The analysis of rotated vector field on the pendulum equation

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Abstract. The driven, damped pendulum equation is a mathematical model of pendulum. It is a nonlinear differential equation which is non-integrable. By the method of rotated vector field, this paper obtains the relation between the external drive $\beta$ and the periodic solution. An other conclusion is that the critical value of $\beta$ remains fixed in the over damping situation. These results is very useful in the study of charge-density wave in physics.

Keyword: pendulum, rotated vector fields, limit-cycle, charge-density wave (CDW)

1. Introduction

The driven,damped pendulum model is an essential model of nonlinear sciences, which displays many manifestations including chaos and appears in many physical subjects such as the Jesophson junctions and the charge-density waves [1–3]. This paper is involves the damped pendulum subject to an external drive, for which the equation of motion is

$$\frac{d^2 \phi}{dt^2} + \gamma \frac{d\phi}{dt} + \sin \phi = \beta, \quad (1)$$

where $\beta$ is the external drive, $\gamma$ is the friction coefficient. Equation $[1]$ is also used in the theory of charge-density waves (CDW) [3]. The relation between $\beta$ and its solution is important to describe the properties of conductivity associated with CDW [4]. This differential equation is nonlinear and non-integrable. It is the purpose of this paper to obtain the relation between $\beta$ and its solutions by the method of rotated vector fields.

In 1881-1886, Poincaré originated a theory named ”vector field” in his papers [5]. With this approach, the solutions of differential equation can be regarded as integral curves in the phase space. The qualitative properties of solutions can be obtained geometrically. After that, many scientists have studied in this area of theory. In 1953, G. F. Duff proposed the rotated vector fields [6], then G. Seifert et al. developed it to the general rotated vector fields [7–9].
By the method of rotated vector field, this paper provides the proofs about the relation between $\beta$ and the periodic solutions of the equation. By analysis, it is found that if the $\gamma$ is bigger enough (over damping), the critical value of $\beta$, with which this equation will have a critical periodic solution, remains fixed all the time, namely, $\beta_0 \equiv 1$. This explicit conclusion is very useful for the paper [4].

2. Preliminary definitions and lemmas

First of all, some statements are introduced in preparation for the proofs. On the $\phi - z$ phase plane, Equation (1) takes the form of vector field:

\[
\begin{align*}
\frac{d\phi}{dt} &= z \\
\frac{dz}{dt} &= \beta - \sin \phi - \gamma z,
\end{align*}
\]

where $\gamma > 0$, $\beta \geq 0$, whose solutions correspond to- trajectories on the plane. Each trajectory, has a direction running as time goes forward, to which the tangents become vectors which constitute the vector field. The points satisfying $\frac{d\phi}{dt} = 0$ and $\frac{dz}{dt} = 0$ is the singularities of Equation (2). When the vectors are rotating owing to the change of some parameter of equation, they constitute the rotated vector field, and accompanying with the moving of singularities as well, the field is named as the general rotated vector field [6–9]. According to $\frac{d\phi}{dt} = z$ indicates that, the trajectories direct from left to right while $z > 0$, and they direct from right to left while $z < 0$. In the theory of differential equation, owing to the uniqueness of the solution, the trajectories don’t intersect each other except at singularities [10]. This is very useful property in the following analysis.

When $\beta > 1$, there exist no singularities. When $0 \leq \beta \leq 1$, there exist singularities on the $\Phi$-axis, denoted by $(\phi_n, 0)$, where

\[
\begin{align*}
\phi_n &= n\pi + (-1)^n \phi_0 \\
\phi_0 &= \arcsin \beta,
\end{align*}
\]

