An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system

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Abstract

In this paper, we provide a much simplified proof of the main result in [11] concerning the global existence and uniqueness of smooth solutions to the Cauchy problem for a 2D incompressible viscous and non-resistive MHD system under the assumption that the initial data are close to some equilibrium states. Beside the classical energy method, the interpolating inequalities and the algebraic structure of the equations coming from the incompressibility of the fluid are crucial in our arguments. We combine the energy estimates with the L^\infty estimates for time slices to deduce the key L^1 in time estimates. The latter is responsible for the global in time existence.

Keywords: MHD system; existence; uniqueness.

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1 Introduction.

In this paper, we consider the global existence of strong solutions to the following 2D incompressible viscous and non-resistive magnetohydrodynamics (MHD) system [10, 11],

\[
\begin{align*}
\partial_t \phi + v \cdot \nabla \phi &= 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t v + v \cdot \nabla v - \Delta v + \nabla p &= -\text{div} [\nabla \phi \otimes \nabla \phi], \\
\text{div} v &= 0, \\
(\phi, v)|_{t=0} &= (\phi_0, v_0),
\end{align*}
\]

(1.1)

where the initial data (\phi_0, v_0) close enough to the equilibrium state (x_2, 0). Here \phi, v = (v_1, v_2)^T and p denote the magnetic potential, velocity and scalar pressure of the fluid respectively.

The system (1.1) is formally equivalent to the following 2D MHD system

\[
\begin{align*}
\partial_t b + v \cdot \nabla b &= b \cdot \nabla v, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla p &= -\frac{1}{2} \nabla (|b|^2) + b \cdot \nabla b, \\
\text{div} v &= \text{div} b = 0, \\
(b, v)|_{t=0} &= (b_0, v_0),
\end{align*}
\]

(1.2)

where \( b = (b_1, b_2)^T, \) \( v = (v_1, v_2)^T \) and \( p \) denote the magnetic field, velocity and scalar pressure of the fluid respectively. In face, the condition \( \text{div} b = 0 \) implies the existence of a scalar function \( \phi \) such that \( b = (\partial_2 \phi, -\partial_1 \phi)^T \), and the system (1.2) becomes the (1.1). Magnetohydrodynamics (MHD) is the study of the dynamics of electrically conducting fluids, such as plasmas, liquid metals, and salt water or electrolytes. The MHD system describes many phenomena such as the geomagnetic dynamo in geophysics and solar winds and solar flares in astrophysics [1, 3, 5, 16].

There are many interesting results on the global regularity problem for the 2D MHD system with dissipation. The viscous and resistive MHD system has a unique global classical solution with the initial data \((v_0, b_0) \in H^m(\mathbb{R}^2), \) \( m \geq 2 \) (e.g. [6, 17]). The inviscid and resistive MHD system has a global weak solution with \((v_0, b_0) \in H^1(\mathbb{R}^2) \) (e.g. [4]). With mixed partial dissipation and additional (artificial) magnetic diffusion in the 2D incompressible MHD system, Cao and Wu [4] proved its global wellposedness results for any initial data in \( H^2(\mathbb{R}^2) \). Considering the ideal MHD system (i.e. inviscid and non-resistive), Bardos, Sulem and Sulem [2] proved the global existence of the classical solution when the initial data...
Remark A

We use the notation \( T > \) for all \( T > \) and \( c \) constant and \( \hat{B}(1.3) \) with the small and smooth initial data. Assume that the initial data satisfy (4.1). In this paper, we will prove the following global existence and uniqueness of the solution of the system (4.1).

It is easy to obtain the local existence and uniqueness of the solution for the system (1.3), see Theorem 4.1. In this paper, we will prove the following global existence and uniqueness of the solution of the system (1.3) with the small and smooth initial data. Assume that the initial data satisfy

\[
\nabla \psi_0, v_0 \in H^2(\mathbb{R}^2), \quad \text{div} v_0 = 0,
\]

and

\[
e^{-|\xi|^2 \hat{f}(\xi)} \in L^2([0, \infty); L^2_{\xi}),
\]

where \( \hat{f} \) is the Fourier transform of the function \( f \). Let \( A_0 = A_{1,0} + A_{2,0} \) with

\[
A_{1,0} = \| \nabla \psi_0 \|_{H^2} + \| v_0 \|_{H^2}, \quad \text{and} \quad A_{2,0} = \| e^{-|\xi|^2 \hat{f}(\xi)} \|_{L^2([0, \infty); L^2_{\xi})},
\]

and \( A_T = A_{1,T} + A_{2,T} \) with

\[
A_{1,T} = \| v \|_{L^\infty([0,T]; H^2)} + \| \nabla \psi \|_{L^\infty([0,T]; H^2)} + \| \text{div} v \|_{L^2([0,T]; H^2)} + \| \partial_t \psi \|_{L^2([0,T]; H^2)},
\]

Denote

\[
E_T^n = \left\{ (\psi, v, \nabla p) \left| \begin{array}{l}
\nabla \psi, v, p \in C([0,T]; H^n), \nabla p \in C([0,T]; H^{n-1}), \\
(\partial_t \nabla \psi, \nabla v) \in L^2([0,T]; H^n \times H^n), \nabla \tilde{\psi} = 0, \partial_t \tilde{\psi} = 0 \in L^2([0,T]; H^2), \end{array} \right. \right\}
\]

We use the notation \( E^n \) if \( T = \infty \) by changing the time interval \([0, T]\) into \([0, \infty)\) in the above definition.

**Theorem 1.1.** Assume that the initial data \((\psi_0, v_0)\) satisfy (1.3), then there exists a positive constant \( c_0 \) such that if

\[
A_0 \leq c_0,
\]

then the system (1.3) has a unique global solution \((\psi, v, \nabla p) \in E^2\) satisfying

\[
A_T \leq C A_0,
\]

and

\[
\| \nabla p \|_{L^\infty([0,T]; H^1)} \leq C(1 + c_0) A_0,
\]

for all \( T > 0 \), where \( C \) is a positive constant independent of \( T \).

**Remark 1.1.** One can see that if \( \nabla \psi_0, v_0 \in B_{2,1}^0 \), then (1.5) holds. Then \( A_{2,0} \) can be replaced by \( A_{2,0} = \| \nabla \psi_0 \|_{B_{2,1}^0} + \| v_0 \|_{B_{2,1}^0} \).
Remark 1.2. Under the assumptions of Theorem 1.1 if \((\nabla \psi_0, v_0) \in H^n(\mathbb{R}^2) \times H^n(\mathbb{R}^2), n \geq 3\), then we can easily obtain that \((\psi, v, \nabla p) \in E^n \) and omit the details.

Remark 1.3. In [12], using the anisotropic Littlewood-Paley analysis, H.F. Lin and P. Zhang proved a global wellposedness result of a 3D toy model. In [13], we provided a new and simple proof. The proof of this paper is similar to that in [13]. The three key technical points in our proof are:

1. interpolating estimates, like Lemma 2.1
2. using algebraic structure: \(\text{div}v = 0\) to inter-changing the derivative estimates for \(\partial_1\) and \(\partial_2\);
3. using the first equation of (1.3) to reduce "\(L^1\) in time estimates" (11) which is the key to the global existence result to "energy estimates and \(L^\infty\) estimates for time slices" that are relatively easy to obtain.

