Non-minimally coupled canonical, phantom and quintom models of holographic dark energy

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We investigate canonical, phantom and quintom models, with the various fields being non-minimally coupled to gravity, in the framework of holographic dark energy. We classify them and we discuss their cosmological implications. In particular, we examine the present value of the dark energy equation-of-state parameter and the crossing through the phantom divide, and we extract the conditions for a future cosmological singularity. The combined scenarios are in agreement with observations and reveal interesting cosmological behaviors.

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I. INTRODUCTION

Nowadays it is strongly believed that the universe is experiencing an accelerated expansion, and this is supported by many cosmological observations, such as SNe Ia [1], WMAP [2], SDSS [3] and X-ray [4]. These observations suggest that the universe is dominated by dark energy with negative pressure, which provides the dynamical mechanism of the accelerating expansion of the universe. Although the nature and origin of dark energy could perhaps understood by a fundamental underlying theory unknown up to now, physicists can still propose some paradigms to describe it. In this direction we can consider theories of modified gravity [5], or field models of dark energy. The field models that have been discussed widely in the literature consider a cosmological constant [6], a canonical scalar field (quintessence) [7], a phantom field, that is a scalar field with a negative sign of the kinetic term [8, 9], or the combination of quintessence and phantom in a unified model named quintom [10]. The quintom paradigm intends to describe the crossing of the dark-energy equation-of-state parameter \( w_\Lambda \) through the phantom divide \(-1\) from above. Additionally, we investigate the possibility of a future \( w_\Lambda \)-divergence [9, 24], and the specification of the time that is it going to happen. The plan of the work is as follows: In section II we construct the cosmological scenarios of non-minimally coupled canonical, phantom and quintom fields, in the framework of holographic dark energy. In particular, we examine the current value of \( w_\Lambda \) and the realization of a recent crossing through the phantom divide \(-1\) from above. Additionally, we investigate the possibility of a future \( w_\Lambda \)-divergence [9, 24], and the specification of the time that is it going to happen. The plan of the work is as follows: In section II we construct the cosmological scenarios of non-minimally coupled canonical, phantom and quintom fields, in the framework of holographic dark energy. In section III we examine their behavior and we discuss their cosmological implications. Finally, in section IV we summarize our results.

II. NON-MINIMALLY COUPLED FIELDS IN THE FRAMEWORK OF HOLOGRAPHIC DARK ENERGY

Let us describe briefly the holographic dark energy framework [14–17]. In this dark-energy model one determines an appropriate quantity to serve as an infrared cut-off for the theory, and imposes the constraint that the total vacuum energy in the corresponding maximum volume must not be greater than the mass of a black hole of the same size. By saturating the inequality one identifies
the acquired vacuum energy as holographic dark energy:

$$\rho_\Lambda = \frac{3c^2}{8\pi GL^2},$$  \hspace{1cm} (1)

with $L$ the IR cut-off and $c$ a constant which can be set to 1. Although the choice of $L$ has raised a discussion in the literature [16, 25], in this work we will use the Hubble scale. Note that the aforementioned choice for the IR cut-off has been found to have problems in conventional, minimally-coupled frameworks [15], but this is not anymore the case if one considers non-minimal coupling, as we do in the present work. Finally, we mention that the extension of holographic dark energy in the presence of non-minimally coupled fields could possibly raise some theoretical questions, but we assume that such an extension is valid. The detailed examination of this subject is left for a future work.

In the following, we are going to investigate holographic dark energy in the presence of canonical, phantom, or both fields, non-minimally coupled to gravity. The space-time geometry will for simplicity be a flat Robertson-Walker:

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2d\Omega^2),$$  \hspace{1cm} (2)

with $a(t)$ the scale factor.

### A. Canonical field

We first consider a canonical scalar field with a non-minimal coupling. This case has been partially investigated in [26], and here we extend it. The action of the universe is

$$S = \int d^4x\sqrt{-g} \left[ \frac{1}{2\kappa^2}R - \frac{1}{2}\xi_\phi \phi'^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + \mathcal{L}_M \right],$$  \hspace{1cm} (3)

where $\kappa^2$ is a gravitational constant. In the action we have added a canonical scalar field $\phi$, which in non-minimally coupled to the curvature with coupling parameter $\xi_\phi$. Although we could include a specific potential (quadratic or exponential), for the scope of the present work and for simplicity we keep the form (3) since our results can be easily generalized to these potential-cases. Lastly, the term $\mathcal{L}_M$ accounts for the matter content of the universe.

