BOWEN-FRANKS GROUPS AS CONJUGACY INVARIANTS FOR $\mathbb{T}^n$ AUTOMORPHISMS

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Abstract. The role of generalized Bowen-Franks groups (BF-groups) as topological conjugacy invariants for $\mathbb{T}^n$ automorphisms is studied.

Using algebraic number theory, a link is established between equality of BF-groups for different automorphisms ($BF$-equivalence) and an identical position in a finite lattice ($L$-equivalence). Important cases of equivalence of the two conditions are proved.

Finally, a topological interpretation of the classical BF-group $\mathbb{Z}^n/\mathbb{Z}^n(I-A)$ in this context is presented.

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1. INTRODUCTION

Bowen-Franks groups arise as a simple but nevertheless important invariant in symbolic dynamics. After its appearance in a paper by R. Bowen and J. Franks [Bw-Fr 77], the second author showed [Fr 84], continuing the work of Parry and Sullivan [Pa-Su 75], that it is a complete invariant (apart from a sign) for flow equivalence of irreducible subshifts of finite type. This was generalized to the reducible case by Huang [Hu 94, Hu 95]. Recently, the subject was treated in a difference perspective in [Bo 02] and [Bo-Hu 03]. Flow equivalence will be considered in the last section of this paper.

In the particular case of subshifts induced -via a Markov partition- by families of piecewise monotone interval maps, the Bowen-Franks groups present rather strong regularities, which have been exploited for instance in [Al-Se-SR 96].

It is worth mentioning, although the subject won’t be touched in this paper, that another important theoretical setting for Bowen-Franks groups is the one of K-theory of C*-algebras (see [Cu-Kr 80]).

In this note we want to address the role of Bowen-Franks groups as invariants in another situation, close to symbolic dynamics: torus automorphisms. The Bowen-Franks groups are, once again, invariants for topological conjugacy and they describe in detail the algebraic structure of periodic orbits.

In [MR 96] and [MR-SR 99] we presented some partial results concerning exclusively the two-dimensional case, where very tight recurrence relations are easily deduced. The generalization of these properties to higher dimensions, as well as other aspects of the subject are the object of current research and we will present new results in a forthcoming paper.
Here, we bring into play algebraic number theory to show how good as an invariant the Bowen-Franks groups are. In fact, Bowen-Franks groups reveal, in a simple and clear manner, quite a lot of the arithmetic structure lying behind these dynamical systems. We will show below that, at least for a large class of automorphisms with the same characteristic polynomial, their Bowen-Franks groups depend only on the position of the automorphism in a certain finite lattice.

Just to illustrate the consequences, and referring the reader to the definitions in the first section, we state the

**Proposition 1.** Let \( p(x) \in \mathbb{Z}[x] \) be an irreducible monic polynomial of degree \( n \), such that \( p(0) = \pm 1 \) and with square-free discriminant. Let \( g(x) \in \mathbb{Z}[x] \). Then all \( \mathbb{T}^n \)-automorphisms with characteristic polynomial \( p(x) \) have the same Bowen-Franks group \( BF_g \).

This will be a corollary of our main result.

The results presented in this paper are related with symbolic dynamics since generalized \( BF \)-groups are also, at least for polynomials satisfying \( g(0) = \pm 1 \), invariants of shift equivalence. In the particular case of primitive, irreducible matrices with \( \det = \pm 1 \), shift equivalence coincides with similarity and therefore the \( BF \)-equivalence classification directly applies. We refer the reader to [Li-Ma 95] for all definitions and results concerning symbolic dynamics.

The main interest of this work lies, from our point of view, on its connection to the general topological classification of torus automorphisms. Although there are algorithms that solve, in principle, this problem ([Bo-Sh 66], [Gr-Se 80]), the goal of finding a complete set of computable and meaningful conjugacy invariants was attained, as far as we know, only in dimension two ([MR-SR 98], [Ad-Tr-Wo 97]).

The links of torus automorphisms with algebraic number theory have also been exploited in the context of \( \mathbb{Z}^d \) actions (cf. [Sc 95], [Li-Sc 99], [Ka-Ka-Sc 00]).

### 2. Basic facts

We briefly recall some basic facts. The \( n \)-torus \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) has \( M_n(\mathbb{Z}) \) as its ring of endomorphisms with group of units \( GL_n(\mathbb{Z}) \). We will use the same notation for a torus endomorphism and its associated matrix. Matrices will act on \( \mathbb{T}^n \) on the left and so torus points will be represented by column vectors. \( \mathbb{Z}^n \) denotes the lattice of points with integer coordinates, and are also represented by column vectors, while the dual group of \( \mathbb{T}^n \), whose elements are represented by row integer vectors, will be denoted \( (\mathbb{T}^n)^* \).

Topological conjugacy of endomorphisms is equivalent to algebraic conjugacy, as one easily deduces by considering the action of homeomorphisms on the one-dimensional homology.

**Remark 1.** Although most of what we will say applies to general endomorphisms, we will consider only irreducible automorphisms; this restriction is not important for classification purposes (cf. the remark 16 below).

The simplest algebraic invariant one can think of is, of course, the characteristic polynomial. This determines the two basic dynamically significant invariants, topological entropy and the zeta function, by well-known results.

A first step beyond the mere counting of periodic points is given by means of the following
Definition 1. Let $A$ be a square integer matrix of dimension $n$. The Bowen-Franks group of $A$ is defined as $BF(A) = \mathbb{Z}^n / (A-I)\mathbb{Z}^n$.

As a simple consequence of the structure theorem for finitely generated abelian groups, we have $BF(A) = \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_m} \oplus \mathbb{Z}^{n-m}$, with $a_1 | a_2 | \cdots | a_m$, where the torsion coefficients $a_i$ and the Betti number $n - m$ may be computed through the reduction of $A - I$ to its Smith normal form.

Bowen-Franks groups give rise not to one but to infinitely many conjugacy invariants by generalizing the polynomial in the definition. Explicitly

Definition 2. Let $g(x) \in \mathbb{Q}[x]$ such that $g(A) \in M_n(\mathbb{Z})$. The (generalized) Bowen-Franks group of $A$ associated to $g(x)$ is $BF_g(A) = \mathbb{Z}^n / g(A)\mathbb{Z}^n$.

