EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Tests for determination of which nonlinear partial differential equations may have exact analytic nonlinear solutions of any of two types of hyperbolic functions or any of three types of Jacobian elliptic functions are presented. The Power Index Method is the principal method employed that extends the calculation of the power index for the most nonlinear terms to all terms in the nonlinear partial differential equations. An additional test is the identification of the net order of differentiation of each term in the nonlinear differential equations. The nonlinear differential equations considered are evolution equations. The tests extend the homogeneous balance condition that is necessary to conditions that may only be sufficient but are very simple to apply. Superposition of Jacobian elliptic functions is also presented with the introduction of a new basis that simplifies the calculations.

1. Introduction. Various direct methods have determined exact analytic solutions of nonlinear partial differential equations (NLPDEs) [3, 5, 6, 8, 13, 15]. These methods may be combined with Lie symmetry methods. Common functions that are chosen are hyperbolic functions, Jacobean elliptic functions or circular functions. The Power Index Method proposed here restricts the NLPDEs to be translationally invariant in the independent variables. Further the solutions are travelling waves. The Power Index Method is a test of which of the nonlinear functions: tanh, sech, sn, cn or dn are possible solutions of the NLPDEs. The aim here as in many direct methods is to convert the NLPDEs to algebraic equations in one function. The parameters of the traveling wave solutions are computed by equating the coefficients of each power of the selected nonlinear function in the algebraic equation to zero. The usual requirements for solving algebraic equations are required.

2. Power index method. The Power Index Method is first stated and later its justification is presented. The method can apply to single or multiple, real or complex NLPDEs in two or more independent variables. The discussion here applies to a single, real NLPDE with two independent variables although examples of complex or multiple NLPDEs are included. Only real solutions are assumed for real NLPDEs.

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The dependent variable $u(x, t)$ is represented by a finite power series in $U$

$$u = \sum_{j=0}^{p} a_{j}U^{j}(x, t)$$  \hspace{1cm} (2.1)

where $U$ is the tanh, sech, sn, cn or dn functions. The power $p$ is a positive definite integer. If $p$ is a fraction or a negative integer, the dependent variable should be changed so that the new power $p$ is positive definite. The power index $P$ of a term in the NLPDE is

$$P = np + d$$  \hspace{1cm} (2.2)

where $n$ is the number of products of $u$ and its derivatives and $d$ is the sum of the order of derivatives. To test for $P$ and $p$ we choose $u \propto U^{p}$ For example $P$ for the term $u^{2}u_{x}u_{xx}$ becomes

$$P = 4p + 3.$$  \hspace{1cm} (2.3)

The value of the power $p$ is calculated by equating the power index of the most nonlinear terms of the NLPDE. This condition, called homogeneous balance, has been widely used before [3, 5, 6, 8, 13, 15]. It is necessary that the two or more of the most nonlinear terms of a NLPDE must cancel if $u$ given by (2.1) can be a solution. With two or more coupled NLPDEs determining the power $p$ for each dependent variable can be more complicated and is not discussed here.

Next the values of the power index of the other terms in the NLPDE are computed. The power index values are ordered in increasing values and the differences are calculated. Then the net order $d$ of the derivatives in each term in the NLPDE is computed. If all the orders of the derivatives are even or all the orders are odd, it is possible to try (2.1) for any of the five functions: tanh, sech, sn, cn or dn.

The matching of the power index $P$ of the most nonlinear terms in a NLPDE is a necessary balance for a possible solution. However the other terms of lower nonlinearity in the NLPDE need to be balanced also. An extended Power Index Method could be required but this is quite complicated. Computing the power index of all terms in the NLPDE may achieve the necessary balance but in some cases it is insufficient. However, cases exist for which the power index values of all terms in the NLPDE are the same where no solution is found with the five functions. Both relation (2.1) and the derivatives of $u$ contribute in each term of the NLPDE to the nonlinear algebraic equation in $U$. Any derivative of the tanh function can be expressed in terms of tanh functions. On the other hand two derivatives of sech, sn, cn or dn functions are required to express the result in terms of the chosen function. As a result if the net order of derivatives is even for all terms in the NLPDE, any of the five functions may be solutions of the NLPDE as each solution when substituted into the NLPDE will produce an algebraic equation in the chosen function with the parameters of the wave number and speed of the wave, the expansion coefficients $a_{j}$ in (2.1) and any constants in the NLPDE. If the net order of the derivatives is odd for all terms in the NLPDE, the resultant algebraic equation for the tanh function will again depend only on the tanh function. On the other hand if the net derivatives for all terms in the NLPDE are odd for sech, sn, cn, or dn functions, then each term in the NLPDE will have a mixed factor that can be divided out. For a sech function it is a sech tanh term, for the Jacobian elliptic functions it is
a product of the other two functions. For NLPDEs with both odd and even net order of derivatives, a solution of the NLPDE is not possible in general for the four functions sech, sn, cn, or dn. An exemption would be if the NLPDE could be split into two parts that are set equal to zero separately. Equation splitting was quite useful for some fluid equations [2] but is unlikely for the five functions.

