On the Hodge theory of the additive middle convolution

by

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Abstract

We compute the behaviour of Hodge data under additive middle convolution for irreducible variations of polarized complex Hodge structures on punctured complex affine lines.

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Introduction

In previous work of Sabbah with one of the authors [4], the effect of the additive middle convolution $\text{MC}_x(V) = V \star L_x$ of a complex polarized Hodge module $V$ on $\mathbb{A}^1$ with a Kummer module $L_x$ on various local and global Hodge data was determined. This leads to an analog of Katz’ algorithm for irreducible rigid local systems [7] in the context of Hodge modules.

It is the aim of this work to extend these results to the case of the middle convolution $V \star L$ (cf. Section 1) of two irreducible and non-constant complex polarized Hodge modules on $\mathbb{A}^1$. It turns out that, to a large extent, the general case can be reduced to the middle convolution with Kummer modules treated in [4].

In Section 2 Theorem 2.1 the global Hodge numbers of tensor products $V \otimes L$ (the degrees of the associated Hodge bundles) are determined, generalizing [4], Prop. 2.3.2. We are indebted to Claude Sabbah for communicating the proof of Theorem 2.1 to us. This result is important in many applications where convolution is applied iteratively in combination with tensor operations (cf. [7], [4], [3]).

In Section 3 we determine the local Hodge data of the vanishing and nearby cycles (cf. Section 1 and [4] for these notions) at the finite singularities of a convolution $V \star L$ (Theorem 3.3). As in [4], the main

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tool for doing this is Saito’s version of the Thom-Sebastiani theorem (cf. [4], Theorem 3.2.3, and its corrigendum [5], where a proof of the Thom-Sebastiani result is provided).

In Section 4 the global Hodge numbers of $V \star L$ are determined. The main observation is that the middle convolution $V \star L$ of an irreducible and nontrivial Hodge module $V$ with a generic Kummer module $L$ is \textit{parabolically rigid}, meaning that the associated parabolic cohomology group

$$H^1_{\text{par}}(V \star L) = H^1(\mathbb{P}^1, j_*\mathcal{H}^{-1}(\mathcal{R}\mathcal{H}(V \star L)))$$

vanishes (where $\mathcal{R}\mathcal{H}(V \star L)$ is the perverse sheaf associated to $V \star L$ via Riemann-Hilbert correspondence and $j$ is the projective embedding of $\mathbb{A}^1$). Using the Riemann-Roch theorem, a formula for the Hodge numbers of $H^1_{\text{par}}(V \star L)$ involving local and global data was given in [4], Proposition 2.3.3. Hence the vanishing of $H^1_{\text{par}}(V \star L)$ gives a method to compute the global Hodge numbers of $V \star L$.

The remaining local Hodge data at $\infty$ of $V \star L$ are determined in Section 5. For this, we make use of hypergeometric Hodge modules with prescribed local behaviour at $\infty$ and reduce the general case to the convolution of these. We believe that a more conceptual proof of these results may be given in the context of irregular Hodge filtrations on twistor modules and their behaviour under Fourier-Laplace transformation (cf. [6]).

In a forthcoming work, the authors prove similar results for the multiplicative convolution (also called Hadamard product).

§1. Preliminary results

Following [4], we review the basic notions of middle convolutions introduced by Katz [7], in the frame of holonomic $\mathcal{D}$-modules. Let $s : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the addition map and let $M,N$ be holonomic $\mathcal{D}(\mathbb{A}^1)$-modules. The \textit{additive $\ast$-convolution} $M \ast N$ of $M$ and $N$ is the object $s_+(M \boxtimes N) \in \mathcal{D}^{b}(\mathbb{A}^1)$. The \textit{additive $\dagger$-convolution} can be defined as $M \dagger N = D(DM \ast DN)$, where $D$ is the duality functor $D_{\text{hol}}^b(\mathcal{D}(\mathbb{A}^1)) \to D_{\text{hol}}^b(\mathcal{D}(\mathbb{A}^1))$. It can also expressed as $s_!(M \boxtimes N)$, if $s_! := DS_+D$ denotes the adjoint by duality of $s_+$, cf. [4] (under the Riemann Hilbert correspondence, the functor $+$ corresponds to the derived $\ast$-functor and $\dagger$ corresponds to $\ast$, explaining the notion).

Let us choose a projectivization $\tilde{s} : X \to \mathbb{A}^1$ of $s$, and let $j : \mathbb{A}^1 \times \mathbb{A}^1 \hookrightarrow X$ denote the open inclusion. Since $\tilde{s}$ naturally commutes with duality, we have $\tilde{s}_! = \tilde{s}_+$ and $s_! = s_+ \circ j_!$. Since there is a natural morphism $j_! \to j_+$ in $D^b_{\text{hol}}(\mathcal{D}X)$, we get a functorial morphism $s_!(M \boxtimes N) \to s_+ (M \boxtimes N)$, that is, $M \ast N \to M \ast N$, in $D^b_{\text{hol}}(\mathcal{D}(\mathbb{A}^1))$. Let $P$ be the full subcategory of $\text{Mod}_{\text{hol}}(\mathbb{A}^1)$ consisting of holonomic $\mathcal{D}(\mathbb{A}^1)$-modules $N$ such that for all holonomic $\mathcal{D}(\mathbb{A}^1)$-modules $M$ both types of convolutions $N \ast M$ and $N \dagger M$ are again holonomic.
**Definition 1.0.1.** For $N$ in $P$ and $M$ holonomic, the *middle convolution* $M \star_{\text{mid}} N$ is defined as the image of $M \star_1 N \to M \star_\ast N$ in $\text{Mod}_{\text{hol}}(\mathcal{D}(\mathbb{A}^1))$. For simplicity we often set $M \star N := M \star_{\text{mid}} N$. As explained in [4], Section 3.3, this notion extends to the category of complex polarized Hodge modules on $\mathbb{A}^1$, using the results of [11] and [12]. If $M$ is smooth on $\mathbb{A}^1 \setminus x$ ($x = \{x_1, \ldots, x_r\} \cup \{\infty\}$) and if $N$ is smooth on $\mathbb{A}^1 \setminus y$ ($y = \{y_1, \ldots, y_s\} \cup \{\infty\}$) then $M \star N$ as well as the other types of convolutions are smooth on $\mathbb{A}^1 \setminus x \star y$, where

$$x \star y = \{x_i + y_j | i = 1, \ldots, r; j = 1, \ldots, s\} \cup \{\infty\}.$$ 

The following result follows from the Riemann-Hilbert correspondence and [7], Cor. 2.6.10 and Cor. 2.6.17:

**Lemma 1.1.** (i) If $N$ is irreducible such that its isomorphism class is not translation invariant then $N$ has the property $P$.

(ii) If $N$ and $M$ are in $P$ then $N \star M$ again is in $P$.

Let $W$ be a complex polarized Hodge module on the complex affine line $\mathbb{A}^1$ which is smooth on $\mathbb{A}^1 \setminus \{z_1, \ldots, z_k\}$. The local system on $\mathbb{A}^1 \setminus \{z_1, \ldots, z_k\}$ which is underlying $W$ is denoted as $\mathcal{W}$. The perverse sheaf on $\mathbb{A}^1$ associated to $W$ via the de Rham functor is denoted by $\mathcal{R}\mathcal{H}(W)$ and we view the $i$-th parabolic cohomology group

$$H^i_{\text{par}}(W) := H^i(\mathbb{P}^1, j_*\mathcal{H}^{-1}(\mathcal{R}\mathcal{H}(W)))$$

to be equipped with its natural Hodge structure.

Throughout the article we will work with Hodge modules $V, L, \delta_x, L_\chi$ which are as follows:

**Assumption 1.2.** (i) We assume that $V = (V, F^\bullet V)$ is a complex polarized Hodge module on the complex affine line $\mathbb{A}^1$ which is the intermediate (minimal) extension of an irreducible nonconstant variation of polarized complex Hodge structures on $\mathbb{A}^1 \setminus x$ (where $x = \{x_1, \ldots, x_r, \infty\} \subset \mathbb{P}^1(\mathbb{C})$). In this situation we sometimes set $x_{r+1} = \infty$.

(ii) Let $L = (L, F^\bullet L)$ be another Hodge module of the same kind which is the minimal extension of a variation of polarized complex Hodge structures on $\mathbb{A}^1 \setminus y$ (where $y = \{y_1, \ldots, y_s, \infty\}$).

(iii) For a point $x \in \mathbb{A}^1$ we write $\delta_x$ for the Hodge module which corresponds to the rank-one skyscraper sheaf on $\mathbb{A}^1$ supported in $x$, having trivial Hodge filtration (so that $h^0(H^0(\mathbb{A}^1, \delta_x)) = 1$).

(iv) As in [4], Section 3.3, we write $L_\chi$ for the Hodge module with trivial Hodge filtration belonging to the Kummer sheaf with residues $(\mu, 1 - \mu)$ ($\mu \in (0, 1)$) such that $\chi = e^{-2\pi i \mu}$, having singular points at $(0, \infty)$. We call $L_\chi$ generic if the monodromy eigenvalues of all sheaves different from $L_\chi$ and $L_{\chi - 1}$ involved in our arguments are different from $\chi^{\pm 1}$.

For the following notions and stated results we refer to [4], Section 1.2 and Sections 2.2, 2.3: on the one hand, one has *global Hodge data* $\delta^p(V)$ given by the degrees of the Hodge bundles. On the other hand,
one has \textit{local Hodge data}: For each point \( x \in \{ x_1, \ldots, x_{r+1} \} \) and each \( \lambda \in S^1 \) one has the notion \( \psi_{x,\lambda}(V) \) of the generalized \( \lambda \)-eigenspace of the nearby cycles \( \psi_x(V) \). We will also use the corresponding notion of vanishing cycles \( \varphi_{x,\lambda}(V) \). These spaces are mixed Hodge structures with associated nilpotent monodromy operator, derived from the local monodromy, which imposes an associated weight filtration \( W \). One has the notion of \( l \)-primitive vectors \( P_{x,\lambda}(V) \) with respect to the Lefschetz decomposition of \( \varphi_{x,\lambda}(V) \).

For \( l \in \mathbb{N} \) we define \( l \)-primitive local Hodge numbers as follows:

\[
\nu_{x,\lambda,l}(V) = \nu_{x,a,l}(V) := \dim \text{gr}_F P_{x,\lambda}(V),
\]

where \( a \in \mathbb{R} \cap [0,1) \) such that \( \lambda = \exp(2\pi i (-a)) \). We set

\[
\nu_{x,a}(V) := \sum_{l \geq 0} \sum_{k=0}^l \nu_{x,a,l}(V) \quad \text{and} \quad \nu_{x,a,\text{prim}} := \sum_{l \geq 0} \nu_{x,a,l}(V)
\]

as well as

\[
h^p(V) := \nu_{x,a,l}(V) := \sum_{a \in [0,1)} \nu_{x,a}(V) \quad \text{and} \quad \nu_{x,\not=0}(V) := \sum_{a \in (0,1)} \nu_{x,a}(V).
\]

One has corresponding notions for vanishing cycles

\[
\mu_{x,\lambda,l}(V) = \mu_{x,a,l}(V) := \dim \text{gr}_F P_{x,\lambda}(V),
\]

and

\[
\mu_{x,a}(V) := \sum_{l \geq 0} \sum_{k=0}^l \mu_{x,a,l}(V) \quad \text{and} \quad \mu_{x,a,\text{prim}} := \sum_{l \geq 0} \mu_{x,a,l}(V).
\]

These notions are related as follows (cf. loc.cit.):

\[
\mu_{x,a,l}(V) = \nu_{x,a,l}(V) \quad \text{if} \quad a \not= 0 \quad \text{and} \quad \mu_{x,0,l}(V) = \nu_{x,0,l+1}(V).
\]

Additionally to \cite{4}, we will use the following further local Hodge numbers, simplifying the computations below:

\textbf{Definition. 1.3.} Let

\[
\omega_{x}^p(V) := \nu_{x}^p(V) - \nu_{x,0,\text{prim}}^p(V) = \nu_{x,\not=0}^p(V) + \mu_{x,0}^{p+1}(V),
\]
cf. [3], (2.2.5*), and
\[
\omega_{ss,x}^p(V) := \nu_{x,\neq 0}^p(V) \\
\omega_{u,x}^p(V) := \mu_{x,0}^{p+1}(V) \\
\omega_{\neq \infty}^p(V) := \sum_{x \in (x \times \infty)} \omega_x^p(V) \\
\omega_{\neq \infty}(V) := \sum_p \omega_{\neq \infty}^p(V) \\
\omega(V) := \sum_{x \in \mathcal{X}} \omega_x^p(V) \\
\omega(V) := \sum_p \omega(V) \\
\kappa_x^p(V) := \nu_{x,0,\text{prim}}^p(V).
\]

Let \( J^p(a,l)(V) \) denote a mixed \( \mathbb{C} \)-Hodge structure which is associated to a nilpotent orbit belonging to a monodromy operator whose Jordan form is a single Jordan block of size \( l \) and having residue \( a \in (0,1) \) such that \( \nu_{a,l-1}^p(V) = 1 \).