In fact, the basic strategy for the proofs is rather clear. Using the basic energy laws, one reduces the problems to estimating certain nonlinear terms of particular forms. For example, one of the difficulties of the proofs would be to control the following type term,

\[
\int_0^T \int_{\mathbb{R}^2} \partial_2 v_2 (\partial_2^3 \psi)^2 dx dt. \tag{1.9}
\]

Since the first derivative of \(\psi, \partial_1 \psi, \) decay fast than \(\partial_2 \psi\) (by energy laws), in [11], H.F. Lin, L. Xu and P. Zhang explored such anisotropic behavior by using the anisotropic Littlewood-Paley theory to conclude the key estimate that \(v_2 \in L^1(\mathbb{R}^+; L^2p(\mathbb{R}^2))\). In this paper, we will show that \(\nabla \psi \in L^4_2(L^\infty)\) in Lemma 2.1 by the interpolating estimate. Then, we use (1.6) \(1\) twice to obtain that

\[
\left| \int_0^T \int_{\mathbb{R}^2} \partial_2 v_2 (\partial_2^3 \psi)^2 dx dt \right| = \left| \int_0^T \int_{\mathbb{R}^2} \partial_2 (\partial_1 \psi + v \cdot \nabla \psi) (\partial_2^3 \psi)^2 dx dt \right| \\
\leq \left| \int_{\mathbb{R}^2} \partial_2 (\partial_2^3 \psi)^2 dx \right|_0^T + \left| \int_0^T \int_{\mathbb{R}^2} \partial_2 v_2 \partial_2 (\partial_2^3 \psi)^2 dx dt \right| + \ldots \\
= \ldots + \left| \int_{\mathbb{R}^2} \partial_2 (\partial_1 \psi + v \cdot \nabla \psi) \partial_2 (\partial_2^3 \psi)^2 dx dt \right| + \ldots \\
\leq \ldots + C \|\nabla \psi\|^2_{L^4(L^\infty(\mathbb{R}^2))} \|\nabla v\|_{L^2(L^2(\mathbb{R}^2)))} \|\nabla \psi\|^2_{L^\infty(L^2(\mathbb{R}^2)))} + \ldots, \tag{1.10}
\]

Refer the details in Lemma 2.5. This is a simple method for the issue concerning the anisotropic dissipative system similar to [1.3]

Remark 1.4. In Lemma 2.5 we want to estimate the following term,

\[
\left| \int_0^T \int_{\mathbb{R}^2} \partial_2 v_1 \partial_1 \psi (\partial_2^3 \psi)^2 dx dt \right| .
\]

Since \(\partial_2^3 \psi \in L^\infty_2(L^2)\) and

\[
\left| \int_0^T \int_{\mathbb{R}^2} \partial_2 v_1 \partial_1 \psi (\partial_2^3 \psi)^2 dx dt \right| \leq \|\nabla v\|_{L^2_2(L^\infty(\mathbb{R}^2)))} \|\partial_1 \psi\|_{L^2_2(L^\infty(\mathbb{R}^2)))} \|\partial_2^3 \psi\|_{L^\infty_2(L^2(\mathbb{R}^2)))},
\]

we need to estimate the term \(\|\partial_1 \psi\|_{L^2_2(L^\infty(\mathbb{R}^2)))}\). One cannot bound it if only have \(\nabla \partial_1 \psi \in L^2_2(H^1(\mathbb{R}^2)))\).

So we add the assumption (1.6) on the initial data, and will get

\[
\|\partial_1 \psi\|_{L^2_2(L^\infty)} \leq C \|\partial_1 \psi(t, \xi)\|_{L^2_2(L^\infty)} \leq CA_{2,0} + CA_{1,T} + CA_{1}. \tag{3}
\]

Please see the details in Section 3.
Remark 1.5. Recently, using the anisotropic Littlewood-Paley analysis techniques, J.H. Wu, Z.Y. Xiang and Z.F. Zhang \cite{15} obtain the global existence and decay estimates of global strong solution when $\|(|\nabla \psi|, v_0)\|_{H^{\sigma} \cap H^{-\sigma}, \cap H^{-\sigma}, \cap H^{-\sigma}} \ll 1$, $s = \frac{1}{2} - \epsilon$. Under the special assumptions on the initial data, by the anisotropic Littlewood-Paley analysis techniques, X.P. Hu and F.H. Lin \cite{8} also obtain the existence of global strong solution when the initial data are in the critical space.

Let us complete this section by the notation we shall use in this paper.

Notation. We shall denote by $(a|b)$ the $L^2$ inner product of $a$ and $b$, $(a|b)_{H^s} = \sum_{k=0}^{\infty} (a|b)_{H^k}$, and $(a|b)_{H^s} = \sum_{k=0}^{\infty} (a|b)_{H^k}$, $C_T(X) = C([0,T];X)$ and $L^p_T(X) = L^p([0,T];X)$.

2 A priori estimates for $A_{1,T}$

In this section, we prove some a priori estimates for $A_{1,T}$ which are crucial for the global existence of the strong solutions for the 2D MHD system \cite{13}. We begin with the following Gagliardo-Nirenberg-Sobolev type estimate, see \cite{15}.

Lemma 2.1. If the function $\psi$ satisfies that $\nabla \psi \in L_T^{p}(H^2(\mathbb{R}^2))$ and $\partial_t \nabla \psi \in L_T^{2}(H^1(\mathbb{R}^2))$, then there holds

$$
\|\nabla \psi\|_{L_T^\infty(L^\infty(\mathbb{R}^2))} \leq C \|\partial_t \nabla \psi\|_{L_T^\infty(H^1(\mathbb{R}^2))}^{\frac{2}{7}} \|\nabla \psi\|_{L_T^\infty(H^1(\mathbb{R}^2))}^{\frac{5}{7}},
$$

where $C$ is a positive constant independent of $T$.

