Topological Gauge Theories
from Supersymmetric Quantum Mechanics
on Spaces of Connections

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Abstract

We rederive the recently introduced $N = 2$ topological gauge theories, representing the Euler characteristic of moduli spaces $\mathcal{M}$ of connections, from supersymmetric quantum mechanics on the infinite dimensional spaces $\mathcal{A}/\mathcal{G}$ of gauge orbits. To that end we discuss variants of ordinary supersymmetric quantum mechanics which have meaningful extensions to infinite-dimensional target spaces and introduce supersymmetric quantum mechanics actions modelling the Riemannian geometry of submersions and embeddings, relevant to the projections $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ and inclusions $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$ respectively. We explain the relation between Donaldson theory and the gauge theory of flat connections in 3d and illustrate the general construction by other 2d and 4d examples.

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1 Introduction

From its historical development it is evident that topological field theory is closely related to supersymmetric quantum mechanics with infinite dimensional target spaces. Nevertheless this aspect of topological field theory has attracted little attention in subsequent developments and the purpose of this paper is to fill this gap and to illustrate the usefulness of this perspective.

In particular, we will use supersymmetric quantum mechanics on the space $\mathcal{A}/G$ of gauge orbits of connections to give a rather pedestrian derivation of the topological gauge theory models introduced recently in [1] and discussed there from several other points of view (e.g. via the Mathai-Quillen formalism [2, 3] and in terms of topological $N = 2$ superfields). The present derivation is aimed at clarifying the origin of the fundamental property of these theories (obviously reminiscent of supersymmetric quantum mechanics): that the partition function of the action $S_M$ associated to a given moduli space $\mathcal{M}$ of connections equals the Euler characteristic $\chi(\mathcal{M})$ of $\mathcal{M}$,

$$Z(S_M) = \chi(\mathcal{M}) \ .$$

Moreover, as a side-result, we will also see in this paper that the action of Donaldson theory [4], the prototype of a cohomological topological field theory (see [5] for a review), can be identified term by term with the standard action $S_{X,V}$ of supersymmetric quantum mechanics on a manifold $X$ coupled to a potential $V$ for $X = A^3/G^3$ (the space of gauge orbits in three dimensions) and $V = CS$ (the Chern-Simons functional). In view of the considerations in [4] this is almost tautologically true, but we have included this result here because we have never seen it spelled out explicitly (i.e. in terms of the Riemannian metric, connection, and curvature of $A^3/G^3$) and a number of things can be learned from this construction.

The main reason why this way of constructing and looking at topological gauge theories has enjoyed only limited popularity at best, is that Donaldson theory has some features which are not available in general. In particular, the existence of a potential on $A^3/G^3$, the Chern-Simons functional, which leads to a description of moduli spaces $\mathcal{M}_I \subset A^4/G^4$ of instantons in $4d$ (via its gradient flow) is a fortuitous coincidence which reflects the richness of Donaldson theory. The fact that, used in this way, supersymmetric quantum
mechanics on $\mathcal{A}^n / \mathcal{G}^n$ leads to a theory in $n + 1$ dimensions, in general prevents one from applying this method to the construction of topological gauge theories based on moduli spaces $\mathcal{M} \subset \mathcal{A}^{n+1} / \mathcal{G}^{n+1}$ as no suitable potential function on $\mathcal{A}^n / \mathcal{G}^n$, nor any other obvious means of exerting control over $\mathcal{M}$ from an $n$-dimensional point of view, will exist.

In this letter we show that the $n$-dimensional topological gauge theory actions $S_\mathcal{M}$ associated with a given moduli space $\mathcal{M} \subset \mathcal{A}^n / \mathcal{G}^n$ can be constructed directly from supersymmetric quantum mechanics on $\mathcal{A}^n / \mathcal{G}^n$, the intermediate theory on $\mathcal{A}^{n+1} / \mathcal{G}^{n+1}$ playing only an auxiliary role. Not only is this method manifestly covariant (in contrast with the above construction which leads to a $(3 + 1)$-dimensional description of Donaldson theory), but also (and more importantly) it frees us from the necessity of having to find an $n$-dimensional description of a moduli space of connections in $n + 1$ dimensions. This also completes the circle of ideas relating Floer cohomology, the Chern-Simons functional, instanton moduli spaces, the Casson invariant, and moduli spaces of flat connections from the point of view of [1].

Our construction will be based on the standard localization and index theory arguments of supersymmetric quantum mechanics. In particular, we will make use of the fact that the theory $S_{X,V}$ localizes to the critical point set $X_V$ of $V$ to derive a topological gauge theory of flat connections in three dimensions (the critical points of $CS$) from Donaldson theory. In order to describe moduli spaces of connections which are not of the form $X_V$ for some $V$ we introduce a new variant of supersymmetric quantum mechanics in finite dimensions which localizes onto an arbitrary given submanifold $Y \subset X$ and describes the Riemannian geometry of embeddings via the Gauss-Codazzi equations. The corresponding action $S_Y \subset X$ also makes sense when applied to $X = \mathcal{A}^n / \mathcal{G}^n$ and any finite-dimensional moduli subspace $Y = \mathcal{M}$ of $\mathcal{A}^n / \mathcal{G}^n$.

