THETA DISTINGUISHED REPRESENTATIONS, INFLATION AND THE
SYMMETRIC SQUARE L-FUNCTION

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Abstract. Let $\Pi_0$ be a representation of a group $H$. We say that a representation $\tau$ is $(H, \Pi_0)$-distinguished, if it is a quotient of $\Pi_0$. It is natural to ask whether this notion “inflates” to larger groups, in the sense that a representation $I(\tau)$ induced from $\tau$ and $H$ to a group $G$, is $(G, \Pi)$-distinguished. We study representations distinguished by theta representations: $H = GL_n$, $\Pi_0$ is a pair of the exceptional representations of Kazhdan and Patterson, $G = \text{GSpin}_{2n+1}$ and $\Pi$ is a pair of the small representations of Bump, Friedberg and Ginzburg. We prove a Rodier-type hereditary property: a tempered representation $\tau$ is distinguished if and only if $I(\tau)$ is distinguished, and the multiplicity in each model is the same. If $\tau$ is supercuspidal and distinguished, we prove that the Langlands quotient of $I(\tau)$ is distinguished. As a corollary, we characterize supercuspidal distinguished representations, in terms of the pole of the local symmetric square $L$-function at $s = 0$.

1. Introduction

Let $\tau$ be an admissible representation of $GL_n(F)$, where $F$ is a local non-Archimedean field. Let $\theta_0$ and $\theta'_0$ be a pair of exceptional representations of a double cover $\overline{GL}_n(F)$ of $GL_n(F)$, in the sense of Kazhdan and Patterson [KP84]. We say that $\tau$ is distinguished if

$$\text{Hom}_{GL_n(F)}(\theta_0 \otimes \theta'_0, \tau^\vee) \neq 0.$$ 

Here $\tau^\vee$ is the representation contragradient to $\tau$. Equivalently, the space of $GL_n(F)$-invariant trilinear forms on $\tau \times \theta_0 \times \theta'_0$ is nonzero.

This space first appeared in a global context. Let $\pi$ be a unitary cuspidal automorphic representation of $GL_n(A)$, where $A$ is the adeles ring of a global field. Assume that $\pi$ has a trivial central character. Bump and Ginzburg [BG92] proved that if the partial symmetric square $L$-function $L^S(s, \pi, \text{Sym}^2)$ has a pole at $s = 1$, the following period integral is nonvanishing

$$\int_{Z GL_n(F) \backslash GL_n(A)} \varphi_\pi(g) \Theta(g) \Theta'(g) dg.$$ 

Here $Z$ is a subgroup of finite index in the center $C_{\text{GL}_n}(A)$ of $GL_n(A)$ ($Z = C_{\text{GL}_n}(A)$ when $n$ is odd); $\varphi_\pi$ is a cusp form in the space of $\pi$; $\Theta$ and $\Theta'$ are automorphic forms in the space of a global exceptional representation of $\overline{GL}_n(A)$. For $n = 2$, an earlier work by Patterson and Piatetski-Shapiro [PPS89] showed that a similar integral characterizes the pole at $s = 1$, for a global field of odd characteristic.

Periods of automorphic forms are often related to poles of $L$-functions and to questions of functoriality. Ginzburg, Jiang and Soudry [GJS10] described such relations in a general setup and also considered several examples. Let $G_n = \text{GSpin}_{2n+1}$ be the odd general spin group of rank $n + 1$. Let $E(g; \rho, s)$ be the Eisenstein series corresponding to an element $\rho$ in the space of the representation of $G_n(A)$ induced from $\tau| \det|^{s} \otimes 1$ ($s \in \mathbb{C}$, $g \in G_n(A)$). The residual representation $E_\pi$ is the space spanned by the residues $E_{1/2}(\cdot; \rho)$ of $E(g; \rho, s)$.
at $s = 1/2$. The following result formulated in [GJS10] (Theorem 3.3) was proved in a series of works ([BG92, BFG03, GJS10, Kap]): the following conditions are equivalent.

1. $L^S(s, \pi, \text{Sym}^2)$ has a pole at $s = 1$.
2. The period integral $\langle \text{L} \rangle$ is nonzero.
3. The residual representation $E_\pi$ is nonzero.
4. $\pi$ is the Langlands functorial transfer of an irreducible generic cuspidal automorphic representation of the split SO$_n(\mathbb{A})$ (if $n$ is even) or Sp$_{(n-1)/2}(\mathbb{A})$ ($n$ is odd).

Note that this was stated in [GJS10] with $G_n$ replaced by SO$_{2n+1}$ (see below).

To prove that the nonvanishing of (1.2) implies the nontriviality of $E_\pi$, one can relate the period to the following co-period integral

$$CP(E_{1/2}(\rho), \Theta, \Theta') = \int_{C_{Gn}(\mathbb{A})(Gn(F)\backslash Gn(\mathbb{A}))} E_{1/2}(g; \rho)\Theta(g)\Theta'(g)dg.$$ 

Here $\Theta$ and $\Theta'$ belong to the exceptional, or small, representation of Bump, Friedberg and Ginzburg [BFG03], whose analog for $G_n$ was described in [Kap14a]. This integral was studied in [Kap] for SO$_{2n+1}$ (extended in [Kap14a] to $G_n$) and we proved the implication $(2) \Rightarrow (3)$. In the setting of $G_n$, one can also study the twisted symmetric square $L$-function. The Rankin-Selberg integral representation for this function has recently been developed by Takeda [Tak14].

The global unfolding of $CP(E_{1/2}(\rho), \Theta, \Theta')$ (in [Kap]) has a local counterpart. Assume that $\tau$ is a distinguished representation. Let

$$I(\tau) = \text{Ind}^{G_n}_{Q_n}(\delta^{1/2}Q_n^1|\text{det}|)^{1/2} \otimes 1),$$

where $Q_n$ is the Siegel parabolic subgroup. Using an integral over $Q_n(F)\backslash G_n(F)$, we show (Proposition 4.1) that for a certain pair $\theta$ and $\theta'$ of exceptional representations of $\tilde{G}_n(F)$ ([BFG03, Kap14a], see below),

$$\text{Hom}_{G_n(F)}(\theta \otimes \theta', I(\tau)^\vee) \neq 0.$$ 

In other words, $I(\tau)$ is a distinguished representation of $G_n(F)$. In fact, depending on the central character of $\tau$, we may need to replace $|\text{det}|^{1/2} \otimes 1$ with $|\text{det}|^{1/2} \otimes \eta$ where $\eta$ is a character of $F^*$. (See the proposition for details.) The local problem is to prove that the Langlands quotient $LQ(I(\tau))$ of $I(\tau)$ is also distinguished.

In the present study we consider an irreducible unitary supercuspidal $\tau$. In this case $I(\tau)$ is either irreducible and generic, or is of length two, has a unique irreducible generic subrepresentation and $LQ(I(\tau))$ is non-generic. Here is our main result, implying that if $\tau$ is distinguished, so is $LQ(I(\tau))$. For more details see Corollary 4.4.

**Theorem 1.** The space of $\theta \otimes \theta'$ as a representation of $G_n(F)$ does not afford a Whittaker functional.

A similar “inflation” phenomena has already been observed by Ginzburg, Rallis and Soudry [GRS99a] (Theorem 2). Assume that $\tau$ is a supercuspidal and self-dual representation, such that the exterior square $L$-function $L(s, \tau, \lambda^2)$ has a pole at $s = 0$. This implies that $\tau$ has a Shalika model and then according to Jacquet and Rallis [JR96], $\tau$ admits a (nontrivial) $GL_n \times GL_n$ invariant functional. In turn, the representation parabolically induced from $\tau|\text{det}|^{1/2}$ to Sp$_{2n}$ has an $Sp_n \times Sp_n$ invariant functional. Ginzburg, Rallis and Soudry [GRS99b] (Theorems 16-17) showed that an irreducible generic representation of Sp$_{2n}$ does not admit such a functional. It follows that the $Sp_n \times Sp_n$ functional factors through the Langlands quotient.
This inflation was one of the ingredients used by Lapid and Mao for the proof of their conjecture on Whittaker-Fourier coefficients, in the case of the metaplectic group ([LM13, LM14a, LM14b, LM15]). Note that their conjecture actually applies to any quasi-split group as well as the metaplectic group. Our results here and their extension, in a forthcoming work, to an arbitrary irreducible generic distinguished representation, are expected to be used in a proof of this conjecture for even orthogonal groups.

Here we have a similar relation between distinguished representations and the symmetric square $L$-function.

**Theorem 2.** Let $\tau$ be an irreducible unitary supercuspidal representation of $GL_n$. Then $\tau$ is distinguished if and only if $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$.

Refer to Shahidi [Sha92] (Theorem 6.2) for a description of the poles of $L(s, \tau, \text{Sym}^2)$ in this setting.

The proof of Theorem 1 is essentially the local analog of the global computation of the co-period in [Kap, Kap14a]. Write $Q_n = M_n \ltimes U_n$ and let $C = C_{U_n}$ be the center of the unipotent radical $U_n$. The global unfolding argument involves a Fourier expansion of $\Theta$ along $C$. Consider a non-generic character of $C$, this means that its stabilizer in $M_n$ contains a unipotent radical $V$ of a parabolic subgroup of $GL_n$. The corresponding Fourier coefficient is constant on $V$, then the cuspidality of $\tau$ is used to prove that the integral vanishes.

When $n$ is odd (and $n > 1$), all characters of $C$ are non-generic. In the even case there is one generic orbit of characters. Its stabilizer is “almost” a Jacobi group, its reductive part is $Sp_{n/2} \times G_0$, where $Sp_{n/2}$ is a symplectic group in $n$ variables. One might attempt to prove invariance of the product of Fourier coefficients under $Sp_{n/2}(A)$, then use the fact that $\tau$ does not admit nontrivial symplectic periods ([JR92]). Albeit the Fourier coefficients do not enjoy this invariance, a certain convolution against Weil theta functions, introduced by Ida [Ike94], can be used instead.

The local argument involves the computation of the twisted Jacquet modules $\theta_{C, \psi_k}$ of $\theta$ with respect to $C$ and a representative $\psi_k$ of some orbit of characters. In contrast with the global setting, the generic character is the crux of the proof. Roughly, this is because $\theta_{C, \psi_k}$ is a tensor of an exceptional representation of $GL_k(F)$ and the Jacquet module of an exceptional representation of $G_{2k}(F)$ along $C_{U_{2k}}$ and a generic character.

When the character is generic, the twisted Jacquet module is (by restriction) a representation of a Jacobi group. The local theory of smooth representations of Jacobi groups [vD78, MVW87, BS98] describes such a representation as a tensor $\kappa \otimes \omega_\psi$, where $\omega_\psi$ is the Weil representation. The Heisenberg group acts trivially on the space of $\kappa$, while the action of the reductive part separates into an action on the space of $\kappa$, and one on the space of $\omega_\psi$. We prove that $\kappa$ is a trivial representation of $Sp_{n/2}(F)$.

**Theorem 3.** Assume $n$ is even and let $\psi_{n/2}$ be a generic character of $C_{U_n}$. As a Jacobi representation $\theta_{C_{U_n}, \psi_{n/2}}$ is the direct sum of (possibly infinitely many) copies of $\omega_\psi$.

Note that the action of the $G_0$ part of the stabilizer on $\theta_{C_{U_n}, \psi_{n/2}}$ is given simply by the central character of $\theta$.

This result underlies Theorem 1. Furthermore, it implies the following multiplicity property. Assume that $\tau$ is irreducible and tempered. We prove that $\tau$ is distinguished if and only if $I(\tau)$ is. Moreover, the dimensions of (1.3) and (1.1) are equal. See Proposition 4.6 for the precise statement. Note that Kable [Kab01] conjectured, and under a certain homogeneity
assumption proved ([Kab01] Corollary 6.1), that (1.1) enjoys multiplicity one. These results motivate the introduction of “exceptional models” over $\text{GL}_n$ and $G_n$.

Theorem 3 may have additional applications. Explicit descriptions of Jacquet modules of exceptional representations have had numerous applications (see below).

Note that when $n = 2$, $Q_n$ is the Heisenberg parabolic subgroup in the notation of Gan and Savin [GS05]. In their terminology, Theorem 3 shows that $\theta$ is “weakly minimal”. In this case it is the minimal representation and $\theta_{U_n, \psi_{n/2}} \cong \psi_{\psi}$ ([GS05] Section 3).

Bump, Friedberg and Ginzburg [BFG03] constructed the small representation $\theta_{SO_{2n+1}}$ for the special odd orthogonal group. This is a representation of a “double cover” $\tilde{SO}_{2n+1}(F)$, obtained by restricting the 4-fold cover of $\text{SL}_{2n+1}(F)$ of Matsumoto [Mat69]. For the low rank cases $n = 2, 3$, it is the minimal representation. In fact for $n = 3$, this representation was already developed by Roskies [Ros96], Sabourin [Sab96] and Torasso [Tor97]. For $n > 3$, there is no minimal representation for a group of type $B_n$ (Vog81).

Bump, Friedberg and Ginzburg [BFG06] showed that when $n > 3$, $\theta_{SO_{2n+1}}$ is attached to one of the possible coadjoint orbits, which is smallest next to the minimal one. This translates into the vanishing of a large class of Fourier coefficients, called generic in [BFG03] (see also CM93, Car93, Gin06). Locally, this means that a large class of twisted Jacquet modules vanish. The representation $\theta_{SO_{2n+1}}$ was used by these authors to construct a lift with certain functorial properties, between covers of orthogonal groups [BFG06]. The representation $\theta_{SO_7}$ was used to construct an integral representation ([BFG00]).

