Pricing in Resource Allocation Games Based on Lagrangean Duality and Convexification

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Abstract

We consider a basic resource allocation game, where the players’ strategy spaces are sub-
sets of $\mathbb{R}^m$ and cost/utility functions are parameterized by some common vector $u \in \mathbb{R}^m$ and, otherwise, only depend on the own strategy choice. A strategy of a player can be interpreted as a vector of resource consumption and a joint strategy profile naturally leads to an aggregate consumption vector. Resources can be priced, that is, the game is augmented by a price vector $\lambda \in \mathbb{R}^m_+$ and players have quasi-linear overall costs/utilities meaning that in addition to the original costs/utilities, a player needs to pay the corresponding price per consumed unit. We investigate the following question: for which aggregated consumption vectors $u$ can we find prices $\lambda$ that induce an equilibrium realizing the targeted consumption profile?

For answering this question, we revisit a well-known duality-based framework and derive several characterizations of the existence of such $u$ and $\lambda$ using convexification techniques. We show that for finite strategy spaces or certain concave games, the equilibrium existence problem reduces to solving a well-structured LP. We then consider a class of monotone aggregative games having the property that the cost/utility functions of players may depend on the induced load of a strategy profile. For this class, we show a sufficient condition of enforceability based on the previous characterizations. We demonstrate that this framework can help to unify parts of four largely independent streams in the literature: tolls in transportation systems, Walrasian market equilibria, trading networks and congestion control in communication networks. Besides reproving existing results we establish new existence results by using methods from polyhedral combinatorics, polymatroid theory and discrete convexity.

1 Introduction

Resource allocation problems appear in several real-world situations. Whenever available resources need to be matched to demands, the goal is to find the most profitable or least costly allocation of the resources. Applications can be found in several areas, including traffic and telecommunication networks. In the above applications, a finite (or infinite) number of players interact strategically, each optimizing their individual objective function. The corresponding allocation of resources in such setting is usually determined by an equilibrium solution of the underlying strategic game. A central question in all these areas concerns the problem of how to incentivize players in order to use the (scarce) resources optimally. One key approach in all named application areas is the concept of pricing resources according to their usage. Every resource comes with an anonymous prices per unit of consumption and defining the “right” prices thus offers the
chance of inducing equilibria with optimal or efficient resource usage. Prominent examples are toll pricing in transportation networks, congestion pricing in telecommunication networks and *market pricing* in economics. A prime example of the latter is the Walrasian competitive equilibrium (cf. Walras [83]), where goods are priced such that there is an allocation of goods to buyers with the property that every buyer gets a bundle of items maximizing her overall utility given the current prices for the goods.

In this paper, we will introduce a generic model of pricing in resource allocation games with quasi-linear costs/utilities that subsumes several of the above mentioned applications as a special case. In the following, we first introduce the model formally, discuss applications and then give an overview on the main results and related work.

### 1.1 The Model

Let $E = \{1, \ldots, m\}$ be a finite and non-empty set of resources and $N = \{1, \ldots, n\}$ be a nonempty finite set of players. For $i \in N$, let $X_i \subset \mathbb{R}^m, X_i \neq \emptyset$ denote the strategy space of player $i$ and define $X := \times_{i \in N} X_i$ as the combined strategy space. The vector $x_i = (x_{ij})_{j \in E} \in X_i$ is a strategy profile of player $i \in N$ and the entry $x_{ij} \in \mathbb{R}$ can be interpreted as the level of resource usage of player $i$ for resource $j$. For every player $i \in N$, there is a function $g_i : \mathbb{R}^m \rightarrow \mathbb{R}^m, x_i \mapsto g_i(x_i)$ mapping the resource usage vector to a vector of the actual resource consumption. The function $g_i$ allows to model player-specific characteristics such as weights. For both $x_i$ and $g_i$ negative values are allowed. We call the vector of resource usage $x = (x_{ij}) \in \mathbb{R}^{m \times n}$ a *strategy distribution*. Given $x \in X$, we can define the load on resource $j \in E$ as $\ell_j(x) := \sum_{i \in N} g_{ij}(x_i)$, where $g_{ij}$ is the $j$-th component of $g_i$. In the following, we introduce properties of utility functions needed for our main results. We will distinguish between cost minimization games and utility maximization games.

**Assumption 1.1.** We assume that cost/utility functions are parameterized by an exogenously given vector $u \in \mathbb{R}^m$ and depend on the own strategy vector only.

1. For minimization games $G^{\text{min}}(u)$ with respect to $u \in \mathbb{R}^m$, the total cost of a player $i \in N$ under strategy distribution $x \in X$ is defined by a function $\text{cost}_i : X \rightarrow \mathbb{R}$, which satisfies
   \[
   \text{cost}_i(x) := \pi_i(u, x_i) \text{ for all } x \in X \text{ for some function } \pi_i : \mathbb{R}^m \times X_i \rightarrow \mathbb{R}.
   \]

2. For maximization games $G^{\text{max}}(u)$, we denote the utility function for $i \in N$ by $\text{utility}_i : X \rightarrow \mathbb{R}$ and we assume that it satisfies
   \[
   \text{utility}_i(x) := v_i(u, x_i) \text{ for all } x \in X \text{ for some function } v_i : \mathbb{R}^m \times X_i \rightarrow \mathbb{R}.
   \]

The vector $u$ can be interpreted as the induced load of an equilibrium, that is, $u = \ell(x)$. We assume for the moment that players are load taking in the sense that they assume not being able to influence the global load vector $u$ by their own strategy $x_i$, thus leading to the prescribed shape of the cost/utility functions – we will later also consider models in which a functional dependency of the strategy choice on the induced load is allowed.

### 1.2 Pricing in Resource Allocation Games

We are concerned with the problem of defining prices $\lambda_j \geq 0, j \in E$ on the resources in order to incentivize an efficient usage of the resources as explained below. If player $i$ uses resource $j$ at consumption level $g_{ij}(x_i)$, she needs to pay $\lambda_j g_{ij}(x_i)$. The quantities $\pi_i(u, x_i)$ and $\lambda^T g_i(x_i)$ are
assumed to be normalized to represent the same unit (say money in Euro) and we assume that the private cost functions are quasi-linear: \( \pi_i(u,x_i) + \lambda^T g_i(x_i) \). If the parameter \( u = (u_j)_{j \in E} \in \mathbb{R}^m \) represents a targeted load vector \( \ell(x^*) \), then, the task is to find prices \( \lambda \in \mathbb{R}^+_m \) so that \( x^* \) becomes an equilibrium of the game with prices.

**Definition 1.2 (Enforceability).** We now introduce three variants of enforceability.

1. A vector \( u \in \mathbb{R}^m \) is enforceable, if there is a tuple \( (x^*,\lambda) \in X \times \mathbb{R}^m_+ \) satisfying 1a. and 1b. for minimization games \( G^{\min}(u) \) or 1a. and 1c. for maximization games \( G^{\max}(u) \):
   - (a) \( \ell_j(x^*) = u_j \) for all \( j \in E \).
   - (b) Minimization: \( x^*_i \in \arg\min_{x_i \in X_i} \{ \pi_i(u,x_i) + \lambda^T g_i(x_i) \} \) for all \( i \in N \).
   - (c) Maximization: \( x^*_i \in \arg\max_{x_i \in X_i} \{ \pi_i(u,x_i) - \lambda^T g_i(x_i) \} \) for all \( i \in N \).

   In this case, we say \( u \) can be enforced by \( (x^*,\lambda) \in X \times \mathbb{R}^m_+ \).

2. A vector \( u \in \mathbb{R}^m \) is called weakly enforceable with market clearing prices, if there is a tuple \( (x^*,\lambda) \) that satisfies the above condition (1b) (or (1c)) but (1a) is replaced with \( \ell(x^*) \leq u \) and \( \lambda \) satisfies the Walrasian law that resources \( j \in E \) with slack capacity have zero price, that is, \( \ell_j(x^*) < u_j \Rightarrow \lambda_j = 0 \) for all \( j \in E \).

3. A vector \( u \in \mathbb{R}^m \) is uniquely enforceable, if there is \( \lambda \in \mathbb{R}^m_+ \) and a unique \( x^* \in X \) satisfying 1a. and 1b. for minimization games \( G^{\min}(u) \) or 1a. and 1c. for maximization games \( G^{\max}(u) \).

Condition 1a. requires that \( x^* \) realizes the capacities \( \ell(x^*) = u \) while Condition 1b. implements \( x^* \) as a pure Nash equilibrium of the minimization game \( G^{\min}(u) \) augmented with prices. Condition 1a. and 1c. refer to a maximization game \( G^{\max}(u) \) augmented with prices. The definition of weakly enforceable capacity vectors (with market prices) is motivated by applications, for which outcomes are interesting that do not use all capacities at equilibrium.

### 1.3 Running Examples

We give four prototypical examples that are used throughout the paper.

**Example 1.3 (Tolls in Network Routing).** There is a directed graph \( G = (V,E) \) and a finite set \( N \) of populations of commuters modeled by tuples \( (s_i,t_i,d_i) \), \( i \in N \), where \( s_i \) is the source, \( t_i \) the sink and \( d_i > 0 \) represents the volume of flow that is traveling from \( s_i \) to \( t_i \). In this setting, we can think of the set \( E \) as being the set of resources and \( X_i \) representing a flow polytope for every population \( i \in N \). In the network routing literature, there are several equilibrium notions known according to whether the underlying model is nonatomic (Wardrop equilibrium) or atomic (Nash equilibrium). Given an equilibrium concept, the goal is to find network tolls \( \lambda_j \geq 0, j \in E \) on edges that enforce a prescribed capacity vector \( u \) via an equilibrium strategy distribution.

Now we turn to the area of *Walrasian market equilibria* which constitutes a central topic in the economics literature, see the original work of Walras [83] and later landmark papers of Kelso and Crawford [56], Gul and Stachetti [42] and Danilov et al. [27].

**Example 1.4 (Market Equilibria).** Suppose there are items \( E = \{1,\ldots,m\} \) for sale and there is a set \( N = \{1,\ldots,n\} \) of buyers interested in buying some of the items. For every subset \( S \subseteq E \) of items, player \( i \) experiences value \( w_i(S) \in \mathbb{R} \) giving rise to a valuation function \( w_i : 2^m \to \mathbb{R}, i \in N \), where \( 2^m \) represents the set of all subsets of \( E \). The market manager wants to determine a price
vector $\lambda \in \mathbb{R}^m_+$ so that all items are sold to the players and every player demands a subset $S_i \subseteq E$ maximizing her quasi-linear utility: $S_i \in \arg \max_{S \subseteq E} \{w_i(S) - \sum_{j \in S} \lambda_j\}$. This is known as a Walrasian competitive equilibrium.

This class of games also belongs to the class $G^\text{max}(u)$ augmented with prices introduced in Assumption 1.1, because the valuation function of a buyer only depends on her own assigned bundle of items. If $X_i, i \in N$ represents the set of incidence vectors of subsets of $E$, we can set $u = (1, \ldots, 1)^T \in \mathbb{R}^m$ and any pair $x \in X, \lambda \in \mathbb{R}^m_+$ that weakly enforces $u$ with market prices corresponds to a competitive equilibrium.

A related application are so-called trading networks as introduced by Hatfield et al. [48].

