Addendum to: Values of zeta functions of varieties over
finite fields, Amer. J. Math. 108, (1986), 297-360.

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October 1, 2013

The original article expressed the special values of the zeta function of a variety over
a finite field in terms of the $\hat{\mathbb{Z}}$-cohomology of the variety. As the article was being com-
pleted, Lichtenbaum conjectured the existence of complexes of sheaves $\mathbb{Z}(r)$ extending the
sequence $\mathbb{Z}, \mathbb{G}_m[-1], \ldots$. The complexes given by Bloch’s higher Chow groups are known
to satisfy most of the axioms for $\mathbb{Z}(r)$. Using Lichtenbaum’s Weil-étale topology, we can
now give a beautiful restatement of the main theorem of the original article in terms of
$\mathbb{Z}$-cohomology groups.

**Notations**

We use the notations of Milne (1986). For example,

$$M^{(n)} = M/nM, \quad TM = \lim_{n} \ker(n: M \to M), \quad z(f) = \frac{[\ker(f)]}{[\text{coker}(f)]},$$

and $\nu_{s}(r)$ denotes the sheaf of logarithmic de Rham-Witt differentials on $X_{et}$ (ibid., p. 307).
The symbol $l$ denotes a prime number, possibly $p$.

**Review of abelian groups**

In this subsection, we review some elementary results on abelian groups. An abelian group
$N$ is said to be **bounded** if $nN = 0$ for some $n \geq 1$, and a subgroup $M$ of $N$ is **pure** if
$M \cap mN = mM$ for all $n \geq 1$.

**Lemma 1.** (a) Every bounded abelian group is a direct sum of cyclic groups.
(b) Every bounded pure subgroup $M$ of an abelian group $N$ is a direct summand of $N$.

**Proof.** (a) Fuchs (1970), 17.2.
(b) Kaplansky (1954), Theorem 7, p. 18, or Fuchs (1970), 27.5.

**Lemma 2.** Let $M$ be a subgroup of $N$, and let $l^n$ be a prime power. If $M \cap l^nN = 0$ and $M$
is maximal among the subgroups with this property, then $M$ is a direct summand of $N$.

**Proof.** The subgroup $M$ is bounded because $l^nM \subset M \cap l^nN = 0$. To prove that it is pure,
one shows by induction on $r \geq 0$ that $M \cap l^rN \subset l^rM$. See Fuchs (1970), 27.7.

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1It was submitted in September 1983. This addendum was originally posted on the author’s website in 2009.
NOTES. [Fuchs (1970), 9.8.] Let $B$ and $C$ be subgroups of an abelian group $A$. Assume that $C \cap B = 0$ and that $C$ is maximal among the subgroups of $A$ with this property. Let $a \in A$. If $pa \in C$ ($p$ prime), then $a \in B + C$.

Proof: We may suppose that $a \notin C$. Then $\langle C, a \rangle$ contains a nonzero element $b$ of $B$, say, $b = c + ma$ with $c \in C$ and $m \in \mathbb{Z}$. Here $(m, p) = 1$ because otherwise $b = c + m_0(pa) \in B \cap C = 0$. Thus $rm + sp = 1$ for some $r, s \in \mathbb{Z}$, and

$$a = r(ma) + s(pa) = rb - rc + s(pa) \in B + C.$$  

[Fuchs (1970), 27.7.] We prove that $M \cap l^nM \subseteq l^nM$ for all $r \geq 0$. This being trivially true for $r = 0$, we may apply induction on $r$. Let $m = l^{r+1}a \neq 0$, $m \in M$, $a \in N$. Then $r \leq n - 1$, because otherwise $l^{r+1}a = M \cap l^nM = 0$. By (9.8), $l^r a \in l^nM + M$, say, $l^r a = l^n c + d$ with $c \in N$, $d \in M$. Then $d = l^r(a - l^n c) \in M \cap l^nN$, which equals $l^nM$ by the induction hypothesis. From $m = l^{r+1}a = l^{n+1}c + ld$, we find that $(m - ld) \in M \cap l^{r+1}N = 0$, and so $m = ld \in l^{r+1}M$.