There is a technical issue when working with $\tilde{SO}_{2n+1}(F)$: the underlying field must contain all 4 4-th roots of unity. This can be remedied using $G_n$. Indeed, one obtains a nontrivial double cover of $G_n(F)$ by restricting the 2-fold cover of $\text{Spin}_{2n,3}(F)$ of Matsumoto [Mat69]. The theory of Bump, Friedberg and Ginzburg [BFG03] can be extended to $\tilde{G}_n(F)$, mainly because both groups are of type $B_n$ and in particular, the unipotent subgroups are isomorphic. The details were carried out in [Kap14a]. Our results here are stated for $G_n$, but apply similarly to $SO_{2n+1}$ and $\theta_{SO_{2n+1}}$ (see Section 3).

Minimal representations have been studied and used extensively, by numerous authors. Knowledge of Fourier coefficients, or Jacquet modules, has proved very useful for applications GRS03, Gin06, GJS11. They have played a fundamental role in the theta correspondence, the descent method and Rankin-Selberg integrals. See, for example Vog81, Kaz90, KS90, Pra93, Sav93, BK94, Sav94, GRS97a, BFG00, GRS01, KPW02, BFG03, JS03, GS05, Sou06, LS08, LS10, GRST11.

We mention that Bump and Ginzburg [BG92] also developed a local theory, where they considered a similar space of equivariant trilinear forms, except that $\theta'_0$ was replaced by a certain induced representation. For $n = 3$, Savin [Sav92] determined the dimension of (1.1) for an arbitrary irreducible quotient of a principal series representation. He also conjectured (and proved for $n = 3$), that the class of distinguished spherical representations of $\text{GL}_n(F)$ is precisely those representations, which are lifts from a certain prescribed classical group. Kable [Kab02] proved that these lifts are distinguished, the other direction was proved in Kap14b.

The group $G_n$ has been the focus of study of a few recent works, among which are the works of Asgari [Asg00, Asg02] on local $L$-functions, Asgari and Shahidi [AS06, AS11] on functoriality and Hundley and Sayag [HS12] on the descent construction.

The rest of this work is organized as follows. In Section 2 we provide notation and definitions. In particular we describe the construction of exceptional representations. Section 3
contains the proof of Theorem 3. Theorems 1 and 2 are proved in Section 4. Section 5 provides the formulation of our results for SO_{2n+1}.

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2. Preliminaries

2.1. General notation. Let $F$ be a local non-Archimedean field of characteristic different from $2$. For an integer $r \geq 1$, let $\mu_r$ be the subgroup of $r$-th roots of unity in $F$. Set $F^{*r} = (F^*)^r$. We usually fix a nontrivial additive character $\psi$ of $F$. Then the normalized Weil factor ([Wei64] Section 14) is denoted by $\gamma_\psi (\gamma_\psi(a) = \gamma_F(a, \psi)$ of [Rao93], $\gamma_\psi(\cdot)^4 = 1$.

The Hilbert symbol of order $r$ is $(,)_r$. If $G$ is a group, $C_G$ denotes its center. For $x, y \in G$ and $Y \subset G$, $xy = yx^{-1}$ and $Y = \{xy : y \in Y\}$. Hereby we omit references to the field, e.g., $\text{GL}_n = \text{GL}_n(F)$.

2.2. The group $\text{GL}_n$ and its cover. Fix the Borel subgroup $B_{\text{GL}_n} = T_{\text{GL}_n} \rtimes N_{\text{GL}_n}$ of upper triangular matrices, where $T_{\text{GL}_n}$ is the diagonal torus. For any $k_1, k_2 \geq 0$ such that $k_1 + k_2 = n$, denote by $P_{k_1, k_2}$ the maximal parabolic subgroup whose Levi part is isomorphic to $\text{GL}_{k_1} \times \text{GL}_{k_2}$. Its unipotent radical is $Z_{k_1, k_2} = \{ \begin{pmatrix} l_{k_1} & z \\ \mathbf{0} & l_{k_2} \end{pmatrix} \}$. The “mirabolic” subgroup $P_{n-1, 1}$ is the subgroup of $\text{GL}_n$ of matrices with the last row $(0, \ldots, 0, 1)$. Let $I_n$ be the identity matrix of $\text{GL}_n$ and $J_n$ be the matrix with 1 on the anti-diagonal and 0 elsewhere. For $g \in \text{GL}_n$, $^t g$ is the transpose of $g$.

We will use the metaplectic double cover $\tilde{\text{GL}}_n$ of $\text{GL}_n$, as constructed by Kazhdan and Patterson [KPS84]. Let $\tilde{SL}_{n+1}$ be the double cover of $\text{SL}_{n+1}$ of Matsumoto [Mat69] and let $\sigma_{\text{SL}_{n+1}}$ be the corresponding cocycle of Banks, Levi and Sepanski [BLS99] (Section 3). We define a 2-cocycle $\sigma_{\text{GL}_n}$ of $\text{GL}_n$ via

$$\sigma_{\text{GL}_n}(a, a') = \sigma_{\text{SL}_{n+1}}(\text{diag}(\det a^{-1}, a), \text{diag}(\det a'^{-1}, a')).$$

This cocycle is related to the cocycle $\sigma_n$ of $\text{GL}_n$ defined in [BLS99] by

$$\sigma_{\text{GL}_n}(a, a') = c(\det a, \det a')\sigma_n(a', a).$$

In particular $\sigma_{\text{GL}_1}(a, a') = (a, a')_2$.

2.3. The group $\text{GSpin}_{2n+1}$. We start by defining the special odd orthogonal group

$$\text{SO}_{2n+1} = \{ g \in \text{SL}_{2n+1} : {^t g} J_{2n+1} g = J_{2n+1} \}.$$

Select its Borel subgroup $B_{\text{SO}_{2n+1}} = B_{\text{GL}_{2n+1}} \cap \text{SO}_{2n+1}$.

Let $\text{Spin}_{2n+1}$ be the simple split simply-connected algebraic group of type $B_n$. It is the algebraic double cover of $\text{SO}_{2n+1}$. We will take the Borel subgroup, which is the preimage of $B_{\text{SO}_{2n+1}}$. The set of simple roots of $\text{Spin}_{2n+1}$ is $\Delta_{\text{Spin}_{2n+1}} = \{ \alpha_i : 1 \leq i \leq n \}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n - 1$ and $\alpha_n = \epsilon_n$.

The group $G_n = \text{GSpin}_{2n+1}$ is an $F$-split connected reductive algebraic group, which can be defined using a based root datum as in [Asg02, AS06, HST12]. It is also embedded in $G'_{n+1} = \text{Spin}_{2n+3}$ as the Levi part of the parabolic subgroup corresponding to $\Delta_{G'_{n+1}} - \{ \alpha_1 \}$ (see [Mat09]). We adapt this identification, which is more natural for the purpose of cover groups, because we will obtain a cover of $G_n$ by restricting the cover of $G'_{n+1}$.
Let $\delta_Q$ be the modulus character of a parabolic subgroup $Q < G_n$.

In the degenerate case $G_0 = \GL_1$.

The Borel subgroup of $G_n$ is denoted $B_n = T_{n+1} \ltimes N_n$, where $N_n$ is the unipotent radical (the rank of the torus is $n + 1$). For $0 \leq k \leq n$, denote by $Q_k = M_k \ltimes U_k$ the standard maximal parabolic subgroup of $G_n$ with $M_k \cong \GL_k \times G_{n-k}$. This isomorphism is not canonical. We describe the choice used in [Kap14a], which is convenient for certain computations (see below).

The derived group $\SL_k$ of $\GL_k$ is generated by the root subgroups of $\{\alpha_i : 2 \leq i \leq k\}$. Let $\eta_1^\gamma, \ldots, \eta_k^\gamma$ be the standard cocharacters of $T_{\GL_k}$ and map $\eta_i^\gamma \mapsto \epsilon_i^\gamma - \epsilon_1^\gamma$ for $1 \leq i \leq k$.

Regarding $G_{n-k}$, the set $\{\alpha_i : k + 2 \leq i \leq n + 1\}$ identifies $G_{n-k}'$ and if $\theta_1, \ldots, \theta_{n-k+1}$ are the characters of $T_{n-k+1}$, define $\theta_i^\gamma \mapsto \epsilon_i^\gamma$ and for $2 \leq i \leq n - k + 1$, $\theta_i^\gamma \mapsto \epsilon_i^{\gamma^+}$. The projection $G_{n-k}' \to \SO_{2n+1}$ is an isomorphism between unipotent subgroups, hence we can identify unipotent subgroups of $G_n$ with those of $\SO_{2n+1}$.

We use the character $\epsilon_1$ to define a “canonical” character $\Upsilon$ of $G_n$. Namely $\Upsilon$ is the extension of $-\epsilon_1$ to $G_n$ (the only other choice would be to use $\epsilon_1$).

The aforementioned embedding of $\GL_k \times G_{n-k}$ in $M_k$ has a few properties, suitable for computations. To compute the conjugation of $b \in \GL_k$ on $U_k$, we can simply look at this action in $\SO_{2n+1}$, where $b$ takes the form $\diag(b, I_{2(n-k)+1}, J_k^t b^{-1} J_k)$. The image of $G_0$ is $C_{G_n}$. The restriction of $\Upsilon$ to $\GL_k$ is $\det$.

2.4. The double cover of $\GSpin_{2n+1}$. Let $\wt{G}_{n+1}'$ be the double cover of $G_{n+1}'$, constructed by Matsumoto [Mat69] using $(,)_2$ as the Steinberg symbol. Restricting the cover to $G_n$, we obtain the exact sequence

$$1 \to \mu_2 \to \wt{G}_n \xrightarrow{p} G_n \to 1.$$ 

Then $\wt{G}_n$ is a double cover of $G_n$. For a subset $X \subset G_n$, $\wt{X} = p^{-1}(X)$. Let $s : G_{n+1}' \to \wt{G}_{n+1}'$ be the block-compatible section constructed by Banks, Levi and Sepanski [BLS99] and $\sigma_{G_{n+1}'}$ be the corresponding cocycle. Denote the restriction of $\sigma_{G_{n+1}'}$ to $\wt{G}_n \times G_n$ by $\sigma_{\wt{G}_n}$. In [Kap14a] we proved that $\sigma_{G_n}$ satisfies the following block compatibility property: if $(a, g), (a', g') \in \GL_k \times G_{n-k} \cong M_k$,

$$\sigma_{G_n}((a, g), (a', g')) = \sigma_{\GL_k}(a, a')\sigma_{G_{n-k}}(g, g')(\Upsilon(g), \det a')_2.$$

We also mention that $C_{\wt{G}_n} = \wt{C}_{\wt{G}_n}$.

2.5. Representations. Let $G$ be an $l$-group ([BZ76] 1.1). Representations of $G$ will be complex and smooth. For a representation $\pi$ of $G$, $\pi^\vee$ is the representation contragradient to $\pi$. We say that $\pi$ is glued from representations $\pi_1, \ldots, \pi_k$, if $\pi$ has a filtration, whose quotients (which may be isomorphic or zero) are, after a permutation, $\pi_1, \ldots, \pi_k$.

Regular induction is denoted $\Ind$ and $\ind$ is the compact induction. Induction is not normalized.

Let $\pi$ be as above and let $U < G$ be a unipotent subgroup, exhausted by its compact subgroups (always the case here). Let $\psi$ be a character of $U$. The Jacquet module of $\pi$ with respect to $U$ and $\psi$ is denoted $\pi_{U, \psi}$. It is a representation of the stabilizer of $\psi$ (and normalizer of $U$). The action is not normalized. We have an exact sequence

$$0 \to \pi(U, \psi) \to \pi \to \pi_{U, \psi} \to 0.$$

The kernel $\pi(U, \psi)$ can be characterized by the Jacquet-Langlands lemma:
Lemma 2.1. (see e.g. [BZ76] 2.33) a vector $v$ in the space of $\pi$ belongs to $\pi(U, \psi)$ if and only if
\[ \int_{O} \pi(u) v \psi^{-1}(u) \, du = 0, \]
for some compact subgroup $O < U$.

When $\psi = 1$, we simply write $\pi(U)$ and $\pi_U$.

Assume that $\tilde{G}$ is a given $r$-th cover of $G$. Let $\varepsilon : \mu_r \to \mathbb{C}^*$ be a faithful character. A representation $\pi$ of $\tilde{G}$ is called $\varepsilon$-genuine if it restricts to $\varepsilon$ on $\mu_r$. When $r = 2$, such a representation is simply called genuine.

Let $\varphi : G \to \tilde{G}$ be a section and assume $\pi$ and $\pi'$ are representations of $\tilde{G}$, such that $\pi$ is $\varepsilon$-genuine and $\pi'$ is $\varepsilon^{-1}$-genuine. Then $\pi \otimes \pi'$ (outer tensor product) is a representation of $G$ via $g \mapsto \pi(\varphi(g)) \otimes \pi'(\varphi(g))$. The actual choice of $\varphi$ does not matter, whence we omit it.

2.6. Representations of Levi subgroups. Levi subgroups of classical groups are direct products. The tensor of representations of the direct factors is usually used, to describe their representations. In passing to a cover group, these factors do not necessarily commute and then the tensor construction fails.

Except for the case of $k = n$, the preimages of $GL_k$ and $G_{n-k}$ in $\widetilde{M}_k$ do not commute. The same phenomena occurs in $GL_n$. The following discussion describes a replacement for the usual tensor product. For more details see [Kap14a].

The metaplectic tensor product in the context of $GL_n$ has been studied by various authors [FK86, Sun97, Kab01, Mez04, Tak13]. Our definitions were motivated by the construction of Kable [Kab01] (see Remark 2.1 below).

