POWERFUL AND MAXIMAL RATIONAL METRIC DIMENSION OF A WHEEL

M. M. PADMA, B. SOORYANARAYANA*, M. JAYALAKSHMI

Department of Mathematics, Dr.Ambedkar Institute of Technology, Bengaluru- 560 056, Karnataka State, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. The rational distance from the vertex $u$ to the vertex $v$ in a graph $G$, denoted by $d(v/u)$, is defined as the average distances from the vertex $u$ to the closed neighbors of $v$ if $u \neq v$, else it is 0. A subset $S$ of vertices of $G$ is called rational resolving set of $G$ if for every pair $u, v$ of distinct vertices in $V - S$, there is a $w \in S$ such that $d(u/w) \neq d(v/w)$ in $G$. In this paper powerful and maximal rational resolving sets are introduced and minimum cardinality of such sets are computed for the wheel graphs.

Keywords: resolving sets; rational resolving sets; rational metric dimension; wheel graphs.

2010 AMS Subject Classification: 05C56, 05C12.

1. INTRODUCTION

Let $G(V, E)$ be a connected simple finite graph. A path $P_{a,b}$ from the vertex $a$ to $b$ is an alternating sequence of distinct vertices and edges starting with $a$ and ending with $b$ in such a way each edge lies between its end vertices. The number of edges in a path $P_{a,b}$ is called the length of the path and is denoted by $l(P_{a,b})$. The distance between two vertices $a$ and $b$ in $G$, denoted by $d_G(a, b)$ (or simply $d(a, b)$), is the minimum length of a path between $a$ and $b$. That is, $d_G(a, b) = \min\{l(P_{a,b})\}$. The number of edges incident with a vertex $v$ of $G$ is the degree of

*Corresponding author

E-mail address: dr.bsnrao@yahoo.co.in

Received June 2, 2021
the vertex $v$ in $G$ and is denoted by $\deg_G(v)$ or simply $\deg(v)$. Further, the closed neighborhood set of a vertex $v \in V$, denoted by $N[v]$, is defined as $N[v] = \{w : d(v, w) \leq 1\}$.

The notion of rational distance is introduced in [12]. The rational distance from the vertex $u$ to the vertex $v \in V$, denoted by $d(v/u)$, is defined as

$$d(v/u) = \begin{cases} 0, & \text{if } v = u, \\
\sum_{w \in N[v]} \frac{d_G(u, w)}{\deg_G(v) + 1}, & \text{otherwise.}
\end{cases}$$

Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ be an arbitrary ordered set. Then, for each vertex $v$ of $G$, we can always associate a vector (called rational code of $v$) with respect to $S$, denoted by $\Gamma(v/S)$, as

$$\Gamma(v/S) = (d(v/s_1), d(v/s_2), \ldots, d(v/s_k)).$$

A subset $S \subseteq V$ is called a rational resolving set if $\Gamma(u/S) \neq \Gamma(v/S)$ for each pair $u, v$ of distinct vertices in $V$. By the definition it follows that the rational resolving property is super hereditary. That is, if $S$ is a rational resolving set of $G$ then so as every super set of $S$. The minimum cardinality of a rational resolving set is called the rational metric dimension of $G$ and is denoted by $rmd(G)$. Further, each rational resolving set with cardinality $rmd(G)$ is called an $rmd$-set of $G$.

The rational resolving sets are defined in [11] and studied by various authors in [6, 7, 8, 9, 10, 12]. We recall that a subset $S \subseteq V$ is a resolving set of $G$ if for each pair $u, v \in V$ there exists a vertex $w \in S$ such that $d(v, w) \neq d(u, w)$. The metric dimension of $G$, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of $G$. A resolving set with minimum cardinality is called a metric basis. The concept of metric dimension was introduced by F. Harary and R. A. Melter [5] and independently by P. J. Slater [16] under the term locating set. For more works on metric dimension, we refer to [3, 4, 7, 13, 14, 15, 17, 18, 19, 21].

The complement of a minimum dominating set is also a dominating set. But, the complement of a rational resolving set of minimum cardinality need not be a rational resolving set. For example every rational resolving set of a triangle should include at least 2 vertices of it and hence its complement is not.

Recently, in 2021, an attempt is made by B. Sooryanarayana, Suma A. S and Chandrakala S. B in [22] to study some special classes of resolving sets as an extension to the earlier work.
of B. Sooryanarayana and Suma A. S [20]. In this paper, we obtain similar results on rational resolving sets of a wheel.

Throughout this paper, $C_m$ denotes a cycle on $m$ vertices with the vertex set $V = \{v_i : 0 \leq i \leq m - 1\}$ and the edge set $E = \{v_iv_{i+1 \pmod m} : 0 \leq i \leq m - 1\}$; $K_m$ denotes the complete graph on $m$ vertices with the vertex set $V = \{v_i : 1 \leq i \leq m\}$ and the edge set $E = \{v_iv_j : i \neq j, 1 \leq i, j \leq m\}$; and $W_{1,m}$ denotes wheel graph on $m + 1$ vertices with vertex set $V = \{v_i : 0 \leq i \leq m - 1\} \cup \{c_0\}$ and edge set $E = \{c_0v_i, v_iv_{i+1 \pmod m} : 0 \leq i \leq m - 1\}$. The vertex $c_0$ is called the central vertex and each $v_i$, $0 \leq i \leq m - 1$, is called a rim vertex of $W_{1,m}$. The terms not defined here may be found in [1, 4].

2. Rational Distances in a Wheel

Since the diameter of the graph $W_{1,m}$ is 2, it follows that $d(u,v) \in \{0, 1, 2\}$ for all $u,v \in V(W_{1,m})$. Hence it is easy to see that each component of the rational vertex code $\Gamma(v_i/S) \in \{0, 1, 3, 7, \frac{3}{2}, \frac{7}{4}\}$ for $0 \leq i \leq m - 1$, and each component of $\Gamma(c_0/S)$ is $\frac{2(m-3)+3}{m+1}$ (Note that $c_0 \notin S \subseteq V(W_{1,m})$) whenever $m \geq 5$. If $m = 3$, then the components of $\Gamma(c_0/S)$ and $\Gamma(v_i/S)$ are in $\{0, \frac{3}{4}\}$ for $0 \leq i \leq 2$. If $m = 4$, then $\Gamma(v_i/S) \in \{0, 1, \frac{5}{4}\}$ for $0 \leq i \leq 3$.

We recall the following results in [11] for immediate reference.

