ON THE APPROXIMATION OF VORTICITY FRONTS BY THE BURGERS-HILBERT EQUATION

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Abstract. This paper proves that the motion of small-slope vorticity fronts in the two-dimensional incompressible Euler equations is approximated on cubically nonlinear timescales by a Burgers-Hilbert equation derived by Biello and Hunter (2010) using formal asymptotic expansions. The proof uses a modified energy method to show that the contour dynamics equations for vorticity fronts in the Euler equations and the Burgers-Hilbert equation are both approximated by the same cubically nonlinear asymptotic equation. The contour dynamics equations for Euler vorticity fronts are also derived.

1. Introduction

The two-dimensional incompressible Euler equations have solutions for vorticity fronts located at \( y = \varphi(x,t) \) that separate two regions with distinct, constant vorticities \(-\alpha_+\) and \(-\alpha_-\) in \( y > \varphi(x,t) \) and \( y < \varphi(x,t) \), respectively. As illustrated in Figure 1.1 these solutions may be regarded as perturbations of a piecewise linear shear flow \((U(y),0)\) with

\[
U(y) = \begin{cases} 
\alpha_+ y & \text{if } y > 0, \\
\alpha_- y & \text{if } y < 0. 
\end{cases}
\]

(1.1)

In his studies of the stability of shear flows, Rayleigh showed that the flow (1.1) is linearly stable [29]. He also showed that the vorticity front supports unidirectional waves and computed the Fourier expansion of a spatially periodic traveling wave on the front up to fifth order in the slope of the front [30]. Rayleigh did not, however, consider the more complex nonlinear dynamics of small-slope fronts with general spatial profiles that is described by the equations analyzed here.

We non-dimensionalize the time variable \( t \) so that

\[
\frac{\alpha_+ - \alpha_-}{2} = 1.
\]

Then, in Appendix A we show that the displacement \( \varphi(x,t) \) of the vorticity front satisfies the following evolution equation

\[
\varphi_t(x,t) + \frac{m}{2} \varphi_x^2(x,t) + \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \varphi_x(x,t) - \varphi_x(x + \zeta,t) \right] \log \left[ 1 + \frac{[\varphi(x,t) - \varphi(x + \zeta,t)]^2}{|\zeta|^2} \right] d\zeta = \mathbf{H}[\varphi](x,t),
\]

(1.2)

where \( \mathbf{H} \) denotes the Hilbert transform with respect to \( x \), which is a Fourier multiplier operator with symbol \(-i \text{ sgn } k\), and

\[
m = \frac{\alpha_+ + \alpha_-}{\alpha_+ - \alpha_-}.
\]

Date: June 16, 2020.

JKH was supported by the NSF under grant numbers DMS-1616988 and DMS-1908947.
Figure 1.1. An illustration of the $x$-velocity field of a vorticity front in a fluid with symmetric constant vorticities ($\alpha_+ = -\alpha_-)$.

The dashed line is the unperturbed shear flow \((1.1)\) plotted versus $y$, and the solid line is the perturbed $x$-velocity. The influence of the front motion on the velocity field decays as $|y| \to \infty$.

In this paper, we prove that small-slope solutions of \((1.2)\) are approximated on cubically nonlinear time scales by solutions of the following Burgers-Hilbert equation (see Theorem 2.1)

$$u_t(x,t) + \sqrt{m^2 + 1} \partial_x \left[ u^2(x,t) \right] = H[u](x,t).$$  \(1.3\)

Moreover, we prove that small-slope solutions of both \((1.2)\) and \((1.3)\) are approximated on cubic time scales by solutions of the following asymptotic equation

$$w_t + m^2 \partial_x \left\{ w^2 |\partial_x w - w| \partial_x w^2 + \frac{1}{3} |\partial_x w^3| \right\} = H[w],$$  \(1.4\)

where $|\partial_x| = H \partial_x$ is the Fourier multiplier with symbol $|k|$ (see Theorem 2.2). A multiple-scale form of this asymptotic solution is given in \((2.3)\) and \((2.5)\).

Equations \((1.3)\) and \((1.4)\) were derived previously as descriptions of vorticity fronts in [1] by means of formal asymptotic expansions of the Burgers-Hilbert and Euler equations; the present paper provides a proof of that result. The proof use a modified energy method introduced in [14] to eliminate the effect of the quadratic terms in \((1.2)\)–\((1.3)\) on energy estimates for the error between solutions of \((1.2)\)–\((1.3)\) and \((1.4)\) on cubic timescales.

The Burgers-Hilbert description is significant because it gives a clear picture of the nonlinear dynamics of small-slope vorticity fronts. Solutions of the linearized Burgers-Hilbert equation $u_t = H[u]$ oscillate with frequency one between an arbitrary spatial profile, its Hilbert transform, and their negatives. The oscillating spatial profile of the front then undergoes a slow, alternate compression and expansion due to the Burgers nonlinearity, leading to a complex deformation of the front profile and an effectively cubic nonlinearity. Numerical solutions show that wave-breaking in small-slope solutions of the Burgers-Hilbert equation corresponds to the formation of multiple, extraordinarily thin filaments in the vorticity front [2], similar to the ones observed in vortex patches [8, 9].
A Burgers-Hilbert equation was written down by Marsden and Weinstein [26] as a quadratic truncation for the motion of the boundary of a vortex patch, which, from (1.2), gives the equation
\[ \varphi_t(x, t) + \frac{m}{2} \partial_x \left[ \varphi^2(x, t) \right] = H[\varphi](x, t). \]

However, this equation does not provide an approximation for front motions on cubic time scales; for example, in the symmetric case \( \alpha_+ = -\alpha_- \), we have \( m = 0 \) and the nonlinear term vanishes in the Burgers-Hilbert equation in [26]. Rather, one has to use the appropriately renormalized nonlinear coefficient given in (1.3).

From the point of view of normal forms, when one uses a near identity transformation to remove the quadratic term from (1.3) (which is nonresonant), one gets the same cubic term as the one that arises from the full Euler front equation (1.2). Dimensional analysis provides some explanation for why the quadratically nonlinear Burgers-Hilbert equation should provide a description of cubically nonlinear vorticity fronts [1]. Further results on the Burgers-Hilbert equation can be found in [3, 4, 12, 20, 23, 24, 31, 33].

2. Statement of the main theorems

For \( n \in \mathbb{N} \), we denote by \( H^n(\mathbb{R}) \) the standard \( L^2 \)-Sobolev space equipped with norm
\[ \|f\|_{H^n(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|^2 \, dx + \int_{\mathbb{R}} |\partial^n f(x)|^2 \, dx, \]
and we abbreviate \( \int_{\mathbb{R}} = \int \) when there is no confusion. For simplicity, we restrict our analysis to Sobolev spaces of integer orders.

The first main result of this paper is the following approximation theorem.

**Theorem 2.1.** Fix an integer \( n \geq 3 \) and suppose that \( \varphi_0 \in H^n(\mathbb{R}) \). Then there exist constants \( \varepsilon_0 > 0 \) and \( C, T > 0 \) depending on \( \|\varphi_0\|_{H^n} \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), there exist unique solutions \( \varphi \in C([0, T/\varepsilon^2]; H^n(\mathbb{R})) \) of the Euler front equation (1.2) and \( u \in C([0, T/\varepsilon^2]; H^n(\mathbb{R})) \) of the Burgers-Hilbert equation (1.3) with initial data
\[ \varphi(\cdot, 0) = u(\cdot, 0) = \varepsilon \varphi_0, \]

which satisfy
\[ \sup_{t \in [0, T/\varepsilon^2]} \|u(\cdot, t) - \varphi(\cdot, t)\|_{H^n} \leq C \varepsilon^2. \]

Here, the existence time \( T \) can be taken as the existence time for the asymptotic equation in Theorem 2.2. Since (1.2) and (1.3) are invariant under time-reversal and reflection \( t \mapsto -t, x \mapsto -x \) the same result holds backwards in time. A similar result and proof also applied to spatially periodic solutions defined on \( \mathbb{T} = \mathbb{R}/(2\pi n \mathbb{Z}) \) instead of \( \mathbb{R} \), but we omit the details. Finally, we remark that this theorem still holds with distinct but close initial conditions \( \varphi(\cdot, 0) = \varepsilon \varphi_0, u(\cdot, 0) = \varepsilon u_0 \), where \( \varphi_0, u_0 \in H^n(\mathbb{R}) \) satisfy \( \|\varphi_0 - u_0\|_{H^n} \leq C \varepsilon \) for any \( C > 0 \). In the proof, we only need to use a smooth approximation of the average \( (\varphi_0 + u_0)/2 \), instead of \( \varphi_0 \), for the initial data of the asymptotic equation.

To prove Theorem 2.1 we show that the cubically nonlinear asymptotics of both (1.2) and (1.3) is given by (1.4). In order to deal with the Euler front equation (1.2) and the Burgers-Hilbert equation (1.3) simultaneously, we consider the equation
\[ \varphi_t(x, t) + \frac{\rho}{2} \partial_x \left[ \varphi^2(x, t) \right] + \frac{\sigma}{2\pi} \int_{\mathbb{R}} \left[ \varphi_x(x, t) - \varphi(x + \zeta, t) \right] \log \left[ 1 + \frac{[\varphi(x, t) - \varphi(x + \zeta, t)]^2}{|\zeta|^2} \right] \, d\zeta = H[\varphi](x, t), \]

where \( \rho, \sigma \in \mathbb{R} \) are parameters. If \( \rho = m, \sigma = 1 \), then (2.1) reduces to (1.2), and if \( \rho = |m|^2 + 1, \sigma = 0 \), then (2.1) reduces to (1.3). Local existence and uniqueness of solutions of the Cauchy problem for this equation with \( \varphi \in C([0, T]; H^n(\mathbb{R})) \) and \( n \geq 3 \) follows by similar arguments to the ones in [19].

Equation (2.1) has the formal multiple-scale asymptotic solution
\[ \varphi(x, t) = \varepsilon e^{tH}[v(x, \varepsilon^2 t) + O(\varepsilon^2)] \quad \text{as } \varepsilon \to 0 \text{ with } t = O(\varepsilon^{-2}), \]
where \(v(x, \tau)\) satisfies
\[
v_{\tau} + \frac{\sigma^2 + \sigma}{2} \partial_x \left\{ v^2 |\partial_x v - v| \partial_x v^2 + \frac{1}{3} |\partial_x v|^3 \right\} = 0. \tag{2.2}
\]
If \(n \geq 3\), the local existence of unique solutions \(v \in C([0, T]; H^n(\mathbb{R}))\) of the Cauchy problem for (2.2) follows by an appropriate modification of the proofs in \([15, 21]\) for spatially periodic solutions. Equation (2.2) also has a complex form (3.16), which is what we use when constructing approximate solutions since it simplifies the algebra.

As stated in the next theorem, the leading order formal asymptotic solution
\[
w(x, t; \varepsilon) = \varepsilon e^{i \mathbf{H} v(x, \varepsilon^2 t)} = \varepsilon \left[ v(x, \varepsilon^2 t) \cos t + \mathbf{H}[v](x, \varepsilon^2 t) \sin t \right] \tag{2.3}
\]
approximates solutions of (2.1) over cubic timescales.

**Theorem 2.2.** Fix an integer \(n \geq 3\) and constants \(C, T > 0\). Let \(n_v \geq n + 5\). Then there exist constants \(C', \varepsilon_0 > 0\) such that for all \(0 < \varepsilon < \varepsilon_0\), all solutions \(v \in C([0, T], H^{n_v}(\mathbb{R}))\) of (2.2) and all \(\varphi_0 \in H^n(\mathbb{R})\) with
\[
\sup_{\tau \in [0, T]} \|v(\cdot, \tau)\|_{H^{n_v}} \leq C, \quad \|\varphi_0 - v(\cdot, 0)\|_{H^n} \leq C \varepsilon,
\]
there exists a unique solution \(\varphi \in C([0, T/\varepsilon^2], H^n(\mathbb{R}))\) of the Cauchy problem for the modified Euler front equation (2.1) with initial data \(\varphi(\cdot, 0) = \varepsilon \varphi_0\), and this solution satisfies
\[
\sup_{t \in [0, T/\varepsilon^2]} \|\varphi(\cdot, t) - \varepsilon \left[ v(\cdot, \varepsilon^2 t) \cos t + \mathbf{H}[v](\cdot, \varepsilon^2 t) \sin t \right]\|_{H^n(\mathbb{R})} \leq C' \varepsilon^2. \tag{2.4}
\]
For the Burgers-Hilbert equation, (2.1) with \(\sigma = 0\), the same result holds for \(n \geq 2\).

Here, we require additional smoothness on the initial data \(v_0 = v(\cdot, 0)\) for the asymptotic equation in order to construct a sufficiently accurate approximate solution, which we can always achieve by smoothing \(\varphi_0\). We remark that the existence of small, smooth \(H^n\)-solutions of the Burgers-Hilbert equation on some cubic life span is proved for \(n \geq 2\) in \([13, 14]\). However, the previous theorem shows that the Burgers-Hilbert solution exists on any time-interval for which the asymptotic solution exists and shows that the solution remains close to the asymptotic solution. For the modified Euler front equation (2.1) with \(\sigma \neq 0\), we need to assume that \(n \geq 3\) in order to estimate an additional error term \(\sigma(J_5 + J_6)\) that appears in Section 5.