For any $H < G_n$, put $H^\circ = \{ h \in H : \Upsilon(h) \in F^* \}$. The subgroup $H^\circ$ is normal in $H$ and the quotient is a finite abelian group. If $\xi$ is a representation of $\tilde{H}$, let $\xi^\circ = \xi|_{\tilde{H}^\circ}$. Assume $0 < k < n$. According to (2.1), the preimages of $GL_k^\circ$ and $G_{n-k}^\circ$ are commuting in the cover. Then if $\rho$ and $\pi$ are genuine representations of $\tilde{H}_1 < \tilde{GL}_k$ and $\tilde{H}_2 < \tilde{G}_{n-k}$, the representation $\rho^\circ \otimes \pi^\circ$ is a genuine representation of
\[ p^{-1}(H_1^\circ \times H_2^\circ) \cong \{(\epsilon, \epsilon) : \epsilon \in \mu_2\}\backslash(\tilde{H}_1^\circ \times \tilde{H}_2^\circ). \]

Put $H = H_1H_2$ and define
\[ \mathcal{I}^\circ(\rho, \pi) = \text{ind}_{p^{-1}(H_1^\circ \times H_2^\circ)}^\tilde{H}(\rho^\circ \otimes \pi^\circ). \]

When $k = n$, $\tilde{GL}_n$ and $\tilde{G}_0$ are commuting, then the metaplectic tensor is defined as usual.

Remark 2.1. The arguments in [Kab01] do not readily extend to $G_n$, mainly because $C_{G_n} < G_n^\circ$ for all $n$, and then $C_{G_n}$ does not play a role similar to that of $C_{GL_n}$ in the cover.

We will use Mackey Theory to relate this induced representation to $\rho$ and $\pi$. We reproduce the following result from [Kap14a], which mimics [Kab01] (Theorem 3.1) in our context.

Lemma 2.2. The representation $\mathcal{I}^\circ(\rho, \pi)$ is a direct sum of $[F^* : F^*^2]$ copies of
\[ \text{ind}_{p^{-1}(H_1^\circ \times H_2^\circ)}^\tilde{H}(\rho^\circ \otimes \pi). \]
Proof of Lemma 2.2. Since \( p^{-1}(H_1^\mathbb{Q} \times H_2^\mathbb{Q}) \) is normal of finite index in \( \tilde{H} \) and \( p^{-1}(H_1^\mathbb{Q} \times H_2) \) modulo \( p^{-1}(H_1^\mathbb{Q} \times H_2^\mathbb{Q}) \) is abelian,

\[
T^\mathbb{Q}(\rho, \pi) = \bigoplus_{\alpha \in H_1^\mathbb{Q} \setminus H_1} \text{ind}_{p^{-1}(H_1^\mathbb{Q} \times H_2)}^{\tilde{H}}(\rho^\mathbb{Q} \otimes \omega_\alpha \pi).
\]

Here \( \omega_\alpha(h) = (\Upsilon(h), \det a)_2 \) ranges over the finite set of characters of \( H_2 \), which are trivial on \( H_1^\mathbb{Q} \). By (2.1) if \( a_0 \in \overline{G}_{n-k} \) and \( h_0 \in \overline{G}_{n-k} \), \( a_0^{-1} h_0 = (\Upsilon(h_0), \det a_0)_2 h_0 \). Hence \( \rho^\mathbb{Q} \otimes \omega_\alpha \pi = a^{-1}(\rho^\mathbb{Q} \otimes \pi) \) and the result follows. \( \square \)

2.7. The Weil representation. We introduce the Weil representation, which plays an important role in this work. Let \( n = 2k \) and \( \lambda \) be the symplectic bilinear form on \( F^n \) defined by \( \lambda(u, v) = u(-J_k)^t v \), where \( u \) and \( v \) are regarded as rows. Let \( H_n \) be the \((n + 1)\)-dimensional Heisenberg group, with the group operation given by

\[
(u_1, u_2; z_1) \cdot (v_1, v_2; z_2) = (u_1 + v_1, u_2 + v_2, z_1 + z_2 + \lambda((u_1, u_2), (v_1, v_2))),
\]

where \( u_i, v_i \in F^k \) and \( z_i \in F \).

Let \( \text{Sp}_k \) be the symplectic group defined with respect to \( \lambda \), i.e., the group of \( g \in \text{GL}_n \) such that \( \lambda(ug, vg) = \lambda(u, v) \) for all \( u, v \in F^n \). Let \( \widetilde{\text{Sp}}_k \) be the metaplectic double cover of \( \text{Sp}_k \), realized using the normalized Rao cocycle \( \text{Rao93} \). The group \( \text{Sp}_k \) acts on \( H_n \) via \( g^{-1}(u_1, u_2; z)g = ((u_1, u_2); g; z) \).

Fix a nontrivial additive character \( \psi \) of \( F \). Let \( \omega_\psi \) be the Weil representation of \( H_n \times \widetilde{\text{Sp}}_k \), realized on the space \( \mathcal{S}(F^k) \) of Schwartz-Bruhat functions on \( F^k \). Recall the following formulas for \( \omega_\psi \) (see \( \text{Persil1} \)): for \( \varphi \in \mathcal{S}(F^k) \),

\[
\omega_\psi( (u \in H_n; z) \varphi(x) = \psi(z) \varphi(x + u_1), \tag{2.3}
\]

\[
\omega_\psi( (0, u \in H_n; 0) \varphi(x) = \psi(x J_k^t u_2) \varphi(x), \tag{2.4}
\]

\[
\omega_\psi( (I_k \begin{pmatrix} u & z \\ I_k \end{pmatrix}, \varepsilon) \varphi(x) = \epsilon \psi(\frac{1}{2} x J_k^t u^t x) \varphi(x) \quad (\varepsilon \in \mu_2). \tag{2.5}
\]

Let \( R = \{ (0, u_1; 0) \} < H_n \) and \( U = \{ (I_k \begin{pmatrix} u & z \\ I_k \end{pmatrix}) \} < \text{Sp}_k \). Since \( U \) normalizes (in fact, commutes with) \( R \), \( (\omega_\psi)_R \) is a \( U \)-module. We will use the following simple observation.

Claim 2.3. The vector spaces \( (\omega_\psi)_R \) and \( (\omega_\psi)_{RU} \) are one dimensional.

Proof of Claim 2.3. According to Lemma 2.1 and (2.4), the space of \( (\omega_\psi)_R \) is \( \mathcal{S}(F^k - 0) \). Hence \( (\omega_\psi)_R \) is one dimensional. Then \( (\omega_\psi)_{RU} = ((\omega_\psi)_R)_U \) is nonzero, because by Lemma 2.1 and (2.5), a function \( \varphi \in \mathcal{S}(F^k) \) such that \( \varphi(0) \neq 0 \) does not belong to \( \omega_\psi( RU) \). \( \square \)

We will also encounter the tensor \( \omega_\psi \otimes \omega_{\psi^{-1}} \) of two Weil representations. This is a representation of \( H_n \), trivial on \( C_{H_n} \), and a representation of \( \text{Sp}_k \). Regarding it as a representation of \( C_{H_n} \), we can compute its twisted Jacquet modules. The group \( \text{Sp}_k \) acts transitively on the nontrivial characters of \( C_{H_n} \setminus H_n \), hence we can consider only one nontrivial character.

Claim 2.4. Let \( \mu \) be a character of \( H_n \), which is trivial on \( C_{H_n} \).

(1) If \( \mu = 1 \), \( (\omega_\psi \otimes \omega_{\psi^{-1}})_{H_n, \mu} \) is the trivial one-dimensional representation of \( \text{Sp}_k \).

(2) If \( \mu(u_1, u_2; z) = \psi((u_1)_1) \), \( (\omega_\psi \otimes \omega_{\psi^{-1}})_{H_n, \mu} \) is the trivial one-dimensional representation of \( P_{n-1, 1} \cap \text{Sp}_k \).
Proof of Claim [2.4]. First assume $\mu = 1$. Equality (2.4) implies that elements $\varphi \otimes \varphi'$ in the space of $\omega_\psi \otimes \omega_\psi^{-1}$, such that the supports of $\varphi$ and $\varphi'$ (as functions in $S(F^k)$) are different, vanish under the Jacquet module along $R$ (use Lemma 2.1). Hence the space of $(\omega_\psi \otimes \omega_\psi^{-1})_R$ is isomorphic to $S(F^k)$. Since the action of $C_{H_n}$ is trivial on $(\omega_\psi \otimes \omega_\psi^{-1})_R = (\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}$. Now $RC_{H_n}$ is a normal subgroup of $H_n$, whence $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n}$ is a quotient of $(\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}$.

It then follows from [2.5] that the action of $U$ is trivial on $(\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}$ and in particular, on $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n}$. The latter is a representation of $Sp_k$, and because $Sp_k$ is generated (as an abstract group) by the subgroups $wU$, $w \in Sp_k$ (it is enough to take Weyl elements $w$), $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n}$ must be a trivial representation of $Sp_k$.

Moreover $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n}$ is one-dimensional. This can be seen as follows. Replace a function $f \in S(F^k) = (\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}$ with its Fourier transform $\widehat{f}(x) = \int_{F^k} f(y) \psi(x'y))dy$. This changes the module structure, but the action of $Sp_k$ remains trivial. The action of $(u_1,0;0) \in H_n$ is now given by

$$(u_1,0;0) \cdot \widehat{f}(x) = \psi^{-1}(x'u_1)\widehat{f}(x)$$

(instead of (2.3)). Next apply Lemma 2.1 the space of $(\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}((RC_{H_n}) \setminus H_n)$ is $S(F^k - 0)$.

We conclude that $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n}$ is the trivial one-dimensional representation of $Sp_k$.

Now consider the case of the nontrivial $\mu$, given in the statement of the claim. Since $\mu|_{R} = 1$, $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n,\mu}$ is a quotient of $(\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}$ and in particular, a trivial representation of $U$. In coordinates,

$$P_{n-1,1}^o \cap Sp_k = \left\{ \begin{pmatrix} 1 & u & v \\ g & * & 1 \end{pmatrix} : g \in Sp_{k-1} \right\}. $$

(In particular it stabilizes $\mu$.) We see that $U < P_{n-1,1}^o \cap Sp_k$ and using conjugations of $U$ by elements from $Sp_{k-1}$, it follows that $(\omega_\psi \otimes \omega_\psi^{-1})_{H_n,\mu}$ is the trivial representation of $P_{n-1,1}^o \cap Sp_k$.

To show this is a one-dimensional representation, argue as above using the Fourier transform, the space of $(\omega_\psi \otimes \omega_\psi^{-1})_{RC_{H_n}}((RC_{H_n}) \setminus H_n, \mu)$ is $S(F^k - (-1,0,\ldots,0))$. \hfill \Box

2.8. Exceptional or small representations. We describe the exceptional representations that appear in this work. Kazhdan and Patterson [KPS4] introduced and studied these representations for $GL_n$. For $G_n$, these are essentially the small representations of Bump, Friedberg and Ginzburg [BFG03], who developed them using a cover of $SO_{2n+1}$ (see Section 5). Their construction was extended to $G_n$ in [Kap14a].

Let $G$ be either $GL_n$ or $G_n$. Let $B$ be the Borel subgroup of $G$, $T$ be the maximal torus and $\Delta$ be the subset of simple roots. For $\alpha \in \Delta$, if $\alpha$ is a long root and $n > 1$, put $l(\alpha) = 2$, otherwise $l(\alpha) = 1$. Denote by $\alpha^\vee$ the coroot of $\alpha$.

Let $\xi$ be a genuine character of $C_{\tilde{T}}$. We say that $\xi$ is exceptional if

$$\xi(s(\alpha^\vee(x^{l(\alpha)}))) = |x|, \quad \forall \alpha \in \Delta, x \in F^\ast. $$

The character $\xi$ corresponds to a genuine irreducible representation $\rho(\xi)$ of $\tilde{T}$ (when $n > 1$, $\tilde{T}$ is a 2-step nilpotent subgroup). Since $\xi$ is exceptional, the representation $\text{Ind}_{G}^G(\delta_{B}^{1/2} \rho(\xi))$ has a unique irreducible quotient $\theta$, called an exceptional representation. Note that $\theta$ is admissible. Occasionally, we use the notation $\theta^{G}$ to record the group.
We appeal to the following explicit description of $C_{\tilde{T}}$: $C_{\tilde{T}_{\GL_n}} = \tilde{T}_{\GL_n}^2 C_{\GL_n}$ with $T_{\GL_n}^2 = \{ t^2 : t \in T_{\GL_n} \}$, $C_{\GL_n} = \{ z \cdot I_n : z \in F^* \}$, where $e$ is 1 if $n$ is odd, otherwise $e = 2$; $C_{\tilde{T}_{n+1}} = C_{\tilde{T}_{\GL_n}} \tilde{C}_0$, and if $T_{n+1}^2 = T_{\GL_n}^2 G_0$, $C_{\tilde{T}_{n+1}} = \tilde{T}_{n+1}^2 C_{\GL_n}$ (see [KP84] p. 57 and [Kap14a] Section 2.1.6). Note that $G_0^e = G_0$. Furthermore, the cocycle $\sigma_G$ satisfies $\sigma_G(z \cdot I_n, z' \cdot I_n) = (z, z')^{[n/2]}$.

