Spectral distance: Results for Moyal plane and Noncommutative Torus

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Overview

▶ Many available examples of Noncommutative (NC) spaces (fuzzy spaces, almost commutative spaces, deformations, Connes-Landi spheres, Connes-Dubois-Violette spaces,...). Relatively few works devoted to study of the spectral distance.
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- On finite dimensional complete Riemann spin manifold, spectral distance between pure states coincides with geodesic distance between corresponding points. In NC case, actual meaning of spectral distance not clear and much more explicit examples are needed.
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- On finite dimensional complete Riemann spin manifold, spectral distance between pure states coincides with geodesic distance between corresponding points. In NC case, actual meaning of spectral distance not clear and much more explicit examples are needed.
- A few past studies [lattice(Dimakis, Müller-Hoissen; Bimonte, Lizzi, Sparano), finite spaces (Iochum, Krajewski, Martinetti), inspired by physics (Martinetti), quantum metric spaces (Rieffel)]
- Study the spectral distance on the noncommutative Moyal plane. Corresponding ST is non compact spectral triple (NCST) proposed by [Gayral, Gracia-Bondia, Iochum, Schücker, Varilly, CMP 2004].
We find explicit formula for the distance between pure states (theorem 9 given below). These are vector states generated by the elements of the matrix base.

Existence of states at infinite distance so that the Moyal plane as described by the NCST proposed recently is not a compact quantum metric space in the sense of Rieffel.

Example of "truncation" of this NCST leading to Rieffel compact quantum metric space is given.

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   - Basic properties
   - Moyal non compact spin geometry
   - Spectral distance on the Moyal plane
   - Spectral distance between pure states
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   - Pure states on noncommutative torus
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2 Noncommutative Torus - preliminaries

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The Moyal product

$S(\mathbb{R}^2) \equiv S$: (Frechet) space of Schwarz functions, $S'(\mathbb{R}^2) \equiv S'$ its topological dual space. $\|\cdot\|_2, \langle \cdot, \cdot \rangle$: $L^2(\mathbb{R}^2)$ norm and inner product.
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**Proposition 1** (see e.g Gracia-Bondia, Varilly, JMP 1988)

*Associative bilinear Moyal ∗-product defined as: \( * : S \times S \rightarrow S \), \( \forall a, b \in S \)

\[
(a \ast b)(x) = \frac{1}{(\pi \theta)^2} \int d^2y d^2z \ a(x + y)b(x + t)e^{-i2y\Theta^{-1}t}
\]

\[
y\Theta^{-1}t \equiv y^\mu \Theta^{-1}_{\mu\nu}t^\nu, \quad \Theta_{\mu\nu} = \theta \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \theta \in \mathbb{R}, \theta \neq 0
\]

*Complex conjugation is an involution for the ∗-product. One has:*

i) \( \int d^2x \ (a \ast b)(x) = \int d^2x \ (b \ast a)(x) = \int d^2x \ a(x)b(x) \)

ii) \( \partial_\mu (a \ast b) = \partial_\mu a \ast b + a \ast \partial_\mu b \).

iii) \( \mathcal{A} \equiv (S, \ast) \) is a non unital involutive Fréchet algebra.
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*Associative bilinear Moyal \( \star \)-product defined as: \( \star : S \times S \to S, \forall a, b \in S \)

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Complex conjugation is an involution for the \( \star \)-product. One has:

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ii) \( \partial_\mu (a \star b) = \partial_\mu a \star b + a \star \partial_\mu b \)

iii) \( A \equiv (S, \star) \) is a non unital involutive Fréchet algebra.

- Set: \( X^n \equiv X \star X \star ... \star X \), \( [a, b]_\star \equiv a \star b - b \star a \). From now on, introduce complex coordinates \( \bar{z} = \frac{1}{\sqrt{2}}(x_1 - ix_2) \), \( z = \frac{1}{\sqrt{2}}(x_1 + ix_2) \).
The matrix base

**Proposition 2 (see e.g Gracia-Bondia, Varilly, JMP 1988)**

The matrix base is the family of functions \( \{ f_{mn} \}_{m,n \in \mathbb{N}} \subset S \subset L^2(\mathbb{R}^2) \) defined by

\[
f_{mn} = \frac{1}{(\theta^{m+n}m!n!)^{1/2}} z^m \star f_{00} \star z^n, \quad f_{00} = 2e^{-2H/\theta}, \quad H = \frac{1}{2}(x_1^2 + x_2^2)
\]

i) One has the relations:

\[
f_{mn} \star f_{pq} = \delta_{np} f_{mq}, \quad f_{mn}^* = f_{nm}, \quad \langle f_{mn}, f_{kl} \rangle = (2\pi\theta)\delta_{mk}\delta_{nl}
\]  

(1)

ii) There is a Frechet algebra isomorphism between \( A \equiv (S, \star) \) and the matrix algebra of decreasing sequences \( (a_{mn}) \), \( \forall m, n \in \mathbb{N} \) defined by \( a = \sum_{m,n} a_{mn} f_{mn} \), \( \forall a \in S \), such that the semi-norms \( \rho_k^2(a) \equiv \sum_{m,n} \theta^{2k}(m + \frac{1}{2})^k(n + \frac{1}{2})^k |a_{mn}|^2 < \infty \), \( \forall k \in \mathbb{N} \).
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\( \ast \)-product can be extended to other subspaces of \( S' \) (use duality and continuity of \( \ast \) on \( S \)). Convenient: Hilbert spaces \( S \subset G_{s,t} \subset S', \ s, t \in \mathbb{R}, \ G_{s,t} = \{ a = \sum a_{mn} f_{mn} \in S' / \|a\|_{s,t}^2 = \sum_{m,n} \theta^{s+t}(m + \frac{1}{2})^s(n + \frac{1}{2})^t |a_{mn}|^2 < \infty \} \)