Definition 3. Two integer matrices $A$ and $B$ with the same characteristic polynomial are called Bowen-Franks-equivalent (or just $BF$-equivalent) if $BF_g(A) = BF_g(B)$, $\forall g(x) \in \mathbb{Q}[x]$ such that $g(A), g(B) \in M_n(\mathbb{Z})$.

Remark 2. It will become clear later that a necessary condition for $A$ and $B$ to be $BF$-equivalent is that $g(A) \in M_n(\mathbb{Z}) \Leftrightarrow g(B) \in M_n(\mathbb{Z}) \forall g(x) \in \mathbb{Q}[x]$.

A standard argument shows the existence of an isomorphism of abelian groups

$$\mathbb{Z}^n / (A^k - I)\mathbb{Z}^n \approx \text{Per}_k(A)$$

where $\text{Per}_k(A)$ is the group of $k$-periodic points of $A$ in $\mathbb{T}^n$. Because of their importance we use the special notation $BF_k(A)$ for $\mathbb{Z}^n / (A^k - I)\mathbb{Z}^n$. This will cause no confusion since $BF_g$ groups for constant polynomials are uninteresting and never used.

Example 1. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -7 & 23 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & 2 & 23 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 6 & -2 & 23 \end{pmatrix}.$$

These matrices have the same characteristic polynomial $p(x) = x^3 - 23x^2 + 7x - 1$. Then $BF_1(A) = \mathbb{Z}_{16}$ while $BF_1(B) = BF_1(C) = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$; but $BF_{x+1}(B) = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$ and $BF_{x+1}(C) = \mathbb{Z}_4 \oplus \mathbb{Z}_{16}$.

The torsion coefficients of $BF_k(A)$ enable us to locate the $k$-periodic orbits: let $T_m = \{ x \in \mathbb{T}^n : mx = 0 \}$; we have

**Proposition 2.** Let $BF_k(A) = \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_n}$ where $k_1 | k_2 | \cdots | k_n$. Then $\text{Per}_k(A)$ is generated by $x_i \in \text{Per}_k(A) \cap T_{k_i}$, $i = 1, \cdots, n$; in particular $T_{k_i} \subset \text{Per}_k(A) \subset T_{k_n}$ and these are the best possible inclusions.

**Remark 3.** In the previous proposition we are implicitly assuming that $\det(A^k - I) \neq 0$, i.e. that $A$ has no $k$-roots of unity among its eigenvalues. Of course the statement is still true in the general case if we extend the definitions in the obvious way: $\mathbb{Z}_0 = \mathbb{Z}$ and $T_0 = \mathbb{T}^n$. In general, we will consider only Bowen-Franks groups $BF_g(A)$ such that $\det(g(A)) \neq 0$.

The above proposition is generalized to the case of an also generalized $BF_g(A)$ if we just replace $\text{Per}_k(A)$ by the group of periodic orbits

$$T_g = \{ x \in \mathbb{T}^n : g(A)x = 0 \}.$$
3. BF-groups and Algebraic Numbers

As we mentioned in the introduction, the study of $\mathbb{T}^n$-automorphisms benefits from the connection with Algebraic Number Theory. To establish this connection in the simplest way, let’s fix an irreducible monic polynomial $p(x) \in \mathbb{Z}[x]$. Consider the ring $\mathbb{Z}[x]/(p(x))$ and its field of fractions $K = \mathbb{Q}[x]/(p(x))$. For later use we recall the basic

**Definition 4.** Let $\theta \in K$ and $\theta_1 = \theta, \theta_2, \ldots, \theta_n$ be its conjugates. The absolute Trace and Norm of $\theta$ are defined resp. as $\text{Tr}(\theta) = \sum_{i=1}^{n} \theta_i$ and $N(\theta) = \prod_{i=1}^{n} \theta_i$.

If the characteristic polynomial of $A \in \mathbb{M}_n(\mathbb{Z})$ is $p(x)$ then, as a $\mathbb{K}^n$-endomorphism, $A$ has the equivalence class of $x$ as a simple eigenvalue (because we assume $p(x)$ to be irreducible). We will denote this equivalence class from now on by $\beta$ and identify $\mathbb{Z}[x]/(p(x))$ with $\mathbb{Z}[^A]$. Let $v = (v_1, \ldots, v_n)$ be an associated row eigenvector; its entries form a $\mathbb{Z}$-basis of a (possibly fractional) ideal $I \triangleleft \mathbb{Z}[^A]$. Recall that a fractional ideal is just a finitely generated $\mathbb{Z}[\beta]$-module contained in $K$ and that these are exactly the sets $zJ$ for some $z \in K$ and $J \subset \mathbb{Z}[^A]$ an (integral) ideal. Two ideals $I, J$ are said to be equivalent if $J = zI$ for some nonzero $z \in K$. Thus, a different eigenvector $w = zv$ gives rise to an equivalent ideal. On the other hand, passing from $A$ to a similar matrix amounts to perform a change of base on $I$.

In short, we have just sketched the proof of the classic:

**Theorem 1.** (Latimer-McDuffee-Taussky) Given an irreducible polynomial $p(x) \in \mathbb{Z}[x]$, there is a bijection between similarity classes of integer matrices with characteristic polynomial $p(x)$ and ideal classes in $\mathbb{Z}[x]/(p(x))$.

This theorem gives us the algebraic number theoretic version of the classification problem (cf. [Nm 72] for a detailed proof).

With the previous notation, the mapping

$$(x_1, \ldots, x_n)^t \rightarrow \sum v_i x_i$$

yields bijections between $\mathbb{Q}^n$ and $K$, $\mathbb{Z}^n$ and $I$, and between $\mathbb{Q}^n/\mathbb{Z}^n$ and the torsion $\mathbb{Z}[\beta]$-module $K/I$, where the $R$-module structure of $\mathbb{Q}^n$ is given by $(\alpha, q) \in R \times \mathbb{Q}^n \rightarrow \alpha(A)q$.

A more geometrical approach involves considering all the real and complex embeddings of $K$ (cf., for instance, [Ke-Ve 98]) but this will be enough for our purposes.

