CONVOLUTIONAL GOPPA CODES

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Abstract. We define Convolutional Goppa Codes over algebraic curves and construct their corresponding dual codes. Examples over the projective line and over elliptic curves are described, obtaining in particular some Maximum-Distance Separable (MDS) convolutional codes.

Index Terms. Convolutional Codes, Goppa Codes, MDS Codes, Algebraic Curves, Finite Fields.

1. Introduction

Goppa codes are evaluation codes for linear series over smooth curves over a finite field \( F_q \). In [1] we proposed a new construction of convolutional codes, which we called Convolutional Goppa Codes (CGC), in terms of evaluation along sections of a family of algebraic curves.

The aim of this paper is to reformulate the results of [1] in a straightforward language. We define CGC as Goppa codes for smooth curves defined over the field \( F_q(z) \) of rational functions in one variable \( z \) over the finite field \( F_q \). These CGC are in fact more general than the codes defined in [1], since there are smooth curves over \( F_q(z) \) that do not extend to a family of smooth curves over the affine line \( \mathbb{A}^1_{F_q} \). With this definition, one has another advantage: the techniques of Algebraic Geometry we need are easier than those used in [1]: we use exactly the same language as is usual in the literature on Goppa codes.

The last two sections of the paper are devoted to illustrating the general construction with some examples. In §4 we construct several CGC of genus zero; that is, defined in terms of the projective line \( \mathbb{P}^1_k \) over the field \( F_q(z) \). Some of these examples are MDS-convolutional codes and are very easy to handle.

In §5 we give examples of CGC of genus one; that is, defined in terms of elliptic curves over \( F_q(z) \). These examples are not so easy to study. In fact, a consequence of this preliminary study of CGC of genus one is that a deeper understanding of the arithmetic properties of elliptic fibrations (see for instance [4]) and of the translation of these properties into the language of convolutional codes, is necessary.

2. Convolutional Goppa Codes

Let \( F_q \) be a finite field and \( F_q(z) \) the (infinite) field of rational functions of one variable. Let \((X, \mathcal{O}_X)\) be a smooth projective curve over \( F_q(z) \) of genus \( g \), and let us denote by \( \Sigma_X \) the field of rational functions of \( X \).

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Given a set $p_1, \ldots, p_n$ of $n$ different $\mathbb{F}_q(z)$-rational points of $X$, if $\mathcal{O}_{p_i}$ denotes the local ring at the point $p_i$, with maximal ideal $m_{p_i}$, and $t_i$ a local parameter at $p_i$, one has exact sequences

$$0 \to m_{p_i} \to \mathcal{O}_{p_i} \to \mathcal{O}_{p_i}/m_{p_i} \cong \mathbb{F}_q(z) \to 0$$

(2.1)

Let us consider the divisor $D = p_1 + \cdots + p_n$, with its associated invertible sheaf $\mathcal{O}_X(D)$. Then, one has an exact sequence of sheaves

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to Q \to 0, \tag{2.2}$$

where the quotient $Q$ is a sheaf with support at the points $p_i$.

Let $G$ be a divisor on $X$ of degree $r$, with support disjoint from $D$. Tensoring the exact sequence (2.2) by the associated invertible sheaf $\mathcal{O}_X(G)$, one obtains:

$$0 \to \mathcal{O}_X(G-D) \to \mathcal{O}_X(G) \to Q \to 0. \tag{2.3}$$

For every divisor $F$ over $X$, let us denote their $\mathbb{F}_q(z)$-vector space of global sections by

$$L(F) \equiv \Gamma(X, \mathcal{O}_X(F)) = \{ s \in \Sigma_X / (s) + F \geq 0 \},$$

where $(s)$ is the divisor defined by $s \in \Sigma_X$. Taking global sections in (2.3), one obtains

$$0 \to L(G-D) \to L(G) \to \mathbb{F}_q(z) \times \cdots \times \mathbb{F}_q(z) \to \cdots$$

$$s \mapsto (s(p_1), \ldots, s(p_n)).$$

**Definition 2.1.** The convolutional Goppa code $\mathcal{C}(D, G)$ associated with the pair $(D, G)$ is the image of the $\mathbb{F}_q(z)$-linear map $\alpha : L(G) \to \mathbb{F}_q(z)^n$.

Analogously, given a subspace $\Gamma \subseteq L(G)$, one defines the convolutional Goppa code $\mathcal{C}(D, \Gamma)$ as the image of $\alpha|_{\Gamma}$.

