ON THE EXACT DEGREE OF MULTI-CYCLIC
EXTENSION OF \( \mathbb{F}_q(t) \)

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Abstract. Let \( q \) be a power of a prime number \( p \), \( k = \mathbb{F}_q(t) \) be the rational function field over finite field \( \mathbb{F}_q \) and \( K/k \) be a multi-cyclic extension of prime degree. In this paper we will give an exact formula for the degree of \( K \) over \( k \) by considering both Kummer and Artin-Schreier cases.

1. Introduction

Let \( S = \{a_1, ..., a_l\} \) be a finite set of nonzero integers. By computing the relative density of the set of prime numbers \( p \) for which all the \( a_i \)'s are simultaneously quadratic residues modulo \( p \), Balasubramanian, Luca and Thangadurai [2] gave an exact formula for the degree of the multi-quadratic field \( \mathbb{Q}(\sqrt{a_1}, ..., \sqrt{a_l}) \) over \( \mathbb{Q} \).

Let \( q \) be a power of a prime number \( p \), \( k = \mathbb{F}_q(t) \) be the rational function field over finite field \( \mathbb{F}_q \) and \( K/k \) be a multi-cyclic extension of prime degree. In this paper we will give an exact formula of \( K \) over \( k \). We consider the following two different situations. The first situation is multi-cyclic Kummer extensions. That is \( K = k(\sqrt[\lambda]{D_1}, ..., \sqrt[\lambda]{D_l}) \) and \( S = \{D_1, ..., D_l\} \) is a finite set of nonzero polynomials in \( \mathbb{F}_q[t] \), where \( \lambda \) is a prime factor of \( q - 1 \).

The second situation is multi-cyclic Artin-Schreier extensions. That is \( K = k(\alpha_1, ..., \alpha_l) \) and there is a finite set \( S = \{D_1, ..., D_l\} \) of nonzero elements in \( \mathbb{F}_q(t) \) such that

\[
\alpha_i^p - \alpha_i = D_i \quad (1 \leq i \leq l).
\]

We follow Balasubramanian, Luca and Thangadurai’s approach to consider the above two situations in Section 2 and 3, respectively. In these two sections, we also assume \( K \) is a geometric extension of \( k \), i.e. the full constant field of \( K \) is \( \mathbb{F}_q \) (see [6, p. 77]). Our main tool is estimations of certain character sums over \( \mathbb{F}_q[T] \) (see Lemma 2.3 and 3.1 and the proof for Theorem 2.5 and 3.4 below). In section 4, using abelian Kummer theory instead of Lemma 2.3 and Lemma 3.1 we give another approach to this problem. Notice that in section 4 we do not assume \( K/k \) is a geometric extension.

Throughout the paper, \( \mathbb{C} \) denotes the complex field, \( P \) denotes the monic irreducible polynomial, \( N \) denotes a positive integer, \( \pi(N) \) denotes the number of monic irreducible polynomial \( P \) such that \( \deg P = N \) and \( S_k \) denotes the set of monic irreducible polynomials which are unramified in \( K \). A set \( S \)
of monic irreducible polynomials is said to have the relative density \( \varepsilon \) with 
\[ 0 \leq \varepsilon \leq 1, \]
if
\[ \varepsilon(S) = \lim_{N \to \infty} \frac{\#\{P \in S \mid \deg P = N\}}{\pi(N)} \]
exists. In this case
\[ \varepsilon(S) = \lim_{N \to \infty} \frac{\#\{P \in S \mid \deg P \leq N\}}{\#\{P \mid \deg P \leq N\}} \]
by Stolz’s theorem.

A set \( S \) of monic irreducible polynomials is said to have the Dirichlet density if
\[ \delta(S) = \lim_{s \to 1^+} \frac{\sum_{P \in S} NP^{-s}}{\sum_{P} NP^{-s}} \]
exists, where \( NP = q^{\deg P} \), see [6, p. 126]. Notice that the existence of the Dirichlet density does not imply the existence of the relative density, see Lemma 4.5 in [7].

The following lemmas will be used in the proof of our results.

**Lemma 1.1.** (Chebotarev’s Density Theorem, first version, see Theorem 9.13A of [6])
Let \( K/k \) be a Galois extension of global function fields and set \( H = \text{Gal}(K/k) \). Let \( C \subset H \) be a conjugacy class in \( H \) and \( S_k \) be the set of primes of \( k \) which are unramified in \( K \). Then
\[ \delta(\{P \in S_k \mid (P, K/k) = C\}) = \frac{\#C}{\#H}, \]
where \( \delta \) denotes the Dirichlet density and \( (P, K/k) \) is the Artin symbol at \( P \).

