IRREGULARITIES OF DISTRIBUTION AND GEOMETRY OF PLANAR CONVEX SETS

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ABSTRACT. We consider a planar convex body $C$ and we prove several analogs of Roth’s theorem on irregularities of distribution. When $\partial C$ is $C^2$ regardless of curvature, we prove that for every set $\mathcal{P}_N$ of $N$ points in $T^2$ we have the sharp bound

$$\int_0^1 \int_{T^2} |\text{card } (\mathcal{P}_N \cap (\lambda C + t)) - \lambda^2 N |C||^2 \, dt d\lambda \geq cN^{1/2}.$$ 

When $\partial C$ is only piecewise $C^2$ and is not a polygon we prove the sharp bound

$$\int_0^1 \int_{T^2} |\text{card } (\mathcal{P}_N \cap (\lambda C + t)) - \lambda^2 N |C||^2 \, dt d\lambda \geq cN^{2/5}.$$ 

We also give a whole range of intermediate sharp results between $N^{2/5}$ and $N^{1/2}$. Our proofs depend on a lemma of Cassels-Montgomery, on ad hoc constructions of finite point sets, and on a geometric type estimate for the average decay of the Fourier transform of the characteristic function of $C$.

1. Introduction

The term Irregularities of distribution, often replaced with (geometric) discrepancy, has been introduced by K. Roth in his seminal paper [On irregularities of distribution (published in Mathematika in 1954, [34]), where the following result has been proved.

**Theorem 1.** Let $N > 1$ be an integer, and let $u_1, u_2, \ldots, u_N$ be $N$ points, not necessarily distinct, in the square $[0, 1)^2$. Then

$$\int_0^1 \int_0^1 (S(x, y) - Nxy)^2 \, dx dy > c \log (N),$$

where

$$S(x, y) = \text{card } \{ j = 1, 2, \ldots, N : u_j \in [0, x) \times [0, y) \}.$$

From here on out, $c$, $c_1$, $\ldots$ are positive constants, independent of $N$, which may change from step to step.

Given any finite point set $\mathcal{P}_N = \{ u_j \}_{j=1}^N \subset [0, 1)^2$, Roth’s theorem concerns the $L^2$ discrepancy between ($N$ times) the area of the rectangle $[0, x) \times [0, y)$ and the number of points $u_j$ that belong to the above rectangle. More generally, Discrepancy Theory concerns the problem of replacing a continuous object with a discrete

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1K. Roth is famous for many results, first of all his solution of the Siegel conjecture concerning approximation of algebraic numbers by rationals. It is known that he considered [34] to be his best work (see [19]).
sampling, and is presently a crossroads between many fields of Mathematics (see e.g. [3], [6], [9], [15], [17], [21], [25], [29], [38]).

Roth’s paper dealt with the van der Corput conjecture, that is a 1-dimensional problem about infinite numerical sequences that turns into a 2-dimensional geometric problem about distributions of finite point sets with respect to a family of rectangles. Roth not only improved the quantitative solution of the van der Corput conjecture previously obtained by T. van Aardenne-Ehrenfest ([1], [2]), but he also introduced a geometric point of view and “started a new field”.

In 1956 H. Davenport [20] proved that the log \( (N) \) lower bound in (1) cannot be improved. He showed that, for every \( N \), there exist \( N \) points \( u_1, u_2, \ldots, u_N \) in the square \( [0, 1)^2 \) such that

\[
\int_0^1 \int_0^1 (S(x, y) - Nxy)^2 \, dx \, dy \leq c \log(N),
\]

where \( S(x, y) \) is as in (2).

H. Montgomery [30, Ch. 6] introduced a different point of view and used Fourier series to prove the following result.

**Theorem 2.** For every finite set \( \mathcal{P}_N \) of \( N \) points in \( \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) we have

\[
\int_0^1 \int_{\mathbb{T}^2} \left| \text{card} \left( \mathcal{P}_N \cap \left( [0, \lambda)^2 + t \right) \right) - \lambda^2 N \right|^2 \, dt \, d\lambda \geq c \log(N).
\]

M. Drmota proved that the LHS in (1) and the LHS in (3) are equivalent (see [23], see also [35]).

It is natural to replace the rectangle \( [0, x) \times [0, y) \) in Roth’s theorem with other geometric objects, first of all suitable families of convex bodies (that is, bounded convex sets with non-empty interiors). Then the lower bound of the discrepancy may be much larger than a logarithm, as W. Schmidt first pointed out considering the case of a ball (see [36]). More generally, we can consider an arbitrary convex body \( C \) and average its discrepancy over translations, dilations and rotations. J. Beck [3] and H. Montgomery (see [30, Ch. 6]) proved independently the following result (which we state only in the planar case).

**Theorem 3.** Let \( C \subset \mathbb{T}^2 \) be a convex body of diameter less than 1. Then for every set \( \mathcal{P}_N \) of \( N \) points in \( \mathbb{T}^2 \) we have

\[
\int_0^1 \int_{SO(2)} \int_{\mathbb{T}^2} \left| \text{card} \left( \mathcal{P}_N \cap \left( \lambda \sigma(C) + t \right) \right) - \lambda^2 N |C| \right|^2 \, dt \, d\sigma d\lambda \geq c N^{1/2},
\]

where \( |C| \) denotes the area of \( C \).

The lower bound in (4) is sharp for every convex body. This follows from a classical result of D. Kendall [28] on lattice points, together with a Fourier analytic result proved in [32] (see also [38, Ch. 8] and [10]).

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2If \( s_1, s_2, s_3, \ldots \) is an infinite sequence of real numbers lying between 0 and 1, then corresponding to any arbitrarily large \( k \), there exist a positive integer \( n \) and two subintervals, of equal length, of \( (0, 1) \), such that the number of \( s_v \) \((v = 1, \ldots, n)\) that lie in one of the subintervals differs from the numbers of such \( s_v \) that lie in the other subinterval by more than \( k \).
The integration over dilations in (1) cannot be avoided (see [39], see also [7], [31], [38, Ch. 11] for results in higher dimensions). The above result of Davenport (see also [18]) shows that also the integration over rotations is necessary in (1). If we replace $C$ with a disk $D$ in Theorem 3 then the integration over rotations is meaningless and (1) reduces to the following (sharp) inequality.

$$\int_0^1 \int_{\mathbb{T}^2} \left| \text{card} \left( P_N \cap (\lambda D + t) \right) - \lambda^2 N \right| |D| \, dt \, d\lambda \geq cN^{1/2}.$$ 

In short, after averaging the discrepancy over translations and dilations, we have log $N$ as a (sharp) lower estimate for the case of the square and $N^{1/2}$ as a (sharp) lower estimate for the case of the disk.

From now on we will always average the discrepancies over translations and dilations.

The above log $N$ lower estimate has been extended from the case of a square to the case of convex polygons in [23], while the $N^{1/2}$ lower estimate has been extended from the case of a disk to the case of convex bodies with $C^{1,2}$ boundary having everywhere positive curvature in [22] (see also [5, Ch. 7]). In [13] we constructed an example of a convex body $C$ whose boundary is $C^2$ with a flat point, such that the $N^{1/2}$ lower bound still holds true.

Related results have been proved by J. Beck in [4], where he obtained lower bounds for the discrepancy in terms of the quality of approximation of $C$ through inscribed polygons. A related point of view has been considered in [11].

We prove that every convex body with $C^2$ boundary, regardless of curvature, satisfies the sharp $N^{1/2}$ lower bound. We also prove that if we are not in the case of a polygon and the boundary is piecewise $C^2$, then the lower bound $N^{2/5}$ holds true. Moreover, for every $2/5 \leq a \leq 1/2$ we give a geometric condition that implies the lower bound $N^a$. We also give an explicit convex body which admits the upper bound $N^a$ for the discrepancy.

2. Main results

Here we state the main results of the paper. The proofs are in the next sections.

The following Theorem 4 is a particular case of both Theorem 7 and Theorem 11 below.

**Theorem 4.** Let $C \subset \mathbb{T}^2$ be a convex body with $C^2$ boundary. Then there exists $c > 0$ such that for every set $P_N$ of $N$ points in $\mathbb{T}^2$ we have

$$\int_0^1 \int_{\mathbb{T}^2} \left| \text{card} \left( P_N \cap (\lambda C + t) \right) - \lambda^2 N \right| |C| \, dt \, d\lambda \geq cN^{1/2}.$$ 

The case of a disk shows that (4) is sharp.

A few definitions and preliminary results are necessary before we state our next theorems.

We are going to consider convex planar bodies which are not polygons and have piecewise $C^2$ boundaries. For the discrepancy associated to these bodies we prove sharp lower bounds depending on the regularity of the boundary, which we measure...
in terms of lengths of chords. In some cases this condition can be interpreted as an “inner disk condition”. See Remark 14.

**Definition 5.** Let \( C \subset \mathbb{T}^2 \) be a convex body. For every unit vector \( \Theta = (\cos \theta, \sin \theta) \) and \( \delta > 0 \) we consider the chord

\[
\gamma_\Theta (\delta) = \left\{ x \in C : x \cdot \Theta = \inf_{y \in C} (y \cdot \Theta) + \delta \right\}
\]

and its length \( |\gamma_\Theta (\delta)| \). See Figure 1.

![Figure 1. The chord \( \gamma_\Theta (\delta) \).](image)

Observe that \( |\gamma_\Theta (\delta)| \) cannot be too small. Namely we have the following result.

**Proposition 6.** Let \( C \) be a planar convex body. Then there exist \( \delta_0, c > 0 \) such that for \( 0 < \delta < \delta_0 \) and every direction \( \Theta \) we have

\[
|\gamma_\Theta (\delta)| \geq c\delta.
\]

Observe that if \( C \) is a polygon, then the above bound is sharp for all but a finite number of directions. If \( \partial C \) is smooth enough, then \( |\gamma_\Theta (\delta)| \) can be much larger. See Remark 9 and Theorem 15 below.

**Theorem 7.** Let \( C \) be a convex body and let \( \Theta = (\cos \theta, \sin \theta) \). Assume the existence of constants \( \delta_0, c_1, c_2 > 0 \), \( 1/2 \leq \sigma \leq 1 \) and an interval \( I \) in \((-\pi, \pi)\) such that for every \( 0 < \delta \leq \delta_0 \) we have

\[
\left\{ \begin{align*}
|\gamma_{-\Theta} (\delta)| + |\gamma_\Theta (\delta)| & \geq c_1 \delta^{1/2} \\
|\gamma_{-\Theta} (\delta)| + |\gamma_\Theta (\delta)| & \geq c_2 \delta^\sigma
\end{align*} \right. \text{ for every } \theta \in I,
\]

\[
|\gamma_{-\Theta} (\delta)| + |\gamma_\Theta (\delta)| \geq c_2 \delta^\sigma \quad \text{for every } \theta \notin I.
\]

Then there exists \( c > 0 \) such that for every set \( \mathcal{P}_N \) of \( N \) points in \( \mathbb{T}^2 \) we have

\[
\int_0^1 \int_{\mathbb{T}^2} \left| \text{card} \left( \mathcal{P}_N \cap (\lambda C + t) \right) - \lambda^2 N |C| \right|^2 \, dt \, d\lambda \geq c N^{2/(2\sigma+3)}.
\]

The following result shows that Theorem 7 is sharp.

