Pseudoconvex and Disprisoning Homogeneous Sprays

L. Del Riego
Facultad de Ciencias
Zona Universitaria
Universidad Autónoma de San Luis Potosí
San Luis Potosí, SLP
78290 MEXICO
bestr1084@bestsd.sdsu.edu

Phillip. E. Parker¹
Mathematics Department
Wichita State University
Wichita KS 67260-0033
USA
pparker@twsuvm.uc.twsu.edu

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Abstract The pseudoconvex and disprisoning conditions for geodesics of linear connections are extended to the solution curves of general homogeneous sprays. The main result is that pseudoconvexity and disprisonment are jointly stable in the fine topology on the space of all homogeneous sprays of any degree of homogeneity.

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1 Introduction

A spray is a natural generalization of the system of geodesics of a linear connection. Pseudoconvexity and disprisonment are two increasingly important properties which such a system may have. Since readers are not likely to be equally familiar with all of these, we shall try in this section to give enough background to provide some motivation.

1.1 Pseudoconvexity and disprisonment

Global hyperbolicity is well known to play an important role in Lorentzian geometry and general relativity. It is a sufficient condition for the existence of maximal length geodesic segments joining causally related points [20]. This gives a partial generalization of the important Hopf-Rinow Theorem to Lorentzian manifolds. A spacetime has a global Cauchy surface $N$ if and only if it is a globally hyperbolic spacetime [15]. Furthermore, these spacetimes are topological products of the form $\mathbb{R} \times N$ and global hyperbolicity is a stable property. One important application of global hyperbolicity is in the singularity theorems of Hawking and Penrose [17]. Many spacetimes have large globally hyperbolic subsets, and these subsets may be used to construct causal geodesics without conjugate points. The timelike convergence condition and the generic condition then imply the incompleteness of such geodesics. Existence of an incomplete causal geodesic is usually taken as indicating a physical singularity.

Beem and Parker [3, 5, 6, 7, 8] have considered a generalization of global hyperbolicity called \textit{pseudoconvexity}, which often can be used in place of global hyperbolicity. A spacetime $(M, g)$ is said to be causally pseudoconvex if and only if given any compact set $K$ in $M$ there is always a larger compact set $K'$ such that all causal geodesics segments joining points of $K$ lie entirely within $K'$. This basic concept can be used for any class of geodesics. For example, null pseudoconvexity is the requirement that all null geodesic segments with endpoints in $K$ lie entirely within $K'$.

Like global hyperbolicity, (causal) pseudoconvexity is a type of completeness requirement. Intuitively, one may think of pseudoconvex spaces as failing to have any “interior” points missing. Thus, Minkowski space less any compact set is neither globally hyperbolic nor causally pseudoconvex. A simple example of a causally pseudoconvex spacetime which is not globally hyperbolic is the open strip $a < x < b$ in the Minkowski $(t, x)$ plane. We say that causal pseudoconvexity generalizes global hyperbolicity since every
globally hyperbolic spacetime is causally pseudoconvex.

In the theory of pseudodifferential equations, the concept of pseudoconvexity is applied to bicharacteristic segments in the study of global solvability. If \((M, g)\) is a Lorentzian manifold with d’Alembertian \(\Box\), then the symbol of \(\Box\) is the metric tensor in the contravariant form. In this case the bicharacteristic segments are the null geodesic segments, and the inhomogeneous wave equation \(\Box u = f\) has global solutions in the distribution sense if (1) \((M, g)\) is null pseudoconvex and (2) each end of each inextendible null geodesic fails to be imprisoned. This second requirement is called disprisonment of null geodesics. In the language of PDE’s, it is the requirement that the operator be of real principal type. Beem and Parker \([3, 5, 6]\) have obtained several powerful results on the stability of solvability of pseudodifferential equations \(\text{via}\) stability theorems for pseudoconvexity and disprisonment. This approach also yielded some new methods in the study of sectional curvature \([4]\).

