The problem is posed of further extending the axiomatic construction proposed in Part 1 for non-local point transformations mapping in each other different curved space times. The new transformations apply to curved space times when expressed in arbitrary coordinate systems. It is shown that the solution permits to achieve an ideal (Gedanken) experiment realizing a suitable kind of phase-space transformation on point-particle classical dynamical systems. Applications of the theory are discussed both for diagonal and non-diagonal metric tensors.

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1 - INTRODUCTION

Following Ref.[1] (hereon referred to as "Part 1"), in this paper further aspects are investigated concerning the extension of the functional setting which lays at the basis of the standard formulation of General Relativity (SF-GR), and Special Relativity (SR) as well [2–7]. More precisely, the issue is about the most general prescription of the class of non-local point transformations (NLPT) and related extended GR-reference frames (extended GR-frames) to be established between two, in principle arbitrary, curved space-times. This will be referred to here as general NLPT-theory, in contrast to the special NLPT-theory earlier developed in Ref.[1].

In previous literature these transformation were identified with local point transformations (LPT) and consequently necessarily mapping a single space-time in itself only [2, 3, 4]. Such a feature, which is actually at the basis of SF-GR, i.e., Einstein’s theory of gravitation, is also of paramount importance in all relativistic theories, ranging from classical to quantum electrodynamics, mechanics and theory of fields. Nevertheless, in certain physical problems such as the Einstein’s Teleparallel approach to GR (or TT-problem), the introduction of a new type of coordinate transformations, identified with the same NLPT indicated above, is found to be mandatory. This refers to the of the theory originally formulated by Einstein in 1928 [5] in order to establish a map between a generic connected and time-oriented curved space-time \((Q^4, g)\) and the flat time-oriented Minkowski space time \((M^4, \eta)\) represented in terms of orthogonal Cartesian coordinates (see Eqs. (1) below). In such a case its metric tensor is identified with the corresponding Minkowski metric tensor \(g^\alpha_{\beta} \equiv \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)\).

As shown in Part 1, in particular, this means that it should always be possible to represent such transformations in terms of real variables (see for example Refs. [12–14]). Also from Part 1 it follows that the solution of the TT-problem involves in particular the following two fundamental notions:

Notion #1: the extended GR-frame.- Departing from the customary notion of GR-reference frame (or briefly GR-frame) traditionally adopted in SF-GR, i.e., of a 4-dimensional real curvilinear coordinate system \(r^\mu\) to be established on 4-dimensional Lorentzian space-times \((Q^4, g)\), the concept of extended GR-frame is introduced. In each of the two space-times \((Q^4, g)\) and \((Q^4, g')\) this corresponds to identify such a notion with the smoothly \(s\)-dependent phase-space state vectors

\[
\mathbf{x}(s) \equiv \left\{ r^\mu(s), u^\mu(s) \equiv \frac{d}{ds}r^\mu(s) \right\}, \quad (1)
\]

\[
\mathbf{x}'(s) \equiv \left\{ r'^\mu(s), u'^\mu(s) \equiv \frac{d}{ds}r'^\mu(s) \right\}, \quad (2)
\]

which are defined at the same prescribed proper time \(s\). Here \(s\) by assumption belongs to a suitable interval \(I \subseteq \mathbb{R}\). Furthermore, \(r^\mu(s) \equiv r^\mu \{ r(s), [r, u] \} \in (Q^4, g)\) and \(r'^\mu(s) \equiv r'^\mu \{ r'(s), [r', u'] \} \in (Q^4, g')\) identify, in terms of in principle arbitrary coordinate systems, the corresponding 4-positions in the two space-times. Finally, \(u^\mu(s) \equiv \frac{d}{ds}r^\mu(s)\) and \(u'^\mu(s) \equiv \frac{d}{ds}r'^\mu(s)\) represent the related 4-velocities which span the corresponding tangent spaces.
**Notion #2: special NLPT-phase-space transformation** - The second one is about the prescription of a suitable phase-space transformation mapping in each other the two extended GR-frames \( \{ r^\mu(s), u^\nu(s) \} \) and \( \{ r'^\mu(s), u'^\nu(s) \} \). For this purpose, in accordance with Einstein’s TT-problem, the curvilinear coordinates \( r^\mu \) and \( r'^\mu \) are preliminarily identified with Cartesian coordinates, i.e., letting respectively
\[
\begin{align*}
 r^\mu &\equiv (r^0, (r \equiv x, y, z)), \\
 r'^\mu &\equiv (r'^0, (r' \equiv x', y', z')).
\end{align*}
\]

Then, the NLPT-phase-space transformation determined in Part 1 is of the form
\[
\begin{align*}
 \{ r^\mu(s), u^\nu(s) \} &\rightarrow \{ r'^\mu(s), u'^\nu(s) \} = \{ r'^\mu \{ r(s), [r, u] \}, (M^{-1})^\mu_\nu (s)u'^\nu(s) \}, \\
 \{ r'^\mu(s), u'^\nu(s) \} &\rightarrow \{ r^\mu(s), u^\nu(s) \} = \{ r^\mu \{ r'(s), [r', u'] \}, M^\nu_\mu(s)u^\nu(s) \}.
\end{align*}
\] (5)

Here, again departing from SF-GR, the coordinate transformation \( r^\mu(s) \leftrightarrow r'^\mu(s) \) rather than being identified with a local point transformation (LPT) acting on the same space-time \((Q^4, g)\), is realized by special NLPT mapping in each other two space-times \((Q^4, g)\) and \((Q'^4, g')\). In Lagrangian form these are of the type
\[
\begin{align*}
P_S: & \quad r'^\mu(s) = r'^\mu(s_o) + \int_{s_o}^{s} d\tau M^\mu_\nu(s)u'^\nu(\tau), \\
P^{-1}_S: & \quad r^\mu(s) = r^\mu(s_o) + \int_{s_o}^{s} d\tau (M^{-1})^\mu_\nu(s)u^\nu(\tau),
\end{align*}
\] (6)

being the two-space-times referred to the same coordinate systems (Assumption \( \alpha \)) and in Part 1 the latter were exclusively identified with the Cartesian coordinates \((3)-(4)\). Here \( M^\mu_\nu(s) \) and \( (M^{-1})^\mu_\nu \) denote the Jacobian matrix and its inverse, both to be assumed of non-gradient type (see related definitions in Part 1). In the case in which the matrix \( M^\mu_\nu \) (and \( (M^{-1})^\mu_\nu \)) are continuously connected to the identity \( \delta^\nu_\mu \) this implies that
\[
\begin{align*}
 M^\mu_\nu & = \delta^\nu_\mu + A^\mu_\nu(r', r), \\
 (M^{-1})^\mu_\nu & = \delta^\nu_\mu + B^\nu_\mu(r, r'),
\end{align*}
\] (7)

with \( A^\mu_\nu \) and \( B^\mu_\nu \) being suitable transformation matrices. Hence, the special NLPT \((8)\) yield the corresponding Lagrangian representations
\[
\begin{align*}
 r'^\mu(s) & = r'^\mu(s_o) + \int_{s_o}^{s} d\tau A^\mu_\nu(r', r)u'^\nu(\tau), \\
 r^\mu(s) & = r^\mu(s_o) + \int_{s_o}^{s} d\tau B^\nu_\mu(r', r)u^\nu(\tau).
\end{align*}
\] (8)

As discussed below (see Section 2), Eqs.\((9)\), or equivalent \((10)\), identify a group of transformations, denoted as special NLPT-group \( \{ P_S \} \), established between \((M^4, \eta)\) and an in principle arbitrary curved space-time \((Q^4, g)\) in validity of Assumption \( \alpha \).

In this paper we intend to investigate the physical bases for the construction of more general non-local transformations, extending the class of special NLPT prescribed by Eqs.\((1)\) and holding in validity of Assumption \( \alpha \). The new transformations, denoted as general NLTP and identifying the general NLPT-group \( \{ P_g \} \), will be achieved by introducing a suitable axiomatic approach. These transformations will be permitted to map two arbitrary distinct curved space-times \((Q^4, g)\) and \((Q'^4, g')\), each one to be represented in terms of arbitrary coordinate systems, in particular generally different from the Cartesian coordinate systems \((3)-(4)\).

The fundamental issue which naturally arises in this connection is, of course, whether these transformations may have a physical interpretation at all. This would require in particular the identification of suitable observable, i.e., classically measurable, dynamical variables. To answer this question in a satisfactory and (hopefully) exhaustive way here we have endeavoured to develop two partially independent routes.

- **First route** - The first one is the search of a suitable *Gedanken experiment* (GDE), namely an ideal measurement experiment, to explicitly construct a general NLPT. As we intend to show, in fact, the same GDE will permit:
  1. The identification of the observable dynamical variables, to be identified with the extended GR-frames \((11)\) and \((12)\) belonging respectively to the curved space-times \((Q^4, g)\) and \((Q'^4, g')\). 2) The conceptual realization, and hence physical interpretation, of an arbitrary transformation of the group of \( \{ P_g \} \).

