MOMENTS OF CHARACTER SUMS TO COMPOSITE MODULUS

BRYCE KERR

Abstract. In this paper we consider the problem of estimating character sums to composite modulus and obtain some progress towards removing the cubefree restriction in the Burgess bound. Our approach is to estimate high order moments of character sums in terms of solutions to congruences with Kloosterman fractions and we deal with this problem by extending some techniques of Bourgain, Garaev, Konyagin and Shparlinski and Bourgain and Garaev from the setting of prime modulus to composite modulus. As an application of our result we improve an estimate of Norton.

1. Introduction

Given an integer $q$ and a primitive character $\chi \mod q$ we consider estimating the sums

$$\sum_{M<n\leq M+N} \chi(n). \tag{1}$$

The first result in this direction is due to Pólya and Vinogradov and states that

$$\sum_{M<n\leq M+N} \chi(n) \ll q^{1/2} \log q. \tag{2}$$

The above bound is nontrivial provided $N \geq q^{1/2+o(1)}$ and a difficult problem is to estimate the sums (1) in the range $N \leq q^{1/2-\delta}$. The first progress in this direction is due to Burgess [7] and in a series of papers [8, 9, 10, 11, 12] the work of Burgess culminated in the following estimate.

**Theorem 1.** Let $q$ be an integer and $\chi$ a primitive character mod $q$. Then we have

$$\sum_{M<n\leq M+N} \chi(n) \ll N^{1-1/r} q^{(r+1)/4r^2+o(1)},$$

for any $r \leq 3$ and any $r \geq 1$ provided $q$ is cubefree.
A well known conjecture states 

$$\sum_{M < n \leq M+N} \chi(n) \ll N^{1/2} q^{o(1)},$$

and a longstanding problem is to improve on Theorem 1 quantitatively and in the range of parameters for which the bound is nontrivial. There has been some progress on this problem for sets of moduli with some arithmetic structure although making progress for general $q$ remains open. See [16, 21, 22, 25] for improvements to smooth modulus with origins in Heath-Brown’s $q$-analogue of Weyl differencing [24] and [1, 20, 26, 29] for improvements to powerful modulus with the first results in this direction due to Postnikov [32, 33]. One of the important consequences of the case $r = 2$ in Theorem 1 is the subconvexity estimate 

$$L\left(\frac{1}{2}, \chi\right) \ll q^{3/16 + o(1)},$$

which has recently been improved by Petrow and Young [31] for cube-free modulus, extending earlier work of Conrey and Iwaniec [17].

The restriction to cubefree modulus arises in many problems when applying the amplification method to estimate exponential sums. In the setting of Theorem 1 removing this restriction would allow the estimation for smaller ranges of the parameter $N$ and have applications to analytic properties of Dirichlet $L$-functions closer to the line $\Re s = 1$. The main difficulty in achieving this lies in the estimation of complete sums modulo prime powers. An important stage in Burgess’ argument is the reduction of estimating the sums (1) to the moments

$$(3) \quad \sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r}.$$ 

These moments were first considered by Davenport and Erdős [19] for prime modulus $q$ who appealed to some earlier work of Davenport [18]. This became obsolete after Weil [34] whose estimates lead to the bound

$$(4) \quad \sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll qV^r + q^{1/2}V^{2r}.$$ 

Extending the estimate (4) to arbitrary composite modulus is the main obstacle in removing the cubefree restriction in Theorem 1. Supposing that $q = p^\alpha$ is a prime power and considering (3), expanding and
interchanging summation gives

\[ \sum_{\lambda=1}^{p^\alpha} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \leq \sum_{1 \leq v \leq V} \left| \sum_{\lambda=1}^{p^\alpha} \chi(F_v(\lambda)) \right|, \]

where

\[ F_v(\lambda) = \frac{(v_1 + \lambda) \ldots (v_r + \lambda)}{(v_{r+1} + \lambda) \ldots (v_{2r} + \lambda)}. \]

If \( \alpha = 1 \) one may partition summation over \( v \) into suitable sets and appeal to the Weil bound

\[ \sum_{\lambda=1}^{p^\alpha} \chi(F_v(\lambda)) \ll p^{1/2}, \]

to get (4). Combining these ideas with the Chinese remainder theorem and the argument of Burgess gives Theorem 1 for squarefree modulus. When \( \alpha > 1 \), considering the sums

\[ \sum_{\lambda=1}^{p^\alpha} \chi(F_v(\lambda)), \]

one may partition \( \lambda \) into residue classes mod \( p^{\alpha/2} \) with the result of transforming into summation over additive characters mod \( p^{\alpha/2} \), see [27, Chapter 12] for some general results related to this technique. This reduces estimating (7) to counting the number of solutions to the congruence

\[ F'_v(\lambda) \equiv 0 \mod p^{\alpha/2}, \quad 0 \leq \lambda < p^{\alpha/2}. \]

If \( \alpha = 2 \) then this is a polynomial congruence mod \( p \) for which there are \( O(1) \) solutions and allows the extension of (4) to cubefree modulus. For arbitrary \( \alpha \) we note that if \( r = 2 \) then (8) is a quadratic congruence whose number of solutions may be estimated via calculations with the discriminant and gives (4) for \( r = 2 \) and arbitrary modulus. The case of \( r = 3 \) is much more difficult and was achieved by Burgess [11, 12] more than 20 years after the \( r = 2 \) case. Since the work of Burgess there has been little progress on extending the estimate (4) apart from some isolated values of \( r \) and \( \alpha \), see [13, 14, 15]. These approaches are based on interpreting the average number of singular solutions to (8) as systems of congruences modulo divisors of \( p^{\alpha/2} \) which are dealt with via a successive elimination of variables and is not clear how to generalize to larger values of \( r \) and \( \alpha \). In this paper we introduce an approach which allows a systematic study of the mean values (3) for arbitrary integers \( q, r \) and in particular give the first nontrivial estimate of the
moments (3) in the cubefull aspect for any \( r \geq 4 \).

Our first step is to take advantage of summation over \( v \) to reduce estimating (8) to counting solutions to congruences with Kloosterman fractions. For integers \( q, \lambda, V, r \) we let \( K_{r,q}(\lambda, V) \) count the number of solutions to the congruence

\[
\frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_r} \equiv \frac{1}{\lambda + v_{r+1}} + \cdots + \frac{1}{\lambda + v_{2r}} \mod q,
\]

with variables satisfying

\[ |v_1|, \ldots, |v_{2r}| \leq V, \]

and note the reduction to \( K_{r,q}(\lambda, V) \) can be seen by using (5), (7), (8) and interchanging summation. We carry out the details of this in Section 3. The problem of estimating \( K_{r,q}(\lambda, V) \) first appears to be considered by Heath-Brown [23] in the case \( r = 2, \lambda = 0 \) who obtained the estimate

\[
K_{2,q}(0, V) \ll \left( \frac{V^{7/2}}{q^{1/2}} + V^2 \right) q^{o(1)}.
\]

