Vibrational Decoherence in Ion Trap Quantum Computers *

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Abstract

The ion trap quantum computer proposed by Cirac and Zoller [Phys. Rev. Lett. 74, 4091 (1995)] is analyzed for decoherence due to vibrations of the ions. An adiabatic approximation exploiting the vast difference between the frequencies of the optical intraionic transition and the vibrational modes is used to find the decoherence time at any temperature $T$. The scaling of this decoherence time with the number of ions is discussed, and compared with that due to spontaneous emission.

I. INTRODUCTION

Since Shor’s discovery [1] of an algorithm for factorization of a composite number of order $2^L$ in $\sim L^3$ steps on an ideal quantum computer (QC), a great deal of effort has gone into developing quantum computational theory, and related ideas in information transfer and cryptography. This work has shed new light on both quantum mechanics and computational complexity theory. As Landauer [2] has so pungently said, however, writing down a Hamiltonian is not the same as specifying an apparatus, and it is also necessary, at some time, to look into building real machines. Landauer’s criticism has not gone completely unanswered, and concrete proposals for implementing QC’s have been put forward. Perhaps the most promising of these is by Cirac and Zoller (CZ) [3] – but see also Ref. [4]. It is based on a linear array of trapped ions driven by a precisely timed sequence of laser pulses. This proposal is being taken seriously enough that at least one group in the world is trying to build a prototype [5].

The CZ QC consists of $N$ identical ions, trapped and cooled in a linear rf Paul trap. The ions form a linear array with nonuniform spacings determined by their mutual Coulomb repulsion and the effective trapping potential (see Fig. 1). Two internal states of each ion, $|g\rangle$ and $|e\rangle$, serve as the quantum bit, and laser pulses ($\pi$, $\pi/2$, etc.) drive $g \leftrightarrow e$ transitions and thus implement one-bit gates. These transitions

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can be driven either by lasers tuned to the direct $g \leftrightarrow e$ transition frequency $\omega_0$, or by Raman pulses in a $\Lambda$ system, where the lower levels form the quantum bit, and are chosen from the ground multiplet of the ion to minimize spontaneous emission. The innovative idea due to CZ is to use the center of mass longitudinal vibrational mode of the array as a bus that enables the execution of two-bit gates. This can be done by a combination of one-ion pulses and pulses detuned by $\omega_z$, the above mentioned vibrational mode frequency, to any pair of ions. Any superposition of the $2^N$ states of the QC (i.e., the internal states of the ionic system) can then be converted to any other superposition by a suitable sequence of one- and two-bit gates.

In this article we shall analyze the decoherence in the Cirac-Zoller (CZ) QC. A brief description of this work has appeared elsewhere [6]. It hardly needs to be said that decoherence is a serious limitation to the functioning of any QC. We will be interested in the intrinsic decoherence. Technical difficulties, such as trapping a large enough number of ions, proper shaping, phase locking, and timing of the laser pulses, optical resolution of individual ions etc., also have the same practical effect as decoherence, and may well turn out to be insurmountable by themselves, but that is a separate matter. We will mainly discuss the decoherence from the vibrations of the ions. Since the discussion in the earlier paper [6] was terse and technical, we will focus in the present article on the physical explanation of why ionic vibration is decohering, and give a qualitative estimate of the decoherence time. Other discussions specific to the CZ scheme are by Plenio and Knight [7], and by Hughes et al. [8]. General discussions of why and how decoherence is detrimental to QC’s have been given by Landauer [2] and by Unruh [9], among others.

Intrinsic decoherence in the CZ QC arises from two sources: spontaneous emission and ionic vibration. It is easy to estimate the decoherence rate from the first. If the spontaneous $|e\rangle \to |g\rangle$ decay time for one ion is $\tau_s$, and we assume that $N/2$ ions are in the excited state on average, we obtain an upper bound of $\tau_{\text{rad}} \approx 2\tau_s/N$ for the window of time in which any computation must be completed, since a single spontaneous decay irretrievably disrupts the wavefunction of the QC as a whole. Our estimate ignores effects like superradiance, or changes in the decay of one ion due to the proximity of the other ions, but this is justified if the inter ion spacing is larger than $2\pi c/\omega_0$, the wavelength of the $g \leftrightarrow e$ spectral line.

