M-affine functions composing Sturm–Liouville families

Lucio R. Berrone and Gerardo E. Sbérghamo

Abstract. Given an $n$-variable mean $M$ defined on a real interval $I$, an $M$-affine function is a solution to the functional equation

$$f(M(x_1, \ldots, x_n)) = M(f(x_1), \ldots, f(x_n)), \quad x_1, \ldots, x_n \in I. \quad (1)$$

When $M$ is a quasilinear mean, the set of continuous $M$-affine functions is a Sturm–Liouville family on every compact interval $[a, b] \subseteq I$; i.e., for every $\alpha, \beta \in [a, b]$, there exists an $M$-affine function $f$ such that $f(a) = \alpha$ and $f(b) = \beta$. The validity of the converse statement is explored in this paper and several consequences are derived from this study. New characterizations of quasilinear means and the solution to Eq. (1) under suitable conditions are among the more important ones.

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1. Introduction and preliminaries

Let $I \neq \emptyset$ be a real interval. An $n$-variable mean $M$ defined on $I$ is a function $M : I^n \rightarrow I$ which is internal; i.e., it satisfies the property

$$\min\{x_1, \ldots, x_n\} \leq M(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}, \quad x_1, \ldots, x_n \in I. \quad (2)$$

$M$ is said to be strict when the inequalities (2) turn out to be strict provided that the variables $x_i$ are not all equal. Immediate consequences of (2) are both the equality

$$M(x, \ldots, x) = x, \quad x \in I,$$

(which shows that means are reflexive functions) and the fact that a mean $M$ is continuous at every point of the diagonal $\{(x, \ldots, x) : x \in I\}$ of $I^n$. A mean invariant under rearrangements of their arguments is said to be a symmetric mean, so that an $n$-variable mean $M$ is symmetric when $M(x_{\sigma_1}, \ldots, x_{\sigma_n}) = M(x_1, \ldots, x_n)$ for every $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$, the symmetric group of order $n$. The restriction to a subinterval $J \subseteq I$ of an $n$-variable mean $M$ defined on $I$ is an $n$ mean on $J$, which will be denoted by $M|_J$. 


The set of all [continuous] $n$-variable means defined on an interval $I$ will be denoted by $\mathcal{M}_n(I)$ [\(\mathcal{CM}_n(I)\)]. When a change of variable $f : I \to J$ is performed, a given mean $M \in \mathcal{CM}_n(I)$ becomes another mean $N \in \mathcal{CM}_n(J)$ and, by identifying the so related means $M$ and $N$, an equivalence relationship is introduced on $\mathcal{CM}_n$. Namely, given $M \in \mathcal{CM}_n(I)$ and $N \in \mathcal{CM}_n(J)$, it is said that $M$ and $N$ are conjugated means when there exists a homeomorphism $f : I \to J$ such that the equality

$$f(M(x_1, x_2, \ldots, x_n)) = N(f(x_1), f(x_2), \ldots, f(x_n)),$$

(3)

holds for every $x_1, x_2, \ldots, x_n \in I$. This relationship decomposes $\mathcal{CM}_n(I)$ into classes named conjugacy classes. For instance, the conjugacy class of the linear mean

$$L(x_1, \ldots, x_n; w) = \sum_{j=1}^{n} w_j x_j,$$

(4)

(where the conditions $w_i > 0$, $i = 1, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$ are satisfied by the coefficients $w_i$, called weights of the mean) is given by the class of quasilinear means; i.e., means of the form

$$L_f(x_1, \ldots, x_n; w) = f^{-1}\left(\frac{1}{n} \sum_{j=1}^{n} f(x_j)\right), \quad x_1, \ldots, x_n \in I,$$

(5)

where $f : I \to \mathbb{R}$ varies on the set of strictly monotonic and continuous functions. The function $f$ is called the generator of the quasilinear mean $L_f$. In the literature (v.g. [8, pg. 266]; [9, pg. 215]; [14, pg. 208]), nonnegative weights are often admitted in definition (4) but, throughout this paper, quasilinear means are always strict means. (Note that the annulation of some weights in (4) simply produces a quasilinear mean in fewer variables). Particularly relevant is the equal weights (or symmetric) case: the conjugacy class of the arithmetic mean $A(x_1, x_2, \ldots, x_n) = (\sum_{j=1}^{n} x_j)/n$, $x_1, x_2, \ldots, x_n \in \mathbb{R}$, is given by means of the form

$$A_f(x_1, \ldots, x_n) = f^{-1}\left(\frac{1}{n} \sum_{j=1}^{n} f(x_j)\right), \quad x_1, \ldots, x_n \in I,$$

(5)

where, as before, $f : I \to \mathbb{R}$ denotes a generic continuous and strictly monotonic function. These means are named quasiarithmetic means. It must be added that, in reference to means defined by (4), a non uniform terminology was employed. In the recent literature, they are frequently named weighted quasiarithmetic means, but in Chap. III of [12], the explicit denomination mean values with an arbitrary function was preferred.

A well known result (cf. [12, Sect. 3.2]; [1, Theorem 2, pg. 67]; [2, Cor. 5, pg. 246]; [14, pg. 382 and ff].) establishes that the generator $f$ of a quasilinear
mean $M$ defined on $I$ is determined only up to an affine homeomorphism by $M$: the equality $L_f = L_g$ holds if and only if $g = mf + h$ for certain real constants $m$ and $h$, $m \neq 0$.

Given two means $M \in \mathcal{CM}_n (I)$ and $N \in \mathcal{CM}_n (J)$, one can look for functions $f$ satisfying equality (3). This type of functional equations (or even a more general one in which $M$ and $N$ are continuous functions) have been studied since the first decades of the past century (for $n = 2$ see [1, pgs. 62, 79, 145], and the corresponding references; [7, pg. 239 and ff.; [4,10,11]), but the problem of finding conditions on the means $M$ and $N$ in order that the functional equation (3) admits nontrivial (non constant) solutions has not been fully solved. When $M = N$, (3) takes the form

$$f (M (x_1, ..., x_n)) = M (f (x_1), ..., f (x_n)), \quad x_1, ..., x_n \in I,$$

(6)
a functional equation which can be seen as a generalization of the Jensen equation

$$f \left(\frac{x_1 + \cdots + x_n}{n}\right) = \frac{f(x_1) + \cdots + f(x_n)}{n}, \quad x_1, ..., x_n \in I,$$

(7)
and whose solutions are, for this reason, named $M$-affine functions ([10,17]). Indeed, for every $n \geq 2$ and every real interval $I$, $A$-affine functions have the form $f(x) = \alpha (x) + h$, where $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function and $b$ is a real constant, but continuous $A$-affine functions reduce to the set of affine functions $f(x) = mx + h$, $m, h \in \mathbb{R}$.

Suppose that $M \in \mathcal{M}_n (J)$ and $\emptyset \neq I \subseteq J$. Along this paper, the general solution $f : I \rightarrow J$ to the equation

$$f (M (x_1, ..., x_n)) = M (f (x_1), ..., f (x_n)), \quad x_1, ..., x_n \in I,$$

(8)
will be denoted by $\mathcal{A}(M; I, J)$, while the notation $\mathcal{AC} (M; I, J)$ is reserved for the general continuous solution to (6). When $I = J$, these notations are to be simplified: $\mathcal{A}(M; I)$ instead of $\mathcal{A}(M; I, I)$ and $\mathcal{AC}(M; I)$ instead of $\mathcal{AC}(M; I, I)$. Accordingly, if $M \in \mathcal{M}_n (I)$, then $\mathcal{AC}(M; [a, b], I)$ stands for the family of continuous $M$-affine functions $f : [a, b] \rightarrow I$ whereas $\mathcal{AC}(M; [a, b])$ denotes the family of continuous $M$-affine functions $f : [a, b] \rightarrow [a, b]$. It is clear that $\mathcal{A}(M; I, J) \supseteq \{ f |_{I} : f \in \mathcal{A}(M; J) \}$ (and, of course, $\mathcal{AC}(M; I, J) \supseteq \{ f |_{I} : f \in \mathcal{AC}(M; J) \}$) and that $\mathcal{A}(M; I)$ and $\mathcal{AC}(M; I)$ are semigroups under "o", the usual composition of functions.

The set constituted by affine functions on an interval $I \subseteq \mathbb{R}$ (i.e., the set $\mathcal{AC}(A; I)$) will be denoted by $\text{Aff} (I)$; i.e.,

$$\text{Aff} (I) = \{ f (t) = mt + h : m, h \in \mathbb{R}, \quad mI + h \subseteq I \}.$$

For instance, $\text{Aff} (\mathbb{R}^+) = \{ f (t) = mt + h : m \in \mathbb{R}^+_+, \quad h \in \mathbb{R}^+ \}$ and $\text{Aff} ([0, 1]) = \{ f (t) = mt + h : m \in [-1, 1], \quad h, m + h \in [0, 1] \}$. The set of parameters $(m,h)$
\[ \in \mathbb{R}^2 \text{ such that the affine function } t \mapsto mt + h \text{ is a member of } \text{Aff}(I) \text{ will be denoted by } \text{AFF}(I); \text{ i.e.,} \]
\[ \text{AFF}(I) = \{ (m, h) \in \mathbb{R}^2 : mI + h \subseteq I \}. \]

For instance,
\[ \text{AFF}(\mathbb{R}^+) = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \text{ and } \text{AFF}([0, 1]) = \{ (0, 0), (1, 0), (0, 1), (-1, 1) \} \wedge, \]
where \( E^\wedge \) denotes the convex hull of the set \( E \). Clearly, \( \text{Aff}(I) \) and \( \text{AFF}(I) \) are convex sets regardless of the interval \( I \). Further properties of these sets are to be considered in Sect. 5, where it will be appreciated that the visual representation \( \text{AFF}(I) \) of \( \text{Aff}(I) \) may help in clarifying some developments.

This paper deals with a sort of inverse problem: deducing properties of the means \( M \) from the knowledge of some properties of \( \mathcal{A}(M; I) \) or \( \mathcal{AC}(M; I) \). For example, if for a strict mean \( M \) defined on \( \mathbb{R} \), the functions \( f(x) = mx + h \), \( m, h \in \mathbb{R} \), are solutions to the equation
\[ f(M(x, y)) = M(f(x), f(y)), \quad x, y \in \mathbb{R}, \quad (9) \]
or, in other words, if the inclusion \( \text{Aff}(\mathbb{R}) \subseteq \mathcal{A}(M; \mathbb{R}) \) holds, then \( M \) must be a linear mean. Indeed, for every \( x, y \in \mathbb{R} \) (cf. [1], Theor. 1, pg. 234)
\[ M(x, y) = M(0(y - x) + x, 1(y - x) + x) = M(0, 1)(y - x) + x. \]

Note that no hypothesis was made on the regularity of \( M \). Furthermore, note that the same result is true whenever \( M \) is a strict mean defined on an interval \( I \) such that the inclusion \( \text{Aff}(I) \subseteq \mathcal{A}(M; I) \) holds. This fact quickly follows from the equality
\[ M(x, y) = \frac{M(x_0, y_0) - x_0}{y_0 - x_0} (y - x) + x, \quad x, y \in I, \]
where \( x_0, y_0 \in I \), \( x_0 < y_0 \). Unfortunately, this is no longer true when the number of variables is greater than 2 (cf. [1], pg. 237]): given the three-variable linear means \( L_1, L_2 \) and \( L_3 \), with \( L_i \neq L_j \) at least for a pair \( i, j \), \( i \neq j \), the (continuous) strict mean \( M(x, y, z) \) defined on \( \mathbb{R} \) by \( M(x, x, x) = x, \quad x \in \mathbb{R} \), and by
\[ M(x, x, y) = L_3(x, x, y), \quad M(x, y, x) = L_2(x, y, x), \quad M(y, x, x) = L_1(y, x, x), \quad (10) \]
when \( x, y \in \mathbb{R} \), \( x \neq y \), and by
\[ M(x, y, z) = \frac{(x - y)^2 e^{-(x - z)^2/4} L_1(x, y, z) + (y - z)^2 e^{-(x - z)^2/4} L_2(x, y, z) + (z - x)^2 e^{-(x - z)^2/4} L_3(x, y, z)}{(x - y)^2 e^{-(x - z)^2/4} + (y - z)^2 e^{-(x - z)^2/4} + (z - x)^2 e^{-(x - z)^2/4}} \quad (11) \]
when \( x, y, z \in \mathbb{R} \), \( x \neq y \), \( y \neq z \), \( z \neq x \), serves as a counterexample. However, one can prove the following:
Proposition 1. Let $I$ be a real interval with $\text{int}(I) \neq \emptyset$ and $M \in \mathcal{CM}_n(I)$ be a strict mean such that $\text{Aff}(I) \subseteq \mathcal{A}(M; I)$. If $M$ is differentiable at a point of the diagonal of $(\text{int}(I))^n$, then $M$ is a linear mean.

Note that a mean fulfilling the hypotheses of the proposition is not only continuous but also differentiable at every point of the diagonal of $I^n$.

Proof. First of all observe that, regardless of the interval $I$, \( \{0\} \times I \subseteq \text{AFF}(I) \) and \((m, h_0) \in \text{AFF}(I)\) provided that $h_0 \in \text{int}(I)$ and $m > 0$ is small enough. Thus, if \((t_0, \ldots, t_0) \in (\text{int}(I))^n\) is the point at which $M$ is differentiable and $x_1, \ldots, x_n \in I$ are fixed, the map

\[
u \mapsto M(ux_1 + t_0, \ldots, ux_n + t_0),
\]

(which is, by the former observation, defined and continuous on an interval of the form \([0, \delta)\) \((\delta > 0)\)) has a right-hand derivative $D^+M$ at $u = 0$ given by

\[
D^+M(ux_1 + t_0, \ldots, ux_n + t_0)|_{u=0} = \sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(t_0, \ldots, t_0) x_i.
\]

Now, by the assumptions we can write

\[
M(ux_1 + t_0, \ldots, ux_n + t_0) = uM(x_1, \ldots, x_n) + t_0, \quad u \in [0, \delta),
\]

and taking (right-hand) derivatives at $u = 0$ in this equality,

\[
\sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(t_0, \ldots, t_0) x_i = M(x_1, \ldots, x_n).
\] (12)

Since $M$ is a strict mean, the weights in the left hand side of equality (12) must satify

\[
\sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(t_0, \ldots, t_0) = 1,
\]

and

\[
\frac{\partial M}{\partial x_i}(t_0, \ldots, t_0) > 0, \quad i = 1, \ldots, n,
\]

so that $M$ is a linear mean. \(\Box\)

In a noteworthy result by J. Matkowski (cf. Theorem 3 in [17]), two-variable quasilinear means are characterized as strict means $M$ such that their corresponding semigroups $\mathcal{A}(M; I)$ are, in a certain sense, extensive. Let us quote this theorem as follows:

Theorem 2. (J. Matkowski, 2003) Let $I \subseteq \mathbb{R}$ be an open interval and $M$ be a two-variable strict mean defined on $I$. Suppose that $\mathcal{A}(M; I)$ contains a continuous (multiplicative) iteration group \(\{f^t : t > 0\}\) with generator $\gamma$. Furthermore, suppose that the function $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by

\[
h(u) = \gamma \left(M(\gamma^{-1}(u), \gamma^{-1}(1))\right), \quad u > 0,
\]

where $\gamma$ is a strictly increasing function on $\mathbb{R}$.
is twice differentiable and satisfies $0 \neq h'(1) \neq 1$. If there exists a continuous
at a point $M$-affine function that is neither constant nor an element of the
iteration group $\{f^t : t > 0\}$, then

$$M(x, y) = \phi^{-1}(w\phi(x) + (1 - w)\phi(y)), \quad x, y \in I,$$

where $\phi : I \to \mathbb{R}^+$ is a continuous and strictly monotonic function and $w = h'(1)$.

