A systemic shock model for too big to fail financial institutions

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Abstract

In this paper we study the distributional properties of a vector of lifetimes in which each lifetime is modeled as the first arrival time between an idiosyncratic shock and a common systemic shock. Despite unlike the classical multidimensional Marshall-Olkin model here only a unique common shock affecting all the lifetimes is assumed, some dependence is allowed between each idiosyncratic shock arrival time and the systemic shock arrival time. The dependence structure of the resulting distribution is studied through the analysis of its singularity and its associated copula function. Finally, the model is applied to the analysis of the systemic riskiness of those European banks classified as systemically important (SIFI).

JEL Classification: C51; G32

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1 Introduction

In this paper we consider a particular generalization of the multidimensional Marshall-Olkin distribution (Marshall and Olkin, 1967) in the specific case in which, apart from the idiosyncratic ones, only one common shock is considered whose occurrence causes the simultaneous end of all lifetimes. More specifically, if \((X_0, X_1, \ldots, X_d)\) are positive random variables that represent the arrival times of some shocks, then we consider, as resulting lifetimes the random variables \(T_1, \ldots, T_d\) defined as \(T_j = \min(X_0, X_j), j = 1, \ldots, d\).

In the Marshall-Olkin model the underlying shocks arrival times are assumed to be independent and exponentially distributed. Many extensions exist in the literature in order to consider marginal distributions different from the exponential one and to include some dependence among the underlying shocks arrival times, even in the more general case where additional systemic shocks involving subsets of the lifetimes \(T_1, \ldots, T_d\) are assumed. Among them, the scale-mixture of the Marshall-Olkin distribution, introduced in Li (2009), is obtained by scaling, through a positive random variable, a random vector distributed according to the Marshall-Olkin distribution: this is equivalent to assume that the underlying shocks

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arrival times have a dependence structure given by an Archimedean copula with a generator that is the Laplace transform of the mixing variable. Scale-mixtures of the Marshall-Olkin distributions have also been considered in Mai et al. (2013) where, in the exchangeable case, a different construction is presented involving Lévy subordinators. On the other side, the approach of allowing for general marginal distributions in place of the exponential one, even preserving the independence, is studied in Li and Pellerey (2011) in the bivariate case and extended to the multidimensional case in Lin and Li (2014): they call their distribution generalized Marshall-Olkin distribution. Scale-mixtures of the generalized Marshall-Olkin distribution are considered in Mulinacci (2015), with the aim, again, to introduce a specific Archimedean dependence (the generator is again given by the Laplace transform of the mixing variable) among the underlying shocks arrival times: the case of an underlying Archimedean dependence with a fully general generator is analyzed in Mulinacci (2017). The union of Marshall-Olkin and Archimedean dependence structures is also studied in Charpentier et al (2014).

The main drawback of all these extensions is that they assume an underlying exchangeable dependence. Aiming at considering an asymmetric underlying dependence, in Pinto and Kolev (2015), when \( d = 2 \), the case in which \( X_1 \) and \( X_2 \) are dependent, while the external shock \( X_0 \) is independent of \((X_1, X_2)\) is studied.

The specific generalization of the Marshall-Olkin distribution presented in this paper is characterized by an asymmetric dependence in the vector \((X_0, \ldots, X_d)\) that goes in the opposite direction with respect to the one considered in Pinto and Kolev (2015): \( X_1, \ldots, X_d \) are assumed to be independent while a particular pairwise dependence is assumed between each \( X_j, j = 1, \ldots, d \) and \( X_0 \). The pairwise dependence results from the assumption \( X_0 = \min_{j=0,1,\ldots,d} Y_j \) where \( Y_0, \ldots, Y_d \) are mutually independent while each \( Y_j \) is correlated with \( X_j, j = 1, \ldots, d \).

A possible branch of application of this model is in the reliability modeling of mechanical or electronic systems, and consequently, in the modeling of the resulting operational and actuarial risk. Consider for example \( d \) working machines (or electronic components) \( M_j, j = 1, \ldots, d \), all separately connected with a same machine \( M_0 \) so that if \( M_0 \) stops to working, immediately the same occurs for all the other machines. Assuming the classical Marshall-Olkin model, the failure of a single machine \( M_j, j = 1, \ldots, d \) does not influence the failure of the machine \( M_0 \) or of the remaining \( M_i, i = 1, \ldots, d, i \neq j \). Conversely, in our model, the failure of one of the \( M_j, j = 1, \ldots, d \) can influence the probability of failure of \( M_0 \), and, consequently, of the collapse of the whole system. This is the case in which some electronic or mechanical disease in one of the \( M_j, j = 1, \ldots, d \), being it connected with \( M_0 \), may worsen or interrupt the functioning status of \( M_0 \) to which can follow its failure.

Another branch of application of this model is credit risk. There is a wide literature on applications of the Marshall-Olkin model and its generalizations to credit and actuarial risk (see Giesecke, 2003, Lindskog and McNeil, 2003, Elouerkhoaoui, 2007, Mai and Scherer, 2009, Baglioni and Cherubini, 2013, Bernhart et al., 2013 and Cherubini and Mulinacci, 2014).
Given the specific type of assumed dependence, the probabilistic model analyzed in this paper looks particularly suitable for the analysis of the joint lifetimes of the so-called Systemically Important Financial Institutions (SIFI) for which the default (or the proximity to it) of one of them, is directly correlated with the collapse of the whole system.

In this paper, we first discuss the survival distribution of the underlying vector of lifetimes \((X_0, \ldots, X_d)\): we study the associated copula function and recover expressions for the Kendall’s function and Kendall’s tau of the pairs \((X_0, X_j), j = 1, \ldots, d\). Then we focus on the resulting joint survival distribution of the lifetimes \((T_1, \ldots, T_d)\): we analyze the probability of simultaneous end of all lifetimes (that is the singularity of the distribution) and the dependence properties through the analysis of the pairwise Kendall’s function and Kendall’s tau. We do not make, in principle, any assumption on the marginal distributions of the underlying shocks arrival times and on the underlying dependence structure: however, in order to obtain closed formulas, we restrict the analysis to particular classes of marginal distributions (that include the exponential one as a particular case) and to Archimedean bivariate copulas. Finally we present and discuss an application to the analysis of the systemic riskiness of European banks classified as SIFI by the Financial Stability Board: of course, the type of systemic risk modeled in this paper is very specific and the method is meant as an additional tool to analyze systemic risk with respect to already existing ones.

The paper is organized as follows. In section \(\S2\) we present and analyze the shocks arrival times model. In section \(\S3\) we derive the distribution of the resulting, subjected to shocks, lifetimes which is, by construction, singular: we compute the probability of the singularity and we analyze the dependence structure through the identification of the pairwise Kendall’s function and Kendall’s tau formulas. In section \(\S4\) we present an application to the analysis of the systemic riskiness of SIFI type European banks while section \(\S5\) concludes.

### 2 The shocks arrival times model

Let us consider a general system whose components’ lifetimes are denoted with \(T_1, \ldots, T_d\). We assume that each lifetime is affected by an idiosyncratic shock causing the default of only that component and by a systemic shock causing the simultaneous default of all the components. The systemic shock arrival time is modeled as the first arrival time among \(d+1\) shocks arrival times: one of them is fully independent (as in the Marshall-Olkin model) while each of the remaining ones is correlated with one of the idiosyncratic lifetimes components.

More formally, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((Y, X) = (Y_0, Y_1, \ldots, Y_d, X_1, \ldots, X_d)\), be a \(2d+1\)-dimensional random vector with strictly positive elements: we interpret the random variable \(X_j\) \((j = 1, \ldots, d)\) in \(X\) as the arrival time of a shock causing the default of only the \(j\)-th element in the system while \(Y_j\) \((j = 0, 1, \ldots, d)\) in \(Y\) represents the arrival time of a shock causing the default of the whole system.
We assume that the random variables in the sub-vector $\mathbf{X}$ are mutually independent as well as those in $\mathbf{Y}$ while some dependence is allowed in the pairs $(Y_j, X_j)$ for $j = 1, \ldots, d$. All random variables $X_j$ in the subvector $\mathbf{X}$ have a survival distribution function denoted with $\bar{F}_{X_j}$, for $j = 1, \ldots, d$, strictly decreasing on $(0, +\infty)$. As for the random variables in the subvector $\mathbf{Y}$, in order to allow for the case of no shock arrival time correlated with the idiosyncratic lifetime component of some element in the system or for the case of no independent shock arrival time, we assume that the survival distribution function $\bar{F}_{Y_j}$ of each $Y_j$, for $j = 0, \ldots, d$, is strictly decreasing or identically equal to 1 on $(0, +\infty)$; however, we assume that there exists $j \in \{0, \ldots, d\}$ so that $\bar{F}_{Y_j}$ is not identically equal to 1 on $(0, +\infty)$.

