Algorithmic approach to cosmological coherent state expectation values in loop quantum gravity

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Abstract

Within the lattice approach to loop quantum gravity on a fixed graph, explicit calculations tend to be involved and are rarely analytically manageable. However, being focused on a particularly interesting setting, concerned with expectation values with respect to coherent states on the lattice (sharply peaked on isotropic and flat cosmology), we are able to provide several simplifications which can facilitate approaching beyond-state-of-the-art problems. We present a step-by-step algorithm resulting in an analytical expression including up-to-first-order corrections in the spread of the coherent state. The algorithm is developed in such a way that it makes the computation straightforward and easily implementable.

Keywords: quantum, gravity, loop quantum gravity, semiclassicity, lattice, coherent states

1. Introduction

With the beginning of the era of gravitational wave astronomy and the possibility at reach to probe physics beyond the cosmic microwave background (CMB), more interest is put in recent years on the potential first effects a theory of quantum gravity can have on isotropic cosmological models. One candidate for quantum gravity comes in the form of loop quantum gravity.
gravity (LQG) [1–3] whose development has pushed consistently towards computability and phenomenology in the past decade. The construction of the theory is not fully finished yet, in the sense that there remain quantisation ambiguities of the scalar constraint operator, which drives the dynamics of LQG. But while current statements about the dynamical predictions of some incarnation of the quantum constraint have to be taken with a grain of salt, renormalisation programmes have entered LQG [4–8] in order to address this caveat. Simultaneously to this development, however, one must ask how to proceed towards observations once a reliable quantum constraint is obtained.

Isotropic cosmologies of Friedmann–Lemaître–Robertson–Walker (FLRW) type [9–12] provide the most promising starting point as the overall large scales of the Universe today as well as in an epoch before formation of the CMB seems to support this model. With the CMB being mostly classical, quantum corrections to the Universe must be small and can only be more dominant in the far past, thus suggesting to use the description of semi-classical, coherent states, whose quantum evolution behaves classical into the future and potentially leads to a resolution of the singularity in the far past. While the construction of such states cannot be completed until the above-mentioned quantisation ambiguities are fixed, good candidate coherent states, so called gauge field theory coherent states (GCS), have already been developed in LQG by Thiemann and Winkler [13–17]. The idea would be to use these states peaked on isotropic cosmology and supported on a finite lattice, to get a first intuition on the quantum effects of general relativity. Since in the limit of vanishing spread of the states, the classical theory on the lattice is recovered (for large momenta), one has to consider a finite spread $t$ in order to get a first intuition on the quantum effects. Thus, one must be interested in a small value of $t$, whose linear corrections to the classical order are not negligible. However, as soon as one considers a lattice, i.e. many countable degrees of freedom, the problem typically becomes very involved. The first investigations in this direction [18] constructed the isotropic cosmological coherent state as a tensor product of Thiemann–Winkler GCS on each edge of the lattice. However, they could only deal with the lattice content when truncating the theory and transcended to the toy model of quantum reduced loop gravity [19, 20]. However, with such simplification, knowledge of the real quantum effects is lost and it thus remained an open issue how to proceed. Lattice gauge theory (to which LQG is closely related) knows the issue of involved lattice computations and relies mostly on numerical methods to tackle it [21, 22]. In LQG, by now understanding of its kinematical objects has developed so far that we can give a clear guidance on how to approach such computations analytically.

The crucial starting point is the set-up of algebraic quantum gravity (AQG) [23] by Giesel and Thiemann, where LQG is restricted to a cubic lattice. In [24], the authors managed to replace inside of coherent state expectation values the paramount Ashtekar–Lewandowski volume operator with another operator, which is polynomial in the basic configuration variables with arbitrary error-control. We refer to this replacement as the Giesel–Thiemann volume as it sparked the investigation of isotropic coherent states of full loop quantum gravity on the lattice. The first computations were performed in [25], proving several important statements about the expectation values of flux-polynomial operators of GCS on single edges, which decreased the computational effort by an exponential factor, when interested in corrections including the first order in the spread $t$ of the states. However, it was not until this year, when [26] delivered a general formula to compute the expectation value of arbitrary polynomials, i.e. including any powers of holonomies, up to first order in $t$. Both of these latter works paved the way to develop the streamlined algorithm of this paper: we give a step-by-step guide on how to compute the expectation values of any polynomial operator in isotropic coherent states of full loop quantum gravity on a lattice.
This article is organized as follows. In section 2, we repeat the construction of cosmological coherent states in LQG (in the appendix A we also state the necessary background of LQG). The basis elements of the kinematical Hilbert space are supported on finite graphs. Therefore, gauge coherent states (GCS) can be used to describe a semi-classical geometry on each graph. Following [25], this is explicitly done for a cubic lattice and isotropic, flat cosmology as underlying geometry. Importantly, we recall the expectation value of polynomial operators on a single edge each up to first order in the spread from [26]. Finally, we recall and introduce new material to facilitate the developments presented in the following section.

In section 4, we construct a step-by-step algorithm, which provides a hand-on tool to compute the expectation value of any polynomial operator in the basic variables (holonomies and fluxes) in the cosmological coherent states of the previous section. Two crucial points simplify the investigation: (i) the realisation from [25] that the right-invariant vector fields only give a leading order contribution when their index equals zero, which removes all but few combinatorial cases to consider and decreases the computational time by an exponential factor and (ii) a strategy to extract the presence of  $\hat{Q}_v$ operators (the building block of the volume operators) out of the expectation value, reducing the number of right-invariant vector fields on each edge, again speeding up the computation.

In section 5, we test our algorithm on $\hat{C}_\epsilon$ the Euclidean part of the scalar constraint in its common quantisation by Thiemann [27] to highlight how much the computation simplifies.

In section 6, we further use the algorithm to extract knowledge about the commutator between $\hat{C}_\epsilon$ and the volume operator. We find that the real quantum corrections do not match with what one would expect from the effective dynamics program: here, one typically interprets the expectation value as a function on the classical phase space and evolves classical observable with respect to it. However, when finite t corrections at first order are present, the two procedures ‘taking the expectation value’ and ‘applying the commutator/Poisson bracket’ do not commute anymore. This indicates potential problems when trying to implement the effective dynamics program in lattice loop quantum gravity with finite t corrections present.

In section 7 we finish with conclusion and outlook for the applications of the algorithm as next steps in the cosmological coherent state expectation value endeavour.

2. Cosmological coherent states

In this section we revise the framework of gauge field coherent states applied to isotropic, flat cosmology. The presentation mostly follows [25], however, we slightly adjust the notation, correct a few typos present in [25] and rely on the, so called, gauge covariant fluxes. Further details can be found in the topical literature. For completeness, in the appendix A we also summarize the underlying formal background relevant for further quantisation of the theory. Experienced readers may wish to jump directly to section 3 and skip the appendix A.

2.1. Preliminaries

Loop quantisation of general relativity promotes the phase space over a graph $\gamma$ (precisely defined through (A.9)–(A.11) in the appendix A) to a quantum Hilbert space $\mathcal{H}_\gamma$ with suitable operators thereon. The union of all single graph Hilbert spaces describes then the Ashtekar–Lewandowski Hilbert space $\mathcal{H}_{AL}$, which is the kinematical Hilbert space of LQG. For the purpose of this article, however, we will later on restrict our attention to a single graph $\gamma$ and study only states in $\mathcal{H}_\gamma$ which have excitations thereon.

Given a graph $\gamma$, one assigns to each edge $e$ a function in $\mathcal{H}_e = L_2(\text{SU}(2), d\mu_H)$ with $\mu_H$ being the unique left- and right-invariant Haar measure over SU(2). The full Hilbert space $\mathcal{H}_\gamma$ is
then simply the tensor product over all square integrable functions on each edge, \( \mathcal{H}_\gamma := \otimes_e \mathcal{H}_e \).

If we label, in the position representation, \( \psi \in \mathcal{H} \), as \( \psi = \psi(\{ g_e \}_{e \in \gamma}) \), then the holonomies get promoted to bounded, unitary multiplication operators

\[
\hat{h}_{\alpha}(e') \psi(\{ g_e \}_{e \in \gamma}) := D_{\alpha}^{(k)}(g_e) \psi(\{ g_e \}_{e \in \gamma}),
\]

and the gauge covariant fluxes become essentially self-adjoint derivation operators

\[
\hat{P}^K(e') \psi(\{ g_e \}_{e \in \gamma}) := \mathcal{E} R^K(e') \psi(\{ g_e \}_{e \in \gamma}),
\]

where \( \mathcal{E} = -i\hbar\kappa\beta/2 \). With \( D_{\alpha}^{(k)}(g_e) \) we denote a Wigner-matrix of a group element \( g_e \) in the \( 2k + 1 \)-dimensional, irreducible representation of SU(2), and the right-invariant vector field \( R^K(e) \) acts as:

\[
R^K(e') \psi(\{ g_e \}_{e \in \gamma}) := \frac{d}{dx} \bigg|_{x=0}\psi(\ldots, e^{i\kappa K} g_{e'}, \ldots).
\]

\( \kappa = 16\pi G \) is the coupling constant while \( \beta \) is the Barbero–Immirzi parameter. (Note that throughout the manuscript we set \( c = 1 \).)

The \( \tau_I \) matrices for \( I = \{ -1, 0, +1 \} \) span the spherical basis of \( \text{su}(2) \) constructed as follows.

Let \( \sigma_1, \sigma_2, \sigma_3 \) denote the Pauli matrices. Then: \( \tau_\pm := \mp i(\sigma_1 \pm i\sigma_2)/\sqrt{2} \) and \( \tau_0 := -i\sigma_3/2 \).

The algebraic relations \( [\tau_+, \tau_-] = i\tau_0 \) and \( [\tau_\pm, \tau_0] = \pm i\tau_\pm \) hold.

Let \( [\tau^K]^{Ij} \) denote the spherical basis in the \( j \)th representation of \( \text{su}(2) \). Exact matrix elements of \( [\tau^K]^{Ij} \) are given in equation (A.12). The matrices \( \tau_I \) defined above are therefore relevant for the fundamental \( j = 1/2 \) representation. The operators \( \hat{h}_0 \) and \( R_j \), with \( j \) labelling the representation, obey the following commutation relations

\[
[R^K(e), \hat{h}_0^{Ij}(e')] = \delta_{e'e} [\tau^K]^{Ij}_{Jj}(e),
\]

\[
[R^I(e), R^J(e')] = -\delta_{e'e} f^{IJ}_{\phantom{IJ}K} R^K(e),
\]

with \( f^{IJ}_{\phantom{IJ}K} \) denoting the structure functions of \( \text{su}(2) \) with respect to the spherical basis used here, i.e. \( f^{IJ}_{\phantom{IJ}K} = (-1)^{I+1}[\tau^I]^{Jk}_{Kk} \).

For completeness, we provide the prescription how the momentum operator acts on the other end of the edge

\[
\hat{P}^K(e'^{-1}) \psi(\{ g_e \}_{e \in \gamma}) := -\mathcal{E} L^K(e') \psi(\{ g_e \}_{e \in \gamma}),
\]

with left-invariant vector fields \( L^K \) defined analogously to (2.3) and obeying similar commutation relations [25].

### 2.2. Cosmological gauge coherent states

In order to define GCS relevant for isotropic, flat cosmology—classically described by the FLRW spacetime—on a fixed graph, we need to adjust the above theoretical framework. For the sake of simplicity, we do not discuss the construction of general GCS in loop quantum gravity. We only note in passing that, as was shown in [16], the general GCS are sharply peaked on the classical configuration.

Typically, flat FLRW is studied in the context of \( \sigma = \mathbb{R}^3 \) as a spatial manifold. However, for the moment we restrict our attention to the compact torus \( \sigma_R = [0, R]^3 \), with cut-off \( R \in \mathbb{R} \). After all, it will be trivial to perform the thermodynamic limit \( R \to \infty \).
Within isotropic, flat cosmology, the Ashtekar–Barbero variables take the simple form (both $c$ and $\rho$ are functions)

$$A^i_a = c \delta^i_a, \quad E^i_a = \bar{\rho} \delta^i_a.$$  

(2.7)

To go further we need to introduce a discretisation of $\sigma_S$ in the form of a cubic lattice $\gamma$ with $M$ points in each direction of the coordinates described by $x^a$. With respect to the fiducial flat metric the coordinate length of the torus was $R$ and we will denote by $\epsilon = R/M$ the regulator of the discretisation, i.e. the coordinate length of each edge.\(^5\) The holonomies $h(\epsilon) \in SU(2)$ of the connection, for an edge $e_k$ along direction $k \in \{ \pm 1, \pm 2, \pm 3 \}$, can directly be computed from (A.6)

$$h(e_k) = \exp(\text{sgn}(k) c \epsilon \tau_k).$$  

(2.8)

with $\tau_k = -i \sigma_k / 2$ (with $\sigma_k$ being the Pauli matrices).\(^6\)

For the gauge-covariant fluxes \([34]\), we have to choose a certain set of paths $\rho_i$, in their construction (A.7). For an edge $e_k$ one splits $\rho_k = \rho_{k,a} \circ \rho'_{k,b}$ where $\epsilon_{ab} = 1$. We choose further that $\rho_{k,a}[0] = e_k \cap S_k$ and $\rho_{k,a}[1] = \rho'_{k,b}[0]$ and $\rho'_{k,b}[1] = x$. Finally, the path $\rho_{k,a}$ starts in direction $\pm a$, stays in $S_k$ with constant tangent vector, and similar for $\rho'_{k,b}$ with constant tangent vector oriented along $\pm b$. For this choice, the gauge-covariant fluxes for the connection and triad (2.7) are found to be

$$P^i(\epsilon_k) = \delta^i_a \rho := \delta^i_a \bar{\rho} \sin^2(c \epsilon / 2).$$  

(2.9)

The cosmological coherent states constructed in accordance with the above setting are \([18, 25]\]

$$\Psi^\gamma_t(\{ g \}) = \prod_{k \in \{1,2,3\}} \prod_{a \in \mathbb{Z}_d} (1)^{-1} \psi^\gamma_t(g_{kk}),$$  

(2.10)

where $(1) := \| \psi^\gamma_t \|$ is the norm (which neither depends on the edge, nor on the group element), so that $\Psi^\gamma_t$ is a tensor product of the cosmological GCS $\psi^\gamma_t$ on each edge. The latter are defined as \([15]\]

$$\psi^\gamma_t(g) := \sum_{j \in \mathbb{N}_0 / 2} (2j + 1) e^{-j^2 \ell / 2} \sum_{m=-j}^j D^\gamma_{mm}(H_I g^I).$$  

(2.11)

The so called semiclassicality parameter $t := \hbar \epsilon / \ell^2 \geq 0$ is dimensionless where $\ell$ is some length scale. For each edge, going in the same direction $I$, we select the same element $H_I \in SL(2,\mathbb{C})$, thus capturing homogeneity. Each element looks as follows in its, so called, holomorphic decomposition \([36, 37]\)

$$H_I = m I e^{-\gamma_0 n_I^4}, \quad \eta = \frac{2c^2 p}{\ell^2 \beta}, \quad \xi = -\epsilon c,$$  

(2.12)

\(^5\) In order to perform the thermodynamic limit, one should keep $\epsilon$ fixed. In this way, $R \to \infty$ also implies that $M \to \infty$, i.e. the numbers of lattice sites is sent to infinity.