The matrix $A$ represents multiplication by $\beta$ in $K$, and in particular in $I$, with respect to the basis $v$. When there is no risk of confusion, we’ll speak simply of the associated matrix (or automorphism) of an ideal or of the associated ideal of a matrix (or automorphism). We stress, however, that this identification depends on the choice of $v$.

Now we have a new, obvious, interpretation for Bowen-Franks groups of $A$:

$$BF_g(A) \approx I/g(\beta)I$$

We will use these identifications without further explanation in the sequel. Besides, as the polynomials $g(x)$ are taken mod $p(x)$, and may be thus identified with elements of $K$, we will also use the notation $BF_\alpha(A)$ if $\alpha = g(\beta)$. 
In particular, proposition 2 translates into the optimal inclusions

\[ k_1 I \supset g(\beta) I \supset k_n I \]

An immediate observation is that for every automorphism \( A \) (equivalently, for every \( \mathbb{Z}[\beta] \)-ideal \( I \))

\[ |BF_\alpha(A)| = |I/\alpha I| = |N(\alpha)|. \]

It’s worthwhile to derive from this that \( I/\alpha I \approx I/\lambda I \implies \frac{|N(\alpha)|}{N(\lambda)} = 1 \) and so if \( \frac{\lambda}{\lambda} \in \mathbb{R} \), we have in fact \( \frac{\lambda}{\lambda} \in \mathbb{R}^* \). Recip., if \( \delta = \frac{\lambda}{\lambda} \in \mathbb{R}^* \), the induced mapping

\[ \begin{align*}
\delta & : I/\lambda I \to I/\alpha I \\
x + \lambda I & \to \lambda x + \alpha I
\end{align*} \]

is an isomorphism. In particular, we conclude that for a given automorphism \( A \) only one group of each possible cardinality may occur as \( BF_\alpha(A) \).

3.1. Strong BF-equivalence. In the definition of BF-equivalence of two automorphisms \( A \) and \( B \) given before, \( BF_g(A) \) and \( BF_g(B) \) are demanded to be isomorphic only as abelian groups. Another equivalence relation is

**Definition 5.** Two automorphisms \( A \) and \( B \) are strongly BF-equivalent if, for every polynomial \( g(x) \), \( BF_g(A) \) and \( BF_g(B) \) are isomorphic \( \mathbb{Z}[\beta] \)-modules.

In fact \( BF_g(A) \approx T_b(A) \) has the \( \mathbb{Z}[\beta] \)-module structure induced by the action of \( A \) making this a natural definition: it requests that the isomorphism between these finite groups of periodic orbits of \( A \) and \( B \) conjugates the actions of the automorphisms.

We will use the notation \( BF_g(A) \approx_R BF_g(B) \) to indicate the existence of such an isomorphism and \( A \approx_R B \) to denote the strong BF-equivalence of the two automorphisms.

If \( BF_g(A) \approx_R BF_g(B) \) then the restriction of the isomorphism to any submodule is again an isomorphism; if this submodule is of the form \( T_h(A) = \{ x \in \mathbb{T}^n : h(A)x = 0 \} \) we conclude that \( BF_h(A) \approx_R BF_h(B) \). This simple observation shows that we may have \( BF_g(A) \approx BF_g(B) \) and not \( BF_g(A) \approx_R BF_g(B) \):

**Example 2.** Consider the two automorphisms

\[
M = \begin{pmatrix}
-1 & -1 & -1 & -4 \\
4 & 1 & 3 & 8 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 7
\end{pmatrix}
\quad \text{and} \quad
M' = \begin{pmatrix}
-5 & -4 & -5 & -12 \\
8 & 5 & 7 & 12 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 7
\end{pmatrix}
\]

with characteristic polynomial \( p(x) = x^4 - 7x^3 - 7x + 1 \); one verifies that

\[
BF_{48}(M) = BF_{48}(M') = \mathbb{Z}_{448} \oplus \mathbb{Z}_{1344} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m
\]

\[
m = 130401445122840192
\]

Let’s suppose that \( BF_{48}(M) \approx_R BF_{48}(M') \), that is, there is an isomorphism

\[ \Psi : T_h(M) = \{ x \in \mathbb{T}^4 : h(M)x = 0 \} \to T_h(M') = \{ x \in \mathbb{T}^4 : h(M')x = 0 \} \]

where \( h(x) = x^{48} - 1 \), such that \( M' \circ \Psi = \Psi \circ M \).

For \( g(x) = x^3 + 4x^2 + 4x + 5 \), we have \( BF_g(M) = \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{64} \); in particular, by proposition 2,

\[ T_g(M) \subset T_{64} = \{ x \in \mathbb{T}^4 : 64x = 0 \} \subset T_h(M) \]
and so $\Psi$ would induce an isomorphism between $T_g(M)$ and $T_g(M')$.

However $BF_g(M') = \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{32} \neq BF_g(M)$.

This argument proves in fact the following

**Lemma 1.** Given automorphisms $A$ and $B$, if $BF_k(A) \simeq_R BF_k(B) \forall k$, then $A \simeq_R B$.

Observe that, in the previous example, the hypothesis

$$BF_{48}(M) \simeq_R BF_{48}(M')$$

would imply that the restrictions of the actions by $M$ and $M'$ to $T_{64}$ would be conjugate; this is the same as to say that the matrices $M$ and $M'$ are conjugate in $GL_4(\mathbb{Z}_{64})$. On the other hand, conjugacy mod 64 would imply $BF_g(M) \simeq_R BF_g(M')$.

Since, given any automorphism $A \in GL_n(\mathbb{Z})$ and any natural number $m$, there exists some $k$ such that every point in $T_m$ is $k$-periodic, the fact that for all $g(x)$,

$$BF_g(A) = \bigoplus_{i=1}^n \mathbb{Z}_{k_i} \Rightarrow T_g(A) \subset T_k,$$

and the previous lemma allow us to conclude that

**Proposition 3.** Two automorphisms $A$ and $B$ (with the same characteristic polynomial) are strongly $BF$-equivalent iff they are conjugate mod $m \forall m \in \mathbb{Z}^+$.