**Example 1.5** (Bilateral Trading Networks). A bilateral trading network is represented by a directed multigraph $G = (N, E)$, where $N$ is the set of vertices and $E = \{e_1, \ldots, e_m\}$ the set of edges. Vertices of the graph correspond to players and edges represent possible bilateral trades that can take place between the pair of incident vertices. For such trade $e = (s, b) \in E$, the source vertex $s$ corresponds to the seller and the sink vertex $b$ corresponds to the buyer. For a set of edge prices $\lambda_e \geq 0, e \in E$, we can associate with each possible trade $e = (s, b) \in E$ a price $\lambda_e \geq 0$ with the understanding that the buyer $b$ pays $\lambda_e$ to the seller $s$. An outcome of the market is a set of realized trades $S \subseteq E$ and a vector of prices $\lambda \in \mathbb{R}^m_+$. Given an outcome, the quasi-linear utility of a player $i \in N$ is defined as the sum of the utility gained from trades plus the income minus the cost of trades, respectively. The utility of realized trades is given by a function $\bar{w}_i : 2^{\delta(E)} \rightarrow \mathbb{R}$. As in market equilibria, the goal is to identify a subset of trades and a price vector so that every player gets a utility maximizing subset of trades. The main difference to market equilibria arises as players can act simultaneously as both, sellers and buyers in the market. As we will show in Section 8, this class of games also belongs to the class $G^\text{max}(u)$ augmented with prices introduced in Assumption 1.1.

The next application resides in the area of congestion control in communication networks.

**Example 1.6** (Congestion Control in Communication Networks). We consider a model of Kelly et al. [54] in the domain of TCP-based congestion control. We are given a directed or undirected capacitated graph $G = (V, E, c)$, where $V$ are the nodes, $E$ the edge set with $|E| = m$ and $c \in \mathbb{R}^m_+$ denotes the edge capacities. There is a set of players $N = \{1, \ldots, n\}$ and each $i \in N$ is associated with an end-to-end pair $(s_i, t_i) \in V \times V$ and a non-decreasing and concave bandwidth utility function $U_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ measuring the received benefit from sending net flow from $s_i$ to $t_i$. The strategy space $X_i$ of a player represents a flow polyhedron and for a flow $x_i \in X_i$ with value $\text{val}(x_i)$ the received bandwidth utility is equal to $U_i(\text{val}(x_i))$. The goal in this setting is to determine a price vector $\lambda \in \mathbb{R}^m_+$ so that a strategy distribution $x^*$ is induced as an equilibrium respecting the network capacities $c$ and, hence, avoids congestion. The equilibrium condition is given by

$$x^*_i \in \arg \max \{U_i(\text{val}(x_i)) - \lambda^T x_i | x_i \in X_i\} \text{ for all } i \in N.$$  

This model fits to the class $G^\text{max}(u)$ augmented with prices: The utility function of a player only depends on the own action and with $u := c$ we obtain the desired structure.

### 1.4 Overview of Results, Main Techniques and Organization of the Paper

In Section 1.1 and 1.2, we introduced a resource allocation model and motivated the question of enforceability of load vectors induced by equilibrium profiles with respect to anonymous resource prices. In Section 2, we will revisit a well-known duality-based framework and we prove a complete characterization of enforceability:
Theorem 2.3: \( u \in \mathbb{R}^m \) is enforceable by \((x^*, \lambda)\) if and only if \((x^*, \lambda)\) yields zero duality gap for the master problem \( \min \{ \sum_{i \in N} \pi_i(u, x) | \ell(x) \leq u, x \in X \} \) and \( x^* \) satisfies \( \ell(x^*) = u \).

The above result only requires the separability structure of utility/cost functions and the conditions on the duality gap.\(^1\) Otherwise, the strategy sets and utility/cost functions are not restricted, for instance, they may be non-convex. However, checking whether or not a non-convex master problem exhibits zero duality gap and admits an optimal solution with the desired property may be very difficult to prove. In this regard, for any \( G_{\min}(u) \), we introduce in Section 3 the notion of a convex relaxation \( G_{\min-\text{conv}}(u) \) using the concept of convex subfunctionals. We derive the following characterization:

Theorem 3.5: \( u \) is enforceable for \( G_{\min}(u) \) via \((x^*, \lambda)\) if and only if \((x^*, \lambda)\) enforces \( u \) for the convexified game \( G_{\min-\text{conv}}(u) \).

While the new game \( G_{\min-\text{conv}}(u) \) is now convex and, thus, more accessible, the complexity of the original non-convex problem is in some sense shifted to the representation of the convex subfunctionals. For two game classes, however, namely (1) games with finite strategy spaces or (2) games for which the convex hull of the strategy space is finite and the cost/utility functions is concave/convex, we can show that the convex subfunctionals are representable as optimal solutions of an underlying LP. This leads to the third main result:

Theorem 3.11 and Theorem 3.15: For (1) and (2): \( u \) is enforceable for \( G_{\min}(u) \) if and only if the master LP for \( G_{\min-\text{conv}}(u) \) admits optimal solutions that are feasible for \( G_{\min}(u) \).

The dual of the master LP can be solved in polynomial time via the ellipsoid method, if there is an efficient separation oracle. The separation oracle for the two problem classes reduces to the so-called demand problem, where for every player one is given prices and the problem is to compute an optimal strategy. The complexity of the demand versus the master problem \( h_{\min}(u) \) can then be used to establish impossibility results for enforceability using complexity-theoretic assumption (like \( P \neq NP \)). The connection between the complexity of the demand problem and that of the master problem (or welfare maximization problem) has been discovered first by Roughgarden and Talgam-Cohen [72] in the context of pricing problems for Walrasian market equilibria.

We then consider in Section 4 the case of integrality of strategy spaces. It follows that for master problems admitting a fractional relaxation with zero duality gap and integer optimal solutions, the sufficiency condition of Theorem 2.3 is satisfied. For polyhedral integral strategy spaces, the powerful methods from polyhedral combinatorics can be used to categorize cases for which such relaxations exist. In this regard, we show two prototypical results:

1. Theorem 4.1 gives an enforceability result for games with homogenous additive linear utilities/costs and (box) totally-dual-integral and decomposable aggregation polytopes.

\(^1\)The if-direction is well-known, see [8, 67, 73]. The only-if direction roughly corresponds to the first welfare theorem in economics saying that every pricing equilibrium maximizes social welfare, however, the theorem asks for a slightly stronger condition (strong duality) and the fact that an inequality must be tight.
2. Theorem 4.2 gives an enforceability result for games with player-specific additive linear utilities/costs on polymatroidal strategy spaces.

In Section 5, we turn to models for which the private cost/utility function intrinsically depends on the aggregated load vector $\ell(x)$, that is, it has the form $\pi_i(\ell(x), x_i), i \in N$ and is not separable anymore. A prime example is an atomic congestion game, where any change of strategies has an effect on the perceived cost since the load vector changes. We identify an expressive class of games that we term **monotone aggregative games** which include among others congestion games with nondecreasing cost functions.

**Theorem 5.3**: Let $G^{\text{min-mag}}$ be a monotone aggregative game. If $u$ is enforceable for the game $G^{\text{min}}(u)$ with $\pi_i(\ell(x), x_i) := \pi_i(u, x_i), i \in N, x \in X$, then $u$ is also enforceable for $G^{\text{min-mag}}$.

This way, we can translate several positive enforceability results to monotone aggregative games including atomic congestion games. With these results and methods at hand, we apply the framework to the four application domains.

**Tolls in traffic networks.** In Section 6, we consider the problem of defining tolls in order to enforce certain load vectors as Wardrop equilibrium. For nonatomic network games, we reprove and generalize in Corollary 6.2 a characterization of enforceable load vectors by Yang and Huang [85], Fleischer et al. [33] and Karakostas and Kolliopoulos [53].

Then we turn to atomic congestion games. For general nondecreasing homogeneous cost functions, we show that polytopal congestion games can be analyzed using Theorem 4.1. It turns out that for a wide classes of congestion games (matroid games, single-source network games, r-arborescences, matching games, and more) the defining aggregation polytope is box-integral and decomposable leading to existence results of enforcing tolls (Corollary 6.4). For all these settings, a congestion vector $u$ minimizing the social cost can be computed in polynomial time (see Del Pia et al. [69] and Kleer and Schäfer [57]) and the space of enforcing prices can be described by a compact linear formulation. It follows that for a fixed enforceable capacity vector $u$, arbitrary linear objective functions (like maximum or minimum revenue) can be efficiently optimized over the price/allocation space. Besides single-source network games (see Fotakis and Spirakis [35] and Fotakis et al. [34]), these results were not known before.

Then, we study the more challenging case of atomic congestion games with nondecreasing player-specific cost functions on the resources. We prove – using Theorem 4.2 – that for polymatroidal strategy spaces, one can obtain existence results (Corollary 6.5). To the best of our knowledge, these are the first existence results of tolls for congestion games with player-specific cost functions.

**Market equilibria.** In Section 7, we study (indivisible) single-, multi-item, or package auctions and show that the existence of Walrasian equilibria can be studied within the framework. Using the fact that in all these models, the quasi-linear utility function is separable and the strategy space of every player (buyer or seller) consists of a finite point set we can use the relationship between a game and its convexified game. This way, we can reprove classical LP-characterization results of the existence of Walrasian equilibria by Bikchandani and Mamer [11], Bikchandani and
Ostroy [12] as well as more recent LP characterizations by Candogan et al. [18] and Roughgarden and Talgam-Cohen [72]. For gross-substitute valuations we reprove the existence of Walrasian equilibria using methods of discrete convexity and $M$-convexity (see Murota [62]).

Then we consider a class of valuations for multi-unit items that we term *separable additive* valuations with *negative externalities*. The idea is that items are of different type but may be sold at a certain multiplicity and the values for received items are additive. The precise item values may depend on the allocation vector. This dependency is assumed to model negative externalities, that is, - roughly speaking - if an item type is sold to more players, the value goes down. For this class of valuations, we prove in Corollary 7.6 that for general polymatroidal environments, Walrasian equilibria exist.

**Trading networks.** In Section 8, we study trading networks as introduced by Hatfield et al. [48]. We show that also this class of games fits into the framework. The main conceptual difference to the previous market equilibrium setting is that players may be both buyers and sellers at the same time. As our model allows negative resource consumption, we just use $\{-1,1\}$ entries in an allocation vector to distinguish buy or sell activities. Using the relationship between a game and its convexified game, we prove an LP-characterization result of the existence of trading equilibria (Corollary 8.2). This characterization was, to the best of our knowledge, not known before. For gross-substitute valuations, we give a simple proof for the existence of Walrasian equilibria using again $M$-convexity arguments.

**Congestion control in communication networks.** In the final Section 9, we consider congestion control problems in communication networks using a flow-based model proposed by Kelly et al. [54]. We first reprove an existence result of enforceable capacity vectors of Kelly et al. [54]. Then, we turn to the much less explored model of *integral flows*, where a discrete unit-packet size is given. With the previous results related to TDI systems, we prove that for single-source networks with identical linearly increasing bandwidth utility functions, every nonnegative capacity vector is weakly enforceable with market prices (Corollary 9.2). We complement this result by showing in Proposition 9.4 that already for two-player instances with different source-sink pairs and linear and identical (capped) bandwidth utility functions, not every $u$ is weakly enforceable by market prices, unless $P = NP$. For this result, we use the LP-characterization of Theorem 3.15 and then show that the demand problem is polynomial time solvable while the master-problem is NP-hard.

### 1.5 Related Work

As outlined in the introduction, the topic of pricing resources concerns different streams of literature and it seems impossible to give a complete overview here. Lagrangian multipliers date back to the 18th century and their use in terms of *shadow prices* measuring the change of the optimal value function for marginal changes of the right-hand sides of constraints is well-known – assuming some constraint qualification conditions, see for instance Boyd and Vandenberghe [13].