\(^6\) Comparison of (2.8) with the literature on loop quantum cosmology (LQC), highlights that we are working with the so-called $\mu_\epsilon$-scheme, i.e. with $\epsilon$ being a constant. In comparison to that the famous $\mu$ scheme of LQC \([43]\) replaces $\epsilon$ with a phase-space dependent and dynamical quantity $\sqrt{\text{const.}} / \bar{\rho}$. However, coming from the full theory of LQG, there is as of today no framework known to obtain a system mimicking cosmology and having the chance to produce the $\mu$-scheme as effective dynamics, see \([33]\) for further discussion on this subject.
with \( z = \xi - i\eta \), and: \( n_I \) given below in (2.13), \( p \) defined\(^7\) in (2.9) and \( c \) together with \( \bar{p} \), being part of \( p \), both originally coming from (2.7). The SU(2) rotations are explicitly given by:

\[
\begin{align*}
    n_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \\
    n_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \\
    n_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\] (2.13)

2.3. Volume

For the sake of further considerations we shall also summarize the known background regarding the volume operator of LQG. The quantisation of the volume, such as the Ashtekar–Lewandowski volume operator \( \hat{V}_{\text{AL}} \) [35], proved vital for many constructions of other important operators such as the quantized scalar constraint (see [27, 28]). Albeit it being a highly complicated object, whose spectrum is not yet under full control, computations of physically relevant quantities are possible due to the following crucial observation from [24]:

**Observation 1.** For any gauge field theory coherent state \( \Psi'_\gamma \) (not only the cosmological one) defined on a graph \( \gamma \), the Ashtekar–Lewandowski volume operator at a vertex \( v \in \gamma \) with only outgoing edges, defined as

\[
\hat{V}_{v, \text{AL}} = \frac{|E|^3/2}{4\sqrt{3}} \sqrt{|\hat{Q}_v|},
\] (2.14)

\[
\hat{Q}_v := \sum_{i,j} \epsilon(i, j, k) \epsilon_{ijk} R^I(e_i) R^J(e_j) R^K(e_k),
\] (2.15)

with

\[
\epsilon(i, j, k) := \text{sgn}(\det(\dot{e}_i, \dot{e}_j, \dot{e}_k)),
\] (2.16)

\[
\epsilon_{ijk} := i\sqrt{6} \begin{pmatrix} 1 & 1 & 1 \\ 1 & J & K \end{pmatrix},
\] (2.17)

and the \( k \)th Giesel–Thiemann volume operator, defined as

\[
\hat{V}_{k,v}^{\text{GT}} = \frac{|E|^{3/2}}{4\sqrt{3}} \sqrt{|\hat{Q}_v|} \left[ 1_{\mathcal{H}_v} + \sum_{n=1}^{2k+1} \frac{(-1)^n}{n!} \left( n - 1 - \frac{1}{4} \right) \left( \frac{\hat{Q}_v^2}{|\hat{Q}_v|^2} - 1_{\mathcal{H}_v} \right)^n \right],
\] (2.18)

with \( \langle \hat{Q}_v \rangle := \langle \Psi'_\gamma, \hat{Q}_v \Psi'_\gamma \rangle \), fulfill

\[
\langle \Psi'_\gamma, F(\hat{V}_{v, \text{AL}}^{\text{AL}}, \hat{h}) \Psi'_\gamma \rangle = \langle \Psi'_\Gamma, F(\hat{V}_{k,v}^{\text{GT}}, \hat{h}) \Psi'_\gamma \rangle + \mathcal{O}(d^{k+1}),
\] (2.19)

for any polynomial \( F \).

In other words, while calculating the expectation values involving \( \hat{V}_{v, \text{AL}}^{\text{AL}} \) up to a given accuracy, one instead can utilize \( \hat{V}_{k,v}^{\text{GT}} \) with an appropriate \( k \). Let us also state a slight reformulation of \( \hat{V}_{k,v}^{\text{GT}} \) which is used in the next section:

\[
\hat{V}_{k,v}^{\text{GT}} = \frac{|E|^{3/2} 4k+2}{4\sqrt{3} \sum_{N=0}^\infty C_N \hat{Q}_v^N},
\] (2.20)

\(^7\) Note, that we have not peaked on \( \bar{p} \), but on \( p \) which includes the gauge-covariant corrections. This is congruent with general relativity on a graph restricted to cosmology, see [38] for further details.
where the coefficients $c_N$ are determined via the series in (2.18).

2.4. Other average values

We finish this section with the important statement, saying how the expectation values of arbitrary polynomial operators on a single edge in $\psi^t_{\text{HI}}$ look like, when including up to next-to-leading order in $t$. The proofs of the formulas provided below are shown in [25, 26] for $\eta \neq 0$.

First of all

$$\langle 1 \rangle := \langle \psi^t_{\text{HI}}, \hat{h}^{(0)}(0) \psi^t_{\text{HI}} \rangle = \sqrt{\frac{2\eta e^{\eta^2/4}}{\pi \sinh(\eta)}} e^{\eta/2},$$

(2.21)

is the normalisation of each state. The main result is expressed using the spherical basis, i.e. $K_1, \ldots, K_N \in \{-1, 0, +1\}$ and $\hat{h} \equiv h(e)$, i.e. going in the same direction as the edge:

$$\langle \psi^t_{\text{HI}}, \hat{h}^{(k)}_{\text{ab}} R^{K_1}_1 \ldots R^{K_N}_N \psi^t_{\text{HI}} \rangle = \langle \psi^t_{\text{HI}}, \hat{h}^{(k)}_{\text{ab}} \psi^t_{\text{HI}} \rangle \left( \frac{i}{t} \right)^N \left[ \sum_{A=1}^{N} \delta_{S_A}^{S_0} \ldots \delta_{S_A}^{S_0} \delta_{AC}^{S_0} \delta_{AC}^{S_0} N \right]^{\mathcal{O}(t^2)}, (2.22)$$

and

$$\langle \psi^t_{\text{HI}}, \hat{h}^{(j)}^{(k)} \psi^t_{\text{HI}} \rangle = \langle 1 \rangle \sum_{c=-j}^{j} D^{(k)}_{ac}(n) e^{i\xi c} \gamma_c^j D^{(k)}_{cb}(n), (2.23)$$

with $\gamma_c^j = 1 - t\tilde{\gamma}_c^j$ where

$$\tilde{\gamma}_c^j = \frac{1}{4} \left( j^2 + j - 3 \right) \frac{\tanh(\eta/2)}{\eta/2} + c^2 \right] + \mathcal{O}(t^2).$$

(2.24)

3. Useful results concerning cosmological coherent state expectation values

Before going to the algorithm, being the major contribution of this paper, we recall some already known properties of cosmological coherent state expectation values, as well as derive a few formulas essential for the main part of this paper. As above, we use the spherical basis $K \in \{-1, 0, +1\}$ for the right-invariant vector fields throughout this section. We first collect former result by means of four observations proven in [25], which we just summarize here for
Observation 2 (From [25]). The leading order contribution only comes from the first line of (2.22) which is proportional to $\delta_{01}^{\delta_{01}} \ldots \delta_{01}^{\delta_{01}} \delta_{ab}$. That is, for the leading order contribution, one does not have to perform any contractions. Likewise, when investigating the next lines, one sees that the first order corrections come from non-trivial contractions of at most a single or one pair of indices.

Observation 3 (From [25]). The total number of right/left-invariant vector fields determines the leading order: each flux being proportional to $t^{-1}$ at most: a monomial with $N$ many right/left-invariant vector fields has an expectation value of order $O(t^{-N})$. Thus, in the presence of such a monomial, other polynomial operators with $N - 2$ or fewer fluxes can be neglected as being higher-order corrections than we are here interested in. Note that, at next-to-leading order, operators may be freely commuted, due to (2.4) and (2.5) each commutation removes a right-invariant vector field.

Observation 4 (From [25]). By extracting the $n_I$ matrices from $H_I$, all three directions $I = 1, 2, 3$ boil down to the sole expectation value on $\psi_{H_I}$. Furthermore, for a polynomial operator $\hat{F}$ we can replace the right-most left-/right-invariant vector fields as follows:

$$\langle \psi_{H_I}, \hat{F} L^K \psi_{H_I} \rangle = e^{i\xi^K} \langle \psi_{H_I}, \hat{F} R^K \psi_{H_I} \rangle,$$

where as before $z = \xi - i\eta$. Moreover, since one can show that

$$[R^K, L^I] = 0,$$

all left-invariant vector fields present in a given expression can be replaced successively, always starting with the right-most one.

Observation 5 (From [25]). Let

$$\epsilon^{(\mu_{\mu_1} \mu_2 \mu_3)}_{ABC} := e^{ijk} D^{(1)}_{i} (n_{|\mu_1|}) D^{(1)}_{j} (n_{|\mu_2|}) D^{(1)}_{k} (n_{|\mu_3|}),$$

where $I, J, K, A, B, C \in \{-1, 0, +1\}$ and $\epsilon^{ijk}$ has been defined in (2.17).

This quantity is anti-cyclic in its indices, in the following sense:

$$\epsilon^{(\mu_{\mu_1} \mu_2 \mu_3)}_{ABC} = -\epsilon^{(\mu_{\mu_2} \mu_1 \mu_3)}_{BAC}.$$

It has been found that

$$D^{(1/2)}_{(n_I)\tau_0} D^{(1/2)}_{(n_I)} = \tau_1,$$

and

$$\epsilon^{(n_{|123|})}_{000} = \epsilon^{(n_{|123|})}_{010} = \epsilon^{(n_{|123|})}_{020} = \epsilon^{(n_{|123|})}_{030} = -\delta_{000}.$$

All the above tools eventually allow to compute the expectation value of $\hat{Q}_v$ in cosmological coherent states $\langle \Psi^v_{t\gamma}, \hat{Q}_v \Psi^v_{t\gamma} \rangle = q_0 + O(t^{-1})$, where

$$q_0 := \frac{48 \eta^3}{t^3} \left[ 1 + t \frac{3}{2 \eta^2} (1 - \eta \coth \eta) \right].$$
Consequently (see [25])

\[
\langle \Psi_t^\gamma, V_{\nu, v}^{GT} \Psi_t^\gamma \rangle = \rho^2 \left[ 1 + \frac{3t}{4\eta} \left( \frac{7}{8\eta} - \coth \eta \right) \right] + O(\rho^2).
\]  

(3.8)

**Theorem 1.** Let \( \hat{F} = \prod_{e} \hat{f}_e \) be an operator with leading order \( O(\rho^0) \) where each \( \hat{f}_e \) is a monomial supported on the edge \( e = (v, I) \) of the form \( \hat{f}_{ab}^{(v)} R_{K} \ldots R_{K_N v} \). We keep appearances of \( Q_v \) separate, in the form of \( \prod_v Q_v^{N_v} \) with \( N_v \) denoting the number of \( Q_v \) at each vertex. We introduce a modified average:

\[
\langle \langle \hat{f}_e \rangle \rangle \equiv \langle \psi_t^{HI_v}, \hat{f}_e \psi_t^{HI_v} \rangle + \langle 1 \rangle \delta_{v}^{K_1 \ldots K_N v} \ Y_{ab}^{(v)} \left[ v, I, N \right],
\]  

(3.9)

with:

\[
Y_{ab}^{(v)} \left[ v, I, N \right] = \frac{i 4}{N_v} \left( \frac{\eta}{t} \right)^{N_v - 1} \left( N_v + N_v + N_v \right) \sum_{i = 0}^{j} c e^{i \xi_c D_{ac}^{(v)} (n_1) D^{(v)}_{cb} (n_1)}. \quad (3.10)
\]

We further define

\[
\tilde{l}_e = t^{-N_v} \lim_{t \to 0} \left( t^{N_v} \langle \psi_t^{HI_v}, \hat{f}_e \psi_t^{HI_v} \rangle \right),
\]  

(3.11)

being the leading order of \( \hat{f}_e \), i.e. \( \langle \langle \hat{f}_e \rangle \rangle / \tilde{l}_e = 1 = O(t) \). Then, the following statement holds at the classical order:

\[
\langle \Psi_t^\gamma, \hat{F} \prod_{v} \hat{Q}_v^{N_v} \Psi_t^\gamma \rangle = P_0 + O \left( t^{1 - \frac{3}{2} \sum_v N_v} \right),
\]  

(3.12)

where

\[
P_0 := \left( \prod_v 2^{3N_v} \left( \frac{\eta}{t} \right)^{3N_v} \left( -6i \right)^{N_v} \right) \prod_e \tilde{l}_e.
\]  

(3.13)

Moreover, first order corrections are given by

\[
\langle \Psi_t^\gamma, \hat{F} \prod_{v} \hat{Q}_v^{N_v} \Psi_t^\gamma \rangle = P_0 \left[ 1 + \sum_e \frac{1}{\tilde{l}_e} \left[ \langle \langle l_e \rangle \rangle - \tilde{l}_e + \frac{t}{4\eta} \left( 1 - \delta_{N_v, 0} \delta_{N_v + e, 0} \right) \times (N_v + N_v + N_v + 3 - 4\eta \coth(\eta) \tilde{l}_e) \right] \right.
\]

\[
+ O \left( t^{1 - \frac{3}{2} \sum_v N_v} \right).
\]  

(3.14)

A tedious proof of theorem 1 can be found in appendix B. In short, it follows from the fact that classically each flux is non-vanishing only along one direction. In the leading order quantum corrections the above rule can be broken by at most one flux in each term. However, such a contribution is at the end simplified by the use of (3.6).

4. Algorithmic approach to expectation values with cosmological coherent states

On a formal ground, the cosmological coherent states \( \psi_t^\gamma \) can serve as a powerful means to describe the semi-classical behaviour of quantum cosmology studied from the perspective of
the full theory of loop quantum gravity put on a lattice. However, one can quickly face obstacles of a purely practical origin, namely, the expressions to be calculated are intractable, or at least seem to be such. As a step towards the solution to this problem we construct an algorithm which allows for a simplification of generically complex expressions, having in mind that at the moment we are only interested in the first-order quantum corrections.

In a nutshell, we classify terms to be further processed (evaluated) as classical or as (first order) quantum, depending on the order of \( t \) they carry. It turns out, that in our approach we split the average value of an operator of interest into a moderate number of classical contributions, which can in principle be cumbersome, and a typically huge number of quantum corrections, which are much simpler to handle. Note that the classical contributions do not solely consist of terms of a lower order with respect to \( t \), in comparison with the quantum ones. They are rather sums of both orders, to be further processed. After we introduce a few pieces of notation necessary for this section we will come back to a formal description of this splitting as well as the whole approach.

4.1. Discretization

From now on, we specialize \( \gamma \) to be a cubic lattice with \( M \) vertices, i.e. each edge \( e \in \gamma \) can be parameterized in the following way: \( e = (v, \mu) \)

\[
\mu \in \{1, 2, 3\}, \quad v \in \mathbb{Z}_M^3 = \{0, \ldots, M - 1\}^3.
\] (4.1)

Of course, we also have the edges \( (v, -\mu) \), with \( \mu \in \{1, 2, 3\} \), pointing in opposite directions, however, they are already contained in the former edges due to the identification \((v, \mu) = (v + e_\mu, -\mu)\).

The vectors \( e_\mu \) are unit with respect to a fiducial geometry and are such that their \( I \)th components \( e^I_\mu \) read

\[
e^I_\mu = \text{sgn}(\mu)\delta_{I,|\mu|}.
\] (4.3)

with \( \delta_{a,b} \) being the Kronecker–Delta, between the objects \( a \) and \( b \). This parameterization allows us to write some ingredients more explicitly, e.g. equation (2.16) takes the form

\[
\epsilon(i, j, k) \equiv \text{sgn}(\mu_\rho \mu_\sigma \mu_\lambda)\epsilon_{|\mu_\rho| |\mu_\sigma| |\mu_\lambda|}.
\] (4.4)

With the help of these vectors we also later employ periodic boundary conditions. To this end we will make the identification of the vertices \( Me_\mu \rightarrow 0e_\mu \), that is, \( v(M) = v(0) \).

Most importantly, the action of the operators involved becomes:

\[
\hat{h}^{(\mu)}(v, \mu) = \mathcal{E} \begin{cases} R^K(v, \mu) & \text{if } \mu > 0 \\ -L^K(v + e_\mu, -\mu) & \text{if } \mu < 0 \end{cases},
\] (4.5)

\[
\hat{h}^{(\mu)}_{ab}(v, \mu) = \begin{cases} D^\mu_{ab}(v, \mu) & \text{if } \mu > 0 \\ \left[D^\mu_{ab}(v + e_\mu, -\mu)\right]^\dagger & \text{if } \mu < 0 \end{cases},
\] (4.6)

where as before \( \mathcal{E} = -i\hbar\kappa_\beta/2 \).

8 In the following we will use the same symbol to denote the Kronecker–Delta between directions \( \mu, \mu' = 1, 2, 3 \) as well as between vertices \( v, v' \).
4.2. Input

In order to develop the algorithm we first need to specify what, on the conceptual level, can serve the role of its main input, as well as which supplementary information needs to be stored for the sake of practical implementations of the algorithm.

Concerning the first, fundamental question, we assume that the operator for which we will compute the expectation value can only depend on the holonomies $\hat{h}_{ab}^{ij}(v, \mu)$ and the volume $\hat{V}^{\text{AL}}_v$. We explicitly exclude dependence on $\hat{P}^k(v, \mu)$ for the moment, as the most interesting operators of LQG feature this behaviour.