**Remark. 1.4.** One has
\[
\psi_{x,j}(V) \simeq \bigoplus_{(1,a,l)} J^i(a,l)_{x,j,a,l-1}(V)
\]
(note that we use complex coefficients, so any pure Hodge structure decomposes into one-dimensional summands).

In the following, let \( j : \mathbb{A}^1 \setminus x \hookrightarrow \mathbb{P}^1 \) be the natural inclusion. Using our above notion of \( \omega^j(V) \) we obtain:

**Proposition. 1.5.**
\[
h^p(H_{\text{par}}^1(V)) = \delta^{p-1}(V) - \delta^p(V) - h^p(V) - h^{p-1}(V) + \omega^{p-1}(V).
\]

**Proof.** By [4], Proposition 2.3.3, we have
\[
h^p(H_{\text{par}}^1(V)) = \delta^{p-1}(V) - \delta^p(V) - h^p(V) - \nu_{\infty,0,\text{prim}}^{p-1}(V) + \sum_{j=1}^r (\nu_{x,j,\neq 0}^{p-1}(V) + \mu_{x,j,0}^p(V))
\]
\[
= \delta^{p-1}(V) - \delta^p(V) - h^p(V) - \nu_{\infty,0,\text{prim}}^{p-1}(V) + \sum_{j=1}^r \omega_{x,j}^{p-1}(V)
\]
\[
= \delta^{p-1}(V) - \delta^p(V) - h^p(V) - h^{p-1}(V) + \omega^{p-1}(V),
\]
using
\[
\nu_{\infty,0,\text{prim}}^{p-1}(V) = \nu_{\infty}^{p-1}(V) - \omega_{\infty}^{p-1}(V) \quad \text{and} \quad h^{p-1}(V) = \nu_{\infty}^{p-1}(V).
\]
\[\square\]
Remark. 1.6. The construction of nearby and vanishing cycles and their basic invariants is carried out for minimal extensions in [4], Section 2.2. The general case can be reduced with this at hand to the case of mixed Hodge-modules with punctual support. The Hodge invariants of these are as follows: Let $V$ be a Hodge module supported on a closed point $x \hookrightarrow \mathbb{A}^1$ (i.e., a complex polarized Hodge structure $V$ placed at $x$) and let $i_+V$ its extension to $\mathbb{A}^1$. By the usual triangle which connects nearby and vanishing cycles, the nearby cycles of $i_+V$ are zero, while the vanishing cycles $\varphi_x(i_+V)$ can be identified with $V$. Note that the natural monodromy operation on $\varphi_x(i_+V)$ is trivial, hence $\mu_{x,a}(i_+V) = 0$ for $a \neq 0$ and $\mu_{x,0}(i_+V) = h^p(V)$.

§2. Degrees of tensor products

We will proceed using the notions of the previous section. The following theorem is a generalization of [4], Proposition 2.3.2. We are indebted to Claude Sabbah for communicating its proof to us.

Theorem. 2.1.

$$\delta^j(V \otimes L) = \sum_p \delta^{j-p}(V) h^p(L) + \sum_p h^{j-p}(V) \delta^p(L) + \sum_{x \in x \cap y} o_x^j(V \otimes L),$$

where

$$o_x^j(V \otimes L) := \sum_p \sum_{a+b \geq 1} V_{x,a}(V) \nu_{x,b}^{j-p}(L).$$

The result depends on the following two lemmata. Let $V^0, L^0, (V \otimes L)^0$ denote the Deligne extensions of $V, L, V \otimes L$ (resp.). There is also $(V \otimes L)^0$. We have the following Hodge filtrations:

- The tensor product filtration $F^j(V^0 \otimes L^0) := \sum_p F^{j-p} V^0 \otimes F^p L^0$.
- Since $V \otimes L$ is a variation of Hodge structures on the punctured $\mathbb{P}^1$, with Hodge filtration equal to the tensor product filtration, we obtain the filtration $F^j(V \otimes L)^0$.

Let $D = x \cup y$ denote the reduced divisor away from which $V$ and $L$ are variations of Hodge structures. A local computation (without using Hodge theory) shows that there are $F$-filtered inclusions

$$(V \otimes L)^0(-D) \subset V^0 \otimes L^0 \subset (V \otimes L)^0$$

which are equalities away from $D$.

Lemma. 2.2. The inclusion $V^0 \otimes L^0 \subset (V \otimes L)^0$ is strict with respect to $F^\ast$.

If this lemma is proved, we find that, for each $\ell$, there is an injective morphism

$$\bigoplus_p gr_{F^\ell}^{j-p} V^0 \otimes gr^p L^0 \hookrightarrow gr_{F}^\ell(V \otimes L)^0$$
whose cokernel is supported on $D$ and has dimension $\dim \gr_F^\ell \left( (V \otimes L)^0 / V^0 \otimes L^0 \right)$. As a consequence, we find

$$\delta^\ell(V \otimes L) = \sum_p (\delta^{p-\ell} V \cdot h^p L + h^{p-\ell} V \cdot \delta^p L) + \dim \gr_F^\ell \left( (V \otimes L)^0 / V^0 \otimes L^0 \right).$$

Lemma. 2.3. We have

$$\dim \gr_F^\ell \left( (V \otimes L)^0 / V^0 \otimes L^0 \right) = \sum_{x \in D} \delta^\ell(V \otimes L).$$

Note that $\delta^\ell(V \otimes L) = 0$ if $x \in D \setminus (x \cap y)$, so the sum is on $x \in x \cap y$.

Proof of Lemma 2.2. The result is local, so the setting is on a small disc with coordinate $t$ around one of the singularities of the variations of Hodge structures. The local computation mentioned above shows that there is an exact sequence

$$(*) \quad 0 \rightarrow V^0 \otimes L^0 \rightarrow (V \otimes L)^0 \rightarrow t^{-1} \bigoplus_{\alpha, \beta \in [0,1]} \gr^\alpha V \otimes \gr^\beta L \rightarrow 0.$$

Moreover, we have $t(V \otimes L)^0 \subset V^0 \otimes L^0$ and

$$(V^0 \otimes L^0) / t(V \otimes L)^0 \cong \bigoplus_{\alpha, \beta \in [0,1]} \gr^\alpha V \otimes \gr^\beta L,$$

giving rise to the exact sequence

$$(**) \quad 0 \rightarrow \bigoplus_{\alpha, \beta \in [0,1], \alpha + \beta < 1} \gr^\alpha V \otimes \gr^\beta L \rightarrow (V \otimes L)^0 / t(V \otimes L)^0 \rightarrow t^{-1} \bigoplus_{\alpha, \beta \in [0,1], \alpha + \beta \geq 1} \gr^\alpha V \otimes \gr^\beta L \rightarrow 0.$$

We also have the following Hodge filtration:

- The tensor product filtration on any $\gr^\alpha V \otimes \gr^\beta L$ considered above.

For the sake of simplicity, we will set $\text{Gr}^0 V := V^0 / tV^0$ (and similarly for $L$ and $V \otimes L$). This space is endowed with the induced filtration $F^* \text{Gr}^0 V$. There is also a filtration $E^* \text{Gr}^0 V$ indexed by $\alpha \in [0,1)$ induced by the decreasing $V$-filtration on $\text{Gr}^0 V$, so that $\text{gr}^\alpha_E \text{Gr}^0 V = \gr^\alpha V$. The Hodge filtration $F^* \gr^\alpha V$ is equal to the filtration induced by $F^* \text{Gr}^0 V$ on $\text{gr}^\alpha_E \text{Gr}^0 V$. We have a natural morphism

$$(2.0.1) \quad \text{Gr}^0 V \otimes \text{Gr}^0 L \rightarrow \text{Gr}^0(V \otimes L)$$

defined as follows:

$$\text{Gr}^0 V \otimes \text{Gr}^0 L = \frac{(V^0 \otimes L^0)}{t(V^0 \otimes L^0)} \rightarrow \frac{V^0 \otimes L^0}{t(V \otimes L)^0} \rightarrow \frac{(V \otimes L)^0}{t(V \otimes L)^0} \rightarrow \text{Gr}^0(V \otimes L).$$
This morphism is compatible with the $F$-filtrations on each term. Grading with respect to $E^\bullet$ gives a morphism
\[ \bigoplus_{\alpha, \beta \in [0, 1)} \text{gr}^\alpha V \otimes \text{gr}^\beta L \longrightarrow \bigoplus_{\gamma \in [0, 1)} \text{gr}^\gamma (V \otimes L). \]
The later morphism is also $F$-filtered, and is moreover a morphism of mixed Hodge structures. It is then $F$-strict. Therefore, (2.0.1) is also $F$-strict. Arguing similarly, we find that for any $k \geq 1$ the natural morphism
\[ (V^0 / t^k V^0) \otimes (L^0 / t^k L^0) \longrightarrow (V \otimes L)^0 / t^k (V \otimes L)^0 \]
is strictly $F$-filtered.

Let us set $\tilde{V}^0 = \varprojlim_k (V^0 / t^k V^0)$, endowed with $F^p \tilde{V}^0 = \varprojlim_k F^p (V^0 / t^k V^0)$. We have $(\tilde{V}^0, F^\bullet \tilde{V}^0) = \hat{\Theta} \otimes (V^0, F^\bullet V^0)$. The previous result implies that the inclusion $\tilde{V}^0 \otimes \hat{L}^0 \hookrightarrow (\tilde{V} \otimes L)^0$ is strictly $F$-filtered, hence, regarding the previous morphism as an inclusion,
\[ F^p (\tilde{V}^0 \otimes \hat{L}^0) = F^p (\tilde{V} \otimes L)^0 \cap (\tilde{V}^0 \otimes \hat{L}^0), \quad \forall p, \]
that is,
\[ \hat{\Theta} \otimes F^p (V^0 \otimes L^0) = \hat{\Theta} \otimes F^p (V \otimes L)^0 \cap \hat{\Theta} \otimes (V^0 \otimes L^0), \quad \forall p. \]

By faithful flatness of $\hat{\Theta}$ over $\Theta$, we conclude that
\[ F^p (V^0 \otimes L^0) = F^p (V \otimes L)^0 \cap (V^0 \otimes L^0), \quad \forall p. \]

\[ \square \]

Proof of Lemma 2.3: We consider the composed $F$-filtered morphism
\[ (2.0.2) \quad \text{Gr}^1 (V \otimes L) \longrightarrow t(V \otimes L)^0 \longrightarrow \frac{t(V^0 \otimes L^0)}{t(V^0 \otimes L^0)} = \text{Gr}^0 V \otimes \text{Gr}^0 L. \]

After grading with respect to the $E^\bullet$ filtration, it becomes
\[ \bigoplus_{\gamma \in [1, 2)} \text{gr}^\gamma (V \otimes L) \longrightarrow \bigoplus_{\alpha, \beta \in [0, 1)} \text{gr}^\alpha V \otimes \text{gr}^\beta L, \]
and has image
\[ \bigoplus_{\alpha, \beta \in [0, 1)} \text{gr}^\alpha V \otimes \text{gr}^\beta L. \]

Being a morphism of mixed Hodge structures, it is also $F$-strict, and so is (2.0.2). Since the isomorphism $t : V^0 \rightarrow V^1$ is $F$-strict (and similarly for $L$ and $V \otimes L$), the isomorphism
\[ t : \frac{(V \otimes L)^0}{V^0 \otimes L^0} \longrightarrow \frac{t(V \otimes L)^0}{t(V^0 \otimes L^0)} \]
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is also $F$-strict. As a consequence,

$$\dim \text{gr}_F (V \otimes L)^0 = \sum_{\alpha, \beta \in (0,1)} \sum_{\alpha + \beta \geq 1} \dim \text{gr}_F^{\ell-p} \text{gr}_V^{\alpha} V \cdot \dim \text{gr}_F^p \text{gr}_L^{\beta}.$$ 

$\square$

§3. Transformation of local Hodge data away from $\infty$ under middle convolution

Recall that in general, the convolution $V \ast L$ is neither irreducible nor an intermediate extension anymore (cf. Assumption [12]). The following definition gives the largest factor of $V \ast L$ which is an intermediate extension:

**Definition.** 3.1. Let $U := \mathbb{A}^1 \smallsetminus x \ast y$ be the smooth locus of $V \ast L$. Define $\tilde{V} \ast L$ to be the intermediate extension of $V \ast L|_U$ to $\mathbb{A}^1$.