Proof. Using Sobolev embedding Theorem and Minkowski’s inequality, we obtain

$$
\|\nabla \psi\|_{L_T^\infty(L^\infty(\mathbb{R}^2))} \leq C \|\nabla \psi\|_{L_T^2}^{\frac{2}{7}} \|\nabla \partial_t \psi\|_{L_T^2}^{\frac{5}{7}} \|\nabla \partial_t \psi\|_{L_T^\infty} \|\nabla \partial_t \psi\|_{L_T^\infty}^{\frac{4}{2}}
$$

similar to \cite{11}, by taking divergence of the $v$ equation of \cite{13}, we can express the pressure function $p$ via

$$
p = -2\partial_2 \psi + \sum_{i,j=1}^{2} (-\Delta)^{-1}[\partial_i v_j \partial_j v_i + \partial_i \partial_j (\partial_i \psi \partial_j \psi)].
$$

Substituting \cite{20} into \cite{13}, we have

$$
\begin{cases}
\partial_t \psi + v \cdot \nabla \psi + v_2 = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,
\partial_t v_1 + v \cdot \nabla v_1 - \Delta v_1 - \partial_t \partial_2 \psi = f_1,
\partial_2 v_2 + v \cdot \nabla v_2 - \Delta v_2 + \partial_2^2 \psi = f_2,
div v = 0,
(\psi, v)|_{t=0} = (\psi_0, v_0),
\end{cases}
$$

where

$$
f = (f_1, f_2)^T = -\sum_{i,j=1}^{2} \nabla(-\Delta)^{-1}[\partial_i v_j \partial_j v_i + \partial_i \partial_j (\partial_i \psi \partial_j \psi)] - \sum_{j=1}^{2} \partial_j [\nabla \psi \partial_j \psi].
$$

The next lemma is a standard energy estimate.
Lemma 2.2. Let \((\psi, v)\) be sufficiently smooth functions which solve \((1.3)\), then there holds

\[
\frac{d}{dt} \left\{ \frac{1}{2} \left( \|v\|^2_{H^2} + \|
abla \psi\|^2_{H^2} + \frac{1}{4} \|\Delta \psi\|^2_{H^1} \right) + \frac{1}{4} (v_2|\Delta \psi)|_{H^1} \right\} + \frac{1}{2} \|v\|^2_{H^2} - \frac{1}{4} \|
abla v_2\|^2_{H^1} + \frac{1}{4} \|\nabla \partial_1 \psi\|^2_{H^1} = -(v \cdot \nabla v|v)|_{H^2} + (v \cdot \nabla \psi|\Delta \psi)_{H^2} - (\text{div}(\nabla \psi \otimes \nabla \psi)|v)|_{H^2} - \frac{1}{4} (v \cdot \nabla v_2|\Delta \psi)|_{H^1}.
\]

Proof. Taking the standard \(H^2\) inner product of \((1.3)\) and \(v\), using the integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{H^2} + (v \cdot \nabla v|v)_{H^2} + \|
abla v\|^2_{H^2} = -(\partial_1 \partial_2 \psi|v_1)_{H^2} - ((\Delta + \partial_2^2) \psi|v_2)_{H^2} - (\text{div}(\nabla \psi \otimes \nabla \psi)|v)_{H^2}.
\]  

Using the fact that \(\text{div} v = 0\) and the integration by parts, we get

\[
- (\partial_1 \partial_2 \psi|v_1)_{H^2} = (\partial_2 \psi|\partial_1 v_1)_{H^2} = -(\partial_2 \psi|\partial_2 v_2)_{H^2} = (\partial_2^2 \psi|v_2)_{H^2}.
\]

From \((1.3)\)\(_1\), we obtain

\[
- (\Delta \psi|v_2)_{H^2} = (\Delta \psi|(\partial_t \psi + v \cdot \nabla \psi))_{H^2} = -\frac{1}{2} \frac{d}{dt} \|
abla \psi\|^2_{H^2} + (\Delta \psi|v \cdot \nabla \psi)_{H^2}.
\]  

From \((2.6)\)\(_2\)\(_3\), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|v\|^2_{H^2} + \|
abla \psi\|^2_{H^2} + \|
abla v\|^2_{H^2} \right) = -(v \cdot \nabla v|v)_{H^2} + (\Delta \psi|v \cdot \nabla \psi)_{H^2} - (\text{div}(\nabla \psi \otimes \nabla \psi)|v)_{H^2}.
\]

Taking the standard \(H^1\) inner product of \((2.4)\)\(_3\) and \(\Delta \psi\), using the integration by parts, we have

\[
(\partial_t v_2|\Delta \psi)_{H^1} + (v \cdot \nabla v_2|\Delta \psi)_{H^1} - (\Delta v_2|\Delta \psi)_{H^1} = -\|\partial_1 \nabla \psi\|^2_{H^1} + (f_2|\Delta \psi)_{H^1}.
\]  

From \((1.3)\)\(_1\), using the integration by parts, we get

\[
(\partial_t v_2|\Delta \psi)_{H^1} = \frac{d}{dt} (v_2|\Delta \psi)_{H^1} - (v_2|\Delta \partial_1 \psi)_{H^1} = \frac{d}{dt} (v_2|\Delta \psi)_{H^1} + (v_2|\Delta (v \cdot \nabla \psi + v_2))_{H^1} = \frac{d}{dt} (v_2|\Delta \psi)_{H^1} - (\nabla v_2|\nabla (v \cdot \nabla \psi))_{H^1} - \|
abla v_2\|^2_{H^1}.
\]

We observe, by \((1.3)\)\(_1\), that

\[
- (\Delta v_2|\Delta \psi)_{H^1} = (\Delta (\partial_t \psi + v \cdot \nabla \psi)|\Delta \psi)_{H^1} = \frac{1}{2} \frac{d}{dt} \|
abla \psi\|^2_{H^1} + (\Delta (v \cdot \nabla \psi)|\Delta \psi)_{H^1}.
\]

From \((2.10)\)\(_2\)\(_2\), we have

\[
\frac{d}{dt} \left\{ \frac{1}{2} \|
abla \psi\|^2_{H^1} + (v_2|\Delta \psi)_{H^1} \right\} + \|
abla \partial_1 \psi\|^2_{H^1} - \|
abla v_2\|^2_{H^1} = -(v \cdot \nabla v_2|\Delta \psi)_{H^1} + (f_2|\Delta \psi)_{H^1} + (\nabla v_2|\nabla (v \cdot \nabla \psi))_{H^1} - (\Delta (v \cdot \nabla \psi)|\Delta \psi).
\]

With \((2.9)\) and \((2.12)\), one can finish the proof. \(\Box\)
The following is the key a priori estimate which is essential to the proof of the main result of this paper.

**Lemma 2.3.** Let \((\psi, v)\) be sufficiently smooth functions which solve (1.3) and satisfy \((\psi, v, \nabla p) \in L^2_T\), then there holds

\[
A_{1,T}^2 \leq C(\|v_0\|_{H^2}(\mathbb{R}^2)^2 + \|\nabla \psi_0\|_{H^2}(\mathbb{R}^2)^2) + CA_1^2(1 + AT)^2. \tag{2.14}
\]

where \(C\) is a positive constant independent of \(T\).