The presence of the non-minimal coupling leads to the effective Newton’s constant:

$$8\pi G_{\text{eff}} = \kappa^2 \left( 1 - \xi_\phi \kappa^2 \phi^2 \right)^{-1}.$$  \hspace{1cm} (4)

The Friedmann equations and the evolution equation for the scalar field are [22]:

$$H^2 - \frac{\kappa^2 \left( \rho_M + \rho_\Lambda + \frac{1}{2}\phi'^2 + 6\xi_\phi H\phi \right)}{3 \left( 1 - \xi_\phi \kappa^2 \phi^2 \right)} = 0$$  \hspace{1cm} (5)

$$\ddot{\phi} + 3H\dot{\phi} + 6\xi_\phi \left( H + 2H^2 \right)\phi = 0$$  \hspace{1cm} (6)

$$\dot{\rho}_M + \dot{\rho}_\Lambda + 3H \left( \rho_M + \rho_\Lambda + p_M + p_\Lambda \right) = 0,$$  \hspace{1cm} (7)

where $H = \dot{a}/a$ is the Hubble parameter. In these expressions, $p_M$ and $\rho_M$ are respectively the pressure and energy density of the matter content of the universe. Finally, $\rho_\Lambda$ and $\rho_\Lambda$ are the corresponding components of dark energy, which as usual is attributed to the scalar field. Since we use the Hubble scale to define holographic dark energy, that is we take $L = H^{-1}$, (1) can be written as $\rho_\Lambda = 3(8\pi G_{\text{eff}})^{-1}H^2$, which, due to the effective nature of the Newton’s constant (4), leads to:

$$\rho_\Lambda = \frac{3}{\kappa^2} \left( 1 - \xi_\phi \kappa^2 \phi^2 \right) H^2.$$  \hspace{1cm} (8)

We are interested in extracting power-law solutions of the cosmological model (5)-(7), in the case of a dark-energy dominated universe ($\rho_M, \rho_M \ll 1$). Thus, we are looking for solutions of the form:

$$a(t) = a_0 t^{w},$$  \hspace{1cm} (9)

$$\phi(t) = \phi_0 t^{w_\phi}.$$  \hspace{1cm} (10)

Insertion of these ansatzes in equations (5),(6) yields:

$$s_\phi (s_\phi - 1) + 3rs_\phi + 6r(2r - 1)\xi_\phi = 0$$
$$s_\phi + 12\xi_\phi r = 0.$$  \hspace{1cm} (11)

As we can easily see, the case of conformal coupling ($\xi_\phi = 1/6$) is not interesting since it leads to the trivial case $r = s_\phi = 0$. Thus, for $\xi_\phi \neq 1/6$ we obtain:

$$r = \frac{1}{4 - 24\xi_\phi},$$
$$s_\phi = -\frac{3\xi_\phi}{1 - 6\xi_\phi}.$$  \hspace{1cm} (12)

leading to:

$$a(t) = a_0 \left( \frac{t^{\frac{1}{4-24\xi_\phi}}}{t^{\frac{1}{1-6\xi_\phi}}} \right),$$
$$\phi(t) = \phi_0 \left( \frac{t^{\frac{3\xi_\phi}{1-6\xi_\phi}}}{t^{\frac{3\xi_\phi}{1-6\xi_\phi}}} \right).$$  \hspace{1cm} (13)

We can use expression (8) in order to acquire $\rho_\Lambda(t)$:

$$\rho_\Lambda(t) = \frac{3}{\kappa^2} \left( t^{-2} - \xi_\phi \kappa^2 \phi_0^2 t^{2s_\phi - 2} \right).$$  \hspace{1cm} (14)

Substitution into (7) then straightforwardly provides $p_\Lambda$:

$$p_\Lambda(t) = \frac{1}{\kappa^2} t^{2s_\phi - 3} \left[ t^{-2}(2 - 3r) + \xi_\phi \phi_0^2 t^{2s_\phi - 2}(3r + 2s_\phi - 2) \right].$$  \hspace{1cm} (15)

In expressions (13) and (14), $r$ and $s_\phi$ are given by (11). Hence, we can calculate the dark energy equation-of-state parameter $w_\Lambda(t)$ as:

$$w_\Lambda(t) = \frac{p_\Lambda(t)}{\rho_\Lambda(t)} = \frac{5}{3} - 8\xi_\phi \left( 2 + \frac{\xi_\phi \kappa^2 \phi_0^2}{t^{2s_\phi - \xi_\phi \kappa^2 \phi_0^2}} \right).$$  \hspace{1cm} (16)


