Coefficients Estimate for a Subclass of Holomorphic Mappings on the Unit Polydisk in $\mathbb{C}^n$

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Abstract. The aim of this paper is to obtain the sharp solutions of Fekete-Szegő problems of high dimensional version for family of holomorphic mappings that are normalized on the unit polydisk $U^n$ in $\mathbb{C}$. The main results unify some recent works, which are closely related to the starlike mappings. Moreover, some previous results are improved.

1. Introduction

Let $\mathbb{C}^n$ be the space of $n$ complex variables $z = (z_1, z_2, ..., z_n)$ with the maximum norm $\|z\| = \max\{|z_1|, |z_2|, ..., |z_n|\}$. Also, let $U^n$ be the unit polydisc in $\mathbb{C}^n$ and let $\mathbb{D}^n = \mathbb{D}$ be the unit disc. Let $\partial_0 U^n = \bigcap_{k=1}^{n} \partial U$ be the distinguished boundary of $U^n$, and $\partial U^n$ be the boundary of $U^n$. We denote by $\mathcal{H}(U^n)$ the family of holomorphic mappings from $U^n$ into $\mathbb{C}^n$ with the standard topology of locally uniform convergence. Let $f \in \mathcal{H}(U^n)$, we say that $f$ is normalized if $f(0) = 0$ and $J_f(0) = I_n$, where $J_f(0)$ is the complex Jacobian matrix of $f$ at the point $0$ and $I_n$ is the identity matrix.

Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded circular domain. The $m$ ($m > 2$)-Fréchet derivative of a mapping $f \in \mathcal{H}(\Omega)$ at point $z \in \Omega$ is written as $D^m f(z)(\partial^{m-1} \cdot, \cdot)$. The matrix representation is (see, e.g. Liu-Xu [13])

$$D^m f(z)(\partial^{m-1} \cdot, \cdot) = \left( \sum_{l_1, l_2, ..., l_m=1}^{n} \frac{\partial^m f_{l_j}(z)}{\partial z_{l_1} \partial z_{l_2} \cdots \partial z_{l_m}} a_{l_1} \cdots a_{l_m} \right)_{1 \leq j, k \leq n},$$

where $f(z) = (f_1(z), f_2(z), ..., f_n(z))^T, a = (a_1, a_2, ..., a_n)^T \in \mathbb{C}^n$.

If $f$ and $g$ are analytic in $\mathbb{D}$, we say that $f$ is subordinate to $g$, written $f(z) < g(z)$, provided there exists an analytic function $w(z)$ defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

Suppose that $\psi$ is a convex Carathéodory function on the unit disk $\mathbb{D}$ such that $\psi(0) = 1, \psi'(0) > 0, \Re(\psi(z)) > 0$ and $\psi(\mathbb{D})$ is symmetric with respect to the real axis. Also, $\psi(z)$ has a series expansion of the form

$$\psi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + ..., (A_1 > 0), \ z \in \mathbb{D}. \quad (1)$$

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Definition 1.1. Let $f : \mathbb{U}^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping. If $f(0) = 0$ and $0 \leq \gamma < 1$, then

$$f \in S_{\psi,\gamma}^a(\mathbb{U}^n) \iff \frac{z_j}{1 - \gamma g_j(z)} - \frac{\gamma}{1 - \gamma} \in \psi(\mathbb{U}), \ z \in \mathbb{U}^n \setminus \{0\},$$

where $g(z) = (g_1(z), g_2(z), ..., g_n(z)) = (D f(z))^{-1} f(z)$, $|z| = \max_{1 \leq k \leq n} |z_k|$, $T_x \in T(x)$ and the function $\psi$ is defined by $(1)$.

Remark 1.2. (I) If $\gamma = 0$, then the definition $S_{\psi,\gamma}^a(\mathbb{U}^n)$ due to Xu-Liu-Liu [31].

(II) If $\gamma = 0$, $n = 1$ and $\psi(\xi) = \frac{1 + \xi}{1 + \xi}$, then the class $S_{\psi,\gamma}^a(\mathbb{U})$ was the usual starlike function.

(III) Let $\alpha \in [0, 1)$, $\epsilon \in (0, 1)$, $n \in \mathbb{Z}^+$, $\xi \in \mathbb{U}$. Define the functions set by

$$M = \left\{ \frac{1 + (1 - 2\alpha)\xi}{1 - \xi}, \frac{1 + \epsilon}{1 + (2\alpha - 1)\epsilon}, \frac{1 + \xi}{1 - \xi}, \frac{1 + \epsilon}{1 - \epsilon} \right\},$$

then for different functions $\psi \in M$ in Definition 1.1, we can get kinds of well-known subclasses of starlike mappings in $\mathbb{U}^n$ (see, e.g., [2, 12, 13, 29, 30]).