So, as we will see in the next sections, it can happen that two automorphisms are conjugate in $GL_n(\mathbb{Z}_m) \forall m$ but not in $GL_n(\mathbb{Z})$. In connection with this observation we quote the following result (cf. [Su 65]):

**Theorem 2.** (D.A. Suprunenko) Let $A$ and $B$ be two square matrices of the same dimension, over a unique factorization domain $R$ with an infinity of nonassociate primes $p$. If $A$ and $B$ are similar over $R/(p)$ for all $p$, then they are similar over the field of fractions of $R$.

### 3.2. The lattice of orders and $\mathcal{L}$–equivalence.

Some more information from algebraic number theory will be needed. We’ll present it along with the development of our results. Although we keep this to a minimum and refer the reader to the many textbooks on the subject (cf. for instance, [Co 78], [Fr-Ta 93], [Nk 99]), most expositions on algebraic number theory overlook the ideal theory of non-integrally closed rings of integers ([Nk 99] is the best exception that we know about) and so some useful results are either missing in the literature or scattered, sometimes “in disguise”, in treatises on commutative algebra as [Ja 80]. Thus, it seems adequate to include a detailed presentation of these.

Recall that $z \in K$ is an integer of the field if its minimal polynomial has rational integer coefficients. The set of all integers in $K$ is a ring, denoted from now on as $\mathbb{Z}_K$. This ring is known to be a Dedekind domain, i.e.

i) all non zero prime ideals are maximal;

ii) the ascending chain condition for ideals is valid;

iii) $\mathbb{Z}_K$ is integrally closed in $K$.

The main consequence of these properties for algebraic number theory is the well known fact that any nonzero ideal of $\mathbb{Z}_K$ is uniquely factorable as a product of prime ideals. In our interest, let’s just point out that all $\mathbb{Z}_K$-ideals are invertible. Concretely, if $I$ is a fractional ideal, define

$$I^{-1} = \{z \in K : zI \subset \mathbb{Z}_K \}$$

which is again a fractional ideal; then $II^{-1} = \mathbb{Z}_K$. 

The ring $\mathbb{Z}[\beta]$ will not be, in general, integrally closed in its field of fractions $K$. In other words, $\mathbb{Z}[\beta] \subset Z_K$ will be a strict inclusion. In fact, $\mathbb{Z}[\beta]$ and $Z_K$ are resp. the minimal and maximal elements in a finite lattice, ordered by inclusion, of orders, i.e. sub-rings of $Z_K$ with finite index and containing 1. Given $\mathbb{Z}[\beta]$, or, more generally, some order, one may construct explicitly its lattice (c.f. [Co 78], for a description of the algorithm).

**Example 3.** Let $p(x) = x^2 - 34x + 1$; the discriminant is $(24)^22$; the lattice associated to $\mathbb{Z}[\beta]$ is

\[
\begin{array}{c}
\mathbb{Z} \left[ \sqrt{2} \right] \\
\mathbb{Z} \left[ 2\sqrt{2} \right] & \mathbb{Z} \left[ 3\sqrt{2} \right] \\
\mathbb{Z} \left[ 4\sqrt{2} \right] & \mathbb{Z} \left[ 6\sqrt{2} \right] \\
\mathbb{Z} \left[ 12\sqrt{2} \right]
\end{array}
\]

where the arrows describe inclusion of orders.

**Example 4.** Let $p(x) = x^3 - 23x^2 + 7x - 1$; the discriminant is $-2^883$; we have in this case

\[
\begin{array}{c}
(1, \frac{\beta + 1}{2}, \frac{\beta^2 + 7}{8}) \\
(1, \frac{\beta + 1}{2}, \frac{\beta^2 + 3}{4}) \\
(1, \beta, \frac{\beta^2 + 3}{4}) \\
(1, \beta, \beta^2 + 1) \\
(1, \beta, \beta^2)
\end{array}
\]

In this case, an integral basis of the corresponding order is displayed in each vertex.

**Remark 4.** Notice that in the quadratic case the lattice is completely determined by the discriminant: if $\text{disc}(p(x)) = f^2\Delta$ where $\Delta$ is square-free, then the lattice will be the one of the divisors of $f$ if $\Delta = 1 \text{mod } 4$, and of $f/2$ otherwise. This doesn’t happen for higher dimensions.

This lattice, denoted as $\mathcal{L}$ in the sequel, will be the fundamental structure to identify BF-equivalence classes and we will pay considerable attention to it.

It will be convenient to state some properties that hold for ideals in any order $R \subset Z_K$, or, in certain cases, even in any noetherian domain.

Let $I$ be an ideal in $R$ and define, just as before, $I^{-1} = \{z \in K : zI \subset R\}$; we say that $I$ is invertible in $R$ if $II^{-1} = R$. Also, $I$ is said to be divisorial if $(I^{-1})^{-1} = I$. Notice that these definitions depend on the ring in question. So, when necessary, we might use the notation $(M : N) = \{z \in K : zN \subset M\}$ valid for any $R$-modules, where $R$ is any ring with field of fractions $K$.

The ring of coefficients of $I$ is defined as $C(I) = \{z \in K : zI \subset I\}$. By the Cayley-Hamilton theorem, $C(I)$ is contained in $Z_K$ and it obviously contains $R$, i.e. it’s one of the orders in the lattice described above.
If \( I \) is the ideal associated to an automorphism (represented by the matrix) \( A \) one easily concludes that

**Lemma 2.** \( R \subset C(I) \) iff \( g(A) \in M_n[\mathbb{Z}] \), for every \( g \in \mathbb{Q}[x] \) such that \( g(\beta) \in R \).

This verification amounts to a trivial computation using the elements in the basis of the different orders above \( \mathbb{Z}[\beta] \).

**Example 5.** Consider again the automorphisms \( A, B, C \) from example 1; the rings of coefficients of the associated ideals are the orders with integral basis \((1, \beta, \beta^2)\), \((1, \beta, \beta^2 + 1)\) and \((1, \beta, \beta^2 + 2)\) resp.

**Definition 6.** Two automorphisms \( A, A' \) with the same characteristic polynomial are \( \mathcal{L} \)-equivalent if their associated (classes of) ideals have the same ring of coefficients.

