Conformal field theory of a space-filling string of gravitational ancestry

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We present a classical conformal field theory on an arbitrary two-dimensional spacetime background. The dynamical object is a space-filling string, and the evolution may be thought as occurring on the manifold of the conformal group. The theory is a “descendant” of the theory of gravitation in two-dimensional spacetime. The discussion is based on the relation of the deformations of the space-filling string with conformal transformations. The realization of the conformal algebra in terms of surface deformations possesses a classical central charge. The action principle, the conformal and Weyl invariances of the action, and the equations of motion are studied. The energy-momentum tensor, the coupling to Liouville matter, and the cancellation of anomalies are analyzed. The quantum theory is not discussed.

Keywords: Conformal field theory, lower dimensional gravity, Hamiltonian formalism.

1. Introduction

It took more than four decades since the publication of the Maxwell equations to establish their conformal invariance. Thereafter the conformal symmetry became a subject of continuous interest in theoretical physics and the realm of its applications vast.

The conformal algebra in two-dimensional spacetime has the special property of having an infinite number of generators. This makes the symmetry especially powerful and makes conformal field theories in two-dimensional spacetime appealing. Much work on them has been done in recent years (see e.g. 3, 4 and references therein).

The purpose of this report is to discuss a novel classical conformal field theory, on an arbitrary two-dimensional Riemannian spacetime background, in which the dynamical object is a space-filling string.

If one considers an arbitrary but fixed system of coordinates $y^\lambda$ on the background, a generic (parametrized) space-filling string is described by a function $y^\lambda(x)$, and an infinitesimal deformation is the change,

$$y^\lambda(x) \rightarrow y^\lambda(x) + \delta y^\lambda(x).$$

In two spacetime dimensions, and only then, that deformation may be described in a conformally invariant manner as,

$$\delta y^\lambda = \delta \xi^\perp \tilde{n}^\lambda + \delta \xi^1 y^\lambda, \quad \text{(2)}$$

provided the normal $\tilde{n}^\lambda$ is chosen to be Weyl invariant, instead of being the customary unit normal.
Then the deformation parameters $\delta \xi^\perp, \delta \xi^1$ describe a conformal transformation, because the general integrability conditions of surface deformations imply that the corresponding generators $s^\perp, s^1$, obey the algebra of the conformal group.

\[ [s^\perp (x), s^\perp (x')] = (s^1 (x) + s^1 (x')) \delta^\prime (x, x'), \]
\[ [s^1 (x), s^1 (x')] = (s^\perp (x) + s^\perp (x')) \delta' (x, x') - \zeta \delta'' (x, x'), \]
\[ [s^1 (x), s^\perp (x')] = (s^1 (x) + s^\perp (x')) \delta^\prime (x, x'), \]

where the central charge $\zeta$ is a constant with the units of an action.

Therefore if one fixes a reference cut $y^\perp_0 (x)$ there is a one-to-one correspondence between the functions $y^\perp (x)$ and the conformal transformation whose effect is to deform $y^\perp_0 (x)$ into $y^\perp (x)$.

We will discuss herein a realization of the algebra (3)–(5) on a two-dimensional spacetime background, endowed with an arbitrary but given metric $\gamma_{\lambda \rho} (y)$, in which the conformal symmetry generators are built out of the $y^\perp (x)$ themselves and their canonical conjugates.

In virtue of the fact that the deformation is described in a conformally invariant manner, one may say that the configuration space of the theory is the group space of the conformal group.

The action and symmetry generators of the theory in question were found in Ref. 11, as those of a particular—and especial—"G-brane." The purpose of the present discussion is to bring out in a self-contained manner the $(1+1)$ G-brane as a classical conformal field theory, focusing on aspects that were not covered or emphasized in Ref. 11.