The exceptional characters can be parameterized in the following way. Start by fixing a character $\xi_0$ of $p(C_{\tilde{T}})$, trivial on $p(C_{\tilde{G}})$. This does not determine $\xi_0$ uniquely. For $G_n$, take $\xi_0 = \delta_{C_n}^{1/4}$. For $G_n$ take $\xi_0$ whose restriction to $p(C_{\tilde{T}_{\GL_n}})$ is $\delta_{C_n}^{1/4} \cdot \det | (n+1)/4$. Given that $\xi_0$ is trivial on $p(C_{\tilde{G}_n})$, this determines $\xi_0$ uniquely for $\tilde{G}_n$ (see [Kap14a] Section 2.3.3 for an explicit formula).

Now for any given $\xi$, there is a character $\chi$ of $F^*$ (called “determinantal character” in [BG92]) and a nontrivial additive character $\psi$ of $F$ such that

$$
(2.6) \quad \xi(e_0(t)) = \epsilon \xi_0(t) \chi(\Upsilon(t)) \gamma_{\psi}^{[n/2]}(z), \quad \forall t \in T^2, d = z^e \cdot I_n \in T, e \in \mu_2.
$$

The corresponding exceptional representation will be denoted $\theta_{\chi, \psi}$.

**Claim 2.5.** We have $\theta_{\chi, \psi} = \chi_{\psi, \psi}$, where on the right-hand side we pull back $\chi$ to a non-genuine character of $\tilde{G}$ via $g \mapsto \chi(\Upsilon(p(g)))$. Additionally, if $\psi_0$ is another additive character of $F$, $\theta_{\chi, \psi} = \eta \theta_{\chi, \psi_0}$ for some square trivial character $\eta$ of $F^*$.

**Proof of Claim 2.5.** The first assertion is clear. Write $\psi(x) = \psi_0(\alpha x)$ for some $\alpha \in F^*$. Then $\gamma_{\psi} = \eta \gamma_{\psi_0}$, where $\eta_0(z) = (\alpha, z)_2$. Since $\eta_0$ is trivial on $F^*$ and $\eta_0(\Upsilon(d)) = \eta_0(d) = \eta_0(z^n) = \eta_0(z)$, (the last equality is trivial if $n$ is even, because then $z \in F^*$), we obtain

$$
\chi(\Upsilon(t)) \gamma_{\psi}^{[n/2]}(z) = (\eta_0^{[n/2]} \chi)(\Upsilon(t)) \gamma_{\psi_0}^{[n/2]}(z).
$$

Hence $\theta_{\chi, \psi} = \theta_{\psi}^{[n/2]} \chi_{\psi_0} = \eta_0^{[n/2]} \theta_{\chi, \psi_0}$. \hfill \Box

Exceptional representations have a useful inductive property. Let $\theta$ be an exceptional representation of $\tilde{G}_n$. Following the arguments of Bump, Friedberg and Ginzburg [BFG03] (Theorem 2.3), we computed $\theta_{U_k}$ ([Kap14a]). For $0 < k < n$,

$$
(2.7) \quad (\theta_{\chi, \psi})_{U_k} \in \mathcal{I}_k(\theta_{\GL_k}^{(2n-k-1)/4, \chi_{\psi}}, \theta_{\GL_k}^{(2n-k)/4, \chi_{\psi}}).
$$

If $k$ (resp. $n-k$) is odd, the exceptional representation of $\GL_k$ (resp. $\tilde{G}_{n-k}$) is unique only up to varying the character $\psi$, or multiplying $\chi$ by a square trivial character of $F^*$. In any case, the space on the right-hand side of (2.7) is unique, because by Claim 2.5, the exceptional representation obtained by such a change to $\psi$ or $\chi$, has the same restriction to $\tilde{G}_{n-k}$ as the original representation.

If $k = n$,

$$
(2.8) \quad (\theta_{\chi, \psi})_{U_n} = \theta_{\GL_n}^{(2n-1)/4, \chi_{\psi}} \otimes \theta_{\GL_n}^{G_0}.
$$

In [Kap14a] we did not find the precise exceptional representations appearing on the right-hand side of (2.8), but this is simple to obtain: since $C_{\tilde{T}_{n+1}} = C_{\tilde{T}_{\GL_n}} \tilde{C}_0$, there is an exceptional character $\xi_1$ of $C_{\tilde{T}_{\GL_n}}$ such that $\xi_0 \xi = w_0^{1} \xi_1 \otimes \xi_1^{G_0}$, where $w_0$ and $w_0'$ are the longest Weyl elements in the Weyl groups of $G_n$ and $\GL_n$ (see [Kap14a] Claim 2.18 for details). It remains to write $\xi_1$ using (2.6).
Remark 2.2. The reason for the imprecise result when \( k < n \) is the lack of a definition for a tensor product. These results are sufficient for our applications.

For \( \text{GL}_n \), Kable proved a result similar to (2.8) for all standard unipotent radicals, with the tensor replaced by his metaplectic tensor ([Kab01] Theorem 5.1 (4)).

Exceptional representations enjoy the vanishing of a large class of twisted Jacquet modules. The following result is the extension of Theorem 2.6 of [BFG03] and Proposition 3 of [BFG06] to \( G_n \) (this extension appeared in [Kap14]). It was used in [BFG03, BFG06] (for \( \text{SO}_{2n+1} \)) to deduce all vanishing properties.

The unipotent radical \( U_1 \) is abelian. A character \( \psi^{(1)} \) of \( U_1 \) takes the form

\[
\psi^{(1)} \left( \begin{pmatrix}
1 & u & \ast \\
I_{2n-1} & \ast & \\
& & 1
\end{pmatrix} \right) = \psi(ua),
\]

where \( a \in F^{2n-1} \) is a column. The length of \( a \) is defined to be \( t^* a \). When \( \psi \) is fixed and clear from the context, we also refer to \( t^* a \) as the length of \( \psi^{(1)} \).

Theorem 4. If the length of \( \psi^{(1)} \) is nonzero, \( \theta_{U_1, \psi^{(1)}} = 0 \).

See [Kap14] (Lemma 2.25) for the details.

3. Twisted Jacquet modules of \( \theta^{G_n} \)

In this section we describe the twisted Jacquet modules of \( \theta = \theta^{G_n} \) with respect to the center of \( U_n \). These modules appear in a filtration of \( \theta \) as a \( \widetilde{Q}_n \)-module and will be used in Section [4]

The group \( \text{GL}_n \), embedded in \( M_n \), acts on the set of characters of \( C = C_{U_n} \) with finitely many orbits. Let \( \psi \) be a nontrivial additive character of \( F \). For any \( 0 \leq k \leq [n/2] \), define a character of \( C \) by

\[
\psi_k(c) = \psi \left( \sum_{i=1}^{k} c_{n-2i+1,n+2i} \right)
\]

(\( c \) is regarded as a \( (2n+1) \times (2n+1) \) unipotent matrix in \( \text{SO}_{2n+1} \)). The stabilizer of \( \psi_k \) in \( \widetilde{M}_n \) is \( \text{St}_{\psi_k} \), with

\[
\text{St}_{\psi_k} = \left\{ \begin{pmatrix} a & z \\ 0 & b \end{pmatrix} : a \in \text{GL}_{n-2k}, b \in \text{Sp}_k \right\} \times G_0.
\]

Here \( \text{Sp}_k \) is the symplectic group in \( 2k \) variables, corresponding to a symplectic bilinear form defined according to \( \psi_k \). Then \( \theta_{C,\psi_k} \) is a representation of \( \text{St}_{\psi_k} \times U_n \).

Regarding \( \text{GL}_{n-2k} \) and \( \text{Sp}_k \) as subgroups of \( \text{St}_{\psi_k} \),

\[
\text{St}_{\psi_k} = \left( (\text{GL}_{n-2k} \times \text{Sp}_k) \times Z_{n-2k,2k} \right) \times G_0.
\]

(\( Z_{n-2k,2k} \) was given in Section [2.2])

We turn to the proof of Theorem 3. Namely, for \( n = 2k \), \( \theta_{C,\psi_k} \) is the direct sum of copies of the Weil representation \( \omega_{\psi} \). Here is an outline of the proof. The theory of smooth representations of Jacobi groups ([vD78], [MVW87], [BS98]) implies that any such representation, with a central character \( \psi \), takes the form \( \kappa \otimes \omega_{\psi} \), where the Heisenberg group acts trivially on the space of \( \kappa \), and the action of the symplectic group separates into an action on the space of \( \kappa \), and one on the space of \( \omega_{\psi} \). The vanishing properties of \( \theta \) - Theorem 4 will show that \( \kappa \) is trivial.
Proof of Theorem 3. Put \( k = n/2 \). The image of \( G_0 \) in \( G_n \) is \( C_{G_n} \). Hence \( \text{St}_{\psi_k} \) is the direct product of a Jacobi group and \( G_0 \). Moreover, \( \widetilde{G}_0 = C_{\widetilde{G}_n} \) (see Section 2.4) and because \( \theta \) is an irreducible representation, \( \widetilde{G}_0 \) acts by the central character of \( \theta \). Therefore in the proof we ignore the \( G_0 \) part of \( \text{St}_{\psi_k} \).

We may replace \( \psi_k \) by any character of \( C \) in the same \( \text{GL}_n \)-orbit, since the Jacquet module will be isomorphic. For convenience, we redefine \( \psi_k(c) = \psi(\sum_{i=1}^{k} c_{i,n+1+i}) \). We use the notation of Section 2.7. The stabilizer \( \text{St}_{\psi_k} \) is now the symplectic group defined with respect to the form \( \lambda \). The cover \( \widetilde{\text{St}}_{\psi_k} \) is a nontrivial double cover. We have an epimorphism \( \ell : \widetilde{\text{St}}_{\psi_k} \times U_n \to \widetilde{\text{Sp}}_k \times H_n : \)

\[
\ell \left( \begin{pmatrix} I_n & u & z \\ 1 & -t u J_n & I_n \end{pmatrix} \right) = \left( t u J_n ; \frac{1}{2} \sum_{i=1}^{k} z_{i,i} - \sum_{i=k+1}^{n} z_{i,i} \right) \in H_n,
\]

\[
\ell \left( \begin{pmatrix} g & 1 \\ J_n^{-1} g^{-1} J_n & \epsilon \end{pmatrix} \right) = \left( J_n t g^{-1} J_n, \epsilon \right) \in \widetilde{\text{Sp}}_k \quad (g \text{ preserves } \lambda, \quad \epsilon \in \mu_2).
\]

The kernel of \( \ell \) is contained in the kernel of \( \psi_k \). Therefore we can regard \( \theta_{C,\psi_{k}} \) as a genuine representation of \( \widetilde{\text{Sp}}_k \times H_n \). As such, it is isomorphic to \( \kappa \otimes \omega_{\psi} \) (see e.g. [BS98] p. 28), where \( \kappa \) is a non-genuine representation on a space \( V \), and the action is given by

\[
( g, \epsilon ) h \cdot ( f \otimes \varphi ) = \kappa(g) f \otimes \omega_{\psi}((g,\epsilon)h) \varphi, \quad g \in \text{Sp}_k, \quad h \in H_n, \quad f \in V, \quad \varphi \in \mathcal{S}(F^k).
\]

We must show that \( \kappa \) is trivial. Consider the subgroup

\[
Y = \left\{ \begin{pmatrix} 1 & 0 & y \\ I_{2(n-1)} & 0 & 1 \end{pmatrix} \right\} < \text{Sp}_k.
\]

Since \( \text{Sp}_k \) is generated by the conjugates \( ^xY \) where \( x \in \text{Sp}_k \), it is enough to prove invariance under \( Y \). That is, we show

(3.1) \( \kappa_Y = \kappa \).

A character of \( Y \) takes the form \( y \mapsto \psi(\alpha y) \) for some \( \alpha \in F \) (on the left-hand side \( y \) is regarded as a matrix, on the right-hand side as an element of \( F \)). The action of the torus of \( \text{Sp}_k \) on the nontrivial characters of \( Y \) has finitely many orbits, namely the different square classes in \( F^* \). Each of these orbits is open. Therefore the kernel of the Jacquet functor \( \kappa_Y \) is filtered by representations induced from \( \kappa_{Y,\psi(\alpha)} \), where \( \alpha \) ranges over the square classes ([BZ76] 5.9-5.12). Hence (3.1) follows if we prove \( \kappa_{Y,\psi(\alpha)} = 0 \) for any \( \alpha \neq 0 \).

Consider the subgroup \( X = Y \cdot \{(0,u_2;0) \in H_n : u_2 = (0,\ldots,0,r)\} \) (a direct product). The epimorphism \( \ell \) splits over \( X \), hence there is a subgroup \( U < \widetilde{\text{St}}_{\psi_k} U_n \) isomorphic to \( X \). In fact,

\[
U = \left\{ \begin{pmatrix} 1 & 0 & y & r & 0 & 0 & -r^2/2 \\ I_{n-2} & 1 & 0 \\ 1 & 0 & -r \\ 1 & -y \\ I_{n-2} & 0 & 1 \end{pmatrix} \right\} < U_1.
\]

\[\]1In [Kap] p. 25 it was incorrectly stated they are equal.
The pullback of $\psi(\alpha \cdot)$ to $U$ is $\psi^*(u) = \psi(-\alpha u_{1,n})$. Observe that $(\theta_{C,\psi_k})_{U,\psi^*} = 0$. Indeed, $(\theta_{C,\psi_k})_{U,\psi^*} = \theta_{C/U,\psi_k \psi^*}$, which is a quotient of $\theta_{U(C \cap U_1),\psi_k \psi^*}$. Since for $u \in U(C \cap U_1)$, $\psi_k \psi^*(u) = \psi(-\alpha u_{1,n} + u_{1,n+2})$, any extension of $\psi_k \psi^*$ to a character of $U_1$ is a character of nonzero length. Thus Theorem 4 yields $\theta_{U(C \cap U_1),\psi_k \psi^*} = 0$ whence $(\theta_{C,\psi_k})_{U,\psi^*} = 0$. Therefore by Lemma 2.1 for any $f \otimes \varphi$ there is a compact $O < Y \cdot R$ ($R = \{(0, u_2);\}$, see Section 2.7) such that

\[(3.2) \quad \int_O yr \cdot (f \otimes \varphi) \psi^{-1}(\alpha y) \, dr \, dy = 0.\]

According to Claim 2.3 and Lemma 2.1 there is $\varphi \in S(F^k)$ such that for all compact subgroups $Y < Y$ and $R < R$,

\[(3.3) \quad \varphi^{Y,R} = \int_Y \int_R \omega_\psi(yr) \varphi \, dy \, dr \neq 0.\]

Take $f \in V$. We will show that for large enough $Y$ and $R$,

\[(3.4) \quad \int_Y \kappa(y_1) f \psi^{-1}(\alpha y_1) \, dy_1 \otimes \varphi^{Y,R} = 0.\]

This along with (3.3) imply that $f$ belongs to the space of $\kappa(Y, \psi(\alpha \cdot))$.

