Constraint Learning for Control Tasks with Limited Duration Barrier Functions

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Abstract—When deploying autonomous agents in unstructured environments over sustained periods of time, adaptability and robustness often times outweigh optimality as a primary consideration. In other words, safety and survivability constraints play a key role and in this paper, we present a novel, constraint-learning framework for control tasks built on the idea of constraints-driven control. However, since control policies that keep a dynamical agent within state constraints over infinite horizons are not always available, this work instead considers constraints that can be satisfied over a sufficiently long time horizon \( T > 0 \), which we refer to as limited-duration safety. Consequently, value function learning can be used as a tool to help us find limited-duration safe policies. We show that, in some applications, the existence of limited-duration safe policies is actually sufficient for long-duration autonomy. This idea is illustrated on a swarm of simulated robots that are tasked with covering a given area, but that sporadically need to abandon this task to charge batteries. We show how the battery-charging behavior naturally emerges as a result of the constraints. Additionally, using a cart-pole simulation environment, we show how a control policy can be efficiently transferred from the source task, balancing the pole, to the target task, moving the cart to one direction without letting the pole fall down.

Index Terms—Constraint learning, Barrier function, Value function, Model-based control, Transfer learning

I. INTRODUCTION

Acquiring an optimal policy that attains the maximum return over some time horizon is of primary interest in the literature of both reinforcement learning [1–3] and optimal control [4]. A large number of algorithms have been designed to successfully control systems with complex dynamics to accomplish specific tasks, such as balancing an inverted pendulum and letting a humanoid robot run to a target location. Those algorithms may result in control strategies that are energy-efficient, take the shortest path to the goal, spend less time to accomplish the task, and sometimes outperform human beings in these senses (cf. [5]). As we can observe in the daily life, on the other hand, it is often difficult to attribute optimality to human behaviors, e.g., the behaviors are hardly the most efficient for any specific task (cf. [6]). Instead, humans are capable of generalizing the behaviors acquired through completing a certain task to deal with unseen situations. This fact casts a question of how one should design a learning algorithm that generalizes across tasks rather than focuses on a specific one.

In this paper, we hypothesize that this can be achieved by letting the agents acquire a set of good enough policies when completing one task instead of finding a single optimal policy, and reuse this set for another task. Specifically, we consider safety, which refers to avoiding certain states, as useful information shared among different tasks, and we regard limited-duration safe policies as good enough policies. Our work is built on the idea of constraints-driven control [7, 8], a methodology for controlling agents by telling them to satisfy constraints without specifying a single optimal path. If feasibility of the assigned constraints is guaranteed, this methodology avoids recomputing an optimal path when a new task is given but instead enables high-level compositions of constraints.

However, state constraints are not always feasible and arbitrary compositions of constraints cannot be validated in general [9]. We tackle this feasibility issue by relaxing safety (i.e., forward invariance [10] of the set of safe states) to limited-duration safety, by which we mean satisfaction of safety over some finite time horizon \( T > 0 \) (see Figure 1.1).

For an agent starting from a certain subset of the safe region, one can always find a set of policies that render this agent safe up to some finite time. To guarantee limited-duration safety, we propose a limited duration control barrier function (LDCBF). The idea is based on local model-based control that constrains the instantaneous control input every time to restrict the growths of values of LDCBFs by solving a computationally inexpensive quadratic programming (QP) problem.

To find an LDCBF, we make use of so-called global value
function learning. More specifically, we assign a high cost to unsafe states and a lower cost to safe states, and learn the value function (or discounted infinite-horizon cost) associated with any given policy. Then, it is shown that the value function associated with any given policy is an LDCBF, i.e., a nonempty set of limited-duration safe policies can be obtained (Section IV-B). Contrary to the optimal control and Lyapunov-based approaches that only single out an optimal policy, our learning framework aims at learning a common set of policies that can be shared among different tasks. Thus, our framework can be contextualized within the so-called lifelong learning [11] and transfer learning [12] (or safe transfer; see Section V-B).

The rest of this paper is organized as follows: Section II discusses the related work, including constraints-driven control and transfer learning. Section III presents notations, assumptions made in this paper, and some background knowledge. Subsequently, we present our main contributions and their applications in Section IV and Section V respectively. In Section VI, we first validate LDCBFs on an existing control problem (see Section II). Then, our learning framework is applied to the cart-pole simulation environment in DeepMind Control Suite [13]; safety is defined as keeping the pole from falling down, and we use an LDCBF obtained after learning the balance task to facilitate learning the new task, namely, moving the cart without letting the pole fall down.

II. RELATED WORK

Finding feasible control constraints that can be translated to a set of state constraints has been of particular interest both in the controls and machine learning communities. Early work includes the study of artificial potential functions in the context of obstacle avoidance, and the construction of so-called navigation functions was studied in [14]. Alternatively, if there exists a control Lyapunov function [15], one can stabilize the agent while keeping it inside a level set of the function. Control Lyapunov functions can be learned through demonstrations [16], for example, and Lyapunov stability was also used in the safe reinforcement learning (see [17–19] for example). As inverse optimality [20] dictates that finding a stabilizing policy is equivalent to finding an optimal policy in terms of some cost function, these approaches can also be viewed as optimization-based techniques.

On the other hand, control barrier functions (CBFs) [21–30] were proposed to guarantee that an agent remains in a certain region of the state space (i.e., forward invariance [10]) by using a locally accurate model of the agent dynamics (i.e., a model that accurately predicts a time derivative of the state at the current state and control input). When the system is linearizable and has a high relative degree, an exponential control barrier function [31] was proposed and was applied to control of quadrotors [32]. When a Lyapunov function is available, the work [33] proposed a sum-of-squares approach to compute a valid barrier function. The idea of constraints-driven controls is in stark contrast to solving the task-specific problem that basically aims at singling out one optimal trajectory. However, although there exist converse theorems for safety and barrier functions which claim that a forward invariant set has a barrier function under certain conditions [25, 34, 35], finding such a set without assuming stability of the system is difficult in general (see [21] for the conditions that a candidate barrier function can be a valid one).

Moreover, our work is also related to safe reinforcement learning, such as Lyapunov-based safe learning (cf. [17, 36]) and constrained Markov decision processes (CMDPs) (cf. [18, 37]). The former is based on the fact that sublevel sets of a control Lyapunov function are forward invariant, and considers stability as safety. The latter is aimed at selecting an optimal policy that satisfies constraints. Note these approaches are designed for one specific task. Our work, on the other hand, does not require stability, and can consider an arbitrarily shaped set of safe states.

Besides, transfer learning (cf. [12]) aims at learning a new task by utilizing the knowledge already acquired via learning other tasks, and is sometimes referred to as “lifelong learning” [11] and “learning to learn” [38]. In transfer learning for reinforcement learning contexts, we first learn a set of source tasks, and speed up learning of a target task (see [39] for example). When the source tasks and the target task have hierarchical structures, it is often called hierarchical reinforcement learning (e.g., [40–42]). Other examples include meta-learning (e.g., [43]) that considers so-called the task distribution. Our work can also be used as a transfer learning technique that uses a set of good enough policies as useful information shared among other tasks.

In the next section, we present some assumptions together with the notations used in the paper.

