AdS/CFT and local renormalization group with gauge fields

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Abstract

We revisit a study of local renormalization group (RG) with background gauge fields incorporated using the AdS/CFT correspondence. Starting with a \((d + 1)\)-dimensional bulk gravity coupled to scalars and gauge fields, we derive a local RG equation from a flow equation by working in the Hamilton-Jacobi formulation of the bulk theory. The Gauss’s law constraint associated with gauge symmetry plays an important role. RG flows of the background gauge fields are governed by vector \(\beta\)-functions, and some interesting properties of them are known to follow. We give a systematic rederivation of them on the basis of the flow equation. Fixing an ambiguity of local counterterms in such a manner that is natural from the viewpoint of the flow equation, we determine all the coefficients uniquely appearing in the trace of the stress tensor for \(d = 4\). A relation between a choice of schemes and a Virial current is discussed. As a consistency check, these are found to satisfy the integrability conditions of local RG transformations. From these results, we are led to a proof of a holographic \(c\)-theorem by finding out a full family of schemes where a trace anomaly coefficient is related with a holographic \(c\)-function.
1 Introduction

Coupling ‘constants’ are literally regarded as constants in ordinary quantum field theories (QFTs). However, it is an interesting question to ask what happens when they have spacetime dependence. A method called local renormalization group (RG) puts this idea in practice. That is, we lift spacetime independent coupling constants $g$ to spacetime dependent coupling functions $g(x)$ \cite{1,2}. The coupling functions can be regarded as external fields. Correlation functions are thus obtained by functional derivatives of the generating functional $\Gamma[g(x)]$ of connected graphs (a.k.a. the Schwinger functional) with respect to $g(x)$. This rule is called the Schwinger’s action principle \cite{3}. In the AdS/CFT correspondence \cite{4}, $\Gamma[g(x)]$ is identified with an on-shell action of a bulk gravity dual with the external fields corresponding to boundary values of bulk fields \cite{5}. Studies of RG flows using the AdS/CFT correspondence have been done extensively so far. In particular, an analysis in this line using a flow equation was first made by de Boer, Verlinde and Verlinde \cite{6}, and then generalized in \cite{7}. For a review, see \cite{8}. In these papers, bulk gravity theories coupled to scalar fields with a generic metric were investigated. It was revealed that the flow equation of a bulk gravity yields a local RG equation of $\Gamma[g(x)]$ with the Weyl anomalies of the boundary QFT reproduced correctly in this framework. For recent developments in local RG, see also \cite{9,10,11,12}. One of the purposes of this paper is to generalize these results by introducing gauge fields in the bulk side. This is partially motivated by a desire to bring a somewhat mysterious quantity called a vector $\beta$-function \cite{1,2} to light. This characterizes how background gauge fields coupled to currents flow under RG transformations. Some of the properties mentioned above were derived from the Wess-Zumino consistency conditions concerning local RG transformations. A paper \cite{13} found that the AdS/CFT correspondence explains these properties nicely. In this paper, we make a systematic rederivation of these results on the basis of the flow equation. One of the advantages of our analysis throughout this paper is to clarify scheme dependence of the results, that is, how they are affected by a choice of local counterterms. In particular, we point out a close relationship between a choice of schemes and a Virial current.

The organization of the paper is as follows. In section 2, we formulate a $(d+1)$-dimensional gravity dual model in the Hamilton-Jacobi formalism. We derive the flow equation and give some comments on its general aspects. Some suggested properties of the vector $\beta$-functions such as (i) gradient property, (ii) orthogonality, (iii) Higgs-like relation between anomalous dimensions and (iv) the relation between a vanishing vector $\beta$-function and non-renormalization of current operators can be readily confirmed. In section 3, we perform explicit calculations in $d = 4$. We obtain closed expressions of anomaly coefficients including central charges \cite{14,15,16} with an emphasis on a scheme choice we made. We can see the coefficient functions satisfy integrability conditions that come from the Wess-Zumino consistency condition associated with the local RG transformation \cite{1,2}. For an earlier work in this line, a paper \cite{12} discusses five-dimensional bulk gravity coupled to scalar fields without gauge fields. It is shown there that the anomaly coefficients computed in that model satisfy the Wess-Zumino consistency conditions. As an application of our results, a holographic
c-theorem in \( d = 4 \) dimensions is proven by finding out a full family of schemes where a monotonically decreasing function under RG flows can be constructed from an anomaly coefficient. This is an extension of a paper [17]. We collect our notations, some useful formulae and lengthy equations in appendices.

2 Formalism

We start with a bulk action in \((d + 1)\)-dimensions:

\[
S \left[ \hat{\gamma}^{\mu \nu} (x, \tau), \hat{\phi}^I (x, \tau), \hat{A}_{\mu} (x, \tau) \right] = \int_{M_{d+1}} d^{d+1}X \sqrt{\hat{\gamma}} \left\{ V(\hat{\phi}) - \hat{R}_{(d+1)} + \frac{1}{2} L_{IJ}(\hat{\phi}) \hat{\gamma}^{\mu \rho} \hat{\nabla}_{\rho} \hat{\phi}^I \hat{\nabla}_{\nu} \hat{\phi}^J + \frac{1}{4} J(\hat{\phi}) \hat{F}_{\mu \nu} \hat{F}^{\alpha \beta} \right\} - 2 \int_{\Sigma_d} d^d x \sqrt{\hat{h}} \hat{K}.
\]

Here \( \hat{\gamma}^{\mu \nu} \) denotes a bulk metric. Using ADM decomposition, it becomes

\[
ds^2 = \hat{\gamma}^{\mu \nu} \, dX^\mu \, dX^\nu = \hat{N}^2 (x, \tau) \, d\tau^2 + \hat{h}_{\mu \nu} (x, \tau) \, (dx^\mu + \hat{\lambda}^\mu (x, \tau) \, d\tau) \, (dx^\nu + \hat{\lambda}^\nu (x, \tau) \, d\tau),
\]

where \( \hat{h}_{\mu \nu} \) is an induced metric on a \( d \)-dimensional hypersurface \( \Sigma_d := \{ X \in M_{d+1} | \tau = \text{const.} \}. \) We have \( \hat{\gamma} = -\det(\hat{\gamma}^{\mu \nu}), \hat{h} = -\det(\hat{h}_{\mu \nu}). \) On each slice, an extrinsic curvature is defined as

\[
\hat{K}_{\mu \nu} := \frac{1}{2N} (\partial_\rho \hat{h}_{\mu \nu} - \hat{\nabla}_\mu \hat{\lambda}_\nu - \hat{\nabla}_\nu \hat{\lambda}_\mu),
\]

and \( \hat{K} := \hat{h}^{\mu \nu} \hat{K}_{\mu \nu}. \) The hatted quantities mean off-shell without the equations of motion imposed. The covariant derivatives are defined as

\[
\hat{\nabla}_{\rho} \hat{\phi}^I := \hat{\nabla}_{\rho} \hat{\phi}^I - i \hat{A}^a_{\rho} (T^a \hat{\phi})^I.
\]

where \( \hat{\nabla}_{\rho} \) denotes a covariant derivative associated with the Levi-Civita connection, \( \hat{\Gamma}_{\rho \sigma}^{\mu} \), constructed from \( \hat{\gamma}^{\mu \nu}. \) \( T^a \) is a generator of the gauge group \( G. \) Since we want to recognize \( \phi \) as real coupling functions, we restrict the symmetry \( G \) to a group which has real representations such as \( SO(N) \). For details, see Appendix [7].

Before proceeding, some comments on earlier works on the AdS/CFT correspondence with bulk gauge fields are in order. This was first investigated by a paper [18] on the basis of holographic renormalization, and since then has been discussed extensively from many perspectives. A systematic algorithm for solving the HJ equations in the cases with Abelian gauge fields and neutral scalar fields coupled is obtained in [19]. Using this algorithm, a paper [20] studies a class of AdS/CMP models that are parametrized with a Lifshitz exponent \( z \). In this section, we aim to give a formalism for analyzing more general systems of bulk gravity.
coupled with non-Abelian gauge fields and charged scalar fields by extending a flow equation that was first developed in [4]. Many of the results we give below coincide with those found already in the papers quoted above.

