Modular Wedge Localization and the d=1+1 Formfactor Program

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Abstract

In this paper I continue the study of the new framework of modular localization and its constructive use in the nonperturbative d=1+1 Karowski-Weisz-Smirnov formfactor program. Particular attention is focussed on the existence of semilocal generators of the wedge-localized algebra without vacuum polarization (FWG-operators) which are closely related to objects fulfilling the Zamolodchikov-Faddeev algebraic structure. They generate a “thermal Hilbert space” and allow to understand the equivalence of the KMS conditions with the so-called cyclicity equation for formfactors which was known to be closely related to crossing symmetry properties. The modular setting gives rise to interesting new ideas on “free” d=2+1 anyons and plektons.

1 Introduction

The historical roots of “Modular Localization” as a kind of inversion of the famous Bisognano-Wichmann observation \[1\] and its relation with the mathematical Tomita-Takesaki modular theory \[2\] of von Neumann algebras in “general position” and their thermal KMS structure \[3\] as well as their constructive power for low-dimensional QFT in general and the formfactor problem in particular, has already been highlighted in previous work of the author \[4\]. In addition there is a recent paper \[5\] in which, similar to the present work, the thermal aspect was applied to a specific problem arising in the formfactor problem.

My original interest in this subject arose from the attempt to understand the loss of unicity when one passes from the unique \((m,s)\)-Wigner representation to free fields: there are as many free fields in the \((m,s)\)-Fock space as there are intertwiners between the \(D^{(s)}(R(\Lambda,\Lambda^{-1}p))\) Wigner rotation matrices and the \(D^{(n+\Lambda,n-\Lambda)}(\Lambda)\) finite dimensional representation matrices of the Lorentz group \[6\]. All of these infinitely many fields describe the same spacetime physics,
even though most of them are not “Eulerian” or Lagrangian. The net of local algebras associated with the \((m, s)\)-Wigner representation is again unique, and the method of its direct construction (bypassing the free fields) is based on modular localization. Although this has been described in the cited literature, we repeat the main arguments in the next section in order to keep this paper reasonably self-contained. The new localization modular applied to wave functions is more suitable for QFT than the old one of Newton and Wigner, who adapted the Born (probability) localization of Schrödinger theory to the Wigner representation spaces. In a similar vein as the existence of the Klein paradox, the difference in localization shows the limitations of analogies between wave functions of relativistic particles and Schrödinger wave functions. Modular localization in the context of the Wigner theory requires us to interpret the complex representation space as a “twice as large” real Hilbert space with an appropriately defined real orthogonal (or equivalently symplectic) inner product. The real orthogonality physically means causally “opposite” in the sense of quantum theoretical localization and this concept in Fock space passes directly to the von Neumann notion of compatibility of measurements in the sense of commutants of von Neumann algebras.

The third section is a continuation of the second section on modular aspects of “Wignerism”; but this time with a stronger emphasis on thermal KMS (Hawking, Unruh) properties of wedge localization and crossing symmetry.

In the context of interacting theories one obtains a framework which is totally intrinsic and characteristic of local quantum physics (LQP). Concepts which are shared with nonrelativistic quantum physics and play a crucial role in perturbation theory, as the interaction picture, time-ordering, euclidean path integrals etc. are not required and unnatural in this approach. No-Go theorems, as the Haag theorem (an obstruction against the validity of the field theoretic interaction picture), are not valid for analogous modular concepts, since one has unitary equivalence between the wedge-restricted incoming (free) and interacting algebras. In the terminology explained in section 4 there is no Haag-theorem obstruction against the existence of the “modular Møller operator”.

There is of course one construction in the literature which also does not use the above standard textbook formalism: the bootstrap formfactor approach to factorizable d=1+1 models of Karowski and Weisz as axiomatized and extended by Smirnov. This has generated the erroneous impression with some physicist that there are two different subjects: standard textbook kind of QFT, and 2-dimensional factorizable- (as well as chiral conformal-) QFT. This paper is also intended to counteract such impressions.

No relativistic QFT allows a physically consistent modifications in the direction of momentum space cutoffs or regularizations. In particular it is not possible to manufacture a factorizing model with such a modification without wrecking its physical interpretation altogether. If one wants to have versions

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1Meaning that, similar as for the Dirac equation, the transformation law of the field follows from the structure of the differential equation. The Wigner representation theory was, among other things, introduced just to avoid the problem of ambiguities in equations of motions and their proliferation.
outside relativistic quantum physics, but with a similar particle interpretation and scattering theory (without Einstein causality), one has to find a lattice version. This is a highly nontrivial task in its own right; the derivation of cluster properties (and hence Haag-Ruelle) scattering theory without Einstein causality is much more difficult [8]. In particular our modular localization concepts would not apply to lattice theories. Our methods in this article are totally characteristic of local quantum physics; they are wrecked by cutoffs, regulators and elementary lengths.

The main part of this paper is the fourth section where we relate modular localization with factorizing models. A useful illustration (despite its physical austerity) is supplied by the Federbush model [23][24]. Its explicitly known solution is similar to the Ising model QFT [26], but its rapidity independent S-matrix depends continuously on a coupling strength. The reason why it did not hitherto appear in the factorizing list is that the S-matrix bootstrap was restricted to parity conserving theories. The main message from this model and the principles of algebraic QFT is that the simplicity of factorizable models does not so much show up in the generalized formfactors of “field coordinates” and correlation functions (which apart from the Ising theory and the Federbush model mostly remain too complicated for explicit analytic calculations), but rather in the computation of the (net of) wedge algebras (and the ensuing double cone algebras). The net of wedge algebras uniquely fixes the total net, and since the wedge generators turn out to be determined uniquely by the factorizing S-matrix, the so called inverse problem of QFT has a unique solution in the context of factorizing models. These algebras and their mutual relations are sufficient for exploring and characterizing the intrinsic physical content of a real time QFT, since the S-matrix and perhaps some distinguished Noether current fields (energy momentum tensor, charge currents) are not only sufficient mathematical characteristics, but are also the only experimentally accessible observables in high energy physics. In statistical mechanics also the individual correlation functions of certain order/disorder fields are measurable and for those one has to work harder than for the structure of the algebras of real time QFT. Fortunately the computation of e.g. the spectrum of critical indices is part of the much simpler real time spin-statistics (the statistical phases [9]) discussion of algebraic QFT.

The concluding remarks offers a scenario for higher dimensional modular constructions in the presence of interactions. In that case one does not have an S-matrix bootstrap program which could be separated from the rest of QFT. Although algebraic QFT offers some hints, one its still very far from covering these ideas with an analytically accessible formalism.

Mathematically the problems at hand are a special case of the so called

\footnote{The constructive aspect of the formfactor approach to renormalizable factorizing models and the present new modular method reveals more structure than the old method of constructive QFT with respect to e.g. $\phi^4$ which was modelled on Lagrangian perturbation theory. The new construction is not impeded by worse than canonical short distance behaviour. In fact the issue of short distance behaviour beyond the limits set by causality does not enter the nonperturbative modular construction.}
inverse problem of modular theory, i.e. the reconstruction of a von Neumann algebra from its modular data \cite{16}. Only in its more physical setting where it passes to the inverse problem of QFT, namely the question whether physically admissable scattering matrices possess a unique local QFT, the uniqueness can be established under reasonable physical assumptions. This will be shown in a subsequent paper \cite{17}. For a mathematical and conceptual background I refer to \cite{31} and to my notes \cite{40} which are intended as part of a planned monograph on “Nonperturbative Approach to Local Quantum Physics”. There the reader also finds (in chapter 2) an account of the unpublished work \cite{13} adapted to the present setting.

2 Liberation from Free Field Coordinates

As explained elsewhere \cite{4}, one may use the Wigner representation theory for positive energy representations in order to construct fields from particle states. For \(d = 3 + 1\) space-time dimensions there are two families of representation; \((m, s)\) and \((0, h)\). Here \(m\) is the mass and designates massive representation and \(s\) and \(h\) are the spins resp. the helicities \(h\). These are invariants of the representations (“Casimirs”) which refer to the Wigner “little” group; in the first case to SU(2) in which case \(s = \) (half) integer, and for \(m = 0\) to the little group (fixed point group of a momentum \(\neq 0\) on the light cone) \(\tilde{E}(2)\) which is the two-fold covering of the euclidean group in the plane. The zero mass representations in turn split into two families. For the “neutrino-photon family” the little group has a non-faithful representation (the “translative” part is trivially represented) whereas for Wigner’s “continuous \(h\) representation” the representation is faithful and infinite component, but allows no identification with known zero mass particles.

In the massive case, the transition to covariant fields is most conveniently done with the help of intertwiners between the Wigner spin \(s\) representations \(D^{(s)}(R(\Lambda, p))\) which involve the \(\Lambda, p\) dependent Wigner rotation \(R\) and the finite dimensional covariant representation of the Lorentz-group \(D^{[A,B]}(\Lambda)\)

\[
u(p)D^{(s)}(R(\Lambda, p)) = D^{[A,B]}(\Lambda)\nu(\Lambda^{-1}p)
\]

The only restriction is:

\[
|A - B| \leq s \leq A + B
\]

which leaves infinitely many \(A, B\) (half integer) choices for a given \(s\). Here the \(\nu(p)\) intertwiner is a rectangular matrix consisting of \(2s + 1\) column vectors \(\nu(p, s, 3) = -s, ..., +s\) of length \((2A+1)(2B+1)\). Its explicit construction using Clebsch-Gordan methods can be found in Weinberg’s book \cite{10}. Analogously there exist antiparticle (opposite charge) \(\nu(p)\) intertwiners: \(D^{(s)}(R(\Lambda, p)) \rightarrow D^{[A,B]}(\Lambda)\). The covariant field is then of the form:

\[
\psi^{[A,B]}(x) = \frac{1}{(2\pi)^{3/2}}\int\left(e^{-ipx}\sum_{s,3}u(p, s, 3)a(p, s, 3) + \ldots \right) +
\]
\[ + e^{ipx} \sum_{sx} v(p_1, s_3) b^*(p_1, s_3) \frac{d^3p}{2\omega} \]

where \(a(p)\) and \(b^*(p)\) are annihilation (creation) operators in a Fock space for particles (antiparticles). The bad news (only at first sight, as it fortunately turns out) is that we lost the Wigner unicity: there are now infinitely many \(\psi^{[A,B]}\) fields with varying \(A, B\) but all belonging to the same \((m, s)\)-Wigner representation and living in the same Fock space. Only one of these fields is “Eulerian” (examples: for \(s = \frac{1}{2}\) Dirac, for \(s = \frac{3}{2}\) Rarita-Schwinger) i.e. the transformation property of \(\psi\) is a consequence of the nature of a linear field equation which is derivable by an action principle from a Lagrangian. Non-Eulerian fields as e.g. Weinberg’s \(D^{(j,0)} + D^{[0,j]}\) fields for \(j \geq \frac{3}{2}\), cannot be used in a canonical quantization scheme or in a formalism of functional integration because the corresponding field equations have more solutions than allowed by the physical degrees of freedom (in fact they have tachyonic solutions). The use of formula (3) with the correct \(u, v\) intertwiners in the (on shell) Bogoliubov-Shirkov approach (which different from off shell approaches as euclidean functional integrals) does not require the existence of free bilinear Lagrangians) based on causality is however legitimate. Naturally from the point of view of the Wigner theory, which is totally intrinsic and does not use quantization ideas, there is no preference of Eulerian versus non Eulerian fields.

