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Topological conjugation of one dimensional maps
Draft of the book

Abstract

Topological conjugateness of one dimensional unimodal dynamical systems, which are generated by interval [0, 1] into itself maps are studied. We study the smoothness and differentiability of the conjugacy of symmetrical and non-symmetrical tent maps. Also we prove the extremal property of the length of the graph of the conjugacy of symmetrical and non-symmetrical tent maps.
1 Preface

During the 10 years, from 2004 till 2013, Volodymyr Fedorenko was a lector of the course of Dynamical systems at Mechanical and mathematical faculty of National Taras Shevchenko University of Kyiv.

All the time, when he considered the topic of the topological conjugation of one dimensional maps, he considered the continuous unimodal $[0, 1] \to [0, 1]$ maps

$$f(x) = \begin{cases} 
2x, & x < 1/2; \\
2 - 2x, & x \geq 1/2,
\end{cases}$$

and $g(x) = 4x(1 - x)$, which are conjugated via $h(x) = \sin^2\left(\frac{\pi x}{2}\right)$.

![Fig. 1: Graphics](image)

Figure 1a) contains the graph of $f$, Figure 1b) contains the graph of $g$, the graph the conjugacy $h$, which defines the conjugacy of $f$ and $g$, i.e. is the solution of the functional equation $h(f) = g(h)$, is given at Figure 1c).

During of one of the lectures, V. Fedorenko mentioned, that the same reasonings, which were used for finding the conjugacy of $f$ and $g$, can be applied for finding the conjugacy $h_v : [0, 1] \to [0, 1]$ of $f$, and $f_v$, where

$$f_v(x) = \begin{cases} 
\frac{x}{v}, & x \leq v; \\
\frac{1-x}{1-v}, & x > v,
\end{cases}$$
and \( v \in (0, 1) \) is a parameter. The finding of \( h_v \) needs the solution of the system of linear functional equations. Methods of solving of linear functional equations are described, for instance, in [47].

This book was inspired by multiple attempts to find the explicit formulas of the conjugacy of \( f \) and \( f_v \). In spite of we was failed with these attempts, we have obtained a list of results, concerning with properties of this conjugacy and the semi conjugacy of \( f \) and \( f_v \). These results are formulated in Section 2.

The conjugation of \( f \) and \( g \) appeared in the first time in the middle of the 20-th century in the von Neumann’s and S. Ulam’s collaborated works, which deal with generators of random numbers. In the same time, the map \( g \), which is know as a logistic map, was discovered for the applications of mathematics by P. Verhulst in the first half of 19-th century in the populational dynamics. Also the one parametrical family of logistic maps was used in 1970-th by M. Feigenbaum in the discovery of universal constants, which are known by his name.

The hardness of studying of the properties of the conjugacy \( h_v \) of \( f \) and \( f_v \), follows, for instance, from the following its property (see [64]): the derivative \( h'_v \) exists and equals either 0, or \( \infty \). Moreover, \( h'_v \) equals 0 almost everywhere in the cense of the Lebesgue measure of \([0, 1]\). The last property yields that \( h_v \) is non-differentiable on any subinterval of \([0, 1]\).

The graph of \( h_v \) for \( v = 3/4 \) is given at Figure 2.

![Graph of \( h_{3/4} \)](image)

**Fig. 2:** Graph of \( h_{3/4} \)
2 The main results

In Section 3 we define the main notions such as dynamical system, fixed and periodical point, trajectory and orbit, topological conjugation and topological semiconjugation of maps. We give examples of topologically conjugated \([0, 1] \to [0, 1]\) mappings and this leads us to the following proposition.

**Proposition** (Proposition 3.5, also Lemma 1 in [32]). Let \(f, g : [0, 1] \to [0, 1]\) be piecewise linear maps, which are topologically conjugated via increasing piecewise linear homeomorphism \(h\). If \(f(0) = 0\), then \(g(0) = 0\) and \(f'(0) = g'(0)\).

In Section 4 we give a historical review of works, which deal with the study of iterations of one-dimensional maps, studying of hat-maps, logistic map, E. Lámerey’s diagrams and topological conjugation of one-dimensional maps (one-dimensional dynamical systems).

Section 5 is devoted to the study of topological conjugation of continuous \([0, 1] \to [0, 1]\) maps, whose semigroup of iterations is a cyclic group call \(C_n\). A map \(f\) with such semigroup of iterations satisfy the functional equation

\[
    f^n = f, \tag{2.1}
\]

and for \(m, 2 \leq m < n\) the equality \(f^m = f\) does not hold. In Section 5.1 we prove the following theorem:

**Theorem** (Theorem 5.1, also Theorem 1 in [16]). If the continuous maps \(f : [0, 1] \to [0, 1]\) satisfy \((2.1)\), then it also satisfy the equation

\[
    f^3 = f.
\]

This theorem is a generalization of the following Theorem.

**Theorem** (Theorem 4.1, also Theorem 2 from [44]). If \(g : [0, 1] \to [0, 1]\) is a continuous function such that \(g^p(x) = x\) for all \(x\), then \(g^2(x) = x\) for all \(x\). In particular, if \(p\) is odd, then \(g(x) = x\) for all \(x\).

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We describe in Section 5.2 the graph of the continuous $f : [0, 1] \to [0, 1]$, whose semigroup of iterations is a finite group.

**Theorem** (Theorem 5.2 also theorem 2 in [16]). For a continuous maps $f : [0, 1] \to [0, 1]$ the following statements are equivalent:

1) $f^n(x) = f(x)$ for all $x \in [0, 1]$ and every $n \in \mathbb{N}$;

2) there exist numbers $a, b$, $0 \leq a \leq b \leq 1$ and the continuous maps $g : [0, a] \to [a, b]$ and $h : [b, 1] \to [a, b]$ such that $f$ can be represented as

$$f(x) = \begin{cases} 
  g(x), & 0 \leq x \leq a, \\
  x, & a \leq x \leq b, \\
  h(x), & b \leq x \leq 1.
\end{cases}$$

![Fig. 3: Graph of $f$](image)

The graph of $f$ from Theorem 5.2 is given at Figure 3.

**Theorem** (Theorem 5.3 also Theorem 3 in [16] and Theorem 10 in [49]). For a continuous maps $f : [0, 1] \to [0, 1]$ the following statements are equivalent:

1) $f^3(x) = f(x)$ for all $x \in I$;

2) there exist numbers $a, b$ with $0 \leq a \leq b \leq 1$ and maps continuous $g : [0, a] \to [a, b]$, $h : [b, 1] \to [a, b]$ and $\varphi : [a, b] \to [a, b]$ with the following properties. the graph of $\varphi$ is symmetrical in the line $y = x$ and $\varphi([a, b]) = [a, b]$. The maps $f$ can be represented as follows:

$$f(x) = \begin{cases} 
  g(x), & 0 \leq x \leq a, \\
  \varphi(x), & a \leq x \leq b, \\
  h(x), & b \leq x \leq 1.
\end{cases}$$
Section 5.3 is devoted to the description of conjugated classes of maps, whose semigroup of iterations is a finite group and which have finitely many intervals of monotony. Let \( f, g : [0, 1] \to [0, 1] \) be continuous. Let \( h : [0, 1] \to [0, 1] \) be a homeomorphism such that \( g = h^{-1}(f(h)) \), also \( p_1, p_2, q_1 \) and \( q_2 \) be such numbers that \( f([0, 1]) = [p_1, q_1] \), and \( g([0, 1]) = [p_2, q_2] \). Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_m \) be extremums of \( f \) and \( g \) correspondingly. Points 0, 1 and ends of intervals of fixed points are also considered as extremums.

**Definition.** Vectors \((v_1, \ldots, v_k)\) and \((w_1, \ldots, w_k)\) are called co-ordered if for every \( i, j \) the inequality \( v_i \leq v_j \) is equivalent to \( w_i \leq w_j \).

**Notation.** Denote the following vectors:
\[
\begin{align*}
v_f &= (f(a_1), \ldots, f(a_n)), \\
\tilde{v}_f &= (f(a_1), f^2(a_1), \ldots, f(a_n), f^2(a_n)), \\
w_g &= (g(b_1), \ldots, g(b_m)), \text{ and} \\
\tilde{w}_g &= (g(b_1), g^2(b_1), \ldots, g(b_m), g^2(b_m)).
\end{align*}
\]

We prove the following two theorems.

**Theorem** (Theorem 5.4, also Lemma 13 in [18]). Continuous maps \( f, g : [0, 1] \to [0, 1] \) are conjugated via increasing homeomorphism if only if \( m = n \) numbers of end points of \( f([0, 1]) \) and \( g([0, 1]) \) coincide and vectors \( \tilde{v}_f \) and \( \tilde{w}_g \) are co-ordered.

**Theorem** (Theorem 5.5, see also Sect. 4 in [18]). Maps \( f \) and \( g \) are conjugated via decreasing homeomorphism if and only if \( m = n \), numbers of end points of \( f([0, 1]) \) and \( g([0, 1]) \) coincide and vectors \( \tilde{v}_f \) and \( w_g^* \) are co-ordered where
\[
w_g^* = (g(b_m), g^2(b_m), \ldots, g(b_1), g^2(b_1)).
\]

In Section 6 we consider the topological conjugation of continuous maps \( f, g : [0, 1] \to [0, 1] \), given by
\[
f(x) = \begin{cases} 
2x, & 0 \leq x < 1/2; \\
2 - 2x, & 1/2 \leq x \leq 1,
\end{cases}
\]
and

\[ g(x) = \begin{cases} 
  g_l(x), & x \leq v; \\
  g_r(x), & x > v,
\end{cases} \quad (2.3) \]

where \( v \in (0, 1) \) is fixed, \( g_l(0) = g_r(1) = 0 \), \( g_l(v) = \lim_{x \to v^-} g_r(x) = 1 \) and functions \( g_l, g_r \) are monotone and continuous. The problem about the conjugation of \( f \) and \( g \) is stated at first at [65] via the following theorem

**Theorem** (Theorem 6.1 also Appendix 1, §3 in [65]). Let \( f \) be a function of the form (2.2) and \( g \) be a convex function of the form (2.3). Consider the integer trajectory of 1 under the action of \( f \), i.e. the smallest set \( M_f \) such that \( 1 \in M_f \) and \( f(x) \in M_f \) is equivalent to \( x \in M_f \). The necessary and sufficient condition of \( f \) and \( g \) be conjugated is combinatorial equivalence of \( M_f \) and \( M_g \) together with that \( M_g = [0, 1] \).

We prove Theorem 6.1 in details. Its proof is constructive and is based on the following notions and reasonings.

**Definition** (Definition 6.1). Let \( f \) be of the form (2.2), and the function \( g \) be of the form (2.3). For every \( x^* \in [0, 1] \) say the value \( y^* \) of the homeomorphism \( h \) at \( x^* \) to be **conditionally found** if the following statement holds: if \( h \) is a topological conjugation of \( f \) and \( g \), then \( h(x^*) = y^* \).

We use the word “conditionally” to notice that in the time when we consider this conditionally found value of the conjugation, the question about the existence of the conjugation is still opened.

**Proposition** (Proposition 6.2). Let \( f \) be of the form (2.2) and \( g \) be of the form (2.3).

If the topological conjugation \( h \) of \( f \) and \( g \) is conditionally found at some point \( x^* \) and equals \( y^* \) then for every \( \tilde{x} \) such that \( f(\tilde{x}) = x^* \), the value at \( \tilde{x} \) is also found as follows:

1. If \( \tilde{x} \leq 1/2 \) then \( h(\tilde{x}) = g_l^{-1}(y^*) \);
2. If \( \tilde{x} > 1/2 \) then \( h(\tilde{x}) = g_r^{-1}(y^*) \).

We use the following notations when prove Theorem 6.1. Denote by \( A_n, n \geq 1 \) the set of all points of \([0, 1]\) such that

\[ f^n(A_n) = 0. \]
Denote by $B_n$, $n \geq 1$ the set of all points of $[0, 1]$ such that
\[ g^n(B_n) = 0. \]

We obtain the following description of $A_n$.

**Proposition** (Proposition 6.3, also Lemma 4 in [17]). $A_n = \left\{ 0, \frac{1}{2n-1}; \ldots; \frac{2^n-1}{2n-1}, 1 \right\}$.

The density of $A = \bigcup_{n=1}^{\infty} A_n$ lets to reduce the proof of Theorem 6.1 to the proof of the density of $B = \bigcup_{n=1}^{\infty} B_n$ in $[0, 1]$. We prove the following theorem.

**Theorem** (Theorem 6.4, also Lemma 1 in [52]). Let functions $f, g : [0, 1] \rightarrow [0, 1]$ be defined by (2.2) and (2.3). If a homeomorphism $h : [0, 1] \rightarrow [0, 1]$ satisfies the functional equation
\[ h(f(x)) = g(h(x)), \]
then it increase and $h(A_n) = B_n$.

Section 6.2 is devoted to the study of topological conjugation of the maps $f$, given by (2.2) and the maps $f_v : [0, 1] \rightarrow [0, 1]$, which is dependent on the parameter $v \in (0, 1)$ and is given by formula
\[ f_v(x) = \begin{cases} \frac{x}{v}, & x \leq v; \\ \frac{1-x}{1-v}, & x > v. \end{cases} \quad (2.4) \]
In other words, we find the homeomorphism $h : [0, 1] \rightarrow [0, 1]$, which satisfy the functional equation
\[ h(f) = f_v(h). \quad (2.5) \]
We prove the following theorem.
Theorem (Theorem 6.5, see also Sect. 2.2 in [17]). For every \( v \in (0, 1) \) the functional equation (2.5) has a solution in the class of homeomorphisms \( h : [0, 1] \to [0, 1] \). Furthermore, this solution is the unique and it increase.

In Section 6.3 we prove the following theorem.

Theorem (Theorem 6.6). Let function \( f : [0, 1] \to [0, 1] \) be given by formula (2.2). For every \( x_0 \in [0, 1] \) and every \( \varepsilon > 0 \) there is a maps \( g : [0, 1] \to [0, 1] \) with the following properties:

1. \( g \) is unimodal;
2. \( g(x) = f(x) \) for every \( x \in [0, 1] \setminus (x_0 - \varepsilon, x_0 + \varepsilon) \);
3. \( f \) and \( g \) are not topologically conjugated.

Section 7 is devoted to the differentiability of the homeomorphism \( h : [0, 1] \to [0, 1] \), which satisfies the functional equation (2.5).

This study is motivated by Theorem 7.1.

Theorem (Theorem 7.1, also Proposition 2 at [64]). The derivative of the homeomorphism \( h : [0, 1] \to [0, 1] \), which is the solutions of the functional equation (2.5), equals 0 almost everywhere and the unique its finite values is 0.

Theorem 7.1 is formulated in [64, Proposition 2] a bit differently: the derivative of \( h : [0, 1] \to [0, 1] \), which is a solutions of (2.5), exists almost everywhere and equals 0 everywhere, where it exists. Nevertheless, it follows from the proof in [64], that authors mean that existing of the derivative is also its finiteness. Following [21, chap. 92, 101], we will assume that function is differentiable at a point if and only if the limit of ratios of its grows over the grows of the argument exists. We will not additionally assume that this limit is finite. In the same time, Theorem 7.1 mans only in that the derivative \( h'(x) \) can not be finite except 0. Another part of the Theorem follows from the Lebesgue theorem about the derivative of monotone function.

Theorem (Theorem 7.2, also Lebesgue Theorem (see [35], or §1.2 in [58])). Every monotone on the interval function has finite derivative almost everywhere on this interval.

Denote elements of \( A_n \) by \( \alpha_{n,k} \) and assume that \( \alpha_{n,k_1} < \alpha_{n,k_2} \) for all \( k_1 < k_2 \). By Proposition 6.3 \( \alpha_{n,k} = \frac{1}{2^n - 1} \). Also denote elements of \( B_n \) by \( \beta_{n,k} \) and assume that \( \beta_{n,k_1} < \beta_{n,k_2} \) for
all $k_1 < k_2$. For every $n \geq 1$ denote by $h_n$ the piecewise linear maps such that all its breaking points belong to $A_n$ and for every $k$, $0 \leq k \leq 2^{n-1}$ the equality
\[
h_n(\alpha_{n,k}) = \beta_{n,k}
\]
holds. We find limits of derivatives $h_n'(x)$ for $x \in (0, 1) \setminus A$ at Section \[7.1\]. Let binary decomposition of $x$ be
\[
x = 0, x_1 x_2 \ldots x_k \ldots .
\]
(2.6)
For a number $x$ of the form (2.6) denote by $x_0 = 0$ and for every $i \geq 2$ denote
\[
\alpha_i(x) = \begin{cases} 
2v & \text{if } x_{i-1} = x_{i-2}, \\
2(1 - v) & \text{if } x_{i-1} \neq x_{i-2}.
\end{cases}
\]
(2.7)
We prove the following theorems

**Theorem** (Theorem \[7.3\] also Lema 7 in \[52\]). For every $n \geq 2$ and $x \not\in A_n$ of the form (7.2) the equality
\[
h_n'(x) = \prod_{i=2}^{n} \alpha_i(x)
\]
holds, where $\alpha_i(x)$ is defined by (7.3).

**Theorem** (Theorem \[7.4\] also Lema 14 in \[52\]). 1. If $v < 1/2$ then for every $x \in A$ the limit $\lim_{n \to \infty} h_n'(x) = 0$ holds.

2. If $v > 1/2$ then for every $x \in A$ the limit $\lim_{n \to \infty} h_n'(x) = \infty$ holds.

3. For every $v \in (0, 1) \setminus \{1/2\}$ limits $\lim_{n \to \infty} \min_{x \in (0, 1) \setminus A} h_n'(x) = 0$ and $\lim_{n \to \infty} \max_{x \in (0, 1) \setminus A} h_n'(x) = \infty$ hold.

The following observation follows from Theorems \[7.1\] and \[7.4\]. Let $\lambda$ be the Lebesgue measure on the interval $[0, 1]$. Denote by $\mathcal{A}^0$ the set, where the derivative of $h$ equals to 0 and denote by $\mathcal{B}$ the set, where the derivative of $h$ equals to infinity. Then $\lambda(\mathcal{A}^0) = 1$, $\lambda(\mathcal{B}) = 0$ and $h$ is non-differentiable on $[0, 1] \setminus (\mathcal{A}^0 \cup \mathcal{B})$. It is evident, that the derivative of the inverse function $h^{-1}$ equals $\infty$ on $h(\mathcal{A}^0)$ and this derivative equals 0 on $h(\mathcal{B})$. Now, it follows from Theorems \[7.1\] and \[7.4\] that $\lambda(h(\mathcal{A}^0)) = 0$ and $\lambda(h(\mathcal{B})) = 1$. These properties of $h$ show how complicated it is.
Section 7.2 is devoted to values of the homeomorphism \( h : [0, 1] \rightarrow [0, 1] \), which satisfies (2.5). We give the proof of Theorem 7.1 as a corollary of Theorem 7.3. Also we prove the following theorem.

**Theorem** (Theorem 7.5, also Lemmas 15-16 in [52] and Theorem 2 in [54]). Let \( x_0 \in [0, 1] \cap \mathbb{Q} \). Then the derivative \( h'(x_0) \) exists. More then this, if \( v < 1/2 \) the \( h'(x_0) = \infty \) and if \( v > 1/2 \) then \( h'(x_0) = 0 \).

Theorem 7.5 was proved in [52] for \( x_0 \in A \) and later generalized at [54] for the case \( x \in \mathbb{Q} \). Theorem 7.5 can be considered as generalization of Theorem 7.4 to rational points set. It follows from Theorem 7.1 that Theorem 7.4 can not be generalized to all all \( x_0 \in [0, 1] \) such that limit \( \lim_{n \to \infty} h_n'(x_0) \) exists. The following statement holds.

**Proposition** (Proposition 7.6). For every \( v \neq 1/2 \) there exists \( x_0 \in [0, 1] \) such that \( h'(x_0) = 0 \) and one of the following statements holds:

1. the limit \( \lim_{n \to \infty} h'_n(x_0) \) does not exists;
2. the limit \( \lim_{n \to \infty} h'_n(x_0) \) exists, but equals infinity \( \infty \).

We construct at Section 8 the formula in terms of electronic tables, which let to find values at \( A_{n+1} \) of the homeomorphic solution of 8. We reformulate Proposition 6.2 as follows.

**Proposition** (Proposition 8.1). Let \( h : [0, 1] \rightarrow [0, 1] \) be the conjugation of maps \( f \) and \( f_v \), which are defined by (2.2) and (2.4) i.e. \( h \) is a solution of the functional equation (2.5). Then the following implications hold.

1. If \( x = 0 \), then \( h(x) = 0 \).
2. If \( x = 1 \), then \( h(x) = 1 \).
3. If \( x \leq \frac{1}{2} \), then
   \[ h(x) = v \cdot h(2x) \]
   and the value \( h(2x) \) appears to be defined earlier.
4. If \( x > \frac{1}{2} \), then
   \[ h(x) = 1 - (1 - v) \cdot h(-2x + 2) \]
   and the value \( h(-2x + 2) \) appears to be defined earlier.
Proposition 8.1 let us to prove the following theorem.

**Theorem** (Theorem 8.2 also Sect. 3.1 in [55]). The value of conjugation $h$ of maps $f$ and $f_v$, which are defined by (2.2) and (2.4) at the set $A_{n+1}$ can be found via the following way:

- Put $n$ into “C1”.
- Put $v$ into “D1”.
- Put 0 into “A1” and the formula $A1+1/(2^C1)$ into “A2”.
- Put the following formula into “B1”:

$$
IF(A1=0; 0; IF(A1=1, 1; IF(A1¡=0,5;
D1^INDIRECT( CONCATENATE("B"; 2*A1*2^C1+1));
1-(1-D1)^INDIRECT( CONCATENATE("B"; -2*A1*2^C1 +1 +2^(1+C1))))))
$$

Copy the values in columns A and B down till the line number $2^n + 1$.

The Figure 4 is prepared with the use of the formula from Theorem 8.2. It contains the values of the conjugation in $A_7$, which correspond to $v = 0.55, 0.6, 0.7, 0.75, 0.8, 0.85, 0.9$ and $0.95$.

Smaller values of $v$ correspond to “graphs”, which are closer to $y = x$. All points of graphs are obtained via electronic tables. It is clear from the picture, that maps, which correspond to $v = 0.55, 0.6, 0.65$ are indeed continuous. Nevertheless, the continuity of the maps, which corresponds to $v = 0.95$ is not so evident.

We reduce in Section 9 the problem of the conjugation of $f$ and $f_v$, which are defined by formulas (2.2) and (2.4) to the solving of a system of functional equations. We prove the following theorem in Section 9.1 we prove the following theorem.

**Theorem** (Theorem 9.1). The system of functional equations

$$
\begin{align*}
&h(2x) = \frac{1}{v} h(x) \quad x \leq 1/2 \quad (2.8a) \\
&h(2-2x) = \frac{1-h(x)}{1-v} \quad x > 1/2 \quad (2.8b)
\end{align*}
$$

has the unique solution $h : [0, 1] \rightarrow [0, 1]$ and it is the solution of functional equation (2.3) i.e. it is a conjugation of maps $f$ and $f_v$, which are defined by formulas (2.2) and (2.4).
We prove the following property of the equation (2.10).

Section 9.2 is devoted to the study of the properties of solutions of (2.8). Section 9.2.1 contains a general methods of solving of linear functional equations. These methods are described in details in [47].

We try to use the general methods of solving linear functional equations, which are presented in Section 9.2.2 and to write the general solution of (2.8). The general solution of the functional equation (2.8) is

\[ h(x) = x^{-\log_2 v} \omega_1(\log_2 x), \quad \text{(2.9)} \]

where \( \omega_1(x) \) is an arbitrary function with period 1. If we plug the function \( h \) of the form (2.9) into the functional equation (2.8), then obtain

\[ (1 - v)(1 - x)^{-\log_2 v} \omega_1(\log_2(1 - x)) = \]

\[ = v(1 - x^{-\log_2 v} \omega_1(\log_2 x)). \quad \text{(2.10)} \]

We prove the following property of the equation (2.10).

Fig. 4: Approximation \( h_7 \) for different values of \( v \)
Proposition (Proposition 9.2). If consider the functional equation (2.10) as given on the whole real line, the unknown function $h$ appears to be constant.

Proposition 9.2 does not contradict to Theorem 6.5 because of the following remark.

Remark (Remark 9.4). If one solve the equation (2.10) for the unknown function $h : [0, 1] \to [0, 1]$, defined by formula (2.9) then reasonings from the proof of the Proposition 9.2 would be incorrect.

Also it is obtained un Section 9.2.1 that the solution of the equation (2.8b) is of the form

$$h(x) = \frac{1}{2-v} + |x - \frac{2}{3}|^{-\log_2(1-v)} \times$$

\[\left\{ \begin{array}{ll}
\omega^+(\log_2 |x - \frac{2}{3}|) & x > \frac{2}{3}; \\
\omega^-(\log_2 |x - \frac{2}{3}|) & x < \frac{2}{3}.
\end{array} \right.\] (2.11)

for functions $\omega^+$ and $\omega^-$, which satisfy the following relations:

\[\left\{ \begin{array}{l}
\omega^-(t + 1) = -\omega^+(t) \\
\omega^+(t + 1) = -\omega^-(t).
\end{array} \right.\] (2.12)

It follows from restrictions (2.12) that functions $\omega^+$ and $\omega^-$ are periodical with period 2. If one plug (2.11) into the functional equation (2.8a), then obtain

$$\frac{v}{2-v} + v |2x - \frac{2}{3}|^{-\log_2(1-v)} \times$$

\[\left\{ \begin{array}{ll}
\omega^+(\log_2 |2x - \frac{1}{3}|) & x > \frac{2}{3}; \\
\omega^-(\log_2 |2x - \frac{1}{3}|) & x < \frac{2}{3}.
\end{array} \right. =\] (2.13)

$$\frac{1}{2-v} + |x - \frac{2}{3}|^{-\log_2(1-v)} \times$$

\[\left\{ \begin{array}{ll}
\omega^+(\log_2 |x - \frac{2}{3}|) & x > \frac{2}{3}; \\
\omega^-(\log_2 |x - \frac{2}{3}|) & x < \frac{2}{3}.
\end{array} \right.\]

and the unknown functions $\omega^+$ and $\omega^-$ would satisfy the relations (2.12). Functional equations (2.10) and (2.13) are not linear and standard methods of solving of functional equations.
can not be applied to (2.10) and (2.13). The complicatedness of equations (2.10) and (2.13) can be explained by the properties of the conjugation \( h \), which are stated by Theorems 7.1 and 7.5. We use in the Section 9.3 the numerical values of \( h : [0, 1] \to [0, 1] \), which is a solutions of the functional equation (2.5). These values are found by the formula from Theorem 8.2. We use these values for studying the properties of the function \( \omega_1 \) from the formula (2.9) and the function

\[
\omega_2(x) = \begin{cases} 
\omega^+ \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x > \frac{2}{3}; \\
\omega^- \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x < \frac{2}{3},
\end{cases}
\]

which in fact appears in the formula (2.11). In other words, \( \omega_2(x) \) is obtained from the equality

\[
h(x) = \frac{1}{2 - v} + \left| x - \frac{2}{3} \right|^{\log_2 (1 - v)} \cdot \omega_2(x).
\]

Also remind that it still follows from (2.12) that \( \omega_2 \) is periodical with period 2. The Figure 5 contains the graph of \( \omega_1(x) \) for \( v = 3/4 \) and \( x \in [0, 1] \). The interval \([0, 1]\) is taken since \( \omega_1 \) is periodical with period 1.

Fig. 5: The graph of \( \omega_1(x) \) for \( v = 3/4 \)

The figure 6 contains the graph of \( \omega_2(x) \) for \( v = 3/4 \) and \( x \in [0, 2] \). The interval \([0, 2]\) is take, since \( \omega_2 \) is periodical with period 2.

Consider examples of “simple” maps \( \omega \) but such that the function \( h \), which is defined by (2.9)
Fig. 6: The graph of $\omega_2(x)$ for $v = 3/4$
is invertible and consider the maps $\tilde{f}_v$, which is defined by commutativity of the diagram

$$
\begin{array}{c}
[0, 1] \xrightarrow{f} [0, 1] \\
\downarrow h \downarrow h \\
[0, 1] \xrightarrow{\tilde{f}_v} [0, 1].
\end{array}
$$

(2.14)

We notice, that if for the invertible maps $h$ of the form (2.9) the diagram (2.14) is commutative, then for $x \in [0, v]$ the equality

$$
\tilde{f}_v(x) = \frac{x}{v}
$$

holds. The simplest case for $\omega_1$ is that when it is constant. The maps $\tilde{f}_v$ for $v = 3/4$, which is defined by (2.14) for the maps $h$ with constant $\omega_1$ is given at Figure 7.

![Fig. 7: Graph of $\tilde{f}_v$ for $v = 3/4$](image_url)

The Figure 8 contains the graph of $\tilde{f}_v$ for $v = 3/4$, which is defined by (2.14) for the maps $h$, defined by (2.9) if $\omega_1(\log_2 x)$ is continuous, whose graph at $[1/2, 1]$ is consisted of two parts of linearity. We construct $\omega_1(\log_2 x)$ in such a way that $h(3/4)$ be equal to the right pre-image of $v$ under the action of $f_v$. Notice, that if $x$ runs through $[1/2, 1]$, then $\log_2 x$ runs through the interval $[-1, 0]$, whose length is 1, which is the interval of periodicity of $\omega_1$.

These examples may be generalized as follows. Take an arbitrary $n$ and use the maps $h_n$, which moves points of $A_n$ to $B_n$ to find the values of $\omega$ on the set $A_n \cap [1/2, 1]$ such that equality $\tilde{f}_v(A_n) = B_n$ hold. Then consider $\omega$ to be linear at all other points and periodical with period 1. Consider the maps

$$
\tilde{h}_n = x^{-\log_2 v} \omega_n(\log_2 x)
$$

(2.15)
as an approximation of $h$. If the constructed $\tilde{h}_n$ would be invertible, then there exists the unique $\tilde{f}_n$, such that diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\tilde{h}_n & \downarrow & \tilde{h}_n \\
B & \xrightarrow{\tilde{f}_n} & B
\end{array}
\]  \hspace{1cm} (2.16)

would be commutative. This $\tilde{f}_n$ can be given via

$$\tilde{f}_n = \tilde{h}_n(f(\tilde{h}_n^{-1})).$$

For example, with the use of these notations the Figure 8 contains the graph $\tilde{f}_2$. Nevertheless, it is possible, that $\tilde{h}_n$ would not be invertible. Then there will not be $\tilde{f}_n$ such that diagram (2.14) would be commutative. Maps $\tilde{h}_n$ are not monotone for $v \approx 0$. Their graphs for different $v$ are given at Figure 9.

Notice as a comment to the Figure 9 that all these maps satisfy the functional equation (2.8a)

$$h(2x) = \frac{1}{v} h(x),$$

i.e. $h$ repeats its graph on each interval of the form $\left[\frac{1}{2^k+1}, \frac{1}{2^k}\right]$, but the graph is $v^k$-times compressed. We prove the following proposition.

**Proposition** (Proposition 9.3). For every $n \in \mathbb{N}$ there exists $v_0 \in (0, 1)$ such that for every $v \in (0, v_0)$ the maps $\tilde{h}_n(x)$ is non-monotone on $\left[\frac{2^{n-1}-1}{2^{n-1}}, 1\right]$.

Now define in the previous way the solution $h$, which is determined by (2.11) and is obtained from (2.8b). Graph of $\tilde{f}_v$, which is determined by commutative diagram (2.14) for the maps $h$ of the form (2.11), if $\omega_2$ is constant is given at Figure 10.
Fig. 9: Graphs of $\tilde{h}_3$ for different $v$

Fig. 10: Graph of $\tilde{f}_v$ for $v = \frac{3}{4}$
Let $\omega^+$ and $\omega^-$ be continuous, piecewise linear with the smallest possible braking points such that $h(1/4) = v^2$. Then the graph of $\tilde{f}_v$, which is defined by diagram (2.14) for the maps $h$ of the form (2.11) is given at Figure 11.

![Fig. 11: Graph of $\tilde{f}_v$ for $v = \frac{3}{4}$](image)

The Figure 12 the result putting the graph from the previous example the the just constructed one (the dots are used for the first graph).

![Fig. 12: The putting of one graph onto another](image)

Section 10 is devoted to explicit formulas of the homeomorphic solution $h : [0, 1] \rightarrow [0, 1]$ of the equation (2.5). In Section 10.1 we prove the following theorem.

**Theorem** (Theorem 10.1 also Sect. 3.2 in [55]). The homeomorphic solution $h : [0, 1] \rightarrow [0, 1]$ of the equation (2.7) can be expressed by formula

$$h(x) = \lim_{n \to \infty} \beta([2^n x], n) = \lim_{n \to \infty} \sum_{t=1}^{[2^n x]} \frac{1}{\zeta_{n,t}}.$$
where
\[
\zeta_{n,k} = \prod_{t=1}^{n} \left( \frac{2^{k}}{2^{n} - t} \right) + \left\{ \left\{ \frac{k}{2^{n} - t + 1} \right\} \right\} + 2 \cdot \left\{ \left\{ \frac{k}{2^{n} - t} \right\} \right\}.
\] (2.17)

**Remark (Remark 10.1).** In spite of the formula for \( h(x) \) is quit complicated and contains a limit, the value of \( h \) is defined at any point. The existence of the limit follows from that the formula is obtained from the same reasonings, which where made in the proof of Theorem 6.5.

**Remark (Remark 10.2).** Notice, that the formula from Theorem 10.1 has the following properties:

1. In the case when \( x \in A \) the finding of \( h(x) \) via this formula leads to \( n \) summand for the approximation \([2^n x] \in A_n\). The case \( x \in A \) means that “approximated” values of \( h(x) \) stabilize after finite number of steps on the exact value.

2. The stabilization via the obtained formula, i.e. the equality
\[
\sum_{t=1}^{[2^n x]} \frac{1}{\zeta_{n,t}} = \sum_{t=1}^{[2^{n+1} x]} \frac{1}{\zeta_{n+1,t}}
\]
for some \( n \in \mathbb{N} \) means that \( x \in A_n \) and the obtained approximation is the exact value of \( h(x) \).

The first of mentioned properties from the Remark 10.2 can be considered as a disadvantage of the formula. We obtain in Section 10.2 another formula for \( \beta_{n,k} \), which is free of this disadvantage. We prove the following theorem.

**Theorem (Theorem 10.2 also Section 3.3 in [55]).** The homeomorphic solution \( h : [0, 1] \to [0, 1] \) of the equation (2.5) can be expressed by the following formula
\[
h(x) = \sum_{i=1}^{\infty} \frac{(2^i - 1)x((-1)^{[-\log_2(2^i x)]} - 1)}{\zeta_{i, [2^i - \log_2(2^i + 1)] + 1, x}},
\]
where \( \zeta_{n,k} \) is expressed by (2.17).

**Remark (Remark 10.3).** Notice, that if \( x \in A \) then the formula from Theorem 10.2 contains only finitely many summands.
We consider in Section 11 the conjugation of maps $f : [0, 1] \to [0, 1]$ of the form (2.2) and a continuous maps $g : [0, 1] \to [0, 1]$ of the form

$$g(x) = \begin{cases} g_l, & \text{if } 0 \leq x < v, \\ g_r, & \text{if } v \leq x \leq 1, \end{cases} \tag{2.18}$$

where $g_l(0) = g_r(1) = 0, g_r(v) = 1$ and functions $g_l$ and $g_r$ are monotone and piecewise linear. We study the conditions of the existence and properties of the homeomorphism $h : [0, 1] \to [0, 1]$ (if it exists), which is the solution of the functional equation

$$h(f) = g(h). \tag{2.19}$$

In Section 11.1 we prove the following theorem.

**Theorem** (Theorem 11.1, also Theorem 1 in [32]). Let the maps $f$, which is given by (2.2), be topologically conjugated a piecewise linear $g$ which maps $[0, 1]$ onto itself and let $h$ be a homeomorphism such that equality (2.19) holds. If $h$ is continuously differentiable on $(\alpha, \beta)$ for some $0 \leq \alpha < \beta \leq 1$, then $h$ is piecewise linear.

The Proposition 3.5 is an example of restrictions for $g$, which should be satisfied, if it is conjugated to $f$ via piecewise linear homeomorphism. The following proposition can be considered as one more restriction for $g$

**Proposition** (Proposition 11.2). If a maps $g : [0, 1] \to [0, 1]$ of the form (2.18) is topologically conjugated to $f : [0, 1] \to [0, 1]$ of the form (2.2), then $g$ only two fixed points: one of which is 0 and the other belongs to $(v, 1)$.

Proposition 11.2 does not assume that conjugation is piecewise linear, but this Proposition is used for the results of the next subsection. In Section 11.2 we consider the conjugation of $f : [0, 1] \to [0, 1]$ of the form (2.2) and the maps $g : [0, 1] \to [0, 1]$ of the form (2.18) via piecewise linear homeomorphism $h : [0, 1] \to [0, 1]$, which satisfy the functional equation (2.19). The main result of the Section 11.2 is the following two theorems.

**Theorem** (Theorem 11.3 also Theorem 2 in [32]). For an arbitrary $v \in (0, 1)$ and an increasing piecewise linear maps $g : [0, v] \to [0, 1]$ such that $g(0) = 0, g(v) = 1$ and $g'(0) = 2$ there exists
and the unique its continuation $\tilde{g} : [0, 1] \to [0, 1]$, which is topologically conjugated to $f$ of the form (2.2) via piecewise linear homeomorphism.

**Theorem** (Theorem 11.4 also Theorem 3 in [32]). For arbitrary $v \in (0, 1)$ and decreasing $g : [v, 1] \mapsto [0, 1]$ such that $g(v) = 1$, $g(1) = 0$ and $(g^2)'(x_0) = 4$, where $x_0$ is a fixed points of $g$, there exists and the unique continuation of $\tilde{g}$ to $[0, 1]$, which is topologically conjugated to $f$ of the form (2.2) via piecewise linear homeomorphism.

In the proof of Theorem 11.4 the following analogue of Proposition 3.5 is used.

**Proposition** (Proposition 11.5 also Lemma 13 in [32]). Let $x_0$ be a fixed point of piecewise linear unimodal maps $g$, which is conjugated with $f$ of the form (2.2) via piecewise linear homeomorphism. Then there exists $\varepsilon > 0$ such that for every $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}$ the equality $(g^2)'(x_0) = 4$ holds, where $g^2$ as in general means the second iteration of $g$.

It is assumed in Theorem 6.1 that the function $g$ of the form (2.18), which is conjugated to $f$ of the from (2.2), is convex. Nevertheless, convexity is not used in the proof of this Theorem. In Section 11.3 we will use in details the techniques from the proof of Theorems 11.3 and 11.4 for obtaining the example of non-convex $g$, which is conjugated to $f$. The graph of this $g$ is given at Figure 13.

![Fig. 13: Graph of $g$](image)

We consider in Section 11.4 the types of linearity of the piecewise linear $g$ of the form (2.18), which is conjugated to $f$ of the from (2.2) via piecewise linear $h : [0, 1] \to [0, 1]$. 23
**Definition** (Definition 11.1). Let for the piecewise linear $g$ of the form (11.2) be topologically conjugated to $f$ of the form (11.1). Let the number of pieces of linearity of $g_1$ and $g_2$ be $p$ and $q$ correspondingly. Call the pair $(p, q)$ the **type of piecewise linearity** of $g$.

**Definition** (Definition 11.2). If for a pair $(p, q)$ there exists a maps $g$ of the form (11.2) with type of piecewise linearity $(p, q)$, then call this type **admissible**. If follows from Proposition 3.5 and 11.5 that pair $(1, q)$ is admissible only if $q = 1$ and the pair $(p, 1)$ is admissible only if $p = 1$. In both cases the equality $g = f$ holds.

We prove the following theorem in Section 11.4

**Theorem** (Theorem 11.6). 1. For any $p \geq 2$ and $q \geq 2$ the type of linearity $(p, q)$ is admissible.

2. A type of linearity $(p, 1)$ and $(1, q)$ is admissible only if it is $(1, 1)$. In this case the maps $g$ coincides with $f$.

In Section 12 we consider the problem of semi conjugation of maps $f$ and $f_v$, which are given by formulas (2.2) and (2.4). Precisely, we consider continuous, but not necessary invertible solutions $\eta : [0, 1] \rightarrow [0, 1]$ of the functional equation

$$\eta(f) = f_v(\eta).$$  \hspace{1cm} (2.20)

In Section 12.1 we prove the following proposition.

**Proposition** (Proposition 12.1). Each continuous monotone solution $\eta : [0, 1] \rightarrow [0, 1]$ of the functional equation (2.20) is a conjugation of $f$ and $f_v$ of the form (2.2) and (2.4).

Let $h : [0, 1] \rightarrow [0, 1]$ be the homeomorphism, which satisfies (2.5) and $\eta : [0, 1] \rightarrow [0, 1]$ be an arbitrary continuous solution of the equation (2.20). Consider the commutative diagram

$$\begin{array}{c}
[0, 1] \xrightarrow{f} [0, 1] \\
\downarrow \eta \downarrow \quad \downarrow \eta \\
[0, 1] \xrightarrow{f_v} [0, 1] \\
\downarrow h^{-1} \downarrow \downarrow h^{-1} \\
[0, 1] \xrightarrow{f} [0, 1]
\end{array}$$  \hspace{1cm} (2.21)
and denote $\xi = h^{-1}(\eta)$. We notice in Section 12.2 that it follows from the commutativity of (2.21) that the problem of finding continuous solutions of (2.20) is equivalent to the problem of finding continuous solutions of the functional equation

$$\xi(f) = f(\xi).$$

(2.22)

Some continuous solutions of the equation (2.22) can be easily found, which is done in the Remark 12.1.

**Remark** (Remark 12.1). The following functions $\xi$ satisfy the functional equation (2.22).

1. $\xi(x) = x$ for all $x \in [0, 1]$;
2. $\xi$ is a constant, which is one of fixed points of $f$;
3. $\xi$ is an arbitrary iteration of $f$.

The following two propositions show that if the solution of (2.22) is in some sense “good” on a subinterval of $[0, 1]$ then it is piecewise “good” on the whole $[0, 1]$.

**Proposition** (Proposition 12.2). If the continuous solution $\xi$ of the functional equation (2.22) is constant on some interval $M = [\alpha, \beta]$, then it is piecewise constant on $[0, 1]$.

**Proposition** (Proposition 12.3). If the graph of the continuous solution $\xi$ of the functional equation (2.22) is a line segment on some set $M = [\alpha, \beta]$ then $\xi$ is piecewise linear on $[0, 1]$.

We consider in Section 12.3 the continuous solutions $\xi : [0, 1] \rightarrow [0, 1]$ of the functional equation (2.22). The main result of this Section is the following theorem.

**Theorem** (Theorem 12.4). Each continuous solution $\xi : [0, 1] \rightarrow [0, 1]$ of the equation (2.22) is piecewise linear.

Section 12.4 is devoted to the properties of the piecewise linear solution $\xi : [0, 1] \rightarrow [0, 1]$ of the functional equation (2.22). We prove the following facts.

**Proposition** (Proposition 12.5). If a continuous solution $\xi$ of the functional equation (2.22) is monotone on some $M = [\alpha, \beta]$ then $\xi$ is piecewise monotone on $[0, 1]$. 25
**Theorem (Theorem 12.6).** If the function $\xi : [0, 1] \to [0, 1]$, which is a solution of a functional equation (2.22), is monotone on an interval $M \subset [0, 1]$, then $\xi$ is piecewise linear on $[0, 1]$.

Theorems 12.4 and 12.6 make natural the consideration of piecewise linear solutions $\xi : [0, 1] \to [0, 1]$ of the functional equation (2.22). Such its solutions are considered in Section 12.5.

We prove the following theorem there.

**Theorem (Theorem 12.7).** Let $\xi$ be piecewise linear solution of the functional equation (12.5). Then $\xi$ is one of the following functions

1. $\xi(x) = x_0$ for all $x \in [0, 1]$, where $x_0 = 0$, or $x_0 = 2/3$;
2. for some $k \in \mathbb{N}$

$$
\xi(x) = \frac{1 - (-1)^{[kx]}}{2} + (-1)^{[kx]}\{kx\},
$$

where $\{\cdot\}$ denotes fractional part and $[\cdot]$ denotes integer part. More then this for any $k \in \mathbb{N}$ the function $\xi(x)$ of the form above is a solution of (12.5).

In Section 13 we consider the length of the graph of the homeomorphic solution $h : [0, 1] \to [0, 1]$ of the functional equation (2.5). In subsection 13.1 we prove the following Theorem.

**Theorem (Theorem 13.1).** Let $h : [0, 1] \to [0, 1]$ be a conjugation of $f$ and $f_v$, defined by (2.2) and (2.4) correspondingly, i.e. $h$ is a solution of (2.5).

Let $h_n$ are piecewise linear approximations of $h$, whose breaking points belong to $A_n$ and which coincide with $h$ on $A_n$. Define $l_n$ the length of the graph of $h_n$. Then

$$
\lim_{n \to \infty} l_n = 2
$$

for all $v \neq 1/2$.

From another hand, we use Theorem 7.3 in Subsection 13.1 to find the explicit formulas of $l_n$ and prove the following proposition.

**Proposition (Proposition 13.2).** The following equality holds

$$
l_{n+1}(v) = \frac{1}{2^n} \cdot \sum_{k=0}^{n} C_n^k \cdot \sqrt{1 + 2^{2n}v^{2k}(1 - v)^{2(n-k)}}
$$

(2.23)

for the length $l_{n+1}(v)$ of $h_{n+1}$.
Proposition 13.2 together with Theorem 13.1 give the following corollary.

Theorem (Theorem 13.3). For every $v \in (0, 1)\setminus\{0.5\}$ the limit $\lim_{n \to \infty} l_n(v) = 2$ holds, where $l_n(v)$ are defined by (2.23).

In Section 13.2 we prove the Proposition 13.2 as a combinatorial fact with the use of probability reasonings. This is, in fact, an alternative proof of Theorem 13.1.

In Section 14 we consider maps $\xi: A_n \to [0, 1]$ for a given $n \geq 1$. We call such $\xi$ admissible, if the equality

$$\xi(f(x)) = f(\xi(x))$$

for all $x \in A_n$ and $f$ is of the form (2.2). Notice, that since $f(A_n) \subseteq A_n$ for all $n$, then (2.24) is defined for all $x \in A_n$. In Section 14.1 we prove the following theorem.

Theorem (Theorem 14.1). There is one to one correspondence between admissible self-semi conjugations $\xi: A_n \to [0, 1]$ and maps $\tilde{\xi}: \bigcup_{i=1}^{n} B_i \to \bigcup_{i=1}^{n} B_i$ with the following properties:

1. For any $m$, $1 \leq m \leq n$, the inclusion $\tilde{\xi}(B_m) \subseteq B_m$ holds.
2. For any $m$, $2 \leq m \leq n$ the equality

$$\tilde{\xi}(j_1, \ldots, j_{m-1}) = (i_1, \ldots, i_{m-1}).$$

holds for some $i_1, \ldots, i_{n}, j_1, \ldots, j_n$, then for any $k$, $1 \leq k \leq n$ the equality $i_k = 0$ yields $j_k = i_0$, where $i_0 \in \{0, 1\}$ is a fixed number.

Corollary (Corollary 14.1). For any $n \geq 1$ the number of admissible self-semi conjugations $\xi: A_n \to [0, 1]$ is

$$\sum_{k=0}^{n} 2^{k+1} \cdot C_n^k.$$
In Section 14.2 we consider continuable maps $\xi_n : A_n \to [0, 1]$. We call the maps $\xi_n : A_n \to [0, 1]$ **continuable**, if there is a self-semiconjugation $h : [0, 1] \to [0, 1]$ of $f$ (i.e. $h$ is surjective and continuous, but not necessary invertible), which coincides with $\xi_n$ on $A_n$. In this case the maps $h$ can be considered as continuous surjective continuation of $\xi_n$. By Theorems 12.4 and 12.7 if $\xi$ is continuable, then either $\xi(x) = 2/3$ for all $x \in A_n$, or $\xi(0) = 0$. From the definition of admissible maps and from the definition of $A_n$ obtain that $\xi(0) = 0$ yields $\xi(A_n) \subseteq A_n$. We prove the following theorem.

**Theorem** (Theorem 14.2). 1. For every $x \in A_n \setminus A_{n-1}$ and for every $y \in A_n$ there exists a continuable $\xi : A_n \to A_n$ such that $\xi(x) = y$.

2. Let $\xi_1, \xi_2 : A_n \to A_n$ be continuable self-semiconjugations of $f$ of the form (2.2) and $\xi_1(x) = \xi_2(x)$ for some $x \in A_n \setminus A_{n-1}$. Then $\xi_1(x) = \xi_2(x)$ for all $x \in A_n$.

**Corollary** (Corollary 14.2). For every $n \geq 1$ there are $2^n - 1$ continuable self-semi conjugations of $f$ of the form (2.2).
3 Introduction

3.1 The main definitions

As integer numbers in arithmetics is a mathematical tool of the description of some objects, the dynamical system in dynamical systems theory is a tool of the description of those objects which are changed dependently on time. As an example of dynamical systems is an experiment in Newtonian mechanics. When a system of points is not under acting of any forces and velocities and coordinates of all the points are known, then they (i.e. coordinates and velocities of all points) can be found at any other time. Dynamical systems have the following property. The further extension of events in dynamical system is dependent only on its state and independent on external factors, for example is independent on previous states of a system or ways with which the system has come to its now state. The properties mentioned above can be easily be formalized (translated into mathematical terminology), which let give the strict definition. Let $X$ be a set of possible states of a system (i.e. in Newtonian mechanics it is a set of arrays of the length 6 with triples of point coordinates triples of its velocities coordinates). Let $T$ be a set of possible values of a variable “time” (for example $\mathbb{R}^+$) but in any way $T$ is a semi group i.e. its elements can be added one to another. Consider a maps

$$f : X \times T \to X,$$  \hspace{1cm} (3.1)

whose acting is that an state $f(x, t)$ of a system is that its state, where is comes after the time $t$ if $x$ is its the former state. Maps $f$ is obviously have some obvious properties: 1) For every $x \in X$ the equality $f(x, 0) = x$ hold i.e. during the time 0 the state of the the system does not change. 2) If the system which at the very beginning (at the time 0) was at the state $x_0$ and after the time $t_1$ it appeared itself in the state $x_1 = f(x, t_1)$ then after the time $t_2$ it will appear in the state $f(x_1, t_2) = f(f(x_0, t_1), t_2)$. Nevertheless the state of the system after the time $t_1$ does not influence the state in which it appeared itself after the time $t_1 + t_2$ after being in the state $x_0$ i.e. fact of defining of the state of the system does not influence its further extension. So, the equality

$$f(x_0, t_1 + t_2) = f(f(x_0, t_1), t_2)$$
Definition 3.1. **Dynamical system** is a triple \((X, T, f)\), where \(X\) is a set, \(T\) is an additive semigroup (i.e. \(0 \in T, \forall s, t \in T, s + t = t + s \in T\)) and \(f\) is a maps which acts \(f : X \times T \to X\) such that the following properties hold:

1) \(f(x, 0) \equiv x, \; \forall x \in X\);

2) \(f(f(x, s), t) = f(x, s + t), \; \forall x \in X, \forall s, t \in T\).