As well known, the coefficient functional $\rho_\alpha(f) = a_3 - \mu a_3^2$ on the normalized analytic functions $f$ plays an important role in one-dimensional function theory, where $f(\xi) = \xi + a_2 \xi^2 + a_3 \xi^3 + \cdots, \ \xi \in \mathbb{U}$. For details, we refer the reader to survey articles of Kanas [10] and Srivastava et al. [18] (also see, e.g., [4, 9, 15, 17, 19–23, 26, 27]). The problem of maximizing the absolute value of the functional $\rho_\alpha(f)$ is called the Fekete-Szegő problem, which is related to the Bieberbach conjecture (see [1]). However, Cartan [3] stated that the Bieberbach conjecture does not hold in several complex variables. Until now, only a few complete results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in $\mathbb{C}^n$ (see, e.g., Graham-Hamada-Kohr [5], Graham-Hamada-Honda-Shon [6], Hamada-Honda-Kohr [8], Kohr [11], Liu-Liu [14], Gong [7]). In 2014, Xu-Liu [30] extended the Fekete-Szegő inequality from the case of one dimension to higher dimensions for a subclass of starlike mappings defined on the unit ball in a complex Banach space or on the unit polydisk in $\mathbb{C}^n$. Furthermore, Luo-Xu [12] and Xu-Fang-Liu [29] consider the results related to strongly starlike mappings of order $\alpha$ and starlike mappings of order $\alpha (0 \leq \alpha < 1)$, respectively. Recently, Liu-Xu [13] established inequalities between the second and the third coefficients of homogeneous expansions for starlike mappings and starlike mappings of order $\alpha$ defined on bounded starlike circular domains in $\mathbb{C}^n$, respectively. Some more general works on coefficients inequalities in several complex variables can be found in Tu-Xiong [25] and Xu-Liu-Liu [31].

In this paper, we will obtain the sharp coefficients bounds on Fekete-Szegő problem for the class $S_{\psi,\gamma}^a(\mathbb{U}^n)$. This is a continuation of the works in [25] and [31]. Our results extend some works that are related starlike mappings in $\mathbb{C}^n$, and give a positive answer to a conjecture proposed by Tu-Xiong [25]. Compare with the recent works on Fekete-Szegő problem(e.g., [12], [31]), the critical processes of proofs are different: our arguments in this paper are heavily based on the subordination techniques.

Throughout the paper, it is assumed that

$$\mathcal{M}_1 = \frac{1}{2} \left[ \frac{1}{(1 - \gamma)^2} \frac{A_2 - A_1}{A_1^2} + 1 \right], \ \mathcal{M}_2 = \frac{1}{2} \left[ \frac{1}{(1 - \gamma)^2} \frac{A_2 + A_1}{A_1^2} + 1 \right],$$

and $\psi(\xi) = 1 + A_1 \xi + A_2 \xi^2 + \cdots + A_n \xi^n + \cdots (A_1 > 0)$ is the function defined as $(1)$.

2. Preliminaries

The following Lemma is needed in the proof of main theorems.
Lemma 2.1 (16). Let $\mathcal{P}$ be the usual class of functions with positive real part in $\mathbb{U}$. Suppose that $p(z) = 1 + c_1z + c_2z^2 + \cdots \in \mathcal{P}$, then $|c_n| \leq 2$ for $n \geq 1$. If $|c_1| = 2$ then $p(z) = p_1(z) = \frac{1 + 2z}{1 - z^2}$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore we have

$$
|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.
$$

If $|c_1| < 2$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$
p_2(z) = \frac{1 + z}{1 - \frac{c_1^2 + 2\gamma_1}{4\gamma_1^2} z^2},
$$

and $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4\gamma_1^2}$. Conversely if $p(z) = p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$, then $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4\gamma_1^2}$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

3. Main results

In this section, we obtain the sharp unified solutions on Fekete-Szegő problem for $\mathcal{S}^\ast_{\psi,\gamma}(\mathbb{U}^n)$ with the the parameters $\mu \in \mathbb{R}$ (also, parameters $\mu \in \mathbb{C}$).

Theorem 3.1. Let $f \in \mathcal{S}^\ast_{\psi,\gamma}(\mathbb{U}^n)$. Suppose that $|z_0| = ||z|| = \max_{1 \leq j \leq n} ||z_j||$, $z_0 = \frac{1}{||z||}$, $z \in \mathbb{U}^n \setminus \{0\}$ and

$$
\frac{1}{2} D^2 f_0(0) \left( z_0, \frac{D^2 f_0(0)(z_0^\ast)}{2!} \right) z_0 = \left( \frac{D^2 f_0(0)(z_0^\ast)}{2!} \right)^2,
$$

then we have

$$
\|\frac{D^3 f_0(0)(z_0^\ast)}{3!} - \mu \frac{1}{2} D^2 f_0(0) \left( z, \frac{D^2 f_0(0)(z_0^\ast)}{2!} \right) \| \leq \begin{cases} 
\frac{1}{4} A_1^2 ||z||^2 \left[ \frac{\alpha_2}{A_1} (1 - \gamma_1^2) + 1 - 2\mu + \frac{\gamma_1^2 - 2\gamma_1}{A_1 (1 - \gamma_1^2)} \right], & \mu < M_1, \\
\frac{1}{4} A_1^2 ||z||^2 \left[ 2\mu - \frac{\alpha_2}{A_1} (1 - \gamma_1^2) - 1 + \frac{\gamma_1^2 - 2\gamma_1}{A_1 (1 - \gamma_1^2)} \right], & \mu > M_1.
\end{cases}
$$

The above estimates are sharp for each real $\mu$.