More precisely, the survival distribution function of $(Y, X)$ is of type

$$
\bar{F}_{(Y, X)}(y_0, y_1, \ldots, y_d, x_1, \ldots, x_d) = \bar{F}_{Y_0}(y_0) \prod_{j=1}^{d} \hat{C}_j \left( \bar{F}_{Y_j}(y_j), \bar{F}_{X_j}(x_j) \right)
$$

where $\left\{ \hat{C}_j(u, v) \right\}_{j=1,\ldots,d}$ is a family of bivariate copula functions: unlike $Y_0$, all other $Y_j$’s, for $j = 1, \ldots, d$, are correlated to the idiosyncratic shock arrival times $X_j$ and $\hat{C}_j$ represents the survival dependence structure of the pair $(Y_j, X_j)$.

Starting from the above setup, we define the random variable

$$
X_0 = \min_{j=0,1,\ldots,d} Y_j
$$

that represents the first arrival time of a shock inducing the collapse of the whole system. As a consequence of the assumptions, its survival distribution function is of type

$$
\bar{F}_{X_0}(x) = \prod_{j=0}^{d} \bar{F}_{Y_j}(x)
$$

and $\bar{F}_{X_0}$ is strictly decreasing on $(0, +\infty)$.

Let us now consider the $d + 1$-dimensional random vector $\mathbf{S} = (X_0, X_1, \ldots, X_d)$ whose survival distribution function is

$$
\bar{F}_{\mathbf{S}}(x_0, x_1, \ldots, x_d) = \mathbb{P} (Y_0 > x_0, Y_1 > x_0, \ldots, Y_d > x_0, X_1 > x_1, \ldots, X_d > x_d) = \bar{F}_{Y_0}(x_0) \prod_{j=1}^{d} \hat{C}_j \left( \bar{F}_{Y_j}(x_j), \bar{F}_{X_j}(x_j) \right)
$$

Thanks to Sklar’s theorem, the induced survival dependence structure is given by the survival copula

$$
\hat{C}(u_0, u_1, \ldots, u_d) = \bar{F}_{Y_0} \circ \bar{F}_{X_0}^{-1}(u_0) \prod_{j=1}^{d} \hat{C}_j \left( \bar{F}_{Y_j} \circ \bar{F}_{X_0}^{-1}(u_0), u_j \right)
$$

(1)
Remark 2.1. Since, for \( j = 0, \ldots, d \), \( g_j = \bar{F}_{Y_j} \circ \bar{F}^{-1}_{X_0} : [0,1] \rightarrow [0,1] \) is strictly increasing or identically equal to 1 and \( \prod_{j=0}^{d} g_j(v) = v \), (1) represents a particular specification of the family of copulas introduced in Lemma 2.1 in Liebscher (2008).

While the idiosyncratic shocks arrival times \((X_1, \ldots, X_d)\) are independent, by construction some dependence may exist only between each idiosyncratic shock arrival time \(X_j, j = 1, \ldots, d\), and the systemic one \(X_0\). More precisely, the survival distribution of each pair \((X_0, X_i)\) for \( i = 1, \ldots, d \) is

\[
\bar{F}_{(X_0, X_i)}(x_0, x_i) = \hat{C}_i \left( \bar{F}_{Y_i}(x_0), \bar{F}_{X_i}(x_i) \right) \prod_{j=0, j \neq i}^{d} \bar{F}_{Y_j}(x_0) = \hat{C}_i \left( \bar{F}_{Y_i}(x_0), \bar{F}_{X_i}(x_i) \right) \frac{F_{X_0}(x_0)}{F_{Y_i}(x_0)}
\]

and the corresponding bivariate survival copulas are

\[
\hat{C}_{0,i}(u_0, u_i) = \hat{C}_i \left( \bar{F}_{Y_i}(u_0), \bar{F}^{-1}_{X_0}(u_0), u_i \right) \frac{u_0}{\bar{F}_{Y_i} \circ \bar{F}^{-1}_{X_0}(u_0)}. \tag{2}
\]

Notice that while copulas \( \hat{C}_i \) parametrize the dependence among the idiosyncratic shock \(X_i\) and the arrival time \(Y_i\) of a shock affecting the whole system, \( \bar{F}_{Y_i} \circ \bar{F}^{-1}_{X_0} \) measures the contribution of the shock \(i\) to the systemic shock arrival time \(X_0\).

In order to analyze the dependence structure induced by (2) we compute the Kendall’s function of a copula \( \tilde{C} \) of type

\[
\tilde{C}(u, v) = C \left( g(u), v \right) \frac{u}{g(u)} \tag{3}
\]

where \( g : [0,1] \rightarrow [0,1] \) is strictly increasing.

We remind that the Kendall’s function of a bivariate copula \( C(u, v) \) is defined as the cumulative distribution function of the random variable \( C(U, V) \) where the random variables \( U \) and \( V \) are uniformly distributed on the interval \([0,1]\) and their joint distribution function is given by the considered copula \( C(u, v) \). More precisely the Kendall’s function of a bivariate copula \( C \) is a function \( K : [0,1] \rightarrow [0,1] \) defined as

\[
K_C(t) = \mathbb{P} \left( C(U, V) \leq t \right), \text{ for } t \in [0,1]
\]

(see Nelsen 2006, p. 127), where \( \mathbb{P} \) is the probability induced by \( C \). The relevance of this notion relies on the fact that it induces, through the corresponding one-dimensional stochastic ordering, a partial ordering in the set of bivariate copulas: notice in particular that if \( C_1(u, v) \leq C_2(u, v) \) for all \((u, v) \in [0,1]^2\), then \( K_{C_1}(t) \geq K_{C_2}(t) \) for all \( t \in [0,1] \) (see Nelsen, 2003, for more details). Let us simplify the notation setting \( \partial_t C(u, v) = \frac{\partial}{\partial u} C(u, v) \) for any copula \( C(u, v) \).
Proposition 2.1. Let $g : [0, 1] \to [0, 1]$ be strictly increasing and differentiable and the copula $C(u, v)$ be strictly increasing with respect to $v$ for any $u$. Then the Kendall’s function of a copula $	ilde{C}$ of type (3) is

$$K(t) = t - t \ln t + t \ln (g(t)) + \int_{t}^{1} \partial_{1}C(u, l_{t}(u)) \frac{g'(u)}{g(u)} u du$$

where $l_{t}(u)$ solves $\tilde{C}(u, l_{t}(u)) = t$.

Proof. Since, for a given $u$, $C(u, v)$ is strictly increasing with respect to $v$, the inverse function $l_{t}(u)$ with respect to $v$ is well defined for all $t \in (0, u]$ and satisfies $\tilde{C}(g(u), l_{t}(u)) = g(u)$. Applying (6) in Genest and Rivest (2001) and after straightforward computations we have that

$$K(t) = t + \int_{t}^{1} \partial_{1} \tilde{C}(u, l_{t}(u)) du =$$

$$= t - t \ln \left( \frac{t}{g(t)} \right) + \int_{t}^{1} \partial_{1}C(u, l_{t}(u)) \frac{g'(u)}{g(u)} u du.$$

Let us now assume that the bivariate copula functions $\hat{C}_{j}$, for $j = 1, \ldots, d$ are of Archimedean type, with strict generator $\phi_{j}$: that is $\phi_{j} : [0, +\infty) \to (0, 1]$ satisfies $\phi_{j}(0) = 1$, $\lim_{x \to +\infty} \phi_{j}(x) = 0$ and it is strictly decreasing and convex on $[0, +\infty)$ (see McNeal and Neslehová, 2009). Hence $\hat{C}_{0,i}$ in (2) takes the form

$$\hat{C}_{0,i}(u_{0}, u_{i}) = \phi_{i} \left( u_{0} \left( \frac{\phi_{i}^{-1}(\hat{F}_{Y_{i}} \circ \hat{F}_{X_{0}}^{-1}(u_{0})) + \phi_{i}^{-1}(u_{i})}{\hat{F}_{Y_{i}} \circ \hat{F}_{X_{0}}^{-1}(u_{0})} \right) \right).$$

According to the general case, this copula is a particular specification of a copula of type

$$\tilde{C}(u, v) = \phi \left( \phi^{-1}(g(u)) + \phi^{-1}(v) \right) \frac{u}{g(u)}.$$  \hspace{1cm} (4)

The expression of the Kendall’s function of a copula of this type can be immediately recovered from Proposition 2.1, taking into account that, now, $l_{t}(u) = \phi \left( \phi^{-1} \left( \frac{g(u)}{u} t \right) - \phi^{-1}(g(u)) \right)$. In fact it is a straightforward computation to verify that

Corollary 2.1. If $g : [0, 1] \to [0, 1]$ is strictly increasing and differentiable and $\tilde{C}(u, v)$ is a copula of type (3) with $\phi$ a strict Archimedean generator, then the Kendall’s function of $\tilde{C}$ is

$$K(t) = t - t \ln t + t \ln (g(t)) + \int_{t}^{1} \frac{h \left( \frac{g(u)}{u} t \right)}{h(g(u))} \frac{g'(u)}{g(u)} u du$$

with $h(x) = \phi' \circ \phi^{-1}(x)$. 