Interestingly, both pseudoconvexity and disprisonment of null geodesics fail to be separately \(C^1\)-stable in the Whitney topology, but the requirement that they hold jointly \(is\) \(C^1\)-stable \([6]\).

As one would expect, disprisonment and pseudoconvexity have important implications for the geodesic structure of a spacetime. Williams \([21]\) found examples of geodesically complete spacetimes with arbitrarily close incomplete metrics in the Whitney \(C^r\)-topology. Hence geodesic completeness fails to be Whitney \(C^r\)-stable for all \(r \geq 1\). On the other hand, Beem and Ehrlich \([2]\) have established the \(C^1\)-fine stability of causal geodesic completeness for Lorentzian manifolds which are both causally pseudoconvex and causally disprisoning.

Beem and Parker \([7]\) established an extension of Seifert’s result \([20]\) to manifolds with a linear connection which is pseudoconvex and disprisoning for \(all\) types of geodesics. In particular, one now has pseudoriemannian versions of the Hopf-Rinow and Hadamard-Cartan Theorems.

1.2 Sprays

One of the most important generalizations of ordinary differential equations to manifolds is the well-known class of second-order differential equations or sprays. For example, they occur as the Hamiltonian vector fields of regular Lagrangians in variational problems \([9, 19]\). A (general) spray on a manifold \(M\) is defined as a projectable section of the second-order tangent bundle \(TTM \to TM\). This is precisely the condition needed to define a second-
order differential equation \([10, 9]\). Recall that an integral curve of a vector field on \(TM\) is the canonical lift of its projection if and only if the vector field is projectable. For any curve \(c\) in \(M\) with tangent vector field \(\dot{c}\), this \(\dot{c}\) is the canonical lift of \(c\) to \(TM\) and \(\ddot{c}\) is the canonical lift of \(\dot{c}\) to \(TTM\). Then each projectable vector field \(S\) on \(TM\) determines a second-order differential equation on \(M\) by \(\ddot{c} = S \circ \dot{c}\) and any such curve with \(\dot{c}(s_0) = v_0 \in T_{c(s_0)}M\) is a solution with initial condition \(v_0\). Solutions are preserved under translations of parameter, they exist for all initial conditions by the Cauchy theorem, and, as our manifolds are assumed to be Hausdorff, each solution will be unique provided we take it to have maximal domain; i.e., to be inextendible \([10, 12, 13]\).

Let \(J\) be the canonical involution on \(TTM\) and \(C\) the Euler (or Liouville) vector field on \(TM\). We recall that in local coordinates, \(J(x, y, X, Y) = (x, X, y, Y)\) and \(C : (x, y) \mapsto (x, y, 0, y)\). Then a section \(S\) of \(TTM\) over \(TM\) is a spray when \(JS = S\); that is, when it can be expressed locally as \(S : (x, y) \mapsto (x, y, Y(x, y))\). We say that a spray \(S\) is (positively) homogeneous of degree \(m\) when \([C, S] = (m - 1)S\) for \(m \geq 0\). In this case the functions \(Y(x, y)\) are homogeneous of degree \(m\) in the fiber component: \(Y(x, ay) = a^m Y(x, y)\). Here \(a\) denotes the induced tangent map of scalar multiplication by \(a\). We denote the set of sprays on \(M\) by \(\text{Spray}(M)\) and those homogeneous of degree \(m\) by \(\text{Spray}_m(M)\). It has been usual to consider only (positive) integral degrees of homogeneity, but we make no such restriction. Previously \([12, 18]\), our (general) sprays have been called semisprays and the name sprays reserved for those homogeneous of degree two. We do not make this restriction either. We do, however, consider only sprays defined on the entire tangent bundle \(TM\); others \([18]\) have used the reduced tangent bundle with the 0-section removed.

For some purposes, it is more convenient to use a different characterization of sprays \([12, 18]\). The vertical endomorphism \(V\), in local coordinates given by \(V(x, y, X, Y) = (x, y, 0, X)\), may be regarded as a vector-valued 1-form on \(TM\). We observe that \(V : TTM \to VTM\), the vertical bundle over \(TM\), and is a nilpotent map: \(V^2 = 0\). Then a spray also can be characterized by \(VS = C\). This version has been used, for example, in stability theory \([13]\).