Working in a Hamilton formalism with \( \tau \) regarded as a time direction rewrites the action (2.1) in a first-order form:

\[
S = \int dt \cdot xd \tau \sqrt{h} \left\{ \hat{\pi}^{\mu \nu} \partial_\tau \hat{h}_{\mu \nu} + \hat{\pi}_I \partial_\tau \hat{\phi}^I + \hat{\pi}^{a \mu} \partial_\tau \hat{A}_\mu^a \right. \\
+ \hat{N} \left[ \frac{1}{d-1} (\hat{\pi}^\mu)^2 - \hat{\pi}_I^2 - \frac{1}{2} L^{IJ}(\hat{\phi}) \hat{\pi}_I \hat{\pi}_J - \frac{1}{2J(\hat{\phi})} \hat{h}^{\mu \nu} \hat{\pi}_a^\mu \hat{\pi}^a_\nu \\
+ V(\hat{\phi}) - \hat{R}(d) + \frac{1}{2} L_{IJ}(\hat{\phi}) \hat{h}^{\mu \nu} \nabla_\mu \hat{\phi}^I \nabla_\nu \hat{\phi}^J + \frac{1}{4} J(\hat{\phi}) \hat{F}_{\mu \nu} \hat{F}^{\mu \nu} \right] \\
+ \hat{\lambda}^\mu \left[ 2 \nabla^\nu \hat{\pi}_{\mu \nu} - \hat{\pi}_I \nabla_\mu \hat{\phi}^I - \hat{F}_{\mu \nu} \hat{\pi}^{a \nu} \right] \\
+ \hat{A}_\tau \left[ \nabla_b \hat{\pi}^{b \nu} - i (T^a \hat{\phi}^I) \hat{\pi}_I \right] \right\} .
\]

(2.5)

Here the canonical momenta conjugate to \( \hat{h}_{\mu \nu} \), \( \hat{\phi}^I \) and \( \hat{A}_\mu^a \) are respectively computed to be

\[
\hat{\pi}^{\mu \nu} := \frac{\partial L_{d+1}}{\partial (\partial_\tau \hat{h}_{\mu \nu})} = \hat{K}^{\mu \nu} - \hat{h}^{\mu \nu} \hat{K} ,
\]

(2.6)

\[
\hat{\pi}_I := \frac{\partial L_{d+1}}{\partial (\partial_\tau \hat{\phi}^I)} = \frac{1}{N} L_{IJ}(\hat{\phi}) \left( \nabla_\tau \hat{\phi}^J - \hat{\lambda}^\mu \nabla_\mu \hat{\phi}^J \right) ,
\]

(2.7)

\[
\hat{\pi}^{a \mu} := \frac{\partial L_{d+1}}{\partial (\partial_\tau \hat{A}_\mu^a)} = \frac{1}{N^3} J(\hat{\phi}) \left[ \hat{N}^2 \hat{h}^{\mu \nu} \hat{F}_{\tau \nu} - \hat{\lambda}^\nu (\hat{N}^2 \hat{h}^{\rho \mu} + \hat{\lambda}^\rho \hat{\lambda}^\mu) \hat{F}_{\nu \rho} \right] ,
\]

(2.8)

with \( L^{IJ} = L_{IJ}^{-1} \). As evident, \( \hat{N} \), \( \hat{\lambda}^\mu \) and \( \hat{A}_\tau \) are the auxiliary fields, and their equations of motion yield the first-class constraints

\[
\hat{H} := \frac{1}{\sqrt{h} \, \delta N} \frac{\delta S}{\delta \hat{N}} = \frac{1}{d-1} (\hat{\pi}^\mu_\mu)^2 - \hat{\pi}_I^2 - \frac{1}{2} L^{IJ}(\hat{\phi}) \hat{\pi}_I \hat{\pi}_J - \frac{1}{2J(\hat{\phi})} \hat{h}^{\mu \nu} \hat{\pi}_a^\mu \hat{\pi}^a_\nu \\
+ V(\hat{\phi}) - \hat{R}(d) + \frac{1}{2} L_{IJ}(\hat{\phi}) \hat{h}^{\mu \nu} \nabla_\mu \hat{\phi}^I \nabla_\nu \hat{\phi}^J + \frac{1}{4} J(\hat{\phi}) \hat{F}_{\mu \nu} \hat{F}^{\mu \nu} \approx 0 ,
\]

(2.9)

\[
\hat{P}_\mu := \frac{1}{\sqrt{h} \, \delta \lambda^\mu} \frac{\delta S}{\delta \lambda^\mu} = 2 \nabla^\nu \hat{\pi}_{\mu \nu} - \hat{\pi}_I \nabla_\mu \hat{\phi}^I - \hat{F}_{\mu \nu} \hat{\pi}^{a \nu} \approx 0 ,
\]

(2.10)

\[
\hat{G}^a := \frac{1}{\sqrt{h} \, \delta A^a_\tau} \frac{\delta S}{\delta A^a_\tau} = \nabla_b \hat{\pi}^{b \mu} - i (T^a \hat{\phi}^I) \hat{\pi}_I \approx 0 .
\]

(2.11)

(2.9) and (2.10) are the Hamiltonian and momentum constraints respectively that result from diffeomorphism in the \( (d + 1) \)-dimensional bulk spacetime. (2.11) is the Gauss’s law constraint due to the gauge symmetry \( G \).
Suppose that we find a solution to the equations of motion of $\hat{h}_{\mu\nu}, \hat{A}_\mu$ and $\hat{\phi}^I$ with the constraints (2.9), (2.10) and (2.11) under a Dirichlet boundary condition at $\tau = \tau_0$:

$$\hat{h}_{\mu\nu}(x, \tau = \tau_0) = h_{\mu\nu}(x), \quad \hat{A}_\mu(x, \tau = \tau_0) = A_\mu(x), \quad \hat{\phi}^I(x, \tau = \tau_0) = \phi^I(x).$$

Here the bulk fields with a bar means on-shell. Substituting the classical solutions into (2.1), we obtain the on-shell action as a functional of the boundary values

$$S[h_{\mu\nu}(x), \phi^I(x), A_\mu(x); \tau_0] := \int d^d x \int_{\tau_0}^\infty d\tau \sqrt{\hat{h}} \left\{ \hat{\pi}^{\mu\nu} \partial_{\tau} \hat{h}_{\mu\nu} + \hat{\pi}_I \partial_{\tau} \hat{\phi}^I + \hat{\pi}^{a\mu} \partial_{\tau} \hat{A}^a_\mu \right\}. \quad (2.12)$$

Following the standard procedure in the Hamilton-Jacobi formalism, it is verified that the variation of the on-shell action under the boundary values and the location of $\Sigma_d$ is given by

$$\delta S[h(x), \phi(x), A(x); \tau_0] = -\int d^d x \sqrt{h} \left\{ \hat{\pi}^{\mu\nu}(x, \tau_0) \delta h_{\mu\nu}(x) + \hat{\pi}_I(x, \tau_0) \delta \phi^I(x) + \hat{\pi}^{a\mu}(x, \tau_0) \delta A^a_\mu(x) \right\}. \quad (2.13)$$

We then obtain the Hamilton-Jacobi equations

$$\hat{\pi}^{\mu\nu}(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S}{\delta h_{\mu\nu}(x)}, \quad \hat{\pi}_I(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \phi^I(x)}, \quad \hat{\pi}^{a\mu}(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S}{\delta A^a_\mu(x)}, \quad \frac{\partial S}{\partial \tau_0} = 0. \quad (2.14)$$

