Some aspects of the critical behavior of the Two-Neighbor Stochastic Cellular Automata

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January 20, 2022

Abstract

Using Pade approximations and Monte Carlo simulations, we study the phase diagram of the Two-Neighbor Stochastic Cellular Automata, which have two parameters $p_1$ and $p_2$ and include the mixed site-bond directed percolation (DP) as a special case. The phase transition line $p_1 = p_{1c}(p_2)$ has endpoints at $(p_1, p_2) = (1/2, 1)$ and at $(0.8092, 0)$. The former point $(1/2, 1)$ is a special point at which Compact DP transition occurs and its critical exponents are known exactly. Results of time-dependent simulation show that in the whole range of parameters, excluding this point $(1/2,1)$, the system belongs to the DP universality class. It is first shown that the shape of the phase transition line near this special point has, asymptotically, a parabolic shape, i.e., $p_{1c}(p_2) - 1/2 \sim (1 - p_2)^\theta$ with $\theta = 1/2$ for $0 < 1 - p_2 \ll 1$. We use the Monte Carlo data to assess the accuracy of rigorous bounds for the line recently reported by Liggett and by Katori and Tsukahara. It is also shown that outside the vicinity of the special point $(1/2,1)$, the curve is well approximated by an interpolation formula, similar to the one proposed by Yanuka and Englman.

PACs number(s): 05.70.Ln, 64.90.+b
1 Introduction

The two-neighbor stochastic cellular automata (SCA), introduced by Domany and Kinzel, can be regarded as a very general model of a spread of influence in 1+1 dimensions. A site on layer $i+1$ is occupied with probability $p_1$ if one and only one of its nearest neighbors on the previous layer is occupied, it is occupied with probability $p_2$ if both of the neighbors are occupied, and it remains vacant if both neighbors are vacant (Fig. 1). Often the system is regarded as a process taking place in one-dimension, with the layer number $i$ replaced by discrete time $t$.

As one can easily see, the case of $p_1 = p_2 = p$ is equivalent to the site directed percolation (DP) with site occupation probability $p$. It is easy to show that $(p_1 = p, p_2 = 2p - p^2)$ corresponds to bond DP with the fraction of open bonds $p$. The mixed site-bond DP with the fraction of open sites $\alpha$ and the fraction of open bonds $\beta$ corresponds to the two-neighbor SCA with $p_1 = \alpha \beta$ and $p_2 = \alpha(2\beta - \beta^2)$ [1, 2, 3].

In the present work, by using Monte Carlo simulations, we evaluate the phase transition line which is characterized as follows. We consider the process starting from a single occupied particle and let $P(t)$ be the probability that at least one particle survives at time $t$. In the subcritical phase $p_1 < p_{1c}(p_2)$, this survival probability decreases exponentially to zero, while in the supercritical phase $p_1 > p_{1c}(p_2)$, it converges to a positive value: $P_\infty \equiv \lim_{t \to \infty} P(t) > 0$. On the phase transition line $p_1 = p_{1c}(p_2)$, $P(t)$ shows a power-law decay and critical phenomena are observed. The ultimate survival probability in the supercritical phase behaves as follows in the vicinity of the phase transition line: $P_\infty(p_1, p_2) \sim (p_1 - p_{1c}(p_2))^{\beta}$ as $p_1 \to p_{1c}(p_2)$ for each $p_2$. It is conjectured for $p_2 < 1$ that the critical exponent $\beta$ is universal, i.e., independent of $p_2$ and is the same as the pure DP, $\beta \simeq 0.276$.

The two-neighbor SCA with $p_2 = 1$ is equivalent to the reaction-limited case of the so-called dimer-dimer model [4], this case (sometimes referred to as Compact Directed Percolation) is understood rather well [1, 2, 3]. In particular, it is easy to see $P_\infty(p_1, 1) = 1 - (1 - p_1)^2/p_1^2 \simeq 8(p_1 - 1/2)$ for $p_1 \simeq 1/2$, which means $p_{1c} = 1/2$ and $\beta(p_2 = 1) = 1$ [3].
The point \((p_1, p_2) = (1/2, 1)\) is an endpoint of the critical line \(p_1 = p_{1c}(p_2)\) in the phase diagram of the two-neighbor SCA. The universality for \(p_2 < 1\) and the above mentioned exact result for \(p_2 = 1\) imply that we will observe the following phenomenon called a crossover when \(p_2 \approx 1\) \footnote{2}. There is a function \(\varepsilon(p_2) \geq 0\) such that if \(0 < p_1 - p_{1c}(p_2) \ll \varepsilon(p_2)\) then \(P_\infty \sim (p_1 - p_{1c}(p_2))^\beta\) with \(\beta \approx 0.276\), while if \(\varepsilon(p_2) < p_1 - p_{1c}(p_2) \ll 1\) then \(P_\infty \sim (p_1 - 1/2)\) and that \(\varepsilon(p_2) \to 0\) as \(p_2 \to 1\).