The case of \( r \geq 3 \) and \( \lambda = 0 \) was considered by Karatsuba [28] who obtained sharp estimates with restricted ranges of the parameter \( V \). Bourgain and Garaev [2, 3] used the Geometry of numbers to remove some restrictions in Karatsuba’s estimate to obtain

\[
K_{r,q}(0, V) \ll \left( \frac{V^{3r-1}}{q} + V^r \right) q^{o(1)}.
\]

We note that both (9) and (10) fall short of the expected bound

\[
K_{r,q}(\lambda, V) \ll \left( \frac{V^{2r}}{q} + V^r \right) q^{o(1)}.
\]

The case of arbitrary \( \lambda \) is much less understood. Bourgain and Garaev [2] have shown for \( q \) prime that

\[
K_{r,q}(\lambda, V) \ll \left( \frac{V^{2r}}{q^{1/4r}} + V^r \right) q^{o(1)}.
\]

The argument of Bourgain and Garaev does not directly apply to composite modulus and builds on a strategy of Bourgain, Garaev, Konyagin and Shparlinski [5] who in a series of papers [4, 5, 6] obtain some estimates and applications for counting the number of solutions to the congruence

\[
(v_1 + \lambda) \cdots (v_r + \lambda) \equiv (v_{r+1} + \lambda) \cdots (v_{2r} + \lambda) \mod q,
\]
with variables satisfying
\begin{equation}
1 \leq v_1, \ldots, v_{2r} \leq V.
\end{equation}

We give a brief overview of the strategy of Bourgain, Garaev, Konyagin and Shparlinski \cite{5} and indicate the ideas required to extend from prime to arbitrary modulus, the details of which are given in Section 5.

Considering solutions to the congruence (13), after removing diagonal terms we are left to consider solutions such that the polynomial

\[ P_v(X) = \prod_{i=1}^{r}(X + v_i) - \prod_{i=r+1}^{2r}(X + v_i), \]

is not constant. Since \( P_v(\lambda) \equiv 0 \mod q \), each solution gives us a point of the lattice

\[ \mathcal{L} = \{(x_0, \ldots, x_{2r-1}) \in \mathbb{Z}^{2r-1} : x_0 + x_1 \lambda + \cdots + x_{2r-1} \lambda^{2r-1} \equiv 0 \mod q\}, \]

and hence a large number of solutions allows us to construct a small lattice point. From this we obtain a polynomial \( Q \) with small coefficients and \( Q(\lambda) \equiv 0 \mod q \). Since \( q \) is prime and each \( P_v \) and \( Q \) have a common root over \( \mathbb{F}_q \), their resultant must vanish

\[ \text{Res}(Q, P_v) \equiv 0 \mod q. \]

If \( V \) is sufficiently small then \( \text{Res}(Q, P_v) < q \) and hence

\[ \text{Res}(Q, P_v) = 0. \]

This implies that for some root \( \sigma \) of \( Q \)

\[ (v_1 + \sigma) \cdots (v_r + \sigma) = (v_{r+1} + \sigma) \cdots (v_{2r} + \sigma), \]

and reduces the problem to counting divisors in some ring of algebraic integers. The same strategy was applied by Bourgain and Garaev \cite{2} to \( K_{r,q}(\lambda, V) \) who required an estimate for the number of solutions to the equation

\[ \frac{1}{v_1 + \sigma} + \cdots + \frac{1}{v_r + \sigma} = \frac{1}{v_{r+1} + \sigma} + \cdots + \frac{1}{v_{2r} + \sigma}, \]

and were able to detect square root cancellation, see \cite[Lemma 6]{2}. The main obstacle in extending this argument to composite modulus is the fact that over a field the resultant of two polynomials vanishes if and only if they have a common root and may not be true for residue rings. We get around this issue by showing some calculations with the resultant also hold for residue rings provided our root is coprime to the modulus then use the fact that any short interval \( I \) contains an integer coprime to \( q \). This allows for a reduction of estimating \( K_{r,q}(\lambda, V) \) to
the case \((\lambda, q) = 1\).

The main obstacle preventing further progress through this method is obtaining a sharp bound for \(K_{r,q}(\lambda, V)\) uniformly over \(q, \lambda, V\) and note the conjectured estimate (11) implies (4) for any integer \(r\) provided \(q\) is a prime power. For the case of arbitrary \(q\) one would need an estimate of the strength (11) when the variables run through intervals of differing side length owing to the use of the Chinese remainder theorem which interferes with lengths of summation when performing the reduction to \(K_{r,q}(\lambda, V)\). One may always apply Hölder’s inequality to reduce to equal side lengths although this is not sufficient for applications to a sharp bound as it loses information about domination of terms \(V^{2r}/q\) and \(V^r\).

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2. Main results

**Theorem 2.** Let \(q\) be an integer with decomposition

\[
q = q_1 s c,
\]

with \(q_1\) squarefree, \(s\) a square with \(s^{1/2}\) squarefree and \(c\) cubefull. For any primitive character \(\chi \mod q\) and integer \(V\) we have

\[
\sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll qV^r + q^{1/2-o(1)}s^{1/4}c^{1/2}V^{r+1/2} + q^{1/2-o(1)}c^{1/2-1/16r(r-1)}V^{2r}.
\]

The estimate of Theorem 2 may be stated in the following less precise form.

**Corollary 3.** Let \(q\) be an integer with cubefull part \(c\). For any primitive character \(\chi \mod q\) and integer \(V\) we have

\[
\sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll qV^r + q^{3/4}c^{1/4}V^{r+1/2}q^{o(1)} + q^{1/2}c^{1/2-1/16r(r-1)}V^{2r}q^{o(1)}.
\]

In applications one usually takes \(V \sim q^{1/2r}\) and in this range the term \(q^{3/4}c^{1/2}V^{r+1/2}\) can be ignored, provided \(c\) is suitably small. Corollary 3...
should be compared with the estimate

\[ \sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll qV^r + q^{1/2}c^{1/2}V^{2r}q^{o(1)}, \]

obtained from the argument of Burgess and treating summation over cubefull terms trivially.

Using Corollary 3 and well known techniques we deduce the following character sum estimate.

**Theorem 4.** Let \( q \) be an integer with cubefull part \( c \). For any primitive character \( \chi \mod q \) and integers \( M, N \) we have

\[ \sum_{M \leq n \leq M+N} \chi(n) \ll N^{1-1/r}q^{(r+1)/4r^2+o(1)}c^{(r-1)/4r^2-1/32r^3} . \]

Comparing the estimate of Theorem 4 with previous results, we note that Norton \cite[Theorem 1.6]{norton} has obtained

\[ \sum_{M \leq n \leq M+N} \chi(n) \ll c^{3/4r}N^{1-1/r}q^{(r+1)/4r^2+o(1)} , \]

and hence our bound is sharper in the \( c \) aspect. We note that the estimate of Norton also contains a factor involving the order of \( \chi \) although this is redundant in our setting from the assumption \( \chi \) is primitive.

