2D Higher order triangular mesh generation in irregular domain for finite element analysis using MATLAB

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Abstract. This paper presents an automated mesh generation for straight and curved sided irregular domains with unstructured two higher order triangular elements. The present higher order (HO) scheme has been implemented on the basis of subparametric transformations which are extracted from the nodal relations of parabolic arcs especially used for the curved domains. This new restructured meshing scheme is based on distmesh2d introduced by Persson and Gilbert Strang. In this work a higher order triangular mesh for two irregular domains star shaped domain and a circle inscribed in a rectangle has been constructed. These in turn is able to find its application in abundant flow problems and thermodynamics. Present innovative meshing scheme provides a refined and improved high quality meshes for these domains and produce accurate results of the node position, boundary edges and element connectivity for the discretized element. This is an advantage in executing finite element method with less computational efforts in practical engineering applications over the irregular domains.

Keywords— Mesh generation, Irregular curved geometry, Subparametric transformations.

1. Introduction

Mesh generation is one of the important aspect of finite element method. In this method, the region under consideration is divided into small element such as triangle or quadrilateral element for a two dimensional region and tetrahedron or hexahedron element for a three dimensional region. The discretization of the domain, into many smaller elements, help in analysing the data perfectly within the domain. In finite element analysis along with the meshing of the domain, the accurate nodal information of the discretized elements is of great importance in solving partial differential equations (PDE) of flow problems on which the qualitative results of nodal solution and element solution are immediately dependent [1 – 5]. Discretization of the domain is performed through MATLAB codes which generate triangular meshes for two dimensional domains that offer great flexibility to construct a large number of elements. Discretization of the domain with linear and quadratic triangular meshes are abundantly available in the literature [6 – 9]. Linear triangular meshes for any domain can be viewed by the work presented by Persson and Strang in [10]. Eventually, Koko has worked on meshing using distmesh2d upto quadratic triangle element for domains shown in [11]. A novel trend of meshing with higher order triangle element has been performed on two dimensional regular and curved domains which is explained on circular domain by Smitha.et.al [12].

Valid information about higher order element discretization using subparametric transformations is imperceptible in literature. Hence, this paper aims in meshing irregular domains with HO finite curved triangle elements. Meshing over these domains are challenging task as irregular geometry contains intrinsic information along its boundaries. The major two issues (i) the interior points in compliance with the prerequisite of element size specification and the (ii) boundary edge recovery so that all the edges should be present in the meshing are needed to be addressed in an irregular geometry. An extension of the discretization with HO elements are achieved in the present paper with an altered MATLAB codes. The data generated from discretization are used for solving PDE for various irregular domains shown in section (3). This precise and best results are very supportive for certain important applications of finite element analysis [4 – 5] such as fluid flow, thermodynamics and structural design.

Meshing on the curved domains is really a challenging task with MATLAB codes. This arduous performance on the formation of higher order meshes for irregular domains, discussed in consequent sections, are especially suitable for the wide range of flow applications. These constructed MATLAB codes help in forming an accurate refined higher order triangular meshes for irregular geometry.
In the present paper, the HO triangular meshing over irregular domains as mentioned earlier are performed. Discretization of irregular domains have been employed. The MATLAB codes HOmesh2d for straight sided element and CurvedHOmesh2d for curved sided element generated by Smitha.et.al in [12], for higher order triangle elements has been reformed for irregular domains discussed in later section. In the present scenario, the renewed MATLAB codes for the different geometry provides a good refined two dimensional triangle meshes. Specified MATLAB code distmesh2d by Persson and Strang in [10] devour few terms such as signed distance function(fd), mesh size function (fh), mesh size (h0), boundary box (bbox) and fixed nodes (pfix) which are inputs for the execution of discretization. These high quality refined meshes constructed for the irregular domains contributes to the information like number of nodes, element connectivity, node coordinates, and boundary data (edges and nodes) of the domain, as an output. These data assist for the solution of finite element analysis, see reference [1 − 5], associated with the flow problems in an efficient manner. Sequentially these data are useful in the field of computational fluid dynamics (CFD) analysis also.