6
The Kendall’s function is strictly related to the widely used concordance measure known as Kendall’s tau (see Section 5.1.1 in Nelsen, 2006). In fact, the Kendall’s tau \( \tau \) can be obtained from the Kendall’s function through

\[
\tau = 3 - 4 \int_0^1 K(t)dt
\]

**Example 2.1.** Let the function \( g \) in (4) be of type \( g(v) = v^\theta \), with \( \theta \in (0, 1] \) (this specific case was firstly introduced in Khoudraji, 1995): notice that this case is recovered in our model (see (2)) when \( \bar{F}_Y(x) = \bar{F}_{X_0}(x) \). In this case we get

\[
K(t) = t - t \ln t + \theta t \ln t + \theta \int_t^1 \frac{h(u^{\theta-1}t)}{h(u^\theta)} du
\]

and

\[
\tau = \theta - 4\theta \int_0^1 \int_t^1 \frac{h(u^{\theta-1}t)}{h(u^\theta)} dudt.
\]

In particular,

- **Clayton case**, that is \( \phi(x) = (1 + x)^{-\frac{1}{\beta}} \), with \( \beta \geq 0 \): since \( h(y) = -\frac{1}{\beta} y^{1+\beta} \), we have

\[
K(t) = t \left(1 + \frac{\theta}{\beta}\right) - (1 - \theta)t \ln t - \frac{\theta}{\beta} t^{1+\beta}
\]

and

\[
\tau = \frac{\beta}{\beta + 2} \theta = \tau^C_\beta \theta
\]

where \( \tau^C_\beta \) is the Kendall’s tau of the Clayton copula with parameter \( \beta \);

- **Gumbel case**, that is \( \phi(x) = e^{-x^{\frac{1}{\beta}}} \), with \( \beta \geq 1 \): since \( h(y) = -\frac{1}{\beta} y (-\ln y)^{1-\beta} \), we have

\[
K(t) = t - t \ln t \left[ 1 - (\beta - 1) \left(\frac{\theta}{1-\theta}\right)^\beta \int_{\theta}^{+\infty} \frac{1}{z^{\beta}(z+1)} dz \right]
\]

and

\[
\tau = \left(1 - \frac{1}{\beta}\right) \left[ \beta \left(\frac{\theta}{1-\theta}\right)^\beta \int_{\theta}^{+\infty} \frac{1}{z^{\beta}(z+1)} dz \right] = \tau^G_\beta \left[ \beta \left(\frac{\theta}{1-\theta}\right)^\beta \int_{\theta}^{+\infty} \frac{1}{z^{\beta}(z+1)} dz \right]
\]

where \( \tau^G_\beta \) is the Kendall’s tau of the Gumbel copula with parameter \( \beta \).

Notice that when \( \theta = 1 \) we recover the Archimedean case: in our model this case corresponds to \( \bar{F}_Y = \bar{F}_{X_0} \), that is the case in which the only admissible fatal common shock is shock \( i \).
3 The lifetimes model

In this section we study the joint distribution of the observed lifetimes \((T_1, T_2, \ldots, T_d)\), each defined as the first arrival time between the corresponding idiosyncratic shock and the systemic one. More precisely, for \(j = 1, \ldots, d\), let

\[ T_j = \min(X_j, X_0) \]

be the lifetime of the \(j\)-th element in the system. If we consider the random variables

\[ Z_j = Y_j \land X_j, \]

then, we can rewrite \(T_j\) as

\[ T_j = \min \left( \min_{i=0, \ldots, d} Y_i, Z_j \right) \tag{7} \]

and each \(T_j\) can also be modeled as the first arrival time among \(d\) independent shocks arrival times. Since the survival distribution of \(Z_j\) is

\[ F_{Z_j}(x) = \hat{C}_j \left( F_{Y_j}(x), F_{X_j}(x) \right), \quad x \geq 0, \]

it follows that the survival distribution of \(T_j\) is

\[ F_{T_j}(x) = \hat{C}_j \left( F_{Y_j}(x), F_{X_j}(x) \right) \frac{F_{X_0}(x)}{F_{Y_j}(x)}, \quad x \geq 0. \]

More in general, the joint survival distribution function of \(T = (T_1, \ldots, T_d)\) can be easily recovered and it turns out to be given by

\[ \bar{F}_T(t_1, \ldots, t_d) = \bar{F}_{Y_0} \left( \max_{i=1, \ldots, d} t_i \right) \prod_{j=1}^{d} \hat{C}_j \left( F_{Y_j} \left( \max_{i=1, \ldots, d} t_i \right), \bar{F}_{X_j}(t_j) \right) \tag{8} \]

for \((t_1, \ldots, t_d) \in (0, +\infty)^d\).

The dependence structure implied by this survival distribution is the result of the joint contribution of the fact that the lifetimes can end simultaneously because of the occurrence of the systemic shock (which is the kind of dependence characteristic of the Marshall-Olkin distribution) and of the fact that each element in the system can influence the occurrence of the systemic shock.

Remark 3.1. If \(\hat{C}_j(u,v) = uv\), for all \(j = 1, \ldots, d\), we get

\[ \bar{F}_T(t_1, \ldots, t_d) = \bar{F}_{X_0} \left( \max_{i=1, \ldots, d} t_i \right) \prod_{j=1}^{d} \bar{F}_{X_j}(t_j) \]

which is a particular specification of the generalized Marshall-Olkin distribution (see Li and Pellerey, 2011 and Lin and Li, 2014) with only one independent shock arrival time \(X_0\).
Example 3.1. Let us assume that the random variables $Y_0, Y_1, \ldots, Y_d, Z_1, \ldots, Z_d$ that generate the random variables $T_j$ (see (1)) have survival distributions that belong to a same specific parametric family. More precisely, we assume that

$$F_j(x) = G^{\gamma_j}(x), \ j = 0, \ldots, d \text{ and } \tilde{F}_j(x) = G^{\eta_j}(x), \ j = 1, \ldots, d$$

where $G$ is the survival distribution function of a strictly positive continuous random variable with support $(0, +\infty)$, $\gamma_j \geq 0$ (with at least one $j$ for which $\gamma_j > 0$) and $\eta_j > 0$. Since $\tilde{F}_{Z_j}(x) \leq \tilde{F}_{Y_j}(x)$ we have that $\eta_j \geq \gamma_j$. We set $\lambda_j = \eta_j - \gamma_j$, for $j = 1, \ldots, d$ and $\lambda_0 = \sum_{j=0}^{d} \gamma_j$. It follows that

$$\tilde{F}_{X_0}(x) = G^{\lambda_0}(x) \text{ and } \tilde{F}_{T_j}(x) = G^{\lambda_0+\lambda_j}(x).$$

As a consequence (8) takes the form

$$\tilde{F}_{T}(t_1, \ldots, t_d) = G^{\lambda_0} \left( \max_{i=1,\ldots,d} t_i \right) \prod_{j=1}^{d} \tilde{C}_j \left( G^{\gamma_j} \left( \max_{i=1,\ldots,d} t_i \right), \tilde{F}_{X_j}(t_j) \right)$$

where $\tilde{F}_{X_j}$ satisfies $\tilde{C}_j \left( G^{\eta_j}(x), \tilde{F}_{X_j}(x) \right) = G^{\gamma_j}(x)$ and the associated survival copula is

$$\tilde{C}_{T}(u_1, \ldots, u_d) = \min_{i=1,\ldots,d} u_i^{\lambda_j} \prod_{j=1}^{d} \tilde{C}_j \left( \min_{i=1,\ldots,d} u_i^{\lambda_0+\lambda_j}, \tilde{F}_{X_j} \left( G^{-1} \left( \frac{1}{u_j^{\lambda_0+\lambda_j}} \right) \right) \right).$$