If either \(\rho = m, \sigma = 1\) or \(\rho = \sqrt{m^2 + 1}, \sigma = 0\), then (2.2) reduces to
\[
v_{\tau} + \frac{m^2 + 1}{2} \partial_x \left\{ v^2 |\partial_x v - v| \partial_x v^2 + \frac{1}{3} |\partial_x v|^3 \right\} = 0, \tag{2.5}
\]
so (1.2) and (1.3) have the same asymptotic equation. Moreover, if \(v(x, \tau)\) satisfies (2.5), then the leading order approximation \(w(x, t; \varepsilon)\) in (2.3) satisfies (1.4) (see Lemma C.1), so (1.4) provides an unscaled version of the asymptotic equation for both (1.2) and (1.3). Theorem 2.1 then follows by comparing \(\varphi\) and \(u\) with \(w\).

The rest of the paper is devoted to the proof of Theorem 2.2. The main idea of the proof is to use a modified energy inspired by a normal form transformation \([14]\) to obtain cubic energy estimates that do not lose derivatives. Related proofs for NLS approximations can be found in \([6, 10, 11, 22, 28]\).

In Section 3, we derive an approximate solution and obtain residual estimates. In Section 4, we define a modified energy for the error equation. In Section 5, we obtain energy estimates for the error and use them to prove Theorem 2.2. In the appendices, we derive the contour dynamics equation for Euler fronts and prove a lemma for a multilinear symbol which arises in the expanded form of the equation.

In the following, we denote the Fourier transform of \(f: \mathbb{R} \to \mathbb{C}\) by \(\hat{f}: \mathbb{R} \to \mathbb{C}\) where
\[
f(x) = \int_{\mathbb{R}} \tilde{f}(\xi) e^{i \xi x} \, d\xi, \quad \hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i \xi x} \, dx.
\]
Throughout the paper, we use $C$ to denote a constant independent of $\varepsilon$, which may change from line to line, and the notation $O(\delta)$ denotes a term satisfying $|O(\delta)| \leq C\delta$.

3. Formal approximation and residual estimates

In this section, we construct an approximate solution of (2.1) and estimate its residual. We first give an expansion of the nonlinear term in the equation.

3.1. Expansion of the nonlinearity. We write (2.1) as

$$\varphi_t + \frac{\rho}{2} \partial_x (\varphi^2) + \sigma N_{\geq 3}[\varphi] = H[\varphi],$$  

where $N_{\geq 3}$ is the cubic term

$$N_{\geq 3}[\varphi](x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} [\varphi(x,t) - \varphi(x + \zeta,t)] \log \left[ 1 + \frac{[\varphi(x,t) - \varphi(x + \zeta,t)]^2}{|\zeta|^2} \right] d\zeta.$$

The following multilinear expansion is proved in [19].

**Lemma 3.1.** Let $N_{\geq 3}[\varphi]$ be given by (3.2). For $k \in \mathbb{N}$, define $c_k \in \mathbb{R}$ and $T_k : \mathbb{R}^{2k+1} \to \mathbb{R}$ by

$$c_k = \frac{(-1)^k}{2\pi k(2k+1)}; \quad T_k(\eta) = \int_{\mathbb{R}} \prod_{j=1}^{2k+1} \frac{(1 - e^{i\eta_j})}{\zeta^{2k}} d\zeta,$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_{2k+1})$. Then for sufficiently small $\varphi \ll 1$

$$N_{\geq 3}[\varphi] = \frac{1}{2} \partial_x \left\{ \varphi^2 |\partial_x| \varphi - \varphi |\partial_x| \varphi^2 + \frac{1}{3} |\partial_x| \varphi^3 \right\} + N_{\geq 5}[\varphi],$$

where

$$N_{\geq 5}[\varphi](x,t) = -\sum_{k=2}^{\infty} c_k \partial_x \int_{\mathbb{R}^{2k+1}} T_k(\eta) \varphi(\eta_1,t) \varphi(\eta_2,t) \cdots \varphi(\eta_{2k+1},t) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2k+1})x} d\eta_k.$$

In view of Lemma [B.1] a rough estimate states that there are at most two derivatives on each $\varphi$ in the term $N_{\geq 5}$, and there exists a constant $C > 0$ such that

$$\|N_{\geq 5}[\varphi]\|_{L^2} \leq C \sum_{k=2}^{\infty} |c_k||\varphi|^{2k+1}_{H^2}.$$

We denote the residual of a function $f(x,t)$ by

$$\text{Res}(f) = -f_t - \frac{\rho}{2} \left( f^2 \right)_x - \sigma N_{\geq 3}[f] + H[f],$$

which measures the extent to which $f$ fails to satisfy (2.1).

3.2. Approximate solution. We look for an approximate solution $\varphi \approx \varepsilon V$ of (2.1) of the form

$$\varepsilon V(x,t;\varepsilon) = \varepsilon V_0(x,t,\varepsilon^2 t) + \varepsilon^2 V_1(x,t,\varepsilon^2 t) + \varepsilon^3 V_2(x,t,\varepsilon^2 t),$$

where $0 < \varepsilon \ll 1$ is a small parameter and the functions $V_n(x,t,\tau)$ are to be determined. The residual of $\varepsilon V$ is given by

$$\text{Res}(\varepsilon V) = -\varepsilon (V_0_t - H[V_0]) - \varepsilon^2 \left( V_{1t} - H[V_1] + \frac{\rho}{2} (V_0^2)_x \right)$$

$$- \varepsilon^3 \left( V_{0\tau} + V_{2t} - H[V_2] + \rho (V_0 V_1)_x + \frac{\sigma}{2} \partial_x \left\{ V_0^3 |\partial_x| V_0 - V_0 |\partial_x| V_0^2 + \frac{1}{3} |\partial_x| V_0^3 \right\} \right) + O(\varepsilon^4).$$
In order to make \( \text{Res}(\varepsilon V) = O(\varepsilon^4) \), we require that \( V_0, V_1, V_2 \) satisfy

\[
V_0 t = H[V_0], \quad V_1 t + \frac{\rho}{2}(V_0^2)_x = H[V_1], \quad V_2 t + V_{0r} + \rho(V_0V_1)_x + \frac{\sigma}{2} \partial_x \left\{ V_0^2 \partial_x V_0 - V_0 \partial_x V_0^2 + \frac{1}{3} \partial_x V_0^3 \right\} = H[V_2].
\]  

Equations for \( \Psi \)

Using (3.10)–(3.13) in (3.9), and equating terms proportional to \( \rho \), we obtain a solution of the homogeneous equation from (3.12). Consider the equation

\[
\frac{\partial}{\partial t} V_1 = \Psi_{12}(x, \tau) e^{-2it} + \Psi_{10}(x, \tau) + \Psi_{12}^\ast(x, \tau) e^{2it},
\]

where

\[
\Psi_{12} = -\frac{i\rho}{2} \left( \Psi^2 \right)_x \quad \text{and} \quad \Psi_{10} = -\rho H \left| \Psi^2 \right|_x.
\]

We omit a solution of the homogeneous equation from \( V_2 \) since we do not need it.

Solution for \( V_0 \). The solution of (3.7) can be written as

\[
V_0(x, t, \tau) = \Psi(x, \tau) e^{-it} + \Psi^\ast(x, \tau) e^{it},
\]

where the complex-valued function \( \Psi \) satisfies \( P[\Psi] = \Psi \). In particular, it follows that \( P[\Psi^2] = \Psi^2 \) and \( H[\Psi^2] = -i\Psi^2 \).

Solution for \( V_1 \). A solution of (3.8) can be written as

\[
V_1(x, t, \tau) = \Psi_{12}(x, \tau) e^{-2it} + \Psi_{10}(x, \tau) + \Psi_{12}^\ast(x, \tau) e^{2it},
\]

where

\[
\Psi_{12} = -\frac{i\rho}{2} \left( \Psi^2 \right)_x \quad \text{and} \quad \Psi_{10} = -\rho H \left| \Psi^2 \right|_x.
\]

Solution for \( V_2 \). To proceed further, we use the following proposition, which is proved by a straightforward computation.

**Proposition 3.2.** Consider the equation

\[
f_t = H[f] + B(x)e^{-int},
\]

where \( n \in \mathbb{Z} \) and \( B \in L^2(\mathbb{R}, \mathbb{C}) \). Then:

1. If \( n^2 \neq 1 \), the equation is uniquely solvable for every \( B \);
2. If \( n = 1 \), then the equation is solvable if and only if \( P[B] = 0 \);
3. If \( n = -1 \), then the equation is solvable if and only if \( Q[B] = 0 \).

Equation (3.9) has solutions of the form

\[
V_2(x, t, \tau) = \Psi_{23}(x, \tau) e^{-3it} + \Psi_{21}(x, \tau) e^{-it} + \Psi_{21}^\ast(x, \tau) e^{it} + \Psi_{23}^\ast(x, \tau) e^{3it},
\]

Using (3.10)–(3.13) in (3.9), and equating terms proportional to \( e^{-3it} \) and \( e^{-it} \), we obtain the following equations for \( \Psi_{23} \) and \( \Psi_{21} \):

\[
-3i\Psi_{23} + \rho(\Psi \Psi_{12})_x + \frac{\sigma}{2} \partial_x \left\{ \Psi^2 \partial_x \Psi - \Psi \partial_x \Psi^2 + \frac{1}{3} \partial_x \Psi^3 \right\} = H[\Psi_{23}],
\]

\[
(\Psi_{21} e^{-it})_t + \Psi_{21} e^{-it} + \rho (\Psi \Psi_{10} + \Psi^\ast \Psi_{12})_x e^{-it} + \frac{\sigma}{2} \partial_x \left\{ 2 |\Psi|^2 \partial_x \Psi + \Psi^2 \partial_x \Psi^\ast - 2 \Psi \partial_x |\Psi|^2 - \Psi^\ast \partial_x |\Psi|^2 + |\partial_x (|\Psi|^2)| e^{-it} = H[\Psi_{21} e^{-it}].
\]
The function $\Psi_{23}$ can be solved from (3.14) directly to give

$$\Psi_{23} = -\frac{3i}{16} \partial_x \left\{ 2\rho \Psi_{12} + \sigma \Psi^2 |\partial_x| \Psi - \sigma \Psi |\partial_x| \Psi^2 + \frac{\sigma}{3} |\partial_x| \Psi^3 \right\}$$

$$+ \frac{1}{16} \partial_x \left\{ 2\rho \Psi_{12} + \sigma \Psi^2 |\partial_x| \Psi - \sigma \Psi |\partial_x| \Psi^2 + \frac{\sigma}{3} |\partial_x| \Psi^3 \right\}.$$

Applying Proposition 3.2 to (3.15) and simplifying the result, we find that the solvability condition for $\Psi_{21} e^{-it}$ is satisfied if

$$\Psi \tau = (\rho^2 + \sigma) \mathbf{P} \left[ i|\Psi|^2 \mathbf{P}_x + \Psi H[|\Psi|^2]_x \right].$$

Equation (3.16) is the complex form of (2.2). Indeed, substituting $\Psi = \mathbf{P}[v] = \frac{1}{2} [v + i H[v]]$ into this equation we find, after some algebra, that $v = \Psi + \Psi^*$ satisfies (2.2).

When (3.16) holds, a solution of (3.15) for $\Psi_{21}$ is given by

$$\Psi_{21} = \frac{1}{2} \mathbf{Q} \left[ (-\rho^2 + \sigma) |\Psi|^2 \mathbf{P}_x + i(\sigma + \rho^2) \Psi H[|\Psi|^2]_x + \sigma |\Psi|^2 \mathbf{P}_x^* \right].$$

In conclusion, given a solution $\Psi$ of (3.16), or equivalently $v$ of (2.2), we have constructed a function $\varepsilon V$ of the form (3.6) that satisfies (2.1) up to a residual of the order $\varepsilon^4$.

3.3. Residual estimates. In this subsection, we obtain estimates for the residual of the approximate solution constructed above. We observe that at each stage in the expansion of $V$ we increase the degree in $\Psi$ by one and introduce one additional $x$-derivative, so $V_k$ is of degree $k+1$ in $\Psi$ and involves $k$ derivatives with respect to $x$. Thus, in order to construct the approximate solution $\varepsilon V \in C([0,T], H^n)$, we require $n_\varepsilon \geq n + 2$ derivatives in the solution $v$ of (2.2). As stated in the next lemma, we also need two further derivatives to estimate the residual of $\varepsilon V$.

