A Fully Quaternion-Valued Capon Beamformer Based on Crossed-Dipole Arrays

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Abstract—Quaternion models have been developed for both direction of arrival estimation and beamforming based on crossed-dipole arrays in the past. However, for almost all the models, especially for adaptive beamforming, the desired signal is still complex-valued and one example is the quaternion-Capon beamformer. However, since the complex-valued desired signal only has two components, while there are four components in a quaternion, only two components of the quaternion-valued beamformer output are used and the remaining two are simply removed. This leads to significant redundancy in its implementation. In this work, we consider a quaternion-valued desired signal and develop a full quaternion-valued Capon beamformer, which has a better performance and a much lower complexity and is shown to be more robust against array pointing errors.

Keywords — quaternion model, crossed-dipole, Capon beamformer, vector sensor array.

I. INTRODUCTION

Electromagnetic (EM) vector sensor arrays can track the direction of arrival (DOA) of impinging signals as well as their polarization. A crossed-dipole sensor array, firstly introduced in [1] for adaptive beamforming, works by processing the received signals with a long polarization vector. Based on such a model, the beamforming problem is studied in detail in terms of output signal-to-interference-plus-noise ratio (SINR) [2]. In [3], [4], further detailed analysis was performed showing that the output SINR is affected by DOA and polarization differences.

Since there are four components for each vector sensor output in a crossed-dipole array, a quaternion model instead of long vectors has been adopted in the past for both adaptive beamforming and direction of arrival (DOA) estimation [5]–[9]. In [10], the well-known Capon beamformer was extended to the quaternion domain and a quaternion-valued Capon (Q-Capon) beamformer was proposed with the corresponding optimum solution derived.

However, in most of the beamforming studies, the signal of interest (SOI) is still complex-valued, i.e. with only two components: in-phase (I) and quadrature (Q). Since the output of a quaternion-valued beamformer is also quaternion-valued, only two components of the quaternion are used to recover the SOI, which leads to redundancy in both calculation and data storage. However, with the development of quaternion-valued wireless communications [11]–[13], it is very likely that in the future we will have quaternion-valued signals as the SOI, where two traditional complex-valued signals with different polarisations arrive at the array with the same DOA.

In such a case, a full quaternion-valued array model is needed to compactly represent the four-component desired signal and also make sure the four components of the quaternion-valued output of the beamformer are fully utilised. In this work, we develop such a model and propose a new quaternion-valued Capon beamformer, where both its input and output are quaternion-valued.

This paper is structured as follows. The full quaternion-valued array model is introduced in Section II and the proposed quaternion-valued Capon beamformer is developed in Section III. Simulation results are presented in IV, and conclusions are drawn in Section V.

II. QUATERNION MODEL FOR ARRAY PROCESSING

A quaternion is constructed by four components [14], [15], with one real part and three imaginary parts, which is defined as 

\[ q = q_a + iq_b + jq_c + kq_d, \]

where \( i, j, k \) are three different imaginary units and \( q_a, q_b, q_c, q_d \) are real-valued. The multiplication principle among such units is

\[ i^2 = j^2 = k^2 = ijk = -1, \]

and

\[ ij = -ji = k, ki = -ik = j, jk = -ki = i \]

The conjugate \( q^* \) of \( q \) is \( q^* = q_a - iq_b - jq_c - kq_d \).

A quaternion number can be conveniently denoted as a combination of two complex numbers \( q = c_1 + ic_2 \), where the complex number \( c_1 = q_a + jq_c \) and \( c_2 = q_b + jq_d \). We will use this form later to represent our quaternion-valued signal of interest.

