Computation of Maxwell’s equations on manifold using implicit DEC scheme

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Abstract

Maxwell’s equations can be solved numerically in space manifold and the time by discrete exterior calculus as a kind of lattice gauge theory. Since the stable conditions of this method is very severe restriction, we combine the implicit scheme of time variable and discrete exterior calculus to derive an unconditional stable scheme. It is an generation of implicit Yee-like scheme, since it can be implemented in space manifold directly. The analysis of its unconditional stability and error is also accomplished.

Keywords: Discrete exterior calculus, Maxwell’s equations, Implicit scheme.

PACS(2010): 41.20.Jb, 02.30.Jr, 02.40.Sf, 02.60.Cb.

1 Introduction

The Yee scheme is a commonly employed efficient approach to solve Maxwell’s equations numerically and so to model wave propagation problems in the time domain [1]. Although it is not a high order method, it is still preferred for many applications because it preserves important structural features of Maxwell’s equations [2–5]. Bossavit et al present the Yee-like scheme to extend the Yee scheme to unstructured grids. This scheme combines the best attributes of the finite element method (unstructured grids) and the Yee

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scheme (preserving geometric structure) [6, 7]. Stern et al [8] generalize the Yee-like scheme to unstructured grids not just in space, but in 4-dimensional spacetime by discrete exterior calculus(DEC) [9–20]. This relaxes the need to take uniform time steps. Based on these results, we generalize the Yee-like scheme to space manifold and the time, which is a kind of lattice gauge theory [21].

The stable conditions for Yee-like scheme and its generation are very severe restriction, and imply that very many time steps will be necessary to follow the solution over a reasonably large time interval. In computer simulations of physical processes, explicit and implicit methods are often used. For some problems, it takes much less computational time to use the implicit method with larger time steps, even taking into account that one needs to solve equations at each step. Based on these considerations, Keräen et al present the unstable conditional implicit Yee-like scheme [22]. In this paper, we show that the implicit scheme and DEC can be united to find an unconditional stable scheme (IDEC) for solving Maxwell’s equations on space manifold and the time. The analysis of IDEC’s unconditional stability and error is also accomplished. This scheme reduces to the implicit Yee scheme, if choosing rectangular mesh for flat space, and reduces to the scheme presented by Keräen et al, if choosing tetrahedral mesh for flat space.

2 Preliminaries

A discrete differential $k$-form, $k \in \mathbb{Z}$, is the evaluation of the differential $k$-form on all $k$-simplices. Dual forms, i.e., forms evaluated on the dual cell. Suppose each simplex contains its circumcenter. The circumcentric dual cell $D(\sigma_0)$ of simplex $\sigma_0$ is

$$D(\sigma_0) := \bigcup_{\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_r} \text{Int}(c(\sigma_0)c(\sigma_1) \cdots c(\sigma_r)),$$

where $\sigma_i$ is all the simplices which contains $\sigma_0, \ldots, \sigma_{i-1}$, and $c(\sigma_i)$ is the circumcenter of $\sigma_i$. In DEC, the exterior derivative $d$ is approximated as the transpose of the incidence matrix of $k$-cells on $k+1$-cells, and the approximated Hodge Star $\ast$ scales the cells by the volumes of the corresponding dual and primal cells.

The 2D or 3D space manifold can be approximated by triangles or tetrahedrons, and the time by line segments. Discrete connection 1–form or gauge field $A$ assigns to each element in the set of edges $E$ an element of the gauge group $\mathbb{R}$:

$$A : E \to \mathbb{R}.$$
Discrete curvature 2–form is the discrete exterior derivative of the discrete connection 1–form
\[ F = dA : \mathcal{P} \to \mathbb{R}. \]

The value of \( F \) on each element in the set of triangular \( \mathcal{P} \) is the coefficient of Holonomy group of this face. The 2–form \( F \) automatically satisfies the discrete Bianchi identity
\[ dF = 0. \tag{1} \]

