MULTI-AGENT SYSTEM FOR TARGET TRACKING ON A SPHERE AND ITS ASYMPTOTIC BEHAVIOR

SUN-HO CHOI, DOHYUN KWON, AND HYOWON SEO

Abstract. We propose a second-order multi-agent system for target tracking on a sphere. The model contains a centripetal force, a bonding force, a velocity alignment operator to the target, and cooperative control between flocking agents. We propose an appropriate regularized rotation operator instead of Rodrigues’ rotation operator to derive the velocity alignment operator for target tracking. By the regularized rotation operator, we can decompose the phase of agents into translational and structural parts. By analyzing the translational part of this reference frame decomposition, we can obtain rendezvous results to the given target. If the multi-agent system can obtain the target’s position, velocity, and acceleration vectors, then the complete rendezvous occurs. Even in the absence of the target’s acceleration information, if the coefficients are sufficiently large enough, then the practical rendezvous occurs.

1. INTRODUCTION

Target tracking refers to designing a dynamical system that agents follow given maneuvering target agents using the information of the targets, such as position, velocity, and acceleration. The target tracking problem is applied in various fields, such as mobile sensor networks, virtual reality, and surveillance systems using unmanned aerial vehicles (UAVs) [18, 22, 24]. Most of the relevant literature focuses on the uncertainty of target motions. From a technical point of view, we can divide the models for this field into measurement models, target motion models, and filtering models. The measurement model deals with target information in a sensor coordinate containing additive noise such as image sensors and radar sensor networks [2, 3, 20]. The target motion model is a coupled dynamical system for target tracking. The filtering model is based on the particle filter method and stochastic frameworks estimating the target state such as nonlinear filtering [13, 15] and adaptive filtering [14, 16].

Depending on the structure of the system, we also divide the models for target tracking into two types of systems: single integrator model and double integrator model. For the single integrator model, one can control the velocity of the agents directly. For example, in [10], the authors proposed a tracking algorithm for a slowly moving target using the target’s position and bearing angle. Many researchers assume agents can obtain only the target’s position and bearing angle for targets maneuvering underwater. From the engineering point of view, it is a reasonable assumption. For the double integrator model, one can control the acceleration of agents. After Olfati-Saber’s seminal work [11], researches for the dynamic tracking system using the double integrator model have been extensively conducted. For this kind of model, the tracking agents can have the position and velocity information of the target. Moreover, to avoid collisions between agents or make a formation flight of the agents, a flocking algorithm and cooperative control are frequently used.

The domain or manifolds of agents are also one of the main topics in this field [1, 18] such as the surveillance system for the restricted area or target tracking system on the whole planet. Our goal is to provide a robust navigational feedback system for the target tracking problem on a sphere. Let
\( \gamma \)-agent be a given target governed by the following system:

\[
\begin{align*}
q_\gamma &= p_\gamma, \\
\dot{q}_\gamma &= -\frac{\|p_\gamma\|^2}{\|q_\gamma\|}q_\gamma + U_\gamma(t),
\end{align*}
\]  

(1.1)

where \( q_\gamma \in S^2, p_\gamma \in T_{q_\gamma} S^2 \), and \( U_\gamma \) are the position, velocity, and control law of the target agent (\( \gamma \)-agent) on sphere, respectively. To conserve the modulus of \( q_\gamma(t) \in S^2 \), we additionally assume that the following condition holds for all \( t \geq 0 \).

\[
q_\gamma(t) \perp U_\gamma(t).
\]

Therefore, the control law \( U_\gamma(t) \) has the following form: for some \( u_\gamma(t) \in \mathbb{R}^3 \),

\[
U_\gamma(t) = \|q_\gamma(t)\|^2 u_\gamma(t) - \langle u_\gamma(t), q_\gamma(t) \rangle q_\gamma(t).
\]

For simplicity, we assume that \( u_\gamma(t) \) is continuous.

For a given \( \gamma \)-agent, we propose a novel multi-agent system for the target tracking on a spherical space:

\[
\begin{align*}
\dot{q}_i(t) &= p_i(t), \\
\dot{p}_i(t) &= -\frac{\|p_i\|^2}{\|q_i\|^2}q_i + \sum_{j=1}^{N} \frac{\sigma_{ij}}{N} \|q_i\|^2 q_j - \langle q_i, q_j \rangle q_i \\
&\quad + c_q \|q_i\|^2 q_\gamma - \langle q_i, q_\gamma \rangle q_i + c_p (P_{q_\gamma \rightarrow q_i}(p_\gamma) - p_i) + U_i,
\end{align*}
\]

(1.2)

where \( q_i \in S^2 \) and \( p_i \in T_{q_i} S^2 \) are the position and velocity of the \( i \)-th agent, respectively. The first term on the right-hand side of the second equation is the centripetal force term to conserve the modulus of \( q_i \). The second term

\[
\sum_{j=1}^{N} \frac{\sigma_{ij}}{N} \|q_i\|^2 q_j - \langle q_i, q_j \rangle q_i
\]

is the cooperative control term between agents where the inter-particle force parameter is given by

\[
\sigma_{ij} = \sigma(\|x_i - x_j\|^2).
\]

The next two terms, \( c_q \|q_i\|^2 q_\gamma - \langle q_i, q_\gamma \rangle q_i \) and \( c_p (P_{q_\gamma \rightarrow q_i}(p_\gamma) - p_i) \), are the bonding force and a velocity alignment term between the target and the \( i \)-th agent, respectively, where \( c_q > 0 \) and \( c_p > 0 \) are target tracking coefficients for the position and velocity, respectively. The last term \( U_i \) is an extra control law based on the target’s information, which will be determined later in (1.5) and (1.6) for each purpose.

Throughout this paper, we assume the initial data satisfies the following admissible conditions on \( S^2 \):

\[
\|q_i(0)\| = 1, \quad \langle p_i(0), q_i(0) \rangle = 0, \quad \forall i \in \{1, \ldots, N\}.
\]

(1.3)

**Definition 1.1.** For a given target \((q_\gamma(t), p_\gamma(t))\), let \( \{(q_i(t), p_i(t))\}_{i=1}^{N} \) be the solution to (1.2). We define the two kinds of rendezvous.

(1) An asymptotic complete rendezvous occurs between the agents and the given target, if

\[
\lim_{t \to \infty} \max_{1 \leq i \leq N} \|q_i(t) - q_\gamma(t)\| = 0.
\]

(2) An asymptotic practical rendezvous occurs between the agents and the given target, if

\[
\lim_{c_q \to 0} \lim_{c_p \to 0} \lim_{t \to \infty} \max_{1 \leq i \leq N} \|q_i(t) - q_\gamma(t)\| = 0.
\]
In what follows, we will show that our model contains many robust properties, including the complete rendezvous. Even in the absence of the target acceleration information, the practical rendezvous occurs when the coefficients are large enough. In particular, we obtain a sharp estimate of the distance between the target and agents. There are many other papers on the dynamics on $\mathbb{R}^n$, but our asymptotic analysis including exponential convergence and practical rendezvous is new on the target tracking problem, to the best of our knowledge.

The derivation of our model is motivated by the decomposition property of flocking dynamics on a flat space. On a flat space, from momentum conservation, the dynamics is represented by the composition of frame reference dynamics and local alignment dynamics as in $[11]$. In contrast to previous results in $\mathbb{R}^n$, it is hard to expect such a decomposition for the flocking model on $S^2$. See Sections 2 and 3 for details. In particular, in our previous papers $[6, 7, 8]$, we used Rodrigues’ rotation operator $R_{\rightarrow}$ to derive a flocking system on a sphere since Rodrigues’ rotation operator $R_{\rightarrow}$ is the most natural flocking operator. However, its composition is complex so that it is difficult to analyze. Moreover, it contains an unavoidable singularity at antipodal points due to its geometric characteristics. From this singularity, even though agents are located on $S^2$, the vanishing point on the communication rate is necessary $[8]$. Due to this difficulty, the target tracking problem on $S^2$ has not been well understood.

We remove the singular term from the natural rotation operator $R_{\rightarrow}$ to obtain a rotation operator in two dimensions:

$$P_{z_1 \rightarrow z_2} := \langle z_1, z_2 \rangle I + z_2 z_1^T - z_1 z_2^T, \quad \text{for } z_1, z_2 \text{ in a unit sphere.}$$

(1.4)

See also Appendix A for the motivation of the non-singularity rotation operator $P$ and its properties. We will prove that its dynamics consists of the composition of the rigid motion part on $S^2$ and the local alignment part. Using this property, we derive an $S^2$-version of the reference frame decomposition in Proposition 3.2 and provide a sufficient condition to obtain a target tracking estimate between multiple agents $\{(q_i(t), p_i(t))\}_{i=1}^N$ and the given target $(q_\gamma(t), p_\gamma(t))$. Moreover, by the regularity of the operator $P$, we can obtain the following global existence result.

**Theorem 1.** Assume that for a continuous function $u_\gamma$, a given target $(q_\gamma(t), p_\gamma(t))$ satisfies (1.1). If the initial data $\{(q_i(0), p_i(0))\}_{i=1}^N$ satisfies (1.3) and $U_i$ is Lipschitz continuous with respect to $\{(q_i, p_i)\}_{i=1}^N$ with $(U_i, q_i) = 0$, then there exists a unique global-in-time solution $\{(q_i(t), p_i(t))\}_{i=1}^N$ to system (1.2) and $\{q_i(t)\}_{i=1}^N$ are located on $S^2$ for all time $t > 0$.

As in $\mathbb{R}^d$, we notice that the velocity alignment operator between the target and the agents plays an important role in target tracking. In particular, the bonding force between the target and the agents, $c_q(\|q\|^2 - \langle q, q_\gamma \rangle q_i)$, alone is not enough to track a target on $S^2$. The velocity alignment operator $c_p(P_{q_\gamma \rightarrow q_i}(p_\gamma) - p_i)$ is crucial for the target tracking algorithm. See the simulations in Section 5. In the next two theorems, we present a quantitative analysis of the velocity alignment operator with two different $U_i$'s:

$$U_i = 2\langle w_\gamma, q_i \rangle (q_i \times p_i) + \dot{w}_\gamma(t) \times q_i \quad (1.5)$$

or

$$U_i = 0, \quad (1.6)$$

where $w_\gamma$ is the angular velocity of the target given by

$$w_\gamma = q_\gamma \times p_\gamma. \quad (1.7)$$

From Theorem 2, if the agents can obtain the exact target information containing acceleration, then the agents can accurately track the target, and the position differences between the target and the agents decay exponentially fast.
**Theorem 2.** Let \((q_{\gamma}(t), p_{\gamma}(t))\) be a target satisfying (1.1) with a continuous target control \(u_{\gamma}\) and \(\{q_{i}(t), p_{i}(t)\}_{i=1}^{N}\) be the solution to (1.2) satisfying (1.3). We assume that \(\sigma_{ij} = \sigma\) is a positive constant and

\[
U_{i} = 2(w_{\gamma}, q_{i})(q_{i} \times p_{i}) + \dot{w}_{\gamma}(t) \times q_{i},
\]

where \(w_{\gamma}\) is the angular velocity defined in (1.7).

If \(c_{q} > \sigma > 0\) or

\[
\frac{1}{N} \sum_{i=1}^{N} ||p_{i}(0) - w_{\gamma}(0) \times q_{i}(0)||^{2} + \frac{\sigma}{2N^{2}} \sum_{i,j=1}^{N} ||q_{i}(0) - q_{j}(0)||^{2} + \frac{c_{q} \sigma}{N} \sum_{i=1}^{N} ||q_{\gamma}(0) - q_{i}(0)||^{2} < \sigma \left( \frac{1 + c_{q} \sigma}{\sigma} \right)^{2},
\]

then the asymptotic complete rendezvous occurs and its convergence rate is exponential, i.e., there are positive constants \(C, D\) such that

\[
||q_{i}(t) - q_{\gamma}(t)||, \quad ||p_{i}(t) - p_{\gamma}(t)|| \leq Ce^{-Dt}.
\]

**Remark 1.1.**

1. If the above sufficient condition in Theorem 2 does not hold, then we can find a steady-state solution. This means that the sufficient condition is almost optimal to lead the convergence result in Theorem 3. See Section 5.

2. The author in [11] does not deal with the estimate of the distance between the target and agents. Our model is inspired by [11], but the target tracking estimate and practical rendezvous are novel.

3. The derivation of \(U_{i}\) in the above theorem is technical, but from the frame decomposition in Proposition 3.2, it is a very natural choice to obtain the complete rendezvous.

The former one in (1.5) corresponds to the case with the target acceleration, while it is unknown in the latter case (1.6). These choices with the different amounts of the target information induce the different accuracies of the target tracking. Since the target information obtained by the agents through observation is usually incomplete, there have been many studies to overcome this incompleteness. For example, many researchers proposed target tracking systems including restricted target information [10, 19], communication-induced delays [12, 17], and additive noise from measurement [9, 23]. The result in Theorem 3 below means that the large coefficients of the system allow the agents to get close enough to the target as needed without acceleration information of the target. In other words, the practical rendezvous occurs.