For convenience, let $W(\phi, z) = z$, $Q(\phi, z) = \beta - \sin \phi - \gamma z$. Because $W(\phi, z) = W(\phi + 2\pi, z)$, $Q(\phi, z) = Q(\phi + 2\pi, z)$, both with a period of $2\pi$ along the $\Phi$-axis, the distribution of the vector field, determined by $W(\phi, z)$ and $Q(\phi, z)$, is also with a period of $2\pi$ along the $\Phi$-axis, the plane can be roll up into a cylindrical surface along the $\Phi$-axis, that is a cylindrical surface system. Owing to the properties of cylindrical surface system, we just discuss the interval $[-\pi - \phi_0, \pi - \phi_0]$ without declaiming and name it main interval. The singularities, denoted by the following symbols as default, have the properties as following.
1) When $0 \leq \beta < 1$, $A_k(\phi_{2k-1},0)$ is the saddle of the equation. The slopes of two separatrices are respectively

$$\lambda_1 = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 + \cos \phi_0},$$

(4)

$$\lambda_2 = -\frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 + \cos \phi_0}.$$  

(5)

When $\gamma > 0$, $B_k(\phi_{2k},0)$ is the stable focus of the equation.

When $\gamma = 0$, $B_k(\phi_{2k},0)$ is the center of the equation. But it is not the subject of the current analysis.

2) When $\beta = 1$, $\phi_0 = \frac{\pi}{2}$, $\phi_{2k+1} = \phi_{2k}$, namely, $A_{k+1}$ and $B_k$ coincide in position. They become combining singularities.

3) When $\beta > 1$, there exist no singularities.

Notice $k \in \mathbb{Z}$ above mentioned.

While $\gamma > 0$, $0 \leq \beta < 1$, in the main interval, there are four special trajectories departing from or going into the two saddles. Let $R$, $V$ denote the trajectory departing from $A_0$ and $A_1$ respectively; Let $S$, $U$ denote the trajectory going into $A_0$ and $A_1$ respectively; as shown in Figure 1. As default symbols, $R$, $V$, $S$, $U$ would represent the four special trajectories given above. In order to demarcate different regions, we draw a curve $G$ as shown in Figure 1, which expression is

$$z = \frac{\beta - \sin \phi}{\gamma}.$$  

(6)

Figure 1: Special trajectories in the main interval.

Let $\Lambda_1$ denotes the domain closed by $G$ and the $\Phi$-axis for $z > 0$;

Let $\Lambda_2$ denotes the domain closed by $G$ and the $\Phi$-axis for $z < 0$;

Let $\Lambda_3$ denotes the domain above $G$ for $z > 0$;

Let $\Lambda_4$ denotes the domain below $G$ for $z < 0$.

From Equation (2), it is easy to obtain:

$$\frac{dz}{d\phi} = \frac{\beta - \sin \phi}{z} - \gamma.$$  

(7)
From Equation (7), it is easy to derive the following properties:

a) \( \frac{dz}{d\phi} > 0 \), while \((\phi, z) \in A_1 \) or \( A_2 \);

b) \( \frac{dz}{d\phi} < 0 \), while \((\phi, z) \in A_3 \) or \( A_4 \);

c) \( \frac{dz}{d\phi} = 0 \), while \((\phi, z) \in G \);

d) \( \frac{dz}{d\phi} = \infty \), while \( z = 0 \) and \( \phi \neq \phi_n \).

On the base of the properties above, it is easy to give the Lemma 1 about the trajectory \( R \).

Lemma 1. Let \( K_0 : \phi = \phi_1 \) expresses a line perpendicular to the \( \Phi \)-axis through point \( A_1 \), as shown in Figure 1. Then the trajectory \( R \) departing from \( A_0 \) must pass over the upper half plane and be bound to reach at point \( C(\phi_c, 0) \) of the \( \Phi \)-axis and \( \phi_c \in [\phi_0, \phi_1] \) as well, or at point \( C(\phi_c, z_c) \) of line \( K_1 \) and \( z_c \in (0, \frac{\beta + 1}{2}) \) as well.

We can also give Lemma 2.