Proof. Fix $z \in \mathbb{U}^n \setminus \{0\}$, and set $z_0 = \frac{1}{||z||}$. We define a function $q_j : \mathbb{U} \rightarrow \mathbb{C}$ by

$$
q_j(\xi) = \begin{cases} 
\frac{1}{1 - \gamma_j} \frac{\xi z}{\|z\|} - \frac{\gamma_j}{1 - \gamma_j}, & \xi \neq 0, \\
1, & \xi = 0,
\end{cases}
$$

where $p(z) = (Df(z))^{-1} f(z)$ and $||z|| = \max_{1 \leq k \leq n} ||z_k||$. It is easy to see that $q_j(\xi) \in H(\mathbb{U})$. Since $f \in \mathcal{S}^\ast_{\psi,\gamma}(\mathbb{U}^n)$, using (4), then we have $q_j(\xi) \in \psi(\mathbb{U})$, $\xi \in \mathbb{U}$. Furthermore, the fact $q_j(0) = \psi(0) = 1$ implies that $q_j(\xi) < \psi(\xi)$, $\xi \in \mathbb{U}$. Taking a function

$$
\mathcal{P}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots < \frac{1 + z}{1 - z}, \quad z \in \mathbb{U},
$$

we note $\mathcal{P}(0) = 1$ and $\mathcal{P}$ is a function with positive real part. By (5), there is a function $w(z)$, such that

$$
q_j(\xi) = \psi(w(\xi)) = 1 + \frac{1}{2} A_1 c_1 \xi + \left( \frac{1}{2} A_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} A_2 c_2^2 \right) \xi^2 + ..., \xi \in \mathbb{U}.
$$

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From (4) and (6), we know that
\[
\left[ 1 + \frac{\gamma}{1 - \gamma} + \frac{1}{2} A_1 c_1 \xi + \left( \frac{1}{2} A_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} A_2 c_1^2 \right) \xi^2 + \cdots \right] x
\]
\[
(1 - \gamma) \left( \xi + \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \left\| z \right\| \xi^2 + \frac{D^3 p_j(0)(z_0^3)}{3! z_j} \left\| z \right\| \xi^3 + \cdots \right) = \xi.
\]
(7)

Comparing with the coefficient of two sides of the (7) in \( \xi^2 \) and \( \xi^3 \), we get
\[
\frac{1}{2} A_1 c_1 = -\frac{1}{1 - \gamma} \frac{D^2 p_j(0)(z_0^3)}{2! z_j}
\]
and
\[
\frac{1}{2} A_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} A_2 c_1^2 = \left( \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right)^2 - \frac{D^3 p_j(0)(z_0^3)}{3! z_j}.
\]
(9)

Using the Lemma 2.1, (8) and (9), then
\[
|c_2 - \frac{1}{2} c_1^2| = \left| \frac{2}{A_1} \left( \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right)^2 - \frac{2}{A_1} \frac{D^3 p_j(0)(z_0^3)}{3! z_j} \right|
\]
\[
- \frac{2 A_2}{A_1^3 (1 - \gamma)^2} \left( \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right)^2 \leq 2 - \frac{1}{2} \frac{A_1}{(1 - \gamma)^2} \left| \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right|^2 \leq \frac{4}{A_1^3}.
\]
(10)

From (10), it shows that
\[
\left| \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right|^2 - \frac{A_2}{A_1^3 (1 - \gamma)^2} \left( \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right)^2 \leq A_1 - \frac{1}{A_1 (1 - \gamma)^2} \left| \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right|^2
\]
\[
\leq A_1 - \frac{1}{A_1 (1 - \gamma)^2} \left| \frac{D^2 p_j(0)(z_0^3)}{2! z_j} \right|^2.
\]
(11)

On the other hand, since \( p(z) = (D f(z))^{-1} f(z) \), then
\[
z + \frac{D^2 f(0)(z^2)}{2!} + \frac{D^3 f(0)(z^3)}{3!} + \cdots
\]
\[
= \left( I + D^2 f(0)(z_0) + \frac{D^2 f(0)(z^2_0)}{2!} + \cdots \right) \times \left( D p(0)z + \frac{D^2 g(0)(z^2)}{2!} + \frac{D^3 g(0)(z^3)}{3!} + \cdots \right).
\]
(12)

Comparing with the homogeneous expansion of two sides of the (12), we have
\[
Dp(0)z = z, \quad \frac{D^2 p(0)(z^2)}{2!} = -\frac{D^2 f(0)(z^2)}{2!}
\]
(13)
and
\[
\frac{D^3 f(0)(z^3)}{3!} = \frac{D^3 p(0)(z^3)}{3!} + \frac{D^3 f(0)(z^3)}{2!} - \frac{D^2 f(0)}{2!} \left( z, \frac{D^2 f(0)(z^3)}{2!} \right).
\]
(14)

