AVERAGE NUMBER OF ZEROS AND MIXED SYMPLECTIC VOLUME OF FINSLER SETS

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Abstract. Let $X$ be an $n$-dimensional manifold and $V_1, \ldots, V_n \subset C^\infty(X, \mathbb{R})$ finite-dimensional vector spaces with Euclidean metric. We assign to each $V_i$ a Finsler ellipsoid, i.e., a family of ellipsoids in the fibers of the cotangent bundle to $X$. We prove that the average number of isolated common zeros of $f_1 \in V_1, \ldots, f_n \in V_n$ is equal to the mixed symplectic volume of these Finsler ellipsoids. If $X$ is a homogeneous space of a compact Lie group and all vector spaces $V_i$ together with their Euclidean metrics are invariant, then the average numbers of zeros satisfy the inequalities, similar to Hodge inequalities for intersection numbers of divisors on a projective variety. This is applied to the eigenspaces of Laplace operator of an invariant Riemannian metric. The proofs are based on a construction of the ring of normal densities on $X$, an analogue of the ring of differential forms. In particular, this construction is used to carry over the Crofton formula to the product of spheres.

1. Introduction

Let $X$ be a differentiable manifold of dimension $n$. We assume that $X$ is connected and has countable base of topology. Let $V \subset C^\infty(X, \mathbb{R})$ be a finite-dimensional vector space with Euclidean metric, such that

$$\forall x \in X \exists f \in V : f(x) \neq 0. \quad (1.1)$$

By a Finsler set or an $F$-set we mean a continuous family $\mathcal{E} = \{\mathcal{E}(x)\}$ of compact convex sets $\mathcal{E}(x) \subset T^*_x$. For a given $V$ we construct the $F$-set $\mathcal{E} = \{\mathcal{E}(x)\}$, in which all $\mathcal{E}(x)$ are ellipsoids in the corresponding cotangent spaces. It turns out that the average number of isolated common zeros of $f_1, \ldots, f_n \in V$, defined below in 2.1, equals the symplectic...
volume of the domain
\[(1.2) \quad \bigcup_{x \in X} \mathcal{E}(x) \subset T^* X,\]
multiplied by a constant depending on \(n\).

This is easily understood in the special case of a submanifold \(X\) of the unit sphere \(S^{N-1} \subset \mathbb{R}^N\), provided \(X\) is not contained in a proper vector subspace and \(V\) consists of linear functionals on \(\mathbb{R}^N\). Namely, give \(X\) a Riemannian metric induced by the Euclidean metric of \(\mathbb{R}^N\). Then Crofton formula for the sphere tells us that the average number of zeros is proportional to the volume of \(X\). More precisely, this number is equal to \(\frac{2}{\sigma_n} \text{vol}(X)\), where \(\sigma_n\) is the volume of the \(n\)-dimensional unit sphere, see e.g. [25]. On the other hand, the Riemannian metric on \(X\) allows us to identify \(T^*_x X\) and \(T_x X\). If the ellipsoid \(\mathcal{E}(x) \subset T^*_x X\) is defined as the unit ball in this metric then the symplectic volume of the domain \(\bigcup_{x \in X} \mathcal{E}(x) \subset T^* X\) differs from \(\text{vol}(X)\) by a coefficient depending only on \(n\).

It is not hard to prove the result on the isolated common zeros of \(f_1, \ldots, f_n \in V\) even if \(X\) is not embedded in \(S^{N-1}\). The main difficulty appears in the case of \(n\) Euclidean vector spaces \(V_1, \ldots, V_n \subset C^\infty(X, \mathbb{R})\). Then we have \(n\) \(F\)-sets, namely, \(F\)-ellipsoids \(\mathcal{E}_1, \ldots, \mathcal{E}_n\) corresponding to \(V_1, \ldots, V_n\). We define the mixed volume of \(F\)-sets and prove Theorem 1 showing that the average number of isolated zeros of \(f_1 \in V_1, \ldots, f_n \in V_n\) equals the mixed volume of \(\mathcal{E}_1, \ldots, \mathcal{E}_n\). Theorem 1 can be viewed as a real geometric counterpart of Bernstein-Kouchnirenko theorem, relating the number of zeros of certain algebraic equations with the mixed volume of attached convex polytopes, see [10]. The mixed volumes of convex polytopes satisfy Alexandrov-Fenchel inequalities, which yield Hodge inequalities for intersection indices on some algebraic varieties, e.g., in the toric case [23]; see also [20] for generalizations. In a special situation, we prove inequalities of this type, which we again call Hodge inequalities, for the average numbers of isolated zeros. Namely, we prove them for homogeneous spaces of compact Lie groups and invariant Euclidean spaces \(V_i\), see Theorem 3. Here too they are a consequence of Alexandrov-Fenchel inequalities.

The proof of Theorem 1 is based on two facts. One of them is Theorem 8 carrying over Crofton formula to the product of spheres. The other one is Theorem 2 calculating the product of some special 1-densities on \(X\). Recall that a \(k\)-density on a vector space is a continuous function \(\delta\) on the cone of decomposable \(k\)-vectors, such that \(\delta(t\xi) = |t|^{\delta} \delta(\xi)\). A \(k\)-density on a manifold \(X\) is a \(k\)-density \(\delta_x\) on each tangent space \(T_x\), such that the assignment \(x \mapsto \delta_x\) is continuous. The main property of \(k\)-densities lies in the fact that they can be integrated along
arbitrary, not necessarily oriented, submanifolds of dimension \( k \). We refer the reader to [15], [3] for other properties and applications of densities.

If \( X \) is equipped with a non-negative quadratic form \( g \) then for any \( k \leq n \) we have a \( k \)-density \( \text{vol}_{k,g} \). The value of \( \text{vol}_{k,g} \) on a \( k \)-vector \( \xi_1 \wedge \ldots \wedge \xi_k \) is the \( g \)-volume of the parallelotope with edges \( \xi_i \). An arbitrary \( k \)-density assigns to such a parallelotope its “\( \delta \)-volume” \( \delta(\xi_1 \wedge \ldots \wedge \xi_k) \). In Section 3 we define a graded subspace \( \mathfrak{n}(X) \) in the space of all densities. The elements of \( \mathfrak{n}(X) \) are called normal densities. For normal densities we define the product making \( \mathfrak{n}(X) \) into a commutative graded ring. Any smooth 1-density is normal. As an example, we have the formula
\[
\text{vol}^k_{1,g} = c(n,k)\text{vol}^g_{k,g}.
\]

"Crofton formula for the product of spheres" is obtained following the pattern of complex projective spaces [21], [22]. In that case \( V_i \) are Hermitian vector spaces of holomorphic functions on a complex manifold \( X \). Let \( \theta_i : X \to V^*_i \) be the mapping assigning to \( x \in X \) the functional \( \theta_i(x)(f) = f(x) \), \( f \in V_i \). Then the average number of isolated common zeros of \( f_i \in V_i \) in a domain \( U \subset X \) equals
\[
j_f(\omega_1 \wedge \ldots \wedge \omega_n), \quad \omega_i = \text{pull-back of the Fubini-Study form on } P(V^*_i).
\]

For a real manifold \( X \) we prove that the average number of isolated common zeros of \( f_i \in V_i \) in \( U \subset X \) is obtained by integrating over \( U \) a certain \( n \)-density \( \Omega \). Namely, if all \( V_i \) are subject to (1.1) then we have a similar mapping \( \theta_i \) from \( X \) to the sphere in the dual space \( V^*_i \) and we can define \( g_i \) as the pull-back of the metric form on that sphere. Then
\[
\Omega = \frac{1}{\pi^n}\text{vol}_{1,g_1} \cdots \text{vol}_{1,g_n}, \quad \text{where } \text{vol}_{1,g_i} \text{ is the 1-density corresponding to the non-negative quadratic form } g_i.
\]

Thus the product of Kähler forms is replaced by the product of 1-densities of the corresponding metrics.

For the proof of Crofton formula we use the standard technique of double fibrations and write \( \Omega \) as the pull-back and push-forward of the volume form on the space of systems of equations, see [15], [22], [3], [27]. The main part of the proof is the presentation of \( n \)-density \( \Omega \) as the product of the abovementioned 1-densities, see Theorem 8.

The deduction of Theorem 1 from Crofton formula relies on the following theorem from convex geometry (Theorem 7): Given \( k \) compact convex sets \( A_1, \ldots, A_k \subset \mathbb{R}^n \), define a \( k \)-density \( d_k(A_1, \ldots, A_k) \) on \( \mathbb{R}^n \) by
\[
d_k(A_1, \ldots, A_k)(\xi_1 \wedge \ldots \wedge \xi_k) = V_k(\pi_H A_1, \ldots, \pi_H A_k) \cdot \text{vol}_k \Pi_\xi,
\]
where \( \xi_i \) are linearly independent vectors generating parallelotope \( \Pi_\xi \), \( \pi_H \) is the orthogonal projection map onto \( H = \mathbb{R}\xi_1 + \ldots + \mathbb{R}\xi_k \) and \( V_k \) is the \( k \)-dimensional mixed volume. Then for a wide class of centrally
symmetric convex sets, including smooth convex bodies and zonoids (see 2.2), one has

\[(1.3) \quad d_1(A_1) \cdot \ldots \cdot d_1(A_k) = k! d_k(A_1, \ldots, A_k).\]

Given a compact convex body in \(\mathbb{R}^n\), its \(k\)-th brightness function is the function on \(\text{Gr}(k, \mathbb{R}^n)\) equal to the \(k\)-volume of the orthogonal projection of \(A\) on a given \(k\)-dimensional subspace \(H\). For \(k = 1\) this function is called the width function. Determination of \(A\) by its brightness functions and relations between these functions are studied in geometric tomography, see [14], [17], [18]. The \(k\)-density \(d_k(A)\) evaluated on a unit \(k\)-vector \(\xi_1 \wedge \ldots \wedge \xi_k\), where \(\xi_i \in H\), is the \(k\)-brightness of \(A\) at \(H \in \text{Gr}(k, \mathbb{R}^n)\). Therefore (1.3) is an assertion from geometric tomography. In particular, for a centrally symmetric smooth convex body \(A\) one has \(d_k(A) = k! d_k(A)\).

The space of normal densities on \(\mathbb{R}^n\) can be identified with a subspace of translation invariant valuations on compact convex sets. Under this identification, the product of smooth normal densities coincides with Alesker product of smooth valuations, see [5]. Therefore (1.3) can be regarded as an identity from the valuation theory, see 3.3.

Theorem 1 and the product theorem for 1-densities related to \(F\)-sets (Theorem 2) are stated in Section 2. Also in this section, we give applications of Theorem 1 to homogeneous spaces of compact Lie groups. Namely, we deduce Hodge inequalities for average numbers of isolated common zeros and consider these numbers in some detail for eigenfunctions of the Laplace operator of an invariant Riemann metric. In Section 3 we construct the ring of normal densities on a vector space and, after that, on a differentiable manifold. Proofs of Theorems 1 and 2 are given in Section 4.

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2. Main results

2.1. Average number of zeros. Let \(X\) be an \(n\)-dimensional manifold and let \(V_1, \ldots, V_n\) be finite dimensional vector subspaces in \(C^\infty(X, \mathbb{R})\). Assume that each \(V_i\) has a fixed scalar product \(\langle \cdot, \cdot \rangle_i\), let \(S_i \subset V_i\) be the sphere of radius 1 with center 0 and let \(\sigma\) be the product of volumes of \(S_i\). For a system of functions \((s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n\) we denote by \(N(s_1, \ldots, s_n)\) the number of isolated common zeros of \(s_i\), \(i = 1, \ldots, n\). We will see later that the following integral exists. We call

\[
\mathcal{M}_X(V_1, \ldots, V_n) = \frac{1}{\sigma} \int_{S_1 \times \ldots \times S_n} N(s_1, \ldots, s_n) \, ds_1 \cdot \ldots \cdot ds_n
\]
the average number of common zeros of \( n \) functions \( s_i \). For any point \( x \in X \) define the functional \( \varphi_i(x) \in V_i^* \) by \( \varphi_i(x)(f) = f(x) \). Assuming \( (1.1) \) for each \( V_i \) we have \( \varphi_i(x) \neq 0 \). Equip \( V_i^* \) with the dual scalar product \( \langle \cdot, \cdot \rangle_i^* \), denote by \( S_i^* \) the unit sphere in \( V_i^* \) and consider the mapping \( \theta_i : X \to S_i^* \) defined by

\[
\theta_i(x) = \frac{\varphi_i(x)}{\sqrt{\langle \varphi_i(x), \varphi_i(x) \rangle_i^*}}.
\]

The pull-back of the Euclidean metric on \( S_i^* \subset V_i^* \) under \( \theta_i \) is a non-negative quadratic form \( g_i \) on the tangent bundle of \( X \).