It turns out that the above family of fields corresponding to \((m, s)\) constitute the linear part of the associated “Borchers class” \([3]\). For bosonic fields the latter is defined as:

\[ B(\psi) = \{ \chi(s) \mid [\chi(x), \psi(y)] = 0, \quad (x - y)^2 < 0 \} \]  

(4)

If we only consider cyclic (with respect to the vacuum) relatively local fields, then we obtain transitivity of the causality for the resulting fields. This class depends only on \((m, s)\) and is generated by the Wick-monomials \(\psi\). A mathematically and conceptually more manageable object which is manifestly independent of the chosen \((m, s)\) Fock-space field, is the local von Neumann algebra \(A(\mathcal{O})\) generated by \(\psi\):

\[ \mathcal{O} \rightarrow A(\mathcal{O}, \psi) = A(\mathcal{O}, \chi) \]  

(5)

Here \(\chi \sim \psi\) is any cyclic (locally equivalent) field in the same Borchers class of \(\psi\).

Now we have reached our first goal: the lack of uniqueness of local \((m, s)\) fields is explained in terms of the arbitrariness in the choice of “field coordinates” which generate the same net of von Neumann algebras. According to the physical interpretation in algebraic QFT this means that the physics does not depend on the concretely chosen (cyclic) field.

Since algebraic QFT shuns inventions and favors discoveries, it is deeply satisfying that there are arguments to the extend that every causal net fulfilling certain spectral properties is automatically “coordinatizable”. For chiral conformal theories there exists even a rigorous proof \([11]\). So one can be confident that the physical content has not been changed as compared to the
standard Wightman approach which in turn was abstracted from a synthesis of
the Heisenberg-Pauli canonical quantization of classical Lagrangian field theory
with the Wigner representation theory of the Poincaré group. The use of local
field coordinates tends to make geometric localization properties of the algebras
manifest. But only if there exist pointlike covariant generators which create
charged states (counter example: for Maxwellian charges they do not exist; the
physical electron field is noncompactly localized) the localization can be en-
coded into classical smearing function. The localization concept is “maximally
classical” for the free Weyl and CAR algebras which in fact are just function
algebras with a noncommutative product structure. For these special cases the
differential geometric concepts as fibre bundles may be directly used in local
quantum physics. Outside of these special context where quantum localization
can be described in terms of support properties of classical functions, the only
sufficiently general reliable concepts are those based on von Neumann algebra
methods of algebraic QFT. In that case the quantum localization may deviate
from the classical (differential-) geometric concepts [31] and the use of “field
coordinates” is less useful than that of local nets. This however does not mean
that geometrical concepts are useless, but rather that they cannot be imposed
but must be derived as a consequence of local quantum physics.

In the following we describe a way to construct the interaction-free nets
directly [4], thus bypassing the use of field coordinates altogether. We use the
d=3+1 Wigner \((m,s)\)-representations as an illustrative example. In case of
charged particles (particles\(\neq\)antiparticles) we double the Wigner representation
space:

\[
H = H_{Wig}^p \oplus H_{Wig}^\bar{p}
\]

in order to incorporate the charge conjugation operation as an (antilinear in the
Wigner theory) operator involving the p-\(\bar{p}\)-flip. On this extended Wigner space
one can act with the full Poincaré group (where those reflections which change
the direction of time are antiunitarily represented). For the modular localization
in a wedge we only need the standard L-boost \(\Lambda(\chi)\) and the standard reflection
\(r\) which (by definition) are associated with the \(t-x\) wedge:

\[
\delta^{\tau} \equiv \pi_{Wig}(\Lambda(\chi = 2\pi \tau))
\]

\[
j \equiv \pi_{Wig}(r)
\]

These operators have a simple action on the p-space (possibly) doubled Wigner
wave functions, in particular:

\[
(j\psi)(p) \simeq \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \tilde{\psi}(p_0, p_1, -p_2, -p_3)
\]

By functional calculus we form \(\delta^{\frac{2}{j}}\) and define:

\[
s \equiv j\delta^{\frac{2}{j}}
\]
This unbounded antilinear densely defined operator $s$ is involutive on its domain: $s^2 = 1$. Its -1 eigenspace is a real closed subspace $H_R$ of $H$ which allows the following characterization of the domain of $s$:

$$\text{dom}(s) = H_R + iH_R$$

$$s(h_1 + ih_2) = -h_1 + ih_2$$

Defining:

$$H_R(W) \equiv U(g)H_R, \ W = gW_{\text{stand}}$$

where $g$ is an appropriate Poincaré transformation, we find the following theorem (D. Guido, private communication):

**Theorem 1** $H_R(W)$ is an isotonous net of real Hilbert spaces i.e. $H_R(W_1) \subsetneq H_R(W_2)$ if $W_1 \subsetneq W_2$.

Its proof, which follows from a theorem by Borchers \cite{12} will not be given here.

If we now define:

$$H_R(O) \equiv \bigcap_{W \supseteq O} H_R(W)$$

then it is easily seen (even without the use of the u,v-intertwiners) that the spaces $H_R(O) + iH_R(O)$ are still dense in $H_{W_{\text{ig}}}$ and that the formula:

$$s(O)(h_1 + ih_2) \equiv -h_1 + ih_2$$

defines a closed involutive operator with a polar decomposition:

$$s(O) = j(O)\delta(O)^{1/2}$$

Although now $j(O)$ and $\delta(O)^{1/2}$ have no obvious geometric interpretation, there is still a bit of geometry left, as the following theorem shows:

**Theorem 2** The $H_R(O)$ form an orthocomplemented net of closed real Hilbert spaces, i.e. the following "duality" holds:

$$H_R(O') = H_R(O)' = iH_R^\perp(O).$$

Here $O'$ denotes the causal complement, $H_R^\perp$ the real orthogonal complement in the sense of the inner product $Re(\psi,\varphi)$ and $H_R'$ is the symplectic complement in the sense of $Im(\psi,\varphi)$. Representing $O'$ as a union of double cones, one defines $H_R(O')$ by additivity.

The direct construction of the interaction-free algebraic bosonic net for $(m, s = \text{integer})$ is now achieved by converting the "premodular" theory of

\footnote{It does not matter whether we take the + or - sign for the characterization of modular localization since we can convert one into the other via multiplication with $i$ and the dense domains $H_R(W) + iH_R(W)$ are the same. This is one of the rare occasions where a sign error remains without serious consequences.}
real subspaces of the Wigner space into the Tomita-Takesaki modular theory for nets of von Neumann algebras using the Weyl functor [13].

The application of the Weyl functor − to the net of real spaces:

\[ H_R(O) \rightarrow \mathcal{A}(O) \equiv \text{alg} \{ W(f) | f \in H_R(O) \} \]  

(17)

leads to a net of von Neumann algebras in \( \mathcal{H}_{\text{Fock}} \) which are in “standard position” with respect to the vacuum state. The ensuing a modular theory restricted to the Fock vacuum \( \Omega \) is geometric:

\[
\Gamma(s) = S, \quad S\Omega = A^*\Omega, \quad A \in \mathcal{A}(W)
\]

\[
S = J\Delta^\frac{1}{2}, \quad J = \Gamma(j), \quad \Delta^\tau = \Gamma(\delta^\tau)
\]

(18)

The proof of this theorem uses the functorial formalism of [13]. It should be evident from the derivation that the wedge localization concept in Fock obtained in this functorial way from the Wigner theory only holds for interaction free situations. The Fock space is also important for interacting QFT, but in that case the wedge localization enters via scattering theory as in section 4, and not just through Wigner’s representation theory.

Clearly the \( W \)- or \( O \)- indexing of the Hilbert spaces corresponds to a localization concept via modular theory. Specifically \( H_R(O) + iH_R(O) \) is a certain closure (in the graph topology of the operator \( \Delta^\frac{1}{2} \)) of the one particle component of the Reeh-Schlieder domain belonging to the localization region \( O \). Although for general localization region the modular operators are not geometric, there is one remaining geometric statement which presents itself in the form of an algebraic duality property [14]:

\[ \mathcal{A}(O') = \mathcal{A}(O)', \quad \text{Haag Duality} \]  

(19)

Here the prime on the von Neumann algebra has the standard meaning of commutant. In the following we make some schematic additions and completions which highlight the modular localization concept for more general interaction-free theories different from the massive bosonic case.

- In the case of \( m \neq 0, s = \text{halfinteger} \), the Wigner theory produces a mismatch between the “quantum” (in the sense of the commutant) and the “geometric” opposite of \( H_R(W) \), which however is easily taken care of by an additional factor \( i \) (interchange of symplectic complement with real orthogonal complement). This (via the physical localization property) requires the application of the CAR-functor instead of the CCR-functor, as well as the introduction of the well-known Klein transformation \( K \) which corrects the above mismatch in Fock space:

\[
J = K\mathcal{F}_{\text{CAR}}(ij)K^{-1}
\]

\[
\mathcal{A}(O') = K\mathcal{A}(O)'K^{-1}
\]

(20)

where the \( K \) is the twist operator of the “twisted” Haag Duality [3] and \( j \) is related to the TCP-operator as before.
For \( m = 0, h = (\text{half}) \text{integer}, \) as a consequence of the nonfaithful representation of the zero mass little group \( E(2) \) (the two-dimensional euclidean group or rather its two-fold covering), the set of possible \( u - v \) intertwiners is limited by the selection rule: \(|A - B| = \pm h\). This means on the one hand that there are e.g. no covariant intertwiners which lead to \( D^{(\frac{1}{2}, \frac{1}{2})} \) (vector-potential of classical Maxwell theory), \( D^{(\frac{1}{2}, \frac{1}{2})} \otimes (D^{(\frac{1}{2}, 0)} + D^{(0, \frac{1}{2})}) \) (Rarita-Schwinger potential for massless particles), gravitational potentials etc. On the other hand, all local bilinear expressions in the allowed covariant intertwiners which could serve as an inner product vanish and hence cannot be used in order to rewrite the Wigner inner product (for e.g. \( h = 1 \) any local inner product in terms of field strength intertwiners \( F_{\mu\nu}(p) \) vanishes as a consequence of the mass shell condition \( p^2 = 0 \)).

