BOUND FOR THE FIRST NON-ZERO STEKLOV EIGENVALUE

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Abstract. Let $\Omega$ be a star-shaped bounded domain in $(\mathbb{S}^n, ds^2)$ with smooth boundary. In this article, we give a sharp lower bound for the first non-zero eigenvalue of the Steklov eigenvalue problem in $\Omega$. This result is the generalization of a result given by Kuttler and Sigillito for a star-shaped bounded domain in $\mathbb{R}^2$. Further we also obtain a two sided bound for the first non-zero eigenvalue of the Steklov problem on the ball in $\mathbb{R}^n$ with rotationally invariant metric and with bounded radial curvature.

1. Introduction

Let $\Omega$ be a bounded domain in $(\mathbb{S}^n, ds^2)$ with smooth boundary $\partial \Omega$. Consider the following problem

$$\begin{align*}
\Delta f &= 0 \quad \text{in } \Omega, \\
\frac{\partial f}{\partial \nu} &= \mu f \quad \text{on } \partial \Omega,
\end{align*}$$

(1)

where $\nu$ is the unit outward normal on the boundary $\partial \Omega$ and $\mu$ is a real number.

This problem is known as Steklov eigenvalue problem and was introduced by Steklov [1] for bounded domains in the plane in 1902. This problem is important as the set of eigenvalues of the Steklov problem is same as the set of eigenvalues of the well known Dirichlet-Neumann map. This map associates to each function $f$ defined on $\partial \Omega$, the normal derivative of its harmonic extension on $\Omega$. The eigenvalues of the Steklov problem are discrete and forms an increasing sequence $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$.

In this article, we are interested in finding bounds for the first non-zero eigenvalue $\mu_1$ of the Steklov problem. The variational characterization of $\mu_1$ is given by

$$\mu_1 = \inf \left\{ \int_\Omega \| \nabla u \|^2 dv : \int_{\partial \Omega} u^2 ds = 0, u(\neq 0) \in C^1(\Omega) \right\}. $$

(2)

There are several results which estimate the first non-zero eigenvalue $\mu_1$ of the Steklov eigenvalue problem [3, 8, 9, 10, 11]. The first upper bound for $\mu_1$ was given by Weinstock [2] in 1954. He proved that among all simply connected planar domains with analytic boundary of fixed perimeter, the circle maximizes $\mu_1$.

Let $\Omega \subset \mathbb{R}^n$ be a star shaped domain with smooth boundary $\partial \Omega$. Let $p$ be a centre of $\Omega$. Let $R_m = \min \{ d(p, x) | x \in \partial \Omega \}$, $R_M = \max \{ d(p, x) | x \in \partial \Omega \}$ and $h_m = \min \{ \langle x, \nu \rangle | x \in \partial \Omega \}$, where $\nu$ is the outward unit normal to $\partial \Omega$. Bramble and Payne [3] proved that

$$\mu_1 \geq \frac{R_m}{R_M^{n+1}} h_m. $$

This bound is sharp when $\Omega$ is a ball. Payne [5] calculated the two sided bound for $\mu_1$ on convex domain $\Omega \subset \mathbb{R}^2$ in terms of minimum and maximum curvature on boundary of $\Omega$. Escobar [8]
generalized this result for two dimensional compact Riemannian manifold \((M, g)\) with non-negative Gaussian curvature such that geodesic curvature on \(\partial M\) is bounded from below.

With a different approach, Kuttler and Sigillito \([4]\) proved the following theorem for a star-shaped bounded domain in \(\mathbb{R}^2\).

**Theorem 1.1** \([4]\). Let \(\Omega\) be a star-shaped bounded domain in \(\mathbb{R}^2\) with smooth boundary. Then

\[
\mu_1(\Omega) \geq \frac{2}{1 + \sqrt{1 + 4 \min \left( \frac{R(\theta)}{\nu(\theta)} \right)^2}} \max \sqrt{R^2(\theta) + R'^2(\theta)}.
\]

where \(R(\theta) := \max \left\{ |x| : x \in \Omega, x = |x|e^{i\theta} \right\}\) and equality holds for a disc.

Garcia and Montano \([13]\) generalized this result for the star-shaped domain in \(\mathbb{R}^n\).

**Theorem 1.2** \([13]\). Let \(\Omega \subset \mathbb{R}^n\) be a star-shaped domain with smooth boundary \(\partial \Omega\) and unit outward normal \(\nu\). If \(0 \leq \theta \leq \alpha < \frac{\pi}{2}\), where \(\cos(\theta) = \langle \nu, \partial_r \rangle\), then the first non-zero eigenvalue of the Steklov problem \(\mu_1(\Omega)\) satisfies

\[
\mu_1(\Omega) \geq \frac{(R_m)^{n-2}}{(R_M)^{n-1}} \left\{ 2 + a - \sqrt{a^2 + 4a} \right\} \frac{2}{2\sqrt{a+1}}.
\]

where \(a := \tan^2 \alpha\), \(R_m := \min R(u)\), \(R_M := \max R(u)\) and \(R(u) := \max \{ r : x \in \Omega, x = (r, u) \}\), \((r, u)\) represents the polar coordinates on \(\mathbb{R}^n\).