Relation (15) allows us to determine both the of $w_\Lambda$-evolution, as well as its current value $w_\Lambda(0)$. In order to express it in a more convenient form for comparison with observations, we can set the current values $t_0 = 1$ and $\sigma_0 = 1$, and use $r \ln t = \ln a = -\ln(1 + z)$ with $z$ the redshift. Therefore, we acquire:

$$w_\Lambda(z) = \frac{5}{3} - 8\xi_\phi \left[ 2 + \frac{\xi_\phi \kappa^2 \phi_0^2}{e^{-24\xi_\phi \ln(1+z)} - \xi_\phi \kappa^2 \phi_0^2} \right].$$  \hspace{1cm} (16)

Expression (16) provides $w_\Lambda(z)$ in terms of the coupling parameter $\xi_\phi$ and the amplitude $\phi_0$. We discuss the cosmological implications for various sub-classes of the present model, in section III.

B. Phantom field

In this subsection we consider a phantom field with a non-minimal coupling, that is a field with an opposite sign in the kinetic term in the Lagrangian [8, 9]. Such models are widely used in order to acquire $w_\Lambda < -1$. The action of the universe is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} \xi_\sigma \kappa^2 \sigma^2 - \frac{1}{2} \frac{g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \mathcal{L}_M}{1 - \xi_\sigma \kappa^2 \sigma^2} \right],$$  \hspace{1cm} (17)

and the presence of the non-minimal coupling leads to the effective Newton’s constant:

$$8\pi G_{eff} = \kappa^2 \left( 1 - \xi_\sigma \kappa^2 \sigma^2 \right)^{-1}. \hspace{1cm} (18)$$

The cosmological equations and the evolution equation for the phantom field are [8]:

$$H^2 - \frac{\kappa^2 \left( \rho_M + \rho_\Lambda - \frac{1}{2} \phi^2 + 6\xi_\sigma H \sigma \right)}{3 \left( 1 - \xi_\sigma \kappa^2 \sigma^2 \right)} = 0 \hspace{1cm} (19)$$

$$\dot{\sigma} + 3H\dot{\sigma} - 6\xi_\sigma \left( \dot{H} + 2H^2 \right) \sigma = 0 \hspace{1cm} (20)$$

$$\dot{\rho}_M + \dot{\rho}_\Lambda + 3H \left( \rho_M + \rho_\Lambda + p_M + p_\Lambda \right) = 0. \hspace{1cm} (21)$$

Similarly to the previous subsection, the use of the Hubble scale in the definition of holographic dark energy, and the effective nature of the Newton’s constant (18), lead to:

$$\rho_\Lambda = \frac{3}{\kappa^4} \left( 1 - \xi_\sigma \kappa^2 \sigma^2 \right) H^2. \hspace{1cm} (22)$$

We examine power-law solutions of equations (19)-(21), in the case of a dark-energy dominated universe ($\rho_M, \rho_M \ll 1$). Thus, we impose:

$$a(t) = a_0 t^r \hspace{1cm} \sigma(t) = \sigma_0 t^{s_\sigma}. \hspace{1cm} (23)$$

Insertion in equations (19),(20) yields:

$$s_\sigma (s_\sigma - 1) + 3r s_\sigma - 6r (2r - 1)\xi_\sigma = 0$$

$$\sigma_\sigma - 12\xi_\sigma r = 0. \hspace{1cm} (24)$$

As we can easily see, the case $\xi_\sigma = -1/6$ leads to the trivial case $r = s_\sigma = 0$. Thus, for $\xi_\sigma \neq -1/6$ we obtain:

$$r = \frac{1}{4 + 24\xi_\sigma}$$

$$s_\sigma = \frac{3\xi_\sigma}{1 + 6\xi_\sigma}, \hspace{1cm} (25)$$

leading to:

$$a(t) = a_0 t^{\frac{1}{1+6\xi_\sigma}}$$

$$\sigma(t) = \sigma_0 t^{\frac{3\xi_\sigma}{1+6\xi_\sigma}}. \hspace{1cm} (26)$$