- **Second route** - The second route followed here, in order to corroborate the GDE-based physical interpretation, is founded on the development of selected applications of the general NLPT-theory, with particular reference to the well-known theoretical issue related to the diagonalization metric tensors associated with curved space-times.
In the literature such a problem is usually treated adopting the so-called Newman-Janis algorithm to diagonalize non-diagonal. Such an algorithm is frequently used in the literature for the purpose of investigating a variety of standard or non-standard GR black-hole solutions (Bambi et al., 2013; Toshmatov et al., 2014; Modesto et al., 2010). These include a number of problems which have remained unsolved to date and appears again of critical importance in GR. In particular:

1. **Problem #P1** - First, the fact that the Newman-Janis algorithm is complex, so that the transformed coordinates are complex too. This inhibits their objective physical interpretation in terms of observables.

2. **Problem #P2** - The fact that again the diagonalization problem at the basis of the same transformation cannot be solved in the framework of the validity of the LPT-GCP.

3. **Problem #P3** - The physical meaning of the transformation: one cannot ignore that fact that there is no clear understanding regarding its physical interpretation and ultimately as to why the algorithm should actually work at all.

4. **Problem #P4** - Finally, despite the obvious fact that the Teleparallel transformation provides in principle also a solution to the diagonalization problem, there is no clear connection emerging between the same transformation and the Newman-Janis algorithm.

The goal of this paper is to address specifically Problems #P1−#P4, a task which has remain essentially unchallenged to date. These problems are investigated based on the adoption of a suitable realization of non-local point transformations (NLPT) acting on appropriate extended GR-frames which are defined with respect to prescribed space-times. For such a purpose the determination is required of the group of general non-local point transformations (general NLPT) connecting subsets of two generic curved space-times \((Q^4, g)\) and \((Q'^4, g')\). The task posed here involves also their physical interpretation based on a suitable Gedanken experiment. This refers, in particular, to three distinct issues:

A) The possible conceptual realization of a measure experiment (Gedanken experiment), simulating the action of a generic NLPT on a GR-reference frame on the physical space-time.

B) The prescription of the family of NLPT, exclusively based on a suitable set of mathematical, i.e., axiomatic, prescriptions, which should be nevertheless physically realizable in principle for arbitrary GR-reference frames which are defined with respect to a prescribed space-time.

C) As an illustration of the theory, the explicit construction of possible physically-relevant transformations of the group \(\{P_g\}\), with special reference to the problem of diagonalization of non-diagonal metric tensors associated with rotating black holes.

As we intend to show, both routes will ultimately enable us to demonstrate the interpretation and physical consistency of the general NLPT-theory developed here, the connection with the analogous formulation holding for special NLPT (presented in Part 1) and - most important in our view - to display the explicit construction method of non-local transformations which mutually map in each other a variety of curved space-times.

**An example from SR**

Consider as a preliminary illustration of the issue the classical dynamical system (CDS) describing the dynamics of single point-particles in the special relativity (SR) setting, i.e., in the time-oriented Minkowski space-time. A possible Gedanken experiment concerns the representation of the same CDS performing a suitable reference-frame transformation. We shall distinguish - in such a process - both the so-called active and passive viewpoints of the transformation, i.e., in which either a point particle evolves in time (“moves”) or the reference frame itself changes, respectively. In order to define properly the two viewpoints let us introduce the 4−displacement and corresponding 4−velocity transformation of the type:

\[
\begin{align*}
\{ dr^\mu = J^\mu_\nu dr'^\nu,
\end{align*}
\]

\[
\begin{align*}
dd'^\mu = (J^{-1})^\mu_\nu dr'^\nu,
\end{align*}
\]

\[
\begin{align*}
\{ u^\mu = J^\mu_\nu u'^\nu,
\end{align*}
\]

\[
\begin{align*}
u'^\mu = (J^{-1})^\mu_\nu u'^\nu.
\end{align*}
\]
Eqs. (10) can be viewed as a Gedanken experiment (GDE) advancing in time separately the states \( \{ r^\mu(s), u^\mu(s) \} \) and \( \{ r'^\mu(s), u'^\mu(s) \} \). To elucidate this point consider the following two CDS’s:

\[
\{ r^\mu(s_o), u^\mu(s_o) \} \leftrightarrow \{ r^\mu(s), u^\mu(s) \},
\]

(11)

\[
\{ r'^\mu(s_o), u'^\mu(s_o) \} \leftrightarrow \{ r'^\mu(s), u'^\mu(s) \},
\]

(12)

which are assumed to be prescribed for all \( s_o, s \in I \). Assuming validity of Eqs. (9) and (10) it follows that the two states \( \{ r^\mu(s), u^\mu(s) \} \) and \( \{ r'^\mu(s), u'^\mu(s) \} \) are manifestly not independent. Indeed, the same CDS’s are not independent, as it follows at once by direct inspection of Eqs. (11) and (12). In particular, the first one (11) (and respectively the second one (12)) are obtained by considering the state \( \{ r^\mu(s), u^\mu(s) \} \) (or correspondingly \( \{ r'^\mu(s), u'^\mu(s) \} \)) as prescribed. As discussed at length in the following Sections, the two choices will be referred to as the active and passive viewpoints in which the GDE can be considered, more precisely: A) In the active viewpoint the state \( \{ r^\mu(s), u^\mu(s) \} \) acting on the curved (“transformed”) space-time evolves in time with \( \{ r'^\mu(s), u'^\mu(s) \} \), the ”background” state defined in the Minkowski space-time being considered a prescribed smooth \( s \)-function and generating the phase-space flow. B) In the passive viewpoint the state \( \{ r^\mu(s), u^\mu(s) \} \) is considered a prescribed smooth function of \( s \), so that the background state \( \{ r'^\mu(s), u'^\mu(s) \} \) must evolve in time accordingly.

For definiteness, let us consider for the Jacobian matrix \( J^\mu_\nu \), with \( (J^{-1})^\mu_\nu \) being its inverse, a realization which corresponds to a boost transformation, i.e., a space-time rotation for which \( J^\mu_\nu \equiv J^\mu_\nu(s) \), where

\[
J^\mu_\nu(s) = \begin{vmatrix} 
\gamma(s) & -\beta(s)\gamma(s) & 0 & 0 \\
-\beta(s)\gamma(s) & \gamma(s) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix},
\]

(13)

while \( \gamma(s) \) and \( \beta(s) \) are the Lorentz and relativistic factors \( \gamma(s) = 1/\sqrt{1 - \beta^2(s)} \), \( \beta(s) \equiv |v(s)|/c \) and \( v(s) \) denotes the spatial components of a local and non-uniform reference velocity. In particular, let us require that \( v(s) \) is parametrized in terms of the arc length \( s \), to be established on a suitable time-like world-line \( r^\mu \) (see below). It follows that by construction \( u^\mu \) and \( u'^\mu \) belong to different tangent spaces defined with respect to the same Minkowskian space-time, since by construction the identity

\[
\eta_{\alpha\beta} \frac{dr^\alpha}{ds} \frac{dr^\beta}{ds} = \eta_{\alpha\beta} \frac{dr'^\alpha}{ds} \frac{dr'^\beta}{ds}
\]

(14)

manifestly holds. The corresponding coordinate transformations \( r^\alpha(s) \rightarrow r'^\alpha(s) \) and its inverse, both defined in \( (M^4, \eta) \) and generated by integrating the 4-velocity transformations \( (16) \) along arbitrary time-like world lines of \( (M^4, \eta) \), are manifestly of the type indicated above (see Eqs. (11)) and therefore identify a particular possible realization of NLPT. It follows that Eqs. (16) can be interpreted as a result of the said Gedanken experiment. More precisely: A) In the active viewpoint a point-particle endowed with a 4-position \( r'^\nu \) (or \( r^\nu \)) acquires a displacement which carries it to the transformed 4-position \( r^\mu \) (or \( r'^\mu \) respectively), by means of a suitable dynamical flow of some kind producing also such a change in the particle 4-position. B) In the passive viewpoint the point-particle 4-position remains invariant, while the reference frame changes in such a way that the 4-position \( r'^\nu \) (respectively, \( r^\nu \)) is transformed to \( r^\mu \) (\( r'^\mu \)).

This simple example further supports the earlier discussion reported in Part 1 regarding the asserted physical inadequacy of the traditional concept of reference frame (the so-called GR-frame) adopted in particular in the context of GR, i.e., of a coordinate system based on the 4-position \( r \equiv \{ r^\mu \} \) only, which is founded - in turn - on the adoption of purely local coordinate transformations. The rationale behind the issue considered here lies on the Einstein equivalence principle (EEP, [3]) itself. This is actually realized by two separate propositions, which in the form presently known must both be ascribed to Albert Einstein’s 1907 original formulation [3] (see also Ref. [1]). In Einstein’s original approach this actually is realized by the following two distinct claims stating: a) the equivalence between accelerating frames and the occurrence of gravitational fields (see also Ref. [1]); b) that “local effects of motion in a curved space (gravitation)” should be considered as “indistinguishable from those of an accelerated observer in flat space” [2, 4].