In Theorem 3.1, we give a lower bound for the first non-zero steklov eigenvalue in a star-shaped domain \(\Omega\) in \((S^n, ds^2)\), following the idea of Kuttler and Sigillito. In particular, given a star-shaped domain \(\Omega\) with respect to a point \(p\) in \((S^n, ds^2)\), let \(R_m\) be the minimum distance of the \(\partial \Omega\) from \(p\) and \(B(R_m)\) be the ball of radius \(R_m\) centered at \(p\). We show that \(\mu_1(\Omega) \geq C \mu_1(B(R_m))\) for some constant \(C\) defined in Section 3.

Let \((B_R, g)\) be a ball of radius \(R\) in \(\mathbb{R}^n\) with a rotationally invariant metric \(g = dr^2 + \sigma^2(r) du^2\), where \(du^2\) is the standard metric on \(S^{n-1}\).

Escobar \([10]\) proved that the first non-constant eigenfunction for the Steklov problem on \(B_R\) has the form \(\phi(r, u) = \psi(r) e(u)\), where \(e(u)\) and \(\psi(r)\) satisfy

\[
\Delta_{S^{n-1}} e + (n - 1) e = 0,
\]

\[
\frac{1}{\sigma^{n-1}(r)} \frac{d}{dr} \left( \sigma^{n-1}(r) \frac{d}{dr} \psi(r) \right) - \frac{(n - 1) \psi(r)}{\sigma^2(r)} = 0 \quad \text{in } (0, R),
\]

\(\psi'(R) = \mu_1(B_R) \psi(R), \quad \psi(0) = 0.\)

Using this result he proved that the first non-zero eigenvalue \(\mu_1\) for two dimensional ball \(B_R\) is \(\mu_1 = \frac{1}{\sigma(R)}\).

For \(n\)-dimensional ball \((B_R, g)\) with rotationally invariant metric \(g = dr^2 + \sigma^2(r) du^2\), Montano \([12]\) proved the following.

- If \(\text{Ric}(g) \geq 0\), then \(\mu_1 \leq \left( \frac{R}{\sigma(R)} \right)^{n+1} \frac{1}{2}.\)
- If \(\text{Ric}(g) \leq 0\), then \(\mu_1 \geq \left( \frac{R}{\sigma(R)} \right)^{n+1} \frac{1}{2}.\)

In Theorem 3.2, we derive a two sided bound for \(\mu_1\) on the ball with rotationally invariant metric and with bounded radial curvature.

2. Preliminaries

In this section we fix some notations which will be used in the proof of the main results.
Definition 2.1. Let $M$ be a Riemannian manifold and $p \in M$. The injectivity radius at $p$, denoted by $\text{inj}(p)$, is defined as

$$\text{inj}(p) := \sup \{ r > 0 | \exp_p : B(0,r) \to B(p,r) \text{ is a diffeomorphism} \}.$$ 

Let $\Omega$ be a star-shaped bounded domain in $(S^n, ds^2)$ with respect to a point $p \in \Omega$ and $\Omega \subset B(p, \text{inj}(p))$, a geodesic ball of radius $\text{inj}(p)$. Let $\partial \Omega$ be the smooth boundary of $\Omega$ with unit outward normal $\nu$. Since $\Omega$ is star-shaped with respect to the point $p$ and with smooth boundary, for every point $q \in \partial \Omega$, there exists unique unit vector $u \in T_p S^n$ and $R_u > 0$ such that $q = \exp_p(R_u u)$. In notation we have the following.

$$\partial \Omega = \{ \exp_p(R_u u) : u \in T_p S^n, \|u\| = 1 \},$$

$$\Omega = \{ \exp_p(t u) : u \in T_p S^n, \|u\| = 1, 0 \leq t < R_u \}.$$

In geodesic normal coordinates, $\Omega$ and $\partial \Omega$ can be written as

$$\partial \Omega = \{ (R_u, u) : u \in T_p S^n, \|u\| = 1 \},$$

$$\Omega \setminus \{ p \} = \{ (r, u) : u \in T_p S^n, \|u\| = 1, 0 < r < R_u \}.$$

Let $R_m := \min R_u, R_M := \max R_u$.