For $t \in \mathbb{A}^1$ let $d_t : \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto t - x$, and write $L(t - x)$ for $d_t^\ast L$. The following result clarifies the relation between $V \ast L$ and $\tilde{V} \ast L$:

**Theorem.** 3.2. One has a short exact sequence of Hodge modules

$$0 \to \tilde{V} \ast L \to V \ast L \to H \to 0,$$

where

$$H = \begin{cases} 
\delta_c(-p-1) & \text{if} \ \exists p \in \mathbb{Z}, c \in \mathbb{A}^1 : V(p) \simeq L^\vee(c-x) \\
0 & \text{otherwise}
\end{cases}.$$

If $H \neq 0$ then $p, c$ are uniquely determined.

**Proof.** As in the proof of [7], Proposition 2.6.9, one finds that $\mathcal{H}^n(\mathcal{R}\mathcal{H}(V \ast L))$ vanishes outside $n = -1, 0$ and that if $\mathcal{H}^0(\mathcal{R}\mathcal{H}(V \ast L)) \neq 0$, then there exists a unique point $c \in \mathbb{A}$ such that $\mathcal{H}^0(\mathcal{R}\mathcal{H}(V \ast L))$ is a rank-one skyscraper sheaf with support at $c$ such that the stalk at $c$ is isomorphic to the Tate-twisted space of invariants $\text{Hom}(\mathcal{V}, \mathcal{L}(c-x)^\vee(-1))$. A necessary condition for the non-vanishing of $\text{Hom}(\mathcal{V}, \mathcal{L}(c-x)^\vee(-1))$ is that one has an isomorphism of local systems $\mathcal{V} \simeq \mathcal{L}(c-x)^\vee$. Since irreducible VPCHS are determined up to a Tate-twist by their local systems, there exists a unique $p$ such that the Tate-twist $\mathcal{V}(p)$ becomes VPCHS-isomorphic to $\mathcal{L}(c-x)^\vee$ in this case. This implies that, taking Hodge structures into account, the stalk $\text{Hom}(\mathcal{V}, \mathcal{L}(c-x)^\vee(-1))$ has weight $p + 1$. Since $V \ast L$ is the image of $V \ast L$ and since $\mathcal{H}^0(\mathcal{R}\mathcal{H}(V \ast L))$ maps isomorphically onto its image inside $\mathcal{R}\mathcal{H}(V \ast L)$, the claim follows. $\square$
Theorem 3.3. If \( t \in \mathbb{A}^1 \setminus \mathbf{x} \ast \mathbf{y} \) (cf. Def. 1.0.1) then
\[
\begin{align*}
\mu_2^x(V \ast L) &= \delta^{i-1}(V \otimes L(t - x)) - \delta^i(V \otimes L(t - x)) - \mu_1^x(V \otimes L(t - x)) - \mu_1^x(V \otimes L(t - x)) + \omega^{i-1}(V \otimes L(t - x)).
\end{align*}
\]

Proof. For \( t \in \mathbb{A}^1 \), let \( d_i(y) = \{ t - y_1, \ldots, t - y_s \} \) and let \( j : \mathbb{A}^1 \setminus (\mathbf{x} \cup d_i(y)) \to \mathbb{P}^1 \) be the natural inclusion. Since \( t \notin \mathbf{x} \ast \mathbf{y} \) one has \( \mathcal{R}H(V) \otimes \mathcal{R}H(L(t - x)) \simeq j_*(\mathcal{V} \otimes \mathcal{L}(t - x)) \) and hence
\[
(V \ast L)_t = (V \ast L)_t = H^1(\mathbb{P}^1, j_*(\mathcal{V} \otimes \mathcal{L}(t - x))).
\]

The claim follows now from Proposition 1.3. \( \square \)

The following result determines the local Hodge data of the vanishing cycles:

Theorem 3.4. Let \( \lambda = \exp(-2\pi i a) (a \in (0, 1]) \) be a fixed element of the unit circle \( S^1 \) and let \( \lambda_1, \lambda_2 \) be variable elements in \( S^1 \) with \( \lambda = \lambda_1 \lambda_2 \). For such \( \lambda_i \in S^1 \) \( (i = 1, 2) \), let \( a_i \in (0, 1] \) with \( \lambda_i = \exp(-2\pi i a_i) \). If \( t \in \mathbf{x} \ast \mathbf{y}, t \neq \infty \), then
\[
\mu_{2, a}(V \ast L) = \mu^p_2(\varphi_{1, \lambda}(V \ast L)) = \sum_{x_i + y_j = t} \left( \sum_{a_1 + a_2 = a} \sum_{l+k=p} \nu^l(\varphi_{x_i, \lambda_1}(V))\nu^k(\varphi_{y_j, \lambda_2}(L)) + \sum_{a_1 + a_2 = 1+a} \sum_{l+k=p} \nu^l(\varphi_{x_i, \lambda_1}(V))\nu^k(\varphi_{y_j, \lambda_2}(L)) \right),
\]

where the expression \( \nu^p \) abbreviates \( \text{dim} \text{gr}^p_F \).

Proof. By Saito’s version of the Thom-Sebastiani theorem (cf. [4] Theorem 3.2.3 and its erratum) one knows that, for all \( (x_i, t - y_j) \) as in the theorem,
\[
\text{gr}^p_F(\varphi_{x_i, t-y_j}, \lambda)(V \otimes L) = \bigoplus_{a_1+a_2=a} \bigoplus_{l+k=p} \text{gr}^l_F(\varphi_{x_i, \lambda_1}(V)) \otimes \text{gr}^k_F(\varphi_{y_j, \lambda_2}(L)) \oplus \bigoplus_{a_1+a_2=1+a} \bigoplus_{l+k=p} \text{gr}^l_F(\varphi_{x_i, \lambda_1}(V)) \otimes \text{gr}^k_F(\varphi_{y_j, \lambda_2}(L)).
\]

Moreover, the support of the vanishing cycles in the fibre over \( t \) is the union of these \((x_i, t - y_j)\). Since middle convolution is afterwards formed via higher direct image along the compactified (hence proper) \( \text{pr}_2 \) and since formation of vanishing cycles is compatible with higher direct images along projective morphisms the claim follows. \( \square \)

Using \( \omega_{\neq \infty, a}(V) = \sum_{x_i \neq \infty} \omega^{p}_{x_i, a}(V) \) and \( \omega_{\neq \infty}(V) = \sum_{a} \omega^{p}_{\neq \infty, a}(V) \) one obtains:

Corollary 3.5. Let \( a \in [0, 1) \). Then the following holds:
(i) For \( t \neq \infty \)
\[
\omega^p_{t,a}(V * L) = \sum_{x_i+y_j=t} \sum_{a_1+a_2=a} \sum_{l+k=p-1} \sum_{a_1+a_2=1+a} \omega^j_{x_i,a_1}(V)\omega^k_{y_j,a_2}(L) + \\
\sum_{a_1+a_2=1+a} \sum_{l+k=p} \omega^j_{x_i,a_1}(V)\omega^k_{y_j,a_2}(L)
\]
\[
\omega^p_{\neq \infty}(V * L) = \sum_{i+j=p-1} \sum_{a_1+a_2=1} \omega^j_{\neq \infty,a_1}(V)\omega^k_{\neq \infty,a_2}(L) + \\
\sum_{i+j=p-1} \sum_{a_1+a_2=1} \omega^j_{\neq \infty,a_1}(V)\omega^k_{\neq \infty,a_2}(L)
\]
and
\[
\sum_{p \leq t} \omega^p_{\neq \infty}(V * L) = \sum_{i+j \leq t-1} \omega^j_{\neq \infty}(V)\omega^k_{\neq \infty}(L) + \sum_{a_1+a_2=1} \omega^j_{\neq \infty,a_1}(V)\omega^k_{\neq \infty,a_2}(L).
\]

(ii) If \( L_\chi \) is generic with respect to \( L \) and \( V^{\neq}L \), then
\[
h^p(V * (L * L_\chi)) - h^p((V^{\neq}L) * L_\chi) = h^p(H^0(\mathcal{R}^0(V * L))) = \omega^p_{\neq \infty}(V * L) - \omega^p_{\neq \infty}(V^{\neq}L).
\]

**Proof.** Let us first treat the case where \( \lambda = 1 \), equivalent to \( a = 0 \) (note that inside Theorem 3.4 the residues \( a \) are contained in \((0,1]\), whereas in the rest of the paper \( a \in [0,1) \), hence we have to adapt our notation to this situation). By Theorem 3.4
\[
\omega^p_{t,0}(V * L) = \mu^{p+1}_{t,0}(V * L)
\]
\[
= \sum_{x_i+y_j=t} \left( \sum_{a_1+a_2=0} \sum_{l+k=p} \mu^j_{x_i,a_1}(V)\mu^k_{y_j,a_2}(L) + \sum_{a_1+a_2=0} \sum_{l+k=p+1} \mu^j_{x_i,a_1}(V)\mu^k_{y_j,a_2}(L) \right)
\]
\[
= \sum_{x_i+y_j=t} \left( \sum_{a_1+a_2=0} \sum_{l+k=p+1} \mu^j_{x_i,a_1}(V)\mu^k_{y_j,a_2}(L) + \sum_{a_1+a_2=1} \mu^j_{x_i,a_1}(V)\mu^k_{y_j,a_2}(L) \right)
\]
\[
= \sum_{x_i+y_j=t} \left( \sum_{a_1+a_2=0} \sum_{l+k=p+1} \omega^{j-1}_{x_i,0}(V)\omega^{k-1}_{y_j,0}(L) + \sum_{a_1+a_2=1} \omega^{j}_{x_i,a_1}(V)\omega^{k}_{y_j,a_2}(L) \right)
\]
\[
= \sum_{x_i+y_j=t} \left( \sum_{a_1+a_2=0} \sum_{l+k=p+1} \omega^{j}_{x_i,0}(V)\omega^{k}_{y_j,0}(L) + \sum_{a_1+a_2=1} \omega^{j}_{x_i,a_1}(V)\omega^{k}_{y_j,a_2}(L) \right).
\]
Note that in the above sum we switch from $a_i \in (0, 1]$ to $a_i \in [0, 1]$ so that the case $a_1 + a_2 = 2$ now corresponds to $a_1 + a_2 = 0$ (and $l + k = p + 1$). Analogously we get for $0 < a < 1$

$$\omega^P_{l,a}(V \ast L) = \mu^P_{l,a}(V \ast L)$$

$$= \sum_{x_1+y_j=t} \left( \sum_{a_1+a_2=a} \sum_{l+k=p-1} \omega^I_{x,j,a_1}(V)\omega^k_{y,a_2}(L) \right) + \sum_{a_1+a_2=1+a} \sum_{l+k=p} \omega^I_{x,a_1}(V)\omega^k_{y,a_2}(L).$$