Elaborating on the supplementary input information, while implementing the algorithm one carefully needs to collect the indices appearing in the operator in question, which shall be contracted at the end of the computation (important also are appropriate summation ranges of these indices). The latter remark is in order, as the holonomy is equipped with an index labelling the representation (spin). From the practical perspective it turns out useful (allows to reduce computational time) to predict and fix the maximal representation index which can appear in a given computation.

Last but not least, there is a remaining question of the lattice size $M$. If the algorithm is used just as a road towards fully analytic calculations, there is no need to fix its value. However, in a scenario such as implementation on a cluster we need to fix $M$ in a way which minimizes the time necessary for the computation and at the same time guarantees that the final result will be independent of the lattice size. The condition which facilitates such a choice stems from the periodic boundary conditions. Because of this geometry, $M$ needs to be sufficiently large in comparison to the ‘correlation length’ of the evaluated operator. For example, if the operator only involves couplings between the nearest vertices, the choice $M = 2$ is sufficient and at the same time optimal.

4.3. Structure of the algorithm

The initial step to be performed in the algorithm is an appropriate representation of the volume by means of tractable quantities following observation 1. In fact, in the assumed approximation (first leading order in $t$) we can set $V^{\text{AL}}_v \equiv V^{\text{GT}}_v$, and replace the latter object by $c_N \hat{Q}^N_v$. Note that $N$ is being added to the list of indices to be contracted at the end of the computation, with its range stemming from equation (2.20), i.e. $N = 0, \ldots, 4k + 2$. One also needs to remember about the constant present in (2.20). We need to use a sufficiently large value of $k$, so that the equivalence of both volume operators holds for the considered order of $t$. Since every commutator involving the volume contributes with $1/t$, to get correct results at order $O(t)$ we need $k = 1 + \#(\text{nested commutators})$, where the number of nested commutators in the evaluated expression refers to the maximal number of nested commutators involving the volume.

As a result, we work with an operator which is a polynomial in $\hat{h}(v, \mu)$ and $\hat{Q}_v$, likely expressed through commutators of these operators.

To go further we create two new objects, $\mathcal{P}_{\text{cl}}$ and $\mathcal{P}_{\text{qu}}$, in which we will collect the aforementioned classical and quantum contributions. The algorithm starts by denoting the whole operator of interest as (for now only) element of $\mathcal{P}_{\text{cl}}$ and we set $\mathcal{P}_{\text{qu}}$ as being empty.

The proper part of our algorithm consists of the following steps:

(a) To resolve commutators appearing in the evaluated operator, using explicit expressions suitable to keep the track of the $O(t^1)$ order;

(b) To use the modified commutation relations from the previous step to shift all the holonomies to the left of every monomial;
(c) To appropriately handle $\hat{P}(v, \mu)$ operators necessarily appearing due to the commutation relations;

(d) To perform a 'link splitting' of every monomial present in both $\mathcal{P}_c$ and $\mathcal{P}_{q_0}$;

(e) To simplify the products of the holonomies at the same edge;

(f) To employ final replacements of the particular expectation values by specific functions.

Each subsection below provides a more detailed description of the above steps.

4.3.1. **Step I: resolving the commutators.** In order to simplify the consideration we introduce two new operators

$$\hat{E}_\pm(v, \mu) = \hat{P}(v, \mu) \pm \hat{P}(v, -\mu), \quad \mu = 1, 2, 3.$$  \hspace{1cm} (4.7)

Note that we restricted the range of $\mu$ to its positive values. We then search $\mathcal{P}_c$, which contains the operator to be processed, for nested commutators. We take the inner most commutator and replace it according to the following three exact rules, which are (2.4) and (2.5) rewritten (summation convention applies to indices which do not appear in original commutators on the left hand sides):

$$[\hat{h}_{ab}^{(j)}(v, \mu), \hat{h}_{cd}^{(j')} (v', \mu')] \rightarrow 0, \quad [\hat{Q}_c \hat{Q}_c^N] \rightarrow 0,$$  \hspace{1cm} (4.8a)

$$[\hat{h}_{ab}^{(j)}(v, \mu), \hat{E}_{\pm}(v', \mu')] \rightarrow \mathcal{E}\delta_{v,v'} (\delta_{\mu,\mu'} \pm \delta_{\mu,-\mu'}) [n \tau n]^{(\mu, jK)}_{(a \mu K)} \hat{h}_{ab}^{(j)}(v, \mu) + \mathcal{E} \delta_{v,v'} (\delta_{\mu,\mu'} \pm \delta_{\mu,-\mu'}) \hat{h}_{ab}^{(j)}(v, \mu) [n \tau n]^{(\mu, jK)}_{(a \mu K)},$$  \hspace{1cm} (4.8b)

$$[\hat{E}_{\pm}(v, \mu), \hat{E}_{\pm}(v', \mu')] \rightarrow -\mathcal{E} \delta_{v,v'} f_{IK} (\delta_{\mu,\mu'} \pm \delta_{-\mu,-\mu'}) \hat{E}_{\pm}(v, \mu),$$  \hspace{1cm} (4.8c)

and two rules (which are approximations following from observation 3):

$$[\hat{h}_{ab}^{(j)}(v, \mu), \hat{Q}_c^N] \rightarrow -6i\mathcal{E}N \text{sgn}(\mu) \left( \hat{A}_{ab}^{(j)}(v, v', \mu) \hat{Q}_c^{N-1} - \frac{N-1}{2} [\hat{A}_{ab}^{(j)}(v, v', \mu), \hat{Q}_c] \hat{Q}_c^{N-2} \right),$$  \hspace{1cm} (4.8d)

$$[\hat{E}_{\pm}(v, \mu), \hat{Q}_c^N] \rightarrow 6i\mathcal{E}N \delta_{v,v'} \left( \hat{B}_{\pm}^{(j)}(v, \mu) \hat{Q}_c^{N-1} - \frac{N-1}{2} [\hat{B}_{\pm}^{(j)}(v, \mu), \hat{Q}_c] \hat{Q}_c^{N-2} \right),$$  \hspace{1cm} (4.8e)

where

$$\hat{A}_{ab}^{(j)}(v, v', \mu) = \epsilon_{ijk}^{(a \mu j k)} \left( \delta_{v,v'} \delta_{\mu,\mu'} [n \tau n]^{(\mu, jK)}_{(i \mu K)} \hat{h}_{ab}^{(j)}(v, \mu) + \delta_{v,v'} \delta_{\mu,\mu'} \hat{h}_{ab}^{(j)}(v, \mu) [n \tau n]^{(\mu, jK)}_{(a \mu K)} \right) \hat{E}_{\pm}^{(j)}(v', \mu_1) \hat{E}_{\pm}^{(j)}(v', \mu_2),$$  \hspace{1cm} (4.8f)

$$\hat{B}_{\pm}^{(j)}(v, \mu) = \epsilon_{ijk}^{(a \mu j k)} \hat{E}_{\pm}^{(j)}(v, \mu_1) \hat{E}_{\pm}^{(j)}(v, \mu_2) f_{IK} \hat{E}_{\pm}^{(j)}(v, \mu),$$  \hspace{1cm} (4.8g)

$$[n \tau n]^{(\mu, k)}_{(a \mu K)} := D^{(1/2)}_{ab} (n_{ab}) \tau^{K (1/2)}_{\mu K} D^{(1/2)}_{c c'} (n_{ab}) (-1)^{c'-c}. $$  \hspace{1cm} (4.8h)

In the above formulas we introduced the following additional pieces of notation:
\[ \tilde{\mu}_s = (|\mu| + s) \mod 3 \equiv (|\mu| + s) - 3 \left(\left\lfloor \frac{|\mu| + s}{3} \right\rfloor \right), \]
where \( \left\lfloor \cdot \right\rfloor \) is the floor function, defined in terms of the modulo operation with offset 1. Note that \( \tilde{\mu}_0 = |\mu|; \)

- The symbol \( \pm \cdot \pm' \) is understood as ‘+’ if \( \pm \) and \( \pm' \) are the same signs and ‘−’ in the opposite case;
- \( \Upsilon \) is an additional, auxiliary parameter indicating the splitting between the classical and the quantum part, further used by the algorithm to distribute the terms between \( P_{cl} \) and \( P_{qu} \). At the end all appearances of \( \Upsilon \) are removed such that its value does not affect the result.

After the replacement of the innermost commutators in \( P_{cl} \) is done we update both objects as follows:

\[ P_{qu} := P_{qu} + \frac{dP_{cl}}{d\Upsilon} \bigg|_{\Upsilon=0}, \quad P_{cl} := \lim_{\Upsilon \to 0} P_{cl}. \]

We repeat the above procedure, successively resolving all the commutators in \( P_{cl} \). Note that in the first application of the commutation rules, only the rules which do not involve \( \hat{E}_\pm^n(v, \mu) \) can enter the game.

After \( P_{cl} \) is free from commutators we need to replace all the commutators inside \( P_{qu} \). We shall do this successively by using the commutation rules with \( \Upsilon = 0 \). We also neither update \( P_{cl} \) nor \( P_{qu} \) at that stage. As a result of the whole first step of the algorithm, both objects \( P_{cl} \) and \( P_{qu} \) are sums of monomials.9

### 4.3.2. Step II: shifting the holonomies.

In the first part of this step we work with monomials inside \( P_{cl} \). We successively replace

\[ \hat{X} h_{ab}^{(j)}(v, \mu) \to \hat{h}_{ab}^{(j)}(v, \mu) \hat{X} + \Upsilon [\hat{X}, \hat{h}_{ab}^{(j)}(v, \mu)], \]

where \( \hat{X} \) can either be \( \hat{E}_\pm \) or \( \hat{Q} \). After each replacement we apply the update rule equation (4.9) and continue the procedure until all the holonomies in \( P_{cl} \) are shifted to the left.10

When the above procedure is over, the object \( P_{qu} \) is complete. We only need to resolve (without the update step) the new commutators which entered \( P_{qu} \). Due to observation 3, the order of the elements in \( P_{qu} \) is irrelevant.

### 4.3.3. Step III: dealing with \( \hat{E}_\pm \).

First, we work with the quantum corrections as this is much easier. We simply replace

\[ \hat{E}_\pm^n(v, \mu) \to e^{2i\eta t} \hat{E}_\pm^n(v, \mu), \]

\[ \hat{E}_\pm^n(v, \mu) \to 0, \]

for all relevant instances inside \( P_{qu} \). This can be done due to being the leading order corrections according to observation 2.

Note that due to the above rule we can freely commute \( \hat{E}_\pm \) with \( \hat{Q} \) also inside \( P_{cl} \). This simplification is possible because even though a generic replacement \( \hat{E}_\pm^n(v, \mu) \hat{Q}^N \to \hat{Q}^N \hat{E}_\pm^n(v, \mu) \)

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9 Replacing the commutator not only bears the advantage of reducing the number of involved terms as compared to expanding \([A, B] \to AB - BA\), but also allows us to neglect those contributions which are already of higher of order in \( t \) which come from \( \hat{Q}' \).

10 This step is basic algebra, but is needed to bring each monomial in the form to ultimately apply (2.22).
performed inside $\mathcal{P}_{cl}$ would potentially bring to $\mathcal{P}_{qu}$ a contribution involving $[\hat{E}_{\pm}^R(v, \mu), \hat{Q}_d^\mu]$, such terms do always vanish due to (4.11). For $\hat{E}_-$ this is immediate as the commutator in question only involves $\hat{E}_+$. For $\hat{E}_-$ we get the factor $\epsilon_{0)(k_1\mu_2}f_{k_30}$, which after summing over $K$ leads to $-f_{000} \equiv 0$ due to (3.6).

We therefore use the above fact to freely shift all $\hat{Q}$ appearing inside $\mathcal{P}_{cl}$ to the right. Therefore, $\mathcal{P}_{cl}$ is a collection of monomials of the form $'\sim h\hbar \hat{E}_- \hat{Q} \hat{Q}'$, where with $\sim$ we cover all non-operator factors to be contracted. For the quantum corrections we already obtained a simplified form $'\sim h\hbar \hat{Q} \hat{Q}'$.

For the monomials constituting $\mathcal{P}_{cl}$ we need to employ a more elaborate replacement:

\[ \hat{E}_-^K(v, \mu) \rightarrow \hat{E} \left( R^K(v, \mu) \mp L^K(v - e_{\pm}, \mu) \right). \]  

(4.13)

Afterwards we ‘abelian’ shift the rightmost $L$ operator to the right (through all $R$), and in this particular position replace it according to the rule (3.1) from observation 4:

\[ L^K(v, m) \rightarrow e^{K_R}R^K(v, m). \]  

(4.14)

We successively repeat this procedure until all operators are of $R$-type.

4.3.4. Step IV: link-splitting. Before we will be able to perform the splitting, we first need to replace the holonomies by $D$ and $D^\dagger$ operators according to the formula (4.6). This is very important as the holonomy with different sign of $\mu$ gives $D$ or $D^\dagger$ on a different vertex.

After the above step is done we are also allowed to impose the periodic boundary conditions, replacing all vertices by its coordinates modulo $M$ (this step is essential when the computer-algebra-methods are involved). If need be, one shall also eliminate trivial products of holonomies according to the formula:

\[ D^{(j)}_{ab}(v, \mu) [D^{(j)}_{ba}(v, \mu)]^\dagger \rightarrow \delta_{ab}/(2j + 1). \]  

(4.15)

which follows from $D^{(j)}$ being unitary matrices.

Finally, we use the identity $[D^{(j)}_{ab}]^\dagger = (-1)^{b-a}D^{(j)}_{b-a}$ to replace all $D^{(j)}$.

At that stage both $\mathcal{P}_{cl}$ and $\mathcal{P}_{qu}$ are collections of monomials, i.e.

\[ \mathcal{P}_{cl/qu} = \sum_s \mathcal{P}_{cl/qu}^{(s)}, \]  

(4.16)

with all $\mathcal{P}_{cl/qu}^{(s)}$ being of the form $'\sim D \cdot DR \cdot R\hat{Q} \cdot \hat{Q}'$. We are therefore ready for the proper part of the link splitting. To this end we write

\[ \mathcal{P}_{cl/qu} = \sum_s \prod_{v \in [2]^3} \left( \prod_{\lambda=1}^3 \xi^{(s)}_{cl/qu}(v, m) \right) \hat{Q}_v^{N_v}, \]  

(4.17)

where each $\xi^{(s)}_{cl/qu}(v, m)$ is a monomial $'\sim D \cdot DR \cdot R$ with all $D$ and $R$ taken at the same $(v, m)$. By $N_v$ we denote the collected power of $\hat{Q}_v$ at $v$.

4.3.5. Step V: simplification of holonomies. After the link splitting, for a fixed monomial and at a given $(v, m)$ we are left with the product of holonomies preceding the product of the $R$ operators. However, we can reduce every product of holonomies to a sum of single $D$ operators\textsuperscript{11}.

\textsuperscript{11} Therefore, each term becomes closer to the form needed for applying (2.22), which involves a single $D$ operator.
To this end we just recursively apply the formula

\[ D_{ab}^{(j_1)}(v, m) D_{cd}^{(j_2)}(v, m) = \sum_{j_{tot} = |j_1 - j_2|}^{j_1 + j_2} Z_{j_1 j_2 j_{tot}}[a, b, c, d] D_{a-c, b-d}^{(j_{tot})}(v, m), \]  

(4.18a)

where

\[ Z_{j_1 j_2 j_{tot}}[a, b, c, d] = (2 j_{tot} + 1)(-1)^{e+c-b-d} \begin{pmatrix} j_1 & j_2 & j_{tot} \\ a & c & a+c \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_{tot} \\ b & d & b+d \end{pmatrix}, \]  

(4.18b)

which simply represents known rules for addition of spin in the fashion of the SU(2) group [39, 40].

As has already been mentioned, knowing the maximal value of \( j_{tot} \) appearing in the whole computation would make the task faster, though this is not a requirement of the algorithm.

Note that after this step, the structure of \( \mathcal{P}_{cl/qu}^{(s)} \) does not change, we just have more terms to be summed over \( s \). In a way we take each \( \mathcal{P}_{cl/qu}^{(s)}(v, m) \) and transform it into many such terms, however, now every monomial has a simpler form \( \sim DR \ldots R \). One just needs to remember about this fact.

