GLOBAL EXISTENCE AND BLOWUP IN INFINITE TIME FOR A FOURTH ORDER WAVE EQUATION WITH DAMPING AND LOGARITHMIC STRAIN TERMS

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ABSTRACT. We consider the well-posedness of solution of the initial boundary value problem to the fourth order wave equation with the strong and weak damping terms, and the logarithmic strain term, which was introduced to describe many complex physical processes. The local solution is obtained with the help of the Galerkin method and the contraction mapping principle. The global solution and the blowup solution in infinite time under sub-critical initial energy are also established, and then these results are extended in parallel to the critical initial energy. Finally, the infinite time blowup of solution is proved at the arbitrary positive initial energy.

1. Introduction. We deal with the fourth order wave equation

\[ u_{tt} + \alpha \Delta^2 u - \beta \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(u_x^i) - \Delta u_t + |u_t|^{r-1} u_t = |u|^{q-1} u \]  

for \((x,t) \in \Omega \times [0,T)\) with the initial data

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \]  

and the boundary conditions

\[ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{or} \quad u = \Delta u = 0, \quad x \in \partial \Omega \times [0,T), \]  

where \(\sigma_i(u_{x_i}) = u_{x_i} \ln |u_{x_i}|^p, \quad \alpha > 0, \quad \beta > 0, \quad \Omega \subset \mathbb{R}^n \quad (n \leq 3)\) is a bounded open domain with a smooth boundary \(\partial \Omega\), \(T > 0\) is the maximum existence time of \(u, \nu\) is the unit outward normal on \(\partial \Omega\) and \(r, q\) satisfy

\[ (H) \quad 1 \leq r < q < \infty \quad \text{if} \quad n \leq 3. \]

The most significant feature of model (1) is the strain term. Generally, the strain term appears in the model as a polynomial [24]. The model is attractive due to its physical meaning and complex structure, which can be used to describe the variety of physical phenomena, such as the nonlinear dynamic beam [18], the interaction of water waves [7], the idealization of the suspension bridge [21] and the

2020 Mathematics Subject Classification. Primary: 35L05, 35A01; Secondary: 35B45.
Key words and phrases. Fourth order wave equation, Galerkin method, global existence, blowup in infinite time, arbitrary positive initial energy.

The first author is supported by the Ph.D. Student Research and Innovation Fund of the Fundamental Research Funds for the Central Universities (3072021CF2404), the second author is supported by NSFC grant 11871017.

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one-dimensional beam suspended by cables [32]. The nonlinear weak damping and strong damping make the model more interesting. Both of these two factors describe respectively the dynamics of a plate that accounts for transverse shear effects [13, 14] and the regularizing effect on the dynamics which give rise to analytic semigroup associated with the linear part of the system [36], also the strong damping term is responsible for dissipation. The fourth-order wave equation combined with the strain term can be used to describe the longitudinal motion of an elasto-plastic bar [3, 4, 5], in its original stress-free state, is of uniform cross-section and unit length [6], which also dealt with a number of problems concerning a model equation for one-dimensional motion of a viscoelastic bar which may undergo phase changes [33]. This type of problem has applications in many branches of physics, such as nuclear physics, optics and geophysics [10, 29, 16]. The physical appeal and importance of the model (1) provided strong motivation for studying the underlying mathematical theory. Next, we shall recall some mathematical results about the above model equations.

For the standard fourth order wave equation

$$u_{tt} + \Delta^2 u = f(u),$$

(4)

Di et al. in [17] obtained the global existence and uniqueness of regular solution and weak solution to the initial boundary value problem of this equation with $f(u) = |u|^p u$ by the Galerkin approximation and potential well method. For the case $f(u) = \int_{\Omega} |x-y|^{-\alpha} u(y)dy + |u|^{p-2} u - u$ ($\alpha \neq 0$) or $f(u) = \int_{\Omega} u(y)dy + |u|^{p-2} u - u$ ($\alpha = 0$), Liu and Zhou in [30] established the local existence and stability of solutions by using the semigroup theory, obtained the global solution and non-global solution in view of the potential well theory and the energy estimate, and further estimated the upper bound of the blowup time. Considering the influence of strong damping, Yang et al. in [43] studied the fourth order damped wave equation

$$u_{tt} + \Delta^2 u - \Delta u - \Delta u_t = f(u),$$

and obtained the finite time blowup of solutions at three different initial energy levels, the existence and the asymptotic behavior of global solutions at critical initial energy level.

Regarding the polynomial strain term, Liu and Xu in [31] considered the following fourth order wave equation with nonlinear strain and source terms

$$u_{tt} + \Delta^2 u - \alpha \Delta u + \sum_{k=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) = f(u),$$

(5)

where $f(u) \leq a_1 |u|^{q_1}, \sigma_i(u_{x_i}) \leq a_2 |u_{x_i}|^{q_2}$ and $a_1$, $a_2$, $q_1$ and $q_2$ are positive constants. They proved the global existence if $I(u_0) > 0$ and blowup in finite time of solution if $I(u_0) < 0$ to the initial boundary value problem of (5) with $E(0) \leq d$, where $d$ is the so-called depth of potential well and will be defined later. Shen et al. in [34] extended the blowup result obtained in [31] to arbitrarily positive initial energy. Wang and Wang in [38] studied the initial boundary value problem of the equation

$$u_{tt} + \Delta^2 u - \alpha \Delta u - \sum_{k=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) + u_t = f(u).$$

(6)

They proved the global existence of solutions and the energy decays exponentially, and also the blowup of solution in finite time with arbitrary initial energy to (6)
with $\sigma_i(s) = |s|^{m-2}u$ and $f(s) = |s|^{p-2}s$. For more nonlinear damping results, we refer to [12, 42].

Some interesting phenomena will occur for the model (5) with logarithmic term $f(s) = s \ln s$ and $\sigma_i(s) = |s|^{m-2}s$

$$u_{tt} + \Delta^2 u - \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|u_{x_i}|^{m-2}u_{x_i}) = u \ln u.$$  

(7)

Xu et al. in [39] proved the local existence and the global existence of the solution to the Dirichlet problem of (7) with sub-critical initial energy and $I(u_0) > 0$. They also obtained the blowup in infinite time of solution under sub-critical, critical and sup-critical initial energy conditions due to the logarithmic source term. For the case of second order version of (7) without $\Delta^2 u$, Lian et al. in [26] obtained the local solution, the global existence of solutions and the energy decays under $I(u_0) < 0$ and the sub-critical and critical initial energy, and they further got the blowup in infinite time to the solution under three initial energy conditions and $I(u_0) < 0$. For more works related to the model with the logarithmic term we refer to [2, 8, 23, 44].

Taking the polynomial strain term $\sigma_i(u_{x_i}) \leq a|u_{x_i}|^q$ in model (1), Lian et al. in [24] gave the local existence of solutions, and proved the global existence and finite time blowup with $E(0) \leq d$, and finite time blowup of the solution at arbitrarily high initial energy level. We pay attention to the effects on the well-posedness of the solution to the problem (1)-(3) when the polynomial strain term in [24] is replaced by the logarithmic strain. In particular, the logarithmic strain term brings us a new and interesting blowup result, that is, blowup in infinite time of the solution instead of the finite time blowup as in [24] due to the polynomial strain term. Such special results will inevitably require new estimation techniques and improved potential well methods. An often indispensable tool is the elliptic theory [20, 22, 35] to establish the potential well theory. The potential well theory is a powerful tool for studying the dependence of the well-posedness of solution to the partial differential equations on the initial data, which can effectively deal with wave equations [28, 45], the pseudo-parabolic equations with non-local source [37], polynomial source [41], singular potential [25], and the coupled parabolic systems [40].

This article is organized as follows. In Section 2, we state some preliminaries needed in the proof of results. In Section 3, we establish the local existence of the solutions to the problem (1)-(3) by the fixed point theory [9]. In Section 4, we introduce the potential well and prove the global existence and blowup of solution for subcritical initial energy. In Section 5, we extend all results obtained in subcritical initial energy to critical initial energy. Section 6 is devoted to the blowup result of solution at arbitrarily high initial energy.

2. Preliminaries and notations. We denote the inner product in $L^2(\Omega)$ by $(u, v) = \int_\Omega uv \, dx$, and define $\| \cdot \|_p = \| \cdot \|_{L^p(\Omega)}$, $\| \cdot \|_{k,p} = \| \cdot \|_{W^{k,p}(\Omega)}$, $\| \cdot \| = \| \cdot \|_2$.

We introduce a new Sobolev space $H := H^2_0(\Omega)$ if $u = \frac{\partial u}{\partial \nu}$ is equal to 0 on the boundary of $\Omega$, $H := H^2_0(\Omega) \cap H^2(\Omega)$ if $u = \Delta u$ is equal to 0 on the boundary of $\Omega$, which is given a norm for $\alpha, \beta > 0$

$$\|u\|_H^2 := \alpha\|\Delta u\|^2 + \beta\|\nabla u\|^2.$$

We use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between $H^{-2}(\Omega)$ and $H^2(\Omega)$.

Here, we show some equivalent relationships between $\|\Delta u\|$, $\|u\|_{2,2}$ and $\|u\|_H$. 

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Lemma 2.1 ([15]). The norms $\|\Delta u\|$ and $\|u\|_{2,2}$ are equivalent provided $u \in H$ with $u = \Delta u = 0$ on $\partial \Omega$.