For a Riemannian manifold $(M, g)$ with constant sectional curvature $k$, we know that

$$g = \begin{cases} 
  dr^2 + r^2 g_{S^{n-1}} & \text{for } k = 0, \\
  dr^2 + \frac{1}{k^2} \sin^2(\sqrt{k} r) g_{S^{n-1}} & \text{for } k > 0, \\
  dr^2 + \frac{1}{|k|} \sinh^2(\sqrt{|k|} r) g_{S^{n-1}} & \text{for } k < 0,
\end{cases}$$

with respect to the geodesic normal coordinates about a point, where $g_{S^{n-1}}$ is the standard metric on the $(n-1)$-dimensional unit sphere and $r \leq \frac{\pi}{\sqrt{k}}$ for $k > 0$. Then for any smooth function $f$ defined on $\overline{\Omega}$,

$$\|\nabla f\|^2 = \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin^2 r} \|\nabla f\|^2,$$

where $\nabla f$ represents the component of $\nabla f$ tangential to $S^{n-1}$, the $(n-1)$-dimensional unit sphere.

Let $\partial_r$ be the radial vector field starting at $p$, the centre of $\Omega$ and $\nu$ be the unit outward normal to $\partial \Omega$. Since $\Omega$ is a star-shaped bounded domain, for any point $q \in \partial \Omega$, $\cos(\theta(q)) = \langle \nu(q), \partial_r(q) \rangle > 0$. Therefore $\theta(q) < \frac{\pi}{2}$ for all $q \in \partial \Omega$. By compactness of $\partial \Omega$ there exist $\alpha$ such that $0 \leq \theta(q) \leq \alpha < \frac{\pi}{2}$ for all $q \in \partial \Omega$.

Let $(B_R, g)$, $R > 0$ be a ball in $\mathbb{R}^n$ centered at a point $p$, with a rotationally invariant metric

$$g = dr^2 + \sigma^2(r) du^2,$$

where $du^2$ is the standard metric on $S^{n-1}$ and $\sigma : [0, R) \to [0, \infty)$ is a non-zero smooth function satisfying $\sigma(0) = 0, \sigma'(0) = 1$ and $\sigma(r) > 0, r \in (0, R]$.

Let $q \in (B_R, g)$ and $X_q \perp \nabla r(q)$ be a unit tangent vector. The sectional curvature $K(\nabla r, X)$ is called the radial curvature and we know that

$$K(\nabla r, X) = -\frac{\sigma''(r)}{\sigma(r)}.$$ 

Since $K(\nabla r, X)$ depends only on $r$, we will denote it by $K(r)$.

3. Main Results

We begin this section by stating the main results.

The following theorem gives a sharp lower bound for the first non-zero Steklov eigenvalue for a star-shaped domain in $(S^n, ds^2)$.
Theorem 3.1. Let $\Omega \subset S^n$, $\nu$ and $\alpha$ are as in Section 2. Let $a = \max \left\{ \| \nabla R_n \|^2 \right\}$. Then the first eigenvalue $\mu_1(\Omega)$ of (1) satisfies the following inequality
\[
\mu_1(\Omega) \geq \left( \frac{R_m}{R_M} \right) \left( \frac{2 + a - \sqrt{a^2 + 4a}}{2} \right) \frac{\sin^{n-1}(R_m)}{\sec(\alpha) \sin^{n-1}(R_M)} \mu_1(B(R_m)),
\]
where $B(R_m)$ is the geodesic ball of radius $R_m$ centered at $p$. Furthermore, equality holds if and only if $\Omega$ is a geodesic ball.

Next theorem gives a sharp lower bound as well as a sharp upper bound for the first non-zero Steklov eigenvalue on the ball with rotationally invariant metric and with bounded radial curvature.

Theorem 3.2. Let $(B_R,g)$, $K(r)$ are as in Section 2 and $(B_R, \text{can}_k)$ be a ball in $\mathbb{R}^n$ of radius $R$, with canonical metric $\text{can}_k = dr^2 + \sin^2(r) du^2$. Then the following holds.

(i) If $K(r) \geq k$. Then
\[
\mu_1(B_R, \text{can}_k) \leq \mu_1(B_R, g) \leq \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, \text{can}_k).
\]

(ii) If $K(r) \leq k$. Then
\[
\left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, \text{can}_k) \leq \mu_1(B_R, g) \leq \mu_1(B_R, \text{can}_k).
\]

Furthermore, equality holds if and only if $(B_R, g)$ is isometric to the $(B_R, \text{can}_k)$.

The following lemma is crucial to prove the Theorem 3.1.

Lemma 3.3. For $x \in [0, \frac{\pi}{2}]$,

(i) $\sin ax \geq a \sin x$ for $0 \leq a \leq 1$,

(ii) $\sin ax \leq a \sin x$ for $a \geq 1$.

Proof. Define $g(x) = \sin ax - a \sin x$. Observe that $g(0) = 0$ for all $a \in \mathbb{R}$.

(i) We show that $g$ is an increasing function on $[0, \frac{\pi}{2}]$, for $0 \leq a \leq 1$. As $0 \leq ax \leq x \leq \frac{\pi}{2}$ and $\cos ax \geq \cos x$, it follows that $g'(x) \geq 0$.