Uses: \( \|a \ast b\|_{s,r} \leq \|a\|_{s,t} \|b\|_{q,r} \), \( t + q \geq 0 \) and \( \|a\|_{u,v} \leq \|a\|_{s,t} \) if \( u \leq s \), \( v \leq t \). Then, for any \( a \in G_{s,t} \) and \( b \in G_{q,r} \), \( b = \sum_{m,n} b_{mn} f_{mn} \), \( t + q \geq 0 \), the sequences \( c_{mn} = \sum_p a_{mp} b_{pn}, \ \forall m, n \in \mathbb{N} \) define the functions \( c = \sum_{m,n} c_{mn} f_{mn}, \ c \in G_{s,r} \) [See e.g Gracia-Bondia, Varilly, JMP 1988].
Relevant elements of the spectral “triple”

\[ D_{L^2} = \{ a \in L^2(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2) / a^{(n)} \in L^2(\mathbb{R}^2), \forall n \in \mathbb{N} \} \].
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\[ \mathcal{A}_\theta = \{ a \in S' / a \star b \in L^2(\mathbb{R}^2), \forall b \in L^2(\mathbb{R}^2) \}. \]
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Proposition 3 (Gayral, Gracia-Bondia, Iochum, Schücker, Varilly, CMP 2004)

i) $A_\theta$ is a unital C*-algebra of operator of $L^2(\mathbb{R}^2)$ with the operator norm $||.||_{op}$, $||a||_{op} = \sup_{0 \neq b \in L^2(\mathbb{R}^2)} \{ \frac{||a \star b||_2}{||b||_2} \}$ for any $a \in A_\theta$, isomorphic to $\mathcal{L}(L^2(\mathbb{R}^2))$.

ii) $A_1$ is a pre C* algebra. One has $A \subset (D_{L^2}, \star) \subset A_1 \subset A_\theta$. 
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Moyal non compact spin geometry

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1) \( \mathcal{A}_\theta \) is a unital \( C^* \)-algebra of operator of \( L^2(\mathbb{R}^2) \) with the operator norm \( \|a\|_{op} = \sup_{0 \neq b \in L^2(\mathbb{R}^2)} \left\{ \frac{\|a \star b\|^2}{\|b\|^2} \right\} \) for any \( a \in \mathcal{A}_\theta \), isomorphic to \( \mathcal{L}(L^2(\mathbb{R}^2)) \).

2) \( \mathcal{A}_1 \) is a pre \( C^* \) algebra. One has \( \mathcal{A} \subset (\mathcal{D}_{L^2}, \star) \subset \mathcal{A}_1 \subset \mathcal{A}_\theta \).

Consider a non compact spectral triple [Gayral, Gracia-Bondia, Iochum, Schücker, Varilly, CMP 2004] as noncommutative generalisation of non compact Riemannian spin manifold:

\( (\mathcal{A}, \mathcal{A}_1, \mathcal{H}, D; J, \chi) \)  \( \text{(2)} \)
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The antiunitary operator \( J \) and involution \( \chi \) will not be relevant here. \( \mathcal{A}_1 \supset \mathcal{A} \) is called a prefered unitization of \( \mathcal{A} \).
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\( \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 \), i.e Hilbert space of integrable square sections of trivial spinor bundle \( S = \mathbb{R}^2 \otimes \mathbb{C}^2 \) with standard Hilbert product

\[
\langle \psi, \phi \rangle = \int d^2 x (\psi_1^* \phi_1 + \psi_2^* \phi_2) \forall \psi, \phi \in \mathcal{H}, \psi = (\psi_1, \psi_2), \phi = (\phi_1, \phi_2).
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Define now \(\partial = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2)\).

Unbounded Euclidean self-adjoint Dirac operator \(D = -i\sigma^\mu \partial_\mu\) (densely defined on \(\text{Dom}(D) = (D_{L^2} \otimes \mathbb{C}^2) \subset \mathcal{H}\)). One has

\[
\sigma^\mu \sigma^\nu + \sigma^\nu \sigma^\mu = 2\delta^\mu_\nu, \quad \forall \mu, \nu = 1, 2 \quad \text{and}
\]

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad D = -i\sqrt{2}\begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix}
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$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad D = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix}$$

A can be represented faithfully on space of bounded operators on $\mathcal{H}$:

$$\pi(a) = L(a) \otimes I_2, \quad \pi(a)\psi = (a \star \psi_1, a \star \psi_2), \quad \forall \psi = (\psi_1, \psi_2) \in \mathcal{H}, \forall a \in \mathcal{A}$$

$L(a)$: left multiplication operator by any $a \in \mathcal{A}$. $\pi(a)$ and $[D, \pi(a)]$ are bounded operators on $\mathcal{H}$ for any $a \in \mathcal{A}$. 
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The spectral distance can be defined as follows

**Definition 4** (see e.g Connes, Landi)

The spectral distance between any two states \( ω_1 \) and \( ω_2 \) of \( \bar{A} \) is defined by

\[
\text{d}(ω_1,ω_2) = \sup_{a∈A} \left\{ |ω_1(a) - ω_2(a)|; \left|\left[D,\pi(a)\right]\right|_{op} ≤ 1 \right\}
\]

where \( \left|\left[D,\pi(a)\right]\right|_{op} \) is the operator norm for the representation of \( A \) in \( B(H) \).

Spectral distance between pure states: noncommutative analog of geodesic distance between two points. Recall: spectral distance for the spectral triple encoding the geometry of compact Riemann spin manifold equals geodesic distance.

(3) extends the notion of distance to non-pure states, i.e objects that are not analogous to points. Determination of spectral distance between 2 pure states not enough to exhaust the full metric information involved in (3) [Rieffel].

Relationship with the Wasserstein distance of order 1 between probability distributions on a metric space and the spectral distance [d'Andrea, Martinetti].
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The spectral distance between any two states $\omega_1$ and $\omega_2$ of $\bar{A}$ is defined by

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \left\{ |\omega_1(a) - \omega_2(a)|; \|[D, \pi(a)]\|_{op} \leq 1 \right\}$$  \hspace{1cm} (3)

where $\| \cdot \|_{op}$ is the operator norm for the representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

- Spectral distance between pure states: noncommutative analog of geodesic distance between two points. Recall: spectral distance for the spectral triple encoding the geometry of compact Riemann spin manifold equals geodesic distance.
- (3) extends the notion of distance to non-pure states, i.e objects that are not analog to points. determination of spectral distance between 2 pure states not enough to exhaust the full metric information involved in (3) [Rieffel].