In section 2 we review the pertinent features of supersymmetric quantum mechanics (path integral representation of the Euler characteristic, evaluation of the partition function of $S_{X,V}$, localization) and introduce the Gauss-Codazzi quantum mechanics action $S_{Y \subset X}$. In section 3 we describe the Riemannian geometry of (moduli) spaces of connections and its field-theoretic realization. We also establish the relation between Donaldson theory and supersymmetric quantum mechanics on $\mathcal{A}^3 / \mathcal{G}^3$, and construct the topological gauge theory of flat connections in three dimensions. In section 4 we
illustrate the use of $S_{Y \subset X}$ by treating moduli spaces of instantons and of flat connections in two dimensions.

## 2 Review of supersymmetric quantum mechanics

The fundamental action of supersymmetric quantum mechanics modelling the de Rham complex of a Riemannian manifold $(X, g)$, is (we are using the conventions of [5])

$$S_X = \int_0^\beta dt \left[ i\dot{x}^\mu B_\mu + \frac{1}{2} g^{\mu\nu} B_\mu B_\nu + \frac{1}{4} R^{\mu\nu}_{\rho\sigma} \bar{\psi}_\mu \psi^\rho \bar{\psi}_\nu \psi^\sigma - i\bar{\psi}_\mu \nabla_t \psi^\mu \right]. \quad (2)$$

Here $x^\mu$ are the coordinates of the Riemannian manifold $(X, g)$ with curvature tensor $R_X = (R^{\mu\nu}_{\rho\sigma})$, $\psi^\mu$ and $\bar{\psi}_\mu$ are Grassmann odd coordinates, and the covariant derivative $\nabla_t$ is the pull-back of the covariant derivative on $X$ to the one dimensional space with Euclidean time coordinate $t$. Upon integrating out the auxiliary field $B$, one recovers the action of [7, 8, 9, 10] with the spinors appearing there decomposed into their components. We choose $\bar{\psi}$ and $\psi$ to be independent real fields (instead of complex conjugates). The supersymmetry of the action (2) is

$$\delta x^\mu = \psi^\mu, \quad \delta \bar{\psi}_\mu = B_\mu - \Gamma^\nu_{\mu\rho} \bar{\psi}_\nu \psi^\rho,$$

$$\delta \psi^\mu = 0, \quad \delta B_\mu = \Gamma^\nu_{\mu\rho} B_\nu \psi^\rho - \frac{1}{2} R^{\nu\rho}_{\mu\sigma} \bar{\psi}_\nu \psi^\rho \psi^\sigma. \quad (3)$$

$S_X$ can be written as the supersymmetry variation of $\int_0^\beta dt \bar{\psi}_\mu (i\dot{x}^\mu + \frac{1}{2} g^{\mu\nu} B_\nu)$ and this has far-reaching consequences. In particular, reinterpreting $\delta$ as a BRST operator, this demonstrates that the ground state reduction of supersymmetric quantum mechanics is topological and that the theory is independent of the coefficient of $B^2$ (cf. [1], pp. 140-176) for a detailed discussion of supersymmetric quantum mechanics in the context of topological field theories).

As is well known, the partition function $Z(S_X)$ of (2) with periodic boundary conditions on all the fields is the Euler number $\chi(X)$ of $X$. The way to see this is to start with the definition of $\chi(X)$ as the Euler characteristic of the de
The Rham complex of \( X \), \( \chi(X) = \sum_k (-1)^k b_k(X) \) (where \( b_k(X) = \dim H^k(X, \mathbb{R}) \) is the \( k \)’th Betti number of \( X \)) and to rewrite this as the Witten index \( \chi(X) = \text{tr}(-1)^F \exp(-\beta H) \) of the Laplace operator \( H \equiv \Delta = d d^* + d^* d \) on differential forms. One then uses the Feynman-Kac formula to represent this as a supersymmetric path integral with the action (2) and periodic boundary conditions on the anticommuting variables \( \psi^\mu \) (due to the insertion of \( (-1)^F \)).

The partition function \( Z(S_X) \) can be evaluated explicitly to give a path integral proof of the Gauss-Bonnet theorem which expresses \( \chi(X) \) as an integral over \( X \) of the Pfaffian of the curvature \( R^X \),

\[
Z(S_X) = \chi(X) = \int_X \text{Pf}(R_X) .
\] (4)

The crucial fact responsible for the reduction of the integral over the loop space \( LX \) of \( X \) (the path integral) to an integral over \( X \) (the Gauss-Bonnet integral) is the \( \beta \)-independence of the Witten index. This permits one to evaluate the partition function in the limit where the radius \( \beta \) of the circle tends to zero. In this limit it can be seen that only the Fourier zero modes (e.g. \( \dot{x} = 0 \)) of the fields are relevant, the contributions from the other modes cancelling identically between the bosonic and fermionic fields. All this is, of course, also an immediate consequence of the BRST symmetry and topological nature of supersymmetric quantum mechanics mentioned above. It is the analogue in infinite dimensions of this observation that allows us to construct topological gauge theories in \( n \) (instead of \( n + 1 \)) dimensions from supersymmetric quantum mechanics on \( \mathcal{A}^n/G^n \).