Our first theorem computes \( M_X(V_1, \ldots, V_n) \) in terms of \( g_i \). Namely, let \( g_{i,x} \) be the quadratic form on the tangent space \( T_x \) corresponding to \( g_i \). Note that \( \sqrt{g_{i,x}} \) is a convex function and consider the convex set \( \mathcal{E}_i(x) \subset T_x^* \) with support function \( \sqrt{g_{i,x}} \). In other words,

\[
\max_{\xi^* \in \mathcal{E}_i(x)} \xi^*(\xi) = \sqrt{g_{i,x}(\xi)}.
\]

Then \( \mathcal{E}_i(x) \) is a centrally symmetric convex body in the orthogonal complement to the kernel of \( g_{i,x} \). We call \( \mathcal{E}_i(x) \) the ellipsoid associated with \( g_i \) at \( x \in X \).

Suppose that for every \( x \in X \) we are given a compact convex set \( \mathcal{E}(x) \subset T_x^* \) depending continuously on \( x \in X \). We call the collection \( \mathcal{E} = \{ \mathcal{E}(x) \mid x \in X \} \) a Finsler set or an \( F \)-set in \( X \). A Finsler set is said to be centrally symmetric if each \( \mathcal{E}(x) \) is centrally symmetric. In particular, the \( F \)-set \( \mathcal{E}_i = \{ \mathcal{E}_i(x) \} \) is centrally symmetric. This \( F \)-set is called the \( F \)-ellipsoid associated to \( g_i \).

The volume of an \( F \)-set \( \mathcal{E} \) is defined as the volume of \( \bigcup_{x \in X} \mathcal{E}(x) \subset T^*X \) with respect to the standard symplectic structure on the cotangent bundle. More precisely, if the symplectic form is \( \omega \) then the volume form is \( \omega^n/n! \). Using Minkowski sum and homotheties, we consider linear combinations of convex sets with non-negative coefficients. The linear combination of \( F \)-sets is defined by

\[
(\sum_i \lambda_i \mathcal{E}_i)(x) = \sum_i \lambda_i \mathcal{E}_i(x).
\]

The symplectic volume of the \( F \)-set \( \lambda_1 \mathcal{E}_1 + \ldots + \lambda_n \mathcal{E}_n \) is a homogeneous polynomial of degree \( n \) in \( \lambda_1, \ldots, \lambda_n \). Its coefficient at \( \lambda_1 \ldots \lambda_n \) divided by \( n! \) is called the mixed volume of \( F \)-sets \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) and is denoted by \( V_n^F(\mathcal{E}_1, \ldots, \mathcal{E}_n) \).
**Theorem 1.** Assume that all spaces $V_i$ are subject to (1.1) and let $\mathcal{E}_i$ be the $F$-ellipsoid associated to $g_i$, $i = 1, \ldots, n$. Then

$$M_X(V_1, \ldots, V_n) = \frac{n!}{(2\pi)^n} \cdot V^F_n(\mathcal{E}_1, \ldots, \mathcal{E}_n)$$

The proof will be given in subsection 4.4.

Introduce a Riemannian metric $h$ on $X$. Let $h_x$ be the corresponding metric on $T_x$ and $h^*_x$ the dual metric on $T^*_x$. For $F$-sets $\mathcal{E}_i$ denote by $V_{\mathcal{E}_1,\ldots,\mathcal{E}_n}(x)$ the mixed volume of the convex sets $\mathcal{E}_1(x), \ldots, \mathcal{E}_n(x)$ measured with the help of $h^*_x$. Define the mixed $n$-density of $F$-sets by

$$D_n(\mathcal{E}_1, \ldots, \mathcal{E}_n) = V_{\mathcal{E}_1,\ldots,\mathcal{E}_n} \cdot dx,$$

where $dx$ is the Riemannian $n$-density on $X$, and note that this definition does not depend on $h$. The mixed volume is related to the mixed density by

$$V^F_n(\mathcal{E}_1, \ldots, \mathcal{E}_n) = \int_X D_n(\mathcal{E}_1, \ldots, \mathcal{E}_n). \quad (2.1)$$

2.2. **Products of mixed densities.** We want to define a mixed $k$-density $D_k(\mathcal{E}_1, \ldots, \mathcal{E}_k)$, generalizing the above definition of a mixed $n$-density. This will lead us to Theorem 2 about the product of mixed densities, which will be used later in the proof of Theorem 1.

For arbitrary tangent vectors $\xi_1, \ldots, \xi_k \in T_x$ let $H \subset T_x$ be the subspace generated by $\xi_i$ and let $H^\perp \subset T^*_x$ be the orthogonal complement to $H$. Given an $F$-set $\mathcal{E}$ in $X$, denote by $\mathcal{E}_{\xi_1,\ldots,\xi_k}(x)$ the image of $\mathcal{E}(x)$ under the projection map $T^*_x \to T^*_x/H^\perp$. Then $\xi_1 \wedge \ldots \wedge \xi_k$ can be considered as a volume form on $T^*_x/H^\perp$, the dual space to $H$. Put

$$D_k(\mathcal{E})(\xi_1 \wedge \ldots \wedge \xi_k) = \left| \int_{\mathcal{E}_{\xi_1,\ldots,\xi_k}(x)} \xi_1 \wedge \ldots \wedge \xi_k \right|.$$ 

Then, by definition, $D_k(\mathcal{E})$ is a $k$-density on $X$. For a linear combination of $F$-sets $\sum \lambda_i \mathcal{E}_i$ the expression $D_k(\lambda_1 \mathcal{E}_1 + \ldots + \lambda_k \mathcal{E}_k)$ is a homogeneous polynomial of degree $k$ in $\lambda_i$. Its coefficient at $\lambda_1 \cdot \ldots \cdot \lambda_k$, divided by $k!$, is denoted by $D_k(\mathcal{E}_1,\ldots,\mathcal{E}_k)$. We call $D_k(\mathcal{E})$ the $k$-density of an $F$-set $\mathcal{E}$ and $D_k(\mathcal{E}_1,\ldots,\mathcal{E}_k)$ the mixed $k$-density of $\mathcal{E}_1, \ldots, \mathcal{E}_k$.

If $X$ is equipped with a Riemannian metric then $T^*_x$ is identified with $T_x$, $T^*_x/H^\perp$ with $H$, and $\mathcal{E}_{\xi_1,\ldots,\xi_k}(x)$ with the orthogonal projection of $\mathcal{E}(x)$ onto $H$. The value $D_k(\mathcal{E})(\xi_1 \wedge \ldots \wedge \xi_k)$ is the volume of this projection multiplied by the length of $\xi_1 \wedge \ldots \wedge \xi_k$.

In what follows, $1$-densities play a special role. It is easy to see that they can be computed in terms of support functions as follows. The
function on the tangent bundle, defined by

$$h_E(x,\xi) = \max_{\eta \in E(x)} \eta(\xi),$$

is called the support function of $E$. The corresponding 1-density is given by $D_1(E)(\xi) = h_E(x,\xi) + h_E(x,-\xi)$.

In Section 3 we introduce the notion of a normal density on an affine space and define the product of normal densities. We also define the ring $n(X)$ of normal densities on $X$ with pointwise product, see Theorem 6.

Recall that a zonoid is a compact convex body that can be approximated, in Hausdorff metric, by a Minkowski sum of segments [26]. In this paper, we use the following notion. A smooth convex body in $\mathbb{R}^n$ is a compact convex set whose support function is of class $C^\infty$ on the unit sphere. A smooth convex body is always $n$-dimensional.

**Theorem 2.** Let $E_1, \ldots, E_k$ be centrally symmetric $F$-sets in $X$. Assume that for every $x \in X$ each $E_i(x) \subset T^*_x X$ is the Minkowski sum of a smooth convex body and a zonoid (one of the two summands can be absent). Then the densities $D_k(E_i)$ are in $n(X)$ and

$$D_1(E_1) \cdot \ldots \cdot D_1(E_k) = k! \cdot D_k(E_1, \ldots, E_k).$$

The proof will be given in 4.2.

**Corollary 2.1.** Let $p + q = k \leq n$. Then

$$D_p(E_1, \ldots, E_p) \cdot D_q(E_{p+1}, \ldots, E_k) = \frac{k!}{p! q!} \cdot D_k(E_1, \ldots, E_k).$$

**Proof.** It suffices to apply Theorem 2 and to write mixed densities as products of 1-densities. ⊓⊔

### 2.3. Hodge inequalities

As a corollary from Theorem 1, we show here that, for a homogeneous space $X$ of a compact Lie group, the average numbers of zeros are subject to certain inequalities. We call them Hodge inequalities because they are similar to the well-known inequalities for intersection indices in algebraic geometry.

**Theorem 3.** Let $X$ be a homogeneous space of a compact Lie group. Assume that the vector spaces $V_i$ and their scalar products $\langle \ldots \rangle_i$ are invariant under the given transitive action. Then one has the following Hodge inequalities:

$$\mathfrak{M}_X^2(V_1, \ldots, V_{n-1}, V_n) \geq \mathfrak{M}_X(V_1, \ldots, V_{n-1}, V_{n-1}) \cdot \mathfrak{M}_X(V_1, \ldots, V_n, V_n).$$

**Proof.** By Theorem 1 it is enough to prove the inequality

$$V_n^F(E_1, \ldots, E_n)^2 \geq V_n^F(E_1, \ldots, E_{n-1}, E_{n-1}) \cdot V_n^F(E_1, \ldots, E_n, E_n).$$
In our situation, the quadratic forms $g_i$ and the ellipsoids $E_i$ are invariant. Choose an invariant Riemannian metric on $X$. Then the mixed density $D_n(E_1, \ldots, E_n)$ is also invariant and the mixed volume $V_{E_1, \ldots, E_n}(x)$ does not depend on $x$. Furthermore,

$$V_n^F(E_1, \ldots, E_n) = V_{E_1, \ldots, E_n}(x) \cdot \vol(X)$$

for any $x \in X$ by (2.1). Fix a point $x \in X$ and let $A_i = E_i(x)$. Then (2.2) turns into

$$V^2_n(A_1, \ldots, A_n) \geq V_n(A_1, \ldots, A_{n-1}, A_{n-1}) \cdot V_n(A_1, \ldots, A_n, A_n),$$

where $V_n$ is the mixed volume of compact convex sets. The latter inequalities follow from Alexandrov-Fenchel inequalities, see [4].

**Corollary 2.2.** In the setting of Theorem 3 one has

$$\mathcal{M}_X^n(V_1, \ldots, V_n) \geq \mathcal{M}_X(V_1) \cdot \ldots \cdot \mathcal{M}_X(V_n).$$

**Proof.** The proof for mixed volumes from [4] applies. \hfill \Box

Recall that a Riemannian homogeneous space $X = K/L$ is called isotropy irreducible if the representation of $L$ in the tangent space at the origin is irreducible, see [13]. Remark that all symmetric spaces of simple compact Lie groups, e.g., the sphere with the special orthogonal group, are isotropy irreducible.

**Corollary 2.3.** If $X$ is isotropy irreducible then we have equalities in Theorem 3 and Corollary 2.2.

**Proof.** The ellipsoids $A_i$ are balls and their mixed volume is the volume of the unit ball multiplied by the product of radii. \hfill \Box

**2.4. Zeros of Laplacian eigenfunctions.** In 2003, V.I.Arnold proposed to apply topological invariants to the study of the zero set of $k \leq n$ eigenfunctions of the Laplace operator, see [8], Problem 2003–10, p.174. He suggested that suitable invariants can be estimated, as in the classical Courant's theorem [11], in terms of the numbers of the corresponding eigenvalues.

Let $\Delta$ be the Laplace operator on a compact Riemannian manifold $X$ and

$$H(\lambda) = \{ f \in C^\infty(X, \mathbb{R}) \mid \Delta(f) + \lambda f = 0 \}$$

the eigenspace of $\Delta$ with eigenvalue $\lambda$, considered with $L^2$ metric. Put

$$\mathcal{M}(\lambda_1, \ldots, \lambda_n) = \mathcal{M}_X(H(\lambda_1), \ldots, H(\lambda_n)), \quad \mathcal{M}(\lambda) = \mathcal{M}(\lambda, \ldots, \lambda).$$

If $X$ is a homogeneous space of a compact Lie group and the metric is invariant then

$$\mathcal{M}(\lambda) \leq \frac{2}{\sigma_n r^{n/2}} \lambda^{n/2} \vol(X),$$

where $r$ is the radius of the unit ball.
where $\sigma_n$ is the volume of the $n$-dimensional sphere of radius 1, see [2]. In this case we have only one $F$-ellipsoid whose shape with respect to the Riemannian metric does not vary with $x \in X$. Its semi-axes $\beta_i$ satisfy $\sum \beta_i^2 = \lambda$, see [2]. Using this fact, one can easily deduce (2.3) from Theorem 1. Furthermore, for isotropy irreducible homogeneous spaces Theorem 1 shows that (2.3) turns into equality obtained in [1], [16].

The right hand side in (2.3) coincides, up to a coefficient depending only on $n$, with the leading term of the asymptotics for the number of $\lambda$ in the celebrated Weyl’s law, see [19]. Therefore (2.3) can be considered as a step in the direction of Arnold’s problem.

We also have the following inequalities of another type.