Consider a uniform linear array with \( N \) crossed-dipole sensors, as shown in Fig. I where the adjacent vector sensor spacing \( d \) equals half wavelength, and the two components of each crossed-dipole are parallel to \( x \)– and \( y \)–axes, respectively. A quaternion-valued narrowband signal \( s_0(t) \) impinges upon the vector sensor array among other \( M \) uncorrelated quaternion-valued interfering signals \( \{s_m(t)\}_{m=1}^M \), with background noise \( n(t) \). \( s_0(t) \) can be decomposed into

\[ s_0(t) = s_{01}(t) + is_{02}(t), \]

where \( s_{01}(t) \) and \( s_{02}(t) \) are two complex-valued sub-signals with the same DOA but different polarizations.

Assume that all signals are ellipse-polarized. The parameters, including DOA and polarization of the \( m \)-th signal are denoted by \( (\theta_m, \phi_m, \gamma_m, \eta_m) \) for the first sub-signal and
A quaternion model can be constructed by combining the two sub-signals, which are equal to each other since the two sub-signals share the same DOA \( \theta_m \). Each crossed-sub-array of the SOI pass through the beamformer and appear in the real and \( j \) components of the beamformer output, while the second complex sub-signal appears in the \( i \) and \( k \) components of the beamformer output. Then, with a quaternion-valued weight vector \( \mathbf{w} \), the constraint can be formulated as

\[
\mathbf{w}^H \mathbf{C} = \mathbf{f}
\]

where \( \{ \}^H \) is the Hermitian transpose of the quaternion-valued conjugate and transpose operation, \( \mathbf{C} = [b_{01} \ b_{02}] \), and \( \mathbf{f} = [1 \ j \ i] \).

With this constraint, the beamformer output \( \mathbf{z}(t) \) is given by

\[
\mathbf{z}(t) = \mathbf{w}^H \mathbf{q}(t) = s_0(t) + is_2(t) + \mathbf{w}^H \mathbf{n}_q(t)
\]

\[
+ \sum_{m=1}^{M} \mathbf{w}^H [b_{m1} s_m1(t) + b_{m2} s_m2(t)]
\]

Clearly, the quaternion-valued SOI has been preserved at the output with the desired unity response.

Now, the full-quaternion Capon (full Q-Capon) beamformer can be formulated as

\[
\min \mathbf{w}^H \mathbf{Rw} \text{ subject to } \mathbf{w}^H \mathbf{C} = \mathbf{f}
\]

where

\[
\mathbf{R} = E \{ \mathbf{q}(t) \mathbf{q}^H(t) \}.
\]

Applying the Lagrange multiplier method, we have

\[
l(\mathbf{w}, \lambda) = \mathbf{w}^H \mathbf{Rw} + (\mathbf{w}^H \mathbf{C} - \mathbf{f})^H \lambda + \lambda (\mathbf{C}^H \mathbf{w} - \mathbf{f}^H)
\]

where \( \lambda \) is a quaternion-valued vector.

The minimum can be obtained by setting the gradient of \( l(\mathbf{w}, \lambda) \) with respect to \( \mathbf{w}^* \) equal to a zero vector [20]. It is given by

\[
\nabla_{\mathbf{w}} l(\mathbf{w}, \lambda) = \frac{1}{2} \mathbf{Rw} + \frac{1}{2} \mathbf{C}^H \lambda = 0
\]

Considering all the constraints above, we obtain the optimum weight vector \( \mathbf{w}_{\text{opt}} \) as follows

\[
\mathbf{w}_{\text{opt}} = \mathbf{R}^{-1} \mathbf{C}(\mathbf{C}^H \mathbf{R}^{-1} \mathbf{C})^{-1} \mathbf{f}^H.
\]

A detailed derivation for the quaternion-valued optimum weight vector can be found at the Appendix.
In the next subsection, we give a brief analysis to show that by this optimum weight vector, the interference part at the beamformer output $z(t)$ in (11) has been suppressed effectively.