In the case in which $\tilde{C}_j$ is Archimedean with strict generator $\phi_j$, we have

$$\tilde{F}_{X_j}(x) = \phi_j \left( \phi_j^{-1} \left( G^{\eta_j}(x) \right) - \phi_j^{-1} \left( G^{\gamma_j}(x) \right) \right)$$

and

$$\tilde{F}_{T}(t_1, \ldots, t_d) = G^{\lambda_0} \left( \max_{i=1,\ldots,d} t_i \right) \prod_{j=1}^{d} \phi_j \left( \phi_j^{-1} \left( G^{\gamma_j} \left( \max_{i=1,\ldots,d} t_i \right) \right) \right) + \phi_j^{-1} \left( G^{\eta_j}(t_j) \right) - \phi_j^{-1} \left( G^{\gamma_j}(t_j) \right)$$

and

$$\tilde{C}_{T}(u_1, \ldots, u_d) = \min_{i=1,\ldots,d} u_i^{\lambda_j} \prod_{j=1}^{d} \phi_j \left( \phi_j^{-1} \left( \min_{i=1,\ldots,d} u_i^{\lambda_0+\lambda_j} \right) \right) + \phi_j^{-1} \left( u_j^{\lambda_0+\lambda_j} \right) - \phi_j^{-1} \left( u_j^{\lambda_0+\lambda_j} \right).$$

Let us set

$$\alpha_j = \frac{\lambda_0}{\lambda_0 + \lambda_j},$$

which represents the ratio between the systemic shock intensity and the marginal one, and

$$\theta_j = \frac{\gamma_j}{\lambda_0},$$

where
which represents the percentage of contribution of the intensity of the shock correlated with each bank to the systemic shock intensity, for \( j = 1, \ldots, d \), while \( \theta_j \) is the percentage of contribution of some completely independent exogenous shock. Then, we can rewrite the copula as

\[
\hat{C}_T(u_1, \ldots, u_d) = \min_{i=1,\ldots,d} u_i^{\alpha\theta_0} \prod_{j=1}^d \phi_j \left[ \phi_j^{-1} \left( \min_{i=1,\ldots,d} u_i^{\alpha\theta_j} \right) + \phi_j^{-1} \left( u_j^{1-\alpha_j(1-\theta_j)} \right) - \phi_j^{-1} \left( u_j^{\alpha_j} \right) \right]
\]

(9)

In particular

- If \( \phi_j \) is for all \( j = 1, \ldots, d \) the Gumbel generator with parameter \( \beta_j \geq 1 \), then \( \bar{F}_{X_j}(x) = G^{\left( \frac{\beta_j}{\gamma_j-\beta_j} \right)^{1/\beta_j}}(x) \) and

\[
\hat{C}_T(u_1, \ldots, u_d) = \min_{i=1,\ldots,d} u_i^{\alpha\theta_0} \exp \left\{ - \sum_{j=1}^d \left[ \phi_j^{\beta_j} \max_{i=1,\ldots,d} \left\{ -\alpha_i \ln u_i \right\}^{\beta_j} + \sigma_j(-\ln u_j)^{\beta_j} \right] \right\}
\]

where \( \sigma_j = (1 - \alpha_j(1 - \theta_j))^{\beta_j} - \alpha_j^{\beta_j} \theta_j^{\beta_j} \)

- If \( \phi_j \) is for all \( j = 1, \ldots, d \) the Clayton generator with parameter \( \beta_j > 0 \), then \( \bar{F}_{X_j}(x) = (1 + G^{-\gamma_j\beta_j}(x) - G^{-\gamma_j\beta_j}(x))^{-1/\beta_j} \) and

\[
\hat{C}_T(u_1, \ldots, u_d) = \min_{i=1,\ldots,d} u_i^{\alpha\theta_0} \prod_{j=1}^d \left[ \max_{i=1,\ldots,d} u_i^{-\alpha_i} \right]^{\theta_j} u_j^{-\alpha_j(1-\theta_j)} - u_j^{-\beta_j\theta_j} \right]^{-1/\beta_j}
\]

(10)

Notice that when \( \bar{F}_{X_j}(x) = \bar{F}_{X_0}^{\gamma_j/\lambda_0}(x) \), we recover the same framework considered in Example 2.7.

From (9) the survival copula associated to \((T_i, T_k)\) is

\[
\hat{C}_{T_i, T_k}(u_i, u_k) = \min_{i=1,\ldots,d} u_i^{\alpha\theta_0} \prod_{j=1,\ldots,k} \phi_j \left[ \phi_j^{-1} \left( \min_{i=1,\ldots,d} u_i^{\alpha_i} u_k^{\alpha_k} \right) + \phi_j^{-1} \left( u_j^{1-\alpha_j(1-\theta_j)} \right) - \phi_j^{-1} \left( u_j^{\alpha_j} \right) \right]
\]

from which, setting \( \alpha_i = 1 \), we recover the survival copula associated to \((X_0, T_k)\)

\[
\hat{C}_{X_0, T_k}(u_i, u_k) = \frac{\min(u_i, u_k^{\alpha_k})}{(\min(u_i, u_k^{\alpha_k}))^{\theta_k}} \phi_k \left[ \phi_k^{-1} \left( \min(u_i, u_k^{\alpha_k}) \right) + \phi_k^{-1} \left( u_k^{1-\alpha_k(1-\theta_k)} \right) - \phi_k^{-1} \left( u_k^{\alpha_k} \right) \right]
\]

Notice that when \( \alpha_k \approx 0 \) then

\[
C_{X_0, T_k}(u_i, u_k) \approx \frac{u_i}{u_i^{\theta_k}} \phi_k \left[ \phi_k^{-1} \left( u_i^{\theta_k} \right) + \phi_k^{-1} \left( u_k \right) \right]
\]

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which is of type (4) with \( g(u) = u^\theta \) (see Example 2.1): the dependence structure of the observed lifetimes with respect to the systemic shock arrival time essentially coincides with that between the idiosyncratic component of risk and the systemic one; in fact, being \( \alpha_k \approx 0 \), \( \lambda_j \) is large with respect to \( \lambda_0 \) and this corresponds to the case in which \( X_j \) has a low survival distribution with respect to that of \( X_0 \).

**Remark 3.2.** In case of perfect dependence between each idiosyncratic shock and the corresponding systemic shock component, that is \( \hat{C}_j(u, v) = \min(u, v) \) for \( j = 1, \ldots, d \), we get

\[
\bar{F}_T(t_1, \ldots, t_d) = \bar{F}_{Y_0} \left( \max_{i=1,\ldots,d} t_i \right) \prod_{j=1}^d \min \left( \bar{F}_{Y_j} \left( \max_{i=1,\ldots,d} t_i \right) , \bar{F}_{X_j}(t_j) \right).
\]

In the particular framework of Example 3.1, we have that, being \( \min \left( \bar{G}_\gamma(x), \bar{F}_{X_j}(x) \right) = \bar{G}_\gamma(x) \), if \( \eta_j > \gamma_j \), then \( \bar{F}_{X_j}(x) = \bar{G}_\gamma(x) \) and, if we assume \( \eta_j > \gamma_j \) for all \( j = 1, \ldots, d \), we can write

\[
\bar{F}_T(t_1, \ldots, t_d) = \bar{G}_\gamma \left( \max_{i=1,\ldots,d} t_i \right) \prod_{j=1}^d \min \left( \bar{G}_\gamma \left( \max_{i=1,\ldots,d} t_i \right) , \bar{G}_\eta(t_j) \right).
\]

Conversely, if \( \eta_j = \gamma_j \), then \( \bar{F}_{X_j}(x) \geq \bar{G}_\gamma(x) \) is not uniquely determined and if \( \eta_j = \gamma_j \) for all \( j = 1, \ldots, d \)

\[
\bar{F}_T(t_1, \ldots, t_d) = \bar{F}_{X_0} \left( \max_{i=1,\ldots,d} t_i \right).
\]

### 3.1 The probability of simultaneous default

By construction, the distribution of \( T \) has a singularity generated by the occurrence of the simultaneous default of all the elements in the system, that is by the fact that the event \( \{ T_1 = T_2 = \cdots = T_d \} \) has positive probability. Both from a theoretical point of view as well as for applications, it is important to measure the probability that the collapse of the whole system has to occur before a given time horizon.