Plugging (3.3) into (3.4) and changing variables leads to

\[
\int_Y \left( \int_Y \int_R \kappa(y) f \otimes \omega_\psi(yr^{-1}) \varphi \, \psi^{-1}(\alpha y) \, dr \, dy \right) \, dy_1.
\]

We will show that the inner $dr \, dy$-integration vanishes for all $y_1 \in Y$. Fix $y_1$. Since $\kappa|_{H_n}$ is trivial, this inner integration equals

\[(3.5) \quad \int_Y \int_R yr \cdot (f \otimes \omega_\psi(y_1^{-1}) \varphi) \psi^{-1}(\alpha y) \, dr \, dy = 0.\]

Again resorting to Claim 2.3

\[(3.6) \quad \omega_\psi(y_1^{-1}) \varphi = c_{y_1} \varphi + \varphi_{y_1}^0,\]

where $c_{y_1} \in \mathbb{C}$ and $\varphi_{y_1}^0$ belongs to the space of $\omega_\psi(R)$. Since $y_1$ varies in a compact subgroup, there is a large enough $R$ such that

\[(3.7) \quad \int_R \omega_\psi(r) \varphi_{y_1}^0 \, dr = 0, \quad \forall y_1 \in Y.\]

Furthermore (3.2) implies that for large $Y$ and $R$,

\[
\int_Y \int_R yr \cdot (f \otimes \varphi) \psi^{-1}(\alpha y) \, dr \, dy = 0
\]

and then for any $c \in \mathbb{C}$,

\[(3.8) \quad \int_Y \int_R yr \cdot (f \otimes c \varphi) \psi^{-1}(\alpha y) \, dr \, dy = 0.\]

Combining (3.6)-(3.8) we conclude that for sufficiently large $R$ and $Y$, the inner $dr \, dy$-integration (3.5) vanishes. Note the order of selecting the compact subgroups: first, choose $R$ and $Y$ which ensure (3.8), they depend only on $f$ and $\varphi$. Then, increase $R$ to have (3.7), it will depend on $\varphi$ and $Y$. This completes the proof of (3.4) and thereby (3.1). We conclude that $\kappa$ is trivial.

$\square$

In the more general case, for arbitrary $k$, we are less precise.
Proposition 3.1. There are exceptional representations \( \theta^{\text{GL}_{n-2k}} \) and \( \theta^{G_{2k}} \) such that \( \theta_{C,\psi_k} \) is embedded in a finite direct sum of copies of the representation

\[
\vartheta \otimes (\theta^{G_{2k}})_{C_{U_{2k}}},\psi_k, \quad \vartheta = \begin{cases} 
\theta^{\text{GL}_n} & k = 0, \\
\text{ind}_{\text{GL}_{n-2k}}^{\text{GL}_{n-2k}}( (\theta^{\text{GL}_{n-2k}}) \Box ) & k > 0.
\end{cases}
\]

Here \( \vartheta \otimes (\theta^{G_{2k}})_{C_{U_{2k}}},\psi_k \) is regarded as a representation of \( \overline{S_t\psi_k} \times U_n \) by extending it trivially on \( U_{n-2k} \). The representations \( \theta^{\text{GL}_{n-2k}} \) and \( \theta^{G_{2k}} \) are related to \( \theta \) via \((2.7)\) and \((2.8)\). If \( k = 0 \), the embedding is in fact an isomorphism and there is only one summand, otherwise there are \([ F^* : F^{*2} ]\) summands.

Proof of Proposition 3.1. For \( n = 1 \), we have \( C = U_1 \) and \( k = 0 \), whence \( \theta_{C,\psi_k} = \theta_{U_1} \) and the result follows immediately from \((2.8)\).

Assume \( n > 1 \). Further assume \( k < n/2 \), otherwise there is nothing to prove. The main part of the proof is to show that \( U_{n-2k} \) acts trivially on \( \theta_{C,\psi_k} \). Of course, this holds for \( U_{n-2k} \cap C \). Let \( V_k = U_{n-2k} \cap U_n \) and note that \( Z_{n-2k,2k} = U_{n-2k} \cap M \). Clearly \( U_{n-2k} = V_k \times Z_{n-2k,2k} \). The following claims imply that the action of \( U_{n-2k} \) is trivial.

Claim 3.2. \( \theta_{C,\psi_k} = \theta_{V_kC,\psi_k} \).

Claim 3.3. \( \theta_{V_kC,\psi_k} = \theta_{U_{n-2k}C,\psi_k} \).

Before proving the claims, let us deduce the proposition. Clearly \( \theta_{U_{n-2k}C,\psi_k} = (\theta_{U_{n-2k}})_{C_{U_{2k}}},\psi_k \). Assume \( k > 0 \). Then by \((2.7)\),

\[
(3.9)
\]

\[
\theta_{U_{n-2k}C,\psi_k} \in \mathcal{I}_C(\theta^{\text{GL}_{n-2k}},\theta^{G_{2k}})_{C_{U_{2k}},\psi_k}.
\]

According to Lemma 2.2, \( \mathcal{I}_C(\theta^{\text{GL}_{n-2k}},\theta^{G_{2k}}) \) is the finite direct sum of \([ F^* : F^{*2} ]\) copies of \( \text{ind}_{p^{-1}(\text{GL}_{n-2k} \times G_{2k})}^{M_{2k}}( (\theta^{\text{GL}_{n-2k}}) \Box \otimes \theta^{G_{2k}} ) \).

Let \( \text{St}'_{\psi_k} \) be the stabilizer of \( \psi_k \) in \( M_{2k} \), when \( \psi_k \) is regarded as a character of \( C_{U_{2k}} \) and \( M_{2k} < G_{2k} < G_n \). The double coset space \( \text{GL}_{n-2k} \times G_{2k} \backslash M_{2k} / \text{GL}_{n-2k} \times \text{St}'_{\psi_k} \) is trivial. Then by virtue of the Geometric Lemma of Bernstein and Zelevinsky \([BZ77]\),

\[
\text{ind}_{p^{-1}(\text{GL}_{n-2k} \times G_{2k})}^{\text{M}_{2k}}( (\theta^{\text{GL}_{n-2k}}) \Box \otimes \theta^{G_{2k}} )_{C_{U_{2k}},\psi_k} = \text{ind}_{p^{-1}(\text{GL}_{n-2k} \times \text{St}'_{\psi_k})}^{\text{G}_{2k}}( (\theta^{\text{GL}_{n-2k}}) \Box \otimes \theta^{G_{2k}} )_{C_{U_{2k}},\psi_k}.
\]

Equally \((2.7)\) implies the subgroups \( \text{GL}_{n-2k} \) and \( \overline{\text{Sp}_k} \) commute. Therefore

\[
\text{ind}_{p^{-1}(\text{GL}_{n-2k} \times \text{St}'_{\psi_k})}^{\text{G}_{2k}}( (\theta^{\text{GL}_{n-2k}}) \Box \otimes \theta^{G_{2k}} )_{C_{U_{2k}},\psi_k} = (\text{ind}_{\text{GL}_{n-2k}}^{\text{G}_{2k}}( \theta^{\text{GL}_{n-2k}} ) \Box ) \otimes (\theta^{G_{2k}} )_{C_{U_{2k}},\psi_k}.
\]

Thus \( \text{ind}_{p^{-1}(\text{GL}_{n-2k} \times \text{St}'_{\psi_k})}^{\text{G}_{2k}}( (\theta^{\text{GL}_{n-2k}}) \Box \otimes (\theta^{G_{2k}} )_{C_{U_{2k}},\psi_k} \) is the direct sum of representations \( \vartheta \otimes (\theta^{G_{2k}} )_{C_{U_{2k}},\psi_k} \). The proposition follows from this. Note that for \( k = 0 \), \( \theta_{U_{n-2k}C,\psi_k} = \theta_{U_n} \) and we can apply \((2.8)\) instead of \((2.7)\), then \((3.9)\) becomes \( \theta_{U_{n}C} = \theta^{\text{GL}_n} \otimes \theta^{G_0} \).

Proof of Claim 3.2. A character \( \psi_k^* \) of \( V_kC \) extending \( \psi_k \) is defined by its restriction to the nontrivial coordinates on the \( (n+1)\)-th column of \( v \in V_k \). We call \( \psi_k^* \) nontrivial if this restriction is nontrivial. We prove that the Jacquet module of \( \theta_{C,\psi_k} \) with respect to \( V_k \)
and $\psi^*$ vanishes for any nontrivial $\psi^*$. The group $GL_{n-2k} < St_{\psi_k}$ acts transitively on these characters. Therefore, it is enough to show

$$ (\theta_{C,\psi_k})_{V_k,\psi^*_k} = \theta_{V_k C, \psi^*_k} = 0, \quad \psi^*_k(v) = \psi(v_{1,n+1}), \quad v \in V_k. $$

This follows immediately from Theorem 3.1 because any character of $U_1$ extending $\psi^*_k|_{U_1}$ has a nonzero length. Thus $\theta_{C,\psi_k} = \theta_{V_k C, \psi^*_k}$. □

**Proof of Claim 3.3.** For $k = 0$ there is nothing to prove ($V_0 = U_n$). Assume $k > 0$. The claim follows once we show that for any nontrivial character $\mu$ of $Z_{n-2k,2k}$,

$$ (\theta_{V_k C, \psi_k})_{Z_{n-2k,2k}, \mu} = 0. \quad (3.10) $$

The group $Z_{n-2k,2k}$ is abelian and $GL_{n-2k} \times Sp_k$ acts on the characters of $Z_{n-2k,2k}$. Write an element $z \in Z_{n-2k,2k}$ in the form

$$ z = z(z_1, z_2, z_3, z_4) = \begin{pmatrix} I_{n-2k-1} & z_1 & z_2 \\ z_3 & z_4 & 1 \\ 1 & 1 & I_{2k-1} \end{pmatrix}. $$

We may assume that $\mu$ does not depend on the coordinates of $z_1$ and $z_4$, and depends on $z_3$. For simplicity, also assume $\mu(z(0,0,z_3,0)) = \psi(z_3)$.

We use the local analog of “exchanging roots”, proved by Ginzburg, Rallis and Soudry [GRS99a] (Lemma 2.2). (For the global setting see [Gin90, GRS01, Sou05, GRST1].) Let $Z_1 < Z_{n-2k,2k}$ be the subgroup of elements $z(z_1,0,0,0)$ and $Z_{2,3,4} < Z_{n-2k,2k}$ be the subgroup consisting of elements $z(0, z_2, z_3, z_4)$. Clearly $Z_{n-2k,2k} = Z_1 \cdot Z_{2,3,4}$ (a direct product). Also consider the subgroup

$$ E = \left\{ \begin{pmatrix} I_{n-2k-1} \\ 1 \\ I_{2k} \end{pmatrix} \right\}. $$

In general, if $\pi$ is a smooth representation of $Z_{n-2k,2k} \rtimes E$, then by [GRS99a] (Lemma 2.2), as $Z_{2,3,4}$-modules

$$ \pi_{Z_{n-2k,2k}, \mu} = \pi_{Z_{2,3,4} \rtimes E, \mu}. $$

Indeed, it is simple to check that the list of properties stated in the lemma are satisfied in this setting (in the notation of [GRS99a], $C = Z_{2,3,4}$, $X = Z_1$ and $Y = E$).

**Remark 3.1.** Lemma 2.2 of [GRS99a] was stated for unipotent subgroups of symplectic groups, but the arguments are general and hold in our setting. See also Section 2.3 of [GRS99a].

It follows that as $Z_{2,3,4}$-modules

$$ (\theta_{V_k C, \psi_k})_{Z_{n-2k,2k}, \mu} = \theta_{(V_k C) \rtimes Z_{2,3,4} E, \psi_k \mu}. $$

Conjugating the right-hand side by a Weyl element of $G_n$, whose action on $N_n$ is given by the action of

$$ \text{diag}(I_{n-2k-1}, I_{2k+1}, 1, I_{2k+1}, I_{n-2k-1}), $$

we obtain that $\theta_{(V_k C) \rtimes Z_{2,3,4} E, \psi_k \mu}$ is a quotient of

$$ (\theta_{U_1 \psi_1})_{U_2, \psi_2}. $$
Here $\psi_1(u) = \psi(u_{1,2})$, $U_2'$ is a certain subgroup of $U_2$ (obtained from the conjugation of $C$) and $\psi_2(u) = \psi(u_{2,2n-1})$. Note that $\psi_1$ corresponds to $\mu$ and the coordinate $z_3$ while $\psi_2$ corresponds to the character $\psi_k$ of $C$ and the $(n - 2k + 1, n + 2k)$-th coordinate of $c \in C$. The character $\psi_2$ is nontrivial on $U_2'$. Finally, by Proposition 4 of Bump, Friedberg and Ginzburg [BFG06] (which is easily extended to $G_n$, given the analog of Theorem 4 in [Kap14a]), $\theta_{U_2',\psi_1}$ is a quotient of $\theta_{U_2'}$ (this is valid for $n \geq 3$, here $0 < k < n/2$ whence $n \geq 3$). Hence, $U_2'$ acts trivially on $\theta_{U_1,\psi_1}$ and therefore $(\theta_{U_1,\psi_1})_{U_2',\psi_2} = 0$ and (3.10) follows.

