It carries by construction a densely-defined representation of $A_{\text{phys}}^{(\ast)}$ in which the quadratic $\mathfrak{o}(3, 3)$ Casimir (3.5) has the value $-3$.

4 Refined algebraic quantisation for $p \geq 3$, $q \geq 3$

We now turn to refined algebraic quantisation. In this section we take $p \geq 3$ and $q \geq 3$. The remaining values of $p$ and $q$ will be treated in section \[5\].
\[ \hat{L}_3 \Psi_{lmn} = m \Psi_{lmn} \]
\[ \hat{J}_3 \Psi_{lmn} = n \Psi_{lmn} \]
\[ \hat{L}_\pm \Psi_{lmn} = \sqrt{(l \pm m + 1)(l \mp m)} \Psi_{l,m \pm 1,n} \]
\[ \hat{J}_\pm \Psi_{lmn} = \sqrt{(l \pm n + 1)(l \mp n)} \Psi_{l,m,n \pm 1} \]
\[ \hat{C}_0 \Psi_{lmn} = \frac{\sqrt{(l - m + 1)(l + m + 1)(l - n + 1)(l + n + 1)}}{2l + 3} \Psi_{l+1,m,n} + \frac{\sqrt{(l - m)(l + m)(l - n)(l + n)}}{2l - 1} \Psi_{l-1,m,n} \]
\[ \hat{C}_1 \Psi_{lmn} = \mp \frac{\sqrt{(l - m + 1)(l + m + 1)(l \pm n + 1)(l \pm n + 2)}}{2l + 3} \Psi_{l+1,m,n \pm 1} \pm \frac{\sqrt{(l - m)(l + m)(l \mp n)(l \mp n - 1)}}{2l - 1} \Psi_{l-1,m,n \pm 1} \]
\[ \hat{C}_2 \Psi_{lmn} = \mp \frac{\sqrt{(l \pm m + 1)(l \pm m + 2)(l - n + 1)(l + n + 1)}}{2l + 3} \Psi_{l+1,m,n} \pm \frac{\sqrt{(l \mp m)(l \mp m - 1)(l - n)(l + n)}}{2l - 1} \Psi_{l-1,m,n \pm 1} \]
\[ \hat{C}_3 \Psi_{lmn} = -\frac{\sqrt{(l \pm m + 1)(l \pm m + 2)(l \pm n + 1)(l \mp n + 2)}}{2l + 3} \Psi_{l+1,m,n \mp 1} - \frac{\sqrt{(l \mp m)(l \pm m - 1)(l \pm n)(l \mp n - 1)}}{2l - 1} \Psi_{l-1,m,n \mp 1} \]
\[ \hat{C}_4 \Psi_{lmn} = +\frac{\sqrt{(l \pm m + 1)(l \pm m + 2)(l \pm n + 1)(l \pm n + 2)}}{2l + 3} \Psi_{l+1,m,n} + \frac{\sqrt{(l \pm m)(l \pm m - 1)(l \mp n)(l \mp n - 1)}}{2l - 1} \Psi_{l-1,m,n \pm 1} \]

Table 1: The action of $\mathcal{A}^{(\ast)}_{\text{phys}}$ on $\mathcal{V}_{\text{phys}}$. Whenever the indices of a $\Psi$ on the right-hand side go outside the allowed range, the numerical coefficient vanishes and the term is understood as zero.
We employ refined algebraic quantisation with group averaging as formulated in [8]. A review can be found in [12] and an outline adapted to the present situation in [10].

4.1 Auxiliary Hilbert space and representation of the gauge group

We introduce the auxiliary Hilbert space \( H_{aux} \cong L^2(\mathbb{R}^{p+q}) \) of square integrable functions \( \Psi(u, v) \) in the inner product

\[
(\Psi_1, \Psi_2)_{aux} := \int d^p u \, d^q v \, \overline{\Psi_1} \Psi_2 ,
\]

where the overline denotes complex conjugation. The quantum constraints (3.2) are essentially self-adjoint on \( H_{aux} \), and exponentiating \(-i\) times their algebra yields a unitary representation of the universal covering group of \( \text{SL}(2, \mathbb{R}) \). Denoting this representation by \( U \), the group elements in the Iwasawa decomposition (A.3) are represented by

\[
U(\exp(-i\mu \hat{H}_2)) = \exp(-i\mu \hat{H}_2) , \quad (4.2a)
\]
\[
U(\exp(i\lambda \hat{D})) = \exp(-i\lambda \hat{D}) , \quad (4.2b)
\]
\[
U(\exp[i(\theta(e^+ - e^-))]) = \exp(-i\theta(\hat{H}_1 - \hat{H}_2)) . \quad (4.2c)
\]

The operators in (4.2a) and (4.2b) act as

\[
[\exp(-i\mu \hat{H}_2)](u, v) = \int \frac{d^q v'}{(2\pi i\mu)^{q/2}} \exp \left[ i \left( \frac{(v - v')^2}{\mu} + \mu u^2 \right) \right] \Psi(u, v') , \quad (4.3a)
\]
\[
[\exp(-i\lambda \hat{D})](u, v) = \exp \left[ \frac{\lambda}{2} (q - p) \right] \Psi(e^{-\lambda} u, e^\lambda v) . \quad (4.3b)
\]

In (4.2c) we have \( \hat{H}_1 - \hat{H}_2 = \hat{H}_{u}^{\text{sho}} - \hat{H}_{v}^{\text{sho}} \), where \( \hat{H}_{u}^{\text{sho}} \) and \( \hat{H}_{v}^{\text{sho}} \) are the harmonic oscillator Hamiltonians of unit mass and angular frequency in respectively \( u \) and \( v \). It follows that \( U(\exp[i(\theta(e^+ - e^-))]) \) is periodic in \( \theta \) with period \( 2\pi \) when \( p + q \equiv 0 \) (mod 2) and with period 4\(\pi \) when \( p + q \equiv 1 \) (mod 2). This means that the gauge group is \( \text{SL}(2, \mathbb{R}) \) when \( p + q \equiv 0 \) (mod 2) and the double cover of \( \text{SL}(2, \mathbb{R}) \) when \( p + q \equiv 1 \) (mod 2). \( U \) is isomorphic to the \((p, q)\) oscillator representation of the double cover of \( \text{SL}(2, \mathbb{R}) \) via the Fourier transform in \( v \).

4.2 Test space

The next step is to introduce a linear space of test states in \( H_{aux} \). The harmonic oscillator Hamiltonians in \( U(\exp[i(\theta(e^+ - e^-))]) \) suggest that we make use of the harmonic oscillator eigenstates in \( u \) and \( v \),

\[
\Psi_{ljmnks}(u, v) := u^l v^m e^{-\frac{i}{2}(u^2 + v^2)} L^l_m(u^2) L^m_n(v^2) Y_{lj}(\theta(u)) Y_{mk}(\theta(v)) , \quad (4.4)
\]
where \( l, j, m \) and \( n \) are non-negative integers, \( \tilde{l} \) and \( \tilde{j} \) are defined by
\[
\tilde{l} := l + (p - 2)/2, \quad \tilde{j} := j + (q - 2)/2, \quad (4.5)
\]

\( u := |u|, v := |v| \), the \( L \)'s are the generalised Laguerre polynomials \[15, 16\] and the \( Y \)'s are the spherical harmonics in the notation of section \[3\]. These states satisfy
\[
\hat{H}_{l}^{sho} \Psi_{ljmnk_{e}} = E_{u} \Psi_{ljmnk_{e}} , \quad E_{u} := 2m + l + (p/2) = 2m + \tilde{l} + 1 , \quad (4.6)
\]

and they are orthogonal in \( \mathcal{H}_{\text{aux}} \):
\[
\left( \Psi_{ljmnk_{e}}, \Psi'_{lj'm'n'k'_{e}} \right)_{\text{aux}} = \frac{\Gamma(l + m + (p/2)) \Gamma(j + n + (q/2))}{4 \Gamma(m + 1) \Gamma(n + 1)} \delta_{ll'} \delta_{jj'} \delta_{mm'} \delta_{nn'} \delta_{k_{e}k'_{e}} . \quad (4.7)
\]

We set \( \Phi_{0} := \text{span}\{\Psi_{ljmnk_{e}}\} = \{P(u, v) \exp \left[ -\frac{1}{2}(u^{2} + v^{2}) \right] \} \), where \( P(u, v) \) is an arbitrary polynomial in \( \{u_{i}\} \) and \( \{v_{i}\} \). \( \Phi_{0} \) is clearly dense in \( \mathcal{H}_{\text{aux}} \) and mapped to itself by the quantum constraints \[8\].