For source case, we need discrete current 1–form \( J \). Let \( A = \sum_{\mathcal{E}} A_i \) and the Lagrangian functional be
\[ L(A, J) = -\frac{1}{2} \langle dA, dA \rangle + \langle A, J \rangle, \]
where
\[ \langle dA, dA \rangle := (A)_{1 \times |\mathcal{E}|} (d | \mathcal{E} | | \mathcal{F} | | \mathcal{F} | | \mathcal{F} | | \mathcal{F} | (d)_{|\mathcal{F}| \times |\mathcal{E}|} (A)^T_{|\mathcal{E}| \times 1}, \]
\[ \langle A, J \rangle := (A)_{1 \times |\mathcal{E}|} (\ast)_{|\mathcal{E}| \times |\mathcal{E}|} (J^T)_{|\mathcal{E}| \times 1}. \]

The Hamilton’s principle of stationary action states that this variation must equal zero for any vary of \( A_i \), implying
\[ d^T \ast dA = \ast J \tag{2} \]

Since \((d^T)^2 = 0\), the discrete continuity equation can express as:
\[ d^T \ast J = 0. \tag{3} \]

The Eqs.(1-3) are called discrete Maxwell’s equations, which are invariant under gauge transformations \( A \rightarrow A + df \) for any 0-forms.

### 3  IDEC for Maxwell’s Equations

If allowing for the possibility of magnetic charges and current discrete 3–form \( \bar{J} \), the symmetric discrete Maxwell’s equations can be written as
\[ dF = \bar{J} \quad d^T \ast F = \ast J, \tag{4} \]
with discrete continuity equations or integrability conditions
\[ d\bar{J} = 0 \quad d^T \ast J = 0. \tag{5} \]
Implicit scheme for TE wave

The discrete current forms, discrete curvature 2–form, and their dual can be written as
\[
\bar{J}^n = (\rho_m^n, -J_{m+\frac{1}{2}} \wedge dt) \quad *J^n = (-(\rho_e dt)^n, *J_{e+\frac{1}{2}}^n)
\]
\[
F^n = E^{n+1} \wedge dt + B^n \quad *F^n = H^{n+1} \wedge dt - D^n,
\]
where \( n \) and \( n + \frac{1}{2} \) denote the coordinate of the time, \( E = \sum E_i e^i \) (electric field) is the discrete 1–form on space, \( B = \sum B_i P^i \) (magnetic field) is the discrete 2–form on space, \( H = \sum H_i *P^i \) (magnetizing field) is the dual of \( B \) on space, \( D = \sum D_i *e^i \) (electric displacement field) is the dual of \( E \) on space, \( \rho_e dt \) (charge density) is the discrete 1–form on time, \( J_e = \sum J_{ei} e^i \) (electric current density) is the discrete 1–form on space. \( \rho_m = \sum \rho_{mi} T^i \) (magnetic charges) is the discrete 3–form on space, \( J_m = \sum J_{mi} P^i \) (current) is the discrete 2–form on space.

The symmetric discrete Maxwell’s equations (4) and (5) can be written as
\[
d_s B^n = \rho_m^n
\]
\[
d_s E^{n+1} \wedge dt = -d_t B^n - J_{m+\frac{1}{2}} \wedge dt
\]
\[
d_t^T D^n = * (\rho_e dt)^n
\]
\[
d_t^T H^{n+1} \wedge dt = d_t^T D^n + *J_{e+\frac{1}{2}}^n,
\]
where \( d_s, d_t^T \) are the restriction of \( d, d_T \) on space, and
\[
d_t B^n := \frac{B^{n+1} - B^n}{\Delta t} \wedge dt \quad d_t^T D^n := \frac{D^{n+1} - D^n}{\Delta t} \wedge dt.
\]

**Proposition 3.1** If the initial condition satisfies the first and third equations in Eqs.(6), the solution of the second and fourth equations in Eqs.(6) automatically satisfy Eqs.(6).

