J-HOLOMORPHIC CYLINDERS BETWEEN ELLIPSOIDS IN DIMENSION FOUR

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ABSTRACT. We establish results concerning the existence and nonexistence of regular J-holomorphic cylinders between nested pairs of ellipsoids in $\mathbb{R}^4$.

1. INTRODUCTION

The foundational paper [EGH] includes a blueprint for proving that Contact Homology is independent of the choice of contact form. If one were to realize an idealized plan for this proof it would imply many generic existence results for regular pseudo-holomorphic curves in trivial symplectic cobordisms. In this note we investigate these existence questions in the simplest nontrivial setting corresponding to nested pairs of ellipsoids in $\mathbb{R}^4$. We first show that the existence results predicted by the naive proof of invariance do not hold in general. We then establish some of the predicted existence theorems under some simple assumptions on the ellipsoids. An additional motivation for the existence results established here is their potential use in the refinement of symplectic embedding obstructions. Such applications will be considered in future works of the present authors and in future work of the first author with Cristofaro-Gardiner and McDuff.

1.1. The Setting. We begin by recalling the features of a standard symplectic ellipsoid in $\mathbb{R}^4 = \mathbb{C}^2$. To a pair of real positive numbers $a < b$ which are rationally independent we associate the symplectic ellipsoid

$$E = E(a, b) = \left\{ \pi|z_1|^2 a + \pi|z_2|^2 b \leq 1 \right\}.$$ 

The notation $cE(a, b)$ will be used for the scaled ellipsoid $E(ca, cb)$.

We denote the boundary of $E$ by $\partial E$ and its interior by $\mathring{E}$. The restriction of the standard Liouville form on $\mathbb{R}^4$ yields a contact form $\lambda = \lambda(a, b)$ on $\partial E$. It has exactly two simple closed Reeb orbits, a faster orbit $\alpha$ with period $a$ and a slower orbit $\beta$ with period $b$. The Conley-Zehnder index of $\alpha^k$, the $k$-th iterate of $\alpha$, is

$$\text{CZ}(\alpha^k) = 2k + 2 \left\lfloor \frac{ka}{b} \right\rfloor + 1$$

and the the Conley-Zehnder index of the $k$-th iterate of $\beta$ is

$$\text{CZ}(\beta^k) = 2k + 2 \left\lfloor \frac{kb}{a} \right\rfloor + 1.$$ 

We also note that for every natural number $k$ exactly one iterate of either $\alpha$ or $\beta$ has Conley-Zehnder index equal to $2k + 1$.

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Now consider a pair of nested ellipsoids, $E_1 = E(a_1, b_1)$ and $E_2 = E(a_2, b_2)$ with $a_1 < a_2$ and $b_1 < b_2$. The set $E_2 \setminus \bar{E}_1$ equipped with the symplectic form inherited from $\mathbb{R}^4$ is a compact symplectic cobordism from the contact manifold $(\partial E_1, \lambda_1)$ to the contact manifold $(\partial E_2, \lambda_2)$ where $\lambda_i = \lambda(a_i, b_i)$. Let $(X^2_i, \omega)$ be its symplectic completion. An almost complex structure on $(X^2_i, \omega)$ is said to be *admissible* if it is tamed by $\omega$ and is (eventually) cylindrical on the two ends of the symplectic completion.

1.2. The Results. We consider the following basic question.

**Question 1.** Is there a generic condition on an admissible almost complex structure $J$ on $(X^2_1, \omega)$ which implies that for each $k \in \mathbb{N}$ there exists a regular $J$-holomorphic cylinder $u_k$ whose ends are asymptotic to the unique closed Reeb orbits of $\lambda_1$ and $\lambda_2$ with Conley-Zehnder index equal to $2k + 1$?

A naive view of contact homology and its invariance under changes in the contact form suggests that the answer to this question is YES. This positive viewpoint is also partially supported by Pardon’s recent work in [Pa]. There, among many other things, Pardon establishes the invariance of contact homology using methods different from those proposed in [EGH]. In the current setting his work implies the following.

**Theorem 1.** (Pa) For any admissible almost complex structure $J$ on $(X^2_1, \omega)$, there exists for each $k \in \mathbb{N}$ a cylindrical $J$-holomorphic building $H_k$ whose ends are asymptotic to the unique closed Reeb orbits of $\lambda_1$ and $\lambda_2$ with Conley-Zehnder index equal to $2k + 1$.

Our first result shows that, despite these suggestions, the answer to Question 1 is NO. We specify a family of nested ellipsoids $E_1 \subset E_2$ and a generic condition on admissible almost complex structures on $(X^2_1, \omega)$ such that some of the expected cylinders can not exist for any $J$ satisfying the generic condition.

Let $g_n$ be the $n^{th}$ odd index Fibonacci number and recall that these numbers are determined by the conditions $g_0 = g_1 = 1$ and the recursion relation

\[ g_{n+2} = 3g_{n+1} - g_n. \]

We say that an admissible almost complex structure $J$ on $(X^2_1, \omega)$ is *simply generic* if every somewhere injective $J$-holomorphic curve of genus zero is regular. It follows from [Di], that the simply generic $J$’s form a residual subset of all smooth and admissible almost complex structures tamed by $\omega$.

**Theorem 2.** Fix a natural number $n \geq 2$ such that $g_{n+1}$ is odd. For a positive real number $c_1 < 1$ and an irrational positive number $\epsilon > 0$ consider the nested pair of irrational ellipsoids

\[ E_1 = c_1 E \left( 1, \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right) \subset E_2 = E \left( 1, \frac{g_{n+2}}{g_n} + \epsilon \right), \]

and denote their faster simple closed Reeb orbits by $\alpha_1$ and $\alpha_2$, respectively.

If $\epsilon > 0$ is sufficiently small then the orbits $\alpha_1^{g_{n+2} - g_n}$ and $\alpha_2^{g_{n+2}}$ have the same Conley-Zehnder index. If, in addition, $J$ is simply generic and $c_1$ is sufficiently close to 1 then there are no (regular) $J$-holomorphic cylinders from $\alpha_1^{g_{n+2} - g_n}$ to $\alpha_2^{g_{n+2}}$.

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1The curves of a holomorphic building in $X^2_1$ have compactifications and these glue together to yield a continuous map to $X$. The building is said to be cylindrical (or planar) if the domain of this continuous map is a cylinder (plane).
Remark 1. The only even $g_n$’s are those of the form $g_{2+3k}$.

Remark 2. The numbers $g_{n+2} - g_n$ and $g_{n+2}$ are coprime, so any pseudo-holomorphic cylinder from $\alpha_1^{g_{n+2} - g_n}$ to $\alpha_2^{g_{n+2}}$ must be somewhere injective.

Our second result runs in the other direction. It establishes the generic existence of regular cylinders, as in Question 1, for certain values of $k$ and certain pairs of ellipsoids.

**Theorem 3.** Suppose that $E_1 = E(a_1, b_1) \subset E_2 = E(a_2, b_2)$ is a nested pair of irrational ellipsoids and denote their faster simple closed Reeb orbits by $\alpha_1$ and $\alpha_2$, respectively. If $k \in \mathbb{N}$ satisfies

$$k < \frac{b_1}{a_1}$$

and the unique closed Reeb orbit of $\lambda_2 = \lambda(a_2, b_2)$ with Conley-Zehnder index equal to $CZ(\alpha_1^k)$ is an iterate, $\alpha_2^\ell$, of the orbit $\alpha_2$, then for a generic choice of admissible almost complex structure $J$ on $(X_2^2, \omega)$ there exists a regular $J$-holomorphic cylinder $u$ from $\alpha_1^{g_{n+2} - g_n}$ to $\alpha_2^{g_{n+2}}$.

**Remark 3.** Condition (4) implies that $CZ(\alpha_1^k) = 2k + 1$ and $CZ(\alpha_1^{g_{n+2}}) = CZ(\alpha_2^\ell)$ if and only if

$$k = \ell + \left\lfloor \frac{\ell a_2}{b_2} \right\rfloor.$$  

1.3. **Structure.** The proof of Theorem 2 is presented in Section 2 and the proof of Theorem 3 given Theorem 1 is presented in Section 3. In Section 4 we present an alternative proof of Theorem 1 under an additional assumption, which might be of some independent interest. This assumption is weaker than the hypotheses assumed in Theorem 3 so altogether we will give a self-contained proof of Theorem 3.

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2. **Proof of Theorem 2**

It follows from the index formula (1) and the recursion relation (3) that for all sufficiently small $\epsilon > 0$ we have

$$CZ(\alpha_1^{g_{n+2} - g_n}) = 2(3g_{n+1}) - 1 = 2(g_{n+2} + g_n) - 1 = CZ(\alpha_2^{g_{n+2}}).$$

To prove the main assertion of Theorem 2 we will argue by contradiction.

**Assumption 1.** For any $\epsilon_0, \delta_0 \in (0, 1)$ there is an irrational $\epsilon \in (0, \epsilon_0)$, a $c_1 \in (1 - \delta_0, 1)$, an admissible, simply generic almost complex structure $J_1^\epsilon$ on $(X_1^2, \omega)$ and a regular $J_1^\epsilon$-holomorphic cylinder $u$ from $\alpha_1^{g_{n+2} - g_n}$ to $\alpha_2^{g_{n+2}}$.