In view of (14), we can obtain
\[
F = \left| \frac{D^3 f(0)(z_0^3)}{3! z_j} - \frac{1}{2} \frac{D^2 f(0)(z_0^3)}{2!} \frac{D^2 f(0)(z_0^3)}{2!} \right| \frac{\| z \|}{z_j}
\]
\[
= -\frac{1}{2} \left| \frac{D^3 p(0)(z_0^3)}{3! z_j} + \frac{D^2 f(0)(z_0^3)}{2!} \right| \frac{\| z \|}{z_j} - \mu \left( \frac{D^2 f(0)(z_0^3)}{2!} \frac{z_0^3}{z_j} \right)^2.
\]
(15)
Furthermore, using (3), (13) and (14) in (15), then we have

\[ F = \frac{1}{2} \left| \frac{D^3 f_j(0)(z_0^2)\|z\|}{3! z_j} + (2 - 2\mu) \left( \frac{D^2 f_j(0)(z_0^2)\|z\|^2}{2! z_j} \right) \right| \]

\[ = \frac{1}{2} \left| \frac{D^3 f_j(0)(z_0^2)\|z\|}{3! z_j} + \left( \frac{D^2 f_j(0)(z_0^2)\|z\|^2}{2! z_j} \right) + A_2 \frac{1}{A_1^2 (1 - \gamma)^2} \right| \]

\[ \leq \frac{1}{2} \left( A_1 - \frac{1}{A_1} \frac{A_2 - A_1}{A_1^2} + 1 \right) \left\| \frac{D^2 p_j(0)(z_0^2)\|z\|^2}{2! z_j} \right\| \left( 1 + \frac{A_2}{A_1^2 (1 - \gamma)^2} + 1 - 2\mu \right) \left\| \frac{D^2 p_j(0)(z_0^2)\|z\|^2}{2! z_j} \right\|^2 \].

(16)

According to the above inequality (16), we consider the following four cases with using the Lemma 6 in Xu-Liu [28]:

**Case 1:** If \( \mu \) satisfies the condition

\[ \mu \leq \frac{1}{2} \left( \frac{1}{1 - \gamma} - \frac{A_2 - A_1}{A_1^2} + 1 \right), \]

then we have

\[ \left| \frac{D^3 f_j(0)(z_0^2)\|z\|}{3! z_j} - \frac{1}{2} D^2 f_j(0)(z_0) \frac{D^2 f_j(0)(z_0^2)\|z\|^2}{2! z_j} \right| \]

\[ \leq \frac{1}{2} \left( A_1 + \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 - 2\mu + \frac{1}{A_1} \frac{1}{(1 - \gamma)^2} \right) \right) \]

\[ \leq \frac{1}{2} \left( A_1 + A_1 \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 - 2\mu + \frac{1}{A_1} \frac{1}{(1 - \gamma)^2} \right) \right) \]

\[ = \frac{1}{2} A_1 \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 - 2\mu + \frac{\gamma^2 - 2\gamma}{1 - \gamma} \frac{1}{A_1} \right). \]

(17)

**Case 2:** If \( \mu \) satisfies the condition

\[ \frac{1}{2} \left( \frac{1}{1 - \gamma} - \frac{A_2 - A_1}{A_1^2} + 1 \right) \leq \mu \leq \frac{1}{2} \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 \right), \]

then we have

\[ \left| \frac{D^3 f_j(0)(z_0^2)\|z\|}{3! z_j} - \frac{1}{2} D^2 f_j(0)(z_0) \frac{D^2 f_j(0)(z_0^2)\|z\|^2}{2! z_j} \right| \]

\[ \leq \frac{1}{2} \left( A_1 + \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 - 2\mu + \frac{1}{A_1} \frac{1}{(1 - \gamma)^2} \right) \right) \]

\[ \leq \frac{1}{2} A_1. \]

(18)

**Case 3:** If \( \mu \) satisfies the condition

\[ \frac{1}{2} \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 \right) \leq \mu \leq \frac{1}{2} \left( \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} + 1 \right), \]

then we have

\[ \left| \frac{D^3 f_j(0)(z_0^2)\|z\|}{3! z_j} - \frac{1}{2} D^2 f_j(0)(z_0) \frac{D^2 f_j(0)(z_0^2)\|z\|^2}{2! z_j} \right| \]

\[ \leq \frac{1}{2} \left( A_1 + \left( 2\mu - \frac{A_2}{A_1^2} \frac{1}{(1 - \gamma)^2} - 1 - \frac{1}{A_1} \frac{1}{(1 - \gamma)^2} \right) \right) \]

\[ \leq \frac{1}{2} A_1. \]

