On the randomized complexity of Banach space valued integration

Stefan Heinrich
Department of Computer Science
University of Kaiserslautern
D-67653 Kaiserslautern, Germany
e-mail: heinrich@informatik.uni-kl.de

Aicke Hinrichs
Institute of Mathematics
University of Rostock
D-18051 Rostock, Germany
e-mail: aicke.hinrichs@uni-rostock.de

Abstract
We study the complexity of Banach space valued integration in the randomized setting. We are concerned with \( r \)-times continuously differentiable functions on the \( d \)-dimensional unit cube \( Q \), with values in a Banach space \( X \), and investigate the relation of the optimal convergence rate to the geometry of \( X \). It turns out that the \( n \)-th minimal errors are bounded by \( cn^{-r/d-1+1/p} \) if and only if \( X \) is of equal norm type \( p \).

1 Introduction

Integration of scalar valued functions is an intensively studied topic in the theory of information-based complexity, see [12], [10], [11]. Motivated by applications to parametric integration, recently the complexity of Banach space valued integration was considered in [2]. It was shown that the behaviour of the \( n \)-th minimal errors \( e_{n}^{\text{ran}} \) of randomized integration in \( C^{r}(Q,X) \) is related to the geometry of the Banach space \( X \) in the following way: The infimum of the exponents of the rate is determined by the supremum of \( p \) such that \( X \) is of type \( p \). In the present paper we further investigate this relation. We establish a connection between \( n \)-th minimal errors and equal norm type \( p \) constants for \( n \) vectors. It follows that \( e_{n}^{\text{ran}} \) is bounded by \( cn^{-r/d-1+1/p} \) if and only if \( X \) is of equal norm type \( p \).
2 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots \}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$. We introduce some notation and concepts from Banach space theory needed in the sequel. For Banach spaces $X$ and $Y$ let $B_X$ be the closed unit ball of $X$ and $\mathcal{L}(X,Y)$ the space of bounded linear operators from $X$ to $Y$, endowed with the usual norm. If $X = Y$, we write $\mathcal{L}(X)$. The norm of $X$ is denoted by $\| \cdot \|$, while other norms are distinguished by subscripts. We assume that all considered Banach spaces are defined over the same scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Let $Q = [0,1]^d$ and let $C^r(Q,X)$ be the space of all $r$-times continuously differentiable functions $f : Q \to X$ equipped with the norm

$$\| f \|_{C^r(Q,X)} = \max_{0 \leq |\alpha| \leq r, t \in Q} \| D^\alpha f(t) \|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$ and $D^\alpha$ denotes the respective partial derivative. For $r = 0$ we write $C^0(Q,X) = C(Q,X)$, which is the space of continuous $X$-valued functions on $Q$. If $X = \mathbb{K}$, we write $C^r(Q)$ and $C(Q)$.

Let $1 \leq p \leq 2$. A Banach space $X$ is said to be of (Rademacher) type $p$, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$

$$\left( \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \right)^{1/p} \leq c \left( \sum_{k=1}^n \| x_i \|^p \right)^{1/p}, \quad (1)$$

where $(\varepsilon_i)_{i=1}^n$ is a sequence of independent Bernoulli random variables with $\mathbb{P}\{ \varepsilon_i = -1 \} = \mathbb{P}\{ \varepsilon_i = +1 \} = 1/2$ on some probability space $(\Omega, \Sigma, \mathbb{P})$ (we refer to [9, 7] for this notion and related facts). The smallest constant satisfying (1) is called the type $p$ constant of $X$ and is denoted by $\tau_p(X)$. If there is no such $c > 0$, we put $\tau_p(X) = \infty$. The space $L_{p_1}(\mathcal{N}, \nu)$ with $(\mathcal{N}, \nu)$ an arbitrary measure space and $p_1 < \infty$ is of type $p$ with $p = \min(p_1, 2)$.