Consider the number

$$K(n) = \inf \left\{ \mu: E \left(1, \frac{g_{n+2} - g_n}{2g_n} \right) \xrightarrow{c_{\epsilon_0}} \mu E \left(\frac{g_{n+2}}{g_{n+1}}, \frac{g_{n+2}}{g_{n+1}} \right) \right\}$$

where we have adopted, from [MS], the notation $A \xrightarrow{c} B$ meaning $A$ embeds symplectically into $B$. In terms of the embedding function $c(\cdot)$ from [MS] we have

$$K(n) = \left( \frac{g_{n+1}}{g_{n+2}} \right) c \left( \frac{g_{n+2} - g_n}{2g_n} \right).$$
The sequence \( \frac{g_{n+2}}{g_n} \) converges monotonically to \( \tau^4 \) where \( \tau = \frac{1+\sqrt{5}}{2} \). Hence for all \( n \geq 1 \) we have

\[
\frac{g_{n+2} - g_n}{2g_n} \in \left[ 2, \frac{\tau^4 - 1}{2} \right] \approx [2, 2.927].
\]

Theorem 1.1.2. of [MS] then implies that

\[
K(n) = \frac{2g_{n+1}}{g_{n+2}}.
\]

With this, the following result yields the contradiction which implies Theorem 2.

Proposition 1. If Assumption 1 holds then

\[
K(n) \geq \frac{g_{n+2} - g_n}{g_{n+2}}.
\]

In particular, it follows from (3) that \( g_{n+2} - g_n > 2g_{n+1} \) for all \( n \geq 2 \).

2.1. Proof of Proposition 1. Suppose we are given a symplectic embedding

\[
\Phi: E \left( 1, \frac{g_{n+2} - g_n}{2g_n} \right) \hookrightarrow \mu E \left( \frac{g_{n+2}}{g_{n+1}}, \frac{g_{n+2}}{g_{n+1}} \right).
\]

Then for all positive constants \( c_1 < 1 \) and \( c_3 > 1 \) and all sufficiently small \( \epsilon > 0 \) we get a symplectic embedding \( \Phi_{c_1,c_3,\epsilon} \) of

\[
E_1 = c_1 E \left( 1, \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right)
\]

into the interior of \( \mu E_3 \) where

\[
E_3 = c_3 E \left( \frac{g_{n+2}}{g_{n+1}}, \frac{g_{n+2}}{g_{n+1}} + \epsilon \right).
\]

Let \( \alpha_1 \) be the faster simple closed Reeb orbits on \( E_1 \) and let \( \beta_{3,\mu} \) be the slower simple closed Reeb orbits on \( \mu E_3 \). It suffices for us to prove the following conditional existence theorem.

Proposition 2. If Assumption 1 holds, then there is an admissible almost complex structure \( J \) on the symplectic completion of \( \mu E_3 \setminus \Phi_{c_1,c_3,\epsilon}(E_1) \) and a \( J \)-holomorphic curve of genus zero with exactly one negative puncture and exactly \( g_{n+1} \) positive punctures. The negative puncture is asymptotic to \( \alpha_1^{g_{n+2}-g_n} \) and each of the positive punctures is asymptotic to \( \beta_{3,\mu} \).

The fact that the curve of Proposition 2 has positive symplectic area implies that

\[
\mu g_{n+1} c_3 \left( \frac{g_{n+2}}{g_{n+1}} + \epsilon \right) > c_1 (g_{n+2} - g_n).
\]

Taking the limits \( c_1 \nearrow 1, c_3 \searrow 1, \epsilon \searrow 0 \) yields the desired inequality, (7).

We now prove Proposition 2 in two steps.

Step 1. (A seed curve) It follows from [MS] that there is a symplectic embedding \( \Psi \) of

\[
E_2 = E \left( 1, \frac{g_{n+2}}{g_n} + \epsilon \right),
\]

into \( \tilde{E}_3 \). Let \( X_2^3 \) be the symplectic completion of \( E_3 \setminus \Psi(\tilde{E}_2) \). Proposition 5 of [CGH] asserts that if \( c_3 \) is close enough to 1 and \( \epsilon \) is small enough then for any admissible almost complex
structure $J_2^3$ on $X_2^3$, there exists a regular $J_2^3$-holomorphic curve $v$ of genus zero with $g_{n+1}$ positive ends each asymptotic to $\beta_3$ and a single negative end asymptotic to $\alpha_2^{g_{n+2}}$.

Let $J_1^3$ be the simply generic, admissible almost complex structure and $u$ the regular $J_1^3$-holomorphic cylinder from $\alpha_1^{g_{n+2} - g_n}$ to $\alpha_2^{g_{n+2}}$ that are promised by Assumption 1. The map $\Psi$ above yields a symplectomorphism $\hat{\Psi}$ from $(X_1^3, \omega)$ to $(\hat{X}_1^3, \hat{\omega})$, the symplectic completion of $\Psi(E_2) \setminus \Psi(E_1)$. Setting $\hat{J}_1^3 = \hat{\Psi}_* J_1^3$, it follows that $\hat{u} = \hat{\Psi} \circ u$ is a $\hat{J}_1^3$-holomorphic cylinder in $\hat{X}_1^3$. Choose $\hat{J}_2^3$ so that on the negative end of $X_2^3$ it eventually matches the cylindrical almost complex structure that $\hat{J}_1^3$ eventually becomes on its positive end. For this choice of $\hat{J}_2^3$ let $v$ be the corresponding regular curve from $[\text{CGH}]$. Gluing this regular curve $v$ to the regular cylinder $\hat{u}$ we arrive at the following existence result.

**Proposition 3.** If Assumption 1 holds, then for all sufficiently small $\epsilon > 0$ and all $c_3 < 1$ sufficiently close to 1 there is an admissible almost complex structure $J_1^3$ on the symplectic completion of $E_3 \setminus \Psi(E_1)$, and a regular $J_1^3$-holomorphic curve $w$ of index zero with $g_{n+1}$ positive ends each asymptotic to $\beta_3$ and a single negative end asymptotic to $\alpha_1^{g_{n+2} - g_n}$.

In fact it remains for us to verify that the curve $w$ of Proposition 3 has index zero for all sufficiently small $\epsilon > 0$. We first recall a standard index formula which we will use several times throughout the paper.

**Aside: Index formula for curves in exact cobordisms between ellipsoids.** For two irrational ellipsoids $E_1$ and $E_2$ let $(X, \omega)$ be the completion of topologically trivial and exact symplectic cobordism from the contact manifold $\partial E_1$ to the contact manifold $\partial E_2$. Let $J$ be an admissible almost complex structure on $(X, \omega)$ and suppose that $u$ is a $J$-holomorphic curve of genus zero such that $u$ has

- $m^+$ positive punctures asymptotic to $\alpha_2$ with multiplicities $r_1^+, \ldots, r_m^+$,
- $n^+$ positive punctures asymptotic to $\beta_2$ with multiplicities $s_1^+, \ldots, s_n^{+}$,
- $m^-$ negative punctures asymptotic to $\alpha_1$ with multiplicities $r_1^-, \ldots, r_m^-$,
- and $n^-$ negative punctures asymptotic to $\beta_1$ with multiplicities $s_1^-, \ldots, s_n^-$.

The index of $u$ is then given by

\begin{equation}
\text{index}(u) = m^+ + n^+ + m^- + n^- - 2 + \sum_{1}^{m^+} \text{CZ}(\alpha_2^{r_i^+}) + \sum_{1}^{n^+} \text{CZ}(\beta_2^{s_i^+}) - \sum_{1}^{m^-} \text{CZ}(\alpha_1^{r_i^-}) - \sum_{1}^{n^-} \text{CZ}(\beta_1^{s_i^-})
\end{equation}

\begin{equation}
= 2 \left[ m^+ + n^+ - 1 + \sum_{1}^{m^+} \left( r_i^+ + \left\lfloor \frac{r_i^+ a_2}{b_2} \right\rfloor \right) + \sum_{1}^{n^+} \left( s_i^+ + \left\lfloor \frac{s_i^+ b_2}{a_2} \right\rfloor \right) \right] - \sum_{1}^{m^-} \left( r_i^+ + \left\lfloor \frac{r_i^- a_1}{b_1} \right\rfloor \right) - \sum_{1}^{n^-} \left( s_i^- + \left\lfloor \frac{s_i^- b_1}{a_1} \right\rfloor \right).\end{equation}
Applying formula (8) to a curve $u$ in the symplectic completion of $E_3 \setminus \Psi(\tilde{E}_1)$ we get

$$\text{index}(u) = 2 \left[ n^+ - 1 + 2 \sum_{i=1}^{m^+} r_i^+ + 2 \sum_{i=1}^{n^+} s_i^+ - \sum_{i=1}^{m^-} \left( r_i^- + \left| \frac{r_i^- - 2g_n}{2g_n + \epsilon} \right| \right) 
- \sum_{i=1}^{n^-} \left( s_i^- + \left| \frac{g_{n+2} - g_n}{2g_n + \epsilon} \right| \right) \right].$$

For the curve $w$ from Proposition 3 we have $m^+ = 0 = n^-$, $n^+ = g_{n+1}$ with $s_i^+ = 1$, and $m^- = 1$ with $r_i^- = g_{n+2} - g_n$. Hence, for all sufficiently small $\epsilon > 0$ we have

$$\text{index}(w) = 2 \left[ g_{n+1} - 1 + 2g_{n+1} - (g_{n+2} - g_n + 2g_n - 1) \right] = 0,$$

as claimed.

**Step 2. (Deformation)** We now deduce Proposition 2 from the existence of the regular seed curve $w$ by deforming the data defining the it. This particular form of deformation argument goes back to [HK]. It has since been refined and extended to different settings in [CGH] and [CGHM].