- Relationship with the Wasserstein distance of order 1 between probability distributions on a metric space and the spectral distance [d'Andrea, Martinetti].
Spectral distance on the Moyal plane

- Spectral distance is related naturally to spectral triple [see e.g Connes, Landi].
- A few past studies [lattice(Dimakis, Müller-Hoissen; Bimonte, Lizzi, Sparano), finite spaces (Iochum, Krajewski, Martinetti), inspired by physics (Martinetti), quantum metric spaces (Rieffel) ]
- The spectral distance can be defined as follows

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- Spectral distance between pure states: noncommutative analog of geodesic distance between two points. Recall: spectral distance for the spectral triple encoding the geometry of compact Riemann spin manifold equals geodesic distance.
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Pure states

- Convenient to use the matrix base (Proposition 2). Start from the very simple observation that any vector state defined by any $f_{mn}$ depends only on the first indice $m \in \mathbb{N}$, thanks to (1).
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Proposition 5

The pure states of $\bar{A}$ are the vector states $\omega_\psi : \bar{A} \to \mathbb{C}$ defined by any unit vector $\psi \in L^2(\mathbb{R}^2)$ of the form $\psi = \sum_{m \in \mathbb{N}} \psi_m f_{m0}$, $\sum_{m \in \mathbb{N}} |\psi_m|^2 = \frac{1}{2\pi \theta}$ and one has

$$\omega_\psi(a) \equiv \langle (\psi, 0), \pi(a)(\psi, 0) \rangle = 2\pi \theta \sum_{m,n \in \mathbb{N}} \psi^*_m \psi_n a_{mn} \quad (4)$$
Pure states

Convenient to use the matrix base (Proposition 2). Start from the very simple observation that any vector state defined by any \( f_{mn} \) depends only on the first indice \( m \in \mathbb{N} \), thanks to (1).

**Proposition 5**

The pure states of \( \bar{\mathcal{A}} \) are the vector states \( \omega_\psi : \bar{\mathcal{A}} \to \mathbb{C} \) defined by any unit vector \( \psi \in L^2(\mathbb{R}^2) \) of the form \( \psi = \sum_{m \in \mathbb{N}} \psi_m f_{m0} \), \( \sum_{m \in \mathbb{N}} |\psi_m|^2 = \frac{1}{2\pi \theta} \) and one has

\[
\omega_\psi(a) \equiv \langle (\psi, 0), \pi(a)(\psi, 0) \rangle = 2\pi \theta \sum_{m,n \in \mathbb{N}} \psi^*_m \psi_n a_{mn}
\]

(4)

**Proof.**

Let \( \mathcal{H}_0 \) be the Hilbert space spanned by the family \( (f_{m0})_{m \in \mathbb{N}} \). For any \( a = \sum_{m,n} a_{mn} f_{mn} \in \mathcal{A} \), one has \( \sum_p \|L(a)f_{p0}\|_2^2 = \sum_{p,m} |a_{pm}|^2 = \|a\|_2^2 < \infty \) so that \( L(a) \) is Hilbert-Schmidt on \( \mathcal{H}_0 \) and therefore compact on \( \mathcal{H}_0 \). Let \( \pi_0 \) be this representation of \( \mathcal{A} \) on \( \mathcal{H}_0 \) and \( \pi_0(\mathcal{A}) \) be the completion of \( \pi_0(\mathcal{A}) \). One has \( \pi_0(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{H}_0) \). \( \pi_0(\mathcal{A}) \) involves all finite rank operators. Then \( \pi_0(\mathcal{A}) \supseteq \mathcal{K}(\mathcal{H}_0) \) and so \( \pi_0(\mathcal{A}) = \mathcal{K}(\mathcal{H}_0) \). This latter has a unique irreducible representation (up to unitary equivalence) and the corresponding pure states are exactly given by vectors states defined by any unit vector \( \psi = \sum_{m \in \mathbb{N}} \psi_m f_{m0} \in \mathcal{H}_0 \).
On the "unit ball"

As algebraic relations:

**Proposition 6**

- The \( f_{mn} \)'s and their derivatives fulfill:

\[
\partial f_{mn} = \sqrt{\frac{n}{\theta}} f_{m,n-1} - \sqrt{\frac{m+1}{\theta}} f_{m+1,n}; \quad \bar{\partial} f_{mn} = \sqrt{\frac{m}{\theta}} f_{m-1,n} - \sqrt{\frac{n+1}{\theta}} f_{m,n+1}, \quad \forall m, n \in \mathbb{N}
\]  
(5)

- \( a = \sum_{m,n} a_{mn} f_{mn} \in \mathcal{A} \), define \( \partial a \equiv \sum_{m,n} \alpha_{mn} f_{mn}, \ \bar{\partial} a \equiv \sum_{m,n} \beta_{mn} f_{mn} \).

  a) The following relations hold:

\[
\alpha_{m+1,n} = \sqrt{\frac{n+1}{\theta}} a_{m+1,n+1} - \sqrt{\frac{m+1}{\theta}} a_{m,n}, \quad \alpha_{0,n} = \sqrt{\frac{n+1}{\theta}} a_{0,n+1}, \quad \forall m, n \in \mathbb{N}
\]  
(6)

\[
\beta_{m,n+1} = \sqrt{\frac{m+1}{\theta}} a_{m+1,n+1} - \sqrt{\frac{n+1}{\theta}} a_{m,n}, \quad \beta_{m,0} = \sqrt{\frac{m+1}{\theta}} a_{m+1,0}, \quad \forall m, n \in \mathbb{N}
\]  
(7)

- One has the inversion formula

\[
a_{p,q} = \delta_{p,q} a_{0,0} + \sqrt{\theta} \sum_{k=0}^{\min(p,q)} \frac{\alpha_{p-k,q-k-1} + \beta_{p-k-1,q-k}}{\sqrt{p-k} + \sqrt{q-k}}, \quad \forall p, q \in \mathbb{N}, p + q > 0
\]  
(8)
On the "unit ball"

- The condition defining the unit ball \( \|[D, \pi(a)]\|_{op} \leq 1 \) can be translated into constraints on the coefficients the expansion of \( \partial a \) and \( \bar{\partial} a \) in the matrix base.
On the "unit ball"

The condition defining the unit ball $||[D, \pi(a)]||_{op} \leq 1$ can be translated into constraints on the coefficients of the expansion of $\partial a$ and $\bar{\partial} a$ in the matrix base.

**Lemma 7**

We set $\partial a = \sum_{m,n} \alpha_{mn} f_{mn}$ and $\bar{\partial} a = \sum_{m,n} \beta_{mn} f_{mn}$, for any $a \in A$ and any unit vector $\varphi = \sum_{m,n} \varphi_{mn} f_{mn} \in L^2(\mathbb{R}^2)$. Assume that $||[D, \pi(a)]||_{op} \leq 1$.