In order to study the universality and the crossover, it is necessary to make precise evaluation of the phase transition line. In particular, we need to know the asymptotic behavior of the phase transition line near the special point \((p_1, p_2) = (1/2, 1)\). By using series expansion technique and comparing the results with the data obtained by Monte Carlo we conclude that the asymptotic shape of the phase transition line near \((1/2, 1)\) is parabolic. In other words, 
\[ p_{1c}(p_2) - 1/2 \sim (1 - p_2)^\theta \] with \(\theta = 1/2\) for \((0 < 1 - p_2 \ll 1)\). We use the Monte Carlo data to assess the accuracy of rigorous bounds recently obtained theoretically \footnote{2} \footnote{4}.

\section{Monte Carlo simulations}

Two points of the phase diagram, corresponding to pure site and bond DP, can be estimated from the corresponding threshold values as \((0.705485,0.705485)\) and \((0.644701,0.873762)\), using the data taken from \footnote{8}. The sector of the phase diagram between them corresponds to the mixed site-bond DP and can be obtained using the data given in \footnote{12}. Therefore, to obtain a complete phase diagram only areas with \(0.5 < p_1 < 0.644701\) and \(p_1 > 0.705485\) need to be covered.

Similarly to \footnote{12}, to determine the location of the phase-transition line, we used the time dependent simulation technique and analyzed the time dependence of the local slopes for the mean number of particles \(n\), for the survival probability \(P\) and for the mean square radius \(R\). Simulations were conducted starting from a single occupied particle, taking 1000 steps and averaging over \(10^5\) configurations. Some of the local slopes curves (for the mean number of
particles) are given in Fig. 2. Our results agree with the assumption that the two-neighbor SCA belong to the DP universality class for \( p_2 < 1 \) \cite{3}, with the dynamical critical exponents defined as

\[
P(t) \propto t^{-\delta} \\
n(t) \propto t^{\eta} \\
R^2(t) \propto t^z
\]

falling within intervals \( \delta = 0.162 \pm 0.020, \eta = 0.308 \pm 0.015 \) and \( z = 1.263 \pm 0.040 \) from the DP values (we take the DP values of the dynamical critical exponents from \cite{3}). On the other hand, the local slopes approach the asymptotic values, corresponding to the values of the critical exponents, in a different way depending on the value of \( p_2 \). While for the part of the phase diagram far enough from the special point \((1/2,1)\) the local slopes stabilize at the asymptotic value relatively fast (Fig. 2a), for the points close to \((1/2,1)\), in the cases we studied, the number of steps taken was not enough to reach the asymptotic value, although the approach to DP values was still quite obvious (Fig. 2b). For \((p_1,p_2) = (1/2,1)\) the exponents agreed with the theoretical predictions of \( \delta = 1/2, \eta = 0 \) and \( z = 1 \) \cite{5}.

The Monte Carlo results for the phase transition line are presented in Tab. 3. In Fig. 3 we compare them with the rigorous lower bound given by Katori and Tsukahara \cite{6} as \( p_2 = (1 - 2p_1^3)/(1 - p_1^2) \) and with the rigorous lower \( (p_2 = 2(1 - p_1)) \) and upper \( (p_2 = 4p_1 - 4p_1^3) \) bounds given by Liggett \cite{7}. Also, we present the rigorous lower bound obtained by averaging the number of particles after 12 steps (obtained by exact enumeration, starting from a single particle) and taking \( p_1 \) and \( p_2 \) values leading to the average equal to 1. The upper bound given by Liggett comes very near to the phase transition line in the vicinity of \((1/2,1)\). It should be remarked that all these bounds can only be proved to be valid in the so-called attractive region, with \( p_1 \leq p_2 \) \cite{10}. However, we see that the Katori and Tsukahara’s lower bound is not only valid over the whole range of \( p_2 \) but, indeed, follows the shape of the Monte Carlo data rather closely.