III. PRELIMINARIES

Throughout, \( \mathbb{R} \), \( \mathbb{R}_{\geq 0} \) and \( \mathbb{Z}_{+} \) are the sets of real numbers, nonnegative real numbers and positive integers, respectively. Let \( \| \cdot \|_{\mathbb{R}^d} := \sqrt{x^T x} \) be the norm induced by the inner product \( (x,y)_{\mathbb{R}^d} := x^T y \) for \( d \)-dimensional real vectors \( x,y \in \mathbb{R}^d \), where \( (\cdot)^T \) stands for transposition. In this paper, we consider an agent with system dynamics described by an ordinary differential equation:

\[
\frac{dx}{dt} = h(x(t), u(t)),
\]

where \( x(t) \in \mathbb{R}^{n_x} \) and \( u(t) \in U \subset \mathbb{R}^{n_u} \) are the state and the instantaneous control input of dimensions \( n_x, n_u \in \mathbb{Z}_{+} \), and \( h : \mathbb{R}^{n_x} \times U \rightarrow \mathbb{R}^{n_x} \). Let \( \mathcal{X} \) be the state space which is a compact subset of \( \mathbb{R}^{n_x} \).

In this work, we make the following assumptions.

**Assumption III.1.** For any locally Lipschitz continuous policy \( \phi : \mathcal{X} \rightarrow U \), \( h \) is locally Lipschitz with respect to \( x \).

**Assumption III.2.** The control space \( U(\subset \mathbb{R}^{n_u}) \) is a polyhedron.

\(^1\)We refer the readers to Appendix E for the case of system dynamics described by a stochastic differential equation.
Given a policy $\phi : \mathcal{X} \to \mathcal{U}$ and a discount factor $\beta > 0$, define the value function associated with the policy $\phi$ by

$$V^{\phi, \beta}(x) := \int_0^\infty e^{-\beta t} \ell(x(t)) dt, \quad \beta > 0,$$

where $\ell(x(t))$ is the immediate cost and $x(t)$ is the trajectory starting from $x(0) = x$. When $V^{\phi, \beta}$ is continuously differentiable over $\text{int}(\mathcal{X})$, namely, the interior of $\mathcal{X}$, we obtain the Hamilton-Jacobi-Bellman (HJB) equation [44]:

$$\beta V^{\phi, \beta}(x) = \frac{\partial V^{\phi, \beta}(x)}{\partial x} h(x, \phi(x)) + \ell(x), \quad x \in \text{int}(\mathcal{X}).$$

Now, if the immediate cost $\ell(x)$ is positive for all $x \in \mathcal{X}$ except for the equilibrium, and that a zero-cost state is globally asymptotically stable with the given policy $\phi : \mathcal{X} \to \mathcal{U}$, one can expect that the value function $V^{\phi, 0}(x) := \int_0^\infty \ell(x(t)) dt$ has finite values over $\mathcal{X}$. In this case, the HJB equation becomes $V^{\phi, 0}(x(t)) := \frac{\partial V^{\phi, 0}(x)}{\partial x} = -\ell(x(t))$, i.e., $V^{\phi, 0}$ is decreasing over time. As such, it is straightforward to see that $V^{\phi, 0}$ is a control Lyapunov function, i.e., there always exists a policy that satisfies the decrease condition for $V^{\phi, 0}$.

However, there are two major limitations to this approach: i) one must assume that the agent stabilizes in a zero-cost state by the given policy $\phi$, and ii) forward invariant sublevel sets of the control Lyapunov function usually become too conservative with respect to the given safe states. To remedy these drawbacks, we present our major contribution in the next section.

IV. Constraint Learning for Control Tasks

In this section, we propose limited duration control barrier functions (LDCBFs), and present their properties and a practical way to find an LDCBF.

A. Limited Duration Control Barrier Functions

Before formally presenting LDCBFs, we give the following definition.

Definition IV.1 (Limited-duration safety). Given an open set of safe states $\mathcal{C}$, let $B_{LD}^T$ be a closed nonempty subset of $\mathcal{C} \subset \mathcal{X}$. The dynamical system (1) is said to be safe up to time $T$, if there exists a policy $\phi$ that ensures $x(t) \in \mathcal{C}$ for all $0 \leq t < T$ whenever $x(0) \in B_{LD}^T$.

Now, we give the definition of the LDCBFs.

Definition IV.2. Let a function $B_{LD} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuously differentiable. Suppose that $h(x, u) = f(x) + g(x)u$, $x \in \mathcal{X}$, $u \in \mathcal{U}$, and that the set of safe states is given by

$$\mathcal{C} := \left\{ x \in \mathcal{X} : B_{LD}(x) < \frac{L}{\beta} \right\}, \quad L > 0, \beta > 0.$$  

(3)

Define the set

$$B_{LD}^T(t) := \left\{ x \in \mathcal{X} : B_{LD}(x) \leq \frac{Le^{-\beta T}}{\beta} \right\} \subset \mathcal{C},$$

for some $T > 0$. Define also $L_f$ and $L_g$ as the Lie derivatives along $f$ and $g$. Then, $B_{LD}$ is called a limited duration control barrier function for $\mathcal{C}$ and for the time horizon $T$ if $B_{LD}^T$ is nonempty and if there exists a monotonically increasing locally Lipschitz continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and

$$\inf_{u \in \mathcal{U}} \{ L_f B_{LD}(x) + L_g B_{LD}(x) u \} \leq \alpha \left( \frac{L e^{-\beta T}}{\beta} - B_{LD}(x) \right) + \beta B_{LD}(x),$$

(4)

for all $x \in \mathcal{C}$. Given an LDCBF, the admissible control space $S_{LD}^T(x), x \in \mathcal{C}$, can be defined as

$$S_{LD}^T(x) := \{ u \in \mathcal{U} : L_f B_{LD}(x) + L_g B_{LD}(x) u \leq \alpha \left( \frac{L e^{-\beta T}}{\beta} - B_{LD}(x) \right) + \beta B_{LD}(x) \}.$$  

(5)

Given an LDCBF, safety up to time $T$ is guaranteed if the initial state is taken in $B_{LD}^T$ and an admissible control is employed as the following theorem claims.

Theorem IV.1. Given a set of safe states $\mathcal{C}$ defined by (3) and an LDCBF $B_{LD}$ defined on $\mathcal{X}$ under Assumption III.1, any locally Lipschitz continuous policy $\phi : \mathcal{X} \to \mathcal{U}$ that satisfies $\phi(x) \in S_{LD}^T(x), \forall x \in \mathcal{C}$, renders the dynamical system (1) safe up to time $T$ whenever the initial state is in $B_{LD}^T$.

Proof. See Appendix A. \hfill $\Box$

When $h(x, u) = f(x) + g(x)u$, $x \in \mathcal{X}$, $u \in \mathcal{U}$, one can constrain the control input within the admissible control space $S_{LD}^T(x), x \in \mathcal{C}$, using a locally accurate model via QPs in the same manner as control barrier functions and control Lyapunov functions. Here, we present a general form of control syntheses via QPs.

Proposition IV.1. Suppose that $h(x, u) = f(x) + g(x)u$, $x \in \mathcal{X}$, $u \in \mathcal{U}$. Given an LDCBF $B_{LD}$ with a locally Lipschitz derivative and the admissible control space $S_{LD}^T(x^*)$ at $x^* \in \mathcal{C}$ defined in (5), consider the QP:

$$\phi(x^*) = \arg\min_{u \in S_{LD}^T(x)} u^T H(x^*) u + 2b(x^*)^T u,$$

(6)

where $H$ and $b$ are Lipschitz continuous at $x^* \in \mathcal{C}$, and $H(x^*) = H^T(x^*)$ is positive definite. If the width of a feasible set is strictly larger than zero, then, under Assumption II.1, the policy $\phi(x)$ defined in (6) is unique and Lipschitz continuous with respect to the state at $x^*$.

Proof. Slight modifications of [45, Theorem 1] proves the proposition. \hfill $\Box$

To see an advantage of considering LDCBFs, we show that an LDCBF can be obtained systematically as described next.

Note $\alpha$ is not necessarily a class-K function [10].

