Generalised Hausdorff measure of sets of Dirichlet non-improvable matrices in higher dimensions

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Abstract

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. A pair $(A, b)$, where $A$ is a real $m \times n$ matrix and $b \in \mathbb{R}^m$, is said to be $\psi$-Dirichlet improvable, if the system

$$\|Aq + b - p\|^m < \psi(T), \quad \|q\|^n < T$$

is solvable in $p \in \mathbb{Z}^m$, $q \in \mathbb{Z}^n$ for all sufficiently large $T$ where $\| \cdot \|$ denotes the supremum norm. For $\psi$-Dirichlet non-improvable sets, Kleinbock–Wadleigh (2019) proved the Lebesgue measure criterion whereas Kim–Kim (2022) established the Hausdorff measure results. In this paper we obtain the generalised Hausdorff $f$-measure version of Kim–Kim (2022) results for $\psi$-Dirichlet non-improvable sets.

1 Introduction

To begin with, we recall the higher dimensional general form of Dirichlet’s Theorem (1842). Let $m, n$ be positive integers and let $X^{mn}$ denotes the space of real $m \times n$ matrices.

Theorem 1.1 (Dirichlet’s Theorem) Given any $A \in X^{mn}$ and $T > 1$, there exist $p \in \mathbb{Z}^m$ and $q \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\|Aq - p\|^m \leq \frac{1}{T} \text{ and } \|q\|^n < T. \quad (1.1)$$

Here $\| \cdot \|$ denotes the supremum norm in $\mathbb{R}^i$, $i \in \mathbb{N}$. Theorem 1.1 guarantees a nontrivial integer solution for all $T$. The standard application of (1.1) is the following corollary, guaranteeing that such a system is solvable for an unbounded set of $T$.

Corollary 1.2 For any $A \in X^{mn}$ there exist infinitely many integer vectors $q \in \mathbb{Z}^n$ such that

$$\|Aq - p\|^m \leq \frac{1}{\|q\|^n} \text{ for some } p \in \mathbb{Z}^m. \quad (1.2)$$

The two statements above give rise to two possible ways to pose Diophantine approximation problems sometimes referred to as uniform vs asymptotic approximation results.
that is, looking for solvability of inequalities for all large enough $T$ vs. for some arbitrarily large $T$. The rate of approximation given in above two statements works for all real matrices $A \in X^{mn}$, which serves as the beginning of the metric theory of Diophantine approximation, a field concerned with understanding sets of matrices is translation invariant under integer vectors, we can restrict attention to elements of a system of affine forms

$$\|Aq - p\|^n < \psi(\|q\|^n) \text{ for some } p \in \mathbb{Z}^m$$

(1.3)

is satisfied for infinitely many integer vectors $q \in \mathbb{Z}^n$. As the set of $\psi$-approximable matrices is translation invariant under integer vectors, we can restrict attention to $(mn)$-dimensional unit cube $[0, 1]^{mn}$. Then the set of $\psi$-approximable matrices in $[0, 1]^{mn}$ will be denoted by $W_{m,n}(\psi)$.

The following result gives the size of the set $W_{m,n}(\psi)$ in terms of Lebesgue measure.

**Theorem 1.3** (Khintchine–Groshev Theorem, [11]) Given a non-increasing $\psi$, the set $W_{m,n}(\psi)$ has zero (respectively full) Lebesgue measure if and only if the series $\sum_k \psi(k)$ converges (respectively, diverges).

Let us now briefly describe what is known in the setting of (1.1). For a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$ with $T_0 > 1$ fixed, consider the set $D_{m,n}(\psi)$ of $\psi$-Dirichlet improvable matrices consisting of $A \in X^{mn}$ such that the system

$$\|Aq - p\|^n \leq \psi(T) \text{ and } \|q\|^n < T$$

has a nontrivial integer solution for all large enough $T$. Elements of the complementary set $D_{m,n}(\psi)^c$, will be referred as $\psi$-Dirichlet non-improvable matrices.

With the notation $\psi_a(x) := x^{-a}$, (1.1) implies that $D_{1,1}(\psi_1) = \mathbb{R}$, and that for any $m, n$ every matrix is $\psi_1$-Dirichlet improvable. It was observed in [8] that for $\min(m, n) = 1$ and in [17] for the general case, that the Lebesgue measure of $D_{m,n}(c\psi_1)$ of the set $c\psi_1$-Dirichlet improvable matrices is zero for any $c < 1$.

The theory of inhomogeneous Diophantine approximation starts by replacing the values of a system of linear forms $Aq$ by those of a system of affine forms $q \mapsto Aq + b$ where $A \in X^{mn}$ and $b \in \mathbb{R}^m$. Following [15], for a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$ a pair $(A, b) \in X^{mn} \times \mathbb{R}^m$ is called $\psi$-Dirichlet improvable if for all $T$ large enough, one can find nonzero integer vectors $q \in \mathbb{Z}^n$ and $p \in \mathbb{Z}^m$ such that

$$\|Aq + b - p\|^n < \psi(T) \quad \text{and} \quad \|q\|^n < T.$$  

Let $\hat{D}_{m,n}(\psi)$ denote the set of $\psi$-Dirichlet improvable pairs in the unit cube $[0, 1]^{mn+m}$. If the inhomogeneous vector $b \in \mathbb{R}^m$ is fixed then let $\hat{D}_{m,n}^b(\psi)$ be the set of all $A \in X^{mn}$ such that (1.4) holds i.e. for a fixed $b \in \mathbb{R}^m$ we have $\hat{D}_{m,n}^b(\psi) = \{A \in X^{mn} : (A, b) \in \hat{D}_{m,n}(\psi)\}$.

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1Here we use the definition as in [14,16], whereas in Sect. 4 we will consider slightly different definition such as in [4] where instead of (1.3) the inequality $\|Aq - b\| < \psi(\|q\|)$ is used.
The Lebesgue measure criterion for the set $\hat{D}_{m,n}(\psi)$ i.e. doubly metric case has been proved by Kleinbock–Wadleigh [16] by reducing the problem to the shrinking target problem on the space of grids in $\mathbb{R}^{m+n}$. The proof of their theorem is based on a correspondence between Diophantine approximation and homogenous dynamics.

**Theorem 1.4** (Kleinbock–Wadleigh, [16]) Given a non-increasing $\psi$, the set $\hat{D}_{m,n}(\psi)$ has zero (respectively full) Lebesgue measure if and only if the series $\sum_{j} \frac{1}{j^s \psi(j)}$ diverges (respectively converges).

Recently (2022), Kim–Kim [12] established the Hausdorff measure analogue of Theorem 1.4.

**Theorem 1.5** (Kim–Kim, [12]) Let $\psi$ be non-increasing with $\lim_{T \to \infty} \psi(T) = 0$ and $0 \leq s \leq mn + m$. Then

$$\mathcal{H}^s(\hat{D}_{m,n}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q) q^s} \left( \frac{q}{\psi(q) q^s} \right)^{mn+m-s} < \infty; \\ \mathcal{H}^s([0,1]^{mn+m}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q) q^s} \left( \frac{q}{\psi(q) q^s} \right)^{mn+m-s} = \infty. \end{cases}$$

In the same article Kim–Kim also provided the Hausdorff measure criterion for the singly metric case.

**Theorem 1.6** (Kim–Kim, [12]) Let $\psi$ be non-increasing with $\lim_{T \to \infty} \psi(T) = 0$. Then for any $0 \leq s \leq mn$

$$\mathcal{H}^s(\hat{D}_{m,n}^b(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q) q^s} \left( \frac{q}{\psi(q) q^s} \right)^{mn-s} < \infty; \\ \mathcal{H}^s([0,1]^{mn}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q) q^s} \left( \frac{q}{\psi(q) q^s} \right)^{mn-s} = \infty, \end{cases}$$

for every $b \in \mathbb{R}^m \setminus \mathbb{Z}^m$.

