Scheme-Theoretic Approach to Computational Complexity. I. The Separation of $P$ and $NP$

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February 20, 2024

Abstract

We lay the foundations of a new theory for algorithms and computational complexity by parameterizing the instances of a computational problem as a moduli scheme. Considering the geometry of the scheme associated to $3$-SAT, we separate $P$ and $NP$. In particular, we show that no deterministic algorithm can solve $3$-SAT in time less than $1.296839^n$ in the worst case.

1 Introduction

This paper introduces the rudiments of a new theory for algorithms and computational complexity via the Hilbert scheme. One of the most important consequences of the theory is the resolution of the conjecture $P \neq NP$.

An easily understood reason for the difficulty of the problem we consider is the superficial similarity between the problems in $P$ and $NP$-complete problems. More concretely, one has not been able to find a metric somehow measuring the time complexity of a problem so that the difference between the values for $3$-SAT and $2$-SAT is large enough. Extracting this intrinsic property from a problem seems out of reach when it is treated by only combinatorial means.

From an elementary point of view, a computational problem is considered to be a language recognized by a Turing machine. Through a slightly refined lens, it is a Boolean function computed by a circuit. We recognize the existence of a much deeper perspective: A computational problem is a (moduli) scheme formed by its instances, and an algorithm is a morphism geometrically reducing it to a single point. This opens the possibility of understanding computational complexity using the language of category theory. In particular, we define a functor from the category of computational problems to the category of schemes parameterizing the instances of a computational problem, albeit currently restricted to $k$-SAT.

For concreteness, consider a satisfiable instance of $3$-SAT represented by the formula $\phi$ with variables $x_1, \ldots, x_n$. We associate with this instance all the solutions that make $\phi$ satisfiable, which can be expressed as the zeros of a polynomial $\phi(x_1, \ldots, x_n)$ over $F_2$. We then identify this information by considering the closed subscheme $\text{Proj} \, F_2[x_0, x_1, \ldots, x_n]/(\phi(x_0, x_1, \ldots, x_n))$. The global scheme corresponding to the computational problem $3$-SAT is the Hilbert scheme parameterizing these closed subschemes together with a set of others to ensure connectedness.

The next step is to unify the notion of a reduction and an algorithm in the new setting. Consider $1$-SAT $\in P$. In order to separate $P$ and $NP$, one needs to rule out a polynomial-time reduction $f$.

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satisfying \( x \in \text{3-SAT} \Leftrightarrow f(x) \in \text{1-SAT} \). We extend this line of thinking by introducing the simplest object in the category of computational problems: the trivial problem defined via an instance with an empty set of variables, which may be represented by a single point. In our new language, solving a problem is nothing but reducing it to the trivial problem. One then needs to show that, in geometric terms we will later formalize, it is impossible to map the scheme of 3-SAT to a single point with polynomial number of unit operations.

2 Computational Problems and the Extended Amplifying Functor

2.1 Computational Problems

A computational problem \((\Pi, \overline{\Pi})\) consists of a set \(\Pi\) of positive instances and a set \(\overline{\Pi}\) of negative instances such that \(\Pi \cap \overline{\Pi} = \emptyset\). In this paper we impose that each instance consists of a finite set of polynomial equations over \(\mathbb{F}_2\). We thus use a polynomial system as a synonym for an instance. The synonym for a single polynomial equation is a clause. One seeks, given an instance, an assignment to the variables in \(\mathbb{F}_2\) satisfying all the equations of the instance. In particular, an instance is in \(\Pi\) if it has such a solution; otherwise it is in \(\overline{\Pi}\). By an instance is meant a positive instance in the rest of the paper, unless otherwise stated. We also briefly denote a given problem by its set of positive instances \(\Pi\). In contrast, we explicitly say if an instance is negative. With an abuse of notation, if \(\Pi\) consists of a single instance \(I\), then \(I\) also denotes the computational problem \(\Pi\). We give below examples of instances and negative instances of some computational problems, both in the classical logical form and in the algebraic form as polynomial systems over \(\mathbb{F}_2\). The simplest problem is what we call TRIVIAL or \(T\) for short, defined via a single instance and a single negative instance, both with an empty set of variables. These instances are denoted by (with an abuse of notation) \(T := \text{True}\) and \(F := \text{False}\). Their algebraic form are \(\{0 = 0\}\) and \(\{1 = 0\}\), respectively.

- **Problem:** TRIVIAL or \(T\)
  - Logical form: \(\{\text{True}\}, \{\text{False}\}\)
  - Algebraic form: \(\{0 = 0\}\), \(\{1 = 0\}\).

- **Problem:** UNIT or \(U\)
  - Logical form: \(\{x\}, \{\overline{x}\}\)
  - Algebraic form: \(\{1 - x = 0\}, \{x = 0\}\).

- **Problem:** 1-SAT
  - Logical form: \(\{x_1 \wedge \overline{x}_2\}, \{x_1 \wedge x_2\}\)
  - Algebraic form: \(\{1 - x_1 = 0, x_2 = 0\}, \{1 - x_1 = 0, x_1 = 0\}\)

- **Problem:** 3-SAT
  - Logical form: \(\{(x_1 \lor \overline{x}_3) \land (x_3 \lor \overline{x}_5)\}\)
  - Algebraic form: \(\{(1 - x_1)x_3(1 - x_4) = 0, x_2(1 - x_3)x_5 = 0\}\).

2.2 Representability of the Hilbert Functor

Let \(S\) be a scheme, and let \(X \subseteq \mathbb{P}^n_S\) be a closed subscheme. Define

\[
H(X/S) := \{Z \subseteq X \text{ is a closed subscheme, } Z \rightarrow S \text{ is flat}\}.
\]
The Hilbert functor $\mathcal{H}_{X/S}$ is the functor $T \mapsto H(X \times_S T/T)$ for any $S$-scheme $T$. We set $S = \text{Spec } \mathbb{F}_2$, and denote $\mathcal{H}_{X/\mathbb{F}_2}$ briefly as $\mathcal{H}_X$.

