The edge effects in layered beams

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Abstract. The article presents the resolving system of equations for solving the problem of the stress-strain state of multilayer beams. The system of resolving equations allows us to solve a wide range of problems, such as shear, bending, normal separation for any number of layers. In the proposed model the interaction between layers is accomplished by the contact layer. The contact layer is an anisotropic medium which can be considered as a "brush" of short elastic rods. The use of contact layer allows avoiding problems such as the endless shear stresses at the boundary of layers near the beam end. Another feature of the proposed model is the strict satisfaction of boundary conditions. Due to the complexity of resolving system of equations we consider a two-layer beam as an example. In this case it is possible to obtain an analytical solution and to produce its analysis. The edge effects arising in a two-layer beam are considered in detail. The article shows that the distribution of tangential stresses in the zone of the edge effect is a complex nonlinear function. At the moment the adhesive interaction is characterized by a single value – the average strength. This value is the ratio of the breaking load to the bonding area. Since the nature of stress distribution is a nonlinear function, the average strength is not true. The obtained solution makes it possible to establish a relationship between the average and true strength of the adhesive interaction, and to define the parameters of the contact layer based on experimental data. The influence of mechanical and geometric characteristics of the model on stress-strain state of the beam is analyzed.

1. Introduction

The multilayer beam model consisting of a set of “external” and contact layers is shown in the figure below.

Figure 1. Multilayer beam with contact layers
In the paper [1] we obtained a system of resolving equations for a multilayer beam. There is a part of this system for the “external” layer.

\[
\begin{align*}
\frac{d^2 N_k}{dx^2} & = \frac{dP_{s,k}^p}{dx} - \frac{dP_{s,k}^b}{dx}, \\
-D_k \frac{d^4 u_h}{dx^4} + \frac{h_k}{2} \left( \frac{dP_{s,k}^p}{dx} + \frac{dP_{s,k}^b}{dx} \right) & = P_{y,k}^p - P_{y,k}^b.
\end{align*}
\]

Equations (1) describe stress-strain state of the "external" layer \( k \). It is assumed that “external” layers obey the classical hypothesis of the theory of plates.

Equations (1) contain the following values: \( k \) – number of external layer; \( h_k \) – layer depth; \( D_k \) – integral bending stiffness; \( N_k \) – axial force; \( u_h \) – normal displacements; \( P_{y,k}^p, P_{y,k}^b \) – normal load; \( P_{s,k}^p, P_{s,k}^b \) – shear load; \( t, b \) – indices of the top and bottom faces of the layer respectively.

In the general case, the forces acting on the “external” layer are the sum of the loads applied to the layer and the forces that arise in the contact layers:

\[
\begin{align*}
P_{y,k}^p & = q_k^p + \sigma_k^*; \quad P_{y,k}^b = q_k^b + \sigma_{k+1}^*; \\
P_{s,k}^p & = s_k^p + \tau_k^*; \quad P_{s,k}^b = s_k^b + \tau_{k+1}^*.
\end{align*}
\]

In expressions (2) \( q_k^p; q_k^b; s_k^p; s_k^b \) – external loads, \( \sigma_k^*; \sigma_{k+1}^*; \tau_k^*; \tau_{k+1}^* \) – normal and tangential stresses in contact layers. Here and further, all values marked with * refer to a contact layer.

In the paper [1] a one-dimensional model of the contact layer is used. This model does not allow to fully satisfy all the boundary conditions. In this paper we will use a two-dimensional model presented in [2]. The stresses (normal and tangential) in the contact layer are determined from the following equations.

\[
\begin{align*}
\sigma_k^* & = -\frac{1}{h_k} \frac{d\tau_k^*}{dx} - \frac{E_k^0}{h_k^0} \left( u_{t,k}^* - u_{b,k}^* \right); \\
\tau_k^* & = \frac{h_k^0}{12E_k^0} \frac{d^4 \varepsilon_k^*}{dx^4} + \frac{1}{2} \left( \frac{du_{t,k}^*}{dx} + \frac{du_{b,k}^*}{dx} \right) - \frac{1}{h_k} \left( u_{t,k}^* - u_{b,k}^* \right).
\end{align*}
\]

Equations (3) include: \( h_k^0 \) – layer depth; \( G_k^0 \) – shear modulus; \( E_k^0 \) – Young’s modulus; \( u_{t,k}^*, u_{b,k}^* \) – normal displacements on the top and bottom faces of the contact layer (represent the normal displacements of the adjacent “external” layers); \( u_{t,k}^*, u_{b,k}^* \) – axial displacements (the expressions are written further in the text); \( y_k^* \) – the variable, counted from the middle line of the contact layer.