Let \( G \) denote the gauge group, and let \( dg \) be the (left and right) invariant Haar measure on \( G \). An \( L^{1} \) function \( h \) on \( G \) defines on \( \mathcal{H}_{\text{aux}} \) the bounded operator \( \hat{h} := \int_{G} dg h(g) U(g) \), and the set of all such operators generates an algebra \( \hat{A}_{G} \). Starting with \( \Phi_{0} \), we first take the closure under the algebra generated by \( \{U(g) \mid g \in G\} \), then take the closure under \( \hat{A}_{G} \), and adopt the resulting space \( \Phi \) as our test space. \( \Phi \) is a dense linear subspace of \( \mathcal{H}_{\text{aux}} \), invariant under both \( \hat{A}_{G} \) and the algebra generated by \( \{U(g) \mid g \in G\} \), and it hence satisfies the test space postulates of \[8\].

### 4.3 Physical Hilbert space

We now construct a rigging map by averaging states in \( \Phi \) over \( G \).

We define on \( \Phi \) the sesquilinear form
\[
(\phi_{2}, \phi_{1})_{ga} := \int_{G} dg \left( \phi_{2}, U(g) \phi_{1} \right)_{\text{aux}} . \quad (4.8)
\]

We show in appendix \[\mathcal{E}\] Theorem \[\mathcal{E.3}\] that the integral in \[\mathcal{E.8}\] is absolutely convergent for all \( \phi_{1}, \phi_{2} \in \Phi \), and \( (\cdot, \cdot)_{ga} \) is hence well defined. We also show that \( (\cdot, \cdot)_{ga} \) vanishes for \( p + q \equiv 1 \) (mod 2). For the rest of this subsection we take \( p + q \equiv 0 \) (mod 2).

Let \( \Phi^{*} \) be the algebraic dual of \( \Phi \) and let \( f[\phi] \) denote the dual action of \( f \in \Phi^{*} \) on \( \phi \in \Phi \). We define the antilinear map \( \eta : \Phi \rightarrow \Phi^{*} \) by
\[
\eta(\phi_{1})[\phi_{2}] := (\phi_{1}, \phi_{2})_{ga} , \quad (4.9)
\]

and we define on the image of \( \eta \) the sesquilinear form \( (\cdot, \cdot)_{\text{RAQ}} \) by
\[
(\eta(\phi_{1}), \eta(\phi_{2}))_{\text{RAQ}} := \eta(\phi_{2})[\phi_{1}] . \quad (4.10)
\]
We need to investigate whether the image of \( \eta \) is nontrivial and whether \((\cdot, \cdot)_{\text{RAQ}}\) is positive definite. If yes, \( \eta \) is a rigging map and the physical Hilbert space is the Cauchy completion of the image of \( \eta \) in \((\cdot, \cdot)_{\text{RAQ}}\).

Note first that if \( \phi_i \in \Phi \) and \( h_i \in L^1(G) \), (4.8) and (4.9) imply

\[
\eta(h_1 \phi_1)[h_2 \phi_2] = \left( \int_G dg h_1(g) \right) \left( \int_G dg h_2(g) \right) \eta(\phi_1)[\phi_2] \tag{4.11}
\]

and \( \eta(\phi_1)[U(g) \phi_2] = \eta(U(g) \phi_1)[\phi_2] = \eta(\phi_1)[\phi_2] \) for all \( g \). Hence it suffices to evaluate \( \eta(\phi_1)[\phi_2] \) for \( \phi_1, \phi_2 \in \Phi_0 \).

By Proposition E.4 in appendix E, Fubini’s theorem implies that we can represent the image of \( \eta \) as functions on \( \mathbb{R}^{p+q} \). The result is (4.13) can be read off from the results in appendix D.1 of [10], by matching our (E.6) to equation (D3) in [10]. The result is

\[
\eta(\Psi_{ljmnk_a k_v}) = 4 \pi^2 (-1)^m \delta_{mn} \frac{\Gamma(l + m + (p/2))}{(2l + p - 2)\Gamma(m + 1)} \Psi_{ljk_a k_v}, \tag{4.14}
\]

where \( \Psi_{ljk_a k_v} \) is as in (3.10). The action of \( \Psi_{ljk_a k_v} \) on \( \Phi_0 \) reads (4.16), p. 244

\[
\Psi_{lj'k'_a k'_v} [\Psi_{ljmnk_a k_v}] = (-1)^m \delta_{2l+p,2j+q} \delta_{l'j'} \delta_{mn} \delta_{k_a k_v} \frac{\Gamma(l + m + (p/2))}{2\Gamma(m + 1)}. \tag{4.15}
\]

Hence the image of \( \eta \) is nontrivial and spanned by \( \{ \Psi_{ljk_a k_v} \} \). From (4.10), (4.14) and (4.13) we find

\[
(\Psi_{lj'm'n'k'_a k'_v}, \Psi_{ljmnk_a k_v})_{\text{RAQ}} = \frac{2l + p - 2}{8\pi^2} \delta_{2l+p,2j+q} \delta_{l'j'} \delta_{k_a k_v} \delta_{k'_a k'_v}. \tag{4.16}
\]

Hence \((\cdot, \cdot)_{\text{RAQ}}\) is positive definite, \( \eta \) is a rigging map, and we have a physical Hilbert space \( \mathcal{H}_{\text{RAQ}} \). The group averaging sesquilinear form on \( \Phi_0 \) reads

\[
(\Psi_{lj'm'n'k'_a k'_v}, \Psi_{ljmnk_a k_v})_{\text{ga}} = 2 \pi^2 (-1)^{m+m'} \delta_{2l+p,2j+q} \delta_{mn} \delta_{m'n'} \delta_{l'j'} \delta_{k_a k_v} \delta_{k'_a k'_v}
\times \frac{\Gamma(l + m' + (p/2)) \Gamma(l + m + (p/2))}{(2l + p - 2)\Gamma(m' + 1)\Gamma(m + 1)} . \tag{4.17}
\]

The uniqueness theorem of [8] shows that every rigging map for our triple \((\mathcal{H}_{\text{aux}}, U, \Phi)\) is a multiple of the group averaging rigging map \( \eta \).
The algebra $A_{\text{phys}}$ is represented on $H_{\text{aux}}$ by (3.4). This representation leaves $\Phi$ invariant and commutes with $U(g)$, and the star-relation in this representation coincides with the adjoint map on $H_{\text{aux}}$. It follows that $H_{\text{RAQ}}$ carries an antilinear representation $\rho$ of $A_{\text{phys}}$, such that the star-relation coincides with the adjoint map on $H_{\text{RAQ}}$. In the notation of (3.4),

$$\rho(\hat{O}_{ij}) : f \mapsto \overline{\hat{O}_{ij}f} .$$

This shows that the algebraic quantisation set up in section 3 yields a quantum theory anti-isomorphically embedded in our group averaging quantum theory whenever $p \geq 3$, $q \geq 3$ and $p+q \equiv 0 \pmod{2}$, even though we were able to complete the algebraic quantisation explicitly only for $p = q = 3$. Apart from $p = q = 3$, we do however not know whether this quantum theory is the only one arising from the algebraic quantisation for $p \geq 3$, $q \geq 3$ and $p+q \equiv 0 \pmod{2}$.

5 Refined algebraic quantisation for $p < 3$ or $q < 3$

In section 4 we assumed $p \geq 3$ and $q \geq 3$. We now discuss refined algebraic quantisation for lower $p$ or $q$. By interchange of $u$ and $v$, it suffices to consider $p \leq q$.

5.1 $p = 1$, $q > 3$

When $p = 1$ and $q > 3$, we define $H_{\text{aux}}$ and $\Phi$ as in section 3. The $u_1$-dependence of the test states (4.1) can be written in terms of Hermite polynomials as $H_{l+2n}(u_1) \exp(-\frac{1}{2}u_1^2)$ (46, p. 240), but the notation in (4.4) covers also $p = 1$, the spherical harmonics on $S^0$ being as described in subsection 3.1. We drop the redundant index $k_u$ and write

$$\phi_{ljmnk}(u_1, v) := \Psi_{ljmn0k} = u^l v^j e^{-\frac{1}{2}(u^2+v^2)}L^l_m(u^2)L^j_n(v^2)Y_l(\theta(u))Y_j(\theta(v)) ,$$

where $l \in \{0, 1\}$ and $\tilde{l} = l - \frac{1}{2}$.