**3. Reduction to equations with Kloosterman fractions**

The main result of this section is a reduction of mean values of character sums to counting solutions to congruences with Kloosterman fractions. Given integers \( q, \lambda, V, r \) we recall that \( K_{r,q}(\lambda, V) \) counts the number of solutions to the congruence

\[ \frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_r} \equiv \frac{1}{\lambda + v_{r+1}} + \cdots + \frac{1}{\lambda + v_{2r}} \mod q, \]

with variables satifying

\[ |v_1|, \ldots, |v_{2r}| \leq V. \]

**Lemma 5.** Let \( q \) be an integer with factorization

\[ q = q_1 \prod_{k \in K} p_k^{2} \prod_{i \in I} p_i^{2\alpha_i} \prod_{j \in J} p_j^{2\beta_j+1} , \]

with \( q_1 \) squarefree, \( K, I, J \) disjoint sets of integers and \( \alpha_i \geq 2, \beta_j \geq 1 \), and define \( q_2, \ldots, q_5 \) by

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where we let $p_i$ denote the $i$-th prime. For any primitive character $\chi \mod q$ and integer $V \leq q$ we have

$$\sum_{\lambda = 1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll q^{r/2 + o(1)} q^{3/2} q V$$

$$\times \sum_{d | q_5 \atop t_2 \ldots t_{2r} | q_1 \atop s_2 \ldots s_{2r} | q_2 \atop t_j s_j \ll V} (t_2 \ldots t_{2r})^{1/2} d^{1/2} s_2 \ldots s_{2r} \max_{\lambda} K_{r,q_3 q_4 d} (\lambda, V/t_j s_j)^{1/2r}.$$  

We adopt the following notation throughout this section. Given a $2r$-tuple of integers $v = (v_1, \ldots, v_{2r})$ we define the polynomials

$$(18) \quad f_{v_1}(x) = \prod_{j=1}^{r} (x - v_j), \quad f_{v_2}(x) = \prod_{j=1}^{r} (x - v_{j+r}).$$

Given an integer $q$ we let $N_v(q)$ count the number of solutions to the congruence

$$(19) \quad f_{v_1}'(\lambda) f_{v_2}(\lambda) - f_{v_1}(\lambda) f_{v_2}'(\lambda) \equiv 0 \mod q,$$

with variable $\lambda$ satisfying

$$(20) \quad (f_{v_1}(\lambda) f_{v_2}(\lambda), q) = 1, \quad 0 \leq \lambda < q.$$

The following is a direct application of the Chinese remainder theorem.

**Lemma 6.** For $q_1$ and $q_2$ coprime we have

$$N_v(q_1) N_v(q_2) \leq N_v(q_1 q_2).$$

We recall some results of Burgess. The following is [8, Lemma 2].

**Lemma 7.** Let $p$ be prime, $\alpha$ an integer and $\chi$ a primitive character mod $p^{2\alpha}$. We have

$$\left| \sum_{\lambda=1}^{p^{2\alpha}} \chi(f_{v_1}(\lambda)) \chi(f_{v_2}(\lambda)) \right| \leq p^\alpha N_v(p^\alpha).$$

The following is [8, Lemma 3].
Lemma 8. Let $\alpha$ be an integer and $\chi$ a primitive character mod $2^{2\alpha+1}$. We have
\[
\left| \sum_{\lambda=1}^{2^{2\alpha+1}} \chi(f_{v_1}(\lambda))\overline{\chi}(f_{v_2}(\lambda)) \right| \leq 2^{\alpha+1}N_v(2^\alpha).
\]

The following is [8, Lemma 4].

Lemma 9. Let $p$ be prime, $\alpha \geq 1$ an integer and $\chi$ a primitive character mod $p^{2\alpha+1}$. We have
\[
\left| \sum_{\lambda=1}^{p^{2\alpha+1}} \chi(f_{v_1}(\lambda))\overline{\chi}(f_{v_2}(\lambda)) \right| \leq p^{\alpha+1/2}N_v(p^\alpha) + p^\alpha N_v(p^{\alpha+1}).
\]

The following is [8, Lemma 7] and is based on the Weil bound and Chinese remainder theorem.

Lemma 10. Let $q$ be squarefree and $\chi$ a primitive character mod $q$. Let
\[
v = (v_1, \ldots, v_{2r}),
\]
be such that
\[
|\{v_1, \ldots, v_{2r}\}| \geq r + 1.
\]
For integer $j$ define
\[
A_j(v) = \prod_{\substack{i=1 \atop i \neq j}}^{2r}(v_j - v_i).
\]
There exists some $j$ with $A_j(v) \neq 0$ such that
\[
\left| \sum_{\lambda=1}^{q} \chi(f_{v_1}(\lambda))\overline{\chi}(f_{v_2}(\lambda)) \right| \leq (4r)^{\tau(q)}q^{1/2}(A_j(v), q)^{1/2}.
\]

The following is [10, Lemma 7].

Lemma 11. Let $p$ be prime and suppose that $v = (v_1, \ldots, v_{2r})$ satisfies $A_j(v) \neq 0$ for some $j$. Then we have
\[
N_v(p) \ll (A_j(v), p).
\]

Lemma 12. Let $q$ be an integer with factorization
\[
q = q_1 \prod_{k \in \mathcal{K}} p_k^{2} \prod_{i \in \mathcal{I}} p_i^{2\alpha_i} \prod_{j \in \mathcal{J}} p_j^{2\beta_j+1},
\]
with \( q_1 \) squarefree, \( \mathcal{K}, \mathcal{I}, \mathcal{J} \) disjoint sets of integers and \( \alpha_i \geq 2, \beta_j \geq 1 \), and define

\[
q_2 = \prod_{k \in \mathcal{K}} p_k, \quad q_3 = \prod_{i \in \mathcal{I}} p_i^{\alpha_i}, \quad q_4 = \prod_{j \in \mathcal{J}} p_j^{\beta_j}, \quad q_5 = \prod_{j \in \mathcal{J}} p_j.
\]

For any primitive character \( \chi \mod q \) and integer \( V \) we have

\[
\sum_{\lambda = 1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll q^{1/2 + o(1)} \sum_{d \mid q_2} \frac{1}{d^{1/2}} \sum_{v \in V_1} (A_1(v), q_1)^{1/2}(A_1(v), q_2) N_v(q_3 q_4 d).
\]

Proof. Let

\[
S = \sum_{\lambda = 1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r}.
\]

Expanding the \( 2r \)-th power and interchanging summation, we have

\[
S \leq \sum_{1 \leq v_1, \ldots, v_{2r} \leq V} \left| \sum_{\lambda = 1}^{q} \chi \left( \frac{(\lambda + v_1) \ldots (\lambda + v_r)}{(\lambda + v_{r+1}) \ldots (\lambda + v_{2r})} \right) \right|,
\]

and with notation as above, this simplifies to

\[
S \leq \sum_{1 \leq v \leq V} \left| \sum_{\lambda = 1}^{q} \chi(f_{v_1}(\lambda)) \overline{\chi(f_{v_2}(\lambda))} \right|.
\]