**Theorem 2.1 ([11]).** For any integer $n \geq 3$,

$$rmd(W_{1,m}) = \begin{cases} 3, & \text{if } n = 3, \\ 2, & \text{if } 4 \leq m \leq 9, \\ \lceil \frac{n}{4} \rceil - 1, & \text{if } n \geq 10 \text{ and } n \equiv 1 \pmod{8}, \\ \lceil \frac{n}{4} \rceil, & \text{otherwise.} \end{cases}$$

Throughout this paper, let $\mathcal{R}(G)$ be the collection of all rational resolving sets of the graph $G$. Then $\mathcal{R}(G)$ is super hereditary, that is for every $S \in \mathcal{R}(G)$, the set $T \in \mathcal{R}(G)$ whenever $S \subseteq T$.

3. Gap in a Wheel

We first define the gap between two vertices in $G$ with respect to a set $S \in \mathcal{R}(G)$.

**Definition 3.1.** Let $G$ be a graph and $S \in \mathcal{R}(G)$. Let $x,y \in V$ and $\bar{S} = V(G) - S$. An $\bar{S}$ path between $x$ and $y$ in $G$ is an $xy$-path of $G$ containing all its internal vertices in $\bar{S}$. The gap between
Let $m$ be the path between $x$ and $y$ with respect to $S$, denoted by $g_S(x, y)$, is defined as the minimum number of vertices of $S$ in an $S$ path (if it exists) between $x$ and $y$, else it is 0.

**Example:** Consider the graph $G$ of Figure 2. Let $S = \{c_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}\}$. Then an $S$ path between $v_1$ and $v_8$ is $P : v_1 - v_2 - v_3 - \cdots - v_7 - v_8$. $g_S(v_1, v_8) = |V(P) \cap S| = 6$. Also, taking $S_1 = \{v_1, v_8, v_{13}, v_{14}, v_{18}, v_{19}, v_{20}\}$, we see that $P : v_1 - c_0 - v_8$ is an $S_1$ path between $v_1$ and $v_8$ with minimum length. Hence, $g_{S_1}(v_1, v_8) = 1$. Similarly, $g_S(v_8, v_{19}) = 0$, $g_S(v_{18}, v_{20}) = 0$.

We now begin with the following lemma which we often use in the proof of next theorems.

**Lemma 3.2.** Let $m \in \mathbb{Z}^+$ and $m \geq 9$. Then a set $S$ containing at least 3 rim vertices is in $\mathcal{J}(W_{1, m})$ if and only if the following hold.

i) $g_{S \cup \{c_0\}}(a, b) \leq 5$ for all $a, b \in S$ and $g_{S \cup \{c_0\}}(a, b) = 5$ for at most one pair $a, b \in S$.

ii) If $3 \leq g_{S \cup \{c_0\}}(a, b) \leq 5$ for any $a, b \in S$, then $g_{S \cup \{c_0\}}(b, c) \leq 2$ and $g_{S \cup \{c_0\}}(a, c) \leq 2$, for every $c \in S$.
Figure 2. The graph between the vertices with respect to the set $S = \{c_0, v_1, v_8, v_{13}, v_{14}, v_{18}, v_{19}, v_{20}\}$.

Proof. Let $S \in \mathfrak{R}(W_{1,m})$ and $|S| = k$. Let $T = S \cup \{c_0\}$. Let us suppose to the contrary that the condition (i) fails. Then either $g_T(a, b) \geq 6$ for some $a, b \in S$ or there are vertices $a, b, c, d \in S$ with $|\{a, b\} \cup \{c, d\}| \geq 3$ such that $g_T(a, b) = 5$ and $g_T(c, d) = 5$.

Case 1: $g_T(a, b) \geq 6$.

In this case, there are at least six consecutive rim vertices, say $v_1, v_2, v_3, v_4, v_5, v_6$, in $\overline{S}$ with $v_1$ adjacent to $a$. But then, $\Gamma(v_3/S) = \Gamma(v_4/S) = (a_1, a_2, \ldots, a_k)$, where $a_i = 7/4$ for $1 \leq i \leq k$, a contradiction to the fact that $S \in \mathfrak{R}(W_{1,m})$.

Case 2: $g_T(a, b) = 5$ and $g_T(c, d) = 5$.

In this case, there are five consecutive rim vertices, say $v_1, v_2, v_3, v_4, v_5$, in $\overline{S}$ with $v_1$ adjacent to $a$, and there are five consecutive rim vertices $u_1, u_2, u_3, u_4, u_5$ in $\overline{S} - \{v_1, v_2, v_3, v_4, v_5\}$ with $u_1$ adjacent to $c$. But then, $\Gamma(u_3/S) = \Gamma(v_3/S) = (a_1, a_2, a_3, \ldots, a_k)$ where $a_i = 7/4$ for all $1 \leq i \leq k$, a contradiction to the fact that $S \in \mathfrak{R}(W_{1,m})$.

In case if the condition (ii) fails, then there are three vertices $a, b, c$ in $S$ such that $g_T(a, b) \in \{3, 4, 5\}$ and, $g_T(b, c) \geq 3$ or $g_T(a, c) \geq 3$. Without loss of generality, we take $g_T(b, c) \geq 3$. Then there are three consecutive rim vertices $v_1, v_2, v_3$ in $W_{1,m}$ with $v_1$ adjacent to $b$. Also there are three consecutive rim vertices $u_1, u_2, u_3$ with $u_1$ adjacent to $b$. But then, $\Gamma(v_1/S) = \Gamma(u_1/S) = (a_1, a_2, \ldots, a_k)$ where $a_i = 7/4$ for all $1 \leq i \leq k$, except for one $i$ for which $a_i = 1$.
(which corresponds to the vertex $b$), a contradiction to the fact that $S \in \mathcal{R}(W_{1,m})$. Hence the conditions $(i)$ and $(ii)$ hold.

Now to prove the converse part, let $S$ be a $k$-element subset of the vertex set of $W_{1,m}$ containing at least three rim vertices such that every pair of vertices in it satisfies the conditions $(i)$ and $(ii)$ of the theorem. We now show that $S \in \mathcal{R}(W_{1,m})$ by the method of contradiction. If $S$ does not belong to $\mathcal{R}(W_{1,m})$, then there are two vertices $u, v \in V - S$ such that $d(u/w) = d(v/w)$ for every $w \in S$.

**Case 1:** $d(u/w) = d(v/w) = 1$ for some $w \in S$.

In this case $u$ and $v$ are the rim vertices of $W_{1,m}$ and $w$ is adjacent to $u$ and $v$ ($\because m \neq 4$).

Subcase 1a: $d(u/w_1) = d(v/w_1) = 1$ for some $w_1 \in S$.

In this case, $w_1$ is also adjacent to $u$ and $v$, and $w_1$ is the rim vertex, and hence $m = 4$, a contradiction.