Furthermore, given $n \in \mathbb{N}$, let $\sigma_{p,n}(X)$ be the smallest $c > 0$ for which (1) holds for any $x_1, \ldots, x_n \in X$ with $\| x_1 \| = \cdots = \| x_n \|$. The contraction principle for Rademacher series, see ([7], Th. 4.4), implies that $\sigma_{p,n}(X)$ is the smallest constant $c > 0$ such that for $x_1, \ldots, x_n \in X$

$$\left( \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \right)^{1/p} \leq c n^{1/p} \max_{1 \leq i \leq n} \| x_i \|. \quad (2)$$

We say that $X$ is of equal norm type $p$, if there is a constant $c > 0$ such that $\sigma_{p,n}(X) \leq c$ for all $n \in \mathbb{N}$. Clearly, $\sigma_{p,n}(X) \leq \tau_p(X)$ and type $p$ implies equal norm type $p$.

Let us comment a little more on the relation of the different notions of type which are used here and in the literature. The concept of equal norm type $p$ was first introduced and used by R. C. James in the case $p = 2$ in [6]. There it is
shown that $X$ is of equal norm type 2 if and only if $X$ is of type 2. This result is attributed to G. Pisier. Later, it even turned out in [1] that the sequence $\sigma_{2,n}(X)$ and the corresponding sequence $\tau_{2,n}(X)$ of type 2 constants computed with $n$ vectors are uniformly equivalent. In contrast, for $1 < p < 2$, L. Tzafriri [13] constructed Tsirelson spaces without type $p$ but with equal norm type $p$. Finally, V. Mascioni introduced and studied the notion of weak type $p$ for $1 < p < 2$ in [8] and showed that, again in contrast to the situation for $p = 2$, a Banach space $X$ is of weak type $p$ if and only if it is of equal norm type $p$.

Throughout the paper $c, c_1, c_2, \ldots$ are constants, which depend only on the problem parameters $r, d$, but depend neither on the algorithm parameters $n, l$ etc. nor on the input $f$. The same symbol may denote different constants, even in a sequence of relations.

For $r, k \in \mathbb{N}$ we let $P_{r,X}^k \in \mathcal{L}(C(Q, X))$ be $X$-valued composite tensor product Lagrange interpolation of degree $r$ with respect to the partition of $[0, 1]^d$ into $k^d$ subcubes of sidelength $k^{-1}$ of disjoint interior, see [2]. Given $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$, there are constants $c_1, c_2 > 0$ such that for all Banach spaces $X$ and all $k \in \mathbb{N}$

$$\sup_{f \in B_{C^r(Q,X)}} \|f - P_{r,X}^k f\|_{C(Q, X)} \leq c_2 k^{-r}$$

(see [2]).

## 3 Banach space valued integration

Let $X$ be a Banach space, $r \in \mathbb{N}_0$, and let the integration operator $S^X : C(Q, X) \to X$ be given by

$$S^X f = \int_Q f(t) dt.$$  

We will work in the setting of information-based complexity theory, see [12, 10, 11]. Below $\epsilon_n^{\text{det}}(S^X, B_{C^r(Q,X)})$ and $\epsilon_n^{\text{ran}}(S^X, B_{C^r(Q,X)})$ denote the $n$-th minimal error of $S^X$ on $B_{C^r(Q,X)}$ in the deterministic, respectively randomized setting, that is, the minimal possible error among all deterministic, respectively randomized algorithms, approximating $S^X$ on $B_{C^r(Q,X)}$ that use at most $n$ values of the input function $f$. The precise notions are recalled in the appendix. The following was shown in [2].

**Theorem 1.** Let $r \in \mathbb{N}_0$ and $1 \leq p \leq 2$. Then there are constants $c_{1-4} > 0$ such that for all Banach spaces $X$ and $n \in \mathbb{N}$ the following holds. The deterministic $n$-th minimal error satisfies

$$c_1 n^{-r/d} \leq \epsilon_n^{\text{det}}(S^X, B_{C^r(Q,X)}) \leq c_2 n^{-r/d}.$$  

Moreover, if $X$ is of type $p$ and $p_X$ is the supremum of all $p_1$ such that $X$ is of type $p_1$, then the randomized $n$-th minimal error fulfills

$$c_3 n^{-r/d-1+1/p_X} \leq \epsilon_n^{\text{ran}}(S^X, B_{C^r(Q,X)}) \leq c_4 \tau_p(X) n^{-r/d-1+1/p}.$$  

3
As a consequence, we obtain

**Corollary 1.** Let \( r \in \mathbb{N}_0 \) and \( 1 \leq p \leq 2 \). Then the following are equivalent:

(i) \( X \) is of type \( p_1 \) for all \( p_1 < p \).