By the Chinese remainder theorem we may factorize

\[
\chi = \chi_1 \prod_{k \in \mathcal{K}} \chi_k \prod_{i \in \mathcal{I}} \chi_i \prod_{j \in \mathcal{J}} \chi_j,
\]

where \( \chi_1 \) is a primitive character mod \( q_1 \), \( \chi_k \) is a primitive character mod \( p_k^2 \), \( \chi_i \) is a primitive character mod \( p_i^{2\alpha_i} \) and \( \chi_j \) is a primitive character mod \( p_j^{2\beta_j + 1} \). A second application of the Chinese remainder theorem to summation over \( \lambda \) gives the decomposition

\[
S \leq \sum_{1 \leq v \leq V} \sigma_1(v) \prod_{k \in \mathcal{K}} \sigma_k(v) \prod_{i \in \mathcal{I}} \sigma_i(v) \prod_{j \in \mathcal{J}} \sigma_j(v),
\]
where

\[
\sigma_1(v) = \sum_{\lambda=1}^{q_1} \chi_1(f_{v_1}(\lambda)) \overline{\chi_1}(f_{v_2}(\lambda)),
\]

\[
\sigma_k(v) = \sum_{\lambda=1}^{p_k^2} \chi_k(f_{v_1}(\lambda)) \overline{\chi_k}(f_{v_2}(\lambda)),
\]

\[
\sigma_i(v) = \sum_{\lambda=1}^{p_i^{2\alpha_i}} \chi_i(f_{v_1}(\lambda)) \overline{\chi_i}(f_{v_2}(\lambda)),
\]

\[
\sigma_j(v) = \sum_{\lambda=1}^{p_j^{2\beta_j+1}} \chi_j(f_{v_1}(\lambda)) \overline{\chi_j}(f_{v_2}(\lambda)).
\]

We partition the outer summation over \( v \) into sets

\[
V_\ell = \{ 1 \leq v \leq V : |\{v_1, \ldots, v_{2r}\}| \geq r + 1, \ A_\ell(v) \neq 0 \}, \quad 1 \leq \ell \leq 2r,
\]

\[
V' = \{ 1 \leq v \leq V : |\{v_1, \ldots, v_{2r}\}| \leq r \},
\]

and note \(|V'| \ll V^r\). Using that

\[
\{ (v_1, \ldots, v_{2r}) : 1 \leq v_i \leq V \} \subseteq \bigcup_{\ell=1}^{2r} V_\ell \cup V',
\]

and estimating terms \( \sigma \) for \( v \in V_2 \) trivially gives

\[
S \ll qV^r + \sum_{\ell=1}^{2r} \sum_{v \in V_\ell} \sigma_1(v) \prod_{k \in K} \sigma_k(v) \prod_{i \in I} \sigma_i(v) \prod_{j \in J} \sigma_j(v) \ll qV^r + S_1,
\]

where

\[
S_1 = \sum_{v \in V_1} \sigma_1(v) \prod_{k \in K} \sigma_k(v) \prod_{i \in I} \sigma_i(v) \prod_{j \in J} \sigma_j(v),
\]

and we have used symmetry to estimate

\[ S_\ell \ll S_1. \]

For \( v \in V_1, k \in K, i \in I \) and \( j \in J \), by Lemmas 7, 8, 9, 10 and 11

\[
\sigma_1(v) \ll q_1^{1/2+o(1)}(A_1(v), q_1)^{1/2},
\]

\[
\sigma_k(v) \ll p_k N_v(p_k) \ll p_k(A_1(v), p_k),
\]

\[
\sigma_i(v) \ll p_i^{\alpha_i} N_v(p_i^{\alpha_i}),
\]

\[
\sigma_j(v) \ll p_j^{\beta_j+1/2} N_v(p_j^{\beta_j}) + p_j^{\beta_j} N_v(p_j^{\beta_j+1}).
\]
and hence
\[ \sigma_1(v) \prod_{k \in K} \sigma_k(v) \prod_{i \in I} \sigma_i(v) \prod_{j \in J} \sigma_j(v) \ll \]
\[ q^{1/2+o(1)}(A_1(v), q_1)^{1/2}(A_1(v), q_2) \prod_{i \in I} N_v(p_i^{\alpha_i}) \prod_{j \in J} N_v(p_j^{\beta_j}) \left( N_v(p_j^{\beta_j+1}) + \frac{N_v(p_j^{\beta_j+1})}{p_j^{1/2}} \right). \]

Recalling (21) and using Lemma 6, we see that
\[ \prod_{i \in I} N_v(p_i^{\alpha_i}) \prod_{j \in J} \left( N_v(p_j^{\beta_j}) + \frac{N_v(p_j^{\beta_j+1})}{p_j^{1/2}} \right) \leq N_v(q_3) \sum_{d \mid q_5} \frac{N_v(q_3d)}{d^{1/2}}, \]

which implies
\[ \sigma_1(v) \prod_{k \in K} \sigma_k(v) \prod_{i \in I} \sigma_i(v) \prod_{j \in J} \sigma_j(v) \ll q^{1/2+o(1)}(A_1(v), q_1)^{1/2}(A_1(v), q_2) \sum_{d \mid q_5} \frac{N_v(q_3q_4d)}{d^{1/2}}. \]

Substituting the above into (23) we get
\[ S_1 \ll q^{1/2+o(1)} \sum_{d \mid q_5} \frac{1}{d^{1/2}} \sum_{v \in V_1} (A_1(v), q_1)^{1/2}(A_1(v), q_2) N_v(q_3q_4d), \]

and the result follows from (22).

4. PROOF OF LEMMA 5

By Lemma 12 we have
\[ \sum_{\lambda=1}^q \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll qV^r + q^{1/2+o(1)} \sum_{d \mid q_5} \frac{1}{d^{1/2}} S_d, \]

where
\[ S_d = \sum_{v \in V_1} (A_1(v), q_1)^{1/2}(A_1(v), q_2) N_v(q_3q_4d). \]

Fix some $d \mid q_5$ and consider $S_d$. Recalling that
\[ A_1(v) = \prod_{i \neq 1} (v_1 - v_i), \]
we partition summation over \( v \) into sets depending on the values of \((A_1(v), q_1)\) and \((A_1(v), q_2)\). For \( d_1 | q_1 \) and \( d_2 | q_2 \) we define

\[
\mathcal{V}_1(d_1, d_2) = \{ v \in \mathcal{V}_1 : (A_1(v), q_1) = d_1, \ (A_1(v), q_2) = d_2 \},
\]

so that

\[
(25) \quad S_d = \sum_{d_1 | q_1, d_2 | q_2} d_1^{1/2} d_2 S_d(d_1, d_2),
\]

where

\[
(26) \quad S_d(d_1, d_2) = \sum_{v \in \mathcal{V}_1(d_1, d_2)} N_v(q_3 q_4 d).
\]