A similar estimate of $\tau_{\text{vib}}$, the computational time limit imposed by vibrational decoherence, is not so easy to obtain. Whatever it is, one simple point should be noted now. Since the two mechanisms of decoherence operate independently and in parallel, we should add their rates to obtain the total decoherence limit on the useful working time of the QC:

$$t_{\text{dec}} = \left(\tau_{\text{rad}}^{-1} + \tau_{\text{vib}}^{-1}\right)^{-1}.$$  \hspace{1cm} (1)

If it should happen that one of the rates, $\tau_{\text{rad}}^{-1}$ and $\tau_{\text{vib}}^{-1}$, is much larger than the other, this would help us relax the design constraints, as we could then ignore the slow decay process to a first approximation. The faster decay may of course still overwhelm us.

The plan of the paper is as follows. The calculation of $\tau_{\text{vib}}$ is done in Sec. II. The results are discussed in Sec. III. Certain mathematical details are relegated to two appendices.
II. VIBRATIONAL DECOHERENCE

The physical origin of vibrational decoherence is as follows. Suppose ion \( j \) (see Fig. 1) is not in its equilibrium position. It creates an excess electric field (or electric field gradient) on a neighboring ion, \( i \). This excess field alters the evolution in the \(|e⟩, |g⟩\) space of ion \( i \) from the desired time evolution, and the accumulation of this effect causes a decay in the probability that the QC will be in the intended state.

To qualitatively estimate the decay time, let us denote the longitudinal position of the \( j \)th ion by \( z_j \), and the deviation from this position by \( u_j \). Let us further suppose that the \( g ↔ e \) transition is of electric dipole (E1) or quadrupole (E2) type, and denote the relevant transition matrix element by \( d_a \), where the index \( a = 1, \) or 2, for the E1 and E2 cases respectively. Ignoring vector and tensor indices, and denoting the ionic charge by \( q \), the change in the \( eg \) matrix element of the Hamiltonian for ion \( i \) is given by

\[
\delta V_i = d_a \sum_{j \neq i} \frac{q}{|z_i - z_j|^{a+2}} u_j. \tag{2}
\]

The key point now is not only that this perturbation is small, i.e., \( |\delta V_i| \ll \hbar \omega_0 \), but that it is also slow. It varies over times set by the periods of the normal modes of the ion array, which are much longer than the optical transition time \( \omega_0^{-1} \). In other words,

\[
\left| \frac{d}{dt} \ln |\delta V_i| \right| \ll \omega_0. \tag{3}
\]

The slowness enables us to treat the perturbation \( \delta V_i \) adiabatically. Let us map each two-state ion onto a spin-\( 1/2 \), with \(|e⟩\) and \(|g⟩\) being the up and down spin states. The \( i \)th spin then sees magnetic fields \( B_z = \hbar \omega_0 \) and \( B_\perp = \delta V_i \) (see Fig. 2). Since \( B_z \gg |B_\perp| \), the instantaneous precession frequency for this spin is given by

\[
\omega_{0i}' = (\omega_0^2 + \delta V_i^2 / \hbar^2)^{1/2} \approx \omega_0 + \frac{\delta V_i^2}{2\hbar^2 \omega_0}. \tag{4}
\]

On the other hand, the precession axis for the spin can be taken to be \( \hat{z} \) at all times to very good approximation. Thus the time evolution of the spin up and spin down states \(|±⟩\) is given by \( \exp \left( ±i \int_0^t dt' \omega_{0i}'(t')/2 \right) |±⟩ \). A more formal derivation of this result is given in Appendix A.