**Proposition 3.1.** If the random vector \( T \) has a survival distribution of type (8), then

\[
P(T_1 = T_2 = \cdots = T_d > t) = - \int_t^{+\infty} \prod_{j=1}^d \bar{F}_{Z_j}(x) d\bar{F}_{Y_0}(x) + \\
- \sum_{j=1}^d \int_t^{+\infty} \bar{F}_{Y_0}(x) \prod_{i \neq j} \bar{F}_{Z_i}(x) \partial_i \hat{C}_j \left( \bar{F}_{Y_j}(x) , \bar{F}_{X_j}(x) \right) d\bar{F}_{Y_j}(x)
\]

(11)
Proof.

\[ P(T_1 = T_2 = \cdots = T_d > t) = \mathbb{E} \left[ P(X_1 > X_0, \ldots, X_d > X_0 | X_0) 1_{\{X_0 > t\}} \right] = \]

\[ -\int_t^{+\infty} \prod_{j=1}^d \hat{C}_j (F_{Y_j}(x), F_{X_j}(x)) dF_{Y_0}(x) + \]

\[ -\int_t^{+\infty} \bar{F}_{Y_0}(x) \sum_{j=1}^d \partial_1 \hat{C}_j (F_{Y_j}(x), F_{X_j}(x)) \prod_{i \neq j} \hat{C}_i (F_{Y_i}(x), F_{X_i}(x)) dF_{Y_j}(x). \]

If each \( \hat{C}_j \) is of Archimedean type with strict generator \( \phi_j \), and \( h_j = \phi_j' \circ \phi_j^{-1} \), we have

\[ P(T_1 = T_2 = \cdots = T_d > t) = -\int_t^{+\infty} \prod_{j=1}^d \bar{F}_{Z_j}(x) d\bar{F}_{Y_0}(x) + \]

\[ -\sum_{j=1}^d \int_t^{+\infty} \bar{F}_{Y_0}(x) \bar{F}_{Z_i}(x) \frac{h_j \circ \bar{F}_{Z_j}(x)}{h_j \circ \bar{F}_{Y_j}(x)} d\bar{F}_{Y_j}(x). \]

In particular, in the framework of Example 3.1, (11) takes the form

\[ P(T_1 = T_2 = \cdots = T_d > t) = \gamma_0 \hat{\lambda} G^\hat{\lambda}(t) + \sum_{j=1}^d \gamma_j \int_0^G(t) y^{\hat{\lambda} - \lambda_j - 1} \frac{h_j(y^{\beta_j})}{h_j(y^\beta_j)} dy \]  \hspace{1cm} \text{(12)}

where \( \hat{\lambda} = \sum_{i=0}^d \beta_j \). Hence

\begin{itemize}
  \item in the Clayton case (that is \( \phi_j(x) = (1 + x)^{-\frac{1}{\beta_j}} \) with \( \beta_j > 0 \) and \( h_j(x) = -\frac{1}{\beta_j} x^{1+\beta_j} \), for \( j = 1, \ldots, d \), we have
  \[ P(T_1 = T_2 = \cdots = T_d > t) = \gamma_0 \hat{\lambda} G^\hat{\lambda}(t) + \sum_{j=1}^d \frac{\gamma_j}{\hat{\lambda} + \beta_j \hat{\lambda}} G^{\hat{\lambda} + \beta_j}(t) \]

and

\[ P(T_1 = T_2 = \cdots = T_d) = \frac{\gamma_0}{\hat{\lambda}} + \sum_{j=1}^d \frac{\gamma_j}{\hat{\lambda} + \lambda_j \beta_j} = \]

\[ = \frac{\theta_0}{\sum_{i=1}^d \alpha_i^{-1} - (d-1)} + \sum_{j=1}^d \frac{\theta_j}{\sum_{i=1}^d \alpha_i^{-1} - d + \beta_j (\alpha_j^{-1} - 1)}, \]
\end{itemize}
in the Gumbel case (that is \( \phi_j(x) = e^{-x^{1/j}} \) with \( \beta_j \geq 1 \) and \( h_j(x) = -\frac{1}{\beta_j} x (-\ln x)^{1-\beta_j} \), for \( j = 1, \ldots, d \), we have

\[
\mathbb{P}(T_1 = T_2 = \cdots = T_d > t) = \left( \frac{\gamma_0}{\lambda} + \frac{1}{\lambda} \sum_{j=1}^{d} \gamma_j \left( 1 + \frac{\lambda_j}{\gamma_j} \right)^{1-\beta_j} \right) \lambda^\lambda(t)
\]

and

\[
\mathbb{P}(T_1 = T_2 = \cdots = T_d) = \frac{\gamma_0}{\lambda} + \frac{1}{\lambda} \sum_{j=1}^{d} \gamma_j \left( 1 + \frac{\lambda_j}{\gamma_j} \right)^{1-\beta_j} = \\
= \frac{\theta_0}{\sum_{i=1}^{d} \alpha_i^{1-\lambda} - (d-1)} + \frac{\theta_j}{\sum_{i=1}^{d} \alpha_i^{1-\lambda} - (d-1)} \left( 1 + \frac{1 - \alpha_j}{\alpha_j \theta_j} \right)^{1-\beta_j}.
\]

### 3.2 The Kendall’s function and the Kendall’s tau

In this section we analyze the pairwise dependence structure of the random vector \( \mathbf{T} \) through the study of the pairwise Kendall’s function and the pairwise Kendall’s tau.

In order to simplify the notation we set, for \( i, k = 1, \ldots, d, i \neq k \),

\[
P_{i,k}(x) = \frac{F_{Z_i}(x)}{F_{Y_i}(x) F_{Y_k}(x)}.
\]

It can be easily checked that the survival distributions of the pairs \((T_i, T_k)\) are

\[
\bar{F}_{i,k}(t_i, t_k) = P_{i,k} \left( \max(t_i, t_k) \right) \prod_{j=i,k} \mathcal{C}_j \left( \bar{F}_{Y_j}(\max(t_i, t_k)), \bar{F}_{X_j}(t_j) \right).
\]

**Proposition 3.2.** Let us assume that \( \mathcal{C}_i \) and \( \mathcal{C}_k \) are strictly increasing with respect to each argument. Then, for \( t \in [0, 1] \),

\[
K_{i,k}(t) = t - t \left( \ln \left( \frac{\bar{F}_{Z_i} \cdot P_{i,k}}{F_{Z_i} \cdot P_{i,k}(z_t)} \right) + \ln \left( \frac{\bar{F}_{Z_k} \cdot P_{k,i}}{F_{Z_k} \cdot P_{k,i}(z_t)} \right) \right) + \\
- \int_{z_t}^{\bar{F}_{i,k}^{-1}(t)} \bar{F}_{Z_i} P_{ik}(x) \partial_t \mathcal{C}_k \left( \bar{F}_{Y_k}(x), \bar{F}_{X_k}(h_t(x)) \right) d\bar{F}_{Y_k}(x) + \\
- \int_{z_t}^{\bar{F}_{k,i}^{-1}(t)} \bar{F}_{Z_k} P_{ik}(x) \partial_t \mathcal{C}_i \left( \bar{F}_{Y_i}(x), \bar{F}_{X_i}(g_t(x)) \right) d\bar{F}_{Y_i}(x)
\]

where \( z_t \) is the solution of \( \bar{F}_{i,k}(z_t, z_t) = t \), \( h_t(\cdot) \) solves \( \bar{F}_{i,k}(x, h_t(x)) = t \) for \( z_t < x \leq \bar{F}_{i,k}^{-1}(t) \) and \( g_t(\cdot) \) solves \( \bar{F}_{i,k}(g_t(y), y) = t \) for \( z_t < y \leq \bar{F}_{k,i}^{-1}(t) \).
Proof. Since \( \bar{F}_{i,k}(x,x) = \hat{F}_{Z_i}(x)\hat{F}_{Z_k}(x)P_{i,k}(x) \) is strictly decreasing, given any \( t \in [0,1] \), the solution of \( \bar{F}_{i,k}(x,x) = t \), denoted with \( z_t \), is well defined.

If we restrict to \( t_i > t_k \), then

\[
\bar{F}_{i,k}(t_i, t_k) = \bar{F}_{Z_i}(t_i)\hat{C}_k \left( \bar{F}_{Y_k}(t_i), \bar{F}_{X_k}(t_k) \right) P_{i,k}(t_i)
\]  

(13)

which is strictly decreasing with respect to \( t_k \) for any given \( t_i \). Hence, for \( x \in (z_t, \bar{F}^{-1}_{T_i}(t)) \) and for any \( t \in [0,1] \), the function \( h_t \) satisfying \( \bar{F}_{i,k}(x,h_t(x)) = t \) is well defined. By similar arguments, the function \( g_t \) of the statement is also well defined.