Our first main result (Theorem 2.3) relies on a decomposition property of the Lagrangian (for separable problems) and the use of Lagrange multipliers for pricing the resources. This approach is by no means new and has been developed in several facets before, see for instance Dantzig and Wolfe [28] and Bertsekas and Ghallager [8]. Dantzig and Wolfe [28] used this principle for their celebrated decomposition framework for solving certain linear (integer) programming problems. Bertsekas and Ghallager [8], Palomar and Chiang [67] and Scutari et al. [73] described how the
Lagrangian of a general separable optimization problem

\[
\max \left\{ \sum_{i \in N} U_i(x_i) \middle| x_i \in X_i, i \in N, \sum_{i \in N} h_i(x_i) \leq u \right\}
\]

can be decomposed into \( n \) independent problems. These works describe the close connection between strong duality and the existence of enforcing dual prices. One subtle difference of this model to ours is the parameterization of the cost/utility functions \( \pi_i(u, x_i) \) with respect to the capacity vector \( u \). This degree of freedom allows to model dependencies of targeted capacity vectors with respect to the intrinsic cost/utilities - a prime example appears in nonatomic congestion games, where the cost function of an agent only depends on the aggregated load vector. Moreover, this dependency allows to model externalities with respect to allocations which are not directly possible in the previous formulations. In contrast to most works in the “dual-decomposition” area, we systematically investigate the impact of non-convexities of the cost/utility functions and the strategy spaces (e.g., integrality of strategies) on the resulting enforceability properties.

**Convexification of Non-Convex Models.** The idea of convexifying a non-convex economic model dates back to the late sixties starting with the work of Shapley and Shubik [75] and Starr [79]. Starr [79] considered a standard Arrow-Debreau exchange economy without convexity assumptions on production or consumption sets nor on the preference ordering. The analysis of the existence of competitive market equilibria is based on a *convexified economy* in which the convex hull of production or consumption sets and the convex hull of the epigraph with respect to the preference orderings are considered (see also later related works of Henry [49], Moore et al. [61] and Svensson [81]). By separation arguments, this convexified economy permits a competitive equilibrium (called a synthetic convex equilibrium). A quasi-equilibrium lives in the original non-convex model and is defined as a closest approximation within w.r.t the synthetic equilibrium. With the Shapley-Folkman Theorem (which appeared inside the paper of Starr) the approximation guarantee can be parameterized in terms of the number of commodities or number of traders involved. For large markets (number of traders tends to be large) this distance vanishes. The approach of convexifying a game in this paper is qualitatively similar to that of Starr. The main difference lies in the representation of the convexified game. Instead of convexifying the epigraph of utility level sets as in Starr, we explicitly use convex envelopes of the utility functions which allow (in the context of separable problems) to define a convex master-problem. This way, we obtain for (1) games with finite strategy spaces or (2) games for which the convex hull of the strategy space is finitely generated and the cost/utility functions is concave/convex a tractable LP formulation for the master problem. The assumption (1) for instance is fulfilled for exchange markets with indivisible items and the representation of the convexified game corresponds to the so-called configuration LP of Bikchandani and Mamer [11] and Bikchandani and Ostroy [12].

**Tolls in Traffic Networks.** A large body of work in the area of transportation networks is concerned with congestion toll pricing. Beckmann et al. [7] showed that for the Wardrop model with homogeneous users, charging the difference between the marginal cost and the real cost in the socially optimal solution (marginal cost pricing) leads to an equilibrium flow which is optimal. Cole et al. [22] considered the case of heterogeneous users, that is, users value latency relative to monetary cost differently. For single-commodity networks, the authors showed the existence

\[\text{The bound was recently improved by Budish and Reny [15].}\]

\[\text{In the spirit of large markets, Aumann [2] derived a very general existence result of competitive equilibria assuming a continuum of traders but without any convexity assumptions.}\]
of tolls that induce an optimal flow as Nash flow. Yang and Huang [85], Fleischer et al. [33] and Karakostas and Kollipoulos [53] proved that there are tolls inducing an optimal flow for heterogeneous users even in general networks - all proofs are based on linear programming duality. Swamy [82] and Yang and Zhang [86] proved the existence of optimal tolls for the atomic splittable model using convex programming duality.

For atomic (unsplitable) network congestion games much less is known regarding the existence of tolls. Caragiannis et al. [20] studied the existence of tolls for singleton congestion games. Fotakis and Spirakis [35] proved the existence of tolls inducing any acyclic integral flow for symmetric s,t network games with homogeneous players. Fotakis et al. [34] further extended this result to heterogeneous players and networks with a common source but different sinks. Marden et al. [59] transferred the idea of charging marginal cost tolls to congestion games and showed the existence of tolls enforcing the load vector of a socially optimal strategy distribution. Very recently, Chandan et al. [21] derived an optimization formulation computing optimal tolls minimizing the resulting price of anarchy.

**Market Equilibria.** For the problem of allocating indivisible single-unit items, there are several characterizations of the existence of competitive equilibria related to the gross-substitute property of valuations, see Kelso and Crawford [56], Gul and Stachetti [42] and Ausubel and Milgrom [4]. Several works established connections of the equilibrium existence problem w.r.t. LP-duality and integrality (see Bikchandani and Mamer [11], Bikchandani and Ostoy [12] and Shapley and Shubik [76]). Murota and Tamura [63, 62] established connections between the gross substitutability property and M-convexity properties of demand sets and valuations. Yokote [87] recently proved that the existence of Walrasian equilibria follows from a duality property in discrete convexity.

For multi-unit items, several recent papers studied the existence of Walrasian equilibria. Danilov et al. [27] investigated the existence of Walrasian equilibria in multi-unit auctions and identified general conditions on the demand sets and valuations related to discrete convexity, see also Milgrom and Struluvici [60] and Ausubel [3]. Baldwin and Klemperer [6] explored a connection with tropical geometry and gave necessary and sufficient conditions for the existence of competitive equilibrium in product-mix auctions of indivisible goods, see also Sun and Yang [80]. For a comparison of the above works especially with respect to the role of discrete convexity, we refer to the excellent survey of Shioura and Tamura [77]. Candogan et al. [18, 19] showed that valuations classes (beyond GS valuations) based on graphical structures also imply the existence of Walrasian equilibria. Their proof also uses integrality of optimal solutions of an associated linear min-cost flow formulation and linear programming formulation, respectively.

Our existence result for polymatroid environments differs to these previous works in the sense that we allow valuations to depend on the allocation of items to other players (negative externalities). Much fewer works allow for externalities in valuation functions, see for instance Zame and Noguchi [88]. Models with positive (network-based) externalities have been considered by Candogan et al. [16], Bhattacharya et al. [9] considered a setting with weighted negative network-based externalities and unit-demand buyers. Bikchandani et al. [10] consider a problem of selling a base of polymatroid. In their model, however, the prices are not anonymous (rather VCG) for several items of the same type. The same holds true for Goel et al. [39] who also consider polymatroids even with budget constraints. Feldman et al. [31] proposed the notion of combinatorial Walrasian equilibria, where items can be packed a priori into bundles. This ensures the existence of equilibria with approximately optimal welfare guarantees.
Trading Networks. Hatfield et al. [47, 48] introduced the model of trading networks and established existence and characterization results for so-called fully-substitutable valuations – a generalization of gross-substitutable valuations. Ikebe et al. [51] generalized the model of Hatfield et al. by using certain discrete concave utility functions for which they derived existence results. Subsequently, Candogan et al. [17] reduced the problem of computing competitive equilibria to a submodular flow problem on a suitably defined network. This way, they established the polynomial time computation of market equilibria for fully substitutable valuations. Further generalizations regarding the inclusion of taxes and other monetary transfers appear in Fleiner et al. [32].

Congestion Control. Kelly et al. [54] proposed to model congestion control via analyzing optimal solutions of a convex optimization problem, where an aggregated bandwidth utility subject to network capacity constraints is maximized. By dualizing the problem and then decomposing terms (as we do in this paper), it is shown that Lagrangian multipliers correspond to equilibrium enforcing congestion prices. For an overview on more related work in this area, we refer to the book by Srikant [78]. Kelly and Vazirani [55] drew connections between market equilibrium computation and the congestion control model of Kelly. Cominetti et al. [23] also studied the convex programming formulation of Kelly et al. and established connections to the Wardrop equilibrium model. The most obvious difference of these work to ours is that they assume convex strategy spaces and concave utility functions. Our framework allows to add integrality conditions or non-convexities to the model.

2 Connection to Lagrangean Duality in Optimization

In the following, we distinguish between cost minimization problems and utility maximization problems. We explicitly prove our main results in the realm of cost minimization but all arguments carry directly over to the maximization case. For later referral, we summarize the results for the maximization case at the end of the section.

For a game \( G^{\min}(u) \), we define the following minimization problem that we call master problem:

\[
\begin{align*}
\min_{x} & \quad \pi(x) \\
\text{s.t.:} & \quad \ell_j(x) \leq u_j, \ j \in E, \\
& \quad x_i \in X_i, \ i = 1, \ldots, n,
\end{align*}
\]

where the objective function is defined as \( \pi(x) := \sum_{i \in N} \pi_i(u, x_i) \).

We assume in the formulation of \( P^{\min}(u) \) that a global minimum actually exists. The Lagrangian function for problem \( P^{\min}(u) \) becomes \( L(x, \lambda) := \pi(x) + \lambda^\top(\ell(x) - u), \ \lambda \in \mathbb{R}^m_+ \), and we can define the Lagrangian-dual as: \( \mu: \mathbb{R}^m_+ \to \mathbb{R}, \ \mu(\lambda) = \inf_{x \in X} L(x, \lambda) = \inf_{x \in X} \{\pi(x) + \lambda^\top(\ell(x) - u)\} \). We assume that \( \mu(\lambda) = -\infty \), if \( L(x, \lambda) \) is not bounded from below on \( X \). The dual problem is defined as:

\[
\sup_{\lambda \geq 0} \mu(\lambda) \quad (D^{\min}(u))
\]

Definition 2.1. Problem \( P^{\min}(u) \) has zero-duality gap, if there is \( \lambda^* \in \mathbb{R}^m_+ \) and \( x^* \in X \) with \( \pi(x^*) = \mu(\lambda^*) \). In this case, we say that the pair \( (x^*, \lambda^*) \) is primal-dual optimal.
If problem $P_{\text{min}}(u)$ has zero-duality gap, the two solutions $\lambda^* \in \mathbb{R}_+^m$ and $x^* \in X$ are optimal for their respective problems $D_{\text{min}}(u)$ and $P_{\text{min}}(u)$ and infima/suprema in the definition of $\mu$ become a minimum/maximum.

We now show a key structure, namely that the Lagrangian dual can be decomposed into $n$ independent subproblems. This decomposition step is classical for separable optimization problems, see Bertsekas and Gallager [8].

**Lemma 2.2.** Let $\lambda \in \mathbb{R}_+^m$. For a problem of type $P_{\text{min}}(u)$, the following holds true:

$$x^* \in \arg\min_{x \in X} L(x, \lambda) \iff x^*_i \in \arg\min_{x_i \in X_i} \{\pi_i(u, x_i) + \lambda^T g_i(x_i)\} \quad \text{for all } i \in N. \quad (2)$$

**Proof.** We calculate:

$$\min_{x \in X} L(x, \lambda) = \min_{x_i \in X_i, i \in N} \left\{ \sum_{i \in N} (\pi_i(u, x_i) + \sum_{j=1}^m \lambda_j (g_{ij}(x_i) - u_j)) \right\}$$

$$= \sum_{i \in N} \min_{x_i \in X_i} \left\{ \pi_i(u, x_i) + \sum_{j=1}^m \lambda_j (g_{ij}(x_i) - u_j) \right\},$$

where the first equality follows by the linearity of $\ell(x)$ w.r.t. $g_{ij}, i \in N$ and the last equality by the assumption that $\pi_i(u, x_i)$ only depends on $x_i \in X_i$. Because taking the minimum is independent of the constant $-\sum_{j=1}^m \lambda_j u_j$, the lemma follows. □

We obtain the following result.

**Theorem 2.3.** The following equivalences hold for $G_{\text{min}}(u)$.

1. A capacity vector $u \in \mathbb{R}^m$ is enforceable via $(x^*, \lambda^*)$ if and only if $(x^*, \lambda^*)$ has zero duality gap for $P_{\text{min}}(u)$ and $x^*$ satisfies (1) with equality.