4.3.6. Step VI: final replacements. According to all previously described simplifications and after looking at the form we arrived at, we can conclude that the average value of the both the classical and the quantum contribution can be calculated with the help of the results from theorem 1. To this end we define

\[ \hat{F}_{cl/qu}^{(s)} = \prod_{v \in \mathbb{Z}^3} \prod_{m=1}^{3} F_{cl/qu}^{(s)}(v, m), \]  

(4.19)

so that

\[ \mathcal{P}_{cl/qu}^{(s)} = \hat{F}_{cl/qu}^{(s)} \prod_{v \in \mathbb{Z}^3} \hat{Q}_{e}^{N_v}. \]  

(4.20)

Therefore \( \langle \Psi_t^{\gamma}, \hat{F}_{cl}^{(s)} \prod_{e=1}^{r} \hat{Q}_{e}^{N_n} \Psi_t^{\gamma} \rangle \) can directly be calculated from equation (3.14), while for \( \langle \Psi_t^{\gamma}, \hat{F}_{cl}^{(s)} \prod_{e=1}^{r} \hat{Q}_{e}^{N_n} \Psi_t^{\gamma} \rangle \) it is sufficient to take only the contribution from (3.12). At the end we need to sum all contributions (i.e. sum over \( s \)) together with their prefactors (not mentioned here explicitly). As stated at the beginning, all indices also need to be contracted.

5. Expectation value of the Euclidean scalar constraint

Having developed the algorithm, we now apply it to the case of Euclidean scalar constraint. We also supply an independent calculation in order to validate the algorithm.

5.1. Discretisation and quantisation of the scalar constraint

The scalar and diffeomorphism constraint encapsulate the dynamical content of general relativity in the continuum. However, in the present work we consider its discretisation on a
spatial lattice and their action cannot straightforwardly be lifted to the lattice, but needs to be approximated by the quantities of the lattice phase space.\(^\text{12}\) Several patterns of discretisation are possible, however we will focus in this work on a single strategy and elucidate on some necessary choices. Moreover, this section considers only the Euclidean part of the scalar constraint \(C_E\) from (A.5) as the Lorentzian part works analogously.

First, we will employ Thiemann’s identity \([27, 28]\) to express:

\[
C_E = \frac{4}{\sqrt{\beta}} f_{ab}^\perp(A) e^{abc} \{ V[\sigma], A_i^\perp \}
\]

with \(V[\sigma] = \int \text{d}x^3 \sqrt{\text{det}(E)}\). We can express this object now in terms of the holonomies and fluxes of the lattice from (A.6) and (A.7) for edges of coordinate length \(\epsilon\) in order to obtain an expression that reduces to \(C_E(x = v)\) in the limit \(\epsilon \to 0\).\(^\text{38}\)

\[
C_E(v) := -\frac{4}{\sqrt{\beta}} \sum_{i,j,k \in L} |(i,j,k)| \text{tr} \left[ (h(A_{e_{i,j}k}) - h(\square_{e_{i,j}k}))h(e_{i})\{ h^\dagger(e_{i}), V'[\sigma] \} \right],
\]

with \(L = \{1, 2, 3, -1, -2, -3\}\) labelling the different directions of the lattice and \(e_i\) indicating an edge starting from \(v\) in direction \(k\). Also \(\square_{e_{i,j}k}\) indicates the minimal plaquette of the lattice starting at \(v\) along direction \(i\) and returning along direction \(j\). Finally, for the cubic lattice \(T_v = 2^3 = 8\) and

\[
V'[\sigma] := \sum_v \sqrt{|Q_v|/(6T_v)},
\]

\[
Q_v = \sum_{v' \cap v \cap v''} \epsilon_{ijk} e_{ijk}^v. \tag{5.3}
\]

In order to quantise this expression, we express holonomies and fluxes by quantum operators, replace \(ih\{ , \} \to [ , ]\) and use the Ashtekar–Lewandowski volume operator for (5.3). Finally, since we are only interested in expectation values including the next-to-leading order we switch the latter one to the Giesel–Thiemann volume, ending up with the quantum operator:

\[
\hat{C}_E = p_{bac} \sum_{i,j,k} \left( \hat{h}(\square_{e_{i,j}k}) - \hat{h}(\square_{e_{i,j}k})\hat{h}(e_{i})\{ \hat{h}^\dagger(e_{i}), \hat{V}_{2i}^G + \hat{V}_{2i+1}^G \} \right) \tag{5.4}
\]

with \(p_{bac} = 1/(i\hbar^2)\). A few remarks are due:

- The appearance of the additional \(\hat{V}_{2i+1}^G\) inside the commutator is different from other choices in the literature [23]. However, we take this choice in order to restore the discrete version (5.2) in the classical limit, which we understand as the leading order of the expectation value in coherent states. Would we not have this term, we would get a discretised constraint on the lattice with only \(V'[\sigma]\) in the Poisson bracket, not \(V[\sigma]\). As it was shown in [38] if not at least one of the arguments of the Poisson bracket is invariant under the symmetries of the lattice, the symmetry restriction to the reduced phase space, spanned by coordinates \((p, c)\), does not work. In particular, to compute the dynamics of the scalar constraint in cosmology, one cannot first restrict and then compute its flow. The latter one

\(^\text{12}\) Notice, that this is different to the situation of the Gauss constraint: (A.2) and its flow on the phase space given by the vector field \(\{G, \cdot, \cdot\}\) can be translated into a discrete version, given by \(G_i(v) = \sum_{x \cap y \cap z \cap v} s_{x,y} P^v(e_i)\), see [38] for further details.
is only possible if one argument of the Poisson bracket is indeed symmetric. However, this is a prerequisite of the effective dynamics conjecture which we will discuss in section 6. Thus, in order to have a chance for it to work, we add the additional term $\tilde{V}_{v+\gamma}$.  

- We use the 2nd Giesel–Thiemann operator $\hat{V}_{2,v}$ albeit being only interested in the first order $\mathcal{O}(t)$. This is due to the fact, that the volume appears inside the commutator: here the $t^0$-order of the expectation value is indeed cancelling. The $t^1$-order becomes the classical order (yielding the classical term due to the $1/t$ in $p_{Riem}$) and the interesting next-to-leading order is indeed at $t^2$.

5.2. Analytic computation of the expectation value of the Euclidean scalar constraint

In order to showcase the above described step-by-step guide, we have implemented it in a Mathematica file. In this section, we will as an example highlight which steps the algorithm takes when evaluating the expectation value of cosmological coherent states for the euclidian part of the scalar constraint. We state the claim produced by the machine and afterwards verify it via a lengthy analytical computation, showcasing the strength of the compact algorithm and providing a consistency check.

The goal is to compute $\mathcal{C}_t^R$ from (5.4) in cosmological coherent states. Upon this task, our algorithm is doing the following:

I. Resolve commutators: there are two commutators at classical order to resolve. After replacing $V^{GT} \rightarrow c_N \hat{Q}^N$ we attack them using equation (4.8d), e.g.

$$\hat{h}(v, 3)[\hat{h}^\dagger(v, 3), \hat{Q}^N_c] \rightarrow -6i\hat{E}N\left(\hat{c}_{iljk}^{(6,123)}[n\tau n]^{(3,1/2,2)} \hat{E}_l^I(v, 1) \hat{E}_k^I(v, 2) \hat{Q}_v^{N-1}
\right.

- \left. \frac{\mathcal{Y}^N}{2} \hat{h}(v, 3)[\hat{A}^0_\alpha(v, 3), \hat{Q}_v]\hat{Q}_v^{N-2}\right)

- \left. -6i\hat{E}N\left(\hat{c}_{iljk}^{(6,123)}[n\tau n]^{(3,1/2,2)} \hat{E}_l^I(v, 1) \hat{E}_k^I(v, 2) \hat{Q}_v^{N-1}
\right.

- \left. \frac{\mathcal{Y}^N}{2} \hat{h}(v, 3)[\hat{A}^0_\alpha(v, 3), \hat{Q}_v]\hat{Q}_v^{N-2}\right)

\left. \times \left( -\hat{A}^0_\alpha(v, 3) \hat{E}_l^I(v, 1) \hat{E}_k^I(v, 2) + \hat{B}_l^I(v, 1) \hat{E}_k^I(v, 2)
\right.

+ \left. \hat{B}_l^I(v, 2) \hat{E}_k^I(v, 1) \right) \hat{Q}_v^{N-2}\right).

II. Shift holonomies to the left: we are in the fortunate situation that all holonomy operators are already on the left from the onset and therefore there is nothing to do.

III. Dealing with vector fields: as discussed above, $\hat{E}_v$ and $\hat{Q}_v$ commute freely thus we can shift all of the $\hat{E}_v$ to the right and replace according to (4.11) and (4.13), respectively. The latter one creates left-invariant vector fields which we immediately turn into a right-invariant vector fields via (3.1).

IV. Link splitting: one has to proceed with each of the terms separately. There are currently 48 in $\mathcal{P}_d$ and 36 in $\mathcal{P}_q$ of which we show one explicitly as an example:

$$\hat{c}_{iljk}^{(6,123)}(\psi^{(\alpha)}_\gamma, \hat{h}(\tilde{\Gamma}_{\alpha,12})) - \hat{h}^\dagger(\tilde{\Gamma}_{\alpha,12}))[n\tau n]^{(3,1/2,2)} R^I(v, 1)(e^{\hat{c}_{iljk}^{(6,123)} \hat{E}_l^I(v, 1)(e^{\hat{c}_{iljk}^{(6,123)} \hat{E}_l^I(v, 1)}) \psi^{(\alpha)}_\gamma}$$

$$\Rightarrow$$
\[ l_{(v,1)} = D_{ab}^{(1/2)} R_{ab}^{(1/2)}, \quad l_{(v+e_1,2)} = D_{bc}^{(1/2)}, \quad l_{(v+e_2,1)} = [D_{cd}^{(1/2)}], \quad l_{(v,2)} = [D_{de}^{(1/2)}], \quad l_{(v-e_2,2)} = R_{ec}^{(2)} \]

(5.6)

V. Simplify products of holonomies: there are no products of holonomies on a single edge appearing, therefore this step finishes after removing the complex conjugation on some Wigner matrices.

VI. Final replacement: we replace each \( l_e \) according to (3.12) and (3.14). As these are several terms we omit them due to lack of space and employ Mathematica implementation of the algorithm. It produces the following result:

Claim: the expectation value of \( \hat{C}_E^{(v)}(v) \) in cosmological coherent states \( \Psi_i \) from (2.10) parameterized by \( \eta = 2e^p/\rho (\ell^2, \beta) \) and \( \xi = cc \) is:

\[
\langle \Psi_i, \hat{C}_E^{(v)}(v) \Psi_i \rangle = \frac{6}{\rho^2} e^{-\rho} \sin(\xi)^2 \left[ 1 + t \left( -\frac{1}{4} - \frac{13}{8} \coth(\eta) + \frac{11}{8} \sinh(\eta) + \frac{9}{32 \eta^2} \right) + \frac{3}{8 \eta} \sin(\xi/2)^2 \right] + O(\ell^2).
\]

(5.7)

Proof. We will now supplement an analytic proof of the claim, thereby presenting a consistency check for the algorithm. Note that during this proof all equations are understood as correct up to \( O(\ell^2) \) even if not indicated explicitly.

As initial step, we make use of the fact that the expression on the lattice is invariant under rotations of ninety degrees and reflections, as well is the state \( \Psi_i \) which mimics an isotropic Universe. Instead of all triples \( j, k, \) we will fix \( i = +1, j = +2, k = +3 \) and use that all \( 3 \times 3 \) triples will result in the same expectation value and can then be summed by linearity.

We split our computation into two parts

\[
\langle \Psi_i, \hat{C}_E^{(v)}(v) \Psi_i \rangle = \rho_{\text{ext}} \frac{(\beta \hbar c)^{3/2}}{2^{3/2} \sqrt{3}} (3 \times 2^3) \left( \langle \Psi_i, \hat{C}_E^{(1)}(v) \Psi_i \rangle + \langle \Psi_i, \hat{C}_E^{(2)}(v) \Psi_i \rangle \right),
\]

(5.8) \hspace{1cm} \langle \Psi_i, \hat{C}_E^{(1)}(v) \Psi_i \rangle = \sum_{N=0}^{\infty} c_N \langle \Psi_i, (\hat{h}_R \hat{\Phi}_{e_1/2} - \hat{\Phi}_{e_1/2} \hat{h}_R) \hat{h}(v_3) \hat{h}(v_3) \hat{\Phi}_{e_1/2} \rangle \Psi_i, (5.10) \]

First, we focus on \( \hat{C}_E^{(1)}(v) \): we denote by \( \langle \hat{F} \rangle \) the expectation value of a polynomial \( \hat{F} \) in the state \( \psi_n \) with \( \hat{h} = \exp(-i(\xi - i) \tau_3) \) that is \( n = \bar{n} = \text{id} \). This shortcut is used when considering all combinations of the product \( \hat{\Phi}_{e_1/2} \) and extracting the \( SU(2) \)-elements \( n_1, n_2, n_3 \). This is done analogously to step IV of the algorithm—link splitting—i.e.:

\[
\langle \Psi_i, \hat{C}_E^{(1)}(v) \Psi_i \rangle = \sum_{N=0}^{\infty} c_N (6 \rho)^{N/2} \epsilon_{\lambda_1, \lambda_1, \lambda_1, \ldots, \lambda_N, \lambda_N, \lambda_N} \sum_{s_1, s_2, s_3} \binom{N}{s_1} \binom{N}{s_2} \binom{N}{s_3} \times D_{ab}^{(1/2)}(n_1) \hat{h}_R^{(1/2)} \ldots \hat{h}^{(1/2)} \hat{h}^{(1/2)}(n_2) \hat{h}_R^{(1/2)}
\]

\[
\times D_{cd}^{(1/2)}(n_2 n_1) \hat{h}_R^{(1/2)} \ldots \hat{h}^{(1/2)} \hat{h}^{(1/2)} \hat{h}_R^{(1/2)} \quad \hat{C}^{(1/2)}(v_3) \hat{C}^{(1/2)}(v_3) \hat{C}^{(1/2)}(v_3)
\]

(5.10)
\[
\times \left( \hat{h}_{ij} \left[ \hat{h}^i_{jkl}, R^{kl} \ldots R^{k_3} \right] \right) - (n_1 \leftrightarrow n_2)
\times \langle R^{k_1+1} \ldots R^{k_N} \rangle \langle R^{j_2+1} \ldots R^{j_N} \rangle \langle R^{k_3+1} \ldots R^{k_N} \rangle.
\]

(5.11)

Following the brute force computation, we need to replace all appearances of \((\ldots)\) with (2.22) and—in theory—sum over all combination of indices \(I_A, J_A, K_A\). However, we will see that in practice most of the summations are contracted with \(\delta_s\), as more overall non-vanishing indices imply a higher powers of \(t\).

First, observe that the fourth line of (2.22) never contributes: \(O(t)\)-correction due to two right-invariant vector fields with non-zero indices, however (3.6) implies that also two other indices need to be non-vanishing, thus of non-leading order. Hence, their product would be \(O(\text{tr})\) and can thus be neglected.

Similarly, after the commutator has been resolved with (2.4), none of the remaining vector field with label \(K\) can have a non-zero index, else (3.6) causes it to be of order \(K\) due to requiring a non-zero index some other edge. The only non-trivial case is an index \(I_A\) or \(J_A\) being non-zero, where \(R^{k_A}\) got absorbed by the commutator.

Therefore, only the following cases are remaining:

- \(A^{CI}\), the purely classical, i.e. leading order contribution, where all indices are zero.
- \(A^{[0]}\), the next-to-leading order corrections due to (2.23) and to the second line of (2.22) for any edge.
- \(A^{[e_1,0]}\), the next-to-leading order correction due to the third line of (2.22) from the edge on which the commutator was supported and all indices vanishing.
- \(A^{[e_1,K_A]}\), the next-to-leading order correction due to the third line of (2.22) from the edge on which the commutator was supported and all but one index vanishing.
- \(A^{[e_1,0]}\) and \(A^{[e_2,0]}\), the next-to leading order corrections from the edges \(e_1\) and \(e_2\), respectively, due to the third line of (2.22) and all indices vanishing.
- \(A^{[e_1,J_A]}\) and \(A^{[e_2,J_A]}\), the next-to-leading order corrections from the edges \(e_1\) and \(e_2\), respectively, due to the third line of (2.22) and all but one index vanishing.