2See Appendix B for the definition.
B. Finding a Limited Duration Control Barrier Function

We present a possible way to find an LDCBF $B_{LD}$ for the set of safe states through global value function learning. Let $\ell(x) \geq 0, \forall x \in \mathcal{X}$, and suppose that the set of safe states is given by

$$\mathcal{C} := \{x \in \mathcal{X} : \ell(x) < L\}, \text{ } L > 0. \tag{7}\label{eq7}$$

Given the dynamical system defined in Definition IV.2, consider the virtual system

$$\hat{x}(t) = f(x(t)) + g(x(t))\phi(x(t)) \text{ if } x(t) \in \mathcal{C} \text{ otherwise } 0, \tag{8}\label{eq8}$$

for a policy $\phi$. Assume that we employ a continuously differentiable function approximator to approximate the value function $V^{\phi,\beta}$ for the virtual system, and let $\hat{V}^{\phi,\beta}$ denote an approximation of $V^{\phi,\beta}$. By using the HJB equation \cite{49}, define the estimated immediate cost function $\hat{\ell}$ as

$$\hat{\ell}(x) = \beta \hat{V}^{\phi,\beta}(x) - L_f \hat{V}^{\phi,\beta}(x) - L_g \hat{V}^{\phi,\beta}(x) \phi(x), \forall x \in \mathcal{C}. \tag{9}\label{eq9}$$

Select $c \geq 0$ so that $\hat{\ell}(x) := \hat{\ell}(x) + c \geq 0$ for all $x \in \mathcal{C}$, and define the function $V^{\phi,\beta}(x) := \hat{V}^{\phi,\beta}(x) + \frac{c}{\beta}$. Then, the following theorem holds.

**Theorem IV.2.** Consider the set

$$\mathcal{B}^T_{LD} = \left\{ x \in \mathcal{X} : \hat{V}^{\phi,\beta}(x) \leq \frac{\hat{\ell}e^{-\beta T}}{\beta} \right\}, \tag{10}\label{eq10}$$

where $\hat{\ell} := \inf_{y \in \mathcal{X} \setminus \mathcal{C}} \beta \hat{V}^{\phi,\beta}(y)$. If $\mathcal{B}^T_{LD}$ is nonempty, then the dynamical system starting from the initial state in $\mathcal{B}^T_{LD}$ is safe up to time $T$ when the policy $\phi$ is employed, and $\hat{V}^{\phi,\beta}(x)$ is an LDCBF for the set

$$\hat{\mathcal{C}} := \left\{ x \in \mathcal{X} : \hat{V}^{\phi,\beta}(x) < \frac{\hat{\ell}}{\beta} \right\} \subset \mathcal{C}. \tag{11}\label{eq11}$$

**Proof.** See Appendix \cite{49}.

**Remark IV.1.** We need to consider $\hat{V}^{\phi,\beta}$ instead of $V^{\phi,\beta}$ because the immediate cost function $\ell$ and the virtual system need to be sufficiently smooth to guarantee that $V^{\phi,\beta}$ is continuously differentiable. In practice, the choice of $c$ and $\hat{\ell}$ affects conservativeness of the set of safe states.

Note, to enlarge the set $\mathcal{B}^T_{LD}$, the immediate cost $\ell(x)$ is preferred to be close to zero for $x \in \mathcal{C}$, and $L$ needs to be sufficiently large.

**Example IV.1.** As an example of finding an LDCBF, we use a deep neural network. Suppose the discrete-time transition is given by $(x_n, u_n, \ell(x_n), x_{n+1})$, where $n \in 0 \cup \mathbb{Z}_+$ is the time instant. Then, by executing a given policy $\phi$, we store the negative data, where $x_{n+1} \notin \mathcal{C}$, and the positive data, where $x_{n+1} \in \mathcal{C}$, separately, and conduct prioritized experience replay \cite{46, 47}. Specifically, initialize a target network $V^{\phi,\beta}_{Target}$ and a local network $V^{\phi,\beta}_{Local}$, and update the local network by sampling a random minibatch of $N$ negative and positive transitions $\{(x_n, u_n, \ell(x_n), x_{n+1})\}_{i \in \{1, 2, \ldots, N\}}$ to minimize

$$\frac{1}{N} \sum_{i=1}^{N} (y_n - V^{\phi,\beta}_{Local}(x_n)), \tag{12}\label{eq12}$$

where

$$y_n = \begin{cases} \ell(x_n) + \gamma V^{\phi,\beta}_{Target}(x_{n+1}) & x_{n+1} \in \mathcal{C}, \\ \ell(x_n) + \frac{1}{1-\gamma} & x_{n+1} \notin \mathcal{C}. \end{cases} \tag{13}\label{eq13}$$

Here, $\gamma \approx -\log (\beta)/\Delta_t$ is a discount factor for a discrete-time case, where $\Delta_t$ is the time interval of one time step.

Although we cannot ensure forward invariance of the set $\mathcal{C}$ using LDCBFs, the proposed approach is still set theoretic. As such, we can consider the compositions of LDCBFs.

C. Compositions of Limited Duration Barrier Functions

The Boolean compositional CBFs were studied in \cite{23, 48}. In \cite{23}, max and min operators were used for the Boolean operations, and nonsmooth barrier functions were proposed out of necessity. However, it is known that, even if two sets $\mathcal{C}_1 \subset \mathcal{X}$ and $\mathcal{C}_2 \subset \mathcal{X}$ are controlled invariant \cite[page 21]{49} for the dynamical system \cite{1}, the set $\mathcal{C}_1 \cap \mathcal{C}_2$ is not necessarily controlled invariant, while $\mathcal{C}_1 \cup \mathcal{C}_2$ is indeed controlled invariant \cite[Proposition 4.13]{49}.

We can make a similar assertion for limited-duration safety as follows.

**Proposition IV.2.** Assume that there exists a limited-duration safe policy for each set of safe states $\mathcal{C}_j \subset \mathcal{X}, j \in \{1, 2, \ldots, J\}, \text{ } J \in \mathbb{Z}_+$, that renders an agent with the dynamical system \cite{1} safe up to time $T$ whenever starting from inside a closed nonempty set $\mathcal{B}_{LD,j} \subset \mathcal{C}_j$. Then, given the set of safe states $\mathcal{C} := \bigcup_{j=1}^{J} \mathcal{C}_j$, there also exists a policy rendering the dynamical system safe up to time $T$ whenever starting from any state in $\mathcal{B}_{LD} := \bigcup_{j=1}^{J} \mathcal{B}_{LD,j}$.

**Proof.** A limited-duration safe policy for $\mathcal{C}_j$ also keeps the agent inside the set $\mathcal{C}$ up to time $T$ when starting from inside $\mathcal{B}_{LD,j}$.

If there exist LDCBFs for $\mathcal{C}_j$s, it is natural to ask if there exists an LDCBF for $\mathcal{C}$. Because of the nonsmoothness stemming from Boolean compositions, however, obtaining an LDCBF for $\mathcal{C}$ requires an additional learning in general (see Appendix \cite{49}). Also, existence of an LDCBF for the intersection of multiple sets of safe states is not guaranteed, and we need an additional learning as well.

So far, we have seen a possible way to obtain an LDCBF for a given set of safe states expressed as in \cite{1}. As our approach is set-theoretic rather than specifying a single optimal policy, it is also compatible with the constraints-driven control and transfer learning as described in the next section.