In this paper, a class of functions $F$ will be considered an extensive one
whenever there exists a function $f \in F$ passing through every pair of points.
More precisely, if $I = [a, b]$, $(a, b \in \mathbb{R},$ $a < b)$ and $J \neq \emptyset$ are two real intervals
and $F \subseteq J^I = \{f | f : I \to J\}$, $F \neq \emptyset$, is a family of functions, then let us
say that $F$ is a Sturm–Liouville family when, for every $\alpha, \beta \in J$, there exists
$f \in F$ such that

$$f(a) = \alpha \text{ and } f(b) = \beta.$$

The terminology comes from the denomination of the boundary conditions
in the theory of boundary value problems for second order linear differential
equations. When $I = [a, b]$ and $J = \mathbb{R}$, the family consisting of all monotonic
functions $f : [a, b] \to \mathbb{R}$ is a Sturm–Liouville family. If $J$ reduces to the single
point $\{c\}$, then the unitary set $F = \{f \equiv c\}$ is also a Sturm–Liouville family
(regardless of the interval $I$). It should be observed that the property of being a
Sturm–Liouville family is invariant under conjugacy: given a homeomorphism
$\phi : [a, b] \to \mathbb{R}$, a family $F$ is a Sturm–Liouville family of functions defined
on $[a, b]$ if and only $\phi \circ F \circ \phi^{-1} = \{\phi \circ f \circ \phi^{-1} : f \in F\}$ is a Sturm–Liouville
family of functions defined on $\phi([a, b])$.

A remarkable example of a Sturm–Liouville family is furnished by the semi-
group $AC(L\phi; J)$ corresponding to the $n$-variable quasilinear mean $L\phi$ with
generator $\phi : J \to \mathbb{R}$ and weights $w_i$, $i = 1, \ldots, n$. In fact, observe that equation
(6) takes, in this case, the form

$$f\left(\phi^{-1}\left(\sum_{i=1}^{n} w_i\phi(x_i)\right)\right) = \phi^{-1}\left(\sum_{i=1}^{n} w_i\phi(f(x_i))\right), \quad x_i \in J, \quad i = 1, \ldots, n,$$

(13)

for an unknown function $f : J \to J$. Replacing $g = \phi \circ f \circ \phi^{-1} : \phi(J) \to \phi(J)$
in (13), reduces it to the equation

$$g\left(\sum_{i=1}^{n} w_it_i\right) = \sum_{i=1}^{n} w_ig(t_i), \quad t_i \in \phi(J), \quad i = 1, \ldots, n,$$

whose general continuous solution is given by

$$g(t) = mt + h, \quad t \in \phi(J),$$

(14)

where $m, h$ are real constants such that $g(t) \in \phi(J), \quad t \in \phi(J)$, (this is a
simple consequence of [1], Theor. 2, pg. 67 or also [14], pg. 382 and ff.) and
hence, a solution $f$ to Eq. (13) must have the form
$$f (x) = \phi^{-1} (m \phi (x) + h), \ x \in J,$$
where $(m, h) \in \text{AFF} (\phi (J))$. A substitution of (15) in (13) shows that (15) really solves this equation, so that (cf. [2], Chap. 15, Prop. 6, for the case $n = 2$
and $M$ symmetric)
$$\text{AC} (L_{\phi}; J) = \{ f : f (x) = \phi^{-1} (m \phi (x) + h), \ x \in J, (m, h) \in \text{AFF} (\phi (J)) \}. \tag{16}$$
A straightforward consequence of this characterization of $\text{AC} (L_{\phi}; J)$ is the following:

**Proposition 3.** Let $J$ be a real interval and $L_{\phi}$ be a quasilinear mean defined on $J$ with generator $\phi : J \to \mathbb{R}$; then, for every compact interval $[a, b] \subseteq J$, the family of restrictions $\{ f|_{[a,b]} : f \in \text{AC} (L_{\phi}; J) \}$ ($= \text{AC} (L_{\phi}; [a, b], J)$) is a Sturm–Liouville family.

**Proof.** It is sufficient to observe that, for any pair of numbers $\alpha, \beta \in J$, the system of equations
$$\begin{align*}
\phi^{-1} (m \phi (a) + h) &= \alpha \\
\phi^{-1} (m \phi (b) + h) &= \beta,
\end{align*} \tag{17}$$
has the solution
$$m = \frac{\phi (\beta) - \phi (\alpha)}{\phi (b) - \phi (a)}, \ h = \phi (\alpha) - \frac{\phi (\beta) - \phi (\alpha)}{\phi (b) - \phi (a)} \phi (a),$$
and that the pair $(m, h)$ given by (17) is really a member of $\text{AFF} (\phi ([a, b])).$ \hfill $\square$

**Remark 4.** In terms of the increasing homeomorphism $\psi : [a, b] \to [0, 1]$ given by
$$\psi (x) = \frac{\phi (x) - \phi (a)}{\phi (b) - \phi (a)}, \ x \in [a, b], \tag{18}$$
the $L_{\phi}$-affine function passing through $(a, \alpha)$ and $(b, \beta)$ takes the form
$$\begin{align*}
\phi^{-1} \left( \frac{\phi (\beta) - \phi (\alpha)}{\phi (b) - \phi (a)} (\phi (x) - \phi (a)) + \phi (\alpha) \right) \\
&= \psi^{-1} \left( \frac{\phi (\beta) - \phi (\alpha)}{\phi (b) - \phi (a)} (\phi (x) - \phi (a)) + \phi (\alpha) - \phi (a) \right) \\
&= \psi^{-1} \left( \frac{\phi (\beta) - \phi (\alpha)}{\phi (b) - \phi (a)} \phi (x) - \phi (a) + \phi (\alpha) - \phi (a) \right) \\
&= \psi^{-1} ((\psi (\beta) - \psi (\alpha)) \psi (x) + \psi (\alpha)),
\end{align*}$$
so that, from the proof of Prop. 3 it is seen that
$$\text{AC} (L_{\phi}; [a, b]) = \{ \psi^{-1} ((\psi (\beta) - \psi (\alpha)) \psi (\cdot) + \psi (\alpha)) : \alpha, \beta \in [a, b] \},$$
where \( \psi : [a, b] \to [0, 1] \) is given by (18).

Now, assume that \( M \) is a strict and continuous mean defined on \( J \) such that, for every compact interval \([a, b] \subseteq J\), the family \( \mathcal{A}C(M; [a, b]) \) (or even \( \mathcal{A}C(L_\psi; [a, b], J) \)) is a Sturm–Liouville family. Must \( M \) be a quasilinear mean? This paper is addressed to answer this question. Concretely, along Sects. 2 and 3, a proof of the following result will be developed.

**Theorem 5.** Let \( M \in \mathcal{C}M_n([a, b]) \) be a strict and continuous mean defined on a compact interval \([a, b]\). If \( \mathcal{A}C(M; [a, b]) \) is a Sturm–Liouville family, then there exists a unique increasing homeomorphism \( \psi : [a, b] \to [0, 1] \) from \([a, b]\) onto \([0, 1]\) such that, for the conjugated mean \( M_\psi \) defined on \([0, 1]\) by

\[
M_\psi (x_1, \ldots, x_n) = \psi \left( M \left( \psi^{-1}(x_1), \ldots, \psi^{-1}(x_n) \right) \right),
\]

the semigroup \( \mathcal{A}C(M_\psi; [0, 1]) \) coincides with \( \text{Aff}([0, 1]). \)

In other words, when for a certain strict continuous mean \( M, \mathcal{A}C(M; [a, b]) \) is a Sturm–Liouville family, then, there exists a unique increasing homeomorphism \( \psi : [a, b] \to [0, 1] \) from \([a, b]\) onto \([0, 1]\) such that every \( f \in \mathcal{A}C(M; [a, b]) \) can be represented in the form

\[
f(t) = \psi^{-1}(m\psi(t) + h), \quad t \in [a, b],
\]

with \((m, h) \in \text{Aff}([0, 1])\). Remarkably, when \( n = 2 \), Theorem 5 implies that \( M_\psi \) is a linear mean, so that \( M \) turns out to be a quasilinear mean and then, the converse of Proposition 3 turns out to be true in this case. Now, what happens if the interval \( I \) is not compact? In Sect. 4, the following result will be shown.

**Theorem 6.** Let \( M \in \mathcal{C}M_2(I) \) be a two-variable mean defined on a real interval \( I \). If \( \{[a_k, b_k] : k \in \mathbb{N}\} \) is a sequence of nested \( ([a_k, b_k] \subseteq [a_{k+1}, b_{k+1}], \ k \in \mathbb{N}) \) and exhaustive \( (\bigcup_k [a_k, b_k] = I) \) compact subintervals of \( I \) such that \( \mathcal{A}C \left( \left(M|_{[a_k, b_k]}, [a_k, b_k]\right) \right) \) is a Sturm–Liouville family for every \( k \in \mathbb{N} \), then \( M \) is a quasilinear mean.

Other consequences of Theorem 5 for two-variable means are explained in Sect. 4. Among them, a special mention is in place of the characterization of two-variable quasilinear means through the theory of bases, which is now presented by setting aside the differentiability hypothesis imposed on the means in [3]. Section 5 is devoted to studying the case of \( n \)-variable means. The following result, which can be considered as an ample generalization of Proposition 1, will be shown there.

**Theorem 7.** Let \( M \in \mathcal{C}M_n(I) \) be an \( n \)-variable, strict and continuous mean defined on a real interval \( I \). If any of the following conditions:

A) \( \mathcal{A}C(M; [a, b]) \) is a Sturm–Liouville family for every compact subinterval \([a, b]\) of \( I; \)

B) \( \mathcal{A}C(M; [a_k, b_k], I) \) is a Sturm–Liouville family for a sequence \( \{ [a_k, b_k] : k \in \mathbb{N} \} \) of nested and exhaustive compact subintervals of \( I \); holds, then there exists a strictly increasing and continuous function \( \phi : I \to \mathbb{R} \) such that

\[
\mathcal{A}C(M_\phi; \phi(I)) = Aff(\phi(I)).
\] (20)

If \( \psi : I \to \mathbb{R} \) is another strictly increasing and continuous function satisfying (20), then \( \psi = m\phi + h \) for certain \( m, h \in \mathbb{R}, m > 0 \).

The final Sect. 6 serves to provide some examples and remarks. In particular, the use of the above results in solving the functional equation (6) will be illustrated there.

2. Continuous \( M \)-affine functions constituting a Sturm–Liouville family

A useful tool in the study of continuous \( M \)-affine functions are the Aczel dyadic iterations of a two-variable mean \( M \). Concretely, denoting the set of dyadic numbers of the interval \([0, 1]\) by \( \mathcal{D}([0, 1]) \), a family of means \( \{ M^d : d \in \mathcal{D}([0, 1]) \} \) is defined on \( I^2 \) as follows: fix \( x, y \in I \) and set

\[
M^{(0)}(x, y) = x, \quad M^{(1)}(x, y) = y;
\]
then, assuming that \( M^{(k)}(x, y) \) is known for \( n \geq 0 \) and every \( 0 \leq j \leq 2^n \), define

\[
M^{\left(\frac{k}{2^n}\right)}(x, y) = \begin{cases} 
M^{\left(\frac{k}{2^n}\right)}(x, y), & k = 2h, \ 0 \leq h \leq 2^n \\
M^{\left(\frac{k}{2^n}\right)}(x, y), M^{\left(\frac{k+1}{2^n}\right)}(x, y), & k = 2h + 1, \ 0 \leq h \leq 2^n - 1.
\end{cases}
\]

In the following result, whose proof can be found in [3] (see also [4] and [5]), the main properties of Aczel dyadic iterations are established.

**Theorem 8.** a) Let \( I \) and \( J \) be two real intervals and \( M \in \mathcal{M}_2(I) \) and \( N \in \mathcal{M}_2(J) \). If the equality

\[
f(M(x, y)) = N(f(x), f(y)), \quad x, y \in I,
\]
holds for any \( x, y \in I \), then, for every \( d \in \mathcal{D}([0, 1]) \),

\[
f(M^{(d)}(x, y)) = N^{(d)}(f(x), f(y)).
\]

b) If \( M \) is a strict continuous mean, the map \( \mathcal{D}([0, 1]) \ni d \mapsto M^{(d)}(x, y) \) can be continuously extended to the interval \([0, 1]\). Moreover, the extension \( \phi_{(x,y)}(\delta) = M^{(\delta)}(x, y) \) is a homeomorphism from \([0, 1]\) onto \([\min(x, y), \max(x, y)]\) which turns out to be increasing when \( x < y \) and decreasing when \( x > y \).
c) For each $\delta \in (0,1)$, $M^{(\delta)}$ is a continuous strict mean defined on $I$ (while $M^{(0)}(x, y) \equiv x$ and $M^{(1)}(x, y) \equiv y$).

For example, if $M(x, y) = A_f(x, y)$ is a quasiarithmetic mean with generator $f$, then it is inductively shown that

$$A_f^{(d)}(x, y) = f^{-1}((1-d)f(x) + df(y)), \ x, y \in I,$$

for every $d \in D ([0,1])$, whence it is easily deduced that

$$A_f^{(\delta)}(x, y) = f^{-1}((1-\delta)f(x) + \delta f(y)), \ x, y \in I, \quad (21)$$

for every $\delta \in [0,1]$.

As a first application of Aczel dyadic iterations, let us prove the following:

**Proposition 9.** Let $M \in \mathcal{CM}_n(J)$ be a continuous and strict mean defined on a real interval $J$ and consider a compact subinterval $[a,b] \subseteq J$. Furthermore, for given $\alpha, \beta \in [a,b]$, let $f$ be a continuous $M$-affine function such that $f(a) = \alpha$ and $f(b) = \beta$. Then $f|_{[a,b]}$ is a homeomorphism from $[a,b]$ onto $[\min(\alpha, \beta), \max(\alpha, \beta)]$.

