Spherical Sherrington–Kirkpatrick Model for Deformed Wigner Matrix with Fast Decaying Edges

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Abstract

We consider the 2-spin spherical Sherrington–Kirkpatrick model whose disorder is given by a deformed Wigner matrix of the form $W + \lambda V$, where $W$ is a Wigner matrix and $V$ is a random diagonal matrix with i.i.d. entries. Assuming that the density function of the entries of $V$ decays faster than a certain rate near the edges of its spectrum, we prove the sharp phase transition of the limiting free energy and its fluctuation. In the high temperature regime, the fluctuation of the free energy converges in distribution to a Gaussian distribution, whereas it converges to a Weibull distribution in the low temperature regime. We also prove several results for deformed Wigner matrices, including a local law for the resolvent entries, a central limit theorem of the linear spectral statistics, and a theorem on the rigidity of eigenvalues.

Keywords Deformed Wigner matrix · Spherical Sherrington–Kirkpatrick model · Free energy

1 Introduction

The Sherrington–Kirkpatrick (SK) model and its variants have been intensively studied in statistical physics and probability theory to understand the behavior of spin glass. Its spherical variant, known as the spherical Sherrington–Kirkpatrick (SSK) model, is defined through the mean-field Hamiltonian of the form

$$-\langle J \sigma, \sigma \rangle,$$

where the disorder $J$ is an $N \times N$ matrix and the spin $\sigma = (\sigma_1, \ldots, \sigma_N) \in S_{N-1} = \{(\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \sum \sigma_i^2 = N\}$. The SSK model is widely used in various fields of study including high-dimensional statistics and learning theory.
One of the key features of the SSK model (and the SK model) is the sharp phase transition of the free energy, defined as

\[ F_N = F_N(\beta) = \frac{1}{N} \log \left[ \int_{S_{N-1}} \exp \left( \beta \langle \sigma, J \sigma \rangle \right) d\omega_N(\sigma) \right], \quad (2) \]

where \( \beta \) is the inverse temperature and \( \omega_N \) is the normalized uniform measure on \( S_{N-1} \). When the disorder \( J \) is a real Wigner matrix, it was proved by Crisanti and Sommers [7], and Talagrand [26] that as \( N \to \infty \) the free energy \( F_N \) converges to

\[ F_N \to F_W(\beta) := \begin{cases} \beta^2 & \text{if } 0 < \beta \leq 1/2 \\ 2\beta - \frac{1}{2} \log(2\beta) - \frac{3}{4} & \text{if } \beta > 1/2 \end{cases} \quad (3) \]

The fluctuation of the free energy is also markedly different in the high temperature case (\( \beta < 1/2 \)) and the low temperature case (\( \beta > 1/2 \)). Baik and the first author [1] studied the fluctuation \( F_N - F_W(\beta) \) and proved that

\[ \begin{align*}
N(F_N - F_W(\beta)) & \to \text{a normal distribution} & \text{if } 0 < \beta < 1/2 \\
2^{2/3}(\beta - \frac{1}{2})N^{2/3}(F_N - F_W(\beta)) & \to \text{the Tracy–Widom distribution} & \text{if } \beta > 1/2
\end{align*} \quad (4) \]

where the convergence is in distribution.

Heuristically, the fluctuation of the free energy in the high temperature regime is affected by all eigenvalues of \( J \) through its linear spectral statistics (LSS), defined by

\[ \sum_{i=1}^{N} f(\lambda_i), \]

where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \) are the eigenvalues of \( J \). On the other hand, in the low temperature regime, the fluctuation of \( F_N \) is dominated by that of the largest \( \lambda_1 \). Since the fluctuations of the LSS and the largest eigenvalue are given by a Gaussian and the Tracy–Widom, respectively, one obtains the phase transition as in (4). Similar argument also holds for other disorders such as the sample covariance matrix and the orthogonal invariant ensemble [1].

One natural question about the free energy of the SSK model is whether the heuristic argument above is universal, i.e., the picture of the all eigenvalues versus the largest eigenvalue is valid even when the disorder is not one of the classical random matrix models (Wigner matrix, sample covariance matrix, and invariant ensemble). To test the universality, we consider the case where the disorder is of the form

\[ J = W + \lambda V, \quad (5) \]

where \( W \) is a Wigner matrix and \( V \) is a random diagonal matrix. Such a matrix is called a deformed Wigner matrix, and with certain choices of the parameters, it is known that several key assumptions in [1] are not satisfied, most notably the square-root decay at the edge of the spectrum and the Tracy–Widom limit of the largest eigenvalue.

1.1 Main Contribution

Under the assumption that the decay of the spectrum of \( J = W + \lambda V \) is convex and that \( \lambda \) is larger than a constant, we prove that there exists a critical inverse temperature \( \beta_c \) such that
• if \( \beta < \beta_c \), the fluctuation of \( F_N \) converges in distribution to a Gaussian distribution, and
• if \( \beta > \beta_c \), the fluctuation of \( F_N \) converges in distribution to a Weibull distribution

with precise formulas for both limiting distributions, where the limiting Weibull distribution is originated from the corresponding (Weibull) distribution of the largest eigenvalue of \( J \). See Theorem 1 and Theorem 2 for the precise statement. This in particular suggests that the dichotomy between the fluctuation given by the LSS and that by the largest eigenvalue holds not only for the classical random matrix models but for more general models. We also prove the limiting free energy \( F(\beta) \) for both regimes.

It should be noted that the order of the fluctuation in the low temperature regime is \( N^{-1/(b+1)} \) for some \( b > 1 \) but that in the high temperature is \( N^{-1/2} \), and hence the fluctuation is larger in the low temperature regime than in the high temperature regime. This was also true for the SSK model with the Wigner disorder in (4), though the exact orders of the fluctuations \( (N^{-2/3} \) in the low temperature regime and \( N^{-1} \) in the high temperature regime) do not coincide with those for our model.

The main technical difficulty in the proof of the main result is the lack of several results for \( J \), which are crucial in the analysis of the free energy in [1]. In this paper, we prove the following for \( J \):

• a local law for resolvent entries,
• a central limit theorem of linear statistics,
• the rigidity of eigenvalues.

These results are not only important for the understanding of the free energy but also significant per se in random matrix theory.

1.2 Related Works

1.2.1 Spherical Sherrington–Kirkpatrick Model

The SK model was introduced by Sherrington–Kirkpatrick [24] as a mean-field version of the Edwards–Anderson model [8], which is an Ising-type model of spin glass. The limiting free energy was first predicted by Parisi [22], which is now known as the Parisi formula, and later proved by Guerra [10] and Talagrand [25].

The spherical Sherrington–Kirkpatrick (SSK) model was introduced by Kosterlitz et al. [12], where the limiting free energy was explicitly computed without a rigorous proof. A formula analogous to the Parisi formula was obtained by Crisanti and Sommers [7] and later proved by Talagrand [26]. For more recent results on the free energy and its fluctuation for the SSK model, we refer to [2–4, 13, 14, 21].

1.2.2 Deformed Wigner Matrix

Deformed Wigner matrix of the form (5) was first introduced by Pastur [23], where it was proved that the empirical distribution of (5) converges to a deterministic probability distribution \( \mu_{fc} \) as \( N \to \infty \). The \( \mu_{fc} \) is known as the free convolution of \( \mu \) and the semicircle distribution, and assuming that the empirical distribution of \( V \) is bounded and exhibits concave decay at the edge of its spectrum, it is known that \( \mu_{fc} \) exhibits square-root decay at the corresponding edge. In this case, several key results for the Wigner matrix, including the local law for the resolvent [15, 19], the delocalization of the eigenvectors and the rigidity
of the eigenvalues [15], the bulk universality [19], the edge universality [18, 19], and the normality of the LSS [11], hold with natural modification.

On the other hand, much less is known for the case where \( \mu_{fc} \) does not exhibit the square-root decay at the edge. It was proved by the first author and Schnelli [15, 16] that \( \mu_{fc} \) decays at the same rates as the empirical distribution of \( V \) if it is convex and \( \lambda \) in (5) is larger than a certain critical value \( \lambda_+ \). (See Lemma 1 for more detail.) In this case, it is also known that the eigenvectors associated to the extreme eigenvalues are partially localized [16].

### 1.3 Relation to a Signal Detection Problem

The SSK model is closely related to the problem of detecting the presence of the rank-one signal in a noisy data matrix. Suppose that the data matrix \( M \) is of the form

\[
M = \sqrt{\lambda}xx^T + H,
\]

where the signal \( x \in \mathbb{R}^N \) and the noise \( H \) is an \( N \times N \) real symmetric random matrix. When the signal-to-noise (SNR) \( \lambda \) is not large, in order to detect the signal, it is common to analyze the largest eigenvalue and its associated eigenvalue, which is the principal component analysis (PCA). In the simplest case where \( H \) is a Wigner matrix and \( \|x\| = 1 \), the following transition for the largest eigenvalue \( \lambda_1 \) of \( M \) is known: if \( \lambda > 1 \), \( \lambda_1 \) is strictly larger than 2 and separates from the bulk of the spectrum, whereas if \( \lambda < 1 \), \( \lambda_1 \) converges to 2, the edge of the spectrum, and cannot be distinguished from the null model (\( \lambda = 0 \)).

If the SNR \( \lambda \) is below the threshold and the noise is Gaussian, it is known that no tests can reliably detect the presence of the signal. For this case, it is natural to consider the hypothesis testing between the null hypothesis \( \lambda = 0 \) and the alternative \( \lambda = \omega \) for some positive constant \( \omega \), which is also known as the weak detection. By the Neyman–Pearson lemma, the likelihood ratio (LR) test is optimal in the sense that it minimizes the sum of the Type-I error and the Type II-error. For the \((i, j)\)-entry of the data matrix with \( i \neq j \), the ratio of the densities under the null and the alternative is

\[
\frac{\exp \left( N(M_{ij} - \sqrt{\lambda}x_ix_j)^2 \right)}{\exp \left( NM_{ij}^2 \right)}.
\]

Assuming that the signal is chosen uniformly from the unit sphere \( S^N \) and the noise is GOE, the likelihood ratio is given by

\[
\frac{d\mathbb{P}_1}{d\mathbb{P}_0} := \int_{S^N} \prod_{i < j} \frac{\exp \left( N(M_{ij} - \sqrt{\lambda}x_ix_j)^2 \right)}{\exp \left( NM_{ij}^2 \right)} \prod_k \frac{\exp \left( N(M_{ij} - \sqrt{\lambda}x_ix_j)^2/2 \right)}{\exp \left( NM_{ij}^2/2 \right)} d\omega^N(\sigma)
\]

\[
= \int_{S^N} \prod_{i \neq j} \exp \left( -N\sqrt{\lambda}M_{ij}x_ix_j + \frac{N}{2} \lambda_i \lambda_j x_i x_j^2 \right) d\omega^N(\sigma),
\]

where \( d\omega^N \) is the uniform measure on \( S^N \). Note that the logarithm of the LR in (6) coincides with the free energy of the SSK model after shifting and rescaling. In the LR test, if the test statistic \( \frac{d\mathbb{P}_1}{d\mathbb{P}_0} < 1 \) the null hypothesis is accepted, while it is rejected if \( \frac{d\mathbb{P}_1}{d\mathbb{P}_0} > 1 \). Since the fluctuation of the LR is equal to the fluctuation of the free energy of the SSK model, it is possible to prove the optimal error for the weak detection.
If the rank-1 signal $xx^T$ is perturbed by $U = (U_{ij})$, the ratio of the densities is changed to
\[
\exp \left( N (M_{ij} - \sqrt{\lambda} U_{ij} - \sqrt{\lambda} x_i x_j)^2 \right)
\]
\[
= \exp \left( -2\sqrt{\lambda} N (M_{ij} - \sqrt{\lambda} U_{ij}) x_i x_j - 2\sqrt{\lambda} N M_{ij} U_{ij} + \lambda N x_i^2 x_j^2 \right).
\]
Thus, for given $U$ the LR in (6) becomes
\[
\prod_{i \neq j} \exp \left( -\sqrt{\lambda} N M_{ij} U_{ij} \right) \int_{\mathbb{S}^N} \prod_{i \neq j} \exp \left( -N \sqrt{\lambda} (M_{ij} - \sqrt{\lambda} U_{ij}) x_i x_j + \frac{N}{2} \lambda x_i^2 x_j^2 \right) d\omega^N (\sigma)
\]
(7)
for which it is required to consider the free energy of the SSK model with deformed Gaussian interaction. Note that while $U$ is not assumed to be diagonal, we may diagonalize $U$ in the integrand in (7) for the analysis since GOE is orthogonally invariant.

1.4 Organization of the Paper

The rest of the paper is organized as follows: In Sect. 2, we precisely define the model and introduce our main results. In Sect. 3, we list several important results needed in the proof of main results. In Sects. 4, 5, and 6, we prove our results on deformed Wigner matrices - local law for the resolvent entries, CLT for the linear spectral statistics, and the rigidity of the eigenvalues, respectively. In Sects. 7 and 8, we prove the main theorems for the low temperature case and the high temperature case, respectively. Some technical details on the results for the steepest descent curve and the proofs of some auxiliary lemmas are collected in Appendices.

2 Model and Main Results

2.1 Definition of the Model

Recall that the disorder $J = W + \lambda V$. Here, $W$ is an $N \times N$ real Wigner matrix for which we use the following definition:

**Definition 1** An $N \times N$ matrix $W = (W_{ij})_{N \times N}$ is a Wigner matrix if
\begin{itemize}
  \item $\{W_{ij}| i \leq j\}$ are independent real-valued random variables.
  \item $W_{ij} = W_{ji}$.
  \item $E[W_{ij}] = 0$, $E[W_{ij}^2] = \frac{1+\delta_{ij}}{N}$.
  \item There exist $\theta > 1$ and $\theta' > 0$ such that
    \[
    \mathbb{P}(\sqrt{N}|W_{ij}| > x) \leq \theta' \exp(-x^{1/\theta}) \quad \forall x \geq 0, N \geq 1 \text{ and } i, j \in \{1, \ldots, N\}
    \] (8)
\end{itemize}
We remark that the subexponential decay condition guarantees the the existence of all (normalized) moments and an overwhelming-probability bound as follows:
1. for any $p \in \{1, 2, \ldots\}$,
\[
\sup_{i,j,N} \mathbb{E}[|\sqrt{N}W_{ij}|^p] < \infty;
\]
(9)
2. If \( \epsilon' > 0 \) and \( D' > 0 \), then for large enough \( N \) we have
\[
P\left( |W_{ij}| \leq N^{\epsilon'-\frac{1}{2}}, \forall i, j \in \{1, \ldots, N\} \right) > 1 - N^{-D'}.
\]

(10)

We assume that \( V \) is a random diagonal matrix whose entries are i.i.d. with centered Jacobi distribution \( \mu \) for which we use the following definitions:

**Definition 2** A probability measure \( \mu \) is a Jacobi measure on \([-1, 1]\) if its density function is given by
\[
d\mu = \frac{d(x)}{Z} (1 + x)^a (1 - x)^b \mathbb{1}_{[-1,1]}(x)
\]
where

- \( a > -1 \) and \( b > -1 \)
- \( d(x) \in C^1([-1, 1]) \) and \( d(x) > 0 \) on \([-1, 1]\).
- \( Z \) is the normalization constant: \( Z = \int_{-1}^1 d(x) (1 + x)^a (1 - x)^b dx \)
- \( \mu \) is centered: \( \int_{-1}^1 x d\mu(x) = 0 \)

We also assume that \( V \) is independent of \( W \). For a given constant \( \lambda > 0 \), if we denote by \( \lambda \mu \) the law of \( \lambda v \) where \( v \) is a random variable with law \( \mu \), then the empirical measure of \( W + \lambda V \) converges to \( \mu_{fc} \) as \( N \to \infty \), which is given by
\[
\mu_{fc} := \mu_{sc} \boxplus (\lambda \mu)
\]
where \( \boxplus \) denotes the additive free convolution and \( \mu_{sc} \) denotes the semicircle distribution.

It is known that \( \mu_{fc} \) has a density function, which we will call \( \rho_{fc} \); see Remark 2.5 in [16] for the detail. In the following lemma, we collect the results on \( \mu_{fc} \),

**Lemma 1** Set
\[
\lambda_{\pm} = \left( \int_{-1}^1 \frac{d\mu(x)}{(1 \mp x)^2} \right)^{1/2}, \quad \tau_{\pm} = \int_{-1}^1 \frac{d\mu(x)}{1 \mp x}.
\]

There exists \( L_- < 0 < L_+ \) such that \( \text{supp}(\mu_{fc}) = [L_-, L_+] \). If \( b > 1 \) and \( \lambda > \lambda_+ \), then

1. \( L_+ = \lambda + \frac{\tau_+}{\lambda} \),
2. \( L_+ + \int_{-L_+}^{\rho_{fc}(x)dx} x \leq \lambda_+ \), and
3. there exists \( C_0 \geq 1 \) such that
\[
\frac{x^b}{C_0} \leq \rho_{fc}(L_+ - x) \leq C_0 x^b \text{ for } x \in [0, L_+]
\]

(11)

If \( a > 1 \) and \( \lambda > \lambda_- \), the statements above hold for \( L_- \), with \( L_+ \) and \( \tau_+ \) replaced by \( L_- \) and \( \tau_- \) respectively. In particular,

\[
\frac{x^a}{C_0} \leq \rho_{fc}(L_- + x) \leq C_0 x^a \text{ for } x \in [0, |L_-|]
\]

(12)

For the proof of Lemma 1, see Lemma 2.3 and Remark 2.6 [16]
2.2 Main Results

Recall that the free energy of the SSK model at inverse temperature $\beta > 0$ is defined by

$$F_N = F_N(\beta) = \frac{1}{N} \log \left[ \int_{S_{N-1}} \exp \left( \beta \langle \sigma, (W + \lambda V) \sigma \rangle \right) d\omega_N(\sigma) \right]$$

where $S_{N-1} = \{(x_1, \ldots, x_N) : \sum_{i=1}^{N} x_i^2 = N\}$ and $\omega_N$ is the (normalized) uniform measure on $S_{N-1}$. We will prove that the constant

$$\beta_c = \frac{1}{2} \int \frac{\rho_{fc}(t)}{L_+ - t} dt.$$  \hspace{1cm} (13)

is the critical inverse temperature of the SSK model, i.e., we study the fluctuation of $F_N$ in two cases: $0 < \beta < \beta_c$ (high temperature regime) and $\beta > \beta_c$ (low temperature regime). We remark that $\beta_c$ is well defined when $b > 1$ and $\lambda > \lambda_+$ (see Eq. (11)).

Our first main result is the following theorem for the free energy in the low temperature regime:

**Theorem 1** (Main theorem: low temperature) Suppose $\beta > \beta_c$, $\lambda > \max(\lambda_-, \lambda_+)$, $b > 11$ and $1 < a < \frac{b^2 - 6b - 7}{4}$. Then the fluctuation of $F_N$ converges in distribution to a Weibull distribution. More precisely,

$$\lim_{N \to \infty} \mathbb{P} \left( N^{\frac{1}{\beta + 1}} \left[ F_N + \frac{1}{2} \log(2e\beta) + \frac{1}{2} \int \frac{\log(L_+ - \tau) d\mu_{fc}(\tau)}{\beta - \beta_c} \right] \leq s \right) = \exp \left( - \frac{C_\mu (-s)^{b+1}}{b+1} \right) \forall s \leq 0$$

where $C_\mu = \left( \frac{\lambda}{\lambda_+} \right)^{b+1} \cdot (1) \cdot 2^a \cdot Z^{-1}$.

Our second main result is for the high temperature regime.

**Theorem 2** (Main theorem: high temperature) Suppose $0 < \beta < \beta_c$, $\lambda > \max(\lambda_-, \lambda_+)$, $a > 1$ and $b > 37/3$. Suppose $\hat{\gamma}$ is the unique point on $(L_+, +\infty)$ such that $\int \frac{1}{\hat{\gamma} - \tau} d\mu_{fc}(\tau) = 2\beta$. Then

$$2\sqrt{N} \left( F_N + \frac{1}{2} \log(2e\beta) - \beta \hat{\gamma} + \frac{1}{2} \int \log(\hat{\gamma} - \tau) d\mu_{fc}(\tau) \right)$$

converges in distribution to a centered Gaussian distribution whose variance is

$$\frac{1}{4\pi^2} \left( \oint_{\mathcal{C}} (1 + m_{fc}(\xi)) m_{fc}(\xi) \log(\hat{\gamma} - \xi) d\xi \right)^2 - \frac{1}{4\pi^2} \int_{-1}^{1} \left( \oint_{\mathcal{C}} \frac{(1 + m_{fc}(\xi)) \log(\hat{\gamma} - \xi)}{(\lambda t - \xi - m_{fc}(\xi))} d\xi \right)^2 d\mu(t)$$

where $m_{fc}(\cdot)$ is the Stieltjes transform of $\mu_{fc}$ and $\mathcal{C}$ is a counterclockwise path which encloses $[L_-, L_+]$ but does not enclose $\hat{\gamma}$. Here we take the analytic branch of $\log(\cdot)$ on $\mathbb{C}\setminus(-\infty, 0]$ such that $\text{Im} \log(\cdot) \in (-\pi, \pi)$.

In Theorems 1 and 2, the conditions on $a$ and $b$ are technical. We expect that they can be weakened. See Sect. 2.4. From Theorems 1 and 2, we immediately obtain the following corollary on the limiting free energy:
Corollary 1 Suppose $\lambda > \max(\lambda_-, \lambda_+)$, $b > 37/3$ and $1 < a < \frac{b^2 - 6b - 7}{4}$. As $N \to \infty$ we have

$$F_N \to F(\beta) = \begin{cases} \frac{1}{2} \log(2e\beta) - \frac{1}{2} \int \log(L_+ - t)d\mu_{fc}(t) + \beta L_+ & \text{if } \beta > \beta_c \\ \frac{1}{2} \log(2e\beta) - \frac{1}{2} \int \log(\hat{\gamma} - t)d\mu_{fc}(t) + \beta \hat{\gamma} & \text{if } 0 < \beta < \beta_c \end{cases}$$

in distribution.

From the definitions of $\beta_c$ and $\hat{\gamma}$, we see that $\lim_{\beta \to \beta_c} \hat{\gamma} = L_+$ and that $\lim_{\beta \to \beta_c} F(\beta) = \lim_{\beta \to \beta_c} F_N(\beta)$.

2.3 Outline of the Proof

In this paper, we study the fluctuation of $F_N$ by following the idea introduced in [1]. In the low temperature case (i.e., $\beta > \beta_c$), we will show that the leading term of $F_N$ is a linear function of $\lambda_1$. Since the fluctuation of $\lambda_1$ has size $O(N^{-1+\epsilon})$ and converges to a Weibull distribution, so does the fluctuation of $F_N$, as in Theorem 1. In the high temperature case (i.e., $0 < \beta < \beta_c$), the leading term of $F_N$ is a linear function of the quantity

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i). \quad (14)$$

for some $N$-independent deterministic function $f$. Thus, by the central limit theorem (see Theorem 4), the fluctuation of $F_N$ has size $O(N^{-1/2})$ and converges to a Gaussian distribution, as in Theorem 2.

For the actual proof, in addition to the known results, we need the local law for resolvent entries, the central limit theorem for linear statistics, and the rigidity of eigenvalues. While we prove these results in the current paper, some of them are not strong enough to directly follow the analysis in [1]. To overcome the difficulty, we introduce several changes in the detail of the proof. Most notably, (1) for the low temperature case, instead of proving a lemma analogous to Lemma 6.4 of [1] that is required to control the integral of an exponential function along the curve of the steepest descent in Lemma 31, we prove a refined result for the curve in Lemma 30, and (2) for the high temperature case, instead of controlling the difference $|\gamma - \hat{\gamma}|$ by applying the rigidity of eigenvalues, we use the local law to control it as in Lemma 33.

In what follows, we list our new results on the deformed Wigner matrices:

Definition 3 For any $z \in \mathbb{C}/\mathbb{R}$, define

$$m_N(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z}, \quad G(z) = \frac{1}{W + \lambda V - z}$$

where $\lambda_1 \geq \cdots \geq \lambda_N$ are eigenvalues of $W + \lambda V$.

Theorem 3 (local law for resolvent entries) Suppose $M > 0$. For any $\epsilon' > 0$ and $D' > 0$, we have for large enough $N$ that

$$\mathbb{P}\left( |G_{ij} - \delta_{ij} \cdot \frac{1}{\lambda_{ij} - z - m_N(z)} | \leq N^{\epsilon' - \frac{1}{2}} |\text{Im}z|^{-3} + N^{\epsilon' - 1} |\text{Im}z|^{-4} + \delta_{ij} N^{\epsilon' - 1} |\text{Im}z|^{-5} \right.$$  

for any $i, j \in \{1, \ldots, N\}, |\text{Re}z| \leq M, |\text{Im}z| \in [N^{-10}, 3]) > 1 - N^{-D'}$$
We remark that the local law for the trace of the resolvent was proved by the first author and Schnelli. See [16] and also Sect. 3.2 of the current paper.

**Theorem 4** (CLT for linear statistics) Let \( f(x) \) be a function which is analytic on a neighborhood of \([L_-, L_+]\). Suppose \( a > 1, b > 37/3 \) and \( \lambda > \max(\lambda_+, \lambda_-) \). Then

\[
\frac{1}{\sqrt{N}} \left( \sum_i f(\lambda_i) - N \int f(t) \rho_{fc}(t) dt \right)
\]

converges in distribution to a centered Gaussian distribution whose variance is

\[
\frac{1}{4\pi^2} \left( \oint_C f(\xi)(1 + m'_{fc}(\xi)) m_{fc}(\xi) d\xi \right)^2 - \frac{1}{4\pi^2} \int_{-1}^1 \left( \oint_C f(\xi)(1 + m'_{fc}(\xi)) \lambda t - \xi - m_{fc}(\xi) d\xi \right)^2 d\mu(t)
\]

where \( C \) is a counterclockwise path enclosing \([L_-, L_+]\) such that \( f \) is analytic on a neighborhood of the region bounded by \( C \).

**Definition 4** Define the deterministic number \( \gamma_x = \gamma_x(N) \) and \( \hat{\gamma}_y = \hat{\gamma}_y(N) \) by

\[
\mu_{fc}([\gamma_x, +\infty]) = \frac{x - \frac{1}{2}}{N} \quad \forall x \in [1, N]
\]

\[
\mu_{fc}([\hat{\gamma}_y, +\infty]) = \frac{y}{N} \quad \forall y \in (0, N)
\]

with the convention that \( \gamma_N = L_- \) and \( \hat{\gamma}_0 = L_+ \). Here \( x \) and \( y \) are not necessarily integers.

**Theorem 5** (Rigidity of eigenvalues) Suppose \( a > 1, b > 3 \) and \( \lambda > \max(\lambda_-, \lambda_+) \). Suppose \( \epsilon \in (\frac{1}{b+1}, \frac{1}{4}) \). There exists an event \( E_N(\epsilon) \) such that

\[
P(E_N(\epsilon)) \geq 1 - \kappa_0(\log N)^{1+2b} N^{-\epsilon}
\]

when \( N \) is large enough. Moreover, if \( E_N(\epsilon) \) holds, then:

1. for any \( \zeta \in (0, \frac{1-\epsilon}{b+1}) \) we have
   \[
   |\lambda_i - \gamma_i| \leq N^{-1+\epsilon+\zeta b} \quad \text{when } N \text{ is large enough and } i \in \mathbb{Z} \cap [\kappa'N^{1-\zeta(b+1)} - \frac{N}{2}, \frac{N}{2}]
   \]

2. for any \( \zeta' \in (0, \frac{1-\epsilon}{a+1}) \) we have
   \[
   |\lambda_i - \gamma_i| \leq N^{-1+\epsilon+\zeta'a} \quad \text{when } N \text{ is large enough and } i \in \mathbb{Z} \cap [\frac{N}{2}, N - \kappa'N^{1-\zeta'(a+1)}]
   \]

Here \( \kappa_0 > 0 \) and \( \kappa' > 0 \) are constants independent of \( \zeta \) and \( \zeta' \).

**2.4 Remarks**

As discussed in Introduction, we expect the existence of the dichotomy between the fluctuation given by the LSS in the high temperature regime and the fluctuation dominated by the largest eigenvalue in the low temperature regime, regardless of the choice of various parameters in the deformed Wigner matrix. The main technical issue is the non-optimality of the local law; if the local law can be improved, the rigidity result will also be improved and it will be possible to relax the condition on \( a \) and \( b \). It is even expected that the fluctuation
of $F_N$ would converge to a Gaussian distribution when $\lambda < \lambda_+$, since the fluctuation of $\lambda_1$ converges to a Gaussian distribution in this case. However, we do not attempt to prove the claim in the current paper.

3 Preliminaries

Definition 5 Suppose $\omega$ is a measure on $\mathbb{R}$. Define its Stieltjes transform by

$$\int \frac{d\omega(t)}{t - z}, \quad \forall z \in \mathbb{C} \setminus \text{supp}(\omega).$$

3.1 Fluctuation of the Largest Eigenvalue

Recall that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are eigenvalues of $W + \lambda V$. The following theorem can be found in [16].