**Proof.** From the energy estimate (2.5) and the definition of \(A_{1,T}\), we obtain for a positive constant \(C\) (independent of \(T\)) that

\[
A_{1,T}^2 \leq C A_{1,0}^2 + C \int_0^T (v \cdot \nabla v_h) dt + C \int_0^T (v \cdot \nabla \psi_1 + (v \cdot \nabla \psi_2) dt)
\]

\[
+ C \int_0^T (\text{div}(\nabla \psi_1 \otimes \nabla \psi)) dt + C \int_0^T (\nabla v_2 \nabla (v \cdot \nabla \psi)) dt + C \int_0^T (\Delta (v \cdot \nabla \psi) dt)
\]

\[
= CA_{1,0}^2 + \sum_{j=1}^7 H_j. \tag{2.15}
\]

We are going to estimate term by term the right hand side of the above inequality. The basic strategies are the similar to [13]. More precisely, we estimate separately terms involving \(\partial_1 \psi\) and terms with \(\partial_2 \psi\). For terms with \(\partial_1 \psi\), one can use the dissipations implied by the energy equality (2.5). For terms containing \(\partial_2 \psi\), we use the algebraic relation (deduced from that \(\text{div} v = 0\)) and the transport equations. The latter reduces space-time estimates to bounds on time-slices and terms with either \(\partial_1 \psi\) or of higher order nonlinearities (hence they are smaller under our smallness assumptions on the initial data). To illustrate the basic idea, we start with the second term \(H_2\). Applying Hölder and Sobolev inequalities, we deduce that

\[
H_2 = C \left| \sum_{k=1}^2 \sum_{i=1}^2 \int_0^T [\partial^\alpha \partial_i (v \cdot \nabla \psi) - v \cdot \nabla \partial^\alpha \partial_i \psi] \partial^\alpha \partial_i \psi dx dt \right|
\]

\[
\leq C \|\nabla \partial_1 \psi\|_{L^2(T; L^2(\mathbb{R}^2))} \|\nabla v\|_{L^2(T; L^2(\mathbb{R}^2))} + \|\nabla \psi\|_{L^2(T; L^2(\mathbb{R}^2))} \|\nabla \psi\|_{L^2(T; L^2(\mathbb{R}^2))}
\]

\[
+ C \|\partial_1 \psi\|_{L^2(T; L^2(\mathbb{R}^2))} \|\nabla v_1\|_{L^2(T; L^2(\mathbb{R}^2))} \|\nabla \psi\|_{L^2(T; L^2(\mathbb{R}^2))} + C \int_0^T \left| \partial_2^2 v_2 \partial_2 \psi + 2 \partial_2 v_2 \partial_2^2 \psi \partial_2^2 \psi dx dt \right|
\]

\[
+ C \|\partial_1^2 \psi\|_{L^2(T; L^2(\mathbb{R}^2))} \|\nabla \partial_1^2 \psi\|_{L^2(T; L^2(\mathbb{R}^2))} + C \int_0^T \left| \partial_1 \psi_1 \partial_1 \psi + \nabla v_1 \partial_1 \psi dx dt \right|
\]

\[
+ C \|\nabla \partial_1 \psi\|_{L^2(T; L^2(\mathbb{R}^2))} \|\nabla \partial_1 \psi\|_{L^2(T; L^2(\mathbb{R}^2))} + C \int_0^T \left| \partial_1 \psi_1 \partial_1 \psi \partial_1^2 \psi \partial_2 \psi dx dt \right|. \tag{2.16}
\]

Then, to estimate the terms in the above inequality, we need the following two lemmas, and give the detail proofs in the later.

**Lemma 2.4.** Under the conditions in Lemma 2.3, then there hold

\[
\left| \int_0^T \partial_2 \psi \partial_2^2 \psi \partial_2^2 v_2 dx dt \right| + \left| \int_0^T \partial_2 \psi \partial_2^2 \psi \partial_2^2 v_2 dx dt \right| + \left| \int_0^T \partial_2 \psi \partial_2^2 \psi \partial_2^2 v_2 dx dt \right| \leq CA_1^2, \tag{2.17}
\]
\[ \left| \int_0^T \partial_2 \psi \partial^3_2 \psi \partial^2_2 v_2 \partial^2_2 \psi dx dt \right| \leq CA^4_T, \quad (2.18) \]
\[ \left| \int_0^T (\partial^2_2 \psi)^2 \partial_2 v_2 dx dt \right| + \left| \int_0^T (\partial^2_2 \psi)^2 \partial^2_2 v_2 dx dt \right| \leq CA^3_T, \quad (2.19) \]

where \( C \) is a positive constant independent of \( T \).

**Lemma 2.5.** Under the conditions in Lemma 2.4, then there holds
\[ \left| \int_0^T \partial_2 v_2 (\partial^3_2 \psi)^2 dx dt \right| \leq CA^3_T(1 + A^2_T), \quad (2.20) \]
where \( C \) is a positive constant independent of \( T \).

Accepting these two lemmas, we proceed with our proof. From (2.10) and Lemmas 2.4-2.5, we get
\[ H_2 \leq CA^2_T(1 + A_T)^2. \quad (2.21) \]
Similarly, one can get
\[ H_7 = C \left| \sum_{i=1}^2 \int_0^T (\partial_i (v \cdot \nabla \psi) | \partial_i \Delta \psi)_{H^1} dt \right| \leq CA^2_T(1 + A_T)^2, \quad (2.22) \]
and
\[ H_3 = C \left| \sum_{k=1}^2 \sum_{|\alpha|=k, \alpha} \int_0^T \partial^\alpha (\partial_2 \psi \partial_2 \psi) \partial^\alpha \partial_2 v_2 dx dt \right| \leq C \left| \sum_{i,j=1}^2 \int_0^T \partial_2 (-\Delta)^{-1} (\partial_i v_j \partial_j \psi) (\Delta \psi)_{H^1} dt \right| \leq C \int_0^T \left| (\partial_2 (-\Delta)^{-1} (\partial_i v_j \partial_j \psi) \Delta \psi)_{H^1} \right| dt \leq C \int_0^T \left| (\nabla v)^2 \right|_{H^1(\mathbb{R}^2)} \left\| \nabla \psi \right\|_{H^1(\mathbb{R}^2)} dt \leq C \left\| \nabla v \right\|^2_{L^2(\mathbb{R}^2)} \left\| \nabla \psi \right\|_{L^2(\mathbb{R}^2)}, \quad (2.23) \]
Apply the same line of arguments, one could deduce that
\[ \left| \sum_{i,j=1}^2 \int_0^T \partial_2 (-\Delta)^{-1} (\partial_i v_j \partial_j \psi) (\Delta \psi)_{H^1} dt \right| \leq C \int_0^T \left| (\nabla v)^2 \right|_{H^1(\mathbb{R}^2)} \left\| \nabla \psi \right\|_{H^1(\mathbb{R}^2)} dt \leq C \left\| \nabla v \right\|^2_{L^2(\mathbb{R}^2)} \left\| \nabla \psi \right\|_{L^2(\mathbb{R}^2)}, \quad (2.24) \]
\[ \left| \int_0^T \left( - \sum_{i,j=1}^2 \partial_2 (-\Delta)^{-1} (\partial_i \psi \partial_j \psi) \Delta \psi \right)_{H^1} dt \right| = \left| \int_0^T (-\partial_2 (-\Delta)^{-1} (\partial^2_1 \psi \partial_1 \psi) \Delta \psi)_{H^1} dt \right| \]
In the same way, we can easily obtain the following estimates and omit the details.

**Proof of Lemma 2.4.**

From (2.15), (2.21)-(2.23) and (2.26)-(2.29), one can obtain (2.14).