(ii) For each \( p_1 < p \) there is a constant \( c > 0 \) such that for all \( n \in \mathbb{N} \)

\[
e_n^{\text{ran}}(S^X, B_{C^r(Q,X)}) \leq cn^{-r/d-1+1/p_1}.
\]

The main result of the present paper is the following

**Theorem 2.** Let \( 1 \leq p \leq 2 \) and \( r \in \mathbb{N}_0 \). Then there are constants \( c_1, c_2 > 0 \) such that for all Banach spaces \( X \) and all \( n \in \mathbb{N} \)

\[
c_1n^{r/d+1-1/p}e_n^{\text{ran}}(S^X, B_{C^r(Q,X)}) \leq \sigma_{p,n}(X) \leq c_2 \max_{1 \leq k \leq n} k^{r/d+1-1/p}e_k^{\text{ran}}(S^X, B_{C^r(Q,X)}).
\]

This allows to sharpen Corollary 1 in the following way.

**Corollary 2.** Let \( r \in \mathbb{N}_0 \) and \( 1 \leq p \leq 2 \). Then the following are equivalent:

(i) \( X \) is of equal norm type \( p \).

(ii) There is a constant \( c > 0 \) such that for all \( n \in \mathbb{N} \)

\[
e_n^{\text{ran}}(S^X, B_{C^r(Q,X)}) \leq cn^{-r/d-1+1/p}.
\]

Recall from the preliminaries that the conditions in the corollary are also equivalent to

(iii) \( X \) is of type 2 if \( p = 2 \) and of weak type \( p \) if \( 1 < p < 2 \), respectively.

For the proof of Theorem 2 we need a number of auxiliary results. The following lemma is a slight modification of Prop. 9.11 of [7], with essentially the same proof, which we include for the sake of completeness.

**Lemma 1.** Let \( 1 \leq p \leq 2 \). Then there is a constant \( c > 0 \) such that for each Banach space \( X \), each \( n \in \mathbb{N} \) and each sequence of independent, essentially bounded, mean zero \( X \)-valued random variables \((\eta_i)_{i=1}^n\) on some probability space \((\Omega, \Sigma, \mathbb{P})\) the following holds:

\[
\left( E \left\| \sum_{i=1}^n \eta_i \right\|_1^p \right)^{1/p} \leq c\sigma_{p,n}(X)n^{1/p} \max_{1 \leq i \leq n} \|\eta_i\|_{L_\infty(\Omega,\mathbb{P},X)}.
\]

**Proof.** Let \((\varepsilon_i)_{i=1}^n\) be independent, symmetric Bernoulli random variables on some probability space \((\Omega', \Sigma', \mathbb{P}')\) different from \((\Omega, \Sigma, \mathbb{P})\). Considering \((\eta_i)_{i=1}^n\) and
$(\varepsilon_i)_{i=1}^n$ as random variables on the product probability space, we denote the expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$ (and the expectation with respect to $\mathbb{P}$, as before, by $\mathbb{E}$). Using Lemma 6.3 of [7] and (2), we get

$$
\left( E \left\| \sum_{i=1}^n \varepsilon_i \right\|_p \right)^{1/p} \leq 2 \left( E \mathbb{E}' \left\| \sum_{i=1}^n \varepsilon_i \eta_i \right\|_p \right)^{1/p} \leq 2 \sigma_{p,n}(X)n^{1/p} \left( E \max_{1 \leq i \leq n} \| \eta_i \| \right)^{1/p} \leq 2 \sigma_{p,n}(X)n^{1/p} \max_{1 \leq i \leq n} \| \eta_i \|_{L^\infty(\Omega, \mathbb{P}, X)}.
$$

Next we introduce an algorithm for the approximation of $S^X f$. Let $n \in \mathbb{N}$ and let $\xi_i : \Omega \rightarrow Q$ ($i = 1, \ldots, n$) be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ uniformly distributed on $Q$. Define for $f \in C(Q, X)$

$$
A_{n,\omega}^{0,X} f = \frac{1}{n} \sum_{i=1}^n f(\xi_i(\omega)) \quad (5)
$$

and, if $r \geq 1$, put $k = \lceil n^{1/d} \rceil$ and

$$
A_{n,\omega}^{r,X} f = S^X(P_k^r f) + A_{n,\omega}^{0,X}(f - P_k^r f). \quad (6)
$$

These are the Banach space valued versions of the standard Monte Carlo method ($r = 0$) and the Monte Carlo method with separation of the main part ($r \geq 1$).