Using (22) we acquire:

$$\rho_\Lambda(t) = \frac{3}{\kappa^2} \sigma^{2} \left( t^{-2} - \xi_\sigma \kappa^2 \sigma_0^{2} t^{2s_\sigma - 2} \right), \hspace{1cm} (27)$$

and thus (21) gives:

$$\rho_\Lambda(t) = \frac{1}{\kappa^2} r \left[ t^{-2} (2 - 3r) + \xi_\sigma \kappa^2 \sigma_0^{2} t^{2s_\sigma - 2} (3r + 2s_\sigma - 2) \right], \hspace{1cm} (28)$$

where $r$ and $s_\sigma$ are given by (25). We can calculate the dark energy equation-of-state parameter $w_\Lambda(t)$ as:

$$w_\Lambda(t) = \frac{\rho_\Lambda(t)}{\rho_\Lambda(t)} = \frac{5}{3} + 8\xi_\sigma \left( 2 + \frac{\xi_\sigma \kappa^2 \sigma_0^2}{t^{-2s_\sigma - 2 - \xi_\sigma \kappa^2 \sigma_0^2}} \right). \hspace{1cm} (29)$$

Finally, similarly to the previous subsection, we can express (29) in terms of the redshift $z$ obtaining:

$$w_\Lambda(z) = \frac{5}{3} + 8\xi_\sigma \left[ 2 + \frac{\xi_\sigma \kappa^2 \sigma_0^2}{\kappa^2 \sigma_0^2 \ln(1+z) - \xi_\sigma \kappa^2 \sigma_0^2} \right]. \hspace{1cm} (30)$$

Relation (30) provides $w_\Lambda(z)$ in terms of the coupling parameter $\xi_\sigma$ and the amplitude $\sigma_0$. We examine it in detail in section III.

C. Quintom model

In this subsection we consider the quintom cosmological scenario [10], that is we consider simultaneously a canonical and a phantom field, both with non-minimally coupling. As we have stated in the introduction, this combined cosmological paradigm has been shown to be capable to describe the crossing of the phantom divide $w_\Lambda = -1$. The action of the model is:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} \xi_\phi \phi^2 - \frac{1}{2} \xi_\sigma \kappa^2 \sigma^2 - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \mathcal{L}_M \right]. \hspace{1cm} (31)$$
and the presence of the non-minimal coupling leads to the effective Newton’s constant:
\[ 8\pi G_{\text{eff}} = \kappa^2 \left[ 1 - \kappa^2 (\xi \phi^2 + \xi_\sigma \sigma^2) \right]^{-1}. \]  
(32)

The cosmological equations and the evolution equation for the canonical and phantom fields are [10]:
\[ H^2 - \frac{\kappa^2 (\rho_M + \rho_\Lambda + \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 + 6 \xi_\phi H \dot{\phi} + 6 \xi_\sigma H \dot{\sigma})}{3 [1 - \kappa^2 (\xi_\phi \phi^2 + \xi_\sigma \sigma^2)]} = 0 \]  
(33)
\[ \ddot{\phi} + 3H \dot{\phi} + 6 \xi_\phi \left( \dot{H} + 2H^2 \right) \phi = 0 \]  
(34)
\[ \ddot{\sigma} + 3H \dot{\sigma} - 6 \xi_\sigma \left( \dot{H} + 2H^2 \right) \sigma = 0 \]  
(35)
\[ \dot{\rho}_M + \dot{\rho}_\Lambda + 3H (\rho_M + \rho_\Lambda + p_M + p_\Lambda) = 0. \]  
(36)

As usual, the use of the Hubble scale in the definition of holographic dark energy, and the effective nature of the Newton’s constant (32), lead to:
\[ \rho_\Lambda = \frac{3}{\kappa^2} \left[ 1 - \kappa^2 (\xi_\phi \phi^2 + \xi_\sigma \sigma^2) \right] H^2. \]  
(37)

We examine power-law solutions of equations (33)-(36), in the case of a dark-energy dominated universe ($\rho_M, p_M \ll 1$). Thus, we impose:
\[ a(t) = a_0 t^r \]
\[ \phi(t) = \phi_0 t^{s_\phi} \]
\[ \sigma(t) = \sigma_0 t^{s_\sigma}. \]  
(38)

Substituting into (33),(34),(35) and requiring a solution for all times we get:
\[ s_\phi (s_\phi - 1) + 3r s_\phi + 6r (2r - 1) \xi_\phi = 0 \]
\[ s_\sigma (s_\sigma - 1) + 3r s_\sigma - 6r (2r - 1) \xi_\sigma = 0 \]
\[ s_\phi + 12 \xi_\phi r = 0 \]
\[ s_\sigma - 12 \xi_\sigma r = 0. \]  
(39)