At the end of this section, we give the proofs of two technical Lemmas 2.4-2.5 that are needed in establishing the key a priori estimates. Using the condition \( \text{div} v = 0 \), we can replace \( \partial_1 v_1 \) by \( \partial_2 v_2 \) in various calculations in the proof of Lemma 2.4.

**Proof of Lemma 2.4.** Using the fact that \( \text{div} v = 0 \), the integration by parts, Hölder's inequality and Sobolev embedding Theorem, we have

\[
+ \left| \int_0^T (2 \partial_2 (\Delta)^{-1} (\partial_2 \partial_1 \psi)) + (\partial_2 \partial_1 \psi) \right| \partial_1 \Delta \psi \right|_{H^1} dt \\
+ \left| \int_0^T (-\Delta)^{-1} \partial_2^2 (\partial_2 \psi)^2 - \partial_2 (\partial_2 \psi)^2 \right| \Delta \psi \right|_{H^1} dt \\
\leq C \int_0^T |\nabla (\partial_1 \psi)|^2_{H^1(R^2)} |\Delta \psi|_{H^1(R^2)} dt + C \int_0^T |\nabla (\partial_1 \psi)| |\partial_1 \psi|_{H^1(R^2)} dt \\
+ \left| \int_0^T ((-\Delta)^{-1} \partial_2^2 (\partial_2 \psi)^2) \Delta \psi \right|_{H^1} dt \\
\leq C |\nabla \partial_1 \psi|_{L^2(H^1(R^2))} (|\nabla \partial_1 \psi|_{L^2(H^1(R^2))} + |\partial_1 \psi|_{L^2(L^\infty(R^2))}) |\nabla \psi|_{L^\infty(H^1(R^2))} \\
+ C \int_0^T |\partial_1 (\partial_2 \psi)^2|_{H^1(R^2)} |\nabla \partial_1 \psi|_{H^1(R^2)} dt \\
\leq C |\nabla \partial_1 \psi|_{L^2(H^1(R^2))} (|\nabla \partial_1 \psi|_{L^2(H^1(R^2))} + |\partial_1 \psi|_{L^2(L^\infty(R^2))}) |\nabla \psi|_{L^\infty(H^1(R^2))}.
\]

With (2.21)-(2.25), we conclude

\[
H_5 = C \left| \int_0^T (f_2 |\Delta \psi|_{H^1}) dt \right| \leq CA_f^2.
\]

In the same way, we can easily obtain the following estimates and omit the details.

\[
H_1 = C \left| \int_0^T (v \cdot \nabla v |v|)_{H^2} dt \right| \leq C |\nabla v|_{L^2(H^1(R^2))}^2 |v|_{L^\infty(H^2(R^2))}.
\]

\[
H_4 = C \left| \int_0^T (v \cdot \nabla v |\Delta \psi|_{H^1}) dt \right| \leq C \sum_{j,k=1}^2 \left| \int_0^T (\partial_k (v_j v_2) |\partial_k \partial_j \psi|_{H^1}) dt \right| \\
\leq C |\nabla v|_{L^2(H^1(R^2))} |v|_{L^2(L^\infty(R^2))} |\nabla^2 v|_{L^\infty(H^1(R^2))} + C |\nabla v|_{L^2(L^4(R^2))} |\nabla^3 \psi|_{L^\infty(L^2(R^2))}
\]

\[
H_6 = C \left| \int_0^T (\nabla v_2 |\nabla (v \cdot \nabla \psi)|)_{H^1} dt \right| \\
\leq C |\nabla v|_{L^2(H^1(R^2))} |\nabla \psi|_{L^\infty(H^1(R^2))} + C |\nabla v_2|_{L^2(H^1(R^2))} |v|_{L^2(L^\infty(R^2))} |\nabla^2 \psi|_{L^\infty(H^1(R^2))}.
\]

From (2.15), (2.21)-(2.23) and (2.26)-(2.29), one can obtain (2.14).
\[
\begin{align*}
\int_0^T \left( -\partial_t^2 v_1 \partial_t \partial^2_2 \psi - \partial_2 \psi \partial_t^2 v_1 \partial_t \partial^2_2 \psi \right) dt \\
= \int_0^T \left( -\partial_2^2 v_1 \partial_t \partial^2_2 \psi + \partial_2^2 \psi \partial^2_2 v_1 \partial_t \partial^2_2 \psi + \partial_2 \psi \partial_2^2 v_1 \partial_t \partial_2^2 \psi \right) dt \\
\leq C\|\partial^2_2 v_1\|_{L^2_T(L^2(\mathbb{R}^2))} \|\partial_t \partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \|\partial^2_2 v_1\|_{L^2_T(L^2(\mathbb{R}^2))} \\
+ C\|\partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \|\partial_t \partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \\
+ C\|\partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \|\partial_t \partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \\
\leq C\|\nabla \psi\|_{L^2_T(H^2(\mathbb{R}^2))} \|\partial_t \nabla \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \|\nabla v\|_{L^2_T(H^2(\mathbb{R}^2))}.
\end{align*}
\]

Similarly, we can obtain (2.17)-(2.18). Using the fact that \(\operatorname{div}v = 0\), the integration by parts, Hölder’s inequality and Sobolev embedding Theorem, we obtain

\[
\begin{align*}
\left| \int_0^T \left( \partial^2_2 \psi \right)^2 \partial_2 v_2 dt \right| \\
= \left| \int_0^T \left( \partial^2_2 \psi \right)^2 \partial_1 v_1 dt \right| \\
= \left| \int_0^T 2 \partial_1 v_1 \partial^2_2 \psi \partial_2 \psi dt \right| \\
\leq C\|v_1\|_{L^2_T(L^\infty(\mathbb{R}^2))} \|\partial_1 \partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))} \|\partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))}.
\end{align*}
\]

Similarly, we can obtain (2.19).

From the idea in Remark 1.3, we can give the proof of the key lemma 2.5. Here, it would be useful (and it may be also necessary) to replace \(v_2\). In fact, via (1.3), we can re-write

\[ v_2 = -\left( \partial_t \psi + v \cdot \nabla \psi \right). \]

The above substitution for \(v_2\) has the advantage that it reduces space-time integral estimates to estimates on time slices and space times integral with higher order nonlinearities and fast dissipation. The latter is smaller by the initial smallness assumptions. To prove Lemma 2.5, we give the following lemma, where we use the equation (2.30) to bounded the term \(\int_0^T \partial_2 \psi \partial_2 v_2 (\partial^2_2 \psi)^2 dt\).

**Lemma 2.6.** Under the conditions of Lemma 2.5, then there holds

\[
\left| \int_0^T \partial_2 \psi \partial_2 v_2 (\partial^2_2 \psi)^2 dt \right| \leq C A_T^4 (1 + AT),
\]

where \(C\) is a positive constant independent of \(T\).