Since \( N_v \) is defined by (19) and (20), we may write

\[
N_v(q_3 q_4 d) = \sum_{\lambda = 0}^{q_3 q_4 d - 1} 1,
\]

where (*) denotes summation with conditions

\[
(27) \quad f'_{v_1}(\lambda) f_{v_2}(\lambda) - f_{v_1}(\lambda) f'_{v_2}(\lambda) \equiv 0 \mod q_3 q_4 d, \quad (f_{v_1}(\lambda) f_{v_2}(\lambda), q_3 q_4 d) = 1.
\]

Substituting into (26) and rearranging summation gives

\[
S_d(d_1, d_2) = \sum_{\lambda = 0}^{q_3 q_4 d - 1} \sum_{v \in \mathcal{V}_1(d_1, d_2)} 1.
\]

Recalling (18), the conditions (27) imply that

\[
\frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_r} \equiv \frac{1}{\lambda + v_{r+1}} + \cdots + \frac{1}{\lambda + v_{2r}} \mod q_3 q_4 d,
\]

hence defining

\[
K_{r, q_3 q_4 d}(\lambda, V, d_1, d_2),
\]

to count the number of solutions to the congruence

\[
\frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_r} \equiv \frac{1}{\lambda + v_{r+1}} + \cdots + \frac{1}{\lambda + v_{2r}} \mod q_3 q_4 d,
\]

with variables satisfying

\[
1 \leq v_1, \ldots, v_{2r} \leq V, \quad (A_1(v), q_1) = d_1, \ (A_1(v), q_2) = d_2,
\]
we have

\[
S_d(d_1, d_2) \leq \sum_{\lambda=0}^{q_3q_4d-1} K_{r,q_3q_4d}(\lambda, V, d_1, d_2).
\]

(28)

Our next step is to estimate \( K_{r,q_3q_4d}(\lambda, V, d_1, d_2) \) in terms of \( K_{r,q_3q_4d}(\lambda, V) \). If \((A_1(v), q_1) = d_1\) and \((A_1(v), q_2) = d_2\), then since both \( q_1 \) and \( q_2 \) are squarefree, there exists a decomposition

\[
d_1 = t_2 \ldots t_{2r}, \quad d_2 = s_2 \ldots s_{2r}, \quad (t_i, t_j) = 1, \quad (s_i, s_j) = 1, \quad i \neq j,
\]

such that

\[
v_j \equiv v_1 \mod t_j, \quad v_j \equiv v_1 \mod s_j,
\]

and since \((q_1, q_2) = 1\) this implies that

\[
v_j \equiv v_1 \mod t_j s_j,
\]

and note that in order for \( A_1(v) \neq 0 \) we must have \( t_j s_j \ll V \). With \( s_2, \ldots, t_{2r} \) as above, let \( K_{r,q_3q_4d}(\lambda, V, s_2, \ldots, t_{2r}) \) count the number of solutions to the congruence

\[
\frac{1}{\lambda + v_1} + \sum_{j=2}^{r} \frac{1}{\lambda + v_1 + u_j t_j s_j} \equiv \sum_{j=r+1}^{2r} \frac{1}{\lambda + v_1 + u_j t_j s_j} \mod q_3q_4d,
\]

with variables satisfying

\[
1 \leq v_1 \leq V, \quad |u_j| \leq \frac{V}{s_j t_j},
\]

so that

\[
(29) \quad K_{r,q_3q_4d}(\lambda, V) \ll \sum_{\substack{t_2 \ldots t_{2r} = d_1 \\ s_2 \ldots s_{2r} = d_2 \\ t_j s_j \ll V}} K_{r,q_3q_4d}(\lambda, V, s_1, \ldots, t_{2r}).
\]

Fix some \( s_2, \ldots, t_{2r} \) and consider \( K_{r,q_3q_4d}(\lambda, V, s_2, \ldots, t_{2r}) \). Estimating the contribution from \( v_1 \) trivially, we see that there exists some \( \lambda^* \) such that \( K_{r,q_3q_4d}(\lambda, V, d_1, d_2) \) is bounded by \( O(V) \) times the number of solutions to the congruence

\[
\frac{1}{\lambda^*} + \sum_{j=2}^{r} \frac{1}{\lambda^* + u_j t_j s_j} \equiv \sum_{j=r+1}^{2r} \frac{1}{\lambda^* + u_j t_j s_j} \mod q_3q_4d,
\]
with variables satisfying $|u_j| \leq V/s_j t_j$. Detecting via additive characters and using Hölder’s inequality, we get

$$K_{r,q_3 q_4 d}(\lambda, V, s_2, \ldots, t_{2r}) \ll \frac{V}{q_3 q_4 d} \prod_{y=1}^{2r} \left| \sum_{j=1}^{\frac{q_3 q_4 d}{V}} e_{q_3 q_4 d}(y(\lambda^* + t_j s_j u_j)^{-1}) \right|$$

$$\ll V \prod_{j=2}^{2r} \left( \frac{1}{q_3 q_4 d} \sum_{y=1}^{\frac{q_3 q_4 d}{V}} \left| \sum_{|u_j| \leq V/s_j t_j} e_{q_3 q_4 d}(y(\lambda^* + t_j s_j u_j)^{-1}) \right| \right)^{1/2r}.$$ 

Hence with $K_{r,q}(\lambda, V)$ defined as in (15) and (16) we have

$$K_{r,q_3 q_4 d}(\lambda, V, s_2, \ldots, t_{2r}) \ll V \prod_{j=2}^{2r} \max_{\lambda} K_{r,q_3 q_4 d}(\lambda, V/t_j s_j)^{1/2r}.$$ 

Substituting the above into (29) gives

$$K_{r,q_3 q_4 d}(\lambda, V) \ll V \sum_{t_2 \ldots t_{2r} = d_1}^{2r} \prod_{j=2}^{2r} \max_{\lambda} K_{r,q_3 q_4 d}(\lambda, V/t_j s_j)^{1/2r},$$

and hence by (28)

$$S_d(d_1, d_2) \ll V q_3 q_4 d \sum_{t_2 \ldots t_{2r} = d_1}^{2r} \prod_{j=2}^{2r} \max_{\lambda} K_{r,q_3 q_4 d}(\lambda, V/t_j s_j)^{1/2r}.$$ 

Combining the above with (24) and (25) gives

$$\sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll q V^r + q^{1/2+\omega(1)} q_3 q_4 V \sum_{d_1 \mid q_3} d_1^{1/2} \sum_{d_2 \mid q_4} d_2^{1/2} \sum_{t_2 \ldots t_{2r} = d_1}^{2r} \prod_{j=2}^{2r} \max_{\lambda} K_{r,q_3 q_4 d}(\lambda, V/t_j s_j)^{1/2r},$$

and the result follows after rearranging summation.

### 5. Equations with Kloosterman fractions

In this section we estimate $K_{r,q}(\lambda, V)$ for arbitrary integer $q$.