Corollary 1. The norms $\|u\|_H$ and $\|u\|_{2,2}$ are equivalent for any $u \in H$, and

$$\|u\|_H^2 \geq C_1\|u\|^2,$$

where $C_1 > 0$ is the best embedding constant.

Corollary 2. Let $s$ satisfy (H), then $H(\Omega) \hookrightarrow L^*(\Omega)$, i.e., $\|u\|_s \leq C_2\|u\|_H$, where $C_2 > 0$ is a constant independence of $u$.

Definition 2.2. (Weak solution) We call $u = u(x,t)$ a weak solution to (1)-(3), if $u \in C([0,T];H) \cap C^1([0,T];L^2(\Omega))$ with $u_t \in L^2([0,T];H^1(\Omega))$ and $u_{tt} \in L^2([0,T];H^{-2}(\Omega))$ satisfies

$$\langle u_{tt}, \varphi \rangle + \alpha \int_{\Omega} \Delta u \varphi dx + \beta \int_{\Omega} \nabla u \nabla \varphi dx + \int_{\Omega} \nabla u_t \nabla \varphi dx + \int_{\Omega} |u_t|^{q-1} u_t \varphi dx = \int_{\Omega} |u|^{q-1} u \varphi dx + \int_{\Omega} \sum_{i=1}^{n} u_{x_i} \ln |u_{x_i}|^p \varphi_{x_i} dx$$

for any $\varphi \in L^\infty([0,T];H)$.

Lemma 2.3 ([11, 19, 27], Logarithmic Sobolev inequality). For $u \in H^1_0(\Omega)$ and any $a > 0$ there holds

$$n(1 + \ln n)\|u\|^2 + 2\int_{\Omega} |u|^2 \ln \left(\frac{|u|}{\|u\|}\right) dx \leq \frac{a^2}{\pi} \|\nabla u\|^2.$$  \hspace{1cm} (9)

Lemma 2.4 ([1], Sobolev imbedding theorem). Assume that $\Omega \subset \mathbb{R}^n$ is bounded. Let $j \geq 0$, $m > 1$ and $1 \leq k < \infty$ be integers, $\Omega$ satisfy the cone condition and if $mk < n$, then

$$W^{j+m,k}(\Omega) \hookrightarrow W^{j,s}(\Omega) \text{ for } k < s < k^* = \frac{nk}{n-mk}.$$  \hspace{1cm} (10)

In particular,

$$W^{m,k}(\Omega) \hookrightarrow L^s(\Omega) \text{ for } k < s < k^* = \frac{nk}{n-mk},$$

and the imbedding constant depends only on $n$, $m$, $k$, $s$ and $j$.

3. Existence of local solution. In this section, we shall study the local well-posedness of the solution by combining the Galerkin method and the contraction mapping principle.

Here, for $T > 0$ we introduce a new space

$$\mathcal{H} := C([0,T];H) \cap C^1([0,T];L^2(\Omega)),$$

endowed with the norm

$$\|u\|^2_{\mathcal{H}} = \max_{t \in [0,T]} (\|u\|^2_H + \|u_t\|^2).$$

The point of proving the local existence is to deal with the nonlinear term, hence we need the following inequality.

Lemma 3.1 ([15]). For any $u_1(x,t)$, $u_2(x,t)$ with $(x,t) \in \Omega \times [0,T]$, $|u_1(x,t)| + |u_2(x,t)| > 0$ and $u_1(x,t) \neq u_2(x,t)$, one has

$$|u_1|^q u_1 - |u_2|^q u_2 \leq q(|u_1| + |u_2|)^{q-1}|u_1 - u_2|.$$
The following lemma is the key to prove the existence of local solution to (1)-(3), which explains why the space dimension considered in this paper is \( n \leq 3 \).

**Lemma 3.2.** Assume \( n \leq 3 \), there must exist two parameters \( m > 2 \) and \( m' > 2 \) both satisfying the imbeddings \( W^{2,2}(\Omega) \hookrightarrow W^{1,m}(\Omega) \), \( W^{2,2}(\Omega) \hookrightarrow W^{1,m'}(\Omega) \) and \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \).

**Proof.** On the one hand, if \( n \leq 2 \), then \( m = 3 \) and \( m' = 6 \) satisfying \( W^{2,2}(\Omega) \hookrightarrow W^{1,m}(\Omega) \), \( W^{2,2}(\Omega) \hookrightarrow W^{1,m'}(\Omega) \) and \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \).

On the other hand, for \( n > 2 \), by Lemma 2.4 and \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \), we need to show that there exist \( m \) and \( m' \) satisfying

\[
\begin{align*}
m, m' & \in \left( \frac{2n}{n-2}, \frac{2n}{n-2} \right). \\
1\ m + 1\ m' &= 1/2.
\end{align*}
\]

From \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \), we derive \( m = \frac{2m'}{m' - 2} \). It is easy to check that \( m(m') \) is decreasing with respect to \( m' \in \left( \frac{2n}{n-2}, \frac{2n}{n-2} \right) \) due to \( \frac{dm}{dm'} = -\frac{4}{(m'-2)^2} < 0 \), which implies that

\[
m \in (n, \infty),
\]

also

\[
m' \in (n, \infty).
\]

In order to ensure (12)-(14), it must be required here

\[
m, m' \in \left( n, \frac{2n}{n-2} \right),
\]

that is \( n \leq 3 \). We complete the proof. \( \square \)

**Remark 1.** Obviously, for any \( n \geq 4 \), (15) is invalid. Therefore, in order to ensure \( W^{2,2}(\Omega) \hookrightarrow W^{1,m}(\Omega), W^{2,2}(\Omega) \hookrightarrow W^{1,m'}(\Omega) \) and \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \), a restriction of dimension \( n \leq 3 \) is necessary. Here are some examples to show that the existence of such \( m \) and \( m' \). The values of \( (m, m') \) given in the Table 1 are not unique in some cases and the symbol “−” means that this situation does not exist.

| \( n \) \( \frac{2n}{n-2} \) | \( m \in \left( \frac{2n}{n-2}, \frac{2n}{n-2} \right) \) | \( m' \in \left( \frac{2n}{n-2}, \frac{2n}{n-2} \right) \) | \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \) | Results |
|---|---|---|---|---|
| \( \leq 2 \) \( (n, \infty) \) | \( (n, \infty) \) | \( (m, m') = (3, 6) \) | valid |
| 3 \( 6 \) \( (3, 6) \) | \( (3, 6) \) | \( (m, m') = (5, \frac{10}{3}) \) | valid |
| 4 \( 4 \) \( \infty \) | \( \infty \) | \( \infty \) | invalid |
| 5 \( \frac{10}{3} \) \( \infty \) | \( \infty \) | \( \infty \) | invalid |

**Table 1.** The values for \( m \) and \( m' \)

In order to prove the local existence of the solution to problem (1)-(3), we first consider the following lemma. In the context, the constant \( C > 0 \) appearing on each line means that we can ignore the specific value, which may include the constants in many inequalities such as Young’s, Sobolev and Poincaré inequalities.
Lemma 3.3. If \((u_0(x), u_1(x)) \in \mathcal{H} \times H^1_0(\Omega)\) and \(u \in \mathcal{H}\), there is only one function \(v \in \mathcal{H} \cap C^2([0,T]; H^{-2}(\Omega))\) and \(v_t \in L^2([0,T]; H^0_0(\Omega))\) for any \(T > 0\) that satisfies

\[
v_{tt} + \alpha \Delta^2 v - \beta \Delta v - \Delta v_t + |v_t|^{r-1} v_t = |u|^{q-1} u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{x_i} \ln |u_{x_i}|^p)
\]

for \((x,t) \in \Omega \times [0,T]\) with the initial data

\[
v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x), \quad x \in \Omega
\]

and the boundary conditions

\[
v = \frac{\partial v}{\partial n} = 0 \text{ or } v = \Delta v = 0, \quad (x,t) \in \partial \Omega \times [0,T).
\]