Lemma 2. If a periodic solution \( z = z(\phi) \) exists for Equation (2), i.e., \( z(\phi) = z(\phi + 2\pi), \phi \in (-\infty, +\infty) \), then it must be satisfied

\[
\int_{\phi}^{\phi + 2\pi} z(\phi) \, d\phi = \frac{2\pi \beta}{\gamma}.
\]

The proof of the Lemma 2 is very simple. According to Equality (7) and noticing the periodicity of \( z(\phi) \), it is easy to obtain Equality (8).

For the proofs in the next section, two definitions are provided as following.

Definition 1. Here \( \beta \) is fixed \( (\beta \geq 0) \), take \( \gamma \) \( (\gamma \geq 0) \) as a parameter, it constructs a rotated vector field. \( \theta \) is the angle between the vector and the \( \Phi \)-axis, the changing ratio of \( \theta \) with respect to \( \gamma \) is

\[
\frac{d\theta}{d\gamma} = -\frac{\gamma^2}{z^2 + (\beta - \sin \phi - \gamma \cdot z)^2} \leq 0.
\]

Definition 2. Here \( \gamma \) is fixed \( (\gamma \geq 0) \), take \( \beta \) \( (\beta \geq 0) \) as a parameter, it constructs a general rotated vector field. \( \theta \) is the angle between the vector and the \( \Phi \)-axis, the changing ratio of \( \theta \) with respect to \( \beta \) is

\[
\frac{d\theta}{d\beta} = \frac{z}{z^2 + (\beta - \sin \phi - \gamma \cdot z)^2}.
\]

While \( \gamma \in [0, +\infty), \beta \in [0, +\infty) \) and \( z > 0 \), we have \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\).

Meanwhile, we give the third lemma.

Lemma 3. Here is \( \beta = 0 \), for arbitrary \( \gamma_1 > 0 \), while \( \gamma = \gamma_1 \), then the trajectory \( R \) departing from \( A_0 \) must pass over upper half of plane and is bound to reach at point \( C(\phi_c, 0) \) of the \( \Phi \)-axis, and the point \( C \) locates between focus \( B_0 \) and saddle \( A_1 \), i.e. \( \phi_c \in [\phi_0, \phi_1] \).

Proof. Let \( \gamma = 0, \beta = 0 \), thus Equation (2) degenerate to be an integrable system. It is a sine-Gordon equation, namely, \( \frac{d^2\phi}{dt^2} + \sin \phi = 0 \). It is easy to give a solution,

\[
z = \sqrt{2(\cos \phi + 1)}.
\]

As shown in Figure 2, its corresponding trajectory \( L_0 \) connects the saddle \( A_0 \) with \( A_1 \). The slope of its tangent line at saddle \( A_0 \), \( A_1 \) is \( \pm 1 \) respectively. While \( \gamma = \gamma_1 \) \( (\gamma_1 \geq 0) \) and \( \beta = 0 \), according to Equality (4) gives that the slope of the tangent of trajectory \( R \) at point
$A_0$ is $\lambda_1$, where $0 < \lambda_1 < 1$. Therefore, $R$ is bound to locate at the inside of curve $L_0$ at the beginning. According to Definition 1 gives that, $\frac{d\theta}{d\gamma} < 0$ for the vectors at each point on the curve $L_0$ except point $A_0$ and $A_1$, so the vectors direct toward the inside of curve $L_0$ while $\gamma = \gamma_1$. According to the uniqueness of the solution, it can be obtained that, the trajectory $R$ runs at inside of curve $L_0$ all the time, or intersect at the singularity $A_1$. If we assume that trajectory $R$ intersects at point $A_1$, thus, the slope of $R$ at point $A_1$ is $\geq -1$ because it reaches point $A_1$ from inside of $L_0$. According to Equality $5$ gives its slope is $\lambda_2 < -1$. Therefore, they contradict each other. So $R$ can not reach point $A_1$. According to of Lemma 1 indicates that $R$ must passe over the upper half plane and reaches point $C(\phi_c,0)$ on the $\Phi$-axis, and point $C(\phi_c,0)$ locates between point $B_0$ and $A_1$, i.e. $\phi_0 \leq \phi_c < \phi_1$.