1. The following property $\mathcal{P}$ holds:

   \[
   (\mathcal{P}) \sum_{p} |\alpha_{mp}| \varphi_{pn} \leq \frac{1}{2\sqrt{\pi \theta}} \quad \text{and} \quad \sum_{p} |\beta_{mp}| \varphi_{pn} \leq \frac{1}{2\sqrt{\pi \theta}}, \quad \forall \varphi \in \mathcal{H}_0, \ ||\varphi||_2 = 1, \ \forall m, n \in \mathbb{N}.
   \]

2. If $\mathcal{P}$ holds, then $|\alpha_{mn}| \leq \frac{1}{\sqrt{2}}$ and $|\beta_{mn}| \leq \frac{1}{\sqrt{2}}$, $\forall m, n \in \mathbb{N}$

3. For any radial function $a \in A$ (i.e $a_{mn} = 0$ if $m \neq n$), $||[D, \pi(a)]||_{op} \leq 1$ is equivalent to $|\alpha_{mn}| \leq \frac{1}{\sqrt{2}}$ and $|\beta_{mn}| \leq \frac{1}{\sqrt{2}}$, $\forall m, n \in \mathbb{N}$. 
On the ”unit ball“

Proof.

If \( ||[D, \pi(a)]||_{op} \leq 1 \), then \( ||\partial a||_{op} \leq \frac{1}{\sqrt{2}} \) and \( ||\bar{\partial} a||_{op} \leq \frac{1}{\sqrt{2}} \). Use matrix base: for any \( \varphi \in \mathcal{H}_0 \),

\[
||\partial a \ast \varphi||_2^2 = 2\pi \theta \sum_{m,n} |\sum_p \alpha_{mp} \varphi_{pn}|^2.
\]

Then (def.of ||\(\partial a||_{op}||), \sum_{m,n} |\sum_p \alpha_{mp} \varphi_{pn}|^2 \leq \frac{1}{4\pi \theta} \) for any \( \varphi \in \mathcal{H}_0 \) with \( \sum_{m,n} |\varphi_{mn}|^2 = \frac{1}{2\pi \theta} \). This implies

\[
|\sum_p \alpha_{mp} \varphi_{pn}| \leq \frac{1}{2\sqrt{\pi \theta}}, \forall \varphi \in \mathcal{H}_0, ||\varphi||_2 = 1, \forall m, n \in \mathbb{N} \tag{10}
\]

Now, \( |\sum_p \alpha_{mp} \varphi_{pn}| \leq \frac{1}{2\sqrt{\pi \theta}} \) true for any \( \varphi \in \mathcal{H}_0 \) with \( ||\varphi||_2 = 1 \) and one can construct \( \tilde{\varphi} \) with

\[
||\tilde{\varphi}||_2 = ||\varphi||_2 \text{ via } \alpha_{mp} \tilde{\varphi}_{pn} = |\alpha_{mp}||\varphi_{pn}|.
\]

Therefore

\[
|\sum_p |\alpha_{mp}||\varphi_{pn}| \leq \frac{1}{2\sqrt{\pi \theta}}, \forall \varphi \in \mathcal{H}_0, ||\varphi||_2 = 1, \forall m, n \in \mathbb{N} \tag{11}
\]

Note: (11) implies (10). Property i) shown. Property ii): direct consequence of the property \( \mathcal{P} \).

To prove iii), show that any radial function \( a \) such that \( |\alpha_{mn}| \leq \frac{1}{\sqrt{2}} \) and \( |\beta_{mn}| \leq \frac{1}{\sqrt{2}} \), \( \forall m, n \in \mathbb{N} \) is in the unit ball. One first observe that if \( a \) is radial, one has \( \alpha_{mn} = 0 \) if \( m \neq n + 1 \) thanks to (6). Then, for any unit vector \( \psi \in \mathcal{H}_0 \)

\[
||\partial a \ast \psi||_2^2 = 2\pi \theta \sum_{p,q} |\sum_r \alpha_{pr} \psi_{rq}|^2 = 2\pi \theta \sum_{p,q} |\alpha_{p,p-1} \psi_{p-1,q}|^2 \leq \pi \theta \sum_{p,q} |\psi_{pq}|^2 \tag{12}
\]

so that \( ||\partial a||_{op}^2 \leq \frac{1}{2} \) and \( a \) is in the unit ball. Similar considerations for \( \beta_{mn} \). 

\[\square\]
Definition 8

We denote by $\omega_m$ the pure state generated by the unit vector $\frac{1}{\sqrt{2\pi\theta}} f_{m0}$, $\forall m \in \mathbb{N}$. For any $a = \sum_{m,n} a_{mn} f_{mn} \in \mathcal{A}$, one has $\omega_m(a) = a_{mm}$. 
Spectral distance between pure states

Definition 8

We denote by $\omega_m$ the pure state generated by the unit vector $\frac{1}{\sqrt{2\pi\theta}}f_{m0}$, $\forall m \in \mathbb{N}$. For any $a = \sum_{m,n} a_{mn}f_{mn} \in A$, one has $\omega_m(a) = a_{mm}$.

Theorem 9

The spectral distance between any two pure states $\omega_m$ and $\omega_n$ is

$$d(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}, \quad \forall m, n \in \mathbb{N}, \quad n < m$$
Proof.

From Proposition 6,

\[ \alpha_{n+1,n} = \sqrt{\frac{n+1}{\theta}} (a_{n+1,n+1} - a_{n,n}) = \sqrt{\frac{n+1}{\theta}} (\omega_{n+1,n+1}(a) - \omega_{nn}(a)), \quad \forall n \in \mathbb{N} \]

Then, use ii) of Lemma 7 implies that, for any \( a \) in the unit ball, one has

\[ |\omega_{n+1,n+1}(a) - \omega_{nn}(a)| \leq \sqrt{\frac{\theta}{2}} \frac{1}{\sqrt{n+1}}, \quad \forall n \in \mathbb{N}. \]

Then, a repeated use of the triangular inequality for the distance gives rise to

\[ d(\omega_m, \omega_n) \leq \sqrt{\frac{\theta}{2}} \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}, \quad \forall m, n \in \mathbb{N}, \quad n < m. \]