**Theorem 8.** For every \( 1/2 \leq \sigma \leq 1 \) there exist \( c > 0 \), an explicit construction of a planar convex body \( C_\sigma \) that satisfies (6), and finite sets \( \mathcal{P}_{N_j} \subset \mathbb{T}^2 \) of cardinality \( N_j \to +\infty \) such that

\[
\int_0^1 \int_{\mathbb{T}^2} \left| \text{card} \left( \mathcal{P}_{N_j} \cap (\lambda C + t) \right) - \lambda^2 N_j |C| \right|^2 \, dt \, d\lambda \leq c N_j^{2/(2\sigma+3)}.
\]
Remark 9. It is not difficult to see that if $\partial C$ is $C^2$ then (7) is true with $\sigma = 1/2$ (therefore Theorem 4 is a consequence of Theorem 2). Indeed, a suitable choice of coordinates allows us to assume that the origin belongs to $\partial C$ and that $(0,1)$ is the inward unit normal at the origin. Hence $\partial C$ coincides locally with the graph of a $C^2$ function $\varphi(x)$ which satisfies $0 \leq \varphi''(x) \leq c$, where $c$ depends only on $C$. A repeated integration of this inequality yields $cx^2 - \varphi(x) \geq 0$, which in turns gives (7) with $\sigma = 1/2$. The above argument can be repeated for every direction. Then (6) holds uniformly for $\sigma = 1/2$ and every $\Theta$.

Observe that the exponent $2/(2\sigma + 3)$ in (7) and (8) takes all values between 2/5 and 1/2. The next proposition is a corollary of Proposition 6 and Theorem 7 and shows that $N^{2/5}$ is a general lower bound for convex planar bodies that are not polygons and have piecewise $C^2$ boundary.

Proposition 10. Let $C$ be a convex planar body which is not a polygon and has piecewise $C^2$ boundary. Then there exists $c > 0$ such that for every set $P_N$ of $N$ points in $\mathbb{T}^2$ we have

$$
\int_0^1 \int_{\mathbb{T}^2} |\text{card}(P_N \cap (\lambda C + t)) - \lambda^2 N |C||^2 \ dt d\lambda \geq c N^{2/5}.
$$

Observe that Theorem 5 shows that the above estimate $N^{2/5}$ cannot be improved.

Theorem 11. Let $C \subset \mathbb{T}^2$ be a convex body. Assume there exist $c, \delta_0 > 0$ and $1/2 \leq \sigma < 1$ such that for $0 \leq \delta \leq \delta_0$ and $\theta \in [0, \pi)$ we have

$$
|\gamma_{-\Theta}(\delta)| + |\gamma_{\Theta}(\delta)| \geq c \delta^\sigma.
$$

Then, there exists $c > 0$ such that for every finite set $P_N$ of $N$ points in $\mathbb{T}^2$ we have

$$
\int_{1/2}^1 \int_{\mathbb{T}^2} |\text{card}(P_N \cap (\tau C + t)) - \tau^2 N |C||^2 \ dt d\tau \geq c N^{1-\sigma}.
$$

Remark 12. We show that there exist planar convex bodies that satisfy the assumptions of Theorem 11 but do not satisfy the assumptions of Theorem 2. Let $0 < \alpha < 1$, let $\{q_r\}_{r=1}^{+\infty} = Q \cap [0,1]$, and let

$$
f(x) = \sum_{r=1}^{+\infty} \frac{1}{r^2} (x - q_r)^{\alpha + 1},
$$

with

$$
x_{+}^{\alpha + 1} = \begin{cases} 
 x^{\alpha + 1} & x \geq 0, \\
 0 & x < 0.
\end{cases}
$$

Since $x_{+}^{\alpha + 1} \in C^{1,\alpha}$ (the space of functions with Hölder continuous derivative of order $\alpha$) we immediately have $f \in C^{1,\alpha}([0,1])$. Let us show that $f \notin C^{1,\beta}([0,1])$ for any $\beta > \alpha$. Indeed, since $x^{\alpha + 1}$ is increasing, for every fixed $r_0 \in Q \cap [0,1]$ we have

$$
\sup_{h > 0} \frac{f'(q_{r_0} + h) - f'(q_{r_0})}{h^\beta} = \sup_{h > 0} \sum_{r=1}^{+\infty} \frac{1}{r^2} (\alpha + 1) \frac{(q_{r_0} + h - q_r)^{\alpha} - (q_{r_0} - q_r)^{\alpha}}{h^\beta} 
\geq \sup_{h > 0} \frac{1}{r_0^2} (\alpha + 1) \frac{h^\beta}{h^\beta} = +\infty.
$$
Since $f$ is convex (note that $x_+^{\alpha+1}$ is convex) we can construct a convex body $C$ such that $\partial C$ is $C^{1,\alpha}$ but not $C^{1,\beta}$ for any $\beta > \alpha$. Theorem 15 below yields our claim.

If $\partial C$ is $C^2$, then Remark 9 yields $|\gamma_{\Theta}(\delta)| \geq c\delta^{1/2}$. One may be tempted to say that the inequality $|\gamma_{\Theta}(\delta)| \geq c\delta^{1/2}$ implies a suitable regularity on $\partial C$. The following example shows that this is not always true. Consider a planar convex body $C$ with $C^1$ boundary $\partial C \ni (0, 0)$, where the inward unit normal is $\Theta_0 = (0, 1)$. Also assume that $\partial C$ coincides locally with the graph of the function 

$$\varphi(x) = \begin{cases} |x|^{3/2} & \text{if } -\varepsilon < x \leq 0, \\ x^2 & \text{if } 0 < x < \varepsilon. \end{cases}$$

Then $|\gamma_{\Theta_0}(\delta)| \approx \delta^{1/2}$, but $\partial C$ is not $C^2$ at the origin. Indeed, $|\gamma_{\Theta_0}(\delta)|$ is the sum of two contributions of different order, one coming from $|x|^{3/2}$ for $x < 0$ and the other coming from $x^2$ for $x > 0$. To obtain information on the regularity of $\partial C$ one has to consider these two contributions separately. This is the motivation of the following definition.

**Definition 13.** Let $C$ be a convex planar body where $\partial C$ is $C^1$. For every unit vector $\Theta$ the chord $\gamma_{\Theta}(\delta)$ is parallel to the tangent line at a point $P \in \partial C$ such that $P \cdot \Theta = \inf_{y \in C} (y \cdot \Theta)$.

Denoting by $n(P)$ the inward unit normal to $\partial C$ at $P$ we have $\Theta = n(P)$ and

$$\gamma_{\Theta}(\delta) = \left\{ x \in C : x \cdot n(P) = \inf_{y \in C} (y \cdot \Theta) + \delta \right\}.$$

The normal $n(P)$ splits the chord into two parts $\gamma_{n(P)}^+(\delta)$ and $\gamma_{n(P)}^-(\delta)$ of lengths $|\gamma_{n(P)}^+(\delta)|$ and $|\gamma_{n(P)}^-(\delta)|$ respectively (see Figure 2).

![Figure 2. Replacing $|\gamma_{n(P)}(\delta)|$ with $\min\left\{|\gamma_{n(P)}^-(\delta)|, |\gamma_{n(P)}^+(\delta)|\right\}$.](image)

**Remark 14.** Let $C$ be a convex planar body where $\partial C$ is $C^1$. Then the existence of a positive constant $c$ such that, for every $\Theta$,

$$\min\left\{|\gamma_{\Theta}^-(\delta)|, |\gamma_{\Theta}^+(\delta)|\right\} \geq c\delta^{1/2}$$

is readily seen to be equivalent the existence of a positive number $R > 0$ such that for every $P \in \partial C$ there exists a disk of radius $R$ contained in $C$ and tangent to $\partial C$ at $P$. This is a uniform version of the inner disk condition that is used in the study of maximum principles for partial differential equations (see e.g. [20 Chapter 6]).
Theorem 15. Let \( C \subset \mathbb{T}^2 \) be a convex body and let \( 0 < \alpha \leq 1 \). Then the following are equivalent.

a) There exist constants \( c > 0 \) and \( \delta_0 > 0 \) such that, for every direction \( \Theta \) and for \( 0 < \delta < \delta_0 \),

\[
\min \{ |\gamma_{\Theta}^{-}(\delta)|, |\gamma_{\Theta}^{+}(\delta)| \} \geq c\delta^{1/(1+\alpha)}.
\]

b) The arc length parameterization \( \Gamma(s) \) of \( \partial C \) is \( C^{1,\alpha} \), that is \( \Gamma(s) \in C^1 \) and there exists \( M > 0 \) such that for every \( s_1, s_2 \)

\[
|\Gamma'(s_1) - \Gamma'(s_2)| \leq M |s_1 - s_2|^\alpha.
\]

3. Geometric estimates of Fourier transforms

Let \( C \) be a planar convex body. The Fourier transform of its characteristic function is defined by

\[
\hat{\chi}_C(\xi) = \int_{\mathbb{R}^2} \chi_C(t) e^{-2\pi i \xi \cdot t} dt.
\]

For a given direction \( \Theta \) we are interested in the decay of \( \hat{\chi}_C(\rho \Theta) \) as \( \rho \to +\infty \).

Without loss of generality we can assume \( \Theta = (1, 0) \) so that

\[
\hat{\chi}_C(\rho, 0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi_C(t_1, t_2) e^{-2\pi i \rho t_1} dt_1 dt_2 = \int_{-\infty}^{+\infty} g(t_1) e^{-2\pi i \rho t_1} dt_1
\]

where

\[
g(t_1) = \int_{-\infty}^{+\infty} \chi_C(t_1, t_2) dt_2.
\]

Since \( C \) is convex then \( g(t_1) \) is supported and concave on a suitable interval \([A, B]\). A change of variables allows us to replace \( g \) with a function \( f \) which is positive, supported and concave on the interval \([-1, 1]\). The Fourier transform of \( f \) is defined as follows

\[
\hat{f}(s) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i s x} dx.
\]

Definition 16. Let \( f : \mathbb{R} \to \mathbb{R} \) supported, nonnegative and concave in the interval \([-1, 1]\). For every \( h \in \left(-\frac{1}{2}, \frac{1}{2}\right) \) define

\[
\mu_f(h) = \max \{ f(-1 + |h|), f(1 - |h|) \}.
\]

The following upper bound of \( \hat{f} \) is due to A. Podkorytov [32] (see also [12]):

\[
|\hat{f}(s)| \leq |s|^{-1} \mu_f \left( |s|^{-1} \right).
\]

This readily implies that

\[
|\hat{\chi}_C(\rho \Theta)| \leq \frac{c}{\rho} \left( |\gamma_{\Theta}^{-}(\rho^{-1})| + |\gamma_{-\Theta}(\rho^{-1})| \right).
\]

In this section we estimate (an average of) \( \hat{f}(s) \) from below. Our approach is based on second order differences.
3.1. Second order differences and moduli of smoothness. Following [24, Chapter 2] we define the modulus of smoothness of a function \( \phi \) as follows.

**Definition 17.** Let \( \phi \in L^2(\mathbb{R}) \) and, for every \( h \in \mathbb{R} \), let \( \Delta_h \) be the difference operator

\[
\Delta_h \phi (x) = \phi (x + h) - \phi (x)
\]

and let \( \Delta^2_h \) be the second order difference operator

\[
\Delta^2_h \phi (x) = \Delta_h \Delta_h \phi (x) = \phi (x + 2h) - 2\phi (x + h) + \phi (x).
\]

**Definition 18.** Let \( \phi \in L^2(\mathbb{R}) \) and let \( \nu \geq 0 \). The second \( L^2 \)-modulus of smoothness of \( \phi \) is given by

\[
\omega^2_2 (\phi, \nu) = \sup_{|h| \leq \nu} \left\{ \int_{-\infty}^{+\infty} |\Delta^2_h \phi (x)|^2 \, dx \right\}^{1/2}.
\]

Let \( f \) be supported, nonnegative and concave in the interval \([-1, 1]\). A relation between \( \Delta^2_h f (x) \) and \( \mu_f (h) \) is proved in the following proposition.