As in Lemma 3.1 of [HK], the two symplectic embeddings

$$\Psi: E_1 \hookrightarrow \tilde{E}_3$$

and

$$\Phi_{c_1, c_2, \epsilon}: E_1 \hookrightarrow \mu \tilde{E}_3$$

can be connected by a smooth family of symplectic embeddings

$$\Phi_t: E_1 \hookrightarrow \mu(t) \tilde{E}_3$$

for $t \in [0, 1]$ where $\mu(t)$ is a smooth function which equals 1 near $t = 0$ and equals $\mu$ near $t = 1$. We may assume also that $E_1 \subset \mu(t) \tilde{E}_3$ for all $t$. Let $(X_t, \omega_t)$ be the symplectic completion of $\mu(t)E_3 \setminus \tilde{E}_1$. Choose a smooth family $J_t$ of almost complex structures such that for each $t$, $J_t$ is admissible and simply generic on $(X_t, \omega_t)$ and $J_0$ is equal to the almost complex structure $J_1^3$ from Step 1. Let $\mathcal{K}_t$ be the moduli space of somewhere injective $J_t$-holomorphic curves of genus zero with exactly one negative puncture and exactly $g_{n+1}$ positive punctures such that the negative puncture is asymptotic to $\alpha_1^{g_{n+2}-g_n}$ and each of the positive punctures is asymptotic to $\beta_{3, \mu(t)}$ the slower simple closed Reeb orbit of $\mu(t)E_3$. Let

$$\mathcal{K} = \{ (t, [u]) \mid t \in [0, 1], [u] \in \mathcal{K}_t \}.$$ 

Proposition 2 clearly follows from the following result.

**Proposition 4.** The oriented cobordism class of $\mathcal{K}_0$ is nontrivial and for a generic choice of the family $J_t$ the set $\mathcal{K}$ is a compact oriented cobordism from $\mathcal{K}_0$ to $\mathcal{K}_1$.

The first assertion of Proposition 4 follows from [We] (see also Proposition 10 of [CGH] or Proposition 3.5 of [HK]). The transversality assertion of Proposition 4 follows from standard arguments. It remains to prove that $\mathcal{K}$ is compact.

Let $F$ be a holomorphic building in the compactification of $\mathcal{K}$ corresponding to $t = t_\infty$. Each curve of $F$ maps to $X_{t_\infty}$ or the symplectization of $\partial E_1$ or the symplectization of $\partial(\mu(t_\infty)E_3)$. In what follows we will denote the symplectizations of $\partial E_1$ by $Y_1$ and, for simplicity, we will identify the symplectization of $\partial(\mu(t_\infty)E_3)$ with that of $\partial E_3$ which we will denote by $Y_3$. 

To analyze $F$ we group its curves into *components*, as in [CGH], using the following two rules (see Step 1 of the proof of Theorem 19 in [CGH]).

- Any pair of curves in $F$ which both map to $Y_1$ or to $Y_3$ and have a matching end belong to the same component.
- Any curve of $F$ without negative ends belongs to the same component as the curves which match its positive ends.

As the name suggests, each component is a connected subbuilding of $F$. A component can have one of three types depending on the type of curves it is required to have and the nature of its ends. The first type of component is required to have a curve that maps to $Y_3$ and its ends are realized by such curves. More precisely, a component of the first type must have exactly one negative end (asymptotic to some iterate of $\alpha_3$ or $\beta_3$) and its positive ends are all asymptotic to $\beta_3$. A component of the second type must contain a curve which maps to $X_{t,\infty}$ and again its ends are realized by such curves. Components of the second type have a single negative end. There is precisely one component of a third type. It consists entirely of (possibly trivial) curves mapping to $Y_1$ and has a single negative end asymptotic to $\alpha_1^{g_{n+2}-g_n}$.

The index of a component $C$ is defined to be the sum of the indices of the curves that comprise it. By design, the index of a component $C_1$ of type one can be computed by applying the standard index formula (8) to $C_1$ as if it was a single curve mapping to $Y_3$. The analogous statements hold for the other two types of components. With this in mind, the following result then follows from Lemma 16 of [CGH].

**Lemma 1.** Suppose that $C_1$ is a component of $F$ of the first type. If the negative end of $C_1$ is asymptotic to $\beta_3^s$ then we have

$$\text{index}(C_1) \geq 2(s^- - 1)$$

with equality if and only if the curves of $C_1$ all map to $Y_3$ and are covers of the trivial cylinder over $\beta_3$. Otherwise we have

$$\text{index}(C_1) \geq 2.$$

The next result is similar in spirit to Lemma 18 of [CGH]. Since its proof differs slightly we include it for the sake of completeness.

**Lemma 2.** Suppose that $C_3$ is a component of $F$ of the third type. If $\epsilon$ is sufficiently small then the index of $C_3$ is nonnegative and equals 0 if and only if $C_3$ consists only of trivial cylinders over $\alpha_1^{g_{n+2}-g_n}$.

**Proof.** It suffices to assume that $C_3$ consists of a single pseudo-holomorphic curve $u$ in $Y_1$ which has exactly one negative end and that this end is asymptotic to $\alpha_1^{g_{n+2}-g_n}$. By (8) we have

$$\text{index}(u) = 2 \left[ m^+ + n^+ - 1 + \sum_{1}^{m^+} \left( r^+_i + \left[ r^+_i + \left( \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right)^{-1} \right] \right) \right.$$

$$\left. + \sum_{1}^{n^+} \left( s^+_i + \left[ s^+_i + \left( \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right) \right] \right) - \sum_{1}^{m^-} \left( g_{n+2} - g_n + \left[ \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right] \right) \right].$$
Hence for all sufficiently small $\epsilon > 0$ we have

\[(9)\]
\[
\text{index}(u) \geq 2 \left[ n^+ + \sum_{i=1}^{m^+} \left( r_i^+ + r_i^- \left( \frac{2g_n}{g_{n+2} - g_n} \right) \right) + \sum_{i=1}^{n^+} \left( s_i^+ + s_i^- \left( \frac{g_{n+2} - g_n}{2g_n} \right) \right) - g_{n+2} - g_n \right]
\]

with equality if and only if

\[
\left[ r_i^+ \left( \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right) \right]^{-1} = r_i^+ \left( \frac{2g_n}{g_{n+2} - g_n} \right) - 1
\]

and

\[
\left[ s_i^+ \left( \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right) \right] = s_i^+ \left( \frac{g_{n+2} - g_n}{2g_n} \right)
\]

for all $i$. In particular, equality holds for all small $\epsilon$ if and only if the numbers $r_i^+ \left( \frac{2g_n}{g_{n+2} - g_n} \right)$ are all integers.

Integrating $d\lambda_1$ over $u$ we get

\[
\sum_{i=1}^{m^+} r_i^+ + \sum_{i=1}^{n^+} \left( s_i^+ \left( \frac{g_{n+2} - g_n}{2g_n} + \epsilon \right) \right) \geq g_{n+2} - g_n.
\]

For $\epsilon$ sufficiently small we may therefore assume that

\[
\sum_{i=1}^{m^+} r_i^+ + \sum_{i=1}^{n^+} \left( s_i^+ \left( \frac{g_{n+2} - g_n}{2g_n} \right) \right) \geq g_{n+2} - g_n.
\]

and hence

\[
\sum_{i=1}^{m^+} r_i^+ \left( \frac{2g_n}{g_{n+2} - g_n} \right) + \sum_{i=1}^{n^+} s_i^+ \geq 2g_n.
\]

Together with (9), these inequalities imply that $\text{index}(u) \geq 0$ with equality only if $n^+ = 0$ and $\sum_{i=1}^{m^+} r_i^+ = g_{n+2} - g_n$.

It remains for us to prove that if $\text{index}(u) = 0$ then $m^+ = 1$ and $r_1^+ = g_{n+2} - g_n$. Since the equality $\text{index}(u) = 0$ implies that $\sum_{i=1}^{m^+} r_i^+ = g_{n+2} - g_n$ is suffices to prove that $r_i^+ \geq g_{n+2} - g_n$ for all $i$. As described above, if equality holds in (9) then for each $i$ we have

\[
r_i^+ = k_i \frac{g_{n+2} - g_n}{2g_n}
\]

for some integer $k_i \in \mathbb{N}$. If we assume that $r_i^+ < g_{n+2} - g_n$, then $g_{n+2} - g_n$ and $2g_n$ must have a common prime factor. However, this implies that $3g_{n+1}$ and $2g_n$ have a common factor. Since $g_{n+1}$ and $g_n$ are coprime and no $g_n$ is divisible by 3, this would contradict our assumption that $g_{n+1}$ is odd.

Let $C_2$ be a component of $F$ of the second type. It consists of curves $u_1, \ldots, u_p$ that map to $X_{t_\infty}$ and subcomponents $W_1, \ldots, W_Q$ which consist of curves mapping to $Y_1$. Suppose that each $u_p$ is a degree $k_p$ cover of a somewhere injective curve $\tilde{u}_p$. The following result is proved as Lemma 20 in [CGH].
Lemma 3. Let $C_2$ be a component of $F$ of the second type. Then

\begin{equation}
\text{index}(C_2) \geq -2 \sum_{i,p} (s_{i,p}^+ - 1),
\end{equation}

where the $s_{i,p}^+$ are the covering numbers of positive ends of $u_p$ that are asymptotic to $\beta_3$. Moreover, if equality holds then $C_2$ has no subcomponents of the form $W_\alpha$ and the covering numbers $s_{i,p}^+$ of the positive ends of $\tilde{u}_p$ asymptotic to $\beta_3$, are all equal to one.

