SELF-ACTION EFFECTS IN THE THEORY OF CLASSICAL SPINNING CHARGE *

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The back-reaction effects for the spinning charge moving through the constant homogeneous electromagnetic field are studied in the context of the mass-shift (MS) method. For the g=2 magnetic moment case we find the (complex) addition to the classical action. Its dependence on the integrals of the unperturbed motion proves to be important in determination of the orbital radiation effects and could assist in understanding the radiation polarization (RP) phenomenon.

1. Introduction

The topic of this note is self-interacting classical charge possessing magnetic moment. The recently renewed interest in the pseudoclassical models of spinning particles stems from the close relations between those models and string theory. With rare exception, the problem of self-interaction for the spinning charge have not been considered there. At that time this problem could find an interesting application in the theory of RP phenomenon.

The effects of self-interaction are usually approached through the Abraham-Lorentz-Dirac (ALD) equation describing the radiation effects for the spinless charge. Being generalized on a non-zero spin case, this approach leads to inappropriately complicated equations (see e.g. [3]). On the other hand, to analyse the self-action effects one can use the complex addition

\[ \Delta W = \frac{1}{2} \int \int J_\mu(x) \Delta_c(x, x'; \mu_{ph}) J_\mu(x') \, dx \, dx' \bigg|_0^F \]  

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*With obvious exceptions we use the system of units with \( c = 1, \hbar = 1, \alpha = e^2/4\pi\hbar c, \) and 4-vector notations \( x_\mu = (x, ix_0). \)
to the classical action functional of the particle. Inspired by QFT, this approach relies on the fact (see e.g. [8]) that \( \exp(i\Delta W) \) is an amplitude and \( \exp(-\frac{2}{\hbar}3\Delta W) \) is the corresponding probability of the photon vacuum to preserve when the classical source \( J_\mu \) is present. The causality of the Green function (GF) \( \Delta_c \) in (1) guarantees the account of radiation \( (\Im \Delta W > 0) \) emitted by the source \( J_\mu \). The dependence of the self-action \( \Delta W \) on the integrals of the unperturbed motion carries an important information. For example, after specific procedure of renormalization [7], one can obtain an exact solution of the ALD equation for the non-relativistic cyclotron motion in the following form:

\[
v_x + iv_y = Ae^{-i\Omega t}, \quad \Omega = \frac{eH}{m + \delta m}, \quad \frac{\delta m}{m} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{i\omega_c}{b}}, \quad \frac{b}{5} = \frac{2}{3} \frac{e^2}{4\pi m}.
\]  

(2)

Here \( \omega_c = eH/m \) is the cyclotron frequency without regard for radiation and \( v_x, v_y \) are velocity’s components orthogonal to the magnetic field \( H \). The negativity of \( \Im \Omega \) originates from the causality of the GF \( \Delta_c \), and the dependence \( \Delta W \) on the (unperturbed by radiation) \( v_\perp^2 \) was the starting point.

This article discusses the simplified version of the polarization effects for the spinning particle with no anomalous magnetic moment moving in the constant homogeneous magnetic or electric field. For such external fields the self-action \( \Delta W \) reduces to the MS according to [5]

\[
\Delta W = -\Delta m T,
\]  

(3)

where the proper time \( T \) corresponds to the interval of the charge’s stay in the external field. For the eq. (3) were meaningful the formation time of the \( \Delta m \) should be much less than \( T \).

The need in adequate quasiclassical interpretation of the RP was pointed out in the book [2]. The wanted explanation would be done i) for the different polarizations of electrons and positrons, ii) for not complete (i.e. <100 p.c.) polarization degree (QED gives 0.924) and iii) for the numerical value of the polarization time [2]

\[
T_{QED} = \frac{8\sqrt{3}}{15} \frac{a_B}{c} \gamma_{\perp}^{-2} \left( \frac{H_c}{H} \right)^3,
\]  

(4)

\( (a_B = 4\pi \hbar^2/m^2, H_c = m^2c^3/\epsilon h \sim 4.4 \times 10^{13} Gs, \gamma_{\perp}^{-2} = 1 - v_{\perp}^2) \). An elementary classical consideration [2] leads within a factor of order 1 to the same value for the characteristic time \( T_{QED} \). This shows that quantum nature of RP might be associated with the relationship \( T_{QED} \sim 1/\mu_B \) only (\( \mu_B \)}}
being the Bohr magneton) and needs quantum description neither for the orbital motion nor for the spin precession.