**Proof.** In the first place, let us consider the case $n = 2$. From parts a)-b) of Theor. 8 and the continuity of $f$, we can write in this case

$$f\left(M^{(\delta)}(x, y)\right) = M^{(\delta)}(f(x), f(y)), \ x, y \in [a,b], \ \delta \in [0,1],$$

whence, setting $x = a$, $y = b$, and using the notation introduced in Theor. 8-b), we derive

$$f\left(\phi_{(a,b)}(\delta)\right) = \phi_{(a,\beta)}(\delta), \ \delta \in [0,1],$$

or, by substituting $u = \phi_{(a,b)}(\delta)$,

$$f(u) = (\phi_{(a,\beta)} \circ \phi_{(a,b)}^{-1})(u), \ u \in [a,b]. \quad (22)$$

This expression and Theor. 8-b) shows that $f|_{[a,b]}$ is a homeomorphism from $[a,b]$ onto $[\min(\alpha, \beta), \max(\alpha, \beta)]$, as stated. Now, if $M \in \mathcal{CM}_n(J)$, let us define a two-variable mean $N$ by

$$N(x, y) = M(x, y, ..., y), \ x, y \in J. \quad (23)$$

Clearly $N$ is a strict and continuous mean, and if $f$ is $M$-affine, then it is also $N$-affine. Hence, the general case follows from the case $n = 2$. This completes the proof. \qed

Like in the previous proposition, consider a continuous and strict mean $M \in \mathcal{CM}_n(J)$ and suppose, for a given compact subinterval $[a,b] \subseteq J$, that $\mathcal{AC}(M; [a,b])$ is a Sturm–Liouville family; i.e., for every $\alpha, \beta \in [a,b]$ there exists $f \in \mathcal{AC}(M; [a,b])$ such that $f(a) = \alpha$ and $f(b) = \beta$. It is asserted that this $f$ is unique. In fact, if $g \in \mathcal{AC}(M; [a,b])$ was another $M$-affine function satisfying $g(a) = \alpha$ and $g(b) = \beta$, then there would exist $t_0 \in (a,b)$ such that
Proof. After the previous discussion, it remains to prove only that $f \neq g$ for every $t$ in a maximal open neighborhood $(c, d)$ of $t_0$. Since the equalities $f(c) = g(c)$ and $f(d) = g(d)$ hold by the maximality of $(c, d)$, we can write

$$f(M(c, c, \ldots, c, d)) = M(f(c), f(c), \ldots, f(c), f(d))$$

$$= M(g(c), g(c), \ldots, g(c), g(d))$$

$$= g(M(c, c, \ldots, c, d)),$$

which, taking into account that $M(c, c, \ldots, c, d) \in (c, d)$ by the strict internality of $M$, is a contradiction. This proves the above assertion and justifies the use of the notation $f_{\alpha, \beta}$ for the unique $f \in AC(M; [a, b])$ satisfying $f(a) = \alpha$ and $f(b) = \beta$. Now, let us show that $\alpha \mapsto f_{\alpha, \beta}(t)$ is a continuous and monotonic function on $[a, b]$. Indeed, in view of (22) in the proof of Proposition 9, it can be written that

$$f_{\alpha, \beta}(t) = \phi_{(\alpha, \beta)} \left( \phi_{(a, b)}^{-1}(t) \right) = M_{\phi_{(a, b)}^{-1}(t)}(\alpha, \beta),$$

so that the continuity of $\alpha \mapsto f_{\alpha, \beta}(t)$ follows from Theor. 8, c). In order to prove the monotonicity, let us consider $\alpha, \alpha' \in [a, b]$, $\alpha < \alpha'$, and suppose that there exists $t_0 \in (a, b)$ such that $f_{\alpha, \beta}(t_0) > f_{\alpha', \beta}(t_0)$. Since $f_{\alpha, \beta}(a) = \alpha < \alpha' = f_{\alpha', \beta}(a)$ and $f_{\alpha, \beta}$ and $f_{\alpha', \beta}$ are both continuous functions, there exists $c \in (a, t_0)$ such that $f_{\alpha, \beta}(c) = f_{\alpha', \beta}(c)$ and therefore, an argument like that used above to prove the uniqueness of $f_{\alpha, \beta}$ shows that $f_{\alpha, \beta}(t) = f_{\alpha', \beta}(t)$, $t \in [c, b]$. Since $t_0 > c$, this is in contradiction to the former assumption and thus $f_{\alpha, \beta}(t) \leq f_{\alpha', \beta}(t)$, $t \in [a, b]$. Since $f_{\alpha, \beta}(b) = \beta = f_{\alpha', \beta}(b)$, $\alpha \mapsto f_{\alpha, \beta}(t)$ is monotonic for every $t \in [a, b]$.

Summarizing the above discussion, the following result can be established.

**Proposition 10.** Let $M \in CM_n(J)$ be a continuous and strictly mean and suppose, for a given compact subinterval $[a, b] \subseteq J$, that $AC(M; [a, b])$ is a Sturm–Liouville family. Then, there exists a unique function $f_{\alpha, \beta} \in AC(M; [a, b])$ such that $f_{\alpha, \beta}(a) = \alpha$ and $f_{\alpha, \beta}(b) = \beta$. Furthermore, the functions $\alpha \mapsto f_{\alpha, \beta}(t)$ and $\beta \mapsto f_{\alpha, \beta}(t)$ turn out to be monotonic and continuous on $[a, b]$.

**Proof.** After the previous discussion, it remains prove only that $\beta \mapsto f_{\alpha, \beta}(t)$ is monotonic and continuous on $[a, b]$. This is an immediate consequence of the representation

$$f_{\alpha, \beta}(t) = f_{\beta, \alpha}(f_{b, a}(t))$$

and the corresponding properties of $\alpha \mapsto f_{\alpha, \beta}(t)$. \quad \Box

Under the hypotheses of Proposition 10 and remembering that $AC(M; [a, b])$ is a semigroup, it turns out that, for every $\alpha_i, \beta_i \in [a, b], i = 1, 2$, there exists a unique pair $\alpha, \beta \in [a, b]$ such that

$$f_{\alpha_1, \beta_1} \circ f_{\alpha_2, \beta_2} = f_{\alpha, \beta}.$$
Since
\[ \alpha = f_{\alpha,\beta} (a) = (f_{\alpha_1,\beta_1} \circ f_{\alpha_2,\beta_2}) (a) = f_{\alpha_1,\beta_1} (f_{\alpha_2,\beta_2} (a)) = f_{\alpha_1,\beta_1} (\alpha_2) \]
and, similarly,
\[ \beta = f_{\alpha,\beta} (b) = f_{\alpha_1,\beta_1} (\beta_2), \]
for every \( \alpha_i, \beta_i \in [a, b], \ i = 1, 2 \), we can write
\[ f_{\alpha_1,\beta_1} \circ f_{\alpha_2,\beta_2} = f_{f_{\alpha_1,\beta_1}(\alpha_2),f_{\alpha_1,\beta_1}(\beta_2)}. \]
(24)

Now, consider the function \( F : [a, b]^3 \to [a, b] \) defined by
\[ F (t, \alpha, \beta) = f_{\alpha,\beta} (t). \]
(25)

**Proposition 11.** Let \( M \in \mathcal{CM}_n (J) \) be a strict and continuous mean and suppose that, for a given compact subinterval \( [a, b] \subseteq J \), \( \mathcal{AC} (M; [a, b]) \) is a Sturm–Liouville family. If \( f_{\alpha,\beta} \in \mathcal{AC} (M_{|[a,b]}; [a, b]) \) is the unique \( M \)-affine function satisfying \( f_{\alpha,\beta} (a) = \alpha \) and \( f_{\alpha,\beta} (b) = \beta \), then the function \( F \) defined by (25) is a solution to the composite functional equation
\[ F (F (t, \alpha_1, \beta_1), \alpha_2, \beta_2) = F (t, F (\alpha_1, \alpha_2, \beta_2), F (\beta_1, \alpha_2, \beta_2)), \]
for every \( t, \alpha_i, \beta_i \in [a, b], \ i = 1, 2 \), (26)
in the class consisting of functions with the following properties:

i) \( F \) is continuous;

ii) \( F (t, \alpha, \beta) \) is monotonic with respect to each variable, and strictly monotonic with respect to the variable \( t \) provided that \( \alpha \neq \beta \) \((t \mapsto F (t, \alpha, \beta) \) is strictly increasing when \( \alpha < \beta \) and strictly decreasing when \( \alpha > \beta \));

iii) \( F (a, \alpha, \beta) = \alpha \) and \( F (b, \alpha, \beta) = \beta \).

**Proof.** The functional equation (26) is a rewriting of (24) using (25). ii) and iii) are immediate consequences of (25) and Propositions 9 and 10 (the strict monotonicity of \( t \mapsto F (t, \alpha, \beta) \) follows from the representation (22) and Theor. 8). Regarding to i), observe that the function \( F : [a, b]^3 \to [a, b] \) is separately continuous and monotonic in each variable and therefore, \( F \) is continuous. In fact, the argument employed by Kruse and Deely in [13], Prop. 2, to prove joint continuity on a given open set can be easily extended to prove joint continuity on the cube \( [a, b]^3 \).

After this result, Proposition 3 and Remark 4 imply that

\[ F (t, \alpha, \beta) = \psi^{-1} ((\psi (\beta) − \psi (\alpha)) \psi (t) + \psi (\alpha)), \]

where \( \psi : [a, b] \to [0, 1] \) is an increasing homeomorphism, must be a solution to the functional equation (26) in the class of functions satisfying the properties i), ii) and iii). A direct checking of this fact is an easy task. As it will be seen in the next section, this expression really provides the general solution to (26).
3. The functional equation (26)

The purpose of this section is to prove the following:

**Theorem 12.** The general solution to the functional equation (26) in the class of functions fulfilling conditions i), ii) and iii) is given by

\[ F(t, \alpha, \beta) = \psi^{-1}((\psi(\beta) - \psi(\alpha)) \psi(t) + \psi(\alpha)), \]

where \( \psi : [a, b] \to [0, 1] \) is an increasing homeomorphism onto \([0, 1]\). \( \psi \) is the unique increasing homeomorphism satisfying (27).

A proof of Theorem 5 will easily follow from this result.

As a first observation note that, since \( f_{a,b} \) must be an increasing homeomorphism onto \([a, b]\) and \( f_{a,b}^2 = f_{a,b} \circ f_{a,b} \) (idempotency), it turns out that \( f_{a,b} = \text{id} \), the identity on \([a, b]\). On the other hand, \( f_{b,a}^2 = f_{b,a} \circ f_{b,a} = f_{a,b} \) (idempotency), so that \( f_{b,a} = \text{id} \), the identity on \([a, b]\). Furthermore, by (24), a generic \( f_{\alpha,\beta} \in \mathcal{A}C(M; [a, b]) \) can be written as a product \( f_{\alpha,b} \circ f_{a,\beta} \) of the “boundary elements” \( f_{\alpha,b} \), \( f_{a,\beta} \), and thus, the whole semigroup \( \mathcal{A}C(M; [a, b]) \) can be reconstructed when \( f_{\alpha,b} \) and \( f_{a,\beta} \) are known for every \( \alpha, \beta \in [a, b] \). This fact is the basis of the following:

**Proposition 13.** If \( F : [a, b]^3 \to [a, b] \) is a solution to the functional equation (26) satisfying conditions i), ii) and iii), then \( F \) can be written in the form

\[ F(t, \alpha, \beta) = \begin{cases} H(H(t, \Psi(\beta, \alpha)), \alpha), & \alpha \leq \beta \\ H(G(t, \Psi(\alpha, \beta)), \beta), & \alpha > \beta \end{cases}, \quad t, \alpha, \beta \in [a, b], \]

where \( H, G : [a, b]^2 \to [a, b] \) are solutions to the system of functional equations

\[ \begin{cases} G(G(t, \alpha), \beta) = G(t, G(\alpha, \beta)) \\ G(H(t, \alpha), \beta) = H(t, G(\alpha, \beta)) \end{cases}, \quad t, \alpha, \beta \in [a, b], \]

which are continuous, monotonic in both variables and strictly monotonic in the first variable when \( \alpha \neq b \), while \( \Psi \) is a continuous function implicitly defined by

\[ H(\Psi(t, \alpha), \alpha) = t, \quad \alpha \leq t \leq b, \quad a \leq \alpha \leq b. \]

**Proof.** If \( F : [a, b]^3 \to [a, b] \) is a solution to (26) satisfying conditions i), ii) and iii), and the functions \( G \) and \( H \) are respectively defined by

\[ G(t, \alpha) = F(t, \alpha, b), \quad t, \alpha \in [a, b], \]

and

\[ H(t, \alpha) = F(t, b, \alpha), \quad t, \alpha \in [a, b], \]

then both \( G \) and \( H \) turn out to be continuous on \([a, b]^2\) by condition i), while condition ii) shows that \( G \) and \( H \) must be monotonic functions in both variables. Moreover, \( t \mapsto G(t, \alpha) \) is strictly monotonic for every \( \alpha \neq b \) and the
same is true for \( t \mapsto H(t, \alpha) \) (but \( G(t, b) \equiv b \equiv H(t, b) \)). For this reason, the (continuous) function \( \Psi \) defined by
\[
\Psi(t, \alpha) = \begin{cases} 
  f_{b, \alpha}^{-1}(t), & a \leq \alpha < b \\
  b, & \alpha = b, \ a \leq t \leq b,
\end{cases}
\]
is the unique solution to Eq. (30). Now, let us prove that system (29) is solved by the above defined functions \( G \) and \( H \). In fact, from (26) and condition iii it is deduced that
\[
G(G(t, \alpha), \beta) = F(F(t, \alpha, b), \beta, b) = F(t, F(\alpha, \beta, b), F(b, \beta, b)) = F(t, F(\alpha, \beta, b), b) = G(t, G(\alpha, \beta)), \ t, \alpha, \beta \in [a, b].
\]
Analogously, it can be written that
\[
G(H(t, \alpha), \beta) = F(F(t, b, \alpha), \beta, b) = F(t, F(b, \beta, b), F(\alpha, \beta, b)) = F(t, b, F(\alpha, \beta, b)) = H(t, G(\alpha, \beta)), \ t, \alpha, \beta \in [a, b].
\]
It remains to prove that, in terms of \( G \) and \( H \), \( F \) is expressed by (28). To this end, first consider the case \( \alpha \leq \beta \); thus, from (26) and condition iii, it is derived that
\[
H(H(t, \beta_1), \alpha) = F(F(t, b, \beta_1), b, \alpha) = F(t, F(b, \beta_1, b, \alpha)) = F(t, \alpha, F(\beta_1, b, \alpha)) = F(t, \alpha, H(\beta_1, \alpha)), \ t, \alpha, \beta_1 \in [a, b];
\]
whence, introducing \( \beta_1 = \Psi(\beta, \alpha) \) and taking (30) into account, we obtain
\[
H(H(t, \Psi(\beta, \alpha)), \alpha) = F(t, \alpha, H(\Psi(\beta, \alpha), \alpha)) = F(t, \alpha, \beta).
\]
Similarly, when \( \alpha > \beta \), it can be written that
\[
H(G(t, \alpha_1), \beta) = F(F(t, \alpha_1, b), b, \beta) = F(t, F(\alpha_1, b, \beta), F(b, b, \beta)) = F(t, F(\alpha_1, b, \beta), \beta) = F(t, H(\alpha_1, \beta), \beta), \ t, \alpha_1, \beta \in [a, b];
\]
and the substitution \( \alpha_1 = \Psi(\alpha, \beta) \) gives
\[
H(G(t, \Psi(\alpha, \beta)), \beta) = F(t, H(\Psi(\alpha, \beta), \beta), \beta) = F(t, \alpha, \beta).
\]
This completes the proof. \(\Box\)
Remark 14. Note that the function $G$ is really increasing in both variables and strictly increasing in the first variable when $\alpha \neq b$. In its turn, $H$ is strictly decreasing in the first variable when $\alpha \neq b$, while it is increasing in the second variable.