**Proof.** From Lemma 2.1 (2.30), the integration by parts, Hölder’s inequality and Sobolev embedding Theorem, we get

\[
\begin{align*}
\left| \int_0^T \partial_2 \psi \partial_2 v_2 (\partial^2_2 \psi)^2 dt \right| \\
= \left| \int_0^T \partial_2 \psi \partial_2 \left( \partial_t \psi + v \cdot \nabla \psi \right) (\partial^2_2 \psi)^2 dt \right| \\
= \left| \int_0^T \frac{1}{2} (\partial^2_2 \psi)^2 (\partial^2_2 \psi)^2 dx \right| + \left| \int_0^T \int_0^T \left[ -(\partial^2_2 \psi)^2 \partial_2^2 \partial_t \psi + \partial_2 \psi \partial_2 (v \cdot \nabla \psi) (\partial^2_2 \psi)^2 \right] dx dt \right| \\
\leq C\|\partial^2_2 \psi\|_{L^2_T(L^\infty(\mathbb{R}^2))} \|\partial^2_2 \psi\|_{L^2_T(L^2(\mathbb{R}^2))}.
\end{align*}
\]

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\[\begin{align*}
&+ \left| \int_0^T \left[ (\partial_2^2 \psi)^2 \partial_2^3 \psi (v \cdot \nabla \psi + v_2) + \partial_2^3 \psi \partial_2 (v \cdot \nabla \psi)(\partial_2^2 \psi)^2 \right] \, dx \, dt \right| \\
\leq & \ C \|\nabla \psi\|_{L^5_T(L^2(\mathbb{R}^2))}^4 + C \|\partial_2 \psi\|_{L^4_T(L^\infty(\mathbb{R}^2))}^4 \|\partial_3^2 \psi\|_{L^5_T(L^2(\mathbb{R}^2))}^4 \|\partial_2^2 (v \cdot \nabla \psi) - v \cdot \nabla \partial_2^2 \psi\|_{L^5_T(L^2(\mathbb{R}^2))} \\
&+ C \|\partial_2 \psi\|_{L^6_T(L^\infty(\mathbb{R}^2))} \|\partial_3^2 \psi\|_{L^5_T(L^2(\mathbb{R}^2))} \|\partial_2^2 v_2\|_{L^5_T(L^2(\mathbb{R}^2))} \\
&+ \left| \int_0^T \left\{ \frac{1}{2} \left[ (\partial_2 \psi)^2 v \cdot \nabla (\partial_2^2 \psi)^2 + v \cdot \nabla (\partial_2 \psi)^2 (\partial_2^2 \psi)^2 \right] \right\} \, dx \, dt \right| \\
&+ \left| \int_0^T \left\{ \partial_2 \psi \partial_2 v \cdot \nabla (\partial_2^3 \psi)^2 \right\} \, dx \, dt \right| \\
\leq & \ CA_T^4(1 + A_T),
\end{align*}\]

where we use the estimates

\[\begin{align*}
\|\partial_2^3 (v \cdot \nabla \psi) - v \cdot \nabla \partial_2^2 \psi\|_{L^5_T(L^2(\mathbb{R}^2))} \\
\leq & \ C \|\nabla^3 \psi\|_{L^2_T(L^\infty(\mathbb{R}^2))} \|\nabla \psi\|_{L^5_T(L^\infty(\mathbb{R}^2))} + C \|\nabla^2 v\|_{L^5_T(L^4(\mathbb{R}^2))} \|\nabla^2 \psi\|_{L^5_T(L^4(\mathbb{R}^2))} \\
&+ C \|\nabla^2 \psi\|_{L^2_T(L^\infty(\mathbb{R}^2))} \|\nabla^2 v\|_{L^2_T(L^2(\mathbb{R}^2))} \\
\leq & \ C \|\nabla v\|_{L^2_T(H^2(\mathbb{R}^2))} \|\nabla \psi\|_{L^5_T(L^\infty(\mathbb{R}^2))} \|\nabla^3 \psi\|_{L^5_T(L^2(\mathbb{R}^2))}^2.
\end{align*}\]

Proof of Lemma 2.5. From Lemmas 2.1, 2.2, 2.3, and Sobolev embedding Theorem, we get

\[\begin{align*}
&\left| \int_0^T \int \partial_2 v_2 (\partial_2^2 \psi)^2 \, dx \, dt \right| \\
= & \left| \int_0^T \int \partial_2 (\partial_1 \psi + v \cdot \nabla \psi)(\partial_2^2 \psi)^2 \, dx \, dt \right| \\
= & \left| \int \partial_2 \psi (\partial_2^2 \psi)^2 \, dx \right|_0^T + \left| \int_0^T \int \left\{-2\partial_2 \psi \partial_2^3 \psi \partial_2 \partial_1 \psi + \partial_2 (v \cdot \nabla \psi)(\partial_2 \psi)^2 \right\} \, dx \, dt \right| \\
\leq & \ C \|\partial_2 \psi\|_{L^5_T(L^\infty(\mathbb{R}^2))} \|\partial_2^2 \psi\|_{L^5_T(L^2(\mathbb{R}^2))}^2 \\
&+ \left| \int_0^T \int \left\{2\partial_2 \psi \partial_2^3 \psi (v \cdot \nabla \psi + v_2) + \partial_2 (v \cdot \nabla \psi)(\partial_2 \psi)^2 \right\} \, dx \, dt \right| \\
\leq & \ C \|\nabla \psi\|_{L^5_T(H^2(\mathbb{R}^2))}^2 + C \|\psi\|_{L^5_T(H^2(\mathbb{R}^2))} \|\nabla v\|_{L^5_T(L^2(\mathbb{R}^2))}^2 \\
&+ \left| \int_0^T \int \left\{2\partial_2 \psi \partial_2^3 \psi \partial_2^2 (v \cdot \nabla \psi) - v \cdot \nabla \partial_2^2 \psi + \partial_2 \psi v \cdot \nabla (\partial_2^3 \psi)^2 \\
+ v \cdot \nabla \partial_2 \psi (\partial_2^3 \psi)^2 + \partial_2 v \cdot \nabla (\partial_2^3 \psi)^2 \right\} \, dx \, dt \right| \\
\leq & \ CA_T^4(1 + A_T),
\end{align*}\]

where we use the estimate

\[\begin{align*}
&\left| \int_0^T \int \left\{2\partial_2 \psi \partial_2^3 \psi [\partial_2^2 (v \cdot \nabla \psi) - v \cdot \nabla \partial_2^2 \psi] + \partial_2 v \cdot \nabla (\partial_2^3 \psi)^2 \right\} \, dx \, dt \right| \\
\]
Proof. Using the Hölder's inequality, we have
\[
\|\hat{\psi}\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\hat{\psi}^2\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} \leq C \|\nabla \psi\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\partial_1 \nabla \psi\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\partial_2 \nabla \psi\|_{L^2_\nu(L^\infty(\mathbb{R}^2))}
\]

where 
\[
C = \left(\int_0^T \int_{\mathbb{R}^2} (\partial_1 \nabla \psi)(\partial_1 \nabla \psi) + (\partial_2 \nabla \psi)(\partial_2 \nabla \psi) + (\partial_1 \nabla \psi)(\partial_2 \nabla \psi)\right) dt.
\]

Lemma 3.1. Under the conditions in Lemma 3.3, then there holds
\[
\|\hat{\nabla}^1_{\{\xi|\geq 1\}}\|_{L^1_\nu(L^2)} + \|\hat{\nabla}^1_{\{\xi|\leq 1\}}\|_{L^1_\nu(L^2)} \leq CA_{1,T},
\]
\[
\|\hat{\nabla}^1_{\{\xi|\geq 1\}}\|_{L^1_\nu(L^2)} + \|\hat{\nabla}^1_{\{\xi|\leq 1\}}\|_{L^1_\nu(L^2)} \leq CA_{1,T},
\]
where \(C\) is a positive constant independent of \(T\).