Naturally one can ask about the generalization of Theorems 1.5 and 1.6 in terms of $f$-dimensional Hausdorff measure. Recall that a natural generalization of the $s$-dimensional Hausdorff measure $\mathcal{H}^s$ is the $f$-dimensional Hausdorff measure $\mathcal{H}^f$ where $f$ is a dimension function, that is an increasing, continuous function $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(r) \to 0$ as $r \to 0$.

In this article we extend the results of Kim–Kim [12] by establishing the zero-full law for the sets $\hat{D}_{m,n}(\psi)$ and $\hat{D}_{m,n}^b(\psi)$ in terms of generalised $f$-dimensional Hausdorff measure. We obtain the following main results.

**Theorem 1.7** Let $\psi$ be non-increasing and $f$ be a dimension function with

$$f(xy) \geq x^\alpha f(y) \quad \forall \quad y^\alpha \leq x \leq y^{\frac{1}{\alpha}} \quad (1.5)$$

where $mn + m - n < s < mn + m$ and $\alpha > 1$ is some absolute constant independent of $x$ and $y$ and suppose that

$$f'(x) = a(x) \frac{f(x)}{x} \quad (1.6)$$
such that \( a(x) \to s \) as \( x \to 0 \). Further, let
\[
(q^{-m})^a \leq \psi(q) \leq (q^{-m})^{1/3}.
\]
(1.7)

Then
\[
\mathcal{H}^f(\mathcal{D}_{m,n}(\psi)^c) = \begin{cases} 
0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)^a} \left( \frac{q^a}{\psi(q)^m} \right)^{mn+m} f \left( \frac{\psi(q)^m}{q^m} \right) < \infty; \\
\mathcal{H}^f([0,1]^{mn+m}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)^a} \left( \frac{q^a}{\psi(q)^m} \right)^{mn+m} f \left( \frac{\psi(q)^m}{q^m} \right) = \infty.
\end{cases}
\]

For the singly metric case we have the following result.

**Theorem 1.8** Let \( \psi \) be non-increasing and \( f \) be a dimension function such that \( r^{-mn} f(r) \to \infty \) as \( r \to 0 \). Suppose that (1.5)–(1.7) holds and \( mn - n < s < mn \).
Then
\[
\mathcal{H}^f(\mathcal{D}_{m,n}(\psi)^c) = \begin{cases} 
0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)^a} \left( \frac{q^a}{\psi(q)^m} \right)^m f \left( \frac{\psi(q)^m}{q^m} \right) < \infty; \\
\mathcal{H}^f([0,1]^{mn}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)^a} \left( \frac{q^a}{\psi(q)^m} \right)^m f \left( \frac{\psi(q)^m}{q^m} \right) = \infty.
\end{cases}
\]

for every \( b \in \mathbb{R}^m \setminus \mathbb{Z}^m \).

We remark that the conditions (1.5) and (1.6) are satisfied in a wide variety of cases, for example \( f(x) = x^t \log^t(x) \) for some \( s > 0 \) and \( t \in \mathbb{R} \). Indeed, (1.5) follows since \( f(xy) = (xy)^t \log^t(xy) \asymp x^s y^t \log^t(y) = x^s f(y) \), and (1.6) follows since
\[
\frac{s x^t f(x)}{f(x)} = x \frac{d}{dx} [s \log(x) + t \log \log(x)] = x \left( \frac{s}{x} + \frac{t}{x \log(x)} \right) \to s \text{ as } x \to 0.
\]

2 Preliminaries and auxiliary results

2.1 Hausdorff measure and dimension

Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be a dimension function i.e. an increasing continuous function such that \( f(r) \to 0 \) as \( r \to 0 \) and let \( V \) be an arbitrary subset of \( \mathbb{R}^n \). For \( \rho > 0 \), a \( \rho \)-cover for a set \( V \) is defined as a countable collection \( \{U_i\}_{i=1}^{\infty} \) of sets in \( \mathbb{R}^n \) with diameters \( 0 < \text{diam}(U_i) \leq \rho \) such that \( V \subseteq \cup_{i=1}^{\infty} U_i \). Then for each \( \rho > 0 \) define
\[
\mathcal{H}_\rho^f(V) = \inf \left\{ \sum_{i=1}^{\infty} f(\text{diam}(U_i)) : \{U_i\} \text{ is a } \rho \text{-cover of } V \right\}.
\]

Note that \( \mathcal{H}_\rho^f(V) \) is non-decreasing as \( \rho \) decreases and therefore approaches a limit as \( \rho \to 0 \). Accordingly, the \( f \)-dimensional Hausdorff measure of \( V \) is defined as
\[
\mathcal{H}^f(V) := \lim_{\rho \to 0} \mathcal{H}_\rho^f(V).
\]

This limit could be zero or infinity, or take a finite positive value.

If \( f(r) = r^s \) where \( s > 0 \), then \( \mathcal{H}^s \) is the \( s \)-dimensional Hausdorff measure and is represented by \( \mathcal{H}^s \). It can be easily verified that Hausdorff measure is monotonic, that is, if \( E \) is contained in \( F \) then \( \mathcal{H}^s(E) \leq \mathcal{H}^s(F) \), countably sub-additive, and satisfies \( \mathcal{H}^s(\emptyset) = 0 \).
The following property
\[ \mathcal{H}^s(V) < \infty \implies \mathcal{H}^{s'}(V) = 0 \quad \text{if } s' > s, \]
implies that there is a unique real point \( s \) at which the Hausdorff \( s \)-measure drops from infinity to zero (unless \( V \) is finite so that \( \mathcal{H}^s(V) \) is never infinite). The value taken by \( s \) at this discontinuity is referred to as the **Hausdorff dimension** of a set \( V \) and is defined as
\[ \dim_H V := \inf\{s > 0 : \mathcal{H}^s(V) = 0\}. \]

For establishing the convergent parts of Theorems 1.7 and 1.8 we will apply the following Hausdorff measure version of the famous Borel–Cantelli lemma [6, Lemma 3.10]:

**Lemma 2.1** Let \( \{B_i\}_{i \geq 1} \) be a sequence of measurable sets in \( \mathbb{R}^n \) and suppose that for some dimension function \( f \),
\[ \sum_{i} f(\text{diam}(B_i)) < \infty. \]
Then \( \mathcal{H}^f(\limsup_{i \to \infty} B_i) = 0. \)

We will use the following principle known as Mass Distribution Principle [10, §4.1] for the divergent part of Theorem 1.7.

**Lemma 2.2** Let \( \mu \) be a probability measure supported on a subset \( V \) of \( \mathbb{R}^k \). Suppose there are positive constants \( c > 0 \) and \( \varepsilon > 0 \) such that
\[ \mu(U) \leq cf(\text{diam}(U)) \quad \forall \text{ sets } U \text{ with } \text{diam}(U) \leq \varepsilon. \]
Then \( \mathcal{H}^f(V) \geq \mu(V)/c. \)

**Theorem 2.3** ([1, Theorem 2]) Let \( \psi : \mathbb{N} \to \mathbb{R}_+ \) be any approximating function and let \( mn > 1 \). Let \( f \) and \( g : r \to g(r) := r^{-m(n-1)}f(r) \) be dimension functions such that \( r \mapsto r^{-m}f(r) \) is monotonic. Then
\[ \mathcal{H}^f(W_{mn}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{m+n-1} g \left( \frac{\bar{\psi}(q)}{q} \right) < \infty; \\ \mathcal{H}^f([0,1]^{mn}) & \text{if } \sum_{q=1}^{\infty} q^{m+n-1} g \left( \frac{\bar{\psi}(q)}{q} \right) = \infty, \end{cases} \]
where \( \bar{\psi}(q) = \psi(q^n)^{\frac{1}{m}}. \)

### 2.2 Ubiquitous systems

To prove the divergent parts of Theorem 1.8 we will use the ubiquity technique developed by Beresnevich, Dickinson, and Velani, see [5, §12.1]. The idea and concept of ubiquity was originally formulated by Dodson, Rynne, and Vickers in [9] and coincided in part with the concept of ‘regular systems’ of Baker and Schmidt [2]. Both have proven to be extremely useful in obtaining lower bounds for the Hausdorff dimension of limsup sets. The ubiquity framework in [5] provides a general and abstract approach for establishing the Lebesgue and Hausdorff measure of a large class of limsup sets.