As a preliminary, let $Y_{j0}(\theta(u))$ denote the zonal spherical harmonics, which depend only on $v_q/v$ and are given by Gegenbauer polynomials 43. The recursion relations for the Gegenbauer polynomials and the generalised Laguerre polynomials 46 allow an explicit computation of the action of $\hat{C}_{1q}$ on $\phi_{ljmn0}$. We find

$$\hat{C}_{1q}\phi_{0ljmn0} = -W_{qj}[(n+j)\phi_{1,j-1,m-1,n,0} + (n+1)\phi_{1,j-1,m,n+1,0}] + W_{q,j+1}[(n+j)\phi_{1,j+1,m-1,n-1,0} + \phi_{1,j+1,m,n0}] ,$$

$$\hat{C}_{1q}\phi_{1ljmn0} = W_{qj}[(m+\frac{1}{2})(n+j)\phi_{0,j-1,m,n0} + (m+1)(n+1)\phi_{0,j-1,m+1,n0}] - W_{q,j+1}[(m+\frac{1}{2})\phi_{0,j+1,m,n-1,0} + (m+1)\phi_{0,j+1,m+1,n0}] ,$$

where

$$W_{qj} := 2 \left[ \frac{j(j+q-3)}{(2j+q-2)(2j+q-4)} \right]^{1/2} \text{ for } j > 0 ,$$

$$W_{q0} := 0 ,$$

16
and any \( \phi_{qmn0} \) on the right-hand side with \( m < 0 \) or \( n < 0 \) is understood as zero.

Now, by Theorem \( \text{E.2} \), the group averaging converges in absolute value. When \( q \) is even, the \( \theta \)-dependence in \( (\text{E.7}) \) shows that the image of \( \eta \) is trivial. In the rest of this subsection we take \( q \) odd and show that the image of \( \eta \) is trivial also in this case.

It suffices to show that \( (\phi_{q'm'n'k'}, \phi_{qjmnmk})_{\text{ga}} \) vanishes. When \( l = l' = 1 \), we can proceed as in subsection \( \text{4.3} \) and the result follows from \( (\text{4.17}) \). When \( l = 0 \) or \( l' = 0 \), \( (\text{E.7}) \) shows that it suffices to consider \( (\phi_{0jm'n'o}, \phi_{0jmnm0})_{\text{ga}} \). The \( \theta \)-dependence in \( (\text{E.7}) \) shows that the integral over \( \theta \) gives zero unless \( 2m = 2n + j + (q - 1)/2 \), and a similar observation with \( U(g) \) conjugated to act on the first argument shows that the integral over \( \theta \) gives zero unless \( 2m' = 2n' + j + (q - 1)/2 \). When \( q = 5 + 4a \), \( a = 0, 1, \ldots \), it therefore suffices to consider \( (\phi_{0,2s,n'+a+1,n'}, \phi_{0,2s,n+a+1,n})_{\text{ga}} \), where \( s, n \) and \( n' \) are non-negative integers and we have suppressed the last index of the \( \phi \)'s, understood to take the value zero. When \( q = 3 + 4b \), \( b = 1, 2, \ldots \), it similarly suffices to consider \( (\phi_{0,2s+1,n'+a+1,n'}, \phi_{0,2s+1,n+a+b+1,n})_{\text{ga}} \), where \( s, n \) and \( n' \) are non-negative integers.

Let \( q = 3 + 4b \), \( b = 1, 2, \ldots \). Recall that \( \hat{C}_{1q} \) is self-adjoint in \( \mathcal{H}_{\text{aux}} \) and commutes with \( U(g) \). We compute

\[
W_{q,2s+1} \left[ (n + s + b + \frac{1}{2})(\phi_{0,2s+1,n'+s+b+1,n'}, \phi_{0,2s+1,n+s+b,n-1})_{\text{ga}} + (n + s + b + 1)(\phi_{0,2s+1,n'+s+b+1,n'}, \phi_{0,2s+1,n+s+b+1,n})_{\text{ga}} \right] = - (\phi_{0,2s+1,n'+s+b+1,n'}, \hat{C}_{1q}\phi_{1,2s,n+s+b,n})_{\text{ga}} \]

\[
= -(\hat{C}_{1q}\phi_{0,2s+1,n'+s+b+1,n'}, \phi_{1,2s,n+s+b,n})_{\text{ga}} = 0 ,
\]

where the first equality follows from \( (\text{5.21}) \) and the last from \( (\text{5.2a}) \) and \( (\text{4.17}) \). By induction in \( n \), \( (\text{5.4}) \) implies \( (\phi_{0,2s+1,n'+s+b+1,n'}, \phi_{0,2s+1,n+s+b+1,n})_{\text{ga}} = 0 \).

Let then \( q = 5 + 4a \), \( a = 0, 1, \ldots \). An argument similar to \( (\text{5.3}) \) shows that \( (\phi_{0,2s,n'+a+1,n'}, \phi_{0,2s,n+a+n+1,n})_{\text{ga}} \) vanishes for \( s > 0 \). When \( s = 0 \), we compute

\[
W_{q2} \left[ (n + a + \frac{3}{2})(n + 2a + \frac{3}{2})(\phi_{0,0,n'+a+1,n'}, \phi_{0,0,n+a+1,n})_{\text{ga}} + (n + a + 2)(n + 1)(\phi_{0,0,n'+a+1,n'}, \phi_{0,0,n+a+2,n+1})_{\text{ga}} \right] = (\phi_{0,0,n'+a+1,n'}, \hat{C}_{1q}\phi_{1,1,n+a+1,n})_{\text{ga}} \]

\[
= (\hat{C}_{1q}\phi_{0,0,n'+a+1,n'}, \phi_{1,1,n+a+1,n})_{\text{ga}} = 0 ,
\]

where the last equality follows from \( (\text{5.2a}) \) and \( (\text{4.17}) \). By induction in \( n \), it therefore suffices to consider \( (\phi_{0,0,n'+a+1,n'}, \phi_{0,0,a+1,0})_{\text{ga}} \). A similar argument in \( n' \) shows that it suffices to consider \( (\phi_{0,0,a+1,0}, \phi_{0,0,a+1,0})_{\text{ga}} \).

In \( (\phi_{0,0,a+1,0}, U(g)\phi_{0,0,a+1,0})_{\text{aux}} \), we use \( (\text{E.7}) \) and perform the elementary integration over \( v \). We then integrate over \( G \) in the Haar measure \( dg = \frac{1}{2}dz \mu d\theta \). The integration over \( \theta \) is elementary. Changing the variables in the inner integral from \( u \) to \( y := u^2/z \) and in the outer integral from \( \mu \) to \( t := \mu z/(z + 1) \), we find that \( (\phi_{0,0,a+1,0}, \phi_{0,0,a+1,0})_{\text{ga}} \)
equals a numerical constant times
\[
\int_0^\infty dz \frac{z^{\alpha+\frac{1}{2}}}{(z+1)^{2\alpha+\frac{3}{2}}} \int_{-\infty}^\infty \frac{dt}{(1 + it)^{2\alpha+\frac{3}{2}}} \times
\]
\[
\int_0^\infty dy \ y^{-\frac{1}{2}} L_{\alpha+1}^{-\frac{1}{2}}(zy) L_{\alpha+1}^{-\frac{1}{2}}(y) \exp\left[-\frac{1}{2}(z + 1)(1 - it)y\right]. \tag{5.6}
\]

We interchange the order of the \(dt\) and \(dy\) integrals in (5.6), justified by the absolute convergence of the double integral, and perform the \(dt\) integral as a contour integral, finding that (5.6) equals a numerical constant times
\[
\int_0^\infty dz \ z^\alpha \int_0^\infty dy \ y^{2\alpha+1} L_{\alpha+1}^{-\frac{1}{2}}(zy) L_{\alpha+1}^{-\frac{1}{2}}(y) \exp\left[-(z + 1)y\right]. \tag{5.7}
\]

In (5.7) we interchange the order of the \(dz\) and \(dy\) integrals, justified by the absolute convergence of the double integral. Changing the variable in the new inner integral from \(z\) to \(x := zy\), we obtain
\[
\int_0^\infty dy \ y^{-\frac{1}{2}} L_{\alpha+1}^{-\frac{1}{2}}(y) \exp\left[-y\right] \int_0^\infty dx \ x^{\alpha+\frac{1}{2}} L_{\alpha+1}^{-\frac{1}{2}}(x) \exp\left[-x\right]. \tag{5.8}
\]

The integrals in (5.8) have factorised, and the integral over \(y\) vanishes by the orthogonality of the generalised Laguerre polynomials [46].

