A Method of Changing the Center Position in Duffing Chaotic System

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Abstract. Under the condition of keeping Duffing system in chaotic state, we present a method of changing the center position by adding a non-smooth periodic signal to Duffing system. In this paper, we first make a theoretical analysis which describe the influence of added signal on center point of Duffing system and provide mathematical proofs, and then build a simulation system which can change the center position of Duffing system. Finally, computer simulations are used to verify the analytical results. Simulation results show that the proposed method can effectively work in secure communication applications.

1 Introduction

Since Carroll and Pecora proposed the principle of chaos self-synchronization and achieved the synchronization of chaotic circuit for the first time [1], chaos has received significant attention. There are many research works such as chaotic masking communication, chaotic parameter modulation, chaotic encryption, chaotic parameter identification and synchronization. On the same time, demodulation methods of chaotic communication have been studied by many researchers. Among many demodulation methods, the parameter estimation method [2] and coherent and non-coherent demodulation method [3] are often used in practice.

Digital message transmission by chaotic system is importantly applied in the secret communication. Several techniques have been proposed and analyzed recently including chaotic switching [4], chaotic shift keying [5], and chaotic frequency modulation [6]. In these techniques, digital message transmission was realized by using the message to modulate system parameters. However, the signal carrying digital message was added to chaotic parameter modulation system; it can form a non-smooth signal disturbance and change this system into a non-smooth system. Many researchers were interested in the chaotic feature of non-smooth system. Nordmark put forward an approximate description method of non-smooth bifurcation based on local singularity analysis [7]. Leine studied the bifurcation problem from a system with discontinuous vector field. He pointed out that there existed not only traditional fork bifurcation and Hopf bifurcation but also grazing bifurcation caused by the non-smoothness [8]. Awrejcewicz gave a method to calculate the Melnikov function of non-smooth system [9]. On the basis of the above researches, we used analytical method to determine the center position of Duffing system. In addition, we applied the mathematical analysis results to design a chaotic secure communication system and gave the demodulation results of transmitted digital signal.

2 The influence of added signal on center point of Duffing system

Definition1. Under the effect of periodic driving force, the state of Duffing system can be chaotic. The dynamic equation of Duffing system can be described as

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x - x^3 - ky + F \cos \omega t
\end{align*}
\]

where \( F, \omega \) are the amplitude and frequency of driving force, \( k \) is damping ratio, \( x - x^3 \) is the nonlinear restoring force.

Definition2. When given added signal \( b \) and the adjustable factor \( m \) to equation (1), Duffing system can be described as

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x - x^3 - ky + m(F \cos \omega t + b)
\end{align*}
\]

In the following, we will prove how added signal \( b \) affects the position of center points and saddle points in Duffing system.

Proof: Consider the equation

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x - x^3 + b
\end{align*}
\]

From equation (3), we can define Hamilton function
as
\[ H(x,y) = \frac{y^2}{2} - \left(\frac{x + b}{2}\right)^2 + \frac{x^4}{4}, \quad H(x,y): \mathbb{R}^2 \to \mathbb{R} \quad (4) \]

Taking the derivatives of \( H(x,y) \) with respect to \( t \), we can obtain
\[
\frac{dH(x,y)}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} = 0 \quad (5)
\]

Where curve \( H(x,y) = c, \quad c \) is a constant.

If point \( P \) is an equilibrium point of \( H(x,y) \), linearized coefficient matrix \( A \) at \( P \) can be expressed as follows.

\[ A = \begin{bmatrix}
\frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\
\frac{\partial^2 H}{\partial y \partial x} & \frac{\partial^2 H}{\partial y^2}
\end{bmatrix}
\]

The characteristic equation of matrix \( A \) is \( \lambda^2 + \det(A) = 0 \). Its eigenvalue is \( \lambda = \pm \sqrt{-\det(A)} \). If \( \exists \lambda \in \mathbb{R} \), equilibrium points are saddle points; if \( \exists \lambda \in \mathbb{C} \), equilibrium points are center points. Let \( f(x) = x - x^3 + b \), at the equilibrium points, we can assume \( f(x) = 0 \) in equation (3). We know that \( f(x) \) is \( n(n \to \infty) \)-times continuously differentiable within a neighborhood \( U(x_0) \) of the point \( x_0 \). The necessary and sufficient condition that \( f(x) \) within \( U(x_0) \) can be expanded into Taylor series is \( R_n(x) = o(|x-x_0|^n), x \to x_0 \). Because \( f(x) \) satisfies the above condition, \( f(x) \) can be described as \( f(x) = f(x_0) + f'(x_0)(x-x_0) \). When given \( x_0 = 0 \), \( x_1 = 1 \) and \( x_2 = -1 \), three equilibrium points in equation (3) are \( d(-b,0), \ e(1+b/2,0) \) and \( f(1-b/2,0) \), where \( d \) is a saddle point, \( e \) and \( f \) are center points.