A reasonable compromise consists in relaxing on strict \( L \)-covariance and compact (double cone) modular localization, but retaining the relation with the Wigner inner product. One then may describe the Wigner space in terms of a vectorpotential which depends in addition on a spacelike direction \( e \) and which has the following affine Lorentz transformation:

\[
(U(\Lambda)A)_\mu(p, e) = \Lambda^\nu_\mu A_\nu(\Lambda^{-1}p, \Lambda^{-1}e)
\]

\[= \Lambda^\nu_\mu A_\nu(\Lambda^{-1}p, e) + p_\mu G(p, \Lambda, e)\]  

where the "gauge" contribution \( G \) by which one has to re-gauge (in order to refer to the original spacelike polarization vector \( e \)) is a nonlocal term which follows from the above definitions. Using the x-t-boost for the definition of \( \delta^d \) and defining \( j \) with the help of the TCP operation as before, one again obtains a wedge localized real subspace \( H_R(W_{st}) \) which contains vectorpotentials with \( e \) pointing into the wedge. This space is the same as if we would have constructed the real modular subspace of the Wigner wave function space of right and left hand polarized photons without vectorpotentials. After applying the Weyl functor, we again obtain a covariant net of wedge algebras which is described in terms of slightly nonlocal semiinfinite stringlike vectorpotential field coordinates whose relation to the local \( F_{\mu\nu}(x) \) field strength can be shown to be given by:

\[
A_\mu(x, e) = \int_0^\infty e'^\nu F_{\nu\mu}(x - es)ds
\]

If we now define the modular localization subspaces as before by starting from the wedge region, we find that the (smoothened versions of) the vectorpotentials are members of these subspaces (or their translates) as long as the spacelike directions \( e \) point inside the wedges. They are lost if we form the localization spaces belonging to e.g. double cone regions. Hence these stringlike localized vector potentials appear in a natural way in our modular localization approach for the wedge regions. Whereas the natural use of such nonpointlike objects in a future interacting theory based on modular localization may be possible, the present formulation of
gauge theories within any known perturbative scheme is based on point-like fields in an Hilbert space involving ghosts and nonunitary (pseudo-unitary) Lorentz-transformations instead of the previous affine action in Wigner space. In this way one formally keeps the classical point localization through the process of deformation by interaction and the ghost removal is only done at the end of the calculation, i.e. the ghosts act like a catalyst for forcing theories involving spin $\geq 1$ vector mesons into the standard renormalizable framework. Such catalysts which leave no intrinsic observable mark in the physical results may be OK in chemistry, but they go a bit against the Bohr-Heisenberg spirit of removing all nonobservable aspects from the formalism. Algebraic QFT therefore considers these BRS cohomological techniques as successful but preliminary.

3 Thermal Aspects of Modular Localization

In modular theory the dense set of vectors which are obtained by applying (local) von Neumann algebras in standard position to the standard (vacuum) vector forms a core for the Tomita operator $S$. The domain of $S$ can then be described in terms of the +1 (or -1) closed real subspace of $S$. In terms of the “premodular” objects $s$ in Wigner space and the modular Tomita operators $S$ in Fock space we introduce the following nets of wedge-localized dense subspaces:

$$H_R(W) + iH_R(W) = \text{dom}(s) \subset H_{\text{Wigner}}$$

$$\mathcal{H}_R(W) + i\mathcal{H}_R(W) = \text{dom}(S) \subset \mathcal{H}_{\text{Fock}}$$

These dense subspaces become Hilbert spaces in their own right if we use the graph norm of the Tomita operators. For the $s$-operators in Wigner space we have:

$$(f, g)_{\text{Wigner}} \rightarrow (f, g)_G = (f, g)_{\text{Wig}} + \overline{(sf, sg)}_{\text{Wig}}$$

The graph topology insures that the wave functions are strip-analytic in the wedge rapidity $\theta$:

$$p_0 = m(p_\perp) \cosh \theta, \quad p_1 = m(p_\perp) \sinh \theta, \quad m(p_\perp) = \sqrt{m^2 + p_\perp^2}$$

where this "G-finiteness" (25) is precisely the analyticity prerequisite for the validity of the KMS property for the two-point function. For scalar Bosons we have for the Wigner inner product restricted to the wedge:

$$(f, g)_{\text{Wig}} = \left\langle A^*(\hat{g}) A^*(\hat{f}) \right\rangle_0^{KMS} + \left\langle A^*(\hat{g}) \Delta A(\hat{f}) \right\rangle_0^{CCR}$$

$$\equiv \left[ A^*(\hat{g}) A(\delta \hat{f}) \right] + \left( f, \delta g \right)_{\text{Wig}}^W, \quad \delta = e^{2\pi K}$$

10
Here we used a field theoretic notation \((A^*(\hat{g})A(\frac{\delta}{1-\delta}f))\) is a smeared scalar complex field of the type (linear in \(\hat{g}\) with \(\text{supp.}\hat{g} \in W\)) in order to emphasize the typical thermal denominator in the c-number commutator on the right hand side and not only implicit as the required restriction of the wave functions to the wedge region on the left hand side. Of course the c-number commutator may be rewritten in terms of p-space Wigner wave functions for particles and (\(\delta^{\pi}\)-transformed) antiparticles in such a way that the localization restriction is guarantied by the property that the resulting expression is finite if the wave functions are finite in the sense of the graph norm. With the localization temperature in this way having been made manifest, the only difference between localization temperatures and heat bath temperatures (for a system enclosed in a box) on the level of field algebras in Fock space corresponds to the difference between hyperfinite type \(\text{III}_1\) and type \(I\) von Neumann algebras. On the level of the generators of the modular group this should correspond to a difference in their spectra. The fact that the boost \(K\) appears instead of the Hamiltonian \(H\) reveals one significant difference between the two situations. For the heat bath temperature of a Hamiltonian dynamics the modular operator \(\delta = e^{-2\beta H}\) is bounded on one particle wave functions whereas the unboundedness of \(\delta = e^{2\pi K}\) in (28) enforces the localization (strip analyticity) of the Wigner wave functions i.e. the boost does not permit a KMS state on the full algebra.

This difference results from the two-sided spectrum of \(K\) as compared to the boundedness from below of \(H\). In fact localization temperatures are inexorably linked with unbounded symmetry operators.

The generalization to fermions as well as to particles of arbitrary spin is easily carried out. The differences between \(K\) and \(H\) also leads to somewhat different energy distribution functions for small energies so that Boson \(K\)-energy distributions may appear as those of \(H\) heat bath Fermions. In this context one is advised to discuss matters of statistics not in Fourier space, but rather in spacetime where they have their unequivocal physical interpretation.

One may of course consider KMS state on the same \(C^*\)-algebra with a different Hawking \(K\)-temperature than \(2\pi\); however, such a situation cannot be obtained by a localizing restriction. Mathematically \(C^*\)-algebras to different \(K\)-temperatures are known to belong to different folia (in this case after von Neumann closure to unitarily inequivalent \(\text{III}_1\)-algebras) of the same \(C^*\)-algebra. Or equivalently, a scaled modular operator \(\Delta^{\alpha_1}\) cannot be the modular operator of the same theory at a different temperature as it would be the case for type I algebras. Localization temperatures are not freely variable.

For those readers who are familiar with Unruh’s work we mention that the Unruh Hamiltonian is different from \(K\) by a factor \(\frac{1}{a}\) where \(a\) is the acceleration.

More generally we may now consider matrix elements of wedge-localized operators between wedge localized multiparticle states. Then the KMS property allows to move the wedge localized particle state as an antiparticle at the
analytically continued rapidity $\theta + i\pi$ from the ket to the bra. The simplest illustration is the two-particle matrix element of a free current of a charged scalar field $j_\mu(x) = \phi^* \overset{\leftrightarrow}{\partial}_\mu \phi$ smeared with the wedge supported function $\hat{h}$ (but any other free field composite would also serve):

$$\langle 0 | \int j_\mu(x) \hat{h}(x) d^4x | f, \delta^\chi g \rangle \overset{KMS}{=} \langle 0 | \left( \Delta \phi^*(\delta^\chi \hat{g}) \Delta^{-1} \right)^* \int j_\mu(x) \hat{h}(x) d^4x | f \rangle$$

(30)

Here $\delta^\chi g^c$ is the charged transformed antiparticle in the Wigner wave function $g$ at the analytically continued rapidity $\theta + i\pi$, whereas $\hat{g}$ denotes as before the wedge-localized spacetime smearing function whose mass shell restricted Fourier transform corresponds to the boundary value of the analytically continuable Wigner wave function $g$. Moving the left hand operator to the left vacuum changes the antiparticle charge to the particle charge. Since the $H_R(W) + iH_R(W)$ complex localization spaces are dense in the Wigner space, the momentum space kernel for both sides of (30) takes the familiar form

$$\langle p' | j_\mu(0) | p \rangle \overset{an\,al.,cont.}{=} \langle 0 | j_\mu(0) | p, p'(z) \rangle$$

(31)

where $p'(z)$ is the rapidity parametrization of above (26). This famous crossing symmetry, which is known to hold also in each perturbative order of renormalizable interacting theories, has never been derived in sufficient generality within a nonperturbative framework of QFT. It is to be thought of as a kind of on shell momentum space substitute for Einstein causality and locality (and its strengthened form, called Haag duality). As such it played an important role in finding a candidate for a nonperturbative S-matrix of the famous Veneziano dual model. Although it stood in this indirect way on the cradle of string theory, the recent string theoretic inventions seem to pay little attention to these physical origins.