Recall that in general a connection only provides a horizontal subbundle of \(TTM\) complementary to the vertical subbundle. The Nijenhuis bracket (e.g., \([13]\)) determines for each spray (or connection) a Lie subalgebra of the Lie algebra of vector fields on \(TM\). This subalgebra consists precisely of those morphisms of \(TTM\) over \(TM\) which preserve the horizontal and
Several important results concerning sprays [1, 10, 14, 18] rely on the facts that each spray $S$ determines a unique torsion-free connection $\Gamma$, and conversely, every spray $S$ arises from a connection $\Gamma$ the torsion of which can be assigned arbitrarily. The solution curves of the differential equation $\ddot{c} = S_\Gamma \circ \dot{c}$ for a connection-induced spray are precisely the geodesics of that connection. The familiar geodesic spray corresponding to a linear connection is a quadratic spray: $[C, S] = S$. In this case its solution curves are not only preserved under translations, but also under affine transformations of the parameter $s \mapsto as + b$ for constants $a, b$ with $a \neq 0$.

Here is perhaps the simplest example of a spray arising from a regular variational problem. If $(M, g)$ is a pseudoriemannian manifold, then the energy function $\epsilon_g$ is defined by
\[
\epsilon_g : TM \to \mathbb{R} : v \mapsto \frac{1}{2}g(v, v).
\]
The canonical spray $S_g$ on $M$ is defined as the vector field on $TM$ corresponding to the 1-form $-d\epsilon_g$ on $T^*M$ with respect to the canonical symplectic structure on $T^*M$. As a derivation on real functions defined on $TM$, $S_g$ annihilates $\epsilon_g$. Thus the lifts $\dot{c}$ of solution curves $c$ are integral curves in $TM$ of $S_g$ along which $\epsilon_g$ is constant. Now the Levi-Civita connection $\Gamma_g$ determines a unique spray which also annihilates $\epsilon_g$. It follows that $S_g$ is the geodesic spray, so the solution curves $c$ for $S_g$ are the geodesics of $\Gamma_g$.

In this paper, we shall study the combination: general homogeneous sprays which are pseudoconvex and disprisoning. Among other things, this may be regarded as a continuation of the program to geometrize the study of PDE’s begun by Beem and Parker [3, 5, 6, 7]. Section 2 contains our definitions, notations and conventions. Section 3 is devoted to the generalization of the main results of [8] to a large class of sprays. Finally, Section 4 gives a generalization of the main stability result of [6] to homogeneous sprays.

Throughout, all manifolds are smooth (meaning $C^\infty$), connected, paracompact, Hausdorff, and usually noncompact (see Section 2).

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2 Preliminaries

We begin with the principal definitions. Let $S$ be a spray on $M$.

**Definition 2.1** We say that a curve $c : (a, b) \to M$ is a geodesic of $S$ or an $S$-geodesic if and only if the natural lifting $\dot{c}$ of $c$ to $TM$ is an integral curve of $S$.

This means that if $\ddot{c}$ is the natural lifting of $\dot{c}$ to $TTM$, then $\ddot{c} = S(\dot{c})$.

**Definition 2.2** We shall say that $S$ is pseudoconvex if and only if for each compact $K \subseteq M$ there exists a compact $K' \subseteq M$ such that each $S$-geodesic segment with both endpoints in $K$ lies entirely within $K'$.

If we wish to work directly with the integral curves of $S$, we merely replace “in” and “within” by “over”.

**Definition 2.3** We shall say that $S$ is disprisoning if and only if no inextendible $S$-geodesic is contained in (or lies over) a compact set of $M$.

In relativity theory [16], such inextendible geodesics are said to be imprisoned in compact sets; hence our name for the negation of this property.