Let $X$ be a projective scheme over $\mathbb{F}_2$, and let $Z \subseteq X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent sheaf on $Z$. The Hilbert polynomial of $Z$ with respect to $\mathcal{F}$ is $P(Z, \mathcal{F})(m) := \chi(Z, \mathcal{F}(m))$, where $\mathcal{F}(m)$ is the twisting of $\mathcal{F}$ by $m$, and $\chi(Z, \mathcal{F})$ denotes the Euler characteristic of $\mathcal{F}$ given by

$$\chi(Z, \mathcal{F}) := \sum_{i=0}^{\dim Z} (-1)^i \dim_{\mathbb{F}_2} H^i(Z, \mathcal{F}).$$

The Hilbert polynomial of $Z$ is

$$P(Z)(m) := \chi(Z, \mathcal{O}_Z(m))$$

where $\mathcal{O}_Z$ is the structure sheaf of $Z$. Let $\mathcal{H}_X^P$ denote the subfunctor of $\mathcal{H}_X$ induced by the closed subschemes of $X$ with a fixed Hilbert polynomial $P \in \mathbb{Q}[x]$. By the following result stated in our context, the Hilbert functor is representable by a projective scheme over $\mathbb{F}_2$.

**Theorem 2.1** ([Π]). Let $X$ be a projective scheme over $\mathbb{F}_2$. Then for every polynomial $P \in \mathbb{Q}[x]$, there exists a projective scheme $\text{Hilb}^P(X)$ over $\mathbb{F}_2$, which represents the functor $\mathcal{H}_X^P$. Furthermore, the Hilbert functor $\mathcal{H}_X$ is represented by the Hilbert scheme

$$\text{Hilb}(X) := \coprod_{P \in \mathbb{Q}[x]} \text{Hilb}^P(X).$$

We will specifically consider the computational problem $\Pi := k\text{-SAT}$ defined via the variable set \{x_1, \ldots, x_n\}. An instance of this problem is a Boolean formula in CNF with each clause containing \(k\) literals, and by our assumptions its corresponding polynomial system over $\mathbb{F}_2$. Note first that given a homogenized polynomial $\phi$, one might consider the closed subscheme

$$\text{Proj } \mathbb{F}_2[x_0, x_1, \ldots, x_n]/(\phi(x_0, x_1, \ldots, x_n)),$$

so that each polynomial equation and hence a polynomial system of $\Pi$ identifies a closed subscheme of $\mathbb{P}_2^n$ via the corresponding ideal. We thus set $X = \mathbb{P}_2^n$ in the theorem above, and refer to the Hilbert polynomial of an instance.

### 2.3 Reductions and Prime Homogeneous Simple Sub-problems

Let $(A, \overline{A})$ and $(B, \overline{B})$ be computational problems, and $f : (A, \overline{A}) \to (B, \overline{B})$ be a set-theoretic map such that $f(A) \subseteq B$ and $f(\overline{A}) \subseteq \overline{B}$.

**Definition 2.2.** A computational procedure $\alpha_f : (A, \overline{A}) \to (B, \overline{B})$ realizing $f$, possibly with an advice string (thus simulating circuits), is called a reduction. We assume that this procedure is executed by a Turing machine, which starts with an element of $(A, \overline{A})$ on its tape, and stops with an element of $(B, \overline{B})$. In this case we disregard the action of the Turing machine on negative instances and briefly denote $\alpha_f$ by $\alpha_f : A \to B$.

**Definition 2.3.** The number of deterministic unit operations performed by a reduction $\alpha_f$ is called the complexity of $\alpha_f$, denoted by $\tau(\alpha_f)$.

**Definition 2.4.** $\tau(f) := \tau(A, B) := \min_{\alpha_f} \tau(\alpha_f)$ is called the complexity of $f$.
Definition 2.5. \( \tau(A) := \tau(A, T) \) is called the complexity of solving \( A \). In this case a computational procedure \( \alpha_f : A \to T \) realizing the unique set-theoretic map \( f : (A, \overline{A}) \to (T, F) \) such that \( f(A) = T \) and \( f(\overline{A}) = F \) is said to solve \( A \).

Definition 2.6. Given an instance \( I \) of \( \Pi \), a reduction \( \alpha : I \to T \) is called a unit operation.

An example of a unit operation is as follows. Suppose \( I \) is \( \{x = 0\} \). A unit operation writes 0 on \( x \), outputting \( \{0 = 0\} \), which is \( T \).

Definition 2.7. Given two instances \( I_1 \) and \( I_2 \) of \( \Pi \), a reduction \( \alpha : I_1 \to I_2 \) is called a unit instance operation.

An example of a unit instance operation is as follows. Suppose \( I_1 \) is \( \{x_1 = 0, 1 - x_2 = 0\} \). Then replacing \( x_1 \) with \( 1 - x_1 \) and \( x_2 \) with \( 1 - x_2 \), we get another instance \( I_2 \), which is \( \{1 - x_1 = 0, x_2 = 0\} \).

Definition 2.8. A reduction is called a unit reduction if it is a unit operation or a unit instance operation.

Definition 2.9. A computational problem defined via a non-empty subset of the instances of \( \Pi \) is called a sub-problem of \( \Pi \).

Definition 2.10. A sub-problem \( \Lambda \) of \( \Pi \) is called a simple sub-problem if the instances of \( \Lambda \) have the same Hilbert polynomial.

Definition 2.11. Instances \( I_1 \) and \( I_2 \) of \( \Pi \) are said to be distinct if they satisfy the following:

1. They have distinct solution sets over \( \mathbb{F}_2 \).
2. \( I_1 \setminus I_2 \neq \emptyset \).
3. \( I_2 \setminus I_1 \neq \emptyset \).

In this case we also say that \( I_1 \) is distinct from \( I_2 \).

Definition 2.12. A sub-problem \( \Lambda \) of \( \Pi \) whose instances are defined via the variable set \( S = \{x_1, \ldots, x_n\} \), is said to be homogeneous if all the variables in \( S \) appear in each instance of \( \Lambda \) and the instances of \( \Lambda \) are pair-wise distinct.

Definition 2.13. Unit reductions \( \alpha : I_1 \to I_2 \) and \( \beta : I_3 \to I_4 \) are said to be distinct if they satisfy the following:

1. \( (I_1 \triangle I_2) \setminus (I_3 \triangle I_4) \neq \emptyset \).
2. \( (I_3 \triangle I_4) \setminus (I_1 \triangle I_2) \neq \emptyset \).

In this case we also say that \( \alpha \) is distinct from \( \beta \).

Remark 2.14. By definition, if \( \alpha \) and \( \beta \) are distinct unit reductions, then \( \alpha \) performs an operation that does not exist in \( \beta \), and \( \beta \) performs an operation that does not exist in \( \alpha \).