The following extends the second part of Proposition 1 of [2].

**Proposition 1.** Let $r \in \mathbb{N}_0$ and $1 \leq p \leq 2$. Then there is a constant $c > 0$ such that for all Banach spaces $X$, $n \in \mathbb{N}$, and $f \in C^r(Q, X)$

$$
\left( E \left\| S^X f - A_{n,\omega}^{r,X} f \right\|_p \right)^{1/p} \leq c \sigma_{p,n}(X)n^{-r/d-1+1/p} \| f \|_{C^r(Q, X)}. \quad (7)
$$

**Proof.** Let us first consider the case $r = 0$. Let $f \in C(Q, X)$ and put

$$
\eta_i(\omega) = \int_Q f(t) dt - f(\xi_i(\omega)).
$$

Clearly, $E \eta_i(\omega) = 0$,

$$
S^X f - A_{n,\omega}^{0,X} f = \frac{1}{n} \sum_{i=1}^n \eta_i(\omega)
$$

and

$$
\| \eta_i(\omega) \| \leq 2 \| f \|_{C(Q, X)}.
$$

An application of Lemma [1] gives (7). If $r \geq 1$, we have

$$
S^X f - A_{n,\omega}^{r,X} f = S^X(f - P_k^r f) - A_{n,\omega}^{0,X}(f - P_k^r f)
$$

and the result follows from (3) and the case $r = 0$. \qed
Lemma 2. Let $1 \leq p \leq 2$. Then there are constants $c > 0$ and $0 < \gamma < 1$ such that for each Banach space $X$, each $n \in \mathbb{N}$, and $(x_i)_{i=1}^n \subset X$ there is a subset $I \subseteq \{1, \ldots, n\}$ with $|I| \geq \gamma n$ and

$$E \left\| \sum_{i \in I} \varepsilon_i x_i \right\| \leq cn^{1/p} \|x\|_{e_\infty(X)} \max_{1 \leq k \leq n} k^{r/d+1-1/p} e_k^{\text{ran}}(S^X, B_{C^r(Q,X)}).$$

Proof. Since for $n \in \mathbb{N}$

$$\max_{1 \leq k \leq n} k^{r/d+1-1/p} e_k^{\text{ran}}(S^X, B_{C^r(Q,X)}) \geq e_1^{\text{ran}}(S^X, B_{C^r(Q,X)}),$$

the statement is trivial for $n < 8^d$. Therefore we can assume $n \geq 8^d$. Clearly, we can also assume $\|(x_i)\|_{e_\infty(X)} > 0$. Let $m \in \mathbb{N}$ be such that

$$m^d \leq n < (m+1)^d,$$

hence

$$m \geq 8. \quad (8)$$

Let $\psi$ be an infinitely differentiable function on $\mathbb{R}^d$ such that $\psi(t) > 0$ for $t \in (0,1)^d$ and supp $\psi \subset [0,1]^d$. Let $(Q_i)_{i=1}^{m^d}$ be the partition of $Q$ into closed cubes of side length $m^{-1}$ of disjoint interior, let $t_i$ be the point in $Q_i$ with minimal coordinates and define $\psi_i \in C(Q)$ by

$$\psi_i(t) = \psi(m(t-t_i)) \quad (i = 1, \ldots, m^d).$$

It is easily verified that there is a constant $c_0 > 0$ such that for all $(\alpha_i)_{i=1}^{m^d} \in [-1,1]^{m^d}$

$$\left\| \sum_{i=1}^{m^d} \alpha_i x_i \psi_i \right\|_{C^r(Q,X)} \leq c_0 m^r \|x_i\|_{e_\infty(X)}.$$

Setting

$$f_i = c_0^{-1} m^{-r} \|x_i\|_{e_\infty(X)}^{-1} x_i \psi_i$$

it follows that

$$\sum_{i=1}^{m^d} \alpha_i f_i \in B_{C^r(Q,X)} \quad \text{for all} \quad (\alpha_i)_{i=1}^{m^d} \in [-1,1]^{m^d}.$$