Proof. For every \(h \geq 1\), let \(W_h\) be the space generated by the \(h\) functions \(\{\omega_1, \cdots, \omega_h\}\), where \(\{\omega_j\}\) is the complete orthogonal system of eigenfunctions of \(-\Delta\) in \(H^1_0(\Omega)\) and \(\|\omega_j\|_1 = 1\) for all \(j\). That is \(\Delta \omega_j + \lambda_j \omega_j = 0\), where \(\lambda_j\) is the related eigenvalues in \(H^1_0(\Omega)\) respectively. Let

\[
u_h^0 = \sum_{j=1}^{h} \left( \int_{\Omega} u_0 \omega_j \right) \omega_j \quad \text{and} \quad u_h^1 = \sum_{j=1}^{h} \left( \int_{\Omega} u_1 \omega_j \right) \omega_j,
\]

so that \(u_h^0 \in W_h\), \(u_h^1 \in W_h\), \(u_h^0 \to u_0\) in \(H\) and \(u_h^1 \to u_1\) in \(H^1_0(\Omega)\) as \(h \to +\infty\). Seeking \(\gamma_j^h\) \((j = 1, \cdots, h)\) in \(C^2[0,T]\) as

\[
v_h(t) = \sum_{j=1}^{h} \gamma_j^h(t) \omega_j,
\]

for every \(\eta \in W_h\) and \(t \geq 0\), which solves

\[
\int_{\Omega} \left( \ddot{v}_h + \alpha \Delta^2 v_h - \beta \Delta v_h - \Delta \dot{v}_h + |\dot{v}_h|^{r-1} \dot{v}_h \right) \eta dx = \int_{\Omega} \left( |u|^{q-1} u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{x_i} \ln |u_{x_i}|^p) \right) \eta dx
\]

with the initial value \(v_h(0) = u_h^0\), \(\dot{v}_h(0) = u_h^1\). Then we can get the Cauchy problem to the ordinary differential equation in term of \(\gamma_j^h\) for \(j = 1, \cdots, h\) and \(\eta = \omega_j\)

\[
\left\{ \begin{array}{ll}
\ddot{\gamma}_j^h + \alpha \lambda_j^2 \gamma_j^h + \beta \lambda_j \gamma_j^h + \lambda_j \dot{\gamma}_j^h + |\dot{\gamma}_j^h|^{r-1} \dot{\gamma}_j^h = \psi_j, \\
\gamma_j^h(0) = \int_{\Omega} u_0 \omega_j dx, \quad \dot{\gamma}_j^h(0) = \int_{\Omega} u_1 \omega_j dx,
\end{array} \right.
\]

here

\[
\psi_j(t) = \int_{\Omega} |u|^{q-1} u \omega_j dx - \int_{\Omega} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{x_i} \ln |u_{x_i}|^p) \omega_j dx, \quad t \in [0,T].
\]

For all \(j\), the solution \(\gamma_j^h \in C^2[0,T]\) of (21) is global and unique, which shows that there exists a unique \(v_h\) satisfying (19) and solving the problem (20). By the Poincaré inequality, there holds

\[
\|\dot{v}_h\| \leq C \|\nabla \dot{v}_h\|, \quad t \in [0,T].
\]
In (20), we take $\eta = \dot{v}_h(t)$ and integrate the result on $[0, t]$ ($t < T$) to have
\begin{equation}
\|\dot{v}_h\|^2 + \|v_h\|^2_{H^2} + 2 \int_0^t \left( \|\nabla \dot{v}_h\|^2 + \|\dot{v}_h\|_{H^1}^2 \right) d\tau \tag{23}
\end{equation}
\begin{equation}
= \|u_1\|^2 + \|u_0\|^2_{H^2} + 2 \int_0^t \int_{\Omega} |u|^{q-1} u\dot{v}_h dx d\tau - 2 \int_0^t \int_{\Omega} \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_x \ln |u_x|^{p}) \dot{v}_h dx d\tau.
\end{equation}

Next, we introduce the space
\[ \mathcal{M}_T := \{ u \in H : u(x, 0) = u_0, \ u_t(x, 0) = u_1, \ \|u\|_H \leq R \}, \tag{24} \]
where
\[ R^2 := 2 \left( \|u_0\|^2_H + \|u_1\|^2 \right). \tag{25} \]

Estimating $2 \int_0^t \int_{\Omega} |u|^{q-1} u\dot{v}_h dx d\tau$ with the help of the Hölder, Young's inequalities, (22) and the Corollary 2 gives
\begin{equation}
2 \int_0^t \int_{\Omega} |u|^{q-1} u\dot{v}_h dx d\tau \\
\leq 2 \int_0^t \int_{\Omega} |u|^q \dot{v}_h dx d\tau \\
\leq 2 \int_0^t \|u\|_{L^q}^2 \dot{v}_h dx d\tau \\
\leq \frac{2}{C} \int_0^t \|u\|_{L^q}^2 dx d\tau + \frac{C}{2} \int_0^t \|\dot{v}_h\|^2 d\tau \tag{26}
\end{equation}
\begin{equation}
\leq \frac{2}{C} \int_0^t \|u\|_{L^q}^2 dx d\tau + \frac{1}{2} \int_0^t \|\nabla \dot{v}_h\|^2 d\tau \\
\leq \frac{2}{C} \int_0^t \|u\|_{H^1}^2 dx d\tau + \frac{1}{2} \int_0^t \|\nabla \dot{v}_h\|^2 d\tau \\
\leq CTR^q + \frac{1}{2} \int_0^t \|\nabla \dot{v}_h\|^2 d\tau.
\end{equation}

Regarding $-2 \int_0^t \int_{\Omega} \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_x \ln |u_x|^{p}) \dot{v}_h dx d\tau$, again using the Hölder and Young’s inequalities, we have
\begin{equation}
-2 \int_0^t \int_{\Omega} \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_x \ln |u_x|^{p}) \dot{v}_h dx d\tau \\
= 2p \int_0^t \int_{\Omega} \sum_{i=1}^n \left( u_x \ln |u_x| \frac{\partial}{\partial x_i} \dot{v}_h \right) dx d\tau \\
\leq 2p \int_0^t \int_{\Omega} \sum_{i=1}^n \left( \left( \int_{\Omega} (u_x \ln |u_x|)^2 dx \right)^{\frac{1}{2}} \frac{\partial}{\partial x_i} \dot{v}_h \right) dx d\tau \\
\leq 2p^2 \int_0^t \int_{\Omega} \sum_{i=1}^n (u_x \ln |u_x|)^2 dx d\tau + \frac{1}{2} \int_0^t \|\nabla \dot{v}_h\|^2 d\tau.
\end{equation}
where \( \Omega := \Omega \cup \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset \), \( \Omega_1 := \{ x \in \Omega \mid |u_{x_i}| < 1 \} \) and \( \Omega_2 := \{ x \in \Omega \mid |u_{x_i}| \geq 1 \}, \ i = 1, \cdots, n \). For \( J_1 \), we in \((0, 1)\) need a minimum of function \( y(k) = k \ln |k| \), then according to the Fermat’s theorem, one has \( \min_{0 < s < 1} k \ln |k| = -\frac{1}{\varepsilon} \).

Hence a simple computation on \( J_1 \) yields

\[
J_1 = \int_{\Omega_1} \sum_{i=1}^{n} (u_{x_i} \ln |u_{x_i}|)^2 \ dx \leq \int_{\Omega_1} \frac{n}{\varepsilon^2} \ dx = \frac{n}{\varepsilon^2} |\Omega_1|. \tag{29}
\]

For \( J_2 \), by the fact that \( k > \ln k \) for \( k > 1 \), \( u \in C([0, T]; H) \), and the Sobolev embedding from \( H \) to \( W^{1,4}_0(\Omega) \) we have

\[
J_2 = \int_{\Omega_2} \sum_{i=1}^{n} (u_{x_i} \ln |u_{x_i}|)^2 \ dx \leq \int_{\Omega_2} \sum_{i=1}^{n} |u_{x_i}|^4 \ dx \leq C \| u \|^4_H := CR^4, \tag{30}
\]

which combining (29) gives

\[
J_0 = J_1 + J_2 \leq C.
\]

The above arguments make (27) become

\[
-2 \int_{0}^{t} \int_{\Omega} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{x_i} \ln |u_{x_i}|^p) \hat{v}_h \, dx \, d\tau \leq CT + \frac{1}{2} \int_{0}^{t} \| \nabla \hat{v}_h \|^2 \, d\tau, \tag{31}
\]

which combining (26) and (23) gives

\[
\| \hat{v}_h \|^2 + \| v_h \|^2_H + \int_{0}^{t} \| \nabla \hat{v}_h \|^2 + 2 \int_{0}^{t} \| \hat{v}_h \|^{p+1}_{r+1} \, d\tau \\
\leq \| u_0 \|^2 + \| u_0^h \|^2_H + CT \leq C_T, \tag{32}
\]

where \( C_T \) does not depend on \( h \). Also in view of (32) it is easy to see that

\[
\int_{0}^{T} \| \hat{v}_h \|_{H^{-2}(\Omega)}^2 \, dt \leq C_T. \tag{33}
\]

Hence the uniform estimates (32) and (33) show that

\[
\{ v_h \} \text{ is bounded in } L^\infty([0, T], H); \\
\{ \hat{v}_h \} \text{ is bounded in } L^\infty([0, T], L^2(\Omega)) \cap L^2([0, T], H^1_0(\Omega)); \\
\{ \tilde{v}_h \} \text{ is bounded in } L^2([0, T], H^{-2}(\Omega)). \tag{34}
\]

We know that the problem (16)-(18) admits a local solution by taking the limit in (20) which satisfies (34). Hence, we obtain the solution \( v \) of (16)-(18).
Uniqueness. Assuming that the solution is not unique, we let \( w \) and \( v \) be solutions to problem (16)-(18) endowing with the same initial data. Then subtracting these two equations about \( w \) and \( v \) gives

\[
\|v_t - w_t\|^2 + \|v - w\|_H^2 + 2 \int_0^t \|\nabla v_r - \nabla w_r\|^2 \, dr
\]

\[
+ 2 \int_0^t \int_\Omega (|v_r|^{-1} v_r - |\omega_r|^{-1} \omega_r) (v_r - \omega_r) \, dx \, dr = 0,
\]

which together with the following inequality ([15], Lemma 2.1)

\[
(|\xi|^{m-2} \xi - |\zeta|^{m-2} \zeta)(\xi - \zeta) \geq C|\xi - \zeta|^m, \ m \geq 2
\]
yields

\[
2 \int_0^t \int_\Omega (|v_r|^{-1} v_r - |\omega_r|^{-1} \omega_r) (v_r - \omega_r) \, dx \, dr \geq C \int_0^t \|v_r - \omega_r\|_{r+1}^2 \, dr \geq 0.
\]

Hence (35) becomes

\[
0 = \|v_t - w_t\|^2 + \|v - w\|_H^2 + 2 \int_0^t \|\nabla v_r - \nabla w_r\|^2 \, dr
\]

\[
+ 2 \int_0^t \int_\Omega (|v_r|^{-1} v_r - |\omega_r|^{-1} \omega_r) (v_r - \omega_r) \, dx \, dr
\]

\[
\geq \|v_t - w_t\|^2 + \|v - w\|_H^2 + 2 \int_0^t \|\nabla v_r - \nabla w_r\|^2 \, dr + C \int_0^t \|v_r - \omega_r\|_{r+1}^2 \, dr \geq 0,
\]

which yields \( v \equiv w \) a.e. in \( H \).