If crossing symmetry is really a general property of local QFT, a conjecture (only proven in perturbation theory, as mentioned before) which nobody seems to doubt, then it should be the on shell manifestation of the off shell KMS property (originating from causality via the Reeh-Schlieder property) for modular wedge localization. In the construction of wedge localized thermal KMS states on the algebra of mass shell operators satisfying the Zamolodchikov-Faddeev algebraic relations in the momentum space rapidity, the derivation of crossing symmetry is similar (albeit more involved) to the previous free field derivation and the argument can be found in section 4 of this work. Recently more general arguments based on the Haag-Ruelle scattering theory which also hold for the case of nonfactorizing QFT’s in $d=1+1$ and higher dimensions were
proposed [5]. In our approach it turns out that the general crossing symmetry in any dimension is indeed related to the wedge KMS condition, but that this statement cannot be derived just from scattering theory alone. It rather follows from the existence and the modular intertwining property of the modular Møller operator $U$ (next section) which, unlike the $S$-matrix, is not just an object of scattering theory as in nonrelativistic physics, but is defined in terms of modular wedge localization. For its existence we have to make an assumption which we presently are not able to derive within the framework of algebraic QFT. We intend to use this object in order to prove [7] the uniqueness of the main inverse problem of QFT: $S_{\text{scat}} \to \text{QFT}$.

In fact the very special free field formalism of the first two sections may be generalized into two directions:

- interacting fields
- curved spacetime

As mentioned before, low-dimensional interacting theories will be discussed in the next section. For the generalization to curved space time (e.g. the Schwarzschild black hole solution) it turns out that only the existence of a bifurcated horizon together with a certain behavior near that horizon (“surface gravitation”) [13][21] is already sufficient in order to obtain the thermal Hawking-Unruh aspect. In the standard treatment one needs isometries in spacetime i.e. classical horizons defined in terms of Killing vectors. The idea of modular localization suggests to consider also e.g. double cones for which there is no spacetime isometry but only an isometry in $H_{Wigner}$ or $H_{Fock}$. Of course such enlargements of spaces for obtaining a better formulation or a generalization of a problem are commonplace in modern mathematics, particularly in noncommutative geometry. The idea is that one replaces the ill-defined isometries by a geometrically “fuzzy” but well-defined symmetry transformations in quantum space, which only near the horizon looses its spacetime fuzziness. The candidates for these noneometric symmetries are the modular automorphisms of von Neumann algebras of arbitrary space time regions together with suitable faithful states from the local folium of admissable states. Although the restriction of the global vacuum state is in that folium, it is not always the appropriate state for the construction of the modular automorphism.

In this context one obtains a good illustration by the (nongeometric) modular theory of, e.g., the double cone algebra of a massive free field. From the folium of states one may want to select that vector, with respect to which the algebra has a least fuzzy (most geometric) behavior under the action of the modular group. Appealing to the net subtended by spheres $S$ at time $t=0$ one realizes that algebras localized in these spheres are independent of the mass. Since $m=0$ leads to a geometric modular situation[11] for the pair $(\mathcal{A}_{m=0}(S), \Omega_{m=0})$, and since the nonlocality of the modular group of the massive theory in the “wrong” (massless) vacuum $(\mathcal{A}_{m \neq 0}(\mathcal{C}(S)), \Omega_{m=0})$ in the subtended double cone $\mathcal{C}(S)$ is

---

\[5\] The modular group is a one-parametric subgroup of the conformal group.
only the result of the fuzzy propagation inside the light cone (the breakdown of Huygens principle or the “reverberation” phenomenon caused by the mass), the fuzziness of the modular group for this pair is a pure propagation phenomenon, i.e., can be understood in terms of the deviation from Huygens principle. In view of the recent micro-local spectrum condition, one expects this nonlocal modular group action even in the correct vacuum i.e. for $\left(A_m \neq 0, \Omega_m \neq 0\right)$ to have modular groups whose generators are pseudo-differential instead of (local) differential operators. In this case the asymptotically local action near the light-like horizon will continue to hold. In order to avoid the pathology of the d=1+1 scalar zero mass field, one should use for the above consideration a massive free spinor field whose massless limit gives a two-component field with the first component only depending on the left light cone and the second on the right hand light cone. In fact the above observation is very much related to the zero mass theory results from Sewell’s restriction to the light cone horizon (boundary). It is my conviction that all the recent speculations about the quantum version of Bekenstein’s classical entropy and in particular the horizon (“holographic”) aspects of the associated degrees of freedom are manifestations of modular properties of generic nonperturbative QFT which however are overlooked in the Lagrangian quantization method. This and similar subjects will be the content of a separate paper with Wiesbrock.

4 Modular Wedge-Localization and Factorizing Theories

In this section we will show that the modular localization can be used as a starting point for constructing for interacting theories. This constructive approach is presently most clear in the case of factorizing d=1+1 QFT’s to which we will limit ourselves in this section. We remind the reader that “factorizable” in the intrinsic physical interpretation of algebraic QFT’s means that the long distance limit of the S-matrix (which only consists of the two-particle elastic part and automatically fulfills the Yang-Baxter relation as a physical consistency condition) defines a QFT model in its own right which we

6 They differ from the previous situation by a Connes cocycle.
call “factorizable”. It has no on-shell particle creation, but it does possess a rich off-shell (or “virtual”) particle structure i.e. it is a full-fledged QFT with nontrivial vacuum polarization caused by interactions. It should be viewed as being the simplest (no real creation) representative of a vast equivalence class of complicated (with real particle creation) models which share the same particle content together with the same long distance S-Matrix $S_{\text{fin}}$. This notion of factorizable is better suited for the present use in local quantum physics than the traditional “integrability” which is defined via quantization.. The idea is somewhat analogous to the construction of the simplest representative in a long-distance equivalence class (in the sense of the S-matrix) of a given superselection class. In other words each general $d=1+1$ field theory has an asymptotic companion which has the same superselection sectors ($\simeq$ same particle structure or incoming Fock space), but vastly simplified dynamics associated to a factorizing S-matrix. In $d=3+1$ this distinguished representative reduces to a free field with the same superselection structure as the other members of the equivalence class. This intrinsic understanding without imposing conservation laws or even Yang-Baxter structures (but rather obtaining them from consistency of the long distance limits of scattering operators) gives an enhanced significance of factorizable models as the simplest representative in a class of general models with the same charge superselection structure. This is of course analogous to the physical significance of the short distance universality class of conformal field theories. The present field theoretic understanding of the bootstrap-formfactor program, which is largely a collection of more or less plausible (but very successful) recipes [7], leaves a certain amount of conceptual clarity to be desired. Our main concern in this paper is therefore the elaboration of a framework which positions the formfactor-bootstrap program within general QFT in such a way that it can be used as a theoretical laboratory for the latter.

All applications of modular localization to interacting theories are based on the observation that in asymptotically complete theories with a mass gap, the full interaction resides in the Tomita operator $J(W)$, whereas the modular group $\Delta^{i\tau}(W)$ for wedges (being equal to Lorentz boosts) as well as the continuous Poincaré group transformations is “blind” against interactions (the physical representation of the Poincaré group are already correctly defined on the free incoming states). In fact the interaction resides in those disconnected parts of the Poincaré group which involve antiunitary time reflections and the Tomita $J$ for the standard wedge (containing the origin in its edge) inherits this from the TCP operator. To be more explicit the Haag Ruelle scattering theory together with the asymptotic completeness easily yield (for each wedge):

\[ J = S_sJ_0, \quad \Delta^{i\tau} = \Delta_0^{i\tau} \]  

where the subscript 0 refers to the free incoming situation and we have omitted the reference to the particular wedge. It is very important here to emphasize that this interpretation of modular data in terms of scattering theories exists only for wedge regions. It enters via the TCP invariance and is not just a consequence of Haag-Ruelle scattering theory, since the latter cannot be formulated within wedges. At this point we differ from the approach in [8] which seeks
to extract the cyclic relation associated with the crossing symmetry from the KMS condition of the wedge algebra of the interacting local fields, whereas we will generate this algebra from semilocal operators without any additional large time limits. In fact the cyclic equation is equivalent to the KMS relation for the interacting wedge algebra written in special nonlocal generators. These nonlocal operators are on-shell\(^7\) and free of vacuum polarizations. Their rapidity space creation- and annihilation- components are forming the Zamolodchikov-Faddeev algebra. We will call them (vacuum-polarization-) free wedge generators, abbreviated FWG, because they are on-shell operators. They are in some sense a nonlocal generalization of free fields and agree with the latter in case of absence of interactions, but they do generate the interacting wedge algebra and therefore merit the attribute “semilocal”, i.e. the FWG’s are special semilocal operators. These objects had been introduced already in previous publications of the author \(^22\)|\(^31\); here we will present them in more details and also use the opportunity to correct some earlier errors.

No physical interpretation is yet known for the modular objects of compactly localized algebras of e.g. double cones. One expects in that case, that different from the wedge case, not only the reflection \(J\), but also the modular group \(\Delta^{\tau}\) will depend on the interaction. As will be seen later, the double cone algebras can be constructed from the wedge algebras.

In order not to change the traditional notation \(S\) for the Tomita involutions, we use the subscript \(s\) whenever we mean the S-matrix of scattering theory. The most convenient form for the previous equation which puts the modular aspect of scattering in evidence is:

\[
S = S_s S_0
\]  

(34)

where \(S\) and \(S_0\) are the antiunitary Tomita operators and \(S_s\) is the unitary scattering operator. Therefore the scattering operator in relativistic QFT has two rather independent aspects: it is a global operator in the sense of large time limits of scattering theory, and it has a modular localization interpretation in measuring the deviation of \(J\) or \(S\) from their free field values \(J_0, S_0\) i.e. it is a relative modular invariant. This modular aspect is characteristic of local quantum physics and has no counterpart in nonrelativistic theory or quantum mechanics.

The modular subspace of \(\mathcal{H}_{\text{Fock}} = \mathcal{H}_{\text{in}}\) for the standard wedge is characterized in terms of the following equations\(^8\):

\[
S_s S_0 \mathcal{H}_R = \mathcal{H}_R
\]

(35)

\[
S_s S_0 \psi = \psi, \quad \psi \in \mathcal{H}_R
\]

There is one more important idea which is borrowed from scattering theory namely the existence of a “modular Möller operator” \(^23\) \(U\) which is related to

\(^7\)For this reason they cannot be used to construct better than wedge localizations. Compactly localized algebras, as double cone algebras, have to be constructed with the help of intersections and have new off-shell generators.