The plan of the paper is the following: Sec. 2 recalls the results obtained in Ref. 11 for the conformal generators of the $(1+1)$ G-brane, while Sec. 3 discusses the action principle, the (lack of) conformal and Weyl invariance of the action, and the equations of motion. Next Sec. 4 is devoted to analyzing the energy-momentum tensor, the coupling to Liouville matter, and the cancellation of anomalies. Finally Sec. 5 contains concluding remarks.

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\[ [L (x), L (x')] = (L (x) + L (x')) \delta' (x, x') \delta'' (x, x') \]

\[ c = 12 \pi \zeta. \]

\[ \text{If one has a spacetime foliation } y^\perp = y^\perp (t, x), \text{ then } \] is equivalent to a change of the coordinates $t, x$. Therefore, a quantity, for example an action integral, will be invariant under changes in the spacetime coordinates if and only if it is conformally invariant. For this reason we will use both terminologies interchangeably in the text, depending on the desired emphasis.

\[ \text{The group space of the conformal group and the related concept of exponentiation of an element of the conformal Lie algebra are not devoid of subtleties, for a lucid discussion see e.g. Ref. 13.} \]
2. Surface deformation algebra and conformal symmetry

2.1. Weyl invariant normal

To relate the infinitesimal deformation (1) to the conformal algebra, it is essential to bring in the metric $\gamma_{\lambda\rho}$ in order to split $\delta y^\lambda$ into its normal and tangential components, $\delta \xi_{\bot}$ and $\delta \xi_{\parallel}$ according to eq. (2). The normal $\tilde{n}^\lambda$ appearing in that equation is defined by,

$$\gamma_{\lambda\rho} \tilde{n}^\lambda \tilde{n}^\rho = -g_{11} \equiv -g,$$  \hspace{1cm} (6)

where,

$$g_{11} = \gamma_{\lambda\rho} y^\lambda_{,1} y^\rho_{,1},$$  \hspace{1cm} (7)

is the metric on the cut $y^\lambda(x)$. On account of (6) and (7) the normal $\tilde{n}^\lambda$ is invariant under Weyl transformations,

$$\gamma_{\lambda\rho}(y) \rightarrow e^{\sigma(y)} \gamma_{\lambda\rho}(y).$$  \hspace{1cm} (8)

The fields $y^\lambda$, $y^\lambda_{,1}$, and $w^\lambda$ are also Weyl invariant because the background metric $\gamma_{\lambda\rho}$ does not enter into their definition.

2.2. Canonical generators

We now recall the expressions for the generators $s_{\bot}$ and $s_1$ obtained in Ref. 11.

If the canonical momenta conjugate to $y^\lambda$ are denoted by $w^\lambda$, so that in terms of Poisson brackets,

$$[y^\lambda(x), w^\rho(x')] = \delta^\lambda_\rho \delta(x,x'),$$

the canonical generators that span the algebra (3)–(5) are realized in terms of the fields $y^\lambda, w^\rho$ as,

$$s_{\bot} = w_{\bot} - \mathcal{L},$$

$$s_1 = w_1.$$

Here,

$$w_{\bot} = w^\lambda \tilde{n}^\lambda,$$

$$w_1 = w^\lambda y^\lambda_{,1},$$

and,

$$\mathcal{L} = \frac{\zeta}{4} \left( 2e^{\varphi} K^2 - \frac{1}{2} \varphi'^2 + 2\varphi'' + \Lambda e^{\varphi} \right).$$  \hspace{1cm} (9)

The fields $\varphi$ and $K$ are the local relative scale and extrinsic curvature of the space-filling string respectively. They are expressed in terms of the $y^\lambda$ as,

$$\varphi = \log (g_{11}) = \log \left( \gamma_{\lambda\rho} y^\lambda_{,1} y^\rho_{,1} \right),$$  \hspace{1cm} (10)
The derivative \( D_1 = y_\lambda D_\lambda \) is the covariant derivative in the external space projected onto the string.