Definition 3.2. Consider a point \(x_0 \in X\) of a phase space. The set \(A \subset X\) is called an **orbit** of a point \(x_0\) if for any point \(a \in A\) there exists a time \(t \in T\) such that after this time the point \(x_0\) goes into the point \(a\), i.e. the equality \(a = f(x_0, t)\) holds.

Definition 3.3. Consider a point \(x_0 \in X\) of a phase space. A function \(N : T \to X\) is called a **trajectory** of the point \(x_0\) if \(N(t) = f(x, t)\).

Note that in the case if dynamical system is a cascade (i.e. if the time may be considered as a natural systems set) then trajectory of a point \(x_0\) is a sequence which is defined with the recurrent equality \(x_{k+1} = f(x_k)\) and the orbit of a point \(x_0\) is the value set of its trajectory.

Definition 3.4. A function \(t \to f(x, t)\) is called a **motion** of a point \(x\), i.e. the motion of a point \(x\) is called a function which for any value of time corresponds the position of \(x\) at this time.

The impotent property of trajectories of phase space points which appears (or not appears) during considering dynamical systems is returning of the point into itself or into its neighborhood (in the case if dynamical system is considered on either metric or topological phase space).

Definition 3.5. Consider a point \(x_0 \in X\) of a phase space. This point is called a **fixed point** of the dynamical system \((X, T, f)\) if for arbitrary \(t \in T\) the equality \(f(x_0, t) = x_0\) holds.

Definition 3.6. Consider a point \(x_0 \in X\) of a phase space. This point is called a **periodic point** of the dynamical system \((X, T, f)\) with period \(t_0\) if for arbitrary \(t \in T\) the equality \(f(x_0, t_0) = x_0\) holds and for any \(t < t_0\) the equality \(f(x_0, t) = x_0\) does not hold.
To make possible the defining the dynamical system it is necessary to demand that elements of the set \( T \) be comparable i.e. that inequality \( t < t_0 \) make sense.

**Definition 3.7.** Consider the maps \( f \) of some set \( A \) into itself. The \( n \)-th **iteration** of \( f \) for arbitrary non negative integer \( n \) is called the maps \( f^n(x) \) which is defined as follows: \( f^0(x) = x, \)
\[
f^k(k) = f(f^{k-1}(x)) \quad \text{for arbitrary } k \in \mathbb{N}.
\]

**Definition 3.8.** For a set \( A \) let \( f : A \to A \) be a map. For every \( x_0 \in A \) the sequence \( \{x_n\}, x \geq 0 \) such that \( x_{n+1} = f(x_n) \) for every \( n \geq 0 \) is called the **trajectory** of \( x_0 \) under the action of \( f \).

**Definition 3.9.** Let a map \( f : A \to A \) be defined on a set \( A \). For any \( x_0 \in A \) the set \( \{x_n\}, x \in \mathbb{Z} \) such that \( x_{n+1} = f(x_n) \) for any \( n \in \mathbb{Z} \) is called the **integer trajectory** of \( x_0 \) under \( f \).

For an arbitrary \( n \geq 1 \) denote the \( n \)-th iteration of a maps \( f \) by \( f^n \), i.e.

\[
f^n(x) = f(f(\ldots(f(x))\ldots)). \quad \text{n times}
\]

We will use this notation not only for the maps \( f \), but also for those maps which are denoted in any other way.

### 3.2 The notion about topological equivalence

George Birkhoff is an American mathematician who lived in the first half of 20-th century. He is one of founders of dynamical systems theory and he has formulated the final problem of dynamical systems theory in the following way: “qualitative determine of all possible trajectories types and stating the interconnections between them” (see. [6], p. 194).

The necessity of considering the equivalent maps naturally implies from the final problem the dynamical systems theory stated by G. Birkhoff. For possibility of talking about trajectories types it is necessary to study ourself to point out those systems which are equal prom the point of view of those questions which are stated in the dynamical systems theory and these
questions after all technical moments need to determine all trajectory types and to state the interconnections between them.

For the case of dynamical systems, which are defined on the interval \( I = [a, b] \) the topological conjugation may by considered as the changing of the scope on \( I \). Let a map \( g \) be topologically conjugated to \( f : I \to I \).

Assume, that we have a spring, whose length equals to the length of \( I \) and this spring is graduated, i.e. the numbers, which correspond to \( I \) are written uniformly on the it. Then ends of the spring are fixed, and some its parts are stretched and some are griped without knots, kinks and self intersections, i.e. the spring is sketched and griped without taking away from the line, where it was at the very beginning.

If the sketched and griped spring is graduated again, then we obtain the monotone continuous increasing map \( h : I \to I \). This \( h \) defines the topological conjugation of then maps \( f \) and \( g \). For every “old” point \( x \in I \) the maps \( h \) sets the “new” point \( h(x) \in I \), which corresponds to the new graduation.

Consider the maps \( f \) as not interval \( I \) into itself maps, but the spring into itself maps. Then different graduations of the spring give different interval \( I \to I \) maps, and in these terms, our new graduation defines the maps \( g \), which is conjugated to \( f \) via \( h \).

The notion of topological conjugation can be introduced not only to interval maps, but for every dynamical systems. Assume that \( h \) is a homeomorphism (i.e. continuous and invertible). Nevertheless, interval homeomorphism can be as increasing, as decreasing. Returning to spring as an interpretation of topological conjugation of the interval maps, we should let to change the ends of the spring.

The exact definition of the topological equivalence is following.

**Definition 3.10.** A maps \( f \) of a set \( A \) into itself is called **topologically conjugated** to a maps \( g \) of a set \( B \) into itself if there exists a homeomorphism \( h \) of the set \( A \) into the set \( B \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{h} & & \downarrow{h} \\
B & \xrightarrow{g} & B
\end{array}
\]
Notice, that topological conjugation is an equivalence relation, whence topologically conjugated maps are also called **topologically equivalent**.

**Definition 3.11.** A maps $f$ of a set $A$ into itself is called **topologically semi conjugated** (or **topologically semi equivalent**) to a maps $g$ of a set $B$ into itself if there exists a continuous surjective $h : A \rightarrow B$ such that $[3.2]$ is commutative.

The following theorem holds.

**Theorem 3.1.** For using the notations of the definition $[3.10]$ the maps $f$ if topologically equivalent to the maps $g$. Let $a \in A$ be a fixed point of $f$. The $h(a)$ is a fixed point of $g$.

**Proof.** The commutativity of the diagram from the definition $[3.10]$ yields that $g(h(a)) = h(f(a))$. Taking into attention that $a$ is a fixed point of $f$ obtain that

$$g(h(a)) = h(a),$$

which finishes the proof. 

In the same manner we prove the following theorem.

**Theorem 3.2.** For using the notations of the definition $[3.10]$ the maps $f$ is topologically equivalent to the maps $g$. Let $a \in A$ be a periodic point of the maps $f$ with period $n$. Then $h(a)$ is a periodic point of the maps $g$ with period $n$.

**Theorem 3.3.** For using the notations of the definition $[3.10]$ let the maps $f$ be topologically equivalent to the maps $h$. For each point $a \in A$ the equality of sets

$$h(f^{-1}(a)) = g^{-1}(h(a))$$

holds.

**Proof.** Consider an arbitrary point $x \in f^{-1}(a)$. The conditions of theorem yield that the following diagram is commutative.

$$
\begin{array}{ccc}
  x & \xrightarrow{f} & f(a) \\
  \downarrow{h} & & \downarrow{h} \\
  h(x) & \xrightarrow{g} & h(a)
\end{array}
$$

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From the commutativity of the diagram obtain that \( h(x) \in g^{-1}(h(a)) \) and this inclusion gives the following sets inclusion

\[
h(f^{-1}(a)) \subseteq g^{-1}(h(a)).
\]

Now consider an arbitrary point \( y \in g^{-1}(h(a)) \). Note that as \( h \) is a homeomorphism then there exists a maps \( h^{-1} \) and so that point \( h^{-1}(y) \) is determined. With using the point \( h^{-1}(y) \) get that following diagram is commutative.

\[
\begin{array}{ccc}
  h^{-1}(y) & \xrightarrow{f} & f(a) \\
  h \downarrow & & \downarrow h \\
  y & \xrightarrow{g} & h(a).
\end{array}
\]

The commutativity of the diagram yields that \( y \in h(f^{-1}(a)) \) and this inclusion gives the sets inclusion

\[
h(f^{-1}(a)) \supseteq g^{-1}(h(a)).
\]

The last finishes the proof. \( \square \)

**Theorem 3.4.** For using the notations of the definition 3.10 the maps \( f \) be topologically equivalent to \( g \). Then for every \( n \geq 2 \) the diagram

\[
\begin{array}{ccc}
  A & \xrightarrow{f^n} & A \\
  h \downarrow & & \downarrow h \\
  B & \xrightarrow{g^n} & B
\end{array}
\]

commutes. Here the power signs mean the correspond iteration of the maps.

**Proof.** Commutative diagram (3.2) can be continued to the right as follows

\[
\begin{array}{ccc}
  A & \xrightarrow{f} & A & \xrightarrow{f} & \ldots & \xrightarrow{f} & A \\
  h \downarrow & & h \downarrow & & \ldots & & h \downarrow \\
  B & \xrightarrow{g} & B & \xrightarrow{g} & \ldots & \xrightarrow{g} & B,
\end{array}
\]

and the obtained diagram is also commutative. If necessary functional equation is obtained is consider the passes from the left top angle by external sides of the rectangle. \( \square \)
The topological equivalence of interval maps can be also considered as their graph transformation.

Let it is known that for a maps \( f \) and for a point \( x_1 \in I \) the equality \( f(x_1) = x_2 \in I \) holds. From the graph definition this means that the point \((x_1, x_2)\) belongs to the graph of the function \( f \). The commutative diagram from the definition 3.10 yields the equality \( g(h(x_1)) = h(x_2) \).

**Definition 3.12.** Call the **graduation** of any given interval \( AB \) of a line call an one to one correspondence between its points and points of some real numbers line interval \([a, b]\) such that number \( a \) corresponds to point \( A \), number \( b \) corresponds to point \( B \) and for every triple of points \( C_1, C_2, C_3 \) such that \( C_2 \) is between \( C_1 \) and \( C_3 \) corresponds a triple of real numbers such that those which corresponds \( C_3 \) is also between images of \( C_1 \) and \( C_3 \).

**Definition 3.13.** If a graduation of an interval \( AB \) has the property that for subintervals of \( AB \) which have equal lengths correspond the pairs of numbers with the same differences then call such graduation a **continuous graduation**.

Consider the graph of the maps \( f \) which is defined on the interval \( I \) and consider this graph to be plotted in uniformly graduated Cartesian coordinate plane. The note that we can understand the in the following way the statement that a point of the graph of \( f \) with coordinates \((x_1, x_2)\) corresponds to some points of the graph of maps \( g \) with coordinates \( g(h(x_1)) = h(x_2) \). The graph of the maps \( f \) (i.e. set o plane points) we will consider as a graph of the maps \( g \) but with usage of the following non uniform graduation. For the points of uniformly graduated interval \( I \) of \( x \)-axis there is a correspondence \( x \leftrightarrow h(x) \) and for uniformly graduated interval \( I \) of the \( y \)-axis there is correspondence which is defined with the rule \( y \leftrightarrow h(y) \). With the use of such graduation that set of points of the plane which was the graph of the maps \( f \) with uniform graduation will become the graph of the maps \( g \) under the described new graduation.
3.3 Examples of topological equivalent maps

Example 3.1. Consider a maps \( f \) of an \( I = [0, 1] \) which is determined with the formula

\[
f(x) = \begin{cases} 
2x, & x \leq 1/2; \\
2 - 2x, & x > 1/2.
\end{cases}
\]

and find a new maps \( g \) which is topologically equivalent to it and is defined with the following maps \( h \)

\[
h(x) = \begin{cases} 
1,5x, & x \leq 1/2; \\
0,5x + 0,5, & x > 1/2.
\end{cases}
\]

from the definition of topological equivalence. In this case during constructing the maps \( g \) we will use transformation of graphs.

The graph of the maps \( f \) is given at the picture 14(a).

Core of the example: Plot the graph of the maps \( f \) and also plot the grid which will describe the uniform graduation (see pict. 14(b)).

![Graph of f](image)

Fig. 14: Graphs of \( f \)

Now with out changing the graph (with out changing the set of points which is the graph) we will change the graduation. Use as at former plot 7 vertical and 7 horizontal lines bet plot them not uniformly but as it is shown on the plot.

For example as \( h(0,5) = 0,75 \) then point 0.75 of \( x \)-axis corresponds to the middle of the real line segment \([0, 1]\). After this both left and right part of the interval will be graduated.
uniformly each. Plot vertical lines which corresponds to graduated values $\frac{1}{8}, \ldots, \frac{7}{8}$. Naturally that in this case right hand part of the $x$-axis will be divided into 2 parts with one additional vertical line and left hand part will be divided to 6 parts with five additional vertical lines.

For obtaining the natural form of the graph of the maps $g$ squeeze the obtained picture (i.e. squeeze the picture with considering it as a geometrical figure which is composed with vertical lines) to make the graduation uniform. The new form of the plot is presented of the picture (see fig. 15a). Now repeat the same with the $y$-axis and obtain the graph of the maps $g$. For doing this just for convenience note points $A$, $B$, $C$ on the picture. Points $A$ and $C$ stay as they are but points $B$ will move vertically down without changing its horizontal position (see fig. 15b).

The graph which is obtained in such a way is the graph of maps $g$.

**Example 3.2.** Consider the maps $f$ and $h$ such as in the example 3.1. Construct the maps $g$ which is topologically equivalent to $f$ but use the definition of topological equivalence and possibility of an analytical representation of $f$ and $h$ and take into attention the commutativity of a diagram from the definition 3.10.
The analytical representation of this maps looks as

\[ h(x) = \begin{cases} 
1, & 5x \quad x \leq 1/2; \\
0, & 5x + 0,5 \quad x > 1/2. 
\end{cases} \]

\[ h^{-1}(x) = \begin{cases} 
2/3x \quad x \leq 3/4; \\
2x - 1 \quad x > 3/4. 
\end{cases} \]

Since for \( x < 3/4 \) the condition \( h^{-1}(x) < 0,5 \) hold then for these values of \( x \) the condition \( f(h^{-1}(x)) = 2h^{-1}(x) = 4/3x \) also holds. It is obvious that \( f(h^{-1}(x)) < 1/2 \) for \( x < 3/8 \). For these values of \( x \) the equality \( h(f(h^{-1}(x))) = 3/2f(h^{-1}(x)) = 2x \) holds. whence the graph of the maps \( g \) passes through the points \((0, 0)\) and \((3/8, 3/4)\). For \( x \in [3/8, 3/4] \) then \( f(h^{-1}(x)) \geq 1/2 \) and whence \( h(f(h^{-1}(x))) = 0,5f(h^{-1}(x)) + 0,5 = 0,5 \cdot 4/3x + 0,5 = 2/3x + 0,5 \). This yields that the graph of the maps \( g \) passes through points \((3/8, 3/4)\) and \((3/4, 1)\). If \( x \geq 3/4 \), then \( h^{-1}(x) = 2x + 0,5 \geq 1/2 \), whence \( f(h^{-1}(x)) = 2 - 2h^{-1}(x) = 2 - 2(2x - 1) = 4 - 4x \). For \( x < 7/8 \) the inequality \( 4 - 4x > 1/2 \) holds whence \( h(f(h^{-1}(x))) = 0,5f(h^{-1}(x)) + 0,5 = 0,5(4 - 4x) + 0,5 = -2x + 2,5 \). So the graph of the maps \( g \) passes through points \((3/4, 1)\) and \((7/8, 3/4)\). If \( x \geq 7/8 \) then the inequality \( 4 - 4x < 1/2 \) holds whence \( h(f(h^{-1}(x))) = 2(f(h^{-1}(x))) = 2(4 - 4x) = 8 - 8x \). So the graph of maps \( g \) will pass through points \((7/8, 3/4)\) and \((1, 0)\). The graph of constructed maps \( g \) is presented of the plot 16. The calculations which are presented above in this example are a bit huge and there are a lot of possibilities for technical mistakes if use of this method. More then this, the presenter method does not
give possibility to catch the mistakes if they would really be made. Nevertheless the presented example is useful because lets to pay attention to some features of topological equivalence and we will discuss them in the proposition 3.5 below.

**Proposition 3.5.** Consider continuous piecewise linear maps $f$ which maps the interval $[0, 1]$ into itself such that $f(0) = 0$. Consider also increasing continuous piecewise linear maps $h$ which maps the interval $[0, 1]$ into itself and defines the topological equivalence of maps $f$ and $g$. Then $g(0) = 0$ and $g'(0) = f'(0)$.

The example 3.2 and correspond huge calculations illustrates the theorem. Nevertheless it is easy to prove the proposition in general case.

**Proof.** Let in the neighborhood of its fixed points 0 the maps $f$ is of the form $f(x) = ax$ and analytical form of the maps $h$ in the neighborhood of this points is of the form $h(x) = bx$.

Then there is some neighborhood of 0 such that analytical for of $h^{-1}$ in it looks as $h^{-1}(x) = 1/bx$ whence in the intersection of these neighborhoods one have

$$g(x) = h(f(h^{-1}(x))) = 1/b \cdot a \cdot bx = ax,$$

which is necessary. \hfill \Box

**Lemma 1.** If a maps $g_1$ is topologically equivalent to a maps $g_2$ and maps $g_2$ is topologically equivalent to a maps $g_3$ then the maps $g_1$ is topologically equivalent to the maps $g_3$.

**Proof.** This lemma is the corollary of the definition of the topological equivalence which can be shown with the following commutative diagram

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{g_1} & [0, 1] \\
p_1 \downarrow & & \downarrow p_1 \\
[0, 1] & \xrightarrow{g_2} & [0, 1] \\
p_2 \downarrow & & \downarrow p_2 \\
[0, 1] & \xrightarrow{g_3} & [0, 1]
\end{array}
$$

In this case the proposition of the lemma and definition of the topological equivalence yield one from another. \hfill \Box
In the study of the population theory Pierre Verhulst proposed in [71] the differential equation
\[ \frac{M dp}{dt} = m - np, \]  
which leads to the difference equation
\[ M_{p_{k+1}} = p_k(m - np_k). \]

It was P. Verhulst, who called the \textit{logistic curve} the graph of the solution of the differential equation (4.1). He explained that the name “logistic” came from that finding the value of this curve needs a lot of mathematical calculations and logists in ancient Greece were those people, who were doing calculations. Also P. Verhulst studied the modelling of population at [70].

Due to Verhulst, the following maps \( \tilde{f}_\lambda : \mathbb{R} \to \mathbb{R} \), which is dependent on a parameter \( \lambda \) is called the \textit{logistic map}, where
\[ g_\lambda(x) = \lambda x(1 - x). \]  
Notice, that if \( \lambda \in [0, 4] \), then \( g_\lambda([0, 1]) \subseteq [0, 1] \) and is interesting from the point of Dynamical Systems Theory point of view for \( x \in [0, 1] \).

According to [31, p. 226], Birkhoff was the first who robust chaos in iterated map in his 1932 paper [7] “Sur quelques courbes fermées remarquables”. The Oxford University mathematicians Theodore Chaundy (1889 - 1971) and Eric Philips were the first to explore the logistic map as a function of the growth parameter \( \lambda \) (see [10]). In 1936 they reported that, in the limit of the large time, \( X_i \) approaches 0 for \( 0 < \lambda < 1 \), approaches \( 1 - 1/\lambda \) for \( 1 < \lambda < 3 \), and “oscillates finitely” for \( 3 < \lambda < 4 \). In 1970-s two groups studied periodical oscillations in this regime: Nicholas Metropolis (born 1915), Myron Stein and Pual Stein at Los Alamos National Laboratory and Robert May (later Lord May, born 1936) at Princeton University.

For \( 0 \leq \lambda \leq 1 \), all orbits converge to 0. For \( 1 < \lambda \leq 3 \), all orbits starting at \( x_0 > 0 \) converge to \( 1 - 1/\lambda \). For \( 3 < \lambda \leq 1 + \sqrt{6} \), the orbits converge to a cycle of period 2.

For \( \lambda > 1 + \sqrt{6} \), the system goes through a whole sequence of period doubling. Let he values \( \lambda_n \) denote the parameter for which the \( n \)-th period doubling occurs. Then \( \lambda_n \) obey the law
\[ \lim_{n \to \infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta = 4.669 \ldots , \]
where $\delta$ is called the Feigenbaum constant. In 1978, Feigenbaum [19] (also see [20]) as well as Coullet and Tresser [66] independently outlined an argument showing that such period doubling cascades should be observable for a large class of systems, and that the constant $\delta$ is universal. For instance, it also appears in the two-dimensional Hénon map

\[
\begin{align*}
    x_{n+1} &= 1 - \lambda x_n^2 + y_n \\
    y_{n+1} &= bx_n
\end{align*}
\]

$0 < b < 1$.

Rigorous proofs of these properties were later worked out by Collet, Eckmann, Koch, Lanford and others in [12].

Consider the difference logistic equation in the form

\[x_{n+1} = \lambda x_n (1 - x_n).\] (4.3)

M. Ranferi Gutiérrez, M.A. Reyes, and H.C. Rosu (see [56] and [72, p. 918]) say, that nowadays evident solutions of (4.3) are known only for $\lambda = -2$, 2 and 4. These solutions can be written in the form

\[x_n = \frac{1}{2} \left( 1 - v_\lambda(r^n v_\lambda^{-1}(1 - 2x_0)) \right)\]

and correspond $g_r$ are as follows:

\[v_{-2}(x) = 2 \cos \left( \frac{1}{3}(\pi - \sqrt{3}x) \right); \]
\[v_2(x) = e^x; \]
\[v_4(x) = \cos x. \]

The logistic equation has since been applied to a wide range of phenomena including spread of technological change, innovations [60] new product diffusion within markets [4] diffusion of social change [11] and diffusion of epidemics [61].

We have found at [31], that it was John Herschel, who obtained at first the topological conjugation of logistic and tent map. R. Kautz write the following: As it happens, that $\lambda = 4$ (i.e. logistic equation $x_{n+1} = 4x_n(1 - x_n)$) is mathematically especially simple in spite of being chaotic. Its simplicity first became apparent in the work of the English mathematician
John Herschel (1792 - 1871), the son of astronomer William Herschel (1738-1822). In 1814, Herschel showed that i-th iterate of the map can be expressed as

\[ X_i = \frac{1}{2}(1 - \cos(2^i \theta)), \]

where

\[ \theta = \cos^{-1}(1 - 2X_0). \]

Nevertheless, studying of his work [27], which is given at the bibliography of correspond section of [31], we have made a conclusion, that it is not so. The only thing, which W. Herschel does at this work correspondingly to our interest is the following.

He considers the iterations of different functions and states the problem of finding the explicit formula \( g^n \) for the \( n \)-th iteration function \( g(x) = 2x^2 - 1 \). He finds the formula

\[ g^n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^{2^n} + \left(x - \sqrt{x^2 - 1}\right)^{2^n}\right). \]  

(4.4)

After obtaining such a solution, Herschel solves the functional equation

\[ \varphi^n(x) = g(x) \]

for \( f(x) = 2x^2 - 1 \). He just plug \( \frac{1}{n} \) instead of \( n \) into (4.4) and get the answer

\[ \varphi(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^{\sqrt{2^n}} + \left(x - \sqrt{x^2 - 1}\right)^{\sqrt{2^n}}\right). \]

After this he writes: “we may here observe, that any one of of the \( n \) values of \( \sqrt{2} \) will equally afford a satisfactory value of \( \varphi(x) \)”. Also Herschel repeats his formula (4.4) for iterations of \( g \) in 1820 at [26, p. 169].

George Bool at his “A Treatice on the Calculus of finite differences” see [9, p. 170, ex. 11] considers the difference equation \( u_{n+1} - 2u_n^1 + 1 = 0 \) and solves it as \( u_n = \cos 2^n x \).

C. Babbage was the first, who in fact used the idea of topological conjugateness. He has paid his attention to the functional equation

\[ \psi^n(x) = x \]  

(4.5)
and mentioned that for any solution $t$ of this equation and for any arbitrary function $\varphi$, the function

$$F(x) = \varphi^{-1}(t(\varphi(x)))$$

(4.6)

would also be solution. He wrote this at his own part, called “Examples of the Solutions of Functional Equations” of the book [26] by J.F.W. Herschel.

Ritt also mentions this Babbage’s work at [59] and cites [26]. Also, Ritt writes there, that it was Babbage, who has made the first attempt to find the general solution of (4.5). Talking about the general solution of (4.5), Ritt uses as he says “well known” periodical transformation

$$w(x) = \frac{\alpha + \beta x}{\gamma + \delta x},$$

where

$$\delta = -\frac{\beta^2 - 2\beta\gamma \cos \frac{2k\pi}{n} + \gamma^2}{2\alpha \left(1 + \cos \frac{2k\pi}{n}\right)},$$

$k$ being any integer prime to $n$. With giving this transformation, Ritt mentions Bool’s Book “Calculus of finite Differences”, whose the first edition was in 1860. Also Ritt criticizes the Babbage statement, that for every solution $t$ and $F$ of (4.5) there exists an invertible function $\varphi$ such that

$$F(x) = \varphi^{-1}(w(\varphi(x))).$$

Suppose, says Ritt, that $\varphi(a) = b$ for some $a$ and $b$. Then for any $n \in \mathbb{N}$ the equality $\varphi(F^n(a)) = w^n(b)$. If some $F^p(a)$ and $F^p(a')$ coincide, then so should $F^p(a)$ and $F^p(a')$, than it may give a contradiction with $\varphi$ being a one to one function. Then Ritt goes Further and say $\varphi$ being defined on some interval $(a, a')$ by a function $\Phi$. Then the function $w(\Phi(F^{-1}))$ defines $\varphi$ on $[F(a), F(a')]$ and, continuing, $w^k(\Phi((F^{-1})^k))$ defines $\varphi$ on $[F^k(a), F^k(a')]$ for all $k \in \mathbb{N}$.

We see, that the mathematicians of 19-th century paid attention to the functional equation of the form $t^2(x) = x$ and to its generalization $t^n(x) = x$. The following theorem is proved in [44] Theor. 2].

**Theorem 4.1.** If $t : [0, 1] \to [0, 1]$ is a continuous function such that $t^p(x) = x$ for all $x$, then $t^2(x) = x$ for all $x$. In particular, if $p$ is odd, then $t(x) = x$ for all $x$. 

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Nowadays it is natural to think about such equations with the use so called Lamerey diagrams, which were discovered almost in the same time.

Nevertheless, we can say, that it was Babbage, who has done the first attempt of graphical interpretation of the solution of the functional equation (4.5) at his [3]. It is the following (see Figure 17).

He wrote: Required the nature of the curve, such that taking any point \( B \) in the abscissa, and drawing the ordinate \( BP \) if we make \( AC \) another abscissa equal \( BP \) the preceding ordinate, and if we continue this \( n \) times, then the \( n \)-th ordinate may be equal to the first abscissa. If \( AB = x \) and the equation of the curve is \( y = \psi(x) \). \( PB = y = \psi(x) \) and \( AC = PB = \psi(x) \), and \( QC = \psi^2(x) \), and generally the \( n \)-th ordinate \( TF \) is equal to \( \psi^n(x) \), hence \( \psi^n(x) = x \), which is the equation whose solution has been just found.

We will introduce below the history of discovering of the Lamerey diagram (see [1]).

In 1914 Pincherle considered a problem of the convergence to a fixed point for trajectories of dynamical systems which are defined by interval into itself maps (see [48]). In other words he considered an increasing maps \( f \) of some interval into itself and proved that the trajectory of each point i.e. the sequence, given by the equation

\[
x_{n+1} = f(x_n),
\]

converges to a solution of \( f(x) = x \).

One of the earliest application of this method in the context of complex dynamics occurs in 1897 in Lameray’s work [37] pp. 315-318. The roots of this method, however, go back
much farther and seem to have their origins in the work of Adrien-Marie Legendre and Évariste Galois. Writing in the Bulletin des Sciences Mathématiques in 1830, Galois remarked, (see [24, p. 413]): “Legendre was the first to notice that, when an algebraic equation was written in, the form $\varphi(x) = x$, where $\varphi$ is a function in $x$ which increases along with $x$, it is easy to find the root of this equation if for a nearby $a$, smaller than the root, $\varphi(a) > a$, or for nearby $a$, larger than the root, $\varphi(a) < a$.

To show this one draws the curve $y = \varphi(x)$ and the line $y = x$. Given an abscissa $= a$, suppose, to fix ideas, that $\varphi(a) > a$. I say that it will be easy to obtain the nearby root which is larger than $a$. In fact the roots of the equation $\varphi(x) = x$ are nothing but the values of the intersection points of the line and of the curve, and it is clear that one approaches the intersection point by substituting $\varphi(a)$ for $a$. One will find a closer and closer value as one assumes $\varphi(a), \varphi(\varphi(a)) = \varphi^2(a), \varphi^3(a)$ and so on. [1830, p. 413]”

The work to which Galois refers is Legendre, which employs a method similar to graphical iteration to solve $\varphi(x) = x$ when $\varphi$ is increasing for $x > 0$ (see [36, p. 32]). Joseph Fourier used a method very close to what Pincherle depicts in Figure 18 as a tool for the solution of equations (see [22]).
The idea of graphical illustration of a topological conjugateness is given at \[64\]. This representation for the maps \( f \), \( \tilde{f} \) and the homeomorphism \( \tilde{h} \) mentioned above, are given at Figure 19. For graphs are given at this picture, i.e. the left top quarter contains “a proper graph” of \( f \); the right top quarter contains the graph of \( \zeta \), but the \( x \)-axis goes up and \( y \)-axis goes right. Due to this, the composition \( y = \zeta(f(x)) \) is a maps of points from the left horizontal segment (of the \( x \)-axis of the graph of \( f \)) to points of the right horizontal segment (of the \( y \)-axis of the graph of \( \zeta \)). In the same way graphs of the bottom part of a picture are organized.

![Graphs illustrating topological conjugateness](image)

Fig. 19: Graphs, which illustrate topological conjugateness

Independently on Verhulst studying of populational dynamics, the logistic map \( g(x) = 4x(1 - x) \) appeared in the study of random generators in \[67\] – an abstract dedicated to the Summer Meeting of the AMS in 1947. There was announced the fact, that for almost all \( x \in \mathbb{R} \) (in the sense of Lebesgue measure) after finite number of steps, iterations of \( x \) belong to \([0, 1]\) and are “randomly” (uniformly) distributed in \([0, 1]\). Nevertheless, J. von Neumann showed, that the function \( 4.2 \) can not be used as random numbers generator (see \[45\]). In fact, he
invented in this work the topological conjugation of \( g \) and the hat map \( f \)

\[
  f(x) = \begin{cases} 
    2x, & \text{if } 0 \leq x < 1/2, \\
    2 - 2x, & \text{if } 1/2 \leq x \leq 1.
  \end{cases}
\]  

(4.8)

J. von Neumann suggested to consider the correspondence \( x_i = \sin^2 \pi \alpha_i \) and pay attention to the sequence \( \{\alpha_i\} \) if \( \{x_i\} \) is generated by \( g \) as random generator. He has written, that in this case the equality \( \alpha_{i+1} = 2\alpha_i \pmod{1} \) will hold. After this, von Neumann concluded, that in some sense the trajectory our number \( x \) will be as random as many random numbers of correspond \( \alpha \) we will take at the very beginning, whence the generator can not be used “in a real world”, possibly being good “in mathematical world”. Also von Neumann mentioned, that if \( \alpha_i \)-th are uniformly distributed, then correspond \( x_i \)-th would be distributed with the probability distribution \( (2\pi)^{-1}[x(x - 1)]^{1/2} \)\( \mathrm{d}x \). This result is close to the invariant measures theory, we will mention below.

Stanislaw Ulam introduced at \[65\] Appendix 1 the homeomorphism

\[
  h = \frac{2}{\pi} \sin^{-1}(\sqrt{x}),
\]

such that

\[
  f(x) = h(g(\lambda(h^{-1}(x)))
\]

for \( \lambda = 4 \).

Characterizing invariant measures for explicit nonlinear dynamical systems is a fundamental problem which connects dynamical theory with statistics and statistical mechanics. In some cases, it would be desirable to to characterize ergodic invariant measures for simple chaotic invariant measures. However, in the cases of chaotic dynamical systems, such attempts to obtain explicit invariant measures have rarely been made.

Stanislaw Ulam also proposed the way of constructing a conjugation of piecewise linear maps. He proved the following Theorem (see \[65\] p. 460 (53)). Let \( f \) be broken-linear function, of equation (4.8). Let \( t(x) \) be a convex function on \([0, 1]\) which transforms the interval into itself, and such that \( t(0) = t(1) = 0 \). For some \( p \) in the interval, we must have \( t(p) = 1 \); by convexity, there is only one such point. Consider the lower tree of 1 (i.e. all integer trajectories of 1). The
necessary and sufficient condition that \( t(x) \) be conjugate to \( f \) is that tree combinatorially the same as that generated by 1 under \( f(x) \), and closure of this points be the whole interval, i.e. that the tree is dense in \([0, 1]\).

O. Rechard has used this result to finding the invariant measure of \( f \) (see [37]). Let \( \tau \) be a transformation (not one to one) of the space \( X \) (or, for example, of the interval \( I = [0, 1] \)) into itself. If the complete inverse image of every measurable set is itself measurable, then \( \tau \) is called a **measurable transformation**, and if, in addition, \( m(\tau^{-1}(A)) = 0 \), whenever \( m(A) = 0 \), the transformation will be said to be **non-singular**. Let \( \mathcal{X} \) be a class of measurable subsets of \( X \).

Assuming \( \tau \) to be a measurable transformation, a finite measure \( m^* \) defined for sets \( \mathcal{X} \) is **invariant** under \( \tau \) if \( m^*(\tau^{-1}(A)) = m^*(A) \) for every measurable set \( A \).

If \( \tau \) be a measurable, non-singular transformation of the space \( X \) (or, for example, of the interval \( I = [0, 1] \)) into itself, and \( v \) is any real valued integrable function on \( X \), then

\[
\mu_v(A) = \int_{\tau^{-1}(A)} v \, dm
\]

is a finite-valued countably additive set function on \( \mathcal{X} \) which is absolutely continuous with respect to \( m \). Consequently, by the Radon-Nikodym theorem, there exists an integrable function \( w \) on \( X \) such that \( \mu_v(A) = \int_A w \, dm \) for every measurable set \( A \). Denoting by \( L(X, m) \) the space of functions integrable over \( X \) with respect to \( m \), it is easy to see that the transformation \( T \) of \( L(X, m) \) into itself, defined by \( Tv = w \) is additive and homogeneous and transforms non-negative functions into non-negative functions. In addition,

\[
\int_X Tv \, dm = \int_{\tau^{-1}(X)} v \, dm = \int_X v \, dm,
\]

and \( ||Tv|| \leq ||v|| \) for \( v \in L(X, m) \).

Thus, O. Rechard defines the function

\[
\phi(x) = h'(x) = \frac{1}{\pi \sqrt{x} \sqrt{1-x}}
\]

for \( \tau(x) = g(x) = 4x(1-x) \) and shows, that for \( 0 \leq x \leq 1/2 \) the equality

\[
\int_0^{4x(1-x)} \phi(t) \, dt = h(\tau(x)) = f(h(x))
\]

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hods. If \(0 \leq x \leq 1/2\), then
\[
f(h(x)) = 2h(x) = \int_{0}^{x} h'(t) \, dt + \int_{0}^{1-x} h'(t) \, dt;
\]
if \(1/2 \leq x \leq 1\), then
\[
f(h(x)) = 2(1 - h(x)) = \int_{x}^{1} h'(t) \, dt + \int_{0}^{1-x} h'(t) \, dt.
\]
This implies, that the function \(h'\) is invariant under the transformation \(T\). Thus, the measure
\[
m^*(A) = \frac{1}{\pi} \int_{A} \frac{dx}{\sqrt{x} \cdot \sqrt{1-x}}
\]
is invariant under \(\tau\).

Katsura and Fukuda studied a dynamical system, which is generated by the maps
\[
x \rightarrow \frac{4x(1-x)(1-lx)}{(1-lx^2)^2}
\]
for \(0 \leq l < 1\). Evidently, it is a generalization of a logistic map for \(\lambda = 4\) (see [30], [68]). It was showed that invariant measure for (4.9) can be given by its density as
\[
\rho(x) = \frac{1}{2K(l)\sqrt{(x(1-x)(1-lx))}}
\]
where \(K(l)\) is the elliptic integral of the first kind given by \(K(l) = \int_{0}^{1} \frac{du}{\sqrt{(1-u^2)(1-lu^2)}}\) (see [68] and [69]).

Dynamical properties of \(g_\lambda\) for different values of \(\lambda\) such as stability of fixed points, the period doubling, chaos appearing etc. were also studied at [41]. It was May, who studied at first the period doubling phenomenon of the logistic map.

The topological conjugation of piecewise linear maps was also studied at [8]. A continuous map \(v\) of a compact interval to itself is linear Markov, if it is piecewise linear, and the set \(P\) of all \(v^k(x)\), where \(k \geq 0\) and \(x\) is endpoint of a linear piece, is finite. Denote elements of \(P\) as \(P = \{x_1 < \ldots < x_N\}\). Let \(P^* = \{1, \ldots, N\}\) be the indexing set of a certain \(v\)-invariant subset \(P\). Denote \(v^* : P^* \rightarrow P^*\) by \(v^*(i) = j\) if \(v(x_i) = x_j\). Also denote \(*v : P^* \rightarrow P^*\)
by \( v(i) = N + 1 - v^*(N + 1 - i) \). Let \( Q = \{ y_1 < \ldots < y_M \} \) and \( w^* \), \( w^* Q^* \to Q^* \) be the corresponding objects for a linear Markov \( w \). According to [8, Theorem 2.6], linear Markov maps \( v \) and \( w \) are topologically conjugate if and only if \( v^* = w^* \), or \( v^* = v^* \).

The review of the results on the iterations of interval maps, which were known at the middle of 20-th century, is given at [12].

One of chapters of [12, §II.6] is devoted to iterations of unimodal interval maps. Due to [12, §II.1] the maps \( f : [0 1] \to [0, 1] \) is called unimodal, if it satisfies the following properties:

1. \( f \) is continuous;

2. \( f(1/2) = 1 \);

3. \( f \) increase on \([0, 1/2]\) and decrease on \([1/2, 1]\).

Notice, that interval maps, which are considered in [12], act on \([-1, 1]\). Due to generality of the style, we reformulate the definitions from [12] to maps, which act on \([0, 1]\). For instance, the (2) in [12] is written as \( f(0) = 1 \) and (3) is considers intervals \([-1, 0]\) and \([0, 1]\). We will reformulate below the definitions from [12] in the same manner.

Let \( f : [0, 1] \to [0, 1] \) be unimodal map such that \( f(1/2) = 1 \). For any \( x \in [0, 1] \) construct the sequence \( \underline{L}(x) \) of symbols \( L, R, C \) as follows.

1. \( \underline{L}(x) \) is either infinite sequence of \( L \) and \( R \), or is a finite sequence of \( L \) and \( R \), which is followed by infinite sequence of \( C \). We will denote by \( \underline{L}_j(x) \) the \( j \)-th symbol of the mentioned sequence.

2. If \( f^j(x) \neq 1/2 \) for all \( j \neq 0 \) then \( \underline{L}_j(x) = L \) for \( f^j(x) < 1/2 \) and \( \underline{L}_j(x) = R \) for \( f^j(x) > 1/2 \).

3. If \( f^k(x) = 1/2 \) for some \( k \), then denote by \( j \) the smallest such \( k \), then set \( \underline{L}_j(x) = C \). Also set \( \underline{L}_l(x) = L \), if \( 0 \leq l < j \) and \( f^l(x) < 1/2 \). Moreover, set \( \underline{L}_l(x) = R \), if \( 0 \leq l < j \) and \( f^l(x) > 1/2 \).

The sequence \( \underline{L}(x) \) is called the itinerary of the point \( x \). The sequence \( \underline{L}(1/2) \) is called the kneading sequence of the \( f \).

The calculus of itineraries is scattered in the literature and usually presented only in a circumstantial context. The most systematic account in Derrida-Gervous-Pomerau [13] and [14], but some precursory use can be found in Metropolis-Stein-Stein [42]. Some lecture notes from
Lanford [33] can be useful to study of this calculus [12 §II.2, p. 81].

It is noticed in [12 §II.6] that if for a preserving orientation homeomorphism \( h : [0, 1] \to [0, 1] \), and for the map \( g : [0, 1] \to [0, 1] \) the equality

\[
f = h^{-1}(g(h)),
\]

holds, then \( g \) is also unimodal and has maximum point at \( x_0 = h(1/2) \). If \( x_0 = 1/2 \), then \( \mathcal{L}_f(1/2) = \mathcal{L}_g(1/2) \), where, naturally, \( \mathcal{L}_f(1/2) \) is a kneading sequence of \( f \), and \( \mathcal{L}_g(1/2) \) is a kneading sequence \( g \).

The following question appeared in the first time at [43]: Is it true, that the kneading sequence of the unimodal function defines it up to topological conjugation. Also it is shown at [43], that maps with the same kneading sequences are semi conjugated, i.e. there exists a surjective \( h : [0, 1] \to [0, 1] \) such that \( h(f) = f(h) \).

Some properties of topologically conjugated \( S \)-unimodal maps are given at [12 §II.6]. Start at first with the definition of a \( S \)-unimodal map.

The unimodal map \( f : [0, 1] \to [0, 1] \) is called \( C^1 \)-unimodal, if it is 1 time differentiable and and \( f'(x) \neq 0 \) for \( x \neq 1/2 \).

Let \( f \) be \( C^1 \)-unimodal and let \( P \) be a periodical orbit with period \( n \). This point is called stable, if \( |f'(x)| \leq 1 \) for every \( x \in P \), where \( f' \) is the derivative. If follows from the formula of the derivative of a composite function, that the value \( f'(x) \) if the same for all the points \( x \in P \), thence the definition is independent on \( x \). The importance of stable periodical orbits of dynamical system is explained by the following observation. If \( P \) is a stable periodical orbit of \( f \) with period \( n \), then there exists an neighborhood \( U \) of \( x \in P \), such that \( \lim_{m \to \infty} f^{mn}(y) = x \) for all \( y \in U \) excepted, possibly, the case, when \( |f'(x)| = 1 \), which will be discussed later. Whence, if the trajectory is stable, then a lot of points have similar habitation under \( f^m \) for \( m \to \infty \). The periodical point \( P \) is super stable, if \( f'(x) = 0 \) for \( x \in P \).

The following example is given at [34, p. 429]. Consider the maps \( g(x) = \lambda x(1-x) \) for \( x \in [0, 1] \), \( \lambda \geq 0 \). Evidently, \( g'(x) = \lambda (1 - 2x) \). The fixed point of \( g \) is fixed if and only if \( g'(x) = 0 \), whence \( x = 1/2 \) should be a fixed point. It is so only in the case, when \( \lambda = 2 \). We can similarly study the super stability of periodical points with period 2. If the cycle \( p \xrightarrow{g} q \xrightarrow{g} p \)
is super stable, then \( x = 1/2 \) belongs to this cycle. Plugging \( x = 1/2 \) into \( g^2(x) = x \) as an equation for \( \lambda \) obtain that \( \lambda = 2, 1 \pm \sqrt{5} \). But \( \lambda = 2 \) corresponds to the fixed point \( 1/2 \) and \( \lambda = 1 - \sqrt{5} < 0 \). The value \( \lambda = 1 + \sqrt{5} \) corresponds to the stable cycle \( \{1/2, 1+\sqrt{5}/2\} \).

Consider below the question on how any periodical orbit may be for a unimodal map. This problem was stated at first by Julia in 1918 at his [29]. He has showed that some unimodal maps, which are the restrictions of functions, which are analytical on \([0, 1]\), can have more then one stable periodical orbit. His theory deals with the maps \( w(x) = 1 - \mu x^2, \quad 0 < \mu \leq 2 \). Nevertheless, the natural discovery was made by Singer in 1978 in [63], when he separated the case of the negative Schwartz derivative as one, when the situation becomes simpler.

Let \( f \in C^3 \). The Schwartz derivative of \( f \) is the expression

\[
Sf(x) = \frac{f'''(x)}{f'(x)} - 3 \left( \frac{f''(x)}{f'(x)} \right)^2.
\]

A maps \( f : [0, 1] \rightarrow [0, 1] \) is called \( S \)-unimodal, if the following conditions hold.

(S1) \( f \) is \( C^1 \)-unimodal;
(S2) \( f \in C^3 \);
(S3) \( Sf(x) < 0 \) for all \( x \in [0, 1] \). Also we admit \( Sf(x) = -\infty \) for \( x = 1/2 \).
(S4) \( f \) maps \( J(f) = [f(1), 1] \) onto itself, i.e. \( f([f(1/2), 1]) = [f(1/2), 1] \)
(S5) \( f''(1/2) < 0 \).

**Theorem 4.2** (Theorem II.4.1 in [12]). If \( f \) satisfy the conditions (S1), (S2) and (S3), then every stable periodical point attracts at least one of the points \( 0, 1/2, 1 \), i.e. ends of the interval and the critical point.

**Corollary 4.1** (Corollary II.4.2 in [12]). If a maps \( f \) satisfies the conditions (S1) ... (S4), then it has at most one stable fixed orbit on \([0, f(1)]\). If \( 1/2 \) is not attracted to the stable periodical orbit, then \( f \) has no stable periodical orbits on \([f(1), 1]\).

**Corollary 4.2** (Corollary II.4.3 in [12]). There exist \( S \)-unimodal functions without stable periodical orbits.

As we have already mentioned, the question about the quantity of stable orbits was appeared at first at [29]. As far as it is known, the role of the negative Schwartz derivative with the
question of the quantity of fixed points was studied the first by Singer at [63]. The connection of the Schwartz derivative and analytical functions is described at [23]. The role of the cross ratio in this question is found by Guckenheimer at [25].

Let maps \( f, g : [0, 1] \to [0, 1] \) be non-topological conjugated, but such that \( L_f(1) = L_g(1) \).

Maps \( f \) and \( g \) on the Figure 20 have the mentioned properties. These maps are piecewise linear, have two braking points each and \( f(0) = f(1) = g(0) = g(1) = 1/2, f(1/2) = g(1/2) = 1 \).

Evidently, \( f \) and \( g \) are not topologically conjugated, because the fixed point \( x_f \) of \( f \) is attracting, but the fixed point \( x_g \) of \( g \) is repelling. In the same time, \( L_f(1) = L_g(0) = RC \).

In the same time, in the following theorems, the topological conjugation follows from the equality of itineraries with some additional conditions.

**Theorem 4.3** (Theorem II.6.1 in [12]). Suppose that \( f \) and \( g \) are \( S \)-unimodal and \( f \) has no stable periodic point. If \( L_f(0) = L_g(0) \) and \( L_f(1) = L_g(1) \) then \( f \) and \( g \) are topologically conjugate.

The following theorem can be considered as a variant or Theorem 4.3.

**Theorem 4.4** (Theorem II.6.1.A in [12]). Suppose that \( f \) and \( g \) are \( S \)-unimodal and \( f \) has no stable periodic point. If \( L_f(1) = L_g(1) \) then \( f|_{f(1), 1} \) and \( g|_{g(1), 1} \) are topologically conjugate through a homeomorphism \( \hat{h} : [f(1), 1] \) onto \([g(1), 1] \).
Theorem 4.5 (Theorem II.6.3 in [12]). Let $f$ and $g$ be $S$-unimodal and assume that $I_f(1) = I_g(1)$.

1. If $I_f(1)$ is finite, then $f|_{I_f}$ and $g|_{I_g}$ are topologically conjugate.

2. If $I_f(1)$ is infinite and periodic of period $n$, i.e. $I_f(1) = D^n$ with $|D| = n$, then there are two possibilities:

   (a) If $|D|$ is odd then $f|_{I_f}$ and $g|_{I_g}$ are topologically conjugate if and only if their stable periodic orbits have the same period (which is $n$ or $2n$).

   (b) If $|D|$ is even, then $f|_{I_f}$ and $g|_{I_g}$ are topologically conjugate if their stable periodic orbits (which have period $n$) are both stable from one side or stable from both sides.

3. If $I_f(1)$ is finite but not periodic, then $f|_{I_f}$ and $g|_{I_g}$ are topologically conjugate.

Guckenheimer has proved at [25], that any $C^1$ unimodal map of the interval is semiconjugate to a quadratic map and that the semi-conjugacy is strictly monotone in the backward orbit of the turning point. The prove of this result uses the assumption that their Schwarzian derivative of the map is negative.

That a quadratic map is described by a very simple mathematical formula is not very useful for the understanding of its dynamics because this property is not preserved under iteration: the $n$-th iterate of the map is a polynomial of degree $2n$. Singer made the following fundamental observation at [63]: if a map has negative Schwarzian derivative then all of its iterates also have this property. Independently Allwright [2] observed something similar. Furthermore, quadratic maps turn out to have negative Schwarzian derivative.

In the same paper, Singer proved that such maps have a finite number of attracting periodic orbits, if they have a finite number of turning points. This, because each of these orbits must attract at least one critical point or one boundary point. Later Guckenheimer showed [25], for unimodal maps with negative Schwarzian derivative, that any interval whose points have the same itinerary must be contained in the basin of the unique attracting periodic orbit. In particular, if the map has no attracting periodic orbit, the backward orbit of its turning point is dense.

Milnor and Thurston proved [43], that a continuous, piecewise monotone map with positive topological entropy is semiconjugate to a continuous, piecewise linear map with constant slope.
and with the same entropy. This result is the following theorem.

**Theorem 4.6.** Assume that \( f : [0, 1] \to [0, 1] \) is a continuous, piecewise (strictly) monotone map with positive topological entropy \( h_t(f) \) and let \( s = \exp(h_t(f)) \). Then there exists a continuous, piecewise linear map \( T : [0, 1] \to [0, 1] \) with slope \( \pm s \), and a continuous, monotone increasing map \( w : [0, 1] \to [0, 1] \) which is a semi-conjugacy between \( f \) and \( T \), i.e.

\[
w(f) = T(w).
\]

Essentially this result was already proved by Parry at [46].

The proof of this theorem gives also a very important relationship between the lap numbers \( l(f^n) \) and the kneading invariants. The definition of lap numbers is the following.