To show that the upper bound is actually reached, consider the radial element

\[ \hat{a} = \sum_{p,q \in \mathbb{N}} \hat{a}_{pq} f_{pq}, \quad \hat{a}_{pq} = \delta_{pq} \sqrt{\frac{\theta}{2}} \sum_{k=p}^{m_0} \frac{1}{\sqrt{k+1}}, \quad m_0 \in \mathbb{N} \text{ fixed} \quad (13) \]

as a linear combination of \( m_0 + 1 - p \) elements of the matrix base, \( \hat{a} \in \mathcal{A} \). By using ii) of Proposition 6 and iii) of Lemma 7, one easily shows that \( \hat{\alpha}_{p+1,p} = -\frac{1}{\sqrt{2}}, \quad 0 \leq p \leq m_0 \) (the other \( \hat{\alpha}'s \) vanish) and \( \hat{a} \) is in the unit ball (for the \( \hat{\beta}_{m,n} \), the proof is similar). \( \square \)
Discussion

- Observe \( d(\omega_m, \omega_n) = d(\omega_m, \omega_p) + d(\omega_p, \omega_n) \) for any \( m < p < n \in \mathbb{N} \).
Observe $d(\omega_m, \omega_n) = d(\omega_m, \omega_p) + d(\omega_p, \omega_n)$ for any $m < p < n \in \mathbb{N}$.

There exist states at infinite distance. Use radial element $\hat{a}$ (13) to get lower bound on spectral distance between 2 pure states Prop 5. Start from

$$\omega_{\psi'}(a) - \omega_\psi(a) = (2\pi \theta)^2 \sum_{m,n,p,q} (da)_{mn,pq} \psi_m^* \psi_p^* \psi'_n \psi_q$$

(14)

$$(da)_{mn,pq} = (a_{mn} \delta_{pq} - a_{pq} \delta_{mn})$$

Using (14) yields

$$d(\omega_{\psi'}, \omega_\psi) \geq (2\pi \theta)^2 \sqrt{\frac{\theta}{2}} \sum_{p \leq k \leq q} \frac{1}{\sqrt{k}} (|\psi_p \psi'_q|^2 - |\psi_q \psi'_p|^2)$$

(15)

Choose now $\psi^0 = \frac{1}{\sqrt{2\pi \theta}} f_{00}$, $\psi'_0 = \frac{1}{\sqrt{2\pi \theta}} \sum_m \sqrt{\frac{\zeta(s)}{(m+1)^s}} f_{m0}$ where $\zeta(s)$ is the Riemann zeta function, $s > 1$. Then

$$d(\omega_{\psi'}, \omega_\psi) \geq \zeta(s) \sqrt{\frac{\theta}{2}} \sum_{1 \leq k \leq m} \frac{1}{(m+1)^s \sqrt{k}}$$

(16)

The right hand side is divergent for $s \leq \frac{3}{2}$. 
Spectral distance on Moyal plane

Discussion

Definition 10 (Rieffel, Contemp. Math. 2004)

A Compact Quantum Metric Space (CQMS) is a order unit space $\mathbb{A}$ equipped with a seminorm $l$ such that $l(1) = 0$ and the distance defined by

$$d(\omega_1, \omega_2) = \sup (|\omega_1 - \omega_2(a)|, / l(a) \leq 1)$$

(17)

induced the weak* topology on the state space of $\mathbb{A}$.

Then, a (unital) ST with $l(a) = ||[D, \pi(a)]||_{op}$ as seminorm is CQMS whenever the spectral distance induces weak* topology on the state space.
Discussion

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- Then, a (unital) ST with $l(a) = ||[D, \pi(a)]||_{op}$ as seminorm is CQMS whenever the spectral distance induces weak* topology on the state space.
- In the Moyal case, existence of states at infinite spectral distance implies that the NCST proposed recently is not a CQMS in the sense of Rieffel.
A truncation of the Moyal ST
1 Spectral distance on Moyal plane

2 Noncommutative Torus - preliminaries
   • basic properties
   • Pure states on noncommutative torus
   • Preliminary results - Spectral distance on NC Torus

3 Conclusion
The noncommutative torus

Definition 11 (For reviews see, e.g Landi, Gracia-Bondia, Varilly)

\( \mathcal{A}_\theta^2 \) universal C*-algebra generated by \( u_1, u_2 \) with \( u_1 u_2 = e^{i 2 \pi \theta} u_2 u_1 \). Algebra of the noncommutative torus \( \mathbb{T}_\theta^2 \) is the dense (unital) pre-C* subalgebra of \( \mathcal{A}_\theta^2 \) defined by

\[
\mathbb{T}_\theta^2 = \left\{ a = \sum_{i,j \in \mathbb{Z}} a_{ij} u_1^i u_2^j / \sup_{i,j \in \mathbb{Z}} (1 + i^2 + j^2)^k |a_{ij}|^2 < \infty \right\}.
\]

- Weyl generators defined by \( U^M \equiv e^{-i \pi m_1 \theta m_2} u_1^{m_1} u_2^{m_2}, \ \forall M = (m_1, m_2) \in \mathbb{Z}^2 \).

For any \( a \in \mathbb{T}_\theta^2 \), \( a = \sum_{m \in \mathbb{Z}^2} a_M U^M \). Let \( \delta_1 \) and \( \delta_2 \): canonical derivations

\[
\delta_a(u_b) = i 2 \pi u_a \delta_{ab}, \ \forall a, b \in \{1, 2\}. \quad \text{One has } \delta_b(a^*) = (\delta_b(a))^*, \ \forall b = 1, 2.
\]

Proposition 12

One has for any \( M, N \in \mathbb{Z}^2 \), \( (U^M)^* = U^{-M} \), \( U^M U^N = \sigma(M, N) U^{M+N} \) where the commutation factor \( \sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C} \) satisfies

\[
\sigma(M+N, P) = \sigma(M, P) \sigma(N, P), \ \sigma(M, N+P) = \sigma(M, N) \sigma(M, P), \ \forall M, N, P \in \mathbb{Z}^2
\]

\[
\sigma(M, \pm M) = 1, \ \forall M \in \mathbb{Z}^2
\]

\[
\delta_a(U^M) = i 2 \pi m_a U^M, \ \forall a = 1, 2, \ \forall M \in \mathbb{Z}^2
\]
The noncommutative torus

Let \( \tau \) be tracial state:
For any \( a = \sum_{M \in \mathbb{Z}^2} a_M U^M \in \mathbb{T}_\theta^2 \), \( \tau : \mathbb{T}_\theta^2 \to \mathbb{C} \), \( \tau(a) = a_{0,0} \).
The noncommutative torus

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  For any \( a = \sum_{M \in \mathbb{Z}^2} a_M U^M \in \mathbb{T}_\theta^2 \), \( \tau : \mathbb{T}_\theta^2 \to \mathbb{C} \), \( \tau(a) = a_{0,0} \).