Consider the example given after Definition 2.7 with \( I_1 : \{x_1 = 0, 1 - x_2 = 0\} \). The unit instance operation permuting the variables \( x_1 \) and \( x_2 \) is not distinct from the aforementioned unit instance operation, as it results in the same instance \( I_2 \).
Definition 2.15. Given a sub-problem \( \Lambda \) of \( \Pi \), let \( T \) be the set of all unit instance operations defined between its distinct instances. The sub-problem \( \Lambda \) is said to be prime if the elements of \( T \) are pair-wise distinct.

Consider the following as an example. Let \( \Lambda \) be defined via the instances

\begin{align*}
I_1 &: \{x_1 = 0, x_2 = 0\}, \\
I_2 &: \{x_1 = 0, 1 - x_2 = 0\}, \\
I_3 &: \{1 - x_1 = 0, 1 - x_2 = 0\}.
\end{align*}

Then \( \Lambda \) is not prime since the unit instance operation from \( I_1 \) to \( I_3 \) contains the unit instance operations from \( I_1 \) to \( I_2 \) and \( I_2 \) to \( I_3 \). In particular, \( (I_1 \Delta I_2) \setminus (I_1 \Delta I_3) = \emptyset \).

Proposition 2.16. Let \( \Lambda \) be a homogeneous sub-problem of \( \Pi \). Let \( I_1 \) and \( I_2 \) be distinct instances of \( \Lambda \). Then a unit operation \( \alpha : I_1 \to T \) is distinct from a unit operation \( \beta : I_2 \to T \).

Proof. Assume \((I_1 \Delta T) \setminus (I_2 \Delta T) = \emptyset \). By definition of \( T \), this is equivalent to \((I_1 \cup T) \setminus (I_2 \cup T) = \emptyset \), and hence \( I_1 \setminus I_2 = \emptyset \), which contradicts the fact \( I_1 \) and \( I_2 \) are distinct. \( \square \)

Proposition 2.17. Let \( \Lambda \) be a homogeneous sub-problem of \( \Pi \). Let \( I_1, I_2, \) and \( I_3 \) be instances of \( \Lambda \). Then a unit operation \( \alpha : I_1 \to T \) is distinct from a unit instance operation \( \beta : I_2 \to I_3 \).

Proof. Assume \((I_1 \Delta T) \setminus (I_2 \Delta I_3) = \emptyset \). By definition of \( T \), this is equivalent to \((I_1 \cup T) \setminus (I_2 \Delta I_3) = \emptyset \), which is a contradiction since neither \( I_2 \setminus I_3 \) nor \( I_3 \setminus I_2 \) contains \( T \). \( \square \)

2.4 The Extended Amplifying Functor

Let \( \Lambda \) be a prime homogeneous simple sub-problem of \( \Pi \) consisting of a set of polynomial systems \( \{P_i\}_{i=1}^\ell \) defined via the variables \( x_1, \ldots, x_n \). Let \( \phi_{ij} \) be the homogenized \( j \)-th polynomial in the polynomial system \( P_i \):

\[ \phi_{ij} := \phi_{ij}(x_0, x_1, \ldots, x_n), \]

for \( j = 1, \ldots, |P_i| \). Define

\[ X_i := \text{Proj } \mathbb{F}_2[x_0, x_1, \ldots, x_n]/(\phi_{i1}, \ldots, \phi_{i|P_i|}), \quad (3) \]

for \( i = 1, \ldots, \ell \). Let \( X_\Lambda := \bigcup_{i=1}^\ell X_i \). In words, \( X_\Lambda \) contains all the closed subschemes identified by the instances of \( \Lambda \). Define the amplifying functor \( \mathcal{A}_\Lambda \) on \( \Lambda \) as

\[ T \mapsto \{ Y \times_{\mathbb{F}_2} T | Y \in X_\Lambda, Y \times_{\mathbb{F}_2} T \to T \text{ is flat} \}, \]

for any scheme \( T \) over \( \mathbb{F}_2 \). It is clear that \( \mathcal{A}_\Lambda \) is a subfunctor of the Hilbert functor. Define \( \text{Hilb}(\Lambda) := \text{Hilb}^P(\mathbb{P}^n_{\mathbb{F}_2}) \), where \( P(\Lambda) \) is the Hilbert polynomial associated to \( \Lambda \). For a fixed Hilbert polynomial \( P \), \( \text{Hilb}^P(\mathbb{P}^n_{\mathbb{F}_2}) \) is connected by a result of Hartshorne [2]. Thus, \( \text{Hilb}(\Lambda) \) is connected.

Our strategy is via an extension of the amplifying functor from the category of computational problems to the category of schemes, which we define implicitly via its representation. We call it the extended amplifying functor. The objects of the source category are certain sub-problems, and the morphisms are reductions between sub-problems. In particular, the extended amplifying functor maps a certain sub-problem \( \Lambda \) to a geometric object \( B(\Lambda) \) whose connectivity is crucial. Recall that in order prove a separation result, one needs to establish a lower bound for any computational procedure, and a computational procedure might produce any set of instances during
its execution. Nevertheless, we are only interested in the complexity of a specific computational problem, which encodes all the necessary information for our purpose. This leads us to the following strategy: We consider the representation of the sub-problem of interest in full detail with the aid of the Hilbert functor. Any other instance that might appear during computation however, is mapped to an object that is devoid of structure. This is enough to establish the main result. A more general functor is needed for a full theory of course, which we do not attempt for the time being.

**Lemma 3.1** Lower Bounds via Prime Homogeneous Simple Sub-problems

Given a morphism \( \Lambda \rightarrow \Lambda \), let \( \kappa(\Pi) \) denote the maximum value of \( b(\Lambda) \), the number of instances of \( \Lambda \). From this point on, fix a single \( \Lambda = \{I_1, \ldots, I_r\} \) with \( \kappa(\Pi) = b(\Lambda) = r \). Let \( \Lambda' = \{I_1, \ldots, I_{r-1}\} \) for \( r \geq 2 \), and \( \Lambda' = \emptyset \) for \( r = 1 \). The objects we consider are \( \Lambda, \Lambda', \{I_r\}, \Gamma, \Lambda \cup \Gamma, \Lambda' \cup \Gamma, \{I_r\} \cup \Gamma \), and \( \emptyset \), where \( \Gamma \) is a computational problem consisting of any non-empty set of instances with \( \Lambda \cap \Gamma = \emptyset \).