**Lemma 3.3.** Let $n \geq 0$ and suppose that $v \in C([0,T], H^{n+2}(\mathbb{R}))$ with $n_\varepsilon \geq n + 4$ is a solution of (2.2). Then for any $0 < \varepsilon_0 \leq 1$, there exists a constant $C > 0$ depending on $\|v(\cdot,0)\|_{H^{n+2}}$ such that for all $0 < \varepsilon < \varepsilon_0$, there is a function $\varepsilon V \in C([0,T], H^{n+2}(\mathbb{R}))$ of the form (3.6) whose residual (3.5) satisfies the estimate

$$\sup_{t \in [0,T/\varepsilon^2]} \| \text{Res}(\varepsilon V)(\cdot,t;\varepsilon) \|_{H^n} \leq C \varepsilon^4.$$

Furthermore,

$$\sup_{t \in [0,T/\varepsilon^2]} \left\| \varepsilon [v(\cdot,\varepsilon^2 t) \cos t + H[v](\cdot,\varepsilon^2 t) \sin t] - \varepsilon V(\cdot,t;\varepsilon) \right\|_{H^n} \leq C \varepsilon^2.$$

**Proof.** We constructed $\varepsilon V$ in Section 3.2 in terms of $\Psi = \mathbf{P}[v]$. We compute that its residual (3.5) is given by

$$\text{Res}(\varepsilon V) = -\varepsilon^4 [V_{\tau} + \frac{\rho}{2} (V_{12}^2)_x + \rho (V_0 V_2)_x] - \varepsilon^5 [V_{2\tau} + \rho (V_1 V_2)_x] - \varepsilon^6 [\rho V_2 V_{2\tau}]$$

$$- \frac{\sigma}{2} \sum_{p=1}^6 \sum_{0 \leq j+k \leq 2 \atop j+k \neq 0} \varepsilon^{p+3} \partial_x \left\{ V_j V_k |\partial_x| V_{\tau} - V_j |\partial_x| (V_k V_{\tau}) + \frac{1}{3} |\partial_x| (V_j V_k V_{\tau}) \right\} - \sigma \mathcal{N}_{25}[\varepsilon V].$$

From the expressions for $V_k$ with $0 \leq k \leq 2$, we see that there are at most four $x$-derivatives on $\Psi$ in all of the terms that involve the $V_k$. The second inequality follows from the construction of $V$.

$\square$
4. A Modified Energy for the Error

Let \( \varphi \) denote the solution of \( (2.1) \) with initial data \( \varphi(0) = \varepsilon \phi_0 \), and let \( \varepsilon V \) denote the approximate solution \( (4.6) \) with initial data \( \varepsilon V(\cdot, 0; \varepsilon) = \varepsilon v_0 \). We denote the error between the solutions by

\[
\varepsilon^\beta R = \varphi - \varepsilon V,
\]

where we will choose \( \beta = 2 \) and show that \( R = O(1) \). To ensure that \( R = O(1) \) at \( t = 0 \), we mollify \( \varphi_0 \in H^n \) to obtain \( v_0 \in H^{n^*} \), such that

\[
\| \varphi_0 - v_0 \|_{H^n} \leq C \varepsilon.
\]

Using \( \varphi = \varepsilon^\beta R + \varepsilon V \) in \( (3.1) \), we obtain that

\[
R_t + \varepsilon^\beta \rho R R_x + \varepsilon \rho (V R)_x + \varepsilon^{-\beta} \sigma (N_{\geq 3} \varepsilon^\beta R + \varepsilon V) = H[R] + \varepsilon^{-\beta} \text{Res}(\varepsilon V),
\]

where \( \text{Res}(\varepsilon V) \) is defined in \( (3.5) \).

We will see that the term \( \varepsilon^{-\beta} \sigma (N_{\geq 3} \varepsilon^\beta R + \varepsilon V) \) is of the order \( \varepsilon^2 \). The most dangerous terms in \( (4.2) \) are \( \varepsilon(V R)_x \) and \( \varepsilon^3 R R_x \). As in \( (3.2) \), they can be removed by a normal form transformation \( R \mapsto \tilde{R} \) where

\[
\tilde{R} = R + \varepsilon \rho H [H[V] H[R]]_x + \frac{1}{2} \varepsilon^\beta \rho H [\{H[R]\}^2]_x,
\]

which yields a cubically nonlinear equation for \( \tilde{R} \). However, this equation contains second-order spatial derivatives in the nonlinearity, resulting in a loss of derivatives in its energy estimates, and the straightforward normal form transformation \( (4.3) \) is not effective.

Following \([6, 10, 14]\), we instead use \( (4.3) \) to define a modified energy functional that is obtained by neglecting the higher-order terms with the most derivatives from \( \int |\partial^n R|^2 \, dx \):

\[
E_n = \int |\partial^n R|^2 \, dx + 2 \varepsilon \rho \int \partial^{n+1} H [H[V] H[R]] \, |\partial^n R| \, dx + \varepsilon^\beta \rho \int \partial^{n+1} H [\{H[R]\}^2] \, |\partial^n R| \, dx.
\]

The first term in \( (4.4) \) is the standard \( \dot{H}^n \)-norm, the second term cancels the leading order effect of \( \varepsilon(V R)_x \) on the time evolution of this norm, and the third term cancels the effect of \( \varepsilon^3 R R_x \). Moreover, as stated in the following lemma, the energy \( E_n \) is equivalent to the standard Sobolev energy \( \| R \|_{\dot{H}^n}^2 \) for small enough \( \varepsilon \).

**Lemma 4.1.** Let \( E_n \) be defined by \( (4.4) \). Then

\[
E_n = \left[ 1 + O(\varepsilon \| H[V_x]\|_{W^{n, \infty}}) + O(\varepsilon^\beta \| H[R_x]\|_{L^\infty}) \right] \| \partial^n R \|_{L^2}^2 \quad \text{as } \varepsilon \to 0.
\]

**Proof.** The \( O(\varepsilon \| H[R_x]\|_{L^\infty}) \) part is proved in Lemma 2 of \([14]\), so we only prove the other part. Using the skew-adjointness of \( H \), integration by parts, and Hölder’s inequality, we find that

\[
\left| \int \partial^{n+1} H [H[V] H[R]] \, |\partial^n R| \, dx \right| = \left| \frac{1}{2} \int H[V_x] |\partial^n H[R]|^2 \, dx - \sum_{j=0}^{n} \binom{n+1}{j} \partial^{n+1-j} H[V] \partial^j H[R] \partial^n H[R] \right|
\]

\[
\leq C \| H[V_x] \|_{W^{n, \infty}} \| \partial^n R \|_{L^2}^2,
\]

and then the lemma follows by the definition of \( E_n \). \( \Box \)

In the following, we fix an integer \( n \geq 3 \) and define the energy

\[
E = E_0 + E_n,
\]

where \( E_0 \) is defined by \( (4.4) \) with \( n = 0 \). By Lemma 4.1, \( E \) is equivalent to the \( H^n \)-energy \( \| R \|_{\dot{H}^n}^2 \) for sufficiently small \( \varepsilon \).
5. Modified energy estimates

In the rest of the paper, we prove that the energy (4.5) satisfies the estimate

\[ E(t) \leq CE(0) \quad \text{for all } 0 \leq t \leq T/\varepsilon^2 \]  

(5.1)

when \( \varepsilon \) is sufficiently small. Then \( \|R\|_{H^n} = O(1) \), and by the definition of \( R \) in (4.1), we obtain that

\[ \|\varphi(\cdot, t) - \varepsilon V(\cdot, t; \varepsilon)\|_{H^n} = O(\varepsilon^\beta) \quad \text{for all } 0 \leq t \leq T/\varepsilon^2, \]

where \( \beta = 2 \). Combining this result with Lemma 3.3, we obtain Theorem 2.2.

In order to study the evolution of \( E = E_0 + E_n \), it suffices to study the evolution of \( E_n \). The term \( E_0 \) can be shown to satisfy the same estimates by replacing \( n \) by \( 0 \). In fact, the estimate for \( E_0 \) is easier. In proving (5.1), we will use the following commutator estimate whose proof can be found in [7].

**Lemma 5.1.** Let \( H \) denote the Hilbert transform. Then for any \( p \in (1, \infty) \), \( \ell_1, \ell_2 \in \mathbb{N} \), \( f \in L^p \), and \( a \in W^{\ell_1, \ell_2, \infty} \), there exists \( C = C(p, \ell_1, \ell_2) > 0 \) such that

\[ \|\partial^{\ell_1}[H, a]\partial^{\ell_2}f\|_{L^p} \leq C\|\partial^{\ell_1+\ell_2}a\|_{L^\infty}\|f\|_{L^p}. \]

Time differentiating (4.4), we obtain that

\[
\frac{1}{2} \frac{d}{dt} E_n = \int \partial^n R \partial^n R\ dx + \varepsilon \rho \int \partial^{n+1} H[H[V]H[R]] \partial^n R \ dx \\
+ \varepsilon \rho \int \partial^{n+1} H[H[V]H[R]] \partial^n R \ dx + \varepsilon \rho \int \partial^{n+1} H[H[V]H[R]] \partial^n R_4 \ dx \\
+ \varepsilon^2 \rho \int \partial^{n+1} H[H[R]H[R]] \partial^{n+1} R \ dx + \varepsilon^2 \rho \int (H[R])^2 \partial^n R \ dx. 
\]

Using (3.5) to eliminate \( \varepsilon V_t \) in terms of \( \text{Res}(\varepsilon V) \) and (4.2) and to eliminate \( R_t \), we get

\[
\frac{1}{2} \frac{d}{dt} E_n = \int \partial^n R \partial^n H[R] \ dx - \varepsilon \rho \int \partial^n R \partial^{n+1}(VR) \ dx + \varepsilon \rho \int \partial^{n+1} H[H[V]H[R]] \partial^n R \ dx \\
+ \varepsilon \rho \int \partial^{n+1} H[H[V]H[R]] \partial^n R \ dx + \varepsilon \rho \int \partial^{n+1} H[H[V]H[R]] \partial^n H[R] \ dx \\
- \varepsilon^2 \rho \int \partial^n R \partial^n (RR_x) \ dx + \varepsilon^2 \rho \int \partial^{n+1} H[H[R]H[R]] \partial^n R \ dx \\
+ \varepsilon^2 \rho \int \partial^{n+1} H[H[R]]^2 \partial^n H[R] \ dx + \rho^2 (I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9) \\
+ \rho (I_{11} + I_{12} + I_{13}) + I_{10} - \sigma (J_1 + J_2 + J_3 + J_4 + J_5 + J_6),
\]  

(5.2)
where, as we will show, $I_k (k = 1, \ldots, 6)$ are terms of order $\varepsilon^2$ or higher. They are given explicitly by

\begin{align*}
I_1 &= -\varepsilon^2 \int \partial^{n+1} H[H[V V_x]] H[R]\partial^n R \, dx, \\
I_2 &= -\varepsilon^2 \int \partial^{n+1} H[H[V] \partial H[V R]] \partial^n R \, dx, \\
I_3 &= -\varepsilon^2 \int \partial^{n+1} H[H[V] H[R]] \partial^{n+1} (V R) \, dx, \\
I_4 &= -\varepsilon^{\beta+1} \int \partial^{n+1} H[H[V] H[R]] \partial^n (R R_x) \, dx, \\
I_5 &= -\varepsilon^{\beta+1} \int \partial^{n+1} H[H[H[V] H[R]]] \partial^n R \, dx, \\
I_6 &= -\varepsilon^{\beta+1} \int \partial^{n+1} H[H[R] H[V R_x]] \partial^n R \, dx, \\
I_7 &= -\varepsilon^{\beta+1} \int \partial^{n+1} H[H[R]]^2 \partial^n (V R) \, dx, \\
I_8 &= -\varepsilon^{2\beta} \int \partial^{n+1} H[H[R] H[R R_x]] \partial^n R \, dx, \\
I_9 &= -\varepsilon^{\beta+1} \int \partial^{n+1} H[H[H[V] H[\text{Res}(V)]]] \partial^n R \, dx, \\
I_{10} &= \varepsilon^\beta \int \partial^n R \partial^n (\text{Res}(V)) \, dx, \\
I_{11} &= \varepsilon^{-\beta+1} \int \partial^{n+1} H[H[V] H[\text{Res}(V)]]] \partial^n R \, dx, \\
I_{12} &= \varepsilon^{-\beta+1} \int \partial^{n+1} H[H[V] H[R]] \partial^n \text{Res}(V) \, dx, \\
I_{13} &= \frac{1}{2} \int \partial^{n+1} H[H[R]]^2 \partial^n \text{Res}(V) \, dx.
\end{align*}

The first term on the right-hand side of (5.2) vanishes due to the skew-adjointness of the Hilbert transform. Making use of the skew-adjointness of $H$, the fact that $H^2 = -I$, and the Cotlar identity

$$H[ab - H[a] H[b]] = a H[b] + b H[a],$$

we find (as a consequence of the choice of the modified energy) that the terms of the order $\varepsilon$ on the right-hand side of in (5.2) also vanish:

\begin{align*}
- \int \partial^n R \partial^{n+1} (V R) \, dx &+ \int \partial^{n+1} H[H^2[V] H[R]] \partial^n R \, dx \\
&+ \int \partial^{n+1} H[H[V] H^2[R]] \partial^n R \, dx + \int \partial^{n+1} H[H[V] H[R]] \partial^n H[R] \, dx \\
&= \int \partial^n R \partial^{n+1} [-V R - H[V H[R]] - H[R H[V]] + H[V] H[R]] \, dx = 0.
\end{align*}

Similarly, we have

\begin{align*}
- \varepsilon^\beta \rho \int \partial^n R \partial^n (R R_x) \, dx &+ \varepsilon^\beta \rho \int \partial^{n+1} H[H[R] R] \partial^n R \, dx + \varepsilon^\beta \rho \int \partial^{n+1} H[H[R]]^2 \partial^n H[R] \, dx \\
&= \frac{\varepsilon^\beta}{2} \rho \int \partial^{n+1} R \partial^n \left[R^2 - 2 H[H[R] R] - (H[R])^2 \right] \, dx = 0.
\end{align*}
Thus, it suffices to estimate the error terms $I_\ell$ $(\ell = 1, \ldots, 15)$ and $J_k$ $(k = 1, \ldots, 6)$. In order to do this, we make a bootstrap assumption

$$E^{1/2} \leq \frac{1}{\varepsilon} \quad \text{for all } 0 \leq t \leq T/\varepsilon^2. \quad (5.3)$$

We then show in the following subsections that all of the error terms can be estimated by $\varepsilon^2E$ or $\varepsilon^2E^{1/2}$, which closes the bootstrap and establishes (5.1).