The resolving system represented by the equations (1) and (3) allows solving a large number of problems on the stress-strain state of layered rods.

As an example, consider a beam provided by two external layers and one contact layer. The general view of this beam is shown in the figure below.
Figure 2. Two-layer beam

We substitute in equations (1) \( k = 0, 1 \) and \( k = 1 \) in equations (3). As a result, we obtain a resolving system for a two-layer beam.

\[
\frac{d^2 N_0}{dx^2} = \frac{dP^0}{dx} - \frac{dP^b}{dx}; \quad -D_0 \frac{d^4 \nu_0}{dx^4} + \frac{h_0}{2} \left( \frac{dP^0}{dx} + \frac{dP^b}{dx} \right) = P^y - P^{b,0};
\]

\[
\frac{d^2 N_1}{dx^2} = \frac{dP^1}{dx} - \frac{dP^b}{dx}; \quad -D_1 \frac{d^4 \nu_1}{dx^4} + \frac{h_1}{2} \left( \frac{dP^1}{dx} + \frac{dP^b}{dx} \right) = P^y - P^{b,1};
\]

\[
\tau^* = \left( \frac{h^*}{2} \right)^2 \frac{d^2 \tau^*}{dx^2} + \frac{1}{2} \left( \frac{du_0}{dx} + \frac{du_1}{dx} \right) - \frac{1}{h^*} (u^*_0 - u^*_1);
\]

\[
\sigma^* = -y^* \frac{d\tau^*}{dx} - E^* \frac{1}{h^*} (u^*_0 - u^*_1).
\]

The values included in this system are recorded below

\[
P^y,0 = q_0; \quad P^{b,0} = \sigma^* \left( -\frac{h^*}{2} \right); \quad P^y,0 = s_0; \quad P^{b,0} = \tau^*;
\]

\[
P^y,1 = \sigma^* \left( \frac{h^*}{2} \right); \quad P^{y,1} = q_1; \quad P^{y,1} = \tau^*; \quad P^{b,1} = s_1;
\]

\[
u^*_0 = u^*_0,0 - \frac{h_0}{2} \frac{du_0}{dx}; \quad u^*_0 = u^*_0,1 + \frac{h_1}{2} \frac{du_1}{dx},
\]

\( u^* \) – the displacements of the “external” layer in the neutral axis. Index 1 for values relating to the contact layer is omitted.

Substituting (5) into (4), we obtain

\[
\frac{d^2 N_0}{dx^2} = -\frac{d\tau^*}{dx}; \quad -D_0 \frac{d^4 \nu_0}{dx^4} + \frac{h_0}{2} \frac{d\tau^*}{dx} = q_0 - \sigma^* \left( -\frac{h^*}{2} \right);
\]

\[
\frac{d^2 N_1}{dx^2} = \frac{d\tau^*}{dx}; \quad -D_1 \frac{d^4 \nu_1}{dx^4} + \frac{h_1}{2} \frac{d\tau^*}{dx} = \sigma^* \left( \frac{h^*}{2} \right) - q_1;
\]

\[
\tau^* = \left( \frac{h^*}{2} \right)^2 \frac{d^2 \tau^*}{dx^2} + \frac{1}{2} \left[ \frac{du_0}{dx} \left( 1 + \frac{h_0}{h^*} \right) + \frac{du_1}{dx} \left( 1 + \frac{h_1}{h^*} \right) \right] - \frac{1}{h^*} (u^*_0 - u^*_1);
\]