4. Distinguished representations

Let $G$ be either $\text{GL}_n$ or $G_n$. Let $\tau$ be an admissible representation of $G$ with a central character $\omega_\tau$. Assume that $\theta$ and $\theta'$ are a pair of exceptional representations of $\widetilde{G}$. We say that $\tau$ is $(\theta, \theta')$-distinguished if

$$\text{Hom}_G(\theta \otimes \theta', \tau^-) \neq 0.$$ 

The following result describes the upper hereditary property of a distinguished representation of $\text{GL}_n$, when induced to a representation of $G_n$. Following the notation of Section 2.8 we denote the exceptional representation of $\widetilde{G}$ corresponding to $\chi$ and $\psi$ by $\theta_{G, \chi, \psi}$. For any representation $\sigma$ of $\text{GL}_n$, $s \in \mathbb{C}$ and a character $\mu$ of $F^*$, one forms a representation $\sigma|\det|^{s} \otimes \mu$ of $M_n$. Put

$$I(\sigma, s, \mu) = \text{Ind}_{Q_n}^{G_n}(\delta_{Q_n}^1/2|^{s} \otimes |^{s} \otimes \mu).$$

**Proposition 4.1.** Let $\tau$ be a $(\theta_{\text{GL}_n}^{\chi, \psi}, \theta_{\text{GL}_n}^{\chi, \psi})$-distinguished representation of $\text{GL}_n$ and set $\eta = (\chi \chi')^{-1}$. Then $I(\tau, 1/2, \eta)$ is a $(\theta_{G_n}^{\chi, \psi}, \theta_{G_n}^{\chi, \psi'})$-distinguished representation of $G_n$.

**Proof of Proposition 4.1.** By definition the space

$$\text{Tri}_{\text{GL}_n}(\tau, \theta_{\text{GL}_n}^{\chi, \psi}, \theta_{\text{GL}_n}^{\chi, \psi})$$

of $\text{GL}_n$-equivariant trilinear forms on $\tau \times \theta_{\text{GL}_n}^{\chi, \psi} \times \theta_{\text{GL}_n}^{\chi, \psi}$ is nonzero. Therefore

$$(4.1) \quad \text{Tri}_{M_n}(\tau \otimes \eta, \theta_{\text{GL}_n}^{\chi, \psi} \otimes \theta_{\text{GL}_n}^{\chi, \psi} \otimes \theta_{\text{GL}_n}^{\chi, \psi'}) \neq 0.$$ 

According to (2.8),

$$(\theta_{\text{GL}_n}^{\chi, \psi})_{U_n} = \theta_{\text{GL}_n}^{\chi, \psi} \otimes \theta_{\text{GL}_n}^{\chi, \psi} \otimes \theta_{\text{GL}_n}^{\chi, \psi'}.$$ 

Applying Frobenius reciprocity we see that $\theta_{\text{GL}_n}^{\chi, \psi}$ is a subrepresentation of

$$(4.2) \quad \text{Ind}_{Q_n}^{\widetilde{G}_n}(\delta_{Q_n}^1 \theta_{\text{GL}_n}^{\chi, \psi} \otimes \theta_{\text{GL}_n}^{\chi, \psi'}).$$

A similar result holds for $\theta_{\text{GL}_n}^{\chi, \psi'}$.

We define

$$T \in \text{Tri}_{G_n}(I(\tau, 1/2, \eta), \theta_{\text{GL}_n}^{\chi, \psi}, \theta_{\text{GL}_n}^{\chi, \psi'})$$

and prove it is nonzero. Let $\varphi$ belong to the space $\theta_{\text{GL}_n}^{\chi, \psi}$, regarded as an element of (4.2), and similarly let $\varphi'$ belong to the space of $\theta_{\text{GL}_n}^{\chi, \psi'}$. Also take $f$ in the space of $I(\tau, 1/2, \eta)$. Now if $L \neq 0$ belongs to (4.1),

$$L(f(q), \varphi(q), \varphi'(q)) = \delta_{Q_n}(q)L(f(1), \varphi(1), \varphi'(1)), \quad q \in Q_n.$$
Thus the following integral is (formally) well defined (see e.g. [BZ76] 1.21),
\[ T(f, \varphi; \varphi') = \int_{Q_n \backslash G_n} L(f(g), \varphi(g), \varphi'(g)) \, dg. \]

It is absolutely convergent according to the Iwasawa decomposition. Since \( T \) satisfies the necessary equivariance properties, it remains to show \( T \neq 0 \). Assume \( L(x, y, y') \neq 0 \) for suitable data. Take \( f \) supported on \( Q_n N' \), for a small compact open neighborhood \( N \) of the identity in \( G_n \), and such that
\[ f((a, b)uv) = \delta_{Q_n}^1(a) |\det a |^{1/2} \eta(b) \tau(a)x, \quad \forall (a, b) \in GL_n \times G_0, u \in U_n, v \in N'. \]

We may assume \( \varphi(1) = y \) (because \( \theta_{\chi, \psi}^{{GL}_n} \otimes \theta_{\chi, 1}^{G_0} \) is irreducible) and \( \varphi'(1) = y' \). Using the Iwasawa decomposition and \( Q_n N \cap K = (Q_n \cap K)N' \) then yields
\[ T(f, \varphi, \varphi') = \int_{(Q_n \cap K)N'} L(f(k), \varphi(k), \varphi'(k)) \, dk. \]

Since \( L \) is invariant with respect to \( Q_n \cap K \), taking a sufficiently small \( N' \) (with respect to \( \varphi \) and \( \varphi' \)), the \( dk \)-integration reduces to a nonzero constant multiple of \( L(f(1), \varphi(1), \varphi'(1)) \), which is nonzero. We conclude that \( I(\tau, 1/2, \eta) = (\theta_{\chi, \psi}^G, \theta_{\chi', \psi'}^G) \)-distinguished.

Let \( \tau \) be a representation of \( G \) as above. Write \( \theta = \theta_{\chi, \psi} \) and \( \theta' = \theta_{\chi', \psi'} \). Since \( \theta_{\chi, \psi} = \chi \theta_{1, \psi} \), we may assume \( \chi = \chi' = 1 \), perhaps twisting \( \tau \) by a character. For simplicity, we then say that \( \tau \) is \( (\psi, \psi') \)-distinguished.

If \( n \) is even, the characters \( \psi \) and \( \psi' \) can be ignored, because \( \theta_{1, \psi} \) does not depend on \( \psi \). If \( n \) is odd and \( \tau \) is \( (\psi_0, \psi'_0) \)-distinguished, then for any \( \psi \) there is \( \psi' \) such that \( \tau \) is \( (\psi, \psi') \)-distinguished. Indeed, write \( \psi(x) = \psi_0(\alpha x) \) for some \( \alpha \in F^* \) and put \( \psi'(x) = \psi'_0(\alpha x) \), then by Claim 2.3 and its proof, \( \theta_{1, \psi_0} \otimes \theta_{1, \psi'_0} = \theta_{1, \psi} \otimes \theta_{1, \psi'} \).

In light of these observations, we say that \( \tau \) is distinguished if for any \( \psi \) there is \( \psi' \) such that \( \tau \) is \( (\psi, \psi') \)-distinguished. Proposition 4.1 implies,

**Corollary 4.2.** Let \( \tau \) be a distinguished representation of \( GL_n \). Then \( I(\tau, 1/2, 1) \) is distinguished.

Now we prove Theorem 4. Namely, for any pair \( \theta \) and \( \theta' \) of exceptional representations of \( \widetilde{G}_n \), and a generic character \( \psi \) of \( N_n \),
\[ (\theta \otimes \theta')_{N_n, \psi} = 0. \]

We consider the filtrations of \( \theta \) and \( \theta' \) corresponding to the Jacquet functor along \( C = C_{U_n} \). The kernel of this functor is glued from representations induced from the Jacquet modules described in Section 3. Taking the twisted Jacquet functor along \( N_n \) truncates some of these quotients and, essentially, reduces the problem to a representation induced from \( (\theta_{G_{2k}})_{U_{2k}, \psi_k} \otimes (\theta_{G_{2k}})_{U_{2k}, \psi'_k} \). Theorem 3 then enables us to further reduce the problem, to the vanishing of \( \text{ind}_{Sp_{2k} U_{n}}^{\text{GL}_{2k}} (\omega_\psi \otimes \omega_{\psi^{-1}})_{N_n, \psi} \), which essentially follows from the results of Offen and Sayag on Klyachko models ([OS08], see also [Kly83]).

**Proof of Theorem 4.** By an analog of the Geometric Lemma of Bernstein and Zelevinsky ([BZ77] Theorem 5.2 and [BZ76] 5.9-5.12), as a \( \widetilde{Q}_n \)-module, \( \theta \) is glued from
\[ \text{ind}_{Sp_{2k} U_n}^{\widetilde{Q}_n} (\theta_{C, \psi_k}), \quad 0 \leq k \leq [n/2]. \]
Claim 4.3. Then as a $Q_n$-module $\theta \otimes \theta'$ is glued from
\begin{equation}
\text{ind}_{St_{\psi_k}U_n}(\theta_{C,\psi_k}) \otimes \text{ind}_{St_{\psi_k}U_n}(\theta'_{C,\psi_k}^{-1}), \quad 0 \leq k, k' \leq \lfloor n/2 \rfloor.
\end{equation}

We prove that the Jacquet functor with respect to $N_n$ and $\psi$ vanishes on each of these representations.

Since functions in $\text{ind}_{St_{\psi_k}U_n}(\theta_{C,\psi_k})$ are compactly supported modulo $St_{\psi_k}U_n$, and $C$ is normal in $Q_n$, by Lemma 2.1
\begin{equation}
\text{ind}_{St_{\psi_k}U_n}(\theta_{C,\psi_k}) \otimes \text{ind}_{St_{\psi_k}U_n}(\theta'_{C,\psi_k}^{-1}))_{N_n,\psi} = \begin{cases} 
(\text{ind}_{St_{\psi_k}U_n}(\theta_{C,\psi_k} \otimes \theta'_{C,\psi_k}^{-1}))_{N_n,\psi} & k = k', \\
0 & k \neq k'.
\end{cases}
\end{equation}

To see this consider $f$ in the space of $\text{ind}_{St_{\psi_k}U_n}(\theta_{C,\psi_k})$ and $f'$ in the space of $\text{ind}_{St_{\psi_k}U_n}(\theta'_{C,\psi_k}^{-1})$, and look at the Jacquet-Langlands integral
\[
\int_C (f \otimes f')(g, g') \, dc = \int_C f(gc)f'(g'c) \, dc = f(g)f'(g') \int_C \psi_k(\vartheta c)\psi_k^{-1}(\vartheta' c) \, dc,
\]
where $C < C$ is a compact subgroup.

Since $\theta_{C,\psi_k} \otimes \theta'_{C,\psi_k}^{-1}$ is a non-genuine representation of $St_{\psi_k}$, we can replace the representation on the right-hand side of (4.5) with
\[
(\text{ind}_{St_{\psi_k}U_n}(\theta_{C,\psi_k} \otimes \theta'_{C,\psi_k}^{-1}))_{N_n,\psi}.
\]

Define $\vartheta$ with respect to $\theta^{\text{GL}_{n-2k}}$ as in Proposition 3.1 and similarly, define $\vartheta'$ with respect to $\theta^{\text{GL}_{n-2k}}$. According to the proposition, $\theta_{C,\psi_k}$ is embedded in a finite direct sum of representations $\vartheta \otimes (\theta^{G_{2k}})_{C_{U_{2k}},\psi_k}$, which are trivial on $U_{n-2k}$. Put
\[
\Pi_k = \vartheta \otimes \vartheta' \otimes (\theta^{G_{2k}})_{C_{U_{2k}},\psi_k} \otimes (\theta^{G_{2k}})_{C_{U_{2k}},\psi_k}^{-1}.
\]

It is enough to prove that for all $0 \leq k \leq \lfloor n/2 \rfloor$,
\begin{equation}
\text{ind}_{St_{\psi_k}U_n}(\Pi_k)_{N_n,\psi} = 0.
\end{equation}
This holds for $k = 0$, simply because $U_n$ is normal in $Q_n$, $\Pi_0$ is trivial on $U_n$ while $\psi$ is not.

The case of $k = n/2$ is handled by the following claim, whose proof is deferred to below.

Claim 4.3. Equality (4.6) holds for $k = n/2$.

Lastly, assume $0 < k < n/2$. Set $Q = Q_{n-2k} \cap Q_n$. By transitivity of induction
\[
(\text{ind}_{St_{\psi_k}U_n}(\Pi_k)_{N_n,\psi} = (\text{ind}_{St_{\psi_k}U_n}(\Pi_k)_{N_n,\psi} = \text{ind}_{St_{\psi_k}U_n}(\Pi_k)_{N_n,\psi}.
\end{equation}

The representation $\text{ind}_{St_{\psi_k}U_n}(\Pi_k)$ is trivial on $U_{n-2k}$. By virtue of the Geometric Lemma of Bernstein and Zelevinsky (BZ77 Theorem 5.2), the representation on the right-hand side is glued from Jacquet modules of $\text{ind}_{St_{\psi_k}U_n}(\Pi_k)$. Note that in general, the quotients are representations induced from Jacquet modules, here the induction is trivial because the stabilizer of the character $\psi$ of $N_n$ is $N_n \times G_0$.