**Proof.** Let \( \Lambda \) and \( \Lambda' \) be the sub-problems defined in the previous section. Since \( \tau(\Pi) \geq \tau(\Lambda) \), it suffices to show \( \tau(\Lambda) \geq r \). We argue by induction on \( r \). For \( r = 1 \), we clearly have \( \tau(\Lambda) \geq 1 \), since

\[
\tau(\Pi) \geq \kappa(\Pi).
\]

**3 Lower Bounds via Prime Homogeneous Simple Sub-problems**

**Lemma 3.1** (Fundamental Lemma).

**Proof.** Let \( \Lambda \) and \( \Lambda' \) be the sub-problems defined in the previous section. Since \( \tau(\Pi) \geq \tau(\Lambda) \), it suffices to show \( \tau(\Lambda) \geq r \). We argue by induction on \( r \). For \( r = 1 \), we clearly have \( \tau(\Lambda) \geq 1 \), since
the complexity of solving a problem other than $T$ is non-zero. For $r \geq 2$, assume $\tau(N') \geq r - 1$. We want to relate the complexity of the map $\Lambda \to T$ to the complexity of the map $N' \to T$. To this aim, consider a factorization of the morphism $f : B(\Lambda) \to \text{Spec} \mathbb{F}_2$ in the image of the amplifying functor as

$$B(\Lambda) \xrightarrow{h} X \to \text{Spec} \mathbb{F}_2,$$

such that $B(N') \subseteq X$ and $h[B(N')] = B(N')$. In this case since $B(\Lambda)$ is connected, $h[B(\Lambda)]$ must be connected. This implies by the requirements of the factorization that $h[B(\Lambda)] = B(\Lambda)$. In words, while reducing $B(\Lambda)$ to $\text{Spec} \mathbb{F}_2$ via a mapping, which fixes $B(N')$, one has to “go through” $B(\Lambda)$ itself. This restriction, provided by the connectedness of $B(\Lambda)$, is crucial for our argument:

$$B(\Lambda) \xrightarrow{h} B(\Lambda) \to \text{Spec} \mathbb{F}_2.$$ 

By our assumption, we also have

$$B(N') \xrightarrow{h} B(N') \to \text{Spec} \mathbb{F}_2,$$

where $h$ is uniquely defined in the image of the extended amplifying functor.

A pre-image of the first factorization above might have the following two reduction sequences applied to $I_r$.

$$\begin{align*}
\Lambda & \to \Lambda \to T, \\
\alpha_1 : I_r & \mapsto I_r \xrightarrow{\alpha_3} T, \\
\alpha_2 : I_r & \xrightarrow{\alpha_4} I_j \mapsto T,
\end{align*}$$

where $j \in \{1, \ldots, r - 1\}$. We also have the following two reduction sequences in a pre-image of the second factorization.

$$\begin{align*}
N' & \to N' \to T, \\
\beta_1 : I_j & \mapsto I_j \xrightarrow{\beta_3} T, \\
\beta_2 : I_j & \xrightarrow{\beta_4} I_k \mapsto T,
\end{align*}$$

where $k \in \{1, \ldots, r - 1\} \setminus \{j\}$. Note that $\alpha_3$ and $\beta_3$ are unit operations, whereas $\alpha_4$ and $\beta_4$ are unit instance operations.

Consider $\alpha_2 \cup \beta_1 : \Lambda \to T$ or $\alpha_2 \cup \beta_2 : \Lambda \to T$, which contains a unit instance operation $\alpha_4$. Since $\Lambda$ is prime, $\alpha_4$ is distinct from all $\beta_4$ in the diagram above. This implies by Remark 2.14 that a reduction containing $\alpha_2$ performs an operation that does not exist in a reduction containing $\beta_2$. Observe next that by Proposition 2.17, $\beta_3$ is distinct from all $\alpha_4$. Likewise, this implies that a reduction containing $\alpha_2$ performs an operation that does not exist in a reduction containing $\beta_1$. We then have the following:

$$\begin{align*}
\tau(\alpha_2 \cup \beta_1) & \geq \tau(\beta_1) + 1, \\
\tau(\alpha_2 \cup \beta_1) & \geq \tau(\beta_2) + 1, \\
\tau(\alpha_2 \cup \beta_2) & \geq \tau(\beta_1) + 1, \\
\tau(\alpha_2 \cup \beta_2) & \geq \tau(\beta_2) + 1.
\end{align*}$$

(4)

Consider next $\alpha_1 \cup \beta_1 : \Lambda \to T$ or $\alpha_1 \cup \beta_2 : \Lambda \to T$, which contains a unit operation $\alpha_3$. Since $\Lambda$ is homogeneous, $\alpha_3$ is distinct from all $\beta_3$ by Proposition 2.16. This implies by Remark 2.14 that a reduction containing $\alpha_1$ performs an operation that does not exist in a reduction containing $\beta_1$. By Proposition 2.17, we also have that $\alpha_3$ is distinct from all $\beta_4$. This implies that a reduction containing $\alpha_1$ performs an operation that does not exist in a reduction containing $\beta_2$. We thus
obtain
\[
\begin{align*}
\tau(\alpha_1 \cup \beta_2) & \geq \tau(\beta_1) + 1. \\
\tau(\alpha_1 \cup \beta_2) & \geq \tau(\beta_2) + 1. \\
\tau(\alpha_1 \cup \beta_1) & \geq \tau(\beta_1) + 1. \\
\tau(\alpha_1 \cup \beta_1) & \geq \tau(\beta_2) + 1.
\end{align*}
\]
(5)

The inequalities (4) and (5) together imply \(\tau(\Lambda) \geq \tau(\Lambda') + 1 \geq (r - 1) + 1 = r\), which completes the induction and the proof.

4 3-SAT: The Separation of P and NP

Denote by \(k\text{-SAT}(n, m)\) the problem \(k\text{-SAT}\) with \(n\) variables and \(m\) clauses.

**Theorem 4.1.** For any constant \(\epsilon > 0\), there exist infinitely many \(n \in \mathbb{Z}^+\) such that
\[\kappa(3\text{-SAT}(n, 2n)) \geq 2^{(\frac{3}{8} - \epsilon)n}.
\]

**Proof.** We construct a prime homogeneous simple sub-problem of 3-SAT with \((\frac{r}{2}) \cdot 2^{r/2}\) instances, each having 4\(r\) variables and 8\(r\) clauses, for \(r \geq 1\).