In the next paragraph, the system of composite Eq. (29) is to be solved. The first equation in this system is no other than the associativity equation. Fortunately, its solution in our setting is furnished by a result due to C. H. Ling (see [15], Main Theorem, or also [16] Theor. 3.2). In the next paragraphs, $\mathbb{R}$ and $[0, +\infty]$ will stand respectively for the sets of extended real numbers and nonnegative extended real numbers.

**Theorem 15.** (C. H. Ling, 1965) Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval. A function $\Gamma : I \times I \to I$ is an associative function satisfying the following conditions: $\Gamma$ is continuous, increasing in both variables, the endpoint $a$ is a left unit (i.e., $\Gamma (a, \alpha) = \alpha$ for all $\alpha$ in $I$) and, for every $\alpha \in (a, b)$, $\Gamma (\alpha, \alpha) > \alpha$, if and only if there exists a continuous and strictly increasing function $f : I \to [0, +\infty]$ with $f (a) = 0$, such that

$$\Gamma (t, \alpha) = f^{-1} (\min (f (t) + f (\alpha), f (b))), \quad t, \alpha \in I.$$  \hspace{1cm} (31)

In order to solve the second equation in (29), let us substitute the expression (31) for $G$ in it to obtain

$$f^{-1} (f (H (t, \alpha)) + f (\beta)) = H (t, f^{-1} (f (\alpha) + f (\beta))), \quad t, \alpha, \beta [a, b].$$

Setting $\xi = f (t), \eta = f (\alpha), \zeta = f (\beta)$ and introducing the function $K : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ defined by $K (\xi, \eta) = f (H (f^{-1} (\xi), f^{-1} (\eta)))$, this equation can be written as

$$K (\xi, \eta) + \zeta = K (\xi, \eta + \zeta), \quad \xi, \eta, \zeta \in \mathbb{R}^+_0,$$

and then

$$K (\xi, \eta) + \zeta = K (\xi, \eta + \zeta) = K (\xi, \zeta + \eta) = K (\xi, \zeta) + \eta,$$

or, equivalently,

$$K (\xi, \eta) - \eta = K (\xi, \zeta) - \zeta, \quad \xi, \eta, \zeta \in \mathbb{R}^+_0.$$
In other words, the function \((\xi, \eta) \rightarrow K(\xi, \eta) - \eta\) depends only on \(\xi\); i.e., there exists \(p : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+\) such that

\[
K(\xi, \eta) = p(\xi) + \eta,
\]

and therefore

\[
H(t, \alpha) = f^{-1}(p(f(t)) + f(\alpha)).
\]  

(32)

The replacement \(\alpha = a\) in the last expression produces

\[
f_{b,a}(t) = H(t, a) = f^{-1}(p(f(t)) + f(a)) = f^{-1}(p(f(t))),
\]

whence it is deduced that \(p\) is a strictly decreasing involutary function satisfying \(p(0^+) = +\infty\) and \(p(+\infty) = 0\). An expression for the function \(\Psi\) of Proposition 13 is promptly derived from (32) in the form

\[
\Psi(t, \alpha) = f^{-1}(p(f(t) - f(\alpha))), \quad \alpha \leq t \leq b, \ a \leq \alpha \leq b.
\]

From the above discussion and Proposition 13, it follows that any solution to Eq. (26) satisfying conditions (i), (ii) and (iii) can be written as

\[
F(t, \alpha, \beta) = \begin{cases} 
  f^{-1}(p(p(f(t)) + p(f(\beta) - f(\alpha))) + f(\alpha)), & \alpha \leq \beta \\
  f^{-1}(p(p(f(t)) + p(f(\alpha) - f(\beta))) + f(\beta)), & \alpha > \beta
\end{cases},
\]  

(33)

where \(f : I \rightarrow [0, +\infty]\) is a continuous and strictly increasing function with \(f(a) = 0\) and \(p\) is a strictly decreasing involutary function satisfying \(p(0^+) = +\infty\) and \(p(+\infty) = 0\). Now, assume that the function \(F\) represented by (33) is a solution to Eq. (26); then, taking \(\alpha_1, \beta_1, \beta_2 \in [a, b]\) with \(\alpha_1 \leq \beta_1\), it can be written that

\[
F(t, \alpha_1, \beta_1) = f^{-1}(p(p(f(t)) + p(f(\beta_1) - f(\alpha_1))) + f(\alpha_1)),
\]

and, in view of \(f(a) = 0\), it follows that

\[
F(F(t, \alpha_1, \beta_1), \alpha_2, \beta_2) = f^{-1}(p(p(p(s) + p(b_1 - a_1)) + a_1) + p(b_2))),
\]  

(34)

where \(s = f(t), a_1 = f(\alpha_1)\) and \(b_i = f(\beta_i), i = 1, 2\). On the other hand,

\[
F(\alpha_1, \alpha_2, \beta_2) = f^{-1}(p(p(f(\alpha_1)) + p(f(\beta_2) - f(\alpha_2))) + f(\alpha_2)),
\]

and

\[
F(\beta_1, \alpha_2, \beta_2) = f^{-1}(p(p(f(\beta_1)) + p(f(\beta_2) - f(\alpha_2))) + f(\alpha_2)),
\]

whence, since \(F(\alpha_1, \alpha_2, \beta_2) \leq F(\beta_1, \alpha_2, \beta_2)\), the following equality is deduced

\[
F(t, F(\alpha_1, \alpha_2, \beta_2), F(\beta_1, \alpha_2, \beta_2)) = f^{-1}(p(p(s) + p(p(b_1) + p(b_2)) - p(p(a_1) + p(b_2)))) + p(p(a_1) + p(b_2))),
\]  

(35)
where again \( s = f(t) \), \( a_1 = f(\alpha_1) \) and \( b_i = f(\beta_i) \), \( i = 1, 2 \). Since the left hand sides of (34) and (35) are equal, their corresponding right hand sides must be equal as well and therefore, the equality

\[
p(p(p(s + p(b_1 - a_1)) + a_1) + p(b_2)) = p(p(s + p(p(b_1) + p(b_2))) - p(p(a_1) + p(b_2)))) + p(p(a_1) + p(b_2)),
\]

must hold for every \( s, a_1, b_1, b_2 \in [a, b] \) or, after the substitutions \( x = b_1 - a_1 \), \( y = a_1 \), \( z = p(b_2) \) and \( s = p(s) \),

\[
p(p(p(s + p(x)) + y) + z) = p(s + p(p(p(x) + y) + z) - p(p(y) + z))) + p(p(y) + z),
\]

where \( x, y, z, s \in \mathbb{R}^+_0 \).

Summarizing the above developments, the following result can be established.

**Proposition 16.** Let \( F : [a, b]^3 \to [a, b] \) be a solution to functional equation (26) satisfying conditions i), ii) and iii). Then, \( F \) can be represented in the form (33), where \( f : I \to [0, +\infty) \) is a continuous and strictly increasing function with \( f(a) = 0 \) (and \( f(+\infty) = +\infty \)) and \( p : [0, +\infty) \to [0, +\infty) \) is a strictly decreasing involutory function with \( p(0^+) = +\infty \) and \( p(+\infty) = 0 \) which solves the functional equation (36).

**Proof.** The proof follows from Proposition 13 and the preceding discussion. \( \square \)

Now, let us pay attention to the functional equation (36). In the first place, observe that in view of the continuity of \( p \) and the fact that \( p(+\infty) = 0 \), taking limits when \( x \uparrow +\infty \) in (36) produces

\[
p(p(p(s) + y) + z) = p(s + p(p(z) - p(p(y) + z))) + p(p(y) + z),
\]

where \( x, y, z, s \in \mathbb{R}^+_0 \). In order to simplify the expressions, let us define a commutative operation (quasisum) \( \Delta : [0, +\infty]^2 \to [0, +\infty] \) by

\[
x \Delta y = p(p(x) + p(y)).
\]

In this way, the substitutions \( s = p(s) \) and \( z = p(z) \) in (37) enables us to write it in the form

\[
(s + y) \Delta z = s \Delta(z - y \Delta z) + y \Delta z.
\]

Note that \( 0 \leq z - y \Delta z \leq z \), with the inequalities strict provided that \( y, z > 0 \).

As it is shown by the following result, the function \( s \mapsto s \Delta z \) has nice properties.

**Lemma 17.** Let \( p : [0, +\infty) \to [0, +\infty) \) be a continuous and strictly decreasing involutory function solving the functional equation (37) and \( \Delta : [0, +\infty]^2 \to [0, +\infty) \) be the quasisum defined by (38). Then, for every \( z \in \mathbb{R}^+_0 \), the function \( s \mapsto s \Delta z \) is strictly subadditive, strictly increasing, strictly concave and continuously differentiable in \( \mathbb{R}^+ \).
By commutativity, the function $z \mapsto s \Delta z$ has the same properties as $s \mapsto s \Delta z$.

**Proof.** Fix $z \in \mathbb{R}^+$ and consider the function $s \mapsto s \Delta z$. Since $p$ is a strictly decreasing function, $s \mapsto s \Delta z$ turns out to be strictly increasing. As a consequence, (39) and the inequality $z - t \Delta z < z$ yields

$$(s + t) \Delta z = s \Delta (z - t \Delta z) + t \Delta z < s \Delta z + t \Delta z, \quad s, t \in \mathbb{R}^+;$$

i.e., $s \mapsto s \Delta z$ is subadditive. To prove the strict concavity of $s \mapsto s \Delta z$, choose a pair $s, t \in \mathbb{R}^+$ with $s \neq t$, say $s < t$; then, a repeated use of (39) produces

$$t \Delta z = \left(\frac{t - s}{2} + \left(\frac{t - s}{2} + s\right)\right) \Delta z$$

$$= \frac{t - s}{2} \Delta \left(z - \left(\frac{t - s}{2} + s\right) \Delta z\right) + \left(\frac{t - s}{2} + s\right) \Delta z$$

$$= \frac{t - s}{2} \Delta \left(z - \frac{t - s}{2} \Delta (z - s \Delta z) - s \Delta z\right) + \frac{t - s}{2} \Delta (z - s \Delta z) + s \Delta z$$

$$< \frac{t - s}{2} \Delta (z - s \Delta z) + \frac{t - s}{2} \Delta (z - s \Delta z) + s \Delta z,$$

where the last inequality holds by the strict monotonicity of $z \mapsto s \Delta z$. On the other hand,

$$\left(\frac{s + t}{2}\right) \Delta z = \left(\frac{s - t}{2} + t\right) \Delta z$$

$$= \frac{s - t}{2} \Delta (z - t \Delta z) + t \Delta z;$$

so that, combining (40) and (41) we deduce

$$t \Delta z < 2 \left(\left(\frac{s + t}{2}\right) \Delta z\right) - s \Delta z,$$

or, equivalently,

$$\left(\frac{s + t}{2}\right) \Delta z > \frac{s \Delta z + t \Delta z}{2}. \quad (42)$$

By symmetry, inequality (42) holds also when $s > t$ and, due to the continuity of $s \mapsto s \Delta z$, it implies the strict concavity of this function.

Now, for every $s \in \mathbb{R}^+$, the existence of the lateral derivatives $D^+_s (s \Delta z)$ and $D^-_s (s \Delta z)$ is ensured by the concavity of $s \mapsto s \Delta z$. In particular, in view of (39), for the right derivative $D^+_s (s \Delta z)$ it can be written that
\[ D^+_s (s\Delta z) = \lim_{t \downarrow 0} \frac{(s + t) \Delta z - s\Delta z}{t} = \lim_{t \downarrow 0} \frac{t \Delta (z - s\Delta z)}{t} = \lim_{t \downarrow 0} \frac{p(t) + p(z - s\Delta z)}{t} = \lim_{u \uparrow +\infty} \frac{p(u + p(z - s\Delta z))}{p(u)}, \quad s \geq 0. \]

The last of these equalities was obtained by replacing \( t = p(u) \). Since \( 0 \leq z - s\Delta z \leq z \) and \( z \in \mathbb{R}^+ \) was arbitrarily chosen, it is concluded that the function

\[ \Phi(\lambda) = \lim_{u \uparrow +\infty} \frac{p(u + \lambda)}{p(u)} \]

is defined for every \( \lambda \geq 0 \). Clearly, \( \Phi \) is decreasing and the equalities

\[ \Phi(\lambda + \mu) = \lim_{u \uparrow +\infty} \frac{p(u + \lambda + \mu)}{p(u)} = \lim_{u \uparrow +\infty} \frac{p(u + \lambda + \mu) p(u + \lambda)}{p(u) p(u + \lambda)} = \lim_{u \uparrow +\infty} \frac{p(u + \lambda + \mu)}{p(u + \lambda)} \lim_{u \uparrow +\infty} \frac{p(u + \lambda)}{p(u)} = \Phi(\lambda) \Phi(\mu), \]

hold for every \( \lambda, \mu \geq 0 \). In other words, \( \Phi \) is a decreasing solution to the exponential Cauchy equation and, in consequence, \( \Phi(\lambda) \equiv 0 \) or \( \Phi(\lambda) \equiv e^{-k\lambda} \) for any \( k \geq 0 \). Indeed, the instances \( \Phi = 0 \) or \( \Phi = 1 \) must be excluded since, in these cases, we would have \( D^+_s (s\Delta z) \equiv 0 \) or \( D^+_s (s\Delta z) \equiv 1 \) and therefore, \( D_s (s\Delta z) \equiv 0 \) or \( D_s (s\Delta z) \equiv 1 \), two identities contradicting the strict concavity of \( s \mapsto s\Delta z \). In this way, there exists \( k > 0 \) such that

\[ D^+_s (s\Delta z) = \lim_{u \uparrow +\infty} \frac{p(u + p(z - s\Delta z))}{p(u)} = e^{-kp(z - s\Delta z)}. \]

This equality shows that \( s \mapsto D^+_s (s\Delta z) \) is continuous on \( \mathbb{R}^+ \) and hence, there exists the standard derivative \( D_s (s\Delta z) \) and

\[ D_s (s\Delta z) = e^{-kp(z - s\Delta z)}, \quad s, z \in \mathbb{R}^+. \quad (43) \]

This completes the proof. \( \square \)

A result on the regularity of solutions to the functional equation (36) is now proved.
Proposition 18. Let \( p : [0, +\infty] \to [0, +\infty] \) be a strictly decreasing involutory function with \( p(0^+) = +\infty \) and \( p(+\infty) = 0 \) which solves the functional equation (36); then \( p \) is continuously differentiable in \( \mathbb{R}^+ \). Moreover, \( p'(0^+) = -\infty \) and \( p'(+\infty) = 0 \).