This motivates us to search, based on physical first principles, for a development of the subject which eventually should/might permit one:

- To determine the most general representation for the phase-space transformations connecting physical reference frames, to be referred to as general NLPT-phase-space transformations.
• To address the main related mathematical and physical implications. For generality these will be investigated in the framework of GR, since the latter by construction encompasses also SR.

Under such premises, and to better elucidate the scope and potential physical relevance of the issue indicated above it must be noticed that the present work belongs to the class of studies aimed at extending in the context of GR and classical field theory the classical notions of local dynamics and local field interactions, with the precise goal of including in these theories various types of non-local phenomena. Recent literature investigations in this category are several. We refer to Part 1 for further discussions on the matter.

However, an instance worth to be mentioned and most relevance in the present discussion concerns the Einstein teleparallelism [5]. In fact, as discussed in Part 1, the teleparallel problem lying at the basis of such an approach cannot be solved in the framework of GCP and actually requires the introduction of a new functional setting for GR (NLPT-functional setting) based on the introduction of suitable NLPT.

As shown below, the new approach proposed here, based on the introduction of suitable NLPTs, permits to cast light on non-local phenomena which can occur in GR due to the choice of the GR-reference frames.

Goals of the paper

Given these premises, we are now in position to state in detail the structure of the present manuscript, pointing out the goals posed in each of the following sections which are accordingly listed below.

1. GOAL #1 - The first one, discussed in Section 2, concerns the development of the theory of general NLPT which permits to map in each other in principle arbitrary space-times. The connection with the special NLTP-theory earlier developed in Part 1 is displayed.

2. GOAL #2 - In Section 3 a Gedanken experiment interpretation and physical implications of the general NLPT-theory developed here are proposed for the phase-space transformations generated by the group of general NLPT.

3. GOAL #3 - In Section 4, the application is considered of the theory of general NLPT to the mapping of diagonal metric tensor expressed in arbitrary curvilinear coordinates.

4. GOAL #4 - In Section 5, the goal is posed of addressing the diagonalization problem for non-diagonal metric tensors occurring in GR.

5. GOAL #5 - Finally, in Section 6 the main conclusions of the paper are drawn.

2 - THEORY OF GENERAL NON-LOCAL POINT TRANSFORMATIONS ON MANIFOLDS

In this section the following representation problem is posed for such a theory: this lies in the search of the most general form which these point transformations and their theory, earlier pointed out in Part 1, can take. In the following these will be referred to as general NLTP and general NLPT-theory respectively.

More precisely, the new transformations should realize a mapping between two arbitrary connected and time-oriented 4–dimensional curved space-times \((Q^4,g)\) and \((Q^4,g')\) when they are referred to arbitrary curvilinear coordinate systems. For this purpose we shall require that by construction the new transformations between \((Q^4,g)\) and \((Q^4,g')\) determine a suitably-prescribed real diffeomorphism of the general form

\[
P_g : r'^\mu(s) \rightarrow r^\mu(s) = P_g^\mu_\nu \left( r', u', \left[ r', u', \frac{D' u'}{D s} \right], s \right),
\]

with inverse transformation

\[
P_g^{-1} : r^\mu(s) \rightarrow r'^\mu(s) = P_g^{-1\mu}_\nu \left( r, u, \left[ r, u, \frac{D u}{D s} \right], s \right),
\]

the square brackets denoting appropriate non-local dependences. In particular, here \(r' \equiv \{ r'^\mu \}, r \equiv \{ r^\mu \}, u' \equiv \{ u'^\mu \}, u \equiv \{ u^\mu \}\), while \(\frac{D' u'}{D s} = \frac{D' u'^\nu}{D s}\) and \(\frac{D u}{D s} = \frac{D u^\nu}{D s}\) identify as usual the covariant derivatives defined in the two space-times \((Q^4,g')\) and \((Q^4,g)\) respectively.
For definiteness, we shall also assume that Eqs. (15) and (16) are also consistent with the requirement that the proper-time $s$ appearing in both equations satisfies the Riemann distance condition

$$ ds^2 = g_{\mu\nu}(r)dr^\mu dr^\nu = g'_{\mu\nu}(r')dr'^\mu dr'^\nu, $$

both prescribed in terms of real and suitably-smooth functions of $s$ and hence warranting also the mass-shell kinematic constraints

$$ u^\mu u^\nu g_{\mu\nu}(r) = 1, \quad u'^\mu u'^\nu g'_{\mu\nu}(r') = 1. $$

It is immediate to notice that an obvious possible realization of the transformations (15) and (16) is obtained simply by considering explicitly $s-$dependent smooth real transformations of the type

$$ P_2 : r^\mu \rightarrow r'^\mu = r^{\mu}(r, s), $$

$$ P_2^{-1} : r'^\mu \rightarrow r^\mu = r^{\mu}(r', s), $$

defined for all $s \in I$. Again, for $dr^\mu$ and $u^\mu \equiv \frac{dr^\mu}{ds}$ transformations of the type (9) and (10) are implied. However, the Jacobians are of the type $\mathcal{J}_\mu^\nu(r', s)$ and $(\mathcal{J}^{-1})_\nu^\mu(r, s)$ and read respectively

$$ \mathcal{J}_\mu^\nu(r', s) \equiv \frac{\partial r'^\mu(r', s)}{\partial r^\mu} + \frac{\partial r'^\mu(r', s)}{\partial s} g_{\alpha\nu} u^\alpha, $$

$$ (\mathcal{J}^{-1})_\nu^\mu(r, s) \equiv \frac{\partial r^\mu(r, s)}{\partial r'^\nu} + \frac{\partial r^\mu(r, s)}{\partial s} g_{\alpha\nu} u^\alpha, $$

thus loosing their gradient form (see Eqs.(4) and (5) in Part 1). Nevertheless, it is obvious that transformations of the type indicated above generally imply the violation of the Riemann-distance constraint (17).

On the other hand, once the implications of the same equation are properly taken into account the representation problem posed here can be readily solved. Consider in fact again Eq. (17). Due to the arbitrariness of $r \equiv \{r^\mu\}$ as well of $s$ and $dr^\mu$ it follows that the same equation requires simultaneously that

$$ \left\{ \begin{array}{l} \exists^\mu = M_{(g)}^\mu(s)dl^\nu, \\ \exists^\nu = \left(M_{(g)}^{-1}\right)_\nu^\mu dl^\mu, \end{array} \right. $$

and

$$ \left\{ \begin{array}{l} g_{\mu\nu}(r) = \left(M_{(g)}^{-1}\right)_\mu^\alpha \left(M_{(g)}^{-1}\right)_\nu^\beta g'_{\alpha\beta}(r'), \\ g'_{\mu\nu}(r') = M_{(g)}^\rho M_{(g)}^{\rho\nu} g_{\mu\nu}(r), \end{array} \right. $$

must hold, with $M_{(g)}^\mu_\nu$ denoting a suitable and still to-be-determined real Jacobian matrix and $\left(M_{(g)}^{-1}\right)_\nu^\mu$ being its inverse. Therefore, Eqs.(23) imply that Eqs.(19) and (20) must recover the form (6), while Eqs.(24) require that necessarily takes the form

$$ \left\{ \begin{array}{l} M_{(g)}^\mu_\nu(s) = M_{(g)}^\mu(r'(s), r(s)), \\ \left(M_{(g)}^{-1}\right)_\nu^\mu(s) = \left(M_{(g)}^{-1}\right)_\nu^\mu(r(s), r'(s)), \end{array} \right. $$

i.e., they can only be functions of $r'(s)$ or respectively $r(s)$. More precisely, on the rhs of the first (second) equation $r(s) (r'(s))$ must be considered as a function of $r'(s)$ (respectively of $r(s)$) determined by means of an equation analogous to that holding for special NLTP, i.e., Eqs.(8), namely

$$ \left\{ \begin{array}{l} P_g : r^\mu(s) = r'^\mu(s_o) + \int_{s_o}^s ds M_{(g)}^\mu_\nu(s_1)u'^\nu(s_1), \\ P_g^{-1} : r'^\mu(s) = r^\mu(s_o) + \int_{s_o}^s ds \left(M_{(g)}^{-1}\right)_\nu^\mu(s_1)u^\nu(s_1). \end{array} \right. $$

This will be referred to as general NLPT. The corresponding phase-space map analogous to Eq.(5), namely

$$ \left\{ \begin{array}{l} \{r^\mu(s), u^\mu(s)\} \rightarrow \{r'^\mu(s), u'^\mu(s)\} = \{r'^\mu \{r(s), [r, u]\}, \left(M_{(g)}^{-1}\right)_\nu^\mu(s)u'^\nu(s)\}, \\ \{r'^\mu(s), u'^\mu(s)\} \rightarrow \{r^\mu(s), u^\mu(s)\} = \{r^\mu \{r'(s), [r', u']\}, M_{(g)}^\rho_\nu(s)u^\rho(s)\}, \end{array} \right. $$

(27)
will be denoted as general NLPT-phase-space transformation. Then the following result holds.