Proof. Using the Hölder's inequality, we have
\[
\|\hat{\nabla}^1_{\{\xi|\geq 1\}}\|_{L^2_\nu(L^2)} + \|\hat{\nabla}^1_{\{\xi|\leq 1\}}\|_{L^2_\nu(L^2)} \leq C \left(\int_{\mathbb{R}^2} \|\nabla \psi\|_{L^2_\nu(L^2)} \left(\int_{\mathbb{R}^2} |\xi|^{-1} d\xi\right) \right)^{\frac{1}{2}}
\]
\[
\leq C \left(\int_{\mathbb{R}^2} \|\nabla \psi\|_{L^2_\nu(L^2)} \right)^{\frac{1}{2}}
\]
which finish the proof of (3.1). Using the Hölder's inequality, we have
\[
\|\hat{\nabla}^1_{\{\xi|\geq 1\}}\|_{L^2_\nu(L^2)} \leq C \left(\int_{\mathbb{R}^2} \|\nabla \psi\|_{L^2_\nu(L^2)} \right)^{\frac{1}{2}}
\]
which finish the proof of (3.2).

3 A priori estimates for \(A_{2,T}\)

Due to Remark 1.4, we need to estimate \(A_{2,T}\) as follows. At first, we estimate the easy one
\[
(\partial_1 \psi, v)_{1\{|\xi|\geq 1\}} \leq C \|\nabla \psi\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\nabla v\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\partial_1 \nabla v\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\partial_2 \nabla v\|_{L^2_\nu(L^\infty(\mathbb{R}^2))} + \|\partial_1 \partial_2 \psi\|_{L^2_\nu(L^\infty(\mathbb{R}^2))}
\]
which finish the proof of (3.1).

\[\square\]
To estimate the difficult part $(\partial_1 \psi, v)1_{|\xi|<1, |\xi_1|>|\xi|^2}$, we need obtain the following expressions of solutions. From (2.4), we have

\[
\begin{aligned}
\partial_t \psi + v_2 &= F, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t v_1 - \Delta v_1 - \partial_1 \partial_2 \psi &= G_1, \\
\partial_t v_2 - \Delta v_2 + \partial_2^2 \psi &= G_2, \\
\text{div} v &= 0, \\
(\psi, v)|_{t=0} &= (\psi_0, v_0),
\end{aligned}
\]

where

\[
F = -v \cdot \nabla \psi,
\]

\[
G = (G_1, G_2)^\top = -v \cdot \nabla v - \sum_{i,j=1}^2 \nabla (-\Delta)^{-1}[\partial_i v_j \partial_j v_i + \partial_i \partial_j (\partial_i \psi \partial_j \psi)] - \sum_{j=1}^2 \partial_j [\nabla \psi \partial_j \psi].
\]

Using the classical Fourier analysis method, we obtain that

\[
\hat{\psi}(t, \xi) = M_{11}(t, \xi)\hat{\psi}_0(\xi) + M_{12}(t, \xi)\hat{\psi}_{0,2}(\xi) + \int_0^t M_{11}(t-s, \xi)\hat{F}(\xi, s)ds + \int_0^t M_{12}(t-s, \xi)\hat{G}_2(\xi, s)ds,
\]

\[
\hat{\nu}_1(t, \xi) = M_{21}(t, \xi)\hat{\nu}_0(\xi) + M_{22}(t, \xi)\hat{\nu}_{0,1}(\xi) + \int_0^t M_{21}(t-s, \xi)\hat{F}(\xi, s)ds + \int_0^t M_{22}(t-s, \xi)\hat{G}_1(\xi, s)ds,
\]

\[
\hat{\nu}_2(t, \xi) = M_{31}(t, \xi)\hat{\nu}_0(\xi) + M_{32}(t, \xi)\hat{\nu}_{0,2}(\xi) + \int_0^t M_{31}(t-s, \xi)\hat{F}(\xi, s)ds + \int_0^t M_{32}(t-s, \xi)\hat{G}_2(\xi, s)ds,
\]

where

\[
M_{11}(t, \xi) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \quad M_{12}(t, \xi) = \frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-},
\]

\[
M_{21}(t, \xi) = \frac{\xi_1 \xi_2 e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \quad M_{22}(t, \xi) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-},
\]

\[
M_{31}(t, \xi) = -|\xi|^2 \frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \quad M_{32}(t, \xi) = M_{22}(t, \xi),
\]

\[
\lambda_\pm = \frac{|\xi|^2}{2} \pm \frac{i}{2} \sqrt{4\xi_1^2 - \xi^4}, \quad \text{when } |\xi_1| > |\xi|^2.
\]

**Lemma 3.2.** Under the conditions in Lemma 2.5, then there holds

\[
\left\| \hat{\partial}_1 \psi 1_{|\xi|<1, |\xi_1|>|\xi|^2} \right\|_{L_t^2(L_\xi^2)} + \left\| \hat{\partial}_1 \psi 1_{|\xi|<1, |\xi_1|>|\xi|^2} \right\|_{L_t^2(L_\xi^2)} \leq CA_{2,0} + CA_T^2,
\]

where $C$ is a positive constant independent of $T$.

**Proof.** When $\xi \in A = \{|\xi_1| \geq |\xi|^2\}$, from (3.7), (3.10), we get

\[
|M_{11}| = \left| e^{-\frac{|\xi|^2}{4} \frac{\sin(\sqrt{\xi_1^2 - \xi^4} t)}{\sqrt{\xi_1^2 - \xi^4}}} \right| \leq C e^{-\frac{|\xi|^2}{4}},
\]

\[
|M_{12}| = \left| e^{-\frac{|\xi|^2}{4} \frac{2 \sin(\sqrt{\xi_1^2 - \xi^4} t)}{\sqrt{\xi_1^2 - \xi^4}}} \right| \leq C|\xi|^2 e^{-\frac{|\xi|^2}{4}},
\]

then we can estimate the linear part as follows,

\[
\left\| M_{11} \hat{\partial}_1 \psi_0 \right\|_{L_t^2(L_\xi^2(A))} \leq C \left\| \xi_1 e^{-\frac{|\xi|^2}{2}} \hat{\psi}_0 \right\|_{L_t^2(L_\xi^2)} \leq CA_{2,0},
\]

\[
\left\| M_{12} \hat{\nu}_{0,2} \right\|_{L_t^2(L_\xi^2(A))} \leq C \left\| e^{-\frac{|\xi|^2}{4}} \hat{\nu}_{0,2} \right\|_{L_t^2(L_\xi^2)} \leq CA_{2,0},
\]