Inserting these into the Hamilton constraint (2.9) gives the flow equation

$$\{S, S\} = L_d, \quad (2.15)$$

where

$$\{S, S\} := \left( \frac{1}{\sqrt{h}} \right)^2 \left[ -\frac{1}{d-1} \left( h_{\mu\nu} \frac{\delta S}{\delta h_{\mu\nu}} \right)^2 + \left( \frac{\delta S}{\delta h_{\mu\nu}} \right)^2 + \frac{1}{2} L_{IJ} (\phi) \frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} + \frac{1}{2} \frac{J(\phi)}{h_{\mu\nu}} \frac{\delta S}{\delta A^a_\mu} \frac{\delta S}{\delta A^a_\nu} \right], \quad (2.16)$$

and

$$L_d := V(\phi) - R(d) + \frac{1}{2} L_{IJ}(\phi) \nabla^\mu \phi^I \nabla_\mu \phi^J + \frac{1}{4} J(\phi) F_{\mu\nu}^a F^{a\mu\nu}. \quad (2.17)$$

Here

$$\nabla_\mu \phi^I := \nabla_\mu \phi^I - i A^a_\mu (T^a \phi)^I. \quad (2.18)$$

$\nabla_\mu$ denotes a covariant derivative associated with the Levi-Civita connection, $\Gamma^\mu_{\nu\rho}$, constructed from the boundary metric $h_{\mu\nu}$. 

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We show that the momentum constraint and the Gauss’s law constraint ensures $d$-dimensional diffeomorphism invariance and gauge invariance of the on-shell action, respectively. First, we note that the Gauss’s law constraint (2.11) and the Hamilton-Jacobi equations give

$$0 = \int d^d x \sqrt{h} \alpha^a \left( \nabla^a_{b\mu} \pi^{b\mu} - i(T^a \phi)^I \pi_I \right) = \int d^d x \left\{ \nabla^a_{\mu} \frac{\delta S}{\delta A^a_{\mu}} + i\alpha^a (T^a \phi)^I \frac{\delta S}{\delta \phi^I} \right\} = \int d^d x \left( \delta^\text{gauge}_a A^a_{\mu} \frac{\delta S}{\delta A^a_{\mu}} + \delta^\text{gauge}_a \phi^I \frac{\delta S}{\delta \phi^I} \right) = \delta^\text{gauge}_a S .$$

(2.19)

Here,

$$\delta^\text{gauge}_a A^a_{\mu} := \nabla^a_{\mu} \alpha^a \equiv \nabla^a_{\mu} \alpha^a + f^a_{bc} A^b_{\mu} \alpha^c , \quad \delta^\text{gauge}_a \phi^I := i\alpha^a (T^a \phi)^I ,$$

(2.20)

denote an infinitesimal gauge transformation. Further, the momentum constraint (2.10) and the Hamilton-Jacobi equations lead to

$$0 = \int d^d x \sqrt{h} \epsilon^{\mu} \left( 2 \nabla^\nu \pi_{\mu\nu} - \pi_I \nabla^\mu \phi^I - F^a_{\mu\nu} \pi^{a\nu} \right)$$

$$= \int d^d x \left\{ \left( \nabla^\mu \epsilon^\nu + \nabla^\nu \epsilon^\mu \right) \frac{\delta S}{\delta h_{\mu\nu}} + \epsilon^{\mu} \nabla^\mu \phi^I \frac{\delta S}{\delta \phi^I} + \epsilon^{\mu} F^a_{\mu\nu} \frac{\delta S}{\delta A^a_{\nu}} \right\}$$

$$= \delta_\epsilon S - \int d^d x \sqrt{h} \epsilon^\mu A^a_{\mu} \left\{ \nabla^a_{b\nu} \pi^{b\nu} - i(T^a \phi)^I \pi_I \right\} .$$

(2.22)

Here,

$$\delta_\epsilon \phi^I := \mathcal{L}_\epsilon \phi^I \equiv \epsilon^\mu \partial_\mu \phi^I , \quad \delta_\epsilon A^a_{\mu} := \mathcal{L}_\epsilon A^a_{\mu} \equiv \epsilon^\nu \partial_\nu A^a_{\mu} + \partial_\mu \epsilon^\nu A^a_{\nu} , \quad \delta_\epsilon h_{\mu\nu} := \mathcal{L}_\epsilon h_{\mu\nu} \equiv \nabla^\mu \epsilon_\nu + \nabla^\nu \epsilon_\mu ,$$

(2.23)

are Lie derivatives with respect to $d$-dimensional diffeomorphism. Noting that the second term in (2.22) vanishes because of (2.11) implies invariance of the on-shell action under $d$-dimensional diffeomorphism.

For the purpose of solving the flow equation (2.16) using systematic derivative expansions, we divide the on-shell action into local and non-local parts:

$$\frac{1}{2\kappa^2_{d+1}} S[h(x), \phi(x), A(x)] \equiv \frac{1}{2\kappa^2_{d+1}} S_{\text{loc}}[h(x), \phi(x), A(x)] - \Gamma[h(x), \phi(x), A(x)] .$$

(2.24)

We next assign an additive number called weight to each ingredient of the action as in a table below. The weight of the gauge field is assigned to be $w = 1$ because of gauge invariance.
We parametrize the local Lagrangian as below
\[ \mathcal{L}_{\text{loc}} = \sum_{w=0,2,4,\ldots} [\mathcal{L}_{\text{loc}}]_w , \] (2.25)
where
\[ [\mathcal{L}_{\text{loc}}]_0 = W(\phi) , \] (2.26)
\[ [\mathcal{L}_{\text{loc}}]_2 = -\Phi(\phi) R(d) + \frac{1}{2} M_{IJ}(\phi) \nabla^\mu \phi^I \nabla^\mu \phi^J . \] (2.27)
It is important here that all the local terms are taken to be gauge invariant. We also define
\[ S_{\text{loc};w-d} := \int d^d x \sqrt{h} \left[ \mathcal{L}_{\text{loc}} \right]_w . \] (2.28)
Note that \( d^d x \) has a weight \(-d\).

Inserting (2.24) into the flow equation and then decomposing it depending on weights, we find for \( w = 0 \)
\[ V(\phi) = -\frac{d}{4(d-1)} W^2(\phi) + \frac{1}{2} L_{IJ}(\phi) \partial_I W(\phi) \partial_J W(\phi) . \] (2.29)
For \( w = 2 \),
\[ \frac{1}{2} L_{IJ}(\phi) = -\frac{d-2}{4(d-1)} W(\phi) M_{IJ}(\phi) - L^{KL}(\phi) \partial_K W(\phi) \Gamma_{L;IJ}(\phi) \]
\[ - W(\phi) \partial_I \partial_J \Phi(\phi) - \frac{1}{2} J(\phi) M_{IK}(\phi) M_{JL}(\phi) (T^a \phi)^K (T^a \phi)^L , \] (2.31)
\[ 0 = W(\phi) \partial_K \Phi(\phi) + L_{IJ}(\phi) \partial_I W(\phi) M_{JK}(\phi) . \] (2.32)
For $w = 4$,
\[
\frac{1}{4} J(\phi) F_{\mu \nu} F^{\mu \nu} = \{\{S, S\}\}_4 .
\]

(2.33)

For $w = d$ with $d \neq 4$,
\[
[\mathcal{L}_d]_d = \frac{2\kappa_{d+1}^2 W(\phi)}{2(d-1)} \frac{\delta \Gamma}{\delta h_{\mu \nu}} - \frac{2\kappa_{d+1}^2}{\sqrt{\hbar}} L^{I J}(\phi) \partial_I W(\phi) \frac{\delta \Gamma}{\delta \phi^J} \\
- \frac{2\kappa_{d+1}^2}{h J(\phi)} \frac{\delta S_{\text{loc};2-d}}{\delta A_\mu^a} \frac{\delta \Gamma}{\delta A_\nu^a} + \{\{S_{\text{loc}}, S_{\text{loc}}\}\}_d .
\]

(2.34)