3. Examples. We start with the KdV and mKdV equations. The KdV equation was originally derived for nonlinear fluid flow in a channel. Consider

\[ u_t + \alpha u^4 u_x + u_{xxx} = 0 \quad (3.5) \]

for \( \alpha \) a constant and where \( l = 1 \) for the KdV equation and \( l = 2 \) for the mKdV equation. Equating the maximum power index \( P \) for the second and third terms, we find \( p = 2 \) for the KdV equation and \( p = 1 \) for the mKdV equation. The power index values are shown above the equation for \( p = 2 \) and below the equation for \( p = 1 \). The power index differences are two in both cases and the net order of the derivatives is odd for all three terms. Both the KdV and mKdV equations may have solutions in tanh, sech, sn, cn or dn functions.

Next consider Burgers equation, a model equation for shock waves,

\[ u_t + uu_x = \gamma u_{xx} \quad (3.6) \]

for \( \gamma \) a constant. Here \( p = 1 \) and the differences are one. Then the tanh function in (2.1) is a solution but the sech function in (2.1) is not a solution. For the sech case one derivative is even and the other derivatives are odd. The odd derivatives have sech tanh terms; the even derivative does not have a sech tanh term. Hence a balance cannot be achieved. Similarly the three Jacobian elliptic functions will not be solutions.

The combined KdV and mKdV equation [3] illustrates another result. Then

\[ u_t + 6\alpha uu_x + 6\beta u^2 u_x + \gamma u_{xxx} = 0 \quad (3.7) \]

where \( \alpha, \beta \) and \( \gamma \) are constants. The power \( p = 1 \) and the differences in the power index are one; the power index values are indicated above the terms in (3.7). Both tanh and sech functions in (3.7) are solutions since the derivatives are all odd. Next look at one example of an extended KdV equation [9],

\[ u_t + u_x + \frac{3}{2} \alpha uu_x + \frac{1}{6} \beta u_{xxx} - \frac{3}{8} \alpha^2 u^2 u_x + \frac{23}{24} \beta \alpha \beta u_x u_{xx} + \frac{5}{12} \alpha^2 \beta uu_{xxx} + \frac{19}{360} \beta^2 u_{xxxxx} = 0. \quad (3.8) \]

Here \( p = 2 \) and the power index differences are two. The power index values are indicated above the terms in (3.8). The published solution was \( A \text{sech}^2(B(x - vt)) \) for (3.8) in [9] but a more general solution is \( A \text{dn}^2(B(x - vt)) \) for (3.8) in [9] with the modulus \( k \neq 1 \) is a new result and was found in unpublished calculations by the author. All net derivatives are odd here.

An example of a NLPDE that has no solution in terms of the five nonlinear functions is the Blasius equation [4] for laminar flow of a fluid past a plate where \( \alpha \)
is a constant. The power \( p = 1 \) but there is no power index difference between the two terms; each is four as seen below in
\[
4 \quad 4
\]
\[u_{xxx} + \alpha uu_{xx} = 0. \quad (3.9)\]
Because the derivative order is even in one term and odd in the other term, only the tanh function could possibly be an analytic solution. However, there are unbalanced lower nonlinear terms in (3.9) as can be seen by the extended Power Index Method but that is not developed here. None of the five functions are analytic solutions for the Blasius equation. The Blasius equation can be solved numerically. That is not surprising since the Blasius equation is invariant under only two Lie symmetries and thus is not reduced to quadratures by those symmetries.

The Zakharov equations [16] in plasmas are
\[
n_{tt} - c_s^2 n_{xx} = \beta (|E|^2)_{xx}, \quad (3.10a)
\]
\[
iE_t + \alpha E_{xx} = \delta n E \quad (3.10b)
\]
where \( n \) is the perturbed ion density, \( E \) is the slow variation amplitude of the electric field intensity; \( c_s^2, \beta, \alpha \) and \( \delta \) are constants. The coupled complex equations have the relation for the electric field
\[
E = \varphi(\bar{x}) \exp[i(kx - \omega t + \delta)], \quad \bar{x} = \beta(x - ct + \delta), \quad (3.11)
\]
for \( \beta = 1 \). Then we have
\[
3 \quad 1 \quad 3
\]
\[\varphi_{xx} + a\varphi + b\varphi^3 = 0. \quad (3.12)\]
The constants \( a \) and \( b \) can be expressed in terms of the parameters. The power \( p = 2 \), the power index values are indicated above the terms in (3.12) and the derivative is of even order. Hyperbolic and Jacobian elliptic functions are possible solutions. The \( G'/G \) method for the Zakharov equations was discussed in [16].