Proof. Because the dimension of spacetime is 3 + 1 or 2 + 1, therefore
\[
d_t^T * (\rho_e dt)^n = 0 \quad d_t^T * J_{e+\frac{1}{2}}^n = 0 \quad d_s \rho_m^n = 0 \quad d_t J_{m+\frac{1}{2}} \wedge dt = 0,
\]
and the continuity equations can be reduced to
\[
d_t^T * (\rho_e dt)^n - d_t^T * J_{e+\frac{1}{2}}^n = 0 \quad -d_s J_{m+\frac{1}{2}} \wedge dt + d_t \rho_m^n = 0.
\]
So we have

\[
\begin{align*}
  d_t^T d_s^T D^n - d_t^T (\rho_c dt)^n &= -d_t^T (\rho_c dt)^n - d_s^T (d_s^T H^{n+1} \wedge dt - *J^{n+\frac{1}{2}}_e) \\
  &= 0 \\
  d_t d_s B^n - d_t \rho_m^n &= -d_t \rho_m^n + d_s (d_s E^{n+1} \wedge dt + J^{n+\frac{1}{2}}_m \wedge dt) \\
  &= 0.
\end{align*}
\]

Now we show the implicit scheme (6) on the 2D discrete space manifold and the time. Take Fig. 1 as an example for a part of 2D space mesh, in which \(e_1, \ldots, e_5\) are edges, \(P_1, P_2\) are triangles, \(*e_1\) is the dual of \(e_1\). The second and fourth equations in Eqs. (6) based on Fig. 1 are

\[
\begin{align*}
  \frac{D_1^{n+1} - D_1^n}{\Delta t} + J^{n+\frac{1}{2}}_{e_1} &= \frac{H_1^{n+1} - H_2^{n+1}}{|*e_1|} \\
  - \frac{B_1^{n+1} - B_1^n}{\Delta t} - J^{n+\frac{1}{2}}_{m_1} &= \frac{E_1^{n+1}|e_1| + E_2^{n+1}|e_2| + E_3^{n+1}|e_3|}{|P_1|} \\
\end{align*}
\]

where \(||\) denotes the measure of forms and dual. The summation on the right is orient, that is to say, inverse the orientation of \(e_i\), then multiply \(-1\) with \(E_i\). Notice that a significant difference from Yee-like scheme is that \(*e_i\) is a polyline. Eqs. (7) can be implemented on 2D discrete space manifold directly (see Fig. 3) so is a generation of implicit Yee-like scheme.

Figure 1: edge and face with direction

In the absence of magnetic or dielectric materials, there are relations

\[
D_i = \varepsilon_0 E_i \quad B_i = \mu_0 H_i,
\]

where \(\varepsilon_0\) and \(\mu_0\) are two universal constants, called the permittivity of free space and permeability of free space, respectively. With relations (8),
Eqs.(7) can be rewritten into an implicit scheme (9) for TE wave.

\[

\begin{align*}
E_0 \frac{E^{n+1}_1 - E^n_1}{\Delta t} + J_{e1}^{n+\frac{1}{2}} &= \frac{H^{n+1}_1 - H^n_2}{|e_1|} \\
\mu_0 \frac{H^{n+1}_1 - H^n_1}{\Delta t} + J_{m1}^{n+\frac{1}{2}} &= -\frac{E^{n+1}_1|e_1| + E_2^{n+1}|e_2| + E_3^{n+1}|e_3|}{|P_1|}
\end{align*}
\]

\text{TE (9)}

Implicit scheme for TM wave

If writing

\[

\begin{align*}
F^n = H^{n+1} \wedge dt - D^n & \quad \ast F^n = -E^{n+1} \wedge dt - B^n \\
J^n = (-\rho^n_e, J_e^{n+\frac{1}{2}} \wedge dt) & \quad \ast J = (-\ast(\rho_m dt)^n, \ast J_m^{n+\frac{1}{2}}),
\end{align*}
\]

where \( H = \sum_E H_i e^i \) is the discrete 1–form on space, \( D = \sum_P H_i P^i \) is the
discrete 2–form on space, \( E = \sum_P D_i \ast P^i \) is the dual of \( D \) on space, \( B = \sum_E B_i \ast e^i \) is the dual of \( H \) on space, \( \rho_e = \sum_{Tet} \rho_e T^i \) is the discrete 3–form on
space, \( J_e = \sum_P J_e P^i \) is the discrete 2–form on space, \( \rho_m dt \) is the discrete
1–form on time, \( J_m = \sum_E J_m e^i \) is the discrete 1–form on space, the discrete
Maxwell’s equations can be rewritten as

\[

\begin{align*}
d_s D^n &= \rho^n_e \\
d_s H^{n+1} \wedge dt &= dt D^n + J_e^{n+\frac{1}{2}} \wedge dt \\
d_s^T B^n &= \ast(\rho_m dt)^n \\
d_s^T E^{n+1} \wedge dt &= -d_t^T B^n - \ast J_m^{n+\frac{1}{2}}.
\end{align*}
\]

Proposition 3.2 If the initial condition satisfies the first and third equations in Eqs.(10), the solution of the second and fourth equations in Eqs.(10) automatically satisfy Eqs.(10).