5.1. **Quadratic terms of order $\varepsilon^2$.** These terms are $I_1 - I_3$. In this subsection, we use $\mathcal{R}_1$ to denote terms that might change from line to line and satisfy the estimate

$$|\mathcal{R}_1| \leq C\varepsilon^2E.$$

Error terms that contain at most $n$ derivatives on $R$ satisfy this estimate by a combination of Sobolev embedding and the Cauchy-Schwarz inequality, while terms that contain strictly fewer than $n$ derivatives on $R$ and $n+1$ derivatives on another $R$ can be converted into terms of the previous type by an integration by parts. Thus, we only need to consider terms with $n$ derivatives on one $R$ and $n+1$ derivatives on another $R$.

After applying the Leibniz rule, the worst term in $I_1$ can be handled by integration by parts as follows:

$$I_1 = -\varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x + \mathcal{R}_1$$

$$= \frac{1}{2} \varepsilon^2 \int \partial H[V] H[V\partial^n R] H[R] d^2x + \mathcal{R}_1$$

$$\leq C\varepsilon^2E.$$

The terms $I_2$ and $I_3$ permit a cancellation when all the derivatives hit $R$ or $H[R]$. Integrating by parts and using the skew-adjointness of $H$, we obtain that

$$I_2 + I_3 = -\varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x - (n + 1)\varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x$$

$$- n\varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x + \varepsilon^2 \int H[V] H[R] H[V\partial^n R] d^2x$$

$$+ (n + 1)\varepsilon^2 \int H[V] H[R] H[V\partial^n R] d^2x + (n + 1)\varepsilon^2 \int H[V] H[R] H[V\partial^n R] d^2x + \mathcal{R}_1 + \mathcal{R}_1$$

$$= \varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x + \mathcal{R}_1.$$

For the first term on the right-hand side, we make use of Lemma 5.1 to get

$$\varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x$$

$$= \varepsilon^2 \int H[V] H[V\partial^n R] H[R] d^2x + \varepsilon^2 \int H[V] \left( H, V \partial^n R \right) H[R] d^2x$$

$$\leq C\varepsilon^2 \left( \|\partial^n R\|^2_{L^2} + \left\| H, V \partial^n R \right\|_{L^2} \|R\|_{L^2} \right)$$

$$\leq C\varepsilon^2E.$$

5.2. **Cubic terms of order $\varepsilon^3$.** We now consider the terms $I_4 - I_7$ of order $\varepsilon^3$. Since these terms are cubic in $R$, we bound them by $E^{3/2}$ and use the bootstrap assumption (5.3). We denote by $\mathcal{R}_2$ terms that satisfy the estimate

$$|\mathcal{R}_2| \leq C\varepsilon^3 E^{3/2}.$$
The second term in the right-hand-side of (5.4) can be estimated similarly by

\[ I_5 = -\varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx + (n+1)\varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx + \mathcal{R}_2, \]

so

\[ I_4 + I_5 = -\varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx + (n+1)\varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx + \mathcal{R}_2. \]  

(5.4)

Using integration by parts and the commutator estimate in Lemma 5.1, we estimate the first term on the right-hand-side of (5.4) by

\[-\varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx = \varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx + \mathcal{R}_2 \]

\[ = -\frac{\beta^2}{2} \int \partial[H(V[R])| \partial^n H[R]|^2 \, dx - \varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx \]

\[ \leq C\varepsilon^\beta + 1 E^{3/2}. \]

The second term in the right-hand-side of (5.4) can be estimated similarly by

\[ -\varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx = \varepsilon^\beta + 1 \int H[V] H[R]\partial^n H[R] \, dx + \mathcal{R}_2 \]

\[ \leq C\varepsilon^\beta + 1 E^{3/2}. \]

The estimates for \( I_6 \) and \( I_7 \) are similar. Observe that

\[ I_6 = -\varepsilon^\beta + 1 \int \partial^n H[R] H[V R]\partial^n H[R] \, dx - \varepsilon^\beta + 1 \int H[R] H[V\partial^n R]\partial^n H[R] \, dx \]

\[ - (n+1)\varepsilon^\beta + 1 \int H[R] H[V\partial^n R]\partial^n H[R] \, dx + (n+1)\varepsilon^\beta + 1 \int H[R] H[V\partial^n R]\partial^n H[R] \, dx + \mathcal{R}_2, \]

\[ I_7 = \varepsilon^\beta + 1 \int H[R]\partial^n H[R] H[V\partial^n R] \, dx + (n+1)\varepsilon^\beta + 1 \int H[R] H[V\partial^n R]\partial^n H[R] \, dx \]

\[ + (n+1)\varepsilon^\beta + 1 \int \partial[H[R]\partial^n H[R] H[V\partial^n R] \, dx + \mathcal{R}_2. \]
We then estimate them together and use integration by parts and Lemma 5.1 to obtain

\[ I_6 + I_7 = -\varepsilon^{\beta+1} \int \partial^n H[R] \partial^{n+1} H[R] \partial V R \, dx - n\varepsilon^{\beta+1} \int \partial H[R] \partial^{n+1} H[R] H[V \partial^n R] \, dx \]
\[ + (n + 1)\varepsilon^{\beta+1} \int \partial H[R] \partial^n H[R] H[V \partial^{n+1} R] \, dx + \mathcal{A}_2 \]
\[ = -\varepsilon^{\beta+1} \int \partial^n H[R] \partial^{n+1} H[R] \partial \left( [H, V][R] \right) \, dx - n\varepsilon^{\beta+1} \int \partial H[R] \partial^{n+1} H[R] \left( [H, V][\partial^n R] \right) \, dx \]
\[ + (n + 1)\varepsilon^{\beta+1} \int \partial H[R] \partial^n H[R] \left( [H, V][\partial^{n+1} R] \right) \, dx + \mathcal{A}_2 \]
\[ \leq C \varepsilon^{\beta+1} E^{3/2}, \]

where the last inequality follows from integration by parts and the commutator estimates in Lemma 5.1.

Using the bootstrap assumption (5.3), and the fact that \( \beta = 2 \), we then have

\[ I_4 + I_5 + I_6 + I_7 \leq C \varepsilon^{\beta+1} E^{3/2} \leq C \varepsilon^2 E. \]

5.3. **Quartic terms of order** \( \varepsilon^{2\beta} \). The only quartic terms of order \( \varepsilon^{2\beta} \) are \( I_8 \) and \( I_9 \), which also need to estimated together. Since these terms are quartic in \( R \), we bound them by \( E^2 \) and use the bootstrap assumption (5.3). Let \( \mathcal{A}_3 \) denote terms that satisfy the estimate

\[ |\mathcal{A}_3| \leq C \varepsilon^{2\beta} E^2. \]

We first observe that

\[ I_8 = -\varepsilon^{2\beta} \int \partial^{n+1} H[R] \partial^n H[R] H[R] R_{Rx} \, dx - \varepsilon^{2\beta} \int H[R] \partial^n H[R] R_{Rx} \partial^{n+1} H[R] \, dx \]
\[ - n\varepsilon^{2\beta} \int \partial H[R] H[R] \partial^n R \partial^{n+1} H[R] \, dx + \mathcal{A}_3, \]

\[ I_9 = \varepsilon^{2\beta} \int H[R] \partial^{n+1} H[R] \partial^n H[R] R_{Rx} \, dx + (n + 1)\varepsilon^{2\beta} \int \partial H[R] \partial^n H[R] \partial^{n+1} H[R] R_{Rx} \, dx + \mathcal{A}_3. \]

Then, absorbing the first term on the right-hand-side of \( I_8 \) into \( \mathcal{A}_3 \) and canceling the identical terms, we obtain

\[ I_8 + I_9 = -\varepsilon^{2\beta} \int \partial^{n+1} H[R] \partial^n H[R] H[R] R_{Rx} \, dx - n\varepsilon^{2\beta} \int \partial H[R] H[R] \partial^{n+1} H[R] \, dx \]
\[ + (n + 1)\varepsilon^{2\beta} \int \partial H[R] H[R] \partial^n H[R] R_{Rx} \, dx + \mathcal{A}_3 \]
\[ = \varepsilon^{2\beta} \int |\partial^n H[R]|^2 \partial[H, R] R_{Rx} \, dx - n\varepsilon^{2\beta} \int \partial^{n+1} H[R] [H, R] \partial^n R \partial H[R] \, dx + \mathcal{A}_3 \]
\[ \leq C \varepsilon^{2\beta} E^2 \]
\[ \leq C \varepsilon^2 E, \]

where the second-to-last inequality follows from integration by parts, the commutator estimates Lemma 5.1 and, in the case when \( n = 2 \) for the Burgers-Hilbert equation, the following pointwise estimate for \( \delta > 0 \)

\[ \| [H, R] R_x \|_{L^\infty} + \| [H, R] \partial^2 R \|_{L^\infty} \leq C \| R_x \|^{\frac{1}{2} + \delta}. \]
5.4. Terms involving the residual. These terms are \( I_{10} - I_{13} \). We can directly use Hölder’s inequality, Sobolev embeddings, and Lemma 3.3 to obtain that

\[
\begin{align*}
I_{10} & \leq \varepsilon^{-\beta} E^{1/2} \| \text{Res}(\varepsilon V) \|_{H^n} \leq C \varepsilon^{4-\beta} E^{1/2}, \\
I_{11} & \leq \varepsilon^{-\beta+1} |V|_{H^{n+1}} \| \text{Res}(\varepsilon V) \|_{H^{n+1} E^{1/2}} \leq C \varepsilon^{5-\beta} E^{1/2}, \\
I_{12} & = -\varepsilon^{-2} \int \partial^n H[H[V] H[R]] \partial^{n+1} \text{Res}(\varepsilon V) \, dx \\
& \leq \varepsilon^{-\beta+1} |V|_{H^n} \| \text{Res}(\varepsilon V) \|_{H^{n+1} E^{1/2}} \leq C \varepsilon^{5-\beta} E^{1/2}, \\
I_{13} & \leq C \| R \|_{H^n} ^2 \| \text{Res}(\varepsilon V) \|_{H^{n+1}} \leq C \varepsilon^5 E.
\end{align*}
\]

Here, since we require the \( H^{n+1} \)-norm of the residual of the asymptotic solution \( \varepsilon V \in C([0, T/\varepsilon^2], H^n) \), we need to take \( n_v \geq n + 5 \) in order to apply Lemma 3.3.

5.5. Higher degree terms. In this subsection, we estimate the terms \( J_k, k = 1, \ldots, 6 \). These terms do not appear for the Burgers-Hilbert equation with \( \sigma = 0 \).

We will use \( \mathcal{A} \) to denote terms that involve lower order derivatives and satisfy a straightforward estimate

\[ |\mathcal{A}| \leq C \varepsilon^2 E. \]

We also use the notation

\[
\Delta_\zeta f(x) = f(x) - f(x + \zeta) \quad \text{and} \quad D_\zeta f(x) = \frac{\Delta_\zeta f(x)}{\zeta}
\]

(5.5) to denote differences and difference quotients, where we show the dependent on the spatial variables explicitly but suppress the time variable.