Theorem 6 If $b > 1$ and $\lambda > \lambda_+$, then

$$\lim_{N \to \infty} \mathbb{P}\left( N \frac{1}{t} (L_+ - \lambda_1) \leq s \right) = 1 - \exp\left( - \frac{C_\mu s^{b+1}}{1 + b} \right), \quad \forall s \geq 0$$

where $C_\mu = \left( \frac{\lambda}{\lambda_2 - \lambda_+} \right)^{b+1} \cdot d(1) \cdot 2^a \cdot Z^{-1}$ as defined in Theorem 1.

If $a > 1$ and $\lambda > \lambda_-$, then

$$\lim_{N \to \infty} \mathbb{P}\left( N \frac{1}{t} (\lambda_N - L_-) \leq s \right) = 1 - \exp\left( - \frac{C'_\mu s^{a+1}}{1 + a} \right), \quad \forall s \geq 0$$

where $C'_\mu = \left( \frac{\lambda}{\lambda - \lambda_-} \right)^{a+1} \cdot d(-1) \cdot 2^b \cdot Z^{-1}$.

Remark 1 For the second conclusion of Theorem 6, see the sentence above Sect. 2.4.1 in [16]. It can also be proved by replacing $W + \lambda V$ by $-W + \lambda (-V)$.

The next lemma is a direct corollary of (3.22) in [16].

Lemma 2 For any constant $r > 0$ we have that

$$\lim_{N \to \infty} \mathbb{P}\left( \max_{1 \leq k \leq N} |\lambda_k| \leq 2 + \lambda + r \right) = 1.$$  

Therefore,

$$[L_-, L_+] \subset [-2 - \lambda, 2 + \lambda].$$  

3.2 Local Law for the Trace of the Resolvent

In this subsection we introduce the local law for the trace of the resolvent obtained in [16].

Definition 6 Suppose $\mu_N$ is the empirical measure of $W + \lambda V$: $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$. Let

- $m_{fc}(z)$ be the Stieltjes transform of $\mu_{fc}$: $m_{fc}(z) = \int \frac{\rho_{fc}(t)}{t - z} dt$ (as mentioned in Theorem 2);
Definition 7 For \( z \in \mathbb{C} \setminus \mathbb{R} \), let
\[
g_i(z) = \frac{1}{\lambda v_i - z - m_{fc}(z)}, \quad \hat{g}_i(z) = \frac{1}{\lambda v_i - z - \hat{m}_{fc}(z)}.
\]

Lemma 3 For \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
m_N(z) = \frac{1}{N} \text{Tr} G(z), \quad \hat{m}_{fc}(z) = \frac{1}{N} \sum_{i=1}^{N} \hat{g}_i(z),
\]
\[
m_{fc}(z) = \mathbb{E}[g_i(z)] = \int \frac{1}{\lambda t - z - m_{fc}(z)} d\mu(t)
\]
\[
(1 + \hat{m}_{fc}'(z))(1 - \frac{1}{N} \sum \delta_i^2(z)) = 1
\]
\[
(1 + m_{fc}'(z))(1 - \int \frac{d\mu(t)}{(\lambda t - z - m_{fc}(z))^2}) = 1.
\]
\[
|g_i(z)| \leq \frac{1}{|\text{Im} z|} \quad \text{and} \quad |\hat{g}_i(z)| \leq \frac{1}{|\text{Im} z|}
\]

Proof The first conclusion is trivial. The second and third conclusions are direct corollaries of (2.3) of [15]. The fourth conclusion can be proved by taking derivatives on both sides of the second conclusion:
\[
\hat{m}_{fc}'(z) = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{\lambda v_i - z - \hat{m}_{fc}(z)} \right)' = \frac{1}{N} \sum \frac{1 + \hat{m}_{fc}'(z)}{(\lambda v_i - z - \hat{m}_{fc}(z))^2}.
\]
The fifth conclusion can be proved by similarly taking derivatives on both sides of the third conclusion. The last conclusion is because both \( \text{Im} m_{fc}(z) \) and \( \text{Im} \hat{m}_{fc}(z) \) have the same sign as \( \text{Im} z \).

Definition 8 Suppose \( \epsilon \in (0, \frac{11b - 9}{2(2b + 1)} \). Let \( \tilde{v}_i \) be the \( i \)-th largest one of \( \{v_1, \ldots, v_N\} \). We define the regions \( \mathcal{D}_\epsilon \), \( \mathcal{D}'_\epsilon \) and the events \( \tilde{\Omega}(\epsilon) \), \( \Omega_\alpha(\epsilon) \) and \( \Omega_0(\epsilon, c_1, c_2) \) by the following.

- \( \mathcal{D}_\epsilon = \{x + iy| -3 - \lambda \leq x \leq 3 + \lambda, N^{-\frac{1}{2} - \epsilon} \leq y \leq N^{-\frac{1}{2(2b + 1)} + \epsilon} \}
- \( \mathcal{D}'_\epsilon = \{z \in \mathcal{D}_\epsilon | \lambda \tilde{v}_i - z - m_{fc}(z) \mid > \frac{1}{2} N^{-\frac{1}{2(2b + 1)} - \epsilon}, \forall i \in [20, N] \}
- \( \tilde{\Omega}(\epsilon) = \{\text{Im} m_N(z) - \hat{m}_{fc}(z) \mid \leq N^{2\epsilon - \frac{1}{2}} \text{ for all } z \in \mathcal{D}'_\epsilon \}
- \( \Omega_\alpha(\epsilon) = \{\text{Im} m_N(z) \leq N^{2\epsilon - \frac{1}{2}}, \forall z \in \mathcal{D}'_\epsilon \}
- \( \Omega_0(\epsilon, c_1, c_2) \) is the event on which the following conditions are satisfied for any \( k \in \{1, \ldots, 19\} \).
  - If \( j \in \{1, \ldots, N\} \setminus \{k\} \) then \( N^{-\epsilon - \frac{1}{2(2b + 1)}} < |\tilde{v}_j - \tilde{v}_k| < (\log N) N^{-\frac{1}{1+\epsilon}} \). Moreover \( N^{-\epsilon - \frac{1}{1+\epsilon}} < |1 - \tilde{v}_1| < (\log N) N^{-\frac{1}{1+\epsilon}} \).
  - If \( z \in \mathcal{D}_\epsilon \) and \( |\text{Re}(z + m_{fc}(z) - \lambda \tilde{v}_k)| = \min_{1 \leq i \leq N} |\text{Re}(z + m_{fc}(z) - \lambda \tilde{v}_i)| \) then
\[
\frac{1}{N} \sum_{i \in \{1, \ldots, N\} \setminus \{k\}} \frac{1}{|\lambda \tilde{v}_i - z - m_{f_c}(z)|^2} < c_1.
\]

- If \( z \in \mathcal{D}_\epsilon \) then \[
\frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\lambda \tilde{v}_i - z - m_{f_c}(z)|^2} - \int \frac{d\mu(t)}{|\lambda \tilde{v}_i - z - m_{f_c}(z)|^2} \leq c_2 N^{\frac{3\epsilon}{2} - \frac{1}{2}}.
\]

Here \( c_1 \in (0, 1) \) and \( c_2 > 0 \) are constants.

**Remark 2**
- Notice that \( \mathcal{D}'_\epsilon \) is random but is independent of \( W \).
- We defined \( \Omega_0(\epsilon, c_1, c_2) \) in the same way as Definition 3.5 in [16]. The condition \( \epsilon \in (0, \frac{11b-9}{2b+2}) \) comes from (3.20) of [16]. Definition 3.5 in [16] involves a constant \( n_0 \) and we set \( n_0 = 20 \) in the current paper.
- [16] requires the entries of the diagonal matrix to be ordered along the diagonal, so in order to use results in [16], we use \( \tilde{v}_i \) instead of \( v_i \) in the definitions of \( \mathcal{D}'_\epsilon \) and \( \Omega_0(\epsilon, c_1, c_2) \).

**Proposition 7** Suppose \( b > 1, \lambda > \lambda_+ \) and \( \epsilon \in (0, \frac{11b-9}{2b+2}) \). There exist constants \( c_1 \in (0, 1), c_2 > 0, v_0 > 0, v_1 > 0 \) and \( N_0 > 0 \) such that:

1. \( \mathbb{P}(\Omega_0(\epsilon, c_1, c_2)) \geq 1 - v_0 (\log N)^{1+2b} N^{-\epsilon} \) for all \( N \);
2. \( \mathbb{P}(\Omega_0(\epsilon, c_1, c_2) \setminus \Omega(\epsilon)) \leq \exp(-v_1 (\log N)^{10\log\log N}) \) if \( N \) is large enough;
3. \( \Omega_0(\epsilon, c_1, c_2) \) is measurable with respect to the sigma algebra generated by the entries of \( V \).

**Proof** The first two conclusions of Proposition 7 are proved in [16]. See (3.30) and Proposition 5.1 there. The last conclusion is from the definition of \( \Omega_0(\epsilon, c_1, c_2) \). \( \square \)

**Definition 9** Let \( \Omega_V(\epsilon) \) be the \( \Omega_0(\epsilon, c_1, c_2) \) with \( c_1 = c_1(\epsilon) \) and \( c_2 = c_2(\epsilon) \) properly chosen such that the conclusions of Proposition 7 hold.

The next lemmas are Lemmas 5.5 and 3.7 in [16].

**Lemma 4** Suppose \( b > 1, \lambda > \lambda_+ \) and \( \epsilon \in (0, \frac{11b-9}{2b+2}) \). There exists a constant \( v_2 > 0 \) such that if \( N \) is large enough then

\[
\mathbb{P}(\Omega_V(\epsilon) \setminus \Omega_\epsilon(\epsilon)) \leq \exp(-v_2 (\log N)^{10\log\log N}).
\]

**Lemma 5** Suppose \( b > 1, \lambda > \lambda_+ \) and \( \epsilon \in (0, \frac{11b-9}{2b+2}) \). If \( \Omega_V(\epsilon) \) holds and \( z \in \mathcal{D}'_\epsilon \), then

\[
|\hat{m}_{f_c}(z) - m_{f_c}(z)| \leq N^{2\epsilon - \frac{1}{2}}.
\]

### 3.3 Integral Representation of the Partition Function of the SSK Model

The following lemma comes from Lemma 1.3 and (5.25) of [1].

**Lemma 6** Suppose \( M \) is an \( N \times N \) real symmetric matrix with eigenvalues \( \lambda_1(M) \geq \cdots \geq \lambda_N(M) \). Suppose \( \beta > 0 \). Then

\[
\int_{S_{N-1}} e^{\beta (\sigma, M \sigma)} d\omega_N(\sigma) = C_N \int_{a_0 + i\infty}^{a_0 + i\infty} e^{\frac{N}{2} R_M(z)} dz
\]

where

- \( a_0 \) is an arbitrary constant satisfying \( a_0 > \lambda_1(M) \);
The integration contour is the vertical line from $a_0 - i\infty$ to $a_0 + i\infty$;

- $R_M(z) = 2\beta z - \frac{1}{N} \sum_i \log(z - \lambda_i(M))$ where we take the analytic branch of the log function such that $\text{Im} \log(z - \lambda_i(M)) \in (-\pi, \pi)$ for all $z$ on the integration contour;

- $C_N = \frac{\Gamma(N/2)}{2\pi i N^\beta} \sqrt{N}$ where $\Gamma(z)$ denotes the Gamma function. Moreover,

$$C_N = \sqrt{N} \beta \frac{i}{\sqrt{\pi} (2\beta e)^{N/2}} (1 + O(N^{-1})).$$

3.4 Helffer–Sjöstrand Formula

The next lemma can be found in Sect. 11.2 of [9].

**Lemma 7** Suppose $\chi : \mathbb{R} \to [0, 1]$ is $C^\infty$ such that $\chi(x) = 1$ when $x \in [-1, 1]$ and $\chi(x) = 0$ when $x \notin [-2, 2]$. If $f \in C^2_c(\mathbb{R})$, then

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{iyf''(x) \chi(y) + if'(y)(f(x) + if(x))}{t - x - iy} dx dy, \quad \forall t \in \mathbb{R}.$$

3.5 Cumulant Expansion

The next lemma is Lemma 3.2 of [17].

**Lemma 8** Suppose $l \in \{1, 2, \ldots\}$ and $F \in C^{l+1}(\mathbb{R}, \mathbb{C})$. Let $Y$ be a real-valued centered random variable with finite moments up to the order $l + 2$. Let $\mathcal{G}$ be a sigma algebra independent of $Y$. Then

$$E[Y F(Y)|\mathcal{G}] = \sum_{r=1}^{l} \frac{k^{(r+1)}(Y)}{r!} E[F^{(r)}(Y)|\mathcal{G}] + E(Y)$$

where $k^{(r+1)}(Y)$ denotes the $(r + 1)$-cumulant of $Y$. The error term $E(Y)$ satisfies:

$$|E(Y)| \leq C_l E[|Y|^{l+2}] \sup_{|t| \leq Q} |F^{(l+1)}(t)| + C_l E[|Y|^{l+2} 1_{|Y| > Q}] \sup_{t \in \mathbb{R}} |F^{(l+1)}(t)|$$

where $Q > 0$ is an arbitrary cutoff and $C_l$ satisfies $C_l \leq \frac{(p_0 - \lambda)^l}{l!}$ for some absolute constant $p_0 > 0$.

3.6 Marcinkiewicz–Zygmund Inequality

The next lemma is copied from Lemmas D.1, D.2 and D.3 of [5]. It is a version of the Marcinkiewicz-Zygmund inequality.

**Lemma 9** Let $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ be independent centered random variables such that for each $p \in \{1, 2, \ldots\}$ there exists a constant $\mu_p > 0$ satisfying

$$E[|X_i|^p]^{1/p} \leq \mu_p, \quad E[|Y_i|^p]^{1/p} \leq \mu_p \quad (1 \leq i \leq N).$$

Then for deterministic families $(a_{ij})$ and $(b_i)$ we have
where \( C_p \) is a constant depending only on \( p \).

### 3.7 Some Results for Symmetric Matrices

Suppose \( M \) is an \( N \times N \) real symmetric matrix. Suppose \( T \) is a subset of \( \{1, \ldots, N\} \).

**Definition 10**  
- We use \( M^{(T)} \) to denote the \((N - |T|) \times (N - |T|)\) matrix:

\[
(M_{ij})_{i, j \in \{1, \ldots, N\} \setminus T}
\]

- For \( z \in \mathbb{C} \setminus \mathbb{R} \) let

\[
R(z) = (M - z)^{-1}, \quad R^{(T)}(z) = (M^{(T)} - z)^{-1}
\]

- We also set

\[
\sum_{i=1}^{(T)} = \sum_{i \in T}, \quad \sum_{i, j} = \sum_{i \in T} \sum_{j \in T}^{(T)}
\]

**Remark 3**  
1. When \( T = \{i\} \), we use \((i)\) instead of \((\{i\})\) in the above definitions. Similarly, we write \((ij)\) instead of \((\{i\}, \{j\})\). We use \((Ti)\) to denote \((T \cup \{i\})\).
2. In \( M^{(T)} \) and \( R^{(T)} \) we use the original values of matrix indices. For example, the indices for the rows and columns of \( M^{(2)} \) are \(1, 3, 4, \ldots, N\).
3. It is easy to see that \( R(z) \) is a symmetric matrix.

**Lemma 10**  
Suppose \( Imz_1 \neq 0 \) and \( Imz_2 \neq 0 \). Then

- \( \frac{d}{dz} R(z_1) = R^2(z_1) \)
- \( \sum_j |R_{ij}(z_1)|^2 = \frac{ImR_{ij}(z_1)}{Imz_1}, \quad \forall i \)
- \( \left| (R^k(z_1)) (R^k(z_2)) \right|_{ij} \leq \frac{1}{|Imz_1|^k|Imz_2|^k}, \quad \text{for any } i, j \in \{1, \ldots, N\}, k_1, k_2 \in \{0, 1, 2, \ldots\} \).

**Proof**  
The first can be proved by directly taking the derivative. The second conclusion is the Ward identity, see (3.6) of [5]. For the last conclusion, suppose

\[
M = O \text{diag}(\lambda_1(M), \ldots, \lambda_N(M)) O^T
\]

where \( \lambda_1(M), \ldots, \lambda_N(M) \) are eigenvalues of \( M \) and \( O \) is an orthogonal matrix. Then

\[
R(z_1) = O \text{diag}((\lambda_1(M) - z_1)^{-1}, \ldots, (\lambda_N(M) - z_1)^{-1}) O^T \quad \text{and}
\]

\[
R'(z_2) = O \text{diag}((\lambda_1(M) - z_2)^{-2}, \ldots, (\lambda_N(M) - z_2)^{-2}) O^T
\]

(by the first conclusion of this lemma).
So

\[ |(R^{k_1}(z_1))(R^{k_2}(z_2))_{ij}| = \left| \sum_{r=1}^{N} \frac{1}{(\lambda_r(M) - z_1)^{k_1}(\lambda_r(M) - z_2)^{k_2}} O_{ir} O_{jr} \right| \]

\[ \leq \frac{1}{|\text{Im}z_1|^k_1 |\text{Im}z_2|^k_2} \exp \left( \sum_{r=1}^{N} |O_{ir} O_{jr}| \right) \leq \frac{1}{|\text{Im}z_1|^k_1 |\text{Im}z_2|^k_2} \sum_{r=1}^{N} \frac{O_{ir}^2 + O_{jr}^2}{2} \]

\[ = \frac{1}{|\text{Im}z_1|^k_1 |\text{Im}z_2|^k_2}. \]

**Lemma 11** (Resolvent identities)

1. If \( i, j, k \not\in T \) and \( i, j \not= k \), then

\[ R_{(T)}^{(T)}_{ij} = R_{ij}^{(Tk)} + \frac{R_{jk}^{(Tk)} R_{ij}^{(T)}}{R_{kk}^{(T)}}. \] (25)

2. if \( i \not= j \) then

\[ R_{ij} = -R_{ii} R_{jj}^{(i)} \left( M_{ij} - \sum_{k,l} M_{ik} R_{kl}^{(i)} M_{lj} \right) \] (26)

3. \[ R_{ii}^{-1} = M_{ii} - z - \sum_{k,l} M_{ik} R_{kl}^{(i)} M_{li} \] (27)

4. \[ \frac{\partial R_{kl}}{\partial M_{ij}} = -\frac{1}{1 + \delta_{ij}} (R_{ki} R_{ij} + R_{kj} R_{li}) \] (28)

**Proof** The first conclusion can be found in (3.4) of [5]. The second conclusion is (5.9) of [5]. The third conclusion is (5.1) of [5]. The fourth conclusion can be proved by definition. \( \square \)

**Lemma 12** If \( M \) is an \( N \times N \) real symmetric random matrix such that \( \{M_{ij} i \leq j\} \) are independent and \( E[M_{ij}^2] = \frac{1}{N} \) for \( i \not= j \), then

\[ \frac{1}{R_{ii}(z)} = -z - \frac{1}{N} \text{Tr} R(z) + M_{ii} + \frac{1}{N} \sum_{k} \frac{(R_{kk}(z))^2}{R_{ii}(z)} - \sum_{k,l} M_{ik} R_{kl}^{(i)} M_{li} + \frac{1}{N} \sum_{k} R_{kk}^{(i)}. \]

**Proof** This lemma is from Lemma 5.2 of [5] and the fact that \( \sum_{k,l} E[M_{ik} R_{kl}^{(i)} M_{li}] = \frac{1}{N} \sum_{k} R_{kk}^{(i)} \) (since \( M_{ik} M_{li} \) is independent of the sigma algebra generated by the entries of \( M^{(i)} \)). \( \square \)
4 Local Law for Resolvent Entries: Proof of Theorem 3

In this section we follow the idea introduced in [5] to prove Theorem 3. Recall that $G(z)$ is defined in Definition 3.

**Lemma 13** Suppose $i \neq j$. For any $\epsilon' > 0$, $D' > 0$, there exists $N_0 = N_0(\epsilon', D') > 0$ such that if $N > N_0$ and $z \in \mathbb{C} \setminus \mathbb{R}$ then

\[
\mathbb{P}\left( \left| \sum_{k,l} W_{ik} G_{kl}^{(ij)} W_{lj} \right| \leq N^{\epsilon'} \sqrt{\frac{1}{N^2} \sum_{k,l} |G_{kl}^{(ij)}|^2} > 1 - N^{-D'} \right) \\
\mathbb{P}\left( \left| \sum_{k,l} W_{ik} G_{kl}^{(ij)} W_{lj} \right| \leq N^{\epsilon'} \sqrt{\frac{1}{N^2} \sum_{k \neq l} |G_{kl}^{(ij)}|^2 + N^{\epsilon' - 1} \sum_{k} |G_{kk}^{(ij)}|} > 1 - N^{-D'} \right) \\
\mathbb{P}\left( \left| \sum_{k,l} W_{jik} G_{kl}^{(ij)} W_{lj} \right| \leq N^{\epsilon'} \sqrt{\frac{1}{N^2} \sum_{k \neq l} |G_{kl}^{(ij)}|^2 + N^{\epsilon' - 1} \sum_{k} |G_{kk}^{(ij)}|} > 1 - N^{-D'} \right) \\
\mathbb{P}\left( \left| \sum_{k} (W_{ik}^2 - \frac{1}{N}) G_{kl}^{(ij)} \right| \leq N^{\epsilon'} \sqrt{\frac{1}{N^2} \sum_{k} |G_{kk}^{(ij)}|^2} > 1 - N^{-D'} \right)
\]

where each of $G_{kl}^{(ij)}$ and $G_{kl}^{(ij)}$ takes value at $z$.

**Proof** 1. For the first conclusion, suppose $G_1$ is the sigma algebra generated by entries of $(W + \lambda V)^{(ij)}$. Then $W_{ik}$ and $W_{lj}$ are independent of $G_1$. Let

\[
B_{kl} = \frac{G_{kl}^{(ij)} \sqrt{\sum_{k,l} |G_{kl}^{(ij)}|^2}}{G_{kl}^{(ij)}}.
\]

For any natural number $p$ and any sample point $\omega$ in the probability space, we have

\[
\mathbb{E}\left[ \left| \sum_{k,l} W_{ik} B_{kl} W_{lj} \right|^{2p} \right| G_1](\omega) = \mathbb{E}\left( \left| \sum_{k,l} W_{ik} B_{kl} W_{lj} \right|^{2p} \left| \sum_{k,l} W_{ik} B_{kl} W_{lj} \right| \right| G_1](\omega) \\
\mathbb{E}\left( \left| \sum_{k,l} W_{ik} B_{kl} W_{lj} \right|^{2p} \right) \\
\mathbb{E}\left( \left| \sum_{k,l} W_{ik} B_{kl}(\omega) W_{lj} \right|^{2p} \right) \\
\leq C \left( \sum_{k,l} \left| B_{kl}(\omega) \right|^2 \right)^p = CN^{-2p}
\]
where $C > 0$ depends only on $p$. We used (9) and the second conclusion of Lemma 9 in the inequality. So,

$$\mathbb{P}\left( \left| \sum_{i,j} W_{ik} G_{kl}^{(ij)} W_{lj} \right| > N^\epsilon \sqrt{\frac{1}{N^2} \sum_{i,j} |G_{kl}^{(ij)}|^2} \right) = \mathbb{P}\left( \left| \sum_{i,j} W_{ik} B_{kl} W_{lj} \right| > N^\epsilon \right)$$

$\leq N^{-2p^\epsilon} \mathbb{E}\left[ \left| \sum_{i,j} W_{ik} B_{kl} W_{lj} \right|^{2p} \right] \leq CN^{-2p^\epsilon}$.

Choosing $p$ large enough such that $2p^\epsilon > D'$ we complete the proof of the first conclusion.

2. For the second conclusion, we use the same argument as above except that the $G_1$ is replaced by the sigma algebra generated by entries of $(W + \lambda V)^{(i)}$, the summation is replaced by $\sum_{k \neq l}$ and $B_{kl}$ is replaced by

$$\frac{G_{kl}^{(i)}}{\sqrt{\sum_{j \neq k} |G_{kl}^{(i)}|^2}}.$$

Then using the third conclusion of Lemma 9 we have for $N > N_0 = N_0(\epsilon', D')$:

$$\mathbb{P}\left( \left| \sum_{k \neq l} W_{ik} G_{kl}^{(i)} W_{li} \right| > N^\epsilon \sqrt{\frac{1}{N^2} \sum_{k \neq l} |G_{kl}^{(i)}|^2} \right) \leq CN^{-2p^\epsilon} \leq N^{-D'}.$$

This together with (10) and the fact that $\sum_{k,l} W_{ik} G_{kl}^{(i)} W_{li} = \sum_{k \neq l} W_{ik} G_{kl}^{(i)} W_{li} + \sum_{l} W_{ik}^2 G_{kk}^{(i)}$ complete the proof of the second conclusion.

3. The third conclusion can be proved in the same way as the second conclusion.

4. For the last conclusion, let $X_k = NW_{ik}^2 - 1$ and $Q_k = G_{kk}^{(i)} / \sqrt{\sum_{k} |G_{kk}^{(i)}|^2}$. Then, similarly as above, with $G_2$ be the sigma algebra generated by entries of $(W + \lambda V)^{(i)}$, for any natural number $p$ and any sample point $\omega$,

$$\mathbb{E}[\sum_{k} X_k Q_k(\omega)|G_2] = \mathbb{E}[\sum_{k} X_k Q_k(\omega)|2^p] \leq C$$

where $C$ depends only on $p$. We used (9) and the first conclusion of Lemma 9 in the last inequality. So for any $\epsilon' > 0, D' > 0$, if $N > N_0 = N_0(\epsilon', D')$, then

$$\mathbb{P}\left( \left| \sum_{k} X_k G_{kk}^{(i)} \right| > N^\epsilon \sqrt{\sum_{k} |G_{kk}^{(i)}|^2} \right) \leq N^{-2p^\epsilon} \mathbb{E}[\sum_{k} X_k Q_k^2] \leq CN^{-2p^\epsilon}.$$

Choosing $p$ large enough such that $2p^\epsilon > D'$, we complete the proof.
Corollary 2 For any $M > 0$, $\epsilon' > 0$, $D' > 0$, if $|\text{Re}z| \leq M$ and $0 < |\text{Im}z| \leq 3$ then
\[
\mathbb{P}\left( \left| \frac{1}{G_{ii}} \right| \leq \frac{3N\epsilon'}{|\text{Im}z|}, \forall i \in [1, N] \right) > 1 - N^{-D'}
\]
\[
\mathbb{P}\left( \left| \frac{1}{G_{jj}^{(i)}} \right| \leq \frac{3N\epsilon'}{|\text{Im}z|}, \forall i \neq j \right) > 1 - N^{-D'}
\]
for large enough $N$.

**Proof** By Lemma 11,
\[
\frac{1}{G_{ii}} = \lambda v_i + W_{ii} - z - \sum_{k,l}^{(i)} W_{ik} G_{kl}^{(i)} W_{li}, \quad \frac{1}{G_{jj}^{(i)}} = \lambda v_j + W_{jj} - z - \sum_{k,l}^{(i)} W_{jk} G_{kl}^{(jj)} W_{lj}.
\]
This together with (10), Lemma 13 and the facts that $|\lambda v_i| \leq \lambda, |G_{kl}^{(i)}| \leq \frac{1}{|\text{Im}z|}, |G_{kl}^{(jj)}| \leq \frac{1}{|\text{Im}z|}, |z| \leq M + 3$ (when $|\text{Re}z| \leq M, |\text{Im}z| \in (0, 3]$)
yield the conclusions. \hfill \Box

**Proof of Theorem 3** Suppose $i \neq j$ and $z \in \mathbb{C}\setminus \mathbb{R}$. By Lemma 10 and (25)
\[
\left| \sum_{k}^{(i)} G_{kk}^{(jj)} \right| = \left| \sum_{k}^{(i)} \left( G_{kk} - \frac{G_{kk} G_{ki} G_{ji}}{G_{ii}} - \frac{G_{kj} G_{ji}}{G_{jj}^{(i)}} \right) \right|
\leq \frac{N}{|\text{Im}z|} + \frac{|G_{ii}^2 - (G_{ij})^2 - (G_{ij})^2|}{|G_{ii}|} + \frac{|(G_{ij})^2 - (G_{jj}^{(i)})^2|}{|G_{jj}^{(i)}|}
\leq \frac{N}{|\text{Im}z|} + \frac{3}{|\text{Im}z|^2 |G_{ii}|} + \frac{2}{|\text{Im}z|^2 |G_{jj}^{(i)}|}
\tag{30}
\]
and
\[
\left( \frac{1}{N^2} \sum_{k,l}^{(i)(j)} |G_{kl}^{(jj)}|^2 \right)^{1/2} = \left( \sum_{k}^{(i)(j)} \frac{\text{Im}G_{kk}^{(jj)}}{N^2 |\text{Im}z|^2} \right)^{1/2}
\leq \left( \frac{1}{N |\text{Im}z|^2} + \frac{3}{N^2 |\text{Im}z|^3 |G_{ii}|} + \frac{2}{N^2 |\text{Im}z|^3 |G_{jj}^{(i)}|} \right)^{1/2}
\]
By Corollary 2, for any $\epsilon' > 0, D' > 0$, if $N$ is large enough and $i \neq j$, then
\[
\mathbb{P}\left( \left( \frac{1}{N^2} \sum_{k,l}^{(i)(j)} |G_{kl}^{(jj)}|^2 \right)^{1/2} \leq \sqrt{\frac{1}{N |\text{Im}z|^2} + \frac{15N\epsilon'}{N^2 |\text{Im}z|^4}} \right) > 1 - N^{-D'}
\tag{31}
\]
provided $|\text{Re}z| \leq M$ and $|\text{Im}z| \in (0, 3]$.