Remark 4. The proof makes crucial use of the assumption that $J_{t_\infty}$ is simply generic.

The building $F$ is connected and the sum of the indices of its constituent curves must be zero. From these facts and Lemma 1, Lemma 2 and Lemma 3 we immediately deduce the following.

- The negative end of each component of the first type is asymptotic to an iterate of $\beta_3$ and the curves of the component all cover the trivial cylinder over $\beta_3$.
- The component of the third type consists entirely of trivial cylinders over $\alpha_1^{g_n+2-g_n}$.
- There is exactly one component of the second type and it consists of a single curve $u_1$ mapping to $X_{t_\infty}$. Moreover, $u_1$ covers a somewhere injective curve $\tilde{u}_1$ whose positive ends are all asymptotic to $\beta_3$.

It suffices to prove that $u_1$ is itself simply connected. For, as $J_{t_\infty}$ is assumed to be simply generic, this would imply that the index of $u_1$ is nonnegative. Thus, all the positive ends of $u_1$ would be asymptotic to $\beta_3$ and each component of the first type would consist of trivial cylinders. In other words, $F$ would belong to $\mathcal{K}_{t_\infty}$.

Arguing by contradiction, let us assume that $u_1$ covers $\tilde{u}_1$ with degree $k_1 > 1$. Since the positive ends of $u_1$ cover $\tilde{u}_1$ with degree $k_1$ times, and the positive ends of $\tilde{u}_1$ are all asymptotic to $\beta_3$ there must be $g_{n+1}/k_1$ positive ends of $\tilde{u}_1$. Since $u_1$ has a single negative end and this is asymptotic to $\alpha_1^{g_n+2-g_n}$, it follows that $\tilde{u}_1$ has a single negative end asymptotic to $\alpha_1^{g_n+2-g_n}/k_1$. Hence, $k_1 > 1$ must be a common factor of both $g_{n+1}$ and $g_{n+2} - g_n$. By (3), it follows that $k_1$ must also be a common factor of both $g_{n+2} + g_n$ and $g_{n+2} - g_n$ and hence of $2g_{n+2}$ and $2g_n$. Since $g_{n+2}$ and $g_n$ are coprime we must have $k_1 = 2$. However, this contradicts the assumption that $g_{n+1}$ is odd.

3. Proof of Theorem 3

3.1. The Plan. Recall that in the setting of Theorem 3 we have nested ellipsoids $E_1 = E(a_1, b_1)$ and $E_2 = E(a_2, b_2)$, the symplectic completion $(X_1^2, \omega)$ of $E_2 \setminus \tilde{E}_1$, and positive integers $k$ and $\ell$ asatisfying (4) and (5). Our proof of the theorem is divided into two propositions.

Proposition 5. Let $J$ be an admissible almost complex structure on $(X_1^2, \omega)$ which is simply generic. Then there is a $J$-holomorphic cylinder $u_k$ from $\alpha_1^k$ to $\alpha_\ell^\ell$.

Given Proposition 5 it remains to prove that we can also, generically, ensure that $u_k$ is regular. To achieve this we will use two results of Wendl from [We]. The first of these is the following implication of Corollary 3.17 in [We].

Theorem 4. ([We]) For generic admissible $J$ on $(X_1^2, \omega)$, every somewhere injective $J$-holomorphic curve of index zero is immersed.

With this we prove the following.
Proposition 6. Suppose that $J$ is an admissible almost complex structure on $(X^2, \omega)$ which is simply generic and is also generic in the sense of Theorem 4. Then the $J$-holomorphic cylinder $u_k$ in the statement of Proposition 5 is regular.

Theorem 3 follows easily from Propositions 5 and 6.

3.2. Proof of Proposition 5. Given an ellipsoid $E = E(a, b)$, we first establish some simple properties of pseudoholomorphic curves mapping to the symplectization of its boundary,

$$(Y, d(e^r \lambda)) = (\mathbb{R} \times \partial E(a, b), d(e^r \lambda(a, b))).$$

Recall that an almost complex structure $J$ on $Y$ is said to be $\lambda$-cylindrical if the restriction of $J$ to $\ker \lambda$ is compatible with $d\lambda$ and $J(\partial_r)$ is equal to the Reeb vector field of $\lambda$.

Lemma 4. Suppose that $J$ is a $\lambda$-cylindrical almost complex structure on $Y$ and that $v$ is a $J$-holomorphic curve with exactly one positive puncture. The index of $v$ is even and nonnegative, and is at least two unless $v$ covers a trivial cylinder. Moreover, if the positive puncture is asymptotic to the closed Reeb orbit $\eta$ of $\lambda$, then for any negative puncture of $v$ asymptotic to some $\gamma$, we have

$$CZ(\gamma) \leq CZ(\eta).$$

Proof. The positive puncture of $v$ is asymptotic to an iterate of either $\alpha$ or $\beta$. The proofs in both cases are essentially the same. So, we assume that the positive puncture of $v$ is asymptotic to $\eta = \alpha^{r^+}$ for some $r^+$, and leave the remaining case to the reader.

Suppose that the curve $v$ has $m^-$ negative punctures asymptotic to $\alpha$ with multiplicities $r^-_1, \ldots, r^-_{m^-}$, and $n^-$ negative punctures asymptotic to $\beta$ with multiplicities $s^-_1, \ldots, s^-_{n^-}$. Denoting the domain of $v$ by $D$, we have

$$0 \leq \int_D v^*(d\lambda) = r^+ a - \sum_{i=1}^{m^-} r^-_i a - \sum_{i=1}^{n^-} s^-_i b. \tag{11}$$

From this we see that $r^+ a$, the period of $\eta$, is at least as large at the period of any closed Reeb orbit corresponding to a negative puncture of $v$. The second assertion above then follows from the fact that for irrational ellipsoids the orderings of closed Reeb orbits by period and by Conley-Zehnder index coincide.

To prove the first assertion we note that (8) implies that the index of $v$ is given by

$$\text{index}(v) = 2 \left( r^+ + \left\lfloor \frac{r^+ a}{b} \right\rfloor - \sum_{i=1}^{m^-} \left\lfloor r^-_i a \right\rfloor - \sum_{i=1}^{n^-} \left\lfloor s^-_i b \right\rfloor - \sum_{i=1}^{n^-} \left\lfloor s^-_i \left( \frac{r^-_i a}{b} \right) \right\rfloor \right). \tag{12}$$

From (11) we have

$$r^+ - \sum_{i=1}^{m^-} r^-_i - \sum_{i=1}^{n^-} \left\lfloor \frac{s^-_i b}{a} \right\rfloor \geq 0 \tag{13}$$

and

$$\frac{r^+ a}{b} - \sum_{i=1}^{m^-} \left\lfloor \frac{r^-_i a}{b} \right\rfloor - \sum_{i=1}^{n^-} s^-_i \geq 0.$$
which, in turn, implies that

\[
\left\lfloor \frac{r^+ a}{b} \right\rfloor - \sum_1^{m^-} \left\lfloor \frac{r^-_i a}{b} \right\rfloor - \sum_1^{n^-} s^-_i \geq 0.
\]

Together with (12), inequalities (13) and (14) imply that index(v) is even and nonnegative. Finally we note that the index of \( v \) is zero only if \( \int_{\mathcal{D}} v^*(d\lambda) = 0 \) in which case \( v \) must cover the trivial cylinder over \( \alpha \).

Now, by Theorem 1 there is a cylindrical \( J \)-holomorphic building \( \mathbf{H} = H_k \) in \( X^2_1 \) of index zero which runs from \( \alpha_{k1} \) to \( \alpha_{k2} \). The curves of \( \mathbf{H} \) map to either \( Y_1, X^2_1 \) or \( Y_2 \) where \( Y_j \) is the symplectization of \( \partial E_j \). Below we prove that the curves of \( \mathbf{H} \) that map to these symplectizations must cover trivial cylinders. Hence \( \mathbf{H} \) has a unique curve mapping to \( X^2_1 \), the desired cylinder \( u_k \).

**Lemma 5.** Each curve of \( \mathbf{H} \) has exactly one positive puncture.

**Proof.** Curves with no positive punctures are precluded by the maximal principle in the symplectizations \( Y_1 \) and \( Y_2 \), and by Stokes’ Theorem in \( (X^2_1, \omega) \). Curves with more than one positive punctures are precluded by the previous fact together with the fact that the building \( \mathbf{H} \) is cylindrical. \( \square \)

**Lemma 6.** Each curve of \( \mathbf{H} \) with image in \( X^2_1 \) has a nonnegative index.

**Proof.** Let \( u \) be a curve of \( \mathbf{H} \) with image in \( X^2_1 \). By Lemma 5, \( u \) has a single positive puncture asymptotic to a closed Reeb orbit \( \gamma \) of \( \lambda_2 \). There are two cases to consider, either \( \gamma \) is an iteration of \( \alpha_2 \) or it is an iteration of \( \beta_2 \). Applying the second assertion of Lemma 4 to the curves of \( \mathbf{H} \) with level above that of \( u \), we see that in either case we have

\[
\text{CZ}(\gamma) \leq \text{CZ}(\alpha^\ell_2).
\]

**Case 1.** \( \gamma \) is an iterate of \( \alpha_2 \).

The curve \( u \) is a (possibly trivial) multiple cover of a somewhere injective \( J \)-holomorphic curve, \( v \). Denote the degree of the covering by \( p \in \mathbb{N} \). In the present case, the positive puncture of \( v \) is asymptotic to \( \alpha^r_2 \) for some \( r \in \mathbb{N} \) and, by (15), we have

\[
pr \leq \ell.
\]