In the AdS/CFT correspondence, $h_{\mu \nu}(x), \phi^I(x)$ and $A_\mu^a(x)$ are identified with a background metric, a coupling function associated with a gauge invariant operator $O_I$, and a background gauge potential of $G$, respectively, in the boundary QFT. Then, we see that (2.34) is equivalent to the local RG equation, which specifies how the coupling functions $\phi^I(x)$ and $A_\mu^a(x)$ flow under local Weyl transformation [1]. In particular, by rewriting (2.34) as
\[
2 h_{\mu \nu} \frac{\delta \Gamma}{\delta h_{\mu \nu}} - \frac{2(d-1)}{W} L^{I J}(\phi) \partial_I W(\phi) \frac{\delta \Gamma}{\delta \phi^J} \\
- \frac{2\kappa_{d+1}^2}{h J(\phi)} \frac{\delta S_{\text{loc};2-d}}{\delta A_\mu^a} \frac{\delta \Gamma}{\delta A_\nu^a} = \frac{1}{2\kappa_{d+1}^2} \frac{2(d-1)}{W} \sqrt{\hbar} \left( [\mathcal{L}_d]_d - \{\{S_{\text{loc}}, S_{\text{loc}}\}\}_d \right),
\]

(2.35)

the scalar $\beta$-function associated with a coupling function $\phi^I$ reads
\[
\beta^I(\phi) := - \frac{2(d-1)}{W(\phi)} L^{I J}(\phi) \partial_J W(\phi),
\]

(2.36)

where $\partial_I = \partial/\partial \phi^I$. Using (2.32), this can be recast as
\[
\beta^I(\phi) = +2(d-1) M^{I J}(\phi) \partial_J \Phi(\phi),
\]

(2.37)

with $M^{I J} = M^{I J}_1$.

The coefficient of $\delta \Gamma/\delta A_\mu^a$ defines a vector $\beta$-function as
\[
\beta^a_\mu(\phi, A) := - \frac{1}{\sqrt{\hbar} W(\phi) J(\phi)} h_{\mu \nu} \frac{\delta S_{\text{loc};2-d}}{\delta A_\nu^a} = +i \frac{2(d-1)}{W(\phi) J(\phi)} M_{I J}(\phi) (T^a \phi)^I \nabla_\mu \phi^J. 
\]

(2.38)

Following [2], we define $\beta^a_\mu \equiv \rho^a_\mu \nabla_\mu \phi^I$ so that
\[
\rho^a_\mu = +i \frac{2(d-1)}{W(\phi) J(\phi)} M_{I J}(\phi) (T^a \phi)^J.
\]

(2.39)
We now show that $W, L, J$ and $V$ can be expressed in terms of $\Phi$ and $M_{IJ}$. From (2.30) and (2.32), we obtain

$$-W^{-1} = M^{IJ} \partial_I \Phi \partial_J \Phi + \frac{d-2}{2(d-1)} \Phi . \quad (2.40)$$

Next, using (2.32), (2.31) becomes

$$L_{IJ} = 2W \left[ -\frac{d-2}{4(d-1)} M_{IJ} - D_I D_J \Phi - \frac{1}{2W} M_{IK} M_{JL} (T^a \phi)^K (T^a \phi)^L \right] . \quad (2.41)$$

Here, $\Gamma^{IJK}$ is the Levi-Civita connection with respect to $M_{IJ}$, and $D_I$ is a covariant derivative defined with the connection. For a consistency check of (2.41), we rewrite (2.32) as

$$-W^{-2} \partial_I W = W^{-1} L_{IJ} M_{JK} \partial_K \Phi . \quad (2.42)$$

Using (2.40) and (2.41), the RHS takes the form

$$-\frac{d-2}{2(d-1)} \partial_I \Phi = 2D_I D_J \Phi \partial_J \Phi - \frac{1}{2W} M_{IK} M_{JL} (T^a \phi)^K (T^a \phi)^L \partial_L \Phi$$

$$= \partial_I \left( -\frac{d-2}{2(d-1)} \Phi - M^{JK} \partial_J \Phi \partial_K \Phi \right) - \frac{1}{2W} M_{IK} (T^a \phi)^K (T^a \phi)^L \partial_L \Phi . \quad (2.43)$$

Because $\Phi(\phi)$ is gauge invariant by definition, we have

$$(T^a \phi)^I \partial_I \Phi = 0 . \quad (2.44)$$

This ensures that (2.41) is consistent indeed. Finally, (2.29) and (2.32) gives

$$V = -W^3 \left[ \partial^I \Phi \partial^J \Phi \left( D_I D_J \Phi - \frac{1}{2(d-1)} M_{IJ} \right) - \frac{d(d-2)}{8(d-1)^2} \Phi \right] . \quad (2.45)$$

Vacuum expectation value of the stress tensor in the presence of the background fields $h, \phi$ and $A$ is defined by

$$\langle T^{\mu\nu}(x) \rangle := \frac{2}{\sqrt{h}} \frac{\delta \Gamma[h, \phi, A]}{\delta h_{\mu\nu}(x)} . \quad (2.46)$$

It follows from (2.35) that the trace of the stress tensor becomes

$$\langle T^\mu_{\mu} \rangle = \frac{2(d-1)}{2\kappa^2_{d+1}} W(\phi) \left( [L^d]_{d} - \{ [S_{loc}, S_{loc}] \}_{d} \right) - \beta^I(\phi) \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta \phi^I} - \beta^a_{\mu} \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta A^a_{\mu}} . \quad (2.47)$$

As explained in [7, 21], the flow equation cannot determine $S_{loc;0}$ uniquely, reflecting an ambiguity of adding local counterterms to $\Gamma$.  

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In a computation of the Weyl anomaly, $S_{\text{loc};0}$ is manifested as a degree of freedom of adding a total derivative [21]. To see this, we note that under an infinitesimal Weyl transformation, $S_{\text{loc};0}$ transforms as

$$\int d^d x \sigma(x) h_{\mu\nu}(x) \frac{\delta}{\delta h_{\mu\nu}(x)} S_{\text{loc};0} = \int d^d x \sqrt{h} \partial_{\mu} \sigma J_{d}^{\mu},$$

because $S_{\text{loc};0}$ is invariant under global scale transformations. Thus,

$$2 h_{\mu\nu}(x) \frac{\delta}{\delta h_{\mu\nu}(x)} S_{\text{loc};0} = -\sqrt{h} \nabla_{\mu} J_{d}^{\mu}.$$ (2.49)

From this relation, we obtain

$$\langle T_{\mu}^{\mu}(x) \rangle = 2(d - 1) 2\kappa_{d+1}^2 W(\phi) \left( [\mathcal{L}]_{d} - [\{S_{\text{loc}}, S_{\text{loc}}\}]_{d} \right) - \frac{1}{2\kappa_{d+1}^2} \nabla_{\mu} J_{d}^{\mu}$$

$$- \beta_{I}(\phi) \frac{1}{\sqrt{h}} \frac{\delta}{\delta \phi^{I}} \left( \Gamma - \frac{1}{2\kappa_{d+1}^2} S_{\text{loc};0} \right) - \beta_{I}(\phi) \frac{1}{\sqrt{h}} \frac{\delta}{\delta \phi^{I}} \left( \Gamma - \frac{1}{2\kappa_{d+1}^2} S_{\text{loc};0} \right).$$ (2.50)

Here, $\{S_{\text{loc}}, S_{\text{loc}}\}$ denotes the bracket $\{S_{\text{loc}}, S_{\text{loc}}\}$ with $[\mathcal{L}_{\text{loc}}]_{d}$ removed from $S_{\text{loc}}$. Hence, the Weyl anomaly, which is defined as

$$W_{d}(x) := \langle T_{\mu}^{\mu}(x) \rangle \bigg|_{\beta=0},$$ (2.51)

contains a total derivative that comes from $S_{\text{loc};0}$. It is important to note that in the AdS/CFT correspondence, $J_{d}^{\mu}$ is the only origin of the Virial current, which might spoil conformal symmetry of scale invariant field theories. For an excellent review on relations between scale and conformal symmetry, see [22]. As an operator, $J_{d}^{\mu}$ is proportional to an identity operator, and therefore gives no obstacle to having CFTs. This is natural because we are working on QFTs with gravity duals. Effects of the Virial current here are only manifested as ambiguities of local counterterms added to $\Gamma$. For the moment, we work in the scheme $S_{\text{loc};0} = 0$ because this is simple and natural in the flow equation. For a discussion on how $S_{\text{loc};0}$ affects the coefficients in $\langle T_{\mu}^{\mu}(x) \rangle$, see [23]. Another choice of the scheme will be discussed below for the purpose of studying a holographic c-theorem.