**Remark 5.** Before we return to \( BF \)-groups, we take the opportunity to show how the classification of irreducible endomorphisms may be reduced to the one of automorphisms. By the Latimer-McDuffee-Taussky theorem, the similarity classes of endomorphisms with characteristic polynomial \( q(x) \) are in bijection with classes of ideals in \( \mathbb{Z}[x]/(q(x)) \).

If \( \beta \) is a unit of this ring, \( \mathbb{Z}[x]/(q(x)) \) is one of the orders above \( \mathbb{Z}[\beta] \) and its classes of ideals are also classes of ideals in \( \mathbb{Z}[\beta] \), as an equivalence of ideals in the former ring may always be turned into an equivalence in the latter. So, in order to identify the ideal classes of the larger ring, one just has to choose among the ideal classes of \( \mathbb{Z}[\beta] \) those with coefficient ring containing \( \mathbb{Z}[x]/(q(x)) \).

**Example 6.** Let \( q(x) = x^2 - 10x - 7 \); a fundamental unit in \( \mathbb{Z}[x]/(q(x)) = \mathbb{Z}[\sqrt{2}] \) is \( \beta = 17 + 12\sqrt{2} \) whose characteristic polynomial is \( p(x) = x^2 - 34x + 1 \) as in example 3.

In dimension two, a complete and effective classification is known; from the 12 ideal classes of \( \mathbb{Z}[\beta] \), one finds that only 3 of them have coefficient ring containing \( \mathbb{Z}[\sqrt{2}] \). The same algorithm enable us to list canonical representatives of the different conjugacy classes of automorphisms with characteristic polynomial \( p(x) \); to get a similar result for the ring of endomorphisms in question, we have only to notice that these endomorphisms represent multiplication by \( 5 + 4\sqrt{2} = \frac{\Delta - 2}{3} \) in their ideal classes; if \( A \) is an automorphism with characteristic polynomial \( p(x) \), associated to one of the 3 ideal classes mentioned above, \( \frac{\Delta - 2}{3} \) is an endomorphism with characteristic polynomial \( q(x) \) associated to the same class.

4. **BF-equivalence and \( \mathcal{L} \)-equivalence**

4.1. **First results.** Suppose that, for two different \( \mathbb{Z}[\beta] \)-ideals \( I \) and \( J \), we have \( C(I) \neq C(J) \). We may take, in that case some \( \frac{a}{m} \in C(I) \backslash C(J) \) with \( \alpha \in R \) and \( m \in \mathbb{Z} \). Then \( \alpha I \subset mI \) and so \( m \mid k_1 \) if \( I/\alpha I \approx \bigoplus_{i=1}^n \mathbb{Z}_{k_i} \). On the other hand, \( \exists y \in J : \frac{a}{m}y \notin J \); let \( J/\alpha J \approx \bigoplus_{i=1}^n \mathbb{Z}_{k_i} \); we have the inclusion \( \alpha J \subset l_1 J \), ie..

\[
\forall z \in J, \exists w_z \in J : \alpha z = l_1 w_z
\]

If \( m \mid l_1 \), then \( \frac{a}{m}y = l_1w_y \in J \), a contradiction. We conclude that

**Lemma 3.** If \( C(I) \neq C(J) \), then \( BF_\alpha(I) \neq BF_\alpha(J) \) for some \( \alpha \in \mathbb{Z}[\beta] \).
Or, in terms of the definitions introduced before,

**Proposition 4.** If two automorphisms $A, A'$ with the same characteristic polynomial are BF-equivalent then they are $L$-equivalent.

A similar reasoning leads us to a partial converse:

**Lemma 4.** If $C(I) = R$ and, for $\alpha \in R$, the Bowen-Franks groups are $BF_\alpha(I) = \bigoplus_{i=1}^n \mathbb{Z}k_i$ and $BF_\alpha (R) = \bigoplus_{i=1}^n \mathbb{Z}l_i$, then $k_1 = l_1$, $k_n = l_n$.

**Proof.** $\alpha R \subset l_1 R \Rightarrow \alpha I \subset l_1 I$, therefore $l_1 \leq k_1$; but $\alpha I \subset k_1 I \iff \frac{\alpha I}{I} \subset C(I) = R \Rightarrow \alpha R \subset k_1 R$ and so $k_1 \leq l_1$. To conclude for the equality of the last coefficients we just have to reverse inclusions. □

For (very) low $n$ this, together with the observation already made about the order of BF-groups, immediately implies equivalence:

**Corollary 1.** Let $n \leq 3$. Two $T^n$-automorphisms $A, A'$ with the same characteristic polynomial are BF-equivalent iff they are $L$-equivalent.

We see that to arrive to a general result of this type, a detailed description of the $\mathbb{Z}[\beta]$-classes of ideals with different coefficient rings is needed. As it will be seen later, the case of invertible ideals is easier to handle.

### 4.2 Invertible ideals

It’s a consequence of the definitions that if $I$ is invertible in $R$, then $C(I) = R$. The converse doesn’t hold in general, as example 7 will show.

We start with two lemmas, the first of which may be found, for instance, in [Ja 80].

**Lemma 5.** If $I \triangleleft R$ is a proper ideal then $I^{-1}$ strictly contains $R$.

**Lemma 6.** Given $I \triangleleft R$ one has $C(I^{-1}) = C((I^{-1})^{-1}) = (II^{-1})^{-1}$.

**Proof.** 1) Let $x \in C(I^{-1})$ i.e. $x I^{-1} \subset I^{-1}$. If $z \in (I^{-1})^{-1}$ then $\forall y \in I^{-1}, xyz \in z I^{-1} \subset R$. Therefore $xz \in (I^{-1})^{-1}$ and so $x \in C((I^{-1})^{-1})$.

2) Now let $x \in C((I^{-1})^{-1})$; given $a \in I$ and $y \in I^{-1}$, since by definition $I \subset (I^{-1})^{-1}$ we have $xa \in (I^{-1})^{-1}$ and $xay \in I^{-1}(I^{-1})^{-1} \subset R$; this means $xII^{-1} \subset R$, that is $x \in (II^{-1})^{-1}$.