5.5.1. Sobolev energy term \( J_1 \). Using (3.2) in the expression for \( J_1 \) and writing the result in terms of the notation in (5.5), we get that

\[
\begin{align*}
J_1 &= \frac{1}{2\pi} \varepsilon^{-\beta} \int \partial^n R(x) \partial^n \int \left[ \varepsilon^\beta R_x(x) - \varepsilon^\beta R_x(x + \zeta) \right] \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \frac{d\zeta}{\zeta} dx \\
& \quad + \frac{1}{2\pi} \varepsilon^{-\beta} \int \partial^n R(x) \partial^n \int \left[ \varepsilon \Delta_\zeta V(x) \right] \left\{ \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] - \log \left[ 1 + \frac{[\varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \right\} \frac{d\zeta}{\zeta} dx \\
& = \frac{1}{2\pi} \left( J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5} \right) + \mathcal{A},
\end{align*}
\]

where

\[
\begin{align*}
J_{1,1} &= \int \partial^n R(x) \partial^n R_x(x) \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \frac{d\zeta}{\zeta} dx, \\
J_{1,2} &= -\int \partial^n R(x) \partial^n R_x(x + \zeta) \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \frac{d\zeta}{\zeta} dx, \\
J_{1,3} &= \int \partial^n R(x) \Delta_\zeta R_x(x) \partial^n \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \frac{d\zeta}{\zeta} dx, \\
J_{1,4} &= \varepsilon^{-\beta+1} \int \partial^n R(x) \partial^n \Delta_\zeta V(x) \left\{ \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] - \log \left[ 1 + \frac{[\varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \right\} \frac{d\zeta}{\zeta} dx, \\
J_{1,5} &= \varepsilon^{-\beta+1} \int \partial^n R(x) \Delta_\zeta V(x) \partial^n \left\{ \log \left[ 1 + \frac{[\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] - \log \left[ 1 + \frac{[\varepsilon \Delta_\zeta V(x)]^2}{|\zeta|^2} \right] \right\} \frac{d\zeta}{\zeta} dx.
\end{align*}
\]
When $\partial^n$ hits $R_x(x)$, we can form a total derivative and integrate by parts

$$J_{1,1} = - \int \left( \frac{\partial^n R(x) R(x)}{2} \partial_0 \log \left[ 1 + \frac{[\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)]}{|\zeta|^2} \right] \right) \partial^0 R(x) dx$$

$$= - \int |\partial^n R(x)|^2 \left[ \frac{\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)}{\zeta^2 + |\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)|^2} \right] \partial^0 R(x) dx$$

$$\leq \int |\partial^n R(x)|^2 \left[ \frac{\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)}{\zeta} \right] \cdot \left[ \frac{\varepsilon^\beta \Delta \zeta R_x(x) + \varepsilon \Delta \zeta V(x)}{\zeta} \right] dx$$

$$= J_{1,1,1} + J_{1,1,2},$$

where

$$J_{1,1,1} = \int_{[\zeta < 1]} |\partial^n R(x)|^2 \left[ \frac{\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)}{\zeta^2} \right] \partial^0 R(x) dx,$$

$$J_{1,1,2} = \int_{|\zeta| > 1} \int_{R} |\partial^n R(x)|^2 \left[ \frac{1}{\zeta^{1/2}} \frac{\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)}{\zeta} \right] \left[ \frac{\varepsilon^\beta \Delta \zeta R_x(x) + \varepsilon \Delta \zeta V(x)}{\zeta^{1/2}} \right] \partial^0 R(x) dx.$$

When $|\zeta|$ is large, we use the Sobolev embedding theorem and the fact that $\zeta \mapsto \zeta^{-2}$ is integrable at infinity to conclude that

$$J_{1,1,1} \leq C \|\partial^n R\|_2^2 (\varepsilon^\beta \|R\|_{L^\infty} + \varepsilon \|V\|_{L^\infty}) (\varepsilon^\beta \|R_x\|_{L^\infty} + \varepsilon \|V_x\|_{L^\infty}).$$

When $|\zeta|$ is small, we use the fact that $\zeta \mapsto |\zeta|^{-1/2}$ is locally integrable, and distribute the remaining $|\zeta|^{3/2}$ in the denominator to form difference quotients and Holder norms. We then bound the difference quotients by Sobolev norms to get

$$J_{1,1,2} \leq C \|\partial^n R\|_2^2 (\varepsilon^\beta \|R_x\|_{L^\infty} + \varepsilon \|V_x\|_{L^\infty}) (\varepsilon^\beta \|R_x\|_{C^{0,1/2}} + \varepsilon \|V_x\|_{C^{0,1/2}}).$$

It follows from the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^{0,1/2}(\mathbb{R})$ that $J_{1,1}$ satisfies the estimate

$$J_{1,1} \leq C(\varepsilon^2 E + \varepsilon \beta + E^{3/2} + \varepsilon^{2\beta} E^2).$$

We now consider the term $J_{1,2}$ that arises when $\partial^n$ hits $R_x(x + \zeta)$, where we convert a derivative in $x$ to a derivative in $\zeta$ and integrate by parts. It follows that

$$J_{1,2} = - \int \frac{\partial^n R(x)}{\partial_0} \partial^0 R(x + \zeta) \partial_0 \log \left[ 1 + \frac{[\varepsilon^\beta \Delta \zeta R(x) + \varepsilon \Delta \zeta V(x)]}{|\zeta|^2} \right] \partial^0 R(x + \zeta) dx$$

$$= -2 \int \frac{\partial^n R(x)}{\partial_0} \partial^n R(x + \zeta)$$

$$\left\{ \varepsilon^\beta \Delta \zeta R(x + \zeta) + \varepsilon \Delta \zeta V(x + \zeta) \right\} \frac{\partial^0 R(x + \zeta)}{\zeta^2 + |\varepsilon^\beta \Delta \zeta R(x + \zeta) + \varepsilon \Delta \zeta V(x)|^2} dx$$

$$\leq C \left( J_{1,2,1} + J_{1,2,2} \right),$$

where

$$J_{1,2,1} = \int_{|\zeta| > 1} |\partial^n R(x)| |\partial^n R(x + \zeta)|$$

$$\left\{ \varepsilon^\beta \Delta \zeta R(x + \zeta) + \varepsilon \Delta \zeta V(x + \zeta) \right\} \frac{\partial^0 R(x + \zeta)}{\zeta^2 + |\varepsilon^\beta \Delta \zeta R(x + \zeta) + \varepsilon \Delta \zeta V(x)|^2} dx,$$

$$J_{1,2,2} = \int_{|\zeta| < 1} |\partial^n R(x)| |\partial^n R(x + \zeta)|$$

$$\left\{ \varepsilon^\beta \Delta \zeta R(x + \zeta) + \varepsilon \Delta \zeta V(x + \zeta) \right\} \frac{\partial^0 R(x + \zeta) - D \partial_0 R(x + \zeta)}{\zeta^{1/2}} + \varepsilon \frac{V(x + \zeta) - D \partial_0 V(x)}{\zeta^{1/2}} dx.$$
The $J_{1,2,1}$ integral over $|\zeta| > 1$ is treated as before.

To treat $J_{1,2,2}$, where the integral is over $|\zeta| < 1$, we form a difference quotient and use the Hölder-norm bound

$$\left\| \frac{f_\zeta(x + \zeta) - D_\zeta f(x)}{\zeta} \right\|_{L^\infty_{x, \zeta}} \leq C\|f\|_{C^{0,1/2}}$$

which follows from the usual definition of the Hölder norm and the mean value theorem applied to the difference quotient $D_\zeta f(x)$. We obtain the bound

$$J_{1,2} \leq C\|\partial^n R\|_{L^2}^2 (\varepsilon \|R\|_{W^{1,\infty}} + \varepsilon \|V\|_{W^{1,\infty}}) (\varepsilon \|R\|_{C^{0,1/2}} + \varepsilon \|V\|_{C^{0,1/2}}).$$

We next consider the term that arises when $\partial^n$ is applied to the logarithm, and in particular $\Delta_\zeta R$ through the chain rule. Letting $g(x) = \varepsilon^\beta R(x) + \varepsilon V(x)$ and $\Delta_\zeta g(x) = \varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)$, we have

$$J_{1,3} = 2 \int \partial^n R(x) \Delta_\zeta R_x(x) \partial^{n-1} \left[ (|\zeta|^2 + |\Delta_\zeta g(x)|)^{-1} \Delta_\zeta g(x) \Delta_\zeta g_x(x) \right] \, d\zeta \, dx$$

$$= 2J_{1,3,1} + \left( (-1)^{n-1} 2^n (n-1)! \right) J_{1,3,2} + \mathcal{P},$$

where

$$J_{1,3,1} = \int \partial^n R(x) \Delta_\zeta R_x(x) \left( |\zeta|^2 + |\Delta_\zeta g(x)| \right)^{-1} \Delta_\zeta g(x) \partial^n \Delta_\zeta g(x) \, d\zeta \, dx,$$

$$J_{1,3,2} = \int \partial^n R(x) \Delta_\zeta R_x(x) \left( |\zeta|^2 + |\Delta_\zeta g(x)| \right)^{-n} (\Delta_\zeta g(x) \Delta_\zeta g_x(x))^n \, d\zeta \, dx.$$

We again split the integrals into regions of large and small $\zeta$. For $J_{1,3,1}$, we have

$$J_{1,3,1} \leq \int \int_{|\zeta| > 1} \left| \partial^n R(x) \right| |\Delta_\zeta R_x(x)| \left| \Delta_\zeta g(x) \right| \left| \partial^n \Delta_\zeta g(x) \right| \frac{1}{|\zeta|^2} \, d\zeta \, dx$$

$$+ \int \int_{|\zeta| < 1} \left| \partial^n R(x) \right| |\Delta_\zeta R_x(x)| \left| \Delta_\zeta g(x) \right| \left| \partial^n \Delta_\zeta g(x) \right| \frac{1}{|\zeta|^{1/2}} \, d\zeta \, dx$$

$$\leq C\left( \|\partial^n R\|_{L^2} \|R\|_{C^{0,1/2}} (\varepsilon^\beta \|R\|_{W^{1,\infty}} + \varepsilon \|V\|_{W^{1,\infty}}) (\varepsilon \|\partial^n R\|_{L^2} + \varepsilon \|\partial^n V\|_{L^2}) \right).$$

For the term $J_{1,3,2}$, we have

$$J_{1,3,2} \leq \int \int_{|\zeta| > 1} \left| \partial^n R(x) \Delta_\zeta R_x(x) \right| \left| \Delta_\zeta g(x) \Delta_\zeta g_x(x) \right| \frac{d\zeta}{\zeta^{2n}} \, dx,$$

$$+ \int \int_{|\zeta| < 1} \left| \partial^n R(x) \right| |\Delta_\zeta R_x(x)| \left( \frac{|\Delta_\zeta g(x)|}{|\zeta|} \right)^n \left( \frac{|\Delta_\zeta g_x(x)|}{|\zeta|} \right)^n \, d\zeta \, dx$$

$$\leq C\left( \|\partial^n R\|_{L^2} \|\partial R\|_{L^2} (\varepsilon^\beta \|R\|_{W^{1,\infty}} + \varepsilon \|V\|_{W^{1,\infty}})^{2n} \right).$$

To handle $J_{1,4}$, we apply the mean value inequality to the difference of the logarithms to obtain

$$\left| \log \left[ 1 + \frac{\varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)^2}{|\zeta|^2} \right] - \log \left[ 1 + \frac{\varepsilon \Delta_\zeta V(x)^2}{|\zeta|^2} \right] \right| \leq \left| \frac{d}{dc} \right|_{c = c_*} \left[ \log \left( 1 + \frac{c^2}{\zeta^2} \right) \right] \cdot \varepsilon^\beta \left| \Delta_\zeta R(x) \right|,$$

where $c_*$ is a value between $c = \varepsilon \Delta_\zeta V(x)$ and $c = \varepsilon^\beta \Delta_\zeta R(x) + \varepsilon \Delta_\zeta V(x)$ that maximizes

$$c \mapsto \frac{d}{dc} \log \left[ 1 + \frac{c^2}{\zeta^2} \right] = -\frac{2c}{\zeta^2 + c^2}.$$
Using $|c| \leq |\varepsilon^\beta \Delta_\zeta R(x)| + |\varepsilon \Delta_\zeta V(x)|$ we find that

$$J_{1,4} \leq 2\varepsilon \int |\partial^n R(x)||\Delta_\zeta \partial^m V(x)| \frac{|\varepsilon^\beta \Delta_\zeta R(x)| + \varepsilon |\Delta_\zeta V(x)|}{\zeta^2} |\Delta_\zeta R(x)| \, d\zeta \, dx$$

$$\leq \varepsilon^2 \|\partial^n R\|_{L^2} \|\partial^m V\|_{L^2} \|R\|_{W^{1,\infty}} \left( \varepsilon^{\beta-1} \|R\|_{W^{1,\infty}} + \|V\|_{W^{1,\infty}} \right),$$

where the last inequality follows by splitting the integration regions as usual.