Next, based on Lemma 3.3 and the contraction mapping principle, we show that the problem (1)-(3) exists a uniqueness local solution.

Theorem 3.4 (Local solution). For \( u_0(x) \in H \) and \( u_1(x) \in H_0^1(\Omega) \), the problem (1)-(3) admits a unique solution \( u \in [0, T] \) for some \( T > 0 \).

Proof. For any \( T > 0 \), similar to Lemma 3.3 we consider the space \( \mathcal{M}_T \) defined in (24). Define \( \Phi: \mathcal{M}_T \to \mathcal{M}_T \), then Lemma 3.3 shows that the problem (16)-(18) exists a unique solution \( v = \Phi(u) \) for \( u \in \mathcal{M}_T \).

We prove the map \( \Phi \) is contractive as follows.

Step I. First, it needs to show \( \Phi(\mathcal{M}_T) \subseteq \mathcal{M}_T \) while \( T > 0 \) small enough, i.e., if \( \|u\|_H \leq R \), then \( \|\Phi(u)\|_H \leq R \). Similar to (23), we get

\[
\|v_t\|^2 + \|v\|_H^2 + 2 \int_0^t \|\nabla v_r\|^2 \, dr + 2 \int_0^t \|v_r\|_{r+1}^2 \, dr
\]

\[
= \|u_1\|^2 + \|u_0\|_H^2 + 2 \int_0^t \int_\Omega |u|^{r-1} u_r \, dx \, dr - 2 \int_0^t \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i} u_{x_i} \ln |u_{x_i}|^p v_r \, dx \, dr. \tag{37}
\]

Next, we will estimate the last two terms in (37). Similar to the estimates (26) and (31), for small enough \( T \) we get

\[
\|v_t\|^2 + \|v\|_H^2 \leq \frac{1}{2} R^2 + CTR^2q + CT \leq R^2,
\]

which says \( \|\Phi(u)\|_H \leq R \), i.e., \( \Phi \) maps \( \mathcal{M}_T \) to itself.

Step II. For \( u_1, u_2 \in \mathcal{M}_T \), we prove \( \|\Phi(u_1) - \Phi(u_2)\|_H \leq \delta \|u_1 - u_2\|_H \) for a positive constant \( \delta \) (0 < \( \delta < 1 \)).
Set \( v_1 = \Phi(u_1), v_2 = \Phi(u_2) \) and \( z = v_1 - v_2 \), then subtracting the two equations gives

\[
z_{tt} + \alpha \Delta^2 z - \beta \Delta z - \Delta z_t + |u_{1t}|^{r-1} u_{1t} - |u_{2t}|^{r-1} u_{2t} = \|u_1|^{q-1} u_1 - |u_2|^{q-1} u_2 - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{1x_i} \ln |u_{1x_i}|^p) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{2x_i} \ln |u_{2x_i}|^p). \tag{38}
\]

Testing the both sides of (38) by \( z_t \) over \((0, t) \times \Omega \) gives

\[
\frac{1}{2} \|z_t\|^2 + \frac{1}{2} \|z\|^2_H + \int_0^t \int_{\Omega} \|\nabla z_{\tau}\|^2 d\tau = \int_0^t \int_{\Omega} (|u_1|^{q-1} u_1 - |u_2|^{q-1} u_2) z_{\tau} d\tau
\]

\[
- p \int_0^t \int_{\Omega} \left( \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{1x_i} \ln |u_{1x_i}|) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{2x_i} \ln |u_{2x_i}|) \right) z_{\tau} d\tau,
\]

which follows from (36) that

\[
\frac{1}{2} \|z_t\|^2 + \frac{1}{2} \|z\|^2_H + \int_0^t \|\nabla z_{\tau}\|^2 d\tau \leq \int_0^t \int_{\Omega} (|u_1|^{q-1} u_1 - |u_2|^{q-1} u_2) z_{\tau} d\tau
\]

\[
- p \int_0^t \int_{\Omega} \left( \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{1x_i} \ln |u_{1x_i}|) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u_{2x_i} \ln |u_{2x_i}|) \right) z_{\tau} d\tau. \tag{39}
\]

For the first term on the right side of (39), by Lemma 3.1, the Hölder inequality, Corollary 2 and (22), we have

\[
\int_0^t \int_{\Omega} (|u_1|^{q-1} u_1 - |u_2|^{q-1} u_2) z_{\tau} d\tau
\]

\[
\leq \int_0^t \int_{\Omega} q \left( |u_1| + |u_2| \right)^{q-1} |u_1 - u_2| z_{\tau} d\tau
\]

\[
\leq q \int_0^t \left( \|u_1\|_{w'} + \|u_2\|_{w'} \right)^{q-1} \|u_1 - u_2\|_w \|z_{\tau}\| d\tau
\]

\[
\leq C \int_0^t \left( \|u_1\|_H + \|u_2\|_H \right)^{q-1} \|u_1 - u_2\|_H \|z_{\tau}\| d\tau \tag{40}
\]

\[
\leq CR^{q-1} \int_0^t \|u_1 - u_2\|_H \|z_{\tau}\| d\tau
\]

\[
\leq \frac{1}{2} CR^{2(q-1)} \int_0^t \|u_1 - u_2\|_H^2 d\tau + \frac{C}{2} \int_0^t \|z_{\tau}\|^2 d\tau
\]

\[
\leq \frac{1}{2} CR^{2(q-1)} \int_0^t \|u_1 - u_2\|_H^2 d\tau + \frac{1}{2} \int_0^t \|\nabla z_{\tau}\|^2 d\tau,
\]

where \( \frac{1}{w} + \frac{1}{w'} = \frac{1}{2} \).
Similar to Lemma 3.1, we estimate the last line of (39), and first give
\[ u_{1x_i} \ln |u_{1x_i}| - u_{2x_i} \ln |u_{2x_i}| \]
\[ = \int_0^1 \frac{d}{dt} \left( (u_{2x_i} + s \psi + \tau \psi) \ln |u_{2x_i} + s \psi + \tau \psi| \right)_{\tau=0} ds \]
\[ = \int_0^1 (\psi \ln |(1-s)u_{2x_i} + s u_{1x_i}| + \psi) ds \]
\[ \leq \int_0^1 (|u_{1x_i}| + |u_{2x_i}|) \ln e) |u_{1x_i} - u_{2x_i}| ds \]
\[ \leq \int_0^1 e |u_{1x_i}| + |u_{2x_i}| |u_{1x_i} - u_{2x_i}| ds \]
\[ = e |u_{1x_i}| + |u_{2x_i}| |u_{1x_i} - u_{2x_i}|, \]
where \( \psi = u_{1x_i} - u_{2x_i} \). Combining the H"older inequality, we derive
\[ -p \int_0^t \int_\Omega \left( \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_{1x_i} \ln |u_{1x_i}|) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_{2x_i} \ln |u_{2x_i}|) \right) z_\tau dxd\tau \]
\[ = p \int_0^t \int_\Omega \left( \sum_{i=1}^n \left( u_{1x_i} \ln |u_{1x_i}| - u_{2x_i} \ln |u_{2x_i}| \right) \frac{\partial}{\partial x_i} z_\tau \right) dxd\tau \]
\[ \leq p e \int_0^t \int_\Omega \left( |u_{1x_i}| + |u_{2x_i}| |u_{1x_i} - u_{2x_i}| \right) \frac{\partial}{\partial x_i} z_\tau \right) d\tau \]
then by Lemma 3.2, the above inequality becomes
\[ -p \int_0^t \int_\Omega \left( \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_{1x_i} \ln |u_{1x_i}|) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_{2x_i} \ln |u_{2x_i}|) \right) z_\tau dxd\tau \]
\[ \leq C \int_0^t \sum_{i=1}^n \left( |u_{1x_i}||u_{m'}| + |u_{2x_i}||m| \right) \frac{\partial}{\partial x_i} z_\tau \right) d\tau \]
\[ \leq C \int_0^t \sum_{i=1}^n \left( |u_{1x_i}| + |u_{2x_i}| |u_{1x_i} - u_{2x_i}| \right) \frac{\partial}{\partial x_i} z_\tau \right) d\tau \]
\[ \leq CR \int_0^t \|u_1 - u_2\|_{\mathcal{H}} \|\nabla z_\tau\| d\tau \]
\[ \leq CR^2 \int_0^t \|u_1 - u_2\|^2_{\mathcal{H}} d\tau + \frac{1}{2} \int_0^t \|\nabla z_\tau\|^2 d\tau, \]
where \( \frac{1}{m} + \frac{1}{m'} = \frac{1}{2} \). Then (39) becomes
\[ \frac{1}{2} \|z_\tau\|^2 + \frac{1}{2} \|z\|^2_{\mathcal{H}} \]
\[ \leq \frac{1}{2} CR^2 (q-1) \int_0^t \|u_1 - u_2\|^2_{\mathcal{H}} d\tau + CR^2 \int_0^t \|u_1 - u_2\|^2_{\mathcal{H}} d\tau, \]
that is
\[ \|z_\tau\|^2 + \|z\|^2_{\mathcal{H}} \leq C \int_0^t \|u_1 - u_2\|^2_{\mathcal{H}} d\tau \leq CT \|u_1 - u_2\|^2_{\mathcal{H}}. \]
Choosing $T$ small enough, then $\Phi$ is contractive. Hence, in view of the contraction mapping principle, we complete the proof of Theorem 3.4. 