\(^8\)The distinction between the \(\pm\) sign is not very important since the multiplication with \(i\) converts one real subspace into the other.
the S-matrix as:

\[ S_s = U J_0 U^* J_0 \]  

(36)

This corresponds to the well-known standard formula \( S_s = (\Omega^{\text{out}})^* \Omega^{\text{in}} \). Writing the S-matrix in terms of an Hermitian phase matrix \( \eta \) as \( S = e^{i\eta} \) it is not difficult to find an \( U \) in terms of \( \eta \). Note however that the Haag-Ruelle scattering theory (as well as its more formal but better known LSZ predecessor) in local quantum physics does not provide a Møller isometry between Heisenberg states and incoming states because the scattering state space and the space for the interacting fields are identical since in local quantum physics, different from the nonrelativistic rearrangement scattering theory, one does not introduce a separate space of asymptotic fragments.

The idea of introducing such an object into our modular approach comes from the unitary equivalence of the interacting and the free hyperfinite type \( \text{III}_1 \) wedge algebras:

\[ \mathcal{A}(W) = U \mathcal{A}^{\text{in}}(W) U^* \]  

(37)

We demand the \( U \)-invariance of the vacuum \( U \Omega = \Omega \) and the one-particles space \( U \mathcal{H}^{(1)} = \mathcal{H}^{(1)} \). The physical idea behind this is that whereas the vacuum and the one-particle states cannot be resolved from the rest of the energy-momentum spectrum in compact regions as e. g. double cones, the semiinfinite wedge region, which is left invariant by the associated Lorentz-boosts, does allow such a resolution. Note that this situation is different from the problem of unitary equivalence of the canonical equal time commutation relations in the free versus the interacting case. A unitary equivalence in this case (of an algebra belonging to a region with trivial spacelike complement) would be forbidden by Haag’s theorem \[3\] on the nonexistence of the interaction picture in QFT.

The above characterization of \( U \) may be replaced by a slightly more convenient one \[^9\] in terms of an intertwining property between modular operators:

\[ US_0 = SU \]  

(38)

In terms of localized spaces, the \( U \) has the property:

\[ U \mathcal{H}^{\text{in}}_{W}(W) = \mathcal{H}_{W}(W) \]  

(39)

Unfortunately one cannot conclude from this transformation of spaces (39) (which is the only property derivable from (38)) in favor of the validity of (37) and therefore the spatial modular property is too weak for the construction of the wedge algebra. In order to find another method for constructing \( \mathcal{A}(W) \), we first study a simple model.

Between the two simplest possibilities, the Ising field theory with \( S^{(2)}_s = -1 \) and the (non-parity invariant) Federbush model with \( S^{(2)}_{I,II} = e^{i\pi g} \) we chose the latter because it allows also a Lagrangian interpretation and hence is simpler to describe to readers familiar with the Lagrangian quantization approach to

[^9]: I am indebted to H.-W. Wiesbrock for emphasizing this intertwining property as the most convenient definition of \( U \).
QFT. The model consists in coupling two species of Dirac fermions via a (parity violating) current-pseudocurrent coupling \[ g : j^I_\mu j^I_\nu : \varepsilon^{\mu\nu}, \quad j^I_\mu =: \bar{\psi}^I \gamma_\mu \psi^I \quad (40) \]

One easily verifies that:

\[
\begin{align*}
\psi_I(x) &= \psi_I^{(0)}(x) e^{ig\Phi^I_l(x)}; \\
\psi_{II}(x) &= \psi_{II}^{(0)}(x) e^{ig\Phi_{II}^r(x)}; \\
\psi^{(0)}_{I,II}(x) &= \frac{1}{\sqrt{2\pi}} \int (e^{-ipx} a_{I,II}(\theta) + e^{ipx} b^*_{I,II}(\theta)) d\theta 
\end{align*}
\]

where \( \Phi^{(l,r)} = \int_{x \leq x'} j_0 dx' \) is a potential of \( j_\mu^5 \) i.e. \( \partial_\mu \Phi \sim \varepsilon_{\mu\nu} j^\nu = j_\mu^5 \) and the superscript \( l, r \) refers to whether we choose the integration region for the line integral on the spacelike left or right of \( x \). The triple ordering is needed in order to keep the closest possible connection with classical geometry and localization and in particular to maintain the validity of the classical field equation in the quantum theory; for its meaning and its conversion into the standard Fermion Wick-ordering we refer to the above papers. This conceptually simpler triple ordering can be recast into the form of the analytically (computational) simpler standard Fermion Wick-ordering in terms of the (anti)particle creation/annihilation operators \( a_{I,II}^\#(\theta), b^\#_{I,II}(\theta) \). Although in this latter description the classical appearance of locality is lost, the quantum exponential do still define local Fermi-fields \[ \psi_I \] with the contributions from the exponential (disorder fields) compensate. This model belongs to the simplest class of factorizing models (those with rapidity independent S-matrix) and its explicit construction via the formfactor program is almost identical to that of the massive Ising field theory \[ [26] \]. The reason why it does not appear under this approach in the literature is that the bootstrap classification was limited to strictly parity conserving theories. For our present purposes it serves as the simplest nontrivial illustration of new concepts arising from modular localization.

Despite the involved looking local fields \[ (11) \], the wedge algebras are easily shown to be of utmost simplicity:

\[
\begin{align*}
\mathcal{A}(W) &= \text{alg} \left\{ \psi^{(0)}_I(f) U_{II}(g), \psi^{(0)}_{II}(h); \text{suppf, h} \in W \right\} \\
\mathcal{A}(W') &= \mathcal{A}(W)_{\text{Klein}} = \text{alg} \left\{ \psi^{(0)}_I(f), \psi^{(0)}_{II}(h) U_I(g); \text{suppf, h} \in W' \right\}
\end{align*}
\]

i.e. the two wedge-localized algebras (\( W \) denotes the right wedge) are generated by free fields “twisted” by global \( U(1) \)-symmetry transformation of angle \( g \) (coupling constant) and the subscript “Klein” denotes the well-known Klein transformation associated with the \( 2\pi \) Fermion rotation. The right hand side follows from the observation that with \( x \) restricted to \( W \), one may replace the exponential in \( \psi_I \) in \[ (11) \] (which represents a left half space rotation) by the
full rotation since the exponential of the right half-space charge is already contained in the right free fermion algebra etc. The following unitarily equivalent description of the pair $A(W), A(W')$ has a more symmetric appearance under the extended parity symmetry $\psi_I(t, x) \leftrightarrow \psi_{II}(t, -x)$:

$$A(W) = \text{alg} \left\{ \psi_I^{(0)}(f) U_{II}(\frac{g}{2}); \text{suppf}, h \in W \right\}$$

$$A(W') = \text{alg} \left\{ \psi_{II}^{(0)}(h) U_I(\frac{g}{2}); \text{suppf}, h \in W' \right\}$$

The verification of all these properties is elementary, and no modular theory is needed. The computation [24] of the scattering matrix $S_s$ from (41) is most conveniently done by Haag-Ruelle scattering theory [3]:

$$S_s |\theta_1, \theta_2 \rangle = e^{i\pi g} |\theta_1, \theta_2 \rangle$$

These formulae (including antiparticles) can be collected into an operator expression [24]:

$$S_s = \exp i\pi g \int \rho_I(\theta_1) \rho_{II}(\theta_2) \varepsilon(\theta_1 - \theta_2) d\theta_1 d\theta_2$$

Where $\rho_I, \rho_{II}$ are the momentum space charge densities in the rapidity parametrization.

The surprising simplicity of the wedge algebra of this model as compared to, say, its double cone algebras consists in the fact that one can choose on-shell generators. We will show that modular wedge localization for factorizing models always leads to on shell generators for the localized spaces, though for rapidity dependent S-matrices they do not generate the wedge algebras.

It would now be easy to solve the n-particle modular localization equation:

$$S_H^{(n)}(W) = H^{(n)}_R(W); H_R(W) = \oplus_n H^{(n)}_R(W)$$

$$H^{(n)}_R(W) = \left\{ \int F(\theta_1, \theta_2, ..., \theta_n) |\theta_1, \theta_2, ..., \theta_n \rangle d\theta_1 d\theta_2 ... d\theta_n \mid F \in H^{(n)}_{strip} \right\}$$

Here $H^{(n)}_{strip}$ denotes the closure of the space of square integrable function which allow an analytic continuation into the strip $0 < \text{Im} z_i < \pi, i = 1...n$ and fulfill certain boundary conditions. This is just the p-space on shell analyticity which comes from the wedge localization. In analogy to Wightman functions in x-space, the $n!$ different boundary prescriptions $\text{Im} z_i > \text{Im} z_{i+1} > .... > \text{Im} z_n \rightarrow$
between species I and II and the independence of S simplicity of the model is reflected in the fact that interactions only take place and the free Fermion anticommutation relations between the same type. The form (48). The use of the Z-basis takes care of the boundary conditions in an check that the solution of the wedge localization equation (47) has in deed the S values on the upper rim. In the free case i.e. for Ss = 1, one just recovers the Bose/Fermi statistics but with the Federbush S-matrix the space consists of strip-analytic functions which are a solution of a Riemann-Hilbert boundary problem. The general solution of this problem (i.e. the characterization of the subspace \( \mathcal{H}_R(W) \) within the full multiparticle wave function space) may be presented as a product of special solution of the Riemann-Hilbert problem with the general solution of the interaction free problem in \( \mathcal{H}_R(W) \). A convenient way to write the solution consists in using auxiliary operators Z as generators for a basis of state vectors:

\[
\int d^2 x_1 \ldots d^2 x_n \mathbf{f}_n(x_1, \ldots, x_n) : Z_{I,II}(x_1) \ldots Z_{I,II}(x_n) : \Omega, \\
\text{supp} \mathbf{f}_n \in W^\otimes n, \mathbf{f}_n \text{ real} \tag{48}
\]

where the Z’s are on shell operators whose frequency positive and negative momentum space components have to fulfill commutation relations which must be compatible with the boundary relations governed by products of two particle S-matrices. One immediately realizes that this leads to the Zamolodchikov-Faddeev algebra relations for the Federbush S-matrix:

\[
Z_{I,II}(x) = \frac{1}{\sqrt{2\pi}} \int \left( e^{-ix_2^2}c_{I,II}(\theta) + e^{ix_2^2}d_{I,II}^*(\theta) \right) d\theta \tag{49}
\]