**THM.1 - Realization of the general NLPT-group \{P_g\}.**

The group \{P_g\} of general NLPT of the type Eqs. (28) can always be realized by means of Jacobians \(M^\mu_{(g)\nu}\) and \((M^{-1})^\mu_{(g)\nu}\) of the form

\[
\begin{align*}
M^\mu_{(g)\nu} &= \frac{\partial g^\mu_A(r')}{\partial r'^\nu} + A^\mu_{(g)\nu}(r', r), \\
(M^{-1})^\mu_{(g)\nu} &= \frac{\partial r^\nu(r)}{\partial s^\mu} + B^\mu_{(g)\nu}(r', r'),
\end{align*}
\]

(28)

with \(A^\mu_{(g)\nu}(r', r)\) and \(B^\mu_{(g)\nu}(r', r')\) being suitable transformation matrices. As a consequence, an arbitrary general NLPT can be represented as

\[
\begin{align*}
P_g: \ r^\mu(s) &= g^\mu_A(r'(s)) + \int_{s_o}^s ds A^\mu_{(g)\nu}(s) u^\nu(\pi), \\
P_g^{-1}: \ r'^\mu(s) &= f^\mu_A(r(s)) + \int_{s_o}^s ds B^{-1\mu}_{(g)\nu}(s) u'^\nu(\pi).
\end{align*}
\]

(29)

**Proof -** In fact, given validity of Eqs. (28) it follows for example that

\[
r^\mu(s) = r'^\mu(s_o) + \int_{s_o}^s ds \left[ \frac{\partial g^\mu_A(r')}{\partial r'^\nu} + A^\mu_{(g)\nu}(r', r) \right] u^\nu(\pi),
\]

(30)

where manifestly \(\int_{s_o}^s ds \frac{\partial g^\mu_A(r')}{\partial r'^\nu} u^\nu(\pi) = g^\mu_A(r'(s)) - g^\mu_A(r'(s_o))\). Now we notice that it is always possible to set the initial condition so that \(r^\mu(s_o) = g^\mu_A(r'(s_o))\). This implies the validity of the first of the Eqs. (29). The proof of the second one is analogous. **Q.E.D.**

Notice that, in difference with Eqs. (30), the transformations (29) (or equivalent (26)) now establish a diffeomorphism between two different, connected and time-oriented space-times \((Q^4, g)\) and \((Q'^4, g')\) under the following assumptions:

- **A1)** \((Q^4, g)\) is an arbitrary curved space-time;
- **A2)** \((Q^4, g')\) is an arbitrary curved space-time;
- **B1)** the space-times \((Q^4, g)\) and \((Q^4, g')\) are referred to as arbitrary GR-frames;
- **B2)** the same space-times \((Q^4, g)\) and \((Q^4, g')\) are referred to as different GR-frames.

Let us consider possible particular realizations of the general-NLPT given above.

The first one is obtained dropping assumption B2), i.e., requiring that the GR-frames of the two space-times \((Q^4, g)\) and \((Q^4, g')\) coincide. In fact, if the coordinate systems for \((Q^4, g)\) and \((Q^4, g')\) are the same ones while still remaining arbitrary, then one obtains that the constraint equations

\[
\begin{align*}
g'^\mu_A(r') &= r'^\mu, \\
f^\mu_A(r) &= r^\mu,
\end{align*}
\]

(31)

must hold identically. In such a case, denoting the transformations matrices as

\[
\begin{align*}
A^\mu_{(g)\nu} &= A^\mu_{\nu}, \\
B^\mu_{(g)\nu} &= B^\mu_{\nu},
\end{align*}
\]

(32)

the transformations (29) recover the same form given by Eqs. (39) and (40) in Part 1. These can be conveniently written as

\[
\begin{align*}
P_g: \ r^\mu(s) &= r'^\mu(s) + \Delta r'^\mu(s), \\
P_g^{-1}: \ r'^\mu(s) &= r^\mu(s) + \Delta r^\mu(s),
\end{align*}
\]

(33)

with \(\Delta r'^\mu(s)\) and \(\Delta r^\mu(s)\) identifying the non-local displacements

\[
\begin{align*}
\Delta r'^\mu(s) &= \int_{s_o}^s ds A^\mu_{\nu}(s) u^\nu(\pi), \\
\Delta r^\mu(s) &= \int_{s_o}^s ds B^\mu_{\nu}(s) u^\nu(\pi).
\end{align*}
\]

(34)
Therefore Eqs. (29) in validity of (31) will be referred to again as special NLPT. Their ensemble realizes manifestly a group (see proof in Part 1), which will be denoted as special NLPT-group \{P_S\}. From this conclusion it is immediate to infer the relationship between general and special NLPT. In fact, it is obvious that for an arbitrary general NLPT the relationship existing between the Jacobians \(M_{(g)\mu}^\nu\) and \(M_\mu^\nu\), as well as the corresponding transformation matrices \(A_{(g)\mu\nu}^r(r', r)\) and \(A_\mu^\nu(r', r)\), is simply provided by the matrix equation

\[
M_{(g)\mu}^\nu = M_\mu^\nu J_\nu^r,
\]

with \(J_\nu^r\equiv \frac{\partial g_{\mu}^r(r')}{\partial r_\nu}\) being the Jacobian of a suitable LPT.

Another interesting realization occurs when the space-time \((Q^4, g')\) is identified with the Minkowski space-time \((Q^4, \eta')\) represented in terms of general curvilinear coordinates \(r'\equiv \{r'^\mu\}\). In such a case its metric tensor is of the form

\[
\eta'_\mu\nu(r') = J_\mu^\alpha(r')J_\nu^\beta(r')\eta_{\alpha\beta},
\]

with \(\eta_{\alpha\beta}\) being the corresponding Minkowski metric tensor in orthogonal Cartesian coordinates. The corresponding tensor transformation laws (24) become now - in difference to the special NLTP introduced in Part 1 - of the general NLPT type (29).

It is interesting to stress that the same conclusions, i.e., in particular Eqs. (29), can actually be recovered following an alternative route. This is obtained by introducing suitable prescriptions on the transformations (15) and (16). Consider in fact the following possible realization of the said maps:

\[
\begin{align*}
gr_\mu^\alpha(r') + \int_{s_0}^s ds g_B^\mu(r', u', \sigma) &\equiv G_k^{\mu\nu}(r, \sigma), \\
r'^\mu &\equiv f_A^\mu(r') + \int_{s_0}^s ds f_B^\mu(r', u(\sigma), \frac{D u(\sigma)}{D s}),
\end{align*}
\]

where the functions \(g_A^\mu(r')\), \(g_B^\mu(r', u', \sigma), \frac{D u'(\sigma)}{D s}\) and \(f_A^\mu(r'), f_B^\mu(r', u(\sigma), \frac{D u(\sigma)}{D s})\) are suitably-defined real and smooth 4-vector functions. Notice that by construction Eqs. (38) are understood as being evaluated along the corresponding world-lines \(r'^\mu(s)\) and \(r'^\mu(s)\), and therefore they realize a Lagrangian representation of the NLPT. In particular, let us assume that the 4-accelerations enter at most linearly, namely

\[
\begin{align*}
gr_B^\mu(r, u', \sigma), \frac{D u'(\sigma)}{D s} &\equiv G_k^{\mu\nu}(r, \sigma), \\
f_B^\mu(r, u(\sigma), \frac{D u(\sigma)}{D s}) &\equiv F_k^{\mu}(r, \sigma).
\end{align*}
\]

being \(G_k^{\mu\nu}\) and \(F_k^{\mu}\) real functions of the form \(G_k^{\mu\nu}(r', [r', u'])\) and \(F_k^{\mu}(r, [r, u])\) respectively. Next, one notices that thanks to the validity of the kinematic constraints (15), the 4-accelerations \(\frac{D u'(\sigma)}{D s}\) and \(\frac{D u(\sigma)}{D s}\) must necessarily satisfy constraint equations of the type

\[
\begin{align*}
D' u'^\mu &\equiv u'^\nu H_{\mu}^{\nu} ds \equiv H_{\mu}^{\nu} dr'^\nu, \\
D u^\mu &\equiv u^\nu H_{\mu}^{\nu} ds \equiv H_{\mu}^{\nu} dr^\nu,
\end{align*}
\]

with \(H_{\mu}^{\nu}\) and \(H_{\mu}^{\nu}\) denoting suitable antisymmetric tensors, yet to be determined. As a consequence, the functional form of \(g_B^\mu\) and \(f_B^\mu\) becomes of the type

\[
\begin{align*}
g_B^\mu &\equiv A_{(g)\mu}^\nu \frac{dr'^\nu}{ds}, \\
f_B^\mu &\equiv B_{(g)\mu}^\nu \frac{dr^\nu}{ds},
\end{align*}
\]
where the real matrices $A^\mu_\nu$ and $B^\mu_\nu$ are defined as

$$A^\mu_\nu = C^\mu_k H^k_\nu,$$

$$B^\mu_\nu = F^\mu_k H^k_\nu. \quad (45)$$

We remark that, despite the matrices $H^\mu_\nu$ and $H^\nu_\mu$ being anti-symmetric in the upper and lower indices, $A^\mu_\nu$ and $B^\mu_\nu$ remain in principle arbitrary, i.e., without definite symmetry (or antisymmetry) index properties. In addition, both matrices $A^\mu_\nu$ and $B^\mu_\nu$ may still retain both local and non-local functional dependences. Therefore, Eqs. $(45)$ manifestly recover the form $(29)$, i.e., once Eqs. $(28)$ are invoked in Eq. $(26)$.