2. The general formulae

The source \( J_\alpha \) in (1) consists of orbit part

\[
j_\alpha(x) = e \int d\tau \dot{x}_\alpha(\tau) \delta^{(4)}(x - x(\tau))
\]

and spin contribution \( \partial_\beta M_{\alpha\beta}(x) \), where

\[
M_{\alpha\beta}(x) = \int d\tau \mu_{\alpha\beta}(\tau) \delta^{(4)}(x - x(\tau))
\]

is the polarization density. The dependence of \( \mu_{\alpha\beta} = i\mu\varepsilon_{\alpha\beta\gamma\delta} \dot{x}_\gamma S_\delta \) on \( \tau \) is determined from the Lorentz and Bargmann-Michel-Telegdi (BMT) equations (see e.g. [6]):

\[
\dddot{x}_\alpha = e m F_{\alpha\beta} \dot{x}_\beta, \quad \hbar^2 \dddot{S}_\alpha = \mu F_{\alpha\beta} S_\beta + \left( \frac{g}{2} - 1 \right) \mu_B \dot{x}_\alpha (\dot{x} \cdot F \cdot S).
\]

Here \( \mu = \frac{g}{2} \mu_B \), \( S_\alpha S_\mu = \zeta^2 = 1 \), the overdots denote the derivatives w.r.t. proper time \( \tau \), and, in what follows, we put \( g = 2 \).

After substitution of the source \( J_\mu = j_\mu + \partial_\nu M_{\mu\nu} \) in the r.h.s. of the eq.(1) and integration by parts, we find that

\[
\Delta W = \Delta W_{or} + \Delta W_{so} + \Delta W_{ss}.
\]

The orbit part \( \Delta W_{or} = -\Delta m_{or} \cdot T \) for the electric or magnetic external fields was considered in [5] and will not be discussed below. The ”spin-orbit” and ”spin-spin” terms are:

\[
\Delta W_{so} = -e \int d\tau \int d\tau' \dot{x}_\alpha(\tau) \mu_{\beta\alpha}(\tau') \partial_\alpha \Delta_c(x, x'; \mu_{ph}) \bigg|_0^F,
\]

\[
\Delta W_{ss} = \frac{1}{2} \int d\tau \int d\tau' \mu_{\alpha\beta} \mu_{\gamma\delta} \partial_\beta \partial_\gamma \Delta_c(x, x'; \mu_{ph}) \bigg|_0^F.
\]

The spin-orbit and spin-spin terms in eq.(9) form the small corrections to orbital one. For example, the magnetic MS ratio \( \Delta m_{so}/\Delta m_{or} \approx (H/H_c)\gamma_\perp \) [7], so that only in the far quantum region those terms could be of the same order. Below we shall focus our attention on \( \Delta m_{so} \) only (see (3)), regarding
the latter as a major contribution w.r.t. $\Delta m_{so}$. Since infrared regulator $\mu_{ph}$ as well as the subtraction $\mid F_0 \rangle$ could be omitted here, we arrive at

$$\Delta W_{so} = -\frac{\mu e}{2\pi^2} \int d\tau \int d\tau' \varepsilon_{\alpha\beta\gamma\delta} (x-x')_{\alpha} \hat{x}_{\beta}(\tau) \hat{x}_{\gamma}(\tau') S_{\delta}(\tau') \frac{(x-x')^2}{[(x-x')^2]^2}, \quad (12)$$