Consider the \( mn \)-dimensional unit cube \([0,1]^{mn}\) with the supremum norm \( \| \cdot \| \). Let \( R = \{R_\kappa \subseteq [0,1]^{mn} : \kappa \in I\} \) be a family of subsets, referred to as resonant sets \( R_\kappa \) of \([0,1]^{mn}\) indexed by an infinite, countable set \( I \). Let \( f : I \to \mathbb{R}_+ : \kappa \mapsto f_\kappa \) be a positive function on \( I \) i.e. the function \( f \) attaches the weight \( f_\kappa \) to the set \( R_\kappa \). Next assume that
the number of terms \( \kappa \) in \( J \) with \( \beta_\kappa \) bounded above is always finite. Following the ideas from [5, §12.1] and [12] let us assume that the family \( \mathcal{R} \) of resonant sets \( R_\kappa \) consists of \((m - 1)n\)-dimensional, rational hyperplanes and define the following notations. For a set \( S \subseteq [0, 1]^mn \), let
\[
\Delta(S, r) := \{ V \in [0, 1]^mn : \text{dist}(V, S) < r \},
\]
where \( \text{dist}(V, S) := \inf \{ \| V - Y \| : Y \in S \} \). Fix a decreasing function \( \Psi : \mathbb{R}_+ \to \mathbb{R}_+ \) let
\[
\Lambda(\Psi) = \{ V \in [0, 1]^mn : V \in \Delta(R_\kappa, \Psi(\beta_\kappa)) \text{ for i.m. } \kappa \in J \}
\]
(2.1)
The set \( \Lambda(\Psi) \) is a lim sup set; it consists of elements of \([0, 1]^mn\) which lie in infinitely many of the thickenings \( \Delta(R_\kappa, \Psi(\beta_\kappa)) \). It is natural to call \( \Psi \) the approximating function as it governs the 'rate' at which the elements of \([0, 1]^mn\) must be approximated by resonant sets in order to lie in \( \Lambda(\Psi) \). Let us rewrite the set \( \Lambda(\Psi) \) in a way which brings its lim sup nature to the forefront.

For \( N \in \mathbb{N} \), let
\[
\Delta(\Psi, N) := \bigcup_{\kappa \in J : 2^{N-1} < \beta_\kappa \leq 2^N} \Delta(R_\kappa, \Psi(\beta_\kappa)).
\]
Thus \( \Lambda(\Psi) \) is the set consisting elements of \([0, 1]^mn\) which lie in infinitely many \( \Delta(\Psi, N) \), that is,
\[
\Lambda(\Psi) := \limsup_{N \to \infty} \Delta(\Psi, N)
\]
(2.2)
Next let \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function with \( \rho(t) \to 0 \) as \( t \to \infty \) and let
\[
\Delta(\rho, N) := \bigcup_{\kappa \in J : 2^{N-1} < \beta_\kappa \leq 2^N} \Delta(R_\kappa, \rho(\beta_\kappa)).
\]
(2.3)
**Definition 2.4** Let \( B \) be an arbitrary ball in \([0, 1]^mn\). Suppose there exist a function \( \rho \) and an absolute constant \( \kappa > 0 \) such that
\[
|B \cap \Delta(\rho, N)| \geq \kappa |B| \text{ for } N \geq N_0(B),
\]
(2.4)
where \( | \cdot | \) denotes the Lebesgue measure on \([0, 1]^mn\). Then the pair \( (\mathcal{R}, \beta) \) is said to be a ‘local ubiquitous system’ relative to \( \rho \) and the function \( \rho \) will be referred to as the ‘ubiquitous function’.

A function \( h \) is said to be 2-regular if there exists a strictly positive constant \( \lambda < 1 \) such that for \( N \) sufficiently large
\[
h(2^{N+1}) \leq \lambda h(2^N).
\]
The next theorem is a simplified version of Theorem 1 and Theorem 2 from [5]. To state the result we define notions similar to those in [5]. Note that with notions in [5], we have \( \Omega := [0, 1]^mn \), the Lebesgue measure on \([0, 1]^mn\) is of type (M2) with \( \delta = mn \) and \( \gamma = (m - 1)n \) and the local ubiquitous system \( (\mathcal{R}, \beta) \) satisfies the intersection conditions with \( \gamma = (m - 1)n \) (see [5, section 12.1]). Given that the Lebesgue measure is comparable with \( \mathcal{H}^\delta \)— a simple consequence of (M2), we have the following combined version of Theorem 1 and Theorem 2 from [5].
Theorem 2.5 Suppose that \((R, \beta)\) is a local ubiquitous system relative to \(\rho\) and that \(\Psi\) is an approximating function. Let \(f\) be a dimension function such that \(r^{-nmf(r)}\) is monotonic, \(r^{-nmf(r)} \to \infty\) as \(r \to 0\) and \(r^{-(m-1)f(r)}\) is increasing. Furthermore, suppose that \(\rho\) is 2-regular and
\[
\sum_{n=1}^{\infty} \frac{(\Psi(2N))^{-n(m-1)f(\Psi(2N))}}{\rho(2N)^n} = \infty.
\] (2.5)

Then
\[
\mathcal{H}^f(\Lambda(\Psi)) = \mathcal{H}^f([0, 1)^{mn}).
\] (2.6)

Proof With \(\delta = mn\), and \(\gamma = (m - 1)n\) the function \(g\) in [5, Theorem 2] becomes
\[
g(r) := f(\Psi(r))\Psi(r)^{-\gamma} \rho(r)^{\gamma - \delta} = f(\Psi(r))\Psi(r)^{-(m-1)n}\rho(r)^{-n}.
\] Also \(\rho\) is 2-regular, thus from [5, Theorem 2] it follows that
\[
\mathcal{H}^f(\Lambda(\Psi)) = \infty \quad \text{if} \quad \sum_{n=1}^{\infty} g(2N) = \infty,
\]
which is same as the divergent sum condition in (2.5).

Note that as the dimension function \(r^{-nmf(r)} \to \infty\) as \(r \to 0\) then \(H^f(\Omega) = \infty\) and Theorem 2.5 leads to the same conclusion as Theorem 2 in [5]. \(\square\)

2.3 Dirichlet improvability and homogenous dynamics

In one dimensional settings, continued fraction expansions have been useful in characterising \(\psi\)-Dirichlet improvable numbers [15]. However this machinery is not applicable in higher dimensions. For general dimensions, building on ideas from [7] (also see [13]), a dynamical approach was proposed in [15], reformulating the homogenous approximation problem as a shrinking target problem and a similar approach was used in [16] to solve an analogous inhomogeneous problem. Following the ideas from [12,16] we will use the standard argument usually known as the ‘Dani correspondence’ which serves as a connection between Diophantine approximation and homogenous dynamics. In order to describe how Dirichlet-improvability is related to dynamics we will start by recalling the dynamics on space of grids. To describe this dynamical interpretation, let us fix some notation.