**Theorem 3.** For \((q_{\gamma}(t), p_{\gamma}(t))\) satisfying (1.1) with a continuous target control \(u_{\gamma}\), let \(\{q_{i}(t), p_{i}(t)\}_{i=1}^{N}\) be the solution to (1.2) subject to the initial data satisfying (1.3) and

\[
U_{i} = 0.
\]

Assume that \(\sigma_{ij} = \sigma\) is a positive constant and the angular velocity of the target and its time derivative are bounded

\[
||w_{\gamma}||, \quad ||\dot{w}_{\gamma}|| < C_{\gamma}.
\]

If \(||p_{i}(0) - p_{\gamma}(0)|| \neq 2\) for all \(i \in \{1, \ldots, N\}\), then the asymptotic practical rendezvous occurs and

\[
||q_{i}(t) - q_{\gamma}(t)|| \leq Ce^{-\frac{4t}{D}} + \frac{C}{D},
\]

where \(C\) is a positive constant depending on the initial data, \(\sigma\), and \(C_{\gamma}\). The constant \(D\) is given by

\[
D := \begin{cases} 
    c_{p} - \sqrt{-4c_{q} + c_{p}^{2}}, & \text{if } c_{p}^{2} \geq -4c_{q}, \\
    c_{p}, & \text{if } c_{p}^{2} < -4c_{q}.
\end{cases}
\]
There are technical issues in the proofs of Theorems 2 and 3. We can obtain the complete rendezvous result in Theorem 2 through Lasalle’s invariance principle with an energy functional. However, Lasalle’s invariance principle does not give a convergence rate. An appropriate Lyapunov functional will be used to obtain the exponential convergence result. In particular, in this case, we derive a closed differential inequality by using six functionals including information on the distance between the target and agents and the distance between agents. The practical rendezvous in Theorem 3 has a more subtle issue. It is necessary to control the distance between the target and agents through the size of the coefficients. However, it is impossible if the coefficients appear in the nonlinear higher-order terms except for the linear terms. If we use a standard functional, the coefficients necessarily occur in the nonlinear terms due to the geometrical characteristics of $S^2$. This problem will be solved by using new functionals inspired by hyperbolic geometry.

The rest of the paper is organized as follows. In Section 2, we present the global-in-time existence and uniqueness of the solution to (1.2) and target tracking results for $\mathbb{R}^3$. Section 3 is devoted to a reference frame decomposition for the main system. From this decomposition, the solution to the main system is represented by the composition of operators for the translational part and the structural part. Next, we reduce the system for the structural part to a linearized system in Section 4. Using this, we prove the complete and practical rendezvous of Theorems 2 and 3 in Section 5. In Section 6, we verify our analytic results using numerical simulations. Section 7 is devoted to the summary of our results.

2. Preliminary: Global well-posedness and Motivations

2.1. The global existence and uniqueness. In this section, we provide the proof of Theorem 3: there is a unique global-in-time solution to (1.2) and this solution is located on the sphere when the initial data satisfies the admissible conditions in (1.3).

For the local existence and uniqueness, we use the same argument in [6, 7]. For given $C^1$ functions $q_\gamma, p_\gamma,$ and $w_\gamma = q_\gamma \times p_\gamma$, we consider the following system of ODEs:

$$
\dot{q}_i(t) = p_i(t),
$$

$$
\dot{p}_i(t) = -\frac{\|p_i\|^2}{\|q_i\|^2} q_i + \sum_{j=1}^N \frac{\sigma(\|x_i - x_j\|^2)}{N} (\|q_i\|^2 q_j - \langle q_i, q_j \rangle q_i)
$$

$$
+ c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma \rangle q_i) + c_p(\langle q_\gamma, q_i \rangle p_\gamma - \langle q_i, p_\gamma \rangle q_\gamma - p_i) + U_i. \tag{2.1}
$$

Here, we will choose $U_i = 2\langle w_\gamma, q_i \rangle (q_i \times p_i) + \dot{w}_\gamma(t) \times q_i$ for the complete rendezvous and $U_i = 0$ for the practical rendezvous.

We assume that the initial data $\{(q_i(0), p_i(0))\}_{i=1}^N$ satisfies the admissible condition in (1.3). Then the right-hand side of (2.1) is Lipschitz continuous with respect to $\{(q_i, p_i)\}_{i=1}^N$ in a small neighborhood of $\{(q_i(0), p_i(0))\}_{i=1}^N$ in $\mathbb{R}^{6N}$. By the Picard-Lindelöf Theorem, there is the maximum time interval $[0, T_M)$ in which a solution of (2.1) exists and it is unique.

We next follow the same argument in [6, 7]. On the maximum time interval $[0, T_M)$, we take the inner product between the second equation of (2.1) and $x_i$ to obtain that

$$
\langle \dot{p}_i, q_i \rangle = -\|p_i\|^2 - c_p \langle p_i, q_i \rangle. \tag{2.2}
$$
By (2.2) and the first equation of (2.1), we obtain that
\[
\frac{d}{dt} \sum_{i=1}^{N} |\langle p_i, q_i \rangle|^2 = 2 \sum_{i=1}^{N} (\langle \dot{p}_i, q_i \rangle + \langle p_i, \dot{q}_i \rangle) \langle p_i, q_i \rangle \\
= 2 \sum_{i=1}^{N} (\langle \dot{p}_i, q_i \rangle + \|p_i\|^2) \langle p_i, q_i \rangle \\
= -2c_p \sum_{i=1}^{N} |\langle p_i, q_i \rangle|^2.
\]
Note that the initial data satisfies \( \sum_{i=1}^{N} |\langle v_i(0), x_i(0) \rangle|^2 = 0 \). Therefore, the Gronwall inequality implies that
\[
\sum_{i=1}^{N} |\langle v_i(t), x_i(t) \rangle| \equiv 0, \quad \text{for} \ t > 0,
\]
and this implies that
\[
\langle v_i(t), x_i(t) \rangle \equiv 0.
\]
We take the inner product between \( \dot{q}_i \) and \( q_i \). By the first equation of (2.1),
\[
\frac{d}{dt} \|q_i\|^2 = 2\langle \dot{q}_i, q_i \rangle = 2\langle p_i, q_i \rangle = 0.
\]
Since initial conditions satisfy \( \|x_i(0)\| = 1 \) and \( \langle v_i(0), x_i(0) \rangle = 0 \) for all \( i \in \{1, \ldots, N\} \), we have
\[
\|x_i(t)\| \equiv 1, \quad \text{for} \ t > 0, \ i \in \{1, \ldots, N\}.
\]
In conclusion, we can apply the extensibility of solutions in [21 Corollary 2.2] to obtain that
\[
T_M = \infty.
\]
Moreover, we can easily check that \( \{(q_i(t), p_i(t))\}_{i=1}^{N} \) is the unique solution to (1.2) with (1.3). Therefore, we can obtain the following proposition.

**Proposition 2.1.** Let \( \{(q_i(t), p_i(t))\}_{i=1}^{N} \) be a solution to (1.2) with (1.3). Then for all \( i \in \{1, \ldots, N\} \) and \( t > 0 \),
\[
\langle q_i(t), p_i(t) \rangle = 0 \quad \text{and} \quad \|q_i(t)\| = 1.
\]

**2.2. Target tracking problem in \( \mathbb{R}^3 \).** In this section, we estimate the distance between the target and agents for the following model in \( \mathbb{R}^3 \):
\[
\dot{q}_i = p_i, \\
\dot{p}_i = \sum_{j=1}^{N} \frac{\psi_{ij}}{N} (p_j - p_i) + \sum_{j=1}^{N} \frac{\sigma_{ij}}{N} (q_j - q_i) + c_q (q_\gamma - q_i) + c_p (p_\gamma - p_i) + u_i,
\]
where \( q_i \in \mathbb{R}^3 \) and \( p_i \in \mathbb{R}^3 \) are the position and velocity of the \( i \)th agent, respectively. Here, \( q_\gamma, p_\gamma, \) and \( u_\gamma \) are the position, velocity, and acceleration of a given target (\( \gamma \)-agent) satisfying
\[
\dot{q}_\gamma = p_\gamma, \\
\dot{p}_\gamma = u_\gamma.
\]
A new input parameter \( u_i \) will be determined later. Depending on the information of the target, we choose two different \( u_i \)’s and analyze the corresponding asymptotic behaviors. The argument is straightforward, and thus the reader familiar with target tracking problems in \( \mathbb{R}^3 \) may skip this section.
Therefore, we can easily check that \( q \) complete rendezvous with an exponential decay rate occurs for any positive \( c_q \). This is a simple linear system of ODEs and it has the following solution:

\[
q_c = \frac{1}{N} \sum_{i=1}^{N} q_i, \quad p_c = \frac{1}{N} \sum_{i=1}^{N} p_i,
\]

and

\[
x_i = q_i - q_c, \quad v_i = p_i - p_c.
\]

Then, the above dynamics can be decomposed into the translational dynamics (2.5) and the structural dynamics (2.4):

\[
\dot{q}_c = p_c, \quad \dot{p}_c = c_q (q_\gamma - q_c) + c_p (p_\gamma - p_c) + u_i,
\]

and

\[
\dot{x}_i = x_i, \quad \dot{v}_i = \sum_{j=1}^{N} \sigma_{ij} (x_j - x_i) - c_q x_i - c_p v_i.
\]

The structural dynamics part in (2.5) has been analyzed in [11].

We focus on the translational dynamics part in (2.4) for two different cases of \( u_i \). We first suppose that all of the position \( p_\gamma \), velocity \( q_\gamma \), and acceleration \( u_\gamma \) of the target are given. In this case, it is natural to choose \( u_i := u_\gamma \). Let

\[
q_d = q_c - q_\gamma, \quad p_d = p_c - p_\gamma.
\]

Then the translational dynamics in (2.4) can be rewritten as

\[
\dot{q}_d = p_d, \quad \dot{p}_d = -c_q q_d - c_p p_d.
\]

This is a simple linear system of ODEs and it has the following solution:

\[
q_d(t) = \frac{1}{2c_q} \left[ c_p q_d(0) e^{\frac{c_p}{2} t} \left( -e^{-\sqrt{c_q^2 - 4c_p} c_p} q_d(0) \sqrt{c_p^2 - 4c_q c_p} + \left( c_p q_d(0) + q_d(0) \right) \sqrt{c_p^2 - 4c_q c_p} \right) \right]
\]

Therefore, we can easily check that \( q_d \) and \( p_d \) converge to zero exponentially. This means that the complete rendezvous with an exponential decay rate occurs for any positive \( c_q \) and \( c_p \).

If we only know the position and velocity of the target, we cannot expect a complete rendezvous. On the other hand, we can control the maximum position difference between the target and agents if the tracking coefficients for the target are sufficiently large. We refer to [4, 5] for related issues.

For \( u_i = 0 \), the translational dynamics is given by

\[
\dot{q}_d = p_d, \quad \dot{p}_d = -c_q q_d - c_p p_d - u_\gamma.
\]
As we mentioned above, we cannot expect the complete rendezvous for this case. Alternatively, to obtain the practical rendezvous estimate, we additionally assume that the acceleration of the target is bounded:

$$\limsup \| u_\gamma \| \leq C_\gamma,$$

(2.6)

for some $C_\gamma > 0$. Then we define auxiliary variables as follows.

$$X_d^1 = \langle q_d, q_d \rangle, \quad X_d^2 = \langle q_d, p_d \rangle, \quad X_d^3 = \langle p_d, p_d \rangle.$$

By the system of the translational dynamics, we can obtain

$$\begin{align*}
\dot{X}_d^1 &= 2X_d^2, \\
\dot{X}_d^2 &= X_d^3 - c_p X_d^1 - c_p X_d^3 - \langle q_d, u_\gamma \rangle, \\
\dot{X}_d^3 &= -2c_q X_d^2 - 2c_p X_d^3 - 2\langle p_d, u_\gamma \rangle.
\end{align*}$$

We rewrite the above system of equations as the following inhomogeneous linear system of ODEs:

$$\dot{X}_d = A_d X_d + F_d,$$

where $X_d = (X_d^1, X_d^2, X_d^3)^T$ and $F_d = (0, -\langle q_d, u_\gamma \rangle, -2\langle p_d, u_\gamma \rangle)^T$, and the coefficient matrix is given by

$$M_d = \begin{bmatrix} 0 & 2 & 0 \\ -c_q & -c_p & 1 \\ 0 & -2c_q & -2c_p \end{bmatrix}.$$ 

Note that $M_d$ has the following eigenvalues.

$$\left\{ -c_p, -c_p - \sqrt{c_p^2 - 4c_q}, -c_p + \sqrt{c_p^2 - 4c_q} \right\}.$$

Let $D_d < 0$ be the greatest real part in the above eigenvalues and let $-\mu_d = D_d$.

Then, we have

$$\frac{d}{dt} \| X_d \|^2 = 2\langle X_d, M_d X_d \rangle + 2\langle X_d, F_d \rangle$$

$$\leq -2\mu_d \| X_d \|^2 + 2\| X_d \| \| F_d \|,$$

this implies that

$$\frac{d}{dt} \| X_d \| \leq -\mu_d \| X_d \| + \| F_d \|.$$

From elementary calculations, it follows that for any $\epsilon > 0$,

$$\| F_d \| \leq \| q_d \| \| u_\gamma \| + 2\| p_d \| \| u_\gamma \|$$

$$\leq \frac{\epsilon \| q_d \|^2}{2} + \frac{1}{2\epsilon} \| u_\gamma \|^2 + \frac{\epsilon \| p_d \|^2}{2} + \frac{2}{\epsilon} \| u_\gamma \|^2$$

$$\leq \epsilon \| X_d \| + \frac{5}{2\epsilon} \| u_\gamma \|^2.$$

We choose $\epsilon = \mu_d/2$ and use the Gronwall inequality and (2.6) to obtain that

$$\| X_d \| \leq e^{-\mu_d t} \| X_d(0) \| + C_2 \frac{5}{2\epsilon} e^{(\mu_d - \epsilon) t} \int_0^t \| u_\gamma(s) \| e^{(\mu_d - \epsilon) s} ds$$

$$\leq e^{-\mu_d t} \| X_d(0) \| + C_2 \frac{5}{2\epsilon} e^{(\mu_d - \epsilon) t} e^{(\mu_d - \epsilon) - 1} / (\mu_d - \epsilon).$$
This implies that
\[ \limsup_{t \to \infty} \|X_d\| \leq \frac{10C_2^2}{\mu_d^2}. \]

Thus, if we choose a sufficiently large tracking coefficients \( c_q, c_p > 0 \), then we obtain that
\[ \limsup_{t \to \infty} \|q_i(t) - q_\gamma(t)\|, \limsup_{t \to \infty} \|p_i(t) - p_\gamma(t)\| < 1. \]

3. Generalized rotation operator on sphere and reference frame decomposition

In this section, we decompose our model (1.2) on \( S^2 \) into structural dynamics and translational dynamics. Due to the complexity of (1.2), the decomposition of agents' positions into a sum of two vectors as the model in \( \mathbb{R}^3 \) is not suitable for our case. Instead, we observe that a rigid body motion on \( S^2 \) can be used as a reference frame. Choosing an appropriate rigid body motion, our model can be represented as the composition of a rigid body motion and local alignment dynamics. The rigid body motion can be derived based on the angular velocity tensor \( W_\gamma(t) \) of the \( \gamma \)-agent and a generalized rotation operator \( S_\gamma \) along the given target described below. Recall the given \( \gamma \)-agent trajectory on \( S^2 \):
\[ \dot{q}_\gamma = p_\gamma, \]
where \( q_\gamma \in S^2 \) and \( p_\gamma \in T_qS^2 \) are the position and velocity of the given \( \gamma \)-agent, respectively.