Proof. Because the dimension of spacetime is 3 + 1 or 2 + 1, therefore

\[

\begin{align*}
d_s^T (\rho_m dt)^n = 0 & \quad d_t^T J_m^{n+\frac{1}{2}} = 0 \\
d_s J_e^{n+\frac{1}{2}} = 0 & \quad d_t J_e^{n+\frac{1}{2}} \wedge dt = 0,
\end{align*}
\]

and the continuity equations can be reduced to

\[

\begin{align*}
d_t^T (\rho_m dt)^n - d_s^T (\ast J_m^{n+\frac{1}{2}}) &= 0 \\
d_s J_e^{n+\frac{1}{2}} \wedge dt - d_t \rho_e^n &= 0.
\end{align*}
\]
So we have

\[ \begin{align*}
    d_t d_s D^n - d_t \rho^n_c &= -d_t \rho^n_c - d_s (d_s H^{n+1} \wedge dt - J^{n+\frac{1}{2}}_c \wedge dt) \\
    &= 0 \\
    d_t^T d_s^T B^n - d_t^T \ast (\rho_m dt)^n &= -d_t^T \ast (\rho_m dt)^n + d_s^T (d_s^T E^{n+1} \wedge dt + \ast J^{n+\frac{1}{2}}) \\
    &= 0.
\end{align*} \]

Now we show the scheme (10) on the 2D discrete space manifold and the time. The second and fourth equations in Eqs.(10) based on Fig.1 are

\[ \begin{align*}
    B^{n+1}_1 - B^n_1 &+ J^{n+\frac{1}{2}}_{m1} = -\frac{E^{n+1}_1 - E^n_2}{|e_1|} \\
    D^{n+1}_1 - D^n_1 &+ J^{n+\frac{1}{2}}_{c1} = \frac{H^{n+1}_1 |e_1| + H^{n+1}_2 |e_2| + H^{n+1}_3 |e_3|}{|P_1|}.
\end{align*} \] (11)

With relations (8), Eqs.(11) can be rewritten into an implicit scheme (12) for TM wave.

\[ \begin{align*}
    \varepsilon \varepsilon_0 \frac{E^{n+1}_1 - E^n_1}{\Delta t} + J^{n+\frac{1}{2}}_{e1} &= \frac{H^{n+1}_1 |e_1| + H^{n+1}_2 |e_2| + H^{n+1}_3 |e_3|}{|P_1|} \\
    \mu_0 \frac{H^{n+1}_1 - H^n_1}{\Delta t} + J^{n+\frac{1}{2}}_{m1} &= -\frac{E^{n+1}_1 - E^n_2}{|e_1|}.
\end{align*} \] TM (12)

General schemes

For real world materials, the constitutive relations are not simple proportionality, except approximately. The relations can usually still be written:

\[ D = \varepsilon E \quad B = \mu H, \]

but \( \varepsilon \) and \( \mu \) are not, in general, simple constants, but rather functions. With Ohm’s law

\[ E = \frac{1}{\sigma} J \quad J_m = \frac{1}{\sigma_m} H, \]

where \( \sigma \) is the electrical conductivity and \( \sigma_m \) is magnetic conductivity. The IDEC schemes can be written as

\[ \begin{align*}
    \varepsilon \frac{E^{n+1}_1 - E^n_1}{\Delta t} + \sigma \frac{E^{n+1}_1 + E^n_1}{2} &= \frac{H^{n+1}_1 - H^n_2}{|e_1|} \\
    \mu \frac{H^{n+1}_1 - H^n_1}{\Delta t} + \sigma_m \frac{H^{n+1}_1 + H^n_1}{2} &= -\frac{E^{n+1}_1 |e_1| + E^{n+1}_2 |e_2| + E^{n+1}_3 |e_3|}{|P_1|}.
\end{align*} \] TE
\[
\begin{align*}
\frac{E_{n+1}^1 - E_n^1}{\Delta t} + \sigma \frac{E_{n+1}^1 + E_n^1}{2} &= H_{n+1}^1|e_1| + H_{n+1}^2|e_2| + H_{n+1}^3|e_3| \\
\mu \frac{H_{n+1}^1 - H_n^1}{\Delta t} + \sigma \frac{H_{n+1}^1 + H_n^1}{2} &= -E_{n+1}^1 - E_{n+1}^2
\end{align*}
\]

\[\text{TM}\]