Let $W$ be a set of representatives to the double cosets $Q\backslash Q_n/(N_nG_0)$. That is, $Q_n = \bigsqcup_{w \in W} Qw^{-1}N_nG_0$. We can take the elements $w$ to be Weyl elements of $\text{GL}_n$. When $\psi|_{U_{n-2k} \cap N_n}$
Here \( \delta \) is some modulus character, hereby ignored, and \( N_{GL_{n-2k}} \times N_{2k} < M_{n-2k} \). As a \( G_0 \)-module, this representation is isomorphic to
\[
(\vartheta \otimes \vartheta')_{N_{GL_{n-2k}}}, \psi \otimes \text{ind}^{Q_{2k}}_{St_{\psi_k}U_{2k}}((\theta^{G_{2k}})_{C_{U_{2k}}} \psi_k \otimes (\theta'^{G_{2k}})_{C_{U_{2k}}} \psi^{-1}_k)_{N_{2k}}, \psi.
\]
Here \( \psi \) is regarded as a generic character of \( N_{GL_{n-2k}} \) and \( N_{2k} \). Since the case \( k = n/2 \) implies
\[
\text{ind}^{Q_{2k}}_{St_{\psi_k}U_{2k}}((\theta^{G_{2k}})_{C_{U_{2k}}} \psi_k \otimes (\theta'^{G_{2k}})_{C_{U_{2k}}} \psi^{-1}_k)_{N_{2k}}, \psi = 0,
\]
Equality (4.6) follows.

**Proof of Claim (4.3)** For \( k = n/2, \Pi_k = \theta_{C, \psi_k} \otimes \theta'_{C, \psi^{-1}_k} \). Now we apply Theorem 3. For simplicity of computations, we can replace \( \psi_k \) with the character defined in the proof of the theorem, then use the epimorphism \( \ell : \text{St}_{\psi_k} \times U_n \rightarrow \text{Sp}_k \times H_n \) given there. By Theorem 3 \( \theta_{C, \psi_k} \) is isomorphic to the direct sum of copies of \( \omega_{\psi} \). Pull \( \omega_{\psi} \) back to a representation of \( \text{St}_{\psi_k} \times U_n \). Note that the \( G_0 \) part of \( \text{St}_{\psi_k} \) was ignored in the proof of Theorem 3, since it acts by a character, so we can ignore this here as well. Equality (4.6) will follow from
\[
(\text{ind}^{Q_{2k}}_{St_{\psi_k}U_{2k}}(\omega_{\psi} \otimes \omega_{\psi^{-1}}))_{N_{2k}, \psi} = 0.
\]

We need some notation. Put
\[
\pi_0 = \omega_{\psi} \otimes \omega_{\psi^{-1}}, \quad G = P^n_{n,1}, \quad V = Z_{n,1}, \quad P = \text{Sp}_k \times V, \quad X = P \backslash G.
\]
Note that \( C \setminus Q_n \cong G = \text{GL}_n \rtimes V \) (in fact, \( C \setminus Q_n \cong G \ltimes G_0 \) but \( G_0 \) was ignored), this isomorphism restricts to an isomorphism \( C \setminus U_n \cong V \). In this manner \( \psi \) is also a character of \( V, \psi(v) = \psi(v_{n,n+1}) \). Since \( \pi_0 \) is trivial on \( C \), we can regard it as a representation of \( P \) and if \( \pi = \text{ind}^G_P(\pi_0) \), \( \text{ind}^{Q_{2k}}_{St_{\psi_k}U_{2k}}(\pi_0) \cong \pi \) as \( G \)-modules.

We apply the theory of \( l \)-sheaves of Bernstein and Zelevinsky ([BZ76] 1.13 and Section 6). In the following, we freely use their terminology and definitions. Let \((X, \mathcal{F})\) be the \( l \)-sheaf corresponding to \( \pi \) ([BZ76] 2.23). The group \( N_n \) acts on \( X \) by \( u \cdot x = xu^{-1} \) and on \( \mathcal{F} \) by \( u \cdot \varphi(x) = \psi^{-1}(u)\varphi(u^{-1} \cdot x) \). An \( N_n \)-invariant \( \mathcal{F} \)-distribution on \( X \) is an element of \( \text{Hom}_{N_n}(\pi, \psi) \). Since \( \text{Hom}_{N_n}(\pi, \psi) \) is the algebraic dual of \( \pi_{N_n, \psi} \), to prove (4.7) we will show that there are no nonzero \( N_n \)-invariant \( \mathcal{F} \)-distributions on \( X \).

The action of \( N_n \) on \( X \) is constructive ([BZ76] Theorem A). If \( x \in X \), let \( P^x = x^{-1}P \cap N_n \) be the stabilizer of \( x \) in \( N_n \). The orbit of \( x \) is \( N_n \cdot x \). The mapping \( u \cdot x \mapsto (P^x)u^{-1} \) induces a homeomorphism \( N_n \cdot x \cong P^x \setminus N_n \) ([BZ76] 1.6). The restriction of \( \mathcal{F} \) to the orbit of \( x \) (this restriction is an \( l \)-sheaf, because the action is constructive) is isomorphic to \( \text{ind}^{N_n}_P(x^{-1}\pi_0) \), where \( x^{-1}\pi_0 \) is the representation of \( P^x \) acting in the space of \( \pi_0 \) by \( x^{-1}\pi_0(z) = \pi_0(xz) \).

By virtue of Theorem 6.9 of [BZ76], to show there are no nonzero \( N_n \)-invariant \( \mathcal{F} \)-distributions on \( X \), it is enough to prove that for each representative \( x \),
\[
\text{Hom}_{N_n}(\text{ind}^{N_n}_P(x^{-1}\pi_0), \psi) = \text{Hom}_P(x^{-1}\pi_0, \psi) = 0.
\]
We can take representatives $x \in P_{n-1,1}^o < \text{GL}_n$. Then $P^x = \text{Sp}_k \ltimes V$, where $\text{Sp}_k = x^{-1} \text{Sp}_k \cap \text{NGL}_n$. Indeed, because $P_{n-1,1}^o$ fixes $\psi|_V$, $(x^{-1} \pi_0)_V \psi = x^{-1}((\pi_0)_V \psi)$. Hence

$$\text{Hom}_{P^x}(x^{-1} \pi_0, \psi) = \text{Hom}_{\text{Sp}_k}(x^{-1}((\pi_0)_V \psi), \psi).$$

Note that $(\pi_0)_V \psi = (\omega_\psi \otimes \omega_{\psi^{-1}})_{H_n,\text{str}}$ for the nontrivial $\mu$ given in Claim 2.4. According to that claim $(\pi_0)_V \psi$ is the trivial one-dimensional representation of $\ell^{-1}(\text{Sp}_k \cap P_{n-1,1})$. Since $x N_{\text{GL}_n} < P_{n-1,1}^o$ for any $x \in P_{n-1,1}^o$, $x^{-1}((\pi_0)_V \psi)$ is trivial on $\text{Sp}_k$ (the epimorphism $\ell$ is easily seen to be harmless here). But Offen and Sayag (OS08 Proposition 2, we use $\mathcal{H}^{r,r'}$ with $r = 0$ and $r' = n$, in their notation) proved that $\psi|_{\text{Sp}_k} \neq 1$ for any $x \in \text{GL}_n$. This implies $\text{Hom}_{\text{Sp}_k}(x^{-1}((\pi_0)_V \psi), \psi) = 0$, as required.

**Corollary 4.4.** Let $\tau$ be an irreducible unitary supercuspidal $(\theta^{\text{GL}_n}_{\chi,\psi}, \theta^{\text{GL}_n}_{\chi',\psi'})$-distinguished representation of $\text{GL}_n$. Then the Langlands quotient of $I(\tau, 1/2, (\chi')^{-1})$ is $(\theta^{\text{GL}_n}_{\chi,\psi}, \theta^{\text{GL}_n}_{\chi',\psi'})$-distinguished. In particular, if $\tau$ is distinguished, so is the Langlands quotient of $I(\tau, 1/2, 1)$. According to Proposition 4.1

$$\text{Hom}_{\text{G}_n}(\theta^{\text{G}_n}_{\chi,\psi} \otimes \theta^{\text{G}_n}_{\chi',\psi'}, I(\tau, 1/2, (\chi')^{-1}) \psi) \neq 0.$$

Theorem 1 implies $I(\tau, 1/2, (\chi')^{-1})$ is reducible (because $\tau$ is generic), then by Casselman and Shahidi (CS08) (Theorem 1), the Langlands quotient $LQ(I(\tau, 1/2, (\chi')^{-1}))$ is non-generic, and the unique irreducible subspace of $I(\tau, 1/2, (\chi')^{-1})$ is generic. Also note that the length of $I(\tau, 1/2, (\chi')^{-1})$ is 2 (because it is reducible and $\tau$ is supercuspidal, see [BZ77] 2.8). Now the result follows from Theorem 1 and the left exactness of the Hom functor.

As a corollary, we can now prove Theorem 2. Namely, for an irreducible unitary supercuspidal $\tau$, being distinguished is equivalent to the occurrence of a pole at $s = 0$ of $L(s, \tau, \text{Sym}^2)$.

**Proof of Theorem 2.** If $\tau$ is distinguished, as in the proof of Corollary 4.4 we see that $I(\tau, 1/2, 1)$ is reducible, then according to Casselman and Shahidi (CS08) (Proposition 5.3, their Conjecture 1.1 was proved for $G_n$ in [Asg02]) $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$.

In the other direction, assume $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$. As an application of the descent method of Ginzburg, Rallis and Soudry (see e.g., [GRS97b, GRS99a, GRS99b, GRS01, GSS02, JS03, JS04, Sou05, Sou06, GRS11]), one can globalize $\tau$ to a cuspidal automorphic representation $\pi$ of $\text{GL}_n(\mathbb{A})$, such that $L^\infty(s, \pi, \text{Sym}^2)$ has a pole at $s = 1$ (see the appendix of [PR12]). Therefore the nonvanishing of (1.2) implies $\tau$ is distinguished ([BG92] Theorem 7.6).

**Corollary 4.5.** If $\tau$ is an irreducible unitary supercuspidal distinguished representation, it must be self-dual.

**Remark 4.1.** The analogous result for an irreducible supercuspidal $(\theta^{\text{GL}_n}_{\chi,\psi}, \theta^{\text{GL}_n}_{\chi',\psi'})$-distinguished $\tau$ should also hold. One needs to verify the applicability of the globalization argument (see [HS12]).

Given exceptional representations $\theta$ and $\theta'$ of $\tilde{G}$ (as in the beginning of this section), we can consider the space of $\theta \otimes \theta'$ as a model for representations of $G$. We refer to the dimension of $\text{Hom}_G(\theta \otimes \theta', \tau^\vee)$ as the multiplicity of $\tau$. The next proposition relates the multiplicities of $\tau$ and $I(\tau, 1/2, \eta)$. 


In the case of $GL_n$, Kable [Kab01] conjectured that the multiplicity of an irreducible representation is at most one. He proved this for $n \leq 3$, and for arbitrary $n$ under a certain homogeneity condition ([Kab01] Corollary 6.1). It is reasonable to believe multiplicity one also holds in the context of $G_n$ (see [LM14a] Remark 4.2).

**Proposition 4.6.** Let $\tau$ be an irreducible tempered representation of $GL_n$, put $\eta = (\chi')^{-1}$ and assume $|\eta| = 1$. Then

$$
(4.8) \quad \dim \text{Hom}_{G_n}(\theta^{G_n}_{\chi,\psi} \otimes \theta^{G_n}_{\chi',\psi'}, I(\tau, 1/2, \eta)^\vee) = \dim \text{Hom}_{GL_n}(\theta^{GL_n}_{\chi,\psi} \otimes \theta^{GL_n}_{\chi',\psi'}, \tau^\vee).
$$

In particular, the representation $\tau$ is $(\theta^{GL_n}_{\chi,\psi}, \theta^{GL_n}_{\chi',\psi'})$-distinguished if and only if $I(\tau, 1/2, \eta)$ is $(\theta^{G_n}_{\chi,\psi}, \theta^{G_n}_{\chi',\psi'})$-distinguished.

**Remark 4.2.** If $\tau$ is $(\theta^{GL_n}_{\chi,\psi}, \theta^{GL_n}_{\chi',\psi'})$-distinguished, in particular $\omega_\tau(z^2 \cdot I_n) = \eta(2z_n)$ for all $z \in F^*$ (the central character of $\tau$ is unitary. In this case by $(4.8)$ $\tau$ enjoys multiplicity one if and only if $I(\tau, 1/2, \eta)$ does.

**Proof of Proposition 4.6.** Set $\theta_0 = \theta^{GL_n}_{\chi,\psi}$, $\theta = \theta^{G_n}_{\chi,\psi}$ and similarly $\theta'_0$ and $\theta'$ (with $\chi'$ and $\psi'$).