Following this definition, we make a convention: all $S$-geodesics are to be regarded as always extended to the maximal parameter intervals (i.e., to be inextendible) unless specifically noted otherwise. When the spray $S$ is clear from context, we refer simply to geodesics. Also, we shall consider only noncompact manifolds because no spray can be disprisoning on a compact manifold. However, Corollary 3.7 may be used to obtain results about compact manifolds for which the universal covering is noncompact.

**Example 2.4** When $S$ is a quadratic spray, we recover the notions previously defined by Beem and Parker [8] for linear connections.

The natural lift $\dot{c}$ of a curve $c$ was denoted by $c'$ in [8]. With this change in notation, the proof of Lemma 3 there applies word-for-word to sprays. Thus we have

**Lemma 2.5** Let $M$ be a manifold with a pseudoconvex and disprisoning spray $S$. If $p \neq q$, $p_n \to p$ and $q_n \to q$, and if for each $n$ there is a geodesic segment from $p_n$ to $q_n$, then there is a geodesic segment from $p$ to $q$. $\square$

As in [8], we obtain
Proposition 2.6 Let $M$ be a manifold with a pseudoconvex and disprisoning spray $S$. If $K \subseteq M$ is compact, then the geodesic convex hull $\lbrack K \rbrack$ is compact.

Proof: Pseudoconvexity implies that $\lbrack K \rbrack$ is contained in a compact set, and the lemma shows $\lbrack K \rbrack$ is closed, hence compact. \hfill \Box

3 Geodesic Systems

In [8], results were obtained concerning geodesic connectedness of manifolds with a linear connection. Since quadratic sprays are equivalent to linear connections, we immediately obtain corresponding results for quadratic sprays. For example [8, Proposition 5]

Proposition 3.1 Let $S$ be a pseudoconvex and disprisoning quadratic spray on $M$. If $S$ has no conjugate points, then $M$ is geodesically connected. In other words, every pair of points in $M$ may be joined by at least one $S$–geodesic segment. \hfill \Box

However, we are primarily interested in more general sprays.

Recall that for each $p \in M$ and each $v \in T_pM$, there is a unique geodesic $c_v$ such that $c_v(0) = p$ and $\dot{c}_v(0) = v$.

Definition 3.2 The exponential map of $S$ at $p$ is given by $\exp_p(v) = c_v(1)$ for all $v \in T_pM$ such that $c_v(1)$ exists.

Thus, as in the usual cases, the domain of exp is an open tubular neighborhood of the 0-section in $TM$. The next lemma follows from Lemma 2 as in [8].

Lemma 3.3 Let $M$ be a manifold with a pseudoconvex and disprisoning spray $S$. Assume $p \neq q$ and $q_n \rightarrow q$. If $(v_n)$ is a sequence in $T_pM$ such that $\exp_p(v_n) = q_n$, then there is a vector $v \in T_pM$ and a subsequence $(v_k)$ of $(v_n)$ such that $v_k \rightarrow v$ and $\exp_p(v) = q$. \hfill \Box

Theorem 3.4 If $S$ is a homogeneous spray, then its exponential map is a local diffeomorphism.

Proof: We proceed as in the usual proof (e.g., [11, p. 116]) except that now $\exp_p(tv) = c_{tv}(1) = c_v(tm)$ for $0 \leq t \leq 1$ when $S$ is homogeneous of degree $m$. But we still obtain $\exp_{ps} = 1$ as usual. \hfill \Box
All examples of general sprays which we have examined have this property. We conjecture that it is true of all sprays, but we cannot prove it yet. Thus we make

**Definition 3.5** A spray is LD if and only if its exponential map is a local diffeomorphism.

What we shall use is that the geodesics of such sprays give normal starlike neighborhoods of each point in $M$. This fact together with Lemma 2.5 yields the next result, as in [8, Prop. 5].

**Proposition 3.6** Let $M$ be a manifold with a pseudoconvex and disprisoning LD spray $S$. If $S$ has no conjugate points, then $M$ is geodesically connected.