where the c and the corresponding anti d can be formally expressed in terms of the incoming (anti)particle creation and annihilation operators:

\[
c_{I,II}(\theta) = a_{I,II}(\theta) : e^{-i\sigma g} \mathbf{f}_\theta^\rho_{I,II}(\theta') d\theta' : \tag{50}
\]

\[
d_{I,II}(\theta) = b_{I,II}(\theta) : e^{i\sigma g} \mathbf{f}_\theta^\rho_{I,II}(\theta') d\theta' :
\]

with the Zamolodchikov-Faddeev relations between the type I and II particles:

\[
c_{I}(\theta_1)c_{I}(\theta_2) = -S^{(2)}(\theta_1 - \theta_2)c_{I}(\theta_2)c_{I}(\theta_1) \tag{51}
\]

\[
c_{I}(\theta_1)c_{I}(\theta_2) = -S^{(2)}(\theta_1 - \theta_2)^{-1}c_{I}(\theta_2)c_{I}(\theta_1)
\]

\[
d_{I}(\theta_1)d_{I}(\theta_2) = -S^{(2)}(\theta_1 - \theta_2)d_{I}(\theta_2)d_{I}(\theta_1)
\]

\[
d_{I}(\theta_1)d_{I}(\theta_2) = -S^{(2)}(\theta_1 - \theta_2)^{-1}d_{I}(\theta_2)d_{I}(\theta_1)
\]

\[
d_{I,I}(\theta_1)c_{I,I}(\theta_2) = -S^{(2)}(\theta_1 - \theta_2)^{-1}c_{I,I}(\theta_2)d_{I,I}(\theta_1)
\]

\[
d_{I,I}(\theta_1)c_{I,I}(\theta_2) = -S^{(2)}(\theta_1 - \theta_2)c_{I,I}(\theta_2)d_{I,I}(\theta_1)
\]

and the free Fermion anticommutation relations between the same type. The simplicity of the model is reflected in the fact that interactions only take place between species I and II and the independence of \( S^{(2)} \) on \( \theta \). It is now easy to check that the solution of the wedge localization equation (48) has indeed the form (48). The use of the Z-basis takes care of the boundary conditions in an analogous way as the (anti)symmetry of the coefficient functions follows from
the commutation relations of the Fermion/Boson operator basis. In particular it takes care of the inversion of the order in $Z$-products under Hermitian conjugation: $S A \Omega = A^* \Omega$, with $A$ an operator as in (48), as well as the coalescence of the $x$-space commutation relations for (49) with those of (43) or (44). The interaction does make a distinction between left and right and parity is only conserved if one also interchanges the two species. The wedge localized states fulfill the following thermal KMS condition

$$\varphi(t) = \langle \psi_2 | \Delta^{it} \psi_1 \rangle, \quad \varphi(t - i) = \langle \Delta^{it} \psi_1 | \psi_2 \rangle$$

where $\psi_{1,2}$ are $n$-particle localized states of the form (48) and $\varphi$ is analytic in the open $t$-strip and continuous on the boundary.

A calculation of the $J$-transformed operators $Z_1^J = JZ_1J$ with

$$c_{l,II}^J(\theta) = a_{l,II}(\theta) : e^{-i\pi g \int_0^\theta \rho_{11,1}(\theta') d\theta'} :$$
$$d_{l,II}^J(\theta) = b_{l,II}(\theta) : e^{i\pi g \int_0^\theta \rho_{11,1}(\theta') d\theta'} :$$

reveals that $Z^J$ commutes with $Z$. This is a speciality of models with rapidity independent $S$-matrices. In such models the wedge locality is manifest since

$$\left[ Z_1^J(\theta), Z_{1II}(\theta') \right] = 0$$

and all other commutators between operators $Z^J$ and $Z$ of the same type are like those of free fields. This means that $Z^J(x)$ commutes with $Z(y)$ for $x, y \in W$, even though the $Z^H(x)'s$ are nonlocal fields.

For more general factorizing models however the verification of wedge locality of the FWG operators is more subtle. We illustrate the procedure for a factorizing model with one interacting (selfconjugate) particle. In that case the $Z$ has the following form in terms of the scattering phase shift $\vartheta$

$$Z(x) = \frac{1}{\sqrt{2\pi}} \int \{ e^{-ipx} Z(\theta) + h.c. \} d\theta, \quad p = m(ch\theta, sh\theta)$$

$$Z(\theta) = a(\theta) e^{-i \int_{-\infty}^{\theta} d(\theta - \theta') a^*(\theta') a(\theta') d\theta'}, \quad S_{sc}(\theta) = e^{i\vartheta(\theta)}$$

$$Z^\#(\theta_1) Z^\#(\theta_2) = S_{sc}(\theta_1 - \theta_2) Z^\#(\theta_2) Z^\#(\theta_1), \quad Z^\# \equiv Z \text{ or } Z^*$$

For the following we find it convenient to introduce the path notation which allows to denote rapidity space creation and annihilation operators by one symbol. We write for free fields $A(x)$ smeared with wedge supported test functions

$$A(\hat{f}) = \int f(\theta) a(\theta) + \int f(\theta - i\pi) a(\theta - i\pi) \equiv \int_C f(\theta)a(\theta)$$

$$a(\theta - i\pi) \equiv a^*(\theta), \quad f(\theta - i\pi) = \hat{f}(\theta)$$
$C$ consists of the real $\theta$–axis and the parallel path shifted down by $-i\pi$ and it is only the function $f$ which is analytic in the strip $-\pi < Im\theta < 0$ and not the operators. The analyticity of $\hat{f}$ is equivalent to the localization property. We will use the same notation for the FWG $Z(\hat{f})$. In the application to the vacuum of course only the creation contribution from the lower rim of the strip survives.

We want to prove that the on-shell $Z$ is a FWG i.e. the smeared operator $Z(\hat{f}) = \int Z(x)\hat{f}(x)d^2x$ with $\text{supp } \hat{f} \in W$ generates the $\ast$-algebra of the interacting theory localized in the wedge. In formula

$$\left[JZ(\hat{f})J, Z(\hat{g})\right] = 0, \quad \text{sup } \hat{f}, \hat{g} \in W$$

(58)

$$Z(\hat{f})\Omega = \int d\theta \hat{f}(\theta) a^\ast(\theta) \Omega$$

To prove this one first notices that the $Z(\theta)\#$ commutes with the $JZ(\theta)\# J$ underneath the Wick-ordering. The reason is that the exponential involve integrals over number density which extend over complementary rapidity regions. The numerical phase factors which originate from the commutation of these exponential factors with the $a^\#s$ mutually compensate. There remains the contraction between the pre-exponential $a^\#s$ which leads to

$$\int \hat{f}(\theta) g(\theta - i\pi) \exp i \int \delta_{sc}(\theta - \theta') n(\theta') d\theta'$$

(59)

Shifting the integration by $i\pi$, and using the crossing symmetry in the form: $\delta_{sc}(\xi + i\pi) = \delta_{sc}(-\xi) + 2\pi n i$, we see that this contraction is equal to that in the opposite order (which has the negative exponential). The pointwise equation $JA(-x)J = PA(x)P$ for the local field $A(x)$, with $P$ being the generator of the parity transformation, is to be replaced by $P \left( \int Z(\theta)\hat{f}(\theta)d\theta \right) P \in JA(W)J$, a relation which is easily checked using the explicit form of the $Z$-operators.

Now we come to the crucial part of the modular localization method, the exploration of consequences of the KMS condition for the $Z$-correlation functions. As a typical case we consider the 4-point function.

$$\langle \Omega, Z(f_{1'})Z(f_{2'})Z(f_2)Z(f_1)\Omega \rangle = \langle Z(f_{1'})Z(f_{2'})Z(f_2)Z(f_1)\rangle_{\text{therm}} \tag{60}$$

$$\langle Z(f_{2'})Z(f_2)Z(f_{1})Z(f_{2''})\rangle_{\text{therm}}$$

Each side is the sum of two terms, the direct term associated with

$$Z(f_2)Z(f_1)\Omega = \int f_2(\theta_2 - i\pi)f_1(\theta_1 - i\pi)Z^\ast(\theta_1)Z^\ast(\theta_2)\Omega + c \text{ number } \cdot \Omega$$

= \int f_2(\theta_2 - i\pi)f_1(\theta_1 - i\pi)S(\theta_2 - \theta_1)a^\ast(\theta_1)a^\ast(\theta_2)\Omega + c\Omega$$

and the analogous formula for the bra-vector. For the inner product there are two contraction terms consisting of direct and crossed contraction (in indices 1 or 2) of the $a^\#s$. Only the second one gives an S-matrix factor in the integrand.
The c-number term on the left hand side cancels the direct term on the right hand side. The equality of the crossed terms on both sides gives (using thedenseness of the analytic wave functions)

\[ S(\theta_2 - \theta_1) = S(\theta_1 - \theta_1' + i\pi) |_{\theta_1' = \theta_2} \]  

(62)
i.e. one obtains the above crossing relation for the two particle S-matrix. Higher inner products involve products of S-matrices, and it is easy to see that the KMS condition for the FWG algebra is equivalent to the crossing property of the S-matrix. The presence of additional local operators \( A \) which can be a fortiori localized in the wedge does not influence the validity of the KMS condition.

\[ \langle Z(f_1')Z(f_2')...Z(f_{m'})AZ(f_n)...Z(f_2)Z(f_1) \rangle_{\text{therm}} = \]  

(63)

\[ = \langle Z(f_2)...Z(f_n)AZ(f_n)...Z(f_2)Z(f_1)Z(f_1') \rangle_{\text{therm}} \]

The rapidity space formulation of this KMS condition is (again using denseness of wave functions, the derivation is completely analogous to the previous case) the desired cyclicity relation for the formfactor of that local operator \( A \) (its coefficient functions in the sense of (65) below):

\[ a_n(\theta_1, \theta_2, ..., \theta_n) = a_n(\theta_2, ..., \theta_n, \theta_1 - 2\pi i) \]  

(64)

This equation was hitherto derived as a special consequence of the crossing symmetry for factorizing systems. The crossing symmetry itself in turn follows from the LSZ scattering theory together with certain analytic assumptions (30) which are presently only controllable in the factorizing setting. Without the mediation of the semilocal FWG \( Z^\# \) it would not have been possible to link the cyclicity equation with the KMS property of \( \mathcal{A}(W) \) i.e. the KMS relation only takes the form of the cyclicity relation in the semilocal FWG field “coordinates”.