(ii) Let $a \geq 1$. Then $ax \geq x$ for $x \in [0, \frac{\pi}{2}]$. We divide the proof into two cases.

- For $ax \leq \frac{\pi}{2}$, proof follows by using the same argument we used for $0 \leq a \leq 1$.
- For $ax \geq \frac{\pi}{2}$, we will show that $a \sin x \geq 1$. Note that $a \sin x$ is an increasing function for $0 \leq x \leq \frac{\pi}{2}$. So it is enough to prove that $a \sin \left( \frac{\pi}{2a} \right) \geq 1$. Since $\frac{1}{a} \leq 1$, we have $\sin \left( \frac{\pi}{2a} \right) \geq \frac{1}{a} \sin \left( \frac{\pi}{2a} \right) = \frac{1}{a}$. Therefore $\sin ax \leq 1 \leq a \sin \left( \frac{\pi}{2a} \right) \leq a \sin x$.

Remark 3.4. If $\sin ax = a \sin x$ for all $x \in [0, \frac{\pi}{2}]$, then it can be easily seen that either $a = 0$ or $a = 1$.

Proof of Theorem 3.1. By variational characterization of $\mu_1$, we have
\[
\mu_1(\Omega) = \inf \left\{ \frac{\int_{\Omega} \| \nabla u \|^2 \, dv}{\int_{\partial \Omega} u^2 \, ds} : \int_{\partial \Omega} u \, ds = 0, u(\neq 0) \in C^1(\Omega) \right\}.
\]

For a continuously differential real valued function $f$ defined on $\Omega$, we find a lower bound for $\int_{\Omega} \| \nabla f \|^2 \, dv$ and an upper bound for $\int_{\partial \Omega} f^2 \, ds$ to find a lower bound for $\frac{\int_{\Omega} \| \nabla f \|^2 \, dv}{\int_{\partial \Omega} f^2 \, ds}$. 


• A Lower bound for $\int_{\Omega} \|\nabla f\|^2 \, dv$.

Let $f$ be a continuously differential real valued function defined on $\Omega$. Then for $q \in \Omega$,

$$
\|\nabla f\|^2 = \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin^2 r} \|\nabla f\|^2.
$$

Therefore

$$
\int_{\Omega} \|\nabla f\|^2 \, dv = \int_{U,\Omega} \int_0^{R_u} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin^2 r} \|\nabla f\|^2 \right] \sin^{n-1} r \, dr \, du.
$$

Let $r = \frac{\rho R_u}{R_m}$. Then the integral is written as

$$
\int_{\Omega} \|\nabla f\|^2 \, dv = \int_{U,\Omega} \int_0^{R_m} \left[ \left( \frac{R_u}{R_m} \right)^2 + \|\nabla R_u\|^2 \left( \frac{\rho}{R_u \sin \left( \frac{\rho R_u}{R_m} \right)} \right)^2 \right] \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{1}{\sin^2 \left( \frac{\rho R_u}{R_m} \right) \|\nabla f\|^2}
$$

$$
- \frac{2 \rho}{R_u \sin^2 \left( \frac{\rho R_u}{R_m} \right)} \left( \frac{\partial f}{\partial \rho} \right) \langle \nabla f, \nabla R_u \rangle \sin^{n-1} \left( \frac{\rho R_u}{R_m} \right) \left( \frac{R_u}{R_m} \right) d\rho \, du
$$

$$
= \int_{U,\Omega} \int_0^{R_m} \left[ \left( \frac{R_m}{R_u} \right)^2 + \|\nabla R_u\|^2 \left( \frac{\rho}{R_u \sin \left( \frac{\rho R_u}{R_m} \right)} \right)^2 \right] \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{R_u}{R_m \sin^2 \left( \frac{\rho R_u}{R_m} \right) \|\nabla f\|^2}
$$

$$
- \frac{2 \rho}{R_m \sin^2 \left( \frac{\rho R_u}{R_m} \right)} \left( \frac{\partial f}{\partial \rho} \right) \langle \nabla f, \nabla R_u \rangle \sin^{n-1} \left( \frac{\rho R_u}{R_m} \right) d\rho \, du.
$$

Next we estimate $\langle \nabla f, \nabla R_u \rangle$. For any function $\beta^2$ on $\Omega$, Cauchy-Schwartz inequality gives

$$
- \frac{2 \rho}{R_m \sin^2 \left( \frac{\rho R_u}{R_m} \right)} \left( \frac{\partial f}{\partial \rho} \right) \langle \nabla f, \nabla R_u \rangle \geq - \frac{1}{\beta^2} \|\nabla R_u\|^2 \left( \frac{\rho}{\sin \left( \frac{\rho R_u}{R_m} \right)} \right)^2 \left( \frac{\partial f}{\partial \rho} \right)^2 - \frac{\beta^2}{R_u \sin^2 \left( \frac{\rho R_u}{R_m} \right) \|\nabla f\|^2}.
$$