Let $M$ be a manifold with a spray $S$ and let $\tilde{M}$ be a covering manifold. If $\phi : \tilde{M} \to M$ is the covering map, then it is a local diffeomorphism. Thus $\tilde{S} = (\phi_*)^*S$ is the unique spray on $\tilde{M}$ which covers $S$, geodesics of $\tilde{S}$ project to geodesics of $S$ and geodesics of $S$ lift to geodesics of $\tilde{S}$. Also, $S$ has no conjugate points if and only if $\tilde{S}$ has none. The fundamental group is simpler, and $\tilde{S}$ may be both pseudoconvex and disprisoning even if $S$ is neither. Proposition 3.6 and simple projection arguments yield (cf. [8, Corollary 6])

**Corollary 3.7** Let $M$ be a manifold with a pseudoconvex and disprisoning LD spray $S$ and let $\tilde{M}$ be a covering manifold with covering homogeneous spray $\tilde{S}$. If $\tilde{S}$ has no conjugate points, then both $\tilde{M}$ and $M$ are geodesically connected.

The next result is the analogue of Theorem 9 of [8].

**Theorem 3.8** Let $S$ be a pseudoconvex and disprisoning LD spray on $M$. If $S$ has no conjugate points, then for each $p \in M$ the exponential map of $S$ at $p$ is a diffeomorphism.

We remark that none of these results require (geodesic) completeness of the spray $S$.

### 4 Stability

In this section we consider the joint stability of pseudoconvexity and disprisonment for homogeneous sprays in the fine topology. Because each linear
connection determines a homogeneous spray, Examples 2.1 and 2.2 of [6] show that neither condition is separately stable. (Although [6] is written in terms of principal symbols of pseudodifferential operators, the cited examples are actually metric tensors). We shall obtain $C^0$-fine stability, rather than $C^1$-fine stability as in [6], due to our effective shift from potentials to fields as the basic objects. The proof requires only minor modifications of that in [6], so we shall concentrate on the changes here and refer to [6] for an outline and additional details.

Rather than considering $r$-jets of functions, we now take $r$-jets of sections in defining the Whitney or $C^r$-fine topology as in Section 2 of [6]. Also, just as was done there, we must modify these topologies to take homogeneity into account. Let $h$ be an auxiliary complete Riemannian metric on $M$. A homogeneous spray is determined by its degree of homogeneity $m$ and its restriction to the $h$-unit sphere bundle $UM$ in $TM$. (Note that our $UM$ replaces $S^*M$ in [6], changing from the cotangent to the tangent bundle.) Thus we actually look at the $C^r$-fine topology on the sections of $TTM|UM$ over $UM$. However, as in [6], we shall say that a set in $\text{Spray}_m(M)$ is open if and only if the corresponding set in the sections over $UM$ is open.

If $\gamma_1$ and $\gamma_2$ are two integral curves of a spray $S$ with $\gamma_1(0) = (x, v)$ and $\gamma_2(0) = (x, \lambda v)$ for some positive constant $\lambda$, then the inextendible geodesics $\pi \circ \gamma_1$ and $\pi \circ \gamma_2$ differ only by a reparametrization. Thus, as in [6], it will suffice to consider only one integral curve for each direction at each point of $M$.

Observe the the equations of geodesics involve no derivatives of $S$. Thus if $\gamma: [0, a] \to TM$ is a fixed integral curve of $S$ in $TM$ with $\gamma(0) = v_0 \in UM$ and if $\gamma': [0, a] \to TM$ is an integral curve of $S'$ in $TM$ with $\gamma'(0) = v$, then $d_h(\pi \circ \gamma(t), \pi \circ \gamma'(t)) < 1$ for $0 \leq t \leq a$ provided that $v$ is sufficiently close to $v_0$ and $S'$ is sufficiently close to $S$ in the $C^0$-fine topology. This and the compactness of $UK_1$ when $K_1$ is compact yield the following result.