Moreover, with $\sigma = \int_Q \psi(t)dt$ we have

$$\left\| \sum_{i=1}^{m^d} \alpha_i S^X f_i \right\| = c_0^{-1} m^{-r} \|x_i\|_{e_\infty(X)}^{-1} \left\| \sum_{i=1}^{m^d} \alpha_i x_i \int_Q \psi_i(t) dt \right\| \leq c_0^{-1} \sigma m^{-r-d} \|x_i\|_{e_\infty(X)}^{-1} \left\| \sum_{i=1}^{m^d} \alpha_i x_i \right\|. \quad (9)$$
Next we use Lemma 5 and 6 of [3] with $K = X$ (although stated for $K = \mathbb{R}$, Lemma 6 is easily seen to hold for $K = X$, as well) to obtain for all $l \in \mathbb{N}$ with $l < m^d/4$

$$e^\text{ran}_l(S^X, B_{C^r(Q,X)}) \geq \frac{1}{4} \min_{I \subseteq \{1, \ldots, m^d\}, |I| \geq m^d - 4l} \mathbb{E} \left\| \sum_{i \in I} \varepsilon_i S^X f_i \right\|_\infty^1 \geq cm^{-r-d} \|(x_i)\|_{\ell^\infty_\infty(X)} \mathbb{E} \left\| \sum_{i \in I} \varepsilon_i x_i \right\|_\infty.
$$

We put $l = \lfloor m^d/8 \rfloor$. Then

$$m^d/16 < l \leq m^d/8. \tag{10}$$

Indeed, by (9) the left-hand inequality clearly holds for $m^d < 16$, while for $m^d \geq 16$ we get $\lfloor m^d/8 \rfloor > m^d/8 - 1 \geq m^d/16$. We conclude that there is an $I \subseteq \{1, \ldots, m^d\}$ with $|I| \geq m^d - 4l \geq m^d/2$ and

$$\mathbb{E} \left\| \sum_{i \in I} \varepsilon_i x_i \right\| \leq cm^{r+d} \|(x_i)\|_{\ell^\infty_\infty(X)} e^\text{ran}_l(S^X, B_{C^r(Q,X)}) \leq cm^{r+d-1} \|(x_i)\|_{\ell^\infty_\infty(X)} \max_{1 \leq k \leq n} k^{r/d+1-p} e^\text{ran}_k(S^X, B_{C^r(Q,X)}) \leq cn^{1/p} \|(x_i)\|_{\ell^\infty_\infty(X)} \max_{1 \leq k \leq n} k^{r/d+1-p} e^\text{ran}_k(S^X, B_{C^r(Q,X)}),$$

where we used (8) and (10). Finally, (8) and (9) give

$$|I| \geq m^d/2 \geq \frac{m^d}{2(m+1)^d} n \geq \frac{8^d}{2 \cdot 9^d} n.$$ 

\[\square\]

**Proof of Theorem 2.** The left-hand inequality of (4) follows directly from Proposition 1 since the number of function values involved in $A^r_{n,\omega}$ is bounded by $ck^d + n \leq cn$, see also (16).

To prove the right-hand inequality of (4), let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. We construct by induction a partition of $K = \{1, \ldots, n\}$ into a sequence of disjoint subsets $(I_l)_{l=1}^{l^*}$ such that for $1 \leq l \leq l^*$

$$|I_l| \geq \gamma \left| K \setminus \bigcup_{j < l} I_j \right| \tag{11}$$

and

$$\mathbb{E} \left\| \sum_{i \in I_l} \varepsilon_i x_i \right\| \leq c \left| K \setminus \bigcup_{j < l} I_j \right|^{1/p} \|(x_i)\|_{\ell^\infty_\infty(X)} \max_{1 \leq k \leq n} k^{r/d+1-p} e^\text{ran}_k(S^X, B_{C^r(Q,X)}), \tag{12}$$

7
where $c$ and $\gamma$ are the constants from Lemma 2. For $l = 1$ the existence of an $I_1$ satisfying (11–12) follows directly from Lemma 2. Now assume that we already have a sequence of disjoint subsets $(I_i)_{i=1}^m$ of $K$ satisfying (11–12). If

$$J := K \setminus \bigcup_{j \leq m} I_j \neq \emptyset,$$

we apply Lemma 2 to $(x_i)_{i \in J}$ to find $I_{m+1} \subseteq J$ with

$$|I_{m+1}| \geq \gamma|J|$$

and

$$\mathbb{E} \left\| \sum_{i \in I_{m+1}} \varepsilon_i x_i \right\| \leq c|J|^{1/p} \| (x_i)_{i \in J} \|_{\ell_\infty(J,X)} \max_{1 \leq k \leq |J|} k^{r/d+1-1/p} e_k^{\text{ran}}(S^X, B_{C^r(Q,X)}).$$