If we denote with \( K_{i,k} \) the Kendall’s function associated to the pair \( (T_i, T_k) \) and we rewrite it in terms of the survival joint distribution function, we get

\[
K_{i,k}(t) = \mathbb{P}(\bar{F}_{i,k}(T_i, T_k) \leq t) = \bar{F}_{T_i}(z_t) - \mathbb{P}((T_i, T_k) \in D_1) + \bar{F}_{T_k}(z_t) - \mathbb{P}((T_i, T_k) \in D_2) - t
\]

where

\[
D_1 = \{(t_i, t_k) : z_t < t_i \leq \bar{F}^{-1}_{T_i}(t), 0 \leq t_k \leq h_t(t_i)\}
\]

and

\[
D_2 = \{(t_i, t_k) : z_t < t_k \leq \bar{F}^{-1}_{T_k}(t), 0 \leq t_i \leq g_t(t_k)\}.
\]

Let us start computing \( \mathbb{P}((T_i, T_k) \in D_1) \). Since here (13) holds, thanks to the definitions of \( z_t \) and \( h_t \), we have

\[
\mathbb{P}((T_i, T_k) \in D_1) = \int_{z_t}^{\bar{F}^{-1}_{T_i}(t)} \mathbb{P}(T_k > h_t(x)|T_i = x) - 1 \) d\( \bar{F}_{T_i}(x) =
\]

\[
= \int_{z_t}^{\bar{F}^{-1}_{T_i}(t)} \mathbb{P}(T_k > h_t(x)|T_i = x) d\bar{F}_{T_i}(x) - t + \bar{F}_{T_i}(z_t) =
\]

\[
= \int_{z_t}^{\bar{F}^{-1}_{T_i}(t)} t \cdot \frac{d(\bar{F}_{Z_i} \cdot P_{i,k})(x)}{\bar{F}_{Z_i} \cdot P_{i,k}(x)} +
\]

\[
+ \int_{z_t}^{\bar{F}^{-1}_{T_i}(t)} \bar{F}_{Z_i} \cdot P_{i,k}(x) \cdot \partial_1 \hat{C}_k \left( \bar{F}_{Y_k}(x), \bar{F}_{X_k}(h_t(x)) \right) d\bar{F}_{Y_k}(x) - t + \bar{F}_{T_i}(z_t) =
\]

\[
= t \ln \left( \frac{(\bar{F}_{Z_i} \cdot P_{i,k}) \circ \bar{F}^{-1}_{T_i}(t)}{(\bar{F}_{Z_i} \cdot P_{i,k})(z_t)} \right)
\]

\[
+ \int_{z_t}^{\bar{F}^{-1}_{T_i}(t)} \bar{F}_{Z_i} \cdot P_{i,k}(x) \cdot \partial_1 \hat{C}_k \left( \bar{F}_{Y_k}(x), \bar{F}_{X_k}(h_t(x)) \right) d\bar{F}_{Y_k}(x) - t + \bar{F}_{T_i}(z_t).
\]
Since \( P((T_i, T_k) \in D_2) \) can be similarly computed, we get
\[
K_{i,k}(t) = t - t \left( \ln \left( \frac{(F_{Z_i} \cdot P_{i,k}) \circ \tilde{F}_{T_i}^{-1}(t)}{(F_{Z_i} \cdot P_{i,k})(z_t)} \right) + \ln \left( \frac{(F_{Z_k} \cdot P_{i,k}) \circ \tilde{F}_{T_k}^{-1}(t)}{(F_{Z_k} \cdot P_{i,k})(z_t)} \right) \right) + \\
\quad - \int_{z_t}^{\tilde{F}_{T_i}^{-1}(t)} F_{Z_i} \cdot P_{i,k}(x) \cdot \partial_1 \tilde{C}_k \left( F_{Y_k}(x), F_{X_k}(h_t(x)) \right) dF_{Y_k}(x) + \\
\quad - \int_{z_t}^{\tilde{F}_{T_k}^{-1}(t)} F_{Z_k} \cdot P_{i,k}(x) \cdot \partial_1 \tilde{C}_i \left( F_{Y_i}(x), F_{X_i}(g_t(x)) \right) dF_{Y_i}(x).
\]

If we consider the case in which \( \tilde{C}_i \) and \( \tilde{C}_k \) are Archimedean copulas with strict generator \( \phi_i \) and \( \phi_k \), respectively, we have
\[
K_{i,k}(t) = t - t \left( \ln \left( \frac{(F_{Z_i} \cdot P_{i,k}) \circ \tilde{F}_{T_i}^{-1}(t)}{(F_{Z_i} \cdot P_{i,k})(z_t)} \right) + \ln \left( \frac{(F_{Z_k} \cdot P_{i,k}) \circ \tilde{F}_{T_k}^{-1}(t)}{(F_{Z_k} \cdot P_{i,k})(z_t)} \right) \right) + \\
\quad - \int_{z_t}^{\tilde{F}_{T_i}^{-1}(t)} F_{Z_i} \cdot P_{i,k}(x) \cdot h_k \left( \frac{t}{F_{Z_i} \cdot P_{i,k}(x)} \right) dF_{Y_k}(x) + \\
\quad - \int_{z_t}^{\tilde{F}_{T_k}^{-1}(t)} F_{Z_k} \cdot P_{i,k}(x) \cdot h_i \left( \frac{t}{F_{Z_k} \cdot P_{i,k}(x)} \right) dF_{Y_i}(x)
\]
with \( \tilde{F}_{Z_j}(x) = \phi_j(\tilde{F}_{Y_j}(x)) + \phi_j^{-1}(\tilde{F}_{X_j}(x)) \) and \( h_j = \phi_j' \circ \phi_j^{-1} \), for \( j = i, k \).

**Example 3.2.** Let us consider the same framework of Example 3.1. We have
\[
{z_t} = G^{-1}(t^{1/(\lambda_0 + \lambda_i + \lambda_k)})
\]
and
\[
\tilde{F}_{T_j}^{-1}(t) = G^{-1}(t^{1/(\lambda_0 + \lambda_j)}) \quad j = i, k.
\]
Under the same notation of Example 3.1, we recover
\[
K_{i,k}(t) = t - t \ln t \left( \frac{\alpha_i (1 - \alpha_k) (1 + \alpha_i \theta_k)}{\alpha_i + \alpha_k - \alpha_i \alpha_k} + \frac{\alpha_k (1 - \alpha_i) (1 - \alpha_k \theta_i)}{\alpha_i + \alpha_k - \alpha_i \alpha_k} \right) - \\
\quad - \theta_k \int_{\frac{1}{t^{\alpha_i}}}^{\alpha_i} y^{\alpha_i} \frac{h_k \left( ty - \frac{1 - \theta_k \alpha_i}{\alpha_i} \right)}{h_k \left( y^{\theta_k} \right)} dy + \\
\quad - \theta_i \int_{\frac{1}{t^{\alpha_k}}}^{\alpha_k} y^{\alpha_k} \frac{h_i \left( ty - \frac{1 - \theta_i \alpha_k}{\alpha_k} \right)}{h_i \left( y^{\theta_i} \right)} dy.
\]
Let us consider Clayton and Gumbel copulas specific cases.
1. Clayton case \( \phi_j(x) = (1 + x)^{\frac{1}{\beta_j}}, \beta_j > 0, j = i, k \).