Fix \(d = m + n\). Let
\[
G_d = SL_d(\mathbb{R}) \text{ and } \hat{G}_d = ASL_d(\mathbb{R}) = G_d \rtimes \mathbb{R}^d
\]
and put
\[
\Gamma_d = SL_d(\mathbb{Z}) \text{ and } \hat{\Gamma}_d = ASL_d(\mathbb{Z}) = \Gamma_d \rtimes \mathbb{Z}^d.
\]

Denote by \(\hat{Y}_d\) the space of affine shifts of unimodular lattices in \(\mathbb{R}^d\) (i.e. space of unimodular grids). Clearly, \(\hat{Y}_d\) is canonically identified with \(\hat{G}_d/\hat{\Gamma}_d\) via
\[
< g, w > \in \hat{G}_d/\hat{\Gamma}_d \longleftrightarrow g\mathbb{Z}^d + w \in \hat{Y}_d
\]
where \(< g, w >\) is an element of \(\hat{G}_d\) such that \(g \in G_d\) and \(w \in \mathbb{R}^d\). Similarly, \(Y_d := G_d/\Gamma_d\) is identified with the space of unimodular lattices in \(\mathbb{R}^d\) (i.e. the space of unimodular grids containing zero vector). Note that \(\Gamma_d\) (respectively, \(\hat{\Gamma}_d\)) is a lattice in \(G_d\) (respectively, \(\hat{G}_d\)).
Denote by \( m_{Y_d} \) the Haar probability measure on \( Y_d \). For any \( t \in \mathbb{R} \), the flow of interest \( a_t \) is given by the diagonal matrix
\[
a_t := \text{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n}).
\]
Let
\[
u_A := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G_d,
\]
\[
u_{A,b} := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix} \in \hat{G}_d
\]
for \( A \in \mathbb{X}^{mn} \) and \((A, b) \in \mathbb{X}^{mn} \times \mathbb{R}^m.\) Let us also denote by
\[
\Lambda_A := \nu_A \mathbb{Z}^d \in Y_d \text{ and } \Lambda_{A,b} := \nu_{A,b} \mathbb{Z}^d \in \hat{Y}_d,
\]
where
\[
\nu_{A,b} \mathbb{Z}^d = \left\{ \begin{pmatrix} Aq + b - p \\ q \end{pmatrix} : p \in \mathbb{Z}^m, q \in \mathbb{Z}^n \right\}.
\]
Following \([16]\), define \( \Delta : \hat{Y}_d \to [-\infty, +\infty) \) by
\[
\Delta(\Lambda) := \log \inf_{v \in \Lambda} \|v\|.
\]

**Lemma 2.6** ([14]) Let \( \psi : [T_0, \infty) \to \mathbb{R}_+ \) be a continuous, non-increasing function where \( T_0 \in \mathbb{R}_+ \) and \( m, n \) be positive integers. Then there exists a continuous function
\[
z = z_\psi : [t_0, \infty) \to \mathbb{R},
\]
where \( t_0 := \frac{m}{m+n} \log T_0 - \frac{n}{m+n} \log \psi(T_0) \), such that
\begin{enumerate}[(i)]
\item the function \( t \mapsto t + nz(t) \) is strictly increasing and unbounded;
\item the function \( t \mapsto t - nz(t) \) is non-decreasing;
\item \( \psi(e^{t+tz(t)}) = e^{-t+tz(t)} \) for all \( t \geq t_0 \).
\end{enumerate}

Note that, properties (i) and (ii) of Lemma 2.6 imply that any \( z = z_\psi \) does not oscillate too wildly. Namely, \( z(s) - \frac{1}{m} \leq z(u) \leq z(s) + \frac{1}{n} \) whenever \( s \leq u \leq s + 1 \).

The following lemma, which rephrases \( \psi \)-Dirichlet improvable properties of \((A, b) \in \mathbb{X}^{mn} \times \mathbb{R}^m\) as the statement about the orbit of \( \Lambda_{A,b} \) in the dynamical space \((\hat{Y}_d, a_t)\), is the general version of the correspondence between the improvability of the inhomogeneous Dirichlet theorem and dynamics on \( \hat{Y}_d \).

**Lemma 2.7** ([16]) Let \( z = z_\psi \) be the function associated to \( \psi \) by Lemma 2.6. Then \((A, b) \in \hat{D}_{m,n}(\psi) \) if and only if \( \Delta(a_t \Lambda_{A,b}) < z_\psi(t) \) for all sufficiently large \( t \).

This equivalence is usually called the Dani Correspondence. In view of this interpretation a pair fails to be \( \psi \)-Dirichlet improvable if and only if the associated grid visits the target \( \Delta^{-1}([z_\psi(t), \infty)) \) at unbounded times \( t \) under the flow \( a_t \). Note that from the above lemma in the definitions \( \hat{D}_{m,n}(\psi) := \limsup_{t \to \infty} \{ (A, b) : \Delta(a_t \Lambda_{A,b}) \geq z_\psi(t) \} \) and \( \hat{D}_{m,n}^b(\psi) := \limsup_{t \to \infty} \{ A : \Delta(a_t \Lambda_{A,b}) \geq z_\psi(t) \} \), the limsup is taken for real values \( t \in \mathbb{R} \).

However to prove the convergent part, we need to use Hausdorff–Cantelli Lemma (Lemma 2.1), therefore we will consider limsup sets taken for \( t \in \mathbb{N} \). Thus we will use the following definitions: there exists a non-zero positive constant \( C_0 \) such that
\[
\hat{D}_{m,n}(\psi) \subseteq \limsup_{t \to \infty, t \in \mathbb{N}} \{ (A, b) : \Delta(a_t \Lambda_{A,b}) \geq z_\psi(t) - C_0 \}, \tag{2.7}
\]
\[ \mathcal{D}_{m,n}(\psi)^c \subseteq \limsup_{t \to \infty, t \in \mathbb{N}} \{ A : \Delta(a_t A_{A, b}) \ge z\psi(t) - C_0 \}. \] (2.8)

The validity of these definitions can be observed by the fact that \( z_\psi \) does not oscillate wildly by [16, Remark 3.3] and \( \Delta \) is uniformly continuous on the set \( \Delta^{-1}(\{z, \infty\}) \) for any \( z \in \mathbb{R} \), ([16, Lemma 2.1]).

### 3 Proof of Theorems 1.7 and 1.8: the convergent case

**Lemma 3.1** Let \( \psi : [T_0, \infty) \to \mathbb{R}_+ \) be a non-increasing function, and let \( z = z_\psi \) be the function associated to \( \psi \) by Lemma 2.6. Let \( f \) be a dimension function satisfying (1.5) and (1.6) where \( nm - n < s \leq nm \). Also suppose that (1.7) holds. Then we have

\[
\sum_{q \in [T_0]} \frac{1}{\psi(q)q^2} \left( \frac{q_1}{\psi(q)} \right)^{mn} f \left( \frac{\psi(q)^{\frac{1}{m}}}{q_1} \right) < \infty
\]

if
\[
\iff \sum_{t = [t_0]}^{\infty} e^{-(m+n)(t)} e^{(m+n)t} f \left( e^{-\frac{(m+n)t}{nm}} \right) < \infty.
\]

**Proof** The proof of this lemma uses ideas introduced in [14, Lemma 8.3] and [16]. Using the monotonicity of \( \psi \) and [16, Remark 3.3], let us replace the sums with integrals

\[
\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2} \left( \frac{x_1}{\psi(x)} \right)^{mn} f \left( \frac{\psi(x)^{\frac{1}{m}}}{x_1} \right) dx \quad \text{and} \quad \int_{T_0}^{\infty} e^{-(m+n)(t)} e^{(m+n)t} f \left( e^{-\frac{(m+n)t}{nm}} \right) dt.
\]

Define

\[
P := -\log \circ \psi \circ \exp : [T_0, \infty) \to \mathbb{R} \quad \text{and} \quad \lambda(t) := t + nz(t).
\]

Since \( \psi(e^\lambda) = e^{-P(\lambda)} \), letting \( \log x = \lambda \) we have

\[
\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2} \left( \frac{x_1}{\psi(x)} \right)^{mn} f \left( \frac{\psi(x)^{\frac{1}{m}}}{x_1} \right) dx = \int_{\log T_0}^{\infty} \frac{1}{\psi(e^{\lambda})} \left( \frac{e^{m\lambda}}{\psi(e^{\lambda})} \right)^{\frac{1}{m}} f \left( \frac{\psi(e^{\lambda})^{\frac{1}{m}}}{e^{\frac{1}{m}}} \right) e^{\lambda} d\lambda.
\]