2. A capacity vector $u \in \mathbb{R}^m$ is weakly enforceable via $(x^*, \lambda^*)$ with market clearing prices $\lambda^*$ if and only if $(x^*, \lambda^*)$ has zero duality gap for $P_{\text{min}}(u)$.

3. A capacity vector $u \in \mathbb{R}^m$ is uniquely enforceable via $(x^*, \lambda^*)$ if and only if $(x^*, \lambda^*)$ has zero duality gap for $P_{\text{min}}(u)$ and $x^*$ is a unique optimal solution for $P_{\text{min}}(u)$ satisfying (1) with equality.

**Proof.** For the proof it suffices to show 2., since 1. satisfies all conditions of 2. except that $\ell(x^*) = u$ holds true for either side of the equivalence in 1. Statement 3. follows directly from 1. as on both sides of 3. uniqueness of $x^*$ is assumed.

For 2.: $\Leftarrow$: Assume there are $\lambda^* \in \mathbb{R}_+^m, x^* \in X$ with $\ell(x^*) \leq u$ so that $\mu(\lambda^*) = \pi(x^*)$. We obtain

$$\mu(\lambda^*) = \min_{x \in X} \{\pi(x) + (\lambda^*)^T (\ell(x) - u)\} \leq \pi(x^*) + (\lambda^*)^T (\ell(x^*) - u) \leq \pi(x^*) = \mu(\lambda^*).$$

Hence, all inequalities must be tight leading to $(\lambda^*)^T (\ell(x^*) - u) = 0$ as claimed. It remains to prove Condition 1b. With $x^* \in \arg\min_{x \in X} L(x, \lambda^*)$ we get

$$x^* \in \arg\min_{x \in X} L(x, \lambda^*) \iff x^*_i \in \arg\min_{x_i \in X_i} \{\pi_i(u, x_i) + \sum_{j \in E} \lambda^*_j g_{ij}(x_i)\} \quad \text{for all } i \in N.$$
\[\Rightarrow: \text{Let } u \in \mathbb{R}^m \text{ be weakly enforceable by some } x^* \in X \text{ with market clearing prices } \lambda^* \in \mathbb{R}^m_+, \text{ that is, } (x^*, \lambda^*) \text{ satisfy } \ell(x^*) \leq u, (\lambda^*)^T(\ell(x^*) - u) = 0 \text{ and } x^*_i \in \arg\min_{x_i \in X_i} \{\pi_i(u, x_i) + (\lambda^*)_i g_i(x_i)\} \text{ for all } i \in \mathbb{N}. \text{ We calculate}
\]

\[
\mu(\lambda^*) = \inf_{x \in X} [\pi(x) + (\lambda^*)^T(\ell(x) - u)] \\
= \pi(x^*) + (\lambda^*)^T\ell(x^*) - (\lambda^*)^Tu \\
= \pi(x^*),
\]

where (3) follows from Lemma 2.2 and (4) uses the market price condition \((\lambda^*)^T(\ell(x^*) - u) = 0\). Hence, strong duality holds for the pair \((x^*, \lambda^*)\). \qed

As mentioned before, the if-direction of the above characterizations are well known in the literature, see, e.g. [8, 54, 67, 73]. We remark here that the theorem does not rely on any assumption on the feasible sets \(X_i\) nor on the functions \(\pi_i(u, x_i), i \in \mathbb{N}\) as long as \(P^{\text{min}}(u)\) has zero duality gap. In the optimization literature, several classes of optimization problems are known to have zero duality gap even without convexity of feasible sets and objective functions, see for instance Zheng et al. [89]. In cost minimization games, the feasible sets \(X_i\) usually contain some sort of covering conditions on the resource consumption. For example in network routing, one needs to send some prescribed amount of flow. In this regard, we introduce a natural candidate set of vectors \(u\) for which we know that any feasible solution satisfying (1) does so with equality.

**Definition 2.4.** A vector \(u \in \mathbb{R}^m\) is called minimal for \(X\), if there are strictly increasing functions \(h_j : \mathbb{R} \to \mathbb{R}, j \in E\) such that \(u \in \arg\min_{u' \in \mathbb{R}^m} \{\sum_{j \in E} h_j(u'_j)\} \ \forall x \in X \text{ with } \ell(x) \leq u'_j\).

The above definition has been previously used by Fleischer et al. [33] in the context of enforcing tolls in nonatomic congestion games.

**Corollary 2.5.** Let \(u \in \mathbb{R}^m\) be minimal for \(X\). Then, the following two statements are equivalent:

1. \(u\) is enforceable via price vector \(\lambda^* \in \mathbb{R}^m_+\) and \(x^* \in X\).
2. \((x^*, \lambda^*)\) satisfies \(\pi(x^*) = \mu(\lambda^*)\).

The only difference to Theorem 2.3 is that by minimality of \(u\), we get \(\ell(x) = u\) for any feasible solution of \(P^{\text{min}}(u)\), therefore, tightness of inequality (1) is already satisfied.

Let us now consider the important special case of convex optimization problems.

**Corollary 2.6.** Let \(X_i, i \in \mathbb{N}\) be nonempty convex sets and assume that \(\pi_i(u, x_i), g_i(x_i), i \in \mathbb{N}\) are convex functions over \(X_i\). Let \(u \in \mathbb{R}^m\) be minimal and suppose there exists \(x^0 \in \text{relint}(\{x \in X | \ell(x) \leq u\})\), where \(\text{relint}(U)\) denotes the relative topological interior of \(U \subset \mathbb{R}^m\). Then, \(u\) is enforceable. If \(\pi(x) = \sum_{i \in \mathbb{N}} \pi_i(u, x_i)\) is strictly convex over \(X\), then \(u\) is uniquely enforceable.

We moved analogous results for maximization problems to the appendix A.

### 3 Convexified Games

So far, the strategy spaces \(X_i, i \in \mathbb{N}\) and the cost functions \(\pi_i, i \in \mathbb{N}\) of a game \(G^{\text{min}}(u)\) were not restricted and are allowed to be non-convex. For instance integrality restrictions in \(X_i \subset \mathbb{Z}^m, i \in \mathbb{N}\) are allowed. In what follows, we connect \(G^{\text{min}}(u)\) with a related convexified game \(G^{\text{min-conv}}(u)\), where \(X_i, i \in \mathbb{N}\) are replaced by their convex hulls and the cost functions \(\pi_i\) are replaced by their
convex envelope or convex subfunctionals. With this convexification, it follows that the duals of the original master problem $P^{\text{min}}(u)$ and that of the convexified game are equal. With this insight, the characterization of enforceable vectors $u$ can (in some cases) be reduced to a more tractable convex problem. The overall idea of convexifying a (nonconvex) optimization problem is quite old and belongs to the broad field of global optimization. Let us refer here to standard textbooks of the late seventies such as that of Horst and Tuy [50, §4.3.] or Shapiro [74, §5]. For an overview on duality theory of general non-convex programs, we refer to the work of Lemaréchal and Renaud [58].

For the general approach to work, we need to make some mild assumptions.

**Assumption 3.1.** We impose the following assumptions.

1. The strategy spaces $X_i \subset \mathbb{R}^m, i \in N$ are compact.
2. The functions $x_i \mapsto \pi_i(u,x_i)$ are lower-semi-continuous (lsc) on $X_i$ for all $i \in N$.
3. The functions $g_i(x_i), i \in N$ are lsc and concave on $X_i$.

**Remark 3.2.** Condition (3) includes the case that $g_i$ is linear, that is, $g_i(x_i) = G_ix_i$, where $G_i \in \mathbb{R}^{m \times m}$ is an $m \times m$ matrix.

For $X_i \subset \mathbb{R}^m$ denote $\text{conv}(X_i) := \cap \{K \supset X_i | K \subset \mathbb{R}^m \text{ convex}\}$ the convex hull of $X_i$, which by Assumption 3.1 is closed and convex. By the theorem of Carathéodory, every $x_i \in \text{conv}(X_i) \subset \mathbb{R}^m$ can be represented as a convex combination of at most $m+1$ points in $X_i$. We thus get

$$
\text{conv}(X_i) = \left\{ \sum_{k=1}^{m+1} \alpha_{ik} y^k \middle| y^k \in X_i, k = 1, \ldots, m+1, \alpha_i \in \Lambda \right\},
$$

where $\Lambda := \{\alpha \in \mathbb{R}^{m+1}_+ | 1^T \alpha = 1\}$. We now define the concept of a convex envelope, see Horst and Tuy [50, §4.3.]

**Definition 3.3.** Let $K \subset \mathbb{R}^m$ be any compact set and let $f : K \to \mathbb{R}$ be lsc. A convex envelope of $f$ on conv($K$) is a function $\phi : \text{conv}(K) \to \mathbb{R}$ satisfying:

1. $\phi$ is convex on conv($K$).
2. $\phi(x) \leq f(x)$ for all $x \in K$.
3. For all convex functions $h : \text{conv}(K) \to \mathbb{R}$ with $h(x) \leq f(x)$ for all $x \in K$ we have $\phi(x) \geq h(x)$ for all $x \in \text{conv}(K)$.

From this definition it is evident that, if the convex envelope exists, it is unique. We will now explicitly describe the (unique) convex envelope of the functions $\pi_i(u,x_i), i \in N$. As shown by Grotzinger [41, Lemma 3.1.], under Assumption 3.1, the convex envelopes of $\pi_i(u,x_i), i \in N$ exist and read as:

$$
\phi_i : \mathbb{R}^m \times \text{conv}(X_i) \to \mathbb{R}
$$

$$(u,x_i) \mapsto \min \left\{ \sum_{k=1}^{m+1} \alpha_{ik} \pi_i(u,x_i^k) \middle| \sum_{k=1}^{m+1} \alpha_{ik} x_i^k = x_i, \alpha_i \in \Lambda, x_i^k \in X_i, k = 1, \ldots, m+1 \right\}. \quad (5)$$

$^4$Rockafellar [70, pp.157] showed that without lsc and compactness of $X_i$, the convex envelope is given by the same formula where min is replaced by inf.
Definition 3.4. For a game $G^{\text{fin}}(u) = (N, X, (\pi_i)_{i \in N})$, the associated convexified game is defined as

$$G^{\text{fin-conv}}(u) = (N, X^{\text{conv}}, (\phi_i)_{i \in N}),$$

where $X^{\text{conv}} := x_{i \in N} \text{conv}(X_i)$.

We obtain the following characterization result connecting $G^{\text{fin}}(u)$ with $G^{\text{fin-conv}}(u)$.

Theorem 3.5. Let $x^* \in X \subseteq X^{\text{conv}}$ and $\lambda \in \mathbb{R}^m_+$. Then, under Assumption 3.1, the following statements are equivalent.

1. $u \in \mathbb{R}^m$ is enforceable for $G^{\text{fin}}(u)$ via $(x^*, \lambda)$.
2. $u \in \mathbb{R}^m$ is enforceable for $G^{\text{fin-conv}}(u)$ via $(x^*, \lambda)$.
3. $(x^*, \lambda)$ is a primal-dual optimal solution of $P^{\text{fin}}(u)$ for $G^{\text{fin}}(u)$ and $x^*$ satisfies $\ell(x^*) = u$.
4. $(x^*, \lambda)$ is a primal-dual optimal solution of $P^{\text{fin}}(u)$ for $G^{\text{fin-conv}}(u)$ and $x^*$ satisfies $\ell(x^*) = u$.

Moreover, all equivalences remain true by replacing the term “enforceable” with “weakly enforceable by market prices” and removing the condition $\ell(x^*) = u$ in Statements 3. and 4.