3 - GEDANKEN EXPERIMENT INTERPRETATION AND PHYSICAL IMPLICATIONS OF NLPT

In this section we analyze certain physical/mathematical implications of the general NLPT determined by $(29)$ (see THM.1) and the related NLPT phase-space transformations $(44)$.

The first one concerns the physical interpretation of the NLPT-phase-space transformation $(44)$ which can be achieved based on the realization of a GDE. As pointed out in the introduction a possible GDE of this type is the one which permits to identify the classical dynamical system (CDS) which is generated by the same phase-space transformation.

The existence of such a CDS is actually immediate. The conclusion follows in a straightforward way, being in fact analogous to the one displayed in the Introduction and realized in the context of SR by means of an $s$–dependent Lorentz boost. For this purpose, let us notice that the NLPT-phase transformation $(44)$ does indeed generate a CDS. In fact, for example, in the case of light cones, NLTPs can be defined for time-like world-lines which in each space-time can be covered by time-like (or if appropriate space-like) world-lines or their limit functions.

The two maps $(11)$ and $(12)$ are immediately determined (they are again not independent), both being prescribed for all $s_o, s \in I$. This realizes the desired GDE. More precisely: a) the first one, i.e., $(11)$ is obtained by considering the state $\{r^\mu(s), u^\nu(s)\}$ as a prescribed function of $s$ in a suitable interval $I$, so that at all $s$ in the same interval, $\{r^\mu(s), u^\nu(s)\}$ is uniquely determined by the same NLPT transformation; b) the second one represented by Eq. $(12)$ is obtained instead by considering the state $\{r^\mu(s), u^\nu(s)\}$ as a prescribed function of $s$, while $\{r^\mu(s), u^\nu(s)\}$ is then determined by the corresponding NLPT transformation. The two cases a) and b) identify respectively to the active and passive viewpoints for the same GDE.

Let us now analyze the conceptual implications of the GDE. For definiteness, let us assume that the two space-times, namely the “current” $(Q_4, g)$ and the “transformed” $(Q_4', g')$ one, are suitably prescribed, together with an arbitrary NLPT phase-space transformation $(5)$. The active viewpoint of the same GDE is realized by first assuming that the transformed phase-state (i.e., the transformed extended GR-frame) $\{r^\mu(s), u^\nu(s)\}$ is prescribed. This means that $\{r^\mu(s), u^\nu(s)\}$ remains in principle an arbitrary, but suitably pre-determined, function of $s$. Thus, for example, $u^\nu(s)$ can always be assumed to be constant for all $s$ in a prescribed interval $I$. Then, the GDE permits one to uniquely measure the time-evolution of the state $\{r^\mu(s), u^\nu(s)\}$ of the current space-time $(Q_4, g)$. In the passive viewpoint, instead, the current state (i.e., the current extended GR-frame) $\{r^\mu(s), u^\nu(s)\}$ is regarded as prescribed. In this case the GDE permits one to measure the behavior of the transformed state $\{r^\mu(s), u^\nu(s)\}$ for the same prescribed NLPT phase-space transformation $(5)$.

Let us now analyze some interesting physical aspects of the theory of NLPT presented here.

The first one concerns the physical domain of existence of NLTPs. As pointed out before, just as in the case of LPT, NLTPs must be defined in the accessible sub-domains of $(Q_4, g)$ and $(Q_4', g')$, namely the connected subsets which in each space-time can be covered by time-like (or if appropriate space-like) world-lines or their limit functions to be suitably defined. In fact, for example, in the case of light cones, NLTPs can be defined for time-like world-lines $r^\mu(s)$ which are endowed with a 4–velocity having arbitrarily-large spatial and/or time components, and therefore arbitrarily close to the same light trajectories. In addition, we stress that the structure of the two space-times themselves remains “a priori” arbitrary. Thus, for example, each of them may be characterized by different ensembles of event horizons, while NLTPs remain defined in the subsets internal or external to the same event horizons such that the mapped subsets have the same signature.

A further aspect to be mentioned concerns the tensor transformation laws with respect to the general NLPT-group $\{P_g\}$. Indeed Eqs. $(23)$–$(24)$ are the prototypes of tensor transformations laws which can be extended to virtually arbitrary higher–rank tensors. Thus, as an illustration, let us consider the case of a 4–scalar field $\Phi(r)$, i.e., a function which remains invariant under the action of an arbitrary transformation of the group $\{P_g\}$, for example identified with the special NLPT

$$r^\mu = r^\mu(r^\nu(s), s) = r^\mu(s) + \Delta r^\mu(s),$$

(47)
with \(\Delta r^{\mu\nu}(s)\) being defined by Eq. (15). Then, denoting as \(\Phi'(r')\) (respectively \(\Phi(r)\)) the realization of the same scalar field in the GR-reference frame \(r^{\mu}\) (respectively \(r'^{\mu}\)), it follows that the Eulerian equation

\[
\Phi'(r') = \Phi(r) \tag{48}
\]

must hold identically. On the other hand, on the rhs of the same equation \(r \equiv \{r^{\mu}\}\) is to be considered a function of \(r' \equiv \{r'^{\mu}\}\) when represented via the the special NLPT given above. It follows that \(\Phi(r(s)) = \Phi(r'^{\mu}(s) + \Delta r'^{\mu}(s))\) when cast in Lagrangian form, i.e., it is parametrized in terms of the world-line \(r^{\mu}(s)\) or \(r'^{\mu}(s)\) respectively and the corresponding proper time \(s\). As a result, Eq. (48) yields also the relationship expressed in Lagrangian form, i.e., in terms of the world-lines \(r(s)\) and \(r'(s)\). Since by construction \(r(s)\) is a non-local function of \(r'(s)\) and the initial and transformed fields \(\Phi(r(s))\) must still coincide identically, i.e.,

\[
\Phi'(r'(s)) = \Phi(r(s)) = \Phi(r'^{\mu}(s) + \Delta r'^{\mu}(s)), \tag{49}
\]

it follows that \(\Phi'(r'(s))\) becomes necessarily a non-local function of \(r'^{\mu}(s)\). To determine the corresponding Eulerian fields in terms of Eq. (48) it is sufficient to represent the proper time \(s\) in terms of the instantaneous 4–position \(r' \equiv \{r'^{\mu}\}\), so that \(s = s(r')\). The way how this can be done, once the world-line \(r'^{\mu}(s)\) is considered prescribed, is discussed in the Appendix. Once the representation \(s = s(r')\) is introduced, it follows that the rhs of Eq. (49) determines actually a function of \(r' \equiv \{r'^{\mu}\}\) only, namely

\[
\Phi(r'^{\mu} + \Delta r'^{\mu}(s)) \equiv \tilde{\Phi}(r'), \tag{50}
\]

so that Eq. (48) implies

\[
\Phi'(r') = \tilde{\Phi}(r') \tag{51}
\]

too. In other words, the scalar field \(\Phi(r)\) and hence \(\Phi'(r')\) become formally a composite and non-local function of \(r' \equiv \{r'^{\mu}\}\).

Finally, a number of comments and suggestions related to the form of the general NLPT, realized in particular in THM.1 and in the subsequent discussion, should be mentioned. These include:

1) The two matrices \(A^\alpha_{(g)\nu}(r', r)\) and \(B^\alpha_{(g)\nu}(r, r')\) identify the acceleration-dependent contributions in the Jacobian matrices.

2) It must be stressed that the two involved metric tensors \(g_{\mu\nu}\) and \(g'_{\mu\nu}\) remain arbitrary. For example, one can always require that both metric tensors are particular solutions of the Einstein equation. In this case Eqs. (21) can be interpreted as equations for the still unknown Jacobian matrix, to be determined accordingly. This includes as a particular case the one in which for example the transformed metric tensor \(g'_{\alpha\beta}\) coincides with the Minkowski metric tensor. If \(g_{\mu\nu}(r)\) and \(g'_{\mu\nu}(r')\) are realizations holding for the two different space-times \((Q^4, g)\) and \((Q'^4, g')\) when they are referred respectively to the coordinate systems \(r^{\mu}\) and \(r'^{\mu}\), the tensor transformation laws (24) must hold. If the vector functions \(g_{\mu\nu}(r')\) and \(f_{\mu\nu}^\alpha(r)\) are considered prescribed, then the first of these equations becomes

\[
g_{\mu\nu}(r) = \left[ \frac{\partial g^\alpha_{(g)\nu}(r')}{\partial r'^\alpha} + A^\alpha_{(g)\mu}(r', r) \right] \left[ \frac{\partial g^\beta_{(g)\nu}(r')}{\partial r'^\beta} + A^\beta_{(g)\nu}(r', r) \right] g_{\alpha\beta}(r'), \tag{52}
\]

which, for special NLPT (see for example Eqs. (53)), reduces simply to

\[
g_{\mu\nu}(r) = \left[ \delta^\alpha_{\mu} + A^\alpha_{(g)\mu}(r', r) \right] \left[ \delta^\beta_{\nu} + A^\beta_{(g)\nu}(r', r) \right] g_{\alpha\beta}(r'). \tag{53}
\]

Eq. (52), or alternatively (53), yields actually a set of implicit, i.e., integral, equations for the components of the same matrix. The explicit construction of the solution for \(A^\alpha_{(g)\mu}\) actually requires representing it in Eulerian form. This involves as before (see related discussion in the previous section) representing the proper-time \(s\) in terms of the instantaneous 4–position \(r' \equiv \{r'^{\mu}\}\), so that \(s = s(r')\). We refer again for this purpose to the discussion reported in the Appendix.