In the present quintom scenario with both fields non-minimally coupled, it becomes clear that the existence of non-trivial solutions requires a relation between the couplings $\xi_\phi$ and $\xi_\sigma$. Thus, for the physically interesting case $\xi_\phi \neq 1/6$, we obtain:
\[ r = \frac{1}{4 - 24 \xi_\phi} \]
\[ s_\phi = -\frac{3 \xi_\phi}{1 - 6 \xi_\phi} \]
\[ s_\sigma = \frac{3 \xi_\phi}{1 - 6 \xi_\phi} \]
\[ \xi_\sigma = -\xi_\phi, \]  
(40)

leading to:
\[ a(t) = a_0 t^{\frac{1}{4 - 24 \xi_\phi}} \]
\[ \phi(t) = \phi_0 t^{-\frac{3 \xi_\phi}{1 - 6 \xi_\phi}} \]
\[ \sigma(t) = \sigma_0 t^{\frac{3 \xi_\phi}{1 - 6 \xi_\phi}}. \]  
(41)

Note also that we could equivalently express solutions (40) in terms of $\xi_\sigma$. Finally, as expected, the choice $\xi_\phi = 1/6$ gives $-1/6 < \xi_\sigma$, which is also non-physical.

Using (37) we obtain:
\[ \rho_\Lambda(t) = \frac{3}{\kappa^2} r^2\left( t^{-2} - \xi_\phi \kappa^2 \phi_0^2 t^{2s_\phi - 2} - \xi_\sigma \kappa^2 \sigma_0^2 t^{2s_\sigma - 2} \right), \]  
(42)
and thus (36) gives:
\[ p_\Lambda(t) = \frac{1}{\kappa^2} r \left[ t^{-2}(2 - 3r) + \xi_\phi \kappa^2 \phi_0^2 t^{2s_\phi - 2} (3r + 2s_\phi - 2) + \xi_\sigma \kappa^2 \sigma_0^2 t^{2s_\sigma - 2} (3r + 2s_\sigma - 2) \right], \]  
(43)
where $r$, $s_\phi$, $s_\sigma$ and $\xi_\sigma$ are given by (40). Thus, we can calculate the dark energy equation-of-state parameter $w_\Lambda(t)$ as:
\[ w_\Lambda(t) = \frac{p_\Lambda(t)}{\rho_\Lambda(t)} = \frac{5}{3} - 16 \xi_\phi + 8 \left[ \frac{\kappa^2 \left( \xi_\phi \phi_0^2 t^{2s_\phi} - \xi_\sigma \sigma_0^2 t^{2s_\sigma} \right) - 1}{\kappa^2 \left( \xi_\phi \phi_0^2 t^{2s_\phi} + \xi_\sigma \sigma_0^2 t^{2s_\sigma} \right) - 1} \right]. \]  
(44)

Finally, expressing (44) in terms of the redshift $z$ we obtain:
\[ w_\Lambda(z) = \frac{5}{3} - 16 \xi_\phi + \frac{8 \left[ \frac{\kappa^2 \left( \xi_\phi \phi_0^2 e^{24s_\phi z} - \xi_\sigma \sigma_0^2 e^{24s_\sigma z} \right) - 1}{\kappa^2 \left( \xi_\phi \phi_0^2 e^{24s_\phi z} + \xi_\sigma \sigma_0^2 e^{24s_\sigma z} \right) - 1} \right]}{\left[ \frac{\kappa^2 \left( \xi_\phi \phi_0^2 e^{24s_\phi z} + \xi_\sigma \sigma_0^2 e^{24s_\sigma z} \right) - 1}{\kappa^2 \left( \xi_\phi \phi_0^2 e^{24s_\phi z} - \xi_\sigma \sigma_0^2 e^{24s_\sigma z} \right) - 1} \right]}. \]  
(45)

Relation (45) provides $w_\Lambda(z)$ in terms of the coupling parameters $\xi_\phi$, $\xi_\sigma$ and the amplitudes $\phi_0$, $\sigma_0$. Note that (45) corresponds to the quintom scenario, and thus expressions (40) are embedded in it. Therefore, one cannot simply set some parameters to zero in order to obtain the simple canonical or simple phantom cases, but he has to solve the problem from the beginning with only one field, that is the procedure we followed in the previous subsections. In the next section we analyze the cosmological implications of the quintom model.