**Lemma 1.2.** (Chebotarev’s Density Theorem, second version, see Theorem 9.13B of [6])
Let \( K/k \) be a geometric, Galois extension of global function fields and set \( H = \text{Gal}(K/k) \). Let \( C \subset H \) be a conjugacy class in \( H \). Suppose the common constant field of \( K \) and \( k \) has \( q \) elements. Let \( S_k \) be the set of primes of \( k \) which are unramified in \( K \). Then for each positive integer \( N \), we have
\[ \#\{P \in S_k \mid \deg P = N, (P, K/k) = C\} = \frac{\#C}{\#H} q^N + O\left(\frac{q^{N/2}}{N}\right). \]

**Lemma 1.3.** (The prime number theorem for polynomials, see Theorem 2.2 of [6])
Let \( N \) be a positive integer and \( \pi(N) \) be the number of monic irreducible polynomial \( P \) in \( \mathbb{F}_q[t] \) of degree \( N \). Then
\[ \pi(N) = \frac{q^N}{N} + O\left(\frac{q^{N/2}}{N}\right). \]

2. Multi Kummer extensions

In this section, let \( q \) be a power of a prime and \( m \) be any prime divisor of \( q - 1 \). Let \( K \) be a multi-\( m \)-cyclic extension of \( k = \mathbb{F}_q(t) \). That is
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Let $K = k(\sqrt[n]{D_1}, \ldots, \sqrt[n]{D_l})$ and $S = \{D_1, \ldots, D_l\}$ be a finite set of nonconstant polynomials in $A = \mathbb{F}_q[t]$. Let $\mathbb{Z}_m$ be the set of integers

$$\mathbb{Z}_m = \{0, 1, 2 \cdots, m - 2, m - 1\}.$$

Let $\gamma_S$ be the cardinality of the following set

$$\{(a_1, a_2, \cdots, a_l) \in \mathbb{Z}_m^l \mid D_1^{a_1} D_2^{a_2} \cdots D_l^{a_l} = F^m \text{ for some } F \in A\}.$$

In this section, we will prove the following result.

**Theorem 2.1.** For a given finite set $S$ of nonzero polynomials with $|S| = l$, we have

$$[K : k] = m^l - r,$$

where $r$ is the non-negative integer given by $m^r = \gamma_S$.

Let $A = \mathbb{F}_q[t]$ and $A^+$ be the set of monic polynomials in $A$. Let $P \in A$ be an irreducible polynomial and $d$ be a divisor of $q - 1$.

**Definition 2.2.** (see [6, p. 24]) If $P$ does not divide $a$, let $(a/P)_d$ be the unique elements of $\mathbb{F}_q^*$ such that

$$a^{N_{P^{-1}}} \equiv \left(\frac{a}{P}\right)_d \pmod{P}.$$

If $P|a$, define $(a/P)_d = 0$. The symbol $(a/P)_d$ is called the $d$-th power residue symbol.

If $m$ is a fixed prime divisor of $q - 1$, then $\mathbb{F}_q^*$ has a unique subgroup $S_m$ of order $m$. Let $\eta$ be a fixed generator of $S_m$, i.e. $\eta \neq \bar{1}$ and $\eta^m = \bar{1}$, then $S_m = \{1, \eta, \eta^2, \ldots, \eta^{m-1}\}$. We define $\chi$ to be the following monomorphism:

$$\chi : \ S_m \to \mathbb{C}$$

$$\eta^k \mapsto \exp\left(\frac{2k\pi i}{m}\right)$$

and we also denote

$$\left(\frac{a}{P}\right) = \chi \cdot \left(\frac{a}{P}\right)_m,$$

for any $a \in A$ such that $P \nmid a$. If $P|a$, denote

$$\left(\frac{a}{P}\right) = 0.$$

We have

$$\left(\frac{ab}{P}\right) = \left(\frac{a}{P}\right) \left(\frac{b}{P}\right),$$

for any $a, b \in A$, that is for any irreducible polynomial $P \in \mathbb{F}_q[t]$, $(\frac{a}{P})$ is a multiplicative character on $\mathbb{F}_q$.}

**Lemma 2.3.** Let $E = k(\sqrt[n]{n})$ be a geometric Kummer extension of $k$. We have

$$\sum_{\deg P = N} \left(\left(\frac{n}{P}\right) + \left(\frac{n}{P}\right)^2 + \cdots + \left(\frac{n}{P}\right)^{m-1}\right) = o(\pi(N)).$$
Proof. Suppose $\text{Gal}(E/k) = < \sigma >$. If $N$ is big enough, from Proposition 10.6 in [6], we have

$$T_1 := \{ \deg P = N \mid \left( \frac{n}{P} \right) = 1 \} = \{ P \in S_k \mid \deg P = N, (P, E/k) = \text{id} \},$$

$$T_2 := \{ \deg P = N \mid \left( \frac{n}{P} \right) \neq 1 \} = \{ P \in S_k \mid \deg P = N, (P, E/k) = \sigma^i, m \nmid i \},$$

where $S_k$ is the set of monic irreducible polynomials which are unramified in $E$. By Chebotarev’s Density Theorem (Lemma 1.2), we have

$$\# T_1 = \frac{1}{m} q^N N + O \left( \frac{q^{N/2}}{N} \right);$$

$$\# T_2 = \frac{m-1}{m} q^N N + O \left( \frac{q^{N/2}}{N} \right).$$

Thus

$$\sum_{P \in T_1} \left( \left( \frac{n}{P} \right) + \left( \frac{n}{P} \right)^2 + \cdots + \left( \frac{n}{P} \right)^{m-1} \right)$$