\[
\sigma^* = -y^* \frac{d\tau^*}{dx} - E^* \frac{1}{h^*} (u^*_0 - u^*_1).
\]
In paper [1] the following relations were obtained

\[
\frac{du^*_1}{dx} = \varepsilon_{0,0} + \frac{h_0}{2} \frac{d^2u_0}{dx^2} = \frac{N_0 + N_{f,0}}{B_0} - \frac{h_0}{2} \frac{d^2u_0}{dx^2};
\]

\[
\frac{du^*_1}{dx} = \varepsilon_{0,1} + \frac{h_1}{2} \frac{d^2u_1}{dx^2} = \frac{N_1 + N_{f,1}}{B_1} + \frac{h_1}{2} \frac{d^2u_1}{dx^2};
\]

\[
B_k = E_k h_k; \quad D_k = \frac{E_k h_k^3}{12}; \quad N_{f,k} = E_k \int_{-h_{f,k}^2}^{h_{f,k}^2} e_{f,k} dy_k.
\]

(7)

The values given in equations (5) and (7): $B_k$ – integral tensile-compression stiffness; $E_k$ – the Young’s modulus of the “external” layer; $N_{f,k}$ – axial forces associated with forced deformations $e_{f,k}$ (temperature, shrinkage etc.).

Further we will assume that the forced deformations are caused by temperature. In this case

\[
e_{f,k} = \alpha_k \Delta T_k; \quad N_{f,k} = E_k h_k \alpha_k \Delta T_k.
\]

(8)

Figure 2 shows the relationship between the longitudinal forces in the “external” layers and the applied longitudinal loads.

\[
N_0 + N_1 = P_1 + P_3; \quad N_0 + N_1 = P_2 + P_4.
\]

(9)

Adding to each other the expressions (9), we find

\[
\begin{align*}
N_1 &= F_p - N_0; \\
F_p &= \frac{1}{2} (P_1 + P_2 + P_3 + P_4).
\end{align*}
\]

(10)

In any case, static equilibrium must be ensured. As result we obtain equation

\[
P_1 + P_3 = P_2 + P_4.
\]

Let us differentiate the fifth equation (6).

\[
\frac{d\tau^*}{dx} = \frac{G^*}{12E^*} \left(\frac{d\tau^*}{dx}\right)^2 - \frac{G^*}{2} \left(\frac{d^2\tau^*}{dx^2} + \frac{d^2\tau^*}{dx^2}ight) \left(\frac{du^*_1}{dx}\right) - \frac{G^*}{h^*} \left(\frac{du^*_1}{dx}\right) - \frac{du^*_1}{dx}.
\]

(11)

Using the first and second relations (7), we transform (11).

\[
\frac{G^*}{12E^*} \left(\frac{d\tau^*}{dx}\right)^2 - \frac{d\tau^*}{dx} + \frac{G^*}{2} \left(\frac{d^2\tau^*}{dx^2} + \frac{d^2\tau^*}{dx^2}\right) \left(1 + \frac{h_0}{h^*}\right) + \frac{d^2\tau^*}{dx^2} \left(1 + \frac{h_1}{h^*}\right) - \frac{G^*}{h^*} \left(\frac{N_0 + N_{f,0}}{B_0} - \frac{N_1 + N_{f,1}}{B_1}\right) = 0.
\]

(12)

We use the first equation (6)

\[
\frac{G^*}{12E^*} \left(\frac{d\tau^*}{dx}\right)^2 - \frac{d^2\tau^*}{dx^2} + \frac{G^*}{h^*} \left(\frac{N_0 + N_{f,0}}{B_0} - \frac{F_p - N_0 + N_{f,1}}{B_1}\right) - \frac{G^*}{2} \left(\frac{d^2\tau^*}{dx^2} + \frac{d^2\tau^*}{dx^2}\right) \left(1 + \frac{h_0}{h^*}\right) + \frac{d^2\tau^*}{dx^2} \left(1 + \frac{h_1}{h^*}\right) = 0.
\]