Hence the first claim follows. In the case where $V \ast L = V \check{\ast}L$, the formula given in (ii) holds obviously true. By Theorem 3.2, if $V \ast L \neq V \check{\ast}L$, then

$$(3.0.2) \quad V \ast L = V \check{\ast}L \oplus \delta_c(-q - 1)$$

and $V(q) \simeq L^\vee(c - x)$. This implies the first equation in (ii). Since $L_\chi$ is generic,

$$(V \check{\ast}L) \ast L_\chi = (V \check{\ast}L) \check{\ast}L_\chi.$$ 

By associativity of the middle convolution (under the assumption that $L, V, L_\chi$ are irreducible and non-trivial)

$$V \ast (L \ast L_\chi) = (V \ast L) \ast L_\chi$$

$$= (V \check{\ast}L \oplus \delta_c(-q - 1)) \ast L_\chi$$

$$= (V \check{\ast}L) \ast L_\chi \oplus L_\chi(x - c)(-q - 1).$$

Therefore

$$h^P(V \ast (L \ast L_\chi)) - h^P((V \check{\ast}L) \ast L_\chi) = h^P(L_\chi(x - c)(-q - 1)) = \delta_{p,q+1},$$

where $\delta_{i,j}$ denotes the usual Kronecker-delta. On the other hand, Rem. 1.6 and (3.0.2) imply that

$$\omega^{P-1}_{\neq \infty}(V \ast L) - \omega^{P-1}_{\neq \infty}(V \check{\ast}L) = \mu^{P}_{c,0}(V \check{\ast}L \oplus \delta_c(-q - 1)) - \mu^{P}_{c,0}(V \check{\ast}L) = \mu^{P}_{c,0}(\delta(-q - 1)) = \delta_{p,q+1}. $$

\[\Box\]

\[\text{§4. Transformation of global Hodge data under middle convolution}\]

The following result transforms a general convolution with a Kummer Hodge module to the standard situation, considered in [H], Assumption 1.1.2:

**Theorem. 4.1.** Let $V \ast L_\chi$ be viewed as Hodge module on $\mathbb{P}^1$ by taking the minimal extension using the canonical map $\mathbb{A}^1 \to \mathbb{P}^1$. Let $x_1, \ldots, x_r \in \mathbb{A}^1(\mathbb{C})$ denote the finite singularities of $V$ and let $\phi_t : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be the Möbius transformation that exchanges $0$ and $\infty$ by inverting the coordinate $t$ of $\mathbb{A}^1$. If $0$ is a smooth point of $V$ then $V \ast L_\chi$ can be obtained as

$$V \ast L_\chi = \phi_t^*(((\phi_t^*(V \otimes L_\chi) \ast L_\chi) \otimes L_{\chi-1}).$$
Let now $V$ along a polynomial map given by

\[(4.0.4)\]

where notion below also for other sheaves and other polynomial maps, so that as above, the Hodge module

\[\text{Proposition. 4.2.}\]

at products of local sections (where the isomorphisms are seen as isomorphisms of the restrictions of Hodge modules to the respective smooth parts):

\[V \ast L_x|_{U} \simeq R^1\mathbb{P}^2_+(j_+(V(x) \otimes L_x(t - x))),\]

where $V(x)$ is the Hodge module on $W$ obtained by pulling back $V|_{U_x}$ along the $x$-projection and where, as above, the Hodge module $L_x(t - x)$ is obtained by pulling back $L_x$ along the map $t - x$. We use a similar notion below also for other sheaves and other polynomial maps, so that $V(f(x, t))$ denotes a pullback of $V$ along a polynomial map given by $f(x, t) \in \mathbb{C}[x, t]$. Then the following isomorphisms follow by looking at products of local sections (where the isomorphisms are seen as isomorphisms of the restrictions of Hodge modules to the respective smooth parts):

\[L_x(t) \simeq L_{x^{-1}}(1/t)\]

and

\[(4.0.1)\]

\[L_x(1/t - x) \simeq L_x(1/t) \otimes L_x(1 - xt)\]

\[(4.0.2)\]

\[\simeq L_{x^{-1}}(t) \otimes L_x(x) \otimes L_x(1/x - t).\]

Let now $U' := \mathbb{A}^1 \setminus \{0, 1/x_1, \ldots, 1/x_r\}$ and consider $\phi_t : U' \to U, t \mapsto 1/t$. Then

\[(4.0.3)\]

\[\phi_t^+(V \ast L_x)|_{U'} \simeq R^1\mathbb{P}^2_+(j_+(V(x) \otimes L_x(1/t - x)))|_{U'},\]

\[(4.0.4)\]

\[\simeq R^1\mathbb{P}^2_+(j_+(V(x) \otimes L_{x^{-1}}(t) \otimes L_x(x) \otimes L_x(1/x - t)))|_{U'},\]

\[(4.0.5)\]

\[\simeq R^1\mathbb{P}^2_+(j_+(V(1/x) \otimes L_x(1/x) \otimes L_{x^{-1}}(t) \otimes L_x(x - t)))|_{U'},\]

\[(4.0.6)\]

\[\simeq R^1\mathbb{P}^2_+(\tilde{j}_+(V \otimes L_x)(1/x) \otimes L_x(x - t)))|_{U'} \otimes L_{x^{-1}}(t)|_{U'},\]

\[(4.0.7)\]

\[\simeq (\phi_t^+(V \otimes L_x) \ast L_x)|_{U'} \otimes L_{x^{-1}}(-t)|_{U'},\]

where we use the following notions and arguments: Throughout we work over the largest smooth locus of the Hodge modules involved. Eq. (4.0.3) holds by the discussion at the beginning of the proof. Eq. (4.0.4) follows from Eq. (4.0.2). In Eq. (4.0.5) we invert fibrewise the coordinate $x$ and $\tilde{j}$ denotes the inclusion of the image of $U$ under this inversion to $\mathbb{P}^1 \times U$. Eq. (4.0.6) follows from the projection formula and Eq. (4.0.7) holds by definition.

**Proposition 4.2.** Let $L_x$ be generic (cf. Assumption [12]). Then $V \ast L_x = V \ast L_x$ is parabolically rigid, meaning that $H^1_{\text{par}}(V \ast L_x) = 0$.

**Proof.** For $t \in \mathbb{A}^1 \setminus x \ast y$,

\[
\text{rk}(V \ast L_x) \leq \omega(V \otimes L_x(t - x)) - 2 \text{rk}(V) \\
= \omega_{\neq \infty}(V),
\]

Proof. For $t \in \mathbb{A}^1 \setminus x \ast y$,
since $L_\chi$ is generic (cf. [2], Proposition 1.2.1 and [7], Corollary 3.3.7). Hence,

\[
\text{rk}(H^1_{\text{par}}(V \ast L_\chi)) = \omega_\infty(V \ast L_\chi) + \omega_{\neq \infty}(V \ast L_\chi) - 2 \text{rk}(V \ast L_\chi) \\
= \omega_\infty(V \ast L_\chi) + \omega_{\neq \infty}(V) - 2 \text{rk}(V \ast L_\chi) \\
= \omega_\infty(V \ast L_\chi) - \text{rk}(V \ast L_\chi) \leq 0,
\]

cf. [2], Proposition 1.2.1(ii), for the equality $\omega_{\neq \infty}(V) = \omega_{\neq \infty}(V \ast L_\chi)$.

The following result was independently proven using different methods by Nicolas Martin in his Dissertation [8], Thm. 6.3.1:

**Proposition 4.3.** (i) Let $\mu \sim 1$ (meaning that $1 - \mu$ is chosen generically and small enough). Then

\[
\begin{align*}
h^i(V \ast L_\chi) &= \delta_{i-1}(V) - \delta^i(V) + \omega_{\neq \infty}^{i-1}(V) \\
\delta^i(V \ast L_\chi) &= \delta^i(V) - \omega_{\neq \infty}^i(V) \\
\omega_{\neq \infty}^i(V \ast L_\chi) &= \omega_{\neq \infty}^{i-1}(V) + \omega_{\neq \infty}^i(V) \\
\omega_\infty^i(V \ast L_\chi) &= h^i(V \ast L_\chi)
\end{align*}
\]

(ii) Let $V \ast L_\chi \neq \delta_x$ for any $x \in \mathbb{A}^1$ (equivalent to $V$ being not isomorphic to a translate of the dual of $L_\chi$). Then

\[
\nu^i_{\infty,a,l}(V) = \begin{cases} 
\nu_{\infty,1-\mu,l+1}^i(V \ast L_\chi), & 0 = a \\
\nu_{\infty,a+1-\mu,l}^i(V \ast L_\chi), & 0 < a < \mu \\
\nu_{\infty,0,l-1}^i(V \ast L_\chi), & a = \mu, l > 0 \\
\nu_{\infty,a-\mu,l}^i(V \ast L_\chi), & a > \mu.
\end{cases}
\]

Moreover, the only other possibly non zero nearby cycle data at infinity of $V \ast L_\chi$ are of the form $\nu_{\infty,1-\mu,0}^i(V \ast L_\chi)$. If $\mu \sim 1$ then the formula simplifies to

\[
\nu^i_{\infty,a,l}(V) = \begin{cases} 
\nu_{\infty,1-\mu,l+1}^i(V \ast L_\chi), & a = 0 \\
\nu_{\infty,a+1-\mu,l}^i(V \ast L_\chi), & a \neq 0.
\end{cases}
\]
Proof. By Theorem 3.3 and Theorem 2.1

\[ h^i(V \ast L_\chi) = h^i(V \ast L_\chi) \]
\[ = \delta^{i-1}(V \otimes L_\chi(t-x)) - \delta^i(V \otimes L_\chi(t-x)) - h^i(V \otimes L_\chi(t-x)) \]
\[ - h^{i-1}(V \otimes L_\chi(t-x)) + \omega_{\chi,L_\chi}^i(V \otimes L_\chi(t-x)) \]
\[ = (\delta^{i-1}(V) - h^{i-1}(V) + 0) - (\delta^i(V) - h^i(V) + 0) \]
\[ - h^{i-1}(V) + \omega_{\chi,L_\chi}^i(V \otimes L_\chi(t-x)) \]
\[ = \delta^{i-1}(V) - h^{i-1}(V) - \delta^i(V) \]
\[ - h^{i-1}(V) + \omega_{\chi,L_\chi}^i(V \otimes L_\chi(t-x)) \]
\[ = \delta^{i-1}(V) - \delta^i(V) + \omega_{\chi,L_\chi}^{i-1}(V), \]

which is the first formula in (i).