The modified KdV-quartic nonlinearity equation [14] is a model equation for the electric potential in the plasma consisting of cold ions and two electron distribution functions at different temperatures. It has a fractional power \( p = 2/3 \). Then
\[
u_t + 2u^3u_x + \frac{1}{2} u_{xxx} = 0. \quad (3.13)
\]
The authors in reference [14] integrated (3.13) directly to find
\[
u = (5c)^{1/3} \text{sech}^{2/3} \left[ \frac{3c}{2} (x - ct) \right]. \quad (3.14)
\]
where \( c \) is a constant. Changing to a new dependent variable with \( p = 1 \), we recover the same solution. A Jacobian elliptic function was tried but the algebraic equations require a modulus \( k = 1 \). Thus a sech function is necessary.

4. **Superposition of Jacobian elliptic functions.** The superposition of Jacobian elliptic functions as solutions of NLPDEs includes two types [1, 7, 10, 11, 12]. One type consists of a particular Jacobian elliptic function with different phases [10]. The second type is a superposition of two Jacobian elliptic functions with the same phase [7, 11, 12] and considered here. For two symmetric NLPDEs any two of \( \text{sn}, \text{cn} \) or \( \text{dn} \) functions may be a solution [7]. Two NLPDEs are symmetric if interchanging the two dependent variables results in the same NLPDEs; asymmetric; two asymmetric NLPDEs are all others. For a single NLPDE or two asymmetric
NLPDEs superposition of cn and dn functions may be a solution [11, 12]. A new basis for the superposition of cn and dn is introduced here.

The first example is the quadratic-cubic nonlinear Schrödinger equation [11],

$$iu_t + u_{xx} + g_1 |u| u + g_2 |u|^2 u = 0,$$

(4.15)

with $g_1$ and $g_2$ are constants. The power is $p = 1$. Once the complex exponential has been factored out, the net order of derivatives is even and the power index values differ by two. The solution has been given previously [11] and is

$$u = \{A[\text{dn}(\bar{x}) \pm k \text{cn}(\bar{x})] + B\} \exp[-i(\omega t - \kappa x + \delta_1)]$$

(4.16)

where $\bar{x} = \beta(x - ct + \delta)$ and $A, B, \omega, c, \beta, \kappa, k^2 = m$ are found if (4.16) is substituted into (4.15).

The superposition of the KdV equation (3.5) is

$$u = \frac{A}{2} [\text{dn}^2(\bar{x}) \pm k \text{dn}(\bar{x})] = \frac{A}{2} [\text{dn}(\bar{x}) \pm k \text{cn}(\bar{x})]^2 + Ak'^2/4.$$

(4.17)

The first equation has been already reported [11]; the second relation was identified in unpublished calculations by the author. The symmetrized solution can appear in other solutions of NLPDEs. Comparing the solutions in (4.16) and (4.17), we can introduce a new basis for Jacobian elliptic functions. These are

$$M(\bar{x}) = \text{dn}(\bar{x}) + k \text{cn}(\bar{x})$$

(4.18a)

$$N(\bar{x}) = \text{dn}(\bar{x}) - k \text{cn}(\bar{x})$$

(4.18b)

$$S(\bar{x}) = k \text{sn}(\bar{x}).$$

(4.18c)

The new basis for Jacobian elliptic functions simplifies the calculation for new solutions of the KdV equation and other nonlinear partial differential equations. It also illustrates that superposition in one basis may not be superposition in another basis for solutions of nonlinear partial differential equations.

5. Conclusion. A simple method for identifying possible exact analytic solutions of nonlinear partial differential equations that are invariant under translations in the independent variables has been presented. The new results in the Power Index Method are the inclusion of the power index of all terms in the NLPDE and the effect of the even and odd character of the net number of derivatives in the NLPDE. The former requirement is more general than the homogeneous balance of the most nonlinear terms as balance is needed for the entire NLPDE. The nonlinear functions considered are the two hyperbolic functions tanh or sech as well as three Jacobian elliptic functions: sn, cn or dn. If the net numbers of derivatives of each term in the NLPDE are all even or are all odd, then any of five of the nonlinear functions may be solutions where the solution is a finite power series in the particular function with the highest power given by the power $p$ determined from the power index of the most nonlinear terms. For net order of derivatives that are mixed even and odd, only the tanh function is a solution. This last result may explain why the tanh-method is frequently employed. Some examples are noted.

Superposition of Jacobian elliptic functions as solutions of NLPDEs is presented for single and two symmetric and asymmetric NLPDEs. A new basis for the cn and dn functions is given that is easier for calculations and the new result that superposition in one basis may not be superposition in another basis.
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