AN ANALOG OF THE FOURIER TRANSFORMATION
FOR A q-HARMONIC OSCILLATOR

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ABSTRACT
A q-version of the Fourier transformation and some of its properties are discussed.

INTRODUCTION
Models of q-harmonic oscillators are being developed in connection with quantum groups and their various applications (see, for example, Refs. [M], [Bi], [AS1], and [AS2]). For a complete correspondence with the quantum-mechanical oscillator problem, these models need an analog of the Fourier transformation that relates the coordinate and momentum spaces. In the present work we fill this gap for one of the models, the one based on the continuous q-Hermite polynomials [M], [AS1] when \(-1 < q < 1\).

In Section I we assemble all those formulas from [AS1], which are necessary for the subsequent exposition. In Section II we discuss the relation between the Mehler bilinear generating function for Hermite polynomials and the kernel \(\exp\left(\frac{1}{\hbar}px\right)\) of the Fourier transformation that connects the coordinate \(x\) and momentum \(p\) spaces [W]. We used the bilinear formula of L. J. Rogers to obtain a reproducing kernel and an analogue of the Fourier transform in the setting determined by the continuous q-Hermite polynomials of Rogers. Some properties of the q-Fourier transformation are discussed in Sections III-IV.

In the following we take \(0 < q < 1\), although most of the formulas remain correct when \(-1 < q < 0\). The limiting case \(q \to 0\) is of some mathematical interest.

1. q-Hermite functions. The continuous q-Hermite polynomials were introduced by Rogers [R]. They can be defined by the three term recurrence relation

\[
2xH_n(x \mid q) = H_{n+1}(x \mid q) + (1 - q^n)H_{n-1}(x \mid q),
\]

\(H_0(x \mid q) = 1,\ H_1(x \mid q) = 2x\). They are orthogonal on \(-1 \leq x = \cos \theta \leq 1\) with respect to a positive measure \(\rho(x)\)
\begin{equation}
\rho(x) = 4 \sin \theta (qe^{2i\theta}, qe^{-2i\theta}, q)_{\infty} = 4 \sqrt{1-x^2} \prod_{j=1}^{\infty} [1 - 2(2x^2 - 1)q^j + q^{2j}] \tag{1.2}
\end{equation}

where the usual notations (see [GR]) are

\begin{equation}
(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n), \tag{1.3}
\end{equation}

\begin{equation}
(a, b; q)_{\infty} = (a; q)_{\infty} (b; q)_{\infty}. \tag{1.4}
\end{equation}

A q-wave function for this q-harmonic oscillator was introduced explicitly in [M], [AS1] as

\begin{equation}
\psi_n(x) = \alpha d_n^{-1} \rho^{1/2}(x) H_n(x \mid q), \quad \alpha = \left( \frac{1-q}{2} \right)^{1/4}, \tag{1.5}
\end{equation}

where \( d_n > 0 \) and

\begin{equation}
d_n^{-2} = \frac{1}{2\pi} (q^{n+1}; q)_{\infty}. \tag{1.6}
\end{equation}

These q-wave functions (1.4) satisfy

\begin{equation}
\int_{-1}^{1} \psi_n(x) \psi_m(x) \, dx = \alpha^2 \delta_{nm}. \tag{1.7}
\end{equation}

See [AI] for a proof of this orthogonality relation.

q-annihilation \( b \) and q-creation \( b^+ \) operators were introduced explicitly in [M] and [AS1]. They satisfy the commutation rule

\begin{equation}
bb^+ - q^{-1} b^+ b = 1 \tag{1.8}
\end{equation}

and act on the q-wave functions defined in (1.4) by

\begin{equation}
b \psi_n(x) = \tilde{e}_n^{1/2} \psi_{n-1}(x), \quad b^+ \psi_n(x) = \tilde{e}_n^{1/2} \psi_{n+1}(x) \tag{1.9}
\end{equation}

where

\begin{equation}
\tilde{e}_n = \frac{1 - q^{-n}}{1 - q^{-1}} = q^{1-n} e_n. \tag{1.10}
\end{equation}

In the limit when \( q \to 1^- \) the functions \( \psi_n(\alpha^2 \xi) \) converge to the classical wave functions

\begin{equation}
\Psi_n(\xi) = \frac{1}{\pi^{1/4}(2n!)^{1/2}} H_n(\xi)e^{-\xi^2/2} \tag{1.11}
\end{equation}
of the linear harmonic oscillator in the coordinate representation.