The fixed space under the local monodromy at \( \infty \) is trivial (otherwise, the last formula in the proof of Proposition 4.2 reads \( \text{rk}(H^1_{\text{par}}(V \ast L_\chi)) < 0 \), a contradiction). This implies \( \omega_{\chi,L_\chi}^i(V \ast L_\chi) = h^i(V \ast L_\chi) \) which is the fourth equation in (i). By Proposition 4.2, \( V \ast L_\chi \) is parabolically rigid. Therefore,

\[ 0 = h^i(H^1_{\text{par}}(V \ast L_\chi)) \]
\[ = \delta^{i-1}(V \ast L_\chi) - \delta^i(V \ast L_\chi) - h^i(V \ast L_\chi) - \omega_{\chi,L_\chi}^{i-1}(V \ast L_\chi) \]
\[ = \delta^{i-1}(V \ast L_\chi) - \delta^i(V \ast L_\chi) - h^i(V \ast L_\chi) + \omega_{\chi,L_\chi}^{i-1}(V \ast L_\chi). \]

Therefore

\[ (4.0.8) \]
\[ \delta^i(V \ast L_\chi) - \delta^{i-1}(V \ast L_\chi) = -h^i(V \ast L_\chi) + \omega_{\chi,L_\chi}^{i-1}(V \ast L_\chi). \]

Using \( \mu \sim 1 \) and the second equality in Corollary 3.3 we obtain

\[ \omega_{\chi,L_\chi}^{i-1}(V \ast L_\chi) = \omega_{\check{\chi},L_\chi}^{i-1}(V) + \omega_{\check{\chi}_{\ast},L_\chi}^{i-1}(V), \]

establishing the third equality in (i). Hence

\[ \delta^i(V \ast L_\chi) - \delta^{i-1}(V \ast L_\chi) = -h^i(V \ast L_\chi) + \omega_{\check{\chi},L_\chi}^{i-1}(V \ast L_\chi) \]
\[ = \delta^i(V) - \delta^{i-1}(V) - \omega_{\check{\chi}_{\ast},\chi}^{i-1}(V) + \omega_{\check{\chi}_{\ast},\chi}^{i-2}(V) \]
\[ = \delta^i(V) - \delta^{i-1}(V) - \omega_{\check{\chi}_{\ast},\chi}^{i-1}(V) + \omega_{\check{\chi}_{\ast},\chi}^{i-2}(V). \]

Summing up we get the second formula of (i):

\[ \delta^i(V \ast L_\chi) = \delta^i(V) - \omega_{\check{\chi},L_\chi}^{i-1}(V). \]
Claim (ii) follows by rewriting $V \star L_\chi$ via Theorem 3.3 as

\[ V \star L_\chi = \phi_\chi^+ (\phi_\chi^+ (V \otimes L_\chi) \star L_\chi) \otimes L_{\chi-1} \]

assuming via a translation that 0 is not a singularity of $V$. Thus the singularity $\infty$ of $V$ becomes the finite singularity 0 of $\phi_\chi^+ (V \otimes L_\chi)$ and we can apply Formula (4.0.11) in order to prove the first claim. Note that $\phi_\chi^+ (V \otimes L_\chi)$ has scalar monodromy $\exp(-2\pi i \mu)$ at $\infty$, which is called the standard situation for the middle convolution with the Kummer sheaf $L_\chi$ in [4]. The first Formula in (ii) follows now from [4], Theorem 3.1.2(2). By Formula (4.0.9) and Theorem 4.4, the only other possibly non zero nearby cycle data at infinity are of the form $\nu_{\infty,1-\mu,0} (V \star L_\chi)$.

\[ \Box \]

**Theorem 4.4.** Let $V \star L \neq 0$. Then

\[ \delta^i (V \star L) = \sum_{i} (\omega_{\neq \infty}^i (V) \delta^{i-1-i}(L) + \delta^i (V) \omega_{\neq \infty}^{i-1-i}(L)) + \sum_{j} \delta^j (V) \delta^{i-1-j}(L) - \sum_{j} \delta^j (V) \delta^{i-j}(L) \]

\[ + \sum_{a+b \geq 1} \omega_{a,b} (V) \omega_{\neq \infty}^{i-1-i}(L) + \sum_{a+b \geq 1} \omega_{a,b} (V) \omega_{\infty}^{i-1-i}(L). \]

\[ \begin{proof} \]

Let $L_\chi$ be generic and $\mu \sim 1$. By Proposition 4.3(i)

\[ \sum_{i \leq l} h^i ((V \star L) \star L_\chi) = -\delta^i (V \star L) + \sum_{i \leq l} \omega_{\neq \infty}^{i-1}(V \star L). \]

By the transformation of residues under convolution, described by Proposition 4.3(ii), the sum of a residue of $L \star L_\chi$ at $\infty$ and a residue of $V$ at $\infty$ is not an integer. Hence the nearby cycles of $V \otimes (L \star L_\chi)(t-x)$ at $\infty$ coincide with the vanishing cycles. Therefore

\[ \omega_{\infty}^{i-1}(V \otimes (L \star L_\chi)(t-x)) = h^{i-1}(V \otimes (L \star L_\chi)(t-x)). \]

This and Theorem 3.3 imply

\[ h^i (V \star (L \star L_\chi)) = \delta^{i-1}(V \otimes (L \star L_\chi)(t-x)) - \delta^i (V \otimes (L \star L_\chi)(t-x)) - h^i (V \otimes (L \star L_\chi)(t-x)) \]

\[ = \delta^{i-1}(V \otimes (L \star L_\chi)(t-x)) - \delta^i (V \otimes (L \star L_\chi)(t-x)) - h^i (V \otimes (L \star L_\chi)(t-x)) + \omega_{\neq \infty}^{i-1}(V \otimes (L \star L_\chi)(t-x)). \]

\[ (4.0.10) \]
One has

\[ h^i(V \otimes (L \ast L) (t - x)) - \omega_{\neq \infty}^{i - 1} (V \otimes (L \ast L) (t - x)) \]

\[ = \sum_j h^j(V) h^{i-j} ((L \ast L) (t - x)) - \omega_{\neq \infty}^{i-j} (V \otimes (L \ast L) (t - x)) \]

\[ = \sum_j h^j(V) h^{i-j} ((L \ast L) (t - x)) - \sum_j h^j(V) \omega_{\neq \infty}^{i-j} ((L \ast L) (t - x)) - \sum_j \omega_{\neq \infty}^j (V) h^{i-j-1} (L \ast L) \]

\[ = \sum_j h^j(V) (h^{i-j} ((L \ast L) (t - x)) - \omega_{\neq \infty}^{i-j} ((L \ast L) (t - x))) - \sum_j \omega_{\neq \infty}^j (V) h^{i-j-1} (L \ast L) \]

\[ = \sum_j h^j(V) (\delta^i - \delta^{i-1} (L) + \omega_{\neq \infty}^{i-j} (L) - \omega_{\neq \infty}^{i-j-2} (L)) \]

(4.0.11)

\[ - \sum_j \omega_{\neq \infty}^j (V) (\delta^{i-j-2} (L) - \delta^{i-j-1} (L) + \omega_{\neq \infty}^{i-j-2} (L)), \]

where we used the following arguments: the first equality uses the usual equality for tensor products, the second follows from basic properties of nearby cycles of tensor products, the third equality is a reorganisation of the sum and in the last equation we use the first and the third equation in Proposition 4.3(i).

Summing up (4.0.10) yields

\[ \sum_{i \leq l} h^i (V \ast (L \ast L)) = -\delta^i (V \otimes (L \ast L) (t - x)) + \sum_i h^i(V) (\delta^{i-i} (L) - \omega_{\neq \infty}^{i-i} (L)) \]

\[ - \sum_i \omega_{\neq \infty}^i (V) \delta^{i-i} (L) + \sum_{i+k \leq l} \omega_{\neq \infty}^k (V) \omega_{\neq \infty}^{i-k} (L) \]

\[ = -\delta^i (V \otimes (L \ast L) (t - x)) + \sum_i h^i(V) (\delta^{i-i} (L) - \omega_{\neq \infty}^{i-i} (L)) \]

\[ - \sum_i \omega_{\neq \infty}^i (V) \delta^{i-i} (L) + \sum_{i \leq l-1} \omega_{\neq \infty}^i (V \ast L) \]

\[ - \sum_j \sum_{a+b \neq 1} \omega_{\neq \infty}^j (V) \omega_{\neq \infty}^{l-1-j} (L), \]

(4.0.12)

where we use (4.0.11) for the first equality and Corollary 3.3(i) for the second.

On the other hand, by the first equality in Proposition 4.3(i),

\[ \sum_{i \leq l} h^i ((V \ast L) \ast L) = -\delta^i (V \ast L) + \sum_{i \leq l} \omega_{\neq \infty}^{i-i} (V \ast L). \]
Thus by Corollary 3.5 (ii) and Eq. (4.0.12)

\[ 0 = \sum_{i \leq l} h^i(V \star (L \bar{L} L_\chi)) - \sum_{i \leq l} h^i((V \bar{L} L) * L_\chi) - \sum_{i \leq l} \omega^i_{\neq \infty}(V * L) + \sum_{i \leq l} \omega^i_{\neq \infty}(V \bar{L} L) \]

\[ = -\delta^i(V \otimes (L * L_\chi)(t - x)) + \sum_{i \leq l} h^i(V)(\delta^{l-i}(L) - \omega^i_{\neq \infty,l}(L)) - \sum_{i \leq l} \omega^i_{\neq \infty}(V)\delta^{l-i-1}(L) \]

\[ + \sum_{i \leq l-1} \omega^i_{\neq \infty}(V * L) - \sum_{i} \sum_{a+b \geq 1} \omega^i_{\neq \infty,a}(V)\omega^i_{\neq \infty,b}(L) \]

\[ + \delta^i(V \bar{L} L) - \sum_{i \leq l-1} \omega^i_{\neq \infty}(V \bar{L} L) - \sum_{i \leq l-1} \omega^i_{\neq \infty}(V * L) + \sum_{i \leq l-1} \omega^i_{\neq \infty}(V \bar{L} L) \]

\[ = \delta^i(V \bar{L} L) - \delta^i(V \otimes (L * L_\chi)(t - x)) + \sum_{i \leq l} h^i(V)(\delta^{l-i}(L) - \omega^i_{\neq \infty,l}(L)) - \sum_{i \leq l} \omega^i_{\neq \infty}(V)\delta^{l-i-1}(L) - \sum_{i \leq l} \omega^i_{\neq \infty,a}(V)\omega^i_{\neq \infty,b}(L). \]

By Theorem 2.1 and Proposition 4.3,

\[ \delta^i(V \otimes (L * L_\chi)(t - x)) = \sum_{i} (\delta^i(V)h^{l-i}(L * L_\chi) + h^i(V)\delta^{l-i}(L * L_\chi)) + \omega^i_{\neq \infty}(V \otimes (L * L_\chi)) \]

\[ = \sum_{i} \delta^i(V)(\delta^{l-i-1}(L) - \delta^{l-i}(L) + \omega^i_{\neq \infty,l}(L)) + \sum_{i \leq l} h^i(V)(\delta^{l-i}(L) - \omega^i_{\neq \infty,l}(L)) \]

\[ + \sum_{i \leq l} \sum_{a+b \geq 1} \omega^i_{\neq \infty,a}(V)\omega^i_{\neq \infty,b}(L). \]

Taking the previous two equations together one has

\[ \delta^i(V \bar{L} L) = \sum_{i} \delta^i(V)(\delta^{l-i-1}(L) - \delta^{l-i}(L) + \omega^i_{\neq \infty,l}(L)) + \sum_{i \leq l} h^i(V)(\delta^{l-i}(L) - \omega^i_{\neq \infty,l}(L)) \]

\[ - \sum_{i \leq l} \sum_{a+b \geq 1} \omega^i_{\neq \infty,a}(V)\omega^i_{\neq \infty,b}(L) - \sum_{i \leq l} h^i(V)(\delta^{l-i}(L) - \omega^i_{\neq \infty,l}(L)) - \sum_{i \leq l} \omega^i_{\neq \infty}(V)\delta^{l-i-1}(L) \]

\[ + \sum_{i \leq l} \omega^i_{\neq \infty}(V)\delta^{l-i-1}(L) \]

\[ = \sum_{i}(\omega^i_{\neq \infty}(V)\delta^{l-i-1}(L) + \delta^i(V)\omega^i_{\neq \infty,l}(L)) + \sum_{i \leq l} \delta^i(V)\delta^{l-i-1}(L) - \sum_{i \leq l} \delta^i(V)\delta^{l-i}(L) \]

\[ + \sum_{i \leq l} \sum_{a+b \geq 1} \omega^i_{\neq \infty,a}(V)\omega^i_{\neq \infty,b}(L) + \sum_{i \leq l} \omega^i_{\neq \infty}(V)\omega^i_{\neq \infty,L}(L), \]

as claimed. \qed
\section{Transformation of local Hodge data at $\infty$ under middle convolution}

In the following, the objects $V, L, M$ satisfy the conditions in Assumption 1.2.