**Definition 4.1.** Let \( f : [0, 1] \to [0, 1] \) be a continuous piecewise monotone map. The lap number \( l(f) \) of \( f \), is the number of maximal intervals on which \( f \) is monotone. In other words, \( l(f) - 1 \) is the number of turning points of \( f \).
5 Maps, whose semigroup of iterations is a finite group

Consider the pair of topologically conjugated maps

\[ y(x) = \begin{cases} 
2x, & x < 1/2; \\
2 - 2x, & x \geq 1/2, 
\end{cases} \]

and

\[ \tilde{y}(x) = 4x(1 - x). \]

We have paid crucial attention to these two maps in Section 4. Maps \( y \) and \( \tilde{y} \) are representors of the following families of maps. For every \( v \in (0, 1) \) consider the piecewise linear maps \( y_v(x) \), whose graph consists on two line segments, which connect points with coordinates \((0, 0), (1/2, v)\) and \((1, 0)\). Also for every \( \alpha \in [0, 4] \) consider the maps \( \tilde{y}_\alpha(x) = \alpha x(1 - x) \). Consider both \( y_v \) and \( \tilde{y}_\alpha \) as maps of \([0, 1]\) into itself.

The maps \( y_v \) for \( v = 1/2 \) is such that \( y_v^2 = y_v \). This example leads to the problem of the description of all \( f : [0, 1] \to [0, 1] \) such that

\[ f^n = f, \tag{5.1} \]

where \( n \) is fixed.

We will assume in this section that \( n \) from the equality \((5.1)\) is the smallest possible. With the use of the algebraic notion of representation we may note, that the maps \( f \), which satisfies \((5.1)\), defines the exact representation of the cyclic group with \( n \) elements. In general, iterations of the map form the cyclic semigroup with respect to compositions. This semigroup will be a group \( C_n \) if and only if the map satisfies \((5.1)\).

5.1 Groups which are exactly represented with iterations of continuous interval maps

Notice, that it follows from the equality \((5.1)\) that cardinalities of orbits of \( f \) are uniformly bounded, or, more precisely, for every \( x_0 \in I \) its orbit has not more then \( n \) elements.
**Lemma 2.** If \( f \in C^0(I, I) \) has a periodical point of period \( m \), \( m > 2 \) then orbits are not uniformly bounded.

**Proof.** Let \( B \) be a periodical point of period \( m \), \( m > 2 \) and let it consists of \( \beta_1 < \beta_2 < ... < \beta_m \). Denote by \( \beta_0 = \max\{ \beta \in B \mid f(\beta) > \beta \} \) (the set \( \{ \beta \in B \mid f(\beta) > \beta \} \) is non-empty). Since \( \beta_0 \neq \beta_m \), then the set \( \{ \beta \in B \mid \beta > \beta_0 \} \) is also non-empty. Let \( \beta_1 = \min\{ \beta \in B \mid \beta > \beta_0 \} \). Since \( m > 2 \) then either \( f(\beta_0) \neq \beta_1 \) or \( f(\beta_1) \neq \beta_0 \). Assume that \( f(\beta_0) \neq \beta_1 \). From the definitions of \( \beta_0 \) and \( \beta_1 \) obtain that \( B \cap (\beta_0, \beta_1) = \emptyset \). Whence either \( f(\beta_1) \leq \beta_0 < \beta_1 < f(\beta_0) \), or \( f[\beta_0, \beta_1] \supset [\beta_0, \beta_1] \). From this property obtain that there exists a sequence \( (y_i, i = 1, \ldots) \) such that

1) \( \beta_0 < y_1 < y_3 < ... < y_{2k+1} < y_{2k+3} < ... < y_{2k+2} < y_{2k} < ... < y_4 < y_2 < \beta_1 \),

2) \( f(y_{i+1}) = y_i, \quad i = 1, 2, \ldots \)

3) \( f(y_1) = \beta_1 \).

Whence for every \( i \), \( \text{card}(\text{orb}(y_i)) = i + m \), whence cardinalities of orbits are not uniformly bounded. \( \square \)

**Theorem 5.1.** If a continuous maps \( f \) of interval into itself satisfies \((5.1)\), then it satisfies the equality

\[ f^3 = f. \] (5.2)

**Proof.** If for the maps \( f \) the equality \((5.2)\) does not hold, but the equality \((5.1)\) holds, then there exists \( x_0 \in [0, 1] \) such that \( f^3(x_0) \neq f(x_0) \), but \( f^n(x_0) = f(x_0) \). It means that the point \( f(x_0) \) is periodical with period 2 for the maps \( f \). It follows from Lemma 2 then cardinalities of orbits for the maps \( f \) are not uniformly bounded. This contradicts to equality \((5.1)\) which means that all these cardinalities are not greater than \( n \). \( \square \)

### 5.2 Graph of the maps with finite group of iterations

We will describe graphs of the maps of the interval into itself, whose semigroup of iterations is a finite group. Let \( f \) satisfies \((5.1)\). Consider cases when \( f^2 = f \) and \( f^2 \neq f \). Describe the graph of \( f \) in each of these cases. Let \( f^2 = f \). Since \( f \) is continuous, then the image of \( I \) under \( f \) is some continuous interval call \([a, b] \), i.e. \( f(I) = [a, b] \). For every \( x_0 \in I \) the condition \( f^2 = f \)
yields that \( f(x_0) \) is a fixed point of \( f \), because

\[
f(f(x_0)) = f(x_0). \tag{5.3}
\]

From the other hand, the condition \( f(I) = [a, b] \) yields that for every \( x_0 \in [a, b] \) there exists \( x^* \) such that \( f(x^*) = x_0 \). Not it follows from (5.3) that \( [a, b] \) is a fixed points set of \( f \). There reasonings prove the following theorem.

**Theorem 5.2.** For a maps \( f \in C^0(I, I) \) the following properties are equivalent:

1) \( f^n(x) = f(x) \) for all \( x \in I \) and every \( n \);

2) there exist real numbers \( a \) and \( b \) and maps \( g \in C^0([0, a], [a, b]) \), \( h \in C^0([b, 1], [a, b]) \) such that \( 0 \leq a \leq b \leq 1 \) and the maps \( f \) can be represented as follows

\[
f(x) = \begin{cases} 
  g(x), & 0 \leq x \leq a, \\
  x, & a \leq x \leq b, \\
  h(x), & b \leq x \leq 1.
\end{cases} \tag{5.4}
\]

![Fig. 21: Graph of \( f \)](image)

The graph of \( f \) from Theorem 5.2 is given at Figure 21.

The case when the maps \( f \) satisfies the equality \( f^3 = f \), but not \( f^2 = f \) can be described by the following equivalent conditions.

**Theorem 5.3.** For a maps \( f \in C^0(I, I) \) the following conditions are equivalent:

1) \( f^3(x) = f(x) \) for all \( x \in I \);

2) There exist reals \( a \) and \( b \) maps \( g \in C^0([0, a], [a, b]) \), \( h \in C^0([b, 1], [a, b]) \) and a maps \( \varphi \in C^0([a, b], [a, b]) \), such that the graph of \( \varphi \) is symmetrical in the line \( y = x \), and \( \varphi([a, b]) = [a, b] \)
and the maps $f$ may be represented as follows.

$$f(x) = \begin{cases} 
g(x), & 0 \leq x \leq a, \\
\varphi(x), & a \leq x \leq b, \\
h(x), & b \leq x \leq 1. 
\end{cases} \quad (5.5)$$

We will need two technical lemmas for the proof of this theorem.

**Lemma 3.** If cardinalities of orbits of $f \in C(I, I)$ are uniformly bounded, then $Fix(f)$ is a closed interval.

**Proof.** Consider the maps $g = f^2$. It follows from Lemma 2 that $Per(f) = Fix(g)$. If the set of fixed points of $g$ contains only one point, then lemma is trivial. Otherwise denote by $\beta$ and arbitrary points of $Fix(g)$ and $I_\beta$ the maximal interval of fixed points of $g$, which contains $\beta$. Assume that $Fix(g) \neq I_\beta$. Let $\beta_1 \in Fix(g) \setminus I_\beta$. Denote $I_\beta = [a, b]$ and assume that $\beta_1 < a$. Since $Fix(g)$ is closed, then without lose of generality assume that the interval $(\beta_1, a)$ does not contain any fixed points of $g$. whence either $g(x) > x$ for all $x \in (\beta_1, a)$ or $g(x) < x$ for all $x \in (\beta_1, a)$. Assume that $g(x) > x$ (see Fig. 22).

![Graph of $f$](image)

**Fig. 22: Graph of $f$**

Let $y_1 \in (\beta_1, a)$. Since $g(\beta_1, y_1) \supset [\beta_1, y_1]$ then there exists a sequence $(y_i, \ i \geq 1)$ such that

1) $\beta_1 < \cdots < y_{i+1} < y_i < \cdots < y_2 < y_1 < a$

2) $g(y_{i+1}) = y_i, \ i \geq 1$. Whence, we see that $card(orb(y_i)) = i$ for every $i \geq 1$. This means that in the case $I_\beta \neq Fix(g)$ the cardinalities of orbits of $g$ are not uniformly bounded. This prove lemma.

**Lemma 4.** If $\{x_1, x_2\}$ is a orbit of period 2 of the continuous maps $f$ of hte interval $I$, then there is a fixed points $x_0$ of $f$ between $x_1$ and $x_2$. 

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Proof. Without loss of generality assume that \(x_1 < x_2\). Consider the function \(h(x) = f(x) - x\). Then \(h(x_1) > 0\) and \(h(x_2) < 0\). Now lemma follows from the known theorem about middle points of a continuous function.

Proof of Theorem 5.3. Notice that each maps, which satisfies the conditions of Theorem 5.2 also satisfies the both conditions of Theorem 5.3.

Evidently, the condition 2. yields the condition 1. Prove the converse implication. In the same way as in the proof of Theorem 5.2 denote \(f(I) = [a, b]\). Consider the maps \(g = f^2\). It follows from the equality \(f^3 = f\) that \(g^2 = g\). If follows from Theorem 5.2 applied to the maps \(g\), that for every \(x \in [a, b]\) the equality \(g(x) = x\) holds, which means that \(f^2(x) = x\). Show that the set of fixed points of \(f\) consists of one point. Otherwise by Lemma 3 the fixed points of \(f\) is some interval \([a_1, b_1]\). From the construction of \(a\) and \(b\) obtain that \([a_1, b_1] \subseteq [a, b]\). If \(a_1 = a\), and \(b_1 = b\) then \(f\) satisfies part 2. of Theorem 5.2. Assume that \(a_1 > a\). Then it follows from continuity of \(f\) that there exists \(\varepsilon\) such that \(f(x) < b_1\) for all \(x \in (a_1 - \varepsilon, a_1)\). Consider as arbitrary point \(x_0 \in (a_1 - \varepsilon, a_1)\). It follows from Lemma 3 and the construction of points \(a, b, a_1\) and \(b_1\) that \(x_0\) is a periodical point of period 2 of the maps \(f\), whence \(x_1 = f(x_0) \in (a, a_1)\) and \(f(x_1) = x_0\). If follows from Lemma 4 that there is a fixed point \(x_0\) of \(f\) between points \(x_0\) and \(x_1\). This contradicts to Lemma 3 because this means that \(\text{Fix}(f)\) is not an interval. The case \(b_1 < b\) should be considered analogically, whence the set of fixed points of \(f\) is consisted of the unique number. Denote it by \(c\). Whence, if the function \(f\) does not satisfy \(f^2 = f\) then its set of fixed points consists of the unique point \(c \in [a, b]\). If the function \(f\) is no monotone on \([a, b]\) then it would contradict to that every point of this interval is either fixed, or periodical. Show that \(f\) decrease on \([a, b]\). Fix an arbitrary point \(\tilde{x} \in (a, c)\). If \(f(\tilde{x}) < c\) then by Lemma 4 there exists a fixed point of \(f\) between \(\tilde{x}\) and \(f(\tilde{x})\), which contradicts to the uniqueness of the fixed point. Whence, \(f(\tilde{x}) > c\). The monotonicity of \(f\) on \([a, b]\) together with \(f(\tilde{x}) > c\) and \(f(c) = c\) means that \(f\) decrease on \([a, b]\). Since \(f\) decrease on \([a, b]\) and \([a, b]\) is a fixed points set of \(f^2\) then \([a, b]\) is a periodical orbit of the period 2. □
5.3 Topological conjugation of maps with finite group of iterations

Let $f, g, h$ be continuous maps $[0, 1] \to [0, 1]$ and iterations of $f$ and $g$ for a finite group. Let $h$ be invertible and $p_1, p_2, q_1$ and $q_2$ be such points that $f([0, 1]) = [p_1, q_1]$ and $g([0, 1]) = [p_2, q_2]$. Let $a_1, \ldots, a_n, b_1, \ldots, b_m$ be extremums of $f$ and $g$ correspondingly. Consider points 0 and 1 as extremums. Notice, that end-points of intervals of fixed points of $f^2$ and $g^2$ are necessarily extremums. Assume that $g = h^{-1}(f(h))$.

Definition 5.1. Call vectors $\langle v_1, \ldots, v_k \rangle$ and $\langle w_1, \ldots, w_k \rangle$ equivalently ordered, if for every $i, j$ the equality $v_i \leq v_j$ is equivalent to $w_i \leq w_j$.

Notation 5.1. Denote the following vectors.

- $v_f = (f(a_1), \ldots, f(a_n))$
- $\tilde{v}_f = (f(a_1), f^2(a_1), \ldots, f(a_n), f^2(a_n))$
- $w_g = (g(b_1), \ldots, g(b_m))$
- $\tilde{w}_g = (g(b_1), g^2(b_1), \ldots, g(b_m), g^2(b_m))$

5.3.1 Idempotent maps with increasing conjugation

Assume that $f$ and $g$ are idempotent maps and $h$ is an increasing conjugation.

Lemma 5. The equalities $h(p_2) = p_1$ and $h(q_2) = q_1$ hold.

Proof. Consider $x_0 \in \text{Fix}(g)$. Assume that $h(x_0) > q_1$. Then $f(h(x_0)) < h(x_0)$, because $f(h(x_0)) \in f([0, 1]) = [p_1, q_1]$ and $h(x_0) > q_1$. Since $h$ increase, then $h^{-1}$ also increase whence $h^{-1}(f(h(x_0))) < h^{-1}(h(x_0))$. But the last equality means that $g(x_0) < x_0$, which contradicts to that $x_0$ is a fixed point of $g$. The case $h(x_0) < p_1$ should be considered analogically. We have that $h([p_2, q_2])$ belongs to $[p_1, q_1]$. Since it follows from $g = h^{-1}(f(h))$ that $f = h(g(h^{-1}))$ then the same reasonings give that $h^{-1}([p_1, q_1]) \subseteq [p_2, q_2]$. The last finishes the proof. \hfill \square

Lemma 6. $m = n$ and for arbitrary $i, 1 \leq i \leq n$ the equality $h(b_i) = a_i$ holds and vectors $v_f$ and $w_g$ are equivalently ordered.

Proof. This lemma follows from that composition of monotone functions is a monotone function. Consider an arbitrary $i \in [1, n - 1]$ and consider the interval of monotonicity of $f$,
call \([a_i, a_{i+1}]\). Let \(f\) increase on this interval. Show that in this case \(g\) would increase on \([h^{-1}(a_i), h^{-1}(a_{i+1})]\). For arbitrary \(x_1, x_2 \in [a_i, a_{i+1}], x_1 < x_2\) we have \(h(x_1) < h(x_2)\); since \(f\) increase on \([a_i, a_{i+1}]\), then \(f(h(x_1)) < f(h(x_2))\), whence \(h^{-1}(f(h(x_1))) < h^{-1}(f(h(x_2)))\), which proves the monotonicity of \(g\) on \([h^{-1}(a_i), h^{-1}(a_{i+1})]\). If the maps \(f\) decrease on \([a_i, a_{i+1}]\) then the prove of degreasing of \(g\) on \([h^{-1}(a_i), h^{-1}(a_{i+1})]\) is analogical to the previous. Consider points \(b_i = h^{-1}(a_i), i = 1, \ldots, n\). It follows from the previous, that these points are extremums of \(g\) and for every \(i = 1, \ldots, n\) the character of extremum (minimum, of maximum) of \(b_i\) coincides with one of \(a_i\). Let for some \(r, s = 1, \ldots, n\) the equality \(g(b_r) \leq g(b_s)\) holds, i.e. \(h^{-1}(f(h(b_r))) \leq h^{-1}(f(h(b_s)))\). Take \(h\) from the both sides of the inequality and it would follow from the increasing of \(h\) that \(f(h(b_r)) \leq f(h(b_s))\). Since \(b_r = h^{-1}(a_r)\) and \(b_s = h^{-1}(a_s)\) then if follows from the last equality that \(f(a_r) \leq f(a_s)\) which is was necessary to prove. \(\square\)

**Lemma 7.** Let \(m = n\), vectors \(v_f\) and \(w_g\) be equivalently ordered and numbers of extremums, which are end-points of intervals \([p_1, q_1]\) and \([p_2, q_2]\) coincide. Then \(f\) and \(g\) are conjugated.

**Proof.** Since for every \(i \in [1, n]\) the equality \(a_i = h(b_i)\) holds, then plugging \(b_i\) into \(g = h^{-1}(f(h))\) obtain \(g(b_i) = h^{-1}(f(a_i))\), i.e. the graph of \(h\) pathes through the point \((g(b_i), f(a_i))\). Notice, that since the condition \(f(a_i) \leq f(a_j)\) is equivalent to \(g(b_i) \leq g(b_j)\) then the obtained restriction for \(h\) does not contradict to its monotonicity. More then this, since \(h(p_2) = p_1\) and \(h(q_2) = q_1\) then the restriction on \(h\) is the restriction only on the interval \([p_2, q_2]\). Take \(h\) to be arbitrary increasing on \([p_2, q_2]\) and passing through the mentioned points. For example, take \(h\) to be piecewise linear. Consider an arbitrary \(i = 1, \ldots, n\) such that \([a_i, a_{i+1}]\) is not \([p_1, q_1]\). Consider an arbitrary \(x_0 \in [b_i, b_{i+1}]\). Then the condition \(g(x_0) = h^{-1}(f(h(x_0)))\) is equivalent to \(h(g(x_0)) = f(h(x_0))\). Since \(h\) is already defined on \([p_2, q_2]\) then \(h(g(x_0))\) is already defined. Since \(x_0 \in [b_i, b_{i+1}]\) then \(h(x_0) \in [a_i, a_{i+1}]\). Since \(f\) increase on \([a_i, a_{i+1}]\), then it has an inverse \(f_i\), which is defined on \([\min(f(a_i), f(a_{i+1})), \max(f(a_i), f(a_{i+1}))]\). Whence, the equality \(h(x_0) = f_i^{-1}(h(g(x_0)))\) is necessary for the equality \(h(g(x_0)) = f(h(x_0))\).

For every \(i = 1, \ldots, n\) define the maps \(h\) on \([b_i, b_{i+1}]\) by the formula \(h = f_i^{-1}(h(g))\). It follows from the construction of \(h\), that if defined the conjugation of \(f\) and \(g\). \(\square\)
5.3.2 Generators of $C_2$ with increasing conjugation

Let $f$ and $g$ be maps, whose iterations for the group $C_2$ each and $h$ be increasing homeomorphism.

**Lemma 8.** If the equality $g = h^{-1}(f(h))$ holds, then the maps $h$ moves end-points of $[p_2, q_2]$ to end-points of $[p_1, q_1]$.

**Proof.** If follows from $g = h^{-1}(f(h))$ that $g^2 = h^{-1}(f^2(h))$.

Assume that $x_0 \in \text{Fix}(g^2)$ and $h(x_0) > q_1$. Then $f^2(h(x_0)) < h(x_0)$, because $f^2([0, 1]) = [p_1, q_1]$. Applying $h^{-1}$ to both sides of the inequality obtain $g^2(x_0) = h^{-1}(f^2(h(x_0))) < h^{-1}(h(x_0)) = x_0$, which contradicts to $x_0 \in \text{Fix}(g^2)$. The analogical consideration of the case $h(x_0) < p_1$ yields that $h([p_2, q_2]) \subset [p_1, q_1]$. It follows from the equality $f = h(g(h^{-1}))$ that $f^2 = h(g^2(h^{-1}))$, whence $h^{-1}([p_1, q_1]) \subset [p_2, q_2]$. Obtain from this that $h([p_2, q_2]) = ([p_1, q_1])$, which means that $h(p_2) = p_1$ and $h(q_2) = q_1$.

**Lemma 9.** If $[p_1, q_1] = [p_2, q_2] = [0, 1]$, then maps $f$ and $g$ are conjugated via the increasing homeomorphism.

**Proof.** Let $x_0^f$ and $x_0^g$ be fixed points of $f$ and $g$ correspondingly. Define the new maps $f_1$ and $g_1$ as follows. $f_1(x) = f(x)$ for $x \leq x_0^f$ and $f_1(x) = x$ for $x > x_0^f$; also define $g_1(x) = g(x)$ for $x \leq x_0^g$ and $g_1(x) = x$ for $x > x_0^g$.

Construct the increasing maps $h$, which defines the conjugation of $f_1$ and $g_1$. The the graph of $h$ passes through the point $(x_0^g, x_0^f)$ and be defined arbitrary on $[x_0^g, 1]$. Then for arbitrary $x \in [x_0^g, 1]$ we have $h(x) \in [x_0^f, 1]$, whence $f_1(h(x)) = h(x)$, which means that $h^{-1}(f_1(h(x))) = h^{-1}(h(x)) = x$ and $g_1 = h^{-1}(f_1(h))$ on $[x_0^g, 1]$. Since $g_1$ is monotone on $[0, x_0^g]$, then for the conjugateness of $f_1$ and $g_1$ it is enough to take $h = f_1^{-1}(h(g_1))$ on $[0, x_0^g]$, where $h$ at the right hand side of defined earlier, because $g_1^{-1}([0, x_0^g]) = [x_0^g, 1]$.

Show that the map $h$, which is constructed in this way, defines the conjugation of maps $f$ and $g$. The equality $g(x) = h^{-1}(f(h(x)))$ for all $x \in [0, x_0^g]$ follows from the construction. Since there are compositions of monotone functions from the left and from the right of the equality $g_1(x) = h^{-1}(f_1(h(x)))$, then write the the equality of there inverses and obtain $g_1^{-1}(x) = h^{-1}(f_1^{-1}(h(x)))$.
for $x \in [0, x_0^g]$. Since $f^2 = id$ and $g^2 = id$ then $g_1^{-1}(x) = g(x)$ and $f_1^{-1}(h(x)) = f(h(x))$ for $x \in [x_0^g, 1]$, which means that $g = h^{-1}(f(h))$. The last proves the Lemma.

**Lemma 10.** Let the maps $g = h^{-1}(f(h))$ be conjugated to $f$ via increasing $h$. Then for arbitrary $i \in [1, n]$ the equality $h(b_i) = a_i$ holds and vectors $v_f$ and $w_g$ are equivalently ordered.

**Proof.** Proof of this lemma is analogical to the proof of Lemma 6.

**Lemma 11.** Let $f$ and $g$ be conjugated via $h$ and $g = h^{-1}(f(h))$. Then vectors $\tilde{v}_f$ and $\tilde{w}_g$ are equivalently ordered.

**Proof.** Plug the value $x = g(b_i)$ into the equality $g(x) = h^{-1}(f(h(x)))$ and obtain $g^2(b_i) = h^{-1}(f(h(g(b_i))))$. Since according to Lemma 7 the graph of $h$ pathes through the points $(g(b_i), f(a_i))$ for all $i, 1 \leq i \leq n$, then the obtained equality is equivalent to $g^2(b_i) = h^{-1}(f(f(a_i)))$. Plugging the left and right part of the obtained equality into $h$, obtain $h(g^2(b_i)) = f^2(a_i)$, which means that $h$ passes through points $(g^2(b_i), f^2(a_i))$ for all $i \in [1, n]$. Whence, the lemma follows from the monotonicity of $h$, and that it passes through $(g(b_i), f(a_i))$ for all $i \in [1, n]$.

**Theorem 5.4.** The maps $f$ and $g$ are conjugated via the increasing homeomorphism if and only if when $m = n$ numbers or extremums, which are end-points of intervals $f([0, 1])$ and $g([0, 1])$ coincide and vectors $\tilde{v}_f$ and $\tilde{w}_g$ are equivalently ordered.

**Proof.** The necessity is proved in Lemma 11.

Prove the conjugateness of $f$ and $g$.

Since vectors $\tilde{v}_f$ and $\tilde{w}_g$ are equivalently ordered, then there exists the maps $h_1 : [p_2, q_2] \rightarrow [p_1, q_1]$, which defined the conjugation of $f$ and $g$ on the set of their periodical points and passes through points $(g(b_i), f(a_i))$ and $(g^2(b_i), f^2(a_i))$ for all $i, 1, \ldots, n$. This maps should be constructed in the same way as in the proof of Lemma 9.

The maps $h$ should be constructed on the set $[0, 1] \setminus [p_2, q_2]$ in the same manner as in the proof of Lemma 7.
5.3.3 Decreasing conjugation

Describe the classes of conjugacy via decreasing homeomorphism of continuous interval maps, whose iterations form a group. An arbitrary decreasing maps $h$ can be represented as $h(x) = 1 - h_1(x)$, where $h_1(x) = 1 - h(x)$ is increasing map. Whence the conjugation vis increasing $h$ is a composition of conjugations via $1 - x$ and via increasing homeomorphism. The action of $1 - x$ on the graph of $f$ can be interpreted as sequent symmetrical reflecting it in the line $x = 1/2$ and symmetrical reflecting in the line $y = 1/2$.

**Theorem 5.5.** Maps $f$ and $g$ are conjugated via decreasing homeomorphism if and only if $m = n$, numbers of extremums, which correspond to end points of $f([0, 1])$ and $g([0, 1])$ coincide and vectors $\tilde{v}_f$ and $w^*_g$ are equally ordered, where

$$w^*_g = (g(b_m), g^2(b_m), \ldots, g(b_1), g^2(b_1)).$$
6 Constructing of the conjugation

6.1 Values of conjugation on the dense set

Consider continuous maps $f, g : [0, 1] \to [0, 1]$, which are defined as follows.

$$f(x) = \begin{cases} 
2x, & 0 \leq x < 1/2; \\
2 - 2x, & 1/2 \leq x \leq 1,
\end{cases} \quad (6.1)$$

and

$$g(x) = \begin{cases} 
g_l(x), & x \leq v; \\
g_r(x), & x > v,
\end{cases} \quad (6.2)$$

where $v \in (0, 1)$ is fixed and functions $g_l$, $g_r$ are continuous monotone such that $g_l(0) = g_r(1) = 0$, $g_l(v) = \lim_{x \to v^-} g_r(x) = 1$. The problem about conjugateness of $f$ and $g$ were stated at first in [65] in the following theorem.

**Theorem 6.1.** [65, Appendix 1, §3] Let $f$ be of the form (6.1) and $g$ be a convex function of the form (6.2). Consider the whole pre image of $1$ under $f$, i.e. the such the smallest set $M_f$ that $1 \in M_f$ and $f(x) \in M_f$ yields $x \in M_f$. Maps $f$ and $g$ are topologically conjugated if and only if sets $M_f$ is combinatorially equivalent, with $M_g$ and additionally $\overline{M}_g = [0, 1]$.

**Proof from [65].** The necessity is obvious, because the whole pre image of $1$ is the the set of binary-rational numbers, i.e. those rational numbers, whose denominator is a power of $2$.

We will prove the existence of conjugation constructively. Tale $h(1/2) = v$. Then take $h(1/4)$ to be the smallest of numbers $g^{-1}(v)$, and $h(3/4)$ be the greatest of $g^{-1}(v)$. Take $h(1/8)$ the smallest of $g^{-1}(h(3/4))$ and so on. Continuing this way, obtain the function which is defined on binary rational numbers of $(0, 1)$. Define $h$ on the whole $[0, 1]$ by the continuity. This definition by continuity is possible, because $M_g$ is dense. The maps $h$ would obviously be monotone and its continuity yields that there exists $h^{-1}$. The equality $h(f) = g(h)$ follows from the construction. \(\square\)

This section is devoted to the proof in details of Theorem 6.1.
Assume that $f$ and $g$ are topologically conjugated and there exists a homeomorphism $h$ such that the diagram
\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, 1] & \xrightarrow{g} & [0, 1]
\end{array}
\] (6.3)
is commutative. The commutativity of this diagram is equivalent to that $h$ is a solution of the functional equation
\[ h(f(x)) = g(h(x)). \] (6.4)

We will obtain the existence of this homeomorphism later, but now we will find some its properties in the assumption that the homeomorphism exists. Precisely, we will find the values of $h$ on the dense set of $[0, 1]$.

**Lemma 12.** If the homeomorphism $h$ defines the topological conjugation of $f$ and $g$, then it increase, i.e. $h(0) = 0$ and $h(1) = 1$.

**Proof.** Since $h$ is a homeomorphism then it is either increase or decrease. Since $h$ maps the interval $[0, 1]$ onto itself then either $h(0) = 0$ and $h(1) = 1$ or $h(0) = 1$ and $h(1) = 0$. Plug the value $x = 0$ into the equality (6.4) and obtain
\[ h(f(0)) = g(h(0)). \]
Notice, that this plugging may illustrated more clearly by the commutative diagram
\[
\begin{array}{ccc}
0 & \xrightarrow{f} & f(0) \\
\downarrow h & & \downarrow h \\
h(0) & \xrightarrow{g} & g(h(0)) = h(f(0)),
\end{array}
\]
which is obtained from (6.3) by plugging $x = 0$ into left top angle.

In any way, since $f(0) = 0$ then equality $h(0) = g(h(0))$ holds, i.e. $x = h(0)$ is a fixed point of $g$.

Nevertheless, the point $x = 1$ is not fixed for $g$, but $x = 0$ is. Whence, $h(0) = 0$, which is necessary. \qed
Definition 6.1. For every point \( x^* \in [0, 1] \) we would say that the value \( y^* \) of the homeomorphism \( h \) at \( x^* \) is **conditionally found**, if the following statement holds. If \( h \) defines a conjugation of \( f \) and \( g \), then \( h(x^*) = y^* \). We use the word “conditionally” for paying the attention to that the question about existence of \( h \) is still open.

For example, in terms of Definition 6.1, Lemma 12 states that the map \( s \) is conditionally found at points 0 and 1. Show that the conditional value of \( h \) at \( x^* = 1/2 \) exists and \( h(1/2) = v \).

Lemma 13. If \( h \) is the conjugation of \( f \) and \( g \), then \( h(1/2) = v \).

Proof. Lemma follows from the commutativity of the diagram

\[
\begin{array}{ccc}
1/2 & \xrightarrow{f} & 1 \\
\downarrow h & & \downarrow h \\
h(1/2) & \xrightarrow{g} & 1,
\end{array}
\]

which is obtained from (6.3) by plugging \( x = 1/2 \) into left top angle.

For an arbitrary point \( x^* \) where the value of \( h \) is conditionally found, consider the pre image \( \tilde{x} \) under \( f \), i.e. \( f(\tilde{x}) = x^* \) and consider the diagram

\[
\begin{array}{ccc}
\tilde{x} & \xrightarrow{f} & x^* \\
\downarrow h & & \downarrow h \\
h(\tilde{x}) & \xrightarrow{g} & h(x^*).
\end{array}
\]

This commutative diagram let us to find the value \( h(\tilde{x}) \) under the assumption that \( \tilde{x}, x^* \) and \( h(x^*) \) are known.

If \( x^* \neq 1 \) then there are two choices for the pre image \( \tilde{x} \) of \( x^* \). Is we search \( \tilde{x} < 1/2 \), then formulas (6.1) give that \( \tilde{x} = x^*/2 \). Searching \( \tilde{x} > 1/2 \), obtain \( \tilde{x} = 2 - x^* \).

Let the new pre image \( \tilde{x} \) of \( x^* \) is found as \( \tilde{x} = x^*/2 \), i.e. \( \tilde{x} < 1/2 \). Since by Lemma 12 the homeomorphism \( h \) increase, then if follows from Lemma 13 that \( h(\tilde{x}) < v \). Since \( h(\tilde{x}) < v \), then \( g(h(\tilde{x})) = g_l(h(\tilde{x})) \). From another hand, it follows from the commutativity of diagram that \( g_l(h(\tilde{x})) = h(x^*) \), whence \( h(\tilde{x}) = g_l^{-1}(h(x^*)) \).

From the analogical reasonings give that if \( \tilde{x} \) is found from the condition \( \tilde{x} > 1/2 \), then \( h(x^*) = g_r(h(\tilde{x})) \), i.e. \( h(\tilde{x}) = g_r^{-1}(h(x^*)) \).

This construction proves the following lemma.
Proposition 6.2. If a homeomorphism $h$ is a conjugation of $f$ and $g$ and $h(x^*) = y^*$ for some $x^*$ and $y^*$ then for any point $\tilde{x}$ such that $f(\tilde{x}) = x^*$ the following implications hold.

1. If $\tilde{x} \leq 1/2$, then $h(\tilde{x}) = g^{-1}_l(y^*)$;
2. If $\tilde{x} > 1/2$, then $h(\tilde{x}) = g^{-1}_r(y^*)$.

Show that, starting from $x^* = 1/2$, one may use Lemma 13 to obtain the set, which is dense in $[0, 1]$ such that $h$ would be conditionally found at each point of this set.

Lemma 14. If the the binary decomposition of the number $x \in [0, 1]$ is

$$x = 0, \alpha_1 \alpha_2 \ldots$$

then the binary decomposition of $f(x)$ is

$$f(x) = \begin{cases} 0, \alpha_2 \alpha_3 \ldots \alpha_n \ldots, & \text{if } \alpha_1 = 0, \\ 0, \overline{\alpha}_2 \overline{\alpha}_3 \ldots \overline{\alpha}_n \ldots, & \text{if } \alpha_1 = 1, \end{cases}$$

where $\overline{\alpha}_i = 1 - \alpha_i$.

Proof. Lemma follows from formulas (6.1) for the function $f$. \qed

Example 6.1. Find the pre images of $1/2$ under $f$.

Deal of the example. The binary decomposition of $1/2$ is 0.1. It follows from Lemma 14 that its pre images are $x_1 = 0.01 = 1/4$ and $x_2 = 0.10(1) = 0.11 = 1/2 + 1/4 = 3/4$.

Pre images of $x_1$ and $x_2$ under $f$ also can be obtained by Lemma 14. Pre images of $x_1$ under $f$ have binary decompositions $x_{11} = 0.001 = 1/8$ and $x_{12} = 0.110(1) = 0.111 = 1/2 + 1/4 + 1/8 = 7/8$.

Pre images of $x_2$ under $f$ have the binary decompositions $x_{21} = 0.011 = 1/4 + 1/8 = 3/8$ and $x_{22} = 0.100(1) = 0.101 = 1/2 + 1/8 = 5/8$.

Whence the set of pre images of $1/2$ is the set $\{0, 1/2, 1\}$. The set of its pre images is $\{k/4, 0 \leq k \leq 4\}$ and its pre images set is $\{k/8, 0 \leq k \leq 8\}$. \qed

The obtained property of pre images of $1/2$ can be generalized as follows.
Notation 6.1. Denote with $A_n$, $n \geq 1$ the set of all those points of the interval $[0, 1]$ such that

$$f^n(A_n) = 0.$$ 

Proposition 6.3.

$$A_n = \left\{ 0, \frac{1}{2^n-1}; \ldots; \frac{2^n-1}{2^n-1}, 1 \right\}.$$ 

Proof. Show by induction at first that the every $n \geq 1$ the set $A_n$ consists of $2^n - 1$ elements. The base of induction for $n = 1$ is obvious. Since for every $n > 1$ each of points of $A_n$ except $x_0 = 1$ has two pre images, then the cardinality of $A_{n+1}$ is $2 \cdot (2^n - 1 - 1) + 1 = 2^n + 1$.

Show that $2^n - 1$ of $A_n$ are those elements, which are mentioned in the Statement. Use the inductive reasonings again. The base for $n = 1$ is clear. Consider

$$A = \left\{ 0, \frac{1}{2^n}; \ldots; \frac{2^n-1}{2^n}, 1 \right\}$$

and prove that $A = A_{n+1}$, assuming that the statement for the previous $n$ is correct.

Consider an arbitrary $x_0$ of the form $x_0 = \frac{p}{2^n}$, where $p$ is as integer between 0 and $2^n$. Find $f(x_0)$. If $x_0 \leq 1/2$, then $f(x_0) = 2x_0 = \frac{p}{2^{n-1}} \in A_{n-1}$. If $x_0 > 1/2$, then $f(x_0) = 2 - 2x_0 = \frac{2^n - p}{2^n - 1} \in A_{n-1}$.

The last finishes the proof.

Show how to find the conditional values of $h$ at $A_n$ under the assumption that values at $A_{n-1}$ are known.

Consider an arbitrary point $\tilde{x} \in A_n$. It follows from the definition of the sets $A_1, \ldots, A_n$ that $\tilde{x} = f(x_0) \in A_{n-1}$, whence the value $y_0 = h(f(x_0))$ is found earlier. Now the value $h(\tilde{x})$ can be found by Proposition [6.2] dependently on whether $\tilde{x} < 1/2$, or $\tilde{x} > 1/2$.

we can apply for $g$ the those reasonings concerning $f$, which were used in the construction of the sets $A_n$.

Notation 6.2. Denote by $B_n$, $n \geq 1$ the set of all points $x \in [0, 1]$ such that

$$g^n(x) = 0.$$ 

The evident lemma holds.
Lemma 15. Independently on \( v, g_l \) and \( g_r \) the equality
\[
B_1 = \{0, 1\}
\]
holds.

Theorem 6.4. If a homeomorphism \( h : [0, 1] \to [0, 1] \) satisfies (6.4), then it increase and \( h(A_n) = B_n \).

Proof. Show that the cardinality of \( B_n \) equals \( 2^{n-1} + 1 \).

Check with the mathematical induction that for any \( n > 1 \) the graph of \( f_v^n \) is piecewise linear and is consisted of \( 2^n \) intervals being linear on each of them such that the image of each of these \( 2^n \) intervals is the whole interval \([0, 1]\). If \( n = 1 \) the the statement follows from the form the graph of \( f_v \). Let for \( B_{n-1} \) the theorem is correct. For each of the monotone intervals \( P \) of the maps \( f_v^{n-1} \) one have \( f_v^{n-1}(P) = [0, 1] \). In this case the graph of the maps \( f_v^n(P) \) in consisted of two monotone intervals and the image of each of them under the acting of \( f_v^n \) will be the whole interval \([0, 1]\). Since the interval \( P \) is arbitrary obtain the statement the necessary cardinality of \( B_n \).

From the construction of \( A \) and (6.4) obtain that for every \( x \in A_n \) the equality \( f_v^n(h(x)) = 0 \) holds, i.e. \( h(A_n) \subseteq B_n \). Now Theorem follows from that cardinalities of \( A_n \) and \( B_n \) coincide.

For every \( n \geq 1 \) denote by \( \alpha_{n,k}, \beta_{n,k}, 0 \leq k \leq 2^{n-1} \) the increasingly ordered elements of \( A_n \) and \( B_n \) correspondingly, i.e. for any \( k_1 < k_2 \) inequalities \( \alpha_{n,k_1} < \alpha_{n,k_2} \) and \( \beta_{n,k_1} < \beta_{n,k_2} \) hold.

Denote \( A = \bigcup_{n \geq 1} A_n \) and \( B = \bigcup_{n \geq 1} B_n \). For every \( n \geq 1 \) denote \( h_n \) the piecewise linear maps, all whose braking points belong to \( A_n \) and such that for every \( k, 0 \leq k \leq 2^{n-1} \) the equality
\[
h_n(\alpha_{n,k}) = \beta_{n,k}
\]
holds.

Lemma 16. For every \( n \geq 1 \) and \( k, 0 \leq k \leq 2^{n-1} \) the equality
\[
h_n(f(\alpha_{n,k})) = g(h_n(\alpha_{n,k})) \tag{6.5}
\]
holds.
Proof. Notice, that $\alpha_{n,2^{n-2}} = \frac{1}{2}$. Indeed, consider the sets $A_{n-1}$, $A_{n}^{-} = A_{n,k} \cap [0, 1/2)$ and $A_{n}^{+} = A_{n,k} \cap (1/2, 1]$. The set $A_{n}^{-}$ is obtained from $A_{n-1}$ by taking smaller pre images under $f$ (i.e. pre images under the maps $x \mapsto 2x$) and $A_{n}^{+}$ is obtained from $A_{n-1}$ by taking greater pre images under $f$ (i.e. pre images under $x \mapsto 2 - 2x$). Whence $\# A_{n}^{-} = \# A_{n}^{+}$ and $1/2$ is the middle point of $A_{n}$.

Notice that if $\alpha_{n,k} \leq 1/2$, then $f(\alpha_{n,k}) = \alpha_{n-1,k}$. This follows from the monotone increasing of $f$ on $[0, 1/2]$ and that $f(A_{n}^{-}) = A_{n-1}$.

Notice, that if $\alpha_{n,k} \geq 1/2$, then $f(\alpha_{n,k}) = \alpha_{n-1,2^{n-2}-k}$. This follows from monotone decreasing of $f$ on $[0, 1/2]$ and that $f(A_{n}^{+}) = A_{n-1}$.

Analogously obtain that $\beta_{n,2^{n-2}} = v$. More then this, if $\beta_{n,k} \leq v$, then $g(\beta_{n,k}) = \beta_{n-1,k}$ and if $\beta_{n,k} \geq v$, then $g(\beta_{n,k}) = \beta_{n-1,2^{n-2}-k}$.

Consider two cases: when $\alpha_{n,k} \leq 1/2$ and when $\alpha_{n,k} > 1/2$.

Assume that $\alpha_{n,k} \leq 1/2$. Then (6.5) follows from the following chain of equalities:

$$h_{n}(f(\alpha_{n,k})) = h_{n}(\alpha_{n-1,k}) = \beta_{n-1,k} = g(\beta_{n,k}) = g(h(\alpha_{n,k})).$$

Assume that $\alpha_{n,k} \geq 1/2$. Then (6.5) follows from the following chain of equalities:

$$h_{n}(f(\alpha_{n,k})) = h_{n}(\alpha_{n-1,2^{n-2}-k}) = \beta_{n-1,2^{n-2}-k} = g(\beta_{n,k}) = g(h(\alpha_{n,k})).$$

\(\square\)

For arbitrary $x_{0} \in [0, 1]$ denote $h^{+}(x_{0}) = \lim_{n \to \infty} h_{n}(x_{0})$ and $h^{-}(x_{0}) = \lim_{n \to \infty} h_{n}(x_{0})$.

**Lemma 17.** If for some $x_{0} \in (0, 1)$ the inequality holds $h^{+}(x_{0}) \neq h^{-}(x_{0})$, then maps $f$ and $g$ are not conjugated. In this case $\mathcal{B} \neq [0, 1]$.

**Proof.** Assume that $h$ is a conjugation of $f$ and $g$, which satisfies the functional equation (6.4).

Let the binary decomposition of $x_{0}$ be

$$x_{0} = 0, x_{1}x_{2} \ldots x_{n} \ldots .$$

For every $k \in \mathbb{N}$ denote by $x_{k}^{-}$ the number, whose binary decomposition is

$$x_{k}^{-} = x_{0} = 0, x_{1}x_{2} \ldots x_{k}$$
and denote
\[ x_k^+ = x_k^- + \frac{1}{2^k}. \]

Since \( \{x_k^-, x_k^+\} \subset A_k \), then \( h(x_k^-) = h_k(x_k^-) \) and \( h(x_k^+) = h_k(x_k^+) \).

Since by Lemma 12 the homeomorphism \( h \) increase then \( h^+(x_0) > h^-(x_0) \).

By Theorem 6.1 for every \( k \) homeomorphism \( h_k \) increase and
\[ h(x^-) = h_k(x^-) \leq h^-(x_0) < h^+(x_0) \leq h_k(x^+) = h(x^+), \]
whence \( h(x^+_k) - h(x^-_k) \geq h^+(x_0) - h^-(x_0) \). But the last inequality contradicts to continuity of \( h \) at \( x_0 \).

The fact that \( B \) is not dense in \([0, 1]\) follows from that \( B \cap (h^-(x_0), h^+(x_0)) = \emptyset \). \( \square \)

**Lemma 18.** If the set \( B \) is dense in \([0, 1]\), then there exists a homeomorphism \( h : [0, 1] \to [0, 1] \), which satisfies the functional equation (6.4).

**Proof.** For every \( x \in [0, 1] \) define \( h(x) \) as
\[ h(x) = \lim_{n \to \infty} h_n(x). \]

The existence of the limit and the continuity of \( h \) follows from the density of \( B \) in \([0, 1]\).

The monotonicity of \( h \) follows from that for every \( n \) homeomorphism \( h_n \) increase.

By Lemma 16 equality (6.4) holds for all \( x \in A \). The equality (6.4) for \( x \in [0, 1] \setminus A \) follows from continuity of the function \( h(f(x)) - g(h(x)) \), which is a composition of continuous ones. \( \square \)

### 6.2 Existing of the conjugation

In this section we continue the study of the problem on the topological conjugacy of maps, which were considered in Section 6 above.

\[ f(x) = \begin{cases} 
2x, & 0 \leq x < 1/2; \\
2 - 2x, & 1/2 \leq x \leq 1.
\end{cases} \] (6.6)
Instead of the maps \( g \) consider more precise one \( f_v : [0, 1] \to [0, 1] \), which depends on \( v \in (0, 1) \) and if defined by the following formulas

\[
f_v(x) = \begin{cases} 
\frac{x}{v}, & x \leq v; \\
\frac{1-x}{1-v}, & x > v.
\end{cases}
\] (6.7)

Denote by \( A, B, A_n \) and \( B_n \) the same sets and denote by \( h_n \) the same maps, which were considered in Section 6.1, but now use the function \( f_v \) instead of \( g \).

Remind that the sets \( A_n \) and \( B_n \) are such that \( f^n(A_n) = f^n(B_n) = 0 \) and by Theorem 6.4 for a homeomorphism \( h \) the equality

\[
h(f) = f_v(h)
\] (6.8)

implies that \( h(A_n) = B_n \).

6.2.1 Find the values of conjugation at the dense set

Conditional values of \( h \) at points of \( A_2 = \{0, 1/2; 1\} \) follow from Lemmas 12 and 13 precisely: \( h(0) = 0, h(1/2) = v, h(1) = 1 \). Correspond three points of the graph of \( h \) for \( v = 3/4 \) are given at Figure 24a).

Example 6.2. Find the conditional values of \( h \) at \( A_3 \).
Deal of the example. As conditional values of $h$ at $A_2$ are found in Lemmas 12 and 13 it is sufficient to find $h$ at $A_3 \setminus A_2 = \{1/4, 3/4\}$.

For the point $1/4$ by i. 1 of Proposition 6.2 one have that $h(1/4) = vh(1/2)$. Since by Lemma 13 $h(1/2) = v$, then $h(1/4) = v^2$.

For the points $3/4$ by i. 2 of Proposition 6.2 have that $h(3/4) = 1 - (1 - v)h(1/2) = 1 - v(1 - v)$.

Values of $h$ on $A_3$, if this homeomorphism exists, are given at Figure 24b).

Fig. 24

Example 6.3. Find conditionally values of $h$ on $A_4$.

Deal of the example. In the same manner as in Example 6.2, find all the conditional values of $h$ only in the set $A_4 \setminus A_3 = \{1/8, 3/8, 5/8, 7/8\}$ and use the data from Example 6.2 and Lemmas 12 and 13.

For points $1/8$ and $3/8$, which are less than $1/2$, we have by i. 1 of Proposition 6.2 that $h(1/8) = vh(1/4)$. Since the conditional value $h(1/4)$ is already found in Example 6.2 then $h(1/8) = v^3$. Analogically, $h(3/8) = vh(3/4) = v - v^2(1 - v)$.

For points $5/8$ and $7/8$, which are greater than $1/2$, we have by i. 1 of Proposition 6.2 that $h(5/8) = 1 - (1 - v)h(f(5/8)) = 1 - (1 - v)h(3/4) = 1 - (1 - v)(1 - v(1 - v))$. Analogically $h(7/8) = 1 - (1 - v)h(1/4)1 - v^2(1 - v)$.

The conditional values of $h$ on $A_3$ are given at Figure 24b).

In the same manner as it was done at Example 6.3 we may find the conditional values of $h$ on $A_5$ and on $A_n$ for $n > 5$. Conditional values of $h$ on $A_5$ are given at Figure 24h).
6.2.2 Density of pre images

By Theorem 6.1 the conjugation of $f$ and $f_v$ follows from the density of $B$ in $[0, 1]$, whence we will concentrate on the proving of this fact. We will need the following technical lemmas.

**Lemma 19.** Assume for some numbers $a, b \in [0, 1]$ the increasing maps $g$ such that $g(a) = 0$ and $g(b) = 1$ is given and its graph on $[a, b]$ is a line segment. Then the graph of $s = f_v(g)$ is a braking line, consisted of two line segments such that $s(a) = s(b) = 0$ and $s(t) = 1$, where

$$t = a + v(b - a).$$

*Proof.* The tangent of $g$ is $k = \frac{1}{b-a}$. The tangent of $s$ of the left segment of monotonicity is $\frac{k}{v}$.

In this case the value of $t$ can be found from $s(t) = 1$ as follows. $t = a + \frac{v}{k} = a + v(b-a)$. □

**Lemma 20.** Assume for some numbers $a, b \in [0, 1]$ the increasing maps $g$ such that $g(a) = 1$ and $g(b) = 0$ is given and its graph on $[a, b]$ is a line segment. Then the graph of $s = f_v(g)$ is a braking line, consisted of two line segments such that $s(a) = s(b) = 0$ and $s(t) = 1$, where

$$t = b - v(b - a).$$

*Proof.* The tangent of $g$ is $k = \frac{-1}{\beta - \alpha}$. The tangent of $s$ on the right segment of monotonicity is $\frac{k}{v}$.

In this case the value $t$ can be found from $s(t) = 1$ as follows. $t = \beta + \frac{v}{k} = \beta - v(\beta - \alpha)$. □

The following corollary follows from Lemmas [19] and [20].

**Corollary 6.1.** Let for some $a, b \in [0, 1]$ the monotonic linear $g : [a, b] \rightarrow [0, 1]$ is given. Then the graph of $s = f_v(g)$ is breaking line, which is consisted of two segments and for the extremum $t$ the following equality of sets

$$\left\{ \frac{t - a}{b - a}, \frac{b - t}{b - a} \right\} = \{v, 1 - v\}$$

holds.

**Lemma 21.** The set $B$ is dance in $[0, 1]$. 

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Proof. For every $n \geq 1$ consider the maximum $d_n$ of distances between points of the set $B_n$. 

to prove the density $B$ is the same as to prove that 

$$
\lim_{n \to \infty} d_n = 0.
$$

Consider two arbitrary neighbor points $\beta_{i,n}, \beta_{i+1,n}$ of $B_n$. 