(19)
Case 4: If $\mu$ satisfies the condition
\[ \mu > \frac{1}{2} \left\{ \frac{1}{(1 - \gamma)^2} A_2 + A_1 + 1 \right\}, \]
then we have
\[
\left| \frac{D^3 f_j(0)(z_0^j)}{3! z_j} - \mu \frac{1}{2} \frac{D^2 f_j(0)(z_0, D^2 f(0)(z_0^j))}{2! z_j} \right| \\
\leq \frac{1}{2} \left( A_1 + 2 \mu - A_2 A_1 \right) \left( \frac{1}{(1 - \gamma)^2} - 1 \right) \left( \frac{1}{A_1 (1 - \gamma)^2} \right)^2 D^2 p_j(0)(z_0^j) \langle z_j \rangle,
\]
\[
\leq \frac{1}{2} \left( A_1 + A^2 \left( \frac{2 \mu - A_2 A_1}{A_1 (1 - \gamma)^2} - 1 \right) \right) \left( \frac{1}{A_1 (1 - \gamma)^2} \right)^2
\]
\[
= \frac{1}{2} A^2 \left[ 2 \mu - A_2 A_1 \right] \left( \frac{1}{(1 - \gamma)^2} - 1 \right) + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^5 A_1}.
\]

From (17)-(20), then we have
\[
\left| \frac{D^3 f_j(0)(z_0^j)}{3! z_j} - \mu \frac{1}{2} \frac{D^2 f_j(0)(z_0, D^2 f(0)(z_0^j))}{2! z_j} \right| \\
\leq \left\{ \frac{1}{2} A^2 \left[ \frac{A_2}{A_1^2 (1 - \gamma)^2} + 1 \right] + 1 - 2 \mu + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^2 A_1} \right\}, \quad \mu \leq M_1,
\]
\[
\left\{ \frac{1}{2} A^2 \left[ 2 \mu - A_2 A_1 \right] \left( \frac{1}{(1 - \gamma)^2} - 1 \right) + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^5 A_1} \right\}, \quad \mu > M_2.
\]

Thus, if $z_0 \in \partial \mathbb{U}^n$, then for $j = 1, 2, ..., n$, (21) implies that
\[
\left| \frac{D^3 f_j(0)(z_0^j)}{3! z_j} - \mu \frac{1}{2} \frac{D^2 f_j(0)(z_0, D^2 f(0)(z_0^j))}{2! z_j} \right| \\
\leq \left\{ \frac{1}{2} A^2 \left[ \frac{A_2}{A_1^2 (1 - \gamma)^2} + 1 \right] + 1 - 2 \mu + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^2 A_1} \right\}, \quad \mu \leq M_1,
\]
\[
\left\{ \frac{1}{2} A^2 \left[ 2 \mu - A_2 A_1 \right] \left( \frac{1}{(1 - \gamma)^2} - 1 \right) + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^5 A_1} \right\}, \quad \mu > M_2.
\]

Also since $D^3 f_j(0)(z, \frac{D^2 f(0)(z)}{2})$ are holomorphic functions on $\mathbb{U}^n$, in view of the maximum modulus theorem of holomorphic functions on $\mathbb{U}^n$, we get
\[
\left| \frac{D^3 f(0)(z)}{3!} - \mu \frac{1}{2} \frac{D^2 f(0)(z, D^2 f(0)(z))}{2!} \right| \\
\leq \left\{ \frac{1}{2} A^2 \left[ \frac{A_2}{A_1^2 (1 - \gamma)^2} + 1 \right] + 1 - 2 \mu + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^2 A_1} \right\}, \quad \mu \leq M_1,
\]
\[
\left\{ \frac{1}{2} A^2 \left[ 2 \mu - A_2 A_1 \right] \left( \frac{1}{(1 - \gamma)^2} - 1 \right) + \frac{\gamma^2 - 2 \gamma + 1}{(1 - \gamma)^5 A_1} \right\}, \quad \mu > M_2,
\]
where $z \in \mathbb{U}^n$.

In order to prove that the sharpness, we need to consider the following mappings.

If $\left| \frac{A_2}{A_1 (1 - \gamma)^2} + 1 - 2 \mu \right| \geq \frac{1}{A_1 (1 - \gamma)^2}$, then
\[
f(z) = z \exp \int_0^z (\psi(t) - 1) \frac{1}{t} dt, \quad z \in \mathbb{U}^n.
\]
Proof. We define the function $f(z)$. Comparing the homogeneous expansions of two sides in (28), we have
\[ \zeta = (1, 0, 0, \ldots, 0)^T, 0 < r < 1, \text{ then the first and third equalities in (23) hold true.} \]

If $A_1 \frac{1}{(1-\gamma)^2} + 1 - 2\mu < \frac{1}{A_1} \frac{1}{(1-\gamma)^2}$, then
\[ f(z) = z \exp \int_0^z (\psi(t^2) - 1) \frac{1}{t} dt, z \in \mathbb{U}^n. \]  
(25)

Also, it is not difficult to verify that $f(z)$. Taking $z = (r, 0, 0, \ldots, 0)^T, 0 < r < 1$, then the second equalities in (23) hold true. This completes the proof of Theorem 3.1. □

We can obtain an interesting result for a subclass of $S_{\psi, \gamma}^n(\mathbb{U}^n)$ by dropping off the condition (3).