3 The influence of non-smooth periodic disturbance on chaotic state of Duffing system

When the injected disturbance \( b \) in equation (2) is a non-smooth periodic function, it can be modified to \( b(t) \).

The period of \( b(t) \) is \( T \), and \( b(t) \) is piecewise smooth. On the basis of Fourier theory, the Fourier series corresponding to \( b(t) \) can be described as

\[ b(t) = \frac{a_n}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)), \omega_0 = \frac{2\pi}{T} \quad (6) \]

Where \( a_n \) and \( b_n \) are Fourier transform coefficients. \( a_n \) and \( b_n \) are defined in (7) and (8).

\[ a_n = \frac{2}{T} \int_0^T b(t) \cos(n\omega_0 t) dt \quad n = 0,1,2,\ldots \quad (7) \]

\[ b_n = \frac{2}{T} \int_0^T b(t) \sin(n\omega_0 t) dt \quad n = 1,2,\ldots \quad (8) \]

By applying (6), (7) and (8) into (2), we can obtain smooth dynamic equation (9) which is equivalent to (2), so the theory of smooth dynamic system can be applied to non-smooth dynamic system.

\[ \begin{cases}
\dot{x} = y \\
\dot{y} = x - x^3 - ky + m(F \cos \omega t + b(t))
\end{cases} \quad (9) \]

Definition3. Smooth dynamic system can be defined as follow.

\[ X = f(X) + mg(X,t), \quad X = (x,y) \in \mathbb{R}^2 \quad (10) \]

Where \( f = (f_1,f_2)^T, g = (g_1,g_2)^T \in \mathbb{C}^n (n \geq 2) \) and bounded. \( g(X,t) \) with respect to \( t \) is on \( T \) cycle. Melnikov function of equation (10) can be described as equation (11).

\[ M(\theta) = \int_{-\infty}^{\infty} f(X(t)) \cdot g(X(t),t+\theta) dt \quad (11) \]

Where \( X(t) = (x(t),y(t)) \) is homoclinic orbit passing the saddle point for \( m=0 \) in (10).

Definition4. The Duffing system with non-smooth periodic disturbance \( b(t) \) can be expressed as

\[ \begin{cases}
\dot{x} = y \\
\dot{y} = x - x^3 - ky + m(F \cos \omega t + b(t))
\end{cases} \quad (12) \]

Where \( b(t) \) is a periodic function. In one period, it can be defined as follow.

\[ b(t) = \begin{cases}
E & |t| \leq \frac{T}{2} \\
-E & |t| > \frac{T}{2}
\end{cases} \quad (13) \]

Proof: Fourier series of \( b(t) \) can be described as

\[ b(t) = \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos n\omega t \quad (13) \]

So equation (2) is equivalent to (14).

\[ \begin{cases}
\dot{x} = y \\
\dot{y} = x - x^3 - ky + m(F \cos \omega t + \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos n\omega t)
\end{cases} \quad (14) \]
Equation (14) is a smooth nonlinear system and equivalent to equation (15)

\[
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= x - x^3 + m(F \cos \omega t + \\
&+ \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos n\omega t - \beta y)
\end{aligned}
\]  

(15)

When \( m = 0 \), equation (15) can be changed into (16).

\[
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= x - x^3
\end{aligned}
\]  

(16)

There exist three fixed points in equation (16), which includes center point (1,0), (-1,0) and saddle point (0,0). Then Hamilton function of equation (16) can be written as

\[
H(x, y) = \frac{1}{2} y^2 - \frac{1}{2} x^2 + \frac{1}{4} kx^4
\]  

(17)

By (17), homoclinic orbit passing the saddle point can be represented by

\[
\dot{x} = x \sqrt{1 - \frac{x^2}{2}}
\]  

(18)