The phase of the \( i \)th ion thus wanders off course by \( \pi \) in a time \( \tau_i \approx \pi / (\omega_{0i} - \omega_0) \). Using Eqs. (2) and (4), we can write

\[
\tau_i^{-1} \approx \frac{q^2 d_a^2}{2\pi \hbar^2 \omega_0} \left( \sum_{j \neq i} \frac{u_j}{(z_i - z_j)^{a+2}} \right)^2. \tag{5}
\]

The angular brackets above denote some kind of average. The motion of different ions is correlated via the normal modes, which we can describe by a density matrix at some temperature \( T \). (This is what one means by the statement that the ions have been cooled to a temperature \( T \).) Additional correlations due to the systematic manipulation of the center of mass mode used to execute the two-bit gates should not
be included since this is part of the designed time evolution and does not represent a decay. It is therefore completely consistent to have left out this part of the time evolution in arriving at Eq. (5). To obtain the best-case answer, we will assume that the ion temperature $T \ll \hbar \omega_z / k_B$, and approximate this by $T = 0$. To obtain an order of magnitude, we ignore the details of the normal modes, and simply take

$$\langle u_j u_k \rangle = \frac{\hbar}{m \omega_t} \delta_{jk},$$  \hspace{1cm} (6)$$

where $\omega_t$ is a typical transverse mode frequency, and $m$ is the ionic mass. This yields

$$\tau_{vib}^{-1} \approx \frac{q^2 d^2 a^2}{2\pi \hbar m \omega_t \omega_z} \sum_{j \neq i} \frac{1}{(z_i - z_j)^2 + 4}.$$  \hspace{1cm} (7)$$

The rationale for using a transverse mode frequency above is two-fold. First, these frequencies are generally higher than the longitudinal ones, and second, it is best to choose the states $|g\rangle$ and $|e\rangle$ to be such that the longitudinal modes cannot excite any transitions due to a $J_z$ selection rule.

The next step is to obtain the decoherence rate $\tau_{vib}^{-1}$ for the QC as a whole. One’s first guess would be that this is obtained by just adding the $\tau_{vib}^{-1}$ for all $i$. The detailed calculation of Ref. [6] shows that this is not quite correct. The correct procedure is to add the squares and then take the square root. In other words,

$$\tau_{vib}^{-2} = \sum_i \tau_{vib}^{-2}.$$  \hspace{1cm} (8)$$

A qualitative justification for this formula is as follows. The overlap between the actual and intended states of the $i$th spin is better approximated by $\cos(t/\tau_i)$ instead of $\exp(-t/\tau_i)$. [See Eq. (21).] The probability $P(t)$ that the QC is in the desired state is thus approximately $\prod_i \cos^2(t/\tau_i) \simeq \exp(-t^2/\tau_{vib}^2)$ with $\tau_{vib}$ as given above. [If we simply added $\tau_{vib}^{-1}$, we would obtain an additional factor of $N^{1/2}$ in the scaling behavior of $\tau_{vib}$, and Eq. (11) below, e.g., would contain an $N$ instead of the $N^{1/2}$.]

By combining Eqs. (7) and (8), we obtain a formal answer for the decoherence rate. The sums over the lattice positions $z_i$, are at worst, numerical problems. When $N \gg 1$, however, we can evaluate these sums by invoking a continuum approximation for the array. This approximation and the lattice sums are discussed in Appendix B. The final result can be written as

$$\tau_{vib}^{-1} \sim N^{1/2} \frac{q^2 d^2 a^2}{2\pi \hbar m \omega_t \omega_z s_0^{2a+4}}.$$  \hspace{1cm} (9)$$

where $s_0$ is the minimum spacing between the ions which occurs at the center of the array. We can make it apparent that $\tau_{vib}^{-1}$ is a rate by noting that [see Eq. (23)], $q^2 = m \omega^2 d_0^2 \tau_{vib}^{-1}$, where $d_0$ is the the trap length scale parameter, and that

$$d_a^2 \propto \hbar / \tau_s k_0^{2a+1},$$  \hspace{1cm} (10)$$

where $k_0 = \omega_0 / c$. It follows that

$$\frac{1}{\tau_{vib}} \sim N^{1/2} \frac{d_0^3}{s_0} \frac{\omega_z^2}{\omega_t \omega_z} \frac{1}{(k_0 s_0)^{2a+1}}.$$  \hspace{1cm} (11)$$
Discussion of this result and comparison with the spontaneous decay rate is given in the next Section.