B. Interference Suppression

Expanding the covariance matrix, we have

$$R = E\{g(t)q^H(t)\} = R_{i+n} + \sigma_1^2 b_{01}b_{01}^H + \sigma_2^2 b_{02}b_{02}^H$$  \hspace{1cm} (17)

where $\sigma_1^2, \sigma_2^2$ are the power of the two sub-signals of SOI and $R_{i+n}$ denotes the covariance matrix of interferences plus noise. Using the Sherman-Morrison formula, we then have

$$w_{opt} = R_{i+n}^{-1} C\beta$$  \hspace{1cm} (18)

where $\beta = (C^HR_{i+n}C)^{-1}f \in \mathbb{H^{2x1}}$ is a quaternion vector.

Applying left eigendecomposition for quaternion matrix $[21]–[23]$, we have

$$R_{i+n} = \sum_{n=1}^{N} \alpha_n u_n u_n^H$$  \hspace{1cm} (19)

with $\alpha_1 \geq \ldots \geq \alpha_{M-2} \geq \alpha_{M-1} = \ldots = \alpha_{N} = 2\sigma_n^2 \in \mathbb{R}$, where $2\sigma_n^2$ denotes the noise power.

With sufficiently high interference to noise ratio (INR), the inverse of $R_{i+n}$ can be approximated by

$$R_{i+n}^{-1} \approx \sum_{n=M+1}^{N} \frac{1}{2\sigma_n^2} u_n u_n^H$$  \hspace{1cm} (20)

Then, we have

$$w_{opt} = \sum_{n=M+1}^{N} \frac{1}{2\sigma_n^2} u_n u_n^H C\beta = \sum_{n=M+1}^{N} u_n \rho_n$$  \hspace{1cm} (21)

where $\rho_n$ is a quaternion-valued constant. Clearly, $w_{opt}$ is the right linear combination of $\{u_{M+1}, u_{M+2}, \ldots, u_N\}$, and $w \in span_R\{u_{M+1}, u_{M+2}, \ldots, u_N\}$.

For those $M$ interfering signals, their quaternion steering vectors belong to the space right-spanned by the related $M$ eigenvectors, i.e. $b_{m1}, b_{n2} \in span_R\{u_1, u_2, \ldots, u_M\}$. As a result,

$$w_{opt}^H b_{m1} \approx 0, w_{opt}^H b_{n2} \approx 0, m = 1, 4, \ldots, M$$  \hspace{1cm} (22)

which shows that the beamformer has eliminated the interferences effectively.

C. Complexity Analysis

In this section, we make a comparison of the computation complexity between the Q-Capon beamformer in [10] and our proposed full Q-Capon beamformer. To deal with a quaternion-valued signal, the Q-Capon beamformer has to process the two complex sub-signals separately to recover the desired signal completely, which means we need to apply the beamformer twice for a quaternion-valued SOI. However, for the full Q-Capon beamformer, the SOI is recovered directly by applying the beamformer once.

For the Q-Capon beamformer, the weight vector is calculated by $w = R^{-1}a_0(a_0^HR^{-1}a_0)^{-1}$, where $a_0$ is the steering vector for the complex-valued SOI. As an example, we use Gaussian elimination to calculate the matrix inversion $R^{-1}$ and $\frac{1}{2}(N^3 - N)$ quaternion-valued multiplications are needed, equivalent to $16(N^3 - N)$ real-valued multiplications. Additionally, $R^{-1}a_0$ requires $16N^2$ real-valued multiplications, while $16(N^2 + N)$ real multiplications are needed for $(a_0^HR^{-1}a_0)^{-1}$. In total, $\frac{16}{3}N^3 + 32N^2 + \frac{16}{3}N$ real multiplications are needed. When processing a quaternion-valued signal, this number will be doubled and the total number of real multiplications becomes $\frac{32}{3}N^3 + 64N^2 + \frac{16}{3}N$.

For the proposed full Q-Capon beamformer, in addition to calculating $R^{-1}$, $32N^2$ real multiplications are required to calculate $R^{-1}C$ and $32M^2 + 32M + 96$ real multiplications for $(C^HR^{-1}C)^{-1}f$. In total, the number of real-valued multiplications is $\frac{16}{3}M^3 + 64M^2 + \frac{22}{3}M + 96$, which is roughly half of that of the Q-Capon beamformer.