Fig. 3 should be compared with the phase diagram shown by Kinzel \cite{2}, which was studied by
the transfer-matrix scaling method. Kinzel showed two possibilities for the behavior of the phase transition line near $p_2 = 0$: the line has an endpoint on the $p_1$-axis at $(p_{10}, 0)$ with $p_{10} < 1$ with $p_{10} < 1$, or the line goes to the corner point $(1, 0)$ in the phase diagram. The former case is concluded from our results with $p_{10} = 0.8092 \pm 0.0004$.

An interpolation formula, proposed by Yanuka and Englman [11] for the ordinary percolation, was shown to perform rather well in the case of the mixed DP [12]. Expanding it to the general case of the two-neighbor SCA leads to

$$p_2 = p_1(2 - (p_1/p_1^a)^{1/(1-\lambda)}).$$

(2)

where $\lambda = \log p_1^s / \log p_2^b$, with $p_1^s$ and $p_2^b$ as ordinates of points on the phase diagram, corresponding to the pure site and pure bond DP, respectively. Although we can not present any theoretical explanation at this moment, one can see that (2) perform rather well even outside the mixed site-bond percolation sector, for which it has been derived. Marked discrepancy begins at $p_2 > 0.9$, as the phase diagram approaches the special point $(1/2, 1)$.

3 Series expansion

While an attempt to directly apply Pade analysis to the series expansion for the survival probability in $p_1$ and $p_2$ around (0,0) fails because all coefficients of the expansion change when the order is increased, changing coordinates to $a$ and $q$ according to

$$p_1 = 1 - aq$$

(3)

$$p_2 = 1 - q$$

(4)

leads to a series with converging coefficients. Computer time required to obtain, using exact enumeration, an expansion to a certain order grows very fast with the order of the expansion because the model includes 2 parameters. In the current work we try to take advantage of the possibility to work analytically with a relatively low order expansion.
Regarding the series expansion for the logarithm of the survival probability in $a$ and $q$ as a polynomial in $q$ we take a $[3,3]$ Pade approximant for its derivative in $q$ and expect that the phase transition line is given by a root of the denominator (for a description of the DLog Pade method we are using, for example, [13]). Solving the corresponding 3-rd order equation exactly and discarding unphysical roots we get a rather complicated expression for the phase diagram in the form of $q$ given as a function of $a$. Assuming that we are in the vicinity of the special point, where, as one can easily see, $a$ goes to infinity, we take an expansion in $1/a^{1/2}$, the two leading terms are obtained as

$$q = \frac{1}{2a} - \frac{1}{4a^{3/2}}$$

where the discarded terms are of the order $0(a^{5/2})$. As one can see in Fig 4, (5) approximates the exact solution surprisingly well. Taking (5) and returning to coordinates $p_1$ and $p_2$ we have for the phase diagram in the vicinity of $(1/2,1)$

$$p_2 = -3 + 20p_1 - 32p_1^2 + 16p_1^3.$$  \hspace{1cm} (6)

Again, taking a Taylor expansion in $p_1 - 1/2$ we have

$$p_2 = 1 - 8(p_1 - 1/2)^2,$$  \hspace{1cm} (7)

discarding terms of the order $0((p_1 - 1/2)^3)$. Surprisingly enough, comparison with the Monte Carlo data (Fig. 3) shows that both (5) and (6) work reasonably well in the vicinity of $(1/2,1)$, with (6), as it should be expected, working over a wider range of $p_1$ values. Moreover, the parabolic asymptotic behavior, predicted by (6), is in agreement with the Ligget’s upper bound and with the Monte Carlo data, as it is shown in Fig. 4

### 4 Conclusions and Future Problems

By using Pade approximations and Monte Carlo simulations, we evaluate the phase transition line $p_1 = p_{1c}(p_2)$ for the two-neighbor SCA. The line has endpoints at $(p_1, p_2) = (1/2, 1)$ and at $(0.8092, 0)$. Using the time dependent simulation technique [12], we confirm that the present
SCA belong to the DP universality class in the whole range of parameters, excluding the point (1/2,1). The special point (1/2,1) is the transition point of the reaction-limited case of the dimer-dimer model [4] and different critical phenomena are observed [4]. The present work show that the shape of the phase transition line near this special point has, asymptotically, a parabolic shape, and we conclude that

\[ p_{1c}(p_2) - \frac{1}{2} \simeq C(1 - p_2)\theta \quad \text{with} \quad \theta = \frac{1}{2} \quad \text{and} \quad C \simeq 0.436 \]  

(8)

for \( 0 < 1 - p_2 \ll 1 \). As discussed in Section 1, this numerical result is a first step in the further study on the crossover which is expected to occur in the vicinity of \( (p_1, p_2) = (1/2, 1) \) [3].