Every abelian group $M$ contains a largest divisible subgroup $M_{div}$, which is obviously contained the first Ulm subgroup of $M$, $U(M) \equiv \bigcap_{n \geq 1} nM$. Note that $U(M/U(M)) = 0$.

NOTES. A sum of divisible subgroups is obviously divisible. For the last statement, let $x \in M$ map to the first Ulm subgroup of $M/U(M)$. Then, for each $n \geq 1$, there exists a $y \in M$ such that $ny - x \in U(M)$, and so $ny - x = ny'$ for some $y' \in M$. Now $x = n(y - y')$, and so $x$ is divisible by $n$ in $M$, i.e., $x \in U(M)$.

PROPOSITION 3. If $M/nM$ is finite for all $n \geq 1$, then $U(M) = M_{div}$.

PROOF. (Cf. Milne (1988), 3.3.) If $U(M)$ is not divisible, then there exists a prime $l$ such that $U(M) \neq IU(M)$. Fix such an $l$, and let $x \in U(M) \setminus IU(M)$. For each $n \geq 1$, there exists an element $x_n$ of $M$ such that $l^n x_n = x$. In fact $x_n$ has order exactly $l^n$ in $M/U(M)$, and so $M/U(M)$ contains elements of arbitrary high $l$-power order.

Let $S$ be a finite $l$-subgroup of $M/U(M)$. As $U(M/U(M)) = 0$ and $S$ is finite, there exists an $n$ such that $S \cap l^n(M/U(M)) = 0$. By Zorn’s lemma, there exists a subgroup $N$ of $M/U(M)$ that is maximal among those satisfying (a) $N \supset S$ and (b) $N \cap l^n(M/U(M)) = 0$. Moreover, $N$ is maximal with respect to (b) alone. Therefore $N$ is a direct summand of $M/U(M)$ (Lemma2). As $N$ is bounded (in fact, $l^nN = 0$), it is a direct sum of cyclic groups (Lemma1). We conclude that $S$ is contained in a finite $l$-subgroup $S'$ of $M/U(M)$ that is a direct summand of $M/U(M)$. Note that

$$S'^{(l)} \hookrightarrow (M/U(M))^{(l)} \cong M^{(l)},$$

and so $\dim_{\mathbb{F}_l} S'^{(l)} \geq \dim_{\mathbb{F}_l} S^{(l)}$. But is clear (from the first paragraph) that $\dim_{\mathbb{F}_l} S^{(l)}$ is unbounded, and so this contradicts the hypothesis on $M$.\]

NOTES. Cf. Fuchs, Vol II, 65.1.

COROLLARY 4. If $TM = 0$ and all quotients $M/nM$ are finite, then $U(M)$ is uniquely divisible (= divisible and torsion-free = a $\mathbb{Q}$-vector space).

PROOF. The first condition implies that $M_{div}$ is torsion-free, and the second that $U(M) = M_{div}$.\]

For an abelian group $M$, we let $M_l$ denote the completion of $M$ with respect to the $l$-adic topology. Every continuous homomorphism from $M$ into a complete separated group factors uniquely through $M_l$. In particular, the quotient maps $M \rightarrow M/l^nM$ extend to homomorphisms $M_l \rightarrow M/l^nM$, and these induce an isomorphism $M_l \rightarrow \varprojlim_n M/l^nM$. The kernel of $M \rightarrow M_l$ is $\bigcap_n l^nM$. See Fuchs (1970), §13.
Lemma 5. Let $N$ be a torsion-free abelian group. If $N/IN$ is finite, then the $l$-adic completion of $N$ is a free finitely generated $\mathbb{Z}_l$-module.