**The Initial Construction: A Homogeneous Simple Sub-problem** Each instance consists of \(r\) blocks. For \(r = 1\), a block of an instance is initially defined via 4 variables \(x_1, x_2, x_3, x_4\), and 8 clauses. We first construct 3 instances with the solution sets over \(\mathbb{F}_2\) consisting of the following points, listed for each instance in a separate column:

| Instance 1       | Instance 2       | Instance 3       |
|------------------|------------------|------------------|
| (0,0,1,0)        | (0,0,1,0)        | (0,1,0,0)        |
| (1,0,0,0)        | (0,1,0,0)        | (0,1,1,0)        |
| (1,1,0,0)        | (1,0,1,0)        | (1,0,0,0)        |

These instances consisting of a single block are shown in Table 1. A block for each instance can be described by a procedure using the truth table of the variables. Each of the 8 clauses is introduced one by one to rule out certain assignments over \(\mathbb{F}_2\) in the tables. We enumerate the rows of the tables for each instance by an indexing of these clauses in Table 2 Table 3 and Table 4. The solution sets over \(\mathbb{F}_2\) are the entries left out by the introduced clauses. The corresponding schemes over \(\mathbb{F}_2\) have isomorphic cohomology groups with respect to any coherent sheaf, so that by (1) and

| Clause | Instance 1       | Instance 2       | Instance 3       |
|--------|------------------|------------------|------------------|
| 1      | \(x_1 \lor x_2 \lor x_3\) | \(x_1 \lor x_2 \lor x_3\) | \(x_1 \lor x_2 \lor x_3\) |
| 2      | \(x_2 \lor x_3 \lor x_4\) | \(x_2 \lor x_3 \lor x_4\) | \(x_2 \lor x_3 \lor x_4\) |
| 3      | \(\overline{x}_2 \lor x_3 \lor \overline{x}_4\) | \(\overline{x}_2 \lor x_3 \lor \overline{x}_4\) | \(\overline{x}_2 \lor x_3 \lor \overline{x}_4\) |
| 4      | \(x_2 \lor x_3 \lor x_4\) | \(x_2 \lor x_3 \lor x_4\) | \(x_2 \lor x_3 \lor x_4\) |
| 5      | \(x_1 \lor x_2 \lor x_3\) | \(x_1 \lor x_2 \lor x_3\) | \(x_1 \lor x_2 \lor x_3\) |
| 6      | \(x_1 \lor x_3 \lor x_4\) | \(x_1 \lor x_3 \lor x_4\) | \(x_1 \lor x_3 \lor x_4\) |
| 7      | \(x_1 \lor x_3 \lor x_4\) | \(x_1 \lor x_3 \lor x_4\) | \(x_1 \lor x_3 \lor x_4\) |
| 8      | \(x_2 \lor x_3 \lor x_4\) | \(x_2 \lor x_3 \lor x_4\) | \(x_2 \lor x_3 \lor x_4\) |

Table 1: The clauses of the 3 instances satisfying Table 2 and Table 3 and Table 4
Table 2: The truth table of a block of Instance 1 with clause-indexing

| Clause | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Clause | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|--------|-------|-------|-------|-------|--------|-------|-------|-------|-------|
| 1      | 0     | 0     | 0     | 0     | 1      | 0     | 0     | 0     | 0     |
| 1      | 0     | 0     | 0     | 1     | 5      | 1     | 0     | 0     | 1     |
|        | 0     | 0     | 1     | 0     | 7      | 1     | 0     | 1     | 0     |
| 2      | 0     | 0     | 1     | 1     | 2      | 1     | 0     | 1     | 1     |
| 6      | 0     | 1     | 0     | 0     |        | 1     | 1     | 0     | 0     |
| 3      | 0     | 1     | 0     | 1     | 3      | 1     | 1     | 0     | 1     |
| 8      | 0     | 1     | 1     | 0     | 8      | 1     | 1     | 1     | 0     |
| 4      | 0     | 1     | 1     | 1     | 4      | 1     | 1     | 1     | 1     |

Table 3: The truth table of a block of Instance 2 with clause-indexing

| Clause | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Clause | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|--------|-------|-------|-------|-------|--------|-------|-------|-------|-------|
| 1      | 0     | 0     | 0     | 0     | 7      | 1     | 0     | 0     | 0     |
| 1      | 0     | 0     | 0     | 1     | 5      | 1     | 0     | 0     | 1     |
|        | 0     | 0     | 1     | 0     |        | 1     | 0     | 1     | 0     |
| 2      | 0     | 0     | 1     | 1     | 2      | 1     | 0     | 1     | 1     |
|        | 0     | 1     | 0     | 0     | 6      | 1     | 1     | 0     | 0     |
| 3      | 0     | 1     | 0     | 1     | 3      | 1     | 1     | 0     | 1     |
| 8      | 0     | 1     | 1     | 0     | 8      | 1     | 1     | 1     | 0     |
| 4      | 0     | 1     | 1     | 1     | 4      | 1     | 1     | 1     | 1     |

Table 4: The truth table of a block of Instance 3 with clause-indexing

| Clause | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Clause | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|--------|-------|-------|-------|-------|--------|-------|-------|-------|-------|
| 1      | 0     | 0     | 0     | 0     |        | 1     | 0     | 0     | 0     |
| 1      | 0     | 0     | 0     | 1     | 5      | 1     | 0     | 0     | 1     |
| 8      | 0     | 0     | 1     | 0     | 8      | 1     | 0     | 1     | 0     |
| 2      | 0     | 0     | 1     | 1     | 2      | 1     | 0     | 1     | 1     |
|        | 0     | 1     | 0     | 0     | 6      | 1     | 1     | 0     | 0     |
| 3      | 0     | 1     | 0     | 1     | 3      | 1     | 1     | 0     | 1     |
| 8      | 0     | 1     | 1     | 0     | 7      | 1     | 1     | 1     | 0     |
| 4      | 0     | 1     | 1     | 1     | 4      | 1     | 1     | 1     | 1     |

The Hilbert polynomials of the instances are the same. In particular, they are the disjoint union of a closed point and a linear subspace as shown below.

The first 5 clauses of the instances are common. Clause 1 forces at least one of $x_1$, $x_2$ and $x_3$ to be 1, as it corresponds to

$$(1 - x_1)(1 - x_2)(1 - x_3) = 0.$$

Given this, the following 4 clauses make $x_4 = 0$, since $x_4 \neq 0$ implies $x_1 = x_2 = x_3 = 0$ by these clauses. In other words, $x_i = 1$ for any $i \in \{1, 2, 3\}$ implies a contradiction in the following system:

$$
\begin{align*}
(1 - x_2)x_3 &= 0, \\
 x_2(1 - x_3) &= 0, \\
x_2x_3 &= 0, \\
x_1(1 - x_2) &= 0.
\end{align*}
$$
Given that $x_4 = 0$ (or more generally $x_4 \neq 1$), we now examine the last 3 clauses of the instances.