Using \( P(\lambda(t)) = t - mz(t) \), we have

\[
\int_{T_0}^{\infty} e^{-(m+n)(t)} e^{(m+n)t} f \left( e^{-\frac{(m+n)t}{nm}} \right) dt
\]

\[
= \int_{T_0}^{\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f \left( e^{-\frac{P(\lambda)}{m}} e^{\frac{1}{m}} \right) d \left[ \frac{m}{m+n} \lambda + \frac{n}{m+n} P(\lambda) \right]
\]

\[
= \frac{m}{m+n} \int_{\log T_0}^{\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f \left( e^{-\frac{P(\lambda)}{m}} e^{\frac{1}{m}} \right) d\lambda
\]

\[
+ \frac{n}{m+n} \int_{\log T_0}^{\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f \left( e^{-\frac{P(\lambda)}{m}} e^{\frac{1}{m}} \right) d(P(\lambda)).
\]

The term in the last line can be expressed by

\[
\frac{n}{m+n} \int_{\log T_0}^{\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f \left( e^{-\frac{P(\lambda)}{m}} e^{\frac{1}{m}} \right) d(P(\lambda))
\]
thereby using (1.5) we can write
\[
(e^{-\frac{P(\lambda)}{m}}) = e^{\frac{-P(\lambda)}{m}}
\]
the second last equation follows from (1.5) and (1.7). Since by using (1.7) and the fact that
\[
\psi(e^{\lambda}) = e^{-P(\lambda)}
\]
we obtain the condition
\[
(e^{-\frac{P(\lambda)}{m}})^\alpha \leq e^{\frac{-P(\lambda)}{m}} \leq (e^{-\frac{P(\lambda)}{m}})^\beta,
\]
therefore by using (1.5) we can write
\[
f(e^{-\frac{P(\lambda)}{m}} \psi) \approx e^{\frac{-P(\lambda)}{m}} f(e^{-\frac{1}{\lambda}}).
\]

Next we will use integration by parts to evaluate the integral in (3.3).

\[
\int_{\log T_0}^{\infty} e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) d\lambda (e^{((1+n)-\alpha)\lambda})
\]
\[
= - \int_{\log T_0}^{\infty} \left((m-1)e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) - \frac{1}{n} e^{(m-1)\lambda} \frac{1}{e^{\frac{1}{\lambda}}} f'(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda}\right) d\lambda
\]
\[
+ e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda}) \bigg|_{\log T_0}^{\infty}
\]
\[
= \int_{\log T_0}^{\infty} \left((1 - m)e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda} + \frac{1}{n} e^{(m-1)\lambda} f'(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda}\right) d\lambda
\]
\[
+ \lim_{\lambda \to \infty} e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda} - T_0^{m-1} f(T_0^{-\frac{1}{\lambda}}) \psi(T_0)^{(1+n)-\alpha}.
\]

by (1.6), we have
\[
f'(e^{-\frac{1}{\lambda}}) = a(e^{-\frac{1}{\lambda}}) f(e^{-\frac{1}{\lambda}})
\]
\[
= \int_{\log T_0}^{\infty} \left(1 - m + \frac{1}{n} a(e^{-\frac{1}{\lambda}}) \right) e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda}) d\lambda
\]
\[
+ \lim_{\lambda \to \infty} e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda} - T_0^{m-1} f(T_0^{-\frac{1}{\lambda}}) \psi(T_0)^{(1+n)-\alpha}.
\]

Note that as \( \lambda \to \infty \), \( e^{-\frac{1}{\lambda}} \to 0 \) thus by assumption \( a(e^{-\frac{1}{\lambda}}) \to s \) and therefore
\[
\left(1 - \frac{mn - a(e^{-\frac{1}{\lambda}})}{n}\right) \to \left(1 - \frac{mn - s}{n}\right),
\]
which is finite and positive for \( T_0 \) large enough (since \( s > mn - n \)). Observe that
\[
\lim_{\lambda \to \infty} e^{(m-1)\lambda} f(e^{-\frac{1}{\lambda}}) e^{((1+n)-\alpha)\lambda}) = 0
\]
if the integral
\[
\int_{\log T_0}^{\infty} e^{(m-1)\lambda} e^{((1+n)\lambda)} f(e^{-\frac{1}{\lambda}}) d\lambda.
\]
converges. Thus the convergence of
\[
\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{n}}}ight)^{mn} \frac{\psi(x)^{\frac{1}{n}}}{x^{\frac{1}{n}}}) \frac{f}{dx} \text{ or } \int_{T_0}^{\infty} e^{-(m+n)x(t)} e^{(m+n)} f(e^{-\frac{1}{\lambda}}) dt
\]
implies the convergence of other since all summands are positive except the finite value 
\[-T_0^{n-1}f(T_0^{-\frac{1}{4}})\psi(T_0)^{-(1+n)}\frac{1}{\Lambda}.
\]

In order to apply the Hausdorff–Cantelli lemma (Lemma 2.1) we need a sequence of coverings for the sets \(D_{m,n}(\psi)^c\) and \(\tilde{D}_{m,n}(\psi)^c\). Recall that we are considering the supremum norm \(\|\cdot\|\) on \([0,1]^{mn}\) and let \(\lambda_j(\Lambda)\) denote the \(j\)-th successive minimum of a lattice \(\Lambda \subseteq \mathbb{R}^d\) i.e. the infimum of \(\lambda\) such that the ball \(B_\lambda^d(0)\) contains \(j\) independent vectors of \(\Lambda\). Then:

**Proposition 3.2** (Kim–Kim, [12, Proposition 3.6]) Let \(C_0\) be the same constant as in (2.7) and (2.8). For \(t \in \mathbb{N}\), let \(Z_t := \{A \in [0,1]^{mn} : \log(d\lambda_d(a_1\Lambda_1)) \geq z_\psi(t) - C_0\}. \) Then \(Z_t\) can be covered with \(KE^{m+n(t-z_\psi(t))}\) balls in \(X^{mn} = M_{m,n}(\mathbb{R})\) of radius \(\frac{1}{2}e^{-(\frac{1}{n} + \frac{1}{\Lambda})t}\) for a constant \(K > 0\) not depending on \(t\).

We are now in a position to prove the following statement.

**Proposition 3.3** Let \(mn - n < s \leq mn\). If
\[
\sum_{q=1}^{\infty} \frac{1}{\psi(q)^s} \left(\frac{q}{\psi(q)m}\right)^{mn} f \left(\frac{\psi(q)^{\frac{1}{m}}}{q}\right) < \infty,
\]
then \(H^f(\limsup_{t \to \infty} Z_t) = 0\) and \(H^{f+m}(\limsup_{t \to \infty} Z_t \times [0,1]^m) = 0\). (Note that \(H^f\) represents the Hausdorff measure of a set when we take \(f + m(r) = r^m f(r)\).)

**Proof** By Lemma 3.1, the assumption
\[
\sum_{q=1}^{\infty} \frac{1}{\psi(q)^s} \left(\frac{q}{\psi(q)m}\right)^{mn} f \left(\frac{\psi(q)^{\frac{1}{m}}}{q}\right) < \infty
\]
is equivalent to
\[
\sum_{t=1}^{\infty} e^{-(m+n)(\varepsilon(t)-t)}f(e^{-(\frac{1}{m} + \frac{1}{\Lambda})t}) < \infty.
\]

For each \(t \in \mathbb{N}\), let \(D_{t,1}, D_{t,2}, \ldots, D_{t,p_t}\) be the balls of radius \(\frac{1}{2}e^{-(\frac{1}{m} + \frac{1}{\Lambda})t}\) covering \(Z_t\) as in Proposition 3.2. Note that \(p_t\), the number of the balls, is not greater than \(KE^{m+n(t-z_\psi(t))}\) by Proposition 3.2. By applying Lemma 2.1 to the sequence of balls \(\{D_{t}\}_{t \in \mathbb{N}}\), we have \(H^f(\limsup_{t \to \infty} Z_t) \subseteq H^f(\limsup_{t \to \infty} D_t) = 0\).