4. The well-posedness for $E(0) < d$. In this section, we give some functionals, and define the stable and unstable sets. Also, we introduce some preliminary lemmas for proving the main results, and then we give the well-posedness of the solution for problem (1)-(3), such as the global existence and infinite time blowup of solutions.

Firstly, we introduce the total energy functional for problem (1)-(3)

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2_H - \frac{p}{2} \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| \, dx + \frac{p}{4} \|\nabla u\|^2 - \frac{1}{q + 1} \|u\|_{q+1}^{q+1}, \quad (43)$$

the potential energy functional

$$J(u) = \frac{1}{2} \|u\|^2_H - \frac{p}{2} \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| \, dx + \frac{p}{4} \|\nabla u\|^2 - \frac{1}{q + 1} \|u\|_{q+1}^{q+1} \quad (44)$$

and the Nehari functional

$$I(u) = \|u\|^2_H - p \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| \, dx - \|u\|_{q+1}^{q+1}, \quad (45)$$

then we know

$$E(t) = \frac{1}{2} \|u_t\|^2 + J(u). \quad (46)$$

We also introduce the sets

$$W = \{ u \in H \mid I(u) > 0 \} \cup \{ 0 \}$$

and

$$V = \{ u \in H \mid I(u) < 0 \} ,$$

the potential well depth

$$d = \inf_{u \in N} J(u), \quad (47)$$

where

$$N = \{ u \in H \mid I(u) = 0 \} .$$

**Lemma 4.1** ($E(t)$ is non-increasing). The total energy defined in (43) to problem (1)-(3) is non-increasing.

**Proof.** Testing the equation (1) by $u_t$ and then integrating on $\Omega \times (0, t)$ yields

$$\frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2_H - \frac{p}{2} \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| \, dx + \frac{p}{4} \|\nabla u\|^2 - \frac{1}{q + 1} \|u\|_{q+1}^{q+1} \right)$$

$$+ \|\nabla u_r\|^2 + \|u_r\|_{r+1}^{r+1} = 0 ,$$

that is

$$E(t) + \int_0^t \|\nabla u_r\|^2 \, d\tau + \int_0^t \|u_r\|_{r+1}^{r+1} \, d\tau = E(0) , \quad (48)$$

then

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} \leq 0. \quad (49)$$
In the following, we give two lemmas to show the properties of the potential energy functional and Nehari functional.

**Lemma 4.2.** If \( u \in H \) and \( \|u\|_H \neq 0 \), then

(i) \( \lim_{\lambda \to 0} J(\lambda u) = 0 \), \( \lim_{\lambda \to +\infty} J(\lambda u) = -\infty \);

(ii) in the interval \( 0 < \lambda < \infty \) has a unique \( \lambda^* = \lambda^*(u) \) satisfying

\[
\frac{d}{d\lambda} J(\lambda u) \bigg|_{\lambda = \lambda^*} = 0;
\]

(iii) on \( 0 \leq \lambda \leq \lambda^* \), \( J(\lambda u) \) is strictly increasing, on \( \lambda^* < \lambda < \infty \), \( J(\lambda u) \) is strictly decreasing, and at \( \lambda = \lambda^* \), \( J(\lambda u) \) has the maximum;

(iv) for \( 0 < \lambda < \lambda^* \), \( I(\lambda u) > 0 \), for \( \lambda^* < \lambda < \infty \), \( I(\lambda u) < 0 \), and \( I(\lambda^* u) = 0 \), where \( \lambda^* \) is a unique root of the equation

\[
\|u\|_H^2 - p \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - p \|\nabla u\|_2^2 \ln \lambda - \lambda^{q-1} \|u\|_{q+1}^{q+1} = 0.
\]

**Proof.** (i) By the definition of \( J(u) \) and (H), we obtain

\[
J(\lambda u) = \frac{1}{2} \lambda^2 \|u\|_H^2 - \frac{p}{2} \int_\Omega \sum_{i=1}^n (\lambda u_{x_i})^2 \ln |\lambda u_{x_i}| dx + \frac{p}{4} \|\nabla (\lambda u)\|^2 - \frac{\lambda^{q+1}}{q+1} \|u\|_{q+1}^{q+1}
\]

\[
= \frac{1}{2} \lambda^2 \|u\|_H^2 - \lambda \frac{p}{2} \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - \frac{\lambda^2 p}{2} \|\nabla u\|_2^2 \ln \lambda + \frac{\lambda^2 p}{4} \|\nabla u\|^2
\]

\[
- \frac{\lambda^{q+1}}{q+1} \|u\|_{q+1}^{q+1}
\]

\[(50)\]

which gives

\[
\lim_{\lambda \to +\infty} J(\lambda u) = -\infty, \quad \lim_{\lambda \to 0} J(\lambda u) = 0
\]

due to \( \|u\|_H \neq 0 \).

(ii) Differentiating the first identity in (50) with respect to \( \lambda \) yields

\[
\frac{d}{d\lambda} J(\lambda u) = \lambda \|u\|_H^2 - \lambda p \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - \lambda p \|\nabla u\|_2^2 \ln \lambda - \lambda^{q-1} \|u\|_{q+1}^{q+1}
\]

\[
= \lambda \left( \|u\|_H^2 - p \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - p \|\nabla u\|_2^2 \ln \lambda - \lambda^{q-1} \|u\|_{q+1}^{q+1} \right)
\]

\[
:= \lambda I_1.
\]

(51)

Combining with \( \lim_{\lambda \to +\infty} J(\lambda u) = -\infty \) and the fact that \( I_1 > 0 \) for a sufficient small \( \lambda \), (51) means that there exists a \( \lambda^* \) which is a unique root to the equation

\[
\|u\|_H^2 - p \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - p \|\nabla u\|_2^2 \ln \lambda - \lambda^{q-1} \|u\|_{q+1}^{q+1} = 0
\]
such that
\[
\frac{d}{d\lambda} J(\lambda u) \bigg|_{\lambda = \lambda^*} = 0. \tag{52}
\]
In fact, suppose that there are two roots \(\lambda_1\) and \(\lambda_2\) with \(0 < \lambda_1 < \lambda_2\) such that
\[
\frac{d}{d\lambda} J(\lambda u) = 0,
\]
then
\[
\|u\|_H^2 - p \int_{\Omega} \sum_{i=1}^{n} u_{x_i}^2 \ln |u_{x_i}| dx - p \|\nabla u\|^2 \ln \lambda_1 - \lambda_1^{q-1} \|u\|_{q+1}^{q+1} = 0
\]
and
\[
\|u\|_H^2 - p \int_{\Omega} \sum_{i=1}^{n} u_{x_i}^2 \ln |u_{x_i}| dx - p \|\nabla u\|^2 \ln \lambda_2 - \lambda_2^{q-1} \|u\|_{q+1}^{q+1} = 0.
\]
Subtracting the two equations, we get
\[
p \|\nabla u\|^2 \ln \frac{\lambda_2}{\lambda_1} + (\lambda_2^{q-1} - \lambda_1^{q-1}) \|u\|_{q+1}^{q+1} = 0,
\]
which is impossible.

(iii) Following from (ii), we get the conclusion.

(iv) From (45) we have
\[
I(\lambda u) = \lambda^2 \|u\|_H^2 - p \lambda^2 \int_{\Omega} \sum_{i=1}^{n} u_{x_i}^2 \ln |u_{x_i}| dx - p \lambda^2 \|\nabla u\|^2 \ln \lambda - \lambda^{q+1} \|u\|_{q+1}^{q+1}
= \lambda^2 \left( \|u\|_H^2 - p \int_{\Omega} \sum_{i=1}^{n} u_{x_i}^2 \ln |u_{x_i}| dx - p \|\nabla u\|^2 \ln \lambda - \lambda^{q+1} \|u\|_{q+1}^{q+1} \right)
= \lambda \frac{d}{d\lambda} J(\lambda u).
\]

Lemma 4.3. If \(u \in H\) and \(\|u\|_H \neq 0\), there hold
\[
J(u) \geq \frac{1}{2} I(u) + \frac{p}{4} \|\nabla u\|^2 \tag{53}
\]
and
\[
E(0) \geq E(t) = \frac{1}{2} \|u_t\|^2 + J(u) \geq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} I(u) + \frac{p}{4} \|\nabla u\|^2. \tag{54}
\]

Proof. From the definitions of \(J(u)\), \(I(u)\) and (H), we have
\[
J(u) \geq \frac{1}{2} \|u\|_H^2 - \frac{1}{2} \left( p \int_{\Omega} \sum_{i=1}^{n} u_{x_i}^2 \ln |u_{x_i}| dx + \|u\|_{q+1}^{q+1} \right) + \frac{p}{4} \|\nabla u\|^2
= \frac{1}{2} I(u) + \frac{p}{4} \|\nabla u\|^2,
\]
which combines (46) to give (54). \(\square\)
4.1. Global solution under sub-critical initial energy. We first give an invariance of the set $W$.