Explicitly, upon plugging (2.22) into (5.11), we obtain:

\[
\langle \hat{\Psi}_1, \hat{C}_E^{(1)} \hat{\Psi}_1 \rangle = (A^{CI} + A^{[0]} + A^{[e_1,0]}) + A^{[e_1,K_A]} + (A^{[e_1,0]} + A^{[e_2,0]}) + (A^{[e_1,J_A]} + A^{[e_2,J_A]})
\]

(5.12)

and attack the individual terms in the following, starting with:

\[
(A^{CI} + A^{[0]} + A^{[e_1,0]}) = -\sum_{N=1}^{10} (6i)^N C_N (-)^N \sum_{s_1,s_2,s_3=0}^{N} \left( \begin{array}{c} N \\ s_1,s_2,s_3 \end{array} \right) \left( \begin{array}{c} N \\ s_2,s_3 \end{array} \right) \left( \begin{array}{c} N \\ s_1 \end{array} \right)
\times 2 \text{Im} \left[ D_{\alpha\beta} \left( U_1 U_2 U_3 \right) \left( \gamma_{1/2} \right)^{1/4} \left( \eta / t \right)^{3N-1} \right]
\times \prod_{i=1}^{2} \left[ 1 + \frac{t}{2\eta} \left( \frac{s_i(s_i+1)}{2\eta} - s_i \coth(\eta) \right) \right]
\times \prod_{i=1}^{3} \left[ 1 + \frac{t}{2\eta} \left( \frac{(N-s_i)(N-s_i+1)}{2\eta} - (N-s_i) \coth(\eta) \right) \right]
\]
\[
\times \sum_{A=1}^{10} \tau_{K_A} \omega \delta_{K_A,0} \left[ 1 + \frac{t}{2\eta} \left( \frac{(s_3 - 1)s_3}{2\eta} - (s_3 - 1) \coth(\eta) \right) \right]
\]  
(5.13)

with \( U_i = D^{(1/2)}(n_i e^{-\tau_0^i} n_i^t) \), the classical expression of the holonomy in cosmology, and using that \( 2\text{Im}(A) := A - A^\dagger \). We introduce

\[
U_{ij} := U_i U_j U_i^\dagger U_j^\dagger \]  
(5.14)

and remember that

\[
\tau_a := D^{(1/2)}(n_a) \tau_0 D^{(1/2)}(n_a^t).
\]  
(5.15)

This enables us to shorten the result, including a final evaluation of the trace and binomial sums (e.g. via Mathematica):

\[
\left( A^C + A^{(1/2)} + A^{(0,0)} \right) = -\sum_{N=1}^{10} (6i)^N c_N (-)^N \text{tr}(U_{12} \tau_3 - U_{21} \tau_3) \
\times \left( \frac{\eta^i}{t} \right)^{3N-1} \sum_{s_3=0}^{2N} \binom{N}{s_3} (s_3) \
\times \left[ 1 + \frac{t}{2\eta} \left( -\frac{\eta}{2} - 2 \tanh \left( \frac{\eta}{2} \right) - (3N - 1) \coth(\eta) \right) \right] \
+ \frac{t^2}{4\eta^2} \sum_{s=0}^{N} \binom{N}{s} (2(N - s)(N - s + 1) + 2s(s + 1) \
+ (N - s_3)(N - s_3 + 1) + s_3(s_3 - 1)) \right] 
\]  
(5.16)

The next contribution is easily handled upon finding that \( \tau_0^2 = 1 \) and \( \text{tr}(U_{ij} - U_{ji}) = 0 \):

\[
A^{(e_3,K_A)} = -\sum_{N=1}^{10} (6i)^N c_N (-)^N \sum_{s_1,s_2,s_3=0}^{N} \binom{N}{s_1} \binom{N}{s_2} \binom{N}{s_3} \text{Im} \ 
\times \left[ D_{(1/2)}(A_1 \tau_3 \tau_2) \left( \frac{\eta^i}{t} \right)^{3N-1} \frac{t}{2\eta} \sum_{A=B}^{10} \tau_{K_A} \tau_{K_B} \omega \delta_{K_A,0} \delta_{K_B,0} \right] 
\]  
(5.17)

\[
= 0.
\]
For the first quantum corrections on the plaquette-holonomies, we use again (5.15) and

$$D^{(1)}_{\gamma^K}\ell(g)\{\tau^{J(1)}_{\mu\nu}\}_{\mu\nu} = D^{(0)}_{\mu\nu}(g)^{\dagger}\{\tau^{K(0)}_{\mu\nu}\}_{\mu\nu}D^{(0)}(g).$$ (5.18)

Then, when doing similar manipulations as before:

$$(A^{[1,0]}_{\gamma^K}\ell(g) + A^{[2,0]}_{\gamma^K}\ell(g))$$

$$= -\sum_{N=1}^{10} (6i)^N c_N(-)^N \frac{\sum_{s_1, s_2}^{N} \left( N \right)_{s_1} \left( N \right)_{s_2} \left( N \right)_{s_3}}{s_3}$$

$$\times \left( -\epsilon_{000}^{(1)} \right) \left( \eta \frac{t}{l} \right)^{3N-1} \sum_{s_3}^{N-1} \left( \sum_{A=0}^{s_3} \tau_{3A} \right)$$

$$\times D_{\text{tr}}(\frac{4}{6}) \left( \tau_1 U_{12} - U_{21}\tau_1 \right) + D_{\text{tr}}(\frac{4}{6}) \left( U_{12}\tau_n \tau_1 U_{21} \right)$$

$$= -\sum_{N=1}^{10} (6i)^N c_N(-)^N \frac{\sum_{s_1, s_2}^{N} \left( N \right)_{s_1} \left( N \right)_{s_2} \left( N \right)_{s_3}}{s_3}$$

$$\times \left( \frac{\eta}{l} \right)^{2N-1} \sum_{s} \left( \frac{N}{s} \right) s \text{tr}(\tau_3 U_{12} - \tau_3 U_{21})$$

$$= -\sum_{N=1}^{10} (6i)^N c_N(-)^N \frac{\sum_{s_1, s_2}^{N} \left( N \right)_{s_1} \left( N \right)_{s_2} \left( N \right)_{s_3}}{s_3}$$

$$\times \frac{\eta}{l} \left[ \frac{N^2}{l} \right] \left[ 2 \sin(\xi/2) \sin(\xi) \right].$$ (5.19)

It only remains the contribution with non-vanishing indices from the right-invariant vector fields:

$$(A^{[1,1]_{\gamma^K}\ell(g)} + A^{[2,1]_{\gamma^K}\ell(g)}) = -\sum_{N=1}^{10} (6i)^N c_N(-)^N 2^N \sum_{s_3}^{N-1} \left( \frac{N}{s_3} \right) s_3 \left( \frac{N-1}{s} \right)$$

$$\times \left( \frac{\eta}{l} \right)^{3N-1} \frac{\eta}{l} \left[ \frac{\tau_{3A}}{l} \right]_{\text{tr}}$$

$$\left[ (-\epsilon_{000}^{(1)} \tau_{3A}) \left( 1 - I_A \tanh \left( \frac{\eta}{l} \right) \right) \right]$$

$$\times D_{\text{tr}}(\frac{4}{6}) \left( n_1 \tau_a \right) D_{\text{tr}}(\frac{4}{6}) \left( \tau_a \right)$$

$$= U_{12} n_1 \left( \tau_a \right) \delta_{a1} \left( 1 - I_A \tanh \left( \frac{\eta}{l} \right) \right)$$

$$\times D_{\text{tr}}(\frac{4}{6}) \left( U_{12} n_1 \left( \tau_n \tau_a \right) \right) \delta_{a1}$$

$$= \frac{\eta}{l} \left[ \frac{N^2}{l} \right] \left[ 2 \sin(\xi/2) \sin(\xi) \right].$$
\[
\times \text{tr} \left( \epsilon_{i,0,K}^{(0)} \tau_K (n_1 \tau n_1^1 U_{12} + U_{21} n_1 \tau n_1^1) \right)
- \epsilon_{i,0,K}^{(0)} \tau_K (U_{12} n_2 \tau n_2^1 + n_2 \tau n_2^1 U_{21})
\]
\[
= - \sum_{N=1}^{10} (6i)^N c_N \left( -\gamma^N \left( \frac{\eta}{r} \right)^{3N-1} 2^{3N-1} \right)
\times \left( -\frac{iN}{\eta} \left[ -i \sin(\xi) - 2 \sin(\frac{\xi}{2}) \sin(\xi) \right] \right) .
\]

(5.20)

Finally, we can collect all contributions together and plug them into (5.12):

\[
\langle \Psi_{\tau}, \hat{\mathcal{C}}_E^{(1)} \Psi_{\tau} \rangle = +i \sqrt{3} \eta \sin(\xi)^2 \left[ 1 - \frac{t}{4} - t \left( 1 - \frac{1}{4} \right) \tanh(\eta/2) \right]
- \frac{t}{4\eta} \coth(\eta) + \frac{9}{8\eta^2} \right] + i \sqrt{3} \eta \left( \frac{\xi}{2} \right) \sin(\xi) \left( \frac{1}{2} + 1 \right)
\]
\[
= +i \sqrt{3} \eta \sin(\xi)^2 \left[ 1 + t \left( -\frac{1}{4} - \frac{\coth(\eta)}{\eta} + \frac{3}{4\eta \sinh(\eta)} + \frac{9}{32\eta^2} \right) \right]
+ i \sqrt{3} \eta t \left( \frac{3i}{4\eta} \sin(\xi /2) \sin(\xi) \right) + O(\eta).
\]

There is still \( \hat{\mathcal{C}}_E^{(2)} \) to compute: it deviates from the previous starting point in the shifted vertex, at which the volume acts, \( \hat{v}_{G,T}^{a,e_1} \). That is, there can be no quantum contributions due to mixing of right-invariant vector fields and the holonomy loop \( \hat{h}(\square, 1) \). All other contributions can repeat. However, on the edge in direction +3 starting at \( v \), we have the commutator of \( \hat{n} \) and left-invariant vector fields.

We will bring this contribution into a form maximally close to the one known so far (as always neglecting contributions of higher order than \( r \)):

\[
\frac{1}{s_3} \left\langle \hat{h}_{ab} \left[ \hat{h}_{ic}, L^a \ldots L^b \right] \right\rangle
= \left\langle \hat{h}_{ab} \tau_{0}^{i} \hat{h}_{i}^{0} L^0 \ldots L^0 \right\rangle
= \left( \hat{h}_{0}^{(0)} R^0 \ldots R^0 \right) \left( \begin{array}{ccc} 1/2 & 1/2 & 0 \\ a & -c & 0 \end{array} \right) \left( \begin{array}{ccc} 1/2 & 1/2 & 0 \\ b & -b' & 0 \end{array} \right) \tau_{0}^{i}(-y)^{-b'}
+ 3 \left( \hat{h}_{0}^{(1)} R^0 \ldots R^0 \right) \left( \begin{array}{ccc} 1/2 & 1/2 & 1 \\ a & -c & -m \end{array} \right) \left( \begin{array}{ccc} 1/2 & 1/2 & 1 \\ b & -b' & -n \end{array} \right) \tau_{0}^{i}(-y)^{-m+n+c-b'}
= \delta_{ab} \frac{1}{23} \sum_{b=-4}^{b} \left( -y^{1/2+a+1/2+b-i/2} \right)^{-1/2-b} \left( -y^{-b} \right)^{-1/2-b} \left( R^0 \ldots R^0 \right)
+ 3 \left( \begin{array}{ccc} 1/2 & 1/2 & 1 \\ a & -c & -m \end{array} \right) \left( \begin{array}{ccc} 1/2 & 1/2 & 1 \\ b & -b' & -n \end{array} \right) (-i/2) \left( -y^{1/2-b-m+c-b'} \right) \left( R^{(1)} R^0 \ldots R^0 \right)
\]

(5.20)
\[ \begin{align*}
= 0 & + 3 \left( \frac{1/2}{a} \left( \frac{1/2}{-c} \frac{1}{-m} \right) \frac{b}{\sqrt{3/2}} \left( -\frac{i}{2} \right) \left( -\frac{1}{2} - 2b - m + c \right) \left( \frac{i\eta}{\tau} \right)^n \right) \\
\times & \left( \delta_{m0} \frac{t}{2\eta} \sum_{\ell=0}^{n-1} \eta^{-\ell} \cdot \frac{\delta_{\ell\eta'}}{\eta} \right) \\
= & 2(-ia/2)\delta_{m0}(-3^{1/2-3b+1/2-b}) \left( \frac{i\eta}{\tau} \right)^n \left( 1 - i \frac{\tanh(\eta/2)}{\eta} \right) \\
\times & \left[ 1 + \frac{t}{2\eta} \left( \frac{n(n+1)}{2\eta} - n \coth(\eta) \right) \right] \\
= & \tau_{\text{recoupling}}^{2} \langle R^0 \ldots R^0 \rangle \left( 1 - i \frac{\tanh(\eta/2)}{\eta} \right), \quad (5.21)
\end{align*} \]

where we used SU(2)-recoupling and that
\[ \binom{j}{m} \binom{j}{n} = \frac{(-1)^{j-m}}{d_{j}} \delta_{m-n}, \quad \binom{1/2}{m} \binom{1/2}{-m} = \frac{m}{\sqrt{3/2}} (-3^{1/2-m}). \quad (5.22) \]

Then, we use (5.21), together with the same steps as before, in
\[ \langle \Psi', \tilde{C}_E^{(2)}(2) \Psi' \rangle = -\sum_{N=0}^{10} c_N (-\eta)^N \epsilon_{i_1J_1K_1} \cdots \epsilon_{i_nJ_nK_n} \sum_{s_1, s_2, s_3 = 0}^{N} \left( \binom{N}{s_1} \binom{N}{s_2} \binom{N}{s_3} \right) \\
\times \left( D \right)_{\text{recoupling}}^{(2)}(n_1)(\tilde{h}_{\rho y})D^{(2)}_{\text{recoupling}}(n_1n_2)(\tilde{h}_{\rho z})D^{(2)}_{\text{recoupling}}(n_1n_2)(\tilde{h}_{\rho z})D^{(2)}_{\text{recoupling}}(n_1n_2)(\tilde{h}_{\rho z}) \\
\times \langle R^1 \ldots R^{n_1+1} \rangle \langle R^{n_1+1} \ldots R^{n_2} \rangle \langle R^{n_1} \ldots R^{n_2} \rangle \langle R^{n_1} \ldots R^{n_2} \rangle \\
\times \langle R^{n_1} \ldots R^{n_2} \rangle \langle R^{n_1} \ldots R^{n_2} \rangle \langle R^{n_1} \ldots R^{n_2} \rangle \\
= -\sum_{N=0}^{10} c_N (-\eta)^N \eta^{3N-1} \left( 2\eta \right)^{N-1} N[-\sin(\xi^2)] \\
\times \left[ 1 + \frac{t}{2\eta} \left( \frac{\eta}{2} - 2 \tanh \left( \frac{\eta}{2} \right) \right) - (3N - 1) \coth(\eta) + \frac{1}{2\eta} N(N + 3) \\
+ \frac{1}{4\eta}(N + 2)(N - 1) - t \frac{\tanh(\eta/2)}{\eta} \right] \\
= -i \sqrt{3\eta} \sin(\xi^2) \left[ 1 - \frac{t}{4} - 2 \frac{\tanh(\eta/2)}{\eta} - \frac{t}{4\eta} \coth(\eta) + \frac{t}{8\eta^2} \right] \\
+ \mathcal{O}(t^2). \quad (5.23) \]
Finally, plugging everything together we end up with

\[ \langle \Psi'_t, \hat{C}_E^{(r)}(r) \Psi'_t \rangle = p_{Euc} \frac{\left( \beta \hbar c \right)^{3/2}}{2^{7/2} \sqrt{3} \eta} 24 (i \sqrt{3} \sin(\xi^2)) \]

\[ \times \left[ 1 + t \left( -\frac{1}{4} - \frac{\tanh(\eta/2)}{\eta} - \frac{\coth(\eta)}{4\eta^2} + \frac{9}{16\eta^2} \right) + O(t^2) \right. \]

\[ + \left. \frac{3}{4\eta \sinh(\eta)} + i \frac{3}{4} \frac{\sin(\xi/2)^2}{\eta \sin(\xi)} \right] + O(t^2) \]

\[ = p_{Euc} \frac{\left( \beta \hbar c \right)^{3/2}}{2^{7/2} \sqrt{3} \eta} 48 (i \sqrt{3} \sin(\xi^2)) \left[ 1 + t \left( -\frac{1}{4} - \frac{13}{8\eta} \right) \right. \]

\[ + \left. \frac{11}{8\eta \sinh(\eta)} + \frac{9}{32\eta^2} \right] + i \frac{3}{8} \frac{\sin(\xi/2)^2}{\eta \sin(\xi)} + O(t^2). \] (5.24)

\[ \square \]

6. On the fate of the effective dynamics program of loop cosmology

The effective dynamics program/conjecture is a tool often used in the field of loop quantum cosmology (LQC) [41, 42]. It makes assumptions concerning the dynamics of the expectation values of certain observables (typically the volume of the Universe/a compact region). In a nutshell, it assumes that a coherent state remains sharply peaked under dynamical evolution and its mean peak follows a trajectory for the variables it is peaked on, which mirrors the trajectory on the classical phase space induced by the (regularised) dynamics. In the isotropic sector of LQC, it has been verified for Gaussian states leading to successful results. Later on it has been assumed without explicit proof in other LQC-like models. However, we demonstrate in this section that if one attempts to do the same in LQG with cosmological coherent states and includes the first order of quantum corrections, it will not work.