3) An alternative interpretation is the one in which one of the two metric tensors, say \(g'_{\alpha\beta}(r')\), is prescribed together with the Jacobian \(M^\alpha_{\mu}(r')\) so that Eqs. (24) provides an explicit representation for the transformed metric tensor \(g_{\mu\nu}(r)\). In this case an interesting remaining issue concerns its possible identification as an admissible particular solution of the Einstein equation corresponding to prescribed physical sources.

4) The problem of the construction of the NLPT - or, better, the corresponding special NLPT to which in principle it should always be possible to refer - amounts therefore to look for the still unknown matrix \(A^\beta_{(g)\nu}(r', r)\).
4 - APPLICATION #1: DIAGONAL METRIC TENSORS

The first application to be considered concerns the construction of a NLPT mapping two connected and time-oriented space-times \((Q^4, g)\) and \((Q^4, g')\) both having diagonal form with respect to suitable sets of coordinates. More precisely we shall require that:

- When \((Q^4, g)\) and \((Q^4, g')\) are referred to the same coordinate systems, both are realized by diagonal metric tensors

\[
\begin{align*}
  g_{\mu\nu}(r) & \equiv \text{diag} (S_0(r), -S_1(r), -S_2(r), -S_3(r)) \\
  g'_{\mu\nu}(r') & \equiv \text{diag} (S'_0(r'), -S'_1(r'), -S'_2(r'), -S'_3(r'))
\end{align*}
\]  

(54)

respectively. The accessible subsets are as follows: a) for \((Q^4, g')\) is that in which for all \(\mu = 0, 3, S'_\mu(r') > 0\); b) for \((Q^4, g)\) is either the set in which for all \(\mu = 0, 3, S_\mu(r) > 0\) or the other one in which \(S_0(r) < 0, S_1(r) < 0, S_2(r) > 0\) and \(S_3(r) > 0\).

- \((Q^4, g)\) and \((Q^4, g')\) are intrinsically different, i.e., that the corresponding Riemann curvature tensors \(R_{\mu\nu}(r)\) and \(R'_{\mu\nu}(r')\) cannot be globally mapped in each other by means of any LPT. This means that a mapping between the accessible subsets of the said space-times can only possibly be established by means of a suitable NLPT.

- Two occurrences are considered: a) the same-signature case in which both \((Q^4, g')\) and \((Q^4, g)\) have the same Lorentzian signature \((+, -, -, -)\); b) the opposite-signature case in which \((Q^4, g')\) and \((Q^4, g)\) have signatures \((+, -, -, -)\) and \((-+, +, +)\) respectively.

In validity of Eqs. (54) the tensor transformation equation (24) take obviously the general form:

\[
\begin{align*}
  S_\mu(r) = \left( M^{-1}_{(g)} \right)_{\mu}^{\alpha} (r, r') \left( M_{(g)}^{-1} \right)^{\alpha}_{(\mu)} (r, r') S'_\alpha(r'), \\
  S'_\mu(r') = M_{(g)\mu}^{\alpha}(r', r) M_{(g)\nu}^{\beta}(r', r) S_\alpha(r),
\end{align*}
\]  

(55)

where manifestly \(M_{(g)\mu}^{\alpha}(r', r) = M_{\mu}^{\alpha}(r', r)\) and \(\left( M_{(g)}^{-1} \right)^{\alpha}_{(\mu)} (r, r') = \left( M^{-1} \right)^{\alpha}_{(\mu)} (r, r')\) as corresponds to the case of a special NLTP. For such a type of space-times in the following we intend to display a number of explicit particular solutions of Eqs. (55) for the Jacobian \(M_{\mu}^{\alpha}\) and its inverse \(\left( M^{-1} \right)^{\alpha}_{\mu}\), and to construct also the corresponding NLPT-phase-space transformations.

**Same-signature diagonal NLTP**

In the case in which \((Q^4, g)\) and \((Q^4, g')\) have the same signatures, it is immediate to show that a particular solution of Eqs. (55) in the accessible subsets of \((Q^4, g)\) and \((Q^4, g')\) is provided by a diagonal Jacobian matrix, i.e., of the form

\[
M_{\mu}^{\alpha}(r', r) = M_{\mu}^{\alpha}(r', r) \delta_{\alpha}^{\mu} = \left[ \delta_{\mu}^{\alpha} + A_{\mu}(r', r) \right] \delta_{\mu}^{\alpha}.
\]  

(56)

Indeed from Eqs. (55) one finds

\[
M_{(\mu)}^{\mu}(r', r) = \frac{1}{\left( M^{-1} \right)^{\mu}_{(\mu)}} = \sqrt{\frac{S'_\mu(r')}{S_{(\mu)}(r)}},
\]  

(57)

where \(\frac{S'_\mu(r')}{S_{(\mu)}(r)} > 0\) in the accessible subsets. In terms of Eqs. (5), or equivalent (26), one then determines the corresponding special NLPT, namely

\[
\begin{align*}
  r^\mu(s) = r'^\mu(s_{o}) + \int_{s_{o}}^{s} d\bar{s} \sqrt{\frac{S'_\mu(r')}{S_{(\mu)}(r)} u^{\mu}(\bar{s})}, \\
  r'^\mu(s) = r^\mu(s_{o}) + \int_{s_{o}}^{s} d\bar{s} \sqrt{\frac{S_{(\mu)}(r)}{S'_\mu(r')} u^{\mu}(\bar{s})},
\end{align*}
\]  

(58)

as well as the corresponding \(4\) — velocity transformation.
Let us now consider a possible physical realizations for the space-times \((Q^4, g)\) and \((Q^4, g')\) and the corresponding metric tensors \(g_{\mu\nu}(r)\) and \(g'_{\mu\nu}(r')\) respectively. Examples are provided by the Schwarzschild or alternatively the Reissner-Nordström space-times, both being characterized by a single event horizon. In terms of the spherical coordinates \((r, \vartheta, \varphi)\) an analogous (Schwarzschild-analog) representation holds of the form \(g_{\mu\nu}(r) \equiv \text{diag}((S_0(r), -S_1(r), -S_2(r), -S_3(r)))\) with

\[
\begin{align*}
S_0(r) &= f(r) \\
S_1(r) &= \frac{r}{f(r)} \\
S_2(r) &= r^2 \\
S_3(r) &= r^2 \sin^2 \vartheta
\end{align*}
\]

and where in the two cases \(f(r)\) is identified respectively with

\[
\begin{align*}
&f(r) = \left(1 - \frac{r_s}{r}\right), \\
&f(r) = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right).
\end{align*}
\]

Here, \(r_s = 2GM/c^2\) is the Schwarzschild radius and \(r_Q = \sqrt{\frac{2GM}{4\pi\varepsilon_0}}\) a characteristic length scale, with \(Q\) being the electric charge and \(1/4\pi\varepsilon_0\) the Coulomb coupling constant. Introducing the curvilinear coordinates \((r^0, r^1 \equiv r, r^2 \equiv r\vartheta, r^3 \equiv r\varphi\sin \vartheta)\), here referred to as pseudo-spherical coordinates, one obtains \(r^2 d\Omega^2 = (dr^2) + (dr^3)^2\). It follows that in Eqs. (60) and (61), \(S_2(r)\) and \(S_3(r)\) are replaced with

\[
\begin{align*}
S_2(r) &= 1, \\
S_3(r) &= 1.
\end{align*}
\]

In both cases, the transformed space-time \((Q^4, g')\) is assumed again Schwarzschild-analog, namely of the type (59). Expressed in the pseudo-spherical coordinates this is prescribed to be

\[
\begin{align*}
S_0'(r') &= f'(r') \\
S_1'(r') &= \frac{1}{f'(r')} \\
S_2'(r') &= 1 \\
S_3'(r') &= 1
\end{align*}
\]

Here \(f'(r')\) is assumed to be an analytic function having \(n > 1\) positive simple roots \(r'_1 < r'_2 < \ldots < r'_n\) in the positive real axis \([0, +\infty)\) and such that \(f'(r') > 0\) for \(r' > r'_n\). In particular, we shall require that the Schwarzschild radius occurs in the interval

\[
r'_1 < r_s < r'_n.
\]

The admissible sub-domains of \((Q^4, g)\) and \((Q^4, g')\), where NLPTs can possibly be established between the two space-times, are therefore defined respectively by the inequalities \(r > r_s\) and \(r' > r'_n\). In these subsets the transformation matrix \(A''_{\mu}(r', r)\) becomes:

\[
\begin{align*}
A''_0(r', r) &= \sqrt{\frac{f'(r')}{f(r)}} - 1, \\
A''_1(r', r) &= \sqrt{\frac{f(r)}{f'(r')}} - 1, \\
A''_2(r', r) &= \sqrt{\frac{1}{1} - 1} = 0, \\
A''_3(r', r) &= \sqrt{\frac{1}{1} - 1} = 0,
\end{align*}
\]

where in the first terms on the rhs of the previous equations the positive values of the square roots have been taken. Therefore, the NLPT corresponding to Eqs. (66)–(69) is the identity transformation as far as the coordinates \(r^2\) and \(r^3\) are concerned. The non-trivial contributions giving rise to non-local terms in Eqs. (26) are produced therefore only by the time and radial components of the 4-velocity, i.e., \(u^0\) and \(u^1\) only. The following physical interpretation is proposed:
The special NLPT corresponding to Eqs. (66)-(69) is only defined in the accessible subset of the space-time namely when \( r' > r_n' \) and \( r > r_s \), respectively occur.