As they stand, the partition function of (2) and the right hand side of (4) do not make sense for infinite dimensional target spaces. There are, however, two generalizations of (4) which turn out to have meaningful counterparts on \( \mathcal{A}/\mathcal{G} \). The first of these involves a choice of potential \( V(x) \) on \( X \). The corresponding action

\[
S_{X,V} = \int dt [i(\dot{x}^\mu + s g^{\mu\nu} \partial_\nu V(x)) B_\mu + \frac{1}{2} g^{\mu\nu} B_\mu B_\nu + \frac{1}{4} R^\mu_{\rho\sigma} \bar{\psi}_\mu \psi^\rho \bar{\psi}_\nu \psi^\sigma \\
- i \bar{\psi}_\mu (\delta^\mu_t \nabla_t + s g^{\mu\nu} \nabla_\nu \partial_\nu V) \psi^\nu] .
\] (5)

(\( s \) is a parameter) arises by replacing the exterior derivative \( d \) by \( d_{sV} \equiv \exp(-sV) d \exp(sV) \). As there is a one-to-one correspondence between \( d^- \) and
$d_s V$-harmonic forms this also represents $\chi(X)$ (independently of $s$). In this case the additional freedom in the choice of $s$ allows one to reduce $Z(S_{X,V})$ to an integral over the set of critical points of $V$ in the limit $s \to \infty$ (alternatively one uses the fact that the partition function is also independent of the coefficient of $B^2$ and observes that $\dot{x}^\mu + sg^{\mu\nu}\partial_\nu V(x) = 0$ implies $\dot{x}^\mu = \partial_\nu V(x) = 0$ for any $s \neq 0$ by squaring and integrating). In the case that the critical points of $V$ are isolated and non-degenerate one arrives at the classical Poincaré-Hopf-Morse theorem

$$\chi(X) = \sum_{x_k : dV(x_k) = 0} (\pm 1) \quad (6)$$

which calculates $\chi(X)$ as the signed sum of critical points of $V$. More generally (6) holds, and can be derived from supersymmetric quantum mechanics for the sum over the zeros of a generic vectorfield on $X$. If the critical points are not isolated then, by a combination of the arguments leading to (4) and (6), one finds

$$\chi(X) = \sum_{(k)} \chi(X_V^{(k)}) \quad , \quad (7)$$

where the $X_V^{(k)}$ are the connected components of the critical point set of $V$. The relevance of this for our purposes is that the right hand side of (7) may be well defined, even if $X$ is infinite dimensional, provided that $X_V$ is finite dimensional. In that case $\chi(X_V)$ is well defined and can be regarded as a regularized Euler number of $X$ (this is the point of view adopted in [3]). The advantage of our construction is that it permits an a priori identification of this $V$-dependent regularized Euler number of $X$ with the Euler number of $X_V$.

Although this looks like a satisfactory state of affairs, we may not always be so fortunate to have a potential at our disposal whose critical points define precisely the (moduli) subspace $Y \subset X$ we are interested in. In fact, it follows from (7) that in finite dimensions $\chi(Y) = \chi(X)$ is a necessary condition for this to be possible. Moreover, it is by now well known that even on a compact four-manifold there are critical points of the vacuum Yang-Mills functional other than instantons. Thus a suitable potential is unlikely to exist e.g. for the instanton moduli spaces $M_I \subset A^4/G^4$. We thus require a generalization of (2) which calculates the Euler number $\chi(Y)$ for any submanifold $Y \subset X$ regardless of whether $Y = X_V$ for some $V$ or not.
In that setting we have the classical Gauss-Codazzi equations which relate the intrinsic curvature $R_Y$ of $Y$ (with the induced metric) to $R_X$ restricted to $Y$ and the extrinsic curvature (second fundamental form) of the embedding $i : Y \hookrightarrow X$. The second fundamental form $K_Y$ of $(Y, i)$ is defined by $K_Y(v, w) = (\nabla_{i_*v}i_*w)^\perp$, where $v, w \in TY$, $\nabla$ is the Levi-Civita connection on $X$, and $(.)^\perp$ denotes projection onto the normal bundle to $TY$ in $TX|_Y$. If $Y$ is a hypersurface in $X$, this reduces to the more mundane statement that the extrinsic curvature is essentially the normal derivative of the induced metric. The Gauss equation now states that

$$\langle R_Y(u, v)z, w \rangle = \langle R_X(u, v)z, w \rangle + (\langle K_Y(v, z), K_Y(u, w) \rangle - (u \leftrightarrow v)) \quad (8)$$

Our construction of an action $S_{Y \subset X}$ calculating $\chi(Y)$ via the Gauss-Bonnet theorem applied to (8) will be modelled on (8) itself. Essentially, it will consist of the action $S_X$ plus a Lagrange multiplier term enforcing the restriction to $Y \subset X$. Provided that this restriction is performed in a way consistent with the supersymmetries of de Rham supersymmetric quantum mechanics this will automatically give rise to the second term of (8).