If we set

\[
\tau_{ik}^{MO} = \frac{\alpha_k \alpha_i}{\alpha_k + \alpha_i - \alpha_k \alpha_i},
\]

which is the Kendall’s tau of the Marshall-Olkin bivariate copula with parameters \( \alpha_i \) and \( \alpha_k \) and

\[
\rho_{rs} = \frac{1 - \alpha_s}{\alpha_s} \tau_{rs}^{MO}, r, s = i, j
\]

we get

\[
K_{i,k}(t) = t \left( 1 + \frac{\theta_k}{\beta_k} \alpha_i + \frac{\theta_i}{\beta_i} \alpha_k \right) - t \ln t \left( (1 - \theta_k \alpha_i) \rho_{ik} + (1 - \theta_i \alpha_k) \rho_{ki} \right) + \frac{\theta_i}{\beta_i} t^{\rho_{ki} \beta_i + 1} - \frac{\theta_k}{\beta_k} t^{\rho_{ik} \beta_k + 1}.
\]

Notice that the above Kendall’s function can be decomposed as

\[
K_{i,k}(t) = K_{0,k}^{0} + K_{0,i}^{0} - 2K'(t)
\]

where

\[
K_{0,k}^{0}(t) = t - (1 - \tau_{ik}^{MO}) t \ln t,
\]

\[
K_{0,k}^{i}(t) = t \left( 1 + \frac{\theta_k}{\beta_k} \alpha_i \right) - (1 - \theta_k \alpha_i \rho_{ik}) t \ln t - \frac{\theta_k \alpha_i}{\beta_k} t^{1 + \beta_k \rho_{ik}},
\]

\[
K_{0,i}^{k}(t) = t \left( 1 + \frac{\theta_i}{\beta_i} \alpha_k \right) - (1 - \theta_i \alpha_k \rho_{ki}) t \ln t - \frac{\theta_i \alpha_k}{\beta_i} t^{1 + \beta_i \rho_{ki}},
\]

and

\[
K'(t) = t - \ln t.
\]

Notice that: (14) is the Kendall’s function of the Marshall-Olkin copula with parameters \( \alpha_i \) and \( \alpha_k \); (15) is a Kendall’s function of type (5) with parameters \( \theta = \theta_k \alpha_i \rho_{ik} \) and \( \beta = \beta_k \rho_{ik} \) (that represents the effect of the dependence between \( Y_k \) and \( X_k \) on the resulting dependence structure of \( (T_i, T_k) \)); symmetrically, (16) is a Kendall’s function of type (5) with parameters \( \theta = \theta_i \alpha_k \rho_{ki} \) and \( \beta = \beta_i \rho_{ki} \); (17) is the Kendall’s function of the independence copula. As a consequence, we get a very meaningful decomposition of the Kendall’s tau:

\[
\tau_{ik} = \tau_{ik}^{MO} + \bar{\tau}_{0,k}^{(i)} + \bar{\tau}_{0,i}^{(k)}
\]

where

\[
\bar{\tau}_{0,k}^{(i)} = \alpha_i \rho_{ik} \theta_k \frac{\rho_{i,k} \beta_k}{\rho_{i,k} \beta_k + 2} \text{ and } \bar{\tau}_{0,i}^{(k)} = \alpha_k \rho_{ki} \theta_i \frac{\rho_{k,i} \beta_i}{\rho_{k,i} \beta_i + 2}.
\]

are Kendall’s tau of type (6) with suitably modified parameters.
It follows that
\[
\tau_{T_k,X_0} = \alpha_k + (1 - \alpha_k)\theta_k \frac{(1 - \alpha_k)\beta_k}{(1 - \alpha_k)\beta_k + 2} = \\
\tau_{T_k,X_0}^{MO} + \tau_{0,k}^{*}
\]
where \(\tau_{T_k,X_0}^{MO}\) is the Kendall’s tau between the observed lifetime and the systemic shock arrival time in the Marshall-Olkin model and \(\tau_{0,k}^{*}\) is a Kendall’s tau of type (6) with parameters rescaled by the coefficient \(1 - \alpha_k\).

2. Gumbel case \((\phi_j(x) = e^{-x^j}, \beta_j \geq 1, j = i, k)\). If
\[
I(a, b, \beta) = \int_a^b \frac{1}{z^\beta(z + 1)} dz,
\]
\[
K_{i,k}(t) = t - t \ln t \left\{ 1 - \tau_{ik}^{MO} + \tau_{ik}^{MO} \left[ \theta_k \left( 1 - \left( \frac{\theta_k \alpha_k}{1 - \alpha_k(1 - \theta_k)} \right) \beta_k \right) - \theta_i \left( 1 - \left( \frac{\theta_i \alpha_i}{1 - \alpha_i(1 - \theta_i)} \right) \beta_i \right) \right] + \right.
\]
\[
- (\beta_k - 1) \left( \frac{\alpha_i \theta_k}{1 - \alpha_i \theta_k} \right) \beta_k I \left( \frac{\alpha_i \theta_k}{1 - \alpha_i \theta_k}, \frac{\alpha_i \theta_k}{\tau_{ik}^{MO} \theta_k(1 - \alpha_i \theta_k)} \right) - 1, \beta_k \right) + \left.
\right) + \left.
\right) - \left( \beta_i - 1 \right) \left( \frac{\alpha_k \theta_i}{1 - \alpha_k \theta_i} \right) \beta_i I \left( \frac{\alpha_k \theta_i}{1 - \alpha_k \theta_i}, \frac{\alpha_k \theta_i}{\tau_{ik}^{MO} \theta_i(1 - \alpha_k \theta_i)} - 1, \beta_i \right) \right\}
\]
and
\[
\tau_{i,k} = \tau_{ik}^{MO} - \tau_{ik}^{MO} \left[ \theta_k \left( 1 - \left( \frac{\theta_k \alpha_k}{1 - \alpha_k(1 - \theta_k)} \right) \beta_k \right) - \theta_i \left( 1 - \left( \frac{\theta_i \alpha_i}{1 - \alpha_i(1 - \theta_i)} \right) \beta_i \right) \right] + \right.
\]
\[
+ (\beta_k - 1) \left( \frac{\alpha_i \theta_k}{1 - \alpha_i \theta_k} \right) \beta_k I \left( \frac{\alpha_i \theta_k}{1 - \alpha_i \theta_k}, \frac{\alpha_i \theta_k}{\tau_{ik}^{MO} \theta_k(1 - \alpha_i \theta_k)} - 1, \beta_k \right) + \left.
\right) + \left.
\right) + \left( \beta_i - 1 \right) \left( \frac{\alpha_k \theta_i}{1 - \alpha_k \theta_i} \right) \beta_i I \left( \frac{\alpha_k \theta_i}{1 - \alpha_k \theta_i}, \frac{\alpha_k \theta_i}{\tau_{ik}^{MO} \theta_i(1 - \alpha_k \theta_i)} - 1, \beta_i \right) \right\}
\]
Even if, as in the Clayton case, we can recognize that the resulting dependence is the sum of the Marshall-Olkin one and two different contributions arising from the assumed dependence between \(Y_i\) and \(X_i\) and between \(Y_k\) and \(X_k\), unlike that case, the latter ones cannot be written in a closed form as modifications of the corresponding ones in Example 2.1.
Moreover, we have

\[
\tau_{T_k,X_0} = \alpha_k - \alpha_k \theta_k \left( 1 - \left( \frac{\theta_k \alpha_k}{1 - \alpha_k (1 - \theta_k)} \right)^{\beta_k - 1} \right) + \\
+ (\beta_k - 1) \left( \frac{\theta_k}{1 - \theta_k} \right)^{\beta_k} \mathcal{I} \left( \frac{\theta_k}{1 - \theta_k}, \frac{1 - \alpha_k (1 - \theta_k)}{\alpha_k (1 - \theta_k)}, \beta_k \right)
\]

4 Application to the analysis of the systemic riskiness in the European banking system

In this section we apply the model presented and discussed in previous sections to the analysis of the riskiness of the so called too-big-to-fail banks in the European banking system. We define systemic riskiness as the capability of a bank to induce a systemic crisis (collapse) in the banking system: in our model, this can be measured thought the degree of dependence between the idiosyncratic component of the risk of default of the bank and the systemic shock arrival time that causes the simultaneous default of all the banks in the system, that is through the Kendall’s tau \( \tau_{X_0,X_j} \) of the vector \((X_0, X_j)\).

For the empirical analysis we restrict to the setup considered in Examples 3.1 and 3.2. More specifically, we consider the case in which all bivariate underlying copulas, modeling the dependence structure between each idiosyncratic component and the associated systemic shock component, are of Clayton type. Since Clayton copula exhibits lower-tail dependence, we are assuming stronger dependence between each idiosyncratic shock arrival time \(X_j\) and the corresponding systemic component arrival time \(Y_j\) when they have a very high probability to occur: this is in line with the well known fact that dependence tends to increase in crisis periods. This choice has also the advantage to let us deal with very nice and meaningful formulas.

In the assumed setup, the bivariate Kendall’s tau of the pairs \((T_i, T_k)\) depend on the the set of parameters

\[\Theta = (\alpha_1, \ldots, \alpha_d, \theta_0, \ldots, \theta_d, \beta_1, \ldots, \beta_d)\]  

\(\alpha_j\) represents the ratio between the systemic shock intensity and the marginal one; \(\theta_j\) measures the contribution of each bank to the systemic shock intensity while \(\theta_0\) measures the contribution of some completely independent shock; the parameters \(\beta_j\) are the parameters of the involved bivariate copulas. As shown in (10), these parameters fully characterize the dependence structure of the vector of observed lifetimes \(T\).