Let
\[ w_\gamma = q_\gamma \times p_\gamma. \]
By elementary calculation, we have \( q_\gamma \times w_\gamma = -p_\gamma \) and
\[ \dot{q}_\gamma = w_\gamma \times q_\gamma. \]
For the angular velocity vector \( w_\gamma = (w^1_\gamma, w^2_\gamma, w^3_\gamma)^T \), we define the angular velocity tensor \( W_\gamma(t) \) of the \( \gamma \)-agent by
\[
W^\gamma_t = \begin{bmatrix}
0 & -w^3_\gamma(t) & w^2_\gamma(t) \\
-w^3_\gamma(t) & 0 & -w^1_\gamma(t) \\
w^3_\gamma(t) & w^1_\gamma(t) & 0
\end{bmatrix}.
\]
From the above notation, the equation for the \( \gamma \)-agent is written by
\[ \dot{q}_\gamma = p_\gamma = W^\gamma_t q_\gamma. \]

Now, we consider the following system of ODEs:
\[ \dot{x}(t) = W^\gamma_t x(t). \quad (3.1) \]
We can define the corresponding solution operator \( S_\gamma(x_0, t) = S^\gamma t x_0 : S^2 \times [0, \infty) \to S^2 \) such that
\[ S^\gamma t x_0 = x(t; x_0), \quad (3.2) \]
where \( x(t; x_0) \) is the solution to (3.1) subject to
\[ x(0; x_0) = x_0 \in S^2. \quad (3.3) \]
One can easily check that \( S^\gamma t \) is a rigid body motion on \( S^2 \).

**Lemma 3.1.** Let \( x_\gamma(t) \in S^2 \) be the position of a \( \gamma \)-agent which is a \( C^2 \) function with respect to \( t \geq 0 \). For the given \( \gamma \)-agent, the solution operator \( S^\gamma_t \) defined above is represented by a matrix and the matrix product. Moreover, for any \( x, y \in \mathbb{R}^3 \),
\[ \|x\|^2 = \|S^\gamma_t x\|^2, \quad (x, y) = (S^\gamma_t x, S^\gamma_t y). \]
Proof. Let \( x_\gamma(t) \) be a given \( C^2 \) function with \( \|x_\gamma(t)\| = 1 \). We define the solution operator \( S^t_\gamma \) by (3.1)-(3.3). Take any two vectors \( x^0_1 \) and \( x^0_2 \) on \( S^2 \). Let

\[
x_1(t) = S^t_\gamma x^0_1, \quad x_2(t) = S^t_\gamma x^0_2.
\]

Equivalently,

\[
\dot{x}_1(t) = W^t_\gamma x_1(t), \quad \dot{x}_2(t) = W^t_\gamma x_2(t),
\]

subject to

\[
x_1(0) = x^0_1, \quad x_2(0) = x^0_2.
\]

Then we have

\[
\dot{x}_1(t) - \dot{x}_2(t) = W^t_\gamma (x_1(t) - x_2(t)).
\]

This implies that

\[
\frac{1}{2} \frac{d}{dt} \|x_1(t) - x_2(t)\|^2 = \langle x_1(t) - x_2(t), W^t_\gamma (x_1(t) - x_2(t)) \rangle.
\]

We note that \( W_\gamma \) is a skew symmetric matrix and this implies that

\[
\langle x_1(t) - x_2(t), W^t_\gamma (x_1(t) - x_2(t)) \rangle = \langle W^T_\gamma (x_1(t) - x_2(t)), x_1(t) - x_2(t) \rangle
\]

\[
= -\langle W_\gamma (x_1(t) - x_2(t)), x_1(t) - x_2(t) \rangle
\]

\[
= -\langle x_1(t) - x_2(t), W_\gamma (x_1(t) - x_2(t)) \rangle.
\]

Therefore, we can obtain that

\[
\langle x_1(t) - x_2(t), W^t_\gamma (x_1(t) - x_2(t)) \rangle = 0
\]

and

\[
\frac{d}{dt} \|x_1(t) - x_2(t)\|^2 = 0.
\]

Since we choose \( x^0_1 \) and \( x^0_2 \) arbitrary, \( S^t_\gamma : S^2 \to S^2 \) is a rigid body motion of \( S^2 \). This implies that \( S^t_\gamma \) is represented by a matrix and the matrix product. Moreover, the following holds.

\[
\|x\|^2 = \|S^t_\gamma x\|^2, \quad \langle x, y \rangle = \langle S^t_\gamma x, S^t_\gamma y \rangle,
\]

for any \( x, y \in \mathbb{R}^3 \). \( \square \)

In \( \mathbb{R}^3 \), the agent’s position can be decomposed into a sum of two vectors as described in (2.3)-(2.5). Similarly, the agent’s position on \( S^2 \) is expressed as the composition of the translational operator \( S^t_\gamma \) and the structural vector \( x_i \): 

\[
q_i(t) = S^t_\gamma x_i(t). \tag{3.4}
\]

Notice that \( x_\gamma(t) := q_i(0) \) is a time-independent fixed point on \( S^2 \) and satisfies

\[
q_i(t) = S^t_\gamma x_\gamma(t). \tag{3.5}
\]

In the proposition below, we derive a second-order system of \( x_i \) in the moving frame.

**Proposition 3.2.** Let \((q_\gamma(t), p_\gamma(t))\) be a given \( \gamma \)-agent satisfying

\[
\dot{q}_\gamma = p_\gamma,
\]

where \( q_\gamma \in S^2 \) and \( p_\gamma \in T_{q_\gamma}S^2 \). Let \( S^t_\gamma \) be the solution operator defined by (3.1)-(3.3). If (3.4) and (3.5) hold, then the followings are equivalent.
Proof. For any $x \in \mathbb{S}^2$, we consider $x(t) = S^t_x x_0$. Then

$$\dot{S^t_x} x_0 = \frac{d}{dt}(S^t_x x_0) = \dot{x}(t) = W^t_{\gamma} x(t) = W^t_{\gamma} S^t_x x_0.$$ (3.8)

Since $x_0$ is arbitrary and $S^t_x$ is a $3 \times 3$ matrix by Lemma 3.1 we have

$$\dot{S^t_x} = W^t_{\gamma} S^t_x.$$ (3.9)

We note that for any $x \in \mathbb{R}^3$,

$$W^t_{\gamma} x = w_{\gamma} \times x.$$ (3.10)

We first prove that if $\{(x_i(t), v_i(t))\}_{i=1}^N$ satisfies (3.6), then $\{(q_i(t), \dot{q}_i(t))\}_{i=1}^N$ is the solution to the main system with (3.7) subject to (3.8) with

$$U_i = 2\langle w_{\gamma}, q_i \rangle (q_i \times p_i) + \dot{w}_{\gamma}(t) \times q_i + S^t_{\gamma} A_i.$$ (3.7)

(1) $\{(x_i(t), v_i(t))\}_{i=1}^N$ satisfies the following structural system of ODEs:

$$\dot{x}_i = v_i, \quad \dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} x_i + \sum_{j=1}^N \sigma_{ij} \left(\|x_j\|^2 x_j - (x_i, x_j)x_i\right) + c_q \left(\|x_i\|^2 x_\gamma - (x_i, x_\gamma)x_i\right) - c_p v_i + A_i,$$ (3.6)

subject to initial data $x_i(0) \in \mathbb{S}^2$, $v_i(0) \in T_{x_i(0)} \mathbb{S}^2$ for all $i \in \{1, \ldots, N\}$.

(2) $\{(q_i(t), p_i(t))\}_{i=1}^N$ is the solution to main system (1.2) subject to (1.3) with

$$U_i = 2\langle w_{\gamma}, q_i \rangle (q_i \times p_i) + \dot{w}_{\gamma}(t) \times q_i + S^t_{\gamma} A_i.$$ (3.7)

Thus, we have

$$\frac{d}{dt} p_i = \dot{S}^t_x x_i + S^t_x \dot{x}_i.$$ (3.11)

By (3.6) and Lemma 3.1

$$S^t_x \dot{x}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} S^t_x x_i + \sum_{j=1}^N \sigma_{ij} \left(\|x_j\|^2 S^t_x x_j - (x_i, x_j)S^t_x x_i\right) + c_q \left(\|x_i\|^2 S^t_x x_\gamma - (x_i, x_\gamma)S^t_x x_i\right) - c_p S^t_x v_i + S^t_{\gamma} A_i.$$ (3.12)

From the property of $S^t_x$ in Lemma 3.1 it follows that

$$\|x_i\|^2 = \|S^t_x x_i\|^2, \quad (x_i, x_j) = (S^t_x x_i, S^t_x x_j).$$

As $[8, 9, 10]$, we can easily prove that

$$x_i(t) \in \mathbb{S}^2, \quad v_i(t) \in T_{x_i(t)} \mathbb{S}^2, \quad \text{for all} \quad t \geq 0, \quad i \in \{1, \ldots, N\}.$$ (3.13)

By this modulus conservation and (3.12),

$$S^t_x \dot{x}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} q_i + \sum_{j=1}^N \sigma_{ij} \left(\|q_j\|^2 q_j - (q_i, q_j)q_i\right) + c_q \left(\|q_i\|^2 q_\gamma - (q_i, q_\gamma)q_i\right) + c_p \left[W^t_{\gamma} q_i - p_i\right] + S^t_{\gamma} A_i.$$ (3.14)
Here, we used (3.19) and (3.211) to obtain

\[-S_{\gamma i}^t v_i = W_{\gamma i}^t q_i - p_i.\]  

(3.15)

By (3.3), (3.19), (3.16) and the definition of $q_i$ and $p_i$,

\[
\begin{aligned}
\tilde{S}_{\gamma i}^t x_i + 2\tilde{S}_{\gamma i}^t \dot{x}_i &= \tilde{W}_{\gamma i}^t S_{\gamma i}^t x_i + W_{\gamma i}^t S_{\gamma i}^t x_i + 2\tilde{S}_{\gamma i}^t \dot{x}_i \\
&= W_{\gamma i}^t q_i + W_{\gamma i}^t W_{\gamma i}^t q_i + 2W_{\gamma i}^t S_{\gamma i}^t \dot{x}_i \\
&= \tilde{W}_{\gamma i}^t q_i + W_{\gamma i}^t W_{\gamma i}^t q_i + 2W_{\gamma i}^t (p_i - W_{\gamma i}^t) q_i \\
&= W_{\gamma i}^t q_i - W_{\gamma i}^t W_{\gamma i}^t q_i + 2W_{\gamma i}^t p_i.
\end{aligned}
\]

(3.16)

Clearly, by the skew symmetric property of $W_{\gamma i}$,

\[
\|p_i\|^2 = \|W_{\gamma i}^t S_{\gamma i}^t x_i\|^2 + 2(W_{\gamma i}^t S_{\gamma i}^t x_i, S_{\gamma i}^t \dot{x}_i) + \|S_{\gamma i}^t \dot{x}_i\|^2
\]

\[
= \|W_{\gamma i}^t q_i\|^2 + 2(W_{\gamma i}^t q_i, p_i - W_{\gamma i}^t) q_i + \|v_i\|^2
\]

\[
= \langle q_i, W_{\gamma i}^t W_{\gamma i}^t q_i \rangle - 2\langle q_i, W_{\gamma i}^t p_i \rangle + \|v_i\|^2.
\]

This implies that

\[-\|v_i\|^2 = -\|p_i\|^2 + \langle q_i, W_{\gamma i}^t W_{\gamma i}^t q_i \rangle - 2\langle q_i, W_{\gamma i}^t p_i \rangle.
\]

(3.17)

By (3.16) and (3.17), we have

\[
\begin{aligned}
\tilde{S}_{\gamma i}^t x_i + 2\tilde{S}_{\gamma i}^t \dot{x}_i - \|v_i\|^2 q_i &= -\|p_i\|^2 q_i + \langle q_i, W_{\gamma i}^t W_{\gamma i}^t q_i \rangle q_i - W_{\gamma i}^t W_{\gamma i}^t q_i \\
&- 2\langle q_i, W_{\gamma i}^t p_i \rangle q_i + 2W_{\gamma i}^t p_i + W_{\gamma i}^t q_i.
\end{aligned}
\]

(3.18)

Thus, by (3.14) and (3.18),

\[
\dot{p} = \tilde{S}_{\gamma i}^t x_i + 2\tilde{S}_{\gamma i}^t \dot{x}_i + S_{\gamma i}^t \dot{x}_i
\]

\[
= -\|p_i\|^2 q_i + \sum_{j=1}^N \sigma_{ij} (\|q_j\|^2 q_j - \langle q_i, q_j \rangle q_i) + c_q (\|q_i\|^2 q_i - \langle q_i, q_i \rangle q_i)
\]

\[
+ c_p (W_{\gamma i}^t q_i - p_i) + \langle q_i, W_{\gamma i}^t W_{\gamma i}^t q_i \rangle q_i - W_{\gamma i}^t W_{\gamma i}^t q_i
\]

\[
- 2\langle q_i, W_{\gamma i}^t p_i \rangle q_i + 2W_{\gamma i}^t p_i + W_{\gamma i}^t q_i + S_{\gamma i}^t A_i.
\]

(3.19)

We note that for any $x \in \mathbb{R}^3$,

\[
W_{\gamma i}^t x = w_{\gamma} \times x.
\]

(3.20)

From (3.19), (3.20) and the modulus conservation property of $S_{\gamma i}^t$ with $x_i(t) \in S^2$, it follows that

\[
\dot{p} = -\frac{\|p_i\|^2}{\|q_i\|^2} q_i + \sum_{j=1}^N \sigma_{ij} (\|q_j\|^2 q_j - \langle q_i, q_j \rangle q_i) + c_q (\|q_i\|^2 q_i - \langle q_i, q_i \rangle q_i)
\]

\[
+ c_p (w_{\gamma} \times q_i - p_i) + 2(w_{\gamma}, q_i)(q_i \times p_i) + \dot{w}_{\gamma} \times q_i + S_{\gamma i}^t A_i
\]

Now, if we choose $A_1$ such as

\[
2\langle w_{\gamma}, q_i \rangle (q_i \times p_i) + \dot{w}_{\gamma}(t) \times q_i + S_{\gamma i}^t A_i = 0,
\]

then our model corresponds to $u_i = 0$ case in the flat space case, and if we choose $A_i = 0$ then our model corresponds to $u_i = u_{\gamma}$ case in the flat space case. From the uniqueness of the solution to the main system, we obtain the desired result.