**Remark 2.2.** The above definition is more general than the one given in [1] in terms of families of curves $X \to \mathbb{A}^1_{\mathbb{F}_q}$. In fact, given such a family, the fibre $X_\eta$, over the generic point $\eta \in \mathbb{A}^1_{\mathbb{F}_q}$, is a curve over $\mathbb{F}_q(z)$. But not every curve over $\mathbb{F}_q(z)$ extends to a family over $\mathbb{A}^1_{\mathbb{F}_q}$.

By construction, $\mathcal{C}(D, G)$ is a convolutional code of length $n$ and dimension

$$k \equiv \dim L(G) - \dim L(G-D).$$

**Proposition 2.3.** Let us assume that $2g - 2 < r < n$. Then, the evaluation map $\alpha : L(G) \hookrightarrow \mathbb{F}_q(z)^n$ is injective, and the dimension of $\mathcal{C}(D, G)$ is

$$k = r + 1 - g.$$

**Proof.** If $r < n$, $\dim L(G-D) = 0$, the map $\alpha$ is injective and $k = \dim L(G)$. If $2g - 2 < r$, $\dim L(G) = 1 - g + r$ by the Riemann-Roch theorem. \qed

### 3. Dual Convolutional Goppa Codes

Let us consider, over the $\mathbb{F}_q(z)$-vectorial space $\mathbb{F}_q(z)^n$, the pairing $\langle \ , \ \rangle$

$$\mathbb{F}_q(z)^n \times \mathbb{F}_q(z)^n \to \mathbb{F}_q(z)$$

$$(u, v) \mapsto \langle u, v \rangle = \sum_{i=1}^n u_i v_i,$$

where $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathbb{F}_q(z)^n$. 
**Definition 3.1.** The dual convolutional Goppa code of the code $C(D,G)$ is the $\mathbb{F}_q(z)$-linear subspace $C^\perp(D,G)$ of $\mathbb{F}_q(z)^n$ given by

$$C(D,G)^\perp = \{ u \in \mathbb{F}_q(z)^n / \langle u, v \rangle = 0 \text{ for every } v \in C(D,G) \}.$$ 

Let us denote by $K$ the canonical divisor of rational differential forms over $X$.

**Theorem 3.2.** The dual convolutional Goppa code $C^\perp(D,G)$ associated with the pair $(D,G)$ is the image of the $\mathbb{F}_q(z)$-linear map $\beta: L(K + D - G) \rightarrow \mathbb{F}_q(z)^n$, given by

$$\beta(\eta) = (\text{Res}_{p_1}(\eta), \ldots, \text{Res}_{p_n}(\eta)).$$

**Proof.** Following the construction of convolutional code of length $n$ by $m$ divisors $D$ via the Riemann-Roch theorem:

$$(3.3) \ 0 \rightarrow \mathcal{O}_X(K) \rightarrow \mathcal{O}_X(K + D - G) \rightarrow Q \rightarrow 0.$$ 

This allows us to define a new convolutional Goppa code associated to the pair of divisors $D = p_1 + \cdots + p_n$ and $G$; tensoring $(2.2)$ by the line sheaf $\mathcal{O}_X(K + D - G)$, one has

$$(3.3) \ 0 \rightarrow \mathcal{O}_X(K - G) \rightarrow \mathcal{O}_X(K + D - G) \rightarrow Q \rightarrow 0.$$ 

Taking global sections, one has:

$$0 \rightarrow L(K - G) \rightarrow L(K + D - G) \xrightarrow{\beta} \mathbb{F}_q(z) \times \cdots \times \mathbb{F}_q(z) \rightarrow \ldots$$

$$\eta \mapsto (\text{Res}_{p_1}(\eta), \ldots, \text{Res}_{p_n}(\eta))$$

The image of $\beta$ is a subspace of $\mathbb{F}_q(z)^n$, whose dimension can be calculated by the Riemann-Roch theorem:

$$\dim L(K + D - G) - \dim L(K - G) =$$

$$= (\dim L(G - D) - (r - n) - 1 + g) - (\dim L(G) - r - 1 + g) = n - k.$$ 

Moreover, $\mathcal{I}m \beta$ is the subspace $C(D,G)^\perp \subset \mathbb{F}_q(z)^n$, since they have the same dimension, and for every $\eta \in L(K + D - G)$ and every $s \in L(G)$ one has

$$\langle \beta(\eta), \alpha(s) \rangle = \sum_{i=1}^{n} s(p_i) \text{Res}_{p_i}(\eta) = \sum_{i=1}^{n} \text{Res}_{p_i}(s \eta) = 0,$$