Now using (26), (10), the first conclusion of Lemma 13 and (31), we have that for any $\epsilon' > 0$ and $D' > 0$, if $N$ is large enough then
\[
\mathbb{P}\left( \max_{i \neq j} |G_{ij}| \leq \sqrt{\frac{N\epsilon'}{N |\text{Im}z|^3} + \frac{N\epsilon'}{N |\text{Im}z|^4}} \right) > 1 - N^{-D'} \quad \text{provided } |\text{Re}z| \leq M \quad \text{and } |\text{Im}z| \in (0, 3].
\]
This together with a classic “lattice” argument proved that for any \( \epsilon' > 0 \) and \( D' > 0 \), if \( N \) is large enough then
\[
\mathbb{P} \left( \max_{i \neq j} |G_{ij}| \leq \frac{N \epsilon'}{\sqrt{N} |\text{Im} z|^3} + \frac{N \epsilon'}{N |\text{Im} z|^4}, \text{ for any } |\text{Re} z| \leq M, |\text{Im} z| \in [N^{-10}, 3] \right) > 1 - N^{-D'}.
\]
By Lemma 12,
\[
|G_{ii} - \frac{1}{\lambda v_i - z - m_N(z)}| = \frac{|G_{ii}|}{|\lambda v_i - z - m_N(z)|} \left| G_{ii} - (\lambda v_i - z - m_N(z)) \right|
= \frac{|G_{ii}|}{|\lambda v_i - z - m_N(z)|} \left| W_{ii} + \frac{1}{N} \sum_{k} (G_{ki}(z))^2 - \sum_{k,l}(G_{kl}(z))^2 - \sum_{k,l} W_{ik}G_{kl} W_{li} + \frac{1}{N} \sum_{k} G_{kk} \right|
\leq |\text{Im} z|^{-2} \left( |W_{ii}| + \frac{|(G_{2i})_{ii}|}{N |G_{ii}|} + |\sum_{k \neq l} W_{ik}G_{kl} W_{li}| + |\sum_{k} (W_{ik}^2 - \frac{1}{N})G_{kk} \right) \tag{33}
\]
where we used Lemma 10 and the fact that \( \text{Im} m_N \) has the same sign as \( \text{Im} z \) in the last step.
By Lemma 10 and (25), we have
\[
\sqrt{\frac{1}{N^2} \sum_{k} |G_{kk}^{(i)}|^2} \leq \frac{1}{|\text{Im} z| \sqrt{N}} \tag{34}
\]
and
\[
\sqrt{\frac{1}{N^2} \sum_{k \neq l} |G_{kl}^{(i)}|^2} \leq \sqrt{\frac{1}{N^2} \sum_{k,l} |G_{kl}^{(i)}|^2} = \left( \sum_{k} \text{Im} G_{kk}^{(i)} \right)^{1/2} \leq \frac{1}{|\text{Im} z| \sqrt{N}} \tag{35}
\]
If \( \epsilon' > 0, D' > 0, |\text{Re} z| \leq M \) and \( |\text{Im} z| \in (0, 3] \), then by (33), (10), Corollary 2, (29) and the last conclusion of Lemma 13 we have for large enough \( N \):
\[
\mathbb{P} \left( |G_{ii} - \frac{1}{\lambda v_i - z - m_N(z)}| \leq \frac{N \epsilon'}{|\text{Im} z|^3} \left[ \frac{1}{\sqrt{N}} + \frac{3N \epsilon'}{N |\text{Im} z|^3} + \sqrt{\frac{1}{N^2} \sum_{k \neq l} |G_{kl}^{(i)}|^2} + \frac{1}{N^2} \sum_{k} |G_{kk}^{(i)}|^2 \right] \right) > 1 - N^{-D'}
\]
thus by (34) and (35),
\[
\mathbb{P} \left( |G_{ii} - \frac{1}{\lambda v_i - z - m_N(z)}| \leq \frac{3N^2 \epsilon'}{N |\text{Im} z|^5} + \frac{5N \epsilon'}{\sqrt{N} |\text{Im} z|^3} \right) > 1 - N^{-D'}
\]
This together with a classic “lattice” argument yields that for any \( \epsilon' > 0 \) and \( D' > 0 \), if \( N \) is large enough then
\[
\mathbb{P} \left( |G_{ii} - \frac{1}{\lambda v_i - z - m_N(z)}| \leq \frac{N \epsilon'}{N |\text{Im} z|^5} + \frac{N \epsilon'}{\sqrt{N} |\text{Im} z|^3}\right),
\]
for any \( |\text{Re} z| \leq M, |\text{Im} z| \in [N^{-10}, 3], 1 \leq i \leq N \) > 1 - N^{-D'}.
This together with (32) completes the proof. □
5 Central Limit Theorem for Linear Statistics: Proof of Theorem 4

Suppose \( f \) is a fixed function satisfying the condition in Theorem 4. In this section we prove Theorem 4, i.e., the fact that the linear statistics

\[
\frac{1}{\sqrt{N}} \left( \sum_i f(\lambda_i) - N \int f(t) \rho f_c(t) dt \right)
\]

converges in distribution to a Gaussian variable. We use the method introduced in [11], but we prove Lemma 14 in a different way. The method we prove Lemma 14 is similar as that in [20].

Throughout this section, we assume that the conditions of Theorem 4 are satisfied.

**Definition 11**
- Suppose \( d \in (0, \frac{1}{2}) \) is a constant which is small enough such that \( f \) is analytic on a neighborhood of the rectangular region whose vertices are \( L_+ + d \pm di \) and \( L_- - d \pm di \).
- Use \( \Gamma \) to denote the boundary of the above rectangular region with counterclockwise orientation.
- Let \( \varpi \in \left( \frac{1}{b + 1}, \frac{1}{8} - \varpi \right) \) and \( \varsigma, \varsigma' \in \left( 0, \min \left( \frac{1}{3} - 4\varpi, \frac{1}{2} - 7\varpi, \frac{1}{2} - 5\varpi - 2\varsigma \right) \right) \).
- Let \( \Gamma_+ = \{ z \in \Gamma | |\text{Im}z| \geq N^{-\varpi} \} \). The orientation of \( \Gamma_+ \) is induced from \( \Gamma \).

**Remark 4** Since \( b > 37/3 \), it is easy to check that the constants \( \varpi, \varsigma, \varsigma' \) exist.

**Definition 12**
- Let \( \sigma(V) \) be the sigma algebra generated by \( V \):
  \[
  \sigma(V) = \sigma(v_1, \ldots, v_N). 
  \]
- Use \( \mathbb{E}_N[\cdot] \) to denote the conditional expectation \( \mathbb{E}[\cdot|\sigma(V)] \).

**Lemma 14** As \( N \to \infty \),

\[
\frac{1}{\sqrt{N}} \int_{\Gamma_+} f(\xi) \left[ TrG(\xi) - \mathbb{E}_N[TrG(\xi)] \right] d\xi \to 0 \quad \text{in distribution.}
\]

We prove Lemma 14 in Sect. 5.3.

**Lemma 15** As \( N \to \infty \),

\[
\frac{1}{\sqrt{N}} \int_{\Gamma_+} f(\xi) \left[ N \hat{m}f_c(\xi) - \mathbb{E}_N[TrG(\xi)] \right] d\xi \to 0 \quad \text{in distribution.}
\]

We prove Lemma 15 in Sect. 5.4.

5.1 Some Auxiliary Lemmas

Recall that \( g_i \) and \( \hat{g}_i \) are defined in Definition 7.
Lemma 16 There is a constant \( C_d > 0 \) depending on \( d \) such that
\[
\min_i |\lambda v_i - z - m_f(z)| \geq C_d \quad \forall z \in \Gamma
\]
Moreover, \( g_i(z) \) is analytic on \( \mathbb{C} \backslash [L_-, L_+] \) for all \( i \in \{1, \ldots, N\} \).

Proof See Appendix B. \( \square \)

Definition 13 Define \( M_d = \max(|L_+ + d + 2|, |L_- - d - 2|) \),
\[
D_{\sigma d}(M_d) = \{ x + iy | x \leq M_d, N^{-\sigma d} < |y| \leq 3 \}
\]
and
\[
B_N = \tilde{\Omega}(\varsigma) \cap \{ \max_{i,j} |G_{ij}(z) - \frac{\delta_{ij}}{\lambda v_i - z - m_N(z)}| \leq N^{\varsigma' - \frac{1}{2}} |\text{Im} z|^{-3}, \forall z \in D_{\sigma d}(M_d) \}
\]
where \( \tilde{\Omega}(\varsigma) \) is defined in Definition 9 and \( D_{\sigma d}(M_d) \) is defined by Definition 3. The parameters \( \sigma, \varsigma \) and \( \varsigma' \) are defined in Definition 11.

Lemma 17 1. there exists \( N_0 > 0 \) such that if \( N > N_0 \) then \( \Gamma_+ \subset \mathcal{D}'_{\varsigma} \)
2. for any \( D' > 0 \) we have that if \( N \) is large enough then
\[
\mathbb{P}(\Omega_V(\varsigma) \backslash B_N) < N^{-D'}.
\]
3. if \( N \) is large enough and \( B_N \cap \Omega_V(\varsigma) \) holds, then the following holds for each \( \xi \in \Gamma_+ : \)
\[
|G_{ii}(\xi) - \hat{g}_i(\xi)| \leq N^{\varsigma' - \frac{1}{2}} \cdot |\text{Im} \xi|^{-3 + N^{2\varsigma - \frac{1}{2}} \cdot |\text{Im} \xi|^{-2}}, \quad |G_{ii}(\xi)| \geq W' |\text{Im} \xi|
\]
\[
|m_f(\xi) - \frac{1}{N} \text{Tr} G^{(i)}(\xi)| \leq N^{2\varsigma - \frac{1}{2}} + \frac{3}{W' N |\text{Im} \xi|^3}
\]
\[
|G_{ii}(\xi) - g_i(\xi)| \leq N^{\varsigma' - \frac{1}{2}} \cdot |\text{Im} \xi|^{-3} + 2N^{2\varsigma - \frac{1}{2}} \cdot |\text{Im} \xi|^{-2}, \quad \forall \xi \in \mathcal{D}_{\sigma d, \varsigma}(\tilde{\Omega}) (\tilde{\Omega})
\]
where \( W' \) is a constant in \( (0, 1) \).

Proof See Appendix B. \( \square \)

5.2 Proof of Theorem 4

Proof of Theorem 4 Let \( \sigma' \) be a positive constant in \( \left[ \frac{1}{b+1} - \sigma, \frac{\sigma}{2} \right] \) and
\[
R_N = \left\{ \lambda_i \in [L_--\frac{d}{10}, L_+ + \frac{d}{10}], \forall i \right\} \cap \Omega_V(\sigma') \cap \tilde{\Omega}(\sigma').
\]
See Sect. 3.2 for the notations. By Definition 8, Definition 9, (19), Lemma 16 and the fact that \( d < 1/2 \), we know that if \( N \) is large enough then
\[
\Gamma \cap \{ z | N^{-\frac{1}{2} - \sigma'} \leq \text{Im} z \leq N^{-\sigma'} \} \subset \mathcal{D}'_{\sigma d, \varsigma}.
\]
By Theorem 6 and Proposition 7,
\[
\mathbb{P}(R_N) \rightarrow 1 \quad \text{as} \quad N \rightarrow \infty.
\]
On \( R_N \) we have
\[
f(\lambda_i) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - \lambda_i} d\xi \quad \text{and} \quad f(t) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - t} d\xi \quad \forall t \in [L_-, L_+]\]
and then

\[
\frac{1}{\sqrt{N}} \left[ \sum f(\lambda_i) - N \int f(t) \rho_{fc}(t) \, dt \right] = \frac{1}{\sqrt{N}} \frac{1}{2\pi i} \oint_{\Gamma} f(\xi) \sum \frac{1}{\xi - \lambda_i} d\xi \\
- N \int f(t) \rho_{fc}(t) \, dt \right]
\]

\[
= \frac{1}{\sqrt{2\pi i}} \int_{\Gamma \setminus \Gamma^+} f(\xi)(Nm_{fc}(\xi) - TrG(\xi)) \, d\xi \\
+ \frac{1}{\sqrt{2\pi i}} \int_{\Gamma^+} f(\xi)(E_N[TrG(\xi)] - TrG(\xi)) \, d\xi \\
+ \frac{1}{\sqrt{2\pi i}} \int_{\Gamma^+} f(\xi)(Nm_{fc}(\xi) - E_N[TrG(\xi)]) \, d\xi \\
+ \frac{1}{\sqrt{2\pi i}} \int_{\Gamma^+} f(\xi)(Nm_{fc}(\xi) - N\hat{m}_{fc}(\xi)) \, d\xi \\
= P_0 + P_1 + P_2 + P_3. 
\]

(41)

**Lemma 18** If \( \xi \in \mathbb{C} \setminus \mathbb{R} \) then

\[
\sqrt{N}(\hat{m}_{fc} - m_{fc}) = (1 + m'_{fc})(\frac{1}{\sqrt{N}} \sum (g_i(\xi) - E[g_i(\xi)]) + \frac{(\hat{m}_{fc} - m_{fc})^2}{\sqrt{N}} \sum \hat{g}_i g_i^2 \\
+ \frac{1}{\sqrt{N}}(\hat{m}_{fc} - m_{fc}) \sum (g_i^2 - E[g_i^2])).
\]

**Proof** See Appendix B. \( \square \)

Using Lemma 18 for \( P_3 \) (i.e., the last term in (41)), we have that if \( R_N \) holds then

\[
\frac{1}{\sqrt{N}} \left[ \sum f(\lambda_i) - N \int f(t) \rho_{fc}(t) \, dt \right] = P_0 + P_1 + P_2 + P_{31} + P_{32} + P_{33} \quad (42)
\]

where \( P_0, P_1, P_2 \) are defined in (41) and

\[
P_{31} := \frac{-1}{2\pi i \sqrt{N}} \sum_i \int_{\Gamma^+} f(\xi)(1 + m'_{fc})(g_i - E[g_i]) d\xi,
\]

\[
P_{32} := \frac{-1}{2\pi i \sqrt{N}} \int_{\Gamma^+} f(\xi)(1 + m'_{fc})(\hat{m}_{fc} - m_{fc}) \sum_i (g_i^2 - E[g_i^2]) d\xi,
\]

\[
P_{33} := \frac{-1}{2\pi i \sqrt{N}} \int_{\Gamma^+} f(\xi)(1 + m'_{fc})(\hat{m}_{fc} - m_{fc})^2 \sum_i \hat{g}_i g_i^2 d\xi.
\]

- **Asymptotic behavior of \( P_0 \).** When \( R_N \) holds, we have:

\[
|P_0| \leq \frac{1}{\sqrt{N} 2\pi} \int_{\Gamma \cap \{|\text{Im}| \leq N^{-(-1 + \sigma')/2}\}} |f(\xi)| Nm_{fc}(\xi) - TrG(\xi) d\xi
\]
\begin{align*}
&+ \frac{1}{\sqrt{N}} \int_{\Gamma \cap \{N^{-(1+\nu')}/2 \leq |\text{Im} \xi| \leq N^{-\nu}\}} |f(\xi)||N m_{f,c}(\xi) - \text{Tr} G(\xi)|d\xi \\
&\leq \frac{1}{\sqrt{N}} 4N^{-\nu - \frac{\nu'}{2}} \sup_{\xi \in \Gamma} |f(\xi)| \cdot 4N/d \quad \text{(since } |m_{f,c}(\xi)| \leq 2/d \text{ and } |G_{ii}(z)| \leq 2/d) \\
&+ \frac{1}{\sqrt{N}} 4N^{-\nu} \sup_{\xi \in \Gamma} |f(\xi)| \cdot 2N^{2\nu' - \frac{1}{2}} \quad \text{(by (39), Lemma 5 and Definition 8)} \\
&= \left( \frac{8}{\pi d} + \frac{4}{\pi} \right) \sup_{\xi \in \Gamma} |f(\xi)|(N^{-\nu'}/2 + N^{2\nu' - \nu}) = o(1) \quad \text{(since } \nu' < \nu/2) \quad (43)
\end{align*}

• Asymptotic behavior of \( P_{33} \). Let

\[
\nu'' \in \left( \frac{1}{1+b}, \frac{11b - 9}{2b + 2} \right) \quad \text{such that } 4\nu'' + \nu < \frac{1}{2}.
\]

By the condition on \( \nu \), such \( \nu'' \) exists. When \( N \) is large enough, we have that \( \Gamma_+ \subset D''_{\nu''} \) and that if \( \Omega_V(\nu'') \) holds then by Lemma 5,

\[
|\tilde{m}_{f,c}(\xi) - m_{f,c}(\xi)| \leq N^{-\frac{1}{2} + 2\nu''} \quad \forall \xi \in \Gamma_+ \\
|P_{33} 1_{\Omega_V(\nu'')}| \leq \frac{1}{2\pi \sqrt{N}} |\Gamma| \sup_{\xi \in \Gamma} |f(\xi)|(1 + \frac{1}{d^2})N^{-1 + 4\nu''} N \cdot N^{\nu} \frac{1}{C_d^2} \quad (44)
\]

where \( C_d \) is defined in Lemma 16. Here we used the fact that \( |m'_{f,c}(\xi)| \) is bounded by \( \frac{1}{d^2} \) for \( \xi \in \Gamma \). The last inequality together with the facts that \( 4\nu'' + \nu < \frac{1}{2} \) and that \( \mathbb{P}(\Omega_V(\nu'')) \rightarrow 1 \) (by Proposition 7) yield:

\[ P_{33} \rightarrow 0 \quad \text{in distribution.} \quad (45) \]

• Asymptotic behavior of \( P_{32} \). Let

\[
W_N = \left\{ \frac{1}{N^\frac{1}{2} + \nu''} \sum_{i=1}^{N} (g_i^2(\xi) - \mathbb{E}[g_i^2(\xi)]) \leq 1, \quad \forall \xi \in \Gamma \right\}.
\]

Lemma 19 Suppose \( a_1 > 0, a_2 > 0 \) are constants. Then

\[
\mathbb{P}\left( \left| \frac{1}{N^{\frac{1}{2} + a_1}} \sum_{i=1}^{N} (g_i^2(\xi) - \mathbb{E}[g_i^2(\xi)]) \right| \leq a_2, \quad \forall \xi \in \Gamma \right) \rightarrow 1 \quad \text{as } N \rightarrow \infty
\]

\[
\mathbb{P}\left( \left| \frac{1}{N^{\frac{1}{2} + a_1}} \sum_{i=1}^{N} (g_i(\xi) - \mathbb{E}[g_i(\xi)]) \right| \leq a_2, \quad \forall \xi \in \Gamma \right) \rightarrow 1 \quad \text{as } N \rightarrow \infty
\]

Proof See Appendix B. \( \square \)

By Lemma 19 and Proposition 7, \( \mathbb{P}(\Omega_V(\nu'') \cap W_N) \rightarrow 1 \) as \( N \rightarrow \infty \). This together with the fact that
Asymptotic behavior of $P_{31}$. Let

$$U_N = \left\{ \frac{1}{N^{1+\frac{\sigma''}{2}}} \sum_{i=1}^{N} (g_i(\xi) - \mathbb{E}[g_i(\xi)]) \right\} \leq 1, \quad \forall \xi \in \Gamma \}.$$ 

By Lemma 19, $\mathbb{P}(U_N) \to 1$ as $N \to \infty$. This together with the fact that

$$\mathbb{1}_{U_N} \int_{\Gamma \setminus \Gamma^+} \frac{-1}{2\pi i \sqrt{N}} \sum_{i} \int_{\Gamma \setminus \Gamma^+} f(\xi)(1 + m'_{f_c})(g_i - \mathbb{E}g_i) d\xi \leq \mathbb{1}_{U_N} \cdot \frac{1}{2\pi} \int_{\Gamma \setminus \Gamma^+} \sup_{\xi} |f(\xi)(1 + \frac{1}{d^2}) N^{\sigma'/2} \cdot \left( \frac{1}{N^{1+\frac{\sigma''}{2}}} \left| \sum_{i=1}^{N} (g_i(\xi) - \mathbb{E}[g_i(\xi)]) \right| \right) d\xi \leq \frac{1}{2\pi} \sup_{\xi} |f(\xi)(1 + \frac{1}{d^2}) \cdot 4 \cdot N^{-\sigma/2} = o(1)$$

since $|\Gamma \setminus \Gamma^+| = 4N^{-\sigma}$ yield:

$$\int_{\Gamma \setminus \Gamma^+} f(\xi)(1 + m'_{f_c})(g_i - \mathbb{E}g_i) d\xi \to 0 \quad \text{in distribution.}$$. (46)

By central limit theorem, (47) converges in distribution to a centered Gaussian distribution whose variance is

$$\text{Var}\left( \frac{-1}{2\pi i} \int_{\Gamma} f(\xi)(1 + m'_{f_c}(\xi)) g_i(\xi) d\xi \right)
= \text{Var}\left( \frac{-1}{2\pi i} \int_{\Gamma} f(\xi) \frac{1 + m'_{f_c}(\xi)}{\lambda v_1 - \xi - m_{f_c}(\xi)} d\xi \right)
= \mathbb{E} \left[ \left( \frac{-1}{2\pi i} \int_{\Gamma} f(\xi) \frac{1 + m'_{f_c}(\xi)}{\lambda v_1 - \xi - m_{f_c}(\xi)} d\xi \right)^2 \right]
- \left( \mathbb{E} \left[ \frac{-1}{2\pi i} \int_{\Gamma} f(\xi) \frac{1 + m'_{f_c}(\xi)}{\lambda v_1 - \xi - m_{f_c}(\xi)} d\xi \right] \right)^2
= \frac{1}{4\pi^2} \left( \int_{\Gamma} f(\xi)(1 + m'_{f_c}(\xi)) m_{f_c}(\xi) d\xi \right)^2.$$
Lemma 20

Suppose the conditions of Lemma 14. Moreover, there exist constants \( r_0 \) and \( N_0 \) such that if \( N > N_0 \) then

\[
\frac{\partial}{\partial W_{ij}} e^{itX_N} = e^{itX_N} \frac{-2it}{\sqrt{N(1+\delta_{ij})}} \int_{\Gamma_+} f(\xi) G'_{ij}(\xi) d\xi \cdot \mathbb{1}_{\Omega_V(\xi)}
\]

Moreover, there exist constants \( r_0 > 0 \) and \( N_0 > 0 \) such that if \( N > N_0 \) then

\[
\left| \frac{\partial}{\partial W_{ij}} e^{itX_N} \right| \leq r_0 \cdot N^{\sigma - \frac{1}{2}}, \quad \left| \frac{\partial^2}{\partial W_{ij}^2} e^{itX_N} \right| \leq r_0 \cdot N^{2\sigma - \frac{1}{2}}, \quad \left| \frac{\partial^3}{\partial W_{ij}^3} e^{itX_N} \right| \leq r_0 \cdot N^{3\sigma - \frac{1}{2}}.
\]

Proof

By Lemma 10,

\[
\frac{\partial}{\partial W_{ij}} e^{itX_N} = e^{itX_N} \frac{it}{\sqrt{N(1+\delta_{ij})}} \int_{\Gamma_+} f(\xi) \frac{G_i(\xi) G_j(\xi)}{1+\delta_{ij}} d\xi \cdot \mathbb{1}_{\Omega_V(\xi)}
\]

\[= e^{itX_N} \frac{-2it}{\sqrt{N(1+\delta_{ij})}} \int_{\Gamma_+} f(\xi) G'_{ij}(\xi) d\xi \cdot \mathbb{1}_{\Omega_V(\xi)}
\]

\[
- \frac{1}{4\pi^2} \int_{\Gamma} \left( \int_{-1}^{1} \frac{f(\xi_1) f(\xi_2)(1+m'_{f,c}(\xi_1))(1+m'_{f,c}(\xi_2))d\mu(t)}{\lambda t - \xi_1 - m_{f,c}(\xi_1) (\lambda t - \xi_2 - m_{f,c}(\xi_2))} d\xi_1 d\xi_2 \right)
\]

\[
= \frac{1}{4\pi^2} \left( \int_{\Gamma} f(\xi)(1+m'_{f,c}(\xi)) m_{f,c}(\xi) d\xi \right)^2
\]

\[
- \frac{1}{4\pi^2} \int_{-1}^{1} \left( \int_{\Gamma} f(\xi)(1+m'_{f,c}(\xi)) d\mu(t) \right) d\xi
\]

(48)

**Conclusion** The asymptotic behaviors of \( P_0, P_{31}, P_{32}, P_{33} \) together with (40), (42), Lemmas 14 and 15 complete the proof of Theorem 4. We remark that the variance of the imaginary part of \( e^{itX_N} \) is bounded as \( N \to \infty \), and the left hand side of (48), must be 0. This is because the above argument show that (47) has the same limit in distribution as the real-valued random variable (36). □

### 5.3 Proof of Lemma 14

**Proof of Lemma 14** According to Proposition 7, it suffices to prove that

\[
X_N := \frac{1}{\sqrt{N}} \int_{\Gamma_+} f(\xi) \left[ \text{Tr} G(\xi) - \mathbb{E}_N \text{Tr} G(\xi) \right] \cdot \mathbb{1}_{\Omega_V(\xi)} d\xi
\]

converges in distribution to zero. Fix \( t \in \mathbb{R} \). We only need to show that

\[
\mathbb{E}[\exp(itX_N)] \to 1 \quad \text{as} \quad N \to \infty.
\]

Notice that

\[
d \frac{d}{dt} \mathbb{E}[\exp(itX_N)] = \frac{d}{dt} \mathbb{E}[\mathbb{E}_N[\exp(itX_N)]]
\]

\[
= \frac{i}{\sqrt{N}} \mathbb{E} \left[ \mathbb{E}_N \left[ \exp(itX_N) \int_{\Gamma_+} f(\xi) \left[ \text{Tr} G(\xi) - \mathbb{E}_N \text{Tr} G(\xi) \right] \cdot \mathbb{1}_{\Omega_V(\xi)} d\xi \right] \right]
\]

\[
= \frac{i}{\sqrt{N}} \int_{\Gamma_+} f(\xi) \cdot \mathbb{E} \left[ \mathbb{1}_{\Omega_V(\xi)} \mathbb{E}_N \left[ \exp(itX_N) [\text{Tr} G(\xi) - \mathbb{E}_N \text{Tr} G(\xi)] \right] \right] d\xi
\]

(50)

**Lemma 20** Suppose the conditions of Lemma 14 are satisfied. Then we have

\[
\frac{\partial}{\partial W_{ij}} e^{itX_N} = e^{itX_N} \frac{-2it}{\sqrt{N(1+\delta_{ij})}} \int_{\Gamma_+} f(\xi) G'_{ij}(\xi) d\xi \cdot \mathbb{1}_{\Omega_V(\xi)}
\]

(51)

Moreover, there exist constants \( r_0 > 0 \) and \( N_0 > 0 \) such that if \( N > N_0 \) then

\[
\left| \frac{\partial}{\partial W_{ij}} e^{itX_N} \right| \leq r_0 \cdot N^{\sigma - \frac{1}{2}}, \quad \left| \frac{\partial^2}{\partial W_{ij}^2} e^{itX_N} \right| \leq r_0 \cdot N^{2\sigma - \frac{1}{2}}, \quad \left| \frac{\partial^3}{\partial W_{ij}^3} e^{itX_N} \right| \leq r_0 \cdot N^{3\sigma - \frac{1}{2}}.
\]

Proof

By Lemma 10,

\[
\frac{\partial}{\partial W_{ij}} e^{itX_N} = e^{itX_N} \frac{it}{\sqrt{N(1+\delta_{ij})}} \int_{\Gamma_+} f(\xi) G_{ij}(\xi) d\xi \cdot \mathbb{1}_{\Omega_V(\xi)}
\]

\[= e^{itX_N} \frac{-2it}{\sqrt{N(1+\delta_{ij})}} \int_{\Gamma_+} f(\xi) G'_{ij}(\xi) d\xi \cdot \mathbb{1}_{\Omega_V(\xi)}
\]
Noticing $|G'_{ij}(\xi)| \leq |\text{Im}\xi|^{-2}$ we complete the proof of the first inequality in\( (52) \). The other two inequalities in \( (52) \) can be proved similarly by directly taking more derivatives of $e^{itX_N}$ with respect to $W_{ij}$.