**Proposition 19.** Let \( f : \mathbb{R} \to \mathbb{R} \) supported, nonnegative and concave in the interval \([-1, 1]\). Then there exist constants \( c_1, c_2 > 0 \), independent of \( f \), such that for every \( h \in (-\frac{1}{2}, \frac{1}{2}) \),

\[
c_1 |h|^{1/2} \mu_f (h) \leq \left\{ \int_{-\infty}^{+\infty} \left| \Delta^2_h f (x) \right|^2 \, dx \right\}^{1/2} \leq c_2 |h|^{1/2} \mu_f (h).
\]

Moreover, for \( 0 < \nu < \frac{1}{2} \)

\[
c_1 \nu^{1/2} \mu_f (\nu) \leq \omega^2_2 (f, \nu) \leq c_2 \nu^{1/2} \mu_f (\nu).
\]

The proof needs the following lemma.

**Lemma 20.** Let \( f \) be supported, nonnegative and concave in \([-1, 1]\). Then, if \( 0 \leq \lambda_1 < \lambda_2 \leq 1 \) or \(-1 \leq \lambda_2 < \lambda_1 \leq 0\),

\[
f (\lambda_2) \leq 2f (\lambda_1),
\]

In particular for every \( x \in \mathbb{R} \)

\[
f (x) \leq 2f (0).
\]

Moreover, for \( 0 \leq \lambda_1 < \lambda_2 < 1 \)

\[
f (\lambda_1) \leq \frac{1 - \lambda_1}{1 - \lambda_2} f (\lambda_2)
\]

and for \(-1 < \lambda_2 < \lambda_1 \leq 0\)

\[
f (\lambda_1) \leq \frac{1 + \lambda_1}{1 + \lambda_2} f (\lambda_2)
\]

**Proof.** We can clearly assume \( 0 \leq \lambda_1 < \lambda_2 \leq 1 \). Since \( f \) is concave in \([-1, \lambda_2]\) we have

\[
f (-1) + \frac{f (\lambda_2) - f (-1)}{\lambda_2 + 1} (\lambda_1 + 1) \leq f (\lambda_1).
\]

This gives

\[
f (\lambda_2) (\lambda_1 + 1) \leq (\lambda_2 + 1) f (\lambda_1)
\]
and since $\frac{\lambda_1 + 1}{\lambda_1 - 1} \leq 2$ we obtain $f(\lambda_2) \leq 2f(\lambda_1)$. Similarly, since $f$ is concave in $[\lambda_1, 1]$, we obtain

$$f(1) + \frac{f(\lambda_1) - f(1)}{\lambda_1 - 1} (\lambda_2 - 1) \leq f(\lambda_2)$$

so that

$$f(\lambda_1) \frac{\lambda_2 - 1}{\lambda_1 - 1} \leq f(\lambda_2) - f(1) \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 - 1} \right) \leq f(\lambda_2).$$

Then we obtain (18).

\[\square\]

**Proof of Proposition 19.** First of all observe that it is enough to consider the case $h > 0$. Indeed, the case $h < 0$ follows applying (14) to the function $f(-x)$. Then we have

$$\int_{-\infty}^{+\infty} |\Delta_h^2 f(x)|^2 \, dx = \int_{-1-2h}^{-1} |\Delta_h f(x+h)|^2 \, dx$$

$$+ \int_{-1}^{-1-2h} |\Delta_h f(x+h) - \Delta_h f(x)|^2 \, dx + \int_{-1-2h}^{-1} |\Delta_h f(x)|^2 \, dx$$

$$= A(h) + B(h) + C(h).$$

For the term $A(h)$, using (16) we obtain

$$A(h) = \int_{-1-2h}^{-1} |f(x+2h) - 2f(x+h)|^2 \, dx$$

$$\leq 2 \int_{-1-2h}^{-1} [f(x+2h)]^2 \, dx + 8 \int_{-1-2h}^{-1} [f(x+h)]^2 \, dx$$

$$= 2 \int_{-1}^{-1+2h} [f(x)]^2 \, dx + 8 \int_{-1}^{-1+h} [f(x)]^2 \, dx.$$  \hspace{1cm} (19)

Observe that, by (16) and (19), for every $x \in [-1,-1+2h]$ we have

$$f(x) \leq 2f(-1+2h) \leq 4f(-1+h).$$

Then

$$A(h) \leq ch \cdot [f(-1+h)]^2.$$  \hspace{1cm} (20)

Similarly for $C(h)$ we have

$$C(h) \leq ch \cdot [f(1-h)]^2.$$  \hspace{1cm} (21)

Now let us consider $B(h)$. Since $f$ is concave in the interval $[-1,1]$, for any given $h > 0$ the function $\Delta_h f(x)$ is decreasing in $[-1,1-h]$. Let $\alpha \in [-1,1-h]$ satisfy $\Delta_h f(x) \geq 0$ for $x \in [-1,\alpha]$ and $\Delta_h f(x) \leq 0$ for $x \in [\alpha,1-h]$. Assume first that

$$-1+h \leq \alpha \leq 1-2h.$$  \hspace{1cm} (22)

Then

$$B(h) = \int_{-1}^{-1+h} |\Delta_h f(x+h) - \Delta_h f(x)|^2 \, dx + \int_{-1}^{\alpha} |\Delta_h f(x+h) - \Delta_h f(x)|^2 \, dx$$

$$+ \int_{\alpha}^{1-2h} |\Delta_h f(x+h) - \Delta_h f(x)|^2 \, dx$$

$$= B_1(h) + B_2(h) + B_3(h).$$
To estimate the term $\mathcal{B}_1 (h)$ we use the inequality
$$|x - y|^2 \leq |x^2 - y^2|$$
that holds for $xy \geq 0$. Thus
$$\mathcal{B}_1 (h) \leq \int_{-1}^{\alpha} \left( [\Delta_h f (x)]^2 - [\Delta_h f (x + h)]^2 \right) dx$$
$$= \int_{-1}^{\alpha} [\Delta_h f (x)]^2 dx - \int_{-1}^{\alpha} [\Delta_h f (x + h)]^2 dx$$
$$\leq \int_{-1}^{\alpha} [\Delta_h f (x)]^2 dx - \int_{-1}^{\alpha} [\Delta_h f (x)]^2 dx$$
$$\leq \int_{-1}^{\alpha} [\Delta_h f (x)]^2 dx = \int_{-1}^{\alpha} [f(x + h) - f(x)]^2 dx.$$Using (16) and (19) the latter can be bounded by
$$2 \int_{-1}^{\alpha} [f(x + h)]^2 dx + 2 \int_{-1}^{\alpha} [f(x)]^2 dx$$
$$\leq 8 \int_{-1}^{\alpha} [f(-1 + 2h)]^2 dx + 8 \int_{-1}^{\alpha} [f(-1 + h)]^2 dx$$
$$\leq 8 \left( [f(-1 + 2h)]^2 + [f(-1 + h)]^2 \right) \leq 40h [f(-1 + h)]^2.$$A similar estimate holds for $\mathcal{B}_3 (h)$. To estimate $\mathcal{B}_2 (h)$ observe $[\Delta_h f (x)]^2$ is decreasing for $-1 \leq x \leq \alpha$ and increasing for $\alpha \leq x \leq 1 - h$. Then, recalling (20), we have
$$\mathcal{B}_2 (h) \leq 2 \int_{\alpha - h}^{\alpha} [\Delta_h f (x + h)]^2 dx + 2 \int_{\alpha - h}^{\alpha} [\Delta_h f (x)]^2 dx$$
$$= 2 \int_{\alpha}^{\alpha + h} [\Delta_h f (x)]^2 dx + 2 \int_{\alpha - h}^{\alpha} [\Delta_h f (x)]^2 dx$$
$$\leq 2 \int_{\alpha - h}^{\alpha} [\Delta_h f(t + \alpha + h - 1)]^2 dt + 2 \int_{\alpha - h}^{\alpha + h} [\Delta_h f(t + \alpha - h + 1)]^2 dt.$$Observe that if $t \in [1 - h, 1]$ we have
$$\alpha \leq t + \alpha - h - 1 < t,$$so that
$$[\Delta_h f(t + \alpha + h - 1)]^2 \leq [\Delta_h f(t)]^2.$$This gives
$$\int_{1 - h}^{1} [\Delta_h f(t + \alpha + h - 1)]^2 dt \leq \int_{1 - h}^{1} [\Delta_h f(t)]^2 dt \leq ch [f(1 - h)]^2.$$Similarly for $t \in [-1, -1 + h]$ we have
$$t \leq t + \alpha - h + 1 \leq \alpha,$$so that
$$\int_{-1}^{-1 + h} [\Delta_h f(t + \alpha - h + 1)]^2 dt \leq \int_{-1}^{-1 + h} [\Delta_h f(t)]^2 dt \leq ch [f(1 + h)]^2.$$
Therefore
\[ B_2(h) \leq c h [f(1-h)]^2 + c h [f(1+h)]^2. \]

Finally observe that when \(-1 \leq \alpha < -1+h\) or \(1-2h < \alpha \leq 1-h\) equation (21) reduces to two terms that can be handled as in the previous case. The second
inequality in (13) is a consequence of the previous computations.

To prove the first inequality in (14) observe that
\[ \int_{-\infty}^{+\infty} |\Delta_h^2 f(x)|^2 \, dx \geq \int_{1-h}^{1} |\Delta_h^2 f(x)|^2 \, dx + \int_{1-2h}^{-1-h} |\Delta_h^2 f(x)|^2 \, dx \]
\[ = \int_{1-h}^{1} |f(x)|^2 \, dx + \int_{1-2h}^{-1-h} |f(x+2h)|^2 \, dx \]
\[ \geq c h \mu_f(h)^2. \]

This completes the proof of (14) and proves also the first inequality in (15). To prove the second inequality in (15) let us fix \(\nu\) and let \(0 \leq h \leq \nu\). Then by (16) we have
\[ \mu_f(h) \leq 2 \mu_f(\nu) \]
so that
\[ \left\{ \int_{-\infty}^{+\infty} |\Delta_h^2 f(x)|^2 \, dx \right\}^{1/2} \leq c_2 |h|^{1/2} \mu_f(h) \leq 2c_2 \nu^{1/2} \mu_f(\nu). \]
Therefore
\[ \omega_2(f, \nu) \leq c \delta^{1/2} \mu_f(\nu). \]

\[ \square \]

3.2. Decay of Fourier transforms. Moduli of smoothness turn out to be a link between Fourier transforms and the chord estimates as introduced in Definition 5. The following result is known (see [8], [27], [14]). We give a proof for completeness.