1. Instance 1:

   \[(1 - x_1)(1 - x_3) = 0.
   \]

   \[x_1x_3 = 0.
   \]

   \[x_2x_3 = 0.
   \]

   $x_1 = 1 \Rightarrow x_3 = 0, x_2 \in \overline{\mathbb{F}}_2$.

   $x_2 = 1 \Rightarrow x_1 = 1, x_3 = 0$.

   $x_3 = 1 \Rightarrow x_1 = 0, x_2 = 0$.

   Thus, the solution set is \{(0, 0, 1)\} ∪ \{(1, α, 0)\}, where $α \in \overline{\mathbb{F}}_2$.

2. Instance 2:

   \[x_1(1 - x_3) = 0.
   \]

   \[(1 - x_2)(1 - x_3) = 0.
   \]

   \[x_2x_3 = 0.
   \]

   $x_1 = 1 \Rightarrow x_2 = 0, x_3 = 1$.

   $x_2 = 1 \Rightarrow x_1 = 0, x_3 = 0$.

   $x_3 = 1 \Rightarrow x_2 = 0, x_1 \in \overline{\mathbb{F}}_2$.

   Thus, the solution set is \{(0, 1, 0)\} ∪ \{(α, 0, 1)\}, where $α \in \overline{\mathbb{F}}_2$.

3. Instance 3:

   \[x_1x_2 = 0.
   \]

   \[x_1x_3 = 0.
   \]

   \[(1 - x_2)x_3 = 0.
   \]

   $x_1 = 1 \Rightarrow x_2 = 0, x_3 = 0$.

   $x_2 = 1 \Rightarrow x_1 = 0, x_3 = 0$.

   $x_3 = 1 \Rightarrow x_2 = 0, x_1 \in \overline{\mathbb{F}}_2$.

   Thus, the solution set is \{(1, 0, 0)\} ∪ \{(0, 1, α)\}, where $α \in \overline{\mathbb{F}}_2$.

Note that all the 4 variables appear in all the instances. Since the instances are also distinct, they form a homogeneous simple sub-problem. Assume now the induction hypothesis that there exists a homogeneous simple sub-problem of size $3^r$, for some $r \geq 1$. In the inductive step, we introduce 4 new variables $x_{4r+1}, x_{4r+2}, x_{4r+3}, x_{4r+4}$, and 3 new blocks on these variables each consisting of 8 clauses with the exact form as in Table I. Appending these blocks to each of the $3^r$ instances of the induction hypothesis, we obtain $3^{r+1}$ instances. The constructed sub-problem is a homogeneous simple sub-problem. We now describe a procedure to make it into a prime homogeneous simple sub-problem.

**Mixing the Blocks: A Prime Homogeneous Simple Sub-problem** For simplicity and the purpose of providing examples, we describe the procedure for $r = 2$. The construction is easily extended to the general case. Suppose that the first block is defined via Instance 1. We perform the following operation: Replace the literals of Clause 4 except $x_7$ with appropriate literals of variables belonging to the second block, depending on which instance it is defined via. If the second block is defined via Instance 1, then Clause 4 becomes $(x_5 \lor x_7 \lor x_{4r+2})$. If it is defined via Instance 2, it becomes $(x_5 \lor x_6 \lor x_{4r+3})$. If it is defined via Instance 3, it becomes $(x_5 \lor x_6 \lor x_{4r+4})$. In extending this to the general case, the second block is generalized as the next block to the current one, and
the variables used for replacement are the ones with the first three indices of the next block in increasing order, respectively corresponding to $x_5, x_6,$ and $x_7$.

If the second block is defined via Instance 2, the same operations are performed, this time considering Clause 5 of the first block. If the second block is defined via Instance 3, we consider Clause 2 of the first block. All possible cases are illustrated in Table 5–Table 10, where the interchanged literals are shown in bold. In the general case, the described operation is also performed for the last block indexed $r$ for which the next block is defined as the first block, completing a cycle.

**The constructed sub-problem is prime:** In mixing the blocks, we force one specific clause

| Clause | Instance 1 | Instance 1 |
|--------|------------|------------|
| 1      | $x_1 \lor x_2 \lor x_3$ | $x_5 \lor x_6 \lor x_7$ |
| 2      | $x_2 \lor \overline{x_3} \lor \overline{x_4}$ | $x_6 \lor \overline{x_7} \lor \overline{x_8}$ |
| 3      | $\overline{x_2} \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor x_7 \lor \overline{x_8}$ |
| 4      | $x_5 \lor x_7 \lor \overline{x_4}$ | $x_1 \lor x_3 \lor \overline{x_8}$ |
| 5      | $\overline{x_1} \lor \overline{x_2} \lor \overline{x_4}$ | $\overline{x_5} \lor x_6 \lor \overline{x_8}$ |
| 6      | $x_1 \lor x_3 \lor x_4$ | $x_5 \lor x_7 \lor x_8$ |
| 7      | $\overline{x_1} \lor \overline{x_3} \lor x_4$ | $\overline{x_5} \lor \overline{x_7} \lor x_8$ |
| 8      | $\overline{x_2} \lor \overline{x_3} \lor \overline{x_4}$ | $\overline{x_6} \lor \overline{x_7} \lor x_8$ |

Table 5: Modification to form a prime sub-problem on Instance 1 and Instance 1 blocks

| Clause | Instance 1 | Instance 2 |
|--------|------------|------------|
| 1      | $x_1 \lor x_2 \lor x_3$ | $x_5 \lor x_6 \lor x_7$ |
| 2      | $x_2 \lor \overline{x_3} \lor \overline{x_4}$ | $x_6 \lor \overline{x_7} \lor \overline{x_8}$ |
| 3      | $\overline{x_2} \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor x_7 \lor \overline{x_8}$ |
| 4      | $x_5 \lor x_7 \lor \overline{x_4}$ | $x_1 \lor x_3 \lor \overline{x_8}$ |
| 5      | $\overline{x_1} \lor \overline{x_2} \lor \overline{x_4}$ | $\overline{x_5} \lor x_6 \lor \overline{x_8}$ |
| 6      | $x_1 \lor x_3 \lor x_4$ | $x_5 \lor x_7 \lor x_8$ |
| 7      | $\overline{x_1} \lor \overline{x_3} \lor x_4$ | $\overline{x_5} \lor \overline{x_7} \lor x_8$ |
| 8      | $\overline{x_2} \lor \overline{x_3} \lor \overline{x_4}$ | $\overline{x_6} \lor \overline{x_7} \lor x_8$ |