Recently rigorous lower and upper bounds of the phase transition line were given for the attractive region \( p_1 \leq p_2 \) [6, 7]. In this paper we present other rigorous lower bound obtained by exactly averaging the number of particles after 12 steps, whose validity can also be proved if and only if \( p_1 \leq p_2 \). We access the accuracy of these rigorous bounds using the Monte Carlo date. The upper bound given by Liggett is excellent in the vicinity of \((1/2,1)\). It is shown that the lower bound by Katori and Tsukahara seems to be valid also in the non-attractive region \( p_1 > p_2 \) and follows the shape of the Monte Carlo date rather closely down to \( p_2 = 0 \). Justification of the "lower bound" for \( p_1 > p_2 \) is a challenging problem, since rigorous things proved for the non-attractive region are still limited [14].

We also show in this paper that far enough from the special point (1/2,1) the phase transition line is well approximated by an interpolation formula [12], similar to the one proposed by Yanuka and Englman for ordinary percolation [11].

**Acknowledgements**

One of the authors (A.T.) acknowledges the support of the Inamori foundation.

**References**
When $p_1 \leq p_2$, if we add a new particle at a vacant site, then the probability with which any of the neighboring vacant sites becomes occupied increases. It follows that the number of particles at time $t > 0$ is a non-decreasing function of the number of particles at time zero. This property is called attractiveness. Using this property, it is proved for $p_1 \leq p_2$ that the survival probability $P_\infty$ is a monotonically non-decreasing function of $p_1$ and $p_2$.
Tables

1. Monte Carlo results for the phase diagram.

Values marked by * were obtained with uncertainty ±0.0002, † corresponds to uncertainty ±0.0004, and ‡ to ±0.0008.

a) the vicinity of $p_2 = 1$.

b) the vicinity of $p_2 = 0$.

Figure captions

1. The Two-Neighbor Stochastic Cellular Automata

2. Local slopes for the mean number of particles $r = \log(n(t)/n(t/5))/\log(5)$ plotted versus the inverse of the time $t$.

Horizontal solid line corresponds to the DP value of the critical exponent $\eta = 0.308$.

a) $p_2 = 0.62$, 1- $p_1 = 0.9175$, 2- $p_1 = 0.9171$, 3- $p_1 = 0.9167$ (critical point), 4- $p_1 = 0.9163$, 5- $p_1 = 0.9159$.

b) $p_2 = 0.52$, 1- $p_1 = 0.9983$, 2- $p_1 = 0.9981$, 3- $p_1 = 0.9979$ (critical point), 4- $p_1 = 0.9977$, 5- $p_1 = 0.9975$.

c) $p_1 = 1/2$, $p_2 = 1$

3. Phase diagram for the Two-Neighbor SCA. Monte Carlo results are represented by points.

1,2- lower and upper bounds by Liggett [7].
3- lower bound by Katori and Tsukahara [3]
4- Pade result.
5- Tailor expansion for the Pade result.
6- Interpolation formula [11, 12].
7- lower bound obtained by averaging the number of particles after 12 steps (obtained by exact enumeration, starting from a single particle) and taking $p_1$ and $p_2$ values leading to the average equal to 1.
4. Phase diagram by Pade approximants in coordinates $q$ and $a$, defined by $p_1 = 1 - q$ and $p_2 = 1 - aq$. The lower line corresponds to the Taylor expansion, given by $q = \left( \frac{1}{2} - \frac{1}{4a} \right)/a$.

5. Phase diagram for the Two-Neighbor SCA in double logarithmic coordinates. Solid line shows the best fit for $q \equiv 1 - p_1 = k(p_1 - 0.5)^2$ over the first 3 points, which corresponds to $k = 5.25 \ldots$. Taylor expansions for the Pade result ($k = 8$) and for the Liggett’s upper boundary ($k = 4$) are also shown.
| $p_1^*$ | $p_2^*$ |
|---------|---------|
| 0.50    | 1.0000  |
| 0.52    | 0.9979* |
| 0.54    | 0.9917* |
| 0.56    | 0.9809* |
| 0.58    | 0.9653* |
| 0.60    | 0.9443* |
| 0.62    | 0.9167† |

| $p_1^*$ | $p_2^*$ |
|---------|---------|
| 0.71    | 0.6877† |
| 0.73    | 0.6010† |
| 0.75    | 0.4933† |
| 0.77    | 0.3590† |
| 0.79    | 0.1916‡ |
| 0.8092† | 0.0000  |