For any $\xi \in \mathbb{C}\setminus\mathbb{R}$, by the definition

$$G(\xi) = \frac{1}{\lambda V + W - \xi}$$

we have $(\xi - \lambda v_i)G_{ii} = -1 + (WG)_{ii} = -1 + \sum_j W_{ij}G_{ij}$. Then by \( (28) \),

$$(\xi - \lambda v_i)\mathbb{E}_N[e^{itX_N}(G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))]$$

$$= \sum_j \left( \mathbb{E}_N[e^{itX_N}W_{ij}G_{ij}(\xi)] - \mathbb{E}_N[e^{itX_N}\mathbb{E}_N[W_{ij}G_{ij}(\xi)]] \right).$$

To use cumulant expansion to study $(\xi - \lambda v_i)\mathbb{E}_N[e^{itX_N}(G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))]$, we write:

$$(\xi - \lambda v_i)\mathbb{E}_N[e^{itX_N}(G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))]$$

$$= \sum_j \frac{1 + \delta_{ij}}{N} \left( \mathbb{E}_N\left[ \frac{\partial e^{itX_N}}{\partial W_{ij}} G_{ij}(\xi) + e^{itX_N} \frac{-1}{1 + \delta_{ij}} (G_{ii}(\xi)G_{jj}(\xi) + (G_{ij}(\xi))^2) \right] - \mathbb{E}_N[e^{itX_N}]\mathbb{E}_N[\frac{-1}{1 + \delta_{ij}} (G_{ii}(\xi)G_{jj}(\xi) + (G_{ij}(\xi))^2)] \right)$$

$$+ \sum_j \frac{\mathbb{E}[|\sqrt{N} W_{ij}|^3]}{2N^{3/2}} \left( \mathbb{E}_N[G_{ij}(\xi) \frac{\partial^2 (e^{itX_N})}{\partial W_{ij}^2}] + 2\mathbb{E}_N[\frac{\partial (e^{itX_N})}{\partial W_{ij}} \frac{\partial G_{ij}(\xi)}{\partial W_{ij}}] \right)$$

$$+ \mathbb{E}_N[(e^{itX_N} - \mathbb{E}_N e^{itX_N}) \frac{\partial^2 G_{ij}(\xi)}{\partial W_{ij}^2}] + e^{i(\xi)}(\xi), \; \forall \xi \in \mathbb{C}\setminus\mathbb{R}. \quad (53)$$

According to Lemma 8, there are constants $r_1 > 0$ and $N_0 > 0$ such that if $N > N_0$ then

$$|E^{(i)}_1(\xi)| \leq r_1 \cdot \frac{1}{N} \cdot \left( N^{3\sigma - \frac{1}{2}} + \frac{1}{|\text{Im}\xi|^4} \right) \leq \frac{r_1}{N|\text{Im}\xi|^4} \quad \text{for any } \xi \in \Gamma_+ \quad (54)$$

Here we used \( (9) \), Lemma 20 and the condition $\sigma < \frac{1}{14}$ to control $E^{(i)}_1(\xi)$. For convenience we let $E^{(i)}_2(\xi)$ be the second summation on the right hand side of \( (53) \):

$$E^{(i)}_2(\xi) := \sum_j \frac{\mathbb{E}[|\sqrt{N} W_{ij}|^3]}{2N^{3/2}} \left( \mathbb{E}_N\left[ G_{ij}(\xi) \frac{\partial^2 (e^{itX_N})}{\partial W_{ij}^2} \right] + 2\mathbb{E}_N\left[ \frac{\partial (e^{itX_N})}{\partial W_{ij}} \frac{\partial G_{ij}(\xi)}{\partial W_{ij}} \right] + \mathbb{E}_N\left[ (e^{itX_N} - \mathbb{E}_N e^{itX_N}) \frac{\partial^2 G_{ij}(\xi)}{\partial W_{ij}^2} \right] \right).$$

Moreover we set:

$$E_3(\xi) := \frac{1}{N} \sum_{i} g_i(\xi)\mathbb{E}_N[e^{itX_N}(G'_{ii}(\xi) - \mathbb{E}_N G'_{ii}(\xi))] + \frac{2it}{N^{3/2}} \int_{\Gamma_+} f(\xi')$$

$$\times \sum_i g_i(\xi)\mathbb{E}_N[e^{itX_N}(G(\xi)G'_{ij}(\xi))_{ij}] d\xi' \mathbb{I}_{\Omega_V(\xi)} - \sum_i g_i(\xi) \left( E^{(i)}_1(\xi) + E^{(i)}_2(\xi) \right).$$
Lemma 21  For any $\xi \in \mathbb{C} \setminus \mathbb{R}$ we have:

$$(1 - \frac{1}{N} \sum_i g_i^2(\xi)) \mathbb{E}_N[e^{itX_N} (\text{Tr}G(\xi) - \mathbb{E}_N\text{Tr}G(\xi))]$$

$$= - \sum_i g_i(\xi) \mathbb{E}_N[e^{itX_N} (G_{ii}(\xi) - g_i(\xi))(m_{fc}(\xi) - \frac{1}{N}\text{Tr}G(\xi))]$$

$$- \mathbb{E}_N[e^{itX_N}] \sum_i g_i(\xi) \mathbb{E}_N[(G_{ii}(\xi) - g_i(\xi))(\frac{1}{N}\text{Tr}G(\xi) - m_{fc}(\xi))] + \mathcal{E}_3(\xi) \ (55)$$

**Proof** See Appendix B. \hfill \Box

Proof Using (28) and Lemma 10, for any $z \in \mathbb{C} \setminus \mathbb{R}$

$$\frac{\partial^2 G_{ij}}{\partial W_{ij}^2} = \frac{6G_{ij}G_{ii}G_{jj} + 2(G_{ij})^3}{(1 + \delta_{ij})^2} \leq \begin{cases} \frac{1}{|\text{Im}z|^3} & \text{if } i = j \\ \frac{6}{|\text{Im}z|^2} \max_{i \neq j} |G_{ij}| + 2 \max_{i \neq j} |G_{ij}|^3 & \text{if } i \neq j \end{cases}$$  \quad (56)

By (9), (28), (56), Lemmas 10 and 20, if $\xi \in \Gamma_+$ and $N > N_0$ then

$$|\mathcal{E}_2^{(i)}(\xi)| \leq r_2 \left( \frac{N^{2\sigma}}{N|\text{Im}\xi|} + \frac{1}{\sqrt{N}|\text{Im}\xi|^2} \mathbb{E}_N[\max_{j \neq i} |G_{ij}(\xi)|]\right)$$  \quad (57)

where $r_2 > 0$ and $N_0 > 0$ are constants.

By Lemma 10,

$$|(G(\xi)G'(\xi'))_{ii}| \leq |\text{Im}\xi|^{-1} |\text{Im}\xi'|^{-2} \quad \forall \xi, \xi' \in \mathbb{C} \setminus \mathbb{R}.$$  \quad (58)

Plugging (58) into the definition of $\mathcal{E}_3(\xi)$, by (54), (57) and the fact that $|g_i(\xi)| \leq |\text{Im}\xi|^{-1}$, we have

$$|\mathcal{E}_3(\xi)| \leq r_3 \left( |\text{Im}\xi|^{-5} + N^{2\sigma} \cdot |\text{Im}\xi|^{-2} + \frac{\sqrt{N}}{|\text{Im}\xi|^3} \mathbb{E}_N[\max_{j \neq i} |G_{ij}(\xi)|]\right), \text{ if } \xi \in \Gamma_+ \text{ and } N > N_0$$  \quad (59)

where $r_3 > 0$ and $N_0 > 0$ are constants.

**Lemma 22** Suppose $\Omega_{\mathcal{V}}(\varsigma)$ holds. If $N$ is large enough, then:

$$\left| 1 - \frac{1}{N} \sum_i g_i^2(\xi) \right| \geq \frac{1}{3} |\text{Im}\xi|^2 \quad \forall \xi \in \Gamma_+.$$  \quad (60)

**Proof** See Appendix B. \hfill \Box

By (55), (59), Lemmas 17 and 22, if $N$ is large enough and $\xi \in \Gamma_+$ then

$$|\mathbb{E}_N[e^{itX_N} (\text{Tr}G(\xi) - \mathbb{E}_N\text{Tr}G(\xi))]| \leq 6N^{1+3\sigma} \left( \mathbb{E}_N[|G_{ii} - g_i|m_{fc} - \frac{1}{N}\text{Tr}G|\mathbb{1}_{\Omega_{\mathcal{V}}(\varsigma)}\setminus B_N] \right.$$}

$$+ \mathbb{E}_N[\max_{i \neq j} |G_{ij}(\xi)| |\mathbb{1}_{\Omega_{\mathcal{V}}(\varsigma)}\setminus B_N] \right) + p_N(\xi) \leq 30N^{1+5\sigma} \mathbb{E}_N[|\mathbb{1}_{\Omega_{\mathcal{V}}(\varsigma)}\setminus B_N] + p_N(\xi)$$  \quad (60)
where \( p_N(\xi) = \frac{8N^2\varsigma + \varsigma'}{|\text{Im}\xi|^6} + \frac{16N^4\varsigma}{|\text{Im}\xi|^3} + \frac{2r_1N^{2\sigma}}{|\text{Im}\xi|^4} + \frac{2r_1N^3\varsigma'}{|\text{Im}\xi|^5} \).

According to (60), (50) and (38), there exist constants \( r_4 > 0 \) and \( N_0 > 0 \) such that if \( N > N_0 \) then
\[
\left| \frac{d}{dt} \mathbb{E}[\exp(itX_N)] \right| \leq \frac{1}{\sqrt{N}} \int_{\Gamma_+} |f(\xi)| \left( 30N^{1+5\sigma} \mathbb{P}(\Omega_V(\xi) \backslash B_N) + p_N(\xi) \right) d\xi
\]
\[
\leq r_4 \left( N^{5\sigma+2\varsigma+\varsigma'-\frac{1}{2}} + N^{4\sigma+4\varsigma-\frac{1}{2}} + N^{7\sigma+\varsigma'-\frac{1}{2}} \right)
\]
and therefore
\[
\left| \mathbb{E}[\exp(itX_N)] - 1 \right| \leq t \cdot r_4 \left( N^{5\sigma+2\varsigma+\varsigma'-\frac{1}{2}} + N^{4\sigma+4\varsigma-\frac{1}{2}} + N^{7\sigma+\varsigma'-\frac{1}{2}} \right). \tag{61}
\]
By the conditions on \( \sigma, \varsigma \) and \( \varsigma' \) in Definition 11, the exponent for each term on the right hand side of (61) must be negative. So (49) is true and we complete the proof of Lemma 14.

\section*{5.4 Proof of Lemma 15}

**Proof of Lemma 15** According to Proposition 7, it suffices to prove that
\[
\frac{1}{\sqrt{N}} \int_{\Gamma_+} f(\xi) \left[ N\hat{m}_{fc}(\xi) - \mathbb{E}_N \text{Tr} G(\xi) \right] \cdot 1_{\Omega_V(\xi)} d\xi \to 0 \quad \text{in distribution.} \tag{62}
\]

Let
\[
Q_i = -\lambda v_i - W_{ii} + \sum_{p,q} W_{ip} G_{pq}(\xi) W_{qi}.
\]

**Lemma 23** For any \( \xi \in \mathbb{C} \backslash \mathbb{R} \), \( \mathbb{E}_N [\text{Tr} G(\xi) - N\hat{m}_{fc}(\xi)] \) equals:
\[
(1 + \hat{m}_f'(\xi))(\frac{1}{N} \sum_{i=1}^{N} \hat{g}_i^2(\xi) \mathbb{E}_N \left[ G_{ii}(\xi) + \frac{1}{G_{ii}(\xi)} \sum_{p} (G_{ip}(\xi))^2 \right] + \mathbb{E}_N \left[ \sum_{i=1}^{N} \frac{(G_{ii}(\xi) - \hat{g}_i(\xi))^3}{(G_{ii}(\xi))^2} \right])
\]
\[
\quad + (1 + \hat{m}_f'(\xi)) \sum_{i=1}^{N} \hat{g}_i^3(\xi) \left( \frac{2}{N} + \frac{2}{N^2} \mathbb{E}_N [(\text{Tr} G(i)(\xi))] + \sum_{p} \mathbb{E}_N [(G_{pp}(\xi))^2] (\mathbb{E}_N[W_{ip}^4] - \frac{3}{N^2}) + \mathbb{E}_N [(\hat{m}_{fc}(\xi) - \frac{1}{N} \text{Tr} G(i)(\xi))^2] \right). \tag{63}
\]

**Proof** See Appendix B.
and thus by (22)

\[ u_1 \cdot |\text{Im} \xi| \leq |\hat{g}_i(\xi)| \leq \frac{1}{|\text{Im} \xi|}. \]  

(64)

By the definition of Stieltjes transform, \( |\hat{m}'_{fc}| \leq \frac{1}{|\text{Im} \xi|^2} \). This together with (9), Lemma 10 and (64) yield:

\[
\begin{align*}
(1 + \hat{m}'_{fc}) \frac{1}{N} \sum_{i=1}^{N} \hat{g}_i^2(\xi) E_N[G_{ii}(\xi)] & \leq 2|\text{Im} \xi|^{-5}, \quad \forall \xi \in \Gamma \backslash \mathbb{R} \\
(1 + \hat{m}'_{fc}) \sum_{i=1}^{N} \hat{g}_i^2(\xi) \left( \frac{2}{N} + \frac{2}{N^2} E_N[(\text{Tr} G^{(i)})'] + \sum_{p} \|E_N[(G^{(i)}_p)^2]/\|E[N(W_{ip}^4] - \frac{3}{N^2}] \right) & \\
& \leq 2|\text{Im} \xi|^{-5} \left( 2 + 2|\text{Im} \xi|^{-2} + |\text{Im} \xi|^{-2} (\max_{a,b} \|E[(\sqrt{N} W_{ab})^4] + 3) \right) & \leq u_2|\text{Im} \xi|^{-7}, \quad \forall \xi \in \Gamma \backslash \mathbb{R}
\end{align*}
\]

(65)

where \( u_2 > 0 \) is a constant. According to Lemma 17, there are constants \( u_3 > 0 \) and \( N_0 > 0 \) such that if \( N > N_0 \) and \( \xi \in \Gamma_+ \), then

\[
\begin{align*}
(1 + \hat{m}'_{fc}) \frac{1}{N} \sum_{i=1}^{N} \hat{g}_i^2(\xi) E_N \left[ \frac{1}{G_{ii}(\xi)} \sum_{p} (G ip(\xi))^2 I_{\Omega V(\xi) \cap B_N} \right] & \\
& \leq \frac{u_3}{|\text{Im} \xi|^5} E_N \left[ \sum_{p} (G ip(\xi))^2 \right] I_{\Omega V(\xi) \cap B_N} \\
& = \frac{u_3}{|\text{Im} \xi|^5} E_N [(G^2(\xi))_ii - (G ii(\xi))^2] I_{\Omega V(\xi) \cap B_N} & \leq u_3 |\text{Im} \xi|^{-7} \quad \text{(by Lemma 10)}
\end{align*}
\]

(66)

\[
\begin{align*}
(1 + \hat{m}'_{fc}) E_N \left[ \sum_{i=1}^{N} \frac{(G_{ii}(\xi) - \hat{g}_i(\xi))^3}{(G_{ii}(\xi))^2} I_{\Omega V(\xi) \cap B_N} \right] & \leq u_3 (\frac{N \xi'}{\sqrt{N} |\text{Im} \xi|^13} + \frac{N^6 \xi}{\sqrt{N} |\text{Im} \xi|^10}) \\
(1 + \hat{m}'_{fc}) \sum_{i=1}^{N} \hat{g}_i^3(\xi) E_N \left[ (\hat{m}'_{fc} - \frac{1}{N} \text{Tr} G^{(i)})^2 I_{\Omega V(\xi) \cap B_N} \right] & \leq u_3 N^4 \xi |\text{Im} \xi|^{-5}.
\end{align*}
\]

(67)

(68)

(69)

By Lemma 23 and the conditions in Definition 11, the terms on the left hand side of (65), (66), (67), (68), (69) all make \( o(1) \) contribution to the quantity in (62). So to prove (62) it suffices to show that

\[
\begin{align*}
\frac{1}{N^{3/2}} \int_{\Gamma_+} f(\xi)(1 + \hat{m}'_{fc}) \sum_{i=1}^{N} \hat{g}_i^2(\xi) E_N \left[ \frac{1}{G_{ii}(\xi)} \sum_{p} (G ip(\xi))^2 I_{\Omega V(\xi) \cap B_N} \right] d\xi & \rightarrow 0 \quad \text{in distribution}
\end{align*}
\]

(70)

\[
\begin{align*}
\frac{1}{\sqrt{N}} \int_{\Gamma_+} f(\xi)(1 + \hat{m}'_{fc}) E_N \left[ \sum_{i=1}^{N} \frac{(G_{ii}(\xi) - \hat{g}_i(\xi))^3}{(G_{ii}(\xi))^2} I_{\Omega V(\xi) \cap B_N} \right] d\xi & \rightarrow 0 \quad \text{in distribution}
\end{align*}
\]

(71)

\[
\begin{align*}
\frac{1}{\sqrt{N}} \int_{\Gamma_+} f(\xi)(1 + \hat{m}'_{fc}) \sum_{i=1}^{N} \hat{g}_i^3(\xi) E_N \left[ (\hat{m}'_{fc} - \frac{1}{N} \text{Tr} G^{(i)})^2 I_{\Omega V(\xi) \cap B_N} \right] d\xi & \rightarrow 0 \quad \text{in distribution}
\end{align*}
\]

(72)

\( \circlearrowleft \) Springer
Notice that $|\hat{m}_{fc}|$ and $\frac{1}{N} \text{Tr} G^{(i)}$ are bounded by $\frac{1}{|\Im \xi|}$. By (38), if $N$ is large enough and $\xi \in \Gamma_+$ then
\[
P\left(\left| \mathbb{E}_N \left[ \frac{1}{N} \text{Tr} G^{(i)}(\xi) \right] - \mathbb{E} \left[ \frac{1}{N} \text{Tr} G^{(i)}(\xi) \right] \right|^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right) > N^{-5}
\]
\[
\leq N^5 \mathbb{E} \left[ \mathbb{E}_N \left[ \left( \frac{1}{N} \text{Tr} G^{(i)}(\xi) \right)^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right] \right]
\]
\[
\leq N^5 \mathbb{E} \left[ \left( \frac{1}{N} \text{Tr} G^{(i)}(\xi) \right)^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right] \leq 4N^{5+2\sigma} \mathbb{P}(\Omega V(\xi) \setminus B_N) < N^{-100}
\]
which together with a classic “lattice” argument yields
\[
P\left( \left| \mathbb{E}_N \left[ \frac{1}{N} \text{Tr} G^{(i)}(\xi) \right] - \mathbb{E} \left[ \frac{1}{N} \text{Tr} G^{(i)}(\xi) \right] \right|^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right) \leq 2N^{-5}, \ \forall \xi \in \Gamma_+ \geq 1 - N^{-20}.
\]
This and the facts that $|\hat{m}_{fc}(\xi)| \leq N^{2\sigma}$ and $|\hat{g}_i(\xi)| \leq N^{\sigma}$ on $\Gamma_+$ complete the proof of (72).

According to (38) and (27), if $N$ is large enough and $\xi \in \Gamma_+$, then
\[
P\left( \left| \mathbb{E}_N \left[ \frac{1}{G_{ii}(\xi)} \sum_p (G_{ip}(\xi))^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right] \right|^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right) > N^{-5}
\]
\[
\leq N^5 \mathbb{E} \left[ \mathbb{E}_N \left[ \frac{1}{G_{ii}(\xi)} \sum_p (G_{ip}(\xi))^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right] \right]
\]
\[
\leq N^5 \mathbb{E} \left[ \left( \frac{1}{G_{ii}(\xi)} \sum_p (G_{ip}(\xi))^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right) \right] \leq N^{6+2\sigma} \mathbb{E} \left[ \frac{1}{G_{ii}(\xi)} \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right]
\]
\[
\leq N^{6+2\sigma} \sqrt{\mathbb{E} \left[ \frac{1}{|G_{ii}(\xi)|^2} \right] \mathbb{P}(\Omega V(\xi) \setminus B_N)}
\]
\[
= N^{6+2\sigma} \sqrt{\mathbb{E} \left[ \lambda_{ii} + W_{ii} - \xi + \sum_{k,l} W_{ik} G_{kl}^{(i)}(\xi) W_{il} \right] \mathbb{P}(\Omega V(\xi) \setminus B_N)}
\]
\[
\leq N^{8+3\sigma} \sqrt{\mathbb{P}(\Omega V(\xi) \setminus B_N)} \leq N^{-100}
\]
which together with (27) and a classic “lattice” argument yields
\[
P\left( \left| \mathbb{E}_N \left[ \frac{1}{G_{ii}(\xi)} \sum_p (G_{ip}(\xi))^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right] \right|^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right) \leq 2N^{-5}, \ \forall \xi \in \Gamma_+ \geq 1 - N^{-20}.
\]
This and the facts that $|\hat{m}_{fc}(\xi)| \leq N^{2\sigma}$ and $|\hat{g}_i(\xi)| \leq N^{\sigma}$ on $\Gamma_+$ complete the proof of (70).

Similarly, according to (38) and (27), if $N$ is large enough and $\xi \in \Gamma_+$, then
\[
P\left( \left| \mathbb{E}_N \left[ \sum_{i=1}^N \left( \frac{G_{ii}(\xi) - \hat{g}_i(\xi)}{G_{ii}(\xi)} \right)^3 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right] \right|^2 \mathbb{1}_{\Omega V(\xi) \setminus B_N} \right) > N^{-5}
\]
\[ \leq N^5 \mathbb{E} \left[ \mathbb{E}_N \left[ \sum_{i=1}^{N} \frac{(G_{ii}(\xi) - \hat{g}_i(\xi))^3}{(G_{ii}(\xi))^2} \mathbb{1}_{\Omega_\gamma(\xi) \setminus B_N} \right] \right] \]
\[ \leq 8N^{5+3\sigma} \sum_i \sqrt{\mathbb{E} \left[ \frac{1}{|G_{ii}(\xi)|^4} \mathbb{P}(\Omega_\gamma(\xi) \setminus B_N) \right]} < N^{-100} \]

which together with (27) and a classic “lattice” argument yields
\[ \mathbb{P} \left( \left| \mathbb{E}_N \left[ \sum_{i=1}^{N} \frac{(G_{ii}(\xi) - \hat{g}_i(\xi))^3}{(G_{ii}(\xi))^2} \mathbb{1}_{\Omega_\gamma(\xi) \setminus B_N} \right] \right| \leq 2N^{-5}, \ \forall \xi \in \Gamma_+ \right) \geq 1 - N^{-20}. \]

This and the facts that \(|\hat{m}'_{fc}(\xi)| \leq N^{2\sigma}\) and \( |\hat{g}_i(\xi)| \leq N^{\sigma}\) on \(\Gamma_+\) complete the proof of (71).

\(\square\)

6 Rigidity of Eigenvalues: Proof of Theorem 5

In this section we prove the rigidity of eigenvalues in the bulk of the spectrum. For an eigenvalue \(\lambda_i\) in the bulk, we show that it is very close to the deterministic number \(\gamma_i\) with high probability. Roughly speaking, the distance between \(\lambda_i\) and \(\gamma_i\) is no more than \(N^{-\frac{1}{4} + \frac{1}{100\sigma}}\) with high probability.

By the definition of \(\gamma_i\) and \(\hat{\gamma}_i\) (see Definition 4) we have that
\[ \int_{\hat{\gamma}_i+1}^{\hat{\gamma}_i} \frac{1}{L_+ - t} d\mu_{fc}(t) \leq \frac{1}{N} \frac{1}{L_+ - \gamma_i} \leq \int_{\hat{\gamma}_i-1}^{\hat{\gamma}_i-2} \frac{1}{L_+ - t} d\mu_{fc}(t), \ \forall i \in [2, N - 1]. \] (73)

Lemma 24 There exists a constant \(C_* \geq 1\) such that

- if \(b > 1\) and \(\lambda > \lambda_+\) then \(C_*^{-1} \left( \frac{i}{N} \right)^{\frac{1}{100\sigma}} \leq |L_+ - \gamma_i| \leq C_* \left( \frac{i}{N} \right)^{\frac{1}{100\sigma}} ;\)
- if \(a > 1\) and \(\lambda > \lambda_-\) then \(C_*^{-1} \left( \frac{N-i}{N} \right)^{\frac{1}{100\sigma}} \leq |L_- - \gamma_i| \leq C_* \left( \frac{N-i}{N} \right)^{\frac{1}{100\sigma}} .\)

Proof Suppose \(b > 1, \lambda > \lambda_+.\) According to Lemma 1, there exists \(C'_* > 1\) such that
\[ \frac{(L_+ - x)^b}{C'_*} \leq \rho_{fc}(x) \leq C'_*(L_+ - x)^b \]
for \(x \in [\gamma_{0.99N}, L_+].\) Therefore if \(i \leq 0.99N,\) then
\[ \frac{(L_+ - \gamma_i)^{b+1}}{C'_*(b+1)} = \int_{\gamma_i}^{L_+} \frac{(L_+ - x)^b}{C'_*} dx \leq \int_{\gamma_i}^{L_+} \rho_{fc}(x) dx = \frac{i - \frac{1}{2}}{N} \]
\[ \leq \int_{\gamma_i}^{L_+} C'_*(L_+ - x)^b dx = \frac{C'_*(L_+ - \gamma_i)^{b+1}}{(b+1)} . \]
If \(i > 0.99N,\) then both \(i/N\) and \(|L_+ - \gamma_i|\) are of order 1, so the inequality also holds. This proves the first conclusion. The second conclusion can be proved in the same way.
Lemma 25 Suppose $b > 1$ and $\epsilon \in (\frac{1}{1+b}, \frac{11b-9}{2b+2})$. Suppose $A_N(\epsilon)$ holds and $z_0$ is in

$$\left\{ x + iy \bigg| |x| \leq 3 + \lambda, y \in \left[ \frac{1}{2} N^{-\frac{1}{1+b}-\epsilon}, N^{-\frac{1}{1+b}+\epsilon} \right] \right\}. \quad (74)$$

If $N$ is large enough, then we have $z_0 \in D'_\epsilon$ and

$$|m_N(z_0) - m_{f,c}(z_0)| \leq 2N^{2\epsilon - \frac{1}{2}}. \quad (75)$$

Remark 5 In the condition $\epsilon \in (\frac{1}{1+b}, \frac{11b-9}{2b+2})$, the interval is not empty since $b > 1$.

Proof By the condition $b > 1$ it’s easy to see that (74) is contained in $D_\epsilon$. We notice that $\text{Im} z_0$ and $\text{Im} m_{f,c}(z_0)$ have the same sign, so $|\lambda v_i - z_0 - m_{f,c}(z_0)| \geq |\text{Im} z_0| \geq \frac{1}{2} N^{-\frac{1}{1+b}-\epsilon}$. Thus $z_0 \in D'_\epsilon$. Finally (75) is from Lemma 5 and the definition of $\tilde{\Omega}(\epsilon)$. \qed

Suppose $\epsilon > 0$ and

- $I$ is an interval contained in $(-2.99 - \lambda, 2.99 + \lambda)$ and it may depend on $N$,
- $\eta_0 = N^{-\frac{1}{4}+\epsilon}$ and $\eta_1 = N^{-\frac{1}{1+b} - \epsilon}$;
- $\chi : \mathbb{R} \to [0, 1]$ is a $C^\infty$ function supported on $[-2, 2]$ such that $\chi(x) = 1$ when $x \in [-1, 1]$;
- $f : \mathbb{R} \to [0, 1]$ is a smooth $N$-depending function such that

$$f(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } \text{dist}(x, I) \geq \eta_0 \end{cases} \quad (76)$$

and

$$\|f'\|_\infty \leq C_a \cdot \eta_0^{-1}, \quad \|f''\|_\infty \leq C_a \cdot \eta_0^{-2} \quad (77)$$

for some absolute constant $C_a > 0$.

Remark 6 It is easy to see that $f$ satisfies the following properties.

1. If $\epsilon < \frac{1}{4}$ then $\text{supp} f \subset (-2.995 - \lambda, 2.995 + \lambda)$ when $N > N_0 = N_0(\epsilon)$.
2. $|\text{supp} f'| \leq 2\eta_0, |\text{supp} f''| \leq 2\eta_0.$

Recall that $\mu_N$ is defined in Definition 6.

Lemma 26 Suppose $b > 3$ and $\epsilon \in (\frac{1}{1+b}, \frac{1}{4})$. Suppose $A_N(\epsilon)$ hold. Then

$$|\int f(t) d\mu_N(t) - \int f(t) d\mu_{f,c}(t)| \leq \alpha_1 (\eta_0^2 + \eta_1^2 \eta_0)$$

for large enough $N$. Here $\alpha_1 > 0$ is a constant depending only on $C_a$.