2. An analog of the Fourier transformation. In proving that the Hermite functions \((1.10)\) are complete in the space \(L_2\) over \((-\infty, \infty)\) and that a Fourier transform of any function from \(L_2\) belongs to the same space, the important role is played by the bilinear generating function (or the Poisson kernel) \([W]\):

\[
K_t(\xi, \eta) = \sum_{n=0}^{\infty} t^n \Psi_n(\xi)\Psi_n(\eta)
= \left[ \pi(1-t^2) \right]^{-1/2} \exp \left[ \frac{4\xi\eta t - (\xi^2 + \eta^2)(1 + t^2)}{2(1 - t^2)} \right].
\]

(2.1)

Since in the limit \(\gamma \to \infty\) the continuously differentiable function

\[
\delta(x, \gamma) = \frac{\gamma}{\sqrt{\pi}} e^{-\gamma^2 x^2}
\]

becomes the Dirac delta function \(\delta(x)\), from (2.1) in the limit \(t \to 1^-\) follows the completeness property of the system \((1.10)\),

\[
\sum_{n=0}^{\infty} \Psi_n(\xi)\Psi_n(\eta) = \delta(\xi - \eta).
\]

(2.2)

On another hand, setting \(t = i\) reduces the right-hand side of (2.1) to the kernel of the Fourier transformation, i.e.

\[
K_i(\xi, \eta) = \frac{1}{\sqrt{2\pi}} e^{i\xi \eta}.
\]

Actually this idea provides the possibility of finding the Fourier transformation for difference analogs of the harmonic oscillator and its \(q\)-generalizations, when a priori it is not clear how one can define the explicit form of the kernels and how these transformations can look like. As first examples we considered Kravchuk and Charlier functions [AAS]. Here we shall give the necessary formulas for the \(q\)-Hermite functions \((1.4)\).

Let us define a \(q\)-analog of (2.1) as

\[
K_t(x, y; q) = \alpha^{-2} \sum_{n=0}^{\infty} t^n \psi_n(x)\psi_n(y).
\]

(2.3)

Substituting the formulas \((1.4)\) and \((1.5)\) into (2.3), we get

\[
K_t(x, y; q) = \frac{(q; q)_\infty}{2\pi} \rho^{1/2}(x)\rho^{1/2}(y) \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} H_n(x|q)H_n(y|q).
\]

(2.4)
This series (the Poisson kernel) was summed as an infinite product by Rogers \cite{R}, and the value is stated in \cite{AI}. A very simple proof was given by Bressoud \cite{Br}. The formula is
\[
\sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} H_n(\cos \theta | q) H_n(\cos \varphi | q) =
\]
\[
= (t^2; q)_{\infty} (te^{i(\theta+\varphi)}, te^{i(\theta-\varphi)}, te^{-i(\theta+\varphi)}, te^{-i(\theta-\varphi)}; q)_{\infty}^{-1},
\]
so we can write (2.4) as
\[
K_t(x, y; q) = \frac{(q, t^2; q)_{\infty} \rho^{1/2}(x) \rho^{1/2}(y)}{2\pi (te^{i(\theta+\varphi)}, te^{i(\theta-\varphi)}, te^{-i(\theta+\varphi)}, te^{-i(\theta-\varphi)}; q)_{\infty}}.
\]

In complete analogy with the case of the harmonic oscillator,
\[
\lim_{t \to 1^-} K_t(x, y; q) = \alpha^{-2} \sum_{n=0}^{\infty} \psi_n(x) \psi_n(y) = \delta(x - y).
\]

This is even easier to prove than in the classical case, since any set of polynomials orthogonal on a finite interval is complete in $L^2$, and this is equivalent to (2.7). Formula (2.7) follows from
\[
t^n \psi_n(x) = \int_{-1}^{1} K_t(x, y; q) \psi_n(y) \, dy, \quad |t| < 1,
\]
which is obvious by integration, which is justified by uniform convergence. An easy corollary of (2.3) is
\[
\int_{-1}^{1} K_t(x, y; q) K_r(y, x'; q) \, dy = K_t r(x, x'; q), \quad |t| < 1, \quad |\tau| < 1.
\]