The effect of the special NLPT produced by Eqs. (66)-(69) is that of mapping the accessible subsets of Schwarzschild or Reissner-Nordström space-time in the corresponding accessible subset of a Schwarzschild-analog space-time. The basic feature of the transformed space-time is that of exhibiting \( n > 1 \) event-horizons instead of a single one as in the initial space-time.

The physical origin for the generation of such an effect is the special NLPT introduced here, which in turn arises when non-local effects are included in Eq. (6) which are carried only by the time and radial components of the 4-velocity. In particular, assuming that the NLPT is of the form determined according to the requirements (29) it follows that Eqs. (66)-(69) correspond to the case in which only a tangential 4-acceleration \( a^\mu = \frac{Dg'^\mu}{Ds} \) can occur, namely in which its only non-vanishing components correspond to \( \mu = 2, 3 \).

A final remark must be made concerning the limit \( \lim_{r' \rightarrow r_n'} \) in Eq. (67) and respectively \( \lim_{r' \rightarrow r_s'} \) in Eqs. (66), where \( r_n' \) and \( r_s' \) are the largest roots of the equations \( f'(r') = 0 \) and \( f(r) = 0 \). In terms of the pseudo-spherical coordinates the previous limits do not exist and therefore the limit NLPT is not defined on the event horizons. Nevertheless, these divergences can be cured by preliminarily recurring to a suitable coordinate system, which in the case of the Schwarzschild metric can be identified with the Kruskal–Szekeres coordinates [9].

### Opposite-signature NLTP

Let us now consider the case in which \((Q^4, g)\) and \((Q'^4, g')\) have opposite signatures, namely respectively \((- , + , + , + )\) and \((+, - , - , - )\) while the metric tensors are still diagonal when expressed with respect to the same coordinate systems, i.e., are of the form [5]. It follows that in the accessible subset of \((Q'^4, g')\) it occurs respectively that

\[
\begin{align*}
S_0(r) &< 0, \\
S_1(r) &< 0.
\end{align*}
\]

In this case it is immediate to show that in the accessible subsets of \((Q^4, g)\) and \((Q'^4, g')\) a particular solution of Eqs. (55) is provided by a Jacobian matrix of the form

\[
\begin{align*}
M_1^0(r', r) &= \frac{1}{(M^{-1})^0} = \sqrt{- \frac{S_0'(r')}{S_0(r)}}, \\
M_1^0(r', r) &= \frac{1}{(M^{-1})^0} = \sqrt{- \frac{S_0'(r')}{S_1(r)}}, \\
M_2^2(r', r) &= \frac{1}{(M^{-1})^3} = \sqrt{ \frac{S_2'(r')}{S_2(r)}}, \\
M_3^3(r', r) &= \frac{1}{(M^{-1})^3} = \sqrt{ \frac{S_3'(r')}{S_3(r)}},
\end{align*}
\]

where \(- \frac{S_3(r)}{S_0'(r)} > 0 \) and \(- \frac{S_0'(r)}{S_1'(r)} \) in the accessible subsets. The corresponding special NLPT follows immediately from Eqs. (60), or equivalent (26). Once again a possible application is provided by Schwarzschild-analog space-times. More precisely let us consider the case in which:

**A** the space-time \((Q'^4, g')\) is assumed again Schwarzschild-analog of the type [5], so that in pseudo-spherical coordinates it is given again by Eqs. (54). In particular in the accessible subset of \((Q'^4, g')\) we shall require

\[
\begin{align*}
S_0'(r') &= f'(r') > 0, \\
S_1'(r') &= \frac{1}{f'(r')} > 0.
\end{align*}
\]

**B** the space-time \((Q^4, g)\) is the Schwarzschild one, the accessible subset being such that

\[
\begin{align*}
S_0(r) &= 1 - \frac{r}{r_s} < 0, \\
S_1(r) &= \frac{1}{1 - \frac{r}{r_s}} < 0.
\end{align*}
\]
As a consequence, the Jacobian becomes
\[
M_0^0(r', r) = \frac{1}{(M^{-1})^0_0} = \sqrt{-\frac{1}{(1 - \frac{r_s}{r}) f'(r')}}, \\
M_0^1(r', r) = \frac{1}{(M^{-1})^0_1} = \sqrt{-\left(1 - \frac{r_s}{r}\right) f'(r')}, \\
M_2^2(r', r) = \frac{1}{(M^{-1})^2_2} = 1, \\
M_3^3(r', r) = \frac{1}{(M^{-1})^3_3} = 1.
\]

Therefore, in this case the resulting special NLPT maps the interior domain of the Schwarzschild space-time, namely its Black Hole domain, onto the exterior domain of a Schwarzschild-analog space-time. As a final comment, it must be stressed that the starting equations adopted in this Section, namely Eqs. (56), can be in principle easily reformulated when arbitrary different coordinate systems are adopted for representing the two space-times \((Q^4, g)\) and \((Q^4, g')\). Although details are here omitted for brevity, it is worth mentioning that this extension can easily be accomplished adopting the general NLPT-theory developed here.

5 - APPLICATION #2: DIAGONALIZATION OF METRIC TENSORS

As a second example, the problem of diagonalization of a non-diagonal metric tensor is posed in the framework of NLPT-theory. More precisely, this concerns the construction of a NLPT mapping two connected and time-oriented space-times \((Q^4, g)\) and \((Q^4, g')\). Here we shall require that when \((Q^4, g)\) and \((Q^4, g')\) are referred to the same coordinate systems they are realized by the metric tensors
\[
g_{\mu\nu}(r) \equiv \text{diag} (S_0(r), -S_1(r), -S_2(r), -S_3(r)),
\]
\[
g'_{\mu\nu}(r') = \begin{bmatrix}
S'_0(r') & S'_{03}(r') \\
-S'_1(r') & -S'_2(r') \\
S'_{03}(r') & -S'_3(r')
\end{bmatrix},
\]
respectively. The accessible subsets are assumed to be both for \((Q^4, g')\) and \((Q^4, g)\) as follows: all \(\mu = 0, 3, \) \(S'_\mu(r') > 0\) and \(S_\mu(r) > 0\).

As before, the realization of the NLPT which maps the two metric tensors is not unique. A possible choice is provided by a special NLPT of the form
\[
dr^0 = \left(1 + A^0_{(g)0}\right) dr^0 + A^0_{(g)3} dr^3,
\]
\[
dr^i = \left(1 + A^i_{(g)i}\right) dr^{(i)},
\]
for \(i = 1, 2, 3\), namely such that
\[
r^0(s) = r^0(s) + \int_{s_0}^{s} ds \left[A^0_{(g)0}(r'(\tau), r(\tau)) u^0(\tau) + A^0_{(g)3}(r'(\tau), r(\tau)) u^3(\tau)\right],
\]
\[
r^i(s) = r^i(s) + \int_{s_0}^{s} ds A^i_{(g)i}(r'(\tau), r(\tau)) u^{(i)}(\tau),
\]
where again the indices in brackets are not subject to the summation rule. The transformation bringing \(r'^{\mu}\) in \(r^{\mu}\) will be referred to as diagonalizing NLPT. The transformation equations for the matrix elements \(A^0_{(g)0}, A^0_{(g)3}\) and \(A^i_{(g)i}\), for \(i = 1, 2, 3\), are therefore
\[
S'_j(r') = \left(1 + A^j_{(g)j}(r', r)\right)^2 S_j(r),
\]
\[
S'_3(r') = \left(1 + A^3_{(g)3}(r', r)\right)^2 S_3(r) - \left(A^0_{(g)3}(r', r)\right)^2 S_0(r),
\]
\[
S'_{03}(r') = A^0_{(g)3}(r', r) A^0_{(g)0}(r', r) S_0(r),
\]
\[
S'_{03}(r') = \left(1 + A^j_{(g)j}(r', r)\right)^2 S_j(r) - \left(A^0_{(g)3}(r', r)\right)^2 S_0(r).
for \( j = 0, 1, 2 \). The first set of equations (97) has a formal solution of the type

\[
A^j_{(g)(j)}(r', r) = \sqrt{\frac{S'_j(r')}{S_j(r)}} - 1. \tag{90}
\]

The third equation (98) gives then

\[
A^0_{(g)3}(r', r) = \frac{S'_0(r')}{S_0(r)} \left[ \sqrt{\frac{S'_0(r')}{S_0(r)}} - 1 \right]^{-1}. \tag{91}
\]

Finally, Eq. (99) delivers

\[
A^3_{(g)3}(r', r) = \sqrt{\frac{S'_3(r') + (A^0_{(g)3}(r', r))^2 S_0(r)}{S_3(r)}} - 1. \tag{92}
\]

The signs of the square roots in the previous equations have been chosen in such a way to recover the correct result for identity transformations.