Notice that $\beta_{n,k}$ are extremums of $f_0^{n-1}$. 

It follows from Corollary 6.1 that 

$$
\left\{ \frac{\beta_{n+1,2k+1} - \beta_{n+1,2k}}{\beta_{n+1,2k+1} - \beta_{n+1,2k+2}}, \frac{\beta_{n+1,2k+2} - \beta_{n+1,2k+1}}{\beta_{n+1,2k+1} - \beta_{n+1,2k}} \right\} \in \{v, 1-v\}.
$$

Whence the following bound 

$$
d_{n+1} \leq \max\{v, 1-v\} \cdot d_n
$$

holds for $d_n$ and $d_{n+1}$. It means the density of $B$ in $[0, 1]$ because $0 < \max\{v, 1-v\} < 1$. \qed

The proof of Lemma 21 contains the proof of the following Lemma.

**Lemma 22.** Let $n > 1$ and $0 = \beta_0, \ldots, \beta_{2n-1} = 1$ be points of $B_n$. Then for every $i$ such that $\beta_i \in B_n \setminus B_{n-1}$ the following statements hold:

1. $i \neq 0, i \neq 2^{n-1}$.
2. $\beta_{i-1} \in B_{n-1}, \beta_{i+1} \in B_{n-1}$.
3. $\left\{ \frac{\beta_i - \beta_{i-1}}{\beta_{i+1} - \beta_{i-1}}, \frac{\beta_{i+1} - \beta_i}{\beta_{i+1} - \beta_{i-1}} \right\} = \{v, 1-v\}$.

**Corollary 6.2.** For $n \in \mathbb{N}$ and $t \geq 2$ consider two neighbor points $\beta_i, \beta_{i+1}$ of $B_{n+t}$, which does not belong to $B_n$. Let $a, b \in B_n$ such neighbor points of $B_n$ that $\beta_i, \beta_{i+1} \in (a, b)$. Then

$$
(\min(v, 1-v))^t(b - a) \leq \beta_{i+1} - \beta_i \leq (\max(v, 1-v))^t(b - a).
$$

The following theorem follows from Lemma 21 and Theorem 6.1.

**Theorem 6.5.** For every $v \in (0, 1)$ the functional equation (6.8) has a solution in the class of homeomorphisms $h : [0, 1] \to [0, 1]$ and this solution of unique and increasing.
6.3 Example of non-conjugated maps

Let the maps $f$ be defined by formulas (6.1) and $g$ be piecewise linear $[0, 1] \rightarrow [0, 1]$, whose graph passes through points $(0, 0), (1/2, 1), (2/3, 2/3), (8/9, 5/9)$ and $(1, 0)$.

The graph of $g$ is given at Figure 25a. In other words, graphs of $f$ and $g$ coincide for $x \in [0, \ 2/3]$ but $g$ has a break for $x \in [2/3, 1]$ in the time, when $f$ is linear for $x \in [2/3, 1]$.

It is easy to prove that these $f$ and $g$ are not topologically conjugated.

Lemma 23. The maps $f$ which is defined by formulas (6.1), is not topologically conjugated to piecewise linear $g : [0, 1] \rightarrow [0, 1]$, all whose braking points are $(0, 0), (1/2, 1), (2/3, 2/3), (8/9, 5/9)$ and $(1, 0)$.

Proof. Consider the second iteration of $g$. All the breaking points of the graph of $g^2$ are

\[ M_2 = \{(0, 0), (1/4, 1), (1/3, 2/3), (7/30, 11/30), (1/2, 1), (11/30, 11/30), (7/9, 7/9), (9/11, 1), (1, 0)\} \]

This graph is given on figure 25b. It is evidently that for every $x \in (11/30, 7/9)$ the equality $g^2(x) = x$ holds. If $\psi_1$ is a homeomorphism, which defines the topological conjugacy of $f$ and $g$ then it follows from the commutative diagram

\[
\begin{array}{ccc}
  & x & x \\
  \psi_1^{-1} & \downarrow & \downarrow \psi_1^{-1} \\
 \psi_1^{-1}(x) & \overset{f^2}{\longrightarrow} & \psi_1^{-1}(x)
\end{array}
\]
that $\psi^{-1}(x)$ is a fixed point of $f^2$. This means that either $\psi^{-1}$ is not a homeomorphism, of $f^2$ has an interval of fixed points. This contradiction finishes the proof.

In spite that we have proved in Lemma 23 that $f$ and $g$ are not topologically conjugated, we may consider the reasonings, which are analogical to those, which were used during the construction of the conjugation of $f$ and $f_v$, given by formulas (6.1) and (6.2). Assume that $\psi_1$ is monotone maps such that the following diagram

\[
\begin{array}{c}
[0, 1] \xrightarrow{f} [0, 1] \\
\downarrow \psi_1 \quad \downarrow \psi_1 \\
[0, 1] \xrightarrow{g} [0, 1].
\end{array}
\] (6.9)

is commutative.

More then this, consider the family of $g_v$ instead of $g$ such that reasonings from the proof of Lemma 23 might be repeated and $g_v$ be possible to considered as approximations of $g$ for some specific $v$. In this time construct $g_v$ to be non-conjugated to $f$ for any $v$.

Let for every $v \in [2/3, 1)$ the maps $g_v$ be piecewise linear on $[0, 2/3]$ and all its breaking points be $(0, 0), (1/2, 1/2)$ and $(2/3, 2/3)$. Let the tangent of $g_v$ on $(2/3, v)$ be equal to $1/2$ and let $g$ be linear on $(v, 1)$ such that $g_v(1) = 0$. The would guarantee, that $g^2$ would have tangent 1 in some neighborhood of $2/3$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig26.png}
\caption{Fig. 26}
\end{figure}

Is is easy to prove that if the diagram (6.9) is commutative for $f, g_v$ and a monotone $\psi_1$ such that $\psi_1(\{0, 1\}) = \{0, 1\}$, then $\psi_1$ increase. The values of $\psi_1$ at $A_7$ for $v = 0, 7, 0.75, 0.9$
are given at Figure 26. It is seen from these Figures that \( \psi_1 \) is discontinuous. The proper proof of the discontinuity of \( \psi_1 \) is similar to the proof of Theorem 23.

Remind that \( g \in [0, 1] \rightarrow [0, 1] \) is called **unimodal** if there exist \( a \in (0, 1) \) such that \( g \) is monotone on \( [0, a] \), also \( g \) is monotone on \( [a, 1] \), but \( g \) is not monotone on \( [0, 1] \).

**Theorem 6.6.** For every \( x_0 \in [0, 1] \) and every \( \varepsilon > 0 \) there exists a maps \( g : [0, 1] \rightarrow [0, 1] \) with the following properties.

1. \( g \) is unimodal;
2. \( g(x) = f(x) \) for every \( x \in [0, 1] \setminus (x_0 - \varepsilon, x_0 + \varepsilon) \);
3. \( f \) and \( g \) are not topologically conjugated.

**Proof.** If \( x_0 = 1/2 \), then take \( g(x_0) = 1 - \varepsilon/2 \). After this take the maps \( g \) to be piecewise linear, whose graph passes through points \((0, 0), (1/2 - \varepsilon, 1 - 2\varepsilon), (1/2, 1 - \varepsilon), (1/2 + \varepsilon, 1 - 2\varepsilon), (1, 0)\).

The constructed \( g \) would not be topologically conjugated to \( f \), because the every point has at last one pre-image under \( f \) in the time, when 1 has no any pre images under \( g \).

Assume that \( x_0 \neq 1/2 \). Without losing the generality assume that \( 1/2 \notin (x_0 - \varepsilon, x_0 + \varepsilon) \), because otherwise just decrease \( \varepsilon \).

Since the set of periodical points of \( f \) is dense in \([0, 1]\), there exists a periodical point \( x^* \in (x_0 - \varepsilon, x_0 + \varepsilon) \). Let \( n \) be its period.

This means that \( x^* \) would be a periodical point of \( f^n \) and, correspondingly, either \( (f^n)'(x^*) = 2^n \), or \( (f^n)'(x^*) = -2^n \). The point \( x^* \) also would be a fixed point of \( f^{2n} \) and \( (f^{2n})'(x^*) = 2^{2n} \).

for the number
\[
\delta = \frac{\min\{x^* - (x_0 - \varepsilon), (x_0 + \varepsilon) - x^*\}}{2}
\]
construct the maps \( g \) as follows.

1. Tangent of \( g \) on \((x^* - \delta, x^* + \delta) \cap [0, 1] \) equals \( \frac{f'(x^*)}{2^n} \);
2. \( g(x) = f(x) \) for every \( x \in [0, 1] \setminus (x_0 - \varepsilon, x_0 + \varepsilon) \);
3. The maps \( g \) is linear on each of two intervals of the set
\[
((x_0 - \varepsilon, x_0 + \varepsilon) \cup [0, 1]) \setminus (x^* - \delta, x^* + \delta).
\]

The neighborhood of of the periodical point \( x^* \) of period 3 is given at Figure 25c.
The fact that $g$ would not be topologically conjugated to $f$ may be proved in the same manner as in the proof of Theorem 23.

If maps $f$ and $g$ are topologically conjugated, then so are (via the same homeomorphism) $f^{2n}$ and $g^{2n}$. In the same time, by construction $(g^{2n})(x) = x$ for all $x \in (x^* - \delta, x^* + \delta)$ and $f^{2n}$ has no an interval of fixed points. This contradiction finishes the proof. □
7 On the differentiability of conjugation

We continue in this section to consider the problem about the topological conjugation of the maps $f, f_v : [0, 1] \to [0, 1]$, where

$$f(x) = \begin{cases} 2x, & x < 1/2; \\ 2 - 2x, & x \geq 1/2 \end{cases}$$

and

$$f_v(x) = \begin{cases} \frac{x}{v}, & x \leq v; \\ \frac{1-x}{1-v}, & x > v. \end{cases}$$

for $v \in (0, 1)$ being a parameter. By Theorem 6.5, there exists and it is unique the homeomorphism $h : [0, 1] \to [0, 1]$ such that the following would commute.

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, 1] & \xrightarrow{f_v} & [0, 1],
\end{array}
$$

(7.1)

This Section is devoted to the differentiability of the homeomorphism $h$. This problem was inspirit by the following result.

**Theorem 7.1.** The derivative of the homeomorphism $h$ such that the diagram (7.1) is commutative, exists almost everywhere and equals to 0 everywhere it is finite.

This result is given at [64, Proposition 2]. It is formulated there as follows: the derivative of $h$, which makes the diagram (7.1) commutative, exists almost everywhere and equals 0 everywhere, where it exists. Nevertheless, it is seen from the proof in [64], that under the assumption about the existence of the derivative, authors mean also the finiteness of the derivative. They consider an arbitrary point $x \in (0, 1)$ and construct the sequence $k_n$ such that $x \in I_n = \left[\frac{k_n}{2^n}, \frac{k_n+1}{2^n}\right]$ and numbers $p_n = h\left(\frac{k_n+1}{2^n}\right) - h\left(\frac{k_n}{2^n}\right)$. They claim that if the derivative $h'(x)$ exists and does not equal to 0, then $\frac{p_{n+1}}{p_n} \to \frac{1}{2}$. But such reasonings are correct only in the case when the derivative $h'(x)$ is also finite.

Following [21, sect. 92, 101], we will assume that the derivative of a function of real argument is the limit of the ratio of the its increment over the correspond increment of the argument.
The derivative (finite or unfinite) is said to exists if and only if the mentioned limit exists and the derivative equals to the value of the limit.

The interest of Theorem 7.1 is because of Lebesgue theorem on the differentiability of the monotone function.

**Theorem 7.2.** Every monotone function on the interval has finite derivative almost everywhere.

Whence, from one hand by Theorem 7.2 the derivative of \( h \) exists almost everywhere, but from another hand by Theorem 7.1 this derivative equals 0 everywhere, where it is finite.

Remind that we have considered in Section 6 the sets \( A_n, B_n, A \) and \( B \), which are defined as follows.

\( A_n, n \geq 1 \) is the set of all points \( x \in [0, 1] \) such that
\[
f^n(x) = 0.
\]

\( B_n, n \geq 1 \) is the set of all \( x \in [0, 1] \) such that
\[
f^n_v(x) = 0.
\]

For every \( n \geq 1 \) we have denoted by \( \alpha_{n,k}, \beta_{n,k}, 0 \leq k \leq 2^{n-1} \) the increasingly ordered elements of \( A_n \) and \( B_n \) correspondingly, i.e. for any \( k_1 < k_2 \) inequalities \( \alpha_{n,k_1} < \alpha_{n,k_2} \) and \( \beta_{n,k_1} < \beta_{n,k_2} \) hold.

We have denoted \( A = \bigcup_{n \geq 1} A_n \) and \( B = \bigcup_{n \geq 1} B_n \). For every \( n \geq 1 \) denote \( h_n \) the piecewise linear maps, all whose braking points belong to \( A_n \) and such that for every \( k, 0 \leq k \leq 2^{n-1} \) the equality
\[
h_n(\alpha_{n,k}) = \beta_{n,k}
\]
holds.

By Theorem 6.4 for every \( n \) the conjugation \( h \) coincides with \( h_n \) on \( A_n \).

### 7.1 Limits of derivatives of approximations of the conjugation

Consider as arbitrary point \( x \in (0, 1) \setminus A \) and find the limit of the sequence \( \lim_{n \to \infty} h'_n(x) \). The condition \( x \not\in A \) guarantee that for every \( n \geq 1 \) the limit \( h'_n(x) \) exists.
Let the binary decomposition of $x$ be as follows.

$$x = 0, x_1x_2 \ldots x_k \ldots \quad (7.2)$$

For the number $x$ of the form (7.2), denote $x_0 = 0$ and for every $i \geq 2$ denote

$$\alpha_i(x) = \begin{cases} 
2v & \text{if } x_{i-1} = x_{i-2}, \\
2(1-v) & \text{if } x_{i-1} \neq x_{i-2}.
\end{cases} \quad (7.3)$$

**Theorem 7.3.** For every $n \geq 2$ and a number $x \notin A_n$ of the form (7.2), the equality

$$h'_n(x) = \prod_{i=2}^{n} \alpha_i(x)$$

holds, where $\alpha_i(x)$ are defined by (7.3).

**Proof.** Consider the maps $f^n_v$. Its set of zeros is $\beta_{n,k}$, $0 \leq k \leq 2^n - 1$. The set of zeros of $f^{n-1}_v$ is $\beta_{n,2k}$, $0 \leq k \leq 2^{n-2}$ and the set of solution of the equation $f^{n-1}_v(x) = 1$ is $\beta_{n,2k+1}$, $0 \leq k \leq 2^{n-2} - 1$.

Consider the maps $f^{n-1}_v$ for $x \in [\beta_{n,2k}, \beta_{n,2k+2}]$. The graph of this maps passes through the points $M(\beta_{n,2k}, 0), Q(\beta_{n,2k+1}, 1)$ and $N(\beta_{n,2k+2}, 0)$ as shown on Figure 27a.

For $x \in (\beta_{n,2k}, \beta_{n,2k+1})$ and the maps $f^{n-1}_v$ it follows from Lemma 19 that the graph of $f_v(f^{n-1}_v)$ passes through points $M(\beta_{n,2k}, 0), S(t_1, 1)$ and $K(\beta_{n,2k+1}, 0)$, where $t_1 = \beta_{n,2k} + v(\beta_{n,2k+1} - \beta_{n,2k})$, as it is shown on the Figure 27b. Nevertheless, since $f^n_v(t_1) = 1$, then $f^{n+1}_v(t_1) = 0$, i.e. $t_1 \in B_{n+1}$, whence

$$\beta_{n+1,4k+1} = \beta_{n,2k} + v(\beta_{n,2k+1} - \beta_{n,2k}). \quad (7.4)$$

In terms of notations of the Figure 27b the equality (7.4) means that

$$\frac{PS}{PQ} = v. \quad (7.5)$$

For $x \in (\beta_{n,2k+1}, \beta_{n,2k+2})$ and the maps $f^{n-1}_v$ is follows from Lemma 20 that graph of $f_v(f^{n-1}_v)$ passes through points $K(\beta_{n,2k+1}, 0), T(t_2, 1)$ and $N(\beta_{n,2k+2}, 0)$, where $t_2 = \beta_{n,2k+1} + (1-v)(\beta_{n,2k+2} - \beta_{n,2k+1})$. Since $f^n_v(t_2) = 1$, then $f^{n+1}_v(t_2) = 0$, i.e. $t_2 \in B_{n+1}$, whence

$$\beta_{n+1,4k+3} = \beta_{n,2k+1} + (1-v)(\beta_{n,2k+2} - \beta_{n,2k+1}). \quad (7.6)$$
In terms of notations of the Figure 27b the equality (7.4) means that
\[
\frac{QT}{RQ} = 1 - v. \tag{7.7}
\]
Denote by \(\gamma_1\) and \(\gamma_2\) correspondingly the the coefficients of \(h_n\) on intervals \(A_{n,2k} = (\frac{2k}{2^n - 1}, \frac{2k+1}{2^n - 1})\) and \(A_{n,2k+1} = (\frac{2k+1}{2^n - 1}, \frac{2k+2}{2^n - 1})\). It means that \(\gamma_1 = 2^{n-1}(\beta_{n,2k+1} - \beta_{n,2k})\) and \(\gamma_2 = 2^{n-1}(\beta_{n,2k+2} - \beta_{n,2k+1})\).

Let \(x \in A_{n+1,4k}\). From (7.4) obtain
\[
h'_{n+1}(x) = 2^n(\beta_{n+1,4k+1} - \beta_{n+1,4k}) = 2^n v(\beta_{n,2k+1} - \beta_{n,2k}) = 2v \gamma_1.
\]
Let \(x \in A_{n+1,4k+1}\). From (7.4) obtain
\[
h'_{n+1}(x) = 2^n(\beta_{n+1,4k+2} - \beta_{n+1,4k+1}) =
= 2^n (\beta_{n,2k+1} - (\beta_{n,2k} + v(\beta_{n,2k+1} - \beta_{n,2k}))) = 2(1 - v) \gamma_1.
\]
Let \(x \in A_{n+1,4k+2}\). From (7.6) obtain
\[
h'_{n+1}(x) = 2^n(\beta_{n+1,4k+3} - \beta_{n+1,4k+2}) = 2^n (\beta_{n,2k+2} - \beta_{n,2k+1}) = 2(1 - v) \gamma_2.
\]
Let \(x \in A_{n+1,4k+3}\). From (7.6) obtain
\[
h'_{n+1}(x) = 2^n (\beta_{n+1,4k+4} - \beta_{n+1,4k+3}) = 2v \gamma_2.
\]

The maps \(h_{n-1}\) divides \([0, 1]\) to \(2^{n-2}\) equal intervals. Each of these intervals is defines by \(n - 1\) the first numbers of the binary decomposition of its points. More then this, if the binary
composition of $x$ is of the form (7.2), then the binary decomposition of a natural $k$ is $x_1 \ldots x_{n-2}$ (where, as in general, the first zeros should be ignored).

The inclusion $x \in A_{n+1,4k}$ is equivalent to that (7.2) is of the form $x = 0.x_1 \ldots x_{n-2}00 \ldots$.

The inclusion $x \in A_{n+1,4k+1}$ is equivalent to that (7.2) is of the form $x = 0.x_1 \ldots x_{n-2}01 \ldots$.

The inclusion $x \in A_{n+1,4k+2}$ is equivalent to that (7.2) is of the form $x = 0.x_1 \ldots x_{n-2}10 \ldots$.

The inclusion $x \in A_{n+1,4k+3}$ is equivalent to that (7.2) is of the form $x = 0.x_1 \ldots x_{n-2}11 \ldots$.

Now evident inductive reasonings finish the proof.

The following theorem 7.3 follows from the proved one.

---

Theorem 7.4. 1. If $v < 1/2$ then for every $x \in A$ the limit $\lim_{n \to \infty} h'_n(x) = 0$ holds.

2. If $v > 1/2$, then for every $x \in A$ the limit $\lim_{n \to \infty} h'_n(x) = \infty$ holds.

3. For every $v \in (0, 1) \setminus \{1/2\}$ the following limits $\lim_{n \to \infty} \min_{x \in (0,1) \setminus A} h'_n(x) = 0$ and $\lim_{n \to \infty} \max_{x \in (0,1) \setminus A} h'_n(x) = \infty$ hold.

The i. 3 of Theorem is illustrated at Figure 28, where are given graphs of $h_4$ for $v = 3/4$ and $v = 3/4$. For an arbitrary $k \in \mathbb{N}$ consider the number, whose the binary decomposition is

$$x_0 = 0.101010 \ldots$$

Then by Theorem 7.3 we have that

$$h'_{2k+1}(x_0) = (2(1-v))^{2k}.$$
In the same time
\[
\lim_{n \to \infty} h_n(x_0) = \lim_{n \to \infty} (2v)^n.
\]

### 7.2 Values of the derivative of conjugation

We will prove Theorem 7.1 in this section, i.e. we will prove the Proposition 2 from [64], which states that the derivative of \( h \) can be equal either 0, or infinity. Also we will find the values of the derivative of conjugation at rational points.

Remind that non triviality of Theorem 7.1 follows from the Lebesgue Theorem (Theorem 7.2) about differentiability of the monotonic function.

**Proof of Theorem 7.1.** Let for the point \( x \in (0, 1) \) the derivative \( h'(x) = t \) exists.

For every \( n \geq 1 \) denote by \( k \) as interval such that \( x \in [\alpha_{n,k}, \alpha_{n,k+1}) \).

The condition \( h'(x) = t \) is equivalent to that
\[
h(x) - h(\alpha_{n,k}) = t(x - \alpha_{n,k}) + m_n^{(1)} \tag{7.8}
\]
and
\[
h(\alpha_{n,k+1}) - h(x) = t(\alpha_{n,k+1} - x) + m_n^{(2)}, \tag{7.9}
\]
where \( m_n^{(1)} \) and \( m_n^{(2)} \) are sequences, which tend to 0 for \( n \to \infty \).

By adding the equalities (7.8) and (7.9) obtain that
\[
h(\alpha_{n,k+1}) - h(\alpha_{n,k}) = t(\alpha_{n,k+1} - \alpha_{n,k}) + m_n^{(3)}; \tag{7.10}
\]
where \( m_n^{(1)} = m_n^{(1)} + m_n^{(2)} \).

Nevertheless, equality (7.10) can be rewritten as
\[
h_n'(x) = t + m_n^{(3)},
\]
which means that \( \lim_{n \to \infty} h_n'(x_0) = t \).

Now theorem follows from parts 1 and 2 of Theorem 7.4.

**Lemma 24.** If \( v < 1/2 \) then for every \( x_0 \in A \) there exists the derivative \( h'(x_0) = \infty \) and if \( v > 1/2 \) then for every \( x_0 \in A \) there exists a derivative \( h'(x_0) = 0 \).
Proof. For \( x_0 \neq 0 \) consider that the left derivative \( h'(x_0 - 0) \) exists and is equal to that, which is stated in the Lemma (the condition of \( x_0 \) to be positive comes from that the left derivative for \( h'(x_0 - 0) \) is undefined for \( x_0 = 0 \)).

Since the sets \( \{ A_n, n > 0 \} \) are embedded, then there exists \( n_0 > 0 \) such that \( x_0 \in A_n \) for all \( n > n_0 \).

For every \( n > n_0 \) denote by \( \alpha_n \) the biggest element of \( A_n \), which is less than \( x_0 \). Since \( A_n \) consists of all rational numbers, whose denominators in the proper form are divisors of \( 2^{n-1} \), then \( \alpha_n = x_0 - 2^{-n+1} \). It follows from the construction of \( h_n \) that for every \( n > n_0 \) the equality \( h_n'(x_0 - 0) = \frac{h(x_0) - h(\alpha_n)}{x_0 - \alpha_n} \) holds, because \( h_n(x_0) = h(x_0) \) and \( h_n(\alpha_n) = h(\alpha_n) \).

It follows from continuity of \( h \) that for every \( x \in (\alpha_n, \alpha_{n+1}] \) the double following inequality holds. This equality can be rewritten as \( 2h_n'(x_0) \leq \frac{h(x_0) - h(\alpha_n)}{x_0 - \alpha_n} \leq \frac{h_n'(x_0)}{2} \).

Consider now an arbitrary increasing sequence \( x_k \), which tends to \( x_0 \) and prove the equality \( \lim_{k \to \infty} \frac{h(x_0) - h(x_k)}{x_0 - x_k} = \lim_{n \to \infty} h_n'(x_0 - 0) \). For every \( k \) there exists an \( n_k \) such that \( x_k \in (\alpha_{n_k}, \alpha_{n_k+1}] \). Then \( 2h_{n_k+1}'(x_0) \leq \frac{h(x_0) - h(x_k)}{x_0 - x_k} \leq \frac{2h_{n_k}'(x_0)}{2} \). The last double inequality proves Lemma independently on whether \( v > 0, 5 \) or \( v < 0, 5 \) and, correspondingly, independently on the value of the limit \( \lim_{n \to \infty} h_n'(x_0) \).

The proof for the right derivative \( h'(x_0 + 0) \) is analogical. \( \square \)

Theorem 7.5. Let \( x_0 \in [0, 1] \cap \mathbb{Q} \). Then the derivative \( h'(x_0) \) exists. More then this, if \( v < 1/2 \) then \( h'(x_0) = \infty \) and if \( v > 1/2 \) then \( h'(x_0) = 0 \).

Proof. Because of Lemma [24] we can restrict our consideration on the case \( x_0 \in \mathbb{Q} \setminus \mathcal{A} \). Consider the binary decomposition of \( x_0 \). Since \( x_0 \in \mathbb{Q} \setminus \mathcal{A} \), then

\[
x_0 = 0, x_1x_2 \ldots x_p(x_{p+1} \ldots x_{p+t}), \tag{7.11}
\]

where \( x_{p+1} \ldots x_{p+t} \) is a periodical part of \( x_0 \) and not all digits of this periodical part equal to 1, because in this case \( x_0 \in \mathcal{A} \).

For an arbitrary sequence \( \{ s_n \} \), which converge to \( x_0 \), consider numbers

\[
k(s_n) = \frac{h(x_0) - h(s_n)}{x_0 - s_n}.
\]

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Consider an arbitrary \( n \in \mathbb{N} \). For every \( s \) denote by \( \alpha_{ns} \) the closest from the left element \( x_0 \) of \( A_s \) and denote by \( \alpha_{ns}^+ \) the element from \( A_s \), which is the closest from the right to \( x_0 \). Choose the maximal \( s_0 \) such that \( s_n \in [\alpha_{ns_0}, \alpha_{ns_0}^+] \). It follows from the maximality of \( s_0 \) and \( x_0 \not\in A \) that numbers \( x_0, s_n \) belong to different “halves” of the interval \([\alpha_{ns_0}, \alpha_{ns_0}^+]\), i.e. it follows from the inclusion \( x_0 \in \left( \alpha_{ns_0}, \frac{\alpha_{ns_0} + \alpha_{ns_0}^+}{2} \right) \) that \( s_n \in \left[ \frac{\alpha_{ns_0} + \alpha_{ns_0}^+}{2}, \alpha_{ns_0}^+ \right] \). We will write \( \alpha_n \) and \( \alpha_n^+ \) instead of \( \alpha_{ns} \) and \( \alpha_{ns}^+ \).

\[
\text{Fig. 29}
\]

The Figure 29 contains points \( A(\alpha_n, h(\alpha_n)), B(\alpha_n^+, h(\alpha_n^+)) \) and \( X(x_0, h(x_0)) \) for the case \( t = 2, x_{k+1} = 1, x_{k+2} = 0 \). Since \( t = 2 \), then the segment \( x \in (\alpha_n, \alpha_n^+) \) is divided by vertical lines to \( 2^t = 4 \) parts. Since \( x_{k+1} = 1 \), then \( x_0 \) is more right than the middle of the interval \((\alpha_n, \alpha_n^+)\) and since \( x_{k+2} = 0 \), then \( x_0 \) is more left then the middle of the interval \((\frac{\alpha_n + \alpha_n^+}{2}, \alpha_n^+)\).

The intermediate vertical and horizontal lines at Figure 29 correspond to values from the sets \( A_{s+t} \) and \( B_{s+t} \).

Denote by \( \tilde{\alpha}_n \) and \( \tilde{\alpha}_n^+ \) the closest left to \( x_0 \) and the closest right to \( x_0 \) points of the set \( A_{p+(n+1)t+1} \). On the Figure 29 points \((\tilde{\alpha}_n, h(\tilde{\alpha}_n))\) and \((\tilde{\alpha}_n^+, h(\tilde{\alpha}_n^+))\) are denoted by circles.

Assume that for some \( n \) the inclusion \( s_n \in [\alpha_n, \tilde{\alpha}_n] \) holds. Find the bounds for \( k(s_n) \). We will use this bounds to prove the theorem for the left derivative \( h'_-(x_0) \). The prove the the right derivative \( h'_+(x_0) \) is analogical.

The figure 30 contains lines, whose tangents bound \( k(s_n) \), because the points with coordinates \((s_n, h(s_n))\) is somewhere in the shades rectangle in the case when \( s_n \in [\alpha_n, \tilde{\alpha}_n] \).

Consider the intersection of the set \( A_{s+2t} \) with the points of the interval \((\tilde{\alpha}_n, \tilde{\alpha}_n^+)\) and consider points of \( B_{s+2t} \), which correspond to this intersection. Put these points to the Figure 31.
in the manner, which was used for the Figure 29.

Denote by $\hat{\alpha}_n$ and $\hat{\alpha}_n^+$ the points of $A_{s+2t}$, which are the closest to $x_0$ from the left and from the right correspondingly. Points $C$, $D$, $R$, $S$ on Figure 31 are of coordinates $C(\tilde{\alpha}_n, h(\alpha_n))$, $D(\hat{\alpha}_n, h(\hat{\alpha}_n))$, $R(\alpha_n, h(\hat{\alpha}_n))$ and $S(\hat{\alpha}_n^+, h(\hat{\alpha}_n))$. Evidently then $k(s_n)$ is bounded by tangents of lines $CD$ and $RS$ which are

$$k_{CD} = \frac{h(\alpha_n^+) - h(\alpha_n)}{\hat{\alpha}_n - \tilde{\alpha}_n}$$  \hspace{1cm} (7.12)

and

$$k_{RS} = \frac{h(\hat{\alpha}_n) - h(\hat{\alpha}_n)}{\alpha_n^+ - \alpha_n}.$$  \hspace{1cm} (7.13)
Find the lower bound for $k_{RS}$ and the upper bound for $k_{CD}$.

Denote by $d$ the number whose the binary decomposition is

$$d = x_{p+1} \ldots x_{p+t},$$

where $x_{p+1}, \ldots, x_{p+t}$ are taken from (7.11). Then it follows from the construction of numbers $\alpha_n, \alpha_n^+, \tilde{\alpha}_n, \tilde{\alpha}_n^+, \hat{\alpha}_n$ and $\hat{\alpha}_n^+$ that the following equalities hold.

$$\frac{\tilde{\alpha}_n - \alpha_n}{\alpha_n^+ - \alpha_n} = \frac{d}{2^t}.$$  

$$\frac{\hat{\alpha}_n - \tilde{\alpha}_n}{\hat{\alpha}_n^+ - \tilde{\alpha}_n} = \frac{d}{2^t}.$$ 

It follows from these two equalities that

$$\hat{\alpha}_n - \tilde{\alpha}_n = \frac{d(\alpha_n^+ - \alpha_n)}{2^{2t}},$$  

(7.14)

$$\hat{\alpha}_n^+ - \alpha_n = (\tilde{\alpha}_n - \alpha_n) + (\hat{\alpha}_n - \tilde{\alpha}_n) + (\tilde{\alpha}_n^+ - \hat{\alpha}_n) =$$

$$= \frac{d(\alpha_n^+ - \alpha_n)}{2^t} + \frac{d(\alpha_n^+ - \alpha_n)}{2^{2t}} + \frac{\alpha_n^+ - \alpha_n}{2^{2t}}.$$  

(7.15)

If follows from Lemma 22 that for every $i \in \mathbb{N}$ the distances between the neighbor points of $B_i$ are not equal. By Corollary 6.2 the following bounds hold.

$$d(\min\{v, 1 - v\})^{2t} \leq \frac{h(\hat{\alpha}_n) - h(\tilde{\alpha}_n)}{h(\alpha_n^+) - h(\alpha_n)} \leq d(\max\{v, 1 - v\})^{2t}.$$  

(7.16)

$$d(\min\{v, 1 - v\})^t + (d + 1)(\min\{v, 1 - v\})^{2t} \leq \frac{h(\hat{\alpha}_n^+) - h(\alpha_n)}{h(\alpha_n^+) - h(\alpha_n)} \leq$$

$$\leq d(\max\{v, 1 - v\})^t + (d + 1)(\max\{v, 1 - v\})^{2t}.$$  

(7.17)

By formulas (7.12), (7.14) and (7.17) we have

$$m_1(d)h_{p+nt+1}'(x_0) \leq k_{CD} \leq M_1(d)h_{p+nt+1}'(x_0),$$  

(7.18)

where
\[ m_1(d) = \frac{2t(d(\min\{v, 1-v\})^t + (d+1)(\min\{v, 1-v\})^{2t})}{d^2} \]

and
\[ M_1(d) = \frac{2t(d(\max\{v, 1-v\})^t + (d+1)(\max\{v, 1-v\})^{2t})}{d^2}. \]

By formulas (7.13), (7.15) and (7.16) we have
\[ m_2(d) = \frac{d(\min\{v, 1-v\})}{2^t} + \frac{d^2}{2^{2t}} + \frac{1}{2^{2t}} \]
\[ M_2(d) = \frac{2^{2t}d(\max\{v, 1-v\})}{2^t d^2 + d^2 + 1}. \]

Notice, that constants \( m_1, m_2, M_1 \) and \( M_2 \) are dependent on \( x_0 \), which is independent on \( n \). These constants are independent on \( d \).

Since the periodical part of \( x_0 \) can be written in different ways (i.e. the first digit of the periodical part of \( x_0 \) can be chosen differently), we have that this periodical part can be given of the form
\[ \sigma^i(x_{p+1} \ldots x_{p+t}), \]
where \( \sigma \) is a cyclic permutation of \( t \) elements and \( i \) is some its iteration. Define by \( d_i \) the natural number, whose the binary decomposition is the sequence (7.20). Clearly, that \( d_i \) is a periodical sequence of period \( t \). Denote by \( m_1(x_0) = \min_{1 \leq i \leq t} m_1(\sigma^i(d_i)), m_2(x_0) = \min_{1 \leq i \leq t} m_2(\sigma^i(d_i)), M_1(x_0) = \max_{1 \leq i \leq t} M_1(\sigma^i(d_i)) \) and \( M_2(x_0) = \max_{1 \leq i \leq t} M_2(\sigma^i(d_i)) \). Then the following restrictions, which are analogical to (7.18) and (7.19) would hold.
\[ m_1(x_0)h'_{p+nt+1}(x_0) \leq k_{CD} \leq M_1(x_0)h'_{p+nt+1}(x_0), \]
\[ m_2(x_0)h'_{p+nt+1}(x_0) \leq k_{RS} \leq M_2(x_0)h'_{p+nt+1}(x_0). \]
Since
\[ k_{RS} \leq k(s_n) \leq k_{CD}. \]
we have that

\[ m_2(x_0)h'_{p+nt+1}(x_0) \leq k(s_n) \leq M_1(x_0)h'_{p+nt+1}(x_0). \] (7.21)

Now Theorem follows from Theorem 7.3 and the evident remark that the limit \( \lim_{n \to \infty} h'_n(x_0) \) exists for all \( x_0 \in \mathbb{Q} \).

Theorem 7.5 can be considered as a generalization of Theorem 7.4 to the set of rational numbers. It follows from Theorem 7.1 that Theorem 7.4 can not be generalized to the set of all real numbers \( x_0 \in [0, 1] \) such that the limit \( \lim_{n \to \infty} h'_n(x_0) \) exists.

The following proposition holds.

**Proposition 7.6.** For every \( v \neq 1/2 \) there exists \( x_0 \in [0, 1] \) such that \( h'(x_0) = 0 \) and one of the following conditions hold.

1. The limit \( \lim_{n \to \infty} h'_n(x_0) \) does not exist;
2. The limit \( \lim_{n \to \infty} h'_n(x_0) \) exists, but equals \( \infty \).

**Proof.** Proposition follows from Theorems 7.1 and 7.4 if consider \( f_{1-v} \) instead of \( f_v \).

The following observation follows from Theorems 7.1 and 7.4. Let \( \lambda \) be the Lebesgue measure on the interval \([0, 1]\). Denote by \( \mathcal{A}^0 \) the set, where the derivative of \( h \) equals to 0 and denote by \( \mathcal{B} \) the set, where the derivative of \( h \) equals to infinity. Then \( \lambda(\mathcal{A}^0) = 1 \), \( \lambda(\mathcal{B}^0) = 0 \) and \( h \) is non-differentiable on \([0, 1]\)\(\setminus(\mathcal{A}^0 \cup \mathcal{B}^0)\). It is evident, that the derivative of the inverse function \( h^{-1} \) equals \( \infty \) on \( h(\mathcal{A}^0) \) and this derivative equals 0 on \( h(\mathcal{B}^0) \). Now, it follows from Theorems 7.1 and 7.4 that \( \lambda(h(\mathcal{A}^0)) = 0 \) and \( \lambda(h(\mathcal{B}^0)) = 1 \). These properties of \( h \) show how complicated it is.
8 Constructing of the conjugation via electronic tables

This section is devoted to the explicit formulas for the conjugation $h$ of maps

\[
f(x) = \begin{cases} 
2x, & x < 1/2; \\
2 - 2x, & x \geq 1/2 
\end{cases} \tag{8.1}
\]

and

\[
f_{v}(x) = \begin{cases} 
\frac{x}{v}, & x \leq v; \\
\frac{1-x}{1-v}, & x > v. 
\end{cases} \tag{8.2}
\]

In other words we will find the explicit formulas for the homeomorphism $h : [0, 1] \rightarrow [0, 1]$, which is the unique solution of the functional equation

\[
h(f) = f_{v}(h). \tag{8.3}
\]

The existence and uniqueness of this $h$ is proved in Theorem 6.5.

In fact, Proposition 6.2 contains the way of constructing of the conjugation $h : [0, 1] \rightarrow [0, 1]$ of maps $f$ and $f_v$ at points of the set

\[A_n = \left\{ 0, \frac{1}{2^{n-1}}, \ldots, \frac{2^{n-1} - 1}{2^{n-1}}, 1 \right\}\]

as a limit of piecewise linear maps $h_n : [0, 1] \rightarrow [0, 1]$, whose breaking points belong to the set $A_n$ and such that $h(A_n) = B_n$, where

\[B_n = f_v^{-n}(0)\]

Under the electronic table we mean the table, whose lines are numbered by arabic numbers (1, 2, ...), and columns are numbered by letters ("A", "B", "C" ...), and the following changes of this table are allowed.

1. Put some number into some cell;
2. Put the formula into some cell. The formula may contain symbols of arithmetical operations the most known mathematical functions and names of another cells. Also formula may contain some specific functions, which are specially deals with electronic tables (we will mention these functions below);
3. To copy the formula from the cell into any fixed number of cells in vertical and (or) horizontal direction. In these case the general agreement on sell copying holds (we will explain this agreement just below).

Now we will explain a bit these rules. Since lines and columns of the table are numbered as they are, then for instance, the left top cell is “A1” and it is above “A2”. The cells to the right from “A1” is “B1” and so on.

If a cells contains a formula, then this cell has two “parameters”: the formula itself and the value of the formula. The simplest example could be the formula “A1+1”, which is put into any cell except “A1” (for instance “A2”). The value of the cell would be a number, which is 1 more then “A1”. It is important (and it is the one of the main deals of the use of electronic tables), that if the cell “A1” (in our case) will be changed, then the value of our cell will be changed immediately and automatically. This rule is transitive. It means, that (for instance in our case) that changing of “A1” leads to changing not only the value of “A2”, but also all the cell, which contain formulas, which use “A2”, because “A2” would be changed because of the change of “A1”.

the agreement about copying the formulas is the following. If we copy the formula $k$ cells down, then all the references to cells (i.e. names of cells) will be changer by increasing the number of lines of cells by $k$. The similar rule is if the cell is copying up of horizontally. Clearly, in the case of copying horizontally, the thing which is changes is numbers of columns of references to cells in the formula. These rules can be not applied to the reference, which contain the symbol “$” before the name of a column and (or) the number of line. For instance, the formula “A1+1” will be transformed to “A2+1” when copying one cell down and it will be transformed to “B1+1” when copying one cell wight. In the same time, the formula “A$1+1” becomes “B$1+1” when copying right and does not transformed when copying vertically. The formula “$A$1+1” does not transforms under copying at all.

Let is return to finding the values of the homeomorphism $h$. For an arbitrary natural $n > 1$ we will construct the table of values of $h$ at all rational points of the form $\frac{k}{2^n}$, where $0 \leq k \leq 2^n$.

For the convenience of the further use, we will reformulate Proposition 6.2 as follows.

**Proposition 8.1.** Let $h : [0, 1] \rightarrow [0, 1]$ be the topological conjugation of the maps $f$ and
\[f_v, \text{ which are defined by formulas (8.1) and (8.2), i.e. } h \text{ is the solution of the functional equation (8.3)}. \text{ Then the following implications hold.}
\]

1. If \(x = 0\), then \(h(x) = 0\).
2. If \(x = 1\), then \(h(x) = 1\).
3. If \(x \leq \frac{1}{2}\), then
   \[h(x) = v \cdot h(2x)\]
   and the value \(h(2x)\) appears to be found earlier.
4. If \(x > \frac{1}{2}\), then
   \[h(x) = 1 - (1 - v) \cdot h(-2x + 2)\]
   and the value \(h(-2x + 2)\) appears to be found earlier.

Put \(n\) into “C1”. We will assume that “C1” contains an integer greater then 1 and will not check the correctness of the data from “C1”. Also put \(v\) into “D1” with the same remark, i.e. we will not check that “D1” contains a number and that this number is between 0 and 1. The deal of the remark of the correctness of data is that we will not check the correctness of numbers from “C1” and “D1” in another formula. Further formulas may appear (and will appear) incorrect in the case if one put “bad” data into “C1” and “D1”.

Construct at first the formulas for obtaining the points of the set \(A_{n+1}\) in the array “A1: A2”’, i.e. obtain the increasing rational numbers from \([0, 1]\) with denominator \(2^n\) in “A1: A2’”.

Put 0 into “A1” and the formula “A1+1/(2\(C\)$1)” into “A2”. Here the symbol \(\wedge\), naturally means that powering. After copying the cell “A2” obtain the necessary set in the first columns of the table.

Make some additional remarks for constructing formulas for “\(B_i\)”. The cell “\(A_i\)” contains the value \(i\) \(\wedge\) \(\frac{1}{2^n}\). This means that the equality
\[i = A_i \cdot 2^n + 1\]
holds.
If $A_i < \frac{1}{2}$, then the line of the table, which contains the value $h(2x)$ is of the number $2i - 1$, or, in terms of $A_i$, its number is

$$2i - 1 = 2 \cdot A_i \cdot 2^n + 1.$$  

Whence, if $A_i < \frac{1}{2}$, then the value $Bi$ should be found by formula

$$v \cdot \text{INDIRECT( CONCATENATE("B"; 2 \cdot A1 \cdot 2^n+1))},$$

where the function CONCATENATE constructs the text line with the array of its arguments and the function INDIRECT returns the value of the cell, whose name if its the unique argument in the format $Aix$, where $A$ is a Latin letter (a number of a column) and $x$ is a natural number (the number of a line).

If $A_i \geq \frac{1}{2}$, then the line of the table, which contains the value $h(-2x + 2)$ has the number $2i - 1$ or, in terms of $A_i$, its number

$$(2 - 2(Ai)) \cdot 2^n + 1.$$  

Whence the values of $h$ on $A_{n+1}$ can be found with the formula, which is introduced in the following theorem.

**Theorem 8.2.** The value of the conjugation $h$ of maps $f$ and $f_v$, which are determined by formulas (8.1) and (8.2), can be determined in the set $A_{n+1}$ with the following way.

- Put $n$ into “C1”.
- Put $v$ into “D1”.
- Put 0 into “A1” and the formula “A1+1/(2^C$1$)” into “A2”.
- Put the following formula into “B1”.

```
IF(A1=0; 0; IF(A1=1, 1; IF(A1=0,5;
D$1*INDIRECT( CONCATENATE("B"; 2*A1*2^C$1+1));
1-(1-D$1)*INDIRECT( CONCATENATE("B"; -2*A1*2^C$1+1 +2^((1+C$1))))))).
```

Copy the formulas in columns A and B down will till the line number $2^n + 1$.  

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The Figure 32 contains the values of conjugation at \( A_7 \) for \( v = 0.55, 0.6, 0.7, 0.75, 0.8, 0.85, 0.9 \) and 0.95.

![Figure 32: Conjugation at \( A_7 \) for different \( v \)](image)

Smaller \( v \) correspond to graphs, which are closer to \( y = x \). All these graphs are obtained from the formula above.

Remind that these graphs are approximations of the conjugation, i.e. the monotone and continuous maps.

It seems from the picture that maps, which correspond to \( v = 0.55, 0.6, 0.65 \) are such as necessary in the time, when it is “hard to believe” in the continuality of the maps, which corresponds to \( v = 0.95 \), because it seems to be “evidently discontinuous”. Nevertheless, it is also obvious that \( h \) is continuously dependent on \( v \). In any way we have proved in Theorem 6.5 that \( h \) is continuous for every \( v \in (0, 1) \).
9 Functional equations which describe the topological conjugation

We will try in this section to apply the methods of solving of linear functional equations to finding the explicit formulas for the conjugation of

\[
\begin{cases}
2x, & x < 1/2; \\
2 - 2x, & x \geq 1/2
\end{cases}
\]  

(9.1)

and

\[
\begin{cases}
\frac{x}{v}, & x \leq v; \\
\frac{1-x}{1-v}, & x > v.
\end{cases}
\]  

(9.2)

Let \( h : [0, 1] \to [0, 1] \) be a homeomorphism such that the diagram

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, 1] & \xrightarrow{f_v} & [0, 1]
\end{array}
\]  

(9.3)

is commutative.

We obtain in this section the system of linear functional equations, whose solution of the necessary homeomorphism \( h \).

**Lemma 25.** If a homeomorphism \( h : [0, 1] \to [0, 1] \) satisfies the commutative diagram (9.3), then it satisfies the functional equation

\[
\begin{cases}
h(2x) = \frac{1}{v} h(x) & x \leq 1/2 \quad (9.4a) \\
h(2 - 2x) = \frac{1 - h(x)}{1 - v} & x > 1/2 \quad (9.4b)
\end{cases}
\]  

(9.4)

**Proof.** By Theorem [6.5] the homeomorphism \( h \) increase, i.e. \( h(0) = 0 \) and \( h(1) = 1 \).

Plug \( x = 1/2 \) into the commutative diagram (9.3) and obtain

\[
\begin{array}{ccc}
1/2 & \xrightarrow{f} & 1 \\
\downarrow h & & \downarrow h \\
h(1/2) & \xrightarrow{f_v} & 1
\end{array}
\]
whence $h(1/2) = v$. Since $h$ is monotone for $x \in [0, 1/2]$ then it follows from the commutative diagram (9.3) that

$$
\begin{array}{ccc}
[0, 1/2] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, v] & \xrightarrow{f_v} & [0, 1].
\end{array}
$$

Because of the form of equations of (9.1) and (9.2) the commutativity of the diagram is equivalent to the functional equation (9.4a).

Functional equation (9.4b) is obtained in the same manner if plug $x \in [1/2, 1]$ into the commutative diagram (9.3) and notice that $h(1/2) = v$.

Section 9 is devoted to the solution of the system of linear functional equations (9.4).

9.1 The uniqueness of the solution of the system of functional equations

We will prove in this section that the continuous solution of the functional equation (9.4) is unique. It would follow from Lemma 25 and Theorem 6.5 that this solution will be the conjugation of $f$ and $f_v$.

Lemma 26. If the maps $h$, $[0, 1] \rightarrow [0, 1]$ satisfies (9.4), then $h(0) = 0$.

Proof. Plug the value $x = 0$ into (9.4a) and obtain $h(0) = 0$.

Lemma 27. If $h$, $[0, 1] \rightarrow [0, 1]$, which satisfies (9.4), then $h\left(\frac{1}{2}\right) = v$.

Proof. Plug $x = 1/2$ into (9.4a) and (9.4b), obtaining

$$
h(1) = \frac{1}{v} h(1/2) = \frac{1 - h(1/2)}{1 - v},
$$

whence

$$
h(1/2) = v.
$$
Notation 9.1. We will say that the value of $h$ at a point $x$ is **unambiguously defined** if there exists $y \in [0, 1]$ such that for every continuous solution $\tilde{h}$ of the system of functional equations (9.4) the equality $\tilde{h}(x) = y$ holds.

For instance, the maps $h$ is unambiguously defined at points $0$, $1$ and $\frac{1}{2}$.

**Lemma 28.** If the maps $h$ is unambiguously defined at a point $\tilde{x}$ then it is unambiguously defined at the integer trajectory of $\tilde{x}$.

**Proof.** Let $\tilde{x}$ be an arbitrary point, where $h$ is unambiguously defined.

Show, that in this case the maps $h$ is unambiguously defined at the point $f(\tilde{x})$. Indeed, if $\tilde{x} \leq \frac{1}{2}$ then plugging of $x = \tilde{x}$, into (9.4a) we can find $h(2\tilde{x}) = \frac{1}{v}h(\tilde{x})$ and this would mean that $h(f\tilde{x})$ is unambiguously defined.

If $\tilde{x} > \frac{1}{2}$, then plugging of $x = \tilde{x}$ into (9.4b) we would find $h(-2\tilde{x} + 2) = \frac{1-h(\tilde{x})}{1-v}$, which means that $h(f\tilde{x})$ is unambiguously defined.

Let $\tilde{\tilde{x}}$ be some pre image of $\tilde{x}$. In this case the fact that $h$ is unambiguously defined at $\tilde{x}$ can be proved in the same manner either by plugging of $x = 2\tilde{\tilde{x}}$ into the equation (9.4a), or by plugging $x = \frac{2-\tilde{\tilde{x}}}{2}$ into (9.4b). $\square$

These results can be generalized in the following theorem.

**Theorem 9.1.** The system of functional equations (9.4) has a unique continuous solution. This solution is the conjugation of $f$ and $f_v$.

**Proof.** Consider the conjugation $h$ of maps $f$ and $f_v$, which exists by Theorem 6.5. By Lemma 28 $h$ is the solution of the system (9.4).