### 5.2 \(p = 1, q = 3\)

When \(p = 1\) and \(q = 3\), we define \(\mathcal{H}_{\text{aux}}\) as in section 4. With \(\Phi\) defined as in section 4, the integral in (5.3) is not absolutely convergent for \(l = j = 0\), and we have not found a weaker unambiguous sense of convergence. The \(\theta\)-dependence in (5.7) however suggests that if group averaging can be made well-defined, it should annihilate states with \(l = j = 0\). We shall achieve this by suitably modifying the test space.

Dropping the redundant index \(k\), we introduce the states \(\phi_{ljmnk}\) by (5.1) with \(q = 3\). We define \(\Phi_{0}^{\text{mod}} := \text{span}\left\{ \phi_{ljmnk} \mid l + j > 0 \right\} \cup \{\psi_{mn}\}\), where
\[
\psi_{mn} := \frac{2}{\sqrt{3}} \left[(m + \frac{1}{2})(n + \frac{3}{2})\phi_{00mn0} + (m + 1)(n + 1)\phi_{00,m+1,n+1,0}\right]. \tag{5.9}
\]

Using the basis (3.4) of \(\mathcal{A}_{\text{phys}}^{(\ast)}\), properties of the spherical harmonics on \(S^2\) [43] [45] and properties of the generalised Laguerre polynomials [46], it can be verified that \(\Phi_{0}^{\text{mod}}\) is invariant under \(\mathcal{A}_{\text{phys}}^{(\ast)}\). In particular, formulas (5.2) and (5.3) hold with \(q = 3\), implying
\[
\hat{C}_{13}\phi_{11mn0} = \psi_{mn} - \frac{1}{\sqrt{3}} \left[(m + \frac{1}{2})\phi_{02,m,n-1,0} + (m + 1)\phi_{02,m+1,n,0}\right], \tag{5.10a}
\]
\[
\hat{C}_{13}\psi_{mn} = \frac{4}{3} (m + \frac{1}{2})(n + \frac{3}{2}) (\phi_{11,m-1,n-1,0} + \phi_{11mn0}) + \frac{4}{3} (m + 1)(n + 1) (\phi_{11mn0} + \phi_{11,m+1,n+1,0}). \tag{5.10b}
\]
We claim that $\Phi^\text{mod}$ is dense in $H^\text{aux}$. If this were not the case, there would exist a nonzero vector $y = \sum_{ljmnk} a_{ljmnk} \phi_{ljmnk} \in H^\text{aux}$ that is orthogonal to all vectors in $\Phi^\text{mod}$. By \eqref{4.7}, orthogonality with each $\phi_{ljmnk}$ with $l + j > 0$ implies $a_{ljmnk} = 0$ for $l + j > 0$. By \eqref{4.1} and \eqref{5.9}, orthogonality with each $\psi_{mn}$ implies $a_{00mn0} + a_{00,m+1,n+1,0} = 0$, from which \eqref{4.7} further shows that $y$ has finite norm only if $y$ is the zero vector. Hence $\Phi^\text{mod}$ is dense in $H^\text{aux}$.

Following section 4 with $\Phi_0$ replaced by $\Phi^\text{mod}$, we first take the closure of $\Phi^\text{mod}$ under the algebra generated by $\{ U(g) \mid g \in G \}$, then take the closure under $A_G$, and adopt the resulting space $\Phi^\text{mod}$ as our test space. $\Phi^\text{mod}$ is a dense linear subspace of $H^\text{aux}$, invariant under $A^{(\ast)}_{\text{phys}}, A_G$ and the algebra generated by $\{ U(g) \mid g \in G \}$, and satisfies hence the test space postulates of \S. We show in appendix E, Theorem E.5, that the integral in \eqref{4.8} converges in absolute value for all $\phi_1, \phi_2 \in \Phi^\text{mod}$.

To evaluate $(\phi_2, \phi_1)_{ga}$ on $\Phi^\text{mod}$, it suffices to consider $\phi_1, \phi_2 \in \Phi^\text{mod}$. When both $\phi_1$ and $\phi_2$ have $l = 1$, we can proceed as in subsection 4.3, arriving at \eqref{4.12}–\eqref{4.17}. When $\phi_1$ and $\phi_2$ have differing values of $l, j$ or $k$, \eqref{E.7} shows that $(\phi_2, \phi_1)_{ga}$ vanishes. What remains is $(\phi_{0jm'n'k}, \phi_{0ljmnk})_{ga}$ with $j > 0$ and $(\psi_{m'n'}, \psi_{mn})_{ga}$. The vanishing of the former follows as in subsection 5.1 noting that \eqref{5.4} holds also for $b = 0$. For the latter, we use \eqref{5.10a}, the self-adjointness of $\hat{C}_{13}$ on $H^\text{aux}$ and the vanishing of $(\psi_{m'n'}, \phi_{0jmnn})_{ga}$ for $j > 0$ and compute
\begin{equation}
(\psi_{m'n'}, \psi_{mn})_{ga} = (\psi_{m'n'}, \hat{C}_{13}\phi_{11mn0})_{ga} = (\hat{C}_{13}\psi_{m'n'}, \phi_{11mn0})_{ga} = 0,
\end{equation}
where the last equality follows from \eqref{5.10b} and \eqref{4.17}.

The evaluation of $(\phi_2, \phi_1)_{ga}$ is now complete. The only nonzero contribution comes from states with $l = 1$, in which case formulas \eqref{4.12}–\eqref{4.17} hold. The image of $\eta$ is one-dimensional, spanned by $\{ \bar{\Psi}_0 \}$, where $\Psi_0$ is the state \eqref{3.10} with $l = 1$ and $j = 0$ and reads explicitly \eqref{4.13}, Section 7.11 $\Psi_0 = v^{-1} \sin(u_1 v)$. The inner product \eqref{4.16} is positive definite, and we obtain a one-dimensional physical Hilbert space $H_{\text{RAQ}}$.

As $\Phi^\text{mod}$ is invariant under $A_{\text{phys}}^{(\ast)}$, $H_{\text{RAQ}}$ carries an antilinear representation of $A_{\text{phys}}^{(\ast)}$. A direct calculation shows that all operators in this representation annihilate $\overline{\Psi}_0$, and the representation is trivial. The quantum theory found in algebraic quantisation in subsection 5.1 is thus anti-isomorphically embedded in the group averaging quantum theory.

### 5.3 $p = 1, q = 2$

When $p = 1$ and $q = 2$, and $H^\text{aux}$ and $\Phi$ are as in section 4, the integral in \eqref{4.8} is not absolutely convergent for $l = j = 0$. It may be possible to modify the $l = j = 0$ sector of $\Phi$ as in subsection 5.2 above, but as now $p + q = 1 \mod 2$, any test space built from linear combinations of the harmonic oscillator eigenfunctions will give an $\eta$ with trivial image.
5.4 \( p = q = 1 \)

When \( p = q = 1 \), and \( \mathcal{H}_{\text{aux}} \) and \( \Phi \) are as in section 4, the integral in (4.8) is not absolutely convergent for \( l = 0 \) or \( j = 0 \) and is unambiguously divergent for example for \( \phi_1 = \phi_2 = \Psi_{0000} \).

We attempt to cure the divergence by modifying the zero angular momentum sector. For technical simplicity, we choose at the outset to work with states that are symmetric under \( (u_1, v_1) \mapsto (-u_1, -v_1) \).

Let \( \mathcal{H}_{\text{aux}}^s \subset \mathcal{H}_{\text{aux}} \) be the Hilbert subspace of vectors symmetric under \( (u_1, v_1) \mapsto (-u_1, -v_1) \). Dropping the redundant indices \( k_u \) and \( k_v \), we write

\[
\Phi_{lmn}(u_1, v_1) := \Psi_{lmn00} = (u^l v^l e^{-\frac{1}{2}(w^2 + v^2)} L^l_m(u^2)L^l_n(v^2)Y_l(\theta(u))Y_l(\theta(v))) ,
\]

where \( l \in \{0, 1\} \) and \( \bar{l} = l - \frac{1}{2} \). \{\( \Phi_{lmn} \)\} is clearly an orthogonal basis for \( \mathcal{H}_{\text{aux}}^s \).