\textbf{Theorem 5.1.} Let $\varepsilon_l := h^{l+1}(H^0(\mathcal{H}^0(\mathcal{H}(V \ast L))))$. Then

\[ h^{l+1}(H^1_{\text{par}}(V \ast L)) + \kappa^l_\infty(V \ast L) + \varepsilon_l - \kappa^l_\infty(V \otimes L(t-x)) = \]

\[ \sum_i (h^i(H^1_{\text{par}}(V)) + \kappa^{i-1}_\infty(V))(h^{l+1-i}(H^1_{\text{par}}(L)) + \kappa^{l-i}_\infty(L)) \]

\textbf{Proof.} By Theorem 1.5

\[ h^{l+1}(H^1_{\text{par}}(V \ast L)) = \delta^l(V \ast L) - \delta^{l+1}(V \ast L) - h^{l+1}(V \ast L) - \kappa^l_\infty(V \ast L) + \omega^l_\infty(V \ast L). \]

Using subsequently Theorem 3.3, Theorem 2.1 and Theorem 1.5 we get

\[ h^{l+1}(V \ast L) = \delta^l(V \otimes L(t-x)) - \delta^{l+1}(V \otimes L(t-x)) - h^{l+1}(V \otimes L(t-x)) - \kappa^l_\infty(V \otimes L(t-x)) \]

\[ + \omega^l_\infty(V \otimes L(t-x)) \]

\[ = \sum_i (\delta^{i-1}(V) - \delta^i(V) - h^i(V) + \omega^{i-1}_\infty(V))h^{l+1-i}(L) \]

\[ + \sum_i h^i(V)(\delta^{l-i}(L) - \delta^{l+1-i}(L) + \omega^{l-i}_\infty(L)) \]

\[ + \omega^l_\infty(V \otimes L(t-x)) - \omega^{l+1}_\infty(V \otimes L(t-x)) - \kappa^l_\infty(V \otimes L(t-x)) \]

\[ = \sum_i (h^i(H^1_{\text{par}}(V)) + \kappa^{i-1}_\infty(V))(h^{l+1-i}(L) \]

\[ + \sum_i h^i(V)(\delta^{l-i}(L) - \delta^{l+1-i}(L) + \omega^{l-i}_\infty(L)) \]

\[ + \omega^l_\infty(V \otimes L(t-x)) - \omega^{l+1}_\infty(V \otimes L(t-x)) - \kappa^l_\infty(V \otimes L(t-x)). \]

By Theorem 4.4

\[ \delta^l(V \ast L) = \sum_i (h^i(V) + \kappa^{i-1}_\infty(V) + h^i(H^1_{\text{par}}(V)))(\delta^{l-i}(L) + \sum_i \delta^{i-1}(V)\omega^{l-i}_\infty(L) \]

\[ + \omega^l_\infty(V \otimes L(t-x)) + \sum_{a+b \geq 1} \omega^i_{\infty,a}(V)\omega^{l-1-i}_{\infty,b}(L). \]

Inserting the last two equations into the first equation we obtain
\[ h_{\text{par}}^{l+1}(V \star L) = \sum_i (\kappa_{\infty}^{i-1}(V) + h^i(\mathcal{H}_{\text{par}}^1(V))) (\delta^{l-i}(L) - \delta^{l+1-i}(L) + h^{l+1-i}(L)) + \]
\[ \sum_i (\delta^{i-1}(V) - \delta^i(V) - h^i(V)) \omega^{l-i}_{\neq \infty}(L) + \]
\[ \kappa_{\infty}^{l}(V \otimes L(t - x)) - \kappa_{\infty}^{l}(V \star L) + \omega_{\neq \infty}^{l}(V \star L) \]
\[ - \sum_i \sum_{a+b \geq 1} \omega^{l-i}_{\#\infty,a}(V) \omega^{l-i}_{\#\infty,b}(L) + \sum_i \sum_{a+b \geq 1} \omega^{l-i}_{\#\infty,a}(V) \omega^{l-i}_{\#\infty,b}(L). \]

By Corollary 3.5(ii),

\[ \omega^{l}_{\neq \infty}(V \star L) - \omega^{l}_{\neq \infty}(V \star L) = \varepsilon_l. \]

Moreover, by Corollary 3.5(i)

\[ \omega^{l}_{\#\infty}(V \star L) = \sum_{i+j=l} \sum_{a_1 + a_2 \geq 1} \omega^{l}_{\#\infty,a_1}(V) \omega^{l}_{\#\infty,a_2}(L) + \sum_{i+j=l-1} \sum_{a_1 + a_2 < 1} \omega^{l}_{\#\infty,a_1}(V) \omega^{l}_{\#\infty,a_2}(L). \]
Inserting this into the above equation yields

\[
 h^{i+1}(H^1_{\text{par}}(V \bar{x} L)) = \sum_i (\kappa^{-1}(V) + h^i(H^1_{\text{par}}(V)))(\delta^{-i}(L) - \delta^{i+1-i}(L) + h^{i+1-i}(L)) + \\
 \sum_i (\delta^{-i}(V) - \delta^i(V) - h^i(V))\omega^{i-i}_{\neq \infty}(L) + \\
 \kappa^1_{\infty}(V \otimes L(t - x)) - \kappa^1_{\infty}(V \bar{x} L) - \varepsilon_l + \sum_i \omega^{i-1}_{\neq \infty}(V)\omega^{i-i}_{\neq \infty}(L)
\]

\[
= \sum_i (\kappa^{-1}(V) + h^i(H^1_{\text{par}}(V)))(\delta^{-i}(L) - \delta^{i+1-i}(L) + h^{i+1-i}(L)) + \\
\sum_i (\delta^{-i}(V) - \delta^i(V) - h^i(V) + \omega^{i-1}_{\neq \infty}(V))\omega^{i-i}_{\neq \infty}(L) + \\
\kappa^1_{\infty}(V \otimes L(t - x)) - \kappa^1_{\infty}(V \bar{x} L) - \varepsilon_l
\]

\[
= \sum_i (h^i(H^1_{\text{par}}(V)) + \kappa^{i-1}(V))(\delta^{-i}(L) - \delta^{i+1-i}(L) - h^{i+1-i}(L) + \omega^{i-i}_{\neq \infty}(L)) + \\
\kappa^1_{\infty}(V \otimes L(t - x)) - \kappa^1_{\infty}(V \bar{x} L) - \varepsilon_l
\]

\[
= \sum_i (h^i(H^1_{\text{par}}(V)) + \kappa^{i-1}(V))(h^{i+1-i}(H^1_{\text{par}}(L)) + \kappa^{i-i}(L)) + \kappa^1_{\infty}(V \otimes L(t - x)) - \kappa^1_{\infty}(V \bar{x} L) - \varepsilon_l,
\]

as claimed.

The following result is also obtained in the Dissertation of Nicolas Martin [3], Thm. 6.3.1:

**Corollary. 5.2.** Let \( V \ast L_x \neq \delta_x \) for any \( x \in k^1 \). Then

\[
\nu^i_{\infty, 1-\mu, 0}(V \ast L_x) = h^i(H^1_{\text{par}}(V)).
\]

**Proof.** Since \( V \bar{x} L_x = V \ast L_x \) we have

\[
(V \ast L_x) \bar{x} L_x = V(-1).
\]
Hence, since $L_\chi$ is parabolically rigid, it follows from Theorem 5.1 that
\[ h^{i+1}(H^1_{\text{par}}(V(-1))) + \kappa_{\infty}^i(V(-1)) - \kappa_{\infty}^i((V \ast L_\chi) \otimes L_\chi(t - x)) = 0. \]

By definition of $\kappa_{\infty}$ and the first formula in Proposition 4.3 (ii)
\[ \kappa_{\infty}^i((V \ast L_\chi) \otimes L_\chi(t - x)) = \nu_{\infty, 1 - \mu, 0}^i(V \ast L_\chi) = \nu_{\infty, 1 - \mu, 0}^i(V \ast L_\chi) + \nu_{\infty}^{i-1}(V \ast L_\chi) = \nu_{\infty, 1 - \mu, 0}^i(V \ast L_\chi) + \kappa_{\infty}^i(V(-1)). \]

Hence we obtain
\[ \nu_{\infty, 1 - \mu, 0}^i(V \ast L_\chi) = h^{i+1}(H^1_{\text{par}}(V(-1))) = h^i(H^1_{\text{par}}(V)). \]

**Remark. 5.3.** The above theorem may also be derived from the well known formula

\[ H^*(\mathbb{A}^1, V \ast L) = H^*(\mathbb{A}^1, V \otimes H^*(\mathbb{A}^1, L)) \]

with
\[
h^{l+1}(H^*(\mathbb{A}^1, V) \otimes H^*(\mathbb{A}^1, L)) = \sum_i (h^i(H^1_{\text{par}}(V)) + \kappa_{\infty}^{i-1}(V))(h^{l+1-i}(H^1_{\text{par}}(L)) + \kappa_{\infty}^{l-i}(L))
\]

and
\[
h^{l+1}(H^*(\mathbb{A}^1, V \ast L)) = h^{l+1}(H^1_{\text{par}}(V \ast L)) + \kappa_{\infty}^l(V \ast L) + \epsilon_l - \kappa_{\infty}^l(V \otimes L(t - x)).
\]

In the last equation one uses the usual long exact cohomology sequence and
\[ H^0(\mathbb{A}^1, V \ast L) = H^0(\mathcal{H}^0(\mathcal{R}(V \ast L))), \]

cf. [7], Lemma 2.6.9.

**Remark. 5.4.** If $V \ast L = \tilde{V} \ast \tilde{L}, \tilde{L} \ast M = \tilde{L} \ast \tilde{M}$ and $(\tilde{V} \ast \tilde{L}) \ast \tilde{M} = (V \ast L) \ast M$ then
\[ (\tilde{V} \ast \tilde{L}) \ast \tilde{M} = \tilde{V} \ast (\tilde{L} \ast \tilde{M}). \]

**Proof.** By assumption we have
\[
(V \ast L) \ast M = (V \ast L) \ast M = V \ast (L \ast M) = V \ast (L \ast M),
\]
using the associativity of the middle convolution under the Assumption 1.2 (cf. [7], Lemma 2.6.5). By the nature of the \( \overline{\ast} \)-convolution, in the term on the the left hand side of the previous equation, there appears no skyscraper sheaf \( \delta_c \) as a summand. Hence we can conclude that \( V \ast (L \overline{\ast} M) = V \overline{\ast} (L \overline{\ast} M) \) and finally

\[
(V \ast L) \overline{\ast} M = V \overline{\ast} (L \overline{\ast} M).
\]

**Corollary 5.5.** Let \( V, L, M \) be parabolically rigid without unipotent Jordan blocks at \( \infty \). Assume further that \( V \otimes L(t - x) \) has also no unipotent Jordan block at \( \infty \).

(i) Then \( V \ast L = V \overline{\ast} L \) is also parabolically rigid (not necessarily irreducible) without unipotent Jordan blocks at \( \infty \).

(ii) Moreover if \( L \ast M = L \overline{\ast} M \) and \( (V \overline{\ast} L) \overline{\ast} M = (V \overline{\ast} L) \ast M \) then

\[
\kappa^l_{\infty}((V \overline{\ast} L) \otimes M(t - x)) = \kappa^l_{\infty}(V \otimes (L \overline{\ast} M)(t - x)).
\]

**Proof.** We have \( \kappa^l_{\infty}(L) = \kappa^l_{\infty}(L) = \kappa^l_{\infty}(M) = 0 \) since there is no unipotent Jordan block at \( \infty \). The assumption that \( V \otimes L(t - x) \) has no unipotent Jordan block at \( \infty \) implies that \( V \) is not dual to a translate of the form \( L(c - x) \). Hence \( V \ast L = V \overline{\ast} L \) by Theorem 3.2. Further \( V \overline{\ast} L \) is parabolically rigid without unipotent Jordan block at \( \infty \) by Theorem 5.1 implying (i).