Subcase 1b: $d(u/w_1) = d(v/w_1) = 3/2$ for some $w_1 \in S$.

This case is possible only if $m = 6$ ($\because d(u,w) = d(v,w) = 1$ and $d(w_1,u) = d(w_1,v) = 2$ on the cycle $C_m$ of $G$), a contradiction to the fact that $m \geq 9$.

From the above two sub cases we see that only possibility is $d(u/w_i) = d(v/w_i) = 7/4$ for every $w_i \in S$ other than $w$ whenever $d(u/w) = d(v/w) = 1$. But then, for the vertices $s_1, s_2$ in $S$ nearer to $w$ in the cycle $C_m$ of $W_{1,m}$ we see that $g_r(s_1,w) \geq 3$ and $g_r(s_1,s_2) \geq 3$, a contradiction to the assumption of the condition $(ii)$.

**Case 2:** $d(u/w) = d(v/w) \neq 1$ for any $w \in S$.

In this case neither $u$ nor $v$ is non-adjacent to $w$, for every $w \in S$. We first show that $d(u/w) = d(v/w) = \frac{7}{4}$ for all $w \in S$. For this, let us assume to the contrary that $d(u/w') = d(v/w') = 3/2$ for some $w' \in S$.

**Claim:** $d(u/w) = d(v/w) \neq 3/2$ for any $w \in S - \{w'\}$.

If possible, suppose to contrary that $d(u/w_1) = d(v/w_1) = 3/2$ for some $w_1 \in S - \{w'\}$. Then $w_1$ can not be in a shortest $uw'$-path or $vv'$-path. Hence $w_1$ should be in a $uv$-path of $C_m$ not containing $w'$. So, $d_{C_m}(u,w_1) = d_{C_m}(v,w_1) = d_{C_m}(u,w') = d_{C_m}(v,w') = 2$. This is possible only if $m = 8$, a contradiction to the fact that $m \geq 9$. Hence the claim.
By the above claim, \( d(u/w) = d(v/w) = 7/4 \) for all \( w \in S - \{w'\} \). So, \( d_{C_m}(u,w_i) \geq 3 \) for all \( w_i \in S - \{w'\} \). Since \( S \) contains at least 3 rim vertices, \( |S - \{w'\}| \geq 3 \). Let \( s_1 \) and \( s_2 \) be the two rim vertices in \( S - \{w'\} \) which are nearer to \( w' \) in \( C_m \). Then \( s_1 \) as well as \( s_2 \) can not be in a shortest \( w'v \) path or \( w'u \) path (else \( d(u/s_1) \neq d(v/s_1) \), a contradiction to the assumption of \( u \) and \( v \)). Without loss of generality, let \( u \) be in the shortest \( w's_1 \) path and \( v \) be in the shortest \( w's_2 \) path. Then, as \( d_{C_m}(s_1,u) \geq 3 \) and \( d_{C_m}(s_2,v) \geq 3 \), we get \( g_{S \cup \{c_0\}}(s_1,w') \geq 4 \) and \( g_{S \cup \{c_0\}}(w,s_2) \geq 4 \), which is a contradiction to the assumption of condition (ii) of the lemma.

Thus, we have arrived at the conclusion that \( d(u/w) = d(v/w) = 7/4 \) for all \( w \in S \). That is \( \Gamma(u/S) = \Gamma(v/S) = (a_1,a_2,...a_k) \) where \( a_i = 7/4 \), for all \( 1 \leq i \leq k \). This is possible only if one of the following hold.

1. Both \( u \) and \( v \) lie in a \( S \cup \{c_0\} \) path between some \( w_1,w_2 \in S \) with \( g_{S \cup \{c_0\}}(w_1,w_2) \geq 6 \).
2. The vertex \( u \) is in the center of the \( S \cup \{c_0\} \) path between \( w_1 \) and \( w_2 \) for some \( w_1,w_2 \in S \), and, \( v \) is in the center of \( S \cup \{c_0\} \) path between \( w_3 \) and \( w_4 \) for some \( w_3,w_4 \in S \) with \( |\{w_1,w_2\} \cap \{w_3,w_4\}| \leq 1 \). But then, \( g_{S \cup \{c_0\}}(w_1,w_2) \geq 5 \) and \( g_{S \cup \{c_0\}}(w_2,w_3) \geq 5 \).

In either of the above possibilities we arrive at a contradiction to the assumption of condition (i). Hence the lemma.

\( \square \)

**Remark 3.3.** The conditions in the above lemma holds for all \( a,b \in V \).

### 4. Powerful Rational Metric Dimension

A rational resolving set \( S \in \mathcal{R}(G) \) is called powerful if \( \overline{S} \in \mathcal{R}(G) \). The least cardinality of a powerful rational resolving set (if it exists) of \( G \) is called powerful rational metric dimension of \( G \) and is denoted by \( rmd_p(G) \).

In this section we determine powerful rational metric dimension of a Wheel.

**Lemma 4.1.** If \( S \in \mathcal{R}(W_{1,m}) \) and \( |S| = \min\{|T| : T \in \mathcal{R}(W_{1,m})\} \geq 3 \), then \( S \) has no three consecutive rim vertices whenever \( m \neq 3 \).

**Proof.** Let \( S \in \mathcal{R}(W_{1,m}) \) be of minimum cardinality and \( m \neq 3 \). If possible, let \( a_1,a_2,a_3 \) be the three consecutive rim vertices in \( S \) and \( |S| \geq 3 \). Then, by Theorem 2.1, \( m \geq 10 \). Let \( S' = S - \{a_2\} \). Let \( a_i \) be the rim vertex at a distance \( i - 1 \) from \( a_1 \), along the circle \( C_m \) of \( W_{1,m} \), in
a shortest path containing the vertex $a_2$. Let $a_{-i}$ be the rim vertex at a distance $i + 1$ from $a_1$, along the circle $C_m$ of $W_{1,m}$, in a shortest path not containing the vertex $a_2$.

**Claim:** $S' \in \mathcal{R}(W_{1,m})$.

We prove the claim by contradiction. Suppose that $S' \notin \mathcal{R}(W_{1,m})$. Then there are two vertices $u$ and $v \in V(W_{1,m})$ such that

\begin{align*}
(1) & \quad d(u/a_i) = d(v/a_i), \text{for } i = 1, 3 \\
(2) & \quad d(u/a_2) \neq d(v/a_2).
\end{align*}

Let $d(u/a_i) = l_i$, $i = 1, 3$. Then we have the following possibilities.

**Case 1:** $l_1 = l_3 = 1$.

In this case, $u, v \in \{a_2, v_0\}$ and hence $n = 4$, a contradiction to the fact that $n \geq 10$.