**Lemma 21.** There exists \(c > 0\) such that for every \(\phi \in L^2(\mathbb{R})\) and \(\rho \geq 1\) we have
\[ \left\{ \int_{\{\rho \leq |s|\}} |\hat{\phi}(s)|^2 \, ds \right\}^{1/2} \leq c \omega_2(\phi, \rho^{-1}) \]
and
\[ \left\{ \int_{\{|s| \geq \rho\}} |s|^4 |\hat{\phi}(s)|^2 \, ds \right\}^{1/2} \leq c \rho^2 \omega_2(\phi, \rho^{-1}). \]

**Proof.** Let \(\eta \in \mathcal{S}(\mathbb{R})\) satisfy \(\hat{\eta}(0) = 1\) and \(\hat{\eta}(s) = 0\) for \(|s| \geq 1\). Let
\[ V_\rho(x) = 2\rho \eta(\rho x) - \frac{\rho}{2} \eta \left( \frac{\rho x}{2} \right). \]

Since
\[ \hat{V}_\rho(s) = 2\hat{\eta}(\rho^{-1} s) - \hat{\eta}(2\rho^{-1} s) \]
it follows that \(\hat{V}_\rho(s) = 0\) if \(|s| \geq \rho\). Then Plancherel Theorem gives
\[ \int_{\{|s| \geq \rho\}} |\hat{\phi}(s)|^2 \, ds \leq \int_{\mathbb{R}} \left| \left(1 - \hat{V}_\rho(s)\right) \hat{\phi}(s) \right|^2 \, ds \]
\[ = \int_{\mathbb{R}} |\phi(x) - V_\rho * \phi(x)|^2 \, dx. \]
Since
\[ V_\rho \ast \phi (x) = \int_\mathbb{R} \phi (x - y) \left( 2\rho \eta (\rho y) - \frac{\rho}{2} \eta \left( \frac{\rho y}{2} \right) \right) dy = \int_\mathbb{R} 2\phi (x - y) \rho \eta (\rho y) dy - \int_\mathbb{R} \phi (x - y) \frac{\rho}{2} \eta \left( \frac{\rho y}{2} \right) dy = \int_\mathbb{R} [2\phi (x - \rho^{-1}z) - \phi (x - 2\rho^{-1}z)] \eta (z) dz \]
and \( \int_\mathbb{R} \eta (z) dz = \tilde{\eta} (0) = 1 \), we have
\[ \phi (x) - V_\rho \ast \phi (x) = \int_\mathbb{R} \phi (x) \eta (z) dz - \int_\mathbb{R} [2\phi (x - \rho^{-1}z) - \phi (x - 2\rho^{-1}z)] \eta (z) dz = \int_\mathbb{R} \left[ \phi (x) - 2\phi (x - \rho^{-1}z) + \phi (x - 2\rho^{-1}z) \right] \eta (z) dz = \int_\mathbb{R} \Delta^2_{\rho^{-1}} \phi (x) \eta (z) dz. \]

Since
\[ (23) \quad \omega_2 \left( \phi, \rho^{-1} |z| \right) \leq (1 + |z|^2) \omega_2 \left( \phi, \rho^{-1} \right), \]
see [24] Chapter 2, §7, then Minkowski integral inequality yields
\[ \left\{ \int_{|s| \geq \rho} \left| \hat{\phi} (s) \right|^2 ds \right\}^{1/2} \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \left| \Delta^2_{\rho^{-1}} \phi (x) \eta (z) \right|^2 dx \right\}^{1/2} \eta (z) dz \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \left| \Delta^2_{\rho^{-1}} \phi (x) \right|^2 dx \right\}^{1/2} |\eta (z)| dz \leq \int_{\mathbb{R}} \omega_2 \left( \phi, \rho^{-1} |z| \right) |\eta (z)| dz \leq \omega_2 \left( \phi, \rho^{-1} \right) \int_{\mathbb{R}} (1 + |z|^2) |\eta (z)| dz \leq c \omega_2 \left( \phi, \rho^{-1} \right). \]
To prove [22] let \( h = (4\pi \rho)^{-1} \). Then, for \(|s| \leq \rho\), we have \(|2\pi sh| \leq c |e^{2\pi is} - 1|\), so that
\[ \int_{|s| \leq \rho} \left| s \right|^4 \left| \hat{\phi} (s) \right|^2 ds = 2^4 \rho^4 \int_{|s| \leq \rho} \left| 2\pi sh \right|^4 \left| \hat{\phi} (s) \right|^2 ds \leq c \rho^4 \int_{|s| \leq \rho} \left| e^{2\pi is} - 1 \right|^4 \left| \hat{\phi} (s) \right|^2 ds = c \rho^4 \int_{\mathbb{R}} \left| (e^{2\pi is} - 1)^2 \hat{\phi} (s) \right|^2 ds = c \rho^4 \int_{\mathbb{R}} \left| \Delta_{\rho}^2 \hat{\phi} (s) \right|^2 ds \leq c \rho^4 \omega_2 \left( \phi, \rho^{-1} \right)^2. \]

**Lemma 22.** There exist four positive constants \( \alpha, \beta, c_1, c_2 \), such that, for every \( f : \mathbb{R} \to \mathbb{R} \) supported, nonnegative and concave in the interval \([-1,1]\) and every
\( \rho \geq 2\alpha^{-1} \), we have
\[
\frac{c_2}{\rho} \mu_f(\rho^{-1})^2 \leq \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds \leq \frac{c_2}{\rho} \mu_f(\rho^{-1})^2
\]
where \( \mu_f(\rho^{-1}) \) comes from Definition (14).

The upper bound follows from (12). The lower bound has been first proved by Podkorytov (33). We provide an alternative proof that depends on the previous lemma and may be of independent interest.

**Proof.** Let \( \alpha < \beta \), by Lemma 21 Proposition 19 and (23) we have
\[
\int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds \leq \int_{\alpha \rho \leq |s|} |\hat{f}(s)|^2 \, ds \leq c \left[ \omega_2(\phi, \alpha^{-1} \rho^{-1}) \right]^2
\]
\[
\leq c \left( 1 + \alpha^{-1} \right)^2 \left[ \omega_2(\phi, \alpha^{-1} \rho^{-1}) \right]^2 \leq c \rho^{-1} \mu_f(\rho^{-1})^2.
\]
Let \( |h| \leq \rho^{-1} \). Using Lemma 21 we obtain
\[
\int_{\mathbb{R}} |\Delta_h^2 f(x)|^2 \, dx = \int_{\mathbb{R}} \left| e^{2\pi i h s} - 1 \right|^4 |\hat{f}(s)|^2 \, ds
\]
\[
\leq c |h|^2 \int_{|s| \leq \alpha \rho} |s|^4 |\hat{f}(s)|^2 \, ds + c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds + c \int_{|s| > \beta \rho} |\hat{f}(s)|^2 \, ds
\]
\[
\leq c \alpha^4 \omega_2(f, \alpha^{-1} \rho^{-1})^2 + c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds + c \left[ \omega_2(f, \beta^{-1} \rho^{-1}) \right]^2.
\]
Hence, being \( |h| \leq \rho^{-1} \),
\[
\omega_2(f, \rho^{-1})^2 \leq c \alpha^4 \omega_2(f, \alpha^{-1} \rho^{-1})^2
\]
\[
+ c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds + c \left[ \omega_2(f, \beta^{-1} \rho^{-1}) \right]^2.
\]
Since \( \alpha^{-1} \rho^{-1} \leq 1/2 \) and \( \beta^{-1} \rho^{-1} \leq 1/2 \), by Proposition 19 we have
\[
\rho^{-1} \mu_f(\rho^{-1})^2 \leq c \omega_2(\rho^{-1})^2 + c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds + c \beta^{-1} \rho^{-1} \mu_f(\beta^{-1} \rho^{-1})^2
\]
\[
\leq c \alpha \rho^{-1} \mu_f(\alpha^{-1} \rho^{-1})^2 + c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds + c \beta^{-1} \rho^{-1} \mu_f(\beta^{-1} \rho^{-1})^2
\]
\[
\leq c \alpha \rho^{-1} \mu_f(\rho^{-1})^2 + c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds + c \beta^{-1} \rho^{-1} \mu_f(\rho^{-1})^2,
\]
because (15) yields
\[
\mu_f(\alpha^{-1} \rho^{-1}) \leq \alpha^{-1} \mu_f(\rho^{-1})
\]
and (16) gives
\[
\mu_f(\beta^{-1} \rho^{-1}) \leq 2 \mu_f(\rho^{-1}).
\]
It follows that
\[
(1 - c \alpha - c \beta^{-1}) \rho^{-1} \mu_f(\rho^{-1})^2 \leq c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds.
\]
Letting $\alpha$ sufficiently small and $\beta$ sufficiently large gives

$$\rho^{-1} \mu_f (\rho^{-1})^2 \leq c \int_{\alpha \rho \leq |s| \leq \beta \rho} |\hat{f}(s)|^2 \, ds$$

for every $\rho \geq 2\alpha^{-1}$.

**Remark 23.** Since in the above lemma $f$ is real we have $|\hat{f}(-s)| = |\hat{f}(s)|$ and therefore we also obtain

$$\rho^{-1} \mu_f (\rho^{-1})^2 \leq c \int_{\alpha \rho}^{\beta \rho} |\hat{f}(s)|^2 \, ds.$$

**Theorem 24.** Let $C \subset \mathbb{T}^2$ be a convex body, let $\sigma \in \left[\frac{1}{2}, 1\right]$ and let $\delta_0 > 0$. Let $I$ be an interval in $\mathbb{T}$ and let $\Theta = (\cos \theta, \sin \theta)$ with $\theta \in I$. Assume there exists a constant $c_1 > 0$ such that every $0 < \delta \leq \delta_0$ we have

$$|\gamma_\Theta (\delta)| + |\gamma_\Theta (\delta)| \geq c_1 \delta^\sigma.$$

Then there exist positive constants $c_2, c_3$, independent of $\theta \in I$, such that for every $\rho \geq 3$,

$$\left\{ \int_{1/2 \leq |\tau| \leq 1} |\tilde{\chi}_C (\tau \rho \Theta)|^2 \, d\tau \right\}^{1/2} \geq c_2 \rho^{-1-\sigma}.$$

**Proof.** Assume, without loss of generality, that $\Theta = (1, 0)$. Then

$$\tilde{\chi}_C (\rho, 0) = \tilde{g}(\rho)$$

with

$$g(t_1) = \int_{\mathbb{R}} \chi_C (t_1, t_2) \, dt_2.$$

Observe that $g$ is nonnegative, supported and concave in a suitable interval $[A, B]$. Let

$$f(x) = g\left(\frac{A + B}{2} + x \frac{B - A}{2}\right)$$

then

$$\hat{f}(s) = \frac{2}{B - A} e^{2\pi is \frac{A + B}{2A}} \tilde{g}\left(\frac{2s}{B - A}\right).$$

By Lemma 22 and Remark 22 we have

$$\frac{c_1}{\rho^2} \mu_f (\rho^{-1})^2 \leq \frac{1}{\rho} \int_{\alpha \rho}^{\beta \rho} |\hat{f}(s)|^2 \, ds,$$

so that

$$\frac{c}{\rho} \left( f(-1 + \rho^{-1}) + f(1 - \rho^{-1}) \right) \leq \left\{ \int_{\alpha \rho}^{\beta \rho} |\hat{f}(\rho t)|^2 \, dt \right\}^{1/2}.$$

Hence

$$\frac{1}{\rho} \left( g\left( \frac{A + B}{2\rho} \right) + g\left( B - \frac{A - B}{2\rho} \right) \right) \leq \frac{c}{B - A} \left\{ \int_{\alpha \rho}^{\beta \rho} \left| \tilde{g}\left( \frac{2\rho t}{B - A} \right) \right|^2 \, dt \right\}^{1/2}$$

$$= \frac{c}{\sqrt{B - A}} \left\{ \int_{\alpha \rho}^{2\beta \rho} \left| \tilde{g}(\rho \omega) \right|^2 \, d\omega \right\}^{1/2}.$$
Since $C$ is a convex body there exists a positive constant $\sigma$ (only depending on $C$) such that

$$\sigma \leq B - A \leq \text{diam}(C).$$

Then for suitable constants $c, \alpha'$ and $\beta'$ we have

$$\frac{1}{\rho} \left( g \left( A + \frac{B - A}{2\rho} \right) + g \left( B - \frac{B - A}{2\rho} \right) \right) \leq c \left\{ \int_{\alpha'}^{\beta'} |\hat{g}(\rho\omega)|^2 \, d\omega \right\}^{1/2}.$$ 

Since

$$g \left( B - \frac{B - A}{2\rho} \right) = \gamma_{\Theta} \left( \frac{B - A}{2\rho} \right)$$

and

$$g \left( A + \frac{B - A}{2\rho} \right) = \gamma_{-\Theta} \left( \frac{B - A}{2\rho} \right)$$

with $\Theta = (1,0)$, we have

$$\frac{c}{\rho} \left( \frac{B - A}{2\rho} \right)^{\sigma} \leq \left\{ \int_{\alpha'}^{\beta'} |\hat{g}(\rho\omega)|^2 \, d\omega \right\}^{1/2}.$$ 

Hence

$$\frac{c}{\rho^{1+\sigma}} \leq \left\{ \int_{\alpha'}^{\beta'} |\hat{g}(\rho\omega)|^2 \, d\omega \right\}^{1/2}.$$ 