This completes the proof.

3. The proofs of theorem

Now we propose the theorem as following and prove it on the basis of the lemmas above.

**Theorem.** Suppose $\gamma = \gamma_1$, where $\gamma_1 > 0$. $R(\beta)$ denotes the trajectory departing from point $A_0$, it must have the propositions as following.

a) Equation (2) must have a value of $\beta_0(\gamma_1)$, $0 < \beta_0 \leq 1$, while $\beta = \beta_0(\gamma_1)$, $R(\beta_0)$ is bound to intersect the $\Phi$-axis at point $A_1$. Here this trajectory is named as a critical periodic solution or pseudo-periodic solution. $\beta_0$ is called the critical value, with which the equation will have a critical periodic solution or pseudo-periodic solution.

b) $\beta_0(\gamma_1)$ is unique.

c) While $\beta > \beta_0$, a periodic solution $z = z_T(\phi)$ exists for equation (2), that is $z_T(\phi) = z_T(\phi + 2\pi)$, moreover $z_T(\phi) > 0$, $\phi \in (-\infty, +\infty)$; while $\beta < \beta_0$, no periodic solution exists for equation (2).

d) The periodic solution of equation (2) is unique and stable.

e) Suppose $\beta_0(\gamma_1) = 1$, and if $\gamma_2 > \gamma_1$, then $\beta_0(\gamma_2) = 1$.

**Proof.**
a) First, let $\beta = 0$, $\gamma = \gamma_1$, thus, $\phi_0 = 0$. Three singularities on the main interval are denoted by $A_0(\phi_{-1}, 0)$, $B_0(\phi_0, 0)$, and $A_1(\phi_1, 0)$ respectively. According to Lemma 3, it can be proved that the trajectory $R(0)$ departing from point $A_0$ must reach the point $C^*(\phi_c^*, 0)$ on the $\Phi$-axis, moreover $0 \leq \phi_c^* < \pi$, as shown in Figure 3.

![Figure 3: The trajectories $R$ as $\beta = 0$ and $\beta > 0$.](image)

Now let $\beta > 0$, but $\beta \leq 1$, thus, $\phi_0 = \arcsin \beta > 0$, moreover $\phi_0 \leq \frac{\pi}{2}$. The new singularities on the main interval are written as $A_0(\phi_{-1}, 0)$, $B_0(\phi_0, 0)$, and $A_1(\phi_1, 0)$ respectively. $R(\beta)$ denotes the trajectory departing from $A_0$. Let $K_1$ is a line perpendicular to the $\Phi$-axis through the point $A_1$. Obviously, point $A_0$ is on the left of point $A_0^*$, the trajectory $R(\beta)$ run at outside of the trajectory $R(0)$ at the beginning. According to Definition 2 gives that $\frac{d\phi}{d\gamma} > 0$ for each point on the curve $R(0)$ except point $A_0^*$ and $C^*$. Therefore, while $\beta > 0$, the vector of field defined by the equation rotates anticlockwise. The direction of the vector at each point of curve $R(0)$ is towards outside. Hence, according to the uniqueness of solution and Lemma 1, it can be proved that the trajectory $R(\beta)$ must run at the outside of $R(0)$, furthermore, it reaches point $C(\phi_c, 0)$ on the $\Phi$-axis, where $\phi_{max} \leq \phi_c \leq \phi_1$, $\phi_{max} = \max\{\phi_c^*, \phi_0\}$; or it reaches point $C(\phi_1, z_c)$ on line $K_1$, where $0 < z_c < \frac{\beta+1}{\gamma_1}$, as shown in Figure 3.

Assume that $R(\beta)$ intersect line $K_1$ at point $C(\phi_1, z_c)$, and $0 < z_c < \frac{\beta+1}{\gamma_1}$. According to the continuing dependence of solution on the parameter $[0, 1]$, there must exist a trajectory $R(\beta_0)$, where $0 < \beta_0 < \beta$, which intersects point $A_1$. Proposition a) is correct.