Proof. Let us denote by \( \text{Diff}(p) \) the set of points where the derivative \( p' \) exists. By Lebesgue’s Theorem, \( \text{Diff}(p) \) contains almost every point of \( \mathbb{R}^+ \) so that, for a given \( s \in \mathbb{R}^+ \), one can chose \( t, z_0 > 0 \) such that \( p(s) + t \) and \( p(p(s) + t) + p(z_0) \) are both in \( \text{Diff}(p) \). Thus, the chain rule applied to \( s \mapsto s\Delta z_0 = p(p(s) + p(z_0)) \) at \( p(s) + t \) yields

\[
D_s (s\Delta z_0)|_{s=p(s)+t} = p' (p(p(s) + t) + p(z_0)) p' (p(s) + t).
\]

Now, in view of (43),

\[
D_s (s\Delta z_0)|_{s=p(s)+t} = e^{-kp(z_0-(p(s)+t)\Delta z_0)} > 0,
\]

so that we must have \( p'(p(s) + t) \neq 0 \), and therefore

\[
\lim_{h \to 0} \frac{p(s+h) - p(s)}{h} = \lim_{h \to 0} \frac{p(p(s+h)+t) - p(p(s)+t)}{p(s+h) - p(s)} = D_s (s\Delta p(t))|_{s} = \frac{D_s (s\Delta p(t))}{p'(p(s) + t)}.
\]

This shows that \( s \in \text{Diff}(p) \), and thus \( \text{Diff}(p) = \mathbb{R}^+ \).

Now, from (38) and (43) it is obtained that

\[
p'(p(s) + p(z)) p'(s) = e^{-kp(z-p(p(s)+p(z)))}, \quad s, z \in \mathbb{R}^+;
\]

or, replacing \( s = p(s) \) and \( z = p(z) \),

\[
p'(s + z) = \frac{e^{-kp(p(z)-p(s+z))}}{p'(p(s))}, \quad (44)
\]

whence the continuity of \( p' \) on \((s, +\infty)\) is easily derived. Since \( s \) can be arbitrarily chosen in \( \mathbb{R}^+ \), \( p' \) turns out to be continuous on \( \mathbb{R}^+ \). Moreover, making \( z \uparrow +\infty \) in (44) yields

\[
p'(+\infty) = \lim_{z \uparrow +\infty} \frac{e^{-kp(p(z)-p(s+z))}}{p'(p(s))} = 0. \quad (45)
\]

Finally, \( p \) being an involutory function, it turns out that

\[
p'(p(s)) p'(s) = 1, \quad s \in \mathbb{R}^+; \quad (46)
\]

whence, in view of (45) and the fact that \( p \) is strictly decreasing, we deduce

\[
p'(0^+) = \lim_{s \downarrow 0} p'(p(s)) = \lim_{s \downarrow 0} \frac{1}{p'(p(s))} = \lim_{z \uparrow +\infty} \frac{1}{p'(z)} = -\infty.
\]

This completes the proof. \( \square \)
It should be noted that an inductive reasoning based on (44) shows that \( p \) is really a \( C^\infty \) function in \( \mathbb{R}^+ \). At this point, the solutions to Eq. (36) can be determined.

**Proposition 19.** Let \( p : [0, +\infty] \to [0, +\infty] \) be a strictly decreasing involutory function with \( p(0^+) = +\infty \) and \( p(+\infty) = 0 \) which solves the functional equation (36), then there exists \( k > 0 \) such that

\[
 p(t) = -\frac{1}{k} \ln \left( 1 - e^{-kt} \right), \quad t > 0. \tag{47}
\]

**Proof.** By Proposition 18, \( p \) is continuously differentiable in \( \mathbb{R}^+ \). Thus, deriving both members of (36) with respect to \( z \) and then taking limits when \( z \downarrow 0 \), it is obtained that

\[
p'(p(p(s + p(x)) + y)) = p'(s + p(x)) p'(x) (p'(p(x + y)) - p'(p(y))) \]

\[
p'(p(y)) + p'(p(y)) = 0. \tag{48}
\]

Observe that \( p'(p(x + y)) - p'(p(x)) \neq 0 \) for every \( x, y \in \mathbb{R}^+ \). In fact, if \( p'(p(x + y)) = p'(p(y)) \) for any pair \( x, y \in \mathbb{R}^+ \); then, \( p'(x + y) = p'(y) \) by (46), an equality which, together with (44) with \( s = y \) and \( z = x \), would imply

\[
1 = p'(y) p'(p(y)) = p'(x + y) p'(p(y)) = e^{-kp(p(x)) - p(x + y)},
\]

whence

\[
p(p(x) - p(x + y)) = 0.
\]

Since \( p(x) - p(x + y) \in \mathbb{R}^+ \), the last equality is an absurdity. In this way, (48) can be rewritten in the form

\[
p'(x) = \frac{p'(p(p(s + p(x)) + y)) - p'(p(y))}{p'(s + p(x))(p'(p(x + y)) - p'(p(y)))},
\]

and then, using (46), we deduce

\[
1 = \lim_{x \uparrow +\infty} p'(x) p'(p(x))
\]

\[
= \lim_{x \uparrow +\infty} \left( \frac{p'(p(p(s + p(x)) + y)) - p'(p(y))}{p'(s + p(x))(p'(p(x + y)) - p'(p(y)))} \right)
\]

\[
= \lim_{x \uparrow +\infty} \frac{p'(p(p(s + p(x)) + y)) - p'(p(y))}{p'(s + p(x))} \lim_{x \uparrow +\infty} \frac{p'(p(x))}{p'(p(x + y)) - p'(p(y))}
\]

\[
= \frac{p'(p(p(x)) + y) - p'(p(y))}{p'(s)} \lim_{x \uparrow +\infty} \frac{p'(p(x))}{p'(p(x + y)) - p'(p(y))},
\]

whence, for every \( s, y \in \mathbb{R}^+ \),

\[
\lim_{x \uparrow +\infty} \frac{p'(p(x))}{p'(p(x + y)) - p'(p(y))} = \frac{p'(s)}{p'(p(p(s) + y)) - p'(p(y))}. \tag{49}
\]
Now, a new application of (46) produces
\[
\lim_{x \to +\infty} \frac{p'(p(x))}{p'(p(x + y)) - p'(p(y))} = \lim_{x \to +\infty} \frac{1}{p'(p(x + y)) - p'(p(y))} = \lim_{x \to +\infty} \frac{p'(x + y)}{p'(x)}, \tag{50}
\]
where the last equality follows from the fact that \(p'(+\infty) = 0\). Moreover, from (46) and (44) with \(s = x\) and \(z = y\), it is obtained that
\[
p'(x + y) \frac{p'(x)}{p'(x)} = p'(x + y) p'(p(x)) = e^{-kp(p(y) - p(x + y))}. \tag{51}
\]
Now, from (49), (50) and (51) we deduce
\[
e^{-ky} = \lim_{x \to +\infty} e^{-kp(p(y) - p(x + y))} = \lim_{x \to +\infty} \frac{p'(x + y)}{p'(x)} = \frac{p'(s)}{p'(p(s + y)) - p'(p(y))},
\]
where \(s, y \in \mathbb{R}^+\) and \(k > 0\) is a constant. Substituting \(s = p(s)\) in the last member of these equalities, gives
\[
e^{-ky} = \frac{p'(p(s))}{p'(p(s + y)) - p'(p(y))},
\]
whence, for every \(s, y \in \mathbb{R}^+\),
\[
\frac{1}{p'(s + y)} = p'(p(s + y)) = p'(p(y)) + p'(p(s)) e^{ky} = \frac{1}{p'(y)} + \frac{1}{p'(s)} e^{ky}.
\]
The first member of these equalities is symmetric in its arguments, which shows that
\[
\frac{1}{p'(y)} + \frac{1}{p'(s)} e^{ky} = \frac{1}{p'(s)} + \frac{1}{p'(y)} e^{ks}
\]
or, equivalently,
\[
p'(s) (1 - e^{ks}) = p'(y) (1 - e^{ky}), \quad s, y \in \mathbb{R}^+.
\]
In other words, there exist a positive constant \(A\) such that
\[
p'(s) = \frac{A}{1 - e^{ks}}, \quad s \in \mathbb{R}^+. \tag{52}
\]
An integration of the equality (52) yields
\[
p(s) = -\frac{A}{k} \ln (1 - e^{-ks}), \quad s \in \mathbb{R}^+. \tag{53}
\]
Let us show that a function \( p \) expressed by (53) is involutory if and only if \( A = 1 \). In fact, this occurs if and only if, for every \( s \in \mathbb{R}^+ \),

\[
1 = p' (p(s)) p'(s) = \frac{A}{1 - (1 - e^{-ks})^{-A}} \frac{A}{1 - e^{ks}},
\]
or, setting \( x = (1 - e^{-ks})^{-1} \),

\[
1 - x^A = A^2 (1 - x), \ x > 1,
\]
which holds if and only if \( A = 1 \). This proves that a solution to Eq. (36) which satisfies the hypotheses of the proposition must be of the form (47). A simple substitution shows that (47) is really a solution to Eq. (36). This completes the proof.

\[\square\]

**Remark 20.** Note that, for \( p \) given by (47), the quasisum \( \Delta \) is expressed by

\[
x \Delta y = - \frac{1}{k} \ln \left( 1 - e^{-k \left( \frac{1}{k} \ln (1 - e^{-kx}) - \frac{1}{k} \ln (1 - e^{-ky}) \right) } \right)
\]

\[
= - \frac{1}{k} \ln \left( 1 - (1 - e^{-kx}) \ (1 - e^{-ky}) \right), \ x, y \in \mathbb{R}^+,
\]
while \( x \Delta 0 = 0 \) and \( x \Delta (+\infty) = x \) for every \( x \in [0, +\infty] \).

Theorems 12 and 5 are now proved.

**Proof of Theorem 12.** Suppose that \( F : [a, b]^3 \to [a, b] \) is a solution to the functional equation (26) in the class of functions fulfilling conditions \( \mathbf{i}) \), \( \mathbf{ii}) \) and \( \mathbf{iii}) \). Then, from Propositions 16 and 19, it turns out that \( F \) can be written in the form

\[
F(t, \alpha, \beta) = \begin{cases} 
    f^{-1} \left( \frac{-1}{k} \ln \left( 1 - (1 - e^{-k f(t)}) (1 - e^{-k (f(\beta) - f(\alpha))}) \right) + f(\alpha) \right), & \alpha \leq \beta \\
    f^{-1} \left( \frac{-1}{k} \ln \left( 1 - e^{-k f(t)} (1 - e^{-k (f(\alpha) - f(\beta))}) \right) \right) + f(\beta), & \alpha > \beta
\end{cases}, \ t \in [a, b]
\]

where \( f : [a, b] \to [0, +\infty] \) is a continuous and strictly increasing function with \( f(a) = 0 \) (\( f(+\infty) = +\infty \)) and \( k > 0 \). In this way, the function \( \psi(t) = 1 - e^{-k f(t)} \), \( t \in [a, b] \), is an increasing homeomorphism from \([a, b]\) onto \([0, 1]\) and, in view of \( f(t) = -k^{-1} \ln (1 - \psi(t)) \) and \( f^{-1}(t) = \psi^{-1}(1 - e^{-kt}) \), (54) is, in terms of \( \psi \), expressed by

\[
F(t, \alpha, \beta) = \begin{cases} 
    \psi^{-1} (\psi(\beta) - \psi(\alpha)) \psi(t) + \psi(\alpha), & \alpha \leq \beta \\
    \psi^{-1} (\psi(\beta) - \psi(\alpha)) \psi(t) + \psi(\alpha), & \alpha > \beta
\end{cases}, \ t \in [a, b],
\]

which is no other than (27).

Now, if \( \psi_\ast \) was another increasing homeomorphism satisfying (27), then, the equality

\[
\psi^{-1}_\ast (\psi_\ast(\beta) - \psi_\ast(\alpha)) \phi(t) + \psi_\ast(\alpha) = \psi^{-1} (\psi(\beta) - \psi(\alpha)) \psi(t) + \psi(\alpha)
\]
would hold for every \( t, \alpha, \beta \in [a, b] \). Setting \( \phi = \psi \circ \psi^{-1} : [0, 1] \to [0, 1] \), \( \alpha = \psi^{-1}(p) \) and \( \beta = \psi^{-1}(q) \) in this equality, it turns out that \( \phi \) is a continuous solution to the functional equation

\[
\phi \left( (1 - t)p + tq \right) = (1 - \phi(t)) \phi(p) + \phi(t) \phi(q), \quad t, p, q \in [0, 1],
\]

whence \([1], \text{or [14, pg. 382 and ff.]}\) \( \phi(t) = t, \quad t \in [0, 1] \). In this way, \( \psi = \psi \) and the proof is complete. \( \square \)

Proof of Theor. 5. Let \( M \) be a mean fulfilling the hypotheses of the theorem and, for \( \alpha, \beta \in [a, b] \), consider \( f_{\alpha, \beta} \in \mathcal{AC}(M; [a, b]) \) such that \( f_{\alpha, \beta}(a) = \alpha \) and \( f_{\alpha, \beta}(b) = \beta \). By Theorem 12, it can be written that

\[
f(t) = \psi^{-1} \left( (\psi(\beta) - \psi(\alpha) \psi(t) + \psi(\alpha)) \right), \quad t \in [a, b],
\]

where \( \psi \) is a uniquely determined increasing homeomorphism from \([a, b] \) onto \([0, 1] \). In this way, the equality

\[
\psi^{-1} \left( (\psi(\beta) - \psi(\alpha) \psi(M(x_1, \ldots, x_n)) + \psi(\alpha)) \right) = M \left( \psi^{-1} \left( (\psi(\beta) - \psi(\alpha) \psi(x_1) + \psi(\alpha)) \right), \ldots, \psi^{-1} \left( (\psi(\beta) - \psi(\alpha) \psi(x_n) + \psi(\alpha)) \right) \right),
\]

which holds for every \( x_1, \ldots, x_n, \alpha, \beta \in [a, b] \), turns out to be equivalent to

\[
mM(\psi(t_1, \ldots, t_n) + k = M(\psi(t_1, \ldots, t_n) + h),
\]

where \( t_1, \ldots, t_n \in [0, 1], \quad m \in [-1, 1], \quad h, m + h \in [0, 1] \) and \( M_{\psi} \) is the mean conjugate of \( M \) by \( \psi \) [defined by (19)]. This shows that \( \text{Aff}([0, 1]) \subseteq \mathcal{AC}(M_{\psi}; [0, 1]) \). To prove the opposite inclusion observe that, if \( g \in \mathcal{AC}(M_{\psi}; [0, 1]) \), then \( \psi^{-1} \circ g \circ \psi \in \mathcal{AC}(M; [a, b]) \), and therefore, there exists \((m, h) \in \text{AFF}([0, 1]) \) such that

\[
(\psi^{-1} \circ g \circ \psi)(t) = \psi^{-1} \left( (m \psi(t) + h) \right), \quad t \in [a, b],
\]

whence \( g(t) = mt + h, \quad t \in [0, 1] ; \) i.e., \( g \in \text{Aff}([0, 1]) \). \( \square \)

4. Two-variable means

The following result, which is not devoid of intrinsic interest, will be the key to derive Theorem 6 from Theorem 5.