Prove the uniqueness of the solution of (9.4). Consider the sets $A_n$, such that $f^n(A_n) = 0$, which were introduced at Section 6.1. By Lemma 28 the solution $h$ is unambiguously defined at $A = \bigcup_{n=1}^{\infty} A_n$. Now the uniqueness of the solution of (9.4) follows from Proposition 6.3, i.e. from the density of $A$. $\square$
9.2 Explicit formulas for the solutions of functional equations

Each from functional equations of (9.4) belongs to the class of so called linear functional equations. Methods of solving of linear functional equations are known, but such a solution is determined up to an arbitrary function.

After finding the general solution of any of the equations of (9.4) (up to arbitrary function) we may plug this solution into another functional equation for obtaining the new equation on the “arbitrary function” from the former equation and the for finding this “arbitrary function” from the new functional equation.

9.2.1 General methods of solving of functional equations.

Consider the functional equation

\[ h(A(x)) = B(x, h(x)), \]  

(9.5)

where \( h \) is unknown function and \( A \) and \( B \) are known functions, which act from the real axes to real axes.

The following so called characteristic transformation \( S : \mathbb{R}^2 \to \mathbb{R}^2 \) can be constructed by (9.5).

\[
S : \begin{cases} 
  t \to A(t), \\
  y \to B(t, y).
\end{cases}
\]  

(9.6)

If a function \( h \) satisfies (9.5), then its graph (i.e. the set \( \{(x, h(x))\} \)) would be invariant under the characteristic transformation \( S \) and conversely, each set, which is invariant under \( S \), corresponds to some solution of (9.5). This let to reduce the problem of solving of functional equation to the problem of finding the invariants of its characteristic transformations, i.e. to finding of functions \( \varphi : \mathbb{R}^2 \to \mathbb{R} \), such that the equality

\[ \varphi(t, y) = \varphi(S(t, y)). \]

holds.

**Notation 9.2.** An equation of the form

\[ h(A(x)) = B(x) \cdot h(x) + C(x), \]  

(9.7)
where \( A, B \) and \( C \) are given functions \( \mathbb{R} \rightarrow \mathbb{R} \) is called a linear functional equation.

**Notation 9.3.** An equation of the form

\[
h(ax) = b \cdot h(t),
\]

where \( a \) and \( b \) are constants is called a linear functional equation with constant coefficients.

For instance the equation (9.4a) is a linear functional equation with constant coefficients, but the equation (9.4b) is a linear functional equation, but it is not with constant coefficients.

The characteristic transformation of the equation (9.8) is as follows.

\[
S: \begin{cases} t \rightarrow at, \\ y \rightarrow by \end{cases}
\]

If \( a > 0 \) and \( b > 0 \) then consider \( \mu = \log a b \) and notice that the function

\[
\varphi(t, y) = \frac{y}{t \log a b}
\]

is invariant under the characteristic transformation, since the equality

\[
\frac{y}{t \log a b} = \frac{by}{(at) \log a b}
\]

from the definition holds.

Since such invariant is known, it is naturally to find the solution of (9.8) in the form

\[
h(x) = \omega(x) \cdot x^{\log a b},
\]

where \( \omega \) is unknown function.

**Remark 9.1.** Notice, that we could write the equality (9.10) for the equation (9.8) without any explanation and say that we want to find the solution of (9.8) in his form.

Plugging (9.10) into (9.8) obtain

\[
\omega(ax) \cdot (ax)^{\log a b} = b \cdot \omega(x) \cdot x^{\log a b},
\]
i.e.

\[ \omega(ax) = \omega(x). \]

We can understand the obtained equality as the dependence of \( \omega \) on \( \log_a x \), such that \( \omega \) is periodical with period 1, i.e. \( h \) is a solution of the equation (9.5) if and only if it is of the form

\[ h(x) = x^{\log_a b} \omega(\log_a x), \quad (9.11) \]

where \( \omega(x) \) is a function with period 1.

If at least one of numbers \( a \) and \( b \) in (9.8) is negative, then the invariant of the characteristic transformation would be

\[ \varphi = \frac{y}{t^{\log_b |b|}}. \]

Since the sign of \( x \) is unknown, then it would be impossible to find the solution in the form (9.10) and its correspond form is

\[ h(x) = |x|^{\log_a |b|} \cdot \begin{cases} 
\omega^+(\log_a x) & x > 0; \\
\omega^-(\log_a |x|) & x < 0,
\end{cases} \quad (9.12) \]

where \( \omega^+ \) and \( \omega^- \) are some functions.

If plug the expression (9.12) into the equation (9.8) then, dependently in the signs of \( a \) and \( b \) obtain the following conditions for \( \omega^+ \) and \( \omega^- \).

\[
\begin{cases}
\omega^+(x + 1) = \omega^+(x); & \text{for } a > 0, \ b > 0. \\
\omega^-(x + 1) = \omega^-(x),
\end{cases}
\]

\[
\begin{cases}
\omega^+(x + 1) = -\omega^+(x); & \text{for } a > 0, \ b < 0. \\
\omega^-(x + 1) = -\omega^-(x),
\end{cases}
\]

\[
\begin{cases}
\omega^-(x + 1) = \omega^+(x); & \text{for } a < 0, \ b > 0. \\
\omega^+(x + 1) = \omega^-(x),
\end{cases}
\]

\[
\begin{cases}
\omega^-(x + 1) = -\omega^+(x); & \text{for } a < 0, \ b < 0. \\
\omega^+(x + 1) = -\omega^-(x),
\end{cases}
\]

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Remark 9.2. Notice, that the first of the forth conditions above is considered earlier, but we have presented it for completeness.

We will present methods, which let to obtain the explicit solution of the equation (9.4b) in spite of that it is not linear functional equation.

Assume that the functional equation (9.5) is such that invariants of its characteristic transformation $S$, which is defined by (9.6), is not obvious.

Lemma 29. For an arbitrary invertible maps $H: \mathbb{R}^2 \to \mathbb{R}^2$ consider the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{S} & \mathbb{R}^2 \\
\downarrow H & & \downarrow H \\
\mathbb{R}^2 & \xrightarrow{\tilde{S}} & \mathbb{R}^2
\end{array}
\]

If $\tilde{\varphi}: \mathbb{R}^2 \to \mathbb{R}$ is the invariant of the characteristic maps $\tilde{S}$, then $\varphi(t, y) = \tilde{\varphi}(H(S(t, y)))$ is an invariant of $S$.

Proof. It is enough to prove that $\varphi(S(t, y)) = \varphi(t, y)$ for proving the lemma.

By the definition of $\varphi$ we have that

\[ \varphi(S(t, y)) = \tilde{\varphi}(H(S(t, y))). \]

It follows from the commutativity of the diagram, obtain that $H(S(t, y)) = \tilde{S}(H(t, y))$, whence

\[ \tilde{\varphi}(H(S(t, y))) = \tilde{\varphi}(\tilde{S}(H(t, y))). \]

But since $\tilde{\varphi}$ is an invariant of $\tilde{S}$ then

\[ \tilde{\varphi}(\tilde{S}(H(t, y))) = \tilde{\varphi}(H(t, y)), \]

and again by the construction of $\varphi$ obtain that

\[ \tilde{\varphi}(H(t, y)) = \varphi(t, y), \]

which proves Lemma.

Notation 9.4. The maps $H$ from Lemma 29 is called the change of variables.
We will show the applying of Lemma 29 to the functional equation (9.4b):

\[ h(2 - 2x) = \frac{-h(x)}{1 - v} + \frac{1}{1 - v}. \]

Characteristic transformation of this equation is as follows

\[ S : \begin{cases} 
  t \rightarrow 2 - 2t, \\
  y \rightarrow \frac{-y}{1 - v} + \frac{1}{1 - v}.
\end{cases} \quad (9.13) \]

Let us find a changing of variables \( H \) such that the characteristic transformation of the equation (9.4b) become of the form (9.9).

We will find the maps \( H \) of the form

\[ H(t, y) = (t - t_0, y - y_0) \]

for fixed \( t_0 \) and \( y_0 \).

The maps \( \tilde{S} \) can be found from the formula

\[ \tilde{S}(t, y) = H(S(H^{-1}(t, y))) = \]

\[ = \left( 2 - 2(t + t_0) - t_0, \frac{-y + y_0}{1 - v} + \frac{1}{1 - v} - y_0 \right). \]

In the assumption that \( \tilde{S} \) has no “free variables”, i.e. saying that there should not be expressions, which are independent on \( t \) and \( y \) obtain

\[ \begin{aligned}
  2 - 3t_0 &= 0 \\
  \frac{-y_0 + 1}{1 - v} - y_0 &= 0.
\end{aligned} \]

**Remark 9.3.** Notice that the point \((x_0, y_0)\) appeared to be a fixed point of (9.13).

For the found \((t_0, x_0)\) the form transformation \( \tilde{S} \) appears to be

\[ \tilde{S} : \begin{cases} 
  \tilde{t} \rightarrow -2\tilde{t}, \\
  \tilde{y} \rightarrow \frac{-\tilde{y}}{1 - v}.
\end{cases} \]

The constructed transformation \( \tilde{S} \) is a characteristic transformation of a linear functional equation with constant coefficients. The solution of this equation can be easily obtained in the form

\[ h(x) = \frac{1}{2 - v} + \left| x - \frac{2}{3} \right|^{\log_2(1 - v)} \times \]

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\[
\begin{cases}
\omega^+ \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x > \frac{2}{3}; \\
\omega^- \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x < \frac{2}{3},
\end{cases}
\]

where functions \( \omega^+ \) and \( \omega^- \) satisfy the condition
\[
\begin{cases}
\omega^- (t + 1) = -\omega^+(t) \\
\omega^+(t + 1) = -\omega^-(t).
\end{cases}
\]

### 9.2.2 Plugging of the solutions of one functional equations into another equation os the system

The equation (9.4a) is a linear functional equation. It is obtained from (9.8) by plugging \( a = 2 \) and \( b = \frac{1}{v} \). The formula for the solution of (9.4a) os obtained by plugging of \( a = 2 \) and \( b = \frac{1}{v} \) into (9.11) and is

\[
h(x) = x^{-\log_2 v} \omega(\log_2 x),
\]

where \( \omega(x) \) is an arbitrary function with period 1. If plug (9.14) into (9.4b), then obtain

\[
(2 - 2x)^{-\log_2 v} \omega(\log_2 (2 - 2x)) =
\]

\[
= \frac{1 - x^{-\log_2 v} \omega(\log_2 x)}{1 - v}.
\]

The periodicity of \( \omega \) let us to rewrite this equation as follows

\[
(1 - v)(1 - x)^{-\log_2 v} \omega(\log_2 (1 - x)) =
\]

\[
= v(1 - x^{-\log_2 v} \omega(\log_2 x)).
\]

**Proposition 9.2.** If consider the equation (9.15) as a functional equation of the function, which is defied on the whole line, then it will appear, that \( h \) it is constant function.

**Proof.** Define \( t = 1 - x \) and obtain

\[
(1 - v)t^{-\log_2 v} \omega(\log_2 t) =
\]

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\[
= v(1 - (1 - t)^{-\log_2 v}\omega(\log_2(1 - t))).
\]

If write \(x\) instead of \(t\) then we can use (9.15) to express \((1 - x)^{-\log_2 v}\omega(\log_2(1 - x))\) from each of the equations and equate them, obtaining

\[
\frac{v}{1 - v} \left(1 - x^{-\log_2 v}\omega(\log_2 x)\right) =
\]

\[
= 1 - \frac{v}{1 - v} x^{-\log_2 v}\omega(\log_2 x),
\]

whence

\[
x^{-\log_2 v}\omega(\log_2 x) = v.
\]

This equality means that the function \(h\) is constant and has no inverse. \(\Box\)

**Remark 9.4.** If solve the equation (9.15) for the \(h : [0, 1] \rightarrow [0, 1]\), which is defined by (9.14), then reasonings from the Proposition 9.2 would be incorrect.

**Explanation of the Remark.** The deal is that equation (9.15) is obtained by plugging of the solution of (9.4a) into (9.4b).

That is why, the substitution \(t = 1 - x\) is, in fact, the substitution in (9.4a).

Nevertheless, the equation (9.4a) is obtained from the commutativity of the diagram

\[
\begin{array}{ccc}
[0, 1/2] & \xrightarrow{f:x\mapsto 2x} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, v] & \xrightarrow{f_v:x\mapsto x/v} & [0, 1],
\end{array}
\]

which is defined only for \(x \in [0, 1/2]\). Since the substitution \(t = 1 - x\) for \(x \in [0, 1/2]\) means that \(t \in [1/2, 1]\), then it leads to a functional equation, which is defined for another set of arguments and expressing of

\[
(1 - x)^{-\log_2 v}\omega(\log_2(1 - x))
\]

from both equations with further equating is incorrect. \(\Box\)

We have obtained in the Section 9.2.1 that the solution of (9.4b) is of the form
\[ h(x) = \frac{1}{2^{-v}} + \left| x - \frac{2}{3} \right|^{-\log_2(1-v)} \times \begin{cases} 
  \omega^+ \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x > \frac{2}{3}; \\
  \omega^- \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x < \frac{2}{3}. 
\end{cases} \]  
\quad (9.16)

for functions \( \omega^+ \) and \( \omega^- \), such that

\[
\begin{align*}
\omega^-(t + 1) &= -\omega^+(t) \\
\omega^+(t + 1) &= -\omega^-(t).
\end{align*}
\]  
\quad (9.17)

In the same manner as in the previous section, we may consider the obtained expressions of \( h \) and try to find \( \omega \).

It follows from (9.17) that functions \( \omega^+ \) and \( \omega^- \) are periodical with period 2.

Plug the solution (9.16) of the functional equation (9.4b) into (9.4a) and obtain

\[ \frac{v}{2^{-v}} + v \left| 2x - \frac{2}{3} \right|^{-\log_2(1-v)} \times \begin{cases} 
  \omega^+ \left( \log_2 \left| 2x - \frac{1}{3} \right| \right) & x > \frac{2}{3}; \\
  \omega^- \left( \log_2 \left| 2x - \frac{1}{3} \right| \right) & x < \frac{2}{3}
\end{cases} = \]

\[ = \frac{1}{2^{-v}} + \left| x - \frac{2}{3} \right|^{-\log_2(1-v)} \times \begin{cases} 
  \omega^+ \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x > \frac{2}{3}; \\
  \omega^- \left( \log_2 \left| x - \frac{2}{3} \right| \right) & x < \frac{2}{3}
\end{cases} \]

for unknown functions \( \omega^+ \) and \( \omega^- \), which are connected by expressions (9.17).

The complicatedness of the obtained equation in comparison with the former one is evident.

The problem on the invertibility of approximations of \( h \), which are obtained in the described way, is so complicated as in the case, when the solution of the first equation was plugged into the second one.
9.3 Numerical experiments

Theorem 8.2 contains the formula, which let to construct $h$ on $A_n$ in terms on electronic tables for arbitrary $n$.

Let maps $h$ be defined by (9.14). With the use of known values of $h$ on $A_n$ for enough huge $n$, we can calculate values of $\omega$ for $x \in [0, 1]$. Remind that $\omega$ is periodical with period 1.

Graph of $\omega$ for $v = 3/4$ and $x \in [0, 1]$ is given at Figure 33.

Fig. 33: Graph of $\omega_1(x)$ for $v = 3/4$

Remark 9.5. Make the remark on the way, how the graph on Figure 33 was obtained.

Deal of the remark. Since $\omega$ is periodical with period 1, then it is enough to find it on any interval of the length 1.

If plug all numbers $x \in [1/2, 1]$ with the step, small enough, into the equation (9.14), then function $\log_2 x$ would be found in all points (with correspond small step) of the interval $[-1, 0]$.

In this case for every $x \in [0, 1]$ we may consider $\log_2 x$ and $\omega(\log_2 x) = h(x) \cdot x^{\log_2 v}$. This procedure will give use the set of points, where the function $\omega$ is found.

The fact, that $h$, which is a solution of (9.4), is very complicated, follows from that $h$ is non-differentiable of the dense set in $[0, 1]$. It follows from the form of the formula (9.14) that it is $\omega$, which is the term, where complexity of $h$ comes from, because $x^{-\log_2 v}$ is differentiable everywhere.
In the same time, we present below the properties of invertible maps $h$, which is of the form (9.14).

**Lemma 30.** If the invertible maps $h : [0, 1] \to [0, 1]$ satisfies (9.14), then the following conditions hold.

1. $h$ increase;
2. For every $n \in \mathbb{N}$ the equality $h \left( \frac{1}{2^n} \right) = v^n$.

Also for all integer $t$ the equality $\omega(t) = 1$ holds.

**Proof.** Prove first, that $\omega$ is bounded. It is so, because $\omega$ is defined by its values, which are obtained from the equation (9.14) for $x \in [1/2, 1]$. But in this case the function $x^{-\log_2 v}$ is bounded by $v$.

Plug $x = 0$ into (9.14) and obtain the product of zero function times bounded, whence $h(0) = 0$. Whence, together with that $h$ is invertible, means that $h$ increase.

If follows from $h(1) = 1$ that plugging of $x = 1$ into (9.14) gives $1 = \omega(\log_2 x)$. Since $\omega$ is periodical with period 1, then for every $t \in \mathbb{Z}$ the equality $\omega(t) = 1$ holds.

Plugging $x = \frac{1}{2^n}$ into (9.14) gives

$$h \left( \frac{1}{2^n} \right) = v^n \cdot \omega(\log_2 2^{-n}) = v^n.$$  

\[
\]

Consider examples of “simple” maps $\omega$, but such that $h$, which is defined by (9.14), is invertible and consider the maps $\tilde{f}_v$, which is defined by commutative diagram

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, 1] & \xrightarrow{\tilde{f}_v} & [0, 1].
\end{array}
\quad (9.18)
$$

**Lemma 31.** If for invertible maps $h$ of the form (9.14) the diagram (9.18) is commutative, then for $x \in [0, v]$ the equality

$$\tilde{f}_v(x) = \frac{x}{v}
$$

holds.
Proof. It follows from the invertibility of $h$ that $\tilde{f}_v$ is well-defined, precisely for all $x \in [0, 1]$ the equality $\tilde{f}_v = h(f(h^{-1}(x)))$ holds. But by Lemma 30 for $x \in [0, v]$ the inclusion $h^{-1}(x) \in [0, 1/2]$ holds, whence $\tilde{f}_v(x) = h(2h^{-1}(x))$.

Since the expression (9.14) is obtained from the functional equation (9.4a), then $h$ from the condition of Lemma satisfies (9.4a). But the equation (9.4a) is equivalent to commutativity of the diagram (9.18) for $\tilde{f}_v(x) = x^v$. Now Lemma follows from the uniqueness of $\tilde{f}_v$. \hfill \square

The simplest case, when $\omega$ is a periodical function with period 1 is that when it is constant. If follows from $h(1) = 1$ that if $\omega$ is constant, then $\omega(x) = 1$ for all $x$.

**Example 9.1.** Plot the graph of $\tilde{f}_v$, which is defined by the commutative diagram (9.18) for the maps $h$ of the form (9.14), if $\omega$ is constant.

**Deal of the example.** If $\omega(x) = 1$ for every $x \in [0, 1]$, then $h(x) = x^{-\log_2 x}$. Then by Lemma 31 (this also can be shown from the direct calculations) follows that for all $x \in [0, v]$ the equality $\tilde{f}_v(x) = \frac{x}{v}$ holds.

For such function $\omega$ the equality $h^{-1}(x) = x^{-\log_2 2}$ holds, whence for $x \in [v, 1]$ obtain

$$\tilde{f}_v(x) = \left(2 - 2x^{-\log_2 v}\right)^{-\log_2 2}.$$  

The graph of $\tilde{f}_v$ for $v = 3/4$ is given at Figure 34. \hfill \square

![Fig. 34: Graph of $\tilde{f}_v$ for $v = 3/4$](image)

**Example 9.2.** Plot the graph of $\tilde{f}_v$, which is defined by the commutative diagram (9.18) for the maps $h$ of the form (9.14), if $\omega$ is a continuous function, whose graph on $[1/2, 1]$ is consisted of two intervals of linearity.
Deal of the example. Plug $x = 3/4$ into the commutative diagram (9.18) and obtain

\[
\begin{array}{ccc}
3/4 & \xrightarrow{f} & 1/2 \\
\downarrow^h & & \downarrow^h \\
h(3/4) & \xrightarrow{\tilde{f}_v} & v = h(1/2).
\end{array}
\]

Define $h(3/4)$ as the bigger pre image of $v$ under $f_v$. This leads to that the values of $f_v$ and $\tilde{f}_v$ would coincide at this bigger pre image.

In other words

\[
h \left( \frac{3}{4} \right) = \left( \frac{3}{4} \right)^{-\log_2 v} \omega \left( \log_2 \left( \frac{3}{4} \right) \right) = v^2 - v + 1.
\]

For $v = \frac{3}{4}$ obtain $\omega(0, 584) \approx 0, 915$.

Define $\omega$ on the interval $[0, 1]$ with the following rule. $\omega$ is a piecewise linear maps, whose graph consists of two intervals of linearity and has breaking point at

\[
\left( \log_2 \frac{3}{4}, \left( v^2 - v + 1 \right) \cdot \left( \frac{3}{4} \right)^{\log_2 v} \right).
\]

Graph of $\tilde{f}_v$ for $v = 3/4$, which is constructed by this $\omega$, is given at Figure 35.

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig35}
\caption{Graph of $\tilde{f}_v$ for $v = 3/4$}
\end{figure}
\]

Examples 9.1 and 9.2 can be generalized as follows.

Consider the approximations $\omega_n$ of $\omega$ and use then for finding approximations $\tilde{h}_n$ of $h$, which is given by (9.14), as follows.

For any approximation of $h$ on $x \in \left[ \frac{1}{2}, 1 \right]$, we can obtain the approximation of $\omega$ on $[-1, 0]$. Then periodicity of $\omega$ would give the approximation of $h$ on the whole $[0, 1]$. 

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The maps $h_n$, which is constructed in previous sections, moves points of $A_n$ to $B_n$. Use these $h_n$ to find $\omega$ on $A_n \cap \left[ \frac{1}{2}, 1 \right]$.

Define by $\omega_n$ the maps, such that $h_n$ defines its values at $\{ \log_2 x, x \in A_n \cap \left[ \frac{1}{2}, 1 \right] \}$. Additionally $\omega_n$ is linear at all points of the set $A_n \cap \left[ \frac{1}{2}, 1 \right]$ and is periodical with period 1.

Consider the maps

$$\tilde{h}_n = x^{-\log_2 v} \omega_n(\log_2 x), \quad (9.19)$$

as the approximation of $h$.

If the constructed maps $\tilde{h}_n$ would be invertible, then there exists a unique maps $\tilde{f}_n$, such that the diagram

$$\begin{array}{ccc}
    A & \xrightarrow{f} & A \\
    \downarrow \tilde{h}_n & & \downarrow \tilde{h}_n \\
    B & \xrightarrow{\tilde{f}_n} & B
\end{array} \quad (9.20)$$

would be commutative. This maps $\tilde{f}_n$ can be given by formula

$$\tilde{f}_n = \tilde{h}_n(f(\tilde{h}_n^{-1})).$$

For instance, the introduced notations give that $\tilde{f}_v$, which is constructed in the example 9.2, is the maps $\tilde{f}_2$ and the correspond $h$ is $\tilde{h}_2$.

Nevertheless, it could happen that the maps $\tilde{h}_n$ would be non invertible, whence there will not exist $\tilde{f}_n$ which would make the diagram (9.20) commutative.

Example 9.3. Consider the case, when the maps $\tilde{h}_3(x)$, (which is, in fact, dependent on $v$), is non monotone for some $v$.

Deal of the example. The Figure 36 contains examples of graphs of maps $\tilde{h}_3$ for $v = 0.01$, $v = 0.025$, $v = 0.1$ and $v = 0.2$.

As a comment to the Figure 36 notice, that all these functions satisfy the functional equation (9.4a)

$$h(2x) = \frac{1}{v} h(x),$$

i.e. for every interval of the form $\left[ \frac{1}{2^{n+k}}, \frac{1}{2^n} \right]$ the graph of the function repeats its form on the interval $[\frac{1}{2}, 1]$, but it is compressed $v^{-k}$ times.
The maps $\tilde{h}_3(x)$, which is calculated for $v = \frac{1}{7}$, is given by formula $\tilde{h}_3(x) = x$.

whence, there are some critical value of $v$, such that decreasing of $v$ after this critical value leads to that $\tilde{h}_3(x)$ appears to be non monotone.

\textbf{Notation 9.5.} Denote by $\hat{h}_n(t)$ the function such that the equality

$$\hat{h}_n(\log_2 x) = \tilde{h}_n(x)$$

holds.

By numbers $\alpha_k = \alpha_{n,k}$ construct $\tilde{\alpha}_k = \log_2 \alpha_k$.

Define by $t_k = t_{k,n}$ the extremum of the maps $\hat{h}_n$ in the interval $(\tilde{\alpha}_k, \tilde{\alpha}_{k+1})$. It follows from the construction that this extremum is unique. We will pay more attention to this fact later.

\textbf{Remark 9.6.} Monotonicity of $\tilde{h}_n$ is equivalent to that for every $k$ the inclusion

$$t_k \in \mathbb{R} \setminus [\tilde{\alpha}_k, \tilde{\alpha}_{k+1}] = \mathbb{R} \setminus [\log_2 \alpha_k, \log_2 \alpha_{k+1}]$$

holds.

The formula \eqref{eq:omega_n} lets to write $\hat{h}_n(t)$ in the form

$$\hat{h}_n(t) = 2^{-\log_2 v} \omega_n(t).$$
It follows from the monotonicity of \( y = \log_2 x \) that the monotonicity of \( \tilde{h}_n \) is equivalent to the monotonicity of \( \hat{h}_n \).

It follows from the equality \( h_n(\alpha_k) = \beta_k \) that the following condition for \( \omega_n \) holds.

\[
\omega_n(\alpha_k) = \beta_k \cdot 2^{\tilde{\alpha}_k \log_2 v}.
\]

Denote by \( \tilde{\beta}_k = \beta_k \cdot 2^{\tilde{\alpha}_k \log_2 v} = \beta_k v^{\tilde{\alpha}_k} \).

Let the maps \( \omega_n \) have the form

\[
\omega_n(t) = a_k \cdot t + b_k
\]
on the interval \((\tilde{\alpha}_k, \tilde{\alpha}_{k+1})\). Then

\[
a_k = \frac{\tilde{\beta}_{k+1} - \tilde{\beta}_k}{\tilde{\alpha}_{k+1} - \tilde{\alpha}_k},
\]

\[
b_k = \tilde{\beta}_k - \frac{\tilde{\alpha}_k (\tilde{\beta}_{k+1} - \tilde{\beta}_k)}{\tilde{\alpha}_{k+1} - \tilde{\alpha}_k} = \frac{\beta_k \tilde{\alpha}_{k+1} - \beta_{k+1} \tilde{\alpha}_k}{\tilde{\alpha}_{k+1} - \tilde{\alpha}_k}.
\]

Find the extremum of \( \hat{h}_n(t) \) on \((\tilde{\alpha}_k, \tilde{\alpha}_{k+1})\). We have that

\[
\hat{h}_n'(t) = 2^{-t \log_2 v} (a_k - \log_2 v \ln 2 \cdot (a_k t + b_k)),
\]

whence the necessary extremum of \( \hat{h}_n(t) \) can be found as

\[
t_k = \frac{a_k - b_k \log_2 v \ln 2}{a_k \log_2 v \ln 2} = \frac{1}{\ln v} - \frac{b_k}{a_k}.
\] (9.22)

It follows from the previous calculations that

\[
\frac{b_k}{a_k} = \frac{\tilde{\beta}_k \tilde{\alpha}_{k+1} - \tilde{\beta}_{k+1} \tilde{\alpha}_k}{\tilde{\beta}_{k+1} - \tilde{\beta}_k}.
\]

After coming back to former notations obtain

\[
\frac{b_k}{a_k} = \frac{\beta_k v^{\log_2 \alpha_k} \log_2 \alpha_{k+1} - \beta_{k+1} v^{\log_2 \alpha_{k+1}} \log_2 \alpha_k}{\beta_{k+1} v^{\log_2 \alpha_{k+1}} - \beta_k v^{\log_2 \alpha_k}}
\]

Since \( \alpha(k, n) = \frac{k}{2^n} \) and \( A_n = \{\alpha(k, n - 1)\} \) we have that \( \alpha_k = \frac{k}{2^n} \), whence

\[
\log_2 \alpha_k = \log_2 k - n + 1.
\]
This means that
\[
\frac{b_k}{a_k} = \frac{\beta_k v^{\log_2 k - n + 1} \log_2 \alpha_{k+1}}{\beta_{k+1} v^{\log_2 (k+1) - n + 1} - \beta_k v^{\log_2 k - n + 1}} - \\
\frac{\beta_{k+1} v^{\log_2 (k+1) - n + 1} \log_2 \alpha_k}{\beta_{k+1} v^{\log_2 (k+1) - n + 1} - \beta_k v^{\log_2 k - n + 1}} = \\
\frac{\beta_k v^{\log_2 k} \log_2 \alpha_{k+1}}{\beta_{k+1} v^{\log_2 (k+1) - \log_2 k} - \beta_k v^{\log_2 k}} - \\
\frac{\beta_{k+1} v^{\log_2 (k+1)} \log_2 \alpha_k}{\beta_{k+1} v^{\log_2 (k+1) - \log_2 k} - \beta_k v^{\log_2 k}} = \\
\frac{\beta_k v^{\log_2 k} (\log_2 (k+1) - n + 1)}{\beta_{k+1} v^{\log_2 (k+1) - \log_2 k} - \beta_k v^{\log_2 k}} - \\
\frac{\beta_{k+1} v^{\log_2 (k+1)} (\log_2 k - n + 1)}{\beta_{k+1} v^{\log_2 (k+1) - \log_2 k} - \beta_k v^{\log_2 k}} = \\
n - 1 + \frac{\beta_k v^{\log_2 k} \log_2 (k+1)}{\beta_{k+1} v^{\log_2 (k+1) - \log_2 k} - \beta_k v^{\log_2 k}} - \\
\frac{\beta_{k+1} v^{\log_2 (k+1) \log_2 k}}{\beta_{k+1} v^{\log_2 (k+1) - \log_2 k} - \beta_k v^{\log_2 k}}.
\]

Example 9.3 leads to the assumption about non monotonicity of \(h_n\) for \(v \to 0\) which also stays for big \(n\). The use of formula (9.22) lets to prove the following proposition.

**Proposition 9.3.** For every \(n \in \mathbb{N}\) there exists \(v_0 \in (0, 1)\) such that for every \(v \in (0, v_0)\) the maps \(\tilde{h}_n(x)\) is non monotonic on \([\frac{2^{n-1} - 1}{2^{n-1}-1}, 1]\).

**Proof.** Find explicit formulas for \(\beta(n, 2^{n-1} - 1)\). We know that
\[
\begin{cases}
  f^n_v(\beta(n, 2^{n-1} - 1)) = 0; \\
  f_v^{n-1}(\beta(n, 2^{n-1} - 1)) = 1.
\end{cases}
\]

The tangent of \(f_v^{n-1}\) at the last interval of its monotonicity is \(\frac{1}{v-1} \cdot \frac{1}{v^{n-2}},\) which mens that

\[
\beta(n, 2^{n-1} - 1) = 1 + v^{n-2} \cdot (v - 1).
\]

To use Formula (9.22) for finding \(t_k\), we should consider \(k = 2^{n-1} - 1, \beta_k = 1 + v^{n-2} \cdot (v - 1),\) \(\beta_{k+1} = 1, \alpha_k = 1 - \frac{1}{2^{n-1}}, \alpha_{k+1} = 1.\) Thus \(\log_2 \alpha_{k+1} = 0,\) whence

\[
t_k = \frac{1}{\ln v}.
\]
\[
\frac{\beta_k v^{\log_2 \alpha_k} \log_2 \alpha_{k+1} - \beta_{k+1} v^{\log_2 \alpha_{k+1}} \log_2 \alpha_k}{\beta_{k+1} v^{\log_2 \alpha_{k+1}} - \beta_k v^{\log_2 \alpha_k}} = \\
= \frac{1}{\ln v} + \frac{\log_2 \alpha_k}{1 - (1 + v^{n-2} \cdot (v - 1)) v^{\log_2 \alpha_k}}.
\]

Since \( v \in (0, 1) \), then \( \frac{1}{\ln v} < 0 \) and \( \frac{1}{\ln v} \to 0 \) if \( v \to 0 \).

Since \( \log_2 \alpha_k < 0 \), then numerator of the second fraction of negative. But the same reason
mean that \( v^{\log_2 \alpha_k} \to \infty \) for \( v \to 0 \), whence denominator of the second fraction is also negative,
whence second fraction is positive, but tends to 0 if \( v \to 0 \), whence \( t_k \to 0 \) for \( v \to 0 \).

By Remark 9.6 to prove the Proposition we need to decide, whether or not the inclusion
\( t \in (\tilde{\alpha}_k, \tilde{\alpha}_{k+1}) \) holds. Since \( \tilde{\alpha}_k = \log_2 \left( 1 - \frac{1}{2^{n-1}} \right) < 0 \) and \( \tilde{\alpha}_{k+1} = \log_2 1 = 0 \), then we have to
understand which the sign of \( t_k \) is for \( v \approx 0 \) and whether this sign is fixed, or \( t_k \) waves about
0, changing its signs, dependently in \( v \).

Consider the limit
\[
\lim_{v \to 0} \frac{\log_2 \alpha_k}{1 - (1 + v^{n-2} \cdot (v - 1)) v^{\log_2 \alpha_k}} : \frac{1}{\ln v} = \\
= \lim_{v \to 0} -\log_2 \alpha_k \cdot \frac{1}{v} \\
= \lim_{v \to 0} \frac{\log_2 \alpha_k v^{\log_2 \alpha_k - 1} + (n - 1 + \log_2 \alpha_k) v^{n-2+\log_2 \alpha_k} - (n - 2 + \log_2 \alpha_k) v^{n-3+\log_2 \alpha_k}}{\log_2 \alpha_k v^{\log_2 \alpha_k} + (n - 1 + \log_2 \alpha_k) v^{n-1+\log_2 \alpha_k} - (n - 2 + \log_2 \alpha_k) v^{n-2+\log_2 \alpha_k}} \\
= \lim_{v \to \infty} \frac{-\log_2 \alpha_k}{\log_2 \alpha_k v^{\log_2 \alpha_k - 1} + (n - 1 + \log_2 \alpha_k) v^{n-1+\log_2 \alpha_k} - (n - 2 + \log_2 \alpha_k) v^{n-2+\log_2 \alpha_k}} \\
\sim \lim_{v \to \infty} \frac{-\log_2 \alpha_k}{\log_2 \alpha_k v^{\log_2 \alpha_k - 1}} = \lim_{v \to \infty} -v^{\log_2 \alpha_k} = 0.
\]

This means that \( t_k < 0 \) for \( v \approx 0 \). Now Proposition follows from Remark 9.6.

We can use Formula (9.22) for studying the Example 9.3 more carefully, precisely study the
numbers \( t_2 \) and \( t_3 \) dependently on \( v \) for \( n = 3 \).

**Example 9.4.** Consider \( t_2(v) \) for \( n = 3 \). We will show an unsuccessful attempt of studying of
the limit \( \lim_{v \to 1} t_2(v) \). Precisely, we will show that this limit exists and equals 1 in the time, when
numerical methods shows like the limit does not exists.
Proof. For $n = 3$ consider the division of the interval $[\frac{3}{2}, 1]$ for two intervals with end points $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{3}{4}$ and $\alpha_4 = 1$. Correspondingly, $\beta_2 = v$, $\beta_3 = \max f^{-1}(v) = 1 - v(1 - v)$ and $\beta_4 = 1$.

Then

$$t_2 = \frac{1}{\ln v} - 2 - \frac{\beta_k v^{\log_2 k} \log_2 (k + 1)}{\beta_{k+1} v^{\log_2 (k+1)} - \beta_k v^{\log_2 k}} +$$

$$+ \frac{\beta_{k+1} v^{\log_2 (k+1)} \log_2 k}{\beta_{k+1} v^{\log_2 (k+1)} - \beta_k v^{\log_2 k}} =$$

$$= \frac{1}{\ln v} - 2 + \frac{(v^2 - v + 1) v^{\log_2 3} - v^2 \log_2 3}{(v^2 - v + 1) v^{\log_2 3} - v^2} =$$

$$= \frac{1}{\ln v} - 1 + \frac{v^2 - v^2 \log_2 3}{(v^2 - v + 1) v^{\log_2 3} - v^2}$$

$$t_2 + 1 = \frac{1}{\ln v} + \frac{v^2 - v^2 \log_2 3}{(v^2 - v + 1) v^{\log_2 3} - v^2}$$

It is necessary and sufficient for being $\hat{h}_2$ continuous on $[\alpha_2, \alpha_3]$ that the following inclusion hold.

$$t_2 \in \mathbb{R} \setminus [-1, \log_2 3 - 2] \approx \mathbb{R} \setminus [-1, -0.415].$$

The numerical calculations about $t_2(v)$ let make the conclusion that it decrease on $v \in [0, 0.5]$ and has the asymptote $v = 0.5$, i.e. $\lim_{v \to 0.5^-} t_2 = -\infty$ in the time, when $\lim_{v \to 0} t_2 = -1$.

If plot the graph of the maps $t_2(v)$ for $v \in [0, 5, 1]$ on the interval $[0.99999, 1]$ then see that it is non non-monotone. Thus, for $v \in [0.6, 0.99999]$ the graph of $t_2(v)$ is given on Figure 37.

In the same time the graph of $t_2(v)$ for $v \in M = [0.99999999, 0.999999995]$ is given at Figure 38. Precisely, this plot is not a graph of $t_2(v)$ but its values at correspond points (looking like braking points), which divide $M$ into equal parts. These points are just connected by line segments to obtain a graph.

Notice, that in spite of the evident continuity of $t_2(v)$ we can see on the graph the points of discontinuity of $\hat{h}_3$ on $[\alpha_2, \alpha_3]$.

Further experiments show that the maps $t_2(v)$ a point of its discontinuity on $[\alpha_2, \alpha_3]$ on each of the intervals of the form $[1 - \frac{1}{10^m+1}, 1 - \frac{1}{10^m}].$
Fig. 37: Graph of $t_2$ for $v \in [0.6, 0.999999]$}

Fig. 38: Graph of $t_2$ for $v \in M$
In the same time these experiments does not make any influence to the mathematical nature of the function under the consideration, because the limit

$$\lim_{v \to 1} \left( \frac{1}{\ln v} + \frac{v^2(1 - \log_2 3)}{(v^2 - v + 1)v^\log_2 3 - v^2} \right)$$

is a limit of the difference of expressions, such that each of them tends to $+\infty$, whence the numerical investigation of this limit can lead to the mistake and is that, what has happened in our case.

Since

$$\lim_{v \to \infty} t_2(v) + 1 = \frac{1}{0} + \frac{0}{0} = \frac{0}{0},$$

we can write this expression with as a proper fraction and apply the L’hopitales rule, obtaining as follows

$$t_2 + 1 = \frac{(v^2 - v + 1)v^\log_2 3 - v^2 + \ln v(v^2 - v^2 \log_2 3)}{\ln v((v^2 - v + 1)v^\log_2 3 - v^2)}.$$

Denote by

$$s(v) = (v^2 - v + 1)v^\log_2 3 - v^2 + \ln v(v^2 - v^2 \log_2 3)$$

and

$$p(v) = \ln v((v^2 - v + 1)v^\log_2 3 - v^2).$$

Then $s'(v) = (2v - 1)v^\log_2 3 + \log_2 3(v^2 - v + 1)v^\log_2 3 - 2v + (v - v \log_2 3) + 2(v - v \log_2 3) \ln v$, whence $s'(1) = 0$. Also $p'(v) = (v - 1)v^\log_2 3 + v^\log_2 3 - 2v + ((2v - 1)v^\log_2 3 + \log_2 3(v^2 - v + 1)v^\log_2 3 - 2v) \ln v$, whence $p'(1) = 0$.

The second derivative $s''(v)$ and $p''(v)$ are as follows

$$s''(v) = 2v^\log_2 3 + (2v - 1)v^\log_2 3 - 2v + (1 - \log_2 3) + 2(1 - \log_2 3) + 2(1 - \log_2 3) \ln v, \text{ whence } s''(1) = 2 + \log_2 3 + \log_2 3(\log_2 3 - 1) + \log_2 3 - 2 + (1 - \log_2 3) + 2(1 - \log_2 3) = 3 + \log_2 3(\log_2 3 - 1) - \log_2 3 = 3 + (\log_2 3)^2 - 2 \log_2 3 \approx 2,3422.$$

$$p''(v) = v^\log_2 3 + \log_2 3(v - 1)v^\log_2 3 - 1 + (\log_2 3 - 1)v^\log_2 3 - 2 - 1 + ((2v - 1)v^\log_2 3 - 2 + (2v - 1)v^\log_2 3 + (2v - 1)v^\log_2 3 - 2) \ln v, \text{ whence } p''(1) = 1 + 0 + (\log_2 3 - 1) - 1 + (1 + \log_2 3 - 2) = 2 \log_2 3 - 2 \approx 1,1699.
That is why,
\[ t_2(v) \to \frac{s''(1)}{p''(1)} - 1 \approx 1,0020. \]

Example 9.5. Consider the function \( t_3(v) \) for \( n = 3 \). We will show, that \( \tilde{h}_3(x) \) becomes to be non monotone for all \( v < v_0 \), where \( v_0 = 0,18867 \pm 0,00001 \). This result is concordant with Example 9.3.

Deal of the example. Consider the monotonicity of \( \tilde{h}_3(x) \) on the interval \([\alpha_3, \alpha_4]\) for \( n = 3 \). In this case
\[
t_3 = \frac{1}{\ln v} - n + 1 - \frac{\beta_k v^{\log_2 k} \log_2 (k + 1)}{\beta_{k+1} v^{\log_2 (k+1)} - \beta_k v^{\log_2 k}} + \frac{\beta_{k+1} v^{\log_2 (k+1)} \log_2 k}{\beta_{k+1} v^{\log_2 (k+1)} - \beta_k v^{\log_2 k}} = \]
\[
= \frac{1}{\ln v} - 2 - \frac{2(v^2 - v + 1)v^{\log_2 3}}{v^2 - (v^2 - v + 1)v^{\log_2 3}} + \frac{v^2 \log_2 3}{v^2 - (v^2 - v + 1)v^{\log_2 3}} = \]
\[
= \frac{1}{\ln v} + \frac{v^2 \log_2 3 - 2v^2}{v^2 - (v^2 - v + 1)v^{\log_2 3}}.
\]
The inclusion
\[ t_3 \in [\log_2 3 - 2, 0] \approx [-0.415, 0] \]
is necessary for violating the monotonicity of \( \tilde{h}_3 \).

The graph of \( t_3(v) \) for \( v \in [0, 0.21] \) is given at Figure 39.

As it is shown in Example 9.3 there is a point near \( v_0 \approx 0,2 \), where \( \tilde{h}_3 \) is non-monotone for all \( \tilde{h}_3 \). Another calculations show that this value belongs to the interval
\[ v_0 \in [0.18868, 0.18869], \]
i.e.
\[ v_0 \approx 0,18867 \pm 0,00001. \]

As about the graph of \( t_3(v) \) on \([0.2, 0.5] \), it has asymptote at the point 0.5 to positive infinity. \qed
Fig. 39: Graph of $t_3$ for $v \in [0, 0, 21]$.

Do the similar calculation for $n = 4$. For this reason for every $v$ and for each of intervals $[\frac{1}{2}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{3}{4}]$, $[\frac{3}{4}, \frac{7}{8}]$ and $[\frac{7}{8}, 1]$ find the value of $t_k$ by Formula (10.22).

For different $v$ and every natural $k \in [4, 7]$ find $t_k(v)$ by obtaining 4 points. These points, just for visibility, connect by line segments, i.e. for every of these $v$ it corresponds one line segments. Also add to the Figure (and plot by bold), two curves, one of which connects numbers $\{\tilde{\alpha}_k\}$ and $\{\tilde{\alpha}_{k+1}\}$ for the same $k \in [4, 7]$. Taking into account the previous explanations, the maps $\hat{h}_{n,v}$ would be monotone if and only if each of the obtained four points would not appear between the bold curves, i.e. appears either higher then the higher, of lower then lowest. This construction is given at Figure 40.

Fig. 40: Graphs of $t_k(v)$ for $k \in [4, 7]$ and $n = 3$

This Figure contains these curves for $v_1 = 0, 15$, $v_2 = 0, 1$, $v_3 = 0, 07$, $v_4 = 0, 03$, $v_5 = 0, 17$, $v_6 = 0, 01$, $v_7 = 0, 001$ and $v_8 = 0, 00001$. The curves, which are top at Figure, correspond to
lower values of $v$.

It is seen from the Figure (and it corresponds to Proposition 9.3), that for $v$ small enough, curves becomes close to the horizontal line $y = 1$ and the last point of each of them (that, which corresponds to the interval $[\alpha_7, 1]$) tends to 1 for $v \to 0$ and that is why, belongs to the interval $(\tilde{\alpha}_7, \tilde{\alpha}_8) = (\tilde{\alpha}_7, 0)$. This means non monotonicity of the maps $\tilde{h}_n$ on the interval $(\alpha_7, \alpha_8) = (\alpha_7, 1)$.

![Graph of $t_k(v)$ for $k \in [4, 7]$ and $n = 5$](image)

Fig. 41: Graphs of $t_k(v)$ for $k \in [4, 7]$ and $n = 5$

The Figure 41 contains the analogical calculations for $n = 5$ and values $v_1 = 0.125$, $v_2 = 0.1$, $v_3 = 0.15$, $v_4 = 0.05$, $v_5 = 0.02$, $v_6 = 0.01$, $v_7 = 0.001$ and $v_8 = 0.000000001$.

Now, in the similar manner to previous, consider the solution $h$, which is defines by (9.16) and is obtained from the equation (9.4b).

Consider the function $h$ for $$x \in \left[\frac{35}{48}, \frac{11}{12}\right].$$

When the value of $x$ runs through this interval, then the value of the function $\log_2 (x - \frac{2}{3})$ runs through the interval $[-4, -2]$, which is of the length 2, which would let to define the function $\omega^+$ on the whole interval.

The graph of the maps $\omega(x)$ for $x \in [0, 2]$ and $v = 3/4$ is given on Figure 42.

**Lemma 32.** If the the invertible maps $h$ of the form (9.16) the diagram (9.18) is commutative, then for $x \in [v, 1]$ the equality

$$\tilde{f}_v(x) = \frac{1 - x}{1 - v}$$
Fig. 42: Graph of $\omega_2(x)$ for $v = 3/4$
holds.

Proof. The proof of This Lemma is similar to one of Lemma 31.

Example 9.6. Plot the graph of the maps $\tilde{f}_v$, which is defined by the commutative diagram (9.18) for the maps $h$ of the form (9.16), if $\omega$ is the function of the most simple form.

Deal of the example. The equations (9.17) except the case, when the $\omega$ is constant non zero function. The most simple case for $\omega$, is that, when $\omega = \omega_0$ for $x < 0$ and $\omega = -\omega_0$ for $x > 0$.

The it follows from the equation $h(0) = 0$ obtain

$$\omega_0 = \frac{-1}{2 - v} \left( \frac{2 \log_2(1-v)}{3} \right).$$

In this case the graph of the maps $\tilde{f}_v(x) = h(f(h^{-1}(x)))$, which is defined for $v = \frac{3}{4}$, if given on the Figure 43.

![Fig. 43: Graph of $\tilde{f}_v$ for $v = \frac{3}{4}$](image)

Example 9.7. Plot the graph of the maps $\tilde{f}_v$, which is defined by the commutative diagram (9.18) for the maps $h$ of the form (9.16), if $\omega$ is the function, for which $\omega^+$ and $\omega^-$ are of the simplest form, when they are not constant.

Proof. Assume that functions $\omega^+$ and $\omega^-$ are continuous.

From the equation $h(0) = 0$ obtain

$$0 = \frac{1}{2 - v} + \left( \frac{2}{3} \right)^{-\log_2(1-v)} \omega^- \left( \log_2 \frac{2}{3} \right),$$

(9.23)
\[ \omega^-(1 - \log_2 3) = \frac{1}{v - 2} \left( \frac{2}{3} \right)^{\log_2(1-v)} \] (9.24)

Since the left pre image of \( \frac{1}{2} \) under \( f \) is mapped under \( h \) to the left pre image of \( v \) under \( f_v \), then

\[ h \left( \frac{1}{4} \right) = v^2, \]

i.e.

\[ v^2 = \frac{1}{2 - v} + \left( \frac{5}{12} \right)^{-\log_2(1-v)} \omega^\left( \log_2 \left( \frac{5}{3} \right) \right), \]

whence it follows from the periodicity of \( \omega^- \) with period 2, that

\[ \omega^- \left( \log_2 \left( \frac{5}{3} \right) \right) = \]

\[ = \left( v^2 - \frac{1}{2 - v} \right) \left( \frac{5}{12} \right)^{\log_2(1-v)} \] (9.25)

Plot the function \( \omega^- \) as follows. The values of \( \omega^- \) at points \( \log_2 \left( \frac{5}{3} \right) \approx 0.737 \) and \( 2 + \log_2 \left( \frac{5}{3} \right) \approx 2.737 \) are equal and are defined by the equality (9.25). Remind that it follows from equation (9.17) that the function \( \omega^- \) is periodical with period 2.

The value of \( \omega^- \) at point \( 3 - \log_2 3 \approx 1.415 \) is defined by the equality (9.24). It follows from periodicity of \( \omega^- \) that the equality (9.24) can be applied for finding the value of \( \omega^- \) at this point.

Take the function \( \omega^- \) to be linear on each of the intervals \([\log_2 \left( \frac{5}{3} \right), 3 - \log_2 3]\) and \([3 - \log_2 3, 2 + \log_2 \left( \frac{5}{3} \right)]\) and make it periodical with period 2.

Construct the function \( \omega^+ \) with the use of \( \omega^- \), using the relation (9.17).

The graph of \( \tilde{f}_v \), which is constructed as above for \( v = 3/4 \), if given on the Figure 44.

The Figure 45 contains the result of imposition of the graph from the previous example for those one, which is constructed now (the dots are used for the first graph).
Fig. 44: Graph of $\tilde{f}_v$ for $v = \frac{3}{4}$

Fig. 45: Result of the imposition of one graph onto another.
10 Explicit formulas for topological conjugation

We will construct in this Section the explicit formulas for the topological conjugacy of the maps $f, f_v : [0, 1] \to [0, 1]$, given as follows

$$ f(x) = \begin{cases} 
2x, & x < 1/2; \\
2 - 2x, & x \geq 1/2 
\end{cases} \tag{10.1} $$

and

$$ f_v(x) = \begin{cases} 
\frac{x}{v}, & x \leq v; \\
\frac{1-x}{1-v}, & x > v, 
\end{cases} \tag{10.2} $$

In other words, we find the homeomorphism $h : [0, 1] \to [0, 1]$, which is a solution of the functional equation

$$ h(f) = f_v(h). \tag{10.3} $$

We have constructed in Section 6 the homeomorphism $h$ as a limit of piecewise linear homeomorphisms $h_n$, whose breaking points belonged to the set $A_n$, which is a solution of the equation $f^n(x) = 0$. For any $n \geq 1$ the equality $h_n(A_n) = B_n$ holds, where $B_n$ is the solution set of the equation $f_n^v(x) = 0$. By Proposition 6.3

$$ A_n = \left\{ 0, \frac{1}{2^{n-1}}, \ldots, \frac{2^{n-1} - 1}{2^{n-1}}, 1 \right\}, $$

By Theorem 6.4 the equality $h(A_n) = B_n$ holds.