Assume that $R(\beta)$ intersect point $C(\phi_c, 0)$ on the $\Phi$-axis, and $\phi_{max} \leq \phi_c \leq \phi_1$, where $\phi_{max} = \max\{\phi_c^*, \phi_0\}$. We notice that point $C(\phi_c, 0)$ must not locate on the left of point $C^*(\phi_c^*, 0)$ at least by increasing of $\beta$, point $A_1$ moving leftward monotonously. And while $\beta = 1$, $\phi_0 = \frac{\pi}{2}$, $B_0$ coincides with $A_1$, $C$ must be between the two point. According to the these situations, it can be declared that there must exist a trajectory $R(\beta_0)$, where $0 < \beta_0 \leq 1$, which intersects point $A_1$. Proposition a) is also correct. Thus the proof of proposition a) is complete.

b) Proof by contradiction. Assume that $\beta_0(\gamma_1)$ is not unique, here is $\beta_0'(\gamma_1) \neq \beta_0(\gamma_1)$, we may set $\beta_0'(\gamma_1) > \beta_0(\gamma_1)$.

While $\beta = \beta_0$, two saddles are denoted by $A_0^{(1)}$ and $A_1^{(1)}$ respectively. The trajectory
connecting them is denoted by $R(β_0)$, as shown in Figure 4.

While $β = β'_0$, two saddles are denoted by $A^{(2)}_0$ and $A^{(2)}_1$ respectively. The trajectory connecting them is denoted by $R(β'_0)$, as shown in Figure 4.

Figure 4: Two critical trajectories which intersect each other.

Because $β'_0 > β_0$, obviously, point $A^{(2)}_0$ is on the left of point $A^{(1)}_0$, point $A^{(2)}_1$ on the left of point $A^{(1)}_1$. Therefore $R(β'_0)$ must intersect with $R(β_0)$, the intersection is not on the Φ-axis, as shown in Figure 4. Referring to the preceding proofs will prove that $R(β'_0)$ cannot intersect $R(β_0)$. The result contradicts with the preceding assumption, hence $β_0(γ_1)$ must be unique.

c) First, we prove for the case of $β > β_0$.

Suppose that $β = β_0$, the equation has a critical periodic trajectory $R(β_0)$, the saddles connecting by it are denoted by $A^*_0$ and $A^*_1$ respectively.

While $β > β_0$, the trajectory departing from the new singularity $A_0$ is denoted by $R(β)$. Referring to the preceding proofs derives that $R(β)$ must run above $R(β_0)$, as shown in Figure 5. The function for $R(β)$ is expressed as following:

$$z = z_β(φ), \quad φ \in (-∞, +∞). \quad (12)$$

Obviously, here is $z_β(−π − φ_0) = 0$, $z_β(π − φ_0) > 0$, thus $z_β(−π − φ_0) < z_β(π − φ_0)$.

According to Equality (7) gives that $\frac{dz}{dφ} \leq 0$ for each point on the line $L_z = \frac{β+1}{γ_1}$. The vectors of field direct from left to right, therefore the vectors of field on line $L$ point downwards of $L$. Moreover, we notice that the vectors of field above line $L$ direct downwards absolutely. So we take a trajectory passing through point $H_0(−π − φ_0, \frac{β+1}{γ_1})$, the expression of function for it is:

$$z = z_m(φ), \quad φ \in (-∞, +∞). \quad (13)$$

Thus there are $z_m(−π − φ_0) = \frac{β+1}{γ_1}$, $z_m(π − φ_0) \leq \frac{β+1}{γ_1}$, and $z_m(−π − φ_0) ≥ z_m(π − φ_0)$. According to the continuing dependence of solutions on the initial values [10], there must exists a point $P$ on the line segment $A_0H_0$, through which the trajectory satisfies that $z_T(−π − φ_0) = z_T(π − φ_0)$, obviously, $z_T(φ) > 0$. According to the properties of cylindrical surface system derives,

$$z_T(φ) = z_T(φ + 2π), \quad φ \in (-∞, +∞), \quad (14)$$
Figure 5: The trajectories as $\beta > \beta_0$.