3) Finally, let $x \in (II^{-1})^{-1}$; for $y \in I^{-1}$ and $\forall a \in I$, $xya \in R$, i.e. $xy \in I^{-1}$ and $x \in C(I^{-1})$. □

This yields a characterization of invertibility for ideals in an order $R$ (which, in fact, is valid for any noetherian domain):

**Proposition 5.** An ideal $I \triangleleft R$ is invertible iff $C(I) = R$ and $I$ is divisorial, iff $C(I^{-1}) = R$.

We show now that in general there may exist $\mathbb{Z}[\beta]$-ideals which are not invertible in its coefficient ring:

**Example 7.** Let $p(x) = x^3 - 23x^2 + 7x - 1$; $(1, \beta, \frac{\beta^2+1}{2})$ is an integral basis of $R$, one of the orders above $\mathbb{Z}[\beta] = \mathbb{Z}[x]/(p(x))$, and the matrix representing multiplication by $\beta$ with respect to it is

$$B = \begin{pmatrix} 0 & -1 & -11 \\ 1 & 0 & -3 \\ 0 & 2 & 23 \end{pmatrix}$$
Consider two other matrices with the same characteristic polynomial
\[
A = \begin{pmatrix} -7 & -7 & -20 \\ 8 & 7 & 0 \\ 0 & 1 & 23 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & -1 & -8 \\ 2 & 1 & -6 \\ 0 & 1 & 23 \end{pmatrix};
\]
the associated ideals \(I\) and \(J\) have integral basis \((8, \beta + 7, \beta^2 + 7)\) and \((2, \beta + 1, \beta^2 + 1)\) resp. Both have \(R\) as ring of coefficients. The inverses are \(I^{-1}\) with basis \((8, \beta + 7, \beta^2 + 7)\) and \(J^{-1}\) with basis \((2, \beta + 1, \beta^2 + 1)\); \(C(I^{-1}) = R\) but \(C(J^{-1})\) is a bigger order. Therefore \(I\) is invertible but \(J\) is not.

A somewhat different approach to invertibility rests on the fundamental concept of localization. Recall that given a prime \(P \triangleleft R\) the localization of \(R\) at \(P\) is the subring of \(K, R_P = \{x/y : x \in R, y \in R \setminus P\}\). \(R_P\) has a single maximal ideal \(P R_P\).

We quote two important results from [Nk 99].

**Proposition 6.** If \(J \neq 0\) is an ideal of \(R\), then
\[
R/J \cong \bigoplus_{P} R_P/JR_P = \bigoplus_{P \geq J} R_P/JR_P.
\]

**Proposition 7.** A fractional ideal \(I\) of \(R\) is invertible iff, for every prime \(P \neq 0,\)
\[
I_P = IR_P
\]

is a fractional principal ideal of \(R_P\).

The following proposition is usually stated only for Dedekind domains. The proof we present here adapts to the general case the one suggested in [Nk 99].

**Proposition 8.** If \(I \triangleleft R\) is an invertible ideal, then \(I\) has rank 2 as a \(R\)-module.

\[
\begin{align*}
\text{Proof. Let } s & \in I; \text{ It suffices to show that } I/(s) \text{ is a principal } R/(s) \text{ ideal; in fact, if } I/(s) = tR/(s) \text{ then } I = sR + tR. \text{ But, taking } J = (s) \text{ in proposition 6 we have that } I/(s) \text{ is isomorphic to } \bigoplus_{P \geq (s)} I_P/sR_P \text{ and by proposition 7 , each } I_P \text{ is a principal } R_P\text{-ideal. Therefore, } I/(s) \text{ is principal.} \\
\end{align*}
\]

4.3. **BF-equivalence: invertible ideals.** Recall the interpretation of Bowen-Franks groups made before: \(BF_g(A) \simeq I/g(\beta)I\).

Given \(\alpha, \lambda \in C(I) = R\) there is an exact sequence
\[
0 \to I/\alpha I \to I/\alpha \lambda I \to I/\lambda I \to 0
\]
where the injection is \(x + \alpha I \to \lambda x + \alpha \lambda I\) and the surjection is the canonical projection.

From the commutative diagram
\[
\begin{array}{cccc}
0 & \to & 0 & \to ((\alpha) \cap I)/\alpha I \\
\downarrow & & \downarrow & \downarrow \\
0 & \to & I & \to I/\alpha I & \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & R & \to R/\alpha R & \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
R/I & \to & R/I & \to & G
\end{array}
\]
and applying the “snake lemma” we get

\[ 0 \to ((\alpha) \cap I)/\alpha I \to R/\alpha R \to \eta R \to R/I \to R/I \to 0 \]

downward arrows indicate monomorphisms and the nodes denote modules.

which tells us that in order to have a \( R \)-isomorphism \( I/I\eta \approx R/\eta R \)

it suffices that \( \eta \) induces a automorphism of \( R/I \) (in other words, that \( \eta \) is a \( R \)-unity mod \( I \)).

Let’s suppose that \( I \) is invertible and \( \alpha \in R \). The proof of proposition 35 shows we may take \( \alpha \) as one of the generators of \( I^{-1} \). If \( \gamma \) is the other one, there exist \( x, y \in I \) such that \( \gamma x + \alpha y = 1 \); the ideal \( J = \gamma I \) is also contained in \( R \) and satisfies \( J + \alpha R = R \) implying that \( J \cap \alpha R = \alpha J \). In consequence the morphism \( J/\alpha J \to R/\alpha R \) is injective and, as the modules are finite, bijective.

Since \( I/\alpha I \approx J/\alpha J \) we proved the following:

**Proposition 9.** If \( A \) and \( B \) are automorphisms associated resp. to the invertible ideal \( I \) and to its coefficient ring \( R \), then \( A \) and \( B \) are strongly \( BF \)-equivalent.

**Remark 6.** The above proof shows that \( BF_{\alpha}(A) \approx_{R} BF_{\alpha}(B) \), if there exists \( \gamma \in I^{-1} \) such that \( \gamma I + \alpha R = R \).

If \( \mathbb{Z}[x]/(p(x)) = \mathbb{Z}[\beta] \) is the ring \( \mathbb{Z}_K \) of all algebraic integers of \( K \), we obtain that all automorphisms with characteristic polynomial \( p(x) \) are strongly \( BF \)-equivalent. That’s what happens, in particular, if the discriminant of \( p(x) \) is free of squares as stated in proposition 1.