To the extent that the derivation in (30) is correct, it must implicitly contain the FGW operators.

Contrary to the previous free case, the sharpening of the support of the test function does not improve the localization within the wedge. This is equivalent to the statement that the reflection with \( J \) does not create an operator which is localized at the geometrically mirrored support region of the test function \( \hat{f} \) in the opposite wedge \( W' \). It only fulfills the commutation relation with respect to the full \( W' \). In fact the breakdown of parity covariance is important for the existence of such nonlocal but wedge-localized fields, since fields which are covariant under all transformations are expected to be either point local or completely delocalized (i.e. not even in a wedge). We will later see that any sharper localization requires the operator to be an infinite power series in the \( Z' \)s:

\[ A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_n, \theta_{n-1}, ..., \theta_1) : Z(\theta_1)....Z(\theta_n) : \]  

(65)

where we again used the previously explained path notation. The sharper localization leads to relations between the \( a'_n \)s. Note that since the commutations of
$Z'$s produce S-matrix phase factors, the $a_n$ must compensate this phase factors upon commuting $\theta'$s.

$$a_n(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S_{sc}(\theta_i - \theta_{i+1})a_n(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n) \quad (66)$$

In this respect the phase factors are like statistics terms. However since the $a_n$ have meromorphic properties in the multi-$\theta$ strip (actually for compact localization the meromorphy region is much bigger), these phase factors must be consistent with the univaluedness in the analytic domain. Together with other requirements, this leads to a multi-variable Riemann-Hilbert problem for the $a_n'$s \cite{[21]}.

Since the d=1+1 double cone algebras $\mathcal{A}(\mathcal{O})$ are obtained by translations and intersections

$$\mathcal{A}(\mathcal{O}_a) = U(\frac{a}{2})\mathcal{A}(W)U(\frac{a}{2}) \cap U(\frac{a}{2})\mathcal{A}(W')U(-\frac{a}{2}) \quad (67)$$

this sharper localization leads to additional restrictions for the $a_n'$s. For a detailed characterization of these algebras in terms of generators I refer to forthcoming joint work \cite{[17]}. It is very important to appreciate the difference between classical and quantum localization. For the FWG fields the quantum localization is defined by these intersection of algebras and cannot be replaced by the classical localization in the sense of support properties of test functions.

It is instructive to reproduce the results of Smirnov and the reformulation of Laskevich \cite{[7],[27]} in the present setting.

**Theorem 3** A sufficient condition for a power series in the $Z$-fields to describe a pointlike local field $A(x)$ is in addition the previous commutation (66) and cyclicity property (64) (in case of our illustrative model without bound states) the presence of poles in the $a_n(\theta_1, \theta_2, \ldots, \theta_n)$ for $\theta$-differences lying on the boundary of the strip:

$$a_{n+2}(\vartheta + i\pi + i\varepsilon, \vartheta, \theta_1, \theta_2, \ldots, \theta_n) \simeq \frac{1}{\varepsilon} \left[ 1 - \prod_{i=1}^{n} S(\vartheta - \theta_i) \right] a_n(\theta_1, \theta_2, \ldots, \theta_n) \quad (68)$$

Here we did not specify the Lorentz transformation property of the field

$$A(x) = \sum_n \frac{1}{n!} \int_C d\theta_1 \ldots \int_C d\theta_n \exp(-iP(\theta_1; \ldots; \theta_n)x) a_n(\theta_n, \theta_{n-1}; \ldots; \theta_1) : Z(\theta_1) \ldots Z(\theta_n) : \quad (69)$$

If the field is irreducible with spin $s$ one must have

$$a_n(\theta_1 + \vartheta, \theta_2 + \vartheta, \ldots, \theta_n + \vartheta) = e^{-s\vartheta}a_n(\theta_1, \theta_2, \ldots, \theta_n), \quad (70)$$

but in the proof this is not needed. The proof is purely structural i.e. independent of the computation (or parametrization of the solutions of all these conditions).

**Proof**
we show that for spacelike separation we can by contour deformation transform of

\[ A(x)B(y) = \sum \frac{1}{m!n!k!} \int_{-\infty}^{\infty} \frac{d^k \xi}{C} \frac{d^m \theta}{\theta} \int d^n \vartheta \times \]

\[ \times \exp -i [P(\xi)(x - y) + iP(\theta)x - iP(\vartheta)y] \]

\[ \times a_{m+k}(\xi_1, \ldots, \xi_k, \theta_1, \ldots, \theta_m) b_{n+k}(\vartheta_n, \ldots, \vartheta_1, \xi_k - i\pi, \ldots, \xi_1 - i\pi) \]

\[ \times : Z(\theta_m) \ldots Z(\theta_1) Z(\vartheta_1) \ldots Z(\vartheta_n) : \]

achieve the opposite order. Since the \( Z' \)'s underneath the Wick-product do not commute (unlike the previous case of \( Z^2 \) with \( Z \)), the compensating terms must come from the \( i\pi \) shift of contour, i.e. from residua of poles encircled in the process of deformation. In order to apply (48) say to \( \xi_i \) together with \( \theta_j \) (i.e. the pole at \( \xi_i = \theta_j + i\pi \)), we have to position the relevant variables at the first two places in \( a_{m+k} \) by successive transpositions. Whereas the contributions from \( \xi \)-permutations in \( a_{m+k} \) compensate those in \( a_{n+k} \), from the \( \theta \) one obtains the phase factor \( \Pi_{\xi \neq \theta} S(\xi_i + i\pi - \theta_j) \). It is convenient to shift simultaneously \( Z(\theta_j) \) to the right and denote the total resulting phase factor:

\[ s(\xi_i) = \Pi_{\xi \neq \theta} S(\xi_i + i\pi - \theta_j) \Pi_{i=1}^n S(\theta_j - \vartheta_i) \]

Now we do the contour shifting which according to (48). We only pick up poles from the coefficient function \( a_{m+k} \). As a result we get the additional factor \(-[1 - \Pi_{\xi \neq \theta} S(\theta_j - \xi_i) \Pi_{\xi \neq \theta} S(\theta_j - \vartheta_i)]\). Note that for space-like distances (we can place the \( x, y \) so that they are in opposite wedges) the exponential factor is damped in \( 0 < Im \xi < \pi \). The integrand of the term \( \langle m, n, k \rangle \) is, apart from the mentioned two factors and the exponential function, as follows

\[ a_{m+k-2}(\xi_1, \ldots, \xi_i, \ldots, \xi_k, \theta_1, \ldots, \theta_j, \ldots, \theta_m) b_{n+k}(\vartheta_n, \ldots, \vartheta_1, \xi_k - i\pi, \ldots, \theta_j, \ldots, \xi_1 - i\pi, \theta_j) \]

\[ \times : Z(\theta_m) \ldots Z(\theta_1) Z(\vartheta_1) \ldots Z(\vartheta_n) Z(\theta_j) : \]

where as usual the roof sign indicates deletion of the variable or the operator.

commuting the operator \( Z(\theta_j) \) to the left, we obtain a phase factor which cancels the previous factor evaluated at the pole \( s(\xi_i = \theta_j + i\pi) \). The important step is now the following renaming (leading to a resummation in \( \sum \)): \( \theta_j \rightarrow \theta_{n+1} \), followed by \( n \rightarrow n - 1 \), \( m \rightarrow m + 1 \) and \( k \rightarrow k + 1 \) and re-numbering \( \xi \) and \( \theta \)

\[ -[1 - \Pi_{\xi \neq \theta} S(\theta_j - \xi_i) \Pi_{\xi \neq \theta} S(\theta_j - \vartheta_i)] a_{m+k}(\xi_1, \ldots, \xi_k, \theta_1, \ldots, \theta_m) \]

\[ \times b_{n+k}(\vartheta_n, \ldots, \vartheta_1, \xi_k - i\pi, \ldots, \xi_1 - i\pi) : Z(\theta_m) \ldots Z(\theta_1) Z(\vartheta_1) \ldots Z(\vartheta_n) : \]

This shifting process has to be repeated with every \( \xi \) and the boundary term has to be taken into account (i.e. the remaining integration variables are placed on the boundary \( \xi \rightarrow \xi + i\pi \)). Using the following identity

\[ \sum_{l=0}^{n} (-1)^{l} \sum_{j_1 < \ldots < j_l} \prod_{r=1}^{l}(1 - t_{j_r}) = \prod_{j=1}^{n} t_{j} \]
one arrives at the desired result

\[ A(x)B(y) = B(y)A(x), \ (x - y)^2 < 0 \]  

(76)

As mentioned before this derivation is not quite in the spirit of our modular approach. The latter requires to study the double cone algebras i.e. to characterize the quantum localization properties of intersections and to derive a necessary and sufficient condition on the \( a_n \) \[^{[17]}\].

One can show that the modular construction of “free” anyons and plektons \[^{[31]}\] in \( d=1+2 \) leads to similar mathematical problems. In this case there is no scattering, but the whole construction takes place in a multiparticle space which, in contradistinction to Fermions and Bosons, has no tensor product structure in terms of Wigner spaces. The braid group commutation relation leads to a Tomita \( J \) which, as in the Federbush model, involves a constant matrix \( S_{twist} \). In this case \( S_{twist} \) does not carry scattering information, but is identical to the braid group representation R-matrix which appears in the exchange algebra of conformal QFT. This statistical R-matrix \( S_{twist} \) has a similar effect as a Klein transformation i.e. the opposite of \( \mathcal{A}(W) \) in the quantum sense of the von Neumann commutant is now different from the geometric opposite \( \mathcal{A}(W)' \neq \mathcal{A}(W) \). Again the model has a rich virtual particle structure, even though no real particle is created in a scattering process. But since this time this vacuum polarization is a result of the nontrivial statistics twist \( S_{twist} \), there is no reason to expect that this goes away in the nonrelativistic limit and the model passes to Schrödinger theory. The fact that the nonrelativistic limit of Fermions and Bosons leads to the Schrödinger QM is related to the existence of relativistic free fields in Fock space. But since the Fock space structure in \( d=1+2 \) cannot support anyons and plektons, there is good reason to expect a kind of nonrelativistic field theory which can incorporate the virtual particle or vacuum polarization structure which is necessary to maintain the relation between spin and (plektonic) statistics in the nonrelativistic limit. Indeed all attempts to incorporate braid group statistics into QM, ever since the time of Leinaas and Myrheim \[^{[34] \ [35]}\] have only led to a deformation of (half)integer spin \[^{[38]}\], but not to nonrelativistic operators with the correct spin-statistics commutation structure. To phrase it into a more mundane fashion: there are good reasons why nobody has succeeded to construct a QM interaction (e.g. Aharonov-Bohm like) which leads to an anyonic spin in the two-particle S-matrix and fulfills those higher particle S-matrix cluster properties which are necessary in order to obtain coherence with the multiparticle statistics. The attempts based on Aharonov-Bohm potentials allow only to mimic an anyonic two-particle spin. I would not expect that the scattering boundary condition for the long range A-B interaction of n-particles can be chosen in such a way that the multiparticle S-matrix fulfills the cluster property which certainly would be a prerequisite for sustaining a nonrelativistic spin-statistics connection. Like e.g. the nonrelativistic Lee model, I rather expect anyons and plektons to be only describable by a nonrelativistic field theory which maintains the vacuum polarization and this is my explanation for why the search for a consistent theory

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of particles with genuine braid group statistics in quantum mechanics has been unsuccessful. At this point it is instructive to compare with a recent No-Go theorem [37].