Equation (2) has a periodic solution, moreover $z_T(\phi) > 0$. The proof is complete for the proposition of $\beta > \beta_0$.

Second, we prove for the case of $\beta < \beta_0$.

Suppose that $\beta = \beta_0$, two saddles are denoted by $A_0^*$ and $A_1^*$ respectively. Let $R(\beta_0)$ denotes the trajectory which connects two saddles, for which the expression of function is $z = z_0(\phi)$, hence $z_0(\phi) = z_0(\phi + 2\pi), \phi \in (-\infty, +\infty)$.

While $\beta < \beta_0$, two saddles are denoted by $A_0$ and $A_1$ respectively. Let $R(\beta)$ denotes the trajectory departing from point $A_0$, Let $U(\beta)$ denotes the one going into point $A_1$, as shown in Figure 6. We will prove by contradiction. Assuming there is a periodic solution, the expression of function for it is $z = z_\beta(\phi)$. Obviously, $z_\beta(\phi) = z_\beta(\phi + 2\pi), \phi \in (-\infty, +\infty)$.

As analogue of the preceding proofs, it is easy to derive the relation of position of trajectory $R(\beta_0)$ with $R(\beta)$ and $U(\beta)$ geometrically. $R(\beta), U(\beta)$ locates by each side of $R(\beta_0)$ respectively, as shown in Figure 6. Obviously, if $z = z_\beta(\phi)$ passes through the line segment $A_0H_0$, it must be through axis $\Phi$. Observing the directions of the vectors of field at each side of axis $\Phi$, we can declare that the trajectory passing through axis $\Phi$ isn’t a periodic trajectory. Therefore, $z = z_\beta(\phi)$ is impossible the one passing through the line segment $A_0H_0$. So it is declared that...
there are only two relations between \( z = z_\beta(\phi) \) and \( z = z_0(\phi) \) as following.

In the first case, \( z_\beta(\phi - 1) < z_0(\phi - 1) \), because the trajectory of \( z = z_\beta(\phi) \) doesn’t crossover the \( \Phi \)-axis, we have \( z_\beta(\phi) \leq 0, \phi \in (-\infty, +\infty) \). While \( \beta > 0, \gamma > 0 \), according to Lemma 2 indicates that this kind of periodic solution is impossible.

In the second case, \( z_\beta(\phi) > z_0(\phi), \phi \in [\phi^*_1, \phi^*_2] \). While \( \beta < \beta_0 \), according to Lemma 2 indicates that this situation is also impossible.

Summarizing the conclusions above, we conclude that the equation does not exists the periodic solution while \( \beta < \beta_0 \). The proof is complete for the proposition of \( \beta < \beta_0 \).

**Figure 7:** The cycle of the second kind on the cylindrical surface.

d) According to Lemma 2, it is easy to prove that the periodic solution is unique. In the proof, it just need to notice the property that trajectories don’t intersect each others.

The periodic solution is a cycle of the second kind on the cylindrical surface, as shown in Figure 7. The characteristic exponent of the limit-cycle is \(-\gamma < 0\), therefore it is a stable limit-cycle [10]. The periodic solution of the equation is a stable periodic solution.

