SOLITON SOLUTIONS IN RELATIVISTIC FIELD THEORIES AND GRAVITATION

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Abstract. We report on some recent results on a class of relativistic lagrangian field theories supporting non-topological soliton solutions and their applications in the contexts of Gravitation and Cosmology. We analyze one and many-components scalar fields and gauge fields.

1 Introduction

The well-known Born-Infeld (BI) model (Born & Infeld 1934) was historically proposed to remove the divergence of the electron’s self-energy in classical electrodynamics. Aside from electrostatic soliton solutions in three space dimensions, the model exhibits dyon and bidyon-like solutions (Chernitskii 1999), electric-magnetic duality (Gibbons & Rasheed 1995) and special properties of wave propagation, belonging to the class of “completely exceptional” theories (Boillat 1970).

Recently there has been a renewed interest in BI theory and its non-abelian extensions since they appear at different levels of string theory (Gibbons 1998). However, there are many other physical contexts where BI-like models have been used, such as the description of dark energy in Cosmology (e.g. Füzfa & Alimi 2006) and the phenomenological description of the nucleon structure (Deser et al 1976; Pavlovski 2002).

On the other hand, many studies on generalized electromagnetic and gauge field theories coupled with gravitation have been devoted to the analysis of particle-like solutions. Although several theorems of the 70’s (Coleman 1977) forbid the existence of static, finite-energy solutions of the pure Yang-Mills theory, such solutions were found in the Einstein-Yang-Mills system (Bartnick & McKinnon 1988). Later it was shown that similar glueball solutions also exist in BI-like models in flat space (Gal’tsov & Kerner 2000) as well as in curved space (e.g. Wirschins et al 2001) (see Volkov & Gal’tsov (1999) for a review and references on non-abelian solitons).

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2 The models

The BI generalization of classical electrodynamics is by no means unique and, from the point of view of soliton solutions, BI theory is only a particular example of a large class, which has been exhaustively determined in Diaz-Alonso & Rubiera-Garcia (2007). Here we deal with the generalized gauge field problem, by considering gauge-invariant lagrangians which are given functions \( L = \phi(X,Y) \) of the two quadratic field invariants

\[
X = -\frac{1}{2} \sum_a F^a_{\mu\nu} F^{a\mu\nu} \quad \text{and} \quad Y = -\frac{1}{2} \sum_a F^a_{\mu\mu} F^*_{\mu\nu} \quad \text{with signals},
\]

defined in a domain \((\Omega \subseteq \mathbb{R}^2)\) which is assumed to be open and connected and including the vacuum \((X = Y = 0)\). In calculating the field invariants we use the ordinary definition of the trace, although other definitions are possible (Tseytlin 1997). We also require the condition \( \phi(X,Y) = \phi(X,-Y) \) to be satisfied, in order to preserve parity invariance. Moreover we restrict our analysis to “physically admissible theories”, which we define by the requirement of the vanishing of the vacuum energy \((\phi(0,0) = 0)\), as well as the positive definiteness of the energy, which demands the minimal necessary and sufficient condition

\[
\rho^* \geq \left( X + \sqrt{X^2 + Y^2} \right) \frac{\partial \phi}{\partial X} + Y \frac{\partial \phi}{\partial Y} - \varphi(X,Y) \geq 0, \tag{2.1}
\]

to be satisfied in the entire domain of definition \((\Omega)\). The field equations read

\[
\sum_b D_{ab\mu} \left[ \frac{\partial \phi}{\partial X} F^{\mu\nu b} + \frac{\partial \phi}{\partial Y} F^{*\mu\nu b} \right] = 0, \tag{2.2}
\]

where \( D_{ab\mu} = \delta_{ab} \partial_\mu + g \sum_c C_{abc} A_{c\mu} \). We next consider the electrostatic spherically symmetric solutions (ESS) \((E_a(r) = -\hat{\nabla}(\phi_a(r)) = -\phi'_a(r) \hat{r}; H_a = 0)\). When this substitution is done in (2.2) we get two sets of equations

\[
\nu = 0 \to \hat{\nabla} \left( \frac{\partial \phi}{\partial X} \hat{\nabla} \phi_a(r) \right) = 0; \quad \nu = i = 1, 2, 3 \to \sum_{bc} C_{abc} \phi_b(r) \phi'_c(r) = 0. \tag{2.3}
\]