Proof. Since $G^{\text{fin-conv}}(u)$ fits into the framework presented so far, Theorem 2.3 implies already $2. \iff 4.$ and $1. \iff 3.$ Thus, we only need to show that the Problems $P^{\text{fin}}(u)$ for $G^{\text{fin-conv}}(u)$ and $G^{\text{fin}}(u)$, respectively, have the same dual. We get

$$
\mu^{\text{conv}}(\lambda) = \min_{x \in X^{\text{conv}}} \left\{ \sum_{i \in N} \phi_i(u, x_i) + \lambda^T(\ell(x) - u) \right\}
= \sum_{i \in N} \min \left\{ \sum_{k=1}^{m+1} \alpha_{ik} \pi_i(u, x_i^k) + \lambda^T \left( \sum_{k=1}^{m+1} \alpha_{ik} x_i^k \right) \right\}
= \min_{x_i \in X_i, i \in N} \left\{ \sum_{i \in N} \pi_i(u, x_i) + \lambda^T(\ell(x) - u) \right\},
$$

(6)

where (6) follows from the concavity of the inner objective functions w.r.t. $\alpha_i, i \in N$. \qed

Under a Slater-type constraint qualification and assuming that $g_i, i \in N$ are linear, we get that $P^{\text{fin}}(u)$ for $G^{\text{fin-conv}}(u)$ always has zero duality gap leading to the following result.

Theorem 3.6. Assume that $g_i, i \in N$ are linear and there is $x^0 \in \text{relint}(X^{\text{conv}}) \cap \{x|\ell(x) \leq u\}$. Then, under Assumption 3.1, the following two statements hold.

1. Any minimal $u$ for $X^{\text{conv}}$ is enforceable for $G^{\text{fin-conv}}(u)$.
2. $u$ is enforceable for $G^{\text{fin}}(u)$ if and only if Problem $P^{\text{fin}}(u)$ for $G^{\text{fin-conv}}(u)$ admits an optimal solution $x^* \in X$ with $\ell(x^*) = u$.

Moreover, the equivalence in 2. remains true by replacing the term “enforceable” by “weakly enforceable with market prices” and removing the condition $\ell(x^*) = u$.

We now discuss two important special cases of Theorem 3.5.
3.1 Finite Point Sets and Concave Extensions

We now consider two special cases: in the first one, \( X_i, i \in N \) consists of a finite collection of points (see Fig. 1 left) and in the second one, we assume that the convex hull of each \( X_i, i \in N \) is assumed to be finitely generated and additionally \( \pi_i(u, x_i) \) is assumed to be concave on \( \text{conv}(X_i) \).

![Figure 1: Left is the scenario of \( X_i \) consisting of a finite point set. Right, \( X_i \) may consist of connected components (in green) and isolated points but the convex hull is assumed to be finitely generated and additionally \( \pi_i(u, x_i) \) is assumed to be concave on \( \text{conv}(X_i) \).]

Assumption 3.7. For all \( i \in N \), \( X_i = \{x_1^i, \ldots, x_k^i\} \) for some \( k_i \in \mathbb{N} \).

With this assumption, the convex envelope \( \phi_i, i \in N \) has a simple form. Let us define the following optimal value function of an associated LP:

\[
\phi_i^{LP}(x_i) := \min \{\pi_i^T \alpha | x_i \alpha_i = x_i, \alpha_i \in \Lambda_i\},
\]

(7)

where \( \pi_i := (\pi_i(u, x_k^i))_{k \in \{1, \ldots, k_i\}} \), \( X_i := (x_1^i, \ldots, x_k^i) \) is a \( m \times k_i \) matrix with columns \( x_k^i, k = 1, \ldots, k_i \), and \( \Lambda_i := \{\alpha_i \in \mathbb{R}^{k_i} | 1^T \alpha_i = 1, \alpha_i \geq 0\} \).

Lemma 3.8. Under Assumption 3.1 and Assumption 3.7, it holds that \( \phi_i(u, x_i) = \phi_i^{LP}(x_i) \) for all \( x_i \in \text{conv}(X_i), i \in N \), where \( \phi_i, i \in N \) is the convex envelope as defined in (5).

Proof. The inequality \( \phi_i^{LP}(x_i) \leq \phi_i(u, x_i) \) follows directly as for any \( x_i = \sum_{k=1}^{m+1} \alpha_{ik} \tilde{x}_i^k \), with \( \tilde{x}_i^k \in X_i, k = 1, \ldots, m + 1 \), the corresponding \( \alpha_i \) is feasible for (7) with the same objective value.

For \( \phi_i^{LP}(x_i) \geq \phi_i(u, x_i) \), we need to show that the LP has optimal solutions with support less or equal than \( m + 1 \). The polytope \( P_i := \{\alpha_i \in \mathbb{R}^{k_i} | x_i \alpha_i = x_i, 1^T \alpha_i = 1, \alpha_i \geq 0\} \) of the LP is non-empty and in standard form. With the theorem of linear programming we get that an optimal solution of the LP is attained at a vertex of \( P_i \). Any vertex \( \alpha_i \) of \( P_i \) has the property that the columns of the defining matrix corresponding to indices \( j \) with \( \alpha_{ij} > 0 \) are linearly independent. Since this matrix has \( m + 1 \) rows, its rank is less than \( m + 1 \) implying the wanted small support representation.

We discuss now another class of concave problems for which we also get an LP representation of the convex envelope.

Assumption 3.9. The sets \( X_i \subset \mathbb{R}^m, i \in N \) satisfy \( \text{conv}(X_i) = \text{conv}\left(\{x_1^i, \ldots, x_{k_i}^i\}\right) \) with \( x_j^i \in X_i \) for \( j = 1, \ldots, k_i, k_i \in \mathbb{N} \), and the functions \( \pi_i(u, x_i), i \in N \) can be extended to the domain \( \text{conv}(X_i) \) so that they are concave on \( \text{conv}(X_i) \).

\[^5\text{For any compact set } S \subseteq \mathbb{R}^n \text{ and concave and lsc function } f : \mathbb{R}^n \rightarrow \mathbb{R}, \text{ we have } \min\{f(x) | x \in S\} = \min\{f(x) | x \in \text{conv}(S)\}.\]
With this assumption, the function \( \phi_i^{LP}(x_i) \) defined in (7) is also equal to \( \phi_i(u, x_i) \).

**Lemma 3.10.** Under Assumption 3.1 and Assumption 3.9, it holds that \( \phi_i(u, x_i) = \phi_i^{LP}(x_i) \) for all \( x_i \in \text{conv}(X_i), i \in N \).

**Proof.** \( \phi_i(u, x_i) \leq \phi_i^{LP}(x_i) \) follows from the second part of the proof of the previous Lemma 3.8.

For the other direction, let \( x_i \in \text{conv}(X_i) \) with \( \phi_i(u, x_i) = \sum_{k=1}^{m+1} a_{ik} \pi_i(u, y_i^k) \) for \( x_i = \sum_{k=1}^{m+1} a_{ik} y_i^k \) with \( y_i^k \in X_i \) for \( k = 1, \ldots, m + 1 \) and \( a_i \in \Lambda \). We first bound the cost of every summand \( \pi_i(u, y_i^k) \) individually. With \( y_i^k \in \text{conv}(X_i) \) we have \( y_i^k = \sum_{j=1}^{k_i} \kappa_i^k x_i^j \) for some \( \kappa_i^k \in \Lambda_i \). With the concavity of \( \pi_i(u, x_i) \), we get

\[
\pi_i(u, y_i^k) = \pi_i\left(u, \sum_{j=1}^{k_i} \kappa_i^k x_i^j \right) \geq \sum_{j=1}^{k_i} \kappa_i^k \pi_i(u, x_i^j) \geq \phi_i^{LP}(y_i^k),
\]

where we use that \( \kappa_i^k \) is feasible for the LP associated with \( \phi_i^{LP}(y_i^k) \). Let us write \( \phi_i^{LP}(y_i^k) = \pi_i^T \beta_i^k \) for some \( \beta_i^k \in \text{arg min}\{\pi_i^T a_i | \chi_i = y_i^k, a_i \in \Lambda_i\} \). We then get

\[
\phi_i(u, x_i) = \sum_{k=1}^{m+1} a_{ik} \pi_i(u, y_i^k) \geq \sum_{k=1}^{m+1} a_{ik} \pi_i^{LP}(y_i^k) = \sum_{k=1}^{m+1} a_{ik} \pi_i^T \beta_i^k = \pi_i^T \sum_{k=1}^{m+1} a_{ik} \beta_i^k \geq \phi_i^{LP}(x_i),
\]

where we used for the last inequality that the vector \( \sum_{k=1}^{m+1} a_{ik} \beta_i^k \) is feasible for the LP corresponding to \( \phi_i^{LP}(x_i) \). To see this, observe

\[
\chi_i \sum_{k=1}^{m+1} a_{ik} \beta_i^k = \sum_{k=1}^{m+1} a_{ik} \chi_i \beta_i^k = \sum_{k=1}^{m+1} a_{ik} y_i^k = x_i, \text{ and } 1^T \sum_{k=1}^{m+1} a_{ik} \beta_i^k = \sum_{k=1}^{m+1} a_{ik} 1^T \beta_i^k = \sum_{k=1}^{m+1} a_{ik} = 1.
\]

\( \square \)

### 3.2 The Master LP and its Dual

Now we will model \( p_{\min}(u) \) for \( G_{\min-\text{conv}}(u) \) for any of the two previous game classes via the following LP in the variables \( a_i, i \in N \). We assume from now on that \( g_i, i \in N \) are linear, that is, \( g_i(x_i) = G_i x_i \), where \( G_i \in \mathbb{R}^{m \times m} \) is an \( m \times m \) matrix. The capacity vector \( u \in \mathbb{R}^m \) is enforceable for \( G_{\min}(u) \).

\[
\begin{align*}
\min & \sum_{i \in N} \pi_i^T a_i \quad & \text{(LP}_{\min}(u)) \\
\ell(a) & \leq u, \\
a_i & \in \Lambda_i \text{ for all } i \in N, 
\end{align*}
\]

where \( \ell(a) := \sum_{i \in N} \sum_{k \in [1, \ldots, k_i]} a_{ik} g_i(x_i^k) \) and \( \Lambda_i := \{a_i \in \mathbb{R}^{k_i} | 1^T a_i = 1\}, i \in N \).

**Theorem 3.11.** Let \( G_{\min}(u) \) be a game for which Assumptions 3.1 and 3.7 hold and assume that \( g_i, i \in N \) are linear. Then, the following statements are equivalent for the respective \( \text{LP}_{\min}(u) \).

1. The capacity vector \( u \in \mathbb{R}^m \) is enforceable for \( G_{\min}(u) \).
2. \( \text{LP}^{\text{min}}(u) \) admits an integral optimal solution \( \alpha^* \) for which (9) is tight. Let \( G^{\text{min}}(u) \) be a game for which Assumptions 3.1 and 3.9 hold true and assume \( g_i, i \in N \) to be linear. Then, the following statements are equivalent for the respective \( \text{LP}^{\text{min}}(u) \).

3. The capacity vector \( u \in \mathbb{R}^m \) is enforceable for \( G^{\text{min}}(u) \).

4. \( \text{LP}^{\text{min}}(u) \) admits an optimal solution \( \alpha^* \) with \( \sum_{j=1}^{k_i} \alpha_{ij}^* x_j^i \in X_i, i \in N \) with (9) being tight. Moreover, the equivalence remains true by replacing the term “enforceable” with “weakly enforceable by market prices” and removing the condition \( \ell(\alpha^*) = u \) in Statements 2. and 4.

**Proof.** By Lemmata 3.8 and 3.10 and the linearity of \( g_i, i \in N \), the \( \text{LP}^{\text{min}}(u) \) is a correct formulation of Problem \( p^{\text{min}}(u) \) for the respective convexified games \( G^{\text{min-conv}}(u) \). Hence, the result follows from Theorem 3.5.