$$= \sum_{P \in T_1} (1 + 1^2 + \cdots + 1^{m-1}) = (m-1) # T_1$$

$$= \frac{m-1}{m} q^N N + O \left( \frac{q^{N/2}}{N} \right);$$

$$\sum_{P \in T_2} \left( \left( \frac{n}{P} \right) + \left( \frac{n}{P} \right)^2 + \cdots + \left( \frac{n}{P} \right)^{m-1} \right)$$

$$= \sum_{P \in T_2} (\zeta_m + \zeta_m^2 + \cdots + \zeta_m^{m-1}) = (-1) # T_2$$

$$= - \frac{m-1}{m} q^N N + O \left( \frac{q^{N/2}}{N} \right),$$

where $\zeta_m$ is some primitive $m$-th root of unity. Therefore,

$$\sum_{\deg P = N} \left( \left( \frac{n}{P} \right) + \cdots + \left( \frac{n}{P} \right)^{m-1} \right)$$

$$= \sum_{P \in T_1} \left( \left( \frac{n}{P} \right) + \cdots + \left( \frac{n}{P} \right)^{m-1} \right) + \sum_{P \in T_2} \left( \left( \frac{n}{P} \right) + \cdots + \left( \frac{n}{P} \right)^{m-1} \right)$$

$$= O \left( \frac{q^{N/2}}{N} \right)$$

when $N$ is big enough. Thus from the prime number theory for polynomials (Lemma 1.3), we have

$$\sum_{\deg P = N} \left( \left( \frac{n}{P} \right) + \left( \frac{n}{P} \right)^2 + \cdots + \left( \frac{n}{P} \right)^{m-1} \right) = o(\pi(N)).$$

\[\square\]

Lemma 2.4. We have $\gamma_S = m^r$ for some $r \leq l$. 

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Proof. The proof is the same as Lemma 2.1 in [2] if we replace $Q^*/Q^{*2}$ with $k^*/k^{*m}$.

Theorem 2.5. Let

$$\mathcal{M} := \left\{ P \mid \left( \frac{D_1}{P} \right)_m = \cdots = \left( \frac{D_l}{P} \right)_m = 1 \right\}.$$ 

The relative density of $\mathcal{M}$ equals to

$$\frac{\gamma_s}{m^l}.$$ 

Proof. In fact

$$\mathcal{M} : = \left\{ P \mid \left( \frac{D_1}{P} \right)_m = \cdots = \left( \frac{D_l}{P} \right)_m = 1 \right\} = \left\{ P \mid \left( \frac{D_1}{P} \right) = \cdots = \left( \frac{D_l}{P} \right) = 1 \right\}.$$ 

Let $\mathcal{P}(S)$ be the set of all distinct prime factors of $D_1D_2...D_l$. Clearly, $\mathcal{P}(S)$ is a finite set. Let $N$ be a positive integer. Considering the following counting function:

$$R_N = \frac{1}{m^l} \sum_{\deg P = N} \sum_{P \notin \mathcal{P}(S)} \left( 1 + \left( \frac{D_1}{P} \right) + \cdots + \left( \frac{D_l}{P} \right)^{m-1} \right) \cdots \left( 1 + \left( \frac{D_1}{P} \right) + \cdots + \left( \frac{D_l}{P} \right)^{m-1} \right).$$ 

Since the $m$-th power residue symbol is completely multiplicative, we have

$$R_N = \frac{1}{m^l} \sum_{\deg P = N} \sum_{P \notin \mathcal{P}(S)} \left( \frac{n}{P} \right)$$

$$= \frac{1}{m^l} \sum_{\deg P = N} \sum_{P \notin \mathcal{P}(S)} \left( \frac{n^2}{P} \right) + \cdots$$

$$+ \sum_{(b_1,...,b_l) \in \mathbb{Z}_m^l} \left( \frac{n^{m-1}}{P} \right)$$

$$= \frac{1}{m^l} \sum_{\deg P = N} \sum_{P \notin \mathcal{P}(S)} \left( \frac{n}{P} + \left( \frac{n}{P} \right)^2 + \cdots + \left( \frac{n}{P} \right)^{m-1} \right)$$

$$= \sum_{(b_1,...,b_l) \in \mathbb{Z}_m^l} \frac{1}{m^l} \sum_{n=D_1^{b_1}...D_l^{b_l}} \left( \frac{n}{P} + \left( \frac{n}{P} \right)^2 + \cdots + \left( \frac{n}{P} \right)^{m-1} \right) \left( \frac{1}{m-1} \right).$$
If \( n \) is a perfect \( m \)-th power, then \( \left( \frac{n}{P} \right) = 1 \) for each \( P \not\in \mathcal{P}(S) \). Thus, for these \( \gamma_S \) values of \( n \), the inner sum is

\[
\frac{1}{m^l} \sum_{\deg P = N\atop P \not\in \mathcal{P}(S)} \left( \left( \frac{n}{P} \right) + \cdots + \left( \frac{n}{P} \right)^{m-1} \right) \left( \frac{1}{m-1} \right) = \frac{1}{m^l} \pi(N),
\]

if \( N \) is large enough. For the remaining values of \( n \) (i.e. when \( n \) is not a \( m \)-th power). From our assumption in this paper, \( K/k \) is a geometric extension. Then we have \( k(\sqrt[m]{n})/k \) is also a geometric \( m \)-cyclic extension. Thus by Lemma 2.3, we have

\[
\frac{1}{m^l} \sum_{\deg P = N\atop P \not\in \mathcal{P}(S)} \left( \left( \frac{n}{P} \right) + \cdots + \left( \frac{n}{P} \right)^{m-1} \right) \left( \frac{1}{m-1} \right) = o(\pi(N)),
\]