IV. Simulations Results

In our simulations, we consider 10 pairs of cross-dipoles with half wave-length spacing. All signals are assumed to arrive from the same plane of $\theta = 90^\circ$ and all interferences have the same polarization parameter $\gamma = 60^\circ$. For the SOI, the two sub-signals are set to $(90^\circ, 1.5^\circ, 90^\circ, 45^\circ)$ and $(90^\circ, 1.5^\circ, 0^\circ, 0^\circ)$, with interferences coming from $(90^\circ, 30^\circ, 60^\circ, -80^\circ)$, $(90^\circ, -70^\circ, 60^\circ, 30^\circ)$, $(90^\circ, -20^\circ, 60^\circ, 70^\circ)$, $(90^\circ, 50^\circ, 60^\circ, -50^\circ)$, respectively. The background noise is zero-mean quaternion-valued Gaussian. The power of SOI and all interfering signals are set equal and SNR (INR) is 20dB.

Fig. 2 shows the resultant 3-D beam pattern by the proposed beamformer, where the interfering signals from $(\phi, \eta) = (30^\circ, -80^\circ), (-70^\circ, 30^\circ), (-20^\circ, 70^\circ)$ and $(50^\circ, -50^\circ)$ have all been effectively suppressed.

In the following, the output SINR performance of the two Capon beamformers (full Q-Capon and Q-Capon) is studied with the DOA and polarization $(90^\circ, 1.5^\circ, 90^\circ, 45^\circ)$ and $(90^\circ, 1.5^\circ, 0^\circ, 0^\circ)$ for SOI and $(90^\circ, 30^\circ, 60^\circ, -80^\circ), (90^\circ, -70^\circ, 60^\circ, 30^\circ), (90^\circ, -20^\circ, 60^\circ, 70^\circ), (90^\circ, 50^\circ, 60^\circ, -50^\circ)$ for interferences. Again, we have set SNR=INR=20dB. All results are obtained by averaging 1000 Monte-Carlo trials.

Fig. 3 shows the output SINR performance versus SNR with 100 snapshots, where the solid-line is for the optimal
beamformer with infinite number of snapshots. For most of the input SNR range, in particular the lower range, the proposed full Q-Capon beamformer has a better performance than the Q-Capon beamformer. For very high input SNR values, these two beamformers have a very similar performance.

Next, we investigate their performance in the presence of DOA and polarization errors. The output SINR with respect to the number of snapshots is shown in Fig. 4 in the presence of $1^\circ$ error for the SOI, where the real DOA and polarization parameters are $(91^\circ, 2.5^\circ, 91^\circ, 46^\circ)$ and $(91^\circ, 2.5^\circ, 1^\circ, 1^\circ)$. It can be seen that the full Q-Capon beamformer has achieved a much higher output SINR than the Q-Capon beamformer, and this gap increases with the increase of snapshot number. Fig. 5 shows a similar trend in the presence of a $5^\circ$ error. Overall, we can see that the proposed full Q-Capon beamformer is more robust against array pointing errors.

V. CONCLUSIONS

In this paper, a full quaternion model has been developed for adaptive beamforming based on crossed-dipole arrays, with a new full quaternion Capon beamformer proposed. Different from previous studies in quaternion-valued adaptive beamforming, we have considered a quaternion-valued desired signal, given the recent development in quaternion-valued wireless communications research. The proposed beamformer has a better performance and a much lower computational complexity than a previously proposed Q-Capon beamformer and is also shown to be more robust against array pointing errors, as demonstrated by computer simulations.