The estimation technique will consist in a moment based approach, through which theoretical bivariate Kendall’s tau will be fitted to empirical ones. Once the parameters are estimated, we can use (6) to estimate the systemic riskiness of each bank.
4.1 Data set

Our data set consists of daily 5 years CDS quotes, from 01/01/2009 to 31/12/2016 of the European banks classified as SIFI by the Financial Stability Board\(^1\). Data were downloaded from Datastream.

We assume that all arrival times are exponentially distributed, that is, in the notation of Examples 3.1 and 3.2, \(G(x) = e^{-x}\): as a consequence, also observable lifetimes are exponentially distributed, with intensities \(\lambda_0 + \lambda_j\) for \(j = 1, \ldots, d\). We also assume a constant interest rate and constant Loss-Given-Default. Thanks to these assumptions, survival probabilities and intensities can be easily extracted from CDS spreads (see Brigo and Mercurio, p.735-6).

Since a sample of default times is not available, we are not in the position to recover the empirical Kendall’s tau from default times data. However, the Kendall’s tau is the difference between the proportion of concordant and discordant pairs of observations and an increase in the intensity of default corresponds to the perception that the default time is going to occur earlier: in the absence of more appropriate data, we recover intensities from the CDS spreads dataset and we assume as empirical Kendall’s tau those estimated from intensities. As a consequence our analysis will be based on the information implied by the CDS.

4.2 Estimation procedure

Let \(\hat{\tau}_{ik}\), \(i = 1, \ldots, d - 1\), \(k = i + 1, \ldots, d\) be the estimated pairwise empirical Kendall’s tau and \(\tau_{ik}(\alpha_i, \alpha_k, \theta_i, \theta_k, \beta_i, \beta_k)\) be the corresponding theoretical ones. Parameters are estimated by solving

\[
\hat{\Theta} = \arg\min_{\Theta} \sum_{i=1}^{d-1} \sum_{k=i+1}^{d} (\hat{\tau}_{i,k} - \tau_{i,j}(\alpha_i, \alpha_k, \theta_i, \theta_k, \beta_i, \beta_k))^2.
\]

(19)

This moment based procedure is a generalization of the Kendall’s tau-based estimation procedure considered in Genest and Rivest (1993) to the multidimensional framework and it has been analyzed and studied in Mazo et al. (2015). The optimization required in (19) is not a trivial task and can only be solved numerically.

4.3 Results

The procedure applied to all SIFI European banks does not provide a good fit and the global minimum remains far from 0. Things work much better if one restrict the analysis to the banks that in the list of globally systemically important banks provided by the Financial Stability Board, are identified as particularly systemically risky since they have associated buckets higher than 1 (higher buckets correspond to higher level of systemic importance): BNP Paribas, Deutsche Bank, HSBC, Barclays.

\(^1\)See the report “2016 list of global systemically important banks (G-SIBs)” published by the Financial Stability Board, [http://www.fsb.org/wp-content/uploads/2016-list-of-global-systemically-important-banks-G-SIBs.pdf](http://www.fsb.org/wp-content/uploads/2016-list-of-global-systemically-important-banks-G-SIBs.pdf)
Table 1: Yearly Kendall’s tau values $\tau_{X_0, X_j}$.

|        | DEUTSCHE BANK | BNP PARIBAS | BARCLAYS | HSBC  |
|--------|---------------|-------------|----------|-------|
| 2009   | 0.66425573    | 0.04839780  | 0.25905985 | 0.01702301 |
| 2010   | 0.1961576     | 0.00000000  | 0.2135096  | 0.5891973  |
| 2011   | 0.00000000    | 0.17627564  | 0.04529224 | 0.77736158 |
| 2012   | 0.00000000    | 0.2586159   | 0.85703128 | 0.11605882 |
| 2013   | 0.00000000    | 0.3797177   | 0.00000000 | 0.6192373  |
| 2014   | 0.07860066    | 0.00000000  | 0.00000000 | 0.92038204 |
| 2015   | 0.00000000    | 0.3477391   | 0.00000000 | 0.6512462  |
| 2016   | 0.1911062     | 0.00000000  | 0.00000000 | 0.8068463  |

The estimation is conducted on a yearly basis and, once the parameters have been estimated, the Kendall’s tau $\tau_{X_0, X_j}$ are evaluated according to (6). In Table 1 we show the obtained values of the Kendall’s tau between each idiosyncratic component $X_j$ and the systemic shock $X_0$.

It worths to mention that the global minimum in (19) is very close to 0 in years 2009-2011. In particular, the fit is particularly good in year 2009: in this year the US banking crises has spread in Europe with its systemically relevant effects and, as shown in Table 1 all banks are systemically risky, in the sense considered in this paper, even if with different degrees. Table 1 shows that, even if the capability of each bank to cause the bankruptcy of the whole banking system changes with time, HCBS is globally the most risky in the analyzed period. Comparing the obtained results with the available 2015 and 2016 reports of the Financial Stability Board (based, respectively, on end 2014 and end 2015 data), we observe the the extraordinary high degree of systemic riskiness estimated for HSBC in 2014 (92%) is in line with the association of this bank to bucket 4 (the highest) in the 2015 report, while its reduced degree of riskiness estimated in 2015 (65%) is in line with the downgrade of HSBC to bucket 3 in the 2016 report. Additionally, according to Table 1 Barclays can be classified as the less risky (in the sense considered in this paper) in recent past years: this is consistent with the fact that, among the considered banks, it is the only one to which it is assigned bucket 2 in the 2016 report.

In Table 2 we list the Kendall’s tau values between each observed lifetime $T_j$ and the systemic shock arrival time $X_0$. We notice that in some cases the values $\tau_{X_0, X_j}$ and $\tau_{X_0, T_j}$ are very close each other: this is the case of Deutsche Bank in 2009, Barclays in 2012 and HSBC in 2011, 2014 and 2016. As noticed at the end of Example 3.1 this is due to the fact that the dependence of the bank lifetime with the systemic risk is essentially given by its capability to induce a systemic shock and in a negligible way by the fact that it is subjected to the systemic shock itself: this is a clear evidence of riskiness.

\[\text{see “2015 list of global systemically important banks (G-SIBs)”, } \text{http://www.fsb.org/wp-content/uploads/2015-update-of-list-of-global-systemically-imp} \]


| Year | DEUTSCHE BANK | BNP PARIBAS | BARCLAYS | HSBC       |
|------|---------------|-------------|----------|------------|
| 2009 | 0.6939868     | 0.6954599   | 0.9342763| 0.9406805  |
| 2010 | 0.8664811     | 0.8623019   | 0.8742886| 0.7039019  |
| 2011 | 0.9176120     | 0.9161709   | 0.8260221| 0.8066270  |
| 2012 | 0.8962147     | 0.8163799   | 0.8570315| 0.7563900  |
| 2013 | 0.8559030     | 0.8049459   | 0.8226217| 0.8344751  |
| 2014 | 0.6916339     | 0.7916272   | 0.9123942| 0.9203822  |
| 2015 | 0.8375908     | 0.7653512   | 0.7463275| 0.8434409  |
| 2016 | 0.2536626     | 0.5585990   | 0.6998114| 0.8206532  |

5 Conclusions

In this paper we have introduced a generalization of the Marshall-Olkin distribution in which some non-exchangeable dependence among the underlying shocks arrival times is assumed. More specifically, we have assumed that each lifetime is the first arrival time between an idiosyncratic and a systemic shock and, unlike the standard Marshall-Olkin model, we have assumed some dependence between each idiosyncratic arrival time and the systemic one: the resulting model is particularly suitable to model situations in which lifetimes influence each other only through the systemic shock arrival time on which they are dependent. The obtained joint distribution of lifetimes is investigated: its singularity analyzed and its dependence properties studied through the induced copula functions and the associated pairwise Kendall’s function and Kendall’s tau. The dependence structure is the composition of a Marshall-Olkin type dependence and the assumed dependence of each idiosyncratic component with the systemic shock arrival time: the higher the second component, the more risky is the considered entity.

The model is applied to the analysis of the systemic riskiness of SIFI type European banks. Results show that the model gives a better fits if one restrict to particularly “big” SIFI banks, according to the Financial Stability Board buckets classification: BNP Paribas, Deutsche Bank, HSBC, Barclays.

The obtained results allow to classify the systemic riskiness of these banks according to their capability to induce the simultaneous default of all the system. This is an information that could be used, in addition to the already used ones, to more completely classify the riskiness of a bank.

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