In the $J_{1,5}$ term, we start by taking one derivative of the difference of logarithms

$$\partial \left\{ \log \left[ 1 + \frac{(\Delta_\zeta g(x))^2}{|\zeta|^2} \right] - \log \left[ 1 + \frac{|\varepsilon \Delta_\zeta V(x)|^2}{|\zeta|^2} \right] \right\} = \frac{2\Delta_\zeta g(x)\Delta_\zeta g_x(x)}{\zeta^2 + (\Delta_\zeta g(x))^2} - \frac{2\varepsilon^2 \Delta_\zeta V(x)\Delta_\zeta V_x(x)}{\zeta^2 + (\varepsilon \Delta_\zeta V(x))^2}.$$ 

When we consider only the terms that are quadratic in $V(x)$, we see that

$$\frac{2\varepsilon^2 \Delta_\zeta V(x)\Delta_\zeta V_x(x)}{\zeta^2 + (\Delta_\zeta g(x))^2} - \frac{2\varepsilon^2 \Delta_\zeta V(x)\Delta_\zeta V_x(x)}{\zeta^2 + (\varepsilon \Delta_\zeta V(x))^2}$$

$$= 2\varepsilon^2 \Delta_\zeta V(x)\Delta_\zeta V_x(x) \cdot \frac{\varepsilon^2 (\Delta_\zeta R(x))^2 - 2\varepsilon^{\beta+1} \Delta_\zeta R(x)\Delta_\zeta V(x)}{\zeta^2 + (\varepsilon \Delta_\zeta V(x))^2}.$$ 

Using these last two equalities in $J_{1,5}$, we have

$$J_{1,5} = \varepsilon^{\beta+1} \int \partial^n R(x) \Delta_\zeta V(x) \partial^{n-1} \left\{ \frac{2\Delta_\zeta V(x)\Delta_\zeta R_x(x)}{\zeta^2 + (\Delta_\zeta g(x))^2} \right\} d\zeta \, dx$$

$$+ \varepsilon^2 \int \partial^n R(x) \Delta_\zeta V(x) \partial^{n-1} \left\{ \frac{2\Delta_\zeta R(x)\Delta_\zeta V_x(x)}{\zeta^2 + (\Delta_\zeta g(x))^2} \right\} d\zeta \, dx$$

$$+ \varepsilon^2 \int \partial^n R(x) \Delta_\zeta V(x) \partial^{n-1} \left\{ \frac{2\Delta_\zeta V(x)\Delta_\zeta R_x(x)}{\zeta^2 + (\Delta_\zeta g(x))^2} \right\} d\zeta \, dx$$

$$- \varepsilon^{\beta+3} \int \partial^n R(x) \Delta_\zeta V(x) \partial^{n-1} \left\{ \frac{2\Delta_\zeta V(x)\Delta_\zeta V_x(x)(\Delta_\zeta R(x))^2}{\zeta^2 + (\varepsilon \Delta_\zeta V(x))^2} \left[ \zeta^2 + (\Delta_\zeta g(x))^2 \right] \right\}$$

$$- \varepsilon^4 \int \partial^n R(x) \Delta_\zeta V(x) \partial^{n-1} \left\{ \frac{4\Delta_\zeta V(x)\Delta_\zeta V_x(x)\Delta_\zeta R(x)\Delta_\zeta V(x)}{\zeta^2 + (\varepsilon \Delta_\zeta V(x))^2} \left[ \zeta^2 + (\Delta_\zeta g(x))^2 \right] \right\}.$$ 

Each of these terms can handled similarly to $J_{1,3}$, and the resulting estimate is

$$J_{1,5} \leq C\varepsilon^2 E(1 + \varepsilon^{\beta-1} E^{1/2} + \varepsilon^{2\beta-2} E^{1/2} + \varepsilon^{2\beta-2} E + \varepsilon^{2\beta n-2} E^n).$$

Putting these estimates together and making use of the bootstrap assumption [5.3], we obtain

$$J_1 \leq C\varepsilon^2 E(1 + \varepsilon^{\beta-1} E^{1/2} + \varepsilon^{2\beta-2} E^{1/2} + \varepsilon^{2\beta-2} E + \varepsilon^{2\beta n-2} E^n) \leq C\varepsilon^2 E.$$ 

5.5.2. Modified energy terms. The higher-order terms that involve the modified energy correction are $J_2 - J_6$. We begin with the term $J_2$. Using the skew-adjointness of the Hilbert transform and considering the term with the most derivatives on $R$, we have

$$J_2 = -\varepsilon \rho \int \partial^{n+1} \left( H[N_{\geq 3}[v] H[R] \right) \partial^n H[R] \, dx$$

$$= -\varepsilon \rho \int H[N_{\geq 3}[v]] H[R] \partial^n H[R] \, dx + \mathcal{R}$$

$$= -\varepsilon \rho \int \partial N_{\geq 3}[v] H \left[ (\partial^n H[R])^2 \right] \, dx + \mathcal{R}.$$
We can obtain a pointwise bound for

\[
\partial N_{\geq 3}[\varepsilon V](x) = \frac{\varepsilon}{2\pi} \int \Delta_\zeta V_2(x) \log \left[ 1 + \frac{[\varepsilon \Delta_\zeta V(x)]^2}{\zeta^2} \right] d\zeta
\]

\[
= \frac{\varepsilon}{2\pi} \int \Delta_\zeta \partial V_2(x) \log \left[ 1 + \frac{[\varepsilon \Delta_\zeta V(x)]^2}{\zeta^2} \right] d\zeta + \frac{\varepsilon}{2\pi} \int \Delta_\zeta V_x(x) \frac{2\varepsilon^2 \Delta_\zeta V(x) \Delta_\zeta V_x(x)}{\zeta^2 + [\varepsilon \Delta_\zeta V(x)]^2} d\zeta
\]

\[
\leq C \varepsilon,
\]

where the last inequality follows from the mean value inequality for the logarithm and \( C > 0 \) depends on \( V \). This implies that

\[
J_2 \leq C \varepsilon^2 (1 + E).
\]

We next consider \( J_3 + J_4 \), which is given by

\[
J_3 + J_4 = \varepsilon^{1-\beta} \rho \left[ \int \mathbf{H}[\partial V] \partial^n \mathbf{H} \left[ N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right] \partial^{n+1} \mathbf{H}[R] \, dx \right.
\]

\[
- \int \mathbf{H}[\partial V] \partial^{n+1} \mathbf{H}[R] \partial^n \mathbf{H} \left[ N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right] \, dx
\]

\[
+ (n+1) \int \mathbf{H}[\partial V] \partial^n \mathbf{H}[R] \partial^n \mathbf{H} \left[ N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right] \, dx + \mathcal{R}
\]

\[
= -(n+1) \varepsilon^{1-\beta} \rho \int \mathbf{H}[\partial V] \partial^n \mathbf{H}[R] \partial^n \mathbf{H} \left[ N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right] \, dx + \mathcal{R}.
\]

The remaining term on the right-hand side of this equation can be estimated in a similar way to before. After using commutators to cancel the Hilbert transforms, we get that

\[
\varepsilon^{1-\beta} \int \mathbf{H}[\partial V] \partial^n \mathbf{H}[R] \partial^n \mathbf{H} \left[ N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right] \, dx
\]

\[
= \varepsilon^{1-\beta} \left[ \int \mathbf{H}[\partial V] \partial^n \mathbf{R} \partial^n \mathbf{H} \left[ N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right] \, dx
\]

\[
- \int \partial^n \mathbf{H}[R] \left[ \mathbf{H}, \mathbf{H}[\partial V] \right] \left[ \partial^n \left( N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right) \right] \, dx \right].
\]

The first term is estimated in a similar way to \( J_1 \); the presence of the factor \( \mathbf{H}[\partial V] \) does not change the method. From Lemma [5.1], the second commutator term satisfies the estimate

\[
\varepsilon^{1-\beta} \int \partial^n \mathbf{H}[R] \left[ \mathbf{H}, \mathbf{H}[\partial V] \right] \left[ \partial^n \left( N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right) \right] \, dx
\]

\[
\leq C \varepsilon^{1-\beta} \left\| \partial^n \mathbf{R} \right\|_{L^2} \left\| \partial^{n+1} \mathbf{H}[V] \right\|_{L^\infty} \left\| N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V] \right\|_{L^2}.
\]
The important quantity to estimate here is the $L^2$ norm of the difference of the nonlinearities

$$
\left\| \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right\|_{L^2}^2
= \frac{1}{4\pi^2} \int \left\{ \int \varepsilon^R \Delta \chi R_x(x) \log \left( 1 + \frac{(\Delta \chi g(x))^2}{\zeta^2} \right) d\zeta 
+ \int \varepsilon \Delta \chi V_x(x) \log \left( 1 + \frac{(\Delta \chi g(x))^2}{\zeta^2} \right) d\zeta - \log \left( 1 + \frac{(\varepsilon \Delta \chi V(x))^2}{\zeta^2} \right) \right\}^2 dx
\leq C \int \left\{ \int \varepsilon^R |\Delta \chi R_x(x)| \log \left( 1 + \frac{(\Delta \chi g(x))^2}{\zeta^2} \right) d\zeta 
+ \int \varepsilon |\Delta \chi V(x)| \log \left( 1 + \frac{(\Delta \chi g(x))^2}{\zeta^2} \right) d\zeta - \log \left( 1 + \frac{(\varepsilon \Delta \chi V(x))^2}{\zeta^2} \right) \right\}^2 dx
\leq C \int \left[ \varepsilon^{R+1} |\Delta \chi R_x(x)| + \varepsilon^{R+1} |\Delta \chi V(x)| \right]^2 dx,
$$

where we use mean value inequalities on the logarithm. Using the Leibniz rule and under the bootstrap assumption (5.3), we can also show that for $m \in \mathbb{N} \cup \{0\}$

$$
\left\| \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right\|_{H^m} \leq C \varepsilon^{R+1} \left\| R \right\|_{H^{m+1}}. \tag{5.6}
$$

It then follows that

$$
J_3 + J_4 \leq C \varepsilon^2 (1 + E).
$$

Finally, we consider $J_5 + J_6$

$$
J_5 + J_6 = \rho \int \partial^n H[R] H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] \partial^{n+1} H[R] dx
+ \rho \int H[R] \partial^n H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] \partial^{n+1} H[R] dx
- \rho \int H[R] \partial^{n+1} H[R] \partial^n H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] dx
- (n + 1) \rho \int \partial H[R] \partial^n H[R] \partial^n H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] dx + \mathcal{R}
= -\frac{\rho}{2} \int \partial^n H[R]^2 \partial^n H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] dx
- (n + 1) \rho \int \partial H[R] \partial^n H[R] \partial^n H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] dx + \mathcal{R}.
$$

For the first term on the right-hand-side, we can use Hölder’s inequality, Sobolev embedding, estimates (5.6), and the bootstrap assumption (5.3) to show that

$$
\left| \int |\partial^n H[R]^2 | \partial \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right| dx
\leq C ||R||_{H^n}^2 \left\| \partial H \left[ \mathcal{N}_{\geq 3}[\varepsilon^R + \varepsilon V] - \mathcal{N}_{\geq 3}[\varepsilon V] \right] \right\|_{L^\infty}
\leq C \varepsilon^{R+1} E^{3/2} \leq C \varepsilon^2 E.
$$
For the remaining term, we use similar commutator estimates to the ones for $J_3 + J_4$, but distribute derivatives differently, to get

\[
\int \partial H[R] \partial^n H[R] \partial^n H[N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V]] \, dx
\leq C \|\partial^n R\|_{L^2} \|\partial^{n-1} H[R]\|_{L^\infty} \left\|N_{\geq 3}[\varepsilon^\beta R + \varepsilon V] - N_{\geq 3}[\varepsilon V]\right\|_{H^2}
\leq C \beta^3 + 1 \|R\|_{H^n} \|R\|_{H^3} \leq C \beta^3 + 1 E^{3/2} \leq C \varepsilon^2 E,
\]

where in the last line follows from Sobolev embeddings, estimates (5.6), and the bootstrap assumption (5.3).

5.6. **Energy estimate.** Using the estimates for $I_{1-13}$ and $J_{1-J_6}$ in (5.2), we find that

\[
\frac{dE}{dt} \leq C \varepsilon^2 (1 + E^{1/2} + E) \leq C \varepsilon^2 (1 + E)
\]

under the bootstrap assumption (5.3). By Gronwall’s inequality,

\[
\sup_{t \in [0,T/\varepsilon^2]} E(t) \leq C (E(0) + T) e^{CT}.
\]

Therefore, if $\varepsilon > 0$ is sufficiently small that

\[
C (E(0) + T) e^{CT} \leq \frac{1}{\varepsilon^2},
\]

the energy estimate (5.1) follows.

Then, using the a priori estimates for $R$ in (4.1) and the local wellposedness of (2.1), we can extend $\varphi$ to obtain a solution $\varphi \in C([0,T/\varepsilon^2]; H^n(\mathbb{R}))$ of (2.1) with $\varphi(\cdot, 0) = \varphi_0$ that satisfies the estimates (2.4). This completes the proof of Theorem 2.2.

**Appendix A. Contour dynamics for Euler vorticity fronts**

In this appendix, we will use contour dynamics to derive equation (1.2) for $\varphi(x,t)$, following the methods used in [16, 17] for SQG and GSQG fronts.

The streamfunction-vorticity formulation for the velocity $u(x,t) = (u(x,y,t), v(x,y,t))$ with $x = (x,y)$ in the two-dimensional incompressible Euler equations is [27]

\[
\alpha_t + u \cdot \nabla \alpha = 0, \quad u = \nabla^\perp \psi, \quad -\Delta \psi = \alpha, \quad \nabla^\perp = (-\partial_y, \partial_x),
\]

where $\psi(x,t)$ is the streamfunction and it is convenient to use the negative vorticity $\alpha(x,t)$.