**Lemma 4.4.** Assume $u_0(x) \in H$, $u_1(x) \in H_1^0(\Omega)$. If $E(0) < d$, the solution $u$ of problem (1)-(3) belongs to $W$ as long as $u_0(x) \in W$.

**Proof.** We use the contradiction method to prove $I(u(t)) > 0$ if $0 < t < T$, i.e., there exists a $t_* \in (0, T)$ satisfying $I(u(t_*)) = 0$ and $I(u) > 0$ for $[0, t_*)$, here $t_*$ is the first time. Then by (47) we obtain

$$d \leq J(u(t_*)) \leq E(u(t_*)) \leq E(0) < d,$$

which is a contradiction. Hence the proof is completed. $lacksquare$

**Theorem 4.5.** Under the assumptions of Lemma 4.4, (1)-(3) exists a global solution. Further, there have two constants $K > 0$ and $k > 0$ such that

$$E(t) < Ke^{-kt}.$$

**Proof.** We will divide into two parts to obtain the results. Firstly we prove the existence of global solution to (1)-(3). Theorem 3.4 shows that the existence of unique local solution $u \in C(0, T; H(\Omega))$ to (1)-(3), where $T$ is the maximum existence time of $u(t)$. From (53) and $I(u) > 0$, for any $t \in (0, T)$ one has

$$J(u) > \frac{p}{4} \|\nabla u\|^2,$$

then from (48), we know

$$\frac{1}{2} \|u_t\|^2 + J(u) + \int_0^t \|\nabla u\|^2 + \int_0^t \|u_t\|^{r+1}_{r+1} = E(0) < d.$$ 

Therefore, the continuation principle implies $T = \infty$, and for any $t \in [0, \infty)$ we also get

$$\|\nabla u\|^2 + \|u_t\|^2 < \max\left\{2d, \frac{4d}{p}\right\},$$

and

$$\int_0^t \|u_t\|^{r+1}_{r+1} \leq C \int_0^t \|\nabla u\|^2 <Cd,$$

where $C$ is an embedding constant from $H^1_0(\Omega)$ to $L^{r+1}(\Omega)$. By the definition of $I(u)$, choosing $a = \sqrt{\frac{\alpha \pi}{p}}$ in (9) and combining (53), for $0 \leq t < \infty$ we have

$$2\|u\|_{H}^2 = 2I(u) + 2p \sum_{i=1}^{n} \ln |u_{x_i}| dx + 2\|u\|^{q+1}_{q+1}$$

$$\leq 4J(u) - p\|\nabla u\|^2 + p \left(2 \ln \|\nabla u\| - n \left(1 + \ln \sqrt{\frac{\alpha \pi}{p}}\right)\right) \|\nabla u\|^2 + \alpha \|\Delta u\|^2$$

$$+ 2C^q + 1 \|\nabla u\|^q + 1$$

$$\leq 4J(u) + p \left(2 \ln \|\nabla u\| - n \left(1 + \ln \sqrt{\frac{\alpha \pi}{p}}\right)\right) \|\nabla u\|^2 + \|u\|_{H}^2$$

$$+ 2C^q + 1 \left(\frac{4d}{p}\right)^{q+1},$$
that is
\[ \|u\|_H^2 \leq 4d + p \left( 2 \ln \|\nabla u\| - n \left( 1 + \ln \left( \frac{\alpha \pi}{p} \right) \right) \right) \|\nabla u\|^2 + 2Cq^+ \left( \frac{4d}{p} \right)^{q+1} \leq C \]
for any \(2 \ln \|\nabla u\| - n \left( 1 + \ln \left( \frac{\alpha \pi}{p} \right) \right) > 0\) (or \(< 0\), where \(C\) is a constant related to \(d\).

Here, we discuss the asymptotic behavior of the global solution. We define
\[ G(t) := \frac{\epsilon}{2} \|\nabla u\|^2 + \epsilon(u, u_t) + E(t), \]
where \(\epsilon\) is small enough such that \(G(0) > 0\) and there are \(\alpha_1\) and \(\alpha_2\) satisfying
\[ \alpha_1 E(t) \leq G(t) \leq \alpha_2 E(t). \] (55)
Setting \(n = u\) in (8), by the definition of \(G(t)\) and (49), we have
\[ G'(t) \leq -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^2 + \epsilon\|u_t\|^2 + \epsilon \int_{\Omega} |u_t|^{r-1} u_t u dx + \epsilon\|u\|_{q+1}^2 \]
+ \(p\epsilon \int_{\Omega} \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - \epsilon\|u\|_H^2. \]

According to the Young’s inequality, taking any positive constant \(\delta\), we get
\[ \left| \int |u_t|^{r-1} u_t u dx \right| \leq \delta\|u_t\|_{r+1}^2 + C(\delta)\|u\|_{r+1}^2. \] (57)

From (53) and (54), we derive
\[ \|u\|_{r+1}^2 \leq \alpha r+1 \|\nabla u\|_{r+1} \leq C r+1 \left( \frac{4\epsilon}{p} E(0) \right)^{\frac{r+1}{q+1}} \frac{4\epsilon}{p} E(t) := C_r E(t). \] (58)

By the definition of \(E(t)\) in (43), we have
\[ \frac{p}{2} \int_{\Omega} \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx + \frac{1}{q+1} \|u\|_{q+1}^2 \]
= \(\frac{p}{2} \int_{\Omega} \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx + \frac{1}{2} \|u\|_{q+1}^2 - \frac{q-1}{2(q+1)} \|u\|_{q+1}^2 \)
= \(\frac{1}{2} \left( p \int_{\Omega} \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx + \|u\|_{q+1}^2 \right) - \frac{q-1}{2(q+1)} \|u\|_{q+1}^2 \)
= \(\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|_{H}^2 - E(t) + \frac{p}{4} \|\nabla u\|^2, \)
then
\[ p \int_{\Omega} \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx + \|u\|_{q+1}^2 \]
= \(\|u_t\|^2 + \|u\|_{H}^2 - 2E(t) + \frac{p}{2} \|\nabla u\|^2 + \frac{q-1}{q+1} \|u\|_{q+1}^2. \) (59)
Substituting (57)-(59) into (56) and from $\|u_t\| \leq C\|\nabla u_t\|$, one has
\[
G'(t) \leq -\|\nabla u\|^2 - \|u_t\|^2 + \epsilon\|u_t\|^{q+1} + \epsilon\|u_t\|^{r+1} + \epsilon\|u_t\|^{q+1} + \epsilon\|u_t\|^{r+1} + \epsilon C(\delta) C_r E(t)
\]
\[
+ \epsilon\|u_t\|^2 - 2\epsilon E(t) + \frac{p\epsilon}{2}\|\nabla u\|^2 + \frac{\epsilon(q-1)}{q+1}\|u\|^{q+1}
\]
\[
\leq (2C\epsilon - 1)\|\nabla u_t\|^2 + (\epsilon\delta - 1)\|u_t\|^{r+1} + \epsilon(C(\delta) C_r - 2) E(t)
\]
\[
+ \epsilon\left(\frac{p}{2}\|\nabla u\|^2 + \frac{q-1}{q+1}\|u\|^{q+1}\right).
\]

And (43) tells us
\[
\frac{1}{2}\left(\|u\|_H^2 - p \int_{\Omega} \sum_{i=1}^{n} u_{x_i} \ln |u_{x_i}| dx - \|u\|^{q+1} + \frac{q-1}{2(q+1)}\|u\|^{q+1} + \frac{p}{4}\|\nabla u\|^2\right)
\]
\[
= \frac{1}{2} I(u) + \frac{q-1}{2(q+1)}\|u\|^{q+1} + \frac{p}{4}\|\nabla u\|^2
\]
\[
\leq E(t),
\]
by $I(u) > 0$, we have
\[
\frac{q-1}{2(q+1)}\|u\|^{q+1} + \frac{p}{4}\|\nabla u\|^2 < E(t).
\]

Observe (61) to see that there exists a continuous function $0 < S(t) < 1$ in $[0, \infty)$ such that
\[
\frac{q-1}{2(q+1)}\|u\|^{q+1} + \frac{p}{4}\|\nabla u\|^2 = S(t) E(t) \leq \max_{t \in [0, \infty)} \{S(t)\} E(t) := \gamma E(t),
\]
where $\gamma := \max_{t \in [0, \infty)} S(t)$. Therefore, substituting (62) into (60) gives
\[
G'(t) \leq (2C\epsilon - 1)\|\nabla u_t\|^2 + (\epsilon\delta - 1)\|u_t\|^{r+1} + \epsilon(C(\delta) C_r - 2 + 2\gamma) E(t).
\]
Obviously, $C_r := \epsilon(C(\delta) C_r - 2) + 2\gamma) = \epsilon(C(\delta) - 2(1 - \gamma)) < 0$ for $0 < \gamma < 1$ and sufficiently large $\delta$. Fix such $\delta > 0$ and take $\epsilon > 0$ small enough such that $2C\epsilon - 1 < 0$ and $\epsilon\delta - 1 < 0$. Then from (55), there holds
\[
G'(t) \leq \epsilon\gamma E(t) \leq \epsilon\gamma G(t)/\alpha_1,
\]
hence
\[
G(t) \leq G(0)e^{\epsilon\gamma t}/\alpha_1.
\]

Again using (55), we get
\[
E(t) \leq \frac{G(t)}{\alpha_1} \leq \frac{G(0)}{\alpha_1} e^{\epsilon\gamma t} = Ke^{-kt},
\]
where $K = \frac{G(0)}{\alpha_1} > 0$ and $k = -\frac{\epsilon\gamma}{\alpha_1} > 0$.