We next prove that if \((q_i(t), p_i(t))\) is the solution to the main system with (3.7), then \((x_i(t), \dot{x}_i(t))\) satisfies (3.9), where $x_i(t) = S_{\gamma i}^{-1}(t)q_i(t)$. By the first equation of (3.2), we have

\[
p_i = \dot{q}_i = S_{\gamma i}^t x_i + S_{\gamma i}^t \dot{x}_i = W_{\gamma i}^t S_{\gamma i}^t x_i + S_{\gamma i}^t \dot{x}_i.
\]

(3.21)
This implies that
\[ \ddot{q}_i = \dot{S}^t_i x_i + 2\dot{S}^t_i \dot{x}_i + S^t_i \ddot{x}_i, \]
\[ = W^t_i S^t_i x_i + W^t_i W^t_i S^t_i x_i + 2W^t_i S^t_i \dot{x}_i + S^t_i \ddot{x}_i. \tag{3.22} \]

By (3.21), we have
\[ \|p_i\|^2 = \|W^t_i S^t_i x_i\|^2 + 2(W^t_i S^t_i x_i, S^t_i \dot{x}_i) + \|S^t_i \dot{x}_i\|^2 \]
\[ = \|W^t_i q_i\|^2 + 2(W^t_i q_i, p_i - W^t_i q_i) + \|v_i\|^2 \]
\[ = \langle q_i, W^t_i W^t_i q_i \rangle - 2\langle q_i, W^t_i p_i \rangle + \|v_i\|^2. \tag{3.23} \]

The second equation in (1.2) and \( q_i(t) \in S^2 \) imply that
\[ \ddot{q}_i = -\|p_i\|^2 q_i + \frac{\sum_{j=1}^N \sigma_{ij}}{N} (\|q_j\|^2 q_j - \langle q_i, q_j \rangle q_i) + c_q(\|q_i\|^2 q_i - \langle q_i, q_j \rangle q_i) + c_p(P_{q_i}, q_i - p_i) + U_i, \]
\[ = -\|p_i\|^2 S^t_i x_i + \frac{\sum_{j=1}^N \sigma_{ij}}{N} (\|x_j\|^2 S^t_j x_j - \langle x_i, x_j \rangle S^t_j x_i) \]
\[ + c_q(\|x_i\|^2 S^t_i x_i - \langle x_i, x_j \rangle S^t_i x_i) + c_p(P_{q_i}, q_i - p_i) + U_i. \]

From the property of \( S^t_i \) in Lemma (3.1), it follows that
\[ \ddot{q}_i = -\|p_i\|^2 S^t_i x_i + \frac{\sum_{j=1}^N \sigma_{ij}}{N} (\|x_i\|^2 S^t_j x_j - \langle x_i, x_j \rangle S^t_j x_i) \]
\[ + c_q(\|x_i\|^2 S^t_i x_i - \langle x_i, x_j \rangle S^t_i x_i) + c_p(P_{q_i}, q_i - p_i) + U_i. \tag{3.24} \]

By (3.22) - (3.21),
\[ S^t_i \ddot{x}_i = -\left[ W^t_i S^t_i x_i + W^t_i W^t_i S^t_i x_i + 2W^t_i S^t_i \dot{x}_i \right] \]
\[ - \|p_i\|^2 S^t_i x_i + \frac{\sum_{j=1}^N \sigma_{ij}}{N} (\|x_i\|^2 S^t_j x_j - \langle x_i, x_j \rangle S^t_j x_i) \]
\[ + c_q(\|x_i\|^2 S^t_i x_i - \langle x_i, x_j \rangle S^t_i x_i) + c_p(P_{q_i}, q_i - p_i) + U_i, \]
\[ = -\left[ W^t_i S^t_i x_i + W^t_i W^t_i S^t_i x_i + 2W^t_i S^t_i \dot{x}_i \right] \]
\[ - \langle q_i, W^t_i W^t_i q_i \rangle - 2\langle q_i, W^t_i p_i \rangle + \|v_i\|^2 \]
\[ + \frac{\sum_{j=1}^N \sigma_{ij}}{N} (\|x_i\|^2 S^t_j x_j - \langle x_i, x_j \rangle S^t_j x_i) \]
\[ + c_q(\|x_i\|^2 S^t_i x_i - \langle x_i, x_j \rangle S^t_i x_i) + c_p(P_{q_i}, q_i - p_i) + U_i. \]

Note that
\[ -\left[ W^t_i S^t_i x_i + W^t_i W^t_i S^t_i x_i + 2W^t_i S^t_i \dot{x}_i \right] - \left( \langle q_i, W^t_i W^t_i q_i \rangle - 2\langle q_i, W^t_i p_i \rangle \right) S^t_i x_i \]
\[ = -\left[ W^t_i q_i + W^t_i W^t_i q_i + 2W^t_i (p_i - W^t_i q_i) \right] - \left( \langle q_i, W^t_i W^t_i q_i \rangle - 2\langle q_i, W^t_i p_i \rangle \right) q_i \]
\[ = -\langle q_i, W^t_i W^t_i q_i \rangle q_i + W^t_i W^t_i q_i + 2(q_i, W^t_i p_i) q_i - 2W^t_i p_i - \dot{W}^t_i q_i \]
\[ = -2\langle w^t_i q_i, q_i \rangle - \dot{w}^t_i \times q_i. \]

Therefore, by the property of \( S^t_i \) and the above two equalities, we obtain that \( \{(x_i(t), v_i(t))\}_{i=1}^N \) satisfies (3.4) with (3.7).
4. REDUCTION TO A LINEARIZED SYSTEM WITH A NEGATIVE DEFINITE COEFFICIENT MATRIX

In this section, we derive a linearized system from the structural system in (3.6). We define auxiliary variables motivated by the flat case in Section 2 and we extract leading order terms using $\|q_i(t)\| = 1$ and $\langle q_i(t), p_i(t) \rangle = 0$ for all $t \geq 0$ and $i \in \{1, \ldots, N\}$. In the system with respect to auxiliary variables, leading order terms form an inhomogeneous linear system of ODEs with a negative definite coefficient matrix.

We consider the following system of ODEs with $\sigma_{ij} = \sigma > 0$ and $c_q, c_p > 0$.

$$\dot{x}_i = v_i,$$

$$\dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} x_i + \sum_{j=1}^{N} \frac{\sigma}{N} (\|x_i\|^2 x_j - \langle x_i, x_j \rangle x_i)$$

$$+ c_q (\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma \rangle x_i) - c_p v_i + A_i. \quad (4.1)$$

For consistency, we additionally assume that for all $t \geq 0$,

$$\langle A_i(t), x_i(t) \rangle = 0, \quad \text{for all } i \in \{1, \ldots, N\},$$

and the initial data satisfies

$$\|x_i(0)\| = 1 \text{ and } \langle v_i(0), x_i(0) \rangle = 0, \quad \text{for all } i \in \{1, \ldots, N\}.$$

We now define the auxiliary variables as follows.

$$X_1^\gamma = \frac{1}{N} \sum_{i=1}^{N} \|x_i - x_\gamma\|^2, \quad X_2^\gamma = \frac{1}{N} \sum_{i=1}^{N} \langle x_i - x_\gamma, v_i \rangle, \quad X_3^\gamma = \frac{1}{N} \sum_{i=1}^{N} \langle v_i, v_i \rangle,$$

and

$$X_1 = \frac{1}{N^2} \sum_{i,k=1}^{N} \langle x_i - x_k, x_i - x_k \rangle, \quad X_2 = \frac{1}{N^2} \sum_{i,k=1}^{N} \langle v_i - v_k, x_i - x_k \rangle,$$

$$X_3 = \frac{1}{N^2} \sum_{i,k=1}^{N} \langle v_i - v_k, v_i - v_k \rangle.$$

We also define the corresponding inhomogeneous terms as follows.

$$F_1^\gamma = 0,$$

$$F_2^\gamma = -\frac{1}{N} \sum_{i=1}^{N} \frac{\|v_i\|^2}{2} \|x_i - x_\gamma\|^2 + \frac{\sigma}{4N^2} \sum_{i,j=1}^{N} \|x_i - x_j\|^2 \|x_i - x_\gamma\|^2$$

$$+ c_q \frac{1}{4N} \sum_{i=1}^{N} \|x_i - x_\gamma\|^4 + \frac{1}{N} \sum_{i=1}^{N} \langle x_i - x_\gamma, A_i \rangle,$$

$$F_3^\gamma = \frac{2}{N} \sum_{i=1}^{N} \langle v_i, A_i \rangle.$$
and
\[ F^1 = 0, \]
\[ F^2 = -\frac{1}{N^2} \sum_{i,k=1}^{N} \frac{\|v_i\|^2 + \|v_k\|^2}{2} \|x_i - x_k\|^2 + \frac{\sigma}{2N^3} \sum_{i,j,k=1}^{N} \|x_i - x_j\|^2 \|x_i - x_k\|^2 \]
\[ + \frac{c_q}{2N^2} \sum_{i,k=1}^{N} \|x\_\gamma - x_i\|^2 \|x_i - x_k\|^2 + \frac{1}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, x_i - x_k \rangle, \]
\[ F^3 = \frac{2}{N^2} \sum_{i,k=1}^{N} (\|v_i\|^2 \langle x_i, v_k \rangle + \|v_k\|^2 \langle x_i, v_i \rangle) + \frac{2\sigma}{N^3} \sum_{i,j,k=1}^{N} \|x_i - x_j\|^2 \langle x_i, v_k \rangle \]
\[ + \frac{c_q}{N^2} \sum_{i,k=1}^{N} \|x\_\gamma - x_i\|^2 \langle x_i, v_k \rangle + \frac{2}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, v_i - v_k \rangle. \]

Let
\[ X = (X^1, X^2, X^3, X^1, X^2, X^3)^T, \quad F = (F^1, F^2, F^3, F^1, F^2, F^3)^T. \]  

(4.2)

**Proposition 4.1.** For the auxiliary variable \( X \) and the inhomogeneous term \( F \), the following holds.
\[ \dot{X} = MX + F, \]  

where the coefficient matrix \( M \) is given by
\[
M = \begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 \\
-c_q & -c_p & 1 & -\sigma/2 & 0 & 0 \\
0 & -2c_q & -2c_p & 0 & \sigma & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -(c_q + \sigma) & -c_p & 1 \\
0 & 0 & 0 & 0 & -2(c_q + \sigma) & -2c_p
\end{bmatrix}.
\]

**Proof.** Clearly,
\[ \frac{d}{dt} X^1 = 2X^2. \]

For \( X^2 \), we have
\[
\frac{d}{dt} X^2 = X^3 + \frac{1}{N} \sum_{i=1}^{N} \langle x_i - x\_\gamma, \dot{v}_i \rangle
\]
\[ = X^3 + \frac{1}{N} \sum_{i=1}^{N} \left( x_i - x\_\gamma, -\|v_i\|^2 x_i + \sum_{j=1}^{N} \frac{\sigma}{N} (\|x_i\|^2 x_j - \langle x_i, x_j \rangle x_j) \right) \]
\[ + c_q (\|x_i\|^2 x\_\gamma - \langle x_i, x\_\gamma \rangle x_i) - c_p v_i + A_i \]
\[ = X^3 - \frac{1}{N} \sum_{i=1}^{N} \frac{\|v_i\|^2}{2} \|x_i - x\_\gamma\|^2 + \frac{\sigma}{N^2} \sum_{i,j=1}^{N} \langle x_i - x\_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle \]
\[ + \frac{c_q}{N} \sum_{i=1}^{N} \langle x_i - x\_\gamma, x_i - \langle x_i, x\_\gamma \rangle x_i \rangle - \frac{c_p}{N} \sum_{i=1}^{N} \langle x_i - x\_\gamma, v_i \rangle + \frac{1}{N} \sum_{i=1}^{N} \langle x_i - x\_\gamma, A_i \rangle. \]
Note that by $x_i \in S^2$ and changing the indices,

\[
\sum_{i,j=1}^{N} \langle x_i - x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle = - \sum_{i,j=1}^{N} \langle x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle \\
= - \sum_{i,j=1}^{N} \|x_i - x_j\|^2 \frac{1}{2} \langle x_\gamma, x_i \rangle \\
= - \frac{1}{2} \sum_{i,j=1}^{N} \|x_i - x_j\|^2 + \frac{1}{2} \sum_{i,j=1}^{N} \|x_i - x_j\|^2 \|x_i - x_\gamma\|^2.
\]

By (4.3), we have

\[
\frac{d}{dt} X_2^\gamma = X_3^\gamma - \frac{1}{N} \sum_{i=1}^{N} \|v_i\|^2 \|x_i - x_\gamma\|^2 - \sigma X_1^\gamma + \frac{\sigma}{2} X_1^\gamma + \sum_{i,j=1}^{N} \|x_i - x_\gamma\|^2 \|x_i - x_j\|^2 - \frac{\sigma}{4} X_1^\gamma + \frac{\sigma}{4} X_3^\gamma + \frac{1}{N} \sum_{i=1}^{N} (x_i - x_\gamma, A_i) \\
= -c_q X_1^\gamma - c_p X_3^\gamma + X_3^\gamma - \frac{\sigma}{2} X_1^\gamma + F_3^\gamma.
\]