e) Assume \( \beta_0(\gamma_1) = 1 \). Let \( \gamma = \gamma_1, \beta = 1, R(\gamma_1) \) denotes the critical periodic trajectory connecting singularity \( A_0 \) and \( A_1 \), as shown in Figure 8. The singularity \( B_0 \) coincides with \( A_1 \). Let \( \gamma = \gamma_2 > \gamma_1, \beta = 1, R(\gamma_2) \) denotes the trajectory departing from \( A_0 \). Referring to the preceding proof derives that, trajectory \( R(\gamma_2) \) must run at the inside of \( R(\gamma_1) \). According to Lemma 1 derives that \( R(\gamma_2) \) must intersect with axis \( \Phi \) at point \( A_1 \) (i.e. \( B_0 \)). it is the critical periodic trajectory yet. Notice that \( \beta = 1 \) still. According to the uniqueness of \( \beta_0 \) gives that \( \beta_0(\gamma_2) = 1, \gamma_2 > \gamma_1 \). In this case, the slope by which \( R(\gamma_2) \) going into point \( A_1 \) is \( \lambda_1 = 0 \).

This completes the proof.

4. Discussion of the conclusion

The theorem above gives a conclusion that, for arbitrary \( \gamma > 0 \), Equation (2) exists a critical value \( \beta_0 \), where \( 0 < \beta_0 \leq 1 \). While \( \beta \geq \beta_0 \), it exists a unique, positive and periodic solution that is a stable limit-cycle of the second kind on the cylindrical surface. According to the Bendixson’s Criteria [10], it is easy to prove that Equation (2) exists no cycle of the first kind on the cylindrical surface for \( \gamma > 0 \). It indicates that, whatever the initial state it is, the final state will be in a periodic motion toward positive. While \( \beta < \beta_0 \), Equation (2) exists no any
periodic solution. In fact, all trajectories on the phase plane will go into the saddles or focuses in this situation. The focuses are stable. The saddles are unstable. If there are some disturbances, it will deviate from the saddles. In the final state, all the trajectories will go into the focuses. In physics, whatever the initial state it is, the system will stay at equilibrium points (focuses) when it reaches the stable state.

Proposition e) of the theorem indicates that, supposing a minimum value \( \gamma_{min} \) such that \( \beta_0(\gamma_{min}) = 1 \), while \( \gamma \geq \gamma_{min} \), the critical value \( \beta_0 \) will be 1 all the time. By numerical calculations, M. Urabe has given the definite value, that is \( \gamma_{min} = 1.193 \). These conclusions above are very useful in the research of CDW. The derivations in detail was presented in another paper [4]. It is unnecessary to go into details in this paper.

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References

[1] E. Ott, Chaos in Dynamical Systems, World Publishing Corporation, Beijing, 2005.

[2] T. Bohr, P. Bak, M.H. Jensen, Transition to chaos by interaction of resonances in dissipative systems. II. Josephson junctions, charge-density waves, and standard maps, Phys. Rev. A 30(1984) 1970-1980.

[3] G. Grüner, A. Zawadowski & P.M. Chaikin, Nonlinear conductivity and noise due to charge-density wave depining in NbSe3, Phys. Rev. Lett. 46(7)(1981) 511-515.

[4] Li L.G., Ruan Y.F., The analysis on the classic model of the charge density wave, Acta Phys. Sin. 55(1)(2006) 441-445 (in Chinese).

[5] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle, J. Math. Pures Appl. 7(3)(1881) 375-422; 8(1882) 251-296; 1(4)(1885) 167-244; 2(1886) 151-217.
[6] G.F.D. Duff, Limit-cycles and rotated vector fields, Ann. Math. 57(1953) 15-31.

[7] G. Seifert, Contributions to the theory of nonlinear oscillations, IV(1958) 125-140.

[8] Chen Xiang-yan, General rotated vector fields, Acta Nanjing Uni. 1(1975) 100-108 (in Chinese).

[9] Ma Zhi-en, The motion of singular closed trajectory in the rotated vector field, J. Xi’an Jiaotong Uni. 4(1978) 49-65 (in Chinese).

[10] L. Perko, Differential Equations and Dynamical Systems, Springer-Verlag, NewYork, 1991.

[11] M Urabe, The least upper bound of a damping coefficient insuring the existence of a periodic motion of a pendulum under constant torque, J. Sci. Hiroshima Uni. A 18(1954) 379-389.