**Theorem 4.** If $X$ is a homogeneous space of a compact Lie group with an invariant Riemannian metric then

$$\mathcal{M}(\lambda_1, \ldots, \lambda_n)^2 \geq \mathcal{M}(\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n-1}) \cdot \mathcal{M}(\lambda_1, \ldots, \lambda_n, \lambda_n)$$

and

$$\mathcal{M}(\lambda_1, \ldots, \lambda_n) \geq (\mathcal{M}(\lambda_1) \cdot \ldots \cdot \mathcal{M}(\lambda_n))^{\frac{1}{n}}.$$

**Proof.** The first inequality follows from Theorem 3, the second one from Corollary 2.2. □

As an application of our results, we obtain another proof of the following theorem due to V.M.Gichev, see [16], Thm. 2, where one has to take $X = M$, $l = r$ and $t_i = 0$ in (31).

**Theorem 5.** If $X$ is an isotropy irreducible homogeneous space of a compact Lie group then

$$\mathcal{M}(\lambda_1, \ldots, \lambda_n) = \frac{2}{\sigma_n n^{n/2}} \sqrt{\lambda_1 \cdot \ldots \cdot \lambda_n \text{vol}(X)}.$$

**Proof.** By Corollary 2.3

$$\mathcal{M}(\lambda_1, \ldots, \lambda_n)^n = \mathcal{M}(\lambda_1) \cdot \ldots \cdot \mathcal{M}(\lambda_n).$$

Also, as we pointed out above, one has the equality in (2.3). This completes the proof. □

3. Ring of normal densities

3.1. Normal densities and normal measures. Let $V$ be a finite-dimensional real vector space. In considerations involving metric properties, we tacitly assume that $V$ has a Euclidean structure and any vector subspace $U \subset V$ carries the induced metric. We will consider translation invariant $k$-densities on $V$. Any of them can be viewed as an even positively homogeneous function of degree 1 on the cone of
decomposable $k$-vectors of $V$. A density of highest degree coincides, up to a scalar factor, with the Lebesgue measure on $V$. Therefore a $k$-density $\delta$ on a vector subspace $U \subset V$ of dimension $k$ is the Lebesgue measure on $U$ multiplied by some constant $c$. Since $\delta$ is translation invariant, we can push it to any shift $v_0 + U$. Given a compact set $B \subset v_0 + U$, we will write $\delta(B)$ for the Lebesgue measure of $B - v_0$ multiplied by $c$.

We denote by $\text{Gr}_a(k,V)$ the Grassmanian of affine subspaces of codimension $k$ in $V$ and identify the affine space $\text{Gr}_a(0,V)$ with the given vector space $V$. The Grassmanian of vector subspaces of dimension $k$ is denoted by $\text{Gr}(k,V)$.

**Definition 3.1.** A translation invariant Borel measure on $\text{Gr}_a(k,V)$, finite on compact sets, is called a normal measure.

**Remark.** Normal measures are Crofton measures as defined in [12]. The pull-back operation for normal measures introduced below coincides with the corresponding operation for Crofton measures considered by D.Faifman and T.Wannerer, see Appendix B in [12].

For $D \subset V$ we put

$$J_{k,D} = \{ H \in \text{Gr}_a(k,V) \mid H \cap D \neq \emptyset \}$$

Let $\mu_k$ be a normal measure on $\text{Gr}_a(k,V)$ and let $\Pi_\xi \subset V$ be a $k$-dimensional parallelootope generated by $\xi_1, \ldots, \xi_k \in V$. Define a function on decomposable $k$-vectors by

$$\chi_k(\mu_k)(\xi_1 \wedge \ldots \wedge \xi_k) = \mu_k(J_{k,\Pi_\xi}).$$

**Proposition 3.1.** $\chi_k(\mu_k)$ is a translation invariant $k$-density on the affine space $V$.

**Proof.** For a $k$-dimensional vector subspace $U \subset V$ the function $\mu_k(J_{k,D})$ on $k$-dimensional domains $D \subset U$ gives rise to a countably additive and translation invariant measure. This measure coincides with the Lebesgue measure of $U$ up to a factor depending continuously on $U$. Thus, for any linear operator $L : U \rightarrow U$ one has $\mu_k(J_{k,L \cdot D}) = |\det(L)| \mu_k(J_{k,D})$. In particular, for a linear operator in the subspace generated by $\xi_i$ and for $D = \Pi_\xi$ we get $\chi_k(\mu_k)(L \cdot \xi_1 \wedge \ldots \wedge L \cdot \xi_k) = |\det(L)| \chi_k(\mu_k)(\xi_1 \wedge \ldots \wedge \xi_k)$.

**Definition 3.2.** A linear combination of densities of the form $\chi_k(\mu_k)$ is called a normal $k$-density on $V$. The space of normal $k$-densities is denoted by $n_k$. 

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Given a linear map \( F : U \to V \) of real vector spaces and a normal measure \( \mu_k \) on \( \text{Gr}_a(k, V) \) we want to define the pull-back \( F^*\mu_k \) on \( \text{Gr}_a(k, U) \). Let \( K = \ker F \) and let
\[
\text{Gr}_a(k, U; K) = \{ H \in \text{Gr}_a(k, U) \mid H \supset u + K \text{ for some } u \in U \}.
\]
The condition defining \( \text{Gr}_a(k, U; K) \) means that the vector subspace associated to \( H \) contains \( K \). Clearly, \( \text{Gr}_a(k, U; K) \) is closed in \( \text{Gr}_a(k, U) \) and translation invariant. For \( T \in \text{Gr}_a(k, U; K) \) and \( \mathcal{T} \subset \text{Gr}_a(k, U; K) \) put
\[
F_*(T) = \{ G \in \text{Gr}_a(k, V) \mid F(T) = G \cap F(U) \}, \quad F_*(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} F_*(T).
\]

**Proposition 3.2.** The function \( \mathcal{T} \mapsto \mu_k(F_*(\mathcal{T})) \) is a normal measure on \( \text{Gr}_a(k, U) \) supported on \( \text{Gr}_a(k, U; K) \).

**Proof.** Write \( F \) as the composition
\[
U \xrightarrow{F'} U/K \hookrightarrow V.
\]
Then \( F'_*(T) = \{ F'(T) \} \), and so we get the diffeomorphism
\[
\text{Gr}_a(k, U; K) \to \text{Gr}_a(k, U/K), \quad T \mapsto F'_*(T).
\]
Thus our statement is reduced to the case of embedding. We may assume that \( U \) is a vector subspace in \( V \), \( F \) is the identity map, and \( T \in \text{Gr}_a(k, U) \). Then \( F_*(T) = \{ G \in \text{Gr}_a(k, V) \mid G \cap U = T \} \).

The closure \( \text{cl}\{F_*(T)\} \) of \( F_*(T) \) in \( \text{Gr}_a(k, V) \) consists of all affine subspaces \( G \) of codimension \( k \), such that \( G \cap U \supset T \). This closure is obviously compact. Now, if \( G_j \in \text{Gr}_a(k, V) \), \( T_j \in \text{Gr}_a(k, U) \) and \( G_j \cap U \supset T_j \), then the convergence of \( \{ T_j \} \) implies that \( \{ G_j \} \) has a convergent subsequence. Moreover, if \( G_j \cap U \neq T_j \) for all \( j \), then a limit point \( G \) of \( \{ G_j \} \) satisfies \( \text{codim}_U G \cap U > k \). This shows that \( \text{cl}\{F_*(T)\} \) and \( \text{cl}\{F_*(T)\} - F_*(T) \) are compact if \( \mathcal{T} \) is compact. On the other hand, the correspondence \( \mathcal{T} \mapsto F_*(T) \), where \( \mathcal{T} \) is any subset of \( \text{Gr}_a(k, U) \), is an injective homomorphism of \( \sigma \)-algebras. Since the image of a compact set in \( \text{Gr}_a(k, U) \) is the difference of two compact sets in \( \text{Gr}_a(k, V) \), it follows that the function \( \mathcal{T} \mapsto \mu_k(F_*(T)) \) is a correctly defined Borel measure. The translation invariance is obvious. \( \square \)

**Definition 3.3.** The measure defined in Proposition\( 3.2 \) is called the pull-back of \( \mu_k \) and is denoted by \( F^*\mu_k \). For \( \mathcal{T} \subset \text{Gr}_a(k, U; K) \) one has
\[
(F^*\mu_k)(\mathcal{T}) = \mu_k(F_*(\mathcal{T})).
\]

Let \( F : Y \to Z \) be a differentiable map. Recall that the pull-back of a \( k \)-density \( \nu \) on \( Z \) is a \( k \)-density \( F^*\nu \) on \( Y \) defined by \( F^*\nu(\xi_1 \wedge \cdots \wedge \xi_k) = \nu(dF_p(\xi_1) \wedge \cdots \wedge dF_p(\xi_k)) \) for any \( p \in Y \) and \( \xi_1, \ldots, \xi_k \in T_p Y \).
Proposition 3.3. For a linear map $F: U \to V$ and a normal measure $\mu_k$ on $\text{Gr}_a(k, V)$ one has $\chi_k(F^*\mu_k) = F^*(\chi_k(\mu_k))$.

Proof. Let $\xi_1, \ldots, \xi_k \in U$ and $\Pi = \Pi_\xi$. Then $\chi_k(F^*\mu_k)(\xi_1 \wedge \ldots \wedge \xi_k) = F^*\mu_k(\mathcal{J}_{k, \Pi}) = \mu_k(F_*\mathcal{J}_{k, \Pi})$ by Definition 3.3. On the other hand,

$$F^*(\chi_k(\mu_k))(\xi_1 \wedge \ldots \wedge \xi_k)) = \chi_k(\mu_k)(F\xi_1 \wedge \ldots \wedge F\xi_k) = \mu_k(\mathcal{J}_{k, F\cdot \Pi})$$

by the definition of pull-back of a density. It follows from the definition of $F_*$ that $F_*(\mathcal{J}_{k, \Pi}) \subset \mathcal{J}_{k, F\cdot \Pi}$. We have to prove that the value of $\mu_k$ on these two sets is the same. If $\dim F^*\cdot \Pi = k$ then the difference $\mathcal{J}_{k, F\cdot \Pi} - F_*(\mathcal{J}_{k, \Pi})$ is formed by subspaces which intersect $F^*\cdot \Pi$ non-transversally. A small shift of such a subspace is another subspace, whose intersection with $F^*\cdot \Pi$ is also non-transversal and disjoint from the initial one. Since $\mu_k$ is translation invariant, it follows from countable additivity of $\mu_k$ that $\mu_k(\mathcal{J}_{k, F\cdot \Pi} - F_*(\mathcal{J}_{k, \Pi})) = 0$. The same argument shows that

$$\mu_k(\{G \in \text{Gr}_a(k, V) \mid G \supset W\}) = 0$$

for an affine subspace $W \subset V$ of arbitrary dimension and, finally, that $\mu_k(\mathcal{J}_{k, F\cdot \Pi}) = 0$ if $\dim F^*\cdot \Pi < k$. \hfill \Box

Let $m_k$ be the vector space, whose elements are differences of normal measures on $\text{Gr}_a(k, V)$, where $k \leq n = \dim V$. We will define a product of normal measures $\mu_p$ on $\text{Gr}_a(p, V)$ and $\mu_q$ on $\text{Gr}_a(q, V)$ as a certain normal measure on $\text{Gr}_a(p + q, V)$. This extends to the product $m_p \times m_q \to m_{p+q}$, where we put $\mu_p\mu_q = 0$ if $p + q > n$, giving the structure of a ring to the graded space

$$m = m_0 \oplus m_1 \oplus \ldots \oplus m_n.$$
Lemma 3.1. If $D \subset \text{Gr}_a(p + q, V)$ is bounded then the images of $D$ under the projection mappings of $P_{p,q}^{-1}(D)$ onto $\text{Gr}_a(p, V)$ and $\text{Gr}_a(q, V)$ are also bounded.

Proof. Let $L \in D$ and $d(L) < R$. Assume that $L = G \cap H$, where $G \in \text{Gr}_a(p, V)$, $H \in \text{Gr}_a(q, V)$. Then $d(G) < R$ and $d(H) < R$. □

For a bounded domain $D \subset \text{Gr}_a(p + q, V)$ put

$$(\mu_p \mu_q)(D) = (\mu_p \times \mu_q)(P_{p,q}^{-1}(D))$$

and note that this is finite by Lemma 3.1. If $\{D_i\}$ is a decreasing sequence of bounded domains then the sequence $\{P_{p,q}^{-1}(D_i)\}$ is also decreasing. Moreover, if $\cap_i U_i = \emptyset$ then $\cap_i P_{p,q}^{-1}(D_i) = \emptyset$. Since $\mu_p, \mu_q$ are countably additive, $\mu_p \mu_q$ is also countably additive. Thus $\mu_p \mu_q$ extends to a Borel measure, which is translation invariant by construction. The resulting normal measure on $\text{Gr}_a(p + q, V)$ is again denoted by $\mu_p \mu_q$.