4 Stability, convergence and accuracy

Now, we analyze the stability for scheme (9). The analysis for scheme (12) can be done in the same way. Suppose the fields to be:

\[
\begin{align*}
E_{n+1}^i &= E_n^i \xi \\
H_{n+1}^i &= H_n^i \cos(k|e_1|) \xi,
\end{align*}
\]

where \(\xi\) is the growth factor of time, and \(k\) is the spatial frequency spectrum. Substituting (13) into scheme (9), we obtain

\[
\begin{align*}
E_i^n \xi &= E_i^n + \frac{\Delta t}{\varepsilon|e_1|} (1 - \cos(k|e_1|)) H_i^n \xi \\
H_i^n \xi &= H_i^n - \frac{\Delta t}{\mu|P_1|} (E_i^n|e_1| + E_2^n|e_2| + E_3^n|e_3|) \xi
\end{align*}
\]

Rewrite the first equation of Eqs.(14) as

\[
E_i^n = \frac{\Delta t}{\varepsilon|e_1|}(1 - \cos(k|e_1|)) H_i^n \xi,
\]

and substitute it into the second equation of Eqs.(14) to obtain

\[
H_i^n \xi = H_i^n - \left( \frac{(c\Delta t)^2}{|P_1|} \left( 1 - \cos(k|e_1|) \right) H_i^n \frac{|e_1|}{*e_1} + \left( 1 - \cos(k|e_2|) \right) H_i^n \frac{|e_2|}{*e_2} \right) \frac{\xi^2}{\xi - 1}
\]

Therefore, we obtain a quadratic equation for \(\xi\) as follows:

\[
(1 + M)\xi^2 - 2\xi + 1 = 0,
\]

where

\[
M = \frac{(c\Delta t)^2}{|P_1|} \left( 1 - \cos(k|e_1|) \right) \phi_1 \frac{|e_1|}{*e_1} + \left( 1 - \cos(k|e_2|) \right) \phi_1 \frac{|e_2|}{*e_2} + \left( 1 - \cos(k|e_3|) \right) \phi_1 \frac{|e_3|}{*e_3} \geq 0
\]
The discriminant of Eq. (15) is

\[ 4 - 4(1 + M) = -4M \leq 0. \]

So

\[ |\xi| = \frac{1}{\sqrt{1 + M}} \leq 1. \]

That is to say scheme (9) is unconditional stability.

By the definition of truncation error, the exact solution of Maxwell’s equations satisfy the same relation as IDEC scheme except for an additional term \( O((\Delta t)^2 + \Delta t|e|) \). This expresses the consistency, and so convergence for IDEC scheme by Lax equivalence theorem (consistency + stability = convergence). The derivative of IDEC scheme are approximated by first order difference. Equivalently, \( H \) and \( E \) are approximated by linear interpolation functions. Consulting the definition about accuracy of finite volume method, we can also say that IDEC scheme has first order temporal and spacial accuracy.

5 Implementation

The IDEC scheme of Maxwell’s equations was implemented on C++ platform, consisting of the following steps:

1. Set the simulation parameters. These are the dimensions of the computational grid and the size of the time step, etc.

2. Set the propagating media parameters.

3. Initialize the mesh indexes.

4. Assign current transmitted signal.

5. Compute the value of all spatial nodes and temporarily store the result in the circular buffer for further computation.

6. Visualize the currently computed grid of spatial nodes.

7. Repeat the whole process from the step 4, until reach the desired time.
Fig. 3 and Fig. 4 exhibit the propagation of Gaussian pulse on rabbit and sphere simulated by IDEC.

![Figure 4](image1)

![Figure 5](image2)

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