A suitable change of variables gives

$$\frac{c}{\rho^{1+\sigma}} \leq \left\{ \int_{1/2}^{1} |\hat{g}(\rho\omega)|^2 \, d\omega \right\}^{1/2}$$

and then

$$\left\{ \int_{1/2}^{1} |\hat{\chi}_C(\rho\omega,0)|^2 \, d\omega \right\}^{1/2} \geq c\rho^{-1-\sigma}.$$ 

$$\blacksquare$$

4. PROOFS OF THE MAIN RESULTS

**Proof of Proposition 2** Let $P_0$ be a fixed interior point of $C$, let

$$d = \inf_{P \in \partial C} \text{dist}(P, P_0)$$

and let $D$ be the disk of radius $d/2$ centered at $P_0$. Clearly $D \subset C$. Let us fix a direction $\Theta$ and let $P \in \partial C$ be such that

$$(24) \quad P \cdot \Theta = \inf_{x \in C} x \cdot \Theta.$$ 

Then

$$\gamma_{\Theta}(\delta) = |\{x \in C : x \cdot \Theta = P \cdot \Theta + \delta\}|$$

Without loss of generality we can assume that $P$ is the origin and that $P_0 = (0,y_0)$ for some $y_0 \geq d$. Let $P_1 = (-d/2,y_0)$, $P_2 = (d/2,y_0)$ and let $T$ be the triangle with vertices $P, P_1, P_2$ (see Figure 3). Since $T \subset C$ we have

$$|\gamma_{\Theta}(\delta)| \geq |\{x \in T : x \cdot \Theta = \delta\}|.$$
From (24) we obtain

\[(25) \quad P_1 \cdot \Theta \geq 0 \quad \text{and} \quad P_2 \cdot \Theta \geq 0.\]

Let us write

\[P_1 = |P_1| (-\sin \gamma, \cos \gamma), \quad P_2 = |P_2| (\sin \gamma, \cos \gamma)\]

and let \(\Theta = (\cos \theta, \sin \theta)\). Using (25) we have \(\gamma \leq \theta \leq \pi - \gamma\). Observe that

\[
\tan \gamma = \frac{d}{2y_0} \geq \frac{d}{2 \sup_{P \in \partial C} d(P, P_0)}
\]

so that \(\gamma \geq \gamma_0\) with \(\gamma_0\) independent of \(\Theta\) and \(\delta\). Since \(T\) is symmetric about the vertical axis it suffices to consider the case \(\gamma \leq \theta \leq \pi - \gamma\). Let \(0 < \delta \leq \frac{d}{2} \sin (\gamma_0)\) (this ensures that the point \(Q_0\) is inside \(T\)). Then

\[
|\{x \in T : x \cdot \Theta = \delta\}| = |Q_1 - Q_2| \geq |Q_2 - Q_0|
\]

\[
= \delta \left[\tan \left(\frac{\pi}{2} - \theta\right) - \tan \left(\frac{\pi}{2} - \theta - \gamma\right)\right] \geq \delta \tan \gamma \geq \delta \tan \gamma_0.
\]

To prove Theorem 7 and Theorem 11 we first need a mild variant of a classical result of J.W.S. Cassels and H. Montgomery (see [30]). The following proof has been inspired by Siegel’s analytic proof of Minkowski’s convex body theorem (see [37]).

**Lemma 25.** Let \(U\) be a neighborhood of the origin. Then there exists a positive constant \(c\) such that for every convex symmetric body \(\Omega\) in \(\mathbb{R}^2\) and every finite set \(\{u(j)\}_{j=1}^N \subset \mathbb{T}^2\) we have

\[
\sum_{m \in (\Omega, U) \cap \mathbb{Z}^2} \left| \sum_{j=1}^N e^{2\pi i m \cdot u(j)} \right|^2 \geq N \text{area}(\Omega) / 4 - cN^2.
\]
Without loss of generality we can assume

\[
\int_{\mathbb{R}^2} \text{card}(\frac{1}{2} \Omega - x) \cap \mathbb{Z}^2) \, dx = \int_{\mathbb{R}^2} \sum_{k \in \mathbb{Z}^2} \chi_{\frac{1}{2} \Omega} (x + k) \, dx = \int_{\mathbb{R}^2} \chi_{\frac{1}{2} \Omega} (x) \, dx = \text{area}(\Omega)/4
\]

we can find \( x \in [-\frac{1}{2}, \frac{1}{2}]^2 \) such that \( \text{card}(\frac{1}{2} \Omega - x) \cap \mathbb{Z}^2) \geq \text{area}(\Omega)/4 \). Let

\[
T(x) = \frac{1}{\text{card}(\frac{1}{2} \Omega - x) \cap \mathbb{Z}^2)} \left| \sum_{m \in (\frac{1}{2} \Omega - x) \cap \mathbb{Z}^2} e^{2\pi i m \cdot x} \right|^2
\]

Clearly \( T \) is a non-negative trigonometric polynomial. Observe that \( \hat{T} \) is non-negative and that the support of \( \hat{T} \) is contained in \( \Omega \) since \( m, k \in (\frac{1}{2} \Omega - x) \cap \mathbb{Z}^2 \) yields \( m - k = \Omega \). Also observe that

\[
T(0) = \text{card}(\frac{1}{2} \Omega - x) \cap \mathbb{Z}^2) \geq \text{area}(\Omega)/4.
\]

Since

\[
0 \leq \hat{T}(m) \leq \hat{T}(0) = \int_{\mathbb{R}^2} T(x) \, dx = 1,
\]

it follows that

\[
\sum_{m \in \Omega \cap \mathbb{Z}^2} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 \geq \sum_{m \in \Omega \cap \mathbb{Z}^2} \hat{T}(m) \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2
\]

\[
= \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{m \in \Omega \cap \mathbb{Z}^2} \hat{T}(m)e^{2\pi i m \cdot (u(j) - u(k))} = \sum_{j=1}^{N} \sum_{k=1}^{N} T(u(j) - u(k)) \geq NT(0) \geq N \text{area}(\Omega)/4.
\]

Finally,

\[
\sum_{m \in (\Omega \setminus U) \cap \mathbb{Z}^2} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 \geq \sum_{m \in U \cap \mathbb{Z}^2} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 - \sum_{m \in U \cap \mathbb{Z}^2} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 \geq N \text{area}(\Omega)/4 - \sum_{m \in U \cap \mathbb{Z}^2} N^2 \geq N \text{area}(\Omega)/4 - cN^2.
\]
Theorem 24 yields, for $\rho$ large enough,

$$
\int_{1/2}^{1} |\hat{\chi}_C(\tau \rho (\cos \theta, \sin \theta))|^2 d\tau \geq \begin{cases} 
\frac{c_0 \rho^{-3}}{} & \text{if } -\alpha < \theta < \alpha \text{ or } -\alpha < \pi - \theta < \alpha, \\
\frac{c_0 \rho^{-2 - 2\sigma}}{} & \text{otherwise.}
\end{cases}
$$

(26)

Let $N$ be a large positive integer. For a positive constant $\kappa$ to be chosen later we consider the following geometric construction. Let $R_0$ be the rectangle having vertices $(\pm X/2, \pm Y/2)$ satisfying $XY = \kappa N$, $X \gg Y$.

Let

$$
\psi = Y/X, \quad M = \left[ \frac{\alpha}{\psi} \right] = \left[ \frac{X}{\alpha Y} \right],
$$

(here $[x]$ denotes the integer part of $x$). For every $-M \leq j \leq M$ we consider the rotated rectangles $R_j := r_j \psi R_0$, where $r_\beta$ is the rotation by angle $\beta$ about the origin. See Figure 4. For every $m = (m_1, m_2) \in \mathbb{Z}^2$ let

$$
\Phi (m) = \sum_{j=-M}^{M} \chi_{R_j}(m)
$$

We want to find a constant $\Gamma = \Gamma (X)$ such that, for every $m \in \mathbb{Z}^2 \setminus \{0\}$,

$$
(27) \quad \Gamma \Phi (m) \leq \begin{cases} 
\frac{c_0 |m|^{-3}}{} & \text{if } -\alpha < \arg (\pm m) < \alpha, \\
\frac{c_0 |m|^{-2 - 2\sigma}}{} & \text{otherwise.}
\end{cases}
$$

(26)

(here $c_0$ is the same constant that appears in (26)). First we assume $m \in \bigcup R_j$ and $|m| \geq Y$. In this case we have

$$
\Phi (m) = \sum_{j=-M}^{M} \chi_{R_j}(m) = \text{card} \{ j \in \mathbb{Z} : -M \leq j \leq M , \ m \in r_j \psi R_0 \}
$$

$$
= \text{card} \{ j \in \mathbb{Z} : -M \leq j \leq M , \ r_{-j} \psi m \in R_0 \} \leq \frac{X}{|m|}.
$$

Figure 4. A nonnegative linear combination of characteristic functions of rotated rectangles gives a function which is smaller than (the average of) $\hat{\chi}_C$. 
Indeed, the points \( r_j - j \psi m \) belong to a circle \( K_m \) of radius \( |m| \), the length of \( K_m \cap R_0 \) is \( \approx Y \) and these points are spaced by \( \psi |m| = \frac{1}{X} |m| \). Then the inequality

\[
\Gamma \Phi (m) \leq c_0 |m|^{-3}
\]

is a consequence of

\[
\Gamma \frac{X}{|m|} \leq c |m|^{-3},
\]

that is

\[
(28) \quad \Gamma \leq \inf_{m \in \cup_j K_j, |m| \geq Y} \frac{c}{X |m|^{\sigma}} \leq \frac{c}{X^3}.
\]

We now assume \( 0 < |m| \leq Y \), regardless of \( \arg(\pm m) \). Then

\[
\Phi (m) \leq 2M + 1 = 2 \left[ \frac{\alpha X}{Y} \right] + 1 \leq c \frac{\alpha X}{Y}.
\]

Since we want

\[
\Gamma \Phi (m) \leq c |m|^{-2-2\sigma},
\]

it suffices

\[
\Gamma \frac{\alpha X}{Y} \leq c |m|^{-2-2\sigma}.
\]

Therefore we need

\[
(29) \quad \Gamma \leq \inf_{0 < |m| \leq Y} \frac{cY}{\alpha X |m|^{2+2\sigma}} = \frac{c}{\alpha X Y^{1+2\sigma}}.
\]

Then by (28) and (29) we set

\[
\Gamma = c \min \left( \frac{1}{X^3}, \frac{1}{\alpha X Y^{1+2\sigma}} \right).
\]

We choose \( \frac{1}{X^3} = \frac{1}{\alpha X Y^{1+2\sigma}} \) and since \( XY = \kappa N \) we obtain

\[
X = c (\alpha, \kappa) N^{\frac{2\sigma+1}{2\sigma+3}},
\]

\[
Y = c (\alpha, \kappa) N^{\frac{2}{2\sigma+3}}.
\]

Then

\[
\Gamma = c (\alpha, \kappa) N^{-3(2\sigma+1)/(2\sigma+3)}
\]

yields (27). Our construction guarantees that for \( |m| \geq c_1 \), we have

\[
\int_{1/2}^{1} |\hat{\chi}_C (m)|^2 d\tau \geq \Gamma \Phi (m) = \Gamma \sum_{j=-M}^{M} \chi_{R_j} (m).
\]

We recall that the periodic function

\[
t \mapsto \text{card} (P_N \cap (\tau C + t)) - \tau^2 N |C|
\]

has Fourier series

\[
\sum_{m \neq 0} \left( \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right) \hat{\chi}_C (m) e^{2\pi i m \cdot t}
\]

(see [9, p. 205]). Then, by Parseval theorem and Lemma 25, we obtain

\[
\int_{1/2}^{1/2} \int_{T^2} \left| \text{card} (P_N \cap (\tau C + t)) - \tau^2 N |C| \right|^2 dtd\tau
\]
Choosing $\kappa$ large enough gives

$$\int_{1/2}^{1} \int_{\mathbb{T}^{2}} \left| \text{card} \left( \mathcal{P} \cap (\tau C + t) \right) \right|^{2} d\tau d\tau$$

$$\geq c(\alpha, \kappa) N^{-3(2\sigma + 1)/(2\sigma + 3)} N^{\frac{2\sigma-1}{2\sigma+3}} N^{2}$$

$$= c(\alpha, \kappa) N^\frac{2\sigma-1}{2\sigma+3}.$$

\[\square\]

We now begin the proof of Theorem 8. Let $\frac{1}{2} \leq \sigma < 1$ (the case $\sigma = 1$ will be addressed later) and let $C_\sigma$ be a convex planar body, symmetric about the axes, such that in a neighborhood of the point $(0, -1)$ the boundary $\partial C_\sigma$ coincides with the graph of the function $y = |x|^{1/\sigma} - 1$ (say for $-\varepsilon \leq x \leq \varepsilon$). We also assume that $\partial C_\sigma$ is $C^2$ and has positive curvature except at the points $(0, -1)$ and $(0, 1)$.