**Lemma 4.1** Assume $K_1$ is a compact set contained in the interior of the compact set $K_2$ and let $S$ be a disprisoning homogeneous spray. There exist tangent vectors $v_1, \ldots, v_m \in TK_1$ and positive constants $\delta_1, \ldots, \delta_m, \alpha_1, \ldots, \alpha_m, \epsilon$ such that if $S'$ is in a $C^0$-fine $\epsilon$-neighborhood of $S$ over $V$, then the following hold:

1. if $c$ is an inextendible $S$-geodesic with $c(0)$ in a $\delta_1$-neighborhood of $v_i$, then $c[0, a_i] \subset V$ and $c(a_i) \in V - K_2$. 

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2. If $c'$ is an inextendible $S'$-geodesic with $\dot{c}'(0)$ in a $\delta_i$-neighborhood of $v_i$, then $c'[0, a_i] \subset V$ and $c'(a_i) \in V - K_2$;

3. Two inextendible geodesics, $c$ of $S$ and $c'$ of $S'$ with $\dot{c}(0)$ and $\dot{c}'(0)$ in a $\delta_i$-neighborhood of $v_i$, remain uniformly close together for $0 \leq t \leq a_i$;

4. The union of all the $\delta_i$-neighborhoods of the $v_i$ is large enough to cover the part of $TK_1$ in which we are interested.

Continuing to follow [6], we construct the increasing sequence of compact sets $\{A_n\}$ which exhausts $M$ and the monotonically nonincreasing sequence of positive constants $\{\epsilon_n\}$. The only changes from [6, p. 17f] are to use integral curves of $S$ in $TM$ instead of bicharacteristic strips in $T^*M$. No additional changes are required for the proof of the next result either.

**Lemma 4.2** Let $S$ be a pseudoconvex and disprisoning homogeneous spray and let $S'$ be $\delta$-near to $S$ on $M$. If $\dot{c}' : (a, b) \to M$ is an inextendible $S'$-geodesic, then there do not exist values $a < t_1 < t_2 < t_3 < b$ with $c'(t_1) \in A_n$, $c'(t_3) \in A_n$, and $c'(t_2) \in A_n + 4 - A_n + 3$.

Now we establish the stability of pseudoconvex and disprisoning homogeneous sprays by showing that the set of all sprays in $Spray_m(M)$ which is pseudoconvex and disprisoning is an open set in the $C^0$-fine topology. The only changes needed from the proof of Theorem 3.3 in [6, p. 19] are replacing principal symbols by sprays, bicharacteristic strips by integral curves, $S^*A_n$ by $UA_n$, and references to Lemma 3.2 there by references to Lemma 4.2 here.

**Theorem 4.3** If $S \in Spray_m(M)$ is a pseudoconvex and disprisoning homogeneous spray, then there is some $C^0$-fine neighborhood $W(S)$ in $Spray_m(M)$ such that each $S' \in W(S)$ is both pseudoconvex and disprisoning.

Since linear connections may be identified with quadratic sprays, we immediately obtain

**Corollary 4.4** Pseudoconvexity and disprisonment are jointly $C^0$-fine stable properties in the space of linear connections.

In particular,

**Corollary 4.5** If $M$ is a pseudoconvex and disprisoning pseudoriemannian manifold, then any linear connection on $M$ which is sufficiently close to the Levi-Civit"a connection is also pseudoconvex and disprisoning.
If we denote by $\text{Spray}_H(M)$ the set of all homogeneous sprays of any degree $m$ on $M$, then we may topologize it by taking the weak topology generated by those on the subsets $\text{Spray}_m(M)$. Now Theorem 4.3 can be given a somewhat more general formulation.

**Theorem 4.6** If $S$ is a pseudoconvex and disprisoning homogeneous spray, then any sufficiently close homogeneous spray (of any degree of homogeneity) is also pseudoconvex and disprisoning. \hfill \Box

**Corollary 4.7** If $(M, g)$ is a pseudoconvex and disprisoning pseudoriemannian manifold, then any homogeneous spray $S$ which is sufficiently close to the geodesic spray $S_g$ is also pseudoconvex and disprisoning. \hfill \Box

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