Not only is the limit $t \to 1^-$ interesting, $t \to i$ is also interesting. The kernel in this case is the special case of (2.6) when $t = i$, i.e.
\[
K_i(x, y; q) = \frac{A}{\pi} (q^2; q^2)_{\infty} \left[ \sin \theta \sin \varphi (qe^{2i\theta}, qe^{-2i\theta}, qe^{2i\varphi}, qe^{-2i\varphi}; q)_{\infty} \right]^{1/2},
\]
\[
x = \cos \theta, \quad y = \cos \varphi.
\]

The classical Fourier transform has a kernel which is bounded, and the fourth power of it is the identity. The fourth power of the $q$-Fourier transform,
\[
F_q[\psi](x) = \lim_{r \to 1^-} \int_{-1}^{1} K_{ir}(x, y; q) \psi(y) \, dy,
\]
\[
\text{(2.11)}
\]

\[
4
\]
is also the identity, from (2.8) with \( t \rightarrow i \) and \( i^4 = 1 \). Since \([-1, 1]\) is compact, the kernel \( K_i(x, y; q) \) cannot be bounded and have \( F_q^4 = I \). It is not, and the singularity comes from the first term in the four products in the denominator of (2.10). The singular part is just the value of (2.10) when \( q = 0 \), i.e.

\[
\frac{-(\sin \theta \sin \varphi)^{1/2}}{\pi \cos(\theta + \varphi) \cos(\theta - \varphi)}.
\]  

(2.12)

Thus this \( q \)-Fourier transform looks more like a weighted Hilbert transform than it looks like the classical Fourier transform. An explicit form of the transformation (2.11) is

\[
F_q[\psi](x) = P \nu \int_{-1}^{1} K_i(x, y; q) \psi(y) \, dy \qquad + \frac{i}{2} \left[ k(x) \psi(\sqrt{1 - x^2}) + k(-x) \psi(-\sqrt{1 - x^2}) \right],
\]

where

\[
k(x) = \frac{x}{(x^2(1 - x^2))^{1/4}} \left( \prod_{k=1}^{\infty} \frac{1 + 4ix\sqrt{1 - x^2}q^k - q^{2k}}{1 - 4ix\sqrt{1 - x^2}q^k - q^{2k}} \right)^{1/2}
\]

and \( P \nu \) denotes Cauchy’s principal value integral.

The inverse of this transformation follows from

\[
\lim_{r, r' \to 1^-} \int_{-1}^{1} K_{ir}(x, y; q) K_{ir'}^*(y, x'; q) \, dy = \delta(x - x'),
\]

(2.14)

where \( * \) denotes the complex conjugate. Formula (2.14) follows from (2.9) when we observe that

\[
K_{ir}^*(y, x; q) = K_{-ir}(y, x; q).
\]  

(2.15)

Another form of (2.14) is

\[
\lim_{r, r' \to 1^-} \int_{-1}^{1} dy K_{ir'}^*(x, y; q) \int_{-1}^{1} K_{ir}(y, z; q) f(z) \, dz = f(x).
\]  

(2.16)

Indeed, by (2.9) and (2.15) we get

\[
\int_{-1}^{1} dy K_{ir'}^*(x, y; q) \int_{-1}^{1} K_{ir}(y, z; q) f(z) \, dz
\]

\[
= \int_{-1}^{1} K_{rr'}(x, z; q) f(z) \, dz \longrightarrow f(x)
\]

with \( r, r' \to 1^- \).
By using the change of variables \( x = \alpha^2 \xi \) and \( y = \alpha^2 \eta \), where \( \alpha^2 = \left( \frac{1-q}{2} \right)^{1/2} \), from (2.3) in the limit \( q \to 1^- \) we can write

\[
\alpha^2 K_t(\alpha^2 \xi, \alpha^2 \eta; q) = \sum_{n=0}^{\infty} t^n \psi_n(\alpha^2 \xi) \psi_n(\alpha^2 \eta)
\]

\[
\longrightarrow \sum_{n=0}^{\infty} \Psi_n(\xi) \Psi_n(\eta) = K_t(\xi, \eta),
\]

or

\[
\lim_{q \to 1^-} \alpha^2 K_t(\alpha^2 \xi, \alpha^2 \eta; q) = K_t(\xi, \eta). \tag{2.17}
\]