More generally, this result completely solves the problem of strong \( BF \)-equivalence for those \( \mathbb{Z}[\beta] \) in which all ideals are invertible in the respective coefficient ring, in particular, in the quadratic case. In fact, for \( n = 2 \), the semigroup of ideal classes of \( \mathbb{Z}[\beta] \) has the structure of a Clifford semigroup, that is, a commutative semigroup \( S \) such that each \( x \in S \) is contained in a subgroup of \( S \). The paper \([Za-Zn 94]\) contains a detailed description of this situation, as well as the following result:

**Theorem 3.** (Zanardo and Zannier) For any field \( K \) of algebraic numbers of degree \( n > 2 \), there is an order \( R \) contained in \( \mathbb{Z}_K \) such that the semigroup of ideal classes of \( R \) is not a Clifford semigroup.

Proposition 9 can not be generalized to all automorphisms. However, we consider the following:

**Problem 1.** Are two \( L \)-equivalent automorphisms necessarily \( BF \)-equivalent?

Although we are unable to prove this for the moment, we will show that this is true for a larger class of ideals in the next section.

4.4. **Ideals in monogenic orders.** We start by considering in detail the properties of ideals in the special case \( C(I) = \mathbb{Z}[\theta] \) for some algebraic integer \( \theta \). Orders of this kind are called monogenic.

We need one more definition:

**Definition 7.** Given an ideal \( I \subset \mathbb{Z}[\beta] \), the dual ideal of \( I \) is

\[ I^{*} = \{ z \in K : Tr(zy) \in \mathbb{Z}, \forall y \in I \} \cdot \]

If \( (v_1, \cdots, v_n) \) is an integral basis of \( I \), the dual basis is the integral basis \( (w_1, \cdots, w_n) \) of \( I^{*} \) defined by \( Tr(v_i w_j) = \delta_{ij} \).
This notion may be extended to equivalence classes of ideals. Notice also that \( I^{**} = I \).

For the ring \( \mathbb{Z}[\beta] \) itself one may choose the entries of \( w_0 = (1, \beta, \ldots, \beta^{n-1}) \) as the canonical integral basis of \( \mathbb{Z}[\beta] \). Its dual basis is then \( \frac{1}{p'(\beta)}(b_0, \cdots, b_{n-1}) \) where the \( b_j \) are another integral basis of \( \mathbb{Z}[\beta] \), and \( p'(\beta) \) is just the usual derivative. To be precise, they are obtained (cf. [Fr-Ta 93]) from the following polynomial identity

\[
\frac{p(x)}{x - \beta} = \sum_{j=0}^{n-1} b_j x^j
\]

Denote by \( C \) the integer matrix with characteristic polynomial \( p(x) \) and column eigenvector \( w_0 \). A direct calculation shows that \( v_0 = \frac{1}{p'(\beta)}(b_0, \cdots, b_{n-1}) \) is exactly the row eigenvector of \( C \) normalized such that \( v_0 w_0 = 1 \).

In the general case, if \( w \) is a column eigenvector of \( A \) associated to \( \beta \), there exists then a non-singular rational matrix \( M \) such that \( w = M w_0 \) and directly from the definition we obtain that the dual basis of \( w \) is given by \( v = v_0 M^{-1} \) which is a row eigenvector for \( A \), since \( MA = CM \); moreover, \( vw = 1 \) as well. So, \( I \) and \( I^* \) are naturally identified with, resp., the lattice \( \mathbb{Z}^n \) and \( (\mathbb{T}^n)^* \).

**Proposition 10.** Given \( I \triangleleft \mathbb{Z}[\beta] \), \( I^{-1} = p'(\beta) I^* \)

**Proof.** Let \( y \in I^{-1} \); then, \( \forall z \in I \), we have \( \text{Tr}(\frac{y}{p'(\beta)}z) \in \mathbb{Z} \), since \( yz \in \mathbb{Z}[\beta] \) and \( \frac{1}{p'(\beta)} \in \mathbb{Z}[\beta]^* = (\mathbb{Z}[\beta])^* \). This shows that \( I^{-1} \subseteq p'(\beta) I^* \). If \( u \in I^* \) then \( \forall z \in I \) and \( \forall u \in \mathbb{Z}[\beta] \) we have \( \text{Tr}(u(za)) \in \mathbb{Z} \), which means that \( uI \subseteq (\mathbb{Z}[\beta])^* = \frac{1}{p'(\beta)} \mathbb{Z}[\beta] \) or \( p'(\beta) u \in I^{-1} \) establishing the reverse inclusion \( \blacksquare \).

As a consequence

\[
(I^{-1})^{-1} = (p'(\beta) I^*)^{-1} = \frac{1}{p'(\beta)} (I^*)^{-1} = \frac{1}{p'(\beta)} (p'(\beta) I^{**}) = I
\]

i.e. any \( \mathbb{Z}[\beta] \)-ideal is divisorial.

**Corollary 2.** An ideal \( I \triangleleft \mathbb{Z}[\beta] \) is invertible iff \( C(I) = \mathbb{Z}[\beta] \).

**Remark 7.** This confirms also that every order \( R \) containing \( \mathbb{Z}[\beta] \) is the ring of coefficients of some \( \mathbb{Z}[\beta] \)-ideal, namely of \( (\mathbb{Z}[\beta] : R) (= R^{-1} \) viewing \( R \) as a \( \mathbb{Z}[\beta] \)-fractional ideal). In fact, being divisorial, \( (R^{-1})^{-1} = R \). This shows that \( C(R^{-1}) \subset R \) and the other inclusion is obvious.

We end this section by describing the place of non-invertible prime ideals of \( \mathbb{Z}[\beta] \).

If \( P \triangleleft \mathbb{Z}[\beta] \) is a non-invertible prime ideal, we have \( C(P) = P^{-1} \) which is, thus, an order above \( \mathbb{Z}[\beta] \). If \( R \) is another order such that \( \mathbb{Z}[\beta] \subset R \subset P^{-1} \), then, by definition, \( P \subset (\mathbb{Z}[\beta] : R) = R^{-1} \). So, either \( R^{-1} = \mathbb{Z}[\beta] \) and \( R = \mathbb{Z}[\beta] \) as well, or \( R^{-1} = P \) and, by divisoriality, \( R = P^{-1} \). The same reasoning shows the converse: if \( R \) is an order immediately above \( \mathbb{Z}[\beta] \), then \( (\mathbb{Z}[\beta] : R) = R^{-1} \) is a prime ideal of \( \mathbb{Z}[\beta] \).