Theorem 3.2. Suppose that $f: \mathbb{U}^n \to \mathbb{C}, F(z) = zf(z) \in S_{\psi, \gamma}^n(\mathbb{U}^n)$, then for $z \in \mathbb{U}^n$, we have
\[ \left\| \frac{D^3 F(0)(z^2)}{3!} - \mu \frac{1}{2} \frac{D^2 F(0)(z)}{2!} \right\| \leq \left\{ \begin{array}{ll} \frac{1}{2} |A_1||z|^3 \left[ \frac{1}{A_1} \frac{1}{(1-\gamma)^2} + 1 - 2\mu + \frac{\gamma^2}{\gamma^2} \right], & \mu < M_1, \\ \frac{1}{2} |A_1||z|^3 \left[ 2\mu - \frac{\gamma^2}{\gamma^2} \right], & \mu > M_2, \end{array} \right. \]
(26)

The above estimates are sharp for each real $\mu$.

Proof. We define the function $q_i(\xi)$ as (4). Since $F(z) = zf(z) \in S_{\psi, \gamma}^n(\mathbb{U}^n)$, then we can deduce that
\[ q_i(\xi)(1 - \gamma) f(\zeta_0) = (1 - \gamma) f(\zeta_0) + Df(\zeta_0)\zeta_0. \]  
(27)

Considering the Taylor series expansions with $\zeta$ in (27), then
\[ \left( 1 + \frac{1}{2} A_1 c_1 \zeta + \left( \frac{1}{2} A_1 \left( c_2 - \frac{2}{2} c_2 \right) + \frac{1}{4} A_2 c_2 \right) \zeta^2 + \ldots \right) (1 - \gamma) \Psi \]
\[ = (1 - \gamma) \Psi + \left( Df(0)\zeta_0 \zeta + D^2 f(0)(z_0^2) \zeta^2 + \ldots \right), \]  
(28)

where \[ \Psi = 1 + Df(0)\zeta_0 + \frac{D^2 f(0)(z_0^2)}{2} \zeta^2 + \ldots. \]

Comparing the homogeneous expansions of two sides in (28), we have
\[ \frac{1}{2} A_1 c_1 (1 - \gamma) = Df(0)\zeta_0 \]  
(29)

and
\[ \frac{1}{2} A_1 \left( c_2 - \frac{2}{2} c_2 \right) + \frac{1}{4} A_2 c_2 = \frac{1}{1 - \gamma} [D^2 f(0)(z_0^2) - (Df(0)\zeta_0)^2]. \]  
(30)

On the other hand, from $F(z) = zf(z)$, we note that
\[ \frac{D^3 F(0)(z_0^2)}{3!} = \frac{D^2 f(0)(z_0^2)}{2!} \frac{z_j}{|z|^2} \cdot \frac{D^2 F(0)(z_0^2)}{2!} = Df(0)\zeta_0 \frac{z_j}{|z|^2}. \]  
(31)
Thus, together with Theorem 3.1, (29), (30) and (31), it shows that
\[
\left| \frac{D^2 F_k(0)(z_0^3)}{3! z_i} - \frac{1}{2} D^2 F_k(0)(z_0, D^2 F(0)(z_0^3)) \right| |z_i| \leq \frac{1}{2} D^2 f(0)(z_0^3) - \mu \frac{1}{2} D^2 F_k(0)(z_0, D^2 F(0)(z_0^3)) \left| \frac{D^2 f(0)(z_0^3)}{2!} \right| |z_i| = \frac{1}{2} D^2 f(0)(z_0^3) - \mu \frac{1}{2} D^2 F_k(0)(z_0, D^2 F(0)(z_0^3)) \left| \frac{D^2 f(0)(z_0^3)}{2!} \right| |z_i| = \frac{1}{2} D^2 f(0)(z_0^3) - \mu (D F(0)(z_0^3))^2 \right| \leq \frac{1}{2} \left| (1 - \gamma) \left[ (1 - \frac{1}{2}) \frac{1}{2} A_2 - \frac{1}{2} A_2 + \frac{1}{4} A_2 c_1^2 + \frac{1}{4} A_2 c_1^2 (1 - \gamma)^2 - 2 \mu \frac{1}{4} A_2 c_1^2 (1 - \gamma)^2 \right] = \frac{A_2}{4} \left[ (1 - \gamma) \left[ (1 - \frac{1}{2}) \frac{1}{2} A_2 - \frac{1}{2} A_2 + \frac{1}{4} A_2 c_1^2 + \frac{1}{4} A_2 c_1^2 (1 - \gamma) - 2 \mu A_2 (1 - \gamma) \right] \right. \leq \frac{A_2}{4} \left. \left( 1 - \gamma \right) \right\} \right. \left. \right. \left( 2 - \frac{1}{2} \right) \left. \left. c_1^2 + \frac{1}{2} c_1^2 A_2 \right| A_1 + A_1 \right) \right) \right). (32)
\]

The rest of the proof is similar to the case in Theorem 3.1 (see, (16)), we omit it. The proof is completed. □

Theorem 3.3. Suppose that the function \( \psi(z) = 1 + A_1 z + A_2 z^2 + \cdots + A_n z^n \), \( A_1 > 0 \) satisfies the condition as (1), then the following results hold true:

(I) If \( f \in \mathcal{S}_\psi(U^n), \|z\| = \max_{1 \leq j \leq n} |z_j|, z_0 = \psi, z \in U^n \setminus \{0\} \) and
\[
\frac{1}{2} D^2 f_k(0)(z_0, \frac{D^2 f(0)(z_0^3)}{2!}) |z_k| = \left( \frac{D^2 f_k(0)(z_0^3)}{2!} \right)^2,
\]

then we have
\[
\left\| L_1 - \mu \mathcal{N}_1 \right\| \leq \frac{1}{2} A_1 \max \left\{ 1, \frac{A_2}{A_1} (1 - \gamma)^2 + A_2 - 2 \mu A_1 + \frac{\gamma^2 - 2 \gamma}{(1 - \gamma)^2} \right\} \left\| z \right\|^3,
\]

where
\[
L_1 = \frac{D^3 f(0)(z_0^3)}{3!}, \quad \mathcal{N}_1 = \frac{1}{2} D^2 f(0)(z_0, \frac{D^2 f(0)(z_0^3)}{2!}).
\]
The above estimates are sharp for each complex \( \mu \).

(II) Suppose that \( f : U^m \to \mathbb{C}, F(z) = z f(z) \in \mathcal{S}_{\psi, z}(U^n) \), then for \( z \in U^n \), we have
\[
\left\| L_2 - \mu \mathcal{N}_2 \right\| \leq \frac{1}{2} A_1 \max \left\{ 1, \frac{A_2}{A_1} (1 - \gamma)^2 + A_2 - 2 \mu A_1 + \frac{\gamma^2 - 2 \gamma}{(1 - \gamma)^2} \right\} \left\| z \right\|^3,
\]

where
\[
L_2 = \frac{D^3 F(z_0^3)}{3!}, \quad \mathcal{N}_2 = \frac{1}{2} D^2 F(0)(z, \frac{D^2 F(0)(z_0^3)}{2!}).
\]
The above estimates are sharp for each complex \( \mu \).

Proof. It is easy to obtain the (I) and (II) by making a straightforward calculation in (16) and (32), respectively. □
Remark 3.4. (a) We note that $A_1 = \psi''(0)$ and $A_2 = \frac{1}{2}\psi'''(0)$. Thus, when $\gamma = 0$ in (II) of Theorem 3.3, the result coincide with the main Theorem proved by Xu-Liu-Liu [31].

(b) When $\gamma = 0$ in (I) of Theorem 3.3, the result is the conjecture proposed by Tu-Xiong [25].

c) By choosing suitable functions $\psi$ and real numbers $\gamma$ as Remark 1.2, the solutions on Fekete-Szegő problems for kinds of subclasses of starlike mappings on $\mathbb{U}^n$ can be deduced by our main Theorems immediately.

4. Conclusion

In this paper, by using the subordination techniques, we obtain the sharp coefficients bounds on Fekete-Szegő problem for a certain subclass of starlike mappings, which are defined on the unit polydisk in $\mathbb{C}^n$. Some previous results are improved. Also, the main works give a positive answer to a conjecture proposed by Tu-Xiong [25].

Basic (or $q$-) series and basic (or $q$-) polynomials are known to have widespread applications. In a recent survey-cum-expository review article, Srivastava [24] applied a fractional $q$-calculus operator to define two subclasses of normalized analytic functions with complex order and negative coefficients. With these subclasses, some current developments involving the usages of the basic (or $q$-) calculus in geometric function theory of complex analysis were investigated. Also, Srivastava [24] exposed the inconsequential nature of the so-called $(p,q)$-variations of the $q$-results by inserting an obviously redundant parameter $p$ in the $q$-results. Subsequently, we might try to consider some Fekete-Szegő problems by using the basic (or $q$-) calculus.