**Case 2:** $l_1 = l_3 = 3/2$.

Since $l_1 = 3/2$, either $u = a_3$ or $v = a_3$. If $u = a_3$ then $v = a_{-1}$ and hence, $d(u/a_3) = 0$ and $d(v/a_3) = 7/4$ ($: m \geq 10$). Similarly, if $v = a_3$ then $u = a_{-1}$ and hence $d(u/a_3) = 7/4$ and $d(v/a_3) = 0$. In either of the cases $l_3 = d(u/a_3) \neq 3/2$, a contradiction.

**Case 3:** $l_1 = l_3 = 7/4$.

Since $l_1 = 7/4$, $u = a_i$ for some $i \geq 4$ or $i \leq -2$. Since $l_3 = 7/4$, $u = a_i$ for some $i \geq 6$ or $i \leq 0$. These two together imply $u = a_i$ for some $i \geq 6$ or $i \leq -2$. In either of the cases, $d(u/a_2) = d(v/a_2) = 7/4$, a contradiction to equation (2).

**Case 4:** $l_1 = 1, l_3 \in \{3/2, 7/4\}$

Since $l_1 = 1$, we have $u, v \in \{a_2, a_0\}$ but then $d(u/a_3) = 1$ if $u = a_2$, or, $d(v/a_3) = 1$ if $u = a_0$. In either of the cases $l_3 = d(u/a_3) = d(v/a_3) = 1 \notin \{3/2, 7/4\}$, a contradiction.

**Case 5:** $l_1 = 3/2$ and $l_3 = 7/4$.

Since $l_1 = 3/2$, $a_3 \in \{u, v\}$. If $a_3 = u$, then $l_3 = d(u/a_3) = 0$. Else if $a_3 = v$ then $d(v/a_3) = 0$.

Therefore, in either of the cases, $l_3 \neq 7/4$, a contradiction.

Other cases follows by symmetry. Hence the Claim.

Therefore, by the above Claim, $S' \in \mathcal{R}(W_{1,m})$, a contradiction to the fact that $S$ is of minimum cardinality in $\mathcal{R}(W_{1,m})$. Hence the lemma. □
Theorem 4.2. For each integer \( m \geq 4 \), every \( rmd \)-set \( S \in \mathcal{R}(W_{1,m}) \) is powerful.

Proof. For \( 4 \leq m \leq 7 \), it is easy to see that a set \( S \) containing exactly two adjacent rim vertices of \( W_{1,m} \) is in \( \mathcal{R}(W_{1,m}) \) and is an \( rmd \)-set of \( W_{1,m} \). Further, \( S \) also contains two adjacent rim vertices and hence by the super hereditary property of \( \mathcal{R}(G) \) it follows immediately that \( S \in \mathcal{R}(W_{1,m}) \).

When \( m = 8 \), every \( rmd \)-set \( S \) of \( W_{1,8} \) is a 2-element set of its rim vertices \( v_i, v_j \) such that \( 2 \leq d_{C_{m}}(v_i, v_j) \leq 3 \). For each of such sets, \( S \) contains a 2-element subset \( S' \) of rim vertices \( v_{i-1}, v_{i+1} \) which is again an \( rmd \)-set of \( W_{1,8} \) in \( \mathcal{R}(W_{1,m}) \) implies that \( S \in \mathcal{R}(W_{1,m}) \) (by super hereditary property). Finally, when \( n \geq 9 \), by Lemma 4.1, for each \( rmd \)-set \( S \in \mathcal{R}(W_{1,m}) \) we have \( g_{S \cup \{c\}}(a,b) \leq 2 \) for every pair of vertices \( a, b \in S \) and hence \( S \in \mathcal{R}(W_{1,m}) \) (by Lemma 3.2). Hence the theorem. \( \square \)

Corollary 4.3. For every integer \( m \geq 4 \), \( rmd_p(W_{1,m}) = rmd(W_{1,m}) \).

5. MAXIMAL AND FOUL RATIONAL METRIC DIMENSION

A rational resolving set \( S \in \mathcal{R}(G) \) is called maximal if \( \overline{S} \notin \mathcal{R}(G) \). The least cardinality of a maximal rational resolving set of \( G \) is called maximal rational metric dimension of \( G \) and is denoted by \( rmd_m(G) \).

A subset \( S \subseteq V(G) \) is called foul rational resolving set if \( S \notin \mathcal{R}(G) \) and \( \overline{S} \notin \mathcal{R}(G) \). The least cardinality of a foul rational resolving set of \( G \) is called foul rational metric dimension of \( G \) and is denoted by \( rmd_f(G) \).

Let \( \hat{\mathcal{R}}(G) \) and \( \neg \mathcal{R}(G) \) be the set of all maximal and foul rational resolving sets of \( G \), respectively. In this section we determine maximal and foul rational metric dimension of a Wheel.

Lemma 5.1. For any integer \( 4 \leq m \leq 8 \), a 2-element set \( S = \{v_i, v_j\} \) of rim vertices is in \( \mathcal{R}(W_{1,m}) \) if and only if one of the following hold.

\begin{itemize}
  \item[(i)] \( m \) is odd.
  \item[(ii)] \( m = 4, 6 \) and \( j \neq i + \frac{m}{2} \).
  \item[(iii)] \( m = 8 \) and \( j \notin \{i+1, i+4\} \).
\end{itemize}

Proof. When \( m \) is odd and \( 4 \leq m \leq 8 \), at most one vertex in \( V - S \) is at a distance at least 3 from both the vertices \( v_i \) and \( v_j \) in \( C_m \). Hence if \( \Gamma(u/S) = \Gamma(v/S) \) for any \( u, v \in V - S \), then
both \( u \) and \( v \) can not be in a common \( v_i v_j \)-path of \( C_m \). But then, \( d_{C_m}(v_i, u) = d_{C_m}(v_i, v) \) and 
\( d_{C_m}(v_j, u) = d_{C_m}(v_j, v) \) implies that \( m = 2(d(v_i, u) + d(u, v_j)) \) is even, a contradiction. Further,
when \( m \) is even, \( \Gamma(v_{i+1} \pmod{m})/S = \Gamma(v_{j-1} \pmod{m})/S = (1, a) \) where \( a = 1, \frac{3}{4}, \frac{7}{4} \) if \( m = 4, 6, 8 \), respectively. So, \( j \neq i + \frac{m}{2} \) whenever \( m \) is even. Also, when \( m = 8 \), \( \Gamma(v_{i+4} \pmod{8})/S = \Gamma(v_{i-4} \pmod{8})/S = (\frac{7}{4}, \frac{7}{4}) \) whenever \( j = i + 1 \) and hence, \( j \neq i + 1 \).