**Lemma 13.** Let $K_{r,q}(\lambda, V)$ be defined by (15) and (16). For any integer $q$, if

$$V \ll q^{1/4k(k-1)},$$

the result follows.
then we have

\[ K_{r,q}(\lambda, V) \ll V^r q^{o(1)}. \]

**Corollary 14.** Let \( K_{r,q}(\lambda, V) \) be defined by (15) and (16). For arbitrary integers \( q \) and \( V \) we have

\[ K_{r,q}(\lambda, V) \ll \left( \frac{V^{2r}}{q^{1/4(r-1)}} + V^r \right) q^{o(1)}. \]

We first recall some basics of linear algebra. Given an \( n \times n \) matrix

\[
A = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{bmatrix},
\]

let \( A_{i,j} \) denote the matrix obtained by deleting the \( i \)-th row and \( j \)-th column from \( A \) and define the adjoint of \( A \), \( \text{adj}(A) \) to be the matrix with \((i, j)\)-th entry \((-1)^{i+j} \det(A_{j,i})\). Then we have

\[ A \times \text{adj}(A) = \text{adj}(A) \times A = \det(A) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \]

Given two polynomials \( f, g \in \mathbb{Z}[X] \) with coefficients

\[ f(X) = a_n X^n + \cdots + a_0, \quad g(X) = b_m X^m + \cdots + b_0, \]

we define the Sylvester matrix \( S(f, g) \) of \( f \) and \( g \) to be \((m+n) \times (m+n)\) matrix

\[
S(f, g) = \begin{bmatrix}
a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 & 0 \\
0 & a_n & a_{n-1} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\
0 & 0 & 0 & \cdots & a_2 & a_1 & a_0 \\
b_m & b_{m-1} & b_{m-2} & \cdots & 0 & 0 & 0 \\
0 & b_m & b_{m-1} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & b_1 & b_0 & 0 \\
0 & 0 & 0 & \cdots & b_2 & b_1 & b_0
\end{bmatrix},
\]

and define the resultant of \( f \) and \( g \) by

\[ \text{Res}(f, g) = \det(S(f, g)). \]
We recall that \( \text{Res}(f, g) = 0 \) if and only if \( f \) and \( g \) have a common root over \( \mathbb{C} \). The following result will be needed to extend the techniques of [2, 5] from prime to composite modulus.

**Lemma 15.** Let \( q \) and \( \lambda \) be integers with \( (\lambda, q) = 1 \). Suppose \( f, g \in \mathbb{Z}[X] \) are polynomials satisfying

\[
(33) \quad f(\lambda) \equiv g(\lambda) \equiv 0 \mod q.
\]

Then we have

\[
\text{Res}(f, g) \equiv 0 \mod q.
\]

**Proof.** We may suppose \( \text{Res}(f, g) \neq 0 \) as otherwise the result is immediate. Let \( f \) and \( g \) have coefficients given by (31) and define

\[
\tilde{\lambda} = \begin{bmatrix} 1 & \lambda & \ldots & \lambda^{m+n} \end{bmatrix}.
\]

The condition \( (\lambda, q) = 1 \) and (33) imply that

\[
S(f, g)\tilde{\lambda} \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \mod q,
\]

and hence by (30) and (32)

\[
\text{Res}(f, g) \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \tilde{\lambda} \equiv 0 \mod q,
\]

which implies \( \text{Res}(f, g) \equiv 0 \mod q. \)

We will require the following resultant estimate of Bourgain, Garaev, Konyagin and Shparlinski [5, Corollary 3].

**Lemma 16.** Let \( P_1(X) \) and \( P_2(X) \) be nonconstant polynomials

\[
P_1(X) = \sum_{i=0}^{M-1} a_i X^{M-1-i}, \quad P_2(X) = \sum_{i=0}^{N-1} b_i X^{N-1-i},
\]

such that

\[
|a_i| < H^{i+\sigma}, \quad |b_i| < H^{i+\theta}.
\]

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Then we have
\[ \text{Res}(P_1, P_2) \ll H^{(M-1+\sigma)(N-1+\theta)-\theta\sigma}. \]

The following is due to Bourgain and Garaev [2, Lemma 6].

**Lemma 17.** For any fixed positive integer \( r \) and all values of \( \sigma \in \mathbb{C} \) the number of solutions to the equation
\[
\frac{1}{\sigma + v_1} + \cdots + \frac{1}{\sigma + v_r} = \frac{1}{\sigma + v_{r+1}} + \cdots + \frac{1}{\sigma + v_{2r}},
\]
with variables satisfying
\[ |x_1|, \ldots, |x_{2r}| \leq V, \]
is bounded by \( V^{r+o(1)} \).

The following is a well known consequence of the sieve of Eratosthenes.

**Lemma 18.** For any integers \( M, N \) and \( q \) we have
\[
\sum_{\substack{M < n \leq M+N \\ (n, q) = 1}} 1 = \frac{\phi(q)}{q} N + O(2^{o(q)}).
\]

The following is a consequence of Lemma 18 and standard estimates for arithmetic functions.

**Corollary 19.** Let \( \varepsilon > 0 \) be an arbitrary positive number and \( q \) an integer. Then any interval \( I \) of length \( |I| \gg q^\varepsilon \) contains an integer coprime to \( q \).

### 6. Proof of Lemma 13

Fix some sufficiently small \( \varepsilon > 0 \) and suppose \( V \gg q^\varepsilon \) as otherwise the result is trivial. By Corollary 19 there exists some \( \lambda^* \) satisfying
\[ |\lambda^* - \lambda| \leq V, \quad (\lambda^*, q) = 1. \]

If \( v_1, \ldots, v_r \) satisfies
\[
\frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_r} \equiv \frac{1}{\lambda + v_{r+1}} + \cdots + \frac{1}{\lambda + v_{2r}} \pmod{q}, \quad |v_i| \leq V,
\]
then
\[
\frac{1}{\lambda^* + u_1} + \cdots + \frac{1}{\lambda^* + u_r} \equiv \frac{1}{\lambda^* + u_{r+1}} + \cdots + \frac{1}{\lambda^* + u_{2r}} \pmod{q},
\]
where
\[ u_i = v_i + (\lambda - \lambda^*), \]
and hence by (34) \(|u_i| \leq 2V\) which implies that
\[ K_{r,q}(\lambda, V) \leq K_{r,q}(\lambda^*, 2V). \]
Hence it is sufficient to show that for any \(\lambda\) satisfying \((\lambda, q) = 1\) and integer \(V\) satisfying
\[ q^r \leq V \ll q^{1/4r(r-1)}, \]
we have
\[ K_{r,q}(\lambda, V) \ll V^{r+o(1)}. \]
We proceed by induction on \(r\) and note that the case \(r = 1\) is trivial. We formulate our induction hypothesis as follows. Let \(k\) be an integer such that for any \(r \leq k - 1\) the estimate (36) holds for any \(V\) satisfying (35). Let \(V\) satisfy
\[ V \ll q^{1/4k(k-1)}, \]
and we aim to show that
\[ K_{k,q}(\lambda, V) \ll V^{k+o(1)}. \]
Let \(K'_{k,q}(\lambda, V)\) count the number of solutions to the congruence
\[ \frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_k} \equiv \frac{1}{\lambda + v_{k+1}} + \cdots + \frac{1}{\lambda + v_{2k}} \mod q, \]
with variables satisfying
\[ |v_i| \leq V, \quad |\{v_1, \ldots, v_{2k}\}| = 2k, \]
and let \(K''_{k,q}(\lambda, V)\) count the number of solutions to the congruence (39) with variables satisfying
\[ |v_i| \leq V, \quad |\{v_1, \ldots, v_{2k}\}| < 2k, \]
so that
\[ K_k(\lambda, V) \leq K'_{k}(\lambda, V) + K''_{k}(\lambda, V). \]
Considering \(K''\), if \((v_1, \ldots, v_{2k})\) satisfy (39) and (41) then \(v_i = v_j\) for some \(i \neq j\) and hence
\[ K''_{k}(\lambda, V) \leq \sum_{1 \leq i < j \leq 2k} K_{i,j}(\lambda, V) \ll K_{i,j}(\lambda, V), \]
for some pair \(i < j\), where \(K_{i,j}(\lambda, V)\) counts the number of solutions to the congruence (39) with variables satisfying (41) and \(v_i = v_j\). Fixing \(v_i\) with \(O(V)\) choices, we see that there exists some sequence
\[ \varepsilon_1, \ldots, \varepsilon_{2k-2} \in \{-1, 1\}, \]
and some integer $b$ such that