An analysis of the Gauss’s law constraint is straightforward. With the HJ equations, (2.11) can be regarded as a constraint on the on-shell action:

$$\nabla_{a}^{\mu} \frac{\delta S}{\delta A_{a}^{\mu}} - i (T^{a}(\phi))^{I} \frac{\delta S}{\delta \phi^{I}} = 0.$$ (2.52)

Inserting (2.24) into this, we see that the local terms give no contribution because they are gauge invariant by definition. Then, (2.52) reduces to

$$\nabla_{a}^{\mu} \frac{\delta \Gamma}{\delta A_{a}^{\mu}} - i (T^{a}(\phi))^{I} \frac{\delta \Gamma}{\delta \phi^{I}} = 0.$$ (2.53)
Because the vev’s of the gauge invariant operators and currents in the presence of the background fields are given by
\[
\langle O_I(x) \rangle := \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta \phi^I(x)} , \quad \langle J^{\mu}{}_a(x) \rangle := \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta A^{a \mu}_I(x)} ,
\]
we obtain an operator identity
\[
\nabla^{\mu} J^{a \mu} = i (T^a \phi)^I O_I .
\] (2.55)

We now give some comments on properties of the vector $\beta$-function that hold for $d$-dimensional QFTs with gravity duals. These were first obtained in [13], and the rest of this section may be regarded as a review of part of that paper. First, we have already observed that (2.38) exhibits the gradient property. Second, an orthogonal relation between scalar and vector $\beta$-functions is easy to verify in the AdS/CFT correspondence thanks to the gauge invariance (2.44):
\[
\rho^a_I = i \frac{4(d-1)^2}{W} (T^a \phi)^f \partial_f \Phi = 0 .
\] (2.56)

In addition, anomalous dimensions receive non-trivial contributions from operator mixing: differentiating the local RG equation
\[
\text{(local terms)} = \int d^d x \left\{ 2 h_{\mu \nu}(x) \frac{\delta}{\delta h_{\mu \nu}(x)} + \beta^f [\phi(x)] \frac{\delta}{\delta \phi^I(x)} + \rho^a_I [\phi(x)] \nabla^{\mu} \phi^f(x) \frac{\delta}{\delta A^{a \mu}_I(x)} \right\} \Gamma[\phi, h, A] ,
\]
with $\phi^f$ and $A^{a \mu}_I$, we obtain the RG equations of correlation functions of $O_I$ and $J^{a \mu}$, of which the anomalous dimensions of $O_I$ and $J^{a \mu}$ are read off as
\[
\gamma^I_J = - \partial_J \beta^I + i \rho^a_I (T^a \phi)^I , \quad \gamma^a_b = i \rho^c_I \delta_{bc} (T^a \phi)^I ,
\] (2.58)
respectively. Here we employ the operator identity (2.55). It then follows
\[
\gamma^I_J = - 2(d-1) \partial_J \partial^I \Phi - \frac{2(d-1)}{W(\phi) J(\phi)} M_{JK}(\phi) (T^a \phi)^K (T^a \phi)^I ,
\] (2.59)
\[
\gamma^a_b = - \frac{2(d-1)}{W(\phi) J(\phi)} M_{IJ}(\phi) (T^a \phi)^J \delta_{bc} (T^c \phi)^I .
\] (2.60)

These expressions exhibit the suggested Higgs-like relation manifestly. Finally, the equivalence
\[
\beta^a_\mu = 0 \iff \nabla^{\mu} J^{a \mu} = 0
\] (2.61)
can also be shown. Recalling (2.55), $\iff$ is obviously true because the conservation of the current implies $(T^a \phi)^I = 0$. On the other hand, if $\beta^a_\mu = 0$, we have two possibilities: (i) $(T^a \phi)^I = 0$ and (ii) $\nabla^{\mu} \phi^I = 0$ because we are assuming $M_{IJ}$ to be invertible (and $d \neq 1$). In case of (i), we have the current conservation via the operator identity (2.55). In case of (ii), since $\phi^I$ does not belong to a singlet, we must have $\phi^I = \text{const.} = 0$, and this again results in the current conservation.
3 Explicit calculations in four dimensions

For $d = 4$, the equation (2.33) should not be imposed, and for $w = 4$ the flow equation (2.15) yields the local RG equation (2.35) instead. Using the formulae given in appendix B together with (2.47), we arrive at the explicit expression of the trace of the stress tensor:

$$\langle T^\mu_\nu \rangle = \frac{6}{2\kappa_5^2} \frac{1}{W(\phi)} I(\phi) F^a_{\mu} F^{a \mu \nu} - \beta^I(\phi) \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta \phi^I} - \beta^a(\phi, A) \frac{1}{\sqrt{h}} \frac{\delta \Gamma}{\delta A^a_\mu}$$

$$- \frac{6\Phi^2}{2\kappa_5^2} R^{\mu \nu} R_{\mu \nu} + \left( \frac{2\Phi^2}{2\kappa_5^2} - \frac{3\kappa_5^2}{2\kappa_5^2} L_{J J} \right) R^2(4)$$

$$+ \frac{12\Phi}{2\kappa_5^2} \partial_\mu \Phi \cdot E^{\mu \nu} \nabla_\mu \Phi^I \cdot \nabla_\nu \phi^J - \frac{6}{2\kappa_5^2} \frac{\kappa_5^2}{2\kappa_5^2} \left( \nabla^2 \phi^I \right) \nabla^2 \phi^J$$

$$+ \frac{6\Phi}{2\kappa_5^2} (2\partial_\mu \partial_\nu \phi^J + M_{IJ}) E^{\mu \nu} \nabla_\mu \phi^I \nabla_\nu \phi^J$$

$$+ \left( \frac{6}{2\kappa_5^2} \partial_\mu \Phi \cdot \partial_\nu \Phi + \frac{3}{2\kappa_5^2} \partial_\mu \Phi M_{JK} - \frac{6}{2\kappa_5^2} \partial_\mu \phi^I \partial_\nu \phi^J \right) \nabla^2 \phi^I \nabla^2 \phi^J$$

$$+ \left( \frac{12}{2\kappa_5^2} \partial_\mu \Phi \cdot \partial_\nu \Phi - \frac{3}{2\kappa_5^2} \partial_\mu \phi^I \partial_\nu \phi^J \right) \nabla^2 \phi^I \nabla^2 \phi^J$$

$$+ \frac{6}{2\kappa_5^2} \partial_\mu \Phi \cdot \nabla^2 \phi^I \nabla^2 \phi^J$$

$$+ \left( \frac{1}{4\kappa_5^2} M_{I J} M_{K L} - \frac{3}{4\kappa_5^2} M_{I K} M_{J L} + \frac{3}{4\kappa_5^2} M_{I L} M_{J K} - \frac{3}{2\kappa_5^2} M_{I J} \partial_\mu \Phi + \frac{3}{2\kappa_5^2} \partial_\mu \phi^I \partial_\nu \phi^J \right) \nabla^2 \phi^I \nabla^2 \phi^J$$

$$- \frac{3}{2\kappa_5^2} \partial_\mu \Phi \cdot \nabla^2 \phi^I \nabla^2 \phi^J$$

$$- \frac{3}{2\kappa_5^2} \frac{\kappa_5^2}{2\kappa_5^2} L^{MN} \Gamma_{M;I J} \Gamma_{N;K L} \nabla^2 \phi^I \nabla^2 \phi^J.$$  \hspace{1cm} \text{(3.1)}