For Euler front solutions with piecewise constant vorticities $-\alpha_+ \neq -\alpha_-$ that jump across $y = \varphi(x,t)$ and approach linear shear flows as $y \to \pm \infty$, we have

\[
\alpha(x,t) = \begin{cases} 
\alpha_+ & \text{if } y > \varphi(x,t), \\
\alpha_- & \text{if } y < \varphi(x,t), 
\end{cases} 
\quad u(x,t) = (\alpha_\pm y, 0) + o(1) \quad \text{as } y \to \pm \infty. \tag{A.1}
\]

We will assume that $\varphi$ satisfies the following conditions on a time interval $0 \leq t \leq T$ with $T > 0$:

(i) $\varphi(\cdot, t) \in C^{1,\gamma}(\mathbb{R})$ for some $\gamma > 0$;

(ii) $\varphi(x,t) = O(|x|^{-(1+\delta)})$ as $|x| \to \infty$ for some $\delta > 0$;

(iii) $\lim_{|x| \to \infty} \varphi(x,t) = c. \tag{A.2}$

In that case, the integrals below converge.
For any $h \in \mathbb{R}$, we denote the negative vorticity and velocity of a planar shear flow for a vorticity front located at $y = h$ by

$$
\tilde{\alpha}_h(y) = \begin{cases} 
\alpha_+ & \text{if } y > h, \\
\alpha_- & \text{if } y < h,
\end{cases}
\tilde{u}_h(y) = (\tilde{u}_h(y), 0),
$$
(A.3)

$$
\tilde{u}_h(y) = \frac{1}{2} \Xi y + \frac{1}{2} \Theta |y - h|,
\Xi = \alpha_+ - \alpha_-.
$$

We then decompose the front solution (A.1) as the sum of a shear flow and a perturbation

$$
\alpha(x, t) = \tilde{\alpha}_h(x) + \alpha^*_h(x, t),
\tilde{u}_h(y) = \tilde{u}_h(y) + u^*_h(x, t),
$$
(A.4)

with

$$
u^*_h = \nabla^\perp \psi^*_h, \quad -\Delta \psi^*_h = \alpha^*_h.
$$
(A.5)

We will use the following orientations for the unit tangent vectors on the front and the line $y = h$:

$$
t(x, t) = \left(1, \varphi(x, t)\right) \frac{1}{\sqrt{1 + \varphi^2(x, t)}} \quad \text{on } y = \varphi(x, t), \quad t(x, t) = (-1, 0) \quad \text{on } y = h.
$$
(A.6)

In the next two sections, we use two different choices of the parameter $h$ to derive (1.2). In the first section, we take $h = c$ to be the limiting displacement of the front in (A.2), which has the advantage that the standard potential representation for $u^*_h$ converges under mild additional assumptions on $\varphi$, but the disadvantage that the line $y = c$ may intersect the front $y = \varphi(x, t)$. In the second section, we choose $h < \inf_{x \in \mathbb{R}} \varphi(x, t)$, which has the advantage that the line $y = h$ does not intersect the front $y = \varphi(x, t)$, but the disadvantage that we have to modify the standard potential representation to get a convergent integral for $u^*_h$.

A.1. Contour dynamics equation I. We make the choice $h = c$ in (A.4), where $c$ is the far-field limit of the function $\varphi$ given in (A.2). Since $\varphi(\cdot, t)$ is continuous, the set $\{x \in \mathbb{R} : \varphi(x, t) \neq c\}$ is open, and, by the structure of open sets in $\mathbb{R}$, it is the disjoint union of countably many open intervals. We denote these open intervals by $I_n = (a_n, b_n)$, with $-\infty \leq a_n < b_n \leq a_{n+1} < b_{n+1} \leq \infty$, $n \in \mathbb{Z}$. Then the set $\Omega^*(t) = \text{supp} \alpha^*_c(\cdot, t)$ can be written as

$$
\Omega^*(t) = \bigcup_{n \in \mathbb{Z}} \Omega^*_n(t),
$$

where each $\Omega^*_n(t)$ has one of the forms

$$
\{(x, y) \in \mathbb{R}^2 : x \in I_n, c < y < \varphi(x, t)\} \quad \text{with } \alpha^*_n(x, t) = -\Theta,
\{(x, y) \in \mathbb{R}^2 : x \in I_n, c > y > \varphi(x, t)\} \quad \text{with } \alpha^*_n(x, t) = \Theta.
$$

Using the Biot-Savart law, we can express the velocity perturbation as

$$
u^*_c(x, t) = \frac{1}{2\pi} \int_{\Omega^*_c(t)} \frac{(x - x')^\perp}{|x - x'|^2} \alpha^*_c(x', t) \, dx' = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{\Omega_n^*(t)} \frac{(x - x')^\perp}{|x - x'|^2} \alpha^*_c(x', t) \, dx',
$$

where $(x, y)^\perp = (-y, x)$. For each $\Omega^*_n(t)$, we apply Green’s theorem to get

$$
\int_{\Omega^*_n(t)} \frac{(x - x')^\perp}{|x - x'|^2} \alpha^*_c(x', t) \, dx' = \Theta \int_{\partial \Omega^*_n(t)} t(x', t) \log |x - x'| \, ds(x'),
$$
where the tangent vector \( \mathbf{t} \) is defined as in (A.6). We then find that
\[
\int_{\Omega_c^+(t)} \frac{(x - x')}{|x - x'|^2} \alpha_c(x', t) \, dx' = \Theta \int_{\mathbb{R}} (1, \varphi_{x'}(x', t)) \log \left| \sqrt{(x - x')^2 + (\varphi(x, t) - \varphi(x', t))^2} \right| - (1, 0) \log \left| \sqrt{(x - x')^2 + (\varphi(x, t) - c)^2} \right| \, dx'.
\]

For unbounded components, a limiting procedure as in [16] can be used, under a mild additional decay condition that \( |\varphi(x', t) - c|/|x'| \) is integrable for large \( |x'| \), but we omit the details here. Summing these contributions, we get that
\[
u_c(x, t) = \frac{\Theta}{2\pi} \int_{\mathbb{R}} (1, \varphi_{x'}(x', t)) \log \left| \sqrt{(x - x')^2 + (\varphi(x, t) - \varphi(x', t))^2} \right| - (1, 0) \log \left| \sqrt{(x - x')^2 + (\varphi(x, t) - c)^2} \right| \, dx'.
\]

(A.7)

Let \( x = (x, \varphi(x, t)) \) be a point on the front and denote by
\[
\mathbf{n}(x, t) = \frac{1}{\sqrt{1 + \varphi_x^2(x, t)}} (\varphi_x(x, t), 1)
\]

(A.8)

the unit upward normal to the front. The front \( y = \varphi(x, t) \) moves with the upward normal velocity \( \mathbf{u} \cdot \mathbf{n} \), so using (A.4), we obtain that
\[
\varphi_t(x, t) = \sqrt{1 + \varphi_x^2(x, t)} \hat{u}_c(x) \cdot \mathbf{n}(x, t) + \sqrt{1 + \varphi_x^2(x, t)} u_c^*(x, t) \cdot \mathbf{n}(x, t).
\]

(A.9)

From (A.7) and (A.8), we have
\[
\sqrt{1 + \varphi_x^2(x, t)} \hat{u}_c(x, t) \cdot \mathbf{n}(x, t)
\]
\[
= -\frac{\Theta}{4\pi} \varphi_x(x, t) \int_{\mathbb{R}} \left\{ \log \left[ 1 + \left( \frac{\varphi(x, t) - \varphi(x', t)}{x - x'} \right)^2 \right] - \log \left[ 1 + \left( \frac{\varphi(x, t) - c}{x - x'} \right)^2 \right] \right\} \, dx'
\]
\[
+ \frac{\Theta}{4\pi} \int_{\mathbb{R}} \varphi_{x'}(x', t) \log \left( (x - x')^2 + (\varphi(x, t) - \varphi(x', t))^2 \right) \, dx'
\]
\[
= -\frac{\Theta}{4\pi} \varphi_x(x, t) \int_{\mathbb{R}} \log \left( 1 + \frac{\varphi(x, t) - \varphi(x', t)}{x - x'} \right)^2 \, dx'
\]
\[
+ \frac{\Theta}{4\pi} \int_{\mathbb{R}} \varphi_{x'}(x', t) \log \left( (x - x')^2 + (\varphi(x, t) - \varphi(x', t))^2 \right) \, dx',
\]

where we have used the identity
\[
\int_{\mathbb{R}} \log \left( 1 + \frac{a^2}{x^2} \right) \, dx = 2\pi|a|.
\]

We observe that the above integrals converge thanks to (A.2).

Moreover, from (A.3) and (A.8), we have
\[
\sqrt{1 + \varphi_x^2(x, t)} \hat{u}_c(x) \cdot \mathbf{n}(x, t) = -\frac{\Xi}{2} \varphi(x, t) \varphi_x(x, t) - \frac{\Theta}{2} \varphi_x(x, t) |\varphi(x, t) - c|.
\]

Using these expressions in (A.9), we get
\[
\varphi_t(x, t) = -\frac{\Xi}{4} \left( \varphi^2(x, t) \right)_x - \frac{\Theta}{4\pi} \varphi_x(x, t) \int_{\mathbb{R}} \log \left[ 1 + \left( \frac{\varphi(x, t) - \varphi(x', t)}{x - x'} \right)^2 \right] \, dx'
\]
\[
+ \frac{\Theta}{4\pi} \int_{\mathbb{R}} \varphi_{x'}(x', t) \log \left( (x - x')^2 + (\varphi(x, t) - \varphi(x', t))^2 \right) \, dx'.
\]
Then, using the identity
\[
\int_{\mathbb{R}} \varphi(x', t) \log((x-x')^2) \, dx' = 2 \text{p.v.} \int_{\mathbb{R}} \frac{\varphi(x', t) - c}{x-x'} \, dx' = 2\pi H[\varphi - c](x, t),
\]
and making the substitution \(x' = x + \zeta\), we find that \(\varphi\) satisfies
\[
\begin{align*}
\varphi_t(x, t) + \frac{\Xi}{4} \partial_x \left[ \varphi^2(x, t) \right] + \frac{\Theta}{4\pi} \int_{\mathbb{R}} & \left[ \varphi_x(x, t) - \varphi_x(x + \zeta, t) \right] \log \left[ 1 + \frac{[\varphi(x, t) - \varphi(x + \zeta, t)]^2}{|\zeta|^2} \right] d\zeta \\
& = \frac{\Theta}{2} H[\varphi - c](x, t).
\end{align*}
\]
(A.11)

Non-dimensionalizing the time variable by \(t \rightarrow \Theta t/2\) and setting \(c = 0\) without loss of generality, we obtain (A.2) with \(m = \Xi/\Theta\). Equation (A.11) agrees with previous results in [15] for the cubic front equation in the symmetric case with \(\Xi = 0\).

We remark that the corresponding dimensional version of the Burgers-Hilbert equation (1.3) is
\[
u_t + \left( \frac{\sqrt{\Xi^2 + \Theta^2}}{4} u^2 \right)_x = \frac{\Theta}{2} H[u].
\]
A.2. Contour dynamics equation II. We choose \(h \in \mathbb{R}\) such that
\[
h < \inf \{\varphi(x, t) : (x, t) \in \mathbb{R} \times [0, T]\}.
\]
The resulting \(\alpha^*_h\) is then
\[
\alpha^*_h(x, t) = \begin{cases} -\Theta & \text{if } h < y < \varphi(x, t) \\ 0 & \text{otherwise} \end{cases}
\]
We denote the support of \(\alpha^*_h(\cdot, t)\) by
\[
\Omega^*_t = \{(x, y) \in \mathbb{R}^2 : h < y < \varphi(x, t)\}.
\]
This choice of \(h\) guarantees that the front \(y = \varphi(x, t)\) does not intersect with the artificial front \(y = h\). However, the velocity integral using the usual Biot-Savart law does not converge. We therefore modify the Biot-Savart law by using a potential that vanishes at a fixed point \(x_0 = (x_0, y_0)\), which can be chosen outside \(\Omega_\ast\) for convenience, rather than at infinity. A class of solutions of (A.5) for \(u^*_h\) then has the Green’s function representation
\[
u^*_h(x, t) = \frac{\Theta}{2\pi} \int_{\Omega^*_t} \left\{ \frac{(x-x^\prime)^\perp}{|x-x^\prime|^2} - \frac{(x_0-x^\prime)^\perp}{|x_0-x^\prime|^2} \right\} dx^\prime + \tilde{u}(t),
\]
(A.12)
where \(\tilde{u}(t)\) is an arbitrary spatially uniform velocity. We will choose \(\tilde{u}(t)\) so that \(u(x, t)\) has the asymptotic behavior in (A.1) as \(|y| \rightarrow \infty\). The integral in (A.12) converges absolutely, since, if \(x^\prime = (x^\prime, y^\prime)\), the integrand is \(O(|x^\prime|^{-2})\) as \(|x^\prime| \rightarrow \infty\) and compactly supported in \(y^\prime\).

We remark that the corresponding integral representation of \(u^*_h\) using the generalized Biot-Savart law in the SQG and GSQG equations converges absolutely, so it is not necessary to modify the standard generalized Biot-Savart kernel in that case [16, 17].

Writing
\[
\frac{(x-x^\prime)^\perp}{|x-x^\prime|^2} - \frac{(x_0-x^\prime)^\perp}{|x_0-x^\prime|^2} = -\nabla_{x^\prime} \left\{ \log |x-x^\prime| - \log |x_0-x^\prime| \right\},
\]
applying Green’s theorem in (A.5) on a truncated region with \(|x-x^\prime| < \lambda\), and taking the limit \(\lambda \rightarrow \infty\), we get that
\[
u^*_h(x, t) = \frac{\Theta}{2\pi} \int_{\partial \Omega^*_t} t(x^\prime, t) \left\{ \log |x-x^\prime| - \log |x_0-x^\prime| \right\} ds(x^\prime) + \tilde{u}(t),
\]
(A.13)
where \(t\) is the negatively oriented unit tangent vector on \(\partial \Omega^*_t\) defined as in (A.6).
If \( \hat{u} = (\hat{u}, \hat{v}) \), then the component form of (A.13) is

\[
\begin{align*}
\mathbf{u}_h^*(x, t) &= (u^*(x, y, t), v^*(x, y, t)) \\
u_h^*(x, y, t) &= \Theta \frac{\partial}{\partial x} \int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] d\mathbf{x}' + \bar{u}(t), \\
v_h^*(x, y, t) &= \Theta \frac{\partial}{\partial y} \int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] \varphi_{x'}(x', t) d\mathbf{x}' + \bar{v}(t).
\end{align*}
\]

The integral for \( u_h^* \) converges since the integrand is \( O(|x'|^{-2}) \) as \( |x'| \to \infty \), while the integral for \( v_h^* \) converges since \( \varphi_{x'}(x', t) = O(|x'|^{-(1+\delta)}) \) as \( |x'| \to \infty \).