On the other hand, suppose that all the conditions in the lemma hold. If \( S \notin \mathcal{R}(W_{1,m}) \) for any \( 1 \leq m \leq 8 \), then there are two vertices \( u, v \in V - S \), such that \( \Gamma(u/S) = \Gamma(v/S) \). If \( \Gamma(u/S) = \Gamma(v/S) = (1, 1) \), then \( u, v \in \{v_{i-1} \pmod{m}, v_{i+1} \pmod{m}\} \), \( j = i + 2 \equiv i - 2 \pmod{m} \) and hence, \( m = 4 \) and \( j = i + \frac{m}{2} \), a contradiction to condition \( (ii) \). If \( \Gamma(u/S) = \Gamma(v/S) = (\frac{3}{4}, \frac{3}{4}) \), then \( u, v \in \{v_{i-2} \pmod{m}, v_{i+2} \pmod{m}\} \), \( j = i + 4 \equiv i - 4 \pmod{m} \) and hence \( m = 8 \), a contradiction to condition \( (iii) \). If \( \Gamma(u/S) = \Gamma(v/S) = (\frac{7}{4}, \frac{7}{4}) \), then length of a longest path on \( C_m \) between \( v_i \) and \( v_j \) to be at least 7 (so \( m \geq 9 \) by condition \( (iii) \) as \( j \neq i + 1 \)), or there are two longest paths of length 6 between \( v_i \) and \( v_j \) in \( C_m \) (so \( m = 12 \)). In either of the cases \( m \geq 9 \), a contradiction to \( m \leq 8 \). If \( \Gamma(u/S) = \Gamma(v/S) = (1, \frac{3}{4}) \), then \( u, v \in \{v_{i-1} \pmod{m}, v_{i+1} \pmod{m}\} \), \( j = i + 3 \equiv i - 3 \pmod{m} \) and hence \( m = 6 \) and \( j = i + \frac{m}{2} \), a contradiction to condition \( (ii) \). If \( \Gamma(u/S) = \Gamma(v/S) = (1, \frac{7}{4}) \), then \( u, v \in \{v_{i-1} \pmod{m}, v_{i+1} \pmod{m}\} \), \( j = i + k \equiv i - l \pmod{m} \) for some \( k, l \geq 4 \). So, \( m = 8 \) and \( j = i + 4 \), or \( m \geq 9 \), a contradiction to condition \( (iii) \) or \( m \leq 8 \), respectively. If 
\( \Gamma(u/S) = \Gamma(v/S) = (\frac{3}{4}, \frac{7}{4}) \), then \( u, v \in \{v_{i-2} \pmod{m}, v_{i+2} \pmod{m}\} \), \( j = i + k \equiv i - l \pmod{m} \) for some \( k, l \geq 5 \), and hence \( m \geq 10 \), again a contradiction. \( \square \)

**Lemma 5.2.** For any \( m \geq 13 \), let \( S \in \mathcal{R}(W_{1,m}) \) be such that \( |S \cap \{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+6}\}| \geq 3 \) for some \( v_i \in S \). Then \( rmd_m(W_{1,m}) \leq |S| + 3 \).

**Figure 3.** Possible \( rmd \)-sets and the corresponding \( rmd_m \)-set as in the proof of Lemma 5.2.
Proof. Let \( a = \min \{ j : j > i, v_j \in S \} \) and \( b = \min \{ k : k > a, v_k \in S \} \). If \( i + 3 \notin \{ a, b \} \), then the set \( S' = S \cup \{ v_i+1, v_i+2 \} \cup \{ v_i+4, v_i+5, v_i+6 \} \in \mathcal{R}(W_{1,m}) \) (by Lemma 3.2, as \( v_i \notin S' \) and \( v_a, v_b \in S' \). Hence, \( rmd_m(W_{1,m}) \leq |S'| \leq |S| + 3 \). If \( i + 3 \in \{ a, b \} \), then the set \( S' = (S - \{ v_i+3 \}) \cup \{ v_i+1, v_i+2 \} \cup \{ v_i+4, v_i+5, v_i+6 \} \in \mathcal{R}(W_{1,m}) \) (by Lemma 3.2, as \( v_i \notin S' \) and \( |S'| \cap \{ v_a, v_b \} | = 1 \). Hence, \( rmd_m(W_{1,m}) \leq |S'| \leq (|S| - 1) + 4 = |S| + 3 \). □

Lemma 5.3. If \( S \) is an rmd-set of a wheel \( W_{1,m} \) and \( m \geq 13 \), then there exist integers \( 0 \leq a < b \) such that \( v_a, v_b \in S \) and \( g_{S \cup \{ v_0 \}}(v_a, v_b) \geq 3 \).

Proof. If not, then \( g_{S \cup \{ v_0 \}}(v_a, v_b) \leq 2 \). But then, as \( rmd(W_{1,m}) \geq 3 \) (since \( m \geq 13 \)), we get an integer \( c > b \) such that \( g_{S \cup \{ v_0 \}}(v_b, v_c) \leq 2 \). Further, by assumption, \( g_{S \cup \{ v_0 \}}(v_x, v_a) \leq 2 \) and \( g_{S \cup \{ v_0 \}}(v_c, v_y) \leq 2 \) for every \( v_x, v_y \in S \) with \( x < a \) and \( y > c \). Hence, the set \( S' = S - \{ v_b \} \) satisfies the conditions of Lemma 3.2. So, \( S' \in \mathcal{R}(W_{1,m}) \) with \( |S'| < |S| = rmd(W_{1,m}) \), a contradiction. □

Corollary 5.4. For every integer \( m \geq 13 \), \( rmd_m(W_{1,m}) \leq rmd(W_{1,m}) + 4 \).