If in (9) the field \( \varphi \) is regarded as a fundamental variable, and \( K \) as its “invariant velocity”; as opposed to being expressed in terms of the \( y_\lambda \) through (10) and (11), then \( L \) appearing in (9) becomes the Lagrangian for an analog in two-dimensional spacetime of Einstein’s theory of gravitation 9, 10, 12, which is related to the Liouville theory. If the replacements (10) and (11) are implemented the “gravitational ancestor” (9) gives rise to its descendant, the space-filling string.

3. Action principle

The action integral is given by,

\[
I = \int dtdx \left( w_\lambda \dot{y}_\lambda - \eta^\perp s_\perp - \eta^1 s_1 \right).
\]

The Weyl invariant lapse and shift functions \( \eta^\perp, \eta^1 \) are considered as external fields which are not varied in the action principle. They parametrize the most general conformal transformation whose unfolding in time is the dynamics of the system, generated by the Hamiltonian,

\[
H = \int dx \left( \eta^\perp s_\perp + \eta^1 s_1 \right).
\]

A particular case is, \( \eta^\perp = 1, \eta^1 = 0 \), which might be termed a “rigid time translation”, the corresponding Hamiltonian is the integral of \( s_\perp \).

3.1. Hamiltonian linear in the momenta.

The Hamiltonian is linear in the momenta because it is a linear combination of the generators \( s_\perp, s_1 \), which have that property. This is a distinctive feature of the (1+1) G-brane. Geometrically, it captures the fact that the motion may be considered as a displacement on the manifold of the conformal group (see eq. (17) below). Dynamically it implies that, independently of the sign of the constant \( \zeta \), the “energy density” \( s_\perp \) can be arbitrarily large and positive or negative. As a consequence of the linearity in the momenta they cannot be expressed in terms of the velocities to obtain an action principle in terms of a Lagrangian which would depend on \( y \) and \( \dot{y} \). This situation changes however when one couples the (1+1) G-brane to matter as discussed in section 4.4.

3.2. Change of the action under a conformal transformation

It will be useful here to employ a standard compact notation to account for both continuous and discrete indices, so that, for example summation over \( a' \) includes
an integration over an accompanying continuous index \(x'\). The conformal algebra \( (3)–(5) \) then reads,

\[
[s_a, s_{b'}] = \kappa_{ab'} s_{c''} + \zeta_{ab'},
\]

where the structure constants \(\kappa\) are products of \(\delta\)-functions and the central term \(\zeta\) is proportional to \(\delta''\).

The conformal transformation of any functional \( F \) of the dynamical fields \(y^\lambda\) and \(w^\lambda\) is given by its Poisson bracket with the corresponding generator:

\[
\delta_F = [F, \xi^a s_a],
\]

(13)

whereas the change of the “conformal lapse” and shift \(\eta^a = (\eta^\perp, \eta^1)\) is given by,

\[
\delta \eta'' = \dot{\xi}^a - \kappa_{bc'} \eta^b \xi^c'.
\]

(14)

If one employs equations \( (13) \) and \( (14) \) to evaluate the change in the action \( (12) \) for a region \(M\) with boundary \(\partial M\), one finds

\[
\delta I = \int_{\partial M} \xi^\perp L + \int_M \xi^a \eta^{b'} \zeta_{ab'},
\]

\[
= \int_{\partial M} \xi^\perp L + \zeta \int_M \left( \eta^\perp \xi^1'' - \eta^1 \xi^\perp'' \right).
\]

(15)

Therefore, due to the central term, even when the boundary is mapped onto itself \((\xi^\perp = 0 \text{ over } \partial M)\), the action is not conformally invariant. This anomaly does not affect the equations of motion because the central term does not include the dynamical fields.