Now we compare these results with those in [2], where a generic form of $\langle T^\mu_\nu \rangle$ is given in accord with symmetry constraints. For details, see Appendix C (C.3), (C.4), (C.5) and (C.6) are easily solved as

$$A = \frac{12\Phi^2}{2\kappa_5^2} W, \quad C = -\frac{3\Phi^2}{2\kappa_5^2} W, \quad B = \frac{216}{2\kappa_5^2} L_{I J} \partial_I \Phi \partial_J \Phi,$$  \hspace{1cm} \text{(3.2)}

$$W_I = \frac{6}{2\kappa_5^2} \Phi \partial_I \Phi = \frac{3}{2\kappa_5^2} \partial_I \Phi^2.$$  \hspace{1cm} \text{(3.3)}
Using \((3.3)\), \((C.7)\) yields

\[
G_{IJ} = -\frac{12}{2\kappa_5^2 W^2} \Phi \partial_I W \partial_J \Phi + \frac{6}{2\kappa_5^2 W} \partial_I \Phi \partial_J \Phi - \frac{6}{2\kappa_5^2 W} \Phi M_{IJ}. \tag{3.4}
\]

From \((C.8)\), \((C.9)\) and \((C.10)\), we find

\[
H_I = 0, \tag{3.5}
\]

\[
E_I = \frac{36}{2\kappa_5^2 W} L^{JK} \partial_{(J} \Phi M_{K)I}, \tag{3.6}
\]

\[
F_{IJ} = -\frac{18}{2\kappa_5^2 W} \partial_I \Phi \partial_J \Phi - \frac{6}{2\kappa_5^2 W} \Phi M_{IJ} + \frac{36}{2\kappa_5^2 W} L^{KL} \partial_{(K} \Phi \Gamma_{L);IJ}. \tag{3.7}
\]

\((C.12)\) and \((C.13)\) give

\[
S_{IJ} = S_{(IJ)} = V_{IJ} = \frac{3}{2\kappa_5^2 W} \partial_I \Phi \partial_J \Phi, \tag{3.8}
\]

from which \((C.11)\) leads to

\[
A_{IJ} = -\frac{6}{2\kappa_5^2 W} L^{KL} M_{IK} M_{JL}. \tag{3.9}
\]

The rest of the equations given in Appendix \(\text{C}\) requires hard work to solve. However, since there are six equations left and six coefficient functions to be determined, viz. \(B_{IJK}, T_{IJK}, C_{IJKL}, \beta_f, Q_I\) and \(P_{IJ}\), thus it is expected that there is a solution.

It is argued in \([2]\) that the quantities appearing in \(\langle T^\mu_\mu \rangle\) must satisfy integrability conditions, that is, Wess-Zumino consistency conditions associated with local RG transformations:

\[
[\Delta_\sigma, \Delta_\sigma'] \Gamma = 0. \tag{3.10}
\]

Here

\[
\Delta_\sigma := \int d^d x \sigma(x) \left( 2h_{\mu\nu} \frac{\delta}{\delta h_{\mu\nu}} + \beta_I^\mu \frac{\delta}{\delta \phi^I} + \beta^a_\mu \frac{\delta}{\delta A^a_\mu} \right). \tag{3.11}
\]

As discussed before, the AdS/CFT correspondence always gives us trivial Virial currents. Then, no nontrivial modification of the scalar and vector \(\beta\)-functions arises that is necessary for gauge invariance of the \(\beta\) functions. Therefore, we can show that a number of integrability conditions derived in \([2]\) still hold on their own. We also obtain additional integrability conditions concerned with the external gauge field. To list some of the integrability conditions that play an important role in this paper, we have

\[
\partial_I A = G_{IJ} \beta^J - \mathcal{L}_\beta W_I, \tag{3.12}
\]

\[
\rho^{\alpha}_I \beta^I = 0. \tag{3.13}
\]
Here $L_\beta$ denotes a Lie derivative associated with the vector $\beta^I$, which acts on $W_I$ as
\[
L_\beta W_I \equiv \beta^K \partial_K W_I + \partial_I \beta^K W_K .
\] (3.14)
The coefficients given in (3.2)-(3.9) should satisfy all the integrability conditions, because the flow equation is formulated on the basis that the effective action $\Gamma$ does exist once a bulk gravity model is given. In fact, a straightforward computation shows that (3.12) holds indeed. Furthermore, (3.13) is nothing but the orthogonality condition (2.56), which is already verified from the gauge invariance of $\Phi$.

We end this paper by making some comments on the $c$-theorem of RG flows in higher dimensions and a holographic $c$-function. As shown by Jack and Osborn in [2], (3.12) can be rewritten as
\[
\beta^I \partial_I \tilde{A} = G_{IJ} \beta^I \beta^J ,
\] (3.15)
with
\[
\tilde{A} := A + W_I \beta^I .
\] (3.16)
Proving positive-definiteness of $G_{IJ}$, if possible, implies that $\tilde{A}$ decreases monotonically under RG flows. $A$ and $G_{IJ}$ in (3.2) and (3.4) are obtained in the scheme $S_{\text{loc}}; 0 = 0$. As evident, this $G_{IJ}$ is not positive definite even if $L_{IJ}$ is assumed to be so. However, $G_{IJ}$ takes a different expression by working in a different scheme with $S_{\text{loc}}; 0 \neq 0$. In fact, it is argued in [2] that adding local counterterms to $\Gamma$ gives rise to a shift
\[
\tilde{A} \to \tilde{A}' := \tilde{A} + g_{IJ} \beta^I \beta^J , \quad G_{IJ} \to G_{IJ}' := G_{IJ} + L_\beta g_{IJ} ,
\] (3.17)
which leaves (3.15) unchanged. Here, we show that an appropriate choice of $g_{IJ}$ maps $\tilde{A}$ to a holographic $c$-function. Relations between a holographic $c$-function and $\tilde{A}$ were first studied in [17]. Our aim in this paper is to make a full identification of schemes where a holographic $c$-function is related directly with a trace anomaly coefficient. As discussed in [24, 25], a holographic $c$-function for $d = 4$ is defined as
\[
c_h := -\frac{27}{2 \kappa_3^2} \frac{1}{W^3} .
\] (3.18)
Here the overall factor is chosen so that the value of $c_h$ at a fixed point equals that of $C$ given in (3.2). It follows from (3.18), (2.40) and (2.41) that
\[
\beta^I \partial_I c_h = \frac{1}{2} c_h L_{IJ} \beta^I \beta^J .
\] (3.19)
This relation was first derived in [26], although we prove it when $\phi^I$ is promoted to space-time dependent couplings. The gradient flow nature becomes more manifest by rewriting the scalar $\beta$-function (2.36) as
\[
\beta^I = \frac{2}{c_h} L_{IJ} \partial_J c_h .
\] (3.20)
The positivity of $c_h$ together with positive definiteness of $L_{IJ}$ guarantees that $c_h$ is indeed a monotonically decreasing function. $c_h L_{IJ}$ is to be identified with a Zamolodchikov metric.

For the purpose of relating $\tilde{A}'$ to $c_h$, we take $g_{IJ}$ to the most general form:

$$2\kappa_5^2 g_{IJ} := X(\phi) \partial_I \Phi \partial_J \Phi + Y(\phi) M_{IJ},$$

where

$$X(\phi) = x_1(\Phi) (\partial \Phi \cdot \partial \Phi) + x_2(\Phi),$$

$$Y(\phi) = (3 - x_1(\Phi)) (\partial \Phi \cdot \partial \Phi)^2 + (4\Phi - x_2(\Phi)) (\partial \Phi \cdot \partial \Phi) + \Phi^2,$$

with $x_1, x_2$ being arbitrary functions of $\Phi = \Phi(\phi_I)$ and $(\partial \Phi \cdot \partial \Phi) = M_{IJ} \partial_I \Phi \partial_J \Phi$. From this mapping, it turns out that $\tilde{A}' = 4c_h$. Furthermore, we can easily show that

$$G'_{IJ} \beta^I \phi^J = 2c_h L_{IJ} \beta^I \phi^J.$$

This implies that (3.15) after a shift (3.17) with (3.21) is identical with (3.19).