Proof. Let \( S \) be an rmd-set of \( W_{1,m} \) for \( m \geq 13 \). Then, Theorem 2.1 and Lemma 5.3, there are suffixes \( a < b < c \) such that \( v_a, v_b, v_c \in S \) with \( g_{S \cup \{ v_0 \}}(v_a, v_b) \geq 3 \). But then, \( g_{S \cup \{ v_0 \}}(v_b, v_c) \leq 2 \) and hence, \( v_c \in \{ v_{b+1}, v_{b+2}, v_{b+3} \} \). Let \( S' = (S - \{ v_b \}) \cup \{ v_{b-3}, v_{b-2}, v_{b-1}, v_{b+1}, v_{b+2}, v_{b+3} \} \). Then, \( S' \) satisfies all the conditions of Lemma 3.2 and hence \( S \in \mathcal{R}(W_{1,m}) \). Further, \( g_{S \cup \{ v_0 \}}(v_a-1, v_b) \geq 3 \). \( g_{S \cup \{ v_0 \}}(v_b, v_{b+4}) = 3 \) and \( |S'| \leq |S| + 4 \). Therefore, by Lemma 3.2, \( S' \notin \mathcal{R}(W_{1,m}) \). Hence \( S' \in \mathcal{R}(W_{1,m}) \). This shows that \( rmd_m(W_{1,m}) \leq |S'| = |S| + 4 \) (since \( v_c \in S \)) = \( rmd(W_{1,m}) + 4 \). □

Theorem 5.5. For any integer \( m \geq 3 \),

\[
\textbf{rmd}_m(W_{1,m}) = \begin{cases} 
3, & \text{if } m = 3, 4. \\
4, & \text{if } m = 5, 6. \\
6, & \text{if } 7 \leq m \leq 11. \\
\left\lceil \frac{m}{4} \right\rceil + 4, & \text{if } m \geq 12 \text{ and } m \equiv 0 \pmod{8}. \\
\left\lceil \frac{m}{4} \right\rceil + 3, & \text{if } m \geq 12 \text{ and } m \not\equiv 0 \pmod{8}.
\end{cases}
\]

Proof. Let \( S \) be an element of minimum cardinality in \( \mathcal{R}(W_{1,m}) \). Then \( S \in \mathcal{R}(W_{1,m}) \). For \( m = 3 \), as \( 3 = rmd(W_{1,3}) = \min \{|T| : T \in \mathcal{R}(W_{1,m})\} \), \( T \notin \mathcal{R}(W_{1,m}) \) for every \( T \in \mathcal{R}(W_{1,4}) \). Hence \( |S| = \)
Claim: $|S| \geq rmd(W_{1,m})$. For $m = 4$, $|S| \leq 1$ (else by condition (ii) of Lemma 5.1 and Theorem 2.1, $S \notin \mathfrak{R}(W_{1,4})$). Also, the set $S_1 = \{v_0, v_1, v_2\}$ is in $\mathfrak{R}(W_{1,4})$ and $S_1 \notin \mathfrak{R}(W_{1,4})$. Hence $rmd_m(W_{1,4}) = 3$.

When $m = 5, 7$, by Lemma 5.1, $|S| \leq 1$ (else $S \notin \mathfrak{R}(W_{1,m})$) and hence $|S| \geq m - 1$. Also, for each odd $m$, $4 \leq m \leq 8$, the set $S' = \{v_0, v_1, \ldots, v_{m-2}\} \in \mathfrak{R}(W_{1,m})$ being the super set of $\{v_0, v_1\}$, which is in $\mathfrak{R}(W_{1,m})$ (by Lemma 5.1) and $S' \notin \mathfrak{R}(W_{1,m})$. Hence $|S| = m - 1$ for $m = 5, 7$.

For $m = 6, 8$, by Lemma 5.1, $|S| \leq 2$ (else $S \notin \mathfrak{R}(W_{1,m})$) and hence $|S| \geq m - 2$. Also, for $m = 6, 8$ the set $\{v_0, v_2, v_4, v_5, \ldots, v_{m-1}\} \in \mathfrak{R}(W_{1,m})$ being the super set of $\{v_0, v_2\}$, which is in $\mathfrak{R}(W_{1,m})$ (by Lemma 5.1). Hence $|S| = m - 2$ for $m = 6, 8$.

When $9 \leq m \leq 12$, by Lemma 3.2, at least one of the conditions (i) or (ii) must fail with respect to the set $S$ ($\because S \notin \mathfrak{R}(W_{1,m})$). Hence, $S$ should contain at least 6 vertices. Therefore, $|S| \geq 6$. On the other hand it is easy to see, by Lemma 3.2, that the set $S = \{v_0, v_1, v_2, v_4, v_5, v_6\} \in \mathfrak{R}(W_{1,m})$ for $9 \leq m \leq 12$ ($\because g_{S,|t|_0}(v_1, v_3) = g_{S,|t|_0}(v_3, v_6) = 3; g_{S,|t|_0}(v_1, v_4) = 1, g_{S,|t|_0}(v_0, v_6) \leq 5$ and for all other pairs $a, b \in V - S$ we get $g_{S,|t|_0}(a, b) = 0$). Therefore, $|S| = 6$ for $9 \leq m \leq 12$.

Let us now consider the cases $m \geq 13$. In these cases, as $S \notin \mathfrak{R}(W_{1,m})$, by Lemma 3.2 we see that $S$ shall contain a 6-element proper subset $T$ which is of the form $T = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ or $T = \{v_i, v_i+1, v_i+2, v_i+4, v_i+5, v_i+6\}$ or a 10-element subset $T = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, v_{i+k+4}\}$ for some $k \geq 6$. For the minimality of $|S|$, we consider the second option (which selects 6 vertices out of 7). Without loss of generality, we take $T = \{v_0, v_1, v_2, v_4, v_5, v_6\} \subseteq S$ and $v_3 \notin S$.

Let $S' = S \cup \{v_3\} - \{v_1, v_2, v_4, v_5\}$. Then $S'$ satisfies all the conditions of Lemma 3.2 ($\because S$ satisfies all the conditions of Lemma 3.2 and by the inclusion of $v_3$) and hence, $S' \in \mathfrak{R}(W_{1,m})$. But then, $|S'| \geq rmd(W_{1,m})$ and $|S'| = |S| - 3$. Thus, $|S| \geq rmd(W_{1,m}) + 3$. We now show that the equality can not be achieved in the cases $m \equiv 0, 1 \pmod{8}$.

Claim: $|S| \geq rmd(W_{1,m}) + 4$, whenever $m \equiv 0, 1 \pmod{8}$.

Let $a$ and $b$ be the least and greatest indices ($a$ may be $b$) such that $v_a, v_b \in S - \{v_1, v_2, v_4, v_5\}$ (such a vertex exists because $m \geq 13$ and by Lemma 3.2, otherwise $g_{S,|t|_0}(v_0, v_6) \geq 6$). Then, by
Lemma 3.2, either $g_{S \cup \{\emptyset\}}(v_6, v_a) \leq 4$ or $g_{S \cup \{\emptyset\}}(v_0, v_b) \leq 4$. Without loss of generality, due to symmetry we take $g_{S \cup \{\emptyset\}}(v_6, v_a) \leq 4$. Let $l = g_{S \cup \{\emptyset\}}(v_6, v_a)$ and $G = W_{1,m}$.

Since $m \geq 13$ and $m \equiv 0, 1 \mod 8$, we have $m = 8k$ or $m = 8k + 1$ for some integer $k \geq 7$.