In the next proposition we estimate the lengths of the chords of $C_\sigma$. Using the symmetry of $C_\sigma$ we can restrict the directions of the inward unit normals $\Theta = (\cos \theta, \sin \theta)$ to the interval $\pi/2 \leq \theta \leq \pi$.

**Proposition 26.** Let $C_\sigma$ as above. Let $\Theta = (\cos \theta, \sin \theta)$ with $\pi/2 \leq \theta \leq \pi$. Then there exists $\delta_0, c > 0$ such that, for $0 < \delta \leq \delta_0$, the chords of $C_\sigma$ satisfy

$$|\gamma_\Theta(\delta)| \approx \begin{cases} 
\delta^\sigma & \text{for } 0 \leq \theta - \frac{\pi}{2} < c \delta^{1-\sigma}, \\
\delta^{1/2} \left( \theta - \frac{\pi}{2} \right)^{\frac{2\sigma-1}{2\sigma+3}} & \text{for } c \delta^{1-\sigma} < \theta - \frac{\pi}{2}.
\end{cases}$$

Observe that $\tan(\theta - \frac{\pi}{2})$ is the slope of the tangent line at the point of $\partial C_\sigma$ where the inward unit normal is $\Theta$.

The notation $A(x) \approx B(x)$ means that there exist constants $c_1$ and $c_2$ depending on $\sigma$, such that

$$c_1 A(x) \leq B(x) \leq c_2 A(x).$$

**Proof.** Let us fix $c_2 > 0$ small. Let $\delta_0$ be small enough and $\theta - \frac{\pi}{2} > c_2$. Then to the chord $\gamma_\Theta(\delta)$ is associated a small arc in $\partial C_\sigma$ that does not contain the origin. Since the curvature in this arc is away from 0 we have $|\gamma_\Theta(\delta)| \approx \delta^{1/2}$.

Let now $\theta - \frac{\pi}{2} \leq c_2$. If we assume $c_2$ and $\delta_0$ small enough, then the arc associated to the chord $\gamma_\Theta(\delta)$ is in the part of $\partial C_\sigma$ that coincides with the graph of $y = |x|^{1/\sigma} - 1$. 

$$\sum_{m \neq 0} \left| \sum_{j=1}^{N} e^{2\pi i m u(j)} \right|^2 \int_{1/2}^{1} |\chi_C(m)|^2 d\tau$$

$$\geq \sum_{|m| \geq c_1} \sum_{j=1}^{N} e^{2\pi i m u(j)} \int_{1/2}^{1} |\chi_C(m)|^2 d\tau$$

$$\geq \Gamma \sum_{j=-M}^{M} \sum_{|m| \geq c_1} \sum_{n \in R_j} e^{2\pi i m u(j)} \geq \Gamma \sum_{j=-M}^{M} \left( (N \text{area}(R_j)/4 - cN^2) \right)$$

$$\geq \Gamma (2M + 1) \left( \kappa N^2 - cN^2 \right).$$

Let us fix $\gamma$ to the chord $\Theta = (\cos \theta, \sin \theta),$ such that in a neighborhood of the point $(0, \gamma)$ we can restrict the directions of the inward unit normals $\sigma$, such that $\sigma$ is associated a small arc in $\partial C_\sigma$ is $C^2$ and has positive curvature except at the points $(0, -1)$ and $(0, 1)$. 

\[\square\]
Let \( (x_0, x_0^{1/\sigma} - 1) \) the point of \( \partial C_\sigma \), where the tangent has slope \( \tan(\theta - \frac{\pi}{2}) \). Then \( \gamma_\theta (\delta) \) coincides with the intersection of \( C_\sigma \) with the line
\[
y = x_0^{1/\sigma} - 1 + \frac{1}{\sigma}x_0^{-1/\sigma} (x - x_0) + \frac{\delta}{\sin(\theta)}.
\]

In the following lemmas we will estimate \( |\gamma_\theta (\delta)| \) by showing that the two solutions \( x_1, x_2 \) of the equation
\[
|x|^{1/\sigma} - x_0^{1/\sigma} - \frac{1}{\sigma}x_0^{-1/\sigma} (x - x_0) = \frac{\delta}{\sin(\theta)}
\]
satisfy
\[
x_2 - x_1 \approx \begin{cases} \frac{\delta^{1/2}}{\sigma^{\frac{2+1}{2}}} & \text{for } 0 < \delta < c x_0^{1/\sigma}, \\ \delta^{\sigma} & \text{for } c x_0^{1/\sigma} < \delta \end{cases}
\]
(observe that \( \sin(\theta) \approx 1 \) so that we can replace \( \frac{\delta}{\sin(\theta)} \) with \( \delta \)). Since \( \theta - \frac{\pi}{2} = \frac{1}{\sigma}x_0^{-1/\sigma} \), this implies \( [33] \).

**Lemma 27.** Let \( 1/2 < \sigma < 1 \) and for \( x \geq -1 \) let \( g(x) = (1 + x)^{1/\sigma} - 1 - \frac{1}{\sigma}x \). Then
\[
g(x) \approx \begin{cases} x^2 & \text{for } |x| \leq 1, \\ x^{1/\sigma} & \text{for } x > 1. \end{cases}
\]

**Proof.** Using the integral form reminder in Taylor’s formula we can write
\[
g(x) = (1 + x)^{1/\sigma} - 1 - \frac{1}{\sigma}x = \frac{1}{\sigma} \left( x - 1 \right) \int_0^x (1 + t)^{-2+1/\sigma} (x - t) \, dt.
\]

Let \( 0 \leq x \leq 1 \), then
\[
\int_0^x (1 + t)^{-2+1/\sigma} (x - t) \, dt \approx \int_0^x (x - t) \, dt = \frac{1}{2}x^2.
\]
Similarly, if \( -\frac{1}{2} \leq x \leq 0 \), we have
\[
\int_0^x (1 + t)^{-2+1/\sigma} (x - t) \, dt \approx \int_x^0 (t - x) \, dt = \frac{1}{2}x^2.
\]
The estimate
\[
g(x) \approx x^2
\]
for \( -1 \leq x < -\frac{1}{2} \) is trivial since \( g(x) \) is positive and bounded away from 0 in this interval. Let now \( x > 1 \). Since \( 1 + \frac{1}{\sigma} > 0 \) we have
\[
\int_{x/2}^{x/2} (1 + t)^{-2+1/\sigma} (x - t) \, dt \approx x \int_{x/2}^{x/2} (1 + t)^{-2+1/\sigma} \, dt = x \left[ \frac{(1 + t)^{-1+1/\sigma}}{-1 + 1/\sigma} \right]_{x/2}^{x/2}
\]
\[
= \frac{x}{-1 + 1/\sigma} \left[ \left( 1 + \frac{x}{2} \right)^{-1+1/\sigma} - 1 \right] \approx x^{1/\sigma}
\]
and
\[
\int_{x/2}^{x/2} (1 + t)^{-2+1/\sigma} (x - t) \, dt \approx x^{-2+1/\sigma} \int_{x/2}^{x/2} (x - t) \, dt = \frac{1}{8} x^2 x^{-2+1/\sigma} \approx x^{1/\sigma}
\]
we obtain
\[
\int_0^x (1 + t)^{-2+1/\sigma} (x - t) \, dx \approx x^{1/\sigma}.
\]
Lemma 28. Let $1/2 < \sigma < 1$ and for $x \geq -1$ let $g(x) = (1 + x)^{1/\sigma} - 1 - \frac{1}{\sigma}x$. For every $y > 0$ the equation $g(x) = y$ has at most two solutions. One of them can be negative. If $\overline{x}$ is a solution then $|\overline{x}| \approx \begin{cases} y^{1/2} & \text{for } 0 \leq y \leq 1, \\ y^\sigma & \text{for } y > 1. \end{cases}$

Proof. Clearly it is enough to show that $|\overline{x}| \approx y^{1/2}$ for $|y|$ small and $|\overline{x}| \approx y^\sigma$ for $y$ large. By the previous lemma there exist constants $c_1, c_2 > 0$ such that
\begin{equation}
(31) \quad c_1 x^2 \leq g(x) \leq c_2 x^2 \text{ for } |x| \leq 1
\end{equation}
and
\begin{equation}
(32) \quad c_1 x^{1/\sigma} \leq g(x) \leq c_2 x^{1/\sigma} \text{ for } x > 1.
\end{equation}
Let $y < c_1$. Since $g(\overline{x}) = y$ by (32) we cannot have $|\overline{x}| \geq 1$. Hence by (31) we have $c_1 \overline{x}^2 \leq y \leq c_2 \overline{x}^2$ and therefore $y \approx \overline{x}^{1/2}$. Let now $y > c_2$. By (31) we cannot have $|\overline{x}| < 1$. Hence $c_1 \overline{x}^{1/\sigma} \leq y \leq c_2 \overline{x}^{1/\sigma}$ and therefore $y \approx \overline{x}^\sigma$. 

Lemma 29. Let $1/2 < \sigma < 1$, let $x_0 > 0$ and, for every $x \in \mathbb{R}$, let $f(x) = |x|^{1/\sigma} - \frac{1}{\sigma}x_0^{-1+1/\sigma}(x - x_0) - x_0^{1/\sigma}$.

Let $y > 0$ and observe that the equation $f(x) = y$ has one solution $x_2 > x_0$ and one solution $x_1 < x_0$. Then
\begin{equation}
(33) \quad x_2 - x_1 \approx \begin{cases} y^{1/2} x_0^{2-1/\sigma} & \text{for } 0 < y < x_0^{1/\sigma}, \\ y^\sigma & \text{for } x_0^{1/\sigma} < y. \end{cases}
\end{equation}

Proof. Let $\overline{x} > 0$ such that $f(\overline{x}) = y$ and let $g(x) = (1 + x)^{1/\sigma} - 1 - \frac{1}{\sigma}x$. Since for $x > 0$
\begin{align*}
f(x) &= x^{1/\sigma} - \frac{1}{\sigma}x_0^{-1+1/\sigma}(x - x_0) - x_0^{1/\sigma} \\
&= x_0^{1/\sigma} \left[ \left(1 + \frac{x - x_0}{x_0}\right)^{1/\sigma} - 1 - \frac{1}{\sigma} \left(\frac{x - x_0}{x_0}\right) \right] \\
&= x_0^{1/\sigma} g \left(\frac{x - x_0}{x_0}\right)
\end{align*}
we have $g\left(\frac{\overline{x} - x_0}{x_0}\right) = y x_0^{-1/\sigma}$. 