by the Residue Theorem. \qed

Under the hypothesis $2g - 2 < r < n$, the map $\beta$ is injective, and $C^\perp(D,G)$ is a convolutional code of length $n$ and dimension

$$\dim L(K + D - G) = n - (1 - g + r).$$
Remark 3.3. Our pairing $\langle \ , \ \rangle : \mathbb{F}_q(z)^n \times \mathbb{F}_q(z)^n \to \mathbb{F}_q(z)$ is $\mathbb{F}_q(z)$-bilinear, whereas the "time reversal" pairing defined by J. Rosenthal in [3] 7.5, given by

$$ [\ , \ ] : \mathbb{F}_q((z))^n \times \mathbb{F}_q((z))^n \to \mathbb{F}_q $$

$$(u,v) \longmapsto \sum_{i=1}^{n} \langle u(i), v(-i) \rangle,$$

where $u = \sum_i u(i)z^i, v = \sum_i v(i)z^i \in \mathbb{F}_q((z))^n$ and $\langle \ , \ \rangle$ is the standard bilinear form on $\mathbb{F}_q^n$, is $\mathbb{F}_q$-bilinear.

The pairing $[\ , \ ]$ can be expressed in the following way:

$$ [u, v] = \text{Res}_{z=0} \left( \langle u, v \rangle \frac{dz}{z} \right) = \sum_{i=1}^{n} \text{Res}_{z=0} \left( u_i v_i \frac{dz}{z} \right). $$

Thus, the duality for convolutional Goppa codes defined in [3.1] is related to the residues in the points of $X$, and the duality with respect to the pairing $[\ , \ ]$ is related to the residues in the variable of the base field. A more precise study of the relationship between both dualities must be done.

4. CONVOLUTIONAL GOPPA CODES OVER THE PROJECTIVE LINE

Let $X = \mathbb{P}_1^{\mathbb{F}_q(z)} = \text{Proj} \mathbb{F}_q(z)[x_0, x_1]$ be the projective line over the field $\mathbb{F}_q(z)$, and let us denote by $t = x_1/x_0$ the affine coordinate.

Let $p_0 = (1,0)$ be the origin point, $p_\infty = (0,1)$ the point at infinity, and $p_1,\ldots,p_n$ be different rational points of $\mathbb{P}_1^{\mathbb{F}_q}$, $p_i \neq p_0, p_\infty$. Let us define the divisors $D = p_1 + \cdots + p_n$ and $G = rp_\infty - sp_0$, with

$$0 \leq s \leq r < n.$$ 

Since $g = 0$, the evaluation map $\alpha : L(G) \to \mathbb{F}_q(z)^n$ is injective and $\text{Im} \alpha$ defines a convolutional Goppa code $C(D, G)$ of length $n$ and dimension $k = r - s + 1$.

Let us choose the functions $t^s, t^{s+1},\ldots,t^r$ as a basis of $L(G)$. If $\alpha_i \in \mathbb{F}_q(z)$ is the local coordinate of the point $p_i, i = 1,\ldots,n$, the matrix of the evaluation map $\alpha$ is,

$$ G = \begin{pmatrix}
\alpha_1^s & \alpha_2^s & \cdots & \alpha_n^s \\
\alpha_1^{s+1} & \alpha_2^{s+1} & \cdots & \alpha_n^{s+1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^r & \alpha_2^r & \cdots & \alpha_n^r
\end{pmatrix}. $$

This is a generator matrix for the code $C(D, G)$.

The dual convolutional Goppa code $C^\perp(D, G)$ also has length $n$, and dimension $n - k = n - r + s - 1$. To construct $C^\perp(D, G)$, let us choose in $L(K + D - G)$ the basis of rational differential forms $\frac{dt}{t} \prod_{i=1}^{n} (t - \alpha_i), \frac{dt}{t} \prod_{i=1}^{n} (t - \alpha_i)^i, \ldots, \frac{dt}{t} \prod_{i=1}^{n} (t - \alpha_i)^{n-r+s-2}$, and let us calculate the residues

$$\text{Res}_{p_j} \left( \frac{t^n dt}{t^s \prod_{i=1}^{n} (t - \alpha_i)} \right) = \text{Res}_{p_j} \left( \frac{(t - \alpha_j + \alpha_j)^m d(t - \alpha_j)}{(t - \alpha_j)(t - \alpha_j + \alpha_j)^s \prod_{i \neq j}^{n} (t - \alpha_j + \alpha_j - \alpha_i)} \right) = \frac{\alpha_j^m}{\alpha_j^s \prod_{i \neq j}^{n} (\alpha_j - \alpha_i)}.$$
If one denotes by $h_j = \frac{1}{\alpha_j} \prod_{i \neq j} (\alpha_j - \alpha_i)$, then the matrix of $\beta : L(K + D - G) \to F_q(z)^n$