\begin{equation}
K_{i,j}(\lambda, V) \ll VK''(\lambda, V),
\end{equation}

where $K''(\lambda, V)$ counts the number of solutions to the congruence

\[
\frac{\varepsilon_1}{\lambda + v_1} + \cdots + \frac{\varepsilon_{2k-2}}{\lambda + v_{2k-2}} \equiv b \mod q,
\]

with variables satisfying $|v_1|, \ldots, |v_{2k-2}| \leq V$. Detecting via additive characters, we have

\[
K''(\lambda, V) = \frac{1}{q} \sum_{y=1}^{q} \prod_{j=1}^{2k-2} \left( \sum_{|v| \leq V} e_q(y\varepsilon_j(\lambda + v)^{-1}) \right) e_q(-yb),
\]

and hence by Hölder’s inequality

\[
K''(\lambda, V) \ll K_{k-1}(\lambda, V).
\]

Hence by (43), (44) and our induction hypothesis

\[
K_k'(\lambda, V) \ll V^{k+o(1)}.
\]

Combining with (42) it is sufficient to show that

\begin{equation}
K_k'(\lambda, V) \ll V^{k+o(1)},
\end{equation}

and hence we may suppose that $K_k'(\lambda, V) \neq 0$. For a 2k-tuple $v = (v_1, \ldots, v_{2k})$ we define the polynomial

\[
P_v(X) = \prod_{i \neq 1} (X + v_i) + \cdots + \prod_{i \neq 2k} (X + v_i) - \prod_{i \neq r+1} (X + v_i) - \cdots - \prod_{i \neq 2k} (X + v_i),
\]

so that $P_v$ has degree at most $2k - 2$. For each $v = (v_1, \ldots, v_{2r})$ satisfying (39) we have

\[
P_v(\lambda) \equiv 0 \mod q,
\]

and the assumption that $|\{v_1, \ldots, v_{2r}\}| = 2r$ implies that

\[
P_v(-v_1) \neq 0.
\]

Since

\[
P_v(-v_1) \ll V^{2r-1} < q,
\]

we see that $P_v(X)$ is not a constant polynomial. Writing

\[
P_v(X) = \sum_{i=0}^{2k-2} a_i Z^{2k-2-i},
\]

the coefficients of $P_v(X)$ satisfy

\begin{equation}
|a_i| \ll V^{i+1}.
\end{equation}
Fixing one point \( v^* = (v_1^*, \ldots, v_{2r}^*) \) counted by \( K'_k(\lambda, V) \), for any other point \( v \) we have
\[
P_{v^*}(\lambda) \equiv P_v(\lambda) \equiv 0 \mod q,
\]
and hence the assumption \((\lambda, q) = 1\) combined with Lemma 15 implies that
\[
(47) \quad \text{Res}(P_{v^*}, P_v) \equiv 0 \mod q.
\]
By (46) and Lemma 16
\[
\text{Res}(P_{v^*}, P_v) \ll V^{4k(k-1)},
\]
and hence by (37) and (47)
\[
\text{Res}(P_{v^*}, P_v) = 0,
\]
so that \( P_{v^*} \) and \( P_v \) have a common root over \( \mathbb{C} \). Let \( \sigma_1, \ldots, \sigma_\ell \) denote the distinct roots of \( P_{v^*} \) over \( \mathbb{C} \). For any \( v = (v_1, \ldots, v_{2k}) \) counted by \( K'_k(\lambda, V) \) we have
\[
P_v(\sigma_j) = 0,
\]
for some \( 1 \leq j \leq \ell \) and note the assumption that the \( v_i \)'s are pairwise distinct implies that \( v_i \neq \sigma_j \) for any \( 1 \leq i \leq 2k \). Hence defining \( J(\sigma) \) to count the number of solutions to the equation
\[
\frac{1}{\sigma + v_1} + \cdots + \frac{1}{\sigma + v_k} = \frac{1}{\sigma + v_{r+1}} + \cdots + \frac{1}{\sigma + v_{2k}},
\]
with variables satisfying \( |v_i| \leq V \) we have
\[
K'_k(\lambda, V) \leq \sum_{j=1}^{\ell} J(\sigma_j),
\]
and hence from Lemma 17
\[
K'_k(\lambda, V) \ll V^{k+o(1)},
\]
which establishes (45) and completes the proof.

7. Proof of Corollary 14

By Lemma 13 we may assume
\[
V \gg q^{1/4r(r-1)}.
\]
We partition the interval \( |v| \leq V \) into disjoint intervals
\[
[-V, V] = \bigcup_{j=1}^{K} I_j, \quad K \ll V/q^{1/4r(r-1)}, \quad |I_j| \ll q^{1/4r(r-1)},
\]
and let $K(I_{j_1}, \ldots, I_{j_{2r}})$ count the number of solutions to the congruence
\[
\frac{1}{\lambda + v_1} + \cdots + \frac{1}{\lambda + v_{2r}} \equiv \frac{1}{\lambda + v_{r+1}} + \cdots + \frac{1}{\lambda + v_{2r}} \mod q,
\]
with variables satisfying $v_i \in I_{j_i}$. By the pigeonhole principle, there exists some tuple $(j_1, \ldots, j_{2r})$ such that
\[
(48) \quad K_r(\lambda, V) \ll \frac{V^{2r}}{q^{1/2(r-1)}} K(I_{j_1}, \ldots, I_{j_{2r}}).
\]
Detecting via additive characters and applying Hölder’s inequality, we have
\[
K(I_{j_1}, \ldots, I_{j_{2r}}) \leq \frac{1}{q} \sum_{y=1}^{q} \prod_{i=1}^{2r} \left| \sum_{v \in I_{j_i}} e_q(y(\lambda + v)^{-1}) \right| ^{2r} 1/2r,
\]
and hence by Lemma 13
\[
K(I_{j_1}, \ldots, I_{j_{2r}}) \ll q^{1/4(r-1) + o(1)}.
\]
Combining with (48) we get
\[
K_r(\lambda, V) \ll \frac{V^{2r} q^{o(1)}}{q^{1/4(r-1)}},
\]
and completes the proof.