Proof. Let $y_1, \ldots, y_r$ be elements of $N$ that form a basis for $N/IN$. Then
\[ N = \sum \mathbb{Z} y_i + IN = \sum \mathbb{Z} y_i + l(\sum \mathbb{Z} y_i + IN) = \cdots = \sum \mathbb{Z} y_i + l^nN, \]
and so $y_1, \ldots, y_r$ generate $N/l^nN$. As $N/l^nN$ has order $l^m$, it is in fact a free $\mathbb{Z}/l^n\mathbb{Z}$-module with basis $\{y_1, \ldots, y_r\}$. Let $a \in N_l$, and let $a_n$ be the image of $a$ in $N/l^{n+1}N$. Then
\[ a_n = c_{n,1}y_1 + \cdots + c_{n,r}y_r \]
for some $c_{n,i} \in \mathbb{Z}/l^{n+1}\mathbb{Z}$. As $a_n$ maps to $a_{n-1}$ in $N/l^nN$ and the $c_{n,i}$ are unique, $c_{n,i}$ maps to $c_{n-1,i}$ in $\mathbb{Z}/l^n\mathbb{Z}$. Hence $(c_{n,i})_{n \in \mathbb{N}} \in \mathbb{Z}_l$, and it follows that $\{y_1, \ldots, y_r\}$ is a basis for $N_l$ as a $\mathbb{Z}_l$-module.

Proposition 6. Let $\phi : M \times N \to \mathbb{Z}$ be a bi-additive pairing of abelian groups whose extension $\phi_l : M_l \times N_l \to \mathbb{Z}_l$ to the $l$-adic completions has trivial left kernel. If $N/IN$ is finite and $\bigcap_l l^nM = 0$, then $M$ is free and finitely generated.

Proof. We may suppose that $N$ is torsion-free. As $\bigcap_l l^nM = 0$, the map $M \to M_l$ is injective. Choose elements $y_1, \ldots, y_r$ of $N$ that form a basis for $N/IN$. According to the proof of Lemma 5, they form a basis for $N_l$ as a $\mathbb{Z}_l$-module. Consider the map
\[ x \mapsto (\phi(x, y_1), \ldots, \phi(x, y_r)) : M \to \mathbb{Z}^r. \]
If $x$ is in the kernel of this map, then $\phi_l(x, y) = 0$ for all $y \in N_l$, and so $x = 0$. Therefore the map $M$ injects into $\mathbb{Z}^r$, which completes the proof.

Review of Bloch’s complex

Let $X$ be a smooth variety over a field $k$. We take $\mathbb{Z}(r)$ to be the complex of sheaves on $X$ defined by Bloch’s higher Chow groups. For the definition of Bloch’s complex, and a review of its basic properties, we refer the reader to the survey article Geisser (2005).

The properties of $\mathbb{Z}(r)$ that we shall need are the following.

(A)$_{n_0}$ For all integers $n_0$ prime to the characteristic of $k$, the cycle class map
\[ \left( \mathbb{Z}(r) \xrightarrow{\cdot n_0} \mathbb{Z}(r) \right) \to \mu_{n_0}^{\otimes r}[0] \]
is a quasi-isomorphism (Geisser and Levine (2001), 1.5).

(A)$_p$ For all integers $s \geq 1$, the cycle class map
\[ \left( \mathbb{Z}(r) \xrightarrow{\nu_s} \mathbb{Z}(r) \right) \to \mathbb{Z}(r)[{-r-1}] \]
is a quasi-isomorphism (Geisser and Levine (2000), Theorem 8.5).

(B) There exists a cycle class map $\text{CH}^r(X) \to H^{2r}(X_{\text{ét}}, \mathbb{Z}(r))$ compatible (via (A)) with the cycle class map into $H^{2r}(X_{\text{ét}}, \widehat{\mathbb{Z}}(r))$. Here $\text{CH}^r(X)$ is the Chow group.

(C) There exist pairings
\[ \mathbb{Z}(r) \otimes^L \mathbb{Z}(s) \to \mathbb{Z}(r+s) \]
compatible (via (A)$_n$) with the natural pairings
\[ \mu_n^{\otimes r} \times \mu_n^{\otimes s} \to \mu_n^{\otimes r+s}, \quad \text{gcd}(n, p) = 1. \]

When $k$ is algebraically closed, there exists a trace map $H^{2d}(X_{\text{ét}}, \mathbb{Z}(d)) \to \mathbb{Z}$ compatible (via (A)$_n$) with the usual trace map in étale cohomology.
Values of zeta functions

Throughout this section, $X$ is a smooth projective variety over a finite field $k$ with $q$ elements, $r$ is an integer, and $p_r$ is the rank of the group of numerical equivalence classes of algebraic cycles of codimension $r$ on $X$.