Similarly, we have

\[
\frac{1}{2} \frac{d}{dt} X_3^\gamma = \frac{1}{N} \sum_{i=1}^{N} \langle v_i, \dot{v}_i \rangle \\
= \frac{1}{N} \sum_{i=1}^{N} \langle v_i, -\|v_i\|^2 x_i + \sum_{j=1}^{N} \|x_i\|^2 x_j - \langle x_i, x_j \rangle x_i \rangle \\
+ c_q (\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma \rangle x_i) - c_p v_i + A_i \rangle \\
= \frac{\sigma}{N^2} \sum_{i,j=1}^{N} \langle v_i, x_j \rangle - \frac{1}{N} \sum_{i=1}^{N} c_q \langle v_i, x_i - x_\gamma \rangle - \frac{1}{N} \sum_{i=1}^{N} c_p \langle v_i, v_i \rangle + \frac{1}{N} \sum_{i=1}^{N} \langle v_i, A_i \rangle.
\]

Thus, we have

\[
\frac{d}{dt} X_3^\gamma = -2c_q X_2^\gamma - 2c_p X_3^\gamma - \sigma X_2^\gamma + F_3^\gamma.
\]

(4.4)

For $X^1$,

\[
\frac{d}{dt} X^1 = 2X^2.
\]
Similar to the previous cases, we use the second equation in (4.1) to obtain

\[ \frac{d}{dt}X^2 = X^3 + \frac{1}{N^2} \sum_{i,k=1}^{N} \langle \dot{v}_i - \dot{v}_k, x_i - x_k \rangle \]

\[ = X^3 + \frac{1}{N^2} \sum_{i,k=1}^{N} \left( -\|v_i\|^2 x_i + \|v_k\|^2 x_k + \sum_{j=1}^{N} \frac{\sigma}{N} \left[ -\langle x_i, x_j \rangle x_i + \langle x_k, x_j \rangle x_k \right] \right. \]

\[ + c_q \left[ -\langle x_i, x_\gamma \rangle x_i + \langle x_k, x_\gamma \rangle x_k \right] - c_p v_i + c_p v_k, x_i - x_k \]

\[ + \frac{1}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, x_i - x_k \rangle. \]

By \( x_i \in S^2 \), we have

\[ \frac{d}{dt}X^2 = X^3 - \frac{1}{N^2} \sum_{i,k=1}^{N} \frac{\|v_i\|^2 + \|v_k\|^2}{2} \|x_i - x_k\|^2 \]

\[ - \sigma X^1 + \frac{\sigma}{4N^2} \sum_{i,j,k=1}^{N} \|x_i - x_j\|^2 \|x_i - x_k\|^2 + \frac{\sigma}{4N^2} \sum_{i,j,k=1}^{N} \|x_k - x_j\|^2 \|x_i - x_k\|^2 \]

\[ - c_q X^1 + \frac{c_q}{4N^2} \sum_{i,j,k=1}^{N} \|x_\gamma - x_i\|^2 \|x_i - x_k\|^2 + \frac{c_q}{4N^2} \sum_{i,j,k=1}^{N} \|x_\gamma - x_k\|^2 \|x_i - x_k\|^2 \]

\[ - c_p X^2 + \frac{1}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, x_i - x_k \rangle. \]

Changing the indices implies that

\[ \frac{d}{dt}X^2 = -\sigma X^1 - c_q X^1 - c_p X^2 + X^3 + F^2. \]
Finally, for $X^3$, we obtain

\[
\frac{1}{2} \frac{d}{dt} X^3 = \frac{1}{N^2} \sum_{i,k=1}^{N} \left( -\|v_i\|^2 x_i + \|v_k\|^2 x_k + \sum_{j=1}^{N} \frac{\sigma}{N} (-\langle x_i, x_j \rangle x_j) \right) \\
+ cj \left[ -\langle x_i, x_k \rangle x_i + \langle x_k, x_j \rangle x_k \right] - c_p v_i + c_p v_k, \ v_i - v_k \right) \\
+ \frac{1}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, v_i - v_k \rangle \\
= \frac{1}{N^2} \sum_{i,k=1}^{N} \left( \|v_i\|^2 \langle x_i, v_k \rangle + \|v_k\|^2 \langle x_k, v_i \rangle \right) \\
- \sigma X^2 + \sum_{i,j,k=1}^{N} \frac{\sigma}{2N^3} \|x_i - x_j\|^2 \langle x_i, v_k \rangle + \sum_{i,j,k=1}^{N} \frac{\sigma}{2N^3} \|x_k - x_j\|^2 \langle x_k, v_i \rangle \\
- c_q X^2 + \frac{c_q}{4N^2} \sum_{i,j,k=1}^{N} \|x_\gamma - x_i\|^2 \langle x_i, v_k \rangle + \frac{c_q}{4N^2} \sum_{i,j,k=1}^{N} \|x_\gamma - x_k\|^2 \langle x_k, v_i \rangle - c_p X^3 \\
+ \frac{1}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, v_i - v_k \rangle.
\]

Thus, we conclude that

\[
\frac{d}{dt} X^3 = -2\sigma X^2 - 2c_q X^2 - 2c_p X^3 + F^3.
\]

\[\square\]

Note that the eigenvalues of the $6 \times 6$ coefficient matrix $M$ have the only negative real part. The above result will be used for the complete rendezvous case.

**Remark 4.2.** In [4], we use $l^\infty$-framework to obtain a uniform decay estimate which is independent of $N$. However, due to $X^2$ term on the right-hand side of (4.4), we cannot use this $l^\infty$-framework. We obtain only the convergence result depending on $N$ by using the $6 \times 6$ system with $l^2$-framework.

For the practical rendezvous result, we use a different framework, weighted $l^\infty$-framework. To obtain $l^\infty$-estimate, we define the following functionals:

\[
X_1^1 = \frac{4}{4 - \|x_i - x_j\|^2}, \quad X_1^2 = \frac{16 \langle x_i - x_j, v_k \rangle}{(4 - \|x_i - x_j\|^2)^2}, \quad X_1^3 = \frac{16 \langle v_i, v_i \rangle}{(4 - \|x_i - x_j\|^2)^2},
\]

and

\[
F_1^1 = 0,
\]

\[
F_1^2 = \frac{\|v_i\|^2}{2} \left( \frac{16 \|x_i - x_j\|^2}{4 - \|x_i - x_j\|^2} + \frac{16\sigma}{N (4 - \|x_i - x_j\|^2)^2} \sum_{j=1}^{N} \langle x_i - x_j, x_j - x_j \rangle x_i \right) \\
+ \frac{16 \langle x_i - x_j, A_i \rangle}{(4 - \|x_i - x_j\|^2)^2} + \frac{64 \langle x_i - x_j, v_i \rangle^2}{(4 - \|x_i - x_j\|^2)^3} \\
F_1^3 = \frac{32\sigma}{N (4 - \|x_i - x_j\|^2)^2} \sum_{j=1}^{N} \langle v_i, x_j \rangle + \frac{32 \langle v_i, A_i \rangle}{(4 - \|x_i - x_j\|^2)^2} + \frac{64 \langle v_i, v_i \rangle \langle x_i - x_j, v_i \rangle}{(4 - \|x_i - x_j\|^2)^3}.
\]

We note that due to the geometric structure of $S^2$, the quartic terms with the coefficient $c_q$ in $F_1^2$ and $F_1^3$ appear. Thus, the standard functional $X(t)$ in the previous argument and Section 2 does not
work for this practical rendezvous case. For the complete rendezvous case, we will use the energy functional method and Lasalle’s invariance principle to control the quartic terms. However, for the practical rendezvous case, we cannot use the same methodology since the system is not autonomous. Thus, if an extra term with the coefficient $c_q$ appears in $F$, then it is hard to obtain the desired result. Alternatively, using the functionals in (4.5), we can remove the quartic term with the coefficient $c_q$ as in (4.6).

By the same argument in Proposition 4.3, we have

$$
\frac{d}{dt} X_i^1 = 2X_i^1.
$$

Using the second equation for the structural system, we obtain the following for $X_i^2$:

$$
\frac{d}{dt} X_i^2 = X_i^3 + \frac{16\langle x_i - x_\gamma, v_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{64\langle x_i - x_\gamma, v_i \rangle^2}{(4 - \|x_i - x_\gamma\|^2)^3}
$$

$$
= X_i^3 + 16\left(\langle x_i - x_\gamma, -\|v_i\|^2 x_i + \sum_{j=1}^N \frac{\sigma}{2N}(\|x_i\|^2 x_j - \langle x_i, x_j \rangle x_i)
\right) + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma \rangle x_i) - c_p v_i + A_i \right) / (4 - \|x_i - x_\gamma\|^2)^2
$$

$$
+ \frac{64(x_i - x_\gamma, v_i)^2}{(4 - \|x_i - x_\gamma\|^2)^3}
$$

$$
= X_i^3 - \frac{\|v_i\|^2}{2} \frac{16\|x_i - x_\gamma\|^2}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{16\sigma}{N (4 - \|x_i - x_\gamma\|^2)^2} \sum_{j=1}^N \langle x_i - x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle
$$

$$
+ \frac{16c_q}{(4 - \|x_i - x_\gamma\|^2)^2} \langle x_i - x_\gamma, x_\gamma - \langle x_i, x_\gamma \rangle x_i \rangle - \frac{16c_p}{(4 - \|x_i - x_\gamma\|^2)^2} \langle x_i - x_\gamma, v_i \rangle
$$

$$
+ \frac{16\langle x_i - x_\gamma, A_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{64(x_i - x_\gamma, v_i)^2}{(4 - \|x_i - x_\gamma\|^2)^3}.
$$

Note that

$$
\langle x_i - x_\gamma, x_\gamma - \langle x_i, x_\gamma \rangle x_i \rangle = \langle x_i - x_\gamma, x_\gamma \rangle - \langle x_i, x_\gamma \rangle \langle x_i - x_\gamma, x_i \rangle
$$

$$
= -\|x_i - x_\gamma\|^2 - \langle x_i - x_\gamma, x_\gamma \rangle \langle x_i - x_\gamma, x_i \rangle
$$

$$
= -\|x_i - x_\gamma\|^2 + \frac{\|x_i - x_\gamma\|^4}{4}.
$$

This implies that

$$
\frac{d}{dt} X_i^2 = X_i^3 - \frac{\|v_i\|^2}{2} \frac{16\|x_i - x_\gamma\|^2}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{16\sigma}{N (4 - \|x_i - x_\gamma\|^2)^2} \sum_{j=1}^N \langle x_i - x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle
$$

$$
- c_q X_i^1 - c_p X_i^2 + \frac{16(x_i - x_\gamma, A_i)}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{64(x_i - x_\gamma, v_i)^2}{(4 - \|x_i - x_\gamma\|^2)^3}.
$$
For $X^3_i$, we have
\[
\frac{d}{dt} X^3_i = \frac{32\langle v_i, \dot{v}_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{64\langle v_i, v_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^3} x_i \cdot (x_i - x_\gamma, v_i)
\]
\[
+ \sum_{j=1}^{N} \frac{\sigma_{ij}}{N} (\|x_i\|^2 x_j - \langle x_i, x_j \rangle x_i)
+ c_q (\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma \rangle x_i - c_p v_i + A_i) / (4 - \|x_i - x_\gamma\|^2)^2 + \frac{64\langle v_i, v_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^3} (x_i - x_\gamma, v_i)
\]
\[
= \frac{32\sigma}{N (4 - \|x_i - x_\gamma\|^2)^2} \sum_{j=1}^{N} \langle v_i, x_j \rangle - 32c_q (4 - \|x_i - x_\gamma\|^2)^2 - 32c_p (4 - \|x_i - x_\gamma\|^2)^2 + 64\langle v_i, v_i \rangle (x_i - x_\gamma, v_i)
\]
\[
+ \frac{32\langle v_i, A_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^2} + 32\langle v_i, A_i \rangle (4 - \|x_i - x_\gamma\|^2)^2 + \frac{64\langle v_i, v_i \rangle}{(4 - \|x_i - x_\gamma\|^2)^3} (x_i - x_\gamma, v_i).
\]
In conclusion, we have
\[
\frac{d}{dt} X^1_i = 2X^2_i + F^1_i,
\]
\[
\frac{d}{dt} X^2_i = -c_q X^1_i - c_p X^2_i + X^3_i + F^2_i,
\]
\[
\frac{d}{dt} X^3_i = -2c_q X^2_i - 2c_p X^3_i + F^3_i.
\]
Therefore, we have proved the following proposition.

**Proposition 4.3.** Let
\[
X_i = (X^1_i, X^2_i, X^3_i)^T, \quad F_i = (F^1_i, F^2_i, F^3_i)^T,
\]
where $X^k_i$, $F^k_i$, $k = 1, 2, 3$ are functionals defined in (4.5) and (4.6).