For brevity, put $d = |\det|$. Applying Frobenius reciprocity,

$$
\text{Hom}_{G_n}(\theta \otimes \theta', I(\tau, 1/2, \eta)^\vee) = \text{Hom}_{GL_n}(I(\theta \otimes \theta') U_n, d^{(n-1)/2} \tau^\vee).
$$

Using the notation of Section 3, the representation $(\theta \otimes \theta')U_n$ is filtered by representations

$$
\mathbb{W}_k = (\text{ind}^{Q_n}_{\text{St}_{\Psi_k} U_n}(\theta_{C,\psi_k} \otimes \theta'_{C,\psi_k^{-1}}))U_n, \quad 0 \leq k \leq [n/2].
$$

The representation $\mathbb{W}_0$ is a quotient of $(\theta \otimes \theta')U_n$ and the kernel of the mapping $(\theta \otimes \theta')U_n \to \mathbb{W}_0$ is filtered by $\mathbb{W}_k$ with $1 \leq k \leq [n/2]$. Regard $\mathbb{W}_k$ as a representation of $GL_n$ (by restriction from $M_n$).

According to Proposition 3.1, $\mathbb{W}_0 = d^{(n-1)/2} \theta_0 \otimes \theta'_0$. Then

$$
\text{Hom}_{GL_n}(I(\mathbb{W}_0, d^{(n-1)/2} \tau^\vee) = \text{Hom}_{GL_n}(\theta_0 \otimes \theta'_0, \tau^\vee).
$$

Thus $(4.8)$ will follow, once we prove that for $k > 0$,

$$
\text{Hom}_{GL_n}(\mathbb{W}_k, d^{(n-1)/2} \tau^\vee) = 0.
$$

If $n$ is even, we claim the following.

**Claim 4.7.** For any irreducible generic representation $\sigma$ of $GL_n$, $\text{Hom}_{GL_n}(\mathbb{W}_{n/2}, \sigma) = 0$. In particular Equality $(4.9)$ holds for $k = n/2$.

The proof will be given below.

Assume $0 < k < n/2$. The representation $\theta_{C,\psi_k}$ is embedded in a finite direct sum of representations $\vartheta \otimes (\theta^{G_k}_{C})_{U_{2k},\psi_k}$ (Proposition 3.1). The representation $\vartheta$ is semisimple, and by Theorem 3 the representation $(\theta^{G_k}_{C})_{U_{2k},\psi_k}$ is also semisimple. Hence $\theta_{C,\psi_k} \otimes \theta'_{C,\psi_k^{-1}}$ is embedded in a semisimple representation, which is the finite direct sum of $\Pi_k (\Pi_k = \vartheta \otimes \vartheta' \otimes (\theta^{G_k}_{C})_{U_{2k},\psi_k} \otimes (\theta^{G_k}_{C})_{U_{2k},\psi_k^{-1}})$. Therefore, using the exactness of induction and Jacquet functors, $\mathbb{W}_k$ is a quotient of $(\text{ind}^{Q_n}_{\text{St}_{\Psi_k} U_n}(\oplus \Pi_k))U_n$. We conclude that we may replace $\mathbb{W}_k$ in $(4.9)$ with $(\text{ind}^{Q_n}_{\text{St}_{\Psi_k} U_n} \Pi_k)U_n$. An application of [BZ77] (Theorem 5.2) yields

$$
(\text{ind}^{Q_n}_{\text{St}_{\Psi_k} U_n} \Pi_k)U_n = \text{ind}^{GL_n}_{F_{n-2k,2k}}(\vartheta \otimes \vartheta' \otimes \mathbb{W}').
$$
where
\[ \mathbb{W}' = (\text{ind}^{Q}_{\text{St}_{\psi_k} U_{n}} (\theta_{C_{U_{k}}, \psi_k} \otimes \theta_{G_{U_{k}}, \psi_k}^G))_{U_{2k}} |_{GL_{2k}}. \]

Note that the relevant double coset space (for [BZ77] Theorem 5.2) is \((Q_{n-2k} \cap Q_{n})/Q_{n}, Q_{n}, \) there is only one representative to consider.

Hence to deduce (4.9), it is enough to prove
\[ \text{Hom}_{GL_{n}} (\text{ind}^{GL_{n}}_{P_{n-2k, 2k}} (\vartheta \otimes \vartheta' \otimes \mathbb{W}'), d^{(n-1)/2} \tau_{1}^\lor) = 0. \]

The left-hand side equals
\[ \begin{align*}
\text{Hom}_{GL_{2k}} (d^{(1-n)/2} \tau, \text{ind}^{GL_{2k}}_{P_{n-2k, 2k}} (\delta_{P_{n-2k, 2k}} (\vartheta \otimes \vartheta' \otimes \mathbb{W}'))) \\
= \text{Hom}_{P_{n-2k, 2k}} (d^{(1-n)/2} \tau_{Z_{n-2k, 2k}}, \delta_{P_{n-2k, 2k}} (\vartheta \otimes \vartheta' \otimes \mathbb{W}')) \\
= \text{Hom}_{P_{n-2k, 2k}} (d^{(1-n)/2} \tau_{P_{n-2k, 2k}} (\vartheta \otimes \vartheta' \otimes \mathbb{W}', \delta_{P_{n-2k, 2k}} (\vartheta \otimes \vartheta' \otimes \mathbb{W}')).
\end{align*} \] (4.10)

Here \( j(\tau) = \delta_{P_{n-2k, 2k}}^{-1/2} \tau_{Z_{n-2k, 2k}} \) (the normalized Jacquet functor). It suffices to show that the last space (4.10) vanishes, when \( j(\tau) \) is replaced by any of its irreducible subquotients. Let \( \tau_1 \otimes \tau_2 \) be one such subquotient. Then (4.10) vanishes if either (4.11)
\[ \text{Hom}_{GL_{2k}} (\mathbb{W}', d^{(2k-1)/2} \tau_{1}^\lor) \]
and (4.12)
\[ \text{Hom}_{GL_{n-2k}} (d^{(1-n-2k)/2} \vartheta \otimes \vartheta', \tau_{1}^\lor) \]
are zero. As explained in the proof of Proposition 4.1 of Lapid and Mao [LM14a], since \( \tau \) is tempered, either \( \tau_2 \) is generic, or the central character \( \omega_{\tau_1} \) of \( \tau_1 \) satisfies \( |\omega_{\tau_1}| = |\det|^{\alpha} \) for some \( \alpha > 0 \). In the former case (4.11) vanishes by Claim 4.7. In the latter case (4.12) vanishes. Indeed, \( z^2 \cdot I_{n-2k} \) acts on \( \vartheta \otimes \vartheta' \) by \( d^{(-1+n+2k)/2} \eta^{-1} (z^{2(n-2k)}) \) and since \( |\eta| = 1 \), this action is unitary on \( d^{(1-n-2k)/2} \vartheta \otimes \vartheta' \), but \( \omega_{\tau_1} \) is not unitary.

Proof of Claim 4.7. Put \( k = n/2 \) and \( \pi = \text{ind}^{Q}_{\text{St}_{\psi_k} U_{n}} (\theta_{C, \psi_k} \otimes \theta_{C, \psi_k}^G) \). We need to prove
\[ \text{Hom}_{GL_{n}} (\pi_{U_{n}}, \sigma) = 0. \]
We will show that \( \pi_{U_{n}} \) has a filtration, whose quotients are all isomorphic to \( \text{ind}^{GL_{n}}_{P_{n+2k}} \). Then since \( \sigma \) is irreducible and generic, \( \text{Hom}_{GL_{n}} (\text{ind}^{GL_{n}}_{P_{n+2k}}, \sigma) = 0 \) ([OS08] Proposition 1, take \( H^{0, n} \)) and the result follows.

We turn to prove the filtration of \( \pi_{U_{n}} \). As in the proof of Claim 4.3, Theorem 5 implies that \( \pi_{U_{n}} \) is filtered by copies of the representation \( \text{ind}^{Q_{n}}_{\text{St}_{\psi_k} U_{n}} (\omega_{\psi} \otimes \omega_{\psi}^{-1}) \). We prove (4.13)
\[ \text{ind}^{Q_{n}}_{\text{St}_{\psi_k} U_{n}} (\omega_{\psi} \otimes \omega_{\psi}^{-1}) U_{n} = \text{ind}^{GL_{n}}_{P_{n+2k}} \cdot 1. \]
(Recall that the left-hand side is regarded as a representation of \( GL_{n} \).)

Since \( U_{n} \) is normal in \( Q_{n} \), Lemma 2.1 implies
\[ \text{ind}^{Q_{n}}_{\text{St}_{\psi_k} U_{n}} (\omega_{\psi} \otimes \omega_{\psi}^{-1}) U_{n} = \text{ind}^{Q_{n}}_{\text{St}_{\psi_k} U_{n}} ((\omega_{\psi} \otimes \omega_{\psi}^{-1}) H_{n}). \]
In more detail, if \( f \) belongs to the space of \( \text{ind}^{Q_{n}}_{\text{St}_{\psi_k} U_{n}} (\omega_{\psi} \otimes \omega_{\psi}^{-1}) \), the Jacquet-Langlands integral takes the form
\[ \int_{U_{n}} f(gu) \, du = \int_{U_{n}} (\omega_{\psi} \otimes \omega_{\psi}^{-1})(\theta(u)) f(g) \, du, \]
for a compact subgroup $U < U_n$. Because $f$ is compactly supported modulo $\text{St}_{\psi_k}U_n$, this integral vanishes for all $g \in Q_n$ if and only if $f(g)$ belongs to the space of $(\omega_\psi \otimes \omega_{\psi^{-1}})(H_n)$ for all $g$. It remains to use the exactness of induction.

According to Claim 2.4, $(\omega_\psi \otimes \omega_{\psi^{-1}})(H_n)$ is the trivial one-dimensional representation of $\text{Sp}_k$ and (4.13) holds.

We can now improve Corollary 4.2 for tempered representations:

**Corollary 4.8.** Let $\tau$ be an irreducible tempered representation of $\text{GL}_n$. Then $\tau$ is distinguished if and only if $I(\tau, 1/2, 1)$ is distinguished.

5. **The small representation of $\text{SO}_{2n+1}$**

Bump, Friedberg and Ginzburg [BFG03] constructed and studied the small representations for $\text{SO}_{2n+1}$. In this section we briefly recall their results and formulate our results for $\text{SO}_{2n+1}$. The cover $\widetilde{SO}_{2n+1}$ was obtained by restricting the 4-fold cover of $\text{SL}_{2n+1}$ of Matsumoto [Mat09]. It is a “double cover” in the sense that the square of the cocycle is trivial on the kernel of the spinor norm.

We use the same notation of $G_n$, e.g., $B_n = B_{\text{SO}_{2n+1}}$ (see Section 2.3), $T_n$ is the diagonal torus and $Q_k$ is a standard maximal parabolic subgroup. If $(a, g) \in \text{GL}_k \times \text{SO}_{2(n-k)+1}$, $(a, g)$ is embedded in $M_k$ as $\text{diag}(a, J_k)$. The block compatibility formula now reads (see [BFG03] (2.20))

$$
\sigma_{\text{SL}_{2n+1}}((a, g), (a', g'))
= \sigma_{\text{SL}_{k+1}}(\text{diag}(a, \det a^{-1}), \text{diag}(a', \det a'^{-1}))^2(\det a, \det a')q_{\text{SL}_{2(n-k)+1}}(g, g').
$$

The benefit of this cover is that the preimages of $\text{GL}_k$ and $\text{SO}_{2(n-k)+1}$ commute, thus the tensor can be used to describe representations of Levi subgroups. Restriction of the cover to $\text{GL}_k$ is a double cover.

The small representation $\theta = \theta_{\text{SO}_{2n+1}}$ is unique; it is the representation of $\widetilde{SO}_{2n+1}$ corresponding to the exceptional character

$$
\xi(\sigma(\text{diag}(t_1^2, \ldots, t_n^2, 1, t_n^{-2}, \ldots, t_1^{-2}))) = \prod_{i=1}^n |t_i|^{n-i+1}.
$$

According to [BFG03] (Theorem 2.3), $\theta_{U_k} = \theta_{\text{GL}_k} \otimes \theta_{\text{SO}_{2(n-k)+1}}$, where $\theta_{\text{GL}_k}$ was explicitly given, and by [BFG03] (Theorem 2.6) and [BFG06] (Proposition 3), $\theta_{U_1, \psi(1)} = 0$ if the length of $\psi(1)$ is nonzero (using the notation of Section 2.8).

Theorems 1 and 3 remain valid as stated. Proposition 3.1 now takes the form

$$
\theta_{C, \psi_k} = \theta_{\text{GL}_{n-2k}} \otimes (\theta_{\text{SO}_{2k+1}})_{C_{U_{2k}}, \psi_k}.
$$

Here $\theta_{\text{GL}_{n-2k}}$ is uniquely determined. Indeed, this equality replaces (3.9) because $\theta_{U_{n-2k}}$ is a tensor of representations.

Regarding Proposition 4.1 assume $\tau$ is $(\psi, \psi')$-distinguished. Twisting $\tau$ by some square trivial character, we obtain a $(\psi, \psi')$-distinguished representation. Then, perhaps after using another twist of $\tau$, it becomes $(\theta_{\text{GL}_n}, \theta_{\text{GL}_n})$-distinguished for the exceptional representation $\theta_{\text{GL}_n}$ such that $\theta_{U_n} = \theta_{\text{GL}_n}$. Now the proof of the proposition proceeds as in the case of $G_n$. So, in this case one must start with a $(\theta_{\text{GL}_n}, \theta_{\text{GL}_n})$-distinguished representation of $\text{GL}_n$, in order
to obtain a distinguished representation \( I(\tau, 1/2, 1) \) of \( \text{SO}_{2n+1} \). Corollary 4.4 is applicable for \( \tau \) such that Proposition 4.1 is valid. Proposition 4.6 remains valid.

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