We list the following conjectures for reference.

$T^r(X)$ (Tate conjecture): The order of the pole of the zeta function $Z(X,t)$ at $t = q^{-r}$ is equal to $p_r$.

$T^r(X,l)$ (l-Tate conjecture): The map $\text{CH}^r(X) \otimes \mathbb{Q}_l \to H^{2r}(\bar{X}_{\text{et}}, \mathbb{Q}_l(r))^T$ is surjective.

$S^r(X)$ (semisimplicity at 1): The map $H^{2r}(\bar{X}_{\text{et}}, \mathbb{Q}_l(r))^T \to H^{2r}(\bar{X}_{\text{et}}, \mathbb{Q}_l(r))_r$ induced by the identity map is bijective.

The statement $T^r(X)$ is implied by the conjunction of $T^r(X,l)$, $T^{d-r}(X,l)$, and $S^r(X,l)$ for a single $l$, and implies $T^r(X)$, $T^{d-r}(X,l)$, $S^r(X,l)$, $S^{d-r}(X,l)$ for all $l$ (see Tate (1994), 2.9; Milne (2007), 1.11).

Let $V$ be a variety over a finite field $k$. To give a sheaf on $\mathbb{V}_{\text{et}}$ is the same as giving a sheaf on $\mathbb{V}_{\text{et}}$ together with a continuous action of $\Gamma \overset{\text{def}}{=} \text{Gal}(k/\mathbb{k})$. Let $\Gamma_0$ be the subgroup of $\Gamma$ generated by the Frobenius element (so $\Gamma_0 \simeq \mathbb{Z}$). The Weil-étale topology is defined so that to give a sheaf on $\mathbb{V}_{\text{Wét}}$ is the same as giving a sheaf on $\mathbb{V}_{\text{et}}$ together with an action of $\Gamma_0$ (Lichtenbaum (2005)). For example, for $V = \text{Spec } k$, the sheaves on $\mathbb{V}_{\text{et}}$ are the discrete $\Gamma$-modules, and the sheaves on $\mathbb{V}_{\text{Wét}}$ are the $\Gamma_0$-modules. In the Weil-étale topology, the Hochschild-Serre spectral sequence becomes

$$H^i(\Gamma_0, H^j(\mathbb{V}_{\text{et}}, F)) \implies H^{i+j}(\mathbb{V}_{\text{Wét}}, F).$$

Since

$$H^i(\Gamma_0, M) = M^\Gamma_0, M_{\Gamma_0}, 0, 0, \ldots \text{ for } i = 0, 1, 2, 3, \ldots,$$

this gives exact sequences

$$0 \to H^{i-1}(\mathbb{V}_{\text{et}}, F)_{\Gamma_0} \to H^i(\mathbb{V}_{\text{Wét}}, F) \to H^i(\mathbb{V}_{\text{et}}, F)^\Gamma_0 \to 0, \quad \text{all } i \geq 0.$$

If $F$ is a sheaf on $\mathbb{V}_{\text{et}}$ such that the groups $H^j(\mathbb{V}_{\text{et}}, F)$ are torsion, then the Hochschild-Serre spectral sequence for the étale topology gives exact sequences

$$0 \to H^{i-1}(\mathbb{V}_{\text{et}}, F)_\Gamma \to H^i(\mathbb{V}_{\text{et}}, F) \to H^i(\mathbb{V}_{\text{et}}, F)^\Gamma \to 0, \quad \text{all } i \geq 0.$$

The two spectral sequences are compatible, and so, for such a sheaf $F$, the canonical maps $H^i(\mathbb{V}_{\text{et}}, F) \to H^i(\mathbb{V}_{\text{Wét}}, F)$ are isomorphisms.