where the use was made of

$$\Delta_c(x,x';0) = i(2\pi)^{-2}/[(x-x')^2 + i0]. \quad (13)$$

3. The spinning charge in magnetic field

Choosing the direction of $\mathbf{H}$ along the z-axis, $\mathbf{H} = (0, 0, H)$, we have the following integrals of the motion: z-component of the four-velocity $v_3 = v_3 \gamma$ and corresponding spin component $S_3$; the energy $\mu u_0$, the scalar product $S_\perp \cdot v_\perp$ of vectors orthogonal to $\mathbf{H}$ and $v_\perp$. With $u_\perp^2 = u_0^2 - u_3^2 = 1 \equiv u_3^2 - 1 = v_\gamma^2 \gamma^2$, $\gamma = (1 - v^2)^{-1/2}$, $v^2 = v_3^2 + v_\perp^2$ and after some algebra the expressions (3) and (12) give rise to:

$$\Delta m_{so} = -\frac{i\mu e}{2\pi^2} (S_3 - v_3 S_0) \omega_c^2 f_m(v_\perp, v_3), \quad (14)$$

where formfactor

$$f_m(v_\perp, v_3) = 2v_\gamma^3 \sqrt{4\sin^2(x/2) - x \sin x} \int_0^\infty \frac{4\sin^2(x/2) - x \sin x}{(4u_\perp^2 \sin^2(x/2) - u_\parallel^2 x^2)^2} \, dx \quad (15)$$

takes the retardation effects into account (variable $x = \omega_c (\tau - \tau')$). Note, that everywhere we put $e = -|e|$ and that, according to ”Frenkel condition”,

$$S_0 = S \cdot v. \quad (16)$$

In calculating the determinant present as a nominator of the integrand in eq. (12), one finds \(^b\)

$$\varepsilon_{\alpha\beta\gamma\delta} (x-x')_{\alpha} \hat{x}_{\beta}(\tau) \hat{x}_{\gamma}(\tau') S_{\delta}(\tau') = -i u_\perp^2 u_{[0] S_3} \omega^{-1}_c (4\sin^2(x/2) - x \sin x), \quad (17)$$

so that $\Delta m_{so}$ would have an opposite sign for positrons because of the factor $\omega^{-1}_c$.

Considering experimental situation \(^2\) we put $v_3 = 0$ and find that spin contribution into the amplitude of the vacuum preservation reduces to the factor $\exp (3\Delta m_{so} \cdot T)$ where, in accordance with (14), $3\Delta m_{so}$ is positive.

\(^b\) $u_{[0] S_3}$ means antisymmetrized combination $u_0 S_3 - u_3 S_0$.\)
when $S_3 < 0$ and negative otherwise. Hence, the probability of radiation from ‘spin-down’ electron as compared with ‘spin-up’ one is by the factor
\[
\frac{\exp (2\Delta m_{so} T + 2\Delta m_{so} (\uparrow) T)}{\exp (2\Delta m_{so} T + 2\Delta m_{so} (\downarrow) T)} = \exp \left( -4T \frac{\mu e}{2\pi^2} \omega^2_c f_m \right)
\]
(18)
suppressed. To obtain the same factor in the positron case one should inverse the directions of arrows in the l.h.s. of eq.(18). The characteristic laboratory time $T_{\text{char}}$ which is deduced from (18),
\[
T_{\text{char}} \simeq \left( \frac{\mu e}{\pi^2} \omega^2_c f_m \right)^{-1} \gamma \simeq 2\sqrt{3} \frac{a_E}{c} \gamma^{-1} (H_c / H)^2,
\]
(19)
differs from $T_{QED}$ in (4) (numerical evaluation for $H \sim 10^4 Gs$ and electron energy about 1 GeV shows $T_{QED} \sim 5 \cdot 10^5 T_{\text{char}}$). It should not be considered as a surprise because time $T_{\text{char}}$ accumulate all of the possibilities for electron to leave the initial state, so that it should be less than $T_{QED}$. Note that formation time of $\Delta m_{so}$ (extracted from (15) at $v_3 = 0$)

\[
\Delta \tau \simeq \frac{x}{\omega_c} = \frac{\Delta \tau}{\omega_c} \gamma^{-1} \gamma_{\perp}^{-1}, \quad \text{and} \quad T_{\text{char}}/\gamma_{\perp} \Delta \tau \simeq \gamma^{-1} (H_c / \alpha H) \gg 1 \text{ (see the text following the eq.(3)).}
\]