Therefore, if \( N \) is large enough,

\[
R_N = \frac{\gamma_S}{m^l} \pi(N) + o(\pi(N))
\]

and hence

\[
\frac{R_N}{\pi(N)} = \frac{\gamma_S}{m^l} + o(1).
\]

So we have

\[
\lim_{N \to \infty} \frac{R_N}{\pi(N)} = \frac{\gamma_S}{m^l}.
\]

This concludes the proof.

Now we can prove the main result in this section.

Proof of Theorem 2.1: Let

\[ f(x) = (x^m - D_1)(x^m - D_2)...(x^m - D_l) \in A[x], \]

then \( K/k \) is the splitting field of \( f(x) \). Let \( S_k \) be the set of monic irreducible polynomials which are unramified in \( K \) and

\[ \mathcal{M} := \left\{ P \in S_k \mid \left( \frac{D_1}{P} \right) = ... = \left( \frac{D_l}{P} \right) = 1 \right\}. \]

By Theorem 2.3, we know that the relative density of \( \mathcal{M} \) is

\[
\frac{\gamma_S}{m^l} = \frac{1}{m^l - r}.
\]

Let \( \sigma_P = (P, K/k) \in \text{Gal}(K/k) \). Since \( P \in \mathcal{M} \), \( D_i \) is a \( m \)-th power residue modulo \( P \) and hence \( P \) splits completely in \( k(\sqrt[m]{D_i}) \). Therefore \( \sigma_P \) restricted to \( k(\sqrt[m]{D_i}) \) is the identity for \( 1 \leq i \leq l \). Suppose \([K : k] = m^l\). From our assumption, \( K \) is a geometric extension of \( k \). By the Chebotarev Density Theorem (Lemma 1.2) and the prime number theorem for polynomials (Lemma 1.3), the relative density of \( \mathcal{M} \) is

(2.2) \[
\frac{1}{[K : k]} = \frac{1}{m^l}.
\]

By comparing equations (2.1) and (2.2), we get \( t = l - r \).
3. Multi Artin-Schreier extensions

In this section, let \( q \) be a power of a prime number \( p \) and \( K \) be a multi-Artin-Schreier extension of \( k \). That is \( K = k(\alpha_1, \ldots, \alpha_l) \) and there is a finite set \( S = \{D_1, \ldots, D_l\} \) of nonconstant elements in \( k = \mathbb{F}_q(t) \) such that

\[
\alpha_i^p - \alpha_i = D_i \quad (1 \leq i \leq l).
\]

In the next subsection we recall the arithmetic of Artin-Schreier extensions (also see [3] and [4]).

3.1. The arithmetic of Artin-Schreier extensions. Let \( q \) be a power of a prime number \( p \). Let \( k = \mathbb{F}_q(t) \) be the rational function field. Let \( L/k \) be a cyclic extension of degree \( p \). Then \( L/k \) is an Artin-Schreier extension, that is, \( L = k(\alpha) \), where \( \alpha^p - \alpha = D \), \( D \in \mathbb{F}_q(t) \) and that \( D \) can not be written as \( x^p - x \) for any \( x \in k \). Conversely, for any \( D \in \mathbb{F}_q(t) \) and \( D \) can not be written as \( x^p - x \) for any \( x \in k \), \( k(\alpha)/k \) is a cyclic extension of degree \( p \), where \( \alpha^p - \alpha = D \). Two Artin-Schreier extensions \( k(\alpha) \) and \( k(\beta) \) such that \( \alpha^p - \alpha = D \) and \( \beta^p - \beta = D' \) are equal if and only if they satisfy the following relations,

\[
\alpha \mapsto x\alpha + B_0 = \beta, \\
D \mapsto xD + (B_0^2 - B_0) = D', \\
x \in \mathbb{F}_p^*, B_0 \in k.
\]

(See [3] or Artin [1] p.180-181 and p.203-206) Thus we can normalize \( D \) to satisfy the following conditions,

\[
D = \sum_{i=1}^{m} \frac{Q_i}{P_i^{e_i}} + f(t),
\]

\[
(P_i, Q_i) = 1, \text{ and } p \nmid e_i, \text{ for } 1 \leq i \leq m,
\]

\[
p \nmid \deg(f(t)), \text{ if } f(t) \notin \mathbb{F}_q,
\]

where \( P_i (1 \leq i \leq m) \) are monic irreducible polynomials in \( \mathbb{F}_q[t] \) and \( Q_i (1 \leq i \leq m) \) are polynomials in \( \mathbb{F}_q[t] \) such that \( \deg(Q_i) < \deg(P_i^{e_i}) \).