**Remark 3.12.** The equivalence between (1) and (2) remains true even for concave \( g_i, i \in N \), since in this case \( \text{LP}^{\text{min}}(u) \) is a relaxation of \( p^{\text{min}}(u) \) w.r.t. the convexified game \( G^{\text{min-conv}}(u) \) with the same dual.

**Remark 3.13.** The game \( G^{\text{min-conv}}(u) \) (assuming Assumption 3.7) can be interpreted as the mixed extension of \( G^{\text{min}}(u) \). By Theorem 2.3, this implies that for any finite strategic game of type \( G^{\text{min}}(u) \), any minimal vector \( u \) can be enforced in mixed strategies for the game \( G^{\text{min-conv}}(u) \) (the LP has zero duality gap). Theorem 3.11 implies that whenever \( \text{LP}^{\text{min}}(u) \) admits integral optimal solutions, \( u \) is also enforceable in pure strategies.

\( \text{LP}^{\text{min}}(u) \) may in general involve (exponentially) many variables \( \alpha_{ij}, i \in N \) depending on the number \( k := \sum_{i \in N} k_i \). A common approach is to dualize \( \text{LP}^{\text{min}}(u) \) to yield an LP with less variables at the cost of obtaining (exponentially) many constraints. In the following we dualize the primal problem in the form \( \min \{ - \sum_{i \in N} \pi^T_i \alpha_i | \ell(\alpha) \leq u, \alpha_i \in \Lambda_i, i \in N \} \). The following steps are reminiscent to the standard dual LP of the Walrasian configuration LP (see e.g., Blumrosen and Nisan [64, §11.3.1]).

\[
\begin{align*}
\min & \sum_{i \in N} \mu_i + \sum_{j \in E} \lambda_j u_j, \\
\sum_{j \in E} g_{ij}(x^k_i) \lambda_j + \mu_i & \geq -\pi_{ik} \text{ for all } i \in N, k = 1, \ldots, k_i \\
\mu_i & \in \mathbb{R}, i \in N, \lambda_j \geq 0, j \in E.
\end{align*}
\]

\( \text{DP}^{\text{min}}(u) \)

Note that \( \mu_i, i \in N \) is not sign-constrained as it is the dual variable to \( \sum_{k} \alpha_{ik} = 1, i \in N \). Moreover, recall that \( g_{ij}(x^k_i) \in \mathbb{R} \) are just parameters in \( \text{DP}^{\text{min}}(u) \). The dual has \( n + m \) many variables but exponentially many constraints, but, if we have a polynomial time separation oracle, we can use the ellipsoid method to obtain a polynomial time algorithm (cf. Groetschel et al. [40]). A standard way to obtain such an oracle is to assume an efficient demand oracle.

**Definition 3.14.** A demand oracle for player \( i \in N \) gets as input prices \( \lambda \in \mathbb{R}^m_+ \) and outputs a cost minimizing vector \( x_i \in X_i \), that is,

\[
x_i(\lambda) \in \arg \min \{ \pi_i(u, x_i) + \lambda^T g_i(x_i)| x_i \in X_i \}.
\]
We obtain the following result for polynomial time computable demand oracles. Let us remark here that we assume that there is a succinct representation of the game \( G^{\min}(u) \) and hence of the \( L^{\min}(u) \).

**Theorem 3.15.** Let \( G^{\min}(u) \) be a game for which the assumptions of Theorem 3.11 are satisfied. If for all \( \lambda \in \mathbb{R}^n \) and \( i \in N \), the demand oracle \( x_i(\lambda) \) can be computed in polynomial time, then, \( L^{\min}(u) \) can be solved in polynomial time.

**Proof.** In order to use the ellipsoid method, we need to check whether we get a polynomial time separation oracle for the constraints:

\[
\sum_{j \in E} g_{ij}(x^k_i) \lambda_j + \mu_i \geq -\pi_{ik}, i \in N, k = 1, \ldots, k_i.
\]

With the demand oracle we can determine the value \( \pi_i^*(\lambda) := \pi_i(u, x_i(\lambda)) + \lambda^T g_i(x_i(\lambda)) \). Now, if \( \pi_i^*(\lambda) \geq -\mu_i \) for all \( i \in N \), the current point \((\mu, \lambda)\) is feasible. Otherwise, suppose \( \pi_i^*(\lambda) < -\mu_i \). If \( G^{\min}(u) \) is a game with finite point sets \( X_i \), we get of course \( x_i(\lambda) = x_i^k \) for some \( k \in \{1, \ldots, k_i\} \) and, hence,

\[
\pi_i^*(\lambda) = \pi_i(u, x_i^k) + \sum_{j \in E} g_{ij}(x_i^k) \lambda_j = \pi_{ik} + \sum_{j \in E} g_{ij}(x_i^k) \lambda_j < -\mu_i,
\]

represents a violated inequality.

If \( G^{\min}(u) \) is a concave game, then, by the linearity of \( g_i \), the function \( x_i \mapsto \sum_{j \in E} g_{ij}(x_i) \lambda_j + \pi_i(u, x_i) \) is concave over \( \text{conv}(X_i) \) as well and attains its minimum over \( \text{conv}(X_i) \) at an extreme point of \( \text{conv}(X_i) \), hence, in \( X_i \).

**Remark 3.16.** All results in this section carry directly over to the case of maximization problems. In this case, a convex envelope becomes a concave envelope, the concave functions \( g_i, i \in N \) become convex and the assumption of having a concave extension changes to a convex extension.

### 3.3 Consequences and Impossibility Results

The characterization result in Theorem 3.11 together with the assumption of a polynomial time demand oracle can be used to establish non-existence results based on complexity-theoretic assumptions like \( P \neq NP \). If the master problem \( p^{\min}(u) \) (which is also called the welfare maximization problem in some applications) is NP-hard but there is a polynomial demand oracle, then, assuming \( P \neq NP \), the guaranteed (weak) enforceability (with market prices) of \( u \) is ruled out since otherwise, we can just compute an integral optimal solution of \( L^{\min}(u) \) in polynomial time (by solving the dual \( D^{\min}(u) \)) which corresponds to an optimal solution of the master problem. This approach has been pioneered by Talgam-Cohen and Roughgarden \cite{72} for the case of pricing equilibria for Walrasian market settings (and generalizations thereof).

### 4 Integral Problems and Compact Linear Relaxations

We assume in the following that for every \( i \in N \), the set \( X_i \) is of the form \( X_i = \{ x_i \in \mathbb{Z}^m | A_i x_i \geq b_i \} \), where \( A_i \) is a rational \( k_i \times m \) matrix and \( b_i \in \mathbb{Q}^{k_i} \) is a rational vector. Thus, the combined set is given by \( X := \{ x \in \mathbb{Z}^{n \cdot m} | A_i x_i \geq b_i, i \in N \} \). We further assume linear resource consumption, that is, \( g_j(x_i) = x_i \) for all \( i \in N \). This assumption implies \( \ell_j(x) = \sum_{i \in N} x_{ij} \) for all \( j \in E \). The private cost function of a player is assumed to be quasi-separable over the resources and depends only on the aggregated load vector and the own load on the resource: \( \pi_i(u, x_i) = \sum_{j \in E} \pi_{ij}(u) \cdot x_{ij} \), where
for a definition) can be reformulated as an integer linear optimization problem. If the polyhedron \( R(X) := \{ x \in \mathbb{R}^{n \times m} | A_i x_i \geq b_i, i \in N \} \) is integral\(^6\), then, it follows directly that \( P_{\text{min}}(u) \) has zero duality gap.

4.1 Aggregation Polytopes and Total Dual Integrality

A powerful tool to recognize integrality of polyhedra is the notion of total-dual-integrality (TDI) of linear systems (see Edmonds and Giles [30]). A rational system of the form \( A z \geq b \) with \( A \in \mathbb{Q}^{r \times m} \) and \( b \in \mathbb{Q}^r \) is TDI, if for every integral \( c \in \mathbb{Z}^m \), the dual of \( \min \{ c^T z | A z \geq b \} \) given by \( \max \{ z^T b | A^T z = c, z \geq 0 \} \) has an integral optimal solution (if the problem admits a finite optimal solution). It is known that for TDI systems, the corresponding polyhedron is integral. A system \( A z \geq b \) is box-TDI, if the system \( A z \geq b, w \leq z \leq u \) is TDI for all rational \( w, u \). A polytope is called box-TDI, if it can be described by a box-TDI system.

Now we assume that for all \( i \in N \), the matrices \( A_i \) are equal to some matrix \( A \in \mathbb{Q}^{r \times m} \). We further assume that the cost functions are linear and homogenous, that is, they have the form \( \pi_i(u, x_i) = \sum_{j \in E} \pi_{ij}(u) \cdot x_{ij} \), where \( \pi_i : \mathbb{Z}^m \rightarrow \mathbb{R} \) denotes the resource-specific per-unit cost on resource \( j \) mapping a load vector \( u \) to the reals. Instead of taking the Cartesian product of the LP-relaxations \( R(X_i) \), we define an aggregation polytope:

\[
P_N = \{ z \in \mathbb{R}^m | A z \geq \sum_{i \in N} b_i, z \geq 0 \}.
\]

This aggregated polytope seems only useful, if it is box-TDI and any solution \( z \) can be decomposed into feasible strategies. This latter property is called the integer decomposition property (IDP). Formally, \( P_N \) has the IDP, if any integral optimal solution \( z \in P_N \) can be decomposed into feasible integral vectors, that is, \( z = \sum_{i \in N} z_i \) with \( z_i \in X_i \) for all \( i \in N \). We remark that Kleer and Schäfer [57] showed - in a different context - that polytopal congestion games (see Section 6 for a definition) with box-integral and IDP aggregation polytopes have nice properties in terms of equilibrium computation and equilibrium welfare properties. Now we have everything together to state the following result.

**Theorem 4.1.** Assume that \( P_N \) is box-TDI and satisfies IDP. Then for homogeneous linear cost functions \( \pi_j, j \in E \), every \( u \in \mathbb{Z}^m \) for which \( P_N \cap \{ y | y \leq u \} \neq \emptyset \) is weakly enforceable by market prices. Moreover under the same condition, every minimal \( u \) w.r.t. \( P_N \) is enforceable.

4.2 Integral Polymatroid Games

We consider now a class of games based on polymatroids which rely on submodular functions defining structured capacity constraints on subsets of resources. An integral set function \( f : 2^E \rightarrow \mathbb{Z} \) is submodular if \( f(U) + f(V) \geq f(U \cup V) + f(U \cap V) \) for all \( U, V \in 2^E \); \( f \) is monotone if \( f(U) \leq f(V) \) for all \( U \subseteq V \subseteq E \); and \( f \) is normalized if \( f(\emptyset) = 0 \). We call an integral, submodular, monotone, and normalized function \( f : 2^E \rightarrow \mathbb{Z} \) an integral polymatroid rank function.

Suppose there is a finite set \( N = \{1, \ldots, n\} \) of players so that each player \( i \) is associated with an integral polymatroid rank function \( f_i : 2^E \rightarrow \mathbb{Z} \) that defines an integral polymatroid \( P_i \) with base polymatroid \( B_{f_i} \). A strategy of player \( i \) in \( N \) is to choose a vector \( x_i = (x_{ij})_{j \in E} \in B_{f_i} \), i.e., player \( i \) chooses an integral resource consumption \( x_{ij} \in \mathbb{Z} \) for each resource \( e \) such that \( f_i(E) \) units are

\(^6\)A polyhedron \( P \subset \mathbb{R}^r \) is integral, if all its vertices are integral.
distributed over the resources and for each $U \subseteq E$ not more than $f_i(U)$ units are distributed over the resources contained in $U$. Formally, the set $X_i$ of feasible strategies of player $i$ is defined as

$$X_i = B_{f_i} = \left\{ x_i \in \mathbb{Z}^m \mid x_i(U) \leq f_i(U) \text{ for all } U \subseteq E, \ x_i(E) = f_i(E) \right\},$$

where, for a set $U \subseteq E$, we write $x_i(U) = \sum_{j \in U} x_{ij}$. We show that the LP-relaxation admits integral optimal solutions - by reformulating $P_{\text{min}}(u)$ as a polymatroid intersection problem whose underlying intersection polytope is known to admit integral optimal solutions.\(^7\) The proof can be found in the appendix.