V. Applications

In this section, we present two practical examples that illustrate benefits of considering LDCBFs, namely, long-duration autonomy and transfer learning.
A. Applications to Long-duration Autonomy

In many applications, guaranteeing particular properties (e.g., forward invariance) over an infinite-time horizon is difficult or some forms of approximations are required. Specifically, when a function approximator is employed, there will certainly be an approximation error. Nevertheless, it is often sufficient to guarantee safety up to a certain finite time, and our proposed LDCBFs act as useful relaxations of CBFs. To see that one can still achieve long-duration autonomy by using LDCBFs, we consider the settings of work in [27]. Suppose that the state $x := [E, p]^T \in \mathbb{R}^3$ has the information of energy level $E \in \mathbb{R}_{\geq 0}$ and the position $p \in \mathbb{R}^2$ of an agent. Suppose also that $E_{\max} > 0$ is the maximum energy level and $\rho(p) \geq 0$ (equality holds only when the agent is at a charging station) is the energy required to bring the agent to a charging station from the position $p \in \mathbb{R}^2$. Then, although we emphasize that we can obtain an LDCBF by value function learning if necessary, let us assume that the function

$$B_{LD}(x) := E_{\max} - E + \rho(p) \geq 0$$

is an LDCBF, for simplicity. Then, by letting $L = \beta(E_{\max} - E_{\min})$ for some $\beta > 0$ and for the minimum necessary energy level $E_{\min}$, $0 \leq E_{\min} < E_{\max}$, the set of safe states can be given by $C := \{x \in \mathcal{X} : B_{LD}(x) < \frac{E_{\min}}{\beta}\}$. Now, under these settings, the following proposition holds.

**Proposition V.1.** Assume that the energy dynamics is lower bounded by $\frac{dE}{dt} \geq -K_d$, $\exists K_d > 0$, which implies that the least exit time $T_{\text{energy}}(E)$ of $E$ being below $E_{\min}$ is

$$T_{\text{energy}}(E) = \frac{(E - E_{\min})}{K_d}.$$

Also, suppose we employ a locally Lipschitz continuous policy $\phi$ that satisfies the LDCBF condition using a battery dynamics model $\frac{dE}{dt} = -K_d$. Then, by taking $T > T_{\text{energy}}(E_0)$ for the initial energy level $E_0 > E_{\min}$ and letting $B'_{LD}(x) := \{x \in \mathcal{X} : B_{LD}(x) \leq \frac{L - \rho(x)}{\beta}\} \subset C$, the agent starting from a state in $B'_{LD}(x)$ will reach the charging station before the energy reaches to $E_{\min}$.

**Proof.** See Appendix [D].

Hence, LDCBFs are shown to be applicable to some cases where it is difficult to guarantee that certain properties hold over infinite horizons, but where limited-duration safety suffices.

One of the benefits of using LDCBFs is that, once a set of limited-duration safe policies or good enough policies is obtained, one can reuse them for different tasks. Therefore, given that we can obtain an LDCBF through global value function learning for a policy that is not necessarily stabilizing the system, it is natural to ask if one can employ LDCBFs to transfer knowledge. Indeed, LDCBFs also have good compatibility with transfer learning (or safe transfer) as discussed below.

B. Applications to Transfer Learning

Given the immediate cost function, reinforcement learning aims at finding an optimal policy. Clearly, the obtained policy is not optimal with respect to a different immediate cost function in general. Therefore, employing the obtained policy straightforwardly in different tasks makes no sense. However, it is quite likely that the obtained policy is good enough even in different tasks because some mandatory constraints such as avoiding unsafe regions of the state space are usually shared among different tasks. Therefore we wish to exploit constraint learning for the sake of transfer learning, i.e., we learn constraints which are common to the target task while learning source tasks.

**Definition V.1** (Transfer learning, [12, modified version of Definition 1]). Given a set of training data $\mathcal{D}_S$ for one task (i.e., source task) denoted by $T_S$ (e.g., an MDP) and a set of training data $\mathcal{D}_T$ for another task (i.e., target task) denoted by $T_T$, transfer learning aims to improve the learning of the target predictive function $f_T$ (i.e., a policy in our example) in $\mathcal{D}_T$ using the knowledge in $\mathcal{D}_S$ and $T_S$, where $\mathcal{D}_S \neq \mathcal{D}_T$ or $T_S \neq T_T$.

For example, when learning an optimal policy for the balance task of the cart-pole problem, one can simultaneously learn a set of limited-duration safe policies that keep the pole from falling down up to certain time $T > 0$. The set of these limited-duration safe policies is obviously useful for other tasks such as moving the cart to one direction without letting the pole fall down.

Here, we present a possible application of limited duration control barrier functions to transfer learning. We take the following steps:

1) Design $J \in \mathbb{Z}_+$ cost functions $\ell_j, j \in \{1, 2, \ldots, J\}$, each of which represents a constraint by defining a set of safe state $C_j := \{x \in \mathcal{X} : \ell_j(x) \leq L\}, L > 0, j \in \{1, 2, \ldots, J\}$.

2) Conduct reinforcement learning for a cost function $\ell$ by using any of the reinforcement learning techniques.

3) No matter the currently obtained policy $\phi$ is optimal or not, one can obtain an LDCBF $B_{LD,j}$ for each set $C_j, j \in \{1, 2, \ldots, J\}$. More specifically, $B_{LD,j}$ is given by

$$B_{LD,j}(x) := V_j^{\beta}(x) := \int_0^\infty e^{-\beta t}\ell_j(x(t))dt, \quad j \in \{1, 2, \ldots, J\},$$

for $x(0) = x$.

4) When learning a new task, policies are constrained by LDCBFs depending on which constraints are common to the new task.

We study some practical implementations. Given LDCBFs $B_{LD,j}, j \in \{1, 2, \ldots, J\}$, define the set $\Phi_j^T$ of admissible policies as

$$\Phi_j^T := \{\phi : \phi(x) \in \mathcal{S}_{LD,j}(x), \forall x \in C_j\} \subset \Phi := \{\phi : \phi(x) \in \mathcal{U}, \forall x \in \mathcal{X}\},$$
where $S_{LD}^T(x)$ is the set of admissible control inputs at $x$ for the $j$th constraint. If an optimal policy $\phi^{Tr}$ for the target task $T_j$ is included in $\Phi_j^T$, one can conduct learning for the target task within the policy space $\Phi_j^T$. If not, one can still consider $\Phi_j^T$ as a soft constraint and can explore the policy space $\Phi \setminus \Phi_j^T$ with a given probability or can just select the initial policy from $\Phi_j^T$.

In practice, a parametrized policy is usually considered; a policy $\phi_\theta$ expressed by a parameter $\theta$ is updated via policy gradient methods [50]. If the policy is in the linear form with a fixed feature vector, the projected policy gradient method [51] can be used. Thanks to the fact that an LDCBF defines an affine constraint on instantaneous control inputs if the system dynamics is affine in control, the projected policy gradient method looks like $\theta \leftarrow \Gamma_j[\theta + \lambda \nabla_\theta F^{Tr}(\theta)]$. Here, $\Gamma_j : \Phi \rightarrow \Phi_j^T$ projects a policy onto the affine constraint defined by the $j$th constraint and $F^{Tr}(\theta)$ is the objective function for the target task which is to be maximized. For the policy not in the linear form with a given probability or can just select the initial policy from $\Phi_j^T$.

The parameters used for this experiment are summarized in Table VI.A. Here, angle threshold stands for the threshold of $\cos \psi$ where $\psi$ is the angle of the pole from the standing position.

VI. EXPERIMENTS

In this section, we validate our learning framework. First, we show that LDCBFs indeed work for the constraints-driven control problem considered in Section V-A by simulation. Then, we apply LDCBFs to a transfer learning problem for the cart-pole simulation environment.