\[(4.2) \quad H = \begin{pmatrix}
    h_1 & h_2 & \ldots & h_n \\
    h_1\alpha_1 & h_2\alpha_2 & \ldots & h_n\alpha_n \\
    \vdots & \vdots & \ddots & \vdots \\
    h_1\alpha_1^{n-r+s-2} & h_2\alpha_2^{n-r+s-2} & \ldots & h_n\alpha_n^{n-r+s-2}
\end{pmatrix},
\]
is a generator matrix for the dual code $C^\perp(D, G)$, and therefore a parity-check matrix for $C(D, G)$. In fact, one has $H \cdot G^T = 0$.

**Remark 4.1.** The matrix (4.2) suggests that $C^\perp(D, G)$ is an alternant code over the field $F_q(z)$, and we can thus apply some kind of Berlekamp-Massey decoding algorithm for convolutional Goppa codes; this will be studied in a forthcoming paper.

**Example 4.2.** Let $a, b \in F_q$ be two different non-zero elements, and

$$\alpha_i = a_i z + b_i, \quad i = 1, \ldots, n, \text{ with } n < q.$$  

We present some examples of convolutional Goppa codes with canonical generator matrices [2], whose free distance $d$ attains the generalized Singleton bound, i.e., they are MDS convolutional codes [4], and we include their encoding equations as linear systems

\[
\begin{align*}
    z^{-1}s &= sA_{\delta \times \delta} + uB_{k \times \delta} \\
    uG &= sC_{\delta \times n} + uD_{k \times n}
\end{align*}
\]

where $\delta$ denotes the degree of the code (in the sense of [2].)

- **Field $F_3(z)$, $F_3 = \{0, 1, 2\}$:**
  
  $G = (z + 1 \quad z + 2)$,
  
  $H = \begin{pmatrix}
    \frac{1}{2(z+1)} & \frac{1}{z+2} \\
  \end{pmatrix}$,
  
  $A = (0)$, $B = (1)$, $C = (1 \quad 1)$, $D = (1 \quad 2)$,
  
  $(n, k, \delta, d) = (2, 1, 1, 4)$.

- **Field $F_4(z)$, $F_4 = \{0, 1, \alpha, \alpha^2\}$ where $\alpha^2 + \alpha + 1 = 0$:***

  $G = \begin{pmatrix}
    1 & 1 & 1 \\
    z + 1 & \alpha z + \alpha^2 & \alpha^2 z + \alpha
  \end{pmatrix}$,
  
  $H = \begin{pmatrix}
    \frac{1}{\alpha^2 z + \alpha} & \frac{1}{\alpha^2 z + \alpha} & \frac{1}{\alpha^2 z + \alpha} \\
    (\alpha^2 z + \alpha)(z+1) & (\alpha^2 z + \alpha)(z+1) & (\alpha^2 z + \alpha)(z+1)
  \end{pmatrix}$,
  
  $A = (0)$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1 \quad 1 \quad 1)$, $D = \begin{pmatrix} 1 & 1 & 1 \\
    1 & \alpha^2 & \alpha
  \end{pmatrix}$,
  
  $(n, k, \delta, d) = (3, 2, 1, 3)$.

- **Field $F_4(z)$:**
\( G = (z + 1 \ z + \alpha \ z + \alpha^2) \),
\( H = \left( \begin{array}{c}
\frac{1}{z+1} \\
\frac{z+\alpha}{\alpha} \\
\frac{\alpha^2}{z+\alpha^2}
\end{array} \right) \),
\( A = (0) \), \( B = (1) \), \( C = (1 \ 1 \ 1) \), \( D = (1 \ \alpha \ \alpha^2) \),
\( (n, k, \delta, d) = (3, 1, 1, 6) \).

- Field \( \mathbb{F}_5(z) \), \( \mathbb{F}_5 = \{0, 1, 2, 3, 4\} \):
  \( G = ((z + 1)^2 \ (z + 2)^2 \ (z + 4)^2) \),
  \( H = \left( \begin{array}{c}
\frac{2}{z+1} \\
\frac{2}{z+2} \\
\frac{1}{z+4}
\end{array} \right) \),
  \( A = (0 \ 1) \), \( B = (1 \ 0) \), \( C = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix} \), \( D = (1 \ 4 \ 1) \),
  \( (n, k, \delta, d) = (3, 1, 2, 9) \).