We prove the second statement by a similar argument. Proposition 3.2 implies that \(Z_t \times [0,1]^m\) can be covered with \(KE^{mn+m(t-z_\psi(t))}\) balls of radius \(\frac{1}{2}e^{-(\frac{1}{m} + \frac{1}{\Lambda})t}\). Applying Lemma 2.1 again, we have \(H^{f+m}(\limsup_{t \to \infty} Z_t \times [0,1]^m) = 0\). □

The convergence parts of Theorems 1.7 and 1.8 follow from this proposition. We will adopt a similar method as in [12].

**Proof** We first prove the singly metric case i.e., the convergent part of Theorem 1.8. We claim that \(\log(d\lambda_d(a_1\Lambda_1)) \geq \Delta(a_1\Lambda_{AB})\) for every \(b \in \mathbb{R}^m\). Let \(v_1, \ldots, v_d\) be linearly independent vectors satisfying \(\|v_i\| \leq \Lambda_d(a_1\Lambda_1)\) for \(1 \leq i \leq d\). The shortest vector of \(a_1\Lambda_{AB}\) can be written as a form of \(\sum_{t=1}^{d} a_i v_i\) for some \(-1 \leq a_i \leq 1\), so the length of the shortest vector is less than \(\sum_{t=1}^{d} ||v_i||\). Thus, \(\Delta(a_1\Lambda_{AB}) \leq \log \sum_{t=1}^{d} ||v_i|| \leq \log(d\lambda_d(a_1\Lambda_1))\).

This implies \(\tilde{D}_{m,n}(\psi)^c \subseteq \limsup_{t \to \infty} \{A \in [0,1]^{mn} : \Delta(a_1\Lambda_{AB}) \geq z_\psi(t) - C_0\} \subseteq \limsup_{t \to \infty} Z_t\) by Lemma 2.7 and Proposition 3.3, thus we obtain \(H^f(\tilde{D}_{m,n}(\psi)^c) \leq H^f(\limsup_{t \to \infty} Z_t) = 0\).

Similarly for the doubly metric case, together with the second statement of Proposition 3.3, \(\tilde{D}_{m,n}(\psi)^c \subseteq \limsup_{t \to \infty} \{A, b \in [0,1]^{mn+m} : \Delta(a_1\Lambda_{AB}) \geq z_\psi(t) - C_0\} \subseteq \limsup_{t \to \infty} Z_t \times [0,1]^m\) provides the proof of the convergent part of Theorem 1.7. □
4 Proof of Theorems 1.7 and 1.8: the divergent case

Recall that $d = m + n$ and assume that $\psi : [T_0, \infty) \to \mathbb{R}_+$ is a decreasing function satisfying $\lim_{T \to \infty} \psi(T) = 0$. Denote by $\| \cdot \|_Z$ and $| \cdot |_Z$ the distance to the nearest integer vector and number, respectively. Define the function $\tilde{\psi} : [S_0, \infty) \to \mathbb{R}_+$ by

$$\tilde{\psi}(S) = \left( \psi^{-1}(S^{-m}) \right)^{\frac{1}{m}}$$

where $S_0 = \psi(T_0)^{\frac{1}{m}}$. The next lemma associates $\psi$-Dirichlet non-improvability with $\tilde{\psi}$-approximability via a transference lemma as follows.

**Lemma 4.1** [12, Lemma 4.2] Given $(A, b) \in X^{mn} \times \mathbb{R}^m$, if the system

$$\|A^t x\|_Z < d^{-1} |b \cdot x|_Z \tilde{\psi}(S) \text{ and } \|x\| < d^{-1} |b \cdot x|_Z S$$

has a nontrivial solution $x \in \mathbb{Z}^m$ for an unbounded set of $S \geq S_0$, then $(A, b) \in \widehat{D}_{m,n}(\psi)^c$.

Following [12] we adopt some notations. Let $W_{S, \varepsilon}$ be the set of $A \in [0, 1]^{mn}$ such that there exists $x_{A,S} \in \mathbb{Z}^m \setminus \{0\}$ satisfying

$$\|A^t x_{A,S}\|_Z < d^{-1}\varepsilon \tilde{\psi}(S) \text{ and } \|x_{A,S}\| < d^{-1}\varepsilon S$$

and let

$$\widehat{W}_{S, \varepsilon} := \{(A, b) \in [0, 1]^{mn+m} : A \in W_{S, \varepsilon} \text{ and } |b \cdot x_{A,S}|_Z > \varepsilon\}.$$  

For fixed $b \in \mathbb{R}^m$, consider the set $W_{b, S, \varepsilon}$ of matrices $A \in [0, 1]^{mn}$ such that there exists $x \in \mathbb{Z}^m \setminus \{0\}$ satisfying

- $|b \cdot x|_Z > \varepsilon$
- $\|A^t x\|_Z < d^{-1}\varepsilon \tilde{\psi}(S)$ and $\|x\| < d^{-1}\varepsilon S$.

Let $W_{b, \varepsilon} := \limsup_{S \to \infty} W_{b, S, \varepsilon}$. Note that $A \in W_{S, \varepsilon}$ if and only if

$$\|A^t x_{A,S}\|_Z < \Psi_{\varepsilon}(U) \text{ and } \|x_{A,S}\| < U \text{ for some } x_{A,S},$$

where

$$\Psi_{\varepsilon}(U) := d^{-1}\varepsilon \tilde{\psi}(d^{-1}U), \quad U = d^{-1}\varepsilon S. \quad (4.1)$$

By Lemma 4.1 $\limsup_{S \to \infty} \widehat{W}_{S, \varepsilon} \subseteq \widehat{D}_{m,n}(\psi)^c$ and $W_{b, \varepsilon} \subseteq \widehat{D}_{m,n}(\psi)^c$.

Further $\limsup_{S \to \infty} W_{S, \varepsilon} = \{ A \in [0, 1]^{mn} : A^t \in W_{n,m}(\Psi_{\varepsilon}) \}$ is the set of matrices whose transposes are $\Psi_{\varepsilon}$-approximable. From here onwards we use a slightly different definition of $\Psi_{\varepsilon}$-approximability; recall from footnote 1 where the inequality $\|A^t x\|_Z < \Psi_{\varepsilon}(|x|)$ is used instead of (1.3). Then, $W_{b, \varepsilon}$ can be considered as the set of matrices whose transposes are $\Psi_{\varepsilon}$-approximable with solutions restricted on the set $\{ x \in \mathbb{Z}^m : |b \cdot x|_Z > \varepsilon \}$.

4.1 Mass distributions on $\Psi_{\varepsilon}$-approximable matrices

In this subsection we prove the divergent part of Theorem 1.7 using mass distributions on $\Psi_{\varepsilon}$-approximable matrices following [1].
Lemma 4.2 For each \( mn - n < s \leq mn \) and \( 0 < \varepsilon < 1/2 \), let \( U_0 = d^{-1} \varepsilon S_0 \) and \( f \) be a dimension function satisfying (1.5) and (1.6). Suppose that (1.7) holds. Then

\[
\sum_{q=1}^{\infty} \frac{1}{\psi(q)q^{s}} \left( \frac{q^{1}}{\psi(q)^{\frac{1}{m}}} \right)^{mn} f\left( \frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{m}}} \right) < \infty
\]

\[
\Longleftrightarrow \sum_{h=1}^{\infty} h^{m+n-1} \left( \frac{\psi_{e}(h)}{h} \right)^{-n(m-1)} f\left( \frac{\psi_{e}(h)}{h} \right) < \infty.
\]

Proof Similar to Lemma 3.1, we may replace the sums with integrals

\[
\int_{T_0}^{\infty} \frac{1}{\psi(x)x^{s}} \left( \frac{x^{\frac{1}{m}}}{\psi(x)^{\frac{1}{m}}} \right)^{mn} f\left( \frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{m}}} \right) \, dx \quad \text{and} \quad \int_{U_0}^{\infty} h^{m+n-1} \left( \frac{\psi_{e}(h)}{h} \right)^{-n(m-1)} f\left( \frac{\psi_{e}(h)}{h} \right) \, dh,
\]

respectively.