III. DISCUSSION

To apply Eq. (11) we must take into account that the scaling of $\tau^{-1}_{\text{vib}}$ with $N$ depends critically on how the trap operating conditions are varied with $N$, and can not be naively taken as $N^{1/2}$. If, for example, $s_0$ is held fixed as $N$ is increased, then $(d_0/s_0)^3 \sim N^2/\ln N$ and $\tau^{-1}_{\text{vib}} \sim N^{5/2}/\ln N$. In this case, however, the longitudinal voltage on the trap electrodes, which is proportional to $\omega_z^2$, varies as $(\ln N)/N^2$. This leads to an increase in the total computational time, since the time for executing a primitive two-bit gate varies as $\omega_z^{-1}$, i.e., as $N$. Also, since the longitudinal trapping is weaker, the ion array becomes more susceptible to patch voltages on the electrodes, and non-linear effects in the trapping potential become more important. If, on the other hand, the trap voltages, and therefore, $\omega_z$ and $\omega_t$, are held fixed as $N$ increases, then $\tau^{-1}_{\text{vib}} \sim N^{(8a+19)/6}(\ln N)^{-2} (2a+4)/3$, i.e., as $N^{19/2} (\ln N)^{-2}$ for an E1 transition, and as $N^{35/6} (\ln N)^{-8/3}$ for an E2 transition. In this case, the minimum spacing $s_0$ varies as $\sim N^{-2/3}$, and it may become difficult to resolve the ions optically as is necessary to execute the basic gates. Obviously, any intermediate variation is possible by allowing both $s_0$ and $\omega_z$ to change with $N$, and the exact manner in which this is done is thus a matter of detailed engineering considerations, which it is premature to discuss.

To get a numerical estimate of $\tau_{\text{vib}}$, we will consider the case of Ba$^+$ ions, which are particularly favorable from the standpoint of minimizing spontaneous emission decoherence. We choose for $|g\rangle$ a state in the ground multiplet $6s^2S_{1/2}$, and for $|e\rangle$ a state in the first excited multiplet $5d^2D_{5/2}$. The frequency $\omega_0 = (2\pi)1.7 \times 10^{14}$ Hz. Since $\Delta L = 2$, the $2D_{5/2} \rightarrow 2S_{1/2}$ decay is an E2 process, and the spontaneous decay time is $\tau_s = 35$ s [10]. (There is some uncertainty over this number. Hughes et al. [8] take it as 47 s, and it may even be as high as 70 s. We have taken an average from Ref. [10].) The index $a$ is 2. Further, we take $\omega_{z}/2\pi = 100$ kHz, and $\omega_{t}/2\pi = 20$ MHz. Then $d_0 = 14 \mu$m, and for $N = 1000$, $\tau_{\text{vib}} \simeq 10^4 \tau_s$, which is surprisingly large. (It is even larger in comparison to $\tau_{\text{rad}} = \tau_s/N$.) On the other hand, $s_0 \simeq 0.5 \mu$m with the same parameters, which runs into the difficulty with optical resolution mentioned above. This suggests that a compromise in which $\omega_z$ is reduced may work better but we have not explored this point further.

That $\tau_{\text{rad}}$ is so much longer than $\tau_{\text{vib}}$ for the example chosen above may make one wonder if one should not have anticipated this fact. We do not believe so. The situation would change completely for larger $N$. Indeed for large enough $N$, our calculation shows that vibrational decoherence will always dominate over spontaneous decay decoherence. Secondly, the scaling with $N$ is quite non-trivial and unexpected.