Then the following holds.
\[
\dot{X}_i = M_\infty X_i + F_i,
\]
where the coefficient matrix $M_\infty$ is given by
\[
M_\infty = \begin{bmatrix}
0 & 2 & 0 \\
-c_q & -c_p & 1 \\
0 & -2c_q & -2c_p
\end{bmatrix}.
\]

5. **Asymptotic Analysis on the Target Tracking Models: Complete and Practical Rendezvous**

In this section, we provide the proofs of Theorems 2 and 3 in Section 4. Let $(q_\gamma, p_\gamma)$ be the phase of the target. We assume that the target satisfies (1.1) for some continuous $u_\gamma(t) \in \mathbb{R}^3$. For the given target $(q_\gamma(t), p_\gamma(t))$, let $(\{q_i(t), p_i(t)\})_{i=1}^{N}$ be the solution to (1.2). By the argument in Section 4, we have the following equivalent system for $x_i(t) = S^{-1}_\gamma(t)p_i(t)$.
\[
\dot{x}_i = v_i,
\]
\[
\dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} x_i + \sum_{j=1}^{N} \frac{\sigma_{ij}}{N} (\|x_i\|^2 x_j - \langle x_i, x_j \rangle x_i) + c_q (\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma \rangle x_i) - c_p v_i + A_i,
\]
where $S_t^\gamma$ is the solution operator defined by (5.1)–(5.3). For the angular velocity $w_\gamma = q_\gamma \times p_\gamma$, $A_i$ is the extra control law given by

$$A_i = S_t^{-1}(t)U_i - 2\langle w_\gamma, q_i \rangle S_t^{-1}(t)[q_i \times p_i] - S_t^{-1}(t)[\dot{w}\gamma(t) \times q_i].$$

5.1. Complete rendezvous. We assume that $\sigma_{ij} = \sigma > 0$ and $A_i = 0$, i.e.,

$$U_i = 2\langle w_\gamma, q_i \rangle q_i \times p_i - \dot{w}\gamma(t) \times q_i.$$

We first use an energy functional method to obtain the convergence result in Theorem 2 without convergence rate. We now define an energy functional $\mathcal{E} = \mathcal{E}(\{(x_i, v_i)\}_{i=1}^N)$ as follows.

$$\mathcal{E} = \mathcal{E}_k + \mathcal{E}_c,$$

where $\mathcal{E}_k$ is the kinetic energy given by

$$\mathcal{E}_k = \frac{1}{2N} \sum_{i=1}^N \|v_i\|^2,$$

and $\mathcal{E}_c$ is the configuration energy given by

$$\mathcal{E}_c = \frac{\sigma}{4N^2} \sum_{i,j=1}^N \|x_i - x_j\|^2 + \frac{c_q}{2N} \sum_{i=1}^N \|x_\gamma - x_i\|^2.$$

This energy functional has a dissipation property. To obtain this, we take the inner product between $v_i$ and $v_i$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|v_i\|^2 = -\frac{\|v_i\|^2}{\|x_i\|^2} \langle x_i, v_i \rangle + \frac{\sigma}{N} \sum_{j=1}^N (\|x_i\|^2 \langle x_j, v_i \rangle - \langle x_i, x_j \rangle \langle x_i, v_i \rangle)$$

$$+ \frac{c_q}{N} (\|x_j\|^2 \langle x_i, v_j \rangle - \langle x_i, x_j \rangle \langle x_i, v_j \rangle) - c_p \|v_i\|^2.$$  

Using the orthogonality $\langle x_i, v_i \rangle = 0$ and $\|x_i\| = 1$ in (5.12), we have

$$\frac{1}{2} \frac{d}{dt} \|v_i\|^2 = \sum_{j=1}^N \frac{\sigma}{N} \langle x_j, v_i \rangle + c_q \langle x_\gamma, v_i \rangle - c_p \|v_i\|^2.$$

Therefore,

$$\frac{d}{dt} \mathcal{E}_k = \sum_{i,j=1}^N \frac{\sigma}{N^2} \langle x_j, v_i \rangle + c_q \sum_{i=1}^N \langle x_\gamma, v_i \rangle - c_p \sum_{i=1}^N \|v_i\|^2.$$

Similarly,

$$\frac{d}{dt} \mathcal{E}_c = \frac{\sigma}{2N^2} \sum_{i,j=1}^N \langle x_i - x_j, v_i - v_j \rangle - \frac{c_q}{N} \sum_{i=1}^N \langle x_\gamma, v_i \rangle$$

$$= -\frac{\sigma}{2N^2} \sum_{i,j=1}^N (\langle x_i, v_j \rangle + \langle x_j, v_i \rangle) - \frac{c_q}{N} \sum_{i=1}^N \langle x_\gamma, v_i \rangle$$

$$= -\frac{\sigma}{N^2} \sum_{i,j=1}^N \langle x_j, v_i \rangle - \frac{c_q}{N} \sum_{i=1}^N \langle x_\gamma, v_i \rangle.$$

Therefore, we have

$$\frac{d}{dt}(\mathcal{E}_k + \mathcal{E}_c) = -\frac{c_q}{N} \sum_{i=1}^N \|v_i\|^2 = -2c_q \mathcal{E}_k.$$

We notice that (5.1) is autonomous, since $x_\gamma$ is a constant vector. Moreover, the energy functional $\mathcal{E}$ is zero if and only if $v_i = 0$ for all $i \in \{1, \ldots, N\}$. 
We can easily prove that the union of the following two sets is the maximum invariant set of $E$.

\[
\left\{ \{(x_i, v_i)\}_{i=1}^N : v_i = 0, \quad x_i = x_\gamma \text{ for all } i \in \{1, \ldots, N\} \right\}
\]

and

\[
\left\{ \{(x_i, v_i)\}_{i=1}^N : v_i = 0, \quad \frac{\sigma}{N} \sum_{j=1}^N x_j + c_q x_\gamma = 0 \text{ for all } i \in \{1, \ldots, N\} \right\}.
\]

If we assume that $c_q > \sigma$ or $E(0) < \frac{\sigma}{2} \left( 1 + \frac{c_q}{\sigma} \right)^2$, then $\frac{\sigma}{N} \sum_{j=1}^N x_j + c_q x_\gamma \neq 0$. Thus, by Lasalle’s invariance principle,

\[
\|v_i(t)\| \to 0 \quad \text{and} \quad x_i(t) \to x_\gamma
\]
as $t \to \infty$. Therefore, we have proved the following proposition.

**Proposition 5.1.** If $c_q > \sigma$ or $E(0) < \frac{\sigma}{2} \left( 1 + \frac{c_q}{\sigma} \right)^2$, then

\[
v_i(t) \to 0
\]

and

\[
x_i(t) \to x_\gamma(t)
\]
as $t \to \infty$ for any initial data satisfying $x_i(0) \neq -x_\gamma(0)$ for all $i \in \{1, \ldots, N\}$.

Next we consider exponential decay estimates for $\|x_i - x_\gamma\|$ and $\|v_i\|$. For notational simplicity, we define the following two functionals.

$$D_x(t) = \max_{1 \leq i \leq N} \|x_i(t) - x_\gamma(t)\|^2$$

and

$$D_v(t) = \max_{1 \leq i \leq N} \|v_i(t)\|^2.$$

**Proposition 5.2.** Assume that $A_i = 0$. Then for the functional $F$ defined in (4.2), the following estimate holds

\[
\|F\| \leq 8(\sigma + c_q)[D_x(t) + D_v(t)]X_\gamma^4.
\]

**Proof.** By elementary calculation, we have

\[
|F_1^1| = 0,
\]

\[
|F_2^1| \leq \left( \frac{D_v(t)}{2} + \frac{\sigma D_x(t)}{4} + \frac{c_q D_x(t)}{4} \right) X_\gamma^4,
\]

\[
|F_3^\gamma| = 0,
\]

and

\[
|F_4^\gamma| = 0.
\]
and
\[ F^1 = 0, \]
\[ F^2 = - \frac{1}{N^2} \sum_{i,k=1}^{N} \left( \frac{1}{2} \|v_i\|^2 + \frac{1}{2} \|v_k\|^2 \right) \|x_i - x_k\|^2 + \frac{\sigma}{2N^2} \sum_{i,j,k=1}^{N} \|x_i - x_j\|^2 \|x_i - x_k\|^2 \]
\[ + \frac{c_q}{2N^2} \sum_{i,k=1}^{N} \|x_i - x_k\|^2 \|x_i - x_k\|^2 + \frac{1}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, x_i - x_k \rangle, \]
\[ F^3 = \frac{2}{N^2} \sum_{i,k=1}^{N} \left( \|v_i\|^2 \langle x_i, v_k \rangle + \|v_k\|^2 \langle x_i, v_i \rangle \right) + \frac{2\sigma}{N^3} \sum_{i,j,k=1}^{N} \|x_i - x_j\|^2 \langle x_i, v_k \rangle \]
\[ + \frac{c_q}{N^2} \sum_{i,k=1}^{N} \|x_i - x_k\|^2 \langle x_i, v_k \rangle + \frac{2}{N^2} \sum_{i,k=1}^{N} \langle A_i - A_k, v_i - v_k \rangle. \]

Note that
\[ \|x_i - x_k\|^2 \leq \|x_i - x_\gamma + x_\gamma - x_k\|^2 \]
\[ \leq 2\|x_i - x_\gamma\|^2 + 2\|x_\gamma - x_k\|^2 \quad (5.2) \]
and
\[ |\langle x_i, v_k \rangle| = |\langle x_i - x_k, v_k \rangle| \]
\[ \leq |\langle x_i - x_\gamma, v_k \rangle| + |\langle x_\gamma - x_k, v_k \rangle| \]
\[ \leq \mathcal{D}_x(t) + \mathcal{D}_v(t). \quad (5.3) \]

By \((5.2)\) and \((5.3)\), we have
\[ |F^2| \leq \frac{\mathcal{D}_v(t)}{N^2} \sum_{i,k=1}^{N} \|x_i - x_k\|^2 + \frac{2\sigma \mathcal{D}_x(t)}{N^2} \sum_{i,k=1}^{N} \|x_i - x_k\|^2 + \frac{c_q \mathcal{D}_x(t)}{2N^2} \sum_{i,k=1}^{N} \|x_i - x_k\|^2, \]
and
\[ |F^3| \leq 4(\mathcal{D}_x(t) + \mathcal{D}_v(t)) \frac{1}{N} \sum_{i=1}^{N} \|v_i\|^2 + \frac{2\sigma(\mathcal{D}_x(t) + \mathcal{D}_v(t))}{N^2} \sum_{i,j=1}^{N} \|x_i - x_j\|^2 \]
\[ + \frac{c_q(\mathcal{D}_x(t) + \mathcal{D}_v(t))}{N} \sum_{i=1}^{N} \|x_i - x_k\|^2. \]

Similarly, we have
\[ \frac{1}{N^2} \sum_{i,k=1}^{N} \|x_i - x_k\|^2 = \frac{1}{N^2} \sum_{i,k=1}^{N} \|x_i - x_\gamma + x_\gamma - x_k\|^2 \]
\[ \leq 4X^1_\gamma. \]

Therefore, we obtain that
\[ |F^1| = 0, \]
\[ |F^2| \leq (4\mathcal{D}_v(t) + 8\sigma \mathcal{D}_x(t) + 2c_q \mathcal{D}_x(t)) X^1_\gamma, \]
\[ |F^3| \leq (8\sigma + c_q)(\mathcal{D}_x(t) + \mathcal{D}_v(t))X^1_\gamma + 4(\mathcal{D}_x(t) + \mathcal{D}_v(t))X^3_\gamma. \]

The above implies the result in this lemma.
We are ready to prove Theorem 2. We first check that the coefficient matrix $M$ has the following six eigenvalues.

\[
\{-c_p, -c_p, -c_p \pm \sqrt{-4c_q + c_p^2}, -c_p \pm \sqrt{-4c_q + c_p^2 - 4\sigma}\}.
\]

Thus, their real parts are all negative. Let $D$ be the greatest real part of the above eigenvalues and we define

\[
\mu := -D > 0.
\]

Then by Proposition 5.1 for any $\epsilon > 0$, there is $t_0 > 0$ such that if $t > t_0$, then

\[
0 \leq D_\omega(t) + D_\nu(t) < \frac{\epsilon}{4(1 + 2\sigma + 2c_q)}.
\]

From Proposition 4.1 and 5.2 it follows that

\[
X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t} e^{A(t-s)}F(s)ds.
\]

This implies that

\[
\|X(t)\| \leq e^{-\mu(t-t_0)}\|X(t_0)\| + \int_{t_0}^{t} e^{-\mu(t-s)}\|F(s)\|ds
\]

\[
\leq e^{-\mu(t-t_0)}\|X(t_0)\| + \epsilon \int_{t_0}^{t} e^{-\mu(t-s)}\|X(s)\|ds.
\]

Therefore, by the Gronwall inequality, if $t > t_0$, then

\[
\|X(t)\| \leq \|X(t_0)\| e^{-(\mu - \epsilon)(t-t_0)}.
\]

5.2. Practical rendezvous. In this part, we consider the target tracking problem without acceleration information of the target. We assume that $\sigma_{ij} = \sigma > 0$ and target speed and acceleration are bounded:

\[
\|\omega_\gamma(t)\|, \|\ddot{\omega}_\gamma(t)\| < C_{\omega}^\gamma, \quad t \geq 0,
\]

where $C_{\omega}^\gamma > 0$ is a positive constant. We assume that $U_i = 0$. We first check that the coefficient matrix $M_\infty$ in Proposition 4.3 has the following eigenvalues.

\[
\{-c_p, -c_p \pm \sqrt{-4c_q + c_p^2}\}.
\]

Thus, their real parts are all negative. Let $D_\infty$ be the greatest real part of the above eigenvalues and we define

\[
\mu_\infty := -D_\infty > 0.
\]

Let

\[
X_\infty = \max_{1 \leq i \leq N} \|X_i\|.
\]

By Proposition 4.3 for any fixed $t > 0$, there is an index $i_t \in \{1, \ldots, N\}$ such that

\[
\|X_{i_t}\| = X_\infty
\]

and

\[
\frac{d}{dt} X_\infty^2 = \frac{d}{dt} \|X_{i_t}\|^2
\]

\[
= (X_{i_t}, M_\infty X_{i_t}) + (X_{i_t}, F_{i_t})
\]

\[
\leq -\mu_\infty \|X_{i_t}\|^2 + \|X_{i_t}\| \|F_{i_t}\|
\]

\[
= -\mu_\infty X_\infty^2 + X_\infty \|F_{i_t}\|.
\]
By direct calculation,
\[ |F_1^3| = 0, \]
\[ |F_1^2| \leq \frac{16 \|x_i - x_\gamma\|^2}{2 (4 - \|x_i - x_\gamma\|^2)^2} + \frac{16\sigma}{N (4 - \|x_i - x_\gamma\|^2)^2} \sum_{j=1}^{N} |\langle x_i - x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle| \]
\[ + \frac{16 \|x_i - x_\gamma, A_i\|}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{64 (x_i - x_\gamma, v_i)^2}{(4 - \|x_i - x_\gamma\|^2)^3} \]
\[ |F_1^3| \leq \frac{32\sigma}{N (4 - \|x_i - x_\gamma\|^2)^2} \sum_{j=1}^{N} |\langle v_i, x_j \rangle| + \frac{32|\langle v_i, A_i \rangle|}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{64\langle v_i, v_i \rangle \|x_i - x_\gamma, v_i\|}{(4 - \|x_i - x_\gamma\|^2)^3}. \]