(14)

Observe that for $l = m+1$, (13) is just (11) and (14) implies (12). Furthermore, (11) implies

$$\left| K \setminus \bigcup_{j \leq l} I_j \right| \leq (1 - \gamma) \left| K \setminus \bigcup_{j \leq l-1} I_j \right|$$

and therefore

$$\left| K \setminus \bigcup_{j \leq l} I_j \right| \leq (1 - \gamma)^l |J|.$$

(15)

It follows that the process stops with $K = \bigcup_{j \leq l} I_j$ for a certain $l = l^* \in \mathbb{N}$. This completes the construction.

Using the equivalence of moments (Theorem 4.7 of [7]), we get from (12) and (15)

$$\left( \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \right)^{1/p} \leq c \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \leq c \sum_{l=1}^{l^*} \mathbb{E} \left\| \sum_{i \in I_l} \varepsilon_i x_i \right\|$$

$$\leq cn^{1/p} \| (x_i) \|_{\ell_\infty(X)} \max_{1 \leq k \leq n} k^{r/d+1-1/p} e_k^{\text{ran}}(S^X, B_{C^r(Q,X)}) \sum_{l=1}^{l^*} (1 - \gamma)^{(l-1)/p}. $$

This gives the upper bound of (4).

Let us mention that results analogous to Theorem 2 and Corollary 2 above also hold for Banach space valued indefinite integration (see [2] for the definition) and for the solution of initial value problems for Banach space valued ordinary
differential equations [5]. Indeed, an inspection of the respective proofs together with Lemma 1 of the present paper shows that Proposition 2 of [2] also holds with $\tau_p(X)$ replaced by $\sigma_{p,n}(X)$, and similarly Proposition 3.4 of [5]. Moreover, in both papers the lower bounds on $e_{\text{ran}}$ are obtained by reduction to (definite) integration and thus the right-hand side inequality of (4) carries over directly.

References

[1] J. Bourgain, N. J. Kalton, L. Tzafriri, Geometry of finite dimensional subspaces and quotients of $L_p$, GAFA 1987/88, Lecture Notes in Mathematics 1376, Springer, 1989, 138–175.

[2] Th. Daun, S. Heinrich, Complexity of Banach space valued and parametric integration, to appear in the Proceedings of Monte Carlo and Quasi-Monte Carlo Methods 2012.

[3] S. Heinrich, Monte Carlo approximation of weakly singular integral operators, J. Complexity 22 (2006), 192–219.

[4] S. Heinrich, The randomized information complexity of elliptic PDE, J. Complexity 22 (2006), 220–249.

[5] S. Heinrich, Complexity of initial value problems in Banach spaces, J. Math. Phys. Anal. Geom. 9 (2013), 73–101.

[6] R. C. James, Nonreflexive spaces of type 2, Israel J. Math. 30 (1978), 1–13.

[7] M. Ledoux, M. Talagrand, Probability in Banach Spaces, Springer, 1991.

[8] V. Mascioni, On weak cotype and weak type in Banach spaces. Note di Mat. (Lecce) 8 (1988), 67–110.

[9] B. Maurey, G. Pisier, Series de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, Stud. Math. 58, 45-90 (1976).

[10] E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics 1349, Springer, 1988.

[11] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume 2, Standard Information for Functionals, European Math. Soc., Zürich, 2010.

[12] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.