### 3.3. Change of the action under a Weyl transformation

Under an infinitesimal Weyl transformations of the background metric,

\[
\delta \gamma_{\lambda\rho} = \sigma \gamma_{\lambda\rho},
\]

the action \( (12) \) transform as:

\[
\delta I = -\frac{\zeta}{4} \int dt dx \eta^\perp e^\sigma (R - \Lambda) \sigma.
\]

(16)

The change \( (16) \) may be obtained by direct calculation, or, better, by realizing that it must be the same as the one of its gravitational ancestor mentioned at the end of the previous section. In that context eq. \( (16) \) is at the heart of the gravitation theory in two spacetime dimensions: demanding that it should vanish gives then the equation of motion,

\[
R - \Lambda = 0,
\]

of the ancestor theory.\(^9,10,12\)
3.4. Equations of motion

For general $\eta^\perp$ and $\eta^1$ the equations of motion are given by,

$$
\dot{y}^\lambda = \eta^\perp \tilde{n}^\lambda + \eta^1 y^\lambda_1, \\
\dot{w}_\perp = 2w_1 \eta^\perp' + \eta^\perp w_1' + 2\eta^1 (w_\perp - \mathcal{L}) + \eta^1 (w_\perp' - \mathcal{L}') - \zeta \eta^1'' + \tilde{\mathcal{L}}, \\
\dot{w}_1 = 2(w_\perp - \mathcal{L}) \eta^\perp' + \eta^\perp (w_1' - \mathcal{L}') + 2w_1 \eta^1' + \eta^1 w_1' - \zeta \eta^1''.
$$

These equations are conformally invariant because, as it was stated at the end of subsection 3.2, the change in the action under a conformal transformation does not depend on the dynamical fields. Comparison of equation (17) with equation (2) shows that the motion may be considered as a displacement on the manifold of the conformal group.

To obtain insight we now consider the case in which the background is Minkowski space and the external coordinate system is cartesian, that is $\gamma_{\alpha \beta} = \eta_{\alpha \beta}$.

Furthermore we take at time $t = 0$

$$
y^0 = 0, \quad y^1 = x, \\
y^0 = t, \quad y^1 = x,
$$

while leaving the $w_\lambda (t = 0, x)$ arbitrary, and further specialize to a “rigid time translation”, i.e.,

$$
\eta^\perp = 1, \quad \eta^1 = 0,
$$

for all times.

Then the solution of the equations of motion is:

$$
y^0 = t, \\
y^1 = x, \\
w_\perp = w_+ (x + t) + w_- (x - t), \\
w_1 = w_+ (x + t) - w_- (x - t).
$$

It is important to emphasize that the above solution is not the most general solution of the equations of motion when $\gamma_{\alpha \beta} = \eta_{\alpha \beta}$ because the initial conditions (20) are very particular corresponding to a simple cut through the spacetime. The most general solution is obtained by acting on (21) with the conformal group through an iteration of transformations generated by $s_\perp$ and $s_1$. The result of this iteration is to go from the cut (20) to a generic spacelike cut $y^\lambda (t, x)$, but the corresponding result for $w_\lambda$ cannot written in a closed form. It is important to keep in mind that different cuts $y^\lambda (t, x)$ do not correspond to different gauge choices, because $s_\perp$ and $s_1$ do not generate a gauge symmetry since they are not constrained to vanish. Note that for the solution (21) the conformal energy density $s_\perp$ does not vanish when $w_\lambda = 0$, but is given by the background energy,

$$
\frac{-\Lambda \zeta}{4}.
$$
4. Energy-momentum tensor. Coupling to matter. Anomaly cancellation

4.1. Energy-momentum tensor

One may define an energy-momentum tensor for the (1+1) G-brane through the standard formula,

\[ T_{\lambda \rho} = \frac{2}{\sqrt{-g}} \delta I. \tag{22} \]

As a consequence of the lack of invariance of the action under changes of the space-time coordinates expressed by (15), \( T_{\lambda \rho} \) defined by (22) not only is not conserved when the equations of motion (17)–(19) hold but its components do not even transform as those of a tensor under a change of the spacetime coordinates. This conformal anomaly is brought in by the presence of the central charge (“Schwinger term”) in the commutation rule for the energy and momentum densities \( s_\perp, s_1 \), which are related to \( T_{\lambda \rho} \) by,

\[ s_\perp = T_{\lambda \rho} \tilde{n}^\lambda \tilde{n}^\rho, \tag{23} \]
\[ s_1 = T_{\lambda \rho} \tilde{n}^\lambda y^\rho_1. \tag{24} \]