Proposition 21. Let \( M \) be a two-variable mean defined on an interval \( I \) and \( \{[a_k, b_k] : k \in \mathbb{N}\} \) be a nested and exhaustive sequence of subintervals of \( I \). If, for every \( k \in \mathbb{N} \), \( M|_{[a_k, b_k]} \) is a quasilinear mean on \([a_k, b_k]\), then \( M \) is a quasilinear mean on \( I \).
Proof. The hypotheses ensure the existence, for every $k \in \mathbb{N}$, of a strictly monotonic and continuous function $\psi_k : [a_k, b_k] \to \mathbb{R}$ and a real number $w_k \in (0, 1)$ such that
\[
M |_{[a_k, b_k]} (x, y) = \psi_k^{-1} ((1 - w_k) \psi_k (x) + w_k \psi_k (y)), \quad x, y \in [a_k, b_k].
\]  
(56)

Since the second member of (56) is not altered by taking $-\psi_k$ instead of $\psi_k$, it can be assumed that $\psi_k$ is strictly increasing. Now, as $M |_{[a_k, b_k]} (x, y) = M |_{[a_l, b_l]} (x, y)$ for every $x, y \in [a_l, b_l], \ l = \min \{k, j\}$, (56) yields
\[
\psi_k^{-1} ((1 - w_k) \psi_k (x) + w_k \psi_k (y)) = \psi_j^{-1} ((1 - w_j) \psi_j (x) + w_j \psi_j (y)), \quad x, y \in [a_l, b_l],
\]
and thus, the function $\psi_{k,j} = \psi_k \circ \psi_j^{-1}$ is a continuous solution to the equation
\[
\psi_{k,j} ((1 - w_j) s + w_j t) = (1 - w_k) \psi_{k,j} (s) + w_k \psi_{k,j} (t), \quad s, t \in [0, 1].
\]
In this way ([1, Theor. 2, pg. 67] or also [14, pg. 382 and ff.]), $w_k = w_j, \ k, j \in \mathbb{N}$, and, for certain $p_{k,j}, q_{k,j} \in \mathbb{R}, p_{k,j} \neq 0, \psi_{k,j} (t) = p_{k,j} t + q_{k,j}, \ t \in [a_l, b_l]$; hence
\[
\psi_k (t) = p_{k,j} \psi_j (t) + q_{k,j}, \quad t \in [a_l, b_l].
\]
(57)

Note on the one hand that, since $\psi_k$ and $\psi_j$ are both strictly increasing functions, it must really occur that $p_{k,j} > 0$ for every $k, j \in \mathbb{N}$, and, on the other, that the equality (56) can be written in the form
\[
M (x, y) = \psi_k^{-1} ((1 - w_1) \psi_k (x) + w_1 \psi_k (y)), \quad x, y \in [a_k, b_k].
\]
(58)

In what follows, a particular instance of (57) will be used; namely, setting $p_k = p_{k+1,k}$ and $q_k = q_{k+1,k}$ for every $k \geq 1$, (57) takes the form
\[
\psi_{k+1} (t) = p_k \psi_k (t) + q_k, \quad t \in [a_k, b_k].
\]
(59)

Now, define a sequence of strictly increasing and continuous functions $\phi_k : [a_k, b_k] \to \mathbb{R}, \ k \in \mathbb{N}$, by $\phi_1 (t) = \psi_1 (t), \ t \in [a_1, b_1]$, and for $k \geq 1,$
\[
\phi_{k+1} (t) = \left( \prod_{j=1}^{k} p_j^{-1} \right) \psi_{k+1} (t) - \sum_{i=1}^{k} \left( q_i \prod_{j=1}^{i} p_j^{-1} \right), \quad t \in [a_{k+1}, b_{k+1}].
\]
From (59) it is deduced that, for every $t \in [a_k, b_k],$
\[
\phi_{k+1} (t) = \left( \prod_{j=1}^{k} p_j^{-1} \right) (p_k \psi_k (t) + q_k) - \sum_{i=1}^{k} \left( q_i \prod_{j=1}^{i} p_j^{-1} \right)
\]
\[
= \left( \prod_{j=1}^{k-1} p_j^{-1} \right)^{-1} \psi_k (t) - \sum_{i=1}^{k-1} \left( q_i \prod_{j=1}^{i} p_j^{-1} \right)
\]
\[
= \phi_k (t),
\]
so that the expression
\[ \phi(t) = \phi_k(t), \ t \in [a_k, b_k], \]
defines a function \( \phi : I \to \mathbb{R} \) which turns out to be strictly increasing and continuous on \( I \). Since a quasilinear mean \( L_f \) does not change when its generator \( f \) is replaced by \( mf + h \) with \( m, h \in \mathbb{R}, \ m \neq 0 \), the equality (58) can be rewritten in the form
\[ M(x, y) = \phi^{-1}_k((1 - w_1) \phi_k(x) + w_1 \phi_k(y)), \ x, y \in [a_k, b_k], \]
whence
\[ M(x, y) = \phi^{-1}((1 - w_1) \phi(x) + w_1 \phi(y)), \ x, y \in I, \]
which shows that \( M \) is quasilinear on \( I \), as stated. \( \square \)

Proof of Theorem 6. Since \( M|_{[a_1, b_1]} \) satisfies the hypotheses of Theorem 5, it follows that there exist a homeomorphism \( \psi_1 : [a_1, b_1] \to [0, 1] \) such that \( \mathcal{AC}(M_{\psi_1}; [0, 1]) = \text{Aff}([0, 1]) \). After what was said in Sect. 1, \( M_{\psi_1} \) is a linear mean, and therefore, there exists \( w_1 \in (0, 1) \) such that
\[ M(x, y) = \psi_1^{-1}((1 - w_1) \psi_1(x) + w_1 \psi_1(y)), \ x, y \in [a_1, b_1]. \]
The same reasoning applied to the interval \( [a_k, b_k] \) yields, for every \( k \in \mathbb{N} \),
\[ M(x, y) = \psi_k^{-1}((1 - w_k) \psi_k(x) + w_k \psi_k(y)), \ x, y \in [a_k, b_k], \]
where \( \psi_k : [a_k, b_k] \to [0, 1] \) is an increasing homeomorphism onto \([0, 1]\) and \( w_k \in (0, 1) \). This proves that, for every \( k \in \mathbb{N} \), the restriction \( M|_{[a_k, b_k]} \) to the subinterval \([a_k, b_k]\) is a quasilinear mean. By Proposition 21, this implies that \( M \) is quasilinear on \( I \). \( \square \)

Remark 22. Under the hypotheses of Theorem 6, consider the mean \( M_{\phi} \), conjugate of \( M \) by \( \phi, \phi \) being the function defined in the proof of Proposition 21. For the \( M_{\phi} \)-affine functions, the equality
\[ \mathcal{AC}(M_{\phi}; \phi(I)) = \text{Aff}(\phi(I)) \]
holds.

A simple consequence of Theorem 6 is the following:

Proposition 23. Let \( I, J \) be non void real intervals and \( M \in \mathcal{CM}_2(I), \ N \in \mathcal{CM}_2(J) \) be a pair of strict continuous means. Suppose that for every \( a, b \in I, \ a \neq b, \) and for every \( c, d \in J, \ c \neq d, \) there exists a strictly monotonic and continuous solution \( f : I \to J \) to the equation
\[ f(M(x, y)) = N(f(x), f(y)), \ x, y \in I, \]
with \( f(a) = c \) and \( f(b) = d; \) then \( M \) and \( N \) are both quasilinear means.
functions $f, g$ given $c, d$ and the entire family $B$ while $f$ defined by $f(a, b) = g^{-1} \circ f$ turns out to be a homeomorphism which solves the equation

$$h(M(x, y)) = M(h(x), h(y)), \quad x, y \in I,$$

and satisfies $f_{\alpha, \beta}(a) = \alpha$ and $f_{\alpha, \beta}(b) = \beta$. In this way, after defining $f_{\alpha, \alpha} = \alpha$, we see that $\mathcal{AC}(M; [a, b])$ is a Sturm–Liouville family on every compact subinterval $[a, b]$ of $I$ and therefore, Theorem 6 implies that $M$ is a quasilinear mean. The quasi-linearity of $N$ follows from a similar argument. \hfill \Box

The remaining of this section is engaged with bases of two-variable means, a concept introduced in [3]. Before stating a result on the characterization of quasi-linear means in terms of bases, an abridged recall will be presented of the involved ideas.

If $M \in \mathcal{CM}_2(I)$ is a strict and continuous mean defined on an interval $I$ and, for a pair $u, v \in [0, 1]$, $M(u), M(v)$ are Aczel dyadic iterations of $M$, then, in view of the monotonicity and continuity of $\delta \mapsto M(\delta(x, y))$ ensured by Theor. 8, there exists a unique real number $\mu \in [0, 1]$ such that

$$M(M^u(x, y), M^v(x, y)) = M^\mu(x, y).$$

Denoting by $P$ the point $(x, y) \in I^2$, this equality can be written in the form

$$M(M^u(x, y), M^v(x, y)) = M^{\mu_P(u, v)}(x, y), \quad (61)$$

in which the dependence of the number $\mu$ on the pair $u, v$ as well as on $P = (x, y)$ has been emphasized. As shown in [3], $\mathcal{B}(M) = \{\mu_P : P \in I^2\}$ is a family of strict and continuous means defined on the unit interval $[0, 1]$. Every member $\mu_P$ belonging to the family $\mathcal{B}(M)$ is called a base mean of the mean $M$, while the entire family $\mathcal{B}(M)$ is said to be the base of $M$.

The base of a quasiarithmetic mean $A_f$ can be easily computed. Indeed, using the expression of $A_f^{(\delta)}$ given by (21) we obtain

$$A_f\left(A_f^{(u)}(x, y), A_f^{(v)}(x, y)\right) = f^{-1}\left(f\left(f^{-1}((1-u)f(x) + vf(y))\right) + f\left(f^{-1}((1-v)f(x) + vf(y))\right)\right)$$

$$= f^{-1}\left(1 - \frac{u+v}{2}\right) f(x) + \frac{u+v}{2} f(y), \quad x, y \in I, u, v \in [0, 1],$$
while
\[ A_f^{\mu_P}(u,v)(x,y) = f^{-1}((1 - \mu_P(u,v))x + \mu_P(u,v)y), \ x, y \in I, \ u, v \in [0,1], \]
and therefore, the base mean
\[ \mu_P(u,v) = \frac{u + v}{2} = A(u,v), \ u, v \in [0,1], \]
does not depend on \( P \in I^2 \). A slightly more involved computation shows that a quasilinear mean \( L_f \) possesses a unitary base as well. Now, what can be said on a two-variable, strict and continuous mean \( M \) when its base is a unitary set? In [3] the following result was established.

**Theorem 24.** Let \( M \in \mathcal{CM}_2(I) \) be a differentiable strict mean. Then, the base of \( M \) is a unitary family if and only \( M \) is a quasilinear mean.

Let us show that, with the help of Theorem 6, the differentiability hypothesis in the above statement can be omitted.

**Theorem 25.** Let \( M \in \mathcal{CM}_2(I) \) be a continuous and strict mean on a real interval \( I \). Then, the base of \( M \) is a unitary family if and only \( M \) is a quasilinear mean.

**Proof.** The “if” part of the proof proceeds along the same lines as in the particular case in which \( M \) is quasiarithmetic. The details can be seen in [3].

To prove the converse, suppose that \( B(M) = \{\mu\} \) is a base of \( M \); then, given \( a, b \in I \), with \( a < b \), (61) yields
\[ M(M^u(a,b), M^v(a,b)) = M^{\mu(u,v)}(a,b), \ u, v \in [0,1], \]
or, using the notation introduced in the statement of Theor. 8,
\[ M(\phi(a,b)(u), \phi(a,b)(v)) = \phi(a,b)(\mu(u,v)), \ u, v \in [0,1]. \]
Similarly, choosing a pair of numbers \( \alpha, \beta \in [a,b] \), we can write
\[ M(\phi(\alpha,\beta)(u), \phi(\alpha,\beta)(v)) = \phi(\alpha,\beta)(\mu(u,v)), \ u, v \in [0,1]. \]

Now well, from (62) and (63) it is deduced that
\[ \phi_{(a,b)}^{-1}(M(\phi(a,b)(u), \phi(a,b)(v))) = \phi_{(\alpha,\beta)}^{-1}(M(\phi(\alpha,\beta)(u), \phi(\alpha,\beta)(v))), \ u, v \in [0,1], \]
or, equivalently
\[ (\phi(\alpha,\beta) \circ \phi_{(a,b)}^{-1})(M(x,y)) = M((\phi(\alpha,\beta) \circ \phi_{(a,b)}^{-1})(x), (\phi(\alpha,\beta) \circ \phi_{(a,b)}^{-1})(y)), \ x, y \in [a,b]. \]
This equality expresses the fact that the function \( f_{\alpha,\beta} : [a,b] \to [a,b] \) given by \( f_{\alpha,\beta} = \phi(\alpha,\beta) \circ \phi_{(a,b)}^{-1} \) is a continuous \( M \)-affine function. Moreover, since \( f_{\alpha,\beta}(a) = \beta \) and \( f_{\alpha,\beta}(b) = \alpha \), the arbitrariness of \( a, b, \alpha \) and \( \beta \) shows that
\( \mathcal{AC}(M;[a_k, b_k]) \) is a Sturm–Liouville family and therefore, Theorem 6 implies that \( M \) is quasilinear. \( \square \)

5. \( n \)-variable means

Due to its usefulness in proving Theorem 7, the following paragraphs dive into the connections existing among \( \operatorname{Aff}(I) \) and \( \operatorname{AFF}(I) \). First of all, note that the algebraic and topological structures of \( \operatorname{Aff}(I) \) and \( \operatorname{AFF}(I) \) find a natural correspondence through the bijective map \( i : \operatorname{Aff}(I) \to \operatorname{AFF}(I) \) given by \( i(m(\cdot) + h) = (m, h) \). In this way, if \( \operatorname{Aff}(I) \) is equipped with the topology of uniform convergence on compact subsets of \( I \) while \( \operatorname{AFF}(I) \) is given the topology induced by the usual topology on \( \mathbb{R}^2 \), then the map \( i \) becomes a homeomorphism. On the other hand, the law defined on \( \operatorname{AFF}(I) \) by \( (m_1, h_1) \bullet (m_2, h_2) = (m_1 m_2, m_1 h_2 + h_1) \) turns out to be an associative operation, and the map \( i : \langle \operatorname{Aff}(I), \circ \rangle \to \langle \operatorname{AFF}(I), \bullet \rangle \) becomes an isomorphism of semigroups.