Let \( \Phi^s_0 := \text{span} \{\psi_{mn}, \phi_{1mn}\} \), where

\[
\psi_{mn} := 2\left[ (m + \frac{1}{2})(n + \frac{1}{2})\phi_{0mn} + (m + 1)(n + 1)\phi_{0,m+1,n+1} \right] .
\]

We then find (46, p. 241)

\[
\hat{C}_{11}\psi_{mn} = 4\left[ (m + \frac{1}{2})(n + \frac{1}{2})(\phi_{1mn} + \phi_{1,m-1,n-1}) + (m + 1)(n + 1)(\phi_{1mn} + \phi_{1,m+1,n+1}) \right] ,
\]

\[
\hat{C}_{11}\phi_{1mn} = \psi_{mn} ,
\]

where \( \hat{C}_{11} \) is the single generator of \( \mathcal{A}^{(s)} \). Hence \( \Phi^s_0 \) is invariant under \( \mathcal{A}^{(s)} \) and it can be shown as in subsection 5.2 that \( \Phi^s_0 \) is dense in \( \mathcal{H}_{\text{aux}}^s \). We build from \( \Phi^s_0 \) a test space \( \Phi^s \) satisfying the postulates of 3 as in subsection 5.2. The integral in (4.8) then converges in absolute value for all \( \phi_1, \phi_2 \in \Phi^s \). The proof is a verbatim adaptation of that of Theorem 3.5.

We need to evaluate \( (\phi_2, \phi_1)_{\text{ga}} \) on \( \Phi^s \). It suffices to consider \( \phi_1, \phi_2 \in \Phi^s_0 \). Clearly \( (\psi_{mn'}, \phi_{1mn})_{\text{ga}} = 0 \). For \( (\phi_{1mn'}, \phi_{1mn})_{\text{ga}} \) we proceed as in subsection 4.3 and arrive at (4.12)–(4.17), the last of which reads

\[
(\phi_{1mn'}, \phi_{1mn})_{\text{ga}} = 2\pi^2(-1)^{m+m'}\delta_{mm'}\delta_{nn'} \frac{\Gamma(m + \frac{3}{2})\Gamma(m' + \frac{3}{2})}{\Gamma(m + 1)\Gamma(m' + 1)} .
\]

To find \( (\psi_{mn'}, \psi_{mn})_{\text{ga}} \), we use the self-adjointness of \( \hat{C}_{11} \) on \( \mathcal{H}_{\text{aux}}^s \) and compute

\[
(\psi_{mn'}, \psi_{mn})_{\text{ga}} = (\psi_{mn'}, \hat{C}_{11}\phi_{1mn})_{\text{ga}} = (\hat{C}_{11}\psi_{mn'}, \phi_{1mn})_{\text{ga}} = -2\pi^2(-1)^{m+m'}\delta_{mm'}\delta_{nn'} \frac{\Gamma(m + \frac{3}{2})\Gamma(m' + \frac{3}{2})}{\Gamma(m + 1)\Gamma(m' + 1)} ,
\]

where the first equality follows from (5.14b) and the last one from (5.14a) and (5.15).
We see that \((\cdot, \cdot)_{ga}\) is an indefinite sesquilinear form. Hence the map \(\eta\) defined by (4.9) is not a rigging map and we do not recover a Hilbert space. The indefiniteness of \((\cdot, \cdot)_{ga}\) further implies, by the uniqueness theorem of [8], that the triple \((\mathcal{H}^{s}_{aux}, U, \Phi^{s})\) admits no rigging maps.

The image of \(\eta\) is two-dimensional, spanned by \(\{\Psi_{00}, \Psi_{11}\}\), where \(\Psi_{00}\) and \(\Psi_{11}\) are given by (3.10) and read explicitly (43, Section 7.11)

\[
\Psi_{00} = \frac{1}{\sqrt{2\pi}} \cos(u_1 v_1), \quad (5.17a)
\]
\[
\Psi_{11} = \frac{1}{\sqrt{2\pi}} \sin(u_1 v_1). \quad (5.17b)
\]

The manifestly indefinite sesquilinear form (4.10) on the image of \(\eta\) is given by (4.16). The representation of \(A^{(s)}_{phys}\) induced on the image of \(\eta\) by (4.12) is anti-isomorphic to the representation obtained in subsection 3.1 on the solution space to the algebraic quantisation constraints.

5.5 \(p = 2, q > 2\)

When \(p = 2\) and \(q > 2\), we define \(\mathcal{H}_{aux}\) and \(\Phi\) as in section 4. Theorem E.3 in appendix E shows that the group averaging converges in absolute value.

When \(q\) is odd, the \(\theta\)-dependence in (E.7) shows that the image of \(\eta\) is trivial. Suppose then that \(q\) is even. When \(l > 0\) and \(l' > 0\), we arrive at equations (4.12)–(4.17) as in subsection 4.3. When \(l = 0\) or \(l' = 0\), it can be shown that \((\Psi_{ijmnk_{u}k_{v}}, \Psi_{ijmnk_{u}k_{v}})_{ga}\) vanishes: The arguments follow those in subsection 5.1 so closely that we will not spell them out here. This means that equations (4.12)–(4.17) hold for all values of the indices in the sense that terms involving \(\delta_{2l+1,2j+q}\) for \(l = 0\) are understood to vanish. Hence the situation is similar to that for \(p \geq 3\), \(q \geq 3\) and \(p+q \equiv 0 (\text{mod} 2)\) in section 4. The image of \(\eta\) is nontrivial, \((\cdot, \cdot)_{RAQ}\) is positive definite, \(\eta\) is a rigging map, and the representation of \(A^{(s)}_{phys}\) on the physical Hilbert space is as described at the end of subsection 4.3.

5.6 \(p = q = 2\)

The case \(p = q = 2\) was analysed in [10]. Group averaging does not converge on the test space of section 4 but the \(l = j = 0\) sector of the test space can be modified so that group averaging converges and the physical observable algebra includes \(A^{(s)}_{phys}\). The physical Hilbert space decomposes into a direct sum of four Hilbert subspaces, each of them carrying a distinct representation of \(A^{(s)}_{phys}\).
6 Discussion

We have discussed the quantisation of a constrained system with unreduced phase space \( \mathbb{R}^{2(p+q)} \), classical gauge group \( \text{SL}(2, \mathbb{R}) \) and a distinguished \( \mathfrak{o}(p,q) \) algebra of classical observables. We employed refined algebraic quantisation, using group averaging on an auxiliary Hilbert space to find the inner product on the physical Hilbert space. We took care to select the quantisation input so that when a quantum theory is recovered, the classical \( \mathfrak{o}(p,q) \) algebra gets promoted into an operator algebra represented on the physical Hilbert space.

When \( p \geq 2, q \geq 2, p + q > 4 \) and \( p + q \equiv 0 \pmod{2} \), we found a quantum theory with a nontrivial representation of the \( \mathfrak{o}(p,q) \) observables. For \( p = q = 2 \), a similar result was obtained in [10]. For \( (p, q) = (1, 3) \) or \( (3, 1) \), we found a quantum theory with a one-dimensional Hilbert space and a trivial representation of the \( \mathfrak{o}(p,q) \) observables. For other values of \( p \) and \( q \) we found no quantum theory.

We also discussed Ashtekar’s algebraic quantisation, solving first the quantum constraints without an inner product and then promoting the classical \( \mathfrak{o}(p,q) \) algebra into operators whose star-relations determine the physical inner product. For all values of \( p \) and \( q \) for which group averaging gave a quantum theory, algebraic quantisation gave a quantum theory that is (anti-)isomorphically embedded in the group averaging theory. For \( p = q = 3 \), we showed that this algebraic quantisation theory is unique.

With both algebraic quantisation and group averaging, qualitative changes emerged depending on whether \( p \) and \( q \) are less than, equal to, or greater than 2. This could be expected from the properties of the classical reduced phase space: The reduced phase space contains a symplectic manifold when and only when \( \min(p,q) \geq 2 \), and this symplectic manifold is connected when and only when \( \min(p,q) \geq 3 \). However, a phenomenon not expected on classical grounds was that neither algebraic quantisation nor group averaging gave a quantum theory for \( p + q \equiv 1 \pmod{2} \). The technical reason was that both quantisation schemes represented the \( \mathfrak{o}(p) \oplus \mathfrak{o}(q) \) subalgebra of \( \mathfrak{o}(p,q) \) by integer-valued rather than half-integer-valued angular momenta. Obtaining quantum theories for \( p + q \equiv 1 \pmod{2} \) by some ‘fermionic’ modification might be an interesting challenge.