**4.5. BF-equivalence: a partial generalization.** By the reasoning preceding the last proposition, in the monogenic case the matrix representing multiplication by \( \beta \), with respect to an appropriate basis of \( I^{-1} \), will be \( A^t \), which implies that they are BF-equivalent. So, just by transposing both sides of the direct sum conjugacy defined in the preceding section, we get the
Proposition 11. If two automorphisms $A$ and $A'$ with associated ideals $I$ and $J$ are $\mathcal{L}$-equivalent and at least one ideal in each of the pairs $(I, (\mathbb{Z} [\beta] : I))$ and $(J, (\mathbb{Z} [\beta] : J))$ is invertible, then $A$ and $A'$ are $BF$-equivalent.

We'll now show that this effectively enlarges the class of $BF$-equivalent automorphisms. We need another bit of information on the role of the ring of coefficients $R$ for $I \triangleleft \mathbb{Z} [\beta]$.

All inverses in the next sentence must be considered with respect to $\mathbb{Z} [\beta]$. Lemma 6, applied to this case, says that $(II^{-1})^{-1} = C(I^{-1}) = C((I^{-1})^{-1}) = C(I) = R$, because $I$ is divisorial as a $\mathbb{Z} [\beta]$-ideal. By the same reason, $II^{-1} = ((II^{-1})^{-1})^{-1} = R^{-1}$, i.e.

Lemma 7. Given $I \triangleleft \mathbb{Z} [\beta]$ and $(\mathbb{Z} [\beta] : I)$ its inverse, $I(\mathbb{Z} [\beta] : I) = (\mathbb{Z} [\beta] : R)$, where $R = C(I)$.

If some $I$ with $C(I) = R$ is not invertible, then $(\mathbb{Z} [\beta] : R)$ is also not invertible ideal with the same coefficient ring. On the other hand, if $(\mathbb{Z} [\beta] : R)$ is non-invertible, for every ideal $I$, with $C(I) = R$, at least one of $I$ and $(\mathbb{Z} [\beta] : I)$ is also not invertible.

So, given an order $R$ above $\mathbb{Z} [\beta]$, there are two possible cases: either all ideals with $R$ as coefficient ring are invertible (this will be the case if $R$ monogenic) and we know that their associated automorphisms are all strongly $BF$-equivalent; or there are non-invertible ideals with coefficient ring $R$; in this case, for each invertible ideal $I$, $(\mathbb{Z} [\beta] : I)$ is not invertible.

In case the converse statement is true (this would be: if $I$ with $C(I) = R$ is not invertible then $(\mathbb{Z} [\beta] : I)$ is invertible), proposition 11 would solve the problem. However, it remains the possibility of both $I$ and $(\mathbb{Z} [\beta] : I)$ being non-invertible.

5. $BF$-groups and flow equivalence

We end this paper by returning, in some way, to the original role of $BF$-groups, i.e. its connection to flow equivalence. We start with the necessary definitions.

Definition 8. Let $f : X \to X$ be a homeomorphism of a topological space. The suspension of $X$ determined by $f$ is the quotient space $Y = (X \times \mathbb{R}) / \sim$ where the equivalence relation is given by $(x, t + 1) \sim (f(x), t)$.

The projection from $X \times \mathbb{R}$ to $Y$ defines a flow in the later space and each flow line contains exactly one orbit of $f$.

Definition 9. Two homeomorphisms $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ are flow-equivalent if there exists an homeomorphism $\varphi : Y_1 \to Y_2$ conjugating the two induced flows and preserving its direction.

The Bowen-Franks group $BF_1$ was introduced by the two authors as a flow-equivalence invariant in the case where the spaces $X_i$ are subshifts of finite type. Later, the second author showed that it is an almost complete invariant:

Theorem 4. Let $(X_A, \sigma)$ and $(X_B, \sigma)$ be two irreducible subshifts of finite type with positive entropy, where $A, B$ are the transition matrices. Then the shifts acting in the two spaces are flow equivalent iff $\det(Id - A) = \det(Id - B)$ and $BF_1(A) = BF_1(B)$. 

We want to show how the Bowen-Franks group appears naturally as a flow-equivalence invariant in the case of torus automorphisms. Let $A$ be a $\mathbb{T}^n$-automorphism and $Y$ the suspension space it determines.

The fundamental group of $\mathbb{T}^n \times \mathbb{R}$, a free abelian group of rank $n$, is embedded into the fundamental group of $Y$. Let $x_1, \ldots, x_n$ be a basis of this subgroup of $\pi_1(Y)$. A new generator $x_0$ must be added, namely the one represented by the closed orbit through 0, i.e. the loop $\{(0, t) : t \in \mathbb{R}\}$.

The presentation of $\pi_1(Y)$ is completed with the relations imposed by the equivalence $(x, t) \sim (Ax, t-1)$. In order to describe these in a simple way, we denote $x_1^{m_1} \cdots x_n^{m_n}$ by $X^m$ where $m = (m_1, \ldots, m_n)$.

We leave to the reader the verification of the following

**Proposition 12.** $\pi_1(Y) = \{x_0, x_1, \ldots, x_n : x_i x_j = x_j x_i, \forall 1 \leq i, j \leq n; x_0 X^m = X^m A x_0, \forall m \in \mathbb{Z}^n\}$.

The abelianization of $\pi_1(Y)$ gives

**Corollary 3.** $H_1(Y) = \mathbb{Z} \oplus \mathbb{Z}^n / \mathbb{Z}^n (A-I) = \mathbb{Z} \oplus BF_1(A)$.

As an homeomorphism between the suspension spaces determined by different $\mathbb{T}^n$-automorphisms induces an isomorphism between homology groups, it’s obvious that the Bowen-Franks group $BF_1$ is an invariant of flow-equivalence.

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