Note that since \( \psi_{e}(h) = d^{-1} \varepsilon \tilde{\psi}(d \varepsilon h) \), if we consider the term \( \int_{U_0}^{\infty} h^{m+n-1} \left( \frac{\psi_{e}(h)}{h} \right)^{-n(m-1)} f\left( \frac{\psi_{e}(h)}{h} \right) \, dh \), then

\[
\int_{U_0}^{\infty} h^{m+n-1} \left( \frac{\psi_{e}(h)}{h} \right)^{-n(m-1)} f\left( \frac{\psi_{e}(h)}{h} \right) \, dh < \infty
\]

\[
\Longleftrightarrow \int_{S_{0}}^{\infty} y^{m+n-1} \left( \tilde{\psi}(q) \right)^{-n(m-1)} f\left( \frac{\tilde{\psi}(q)}{y} \right) \, dy < \infty.
\]

Also, since \( \tilde{\psi}(y) = \psi^{-1}(y^{-m})^{-\frac{1}{m}} \), we have

\[
\int_{S_{0}}^{\infty} y^{m+n-1} \left( \frac{\tilde{\psi}(y)}{y} \right)^{-n(m-1)} f\left( \frac{\tilde{\psi}(y)}{y} \right) \, dy
\]

\[
= \int_{S_{0}}^{\infty} y^{mn+m-1}(\psi^{-1}(y^{-m}))^{m-1} f\left( \frac{(\psi^{-1}(y^{-m}))^{-\frac{1}{m}}}{y} \right) \, dy
\]

\[
= \frac{1}{m} \int_{S_{0}^{m}} t^{n}(\psi^{-1}(t^{-1}))^{m-1} f\left( \frac{(\psi^{-1}(t^{-1}))^{-\frac{1}{m}}}{t^{rac{1}{m}}} \right) \, dt
\]

\[
= \frac{1}{m} \int_{\psi^{-1}(S_{0}^{m})} x^{n-1}(\psi(x)^{-1})^{m} f\left( \frac{x^{-\frac{1}{m}}}{(\psi(x)^{-1})^{\frac{1}{m}}} \right) d\psi(x)^{-1}
\]

\[
\approx \frac{1}{m} \left( n - \frac{s}{m} + 1 \right)^{-1} \int_{T_{0}}^{\infty} x^{m-1} f(x^{-\frac{1}{m}} d(\psi(x)^{-1})^{n-\frac{1}{m}+1},
\]

where in the second last line we used the change of variables \( x = \psi^{-1}(t^{-1}) \), \( t = \psi(x)^{-1} \) and in the last line we used (1.5) and (1.7). Since it follows from (1.7) that \( (x^{-\frac{1}{m}})^{n} \leq (\psi(x)^{-1})^{-\frac{1}{m}} \leq (x^{-\frac{1}{m}})^{\frac{1}{m}} \). Therefore by using (1.5) we can write

\[
 f((\psi(x)^{-1})^{-\frac{1}{m}} x^{-\frac{1}{m}}) \approx (\psi(x)^{-1})^{-\frac{1}{m}} f(x^{-\frac{1}{m}})
\]

Using integration by parts

\[
\int_{T_{0}}^{\infty} x^{m-1} f(x^{-\frac{1}{m}} d(\psi(x)^{-1})^{n-\frac{1}{m}+1}
\]
\[
\begin{align*}
&= \left( \lim_{x \to \infty} x^{n-1} \psi(x)^{-n-1+\frac{s}{x}} f(x^{-\frac{1}{x}}) - T_0^{m-1} \psi(T_0)^{-n-1+\frac{s}{T_0^{\frac{1}{T}}}} \right) \\
&\quad + \int_{T_0}^{\infty} \left[ -(m-1)x^{m-2}f(x^{-\frac{1}{x}}) + \frac{1}{n} x^{m-1} x^{-\frac{1}{n}} f'(x^{-\frac{1}{x}}) \right] \psi(x)^{-n-1+\frac{s}{x}} dx \\
&= \lim_{x \to \infty} x^{m-1} \psi(x)^{-n-1+\frac{s}{x}} f(x^{-\frac{1}{x}}) - T_0^{m-1} \psi(T_0)^{-n-1+\frac{s}{T_0^{\frac{1}{T}}}} \\
&\quad + \int_{T_0}^{\infty} \left[ -(m-1) + \frac{1}{n} a(x^{-\frac{1}{x}}) \right] x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx, \quad \text{by (1.6)} \\
&\succ \lim_{x \to \infty} x^{m-1} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) - T_0^{m-1} \psi(T_0)^{-n-1+\frac{s}{T_0^{\frac{1}{T}}}} \\
&\quad + \int_{T_0}^{\infty} \left[ \frac{1}{n} a(x^{-\frac{1}{x}}) - (m-1) \right] x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx.
\end{align*}
\]

Note that
\[
\int_{T_0}^{\infty} x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx = \int_{T_0}^{\infty} x^{m-1} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) d \log x. \quad (4.2)
\]

Thus the convergence of \( \int_{T_0}^{\infty} x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx \) gives that
\[
\lim_{x \to \infty} x^{m-1} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) < \infty.
\]

Also observe that as \( x \to \infty, a(x^{\frac{1}{x}}) \to s \). Therefore
\[
\frac{1}{n} a(x^{\frac{1}{x}}) - (m-1) \to \frac{s-n(m-1)}{n}
\]
which is finite and positive (since \( s > mn - n \)). Therefore the convergence of \( \int_{T_0}^{\infty} x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx \) gives the convergence of
\[
\frac{1}{n} \int_{T_0}^{\infty} a(x^{\frac{1}{x}}) x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx - (m-1) \int_{T_0}^{\infty} x^{m-2} \psi(x)^{-n-1} f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx.
\]

Hence the convergence of
\[
\int_{T_0}^{\infty} \frac{1}{\psi(x) x^2} \left( \frac{x^{\frac{1}{\pi}}}{\psi(x)^{\frac{1}{\pi}}} \right)^m f \left( \frac{\psi(x)^{\frac{1}{x}}}{x^{\frac{1}{\pi}}} \right) dx
\]
implies the convergence of other one since for \( T_0 \) large enough all summands in (4.2) are positive except the finite value
\[
-T_0^{m-1} \psi(T_0)^{-n-1+\frac{s}{T_0^{\frac{1}{T}}}} f \left( \frac{\psi(T_0)^{\frac{1}{x}}}{T_0^{\frac{1}{T}}} \right). \]

\[ \square \]
Lemma 4.3 (1, Section 5) Assume that \( \frac{\sum_{q=1}^{\infty} \psi(q)q^{\frac{1}{m}}}{\psi(q)q^{\frac{1}{m}}} \left( \frac{q^m}{\psi(q)q^{\frac{1}{m}}} \right)^m f \left( \frac{\psi(q)q^{\frac{1}{m}}}{q^s} \right) \) is a natural measure. For any \( \eta > 0 \) and a sufficiently small radius \( r(D) \) we have

\[
\mu(D) \ll \frac{f(r(D))}{\eta},
\]

where the implied constant does not depend on \( D \) or \( \eta \).

Proof Note that \( \limsup_{S \to \infty} W_{S,\epsilon} = \{ A \in [0,1]^{mn} : A^t \in W_{n,m}(\psi) \} \). By Lemma 4.2

\[
\sum_{h=1}^{\infty} h^{m+n-1} \left( \frac{\psi(h)}{h} \right)^{-n(m-1)} f \left( \frac{\psi(h)}{h} \right) = \infty,
\]

which is the divergent assumption of Theorem 2.3 for \( W_{n,m}(\psi) \). From the proof of Jarnik’s Theorem in [1] and the construction of probability measure in [1, Section 5] we can obtain a probability measure \( \mu \) on \( \limsup_{S \to \infty} W_{S,\epsilon} \) satisfying the above condition. \( \square \)

Let us prove the divergent part of Theorem 1.7.