Let $X$ be a smooth projective variety over a finite field, and let

$$e^{2r} : H^{2r}(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r)) \to H^{2r+1}(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))$$

denote cup-product with the canonical element of $H^1(\Gamma_0, \mathbb{Z}) = H^1(k_{\text{Wét}}, \mathbb{Z})$, and let

$$\chi(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r)) = \prod_{i \neq 2, 2r+1} [H^i(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))]^{-1} \mathbb{Z}(e^{2r})$$

to which all terms are defined and finite. Let

$$\chi(X, O_X, r) = \sum_{i \leq r} \sum_{j \leq r} (-1)^{i+j} (r - i) \dim H^j(X, O_X^i/j).$$

We define $\chi'(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))$ as for $\chi(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))$, but with each group $H^i(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))$ replaced by its quotient

$$H^i(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))' \overset{\text{def}}{=} \frac{H^i(\mathbb{V}_{\text{Wét}}, \mathbb{Z}(r))}{U(H^i(\mathbb{Z}(\mathbb{Z}(r))}.$$
Theorem 7. Let $X$ be a smooth projective variety over a finite field such that the Tate conjecture $T'(X)$ is true for some integer $r \geq 0$. Then $\chi'(X_{\text{ét}}, \mathbb{Z}(r))$ is defined, and

$$
\lim_{t \to q^{-r}} Z(X, t) \cdot (1 - q^t)^{p^r} = \pm \chi'(X_{\text{ét}}, \mathbb{Z}(r)) \cdot q^{\chi'(X, \mathcal{O}_X, x)}.
$$

(3)

In particular, the groups $H^i(X_{\text{ét}}, \mathbb{Z}(r))'$ are finite for $i \neq 2r, 2r + 1$. For $i = 2r, 2r + 1$, they are finitely generated. For all $i$, $U(H^i(X_{\text{ét}}, \mathbb{Z}(r)))$ is uniquely divisible.

Proof. We begin with a brief review of Milne (1986). For an integer $n = n_0 p^r$ with $\gcd(p, n_0) = 1$,

$$
H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r)) \overset{\text{def}}{=} H^i(X_{\text{ét}}, \mu_n^{\otimes r}) \times H^{i-r}(X_{\text{ét}}, V_s(r)),
$$

and

$$
H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \overset{\text{def}}{=} \lim_{\longleftarrow n} H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r))
$$

(ibid. p. 309). Let

$$
e^{2r} : H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \to H^{2r+1}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))
$$

denote cup-product with the canonical element of $H^1(\Gamma, \hat{\mathbb{Z}}) \simeq H^1(k_{\text{ét}}, \hat{\mathbb{Z}})$, and let

$$
\chi(X, \hat{\mathbb{Z}}(r)) \overset{\text{def}}{=} \prod_{i \neq 2r, 2r+1} [H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r))]^{-1} z(e^{2r})
$$

denote the number of roots of $\chi(X, \hat{\mathbb{Z}}(r))$. Theorem 0.1 (ibid. p. 298) states that $\chi(X, \hat{\mathbb{Z}}(r))$ is defined if and only if $S'(X, l)$ holds for all $l$, in which case

$$
\lim_{t \to q^{-r}} Z(X, t) \cdot (1 - q^t)^{p^r} = \pm \chi(X, \hat{\mathbb{Z}}(r)) \cdot q^{\chi(X, \mathcal{O}_X, x)}.
$$

(4)

In particular, if $S'(X, l)$ holds for all $l$, then the groups $H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$ are finite for all $i \neq 2r, 2r + 1$.

For each $n \geq 1$ and $i \geq 0$, property (A) of $\mathbb{Z}(r)$ gives us an exact sequence

$$
0 \to H^i(X_{\text{ét}}, \mathbb{Z}(r))^{(n)} \to H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r)) \to H^{i+1}(X_{\text{ét}}, \mathbb{Z}(r)) \to 0.
$$

The middle term is finite, and so $H^i(X_{\text{ét}}, \mathbb{Z}(r))^{(n)}$ is finite for all $i$ and $n$. On passing to the inverse limit, we obtain an exact sequence

$$
0 \to H^i(X_{\text{ét}}, \mathbb{Z}(r)) \to H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \to TH^{i+1}(X_{\text{ét}}, \mathbb{Z}(r)) \to 0
$$

(5)

in which the middle term is finite for $i \neq 2r, 2r + 1$. As $TH^{i+1}(X_{\text{ét}}, \mathbb{Z}(r))$ is torsion-free, it must be zero for $i \neq 2r, 2r + 1$. In other words, $TH^i(X_{\text{ét}}, \mathbb{Z}(r)) = 0$ for $i \neq 2r + 1, 2r + 2$.