The method advocated here by analogy to factorizing d=1+1 models of using the Wigner one particle representations combined with the correct non-tensor-product multiparticle structure from scattering theory [36] together with the present modular localization method looks very promising (but still needs to be carried out). The results should be of great practical relevance for unraveling the structure of (nonrelativistic) d=1+2 quasi-particles in many body systems with spatial layer structures [39]. As with Fermions and Bosons, the derivation of the spin-statistics properties and the classification of the possible statistics (+internal symmetry) is done in the relativistic setting, in order then to be used e.g. for the (nonrelativistic) quasi-particles in condensed matter physics and statistical mechanics.

A similar idea of using an \( S_{\text{twist}} \) in a constructive modular approach should also be helpful to complete the classification and construction of chiral conformal QFT's. In this connection one should recall that presently the construction of these models is done in most cases by studying the representation theory of affine algebras, in other words by ideas which are quite different from those of standard QFT. The charge-carrying fields which fulfill braid group exchange algebras are then constructed as intertwiners between the vacuum and charged representations of these algebras. It would be more in the spirit of QFT to first classify all plektonic statistics (the structure constants of the exchange algebras) and then to construct the vacuum representation (expected to be unique) of the associated exchange algebras. There is however one caveat: the exchange algebra (unlike the CCR or CAR algebras) is incomplete since the distributional behavior at coalescent points is left undetermined. This is the reason why the attempts in e.g. [41] which were based on monodromy properties are more “artistic” (in the sense of depending on hindsight and luck) than systematic. Momentum space algebras, as the Z-F algebra on the other hand, are complete. But can one use the R-matrices as a Klein twist \( S_{\text{twist}} \) in momentum space using the previous method of wedge localization? Since conformal theories have an infraparticle- but not a particle-structure, this problem is not trivial. The normalization for obtaining finite form factors is infrared divergent with respect to that of finite correlation functions. We hope to present an affirmative answer in a separate work. The importance of such a new approach to chiral conformal QFT results from the fact that it places it back into the mainstream of QFT where it belongs. The statistics input in the absence of genuine interaction (chiral conformal QFT's are expected to belong to this kind since there can be no genuine coupling parameter dependent interaction on one light cone) and for given internal symmetry groups (in order to have a distinction between charged fields for current algebras and the W-fields) should uniquely determine the vacuum representation of the completed exchange algebra. In more technical terms and only for experts on conformal QFT: the Friedan-Qiu-Shenker-quantization of the energy-momentum tensor algebra and similar quantizations for W-algebra generalizations should follow from the more
fundamental DHR-Jones-Wenzl quantization which has a direct interpretation in terms of particle statistics. In this way chiral conformal QFT would change from a mathematical playground back to a valuable theoretical laboratory of QFT since it can then be formulated as simple analytic realization of principles which have validity in higher dimension.

In the remaining part of this section I would like to make some pedagogical remarks for readers with an incomplete knowledge of general structural properties of nonperturbative QFT. In connection with the analytic aspects of the rapidity-dependent Zamolodchikov-Faddeev algebra operators and their analogy with the x-space chiral conformal operators of the exchange algebras one often finds the erroneous concept of “analytic field operators” and “holomorphic algebras”. Since their use is so widespread (the few articles where this misleading terminology is not used are rather the exception than the rule), it is interesting to ask where such ideas are coming from. I am not an expert on string theory, therefore I have limited my search to QFT. The oldest paper which could be interpreted as alluding to “analytic operators” \( A(z) \), \( z \in \mathbb{C} \) seems to be the famous BPZ \(^{11}\) paper on minimal models. Although the authors do not use such terminology in print, the notation used in that paper may have caused misunderstandings (and has been misunderstood by physicist whose first experience with nonperturbative QFT came through that famous work or was influenced by string theory). The truth is that field algebras never have holomorphic properties. The analytic properties of correlation functions and state vectors depend entirely on the nature of states one puts on those algebras. Whereas vacuum ground states lead to the famous BHW-domain \(^{18}\) (in chiral conformal QFT equal to a uniformization region with poles for coalescing coordinates), KMS states will only lead to strip analyticity. It is in any case the state which generates the analytic continuation holomorphic properties, and the higher its symmetry property, the bigger the holomorphic regions.

It is a fact that the associated analytic Bargman-Hall-Wightman domain for a Moebius invariant vacuum state in conformally covariant QFT is larger than that of the corresponding Poincaré covariant massive theories and as a result of braid group statistics, one looses the univaluedness of these analytic continuations (but of course not the univaluedness in the real time physical localization points!). The effect of restriction of the vacuum state of massive theories to the wedge algebra yields the momentum space analytic properties which together with the factorization property lead to the rapidity being an analytic uniformization variable. Contrary to the use of field theoretic terminology in the contemporary literature neither the old nor the new BHW domains nor the analytic continuation in rapidity have anything to do with the “living space” of fields in the sense of quantum localization of operators in this article. “Localization” is indeed a property of the algebras whereas holomorphy is not.

\(^{11}\)In an older paper on conformal blocks e.g. \(^{29}\) (called nonlocal components in a conformal decomposition theory with respect to the center of the universal covering), such “holomorphic” terminology was never used.
structive algebraic meaning to the above unfortunate but nevertheless popular terminology, one could point to the following property which has no analogue in higher dimension (not even in the conformal invariant limit of higher dimensional theories). Whereas generally with vacuum expectation values one can relate at most two physical theories: a noncommutative real time QFT and a commutative Euclidean field theory (a candidate for a continuous statistical mechanics), a chiral conformal theory on one light cone has infinitely many noncommutative boundary values (and no commutative chiral boundary value) each of which defines a set of positive definite correlation functions and hence a theory. This is to say the restriction of the analytically continued correlation function defines a positive QFT not only on the circle (the standard living space of chiral theories) but also on each boundary encircling the origin (with the right \(i\varepsilon\) Wightman boundary prescription)\(^{12}\). However all this does not legitimize “holomorphic operators” in the literal sense but rather the existence of a operator conformal QFT for each chosen boundary circumference. The reader recognizes easily that this structure is equivalent to the existence of the infinite dimensional diffeomorphism group which is related to the Virasoro algebra structure. The application of any symmetry, which does not leave the vacuum reference state invariant, defines another set of positive Wightman functions which at the case at hand belong to the deformed boundary. If one prefers a geometrical to a quantum physical terminology, one may emphasize the diffeomorphism group, whereas for a physicist who prefers the setting of local quantum physics the existence of an interesting generalization of the higher dimensional dichotomy between real time-imaginary time theories may be the interesting aspect.

5 General Interactions and Outlook

The factorizing models of the bootstrap formfactor approach belong to a class of models which are “real particle” (on shell) conserving but “virtual particle” (off shell) nonconserving. In the context of our modular localization approach this means that although the wedge localization Hilbert spaces can be generated by on shell FWG operators, any sharper localization as e.g. the double cone localization and in particular the state vector obtained by applying smeared pointlike local fields to the vacuum lead to “virtual particle clouds”. Although absence of real particle creation according to well-known theorems is impossible in \(d=1+3\) interacting theories, we expect the three-dimensional theories \(d=1+2\) to form an exception if the associated particles obey genuine braid group statistics (anyons and plektons) and hence their charge-carrying field have a noncompact seminfinitive string-like extension \(^{13}\). Namely we expect the existence of “free”\(^ {13}\) anyons and plektons which similar to free Fermions and Bosons have

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\(^{12}\)This is probably what string theorist have in mind when they draw their pictures.

\(^{13}\)If "free" in \(d=1+2\) implies the existence of generating fields \(A\) in Fock space without vacuum polarization clouds in \(A\Omega\) then plektons are never free because the vacuum polarization is needed to sustain the braid group statistics and the associated string-like extension. We use
no on-shell creation but (different from free Fermions and Bosons) possess a rich virtual particle structure \[37\].

Let me finally address the important question of whether the concept of modular localization can be expected to lead to a nonperturbative approach for d=1+3 interacting theories of Fermions and Bosons. This depends on whether it is possible to relate an admissible S-matrix with an auxiliary semilocal (wedge localized) operator \(Z\) which is free of vacuum polarization and creates the interacting localized wedge spaces \(\mathcal{H}_R(W)\). The existence of such an “on-shell” operator would be the correct substitute for the FWG operators in the case of non-factorizing situations. On a formal level such operators have appeared in the light cone quantization of interacting theories \[42\] but in that formalism the relation to the original local variables (with vacuum polarization) gets lost and without relation to local operators one has no physical interpretation beyond global spectral properties \[40\]. Our modular wedge localization approach on the other hand does not depend on assumptions about canonical commutation relations and therefore could serve as a rigorous substitute for the ill-defined light cone quantization.

A prerequisite for all modular constructions including the modular approach to the formfactor bootstrap program is its uniqueness i.e. to one admissible S-matrix there should be only one local algebraic net. This is part of the more general question whether modular data \((J, \Delta^u)\) determine uniquely an algebra and a state. Whereas this modular inverse problem has many solutions \[16\], the corresponding inverse problem of QFT

\[
S_{sc} \text{ in } \mathcal{H}_{Fock} \to \{A(W)\}_{all \ wedges}
\]

(77)

can be shown to have a unique solution even beyond the special factorizing d=1+1 models \[17\] if one accepts certain plausible but presently unproven) vacuum and one-particle properties of the modular Møller operator \(U\) \[17\].

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