(13)
Let us write the final equation (13).
\[
\begin{align*}
G^* \left( h^* \right)^2 \frac{d^4 N_0}{dx^4} - \frac{d^2 N_0}{dx^2} + N_0 \left( \frac{1}{h^*} + \frac{1}{B_0} \right) + G^* \left( \frac{N_{f,0}}{h^*} \right) & \left( \frac{1}{B_0} - \frac{F_p + N_{f,1}}{B_1} \right) - \\
- \frac{G^*}{2} \left[ \frac{d^2 \psi_0}{dx^2} \left( 1 + \frac{h_0}{h^*} \right) + \frac{d^2 \psi_1}{dx^2} \left( 1 + \frac{h_1}{h^*} \right) \right] &= 0.
\end{align*}
\]  
(14)

We transform the sixth equation (6) using the first
\[
\sigma^* = \gamma^* \frac{d^2 N_0}{dx^2} - \frac{E^*}{h^*} \left( v_0 - v_1 \right).
\]  
(15)

Finally, the resolving system of equations takes the form
\[
\begin{align*}
-D_0 \frac{d^4 \psi_0}{dx^4} - \frac{d^2 \psi_0}{dx^2} \left( \frac{h_0}{2} - \frac{h^*}{2} \right) &= q_0 + \frac{E^*}{h^*} \left( v_0 - v_1 \right); \\
-D_1 \frac{d^4 \psi_1}{dx^4} - \frac{d^2 \psi_1}{dx^2} \left( \frac{h_1}{2} + \frac{h^*}{2} \right) &= E^* \left( v_0 - v_1 \right) - q_1; \\
G^* \left( h^* \right)^2 \frac{d^4 N_0}{dx^4} - \frac{d^2 N_0}{dx^2} + N_0 \left( \frac{1}{h^*} \frac{1}{B_0} \right) + G^* \left( \frac{N_{f,0}}{h^*} \right) & \left( \frac{1}{B_0} - \frac{F_p + N_{f,1}}{B_1} \right) - \\
- \frac{G^*}{2} \left[ \frac{d^2 \psi_0}{dx^2} \left( 1 + \frac{h_0}{h^*} \right) + \frac{d^2 \psi_1}{dx^2} \left( 1 + \frac{h_1}{h^*} \right) \right] &= 0.
\end{align*}
\]  
(16)

2. Lap joint

In considering such problems it is possible to neglect the vertical displacements of the layers. This neglect is justified due to the low flexibility of the used samples.

As a result, only one equation remains of the resolving system (16).
\[
\begin{align*}
G^* \left( h^* \right)^2 \frac{d^4 N_0}{dx^4} - \frac{d^2 N_0}{dx^2} + N_0 \left( \frac{1}{h^*} \frac{1}{B_0} \right) + G^* \left( \frac{N_{f,0}}{h^*} \right) & \left( \frac{1}{B_0} - \frac{F_p + N_{f,1}}{B_1} \right) - \\
- \frac{G^*}{2} \left[ \frac{d^2 \psi_0}{dx^2} \left( 1 + \frac{h_0}{h^*} \right) + \frac{d^2 \psi_1}{dx^2} \left( 1 + \frac{h_1}{h^*} \right) \right] &= 0.
\end{align*}
\]  
(17)

We write (17) in a more compact form
\[
\frac{d^4 N_0}{dx^4} - 2 \omega^2 \frac{d^2 N_0}{dx^2} + \lambda^2 N_0 + \eta = 0,
\]  
(18)

where
\[
\omega = \frac{6E^*}{G^* \left( h^* \right)^2} \gamma^*; \quad \lambda = \left[ \frac{12E^*}{G^* \left( h^* \right)^2} \left( \frac{1}{B_0} + \frac{1}{B_1} \right) \right]^{1/2}; \quad \eta = \frac{12E^*}{G^* \left( h^* \right)^2} \left( \frac{N_{f,0}}{B_0} - \frac{F_p + N_{f,1}}{B_1} \right).
\]  
(19)

The general solution of equation (18) is written below
\[
N_0 = -\frac{\eta}{\lambda^2} + C_1 \exp(\psi_1 x) + C_2 \exp(-\psi_1 x) + C_3 \exp(\psi_2 x) + C_4 \exp(-\psi_2 x),
\]  
(20)

\(C_1\) – unknown constants of integration, determined from the boundary conditions.