For \( p = q = 1 \), both algebraic quantisation and group averaging failed to give a quantum theory, for closely related reasons. Algebraic quantisation led to a two-dimensional vector space of solutions to the constraints, but requiring the \( \mathfrak{o}(1,1) \) generator to be symmetric forced the sesquilinear form on this vector space to be indefinite. In group averaging, a judicious choice of the test space ensured convergence of the averaging and the inclusion of the \( \mathfrak{o}(1,1) \) generator in the would-be physical observable algebra, but the outcome was the same indefinite sesquilinear form on the same two-dimensional vector space as in algebraic quantisation. It is not clear whether the case \( p = q = 1 \) has physical interest, especially as the reduced phase space consists of just three points, non-Hausdorff close to each other, but from the mathematical point of view this provides the first example known to us where group averaging fails to produce a Hilbert space owing to indefiniteness of the would-be inner product. As the uniqueness theorem of [8]...
does not assume positive definiteness, the theorem is applicable here and implies that our test space admits no rigging maps.

We assumed throughout \( p \geq 1 \) and \( q \geq 1 \). If either \( p \) or \( q \) vanishes, the action (2.1) still defines a classical theory, but the reduced phase space then consists of a single point. Algebraic quantisation in the representation of section 3 gives no solutions to the constraints, and when group averaging based on the harmonic oscillator eigenstates converges, it gives an identically-vanishing sesquilinear form owing to the \( \theta \)-dependence in \( U(g)\Psi \) (E.7).

Finally, one would like to characterise the representations of \( \mathfrak{o}(p, q) \) on our physical Hilbert spaces in terms of invariants [42], as done in [10, 16, 21] for \( p = q = 2 \). The value of the quadratic Casimir operator can be read off from (3.5). As our representation of the gauge group on the auxiliary Hilbert space is isomorphic to the oscillator representation of \( \text{SL}(2, \mathbb{R}) \) [38], the joint representation theory of the dual pair \((\text{O}(p, q), \text{SL}(2, \mathbb{R}))\) [47, 48] may be useful with this question.

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**A Appendix: SL(2, \mathbb{R})**

In this appendix we collect some relevant properties of \( \text{SL}(2, \mathbb{R}) \). The notation follows [38].

\( \text{SL}(2, \mathbb{R}) \) consists of real \( 2 \times 2 \) matrices with unit determinant. The Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \) is spanned by the matrices

\[
\begin{align*}
  h &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & e^+ &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & e^- &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

(A.1)

whose commutators are

\[
\begin{align*}
  [h, e^+] & = 2e^+, \\
  [h, e^-] & = -2e^-, \\
  [e^+, e^-] & = h.
\end{align*}
\]

(A.2)

Elements of \( \text{SL}(2, \mathbb{R}) \) have the unique Iwasawa decomposition

\[
g = \exp(\mu e^-) \exp(\lambda h) \exp[\theta(e^+ - e^-)] ,
\]

(A.3)
or explicitely
\[ g = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \] (A.4)
where \( \mu \in \mathbb{R}, \lambda \in \mathbb{R} \) and \( 0 \leq \theta < 2\pi \). The unique Iwasawa decomposition of the universal covering group of \( SL(2, \mathbb{R}) \) is given by (A.3) with \( -\infty < \theta < \infty \), and that of the double cover by \( 0 \leq \theta < 4\pi \). The left and right invariant Haar measure reads \( dg = e^{2\lambda} d\lambda d\mu d\theta \).

### B Appendix: Linear independence of the constraints

In this appendix we show that the gradients of the constraints are all vanishing on \( \Gamma_0 \), linearly dependent but not all vanishing on \( \Gamma_{\text{ex}} \), and linearly independent on \( \Gamma_{\text{reg}} \).

From (2.3), the gradients of the constraints read
\[
\begin{align*}
    dH_1 &= \sum_i (p_i dp_i - v_i dv_i), \\
    dH_2 &= \sum_i (\pi_i d\pi_i - u_i du_i), \\
    dD &= \sum_i (u_i dp_i + p_i du_i - \pi_i dv_i - v_i d\pi_i). \quad (B.1)
\end{align*}
\]
For \( \alpha, \beta, \gamma \in \mathbb{R} \), the equation \( \alpha dH_1 + \beta dH_2 + \gamma dD = 0 \) is equivalent to
\[
\begin{align*}
    \gamma u + \alpha p &= 0 = -\beta u + \gamma p, \\
    \gamma \pi + \alpha v &= 0 = -\beta \pi + \gamma v. \quad (B.2)
\end{align*}
\]
\( \Gamma_0 \) is clearly the set where the gradients of all the constraints vanish.

On \( \Gamma_{\text{ex}} \), we saw in section 2 that each point can be brought to the form (2.11) by a gauge transformation (2.5) with some \( g \in SL(2, \mathbb{R}) \). Given such a \( g \), (B.2) is satisfied by \( \alpha = (g_{12})^2, \beta = -(g_{11})^2 \) and \( \gamma = g_{11}g_{12} \), where at least one of \( \alpha \) and \( \beta \) must be nonvanishing since \( \det(g) \neq 0 \). Hence the gradients of the constraints are linearly dependent on \( \Gamma_{\text{ex}} \).

On \( \Gamma_{\text{reg}} \), the pair \( (u, p) \) (as well as the pair \( (v, \pi) \)) is linearly independent, and (B.2) implies \( \alpha = \beta = \gamma = 0 \). Hence the gradients of the constraints are linearly independent on \( \Gamma_{\text{reg}} \).

### C Appendix: Separation of \( M_{\text{reg}} \) by \( A_{\text{class}} \)

In this appendix we show that the classical observable algebra \( A_{\text{class}} \) separates \( M_{\text{reg}} \). We assume \( p \geq 2 \) and \( q \geq 2 \), which is necessary and sufficient for \( M_{\text{reg}} \) to be nonempty. The case \( p = q = 2 \) was treated in [10, 16, 21].
Let \( \mathcal{M}_i, 1 \leq i \leq p, \) be the subset of \( \mathcal{M}_{\text{reg}} \) whose points have a representative in \( \Gamma_{\text{reg}} \) satisfying the gauge conditions (2.13) with \( u_i^2 + p_i^2 > 0 \). It follows that \( \mathcal{M}_i \cap \mathcal{M}_j \neq \emptyset \) for all \( i \) and \( j \) and

\[
\mathcal{M}_{\text{reg}} = \bigcup_{i=1}^{p} \mathcal{M}_i .
\]

Lemma C.1 Let \( \tilde{a} \in \mathcal{M}_{\text{reg}} \) and \( \tilde{b} \in \mathcal{M}_{\text{reg}} \) such that there is no \( \mathcal{M}_i \) containing both \( \tilde{a} \) and \( \tilde{b} \). Then \( \mathcal{A}_{\text{class}} \) separates \( \tilde{a} \) and \( \tilde{b} \).

Proof. Let \( a = (u, p, v, \pi) \in \Gamma_{\text{reg}} \) and \( b = (u', p', v', \pi') \in \Gamma_{\text{reg}} \) be representatives of respectively \( \tilde{a} \) and \( \tilde{b} \), each satisfying (2.13). As the pair \( (u, p) \) is linearly independent, there exist \( i \neq j \) such that \( u_ip_j - u_jp_i \neq 0 \). It follows that \( \tilde{a} \in \mathcal{M}_i \cap \mathcal{M}_j \). By assumption then \( \tilde{b} \notin \mathcal{M}_i \cup \mathcal{M}_j \), which implies \( u_i' = u'_j = u_j' = p_j' = 0 \). Hence \( A_{ij}(\tilde{a}) = u_ip_j - u_jp_i \neq 0 \) but \( A_{ij}(\tilde{b}) = u'_i p'_j - u'_j p'_i = 0 \), which shows that the observable \( A_{ij} \) (2.9) separates \( \tilde{a} \) and \( \tilde{b} \).

Remark. Repeating the proof with \( \tilde{a} \) and \( \tilde{b} \) interchanged shows that points satisfying the conditions of Lemma C.1 exist only for \( p \geq 4 \).

Theorem C.1 \( \mathcal{A}_{\text{class}} \) separates \( \mathcal{M}_{\text{reg}} \).

Proof. By Lemma C.1 it suffices to consider individually each \( \mathcal{M}_k \).

From now on let \( \mathcal{M}_k \) be fixed. We saw in subsubsection 2.3.3 that \( \mathcal{M}_{\text{reg}} \) can be represented as the quotient of the set (2.13) under the \( U(1) \) action given by (2.5) with (2.14). Within \( \mathcal{M}_k \), each \( U(1) \) equivalence class in (2.13) has a unique representative that satisfies \( p_k = 0 \) and \( u_k > 0 \). Performing on this representative a gauge transformation (2.5) with \( g = \text{diag}(u_k^{-1}, u_k) \), we obtain a point in \( \Gamma \) satisfying

\[
\begin{align*}
  u^2 &= \pi^2 > 0 , \quad p^2 = v^2 > 0 , \\
  u \cdot p &= v \cdot \pi = 0 , \\
  p_k &= 0 , \quad u_k = 1 .
\end{align*}
\]

It follows that \( \mathcal{M}_k \) can be represented as the subset of \( \Gamma \) satisfying (C.2).