4. The spinning charge in electric field

The list of the integrals of the motion in electric field $E = (0, 0, E)$ is following: $u_1, u_2, S_1, S_2$ being the $(x, y)$-components of the 4-velocity $\vec{x}_\mu$ and the spin $S_\mu$ respectively. The substitution of the solutions of eqs.(8) ($g = 2, e = -|e|$) into r.h.s. of (12) leads to
\[
\Delta m_{so} = -i \frac{\mu e}{2\pi^2} w^2 \bar{v}_1 S_2 f_e(u^2_{\perp}),
\]
(20)
\[
f_e(u^2_{\perp}) = 2\gamma^3_0 \int_0^\infty \frac{x \sinh x - 4 \sinh^2 (x/2)}{[u^2_{\perp} x^2 - 4 u^2_{\parallel} \sinh^2 (x/2)]^2} dx,
\]
(21)
with $w = eE/m$, $\gamma^3_0 = 1 + u^2_{\perp} = u^2_{\parallel}$, $u^2_{\perp} = u^2_1 + u^2_2$. $\bar{v}_{1,2} \equiv v_{1,2}/\gamma_0$ are the $(x, y)$-components of velocity $\vec{v}$ taken at the moment when the component $v_3 = 0$.

The wanted asymmetry between polarizations is associated with the factor $\langle \vec{v} \times \vec{S} \rangle \cdot \vec{E}$. One can see that $\Delta \Delta m_{so} > 0$ when $\langle \vec{v} \times \vec{S} \rangle_3 < 0$, i.e. when the system $\vec{S}, \vec{v}, \vec{E}$ forms the right-handed triple. This orientation corresponds to the suppressed radiation probability by the factor $\exp \left( -2\Delta m_{so} T \right)$ (cf. with (18)). For positrons the primary direction is,
of course, opposite. Notice that orbital part of the MS in electric field involves the infrared singularity signalling about the large formation time of the MS\textsuperscript{5}. This does not affect the relative quantities like l.h.s of eq. (18), but makes it necessary to weaken an ‘orbital background’ of the spin radiation\textsuperscript{2}. That could in principle be done by the employment of boundaries\textsuperscript{8}.

5. Conclusion

The primary purpose of this note was to demonstrate a new possibility to account for radiation effects in the dynamics of the spinning charge. The present consideration is certainly not complete. One could ask about the role of \((g - 2)\)- term in BMT equation, see (8). It is known\textsuperscript{11} that in the relativistic limit this term might be of first importance. The next point is \(\Delta W_{ss}\) in (11) which we are going to discuss in a subsequent publication. Its dependence on \(S_3\) would be expected to clarify the non-complete polarization degree\textsuperscript{4}. Of some practical interest could be the exact dependences of the formfactors \(f_m\) and \(f_e\) on dynamical invariants as well as not discussed yet the possibility to observe RP in electric field.

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References

1. V.N. Baier. Uspekhi Fizicheskikh Nauk,\textbf{105} (1971) 441.
2. I.M. Ternov, Introduction to spin Physics of relativistic particles [in Russian], Moscow Univ., Moscow, 1997.
3. E.G.P. Rowe, G.T. Rowe, Phys. Rep.,\textbf{149} (1987) 287.
4. V.I. Ritus, ZhETF, \textbf{75} (1978) 1560.
5. V.I. Ritus, ZhETF, \textbf{80} (1981) 1288.
6. C. Itzykson, J.-B. Zuber. \textit{Quantum Field Theory}. McGraw-Hill, 1980.
7. S.L. Lebedev. In: \textit{The 3-d Saharov Conference. Proceedings}. Scientific World, Moscow, 2002; [hep-th/0211078]
8. S.L. Lebedev. ZhETF, \textbf{106} (1994) 956.

\textsuperscript{4} Que can guess that the retardation interaction term \(\propto \mu_{\alpha\beta} \mu_{\alpha\gamma}^\prime\) in (11) corresponds to a recoil effects of spin radiation.