8. Proof of Theorem 2

Assuming $q$ has factorization
\[
q = q_1 \prod_{k \in \mathcal{K}} p_k^{2} \prod_{i \in \mathcal{I}} p_i^{2\alpha_i} \prod_{j \in \mathcal{J}} p_j^{2\beta_j+1},
\]
for some sets of disjoint integers $\mathcal{K}, \mathcal{I}, \mathcal{J}$, integers $\alpha_j \geq 2, \beta_j \geq 1$ and $q_1$ squarefree, we have
\[
(49) \quad s = \prod_{k \in \mathcal{K}} p_k^{2}, \quad c = \prod_{i \in \mathcal{I}} p_i^{2\alpha_i} \prod_{j \in \mathcal{J}} p_j^{2\beta_j+1}.
\]
With notation as in Lemma 5
\[
(50) \quad \sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v)^{2r} \right| \ll V q^{r} + q^{1/2 + o(1)} q_3 q_4 V S,
\]
where

\[
S = \sum_{d | q_5, \ t_2 \ldots t_{2r} | q_1, s_2 \ldots s_{2r} | q_2, t_j s_j \ll V} (t_1 \ldots t_{2r})^{1/2} d^{1/2} s_2 \ldots s_{2r} \prod_{j=2}^{2r} \max \lambda K_{r,q_3q_4d}(\lambda, V/t_j s_j)^{1/2r}
\]

\begin{equation}
S = \sum_{d | q_5, \ t_2 \ldots t_{2r} | q_1, s_2 \ldots s_{2r} | q_2, t_j s_j \ll V} \frac{1}{2r} (t_1 \ldots t_{2r})^{1/2} d^{1/2} s_2 \ldots s_{2r} \prod_{j=2}^{2r} \max \lambda K_{r,q_3q_4d}(\lambda, V/t_j s_j)^{1/2r}
\end{equation}

and

\[
S(d, t_2, s_2, \ldots, t_{2r}, s_{2r}) = (t_2 \ldots t_{2r})^{1/2} d^{1/2} s_2 \ldots s_{2r} \prod_{j=2}^{2r} \max \lambda K_{r,q_3q_4d}(\lambda, V/t_j s_j)^{1/2r}.
\]

We recall that \(q_2, \ldots, q_5\) are given by

\[
q_2 = \prod_{k \in K} p_k, \quad q_3 = \prod_{i \in I} p_i^{\alpha_i}, \quad q_4 = \prod_{j \in J} p_j^{\beta_j}, \quad q_5 = \prod_{j \in J} p_j.
\]

Fix some \(d, t_2, \ldots, t_{2r}, s_2, \ldots, s_{2r}\) satisfying

\[
d | q_5, \quad t_2 \ldots t_{2r} | q_1, \quad s_2 \ldots s_{2r} | q_2, \quad t_j s_j \ll V,
\]

and consider \(S(d, t_2, s_2, \ldots, t_{2r}, s_{2r})\). We partition the indices \(\{2, \ldots, 2r\}\) into sets

\[
S_1 = \{2 \leq j \leq 2r : t_j s_j < V/(q_3q_4d)^{1/4r(r-1)}\},
\]

\[
S_2 = \{2 \leq j \leq 2r : V/(q_3q_4d)^{1/4r(r-1)} \leq t_j s_j \ll V\},
\]

and write

\begin{equation}
|S_1| = k_1, \quad |S_2| = k_2, \quad k_1 + k_2 = 2r - 1.
\end{equation}

By Lemma 14, for any \(0 \leq \lambda < q_3q_4d\) we have

\begin{equation}
K_{r,q_3q_4d}(\lambda, V/t_j s_j) \ll \begin{cases}
q^{o(1)}(V/t_j s_j)^{2r} / (q_3q_4d)^{1/4r(r-1)}, & j \in S_1, \\
q^{o(1)}(V/t_j s_j)^r, & j \in S_2,
\end{cases}
\end{equation}
which implies that

\[
S(d, t_2, s_2, \ldots, t_{2r}, s_{2r}) \ll q^{o(1)} d^{1/2} \prod_{j \in S_1} t_j^{1/2} \left( \frac{\lambda}{t_j s_j} \right) \prod_{j \in S_2} t_j^{1/2} \left( \frac{V}{t_j s_j} \right)^{1/2} \\
\ll q^{o(1)} \frac{d^{1/2}}{(q^3 q_4 d)^{k_1/8(r-1)}} \prod_{j \in S_2} s_j^{1/2} \prod_{j \in S_1} \frac{1}{t_j^{1/2}} V^{k_1 + k_2/2} \\
\ll q^{o(1)} \frac{q_5^{1/2}}{(q^3 q_4 q_5)^{k_1/8(r-1)}} \prod_{j \in S_2} s_j^{1/2} V^{r - 1/2},
\]

using that \(d | q_5\). Since each \(s_j \ll V\) and \(s_2 \ldots s_{2r} | q_2\), we have

\[
\prod_{j \in S_2} s_j^{1/2} \ll \min \{ V^{k_2}, q_2 \},
\]

and hence

\[
S(d, t_2, s_2, \ldots, t_{2r}, s_{2r}) \ll q^{o(1)} \frac{q_5^{1/2} V^{2r - 1}}{(q^3 q_4 q_5)^{k_1/8(r-1)}},
\]

and

\[
S(d, t_2, s_2, \ldots, t_{2r}, s_{2r}) \ll q^{o(1)} (q_5 q_2)^{1/2} \left( \frac{V^{k_1/2}}{(q^3 q_4 q_5)^{k_1/8(r-1)}} \right) V^{r - 1/2}.
\]

If \(k_1 = 0\) then we use (55), while if \(k_1 > 0\) then we use (54). This gives

\[
S(d, t_2, s_2, \ldots, t_{2r}, s_{2r}) \ll q^{o(1)} \frac{q_5^{1/2}}{q_2^{1/2} V^{r - 1/2} + \frac{V^{2r - 1}}{(q^3 q_4 q_5)^{1/8(r-1)}}},
\]

and hence from (50), (51) and the estimate \(d(n) = n^{o(1)}\) we get

\[
\sum_{\lambda=1}^q \sum_{1 \leq v < V} \chi(\lambda + v) 2r \ll q V^r \\
+ q^{1/2+o(1)} q_2^{1/2} q_3 q_4 q_5^{1/2} V^{r+1/2} + q^{1/2+o(1)} (q_3 q_4 q_5^{1/2})^{1-1/8(r-1)} V^{2r},
\]

and the result follows since

\[
q_3 q_4 q_5^{1/2} = c^{1/2}, \quad q_2 = s^{1/4}.
\]

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School of Mathematical Sciences, The University of New South Wales Canberra, Australia

E-mail address: b.kerr@adfa.edu.au