When $m = 8k$, by Theorem 2.1, $rmd(G) = rmd(W_{1,8k}) = \lceil \frac{8k}{4} \rceil = 2k$. When $m = 8k + 1$, by Theorem 2.1, $rmd(G) = rmd(W_{1,8k+1}) = \lceil \frac{8k+1}{4} \rceil - 1 = 2k + 1 - 1 = 2k$. Thus, in either of the cases, it suffices to show that $|S| \geq 2k + 4$ whenever $m = 8k$ or $8k + 1$.

**Case 1:** $l = 3, 4$.

In this case, $a = 10$ or $11$ if $l = 3$ or $4$ respectively, and $g_{S \cup \{\emptyset\}}(v_a, v_{14}) \leq 2$. Let $G' = (G - \{v_1, v_2, \ldots, v_a\}) + v_0v_{a+1}$. Then, $G' \equiv W_{1,m-a}$ and $S' = S - \{v_1, v_2, v_4, v_5, v_6, v_a\} \in \mathcal{R}(G')$ (since $S'$ satisfies all the conditions of Lemma 3.2 as $S$ fulfilled the conditions and by the construction of $G'$). Therefore, $|S'| \geq rmd(G')$. But $rmd(G') = \lceil \frac{m-a}{4} \rceil = 2k - 2$ (by Theorem 2.1 as $m-a \neq 1 \mod 8$ for $m = 8k$ or $8k + 1$, $10 \leq a \leq 11$) and $|S| = |S'| + 6$. Hence, $|S| = |S'| + 6 \geq 2k - 2 + 6 = 2k + 4$.

**Case 2:** $l = 2$.

In this case, $a = 19$. Let $G' = (G - \{v_1, v_2, \ldots, v_6\}) + v_0v_7$. Then, $G' \equiv W_{1,m-6}$ and $S' = S - \{v_1, v_2, v_4, v_5, v_6\} \in \mathcal{R}(G')$. Therefore, $|S'| \geq rmd(G') = \lceil \frac{m-6}{4} \rceil = 2k - 1$ (by Theorem 2.1 as $m - 6 \neq 1 \mod 8$ for $m = 8k$ or $8k + 1$) and hence $|S| = |S'| + 5 \geq 2k - 1 + 5 = 2k + 4$.

**Case 3:** $l = 0, 1$. 

---

**Figure 4.** The subset $T$ of $S \in \mathcal{R}(W_{1,m})$. 

---
In this case, $a \in \{7, 8\}$. Let $G' = (G - \{v_1, v_2, \ldots, v_5\}) + v_0v_6$. Then, $G' \equiv W_{1,m-5}$ and $S' = S - \{v_1, v_2, v_4, v_5, v_6\} \in \mathfrak{S}(G')$. Therefore, $|S'| \geq \text{rmd}(G') = \left\lceil \frac{m-5}{4} \right\rceil = 2k - 1$ (by Theorem 2.1 as $m - 5 \not\equiv 1 \pmod{8}$ for $m = 8k$ or $8k + 1$) and hence $|S| = |S'| + 5 \geq 2k - 1 + 5 = 2k + 4$.

Hence the Claim.

By the above Claim, Theorem 2.1, and above explanation, we now conclude that

$$\text{rmd}_m(W_{1,m}) \geq \begin{cases} \left\lceil \frac{m}{4} \right\rceil + 3 & \text{if } m \not\equiv 0 \pmod{8}, \\ \left\lceil \frac{m}{4} \right\rceil + 4 & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$

To prove the reverse inequality we execute $\text{rmd}_m$-sets of $W_{1,m}$ in different cases as follows:

Let $m = 8k + q$, where $0 \leq q \leq 7$. Let $S_1 = \{v_{8j} : 0 \leq j \leq k\}$ and $S_2 = \{v_{8j} : 0 \leq j \leq k - 1\}$. Then $|S_1 \cup S_2| = 2k + 1$ if $m \not\equiv 0 \pmod{8}$ and $S_1 \cap S_2 = \emptyset$. Let $S = S_1 \cup S_2$.

**Case 1:** $q = 0, 1$.

In this case, by Corollary 5.4, $\text{rmd}_m(W_{1,m}) \leq \text{rmd}(W_{1,m}) + 4$. Hence, by Theorem 2.1, $\text{rmd}(W_{1,m}) \leq \left\lceil \frac{m}{4} \right\rceil + 4$ if $q = 0$, and $\text{rmd}(W_{1,m}) \leq \left\lceil \frac{m}{4} \right\rceil + 4$ if $q = 1$.

**Case 2:** $q = 2$.

By Lemma 3.2, the set $S \in \mathfrak{S}(W_{1,m})$. Also, $\{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_8, v_{8k+1}, v_0, v_1\} \subseteq S$ and hence $|S \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_8, v_{8k+1}, v_0, v_1\}| \geq 3$. Therefore, by Lemma 5.2, $\text{rmd}_m(W_{1,m}) \leq |S| + 3 = 2k + 1 + 3 = \left\lceil \frac{m}{4} \right\rceil + 3$.

**Case 3:** $q = 3$.

By Lemma 3.2, the set $S \in \mathfrak{S}(W_{1,m})$. Also, $\{v_{8k+2}, v_8, v_{8k+1}, v_{8k+2}, v_0\} \subseteq S$ and hence $|S \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_8, v_{8k+1}, v_{8k+2}, v_0\}| \geq 3$. Therefore, by Lemma 5.2, $\text{rmd}_m(W_{1,m}) \leq |S| + 3 = 2k + 1 + 3 = \left\lceil \frac{m}{4} \right\rceil + 3$.

**Case 4:** $q = 4$.

By Lemma 4.1, the set $S' = (S - \{v_0\}) \cup \{v_{8k+3}\} \in \mathfrak{S}(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_{8k+3}\} \subseteq S'$ and hence $|S' \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_{8k+3}\}| \geq 3$. Therefore, by Lemma 5.2, $\text{rmd}_m(W_{1,m}) \leq |S'| + 3 = \left\lceil |S| + 1 \right\rceil + 3 = |S| + 3 = 2k + 1 + 3 = \left\lceil \frac{m}{4} \right\rceil + 3$.

**Case 5:** $q = 5, 6$.

By Lemma 3.2, the set $S' = S \cup \{v_{8k+3}\} \in \mathfrak{S}(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_{8k+3}\} \subseteq S'$ and hence $|S' \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_{8k+3}\}| \geq 3$. Therefore, by Lemma 5.2, $\text{rmd}_m(W_{1,m}) \leq |S'| + 3 = |S| + 1 + 3 = 2k + 2 + 3 = \left\lceil \frac{m}{4} \right\rceil + 3$. 