Since \( J \) is simply generic, by assumption, the index of curve \( v \) is nonnegative. If \( v \) has \( m^- \) negative punctures asymptotic to \( \alpha_1 \) with multiplicities \( r^-_1, \ldots, r^-_{m^-} \), and \( n^- \) negative punctures asymptotic to \( \beta_1 \) with multiplicities \( s^-_1, \ldots, s^-_{n^-} \), this implies that

\[
r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor - \sum_1^{m^-} \left( r^-_i + \left\lfloor \frac{r^-_i a_1}{b_1} \right\rfloor \right) - \sum_1^{n^-} \left( s^-_i + \left\lfloor \frac{s^-_i b_1}{a_1} \right\rfloor \right) \geq 0.
\]

Invoking hypotheses (4) and (5) together with (15) we also have

\[
r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor \leq \ell + \left\lfloor \frac{la_2}{b_2} \right\rfloor = k < \frac{b_1}{a_1}.
\]
It then follows from (17) and (18), that \( v \) (and hence \( u \)) can have no negative punctures asymptotic to multiples of \( \beta_1 \). Thus, (17) simplifies to

\[
(19) \quad r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor - \sum_{i=1}^{m^-} \left( r_i^- + \left\lfloor \frac{r_i^- a_1}{b_1} \right\rfloor \right) \geq 0.
\]

Suppose that the sum \( \sum_{i=1}^{m^-} r_i^- \) was greater than \( \frac{k}{p} \). Then inequality (19) would imply

\[
r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor > \frac{k}{p}
\]

which, together with (16), would yield

\[
\ell + \left\lfloor \frac{\ell a_2}{b_2} \right\rfloor \geq pr + \left\lfloor \frac{pra_2}{b_2} \right\rfloor \geq p \left( r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor \right) > k.
\]

This would contradict the index equality (5). Hence, we must instead have

\[
\sum_{i=1}^{m^-} r_i^- \leq \frac{k}{p}.
\]

This implies that

\[
(20) \quad pr_i^- < \frac{b_1}{a_1} \quad \text{for all} \quad i = 1, \ldots, m^-
\]

and so (19) simplifies to

\[
(21) \quad r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor - \sum_{i=1}^{m^-} r_i^- \geq 0.
\]

Invoking inequalities (20) and (21), in succession we then get

\[
\text{index}(u) = 2 \left( pr + \left\lfloor \frac{pra_2}{b_2} \right\rfloor - \sum_{i=1}^{m^-} \left( pr_i^- + \left\lfloor \frac{pr_i^- a_1}{b_1} \right\rfloor \right) \right)
\]

\[
\geq 2p \left( r + \left\lfloor \frac{ra_2}{b_2} \right\rfloor - \sum_{i=1}^{m^-} r_i^- \right)
\]

\[
\geq 0,
\]

as desired.

**Case 2.** \( \gamma \) is an iterate of \( \beta_2 \).

Define \( p \) and \( v \) as in the previous case. The positive puncture of \( v \) is now asymptotic to \( \beta_2^r \) for some \( r \in \mathbb{N} \). We label the negative punctures of \( v \) as before. The fact that the index of \( v \) is nonnegative now implies that

\[
r + \left\lfloor \frac{rb_2}{a_2} \right\rfloor - \sum_{i=1}^{m^-} \left( r_i^- + \left\lfloor \frac{r_i^- a_1}{b_1} \right\rfloor \right) - \sum_{i=1}^{n^-} \left( s_i^- + \left\lfloor \frac{s_i^- b_1}{a_1} \right\rfloor \right) \geq 0.
\]
the second assertion of Lemma 4 implies that \( CZ(\beta_{pr}^2) \leq CZ(\alpha_{pr}^2) \) and hence

\[
pr + \left\lfloor \frac{pr b}{a} \right\rfloor \leq \ell + \left\lfloor \frac{\ell a}{b} \right\rfloor.
\]

As in the previous case, this together with (4) and (5) precludes \( v \) from having negative punctures asymptotic to multiples of \( \beta_1 \). So, we now have

(22)

\[ r + \left\lfloor \frac{rb}{a} \right\rfloor - \sum_{i=1}^{m^-} \left( r_i^- + \left\lfloor \frac{r_i^- a}{b} \right\rfloor \right) \geq 0. \]

If the inequality \( \sum_{i=1}^{m^-} r_i^- > \frac{k}{p} \) held, then (22) would yield

\[ r + \left\lfloor \frac{rb}{a} \right\rfloor > \frac{k}{p}, \]

and imply that

\[ \ell + \left\lfloor \frac{\ell a}{b} \right\rfloor \geq pr + \left\lfloor \frac{pr b}{a} \right\rfloor \geq p \left( r + \left\lfloor \frac{rb}{a} \right\rfloor \right) > k. \]

Since this contradicts (5) we must have \( \sum_{i=1}^{m^-} r_i^- \leq \frac{k}{p} \) instead. From this it follows that

\[ pr_i^- < \frac{b_i}{a_i} \text{ for all } i = 1, \ldots, m^- \]

and

\[ r + \left\lfloor \frac{rb}{a} \right\rfloor - \sum_{i=1}^{m^-} r_i^- \geq 0. \]

Thus,

\[
\text{index}(u) = 2 \left( pr + \left\lfloor \frac{pr b}{a} \right\rfloor - \sum_{i=1}^{m^-} \left( pr_i^- + \left\lfloor \frac{pr_i^- a}{b} \right\rfloor \right) \right)
\geq 2p \left( r + \left\lfloor \frac{rb}{a} \right\rfloor - \sum_{i=1}^{m^-} r_i^- \right)
\geq 0,
\]

again as desired. \( \square \)

At this point we can complete the proof of Proposition 5. The total index of all the curves of \( \mathbf{H} \) is zero. It therefore follows from Lemma 4, Lemma 5 and Lemma 6 that each curve of \( \mathbf{H} \) has index zero and those curves mapping to \( Y_1 \) or \( Y_2 \) cover trivial cylinders. The compactifications of all the curves of \( \mathbf{H} \) glue together to yield a continuous map \( \bar{\mathbf{H}} \) from the cylinder to \( E_2 \setminus \bar{E}_1 \) that connects \( \alpha_{pr}^1 \) to \( \alpha_{pr}^2 \). Since the curves mapping to \( Y_1 \) or \( Y_2 \) cover trivial cylinders, the compactifications of each of these curves map to single closed Reeb orbits. Hence, \( \mathbf{H} \) can have only one curve with image in \( X_{pr}^2 \), and this curve must have exactly one negative and one positive end asymptotic to \( \alpha_{pr}^1 \) and \( \alpha_{pr}^2 \), respectively. This is the desired \( J \)-holomorphic cylinder \( u_k \).
3.3. Proof of Proposition 6 Assume that \( J \) is simply generic and generic in the sense of Theorem 4. The curve \( u_k \) from Proposition 5 is the \( p \)-fold cover of a somewhere injective \( J \)-holomorphic cylinder, \( v \), for some \( p \) in \( \mathbb{N} \).

**Lemma 7.** The index of \( v \) is zero.

**Proof.** The curve \( v \) is a cylinder from \( \alpha_1^k \) to \( \alpha_2^\ell \) where \( rp = k \) and \( sp = \ell \). Invoking inequality (4) again we get

\[
\text{index}(u_k) - p(\text{index}(v)) = 2 \left( \ell + \left\lfloor \frac{\ell a_2}{b_2} \right\rfloor - k - \left\lfloor \frac{ka_1}{b_1} \right\rfloor \right) - 2p \left( s + \left\lfloor \frac{sa_2}{b_2} \right\rfloor - r - \left\lfloor \frac{ra_1}{b_1} \right\rfloor \right)
\]

\[
= 2 \left( \left\lfloor \frac{psa_2}{b_2} \right\rfloor - p \left\lfloor \frac{sa_2}{b_2} \right\rfloor \right)
\]

\[
\geq 0.
\]

It follows that the index of \( v \) is at most that of \( u_k \), zero. On the other hand, by our choice of \( J \), \( v \) is regular and hence has nonnegative index. Thus, the index of \( v \) must be equal to zero. \( \square \)

By Theorem 4, the curve \( v \) must be immersed. From this, it follows that the original curve \( u_k \) has no critical points since both \( u_k \) and \( v \) are cylinders. It then follows from Theorem 1 of [We], that \( u_k \) is automatically regular for any admissible \( J \) on \( X_2^2 \). This completes the proof of Proposition 6 and hence Theorem 3.

4. Another Path to Holomorphic Buildings

In this section we present an alternative proof of Theorem 1 under some additional assumptions which include those of Theorem 3. More precisely, we prove the following.

**Theorem 5.** Given ellipsoids \( E_1 \subset \hat{E}_2 \), suppose that \( k \in \mathbb{N} \) satisfies

\[\text{CZ}(\alpha_1^k) = \text{CZ}(\alpha_1^{k-1}) + 2\]

and that

\[\text{CZ}(\alpha_2^\ell) = \text{CZ}(\alpha_1^k) \text{ for some } \ell \in \mathbb{N}.\]

Then for any admissible almost complex structure \( J \) on \((X_2^2, \omega)\) there exists a cylindrical \( J \)-holomorphic building \( H \) of index 0 from \( \alpha_1^k \) to \( \alpha_2^\ell \).