References

[1] L. Bieberbach, Uber die Koeffizienten der einigen Potenzreihen welche eine schlichte Abbildung des Einheitskreises vermitteln, S. B. Preuss. Akad. Wiss (1916).
[2] E.C. Bi, G.C.Su, Z.H. Tu, The Kobayashi pseudometric for the Fock-Bargmann-Hartogs domain and its application, J. Geom. Anal. 30 (2020), 86-106.
[3] H. Cartan, Sur la possibilité d’étendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalentes, in: Montel P. (Ed.), Lecons sur les Fonctions Univalentes ou Multivalentes, Gauthier-Villars, Paris (1933).
[4] Y. L. Chung, M. H. Mohd, S. K. Lee, On a Subclass of Close-to-Convex Functions, Bull. Iran. Math. Soc. 44 (2018), 611-621.
[5] I. Graham, H. Hamada, G. Kohr, Parametric representation of univalent mappings in several complex variables, Canad. J. Math. 54 (2002), 324-351.
[6] I. Graham, H. Hamada, T. Honda, K. H. Shon, Growth, distortion and coefficient bounds for Carathéodory families in $\mathbb{C}^n$ and complex Banach spaces, J. Math. Anal. Appl. 416 (2014), 449-469.
[7] S. Gong, The Bieberbach Conjecture, Amer. Math. Soc., International Press, Providence, RI (1999).
[8] H. Hamada, T. Honda, G. Kohr, Growth theorems and coefficient bounds for univalent holomorphic mappings which have parametric representation, J. Math. Anal. Appl. 317 (2006), 302-319.
[9] S. Kanas, H. E. Darwish, Fekete-Szegő problem for starlike and convex functions of complex order, Appl. Math. Lett. 23 (2010), 777-782.
[10] S. Kanas, An unified approach to the Fekete-Szegő problem, Appl. Math. Comput. 218 (2012), 8453-8461.
[11] G. Kohr, On some best bounds for coefficients of subclasses of univalent holomorphic mappings in $\mathbb{C}^n$, Complex Var. 56 (2012), 8453-8461.
[12] H. Luo, Q. H. Xu, On the Fekete and Szegö inequality for the subclass of strongly starlike mappings of order $\alpha$, Results Math. 72 (2017), 343-357.
[13] T. S. Liu, Q. H. Xu, Fekete and Szegö inequality for a subclass of starlike mappings of order $\alpha$ on the bounded starlike circular domain in $\mathbb{C}^n$, Acta Math. Sci. 37B (2017), 722-731.
[14] X. S. Liu, T. S. Liu, The estimates of all homogeneous expansions for a subclass of biholomorphic mappings which have parametric representation in several complex variables, Acta Math. Sin. (Engl. Ser.) 33 (2017), 287-300.
[15] H. Orhan, E. Deniz, D. Răducanu, The Fekete-Szegő problem for subclasses of analytic functions defined by a differential operator related to conic domains, Comput. Math. Appl. 59 (2010), 283-295.
[16] C. Pommerenke, Univalent Functions, in: Studia Mathematica Mathematische Lehrbucher, Vandenhoeck and Ruprecht (1975).
[17] R. K. Raina, J. Sokol, Fekete-Szegő problem for some starlike functions related to shell-like curves, Math. Slovaca 66 (2016), 135-140.
[18] H. M. Srivastava, A. K. Mishra, and P. Goehrayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010), 1188-1192.
[19] H. M. Srivastava, S. Gaboury, F. Ghanim, Initial coefficient estimates for some subclasses of $m$-Fold symmetric bi-univalent functions, Acta Math. Sci. 36B (2016), 863-871.
[20] H. M. Srivastava, S. Hussain, A. Raziq, and M. Raza, The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination, Carpathian J. Math. 34 (2018), 103-113.
[21] H. M. Srivastava, N. Raza, E. S. A. Abujarad, G. Srivastava, and M. H. Abujarad, Fekete-Szegö inequality for classes of \((p, q)\)-starlike and \((p, q)\)-convex functions, Rev. Real Acad. Cienc. Exactas Fas. Natur. Ser. A Mat. (RACSAM) 113 (2019), 3563-3584.

[22] H. M. Srivastava, A. O. Mostafa, M. K. Aouf, and H. M. Zayed, Basic and fractional \(q\)-calculus and associated Fekete-Szegö problem for \(p\)-valently \(q\)-starlike functions and \(p\)-valently \(q\)-convex functions of complex order, Miskolc Math. Notes 20 (2019), 489-509.

[23] H. M. Srivastava, N. Khan, M. Darus, S. Khan, Q. Z. Ahmad, and S. Hussain, Fekete-Szegö type problems and their applications for a subclass of \(q\)-starlike functions with respect to symmetrical points, Mathematics 8 (2020), Article ID 842, 1-18.

[24] H. M. Srivastava, Operators of basic (or \(q\)-) calculus and fractional \(q\)-calculus and their applications in geometric function theory of complex analysis, Iran, J. Sci. Technol. Trans. A: Sci. 44 (2020), 327-344.

[25] Z. H. Tu, L. P. Xiong, Unified Solution of Fekete-Szegö Problem for Subclasses of Starlike Mappings in Several Complex Variables, Math. Slovaca. 69(2019), 1775-1788.

[26] Z. H. Tu, L. P. Xiong, Coefficient problems for unified starlike and convex classes of \(m\)-fold symmetric bi-univalent functions, J. Math. Inequal. 12(2018), 921-932.

[27] H. Tang, H. M. Srivastava, S. Sivasubramanian, and P. Gurusamy, The Fekete-Szegö functional problems for some classes of \(m\)-fold symmetric bi-univalent functions, J. Math. Inequal. 10 (2016), 1063-1092.

[28] Q. H. Xu, T. S. Liu, Biholomorphic mappings on bounded starlike circular domains, J. Math. Anal. Appl. 366(2010), 153-163.

[29] Q. H. Xu, F. Fang, T. S. Liu, On the Fekete and Szegö problem for starlike mappings of order \(\alpha\), Acta Math. Sin. (Engl. Ser.)33(2017), 554-564.

[30] Q. H. Xu, T. S. Liu, On the Fekete and Szegö problem for the class of starlike mappings in several complex variables, Abstr. Appl. Anal. ID 807026(2014), 1-6.

[31] Q. H. Xu, T. S. Liu, X. S. Liu, Fekete and Szegö problem in one and higher dimensions, Sci. China Math. 61(2018), 1775-1788.