If \( D \) has the above normalized forms, then the infinite place \((1/t)\) is split, inert, or ramified in \( L \) respectively when \( f(t) = 0 \); \( f(t) \) is a constant and the equation \( x^p - x = f(t) \) has no solutions in \( \mathbb{F}_q \); \( f(t) \) is not a constant. Then the field \( K \) is called real, inert imaginary, or ramified imaginary respectively. Moreover, the finite places of \( k \) which are ramified in \( K \) are \( P_1, \ldots, P_m \) (see [3, p.39]). Let \( P \) be a finite place of \( k \) which is unramified in \( L \), i.e. \( P \) does not equal to \( P_1, \ldots, P_m \). Let \((P, L/k)\) be the Artin symbol at \( P \). Then

\[
(P, L/k)\alpha = \alpha + \{D/P\}
\]
and the Hasse symbol $\{ D/P \}$ is defined for $\text{ord}_P(D) \geq 0$ by the following equalities:

\[
\{ D/P \} \equiv D + D^p + \cdots + D^{N(P)/p} \mod P \\
\equiv (D + D^q + \cdots + D^{N(P)/q}) \\
+ (D + D^q + \cdots + D^{N(P)/q})^p \\
+ \cdots \\
+ (D + D^q + \cdots + D^{N(P)/q})^{p^j} \mod P,
\]

(3.2)

\[
\{ D/P \} = \text{tr}_{F_q/F_p} \text{tr}_{(O_K/P)/F_q}(D \mod P)
\]

(see [3, p. 40]). The Artin-Schreier operator $\mathcal{P}$ is defined by

\[
\mathcal{P}(x) = x^p - x,
\]

for $x \in L$, and obviously $\mathcal{P}$ is an additive operator. A root of a polynomial $x^p - x - a$ with $a \in k$ will be denoted by $\mathcal{P}^{-1}(a)$ (see [5, p. 296]). Let

\[
\mathcal{P}k = \{ \mathcal{P}(a) \mid a \in k \} \quad \text{and} \quad \mathcal{P}^{-1}k = \{ \mathcal{P}^{-1}(a) \mid a \in k \}.
\]

We define $\varphi$ to be the following monomorphism:

\[
\varphi : \ F_p \to \mathbb{C} \\
\quad x \mapsto \exp \left( \frac{2\pi i x}{p} \right)
\]

and we also denote

\[
\{ \{ D/P \} \} = \varphi \cdot \{ D/P \};
\]

for any $D \in K$ such that $\text{ord}_P(D) \geq 0$. We have

(3.3)

\[
\{ \left\{ \frac{D_1 + D_2}{P} \right\} \} = \{ \{ D_1/P \} \} \{ \{ D_2/P \} \},
\]

for any $D_1, D_2 \in K$ such that $\text{ord}_P(D_1) \geq 0$ and $\text{ord}_P(D_1) \geq 0$.

**Lemma 3.1.** Let $E = k(\mathcal{P}^{-1}(n))$ be a geometric Artin-Schreier extension of $k$. We have

\[
\sum_{\text{deg}P = N} \left( \left\{ \left\{ \frac{n}{P} \right\} \right\} + \left( \left\{ \frac{n}{P} \right\} \right)^2 + \cdots + \left( \left\{ \frac{n}{P} \right\} \right)^{p-1} \right) = o(\pi(N)).
\]

**Proof.** Suppose $\text{Gal}(E/k) = \langle \sigma \rangle$. If $N$ is big enough, from equation (3.1), we have

\[
T_1 : = \left\{ \text{deg}P = N \mid \left\{ \left\{ \frac{n}{P} \right\} \right\} = 1 \right\} = \left\{ \text{deg}P = N \mid \left\{ \frac{n}{P} \right\} = 0 \right\}
\]

\[
= \{ P \in S_k \mid \text{deg}P = N, (P, E/k) = \text{id} \};
\]

\[
T_2 : = \left\{ \text{deg}P = N \mid \left\{ \left\{ \frac{n}{P} \right\} \right\} \neq 1 \right\} = \left\{ \text{deg}P = N \mid \left\{ \frac{n}{P} \right\} \neq 0 \right\}
\]

\[
= \{ P \in S_k \mid \text{deg}P = N, (P, E/k) = \sigma^i, p \nmid i \},
\]
where $S_k$ is the set of monic irreducible polynomials which are unramified in $E$. By Chebotarev’s Density Theorem (Lemma 1.2), we have

$$
\# T_1 = \frac{1}{p} q^N + O \left( \frac{q^{N/2}}{N} \right); \\
\# T_2 = \frac{p - 1}{p} q^N + O \left( \frac{q^{N/2}}{N} \right).
$$