APPENDIX

The gradient of a quaternion vector $\mathbf{u} = w^H \mathbf{C} \lambda^H$ with respect to $\mathbf{w}^*$ can be calculated as below:

$$\nabla_{\mathbf{w}^*} \mathbf{u} = \left[ \nabla_{w_{1a}} \mathbf{u} \quad \nabla_{w_{1b}} \mathbf{u} \quad \ldots \quad \nabla_{w_{1d}} \mathbf{u} \right]^T$$

(23)

where $w_{1a}$, $n = 1, 2, \ldots, N$ is the $n$-th quaternion-valued coefficient of the beamformer. Then,

$$\nabla_{w_{1a}} \mathbf{u} = \frac{1}{4} (\nabla_{w_{1a}} \mathbf{u} + \nabla_{w_{1b}} \mathbf{u} i + \nabla_{w_{1c}} \mathbf{u} j + \nabla_{w_{1d}} \mathbf{u} k)$$

(24)

where

$$w_{1a}^* = w_{1a} - w_{1b} i - w_{1c} j - w_{1d} k$$

(25)

Since $w_{1a}$ is real-valued, with the chain rule [20], we have

$$\nabla_{w_{1a}} \mathbf{u} = \nabla_{w_{1a}} (w^H) \mathbf{C} \lambda^H + w^H \nabla_{w_{1a}} (\mathbf{C} \lambda^H)$$

$$= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \mathbf{C} \lambda^H$$

(26)

Similarly,

$$\nabla_{w_{1b}} \mathbf{u} = [i \ 0 \ 0 \ \ldots \ 0] \mathbf{C} \lambda^H$$

$$\nabla_{w_{1c}} \mathbf{u} = [j \ 0 \ 0 \ \ldots \ 0] \mathbf{C} \lambda^H$$

$$\nabla_{w_{1d}} \mathbf{u} = [k \ 0 \ 0 \ \ldots \ 0] \mathbf{C} \lambda^H$$

Hence,

$$\nabla_{w_{1a}} \mathbf{u} = \frac{1}{4}(4 \text{Real}(\mathbf{C} \lambda^H))_1 = \text{Real}(\mathbf{C} \lambda^H)_1$$

(27)

where the subscript $\{ \}_{1}$ in the last item means taking the first entry of the vector.

Finally,

$$\nabla_{w^*} \mathbf{u} = \text{Real}(\mathbf{C} \lambda^H)$$

(28)

The gradient of the quaternion vector $\mathbf{v} = \mathbf{C} \lambda^H \mathbf{w}$ with respect to $\mathbf{w}^*$ can be calculated in the same way:

$$\nabla_{w_{1a}} \mathbf{v} = \mathbf{C} \lambda^H \nabla_{w_{1a}} \mathbf{w} + \nabla_{w_{1a}} (\mathbf{C} \lambda^H) \mathbf{w}$$

$$= \mathbf{C} \lambda^H [1 \ 0 \ 0 \ \ldots \ 0]^T$$

(29)
Thus, the gradient can be expressed as
\[ \nabla \mathbf{w}_i \mathbf{v} = \mathbf{C} \mathbf{H}[i] 0 0 \ldots 0 \mathbf{T} \]
\[ \nabla \mathbf{w}_j \mathbf{v} = \mathbf{C} \mathbf{H}[j] 0 0 \ldots 0 \mathbf{T} \]
\[ \nabla \mathbf{w}_k \mathbf{v} = \mathbf{C} \mathbf{H}[k] 0 0 \ldots 0 \mathbf{T} \] (30)

Similarly,
\[ \nabla \mathbf{w}_i^* \mathbf{v} = -\frac{1}{2} \mathbf{C} \mathbf{H}^* \]
\[ \nabla \mathbf{w}_j^* \mathbf{v} = -\frac{1}{2} \mathbf{C} \mathbf{H}^* \]
\[ \nabla \mathbf{w}_k^* \mathbf{v} = -\frac{1}{2} \mathbf{C} \mathbf{H}^* \] (31)

Finally,
\[ \nabla \mathbf{w}_i \mathbf{v} = -\frac{1}{2} (\mathbf{C} \mathbf{H}^*)^* \]
\[ \nabla \mathbf{w}_j \mathbf{v} = -\frac{1}{2} (\mathbf{C} \mathbf{H}^*)^* \]
\[ \nabla \mathbf{w}_k \mathbf{v} = -\frac{1}{2} (\mathbf{C} \mathbf{H}^*)^* \] (32)