Thus

$$
\int_{\Omega} \|\nabla f\|^2 \, dv \geq \int_{U,\Omega} \int_0^{R_m} \left[ \left( \frac{R_m}{R_u} \right)^2 - \left( \frac{1}{\beta^2} - 1 \right) \|\nabla R_u\|^2 \left( \frac{\rho}{\sin \left( \frac{\rho R_u}{R_m} \right)} \right)^2 \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{R_u}{R_m \sin^2 \left( \frac{\rho R_u}{R_m} \right) \|\nabla f\|^2} \sin^{n-1} \left( \frac{\rho R_u}{R_m} \right) \right] d\rho \, du.
$$

(5)

Note that $0 \leq \frac{\rho}{R_m} \leq 1 \leq \frac{\rho}{R_u}$ and $0 \leq R_u, R_m \leq \frac{\pi}{2}$. Hence it follow from Lemma 3.3 that

$$
\frac{\rho}{R_m \sin(R_u)} \leq \sin \left( \frac{\rho R_u}{R_m} \right) \leq \frac{R_u}{R_m} \sin(\rho).
$$

This gives

$$
\frac{\sin(\rho)}{\sin \left( \frac{\rho R_u}{R_m} \right)} \geq \frac{R_m}{R_u} \quad \text{and} \quad - \left( \frac{\rho}{\sin \left( \frac{\rho R_u}{R_m} \right)} \right)^2 \geq - \left( \frac{R_m}{\sin(R_u)} \right)^2, \quad \text{for } \rho > 0.
$$

(6)

Since $0 \leq \rho \leq \frac{\rho R_u}{R_m} \leq R_u \leq \frac{\pi}{2}$, we have

$$
0 \leq \sin^{n-1}(\rho) \leq \sin^{n-1} \left( \frac{\rho R_u}{R_m} \right) \leq 1.
$$

(7)
We assume $\beta^2 < 1$ and by substituting above inequalities in (5), we get
\[
\int_{\Omega} \|\nabla f\|^2 \, dv \geq \int_{U_{\rho} \Omega} \int_0^{R_m} \left[ \left\{ \left( \frac{R_m}{R_u} \right)^2 - \left( \frac{1}{\beta^2} - 1 \right) \|\nabla R_u\|^2 \left( \frac{R_m}{R_u} \sin(R_u) \right) \right\} \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{(1 - \beta^2)}{\sin^2(\rho)} \|\nabla f\|^2 \right] \sin^{-1} (\rho) \, d\rho \, du
\]
\[
\geq \int_{U_{\rho} \Omega} \int_0^{R_m} \left[ \left\{ 1 - \left( \frac{1}{\beta^2} - 1 \right) a \right\} \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{(1 - \beta^2)}{\sin^2(\rho)} \|\nabla f\|^2 \right] \sin^{-1} (\rho) \, d\rho \, du.
\]
By solving the equation $1 - \left( \frac{1}{\beta^2} - 1 \right) a = 1 - \beta^2$ for $\beta^2$ we see that
\[
\beta^2 = -a + \frac{\sqrt{a^2 + 4a}}{2}.
\]
Therefore
\[
1 - \left( \frac{1}{\beta^2} - 1 \right) a = 1 - \beta^2 = \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} > 0.
\]
From this it follows that
\[
\int_{\Omega} \|\nabla f\|^2 \, dv \geq \left( \frac{R_m}{R_M} \right) \left( \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} \right) \int_{U_{\rho} \Omega} \int_0^{R_m} \left[ \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{1}{\sin^2(\rho)} \|\nabla f\|^2 \right] \sin^{-1} (\rho) \, d\rho \, du
\]
\[
= \left( \frac{R_m}{R_M} \right) \left( \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} \right) \int_{B(R_m)} \|\nabla f\|^2 \, dv. \tag{8}
\]