So far we have used only conjecture $S'(X, l)$ (all $l$) and property (A) of $\mathbb{Z}(r)$. To continue, we need to use $T'(X, l)$ (all $l$) and the property (B) of $\mathbb{Z}(r)$. The $l$-Tate conjecture $T'(X, l)$ for all $l$ implies that the cokernel of the map $CH^i(X) \otimes \mathbb{Z} \to H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$ is torsion. As this map factors through $H^{2r}(X_{\text{ét}}, \mathbb{Z}(r))^\vee$, it follows that $TH^{2r+1}(X_{\text{ét}}, \mathbb{Z}(r)) = 0$ and $H^{2r}(X_{\text{ét}}, \mathbb{Z}(r))^\vee \simeq H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$. Consider the commutative diagram

$$
\begin{array}{ccc}
H^{2r}(X_{\text{ét}}, \mathbb{Z}(r))^\vee & \overset{\sim}{\longrightarrow} & H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \\
\downarrow & & \downarrow \hat{\mathbb{Z}} \\
H^{2r+1}(X_{\text{ét}}, \mathbb{Z}(r))^\vee & \longrightarrow & H^{2r+1}(X_{\text{ét}}, \hat{\mathbb{Z}}(r)).
\end{array}
$$
As \( e^{2r} \) has finite cokernel, so does the bottom arrow, and so \( TH^{2r+2}(X_{\text{Wét}}, \mathbb{Z}(r)) = 0 \). We have now shown that
\[
TH^i(X_{\text{Wét}}, \mathbb{Z}(r)) = 0 \quad \text{for all } i
\]
and so (5) and Corollary 4
\[
\begin{cases} 
H^i(X_{\text{Wét}}, \mathbb{Z}(r)) \cong H^i(X_{\text{ét}}, \widehat{\mathbb{Z}}(r)) \\
U(H^i(X_{\text{Wét}}, \mathbb{Z}(r))) \text{ is uniquely divisible for all } i.
\end{cases}
\]

In particular, we have proved the first statement of the theorem except that each group \( H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \) has been replaced by its completion. It remains to prove that \( H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \) is finite for \( i \neq 2r, 2r+1 \) and is finitely generated for \( i = 2r, 2r+1 \) (for then \( H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \cong H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \otimes \widehat{\mathbb{Z}}) \).

The kernel of \( H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \to (H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \cong U(H^i(X_{\text{Wét}}, \mathbb{Z}(r))') = 0 \), and so \( H^i(X_{\text{Wét}}, \mathbb{Z}(r))' \) is finite for \( i \neq 2r, 2r+1 \).

It remains to show that the groups \( H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))' \) and \( H^{2r+1}(X_{\text{Wét}}, \mathbb{Z}(r))' \) are finitely generated. For this we shall need property (C) of \( \mathbb{Z}(r) \). For a fixed prime \( l \neq p \), the pairings in (C) give rise to a commutative diagram
\[
\begin{array}{ccc}
H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))'/\text{tors} & \times & H^{2d-2r+1}(X_{\text{Wét}}, \mathbb{Z}(d-r))'/\text{tors} \\
\downarrow & & \downarrow \\
H^{2r}(X_{\text{ét}}, \mathbb{Z}/l(r))'/\text{tors} & \times & H^{2d-2r+1}(X_{\text{ét}}, \mathbb{Z}/l(d-r))'/\text{tors} \\
\end{array}
\]
\( \to \mathbb{Z}/l \)
to which we wish to apply Proposition 6. The bottom pairing is nondegenerate, the group \( U(H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))') \) is zero, and the group \( H^{2d-2r+1}(X_{\text{Wét}}, \mathbb{Z}(d-r))(l) \) is finite, and so the proposition shows that \( H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))'/\text{tors} \) is finitely generated. Because \( U(H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))') = 0 \), the torsion subgroup of \( H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))' \) injects into the torsion subgroup of \( H^{2r}(X_{\text{ét}}, \widehat{\mathbb{Z}}(r)) \), which is finite (Gabber [1983]). Hence \( H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))' \) is finitely generated. The group \( H^{2r+1}(X_{\text{Wét}}, \mathbb{Z}(r))' \) can be treated similarly.

\[ \square \]

Remark 8. In the proof, we didn’t use the full force of \( T^r(X) \).