A. Constraints-driven coverage control of multi-agent systems

Let the parameters be $E_{max} = 1.0$, $E_{min} = 0.55$, $K_d = 0.01$, $\beta = 0.005$ and $T = 50.0 > 45.0 = (E_{max} - E_{min})/K_d$. We consider six agents (robots) with single integrator dynamics. An agent of the position $p_i := [x_i, y_i]^T$ is assigned a charging station of the position $p_{\text{charge,}i}$, where $x$ and $y$ are the X position and the Y position, respectively. When the agent is close to the station (i.e., $\|p_i - p_{\text{charge,}i}\|_{\ell_2} \leq 0.05$), it remains there until the battery is charged to $E_{\text{ch}} = 0.92$. Actual battery dynamics is given by $dE/dt = -0.01E$. The coverage control task is encoded as Lloyd’s algorithm [54] aiming at converging to the Centroidal Voronoi Tessellation, but with a soft margin so that the agent prioritizes the safety constraint. The locational cost used for the coverage control task is given as follows [55]:

$$\sum_{i=1}^{6} \int_{V_i(p)} \|p_i - \hat{p}\|^2 \varphi(\hat{p})d\hat{p},$$

where $V_i(p) = \{\hat{p} \in \mathcal{X} : \|p_i - \hat{p}\| \leq \|p_j - \hat{p}\|, \forall j \neq i\}$ is the Voronoi cell for the agent $i$. In particular, we used $\varphi([x, y]^T) = e^{-\{(x - 0.2)^2 + (y - 0.3)^2\}/0.06 + 0.5e^{-\{(x + 0.2)^2 + (y + 0.1)^2\}/0.03}$. In MATLAB simulation (the simulator is provided on the Robotarium [56] website: www.robotarium.org), we used the random seed rng(5) for determining the initial states. Note, for every agent, the energy level and the position are set so that it starts from inside the set $B^T_{LD}$.

Figure VI.1 shows (a) the images of six agents executing coverage tasks and (b) images of the agents three of which are charging their batteries. Figure VI-B shows the simulated battery voltage data of the six agents, from which we can observe that LDCBFs worked effectively for the swarm of agents to avoid depleting their batteries.

B. Transfer from Balance to Move: Cart-pole problem

Next, we apply LDCBFs to transfer learning. The simulation environment and the deep learning framework used in this experiment are “Cart-pole” in DeepMind Control Suite and PyTorch [57], respectively. We take the following steps:

1) Learn a policy that successfully balances the pole by using DDPG [52].
2) Learn an LDCBF by using the obtained actor network.
3) Try a random policy with the learned LDCBF and a (locally) accurate model to see that LDCBF works reasonably.
4) With and without the learned LDCBF, learn a policy that moves the cart to left without letting the pole fall down, which we refer to as move-the-pole task.

The parameters used for this experiment are summarized in Table VI.B. Here, angle threshold stands for the threshold of $\cos \psi$ where $\psi$ is the angle of the pole from the standing position.
position, and position threshold is the threshold of the cart position \( p \). The angle threshold and the position threshold are used to terminate an episode. Note that the cart-pole environment of MuJoCo [58] xml data in DeepMind Control Suite is modified so that the cart can move between \(-3.8\) and \(3.8\). As presented in Example [VI.1], we use prioritized experience replay when learning an LDCBF. Specifically, we store the positive and the negative data, and sample 4 data points from the positive one and the remaining 60 data points from the negative one. In this experiment, actor, critic and LDCBF networks use ReLU nonlinearities. The actor network and the LDCBF network consist of two layers of 300 → 200 units, and the critic network is of two layers of 400 → 300 units. The control input vector is concatenated to the state vector from the second critic layer.

**Step 1:** The average duration (i.e., the first exit time, namely, the time when the pole first falls down) out of 10 seconds (corresponding to 1000 time steps) over 10 trials for the policy learned through the balance task by DDPG was 10 seconds.

**Step 2:** Then, by using this successfully learned policy, an LDCBF is learned by assigning the cost \( \ell(x) = 1.0 \) for \( \cos \psi < 0.2 \) and \( \ell(x) = 0.1 \) elsewhere. Also, because the LDCBF is learned in a discrete-time form, we transform it to a continuous-time form via multiplying it by \( \Delta_t = 0.01 \). When learning an LDCBF, we initialize each episode as follows: the angle \( \psi \) is uniformly sampled within \(-1.5 \leq \psi \leq 1.5\), the cart velocity \( \dot{p} \) is multiplied by 100 and the angular velocity \( \dot{\psi} \) is multiplied by 200 after being initialized by DeepMind Control Suite. The LDCBF learned by using this policy is illustrated in Fig. VI.2, which agrees with our intuitions. Note that \( \frac{T}{\beta} \) in this case is \( -\frac{\log(0.999)}{0.01} \approx 10.0 \).

**Step 3:** To test this LDCBF, we use a uniformly random policy \( \phi(x) \) takes the value between \(-1 \) and \(1\) constrained by the LDCBF with the function \( \alpha(q) = \max \{0.1q, 0\} \) and with the time constant \( T = 5.0 \). When imposing constraints, we use the (locally accurate) control-affine model of the cart-pole in the work [59], where we replace the friction parameters by zeros for simplicity. The average duration out of 10 seconds over 10 trials for this random policy was 10 seconds, which indicates that the LDCBF worked sufficiently well. We also tried this LDCBF with the function \( \alpha(q) = \max \{3.0q, 0\} \) and \( T = 5.0 \), which resulted in the average duration of 5.58 seconds. Moreover, we tried the fixed policy \( \phi(x) = 1.0 \), \( \forall x \in X \), with the function \( \alpha(q) = \max \{0.1q, 0\} \) and \( T = 5.0 \), and the average duration was 4.73 seconds, which was sufficiently close to \( T = 5.0 \).

**Step 4:** For the move-the-pole task, we define the success by the situation where the cart position \( p \), \(-3.8 \leq p \leq 3.8\), ends up in the region of \( p \leq -1.8 \) without letting the pole fall down. The angle \( \psi \) is uniformly sampled within \(-0.5 \leq \psi \leq 0.5\) and the rest follow the initialization of DeepMind Control Suite. The reward is given by \((1 + \cos \psi)/2\times\text{utils.rewards.tolerance}(\dot{p} + 1.0, \text{bounds} = (-2.0, 0.0), \text{margin} = 0.5))\), where \text{utils.rewards.tolerance} is the function defined in [13]. In other words, to move the pole to left, we give high rewards when the cart velocity is negative and the pole is standing up. To use the learned LDCBF for DDPG, we store matrices and vectors used in linear constraints along with other variables such as control inputs and states, which are then used for experience replay. Then, the log-barrier extension cost proposed in [53] is added when updating policies. Also, we try DDPG without using LDCBF for the move-the-pole task. Both approaches initialize the policy by the one obtained after the balance task. The average success rates of the policies obtained after the numbers of episodes up to 15 over 10 trials are given in Table VI.2 for DDPG with the learned LDCBF and DDPG without LDCBF. This result implies that our proposed approach successfully transferred information from the source task to the target task by sharing a common safety constraint.

**VII. CONCLUSION**

In this paper, we presented a notion of limited-duration safety as a relaxation of forward-invariance of a set of safe
It thus follows that $T$ successfully applied to a transfer learning problem via sharing validated through persistent coverage control tasks and were learning, and analyzed some of their properties. LDCBFs were that LDCBFs can be obtained through global value function

and $T_{p} + T$ is the first time at which the trajectory $s(t), t \geq T_{p}$, exists $\mathcal{C}$. Because $\alpha (L e^{-\beta T_{p}} - r(t)) \leq 0$, $T_{p} \leq t \leq T_{e}$, we obtain, by the Comparison Lemma [10], [60, Theorem 1.10.2], $B_{LD}(x(t)) \leq r(t) \leq s(t)$ for all $t$, $T_{p} \leq t \leq T_{e}$. If we assume $T_{e} < T_{p} + T$, it contradicts the fact that $B_{LD}(x(T_{e})) \leq s(T_{e}) < s(T_{p} + T) = \frac{L}{\beta}$, and hence $T_{e} \geq T_{p} + T$. Therefore, we conclude that any Lipschitz continuous policy $\phi : \mathcal{X} \to \mathcal{U}$ such that $\phi(x) \in \mathcal{S}_{LD}(x), \forall x \in \mathcal{C}$, renders the dynamical system safe up to time $T_{p} + T(\geq T)$ whenever the initial state $x(0)$ is in $B_{LD}^{T}$.