We note that
\[ \langle x_i - x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle = \langle x_i - x_\gamma, x_j - x_\gamma \rangle + \langle x_i - x_\gamma, x_\gamma \rangle - \langle x_i - x_\gamma, \langle x_i, x_j \rangle x_i \rangle \]
\[ = \langle x_i - x_\gamma, x_j - x_\gamma \rangle - \frac{1}{2} \|x_i - x_\gamma\|^2 - \frac{\langle x_i, x_j \rangle}{2} \|x_i - x_\gamma\|^2. \]

This implies that
\[ |\langle x_i - x_\gamma, x_j - \langle x_i, x_j \rangle x_i \rangle| \leq 2 \max_{1 \leq i \leq N} \|x_i - x_\gamma\|^2. \]

Similarly,
\[ |\langle v_i, x_j \rangle| = |\langle v_i, x_j - x_\gamma \rangle| + |\langle v_i, x_\gamma - x_i \rangle| \leq \|v_i\|^2 + \max_{1 \leq i \leq N} \|x_i - x_\gamma\|^2, \]
\[ \langle x_i - x_\gamma, v_i \rangle^2 \leq 4\|v_i\|^2. \]

Thus,
\[ |F_1^3| = 0, \]
\[ |F_1^2| \leq 2X_{\infty} + \frac{32\sigma \max_{1 \leq i \leq N} \|x_i - x_\gamma\|^2}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{16 \|x_i - x_\gamma, A_i\|}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{256\|v_i\|^2}{(4 - \|x_i - x_\gamma\|^2)^3}, \]
\[ |F_1^3| \leq 2\sigma X_{\infty} + \frac{32\sigma \max_{1 \leq i \leq N} \|x_i - x_\gamma\|^2}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{32|\langle v_i, A_i \rangle|}{(4 - \|x_i - x_\gamma\|^2)^2} + \frac{256\|v_i\|^3}{(4 - \|x_i - x_\gamma\|^2)^3}. \]

By elementary calculation, we have
\[ |\langle x_i - x_\gamma, A_i \rangle| \leq \|x_i - x_\gamma\|^2 + \frac{\|A_i\|^2}{4}. \]

Note that
\[ \|A_i\|^2 \leq 6\|\omega_\gamma\|^2\|S_\gamma^{-1}(t)\|p_i\|^2 + 3\|\omega_\gamma\|^2. \]

Since \(p_i(t) = W_\gamma S_\gamma^i x_i(t) + S_\gamma^i \dot{x}_i(t)\),
\[ \|S_\gamma^{-1}(t)q_i\|^2 \leq 2\|\omega_\gamma\|^2 + 2\|v_i\|^2 \]

and
\[ \|A_i\|^2 \leq 12(C^w_\gamma)^2\|v_i\|^2 + 12(C^w_\gamma)^4 + 3(C^w_\gamma)^2. \]

Therefore, we have
\[ |\langle x_i - x_\gamma, A_i \rangle| \leq \|x_i - x_\gamma\|^2 + 3(C^w_\gamma)^2\|v_i\|^2 + 3(C^w_\gamma)^4 + \frac{3(C^w_\gamma)^2}{4}. \] (5.4)

Similarly, we have
\[ |\langle v_i, A_i \rangle| \leq \|v_i\|^2 + 3(C^w_\gamma)^2\|v_i\|^2 + 3(C^w_\gamma)^4 + \frac{3(C^w_\gamma)^2}{4}. \] (5.5)
By (5.4)-(5.5) and the above argument, if \( \max_{1 \leq i \leq N} \|x_i - x_\gamma\| < \frac{2\sqrt{C_1 - 1}}{\sqrt{C_1}} < 2 \), then

\[
|F^1_i| = 0,
\]

\[
|F^2_i| \leq (2 + 2\sigma C_1 + 5C_1 + 3(C^w_\gamma)^2)X_\infty + 3(C^w_\gamma)^4C_1^2 + \frac{3(C^w_\gamma)^2C_1^2}{4},
\]

\[
|F^3_i| \leq (2 + 2\sigma + 2\sigma C_1 + 6(C^w_\gamma)^2)X_\infty + 6(C^w_\gamma)^4C_1^2 + \frac{6(C^w_\gamma)^2C_1^2}{4} + 4X_\infty^{3/2}.
\]

We conclude that

\[
\|F_i\| \leq (4 + 2\sigma + 4\sigma C_1 + 5C_1 + 9(C^w_\gamma)^2)X_\infty + 9(C^w_\gamma)^4C_1^2 + \frac{9(C^w_\gamma)^2C_1^2}{4} + 4X_\infty^{3/2}.
\]

Therefore, we obtain that

\[
\dot{X}_\infty \leq -\mu_\infty X_\infty + (4 + 2\sigma + 4\sigma C_1 + 5C_1 + 9(C^w_\gamma)^2)X_\infty + 9(C^w_\gamma)^4C_1^2 + \frac{9(C^w_\gamma)^2C_1^2}{4} + 4X_\infty^{3/2}.
\]

We choose \( c_q \) and \( c_p \) sufficiently large and take

\[
X_\infty(0) = \frac{\sqrt{C_1 - 1}}{\sqrt{C_1}}.
\]

Let \( T > 0 \) be a maximal number such that on \( t \in [0, T) \),

\[
\max_{1 \leq i \leq N} \|x_i(t) - x_\gamma(t)\| < 2X_\infty(0), \quad t \in [0, T).
\]

By the initial condition and the continuity of the solution, there is a positive number \( T > 0 \) satisfying (5.7). We claim that if \( c_q \) and \( c_p \) are sufficiently large, then \( T = \infty \). We note that for a given initial data, \( \sigma, C_1, C^w_\gamma \) are fixed constants. Therefore, on \( t \in [0, T) \),

\[
\dot{X}_\infty \leq -\mu_\infty X_\infty + C X_\infty + C.
\]

(5.8) implies

\[
\dot{X}_\infty \leq -\mu_\infty X_\infty + C,
\]

if \( c_q \) and \( c_p \) are sufficiently large. Therefore, by the Gronwall inequality and (5.5),

\[
X_\infty(t) \leq e^{-\frac{\mu}{2}t}X_\infty(0) + e^{-\frac{\mu}{2}t} \frac{2Ce^{\frac{\mu}{2}t} - 2C}{\mu} = e^{-\frac{\mu}{2}t} \left( X_\infty(0) - \frac{2C}{\mu} \right) + \frac{2C}{\mu}.
\]

If \( c_q \) and \( c_p \) are sufficiently large, then \( \mu \) is sufficiently large and \( X_\infty \leq X_\infty(0) \). These imply that on \( t \in [0, T) \),

\[
\max_{1 \leq i \leq N} \|x_i(t) - x_\gamma(t)\| \leq X_\infty \leq X_\infty(0) < 2X_\infty(0).
\]

By the continuity of the solution, we obtain that

\[
T = \infty.
\]

Finally, by the above, we obtain the following practical rendezvous estimate.

\[
X_\infty(t) \leq e^{-\frac{\mu}{2}t} \left( X_\infty(0) - \frac{2C}{\mu} \right) + \frac{2C}{\mu}.
\]

Thus, we complete the proof of Theorem 3.
6. Simulation results

In this section, we present several numerical simulations for the target tracking problem on the unit sphere and the flat space to verify the asymptotic complete rendezvous and practical rendezvous. We use the fourth-order Runge-Kutta method. We consider six α-agents \{(q_i, p_i)\}_{i=1}^6 chasing one target \((q_\gamma, p_\gamma)\). We assume that the control law for the target \((q_\gamma, p_\gamma)\) is given by

\[ u_\gamma(t) = a(\cos t, \sin t, 1), \]

where \(a\) is a constant. Throughout this section, we assume that the inter-particle bonding force parameter is given by

\[ \sigma = 1. \]

With the extra control law for agents

\[ U_i = 2\langle \omega_i, q_i \rangle (q_i \times p_i) + \hat{\omega}_i(t) \times q_i, \]

the initial positions and velocities for the agents are randomly chosen in

\[(q_i(0), p_i(0)) \in TS^2 \cap [-1,1]^3 \times [-1,1]^3 \]

as follows:

\[
\begin{align*}
q_1(0) &= (0.8132, 0.4989, -0.2993), & q_2(0) &= (0.7198, 0.4908, 0.4908), \\
q_3(0) &= (-0.6758, -0.6991, 0.2330), & q_4(0) &= (-0.7878, 0.5627, -0.2501), \\
q_5(0) &= (-0.5440, -0.7504, 0.3752), & q_6(0) &= (-0.8599, -0.3608, 0.3608),
\end{align*}
\]

and

\[
\begin{align*}
p_1(0) &= (0.1028, -0.1884, -0.0347), & p_2(0) &= (-0.1168, 0.5118, -0.3405), \\
p_3(0) &= (-0.0821, 0.0857, 0.0191), & p_4(0) &= (-0.1454, -0.1506, 0.1189), \\
p_5(0) &= (0.2220, -0.1040, 0.1137), & p_6(0) &= (-0.0003, 0.3768, 0.3759).
\end{align*}
\]

The initial data for the target is

\[ q_\gamma(0) = (-0.6451, 0.6605, -0.3840) \quad \text{and} \quad p_\gamma(0) = (0.1761, 0.3646, 0.3311). \]

Note that all the initial positions and velocities satisfy the admissible conditions in [153]. Since \(\omega_\gamma = q_\gamma \times p_\gamma\), we can check that

\[
\begin{align*}
U_i &= 2\langle \omega_i, q_i \rangle (q_i \times p_i) + \hat{\omega}_i(t) \times q_i \\
&= 2\langle \omega_i, q_i \rangle (q_i \times p_i) + (q_\gamma \times p_\gamma + q_\gamma \times \hat{p}_\gamma) \times q_i \\
&= 2\langle \omega_i, q_i \rangle (q_i \times p_i) + (q_\gamma \times \left( \frac{\|p_\gamma\|^2}{\|q_\gamma\|^2} q_\gamma + \|q_\gamma\|^2 u_\gamma - \langle u_\gamma, q_\gamma \rangle q_\gamma \right) \times q_i \\
&= 2\langle \omega_i, q_i \rangle (q_i \times p_i) + (q_\gamma \times \|q_\gamma\|^2 u_\gamma) \times q_i. \tag{6.1}
\end{align*}
\]

We fix

\[ \sigma = 1, \ c_q = 5, \ c_p = 0.1 \quad \text{and} \quad a = 0.5. \]

For this case, the time evolution of (1.2) is given in Figure 11. The red points and blue lines stand for the position \(q_i(t)\) at \(t = t_0\) and trajectories for the time interval \([t_0 - 3, t_0]\), respectively. The yellow one is for the target agent \(q_\gamma(t)\). In addition, we can check that the asymptotic complete rendezvous occurs as we proved in Theorem 2. See Figure 2. Here, the exponential function is \(2 e^{(-c_\gamma + 0.05)(t-t_0)}\).

For the zero extra control law, i.e. \(U_i = 0\), we fix the parameters such that

\[ \sigma = 1, \ c_q = 4, \ c_p = 4, \ a = 0.5. \]
The initial data of agents are randomly chosen but near the target as follows:

\begin{align*}
q_1(0) &= (-0.8147, -0.5366, \ 0.2193), \\
q_2(0) &= (-0.4335, -0.8173, \ 0.3794), \\
q_3(0) &= (-0.4420, -0.7998, \ 0.4060), \\
q_4(0) &= (-0.8645, -0.2373, \ 0.4429), \\
q_5(0) &= (-0.4312, -0.6004, \ 0.6734), \\
q_6(0) &= (-0.4084, -0.0987), \\
p_1(0) &= (0.0228, -0.0750, \ -0.0987), \\
p_2(0) &= (0.2519, -0.1263, \ 0.0383), \\
p_3(0) &= (0.0200, \ 0.0169, \ 0.0594), \\
p_4(0) &= (0.0388, -0.1447, \ -0.0017), \\
p_5(0) &= (0.0365, \ 0.1109, \ 0.2583), \\
p_6(0) &= (0.0081, \ 0.0050, \ 0.0097).
\end{align*}

The initial data for the target is given by

The initial data of agents are randomly chosen but near the target as follows:
Figure 3. The time evolution of (1.2) with control law

\[ q_\gamma(0) = (-0.6324, -0.6324, 0.4472) \quad \text{and} \quad p_\gamma(0) = (0.4712, -0.1742, 0.4199). \]

Figure 3 shows the time evolution of (1.2) without extra control law.

We can see that the maximum distance

\[ \max_{1 \leq i \leq 6} \| q_i(t) - q_\gamma(t) \| \]

between agents and the target is bounded by \( 2/\sqrt{c_p} \). See Figure 4(A). Let

\[ d(t) = \max_{1 \leq i \leq 6} \| q_i(t) - q_\gamma(t) \|. \]

Figure 4(B) displays \( d(t) \) at \( t = 100 \) with respect to \( c_p \). As \( c_p \) increases, the maximum distance between agents and target decreases. Therefore, we observe that the asymptotic practical rendezvous occurs.

Figure 4. The asymptotic practical rendezvous
With the extra control law, we observed the asymptotic complete rendezvous in Figure 1 and Figure 2. However, if we choose the parameter $c_p$ as zero, then the agents are not able to track the target. See Figure 3. Here, other parameters and initial data are the same as the case in Figure 1. In the absence of the velocity alignment term, the agents easily escape the sphere due to the accumulation of errors. To overcome this, as in [8], we add the following feedback term $f^0_i$ on the second equation of (1.2):

$$f^0_i = -k_0 \left( q_i - \frac{q_i}{\|q_i\|} \right),$$

where $k_0 = 10^4$. From this, we conclude that the velocity alignment operator is crucial in this target tracking algorithm.