- \( \mathcal{H}_\tau \): GNS Hilbert space (completion of \( \mathbb{T}_\theta^2 \) in the Hilbert norm induced by \( \langle a, b \rangle \equiv \tau(a^* b) \)). One has \( \tau(\delta_b(a)) = 0 \), \( \forall b = 1, 2 \).
The noncommutative torus

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For any $a = \sum_{M \in \mathbb{Z}^2} a_M U^M \in \mathbb{T}_\theta^2$, $\tau : \mathbb{T}_\theta^2 \rightarrow \mathbb{C}$, $\tau(a) = a_{0,0}$.

$\mathcal{H}_\tau$: GNS Hilbert space (completion of $\mathbb{T}_\theta^2$ in the Hilbert norm induced by $\langle a, b \rangle \equiv \tau(a^* b)$). One has $\tau(\delta_b(a)) = 0$, $\forall b = 1, 2$.

The even real spectral triple:

$$(\mathbb{T}_\theta^2, \mathcal{H}, D; J, \Gamma)$$

$\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$. One has $\delta_b^\dagger = -\delta_b$, $\forall b = 1, 2$, in view of

$\langle \delta_b(a), c \rangle = \tau((\delta_b(a)^* c) = \tau(\delta_b(a^*) c) = -\tau(a^* \delta_b(c)) = -\langle a, \delta_b(c) \rangle$ for any $b = 1, 2$ and $\delta_b(a^*) = (\delta_b(a))^*$. 

The noncommutative torus

Let $\tau$ be tracial state:
For any $a = \sum_{M \in \mathbb{Z}^2} a_M U^M \in \mathbb{T}_\theta^2$, $\tau : \mathbb{T}_\theta^2 \to \mathbb{C}$, $\tau(a) = a_{0,0}$.

$H_\tau$: GNS Hilbert space (completion of $\mathbb{T}_\theta^2$ in the Hilbert norm induced by $<a, b> \equiv \tau(a^* b)$). One has $\tau(\delta_b(a)) = 0$, $\forall b = 1, 2$.

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Define $\delta = \delta_1 + i\delta_2$ and $\bar{\delta} = \delta_1 - i\delta_2$. $D$: unbounded self-adjoint Dirac operator $D = -i \sum_{b=1}^{2} \delta_b \otimes \sigma^b$, densely defined on $\text{Dom}(D) = (\mathbb{T}_\theta^2 \otimes \mathbb{C}^2) \subset H$.

$$D = -i \begin{pmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{pmatrix}$$
The noncommutative torus

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  For any $a = \sum_{M \in \mathbb{Z}^2} a_M U^M \in \mathbb{T}_\theta^2$, $\tau : \mathbb{T}_\theta^2 \to \mathbb{C}$, $\tau(a) = a_{0,0}$.

- $\mathcal{H}_\tau$: GNS Hilbert space (completion of $\mathbb{T}_\theta^2$ in the Hilbert norm induced by $\langle a, b \rangle \equiv \tau(a^* b)$). One has $\tau(\delta_b(a)) = 0$, $\forall b = 1, 2$.

- The even real spectral triple:
  $$(\mathbb{T}_\theta^2, \mathcal{H}, D; J, \Gamma)$$
  $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$. One has $\delta_b^\dagger = -\delta_b$, $\forall b = 1, 2$, in view of
  $\langle \delta_b(a), c \rangle = \tau((\delta_b(a)^* c) = \tau(\delta_b(a^*) c) = -\tau(a^* \delta_b(c)) = -\langle a, \delta_b(c) \rangle$ for any $b = 1, 2$ and $\delta_b(a^*) = (\delta_b(a))^*$.

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  $$D = -i \begin{pmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{pmatrix}$$

- Faithfull representation $\pi : \mathbb{T}_\theta^2 \to \mathcal{B}(\mathcal{H}) : \pi(a) = L(a) \otimes I_2$,
  $\pi(a)\psi = (a\psi_1, a\psi_2)$, $\psi = (\psi_1, \psi_2) \in \mathcal{H}$, $\forall a \in \mathbb{T}_\theta^2$. $L(a)$: left multiplication operator by any $a \in \mathbb{T}_\theta^2$. $\pi(a)$ and $[D, \pi(a)]$ bounded on $\mathcal{H}$ for any $\mathbb{T}_\theta^2$.
  $$[D, \pi(a)]\psi = -i(L(\delta_b(a)) \otimes \sigma^b)\psi = -i \begin{pmatrix} L(\delta(a)) & 0 \\ 0 & L(\bar{\delta}(a)) \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$ (18)
Pure states on noncommutative torus

Classification of the pure states in the irrational case is lacking.

Consider rational case: \( \theta = \frac{p}{q} \), \( p < q \), \( p \) and \( q \) relatively prime, \( q \neq 1 \). Set \( \mathbb{T}_{p/q} \equiv \mathbb{T}_p / q \) [see e.g. Connes, Landi, Rieffel].

Unitary equivalence classes of irreps. \( \mathbb{T}_{p/q} \) classified by a torus parametrized by \((\alpha, \beta)\). Irreps. given by \( \pi_{\alpha, \beta} : \mathbb{T}_{p/q} \to \mathbb{C}^q \), \( \alpha, \beta \in \mathbb{C} \) unitaries and \( \pi_{\alpha, \beta}(u_1), \pi_{\alpha, \beta}(u_2) \in M_q(\mathbb{C}) \) are the usual clock and shift matrices in the basis defined by \( \hat{e}_k = \beta - k / q u_k \), \( \forall k \in \{0, 1, \ldots, q - 1\} \) and \( u_1 e_0 = \alpha 1 / q e_0 \).

Proposition 13

The set of pure states of the rational noncommutative torus is exactly the set of vector states \( \omega_{\psi_{\alpha, \beta}} : \mathbb{T}_{p/q} \to \mathbb{C} \)

\[
\omega_{\psi_{\alpha, \beta}}(a) = (\psi, \pi_{\alpha, \beta}(a)\psi),\quad \forall \psi \in \mathbb{C}^q, ||\psi|| = 1
\]

where \( \psi \) is given up to an overall phase. The pure states are then classified by a bundle over a commutative torus parametrized by \((\alpha, \beta)\) with fiber \( \mathbb{P}(\mathbb{C}^q) \).
Pure states on noncommutative torus

Classification of the pure states in the irrational case is lacking.