Without loss of generality we will talk in this section about the flow of a physical Hamiltonian, which for example can be realized in general relativity by deparameterizing with suitable dust reference fields [49].

The effective dynamics conjecture has several levels, which can be summarized as follows:

(a) The state stays sharply peaked under dynamical evolution with respect to some Hamiltonian for the considered observable.

(b) The trajectory of the expectation value of the observable follows the trajectory of an effective Hamiltonian, which is the expectation value of the quantum Hamiltonian, evolved on the classical phase space.

(c) One can restrict the effective Hamiltonian and its flow on the classical phase space to a small subspace of ‘relevant’ degrees of freedom. On this subspace the restricted Hamiltonian and its flow with respect to the reduced Poisson brackets matches the evolution of the full effective Hamiltonian.

13 Note that the appearance of the imaginary part is simply due to \( \hat{C}_E \) not being self-adjoint. A common extension in the literature is therefore to work with \( \hat{C}_E + (\hat{C}_E)^\dagger \), whose expectation value follows straightforward from this result.
First, we want to stress that the program has been verified with great success in certain quantum models of the isotropic sector of cosmology, i.e. LQC [43, 50]. Here, one deals with a quantum theory of a single degree of freedom, thus the last bullet point (C) of the above mentioned list becomes void. The classical phase space is parameterized by two parameters $p, c$ and endowed with the symplectic structure $\{p, c\} = \kappa \beta / 6$. A quantisation of a symmetry restricted scalar constraint, coupled to a scalar field as clock content, could be analytically investigated: its quantum flow on the corresponding coherent state was directly computed and verified to agree with the flow of some effective generator $C_{\text{cos}}(c, p)$ with the above mentioned Poisson bracket [43]. Moreover, it was turned out in [44] that the expectation value of the scalar constraint operator in coherent states peaked sharply on classical values $p, c$ results approximately in said function $C_{\text{cos}}(c, p)$. This success sparked the effective dynamics program: even in other LQC-like scenarios agreement of an effective constraint and the expectation value of the operator was assumed, without being thoroughly proven (see [45–48] for some recent applications).

Of course, there is a downside to the LQC quantisation strategy of only a single degree of freedom, i.e. the scale factor: it is unclear how its relation to an actual field theoretical quantisation of GR does look like. In this section, we explore a bit in this direction, as—albeit not having access to quantum field theory—we have a quantum theory with many degrees of freedom on the lattice. In contrast to LQC, we do not work with Gaussian states, but with their $SU(2)$-pendant: being the GCS.

Therefore, the third bullet point becomes paramount: in which situations is it possible to restrict the full dynamics of a Hamiltonian on the lattice phase space to some subspace, without losing information? In [38], this procedure was proven on a classical level to work, given the reduced phase space consists of those points that are left invariant by the action of some group of symplectomorphisms and if the Hamiltonian is invariant under action of the same group as well. In [38] it was presented how gravity on the graph can be symmetry restricted to the phase space of isotropic cosmology parameterised by $p, c$ as outlined before. However, this only proves the validity of (C) in the limit $t \to 0$. This section shall therefore consider, how it fares if the first order in $t$ is included and thus elucidates on the fate of the effective dynamics program.14

The validity of restricting the dynamics can be tested at the infinitesimal level with the toy model outlined in this paper so far: instead of investigating the whole flow of the Hamiltonian, we ask whether at small times the change of the expectation value of a time-evolved observable agrees with the classical flow induced on the classical observable on the restricted phase space. In other words: for some observable $\hat{O}$ and some generator $\hat{H}$ the claim of the effective dynamics that $\langle e^{i \hat{H} t} \hat{O} e^{-i \hat{H} t} \rangle = e^{i \{H_{\text{eff}}, \langle \hat{O} \rangle \}}$ translates to the infinitesimal level as: $\langle [\hat{H}, \hat{O}] \rangle = \{H_{\text{eff}}, \langle \hat{O} \rangle \}$ with $H_{\text{eff}} = \langle \hat{H} \rangle$. This claim can be check explicitly with our algorithm.

As this is a necessary condition for the effective dynamics program, it suffices to show that this is in general not the case by considering the following: we take the Euclidean part of the scalar constraint15 as generator of the flow and the volume of the whole spatial manifold as observable. We choose an isotropic and homogeneous lapse function $N(v) = 1$, denoting $C_{\epsilon}^{(1)}[1] = \sum_{v} [C_{\epsilon}^{(1)}(v) + (C_{\epsilon}^{(1)}(v))^\dagger] \bar{N}(v)$ and a lattice $\gamma$ of $M$ vertices along each direction. Also, we fix the total coordinate length of the torus to be 1 such that $\epsilon = 1 / M$. With the computer

14 We emphasize that in [38] invariance under the action of the group is necessary in order to reduce Poisson brackets. Thus, it is necessary in (5.4) to include the contribution of all touching vertices if we want to have a chance for the effective dynamics program to work.

15 This could be imagined as a toy model Hamiltonian if deparameterised with Gaussian dust [49] and choosing the Barbero–Immirzi parameter $\beta = i$. 

25
Algorithm described in this paper it is now easy to compare the following two quantities:

\[ K_{\text{qu}} := \left\langle \Psi^f_{\gamma}, \frac{1}{i\hbar} \left[ \hat{C}_E^\gamma[1], \sum_v \hat{V}^{\text{AL}}_v \right] \right\rangle \quad \text{and} \quad K_{\text{eff}} := \left\{ \left\langle \Psi^f_{\gamma}, \hat{C}_E^\gamma[1] \Psi^f_{\gamma} \right\rangle, \left\langle \Psi^f_{\gamma}, \sum_v \hat{V}^{\text{AL}}_v \Psi^f_{\gamma} \right\rangle \right\}. \]  

(6.1)

Equality between both quantities is already known up to order \( O(t) \), but so far nothing is known when including the next-to-leading order.

With the result from section 5 and the formula of the volume (3.8), the right side is easily calculated:

\[
K_{\text{eff}} := -M^2 \frac{3}{2} \beta \rho e^2 \sin(2\omega \epsilon) - t M^2 \frac{3 \beta \ell^2}{128 \eta} \frac{\sin(2\ell)}{\sinh(\eta)} \times \left[ 1 - 20\eta^2 + (4\eta^2 - 1) \cosh(2\eta) - 44\eta \sinh(\eta) + 30\eta \sinh(2\eta) \right] + O(t^2).
\]

(6.2)

The left side, \( K_{\text{qu}} \), is (with the help of Mathematica) evaluated to be (recall that \( \eta = \frac{2\rho e^2 \epsilon}{\beta m} \), \( \xi = -\epsilon \epsilon \))

\[
K_{\text{qu}} = \left\langle \Psi^f_{\gamma}, \frac{1}{i\hbar} \left[ \hat{C}_E^\gamma[1], \sum_v \hat{V}^{\text{AL}}_v \right] \right\rangle = -M^2 \frac{3}{2} \beta \rho e^2 \sin(2\omega \epsilon) - t M^2 \frac{3 \beta \ell^2}{512 \eta} \frac{\sin(\xi)}{\sin(\eta)} \times \left[ \sinh(\eta) \left( (64\eta^2 - 141) \cos(\xi) - 5 \right) + 2\eta (-168 \cos(\xi) + \cos(2\xi) + 3) + 8\eta \cosh(\eta)(57 \cos(\xi) - 1) \right] + O(t^2).
\]

(6.3)

However, while both expressions cancel in their leading order, the same is not true when including quantum corrections:

\[
\left\langle \Psi^f_{\gamma}, \frac{1}{i\hbar} \left[ \hat{C}_E^\gamma[1], \sum_v \hat{V}^{\text{AL}}_v \right] \right\rangle - \left\{ \left\langle \Psi^f_{\gamma}, \hat{C}_E^\gamma[1] \Psi^f_{\gamma} \right\rangle, \left\langle \Psi^f_{\gamma}, \sum_v \hat{V}^{\text{AL}}_v \Psi^f_{\gamma} \right\rangle \right\} = t f(\eta, \xi) + O(t^2) \neq 0.
\]

(6.4)

This shows, that upon including the quantum corrections, an ideally irrelevant choice (computing \( V \) directly on the quantum level versus first computing \( \langle \hat{C}_E \rangle \) and then deriving \( \hat{V} \) via it) bears significant differences and affects the effective dynamics program. The additional \( O(t) \) corrections of the generator of dynamics if understood as a classical phase space functions should not be used in Hamilton’s equation in order to get reliable insight into the real quantum dynamics.

Indeed, the difference becomes quit drastic for the several cases: first the limit \( \eta \to 0 \) is not faithfully captured. However, we point out that in this limit the results obtained as expectation values from our algorithm loose their validity, as the approximation of using (2.18) fails. Encouragingly, it turns out that for \( \xi \to 0 \) the function \( f(\eta, \xi) \) approaches zero quickly, without impacting the classical limit. More importantly however, it turns out that this is not the
case for $\xi \approx \pi/2$: here the difference between $K_{\text{eff}}$ and $K_{\text{qu}}$ is maximally, as one can easily see that at this point the quantum corrections of the effective theory would vanish (as $\sin \pi = 0$) while the same is not true for the corrections of the quantum model. However, this regime describes the area of most importance for quantum gravity, as it describes the situation close to the initial singularity. Consequently, we deduce that the effective dynamics program should not be employed in order to draw information about the relevant quantum gravity corrections in isotropic flat cosmology.\(^\ref{16}\)

Therefore, as long as the quantum corrections of the spread are small compared against the discretisation effects due to the lattice, the effective dynamics conjecture has still a chance to find application (although item (A) and (B) remain + to be carefully investigated even in the limit $t \to 0$). However, if one is interested in the effects that appear due to finite $t$ parameters, we found the effective dynamics program to yield inconsistent results (if not artificially restricted to the $O(t^0)$-level)!

Remark 6.1. One may propose to get at least a first intuition on the small time effects that a self-adjoint quantum Hamiltonian $\hat{H}$ enforces on some observable $\hat{O}$, by looking at the power series expansion of $e^{-i\hat{O}/\epsilon}$. That is to investigate the expectation values of the $n$th-momenta $[\hat{H}, \hat{V}]_n$ with the iterated commutator. However, such a procedure can only be trustful, if suitable convergence properties of the power series can be shown.

Remark 6.2. Currently, the corrections which drive the dynamics in models such as LQC [43, 50] are mainly driven by ambiguous discretisation choices. Although these feature a bounce, one can see by our investigations that said bounce is not due to quantum effects but due to discretisation artefacts. In other words, the bouncing behaviour can only be accepted if one takes the premise of a fundamentally discrete spacetime for granted. Conversely, one is typically interested in quantum field theories where the lattice regulator is removed, $\epsilon \to 0$. Next to the fact that this worsens the above situation for the effective dynamics program as soon as $t \gg \epsilon$, the above framework does not give reliable results in this limit: in the context of quantum gravity the replacement of the Ashtekar–Lewandowski volume with the Giesel–Thiemann volume was necessary for the algorithm to work. However, said replacement is only valid in the limit $\epsilon \gg t$, therefore one should not interpret the $\epsilon \to 0$-limit of the above stated expressions as containing valid physical information.

7. Conclusions and outlook

This paper presented a step-by-step ‘computer algorithm’ for expectation values of polynomial operators in cosmological coherent states including the first order in the spread of the states. The cosmological coherent states are a tensor product of suitable GCS which are suitable for all LGTs from [15–17]. These GCS are labeled by classical phase space data $p, c$ of isotropic flat cosmology and sharply peaked in the sense, that the expectation value of any operator, corresponding to a classical function on the phase space, results in precisely the evaluation

\(^{16}\)The following test highlights the difference between both $K_{\text{qu}}$ and $K_{\text{eff}}$: one could use $\langle \Psi', C_\epsilon^0(\Psi') + \pi^2/(2p^2) \rangle$ as classical generator of the effective dynamics minimally coupled scalar field $\phi$. Hamilton equations of motions [e.g. $p = 2/(3, \sqrt{\beta}/K_{\text{eff}})$] one obtains a symmetric bounce as in standard LQC. However, if we were to replace (as a toy-model) the above mentioned Hamilton equation by $p = 2/(3, \sqrt{\beta}/K_{\text{qu}}$, this is not the only possible solution: e.g. for $\epsilon = 0.01, \epsilon = 16\times10^{-7}, \beta = 0.2575, p(0) = 60,000, \pi_p(0) = 300$ one finds shortly before the first bounce a recollapse preceded by a second bounce. This showcases that the quantum corrections can even indeed lead to qualitative differences!
of its classical function in said phase space data in zeroth order of the spread of the states $t$. However, the first order is far from non-trivial, but captures important modifications due to the quantum nature of space time. In order to compute those, our algorithm enters the stage:

As input one can investigate any polynomial operator in the basic variables (holonomies and fluxes). Due to several crucial observations the computation can be drastically simplified, presenting a further step towards bringing LQG into the realm of computability. In that regard, it is to be observed that LQG also starts to benefit from the field of quantum simulations [51, 52]. Platforms based on linear optics [53], nuclear magnetic resonance (NMR) [54], adiabatic methods [55] used e.g. by D-wave machine, or famous IBM devices utilizing superconducting qubits [56] are now being conceptually exploited. However, while quantum simulations aim to use speedup offered by quantum computers, severe technical limitations will necessitate a clever pre-processing of complex inputs. Our results are exactly aimed at reaching this latter objective, i.e. to make quantities in LQG computable by classical or quantum machines.

Summarizing the technical details, complicated operators such as the volume operator can be tackled by manipulations such as the replacement from [24] and then observing that products of $\hat{Q}_v$ decouple from the remaining operator. The final simplifications were due to the observations from [25] that products of right-invariant vector fields contribute to the classical order when all indices are zero, and to the first order with at most one pair of non-zero indices. This removed the major part of contractions that need to be summed over, once the formula of [26] is employed, which translates the expectation value into a function of $p, c, t$.

The main steps of the algorithm (detailed in section 4) can be summarized as follows:

(a) All commutators appearing in the operator in question are resolved, using explicit expressions suitable to keep the track of the leading order in spread $t$. Then, we split the operator into a sum of monomials.

(b) Using the same commutation relations from the previous step, for each monomial we shift all the holonomy operators to the left, thus obtaining the form suitable for the general formula (2.22).

(c) We employ several identities for right/left-invariant vector fields to simplify many typical combinations (some of them necessarily appearing due to the commutation relations).

(d) In the present form the contributions over all edges decouple, thus we split each monomial into a product of sub-operators (supported on each edge) and compute the expectation value separately for each of them.

(e) Using SU(2) recoupling theory, the products of holonomy operators on the same edge is simplified by combining them into a sum over terms, each of which involving only a single holonomy operator.

(f) We employ final replacements of the particular operators by their expectation values utilizing the simplifications arising due to theorem 1. Lastly, all indices can again be contracted to yield the final result.

A few words of caution have to be made at this point: the coherent states are of kinematical nature in the sense that—albeit their exceptional peakedness properties—they are not in the kernel of a known quantisation of the constraints, neither the scalar/diffeomorphism constraint nor the Gauss constraint. Next to considering different proposals (e.g. see [57, 58]), it became popularized in quantum gravity approaches on the lattice, to utilize the so called group averaging procedure [3, 59, 60], by which e.g. a simple tensor product over coherent states can be projected to a gauge invariant state [61]. It is worthwhile to note, that the expectation value of gauge-invariant operators, like the volume, do not get afflicted by this projection in its leading order in $t \sim \hbar$, but in its next-to-leading order. In future, one may adapt the present algorithm
to include the additional corrections stemming from such a group averaging procedure. See the appendix of [25] for a strategy how these corrections can in general be computed.