Now, if \( \phi \in \operatorname{Aff}(\mathbb{R}) \), it is clear that \( f \in \operatorname{Aff}(I) \) if and only if \( \phi \circ f \circ \phi^{-1} \in \operatorname{Aff}(\phi(I)) \), which is compactly expressed by the equality

\[
\operatorname{Aff}(\phi(I)) = \phi \circ \operatorname{Aff}(I) \circ \phi^{-1}.
\]

Correspondingly,

\[
\operatorname{AFF}(\phi(I)) = i(\phi) \bullet \operatorname{AFF}(I) \bullet i(\phi^{-1}).
\]

In particular, taking \( I = [0, 1] \) and \( \phi(t) = (b - a) t + a \), we obtain

\[
\operatorname{AFF}([a, b]) = (b - a, a) \bullet \operatorname{AFF}([0, 1]) \bullet \left( \frac{1}{b - a}, -\frac{a}{b - a} \right)
\]

\[
= \left\{ (b - a, a) \bullet (m, h) \bullet \left( \frac{1}{b - a}, -\frac{a}{b - a} \right) : (m, h) \in \operatorname{AFF}([0, 1]) \right\}
\]

\[
= \{ (m, -am + (b - a) h + a) : (m, h) \in \operatorname{AFF}([0, 1]) \}
\]

\[
= T_{a,b} (\operatorname{AFF}([0, 1])),
\]

where \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is the transformation given by

\[
T_{a,b}(x, y) = \begin{pmatrix} 1 & 0 \\ -a & b - a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}.
\]

Since \( T_{a,b} \) is affine and, as noted in Sect. 1, \( \operatorname{AFF}([0, 1]) = \{ (0, 0), (1, 0), (0, 1), (-1, 1) \} \), it turns out that

\[
\operatorname{AFF}([a, b]) = \{ T_{a,b}(0, 0), T_{a,b}(1, 0), T_{a,b}(0, 1), T_{a,b}(-1, 0) \}^\wedge
\]

\[
= \{ (0, a), (1, 0), (0, b), (-1, a + b) \}^\wedge.
\]

(65)
The three propositions proved below will play a relevant role in the proof of Theorem 7.

**Proposition 26.** Let be $I \neq \emptyset$ a real interval and $S \neq \emptyset$ be a closed subset of $\text{Aff} \ (I)$ such that, for a given nested and exhaustive sequence $\{[a_k, b_k] : k \in \mathbb{N}\}$ of compact subintervals of $I$, the inclusion

$$\text{Aff} \ ([a_k, b_k]) \cap \text{Aff} \ (I) \subseteq S,$$

holds for every $k \in \mathbb{N}$. Then, the equality

$$S = \text{Aff} \ (I)$$

holds provided that

i) $I$ is bounded or,

ii) $I$ is unbounded and $S$ is a subsemigroup of $\text{Aff} \ (I)$ with the property that, if $f \in S$ and $f^{-1} \in \text{Aff} \ (I)$, then $f^{-1} \in S$.

**Proof.** By equality (64), it will be sufficient to prove the proposition for the instances $I = [0, 1], (0, 1), [0, +\infty), (0, +\infty), \mathbb{R}$. Now, when $I = [0, 1]$ there is nothing to prove, so that we must consider only the five remaining cases. From these, the treatment of the instances $I = [0, 1), (0, 1)$ turns out to be very similar, and the same occurs when $I = [0, +\infty), (0, +\infty)$, so that a detailed argument is to be given below only for the three cases $I = [0, 1), [0, +\infty), \mathbb{R}$. On the other hand, the isomorphism $i : \langle \text{Aff} \ (I), \circ \rangle \to \langle \text{AFF} \ (I), \bullet \rangle$ can be applied to derive an equivalent formulation of the proposition in terms of subsets of $\mathbb{R}^2$. After all these simplifications, the statement to be proved is the following: let $I$ be one of the real intervals $[0, 1), [0, +\infty), \mathbb{R}$, and let $S \neq \emptyset$ be a closed subset of $\text{AFF} \ (I)$ such that, for a certain nested and exhaustive sequence $\{[a_k, b_k] : k \in \mathbb{N}\}$ of compact subintervals of $I$, the inclusion

$$\text{AFF} \ ([a_k, b_k]) \cap \text{AFF} \ (I) \subseteq S,$$  \hfill (66)

holds for every $k \in \mathbb{N}$. Then, the equality

$$S = \text{AFF} \ (I)$$

holds provided that $I = [0, 1)$ or $I = [0, +\infty), \mathbb{R}$ and $S$ is a subsemigroup of $\text{AFF} \ (I)$ with the property that, if $f \in S$ and $f^{-1} \in \text{AFF} \ (I)$, then $f^{-1} \in S$. A separate analysis of the three cases follows.

$I = [0, 1):$ In this case, a nested and exhaustive sequence of compact subintervals of $I$ has the form $\{[0, b_k] : k \in \mathbb{N}\}$, where $(b_k)$ is a sequence of numbers satisfying $0 < b_k < 1$, $k \in \mathbb{N}$, and $b_k \uparrow 1$. Thus, (65) yields

$$\text{AFF} \ ([0, b_k]) = \{(0, 0), (1, 0), (0, b_k), (-1, b_k)\} \wedge,$$  \hfill (67)
and taking into account that $\text{AFF}([0, 1)) = \text{AFF}([0, 1]) \cap \{(m, h) \in \mathbb{R}^2 : h < 1\}$, the inclusion (66) gives

$$S \supseteq \bigcup_{k=1}^{+\infty} (\text{AFF}([0, b_k]) \cap \text{AFF}([0, 1))) = \text{AFF}([0, 1]) \cap \{(m, h) \in \mathbb{R}^2 : m + h < 1\},$$

or, since $S$ is closed in $\text{AFF}([0, 1))$,

$$S \supseteq \bigcup_{k=1}^{+\infty} (\text{AFF}([0, b_k]) \cap \text{AFF}([0, 1))) = \text{AFF}([0, 1])\times[0, +\infty).$$

The closure operator in the above equalities is taken with respect to the relative topology induced on $\text{AFF}([0, 1))$ by the usual topology of $\mathbb{R}^2$.

$I = [0, +\infty)$ : Here, a generic nested and exhaustive sequence of compact subintervals of $I$ is given by $\{[a_k, b_k] : k \in \mathbb{N}\}$, where $(b_k)$ is a sequence satisfying $0 < b_k, k \in \mathbb{N},$ and $b_k \uparrow +\infty.$ Accordingly, $\text{AFF}([0, b_k])$ is also given by (67), and taking into account that

$$\text{AFF}([0, +\infty)) = [0, +\infty)^2,$$

from (66) we deduce

$$S \supseteq \bigcup_{k=1}^{+\infty} (\text{AFF}([0, b_k]) \cap \text{AFF}([0, +\infty))) \supseteq \{(−1, a_1 + b_1)\cup([0, 1] \times \mathbb{R})\}.$$ 

As a consequence, $(m, 0) \in S$ for every $0 < m < 1$ and, since $(m, 0)^{-1} = (m^{-1}, 0) \in \text{AFF}([0, +\infty))$, we must have $(m^{-1}, 0) \in S$, so that $(m, 0) \in S$ for every $m \geq 0.$ Finally, if $(m, h) \in \text{AFF}([0, +\infty))$, it can be written that

$$(m, h) = (m, 0) \bullet (1, h/m)$$

where $(m, 0), (1, h/m) \in S,$ and thus $(m, h) \in S$.

$I = \mathbb{R}$ : $\{(a_k, b_k) : k \in \mathbb{N}\}$, where $(a_k)$ and $(b_k)$ are sequences of real numbers satisfying $a_k < b_k, k \in \mathbb{N},$ and $a_k \downarrow -\infty, b_k \uparrow +\infty$, is a generic nested and exhaustive sequence of compacts, subintervals of $I$. For this sequence, (65) yields

$$\text{AFF}([a_k, b_k]) = \{(0, a_k), (1, 0), (0, b_k), (-1, a_k + b_k)\}.$$ 

It can be seen that

$$\bigcup_{k=1}^{+\infty} (\text{AFF}([a_k, b_k]) \cap \text{AFF}(\mathbb{R})) \supseteq \{−1, a_1 + b_1\} \cup ([0, 1] \times \mathbb{R}),$$

where the closure operator is the closure on $\mathbb{R}^2,$ so that

$$S \supseteq \{−1, a_1 + b_1\} \cup ([0, 1] \times \mathbb{R}).$$
It is shown like in the case \( I = [0, +\infty) \) that \((m, h) \in S\) for every \( m \geq 0 \) and \( h \in \mathbb{R} \). Now, if \( m < 0 \) and \( h \in \mathbb{R} \), then \((-m, 0), (-1, a_1 + b_1), (1, h/m + a_1 + b_1) \in S\), and the equality
\[
(m, h) = (-m, 0) \circ (-1, a_1 + b_1) \circ (1, h/m + a_1 + b_1),
\]
shows that \((m, h) \in S\). It has thus been proved that \( S = \mathbb{R}^2 = AFF(\mathbb{R}) \).

\[\square\]

**Proposition 27.** Let \( I \neq \emptyset \) be a real interval, then there exists a nested and exhaustive sequence \( \{[a_k, b_k] : k \in \mathbb{N}\} \) of compact subintervals of \( I \) such that the inclusion

\[
AFF([a_k, b_k]) \cap AFF(I) \subseteq AFF([a_{k+1}, b_{k+1}]) \cap AFF(I)
\]
holds for every \( k \in \mathbb{N} \).

**Proof.** Like in the proof of Prop. 26, it will be sufficient to prove the proposition for the intervals of the list \([0, 1), \, [0, 1), \, (0, 1), \, [0, +\infty), \, (0, +\infty), \, \mathbb{R} \). Indeed, for every bijective affine function \( \phi \), the inclusion (68) is clearly equivalent to

\[
\phi \circ (AFF([a_k, b_k]) \cap AFF(I)) \circ \phi^{-1} \subseteq \phi \circ (AFF([a_{k+1}, b_{k+1}]) \cap AFF(I)) \circ \phi^{-1},
\]
which, by equality (64), can be rewritten in the form

\[
AFF(\phi([a_k, b_k])) \cap AFF(\phi(I)) \subseteq AFF(\phi([a_{k+1}, b_{k+1}])) \cap AFF(\phi(I)).
\]

The assertion follows from equating \( \phi \) to the affine function transforming the interval \( I \) in any of above listed intervals. Now well, consider the sequences \( \{[a_k, b_k] : k \in \mathbb{N}\} \) explicitly given by:

- \( I = [0, 1) : [a_k, b_k] = [0, 1], \, k \in \mathbb{N} \);
- \( I = [0, 1) : [a_k, b_k] = [0, b_k], \, k \in \mathbb{N} \), where \( 0 < b_k \uparrow 1 \);
- \( I = (0, 1) : [a_k, b_k] = [\delta_k, 1 - \delta_k], \, k \in \mathbb{N} \), where \( 0 < \delta_k < 1/2 \) and \( \delta_k \downarrow 0 \);
- \( I = [0, +\infty) : [a_k, b_k] = [0, b_k], \, k \in \mathbb{N} \), where \( 0 < b_k \uparrow +\infty \);
- \( I = (0, +\infty) : [a_k, b_k] = [1/b_k, b_k], \, k \in \mathbb{N} \), where \( 0 < b_k \uparrow +\infty \);
- \( I = \mathbb{R} : [a_k, b_k] = [\sigma - b_k, b_k], \, k \in \mathbb{N} \), where \( \sigma \in \mathbb{R} \), \( \sigma < 2b_1 \), and \( 0 < b_k \uparrow +\infty \).

By (65), it is easily checked that (68) holds for the sequences in the list.

\[\square\]

**Proposition 28.** Let \( I \) be a real interval with \( \text{int}(I) \neq \emptyset \). The inclusion

\[
\phi \circ AFF(I) \subseteq AFF(\phi(I)) \circ \phi
\]
holds for a continuous function \( \phi : I \rightarrow \mathbb{R} \) if and only if \( \phi \) is affine.
**Proof.** First assume that \( \phi \) is affine; i.e., for a certain pair \( p, q \in \mathbb{R} \), \( \phi(t) = pt + q \), \( t \in I \). A generic \( f \in \text{Aff}(I) \) has the form \( f(t) = p(mt + h) + q \), with \( (m,h) \in \text{AFF}(I) \) and thus, setting \( \mu = m \) and \( \nu = ph - mq + q \), it turns out that

\[
f(t) = p(mt + h) + q = \mu(pt + q) + \nu.
\]

Let us show that \((\mu, \nu) \in \text{AFF}(\phi(I))\). In fact, since \((m,h) \in \text{AFF}(I)\), it can be written that

\[
\mu \phi(I) + \nu = m(pI + q) + ph - mq + q = p(mI + h) + q \subseteq pI + q = \phi(I).
\]

In consequence, \( f \in \text{Aff}(\phi(I)) \circ \phi \) and the inclusion (69) follows. Conversely, if (69) holds for a continuous function \( \phi : I \to \mathbb{R} \), then for every \((m,h) \in \text{AFF}(I)\) there exists \((\mu(m,h), \nu(m,h)) \in \text{AFF}(\phi(I))\) such that

\[
\phi(mt + h) = \mu(m,h) \phi(t) + \nu(m,h), \ t \in I. \tag{70}
\]

Since \( \text{int}(I) \neq \emptyset \), (70) can be evaluated at two different points \( t_0, t_1 \in I \), \( t_0 < t_1 \), to obtain

\[
\begin{align*}
\phi(mt_0 + h) &= \mu(m,h) \phi(t_0) + \nu(m,h), \\
\phi(mt_1 + h) &= \mu(m,h) \phi(t_1) + \nu(m,h),
\end{align*}
\]

whence

\[
\begin{align*}
\mu(m,h) &= \frac{\phi(mt_1 + h) - \phi(mt_0 + h)}{\phi(t_1) - \phi(t_0)}, \\
\nu(m,h) &= \phi(mt_0 + h) - \frac{\phi(mt_1 + h) - \phi(mt_0 + h)}{\phi(t_1) - \phi(t_0)} \phi(t_0).
\end{align*}
\]