Remark 7 The condition $\epsilon \in (\frac{1}{1+b}, \frac{1}{4})$ is stronger than the condition $\epsilon \in (\frac{1}{1+b}, \frac{11b-9}{2b+2})$ in Lemma 25 because the condition $b > 3$ ensures that $\frac{1}{4} < \frac{11b-9}{2b+2}$. \&
According to the Helffer-Sjöstrand formula (see Lemma 7),

\[
\int f(t) d\mu_N(t) - \int f(t) d\mu_f(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( \frac{iyf''(\chi(y) + i(f(x) + iyf'(\chi(y) }{t - x - iy} \right) (d\mu_N(t) - d\mu_f(t))
\]

where

\[
K_1 = \text{Re} \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi'(y) (f(x) + iyf'(\chi(y))(m_N(x + iy) - m_f(x + iy)) dy dx \right)
\]

\[
K_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{|y| \geq \eta_1} yf''(\chi(y)) \text{Im} (m_N(x + iy) - m_f(x + iy)) dy dx
\]

\[
K_3 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{|y| < \eta_1} yf''(\chi(y)) \text{Im} (m_N(x + iy) - m_f(x + iy)) dy dx
\]

Recall \( \epsilon > \frac{1}{1 + B} \) and that \( A_N(\epsilon) \) holds. For simplicity let \( \tilde{m}(z) = m_N(z) - m_f(z) \).

1. We first estimate \( K_1 \). By (75), Remark 6 and the definition of \( \chi \), if \( N \) is large enough then:

\[
|K_1| \leq 2N^{2\epsilon - \frac{1}{2}} \cdot \frac{1}{2\pi} \cdot 2\|\chi'\|_\infty \int |f(x)| + 2|f'(x)| dx \leq CN^{2\epsilon - \frac{1}{2}} = C\eta_0^2
\]

where \( C > 0 \) is a constant depending only on \( C_a \).

2. Then we estimate \( K_2 \). Suppose \( x + iy \) is in the support of \( yf''(\chi(y)) \text{I}_{[|y| \geq \eta_1]}(x + iy) \). Let \( L \) be the counterclockwise circle centered at \( x + iy \) with radius \( \min(\frac{x}{2}, 0.005) \). If \( N > N_0 = N_0(\epsilon) \), then \( L \) is contained in (74) and we have by Cauchy’s Theorem and (75) that

\[
|\partial_y \tilde{m}(x + iy)| = \left| \frac{1}{2\pi i} \oint_L \frac{\tilde{m}(r)}{(r - x - iy)^2} dr \right| \leq 4N^{2\epsilon - \frac{1}{2}} \max \left( \frac{1}{y}, 100 \right)
\]

and therefore

\[
|K_2| \leq \frac{1}{2\pi} \int_{|y| \leq \eta_1, 2} \text{Im} \left[ - \int L f'(x) \partial_y \tilde{m}(x + iy) dx \right] y \chi(y) dy \quad \text{(integral by parts for \( x \))}
\]

\[
\leq \frac{1}{2\pi} \cdot 4N^{2\epsilon - \frac{1}{2}} \cdot 2C_a \cdot 2 \int_{\eta_1}^{2} y \chi(y) \max \left( \frac{1}{y}, 100 \right) dy \leq \frac{3200C_a}{\pi} N^{2\epsilon - \frac{1}{2}} \frac{1}{\pi} = \frac{3200C_a \cdot \eta_0^2}{\pi}
\]

3. Finally we estimate \( K_3 \). Notice that both \( y \text{Im} m_N(x + iy) \) and \( y \text{Im} m_f(x + iy) \) are nonnegative and increasing when \( y \in [0, \eta_1] \). So,
We remark that if \( f \) is a function such that \( \alpha \geq 1 \), and satisfies (77). From Lemma 26, if \( N \) is large enough then

\[
|K_3| = \frac{1}{\pi} \left| \int_{\mathbb{R}} f''(x) \left( y \text{Im} N(x + iy) - y \text{Im} f_c(x + iy) \right) dy dx \right|
\]

\[
\leq \frac{1}{\pi} \int_{\mathbb{R}} |f''(x)| \left| \text{Im} N(x + iy) + y \text{Im} f_c(x + iy) \right| dy dx
\]

\[
\leq \frac{1}{\pi} \int_{\mathbb{R}} |f''(x)| \left( \eta_1 \text{Im} N(x + i \eta_1) + \eta_1 \text{Im} f_c(x + i \eta_1) \right) dy dx
\]

\[
= \frac{\eta_1^2}{\pi} \int_{\mathbb{R}} |f''(x)| (\text{Im} N(x + i \eta_1) + \text{Im} f_c(x + i \eta_1)) dx
\]

If \( x \in \text{supp} f \), then \( x + i \eta_1 \) is in (74) and thus also in \( D'_e \), provided \( N \) is large enough. So we know by (75) and the definition of \( \Omega_a(\epsilon) \) that if \( N \) is large enough then

\[
\text{Im} N(x + i \eta_1) + \text{Im} f_c(x + i \eta_1) \leq 4N^2\epsilon - \frac{1}{2}
\]

and thus

\[
|K_3| \leq \frac{4\eta_1^2}{\pi} N^{2\epsilon - \frac{1}{2}} \int_{\mathbb{R}} |f''(x)| dx \leq \frac{8C_a \cdot \eta_1^2}{\pi \eta_0} N^{2\epsilon - \frac{1}{2}} = \frac{8C_a \cdot \eta_1^2 \cdot \eta_0}{\pi}
\]

The estimates for \( K_1, K_2 \) and \( K_3 \) together complete the proof of the lemma. \( \square \)

**Lemma 27** Suppose \( b > 3 \) and \( \epsilon \in (\frac{1}{1+\beta}, \frac{1}{2}) \). Suppose \( A_N(\epsilon) \) holds. If \( N \) is large enough, then for any (possibly \( N \)-depending) interval \( J \subset \mathbb{R} \),

\[
|\mu_N(J) - \mu_{f_c}(J)| \leq 4\eta_0(\|\rho_{f_c}(x)\|_\infty + \alpha_2)
\]

where \( \alpha_2 > 0 \) is a constant depending only on \( C_a \).

**Proof** First, suppose \( J \subset (-2.99 - \lambda, 2.99 + \lambda) \). Define \( h: \mathbb{R} \to [0, 1] \) to be a smooth function such that

\[
h(x) = \begin{cases} 
1 & \text{if } x \in J \\
0 & \text{if } \text{dist}(x, J) \geq \eta_0
\end{cases}
\]

and satisfies (77). From Lemma 26, if \( N \) is large enough then

\[
\mu_N(J) \leq \int h(x) d\mu_N(x) \leq \int h(x) d\mu_{f_c}(x) + \alpha_1(\eta_0^2 + \eta_1^2 \eta_0)
\]

\[
\leq \mu_{f_c}(J) + 2\eta_0(\rho_{f_c}(x)\|_\infty + \alpha_1(\eta_0^2 + \eta_1^2 \eta_0)
\]

(79)

where \( \alpha_1 > 0 \) is defined in Lemma 26. On the other hand, let

\[
J' = \{ x \in J | \text{the distance between } x \text{ and the edges of } J \text{ is no less than } \eta_0 \}
\]

and define \( \tilde{h} \) in the same way as \( h \), except that \( J \) in (78) is replaced by \( J' \). So by Lemma 26, if \( N \) is large enough then

\[
\mu_N(J) \geq \int \tilde{h}(x) d\mu_N(x) \geq \int \tilde{h}(x) d\mu_{f_c}(x) - \alpha_1(\eta_0^2 + \eta_1^2 \eta_0)
\]

\[
\geq \mu_{f_c}(J) - 2\eta_0(\rho_{f_c}(x)\|_\infty - \alpha_1(\eta_0^2 + \eta_1^2 \eta_0)
\]

(80)

We remark that if \( |J| < 2\eta_0 \), then \( J' = \emptyset \), but (80) is trivial in this case. By (80) and (79),

\[
|\mu_N(J) - \mu_{f_c}(J)| \leq \eta_0(2(\|\rho_{f_c}(x)\|_\infty + \alpha_1(\eta_0 + \eta_1^2))
\]

(81)
so we complete the proof in the case that $J \in (-2.99 - \lambda, 2.99 + \lambda)$.

Now suppose $J$ is not necessarily contained in $(-2.99 - \lambda, 2.99 + \lambda)$. By Lemma 2,

$$\mu_{f_c}([-2 - \lambda, 2 + \lambda]) = 1.$$ 

So by (81) we have

$$\mu_N((-2.99 - \lambda, 2.99 + \lambda)) = 1 - \mu_N((-2.99 - \lambda, 2.99 + \lambda)) \leq 1 - \left(\mu_{f_c}((-2.99 - \lambda, 2.99 + \lambda)) - \eta_0(2\|\rho_{f_c}(x)\|_\infty + \alpha_1(\eta_0 + \eta_1^2))\right) = \eta_0(2\|\rho_{f_c}(x)\|_\infty + \alpha_1(\eta_0 + \eta_1^2)).$$

Let $J_1 = J \cap (-2.99 - \lambda, 2.99 + \lambda)$ and $J_2 = J \setminus J_1$. So by (81) and the above inequality,

$$|\mu_N(J) - \mu_{f_c}(J)| \leq |\mu_N(J_1) - \mu_{f_c}(J_1)| + |\mu_N(J_2) - \mu_{f_c}(J_2)| \leq \eta_0(2\|\rho_{f_c}(x)\|_\infty + \alpha_1(\eta_0 + \eta_1^2)) + \mu_N(J_2) \leq 2\eta_0(2\|\rho_{f_c}(x)\|_\infty + \alpha_1(\eta_0 + \eta_1^2)).$$

\[\square\]

**Proof of Theorem 5** By Proposition 7 and Lemma 4, there exist constants $N_0 > 0$ and $\nu_0 > 0$ such that if $N > N_0$ then

$$P(A_N(\epsilon)) \geq 1 - 2\nu_0(\log N)^{1+2b}N^{-\epsilon}.$$ 

Suppose $A_N(\epsilon)$ holds. Now it suffices to prove (16) and (17).

Suppose $L_0$ is the unique point in $[L_-, L_+]$ such that $\mu_{f_c}([L_0, L_+]) = 2/3$. If $N$ is large enough, then by Lemma 27,

$$\mu_N([L_0, L_+]) \geq \frac{1}{2}$$

thus

$$\lambda_i \geq L_0, \quad \forall i \in \left[1, \frac{N}{2}\right].$$ 

(82)

Define $g(x)$ by

$$g(x) = \mu_{f_c}([x, +\infty)).$$

According to Lemma 27, if $N$ is large enough, then

$$|g(\lambda_i) - g(\gamma_i)| \leq |g(\lambda_i) - \frac{i}{N}| + |\frac{i}{N} - g(\gamma_i)| \leq |\mu_{f_c}([\lambda_i, +\infty)) - \mu_N([\lambda_i, +\infty))]| + \frac{1}{2N} \leq 5\eta_0(\|\rho_{f_c}(x)\|_\infty + \alpha_2), \quad \forall i \in [1, N]$$

(83)

where $\alpha_2 > 0$ is defined in Lemma 27.

By (11) there is a constant $C > 1$ such that

$$\frac{(L_+ - x)^b}{C} \leq \rho_{f_c}(x) \leq C(L_+ - x)^b, \quad \forall x \in [L_0, L_+]$$

(84)

and therefore

$$\int_{\gamma_i}^{L_+} \rho_{f_c}(x)dx \leq C \int_{\gamma_i}^{L_+} (L_+ - x)^b dx = \frac{C}{b + 1}|L_+ - \gamma_i|^{b+1}, \quad \forall 1 \leq i \leq \frac{N}{2}.$$
Then we have
\[ |γ_i - L_+| \geq \left( \frac{i^b + 1}{2N^b C} \right)^{\frac{1}{1-\zeta}} \geq N^{-\xi}, \quad \forall i \in \left[ \frac{2C}{1+b} N^{1-\xi(b+1)}, \frac{N}{2} \right]. \] (85)

We control \(|γ_i - λ_i|\) in two cases.

**Case 1.** Suppose \(i \in \left[ \frac{2C}{1+b} N^{1-\xi(b+1)}, \frac{N}{2} \right]\) and \(λ_i \leq γ_i\). By (82), (83), (84) and (85), when \(N\) is large enough, there exists \(s \in (λ_i, γ_i)\) such that
\[
|γ_i - λ_i| = \frac{|g(γ_i) - g(λ_i)|}{|g'(s)|} \leq \frac{5η_0(\|ρ_f(x)\|_\infty + α_2)}{ρ_f(x)} \leq \frac{5η_0(\|ρ_f(x)\|_\infty + α_2)}{(L_+ - s)^b/C} \leq U_1 N^{-\frac{1}{4} + \epsilon + ξ b}.
\] (86)
where \(U_1 > 0\) is a constant.

**Case 2.** Suppose \(i \in \left[ \frac{2C}{1+b} N^{1-\xi(b+1)}, \frac{N}{2} \right]\) and \(λ_i > γ_i\). By (83), (84), (85) and the definition of \(ξ\), if \(N\) is large enough then
\[
g(λ_i) \geq g(γ_i) - 5η_0(\|ρ_f(x)\|_\infty + α_2)
\geq \int_{γ_i}^{L_+} C^{-1}(L_+ - x)^b dx - 5η_0(\|ρ_f(x)\|_\infty + α_2)
= \frac{C^{-1}(L_+ - γ_i)^{b+1}}{b + 1} - 5η_0(\|ρ_f(x)\|_\infty + α_2) \geq \frac{C^{-1}}{2(b + 1)} N^{-ξ(b+1)}
\] (87)
which implies \(λ_i < L_+\) (otherwise \(g(λ_i) = 0\)) and thus
\[
g(λ_i) \leq \int_{λ_i}^{L_+} C(L_+ - x)^b dx \leq \frac{C}{1+b}(L_+ - λ_i)^{b+1}.
\] (88)

By (87) and (88), if \(N\) is large enough then
\[ L_+ - λ_i \geq (2C^2)^{\frac{1}{1-\xi}} N^{-ξ}, \]
so
\[
|γ_i - λ_i| = \frac{|g(γ_i) - g(λ_i)|}{|g'(t)|} \leq \frac{5η_0(\|ρ_f(x)\|_\infty + α_2)}{ρ_f(x)} \leq \frac{5η_0(\|ρ_f(x)\|_\infty + α_2)}{(L_+ - λ_i)^b/C} \leq U_2 N^{-\frac{1}{4} + \epsilon + ξ b}.
\] (89)

Here \(t \in (γ_i, λ_i)\) and \(U_2 > 0\) is a constant.

(86), (89) and the fact that \(ξ\) can be arbitrarily small complete the proof of (16), (17) can be proved in the same way.

\[ \Box \]

**7 SSK Model in Low Temperature: Proof of Theorem 1**

In this section we follow the idea introduced in [1] to prove Theorem 1. Because of the results in Lemma 30, we know that if a particle is moving along the curve of steepest-descent defined in Definition 17, then its \(y\)-coordinate is monotone, therefore we do not need a lemma like Lemma 6.4 in [1].

Throughout this section we suppose the conditions of Theorem 1 hold.
Definition 15 Suppose \( \epsilon_0 > 0 \) is a constant. Let \( s_0 = s_0(\epsilon_0) > 0 \) be a constant such that
\[
\mathbb{P}(|\lambda_1 - L_+| < s_0 N^{-\frac{1}{b+1}} \text{ and } |\lambda_N - L_-| < s_0 N^{-\frac{1}{b+1}}) > 1 - \epsilon_0.
\] (90)
for large enough \( N \). Set
\[
\Omega^0_N(\epsilon_0) = \left\{ |\lambda_1 - L_+| < s_0 N^{-\frac{1}{b+1}} \text{ and } |\lambda_N - L_-| < s_0 N^{-\frac{1}{b+1}} \right\}.
\]

Remark 8 By Theorem 6 the constant \( s_0 \) exists.

Definition 16 Let \( R(z) \) be an analytic function defined on \( \mathbb{C} \setminus (-\infty, \lambda_1] \) by
\[
R(z) = 2\beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i).
\]
Here we take the analytic branch of the log function such that \( \text{Im} \log(z - \lambda_i) \in (-\pi, \pi) \) for all \( z \in \mathbb{C} \setminus (-\infty, \lambda_1] \). Let \( \gamma \) denote the unique number in \( (\lambda_1, +\infty) \) such that \( R'(\gamma) = 0 \). Equivalently, \( \gamma \) is the unique number on \( (\lambda_1, +\infty) \) satisfying \( 2\beta = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} \).

Lemma 28 Suppose
\begin{itemize}
  \item \( \epsilon \) is a constant in \( \left( \frac{1}{1+b}, \min\left( \frac{1}{4} - \frac{2}{b+1}, \frac{1}{4} - \frac{1}{b+1} - \frac{a}{(b+1)^2} \right) \right) \);
  \item \( \tau \) is a constant in \( \left( \frac{1}{(b+1)^2}, \min\left( \frac{1-a}{a+b}, \frac{1-b}{b+1}, \frac{1}{b+1} \right) \right) \);
  \item \( \tau_1 < \tau_0 \) are two constants both in \( (1 - \tau(b+1), 1 - \frac{1}{b+1}) \).
\end{itemize}
Suppose \( E_N(\epsilon) \cap \Omega^0_N(\epsilon_0) \) holds. There exists a constant \( N_0 > 0 \) such that if \( N > N_0 \), then
\[
\lambda_1 + \frac{1}{3\beta N} < \gamma < \lambda_1 + N^{-1+\tau_0}.
\]

Remark 9 According to the conditions \( b > 11 \) and \( 1 < a < \frac{b^2 - 6b - 7}{4} \), it is easy to check that the constants \( \epsilon, \tau, \tau_0 \) and \( \tau_1 \) exist. The event \( E_N(\epsilon) \) is defined in Theorem 5.

Proof Notice that \( R'(x) = 2\beta - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x - \lambda_i} \) is increasing on \( (\lambda_1, +\infty) \). Since
\[
R'(\lambda_1 + \frac{1}{3\beta N}) < 2\beta - \frac{1}{N} \left( \lambda_1 + \frac{1}{3\beta N} \right) - \lambda_1 < 0 = R'(\gamma),
\]
we have that \( \lambda_1 + \frac{1}{3\beta N} < \gamma \).

Suppose that \( E_N(\epsilon) \cap \Omega^0_N(\epsilon_0) \) holds. Since \( \frac{1-a}{a+b} - \frac{b+2}{(1+b)^2} < \frac{1-a}{a+b} \), the \( \tau \) satisfies the conditions for \( \zeta \) and \( \zeta' \) in Theorem 5. According to Theorem 5, when \( N \) is large enough, we have
\[
\begin{cases}
|\lambda_i - \gamma_i| \leq N^{-\frac{1}{2}+\epsilon + \tau b} & \text{if } i \in [\kappa' N^{1-\tau(b+1)}, \frac{N}{2}] \\
|\lambda_i - \gamma_i| \leq N^{-\frac{1}{2}+\epsilon + \tau a} & \text{if } i \in [\frac{N}{2}, N - \kappa' N^{1-\tau(a+1)}]
\end{cases}
\] (91)
where \( \kappa' > 0 \) is defined in Theorem 5.
To prove \( \gamma < \lambda_1 + N^{-1+\tau_0} \) we need
\[
\left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_1 + N^{-1+\tau_0} - \lambda_i} - \int \frac{d\mu_{fc}(t)}{L_+ - t} \right| \leq I + II + III
\]
where
\[
I = \left| \frac{1}{N} \sum_{i=N^\tau_1}^{N} \frac{1}{\lambda_1 + N^{-1+\tau_0} - \lambda_i} - \frac{1}{N} \sum_{i=N^\tau_1}^{N-k'N^{-1-\tau(a+1)}} \frac{1}{L_+ - \gamma_i} \right|
\]
\[
II = \left| \frac{1}{N} \sum_{i=N^\tau_1}^{N-k'N^{-1-\tau(a+1)}} \frac{1}{L_+ - \gamma_i} - \int \frac{d\mu_{fc}(t)}{L_+ - t} \right|
\]
\[
III = \left| \frac{1}{N} \sum_{i<N^\tau_1} \frac{1}{\lambda_1 + N^{-1+\tau_0} - \lambda_i} + \frac{1}{N} \sum_{i>N-k'N^{-1-\tau(a+1)}} \frac{1}{\lambda_1 + N^{-1+\tau_0} - \lambda_i} \right|
\]

**Estimation of I.** If \( N \) is large enough, then for any \( i \in [N^\tau_1, N/2] \subset [k'N^{-1-\tau(b+1)}, N/2] \),
\[
\left| \frac{1}{\lambda_1 + N^{-1+\tau_0} - \lambda_i} - \frac{1}{L_+ - \gamma_i} \right| |L_+ - \gamma_i| \\
\leq \frac{|\lambda_1 - L_+| + |\gamma_i - \lambda_i| + N^{-1+\tau_0} - |L_+ - \gamma_i| - |L_+ - \lambda_1| - |\lambda_i - \gamma_i| - N^{-1+\tau_0}}{s_0 N^{-\frac{\tau_1}{1+\tau_b}} + N^{\frac{1}{1+\tau_b} - 1} + \epsilon + \tau b + N^{1-1+\tau_0}} \\
\leq \frac{s_0 N^{-\frac{\tau_1}{1+\tau_b}} - s_0 N^{-\frac{1}{1+\tau_b} - \frac{1}{2} + \epsilon + \tau b} - N^{-1+\tau_0}}{C_\ast^{-1} \cdot i^{1+\tau_b} - s_0 - N^{\frac{1}{1+\tau_b} - \frac{1}{2} + \epsilon + \tau b} - N^{1-1+\tau_0}} \\
\leq \frac{s_0 - N^{\frac{1}{1+\tau_b} - \frac{1}{2} + \epsilon + \tau b} + N^{\frac{1}{1+\tau_b} - 1 + \tau_0}}{C_\ast^{-1} \cdot N^{\frac{1}{1+\tau_b}} - s_0 - N^{\frac{1}{1+\tau_b} - \frac{1}{2} + \epsilon + \tau b} - N^{1-1+\tau_0}} \leq 2s_0 C_\ast N^{-\frac{\tau_1}{1+\tau_b}}. \quad (92)
\]

Here we used Lemma 24, (91) and the definition of \( \Omega^0_N(\epsilon_0) \) in the second inequality and we used the conditions on \( \tau \) and \( \tau_0 \) in the last inequality. (In particular, the condition \( \tau < \frac{1}{b} \cdot \frac{b+2}{(b+1)^2} \) implies \( \frac{1}{1+\tau_b} - \frac{1}{2} + \epsilon + \tau b < 0 \).) The constant \( C_\ast > 0 \) is defined in Lemma 24.

By a similar argument we can prove (92) for \( i \in [N/2, N-k'N^{-1-\tau(a+1)}] \). So we have that if \( N \) is large enough, then
\[
I \leq 2s_0 C_\ast N^{-\frac{\tau_1}{1+\tau_b}} \cdot \sum_{i=N^\tau_1}^{N-k'N^{-1-\tau(a+1)}} \frac{1}{L_+ - \gamma_i} \\
\leq 2s_0 C_\ast N^{-\frac{\tau_1}{1+\tau_b}} \cdot \sum_{i=N^\tau_1}^{N-k'N^{-1-\tau(a+1)}} \int_{\gamma_i-1}^{\gamma_i-\tau_2} \frac{1}{L_+ - t} d\mu_{fc}(t) \leq 4\beta_c \cdot s_0 C_\ast N^{-\frac{\tau_1}{1+\tau_b}}
\]

where we used (73) in the second inequality and used the definition of \( \beta_c \) in the last inequality.

**Estimation of II.** If \( N \) is large enough, then by (73), (11) and Lemma 24,
\[
II \leq \int_{L_-}^{\hat{\gamma}_{N-k'N^{-1-\tau(a+1)}-2}} \frac{d\mu_{fc}(t)}{L_+ - t} + \int_{\hat{\gamma}_{N^\tau_1}^1}^{L_+} \frac{d\mu_{fc}(t)}{L_+ - t}
\]
therefore
\[ \gamma < \lambda \]

satisfying:
\[ \text{of} \]

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Suppose the assumptions in Lemma 1
\[ \max \left\{ 1 - \tau (1 + b), 1 - \tau (1 + a) \right\} < \tau_2 < 1 \]
\[ \max \left( 2 + (1 + b)(-t) + \epsilon + \tau b, 1 - \tau (b + 1), 1 - \frac{a+1}{a}(1 \epsilon - \frac{1}{b+1}) \right) < \tau_3 < \frac{b}{b+1} \]
\[ \frac{1 - \tau_3}{a+1} \tau' < \min \left( \frac{\tau_3 - \epsilon}{a+1}, \frac{1 - \epsilon}{a+1} \right) \]

Then we have the following conclusions.
\[ \text{If } N \text{ is large enough, then} \]
\[ \left| \frac{1}{N} \sum_{i=1}^{N} \log(\gamma - \lambda_i) - \int \log(L_+ - t) d\mu_{f_c}(t) - 2\beta_c(\gamma - L_+) \right| \leq W_1 \Phi_N \]
where $W_1 > 0$ is a constant and
\[
\Phi_N = N^{-2+2\tau_0-2\tau_1-1\frac{1}{\pi+\tau}} + N^{-2\tau_1} + N^{-\frac{1}{3}+\varepsilon+\tau b + \frac{1}{3}\tau_1} + N^{-\frac{1}{3}+\varepsilon+\tau a} + N^{-\frac{b(\tau_1-1)}{b+1}} \left( N^{-\frac{1}{3}+\gamma} + N^{-1+\tau_0} + N^{\tau_3-1}\log N \right).
\]

- If $N$ is large enough, then
\[
N^{l-l_{\tau_0}-1} \leq \frac{R(l)(\gamma)(-1)^l}{(l-1)!} \leq W_2N^{-1+\tau_2+l}, \quad l = 2, 3, \ldots
\]
where $W_2 > 0$ is a constant.

**Remark 10**

- From the definition of $\epsilon$ and $\tau$ (see Lemma 28), we see that the $\tau_2$ and $\tau_3$ satisfying the conditions exist. Since $\tau < \frac{1 - \frac{\epsilon - 1}{b+1}}{b+2}$, we have from the definition of $\tau_3$:
\[
\tau_3 > 1 - \tau(b + 1) > \frac{3}{4} + \epsilon \tag{94}
\]
(94) and the definition of $\tau_3$ yields $\frac{1 - \tau_3}{a+1} < \min\left( \frac{1 - \frac{\epsilon - 1}{a+1}}{b+2}, \frac{1 - \epsilon}{b+1} \right)$, thus $\tau'$ is well defined.

- By (94) and the definitions of $\tau_0$ and $\tau_3$, we have
\[
\tau_3 > 1 - \tau(1 + b) > \frac{3}{4} > (\tau_0 - 1)(b + 1) + \frac{3}{2} \tag{95}
\]
so
\[
\frac{1}{b+1} - 2 + 2\tau_0 - 2\frac{\tau_3 - 1}{1+b} < 0. \tag{96}
\]
By $\tau_0 < \frac{b}{b+1}$ and the definition of $\tau_3$ we have $\tau_3 < \frac{b}{b+1} < 2 - \frac{b+1}{b}\tau_0$, so
\[
\frac{1}{b+1} + \frac{b(\tau_3 - 1)}{b+1} - 1 + \tau_0 < 0 \tag{97}
\]
By (96), (97) and the definition of $\tau_3$ we see that
\[
\lim_{N \to \infty} N^{\frac{1}{\pi+\tau}} \cdot \Phi_N = 0. \tag{98}
\]

**Proof**

By Lemma 24, Theorem 5, Lemma 28 and the definition of $\Omega^0_N(\epsilon_0)$, if $N$ is large enough and $i \in [1 + N^{\tau_3}, N - N^{\tau_3}]$, then
\[
\left| \frac{\gamma - L_+}{L_+ - \gamma_i} \right| \leq \left| \frac{\gamma - \lambda_1}{L_+ - \gamma_i} \right| + \left| \frac{\lambda_1 - L_+}{L_+ - \gamma_i} \right| \leq \frac{N^{1+\tau_0} + s_0 N^{-\frac{1}{1+\tau}}}{C_* N^{\frac{\tau_3-1}{\pi+\tau}}} N \to \infty 0 \quad \text{(by (123))} \tag{99}
\]
\[
\left| \frac{\gamma_i - \lambda_i}{L_+ - \gamma_i} \right| \leq \frac{N^{-\frac{1}{3}+\varepsilon+\tau b}}{C_* N^{\frac{\tau_3-1}{\pi+\tau}}} \sum_{i \leq N/2} + \frac{N^{-\frac{1}{3}+\varepsilon+\tau a}}{C_* N^{\frac{\tau_3-1}{\pi+\tau}}} \sum_{i \geq N/2} N \to \infty 0 \quad \text{(by definitions of $\tau_3$ and $\tau'$)} \tag{100}
\]
and thus
\[
\log(\gamma - \lambda_i) - \log(L_+ - \gamma_i) = \log(1 + \frac{\gamma - L_+}{L_+ - \gamma_i} + \frac{\gamma_i - \lambda_i}{L_+ - \gamma_i}) = \frac{\gamma - L_+}{L_+ - \gamma_i} + B_1 + B_2 \tag{101}
\]
where $|B_1| \leq 2 \left| \frac{y_i - \lambda_i}{L_i - y_i} \right|$ and $|B_2| \leq \left| \frac{\gamma - L_+}{L_i - y_i} \right|^2$.