The first set leads to the first integrals \( r^2 \frac{\partial \phi}{\partial X} \phi'(r) = Q_a \) \((Q_a \text{ being integration constants, identified as color charges})\) which coincide with the set of first integrals of the field equations for static, spherically symmetric solutions (SSS) of a multicomponent scalar field theory with a lagrangian density given by

\[
L = f \left( \sum_a \partial_\mu \phi_a \partial^\mu \phi_a \right) \equiv f(X) = -\varphi(-X, Y = 0). \tag{2.4}
\]

The solutions of Eqs. (2.3) for the multiscalar or gauge cases take the form \( \phi_a(r) = \frac{Q_a}{Q} (\phi(r, Q) + \alpha_a) \), where \( Q = \sqrt{Q_0^2} \) is the mean-square (scalar or color) charge and \( \alpha_a \) are integration constants. The function \( \phi(r, Q) \) is the solution of the one-component scalar field theory resulting from the restricted lagrangian \( f(X) = \partial_\mu \phi \partial^\mu \phi \). In the gauge case the second set of equations (2.3) restricts the possible
values of the integration constants $\alpha_n = \frac{Q}{\phi} \alpha$ and then the ESS solutions take the form $\phi_n(r) = \frac{Q}{\phi} (\phi(r, Q) + \alpha)$ where now $Q$ is the mean-square color charge.

Thus, the form of the ESS solutions of generalized gauge field theories coincides with that of the SSS solutions of an associated one-component scalar field theory. The energy of the ESS solutions, when finite, is twofold the energy of the associated SSS solutions, which reads

$$
\varepsilon(Q) = -4\pi \int_0^\infty r^2 f \left(-\phi'^2(r)\right) dr = Q^{3/2}\varepsilon(Q = 1).
$$

The requirement of convergence of this integral leads to a classification of the admissible models according to the central (A-cases) and asymptotic (B-cases) behaviors of the soliton field strength (see Fig. 1). Cases A-1 and A-2 correspond to infinite (but integrable) and finite soliton field strengths, respectively. Cases B-1, B-2 and B-3 correspond to asymptotic dampings of the soliton field strength which are slower than coulombian, coulombian, or faster than coulombian, respectively (Diaz-Alonso & Rubiera-Garcia 2007).

For each admissible scalar model supporting finite-energy SSS solutions there exists an infinite family of admissible gauge field theories supporting similar finite-energy ESS solutions, obtained from (2.4) and the admissibility conditions (2.1).

The linear stability of the SSS and ESS solutions requires the energy (2.5) to be a minimum against small charge-preserving perturbations of the fields and potentials in the gauge and scalar cases. In the latter we found that all finite-energy SSS solutions of admissible models are always linearly stable while in the former the following supplementary condition for stability must be satisfied

$$
\frac{\partial \phi}{\partial X} - 2X \frac{\partial^2 \phi}{\partial Y^2} > 0 , \forall(X, Y = 0).
$$

As a non-trivial example of this class of soliton-supporting theories we introduce here the family of generalized electromagnetic field models

$$
\varphi(X,Y) = X/2 + \lambda X^\alpha + \beta Y^2 (\alpha > 3/2, \beta > 0),
$$

Fig. 1. Different possible central and asymptotic behaviors of the admissible models
which is the simplest generalization of a family of scalar models (defined from (2.7) by setting $\beta = 0$) supporting SSS soliton solutions, which has been analyzed in Diaz-Alonso & Rubiera-Garcia (2007). This generalized electromagnetic family fulfills the admissibility conditions (2.1) and supports ESS solitons. For certain values of the parameters in the electromagnetic case ($\lambda < 0$, $a = 2$) the non-linear term has the form of the decoupled part of the four-vertex contribution to the effective lagrangian of quantum electrodynamics.

3 Conclusions and outlook

In summary, we have analyzed scalar and generalized gauge field theories supporting soliton solutions. Aside from BI-like models which have been widely used during the last few years, other models have also recently attracted attention in search of self-gravitating scalar, electromagnetic and gauge field solitons as well as regular charged solutions (Ayon-Beato & Garcia 1999). The class of models considered here can also be useful in other physical contexts. For instance, in the extension of the description of dark energy in Cosmology, as a non-canonical scalar field (Armendariz-Picon & Lim 2005) or as a gauge field governed by generalized actions (Dyadichev et al 2002), in the phenomenological description of nucleon structure, in generalized gauge theories in higher dimensions, glueballs, etc.

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