- **An upper bound for $\int_{\partial \Omega} f^2 \, ds$.**

Recall that the Riemannian volume measure on $\partial \Omega$, denoted $ds$, is given by $ds = \sec(\theta) \sin^{n-1}(R_u) \, du$ (see [7]), where $\theta$ and $R_u$ are defined in Section 2. Then
\[
\int_{\partial \Omega} f^2 \, ds = \int_{U_{\rho} \Omega} f^2 \sec(\theta) \sin^{n-1}(R_u) \, du.
\]
By using the fact that $\sin^{n-1} R_m \leq \sin^{n-1} R_u \leq \sin^{n-1} R_M$ and substituting $r = \frac{R_m}{R_m}$, this integral becomes
\[
\int_{\partial \Omega} f^2 \, ds \leq \frac{\sec(\alpha) \sin^{n-1}(R_M)}{\sin^{n-1}(R_m)} \int_{U_{\rho} \Omega} f^2 \sin^{n-1}(R_m) \, du
\]
\[
= \sec(\alpha) \sin^{n-1}(R_M) \int_{S(R_m)} f^2 \, ds. \tag{9}
\]
If \( f \) satisfies \( \int_{S(R_m)} f \, ds = 0 \), by inequalities (8) and (9) we have
\[
\frac{\int_{\Omega} \| \nabla f \|^2 \, dv}{\int_{\partial \Omega} f^2 \, ds} \geq \left( \frac{R_m}{R_M} \right) \left( \frac{2 + a - \sqrt{a^2 + 4a}}{2} \right) \frac{\sin^{-1}(R_m)}{\sec(\alpha) \sin^{-1}(R_M)} \frac{\int_{B(R_m)} \| \nabla f \|^2 \, dv}{\int_{S(R_m)} f^2 \, ds}
\geq \left( \frac{R_m}{R_M} \right) \left( \frac{2 + a - \sqrt{a^2 + 4a}}{2} \right) \frac{\sin^{-1}(R_m)}{\sec(\alpha) \sin^{-1}(R_M) \mu_1(B(R_m))}.
\]

Next we find a test function for the Steklov problem on \( B(R_m) \) in terms of a first eigenfunction for the Steklov problem on \( \Omega \) and use the above inequality to get the desired bound.

Let \( \psi_1 \) be a first eigenfunction for the Steklov problem (1) on \( \Omega \). Define
\[
\psi := \psi_1 - \overline{\psi_1} \quad \text{where} \quad \overline{\psi_1} = \frac{\int_{S(R_m)} \psi_1 \, ds}{Vol(S(R_m))}.
\]

Observe that \( \int_{S(R_m)} \psi \, ds = 0 \). Therefore from (10), we have
\[
\frac{\int_{\Omega} \| \nabla \psi \|^2 \, dv}{\int_{\partial \Omega} \psi^2 \, ds} \geq \left( \frac{R_m}{R_M} \right) \left( \frac{2 + a - \sqrt{a^2 + 4a}}{2} \right) \frac{\sin^{-1}(R_m)}{\sec(\alpha) \sin^{-1}(R_M) \mu_1(B(R_m))}.
\]

Next by using the fact that \( \int_{\partial \Omega} \psi_1 \, ds = 0 \) we obtain
\[
\frac{\int_{\Omega} \| \nabla \psi \|^2 \, dv}{\int_{\partial \Omega} \psi^2 \, ds} = \frac{\int_{\Omega} \| \nabla \psi_1 \|^2 \, dv}{\int_{\partial \Omega} (\psi_1^2 + \overline{\psi_1}^2) \, ds}
\leq \frac{\int_{\Omega} \| \nabla \psi_1 \|^2 \, dv}{\int_{\partial \Omega} \overline{\psi_1}^2 \, ds}
= \mu_1(\Omega).
\]

Combining the inequality (12) and (13) we obtain
\[
\mu_1(\Omega) \geq \left( \frac{R_m}{R_M} \right) \left( \frac{2 + a - \sqrt{a^2 + 4a}}{2} \right) \frac{\sin^{-1}(R_m)}{\sec(\alpha) \sin^{-1}(R_M)} \mu_1(B(R_m)).
\]

If \( \Omega \) is a geodesic ball of radius \( R \), then \( R_m = R_M = R \) and \( a = \alpha = 0 \), hence equality holds in (3). Next assume that equality hold in (3), then equality holds in (6) and (7). This implies \( R_a = R_m \). Hence \( \Omega \) is a geodesic ball. This proves the theorem.

The following lemma is used to prove the Theorem 3.2.

Let
\[
\sin_k r = \begin{cases} r & \text{for } k = 0, \\ \sin \sqrt{k \epsilon_r} & \text{for } k > 0, \\ \sin \sqrt{\sqrt{-k} r} & \text{for } k < 0. \end{cases}
\]

**Lemma 3.5.** Let \((B_R, g)\), \( R > 0 \) be a ball in \( \mathbb{R}^n \) with a rotationally invariant metric \( g = dr^2 + \sigma^2(r) \, du^2 \) and \( K(r) \) be the radial sectional curvature such that \( K(r) \geq k \) for some \( k \in \mathbb{R} \). We assume \( R \leq \frac{r}{\sqrt{k}} \) when \( k > 0 \). Then the following holds.

(i) \( \sigma(r) \leq \sin_k(r) \) for \( r \in [0, R] \),
(ii) \( \lim_{r \to 0} \frac{\sigma(r)}{\sin_k(r)} = 1 \) and
(iii) \( \frac{\sigma(r)}{\sin_k(r)} \) is a decreasing function.