Table 6: Modification to form a prime sub-problem on Instance 1 and Instance 2 blocks

| Clause | Instance 1 | Instance 3 |
|--------|------------|------------|
| 1      | $x_1 \lor x_2 \lor x_3$ | $x_5 \lor x_6 \lor x_7$ |
| 2      | $x_2 \lor \overline{x_3} \lor \overline{x_4}$ | $x_1 \lor x_3 \lor \overline{x_8}$ |
| 3      | $\overline{x_2} \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor x_7 \lor \overline{x_8}$ |
| 4      | $x_5 \lor x_6 \lor \overline{x_4}$ | $x_6 \lor \overline{x_7} \lor \overline{x_8}$ |
| 5      | $\overline{x_1} \lor \overline{x_2} \lor \overline{x_4}$ | $x_5 \lor x_6 \lor \overline{x_8}$ |
| 6      | $x_1 \lor x_3 \lor x_4$ | $x_5 \lor x_7 \lor x_8$ |
| 7      | $\overline{x_1} \lor \overline{x_3} \lor x_4$ | $\overline{x_5} \lor x_7 \lor x_8$ |
| 8      | $\overline{x_2} \lor \overline{x_3} \lor \overline{x_4}$ | $\overline{x_6} \lor \overline{x_7} \lor x_8$ |

Table 7: Modification to form a prime sub-problem on Instance 1 and Instance 3 blocks
of a block depending on its type to contain variables belonging to the next block in a way distinctive to the type of the next block. In particular, suppose we represent an instance as a sequence of blocks numbered according to their types. Then any unit instance operation from the instance 22 to the instance 23 is distinct from a unit instance operation from the instance 32 to the instance 33. The first operation can in fact be labeled as one from $(2_2, x_7)$ to $(2_3, x_8)$, since a block is distinguished by itself together with the next block. The second operation is from $(3_2, x_8)$ to $(3_3, x_8)$, which better indicates that it is distinct from the first one. The same applies to the general case, where there are arbitrarily many blocks, ensuring that we have a prime sub-problem.

Table 8: Modification to form a prime sub-problem on Instance 2 and Instance 2 blocks

| Clause | Instance 2 | Instance 2 |
|--------|------------|------------|
| 1      | $x_1 \lor x_2 \lor x_3$ | $x_5 \lor x_6 \lor x_7$ |
| 2      | $x_2 \lor x_3 \lor x_4$ | $x_6 \lor x_7 \lor x_8$ |
| 3      | $x_2 \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor x_7 \lor \overline{x_8}$ |
| 4      | $x_2 \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor \overline{x_7} \lor x_8$ |
| 5      | $\overline{x_5} \lor \overline{x_6} \lor x_4$ | $\overline{x_5} \lor x_6 \lor \overline{x_8}$ |
| 6      | $\overline{x_1} \lor x_2 \lor x_4$ | $x_5 \lor x_7 \lor x_8$ |
| 7      | $\overline{x_1} \lor x_3 \lor x_4$ | $\overline{x_5} \lor \overline{x_7} \lor x_8$ |
| 8      | $x_2 \lor x_3 \lor x_4$ | $x_6 \lor \overline{x_7} \lor x_8$ |

Table 9: Modification to form a prime sub-problem on Instance 2 and Instance 3 blocks

| Clause | Instance 3 | Instance 3 |
|--------|------------|------------|
| 1      | $x_1 \lor x_2 \lor x_3$ | $x_5 \lor x_6 \lor x_7$ |
| 2      | $\overline{x_6} \lor \overline{x_7} \lor x_4$ | $\overline{x_2} \lor \overline{x_3} \lor x_8$ |
| 3      | $x_2 \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor x_7 \lor \overline{x_8}$ |
| 4      | $x_2 \lor x_3 \lor \overline{x_4}$ | $\overline{x_6} \lor \overline{x_7} \lor x_8$ |
| 5      | $\overline{x_1} \lor x_2 \lor x_4$ | $x_5 \lor x_7 \lor x_8$ |
| 6      | $\overline{x_1} \lor x_3 \lor x_4$ | $\overline{x_5} \lor \overline{x_7} \lor x_8$ |
| 7      | $\overline{x_1} \lor x_3 \lor x_4$ | $\overline{x_5} \lor \overline{x_7} \lor x_8$ |
| 8      | $x_2 \lor x_3 \lor x_4$ | $x_6 \lor \overline{x_7} \lor x_8$ |

Table 10: Modification to form a prime sub-problem on Instance 3 and Instance 3 blocks
**Selecting a simple sub-problem:** We next establish facts about the solution sets. We observe the following for the first block, which also holds for all the other blocks by the construction. Assume \( x_4 \neq 0 \) and \( x_4 \neq 1 \). We will show that this leads to a contradiction, so that \( x_4 \neq 0 \) implies \( x_4 = 1 \). Consider the case in which the first block is defined via Instance 1. By the equations numbered 2, 3 and 5 of the first block, we then have

\[
(1 - x_2)x_3 = 0, \\
x_2(1 - x_3) = 0, \\
x_1(1 - x_2) = 0.
\]

Since at least one of \( x_1, x_2, \) and \( x_3 \) is 1 by Equation 1, by checking each case, we have that the solution set to these equations is \( \{(\alpha, 1, 1)\} \). As computed previously, this contradicts the solution set implied by the last 3 equations of the first block for \( x_4 \neq 1 \): \( \{(0, 0, 1)\} \cup \{(1, \alpha, 0)\} \).

Suppose now that the first block is defined via Instance 2. By looking at the equations numbered 2, 3 and 4 of the first block, we get

\[
(1 - x_2)x_3 = 0, \\
x_2(1 - x_3) = 0, \\
x_2x_3 = 0.
\]

Since at least one of \( x_1, x_2, \) and \( x_3 \) is 1 as noted, the solution set to these equations is \( \{(1, 0, 0)\} \). This contradicts the solution set implied by the last 3 equations of Instance 2 for \( x_4 \neq 1 \): \( \{(0, 1, 0)\} \cup \{(\alpha, 0, 1)\} \).

Finally, suppose that the first block is defined via Instance 3. By looking at the equations numbered 3, 4 and 5 of the first block, we obtain

\[
x_2(1 - x_3) = 0, \\
x_2x_3 = 0, \\
x_1(1 - x_2) = 0.
\]

With the requirement that at least one of \( x_1, x_2, \) and \( x_3 \) is 1, the solution set to these equations is \( \{(0, 0, 1)\} \). This contradicts the solution set implied by the last 3 equations of Instance 3 for \( x_4 \neq 1 \): \( \{(1, 0, 0)\} \cup \{(0, 1, \alpha)\} \). Thus, either \( x_4 = 0 \) or \( x_4 = 1 \).