Lemma 3.2. Suppose $p_1 + \ldots + p_k \leq n$ and let $\mu_i$ be a normal measure on $\text{Gr}_a(p_i, V)$, $i = 1, \ldots, n$. Denote by $\mathcal{D}_{p_1, \ldots, p_k}$ the subset of $\text{Gr}_a(p_1, V) \times \ldots \times \text{Gr}_a(p_k, V)$ formed by all $k$-tuples $(G_1, \ldots, G_k)$, such that codim $(G_1 \cap \ldots \cap G_k) < p_1 + \ldots + p_k$. Then

$$(\mu_1 \times \ldots \times \mu_k)(\mathcal{D}_{p_1, \ldots, p_k}) = 0.$$ 

Proof. For $k = 1$ the assertion is trivial. Let $k \geq 2$, $q = p_1 + \ldots + p_{k-1}$. For $G \in \text{Gr}_a(q, V)$ put

$$\mathcal{D}_G = \{H \in \text{Gr}_a(p_k, V) \mid \text{codim } (H \cap G) < p_k + q\}.$$ 

Assume by induction that $(\mu_1 \times \ldots \times \mu_{k-1})(\mathcal{D}_{p_1, \ldots, p_{k-1}}) = 0$. Then, by Fubini’s theorem, it suffices to prove that $\mu_k(\mathcal{D}_G) = 0$. Now, $\mu_k$ is translation invariant, so we have $\mu_k(\mathcal{D}_{\epsilon + G}) = \mu_k(\mathcal{D}_G)$ for any $\epsilon \in V$. Assume first that $q = 1$. Then $\mathcal{D}_G$ consists of all those $H \in \text{Gr}_a(p_k, V)$ which are contained in $G$. For a generic sequence $\epsilon_i$ the non-transversality condition implies $\mathcal{D}_{\epsilon_i + G} \cap \mathcal{D}_{\epsilon_j + G} = \emptyset$. Choose $\epsilon_i$ so that the series $\sum \epsilon_i$ is normally convergent. Then $\cup \mathcal{D}_{\epsilon_i + G}$ is relatively compact. Since $\mu_k$ is countably additive, we conclude that $\mu_k(\mathcal{D}_G) = 0$. Assume now that $q > 1$. Choose $\epsilon$ so that $\epsilon + G \neq G$ and put $\epsilon_i = t_i \epsilon$, where $t_i \neq t_j$ and $\sum |t_i| < \infty$. Then $\mathcal{D}_{\epsilon_i + G} \cap \mathcal{D}_{\epsilon_j + G} \subset \mathcal{D}_{\epsilon t_i + G}$, hence $\mu_k(\mathcal{D}_{\epsilon + G} \cap \mathcal{D}_{\epsilon_j + G}) = 0$ by induction on $q$. Also, $\cup \mathcal{D}_{\epsilon_i + G}$ is relatively compact. Applying countable additivity of the measure to the increasing sequence of finite unions $\cup_{i \leq j} \mathcal{D}_{\epsilon_i + G}$, we obtain our assertion. □
**Corollary 3.1.** Let \( D \subset \operatorname{Gr}_a(p_1 + \ldots + p_k, V) \) be a bounded domain and let
\[
T_D = \{(G_1, \ldots, G_k) \in \prod_i \operatorname{Gr}_a(p_i, V) \mid D \cap G_1 \cap \ldots \cap G_k \neq \emptyset\}.
\]
The product of normal measures is associative and
\[
(\mu_1 \cdot \ldots \cdot \mu_k)(D) = (\mu_1 \times \ldots \times \mu_k)(T_D).
\]

**Lemma 3.3.** The subspace \( I = \ker(\chi_0) \oplus \ldots \oplus \ker(\chi_n) \) is a homogeneous ideal of the ring \( m \).

**Proof.** If \( \mu_p \in m_p, \chi_p(\mu_p) = 0 \) and \( \mu_q \in m_q \) then Fubini’s theorem implies \((\mu_p \mu_q)(T_{p+q},1) = 0\) for any \((p+q)\)-dimensional paralleloope \( \Pi \). Therefore \( \chi_{p+q}(\mu_p \mu_q) = 0 \). \( \Box \)

Let \( n = n_0 \oplus n_1 \oplus \ldots \oplus n_n \) denote the graded space of normal densities. Consider the linear map \( \chi = \oplus \chi_i : m \to n \) of graded spaces.

**Corollary 3.2.** There is a unique structure of a graded ring on \( n \), such that \( \chi \) is a ring homomorphism. In the notations of Proposition 3.3 the pull-back operations on measures and densities are ring homomorphisms.

Given a manifold \( X \), the graded ring of normal densities \( n_x \) is defined for every tangent space \( T_x \). A density \( \delta \) on \( X \) is called normal if \( \delta_x \in n_x \) for every \( x \in X \). The set of normal densities on \( X \) is denoted by \( n(X) \). With respect to pointwise multiplication, \( n(X) \) is a commutative graded algebra over \( C(X) \).

**Theorem 6.** For any differentiable map \( F : X \to Y \) the pull-back operation \( \delta \mapsto F^*\delta, (F^*\delta)_x = (dF_x)^*\delta_{F(x)} \), defines a homomorphism \( n(Y) \to n(X) \). The assignment \( X \to n(X) \) is a contravariant functor from the category of differentiable manifolds to the category of commutative graded rings. For any \( \delta \in n(Y) \) and \( f \in C(Y) \) one has \( F^*(f \delta) = (f \circ F) \cdot F^*\delta \).

**Proof.** The pull-back of a normal density is normal by Proposition 3.3. The fact that above map \( n(Y) \to n(X) \) is a ring homomorphism follows from Corollary 3.2. \( \Box \)

### 3.2. Normal densities and the cosine transform.
For a Euclidean structure on \( V \) let \( \operatorname{vol}_k \) denote the corresponding Riemannian \( k \)-density, i.e., the translation invariant \( k \)-density, whose value on the \( k \)-vector \( \xi_1 \wedge \ldots \wedge \xi_k \) equals the \( k \)-dimensional volume of the paralleloope generated by \( \xi_1, \ldots, \xi_k \). Starting with a continuous real function \( \phi \) on \( \operatorname{Gr}(k, V) \), we will define a translation invariant \( k \)-density \( \delta_{k,\phi} \) on \( V \) and a normal
measure $\mu_{k,\phi} \in m_k$. The function $\phi$ will be called the gauge function of $\delta_{k,\phi}$ and $\mu_{k,\phi}$.

The density $\delta_{k,\phi}$ is defined by

$$(3.2) \quad \delta_{k,\phi}(\xi_1 \wedge \ldots \wedge \xi_k) = \phi(H) \cdot \text{vol}_k(\xi_1 \wedge \ldots \wedge \xi_k),$$

where $H$ is the subspace generated by linearly independent vectors $\xi_1, \ldots, \xi_k$. Note that any translation invariant $k$-density can be written in this form. A translation invariant $k$-density is said to be of class $C^\infty$ if the associated gauge function $\phi : \text{Gr}(k, V) \to \mathbb{R}$ is of class $C^\infty$.

In the special case of normal densities, we attach the gauge function to a density and identify $n_k$ with a (non-closed) vector subspace of $C(\text{Gr}(k, V))$ considered with the topology of uniform convergence. We will sometimes indicate the ambient space and write $J_{p,C,V}$ in place of $J_{p,C}$ (see (3.1)).

**Lemma 3.4.** Assume that the normal measure $\mu_p \in m_p$ is non-negative. If $C \subset V$ is a cube in some coordinate system then

$$\mu_p(J_{p,C}) \leq a(C) \cdot m,$$

where $m$ is the maximum of the gauge function of $\chi_p(\mu_p)$ and the constant $a(C)$ does not depend on $\mu_p$.

**Proof.** If $p = n = \dim V$ then $\mu_p$ is a Lebesgue measure and the estimate holds true with $a(C)$ being the volume of $C$. In particular, the assertion of the lemma is obvious if $n = 1$. Suppose $n > 1$. Let $W_1, \ldots, W_n \subset V$ be the coordinate subspaces of codimension 1 and let $C_1, \ldots, C_{2n}$ be the $(n-1)$-dimensional faces of $C$. We can assume that the numbering is chosen so that $C_i \subset W_i$ for all $i = 1, \ldots, n$. Clearly,

$$J_{p,C,V} = \bigcup_{i=1}^{2n} J_{p,C_i,V}.$$

On the other hand, denote by $\mu^{(i)}_p$ the pull-back of $\mu_p$ under the embedding $W_i \to V$. By Definition 3.3 we have

$$\mu_p(J_{p,C_i,V}) = \mu^{(i)}_p(J_{p,C_i,W_i}), \quad i = 1, \ldots, n.$$

By Proposition 3.3 the gauge functions of $\chi_p(\mu^{(i)}_p)$ do not exceed $m$. Thus

$$\mu_p(J_{p,C,V}) \leq \sum_{i=1}^{2n} \mu_p(J_{p,C_i,V}) = 2 \sum_{i=1}^{n} \mu^{(i)}_p(J_{p,C_i,W_i}) \leq 2n \cdot \max a(C_i) \cdot m$$

by induction. □
The following property of the product $n_p \times n_q \to n_{p+q}$ will be useful. Let $\delta = \chi_p(\mu_p)$ and $\delta' = \chi_q(\mu_q)$ be a normal $p$-density and a normal $q$-density, respectively.

**Proposition 3.4.** Suppose $\{\delta_i\} \subset n_p$ and $\{\delta'_i\} \subset n_q$ are two sequences of normal densities converging to $\delta$ and $\delta'$, respectively. Assume that $\delta'_i = \chi_q(\mu_{q,i})$, where all normal measures $\mu_{q,i}$ are non-negative. Then $\delta_i \delta'_i$ tends to $\delta \delta'$.

**Proof.** Let $A$ be a compact convex $(p+q)$-dimensional set in $V$, $G$ an arbitrary point in $Gr_a(q,V)$ and $A_G = A \cap G$. Then

$$(\delta \delta')(A) = \int_{G \in Gr_a(q,V)} \delta(A_G) d\mu_q(G)$$

by the definition of the product of normal densities and by Fubini’s theorem. Suppose now that $\{\delta_i\} \subset n_p$ and $\{\delta'_i\} \subset n_q$ are two sequences of normal densities converging to $\delta$ and $\delta'$, respectively. Then $\delta_i(A_G)$ tends to $\delta(A_G)$ uniformly in $G$ with $A_G$ non-empty, hence $\delta_i \delta' \to \delta \delta'$ and, similarly, $\delta'_i \to \delta \delta'$. Finally, write $\delta_i \delta'_i = (\delta_i - \delta)(\delta'_i + \delta \delta')$. We have to show that the first summand tends to 0. Since the sequence of gauge functions of $\chi_q(\mu_{q,i})$ is convergent, Lemma 3.4 shows that the measures $\mu_{q,i}(\{G \in Gr_a(q,V) | A_G \neq \emptyset\})$ are bounded by the same constant. This completes the proof. \qed

An affine subspace $G \in Gr_a(k,V)$ has a unique presentation as the sum $G = h + H^\perp$, where $H \in Gr(k,V)$, $H^\perp$ is the orthogonal complement to $H$ and $h \in H$. The metric on $V$ induces a metric on $H$ and the associated Lebesgue measure on $H$ is denoted by $dH$. The measure $\mu_{k,\phi}$ is given by integration against compactly supported functions. Namely, if $\psi$ is such function on $Gr_a(k,V)$ then

$$(3.3) \quad \int_{Gr_a(k,V)} \psi \cdot d\mu_{k,\phi} = \int_{Gr(k,V)} \phi(H) \left( \int_H \psi(h + H^\perp) \cdot dh \right) \cdot dH,$$

where $dH$ is the Haar measure on the Grassmanian.

Recall the definition of the cosine transform. Given $E \in Gr(k,V)$, $F \in Gr(l,V)$, where $k \leq l$, let $A \subset E$ be any subset of non-zero volume. The cosine of the angle between $E$ and $F$ is the ratio of the $k$-dimensional volume of the orthogonal projection of $A$ onto $F$ to the $k$-dimensional volume of $A$. The ratio is denoted here by $\cos(E,F)$ (some authors write $|\cos(E,F)|$ in a more classical way). The cosine transform $T_k : C(Gr(k,V)) \to C(Gr(k,V))$ is the integral operator

$$T_k(f)(G) = \int_{Gr(k,V)} f(H) \cos(H,G) \, dH.$$
Proposition 3.5. Let $\Phi = T_k(\phi)$. Then $\delta_{k,\Phi} = \chi_k(\mu_{k,\phi})$, i.e., the functions from the image of $T_k$ are gauge functions of normal densities and the diagram

$$
\begin{align*}
C(\text{Gr}(k, V)) & \xrightarrow{T_k} \text{Im } T_k \subset C(\text{Gr}(k, V)) \\
\downarrow & \quad \downarrow \\
m_k & \xrightarrow{\chi_k} n_k
\end{align*}
$$

commutes, where the mappings denoted by vertical arrows attach the measure and, respectively, the density to a gauge function.