The remaining component,

\[ T_{11} = T_{\lambda \rho} y^\lambda y^\rho_1 = gT + s_\perp, \]

where,

\[ T = \gamma^\lambda \nu T_{\lambda \rho}, \]

may be obtained from (16) to be,

\[ T = \frac{\zeta}{2} (R - \Lambda), \tag{25} \]

and it is a world scalar although \( T_{\lambda \rho} \) is not a tensor.

4.2. Coupling to Liouville field

One may couple the (1+1) G-brane by adding to (12) the action of a “matter field” also described by a conformal field theory. That matter action will have a form similar to (12):

\[ I_{\text{matter}} = \int \pi \dot{\psi} - \eta \dot{s}_\perp \text{matter} - \eta_1 \dot{s}_1 \text{matter}. \]

The generators \( s^{\text{Liouv}} \) will obey the conformal algebra (3)–(5). Since they will be built out of \( \pi \) and \( \psi \) they will commute with those of the (1+1) G-brane. Therefore the generators of the complete theory,

\[ s^{\text{total}} = s + s^{\text{matter}}, \]
will obey the conformal algebra with a total central charge which is the sum of the central charges of the (1+1) G-brane and that of the matter theory.

There is one theory for a matter field which has a central charge at the classical level, that is the Liouville theory for which,

\[ s_{\text{Liouv}}^\perp = \frac{1}{2} \left( k\pi^2 + k^{-1}\psi'^2 \right) - 2k^{-1}\psi'' + \frac{1}{2k} m^2 \psi, \]

\[ s_{\text{Liouv}}^1 = \pi \psi' - 2\pi', \]

obey the algebra (3)–(5) with,

\[ \zeta_{\text{Liouv}} = \frac{4}{k}. \]

The total central charge is then

\[ \zeta_{\text{total}} = \zeta + \frac{4}{k}. \] (26)

### 4.3. Anomaly cancellation

An interesting possibility now appears, namely that of adjusting the (1+1) G-brane central charge \( \zeta \) for a given \( k \) so that the sum (26) vanishes,

\[ \zeta_{\text{total}} = 0. \] (27)

For physical reasons, such as the positivity of the energy of the Liouville field, it is desirable to have \( k > 0 \), however no similar restriction appears for \( \zeta \) because, as it was pointed out above, for any sign of \( \zeta \) the (1+1) G-brane energy is unbounded from above and from below.

When (27) holds, and only then, one has the option of constraining the total generators to vanish,

\[ s_{\text{total}}^\perp = s_{\perp} + s_{\text{Liouv}}^\perp \approx 0, \] (28)

\[ s_{\text{total}}^1 = s_1 + s_{\text{Liouv}}^1 \approx 0. \] (29)

If the total central charge were not to vanish equations (28) and (29) would not be preserved under the action of \( s_{\text{total}}^\perp \) and \( s_{\text{total}}^1 \).

Before the constraints are imposed the complete theory is invariant under two independent conformal symmetries, one acting on the (1+1) G-brane, the other on the Liouville field. There are independent parameters \( \eta \) for each, which are not varied in the action principle. The two theories are decoupled and there are three degrees of freedom per space point.

If the constraints are imposed the only symmetry is a simultaneous conformal transformation of the same magnitude on both the brane and the Liouville field. The common parameter \( \eta \) is now varied in the action principle in order to yield the constraint. The number of degrees of freedom per point of the combined theory then decreases from three to one, and the brane and the Liouville field become coupled through the constraint, although their equations of motion remain uncoupled.
When the total central charge vanishes, according to eq. (27) the complete action is invariant under spacetime reparametrizations because the anomalous term in (15) is proportional to $\xi_{\text{total}}$. On the other hand the change of the complete action under a Weyl transformation is still given by (16) because the Liouville action is Weyl invariant if one assumes $\psi$ to have that property.