□
By Lemma 28 we have
\[
\left| \chi - x_0 \right| \approx \begin{cases} y^{1/2}x_0^{-1/(2\sigma)} & \text{for } 0 < y < x_0^{1/\sigma}, \\ y^\sigma x_0^{-1} & \text{for } y > x_0^{1/\sigma}. \end{cases}
\]

If both \( x_1 \) and \( x_2 \) are positive, from the above estimate we easily obtain
\[
x_2 - x_1 \approx \begin{cases} y^{1/2}x_0^{-1/(2\sigma)} & \text{for } 0 < y < x_0^{1/\sigma}, \\ y^\sigma & \text{for } y > x_0^{1/\sigma}. \end{cases}
\]
Assume now \( x_1 \leq 0 \) and \( x_2 > 0 \). Then \( f(0) \leq y = f(x_2) \). This implies
\[
x_0 < \sigma^{\sigma/(1-\sigma)} x_2
\]
with \( \sigma^{\sigma/(1-\sigma)} < 1 \). Then \( x_2 - x_0 \approx x_2 \). Let us show that \( |x_1| < x_2 \). This means that
\[
f(-x_2) > f(x_2) = y.
\]
Indeed,
\[
-|x_2|^{1/\sigma} - \frac{1}{\sigma} x_0^{-1+1/\sigma} (-x_2 - x_0) - x_0^{1/\sigma} > |x_2|^{1/\sigma} - \frac{1}{\sigma} x_0^{-1+1/\sigma} (x_2 - x_0) - x_0^{1/\sigma} = y.
\]
This gives (33) also in this case. \( \square \)

The proof of Proposition 26 is now complete.

**Proposition 30.** Let \( \frac{1}{2} \leq \sigma < 1 \). Then there exists \( c > 0 \) such that for every \( \theta \in [0, \pi] \) we have
\[
\left| \bar{\chi}_{C_\sigma} (\pm \rho \Theta) \right| \leq c \begin{cases} \rho^{-1-\sigma} & \text{for } \left| \theta - \frac{\pi}{2} \right| \leq \rho^{-1+\sigma}, \\ \rho^{-3/2} \left| \rho - \frac{\pi}{2} \right|^{2(1-\sigma)} & \text{for } \rho^{-1+\sigma} \leq \left| \theta - \frac{\pi}{2} \right| \leq \frac{\pi}{2}. \end{cases}
\]

**Proof.** It is a consequence of Proposition 26 (13) and the symmetries of \( C_\sigma \). \( \square \)

We now consider the case \( \sigma = 1 \) which requires a slightly different construction.
Let \( C_1 \) be a convex planar body, symmetric about the axes, such that in a neighborhood of the point \( (0, -1) \) the boundary \( \partial C_1 \) coincides with the graph of the function \( y = \frac{1}{2} x^2 + \frac{1}{4} |x| - 1 \). We also assume that \( \partial C_1 \) has positive curvature away from the points \( (0, \pm 1) \). We have the following result.

**Lemma 31.** The above convex body \( C_1 \) satisfies the following estimates. There exists \( c > 0 \) such that for every \( \theta \in [0, \pi] \) we have
\[
\left| \bar{\chi}_{C_1} (\pm \rho \Theta) \right| \leq c \begin{cases} \rho^{-2} & \text{for } \left| \theta - \frac{\pi}{2} \right| \leq c_2, \\ \rho^{-3/2} & \text{for } c_2 \leq \left| \theta - \frac{\pi}{2} \right|. \end{cases}
\]

**Proof.** In view of (13) and the symmetries of \( C_1 \) it is enough to estimate \( |\gamma_\Theta (\delta)| \) for \( \pi/2 \leq \theta \leq \pi \). Observe that for \( \frac{1}{4} \leq \tan \left( \theta - \frac{\pi}{2} \right) \leq 1 \) there exists a point \( P = (x_0, y_0) \in \partial C_1 \), with \( x_0 > 0 \) such that \( n(P) = \Theta \). If \( \tan \left( \theta - \frac{\pi}{2} \right) = \frac{1}{4} \) we do not have a unique normal at \( (0, -1) \) and in this case \( \Theta \) is the limit of the inward normal as \( x_0 \to 0^+ \). Recall that
\[
\gamma_\Theta (\delta) = \left\{ x \in C_1 : x \cdot n(P) = \inf_{y \in C} (y \cdot n(P)) + \delta \right\}
\]
(see Definition 13). Since the curvature at $P$ is positive we have $|\gamma_\Theta (\delta)| \approx |\gamma_\Theta^+ (\delta)| \approx \delta^{1/2}$. Let now $0 \leq \tan \left( \theta - \frac{\pi}{2} \right) < \frac{1}{2}$. A computation shows that the chord $\gamma_\Theta (\delta)$ satisfies

$$|\gamma_\Theta (\delta)| \approx \frac{\delta}{\sqrt{\left[ \frac{1}{4} - \tan \left( \theta - \frac{\pi}{2} \right) \right]^2 + 3\delta}}$$

$$\approx \begin{cases} 
\delta^{1/2} & \text{for } 0 \leq \frac{1}{4} - \tan \left( \theta - \frac{\pi}{2} \right) < \delta^{1/2}, \\
\frac{\delta}{\tan \left( \theta - \frac{\pi}{2} \right)} & \text{for } \frac{1}{4} - \tan \left( \theta - \frac{\pi}{2} \right) \geq \delta^{1/2}.
\end{cases}$$

In particular, we have the estimate $|\gamma_\Theta (\delta)| \leq c \delta^{1/2}$ for every $\theta$. It follows that for a suitable $c_2 < \frac{1}{4}$ we have

$$|\gamma_\Theta (\delta)| \leq c_1 \begin{cases} 
\delta^{1/2} & \text{for } |\tan \left( \theta - \frac{\pi}{2} \right)| > c_2, \\
\delta & \text{for } |\tan \left( \theta - \frac{\pi}{2} \right)| \leq c_2.
\end{cases}$$

\[\square\]

**Proof of Theorem 5.** Assume first $\frac{1}{2} \leq \alpha < 1$ and let $C_\alpha$ be as above. For every positive integer $j$ let

$$N_j = \left[ j^{\frac{2\alpha+1}{2\alpha}} \right] \left[ j^{\frac{2\alpha}{2\alpha+1}} \right].$$

Here $[x]$ denotes the integer part of $x$. Also let $K = \left[ j^{\frac{2\alpha}{2\alpha+1}} \right]$, $L = \left[ j^{\frac{2\alpha+1}{2\alpha}} \right]$, 

$$u_{k,\ell} = \left( \frac{k}{K}, \frac{\ell}{L} \right),$$

and

$$P_{N_j} = \{ \{ u_{k,\ell} \} \}_{k=0, \ldots, K-1, \ell=0, \ldots, L-1}.$$

Then

$$\int_{\mathbb{T}^2} \left| \text{card} \left( P_{N_j} \cap (C_\alpha + t) \right) - N_j |C_\alpha| \right|^2 \, dt = \sum_{m \neq 0} \left| \hat{\chi}_{C_\alpha} (m) \right|^2 \sum_{k=0}^{K-1} \sum_{\ell=0}^{L-1} e^{2\pi im \cdot u_{k,\ell}}.$$

Observe that

$$\sum_{k=0}^{K-1} \sum_{\ell=0}^{L-1} e^{2\pi im \cdot u_{k,\ell}} = \begin{cases} 
KL & \text{if } m_1 = Kn_1 \text{ and } m_2 = Ln_2 \text{ with } n_1, n_2 \in \mathbb{Z}, \\
0 & \text{otherwise}.
\end{cases}$$

Therefore

$$\sum_{m \neq 0} \left| \hat{\chi}_{C_\alpha} (m) \right|^2 \left| \sum_{k=0}^{K-1} \sum_{\ell=0}^{L-1} e^{2\pi im \cdot u_{k,\ell}} \right|^2 = \sum_{(n_1, n_2) \neq (0, 0)} \left| \hat{\chi}_{C_\alpha} (Kn_1, Ln_2) \right|^2 K^2 L^2$$

$$= K^2 L^2 \sum_{(n_1, n_2) \in \Gamma_1} \left| \hat{\chi}_{C_\alpha} (Kn_1, Ln_2) \right|^2 + K^2 L^2 \sum_{(n_1, n_2) \in \Gamma_2} \left| \hat{\chi}_{C_\alpha} (Kn_1, Ln_2) \right|^2$$

$$+ K^2 L^2 \sum_{(n_1, n_2) \in \Gamma_3} \left| \hat{\chi}_{C_\alpha} (Kn_1, Ln_2) \right|^2$$

where

$$\Gamma_1 = \{ (n_1, n_2) \in \mathbb{Z}^2 \setminus \{ (0, 0) \} : |Kn_1| \geq c_2 |Ln_2| \}$$
\[ \Gamma_2 = \{ (n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : c_1 |Ln_2|^{\sigma} < |Kn_1| < c_2 |Ln_2| \} \]
\[ \Gamma_3 = \{ (n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : |Kn_1| \leq c_1 |Ln_2|^{\sigma} \} . \]
Let \((n_1, n_2) \in \Gamma_1\), then (34) yields
\[ |\hat{\chi}_{C_n} (Kn_1, Ln_2)|^2 \leq c |Kn_1|^{-3} . \]
It follows that
\[ K^2L^2 \sum_{(n_1, n_2) \in \Gamma_1} |\hat{\chi}_{C_n} (Kn_1, Ln_2)|^2 \leq cK^2L^2 \sum_{n_1=1}^{+\infty} \sum_{n_2 \leq cL} |Kn_1|^{-3} \]
\[ = cK^{-1}L^2 \sum_{n_1=1}^{+\infty} |n_1|^{-3} \left( \frac{K |n_1|}{L} + 1 \right) \leq cL + cL^2K^{-1} \leq cj^{\frac{3-\sigma}{2\sigma}} . \]
Let \((n_1, n_2) \in \Gamma_2\). Then (34) yields
\[ |\hat{\chi}_{C_n} (Kn_1, Ln_2)|^2 \leq c |Ln_2|^{-3} \left| \frac{Kn_1}{Ln_2} \right|^{\frac{3-\sigma}{2\sigma}} = c |Ln_2|^{-\frac{3-\sigma}{2\sigma}} |Kn_1|^{\frac{3-\sigma}{1-\sigma}} , \]
because \(\tan(\theta - \frac{\pi}{2}) = (Kn_1) / (Ln_2)\). Since \((n_1, n_2) \neq 0\) we also have \(c_2 \frac{|Ln_2|}{K} \geq 1\) and therefore \(|n_2| \geq c_2 K\). Hence
\[ K^2L^2 \sum_{(n_1, n_2) \in \Gamma_2} |\hat{\chi}_{C_n} (Kn_1, Ln_2)|^2 \]
\[ \leq K^2L^2 \sum_{n_2 \geq cK} \sum_{n_1 \leq c\frac{|Ln_2|}{L}} c \left| \frac{Ln_2}{n_1} \right|^{-\frac{3-\sigma}{2\sigma}} |Kn_1|^{\frac{3-\sigma}{1-\sigma}} \]
\[ \leq cL^{-\frac{3\sigma}{2\sigma}} K^{\frac{1-\sigma}{\sigma}} \sum_{n_2 \geq cK} |n_2|^{-\frac{3-\sigma}{2\sigma}} \sum_{n_1 \leq c\frac{|Ln_2|}{L}} |n_1|^{\frac{3-\sigma}{1-\sigma}} \]
\[ \leq cL^{-\frac{3\sigma}{2\sigma}} K^{\frac{1-\sigma}{\sigma}} \sum_{n_2 \geq cK} |n_2|^{-\frac{3\sigma}{2\sigma}} \left( \frac{|Ln_2|}{K} \right)^{\frac{3-\sigma}{1-\sigma}} = cK \sum_{n_2 \geq cK} c |n_2|^{-2} \]
\[ \leq cL \leq cj^{\frac{3-\sigma}{2\sigma}} . \]
Let \((n_1, n_2) \in \Gamma_3\). Then we have
\[ K |n_1| \leq cL |n_2| , \]
so that (34) yields
\[ |\hat{\chi}_{C_n} (Kn_1, Ln_2)|^2 \leq c |Ln_2|^{-2-2\sigma} . \]
Hence
\[ K^2L^2 \sum_{(n_1, n_2) \in \Gamma_3} |\hat{\chi}_{C_n} (Kn_1, Ln_2)|^2 \leq cK^2L^2 \sum_{n_2=1}^{+\infty} \sum_{n_1 \leq c\frac{|Ln_2|}{K}} |Ln_2|^{-2-2\sigma} \]
\[ \leq cK^2L^{-2\sigma} \sum_{n_2=1}^{+\infty} |n_2|^{-2-2\sigma} \left( \frac{|Ln_2|^{\sigma}}{K} + 1 \right) \]
\[ \leq cK^2L^{-2\sigma} \left( \frac{L^{\sigma}}{K} + 1 \right) \leq cKL^{-\sigma} + cK^2L^{-2\sigma} \leq cj^{\frac{3-\sigma}{2\sigma}} . \]
Then
\[ \int_{\mathbb{T}^2} \left| \text{card} \left( \mathcal{P}_{N_j} \cap (C_{\sigma} + t) \right) - N_j \right| C_{\sigma} \right|^2 \, dt \leq c j^{2/5} \leq c N_j^{2/5}. \]