Consider rational case: \( \theta = \frac{p}{q} \), \( p < q \), \( p \) and \( q \) relatively prime, \( q \neq 1 \). Set \( T^2_{p/q} \equiv T_{p/q} \) [see e.g Connes, Landi, Rieffel]. Unitary equivalence classes of irreps. \( T_{p/q} \) classified by a torus parametrized by \((\alpha, \beta)\). Irreps. given by \( \pi_{\alpha, \beta} : T_{p/q} \to \mathbb{C}^q \), \( \alpha, \beta \in \mathbb{C} \) unitaries and \( \pi_{\alpha, \beta}(u_1), \pi_{\alpha, \beta}(u_2) \in \mathbb{M}_q(\mathbb{C}) \) are the usual clock and shift matrices in the basis defined by \( \{ e_k = \beta^{-k/q}u_2^k e_0 \} \), \( \forall k \in \{0, 1, \ldots, q - 1\} \) and \( u_1 e_0 = \alpha^{1/q} e_0 \).

**Proposition 13**

The set of pure states of the rational noncommutative torus is exactly the set of vector states \( \omega^\psi_{\alpha, \beta} : T_{p/q} \to \mathbb{C} \)

\[
\omega^\psi_{\alpha, \beta}(a) = (\psi, \pi_{\alpha, \beta}(a)\psi), \quad \forall \psi \in \mathbb{C}^q, \quad ||\psi|| = 1
\]  

(19)

where \( \psi \) is given up to an overall phase. The pure states are then classified by a bundle over a commutative torus parametrized by \((\alpha, \beta)\) with fiber \( P(\mathbb{C}^q) \).

**Proof.**

By standard results on C*-algebras, any irrep. \( (\pi_{\alpha, \beta}, \mathbb{C}^q) \) is unitarily equivalent to the GNS representation \( (\omega_\psi, \pi_{\alpha, \beta}) \) for any \( \psi \in \mathbb{C}^q \). Then, the \( \omega_\psi \) are pure states. Write now

\[
\omega^\psi_{\alpha, \beta}(a) = (\psi, \pi_{\alpha, \beta}(a)\psi) \quad \text{for any} \quad a \in T_{p/q}.
\]
Lemma 14

Set $\delta(a) = \sum_{N \in \mathbb{Z}^2} \alpha_N U^N$. One has $\alpha_N = i2\pi(n_1 + in_2)a_N$, $\forall N = (n_1, n_2) \in \mathbb{Z}^2$.

i) For any $a$ in the unit ball, $\|[D, \pi(a)]\|_{op} \leq 1$ implies $|\alpha_N| \leq 1$, $\forall N \in \mathbb{Z}^2$. Similar results hold for $\bar{\delta}(a)$.

ii) The elements $\hat{a}^M \equiv \frac{U^M}{2\pi(m_1 + im_2)}$ verify $\|[D, \pi(\hat{a}^M)]\|_{op} = 1$, $\forall M = (m_1, m_2) \in \mathbb{Z}^2$, $M \neq (0, 0)$.
Lemma 14

Set $\delta(a) = \sum_{N \in \mathbb{Z}^2} \alpha_N U^N$. One has $\alpha_N = i2\pi(n_1 + in_2)a_N$, $\forall N = (n_1, n_2) \in \mathbb{Z}^2$.

i) For any $a$ in the unit ball, $\|[D, \pi(a)]\|_{op} \leq 1$ implies $|\alpha_N| \leq 1$, $\forall N \in \mathbb{Z}^2$. Similar results hold for $\bar{\delta}(a)$.

ii) The elements $\hat{a}^M \equiv \frac{U^M}{2\pi(m_1+im_2)}$ verify $\|[D, \pi(\hat{a}^M)]\|_{op} = 1$, $\forall M = (m_1, m_2) \in \mathbb{Z}^2$, $M \neq (0, 0)$.

Proof.

The relation involving $\alpha_N$ obvious. Then, $\|[D, \pi(a)]\|_{op} \leq 1$ is equivalent to $\|\delta(a)\|_{op} \leq 1$ and $\|\bar{\delta}(a)\|_{op} \leq 1$ in view of (18). For any $a \in \mathfrak{A}_\theta^2$ and any unit $\psi = \sum_{N \in \mathbb{Z}^2} \psi_N U^N \in \mathcal{H}_\tau$, one has $\|\delta(a)\psi\|^2 = \sum_{N \in \mathbb{Z}^2} |\sum_{P \in \mathbb{Z}^2} \alpha_P \psi_{N-P} \sigma(P, N)|^2$. Then $\|\delta(a)\|_{op} \leq 1$ implies $|\sum_{P \in \mathbb{Z}^2} \alpha_P \psi_{N-P} \sigma(P, N)| \leq 1$, for any $N \in \mathbb{Z}^2$ and any unit $\psi \in \mathcal{H}_\tau$. By a straightforward adaptation of the proof carried out for ii) of Lemma 7, this implies $|\alpha_M| \leq 1$, $\forall M \in \mathbb{Z}^2$. This proves ii). Finally, iii) stems simply from an elementary calculation.
Proposition 15

Let the family of unit vectors \( \Phi_M = \left( \frac{1+U^M}{\sqrt{2}}, 0 \right) \in \mathcal{H}, \ \forall M \in \mathbb{Z}^2, \ M \neq (0,0) \)
generating the family of vector states of \( T^2_\theta \)

\[
\omega_{\Phi_M} : T^2_\theta \rightarrow \mathbb{C}, \ \omega_{\Phi_M}(a) \equiv (\Phi_M, \pi(a)\Phi_M)_\mathcal{H} = \frac{1}{2} < (1 + U^M), (a + aU^M) >
\]  

The spectral distance between any state \( \omega_{\Phi_M} \) and the tracial state is

\[
d(\omega_{\Phi_M}, \tau) = \frac{1}{2\pi|m_1 + im_2|}, \ \forall M = (m_1, m_2) \in \mathbb{Z}^2, \ M \neq (0,0)
\]
Preliminary results - Spectral distance on NC Torus