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**A Notations**

Let $\phi^I$ be a charged scalar. We divide the index $I$ into two parts: $I = (i, \alpha_i)$. For each $i$, the charged field transforms as a representation $R_i$ under the gauge group $G$. $\alpha_i = 1, 2, \cdots, \dim R_i$ is an index of $R_i$. The generator of $G$, $(T^a)^I_J$, defined in this paper refers to

$$(T^a)^I_J = \delta^I_J (t^a)_{\alpha_i \beta_i}.$$

with $I = (i, \alpha_i), J = (j, \beta_j)$. $t^a_{\alpha_i \beta_i}$ is the generators of $G$ that belong to the representation $R_i$. The covariant derivative $\nabla$ acts on $\phi^I$ as

$$\nabla_\mu \phi^I = \nabla_\mu \phi^{i,\alpha_i} := \nabla_\mu \phi^{i,\alpha_i} - iA^a_\mu (T^a)^I_J \phi^J$$

$$= \nabla_\mu \phi^{i,\alpha_i} - iA^a_\mu \sum_\beta_i (t^a)_{\alpha_i \beta_i} \phi^{i,\beta_i}.$$

In addition, we define a symbol ( ) to denote symmetric parts of tensors:

$$2X(\phi) := X_{IJ} + X_{JI},$$

and denote the Levi-Civita connection in the theory space constructed from $M_{IJ} \Gamma_{IJK}$, i.e.

$$\Gamma_{IJK} := \frac{1}{2} (\partial_J M_{IK} + \partial_K M_{IJ} - \partial_I M_{JK}).$$
B Some Useful Formulae

The following formulae are useful and valid for any $d$:

\[
\frac{\delta}{\delta h_{\mu\nu}} S_{\text{loc;}-d} = \frac{1}{2} \sqrt{h} h^{\mu\nu} W(\phi), \quad (B.1)
\]

\[
\frac{\delta}{\delta \phi^I} S_{\text{loc;}-d} = \sqrt{h} \partial_I W(\phi), \quad (B.2)
\]

\[
\frac{\delta}{\delta h_{\mu\nu}} S_{\text{loc;}-d} = \sqrt{h} \left\{ \Phi(\phi) (R^{\mu\nu} - \frac{1}{2} h^{\mu\nu} R_{(d)}) - \nabla^\mu \nabla^\nu \Phi(\phi) + h^{\mu\nu} \nabla^2 \Phi(\phi) \\
+ \frac{1}{2} M_{IJ}(\phi) \left[ \frac{1}{2} h^{\mu\nu} \nabla^\rho \phi^I \nabla_\rho \phi^J - \nabla^\mu \phi^I \nabla^\nu \phi^J \right] \right\}, \quad (B.3)
\]

\[
\frac{\delta}{\delta \phi^I} S_{\text{loc;}-d} = \sqrt{h} \left\{ - \partial_I \Phi(\phi) R_{(d)} - \Gamma_{IJK}(\phi) \nabla^\mu \phi^J \nabla_\mu \phi^K - M_{IJ}(\phi) \nabla^2 \phi^J \right\}. \quad (B.4)
\]
From these results, we find

$$\left\{ S_{\text{loc};2-d}, S_{\text{loc};2-d} \right\}_4 = R_{\mu\nu} \Phi^2 + R_{(d)}^2 \left( -\frac{d}{4(d-1)} \Phi^2 + \frac{1}{2} L^{IJ} \partial_I \Phi \partial_J \Phi \right)$$

$$+ E_{\mu \nu} \nabla_\mu \nabla_\nu \phi^I ( -2 \partial_I \Phi )$$

$$+ E_{\mu \nu} \nabla_\mu \phi^I \nabla_\nu \phi^J [ - \Phi (2 \partial_I \partial_J \Phi + M_{IJ}) ]$$

$$+ R_{(d)}^2 \phi^I \left[ L^{JK} \partial_J ( \Phi M_{K})_I \right]$$

$$+ R_{(d)} \phi^I \nabla_\mu \phi^J \left[ - \frac{d - 2}{4(d-1)} \Phi M_{IJ} + L^{KL} \partial_K ( \Phi \Gamma_{L})_J \right]$$

$$+ \nabla^2 \phi^I \nabla^2 \phi^J \left[ - \partial_I \Phi \partial_J \Phi + \frac{1}{2} L^{KL} M_{IK} M_{JL} \right]$$

$$+ \nabla^2 \phi^I \nabla^2 \phi^J \left[ \partial_I \Phi (2 \partial_J \partial_K \Phi + M_{JK}) \right]$$

$$+ \nabla^2 \phi^I \nabla_\mu \phi^J \nabla^\nu \phi^K \left[ \partial_I \Phi (2 \partial_J \partial_K \Phi + M_{JK}) \right]$$

$$+ \nabla^2 \phi^I \nabla_\mu \phi^J \nabla_\nu \phi^K \left[ \frac{d}{16(d-1)} M_{IJ} M_{KL} + \frac{1}{4} M_{IK} M_{JL} - \frac{1}{4} (M_{IJ} \partial_K \partial_L \Phi + M_{KL} \partial_I \partial_J \Phi) \right]$$

$$+ \frac{1}{2} (M_{IK} \partial_J \partial_L \Phi + M_{JL} \partial_I \partial_K \Phi) - (\partial_I \partial_J \Phi \partial_K \partial_L \Phi - \partial_I \partial_K \Phi \partial_J \partial_L \Phi)$$

$$+ \frac{1}{2} L^{MN} \Gamma_{M;IJ} \Gamma_{N;KL} \right].$$
C Trace of stress tensor defined in [2] and its relation to that obtained from bulk gravity

In [2], Jack and Osborn wrote down the explicit form of the trace of the stress tensor as

\[
\langle T^\mu_\mu \rangle = CW^2_{\mu
u\rho\sigma} - \frac{1}{4} AE_4 - \frac{1}{72} BR^2_{(4)} - E^{\mu\nu} G_{IJ} \nabla_\mu \phi^I \nabla_\nu \phi^J \\
- \frac{1}{6} R_{(4)} (E_I \nabla^2 \phi^I + F_{IJ} \nabla^\mu \phi^J \nabla_\mu \phi^J) \\
+ \frac{1}{2} A_{IJ} \nabla^2 \phi^I \nabla^2 \phi^J + B_{IJK} \nabla^2 \phi^I \nabla^\mu \phi^J \nabla_\mu \phi^K + \frac{1}{2} C_{IJKL} \nabla^\mu \phi^J \nabla^\nu \phi^J \nabla_\nu \phi^K \\
+ \frac{1}{4} (F^{\mu\nu} F_{\mu\nu}) \beta_I + F^{\mu\nu} \cdot P_{IJ} \nabla_\mu \phi^I \nabla_\nu \phi^J \\
+ 2 \nabla_\mu \left( E^{\mu\nu} W_I \nabla_\nu \phi^I + \frac{1}{6} R_{(4)} H_I \nabla^\mu \phi^I + S_{IJ} \nabla^\mu \phi^I \nabla^2 \phi^J + T_{IJK} \nabla^\mu \phi^I \nabla^\nu \phi^J \nabla_\nu \phi^K \\
+ F^{\mu\nu} \cdot Q_{IJ} \nabla_\nu \phi^J \right) \\
- \nabla^2 \left( \frac{1}{6} R_{(4)} D + U_I \nabla^2 \phi^I + V_{IJ} \nabla^\mu \phi^I \nabla_\mu \phi^J \right) + \text{(terms proportional to } \beta\text{-functions)} \right) .
\]

(C.1)