By (73), Lemmas 1 and 24, if $N$ is large enough, then

$$
\left| \frac{1}{N} \sum_{i=N^{-T_3}+1}^{N-N^{-T_3}} \frac{1}{L_i - y_i} - 2\beta_c \right| \leq \left| \int_{\hat{\gamma}_{N^{-T_3}+2}}^{\hat{\gamma}_{N^{-T_3}+1}} \frac{d\mu_f(t)}{L_+ - t} - 2\beta_c \right|
$$

$$
= \int_{\hat{\gamma}_{N^{-T_3}+1}}^{\hat{\gamma}_{N^{-T_3}+2}} \frac{d\mu_f(t)}{L_+ - t} + \int_{\hat{\gamma}_{N^{-T_3}+2}}^{\hat{\gamma}_{N^{-T_3}+2}} \frac{d\mu_f(t)}{L_+ - t}
$$

$$
\leq W_3 \int_{\hat{\gamma}_{N^{-T_3}+1}}^{\hat{\gamma}_{N^{-T_3}+2}} (t - L_-)^a dt + C_0 \int_{\hat{\gamma}_{N^{-T_3}+2}}^{\hat{\gamma}_{N^{-T_3}+2}} (L_+ - t)^b dt
$$

$$
\leq W_3 |\gamma_{N^{-T_3}+1} - L_-|^a + C_0 \frac{b}{b}|L_+ - \gamma_{N^{-T_3}+2}|b \leq W_3 \cdot \frac{N^{T_3}}{N} + C_0 \frac{b}{b}|L_+ - \gamma_{N^{-T_3}+2}|b
$$

$$
\leq W_3 \frac{N^{T_3}}{N} + C_0 \frac{b}{b} \cdot \gamma_{N^{-T_3}} \left( \frac{N^{T_3} + 3}{N} \right)^{\frac{b}{b+1}} \leq W_3 N^{\frac{b(T_3-1)}{b+1}}
$$

(102)

where $W_3 > 0$ is a constant, $C_0 > 0$ is defined in Lemma 1 and $C_\ast$ is defined in Lemma 24.

According to Lemmas 1, 24, the definitions of $\gamma_i$ and $\hat{\gamma}_i$ and the fact that

$$
\int_{\hat{\gamma}_{i-1}}^{\hat{\gamma}_i} \log(L_+ - t) d\mu_f(t) \leq \frac{1}{N} \log(L_+ - \gamma_i) \leq \int_{\hat{\gamma}_{i+1}}^{\hat{\gamma}_i} \log(L_+ - t) d\mu_f(t) \quad (2 \leq i \leq N - 1)
$$

we know that if $N$ is large enough, then

$$
\left| \frac{1}{N} \sum_{i=N^{-T_3}+1}^{N-N^{-T_3}} \log(L_+ - \gamma_i) - \int \log(L_+ - t) d\mu_f(t) \right|
$$

$$
\leq \left| \int_{\hat{\gamma}_{N^{-T_3}-2}}^{\hat{\gamma}_{N^{-T_3}-1}} \log(L_+ - t) d\mu_f(t) + \int_{\hat{\gamma}_{N^{-T_3}+2}}^{\hat{\gamma}_{N^{-T_3}+1}} \log(L_+ - t) d\mu_f(t) \right|
$$

$$
\leq W_4 \int_{\hat{\gamma}_{N^{-T_3}-2}}^{\hat{\gamma}_{N^{-T_3}-1}} (t - L_-)^a dt + C_0 \int_{\hat{\gamma}_{N^{-T_3}+2}}^{\hat{\gamma}_{N^{-T_3}+1}} (L_+ - t)^b d\log(L_+ - t) dt
$$

$$
\leq W_4 N^{T_3-1} + C_0 \frac{|L_+ - \gamma_{N^{-T_3}+3}|^{b+1}}{b+1} \left| \log \left( \frac{L_+ - \gamma_{N^{-T_3}+3}}{b+1} \right) - \frac{1}{b+1} \right|
$$

$$
\leq W_4 N^{T_3-1} \log N
$$

(103)

where $C_0 > 0$ is defined in Lemma 1 and $W_4 > 0$ is a constant.

By Lemma 28 and the definition of $\Omega_N^{0}(\epsilon_0)$, if $N$ is large enough, then

$$
\left| \frac{1}{N} \sum_{i \leq 1+N^{-T_3}} \log(\gamma - \lambda_i) + \frac{1}{N} \sum_{i \geq N-N^{-T_3}} \log(\gamma - \lambda_i) \right| \leq 4N^{T_3-1} \log N.
$$

(104)

According to (99), (100), (101), (102), (103), (104), if $N$ is large enough, then

$$
\left| \frac{1}{N} \sum_{i=1}^{N} \log(\gamma - \lambda_i) - \int \log(L_+ - t) d\mu_f(t) - 2\beta_c (\gamma - L_+) \right|
$$

$$
\leq \left| \frac{1}{N} \sum_{i=1+1+N^{-T_3}}^{N-N^{-T_3}} \log(\gamma - \lambda_i) - \frac{1}{N} \sum_{i=1+1+N^{-T_3}}^{N-N^{-T_3}} \log(L_+ - \gamma_i) - \frac{\gamma - L_+}{N} \sum_{i=1+1+N^{-T_3}}^{N-N^{-T_3}} \frac{1}{L_+ - \gamma_i} \right|
$$

\(\Box\)
+ \left| \frac{\gamma - L_+}{N} - \sum_{i=1+NT^3}^{N-NT^3} \frac{1}{L_+ - \gamma_i} - 2\beta_\epsilon (\gamma - L_+) \right| \leq \frac{1}{N} \left| \sum_{i=1+NT^3}^{N-NT^3} \log(L_+ - \gamma_i) \right|

- \int \log(L_+ - t) d\mu_{f_\epsilon}(t) \biggr| + \frac{1}{N} \sum_{i \geq 1+NT^3} \log(\gamma - \lambda_i) + \frac{1}{N} \sum_{i \geq N-NT^3} \log(\gamma - \lambda_i)

\leq \left( \frac{N^{-1+\tau_0} + s_0 N^{-\frac{1}{\beta + \gamma}}}{C_s^{-1} N^{-\frac{\beta - 1}{\beta + \gamma}}} \right)^2 + 2 \frac{N^{-\frac{1}{4} + \epsilon + \tau b}}{C_s^{-1} N^{-\frac{\beta - 1}{\beta + \gamma}}} + 2 \frac{N^{-\frac{1}{4} + \epsilon + \tau a}}{|L_+ - \gamma N/2|}

+ W_3 N^{\frac{b(\tau - 1)}{\beta + \gamma}} |\gamma - L_+| + W_4 N^T-1 \log N + 4N^T-1 \log N.

The above inequality together with

|\gamma - L_+| \leq |\gamma - \lambda_1| + |\lambda_1 - L_+| < N^{-1+\tau_0} + s_0 N^{-\frac{1}{\beta + \gamma}}

and the fact that |L_+ - \gamma N/2| is bounded below by a constant completes the proof of the first conclusion of the lemma.

For the second conclusion, notice that

R^{(l)}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{(-1)^l (l - 1)!}{(z - \lambda_i)^l} \quad l = 2, 3, \ldots,

so for large enough N we have by Lemma 28 that

\frac{R^{(l)}(\gamma)(-1)^l}{(l - 1)!} \geq \frac{1}{N} \frac{1}{(\gamma - \lambda_1)^l} \geq N^{l-1+\tau_0-1}, \quad l = 2, 3, \ldots

(105)

From the conditions on \tau_2 and \tau we see

1 > \tau_2 > 1 - \tau(1 + b) > 1 + (b + 1)(-\frac{1}{4} + \epsilon + \tau b).

(106)

By Lemma 28, if \textit{N} is large enough, then

\frac{1}{N} \left( \sum_{i=1}^{N^{T_2}} + \sum_{i=N^{T_2}}^{N} \right) \frac{1}{(\gamma - \lambda_i)^l} \leq 3N^{-1+\tau_2} \cdot (3\beta N)^l = 3(3\beta)^l N^{-1+\tau_2+l}, \quad l = 2, 3, \ldots

(107)

By (106) and Lemma 24 if \textit{N} is large enough and \textit{i} \in [N^{T_2}, N - N^{T_2}], then |L_+ - \gamma_i| \geq C_s^{-1} N^{-\frac{\beta - 1}{\beta + \gamma}}

|L_+ - \gamma| \leq |L_+ - \lambda_1| + |\gamma - \lambda_1|

\leq s_0 N^{-\frac{1}{\beta + \gamma}} + N^{-1+\tau_0} \leq (1 + s_0) N^{-\frac{1}{\beta + \gamma}} \quad \text{(since } \tau_0 < \frac{b}{b + 1})

and |\lambda_i - \gamma_i| \leq \begin{cases} N^{-\frac{1}{4} + \epsilon + \tau b} & \text{if } N^{T_2} \leq i \leq N/2 \\ N^{-\frac{1}{4} + \epsilon + \tau a} & \text{if } N/2 \leq i \leq N - N^{T_2} \end{cases}

(108)

The above estimations and the fact that |L_+ - \gamma_j| is of order 1 for \textit{j} \geq N/2 imply that if \textit{N} is large enough then
\[ |\gamma - \lambda_i| \geq |L_+ - \gamma_i| - |\gamma_i - \lambda_i| - |L_+ - \gamma| \geq \frac{|L_+ - \gamma_i|}{2} \]

\[ \geq \frac{1}{2C_*} \left( \frac{i}{N} \right)^\frac{i}{1+\beta}, \text{ if } N \tau_2 \leq i \leq N - N \tau_2 \]
and thus

\[ \frac{1}{N} \sum_{N \tau_2 < i < N-N \tau_2} \frac{1}{(\gamma - \lambda_i)^l} \leq \frac{1}{N} \sum_{N \tau_2 < i < N-N \tau_2} \frac{(2C_*)^l N \tau_2^l}{i^{l/(b+1)}} \]

\[ \leq (2C_*)^l N \tau_2^{l-1} \int_{\frac{1}{N}N \tau_2}^N x^{-\frac{l}{b+1}} dx \]

\[ \leq \begin{cases} (2C_*)^l \log N & \text{if } b \text{ is an integer and } l = b + 1 \\ \frac{b+1}{b+1-l} (2C_*)^l & \text{if } l < b + 1 \\ \frac{b+1}{l-(b+1)} (4C_*)^l N^{(1-\tau_2)(\frac{l}{b+1}-1)} & \text{if } l > b + 1 \end{cases} \tag{109} \]

where \( C_* \) is defined in Lemma 24. In (109), the coefficient \( \frac{b+1}{b+1-l} \) for the case \( l \neq b + 1 \) is bounded by:

\[ \left| \frac{b + 1}{b + 1 - l} \right| \leq \begin{cases} \frac{b + 1}{\text{dist}(b, Z)} & \text{if } b \in \mathbb{Z} \\ \frac{b + 1}{\text{dist}(b, \mathbb{Z})} & \text{if } b \notin \mathbb{Z} \end{cases} \]

and this bound is independent of \( l \). So by the fact \(-1 + \tau_2 + l > (1 - \tau_2)(\frac{l}{b+1} - 1)\) and (105), (107) and (109) we complete the proof. \( \square \)

**Definition 17**

- Set \( S = \{ x + iy \in \mathbb{C} \setminus (-\infty, \lambda_1] \mid \text{Im} R(x + iy) = 0 \} \).
- Set \( S^+ = \{ x + iy \in S \mid y > 0 \} \), \( S^- = \{ x + iy \in S \mid y < 0 \} \).
- For \( y \) satisfying \( 0 < |y| < \frac{\pi}{2\beta} \), let \( h(y) \) be the unique real number such that \( h(y) + iy \in S \). Set \( h(0) = y \).

**Lemma 30**

- \( h(y) \) is well defined. In other words, for any \( y \) satisfying \( 0 < |y| < \frac{\pi}{2\beta} \), there is a unique real number \( x \) such that \( x + iy \in S \).
- \( S = \{ h(y) + iy \mid \frac{\pi}{2\beta} < y < \frac{\pi}{2\beta} \} \) and \( h(y) \in C^1((\frac{\pi}{2\beta}, \frac{\pi}{2\beta})) \).
- \( h(y) \leq y \) and the identity holds only when \( y = 0 \).
- If \( \frac{1}{\beta} < c_0 < \frac{\pi}{2} \), then \( h(y) \) is strictly decreasing on \([c_0, \frac{\pi}{2\beta}] \).

**Proof** See Appendix A. \( \square \)

**Remark 11** Since \( h(y) \) is \( C^1 \), we can define the integral of continuous functions along \( S \).

**Lemma 31** Suppose the conditions in Lemma 29 are satisfied. Set

\[ K = -ie^{-\frac{N}{2} R(y)} \int_{y-i\infty}^{y+i\infty} e^{\frac{N}{2} R(z)} dz. \tag{110} \]

There exist constants \( N_0 > 0 \) and \( W_0 > 0 \) such that if \( N > N_0 \) then

\[ N^{-10} \leq K \leq W_0. \]
\textbf{Proof} From Lemma 6 we know $K > 0$. By the same argument as (6.31) of [1],
\[
    K = -i \int_{\gamma - i\infty}^{\gamma + i\infty} e^{N \frac{1}{2} \log (1 + \frac{t^2}{(1 + 2\lambda_0 + L_+ - L_-)^2})} dt = \int_{-\infty}^{\infty} \exp \left( -\frac{N}{4} \log \left( 1 + \frac{t^2}{(1 + 2\lambda_0 + L_+ - L_-)^2} \right) \right) dt
\]
where we take the absolute value of the integrand to obtain the last inequality. Since
\[
    0 < \gamma - \lambda_i \leq \gamma - \lambda_N \leq |\gamma - \lambda_1| + |L_+ - L_-| + |L_+ - \lambda_N| \leq 1 + 2\lambda_0 + L_+ - L_-,
\]
we have for $N > 4$:
\[
    K \leq \int_{-\infty}^{\infty} \exp \left( -\frac{N}{4} \log \left( 1 + \frac{t^2}{(1 + 2\lambda_0 + L_+ - L_-)^2} \right) \right) dt \leq \int_{\mathbb{R}} \left( 1 + \frac{t^2}{(1 + 2\lambda_0 + L_+ - L_-)^2} \right)^{-1} dt < \infty
\]
and therefore the right hand side of the conclusion is proved.

To prove the left hand side of the conclusion we need Lemma 30. We first claim that
\[
    \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{N}{2} R(z)} dz = \int_{S} e^{\frac{N}{2} R(z)} dz \tag{111}
\]
where the direction on $S$ is from $-\infty - \frac{\pi}{2\beta} i$ to $-\infty + \frac{\pi}{2\beta} i$. In fact, suppose $r > 0$ such that $|z - \lambda_i| > r/2$ for all $|z| = r$, then for $C_r := \{ z \in \mathbb{C} | |z| = r, \text{Re} z \leq \gamma \}$ we have
\[
    \text{Re} R(z) \leq 2\beta \gamma - \log(r/2), \quad \forall z \in C_r
\]
and thus
\[
    \left| \int_{C_r} e^{\frac{N}{2} R(z)} dz \right| \leq 2\pi (2\beta \gamma)^{N - \frac{N}{2}} \to 0 \quad \text{as } r \to \infty.
\]
Moreover, by the last conclusion of Lemma 30 we have
\[
    \int_{0}^{\pi/(2\beta)} \left| e^{\frac{N}{2} R(h(y) + iy)} \sqrt{1 + (h'(y))^2} \right| dy < \infty. \text{ So (111) is true.}
\]
Notice that $R(z) = R(\bar{z})$ for $z \in S$, so by (111)
\[
    K = -i \int_{S} e^{\frac{N}{2} (R(z) - R(\bar{z}))} dz = -i \left( \int_{S^+} e^{\frac{N}{2} (R(z) - R(\bar{z}))} (dx + idy) + \int_{S^-} e^{\frac{N}{2} (R(z) - R(\bar{z}))} (dx + idy) \right)
\]
\[
    = 2 \int_{S^+} \exp \left( \frac{N}{2} (R(z) - R(\bar{z})) \right) dy \tag{112}
\]
Now we define
\[
    Q_N = \{ z \in \mathbb{C} | |z - \gamma| < N^{-\theta} \}.
\]
By Lemma 28, \( R(z) \) is analytic on a neighborhood of \( \tilde{Q}_N \), so by \( R'(\gamma) = 0 \) we have for large enough \( N \):
\[
R(z) - R(\gamma) = \sum_{j=2}^{\infty} \frac{R^{(j)}(\gamma)}{j!} (z - \gamma)^j, \quad \text{if} \ z \in Q_N.
\]
(113)

The next lemma shows that \( S \) does not leave \( \gamma \) too fast when the \( y \) coordinate is small enough.

**Lemma 32** When \( N \) is large enough, we have
\[
\{ z \in S^+ | Imz \in (0, N^{-10}) \} \subset Q_N.
\]
(114)

**Proof** See Appendix B. \( \square \)

By Lemma 29, (113) and the definition of \( Q_N \), if \( N \) is large enough, then:
\[
|R(z) - R(\gamma)| \leq \sum_{j=2}^{\infty} \frac{|R^{(j)}(\gamma)|}{j!} |z - \gamma|^j \leq 2W_2^2N^{-16}, \quad \forall z \in Q_N
\]
(115)

which together with (112) and (114) imply:
\[
K = 2 \int_{S^+} \exp \left( \frac{N}{2} (R(z) - R(\gamma)) \right) dy \geq 2 \int_0^{N^{-10}} \exp \left( \frac{N}{2} \cdot (-2W_2^2N^{-16}) \right) dy > N^{-10}.
\]

So the proof is complete. \( \square \)

**Proof of Theorem 1** Recall the definition of \( K \) in (110). According to Lemma 6,
\[
\int_{S_{N-1}} e^{\beta (\sigma (W+\lambda V)\sigma)} d\omega_N(\sigma) = \frac{\sqrt{N} \beta}{\sqrt{\pi} (2\beta e)^{N/2}} \cdot e^{N/2 R(\gamma)} \cdot K \cdot (1 + O(N^{-1})).
\]

Now we choose the constants \( s_0, \epsilon, \tau, \tau_0, \tau_1, \tau_2, \tau_3, \tau' \) in the same way as in (90), Lemma 28 and Lemma 29. Suppose \( E_N(\epsilon) \cap \Omega_0^0(\epsilon_0) \) holds. By Lemma 31, if \( N \) is large enough, then
\[
F_N = \frac{1}{N} \log \left( \int_{S_{N-1}} e^{\beta (\sigma (W+\lambda V)\sigma)} d\omega_N(\sigma) \right) = \frac{R(\gamma)}{2} - \frac{1}{2} \log(2\beta e) + e_N
\]
where
\[
|e_N| \leq C \log N / N
\]
(116)

for some constant \( C > 0 \). By Lemmas 28 and 29, if \( N \) is large enough and \( E_N(\epsilon) \cap \Omega_0^0(\epsilon_0) \) holds, then
\[
\left| F_N + \frac{1}{2} \log(2\beta e) - \beta \lambda_1 + \frac{1}{2} \int \log(L_+ - t) d\mu_{fc}(t) + \beta_c(\lambda_1 - L_+) \right|
\]
\[
\leq |e_N| + (\beta + \beta_c) |\gamma - \lambda_1| + W_1 \Phi_N \leq |e_N| + (\beta + \beta_c) N^{-1+\tau_0} + W_1 \Phi_N
\]

and thus
\[
\left| I_N - N \frac{1}{|\beta - \beta_c|} (\lambda_1 - L_+) \right| \leq \frac{N}{|\beta - \beta_c|} \left[ |e_N| + (\beta + \beta_c) N^{-1+\tau_0} + W_1 \Phi_N \right]
\]
(117)
where

\[ I_N = N^{\frac{1}{N+1}} \left[ F_N + \frac{1}{2} \log(2e\beta) + \frac{1}{2} \int \log(L_+ - t) d\mu_{fc}(t) - \beta L_+ \right]. \]

Fix \( s < 0 \). Choose \( \epsilon_0 \in (0, |s|/10) \). There exists \( \delta_s > 0 \) such that if \( |s' - s| \leq \delta_s \) then

\[ |\exp(-\frac{C_{\mu}(-s')^{b+1}}{b+1}) - \exp(-\frac{C_{\mu}(-s)^{b+1}}{b+1})| < \epsilon_0. \]  

Write \( E_N = N^{\frac{1}{N+1}} (\lambda_1 - L_+) - I_N \). Notice that

\[
\mathbb{P}(I_N \leq s) = \mathbb{P}([I_N \leq s] \cap (E_N(\epsilon) \cap \Omega^0_N(\epsilon_0)^c)) + \mathbb{P}([I_N \leq s] \cap (E_N(\epsilon) \cap \Omega^0_N(\epsilon_0)))
\]

\[ = \mathbb{P}([I_N \leq s] \cap (E_N(\epsilon) \cap \Omega^0_N(\epsilon_0)^c)) + \mathbb{P}([N^{\frac{1}{N+1}} (\lambda_1 - L_+) \leq s + E_N] \cap (E_N(\epsilon) \cap \Omega^0_N(\epsilon_0)) \cap (E_N(\epsilon) \cap \Omega^0_N(\epsilon_0)^c))
\]

\[ + \mathbb{P}([I_N \leq s] \cap (E_N(\epsilon) \cap \Omega^0_N(\epsilon_0))^c). \]  

(119)

If \( N \) is large enough, then by (98), (116), (117) and the definition of \( \tau_0 \), we have \( E_N \in (-\delta_s, \delta_s) \) and then

\[
\mathbb{P}(N^{\frac{1}{N+1}} (\lambda_1 - L_+) \leq s + E_N) \in \left( \mathbb{P}(N^{\frac{1}{N+1}} (\lambda_1 - L_+) \leq s - \delta_s), \mathbb{P}(N^{\frac{1}{N+1}} (\lambda_1 - L_+) \leq s + \delta_s) \right)
\]

(120)

By (118), (119), (120), (15) and (90) we have:

\[ |\mathbb{P}(I_N \leq s) - \mathbb{P}(N^{\frac{1}{N+1}} (\lambda_1 - L_+) \leq s)| \leq 5\epsilon_0 \]  

when \( N \) is large enough.

Since \( \epsilon_0 \) can be arbitrarily small, we have by Theorem 6,

\[
\lim_{N \to \infty} \mathbb{P}(I_N \leq s) = \lim_{N \to \infty} \mathbb{P}(N^{\frac{1}{N+1}} (\lambda_1 - L_+) \leq s) = 1 - \mathbb{P}(N^{\frac{1}{N+1}} (L_+ - \lambda_1) \leq -s)
\]

\[ = \exp(-\frac{C_{\mu}(-s)^{b+1}}{b+1}). \]

It is easy to check that the above identity is also true when \( s = 0 \).

\[ \square \]

8 SSK Model in High Temperature: Proof of Theorem 2

In this section we use the method introduced in [1] and Theorem 4 to prove Theorem 2, but we follow a different way to control \(|\gamma - \hat{\gamma}|\). In [1], the tool used to control \(|\gamma - \hat{\gamma}|\) is the rigidity of eigenvalues, but we will use the local law to control \(|\gamma - \hat{\gamma}|\) because the rigidity we have here is not strong enough to provide a proper estimation of \(|\gamma - \hat{\gamma}|\).

Throughout this section we assume that the conditions of Theorem 2 are satisfied.

Definition 18  
• Set \( \hat{R}(z) = 2\beta z - \int \log(z - t) d\mu_{fc}(t) \) analytically defined on \( \mathbb{C} \setminus (-\infty, L_+) \) such that \( \text{Im} \log(z - t) \) for all \( t \in \text{supp}(\mu_{fc}) \).

• Suppose \( \epsilon \in (\frac{1}{b+1}, \frac{1}{12}) \) and

\[ \square \] Springer
\[ \Omega_1(\epsilon) = \left\{ |\lambda_1 - L_+| < \frac{\min(\hat{\gamma} - L_+, 1)}{20} \text{ and } |\lambda_N - L_-| < \frac{\min(\hat{\gamma} - L_+, 1)}{20} \right\} \cap \Omega_V(\epsilon) \cap \tilde{\Omega}(\epsilon). \]

**Remark 12** \( \hat{R}(z) \) is an analogue of the \( R(z) \) defined in Definition 16. Obviously the \( \hat{\gamma} \) defined in Theorem 2 is the unique point on \((L_+, +\infty)\) such that \( \hat{R}'(\hat{\gamma}) = 0 \). According to Theorem 6 and Proposition 7 we have

\[ \mathbb{P}(\Omega_1(\epsilon)) \rightarrow 1 \text{ as } N \rightarrow \infty. \]  

**Lemma 33** There exists a constant \( N_0 > 0 \) such that if \( N > N_0 \) and \( \Omega_1(\epsilon) \) holds, then

\[ |\gamma - \hat{\gamma}| \leq N^{3e - \frac{1}{2}}, \quad |R(\hat{\gamma}) - R(\gamma)| \leq \frac{8}{(\hat{\gamma} - L_+)^2} N^{6e - 1} \]

where \( \gamma \) was defined in Definition 16 and \( \hat{\gamma} \) was defined in Theorem 2.

**Proof** Let \( L \) be the boundary of the rectangle with vertices

\[ L_+ = \frac{\min(\hat{\gamma} - L_+, 1)}{3} \pm \frac{\min(\hat{\gamma} - L_+, 1)}{3} \cdot i \quad \text{and} \quad L_- = \frac{\min(\hat{\gamma} - L_+, 1)}{3} \pm \frac{\min(\hat{\gamma} - L_+, 1)}{3} \cdot i \]

with counterclockwise orientation. By Lemmas 2 and 16, if \( N \) is large enough, then

\[ L \cap \{z : \text{Im} z \geq N^{-\frac{1}{2} - \epsilon}\} \subset D_\epsilon. \]

Notice that if \( x > (\hat{\gamma} + L_+)/2 \), \( N \) is large enough and \( \Omega_1(\epsilon) \) holds then

\[ |R^{(l)}(x)| = \frac{1}{N} \left| \sum_{i=1}^{N} \frac{(l - 1)!}{(x - \lambda_i)^l} \right| \leq \frac{(l - 1)!}{((\hat{\gamma} - L_+)/4)^l} \quad (l = 2, 3, \ldots) \quad (122) \]

\[ \frac{1}{x - t} = \frac{1}{2\pi i} \int_L \frac{1}{(x - \xi)(\xi - t)} \, d\xi \quad \text{for any } t \text{ enclosed by } L \quad (123) \]

\[ |R'(x) - \hat{R}'(x)| \]

\[ = \left| \int \frac{1}{x - t} d\mu_{fc}(t) - \frac{1}{N} \sum \frac{1}{x - \lambda_i} \right| \]

\[ = \frac{1}{2\pi} \left| \int \frac{m_N(\xi) - m_{fc}(\xi)}{x - \xi} \, d\xi \right| \quad \text{(by (123))} \]

\[ \leq \frac{1}{2\pi} \int_{L \cap |\text{Im} \xi| \leq N^{-\frac{1}{2} - \epsilon}} \frac{|m_N(\xi) - m_{fc}(\xi)|}{|x - \xi|} \, d\xi \]

\[ + \frac{1}{2\pi} \int_{L \cap |\text{Im} \xi| > N^{-\frac{1}{2} - \epsilon}} \frac{|m_N(\xi) - m_{fc}(\xi)|}{|x - \xi|} \, d\xi \]

\[ \leq \frac{100}{\pi \min(\hat{\gamma} - L_+, 1)^2} N^{\frac{1}{2} - \epsilon} + \frac{6|L|}{\pi \cdot \min(\hat{\gamma} - L_+, 1)} N^{2\epsilon - \frac{1}{2}} \leq CN^{2\epsilon - \frac{1}{2}} \quad (124) \]
for some constant $C > 0$. Here we used Lemma 5 and Definition 8 to get the second inequality.