We conclude that ionic vibrations are not a significant source of decoherence in the original scheme envisaged by CZ, at least for $N \leq 1000$. This should not be taken to mean that the problems posed by radiative decoherence by themselves are not serious. Indeed, this could well make the whole scheme unworkable. Similarly, the challenges of trapping 1000 ions, and of addressing them individually are at the moment quite daunting. Nevertheless, our conclusions are encouraging in that they enable us to focus on the radiative decay problem. Several authors [7,8] have suggested working
with a Λ system and Raman pulses as a way of dealing with this. An evaluation of the vibrational decoherence in this setup remains to be done.

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APPENDIX A: SPIN-1/2 IN SLOW AND WEAK TRANSVERSE FIELD

We consider in this Appendix a spin-1/2 system with the following Hamiltonian

\[ H = \frac{1}{2} \omega_0 \sigma_z + \mathbf{f}(t) \cdot \mathbf{\sigma}, \]

(12)

where \( \mathbf{f}(t) = (f_x, f_y, 0) \) has no z component. Further \( \mathbf{f} \) is small and slow as explained in Sec. II. (The field \( B_\perp \) introduced in Sec. II is just twice \( \mathbf{f} \).) We are interested in solving for the time evolution of an arbitrary state for general \( \mathbf{f}(t) \).

Denoting the eigenstates of \( S_z \) with eigenvalues \( \pm \frac{1}{2} \) by \( |\pm\rangle \), let us write a general state of the spin as

\[ |\psi(t)\rangle = u^+(t)e^{-i\omega_0 t/2}|+\rangle + u^-(t)e^{i\omega_0 t/2}|-\rangle. \]

(13)

Schrödinger’s equation then takes the form

\[ i\dot{u}^\pm = \pm e^{\pm i\omega_0 t} f^\mp(t) u^\mp(t), \]

(14)

where \( f^\pm = f_x \pm if_y \).

Since \( f^\pm \) varies very slowly, we seek the answer to Eq. (14) in the form

\[ u^\pm(t) = \alpha^\pm(t) + \beta^\pm(t), \]

(15)

where \( \alpha^\pm \) and \( \beta^\pm \) are fast and slow parts, the latter varying little over a period \( 2\pi/\omega_0 \), and the former averaging to zero over several such periods.

Substituting Eq. (15) in (14), and separating the fast and slow parts, we obtain

\[ i\dot{\alpha}^\pm = e^{\pm i\omega_0 t} f^\mp(t) \beta^\mp(t), \]

(16)

\[ i\dot{\beta}^\pm = e^{\pm i\omega_0 t} f^\mp(t) \alpha^\mp(t). \]

(17)

To integrate Eq. (16), it is a good approximation to treat the slowly varying functions \( f^\pm \) and \( \beta^\pm \) as constants. In this way, we obtain

\[ \alpha^\pm = \mp \omega_0^{-1} e^{\pm i\omega_0 t} f^\mp(t) \beta^\mp(t). \]

(18)

We now put this solution in Eq. (17), and average the resulting equation over several periods \( 2\pi/\omega_0 \). This yields

\[ i\dot{\beta}^\pm = \pm \frac{|f(t)|^2}{\omega_0} \beta^\pm(t). \]

(19)
Integrating this, we obtain $\beta_{\pm}(t) = \exp(\mp i \Phi(t)) \beta_{\pm}(0)$, where
\[ \Phi(t) = \int_0^t dt' \frac{|f(t')|^2}{\omega_0}. \] (20)

We can take $u_{\pm} \approx \beta_{\pm}$, since $|\alpha_{\pm}| \ll |\beta_{\pm}|$. The quantity $\Phi(t)$ is precisely the excess angle through which the spin precesses about the $\hat{z}$ axis as discussed in Sec. II, i.e., it is half the difference between $\int_0^t \omega_0(t') dt'$ and $\omega_0 t$.