4.2. **Infinite time blowup of the solution.** Here, we consider the infinite time blowup of solution to the problem (1)-(3) for the case $r = 1$,
\[
u_{tt} + \alpha \Delta^2 u - \beta \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u_{x_i} \ln |u_{x_i}|^p - \Delta u_t + u_t = |u|^{q-1} u
\]
for $(x, t) \in \Omega \times [0, T)$ with the initial data
\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega
\]
and the boundary conditions

\[ u = \frac{\partial u}{\partial \nu} = 0 \text{ or } u = \Delta u = 0, \quad (x, t) \in \partial \Omega \times [0, T). \]  

(65)

Recalling (48), the energy identity of problem (63)-(65) becomes

\[ E(t) + \int_0^t (||\nabla u_t||^2 + ||u_t||^2) d\tau = E(0), \]

(66)

where \( E(t) \) is defined in (43). For convenience, we introduce the star inner product

\[(u, v)_* = \int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx\]

and the norm \( ||u||_*^2 = (u, v)_* \) for the Sobolev space \( H^1_0(\Omega) \).

We first give a relationship between \( I(u) \) and \( J(u) \).

**Lemma 4.6.** Assume that \( u \in V \), there hold

\[ d < \frac{p}{4} ||\nabla u||^2 + \frac{q - 1}{2(q + 1)} ||u||^{q+1}_{q+1} \]  

(67)

and

\[ I(u) < 2(J(u) - d). \]  

(68)

**Proof.** The fact \( u \in V \) and Lemma 4.2 indicate that \( 0 < \lambda^* < 1 \) satisfying \( I(\lambda^* u) = 0 \). By the definitions of \( J(u) \) and \( I(u) \), we have

\[ J(u) = \frac{1}{2} ||u||_H^2 - \frac{p}{2} \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx + \frac{p}{4} ||\nabla u||^2 - \frac{1}{q + 1} ||u||^{q+1}_{q+1} \]

\[ = \frac{1}{2} (||u||_H^2 - p \int_\Omega \sum_{i=1}^n u_{x_i}^2 \ln |u_{x_i}| dx - ||u||^{q+1}_{q+1}) + \frac{p}{4} ||\nabla u||^2 \]

\[ + \left( \frac{1}{2} - \frac{1}{q + 1} \right) ||u||^{q+1}_{q+1} \]  

(69)

From the definition of \( d \) and (69), we have

\[ d \leq J(\lambda^* u) \]

\[ = \frac{p}{4} ||\lambda^* \nabla u||^2 + \frac{q - 1}{2(q + 1)} ||\lambda^* u||^{q+1}_{q+1} \]

\[ < \frac{p}{4} ||\nabla u||^2 + \frac{q - 1}{2(q + 1)} ||u||^{q+1}_{q+1} \]

\[ = J(u) - \frac{1}{2} I(u), \]

which implies (68). \( \square \)

**Lemma 4.7.** Let \( u_0(x) \in H \) and \( u_1(x) \in H^1_0(\Omega) \). Suppose that \( E(0) < d \). Then the solution of problem (63)-(65) belongs to \( V \), provided that \( u_0(x) \in V \).

**Proof.** We can omit the proof in view of Lemma 4.4. \( \square \)

**Theorem 4.8.** Under the assumptions of Lemma 4.7 and \( (u_0, u_1) \geq 0 \), the solution to problem (63)-(65) blows up in infinite time.
Proof. Theorem 3.4 shows that (63)-(65) exists a solution \( u(t) \) in \((0, T)\), \( T \) is the maximum existence time. We prove that the solution \( u(t) \) blows up in infinite time under \( E(0) < d \) and \( I(u_0) < 0 \). We first introduce the auxiliary function \( \theta(t) : [0, T) \to \mathbb{R}^+ \) for \( 0 < T_0 < T \)

\[
\theta(t) := \|u\|^2 + \int_0^t \|u\|^2 \, dt + (T_0 - t) \|u_0\|^2,
\]

which gives

\[
\theta'(t) = 2(u, u_t) + 2 \int_0^t (u, u_{\tau}) \, d\tau
\]

and

\[
\theta''(t) = 2\langle u_{tt}, u \rangle + 2\|u_t\|^2 + 2\langle u_t, u \rangle.
\]

Multiplying the both side of (63) by \( u(x,t) \), we obtain

\[
\langle u_{tt}, u \rangle + \|u\|^2_H + \int_\Omega \nabla u_t \nabla udx + \int_\Omega u_t udx = \int_\Omega \sum_{i=1}^n u_x^2 i |u_{x_i}|^2 dx + \|u\|_{q+1}^2,
\]

which indicates

\[
\theta''(t) = 2\|u_t\|^2 - 2 \left( \|u\|^2_H - p \int_\Omega \sum_{i=1}^n u_x^2 i \ln |u_{x_i}| dx - \|u\|_{q+1}^q \right)
\]

\[
= 2\|u_t\|^2 - 2I(u) > 0,
\]

where \( I(u) < 0 \) due to Lemma 4.6. Combining (69), (46) and (66), we can obtain

\[
\theta''(t) = 2\|u_t\|^2 - 4J(u) + p\|\nabla u\|^2 + 2\|a\|_{q+1}^q
\]

\[
= 2\|u_t\|^2 - 4E(t) + 2\|u_t\|^2 + p\|\nabla u\|^2 + 2\|a\|_{q+1}^q
\]

\[
= 4\|u_t\|^2 - 4E(0) + 4 \int_0^t \|u_{\tau}\|^2 d\tau + p\|\nabla u\|^2 + 2\|a\|_{q+1}^q.
\]

From (71) we get

\[
(\theta'(t))^2 = 4 \left( (u, u_t)^2 + 2(u, u_t) \int_0^t (u, u_{\tau}) \, d\tau + \left( \int_0^t (u, u_{\tau}) \, d\tau \right)^2 \right),
\]

which together with the following inequalities derived by Cauchy-Schwarz and the Young inequalities

\[
(u, u_t) \leq \|u\| \|u_t\|,
\]

\[
\left( \int_0^t (u, u_{\tau}) \, d\tau \right)^2 \leq \int_0^t \|u\|^2 \, dt \int_0^t \|u_{\tau}\|^2 \, d\tau
\]

and

\[
2(u, u_t) \int_0^t (u, u_{\tau}) \, d\tau \leq \|u\|^2 \int_0^t \|u_{\tau}\|^2 \, d\tau + \|u_t\|^2 \int_0^t \|u\|^2 \, d\tau
\]

to give

\[
(\theta'(t))^2 \leq 4\theta(t) \left( \|u_t\|^2 + \int_0^t \|u_{\tau}\|^2 \, d\tau \right).
\]

(74)
Then combining (73) and (74) we derive
\[
\theta''(t)\theta(t) - (\theta'(t))^2 \\
\geq \theta''(t)\theta(t) - 4\theta(t) \left( ||u_t||^2 + \int_0^t ||u_r||^2_2 \,dr \right) \\
= \theta(t) \left( \theta''(t) - 4 \left( ||u_t||^2 + \int_0^t ||u_r||^2_2 \,dr \right) \right) \\
\geq \theta(t) \left( -4E(0) + p||\nabla u||^2 + \frac{2(q-1)}{q+1} ||u||^2_{q+1} \right).
\]
By \(0 < E(0) < d\), (75) becomes
\[
\theta''(t)\theta(t) - (\theta'(t))^2 \geq 4\theta(t) \left( -d + \frac{q-1}{2(q+1)} ||u||^2_{q+1} + \frac{p}{4} ||\nabla u||^2 \right).
\]
From (67) and the definition of \(\theta(t)\), we have
\[
\theta''(t)\theta(t) - (\theta'(t))^2 > 0.
\]
A direct calculation gives
\[
(ln |\theta(t)|)' = \frac{\theta'(t)}{\theta(t)} \tag{76}
\]
and
\[
(ln |\theta(t)|)'' = \left( \frac{\theta'(t)}{\theta(t)} \right)' = \frac{\theta''(t)\theta(t) - (\theta'(t))^2}{\theta^2(t)} > 0. \tag{77}
\]
By (77), the increase of \(\ln |\theta(t)|\)' is obvious. Integrating (76) on \((t_0, t)\) \((t_0 \in [0, t])\), which says
\[
\theta(t) \geq \theta(t_0) \exp \left( \frac{\theta'(t_0)}{\theta(t_0)} (t - t_0) \right). \tag{78}
\]
Due to \((u_0, u_1) \geq 0\), (71) and (72) implies that for a \(t_0\) sufficient small, we have \(\theta'(t_0) > 0, \theta(t_0) > 0\). Then, (78) shows that \(\lim_{t \to +\infty} \theta(t) = \infty\), i.e., the solution to problem (63)-(65) blows up in infinite time.