**Proposition 15**

Let the family of unit vectors \( \Phi_M = \left( \frac{1+U^M}{\sqrt{2}}, 0 \right) \in \mathcal{H}, \forall M \in \mathbb{Z}^2, \ M \neq (0,0) \) generating the family of vector states of \( T^2_\theta \)

\[
\omega_{\Phi_M} : T^2_\theta \to \mathbb{C}, \ \omega_{\Phi_M}(a) \equiv (\Phi_M, \pi(a)\Phi_M)_{\mathcal{H}} = \frac{1}{2} < (1 + U^M), (a + aU^M) >
\]

The spectral distance between any state \( \omega_{\Phi_M} \) and the tracial state is

\[
d(\omega_{\Phi_M}, \tau) = \frac{1}{2\pi|m_1 + im_2|}, \ \forall M = (m_1, m_2) \in \mathbb{Z}^2, \ M \neq (0,0)
\]

**Proof.**

Set \( a = \sum_{N \in \mathbb{Z}^2} a_N U^N \). Using Proposition 12 yields \( \omega_{\Phi_M}(a) = \tau(a) + \frac{1}{2} (a_M + a_{-M}) \). This, combined with Lemma 14 yields \( d(\omega_{\Phi_M}, \tau) \leq \frac{1}{2\pi|m_1 + im_2|} \). Upper bound obviously saturated by the element \( \hat{a}^M \) of iii) of Lemma 14 which belongs to the unit ball.
Proposition 16

For any two pure states in the family \( \{ \omega_{k}^{\alpha,\beta} \} \), one has

\[
\begin{align*}
\text{d}
(\omega_{e_{k}}^{\alpha,\beta},\omega_{e_{l}}^{\alpha',\beta'}) & \geq \\
\sup_{(m_{1},m_{2}) \neq (0,0)} \\
\frac{1}{4} \pi |m_{1} + im_{2}q| & \sin 2\pi \left( m_{1}(\theta_{k} - l_{2} - \phi - \phi'q) - m_{2}\psi - \psi' \right)
\end{align*}
\]

where \( k, l \in \{ 0, 1, \ldots, q - 1 \} \) and \( \alpha = e^{2i\pi\phi}, \beta = e^{2i\pi\psi}, \alpha' = e^{2i\pi\phi'}, \beta' = e^{2i\pi\psi'} \).

Proof.

Set \( a = \mathbb{P}_{m_{1},m_{2}} \in \mathbb{Z} \). One first obtains by standard calculation

\[
\omega_{e_{k}}^{\alpha,\beta}(a) = \sum_{M \in \mathbb{Z}_{2}^{a}} \alpha_{m_{1}}e^{2i\pi\theta_{m_{1}}k} \beta_{m_{2}}e^{-2i\pi\phi} \exp \left( -2i\pi \theta_{m_{1}}l_{2} - m_{2}\psi - m_{1}\phi \right)
\]

For \( \hat{a} M \) defined in ii) of Lemma 14 such that \( M \neq (0,0) \), we have

\[
|\omega_{e_{k}}^{\alpha,\beta}(\hat{a} M) - \omega_{e_{l}}^{\alpha',\beta'}(\hat{a} M)| = \frac{1}{4} \pi |m_{1} + im_{2}q| \sin 2\pi \left( m_{1}(\theta_{k} - l_{2} - \phi - \phi'q) - m_{2}\psi - \psi' \right)
\]

then larger than supremum of these quantities for \( (m_{1},m_{2}) \neq (0,0) \) and \( \phi, \psi, \phi', \psi' \mod 1 \).

\[\Box\]
Proposition 16

For any two pure states in the family \( \{ \omega_{\alpha, \beta}^{e_k} \} \), one has

\[
\text{d}(\omega_{\alpha, \beta}^{e_k}, \omega_{\alpha', \beta'}^{e_l}) \geq \sup_{(m_1, m_2) \neq (0,0)} \frac{1}{4\pi |m_1 + im_2 q|} \left| \sin 2\pi \left( m_1 \left( \frac{\theta k - l}{2} - \frac{\phi - \phi'}{2q} \right) - m_2 \frac{\psi - \psi'}{2} \right) \right| \text{ where }
\]

\( k, l \in \{0, 1, ..., q - 1\} \) and \( \alpha = e^{2i\pi \phi}, \beta = e^{2i\pi \psi}, \alpha' = e^{2i\pi \phi'} \) and \( \beta' = e^{2i\pi \psi'} \).

Proof.

Set \( a = \sum_{m_1, m_2 \in \mathbb{Z}} a_{m_1, m_2} u_1^{m_1} u_2^{m_2} \). One first obtains by standard calculation

\[
\omega_{\alpha, \beta}^{e_k}(a) = \sum_{M \in \mathbb{Z}^2} a_{(m_1, m_2 q)} \alpha^{m_1} \beta^{m_2} e^{-2i\pi \theta m_1 k} = \sum_{M \in \mathbb{Z}^2} a_{(m_1, m_2 q)} \exp \left( -2i\pi \left( \theta m_1 k - \frac{m_1}{q} \phi - m_2 \psi \right) \right)
\]

For \( \hat{a}^M \) defined in ii) of Lemma 14 such that \( M = (m_1, m_2 q) \neq (0,0) \), we have

\[
|\omega_{\alpha, \beta}^{e_k}(\hat{a}^M) - \omega_{\alpha', \beta'}^{e_l}(\hat{a}^M)| = \frac{1}{4\pi |m_1 + im_2 q|} \left| \sin 2\pi \left( m_1 \left( \frac{\theta k - l}{2} - \frac{\phi - \phi'}{2q} \right) - m_2 \frac{\psi - \psi'}{2} \right) \right| \tag{22}
\]

\( d(\omega_{\alpha, \beta}^{e_k}, \omega_{\alpha', \beta'}^{e_l}) \) then larger than supremum of these quantities for \( (m_1, m_2) \neq (0,0) \) and \( \phi, \psi, \phi', \psi' \) mod 1.
Conclusion

1 Spectral distance on Moyal plane

2 Noncommutative Torus - preliminaries

3 Conclusion
Conclusion

- Determination of distance between arbitrary pure states for Moyal plane difficult. In progress
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- Determination of distance between arbitrary pure states for Moyal plane difficult. In progress
- Noncommutative torus has been undertaken (ways "inspired by the Moyal case")
Conclusion

- Determination of distance between arbitrary pure states for Moyal plane difficult. In progress
- Noncommutative torus has been undertaken (ways ”inspired by the Moyal case“)
- Other examples of noncommutative spaces: $SU(2)_q$, Connes-Landi