We shall need the following standard result.

Lemma 9. Let \( A \) be a (noncommutative) ring and let \( \bar{A} \) be the quotient of \( A \) by a nil ideal \( I \) (i.e., a two-sided ideal in which every element is nilpotent). Then:

(a) an element of \( A \) is invertible if it maps to an invertible element of \( \bar{A} \);
(b) every idempotent in \( \bar{A} \) lifts to an idempotent in \( A \), and any two liftings are conjugate by an element of \( A \) lying over \( 1_A \);
(c) let \( a \in A \); every decomposition of \( a \) into a sum of orthogonal idempotents in \( \bar{A} \) lifts to a similar decomposition of \( a \) in \( A \).

Notes. We denote \( a + I \) by \( \bar{a} \).

(a) It suffices to consider an element \( a \) such that \( \bar{a} = 1_A \). Then \( (1-a)^N = 0 \) for some \( N > 0 \), and so
\[
\frac{a}{(1-(1-a))(1+(1-a)+(1-a)^2+\cdots+(1-a)^{N-1})} = 1.
\]
(b) Let \( a \) be an element of \( A \) such that \( \bar{a} \) is idempotent. Then \( (a-a^2)^N = 0 \) for some \( N > 0 \), and we let \( a' = (1-(1-a)^N)^N \). A direct calculation shows that \( \bar{a'} = a' \) and that \( \bar{a} = \bar{a} \).

Let \( e \) and \( e' \) be idempotents in \( A \) such that \( \bar{e} = \bar{e}' \). Then \( \bar{a} = e'e + (1-e')(1-e) \) lies above \( 1_A \) and satisfies \( e'a = e'e = ae \).

(c) Follows easily from (b).
**Proposition 10.** Let $X$ be a smooth projective variety over a finite field $k$, and let $r$ be an integer. Assume that for some prime $l$ the ideal of $l$-homologically trivial correspondences in $\text{CH}^{\dim X}(X \times X)_Q$ is nil. Then $H^i(X_{\text{et}}, \mathbb{Z}(r))$ is torsion for all $i \neq 2r$, and the Tate conjecture $T'(X)$ implies that $H^{2r}(X_{\text{et}}, \mathbb{Z}(r))$ is finitely generated modulo torsion.

**Proof.** This is essentially proved in Jannsen (2007), pp. 131–132, and so we only sketch the argument. Set $d = \dim X$ and let $k = \mathbb{F}_q$.

According to Lemma 9 there exist orthogonal idempotents $\pi_0, \ldots, \pi_{2d}$ in $\text{CH}^{\dim X}(X \times X)_Q$ lifting the Künneth components of the diagonal in the $l$-adic topology. Let $h^j X = (hX, \pi_j)$ in the category of Chow motives over $k$. Let $P_i(T)$ denote the characteristic polynomial $\det(T - \sigma_X|H^i(X_{\text{et}}, \mathbb{Q}_l))$ of the Frobenius endomorphism $\sigma_X$ of $X$ acting on $H^i(X_{\text{et}}, \mathbb{Q}_l)$. Then $P_i(\sigma_X)$ acts as zero on the homological motive $h^i_m X$, and so $P_i(\sigma_X)^N$ acts as zero on $h^i X$ for some $N \geq 1$ (from the nil hypothesis). We shall need one last property of Bloch’s complex, namely, that $H^i(X_{\text{et}}, \mathbb{Z}(r))_Q \simeq K_{2r-i}(X)^{(r)}$ where $K_{2r-i}(X)^{(r)}$ is the subspace of $K_{2r-i}(X)_Q$ on which the $r$th Adams operator acts as $r^N$ for all $r$.