### III. COSMOLOGICAL IMPLICATIONS

In the previous subsections we have obtained the equation-of-state parameter of dark energy $w_\Lambda(z)$, in terms of the coupling parameters $\xi_\phi$, $\xi_\sigma$ and the amplitudes $\phi_0$, $\sigma_0$. In the present section we investigate the cosmological implications for each case.
A. Canonical field

In the case of a simple canonical field, non-minimally coupled to gravity, \( w_\Lambda(z) \) is given by relation (16). In fig. 1 we depict \( w_\Lambda(z) \) for four different values of the coupling \( \xi_\phi \) and for three different values of the combination \( \kappa^2 \phi_0^2 \). Note that the physical requirement of an expanding universe, results to an upper limit for \( \xi_\phi \), namely \( \xi_\phi < 1/6 \), as can be seen in the first relation (12). In addition, we mention that in general \( \xi_\phi \) could be also negative, but since it leads to non-physical behavior of \( w_\Lambda(z) \) we neglect this case in this subsection.

![Graph](image-url)

**FIG. 1:** (Color online) \( w_\Lambda(z) \) vs \( z \) in the canonical field case, for \( \xi_\phi = 1/20, \xi_\phi = 1/9, \xi_\phi = 1/8, \xi_\phi = 1/7 \), where in each case the combination \( \kappa^2 \phi_0^2 \) is taken equal to 10, 0.1, respectively. The divergence of \( w_\Lambda(z) \) is a direct consequence of the singularity of (16), and thus the corresponding combinations of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \) must be excluded.

As we observe, the value of \( w_\Lambda(z) \) at \( z = 0 \), that is its current value \( w_{\Lambda 0} \), decreases as \( \xi_\phi \) increases, while its dependence on \( \kappa^2 \phi_0^2 \) is non-monotonic. However, in this simple canonical field case \( w_{\Lambda 0} \) is always greater than \(-1\), independently of the values of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \). This was expected since this case is well known to be insufficient to describe the crossing of the phantom divide \( w_\Lambda = -1 \) from above [7].

Secondly, we can see that for not so small \( \xi_\phi \), and for \( \kappa^2 \phi_0^2 \) of the order of 1, we obtain a divergence of \( w_\Lambda(z) \). This behavior is a clear prediction of relation (16), since it possesses a singularity at:

\[
z_\ast = -1 + (\xi_\phi \kappa^2 \phi_0^2)^{-1}.
\]

Therefore, the combinations of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \) that satisfy this transcendental equation giving a positive \( z_\ast \), must be excluded. However, focusing on the future instead of the past, this behavior of \( w_\Lambda \) has a very important cosmological implication. Using directly the form (15), which allows us to investigate the future evolution, we conclude that there are some combinations of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \) that lead to a future divergence of \( w_\Lambda \). Thus, the non-minimally coupled canonical field model of holographic dark energy predicts a cosmological singularity at a future time \( t_{CS} \), for combinations of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \) that satisfy:

\[
t_{CS} = (\xi_\phi \kappa^2 \phi_0^2)^{-1} > 1,
\]

and since \( \xi_\phi < 1/6 \), the \( w_\Lambda \)-divergence realization condition reads simply:

\[
\xi_\phi \kappa^2 \phi_0^2 > 1.
\]

Fortunately, this condition leads to a negative effective Newton’s constant in (4), and thus the corresponding parameter combinations must be excluded, leaving the model free of a future \( w_\Lambda \)-divergence. In any case, we have to mention that in the model at hand the \( w_\Lambda \)-divergence at \( t_{CS} \) is not accompanied by a divergence in the scale factor, in its time-derivative and in the dark energy density and pressure. Thus, technically, it does not correspond to the Big Rip of the literature [9, 24], but rather to some new singularity family.

For reasons of completeness we present explicitly the behavior of \( w_\Lambda(z) \) for \( \kappa^2 \phi_0^2 \ll 1 \), that is for very small current value of the scalar field. Specifically, we find that the present value \( w_{\Lambda 0} \) is:

\[
w_{\Lambda 0} |_{\kappa^2 \phi_0^2 \ll 1} \approx \frac{5}{3} - 16\xi_\phi > -1,
\]

with the last inequality arising from the upper bound of \( \xi_\phi < 1/6 \).

Finally, we mention that the model at hand should receive additional constraints through the observations of the time variation of gravitational constant [27]. In particular, differentiating (4) with respect to \( t \) and setting \( t_0 = 1 \) for the present time, we acquire:

\[
\lim_{t \to 0} \left| \frac{\dot{G}}{G} \right| = - \frac{6\xi_\phi^2 \kappa^2 \phi_0^4}{(1 - 6\xi_\phi)(1 - \xi_\phi \kappa^2 \phi_0^2)}.
\]

where we have also used (11) and (12). This combination must be less than 4% [27].