5. The well-posedness of solution for \(E(0) = d\). Firstly, an essential lemma as following is introduced to get the main conclusion.

**Lemma 5.1.** Assume that \(u_0 \in H, u_1 \in H^1_0(\Omega)\), and the condition (H) holds. Then for some \(\tau \in (0, T)\) there holds
\[
\int_0^\tau (||\nabla u_r||^2 + ||u_r||^2_{r+1}) \,dr > 0 \tag{79}
\]
if \(u\) is not a steady solution of (1)-(3).

**Proof.** Let \(u\) be a weak solution to the problem (1)-(3), \(T\) be the maximum existence time of \(u\). Arguing by contradiction, for any \(t \in (0, T)\), we have
\[
\int_0^t (||\nabla u_r||^2 + ||u_r||^2_{r+1}) \,dr = 0,
\]
then \(||\nabla u_r||^2 + ||u_r||^2_{r+1} = 0\) for any \(t \in (0, T)\) and \(x \in \Omega\), which says \(u = u_0\). Hence, the solution \(u\) is steady, which contradicts the assumption. \(\square\)
Next, we introduce the following lemma to show the invariance of $W$ under critical initial energy.

**Lemma 5.2.** Suppose that $E(0) = d$ and $(u_0, u_1)$ in $H \times H^1_0(\Omega)$, then $u$ is in $W$ provided $u_0$ is in $W$.

**Proof.** Assume that (1)-(3) admits a weak unsteady solution $u$. We use the contradiction method to reach $I(u) > 0$ for any $t \in (0, T)$. Suppose that $t_0$ in $(0, T)$ leads to $I(u(t_0)) = 0$, also $t$ in $[0, t_0)$ induces $I(u(t)) < 0$, here $t_0$ is the first such time. Then $J(u(t_0)) \geq d$. From $E(0) = d$, (48) and (46), we obtain

$$J(u(t_0)) + \frac{1}{2} \|u_t(t_0)\|^2 + \int_0^{t_0} \left(\|u_\tau\|_{r+1} + \|\nabla u_\tau\|^2\right) d\tau = d,$$

hence

$$\frac{1}{2} \|u_t(t_0)\|^2 + \int_0^{t_0} \left(\|u_\tau\|_{r+1} + \|\nabla u_\tau\|^2\right) d\tau = 0,$$

that is $\frac{dW}{dt} = 0$. Hence, $u = u_0$ for any $(x, t) \in \Omega \times [0, t_0]$, also $I(u(t_0)) = I(u_0) > 0$, which is a contradiction. \hfill \Box

In this section, the existence of the global solution is proved for critical initial energy.

**Theorem 5.3.** Under the assumptions of Lemma 5.2, one gets a global weak solution $u$ of (1)-(3). Further, there exist two constants $K > 0$ and $k > 0$ satisfying

$$E(t) < Ke^{-kt}.$$  

**Proof.** If $u$ is steady, then the maximum existence time is infinite. If $u$ is not a steady solution, we know that $E(\tilde{t}) < E(0) = d$ by Lemma 4.1, $u(\tilde{t}) \in W$ due to Lemma 5.2, and $I(u(\tilde{t})) > 0$ for some $\tilde{t} \in (0, T)$ satisfying (79) by Lemma 5.1. For any $t \geq 0$, obviously $v(t) := u(t + \tilde{t})$ is a solution of (1)-(3). Hence, we have the stability of the maximum existence time of $v(t)$ in view of Theorem 4.5, i.e., $T = +\infty$. \hfill \Box

**Lemma 5.4.** Let $u_0(x) \in H$, $u_1(x) \in H^1_0(\Omega)$, $E(0) = d$ and $(u_0(x), u_1(x)) \geq 0$. We have the invariance of $V$ as long as $u_0$ in $V$.

**Proof.** The proof is similar to that of Lemma 5.2. \hfill \Box

**Theorem 5.5.** Under the assumptions of Lemma 5.4, the solution to (63)-(65) blows up in infinite time.

**Proof.** Theorem 4.8, Lemma 5.1 and Lemma 5.4 can directly give the conclusion. \hfill \Box

6. **Blow up in infinite time of solution** for $E(0) > 0$. In this section, in the absence of strongly damping and for $r = 1$ in (1), we show a infinite time blowup result of (1)-(3) under arbitrary positive initial energy.

**Lemma 6.1.** For $\beta \geq 0$ and $T > 0$, assume that $f(t)$ is the Lipschitz continuous function and $f(0) \geq 0$. If $f'(t) + \beta f(t) > 0$, then $f(t) > 0$ for any $t \in (0, T)$.

**Proof.** Define $V(t) := e^{\beta t} f(t)$, then

$$\frac{dV(t)}{dt} = \beta e^{\beta t} f(t) + e^{\beta t} f'(t) = e^{\beta t} \left(f'(t) + \beta f(t)\right) > 0,$$

that is $e^{\beta t} f(t) > e^{0} f(0) \geq 0$. By the fact $e^{\beta t} > 0$, we derive that $f(t) > 0$. \hfill \Box
Lemma 6.2. If $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $(u_0, u_1) \geq 0$ and

$$\|u_0\|^2 > \frac{4}{pC} E(0),$$

(80)

then $I(u) < 0$ as long as $I(u_0) < 0$, where $C$ is a constant provided by $\|\nabla u\|^2 \geq C\|u\|^2$. Further, $\{t \to \|u\|^2\}$ is strictly monotonically increasing.

Proof. We use the contradiction method to reach this conclusion. Assuming that $I(u(t_0)) = 0$ for the first time $t_0$ ($0 < t_0 < T$) and in the interval $[0, t_0)$ there holds $I(u(t)) < 0$. Taking $\eta = u$ in (8), setting $r = 1$ and ignoring the strong damping, we obtain

$$\frac{d}{dt} \left( (u, u_t) + \frac{1}{2}\|u\|^2 \right) = \|u_t\|^2 - I(u),$$

that is

$$\frac{d^2}{dt^2}\|u\|^2 + \frac{d}{dt}\|u\|^2 = 2\|u_t\|^2 - 2I(u).$$

By $I(u(t)) < 0$ for $t \in [0, t_0)$, we get

$$H''(t) + H'(t) > 0,$$

where $H(t) := \|u\|^2$. From Lemma 6.1 and $H'(0) = (u_0, u_1) \geq 0$, we have $H'(t) > 0$, i.e. $\{t \to \|u\|^2\}$ is strictly increasing for $t \in [0, t_0)$, then $\|u\|^2 \geq \|u_0\|^2 > \frac{1}{pC} E(0)$, also

$$\|u(t_0)\|^2 > \frac{4}{pC} E(0).$$

(81)

By (48), (54), $I(u(t_0)) = 0$ and the Poincaré inequality, we obtain

$$E(0) \geq \frac{1}{2}\|u_t(t_0)\|^2 + \frac{p}{4}\|\nabla u(t_0)\|^2 \geq \frac{pC}{4}\|u(t_0)\|^2,$$

(82)

which contradicts (81).

\[]

Theorem 6.3. Under the assumptions of Lemma 6.2, the solution of (1)-(3) blows up in infinite time.

Proof. Firstly, the existence of local solution $u \in C([0, T]; H)$ is proved in Theorem 3.4. Next we prove that the solution $u(t)$ blows up at $\infty$ for the condition (80). For any $0 < T_0 < T$, define

$$M(t) := (T_0 - t)\|u_0\|^2 + \|u\|^2 + \int_0^t \|u\|^2 d\tau,$$

(83)

then $M(t) > 0$ for any $t \in [0, T_0]$. Similar to the calculation of (71)-(74), one has

$$M''(t)M(t) - (M'(t))^2 \geq 2M(t) \left( \|u_t\|^2 - I(u) - 2\int_0^t \|u_\tau\|^2 d\tau \right).$$

(84)

Let

$$\xi(t) = -2\int_0^t \|u_\tau\|^2 d\tau + \|u_t\|^2 - I(u).$$
The Poincaré inequality, (48) and (54) give
\[ \xi(t) \geq -2E(t) - 2 \int_0^t \|u_\tau\|^2 d\tau + \frac{p}{2} \|\nabla u\|^2 \]
\[ = -2E(0) + \frac{p}{2} \|\nabla u\|^2 \]
\[ \geq -2E(0) + \frac{pC}{2} \|u\|^2. \]
According to Lemma 6.2 and \( \|u_0\|^2 > \frac{4}{pC} E(0) \), we have
\[ \xi(t) > -2E(0) + \frac{pC}{2} \|u_0\|^2 > 0, \]
that is
\[ M''(t)M(t) - (M'(t))^2 > 0 \quad t \in [0, T_0]. \tag{85} \]
It is obvious that \( M'(t_0) \) and \( M(t_0) \) are positive for small enough \( t_0 \). Then similar to (76)-(78), (85) indicates that \( \lim_{t \to \infty} M(t) = \infty. \)

Acknowledgments. The authors gratefully thank the anonymous referees for their insightful comments and suggestions.

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Received August 2021; revised September 2021; early access October 2021.

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