With the aid of the same consideration, from (2.11) we get

\[
F_q[\psi](x) = \lim_{r \to 1^-} \int_{-1/\alpha^2}^{1/\alpha^2} \alpha^2 K_{it}(\alpha^2 \xi, \alpha^2 \eta; q) \psi(\alpha^2 \eta) \, d\eta
\]

\[
\longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi \eta} f(\eta) \, d\eta = F[f](\xi),
\]

where \( f(\eta) = \lim_{q \to 1^-} \psi(\alpha^2 \eta) \). Therefore, the classical Fourier transformation can be realized as a limiting case of the \( q \)-Fourier transform.

3. “Momentum and position” operators. As is well known from quantum mechanics, the kernel of the classical Fourier transformation is also an eigenfunction of the momentum and position operators, i.e.

\[
Q_\xi K_i(\xi, \eta) = \xi K_i(\xi, \eta), \tag{3.1}
\]

\[
P_\xi K_i(\xi, \eta) = i^{-1} d d\xi \left( \frac{1}{\sqrt{2\pi}} e^{i\xi \eta} \right) = \eta K_i(\xi, \eta).
\]

We can retain this property in the case of the \( q \)-Fourier transformation, if we choose the corresponding operators for the kernel (2.10) as

\[
Q = \frac{\sqrt{1-q}}{2} \left( q_{N/2} b + b^+ q_{N/2} \right),
\]

\[
P = \frac{\sqrt{1-q}}{2i} \left( q_{N/2} b - b^+ q_{N/2} \right), \tag{3.2}
\]

where \( N = \log[1 - (1 - q^{-1}) b^+ b]/\log q^{-1} \) is the “particle number” operator. In fact, the action of the operator \( Q \) on the kernel (2.10) gives

\[
\alpha^2 Q_x K_i(x, y; q) = \frac{\sqrt{1-q}}{2} \left( q_{N/2} b_x + b_x^+ q_{N/2} \right) \sum_{n=0}^{\infty} i^n \psi_n(x) \psi_n(y)
\]

\[
= \frac{\sqrt{1-q}}{2} \sum_{n=0}^{\infty} i^n \left[ e^{1/2} \psi_{n-1}(x) + e^{1/2} \psi_{n+1}(x) \right] \psi_n(y). \tag{3.3}
\]
From the three-term recurrence relation (1.1) for the continuous $q$-Hermite polynomials, it follows that

$$e^{1/2} \psi_{n+1}(x) + e^{1/2} \psi_{n-1}(x) = \frac{2}{\sqrt{1-q}} x \psi_n(x). \quad (3.4)$$

Therefore for the “position” operator $Q$ we have, indeed,

$$Q_x K_i(x, y; q) = x K_i(x, y; q). \quad (3.5)$$

Exactly in the same manner, one can obtain directly from definitions (3.2) and expansion (2.3) for $t = i$, that

$$P_x K_i(x, y; q) = Q_y K_i(x, y; q), \quad (3.6)$$

and, consequently,

$$P_x K_i(x, y; q) = y K_i(x, y; q). \quad (3.7)$$

Equations

$$Q_x K_t(x, y; q) = x K_t(x, y; q), \quad (3.8)$$

$$P_x K_t(x, y; q) = \frac{2ty - (1 + t^2)x}{i(1 - t^2)} K_t(x, y; q) \quad (3.9)$$

are an extension of (3.5) and (3.7).

The commutation rule of the operators (3.2) is

$$[Q, P] = i \frac{1 - q}{2} q^N \quad (3.10)$$

and, therefore, the Hamiltonian of the $q$-oscillator,

$$\tilde{H} = b^+ b = \frac{1 - q^{-N}}{1 - q^{-1}}, \quad (3.11)$$

has the following form

$$\tilde{H}(P, Q) = \frac{i - 2(1-q)^{-1} [Q, P]}{2q^{-1} [Q, P]} \quad (3.12)$$

in terms of these momentum and position operators.