Here the most general total derivative terms are added. For the purpose of matching with those results computed from the bulk gravity, however, it is sufficient to set \( D = 0 = U_I \) because there is no term proportional to \( \nabla^2 R_{(4)} \) or \( \nabla^4 \phi^I \) there with \( S_{\text{loc};0} = 0 \). It is then straightforward to verify
\[ (T^\mu_\mu) = \left( C - \frac{1}{4} A \right) R^2_{\mu
u\rho\sigma} + \left( -2C + A \right) R^2_{\mu\nu} + \left( \frac{C}{3} - \frac{A}{4} - \frac{B}{72} \right) R^2_{(4)} \]

\[ + E^{\mu\nu} \nabla_\mu \nabla_\nu \phi^I \cdot 2W_I + E^{\mu\nu} \nabla_\mu \phi^J \nabla_\nu \phi^J \left[ -G_{IJ} + 2\partial(IW_J) - 2S_{(IJ)} \right] \]

\[ + R_{(4)} \nabla^2 \phi^I \left( -\frac{1}{6} E_I + \frac{1}{3} H_I \right) + R_{(4)} \nabla^\mu \phi^I \nabla_\mu \phi^J \left( -\frac{1}{6} F_{IJ} + \frac{1}{3} \partial(IH_J) \right) \]

\[ + \nabla^\mu R_{(4)} \nabla_\mu \phi^I \frac{1}{3} H_I + \nabla^2 \phi^I \nabla^2 \phi^I \left( \frac{1}{2} A_{IJ} + 2S_{(IJ)} \right) + \nabla^\mu \nabla^\nu \phi^I \nabla_\mu \nabla_\nu \phi^J (-2V_{IJ}) \]

\[ + \nabla^\mu \nabla^\alpha \nabla^\nu \phi^{i,\alpha} \nabla^\beta \phi^{i,\beta} \left[ 2(S_{ij})_{\alpha_i\beta_j} - 2(V_{ij})_{\alpha_i\beta_j} \right] \]

\[ + \nabla^\mu \nabla^\alpha \nabla^\nu \phi^K \nabla_\nu \phi^K \left[ 4T_{JKI} - 4\partial_J V_{KI} \right] \]

\[ + \nabla^\mu \phi^I \nabla_\mu \phi^J \nabla^\nu \phi^K \left( \frac{1}{2} C_{IKL} + \partial(KT_{JI}) + \partial(KL_{II}) - \frac{1}{2} \partial_I \partial_J V_{KL} - \frac{1}{2} \partial_K \partial_L V_{IJ} \right) \]

\[ + \left( F^{\mu\nu} F_{\mu\nu} \right) \frac{1}{4} \left( \beta_f \right)_{\alpha_i} + \phi^{i,\beta} Q_{i,\alpha_i} \]

\[ + \nabla^\mu \phi^{i,\alpha_i} \left( P_{ij} \right)_{\alpha_i\beta_j} + 2(\partial_i Q_J)_{\alpha_i\beta_j} + 2(S_{ij})_{\alpha_i\beta_j} \left( F^{\mu\nu} \right)_{\beta_j \gamma_j} \nabla_\nu \phi^{i,\gamma_j} \]

\[ + \nabla^\mu \phi^{i,\alpha_i} \left( F_{\mu\nu} \right)_{\beta_j \gamma_j} \left[ 2\delta_{\alpha_i}^{\gamma_j} Q_{j,\beta_j} - 2(S_{ij})_{\alpha_i\beta_j} \phi^{i,\gamma_j} \right] \]

\[ + \text{(terms proportional to } \beta\text{-functions).} \]

(C.2)
Comparing the coefficients of operators appearing in (3.1) and those in (C.2) gives

\[
C - \frac{1}{4} A = 0 ,
\]

\[
-2C + A = -\frac{6\Phi^2}{2\kappa^2 W} ,
\]

\[
C - \frac{A}{4} - \frac{B}{72} = \frac{2\phi^2}{2\kappa^2 W} - \frac{3}{2\kappa^2 W} L^{IJ} \partial_I \Phi \partial_J \Phi ,
\]

\[
2W_I = \frac{12}{2\kappa^2 W} \Phi \partial_I \Phi ,
\]

\[
-G_{IJ} + 2\partial_I W_J - 2S_{(IJ)} = \frac{12}{2\kappa^2 W} \Phi \partial_I \partial_J \Phi + \frac{6}{2\kappa^2 W} \Phi M_{IJ} ,
\]

\[
-\frac{1}{6} E_I + \frac{1}{3} H_I = -\frac{6}{2\kappa^2 W} L^{JK} \partial_J (\Phi M_K)I ,
\]

\[
-\frac{1}{6} F_{IJ} + \frac{1}{3} \partial_I H_J - S_{(IJ)} = \frac{1}{2\kappa^2 W} \Phi M_{IJ} - \frac{6}{2\kappa^2 W} L^{KL} \partial_K (\Phi \Gamma_L)IJ ,
\]

\[
H_I = 0 ,
\]

\[
\frac{1}{2} A_{IJ} + 2S_{(IJ)} = \frac{6}{2\kappa^2 W} \partial_I \Phi \partial_J \Phi - \frac{3}{2\kappa^2 W} L^{KL} M_{IK} M_{JL} ,
\]

\[
-2V_{IJ} = -\frac{6}{2\kappa^2 W} \partial_I \Phi \partial_J \Phi ,
\]

\[
2(S_{ij})_{\alpha,\beta_j} - 2(V_{ij})_{\alpha,\beta_j} = 0 ,
\]

\[
B_{IJK} + 2\partial_J S_K I + 2T_{IJK} - \partial_I V_{JK} = \frac{12}{2\kappa^2 W} \partial_I \Phi \partial_J \Phi \partial_K \Phi + \frac{3}{2\kappa^2 W} \partial_I M_{JK} - \frac{6}{2\kappa^2 W} L^{LM} M_{I(L} \Gamma_{MN)JK} ,
\]

\[
4T_{JKI} - 4\partial_J V_{KI} = -\frac{12}{2\kappa^2 W} \partial_I \Phi \partial_J \Phi \partial_K \Phi - \frac{6}{2\kappa^2 W} \partial_I \Phi M_{JK} ,
\]

\[
\frac{1}{2} C_{IJKL} + \partial_I T_{JKL} + \partial_K T_{IJL} - \frac{1}{2} \partial_I \partial_J V_{KL} - \frac{1}{2} \partial_K \partial_L V_{IJ}
\]

\[
= \frac{1}{4\kappa^2 W} M_{IJ} M_{KL} - \frac{3}{4\kappa^2 W} M_{IK} M_{JL} + \frac{3}{4\kappa^2 W} (M_{IJ} \partial_K \partial_L \Phi + M_{KL} \partial_I \partial_J \Phi)
\]

\[
- \frac{3}{2\kappa^2 W} (M_{IK} \partial_J \partial_L \Phi + M_{JL} \partial_I \partial_K \Phi) + \frac{6}{2\kappa^2 W} (\partial_I \partial_J \Phi \partial_K \partial_L \Phi - \partial_I \partial_K \Phi \partial_J \partial_L \Phi)
\]

\[
- \frac{3}{2\kappa^2 W} L^{M} \Gamma_{M;IJ} \Gamma_{N;KL} ,
\]

\[
\frac{1}{4} (\beta_I)^{\beta_{\alpha_i}} + \phi^{i,\beta_i} Q_{i,\alpha_i} = \frac{6}{2\kappa^2 W} \frac{1}{4} B(\phi) \delta_{\alpha_i}^{\beta_i} ,
\]

\[
(P_{ij})_{\alpha,\beta_j} + 2(\partial_i Q_j)_{\alpha,\beta_j} + 2(S_{ij})_{\alpha,\beta_j} = 0 ,
\]

\[
2\delta^i_{\alpha_i} \delta^j_{\beta_j} Q_{i,\beta_j} - 2(S_{ij})_{\alpha,\beta_j} \phi^{j,\gamma_j} = 0 .
\]
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