More concretely, assume that $Y \subset X$ is (locally) given by

$$Y = \{ x \in X : F^a(x) = 0, \ a = 1, \ldots, \text{dim}(X) - \text{dim}(Y) \}$$

(the relation between the formulae arising from this implicit description and that in terms of an explicit embedding $y^k(x^\mu)$ is explained e.g. in [11]). We then group the fields appearing in (2) into a topological $N = 2$ superfield

$$X^\mu(t, \theta, \bar{\theta}) = x^\mu(t) + \theta \psi^\mu(t) + \bar{\theta} g^{\mu\nu} \bar{\psi}_\nu(t) - \theta \bar{\theta} (g^{\mu\nu} B_\nu(t) + g^{\beta\nu} \Gamma^\mu_{\beta\lambda} \bar{\psi}_\nu \psi^\lambda) \quad (9)$$

($\theta$ and $\bar{\theta}$ are Grassmann odd scalars). This choice of superfields is designed to reproduce the supersymmetry transformations (3). As we will see below, the second term of the $\theta \bar{\theta}$-component moreover leads to a Taylor expansion of superfields in terms of covariant derivatives so that superspace actions are manifestly covariant. We also introduce $N = 2$ Lagrange multiplier fields

$$\Lambda_a(t, \theta, \bar{\theta}) = \lambda_a(t) + \theta \sigma_a(t) + \bar{\theta} \bar{\sigma}_a(t) + \theta \bar{\theta} b_a(t) \quad , \quad (10)$$

and choose the action to be

$$S_{Y \subset X} = S_X + \alpha \int dt \int d\theta d\bar{\theta} \Lambda_a(t, \theta, \bar{\theta}) F^a(X(t, \theta, \bar{\theta})) \quad , \quad (11)$$
so that the integration over the $\Lambda_a$ imposes the superconstraints $F^a(X) = 0$. The argument given above leading to the elimination of the non-constant modes is not affected by the addition of this term and thus, upon Taylor expanding $F^a(X)$, (11) becomes (all ‘fields’ are now time independent)

$$S_{Y \subset X} = \beta\left[ \frac{1}{2} g^{\mu\nu} B_\mu B_\nu + \frac{1}{4} R^{\rho\sigma}_\mu \bar{\psi}_\mu \psi^\rho \bar{\psi}_\nu \psi^\sigma \right] + \alpha \left[ b_a F^a - \sigma_a \bar{\psi}^\mu \partial_\mu F^a + \bar{\sigma}_a \psi^\mu \partial_\mu F^a + \lambda_a (B^a \partial_\mu F^a - g^{\beta\sigma} \nabla_\beta \partial_\lambda F^a \bar{\psi}_\nu \psi^\lambda) \right].$$

We see that the integral over $b$ restricts the bosonic coordinates to $Y$ while the integrals over $\sigma$ and $\bar{\sigma}$ constrain the fields $\psi^\mu$ and $\bar{\psi}_\mu$ to be tangent to $Y$. It is now a simple matter to perform the Gaussian integrals over the remaining auxiliary fields $B_\mu$ and $\lambda_a$ with the result

$$S_{Y \subset X} = \left( \frac{1}{4} R^{\mu\nu}_\rho_\sigma + \frac{1}{2} g^{\mu\nu} g^{\nu\lambda} \nabla_\alpha \partial_\mu F^a (F^{-1})_{ab} \nabla_\beta \partial_\sigma F^b \right) \bar{\psi}_\mu \psi^\mu \bar{\psi}_\nu \psi^\sigma.$$

Here $F^{ab}$ is the matrix $F^{ab} = \partial_\mu F^a \partial_\nu F^b g^{\mu\nu}$ and the description of $Y \subset X$ in terms of the $F^a$ is valid at points where $\det(F^{ab}) \neq 0$ so that $F^{ab}$ is indeed invertible there. We see that $\alpha$ has dropped out (as it should) and we have rescaled the $\bar{\psi}$’s by $\beta^{1/2}$ to eliminate all $\beta$-dependence from both the measure and the action. Equation (13) is precisely the Gauss equation (8) which we have thus derived from supersymmetric quantum mechanics. Therefore, upon expanding the path integral to soak up the $\text{dim}(Y)$ fermionic $\psi$ and $\bar{\psi}$ zero modes, we will indeed find

$$Z(S_{Y \subset X}) = \chi(Y),$$

now valid for arbitrary submanifolds $Y \subset X$ (not necessarily of the form $X_V$). This is the generalization we need to be able to apply supersymmetric quantum mechanics to spaces of connections. We also see that, in a certain sense, the action $S_{X,V}$ (3) is a special case of the action $S_{Y \subset X}$ (11), the zero mode of $B$ playing the role of the multiplier $b$.