**APPENDIX A**

**PROOF OF THEOREM [IV.1]**

Under Assumption [III.1] the trajectories $x(t)$ with an initial condition $x(0) \in B_{LD}^{T}$ exist and are unique over $t \geq 0$. Let the first time at which the trajectory $x(t)$ exits $\mathcal{C}$ be $T_{e} > 0$ and let $T_{p}, 0 \leq T_{p} < T_{e}$, denote the last time at which the trajectory $x(t)$ passes through the boundary of $B_{LD}^{T}$, from inside before first exiting $\mathcal{C}$. Since $\alpha$ is locally Lipschitz continuous and $B_{LD}$ is continuously differentiable, the right hand side of (4) is locally Lipschitz continuous. Thus solutions to the differential equation

$$
\dot{r}(t) = \alpha \left( \frac{L e^{-\beta T}}{\beta} - r(t) \right) + \beta r(t),
$$

where the initial condition is given by $r(T_{p}) = B_{LD}(x(T_{p}))$, exist and are unique for all $t$, $T_{p} \leq t \leq T_{e}$. On the other hand, the solution to

$$
s(t) = \beta s(t),
$$

where the initial condition is given by $s(T_{p}) = B_{LD}(x(T_{p})) = \frac{L e^{-\beta T}}{\beta}$, is

$$
s(t) = B_{LD}(x(T_{p})) e^{\beta(t - T_{p})}, \forall t \geq T_{p}.
$$

It thus follows that

$$
s(T_{p} + T) = \frac{L}{\beta} e^{-\beta T} e^{\beta T} = \frac{L}{\beta},
$$

and $T_{p} + T$ is the first time at which the trajectory $s(t), t \geq T_{p}$, exists $\mathcal{C}$. Because $\alpha (L e^{-\beta T_{p}} - r(t)) \leq 0$, $T_{p} \leq t \leq T_{e}$, we obtain, by the Comparison Lemma [10], [60, Theorem 1.10.2], $B_{LD}(x(t)) \leq r(t) \leq s(t)$ for all $t$, $T_{p} \leq t \leq T_{e}$. If we assume $T_{e} < T_{p} + T$, it contradicts the fact that $B_{LD}(x(T_{e})) \leq s(T_{e}) < s(T_{p} + T) = \frac{L}{\beta}$, and hence $T_{e} \geq T_{p} + T$. Therefore, we conclude that any Lipschitz continuous policy $\phi : \mathcal{X} \to \mathcal{U}$ such that $\phi(x) \in \mathcal{S}_{LD}(x), \forall x \in \mathcal{C}$, renders the dynamical system safe up to time $T_{p} + T(\geq T)$ whenever the initial state $x(0)$ is in $B_{LD}^{T}$.

**APPENDIX B**

**ON PROPOSITION [IV.1]**

The width of a feasible set is defined as the unique solution to the following linear program:

$$
u_{\omega}(x) = \max_{[u, \omega] \in \mathbb{R}^{n+1}} \omega (B_{LD}(x) + L_{u} B_{LD}(x) u + \omega)^{T} \in \mathcal{U}
$$

**APPENDIX C**

**PROOF OF THEOREM [IV.2]**

Because, by definition,

$$
\hat{V}_{c}^{\phi, \beta}(x) \geq \frac{\hat{L}}{\beta}, \forall x \in \mathcal{X} \setminus \mathcal{C},
$$

it follows that

$$
\hat{C} = \left\{ x \in \mathcal{X} : \hat{V}_{c}^{\phi, \beta}(x) < \frac{\hat{L}}{\beta} \right\} \subset \mathcal{C}.
$$
Because the continuously differentiable function $\hat{V}_c^{\phi,\beta}(x)$ satisfies
\[
\dot{\hat{V}}^{\phi,\beta}_c(x) = L_f \hat{V}^{\phi,\beta}_c(x) + L_g \hat{V}^{\phi,\beta}_c(x) \phi(x) = \beta \hat{V}^{\phi,\beta}_c(x) - \dot{\hat{c}}(x), \quad \forall x \in C,
\]
and $\dot{\hat{c}}(x) \geq 0$, $\forall x \in C$, there exists at least one policy $\phi$ that satisfies
\[
\phi(x) \in S^T_{\text{LD}}(x) = \{ u \in U : L_f \hat{V}^{\phi,\beta}_c(x) + L_g \hat{V}^{\phi,\beta}_c(x) u \leq \alpha \left( \frac{Le^{-\beta T}}{\beta} - \hat{V}^{\phi,\beta}_c(x) \right) + \beta \hat{V}^{\phi,\beta}_c(x) \},
\]
for all $x \in C$ and for a monotonically increasing locally Lipschitz continuous function $\alpha$ such that $\alpha(q) = 0$, $\forall q \leq 0$. Therefore, $V^{\phi,\beta}_c$ is an LDCBF for the set $\hat{C}$.

Remark C.1. A sufficiently large constant $c$ could be chosen in practice. If $\hat{f}_c(x) > 0$ for all $x \in C$ and the value function is learned by using a policy $\phi$ such that $\phi(x) + [c_0, c_0, \ldots, c_0]^T \in U$ for some $c_0 > 0$, then the unique solution to the linear program (B.1) satisfies $u_\omega(x) > 0$ for any $x \in C$.

APPENDIX D

PROOF OF PROPOSITION [V.1]

Under Assumption [III.1] the trajectories $x(t)$ with an initial condition $x(0) \in B^T_{\text{LD}}$ satisfying the given LDCBF condition (that uses the battery dynamics model $dE/dt = -K_d$) exist and are unique over $t \geq 0$. Let the actual exit time of $E$ being below $E_{\text{min}}$ be $T_{\text{energy}} > 0$, and let the first time at which the trajectory $x(t)$ exits $C$ be $T_e > 0$ ($T_e = \infty$ if the agent never exits $C$). Note it holds that $T_e \leq T_{\text{energy}}$. Also, let $E_t$ and $\rho_t$ be the trajectories of $E$ and $\rho(p)$. Define
\[
\hat{T}_e := \min \{ T_e, \inf \{ t : \rho_t = 0 \land x(t) \notin B^T_{\text{LD}} \} \},
\]
and
\[
T_p := \max \{ t : x(t) \in \partial B^T_{\text{LD}} \land t \leq \hat{T}_e \},
\]
where $\partial B^T_{\text{LD}}$ denotes the boundary of $B^T_{\text{LD}}$. Now, if $E_t = E_{\text{min}}$ and $\rho_t > 0$ for some $t < T_p$, it contradicts the fact that $B_{\text{LD}}(x(t)) = E_{\text{max}} - E_t + \rho_t < E_{\text{max}} - E_{\text{min}}$ ($\forall x(t) \in \hat{C}$, $\forall t < T_p$). Therefore, it follows that $E_t > E_{\text{min}}$ or $\rho_t = 0$ for all $t < T_p$. This implies that we should only consider the time $t \geq T_p$ by assuming that $\rho_{T_p} \geq 0$.