By mean value theorem and the fact that $\hat{R}'(\hat{\gamma}) = 0$,

$$
\hat{R}'(\hat{\gamma} + N^{3e - \frac{1}{2}}) = \hat{R}''(\hat{\gamma} + t_1 N^{3e - \frac{1}{2}}) \cdot N^{3e - \frac{1}{2}} = \hat{R}''(\hat{\gamma}) \cdot N^{3e - \frac{1}{2}} + \hat{R}'''(\hat{\gamma} + t_1 t_2 N^{3e - \frac{1}{2}}) \cdot t_1 N^{6e - 1} \quad \text{(125)}
$$

$$
\hat{R}'(\hat{\gamma} - N^{3e - \frac{1}{2}}) = -\hat{R}''(\hat{\gamma} - t'_1 N^{3e - \frac{1}{2}}) \cdot N^{3e - \frac{1}{2}} = -\hat{R}''(\hat{\gamma}) \cdot N^{3e - \frac{1}{2}} + \hat{R}'''(\hat{\gamma} - t'_1 t'_2 N^{3e - \frac{1}{2}}) \cdot t'_1 N^{6e - 1} \quad \text{(126)}
$$

where $t_1, t_2, t'_1, t'_2$ are all in $[0, 1]$. According to (122), (124), (125), (126) and the fact that $\hat{R}''(\hat{\gamma}) > 0$, we have that if $N$ is large enough and $\Omega_1(\epsilon)$ holds, then

$$
R'(\hat{\gamma} + N^{3e - \frac{1}{2}}) \geq \hat{R}'(\hat{\gamma} + N^{3e - \frac{1}{2}}) - CN^{2e - \frac{1}{2}} \geq \hat{R}''(\hat{\gamma}) \cdot N^{3e - \frac{1}{2}} - 128 \left(\frac{\hat{\gamma} - L_+}{3}\right)^3 \cdot t_1 N^{6e - 1} - CN^{2e - \frac{1}{2}} > 0
$$

$$
R'(\hat{\gamma} - N^{3e - \frac{1}{2}}) \leq \hat{R}'(\hat{\gamma} - N^{3e - \frac{1}{2}}) + CN^{2e - \frac{1}{2}} \leq -\hat{R}''(\hat{\gamma}) \cdot N^{3e - \frac{1}{2}} + \frac{128}{(\hat{\gamma} - L_+)^3} \cdot t'_1 N^{6e - 1} + CN^{2e - \frac{1}{2}} < 0
$$

thus

$$
|\gamma - \hat{\gamma}| \leq N^{3e - \frac{1}{2}}
$$

because $R'(\gamma) = 0$ and $R'$ is increasing on $(\lambda_1, +\infty)$.

For the second conclusion, according to Taylor’s formula and the fact that $R'(\gamma) = 0$, we have:

$$
R(\hat{\gamma}) - R(\gamma) = \frac{1}{2} R''(\gamma + s(\hat{\gamma} - \gamma))(\hat{\gamma} - \gamma)^2
$$

for some $s \in [0, 1]$. This together with (122) and the first conclusion yields the second conclusion. $\square$

**Lemma 34** Suppose $c_3 \in (0, 1/10)$. There exists constants $c_4 > 0$ and $N_0 > 0$ such that if $N > N_0$ and $\Omega_1(\epsilon)$ holds, then

$$
\int_{\gamma_i}^{\gamma + i\infty} e^{\frac{N}{2} R(z)} dz = i e^{\frac{N}{2} R(\gamma)} \sqrt{\frac{4\pi}{NR''(\gamma)}} (1 + w_N)
$$

where $|w_N| \leq c_4 N^{4c_3 - \frac{1}{2}}$.

**Proof** Suppose $\Omega_1(\epsilon)$ holds. Notice that

$$
\int_{\gamma_i}^{\gamma + i\infty} e^{\frac{N}{2} R(z)} dz = \frac{i}{\sqrt{N}} \int_{-\infty}^{+\infty} \exp\left(\frac{N}{2} R(\gamma + \frac{it}{\sqrt{N}})\right) dt
$$

$$
= \frac{i e^{\frac{N}{2} R(\gamma)}}{\sqrt{N}} \int_{-\infty}^{+\infty} \exp\left(\frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma)\right)\right) dt.
$$

Using the Taylor’s formula (for complex analytic functions), if $N$ is large enough and $|t| \leq \frac{1}{4N}$, then

$$
\int_{-\infty}^{+\infty} \exp\left(\frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma)\right)\right) dt \leq \exp\left(\frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma)\right)\right)
$$

$$
= \exp\left(\frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma)\right)\right)
$$

where $|R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma)| \leq \frac{C}{2} t^2 N^{-\frac{3}{2}}$. Therefore,

$$
\int_{\gamma_i}^{\gamma + i\infty} e^{\frac{N}{2} R(z)} dz = i e^{\frac{N}{2} R(\gamma)} \sqrt{\frac{4\pi}{NR''(\gamma)}} (1 + w_N)
$$

where $|w_N| \leq c_4 N^{4c_3 - \frac{1}{2}}$. $\square$
\[ N^{c_3} \text{ then} \]
\[ R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma) = \frac{R''(\gamma)}{2} \left( \frac{it}{\sqrt{N}} \right)^2 + \frac{R'''(\gamma)}{6} \left( \frac{it}{\sqrt{N}} \right)^3 + r_N(t) \]
and the remaining term \( r_N(t) \) satisfies:
\[
|r_N(t)| = \left| \left( \frac{it}{\sqrt{N}} \right)^4 \frac{1}{2\pi i} \oint_{|w-\hat{\gamma}|=|\hat{\gamma}-L_+|/2} \frac{R(w)}{(w-\gamma)^4(w-\gamma - \frac{it}{\sqrt{N}})} \, dw \right| \\
\leq C_1 t^4/N^2 \leq C_1 N^{4c_3-2} \tag{127}
\]
for some \( t \)-independent constant \( C_1 > 0 \). By (122), if \( N \) is large enough and \( |t| \leq N^{c_3} \) then
\[
|N/2 \cdot \frac{R'''(\gamma)}{6} \left( \frac{it}{\sqrt{N}} \right)^3| \leq C_2 N^{3c_3-\frac{1}{2}} \tag{128}
\]
for some \( t \)-independent constant \( C_2 > 0 \), therefore we have
\[
\left| \int_{-N^3}^{N^3} \exp \left( \frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma) \right) \right) dt - \int_{-N^3}^{N^3} \exp \left( -\frac{t^2}{4} R''(\gamma) \right) dt \right|
\leq N^3 \exp \left( -\frac{t^2}{4} R''(\gamma) \right) \cdot 2 \cdot \left| \frac{N}{2} \left( \frac{R'''(\gamma)}{6} \left( \frac{it}{\sqrt{N}} \right)^3 + r_N(t) \right) \right| dt
\leq 5C_2 N^{4c_3-\frac{1}{2}} \tag{129}
\]
where we used (127), (128) and the fact that \( R''(\gamma) > 0 \) in the last inequality.
Since \( \Omega_1(\epsilon) \) holds, Lemma 33 yields \((\hat{\gamma} - L_+)/2 \leq |\gamma - \lambda_i| \leq 2(\hat{\gamma} - L_+) + (L_+ - L_-)\), so
\[
\int_{N^3}^{\infty} \left| \frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma) \right) \right| dt = \int_{N^3}^{\infty} \exp \left( -\frac{1}{2} \sum_{i=1}^{N} \log \sqrt{1 + \frac{t^2}{N(\gamma - \lambda_i)^2}} \right) dt
\leq \int_{N^3}^{\infty} \exp \left( -\frac{N}{4} \log(1 + \frac{C_3 t^2}{N}) \right) dt \tag{130}
\]
for some constant \( C_3 > 0 \). Plugging
\[
\log(1 + \frac{C_3 t^2}{N}) \geq \begin{cases} \frac{C_3 t^2}{2N} \geq \frac{C_3 N^{2c_3}}{2N} & \text{if } t \in [N^{c_3}, \sqrt{N/C_3}] \text{ (since } \log(1 + x) \geq \frac{x}{2} \text{ on } [0, 1]) \\
\log 2 \geq \frac{C_3 N^{2c_3}}{2N} & \text{if } t \in [\sqrt{N/C_3}, N] \text{ and } N > N_0
\end{cases}
\]
into (130), we know if \( N \) is large enough, then
\[
\int_{N^3}^{\infty} \left| \frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma) \right) \right| dt \leq \int_{N^3}^{N} \exp \left( -\frac{N}{4} \log(1 + \frac{C_3 t^2}{N}) \right) dt + \int_{N}^{\infty} \exp \left( -\frac{N}{4} \log(1 + \frac{C_3 t^2}{N}) \right) dt \leq \frac{1}{N}. \tag{131}
\]
Similarly we have\[\int_{-\infty}^{-N^c} \left| \exp\left( \frac{N}{2} \left( R(\gamma + \frac{it}{\sqrt{N}}) - R(\gamma) \right) \right) \right| \, dt \leq \frac{1}{N} \] when \( N \) is large enough. This together with (129) and (131) imply that if \( N \) is large enough, then:

\[
\left| \int_{-N^c, -N^c} \exp\left( -\frac{t^2}{4} R''(\gamma) \right) \, dt \right| \leq \frac{5 C_2 N^{4c_3 - \frac{1}{2}}}{N} + \frac{2}{N} + \frac{1}{N} \leq 6 C_2 N^{4c_3 - \frac{1}{2}} \tag{132}
\]

Since \( \int_{\mathbb{R}} \exp\left( -\frac{t^2}{4} R''(\gamma) \right) \, dt = \sqrt{4\pi/R''(\gamma)} \), we complete the proof by (122).

**Proof of Theorem 2** According to Lemmas 6 and 34, if \( N \) is large enough and \( \Omega_1(\epsilon) \) holds, then

\[
\int_{S_{N-1}} e^{\beta \langle \sigma, (W + \lambda V) \sigma \rangle} \, d\omega_N(\sigma) = \sqrt{N \beta} \frac{\sqrt{2\pi} e^{N R(\gamma)}}{\sqrt{2\pi} e^{N R(\gamma)}(1 + u_N)}
\]

where \(|u_N| \leq N^{-1/3}\) and thus by Lemma 33

\[
F_N = \frac{1}{N} \log \int_{S_{N-1}} e^{\beta \langle \sigma, (W + \lambda V) \sigma \rangle} \, d\omega_N(\sigma)
= -\frac{1}{2} \log(2\beta e) + \beta \hat{\gamma} - \frac{1}{2N} \sum_{i=1}^{N} \log(\hat{\gamma} - \lambda_i) + t_N
\]

where

\[
|t_N| \leq CN^{6\epsilon - 1}\tag{133}
\]

for some constant \( C > 0 \). Therefore, if \( N \) is large enough and \( \Omega_1(\epsilon) \) holds, then

\[
-2\sqrt{N} \left( F_N + \frac{1}{2} \log(2\beta e) - \beta \hat{\gamma} \right) - \sqrt{N} \int \log(\hat{\gamma} - t) \, d\mu_{f_c}(t)
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \log(\hat{\gamma} - \lambda_i) - \sqrt{N} \int \log(\hat{\gamma} - t) \, d\mu_{f_c}(t) - 2\sqrt{N} \cdot t_N.
\]

According to Theorem 4, (133), (121) and the assumption that \( \epsilon < 1/12 \), we complete the proof.

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**Data Availability** We do not use any data.

**Declarations**

**Conflict of interest** We do not have any conflict of interest.
Appendix A: Analysis on the Curve of Steepest-Descent: Proof of Lemma 30

Now we study the curve of steepest-descent. In this section,

- \( N \) is fixed;
- we do not consider randomness. In other words, we can imagine that the sample point in the probability space is fixed.

**Lemma 35** If \( \Im R(x + iy) = 0 \) and \( y > 0 \), then \( y \in (0, \frac{\pi}{2\beta}) \). On the other hand, for any \( y \in (0, \frac{\pi}{2\beta}) \) there is a unique \( x \in \mathbb{R} \) such that \( \Im R(x + iy) = 0 \).

**Proof** By definition we have that if \( x \in \mathbb{R} \) and \( y > 0 \) then

\[
\Im R(x + iy) = 2\beta y - \frac{1}{N} \sum_{i=1}^{N} \arccos \frac{x - \lambda_i}{\sqrt{(x - \lambda_i)^2 + y^2}}. \tag{A1}
\]

So if \( \Im R(x + iy) = 0 \) then \( 2\beta y = \frac{1}{N} \sum_{i=1}^{N} \arccos \frac{x - \lambda_i}{\sqrt{(x - \lambda_i)^2 + y^2}} < \pi \), thus \( y < \pi/(2\beta) \).

On the other hand, suppose \( y \in (0, \pi/(2\beta)) \). Let

\[
f_y(x) = \Im R(x + iy) = 2\beta y - \frac{1}{N} \sum_{i=1}^{N} \arccos \frac{x - \lambda_i}{\sqrt{(x - \lambda_i)^2 + y^2}}.
\]

Then \( \lim_{x \to -\infty} f_y(x) = 2\beta y - \pi < 0 \) and \( \lim_{x \to +\infty} f_y(x) = 2\beta y > 0 \). By continuity there exists \( x \in \mathbb{R} \) such that \( f_y(x) = 0 \) so \( \Im R(x + iy) = 0 \). Moreover, if \( x_1 < x_2 \) such that \( f_y(x_1) = f_y(x_2) = 0 \) then there is \( x_3 \in (x_1, x_2) \) with \( f'_y(x_3) = 0 \). But

\[
f'_y(x) = \frac{y N}{\sqrt{(x - \lambda_i)^2 + y^2}} > 0,
\]

so such \( x_3 \) cannot exist. In summary, there is a unique \( x \in \mathbb{R} \) such that \( \Im R(x + iy) = 0 \).

**Lemma 36** \( h(y) < \gamma \) for all \( y \in (0, \frac{\pi}{2\beta}) \)

**Proof** Suppose \( y_0 \in (0, \frac{\pi}{2\beta}) \). If \( h(y_0) = \gamma \), then \( \Im R(\gamma + iy_0) = 0 \). Since \( \Im R(\gamma) = 0 \), there must be \( y_1 \in (0, y_0) \) such that

\[
\frac{\partial}{\partial y} \bigg|_{y=y_1} \Im R(\gamma + iy) = 0.
\]

But

\[
\frac{\partial}{\partial y} \bigg|_{y=y_1} \Im R(\gamma + iy) = 2\beta - \frac{1}{N} \sum_{i=1}^{N} \frac{\gamma - \lambda_i}{(\gamma - \lambda_i)^2 + y^2} > 2\beta - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} = 0.
\]

The definition of \( \gamma \) is used in the last identity. This means \( h(y) \neq \gamma \) for all \( y \in (0, \frac{\pi}{2\beta}) \).

Since

\[
\lim_{y \to \frac{\pi}{2\beta}} \ h(y) = -\infty,
\]

we know by continuity that \( h(y) < \gamma \) for all \( y \in (0, \frac{\pi}{2\beta}) \).
Lemma 37 If $y \in [\frac{1}{4\beta}, \frac{\pi}{2\beta})$ then

$$h'(y) \leq \frac{1}{2} - 2\beta y.$$ \hfill (A2)

Proof Suppose $y \in (0, \frac{\pi}{2\beta})$. According to (A1) and the implicit function theorem,

$$-h'(y) = \left. \frac{\partial \ln R(x+iy)}{\partial y} \right|_{x=h(y)} = \frac{2\beta - \frac{1}{N} \sum_{i=1}^{N} \frac{h(y)-\lambda_{i}}{(h(y)-\lambda_{i})^2+y^2}}{y \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(h(y)-\lambda_{i})^2+y^2}}.$$ \hfill (A3)

Since

$$\frac{h(y) - \lambda_{i}}{(h(y) - \lambda_{i})^2 + y^2} \leq \frac{1}{2y}, \quad \frac{1}{(h(y) - \lambda_{i})^2 + y^2} \leq \frac{1}{y^2},$$

we see that on the right hand side of (A3), the numerator is larger than or equal to $2\beta - \frac{1}{2\beta}$ and the denominator is less than or equal to $1/y$. So the lemma is proved.

Corollary 3 If $\frac{1}{4} < c_0 < \frac{\pi}{2\beta}$, then $h$ is a bijection from $[\frac{c_0}{\beta}, \frac{\pi}{2\beta})$ to $(-\infty, h(\frac{c_0}{\beta}))$. The inverse function satisfies:

$$\frac{2}{1 - 4c_0} \leq (h^{-1})'(x) < 0, \quad \forall x \in (-\infty, h(\frac{c_0}{\beta})].$$

Proof By (A2) we know $h$ is strictly decreasing on $[\frac{c_0}{\beta}, \frac{\pi}{2\beta})$, so is bijective on this interval. Then using $(h^{-1})' = (h')^{-1}$ we complete the proof.

Lemma 38 We have

$$\lim_{y \to 0^+} h(y) = \gamma, \quad \lim_{y \to 0^+} \frac{h(y) - \gamma}{y} = 0 \quad \text{and} \quad \lim_{y \to 0^+} h'(y) = 0.$$

Proof We put this proof at the end of this section.

Now we are ready to prove Lemma 30.

Proof of Lemma 30 The first, third and last conclusions of Lemma 30 come from Lemma 35, Lemma 36 and Lemma 37 respectively. Notice that $\overline{R(\overline{z})} = \overline{R(\overline{z})}$ and that $\gamma$ is the only number in $(\lambda_{1}, \infty)$ where $R' = 0$. Moreover, according to Lemma 30, the $y$-coordinate of any point on $S$ is in $(-\frac{\pi}{2\beta}, \frac{\pi}{2\beta})$. So we have $S = \{h(y) + iy | -\frac{\pi}{2\beta} < y < \frac{\pi}{2\beta}\}$. Finally, by Lemma 38 we have $h(y) \in C^1((-\frac{\pi}{2\beta}, \frac{\pi}{2\beta}))$.

Proof of Lemma 38 We notice that $N$ is a fixed number in this lemma. If $\lambda_1 = \cdots = \lambda_N$ then the lemma is trivial. Now we assume that

$$\lambda_1 = \cdots = \lambda_M > \lambda_{M+1} \geq \cdots \geq \lambda_N$$

for some $M \in [1, N-1]$.

Lemma 39 • If $0 < t < 1$ then there exists $t_1 \in [0, t]$ such that

$$\arccos \sqrt{1 - t^2} = t + \frac{t^3}{6} \cdot \frac{1 + 2t_1^2}{(1 - t_1^2)^{5/2}}.$$
• There exists \( w_0 > 0 \) such that \( \frac{\arccos \sqrt{1-t^2}}{t} \geq w_0 \) for all \( t \in [0, 1] \).

**Proof** The first conclusion is from Taylor’s formula. The other conclusion is trivial. \( \square \)

Now we use Lemma 39 to prove Lemma 38.

1. According to (A1),
   \[
   2\beta y = \frac{1}{N} \sum_{i=1}^{N} \arccos \frac{h(y) - \lambda_i}{\sqrt{(h(y) - \lambda_i)^2 + y^2}} \quad \forall y \in (0, \frac{\pi}{2\beta})
   \]
   so we have
   \[
   h(y) > \lambda_1 \quad \forall y \in (0, \frac{\pi}{4N\beta})
   \]
   otherwise the right hand side of (A4) is larger than \( \frac{1}{N} \arccos \frac{h(y) - \lambda_1}{\sqrt{(h(y) - \lambda_1)^2 + y^2}} \geq \frac{\pi}{2N} \geq 2\beta y. \)

   For \( y \in (0, \frac{\pi}{4N\beta}] \), define
   \[
   f(y) := 2\beta y - \frac{1}{N} \sum_{i=M+1}^{N} \arccos \frac{h(y) - \lambda_i}{\sqrt{(h(y) - \lambda_i)^2 + y^2}}
   \]

   By Lemma 39 there exists \( \delta_0 \in (0, \frac{\pi}{4N\beta}) \) such that if \( y \in (0, \delta_0) \) then
   \[
   |f(y)| = \left| 2\beta y - \frac{1}{N} \sum_{i=M+1}^{N} \arccos \left( 1 - \frac{y}{\sqrt{(h(y) - \lambda_i)^2 + y^2}} \right)^2 \right|
   \]
   \[
   \leq 2\beta y + \frac{1}{N} \sum_{i=M+1}^{N} \frac{2y}{\sqrt{(h(y) - \lambda_i)^2 + y^2}} \leq 2\beta y + \frac{2y}{\lambda_1 - \lambda_{M+1}} = w_1 y
   \]
   where \( w_1 := 2\beta + \frac{2y}{\lambda_1 - \lambda_{M+1}} \). By Lemma 39, (A5), (A4) and the definition of \( f(y) \), if \( y \in (0, \delta_0) \) then
   \[
   w_1 y \geq |f(y)| \geq \frac{M}{N} \arccos \frac{h(y) - \lambda_1}{\sqrt{(h(y) - \lambda_1)^2 + y^2}}
   \]
   \[
   = \frac{M}{N} \arccos \left( 1 - \frac{y}{\sqrt{(h(y) - \lambda_1)^2 + y^2}} \right)^2 \geq \frac{M}{N} \frac{w_0 y}{\sqrt{(h(y) - \lambda_1)^2 + y^2}}
   \]
   which implies
   \[
   \sqrt{(h(y) - \lambda_1)^2 + y^2} \geq \frac{M w_0}{N w_1}
   \]
   So there exists \( \delta_1 \in (0, \delta_0) \) such that
   \[
   h(y) - \lambda_1 \geq \frac{M w_0}{2N w_1} \quad \forall y \in (0, \delta_1)
   \]
According to (A1) and (A3), if \( y \in (0, \delta_1) \) then

\[
-h'(y) \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(h(y) - \lambda_i)^2 + y^2} = \frac{1}{y} \left( 2\beta - \frac{1}{N} \sum_{i=1}^{N} \frac{h(y) - \lambda_i}{(h(y) - \lambda_i)^2 + y^2} \right) 
\]

\[
= \frac{1}{y} \left( \frac{1}{yN} \sum_{i=1}^{N} \arccos \frac{h(y) - \lambda_i}{\sqrt{(h(y) - \lambda_i)^2 + y^2}} - \frac{1}{N} \sum_{i=1}^{N} \frac{h(y) - \lambda_i}{(h(y) - \lambda_i)^2 + y^2} \right) 
\]

\[
= \frac{1}{y^2N} \sum_{i=1}^{N} A_i
\]

(A7)

where

\[
A_i = \arccos \sqrt{1 - \left( \frac{y}{\sqrt{(h(y) - \lambda_i)^2 + y^2}} \right)^2 - \frac{y(h(y) - \lambda_i)}{(h(y) - \lambda_i)^2 + y^2}}.
\]

By Lemma 39 and (A6), there exist constants \( w_2 > 0 \) and \( \delta_2 \in (0, \delta_1) \) such that if \( y \in (0, \delta_2) \) then \( |A_i| \leq w_2y^3 \) and thus by (A7) and Lemma 36:

\[
|h'(y)| \leq \left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(h(y) - \lambda_i)^2 + y^2} \right| \leq w_2y \leq y \cdot w_2 \left( (y - \lambda_N)^2 + \left( \frac{\pi}{2\beta} \right)^2 \right). \quad (A8)
\]

This tells us the boundedness of \( h'(y) \) on \( (0, \delta_2) \). So by the completeness of \( \mathbb{R} \) we know \( \lim_{y \to 0^+} h(y) \) exists. Now multiplying both sides of the first identity in (A7) by \( y \), letting \( y \to 0^+ \), using the boundedness of \( h'(y) \) on \( (0, \delta_2) \), we have:

\[
2\beta - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{h(y) - \lambda_i} = 0.
\]

This together with the definition of \( \gamma \) completes the proof of the first conclusion.

2. Plugging \( 2\beta = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} \) into the first identity of (A7), we have for \( y \in (0, \pi/(2\beta)) \):

\[
-h'(y) \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(h(y) - \lambda_i)^2 + y^2} = \frac{1}{y} \left( \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} - \frac{1}{N} \sum_{i=1}^{N} \frac{h(y) - \lambda_i}{(h(y) - \lambda_i)^2 + y^2} \right) 
\]

\[
= \frac{y}{N} \sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)((h(y) - \lambda_i)^2 + y^2)} 
\]

\[
+ \frac{h(y) - \gamma}{y} \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{h(y) - \lambda_i}{(\gamma - \lambda_i)((h(y) - \lambda_i)^2 + y^2)} \quad (A9)
\]

Taking \( y \to 0^+ \) on both sides of (A9), using (A8) and the first conclusion of this lemma, we have

\[
\left( \lim_{y \to 0^+} \frac{h(y) - \gamma}{y} \right) \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)^2} = 0
\]

which completes the proof of the second conclusion.
3. Now we use the first two conclusions to prove the third conclusion. According to (A3) and the fact that \( \lim_{y \to 0^+} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(h(y)-\lambda_i)^2+y^2} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\gamma-\lambda_i)^2} > 0 \), we know from the formula \( \frac{\partial \text{Im} \, R}{\partial y} = \text{Re} R' \) that it suffices to show that
\[
\lim_{y \to 0^+} \frac{1}{y} \cdot \text{Re}(R'(h(y) + iy)) = 0 \tag{A10}
\]
Notice that \( R'(\gamma) = 0 \). So
\[
\frac{1}{y} \cdot R'(h(y) + iy) = \frac{R'(h(y) + iy) - R'(\gamma)}{h(y) + iy - \gamma} \cdot \frac{h(y) + iy - \gamma}{y} \tag{A11}
\]
According to the first two conclusions of this lemma and the mean-value theorem, (A11) must converges to \( i \cdot R''(\gamma) = \frac{i}{N} \sum_{i=1}^{N} \frac{1}{(\gamma-\lambda_i)^2} \) as \( y \to 0^+ \), so (A10) is true.

\[\square\]

**Appendix B: Proofs of Auxiliary Lemmas**

**Proof of Lemma 16** If \( z \) is on the upper or lower edge of \( \Gamma \) then \( |\text{Im}(\lambda v_i - z - m_{fc}(z))| \geq |\text{Im} z| = d \). So we only need to prove the lemma for \( z \) on the left and right edges. Now let \( z \) be on the right edge of \( \Gamma \). The case that \( z \) is on the left edge can be proved by the same method. Notice that
\[
\frac{d}{dx}(x + m_{fc}(x)) > 0 \quad \text{on} (L_+, \infty),
\]
so by Lemma 1
\[
C := (L_+ + d) + m_{fc}(L_+ + d) - \lambda > L_+ + m_{fc}(L_+) - \lambda = 0.
\]
Since \( v_i \in [-1, 1] \), we have that
\[
\min_i |L_+ + d + m_{fc}(L_+ + d) - \lambda v_i| \geq C.
\]
By continuity there is \( y_0 > 0 \) such that if \( z = L_+ + d + iy \) with \( y \in [-y_0, y_0] \) then
\[
\min_i |z + m_{fc}(z) - \lambda v_i| \geq C/2.
\]
If \( z = L_+ + d + iy \) with \( y \not\in [-y_0, y_0] \) then
\[
\min_i |z + m_{fc}(z) - \lambda v_i| \geq |\text{Im}(z + m_{fc}(z))| \geq |\text{Im} z| \geq y_0.
\]
Taking \( C_d = \min(\frac{C}{2}, y_0) \) we complete the proof of the first conclusion. Since \( d \) can be arbitrarily small, the second conclusion follows from the first conclusion. \[\square\]

**Proof of Lemma 17** The first conclusion is from (19), Lemma 16 and the facts that \( \varsigma \geq \frac{1}{1+b} \), \( \varsigma' \leq \frac{1}{2} + \varsigma \) and \( d < 1 \). The second conclusion is from Proposition 7 and Theorem 3. For the third conclusion, notice that if \( N \) is large enough and \( B_N \cap \Omega V(\varsigma) \) holds, then the following statements hold for each \( \xi \in \Gamma_+ \)
\[
|G_{ii}(\xi) - \hat{g}_i(\xi)| \leq |G_{ii}(\xi) - \frac{1}{\lambda v_i - \xi - m_N(\xi)}| + \frac{|m_N - \hat{m}_{fc}|}{|\lambda v_i - \xi - m_N| |\lambda v_i - \xi - \hat{m}_{fc}|}
\]
\[
\leq N^{\varsigma' - \frac{1}{2}} \cdot |\text{Im} \xi|^{-3} + N^{2\varsigma - \frac{1}{2}} \cdot |\text{Im} \xi|^{-2} \quad \text{(by definitions of \( B_N \) and \( \hat{\Omega}(\varsigma) \))}.
\]
Proof of Lemma 18

By Lemma 3 and the definitions of \( g_i \) and \( \hat{g}_i \),

\[
\sqrt{N} \left( \hat{m}_{fc} - m_{fc} \right) = \sqrt{N} \left( \frac{1}{N} \sum \hat{g}_i(\xi) - \int \frac{d\mu(t)}{\lambda t - \xi - m_{fc}(\xi)} \right)
\]

\[
= \frac{1}{\sqrt{N}} \sum \left( \hat{g}_i(\xi) - \mathbb{E}[g_i(\xi)] \right)
\]

\[
= \frac{1}{\sqrt{N}} \sum \left( g_i(\xi) - \mathbb{E}[g_i(\xi)] \right) + \frac{1}{\sqrt{N}} \sum \left( \hat{m}_{fc} - m_{fc} \right) g_i(\xi) \hat{g}_i(\xi)
\]

\[
= \frac{1}{\sqrt{N}} \sum \left( g_i(\xi) - \mathbb{E}[g_i(\xi)] \right) + \frac{\hat{m}_{fc} - m_{fc}}{\sqrt{N}} \left( \sum g_i^2 + \sum \hat{m}_{fc} - m_{fc} \right) \hat{g}_i^2
\]

\[
+ \sqrt{N} \left( \hat{m}_{fc} - m_{fc} \right) \left( \frac{1}{N} \sum (g_i^2 - \mathbb{E}[g_i^2]) \right)
\]

\[
+ \sqrt{N} \left( \hat{m}_{fc} - m_{fc} \right) \int \frac{d\mu(t)}{(\lambda t - \xi - m_{fc})^2} \quad \text{(since } \mathbb{E}[g_i^2] = \int \frac{d\mu(t)}{(\lambda t - \xi - m_{fc})^2})
\]