**Theorem 4.2.** For polymatroid games, every $u \in \mathbb{Z}^m$ for which $P_{\text{min}}(u)$ admits a finite optimal solution is weakly enforceable with market prices. If $u \in \mathbb{Z}^m$ is minimal for $X$, then $u$ is enforceable.

For the maximization variant, we refer to Appendix C.

## 5 Monotone Aggregative Games

In the previous sections, we assumed that the private cost function $\pi_i(u, x_i)$ of every player $i \in N$ is parameterized in $u$ and, otherwise, only depends on $x_i \in X_i$. This separability condition allowed to decompose the Lagrangian of Problem $P_{\text{min}}(u)$ leading to the subsequent characterizations. Several games of interest, however, do not fulfill this assumption. A prime example are atomic congestion games, where the private cost depends on the aggregated load vector $\ell(x)$ of all players and changes, if player $i$ changes her strategy to some $y_i \neq x_i \in X_i$. We introduce a class of **monotone aggregative games** (mag), where the cost function of a player $i \in N$ is allowed to depend on both, $x_i \in X_i$ and the aggregate $\ell(x), x \in X$. Throughout this section, we assume that $X_i \subseteq \mathbb{R}^m_+$ and $g_i: X_i \to \mathbb{R}^m_+$ for all $i \in N$. For $x_i \in X_i$, we use the notation $E(x_i) := \{ j \in E \mid x_{ij} > 0 \}$.

**Assumption 5.1.** Let $X_i \subseteq \mathbb{R}^m_+$ for all $i \in N$. We assume that cost/utility functions depend on the vector of loads and on the own strategy vector only – this structure is known in the literature as aggregative games, see Harks and Klimm [44], Jensen [52] and Paccagnan et al [66].

1. For minimization games $G_{\text{min-mag}}$, the total cost of a player $i \in N$ under strategy distribution $x \in X$ is defined by a function $\text{cost}_{\text{mag}}^i: X \to \mathbb{R}$, which satisfies

$$\text{cost}_{\text{mag}}^i(x) := \pi_{\text{mag}}^i(\ell(x), x_i) \text{ for all } x \in X,$$

for some function $\pi_{\text{mag}}^i: \mathbb{R}^m_+ \times X_i \to \mathbb{R}$. For maximization games $G_{\text{max-mag}}$, we denote the utility function for $i \in N$ by $\text{utility}_{\text{mag}}^i: X \to \mathbb{R}$ and we assume that it satisfies

$$\text{utility}_{\text{mag}}^i(x) := v_{\text{mag}}^i(\ell(x), x_i) \text{ for all } x \in X,$$

for some function $v_{\text{mag}}^i: \mathbb{R}^m_+ \times X_i \to \mathbb{R}$.

2. We further assume that the indirect cost/utility functions exhibit negative externalities in the load vector, that is, the following monotonicity condition holds:

$$\pi_{\text{mag}}^i(u, x_i) \leq \pi_{\text{mag}}^i(w, x_i) \text{ for all } u \leq w, u, w \in \mathbb{R}^m_+ \quad \text{(11)}$$

$$v_{\text{mag}}^i(u, x_i) \geq v_{\text{mag}}^i(w, x_i) \text{ for all } u \leq w, u, w \in \mathbb{R}^m_+.$$

---

\(^7\)The intersection of two polymatroid base polytopes, however, need not be a polymatroid.
We call a game satisfying
\[
\pi_i^{\text{mag}}(u, x_i) = \pi_i^{\text{mag}}(w, x_i) \quad \text{for all } u, w \in \mathbb{R}_+^m \quad \text{with } u_j = w_j \quad \text{for all } j \in E(x_i), x_i \in X_i, i \in N
\]
\[
v_i^{\text{mag}}(u, x_i) = v_i^{\text{mag}}(w, x_i) \quad \text{for all } u, w \in \mathbb{R}_+^m \quad \text{with } u_j = w_j \quad \text{for all } j \in E(x_i), x_i \in X_i, i \in N.
\]

4. The strategy spaces \(X_i \subseteq \mathbb{R}_+^m, i \in N\) exhibit an overlapping structure, that is, for all \(i \in N\):
\[
g_{ij}(x_i) = g_{ij}(y_i) \quad \text{for all } j \in E(x_i) \cap E(y_i), x_i, y_i \in X_i.
\]

We call a game satisfying 1.-4. a monotone aggregative game.

Condition 4. implies that for any resource \(j \in E\) with \(j \in E(x_i) \cap E(y_i)\) for at least two \(x_i \neq y_i \in X_i\), the resource consumption level of player \(i\) on resource \(j\) is fixed to some \(x_{ij} \geq 0\) for every strategy \(x_i \in X_i\) with \(j \in E(x_i)\). On every other resource \(j \in E\), however, the resource usage level may be arbitrary. We need to redefine the concept of enforceability for this more general class of games.

**Definition 5.2 (Enforceability).** A vector \(u \in \mathbb{R}^m\) is enforceable by prices \(\lambda \in \mathbb{R}^m_+\), if there is \(x^* \in X\) satisfying 1. and 2. for minimization games \(G^{\text{min-mag}}\) or 1. and 3. for maximization games \(G^{\text{max-mag}}\):

1. \(\ell_j(x^*) = u_j\) for all \(j \in E\).

2. Minimization: \(x^*_i \in \arg \min_{x_i \in X_i} \{\pi_i(\ell(x_i, x^*_i), x_i) + \lambda^T g_i(x_i)\} \) for all \(i \in N\).

3. Maximization: \(x^*_i \in \arg \max_{x_i \in X_i} \{\pi_i(\ell(x_i, x^*_i), x_i) - \lambda^T g_i(x_i)\} \) for all \(i \in N\).

We obtain the following result for monotone aggregative games.

**Theorem 5.3.** Let \(G^{\text{min-mag}} = (N, X, (\pi_i^{\text{mag}})_{i \in N})\) be a monotone aggregative minimization game and let \(G^{\text{min}}(u) = (N, X, (\pi_i(u))_{i \in N})\) be an associated game with \(\pi_i(u, x_i) := \pi_i^{\text{mag}}(u, x_i), x_i \in X_i, i \in N\). Denote the same assumption for a maximization game by exchanging \(\min\) with \(\max\) and \(\pi\) with \(v\). Then, the following holds true.

1. If \((x^*, \lambda) \in X \times \mathbb{R}^m_+\) enforces \(u\) for \(G^{\text{min}}(u)\), then \((x^*, \lambda)\) also enforces \(u\) for \(G^{\text{min-mag}}\).

2. If \((x^*, \lambda) \in X \times \mathbb{R}^m_+\) enforces \(u\) for \(G^{\text{max}}(u)\), then \((x^*, \lambda)\) also enforces \(u\) for \(G^{\text{max-mag}}\).

**Proof.** We only prove 1. as 2. follows by the same arguments. For 1., we only need to verify the Nash equilibrium conditions of a given tuple \((x^*, \lambda) \in X \times \mathbb{R}^m_+\) for the game \(G^{\text{min-mag}}\). We obtain for any \(y_i \in X_i, i \in N:\)

\[
\pi_i^{\text{mag}}(\ell(x^*, x^*_i) + \lambda^T g_i(x^*_i)) = \pi_i(u, x^*_i) + \lambda^T g_i(x^*_i)
\]
\[
\leq \pi_i(u, y_i) + \lambda^T g_i(y_i)
\]
\[
= \pi_i^{\text{mag}}(w^1, y_i) + \lambda^T g_i(y_i) \quad \text{for } w^1_j := \begin{cases} u_j, & \text{for } j \in E(y_i) \\ 0, & \text{otherwise} \end{cases}
\]
\[
\leq \pi_i^{\text{mag}}(w^2, y_i) + \lambda^T g_i(y_i) \quad \text{for } w^2_j := \begin{cases} u_j + g_{ij}(y_i), & \text{for } j \in E(y_i) \setminus E(x^*_i) \\ w^1_j, & \text{otherwise} \end{cases}
\]
\[
= \pi_i^{\text{mag}}(\ell((x^*_{-i}, y_i)), y_i) + \lambda^T g_i(y_i),
\]
where (14) follows, because \(u\) is enforced by \((x^*, \lambda)\) for \(G^{\text{min}}(u)\). Equality (15) follows from the independence of irrelevant choice condition (12). Inequality (16) follows from the monotonicity condition (11). Equality (17) follows from the overlapping condition (13).
6 Applications in Congestion Games

We now demonstrate the applicability of our framework by deriving new existence results of tolls enforcing certain load vectors in congestion games. Moreover, we show how several known results in the literature follow directly.

6.1 Nonatomic Congestion Games

We first present results for the case that the strategy spaces of players are convex subsets of $\mathbb{R}^n$. We are given a directed graph $G = (V, E)$ and a set of populations $N := \{1, \ldots, n\}$, where each population $i \in N$ has a demand $d_i > 0$ that has to be routed from a source $s_i \in V$ to a destination $t_i \in V$. In the nonatomic model, the demand interval $[0, d_i]$ represents a continuum of infinitesimally small agents each acting independently choosing a cost minimal $s_i, t_i$ path. There are continuous cost functions $c_{ij} : \mathbb{R}^n \to \mathbb{R}_+$, $i \in N, j \in E$ which may depend on the population identity and also on the aggregate load vector – thus allowing for modeling non-separable latency functions. A flow for population $i \in N$ is a nonnegative vector $x_i \in \mathbb{R}_+^{|E|}$ that lives in the flow polytope:

$$X_i = \left\{ x_i \in \mathbb{R}_+^{|E|} \left| \sum_{j \in \delta^+(v)} x_{ij} - \sum_{j \in \delta^-(v)} x_{ij} = \gamma_i(v), \text{ for all } v \in V \right\},$$

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering $v$, and $\gamma_i(v) = d_i$, if $v = s_i$, $\gamma_i(v) = -d_i$, if $v = t_i$, and $\gamma_i(v) = 0$, otherwise. We assume that every $t_i$ is reachable in $G$ from $s_i$ for all $i \in N$, thus, $X_i \neq \emptyset$ for all $i \in N$. Given a combined flow $x \in X$, the cost of a path $P \in \mathcal{P}_i$, where $\mathcal{P}_i$ denotes the set of simple $s_i, t_i$ paths in $G$, is defined as

$$c_{i,P}(\ell(x)) := \sum_{j \in P} c_{ij}(\ell(x)).$$

A Wardrop equilibrium $x$ with path-decomposition $(x_i, P)_{i \in N, P \in \mathcal{P}_i}$ is defined as follows:

$$c_{i,P}(\ell(x)) \leq c_{i,Q}(\ell(x)) \text{ for all } P, Q \in \mathcal{P}_i \text{ with } x_{i,P} > 0.$$

The interpretation here is that any agent is traveling along a shortest path given the overall load vector $\ell(x)$. One can reformulate the Wardrop equilibrium conditions using load vectors $u$ stating that - given the load vector of a Wardrop equilibrium - every agent is traveling along a shortest path.

Lemma 6.1 (Dafermos [25, 26]). A strategy distribution $x^* \in X$ with overall load vector $u := \ell(x^*)$ is a Wardrop equilibrium if and only if

$$x_i^* \in \arg\min_{x_i \in X_i} \left\{ \sum_{j \in E} c_{ij}(u)x_{ij} \left| \right. x_i \in X_i \right\} \text{ for all } i \in N.$$

With this characterization, the model fits in our framework and we can apply our general existence result.