Thus

$$
\sum_{P \in T_1} \left( \{ \frac{n}{P} \} + \{ \frac{n}{P} \}^2 + \cdots + \{ \frac{n}{P} \}^{p-1} \right) \\
= \sum_{P \in T_1} (1 + 1^2 + \cdots + 1^{p-1}) = (p - 1) \# T_1 \\
= \frac{p - 1}{p} q^N + O \left( \frac{q^{N/2}}{N} \right); \\
\sum_{P \in T_2} \left( \{ \frac{n}{P} \} + \{ \frac{n}{P} \}^2 + \cdots + \{ \frac{n}{P} \}^{p-1} \right) \\
= \sum_{P \in T_2} (\zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1}) = (-1) \# T_2 \\
= - \frac{p - 1}{p} q^N + O \left( \frac{q^{N/2}}{N} \right),
$$

where $\zeta_p$ is some primitive $p$-th root of unity. Therefore,

$$
\sum_{\deg P = N} \left( \{ \frac{n}{P} \} + \cdots + \{ \frac{n}{P} \}^{p-1} \right) \\
= \sum_{P \in T_1} \left( \{ \frac{n}{P} \} + \cdots + \{ \frac{n}{P} \}^{p-1} \right) + \sum_{P \in T_2} \left( \{ \frac{n}{P} \} + \cdots + \{ \frac{n}{P} \}^{p-1} \right) \\
= O \left( \frac{q^{N/2}}{N} \right)
$$

when $N$ is big enough. Thus from the prime number theory for polynomials (Lemma 1.3), we have

$$
\sum_{\deg P = N} \left( \{ \frac{n}{P} \} + \{ \frac{n}{P} \}^2 + \cdots + \{ \frac{n}{P} \}^{p-1} \right) = o(\pi(N)).
$$

\[ \square \]

3.2. Main result. In this subsection we state and prove our main result.

Let $\gamma_S$ be the cardinality of the following set

$$
\{(a_1, a_2, \cdots, a_l) \in \mathbb{F}_p^l \mid a_1 D_1 + a_2 D_2 + \cdots + a_l D_l = F^p - F \text{ for some } F \in k\}.
$$

We will prove the following result.

**Theorem 3.2.** For a given finite set $S$ of nonconstant elements of $k = \mathbb{F}_q(t)$ with $|S| = l$, we have

$$
[K : k] = p^{l-r},
$$
where \( r \) is the non-negative integer given by \( p^r = \gamma_S \).

**Lemma 3.3.** We have \( \gamma_S = p^r \) for some \( r \leq l \).

**Proof.** We extend the proof of Lemma 2.1 in [2] to our case. Let \( V = \mathbb{F}_p' \) be the vector space having \( v_1, \ldots, v_t \) as a basis. Let \( W = k/Pk \). Then \( W \) is a \( \mathbb{F}_p' \)-vector space. Let \( \tau : V \to W \) be given by \( \tau(v_i) = D_i (\text{mod } Pk) \) and extended by linearity. It is then clear that \( (a_1, a_2 \cdots, a_t) \in \mathbb{F}_p' \) satisfies \( a_1D_1 + a_2D_2 + \cdots + a_tD_t \in Pk \) if and only if \( (a_1, a_2 \cdots, a_t) \in \ker(\tau) \). Thus \( \gamma_S = p^r \), where \( r \) is the dimension of \( \ker(\tau) \). \( \square \)

**Theorem 3.4.** Let

\[
\mathcal{M} := \left\{ P \mid \left\{ \frac{D_1}{P} \right\} = \cdots = \left\{ \frac{D_t}{P} \right\} = 0 \right\}.
\]

The relative density of \( \mathcal{M} \) equals to

\[
\frac{\gamma_S}{p^r}.
\]

**Proof.** In fact

\[
\mathcal{M} := \left\{ P \mid \left\{ \frac{D_1}{P} \right\} = \cdots = \left\{ \frac{D_t}{P} \right\} = 0 \right\} = \left\{ P \mid \{ \left\{ \frac{D_1}{P} \right\} \} = \cdots = \{ \left\{ \frac{D_t}{P} \right\} \} = 1 \right\}.
\]

Let \( \mathcal{P}(S) = \bigcup_{i=1}^{t} S(D_i) \), where \( S(D_i) \) is defined as the set of prime factors of the denominator of \( D_i \). Clearly, \( \mathcal{P}(S) \) is a finite set. Let \( N \) be a positive integer. Considering the following counting function:

\[
R_N = \frac{1}{p^l} \sum_{\substack{\deg P = N \\
P \notin \mathcal{P}(S)}} \left( 1 + \{\frac{D_1}{P}\} \right) \cdots \left( 1 + \{\frac{D_{t-1}}{P}\} \right) \left( 1 + \{\frac{D_t}{P}\} \right).
\]