The compact cobordism \( E_2 \setminus \hat{E}_1 \) can be identified with a domain in the symplectization \((Y_1 = \mathbb{R} \times \partial E_1, d(e^\tau \lambda_1))\) of the form

\[\{1 \leq \tau \leq h(z)\}\]

where \( h : \partial E_1 \to \mathbb{R}_+ \) is a (simple) smooth function. The hypersurface \( \Sigma_2 = \{\tau = h(z)\} \) admits an open neighborhood \( U_2 \subset Y_1 \) which is symplectomorphic to a neighborhood of the form \((-\delta, +\delta) \times \partial E_2, d(e^\tau \lambda_2)) \subset Y_2 \) for some \( \delta > 0 \), where \( Y_2 \) is the symplectization of \( \partial E_2 \).

With these identifications in place, the structure of our proof of Theorem 5 can be described as follows. We start by establishing the existence of a nontrivial compact class of curves in \( Y_1 \) which are pseudo-holomorphic with respect to a domain dependent family of almost complex structures. Then we split \( Y_1 \) along \( \Sigma_2 \) and obtain from our original class of curves, a pseudo-holomorphic building one of whose curves is a cylinder which maps to \( X_2^2 \) with the required asymptotics and is pseudo-holomorphic with respect to a domain dependent family of almost complex structures. This is the content of Proposition 8. Finally
we consider a limit of such cylinders for a sequence of domain dependent families of almost complex structures that converges to a domain independent family. This last limit gives the desired holomorphic building, $H$.

4.1. Almost complex structures. Fix a $\tau_2 > 0$ such that the closure of the neighborhood $U_2 \subset Y_1$ of the hypersurface $\Sigma_2$ is contained in $\{\tau \leq \tau_2\} = (-\infty, \tau_2) \times \partial E_1$. Fix a $\lambda_1$-cylindrical almost complex structure $J_1$ on $Y_1$ and a $\lambda_2$-cylindrical almost complex structure $J_2$ on $Y_2$.

**Definition 1.** Let $\mathcal{J}$ be the set of admissible almost complex structures $J$ on $Y_1$ for which there exists a compact subset $K \subset Y_1$ contained in $\{\tau \leq \tau_2\}$ such that

- $J = J_1$ away from $K$.
- $J$ is compatible with an exact symplectic structure on $Y_1$ that is equal to $d(e^{\tau} \lambda_1)$ away from $K$.

**Definition 2.** Let $\mathcal{J}_{\Sigma_2} \subset \mathcal{J}$ be the set of admissible almost complex structures $J$ on $Y_1$ for which there is a compact subset $K \subset Y_1$ that contains $U_2$ and is contained in $\{\tau \leq \tau_2\}$ such that

- $J = J_1$ away from $K$ and is equal to $J_2$ in $U_2$, identifying $U_2$ with a subset of $Y_2$ as above.
- $J$ is compatible with an exact symplectic structure on $Y_1$ that is equal to $d(e^{\tau} \lambda_1)$ away from $K$ and is equal to $d(e^{\tau} \lambda_2)$ on $U_2$.

**Definition 3.** For $N \in [0, \infty)$, let $\mathcal{J}^N_{\{0,1,\infty\}} \subset \mathcal{J}$ be the set of almost complex structures on $Y_1$ obtained from those of $\mathcal{J}_{\Sigma_2}$ by the standard stretching procedure to a length $N$.

**Definition 4.** Let $\mathcal{J}^N_{\Sigma_2}$ be the set of almost complex structures on $Y_1$ for which there exists a compact subset $K \subset Y_1$ contained in $\{\tau \leq \tau_2\}$ such that in disjoint open discs around 0, 1 and $\infty$ the complex structure $J_z$ only depends on the (local) angular coordinate.

4.2. Conventions. To simplify and clarify the arguments below we now state two conventions. The first concerns notation and the other concerns the meaning of term index.

**Convention 1.** Given a subset of domain independent almost complex structures $\mathcal{J}$ on $Y_1^N$ the symbol

$$\mathcal{J}^N_{\{0,1,\infty\}} \cap \mathcal{J}$$

will be used to denote the set of families $J_z$ in $\mathcal{J}^N_{\{0,1,\infty\}}$ such that $J_z$ belongs to $\mathcal{J}$ for each $z \in \mathbb{CP}^1$.

**Convention 2.** Since we deal with domain dependent almost complex structures we must clarify what we will mean by the index of a pseudo-holomorphic curve. Let $u$ be a pseudo-holomorphic curve in a topologically trivial and exact symplectic cobordism. In what follows, by the index of $u$, we will always mean the number given by the standard index formula recalled below, (8). This is a convenient formula because it is preserved under limits as in formulas (23) and (24). If $u$ is pseudo-holomorphic with respect to a domain independent almost complex structure, then the index of $u$ is the virtual dimension of the moduli space of holomorphic curves containing $u$, modulo biholomorphisms of the domain, as usual. If however $u$ is pseudo-holomorphic with respect to a family of domain dependent almost complex structures then the reparameterization group may have smaller dimension. In this
case, the dimension of the moduli space of holomorphic curves containing \( u \) will typically exceed the index of \( u \) by the defect in the dimension of the reparameterization group.

### 4.3. A moduli space of curves.

Fix a point \( p_1 \) on the short closed Reeb orbit \( \alpha_1 \) on \( \partial E_1 \). Given an \( N \geq 0 \) and a \( J_z \) in \( \mathcal{J}_{\{0,1,\infty\} \cap \mathcal{J}} \) let \( \mathcal{M}^N(J_z) \) be the space of maps

\[
u: CP^1 \setminus \{0, 1, \infty\} \to Y_1^N
\]

which satisfy \(^3\)

\[
du(z) \circ i = J_z \circ du(z)
\]

for all \( z \in CP^1 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\} \), and have the following additional properties:

- \((u1)\) \( u \) has a positive puncture at 0 which is asymptotic to \( \alpha_1 \).
- \((u2)\) \( u(z) \) approaches \( (+\infty, p_1) \) as \( z \) approaches 0 along the positive real axis.
- \((u2)\) \( u \) has a positive puncture at 1 which is asymptotic to \( \alpha_1^{k-1} \).
- \((u4)\) \( u \) has a negative puncture at \( \infty \) which is asymptotic to \( \alpha_1^k \).
- \((u5)\) \( u(2) \) belongs to \( \{\tau_2\} \times \partial E_1 \subset Y_1^N \).

It follows from the index formula (8) and the hypothesis of Theorem 5 that each curve \( u \) in \( \mathcal{M}^N(J_z) \) has index equal two according to Convention 2. Taking the constraints \((u2)\) and \((u5)\) into consideration, it follows that each space \( \mathcal{M}^N(J_z) \) has virtual dimension equal to zero.

### 4.4. Seed curves.

Consider a family \( J_z^{0} \in \mathcal{J}_{\{0,1,\infty\} \cap \mathcal{J}} \) whose elements are all \( \lambda_1 \)-cylindrical.

**Lemma 8.** For a generic choice of \( J_z^{0} \) the space \( \mathcal{M}^0(J_z^{0}) \) contains exactly one curve and this curve is regular.

**Proof.** By \((u1)\), \((u2)\) and \((u4)\), every curve in \( \mathcal{M}^0(J_z^{0}) \) has zero \( d\lambda_1 \)-area. Hence each curve in \( \mathcal{M}^0(J_z^{0}) \) covers the trivial cylinder over \( \alpha_1 \). In fact, this covering map is uniquely determined by the conditions \((u1)-(u5)\). The poles and zeros of the covering map are fixed by conditions \((u1)\), \((u3)\) and \((u4)\). The remaining \( C^* \) ambiguity is then resolved by conditions \((u2)\) and \((u5)\). Thus, the space \( \mathcal{M}^0(J_z^{0}) \) contains a single curve. Choosing \( J_z^{0} \) to be genuinely domain dependent, we may also ensure that this curve is regular. Indeed, there are no infinitesimal deformations tangent to the trivial cylinder, and deformations in the normal direction can be generically excluded using the domain dependence of \( J_z^{0} \).

### 4.5. Deformation.

Let \( J_z \) be a family in \( \mathcal{J}_{\{0,1,\infty\} \cap \mathcal{J}} \) such that \( \mathcal{M}^N(J_z) \) is regular. Consider a smooth path

\[
s \mapsto J_z^s \in \mathcal{J}_{\{0,1,\infty\} \cap \mathcal{J}}^{sN}
\]

connecting the family \( J_z^{0} \) from Lemma 8 to \( J_z \).

**Proposition 7.** For a generic choice of the path \( J_z^s \), the space

\[
\{(s,u) | u \in \mathcal{M}^{sN}(J_z^s)\}
\]

is a compact 1-dimensional manifold.

\(^2\)Here it is understood that \( CP^1 \setminus \{0,1,\infty\} \) is equipped with the standard complex form \( i \) inherited from \( CP^1 \).
Since the space $\mathcal{M}^0(J^*_z)$ is regular, the proof of the relevant transversality assertion follows from standard arguments. Turning then to compactness, we consider a sequence $(s_n, u_n)$ in the space $\{(s, u)| u \in \mathcal{M}^{sN}(J^*_z)\}$ such that the $s_n$ converge to some $\hat{s} > 0$ and the $u_n$ converges in the sense of $[BEHWZ]$ to a holomorphic building $F$. To prove Proposition 7 we must show that all but one of the curves of $F$ are trivial cylinders and the remaining (nontrivial) curve belongs to $\mathcal{M}^{sN}(J^*_\hat{s})$.

We begin our analysis of the curves of $F$ with two simple observations. First we mention the following result whose proof is analogous to that of Lemma 5.