Finally we mention that one can also construct supersymmetric quantum mechanics actions $S_{Z \rightarrow X}$ for Riemannian submersions $Z \rightarrow X$ instead of embeddings, deriving the O’Neill equations (12) in this case instead of the Gauss-Codazzi equations. This is most effortlessly done when the submersion is actually a fibration. Instead of developing the full machinery here, we will illustrate this in passing in the following section.
3 Donaldson theory and flat connections in 3d

We will now introduce the data entering into the construction of the supersymmetric quantum mechanics actions $S_{X,V}$ and $S_{Y \subset X}$ on spaces of connections (see e.g. [13, 14, 15]). Let $(N, g)$ be a compact, oriented, Riemannian $n$-manifold, $P \to N$ a principal $G$ bundle over $N$, $G$ a compact semisimple Lie group and $g$ its Lie algebra. We denote by $A$ the space of (irreducible) connections on $P$, by $G$ the infinite dimensional gauge group of vertical automorphisms of $P$ (modulo the center of $G$), by $\Omega^k(N, g)$ the space of $k$-forms on $N$ with values in the adjoint bundle $\text{ad}P := P \times \text{ad} g$ and by $d$ the covariant exterior derivative. The spaces $\Omega^k(N, g)$ have natural scalar products defined by the metric $g$ on $N$ (and the corresponding Hodge operator $*$) and an invariant scalar product $\text{tr}$ on $g$, namely

$$\langle X, Y \rangle = \int_M \text{tr}(X * Y) , \quad X, Y \in \Omega^k(N, g) .$$

(15)

The tangent space $T_A A$ to $A$ at a connection $A$ can be identified with $\Omega^1(N, g)$ and (15) thus defines a metric $g_A$ on $A$. At each point $A \in A$, $T_A A$ can be split into a vertical part $V_A = \text{Im}(d_A)$ (tangent to the orbit of $G$ through $A$) and a horizontal part $H_A = \text{Ker}(d^*_A)$ (the orthogonal complement of $V_A$ with respect to the scalar product (15)). Explicitly this decomposition of $X \in \Omega^1(N, g)$ into its vertical and horizontal parts is

$$X = d_A G^0_A d^*_A X + (X - d_A G^0_A d^*_A X) , \equiv v_A X + h_A X ,$$

(16)

where $G^0_A = (d^*_A d_A)^{-1}$ is the Greens function of the scalar Laplacian (which exists if $A$ is irreducible). We will identify the tangent space $T_{[A]} A/G$ with $H_A$ for some representative $A$ of the gauge equivalence class $[A]$. Then $g_A$ induces a metric $g_{A/G}$ on $A/G$ via

$$g_{A/G}([X], [Y]) = g_A(h_A X, h_A Y) ,$$

(17)

where $X, Y \in \Omega^1(N, g)$ project to $[X], [Y] \in T_{[A]} A/G$. With the same notation the Riemannian curvature of $A/G$ is

$$\langle R_{A/G}([X], [Y])[Z], [W] \rangle = \langle *[h_A X, *h_A W], G^0_A * [h_A Y, *h_A Z] \rangle - (X \leftrightarrow Y) + 2 \langle *[h_A W, *h_A Z], G^0_A * [h_A X, *h_A Y] \rangle .$$

(18)
The last ingredient we would need to be able to write down the action (2) or (3) is the Christoffel symbols of $g_{A/G}$, or, rather, particular components thereof. We will sketch the required calculation below. Equipped with all this we can now exhibit the relation between Donaldson theory and supersymmetric quantum mechanics on $A^3/G^3$. 

The action of Donaldson theory on a four-manifold $N$ in equivariant form (i.e. prior to the introduction of gauge ghosts) is

$$S = \int_N \left( B_+ F_A + \chi_+ d_A \psi - B_+^2 / 2 + \eta d_A \ast \psi \right)$$

Here $F_A = dA + \frac{1}{2}[A, A]$ is the curvature of the connection $A$, $\psi \in \Omega^1(N, g)$, the superpartner of $A$, is a Grassmann odd Lie algebra valued one-form with ghost number 1, $(B_+, \chi_+)$ are self-dual two-forms with ghost numbers $(0, -1)$ (Grassmann parity (even,odd)), and $(\phi, \bar{\phi}, \eta)$ are elements of $\Omega^0(N, g)$ with ghost numbers $(2, -2, -1)$ and parity (even,even,odd). The equivariantly nilpotent BRST-symmetry of (19) is

$$\begin{align*}
\delta A &= \psi \quad \delta \psi = -d_A \phi \\
\delta \chi_+ &= B_+ \quad \delta B_+ = [\phi, \chi_+] \\
\delta \bar{\phi} &= \eta \quad \delta \eta = [\phi, \bar{\phi}] \\
\delta \phi &= 0 \quad \delta^2 = \delta_\phi
\end{align*}$$

where $\delta_\phi$ denotes a gauge variation with respect to $\phi$. The action (19) is far from being unique. In particular, by standard arguments of topological field theory many $\delta$-exact terms can be added to the action without changing the partition function or correlation functions (the Donaldson invariants in this case). We will make use of this freedom below. For many of the other things that can and should be said about (19,20) we refer to [4] and [5, pp. 199-235].