**Proof.**

(i) Let \( k \in \mathbb{R} \) and observe that \( \sin_k r \) is the solution of
\[
y''(r) + k y(r) = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad r \in [0, R].
\]
Since \( K(r) \geq k \) and
\[
\sigma''(r) + K(r) \sigma(r) = 0, \quad \sigma(0) = 0, \quad \sigma'(0) = 1, \quad r \in [0, R],
\]
it follows from Sturm Comparison Theorem [6] that \( \sigma(r) \leq \sin_k(r) \) for \( r \in [0, R] \).

(ii) Since \( \sigma'(0) = 1 = \sin_k'(0) \), it follows from L'Hospital Rule that
\[
\lim_{r \to 0} \frac{\sigma(r)}{\sin_k(r)} = \lim_{r \to 0} \frac{\sigma'(r)}{\sin_k'(r)} = 1.
\]

(iii) For \( k < 0 \),
\[
\left( \frac{\sigma(r)}{\sinh \sqrt{-kr}} \right)' = \frac{\sigma'(r) \sinh \sqrt{-kr} - \sqrt{-k} \sigma(r) \cosh \sqrt{-kr}}{\sinh^2 \sqrt{-kr}}
\]
and
\[
\left( \sigma'(r) \sinh \sqrt{-kr} - \sqrt{-k} \sigma(r) \cosh \sqrt{-kr} \right)'
= \sigma''(r) \sinh \sqrt{-kr} + \sqrt{-k} \sigma'(r) \cosh \sqrt{-kr} - \sqrt{-k} \sigma'(r) \cosh \sqrt{-kr} + k \sigma(r) \sinh \sqrt{-kr}
= (\sigma''(r) + k \sigma(r)) \sinh \sqrt{-kr}
\leq 0.
\]
Therefore \( \sigma'(r) \sinh \sqrt{-kr} - \sqrt{-k} \sigma(r) \cosh \sqrt{-kr} \) is a decreasing function of \( r \) and
\[
\sigma'(r) \sinh \sqrt{-kr} - \sqrt{-k} \sigma(r) \cosh \sqrt{-kr} \leq 0 \text{ for } r \geq 0.
\]
Hence \( \left( \frac{\sigma(r)}{\sinh \sqrt{-kr}} \right)' \leq 0 \). This implies that \( \frac{\sigma(r)}{\sinh \sqrt{-kr}} \) is a decreasing function.

A similar computation as above proves that \( \frac{\sigma(r)}{\sinh \sqrt{-kr}} \) is a decreasing function for \( k \geq 0 \) as well.

\( \square \)

**Proof of Theorem 3.2.** First we prove the theorem for \( K(r) \geq k \).
Let \( f \) be a continuously differential real valued function defined on \( B_R \). Then
\[
\| \nabla f \|^2 = \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sigma^2(r)} \| \nabla f \|^2,
\]
where \( \nabla f \) is the component of \( \nabla f \) tangential to \( S^{n-1} \). Therefore
\[
\int_{B_R} \| \nabla f \|^2 dv_g = \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sigma^2(r)} \| \nabla f \|^2 \right] \sigma^{n-1}(r) dr du.
\]
By Lemma 3.5, \( \frac{\sigma(r)}{\sin_k(r)} \leq \frac{\sigma(r)}{\sin_k(r)} \leq 1 \) for \( 0 < r \leq R \), we have
\[
\int_{B_R} \| \nabla f \|^2 dv_g \geq \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{\sigma(r)}{\sin_k R} \right] \frac{1}{\sigma^2(r)} \| \nabla f \|^2 \left( \frac{\sigma(R)}{\sin_k R} \right)^{n-1} \sin_k^{n-1} r dr du
= \left( \frac{\sigma(R)}{\sin_k R} \right)^{n-1} \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin_k^2 R} \| \nabla f \|^2 \right] \sin_k^{n-1} r dr du
= \left( \frac{\sigma(R)}{\sin_k R} \right)^{n-1} \int_{B_R} \| \nabla f \|^2 dv_{\text{can}_k},
\]
and

\[ \int_{\partial B_R} f^2 \, ds_g = \int_{\partial B_R} f^2 \sigma^{n-1}(R) \, du \]
\[ = \left( \frac{\sigma(R)}{\sin_k R} \right)^{n-1} \int_{\partial B_R} f^2 \sin_k^{n-1}(R) \, du \]
\[ = \left( \frac{\sigma(R)}{\sin_k R} \right)^{n-1} \int_{\partial B_R} f^2 \, ds_{\text{can}}. \]

Hence

\[ \frac{\int_{B_R} \|\nabla f\|^2 \, dv_g}{\int_{\partial B_R} f^2 \, ds_g} \geq \frac{\int_{B_R} \|\nabla f\|^2 \, dv_{\text{can}}}{{\int_{\partial B_R} f^2 \, ds_{\text{can}}}}, \quad (14) \]

Now we obtain a test function for the Steklov problem on \((B_R, \text{can}_k)\) in terms of a first eigenfunction for the Steklov problem on \((B_R, g)\).