Next, while it may be tempting to straightforwardly compute the expectation value of the scalar constraint (note that in this paper we have only shown the Euclidean part as a toy model, while the Lorentzian part can be computed analogously albeit with more computational effort) and to extract physical predictions from it, we emphasise again that the framework of LQG is plagued by many quantisation ambiguities stemming from several choices of how to approximate the continuum scalar constraint $C$ as a function on the lattice $C'$ [27, 28, 62]. In the presence of a finite lattice, these ambiguities will overshadow the quantum corrections—as they appear on the leading order level—and therefore need to be dealt with first. Neither is any known quantisation of the scalar constraint cylindrical consistent (which is the necessary condition to promote it to a continuum operator), nor can one simply take the limit of $\epsilon \to 0$ as the replacement of the Ashtekar–Lewandowski volume with the Giesel–Thiemann volume only works at finite lattice spacing $\epsilon \gg t$. These issues will be attacked in future publications which now gain the support of being easier testable due to our algorithm.

Leaving the above mentioned technicalities aside, even at the current stage, one can use the algorithm to learn something about the dynamics of LQG: namely, we investigated in section 6 the fate of the effective dynamics program. In loop cosmology-like models of reduced systems, one often takes the expectation value of the scalar constraint in said system (or a function which is hoped to approximate it at leading order in $t$) and evolves physical quantities with respect to the flow generated by said expectation value due to the classical Poisson brackets. We asked whether such a strategy can have the potential to be viable once the many degrees of freedom of the lattice are considered and found the answer to be in the negative once the next-to-leading order is considered! A quick consistency check revealed that—although agreeing in the leading order—the $O(t)$-modifications of $\langle [\cdot, \cdot] \rangle$ are different then those of $\{ \langle \cdot \rangle, \langle \cdot \rangle \}$ even for the simple case isotropic, flat cosmology on the lattice. In other words, computing the infinitesimal level of evolution (i.e. the velocity) on the quantum level or obtaining it via the effective dynamics program gives two different expressions and provides a correction that is of the same order as the new linear $t$ corrections whose influence one wants to investigate. Therefore, the effective dynamics conjecture yields inconsistent results when applied to lattice LQG using as generator of evolution the expectation value of the scalar constraint including next-to-leading order corrections. Instead one should rather be interested in a power series of higher order momenta of the quantum observables in question—which poses a problem that can be addressed with the present algorithm. We reserve this part for future research.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. General relativity as SU(2) gauge theory and its kinematics on a graph

The Hamiltonian or ADM formulation of general relativity \([29]\) requires that one can split the four-dimensional manifold \(M\) in the following way: \(M \cong \mathbb{R} \times \sigma\) where \(\sigma\) is a smooth three-dimensional manifold admitting Riemannian metrics. Extending the findings of Sen \([30]\), it transpired that the ADM formulation can be cast into the form of a gauge theory with gauge group SU(2). The phase space is coordinatised by the Lie algebra valued connection \(A_a(x) = A^I_a(x)\tau^I\) and the electric field \(E_a(x) = E^I_a(x)\tau^I\) commonly called Ashtekar–Barbero variables \([31,32]\) with elementary Poisson brackets:

\[
\{E^I_a(x), E^J_b(y)\} = \{A^I_a(x), A^J_b(y)\} = 0,
\]

\[
\{E^I_a(x), A^J_b(y)\} = \frac{\kappa\beta}{2} \delta^I_b \delta^J_a \delta^{(3)}(x, y).
\] (A.1)

Here \(\{\tau_J\}\) represent an unspecified basis of \(su(2)\). The parameter \(\kappa = 16\pi G\) is the gravitational coupling constant and \(\beta \in \mathbb{R} - \{0\}\) is the Barbero–Immirzi parameter. Moreover, the phase space is subject to the Gauss constraint

\[
G_J = \partial_a E^J_a + \epsilon_{JKL} A^k_b E^L_b = 0, \quad (A.2)
\]

in addition to the usual constraints of the ADM formulation, namely the scalar constraint \(C\) and the diffeomorphism constraint \(C^E\). The latter ones, in terms of the Ashtekar–Barbero variables, read (assuming positive orientation of the triad from here on):

\[
D_a = \frac{2}{\kappa \beta} F^I_{ab}(A) E^J_b, \quad (A.3)
\]

\[
C = C^E = \frac{1 + \beta^2}{\kappa} K^M_a K^N_b \epsilon_{MNJLK} \frac{E^I_a E^b_J}{\sqrt{\det E}}, \quad (A.4)
\]

\[
C^E := \frac{1}{\kappa} F^I_{ab}(A) \epsilon_{JKL} \frac{E^I_a E^b_J}{\sqrt{\det E}}. \quad (A.5)
\]

Here \(K^I_a\) is the extrinsic curvature and \(F^I_{ab}(A) = 2\partial_a A^I_b + \epsilon_{JKL} A^k_a A^I_b\) is the curvature of the connection \(A\).

To facilitate quantisation of the above theory, one considers discretisations of the spatial slice \(\sigma\), in form of graphs \(\gamma\) that allow an associated dual cell complex. Each graph is a collection of finitely many edges \(e\), i.e. paths in \(\sigma\), and vertices \(v\), i.e. ending points of multiple edges. For each \(\gamma\) one introduces the corresponding phase space by considering the discretised phase space variables, namely \([34]\)

- The holonomies \(h(e) \in SU(2)\) of the connection along the edges \(e\)

\[
h(e) := \mathcal{P} \exp \left( \int_0^1 dt A^I_a(e(t)) \tau^I \epsilon^a(t) \right), \quad (A.6)
\]
The gauge covariant fluxes $P(S_e) := P^i(S_e) \tau_j$, along the associated faces $S_e$ of the dual cell complex (whose intersection with the edges we choose to be in the middle of each edge)

$$P^i(e) := -2 \text{tr} \left[ \tau_j h(e_{[0,1/2]}) \right] \times \int_{S_e} dx \ h(\rho_x) (\ast E(x) h(\rho_x)^{\dagger} h(e_{[0,1/2]}))^{\dagger} \right], \quad (A.7)$$

where later values are ordered to the right in the path-ordered exponential, and $\rho_x$ is some, arbitrary path inside of $S_e$ such that $\rho_x(0) \in e_k$ and $\rho_x(1) = x$ and $e_{[0,1/2]} \subset e$ denotes the part of the path from $e[0]$ until $\rho_x(0)$.

A few comments are in order here, as the choice for the fluxes in (A.7) deviates from the standard one $E(e) = \{e \ast E(x)\}$ which was used in [25]. Firstly, upon taking a suitable continuum limit, in the sense of considering a family of graphs $\{\gamma\}$, that fill out $\sigma$ for $c \to 0$, both operators $P(e) := P^i(e) \tau_j$ and $E(S_e)$ reduce to $E_a^i(e[0]) \tau_i \gamma^a[0]$, i.e. to the electric field. Therefore, when considering the continuum limit, it does not make a difference which flux-regularisation is being considered. However, in presence of finite regularisation, i.e. for every finite graph $\gamma$, the situation is drastically different. This setting favours the choice $P(e)$ in (A.7) since only this operator transforms covariantly under gauge transformations, i.e.

$$P(e) \mapsto g(e[0])P(e)g(e[0])^{-1}, \quad (A.8)$$

for $g(x) \in \text{SU}(2)$.

The above choice allows to find regularisations of the constraints $D_a$ and $C$ from (A.3) and (A.4) such that they are SU(2) gauge invariant and similarly enables the construction of SU(2) gauge invariant observables using only the basic building blocks (A.6) and (A.7) of the graph $\gamma$. If one wants to look at general relativity restricted to one single graph $\gamma$ it is therefore necessary to work with the gauge covariant fluxes in order to obtain physically meaningful results.

Further, it is to be noted that the set of basic building blocks on a graph $\gamma$ form the phase space of a discretised theory only upon using fluxes, that do not have vanishing Poisson brackets, in order to avoid problems with the Jacobi identity, when (A.10) below is assumed. As it is complicated to determine the Poisson brackets between the $E^i(e)$, the gauge covariant fluxes $P(e)$ are once again favoured as their Poisson bracket on $\gamma$ is uniquely found to be:

$$\{h(e), h(e')\}_\gamma = 0, \quad (A.9)$$

$$\{P^i(e), h(e')\}_\gamma = \kappa \beta \delta(e, e') \gamma h(e), \quad (A.10)$$

$$\{P^i(e), P^j(e')\}_\gamma = -\frac{\kappa \beta}{2} \delta(e, e') f^j_k P^k(e), \quad (A.11)$$

and obeys the Jacobi identity. Note that $f^j_k$ are the structure functions of the chosen basis of $\text{su}(2)$.

In the main body of the paper we use the spherical basis $\tau_j$ for which, as already mentioned, $f^j_k = (-1)^j [\tau_j^{(0)}]_{jk}^{(1)}$, where

$$[\tau^K]_{ab} = i \sqrt{j(j+1)(2j+1)}(-1)^{a+b} \begin{pmatrix} j & 1 & j \\ b & K & -a \end{pmatrix}. \quad (A.12)$$

The degenerate case $j = 0$ is actually defined as $[\tau^K]_{ab}^{(0)} \equiv 0$. 


Appendix B. Proof of theorem 1

We are now going to derive the main technical ingredient of the algorithm presented in this paper, namely, theorem 1. At the beginning let us point out that, given the operator $\hat{F} = \prod_{I} \hat{L}_{I}$, the leading order of its expectation value is just $\prod_{I} \langle \hat{L}_{I} \rangle$, with $\hat{L}_{I}$ from (3.11). Therefore, taking into account the leading order of $\hat{Q}_{e}$, equation (3.12) follows immediately.

Thus, it only remains to compute the quantum corrections. For said purpose, we first consider the following special case:

**Lemma**: let $\hat{F} = \prod_{I} \hat{L}_{I}$ be an operator with leading order $\mathcal{O}(\ell^{0})$ where each $\hat{L}_{I}$ is a monomial supported on the edge ‘$e$‘, and is of the form $R^{K_{I}} \ldots R^{K_{0}}$ (without any $\hat{h}^{0}$ for the moment, i.e. $j = 0$). With the notation from (3.9) the expectation value is:

$$
\langle \Psi_{\gamma}, \hat{F} \hat{Q}_{e} \Psi_{\gamma} \rangle = \left[ \prod_{e', \ell e'=\emptyset} \langle \langle \hat{L}_{e} \rangle \rangle \right] \sum_{\kappa_{1} \neq \kappa_{2} = 0}^{3} \prod_{j=1}^{3} \left( \frac{i}{2} \partial_{\eta_{j}} \right)^{\kappa_{j}} \langle \langle \hat{L}_{(\ell + e',-\ell)} \rangle \rangle + \mathcal{O}(\ell^{-1}), \tag{B.1}
$$

where we note that the leading order of $\hat{Q}_{e}$ is $\mathcal{O}(\ell^{-3})$. Notation describing the edges, $e = (v,i)$, has in detail been introduced in 4.1. Note that for $\hat{F}$ independent of holonomies, we have $\langle \langle \hat{L}_{e} \rangle \rangle = \langle \langle \psi_{\hat{h}^{0},e}^{\gamma}, \hat{t}_{e} \psi_{\hat{h}^{0}}^{\gamma} \rangle \rangle$.

**Proof.** On each vertex we have a product of the expectation values on the six adjacent links. For each of these links it follows from the first line (i.e. $\mathcal{O}(\ell^{0})$ order) of (2.22) that [recall that the leading order of $R^{i}$ is $\mathcal{O}(1/\ell)$]

$$
D^{(1)}_{-K-\ell}(n_{e}^{i}) \langle \Psi_{\gamma}, \hat{F} R^{K}(e) \Psi_{\gamma} \rangle = \delta^{i}_{0} D^{(1)}_{0-K}(n_{e}^{i}) \langle \Psi_{\gamma}, \hat{F} R^{K}(e) \Psi_{\gamma} \rangle + \mathcal{O}(\ell^{0}), \tag{B.2}
$$

where $n_{e}$ is the $SU(2)$ element corresponding to the direction of the edge $e$. Secondly, being concerned with the definition of $Q_{e}$ we observe that the contraction of Lie algebra indices in (2.15) allows for the following manipulations which utilize properties of the SU(2) group:

$$
\epsilon_{\ell K R}(e_{i}) R^{i}(e_{j}) R^{K}(e_{k}) = \epsilon_{PFKF} \delta_{IP} R^{i}(e_{j}) \delta_{JP} R^{j}(e_{j}) \delta_{KK} R^{K}(e_{j})
$$

where $n_{e}$ is the $SU(2)$ element corresponding to the direction of the edge $e$. Secondly, being concerned with the definition of $Q_{e}$ we observe that the contraction of Lie algebra indices in (2.15) allows for the following manipulations which utilize properties of the SU(2) group:

$$
\epsilon_{PFKF} \delta_{IP} R^{i}(e_{j}) \delta_{JP} R^{j}(e_{j}) \delta_{KK} R^{K}(e_{j}) = \epsilon_{PIK} \delta_{IP} R^{i}(e_{j}) \delta_{PK} \epsilon_{FK} \delta_{JK} R^{K}(e_{j})
$$

i.e. we obtain three terms compatible with (B.2), as well as the object $\epsilon^{(0,ijk)}_{\ell K}$ defined in observation 5. Due to the property (B.2) applied sequentially, at classical order in $t$, we encounter only $\epsilon^{(0,ijk)}_{\ell K}$. Moreover, the first order quantum corrections appear only on one single edge, so that we only get additional terms of the form $\epsilon^{(0,ijk)}_{\ell K}$ (or lower indices given by $0K0$ and $K00$). Now, we use (3.6) from observation 5, together with the fact that we are only concerned with the next to leading order, to see that all terms with $K \neq 0$ vanish. Therefore, at the order of
interest it is sufficient to evaluate \( D_{0→k}^{(1)}(n_i^j) \langle \Psi_i^j, F R^k \Psi_i^j \rangle \). This simplified task, however, can be performed by resorting to equation (4.2) from [25]\(^{17}\), so that we obtain

\[
D_{0→k}^{(1)}(n_i^j) \langle \Psi_i^j, F R^k \Psi_i^j \rangle = \left( \frac{i}{2} \partial_\eta \right) \langle \Psi_i^j, F \Psi_i^j \rangle + O(\eta). \tag{B.4}
\]

Finally, we recall that on a cubic graph with all edges oriented along its main axes we have explicitly

\[
\hat{Q}_v = \epsilon_{ijk}(R^i(v, 1) + L^i(v - e_1, 1))(R^j(v, 2) + L^j(v - e_2, 2))
\times (R^k(v, 3) + L^k(v - e_3, 3)). \tag{B.5}
\]

Upon employing the necessary combinatorics of distributing \( R^0 \) on all possible edges, together with the rule replacing \( L \) operators by \( R \) operators inside the average value, (B.1) follows. \( \square \)

We must also include the case where non-vanishing holonomies \( \hat{h}_{ab}(e) \) present on a given link. Then, it is clear that if a quantum correction on the link \( e \) appears due to \( \hat{h}_{ab} R \ldots R, \) then all other edges contribute with their classical order. Therefore, without loss of generality let \( v \) be a vertex and \( e \) an edge outgoing at \( v. \) Then, the additional contribution from the third line of (2.22) is for the following operator:

\[
\hat{h}_{ab}(e) R^k(e) \ldots R^{k_n}(e) Q_{v}^{N_v} \mapsto (-6i) \eta \left( \frac{\eta i}{2} \right)^{2N_v} 2^{N_v}
\times \left( \prod_{i=1}^{N_v} \epsilon_{000}^{(n_i^j)} D_{0→k}^{(1)}(n_i^j) \right) \frac{N_v}{N_v}
\times \langle \psi_{h_{ab}}', \hat{h}_{ab} R^k \ldots R^{k_n} \psi_{h_{ab}}' \rangle
\times \langle \psi_{h_{ab}}', L_{\tau_{\hat{h}_{ab}}}^{\tau_{\hat{h}_{ab}}} \ldots L_{\tau_{\hat{h}_{ab}}}^{\tau_{\hat{h}_{ab}}} \psi_{h_{ab}}' \rangle, \tag{B.6}
\]

with \( \tilde{e} \) also being the edge adjacent \( v, \) but at the same time opposite to \( e. \) To arrive at this replacement, which encodes quantum corrections due to the correlations of \( R(e) \) and \( Q_v, \) we again used the combinatorics of distribution (B.5) together with the fact that all other edges (except \( e \) and \( \tilde{e} \)) contribute in this case only with their classical order. Thus, analogously to the above proof, the \( \epsilon_{000}^{(n_i^j)} \) enforces the \( \delta_{00}^j \) for all the indices in \( D_{0→k}^{(1)}(n_i^j) \) of \( Q_v^{N_v}. \)