Integrating \( \dot{x} \) with respect to \( t \), we can obtain homoclinic orbit equation (19)

\[
\begin{aligned}
x_1(t) &= \sqrt{2} \sec h(t) \\
y_1(t) &= -\sqrt{2} \sec h(t) \sin(\omega t)
\end{aligned}
\]  

(19)

Comparing equation (10), we get

\[
f = (y, x - x^3)^T \\
g = (0, F \cos \omega t + \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos n\omega t - \beta y)^T
\]

By substituting the above expressions into (11), we then have

\[
M(\theta) = -\int_{-\infty}^{\infty} \sqrt{2} \sec h(t) \sin(\omega t) \cos(\omega t + \theta) dt
+ \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos n\omega t \sec h(t) \cos(\omega t) dt
\]

(20)

By using the residue method in term 2, we can obtain that

\[
\begin{aligned}
F &\sqrt{2} \sin(\omega \theta) \int_{-\infty}^{\infty} \sec h(t) \sin(\omega t) \sin(\omega t) dt \\
&= \pi \omega F \sqrt{2} \sin(\omega \theta) \sec h(\omega \theta/2)
\end{aligned}
\]

(22)

So equation (20) can be rewritten as

\[
M(\theta) = \pi \omega F \sqrt{2} \sin(\omega \theta) \sec h(\omega \theta/2) - \frac{4\beta}{3}
\]  

(23)

Remark. Let \( F' = \frac{2\sqrt{2}\beta}{3\pi \omega \sec h(\omega \theta/2)} \), then if \( F > F' \), \( M(\theta) \) of Melnikov function can appear simple zero from Smale-Birkhoff theorem. If \( m \) in equation (10) is sufficiently small, transverse intersections of the stable and unstable manifolds will occur, and a chaotic behaviour with the Smale horseshoe will appear in this system. If \( F < F' \), \( M(\theta) \) can not appear simple zero, transverse intersections of the stable and unstable manifolds will not occur and the motion of the system will be periodic.

4 Simulation

We will verify the above research results in this section. From equation (23), we can know that \( F' = 0.3765 \) for \( k = 0.5, \omega = 1, m = 1, \beta = 0.5 \) in (14) and (15). Then when \( F = 0.72 \), the motion of the system is chaotic. In order to verify the influence of added signal on the position of center points, let \( b(t) = 0, b(t) = -0.2 \) and \( b(t) = 0.2 \). When \( b(t) = 0 \), namely, not including added signal in the system, the phase diagram of chaotic system is shown in figure 1. There exist center point (1,0) and (-1,0) in the phase diagram. When added signal \( b(t) = -0.2 \), the phase diagram of chaotic system is shown in figure 2. The position of center points shift towards the negative x-axis. The x-axis position value of center points is separately smaller than 1 and -1. When added signal \( b(t) = 0.2 \), the phase diagram of chaotic system is shown in figure 3. The position of center points shift towards the positive x-axis. The x-axis position value of center points is separately bigger than 1 and -1.

The characteristic that added signal can affect the position of center points is useful. So based on the mathematical theory in section II and section III, we have designed a communication simulation system by using MATLAB. The simulation system diagram is shown in figure 4. When a periodic square wave is added to the system, the chaotic modulation signal is shown in figure 5.
Transmitted digital signal and demodulated signal are shown in figure 6. The square curves represent transmitted digital signal and the random curves represent demodulated signal in these two figures. The transmitted digital signal in figure 6 represents digital message 110110111101010101. Demodulation results show that receiving system can demodulate transmitted digital signal. When transmitted message is 0, the probability that the state of Duffing system for sending signal appears at the negative x-axis is bigger. The integral of $x$ is negative. When transmitted message is 1, the probability that the state of Duffing system for sending signal appears at the positive x-axis is bigger. The integral of $x$ is positive. So we can demodulate transmitted message based on this integral.

![Figure 1. Phase diagram of chaotic system ($b(t) = 0$).](image1)

![Figure 2. Phase diagram of chaotic system ($b(t) = -0.2$).](image2)

![Figure 3. Phase diagram of chaotic system ($b(t) = 0.2$).](image3)

5 Conclusions

Under the influence of non-smooth periodic disturbance, the state changes of Duffing system have been studied in this paper. We analyzed the position changes of center points of Duffing system and proved the conditional expression when a chaotic behaviour occurred in this case. According to the theoretical analysis results, we built a simulation system and carried on simulation test. Simulation results showed that the proposed method in this paper was practicable and can be used in the secret communication.

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