By Theorem 6.4 the equality $h(x) = h_n(x)$ holds for every $x \in A_n$ and the conjugacy $h$ increase.

We have denoted in Section 7 the elements of $A_n$ by $\alpha_{n,k}$, $0 \leq k \leq 2^{n-1}$ such that $\alpha_{n,k_1} < \alpha_{n,k_2}$ for $k_1 < k_2$. By Proposition 6.3 the equality

$$ \alpha_{n,k} = \frac{k}{2^{n-1}} $$

holds. Also we have in Section 7 the elements of $B_n$ by $\beta_{n,k}$, $0 \leq k \leq 2^{n-1}$ such that $\beta_{n,k_1} < \beta_{n,k_2}$ for $k_1 < k_2$. In these notations, if follows from Theorem 6.4 that $h(\alpha_{n,k}) = \beta_{n,k}$ for all $n \in \mathbb{N}$ and all $k$, $0 \leq l \leq 2^{n-1}$. 

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Notice that the following evident property of $\alpha_{n,k}$ and $\beta_{n,k}$ holds.

\[
\begin{aligned}
\alpha_{n+t,2^k} &= \alpha_{n,k}, \\
\beta_{n+t,2^k} &= \beta_{n,k}.
\end{aligned}
\] (10.4)

Notice, that the number, which is formed of the first $n$ digits of the binary decomposition of $x$ is $\alpha_{n,k}$ (for some $k$) and $k = [2^n x]$, where brackets denote the integer part of a number, i.e. $[2^n x]$ is the biggest integer, which is not grater than $2^n x$. Thus, the following equality holds for $h$.

\[h(x) = \lim_{n \to \infty} \beta_{n,[2^n x]}.\] (10.5)

### 10.1 The first way of finding of explicit formulas

Remind, that the graph of $f^n_v$ is piecewise linear and consists of $2^n$ intervals of monotonicity, each of them maps some subinterval of $[0, 1]$ to the whole $[0, 1]$.

Consider an integer $x = 1, \ldots, 2^n$ and calculate the tangent of the branch of monotonicity number $k$, $0 \leq k \leq 2^n - 1$ of the maps $f^n_v$. Let the binary decomposition of $k$ be

\[k = x_1 x_2 \ldots x_n.\]

Let $\bar{x} \in [0, 1]$ be a point of the interval of monotonicity under consideration, i.e. $\bar{x} \in [\alpha_{n,k}, \alpha_{n,k+1})$.

The graph of $f^n_v$ increase on the intervals with even numbers (the numeration starts with zero) and decrease on the intervals with odd number. In other words, $(f^n_v)'(x) > 0$ for $x_n = 0$ and $(f^n_v)'(x) < 0$ for $x_n = 1$.

The maps $f_v$ acts on each of the branch of the linearity of it’s correspond iteration as follows. Independently on increasing or decreasing of the branch, it transforms to two branches such that the left one increase and the right one decrease.

1. If the branch increased, then the tangent of the new increasing branch if the former tangent, multiplied by $\frac{1}{v}$. The tangent of the new decreasing is obtained from the former tangent by multiplying it by $\frac{1}{v-1}$.
2. If the branch increased, then the tangent of the new increasing branch if the former tangent, multiplied by \( \frac{1}{v-1} \). The tangent of the new decreasing is obtained from the former tangent by multiplying it by \( \frac{1}{v} \).

Notice that the \( n \)-th iteration of \( f_v \) at \( \tilde{x} \) is a composition of linear maps, whence the tangent of \( f^n_v \) at this point would be the product of tangents os correspond branches of linearity of \( f_v \) at points \( \alpha_k = f^k_v(\tilde{x}) \), \( k = 1, \ldots n \). In other words,

\[
(f^n_v)'(\tilde{x}) = \prod_{k=1}^{n} \alpha_k,
\]

where \( \alpha_k \) can be found as follows.

\[
\begin{aligned}
\alpha_1 &= \frac{1}{v} & \text{for } x_1 = 0 \\
\alpha_1 &= \frac{1}{v-1} & \text{for } x_1 = 0 \\
\alpha_k &= \frac{1}{v} & \text{for } x_{k-1} + x_k \equiv 0 \pmod{2} \\
\alpha_k &= \frac{1}{v-1} & \text{for } x_{k-1} + x_k \equiv 1 \pmod{2}
\end{aligned}
\]

For bigger \( k \) we have

Take \( x_0 = 0 \) for making this formula correct for \( k = 1 \).

We will need two additional notations for the following result. Denote \( \psi_1(x) \) as follows

\[
\psi_1(x) = \begin{cases} 
0 & \text{for } x \equiv 0 \pmod{2} \\
1 & \text{for } x \equiv 1 \pmod{2},
\end{cases}
\]

and denote

\[
\psi_2(x) = 1 - \psi_1(x).
\]

**Lemma 33.** Functions \( \psi_1(x) \) and \( \psi_2(x) \) can be given as

\[
\psi_1(x) = 2 \left\{ \frac{x}{2} \right\} ; \quad \psi_2(x) = 2 \left\{ \frac{x + 1}{2} \right\},
\]

where figure brackets denote the fractional part of a number.
Proof. Lemma is evident.

With the use of \( \psi_1 \) and \( \psi_2 \) we can rewrite the formula for \( \alpha_k \) as follows

\[
\alpha_k = \frac{1}{v-1} \cdot \psi_1(x_k + x_{k-1}) + \frac{1}{v} \cdot \psi_2(x_k + x_{k-1}) = \\
= \frac{2}{v-1} \cdot \left\{ \frac{x_k + x_{k-1}}{2} \right\} + \frac{2}{v} \cdot \left\{ \frac{x_k + x_{k-1} + 1}{2} \right\} .
\]

Thus, the tangent of the \( k \)-th branch of monotonicity of \( f^n_v \) (for \( k = 1, \ldots, 2^n \)), can be calculated as

\[
\zeta_{n,k} = \prod_{k=1}^{n} \left( \frac{2}{1-v} \cdot \left\{ \frac{x_k + x_{k-1}}{2} \right\} + \frac{2}{v} \cdot \left\{ \frac{x_k + x_{k-1} + 1}{2} \right\} \right) . \tag{10.6}
\]

Since the graph of \( f^n_v \) consists of \( 2^n \) branches, and each of them maps some subinterval of \([0, 1]\) into the whole \([0, 1]\), then the following expressions for \( \beta_{n,k} \) hold.

\[
\beta_{n,k} = \sum_{t=1}^{k} \frac{1}{\zeta(t,n)} = \exp \left( \prod_{t=1}^{k} \frac{1}{\zeta(t,n)} \right) . \tag{10.7}
\]

Remind that \( x_1, \ldots, x_n \) in the formula for \( \zeta_{n,k} \) if the binary decomposition of \( k \). We will obtain the formulas, which would express each of these digits in terms of \( k \).

The number, which is consisted of the last \( t \) binary digits of \( k \), can be found by the formula \([\frac{k}{2^t}]\).

If one change the last \( t \) digits of \( x \) to zeros, then would obtain the number

\[
k - 2^t \cdot \left[ \frac{k}{2^t} \right].
\]

If delete the last \( t \) digits of the obtained number, then obtain

\[
\frac{k - 2^t \cdot \left[ \frac{k}{2^t} \right]}{2^t} = k - \left[ \frac{k}{2^t} \right] = \left\{ \frac{k}{2^t} \right\} .
\]

The last \( t+1 \)-st digit of \( x \) can be calculated as the last digit of the the number, which is obtained from \( x \) by deleting the last \( t \) digits. In other words, it ca be found as

\[
\psi_1 \left( \left\{ \frac{k}{2^t} \right\} \right) = 2 \left\{ \frac{\left\{ \frac{k}{2^t} \right\}}{2} \right\} .
\]
If the number \( k \) is consisted of \( n \) digits, then each of them can be found as

\[
x_p = 2 \left\{ \frac{k}{2^{n-p}} \right\}.
\]

These computations let us to rewrite the formula (10.6) for \( \zeta_{n,k} \) as follows

\[
\zeta_{n,k} = \prod_{t=1}^{\lfloor k/2 \rfloor} \left( \frac{2}{1-v} \cdot \left\{ \left\{ \frac{k}{2^{n-t}} \right\} / 2 + \left\{ \frac{k}{2^{n-t+1}} / 2 \right\} \right\} + \frac{2}{v} \left\{ \left\{ \frac{k}{2^{n-t}} / 2 \right\} + \left\{ \frac{k}{2^{n-t+1}} / 2 + \frac{1}{2} \right\} \right\} \right).
\]

(10.8)

Thus, using (10.7), we have the following theorem.

**Theorem 10.1.** The homeomorphic solution \( h : [0, 1] \to [0, 1] \) of the equation (10.3) can be expressed by the formula

\[
h(x) = \lim_{n \to \infty} \beta ([2^n x] , n) = \lim_{n \to \infty} \sum_{t=1}^{[2^n x]} \frac{1}{\zeta_{n,t}},
\]

where \( \zeta(k,n) \) is given by (10.8).

**Remark 10.1.** In spite that the formula for \( h(x) \) is quite complicated and contains a limit, this limit exists and the function is defined correctly.

**Proof.** The existence of the limit follows from the same reasonings, which where done in the proof of Theorem 6.5.

**Remark 10.2.** Notice that the formula from Theorem 10.1 has the following properties.

1. Even in the case when \( x \in A \) the calculation of \( h \) by this formula would need the \( n \) summands for obtaining the approximation \( [2^n x] \in A_n \). The condition \( x \in A \) yield that the approximate values of \( h(x) \) would not change with increasing \( n \) after some \( n \), huge enough.

2. The stabilizing of \( h(x) \), which is under consideration, means that for some \( n \in \mathbb{N} \) the equality

\[
\sum_{t=1}^{[2^n x]} \frac{1}{\zeta_{n,t}} = \sum_{t=1}^{[2^{n+1} x]} \frac{1}{\zeta_{n+1,t}}
\]

holds. From another hand, this equality means that the \( x \in A_n \) and the exact value of \( h(x) \) is found.
The first of the properties from the Remark 10.2 can be considered as its deficiency. We will find in the next section another formula for $\beta_{n,k}$, which would not have this deficiency.

### 10.2 The second way of finding of explicit formulas

Clearly, the values of $\beta_{1,k}$, $0 \leq k \leq 2$ as as follows: $\beta_{1,0} = 0$, $\beta_{1,1} = v$ and $\beta_{1,2} = 1$.

Similarly to as it was done in Section 10.1 write $k$ as follows

$$k = \sum_{i=1}^{n} x_i 2^{n-i}.$$  

Consider the sequence $\xi_t = \xi_{t,n,k}$ of the left ends of the interval $[\beta_{t,s}, \beta_{t,s+1}) (t < n)$, if is given that that is contains the point $\beta(n,k)$.

Let for some $t$ the number $\xi_t$ is found and let $p_t$ be the tangent from the right of the maps $f^t$. Precisely, $\xi_0 = 0$, $p_0 = 1$.

**Lemma 34.** If $x_1 = 1$, then $\xi_1 = v$, and the maps $f^1$ decrease at the fight neighborhood of $\xi_1$ and $p_1 = \frac{1}{v-1}$.

If $x_1 = \ldots = x_{t-1} = 0$ and $x_t = 1$, then $\xi_t = v^t$ and the maps $f^t$ decrease in the right neighborhood of $\xi_t$ and $p_t = \left(\frac{1}{v}\right)^{t-1} \frac{1}{v-1}$.

Let $x = \frac{k}{2^n}$, then

$$x = \sum_{i=1}^{n} x_i 2^{-i}.$$  

Notice, that the number of the t-th first digit 1 of the expression for $x$ can be found as follows

$$t = \lfloor - \log_2 x \rfloor.$$  

If $\{\log_2 x\} = 0$ then $x$ is a power of 2, i.e.

$$((-1)^{\lfloor - \log_2 x \rfloor} - 1)/2 = \begin{cases} 0 & x = 2^t \\ -1 & x \neq 2^t \end{cases}$$  

**Lemma 35.** Let for some $t$ the maps $f^t$ decrease in the right neighborhood of $\xi_t$ and $x_t = 1$.

Then the following implications hold.
If \( x_{t+1} = 1 \), then \( \xi_{t+1} = \xi_t - \frac{1}{p_{t+1}} \) and \( p_{t+1} = \frac{1}{p_t} \).

If \( x_{t+1} = x_{t+2} = \ldots = x_{t+s} = 0 \) \((s \geq 1)\) and \( x_{t+s+1} = 1 \), then \( \xi_{t+s+1} = \xi_t - \frac{1}{p_{t+s}} \) and \( p_{t+s+1} = p_t \left( \frac{1}{p_t - 1} \right)^2 \left( \frac{1}{2} \right)^{s-1} \).

**Lemma 36.** Let for some \( t \) the maps \( f^t \) decrease in the right neighborhood of \( \xi_t \) and \( x_t = 1 \).

Let also \( x_{t+1} = x_{t+2} = \ldots = x_{t+s} = 0 \) \((s \geq 1)\) and \( x_{t+s+1} = 1 \). Then

\[
\xi_{t+s+1} = \xi_t - \frac{1}{\zeta([2^{t+s+1}x], t)},
\]

where \( \zeta(k, n) \) is expressed by (10.8).

The number of the second digit 1 of the expression of \( x \) is

\[
t_2 = \left\lfloor -\log_2(x - 2^{-t}) \right\rfloor.
\]

Continuing this way, we can obtain the formula for the value \( h(x) \) for every \( x \in [0, 1] \).

Thus, we have proved the following theorem.

**Theorem 10.2.** The homeomorphic solution \( h : [0, 1] \to [0, 1] \) of (10.3) can be expressed by formula

\[
h(x) = \sum_{i=1}^{\infty} \frac{(2^{i-1}x)((-1)^{[-\log_2(2^{i+1}x)]} - 1)}{\zeta_i([2^{i+1}x], t)}
\]

where \( \zeta_{m,k} \) expressed by (10.8).

**Remark 10.3.** Notice, that if \( x \in A \), then the formula from Theorem 10.2 contains only finite number of summands.
11 Conjugateness of piecewise linear unimodal maps

We will consider in this Section the map

\[ f(x) = \begin{cases} 
2x, & \text{if } 0 \leq x < 1/2, \\
2 - 2x, & \text{if } 1/2 \leq x \leq 1.
\end{cases} \tag{11.1} \]

and a continuous map \( g : [0, 1] \to [0, 1] \) of the form

\[ g(x) = \begin{cases} 
g_l, & \text{if } 0 \leq x < v, \\
g_r, & \text{if } v \leq x \leq 1,
\end{cases} \tag{11.2} \]

where \( g_l(0) = g_r(1) = 0, g_r(v) = 1 \) and functions \( g_l \) and \( g_r \) are monotone piecewise linear.

We will consider a homeomorphism \( h : [0, 1] \to [0, 1] \), such that the following diagram

\[ \begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, 1] & \xrightarrow{g} & [0, 1],
\end{array} \]

is commutative, i.e. the equality

\[ h(f(x)) = g(h(x)) \tag{11.3} \]

holds for every \( x \in [0, 1] \).

11.1 Continuous differentiability of the conjugation

The main result of this section is the following theorem.

**Theorem 11.1.** Let the map \( f \), which is given by (11.1), be topologically conjugated with piecewise linear map \( g, [0, 1] \to [0, 1] \), and let \( h \) be the conjugacy such that (11.3) holds. If \( h \) is continuously differentiable on \( (\alpha, \beta) \) for some \( 0 \leq \alpha < \beta \leq 1 \), then \( h \) is piecewise linear on \([0, 1] \).

We will formulate some lemmas at the very beginning. These lemmas are, in fact, the steps of the proof of Theorem 11.1.
Lemma 37. If the homeomorphism $h$ is piecewise differential on some interval $(\alpha, \beta) \subset (0, 1)$ and is a conjugacy of $f$ and the piecewise linear $g$, then there exists open disjoint intervals $A_1, \ldots, A_s$ such that $\bigcup_{i=1}^s A_i = [0, 1]$ and $h$ is continuously differentiable on each $A_i$.

Lemma 38. If the homeomorphism $h$ is continuously differentiable on some interval $(\alpha, \beta) \subset (0, 1)$ and $A_1, \ldots, A_s$ are open intervals from Lemma 37, then for every $i, 1 \leq i \leq s$ there exists $k_i \in \mathbb{R}$ such that for every $x \in A_i$ the maps $h$ is differentiable at $f(x)$ and the equality

$$h'(f(x)) = k_i h'(x)$$

holds.

Lemma 39. Let $x^*$ be a periodical point of $f$ of the period $n$, whose orbit belongs to $\bigcup_{i=1}^s A_i$ and $h'(x^*) \neq 0$. For every $i, 1 \leq i < n$ denote by $x_i = f^i(x^*)$ the trajectory of $x^*$. Let $A_{c_0}, A_{c_1}, \ldots, A_{c_{n-1}}$ be sets from Lemma 37, which contain the trajectory (i.e. $x_i \in A_{c_i}$). Then the equality $k_{c_0} \cdot k_{c_1} \cdot \ldots \cdot k_{c_{n-1}} = 1$ holds, where $k_{c_0}, k_{c_1}, \ldots, k_{c_{n-1}}$ are constructed for intervals $A_{c_0}, A_{c_1}, \ldots, A_{c_{n-1}}$ as in Lemma 38.

Lemma 40. There exists an interval $[a, b] \subset [0, 1]$, where the derivative $h'$ is constant.

Lemma 41. Let the maps $f$ be of the form (11.3) and let $h$ be the conjugacy of $f$ with the piecewise linear unimodal $g$. Assume that $h$ is piecewise continuously differentiable on an interval $(\alpha, \beta) \subset (0, 1)$. Then there exists numbers $\alpha = \alpha_1 < \alpha_2 < \ldots < \alpha_t = \beta$ such that for every $p \in 1, \ldots, t - 1$ there exists $k_p$ such that $h'(w) = k_p h'(x)$ for all $x \in (\alpha_p, \alpha_{p+1})$, where $w = f(x)$.

Proof. Without loss of generality assume that $1/2 \not\in (\alpha, \beta)$. For avoiding two cases whether $\beta < 1/2$, of $\alpha > 1/2$ denote $f(x) = ax + b$ for $x \in (\alpha, \beta)$, as $f$ is linear on $(\alpha, \beta)$. Let $\alpha = \alpha_1 < \alpha_2 < \ldots \alpha_t = \beta$ be such numbers, that all braking points of $g$ belong to $h(\alpha_1), \ldots, h(\alpha_t)$. Let $g(x) = a_p x + b_p$ be the formula for $g$ for $x \in (h(\alpha_p), h(\alpha_{p+1}))$.

Let $x \in (\alpha, \beta) \setminus \{\alpha_2, \ldots, \alpha_{t-1}\}$ be fixed. Consider a sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x$ and $x_n \neq x$ for all $n$. Consider the following equalities

$$\begin{cases} h(f(x)) = g(h(x)), \\ h(f(x_n)) = g(h(x_n)). \end{cases} \quad (11.4)$$
Since \( x \in (\alpha_p, \alpha_{p+1}) \) for some \( p, 1 \leq p \leq t - 1 \), then equalities (11.4) can be rewritten as

\[
\begin{align*}
    x & \quad \xrightarrow{f} \quad ax + b \\
    h(x) & \quad \xrightarrow{g} \quad a_p h(x) + b_p
\end{align*}
\]
and

\[
\begin{align*}
    x_n & \quad \xrightarrow{f} \quad ax_n + b \\
    h(x_n) & \quad \xrightarrow{g} \quad a_p h(x_n) + b_p
\end{align*}
\] (11.5)

Without loss of generality assume that \( x_n \in (\alpha_p, \alpha_{p+1}) \) for all \( n \). Consider a number \( w = ax + b \) and the sequence \( w_n = ax_n + b_n \). Since \( \lim_{n \to \infty} x_n = x \), then \( \lim_{n \to \infty} w_n = w \). Since \( x_n \neq x \) for all \( n \), then \( w_n \neq w \) for all \( n \). Prove the existence of the limit \( \lim_{n \to \infty} \frac{h(w) - h(w_n)}{w - w_n} = \frac{a_p (h(x) - h(x_n))}{a (x - x_n)} \), which prove Lemma if take \( k_p = \frac{a_p}{a} \).

**Proof of Lemma 38.** Consider numbers \( \alpha_1, \ldots, \alpha_t \) from Lemma 41. Since for every \( p \) the maps \( h \) is differentiable on \( f(\alpha_p, \alpha_{p+1}) \), we may repeat the proof of Lemma 41 for the intervals, whose union is \( f(\alpha, \beta) \). Evidently, there is a finite \( k \) such that \( f^k(\alpha, \beta) = (0, 1) \) and we will divide the each interval into a finitely many sub intervals, whence the necessary \( A_1, \ldots, A_s \) would appear.

We will need the following two remarks for the further reasonings.

**Remark 11.1.** The set of periodical points of \( f \) is dense in \([0, 1]\).

**Remark 11.2.** None of numbers \( k_1, \ldots, k_t \) from Lemma 38 does not equal to 0.

**Proof of Remark 11.1.** The graph of the \( n \)-th iteration of \( f \) consists of \( 2^n \) line segments, each of them has tangent either \( 2^n \), or \(-2^n \) and maps some subinterval of \([0, 1]\) onto the whole \([0, 1]\). The lengths of “domains” of these line segments are \( \frac{1}{2^n} \) and tend to 0 if \( n \to \infty \). Since each of these line segments intersects the line \( y = x \), then correspond domain contains a fixes point of \( f^n \), which is periodical for \( f \).

**Proof of Remark 11.2.** The equality \( k_i = 0 \) yields that \( h \) is constant on \( f(A_i) \), which contradicts to that \( h \) is a homeomorphism.
Denote \( A = \bigcup_{i=1}^{s} A_i \). Since the set \([0, 1]\setminus A\) is finite, then the following corollary follows from Remark 11.1.

**Corollary 11.1.** The set of periodical points of \( f \), whose trajectories belong to \( A \), is dense in \([0, 1]\).

**Lemma 42.** Let \( x^* \) be a periodical point of \( f \), such that its trajectory belongs to \( A \) and \( h'(x^*) \neq 0 \). Then there is a neighborhood of \( x^* \), where the derivative \( h' \) is constant.

**Proof.** Let \( n \) be a period of \( x^* \). Denote by \( x_i \) the trajectory of \( x^* \), i.e. \( x_i = f^i(x^*) \), and \( x_0 = x^* = x_n \). Since the trajectory of \( x^* \) belongs to \( A \), then for every \( i \) the derivative \( h'(x_i) \) exists. It follows from Lemma 38 and Remark 11.2 that \( h'(x_i) \neq 0 \) for every \( i \). Let \( A_{c_0}, A_{c_1}, \ldots, A_{c_{n-1}} \) be the sets from Lemma 37, which contains the points of trajectory of \( x^* \), i.e. \( x_i \in A_{c_i} \). Then it follows from Lemma 38 that \( h'(x^*) = k_{c_0} \cdot k_{c_1} \cdot \ldots \cdot k_{c_{n-1}} h'(x^*) \). Since \( h'(x^*) \neq 0 \), then

\[
k_{c_0} \cdot k_{c_1} \cdot \ldots \cdot k_{c_{n-1}} = 1. \quad (11.6)
\]

Since the set \( A_{c_0} \) is open and \( x^* \in A_{c_0} \), then there exists \( \varepsilon > 0 \), such that \((x^* - \varepsilon, x^* + \varepsilon) \subset A_{c_0}\). Since the trajectory of \( x^* \) belongs to \( A \) then without loss of generality we may assume that the first \( n \) points of the trajectory of each point from \((x^* - \varepsilon, x^* + \varepsilon)\) also belong to \( A \).

Precisely, for each point \( \bar{x} \in (x^* - \varepsilon, x^* + \varepsilon) \) and for every \( i, 1 \leq i \leq n \) the inclusion \( f^i(\bar{x}) \in A_{c_i} \) holds. Without loss of generality assume that \( f^n \) is linear on \((x^* - \varepsilon, x^* + \varepsilon)\), because otherwise decrease \( \varepsilon \). Notice that \( f^n(x^*) = x^* \). There is a neighborhood of \( x^* \), where \( f^n \) is given either by \( f^n(x) = 2^n(x - x^*) \), or \( f^n(x) = -2^n(x - x^*) \).

Assume that \( f^n(x) = 2^n(x - x^*) \) in some neighborhood of \( x^* \). Consider an arbitrary \( \bar{x} \in (x^* - \varepsilon, x^*) \) and the sequence of its pre images \( \bar{x}_i \), which is given by the equalities \( \bar{x}_0 = x^* \) and \( \bar{x}_{i+1} = (f^n)^{-1}(\bar{x}_i) \), where \((f^n)^{-1}\) means the maps, which is inverse to \( f^n \) on \((x^* - \varepsilon, x^* + \varepsilon)\). Since the derivative of the maps \( f^n \) on the correspondent interval is \( > 1 \), then the sequence \( \bar{x}_i \) increase and tends to \( x^* \). It follows from the equality \((11.6)\) andLemma 38 that \( h'(\bar{x}_i) = h'(\bar{x}_{i-1}) \) for every \( i \geq 0 \). Since \( h \) is continuously differentiable, then it follows from that \( h'(\bar{x}_i) = h'(x^*) \) for every \( i \geq 0 \) and \( \lim_{i \to \infty} \bar{x}_i = x^* \) that \( h'(\bar{x}) = h'(x^*) \). The arbitrariness of \( \bar{x} \in (x^* - \varepsilon, x^*) \) yields that derivative of \( h \) is constant on \((x^* - \varepsilon, x^*)\).
The case when $f^n(x) = -2^n(x - x^*)$ is similar if consider the images instead of pre images of $\tilde{x} \in (x^* - \varepsilon, x^*)$.

Now Theorem 11.1 follows from the proved lemmas.

Proof of Theorem 11.1. By Lemma 40 let $[a, b] \subset [0, 1]$ be an interval, where the derivative $h'$ is constant, denote $p_1$. Without loss of generality assume that $[a, b] \subset A_i$ for some $i$, $1 \leq i \leq s$ where $\{A_i\}$ are defined in Lemma 37. By Lemma 38 there exists $k_i \in \mathbb{R}$ (which is dependent only on $i$), such that $h'(f(x)) = k_i f(x)$ for every $x \in A_i$. Now we can write $f([a, b]) = \left( \bigcup_{i=1}^{s} [A_i \cap f([a, b])] \right) \cup C$, where $C$ is a finite set whence $h$ is piecewise linear on $f([a, b])$. Since tangents of $f$ are 2 and $-2$ only, then $\{f(x) | x \in [a, b]\}$ is of Lebesgue’s measure $\min\{2(b-a), 1\}$ and Theorem follows from that $[0, 1]$ is an image of $[a, b]$ under finitely many iterations.

Lemma below follows from Theorem 6.4.

Lemma 43. 1. Homeomorphism $h$ increase and $h(1/2) = v$;

2. The equation (11.3) can be rewritten as

\[
\begin{align*}
  h(f_i(x)) &= g_l(h(x)) \quad \text{for } x \leq 1/2, \\
  h(f_r(x)) &= g_l(h(x)) \quad \text{for } x \geq 1/2.
\end{align*}
\]

The following propositions contain some properties of $g$, which follow from that conjugacy is piecewise linear.

Proposition 11.2. The maps $g$ has exactly two fixed points: one of these points is zero and another belongs to $(v, 1)$.

Proof. The existence of the mentioned points follows from the construction of $g$ and the theorem about the mean value of a continuous function.

Prove that there is no other fixed points of $g$.

Non-existence of other fixed points of $g$ on $(v, 1)$ follows from that $g$ decrease on $(v, 1)$ and the fixed point if the point of intersection of the graph of $g$ with $y = x$.

Assume that $g$ has a fixed point $x_0$ on $(0, v)$. Applying (11.3) for $x = h^{-1}(x_0)$, obtain that $h^{-1}(x_0)$ is a fixed point of $f_i$ on $(0, 1/2)$, which contradicts to that $f$ has no fixed points on this interval.
Corollary 11.2. Trajectory of an arbitrary \( x \in (0, 1) \) tends to 0 under \( g_l^{-1} \).

11.2 Piecewise linearity of the conjugation

Remind that it is obtained in Theorem 3.5 that if the map \( g : [0, 1] \to [0, 1] \) of the form (11.2) is conjugated to \( f : [0, 1] \to [0, 1] \) of the form (11.1) via piecewise linear conjugacy then \( g'(0) = 2 \). Theorem 3.5 was the inspiration of this Section.

Theorem 11.3. For an arbitrary \( v \in (0, 1) \) and increasing piecewise linear maps \( g : [0, v] \to [0, 1] \) such that \( g(0) = 0 \), \( g(v) = 1 \) and \( g'(0) = 2 \) there exists its unique continuation \( \tilde{g} : [0, 1] \to [0, 1] \), which is conjugated to \( f \) of the form (11.1) via piecewise linear conjugation.

Theorem 11.4. For an arbitrary \( v \in (0, 1) \) and arbitrary decreasing piecewise linear maps \( g : [v, 1] \mapsto [0, 1] \) such that \( g(v) = 1 \), \( g(1) = 0 \) and \( (g^2)'(x_0) = 4 \), where \( x_0 \) is a positive fixed points of \( g \), there exists its unique continuation \( \tilde{g} : [0, 1] \to [0, 1] \), which is topologically conjugated to \( f \) of the form (11.1) via piecewise linear conjugation.

We will prove Theorems 11.3 and 11.4 constructively. In the proof of Theorem 11.3 we will show that the increasing part of \( g \) defines uniquely the piecewise linear conjugacy \( h \), which satisfies (11.3). After this, equation (11.3) defines uniquely the map \( g \) via \( f \) and the found conjugacy. Theorems 11.3 and 11.4 also can be understood in the manner that \( g_l \) itself and \( g_r \) itself defines the conjugacy \( h \). Nevertheless, Theorem 3.5 states, that \( g \) can not be arbitrary, for instance, \( g'_l(0) = 2 \).

Notation 11.1. For the piecewise linear conjugacy of \( f \) of the form (11.1) and unimodal piecewise linear map \( g \), denote by \( k \) the derivative of \( h \) at 0. Also denote by \( \varepsilon \) the \( x \)-coordinate of the first break of \( g \), i.e. a). \( h(x) = kx \) at some neighborhood of \( x = 0 \); b). \( g(x) = 2x \) for all \( x \in (0, \varepsilon) \) and c). \( x = \varepsilon \) is a break point of \( g \).

Let \( A_1, \ldots, A_s \) be open intervals of linearity of \( g \) on \([0, v]\) and \( \bigcup_{i=1}^{s} \overline{A_i} = [0, v] \).

Lemma 44. For every \( x \in [0, \frac{v}{k}] \) the equality \( h(x) = kx \) holds. Moreover, \( \frac{v}{k} \) is a breakpoint of \( h \).
Proof. Since $h$ is piecewise linear, then there exists $\delta$ such that the equality (11.3) is of the form $h(2x) = 2 \cdot kx$ for $x \in [0, \delta]$ and $\delta$ is determined from that $g(x) = 2x$ for all $x \in (0, k\delta)$. From another hand, it follows from the formula for $h$ on $x \in [0, \delta]$ that $h(x) = kx$ for all $x \in (0, 2\delta)$ and this lets to consider the functional equation $h(2x) = g(kx)$ for $x \in (0, 2\delta)$. For these $x$ the range of $g(kx)$ is $[0, y_0]$, where $y_0 = h(4\delta) = g(2k\delta)$. The maximum length of the interval for $x$, such that functional equation (11.3) is of the form $h(2x) = 2 \cdot kx$ is $x \in [0, \varepsilon/2]$. Further increasing of the interval for $x$ leads to that functional equation (11.3) transforms to $h(2x) = a_2 \cdot kx + b_2$, where $g(x) = a_2x + b_2$ is the formula for $g$ on the next interval of linearity of $g$ after $(0, \varepsilon)$.

In fact, Lemma 44 states that $g_l$ defines the length of the first interval of linearity of the conjugacy $h$ dependently on its tangent on its the first interval of linearity.

Lemma 45. Let for some $v \in (0, 1)$ and piecewise linear maps $g, \tilde{g} : [0, 1] \to [0, 1]$, which are conjugated to $f$, equalities $g(v) = \tilde{g}(v) = 1$ and $g(0) = g(1) = \tilde{g}(0) = \tilde{g}(1) = 0$ holds and, furthermore, $g(x) = \tilde{g}(x)$ for all $x \in [0, v]$. Then tangents of correspond conjugacies at 0 coincide.

This lemma is a partial case of Theorem 11.3. We will prove Lemma 45 by calculating the tangent at 0 for the conjugacy $h$ of $f$ and $g$. We will use in our calculations only maps $g_l$, but not $g_r$. We will use one more notation.

Notation 11.2. Let $G_l(\alpha, \beta) = (f_l^{-1}(\alpha), g_l^{-1}(\beta))$ and $G_r(\alpha, \beta) = (f_r^{-1}(\alpha), g_r^{-1}(\beta))$ be maps of the square $[0, 1]^2$ to itself.

The following lemma follows from the increasing of $h$.

Lemma 46. Let numbers $\alpha, \beta \in [0, 1]$ be such that $h(\alpha) = \beta$. Then the graph of $h$ is invariant under the action of $G_l$ and $G_r$, i.e. $h(f_l^{-1}(\alpha)) = g_l^{-1}(\beta)$ and $h(f_r^{-1}(\alpha)) = g_r^{-1}(\beta)$.

Proof of Lemma 46. Notice that $h(1) = 1$ and consider the trajectory of $(1, 1)$ under $G_l$, obtaining $G_l^n(1, 1) = (1/2^n, g_l^{-n}(1))$. By Lemma 46 obtain that $h(1/2^n) = g_l^{-n}(1)$ for all $n$. By Corollary 11.2 the sequence $\{g_l^{-n}(1)\}$ decrease to 0. Find the tangent $k$ of the conjugacy $h$.
at the neighborhood of 0. By Lemma 44, \( h(x) = kx \) for \( x \in (0, \varepsilon/k) \), where \( \varepsilon \) is taken from Notation 11.1. For \( n \) huge enough the inclusion \( 1/2^n \in (0, \varepsilon/k) \), which means that the sequence \( k_n = 2^n g_{-n}(1) \) stabilizes on \( k \).

\[ \square \]

**Remark 11.3.** Notice that in the same way as it was done in the proof of Lemma 45, we can show that for every \( \alpha, \beta \in (0, 1) \) such that \( h(\alpha) = \beta \) the sequence \( k_n = 2^n g_{-n}(\beta) \) stabilizes independently on whether \( h \) is piecewise linear, or not. Piecewise linearity of \( h \) means that for every pair \( \alpha, \beta \in (0, 1) \) the obtained sequence stabilizes on those \( k \), which is obtained in the proof of Lemma 45 and is independent on \( \alpha \) and \( \beta \).

**Proof of Theorem 11.3.** Let \( h \) be the conjugacy of \( f \) and \( g \). By reasonings from the proof of Lemma 45, find by \( g \) the tangent \( k \) of \( h \) at 0. Then by Lemma 44 the conjugacy is expressed by \( h(x) = kx \) for \( x \in [0, \varepsilon/k] \), where \( \varepsilon \) is the first break point of \( g \). Consider \( n \), such that \( 1/2^n < \varepsilon/k \).

For every \( t \geq 0 \) from the equation (11.3) for \( x \in [1/2^n - t, 1/2^n + t] \) and from values of \( h \), known earlier on \([1/2^n - t, 1/2^n + t]\), find the values of \( h \) on \([1/2^n - t, 1/2^n + t]\). For \( t = n - 1 \) the equation (11.3) will be defined for \( x \in [1/2^n, 1/2] \), whence we find \( h \) on the whole \([0, 1]\). After this the map \( g_r \) can be found from \( h(f_r(x)) = g_r(h(x)) \) for \( x \in [1/2^n, 1] \) by the equation \( g_r(x) = h(f_r(h^{-1}(x))) \). This finishes the proof. \[ \square \]

Proposition 11.5 contains the restriction on \( g \) in the neighborhood of 0 for existing of piecewise linear conjugacy \( h \) of \( f \) and \( g \). The analog of this proposition holds for the positive fixed point of \( g \).

**Proposition 11.5.** Let \( x_0 \) be a fixed point of unimodal piecewise liner map \( g \), which is conjugated to \( f \) of the form (11.1) via piecewise linear conjugacy. Then there exists \( \varepsilon > 0 \), such that for every \( x \in (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \) the equality \( (g^2)'(x_0) = 4 \) holds, where \( g^2 \) is the second iteration of \( g \).

We know that \( h(2/3) = x_0 \), since \( 2/3 \) is a fixed point of \( f \). Let at the left neighborhood of \( x_0 \) the map \( g \) is given by \( g_1(x) = a_1x + b_1 \), and in the right neighborhood of \( x_0 \) it is given by \( g_2(x) = a_2x + b_2 \).

**Remark 11.4.** Using this notations claim that Proposition 11.3 is equivalent to \( a_1a_2 = 4 \).
**Explanation of Remark 11.4.** Since \( g \) decrease on \([v, 1]\), then it is enough to prove that there exists \( \varepsilon > 0 \) such that \( g(x_0, x_0 + \varepsilon) \subset (v, x_0) \) and \( g(x_0 - \varepsilon, x_0) \subset (x_0, 1) \).

Then for the left neighborhood \((x_0 - \varepsilon, x_0)\) the maps \( g^2 \) is defined by \( g^2(x) = g_2(g_1(x)) \), and in the right neighborhood \((x_0, x_0 + \varepsilon)\) it is defined by \( g^2(x) = g_1(g_2(x)) \). Nevertheless, in both first and second case the maps is linear in \( x_0 \) and its tangent is \( a_1a_2 \).

\[ \square \]

**Proof of Proposition 11.5.** Left in the left neighborhood of \( 2/3 \) the maps \( h \) is given by \( h_1(x) = a_3x + b_3 \), and in the right neighborhood of \( 2/3 \) it is given by \( h_2(x) = a_4x + b_4 \).

For an \( \varepsilon \) small enough consider an arbitrary number \( \beta \in (x_0, x_0 + \varepsilon) \) and consider the sequence \( G^0_r(\alpha, \beta) \), where \( \beta = h(\alpha) \). This sequence converges to \((2/3, x_0)\). The first three elements of the sequence are \((\alpha, \beta) \to (f_r^{-1}(\alpha), g_r^{-1}(\beta)) \to (f_r^{-2}(\alpha), g_r^{-2}(\beta))\), i.e. \((\alpha, \beta) \to (f_r^{-1}(\alpha), g_1^{-1}(\beta)) \to (f_r^{-2}(\alpha), g_2^{-2}(\beta))\), because \( \beta > x_0 \) and the map \( g \) decrease in the neighborhood of \( x_0 \).

Express the maps \( g_r^{-1}(\beta) \) and \( g_r^{-2}(\beta) \) as \( g_r^{-1}(\beta) = \frac{\beta - b_1}{a_1} \) and \( g_r^{-2}(\beta) = \frac{\beta - b_1}{a_2} \) = \( \frac{\beta - b_1 - a_1b_2}{a_1a_2} \). Moreover, \( f_r^{-1}(\alpha) = 1 - \frac{\alpha}{2} \) and \( f_r^{-2}(\alpha) = \frac{1}{2} + \frac{\alpha}{4} \). Also the following equalities hold.

\[
\begin{cases}
\beta = a_4\alpha + b_4, \\
g_r^{-1}(\beta) = a_3f_r^{-1}(\alpha) + b_3, \\
g_r^{-2}(\beta) = a_4f_r^{-2}(\alpha) + b_4.
\end{cases}
\]

By plugging of the first two equalities to the third one gives

\[
\frac{(a_4\alpha + b_4) - b_1 - a_1b_2}{a_1a_2} = a_4 \left( \frac{1}{2} + \frac{\alpha}{4} \right) + b_4. \tag{11.7}
\]

If follows from the arbitrariness of \( \beta \) in correspond neighborhood of \( x_0 \) and from the continuity of \( h \) that the set of \( \alpha \), such that \( h(\alpha) = \beta \) is some neighborhood of \( 2/3 \). That is why, the equality \((11.7)\) is the identity in correspond neighborhood of \( \alpha \). Saying that coefficients near \( \alpha \) in the left and right hand side of \((11.7)\) are equal, obtain that \( a_1a_2 = 4 \).

\[ \square \]

**Proof of Theorem 11.4.** Let \( x_0 \) be positive fixed point of \( g \) and \( \tilde{g} \). Let \( h \) be a conjugation of \( f \) and \( g \). Since \( h(2/3) = x_0 \) by Lemma 43 then there exists \( \varepsilon > 0 \) such that \( h(x) = k(x - 2/3) + x_0 \) for all \( x \in (2/3, 2/3 + \varepsilon) \). Consider the trajectory of \((1, 1)\) under \( G_t \). Denote by \((x_n, y_n)\) the
$n$-th iteration of this point, i.e. $G^n_i(1, 1) = (x_n, y_n)$. It follows from Lemma 16 that for each point of the trajectory one have that

$$h(x_n) = y_n.$$  \hspace{1cm} (11.8)

The sequence $x_n$ can be given by $x_n = f^{-1}_r(x_{n-1})$ and tends to $2/3$. Furthermore, for every $k \in \mathbb{N}$ the inclusions $(f^{-1}_r)^{2k}(1) \in (1/2, 2/3)$ and $(f^{-1}_r)^{2k+1}(1) \in (2/3, 1)$ holds and $\lim_{i \to \infty} x_n = 2/3$. Since for huge odd $n$ the inclusion $x_n \in (2/3, 2/3 + \varepsilon)$ holds, then the equality (11.8) can be rewritten as $y_n = k(x_n - 2/3) + x_0$, whence

$$k = \frac{y_n - x_0}{x_n - 2/3}.$$ \hspace{1cm} (11.9)

Moreover, by Proposition 11.5 without loss of generality we may assume that for these $n$ and all $y_n$ then maps $g^2$ is given by equality

$$g^2(x) = 4(x - x_0) + x_0.$$ \hspace{1cm} (11.10)

Similarly as in the proof of Proposition 11.5 we can show that if $x_n \in (2/3, 2/3 + \varepsilon)$ then

$$\frac{y_n - x_0}{x_n - 2/3} = \frac{y_{n+2} - x_0}{x_{n+2} - 2/3}.$$\hspace{1cm} (11.10)

It follows from equation (11.10) that $y_{n+2} = \frac{y_n + 3x_n}{4}$. Also $x_{n+2}$ can be found from the formula $x_{n+2} = (f^{-1}_r)^2(x_n)$, i.e. $x_{n+2} = \frac{1}{2} + \frac{x_n}{4}$. Plugging of these expressions into $\frac{y_{n+2} - x_0}{x_{n+2} - 2/3}$ gives, after simplifications that $\frac{y_{n+2} - x_0}{x_{n+2} - 2/3} = \frac{y_n - x_0}{x_n - 2/3}$. Thus, $h(2/3) = x_0$ and $h$ has the tangent $k$, defined by (11.9) for $x \in (2/3, 2/3 + \varepsilon)$.

Denote $A_1 = (2/3, 2/3 + \varepsilon)$ and consider the sequence of sets $A_{i+1} = f(A_i)$. Clearly, $f^t(A_1) = [0, 1]$ for some finite $t \in \mathbb{N}$. For every $x \in A_i$ the equation (11.3) lets to determine the values of $h$ on $A_{i+1}$ in the assumption that its values on $A_i$ is known. Sequent applying of these reasonings let to find $h$ on the whole $[0, 1]$. In fact, we have proved that the maps $g$ defines $h$ by it values on $[v, 1]$ and, then $h$ defines the increasing part of $g$. The last finishes the proof. □
11.3 Example of non convex maps, which is conjugated to standard hat-map

It is assumed in Theorem 6.1 that the function $g$, which is conjugated to $f$, is convex. Nevertheless, convexity is not used in the proof of this Theorem.

In this section we will use in details the techniques from the proof of Theorems 11.3 and 11.4 for obtaining the example of non-convex $g$, which is conjugated to $f$.

Let $g_1 : [0, 5/8] \to [0, 1]$ be monotone, increasing, piecewise linear such that $(1/4, 1/2), (1/2, 5/8), (5/8, 1)$ are all breaking points of $g_1$. Evidently, this $g_1$ is not convex. Since the proof of Theorem 11.3 is constructive, we may apply it to find the maps $g_2 : [5/8, 1] \to [0, 1]$ such that $g$ of the form (11.2), which is constructed with these $g_1$ and $g_2$ would be conjugated to $f$. Assume, that $g$ is already constructed and $h : [0, 1] \to [0, 1]$ is a homeomorphic solution of the functional equation (11.3).

Notice, that it follows from Lemma 43, that $h$ increase.

**Lemma 47.** $h(x) = 2x$ for all $x \in [0, 1/4]$ and $(1/4, 1/2)$ is a breaking points of $h$.

**Proof.** In fact, this Lemma means that $k$ from Lemma 44 equals to 2.

With the use of Lemma 46, consider the trajectory of 1 under $g_t^{-1}(1)$ and notice, that since

\[
\frac{1}{2^n} = f_t^{-n}(1), \text{ whence } h\left(\frac{1}{2^n}\right) = g_t^{-n}(1).
\]

![Graph of $g_t$](image)

The trajectory of 1 under $g_t^{-1}$ is given at Figure 46. It is clear, that $g^{-3}(1)$ is the x-coordinate if the first breaking points of $g_t$, whence $h(1/8) = 1/4$ and $k = 2$. 

\[\square\]
Let $A, B \subseteq [0, 1]$ are closed intervals. Denote by $A \xrightarrow{\varphi} B$ the fact the maps $\varphi$ is linear and increasing on $A$ such that $\varphi(A) = B$. Also denote by $A \xrightarrow{\varphi} B$ that $\varphi$ is linear on $A$, but decreasing.

In this notation Lemma 47 means that following diagram

\[
\begin{array}{ccc}
[0, 1/8] & \xrightarrow{f} & [0, 1/4] \\
\downarrow h & & \downarrow h \\
[0, 1/4] & \xrightarrow{g} & [0, 1/2]
\end{array}
\]

is commutative. The following technical lemma is obvious.

**Lemma 48.** Assume that for intervals $A$, $B$, $C$ and $D$ and maps $\varphi_1$, $\varphi_2$, $\varphi_3$ and $\varphi_4$ the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi_1} & B \\
\varphi_2 \downarrow & & \varphi_4 \downarrow \\
C & \xrightarrow{\varphi_3} & D
\end{array}
\]

is commutative. Then $B \xrightarrow{\varphi_4} D$, i.e. $\varphi_4$ is linear on $B$.

**Lemma 49.** All breaking points of $h$ are $(1/4, 1/2)$ and $(1/2, 5/8)$.

**Proof.** By Lemma 47 the following diagram

\[
\begin{array}{ccc}
[1/8, 1/4] & \xrightarrow{f} & [1/4, 1/2] \\
\downarrow h & & \downarrow h \\
[1/4, 1/2] & \xrightarrow{g_1} & [1/2, 5/8]
\end{array}
\tag{11.11}
\]

is commutative. By Lemma 48 the diagram (11.11) can be continued as

\[
\begin{array}{ccc}
[1/8, 1/4] & \xrightarrow{f} & [1/4, 1/2] & \xrightarrow{f} & [1/2, 1] \\
\downarrow h & & \downarrow h & & \downarrow h \\
[1/4, 1/2] & \xrightarrow{g_1} & [1/2, 5/8] & \xrightarrow{g_1} & [5/8, 1].
\end{array}
\]

Repeating application of Lemma 48 finishes the proof.

**Lemma 50.** All the breaking points of $g_2$ are $(13/16, 5/8)$ and $(29/32, 1/2)$. 

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**Proof.** The maps $g_2$ can be found from the commutative diagram

$$
\begin{align*}
[1/2, 1] & \xrightarrow{f} [0, 1] \\
[5/8, 1] & \xrightarrow{g_2} [0, 1]
\end{align*}
$$

(11.12)

By Lemma 49, diagram (11.12) may be rewritten as the following three diagrams.

$$
\begin{align*}
[1/2, 3/4] & \xrightarrow{f} [1/2, 1] \\
[3/4, 7/8] & \xrightarrow{-f} [1/4, 1/2] \\
[7/8, 1] & \xrightarrow{-f} [0, 1/4]
\end{align*}
$$

$$
\begin{align*}
[5/8, 13/16] & \xrightarrow{g_2} [5/8, 1] \\
[13/16, 29/32] & \xrightarrow{g_2} [1/2, 5/8] \\
[29/32, 1] & \xrightarrow{g_2} [0, 1/2]
\end{align*}
$$

This finishes the proof.

The graph of the maps $g$, constructed above, is given at Figure 47.

### 11.4 Types of piecewise linearity

The example in Section 11.3 contained the maps $g : [0, 1] \rightarrow [0, 1]$ of the form (11.2), whose increasing and decreasing part contained 3 parts of linearity each.

**Definition 11.1.** Let for the piecewise linear $g$ of the form (11.3) be topologically conjugated to $f$ of the form (11.1). Let the number of pieces of linearity of $g_1$ and $g_2$ be $p$ and $q$ correspondingly. Call the pair $(p, q)$ the **type of piecewise linearity** of $g$.  

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Definition 11.2. If for a pair \((p, q)\) there exists a maps \(g\) of the form (11.2) with type of piecewise linearity \((p, q)\), then call this type admissible. If follows from Proposition 3.5 and 11.5 that pair \((1, q)\) is admissible only if \(q = 1\) and the pair \((p, 1)\) is admissible only if \(p = 1\). In both cases the equality \(g = f\) holds.

Lemma 51. For any \(q \geq 2\) the type \((2, q)\) is admissible.

Proof. For an arbitrary \(n \in \mathbb{N}\), \(n > 2\) consider the maps \(g_1\), whose graph pass through points \((0, 0), \left(\frac{1}{n}, \frac{2}{n}\right)\) and \(\left(\frac{n-1}{n}, 1\right)\). Let \(h : [0, 1] \to [0, 1]\) be a piecewise linear homeomorphism, which defines he conjugation of \(f\) and \(g\) with \(g_1\). The existence and uniqueness of this \(h\) follows from Theorem 11.3. We will find the correspond \(g_2\) and whence calculate the type of linearity of \(g\).