If $N$ is of the form $N = M \times S^1$ (where we think of $S^1$ as the ‘time’ direction) we can perform a $(3+1)$-decomposition of the action. Identifying the self-dual two-forms $B_+$ and $\chi_+$, with (time-dependent) elements $B$ and $\psi$ of $\Omega^1(M, g)$, reserving henceforth the notation $A$ for the spatial part of the connection, and renaming $A_0 \to u$ and $\psi_0 \to \bar{\eta}$ we find that (19) takes the form

$$S = \int_M \int dt \left( B \ast (A - d_A u - \ast F_A) - B \ast B / 2 + \bar{\psi} d_A \psi + \psi \ast \bar{\psi} + \bar{\phi} d_A \ast d_A \phi \right)$$
\[ + \left( u[\psi, \ast \bar{\psi}] + \eta d_A \ast \psi + \bar{\eta} d_A \ast \bar{\psi} + \bar{\phi}[\psi, \ast \psi] - \phi[\bar{\psi}, \ast \bar{\psi}]/2 \right) \]  

(21)

In going from (19) to (21) we have, for later convenience, also subtracted the BRST exact term \((D_0 \equiv \partial_0 + [u, ])\)

\[ \delta(\bar{\phi}D_0\bar{\eta}) = \eta D_0\bar{\eta} + \bar{\phi}[\bar{\eta}, \bar{\eta}] - \bar{\phi}D_0D_0\phi \] .

We now perform the following elementary manipulations (Gaussian integrals):

- Integration over \(\eta\) and \(\bar{\eta}\) forces \(\psi\) and \(\bar{\psi}\) to be horizontal, \(h_A\psi = \psi\), \(h_A\bar{\psi} = \bar{\psi}\), i.e. to represent tangent vectors to \(A/G\)

- Integration over \(\bar{\phi}\) yields \(\phi = -G_A^0 \ast [\psi, \ast \psi]\), giving rise to a term

\[ \frac{1}{2} \langle \ast[\bar{\psi}, \ast \bar{\psi}], G_A^0 \ast [\psi, \ast \psi] \rangle \]

in the action

- The equation of motion for \(u\) reads

\[ u = G_A^0(d_A^\ast \dot{A} + \ast[\psi, \ast \bar{\psi}]) \]

and plugging this back into (21) one obtains

\[ \frac{1}{2} \langle h_A(\dot{A} - \ast F_A), h_A(\dot{A} - \ast F_A) \rangle + \frac{1}{2} \langle \ast[\psi, \ast \bar{\psi}], G_A^0 \ast[\psi, \ast \bar{\psi}] \rangle + \langle \ast[\psi, \ast \bar{\psi}], G_A^0 d_A^\ast \dot{A} \rangle \]

Putting all this together we see that the combination of Greens functions appearing is precisely that entering the equation (18) for the curvature tensor \(R_{A/G}\) while the kinetic term for the gauge fields is exactly \(g_{A/G}([\dot{A} - \ast F_A], [\dot{A} - \ast F_A])/2\) (eq. 17). Recalling that \(\ast F_A\) is the (automatically horizontal) gradient vector field of the Chern-Simons functional \(CS(A)\) and \(\ast d_A\) its second derivative, we find perfect term by term agreement with the action \(S_{X,V} = S_{A/G,CS}\) provided that \(\langle \ast[\psi, \ast \bar{\psi}], G_A^0 d_A^\ast \dot{A} \rangle\) correctly reproduces the affine terms appearing in the second line of (5). That this is indeed the case can be seen by noting that the variation of the metric \(g_{A/G}\) with respect to \(A\) arises solely from the variation of the projectors \(h_A\). As \(\psi\) and \(\bar{\psi}\) are horizontal the only variation that will therefore contribute is that of the first \(A\) in
the vertical projector $d_A G^0_A d_A$ giving rise to the above term when contracted with $\dot{A}$ as in $\langle \dot{\psi}, \nabla_t \psi \rangle$. By the same argument the affine term in $\nabla \partial V$ does not contribute as $* F_A$, $\psi$ and $\bar{\psi}$ are all horizontal and one of them will be annihilated by a $d_A^*$ appearing in the variation of the metric.

In summary, we have seen thus far that the standard action (19) of Donaldson theory on a four-manifold of the form $M \times S^1$ is precisely the quantum mechanics action (3) on $A^3/G^3$ rewritten, as in [3], in local form with the help of auxiliary fields. Conversely, the action (19), for which several other constructions are also available [5], could have been used to derive the metric and curvature on $A/G$ from those on $A$. It is in this sense that (21), without the (model dependent) terms coming from the potential, provides a realization $S_{A \rightarrow A/G}$ of the Riemannian submersion action $S_{Z \rightarrow X}$ mentioned at the end of section 2. This part of the action is universal, i.e. common to all supersymmetric quantum mechanics actions on $A/G$, and is the counterpart of the universal action of $N = 2$ topological gauge theory describing the Riemannian geometry of $A/G$ and discussed in [1].