Since \( \varphi(x, t) \to c \) as \( |x| \to \infty \), we have as \( |y| \to \infty \) that

\[
\begin{align*}
\int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] d\mathbf{x}' &= |y| \int_{\mathbb{R}} \log \left[ \frac{\eta^2}{\eta^2 + (1-h/y)^2} \right] d\eta \\
&= |y| \int_{\mathbb{R}} \log \left[ \frac{\eta^2}{\eta^2 + (1-h/y)^2} \right] d\eta + o(1) \\
&= -2(c+h) \text{sgn} y \int_{\mathbb{R}} \frac{1}{1+\eta^2} d\eta + o(1) \\
&= -2\pi(c+h) \text{sgn} y + o(1).
\end{align*}
\]

We also have that

\[
\begin{align*}
\int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] \varphi_{x'}(x', t) d\mathbf{x}' &= \log |y| \int_{\mathbb{R}} \varphi_{x'}(x', t) d\mathbf{x}' + \int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] \varphi_{x'}(x', t) d\mathbf{x}' \\
&= \log |y| \int_{\mathbb{R}} \varphi_{x'}(x', t) d\mathbf{x}' + o(1) \quad \text{as } |y| \to \infty,
\end{align*}
\]

so the \( y \)-component of the velocity approaches zero if \( \int \varphi_{x'}(x', t) d\mathbf{x}' = 0 \), which is the case if \( \varphi \) satisfies (A.2)

Under the assumptions in (A.2), it follows that the velocity perturbations have the asymptotic behavior as \( |y| \to \infty \)

\[
\begin{align*}
u^*_h(x, y, t) &= \frac{1}{2} \Theta(c + h) \text{sgn} y + \bar{u}(t) - u_\infty(t) + o(1), \\
v^*_h(x, y, t) &= \bar{v}(t) - v_\infty(t) + o(1), \\
u_\infty(t) &= \Theta \frac{\partial}{\partial x} \int_{\mathbb{R}} \log \left[ \frac{(x_0-x')^2 + (y_0 - \varphi(x', t))^2}{(x_0-x')^2 + (y_0 - h)^2} \right] d\mathbf{x}', \\
v_\infty(t) &= \Theta \frac{\partial}{\partial y} \int_{\mathbb{R}} \log \left[ \frac{(x_0-x')^2 + (y_0 - \varphi(x', t))^2}{(x_0-x')^2 + (y_0 - h)^2} \right] \varphi_{x'}(x', t) d\mathbf{x}'.
\end{align*}
\]

We choose \( \hat{u} = (u_\infty, v_\infty) \) in (A.13), in which case, using the integral

\[
\int_{\mathbb{R}} \log \left[ \frac{x^2 + a^2}{x^2 + b^2} \right] dx = 2\pi \log (|a| - |b|) \quad \text{(A.14)}
\]

\( \Theta \) to replace \( h \) by \( c \) in \( u \), we find that the full velocity field \( \mathbf{u} = (u, v) \) can be written as

\[
\begin{align*}
u(x, y, t) &= \Theta \frac{\partial}{\partial x} \int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] d\mathbf{x}' + \frac{1}{2} \text{sgn} y + \frac{1}{2} \Theta|y - c|, \\
v(x, y, t) &= \Theta \frac{\partial}{\partial y} \int_{\mathbb{R}} \log \left[ \left( \frac{x-x'}{y} \right)^2 + \left( \frac{y-y'}{y} \right)^2 \right] \varphi_{x'}(x', t) d\mathbf{x}'.
\end{align*}
\]
This velocity has the far-field behavior in (A.1) as \(|y| \to \infty\).

If \(x = (x, \varphi(x,t))\) is a point on the front and \(\varphi = \varphi(x,t)\), then
\[
\begin{align*}
u(x, \varphi, t) &= \frac{\Theta}{4\pi} \int_{\mathbb{R}} \log \left[ \frac{(x - x')^2 + (\varphi(x,t) - \varphi(x',t))^2}{(x - x')^2 + (\varphi(x,t) - c)^2} \right] dx' + \frac{1}{2} \Xi \varphi + \frac{1}{2} \Theta |\varphi(x,t) - c|, \\
v(x, \varphi, t) &= \frac{\Theta}{4\pi} \int_{\mathbb{R}} \log \left[ \frac{(x - x')^2 + (\varphi(x,t) - \varphi(x',t))^2}{(x - x')^2 + (\varphi(x,t) - c)^2} \right] \varphi_x(x', t) dx'.
\end{align*}
\]

Imposing the condition that the front \(y = \varphi(x,t)\) moves with the upward normal velocity \(\mathbf{u} \cdot \mathbf{n}\), we get that
\[
\varphi_t + \frac{1}{2} (\Xi \varphi + \Theta |\varphi - c| + \Theta F_1) \varphi_x = \frac{1}{2} \Theta F_2,
\]
where
\[
F_1(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \log \left[ \frac{(x - x')^2 + (\varphi(x,t) - \varphi(x',t))^2}{(x - x')^2} \right] dx',
\]
\[
F_2(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_x(x', t) \log \left[ \frac{(x - x')^2 + (\varphi(x,t) - \varphi(x',t))^2}{(x - x')^2} \right] dx'.
\]

Using (A.10) and (A.14), we can write
\[
\begin{align*}
F_1(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \log \left[ \frac{(x - x')^2 + (\varphi(x,t) - \varphi(x',t))^2}{(x - x')^2} \right] dx' - |\varphi(x,t) - c|, \\
F_2(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_x(x', t) \log \left[ \frac{(x - x')^2 + (\varphi(x,t) - \varphi(x',t))^2}{(x - x')^2} \right] dx' + \mathbf{H}|\varphi - c|(x,t).
\end{align*}
\]

Using these expressions in (A.15), simplifying the result, and substituting \(x' = x + \zeta\), we get (A.11) as before.

**Appendix B. Multilinear symbol estimates**

In this appendix, we state a lemma that estimate the multilinear symbol in \(\mathcal{N}_{\geq 5}\), and the proof is similar to the Case II in the proof of Proposition 3.1 in [18].

**Lemma B.1.** Let \(T_n\) be defined by (3.3) for an integer \(n \geq 2\). Then
\[
|T_n(\eta_n)| \leq 16|\eta_{2n}| \cdot \prod_{j=1}^{2n-2} |\eta_j| \text{ for all } |\eta_1| \leq |\eta_2| \leq \cdots \leq |\eta_{2n+1}|.
\]

**Proof.** Without loss of generality, we assume that \(|\eta_{j_1}| \leq |\eta_{j_2}| \leq \cdots \leq |\eta_{j_{2n}}| \leq |\eta_{2n+1}|\), define \(\eta_{j_{2n+1}} = \eta_{2n+1}\), and split the integral of \(T_n\) (3.3) into three parts.

\[
T_n(\eta_n) = T_n^{low}(\eta_n) + \sum_{k=1}^{2n-1} T_n^{med,(k)}(\eta_n) + T_n^{high}(\eta_n),
\]

\[
T_n^{low}(\eta_n) = \int_{|\eta_{j_{2n}}| \zeta < 2} \frac{\prod_{j=1}^{2n+1} (1 - e^{i\eta_j \zeta})}{\zeta^{2n}} d\zeta,
\]

\[
T_n^{med,(k)}(\eta_n) = \int_{|\eta_{j_{k+1}}| \leq |\zeta| \leq |\eta_{j_k}|} \frac{\prod_{j=1}^{2n+1} (1 - e^{i\eta_j \zeta})}{\zeta^{2n}} d\zeta,
\]

\[
T_n^{high}(\eta_n) = \int_{|\eta_{j_1}| > 2} \frac{\prod_{j=1}^{2n+1} (1 - e^{i\eta_j \zeta})}{\zeta^{2n}} d\zeta.
\]
To estimate (B.1), we notice that
\[
|T_n^{low}(\eta_n)| \leq 2 \prod_{k=1}^{2n} |\eta_k| \cdot \int_{|\eta_{j_2n}| < 2} \left( \prod_{k=1}^{2n} \left| 1 - e^{i\eta_k \zeta} \right| \right) \left| 1 - e^{i\eta_{2n+1}\zeta} \right| d\zeta
\]
\[
\leq 2 \prod_{k=1}^{2n} |\eta_k| \cdot \int_{|\eta_{j_2n}| < 2} 2 d\zeta
\]
\[
\leq 8 \cdot 2^{n-1} \prod_{k=1}^{2n-1} |\eta_k|.
\]
If \( k = 2n - 1 \), we have
\[
|T_n^{med, (2n-1)}(\eta_n)| \leq 8 \prod_{\ell=1}^{2n} |\eta_{j_\ell}| \cdot \left( |\eta_{j_{2n-1}}| - 1 - |\eta_{j_{2n}}|^{-1} \right)
\]
\[
\leq 16 |\eta_{2n}| \cdot \prod_{\ell=1}^{2n-2} |\eta_{j_\ell}|.
\]
For each \( 1 \leq k \leq 2n - 2 \), we estimate (B.2) as
\[
|T_n^{med, (k)}(\eta_n)| \leq \prod_{\ell=1}^{k} |\eta_{j_\ell}| \cdot \int_{\frac{2}{\eta_{j_{k+1}}}}^{\frac{2}{|\eta_k|}} \left( \prod_{\ell=1}^{k} \left| 1 - e^{i\eta_{j_\ell} \zeta} \right| \right) \cdot \frac{\prod_{\ell=k+1}^{2n+1} |1 - e^{i\eta_{j_\ell} \zeta}|}{|\zeta|^{2n-k}} d\zeta
\]
\[
\leq 2^{2n-k} k \prod_{\ell=1}^{k} |\eta_{j_\ell}| \cdot \int_{\frac{2}{|\eta_k|}}^{2} |\zeta|^{-2n+k} d\zeta
\]
\[
\leq \frac{8}{2n-k-1} \left( |\eta_{j_k}|^{2n-k-1} + |\eta_{j_{k+1}}|^{2n-k-1} \right) \prod_{\ell=1}^{k} |\eta_{j_\ell}|
\]
\[
\leq 16 \cdot 2^{n-1} \prod_{k=1}^{2n-1} |\eta_k|.
\]
As for (B.3), we have
\[
|T_n^{high}(\eta_n)| \leq |\eta_{j_1}| \int_{|\eta_{j_1}| > 2} \left( \prod_{k=2}^{2n+1} \left| 1 - e^{i\eta_j \zeta} \right| \right) \cdot \frac{|1 - e^{i\eta_{j_1} \zeta}|}{|\eta_{j_1} \zeta|} |\zeta| d\zeta
\]
\[
\leq \frac{2^{2n} |\eta_{j_1}|}{|\eta_{j_1}|} \int_{|\eta_{j_1}| > 2} \frac{d\zeta}{|\zeta|^{2n-1}}
\]
\[
\leq \frac{8}{n-1} \prod_{k=1}^{2n-1} |\eta_k|.
\]
Collecting these estimates and using the fact that \( |\eta_{2n}| \geq |\eta_{2n-1}| \), we conclude the lemma. \( \square \)

**APPENDIX C. A LEMMA**

In this section, we show that the removal of the linear evolution \( e^{tH} \) from (1.4) leads to (2.5).
Lemma C.1. Suppose that $v(x, \tau)$ and $w(x, t)$ are related by \eqref{2.3}. Then $v(x, \tau)$ satisfies \eqref{2.5} if and only if $w(x, t)$ satisfies \eqref{1.4}.

Proof. We write \eqref{1.4} as

$$w_t + \frac{m^2 + 1}{6} \partial_x M(w, w, w) = H[w], \quad M(w, w, w) = 3w^2|\partial_x|w - 3w|\partial_x|w^2 + |\partial_x|w^3,$$

where $M$ is a symmetric trilinear operator with

$$M(e^{ikx}, e^{i\xi x}, e^{i\eta x}) = m(k, \xi, \eta)e^{i(k+\xi+\eta)x},$$

$$m(k, \xi, \eta) = |k| + |\xi| + |\eta| - |k + \xi| - |\xi + \eta| - |k + \eta| + |k + \xi + \eta|.$$ 

Then, using \eqref{2.3} in \eqref{1.4}, we find that

$$v_t + \frac{m^2 + 1}{6} e^{-tH} \partial_x M(e^tHv, e^tHv, e^tHv) = 0,$$

so we just have to show that $M(e^tHv, e^tHv, e^tHv) = e^tH M(v, v, v)$, but this follows from the fact that

$$\text{sgn } k + \text{sgn } \xi + \text{sgn } \eta = \text{sgn } (k + \xi + \eta)$$

whenever $m(k, \xi, \eta) \neq 0$. \qed

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