Let \(\psi_1\) be the first eigenfunction for the Steklov problem \((1)\) on \((B_R, g)\). Then \(\int_{\partial B_R} \psi_1 \, ds_g = 0\).

Define

\[ \psi := \psi_1 - \overline{\psi_1} \quad \text{where} \quad \overline{\psi_1} = \frac{\int_{\partial B_R} \psi_1 \, ds_{\text{can}}}{\int_{\partial B_R} ds_{\text{can}}}. \]

Observe that \(\int_{\partial B_R} \psi \, ds_{\text{can}} = 0\). From \((2)\) and \((14)\) it follows that

\[ \mu_1(B_R, \text{can}_k) \leq \frac{\int_{B_R} \|\nabla \psi\|^2 \, dv_{\text{can}}}{\int_{\partial B_R} \psi^2 \, ds_{\text{can}}}, \]
\[ \leq \frac{\int_{B_R} \|\nabla \psi\|^2 \, dv_g}{\int_{\partial B_R} \psi^2 \, ds_g}, \]
\[ \leq \frac{\int_{B_R} \|\nabla \psi_1\|^2 \, dv_g}{\int_{\partial B_R} \psi_1^2 \, ds_g} = \mu_1(B_R, g). \]

Next we prove that \(\mu_1(B_R, g) \leq \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, \text{can}_k)\).

Note that

\[ \int_{B_R} \|\nabla f\|^2 \, dv_g = \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sigma^2(r)} \|\nabla f\|^2 \right] \sigma^{n-1}(r) \, dr \, du. \]

(16)

Lemma 3.5 gives \(\frac{\sigma(R)}{\sin_k(R)} \leq \frac{\sigma(r)}{\sin_k(r)} \leq 1\) for \(0 < r \leq R\). By substituting this in \((16)\), we get

\[ \int_{B_R} \|\nabla f\|^2 \, dv_g \leq \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \left( \frac{\sin_k r}{\sigma(r)} \right)^2 \frac{1}{\sin_k^2(r)} \|\nabla f\|^2 \right] \sin_k^{n-1} r \, dr \, du \]
\[ \leq \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \left( \frac{\sin_k R}{\sigma(R)} \right)^2 \frac{1}{\sin_k^2(r)} \|\nabla f\|^2 \right] \sin_k^{n-1} r \, dr \, du \]
\[ = \left( \frac{\sin_k R}{\sigma(R)} \right)^2 \int_{B_R} \left[ \left( \frac{\sigma(R)}{\sin_k R} \right)^2 \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin_k^2(r)} \|\nabla f\|^2 \right] \sin_k^{n-1} r \, dr \, du. \]

Since \(\frac{\sigma(R)}{\sin_k(R)} \leq 1\), it follows that

\[ \int_{B_R} \|\nabla f\|^2 \, dv_g \leq \left( \frac{\sin_k R}{\sigma(R)} \right)^2 \int_{B_R} \left[ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin_k^2(r)} \|\nabla f\|^2 \right] \sin_k^{n-1} r \, dr \, du \]
\[ = \left( \frac{\sin_k R}{\sigma(R)} \right)^2 \int_{B_R} \|\nabla f\|^2 \, dv_{\text{can}}. \]
Hence
\[ \frac{\int_{B_R} \|\nabla f\|^2 \, dv_g}{\int_{\partial B_R} f^2 \, ds_g} \leq \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \frac{\int_{B_R} \|\nabla f\|^2 \, dv_{can_k}}{\int_{\partial B_R} f^2 \, ds_{can_k}}. \]

By similar computation as in the previous case, we get
\[ \mu_1(B_R, g) \leq \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, can_k). \]

By combining (15) and (17), we have
\[ \mu_1(B_R, can_k) \leq \mu_1(B_R, g) \leq \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, can_k). \]

If \( K_X \equiv k \), then by the uniqueness theorem of ordinary differential equation, \( \sigma(r) = \sin_k r \) for \( r \in [0, R] \) and equality holds for all inequalities in the proof. Hence
\[ \mu_1(B_R, can_k) = \mu_1(B_R, g) = \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, can_k). \]

Next we assume that
\[ \mu_1(B_R, can_k) = \mu_1(B_R, g) = \left( \frac{\sin_k R}{\sigma(R)} \right)^{n+1} \mu_1(B_R, can_k). \]

Then \( \frac{\sigma(R)}{\sin_k(R)} = \frac{\sigma(r)}{\sin_k(r)} = \lim_{r \to 0} \frac{\sigma(r)}{\sin_k(r)} = 1 \) and \( \psi_1 = 0 \). Therefore \( \sigma(r) = \sin_k r \) for \( r \in [0, R] \). Hence the theorem follows.

Proof for \( K(r) \leq k \) follows from similar computation as the proof for the case \( K(r) \geq k \).

\[ \square \]

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