A complex integrand of rational functions is observed in solving partial differential equations in FEM especially for curved domains [13 − 15]. To recover these complexities, subparametric transformation with parabolic arcs, briefly explained in section (2), has been introduced to the method and even in coding the meshes of higher order triangular element. A fabulous work on subparametric transformation for HO triangular elements which is available in literature by Nagraja.et.al can be seen in the reference [15 − 17]. Assembling these ideas together a technologically improved automatic HO mesh generator was introduced by Simtha.et.al in [12], Finite element analysis can be excellently carried out by reducing the computation efforts. In the present paper, a modification has been implemented in MATLAB codes to accomplish the approach in a prudent way on different irregular domain boundary.

This augmented work discusses unstructured triangle meshes with higher order elements for the domain. Brief introduction of the mathematical formulation of cubic order triangle mesh and detailed explanation of discretization of Star shaped and a circle inscribed in a rectangle domain is given in section (2) and (3). The output generated by discretization, results a high quality unstructured triangle meshes. The mesh efficiency has been calculated for one of the irregular domain in the following sub-section (3.2).

2. Mathematical formulation for cubic-order triangle meshing

In finite element analysis, the main task is to mesh the arbitrary irregular domains and to solve the PDE where we encounter complex numerical integration which arises in these domains. To simplify the complexities arising in the numerical integrations in FEM, we map each unmapped element from \((x, y)\) Cartesian coordinate system to a standard right-angled isosceles triangle in natural coordinate system \((\xi, \eta)\) as shown in figure (1). A continuous work on mathematical theory developed on the point transformation for HO regular and curved triangular elements with parabolic matching curved boundaries can be viewed in literature by various authors such as Mitchell et.al [18 − 19], Rathod.et.al and Nagaraja.et.al [13 − 17]. Point transformation formulae defines the nodes on the boundary and the interior of the element for any domain. It is represented by

\[
t = \sum_{i=1}^{[(n+1)\times(n+2)]/2} N_i^{(n)}(\xi, \eta) t_i, \quad (t = x, y)
\]

where \(N_i\) and \(t_i\) denotes the conventional triangular element Lagrange shape functions and the nodal values for each element at the \(i^{th}\) node respectively. For any regular straight sided geometry, equation (1) for HO triangular elements can be approximated,

\[
t = t_3 + (t_1 - t_3) \xi + (t_2 - t_3) \eta, \quad (t = x, y)
\]

And for any curved geometry, equation (1) can be approximated to:

\[
t = t_3 + (t_1 - t_3) \xi + (t_2 - t_3) \eta + \sum_{n=1}^{n} [n t_4 - (n - 1) t_1 + t_2] \xi \eta, \quad (t = x, y)
\]

The subparametric transformation which maps the global coordinates of one side curved cubic triangular element to the local coordinates in the standard triangle is shown below.
Proceeding further with the same concepts of subparametric transformations of triangle, a higher order triangle element meshing of two irregular domains has executed in next section (3).

3. Mesh generation for irregular domains
Domains which are not even or balanced are irregular domains. They possess straight sided edges or curved edges depending on the shape of the domain. Two irregular shapes star shape and circle inscribed in a rectangle shape has been meshed with HO finite elements. Meshing of these shapes are explained in the following section with higher order triangular element. Both types of irregular domain are vividly described below and illustrated with different order triangular meshes.

3.1. Star shaped domain
A star shaped irregular domain has all straight sided edges and the meshing over it is quite faster than the curved edges. It is meshed with the help of HOMesh2d introduced in [12]. This code was altered and implemented for the star shape boundary. The geometry of the polygonal is defined by signed distance function (fd) in MATLAB code as mentioned in section (1). The advantage of this scheme is that it can be meshed with different mesh size expressed as h0 depending on the requirements. Figure 2(a) – 2(f) depicts the discretization over this geometry by considering the mesh size as h0 = 0.25 with linear order to sextic order triangle element has been shown mentioned below:
Figure: 2(c) Cubic order element

Figure: 2(d) Quartic order element

Figure: 2(e) Quintic order element
3.2. A Circle inscribed in a rectangle

A circle inscribed in a rectangle geometry can be viewed in many fluid flow apparatus and even in mechanical parts. Discretization for such a geometry require a curved triangle element with two straight sides and one curved side along the curved boundary. Subparametric transformation stated in equation (3) satisfies the requirement of the curved domain meshing. The nodes on the curved boundary of the circle inside this rectangle are placed by using these transformations.