Proof. Let $\psi$ denote the characteristic function of $T_{k,\Pi_\xi}$ and let $G_\xi$ be the vector subspace generated by $\xi_1, \ldots, \xi_k$. Then

$$
\chi_k(\mu_{k,\phi})(\Pi_\xi) = \int_{G \in \text{Gr}(k, V)} \phi(G) \left( \int_G \psi(g + G^\perp) dg \right) dG =
$$

$$
= \int_{\text{Gr}(k, V)} \phi(G) \text{vol}_k(\Pi_\xi) \cos(G, G_\xi) dG = \Phi(G_\xi) \text{vol}_k(\Pi_\xi) = \delta_{k,\phi}(\Pi_\xi).
$$

One can mimic the definition (3.3) of $\mu_{k,\phi}$ replacing the function $\phi$ by a Borel measure $\nu$ on $\text{Gr}(k, V)$. Namely, define the normal measure $\mu_{k,\nu}$ by

$$
\int_{\text{Gr}(k, V)} \psi \cdot d\mu_{k,\nu} = \int_{\text{Gr}(k, V)} \left( \int_H \psi(h + H^\perp) \cdot dh \right) \cdot d\nu(H).
$$

Proposition 3.6. Let

$$
T_k(\nu)(G) = \int_{\text{Gr}(k, V)} \cos(H, G) \cdot d\nu(H).
$$

If $\Phi = T_k(\nu)$ then $\delta_{k,\Phi} = \chi_k(\mu_{k,\nu})$.

Proof. It suffices to replace $\phi(G) dG$ by $d\nu(G)$ in the proof of Proposition 3.5.

Proposition 3.7. Every $C^\infty$ translation invariant 1-density is normal.

Proof. By Proposition 3.5 a density of the form $\delta_{1, T_1(\phi)}$ is normal. On the other hand, the restriction of the cosine transform to $C^\infty$ functions is an automorphism

$$
T_1 : C^\infty(\text{Gr}(1, V)) \to C^\infty(\text{Gr}(1, V)).
$$

In particular, any $C^\infty$ function on $\text{Gr}(1, V)$ is in the image of the cosine transform, see [7].
We will need the notion of push-forward for densities. Let $F : Y \to Z$ be a fibration with $l$-dimensional fiber $Y_z$, $z \in Z$, and let $\mu$ be a $k$-density on $Y$. Given $e_1 \wedge \ldots \wedge e_{k-l} \in \bigwedge^{k-l} T_z Z$, choose tangent vectors $h_1, \ldots, h_{k-l} \in T_y Y$, such that $dF_y(h_i) = e_i$. Then the $l$-density on the fiber $F_z$, given by

$$i_{h_1 \wedge \ldots \wedge h_{k-l}} : v_1 \wedge \ldots \wedge v_l \mapsto \mu(h_1 \wedge \ldots \wedge h_{k-l} \wedge v_1 \wedge \ldots \wedge v_l),$$

is independent of the choice of $h_i$. The push-forward of $\mu$ is defined as the $(k-l)$-density

$$F_* \mu(e_1 \wedge \ldots \wedge e_{k-l}) = \int_{F_z} i_{h_1 \wedge \ldots \wedge h_{k-l}} \nu,$$

provided the integral is finite.

Let $U \subset V$ be a vector subspace of dimension $k$. Define the mapping

$$\nu : \text{Gr}(1,V) \setminus \text{Gr}(1,U^\perp) \to \text{Gr}(1,U)$$

by $\nu(H) = (H + U^\perp) \cap U$. The $\nu$ is a fibration with the fiber

$$\nu^{-1}(L) = \text{Gr}(1, L + U^\perp) \setminus \text{Gr}(1, U^\perp).$$

Let $dH$ be the normalized density of highest degree on $\text{Gr}(1,V)$, i.e.,

the density of the Haar measure on the projective space. The push-forward of a continuous real function $g$ on $\text{Gr}(1,V)$ is defined by the equality

$$\nu_*(g \cdot dH) = (\nu_* g) \cdot dL,$$

where $dL$ is the normalized density of highest degree on $\text{Gr}(1,U)$. Note that if $\phi : \text{Gr}(1,V) \to \mathbb{R}$ is constant on the fibers of $\nu$ then $\nu_*(\phi g) = \phi \cdot \nu_* g$.

For future use we prove the following proposition.

**Proposition 3.8.** Let $f \in C^\infty(\text{Gr}(1,V)))$, $f_U$ the restriction of $f$ to $\text{Gr}(1,U)$, and $g(H) = T_1^{-1} f(H) \cdot \cos(H,U)$. Then

$$T_{1,U}^{-1} f_U = \nu_* g,$$

where $T_{1,U}$ is the cosine transform on $\text{Gr}(1,U)$.

**Proof.** For $L \in \text{Gr}(1,U)$ one has $\cos(H,L) = \cos(H,U) \cos(\nu(H),L)$, hence

$$f_U(L) = \int_{\text{Gr}(1,V)} \{T_1^{-1} f(H)\} \cos(H,L) \, dH = \int_{\text{Gr}(1,V)} g(H) \cos(\nu(H),L) \, dH.$$

From the definition of $\nu_* g$, it follows that

$$f_U(L) = \int_{\text{Gr}(1,U)} (\nu_* g)(K) \cos(K,L) \, dK = T_{1,U} (\nu_* g)(L).$$
3.3. Normal densities and valuations of convex bodies. This small subsection contains a number of remarks that are neither proved, nor used in the rest of the paper. Let \( v \) be a function on the set of compact convex bodies in \( \mathbb{R}^n \). Then \( v \) is called a valuation if \( v(A \cap B) + v(A \cup B) = v(A) + v(B) \) for any two compact convex bodies \( A, B \), such that \( A \cup B \) is also convex. For the theory of valuations the reader is referred to [5] and references therein. We assume that \( v \) is continuous in Hausdorff metric, translation invariant and even, i.e., \( v(-A) = v(A) \).

The valuation \( v \) is called \( k \)-homogeneous if \( v(tA) = |t|^k v(A) \). A normal \( k \)-density \( \delta \), as a function on \( k \)-dimensional parallelotopes, extends to a \( k \)-homogeneous valuation \( v_\delta \). Indeed, following Definition 3.2, we may assume that \( \delta = \chi_k(\mu_k) \), where \( \mu_k \) is a normal measure. Then we put \( v_\delta(A) = \mu_k(J_{k,A}) \), where \( J_{k,A} \) is defined by (3.1). In other words, normal densities are contained in the image of Klain map, see [24].

Remark 1. If a normal \( k \)-density \( \delta \) is smooth (of class \( C^\infty \)) then the valuation \( v_\delta \) is also smooth in the sense of [5]. The assignment \( \delta \mapsto v_\delta \) defines a one-to-one correspondence between the sets of smooth normal \( k \)-densities and smooth \( k \)-homogeneous valuations.

Remark 2. The product of smooth normal densities agrees with Alesker product of smooth valuations. Therefore the equality

\[
d_p(A_1, \ldots, A_p) \cdot d_q(A_{p+1}, \ldots, A_{p+q}) = \frac{(p+q)!}{p!q!} d_{p+q}(A_1, \ldots, A_{p+q}),
\]

following from Theorem 7 in Section 4, can be regarded as a computation of the product of valuations related to smooth centrally symmetric bodies \( A_i \).

Remark 3. The product of measures on affine Grassmanians and its connection with Alesker product of valuations is considered in [9].

Remark 4. The pull-back operation on valuations of convex bodies is defined in [6]. This operation agrees with our pull-back operation for normal measures.

4. Proofs of main results

4.1. Some facts from convex geometry. Let \( V = \mathbb{R}^n \) be a Euclidean space, \( S^{n-1} \subset V \) the unit sphere. For a vector subspace \( M \subset V \) we denote by \( M^\perp \) its orthogonal complement and by \( \pi_M \) the projection map \( V \to M \). For \( x \in S^{n-1} \) we write \( x^\perp \) instead of \( (\mathbb{R} \cdot x)^\perp \).

Let \( A \subset V \) be a compact convex set of dimension \( k \), \( V_k(A) \) its \( k \)-dimensional volume and \( h_A \) the support function. Later on, it will be also convenient to define the width function of \( A \) on the projective
space $\text{Gr}(1, V)$ by

$$s_A(H) = h_A(x) + h_A(-x) = V_1(\pi_H A),$$

where $H \in \text{Gr}(1, V)$ and $x$ is a unit vector in $H$. The cosine transform on the unit sphere

$$f \mapsto (Tf)(x) = \int_{S^{n-1}} f(s)(x, s)ds$$

will be considered as a linear operator on even functions which are identified with functions on $\text{Gr}(1, V)$. Then $T$ is invertible on the space of even $C^\infty$ functions, see [7]. Furthermore, if $A$ is centrally symmetric with center 0 and $h_A$ is smooth, then $h_A$ is contained in the image of $T$. The proof of the following result is due to S.Alesker.

**Lemma 4.1.** For a smooth centrally symmetric body $A$ with center 0 one has

$$\int_{S^{n-1}} T^{-1} h_A(x) V_{n-1}(\pi_x A)dx = \frac{n}{2} V_n(A).$$

**Proof.** We may assume that the Gaussian curvature $K$ of $\partial A$ does not vanish. Indeed, one can approximate $A$ by a convex centrally symmetric smooth body having this property and then use the continuity of $T^{-1}$ in $C^\infty$-topology. Recall that $V_n(A) = \frac{1}{n} \int_{S^{n-1}} h_A(x) K(x)^{-1}dx$. Since $T$ is a self-adjoint operator in $L^2(S^{n-1}, dx)$, we have

$$nV_n(A) = \int_{S^{n-1}} h_A(x) K^{-1}(x)dx = \int_{S^{n-1}} T^{-1} h_A(x) TK^{-1}(x)dx =$$

$$= 2 \int_{S^{n-1}} T^{-1} h_A(x) V_{n-1}(\pi_x A)dx,$$

where we used the identity $TK^{-1}(x) = 2V_{n-1}(\pi_x A)$. □

We now want to restate the assertion of Lemma 4.1 in terms of the Haar measure $dH$ on the projective space $\text{Gr}(1, V)$. Note that for $A$ symmetric $s_A(H) = 2h_A(x)$, where $x$ is a unit vector in $H$.

**Corollary 4.1.** Let $T_1$ be the cosine transform on $\text{Gr}(1, V)$. Under the above assumptions

$$\int_{\text{Gr}(1, V)} T_1^{-1} s_A(H) V_{n-1}(\pi_{H^*} A) dH = nV_n(A).$$

The next proposition deals with a vector subspace $D \subset V$. For $D = V$ we retrieve Corollary 4.1.
**Proposition 4.1.** Let \( D \subset V \) be a \( k \)-dimensional vector subspace. Then under the above assumptions

\[
\int_{\text{Gr}(1,V)} T_1^{-1}s_A(H) \cos(H,D)V_{k-1}(\pi_{H\perp\cap D}A)dH = kV_k(\pi_{D}A).
\]

*Proof.* For \( H \in \text{Gr}(1,V) \setminus \text{Gr}(1,D) \) put \( \nu(H) = (H + D) \cap D \). By Proposition 3.8 we obtain \( T_1^{-1}(s_A)_D = \nu g \), where

\[
g(H) = T_1^{-1}s_A(H) \cos(H,D).
\]

Let \( \phi(H) = V_{k-1}(\pi_{H\cap D}A) = V_{k-1}(\pi_{L_H} \pi_{D}A) \), where \( L_H^\perp \) is the orthogonal complement to \( L_H = \nu(H) \) in \( D \). Since \( \phi \) is constant along the fibers of \( \nu \), we have \( \nu_*(\phi g) = \phi \nu_*(g) \). It follows that the integral on the left hand side equals

\[
\int_{\text{Gr}(1,D)} (T_1^{-1}(s_A)_D)(L) V_{k-1}(\pi_{L_H \cap D}A) dL.
\]

Applying Corollary 4.1 to the convex body \( \pi_{D}A \subset D \), we get the desired equality. \( \square \)

### 4.2. Densities \( d_k(A_1, \ldots, A_k) \). Suppose we are given \( k \) compact convex sets \( B_1, \ldots, B_k \) in a \( k \)-dimensional vector subspace of \( V = \mathbb{R}^n \). Then their mixed volume is denoted by \( V_k(B_1, \ldots, B_k) \). For any compact convex sets \( A_1, \ldots, A_k \subset V \) we have the associated translation invariant \( k \)-density \( d_k(A_1, \ldots, A_k) \), defined as follows. Let \( \xi_1, \ldots, \xi_k \in V \) and let \( H \) be the vector subspace generated by \( \xi_1, \ldots, \xi_k \). Then

\[
d_k(A_1, \ldots, A_k)(\xi_1 \land \ldots \land \xi_k) = V_k(\pi_H A_1, \ldots, \pi_H A_k) \cdot \text{vol}_k(\xi_1 \land \ldots \land \xi_k)
\]

if \( \dim H = k \) and \( d_k(A_1, \ldots, A_k) = 0 \) if \( \dim H < k \). We also use the notation \( d_k(A) = d_k(A, \ldots, A) \), where \( A \) appears \( k \) times on the right hand side.