Observe next that the replaced clauses in each block are satsifiable. Assume \( x_4 \neq 0 \). If the second block is defined via Instance 1, \( x_5 \lor x_7 \) does not contradict the solution set for Instance 1, which is \( \{(0, 0, 1)\} \cup \{(1, \alpha, 0)\} \cup \{(\alpha, 1, 1)\} \). Similarly, if the second block is defined via Instance 2, \( \overline{x_6} \lor \overline{x_9} \) does not contradict the solution set for Instance 2, which is \( \{(1, 0, 0)\} \cup \{(0, 1, 0)\} \cup \{(\alpha, 0, 1)\} \). If the second block is defined via Instance 3, \( \overline{x_5} \lor \overline{x_6} \) does not contradict the solution set for Instance 3, which is \( \{(0, 0, 1)\} \cup \{(1, 0, 0)\} \cup \{(0, 1, \alpha)\} \).

We have already shown that for \( x_4 = 0 \), the solution sets associated to three different types of blocks have the same cohomology. Notice that for \( x_4 = 1 \), the solution sets associated to these blocks are the ones computed in the discussion above. For Instance 1, it is \( (\alpha, 1, 1, 1) \). For Instance 2, it is \( (1, 0, 0, 1) \). For Instance 3, it is \( (0, 0, 1, 1) \). Thus, the Hilbert polynomials associated to Instance 2 and Instance 3 are the same, whereas Instance 1 differs from them. We consider the following set of instances with uniform Hilbert polynomial. Select out of all instances having \( r/2 \) blocks defined via Instance 1 and \( r/2 \) blocks defined via either Instance 2 or Instance 3, where we assume \( r \) is even. The number of such instances is \( \binom{r}{r/2} \cdot 2^{r/2} \). Using the Stirling approximation, we have for all \( \epsilon > 0 \)

\[
\binom{r}{r/2} \cdot 2^{r/2} > 2^{\left(\frac{r}{2} - \epsilon\right)r},
\]

as \( r \) tends to infinity. Since \( r = n/4 \), the proof is completed. \( \square \)
By Theorem 4.1, Lemma 3.1, and the NP-completeness of 3-SAT \(^5\):

**Corollary 4.2.** \( P \neq NP \).

The definition of \( \tau \) also implies

**Corollary 4.3.** \( NP \not\subseteq P/\text{poly} \).

Furthermore, by the specific lower bound derived for 3-SAT:

**Corollary 4.4.** \textit{The exponential time hypothesis} \(^3\) \textit{is true against deterministic algorithms}.

Finally, this exponential lower bound implies the following by \(^4\).

**Corollary 4.5.** \( \text{BPP} = P \).

## 5 Final Remarks

We first note that the base of the exponential function in Theorem 4.1 is \( 2^{3/8} \approx 1.296839 \). In contrast, the best deterministic algorithm for 3-SAT runs in time \( O(1.32793^n) \) \(^6\). We next show that the strategy developed in the previous section cannot establish a strong lower bound for 2-SAT. This partially explains, from a technical standpoint, why 3-SAT is hard but 2-SAT is easy. In brief, the strategy was as follows:

1. Define 3 instances on 4 variables, each via a single block, and forming a homogeneous simple sub-problem.

2. Introduce \( n \) blocks, each with a new set of variables, to attain an exponential number of instances forming a homogeneous simple sub-problem.

3. Mix the consecutive blocks in a distinctive way depending on their types, so that we have a prime homogeneous sub-problem. Select a further sub-problem, which is simple.

Let us now try to imitate our strategy for 2-SAT by defining 2 distinct instances on 3 variables. Consider the two instances given in Table 11. The first 3 clauses imply that at least one of \( x_1 \) and \( x_2 \) is 1, and \( x_3 \) is 0. These are analogous to the first 5 clauses of the blocks constructed for 3-SAT. Suppose we want to fix \( x_1 = 0 \) in the first instance so that the last clause is \( \overline{x}_1 \lor x_3 \). The solution set of this instance over \( \mathbb{F}_2 \) consists of the single closed point \((0,1,0)\), with the Hilbert polynomial 1. For the second instance, we must analogously use \( \overline{x}_2 \lor x_3 \) as the last clause, as there is no other option for the first literal. Observe that we can relate consecutive blocks via the second and the third clauses to create a prime sub-problem. In particular, Instance 1 has two possible forms in

| Clause | Instance 1 | Instance 2 |
|--------|------------|------------|
| 1      | \( x_1 \lor x_2 \) | \( x_1 \lor x_2 \) |
| 2      | \( \overline{x}_1 \lor \overline{x}_3 \) | \( \overline{x}_1 \lor \overline{x}_3 \) |
| 3      | \( \overline{x}_2 \lor x_3 \) | \( \overline{x}_2 \lor x_3 \) |
| 4      | \( \overline{x}_1 \lor x_3 \) | \( \overline{x}_2 \lor x_3 \) |

Table 11: Two instances of 2-SAT forming a homogenous sub-problem

\[ \]
a prime sub-problem, the first including the clauses numbered 1, 2, and 4, the second including clauses numbered 1, 3, and 4. The first one corresponds to the following system:

\[(1 - x_1)(1 - x_2) = 0.\]
\[x_1x_3 = 0.\]
\[x_1(1 - x_3) = 0.\]

The solution set of this system is \{\((1, 0, 1), (0, 1, 0)\)\}. The second one corresponds to the following system:

\[(1 - x_1)(1 - x_2) = 0.\]
\[x_2x_3 = 0.\]
\[x_1(1 - x_3) = 0.\]

The solution set of this system is \{\((0, 1, \alpha)\)\}, where \(\alpha \in \mathbb{F}_2\). This however forbids a simple sub-problem, since the cohomology of the solution set of a fixed type of block is not fixed.

A clause of 2-SAT puts a more stringent requirement on the variables than 3-SAT, resulting in only one clause that is not common between the instances. Furthermore, there is not enough “room” in a clause of 2-SAT letting us consider different variations so as to ensure even a homogeneous simple sub-problem. In contrast, the freedom of having 4 variables and 3 non-common clauses between instances in the case of 3-SAT allows us to consider many more combinations, and we were able to show that one of them leads to a sub-problem that is both homogeneous, prime and simple.

**Acknowledgment**

We would like to thank Sinan Ünver for informing us on schemes and morphisms.

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