\( (B12) \)

Moving the last term on the right hand side of (B12) to the left hand side, multiplying both sides by \( 1 + m_{fc}(\xi) \), using (21), we complete the proof. \( \square \)

Proof of Lemma 19

Let

\[ Y_N(\xi) = N^{-\frac{1}{2} - a_1} \sum_i (g_i^2(\xi) - \mathbb{E}[g_i^2(\xi)]) \]

By the Cramer–Wold Theorem, for any \( \xi_1, \ldots, \xi_k \in \Gamma \), if \( N \to \infty \) then

\[ (Y_N(\xi_1), \ldots, Y_N(\xi_k)) \to 0 \quad \text{in distribution} \]

\( (B13) \)

where \( 0 \) is the zero vector in \( \mathbb{R}^k \). For any \( \xi_1, \xi_2 \in \Gamma \), by Lemma 16 and the definition of \( \Gamma \), we have \( |g_i(\xi_1)| \leq \frac{1}{C_\epsilon}, |g_i(\xi_2)| \leq \frac{1}{C_\epsilon} \) and

\[
|g_i^2(\xi_1) - g_i^2(\xi_2)| \leq \frac{2}{C_d^3} |\xi_1 - \xi_2| + |m_{fc}(\xi_1) - m_{fc}(\xi_2)| \leq \frac{2}{C_d} (1 + \frac{1}{d}) |\xi_1 - \xi_2|
\]

\( (B14) \)
where \( C_d \) is defined in Lemma 16. So

\[
\mathbb{E}[|Y_N(\xi_1) - Y_N(\xi_2)|^2] = \frac{1}{N^{1+2a_1}} \mathbb{E}\left[ \sum_{i,j=1}^{N} \left( g_i^2(\xi_1) - g_i^2(\xi_2) - \mathbb{E}[g_i^2(\xi_1)] + \mathbb{E}[g_i^2(\xi_2)] \right) \right.
\]

\[
\left. \left( g_j^2(\xi_1) - g_j^2(\xi_2) - \mathbb{E}[g_j^2(\xi_1)] + \mathbb{E}[g_j^2(\xi_2)] \right) \right]
\]

\[
= \frac{1}{N^{1+2a_1}} \mathbb{E}\left[ \sum_{i=1}^{N} \left( g_i^2(\xi_1) - g_i^2(\xi_2) - \mathbb{E}[g_i^2(\xi_1)] + \mathbb{E}[g_i^2(\xi_2)] \right)^2 \right]
\]

(by independence of \( g_1, \ldots, g_N \))

\[
\leq \frac{1}{N^{2a_1}} \left( \frac{4}{C_d^3} \right) \left( 1 + \frac{1}{d^2} \right)|\xi_1 - \xi_2|^2.
\]

According to Theorem 12.3 of [6],

\[
\{Y_N(\xi) | \xi \in \Gamma \} \quad N = 1, 2, \ldots
\]

is a tight sequence of random functions on \( \Gamma \). This together with (B13) and Theorem 8.1 of [6] imply that

\[
\{Y_N(\xi) | \xi \in \Gamma \}
\]

(as random elements in the space of continuous functions on \( \Gamma \)) converges in distribution to 0 as \( N \to \infty \). By Portmanteau’s Theorem (see, for example, Theorem 2.1 of [6]) the proof of the first conclusion is complete. The second conclusion can be proved in the same way. \( \square \)

**Proof of Lemma 21** By (51) we have

\[
\mathbb{E}_N\left[ G_{ij}(\xi) \frac{\partial e^{tX_N}}{\partial W_{ij}} \right] = \frac{\mathbb{E}_N\left[ e^{itX_N} G_{ij}(\xi) \int_{\Gamma^+} f(\xi') G_{ij}'(\xi') d\xi' \right] \cdot 1_{\Omega_V(\xi)}}{\sqrt{N} 1 + \delta_{ij}}
\]

Putting (B15) into (53), we have:

\[
(\xi - \lambda v_i) \mathbb{E}_N[e^{itX_N}(G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))]
\]

\[
= -\frac{1}{N} \sum_{j} \left( \mathbb{E}_N[e^{itX_N}(G_{ii}(\xi) G_{jj}(\xi) + (G_{ij}(\xi))^2) \right]
\]

\[
-\mathbb{E}_N[e^{itX_N}] \mathbb{E}_N[(G_{ii}(\xi) G_{jj}(\xi) + (G_{ij}(\xi))^2)]
\]

\[
-\frac{2it}{N^{3/2}} \sum_{j} \int_{\Gamma^+} f(\xi') \mathbb{E}_N\left[ e^{itX_N} G_{ij}(\xi) G_{ij}'(\xi') d\xi' \right] 1_{\Omega_V(\xi)} + \varepsilon'^{(i)}(\xi) + \varepsilon'^{(j)}(\xi)
\]

\[
= -\frac{1}{N} \mathbb{E}_N[e^{itX_N}(G_{ii}(\xi) Tr G(\xi) - \mathbb{E}_N[G_{ii}(\xi) Tr G(\xi)])] + \varepsilon'^{(i)}(\xi) + \varepsilon'^{(j)}(\xi)
\]
\[- \frac{1}{N} \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))] \]
\[- \frac{2it}{N^{3/2}} \int_{\Gamma_+} f(\xi') \mathbb{E}_N \left[ e^{itX_N} (G(\xi) G'(\xi'))_{ij} \right] d\xi' \mathbb{I}_{\Omega_V(\xi)} \]  
(B16)

Notice

\[- \frac{1}{N} \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))] = -m_f e \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))] \]
\[- \frac{g_i}{N} \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))] + \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))(m_f e - \frac{1}{N} \text{Tr}(G))] \]
\[+ \frac{1}{N} \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))] + \frac{1}{N} \mathbb{E}_N [e^{itX_N}] \mathbb{E}_N [G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi)] \]  
(B17)

Plugging (B17) into (B16), moving the term \(-m_f e \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))]\) to the left hand side, multiplying both sides by \(-g_i(\xi)\) and taking \(\sum_i\), using the definition of \(\mathcal{E}_3(\xi)\), we have

\[\mathbb{E}_N [e^{itX_N} (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] = - \sum_i g_i(\xi) \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))(m_f e - \frac{1}{N} \text{Tr}(G))] \]
\[- \frac{1}{N} \sum_i g_i(\xi)(g_i(\xi) - \mathbb{E}_N G_{ii}(\xi)) \mathbb{E}_N [e^{itX_N} (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] \]
\[- \frac{1}{N} \mathbb{E}_N [e^{itX_N}] \sum_i g_i(\xi) \mathbb{E}_N [G_{ii}(\xi) (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] \]
\[+ \frac{1}{N} \sum_i g_i^2(\xi) \mathbb{E}_N [e^{itX_N} (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] + \mathcal{E}_3(\xi). \]  
(B18)

Moving the term \(\frac{1}{N} \sum_i g_i^2(\xi) \mathbb{E}_N [e^{itX_N} (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))]\) to the left hand side:

\[(1 - \frac{1}{N} \sum_i g_i^2(\xi)) \mathbb{E}_N [e^{itX_N} (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] \]
\[= - \sum_i g_i(\xi) \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - \mathbb{E}_N G_{ii}(\xi))(m_f e - \frac{1}{N} \text{Tr}(G))] \]
\[- \frac{1}{N} \sum_i g_i(\xi)(g_i(\xi) - \mathbb{E}_N G_{ii}(\xi)) \mathbb{E}_N [e^{itX_N} (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] \]
\[- \frac{1}{N} \mathbb{E}_N [e^{itX_N}] \sum_i g_i(\xi) \mathbb{E}_N [G_{ii}(\xi) (\text{Tr}(\xi) - \mathbb{E}_N \text{Tr}(\xi))] + \mathcal{E}_3(\xi) \]
\[= - \sum_i g_i(\xi) \mathbb{E}_N [e^{itX_N} (G_{ii}(\xi) - g_i(\xi))(m_f e - \frac{1}{N} \text{Tr}(G))] \]
\[- \mathbb{E}_N [e^{itX_N}] \sum_i g_i(\xi) \mathbb{E}_N [(G_{ii}(\xi) - g_i(\xi))(\frac{1}{N} \text{Tr}(\xi) - m_f e(\xi))] + \mathcal{E}_3(\xi) \]  
(B19)

where we used the fact that \(g_i\) and \(\mathbb{E}_N G_{ii}\) are \(\sigma(V)\)-measurable. \(\square\)
Proof of Lemma 22  First we notice that the conditions in Definition 11 imply:

\[ 5\sigma + 2\xi - \frac{1}{2} < 0. \]  
\(*20\)

According to (23), for any \( \xi \in \mathbb{C}\setminus\mathbb{R} \):

\[
|\hat{m}'_{fc} - (1 + \hat{m}'_{fc})(\frac{1}{N} \sum g_i^2)| = \frac{1}{N} \left| \sum \frac{1}{(\lambda v_i - \xi - \hat{m}_{fc})^2} \right| - \frac{1}{N} \sum \frac{1}{(\lambda v_i - \xi - m_{fc})^2} 
\leq \frac{1}{N} \sum \left| \frac{(\lambda v_i - \xi - m_{fc}(\xi)) + (\lambda v_i - \xi - \hat{m}_{fc}(\xi))}{(\lambda v_i - \xi - m_{fc}(\xi))^2(\lambda v_i - \xi - \hat{m}_{fc}(\xi))^2} \right| |m_{fc} - \hat{m}_{fc}| 
\leq |1 + \hat{m}'_{fc}| \cdot \frac{2}{|\text{Im}\xi|^3} |m_{fc} - \hat{m}_{fc}|. \tag{B21} 
\]

If \( \xi \in \Gamma_+ \), then |\text{Im}\xi| \leq d < 1 and

\[
|1 + \hat{m}'_{fc}| \leq 1 + |\text{Im}\xi|^2 \leq 2|\text{Im}\xi|^2. \tag{B22} 
\]

Now suppose \( N \) is large enough and \( \Omega_V(\xi) \) holds. By Lemma 17 we have \( \Gamma_+ \subset \mathcal{D}_\xi' \).

According to Lemma 5, (B21) and (B22), if \( \xi \in \Gamma_+ \), then

\[
|\hat{m}'_{fc} - (1 + \hat{m}'_{fc})(\frac{1}{N} \sum g_i^2)| \leq \frac{4}{|\text{Im}\xi|^3} N^{2\xi - \frac{1}{2}} \leq 4N^{5\sigma + 2\xi - \frac{1}{2}} = o(1) \quad \text{(by (B20))} 
\]

and therefore

\[
|1 + \hat{m}'_{fc}| \left| 1 - \frac{1}{N} \sum g_i^2(\xi) \right| = \left| 1 + \left( \hat{m}'_{fc} - (1 + \hat{m}'_{fc})(\frac{1}{N} \sum g_i^2) \right) \right| \geq \frac{2}{3}. 
\]

The last inequality together with (B22) completes the proof. \( \Box \)

Proof of Lemma 23  According to (27),

\[ G_{ii}(\xi) = \frac{-1}{\xi + Q_i(\xi)} \quad \text{and} \quad Q_i - \hat{m}_{fc} + \lambda v_i = \frac{1}{g_i} - \frac{1}{Q_i}. \tag{B23} \]

Using \( \frac{-1}{a} = \frac{a+b}{b^2} + \frac{(a+b)^2}{b^3} + \frac{-1}{a} \frac{(a+b)^3}{b^4} \) with \( a = \xi + Q_i \) and \( b = -\xi - \hat{m}_{fc} + \lambda v_i \) we have

\[
\frac{-1}{\xi + Q_i} = \frac{1}{-\xi - \hat{m}_{fc} + \lambda v_i} + \frac{Q_i - \hat{m}_{fc} + \lambda v_i}{(-\xi - \hat{m}_{fc} + \lambda v_i)^2} + \frac{(Q_i - \hat{m}_{fc} + \lambda v_i)^2}{(-\xi - \hat{m}_{fc} + \lambda v_i)^3} + \frac{-1}{\xi + Q_i} \frac{(Q_i - \hat{m}_{fc} + \lambda v_i)^3}{(-\xi - \hat{m}_{fc} + \lambda v_i)^4} 
\]

\[
\hat{g}_i(\xi) + \hat{g}_i^2(\xi)(Q_i - \hat{m}_{fc} + \lambda v_i) + \hat{g}_i^3(\xi)(Q_i - \hat{m}_{fc} + \lambda v_i)^2 
\]

\[
- \left( \frac{\hat{g}_i^3(\xi)}{\xi + Q_i} \right)(Q_i - \hat{m}_{fc} + \lambda v_i)^3 
\]

and therefore

\[
\text{Tr}G(\xi) - N\hat{m}_{fc}(\xi) = \left( \sum_{i=1}^N G_{ii}(\xi) \right) - N\hat{m}_{fc}(\xi) = \left( \sum_{i=1}^N \frac{-1}{\xi + Q_i} \right) - N\hat{m}_{fc}(\xi) 
\]

\[
= \sum_{i=1}^N \left( \hat{g}_i(\xi) + \hat{g}_i^2(\xi)(Q_i - \hat{m}_{fc} + \lambda v_i) + \hat{g}_i^3(\xi)(Q_i - \hat{m}_{fc} + \lambda v_i)^2 \right) 
\]

\[ \ddagger \text{Springer} \]
\[-\frac{\hat{g}_i^3(\xi)}{\xi} \left( Q_i - \hat{m}_{fc} + \lambda v_i \right)^3 - N \hat{m}_{fc}(\xi)\]
\[= \sum_{i=1}^N \left( \hat{g}_i^2(\xi) (Q_i - \hat{m}_{fc} + \lambda v_i) + \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i)^2 \right) \]
\[= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i)^3 \quad \text{(by Lemma 3)}\]
\[= \sum_{i=1}^N \left( \hat{g}_i^2(\xi) (Q_i - \hat{m}_{fc} + \lambda v_i) + \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i)^2 \right) \]
\[+ \frac{\hat{g}_i^3(\xi) G_{ii}(\xi)}{(G_{ii}(\xi))^2} \quad \text{(by } (B23))\]
\[= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i) + \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i)^2 \]
\[+ \sum_{i=1}^N \frac{(G_{ii}(\xi) - \hat{g}_i(\xi))}{(G_{ii}(\xi))^2} (B24)\]

Notice that \( W_{ij} \) is independent of the sigma algebra generated by \( V \) and \( \{ G_{pq}^{(i)} | p, q \neq i \} \). So if \( j_1, \ldots, j_k \in \{1, \ldots, N\} \) and \( p, q, r, t \in \{1, \ldots, N\} \backslash \{i\} \), then
\[
\begin{cases}
\mathbb{E}_N[\sum_{i=1}^N (g_{ii}^3(\xi)) (Q_i - \hat{m}_{fc} + \lambda v_i)] = \sum_{i=1}^N \mathbb{E}_N[\sum_{p} W_{ip} (q_{ip}^{(i)} W_{qi} \mid \xi)] - \hat{m}_{fc}(\xi) \\
\mathbb{E}_N[\sum_{i=1}^N (g_{ii}^3(\xi)) (Q_i - \hat{m}_{fc} + \lambda v_i)^2] = \sum_{i=1}^N \mathbb{E}_N[\sum_{p} W_{ip} (q_{ip}^{(i)} W_{qi} \mid \xi)^2] - \hat{m}_{fc}(\xi)^2 \quad \text{(by } (B25))\end{cases}
\]

Notice that \( \hat{g}_i \) and \( \hat{m}_{fc} \) are \( \sigma(V) \)-measurable. By (B25) we have
\[
\begin{align*}
\mathbb{E}_N[\sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i)] &= \sum_{i=1}^N \mathbb{E}_N[\sum_{p} W_{ip} (q_{ip}^{(i)} W_{qi} \mid \xi)] - \hat{m}_{fc}(\xi) \\
&= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} \left[ \frac{1}{N} \mathbb{E}_N[\sum_{p} G_{pp}^{(i)}] - \hat{m}_{fc}(\xi) \right] \\
&= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} \left[ \frac{1}{N} \mathbb{E}_N[\sum_{p} G_{pp} - \frac{(G_{ip})^2}{G_{ii}(\xi)}] - \hat{m}_{fc}(\xi) \right] \quad \text{(by } 25) \\
&= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} \mathbb{E}_N[\frac{1}{N} \text{Tr} G(\xi) - \hat{m}_{fc}(\xi)] - \frac{1}{N} \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} \mathbb{E}_N[G_{ii}(\xi)] \\
&\quad + \frac{1}{G_{ii}(\xi)} \sum_{p} (G_{ip}(\xi))^2 \quad \text{(B26)}
\end{align*}
\]

Similarly, using (B25),
\[
\begin{align*}
\mathbb{E}_N[\sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} (Q_i - \hat{m}_{fc} + \lambda v_i)^2] &= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} \mathbb{E}_N\left[ \left( - W_{ii} - \hat{m}_{fc} + \sum_{p} W_{ip} (q_{ip}^{(i)} W_{qi} \mid \xi) \right)^2 \right] \\
&= \sum_{i=1}^N \frac{\hat{g}_i^3(\xi)}{\xi} \left( \frac{2}{N} \hat{m}_{fc} - \frac{2}{N} \hat{m}_{fc} \mathbb{E}_N[\text{Tr} G^{(i)}] + \mathbb{E}_N\left[ \left( \sum_{p} W_{ip} (q_{ip}^{(i)} W_{qi} \mid \xi) \right)^2 \right] \right) \
&\quad \text{(B27)}
\end{align*}
\]
Considering the cases \( p = q = r = t, p = q \neq r = t, p = t \neq q = r \) and \( p = r \neq q = t \) we have

\[
\mathbb{E}_N \left[ \left( \sum_{p,q} W_{ip} G_{pq}^{(i)} W_{qi} \right)^2 \right] = \sum_{p,q,r,t} \mathbb{E}_N \left[ W_{ip} W_{qi} W_{ir} W_{ti} G_{pq}^{(i)} G_{rt}^{(i)} \right]
\]

\[
= \sum_{p} \mathbb{E}[W_{ip}^4] \mathbb{E}_N \left[ (G_{pp}^{(i)})^2 \right] + \frac{1}{N^2} \sum_{p \neq r} \mathbb{E}_N \left[ G_{pp}^{(i)} G_{rr}^{(i)} \right] + \frac{2}{N^2} \sum_{p \neq q} \mathbb{E}_N \left[ (G_{pq}^{(i)})^2 \right] \quad \text{(by B25)}
\]

\[
= \sum_{p} \frac{(i)}{\mathbb{E}[W_{ip}^4] \mathbb{E}_N \left[ (G_{pp}^{(i)})^2 \right]} + \frac{1}{N^2} \left( \mathbb{E}_N \left[ (\text{Tr}G^{(i)})^2 \right] - \sum_{p} \mathbb{E}_N \left[ (G_{pp}^{(i)})^2 \right] \right)
\]

\[
+ \frac{2}{N^2} \left( \mathbb{E}_N \left[ \text{Tr}(G^{(i)}G^{(i)}) \right] - \sum_{p} \mathbb{E}_N \left[ (G_{pp}^{(i)})^2 \right] \right)
\]

\[
= \sum_{p} \frac{(i)}{\mathbb{E}[W_{ip}^4] \mathbb{E}_N \left[ (G_{pp}^{(i)})^2 \right]} + \frac{1}{N^2} \mathbb{E}_N \left[ (\text{Tr}G^{(i)})^2 + 2(\text{Tr}G^{(i)})' - 3 \sum_{p} (G_{pp}^{(i)})^2 \right] \quad \text{(B28)}
\]

Plug (B26), (B27) and (B28) into (B24), then we have:

\[
\mathbb{E}_N \left[ \text{Tr}G(\xi) - N\hat{m}_{fc}(\xi) \right] = \sum_{i=1}^{N} \tilde{g}_i^2(\xi) \mathbb{E}_N \left[ \frac{1}{N} \text{Tr}G(\xi) - \hat{m}_{fc}(\xi) \right] - \frac{1}{N} \sum_{i=1}^{N} \tilde{g}_i^2(\xi) \mathbb{E}_N \left[ G_{ii}(\xi) + \frac{1}{G_{ii}(\xi)} \sum_{p} (G_{ip}(\xi))^2 \right]
\]

\[
+ \sum_{i=1}^{N} \tilde{g}_i^3(\xi) \left( \frac{2}{N} + \frac{2}{N^2} \mathbb{E}_N \left[ (\text{Tr}G^{(i)})' \right] + \sum_{p} \mathbb{E}_N \left[ (G_{pp}^{(i)})^2 \right] + \frac{3}{N^2} \right)
\]

\[
+ \mathbb{E}_N \left[ (\hat{m}_{fc} - \frac{1}{N} \text{Tr}G^{(2)}(\xi)) \right] + \mathbb{E}_N \left[ \sum_{i=1}^{N} \frac{(G_{ii}(\xi) - \tilde{g}_i(\xi))^3}{(G_{ii}(\xi))^2} \right]
\]

Moving the first term on the RHS to the LHS and using (20), we complete the proof. \( \square \)

**Proof of Lemma 32** Comparing the imaginary part of both sides of (113):

\[
0 = R''(\gamma) \text{Re}(z - \gamma) \text{Im}z + \frac{R'''(\gamma)}{2} (\text{Re}(z - \gamma))^2 \text{Im}z - \frac{R'''(\gamma)}{6} (\text{Im}z)^3
\]

\[
+ \sum_{j=4}^{\infty} \frac{R^{(j)}(\gamma)}{j!} \text{Im}((z - \gamma)^j)
\]

for all \( z \in S^+ \cap Q_N \). In the above equation \( \text{Im}z \cdot R''(\gamma) \neq 0 \), so we can divide both sides by \( \text{Im}z \cdot R''(\gamma) \) and have:

\[
X - \frac{\alpha}{2} X^2 + \frac{\alpha}{6} Y^2 + H(X, Y) = 0, \quad \forall z \in S^+ \cap Q_N \quad \text{(B29)}
\]

where
According to (6.46) of [1], we have

\[ |\text{Im}((x + iy)^j)| \leq j \cdot |x + iy|^{j-1} \cdot |y| \quad \forall x, y \in \mathbb{R}, j \in \{1, 2, \ldots\}. \]  \hspace{1cm} (B30)

By Lemma 29 and (B30), if \( N \) is large enough then

\[ \frac{2}{W_2} N^{1-3t_0-t_2} \leq \alpha \leq 2W_2^3 N^{1+2t_0+t_2} \]  \hspace{1cm} (B31)

and

\[ |H(X, Y)| \leq \sum_{j=4}^\infty W_2^j N^{-2+2t_0+t_2+j} |X + iY|^{j-1} \leq 2W_2^4 N^{2t_0+t_2+2} |X + iY|^3 \]
\[ \leq 2W_2^2 N^{2t_0+t_2-7} (X^2 + Y^2), \quad \forall z = X + \gamma + iY \in S^+ \cap Q_N \]
where \( W_2 > 0 \) is defined in Lemma 29. We used the condition \( |X + iY| < N^{-9} \) in the last inequality. So if \( z = X + \gamma + iY \in S^+ \cap Q_N \) and \( N \) is large enough, then we can write

\[ H(X, Y) = H^{(1)}(X, Y) + H^{(2)}(X, Y) \]  \hspace{1cm} (B32)

where

\[ |H^{(1)}(X, Y)| \leq 2W_2^4 N^{2t_0+t_2-7} X^2 \quad \text{and} \quad |H^{(2)}(X, Y)| \leq 2W_2^2 N^{2t_0+t_2-7} Y^2. \]  \hspace{1cm} (B33)

By (B29), (B32) and (B33), if \( z = X + \gamma + iY \in S^+ \cap Q_N \) and \( N \) is large enough, then

\[ X(1 - \frac{\alpha}{2} X + \frac{H^{(1)}(X, Y)}{X}) + \frac{\alpha}{6} Y^2 (1 + \frac{6H^{(2)}(X, Y)}{\alpha Y^2}) = 0 \]

where (by (B31), (B33) and the definition of \( Q_N \))

\[ | - \frac{\alpha}{2} X + \frac{H^{(1)}(X, Y)}{X} | \leq 2W_2^3 N^{-5} < \frac{1}{2} \quad \text{and} \quad \left| \frac{6H^{(2)}(X, Y)}{\alpha Y^2} \right| \leq 6W_2^6 N^{-1} < \frac{1}{2} \]

thus we have

\[ \frac{-X}{Y^2} \in \left[ \frac{\alpha}{18}, \frac{\alpha}{2} \right] \subseteq \left[ \frac{1}{9W_2^2} N^{1-3t_0-t_2}, W_2^3 N^{1+2t_0+t_2} \right] \quad \text{(by (B31))} \]

and

\[ |X| \leq Y^2 \cdot W_2^3 N^{1+2t_0+t_2} \leq Y \]

which implies (114).

\[ \square \]

References

1. Baik, J., Lee, J.O.: Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model. J. Stat. Phys. 165, 185–224 (2016)
2. Baik, J., Lee, J.O.: Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model with ferromagnetic interaction. Ann. Henri Poincaré 18, 1867–1917 (2017)
3. Baik, J., Lee, J.O.: Free energy of bipartite spherical Sherrington–Kirkpatrick model. Ann. Inst. H. Poincaré Probab. Stat. 56, 2897–2934 (2020)
4. Baik, J., Lee, J.O., Wu, H.: Ferromagnetic to paramagnetic transition in spherical spin glass. J. Stat. Phys. 173, 1484–1522 (2018)
5. Benaych-Georges, F., Knowles, A.: Lectures on the local semicircle law for Wigner matrices. In: Advanced Topics in Random Matrices. Panoramas et Synthèses 53, Société Mathématique de France (2016)
6. Billingsley, P.: Convergence of Probability Measures. Wiley, New York (1968)
7. Crisanti, A., Sommers, H.-J.: The spherical p-spin interaction spin glass model: the statics. Z. Phys. B Condens. Matter 87, 341–354 (1992)
8. Edwards, S.F., Anderson, P.W.: Theory of spin glasses. J. Phys. F 5(5), 965 (1975)
9. Erdős, L., Yau, H.-T.: A Dynamical Approach to Random Matrix Theory. American Mathematical Society, Providence (2017)
10. Guillera, F.: Broken replica symmetry bounds in the mean field spin glass model. Commun. Math. Phys. 233(1), 1–12 (2003)
11. Ji, H.C., Lee, J.O.: Central limit theorem for linear spectral statistics of deformed Wigner matrices. Random Matrices Theory Appl. 9, 2050011 (2020)
12. Kosterlitz, J., Thouless, D., Jones, R.C.: Spherical model of a spin-glass. Phys. Rev. Lett. 36, 859–860 (1976)
13. Landon, B., Sosoe, P.: Fluctuations of the overlap at low temperature in the 2-spin spherical SK model. Ann. Inst. H. Poincaré Probab. Stat. 58, 1426–1459 (2022)
14. Landon, B., Sosoe, P.: Fluctuations of the 2-spin SSK model with magnetic field. Preprint (2020). arXiv:2009.12514
15. Lee, J.O., Schnelli, K.: Local deformed semicircle law and complete delocalization for Wigner matrices with random potential. J. Math. Phys. 54, 103504 (2013)
16. Lee, J.O., Schnelli, K.: Extremal eigenvalues and eigenvectors of deformed Wigner matrices. Probab. Theory Relat. Fields 164, 165–241 (2016)
17. Lee, J.O., Schnelli, K.: Local law and Tracy–Widom limit for sparse random matrices. Probab. Theory Relat. Fields 171, 543–616 (2018)
18. Lee, J.O., Schnelli, K.: Edge universality for deformed Wigner matrices. Reviews in Mathematical Physics 27, 1550018 (2015)
19. Lee, J.O., Schnelli, K., Stetler, B., Yau, H.-T.: Bulk universality for deformed Wigner matrices. Ann. Probab. 44, 1601–1646 (2016)
20. Li, Y., Schnelli, K., Xu, Y.: Central limit theorem for mesoscopic eigenvalue statistics of deformed Wigner matrices and sample covariance matrices. Ann. Inst. H. Poincaré Probab. Statist. 57, 506–546 (2021)
21. Nguyen, V.L., Sosoe, P.: Central limit theorem near the critical temperature for the overlap in the 2-spin spherical SK model. J. Math. Phys. 60, 103302 (2019)
22. Parisi, G.: A sequence of approximated solutions to the SK model for spin glasses. J. Phys. A 13(4), L115 (1980)
23. Pastur, L.: On the spectrum of random matrices. Theor. Math. Phys. 10, 67–74 (1972)
24. Sherrington, D., Kirkpatrick, S.: Solvable model of a spin-glass. Phys. Rev. Lett. 35(26), 1792 (1975)
25. Talagrand, M.: The Parisi formula. Ann. Math. 163(1), 221–263 (2006)
26. Talagrand, M.: Free energy of the spherical mean field model. Probab. Theory Relat. Fields 134, 339–382 (2006)

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