By Theorem 5.1

\[
h^{l+1}(H^1_{\text{par}}((V \overline{\ast} L) \overline{\ast} M)) + \kappa^l_{\infty}((V \overline{\ast} L) \overline{\ast} M) + \epsilon_i((V \overline{\ast} L) \overline{\ast} M) - \kappa^l_{\infty}((V \overline{\ast} L) \otimes M(t - x)) = 0
\]

and

\[
h^{l+1}(H^1_{\text{par}}(V \overline{\ast} (L \overline{\ast} M))) + \kappa^l_{\infty}(V \overline{\ast} (L \overline{\ast} M)) + \epsilon_i(V \overline{\ast} (L \overline{\ast} M)) - \kappa^l_{\infty}((V \otimes (L \overline{\ast} M)(t - x)) = 0.
\]

Since by Remark 5.4(ii), \( (V \overline{\ast} L) \overline{\ast} M = V \overline{\ast} (L \overline{\ast} M) \) we deduce

\[
\kappa^l_{\infty}((V \overline{\ast} L) \otimes M(t - x)) = \kappa^l_{\infty}(V \otimes (L \overline{\ast} M)(t - x)).
\]

**Remark 5.6.** Let \( m, n \in \mathbb{N}_{>0} \) and let \( a_m, b_n \in \mathbb{R} \cap (0, 1) \).
(i) Let \( V, L = L(n, b_n) \) be as in Assumption \([1.2]\) such that \( \psi_{\infty}(V) \simeq J^{m-1}(a_m, m) \) and \( \psi_{\infty}(L) \simeq J^{n-1}(b_n, n) \). Then the non zero Hodge numbers of \( V \otimes L \) are

\[
h^p(V \otimes L) = \begin{cases} 
  p + 1, & 0 \leq p \leq \min\{m, n\} - 1 \\
  \min\{m, n\}, & \min\{m, n\} \leq p < m + n - \min\{m, n\} \\
  m + n - p - 1, & m + n - \min\{m, n\} \leq p < m + n \end{cases}
\]

and \( \psi_{\infty}(V \otimes L) \) is isomorphic to

\[
J^{m+n-2}(a_m + b_n - a_m + b_n, m+n-1) \oplus \cdots \oplus J^{m+n-1-\min\{m,n\}}(a_m + b_n - a_m + b_n, m+n-1-2\min\{m,n\}).
\]

(ii) Let \( V, L = L(n, b_n) \) be as in Assumption \([1.2]\). Then the structure of \( \psi_{\infty}(V) \) is uniquely determined by \( \kappa_{\infty}(V \otimes L(n, b_n)) \) for all \( n, b_n \).

**Proof.** The tensor decomposition of \( J(n) \otimes J(m) \) of the tensor product of two unipotent Jordan blocks of size \( n \), resp. \( m \), in characteristic zero is given by

\[
J(m + n - 1) \oplus J(m + n - 3) \oplus \cdots \oplus J(m + n + 1 - 2\min\{m, n\}),
\]

cf. Reference Chapter, Table 5, A1, \([9]\). Moreover, if \( \psi_{\infty}(V) \simeq J^{m-1}(a_m, m) \) and \( \psi_{\infty}(L) \simeq J^{n-1}(b_n, n) \) then

\[
h^p(V) = \ldots = h^{m-1}(V) = 1, \quad h^0(L) = \ldots = h^{n-1}(L) = 1
\]

and the non zero Hodge numbers are

\[
h^p(V \otimes L) = \sum_{i+j=p} h^i(V) h^j(L) = \begin{cases} 
  p + 1, & 0 \leq p \leq \min\{m, n\} - 1 \\
  \min\{m, n\}, & \min\{m, n\} \leq p < m + n - \min\{m, n\} \\
  m + n - p - 1, & m + n - \min\{m, n\} \leq p < m + n \end{cases}
\]

Since

\[
\#\{p \mid h^p(V \otimes L) \geq 1\} = m + n - 1
\]

and \( \nu_{\infty, a_m + b_n - [a_m + b_n], m+n-2}(V \otimes L) \geq 1 \) for some \( i_1 \) we obtain \( i_1 = m + n - 2 \). Since

\[
\#\{p \mid h^p(V \otimes L) \geq 2\} = m + n - 3
\]

and \( \nu_{\infty, a_m + b_n - [a_m + b_n], m+n-3}(V \otimes L) \geq 1 \) we get \( i_2 = m + n - 3 \). It follows now iteratively by repeating this argument that the only possibility that the above derived Hodge numbers match this Jordan decomposition is given as follows:

\[
\psi_{\infty}(V \otimes L) \simeq J^{m-1}(a_m, m) \otimes J^{n-1}(b_n, n) = J^{m+n-2}(a_m + b_n - [a_m + b_n], m+n-1) \oplus \cdots \oplus J^{m+n-1-\min\{m,n\}}(a_m + b_n - [a_m + b_n], m+n-1-2\min\{m,n\}).
\]
This proves (i).

Let \( \psi_\infty(V) \cong \bigoplus_{i,l} J^i(0,1)^{\nu_{\infty,0,1-l}(V)} \oplus \bigoplus_{(i,a,l), a \in (0,1)} J^i(a,1)^{\nu_{\infty,a,1-l}(V)} \).

Then
\[
\kappa^p_\infty(V \otimes L(1,0)) = \kappa^p_\infty(V) = \nu_{\infty,0,0,l}(V) = \sum_l \nu^{l^p}_{\infty,0,0,l}(V)
\]
and by (i)
\[
\kappa^{p+1}_\infty(V \otimes L(2,0)) = \nu^{p+1}_{\infty,0,0,l}(V \otimes L(2,0)) = \sum_{l>0} \nu^{p+1}_{\infty,0,0,l}(V) + \sum_l \nu^{p}_{\infty,0,0,l}(V)
\]
and
\[
\kappa^{p+2}_\infty(V \otimes L(0,3)) = \nu^{p+2}_{\infty,0,0,l}(V \otimes L(0,3)) = \sum_{l>1} \nu^{p+2}_{\infty,0,0,l}(V) + \sum_{l>0} \nu^{p+1}_{\infty,0,0,l}(V) + \sum_l \nu^{p}_{\infty,0,0,l}(V).
\]

Iterating this argument, one obtains
\[
\kappa^{p+r}_\infty(V \otimes L(r+1,0)) = \nu^{p+r}_{\infty,0,0,l}(V \otimes L(r+1,0)) = \sum_{l>r-1} \nu^{p+r}_{\infty,0,0,l}(V) + \sum_{l>r-2} \nu^{p+r-1}_{\infty,0,0,l}(V) + \cdots + \sum_l \nu^{p}_{\infty,0,0,l}(V).
\]

Hence one can recursively determine \( \nu^{p+r}_{\infty,0,0,l}(V) \) starting with the \( \nu^{j}_{\infty,0,0,l}(V) \) for all \( j \). Analogously we proceed in case where \( a \in (0,1) \).

**Lemma. 5.7.** Let \( m, n \in \mathbb{N}_{>0} \) and let \( a_m, b_n \in \mathbb{R} \cap (0,1) \). Let \( M_m, N_n \) be irreducible hypergeometric Hodge modules of rank \( m \), resp. \( n \), such that \( \omega_0(M_m) = m - 1 \) and \( \omega_0(N_n) = n - 1 \) and the local monodromy at \( \infty \) is a maximal Jordan block of the form \( \psi_\infty(M_m) \cong J^{m-1}(a_m, m) \) and \( \psi_\infty(N_n) \cong J^{n-1}(b_n, n) \).

Then \( M_m \) and \( N_n \) are parabolically rigid, i.e.
\[
H^1_{\text{par}}(M_m) = 0, \quad H^1_{\text{par}}(N_n) = 0,
\]
and
\[
\psi_\infty(M_m \otimes \bar{N}_n) = \begin{cases} 
\psi_\infty(M_m) \otimes \psi_\infty(N_n) & \cong J^{m-1}(a_m, m) \otimes J^{n-1}(b_n, n), \quad 0 < a_m + b_n < 1 \\
(\psi_\infty(M_m) \otimes \psi_\infty(N_n))(-1) & \cong (J^{m-1}(a_m, m) \otimes J^{n-1}(b_n, n))(-1), \quad 1 < a_m + b_n < 2
\end{cases},
\]
where
\[
J^{m-1}(a_m, m) \otimes J^{n-1}(b_n, n) = J^{m-1+n-1}(a_m + b_n - \lfloor a_m + b_n \rfloor, m + n - 1) \oplus \cdots \oplus J^{m+n-1-\min\{m,n\}}(a_m + b_n - \lfloor a_m + b_n \rfloor, m + n + 1 - 2 \min\{m,n\}).
\]

**Proof.** A hypergeometric Hodge module \( H \) has singularities at 0, 1 and \( \infty \) (up to a Moebius transformation), where the local monodromy at 1 is a pseudo reflection, i.e. \( \omega_1(H) = 1 \), cf. Section 2. [1]. If
$\omega_0(H) = \text{rk}(H) - 1$ we get

$$\text{rk}(H^1_{\text{par}}(H)) = \omega_0(H) + \omega_1(H) + \omega_\infty(H) - 2 \text{rk}(H) = 0,$$

which implies the first claim.

Assume first $0 < a_m + b_n < 1$. Then $M_m \ast N_n = M_m \tilde{\otimes} N_n$ by Corollary 5.5(i). In the proof of Theorem 5.1 it was shown that

$$h^k(M_m \tilde{\otimes} N_n) = \sum_i (h^i(H^1_{\text{par}}(M_m)) + \kappa^{-1}_\infty(M_m)h^{k-i}(N_n))$$

$$+ \sum_i h^i(M_m)(\delta^{k-1-i}(N_n) - \delta^{k-i}(N_n) + \omega^{k-1-i}_\infty(N_n))$$

$$+ \psi^k_\infty(M_m \otimes N_n(t-x)) - \psi^k_\infty(M_m \otimes N_n(t-x)) - \kappa^{-1}_\infty(M_m \otimes N_n(t-x))$$

$$= 0 + \sum_i h^i(M_m)(h^{k-i}(H^1_{\text{par}}(N_n)) + h^{k-i}(N_n)) + 0 - 0 - 0$$

$$= \sum_i h^i(M_m)h^{k-i}(N_n)$$

$$= h^k(M_m \otimes N_n(t-x)).$$

The stationary phase formula (cf. [10]), Theorem 5.1, implies that the Jordan blocks of $M_m \ast N_n$ at infinity are $J(a_m + b_n, m + n - 1), J(a_m + b_n, m + n - 3), \ldots, J(a_m + b_n, m + n + 1 - 2 \min\{m, n\})$ which are exactly the Jordan blocks of the tensor product $J(a_m, m) \otimes J(b_n, n)$ by Remark 5.6. The only possibility that the above derived Hodge numbers match this Jordan decomposition is given as follows:

$$\psi_\infty(M_m \ast N_n) = J^{m-1}(a_m, m) \otimes J^{n-1}(b_n, n)$$

$$= J^{m+n-2}(a_m + b_n, m + n - 1) \oplus \cdots \oplus J^{m+n-\min\{m,n\}}(a_m + b_n, m + n + 1 - 2 \min\{m, n\}).$$
Assume now $1 < a_m + b_n < 2$. As in the proof of Theorem 5.1 and using $\delta_\infty(M_m \otimes N_n(t - x)) = h^i(M_m \otimes N_n(t - x))$ (cf. Theorem 2.1) one finds

\[ h^k(M_m \tilde{\otimes} N_n) = \sum_i (h^i(H^\par(M_m)) + \kappa^i_{i-1}(M_m))h^{k-i}(N_n) \]

\[ + \sum_i h^i(M_m)(\sigma^{k-1-i}(N_n) - \sigma^{k-i}(N_n) + \omega_{k-i}(N_n)) \]

\[ + \sigma_\infty(M_m \otimes N_n(t - x)) - \sigma_\infty(M_m \otimes N_n(t - x)) - \kappa^{k-1}(M_m \otimes N_n(t - x)) \]

\[ = 0 + \sum_i h^i(M_m)(h^{k-i}(H^\par(N_n)) + h^{k-i}(N_n)) \]

\[ + h^{k-1}(M_m \otimes N_n(t - x)) - h^k(M_m \otimes N_n(t - x)) - 0 \]

\[ = \sum_i h^i(M_m)h^{k-i}(N_n) + h^{k-1}(M_m \otimes N_n(t - x)) - h^k(M_m \otimes N_n(t - x)) \]

\[ = h^{k-1}(M_m \otimes N_n(t - x)). \]

Using the stationary phase as before the claim follows as in the case $0 < a_m + b_n < 1$. 