The equation

$$P_x^2 K_i(x, p; q) = p^2 K_i(x, p; q) \quad (3.13)$$

may be considered as an equation of motion for a “$q$-free particle”.

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4. Some properties of the $q$-Fourier transformation. The kernel $K_i(\xi, \eta)$ corresponds to the classical Fourier transformation

$$f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi \eta} g(\eta) \, d\eta = F[g](\xi) ,$$

which has well-known properties [W]. With the aid of the kernel (2.6) we defined a $q$-version of the Fourier transformation by

$$\psi(x) = \lim_{r \to 1^-} \int_{-1}^{1} K_{ir}(x, y) \varphi(y) \, dy = F_q[\varphi](x) .$$

We can establish the simple properties of this generalization. The orthogonality property (2.14) of the kernel (2.6) results in the inversion formula

$$\varphi(y) = \lim_{r' \to 1^-} \int_{-1}^{1} K_{ir'}^*(x, y) \psi(x) \, dx ,$$

as well as in the relation $\|\varphi\|^2 = \|\psi\|^2$.

Analogs of the properties

$$i^{-1} \frac{d}{d\xi} F[g](\xi) = F[\eta g](\xi) ,$$

$$iF \left[ \frac{d}{d\eta} \right] (\xi) = \xi F[g](\xi) ,$$

have the forms

$$P_x F_q[\varphi](x) = F_q[y \varphi](x) ,$$

$$F_q[P_y \varphi](x) = -xF_q[\varphi](x) .$$

Moreover, the following properties of the Fourier transform,

$$F[f](\xi + \xi_0) = e^{i\xi_0 P_\xi} F[f](\xi) = F[e^{i\xi_0 \eta} f](\xi) ,$$

$$F[f(\eta - \eta_0)](\xi) = F[e^{-i\eta_0 P_\eta} f](\xi) = e^{i\xi \eta_0} F[f](\xi) ,$$

admit a generalization

$$K_i(x_0, P_x) F_q[\varphi](x) = F_q[K_i(x_0, y) \varphi](x) ,$$

$$F_q[K_i(y_0, -P_y) \varphi](x) = K_i(x, y_0) F_q[\varphi](x) .$$
To define a $q$-version of the convolution $\varphi \ast \psi$ let us retain the property
\[ F_q[\varphi \ast \psi] = F_q[\varphi] \cdot F_q[\psi] , \tag{4.8} \]
or
\[ \lim_{r \to 1^-} \int_{-1}^{1} K_{ir}(x, z)(\varphi \ast \psi)(z) \, dz = \]
\[ \lim_{r', r'' \to 1^-} \int_{-1}^{1} \int_{-1}^{1} K_{ir'}(x, y)K_{ir''}(x, y') \varphi(y)\psi(y') \, dy \, dy' . \tag{4.9} \]

Using (2.14) and (4.9) we arrive at the definition
\[ (\varphi \ast \psi)(z) = \lim_{r, r', r'' \to 1^-} \int_{-1}^{1} \int_{-1}^{1} dy \, dy' \varphi(y)\psi(y') \]
\[ \int_{-1}^{1} dx K_{ir'}(x, y)K_{ir''}(x, y')K^*_{ir}(x, z) . \tag{4.10} \]

The usual properties,
\[ \varphi \ast \psi = \psi \ast \varphi , \]
\[ (\varphi \ast \psi) \ast \chi = \varphi \ast (\psi \ast \chi) , \]
are valid.

In the limit $q \to 1^-$ we can write
\[ \alpha^4 \int_{-1}^{1} K_{ir'}(x, y)K_{ir''}(x, y')K^*_{ir}(x, z) \, dx \]
\[ \longrightarrow \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{i\xi(\eta + \eta' - \zeta)} \, d\xi = \frac{1}{\sqrt{2\pi}} \delta(\eta + \eta' - \zeta) , \tag{4.11} \]
and, therefore,
\[ (\varphi \ast \psi)(z) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)g(\zeta - \eta) \, d\eta . \]

We think the transformation (4.2) deserves a more detailed consideration.

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