For curved elements, positioning the nodes on its boundary requires suitable coding. The MATLAB code CurvedHOmesh2d provided by Smitha.et.al in [12] is for a simple circular domain. For meshing over a circle inside the rectangle, additional inputs in CurvedHOmesh2d has been implemented. Excellent high quality meshes are constructed with higher order refined triangular element which is extensively used in solving FEM in fluid problems.

A linear triangle element meshing on hollow cylinder is carried out by S.H. Lo in [8] and has performed a refinement of meshes with linear order element only. We have customised the code with higher order curved triangle element for meshing irregular domain. The modified code in this paper can be extended up to higher order sextic triangle mesh.

Mesh generated over a circle inscribed in a rectangle has been shown below with linear order triangle element (n=1) with specified node numbering on each discretized element see figure 3(a) and quadratic order curved triangle element (n=2) to sextic order curved triangular element (n=6) with dots represent the nodes as given in figure 3(b) - figure 3(f).
Figure 3(a) Linear triangle element

Figure 3(b) Quadratic triangle element

Figure 3(c) Cubic order element

Figure 3(d) Quartic order element
Figure: 3(e) Quintic order element

Figure: 3(f) Sextic order element

Figure: 3(a) – 3(f) Meshing on the circle inscribed in a rectangle with linear to sextic order triangle element.
Mesh quality can be analysed on the basis of the CPU time factor. CPU time taken in meshing the hollow cylinder has been calculated using linear triangle element \((n=1)\) to higher order element \((n=6)\), until sextic order and the number of nodes formed in each case has been given below:

**Table 1**: CPU time computed in meshing Circle inscribed in a rectangle domain for higher order elements with mesh size \(h=0.075\).

| Higher order elements | CPU time (in seconds) | Number of nodes |
|-----------------------|-----------------------|-----------------|
| \(n=1\)              | 0.632                 | 138             |
| \(n=2\)              | 1.096                 | 225             |
| \(n=3\)              | 1.817                 | 376             |
| \(n=4\)              | 2.227                 | 534             |
| \(n=5\)              | 3.157                 | 762             |
| \(n=6\)              | 3.761                 | 945             |

For many manufacturing and modelling applications, this approach of HO schemes implemented for the domain will help in deriving compact terrain data, as it extracts the data of the element connectivity, node coordinates, and boundary data (edges and nodes) of all the elements of the domain, as an output. These domains play an important role in machinery parts helpful for other researchers involved in manufacturing industries and computational fluid dynamics. This paper provides a methodology for HO meshing of commonly found irregular domain. A well-organized meshing reduces the hardship of many research areas which require the simplification of the numerical solutions in FEM. The proposed simple and modified scheme could drastically reduce the computation time and efforts over the HO mesh generation front for any irregular domain. CPU time taken and number of nodes formed in meshing the above domain for higher order element has been tabulated which can be viewed in table (1). It is clear from the table that the data predicts a refined quality meshes and they are formed faster. This method is found to be robust and reliable for arbitrary geometry.

**Conclusion**

Efficient HO meshing implemented on the irregular domains results in good refined triangular elements and generates a precise output for the element and nodal information. Several outlooks and fluid flow applications can be envisaged from this work. This HO meshing technique for irregular curved domains is a customised version of CurvedHOmesh2d. This paper provides a wider and clear output data regarding the same. Because of complicated geometry of the irregular domain, it leads to higher discrepancies between experimental data and numerical solution. It allows the users to focus on the practical aspects of structural optimization in an efficient way. Present automated mesh generation and the output extracted from it can ease many of the discrepancies to an extent for a two-dimensional flow analysis over the domain.

Moreover, our approach extends to mesh on any irregular geometry with specific facts that incorporates an approximation to the boundary. Literature has interestingly pointed out that the FE mesh is required for a numerical solution for most of the physical problems governed by PDE. Few of the fluid flow applications needs contour plot at particular points of the domain. Those particular points can be very easily extracted through this advanced meshing approach for contour formation in various flow problems. Quality of refined meshing has been analysed by computing the CPU time of one of the irregular domain in the present work. Hence, the precise and best results are useful and convenient while performing certain other applications of finite element analysis as it gives output in an expeditious manner.

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Acknowledgments
The authors have extended their sincere thanks to the respectful referees for their suggestions and modifications which transformed the presentation of the paper in an enhanced form.