From equation \((3.3)\), we have

\[
R_N = \frac{1}{p^l} \sum_{\substack{\deg P = N \\
P \notin \mathcal{P}(S)}} \sum_{(b_1, \ldots, b_t) \in \mathbb{F}_p^t} \{\frac{n}{P}\} + \sum_{(b_1, \ldots, b_t) \in \mathbb{F}_p^t} \{\frac{2n}{P}\} + \cdots
\]

\[
= \frac{1}{p^l} \sum_{\substack{\deg P = N \\
P \notin \mathcal{P}(S)}} \frac{1}{p-1} \left( \sum_{(b_1, \ldots, b_t) \in \mathbb{F}_p^t} \{\frac{n}{P}\} + \sum_{(b_1, \ldots, b_t) \in \mathbb{F}_p^t} \{\frac{2n}{P}\} + \cdots \right)
\]

\[
= \frac{1}{p^l} \sum_{\substack{\deg P = N \\
P \notin \mathcal{P}(S)}} \frac{1}{p-1} \left( \sum_{(b_1, \ldots, b_t) \in \mathbb{F}_p^t} \{\frac{n}{P}\} + \cdots + \{\frac{n}{P}\} \right)
\]

\[
= \frac{1}{p^l} \sum_{\substack{\deg P = N \\
P \notin \mathcal{P}(S)}} \left( \sum_{(b_1, \ldots, b_t) \in \mathbb{F}_p^t} \{\frac{n}{P}\} + \cdots + \{\frac{n}{P}\} \right) \left( \frac{1}{p-1} \right).
\]
If \( n \in \mathcal{P}k \), then \( \{\{\frac{n}{P}\}\} = 1 \) for each \( P \notin \mathcal{P}(S) \). Thus, for these \( \gamma_S \) values of \( n \), the inner sum is
\[
\frac{1}{p^l} \sum_{\deg P = N \atop P \notin \mathcal{P}(S)} \left( \{\{\frac{n}{P}\}\} + \ldots + \{\{\frac{n}{P}\}\}^{p-1} \right) \left( \frac{1}{p-1} \right) = \frac{1}{p^l} \pi(N),
\]
if \( N \) is large enough. For the remaining values of \( n \) (i.e. \( n \notin \mathcal{P}k \)). From our assumption in this paper, \( K/k \) is a geometric extension. we have \( k(\mathcal{P}^{-1}(n))/k \) is also a geometric extension. Thus by Lemma 3.1 we have
\[
\frac{1}{p^l} \sum_{\deg P = N \atop P \notin \mathcal{P}(S)} \left( \{\{\frac{n}{P}\}\} + \ldots + \{\{\frac{n}{P}\}\}^{p-1} \right) \left( \frac{1}{p-1} \right) = o(\pi(N)),
\]
Therefore, if \( N \) is large enough,
\[
R_N = \frac{\gamma_S}{p^l} \pi(N) + o(\pi(N))
\]
and hence
\[
\frac{R_N}{\pi(N)} = \frac{\gamma_S}{p^l} + o(1).
\]
So we have
\[
\lim_{N \to \infty} \frac{R_N}{\pi(N)} = \frac{\gamma_S}{p^l}.
\]
This concludes the proof. \( \square \)

Now we can proof the main result in this section.

Proof of Theorem 3.2. Let
\[
f(x) = (x^p - x - D_1)(x^p - x - D_2)\ldots(x^p - x - D_l) \in k[x],
\]
then \( K/k \) is the splitting field of \( f(x) \). Let \( S_k \) be the set of monic irreducible polynomials which are unramified in \( K \) and
\[
\mathcal{M} := \left\{ P \in S_k \mid \{\frac{D_1}{P}\} = \ldots = \{\frac{D_l}{P}\} = 0 \right\}.
\]
By Theorem 3.4, we know that the relative density of \( \mathcal{M} \) is
\[
\frac{\gamma_S}{p^l} = \frac{1}{p^l - r}.
\]
Let \( \sigma_P = (P, K/k) \in \text{Gal}(K/k) \). Since \( P \in \mathcal{M} \), Hasse symbol \( \{\frac{D_j}{P}\} = 0 \), hence from equation (3.1), \( \sigma_P \) restricted to \( k(\alpha_i) \) is the identity, where
\[
\alpha_i^p - \alpha_i = D_i,
\]
for \( 1 \leq i \leq l \). Suppose \( [K:k] = p^l \). For our assumption, \( K \) is a geometric extension of \( k \). By the Chebotarev’s Density Theorem (Lemma 1.2) and the prime number theorem for polynomials (Lemma 1.3), the relative density of \( \mathcal{M} \) is
\[
\frac{1}{[K:k]} = \frac{1}{p^l}.
\]
By comparing equations (3.4) and (3.5), we get \( t = l - r \). This finishes our proof.

4. Another approach

In this section, using abelian Kummer theory instead of Lemma 2.3 and Lemma 3.1 we give another approach to this problem. Notice that in this section we do not assume \( K/k \) is a geometric extension.

4.1. Multi-Kummer case: Let \( m \) be any prime divisor of \( q - 1 \). Let \( K \) be a multi-\( m \)-cyclic extension of \( k = \mathbb{F}_q(t) \). That is \( K = k(\sqrt[m]{D_1}, \ldots, \sqrt[m]{D_l}) \) and \( S = \{D_1, \ldots, D_l\} \) is a finite set of nonzero polynomials in \( A = \mathbb{F}_q[t] \).