**Lemma 9.** Each curve of $F$ has at least one and at most two positive punctures.

Next we note that the compactness results of $[BEHWZ]$ imply that curves of $F$ mapping to $Y^{sN}_1$ all lie in the same level of the building $F$. We will henceforth refer to this as the $\tau_2$-level since one of the curves at this level must satisfy constraint (u5). The remaining curves map to $Y_1$.

Now, away from $\{0, 1, \infty\}$ and a finite (possibly empty) set of additional punctures, $\{z_1, \ldots, z_k\}$, the curves $u_n$ converge in the $C^{sN}_{loc}$-topology to a unique curve $C$ of $F$. Eventually, we will show that $C$ is the only nontrivial curve of $F$. To this end we use $C$ to organize our analysis of the other curves. By the convergence theorem, the curves of $F$ other than $C$ can be sorted into disjoint subsets indexed by the punctures of $C$. Denote by $\eta_0, \eta_1, \eta_\infty$ and $\eta_{z_j}$ the closed Reeb orbits of $\lambda_1$ corresponding to the punctures of $C$ at $0, 1, \infty$, and $z_j$, respectively. We then define $C_0, C_1$ and $C_\infty$ to be the cylindrical holomorphic subbuildings of $F$ asymptotic to $\eta_0, \eta_1$ and $\eta_\infty$ respectively at one end, and $\alpha_1, \alpha_1^{-1}$ and $\alpha_1^k$ at the other end. Similarly $D_j$ is the collection of curves which fit together to form a planar component asymptotic to $\eta_{z_j}$.

Our definition of the space $\mathcal{J}^N_{\{0,1,\infty\}}$, in particular the prescribed nature of the domain dependence near the punctures, implies the following.

**Lemma 10.** Suppose that $C$ is a curve in $C_0, C_1$ or $C_\infty$ which maps to the $\tau_2$-level and is not part of a planar component in the complement of $C$. Then $C$ has two distinguished punctures which do not match with planar components, say at $0$ and $\infty$. Let $\Gamma$ denote the remaining punctures. The corresponding map with domain $C^s \setminus \Gamma$ is then pseudo-holomorphic with respect to a domain dependent family of almost complex structures that only depend on $\arg z$.

**Remark 5.** This subtle feature of the curves in $C_0, C_1$ or $C_\infty$ at the $\tau_2$-level will play a crucial role in the proof. The precise nature of domain dependence will allow us to quantify the difference between the indices of curves and the virtual dimensions of the moduli spaces containing them (see the proof of Lemma 15).

The following property of $C$ will be used several times.

**Lemma 11.** The curve $C$ lies at or above the $\tau_2$-level.

**Proof.** If $C$ was below the $\tau_2$-level then the point constraint (u5) would force $C$ to have a positive puncture at $2$. This, in turn, would force $F$ to include a curve with only negative punctures. Stokes' Theorem precludes such curves.

Now we begin our analysis of the collections $C_0, C_1, C_\infty$, and $D_j$. We start with the collections $D_j$ corresponding to the additional punctures of $C$. Each $D_j$ is a planar building.
with a positive puncture asymptotic to $\eta_{z_j}$. The index formula (8) (which applies equally well to buildings) yields the following.

**Lemma 12.** The sum of the indices of the curves in $\mathcal{D}_j$ is at least 2.

Next we analyze the collections $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_\infty$, in turn.

**Lemma 13.** If the puncture of $\mathcal{C}$ at 1 is positive, then the curves of $\mathcal{C}_1$ either all cover trivial cylinders or their collective index is at least two. If the puncture of $\mathcal{C}$ at 1 is negative, then the curves of $\mathcal{C}_1$ have a collective index of at least six.

**Proof.** The curves of $\mathcal{C}_1$ form a cylindrical building that connects $\eta_1$ to $\alpha_1^{-1}$. Let $\mathcal{C}_1^+$ be the subset of curves of $\mathcal{C}_1$ that lie above the level of $\mathcal{C}$. By the maximum principle, the curves of $\mathcal{C}_1^+$ comprise a connected building whose total domain is a sphere with at least two punctures. Exactly one of these punctures is positive and it is asymptotic to $\alpha_1^{-1}$. By Lemma 11 the curves of $\mathcal{C}_1^+$ are all $J_1$-holomorphic. Arguing as in Lemma 4 it then follows that the curves in $\mathcal{C}_1^+$ either all cover a trivial cylinder or their collective index is at least two.

If the puncture of $\mathcal{C}$ at 1 is positive, then and at least one of the negative punctures of $\mathcal{C}_1^+$ is asymptotic to $\eta_1$. The rest of the curves of $\mathcal{C}_1$ can then be sorted into subsets indexed by the negative punctures of the building $\mathcal{C}_1^+$ other than the one that is asymptotic to $\eta_1$. Each of these subsets is a planar building (which caps a negative puncture of $\mathcal{C}_1^+$). Arguing as in Lemma 12 it then follows that these curves also contribute at least two to the total index. This settles the first assertion of Lemma 13.

Suppose that the puncture of $\mathcal{C}$ at 1 is negative. Then the curves of $\mathcal{C}_1$ not in $\mathcal{C}_1^+$ can all be sorted into subsets indexed by the negative punctures of $\mathcal{C}_1^+$. Exactly one of these subsets forms a cylindrical subbuilding with two positive ends. The index formula (8) implies that this cylindrical subbuilding contributes at least six to the total index. The other subsets form planar buildings which, as above, contribute at least 2 to the total index. \hfill \Box

**Lemma 14.** If the puncture of $\mathcal{C}$ at 0 is positive, then the curves of $\mathcal{C}_0$ are all trivial. If the puncture of $\mathcal{C}$ at 0 is negative, then the curves of $\mathcal{C}_0$ have a collective index of at least six.

**Proof.** Let $\mathcal{C}_0^+$ be the subset of curves of $\mathcal{C}_0$ that lie above the level of $\mathcal{C}$. As the unmatched positive puncture of $\mathcal{C}_0^+$ is asymptotic to $\alpha_1$, which has minimal action, we see that $\mathcal{C}_0^+$ is just a trivial cylinder over $\alpha_1$. If the puncture of $\mathcal{C}$ at 0 is positive then there can be no other curves of $\mathcal{C}_0$ and we are done.

If the puncture of $\mathcal{C}$ at 0 is negative then the curves of $\mathcal{C}_0$ at and below the level of $\mathcal{C}$ form a cylindrical building with two positive ends connecting $\eta_1$ to $\alpha_1$. Appealing again to the index formula (8), it follows that these curves contribute at least six to the total index. \hfill \Box

**Lemma 15.** If $\mathcal{C}$ is in the $\tau_2$-level, then the curves of $\mathcal{C}_\infty$ either all cover trivial cylinders or their collective index is at least 2. If $\mathcal{C}$ is above the $\tau_2$-level, then $\mathcal{C}_\infty$ has exactly one curve, $\mathcal{C}_\infty$, that is at the $\tau_2$-level and is not part of a planar component. It has index at least $-2$ and the other curves of $\mathcal{C}_\infty$ either all cover trivial cylinders or their collective index is at least 2.

**Proof.** When $\mathcal{C}$ is in the $\tau_2$-level the maximum principle implies that all the curves of $\mathcal{C}_\infty$ must lie below the $\tau_2$-level. (Otherwise $\mathcal{C}_\infty$ would include a curve with no positive ends.)
Thus, each curve of $\mathcal{C}_\infty$ has one positive end and is pseudo-holomorphic with respect to a $\lambda_1$-cylindrical almost complex structure. The first assertion then follows as in Lemma 4.

If $\mathcal{C}$ lies above the $\tau_2$-level, then by the maximum principle $\mathcal{C}_\infty$ contains a unique curve, $\mathcal{C}_\infty$, in the $\tau_2$-level which is not part of a planar subset. By Lemma 10, it can be represented as a map with domain $\mathbb{C}^* \setminus \Gamma$ and satisfies an $\arg z$ dependent holomorphic curve equation. To appear as a limit in a generic 1-parameter family we expect $\mathcal{C}_\infty$ to have virtual deformation index of at least $-1$, modulo reparameterization. There is a 1-dimensional reparameterization group given by scaling, whereas domain independent curves have a 2-parameter reparameterization group given by rotation and scaling. Hence, the virtual deformation index of $\mathcal{C}_\infty$ modulo reparameterization, is one greater than its index as defined by Convention 2. So, the (conventional) index of $\mathcal{C}_\infty$ is at least $-2$. By Lemma 4 the other curves of $\mathcal{C}_\infty$ must either all cover trivial cylinders or their collective index is at least 2. □

At this point we can complete the proof of Proposition 7. Overall, we have the index equality

$$2 = \text{index}(\mathcal{F}) = \text{index}(\mathcal{C}) + \text{index}(\mathcal{C}_0) + \text{index}(\mathcal{C}_1) + \text{index}(\mathcal{C}_\infty) + \sum_j \text{index}(\mathcal{D}_j).$$

By Lemma 11 $\mathcal{C}$ is either at or above the $\tau_2$-level.

Assume that $\mathcal{C}$ lies at the $\tau_2$-level. With this, Lemmas 12, 13, 14 and 15 and the fact that all indices are even imply that either $\mathcal{C}$ has index 2 or its index is at most 0. In the first case it follows from these same Lemmas that the curves of $\mathcal{F}$, other than $\mathcal{C}$, must all cover trivial cylinders and so we are done. As we now describe, the remaining case can, generically, be excluded.