Suppose the initial state of the spin is $\frac{1}{\sqrt{2}} (|+\rangle + |\rangle)$, i.e., $u_{\pm}(0) = \frac{1}{\sqrt{2}}$. Let us denote the state at time $t$ that would be obtained in the absence of the transverse field $f$ by $|\psi_0(t)\rangle$. In the context of the QC, the states $|\psi_0(t)\rangle$ and $|\psi(t)\rangle$ are analogous to the states of the ideal and actual QC, without and with decoherence, respectively. The extent of decoherence is given by the overlap
\[ \langle \psi_0(t) | \psi(t) \rangle = \cos(\Phi(t)). \] (21)

A more careful calculation shows that $\Phi(t)$ also has a Berry phase part $\hat{z} \cdot (f \times \dot{f})$. For the problem of interest to us, this is much smaller than the dynamical phase and may be neglected. Further, the adiabatic approximation breaks down due to secular effects for $t \sim \omega_0^3 / |f|^2$. It is not hard to see that this breakdown time is much larger than $\tau_{\text{vib}}$, and so the approximations (4) or (20) are completely adequate.

**APPENDIX B: CONTINUUM APPROXIMATION FOR ION ARRAY**

We wish in this Appendix to quantitatively understand the structure of the linear array of trapped ions when $N \gg 1$. We do this via a continuum approximation based on the expectation that the local spacing $s(z_i)$ between ions in the vicinity of ion $i$ will vary slowly with $i$. Our goal is to find the function $s(z)$, where we regard $z$ as a continuous variable. We will also find how the total length of the array, $2L$, varies with $N$.

A simple-minded argument for $s(z)$ is as follows. Consider the Coulomb forces on an ion at position $z$ from its nearest neighbours to the left and right, which we take to be at positions $z - s_-$ and $z + s_+$ respectively. The net force is then $q^2(s_-^2 - s_+^2) \approx 2q^2 s^2 (ds/dz)$, where we have approximated $s_+ - s_-$ by $s(z)(ds/dz)$. The force from successively distant pairs of neighbours is smaller than this expression by factors of 4, 9, 16, etc., since the distances are approximately doubled, tripled and so on. Thus the net Coulomb force is $(\pi^2 q^2 / 3s^2)(ds/dz)$, since $\sum_n n^{-2} = \pi^2 / 6$. Equating this to the opposing spring force $m \omega_0^2 z$ from the trapping potential, we obtain
\[ \frac{\pi^2}{3s^2} \frac{ds}{dz} = \frac{z}{d_0^3}, \] (22)

where
\[ d_0 = (q^2 / m \omega_0^2)^{1/3} \] (23)
is a natural length scale for the trap. (It is easy to show that the ion spacing is of the order of $d_0$ for 2 or 3 ions in the trap.) Denoting the total length of the array by $2L$
and placing the center at \( z = 0 \), integration of Eq. (22) gives

\[
\frac{1}{s(L)} - \frac{1}{s(z)} = -\frac{3}{2\pi^2d_0^3}(L^2 - z^2).
\]

(24)

By balancing the forces on the ion at the end of the array in the same way as was done above, we obtain

\[
s(L) \approx \pi(d_0^3/6L)^{1/2}.
\]

We can thus ignore \( s^{-1}(L) \) compared to \( 3L^2/2\pi^2d_0^3 \) in Eq. (24), which yields

\[
s(z) = s_0(1 - z^2/L^2)^{-1},
\]

(25)

where \( s_0 \equiv s(0) = 2\pi^2d_0^3/L^2 \) is the minimum ion spacing (attained at \( z = 0 \)).

To obtain an expression for \( L(N) \), let us denote the ion number at position \( z \) by \( n(z) \). Then, integration of the approximate relation \( dn/dz = 1/s(z) \) using Eq. (25) gives

\[
L = d_0(\pi^2N/2)^{1/3}.
\]

This implies \( s_0 \sim N^{-2/3} \). The mean spacing can be found to vary as \( N^{-2/3} \ln N \).