Introducing these expressions for \( \mu \) and \( \nu \) in (70), we obtain

\[
\phi(mt + h) = \frac{\phi(mt_1 + h) - \phi(mt_0 + h)}{\phi(t_1) - \phi(t_0)} (\phi(t) - \phi(t_0)) + \phi(mt_0 + h), \ t \in I,
\]

an equality which, when expressed in terms of the function \( \psi : I \to \mathbb{R} \) defined by

\[
\psi(t) = \frac{\phi(t) - \phi(t_0)}{\phi(t_1) - \phi(t_0)}, \tag{71}
\]

becomes

\[
\psi(mt + h) = (1 - \psi(t)) \psi(mt_0 + h) + \psi(t) \psi(mt_1 + h), \ t \in I, \ (m,h) \in \text{AFF}(I). \tag{72}
\]

Now, in view of the identity,

\[
mt + h = \left(1 - \frac{t - t_0}{t_1 - t_0}\right) (mt_0 + h) + \frac{t - t_0}{t_1 - t_0} (mt_1 + h),
\]
and the fact that \([t_0, t_1] \subseteq I\), the equality
\[
\psi \left( (1 - \frac{t - t_0}{t_1 - t_0}) x_0 + \frac{t - t_0}{t_1 - t_0} x_1 \right) = (1 - \psi(t)) \psi(x_0) + \psi(t) \psi(x_1), \quad t \in [t_0, t_1], \; x_0, x_1 \in I,
\]
can be easily derived from (72), and then ([1, Theor. 2, pg. 67], or [14, pg. 382 and ff.])
\[
\psi(t) = \frac{t - t_0}{t_1 - t_0}, \quad t \in [t_0, t_1].
\]
This equality and (71) yields
\[
\frac{\phi(t) - \phi(t_0)}{\phi(t_1) - \phi(t_0)} = \frac{t - t_0}{t_1 - t_0}, \quad t \in [t_0, t_1],
\]
so that \(\phi\) is affine in every subinterval \([t_0, t_1] \subseteq I\), which implies that \(\phi\) is affine.

Proof of Theorem 7. Let \(M \in \mathcal{CM}_n(I)\) be a continuous and strict mean defined on \(I\) such that either of the conditions \(A\) or \(B\) is satisfied. Then, there exists a nested and exhaustive sequence \([a_k, b_k] : k \in \mathbb{N}\) of compact subintervals of \(I\), such that \(\mathcal{AC}(M; [a_k, b_k])\) is a Sturm–Liouville family. Every nested and exhaustive sequence \([a_k, b_k] : k \in \mathbb{N}\) serves this purpose in case \(A\), while the prescribed sequence \([a_k, b_k] : k \in \mathbb{N}\) must be taken in case \(B\). In this way, the hypotheses of Theorem 6 are satisfied by the two-variable mean \(N\) defined on \(I\) by (23), and therefore there exist both a strictly increasing and continuous function \(\phi : I \to \mathbb{R}\) and a number \(w \in (0, 1)\) such that
\[
N(x, y) = \phi^{-1}((1 - w)\phi(x) + w\phi(y)), \quad x, y \in I.
\]
Now, consider the mean \(M_\phi\) the conjugate of \(M\) by \(\phi\); i.e.,
\[
M_\phi(y_1, \ldots, y_n) = \phi(M(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_n))), \quad y_1, \ldots, y_n \in \phi(I).
\]
The inclusion
\[
\mathcal{AC}(M_\phi; \phi(I)) \subseteq \mathcal{AC}(N_\phi; \phi(I)) = \text{Aff}(\phi(I))
\]
follows from (16) by a simple specialization of the variables in equation (6). Let us show that the above inclusion is really an equality. In case \(B\), \([\phi(a_k), \phi(b_k)] : k \in \mathbb{N}\) turns out to be a nested and exhaustive sequence of compact subintervals of \(\phi(I)\) and \(\mathcal{AC}(M_\phi; [\phi(a_k), \phi(b_k)]; \phi(I))\) is clearly a Sturm–Liouville family for every \(k \in \mathbb{N}\). Thus, if \(f \in \text{Aff}(\phi(I))\) and \(k \in \mathbb{N}\), then there exists \(g_k \in \mathcal{AC}(M_\phi; [\phi(a_k), \phi(b_k)]; \phi(I))\) such that \(g_k(\phi(a_k)) = f(\phi(a_k))\) and \(g_k(\phi(b_k)) = f(\phi(b_k))\). Clearly \(g_k = f|_{\phi(a_k), \phi(b_k)}\) and, taking into account that the equality
\[
g_k(M_\phi(y_1, \ldots, y_n)) = M_\phi(g_k(y_1), \ldots, g_k(y_n)), \quad y_1, \ldots, y_n \in [\phi(a_k), \phi(b_k)],
\]
holds for every $k \in \mathbb{N}$, it turns out that
\[ f(M_\phi(y_1, \ldots, y_n)) = M_\phi(f(y_1), \ldots, f(y_n)), \quad y_1, \ldots, y_n \in \phi(I); \]
i.e., $f \in AC(M_\phi; \phi(I))$.

In case A), consider a sequence $\{[A_k, B_k] : k \in \mathbb{N}\}$ of compact subintervals of $\phi(I)$ satisfying the inclusions (68) of Proposition 27. Since $AC(M; [\phi^{-1}(A_k), \phi^{-1}(B_k)])$ is a Sturm–Liouville family for every $k \in \mathbb{N}$, $AC(M_\phi; [A_k, B_k])$ also turns out to be a Sturm–Liouville family for every $k \in \mathbb{N}$. It is affirmed that $S = AC(M_\phi; \phi(I))$ satisfies the hypotheses of Proposition 26 and thus $S = Aff(\phi(I))$. In fact, if $f \in Aff([A_\nu, B_\nu]) \cap Aff(\phi(I))$ for a certain $\nu \in \mathbb{N}$, and $y_1, \ldots, y_n \in \phi(I)$, then $f \in Aff([A_k, B_k]) \cap Aff(\phi(I))$ for every $k \geq \nu$ by (68), and there exists $\mu \in \mathbb{N}$, such that $y_1, \ldots, y_n \in [A_k, B_k]$ for every $k \geq \mu$. In this way, $f \in AC(M_\phi; [A_k, B_k])$ for every $k \geq \max\{\nu, \mu\}$, whence the equality
\[ f(M_\phi(y_1, \ldots, y_n)) = M_\phi(f(y_1), \ldots, (y_n)), \]
holds. From the arbitrariness of $y_1, \ldots, y_n \in \phi(I)$ it follows that $f \in S$. In other words, the inclusion
\[ Aff([A_k, B_k]) \cap Aff(\phi(I)) \subseteq S, \]
holds for every $k \in \mathbb{N}$. Now, $S$ is clearly a closed subsemigroup of $Aff(\phi(I))$ and, moreover, if $(m, h) \in S$ for a certain $(m, h) \in AFF(\phi(I))$ such that $(m, h)^{-1} = (1/m, -h/m) \in AFF(\phi(I))$, then the equality
\[ M_\phi(mx_1 + h, \ldots, mx_n + h) = mM_\phi(x_1, \ldots, x_n) + h, \quad x_1, \ldots, x_n \in \phi(I), \]
holds for every $x_1, \ldots, x_n \in \phi(I)$, so that the substitutions $x_i = (y_i - h)/m$, $i = 1, \ldots, n$, yield
\[ \frac{1}{m}M_\phi(y_1, \ldots, y_n) - \frac{h}{m} = M_\phi((y_1 - h)/m, \ldots, (y_n - h)/m), \quad y_1, \ldots, y_n \in \phi(I), \]
which shows that $(m, h) \in S$.

Now, suppose that $\psi : I \to \mathbb{R}$ is another strictly increasing and continuous function satisfying (20). Then
\[ \psi^{-1} \circ Aff(\psi(I)) \circ \psi = AC(M; I) = \phi^{-1} \circ Aff(\phi(I)) \circ \phi, \]
whence
\[ (\psi \circ \phi^{-1}) \circ Aff(\phi(I)) = Aff(\psi(I)) \circ (\psi \circ \phi^{-1}), \]
so that Proposition 28 applies to conclude that $\psi \circ \phi^{-1} : \phi(I) \to \mathbb{R}$ is an affine function; i.e., there exist $m, h \in \mathbb{R}$ such that $\psi \circ \phi^{-1}(t) = mt + h$, $t \in \phi(I)$, or
\[ \psi(t) = m\phi(t) + h, \quad t \in I. \]
Since $\psi$ is strictly increasing, we must have $m > 0$. This finishes the proof. □
Remark 29. A two-variable specialization of a mean $M \in \mathcal{M}_n(I)$ is a mean $N \in \mathcal{M}_2(I)$ defined by

$$N(x, y) = M(x_1, \ldots, x_n)$$

where $x_i = x$ or $x_i = y$ for every $i = 1, \ldots, n$ and $x_i \neq x_j$ for a pair $i, j$ of indices. If $M \in \mathcal{CM}_n(I)$ is an $n$-variable, strict and continuous mean defined on a real interval $I$ such that $\mathcal{AC}(M; [a_k, b_k])$ is a Sturm–Liouville family for a certain sequence $\{[a_k, b_k] : k \in \mathbb{N}\}$ of nested and exhaustive compact subintervals of $I$; then a two-variable specialization $N$ of $M$ is always a quasilinear mean. This is the content of the first part of the proof of Theorem 7. Furthermore, if $M$ satisfies either of the hypotheses (A) or (B) of Theorem 7 and $N_1, N_2$ is a pair of two-variable specializations of $M$, then $N_1$ and $N_2$ are quasilinear means with the same generator (i.e., $N_1$ and $N_2$ may differ from each other only by their weights). This fact is a simple consequence of Theorem 7.

6. Final remarks

Note that the result established by Theorem 12 gives an explicit representation for continuous and monotonic two-parameter semigroups of homeomorphisms $\{f_{\alpha, \beta} : [a, b] \to [a, b] : \alpha, \beta \in [a, b]\}$ which satisfy the conditions $f_{\alpha, \beta}(a) = \alpha$ and $f_{\alpha, \beta}(b) = \beta$.

The nice properties of Aczéľ’s iterations ensured by Theor. 8 can not be extended to non strict means (even though the continuity can be somewhat relaxed). The reader is referred to [6] for further studies on Aczéľ’s iterations in the absence of strictness or continuity. In view of the basic role played by Theor. 8, there is no hope that the main results stated in Sect. 1 continue to be true for non strict means. For instance, if $M$ is a two-variable continuous mean defined on the compact interval $[a, b]$ such that $M(a, b) = a$ and $\mathcal{AC}(M; [a, b])$ is a Sturm–Liouville family, then, for every $\alpha, \beta \in I$, there exists $f \in \mathcal{AC}(M; [a, b])$ satisfying $f(a) = \alpha$ and $f(b) = \beta$, and therefore

$$M(\alpha, \beta) = M(f(a), f(b)) = f(M(a, b)) = f(a) = \alpha, \alpha, \beta \in I,$$

which shows that $M$ is not quasilinear. The case of the maximal means max and min, for which the set of continuous $M$-affine functions is not a two-parameter semigroup, seems to be the general rule in the absence of strictness.

The generalization of other results in this paper is possible like for example, Proposition 21, which can be extended without difficulty to $n$-variable means. Prop. 23 is no longer valid for $n$-variable means when $n > 2$, whereas a generalization of the concept of base to $n$-variable means is a dubious question. On the other hand, the main theorems can be restated for symmetric means by simply introducing the word “quasiarithmetic” instead of the word “quasilinear” wherever it appears in a statement.
Results like Matkowski’s Theorems 2 or 5, 6 and 7 in this paper can be applied in determining whole families of continuous affine functions of a given mean. This fact is illustrated by the following examples.

**Example 30.** The (two-variable) counterharmonic mean is defined by

\[
CH(x, y) = \frac{x^2 + y^2}{x + y}, \quad x, y > 0.
\]

Note that the \(CH\)-affine functions \(f^t(u) = tu, \ t > 0\), make up a continuous iteration group (with generator \(\gamma(u) = u, \ u > 0\)), and that \(h(u) = CH(u, 1), \ u > 0\), turns out to be a rational infinite differentiable function with \(0 \neq h'(1) \neq 1\). Then, the hypotheses of Matkowski’s Theorem 2 are satisfied by \(CH\) and, taking into account that \(CH\) is not a quasiarithmetic mean, it follows that \(\mathcal{AC}(CH; \mathbb{R}^+) = \{f^t : t > 0\} \cup \{f = c : c > 0\}\).

**Example 31.** Consider a strict and continuous mean \(M\) defined on \(\mathbb{R}\) for which the inclusion

\[
\mathcal{Aff}(\mathbb{R}) \subseteq \mathcal{AC}(M; \mathbb{R}) \tag{76}
\]

is satisfied. The mean \(M\) defined in Sect. 1 by (10) and (11) serves as an example of this kind of means. Let us show that the inclusion (76) must really be an equality. Since \(\mathcal{Aff}([a, b]) \subseteq \mathcal{Aff}(\mathbb{R})\) for every compact interval \([a, b]\), the hypotheses of Theorem 7 are fulfilled by \(M\) and then, there exists a strictly increasing and continuous function \(\phi : \mathbb{R} \to \mathbb{R}\) such that

\[
\mathcal{AC}(M; \mathbb{R}) = \mathcal{Aff}(\phi(\mathbb{R})) \tag{77}
\]

or, equivalently,

\[
\mathcal{AC}(M; \mathbb{R}) = \phi^{-1} \circ \mathcal{Aff}(\phi(\mathbb{R})) \circ \phi.
\]

From (76) and (77) it is deduced that

\[
\phi \circ \mathcal{Aff}(\mathbb{R}) \subseteq \mathcal{Aff}(\phi(\mathbb{R})) \circ \phi,
\]

an inclusion which, by Proposition 28, implies that \(\phi\) is affine. Clearly, \(\phi\) does not reduce to a constant, so that \(\phi^{-1} \circ \mathcal{Aff}(\phi(\mathbb{R})) \circ \phi = \mathcal{Aff}(\mathbb{R})\) and then,

\[
\mathcal{AC}(M; \mathbb{R}) = \mathcal{Aff}(\mathbb{R}),
\]

as stated.

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Lucio R. Berrone and Gerardo E. Sbérghamo
Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Laboratorio de Acústica y Electroacústica, Facultad de Cs. Exactas, Ing. y Agrim. Univ. Nac. de Rosario
Riobamba 245 Bis
2000 Rosario
Argentina
e-mail: berrone@fcea.unr.edu.ar

Gerardo E. Sbérghamo
e-mail: gerardo@fcea.unr.edu.ar

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