Recall that a translation invariant \( k \)-density can be evaluated on compact subsets contained in a shift of a \( k \)-dimensional vector subspace, see 3.1. Recall also that normal densities can be multiplied, see Corollary 3.2.

**Proposition 4.2.** Let \( A \) be a smooth convex body. Then the density \( d_1(A) \) is normal.

*Proof.* Let \( \xi \in V, \xi \neq 0 \), and \( H = \mathbb{R}\xi \). Then

\[
d_1(A)(\xi) = V_1(\pi_H A) \cdot \text{vol}_1(\xi) = s_A(H) \cdot \text{vol}_1(\xi) = \delta_{1,s_A}(\xi)
\]

by 3.2. Since \( s_A \) is a smooth function, we can put \( \phi = T_1^{-1}(s_A) \). Then

\[
d_1(A) = \delta_{1,s_A} = \chi_1(\mu_1,\phi)
\]

by Proposition 3.5, showing that \( d_1(A) \) is normal. \( \square \)
Proposition 4.3. Let \( A \) be as in Proposition 4.2 and centrally symmetric. If \( B \subset V \) is a compact convex set of dimension \( k \), contained in a shift of a \( k \)-dimensional vector subspace \( D \subset V \), then

\[
(d_1(A))^k(B) = k! d_k(A)(B).
\]

Proof. We will prove our statement by induction. For \( k = 1 \) there is nothing to prove, so let \( k > 1 \). By the definition of the product of normal densities \( \delta_p = \chi_{p}(\mu_p) \), \( \delta_q = \chi_{q}(\mu_q) \) and by Fubini’s theorem

\[
\delta_p \delta_q(B) = (\mu_p \times \mu_q)\{(I, J) \in \text{Gr}_a(p, V) \times \text{Gr}_a(q, V) | I \cap J \cap B \neq \emptyset \} =
\int_{I \in \text{Gr}_a(p, V)} \mu_q\{J \in \text{Gr}_a(q, V) | I \cap J \cap B \neq \emptyset \} d\mu_p,
\]

where \( B \) is a compact convex set of dimension \( p + q \). Take \( p = 1, q = k - 1 \), \( \delta_1 = d_1(A), \delta_{k-1} = d_1(A)^{k-1} \) and apply the induction hypothesis. We get

\[
(d_1(A))^k(B) = \int_{I \in \text{Gr}_a(1, V)} d_1(A)^{k-1}(B \cap I) d\mu_1,\phi =
\]

\[
= (k-1)! \int_{I \in \text{Gr}_a(1, V)} V_{k-1}(B \cap I) V_{k-1}(\pi_{D_I} A) d\mu_1,\phi,
\]

where \( D_I \) is the intersection of \( D \) with the hyperplane through the origin parallel to \( I \). Using (3.3) rewrite this as

\[
(d_1(A))^k(B) =
\]

\[
= (k-1)! \int_{H \in \text{Gr}(1, V)} \phi(H) V_{k-1}(\pi_{H^\perp \cap D} A) \left( \int_H V_{k-1}((h+H^\perp) \cap B) dh \right) dH.
\]

Now notice that

\[
\int_H V_{k-1}((h+H^\perp) \cap B) dh = V_k(B) \cos(H, D),
\]

hence

\[
(d_1(A))^k(B) =
\]

\[
= (k-1)! V_k(B) \int_{H \in \text{Gr}(1, V)} T_{A/H}^{-1} s_A(H) V_{k-1}(\pi_{H^\perp \cap D} A) \cos(H, D) dH.
\]

A parallel shift of \( A \) does not change the density \( d_1(A) \). Therefore we may assume that the center of symmetry of \( A \) is 0. Then Proposition 4.1 applies and our assertion follows. \( \square \)
By definition, a zonotope is a Minkowski sum of finitely many segments and a zonoid is a convex body that can be approximated, in Hausdorff metric, by a sequence of zonotopes. Zonoids are known to have a center of symmetry. The zonoids with center 0 are characterized by the fact that their support functions are obtained via cosine transform from non-negative even measures on the sphere, see [26].

Lemma 4.2. A zonoid can be approximated by a sequence of smooth zonoids.

Proof. The support function of the Minkowski sum of finitely many convex bodies is the sum of support functions of the summands. Thus, if all bodies are smooth then their Minkowski sum is also smooth. Therefore it suffices to prove the lemma for a segment. Consider any plane containing the segment \([a, b]\) and take the ellipse with focuses \(a, b\) in that plane. The segment can be approximated by ellipsoids obtained by rotating the ellipse around the axis through \(a\) and \(b\). It remains to show that any ellipsoid is a zonoid. This is clear for the sphere with center 0 because its support function is the cosine transform of a constant function. Since zonoids form an affine-invariant class of convex bodies, all ellipsoids are zonoids. \(\square\)

Lemma 4.3. If a compact convex body \(A\) is a zonoid then the density \(d_1(A)\) is normal.

Proof. We may assume that \(A\) has center 0. Let \(\nu\) be the even measure on the sphere, i.e., the measure on \(\text{Gr}(1, V)\), such that \(T_1(\nu) = s_A\). Then
\[
d_1(A) = \delta_{1,s_A} = \chi_1(\mu_1, \nu)
\]
by Proposition 3.6. \(\square\)

Theorem 7. Let \(A_1, \ldots, A_k\) be centrally symmetric compact convex sets in \(V\). Assume that each \(A_i\) is either smooth, or is a zonoid, or else is the Minkowski sum of the bodies of those two types. Then
\[
d_1(A_1) \cdot \ldots \cdot d_1(A_k) = k! d_k(A_1, \ldots, A_k).
\]

Proof. Note that \(d_1(A + B) = d_1(A) + d_1(B)\), so all densities \(d_1(A_i)\) are normal by Proposition 4.2 and Lemma 4.3. Thus the multiplication of \(d_1(A_i)\) makes sense. Assume first that all \(A_i\) are smooth convex bodies. The expression \((d_1(\lambda_1 A_1 + \ldots + \lambda_k A_k))^k\) is a homogeneous polynomial of degree \(k\) in \(\lambda_i\). The polarization formula for this polynomial reduces the theorem to Proposition 4.3.

More generally, assume that each convex body is of the form \(A_i + Z_i\), where \(Z_i\) is a zonoid. Using Lemma 4.2, we can approximate \(Z_i\) by
smooth zonoids $Z_{ij}$. Then
\[ d_1(A_1 + Z_{ij}) \cdots d_1(A_k + Z_{kj}) = k! d_k(A_1 + Z_{ij}, \ldots, A_k + Z_{kj}) \]
for any $j$. As $j \to \infty$, we have $d_1(A_i + Z_{ij}) \to d_1(A_i + Z_i)$ and $d_k(A_1 + Z_{1j}, \ldots, A_k + Z_{kj}) \to d_k(A_1 + Z_1, \ldots, A_k + Z_k)$. Moreover, the measures defining the densities $d_1(Z_{ij})$ are non-negative. Thus the assumptions of Proposition 3.4 are fulfilled for all sequences $\{d_1(Z_{ij})\}$, where $i = 1, \ldots, k$. It follows that the product on the left hand side tends to $d_1(A_1 + Z_1) \cdots d_1(A_k + Z_k)$.

**Proof of Theorem** Using an arbitrary Euclidean metric in the tangent space $T_x X$, identify $T_x X$ with its dual $T^*_x X$. Let $H$ be the subspace generated by $\xi_1, \ldots, \xi_k \subset T_x X$. The definition of $D_k(E)$ in 2.2 implies
\[ D_k(E)(\xi_1 \wedge \ldots \wedge \xi_k) = V_k(\pi_H E(x)) \cdot \text{vol}_k(\xi_1 \wedge \ldots \wedge \xi_k). \]
This shows that the density $D_k(E)$ and the mixed density $D_k(E_1, \ldots, E_k)$ on $T_x X$ coincide with $d_k(E(x))$ and $d_k(E_1(x), \ldots, E_k(x))$, respectively. Therefore the required assertion follows from Theorem 7.

4.3. **Crofton formula.** For $i = 1, \ldots, n$, let $E_i$ be a finite-dimensional real vector space with scalar product $\langle \cdot, \cdot \rangle_i$ and let $E^*_i$ be the dual Euclidean space. We denote by $S_i$ and $S^*_i$ the unit spheres in $E_i$ and $E^*_i$, respectively. Put $E = E_1 \times \ldots \times E_n$, $E^* = E^*_1 \times \ldots \times E^*_n$, $S = S_1 \times \ldots \times S_n$, and $S^* = S^*_1 \times \ldots \times S^*_n$. We consider a point $s^*_i \subset S^*_i$ as a linear function on $E_i$ and, also, as a linear function on $E$ obtained by an obvious lifting using the projection $E \to E_i$. Thus, the $n$-tuple $s^* = (s^*_1, \ldots, s^*_n) \subset S^*$ is a system of functions on $E$. Let $ds_i$ and $ds^*_i$ be the Euclidean volume densities on $S_i$ and $S^*_i$, respectively. We denote by $ds$ and $ds^*$ their normalized products, i.e., the densities on $S$ and $S^*$ equal to
\[ ds = \frac{1}{\sigma} \prod_i ds_i, \quad ds^* = \frac{1}{\sigma} \prod_i ds^*_i, \]
where $\sigma$ is the product of volumes of the unit spheres $S_i$ (or $S^*_i$).

Let $X \subset S$ be an embedded submanifold of dimension $n$. The number of isolated common zeros of the system of functions $s^*$ on $X$ is denoted by $N_X(s^*)$. By definition, the average number of isolated common zeros of all such systems is the integral
\[ \mathcal{M}_X = \int_{S^*} N_X(s^*) \, ds^*. \]

We now state a theorem showing that the integral exists and computing its value. For $x = (x_1, \ldots, x_n) \in S$ consider the tangent spaces $T_x = T_{x_i}(S_i) = \{ \xi_i \in E_i \mid \langle \xi_i, x_i \rangle_i = 0 \} \subset E_i$. For $\xi = (\xi_1, \ldots, \xi_n) \in$
\[ T = T_1 \oplus \ldots \oplus T_n, \] write \( g_i(\xi_i) = \langle \xi_i, \xi_i \rangle \) and \( \text{vol}_{1,i}(\xi) = \sqrt{g_i(\xi)} \). Then \( \text{vol}_{1,i} \) are 1-densities on \( S \). We recall that the product of 1-densities is defined in 3.1 and use the notations introduced there.

**Theorem 8.** (Crofton formula for the product of spheres).

\[ \mathcal{M}_X = \frac{1}{\pi^n} \int_X \text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n}. \]

In what follows, we use some standard notations. Namely, \( \sigma_p \) denotes the volume of the \( p \)-dimensional unit sphere and \( v_q \) the volume of the \( q \)-dimensional unit ball.

**Example 4.1.** Let \( C_i \subset S_i, i = 1, \ldots, n, \) be a circle with center 0 and let \( X = C_1 \times \ldots \times C_n \subset S_1 \times \ldots \times S_n \) be the \( n \)-dimensional torus. For \( x = (x_1, \ldots, x_n) \in X \) and \( \xi = (\xi_1, \ldots, \xi_n) \in T_x X \) we have \( \text{vol}_{1,i}(\xi) = |\xi_i| \).

Observe that for \( n \) pairwise orthogonal segments \( A_1, \ldots, A_n \) of length 1 in \( \mathbb{R}^n \) one has \( V(A_1, \ldots, A_n) = 1/n! \), hence \( d_1(A_1) \cdot \ldots \cdot d_1(A_n) \) is the Euclidean volume density by Theorem 7. Apply this to \( \mathbb{R}^n = T_x X \) and take \( A_i \) tangent to \( C_i \), so that \( d_1(A_i)(\xi) = |\xi_i| \). It follows that the restriction of the \( n \)-density \( \text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n} \) to \( X \) is the Riemannian volume density. Therefore Theorem 8 implies

\[ \mathcal{M}_X = \frac{1}{\pi^n} \int_X \text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n} = 2^n. \]

**Example 4.2.** (Crofton formula for the sphere). Let \( g \) be the Euclidean metric on \( E = \mathbb{R}^N \), \( B \) the unit ball in \( E \), \( S = \partial B \). The \( k \)-density on \( S \) defined by the induced metric is denoted by \( \text{vol}_{k,g} \). Put \( E_1 = \ldots = E_n = E \) and consider a manifold \( X \subset S \), \( \dim X = n \), embedded diagonally in the product of \( n \) spheres \( S \times \ldots \times S \). The classical Crofton formula for the sphere

\[ \mathcal{M}_X = \frac{2}{\sigma_n} \int_X \text{vol}_{n,g} \]

follows from Theorem 8. Indeed, restricting all densities to \( X \) we get \( \text{vol}_{1,i} = \text{vol}_{1,g} = \frac{1}{2} d_1(B) \) for all \( i \), where \( d_1(B) \) is a 1-density in \( E \). Hence, from Theorem 7 we obtain

\[ \frac{1}{\pi^n} (\text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n}) = \frac{d_1(B)^n}{(2\pi)^n} = \frac{n! d_n(B)}{(2\pi)^n} = \frac{n! v_n}{(2\pi)^n} \text{vol}_{n,g} = \frac{2}{\sigma_n} \text{vol}_{n,g}, \]

where the last equality follows from the relations

\[ v_n \sigma_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \cdot (n + 1) \frac{\pi^{n+1}}{\Gamma(n+1/2 + 1)} = \frac{2(2\pi)^n}{n!}. \]
The proof of Theorem 8 requires some preparations. In integral geometry, there is a standard technique producing an $n$-density $\Omega$ on $S$, such that

$$M_X = \int_X \Omega$$

for any $X \subset S$. Namely, $\Omega$ is obtained from $ds^*$ by the pull-back and push-forward operations in a certain double fibration

$$\begin{array}{c}
\pi_1 \\
\downarrow \\
S \\
\downarrow \\
\pi_2 \\
\downarrow \\
S^*
\end{array}$$

see [3], [15], [22], [27].