The general strategy for the construction of such actions, at least in the case of fibrations, should now also be clear: one modifies the nilpotent BRST symmetry (3) to an equivariantly nilpotent symmetry squaring (as in (20)) to translations along the fibers parametrized by a new (ghost number 2) field $\phi$. Rest as before.

Our next goal is to show how to obtain a topological gauge theory of flat connections in 3d from the supersymmetric quantum mechanics action $S_{A/G, CS}$ in 3d. This is straightforward since, by the arguments of section 2, only the time-independent modes contribute to the partition function of (21) which is thus the same as the partition function of the action $S_{\mathcal{M}}$

$$S_{\mathcal{M}} = \int_M \left( \frac{1}{2} F_A * F_A + \frac{1}{2} d_A u * d_A u - d_A \phi * d_A \phi + \bar{\psi} d_A \psi \right)$$

$$+ \left( u[\psi, * \bar{\psi}] + \eta d_A * \psi + \bar{\eta} d_A * \bar{\psi} + \bar{\phi}[\psi, * \psi] - \phi[\bar{\psi}, * \bar{\psi}] / 2 \right) . \quad (22)$$

This is precisely the action obtained in [1] as a field theoretic realization of the Euler number $\chi(\mathcal{M})$ of the moduli space $\mathcal{M} = \mathcal{M}(M, G)$ of flat connections.

\[^1\text{To arrive at this action, multiply (21) by } 1/\beta \text{ and scale } t \text{ by } \beta \text{ so that the circle has unit-radius. Then } \beta \text{ will appear only in terms with time-derivatives. To eliminate these, scale the non-constant modes of } A \text{ and } \psi \text{ by } \beta. \text{ In the limit } \beta \rightarrow 0 \text{ these modes decouple and one is left with (22).}\]
From the present derivation of this action it is obvious that \( Z(S_M) = \chi(M) \)
while in \([1]\) we verified this by calculating the partition function. To that end we integrate over \( \phi, \bar{\phi}, u, \eta, \) and \( \bar{\eta} \) as above to obtain \( R_{A/G} \). To evaluate the integral over the remaining fields \( A, \psi, \) and \( \bar{\psi} \) we expand them about their classical configurations. By standard arguments we may restrict ourselves to a one-loop approximation and to this order the remaining terms in the action become

\[
\int_M \left( \frac{1}{2} F_A \ast F_A + \bar{\psi} d_A \psi \right) \rightarrow \int_M \left( \frac{1}{2} d_{A_c} A_q \ast d_{A_c} A_q + [\bar{\psi}_c, \psi_c] A_q \right) .
\] (23)

where we can choose \( F_{A_c} = 0 \) as the coefficient of \( B^2 \) in \([21]\) is arbitrary. Integration over \( A_q \) yields

\[
\frac{1}{2} \langle [\bar{\psi}_c, \psi_c] , G_{A_c}^1 \ast [\bar{\psi}_c, \psi_c] \rangle ,
\] (24)

where \( G_{A_c}^1 \) is the Greens function of the Laplacian \( d_{A_c} d_{A_c}^* + d_{A_c}^* d_{A_c} \) on one-forms, composed with a projector onto the orthogonal complement of the space of \( d_{A_c} \)-harmonic one-forms. Now the extrinsic curvature of \( M \subset A^3/G^3 \) is

\[
K_M([X], [Y]) = -d_A^* G_{A_c}^1 \ast [\bar{X}, \bar{Y}]
\]
(see \([16, 1]\)) with \( \bar{X} \) and \( \bar{Y} \) satisfying the linearized flatness and horizontality equations \( d_A \bar{X} = d_A^* \bar{X} = d_A \bar{Y} = d_A^* \bar{Y} = 0 \). Therefore the above term \((24)\) is precisely the \( K^2_M \) contribution to the Gauss equation \((8)\) for \( R_M \) and the partition function of \( S_M \) is indeed the Euler characteristic \( \chi(M) \).

We want to draw attention to the double-role played by the multiplier field \( B \): its exact part couples to \( u \) and gives rise to the submersion action \( S_A \rightarrow A/G \) and the O’Neill equations, while its coexact part couples to the gradient \( \ast F_A \) of the potential and is responsible for the \( K \)-part of the Gauss-Codazzi equations.

The reader may be puzzled at this point by the fact that Donaldson theory on \( M \times S^1 \) apparently calculates the Euler number of the moduli space \( \mathcal{M}(M,G) \) of flat connections in three dimensions although it is known (and was invented) to describe moduli spaces of instantons in four dimensions. How this happens is explained in detail in \([1]\). It is essentially due to the fact that the index of the instanton deformation complex (the formal dimension
of the instanton moduli space) on a four-manifold of the form $M \times S^1$ is non-zero in the topologically non-trivial sector so that the partition function vanishes there, while irreducible ‘instantons’ in the trivial sector correspond to flat connections in 3d.