We still have to prove the case \( \sigma = 1 \). Let \( C_1 \) be as in Lemma 23 and for every integer \( j > 0 \) let \( K = \left[ j^{3/5} \right] \), \( L = \left[ j^{2/5} \right] \), \( N_j = \left[ j^{3/5} \right] \left[ j^{2/5} \right] \),

\[ u_{k,\ell} = \left( \frac{k}{K}, \frac{\ell}{L} \right), \]

and
\[ \mathcal{P}_{N_j} = \{ u_{k,\ell} \}_{k=0, \ldots, K-1} \quad \ell=0, \ldots, L-1. \]

Then, as in the case \( \frac{2}{5} \leq \sigma < 1 \),
\[ \int_{\mathbb{T}^2} \left| \text{card} \left( \mathcal{P}_{N_j} \cap (C_{1} + t) \right) - N_j \right| C_{1} \right|^2 \, dt \]
\[ = K^2 L^2 \sum_{(n_1,n_2) \in \Gamma_1} |\hat{\chi}_{C_1}(K_{n_1}, L_{n_2})|^2 + K^2 L^2 \sum_{(n_1,n_2) \in \Gamma_2} |\hat{\chi}_{C_1}(K_{n_1}, L_{n_2})|^2, \]
where
\[ \Gamma_1 = \{(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} : |K_{n_1}| \leq c |L_{n_2}| \}, \]
\[ \Gamma_2 = \{(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} : |K_{n_1}| > c |L_{n_2}| \}. \]

For the first series we have
\[ K^2 L^2 \sum_{(n_1,n_2) \in \Gamma_1} |\hat{\chi}_{C_1}(K_{n_1}, L_{n_2})|^2 \leq c K^2 L^2 \sum_{n_2=1}^{+\infty} \sum_{n_1 < \frac{|L_{n_2}|}{K}} |L_{n_2}|^{-4} \]
\[ \leq c K^2 L^{-2} \sum_{n_2=1}^{+\infty} |n_2|^{-4} \left( \frac{|L_{n_2}|}{K} + 1 \right) \leq c K L^{-1} + K^2 L^{-2} \approx j^{2/5} \approx N_j^{2/5}. \]

For the second series we have
\[ K^2 L^2 \sum_{(n_1,n_2) \in \Gamma_2} |\hat{\chi}_{C_1}(K_{n_1}, L_{n_2})|^2 \leq c K^2 L^2 \sum_{n_1=1}^{+\infty} \sum_{n_2 < \frac{|K_{n_1}|}{L}} |K_{n_1}|^{-3} \]
\[ \leq c K^{-1} L^2 \sum_{n_1=1}^{+\infty} |n_1|^{-3} \left( \frac{|K_{n_1}|}{L} + 1 \right) \leq c j^{2/5} \approx N_j^{2/5}. \]

\[ \square \]

**Proof of Proposition 7**. We show that we can apply Theorem 7 with \( \sigma = 1 \) to the convex \( C \). By Proposition 6 we have
\[ |\gamma_{\Theta}(\delta)| \geq c \delta \]
for every \( \Theta \). Since \( \partial C \) is piecewise \( C^2 \) and \( C \) is not a polygon there is an arc \( \Gamma \) in \( \partial C \) which is \( C^2 \) and where the curvature is away from zero. For every \( P \in \Gamma \) let \( \Theta \) be the inward unit normal at \( P \). Then, by the argument in Remark 10 we have
\[ |\gamma_{\Theta}(\delta)| \geq c \delta^{3/2}. \]

\[ \square \]
Proof of Theorem 17. By Theorem 14 if \( \xi \in \mathbb{R}^2, |\xi| \geq c_1 \) we have
\[
\int_{1/2}^{1} |\hat{\chi}_C(\tau \xi)|^2 \, d\tau \geq c_2 |\xi|^{-2-2\sigma}.
\]
To apply Lemma 25 let \( \Omega = \left\{ t \in \mathbb{R}^2 : |t| \leq c_3 \sqrt{N} \right\} \), where \( c_3 \) is a constant that will be chosen later and let \( U = \left\{ t \in \mathbb{R}^2 : |t| \leq c_1 \right\} \). Then, as in the proof of Theorem 7, we have
\[
\int_{1/2}^{1} \int_{\tau^2} \left| \text{card} (P_N \cap (\tau C + t)) - \tau^2 N |C| \right|^2 \, dt \, d\tau
\]
\[
= \int_{1/2}^{1} \sum_{m \neq 0} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 \left| \hat{\chi}_{\tau C}(m) \right|^2 \, d\tau
\]
\[
\geq \sum_{m \in \Omega \setminus U} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 \left| \hat{\chi}_{\tau \Omega}(m) \right|^2 \, d\tau
\]
\[
\geq c_2 \sum_{m \in \Omega \setminus U} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2 \left| m \right|^{-2-2\sigma}
\]
\[
\geq c_2 \left( c_3 \sqrt{N} \right)^{-2-2\sigma} \sum_{m \in \Omega \setminus U} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot u(j)} \right|^2
\]
\[
\geq c_2 \left( c_3 \sqrt{N} \right)^{-2-2\sigma} (N \text{area}(\Omega)/4 - cN^2)
\]
\[
= c_2 (c_3)^{-2-2\sigma} N^{-1-\sigma} (N\pi c_3^2 N/4 - cN^2) = cN^{1-\sigma},
\]
provided that \( \pi c_3^2/4 > c \).

Proof of Theorem 17. Assume that condition a) holds true. Since \( C \) is convex, at every point \( P \in \partial C \) there exist a left and a right tangent. Observe that if they differed, then we would have \( |\gamma^\circ_0(\delta)| + |\gamma^\circ_1(\delta)| \leq c\delta \) which is incompatible with (11). This means that \( \Gamma'(s) \) exists for every \( s \). We denote by \( n(s) \) the inward unit normal at \( \Gamma(s) \). Let us fix \( s_1, s_2 \). We can clearly assume that \( |s_1 - s_2| \) is small. Let \( \Theta = n(s_1) \) and let \( \delta = [\Gamma(s_2) - \Gamma(s_1)] \cdot n(s_1) \). Observe that \( ||\Gamma(s_2) - \Gamma(s_1)|| \cdot \Gamma'(s) \) is the length of \( \gamma^\circ_0(\delta) \) or \( \gamma^\circ_1(\delta) \) (according to the orientation of the curve). See Figure 5. Then (11) yields
\[
||[\Gamma(s_2) - \Gamma(s_1)] \cdot \Gamma'(s)|| \geq c\delta^{1/(1+\alpha)},
\]
so that
\[
0 \leq [\Gamma(s_2) - \Gamma(s_1)] \cdot n(s_1) \leq c ||[\Gamma(s_2) - \Gamma(s_1)] \cdot \Gamma'(s_1)||^{1+\alpha}
\]
\[
\leq c ||\Gamma(s_2) - \Gamma(s_1)||^{1+\alpha}.
\]
Similarly
\[
0 \leq [\Gamma(s_1) - \Gamma(s_2)] \cdot n(s_2) \leq c ||\Gamma(s_2) - \Gamma(s_1)||^{1+\alpha}.
\]
We claim that
\[
|n(s_1) - n(s_2)| \leq c |s_2 - s_1|^\alpha.
\]
Indeed, let \( \rho = |\Gamma (s_1) - \Gamma (s_2)| \) and let us choose coordinates so that \( \Gamma (s_2) - \Gamma (s_1) = (\rho, 0) \).

Then from (36) and (37) we obtain, writing \( n (s) = (n_1 (s), n_2 (s)) \), that

\[
0 \leq n_1 (s_1) \leq c\rho^\alpha \\
0 \leq -n_1 (s_2) \leq c\rho^\alpha
\]

and therefore

\[
0 \leq n_1 (s_1) - n_1 (s_2) \leq c\rho^\alpha.
\]

Since \( |n_1 (s_1)| = |n_1 (s_2)| = 1 \) and \( n_2 (s_1), n_2 (s_2) > 0 \) we have

\[
|n_2 (s_1) - n_2 (s_2)| = \left| \sqrt{1 - [n_1 (s_1)]^2} - \sqrt{1 - [n_1 (s_2)]^2} \right| \\
\leq c |n_1 (s_1) - n_1 (s_2)| \leq c\rho^\alpha.
\]

It follows that

\[
|n (s_1) - n (s_2)| \leq c\rho^\alpha \leq c |s_2 - s_1|^{\alpha'},
\]

so that

\[
|\Gamma' (s_1) - \Gamma' (s_2)| \leq M |s_1 - s_2|^{\alpha'}.
\]

Assume now that b) holds true. Let us fix a direction \( \Theta \) and let \( \Gamma (s_0) = \Theta \). For \( \delta \) small enough there exist two points \( \Gamma (s_1) \) and \( \Gamma (s_2) \) on \( \partial C \) such that

\[
|\gamma_{\Theta}^+ (\delta)| = |[\Gamma (s_1) - \Gamma (s_0)] \cdot \Gamma'(s_0)|,
\]

\[
|\gamma_{\Theta}^- (\delta)| = |[\Gamma (s_1) - \Gamma (s_0)] \cdot \Gamma'(s_0)|.
\]

and

\[
[\Gamma (s_j) - \Gamma (s_0)] \cdot n (s_0) = \delta, \quad j = 1, 2.
\]

Then (see Figure 6), for \( j = 1, 2 \) we have

\[
[\Gamma (s_j) - \Gamma (s_0)] \cdot n (s_0) = \left| \int_{s_0}^{s_j} \Gamma' (\tau) \cdot n (s_0) \, d\tau \right| \\
= \left| \int_{s_0}^{s_j} [\Gamma' (\tau) - \Gamma' (s_0)] \cdot n (s_0) \, d\tau \right| \\
\leq \left| \int_{s_0}^{s_j} |\Gamma' (\tau) - \Gamma' (s_0)| \, d\tau \right|
\]
Figure 6. Proof of Theorem 15, second part.

\[ \int_{s_0}^{s_j} |\tau - s_0|^\alpha \, d\tau \leq c |s_j - s_0|^{\alpha+1}. \]

Hence

\[ \delta^{1/\alpha+1} \leq c |s_1 - s_0| \leq c |\Gamma(s_1) - \Gamma(s_0)| \leq |\gamma^-_{\Theta}(\delta)|, \]

and similarly \( \delta^{1/\alpha+1} \leq |\gamma^+_{\Theta}(\delta)|. \)

\[ \square \]

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