It is clear that the above argument does not treat the ends of the chain properly, and also underestimates the Coulomb forces due to the more distant neighbors. A more sophisticated approach is due to Dubin [11], who treats the ion array as a fluid of total charge \( Nq \). In a harmonic trapping potential, the solution to this problem is known: the fluid forms a blob of uniform charge density in the shape of an ellipsoid of revolution of total volume \( 4\pi Nd_0^3 \) provided \( \omega_t \gg \omega_z \). If the semi major axis of this ellipsoid is \( L \), the semi minor axis is therefore \((3Nd_0^3/L)^{1/2}\). The inverse spacing \( 1/s(z) \) is clearly the number of charges per unit length along the major axis of the ellipsoid, and this in turn is given by its cross-sectional area. In this way we obtain

\[
\frac{1}{s(z)} = \frac{3N}{4L} \left(1 - \frac{z^2}{L^2}\right).
\]

(26)

This is identical to our approximate form (25), but \( s_0 \) is different.

We still need to find the length \( L \). The fluid approximation breaks down over here, as the answers it gives depend on the ratio \( \omega_z/\omega_t \), which is clearly wrong as long as the linear structure is stable. Dubin uses a local density functional theory to estimate the correction to the Coulomb energy due to the discreteness of the array, and minimizes the sum of this correction, the fluid drop self energy, and the trapping potential energy, with respect to \( L \). The result for \( L \) is then independent of \( \omega_z/\omega_t \) and is given by

\[
L^3 = 3N \ln(0.8N)d_0^3.
\]

(27)

(The 0.8 is actually \( 6e^{-13/5} \), with \( e \) being Euler’s constant.) This result differs from our previous one by logarithmic factors. We also obtain

\[
s_0 = 4L/3N = 1.92N^{-2/3}[\ln(0.8N)]^{1/3}d_0.
\]

(28)

This result should also be compared to that of Hughes et al. [8], who find on the basis of a numerical fit that \( s_0 = 2N^{-0.56}d_0 \). Since for moderate values of \( N \), the logarithmic factor in Eq. (28) will have the effect of increasing the apparent exponent of \( N \), these two results are quite comparable. We do not know if Hughes et al. did
numerics for large enough $N$ to discern the presence or absence of logarithmic factors, so it is hard to say which is better.

The above results can be used to perform the sums over lattice positions that appear in Eqs. (7) and (8). There are two types of sums. The first,

$$S_n(i) \equiv \sum_{j \neq i} \frac{1}{|z_i - z_j|^n},$$

(29)

can be very simply evaluated as

$$S_n(i) \approx 2s^n(z_i) \sum_{j=1}^{\infty} \frac{1}{j^n} = \frac{2\zeta(n)}{s^n(z_i)}.$$  \hspace{1cm} (30)

This approximation is actually fairly good for all $i$ except very close to the ends, since the exponent $n$ is at least 3 or 4 in all cases that we encounter.

The second type of sum is $T_n = \sum_i s^{-n}(z_i)$. This can be approximated by an integral. Writing $\Delta i \approx dz/s(z)$, we obtain

$$T_n = \sum_i \frac{1}{s^n(z_i)} \approx \int_{-L}^{L} \frac{dz}{s^{n+1}(z)}.$$ \hspace{1cm} (31)

With $s(z)$ given by Eq. (25), the integral is elementary, and we obtain

$$T_n \approx \frac{L}{s_0^{n+1}} \left( \frac{4\pi}{4n + 7} \right)^{1/2},$$ \hspace{1cm} (32)

where the last form comes from an asymptotic formula for $\beta(n + 2, 1/2)$.

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Fig. 1. Schematic of the array of trapped ions in the Cirac-Zoller quantum computer.

Fig. 2. Equivalent magnetic fields acting on the internal states of the $i$th ion in the spin representation.
Driving Lasers

Ions

$\pi w_0$

$\left| e \right>$

$\left| g \right>$

$i$

$j$

$u_j$
\( B_{\text{tot}} \quad \quad B_z \propto \hbar \omega_0 \quad \quad B_{\perp} \propto \delta V_i \)