Let $\Gamma \subset S \times S^*$ be the submanifold defined by the equations $x_i^* (x_i) = 0$, $i = 1, \ldots, n$, where $x = (x_1, \ldots, x_n) \in S$, $x^* = (x_1^*, \ldots, x_n^*) \in S^*$. The projection mappings $S \times S^* \to S$ and $S \times S^* \to S^*$, restricted to $\Gamma$, are denoted by $\pi_1$ and $\pi_2$, respectively. With these notations, one has the following proposition.

**Proposition 4.4.** Let $\Omega = \pi_1^* \pi_2^* (ds^*)$. Then

$$M_X = \int_X \Omega.$$

**Proof.** See [3], [15].

From now on we identify $E_i$ with $E_i^*$ using given scalar products $\langle \cdot, \cdot \rangle_i$. As a consequence, $S_i$ is identified with $S_i^*$, $E$ with $E^*$ and $S$ with $S^*$, respectively. Furthermore,

$$\Gamma = \{(x, x^*) \in S \times S^* | \langle x_i, x_i^* \rangle_i = 0, \ i = 1, \ldots, n\}.$$

Let $\xi_i, \eta_i \in E_i$ be the components of $\xi, \eta \in E$, respectively.

**Lemma 4.4.** If $\xi_i \in T_i$ and $\eta_i = -\langle \xi_i, x_i^* \rangle_i x_i$ for all $i$, $i = 1, \ldots, n$, then $(\xi, \eta) \in T_{(x,x^*)}\Gamma$.

**Proof.** By the definition of $\Gamma$, the subspace $T_{(x,x^*)}\Gamma \subset E \oplus E$ is given by the equations

$$\langle \xi_i, x_i^* \rangle_i + \langle x_i, \eta_i \rangle_i = \langle \xi_i, x_i \rangle_i = \langle \eta_i, x_i^* \rangle_i = 0$$

for all $i$, $i = 1, \ldots, n$. If $\xi_i \in T_i$ and $\eta_i = -\langle \xi_i, x_i^* \rangle_i x_i$ for all $i$, then these equations hold true. □

The vector space $T = T_x S = T_1 \oplus \ldots \oplus T_n$ is equipped with the scalar product $\langle \cdot, \cdot \rangle_1 + \ldots + \langle \cdot, \cdot \rangle_n$. For each $i$ denote by $W_i$ the unit sphere with center 0 in $T_i$, put $W = W_1 \times \ldots \times W_n$ and take a point $w = (w_1, \ldots, w_n) \in W$. Let $\tilde{w}_i$ be an element of unit length in the highest
component of the exterior algebra of $T_w W_i$, i.e., the dual to a volume form of $W_i$.

**Lemma 4.5.** $i_{x_1 \wedge \ldots \wedge x_n} ds^*$ is the density of Riemannian volume of $W$ divided by vol($S$).

*Proof.* Note that $ds^*_i(x_i \wedge \tilde{w}_i) = 1$. Therefore, by the definition of $ds^*$, we have $ds^*(x_1 \wedge \ldots \wedge x_n \wedge \tilde{w}_1 \ldots \wedge \tilde{w}_n) = 1 / \text{vol}(S)$. □

In the following proposition, we compute the value of density $\Omega$ on the wedge product $\theta_1 \wedge \ldots \wedge \theta_n$, where $\theta_1, \ldots, \theta_n \in V$. As a warning, we remark that $\theta_i$ are not necessarily contained in $T_i$.

**Proposition 4.5.** Let $\Pi_\theta$ be the parallelotope generated by $\theta_i$. Take the image of $\Pi_\theta$ under the projection map $\pi_w : T \to \mathbb{R}w_1 + \ldots + \mathbb{R}w_n$ and denote by $\text{vol}_w(\Pi_\theta)$ the volume of $\pi_w(\Pi_\theta)$. Then

$$\Omega(\theta_1 \wedge \ldots \wedge \theta_n) = \frac{1}{\text{vol}(S)} \int_W \text{vol}_w(\Pi_\theta) \cdot dw_1 \cdot \ldots \cdot dw_n,$$

where $dw_i$ is the Euclidean volume form on the sphere $W_i$.

*Proof.* Let $\theta_i = (\theta_{i_1}, \ldots, \theta_{i_n})$, where $\theta_{ij} \in T_j$ and put

$$\eta_{x,w,\theta_i} = -\sum_{j=1}^n \langle \theta_{ij}, w_j \rangle_j \cdot x_j.$$

Then

$$h_i = (\theta_i, \eta_{x,w,\theta_i}) \in T_{(x,w)} \Gamma$$

by Lemma 4.4. Recall that $\Omega = \pi_1 \pi_2^*(ds^*)$. Since $\pi_1^{-1}(x) = \{x\} \times W$, we have

$$\Omega(\theta_1 \wedge \ldots \wedge \theta_n) = \int_{\{x\} \times W} i_{h_1 \wedge \ldots \wedge h_n} \pi_2^*(ds^*)$$

by the definition of push-forward. The projection map $\pi_2$ sends $h_i$ to $\eta_{x,w,\theta_i}$ and defines a diffeomorphism between $\pi_1^{-1}(x)$ and $W$. From the previous equality it follows that

$$\Omega(\theta_1 \wedge \ldots \wedge \theta_n) = \int_W i_{\eta_{x,w,\theta_1} \wedge \ldots \wedge \eta_{x,w,\theta_n}} ds^*.$$

Let $A$ be the matrix with entries $a_{ij} = \langle \theta_{ij}, w_j \rangle_j$. The definition of $\eta_{x,w,\theta_i}$ shows that

$$\eta_{x,w,\theta_1} \wedge \ldots \wedge \eta_{x,w,\theta_n} = \pm \det(A) x_1 \wedge \ldots \wedge x_n = \pm \text{vol}_w(\Pi_\theta) x_1 \wedge \ldots \wedge x_n.$$

The density $ds^*$ is non-negative, so the above formula for $\Omega(\theta_1 \wedge \ldots \wedge \theta_n)$ yields

$$\Omega(\theta_1 \wedge \ldots \wedge \theta_n) = \int_W \text{vol}_w(\Pi_\theta) \cdot i_{x_1 \wedge \ldots \wedge x_n} ds^*.$$
Together with Lemma 4.5 this completes the proof.

**Proof of Theorem 8** Let $D \subset T$ be a compact convex set of dimension $n$. Proposition 4.5 gives an expression for the value of $\Omega$ on $D$. If $m_i = \dim T_i$ then

$$\Omega(D) = \frac{1}{\prod_i \sigma_{m_i}} \cdot \int_{W_1 \times \ldots \times W_n} \operatorname{vol}_w(D) \cdot dw_1 \cdot \ldots \cdot dw_n. \quad (4.1)$$

We have to prove that

$$\Omega(D) = \frac{1}{\pi^n} (\operatorname{vol}_{1,1} \cdot \ldots \cdot \operatorname{vol}_{1,n})(D). \quad (4.2)$$

The density $\operatorname{vol}_{1,i}$ is the pull-back of the Riemannian 1-density $\operatorname{vol}_{1,g_i}$ on $T_i$ under the projection map $\pi_i : T \to T_i$. According to (3.2), the gauge function $\phi$ on $\operatorname{Gr}(1,T_i)$, associated with $\operatorname{vol}_{1,g_i}$, equals 1. If $\Phi$ is the preimage of $\phi$ under the cosine transform then

$$\Phi = \sigma \frac{m_{i-1}}{2\nu_{m_i-1}}$$

by a direct calculation. Applying Proposition 3.3 we get $\operatorname{vol}_{1,g_i} = \chi_1(\mu_i)$ for the normal measure $\mu_i = \mu_{1,\phi}$ on $\operatorname{Gr}(1,T_i)$ defined by (3.3). Namely, for a subset $A \subset \operatorname{Gr}(1,T_i)$ and for $H \in \operatorname{Gr}(1,T_i)$ put $H_A = \{h \in H \mid h + H^\perp \in A\}$. Then

$$\mu_{g_i}(A) = \sigma \frac{m_{i-1}}{2\nu_{m_i-1}} \int_{\operatorname{Gr}(1,T_i)} \lambda_i(H_A) dH, \quad (4.3)$$

where $\lambda_i$ is the Lesbesgue measure corresponding to $g_i$ on the line $H$.

Assume first that $n = 1$, so that $\operatorname{vol}_{1,1} = \operatorname{vol}_{1,g_i}$. If $A \subset \operatorname{Gr}(1,T)$ is formed by affine hypersurfaces intersecting $D$, then

$$\Omega(D) = \frac{1}{\sigma_{m_1}} \int_{W_1} \operatorname{vol}_w(D) dw = \frac{\sigma_{m_1-1}}{\sigma_{m_1}} \int_{\operatorname{Gr}(1,T_1)} \lambda_1(H_A) dH =$$

$$= \frac{\sigma_{m_1-1}}{\sigma_{m_1}} \cdot \frac{2\nu_{m_1-1}}{\sigma_{m_1-1}} \cdot \mu_{g_1}(A) = \frac{\mu_{g_1}(A)}{\pi} = \frac{\operatorname{vol}_{1,1}(D)}{\pi}.$$
where
\[ J_n(D) = \{(H_1, \ldots, H_n) \in (\text{Gr}_a(1, T))^n \mid D \cap H_1 \cap \ldots \cap H_n \neq \emptyset\}. \]

The support of \( \mu_i \) is the set \( \mathcal{G}_i \) of affine hyperplanes containing affine shifts of \( \bigoplus_{j \neq i} T_j \). For any set \( A \subset \mathcal{G}_i \) its measure \( \mu_i(A) \) is equal to the value of \( \mu_i \) on the projection of \( A \) onto \( \text{Gr}_a(1, T_i) \). The product measure \( \mu_1 \cdot \ldots \cdot \mu_n \) is supported on the surface \( \mathcal{G} \subset \text{Gr}_a(n, T) \) formed by affine subspaces \( G_1 \cap \ldots \cap G_n \), where \( G_i \in \mathcal{G}_i \). For \( G \in \text{Gr}_a(1, T_n) \) put \( \bar{G} = \pi_{n-1}(G) \). Let
\[ I(D \cap \bar{G}) = \{(G_1, \ldots, G_{n-1}) \mid G_i \in \mathcal{G}_i, D \cap \bar{G} \cap G_1 \cap \ldots \cap G_{n-1} \neq \emptyset\}. \]