### 4.4. Lagrangian

For the coupled action one may pass from the Hamiltonian formulation to a Lagrangian formulation by eliminating the constraints and the momenta from their equations of motion. The resulting Lagrangian density which is a function of $y^\lambda$, $\dot{y}^\lambda$, $\psi$, $\dot{\psi}$ and $\gamma_{\lambda\rho}$ has the form,

$$
L_{\text{total}} = L_{\text{string}} + L_{\text{Liouv}},
$$

where,

$$
L_{\text{string}} = \eta^\perp L,
$$

with $L$ given by (11), and

$$
L_{\text{Liouv}} = \frac{1}{2k} \left\{ \left( \eta^\perp - \frac{1}{2} \eta^1 \right)^2 - \eta^\perp \eta^{1,2} + 4 \eta^\perp \eta^{1,3} + 2 \eta^\perp \Lambda e^\psi \right\}.
$$

(See Ref. 9).

Here $\eta^\perp$ and $\eta^1$ stand as abbreviations for,

$$
\eta^\perp = -g^{-1} \tilde{n}^\lambda \dot{y}^\lambda, \quad \eta^1 = g^{-1} \gamma_{\lambda\rho} y^\lambda \dot{y}^\rho,
$$

and the constant $k$ is related to the (1+1) G-brane central charge $\zeta$ through (15) and (27).

Since the conformal anomalies of the (1+1) G-brane and the Liouville field mutually cancel the action,

$$
I_{\text{total}} = \int dtdx L_{\text{total}},
$$

is invariant under spacetime reparametrizations, therefore the total energy-momentum tensor,

$$
T_{\lambda\rho} = \frac{2}{\sqrt{-\gamma}} \frac{\delta I_{\text{total}}}{\delta \gamma^{\lambda\rho}},
$$

is indeed a tensor and it is conserved. The energy and momentum densities (of weight two) are given by

$$
T_{\perp\perp} = T^{\text{total}}_{\lambda\rho} \tilde{n}^\lambda \tilde{n}^\rho = T_{\perp\perp}^{\text{Liouv}} - L, \quad T_{\perp\perp} = T^{\text{total}}_{\lambda\rho} \tilde{n}^\lambda \gamma^\rho_{\perp1} = T_{\perp\perp}^{\text{Liouv}},
$$

whereas the trace is still given by (25).

One sees that the energy of the system acquires a “background contribution” $-L$ in addition to the Liouville energy density. Furthermore it is only the sum of
the two which has covariant meaning. If one changes the spatial coordinate on the surface $y^\lambda(x)$, the two pieces do not transform separately as a density because of the conformal anomaly, but the sum does.

In the above discussion the Liouville field $\psi$ is unrelated to the field $\varphi$ appearing in the ancestor gravitation theory referred to after eq. (11). If it where identified with it, so as to couple the descendant to its ancestor, the variation of the combined action with respect to $\psi$ would be,

$$\delta I = - \int dtdxk^{-1} \eta^\perp e^{\psi} \left( \Lambda + \frac{m^2}{2} \right) \delta \psi,$$

so that the action principle would either be contradictory, if $\Lambda \neq -m^2/2$, or empty, if $\Lambda = -m^2/2$. Incest is either forbidden or it does not happen.

5. Concluding remarks

We have presented a conformal field theory on an arbitrary two-dimensional space-time background. The dynamical object is a space-filling string, and the evolution may be thought as occurring on the group space of the conformal group. The theory is a “descendant” in the sense of Ref. 11, of an analog of the theory of gravitation proposed in Refs. 9, 10 for two dimensional spacetime. The discussion has remained classical throughout, including the treatment of the central charge which already appears at that level. One would expect many delicate issues to appear in the passage to the quantum theory, not least in the discussion of anomalies.

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