Let now \( \tilde{a}, \tilde{b} \in \mathcal{M}_k \) such that \( A(\tilde{a}) = A(\tilde{b}) \) for all \( A \in \mathcal{A}_{\text{class}} \). Let \( a = (u, p, v, \pi) \) and \( b = (u', p', v', \pi') \) be the respective representatives of \( \tilde{a} \) and \( \tilde{b} \) in the gauge (C.2). We shall show that \( a = b \). We use the basis (2.9) of \( \mathcal{A}_{\text{class}} \).

Consider the observables \( A_{ij} \). From \( A_{ij}(\tilde{a}) = A_{ij}(\tilde{b}) \) we obtain

\[
u_i p_j - u_j p_i = u_i' p_j' - u_j' p_i' ,
\]

where \( 1 \leq i \leq p \) and \( 1 \leq j \leq p \). With \( i = k \) and \( j \neq k \), the gauge conditions (C.2) show that (C.3) reduces to \( p_j = p_j' \). The gauge conditions (C.2) imply directly that \( p_k = p_k' \). Hence \( p = p' \).
Multiplying \((C.3)\) by \(p_j\) and summing over \(j\) gives
\[
p^2 u_i - (u \cdot p)p_i = (p \cdot p')u'_i - (u' \cdot p)p'_i.
\] (C.4)

Using \(p = p'\) and \((C.2), (C.3)\) reduces to \(u_i = u'_i\). Hence \(u = u'\).

Consider then the observables \(C_{ij}\). From \(C_{ij}(\hat{a}) = C_{ij}(\hat{b})\) we obtain
\[
u_i v_j - p_i \pi_j = u'_i v'_j - p'_i \pi'_j,
\] (C.5)
where \(1 \leq i \leq p\) and \(1 \leq j \leq q\). With \(i = k\), \((C.2)\) shows that \((C.5)\) reduces to \(v_j = v'_j\).

Hence \(v = v'\).

Substituting \(u = u', p = p'\) and \(v = v'\) in \((C.5)\) gives \(p_i(\pi_j - \pi'_j) = 0\). As \(p^2 > 0\), this implies \(\pi_j = \pi'_j\). Hence \(\pi = \pi'\). ☐

D Appendix: \(A^{(\ast)}_{\text{phys}}\) on \(V_{\text{phys}}\) for \(p = q = 3\)

In this appendix we analyse the representation of \(A^{(\ast)}_{\text{phys}}\) on \(V_{\text{phys}}\) for \(p = q = 3\), displayed in Table I. We show first that this representation is irreducible. We then show that the only inner products in which the star-relations \((3.13)\) become adjoint relations are multiples of \((3.14)\).

**Proposition D.1** Let \(U \subset V_{\text{phys}}\) be a linear subspace invariant under \(A^{(\ast)}_{\text{phys}}\), \(U \neq \{0\}\). Then \(U = V_{\text{phys}}\).

**Proof.** Recall that the operator \(\hat{L}^2 := \hat{L}_0^2 + \frac{1}{2}(\hat{L}_+ \hat{L}_+ - \hat{L}_- \hat{L}_-)\) satisfies \(\hat{L}^2 \Psi_{lmn} = l(l+1) \Psi_{lmn}\). Let \(u \in U\), \(u \neq 0\). Then \(u = \sum a_{lmn} \Psi_{lmn}\), where only finitely many \(a_{lmn}\) are nonzero. Let \(l_0\) be the largest \(l\) for which some \(a_{lmn}\) is nonzero. Then \(u^{(1)} := \prod_{l < l_0} [\hat{L}^2 - l(l+1)] u = k \sum_{mn} a_{lmn} \Psi_{lmn}\), where \(k \neq 0\). Acting on \(u^{(1)}\) finitely many times with \(\hat{L}_+\) and \(\hat{J}_+\) gives the vector \(u^{(2)} = a^{(2)} \Psi_{l_0 l_0} \neq 0\), and \(u^{(3)} := (\hat{L}_-)^{l_0} (\hat{J}_-)^{l_0} u^{(2)} = a^{(3)} \Psi_{l_00} \neq 0\). Hence \(\Psi_{l_00} \in U\).

A direct computation from Table I shows that \(\hat{J}_- \hat{C}_{l_1}^+ \Psi_{l_00} - (l - 1) \hat{C}_0 \Psi_{l_00}\) is a nonzero multiple of \(\Psi_{l+1,00}\) for all \(l\) and \(\hat{J}_- \hat{C}_{l_1}^+ \Psi_{l_00} + (l + 1) \hat{C}_0 \Psi_{l_00}\) is a nonzero multiple of \(\Psi_{l-1,00}\) for \(l > 0\). It follows by induction that \(\Psi_{l_00} \in U\) for all \(l\). Acting on \(\Psi_{l_00}\) with \(\hat{L}_\pm\) and \(\hat{J}_\pm\) shows that \(\Psi_{lmn} \in U\) for all values of the indices. ☐

**Proposition D.2** Let \((\cdot, \cdot)\) be an inner product in which the star-relations \((3.13)\) become adjoint relations. Then \((\Psi_{lmn}, \Psi_{l'm'n'}) = r(2l+1)\delta_{w} \delta_{mn'} \delta_{nn'},\) where \(r\) is a positive constant.

**Proof.** The adjointness relations imply that the operator \(\hat{L}^2\) introduced in the proof of Proposition D.1 is self-adjoint. Hence \((l'+1)(\Psi_{lmn}, \Psi_{l'm'n'}) = (\Psi_{lmn}, \hat{L}^2 \Psi_{l'm'n'}) = (\hat{L}^2 \Psi_{lmn}, \Psi_{l'm'n'}) = l(l+1)(\Psi_{lmn}, \Psi_{l'm'n'})\), which shows that \((\Psi_{lmn}, \Psi_{l'm'n'})\) vanishes for
\[ l \neq l'. \] By standard angular momentum techniques in the \( \mathfrak{o}(3) \) subalgebras generated respectively by the \( \hat{L} \)’s and the \( \hat{J} \)’s (see for example [49]), we then find

\[ (\Psi_{lmn}, \Psi_{l'm'n'}) = A_l \delta_{ll'} \delta_{mm'} \delta_{nn'}, \tag{D.1} \]

where \( A_l \) depends only on \( l \).

To determine \( A_l \), we use the self-adjointness of \( \hat{C}_0 \). Writing \( \Psi_l := \Psi_{l00} \) and using the action of \( \hat{C}_0 \) from Table I and (D.1), we compute

\[ \frac{(l + 1)^2}{2l + 1} A_l = \frac{(l + 1)^2}{2l + 1} (\Psi_l, \Psi_l) = (\Psi_l, \hat{C}_0 \Psi_{l+1}) = (\hat{C}_0 \Psi_l, \Psi_{l+1}) = \frac{(l + 1)^2}{2l + 3} (\Psi_{l+1}, \Psi_{l+1}) = \frac{(l + 1)^2}{2l + 3} A_{l+1}, \tag{D.2} \]

from which by induction \( A_l = (2l + 1)A_0 \). ■

### E Appendix: Convergence of the group averaging

In this appendix we provide the group averaging convergence results needed in the main text. When not mentioned otherwise, \( p \) and \( q \) are arbitrary positive integers.

To begin, consider \( U(g) \Psi_{ljmnk_u v_k} \). Writing \( g \) in the Iwasawa decomposition (A.3), (4.2) gives

\[ U(g) = \exp(-i\mu \hat{H}_2) \exp(-i\lambda \hat{D}) \exp(-i\theta(\hat{H}_1 - \hat{H}_2)) \] . \tag{E.1}

As \( \Psi_{ljmnk_u v_k} \) is an eigenstate of \( \hat{H}_1 - \hat{H}_2 \) with eigenvalue \( E_u - E_v \), (E.1) yields

\[ U(g) \Psi_{ljmnk_u v_k} = \frac{z^{(l-j)/2} e^{-i\theta(E_u - E_v)}}{(2\pi i \mu)^{q/2}} Y_{lk_u}(\theta(u)) \int \int_0^\infty dv' v'(v')^{j+q-1} \frac{L^l_m(u^2/v) L^j_n(v'(v')^2)}{\mu} \times \exp \left[ -\frac{1}{2} \left( \frac{u^2}{z} + \frac{z}{v'} \right)^2 + i \theta(u') \right] \exp \left( -i\mu (v \cdot v') \right) Y_{jk_v}(\theta(v')) , \tag{E.2} \]

where \( z := e^{2\lambda} \) and we are assuming \( \mu \neq 0, v \neq 0 \) and \( u \neq 0 \).