Then
\[ (\mu_1 \cdot \ldots \cdot \mu_n)(J_n(D)) = \int_{G \in \text{Gr}_a(1, T_n)} (\mu_1 \cdot \ldots \cdot \mu_{n-1})(I(D \cap \bar{G})) \, d\mu_{g_n} \]
by Fubini’s theorem. Let \( U = T_1 + \ldots + T_{n-1} \) and let \( \pi_U : T \to U \) be the projection map. We use the same notation \( \mu_1 \cdot \ldots \cdot \mu_{n-1} \) for the product of measures \( \mu_i \) on \( \text{Gr}_a(n-1, U) \) and for its pull-back under \( \pi_U \) on \( \text{Gr}_a(n-1, T) \). Keeping this in mind, we get
\[ (\mu_1 \cdot \ldots \cdot \mu_{n-1})(I(D \cap \bar{G})) = (\mu_1 \cdot \ldots \cdot \mu_{n-1})(J_{n-1}(\pi_U(D \cap \bar{G}))). \]
Applying the definition of \( \Omega \) to \( U \), we get an \( (n-1) \)-density, to be denoted by \( \Omega_U \). For \( H \in \text{Gr}(1, T_n) \) we denote by \( H^\perp \) the orthogonal complement to \( H \) in \( T \). By induction and by (4.3) for \( i = n \) we have
\[ (\text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n})(D) = \int_{G \in \text{Gr}_a(1, T_n)} (\text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n-1})(\pi_U(D \cap \bar{G})) \, d\mu_{g_n} = \]
\[ = \pi^{n-1} \int_{G \in \text{Gr}_a(1, T_n)} \Omega_U(\pi_U(D \cap \bar{G})) \, d\mu_{g_n}. \]
Finally, using the expression (4.3) for \( \mu_{g_n} \), we obtain
\[ (\text{vol}_{1,1} \cdot \ldots \cdot \text{vol}_{1,n})(D) = \]
\[ = \frac{\sigma_{m_{n-1}} \pi^{n-1}}{2^{m_{n-1}}} \int_{H \in \text{Gr}(1, T_n)} \int_{h \in H} \Omega_U(\pi_U(D \cap (h + H^\perp))) \, dh \, dH. \]
On the other hand, by Fubini’s theorem we can write (4.1) in the form
\[ \Omega(D) = \frac{1}{\prod_{i=1}^n \sigma_{m_i}} \cdot \int_{W_n} F(w_n) \, dw_n, \]
where
\[ F(w_n) = \int_{W_1 \times \ldots \times W_{n-1}} \text{vol}_W(D) \, dw_1 \cdot \ldots \cdot w_{n-1}. \]
Let $\omega$ and $\omega'$ be the subspaces generated by $w_1, \ldots, w_{n-1}, w_n$ and $w_1, \ldots, w_{n-1}$, respectively. Denote by $H$ the line generated by $w_n$. Then

$$\text{vol}_w(D) = \int_H V_{n-1}(\pi_\omega(D) \cap (h + H^\perp)) \, dh.$$ 

Plug this in the integral $F(w_n)$ and observe that $\pi_\omega(D) \cap (h + H^\perp) = h + \pi_{\omega'}(D \cap (h + H^\perp)) = h + \pi_{\omega'}(D \cap (h + H^\perp)).$ As a result, we get

$$F(w_n) = \int_H \int_{W_1 \times \ldots \times W_{n-1}} V_{n-1}(\pi_{\omega'}(D \cap (h + H^\perp))) = \prod_{i=1}^{n-1} \sigma_m \cdot \int_H \Omega_U(\pi_U(D \cap (h + H^\perp)) \, dh)$$

by the definition of $\Omega_U$. Therefore

$$\Omega(D) = \frac{1}{\sigma_m} \int_{W_n} \int_H \Omega_U(\pi_U(D \cap (h + H^\perp)) \, dh \, dw_n.$$ 

Passing from the sphere to the projective space, we obtain

$$\Omega(D) = \frac{\sigma_m^{-1}}{\sigma_m} \int_{\text{Gr}(1,T_n)} \int_H \Omega_U(\pi_U(D \cap (h + H^\perp)) \, dh \, dH.$$ 

Since $\sigma_m = 2\pi \cdot v_{m-1}$, equality (4.2) follows from (4.3). \qed

### 4.4. Proof of Theorem 1

We use the notations introduced in 4.3 with $E_k = V_k^*$. Then $S_k \subset V_k^*$, $S_k^* \subset V_k$, and $g_k$ is the pull-back of the metric form on $V_k$ under $\theta_k : X \to S_k^* \subset V_k$. For $s^* = (s_1^*, \ldots, s_n^*) \in S^* = S_1^* \times \ldots \times S_n^*$ we denote by $N_U(s_1^*, \ldots, s_n^*)$ the number of isolated common zeros of $s_1^*, \ldots, s_n^*$ in an open set $U \subset X$. We want to prove that $N_X(s_1^*, \ldots, s_n^*)$ is integrable and compute the integral whose value divided by $\sigma$ is denoted by $\mathcal{M}_X(V_1, \ldots, V_n)$, see 2.1. Theorem 1 is a consequence of the equality

$$\mathcal{M}_X(V_1, \ldots, V_n) = \frac{1}{\pi^n} \int_X \text{vol}_{1,g_1} \cdot \ldots \cdot \text{vol}_{1,g_n}. $$

Indeed, if $E_i$ is the ellipsoid corresponding to $g_i$ then for all $x \in X, \xi \in T_x(X)$ one has

$$\text{vol}_{1,g_i}(\xi) = \frac{d_1(E_i(x))(\xi)}{2},$$

where $d_1(E_i(x))$ is considered as a translation invariant 1-density. By Theorem 7 we have

$$\frac{1}{\pi^n} \text{vol}_{1,g_1} \cdot \ldots \cdot \text{vol}_{1,g_n} = \frac{n!}{(2\pi)^n} d_n(E_1(x), \ldots, E_n(x)).$$
By the definition of $d_n$, it follows that
\[ \frac{1}{\pi^n} (\text{vol}_{1,g_1} \ldots \text{vol}_{1,g_n}) (\xi_1 \land \ldots \land \xi_n) = \frac{n!}{(2\pi)^{n}} V_n(\xi_1(x), \ldots, \xi_n(x)) \text{vol}_n(\Pi \xi), \]
where $\text{vol}_n$ is taken with respect to $g$ on $T_x(X)$ and the mixed volume $V_n$ with respect to $g^*$ on $T^*_x(X)$. Combining this equality with (2.1), we get Theorem 11 from (1.5).

Now, if the mapping $\theta = \theta_1 \times \ldots \times \theta_n$ is an embedding of $X$ into $S^*_1 \times \ldots \times S^*_n$ then (1.5) is a direct consequence of Crofton formula (Theorem 8) and equality (2.1). In the general case, it is enough to prove (1.5) for an arbitrary relatively compact domain $U \subset X$. In fact, $X$ is the union of an increasing sequence of such domains $U_i$. Thus, if
\[ (4.6) \quad \mathfrak{M}_U(V_1, \ldots, V_n) = \frac{1}{\pi^n} \int_U \text{vol}_{1,g_1} \ldots \text{vol}_{1,g_n} \]
for each $U$ then we get two increasing sequences on the both sides of (4.6) for $U = U_i$ and (1.5) is obtained by passing to the limit. We will now prove (1.6). Let $D$ be the set of critical points of $\theta$ in the closure $\bar{U}$ of a relatively compact domain $U \subset X$. If $D \neq \emptyset$ then take a decreasing sequence of relatively compact neighborhoods $D_i$ of $D$, such that $\cap D_i = D$. To finish the proof, we need three lemmas.

**Lemma 4.6.** If $\theta$ has no critical points in $U$ then (4.6) holds true.

**Proof.** Assume first that $D = \emptyset$. Then there is a finite covering $\{U_i\}$ of $U$ such that $\theta|_{U_i}$ is a closed embedding for every $i$. We know that (4.6) is valid for all intersections $U_{i_1} \cap \ldots \cap U_{i_k}$. Since the both parts are additive functions of a domain, (4.6) for the union of $U_i$ follows by inclusion-exclusion principle. In case $D \neq \emptyset$ one can apply (4.6) to $U \setminus D_i$ and get the required assertion as $i \to \infty$. 

**Lemma 4.7.** One has
\[ \lim_{i \to \infty} \int_{D_i} \text{vol}_{1,g_1} \ldots \text{vol}_{1,g_n} = 0. \]

**Proof.** Choosing a Riemannian metric $g$ on $X$ we can write
\[ (\text{vol}_{1,g_1} \ldots \text{vol}_{1,g_n})(\xi_1 \land \ldots \land \xi_n) = f(x) \cdot \text{vol}_{n,g}(\xi_1 \land \ldots \land \xi_n), \]
where $\xi_i \in T_x$. Then $f(x)$ is a non-negative continuous function and $\max_{x \in D_i} f(x) \to 0$ as $i \to \infty$, hence the assertion. 

For an arbitrary subset $A_k \subset S_k$ denote by $A^*_k \subset S^*_k$ the set of linear functions with a zero in $A_k$. Consider $s^* = (s^*_1, \ldots, s^*_n) \in S^*$ as an $n$-tuple of functions on $S$. For $A \subset S$ put $A^* = \{s^* \in S^* \mid \exists x \in A : \forall i s^*_i(x) = 0\}$. Note that if $A = A_1 \times \ldots \times A_n$ then $A^* = A^*_1 \times \ldots \times A^*_n$. 

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Lemma 4.8. Let $D \subset S$ be the set of critical values of $\theta$ on $U$. Then $D^* \subset S^*$ has measure 0.

Proof. By Sard’s lemma the $n$-dimensional Hausdorff measure of $D$ is 0. Consider a covering of $D$ by open sets $B_i = B_{i,1} \times \ldots \times B_{i,n}$, where $B_{i,k} \subset S_k$ are open balls of the same radius $r(B_i)$ for any given $i$. For $\epsilon > 0$ small enough there exists such a covering with $r(B_i) < \epsilon$ and $\sum_i r^n(B_i)$ arbitrarily small. Let $D^*(\epsilon) = \cup_i B_i^*$ for such a covering. Then $B_i^* = B_{i,1}^* \times \ldots \times B_{i,n}^*$ by construction, hence

$$\text{vol}_{S^*}(B_i^*) = \prod_{k=1}^n \text{vol}_{S_k^*} B_{i,k}^* < C r^n(B_i).$$

By the definition of Hausdorff measure $\sum r^n(B_i) \to 0$ as $\epsilon \to 0$. Therefore $\text{vol}(D^*(\epsilon)) \to 0$. □

End of the proof. Let $U$ be a relatively compact domain in $X$. Then $N_U^*(s_1^*, \ldots, s_n^*) = N_{U \setminus D}(s_1^*, \ldots, s_n^*)$ almost everywhere on $S^*$ by Lemma 4.8. But $N_{U \setminus D}$ is integrable by Lemma 4.6. Thus $N_U$ is also integrable and the integrals of $N_U$ and $N_{U \setminus D}$ are equal. Finally, by Lemma 4.7 one can replace the integration domain $U$ on the right hand side of (4.6) by $U \setminus D$. □

References

[1] D. Akhiezer, B. Kazarnovskii, On common zeros of eigenfunctions of the Laplace operator, Abh. Math. Sem. Univ. Hamburg 87, no.1 (2017), 105 – 111, DOI:10.1007/s12188-016-0138-1.

[2] D. Akhiezer, B. Kazarnovskii, An estimate for the average number of common zeros of Laplacian eigenfunctions, Trudy Mosk. Matem. Obshchestva, vol. 78, no. 1 (2017), pp. 145 – 154 (in Russian); Trans. Moscow Math. Soc. 2017, pp. 123 – 130 (English translation).

[3] J.-C. Álvarez Paiva, E. Fernandes, Gelfand transforms and Crofton formulas, Selecta Math., vol. 13, no.3 (2008), pp.369 – 390.

[4] A.D. Aleksandrov, To the theory of mixed volumes of convex bodies. Part II: New inequalities for mixed volumes and their applications, Matem. Sbornik, vol. 2, no. 6 (1937), pp. 1205 – 1235 (in Russian); pp. 61 – 98 in: Selected Works, Part I, ed. by Yu.G. Reshetnyak and S.S. Kutateladze, Amsterdam, Gordon and Breach, 1996 (English translation).

[5] S. Alesker, Theory of valuations on manifolds: a survey, Geom. Funct. Anal. 17, no.4 (2007), 1321–1341.

[6] S. Alesker, A Fourier type transform on translation invariant valuations on convex sets, Israel J. Math. 181, no.1 (2011), pp. 189–294.

[7] S. Alesker, J. Bernstein, Range characterization of the cosine transform on higher Grassmanians, Advances in Math., vol. 184, no.2 (2004), pp.367 – 379
V.I. Arnold (ed), *Arnold’s Problems*, Springer, 2005.

A. Bernig, *Valuations with Crofton formula and Finsler geometry*, Advances in Math., 210, no.2 (2007), pp.733–753.

D.N. Bernstein, *The number of roots of a system of equations*, Funct. Anal. Appl. 9, no.2 (1975), pp.95–96 (in Russian); Funct. Anal. Appl. 9, no.3 (1975), pp.183–185 (English translation).

R. Courant & D. Hilbert, *Methods of Mathematical Physics, I*, Interscience Publishers, New York, 1953.

D. Faifman, *Crofton formulas and indefinite signature*, Geom. Funct. Anal. 27, no. 3 (2017), pp. 489–540.

S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, Third edition, Springer, 2004.

R. Gardner, *Geometric tomography*, Cambridge Univ. Press, 1995.

I.M. Gelfand, M.M. Smirnov, *Lagrangians satisfying Crofton formulas, Radon transforms, and nonlocal differentials*, Advances in Math. 109, no.2 (1994), pp. 188 – 227.

V.M. Gichev, *Metric properties in the mean of polynomials on compact isotropy irreducible homogeneous spaces*, Anal. Math. Phys. 3, no.2 (2013), pp. 119 – 144.

P. Goodey, R. Schneider, W. Weil, *On the determination of convex bodies by projection functions*, Bulletin of London Math. Soc., 29, no.1 (1997), pp.82–88.

B. Kazarnovskii, *On the zeros of exponential sums*, Doklady AN SSSR, vol. 257, no.4 (1981), pp.804–808 (in Russian); Soviet Math. Dokl., vol. 23, no.2 (1981), pp.347–351.

B. Kazarnovskii, *Newton polyhedra and zeros of systems of exponential sums*, Funct. Anal. Appl., vol. 18, no.4 (1984), pp.40–49 (in Russian); Funct. Anal. Appl., vol. 18, no.4 (1984), pp.299–307 (English translation).

A.G. Khovanskii, *Algebra and mixed volumes*, pp.182–207 in: Y.D. Burago, V.A. Zalgaller, *Geometric inequalities*, Springer, 1988.