The gradient of \( \mathbf{c}_w = \mathbf{w}^H \mathbf{R} \mathbf{w} \) can be calculated as follows.
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \nabla \mathbf{w}_j \mathbf{c}_w = \nabla \mathbf{w}_k \mathbf{c}_w = \nabla \mathbf{w}_l \mathbf{c}_w \]
\[ = \frac{1}{4} (\nabla \mathbf{w}_i \mathbf{w}^H + \nabla \mathbf{w}_j \mathbf{w}^H + \nabla \mathbf{w}_k \mathbf{w}^H + \nabla \mathbf{w}_l \mathbf{w}^H) \]
\[ = \frac{1}{4} (\nabla \mathbf{w}_i \mathbf{w}^H \mathbf{R} + \nabla \mathbf{w}_j \mathbf{w}^H \mathbf{R} + \nabla \mathbf{w}_k \mathbf{w}^H \mathbf{R} + \nabla \mathbf{w}_l \mathbf{w}^H \mathbf{R}) \] (33)

Now we calculate the gradient of \( \mathbf{c}_w \) with respect to the four components of \( \mathbf{w}_i \).
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \nabla \mathbf{w}_j \mathbf{c}_w = \nabla \mathbf{w}_k \mathbf{c}_w = \nabla \mathbf{w}_l \mathbf{c}_w \]
\[ = \mathbf{w}^H \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w} \]
\[ = [0 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = [i 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = [j 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = [k 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = \lambda \mathbf{C} \mathbf{H} \]
\[ = \lambda \mathbf{C} \mathbf{H} \] (34)

The other three components are,
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \mathbf{w}^H \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w} \]
\[ = [0 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = [0 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = [0 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = [0 0 0 \ldots 0 \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}]^T \]
\[ = \lambda \mathbf{C} \mathbf{H} \] (35)

Hence,
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \mathbf{w}^H \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w} \]
\[ = \mathbf{w}^H \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w} \]
\[ = \mathbf{w}^H \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w} \]
\[ = \lambda \mathbf{C} \mathbf{H} \]
\[ = \lambda \mathbf{C} \mathbf{H} \] (36)

Finally,
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \mathbf{w}^H \mathbf{R} \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w} \]
\[ = \lambda \mathbf{C} \mathbf{H} \]
\[ = \lambda \mathbf{C} \mathbf{H} \] (37)

Combining (28), (32), and (37) with (14), we have
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \frac{1}{2} (\mathbf{R} \mathbf{w} + \mathbf{C} \mathbf{H}^*) = 0 \]
\[ \nabla \mathbf{w}_i \mathbf{c}_w = \frac{1}{2} (\mathbf{R} \mathbf{w} + \mathbf{C} \mathbf{H}^*) = 0 \] (38)

Further, \( \mathbf{w} = -2 \mathbf{R}^{-1} \mathbf{C} \mathbf{H} \)
\[ \mathbf{w} = -2 \mathbf{R}^{-1} \mathbf{C} \mathbf{H} \] (39)

Substituting (39) into (10),
\[ \lambda = -\frac{1}{2} \mathbf{f}(\mathbf{C} \mathbf{H} \mathbf{R}^{-1} \mathbf{C})^{-1} \]
\[ \lambda = -\frac{1}{2} \mathbf{f}(\mathbf{C} \mathbf{H} \mathbf{R}^{-1} \mathbf{C})^{-1} \] (40)

Finally,
\[ \mathbf{w} = \mathbf{R}^{-1} \mathbf{C} \mathbf{H} \mathbf{R}^{-1} \mathbf{C}^{-1} \mathbf{f}^{-1} \]
\[ \mathbf{w} = \mathbf{R}^{-1} \mathbf{C} \mathbf{H} \mathbf{R}^{-1} \mathbf{C}^{-1} \mathbf{f}^{-1} \] (41)