We need to evaluate the angular integral in (E.2). Suppose \( q > 2 \). We write \( v \cdot v' = vv' \cos \gamma \) and expand the exponential under the angular integral by ([44], page 98)

\[ e^{it \cos \gamma} = \frac{1}{2} \Gamma \left( \frac{q - 2}{2} \right) \sum_{a=0}^{\infty} i^a (2a + q - 2) J_{(q-2+2a)/2}(t) \frac{J_{(q-2+2a)/2}(t/2)}{(t/2)^{(q-2)/2}} C_a^{(q-2)/2}(\cos \gamma) . \tag{E.3} \]
We then expand the Gegenbauer polynomial $C^{(q-2)/2}_a(\cos \gamma)$ as

$$C^{(q-2)/2}_a(\cos \gamma) = \frac{4\pi^{q/2}}{\Gamma((q-2)/2)(2a+q-2)} \sum_k Y_{ak}^{(\theta(v))} Y_{ak}^{(\theta(v'))}, \quad (E.4)$$

which follows from formula 11.4(2) in [43] (correcting a typographical error in the normalisation factor, as seen from the final step of the proof on page 247). Using the orthonormality of the spherical harmonics, we obtain

$$\int \! d\Omega_{v'} \exp \left( -\frac{i}{\mu} (\mathbf{u} \cdot \mathbf{v'})^2 \right) Y_{jkv}^{(\theta(v'))} = (2\pi)^{q/2} \frac{1}{\mu} J_{(q-2+2j)/2}(\mu v'/\mu) Y_{jkv}^{(\theta(v))} . \quad (E.5)$$

For $q = 2$, (E.5) follows by recognising the angular integral as a representation of $J_j$, and for $q = 1$ it follows from the relation of $J_{\pm 1/2}$ to trigonometric functions ([43], Sections 7.3.1 and 7.11). Hence, for all $p \geq 1$ and $q \geq 1$, we have

$$U(g) \Psi_{ljmnk_v} = \frac{i^{-j-1} z^{j-\tilde{j}/2} e^{-i\theta(E_u - E_v)}}{\mu} Y_{lk_v}^{(\theta(u))} Y_{jk_v}^{(\theta(v))} u^{(2-p)/2} v^{(2-q)/2}$$

$$\times \int_0^\infty dv' u'(v')^{\tilde{j}+1} J_j(vv'/\mu) L_m^j(u^2/z) L_n^j(z(v')^2)$$

$$\times \exp \left[ -\frac{1}{2} \left( \frac{u^2}{z} + z(v')^2 \right) + \frac{i}{2} \left( \mu u^2 + v^2 + (v')^2 \right) \right]. \quad (E.6)$$

Performing the integral in (E.6) gives ([50], formula 7.421.4)

$$U(g) \Psi_{ljmnk_v} = e^{-i\theta(E_u - E_v)} z^{j-\tilde{j}/2} (1 + i\mu z)^{-j-1} \left( \frac{1 - i\mu z}{1 + i\mu z} \right)^n Y_{lk_v}^{(\theta(u))} Y_{jk_v}^{(\theta(v))}$$

$$\times u'^j L_m^j(u^2/z) L_n^j(z(v')^2)$$

$$\times \exp \left[ -\frac{1}{2} \left( \frac{1}{z} - i\mu \right) u^2 - \frac{1}{2} \left( \frac{z}{1 + i\mu z} \right) v^2 \right]. \quad (E.7)$$

We can now use (E.7) to prove the convergence results.

**Proposition E.1** Let $\tilde{l} + \tilde{j} > 0$. Then $(\Psi_{ljmnk_v}, U(g) \Psi_{ljmnk_v})_{\text{aux}}$ is integrable in absolute value over $G$.

**Proof.** It suffices to consider $l' = l$, $j' = j$, $k'_u = k_u$ and $k'_v = k_v$, for otherwise the integrand vanishes.
In $\Psi_{ljmn'kukv}U(g)\Psi_{ljmnkukv}$, we use (4.4) and (E.7) and expand the product of the generalised Laguerre polynomials as a sum of numerical constants times terms of the form
\[
\left(\frac{u^2}{z}\right)^r \left(\frac{v^2}{1+\mu^2 z^2}\right)^s,
\]
where $r$, $s$, $r'$ and $s'$ are non-negative integers. Integrating over $u$ and $v$ term by term, we find that $(\Psi_{ljmn'kukv},U(g)\Psi_{ljmnkukv})_{aux}$ is a sum of terms whose respective absolute values are numerical constants times
\[
\frac{z^{(l+j)/2+1+s+r'} (1+\mu^2 z^2)^{(s'-s)/2}}{[(z+1)^2 + \mu^2 z^2]^{1+(l+j+r'+r'+s')/2}}.
\]
An elementary analysis shows that sufficient conditions for (E.9) to be integrable over $G$ in the Haar measure $e^{2\lambda} d\lambda d\mu d\theta = \frac{1}{2} dz d\mu d\theta$ are
\[
\tilde{l} + \tilde{j} + 2r + 2s > 0, \quad \tilde{l} + \tilde{j} + 1 + r + 2s > 0,
\]
which hold since $\tilde{l} + \tilde{j} > 0$ by assumption. ■

**Proposition E.2** Let $\tilde{l} + \tilde{j} > 0$ and $p + q \equiv 1 \pmod{2}$. Then the value of the integral in Proposition E.1 is zero.

**Proof.** As $p + q \equiv 1 \pmod{2}$, $G$ is the double cover of SL(2,$\mathbb{R}$) and the range of $\theta$ in (E.1) is $0 \leq \theta < 4\pi$. By Proposition E.1 we may perform the integral over $\theta$ first, and the $\theta$-dependence in (E.7) shows that this integral evaluates to zero. ■

**Theorem E.3** Let $p \geq 2$, $q \geq 2$ and $p + q > 4$. Then the integral in (4.8) converges in absolute value for all $\phi_1, \phi_2 \in \Phi$. If $p + q \equiv 1 \pmod{2}$, the value of the integral is zero.

**Proof.** It suffices to consider $\phi_1, \phi_2 \in \{\Psi_{ljmnkukv}\}$. The inequalities on $p$ and $q$ imply that the conditions of Propositions E.1 and E.2 are satisfied. ■

**Proposition E.4** Let $\tilde{l} > 0$ and $\tilde{j} > 0$. Then $\Psi_{ljmn'kukv}U(g)\Psi_{ljmnkukv}$ is integrable in absolute value over $G \times \mathbb{R}^{p+q}$.

**Proof.** In $\Psi_{ljmn'kukv}U(g)\Psi_{ljmnkukv}$, we use (4.4) and (E.7), expand the product of the generalised Laguerre polynomials as in the proof of Proposition E.1 and consider the individual terms in this expansion. We now first take the absolute value and then integrate. The integrals over $\theta$, $\theta^{(u)}$ and $\theta^{(v)}$ are bounded by constants, the integrals over $u$ and $v$ are convergent and easily performed, and an elementary analysis shows that the remaining $\int dz d\mu$ integral is convergent provided $\tilde{l} > 0$ and $\tilde{j} > 0$. ■

**Theorem E.5** Let $p = 1$, $q = 3$ and let $\Phi^{mod}$ be as in subsection 5.2. Then the integral in (4.8) converges in absolute value for all $\phi_1, \phi_2 \in \Phi^{mod}$.
Proof. The only case not covered by Proposition E.1 is \( \phi_1 = \psi_{mn}, \phi_2 = \psi_{m'n'} \).

In \( \psi_{m'n'} U(g) \psi_{mn} \), we use (5.12), (5.13) and (E.7) and expand the generalised Laguerre polynomials of argument \( u^2/z \) and \( vz^2/(1 + \mu^2 z^2) \) as polynomials in their respective arguments. Inequalities (E.10) in the proof of Proposition E.1 show that it suffices to keep only the constant terms of these polynomials. Doing this, and integrating over \( u_1 \) and \( v \) by 7.414.8 in [50], we obtain two terms whose absolute values are numerical constants times

\[
\frac{z^2}{[(z + 1)^2 + \mu^2 z^2]^2} \times \left[ (z - 1)^2 + \mu^2 z^2 \right]^{(m' + n')/2},
\]

which is integrable in the measure \( dz \, d\mu \). \( \blacksquare \)
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