Let $\hat{E}_t$ be the trajectory following the virtual battery dynamics $dE/dt = -K_d$ and $\hat{E}_{T_p} = E_{T_p}$, and let $s(t)$ be the unique solution to
\[
\dot{s}(t) = \beta s(t), \quad t \geq T_p,
\]
where $s(T_p) = B_{\text{LD}}(x(T_p)) = (E_{\text{max}} - E_{\text{min}}) e^{-\beta T_p}$. Also, let $\dot{g}(t) = s(t) + \hat{E}_t - E_{\text{max}}, \quad t \geq T_p$. Then, the time at which $s(t)$ becomes $E_{\text{max}} - E_{\text{min}}$ is $T_p + T$ because $s(T + T_p) = B_{\text{LD}}(x(T_p)) e^{\beta(t + T_p - T_p)} = (E_{\text{max}} - E_{\text{min}}) e^{-\beta T_p} e^{\beta(T + T_p - T_p)} = E_{\text{max}} - E_{\text{min}}$. Since $T_{\text{energy}}(E_{T_p}) \leq T_{\text{energy}}(E_0) < T$ and
\[
g(t) = B_{\text{LD}}(x(T_p)) e^{\beta(t - T_p)} + \hat{E}_{T_p} - K_d(t - T_p) - E_{\text{max}},
\]
we obtain
\[
\hat{T}_0 := \inf \{ t : g(t) = 0 \land t \geq T_p \} \leq T_p + \hat{T}_{\text{energy}}(E_{T_p}).
\]

On the other hand, the actual battery dynamics can be written as $dE/dt = -K_d + \Delta(x)$, where $\Delta(x) \geq 0$. Therefore, we see that the trajectory $x(t)$ satisfies
\[
\frac{d B_{\text{LD}}(x(t))}{dt} \leq \beta B_{\text{LD}}(x(t)) - \Delta(x), \quad \forall t, \quad T_p \leq t \leq \hat{T}_e.
\]

Then, because
\[
\frac{d (B_{\text{LD}}(x(t)) - s(t))}{dt} \leq \beta (B_{\text{LD}}(x(t)) - s(t)) - \Delta(x(t)) \leq \beta (B_{\text{LD}}(x(t)) - s(t)) - s(t) \leq -\int_0^t \Delta(x(t)) dt, \quad \forall t, \quad T_p \leq t \leq \hat{T}_e.
\]

and $\beta (B_{\text{LD}}(x(T_p)) - s(T_p)) = 0$, we obtain $B_{\text{LD}}(x(t)) - s(t) \leq -\int_0^t \Delta(x(t)) dt$, $\forall t$, $T_p \leq t \leq \hat{T}_e$. Also, it is straightforward to see that $T_{\text{energy}} \geq T_e \geq T_p + T \geq T_p + \hat{T}_{\text{energy}}(E_{T_p})$. From the definitions of $B_{\text{LD}}$ and $g(t)$, it follows that
\[
\rho_t - g(t) = B_{\text{LD}}(x(T_p)) - s(t) + E_t - \hat{E}_t \leq -\int_0^t \Delta(x(t)) dt + \int_0^t \Delta(x(t)) dt = 0, \quad \forall t, \quad T_p \leq t \leq \hat{T}_e,
\]

which leads to the inequality $\rho_t \leq g(t)$ for all $t$, $T_p \leq t \leq \hat{T}_e$. Hence, we conclude that
\[
\hat{T} := \inf \{ t : \rho_t = 0 \land t \geq T_p \} \leq \hat{T}_0 \leq T_p + \hat{T}_{\text{energy}}(E_{T_p}) \leq T_{\text{energy}},
\]
and $\hat{T}_e = \hat{T}$, which proves the proposition.

APPENDIX E

STOCHASTIC LIMITED DURATION CONTROL BARRIER FUNCTIONS

For the system dynamics described by a stochastic differential equation:
\[
dx = h(x(t), u(t)) dt + \eta(x(t)) dw, \quad (E.1)
\]
where $h : \mathbb{R}^{n_x} \times U \rightarrow \mathbb{R}^{n_x}$ is the drift, $\eta : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times \mathbb{R}}$ is the diffusion, $w$ is a Brownian motion of dimension $n_w \in \mathbb{Z}_+$, it is often impossible to guarantee safety with probability one without making specific assumptions on the dynamics. Therefore, we instead consider an upper bound on the probability that a trajectory escape from the set of safe states within a given finite time. We give the formal definition below.

Definition E.1 (Limited-duration safety for stochastic systems). Let $B_{\text{SLD}}^{T,\beta}$ be a closed nonempty subset of a open set of safe states $C_S$ and $\tau$ the first exit time $\tau := \inf \{ t : x(t) \notin C_S \}$. Then, the stopped process $\tilde{x}(t)$ defined by
\[
\tilde{x}(t) := \begin{cases} x(t), & t < \tau, \\ x(\tau), & t \geq \tau, \end{cases} \quad (E.2)
\]
where $x(t)$ evolves by (E.1), is safe up to time $T > 0$ with probability $\delta$, $0 \leq \delta \leq 1$, if there exists a policy $\phi$ which ensures that

$$P \left\{ \hat{x}(t) \neq C_S \text{ for some } t, \ 0 \leq t \leq T : \hat{x}(0) \in B^{T,\delta}_{\text{SLD}} \right\} \leq 1 - \delta.$$

To present stochastic limited duration control barrier functions (SLDCBFs) that are stochastic counterparts of LDCBFs, we define the infinitesimal generator $G$, for a function $B_{\text{SLD}} : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, by

$$G(B_{\text{SLD}})(x) := \frac{1}{2} \text{tr} \left[ \frac{\partial^2 B_{\text{SLD}}(x)}{\partial x^2} \eta(x) \eta(x)^T \right] - \frac{\partial B_{\text{SLD}}(x)}{\partial x} h(x, \phi(x)), \ x \in \int(\mathcal{X}),$$

where $\text{tr}$ stands for the trace. Also, we make the following assumption.

**Assumption E.1.** For any Lipschitz continuous policy $\phi$, both $h(x, \phi(x))$ and $\eta(x, \phi(x))$ are Lipschitz continuous with respect to $x$, i.e., the stochastic process defined in (E.1) is an Itô diffusion [44, Definition 7.1.1], which has a pathwise unique solution for $t, \ 0 \leq t < \infty$.

Then, the following theorem holds.

**Theorem E.1.** Given $T > 0$ and $\delta$, $0 \leq \delta \leq 1$, define a set of safe states

$$C_S := \left\{ x \in \mathcal{X} : B_{\text{SLD}}(x) < \frac{L}{\beta} \right\}, \ L > 0, \ \beta > 0,$$

for a twice continuously differentiable function $B_{\text{SLD}} : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$. Define also the set $B^{T,\delta}_{\text{SLD}}$ as

$$B^{T,\delta}_{\text{SLD}} := \left\{ x \in \mathcal{X} : B_{\text{SLD}}(x) \leq (1 - \delta) \frac{L e^{-\beta T}}{\beta} \right\} \subset C_S.$$

If $B^{T,\delta}_{\text{SLD}}$ is nonempty and if there exists a Lipschitz continuous policy $\phi : \mathcal{X} \rightarrow \mathcal{U}$ satisfying

$$\phi(x) \in \mathcal{S}^{T}_{\text{SLD}} := \left\{ u \in \mathcal{U} : -G(B_{\text{SLD}})(x) \leq \beta B_{\text{SLD}}(x) \right\},$$

for all $x \in C_S$, then, under Assumption E.1, the policy $\phi$ renders the stopped process $\hat{x}(t)$ in (E.2) safe up to time $T$ with probability $\delta$.