Notice, that
\[
g_1^{n-1}\left(\frac{1}{n}\right) = 1.
\]

Since \(h\) increase and satisfy (11.3), we have
\[
h \left(\frac{1}{2t}\right) = 1 - \frac{t}{n}, \quad 1 \leq t \leq n - 1.
\]

(11.13)

For instance, for \(t = n - 1\) we have
\[
h \left(\frac{1}{2^{n-1}}\right) = \frac{1}{n}.
\]

The reasonings, which are similar to those in the prove of Lemmas 43 and 47 give that
\[
h(x) = \frac{2^{n-1}}{n} x, \quad x \in \left[0, \frac{1}{2^{n-2}}\right].
\]

It is evident by induction by \(k\), that the following diagram
\[
\begin{array}{ccc}
\left[\frac{2^k}{2n-1}, \frac{2^{k+1}}{2n-1}\right] & \xrightarrow{f} & \left[\frac{2^{k+1}}{2n-1}, \frac{2^{k+2}}{2n-1}\right] \\
\downarrow h & & \downarrow h \\
\left[\frac{k+1}{n}, \frac{k+2}{n}\right] & \xrightarrow{g_1} & \left[\frac{k+2}{n}, \frac{k+3}{n}\right]
\end{array}
\]

is commutative for all \(k, 0 \leq k \leq n - 3\). This prove that (11.13) is the complete set of breaking points of \(h\), precisely, \(h\) is linear on \([1/2, 1]\).
Define \( g_2 \) from the commutative diagram

\[
\begin{array}{ccc}
\frac{1}{2}, 1 & \to & [0, 1] \\
h & & h \\
\frac{n-1}{n}, 1 & \to & [0, 1]
\end{array}
\] (11.14)

Since \( h \) has \( n-2 \) break points, then it has \( n-1 \) pieces of linearity, call \( P_i, 1 \leq i \leq n-1 \). Whence, diagram (11.14) break into \( n \) diagrams of the form

\[
\begin{array}{ccc}
Q_i & \to & P_i \\
h & & h \\
h(Q_i) & \to & h(P_i)
\end{array}
\] (11.15)

where \( Q_i \) are pre-images of \( P_i \) under \( x \mapsto 2 - 2x \) and evidently, \( Q_i \subset [1/2, 1] \), and \( \bigcup_{i=1}^{n-1} Q_i = [1/2, 1] \).

These reasonings show, that the type of piecewise linearity of \( g \) is \((2, n)\).

The notice, that \( n > 1 \) is arbitrary and taking \( q = n - 1 \) finishes the proof.

Lemma 52. For any \( p, q \) such that \( 2 \leq p \leq q \) the type \((p, q)\) is admissible.

Proof. Analogically to it was done at the proof of Lemma 51 denote \( n = q + 1 \). Consider the following intervals. \( P_0 = [0, \frac{1}{2n-2}] \), \( P_t = \left[ \frac{2^t}{2n}, \frac{2^{t+1}}{2n} \right] \) for \( 0 \leq t \leq n-2 \). Observe, that \( |P_t| = \frac{2^{t+1}}{2n} \) for \( 0 \leq t \leq n-2 \) and \( f(P_t) = P_{t+1} \) for \( 0 \leq t \leq n-3 \).

Let \( k_t, 1 \leq t \leq n-2 \) be positive numbers such that

\[
2k_0 + \sum_{t=1}^{n-2} k_t 2^t = 2^{n-1}
\]

and \( k_t \neq k_{t+1} \) for all \( t, 0 \leq t \leq n-2 \). Notice, that these conditions let us to consider an increasing piecewise linear homeomorphism \( h : [0, 1] \to [0, 1] \) which is linear on \( P'_0 \) and such that \( k_t \) is the tangent of \( h \) on \( P_t \) for all \( t \). Denote the maps \( g = h^{-1}(f(h)) \). Evidently \( g : [0, 1] \to [0, 1] \) and \( g \) is of the form (11.2). clearly, \( g \) is dependent on \( K = \{ k_t, 0 \leq t \leq n-2 \} \).

Since in this case the diagram (11.15) will be commutative, then the type of piecewise linearity of \( g \) will be \((\tilde{p}, q)\) for some \( \tilde{p} \).
Denote by $\tilde{k}_t$ the tangent of $g_1$ on $h(P_t)$. It follows from Lemma 11.3 that $\tilde{k}_0 = 2$.

For any $t$, $0 \leq t \leq n - 3$ it follows from commutative diagram

\[
\begin{array}{ccc}
P_t & \xrightarrow{f} & P_{t+1} \\
h | & & h \\
\downarrow & & \downarrow \\
h(P_t) & \xrightarrow{g_1} & h(P_{t+1})
\end{array}
\]

that

\[
\tilde{k}_t = \frac{2k_{t+1}}{k_t}.
\]

Since, $k_2 \neq k_1$, then $\tilde{k}_2 \neq 2$. Evidently, the condition $\tilde{k}_t = \tilde{k}_{t+1}$ is equivalent to

\[
k_{t+2} = \frac{k_{t+1}^2}{k_t}.
\]

Assume that $p = q$. Take an arbitrary positive reals $r_0, \ldots, r_{p-1}$ such that:

\[
r_t \neq r_{t+1} \text{ for } 0 \leq t \leq p - 2 \text{ and } r_{t+2} \neq \frac{r_{t+1}^2}{r_t} \text{ for } 1 \leq t \leq p - 3
\]

Denote

\[
r = \frac{r_0}{2n-2} + \sum_{t=1}^{n-2} \frac{r_t^2}{2n-1}
\]

and consider

\[
k_t = \frac{r_t}{r}, \text{ for } 0 \leq t \leq q - 1
\]

as tangents of $h : [0, 1] \to [0, 1]$ on $P_t$ such that $h$ is linear on $P'_0$. Then the type of piecewise linearity of $g = h^{-1}(f(h))$ would be $(p, q)$.

Assume now that $1 < p < q$. Then construct the maps $g$ in the same way as in the previous case, but with $\tilde{k}_0 = 2$, $\tilde{k}_1 \neq \tilde{k}_0$, $\tilde{k}_1 = \tilde{k}_2 = \ldots = \tilde{k}_{q-p+1}$, $\tilde{k}_t \neq \tilde{k}_{t+1}$ for $q - p + 1 \leq t \leq q - 1$ Remark, that $q - 1 = n - 2$. For this reason take an arbitrary $r_0, \ldots, r_{n-2}$ such that

\[
r_t \neq r_{t+1} \text{ for } 0 \leq t \leq n - 3,
\]

\[
r_{t+2} = \frac{r_{t+1}^2}{r_t}, \text{ for } 1 \leq t \leq q - p,
\]

and $r_{t+2} \neq \frac{r_{t+1}^2}{r_t}$, for $q-p+1 \leq t \leq n-4$. Notice, that there is no contradiction between (11.18) and (11.19), because $r_{t+1} \neq r_t$ implies $\frac{r_{t+1}^2}{r_t} \neq r_{t+1}$, whence $r_{t+2} \neq r_{t+1}$ for $r_{t+2}$ found by (11.18).
Now define $r$ and $k_t$, $0 \leq t \leq n - 2$ by (11.16) and (11.17) and consider $k_t$ as tangents of $h : [0, 1] \to [0, 1]$ on $\mathcal{P}_t$ such that $h$ is linear on $\mathcal{P}_0'$. Then the type of piecewise linearity of $g = h^{-1}(f(h))$ would be $(p, q)$.

\begin{center}\textbf{Lemma 53.} For any $p, q$ such that $2 \leq q < p$ the type $(p, q)$ is admissible.\end{center}

Before the proof of Lemma 53 make some observation about the proof of Lemma 52.

Denote by $f_l(x) = 2x$ for $x \in [0, 1/2]$, which is the decreasing part of $f$ and denote by $f_r(x) = 2 - 2x$ for $x \in [1/2, 1]$, which is the decreasing part of $f$. For every $t$ denote by $f_l^{-t}$ and $f_r^{-t}$ the $t$-th iteration of the maps $f_l^{-1} : [0, 1] \to [0, 1/2]$ and $f_r^{-1} : [0, 1] \to [1/2, 1]$ correspondingly. Naturally, $f_l^{-1}$ and $f_r^{-1}$ are inverse maps to $f_l$ and $f_r$ correspondingly. Then it is evident, that in the notations of the proof of Lemma 52 we have that $\mathcal{P}_{n-2} = [1/2, 1]$ and $\mathcal{P}_{n-2-k} = f_l^{-k}(\mathcal{P}_{n-2})$ for all $k$, $0 \leq k \leq n - 2$. Denoting $t = n - 2 - k$ obtain that $\mathcal{P}_t = f_l^{t+2-n}(\mathcal{P}_{n-2}) = f_l^{t+2-n}([1/2, 1]).$

\begin{center}\textbf{Proof of Lemma 53.} Denote $n = p + 1$ and $Q_t = f_l^{t+2-n}[0, 1/2]$ for $0 \leq t \leq n - 2$ and $Q'_0 = [1/2, 1]\bigcup \limits_{t=0}^{n-2} Q_t$. Notice, that the fixed point of $f_r$ belong to $Q'_0$ and the length of $Q_t$ is $\frac{2^t}{2^{n-t}}$ for all $t$, $0 \leq t \leq n - 2$.

The continuation of the proof is analogical to the proof of Lemma 52.\end{center}

The following theorem follows from Proposition 3.5 and 11.5 and Lemmas 52 and 53.

\begin{center}\textbf{Theorem 11.6.} 1. For any $p \geq 2$ and $q \geq 2$ the type of linearity $(p, q)$ is admissible.

2. A type of linearity $(p, 1)$ and $(1, q)$ is admissible only if it is $(1, 1)$. In this case the maps $g$ coincides with $f$.\end{center}
12 Semi conjugation of unimodal maps

We will consider in this section the semi conjugation of maps \( f : [0, 1] \to [0, 1] \), which is given by

\[
    f(x) = \begin{cases} 
        2x, & x < 1/2; \\
        2 - 2x, & x \geq 1/2,
    \end{cases}
\]  

(12.1)

and the map \( f_v : [0, 1] \to [0, 1] \), which is given by

\[
    f_v(x) = \begin{cases} 
        \frac{x}{v}, & 0 \leq x \leq v; \\
        \frac{1-x}{1-v}, & v < x \leq 1,
    \end{cases}
\]  

(12.2)

for an arbitrary \( v \in [0, 1] \setminus \{1/2\} \).

Consider the functional equation

\[
    \psi(f(x)) = f_v(\psi(x))
\]  

(12.3)

for the unknown continuous function \( \psi : [0, 1] \to [0, 1] \). In this section we will study the properties of the solutions of (12.3), and will not assume the invertibility of them.

Remind that continuous surjective solution of (12.3) is called the semi conjugation of \( f \) and \( f_v \).

12.1 Monotone solutions of functional equation

Let \( \eta : [0, 1] \to [0, 1] \) be continuous (but not necessary surjective) solution of functional equation (12.3). We will prove, that if \( \eta \) monotone, then it is a conjugation between \( f \) and \( f_v \), defined by (12.1) and (12.2).

Lemma 54. \( \eta(0) = 0 \), or \( \eta(0) = \frac{1}{2-v} \).

Proof. Plug \( x = 0 \) into (12.3), and obtain

\[
    \eta(f(0)) = f_v(\eta(0)).
\]

Whence, \( \eta(0) \) is a fixed point of \( f_v \). Since \( f_v \) has two branches of monotonicity, which are defined by formulas \( f_v(x) = \frac{x}{v} \) and \( f_v = \frac{1-x}{1-v} \), then fixed points of \( f_v \) are 0 and \( \frac{1}{2-v} \). \qed
Lemma 55. If \( \eta(0) = 0 \), then either \( \eta(1) = 0 \), or \( \eta(1) = 1 \).

Proof. Plug \( x = 1 \) into (12.3) and obtain \( \eta(f(1)) = f_v(\eta(1)) \). Now lemma follows from \( f(1) = 0 \) and the fact that \( f_v \) equals 0 only at 0 and 1. \( \square \)

Corollary 12.1. If \( \eta(1) = 1 \), then \( \eta \) is a homeomorphism.

Lemma 56. If \( \eta(0) = \frac{1}{2-v} \), then either \( \eta(1) = \frac{1}{2-v} \), or \( \eta(1) = \frac{v}{2-v} \).

Proof. Lemma follows from the equality \( f_v(\eta(1)) = \frac{1}{2-v} \), which is obtained from (12.3) by plugging \( x = 1 \). \( \square \)

Corollary 12.2. If \( \eta \) is monotone and \( \eta(0) = \frac{1}{2-v} \), then \( \eta(1) = \frac{v}{2-v} \).

Lemma 57. If \( \eta(0) = \frac{1}{2-v} \) and \( \eta(1) = \frac{v}{2-v} \), then \( \eta \) is non monotone.

Proof. It follows from the functional equation (12.3) for \( x = 1/2 \) that conditions of Lemma imply \( f_v(\eta(1/2)) = \frac{v}{2-v} \).

Remind that graph of \( f_v \) consists of two branches of monotonicity and \( x = \frac{v}{2-v} \) is its fixed point. In other words, \( \eta(1/2) \) is a fixed point of \( f_v \). But \( \eta(1/2) = \frac{v}{2-v} \) contradicts to monotonicity of \( \eta \), whence \( \eta(1/2) = \frac{v^2}{2-v} \). But this also contradicts to monotonicity of \( \eta \), because implies that \( \eta(1/2) < \eta(0) \) and \( \eta(1/2) < \eta(1) \). The last proved Lemma. \( \square \)

Lemma 58. If \( \eta(0) = \frac{1}{2-v} \) and \( \eta(1) = 0 \), then \( \eta \) is non monotone.

Proof. It follows from (12.3) for \( x = 1/2 \) and from the condition of Lemma that

\[ f_v(\eta(1/2)) = 0, \]

whence \( \eta(1/2) \in \{0, 1\} \). This contradicts to monotonicity of \( \eta \). \( \square \)

The next proposition follows from the proved lemmas.

Proposition 12.1. Every continuous monotone solution \( \eta: [0, 1] \rightarrow [0, 1] \) of the functional equation (12.3) is the conjugacy of \( f \) and \( f_v \) forms (12.1) and (12.2) correspondingly.
12.2 Problem of the self-semiconjugation

Let $h : [0, 1] \rightarrow [0, 1]$ be the conjugacy of $f$ and $f_v$, i.e. the solution of the functional equation (12.3). By Theorem 6.5 this solution exists and is the unique. Also $h$ increase.

For an arbitrary (not necessary homeomorphic) solution $\eta : [0, 1] \rightarrow [0, 1]$ of the functional equation (12.3) consider a commutative diagram

![Diagram](image)

and denote $\xi = h^{-1}(\eta)$.

Let $\xi$ be an arbitrary (not necessary homeomorphic) solution of a functional equation

$$\psi(f) = f(\psi).$$

for an unknown function $\psi : [0, 1] \rightarrow [0, 1]$.

Notice, that (12.5) means that diagram

![Diagram](image)

is commutative.

From commutative diagram (12.4) obtain, that for any solution $\xi$ of functional equation (12.5), the function $\eta = h(\psi)$ is a solution of functional equation (12.3). Whence, the uniqueness of homeomorphic solutions of (12.3) yields the one to one correspondence between non-homeomorphic solutions of (12.3) and non-homeomorphic solutions of more simple equation (12.5).

Let $\xi : [0, 1] \rightarrow [0, 1]$ be a continuous solution of (12.5).

Remark 12.1. Consider some examples of solutions $\xi$ of the functional equation (12.5):
1. $\xi(x) = x$ for all $x \in [0, 1]$;
2. $\xi$ is constant, which is one of fixed points of $f$;
3. $\xi$ is some iteration of $f$.

**Proposition 12.2.** If a continuous solution $\xi$ of the functional equation (12.5) is constant on some $M = [\alpha, \beta]$, then it is constant on the whole $[0, 1]$.

**Proof.** Prove that if $\xi$ is constant on $M$, then it is piecewise constant on some interval $[\tilde{\alpha}, \tilde{\beta}]$, which is either of the length $2(\beta - \alpha)$, or $\tilde{\alpha} = 0$ is $[0, \gamma]$.

Consider the commutative diagram

$$
\begin{array}{ccc}
[\alpha, \beta] & \xrightarrow{f} & f([\alpha, \beta]) \\
\downarrow{\xi} & & \downarrow{\xi} \\
\xi([\alpha, \beta]) & \xrightarrow{f} & \xi(f([\alpha, \beta])) = f(\xi([\alpha, \beta]))
\end{array}
$$

and consider two cases, whether $[\alpha, \beta]$ contains $0.5$, or not. If $0.5 \in [\alpha, \beta]$, then change $M$ into either $M_1 = [\alpha, 0.5]$ or $M_2 = [0.5, \beta]$. For both $M_1$ and $M_2$ we have

$$
\xi(f(M_i)) = f(\xi(M_i)),
$$

whence $\xi$ is constant on $f(M_i)$.

If $M$ contains $\frac{1}{2}$, then $\xi$ is constant on $f(M)$, which is of the form $[\delta, 1]$ and, applying the same commutative diagram, obtain that $\xi$ is constant on $[0, \tilde{\beta}]$. Applying finitely many times the same reasonings for $[0, \tilde{\beta}]$ obtain that $\xi$ is constant of the whole $[0, 1]$.

If $1/2 \not\in M$, then $f(M)$ is an interval, whose length is two times more than one of $M$ and $\xi$ is constant on $f(M)$. Repeating these reasonings finitely many times, we obtain either an interval, which contains $1/2$. \qed

**Proposition 12.3.** If the graph of $\xi$ is a segment of a line on some $M = [\alpha, \beta]$ then $\xi$ is piecewise linear on $[0, 1]$.

**Proof.** Proof of this proposition is analogical to the proof of Proposition 12.2. We just have to change the word “constant” to “is piecewise linear” through the whole proof. \qed
12.3 Piecewise linearity of continuous self-semiconjugation

Since interval \([0, 1]\) is compact, then continuity of \(\xi\) implies the uniformly continuity. For every \(n \in \mathbb{N}\) it follows from the uniformly continuity of \(\xi\) that there exists \(m\) such that if the first \(m\) binary digits of \(a, b \in [0, 1]\) coincide, then

\[|\xi(a) - \xi(b)| < 2^{-n}.\]  \hspace{1cm} (12.6)

Denote this \(m\) by \(m_\xi(n)\).

**Lemma 59.** For every \(n \in \mathbb{N}\) if numbers \(a, b \in [0, 1]\) have the same \(m_\xi(n) + 1\) the first binary digits, then \(|\xi(a) - \xi(b)| < 2^{-(n+1)}\).

**Proof.** Let \(m = m_\xi(n)\) and the binary decompositions of \(a\) and \(b\) are

\[a = 0, c_1 c_2 \ldots c_{m+1} x_1 x_2 \ldots\]

and

\[b = 0, c_1 c_2 \ldots c_{m+1} y_1 y_2 \ldots\]

Since the first \(m\) digits of \(a\) and \(b\) coincide, then the inequality \(|\xi(a) - \xi(b)| < 2^{-n}\) holds.

Without loss of generality assume that \(\xi(a) \geq \xi(b)\). Assume that \(\xi(a) - \xi(b) \geq 2^{-(n+1)}\).

This assumption leads to the following two cases of the form of \(\xi(a)\) and \(\xi(b)\).

Case 1:

\[\xi(a) = 0, M^{\frac{1}{n}} 0 A,\]
\[\xi(b) = 0, M^{\frac{0}{n}} 1 B,\]

where \(M, A\) and \(B\) are blocks of digits, the length of \(M\) is \(n - 1\) and \(0, A \geq 0, B\). Notice, that in these notations \(\xi(a) - \xi(b) = 2^{-(n+1)} + 2^{-(n+2)} \cdot (0.A - 0.B)\).

Case 2:

\[\xi(a) = 0, M^{\frac{1}{n}} 1 A,\]
\[\xi(b) = 0, M^{\frac{0}{n}} 0 B,\]

where \(M, A\) and \(B\) are blocks of digits, the quantity of numbers in the block \(M\) is \(n\) and \(0, A \geq 0, B\).
If the first \( m + 1 \) binary digits of \( a \) and \( b \) coincide, then the first \( m \) digits of \( f(a) \) and \( f(b) \).

From the equality

\[ \xi(f(x)) = f(\xi(x)), \]

obtain that (12.6) implies

\[ |f(\xi(a)) - f(\xi(b))| < 2^{-n}. \quad (12.9) \]

Denote by \( M' \) the block \( M \) without its the first digit. By the name of block with a line above (for instance \( \overline{A}, \overline{M}' \) etc.) denote the block, which is obtained from the former after the inversion of all its digits.

Consider the first case, i.e. \( \xi(a) \) and \( \xi(b) \) are of the form (12.7). If the first digit of \( M \) is zero, then

\[
\begin{align*}
  f(\xi(a)) &= 0, M'_{n-1} 1 0 A, \\
  f(\xi(b)) &= 0, M'_{n-1} 0 1 B.
\end{align*}
\]

This contradicts to (12.9), because it appears to be that

\[ f(\xi(a)) - f(\xi(b)) = 2^{-n} + 2^{-n-1} \cdot (0, A - 0, B). \quad (12.10) \]

If the first digit of \( M \) equals 1, then

\[
\begin{align*}
  f(\xi(a)) &= 0, \overline{M'}_{n-1} 0 1 \overline{A}, \\
  f(\xi(b)) &= 0, \overline{M'}_{n-1} 0 1 \overline{B}.
\end{align*}
\]

This means that

\[ f(\xi(b)) - f(\xi(a)) = 2^{-n} + 2^{-n-1} \cdot (0, \overline{B} - 0, \overline{A}) \quad (12.11) \]

and also contradicts (12.9).

Consider the second case, i.e. when \( \xi(a) \) and \( \xi(b) \) are of the form (12.8). Similarly to the first case consider whether the first digits of \( M \) is 0 or 1.
If the first digit of $M$ equals 0, then
\[ f(\xi(a)) = 0, \underbrace{M'}_{n-1} 1 A, \]
\[ f(\xi(b)) = 0, \underbrace{M'}_{n-1} 0 B, \]
whence the difference $f(\xi(a)) - f(\xi(b))$ is of the form (12.10) and we have a contradiction with (12.9).

If the first digit of $M$ is 1, then
\[ f(\xi(a)) = 0, \underbrace{M'}_{n} 0 A, \]
\[ f(\xi(b)) = 0, \underbrace{M'}_{n} 1 B, \]
and the difference $f(\xi(b)) - f(\xi(a))$ is of the form (12.11) and it is also contradicts to (12.9). \qed

**Corollary 12.3.** Let $t, n \in \mathbb{N}$ and $m = m_\xi(n)$. If the first $m + t$ binary digits of $a, b \in [0, 1]$ coincide, then $|\xi(a) - \xi(b)| < 2^{-(n+t)}$.

**Proof.** This corollary should be proved by induction of $t$ with the same reasonings an in Lemma 59 \qed

Corollary 12.3 admits the following geometrical interpretation.

Let the number $n$ be fixed. Notice, that the coincidence of the first $m$ digits of $a$ and $b$ means that these numbers are between two neighbor points of $A_{m+1}$. Fix arbitrary neighbor points of $A_m$, say $\alpha(t, m)$ and $\alpha(t + 1, m)$.

It follows from the construction of $m$ by $n$ that for
\[ x \in [\alpha(t, m), \alpha(t + 1, m)] \]
the graph of $h$ belongs to the rectangle of the height $2^{-n}$, whose sides are parallel to coordinate axes.

Lemma 59 means that if we divide this rectangle into four ones by the lines, which are parallel to sides and pass through the middle points of sides, then the graph of $\xi$ would belong to exactly two of the obtained rectangles.

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Corollary 12.3 means that if each of the smaller rectangles, which contains the graph of \( \xi \), will be divided in the same manner, then the the graph would be contained exactly in two new rectangles and the process can be continued to infinity.

**Definition 12.1.** Let \( n \in \mathbb{N}, m = m_\xi(n) \) and \( t, 0 \leq t \leq 2^{m-1} \) be a number. We shall call the following line segments the **lines of the net**.

1. Lines, which are parallel to \( x \)-axis for \( x \in [\alpha(t, m), \alpha(t + 1, m)] \) and bound the graph of \( \xi \), if the distance between them is \( 2^{-n} \);

2. Vertical line segments, which connect the ends of lines, which are described in the item 1 above.

3. Each of line segments, which is constructed in the geometrical interpretation of Lemma 59 and Corollary 12.3.

The points of intersection of the lines of the net will be called the **knots of the net**.

We shall call the rectangles, which are obtained in the item 3 above by new lines of the net and old lines of the net, the **rectangles of the net**. Notice, that new lines of the net are constructed only in the case when the old rectangle contains the graph of \( \xi \).

**Remark 12.2.** Notice, that lines of the net are defines in item 1. of Definition 12.1 not unambiguously. The only thing, which is unambiguous, is the distance between them.

All other is defined unambiguously.

The following lemma follows from Corollary 12.3 and the notions above.

**Lemma 60.** If after the recurrent division of the rectangle of the net, mentioned at item 3 of Definition 12.1, the graph does not contains in two new neighbor vertical rectangles, then it passes through the knot of the net, which is the intersection of new lines of the net.

Figure 48 contains the interval \([\alpha(t, m), \alpha(t + 1, m)]\) and the graph of \( \xi \).

We consider the upper and lower bound of the rectangle as the bounds, which some from the uniformly continuity of \( \xi \).

The rectangle is divided into 4 rectangles, which are called \( A, B, C, D \). The graph of \( \xi \) contains in the rectangles \( A \) and \( D \).
Rectangle $A$ is divided into 4 parts, which are named with numbers from 1 till 4. It is clear that the graph of $\xi$ is contained in rectangles 1 and 3. By Lemma 12.1 and Corollary 12.3 this means that this graph passes through the point of intersection of new lines of the net and the graph of $\xi$ has no points in the rectangles 2 and 4.

**Lemma 61.** If the map $\xi$ is not constant, then for any interval $J$ of the form (12.12) the graph of $\xi$ passes through the new knot of the net.

*Proof.* By Proposition 12.2 the maps $\xi$ is not constant on $J$. Then denote $a = \min_{x \in J} \xi(x)$ and $b = \max_{x \in J} \xi(x)$ and notice that $a \neq b$.

Since the height of each rectangle is divided by 2 on each step, then there will be a step, when this height would be less then $b - a$. By Lemma 60 is means that not later then after this number of steps the graph of $\xi$ would pass through a knot of the net. \hfill $\square$

**Corollary 12.4.** If the maps $\xi$ is not constant, then on any $J$ of the form (12.12) there is at least two different knots of the net, which belong to the graph of $\xi$.

*Proof.* Proof the this corollary is the same as one of Lemma 61. \hfill $\square$

**Lemma 62.** If the graph of $\xi$ is not constant on some interval $J$ of the form (12.12) and passes through different knots of the net, then $\xi$ is piecewise linear on the whole $[0, 1]$ with finite number of intervals of linearity.

*Proof.* Let $a, b$, $(a < b)$ be points of $J$, where the graph of $\xi$ passes through knots of the net.
By Corollary 12.3 graph of the map $\xi$ passes through the point $(c, \xi(c))$, which is the middle of the line segment with ends at $(a, \xi(a))$ and $(b, \xi(b))$.

We can apply Corollary 12.3 to line segments $(a, c)$ and $(c, b)$ and continue the reasonings infinitely, whence obtain that the graph of $\xi$ has a dense set of point on the line segment with ends at $(a, \xi(a))$ and $(b, \xi(b))$.

It follows from continuity of $\xi$ that its graph is a line segment for $x \in [a, b]$.

Now lemma follows from Proposition 12.3. □

Lemma 62 the following Theorem.

**Theorem 12.4.** Every continuous solution $\xi : [0, 1] \rightarrow [0, 1]$ of equation (12.4) is piecewise linear.

### 12.4 Piecewise linearity of monotone self-semiconjugation

Assume that $\xi : [0, 1] \rightarrow [0, 1]$ is a piecewise linear solution of (12.5).

In the same way as we have proved Proposition 12.2 we can prove the following proposition.

**Proposition 12.5.** If $\xi$ is monotone on some interval $M = [\alpha, \beta] \subset [0, 1]$, then $\xi$ is piecewise monotone on $[0, 1]$.

**Proof.** Assume, that $\xi$ is piecewise monotone on $[0, b]$ for some $b, b \leq 1/2$. Then equation (12.5) can be rewritten as

$$\xi(2x) = f(\xi(x)).$$

As $f$ is piecewise monotone, then $\xi$ follows to be piecewise monotone on $[0, 2b]$. Repeating these reasonings for several times if necessary, obtain the proof of the theorem.

Assume, that $\xi$ is piecewise monotone on $[a, 1]$, where $a \geq 1/2$. Then equation (12.5) can be rewritten as

$$\xi(2 - 2x) = f(\xi(x)).$$

Nevertheless, this equation determines $\xi$ to be piecewise monotone on $f([a, 1]) = [0, 2 - 2a]$ and the case is reduced to previous one.
Case of $\xi$ being piecewise monotone on $[a, b] \subseteq [0, 1/2]$ and on $[a, b] \subseteq [1/2, 1]$ should be considered similarly. \hfill \Box

**Lemma 63.** The inclusion $\xi(0) \in \{0, 2/3\}$ holds.

Notice, that $2/3$ is positive fixed point of $f$.

**Proof.** It follows from the functional equation (12.5), that the following diagram is commutative

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & 0 \\
\downarrow{\xi} & & \downarrow{\xi} \\
\xi(0) & \xrightarrow{f} & \xi(0).
\end{array}
\]

It means, that $\xi(0)$ is a fixed point of $f$, and proves the lemma. \hfill \Box

**Lemma 64.** Either $\xi(x) = 2/3$ for all $x \in [0, 1]$, or $\xi(0) = 0$.

**Proof.** Prove this lemma by contradiction. By Lemma 63 assume, that $\xi(0) = 2/3$.

Consider the interval call $[0, a]$ such that $\xi$ is monotone on it. If $\xi$ increase on $[0, a]$, then for any $x \in (0, \min\{a/2, 1/2\})$ consider the commutative diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & 2x \\
\downarrow{\xi} & & \downarrow{\xi} \\
\xi(x) & \xrightarrow{f} & \xi(2x).
\end{array}
\]

As $\xi(x) > 2/3$, then $\xi(2x) = 2 - 2\xi(x)$. Next, as $\xi(x) > 2/3$, then $2 - 2\xi(x) < 2/3$, whence $\xi(2x) < 2/3$. It contradicts to that $\xi(x) > 2/3$ for all $x \in (0, a)$.

Consider now the case, when $\xi$ decrease on $[0, a]$.

Assume, that there exists $x_0 \in (0, a]$ such that $\xi(x_0) \geq 1/2$. Then for any $x \in (0, \min\{x_0/2, 1/2\})$ consider the commutative diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & 2x \\
\downarrow{\xi} & & \downarrow{\xi} \\
\xi(x) & \xrightarrow{f} & \xi(2x).
\end{array}
\]

As $\xi(x) < 2/3$, then $\xi(2x) = 2 - 2\xi(x)$. Next, as $\xi(x) < 2/3$, then $2 - 2\xi(x) > 2/3$, whence $\xi(2x) > 2/3$. It contradicts to that $\xi(x) < 2/3$ for all $x \in (0, a)$. 

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Assume at last, that $\xi(x) < 1/2$ for all $x \in (0, a]$, $x(0) = 2/3$ and $\xi$ decrease on $[0, a]$. For any $x_0 \in (0, \min\{a/2, 1/2\})$ consider the commutative diagram

$$
\begin{array}{ccc}
x_0/2 & \xrightarrow{f} & x_0 \\
\uparrow & & \uparrow \\
\xi(x_0/2) & \xrightarrow{f} & \xi(x_0).
\end{array}
$$

As $\xi(x_0/2) < 1/2$, then $\xi(x_0) = 2\xi(x_0/2)$, whence $\xi(x_0/2) > \xi(x_0)$, which contradicts to that $\xi$ decrease on $[0, a]$. \qed

**Lemma 65.** For any $n \geq 2$ the inclusion $\xi(A_n) \subseteq A_n$ holds.

**Proof.** Consider the continuation to the right of the commutative diagram, which is equivalent to functional equation (12.5), and obtain

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{f^n} & [0, 1] \\
\uparrow & & \uparrow \\
\xi & \xrightarrow{f^n} & \xi
\end{array}
$$

The corollary of this diagram is

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{f^n} & [0, 1] \\
\uparrow & & \uparrow \\
\xi & \xrightarrow{f^n} & \xi
\end{array}
$$

Using Lemma 64 it follows from the last diagram, that for any $x \in A_n$ the following diagram commutes

$$
\begin{array}{ccc}
x & \xrightarrow{f^n} & 0 \\
\uparrow & & \uparrow \\
\xi & \xrightarrow{f^n} & \xi
\end{array}
$$

and this proves the lemma. \qed

**Lemma 66.** Let $\xi$ be monotone on some $A_{n,k}$ and for any $m \geq n$ the equality

$$
\xi(\alpha_{m,p}) + \xi(\alpha_{m,p+1}) = 2\xi(\alpha_{m+1,2p+1}) \quad (12.13)
$$

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follows from the inclusion $A_{m,p} \subset A_{n,k}$. Then $\xi$ is linear on $A_{n,k}$.

**Proof.** From the induction on $m$ and Proposition 6.3 it follows, that $\xi$ is linear on $A \cap A_{n,k}$. More precisely, for any $t \in \mathbb{N}$ denote by $\xi_t$ the linear approximation of $\xi$ such that $\xi_t(x) = \xi(x)$ for every $x \in A_t$ and for every $s$, $0 \leq s \leq 2^{t-1} - 1$ the maps $\xi_t$ are linear on $A_{t,s}$. Then equation (12.13) mean that tangents of $\xi_{m+1}$ and $\xi_m$ coincide on $A_{m,p}$, i.e. $\xi_{m+1}$ is linear in $A_{m,p}$ for all $p$ such that $A_{m,p} \subset A_{n,k}$. Application of induction on $m$ leads to that $\xi_{m+1}$ is linear on $A_{n,k}$.

Now the lemma follows from density of $A \cap A_{n,k}$ in $A_{n,k}$. \hfill \Box

**Lemma 67.** There is $A_{n,k}$ such that $\xi$ is linear on $A_{n,k}$.

**Proof.** If $\xi$ is monotone on some interval, then this interval contains a set $A_{n,k}$ for some $n$ and $k$. By Lemma 65 there exists $s$, $t$ such that $\xi(\alpha_{n,k}) = \alpha_{n,t}$ and $\xi(\alpha_{n,k+1}) = \alpha_{n,s}$. Denote $d = |\alpha_{n,s} - \alpha_{n,t}| \cdot 2^{n-1}$ and prove the lemma by induction on $d$. Notice, that $d$ equals to the quantity of intervals of the form $A_{n,p}$ such that $\xi(A_{n,k})$ is union of these intervals, i.e.

$$A_{n,k} = \bigcup_{i=1}^{d} A_{n,k_i},$$

where $k_1, \ldots, k_d$ are sequent integers. In other words, $d+1$ is the number of points in $A_{n,k} \cap A_n$.

If $d = 1$ then the lemma follows from Proposition 6.3 and Lemmas 65, 66.

For every $t \geq 1$ consider the sets $A_{n+t,2^t k}, \ldots, A_{n+t,2^t k+2^t-1}$. By Proposition 6.3, the equality

$$A_{n,k} = \bigcup_{s=0}^{2^t-1} A_{n+t,2^t k+s}$$

holds. For any $t$ and $s$ calculate

$$d_{t,s} = |\xi(\alpha_{n+t,2^t k+s}) - \xi(\alpha_{n+t,2^t k+s+1})| \cdot 2^{n+t-1}.$$

Evidently, it follows from the monotonicity of $\xi$ on $A_{n,k}$, that

$$2^t \cdot d = \sum_{s=0}^{2^t-1} d_{t,s}.$$

If for all $t$, $s$ the equality $d = d_{t,s}$ holds, then the Lemma follows from Lemma 66. Otherwise, there is $t$, $s$, such that $d_{t,s} < d$ and we can apply the induction to $A_{n+t,2^t k+s}$. \hfill \Box
The following theorem follows from Proposition 12.5 and Lemma 67.

**Theorem 12.6.** If the function \( \xi : [0, 1] \to [0, 1], \) which is a solution of a functional equation (12.5), is monotone on an interval \( M \subset [0, 1], \) then \( \xi \) is piecewise linear on \([0, 1]\).

### 12.5 Piecewise linear self-semiconjugation

Assume that \( \xi : [0, 1] \to [0, 1] \) is a piecewise linear solution of equation (12.5). The main result of this subsection of the following theorem. Because of Lemma 64 assume, that \( \xi(0) = 0. \) Denote by \( k \) the tangent of \( \xi \) at 0.

**Theorem 12.7.** Let \( \xi \) be piecewise linear solution of the functional equation (12.5). Then \( \xi \) is one of the following functions

1. \( \xi(x) = x_0 \) for all \( x \in [0, 1], \) where \( x_0 = 0, \) or \( x_0 = 2/3; \)
2. for some \( k \in \mathbb{N} \)
   \[
   \xi(x) = \frac{1 - (-1)^{[kx]}}{2} + (-1)^{[kx]}\{kx\},
   \]
   where \( \{\cdot\} \) denotes fractional part and \( [\cdot] \) denotes integer part. More then this for any \( k \in \mathbb{N} \) the function \( \xi(x) \) of the form above is a solution of (12.5).

The fact, that all the maps, mentioned in Theorem 12.7 satisfy the functional equation (12.5) is evident.

**Lemma 68.** Let \( \xi(1) \in \{0, 1\}. \)

**Proof.** It follows from Lemma 64 and functional equation (12.5), that the following diagram is commutative

\[
\begin{array}{ccc}
1 & \xrightarrow{f} & 0 \\
\downarrow \xi & & \downarrow \xi \\
\xi(1) & \xrightarrow{f} & 0.
\end{array}
\]

This proves lemma. \( \square \)

**Lemma 69.** \( k \geq 1 \) and \( \xi \) is linear on the interval \([0, \frac{1}{k}]. \)
Proof. Evidently, there exists \( n \in \mathbb{N} \) such that \( \xi \) is linear on \( M = [0, \frac{1}{k^2n}] \). Whence, for every \( x \in M \) functional equation (12.5) is equivalent to commutativity of the following diagram

\[
\begin{array}{c}
x \\
| \quad f^n \quad 2^n \cdot x \quad | \quad 2^n \cdot kx \\
| \quad \xi \quad \downarrow \quad \xi \quad \downarrow \quad kx \\
| \quad f^n \quad 2^n \cdot kx \\
\end{array}
\]

Linearity of \( \xi \) on \([0, \frac{1}{k}]\) follows from the diagram. From this and Lemma 68 obtain the proof. \( \square \)

Lemma 70. Let for \( a < b \leq 1/2 \) the following holds: \( \xi(a) = 0, \xi(b) = 1 \) and \( \xi \) is linear with tangent \( k \) for \( x \in [a, b] \). Then \( \xi'(x) = k \) for all \( x \in (2a, a + b) \) and \( \xi'(x) = -k \) for all \( x \in (a + b, 2b) \).

Proof. For any \( x \in (a, \frac{a+b}{2}) \) functional equation (12.5) is equivalent to commutativity of the following diagram

\[
\begin{array}{c}
x \\
| \quad f \quad \quad \downarrow \quad \xi \\
| \quad 2x \\
| \quad k(x-a) \quad 2k(x-a) \\
\end{array}
\]

Whence, \( \xi'(x) = k \) for all \( x \in (2a, a + b) \).

For any \( x \in (\frac{a+b}{2}, b) \) functional equation (12.5) is equivalent to commutativity of the following diagram

\[
\begin{array}{c}
x \\
| \quad f \quad \quad \downarrow \quad \xi \\
| \quad 2x \\
| \quad k(x-a) \quad 2 - 2k(x-a) \\
\end{array}
\]

Whence, \( \xi'(x) = -k \) for all \( x \in (2a, a + b) \). \( \square \)

Lemma 71. \( k \in \mathbb{N} \).

Proof. If follows from Lemmas 69 and 70, that the graph of \( \xi \) consists of consequent line segments, which correspond to intervals of the form \( M = \left[ \frac{s}{k}, \frac{s+1}{k} \right] \) for integer \( s \), such that \( \xi(M) = [0, 1] \) and \( |\xi'(x)| = k \) for all \( x \in M \). Because of these reasonings, the lemma follows from Lemma 68. \( \square \)

Now Theorem 12.7 follows from Lemmas 68 – 71.
13 The length of the graph of the conjugacy

13.1 Finding of the length of the graph of the conjugation by dynamical reasonings

We find in this section the length of the graph of the conjugation \( h : [0, 1] \rightarrow [0, 1] \) which is a solutions of the functional equation

\[
h(f) = f_v(h), \tag{13.1}
\]

where \( f : [0, 1] \rightarrow [0, 1] \) is given by

\[
f(x) = \begin{cases} 
2x, & x < 1/2; \\
2 - 2x, & x \geq 1/2,
\end{cases} \tag{13.2}
\]

and \( f_v : [0, 1] \rightarrow [0, 1] \) is given by

\[
f_v(x) = \begin{cases} 
\frac{x}{v}, & 0 \leq x \leq v; \\
\frac{1-x}{1-v}, & v < x \leq 1,
\end{cases} \tag{13.3}
\]

for arbitrary \( v \in (0, 1) \).

The existence of the conjugation \( h \) was proved in Theorem 6.5. Remind, that we have introduced in Section 6.1 the set \( A_n, n \geq 1 \), which is a solution of the equation \( f^n(x) = 0 \). By Proposition 6.3 the equality

\[ A_n = \left\{ \frac{k}{2^{n-1}}, 0 \leq k \leq 2^n-1 \right\} \]

holds. Denote

\[ A_{n,k} = \left[ \frac{k}{2^{n-1}}, \frac{k+1}{2^n-1} \right] \]

Remind, that we have introduced in Section 6.1 the sets \( B_n \) which consists of solution of the equation \( f_v^n(x) = 0 \) and by Theorem 6.4 the equality

\[ h(A_n) = B_n \]

holds for all \( n \geq 1 \). As in Section 6.1 denote by \( h_n : [0, 1] \rightarrow [0, 1] \) the piecewise linear maps which coincide with \( h \) on \( A_n \) and all whose breaking points belong to \( A_n \). In other words, \( h_n \) in linear on \( A_{n,k} \) for all admissible \( k \).
Let \( n \in \mathbb{N} \) and \( k, 0 \leq k \leq 2^{n-1} - 1 \) be given. Let \( \alpha \) be a tangent of \( h_n \) in \( A_{n,k} \).

As in the proof of Theorem 7.3, for any \( x \in [0, 1] \) denote its binary decomposition by

\[
x = 0, x_1x_2 \ldots x_k \ldots
\]

and the only case when there exists \( n_0 \) such that \( x_n = x_{n+1} = 1 \) for all \( n > n_0 \) is if \( x = 1 \).

For a number \( x \) of the form (13.4) denote \( x_0 = 0 \) and for every \( i \geq 2 \) denote

\[
\alpha_i(x) = \begin{cases} 
2v & \text{if } \bar{x}_{i-1}x_{i-2} \in \{00, 11\} \\
2(1-v) & \text{if } \bar{x}_{i-1}x_{i-2} \in \{01, 10\}.
\end{cases}
\]

It follows from Theorem 7.3 that tangents \( \tau_1 \) and \( \tau_2 \) on \( h_{n+1} \) on intervals \( A_{n+1,2k} \) and \( A_{n+1,2k+1} \) are determined only by \( \alpha \) and evenness of \( k \) and are independent on \( n \) and exact value of \( k \). Precisely, one of \( \tau_1 \) and \( \tau_2 \) equals \( 2v \cdot \tan \alpha \) and another equals \( 2(1-v) \cdot \tan \alpha \). The only thing which depends on evenness of \( k \) is that which of mentioned values is equal to \( 2v \cdot \tan \alpha \). Whence the length of approximation \( h_{n+1} \) on \( A_{n,k} \) is determined by \( v, n \) and the length of \( h_n \) on this interval.

Consider points \( F_1(\frac{k}{2^n-1}, \beta_{n,k}) \) and \( F_2(\frac{k+1}{2^n-1}, \beta_{n,k+1}) \) of the graph of \( h_n \). Denote

\[
t_{n,k} = \frac{F_1F_2}{2^{n-1} + (\beta_{n,k+1} - \beta_{n,k})}.
\]

Evidently, \( t < 1 \). Since \( 2^{n-1} = l \cos \alpha \) and \( \beta_{n,k+1} - \beta_{n,k} = l \sin \alpha \), then

\[
t_{n,k} = \frac{1}{\sin \alpha + \cos \alpha} = \frac{1}{\sqrt{2} \sin(\alpha + \pi/4)},
\]

whence \( t \in [\frac{\sqrt{2}}{2}, 1] \). Denote by \( \xi_{n,k}(t_{n,k}) \) the ratio of the length of the graph of \( h_{n+1} \) on \( A_{n+1,2k} \) over the sum \( 2^{n-1} + (\beta_{n,k+1} - \beta_{n,k}) \).

The increasing function \( \xi(t) \), which can be constructed in the following Lemma, can be considered as lower bound of \( \xi_{n,k} \).

**Lemma 72.** There exists a continuous function \( \xi_v : [\frac{\sqrt{2}}{2}, 1] \to [\frac{\sqrt{2}}{2}, 1] \), dependent on \( v \), but independent on \( n, k \), which satisfy the following properties:

1. \( \xi_v(t) > t \) for every \( t \in [\frac{\sqrt{2}}{2}, 1] \);
2. \( \xi_v(1) = 1 \);
3. For every \( n, k \) the inequality \( \xi_v(t_{n,k}) > \xi_{n,k}(t_{n,k}) \) holds.
Proof. Let for given \( n, k \) the points \( F_1, F_2 \) and the number \( t = t_{n,k} \) is described in the beginning of the chapter. For making the further reasonings more short, denote \( a = 2^{n-1}, b = \beta_{n,k+1} - \beta_{n,k} \) and \( l = F_1F_2 \). Then \( a = l \cos \alpha \) and \( b = l \sin \alpha \).

Denote \( H(\frac{k+1}{2^n}, \beta_{n+1,2k+1}) \) and \( C(\frac{2k+1}{2^n}, \beta_{n,k}) \). By Lemmas 19 and 20 the point \( H \) coincides either with \( H_1 \) or \( H_2 \) such that \( H_1C = (1 - v)b \) and \( H_2C = vb \) (see Fig. 49).

![Diagram](image)

Fig. 49: Finding of the length of \( h_{n+1} \) by given \( h_n \)

Since \( F_1H_2F_2H_1 \) is a parallelogram, then \( F_1H_2 + H_2F_2 = F_1H_1 + H_1F_2 \). In other words the length of the graph of \( h_{n+1} \) on \( \left( \frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}} \right) \) is independent on those which of equalities \( H = H_1 \) or \( H = H_2 \) holds.

The length of the graph \( h_{n+1} \) on \( \left( \frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}} \right) \) can be calculated as

\[
F_1H_2 + H_2F_2 = \sqrt{\frac{a^2}{4} + b^2v^2} + \sqrt{\frac{a^2}{4} + b^2(1 - v)^2} = \\
l \left( \sqrt{\frac{\cos^2 \alpha}{4} + v^2 \sin^2 \alpha} + \sqrt{\frac{\cos^2 \alpha}{4} + (1 - v)^2 \sin^2 \alpha} \right).
\]

Then

\[
\xi_n(t) = \frac{AC + CB}{a + b} = \frac{AC + CB}{l/t} = \\
t \cdot \left( \sqrt{\frac{\cos^2 \alpha}{4} + v^2 \sin^2 \alpha} + \sqrt{\frac{\cos^2 \alpha}{4} + (1 - v)^2 \sin^2 \alpha} \right)
\]

\[
t^2 = \frac{1}{1 + \sin 2\alpha} = 1
\]

\[
\sin 2\alpha = \frac{1}{t^2} - 1
\]

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\[ \cos 2\alpha = \pm \sqrt{1 - \left(\frac{1}{t^2} - 1\right)^2} = \pm \sqrt{\frac{2}{t^2} - \frac{1}{t^4}} = \pm \frac{\sqrt{2t^2 - 1}}{t^2}. \]

Consider two cases, when
\[ \cos 2\alpha_1 = \frac{\sqrt{2t^2 - 1}}{t^2} \]
and
\[ \cos 2\alpha_2 = -\frac{\sqrt{2t^2 - 1}}{t^2}. \]

Since
\[ \cos 2\alpha = 2 \cos^2 \alpha - 1, \]
then
\[ \cos^2 \alpha_1 = \frac{1 + \frac{\sqrt{2t^2 - 1}}{t^2}}{2} = \frac{t^2 + \sqrt{2t^2 - 1}}{2t^2}; \]
\[ \cos^2 \alpha_2 = \frac{1 - \frac{\sqrt{2t^2 - 1}}{t^2}}{2} = \frac{t^2 - \sqrt{2t^2 - 1}}{2t^2}. \]

Since
\[ \cos 2\alpha = 1 - 2 \sin^2 \alpha, \]
then
\[ \sin^2 \alpha_1 = \frac{1 - \frac{\sqrt{2t^2 - 1}}{t^2}}{2} = \frac{t^2 - \sqrt{2t^2 - 1}}{2t^2} \]
and
\[ \sin^2 \alpha_2 = \frac{t^2 + \sqrt{2t^2 - 1}}{2t^2}. \]

Now for the case \( \alpha_1 \) we have
\[ \xi_v^{(1)}(t) = t \cdot \left( \sqrt{\frac{t^2 + \sqrt{2t^2 - 1}}{8t^2} + \frac{v^2(t^2 - \sqrt{2t^2 - 1})}{2t^2}} + \right. \]
\[ + \sqrt{\frac{t^2 + \sqrt{2t^2 - 1}}{8t^2} + \frac{(1 - v)^2(t^2 - \sqrt{2t^2 - 1})}{2t^2}} \right) = \]
\[ = \left( \sqrt{\frac{(1 + 4v^2)t^2}{8} + \frac{(1 - 4v^2)\sqrt{2t^2 - 1}}{8}} + \right. \]
\[ + \sqrt{\frac{(1 + 4v^2)t^2}{8} + \frac{(1 - 4v^2)\sqrt{2t^2 - 1}}{8}} \right) = \]
\[ + \sqrt{\left(\frac{1 + 4(1-v)^2}{8} \right)^2 + \left(\frac{1 - 4(1-v)^2}{8}\right)\sqrt{2t^2 - 1}} \].