B. Phantom field

In the case of a phantom field, non-minimally coupled to gravity, \( w_\Lambda(z) \) is given by relation (30). In fig. 2 we depict \( w_\Lambda(z) \) for four different values of the coupling \( \xi_\phi \) and for three different values of the combination \( \kappa^2 \phi_0^2 \). Note that in this case the physical requirement of an expanding universe, results to a lower limit for \( \xi_\phi \), namely \(-1/6 < \xi_\phi \), as it is implied by the first relation (26).

As we can see, the value of \( w_{\Lambda 0} \) is now a non-monotonic function of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \). Furthermore, we observe that for some particular combinations of \( \xi_\phi \) and \( \kappa^2 \phi_0^2 \), as a
κ²σ₀², which is not what is expected for a phantom field. This behavior is a clear result of the non-minimal coupling in the holographic dark energy framework. However, contrary to the canonical field case where negative values of the coupling lead to non-physical behavior (wₐ₀ > 1), in this phantom field scenario such a choice leads to interesting cosmological implications. In fig. 3 we depict wₐ(z) for four different parameter choices with negative values of ξ σ. As we observe, negative values of the coupling produce cosmological behaviors with decreasing wₐ(z) and wₐ₀ very close to -1. Furthermore, for ξ σ < -1/7 we obtain a wₐ₀ inside the observational limits [1-4], although we cannot acquire a clear phantom divide crossing. It is known that under specific potential choices, a non-minimally coupled phantom scenario can achieve the -1-crossing [28]. It seems that the holographic dark energy framework does not allow for such a behavior.

Additionally, taking the limit κ²σ₀² ≪ 1 we find that:

\[ w_{a|\kappa^2\sigma_0^2\ll 1} \simeq \frac{5}{3} + 16\xi_\sigma > -1, \]

(54) with the last inequality arising from the lower bound of -1/6 < ξ σ.

Finally, the present scenario should also receive additional constraints through the observations of the time variation of gravitational constant [27]. In particular, we acquire:

\[ \left| \frac{G}{G}_0 \right| = \frac{6\xi_\sigma^2\kappa^2\sigma_0^2}{(1 + 6\xi_\sigma)(1 - \xi_\sigma\kappa^2\sigma_0^2)}, \]

(55)

where we have also used (25) and (26), and thus this combination must be less than 4% [27].
C. Quintom model

In the case of the combined quintom model, that is when both the canonical and phantom fields are considered to be non-minimally coupled to gravity simultaneously, \( w_\Lambda(z) \) is given by relation (45). In fig. 4 we depict \( w_\Lambda(z) \) for four different values of the coupling \( \xi_\phi \) and for three different combinations \( \kappa^2 \phi_0^2 \) and \( \kappa^2 \sigma_0^2 \). Note that in this case the physical requirement of an expanding universe, results to an upper limit for \( \xi_\phi \), namely \( \xi_\phi < 1/6 \), as it is implied by the first relation (41). The value of \( w_\Lambda \) is a monotonic function of \( \xi_\phi \). As in the previous cases, for some particular combinations of \( \xi_\phi, \kappa^2 \phi_0^2 \) and \( \kappa^2 \sigma_0^2 \), as a consequence of (45), there is a singularity of \( w_\Lambda(z) \) at a specific \( z_\phi \). The form of the denominator of (45) does not allow for an explicit expression of \( z_\phi \), but numerical investigation provides the specific excluded parameter values.

Similarly to the previous subsections, there are some combinations of \( \xi_\phi, \kappa^2 \phi_0^2 \) and \( \kappa^2 \sigma_0^2 \) that lead to a future divergence of \( w_\Lambda \). Thus, the non-minimally coupled quintom model of holographic dark energy predicts a future cosmological singularity, for parameter combinations that make (44) diverge for \( t_{CS} > 1 \), that is in the future. We mention that the transcendental form of the denominator forbids the extraction of an explicit relation for \( t_{CS} \), but the corresponding values can be provided by simple numerical calculations.