A number of remarks must be made:

1) Also the present application can be in principle reformulated adopting arbitrary different coordinate systems for the representation of the space-times \((Q^4, g)\) and \((Q'^4, g')\). This ultimately involves adopting the general NLPT-theory developed here.

2) The transformation (85)-(86) is defined provided the inequality

\[
\sqrt{\frac{S'_0(r')}{S_0(r)}} - 1 \neq 0 \tag{93}
\]

holds. In this case in fact all the matrix elements \( A^i_0 \) determined above are real and smooth functions.

3) A solution satisfying the inequality (93) can always be found by suitably prescribing \( S_0(r) \) once \( S'_0(r') \) is considered fixed.

4) As an alternate possibility, in case the condition (93) is not satisfied, is to look for another possible realization of the transformation (85)-(86). The general solution can be cast in the form

\[
dr^0 = \left(1 + A^0_{(g)0} \right) dr^0 + A^0_{(g)3} dr^3, \tag{94}
\]

\[
dr^i = \left(1 + A^0_{(g)i} \right) dr^{r(i)}, \tag{95}
\]

\[
dr^3 = \left(1 + A^3_{(g)3} \right) dr^3 + A^3_{(g)0} dr^0, \tag{96}
\]

for \( i = 1, 2 \), namely such that

\[
r^0(s) = r^0(s) + \int_{s_0}^{s} ds \left[ A^0_{(g)0}(r^0(\varphi), r(\varphi)) u^0(\varphi) + A^0_{(g)3}(r^0(\varphi), r(\varphi)) u^3(\varphi) \right], \tag{97}
\]

\[
r^i(s) = r^i(s) + \int_{s_0}^{s} ds A^i_{(g)i}(r^0(\varphi), r(\varphi)) u^{r(i)}(\varphi), \tag{98}
\]

\[
r^3(s) = r^3(s) + \int_{s_0}^{s} ds A^3_{(g)3}(r^0(\varphi), r(\varphi)) u^3(\varphi) + A^3_{(g)0}(r^0(\varphi), r(\varphi)) u^0(\varphi) \tag{99}
\]

The resulting equations can be immediately solved.

5) The diagonalization of the Kerr metric tensor expressed in spherical coordinates, as well as the Kerr-Newman and analogous Kerr-like solutions, can be carried out in terms of either a transformation of the type (85)-(86) or (97)-(99).

6) Regarding the physical interpretation of the differential equations (83)-(84) we notice that the first equation implies that the time component of the 4-velocity in the initial frame is modified by the combined effects of time- and 3-components of the 4-velocity in the transformed frame. In the case of the Kerr metric, in particular, the latter corresponds to an azimuthal component of the 4-velocity. Therefore, the corresponding non-local coordinate
transformation (85)-(86) produces a modification of the coordinate time $r^0(s)$ taking into account also the contribution of the azimuthal velocity.

7) Also for the diagonalizing NLPT a teleparallel realization can be given. This follows by identifying now the space-time $(Q^4, g)$ with the Minkowski space-time. The solution for the Jacobian of a such a transformation is obtained from Eqs.(30)-(32) by setting $S_\mu(r) = 1$ identically. This means that it is always possible to transform a non-diagonal metric tensor into the Minkowski one by means of the inverse diagonalizing NLPT transformation.

8) Finally, an interesting comparison is possible with the so-called Newman-Janis algorithm [12, 15, 16]. As is well known (see also related discussion in Part 1) this algorithm can be used to diagonalize non-diagonal metric tensors and is frequently used in the literature for the purpose of investigating a variety of standard or non-standard GR black-hole solutions [13, 14]. Its basic feature is that adopting a complex coordinate transformation, a feature which effectively inhibits its physical interpretation and puts in doubt its very validity. In contrast, within the present NLPT approach, the physical consistency of the transformation approach is preserved. Hence, the present conclusions seem particularly rewarding. Indeed, based on the NLPT-approach indicated above, the difficulties and physical limitations of the complex Newman-Janis algorithm are effectively avoided by adopting the NLPT-theory. This is of paramount importance for theoretical and astrophysical applications, such as the physics around rotating black holes and gravitational waves.

6 - CONCLUSIONS

In this paper the problem has been posed of extending the class of local point transformations (LPT) on which the general covariance principle (GCP) lying at the bases of General Relativity (GR) is based. Such transformations in the customary formulation of GR map in each other different reference frames, i.e., coordinate systems. However, theoretical motivations suggest the extension of the traditional concept of reference frame adopted previously in GR based on the identification of an extended class of point transformations. These have been constructed relying on a number of physical requirements (Requirements #1-#3), prescribing in particular the functional form of the corresponding Jacobians, and referred to as non-local point transformations (NLPT). While extending the class of local point transformations (LPT) on which both the differential geometry and the GCP rely, NLPT permit one to map intrinsically physically-different space-times, i.e., characterized by different metric and curvature Riemann tensors. Two characteristic features of these transformations emerge. The first one is their non-locality, which appears both in their Lagrangian and Eulerian forms. This is due to a non-local linear dependence with respect to the transformed 4-velocity. The second one lies in their Jacobians. In difference with the case of LPT, the latter by construction cannot be identified with gradient operators. Nevertheless, since the same Jacobians remain velocity-independent, tensor transformation laws can be still determined, which are based on the transformation properties holding for the infinitesimal displacements and the corresponding 4-velocities. In addition, a physical interpretation of NLPT has been pointed out which is based on an ideal (Gedanken) experiment.

Two different applications of the theory have been proposed, which concern the mapping between diagonal metric tensors and the diagonalization of non-diagonal metric tensors. Both these problems cannot be approached in the framework of customary LPT, while their solution becomes straightforward and physically-consistent when the theory of NLPT developed here is invoked.

These features, in our view, suggest the theory presented here as an extremely promising and innovative research topic, which might eventually give rise to a novel scientific mainstream in GR. The theory developed here is in fact susceptible of a plethora of potential applications, besides its natural framework, i.e., GR. In particular, general NLPT-theory provides the theoretical basis for important possible subsequent developments ranging from classical relativistic mechanics and electrodynamics [17, 22], quantum theory of extended particle dynamics [24], relativistic kinetic theory [21], to cosmology as well as relativistic quantum mechanics and quantum gravity.

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### APPENDIX - EULERIAN AND LAGRANGIAN FORMS OF TENSOR FIELDS

From the definition of the Riemannian distance \( ds \) (see Eq. (17)) it follows that

\[
 ds = g_{\mu\nu}(r) \frac{dr^\mu(s)}{ds} dr^\nu(s),
\]

or equivalently

\[
 ds = g'_{\mu\nu}(r) \frac{dr'^\mu(s)}{ds} dr'^\nu(s).
\]

Hence, integrating and letting \( s_o = 0 \) one obtains:

\[
 s - s_o \equiv s = \int_{r(s_o)}^{r'(s)} g_{\mu\nu}(r) \frac{dr^\mu(s')}{ds'} dr^\nu,
\]

where the integration variable \( r' \) belongs to the 4-dimensional subset of \((Q^4, g)\), the set having boundaries \( r'(s_o) \) and \( r'(s) \). Notice furthermore that in the integrand on the rhs of the previous equation the variable \( s' \) is to be considered as dependent from the integration variable, i.e., of the form \( r'(s') \). Indeed, from Eq. (102) it follows manifestly also that

\[
 s' = \int_{r'(s_o)}^{r'(s)} g_{\mu\nu}(r) \frac{dr^\mu(s)}{ds} dr^\nu.
\]

Hence, Eq. (102) implies necessarily that

\[
 s = s(r),
\]

where \( r \equiv r'(s) \) and similarly form Eq. (103) it follows that \( s' = s'(r'(s)) \).

Let us now consider an arbitrary tensor field \( A_{\mu\nu} \) - for example to be identified with the metric tensor \( g_{\mu\nu} \) as in Section 3 - which when expressed in Lagrangian form is assumed to take the form \( A_{\mu\nu}(r, s) \). Here \( r \) denotes \( r = r(s) \equiv \{r'(s)\} \), while for all \( s \in I \equiv \mathbb{R} \), \( s \) is the proper-time which is associated with the time-like world line.

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