![Figure 5](image1.png)

**Figure 5.** The time evolution of (1.2) with extra control law (6.1) and $c_p = 0$

As we mentioned in Subsection 2.2, the flocking term is negligible for the target tracking problem (1.2). With the same parameters of Figure 1 and Figure 3, the numerical results of (1.2) including the rotational flocking term

$$\sum_{j=1}^{N} \psi_{ij} (R_{q_j} - q_i) (p_j - p_i),$$

where $\psi_{ij} = 1$ is given in Figure 6. It is confirmed that the flocking term does not affect the results. See also Figure 7.

![Figure 6](image2.png)

**Figure 6.** The numerical results with flocking term and the same parameters with Figure 2

Finally, we compare the target tracking problems on a sphere and flat space numerically. To compare the two cases, we impose the periodic boundary for the flat space and fix parameters such as $\sigma = 1$, $c_q = 5$, and $c_p = 0.1$. Let

$$u_\gamma = (a \cos t, a \sin t, a),$$
Figure 7. The numerical results with flocking term and the same parameters with Figure 4 where $a = 0.5$ and $u_i = u_i$. Then we can observe that the complete rendezvous occurs. See Figure 8. If $u_i = 0$, then we observe the practical rendezvous. See Figure 9.

Figure 8. The snapshots of complete rendezvous on flat space

7. Conclusion

In this paper, we proposed a novel model for target tracking on spherical geometry. With the target’s position, velocity, and acceleration, if the initial energy of agents is small or the bonding force between the target and each agent is larger than the one between agents, the complete rendezvous occurs. When only the information of position and velocity is known and the target’s angular velocity and its time derivative are bounded, the practical rendezvous is obtained for relatively large intra-bonding forces. The target tracking problems on $S^2$ with time delay, white noises from the observation, and measurement are also interesting topics. These issues will be discussed in our future researches.
Figure 9. The snapshots of practical rendezvous on flat space

APPENDIX A. PROPERTIES OF THE ADMISSIBLE ROTATION OPERATOR

In this part, we consider admissible rotation operators on a sphere and their properties. The rotation operator appears naturally for defining the flocking on a sphere [6]. Let $R_{\rightarrow}$ be Rodrigues’ rotation operator given by

$$R_{x_k \rightarrow x_i}(v_k) = R(x_k, x_i) \cdot v_k$$

and for $x_k \neq x_i$,

$$R(x_k, x_i) := \langle x_k, x_i \rangle I + x_i x_i^T - x_k x_k^T + (1 - \langle x_k, x_i \rangle) \left( \frac{x_k \times x_i}{|x_k \times x_i|} \right) \left( \frac{x_k \times x_i}{|x_k \times x_i|} \right)^T.$$

Here, $x_k$, $x_i$, and $v_j$ are three dimensional column vectors. The rotation operator $R_{\rightarrow}$ has many good properties we desired or needed to be physically established and we can construct a flocking model by replacing the velocity difference term $v_i - v_j$ in the flat space to $R_{x_j \rightarrow x_i}(v_j(t) - v_i(t))$. See [6] for the details. However, there are some inconvenient points due to the presence of singularity on $R_{\rightarrow}$. Therefore, we can naturally ask whether such alternatives can be found.

The idea to find the alternative is as follows. First, classify the properties that the rotation operators must satisfy, and find all the operators that satisfy the properties. Next, we will choose one of those operators that meets our needs. Our option will be the simplest of the possible operators. This form has various advantages. It is convenient to calculate, and it shares most of the good properties of the rotation operator $R_{\rightarrow}$ previously defined. By removing the singularity, we easily show the global-in-time existence and uniqueness of the new model in (1.2). See [6] for the existence and uniqueness of the model with $R_{\rightarrow}$.

To construct a unit sphere model with the Newtonian equation, we need a modification of $v_j - v_i$ terms, which is the first motivation of the operators $R_{x_j \rightarrow x_i}$ in [6]. As we compute the velocity difference between $v_i$ and $v_j$ at the point $x_i$, we should transform $v_j$ into a tangential vector of the sphere at $x_i$. We note that the typical ansatz for the flocking motion on a sphere is circle motions. In order to include circle motions along one great circle, the operator should coincide with a rotation operator in two dimensions, a $(x_i, x_j)$-plane. In other words, an admissible rotation operator $M$ from
\(z_1\) to \(z_2\) can be a \(3 \times 3\) matrix such that
\[
Mz_1 = z_2, \quad Mz_2 = 2(z_1, z_2)z_2 - z_1, \quad (A.1a)
\]
\[
\langle Mv, z_2 \rangle = 0 \text{ for any } z_1, z_2 \in \mathcal{D} \text{ and } v \in T_{z_1} \mathcal{D}. \quad (A.1b)
\]

In the next proposition, we can prove that the admissible choices in \((A.1)\) for the rotation operator are equivalent to the following set.
\[
A_{z_1 \rightarrow z_2} := \left\{ P_{z_1 \rightarrow z_2} + a(z_1 \times z_2)(z_1 \times z_2)^T + b(z_1 - (z_1, z_2)z_2)(z_1 \times z_2)^T : a, b \in \mathbb{R} \right\}, \quad (A.2)
\]
where \(P_{z_1 \rightarrow z_2}\) is the operator defined in \((1.4)\).

**Proposition A.1.** Suppose that unit vectors \(z_1\) and \(z_2\) are linearly independent. Then, a \(3 \times 3\) matrix \(M\) satisfies \((A.1)\) if and only if \(M \in A_{z_1 \rightarrow z_2}\).

**Proof.** As two vectors \(z_1\) and \(z_2\) are perpendicular to \(z_1 \times z_2\), operator \(P_{z_1 \rightarrow z_2}\) satisfies \((A.1)\) from the direct computation. Note that \(\langle z_1 \times z_2, z_i \rangle = 0\) for \(i = 1, 2\). From this motivation, we naturally define
\[
M := P_{z_1 \rightarrow z_2} + a(z_1 \times z_2)(z_1 \times z_2)^T + b(z_1 - (z_1, z_2)z_2)(z_1 \times z_2)^T \quad (A.3)
\]
for any \(a, b \in \mathbb{R}\). Then \(M\) satisfies \((A.1a)\). Also, as \(z_2\) is perpendicular to both \(z_1 \times z_2\) and \((z_1 - (z_1, z_2)z_2)\), we conclude \((A.1b)\).

Conversely, choose any \(3 \times 3\) matrix \(M'\) satisfying \((A.1)\). As \(z_1\) and \(z_2\) are linearly independent, \(\{z_2, z_1 - (z_1, z_2)z_2, z_1 \times z_2\}\) are a basis of \(\mathbb{R}^3\). Therefore, there are \(a, b, c \in \mathbb{R}\) such that
\[
M' \frac{z_1 \times z_2}{\|z_1 \times z_2\|^2} = a(z_1 \times z_2) + b(z_1 - (z_1, z_2)z_2) + cz_2. \quad (A.4)
\]
From \((A.1b)\) and \(z_1 \times z_2 \in T_{z_1} \mathcal{D}\), it follows that \(c = 0\). Therefore, we conclude that
\[
Mz_1 \times z_2 = M'z_1 \times z_2
\]
for \(M\) given in \((A.3)\). On the other hand, \((A.1a)\) show that
\[
M(z_2) = M'(z_2) \quad \text{and} \quad M(z_1 - (z_1, z_2)z_2) = M'(z_1 - (z_1, z_2)z_2). \quad (A.5)
\]
From \((A.1)\) and \((A.5)\), we obtain that \(M = M'\).

The set \(A_{z_1 \rightarrow z_2}\) includes the rotation operators \(R_{z_1 \rightarrow z_2}\) and \(P_{z_1 \rightarrow z_2}\) given in \(\mathbb{[1]}\) and \((1.4)\), respectively. Here, if we take the following values in \((A.3)\):
\[
a = \frac{1 - \langle z_1, z_2 \rangle}{\|z_1 \times z_2\|^2} \quad \text{and} \quad b = 0,
\]
then the matrix coincides with \(R_{z_1 \rightarrow z_2}\), which preserves the modulus of each vectors. See Lemma 2.3 in \([6]\). Among several choices in the admissible set in \((A.2)\), \(P_{z_1 \rightarrow z_2}\) can be regarded as the simplest choice such that \(a = b = 0\) in \((A.2)\). Moreover, there is no singularity compared to the previous rotation operator \(R_{\rightarrow}^\rightarrow\). In addition to this simplicity, the rotation operator \(P_{z_1 \rightarrow z_2}\) also share the following desired transport properties.

**Lemma A.2.** For \(z_1, z_2 \in \mathcal{D}, P_{z_1 \rightarrow z_2}\) given in \((1.4)\) satisfies \((A.1)\). Furthermore, we have
\[
P_{z_1 \rightarrow z_2}^T = P_{z_2 \rightarrow z_1} \quad (A.6)
\]
and
\[
P_{z_1 \rightarrow z_2}^T P_{z_1 \rightarrow z_2}(z_1) = z_1, \quad P_{z_1 \rightarrow z_2}^T P_{z_1 \rightarrow z_2}(z_2) = z_2.
\]
Proof. As two vectors $z_1$ and $z_2$ are perpendicular to $z_1 \times z_2$, the properties in (A.4) follow from the direct computation. Also, since the transpose is the linear operator, we have

$$P^T_{z_1 \rightarrow z_2} = \langle z_1, z_2 \rangle I - z_2 z_2^T + z_1 z_1^T,$$

and we conclude (A.6). From (A.1) and (A.6), it holds that

$$P^T_{z_1 \rightarrow z_2} P_{z_1 \rightarrow z_2} (z_1) = P^T_{z_1 \rightarrow z_2} (z_2) = z_1$$

and

$$P^T_{z_1 \rightarrow z_2} P_{z_1 \rightarrow z_2} (z_2) = P^T_{z_1 \rightarrow z_2} (2\langle z_1, z_2 \rangle z_2 - z_1) = 2\langle z_1, z_2 \rangle z_1 - (2\langle z_1, z_2 \rangle z_2 - z_2) = z_2.$$

□

While the two operators $R_{z_1 \rightarrow z_2}$ and $P_{z_1 \rightarrow z_2}$ coincide on the $(z_1, z_2)$-plane from Lemma A.2, the following lemma gives us one difference between the two operators. We can show that $P_{z_1 \rightarrow z_2}$ gives a map between two tangent spaces although the operator is not a bijection if $\langle z_1, z_2 \rangle = 0$.

Lemma A.3. $P_{z_1 \rightarrow z_2} \mid_{T_{z_1} \mathcal{D}}$ is a map from $T_{z_1} \mathcal{D}$ to $T_{z_2} \mathcal{D}$. Furthermore, if $\langle z_1, z_2 \rangle \neq 0$, then $P_{z_1 \rightarrow z_2} \mid_{T_{z_1} \mathcal{D}}$ is a bijection from $T_{z_1} \mathcal{D}$ to $T_{z_2} \mathcal{D}$.

Proof. As $\mathcal{D}$ is a unit sphere, $v \in T_y \mathcal{D}$ if and only if $\langle v, y \rangle = 0$ for any $y \in \mathbb{R}^3$. Thus, we have

$$\langle v, z_1 \rangle = 0 \quad \text{for any vector } v \in T_{z_1} \mathcal{D}. \quad (A.7)$$

From (A.1) and (A.6), it holds that for any $v \in \mathbb{R}^3$,

$$(P_{z_1 \rightarrow z_2} v) \cdot z_2 = v^T P_{z_1 \rightarrow z_2} z_2 = v^T P_{z_1 \rightarrow z_2} rz_2 = v^T z_1 = \langle v, z_1 \rangle. \quad (A.8)$$

By (A.7) and (A.8), we conclude that

$$(P_{z_1 \rightarrow z_2} v) \cdot z_2 = 0 \quad \text{and thus } P_{z_1 \rightarrow z_2} v \in T_{z_2} \mathcal{D} \quad \text{for any vector } v \in T_{z_1} \mathcal{D}.$$

We now assume that $\langle z_1, z_2 \rangle \neq 0$ and show that $P_{z_1 \rightarrow z_2} \mid_{T_{z_1} \mathcal{D}}$ is bijective between two tangent spaces. First, if $z_1 = z_2$ or $z_1 = -z_2$, we get $P_{z_1 \rightarrow z_2} = I$ and $P_{z_1 \rightarrow z_2} = -I$. If not, $z_1$ and $z_2$ are linearly independent. From the assumption, $P_{z_1 \rightarrow z_2} (z_1 \times z_2) = \langle z_1, z_2 \rangle (z_1 \times z_2)$ is a nonzero vector. Combining this with (A.1), we conclude that $P_{z_1 \rightarrow z_2} \mid_{T_{z_1} \mathcal{D}}$ is surjective in $T_{z_2} \mathcal{D}$ and thus the determinant of $P_{z_1 \rightarrow z_2}$ is nonzero. As the inverse function of $P_{z_1 \rightarrow z_2}$ exists, we conclude that this lemma holds.

□

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(Sun-Ho Choi) DEPARTMENT OF APPLIED MATHEMATICS AND THE INSTITUTE OF NATURAL SCIENCES, KYUNG HEE UNIVERSITY, 1732 DEOGYEONG-DAERO, GHEUNG-GU, YONGIN 17104, REPUBLIC OF KOREA
Email address: sunhochoi@khu.ac.kr

(Dohyun Kwon) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DR., MADISON, WI 53706, USA
Email address: dkwon7@wisc.edu

(Hyowon Seo) DEPARTMENT OF APPLIED MATHEMATICS AND THE INSTITUTE OF NATURAL SCIENCES, KYUNG HEE UNIVERSITY, 1732 DEOGYEONG-DAERO, GHEUNG-GU, YONGIN 17104, REPUBLIC OF KOREA
Email address: hyowseo@gmail.com