For the case \(\alpha_2\) we have

\[ \xi_{\nu}^{(2)}(t) = \left( \sqrt{\left(\frac{1 + 4v^2}{8}\right)^2} - \frac{1 - 4v^2}{8}\sqrt{2t^2 - 1} + 
+ \sqrt{\left(\frac{1 + 4(1-v)^2}{8}\right)^2} - \frac{1 - 4(1-v)^2}{8}\sqrt{2t^2 - 1} \right). \]

Consider the expression

\[ \varphi_{\nu}(t) = (1 + 4v^2)t^2 + (1 - 4v^2)\sqrt{2t^2 - 1}. \]

\[ \varphi'_{\nu}(t) = 2(1 + 4v^2)t \cdot \frac{2(1 - 4v^2)t}{\sqrt{2t^2 - 1}} = \frac{2t \cdot (1 + 4v^2)\sqrt{2t^2 - 1} + (1 - 4v^2)}{\sqrt{2t^2 - 1}} = \frac{2t \cdot (1 + 4v^2) \cdot \left(\sqrt{2t^2 - 1} + \frac{1 - 4v^2}{1 + 4v^2}\right)}{\sqrt{2t^2 - 1}} > 0, \]

As \(\sqrt{2t^2 - 1} > 1\) and \(\frac{1 - 4v^2}{1 + 4v^2} < 1\), then \(\varphi'_{\nu}(t) > 0\).

Since \(\xi_{\nu}^{(1)}(t) = \sqrt{\varphi_{\nu}(t)} + \sqrt{\psi_{1-\nu}(t)}\), then \(\xi_{\nu}^{(1)}(t)\) increase.

Consider the expression

\[ \psi_{\nu}(t) = (1 + 4v^2)t^2 - (1 - 4v^2)\sqrt{2t^2 - 1}. \]

\[ \psi'_{\nu}(t) = 2t(1 + 4v^2) - \frac{2(1 - 4v^2)t}{\sqrt{2t^2 - 1}} = \frac{2t \cdot ((1 + 4v^2)\sqrt{2t^2 - 1} - (1 - 4v^2))}{\sqrt{2t^2 - 1}} = \frac{2t \cdot (1 + 4v^2) \cdot \left(\sqrt{2t^2 - 1} - \frac{1 - 4v^2}{1 + 4v^2}\right)}{\sqrt{2t^2 - 1}} > 0, \]

because \(\sqrt{2t^2 - 1} > 1\) and \(\frac{1 - 4v^2}{1 + 4v^2} < 1\).

Since \(\xi_{\nu}^{(2)}(t) = \sqrt{\psi_{\nu}(t)} + \sqrt{\psi_{1-\nu}(t)}\), then \(\xi_{\nu}^{(2)}(t)\) increase.

Now let \(\xi_{\nu}(t) = \min\{\xi_{\nu}^{(1)}(t), \xi_{\nu}^{(2)}(t)\}\) and this finishes the proof. \(\square\)
Evidently, that $\xi_v(t) = \xi_{1-v}(t)$ for all $v$ and $t$.

Graphs of $y = \xi_v(t) - t$ for $v = 0.1$, $v = 0.2$, $v = 0.3$ and $v = 0.4$ are given on Figure 50. Notice, that for $\{v_1, v_2\} < 0.5$ and all $t \in t \in \left[\frac{\sqrt{2}}{2}, 1\right)$ the inequality $\xi_{v_1}(t) > \xi_{v_2}(t)$ holds for $v_1 < v_2$. More then this $\xi_{0.5}(t) - t = 0$ for all $t$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig50}
\caption{Graphs of $y = \xi_v(t) - t$ for $v = 0.1$, $v = 0.2$, $v = 0.3$ and $v = 0.4$}
\end{figure}

**Theorem 13.1.** The length of the graph of $h_n$ tends to 2 for $n \to \infty$.

*Proof.* As the sequence $l_n$ of lengths of the graphs of $h_n$ increase and is bounded by 2, then limit $L = \lim_{n \to \infty} l_n$ exists and $L \leq 2$.

Assume, that $L < 2$. For any $n \geq 2$ consider the graph of $h_n$, which is linear on each of intervals $A_{n,k}$ for $0 \leq k \leq 2^{n-1} - 1$.

Consider all the numbers $t_{n,k}$ for $0 \leq k \leq 2^{n-1} - 1$, defined by (13.6). If for some $k$ the inequality $t_{n,k} > \frac{k}{2}$ holds, then for every $n' > n$ the length of $h_{n'}$ on $A_{n,k}$ is more than $\frac{k}{2}$ times more, than $2^{n-1} + \beta_{n,k+1} - \beta_{n,k}$.

Denote by $C_n$ the union of all $A_{n,k}$ such that $t_{n,k} > \frac{k}{2}$ and denote by $D_n = \bigcup_{n=2}^{n} C_n$ and $D = \bigcup_{n=2}^{\infty} C_n$. 

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We will show, that \( \lambda(D) \), where \( \lambda \) is the Lebesgue measure.

For any \( i \geq 2 \) consider the function \( \gamma_i(x) \) such that \( \gamma_i(x) = 1 \) if \( \alpha_i(1) = 2v \) and \( \gamma_i(x) = 0 \) if \( \alpha_i(1) = 2(1 - v) \), where \( \alpha_i \) is defined by (13.5). Consider the function

\[
\psi_n(x) = \sum_{i=2}^{n} \gamma_i(x) \frac{2i}{2^n} + 2^n \cdot \left\lfloor \frac{x}{2^n} \right\rfloor.
\]

Consider the function \( \psi(x) = \lim_{n \to \infty} \psi_n(x) \). As for any \( n \) the function \( \psi_n \) permutes intervals of \([0, 1]\), then for any Lebesgue measurable set \( M \subseteq [0, 1] \) the equality \( \lambda(\psi_n(M)) = \lambda(M) \), whence \( \lambda(\psi(M)) = \lambda(M) \). Notice, that this fact, together with that \( \psi \) and all \( \psi_n \) are bijections means that \( \psi \) and \( \psi_n \) preserve the Lebesgue measure.

There is a correspondence between a point \( x \in [0, 1] \) (or \( \psi(x) \in [0, 1] \)) and the following paths in the quadrant \( K_1 = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+: x \geq 0, y \geq 0\} \). We start from \((0, 0)\) and do 1 right, if the first binary digit of \( \psi(x) \) is 1 and go 1 up, if this digit is 0. Then consider the next digit of \( \psi(x) \) and repeat infinitely the process.

This leads us to considering \( x \) as a Markov process (see [39] and [40]), with 2 states \( p_0, p_1 \) and transition matrix

\[
\begin{pmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{pmatrix}.
\]

The state \( p_1 \) means moving right and \( p_2 \) means moving up.

For any \( n \geq 2 \) the chain of the first \( n - 1 \) states corresponds to a set \( A_{n,k} \) for some \( k \), dependent on \( n \). Let \( s \) be sum of first \( n - 1 \) digits of \( \psi(x) \). Then \( h'(x) = (2v)^s \cdot (2 - 2v)^{n-s-1} \) for \( x \in A_{n,k} \). This point corresponds to the point \( P_n(x) \in K_1 \) with coordinates \((s, n - s - 1)\), which belong to the line \( y + x = n - 1 \).

Describe the conditions on \( s \) such that \( A_{n,k} \not\in C_n \). Denote the angle between the graph of \( h_n \) on \( A_{n,k} \) and the \( x \)-axis by \( \alpha \). Then the condition

\[
t_{n,k} = \frac{1}{\sin \alpha + \cos \alpha} < \frac{L}{2}
\]

is equivalent to \( A_{n,k} \not\in C_n \). Transformations of the last inequality give

\[
\sin \left( \alpha + \frac{\pi}{4} \right) > \frac{\sqrt{2}}{L}.
\]
\[
\frac{\pi}{2} > \frac{3\pi}{4} - \arcsin \frac{\sqrt{2}}{L} > \alpha > \arcsin \frac{\sqrt{2}}{L} - \frac{\pi}{4} > 0
\]

\[
\sin \left( \frac{3\pi}{4} - \arcsin \frac{\sqrt{2}}{L} \right) = \frac{\sqrt{2}}{2} \cos \left( \arcsin \frac{\sqrt{2}}{L} \right) + \frac{\sqrt{2}}{2} \sin \left( \arcsin \frac{\sqrt{2}}{L} \right) =
\]

\[
= \frac{\sqrt{2}}{2} \left( \sqrt{1 - \frac{2}{L^2}} + \frac{\sqrt{2}}{L} \right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{L^2 - 2} + \sqrt{2}}{L}
\]

\[
\cos \left( \frac{3\pi}{4} - \arcsin \frac{\sqrt{2}}{L} \right) = -\frac{\sqrt{2}}{2} \cos \left( \arcsin \frac{\sqrt{2}}{L} \right) + \frac{\sqrt{2}}{2} \sin \left( \arcsin \frac{\sqrt{2}}{L} \right) =
\]

\[
= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 - \sqrt{L^2 - 2}}}{L}
\]

\[
\frac{\sqrt{L^2 - 2} + \sqrt{2}}{\sqrt{2} - \sqrt{L^2 - 2}} > \tan \alpha > \frac{\sqrt{2 - \sqrt{L^2 - 2}}}{\sqrt{L^2 - 2} + \sqrt{2}}
\]

Denote by

\[
T^+ = \frac{\sqrt{L^2 - 2} + \sqrt{2}}{\sqrt{2} - \sqrt{L^2 - 2}}
\]

and

\[
T^- = \frac{\sqrt{2} - \sqrt{L^2 - 2}}{\sqrt{L^2 - 2} + \sqrt{2}}.
\]

Whence,

\[
T^+ > 2^n \cdot v^s \cdot (1 - v)^{n-s-1} > T^-
\]

\[
\ln(T^+) > n \ln 2 + s \ln v + (n - s - 1) \ln(1 - v) > \ln(T^-)
\]

\[
\ln(T^+) > s(\ln v - \ln(1 - v)) + n \ln 2 + (n - 1) \ln(1 - v) > \ln(T^-)
\]

If \(v > 1/2\), then \(\ln v - \ln(1 - v) > 0\), whence

\[
\frac{\ln(T^+) - n \ln 2 - (n - 1) \ln(1 - v)}{\ln v - \ln(1 - v)} > s > \frac{\ln(T^-) - n \ln 2 - (n - 1) \ln(1 - v)}{\ln v - \ln(1 - v)}.
\]

If \(v < 1/2\), then \(\ln v - \ln(1 - v) < 0\), whence

\[
\frac{\ln(T^+) - n \ln 2 - (n - 1) \ln(1 - v)}{\ln v - \ln(1 - v)} < s < \frac{\ln(T^-) - n \ln 2 - (n - 1) \ln(1 - v)}{\ln v - \ln(1 - v)}.
\]
Nevertheless, independently on the sign of the expression $v - 1/2$, the condition $A_{n,k} \notin \mathbb{C}_n$ is equivalent to that $s$ belongs to some interval of the length not more then

$$T = \left[ \frac{\ln(T^+) - \ln(T^-)}{\| \ln v - \ln(1 - v) \|} \right],$$

where $||x||$ denotes the absolute value of $x$ and $[x]$ denotes integer part of $x$.

Resume, that we know, that $[0, 1] \setminus \mathbb{C}_n$ consists of not more then $T$ intervals. Consider all the passes, which end at $y = n - 1 - x$, i.e. which have the length $n - 1$. There are $2^{n-1}$ such pathes. For any $s$, $0 \leq s \leq n - 1$ there are $C_{n-1}^s$ pathes, which end at $(s, n - s - 1)$, where $C_{n-1}^s$ is a binomial coefficient. If $s_1, \ldots, s_T$ are numbers of these sets, then

$$\lambda(C_n) = \frac{\sum_{i=1}^{T} C_{n-1}^{s_i}}{2^{n-1}}.$$

As $C_{n-1}^{[(n-1)/2]}$ is the biggest binomial coefficient of the form $C_{n-1}^s$, then we have the restriction

$$\lambda(C_n) < \frac{T \cdot C_{n-1}^{[n-1/2]}}{2^{n-1}}.$$

Assume, that $n = 2m + 1$ is odd. Then Stirling approximation gives

$$\frac{T \cdot C_{n-1}^{[n-1/2]}}{2^{n-1}} = \frac{T \cdot C_{2m}^m}{2^{2m}} \sim \frac{T \cdot \sqrt{2\pi \cdot 2m \cdot (\frac{2m}{e})^{2m}}}{2\pi m \cdot (\frac{m}{e})^{2m} \cdot 2^{2m}} =$$

$$= \frac{T \cdot C_{2m}^m}{2^{2m}} \sim \frac{T \cdot \sqrt{2\pi \cdot 2m \cdot 2^{2m}}}{2\pi m \cdot 2^{2m}} \sim \frac{1}{\sqrt{m}}.$$

The result for the case, when $n$ is even is the same, whence

$$\lim_{n \to \infty} \lambda(C_n) = 1,$$

whence

$$\lambda(D) = 1,$$

which contradicts to the fact, that lengths of the graphs $h_n$ are bounded by $L < 2$. \qed
13.2 Finding of the length of the graph of the conjugation by combinatorial reasonings

Now we will find the evident formula for the length of the graph of $h_n$, which approximates the conjugation $h_n$ of $f$ and $f_v$. The graph of $h_{n+1}$ divides $[0, 1]$ into $2^n$ equal parts, where it is linear. At each of these parts the derivative of $h_{n+1}$ equals $\prod_{i=2}^{n+1} \alpha_i$. Each of these $n$ multipliers can be either $2v$, or $2(1 - v)$.

Let for some interval of the length $2^{-n}$ the derivative is $t$. It means, that $\tan \alpha = t$, where $\alpha$ is an angle between the graph and $x$-axis. Then $\cos \alpha = \frac{1}{\sqrt{1 + t^2}}$ and the length of the graph on this interval is $\frac{1}{2^n \cos \alpha} = \frac{1}{2^n} \sqrt{1 + t^2}$.

As all the both values of $\alpha_i$ are equally expected (for the random uniformly distributed number), then the probability, that $\prod_{i=2}^{n+1} \alpha_i = (2v)^k(2 - 2v)^{n-k}$ equals $\frac{C_n^k}{2^n}$, where $C_n^k$ is the binomial coefficient.

These reasonings prove the following proposition.

**Proposition 13.2.** The following equality holds

$$l_{n+1}(v) = \frac{1}{2^n} \cdot \sum_{k=0}^{n} C_n^k \cdot \sqrt{1 + \frac{2^{2n}v^{2k}(1 - v)^{2(n-k)}}{2^n}}$$

(13.7)

for the length $l_{n+1}(v)$ of $h_{n+1}$.

The following combinatorial fact follows from Theorem 13.1.

**Theorem 13.3.** For every $v \in (0, 1) \backslash \{0.5\}$ the limit $\lim_{n \to \infty} l_n(v) = 2$ holds, where $l_n(v)$ are defined by (13.7).

Notice, that expression (13.7) has sense also for $v = 0, 0.5$ and 1. Obviously, $l_n(0) = l_n(1) = 1$ and $l_n(0.5) = \sqrt{2}$ for $l_n(v)$ be defined by (13.7). The case $l_n(0.5)$ corresponds to the trivial conjugation $y = x$ of the maps $f$ with itself.

Prof. Georgiy Shevchenko from Taras Shevchenko National University of Kyiv noticed us that Theorem 13.3 can be simply proven with the use of probability theory reasonings.

Let $l_{n+1}(v)$ be given by (13.7). The graphs of $y(n) = l_{n+1}(v)$ for

$$v \in \{0.52; 0.54; 0.56; 0.57; 0.58; 0.6; 0.62; 0.65; 0.7; 0.8\}$$

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are given at Figure 51. Here $l_{n+1}(v)$ is the length of the graph of $h_{n+1}$, dependent on $v$.

Clearly, it is not evident from the graph at Figure 51 that, for instance, $\lim_{n \to \infty} l_n(0.52) = 2$. Nevertheless, graphs of $y = l_{n+1}(0.51)$, $y = l_{n+1}(0.512)$, $y = l_{n+1}(0.515)$ and $y = l_{n+1}(0.52)$ are given at Figure 52 where $n$ is considered from 20,000 (for $y = l_{n+1}(0.51)$ with $l_{20,001}(0.52) = 1.99611$) to 50,000 (for $y = l_{n+1}(0.51)$ with $l_{50,001}(0.51) = 1.97903$).
14 Piecewise linear approximations of self semi conjugation

In this section we consider the topological self-semiconjugation of the maps \( f : [0, 1] \rightarrow [0, 1] \), given by

\[
    f(x) = \begin{cases} 
    2x, & 0 \leq x < 1/2; \\
    2 - 2x, & 1/2 \leq x \leq 1,
    \end{cases}
\]  

(14.1)

i.e. we consider the properties of continuous surjective \( h : [0, 1] \rightarrow [0, 1] \) of the functional equation

\[
    \xi(f) = f(\xi).
\]  

(14.2)

We have considered in the Section 6.1 the sets

\[
    A_n = \left\{ \frac{t}{2^{n-1}} : 0 \leq t \leq 2^n \right\}
\]

and according to Proposition 6.3, these \( A_n \) are solutions of the equation \( f^n(x) = 0 \).

For any \( n \geq 1 \) consider the map \( \xi_n : A_n \rightarrow A_n \) consider the possibility of its continuation to a self-semiconjugation of \( f \).
Notation 14.1. The maps $\xi_n : A_n \to [0, 1]$ is called admissible, if the equality

$$\xi(f(x)) = f(\xi(x))$$

(14.3)

for all $x \in A_n$.

Notice, that since $f(A_n) \subseteq A_n$ for all $n$, then (14.3) is defined for all $x \in A_n$.

Notation 14.2. The maps $\xi_n : A_n \to A_n$ is called continuable, if the there is a self-semiconjugation $h : [0, 1] \to [0, 1]$ of $f$, which coincides with $\xi_n$ on $A_n$.

In this case the maps $h$ can be considered as continuous surjective continuation of $\xi_n$.

14.1 Admissible self-semiconjugations

Let $\xi_n : A_n \to [0, 1]$ be an admissible maps. Denote

$$x_0 = \xi(0).$$

From the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & 0 \\
\downarrow{h} & & \downarrow{h} \\
x_0 & \xrightarrow{f} & x_0 \\
\end{array}
\]

obtain that $x_0$ is a fixed point of $f$, whence

$$x_0 \in \{0, 2/3\}. \quad (14.4)$$

Denote $\varphi_0(x) = 2x$ for $x \in [0, 1/2]$ and denote $\varphi_2(x) = 2 - 2x$ for $x \in [1/2, 1]$. This notation let us to rewrite (14.1) as

$$f(x) = \begin{cases} 
\varphi_0, & 0 \leq x < 1/2; \\
\varphi_1, & 1/2 \leq x \leq 1.
\end{cases} \quad (14.5)$$

Notice, that maps $\varphi_i, i = 0, 1$ are invertible.

Consider the commutative diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{f} & 0 \\
\downarrow{h(1)} & & \downarrow{h(1)} \\
h(1) & \xrightarrow{f} & x_0 \\
\end{array}
\]

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and conclude that
\[ h(1) = \varphi_{i_1}^{-1}(x_0) \text{ for some } i_1 \in \{0, 1\}. \] (14.6)

From equalities (14.4) and (14.6) we obtain the following lemma.

**Lemma 73.** For any \( x \in A_1 \) there exist \( i_1 \in \{0, 1\} \) such that \( \xi_n(x) = \varphi_{i_1}^{-1}(x_0) \).

Denote by \( i_0 \in \{0, 1\} \) such number that
\[ x_0 = \varphi_{i_0}(x_0). \] (14.7)

This lemma can be inductively generalized as follows.

**Lemma 74.** For any \( m \leq n \) and any \( x \in A_m \) there exist \( i_1, \ldots, i_m \in \{0, 1\} \) such that
\[ \xi_n(x) = \varphi_{i_m}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots). \]

**Proof.** The base of induction (the case \( m = 1 \)) follows from Lemma 73.

Assume that for \( m = k \) the lemma is proved. For any \( x \in A_{k+1} \) consider the commutative diagram
\[
\begin{array}{ccc}
x & \xrightarrow{f} & f(x) \\
\downarrow & & \downarrow \\
\xi(x) & \xrightarrow{f} & \xi(f(x))
\end{array}
\]

Since \( f(x) \in A_k \), then by the assumption of induction obtain \( \xi(f(x)) = \varphi_{i_k}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots) \), whence
\[ f(\xi(x)) = \varphi_{i_k}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots). \]

The last equality means that there exists \( i_{k+1} \) such that
\[ \xi(x) = \varphi_{i_{k+1}}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots). \]

\[ \square \]

**Remark 14.1.** From the definition of \( A_n \) obtain that \( x \in A_n \) if and only if there exist \( j_1, \ldots, j_n \) such that
\[ x = \varphi_{j_n}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots). \] (14.8)
Notation 14.3. For any $m \geq 1$ denote by $\mathcal{B}_m$ the set of sequences the the length $m$, consisted of 0-s and 1-s. Also denote $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$.

If follows from Lemma 74 that for any $m \geq 1$ and $j_1, \ldots, j_m$ there exist $i_1, \ldots, i_m$ such that

$$\xi(\varphi_{j_n}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots)) = \varphi_{i_m}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots).$$ (14.9)

This equality means that $\xi$ generates the maps $\tilde{\xi} : \mathcal{B} \rightarrow \mathcal{B}$ such that for any $m \geq 1$ the inclusion $\tilde{\xi}(\mathcal{B}_m) \subseteq \mathcal{B}_m$ holds and

$$\tilde{\xi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m)$$ (14.10)

for $1 \leq m \leq n$, and $i_1, \ldots, i_m, j_1, \ldots, j_m$ such that (14.9) holds.

Lemma 75. Let $\tilde{\xi}$ be defined by (14.10) for an admissible $\xi$. For $m \leq n$ and $t < m$ the equality

$$\tilde{\xi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m)$$

yields

$$\tilde{\xi}(j_1, \ldots, j_{m-t}) = (i_1, \ldots, i_{m-t}).$$

Proof. Prove the lemma for $t = 1$. Consider the commutative diagram

$$\begin{array}{ccc}
\varphi_{j_m}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots) & \xrightarrow{f} & \varphi_{j_{m-1}}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots) \\
\downarrow{\xi} & & \downarrow{\xi} \\
\varphi_{i_m}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots) & \xrightarrow{f} & \varphi_{i_{m-1}}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots).
\end{array}$$

For $t > 1$ the reasonings are the same. \qed

Remark 14.2. Notice, that (14.8) is not one to one correspondence between $A_n$ and finite sequences $j_n, \ldots, j_n$, whence (14.10) does not define the one to one correspondence between admissible $\xi$ and the the functions, whose domain and gange is the finite sequence of 0-s and 1-s. Indeed, if $j_1, \ldots, j_k = 0$ and $j_{k+1} \neq 0$, then

$$\varphi_{j_n}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots) = \varphi_{j_n}^{-1}(\ldots(\varphi_{j_2}^{-1}(0))\ldots) = \ldots = \varphi_{j_n}^{-1}(\ldots(\varphi_{j_k}^{-1}(0))\ldots),$$

and

$$\varphi_{j_1}^{-1}(\ldots(\varphi_{j_k}^{-1}(0))\ldots) \neq \varphi_{j_n}^{-1}(\ldots(\varphi_{j_{k+1}}^{-1}(0))\ldots).$$
These reasonings prove the following lemma.

**Lemma 76.** Let \( \tilde{\xi} \) be defined by an admissible \( \xi \). Let for some \( m \geq 1 \) and \( j_1, \ldots, j_m, i_1, \ldots, i_m \) the equality (14.10) holds. Let \( i_0 \) be defined by (14.7). Then for any \( k, 1 \leq k \leq m \) the equality \( j_k = 0 \) yields \( i_k = i_0 \).

**Proof.** Prove the lemma for \( k = 1 \). Consider \( x \in [0, 1] \) defined by (14.8). The case \( k = 1 \) means that \( j_1 = 0 \), whence \( \varphi^{-1}_{j_1}(0) = 0 \), which means that

\[
\varphi^{-1}_{j_m}(\ldots(\varphi^{-1}_{j_1}(0))\ldots) = \varphi^{-1}_{j_m}(\ldots(\varphi^{-1}_{j_2}(0))\ldots).
\]

The equality

\[
\xi(\varphi^{-1}_{j_m}(\ldots(\varphi^{-1}_{j_1}(0))\ldots)) = \xi(\varphi^{-1}_{j_m}(\ldots(\varphi^{-1}_{j_2}(0))\ldots))
\]

means that

\[
\varphi^{-1}_{i_m}(\ldots(\varphi^{-1}_{i_1}(x_0))\ldots) = \varphi^{-1}_{i_m}(\ldots(\varphi^{-1}_{i_2}(x_0))\ldots).
\]

Applying \( f^{m-1} \) to both sides of the obtained equality obtain

\[
f(\varphi^{-1}_{i_1}(x_0)) = x_0,
\]

which means that \( i_1 = i_0 \).

The case \( k > 1 \) follows from the case \( k = 1 \) and Lemma 75. \( \square \)

**Theorem 14.1.** There is one to one correspondence between admissible self-semi conjugations \( \xi : A_n \to [0, 1] \) and maps \( \tilde{\xi} : \bigcup_{i=1}^n B_i \to \bigcup_{i=1}^n B_i \) with the following properties:

1. For any \( m, 1 \leq m \leq n \), the inclusion \( \tilde{\xi}(B_m) \subseteq B_m \) holds.
2. For any \( m, 2 \leq m \leq n \) the equality

\[
\tilde{\xi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m)
\]

yields

\[
\tilde{\xi}(j_1, \ldots, j_{m-1}) = (i_1, \ldots, i_{m-1}).
\]

3. If the equality

\[
\tilde{\xi}(j_1, \ldots, j_n) = (i_1, \ldots, i_n)
\]
holds for some \( i_1, \ldots, i_n, j_1, \ldots, j_n \), then for any \( k, 1 \leq k \leq n \) the equality \( i_k = 0 \) yields \( j_k = i_0 \), where \( i_0 \in \{0, 1\} \) is a fixed number.

**Proof.** If follows from Lemmas 74, 75 and 76 that maps \( \tilde{\xi} \), which is defined by \( \xi \) via the equalities (14.9) and (14.10) satisfies the conditions (1), (2) and (3) of Theorem for \( i_0 \) such that \( \varphi_{i_0}(\xi(0)) = \xi(0) \).

Let \( \tilde{\xi} \) satisfy conditions (1), (2) and (3) of Theorem. Define an admissible \( \xi : A_n \to [0, 1] \) as follows.

If \( i_0 = 0 \) then let \( x_0 = 0 \), otherwise let \( x_0 = 1 \).

For any \( x \in A_n \) define \( \xi(x) \) as follows. By the Remark (14.1) there exist \( j_1, \ldots, j_n \) such that

\[
x = \varphi^{-1}_{j_n}(\ldots(\varphi^{-1}_{j_1}(x_0))\ldots).
\]

Let \( i_1, \ldots, i_n \) be such that

\[
\tilde{\xi}(j_1, \ldots, j_n) = (i_1, \ldots, i_n).
\]

(14.11)

Now define \( \xi(x) \) as

\[
\xi(x) = \varphi^{-1}_{i_n}(\ldots(\varphi^{-1}_{i_1}(x_0))\ldots).
\]

Prove that the obtained \( \xi \) would be admissible.

Evidently \( f(\xi(x)) = \varphi^{-1}_{i_{n-1}}(\ldots(\varphi^{-1}_{i_1}(x_0))\ldots) \). From the other hand,

\[
f(x) = \varphi^{-1}_{j_{n-1}}(\ldots(\varphi^{-1}_{j_1}(0))\ldots).
\]

(14.12)

Consider

\[
x^* = \varphi^{-1}_{j_{n-1}}(\ldots(\varphi^{-1}_{j_1}(0))\ldots) \in A_{n-1}
\]

.

From (14.11) and from condition (2) of Theorem obtain that

\[
\tilde{\xi}(j_1, \ldots, j_{n-1}) = (i_1, \ldots, i_{n-1}),
\]

whence

\[
\xi(x^*) = \varphi^{-1}_{j_{n-1}}(\ldots(\varphi^{-1}_{j_1}(0))\ldots).
\]

(14.13)

Now Theorem follows from the comparing of equations (14.12) and (14.13).  \( \square \)
Corollary 14.1. For any $n \geq 1$ the number of admissible self-semi conjugations $\xi : A_n \to [0, 1]$ is
\[
\sum_{k=0}^{n} 2^{k+1} \cdot C_n^k
\].

**Proof.** By Theorem [14.1] we can calculate the maps $\tilde{\xi}$ from the Theorem instead of admissible maps.

By condition (2) of Theorem [14.1] it is enough to define $\tilde{\xi}$ only on $B_n$.

Let $J = (j_1, \ldots, j_n) \in B_n$ be a typical element of $B_n$. Let $k$ be the quantity of 1-s in $J$. There are $C_n^k$ possibilities to choose positions $j_{s_1}, \ldots, j_{s_k}$ of these 1-s and there are $2^k$ possibilities to choose correspond $i_{s_1}, \ldots, i_{s_k}$. All another elements of $J$ will be $i_0$, and we have 2 possibilities for $i_0$.

Whence, there are
\[
\sum_{k=0}^{n} 2^{k+1} \cdot C_n^k
\]
possibilities to choose maps $\tilde{\xi}$, which satisfy the conditions of Theorem [14.1].

\[\square\]

14.2 Continuable self-semiconjugations

We will pay our attention to continuable self-semiconjugations $\xi : A_n \to [0, 1]$ in this section, where $n$ is considered to be arbitrary fixed.

**Lemma 77.** Let $\xi_1, \xi_2 : A_n \to [0, 1]$ be admissible self-semiconjugations. If $\xi_1(x) = \xi_2(x)$ for all $x \in A_n \setminus A_{n-1}$, then $\xi_1(x) = \xi_2(x)$ for all $x \in A_n$.

**Proof.** According to Proposition [6.3]

\[
A_n \setminus A_{n-1} = \left\{ \frac{2t + 1}{2^{n-1}}, \ t \in [0, 2^{n-2} - 1] \right\}.
\]

For $t \in [0, 2^{n-2} - 1]$ and $x = \frac{2t+1}{2^n}$ consider the following commutative diagram for $\xi_i, i = 1, 2$:

\[
\begin{array}{ccc}
x & \xrightarrow{f} & f(x) \\
\downarrow{\xi_i} & & \downarrow{\xi_i} \\
\xi(x) & \xrightarrow{f} & f(\xi_i(x)).
\end{array}
\]

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Since $\xi_1(x) = \xi_2(x)$ for all $x \in A_n \setminus A_{n-1}$, then it follows from the commutative diagram, that $\xi_1(x) = \xi_2(x)$ for all $x \in f(A_n \setminus A_{n-1})$. It follows from the definition of $A_n$, that $f(A_k) = A_{k-1}$ for all $k \geq 1$, whence $f(A_n \setminus A_{n-1}) = A_{n-1} \setminus A_{n-2}$.

Applying $n - 1$ times the reasonings above obtain that $\xi_1(x) = \xi_2(x)$ for all $x \in A_n$. □

If follows from Theorems 12.4 and 12.7 that if $\xi : A_n \to [0, 1]$ is continuable, then either $\xi(x) = 2/3$ for all $x \in A_n$, or $\xi(0) = 0$. More then this, the maps $h : [0, 1] \to [0, 1]$ from Theorem 12.7 are all possible continuations of admissible $\xi : A_n \to [0, 1]$.

Lemma 78. If $\xi : A_n \to [0, 1]$ is a continuous self-semiconjugation of $f$, then $\xi(A_n) \subseteq A_n$.

Proof. Since $\xi(0) = 0$ by Theorems 12.4 and 12.7 then for any $x \in A_n$ it follows from the definition of admissibility of $\xi$ that the following diagram

$$
\begin{array}{ccc}
x & \xrightarrow{f^n} & 0 \\
\downarrow{\xi} & & \downarrow{\xi} \\
\xi(x) & \xrightarrow{f^n} & 0
\end{array}
$$

is commutative. This proves the Lemma. □

Lemma 79. Let $\xi : A_n \to A_n$ be continuable self-semiconjugation of $f$ and $\xi(\alpha_{n,2s+1}) = \alpha_{n,p}$ for some $\alpha_{n,2s+1}$ and $\alpha_{n,p}$.

1. If $h$ is a continuation of $\xi$ and $k$ is its tangent at 0, then either

$$k(2s + 1) - p \equiv 0 \mod 2^n; \tag{14.14}$$

or

$$k(2s + 1) + p \equiv 0 \mod 2^n. \tag{14.15}$$

2. If $k$ satisfies either (14.14) or (14.15), then there is a continuation $h : [0, 1] \to [0, 1]$ of $\xi$, whose tangent at 0 is $k$.

Proof. There exist $q \in \mathbb{N} \cup \{0\}$, $q \leq 2^{n-1} - 1$ such that $\alpha_{n,2s+1} \in [\alpha_{n,q}, \alpha_{n,q+1}]$.

Consider the possibilities when $q$ is even and when it is odd.

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If \( q = 2t \) for some \( t \), then \( h \) increase on \([\alpha_{n,q}, \alpha_{n,q+1}]\), whence

\[
\frac{\alpha_{n,2s+1} - \frac{2t}{k}}{\frac{1}{k}} = \alpha_{n,p}.
\]  

(14.16)

We can simplify this equality as

\[
k(2s + 1) \frac{1}{2^{n-1}} - 2t = \frac{p}{2^{n-1}},
\]

\[
k(2s + 1) - p = 2t \cdot 2^{n-1}.
\]

Notice, that the last equation means that (14.16) is equivalent to (14.14).

Consider another case, i.e. \( q = 2t + 1 \) for some \( t \). Then \( h \) decrease on \([\alpha_{n,q}, \alpha_{n,q+1}]\), whence

\[
\frac{\alpha_{n,2s+1} - \frac{2t+1}{k}}{\frac{1}{k}} = 1 - \alpha_{n,p}.
\]  

(14.17)

We can simplify this equality as

\[
k(2s + 1) \frac{1}{2^{n-1}} - (2t + 1) = 1 - \frac{p}{2^{n-1}},
\]

\[
k(2s + 1) + p = 2(t + 1) \cdot 2^{n-1}.
\]

The last equation means that (14.17) is equivalent to (14.15). \(\square\)

**Lemma 80.** 1. The set of natural \( k \), which satisfy the congruence (14.14) is \( k = k_1 + 2^nt, \ t \in \mathbb{N} \) for some \( k_1, 1 \leq k_1 \leq 2^n \).

2. The set of natural \( k \), which satisfy the congruence (14.15) is \( k = k_2 + 2^nt, \ t \in \mathbb{N} \) \( k_2, 1 \leq k_2 \leq 2^n \).

3. \( k_1 + k_2 = 2^n \), or \( k_1 + k_2 = 2^{n+2} \) for \( k_1 \) and \( k_2 \) from the previous items of this corollary.

**Proof.** 1. Since \( 2s + 1 \) and \( 2^n \) are reciprocal simple then there exist integers \( \gamma_1 \) and \( \gamma_2 \) such that

\[
\gamma_1 \cdot (2s + 1) + \gamma_2 \cdot 2^n = 1,
\]

whence

\[
p \cdot \gamma_1 \cdot (2s + 1) \equiv p \mod 2^n.
\]

Since for every \( p, 1 \leq p \leq 2^n \) there exists \( k_1, 1 \leq k_1 \leq 2^n \), which satisfy (14.14), then for each \( p \) the correspond \( k \) is unique up to adding a number, which is divisible by \( 2^n \).
2. The proof of second item is the same as of the first one.

3. If $k_1$ and $k_2$, $1 \leq k_1, k_2 \leq 2^n$ are solutions of (14.14) and (14.15) correspondingly then there exist $t_1$ and $t_2$ such that

$$\begin{cases} 
  k_1(2s + 1) - p = 2^n \cdot t_1 \\
  k_2(2s + 1) + p = 2^n \cdot t_2.
\end{cases}$$

Adding these two equalities obtain that $k_1 + k_2$ is divisible by $2^n$.

The main result of this section is the following theorem.

**Theorem 14.2.**

1. For every $x \in A_n \setminus A_{n-1}$ and for every $y \in A_n$ there exists a continuable $\xi : A_n \to A_n$ such that $\xi(x) = y$.

2. Let $\xi_1, \xi_2 : A_n \to A_n$ be continuable self-semiconjugations of $f$ of the form (14.1) and $\xi_1(x) = \xi_2(x)$ for some $x \in A_n \setminus A_{n-1}$. Then $\xi_1(x) = \xi_2(x)$ for all $x \in A_n$.

**Proof.** The first part of Theorem follows from Lemma 79 and reasonings, which are similar to those from the proof of item 1 of Lemma 80.

The second part of Theorem follows from the following observations. For any $\alpha_{n,2s+1} \in A_n$ consider all possible $\alpha_{n,p}$ such that $\xi(\alpha_{n,2s+1}) = \alpha_{n,p}$. According to item 3. of Lemma 80, each $p$ (of all $2^{n-1}$ possible) divides the set of all possible $k$ (up to adding a number, which is divisible by $2^n$) to pairs $\{k_1, k_2, 1 \leq k_1, k_2 \leq 2^n\}$ such that $k_1 + k_2$ is divisible by $2^n$. This gives $2^{n-1}$ pairs. From another hand, by Lemma 79 each pair defines $k$ as a tangent of the continuation $h$ of $\xi$ uniquely up to adding a number, which is divisible by $2^n$.

Now if $\xi_1(x) = \xi_2(x)$ for continuable $\xi_1, \xi_2$, then there exists a pair $\{k_1, k_2\}$, mentioned above and this pair defines $\xi_1$ and $\xi_2$ in the unique way at all points of $A_n \setminus A_{n-1}$. Applying Lemma 77 finishes the proof.

**Corollary 14.2.** For every $n \geq 1$ there are $2^{n-1}$ continuable self-semi conjugations of $f$ of the form (14.1).

**Proof.** By item 1 of Theorem 14.2 for every $x \in A_n \setminus A_{n-1}$ and for every $y \in A_n$ there exist a continuable $\xi : A_n \to A_n$ such that $\xi(x) = y$. 

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If consider \( x \in A_n \setminus A_{n-1} \) to be fixed, then by item 2 of Theorem 14.2 every \( y \in A_n \) defines a continuable \( \xi \) in the unique way.
References

[1] *Daniel S. Alexander, Felice Iaverano, Alessandro Rosa*, Early Days in Complex Dynamics in One Variable During 1906-1942. AMS, London Math. Soc., Vol. 32, 2010, 454p.

[2] D.J. Allwright, Hypergraphic functions and bifurcations in recurrence relations. SIAM J. App. Math. 34, 687-691. (1978)

[3] C. Babbage, *An Essay towards the Calculus of Functions*, Philosophical Transactions of the Royal Society of London, Vol. 105 (1815), pp. 389-423.

[4] F.M. Bass, A New Product Growth Model for Consumer Durables, Management Science 15 No 5 Theory Series (1969), 215-227.

[5] Nils Berglund, Geometrical Theory of Dynamical Systems, Lecture Notes, 85 p. (2001).

[6] G.D. Birkhoff, Dynamical systems, Igevsk, RHD, 1999, 408 p. (in Russian)

[7] George D. Birkhoff, Sur quelques courbes fermées remarquables, Bulletin de la S.M.F., tomme 60, (1932), p. 1-26.

[8] Louis Block and Ethan M. Coven, topological conjugacy and transitivity for a class of piecewise monotone maps of the interval, Transactions of the American Mathematical Society, 300, No 1 (1987), 297-306.

[9] George Bool, “A Treatise on the Calculus of finite differences”, London, 1880 (the Third edition). [The first edition was at 1860, second at 1872].

[10] T.W. Chaudy and E. Philips, The convergence of sequences defined by quadratic recurrence-formule. The quarterly Journal of Mathematics, Oxford, 7, 74-80 (1936).

[11] J.S. Coleman, Introduction to Mathematical Sociology, The Free Press of Glencoe, Collier-Macmillan, London (1964).

[12] Pierre Collet, Jean-Pierre Eckmann. Iterated maps on the interval as dynamical systems. Basrel- Boston- Stuttgart, 1948, 248 p., (2-nd edition 1981, 30-rd edition 1983).
[13] B. Derrida, A. Gervois, Y. Pomeau, Iteration of endomorphisms of the real axis and representations of numbers. Annales de l’Institut Henri-Poincaré 29, 305 (1978).

[14] B. Derrida, A. Gervois, Y. Pomeau, Universal metric properties of bifurcations of endomorphisms. J. Phys. A12, 269 (1979).

[15] Yu.V. Fedorenko and M.V. Plakhotnyk, Continuous and smooth maps, which are not changed under iterations. Differential equations and their applications. Thesis of the Conference, Kyiv, June 2005, p. 86-87. (in Ukrainian).

[16] V.V. Fedorenko, Yu.V. Fedorenko and M.V. Plakhotnyk, One Dimensional Dynamical Systems, whose orbits are uniformly bounded, Bull. of Kyiv University, Ser. Phis.-Math. Sci., No 4, 2006, p. 119-128. (in Ukrainian).

[17] Fedorenko V.V. and Plakhotnyk M.V., Topological conjugation of piecewise linear unimodal maps. Collected articles of Kyiv institute of Mathematics of National Academy of Sciences of Ukraine 2014, Vol. 11, No. 5, pp. 115-127. (in Ukrainian)

[18] V. Fedorenko, V. Kyruchenko, M. Plakhotnyk, Exponent Matrices and Topological Equivalence of Maps, Algebra and Discrete Math. N 4, 2007, pp. 45-58.

[19] Mitchell J. Feigenbaum, Quantitative Universality for a Class of Nonlinear Transformations, Journal of Statistical Physics, V. 19, No. 1, 25 – 52, 1978.

[20] Mitchell J. Feigenbaum, The Universal Metric Properties of Nonlinear Transformations, Journal of Statistical Physics, V. 21, No. 6, 669-706, 1979.

[21] G.M. Fichtengolts, Course of Differential Calculus, Vol. 1. Moskow, FML, 1962, 608 p. (in Russian)

[22] J.B.J. Fourier, Théorie de la chaleur. Paris, Firmin Didot, 1822, Grand in-4°.

[23] E. Hille, Ordinary Differential Equations in the Complex Domain. John Wiley and Sons, New York, London, Sidney, Toronto (1976).
[24] E. Galois, Note sur la resolusion des equations numeriques. Bull. Sci. Math., Physiques et Chimiques, XIII, §216, Juin, 1830, pp. 413-414.

[25] J. Guckenheimer, Sensitive dependence on initial conditions for one-dimensional maps. Comm. Math. Phys. 70 133-160 (1979).

[26] F. W. Herschel, a Collectinon of Exampples of the applications of the Calculus of Finite Differences. Cambridge, 1820.

[27] F. W. Herschel, Consideration of Various Points of Analysis, Philosophical Trans- actions of the Royal Society of London, Vol. 104 (1814), pp. 440-468. Also see http://www.jstor.org/stable/107442

[28] J. Guckenheimer. 1978: Bifurcations of dynamical systems, CIME Lecture (1978), Progress in Mathematics, Vol 8. Birkhäuser Boston-Basel-Stuttgart 1980.

[29] G. Julia. 1918: Mémoire sur l’iteration des fonctions rationelles. J. de Math. Ser. 7, 4, 47-245 (1918).

[30] S. Katsura and W. Fukuda, Physica (Amsterdam) 130 A, 597 (1985).

[31] R. Kautz, Chaos : the science of predictable random motion. Oxford: Oxford University Press, 369 p., 2011.

[32] V.V. Kyrychenko, and M.V. Plakhotnyk, Topologically conjugated piecewise linear uni- modal maps of a interval into itself, Ukrainian Mathematical Journal, (in Ukrainian), (Will be soon).

[33] O.E. Lanford III., Lecture Notes, Zürich, 1979.

[34] G.C. Layek, An introduction to Dynamical Systems and Chaos, Springer India, 2015, 622p.

[35] H. Lebesgue, / Lecons sur l’integration et le recherche des fonctions primitives, 2-me ed., Paris, 1928.
[36] Legendre, A.M. Méthodes nouvelles pour la résolution approchée des équations numériques. §III, of Supplément á l’Essai sur la théorie des nombres, seconde édition. Bound with Essai sur la théorie des nombres, 2nd edition, Courcier, Paris, 1808.

[37] Lemerai, E. Sur la convergence des substitutions unifermes. Nouv. Ann. Math. 16, 1897, pp. 306-319

[38] Welington de Melo and Sebastian van Strien, One-Dimensional Dynamics, Springer, 1993, ISBN 0387564128, 586 p.

[39] Markov A.A., Extension of the law of large numbers to dependent quantities, Izvestiia Fiz.-Matem. Obsch. Kazan Univ., (2nd Ser.), 15(1906), p. 135-156. (in Russian)

[40] Markov A.A., Selected Works, A.N.S.S.S.R., Leningrad, 1951. (in Russian)

[41] Robert M. May, Simple mathematical models with very complicated dynamics, Nature, 261, 459-67, pp. 85-93, (1976).

[42] M. Metropolis, M.L. Stein, P.R. Stein. 1973: On finite limit sets for transformations of the unit interval. J. Combinatorial Theory 15, 25-44 (1973).

[43] J. Milnor, P. Thurston. 1977: On iterated maps of the interval, I, II. Preprint, Princeton, 1977. Published in: "Dynamical Systems: Proc. Univ, of Maryland 1986-87", (1988), Lect. Notes in Math., vol. 1342. Springer, Berlin New York, pp. 465-563.

[44] Melvyn B. Natahson, Piecewise linear functions with almost all points eventually periodic, Proc. of the Amer. Math. Soc. Vol. 60, p. 75-81, Oct. 1976.

[45] John von Neumann, Various techniques used in connection with random digits, Monte Carlo Method, National Bureau of Standards Applied Math. Series 12, 1951, p. 36-38.

[46] W. Parry: Symbolic dynamics and transformations of the unit interval. Trans. A.M.S. 122, 368-378 (1966).
[47] G.P. Peluh, A.N. Sharkovsky, Introduction to Functional Equations Theory, Kyiv, Naukova Dumka, 1974, 120 p. (in Russian)

[48] Pincherle, S. Alcune osservazioni sulla iterata di una funzione data. Rend. Reale Acad. Sci. Instit. Bologna (Nuova Serie), 18, (1913-1914), pp.75-88

[49] M. Plakhotnyk, Representation of the finite cyclic semigroup by continuous interval maps, Bull. of Kyiv University, Ser. Phis.-Math. Sci., No 3, 2006, p. 116-124. (in Ukrainian)

[50] M.V. Plakhotnyk, Systems of linear functional equations in the problem of topological conjugation of maps, Bull. of Kyiv University, Ser. Phis.-Math. Sci., No 4, 2014, p. 40-52. (in Ukrainian)

[51] M. Plakhotnyk, Differentiability of the conjugation for the pair of tent-like maps, Bull. of Kyiv University, Ser. Mathematics and Mechanics, 34, 2015, p. 28-34. (in Ukrainian)

[52] M. Plakhotnyk, Differentiability of the Homeomorphism of conjugateness for the pair of tent-like interval itself maps, Taras Shevchenko National University of Kyiv, Bulletin. Ser. Mathematics, Mechanics. Vol. 34, 2015, p. 28-34. (in Ukrainian)

[53] M. Plakhotnyk, Non-invertible analogue of the maps of conjugacy for the pair of tent-like maps. Bulletin of Odesa Mechnikov Univ., Ser. Phys.-Math. Sciences, (in Ukrainian), (Will be soon)

[54] M.V. Plakhotnyk, Differentiability of the homeomorphism of conjugateness for the pair of tent-like interval itself maps, Matematychni Studii. (in Ukrainian), (Will be soon).

[55] M.V. Plakhotnyk, Explicit formulas for the homeomorphism which determines the topological conjugateness of one dimensional unimodal maps. Algebra and discrete mathematics, (Will be soon).

[56] M. Gutiérrez Ranferi, M.A. Reyes and and H.C. Rosu, A note on Verhulst’s logistic equation and related logistic maps, arXiv:0910.1560v2 [math-ph] 2 May 2010.
[57] O. W. Rechard, Invariant measures for many-one transformations, Duke Math. J. 23 (1956), 477-488.

[58] Riesz F., Sz.-Nagy B., Lecons d’Analyse Functionelle, Budapest, 1972.

[59] J. F. Ritt, On Certain Real Solutions of Babbage’s Functional Equation, Annals of Mathematics, Second Series, Vol. 17, No. 3 (Mar., 1916), pp. 113-122.

[60] E.M. Rogers, The Diffusion of Innovations, New York, Free Press, (1962).

[61] R. Ross, The prevention of malaria, John Murray, London 1911.

[62] A.N. Sharkovskiy, S.F. Koliada, A.G. Sivak, V.V. Fedorenko, Introduction to the Functional Equations Theory, Kyiv, Naukova Dumka, 1989. 216 p. (in Russian)

[63] D. Singer. Stable orbits and bifurcations of maps of the interval. SIAM J. Appl. Math. 35, 260 (1978).

[64] Joseph D. Skufca, Erik M. Bolt. A concept of homeomorphic defect for defining mostly conjugate dynamical systems // Chaos, 2008 No 03118 p. 1-18.

[65] P. R. Stein and S. M. Ulam, Non-linear transformation studies on electronic computers, Rozprawy Mat. 39 (1964), l-66. (also in Stanislaw Ulam: Sets, Numbers, and Universes, edited by W. A. Beyer, J, Mycielski, and G.-C. Rota. Cambridge, Massachusetts: The MIT Press, 1974 – pp. 401-484).

[66] C. Tresser and P. Coullet, Itérations d’endomorphismes et groupe de renormalisation, C. R. Acad. Sc. Paris 287A 577-580 (1978).

[67] S.N. Ulam and J. von Neumann, On combination of stochastic and deterministic processes, Bull. Amer. Math. Soc. vol. 53 (1947) p. 1120 (Summer Meeting of the AMS in 1947) (URL: http://www.ams.org/journals/bull/1947-53-11/S0002-9904-1947-08918-7/S0002-9904-1947-08918-7.pdf)
[68] K. Umeno, Method of constructing exactly solvable chaos, Phys. Rev.E., 55, (1997), 5280-5284. (URL: http://cds.cern.ch/record/312068/files/9610009.pdf)

[69] K. Umeno, Inferring invariant measures of dynamical systems, IEICE Technical Report NC96-13 (1996).

[70] Pierre-Francois Verhulst, Recherches mathematiques sur la loi d’accroissement de la population. [Mathematical Researches into the Law of Population Growth Increase]. Nouveaux Memoires de l’Academie Royale des Sciences et Belles-Lettres de Bruxelles 18: 1-42. (URL: http://gymportalen.dk/sites/lru.dk/files/lru/124.kap6.verhulst_artikel_1844.pdf), (1845)

[71] P. Verhulst, Notice sur la loi que la population suit dans son accroissement, Correspondance mathèmatique et physique, 10 (1838), 113-125 (URL: http://link.springer.com/article/10.1007%2FBF02309004)

[72] S. Wolfram, A New Kind of Science (Champaign: Wolfram Media), Wolfram Media, (2002).
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