[13] L. Tzafriri, On the type and cotype of Banach spaces, Israel J. Math. 32 (1979), 32–38.
4 Appendix

In this appendix we recall some basic notions of information-based complexity – the framework we used above. We refer to [10, 12] for more on this subject and to [3, 4] for the particular notation applied here. First we introduce the class of deterministic adaptive algorithms of varying cardinality \( A^\text{det}(C(Q, X), X) \). It consists of tuples \( A = ((L_i)_{i=1}^\infty, (g_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty) \), with \( L_1 \in Q, g_0 \in \{0, 1\}, \varphi_0 \in X \)
and
\[
L_i : X^{i-1} \to Q \quad (i = 2, 3, \ldots), \quad g_i : X^i \to \{0, 1\} \quad \varphi_i : X^i \to X \quad (i = 1, 2, \ldots)
\]
being arbitrary mappings. To each \( f \in C(Q, X) \), we associate a sequence \((t_i)_{i=1}^\infty\) with \( t_i \in Q \) as follows:
\[
t_1 = L_1, \quad t_i = L_i(f(t_1), \ldots, f(t_{i-1})) \quad (i \geq 2).
\]
Define \( \text{card}(A, f) \), the cardinality of \( A \) at input \( f \), to be 0 if \( g_0 = 1 \). If \( g_0 = 0 \), let \( \text{card}(A, f) \) be the first integer \( n \geq 1 \) with \( g_n(f(t_1), \ldots, f(t_n)) = 1 \), if there is such an \( n \), and \( \text{card}(A, f) = +\infty \) otherwise. For \( f \in C(Q, X) \) with \( \text{card}(A, f) < \infty \) we define the output \( Af \) of algorithm \( A \) at input \( f \) as
\[
Af = \begin{cases}
\varphi_0 & \text{if } n = 0 \\
\varphi_n(f(t_1), \ldots, f(t_n)) & \text{if } n \geq 1.
\end{cases}
\]
Let \( r \in \mathbb{N}_0 \). Given \( n \in \mathbb{N}_0 \), we let \( A^\text{det}_n(B_{C^r(Q, X)}, X) \) be the set of those \( A \in A^\text{det}(C(Q, X), X) \) for which
\[
\max_{f \in B_{C^r(Q, X)}} \text{card}(A, f) \leq n.
\]
The error of \( A \in A^\text{det}_n(B_{C^r(Q, X)}, X) \) as an approximation of \( S^X \) is defined as
\[
e(S^X, A, B_{C^r(Q, X)}) = \sup_{f \in B_{C^r(Q, X)}} \|S^X f - Af\|.
\]
The deterministic \( n \)-th minimal error of \( S^X \) is defined for \( n \in \mathbb{N}_0 \) as
\[
e^\text{det}_n(S^X, B_{C^r(Q, X)}) = \inf_{A \in A^\text{det}_n(B_{C^r(Q, X)})} e(S^X, A, B_{C^r(Q, X)}).
\]
It follows that no deterministic algorithm that uses at most \( n \) function values can have a smaller error than \( e^\text{det}_n(S^X, B_{C^r(Q, X)}) \).

Next we introduce the class of randomized adaptive algorithms of varying cardinality \( A^\text{ran}_n(B_{C^r(Q, X)}, X) \), consisting of tuples \( A = ((\Omega, \Sigma, \mathbb{P}), (A_\omega)_{\omega \in \Omega}) \), where \((\Omega, \Sigma, \mathbb{P})\) is a probability space, \( A_\omega \in A^\text{det}(C(Q, X), X) \) for all \( \omega \in \Omega \), and for each \( f \in B_{C^r(Q, X)} \) the mapping \( \omega \in \Omega \to \text{card}(A_\omega, f) \) is \( \Sigma \)-measurable and satisfies \( \mathbb{E} \text{card}(A_\omega, f) \leq n \). Moreover, the mapping \( \omega \in \Omega \to A_\omega f \in X \) is \( \Sigma \)-to-Borel measurable and essentially separably valued, i.e., there is a separable
subspace \( X_0 \subseteq X \) such that \( A_\omega f \in X_0 \) for \( \mathbb{P} \)-almost all \( \omega \in \Omega \). The error of \( A \in \mathcal{A}_n(C(Q,X),X) \) in approximating \( S^X \) on \( B_{C^r}(Q,X) \) is defined as

\[
e(S^X,A,B_{C^r}(Q,X)) = \sup_{f \in B_{C^r}(Q,X)} \mathbb{E} \| S^X f - A_\omega f \|,
\]

and the randomized \( n \)-th minimal error of \( S^X \) as

\[
e_n^{\text{ran}}(S,B_{C^r}(Q,X)) = \inf_{A \in \mathcal{A}^{\text{ran}}(B_{C^r}(Q,X))} e(S^X,A,B_{C^r}(Q,X)).
\]

Consequently, no randomized algorithm that uses (on the average) at most \( n \) function values has an error smaller than \( e_n^{\text{ran}}(S,B_{C^r}(Q,X),X) \).