Case 6: $q = 7$.

By Lemma 3.2, the set $S' = (S - \{v_0\}) \cup \{v_{8k+6}, v_{8k+3}\} \in \mathcal{R}(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_{8k+3}\} \subseteq S'$ and hence $|S| = |S' \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_{8k+3}\}| \geq 3$. Therefore, by Lemma 5.2, $rmd_m(W_{1,m}) \leq |S'| + 3 = |S| + 3 = [2k + 3] = \lceil \frac{m}{4} \rceil + 3$. \hfill \Box

**Theorem 5.6.** For any integer $m \geq 3$,

$$rmd_f(W_{1,m}) = \begin{cases} 2, & \text{if } m = 3, 4, \\ 6, & \text{if } m \geq 12. \end{cases}$$

Further, for $5 \leq m \leq 12$, $\neg \mathcal{R}(W_{1,m}) = \emptyset$.

**Proof.** When $m = 3$, by Theorem 2.1, $rmd(W_{1,3}) = 3$ and hence the set $S = \{v_0, v_1\} \in \neg \mathcal{R}(W_{1,3})$ and is of minimum cardinality. So, $rmd_f(W_{1,3}) = 2$. When $m = 4$, by Lemma 5.1, the sets $S = \{v_0, v_2\} \notin \mathcal{R}(W_{1,4})$ and $S = \{c_0, v_1, v_3\} \notin \mathcal{R}(W_{1,4})$. Further, by Lemma 5.1, every 4-element subset of vertices of $W_{1,4}$ is a super set of an $rmd$, and hence it is in $\mathcal{R}(W_{1,4})$ implies that $rmd_f(W_{1,4}) = |S| = 2$. When $5 \leq m \leq 11$, each subset $S$ of vertices of $W_{1,m}$ shall contain at least 6 vertices in $S$ to make $S \notin \mathcal{R}(W_{1,m})$, but then $S \in \mathcal{R}(W_{1,m})$. Hence $\neg \mathcal{R}(W_{1,m}) = \emptyset$ for $5 \leq m \leq 11$.

For $m \geq 12$, by Lemma 3.2, $S$ shall contain at least 6 rim vertices to make $S \notin \mathcal{R}(W_{1,m})$. Thus, $|S| \geq 6$ for every $S \in \neg \mathcal{R}(W_{1,m})$. To prove the reverse inequality, let $S = \{v_1, v_2, \ldots, v_6\}$. Then $g_{S,\{c_0\}}(v_1, v_6) \geq 6$ and $g_{S,\{c_0\}}(v_0, v_7) \geq 6$. So, by Lemma 3.2, $S \notin \mathcal{R}(W_{1,m})$ and $S \notin \mathcal{R}(W_{1,m})$ implies that $S \in \neg \mathcal{R}(W_{1,m})$ with $|S| = 6$. Hence the theorem. \hfill \Box

**ACKNOWLEDGMENT**

Authors are very much thankful to the Management and the Principal of Dr. Ambedkar Institute of Technology, Bengaluru, for their constant support and encouragement during the preparation of this paper. Also special thanks to anonymous referees for their valuable suggestions for the improvement of the paper.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.
REFERENCES

[1] F. Buckley and F. Harary, Distance in graphs, Addison-Wesley, (1990).
[2] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, and D.R. Wood, On the metric dimension of some families of graphs, Electron. Notes Discrete Math. 22(2005), 129-133.
[3] G. Chartrand, Linda Eroh, Mark A. Johnson, and Ortrud R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105(1-3)(2000), 99-113.
[4] H. Gerhard and Ringel, Pearls in graph theory, Academic Press, USA, (1994).
[5] F. Harary and R.A. Melter, The metric dimension of a graph, Ars Combin. 2(1976), 191-195.
[6] M. Jayalakshmi and Padma M.M, Boundary values of $r_r$, $r^*_r$, $R_r$, $R^*_r$ sets of certain classes of graphs, J. Theor. Math. Appl. 7(1)(2017), 29-39.
[7] S. Khuller, B. Raghavachari, and A. Rosenfied, Landmarks in graphs, Discrete Appl. Math. 70(1996), 217-229.
[8] M. M. Padma and Jayalakshmi M, Variety of Rational Resolving Sets of Power of a Cycle, Test Eng. Manage. 83(2020), 4162-4167.
[9] M. M. Padma and Jayalakshmi M, Variety of rational resolving sets of corona product of graphs, Adv. Math.: Sci. J. 9(10)(2020), 1857-8438.
[10] M. M. Padma and Jayalakshmi M, k-local resolving and rational resolving sets of graphs, Int. J. Eng. Sci. Manage. 2(2)(2020), 15-20.
[11] A. Raghavendra, B. Sooryanarayana, and C. Hegde, Rational metric dimension of graphs, Commun. Optim. Theory, 2014 (2014), Article ID 8.
[12] A. Raghavendra, B. Sooryanarayana, Raksha Poojary, A study on rational metric energy of a graph, Int. J. Math. Sci. Eng. Appl. 2(3)2017, 155-163.
[13] V. Saenpholphat and Ping Zhang, Connected resolvability of graphs, Czechoslovak Math. J. 53(4)(2003), 827-840.
[14] A. Sebo and E. Tannier, On metric generators of graphs, Math. Oper. Res. 29(2) (2004), 383-393.
[15] B. Shanmukha, B. Sooryanarayana and K.S. Harinath, Metric dimension of wheels, Far East J. Appl. Math. 8(3)(2002), 217-229.
[16] P.J. Slater, Leaves of trees, In Proc. 6th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Congr. Numer. 14(1975), 549-559.
[17] B. Sooryanarayana, On the metric dimension of graph, Indian J. Pure Appl. Math. 29(4)(1998), 413-415.
[18] B. Sooryanarayana and B. Shanmukha, A note on metric dimension, Far East J. Appl. Math. 5(3)(2001), 331-339.
[19] B. Sooryanarayana, K. Shreedhar and N. Narahari, Metric dimension of generalized wheels, Arab J. Math. Sci. 25(2)(2019), 131-144.
[20] B. Sooryanarayana and A. S. Suma, On classes of neighborhood resolving sets of a graph, Electron. J. Graph Theory Appl. 6(1)(2018), 29-36.

[21] B. Sooryanarayana and A. S. Suma, Graphs of Neighborhood Metric Dimension Two, J. Math. Fund. Sci. 53(1)(2021), 118-133.

[22] B. Sooryanarayana, A. S. Suma, S. B. Chandrakala, Certain varieties of resolving sets of a graph, J. Indones. Math. Soc. 27(1)(2021), 103-114.