\[
\left\| M_{31} \hat{\nu}_0 \right\|_{L_t^2(L_\xi^2(A))} \leq C \left\| M_{32} \hat{\nu}_0 \right\|_{L_t^2(L_\xi^2(A))} \leq CA_{2,0},
\]

\[
\left\| M_{31} \hat{\nu}_0 \right\|_{L_t^2(L_\xi^2(A))} \leq C \left\| \hat{\nu}_{0,2} \right\|_{L_t^2(L_\xi^2)} \leq CA_{2,0}.
\]
From (3.12)-(3.13), we obtain
\[ \| \xi_1 M_{11}(t) 1_A \|_{L^2_t(L^2_{\xi_1})} \leq C \| \xi_1 e^{-\frac{L^2_\xi}{2} t} \|_{L^2_t(L^2_{\xi_1})} \leq C t^{-\frac{1}{2}} \| e^{-\frac{L^2_\xi}{2} t} \|_{L^2_t} \leq C t^{-\frac{1}{2}}, \] (3.16)
\[ \| \xi_1 M_{12}(t) 1_A \|_{L^2_t(L^2_{\xi_1})} \leq C \| e^{-\frac{L^2_\xi}{4} t} \|_{L^2_t(L^2_{\xi_1})} \leq C \| \xi_1 e^{-\frac{L^2_\xi}{4} t} \|_{L^2_t} \leq C t^{-\frac{1}{2}}. \] (3.17)

Using Young’s inequality and the interpolation inequality, we have
\[ \| \hat{F} \|_{L^p_t(L^p_\xi \xi_1)} \leq C \| \hat{\psi} \|_{L^q_t(L^q_\xi)} \| \nabla \hat{\psi} \|_{L^r_t(L^r_\xi)} \leq C \| \hat{\psi} \|_{L^q_t(L^q_\xi)} \| \xi_1 \|_{L^4_t(L^4_\xi)} \leq CA_{1,T} A_{2,T}, \] (3.18)
and combining the idea in (2.25),
\[ \| \hat{G}_2 \|_{L^p_t(L^p_\xi \xi_1)} \leq C \| \hat{\psi} \|_{L^q_t(L^q_\xi)} \| \nabla \hat{\psi} \|_{L^r_t(L^r_\xi)} + C \| \nabla_1 \psi \|_{L^q_t(L^q_{\xi_1})} \| \nabla_1 \|_{L^r_t(L^r_{\xi_1})} \] (3.19)
\[ \leq CA^2_{1,T}. \]

From (3.10)-(3.19), using the Riesz potential inequality ([18], Theorem 1, P119), we have
\[ \left\| 1_A \xi_1 \int_0^t M_{11}(t-s, \xi) \hat{F}(\xi, s) ds \right\|_{L^p_t(L^1_\xi)} \leq \left\| \int_0^t \| 1_A \xi_1 M_{11}(t-s, \xi) \|_{L^2_t(L^\infty_\xi)} \| \hat{F}(\xi, s) \|_{L^2_t(L^1_\xi)} ds \right\|_{L^p_t} \leq C \left\| \int_0^t (t-s)^{-\frac{1}{2}} \| \hat{F}(\xi, s) \|_{L^2_t(L^1_\xi)} ds \right\|_{L^p_t} \leq C \left\| \hat{F}(\xi, s) \right\|_{L^p_t(L^2_t(L^1_\xi))} \leq CA_{1,T} A_{2,T}, \] (3.20)
and
\[ \left\| 1_A \xi_1 \int_0^t M_{12}(t-s, \xi) \hat{G}_2(\xi, s) ds \right\|_{L^p_t(L^1_\xi)} \leq \left\| \int_0^t \| 1_A \xi_1 M_{12}(t-s, \xi) \|_{L^1_t(L^2_{\xi_1})} \| \hat{G}_2(\xi, s) \|_{L^\infty_t(L^2_{\xi_1})} ds \right\|_{L^p_t} \leq C \left\| \int_0^t (t-s)^{-\frac{1}{2}} \| \hat{G}_2(\xi, s) \|_{L^\infty_t(L^2_{\xi_1})} ds \right\|_{L^p_t} \leq C \left\| \hat{G}_2(\xi, s) \right\|_{L^p_t(L^2_t(L^1_{\xi_1}))} \leq CA^2_{1,T}. \] (3.21)

Combining (3.14)-(3.17) and (3.20)-(3.21), we get
\[ \left\| \nabla_1 \hat{\psi} 1_{\{ \xi \in [1, \xi_1 \cap I]\}} \right\|_{L^p_t(L^1_\xi)} \leq CA_{2,0} + CA^2_{T}. \] (3.22)

Similarly, we can estimate \( \| \hat{\psi} 1_{\{ \xi \in [1, \xi_1 \cap I]\}} \|_{L^2_t(L^1_\xi)} \) and omit the details. \( \square \)

From Lemmas 3.1 and 3.2 we can immediately obtain the following lemma.

**Lemma 3.3.** Under the conditions in Lemma 2.3 then there holds
\[ A_{2,T} \leq CA_{2,0} + CA_{1,T} + CA^2_T, \] (3.23)
where \( C \) is a positive constant independent of \( T \).
4 Proof of Theorem 1.1

Via the analysis in [14], we can get the following local existence result following now the standard argument.

**Theorem 4.1.** Assume that the initial data \((\psi_0, v_0)\) satisfy (1.4)-(1.5), then there exists \(T_0 > 0\) such that the system (1.3) has a unique local solution \((\psi, v, \nabla p) \in E_{T_0}^2\) on \([0, T_0]\).

**Proof of Theorem 1.1.** Theorem 4.1 implies that the system (1.3) has a unique local strong solution \((\psi, v, \nabla p)\) on \([0, T^*]\), where \([0, T^*]\) is the maximal existence interval of the above solution. The goal of this section is to prove that \(T^* = \infty\) provided that the initial data \((\psi_0, v_0)\) satisfy (1.6).

Assume that \((\psi, v, \nabla p)\) is the unique local strong solution of (1.3) on \([0, T^*]\), and satisfies \((\psi, v, \nabla p) \in E_{T}^2\), for all \(T \in (0, T^*)\). From (2.14) and (3.23), we have
\[
A_T^2 \leq CA_0^2 + CA_0^3(1 + A_T^2)
\] (4.1)
for all \(T \in (0, T^*)\). If the initial data \((\psi_0, v_0)\) satisfy (1.6), where \(c_0\) satisfies
\[
C\sqrt{2C_0}(1 + 2C_0^2) \leq \frac{1}{2},
\] (4.2)
one can easily obtain
\[
A_T^2 \leq 2CA_0^2\quad \text{for all } T \in (0, T^*).
\] (4.3)
Thus, we have that \(T^* = \infty\), and hence (1.7) holds. From (2.3), we see that \(\nabla p \in C([0, \infty); H^1)\) and (1.8) holds. This finish proof of Theorem 1.1.

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