**Proof.** Define $B_{\text{SLD}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ as

$$B_{\text{SLD}}(t) := e^{-\beta t} B_{\text{SLD}}(\hat{x}(t)).$$

Because $\hat{x}(t)$ is an Itô diffusion, from Dynkin’s formula (cf. [61, Chapter III.3]), we obtain

$$E_x \left[ B_{\text{SLD}}(\hat{x}(t)) \right] - B_{\text{SLD}}(\hat{x}(0)) = E_x \left[ \int_0^t -e^{-\beta s} \left[ G(B_{\text{SLD}}) + \beta B_{\text{SLD}}(\hat{x}(s)) \right] ds \right],$$

for any $t, \ 0 \leq t < \infty$, from which it follows that

$$E_x \left[ B_{\text{SLD}}(\hat{x}(t)) \right] \leq B_{\text{SLD}}(\hat{x}(0)).$$

Therefore, $\tilde{B}_{\text{SLD}}$ is a supermartingale with respect to the filtration $\{ M_t : t \geq 0 \}$ generated by $\hat{x}(t)$ because $B_{\text{SLD}}(\hat{x}(t))$ is twice continuously differentiable and $C_S$ is bounded. We thus obtain [62, p.25]

$$P \left\{ \sup_{0 \leq t \leq T} B_{\text{SLD}}(\hat{x}(t)) \geq \frac{L}{\beta} \right\} \leq P \left\{ \sup_{0 \leq t \leq T} \frac{\tilde{B}_{\text{SLD}}(t) - \tilde{B}_{\text{SLD}}(0)}{L} \geq \frac{L e^{-\beta T}}{L} \right\} \leq \frac{\beta e^{\beta T} B_{\text{SLD}}(\hat{x}(0))}{L} \leq 1 - \delta.$$

To make a claim similar to Theorem V.2 we consider the following value function associated with the policy $\phi$:

$$V^{\phi,\beta}(x) := E_x \int_0^\infty e^{-\beta t} \ell(x(t)) dt, \ \beta \geq 0,$$

where $\ell(x(t))$ is the immediate cost and $E_x$ is the expectation for all trajectories (time evolutions of $x(t)$) starting from $x = x(0)$. When $V^{\phi,\beta}$ is twice continuously differentiable over $\int(\mathcal{X})$, we obtain the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation [44]:

$$\beta V^{\phi,\beta}(x) = -G(V^{\phi,\beta})(x) + \ell(x), \ x \in \int(\mathcal{X}). \quad (E.3)$$

Given the set of safe states

$$C_S := \left\{ x \in \mathcal{X} : \ell(x) < L \right\}, \ L > 0,$$

for $\ell(x) \geq 0, \ \forall x \in \mathcal{X}$, and the stopped process (E.2), assume that we employ a twice continuously differentiable function approximator to approximate the value function $V^{\phi,\beta}$ for the stopped process, and let $V_c^{\phi,\beta}$ denote the approximation of $V^{\phi,\beta}$. By using the HJBI equation (E.3), define the estimated immediate cost function $\hat{\ell}$ as

$$\hat{\ell}(x) = \beta \hat{V}^{\phi,\beta}(x) + G(\hat{V}^{\phi,\beta})(x), \ \forall x \in C_S.$$

Select $c \geq 0$ so that $\hat{\ell}(x) := \hat{\ell}(x) + c \geq 0$ for all $x \in C_S$, and define the function $V_c^{\phi,\beta}(x) := V^{\phi,\beta}(x) + \frac{c}{\beta}$. Now, the following theorem holds.

**Theorem E.2.** Consider the set

$$B^{T,\delta}_{\text{SLD}} = \left\{ x \in \mathcal{X} : V_c^{\phi,\beta}(x) \leq (1 - \delta) \frac{L e^{-\beta T}}{\beta} \right\},$$

where $\hat{L} := \inf_y \in \mathcal{X} \setminus C_S \beta V_c^{\phi,\beta}(y)$. If $B^{T,\delta}_{\text{SLD}}$ is nonempty, then the stopped process starting from the initial state in $B^{T,\delta}_{\text{SLD}}$ is safe up to time $T$ with probability $\delta$ when the policy $\phi$ is employed, and $\tilde{V}_c^{\phi,\beta}(x)$ is an SLDCBF for the set

$$C_S := \left\{ x \in \mathcal{X} : V_c^{\phi,\beta}(x) < \frac{\hat{L}}{\beta} \right\} \subset C_S.$$

**Proof.** Because, by definition,

$$V_c^{\phi,\beta}(x) \geq \frac{\hat{L}}{\beta}, \ \forall x \in \mathcal{X} \setminus C_S,$$
it follows that
\[ \hat{C}_S = \left\{ x \in \mathcal{X} : \hat{V}_c^{\phi, \beta}(x) < \frac{L}{\beta} \right\} \subseteq C_S. \]

Because the twice continuously differentiable value function \( \hat{V}_c^{\phi, \beta} \) satisfies
\[ -\mathcal{G}(\hat{V}_c^{\phi, \beta})(x) = \beta \hat{V}_c^{\phi, \beta}(x) - \hat{\ell}_c(x), \quad x \in C_S, \]
and \( \hat{\ell}_c(x) \geq 0, \forall x \in C_S \), there exists at least one policy \( \phi \) that satisfies
\[ \phi(x) \in \mathcal{S}_{LD}(x) = \left\{ u \in U : -\mathcal{G}(\hat{V}_c^{\phi, \beta})(x) \leq \beta \hat{V}_c^{\phi, \beta}(x) \right\} \]
for all \( x \in C_S \). Therefore, \( \hat{V}_c^{\phi, \beta} \) is an SLDCBF for the set \( \hat{C}_S \).

\[ \square \]

**APPENDIX F COMPOSITIONS OF LDCBFs**

From Proposition [IV.2] we know that an agent can be made safe up to time \( T \) for the set of safe states \( C := \bigcup_{j=1}^{J} C_j \) if starting from inside \( B_{LD_j}^T \), \( j \), \( j \) for the sets \( C_j \), \( j \). For simplicity, in this section, we focus on the case where \( J = 2 \) without losing generality. First, let us consider the following function denoted as \( B_{LD_1} \lor B_{LD_2} \):
\[ B_{LD_1} \lor B_{LD_2}(x) := \min \{ B_{LD_1}(x), B_{LD_2}(x) \}. \]

Then, we obtain
\[ B_{LD}^T = \left\{ x \in \mathcal{X} : B_{LD_1} \lor B_{LD_2}(x) \leq \frac{L e^{-\beta T}}{\beta} \right\} \subseteq C, \]
\[ C = \left\{ x \in \mathcal{X} : B_{LD_1} \lor B_{LD_2}(x) < \frac{L}{\beta} \right\}. \]

The function \( B_{LD_1} \lor B_{LD_2} \) is, however, nonsmooth in general. Therefore, even if we consider differential inclusion and associated Carathéodory solutions as in [23, 63], there might exist sliding modes that violate inequalities imposed by a function \( B_{LD_1} \lor B_{LD_2} \), \( x \in \Omega_{f(x, \theta)} \). Here, \( \Omega_{f(x, \theta)} \) represents the zero-measure set where the dynamics is nondifferentiable (see [64] for detailed arguments, for example). Nevertheless, to obtain a smooth LDCBF for \( C \), we may obtain a smooth approximation of a possibly discontinuous policy \( \phi \) that satisfies
\[ \frac{dB_{LD_j}(x^*)}{dt} \leq \alpha \left( \frac{Le^{-\beta T}}{\beta} - B_{LD_j}(x^*) \right) + \beta B_{LD_j}(x^*), \]
for all \( x^* \) such that \( B_{LD_1}(x^*) < B_{LD_2}(x^*) \) for \( j = 1 \) and \( B_{LD_2}(x^*) < B_{LD_1}(x^*) \) for \( j = 2 \). Then, we can conduct value function learning to obtain an LDCBF for \( C \) with a set \( B_{LD}^T \) that is possibly smaller than \( B_{LD}^T \).

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