**Corollary 5.8.** Let $m \in \mathbb{N}_{\geq 0}$ and let $a_m \in \mathbb{R} \cap (0, 1)$. Let further $M_m$ be a parabolically rigid hypergeometric Hodge module of rank $m$ with one non unipotent Jordan block $J^{m-1}(a_m, m)$ at $\infty$ of size $m$ and $L$ be a Hodge module underlying a parabolically rigid local system without unipotent Jordan blocks at $\infty$. If $L \otimes M_m(t - x)$ has no unipotent Jordan block at infinity then

\[ \psi_\infty(L \tilde{\otimes} M_m) \simeq \bigoplus_{(k, a, l)} \left( J^k(a, l) \nu^{\infty, a, i-1}(L) \otimes J^{m-1}(a_m, m) \right) (-[a + a_m]). \]

**Proof.** The claim is settled if $M_m = M_1$ a Kummer sheaf and therefore hypergeometric or if $L$ is hypergeometric by the previous result. Let now $L$ be non-hypergeometric and $m > 1$. Let $N_n$ be as in Lemma 5.7 such that $a_m + b_n \notin \mathbb{Z}$. Then $M_m \ast N_n = M_m \tilde{\otimes} N_n$ by Corollary 5.5. If $N_n = N_1$ is a Kummer sheaf then

\[ (L \ast M_m) \ast N_1 = L \ast (M_m \ast N_1) \]

\[ = L \ast (M_m \tilde{\otimes} N_1) \]

\[ = L \tilde{\otimes} (M_m \tilde{\otimes} N_1), \]

where the second equality uses that $M_m$ is not a Kummer sheaf and the third equality uses that $M_m \ast N_1 = M_m \tilde{\otimes} N_1$ is a parabolically rigid irreducible hypergeometric Hodge module and $L$ is not hypergeometric.

On the right hand side of the last equation there appears no skyscraper sheaf as a direct summand. Hence $(L \ast M_m) \ast N_1 = (L \ast M_m)\tilde{\otimes} N_1 = (L \tilde{\otimes} M_m)\tilde{\otimes} N_1$, where $L \ast M_m = L \tilde{\otimes} M_m$ by Corollary 5.5. If $n > 1$ we
choose a $N_n$ with a residue $\mu$ at 0 such that $-\mu$ is not a residue of $L\tilde{\star}M_m$. Hence by Theorem 3.2

$$(L\tilde{\star}M_m) \ast N_n = (L\tilde{\star}M_m)\tilde{\star}N_n,$$

also for $n > 1$.

Let $H_{m+n+1-2k}$ be hypergeometric with

$$\psi_\infty(H_{m+n+1-2k}) = J^{m+n-2k}(a_m+b_n-[a_m+b_n],m+n+1-2k).$$

By Corollary 5.5 and Lemma 5.7 if $0 < a_m+b_n < 1$,

$$\kappa^l_\infty((L\tilde{\star}M_m)\otimes N_n(t-x)) = \kappa^l_\infty(L \otimes (M_m \ast N_n)(t-x))$$

$$= \sum_{k=1}^{\min(m,n)} \kappa^l_\infty((L \otimes H_{m+n+1-2k}(t-x))(-k+1))$$

$$= \kappa^l_\infty(L \otimes (M_m \otimes N_n))$$

$$= \kappa^l_\infty((L \otimes M_m)\otimes N_n)$$

and if $1 < a_m+b_n < 2$,

$$\kappa^l_\infty((L\tilde{\star}M_m)\otimes N_n(t-x)) = \kappa^l_\infty(L \otimes (M_m \ast N_n)(t-x))$$

$$= \sum_{k=1}^{\min(m,n)} \kappa^l_\infty((L \otimes H_{m+n+1-2k}(t-x))(-k))$$

$$= \kappa^l_\infty(L \otimes (M_m \otimes N_n)(-1))$$

$$= \kappa^l_\infty((L \otimes M_m)(-1)\otimes N_n).$$

The claim follows now from Remark 5.6.

\[ \square \]

**Theorem 5.9.** Let $V, L$ be the Hodge modules underlying irreducible nonconstant variations of complex polarized Hodge structures with

$$\psi_\infty(V) \simeq \bigoplus_{(i,a,l)} J^i(a, l)^{\nu_{\infty, a, i-1}(V)}$$

and

$$\psi_\infty(L) \simeq \bigoplus_{(j,b,m)} J^j(b, m)^{\nu_{\infty, b, m-1}(L)}.$$
Then there is an isomorphism of nilpotent orbits

\[
\psi_\infty(V \tilde{\ast} L) \simeq \bigoplus_{(i,a,b,l,m):a \neq 0, b \neq 0, a+b \neq 1} J^i(a,l)^{\nu^i_{\infty,a,l-1}(V)} \otimes J^j(b,m)^{\nu^j_{\infty,b,m-1}(L)}(-[a+b])
\]

\[
\bigoplus_{(i,a,b,l,m):a=0, b \neq 0} \varphi\left(J^i(a,l)^{\nu^i_{\infty,a,l-1}(V)} \otimes J^j(b,m)^{\nu^j_{\infty,b,m-1}(L)}\right)(-1)
\]

\[
\bigoplus_{(i,a,b,l,m):a \neq 0, b=0} J^{i+1}(a,l+1)^{\nu^i_{\infty,a,l-1}(V)} \otimes J^{j+1}(b,m+1)^{\nu^j_{\infty,b,m-1}(L)}
\]

\[
\bigoplus_{(i,a,l)} J^i(a,l)^{\nu^i_{\infty,a,l-1}(V)} \otimes H^1_{par}(L)
\]

\[
\bigoplus_{(j,b,m)} J^j(b,m)^{\nu^j_{\infty,b,m-1}(L)} \otimes H^1_{par}(V)
\]

where \(\varphi(J^i(0,l)) := J^{i-1}(0,l-1)\) where \(\varphi(J^i(a,l)) = J^i(a,l)\) for \(a \neq 0\) and the notion is extended using direct sums, and where moreover

\[
J^i(a,l) \otimes J^j(b,m) = J^{i+j}(a+b-|[a+b]|, l+m-1) \oplus \cdots \oplus J^{i+j+1-min\{l,m\}}(a+b-|[a+b]|, l+m+1-2\min\{l,m\}).
\]

**Proof.** Assume first that \(V, L\) are parabolically rigid without unipotent Jordan block at \(\infty\) such that \(V \otimes L(t-x)\) has also no unipotent Jordan block at \(\infty\). By Corollary 5.8 there exists for each \(n, a_n \in (0, 1)\) a parabolic rigid irreducible hypergeometric \(H_m(a_m)\) such that \(\psi_\infty(H_m(a_m)) \simeq J^{n-1}(a_n, n)\) and

\[
(V \tilde{\ast} L) \tilde{\ast} H_n(a_n) = V \tilde{\ast} (L \tilde{\ast} H_n(a_n)).
\]

Hence, by Corollary 5.5(ii)

\[
\kappa^\ell_\infty((V \tilde{\ast} L) \otimes H_n(a_n)(t-x)) = \kappa^\ell_\infty(V \otimes (L \tilde{\ast} H_n(a_n)(t-x))).
\]

Since these numbers determine uniquely the vanishing cycle structure of \(V \tilde{\ast} L\) at infinity by Remark 5.10, we obtain using Corollary 5.8

\[
(5.0.1) \quad \psi_\infty(V \tilde{\ast} L) \simeq \bigoplus_{(i,a,b,l,m)} J^i(a,l)^{\nu^i_{\infty,a,l-1}(V)} \otimes J^j(b,m)^{\nu^j_{\infty,b,m-1}(L)}(-[a+b]),
\]

as claimed.

In the general situation we proceed as follows: Let \(L_{\chi_1}, L_{\chi_2}\) be generic and \(\mu_1, \mu_2 \sim 1\). Then \(V \ast L_{\chi_1}\) and \(L \ast L_{\chi_2}\) are parabolically rigid by Proposition 4.2 without unipotent Jordan block at \(\infty\) by Proposition 4.3.
(i). Hence, by Corollary 4.3(ii)
\[
\psi_\infty(V \ast L_{\lambda_1}) \simeq \bigoplus_{(i,o,l)} J^{i+1}(1 - \mu_1, l + 1) \nu_{\infty,0,l-1}(V) \bigoplus_{(i,a,l):a \neq 0} J^i(a + 1 - \mu_1, l) \nu_{\infty,a,l-1}(V) \\
\bigoplus J^0(1 - \mu_1, 1) \otimes H^1_{\text{par}}(V)
\]

and
\[
\psi_\infty(L \ast L_{\lambda_2}) \simeq \bigoplus_{(j,0,m)} J^{j+1}(1 - \mu_2, m + 1) \nu_{\infty,0,m-1}(L) \bigoplus_{(j,b,m):b \neq 0} J^j(b + 1 - \mu_2, m) \nu_{\infty,b,m-1}(L) \\
\bigoplus J^0(1 - \mu_2, 1) \otimes H^1_{\text{par}}(L).
\]

By what was said above, the assumptions for Equation (5.0.1) are now fulfilled with \(V\) replaced by \(V \ast L_{\lambda_1}\) and with \(L\) replaced by \(L \ast L_{\lambda_2}\) which proves the claim of the theorem for \(W := (V \ast L_{\lambda_1}) \ast (L \ast L_{\lambda_2})\).

Thus
\[
\psi_\infty(W) \simeq \bigoplus_{(i,j,0,l,m)} J^{i+1}(1 - \mu_1, l + 1) \nu_{\infty,0,l-1}(V) \otimes J^{j+1}(1 - \mu_2, m + 1) \nu_{\infty,0,m-1}(L) \\
\bigoplus_{(i,j,0,b,l,m):b \neq 0} J^i(a + 1 - \mu_1, l) \nu_{\infty,a,l-1}(V) \otimes J^j(b + 1 - \mu_2, m) \nu_{\infty,b,m-1}(L) \\
\bigoplus_{(i,o,l)} J^i(a + 1 - \mu_1, l) \nu_{\infty,a,l-1}(V) \otimes J^0(1 - \mu_2, 1) \otimes H^1_{\text{par}}(L) \\
\bigoplus_{(i,a,l):a \neq 0} J^j(a + 1 - \mu_1, l) \nu_{\infty,a,l-1}(V) \otimes J^0(1 - \mu_2, 1) \otimes H^1_{\text{par}}(L) \\
\bigoplus J^0(1 - \mu_1, 1) \otimes H^1_{\text{par}}(V) \otimes J^{j+1}(1 - \mu_2, m + 1) \nu_{\infty,0,m-1}(L) \\
\bigoplus_{(j,b,m):b \neq 0} J^0(1 - \mu_1, 1) \otimes H^1_{\text{par}}(V) \otimes J^j(b + 1 - \mu_2, m) \nu_{\infty,b,m-1}(L) \\
\bigoplus J^0(1 - \mu_1, 1) \otimes H^1_{\text{par}}(V) \otimes J^0(1 - \mu_2, 1) \otimes H^1_{\text{par}}(L).
\]

Hence, since \(\mu_1, \mu_2 \sim 1\) are generic the only residues (contained in \([0,1]\)) that contribute to \(\psi_\infty(W)\) are by Formula (5.0.1)
\[
a + b + 1 - \mu_1 + 1 - \mu_2 - [a + b], \quad a + 1 - \mu_1 + 1 - \mu_2, \quad b + 1 - \mu_1 + 1 - \mu_2, \quad 1 - \mu_1 + 1 - \mu_2
\]
with \(a, \text{resp. } b\), a non-zero residue of \(V\), resp. \(L\), at infinity.
Using commutativity and associativity of the middle convolution together with Theorem 3.2 one finds
\[ W \star L_{\frac{V}{Z}} = (V \star L)(-1). \]

By Proposition 4.3(ii) we deduce that a Jordan block \( J^i(c, l) \) of \( \psi_{\infty}(W) \) is transformed to a Jordan block of \( \psi_{\infty}(W \star L_{\frac{V}{Z}}) \) as follows:

\[
J^i(c, l) \mapsto \begin{cases} 
J^i(0, l - 1), & c = 2 - \mu_1 - \mu_2 \\
J^{i+1}(c + \mu_1 + \mu_2 - 2, l), & c \neq 0, c \neq 2 - \mu_1 - \mu_2
\end{cases}
\]

which implies the expression for \( \psi_{\infty}(V \star L) = \psi_{\infty}(V_{\frac{V}{Z}}L) \) in the theorem. □

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