We end this section with the remark that, by a result of Taubes [17], the partition function of (22) formally equals the Casson invariant of $M$ if $M$ is a homology three-sphere [18]. This, combined with the above considerations, has led us to propose $\chi(M)$ as a candidate for the definition of the Casson invariant of more general three-manifolds (see [1] for some preliminary considerations).

## 4 Other examples

We will now briefly discuss the corresponding constructions for those moduli spaces $M$ which are not of the form $X_V$ for some potential $V$ on $X = A/G$. In that case we will, as discussed in section 2, use the Gauss-Codazzi supersymmetric quantum mechanics action $S_{M \subset A/G}$ which will be the sum of $S_{A/G}$ (or its local counterpart $S_{A \to A/G}$), and the $N = 2$ supersymmetric delta function constraints onto $M \subset A/G$. Alternatively, to construct e.g. the action associated to the moduli space $M^2$ of flat connections in two dimensions we can simply dimensionally reduce the action (21) assuming that $M = \Sigma \times S^1$. This makes the double role played by $B$ particularly transparent: the action $S_{A^3/G^3,CS}$ reduces to the Gauss-Codazzi action $S_{M_2 \subset A^2/G^2}$ via

$$
S_{A^3/G^3,CS} = \int B \ast (\dot{A} - d_Au - *F_A) + \ldots
$$

$$
\rightarrow S_{M_2 \subset A^2/G^2} = \int B \ast (\dot{A} - d_Au) + bF_A + \ldots
$$

(25)

(b is the scalar (time) component of $B$). Now $B$ evidently represents the submersion (O’Neill) part of the action, while $b$ represents the embedding (Gauss-Codazzi) part. Proceeding as above to extract the two-dimensional (zero mode) action from (25) one finds

$$
S_{M_2} = \int_{\Sigma} \left( bF_A + \frac{1}{2}d_Au \ast d_Au + u[\psi, *\bar{\psi}] - d_A\vec{\phi} \ast d_A\phi + \chi d_A\psi - \chi d_A\bar{\psi} + \frac{1}{2}B \ast B \right) + \left( \eta d_A \ast \psi + \bar{\eta} d_A \ast \bar{\psi} + \bar{\phi}[\psi, *\bar{\psi}] - \phi[\bar{\psi}, *\psi] / 2 + u(d_AB + [\psi, \bar{\psi}]) \right) .
$$

(26)
Once again, this is precisely the action derived in \[1\] satisfying $Z(S_{M_2}) = \chi(M_2)$. This can, of course, also be established by direct calculation. This calculation is somewhat simpler here than in three dimensions as we have a delta function constraint instead of a Gaussian around $M_2$ so that the partition function can be calculated directly without performing a classical-quantum split. In particular, it is now the integration over $u$ and $B$ that will produce the extrinsic curvature contribution which is (cf. (24)) $\frac{1}{2}(\ast [\bar{\psi}, \psi], G^0_A \ast [\bar{\psi}, \psi])$, in agreement with the calculations in \[1\].

We also want to mention that the non-degeneracy condition $\det(\mathcal{F}^{ab}) \neq 0$ we encountered in our discussion of Gauss-Codazzi supersymmetric quantum mechanics in section 2 is just the condition that the Laplacian on zero-forms be invertible, i.e. that the connection be irreducible, as we have assumed all along. It is, of course, only at those points that the condition $F_A = 0$ gives a non-singular description of $M_2$. It appears likely that a non-singular description of the reducible points (automatically gauge fixing the residual symmetry there) can be obtained by adding a term $\gamma A^2$ \[1\] to the action $S_{M \subset \mathcal{A}/G}$ \[1\] and carefully taking the limit $\gamma \to 0$. This, as well as the other suggestions for dealing with reducible connections put forward in \[1\], is currently under investigation.

A related issue is the question, what the path integral calculates if the target spaces $X, Y \subset X$ or $X_V \subset X$ (and, in particular, the moduli spaces $\mathcal{M} \subset \mathcal{A}/G$) are not smooth manifolds but perhaps orbifolds or orbifold stratifications. In the case of orbifolds one expects to obtain the virtual Euler characteristic via Satake’s Gauss-Bonnet theorem for $V$-manifolds \[1\]. On the other hand, the equivariant orbifold Euler characteristic familiar from string theory appears to arise upon reduction of supersymmetric quantum mechanics on $X = Y \times S^1$ to $Y$ with twisted boundary conditions on both the ‘temporal’ and ‘spatial’ circles.

Finally, in order to obtain a theory modelled on the moduli spaces $\mathcal{M}_I$ of instantons in four dimensions, we can construct the corresponding Gauss-Codazzi quantum mechanics action on $\mathcal{A}^4/G^4$. The construction is almost identical to that for $\mathcal{M}_2$ (essentially because the deformation complex is ‘short’ in both examples so that no additional gauge fixing is required), and the resulting 4d action is obtained from (26) by replacing the scalar $b$-multiplet by a multiplet of self-dual two-forms.
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