Suppose that $\mathcal{C}$ has index at most 0. The Lemmas listed above in fact imply that the index of $\mathcal{C}$ is at most at most $-4$ if $\mathcal{C}_0$ is nontrivial. Since $\mathcal{C}$ is at the $\tau_2$-level, it satisfies a domain dependent equation and so has a 0-dimensional reparameterization group. Hence, $\mathcal{C}$ belongs to a moduli space whose unconstrained virtual dimension is at most 6 when $\mathcal{C}_0$ is trivial, and at most 2 when $\mathcal{C}_0$ is nontrivial. However, $\mathcal{C}$ has punctures at 0, 1, and $\infty$ and it must satisfy (u5). This is a 7-dimensional family of constraints and so will not generically be satisfied (for curves appearing in a 1-parameter family) if $\mathcal{C}_0$ is nontrivial. But if $\mathcal{C}_0$ is trivial, then $\mathcal{C}$ would also have to satisfy the constraint (u2). Thus, $\mathcal{C}$ would belong to a moduli space whose constrained virtual dimension is $-2$. Such curves can again be avoided in generic 1-parameter families.

Assume that $\mathcal{C}$ lies above the $\tau_2$-level. We show that this situation can also be precluded. Since $\mathcal{C}$ lies above the $\tau_2$-level, constraint (u5) implies that $\mathcal{C}$ has a puncture at 2 in addition to its punctures at 0, 1, and $\infty$. By Lemma 12 we then have

$$\sum_j \text{index}(\mathcal{D}_j) \geq 2.$$ 

It also follows from Lemma 13 that $\text{index}(\mathcal{C}_\infty) \geq -2$.

Now, if $\mathcal{C}_0$ is nontrivial or the puncture at 1 is negative, then Lemmas 13 and 14 imply that $\text{index}(\mathcal{C}) \leq -4$ and $\mathcal{C}$ has a single positive end. Taken together, these statements contradict Lemma 4.

If $\mathcal{C}_0$ is trivial and the puncture at 1 is positive, then $\mathcal{C}$ has a positive end at 0 asymptotic to $\alpha_1$, and a positive end at 1 of action at most that of $\alpha_1^{k-1}$. By the maximum principle there are no more positive punctures. As $\text{index}(\mathcal{C}_\infty) \geq -2$ we see that $\mathcal{C}$ has a negative
puncture at \( \infty \) of action at least that of \( \alpha_1^{k-1} \). Together with the negative puncture at 2, we observe that \( C \) necessarily has nonpositive action, and indeed can only be a cover of the trivial cylinder with positive ends at 0, 1 asymptotic to \( \alpha_1 \) and \( \alpha_1^{k-1} \) respectively, and negative punctures at 2, \( \infty \) asymptotic to \( \alpha_1 \) and \( \alpha_1^{k-1} \) respectively. There is a unique such curve up to scale (translation) which satisfies constraint \((u2)\). In particular the asymptotic approach at \( \infty \) is determined, that is, approaching along the real axis we limit at a specific point on \( \alpha_1 \). On the other hand, by Lemma [15] in a 1-parameter family we expect to see only finitely many buildings \( \mathcal{C}_\infty \) with asymptotes at \( \alpha_1^{k-1} \) and \( \alpha_1^k \), at least up to reparameterization by scaling. This determines finitely many possible asymptotic limits at the positive end and generically we do not expect the limits to coincide with that of \( C \), contradicting the matching condition for limiting buildings.

4.6. Splitting along \( \Sigma_2 \). Choose a regular \( J_z \) in \( J^0_{[0,1,\infty]} \cap J_{\Sigma_2} \) such that the stretched families \( J_z^N \) in \( J^N_{[0,1,\infty]} \cap J^N_{\Sigma_2} \) are regular for all \( N \in \mathbb{N} \). Together, Lemma [8] and Proposition [7] imply that for all \( N \in \mathbb{N} \) there is a curve \( u_N \) in \( \mathcal{M}^N(J_z^N) \). After passing to a subsequence, if necessary, we may assume that the \( u_N \) converge to a limiting building \( G \) in the resulting split manifold. This split manifold is diffeomorphic to \( (\mathbb{R} \times \partial E_1) \setminus \Sigma_2 \) and so has two components. One is the completion \((X_1^2, \omega)\) that appears in the statement of Theorem [11]. The other, which we denote by \((\tilde{Z}, \Omega)\), can be described as the completion of the obvious symplectic cobordism from \( \Sigma_2 \) to \( \{\tau_2\} \times \partial E_1 \). In this section we prove the following.

**Proposition 8.** For \( J_z \) as above, there is exactly one curve of \( G \) that maps to \( X_1^2 \). This curve, \( \tilde{u} \), can be parameterized with domain \( \mathbb{C}P^1 \setminus \{0, \infty\} \) with a positive puncture at 0 asymptotic to \( \alpha_2^k \) and a negative puncture at \( \infty \) asymptotic to \( \alpha_1^k \). It is pseudo-holomorphic with respect to a family of almost complex structures \( \tilde{J}_z \) on \( X_1^2 \) depending only on \( \arg z \).

**Proof.** We analyze the building \( G \) as we analyzed \( F \). The curves of \( G \) map to \( X_1^2 \), \( Z \), \( Y_1 \) or \( Y_2 \). Those mapping to \( Z \) are at the same level, which we refer to again as the \( \tau_2 \)-level. The curves \( u_N \) converge to a distinguished curve \( C \) of \( G \) which is either at or above the \( \tau_2 \)-level. It has punctures at 0, 1 and \( \infty \), and additional punctures at the possibly empty set of points \( \{z_1, \ldots, z_l\} \). We then partition the curves of \( G \), other that \( C \), into subsets \( \mathcal{C}_0 \), \( \mathcal{C}_1 \), \( \mathcal{C}_\infty \) and \( \mathcal{D}_j \) as before. Lemma [9] holds with \( F \) replaced by \( G \) while Lemmas [12] [13] and [14] hold as stated. Lemma [10] holds for any curves of \( G \), other than \( C \), which map to either \( X_1^2 \) or \( Z \). The only significant difference involves our analysis of the subbuilding \( \mathcal{C}_\infty \). We now have the following.

**Lemma 16.** The curves of \( \mathcal{C}_\infty \) mapping to \( X_1^2 \) or \( Z \) all have nonnegative index. The other curves of \( \mathcal{C}_\infty \) either all cover trivial cylinders or their collective index is at least 2.

**Proof.** By Lemma [10], the curves of \( \mathcal{C}_\infty \) mapping to \( Z \) and to \( X_1^2 \) and not part of a planar component are pseudo-holomorphic with respect to a domain dependent family of almost complex structures that only depends on the natural \( S^1 \)-parameter on their domain. As we are no longer considering a 1-parameter family of almost-complex structures, these curves are now expected to have virtual deformation index of at least 0, modulo reparameterization. Again, their virtual deformation index modulo reparameterization, is one greater than their index as defined by Convention 2. So, the index of each of these curves of \( C_\infty \) is at least \(-1\). As the indices are all even they are therefore all nonnegative. The other assertions concerning the curves mapping to \( Y_1 \) and \( Y_2 \) follow as before from Lemma [4]. □
Lemma 17. The curve $C$ maps to the $\tau_2$-level.

Proof. Starting with

$$2 = \text{index}(G) = \text{index}(C) + \text{index}(E_0) + \text{index}(E_1) + \text{index}(E_\infty) + \sum_j \text{index}(D_j),$$

we observe again that if $C$ lies above the $\tau_2$-level then constraint $(u5)$ implies that

$$\sum_j \text{index}(D_j) \geq 2.$$ 

This together with Lemmas [13] [14] and [16] imply that $\text{index}(C) \leq 0$. Since $C$ must have punctures at $0, 1, 2, \infty$ this is a contradiction.

Arguing as in the proof of Proposition [7] it follows that $C$ must have index 2. In particular, the curves of $G$ mapping to $Y_1$ and $Y_2$ must all cover trivial cylinders, and $C$ must have exactly three punctures, two positive punctures at 0 and 1, asymptotic to $\alpha_1$ and $\alpha_1^{k-1}$ respectively, and one negative puncture at $\infty$. From this it follows that the only other curve of $G$ is a cylinder mapping to $X_2^2$ with index 0 and a negative puncture at $\infty$ asymptotic to $\alpha_1^k$. As the cylinder has index 0 its positive puncture is asymptotic to $\alpha_1^k$. This is the desired curve $\tilde{u}$ of Proposition [8].

□

4.7. Domain Independence. Choose a sequence $J^n_z$ of families in $J_{\{0,1,\infty\}} \cap J_{\Sigma_2}$ which are regular as in the statement of Proposition [8] and which converge to some $J_\infty$ in $J^0_{\{0,1,\infty\}} \cap J_{\Sigma_2}$ whose domain dependence is trivial. By Proposition [8] this sequence yields

- a sequence of almost complex structures $\tilde{J}_z$ on $X_2^2$ that each only depend on $\text{arg } z$ and which converge to a fixed almost complex structure $J_\infty$ on $X_1^2$
- a sequence of $\tilde{J}_z$-holomorphic curves $\tilde{u}_n: \mathbb{CP}^1 \setminus \{0, \infty\} \to X$ whose punctures at 0 are positive and asymptotic to $\alpha_1^k$ and whose punctures at $\infty$ are negative and asymptotic to $\alpha_1^k$.

After passing to a subsequence we may assume that the curves $u_n$ converge to a $\tilde{J}_\infty$-holomorphic cylindrical building $H$ in $X_1^2$ of index zero and genus zero with the desired asymptotics. This completes the proof of Theorem [5].

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