Let \( Z_m \) be the set of integers

\[
Z_m = \{0, 1, 2, \ldots, m - 2, m - 1\}.
\]

Let \( \gamma_S \) be the cardinality of the following set

\[
\{(a_1, a_2, \ldots, a_l) \in Z_m^l \mid D_1^{a_1}D_2^{a_2} \cdots D_l^{a_l} = F^m \text{ for some } F \in \mathbb{F}_q[t]\},
\]

We have

**Theorem 4.1.**

\[
[K : k] = m^{l-r},
\]

where \( r \) is the non-negative integer given by \( m^r = \gamma_S \).

**Proof.** Let \( B \) be a subgroup of \( k^* \) generated by \( k^m \) and \( S \). From Chapter VI, Theorem 8.1 in [5] and the definition of \( S \), we have

\[
[K : k] = [k(B^m) : k] = [B : k^m] = m^{l-r}.
\]

\( \square \)

**Theorem 4.2.** Let

\[
\mathcal{M} := \left\{ P \mid \left( \frac{D_1}{P} \right)_m = \cdots = \left( \frac{D_l}{P} \right)_m = 1 \right\}.
\]

The Dirichlet density of \( \mathcal{M} \) equals to \( 1/m^{l-r} \), where \( r \) is the non-negative integer given by \( m^r = \gamma_S \). In particular, if \( k/k \) is a geometric extension, then the relative density of \( \mathcal{M} \) also equals to \( 1/m^{l-r} \).

**Proof.** Let \( \sigma_P = (P, K/k) \in \text{Gal}(K/k) \). Since \( P \in \mathcal{M} \), \( D_i \) is an \( m \)-th power residue modulo \( P \) and hence \( P \) splits completely in \( k(\sqrt[m]{D_i}) \) from Proposition 10.6 in [6]. Therefore \( \sigma_P \) restricted to \( k(\sqrt[m]{D_i}) \) is the identity for \( 1 \leq i \leq l \). From Theorem 4.1, we have \([K : k] = m^{l-r}\). By Chebotarev’s Density Theorem (Lemma 1.1), the Dirichlet density of \( \mathcal{M} \) equals to \( \frac{1}{[K : k]} = \frac{1}{m^{l-r}} \).

If \( K/k \) is a geometric extension, then by Chebotarev’s Density Theorem (Lemma 1.2) and the prime number theorem for polynomials (Lemma 1.3), the relative density of \( \mathcal{M} \) also equals to \( \frac{1}{[K : k]} = \frac{1}{m^{l-r}} \). \( \square \)
4.2. Multi-Artin-Schreier case: Let $q$ be a power of prime $p$ and $K$ be a multi-Artin-Schreier extension of $k = \mathbb{F}_q(t)$. $K = k(\alpha_1, ..., \alpha_l)$ and there is a finite set $S = \{D_1, ..., D_l\}$ of nonzero elements in $k$ such that

$$\alpha_i^p - \alpha_i = D_i \quad (1 \leq i \leq l).$$

Let $\gamma_S$ be the cardinality of the following set

$$\{(a_1, a_2, \cdots, a_l) \in \mathbb{F}_p^l \mid a_1D_1 + a_2D_2 + \cdots + a_lD_l = F^p - F \text{ for some } F \in k\}.$$

We have

**Theorem 4.3.**

$$[K : k] = p^{l-r},$$

where $r$ is the non-negative integer given by $p^r = \gamma_S$.

**Proof.** Let $B$ be a subgroup of $k$ generated by $\mathcal{P}k$ and $S$. From Chapter VI, Theorem 8.3 in [5] and the definition of $S$, we have

$$[K : k] = [k(\mathcal{P}^{-1}B) : k] = [B : \mathcal{P}k] = p^{l-r}. \quad \square$$

**Theorem 4.4.** Let

$$\mathcal{M} := \left\{ P \mid \left\{ \frac{D_1}{P} \right\} = \cdots = \left\{ \frac{D_l}{P} \right\} = 0 \right\}.$$

The Dirichlet density of $\mathcal{M}$ equals to $1/p^{l-r}$, where $r$ is the non-negative integer given by $p^r = \gamma_S$. In particular, if $k/k$ is a geometric extension, then the relative density of $\mathcal{M}$ also equals to $1/p^{l-r}$.

**Proof.** The proof is similar to the proof of Theorem 3.2. Let $\sigma_P = (P, K/k) \in \text{Gal}(K/k)$. Since $P \in \mathcal{M}$, If $\left\{ \frac{D_i}{P} \right\} = 1$, then $P$ splits completely in $k(\alpha_i)$ by equation 3.1 where

$$\alpha_i^p - \alpha_i = D_i \quad (1 \leq i \leq l).$$

Therefore $\sigma_P$ restricted to $k(\alpha_i)$ is the identity for $1 \leq i \leq l$. From Theorem 1.3, we have $[K : k] = p^{l-r}$. By Chebotarev’s Density Theorem (Lemma 1.1), the Dirichlet density of $\mathcal{M}$ equals to $\frac{1}{[K:k]} = p^{l-r}$. If $K/k$ is a geometric extension, then by Chebotarev’s Density Theorem (Lemma 1.2) and the prime number theorem for polynomials (Lemma 1.3), the relative density of $\mathcal{M}$ also equals to $\frac{1}{m^{l-r}}$. \quad \square

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