Define for \( \varepsilon > 0 \) the information complexity as

\[
n_\varepsilon^{\text{ran}}(S,B_{C^r}(Q,X)) = \min \{ n \in \mathbb{N}_0 : e_n^{\text{ran}}(S,B_{C^r}(Q,X)) \leq \varepsilon \};
\]

if there is such an \( n \), and \( n_\varepsilon^{\text{ran}}(S,B_{C^r}(Q,X)) = +\infty \), if there is no such \( n \). Thus, if \( n_\varepsilon^{\text{ran}}(S,B_{C^r}(Q,X)) < +\infty \), it follows that any algorithm with error \( \leq \varepsilon \) needs at least \( n_\varepsilon^{\text{ran}}(S,B_{C^r}(Q,X)) \) function values, while \( n_\varepsilon^{\text{ran}}(S,B_{C^r}(Q,X)) = +\infty \) means that no algorithm at all has error \( \leq \varepsilon \). The information complexity is essentially the inverse function of the \( n \)-th minimal error. So determining the latter means determining the information complexity of the problem.

Let us also mention the subclasses consisting of quadrature formulas. Let \( n \geq 1 \). A mapping \( A : C(Q,X) \rightarrow X \) is called a deterministic quadrature formula with \( n \) nodes, if there are \( t_i \in Q \) and \( a_i \in \mathbb{K} \) \( (1 \leq i \leq n) \) such that

\[
Af = \sum_{i=1}^{n} a_i f(t_i) \quad (f \in C(Q,X)).
\]

In terms of the definition of \( \mathcal{A}^{\text{det}}(C(Q,X),X) \) this means that the respective functions \( L_i \) and \( g_i \) are constant, \( g_0 = g_1 = \cdots = g_{n-1} = 0 \), \( g_n = 1 \), and \( \varphi_n \) has the form \( \varphi_n(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i x_i \). Clearly, \( A \in \mathcal{A}^{\text{det}}_n(B_{C^r}(Q,X),X) \).

A tupel \( A = ((\Omega,\Sigma,\mathbb{P}), (A_\omega)_{\omega \in \Omega}) \) is called a randomized quadrature with \( n \) nodes if there exist random variables \( t_i : \Omega \rightarrow Q \) and \( a_i : \Omega \rightarrow \mathbb{K} \) \( (1 \leq i \leq n) \) with

\[
A_\omega f = \sum_{i=1}^{n} a_i(\omega) f(t_i(\omega)) \quad (f \in C(Q,X), \omega \in \Omega).
\]

For each such \( A \) we have \( A \in \mathcal{A}^{\text{ran}}_n(B_{C^r}(Q,X),X) \). Finally we note that the algorithms \( A^{\text{ran}}_{n,\omega} \) defined in \( \text{(3)} \) and \( \text{(4)} \) are quadratures. Indeed, for \( A^{\text{ran}}_{0,\omega} \) given by \( \text{(3)} \) this is obvious. For \( r \geq 1 \) we represent \( P^{r,X}_k \in \mathcal{L}(C(Q,X)) \) as

\[
P^{r,X}_k f = \sum_{j=1}^{M} f(u_j) \psi_j(t)
\]
with $M \leq ck^d$, $u_j \in Q$, $\psi_j \in C(Q)$ ($1 \leq i \leq M$), and obtain, setting $b_j = \int_Q \psi_j(t) dt$,

$$A_{n,\omega}^{r,X} f = S^X(P_k^{r,X} f) + A_{n,\omega}^{0,X}(f - P_k^{r,X} f)$$

$$= \sum_{j=1}^{M} b_j f(u_j) + \frac{1}{n} \sum_{i=1}^{n} \left( f(\xi_i(\omega)) - \left( P_k^{r,X} f \right)(\xi_i(\omega)) \right)$$

$$= \sum_{j=1}^{M} b_j f(u_j) + \frac{1}{n} \sum_{i=1}^{n} f(\xi_i(\omega)) - \sum_{j=1}^{M} \left( \frac{1}{n} \sum_{i=1}^{n} \psi_j(\xi_i(\omega)) \right) f(u_j). \quad (16)$$