Proof Assume that \( mn + m - n < s < mn + m \) and fix \( 0 < \epsilon < \frac{1}{2} \). For any fixed \( \eta > 0 \), let \( \mu \) be a probability measure on \( \limsup_{S \to \infty} W_{S,\epsilon} \) as in Lemma 4.3 with \( f(r(D)) \) replaced by \( r(D)^{-mf(r(D))} \).

Here we remark that since \( f(r) \) satisfies (1.5) and (1.6) it is not hard to check that the new function \( f^*(r) := f^*(r) - \frac{1}{m}f(r) \) satisfies conditions (1.5) and (1.6) with \( s \) replaced by \( s - m \). Indeed, (1.5) (with \( s \) replaced by \( s - m \)) follows since \( f^*(xy) = \frac{f(xy)}{f(y)} \), and (1.6) (with \( s \) replaced by \( s - m \)) follows since

\[
\frac{r^m f^*(r)}{f^*(r)} = \frac{r}{f^*(r)} \left[ r^{m-1} f(r) - mf(r) \right] = \frac{f'(r)}{f(r)} \to (s - m) \text{ as } r \to 0.
\]

Now consider the product measure \( \nu = \mu \times m_{\mathbb{R}^m} \), where \( m_{\mathbb{R}^m} \) is the canonical Lebesgue measure on \( \mathbb{R}^m \) and let \( \pi_1 \) and \( \pi_2 \) be the natural projections from \( \mathbb{R}^{mn} \) to \( \mathbb{R}^{mn} \) and \( \mathbb{R}^m \), respectively.

For any fixed integer \( N \geq 1 \), let \( V_{S,\epsilon} = W_{S,\epsilon} \setminus \bigcup_{k=N}^{N-1} W_{k,\epsilon} \) and \( \tilde{V}_{S,\epsilon} = \{ (A, b) \in \tilde{W}_{S,\epsilon} : A \in V_{S,\epsilon} \} \). Then \( \nu(\bigcup_{S \geq N} \tilde{W}_{S,\epsilon}) = \nu(\bigcup_{S \geq N} \tilde{V}_{S,\epsilon}) \geq 1 - 2\epsilon \), see [12, p.21].

Since \( N \geq 1 \) is arbitrary, we have \( \nu(\limsup_{S \to \infty} \tilde{W}_{S,\epsilon}) \geq 1 - 2\epsilon \). For an arbitrary ball \( B \subseteq \mathbb{R}^{mn} \) of sufficiently small radius \( r(B) \), we have

\[
\nu(B) \leq \mu(\pi_1(B)) \times m_{\mathbb{R}^m}(\pi_2(B)) \ll \frac{f(r(B))}{\eta},
\]

where the implied constant does not depend on \( B \) or \( \eta \). By using the Mass Distribution Principle i.e. Lemma 2.2 and the Transference Lemma i.e. Lemma 4.1, we have

\[
\mathcal{H}(\beta_{m,n}(\psi)^{\epsilon}) \geq \mathcal{H}(\limsup_{S \to \infty} \tilde{W}_{S,\epsilon}) \gg (1 - 2\epsilon)\eta,
\]

and by letting \( \eta \to \infty \) we obtain the desired result. \( \square \)
4.2 Local ubiquity for $W_{b,e}$

We will use the idea of local ubiquity for $W_{b,e}$ to prove the divergent part of Theorem 1.8. Following [12] we define

$$
\varepsilon(b) = \min_{1 \leq j \leq m, b_j \neq 0} \frac{|b_j|}{4}, \quad (4.3)
$$

for $b = (b_1, \cdots, b_m) \in \mathbb{R}^m \setminus \mathbb{Z}^m$. Note that $\varepsilon(b) > 0$ is due to the fact that $b \in \mathbb{R}^m \setminus \mathbb{Z}^m$.

The following lemma is used when we count the number of integral vectors $z \in \mathbb{Z}^m$ such that

$$
|b \cdot z|_\mathbb{Z} \leq \varepsilon(b). \quad (4.4)
$$

**Lemma 4.4** ([12, Lemma 4.4]) For $b = (b_1, \cdots, b_m) \in \mathbb{R}^m \setminus \mathbb{Z}^m$, let $\varepsilon(b)$ be as in (4.3) and $1 \leq i \leq m$ be an index such that $\varepsilon(b) = \frac{|b_i|}{4}$. Then, for any $x \in \mathbb{Z}^m$, at most one of $x$ and $x + e_i$ satisfies (4.4) where $e_i$ denotes the vector with a 1 in the $i$th coordinate and 0’s elsewhere.

For a fixed $b \in \mathbb{R}^m \setminus \mathbb{Z}^m$, let $\varepsilon_0 := \varepsilon(b)$, $\Psi_0 := \Psi_{\varepsilon_0}$ and $\Psi(h) = \frac{\Psi_0(h)}{h}$. With notions in the Subsection 2.2, which are defined for the ubiquitous system construction, let

$$
J := \{(x, y) \in \mathbb{Z}^m \times \mathbb{Z}^m : \|y\| \leq m\|x\| \text{ and } |b \cdot x|_\mathbb{Z} > \varepsilon_0\} \quad (4.5)
$$

for $k := (x, y) \in J$ denote $\beta_k := \|x\|$ and $R_k := \{A \in [0,1)^m : A^T x = y\}$. \quad (4.6)

Note that $W_{b,0} \subset \Lambda(\Psi)$ and the family $R$ of resonant sets $R_k$ consists of $(m - 1)n$-dimensional, rational affine subspaces.

By Lemma 4.2, now we assume that the divergence part of Theorem 1.8 is satisfied. Then we can find a strictly increasing sequence of positive integers $\{h_i\}_{i \in \mathbb{N}}$ such that

$$
\sum_{h_{i-1} < h < h_i} h^{m+n-1} \left(\frac{\Psi_0(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_0(h)}{h}\right) \omega(h)^{-n} > 1 \quad (4.7)
$$

and $h_i > 2h_{i-1}$. Put $\omega(h) := i^\frac{1}{n}$ if $h_{i-1} < h \leq h_i$. Then

$$
\sum_{h=1}^{\infty} h^{m+n-1} \left(\frac{\Psi_0(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_0(h)}{h}\right) \omega(h)^{-n} = \infty.
$$

For a constant $c > 0$, define the ubiquitous function $\rho_c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$
\rho_c(h) = \begin{cases} 
ch^{-\frac{1+n}{m}} & \text{if } m = 1; \\
ch^{-\frac{m}{m+n}} \omega(h) & \text{if } m \geq 2.
\end{cases} \quad (4.8)
$$

Clearly the ubiquitous function is 2-regular.

**Theorem 4.5** ([12, Theorem 4.5]) The pair $(\mathcal{R}, \beta)$ is a locally ubiquitous system relative to $\rho = \rho_c$ for some constant $c > 0$.

The divergent part of Theorem 1.8.
Assume that \((m - 1)n < s \leq mn\) and \(r^{-mn}f(r) \to \infty\) as \(r \to 0\). It follows from Theorem 2.5 and Theorem 4.5 that
\[
\mathcal{H}^f(D_{m,n}(\psi))^c \geq \mathcal{H}^f(W_{h_0}) = \mathcal{H}^f([0, 1]^{mn}).
\]

Similar as in [12] here we have used the fact that the divergence and convergence of the sums
\[
\sum_{N=1}^{\infty} \frac{2^N \mathcal{F}(2^N)}{N} \quad \text{and} \quad \sum_{h=1}^{\infty} h^{-1} \mathcal{F}(h)
\]

coincide for any monotonic function \(\mathcal{F} : \mathbb{Z}_+ \to \mathbb{R}_+\) and \(\kappa \in \mathbb{R}\). This completes the proof of the divergent part of Theorem 1.8.

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