\(\psi_i\) – roots of the characteristic equation \(\psi^4 - 2 \omega \psi^2 + \lambda^2 = 0\). Below are the given roots
\[ \psi_1 = \sqrt{\omega + (\omega^2 - \lambda^2)^2} \; ; \; \psi_2 = \sqrt{\omega - (\omega^2 - \lambda^2)^2}. \]  

(21)

There are the boundary conditions for determining the unknown constants of integration

\begin{align*}
N_0 \left( \frac{-l}{2} \right) &= P_1; \quad \frac{d}{dx} \left[ N_0 \left( \frac{-l}{2} \right) \right] = -\tau' \left( \frac{-l}{2} \right) = 0; \\
N_0 \left( \frac{l}{2} \right) &= P_2; \quad \frac{d}{dx} \left[ N_0 \left( \frac{l}{2} \right) \right] = -\tau' \left( \frac{l}{2} \right) = 0.
\end{align*}

(22)

The general solution obtained from these boundary conditions is not given in this article, as it is cumbersome.

Further, we will consider two options for load case in accordance with the figure below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{a) – Load case 1; b) – Load case 2}
\end{figure}

In all these cases \( P_2 = 0 \).

To analyze the effect of geometric and physical and mechanical characteristics on the stress-strain state of the model, we will use reference values. Below are the reference values.

\begin{itemize}
  \item \( h_0 = 10\text{mm}; \; h_1 = 10\text{mm}; \; h^* = 1\text{mm}; \; l = 100\text{mm}; \)
  \item \( E_0 = 2 \cdot 10^5\text{MPa}; \; E_1 = 2 \cdot 10^5\text{MPa}; \; E^* = 10^3\text{MPa}; \; P = 1 \cdot \text{kN/mm}. \)
\end{itemize}

The results of calculating the longitudinal forces and tangential stresses for the reference parameters of the model are given below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Axial forces in external layers. Load case 1}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Axial forces in external layers. Load case 2}
\end{figure}
Figure 6. Tangential stresses in contact layer for load cases 1 and 2

Figure 6 shows the edge effect that occurs in the contact layer in a narrow zone. By testing shear samples, the adhesive strength is calculated as the breaking load divided by the gluing area. The value obtained this way should be called the average adhesion strength $\tau_{mid}$. The true adhesion strength is the maximum value of the tangential stresses $\tau_{max}$ at the moment of destruction. Further, a series of graphs will be presented reflecting the correlation between the true and average adhesion strength.

Figure 7. The ratio between the true and average adhesion strength, depending on the length of the joint

As we can see, as the length of the joint increases, the ratio of the true adhesion strength to the average strength increases. Starting with a certain value, these dependencies are linear. This is due to the fact that the increase in the length of the joint ceases to affect the maximum value of tangential stresses. At small sizes the true strength almost coincides with the average one.
Further is a series of graphs reflecting the dependence of the magnitude of the maximum tangential stresses on various model parameters.

![Graphs showing dependence of maximum tangential stresses on various parameters.](image)

**Figure 8.** Dependence true adhesion strength/length of the joint

**Figure 9.** Dependence true adhesion strength/Young’s modulus of contact layer

**Figure 10.** Dependence true adhesion strength/depth of contact layer

**Figure 11.** Dependence true adhesion strength/Young’s modulus of “external” layers

3. Conclusion

In this article we obtain the system of resolving equations to solve numerous problems of the theory of multilayered beams. These equations predict the appearance of the edge effect. The results of solving the problem of the stress-strain state of the lap joint are also obtained. The influence of mechanical characteristics on the true adhesion strength is analyzed.

It is obvious that the volume of the article is not enough for a detailed analysis of the results. However, it can be said that the proposed model of adhesion interaction using the contact layer
method reflects the influence of many factors on strength and allows to qualitatively and quantitatively analyze.

4. References

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