Now, being interested in the third line of (2.22) we can replace \( R \mapsto \tau_{\hat{h}_{ab}}^l \) and the general identity \( D_{0→k}^{(1)} - L(g)[\tau^r \tau^j]_{lm} = D_{0→k}^{(0)}(g')[\tau^r \tau^j]_{lm} D_{0→k}^{(0)}(g) \) in order to move the \( n_r, \) Lastly, as only \( \tau^b \) appears we use that \( [\tau^0]_{b}^{c,d} = i b \delta_{bc} \) due to (A.12). As we have said, we are only interested in the third line of (2.22) and can therefore w.l.o.g set \( L = 1 \) (which is by assumption independent of \( \hat{h}. \)) We obtain:

\(^{17}\) Note that here \( \tau_l = -i \sigma_l / 2 \) instead of just the Pauli matrices in [25] causing the additional factor of 1/2.
Note that on the classical level

\[ \frac{(i\hbar)^N_i}{t} 2^{N_i} \sum_{j=0}^{N_i} \left( \begin{array}{c} N_i \\ j \end{array} \right) (\hat{h}_{j}^{(1)} R^{k_1} \ldots R^{k_N}) \]

\times \left( 1 + \frac{t}{2\eta} [D_{cd}^{(1)}(n_e) d D_{ab}^{(1)}(n_j)] + O(t^2) \right) \quad (B.7)

\[ = (6\eta)^N_i \left( \frac{(i\hbar)^N_i}{t} 2^{3N_i} \left( \hat{h}_{j}^{(1)} R^{k_1} \ldots R^{k_N} \right) \left( 1 + \frac{tN_e}{4\eta} [D_{cd}^{(1)}(n_e) d D_{ab}^{(1)}(n_j)] + O(t^2) \right) \right) . \quad (B.8) \]

Note that analogously, in the case of \( \hat{h}_{j}^{(1)}(e) \ldots Q_{\frac{N}{N_{+e}}}' \) one obtains a contribution of the form 

\[ + tN_{t+e} / (4\eta) \ldots \]. Together, these additional contributions are exactly \( Y_{j}^{(1)} \) from (3.10)\(^{18}\). Thus:

Let \( \hat{F} = h(e)_{ab} \prod_{e} \hat{t}_e \) an operator with leading order \( O(h^0) \) where each \( \hat{t}_e \) is a monomial-free non-trivial \( \hat{h}_{j}^{(1)} \). Then—due to all the above considerations—\( \langle \psi_i', \hat{F} Q_{e} \psi_i' \rangle \) is again given by (B.1)—with non-trivial \( Y \) contribution in \( \langle \langle \hat{h}_v(e) \rangle \rangle \).

We can generalize this to multiple appearances of the operator \( Q_{e} \), via denoting each edge on a lattice by \( e = e_m = (v, m) \) with starting vertex \( v \) and direction \( m \). Then, from above considerations it is immediate for \( \hat{F} = \prod_{e} \hat{t}_e \) with, finally, \( \hat{t}_e = \hat{h}_{j}^{(1)} R^{k_1} \ldots R^{k_N} \) depending on non-trivial holonomies:

\[ \langle \psi_i', \hat{F} \prod_{e} Q_{e}^{N_{i}} \psi_i' \rangle = \left( \prod_{e \in Z_3^0} (6\eta)^N_i \sum_{\kappa_{i,1} \kappa_{i,2} \kappa_{i,3} = 0} \left( \begin{array}{c} N_i \\ \kappa_{i,1} \kappa_{i,2} \kappa_{i,3} \end{array} \right) \left( \begin{array}{c} N_i \\ \kappa_{i,1} \kappa_{i,2} \kappa_{i,3} \end{array} \right) \left( \begin{array}{c} N_i \\ \kappa_{i,1} \kappa_{i,2} \kappa_{i,3} \end{array} \right) \right) \]

\[ \times \prod_{e \in Z_3^1} \prod_{m=1,2,3} \left( \frac{i}{2} \partial_{n_e} \right)^{N_{e} - \kappa_{e,m} + \kappa_{e,m} \kappa_{e,m}} \langle \langle \hat{h}_v(e) \rangle \rangle \right) \]  

(B.9)

Note that the binomial sums at each vertex do exactly represent all possible combinations of triples that can emerge from \( Q_{e}^{N_{i}} \). The rest of the appendix is now devoted to simplifying this expression further. All equal signs are understood up to next-to-leading order in \( t \).

We split (B.9) into two parts: the ‘classical’ contribution \( C_{l}^{(N_{i})} \), i.e. for each edge the leading order expression of \( \hat{F} \) contributes, and its quantum corrections \( O_{l}^{(N_{i})} \), i.e. summing over all edges and considering only the quantum contribution of said edge. This accounts to

\[ \langle \psi_i', \hat{F} \prod_{e} Q_{e}^{N_{i}} \psi_i' \rangle = C_{l}^{(N_{i})} + O_{l}^{(N_{i})} \]

(B.10)

with

\[ C_{l}^{(N_{i})} = \left( \prod_{e \in Z_3^0} (6\eta)^N_i \sum_{\kappa_{i,1} \kappa_{i,2} \kappa_{i,3} = 0} \left( \begin{array}{c} N_i \\ \kappa_{i,1} \kappa_{i,2} \kappa_{i,3} \end{array} \right) \right) \]

\[ \times \prod_{e \in Z_3^1} \prod_{m=1,2,3} \left( \frac{i}{2} \partial_{n_e} \right)^{N_{e} - \kappa_{e,m} + \kappa_{e,m} \kappa_{e,m}} \hat{l}_{(v,m)} \]  

(B.11)

\(^{18}\) Note that on the classical level \( h_{j}^{(1)}(e) = D_{j}^{(1)}(n_e) e^{\alpha \beta} D_{j}^{(1)}(n_j) \) for cosmology, giving the exact for equation (3.10).
and
\[
Qu_{(N)} = \left( \prod_{i \in \mathbb{Z}_3} (6i)^{N_i} \sum_{\kappa_{1,2,3} = 0}^{N_i} \frac{N_i}{\kappa_{1,2,3}} \left( \begin{array}{ccc} N_i & N_i & N_i \\ \kappa_{1,2,3} & \kappa_{1,2} & \kappa_{1,3} \end{array} \right) \right) \times \sum_{v' \in \mathbb{Z}_3 P_m = 1,2,3} \left( \frac{i}{2} \partial_t \right)^{N_v - \kappa_v - \kappa_{v' + v'' m'}} \left( \langle \langle l_{v' v'' m'} \rangle \rangle - \tilde{l}_{v' v'' m'} \right) \times \prod_{(v,m) \neq (v',m')} \left( \frac{i}{2} \partial_t \right)^{N_v - \kappa_{v,m} + \kappa_{v' + v'' m'}} \tilde{l}_{(v,m)}. \tag{B.12}
\]

Note that \(Qu_{(N)}\) is already linear in \(t\), therefore all derivatives can be replaced by \(\frac{2\alpha}{t}\), as the error we are making is of even higher order in \(t\). Further recall that \(\sum_i \binom{N}{i} = 2^N\), thus:

\[
Qu_{(N)} = \left( \prod_{i \in \mathbb{Z}_3} (6i)^{N_i} \sum_{\kappa_{1,2,3} = 0}^{N_i} \frac{N_i}{\kappa_{1,2,3}} \left( \begin{array}{ccc} N_i & N_i & N_i \\ \kappa_{1,2,3} & \kappa_{1,2} & \kappa_{1,3} \end{array} \right) \right) \sum_{v' \in \mathbb{Z}_3 P_m = 1,2,3} \left( \frac{2\alpha}{t} \right)^{3N_v} \left( \langle \langle l_{v' v'' m'} \rangle \rangle - \tilde{l}_{v' v'' m'} \right) \times \prod_{(v,m) \neq (v',m')} \left( \frac{2\alpha}{t} \right)^{3N_v} \prod_{m=1,2,3} \tilde{l}_{(v,m)}. \tag{B.13}
\]

For manipulating the ‘classical’ contributing (in which still quantum effects due to \(Qu\) appear), we ease notation by calling \(\alpha := N_v - \kappa_{v,m} + \kappa_{v' + v'' m'}\). Then, we split again the leading order—turning out to be exactly \(P_0\) from (3.12)—from the remaining part:

\[
Cl_{(N_v)},(\tilde{l}_{v}) = \left( \prod_{i \in \mathbb{Z}_3} (6i)^{N_i} \sum_{\kappa_{1,2,3} = 0}^{N_i} \frac{N_i}{\kappa_{1,2,3}} \left( \begin{array}{ccc} N_i & N_i & N_i \\ \kappa_{1,2,3} & \kappa_{1,2} & \kappa_{1,3} \end{array} \right) \right) \times \prod_{v' \in \mathbb{Z}_3 P_m = 1,2,3} \left( \frac{i}{2} \partial_t \right)^{N_v} \tilde{l}_{v,m} \times \prod_{v' \neq v''} \left( \frac{2\alpha}{t} \right)^{3N_v} \prod_{m=1,2,3} \tilde{l}_{(v,m)} = \left( \prod_{i \in \mathbb{Z}_3} (6i)^{N_i} \sum_{\kappa_{1,2,3} = 0}^{N_i} \frac{N_i}{\kappa_{1,2,3}} \left( \begin{array}{ccc} N_i & N_i & N_i \\ \kappa_{1,2,3} & \kappa_{1,2} & \kappa_{1,3} \end{array} \right) \right) \prod_{v' \in \mathbb{Z}_3 P_m = 1,2,3} \left( \frac{2\alpha}{t} \right)^{N_v} \tilde{l}_{(v,m)} + \left( \prod_{i \in \mathbb{Z}_3} (6i)^{N_i} \sum_{\kappa_{1,2,3} = 0}^{N_i} \frac{N_i}{\kappa_{1,2,3}} \left( \begin{array}{ccc} N_i & N_i & N_i \\ \kappa_{1,2,3} & \kappa_{1,2} & \kappa_{1,3} \end{array} \right) \right) \prod_{v' \neq v''} \left( \frac{2\alpha}{t} \right)^{3N_v} \prod_{m=1,2,3} \tilde{l}_{(v,m)}
\]
\[
\begin{align*}
&\times \left( \begin{array}{c}
N_i \\
\kappa_{i1} \\
N_j \\
\kappa_{i2} \\
N_l \\
\kappa_{i3}
\end{array} \right) \sum_{\nu'\in\mathbb{Z}_{2d}^m} \sum_{m'=1,2,3} (1 - \delta_{\nu',0} \delta_{N_{\nu' + e\nu},0}) \\
&\times \left( \frac{\eta_i}{t} \right)^{\alpha - 1 - \varepsilon} \left( \frac{i}{2} \partial_\eta \right) \left( \frac{\eta_i}{t} \right)^{s} \tilde{l}(\nu',m) - \left( \frac{\eta_i}{t} \right)^{N_{\nu'}} \alpha \tilde{l}(\nu',m) \\
&\times \prod_{(v,m) \neq (v',m')} \left( \frac{\eta_i}{t} \right)^{N_{\nu'}} \tilde{l}_{v,m}.
\end{align*}
\]

Note that we took advantage of the fact that at next-to-leading orders all but one derivative must be replaced by \((2\eta/t)\). Now, in the last two lines all non-trivial appearances of \(\kappa_{i,m}\) are inside the square brackets \([\ldots]\). Thus, we perform this summation first, using the explicit form of \(\partial_\eta \tilde{l}\):

\[
\sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left( \begin{array}{c}
N_i \\
\kappa_{i1} \\
N_j \\
\kappa_{i2} \\
N_l \\
\kappa_{i3}
\end{array} \right) \left( \frac{\eta_i}{t} \right)^{\alpha - 1 - \varepsilon} \left( \frac{i}{2} \partial_\eta \right) \left( \frac{\eta_i}{t} \right)^{s} \tilde{l}(\nu',m) \\
= \sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left( \begin{array}{c}
N_i \\
\kappa_{i1} \\
N_j \\
\kappa_{i2} \\
N_l \\
\kappa_{i3}
\end{array} \right) \left( \frac{\eta_i}{t} \right)^{\alpha - 1} \left( \frac{i}{2} \partial_\eta \right) \tilde{l}(\nu',m) \\
+ \sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left( \begin{array}{c}
N_i \\
\kappa_{i1} \\
N_j \\
\kappa_{i2} \\
N_l \\
\kappa_{i3}
\end{array} \right) \left( \frac{\eta_i}{t} \right)^{\alpha - 1} \left( \frac{i}{2} \partial_\eta \right) \tilde{l}(\nu',m) \\
= \frac{\eta_i}{t} \sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left( \begin{array}{c}
N_i \\
\kappa_{i1} \\
N_j \\
\kappa_{i2} \\
N_l \\
\kappa_{i3}
\end{array} \right) \left( \frac{\eta_i}{t} \right)^{\alpha - 1} \left( \frac{i}{2} \partial_\eta \right) \tilde{l}(\nu',m) \\
= \frac{\eta_i}{t} \sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left[ \frac{t}{2\eta} 2^{3N_i - 2} (N_{\nu'} + N_{\nu' + e\nu}) \partial_\eta \\
+ \frac{t}{2\eta} 2^{3N_i - 3} ((N_{\nu'} + N_{\nu' + e\nu})^2 - (N_{\nu'} + N_{\nu' + e\nu})) \\
- 2^{3N_i - 1} (N_{\nu'} + N_{\nu' + e\nu}) \tilde{l}(\nu',m) \\
= \frac{\eta_i}{t} \sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left[ \frac{t}{16\eta^2} (N_v + N_{v+e})(4N_v + N_v + N_{v+e} + 3 - 4\eta \coth(\eta)) \right] \tilde{l}(\nu',m) \\
= \frac{\eta_i}{t} \sum_{\kappa_{i1}\kappa_{i2}\kappa_{i3}=0}^{N_i} \left[ \frac{t}{16\eta^2} (N_v + N_{v+e})(4N_v + N_v + N_{v+e} + 3 - 4\eta \coth(\eta)) \right] \tilde{l}(\nu',m).
\]

(B.15)
Plugging this into (B.14), we get:

\[
\text{Cl}_{\{N_v\}} = P_0 + \sum_{v', m'} \sum_{m = 1, 2, 3} Z[\mathcal{N}(v', m')] N_{v'} N_{v'+e_m} \tilde{I}_{(v', m')}
\times (6i)^N_v 2^{3N_v} (1 - \delta_{N_v, 0}) Z[\mathcal{N}(v', m')] N_{v'} N_{v'+e_m} \tilde{I}_{(v', m')}
\]

With this and (B.12), we can do the final manipulations to arrive at (3.14):

\[
\left\langle \Psi', F \prod_v Q_v^{N_v} \Psi \right\rangle = \text{Cl}_{\{N_v\}} + \text{Qu}_{\{N_v\}}
\]

\[
= P_0 + \left( \prod_v \left( \frac{2\eta_i}{l} \right)^{N_v} (6i)^N_v \prod_{m = 1, 2, 3} \tilde{I}_{(v, m)} \right) \sum_{v', m'} \sum_{m = 1, 2, 3} Z[\mathcal{N}(v', m')] N_{v'} N_{v'+e_m} \tilde{I}_{(v', m')}
\]

\[
\times (1 - \delta_{N_v, 0}) Z[\mathcal{N}(v', m')] N_{v'} N_{v'+e_m} \tilde{I}_{(v', m')}
\]

\[
+ \left( \prod_v \left( \frac{2\eta_i}{l} \right)^{N_v} (6i)^N_v \prod_{m = 1, 2, 3} \tilde{I}_{(v, m)} \right) \sum_{v', m'} \sum_{m = 1, 2, 3} \left( \langle \langle \tilde{I}_{(v', m')} \rangle \rangle - \tilde{I}_{v', m'} \right) \tilde{I}_{v', m'}
\]

\[
= P_0 \left[ 1 + \sum_{v', m'} \sum_{m = 1, 2, 3} \frac{1}{\tilde{I}_{v', m'}} \left( \langle \langle \tilde{I}_{(v', m')} \rangle \rangle - \tilde{I}_{v', m'} \right)
\]

\[
+ (1 - \delta_{N_v, 0}) Z[\mathcal{N}(v', m')] N_{v'} N_{v'+e_m} \tilde{I}_{(v', m')} \right]
\]

(B.17)

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