As we observe in fig. 4, \( w_\Lambda \) is greater than \(-1\), independently of the values of \( \xi_\phi, \kappa^2 \phi_0^2 \) and \( \kappa^2 \sigma_0^2 \). However, for a class of parameter combinations we obtain cosmological evolutions in agreement with observations. In fig. 5 we depict \( w_\Lambda(z) \) for four such combinations of \( \xi_\phi, \kappa^2 \phi_0^2 \) and \( \kappa^2 \sigma_0^2 \). As we can see, we can obtain a decreasing form of \( w_\Lambda \) with its current values inside the observational limits [1–4]. It is interesting that we cannot acquire a clear \(-1\)-crossing, which was the basic motive of the construction of quintom scenario [10]. It seems that the holographic dark energy framework refutes such an eventuality. Furthermore, we mention that in the case where \( \kappa^2 \phi_0^2 = \kappa^2 \sigma_0^2 \) the effects of the canonical and phantom fields cancel each other, as expected by relation (45), and the dark energy in the model at hand behaves like a cosmological constant (dotted curve of fig. 5). Lastly, numerical investigations show that the parameter subspace that leads to \( w_\Lambda \approx -1 \), cannot lead to a future cosmological singularity, which is also an advantage of the model.

Taking the limit \( \kappa^2 \phi_0^2 \sigma_0^2 \ll 1 \) we find that:

\[
w_\Lambda|_{\kappa^2 \phi_0^2 \sigma_0^2 \ll 1} \approx \frac{5}{3} - 16 \xi_\phi > -1, \tag{56}\]

with the last inequality arising from the upper bound of \( \xi_\phi < 1/6 \). Finally, we close this subsection with the external constraints to the model by the observations of the time variation of gravitational constant [27]. In particular, we acquire:

\[
\frac{\dot{G}}{G_0} = -\left[1 - \kappa^2 (\xi_\phi \phi_0^2 + \xi_\sigma \sigma_0^2)\right]^{-1} \left[\frac{6 \xi_\phi^2 \kappa^2 \phi_0^2}{(1 - 6 \xi_\phi)} - \frac{6 \xi_\sigma^2 \kappa^2 \sigma_0^2}{(1 + 6 \xi_\sigma)}\right], \tag{57}\]

where we have also used (40) and (41), and thus this combination must be less than 4% [27].
IV. CONCLUSIONS

In this work we construct various field models of dark energy, such as simple canonical and phantom fields, and their simultaneous consideration into a combined model called quintom. All fields are non-minimally coupled to gravity through extra terms in the action, and the investigation has been performed in the framework of holographic dark energy. In each case we extract \( w_\Lambda \), that is the dark energy equation-of-state parameter, as a function of the redshift and using as parameters the couplings and the amplitudes of the fields, and we analyze it in order to obtain its cosmological implications. In particular we examine the present value \( w_\Lambda_0 \), the crossing through the phantom divide \(-1\), and we extract the conditions for a future cosmological singularity.

For the simple canonical field we find that \( w_\Lambda_0 \) cannot be less than \(-1\), thus this model cannot describe the transition through the phantom divide. In addition, we give the parameter subspace that has to be excluded since it leads to a singular behavior in the past. Furthermore, we find that for a specific parameter subspace the universe will result in a future cosmological singularity, and we extract a specific relation for the time that it is going to be realized. Fortunately, the physical requirement for a positive effective Newton’s constant makes the model free of such a singularity. Finally, we give a constraint for the model parameters in order for the time variation of the gravitational constant to be consistent with observations.

For the simple phantom field we provide the parameter subspace that has to be excluded in order to acquire a regular evolution in the past. We extract the conditions and the time of a future cosmological singularity. For the case of negative couplings we find a decreasing \( w_\Lambda \) with a current value inside the observational limits, in agreement with cosmological observations. The fact that \( w_\Lambda \) lies above the phantom divide is a clear result of the non-minimal coupling in the holographic dark energy framework. Furthermore, in the single phantom field case, the future cosmological singularity cannot be excluded. Lastly, we present the constraints to the model by the time variation of the gravitational constant.

For the quintom model, that is the combined case of both canonical and phantom fields, we give the conditions for physical evolutions, that is without divergencies in the past, and we provide the requirements for a future cosmological singularity. We find that a clear crossing of the phantom divide cannot be obtained, in contrast to what is expected for a quintom scenario. It seems that the holographic dark energy framework refutes such a behavior. However, we do obtain a decreasing \( w_\Lambda \) with a current value inside the observational limits. In addition, these solutions do not possess a future cosmological singularity and these features make them a good candidate for the description of dark energy. Finally we provide the parameter constraints in order for the model to be consistent with the observed time variation of the gravitational constant.

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