- Field \( \mathbb{F}_5(z) \):
  \( G = \begin{pmatrix} z + 1 & 2z + 3 & 4z + 4 & 3z + 2 \\ (z + 1)^2 & (2z + 3)^2 & (4z + 4)^2 & (3z + 2)^2 \end{pmatrix} \),
  \( H = \begin{pmatrix}
\frac{4}{z+1} & \frac{4}{z+2} & \frac{4}{z+3} & \frac{4}{z+4} \\
\frac{4}{(z+1)(z+2)(z+3)} & \frac{4}{(z+2)(z+3)(z+4)} & \frac{4}{(z+1)(z+2)(z+3)} & \frac{4}{(z+2)(z+3)(z+4)}
\end{pmatrix} \),
  \( A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), \( C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \), \( D = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 1 & 4 & 1 & 4 \end{pmatrix} \),
  \( (n, k, \delta, d) = (2, 1, 3, 8) \).

5. Convolutional Goppa Codes Associated with Elliptic Curves

We can obtain convolutional codes from elliptic curves in the same way. Let \( X \subseteq \mathbb{P}^2_{\mathbb{F}_q(z)} \) be a plane elliptic curve over \( \mathbb{F}_q(z) \), and let us denote by \( (x, y) \) the affine coordinates in \( \mathbb{P}^2_{\mathbb{F}_q(z)} \). Let \( p_\infty \) be the infinity point, and \( p_1, \ldots, p_n \) rational points of \( X \), with \( p_i = (x_i(z), y_i(z)) \). Let us define \( D = p_1 + \cdots + p_n \) and \( G = r p_\infty \).

The “canonical” basis of \( L(G) \) is \( \{1, x, y, \ldots, x^n y^b\} \), with \( 2a + 3b = r \). Thus, the evaluation map \( \alpha: L(G) \rightarrow \mathbb{F}_q(z)^n \) is

\[ \alpha(x^i y^j) = (x_1^i(z)y_1^j(z), \ldots, x_n^i(z)y_n^j(z)). \]

The image of a subspace \( \Gamma \subseteq L(G) \) under the map \( \alpha \) provides a Goppa convolutional code.

We present a couple of examples obtained from elliptic curves that, although not MDS, have free distance approaching that bound.

Example 5.1. We consider the curve over \( \mathbb{F}_2(z) \)

\[ y^2 + (1 + z)xy + (z + z^2)y = x^3 + (z + z^2)x^2 \]
and the points
\[ p_1 = (z^2 + z, z^3 + z^2) \]
\[ p_2 = (0, z^2 + z) \]
\[ p_3 = (z, z^2) \]

Let \( \Gamma \subset L(G) \) be the subspace generated by \( \{1, x\} \). Accordingly, the valuation map \( \alpha \) over \( \Gamma \) is defined by the matrix
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & z^2 + z & z
\end{pmatrix}.
\]

This code has free distance \( d = 2 \). The maximum distance for its parameters is 3.

Example 5.2. Let us now consider the curve over \( \mathbb{F}_2(z) \)
\[ y^2 + (1 + z + z^2)xy + (z^2 + z^3)y = x^3 + (z^2 + z^3)x^2 \]
and the points
\[ p_1 = (z^3 + z^2, 0) \]
\[ p_2 = (0, z^3 + z^2) \]
\[ p_3 = (z^3 + z^2, z^5 + z^3) \]
\[ p_4 = (z^2 + z, z^3 + z) \]
\[ p_5 = (z^2 + z, z^4 + z^2) \]

Let \( \Gamma \subset L(G) \) be the subspace generated by \( \{1, x\} \). Then, the valuation map \( \alpha \) over \( \Gamma \) is defined by the matrix
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
z^3 + z^2 & z^3 + z^2 & z^2 + z & z^2 + z
\end{pmatrix}.
\]

This code has free distance \( d = 4 \). The maximum distance for its parameters is 5.

Remark 5.3. Every elliptic curve \( X \) over \( \mathbb{F}_q(z) \) can be considered as the generic fibre of a fibration \( X \to U = \text{Spec} \mathbb{F}_q[z] \), with some fibres singular curves of genus 1. The global structure of this fibration is related to the singular fibres (see [5]); the translation into the language of coding theory of the arithmetic and geometric properties of the fibration is the first step in the program of applying the general construction to the effective construction of good convolutional Goppa codes of genus 1.

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