The $q$th Adams operator acts as the Frobenius endomorphism, and so $\sigma_X$ acts as multiplication by $q^r$ on $K_{2r-i}(X)^{(r)}$. Therefore $H^i(X_{\text{et}}, \mathbb{Z}(r))_Q$ is killed by $P_i(q^r)$, which is nonzero for $i \neq 2r$ (by the Weil conjectures), and so $H^i(X_{\text{et}}, \mathbb{Z}(r))$ is torsion for $i \neq 2r$.

The Tate conjecture implies that $P_2(T) = Q(T) \cdot (T - q^r)^{N_2}$ where $Q(q^r) \neq 0$, and so

$$1 = q(T)Q(T)^N + p(T)(T - q^r)^{N_2}, \quad \text{some } q(T), \ p(T) \in \mathbb{Q}[T].$$

As before, $P_2(\sigma_X)^N$ acts as zero on $h^{2r} X$ for some $N \geq 1$. Therefore $q(\sigma_X)Q(\sigma_X)^N$ and $p(\sigma_X)(\sigma_X - q^r)^{N_2}$ are orthogonal idempotents in $\text{End}(h^{2r} X)$ with sum 1, and correspondingly $h^{2r} X = M_1 \oplus M_2$. Now $H^{2r}(M_1, \mathbb{Z}(r))_Q = 0$ because $Q(\sigma_X)^N$ is zero on $M_1$ and $Q(q^r) \neq 0$. On the other hand, $M_2$ is isogenous to $(\mathbb{L}^{\otimes r})^{N_2}$ where $\mathbb{L}$ is the Lefschetz motive (Jannsen (2007), p. 132), and so $H^{2r}(M_2, \mathbb{Z}(r))$ differs from $H^{2r}(\mathbb{L}^{\otimes r}, \mathbb{Z}(r))^{N_2} \simeq H^{2r}(\mathbb{P}^d, \mathbb{Z}(r))^{N_2} \simeq \mathbb{Z}^{N_2}$ by a torsion group.

**Notes.** When $k = \mathbb{F}_q$, the $q$th Adams operator acts as $\sigma$ (Hiller 1981, §5; Soulé 1985, 8.1), and so $K_i(X)^{(j)}$ is the subspace on which $\sigma$ acts as $q^j$ (because the $m^j$-eigenspace of the $m$th Adams operators is independent of $m$, Seiler 1988, Theorem 1).

**Theorem 11.** Let $X$ be a smooth projective variety over a finite field such that the Tate conjecture $T'(X)$ is true for some integer $r \geq 0$. Assume that, for some prime $l$, the ideal of $l$-homologically trivial correspondences in $\text{CH}^{\dim X}(X \times X)_Q$ is nil. Then $\chi(X_{\text{et}}, \mathbb{Z}(r))$ is defined, and

$$\lim_{t \to q^r} Z(X, t) \cdot (1 - q^r t)^{N_2} = \pm \chi(X_{\text{et}}, \mathbb{Z}(r)) \cdot q^r \chi(X, \ell^{r, r}).$$

In particular, the groups $H^i(X_{\text{et}}, \mathbb{Z}(r))$ are finite for $i \neq 2r, 2r + 1$. For $i = 2r, 2r + 1$, they are finitely generated.

**Proof.** This will follow from Theorem 7 once we show that the groups $U^i \equiv U(H^i(X_{\text{et}}, \mathbb{Z}(r)))$ are zero. Because $H^i(X_{\text{et}}, \mathbb{Z}(r))$ is finitely generated modulo torsion (Proposition 10), it does not contain a nonzero $\mathbb{Q}$-vector space, and so $U^i = 0$ (Corollary 5).
Remark 12. For a smooth projective algebraic variety $X$ whose Chow motive is finite-dimensional, the ideal of $l$-homologically trivial correspondences in $\text{CH}^{\dim X}(X \times X) \otimes \mathbb{Q}$ is nil for all prime $l$ (Kimura). It is conjectured (Kimura and O'Sullivan) that the Chow motives of algebraic varieties are always finite-dimensional, and this is known for those in the category generated by the motives of abelian varieties. On the other hand, Beilinson has conjectured that, over finite fields, rational equivalence with $\mathbb{Q}$-coefficients coincides with with numerical equivalence, which implies that the ideal in question is always null (not merely nil).

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