Corollary 6.2 (Yang and Huang [85], Fleischer et al. [33], Karakostas and Kolliopoulos [53]). Every minimal capacity vector $u$ is enforceable.
and the fact that Slater’s constraint qualification condition is satisfied, the problem has zero duality gap. Note that Problem $\text{P}_{\min}(u)$ has the following structure:

$$\min \left\{ \sum_{i \in N} \sum_{j \in E} \pi_{ij}(u)x_{ij} \mid \ell(x) \leq u, x \in X \right\},$$

where we assume linear resource consumption $g_i(x_i) = x_i$ for all $x_i \in X_i, i \in N$.

**Homogeneous Cost Functions.** We first assume that cost functions are homogeneous, thus, the private cost of player $i \in N$ has the form $\pi_i(\ell(x), x_i) := \sum_{j \in E} c_j(\ell_j(x))x_{ij}$. We use in the following the more general model of so-called polytopal congestion games introduced by Del Pia et al. [69] and further studied by Kleer and Schäfer [57]. In this model, the strategy spaces are defined as

$$X_i := P_i \cap \{0,1\}^m, i \in N,$$

where $P_i$ are polyhedrons of the form $P_i = \{x_i \in \mathbb{R}_+^m \mid Ax_i \geq b_i\}$ for some rational matrix $A$ and integral vector $b_i$ of appropriate dimension. We remark here that all characterizations regarding box-TDI
and IPD also work for systems $Ax_i = b_i$ or $Ax_i \leq b_i$, assuming that $A$ and $b_i$ carry the desired structure (see for instance Kleer and Schäfer [57, Prop. 2.1]). For homogeneous cost functions and polytopal strategy spaces, we can use an LP formulation of Problem $P_{\text{min}}(u)$ using the aggregation polyhedron $P_N$ as defined in Section 4.1. Thus, Theorem 4.1 implies the following result.

**Corollary 6.4.** Let $(N, X, (\pi_i)_{i \in N})$ be a congestion game with homogeneous nondecreasing cost functions and polytopal strategy spaces with aggregation polyhedron $P_N$. Let $u$ be minimal for $X$. If $P_N$ is box-TDI and satisfies IDP, $u$ is enforceable. In particular, box-TDI and IDP holds for:

1. Network games with a common source and multiple sinks,
2. $r$-arborescence congestion games (see Harks et al. [43] and Kleer and Schäfer [57] for a definition)
3. Intersection of strongly base-orderable matroids (see Kleer and Schäfer [57] for a definition)
4. symmetric totally-unimodular games (see Del Pia et al. [69] for a definition) including matching games,
5. Asymmetric matroid games (see Ackermann et al. [1]).

Note that by box-integrality and IDP of $P_N$, any $u \in \mathbb{Z}_+^m$ that minimizes a strictly component-wise monotonically increasing function is enforceable. In particular, for monotonically increasing functions $c_j(\ell_j(x)), j \in E$, a vector $u \in \mathbb{Z}_+^m$ corresponding to a minimum cost solution, that is, $u = \ell(x^*)$ for some

$$x^* \in \arg\min \left\{ \sum_{j \in N} c_j(\ell_j(x))\ell_j(x) \mid x \in X \right\}$$

is enforceable. The congestion vector $u$ minimizing the (weakly convex) social cost can be computed in polynomial time (see Del Pia et al. [69] and Kleer and Schäfer [57]) and additionally the space of enforcing prices can be described by a compact linear formulation. This allows for optimizing arbitrary linear objective functions (like maximum or minimum revenue) over the price/allocation space. To the best of our knowledge, the only previous results for the existence of (optimal) tolls are due to Marden et al. [59], Fotakis and Spirakis [35] and Fotakis et al. [34]. Marden et al. [59] proved that marginal cost tolls enforce the minimum cost solution by charging the difference between the social cost and the cost of Rosenthal’s potential. With this approach, there is no control on the magnitude of price and no structure for optimizing secondary objectives over prices. In addition, for other (non-optimal) vectors $u$, this approach does not work. Fotakis and Spirakis [35] proved that any acyclic integral flow in an $s,t$ digraph can be enforced. It is not hard to see that the notion of minimality of $u$ for $X$ exactly corresponds to the set of acyclic integral $s,t$ flows. Fotakis et al. [34] generalized this result to single source multi-sink network games allowing even for heterogeneous players.

**Congestion Games with Player-Specific Cost Functions.** Now we turn to the general model of player-specific non-decreasing separable cost functions $c_{ij}(\ell_{ij}(x)), i \in N, j \in E$ and consider the case of integral polymatroid congestion games, see Harks et al. [46]. In this model, for every $i \in N$, the strategy space $X_i$ is the integral base polyhedron $B_{f_i} \subset [0,1]^m$ of a polymatroid $P_{f_i}$. Theorem 5.3 together with Theorem 4.2 imply the following result.

**Corollary 6.5.** Consider an integral polymatroid congestion game with $X_i = B_{f_i} \subset [0,1]^m, i \in N$ and nondecreasing player-specific separable cost functions. Let $u \in \mathbb{Z}^m$ be minimal for $X$. Then, $u$ is enforceable.
To the best of our knowledge, this is the first existence result of enforceable tolls in congestion games with player-specific cost functions.

One might be tempted to think that - as in Corollary 6.4 - one can use the aggregation polytope $P_N$ and also show integrality of optimal solutions to $LP_{\min}^u$ for natural classes such as $s,t$ network games with player-specific cost functions. Note that this approach does not work, because the objective function depends on the player’s identities on the elements and not on an aggregate. One can show that, unless $P = NP$, even for $s,t$ network games with homogeneous cost functions but heterogeneous players, $LP_{\min}^u$ is not integral in general. The proof is a straightforward reduction from the disjoint path problem and omitted here.

7 Application to Market Equilibria

We now consider market games and first present a classical model of Walrasian market equilibria with indivisible items. Then, we study a class of valuations for multi-item settings that allows for some degree of exterannualities of allocations.

7.1 Linear Pricing without Externalities

We are given a finite set $E = \{1, \ldots, m\}$ of items and there is a finite set of players $N = \{1, \ldots, n\}$ interested in buying some of the items. For every subset $S \subseteq E$ of items, player $i$ derives value $w_i(S) \in \mathbb{R}$ giving rise to a valuation function $w_i : 2^m \rightarrow \mathbb{R}, i \in N$, where $2^m$ represents the set of all subsets of $E$. The seller wants to determine a price vector $\lambda \in \mathbb{R}^m$ so that every player $i \in N$ gets a subset $S_i$ of items that are maximizers of her quasi-linear utility, that is, $S_i \in \arg\max_{S \subseteq E} [w_i(S) - \sum_{j \in S} \lambda_j]$ and every item is sold to at most one player. Such a tuple $((S_i)_{i \in N}, \lambda)$ is known as a competitive Walrasian equilibrium. A frequently assumed condition on valuations is normalization and monotonicity as stated below.

1. $w_i(\emptyset) = 0$ for all $i \in N$.
2. $w_i(S) \leq w_i(T)$ for any $S \subseteq T \subseteq E$ and all $i \in N$.

We can derive an equivalent game $G_{\max}(1)$ as follows. Let $X_i = \{0, 1\}^m, i \in N$ be the set of incidence vectors of the set $E$. The valuation function is given by $v_i : \mathbb{R}^m \times X_i \rightarrow \mathbb{R}, (1, x_i) \mapsto w_i(E(x_i))$, where $E(x_i) := \{j \in E : x_{ij} = 1\}$. The resource consumption function are linear $g_i(x_i) = x_i, i \in N$. We get the following characterization.

**Lemma 7.1.** The tuple $((S_i)_{i \in N}, \lambda)$ is a competitive Walrasian equilibrium if and only if the tuple $(x^*, \lambda)$ with $E(x_i) = S_i, i \in N$ weakly enforces 1 with market prices $\lambda$ for $G_{\max}(1)$.

**Proof.** Note that the condition $\ell(x^*) \leq 1$ ensures that every item goes to at most one player. Thus, we get

$$x^*_i \in \arg\max \{v_i(1, x_i) - \lambda^T x_i | x_i \in X_i\}$$

$$\Leftrightarrow x^*_i \in \arg\max \{w_i(E(x_i)) - \lambda^T x_i | x_i \in X_i\} \quad \text{(By definition of } v_i)$$

$$\Leftrightarrow E(x^*_i) \in \arg\max \left\{w_i(E(x_i)) - \sum_{e \in E(x_i)} \lambda_e \big| E(x_i) \subseteq E\right\}$$

$$\Leftrightarrow S_i \in \arg\max \left\{w_i(S_i) - \sum_{e \in S_i} \lambda_e \big| S_i \subseteq E\right\}.$$

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With this analogy to $G^{\max}(1)$ we can analyze Problem $P^{\max}(u)$ in more detail:

$$\max \left\{ \sum_{i \in N} v_i(1, x_i) \left| x_i \in \{0, 1\}^m, i \in N, \ell(x) \leq 1 \right. \right\}.$$ 

Clearly, $X_i, i \in N$ consists of finitely many ($k_i = 2^m$) points and thus we can apply Theorem 3.11 to obtain a full characterization of the existence of Walras market equilibria (which leads precisely to the characterization of Bikchandani and Mamer [11]).

**Corollary 7.2** (Bikchandani and Mamer [11]). Competitive Walrasian equilibria exist if and only if the following LP admits integral optimal solutions:

$$\max \sum_{i \in N} v_i^\top \alpha_i, \ell(\alpha) \leq 1, \alpha_i \in \Lambda_i \text{ for all } i \in N, \quad (LP^{\max}(1))$$

where $v_i := (v_i(1, x_i))_{x_i \in X_i}$ and $\ell(\alpha) := \sum_{i \in N} \sum_{j \in \{1, \ldots, k_i\}} \alpha_{ij}$.

A fundamental property of valuations $w_i, i \in N$ is the so-called *gross-substitutes (GS)* condition, requiring that whenever the prices of some items increase and the prices of other items remain constant, the agent’s optimal demand for the items whose price remain constant only increases. Let us recall an existence theorem by Kelso and Crawford [56].

**Theorem 7.3** (Kelso and Crawford [56]). For GS valuations, there exists a competitive Walrasian equilibrium.

We can use Theorem A.1 by showing that problem $P^{\max}(u)$ has zero duality gap for the supply vector $u = (1, \ldots, 1)^\top \in \mathbb{R}^m$. One can show this property by using insights from discrete convexity and the special form of $M^\natural$-concave functions, see Murota [62], Sec. 11.3 for a definition and an exhaustive overview of the topic. Let us now recall a characterization of Fujishige and Yang [38].

**Theorem 7.4** (Fujishige and Yang [38]). A normalized and monotone valuation function is GS if and only if it is $M^\natural$-concave.

It is known that if all $v_i, i \in N$ are $M^\natural$-concave, so is $\sum_{i \in N} v_i$. Altogether, problem $P^{\max}(u)$ is a very special discrete optimization problem with an $M^\natural$-concave function over $\{0, 1\}^n \times m$ involving the special constraint $\ell(x) \leq 1$ which constitutes a *laminar system or hierarchy*, see Yokote [87] and Budish et al. [14] for further details. Yokote [87] proved that such a problem has zero duality gap. Thus, with the zero duality gap property and the monotonicity of $v(x)$, Theorem A.1 implies Theorem 7.3. We remark here that Yokote [87, Sec. 4] described in his paper the connection of his strong duality theorem (in the realm of discrete convexity) with the existence of market equilibria for GS valuations.

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8 Gul and Stachetti [42] even showed that in some sense GS is the maximal condition on valuations for which equilibria exist.
7.2 Nonlinear Package Pricing

The auction model so far assumes a single seller that uses linear anonymous pricing functions, that is, every item \( j \) comes with a price \( \lambda_j \geq 0 \) and the price of every subset of items \( S \) is linear in item prices, that is, \( \lambda(S) = \sum_{j \in S} \lambda_j \). Bikhchandani and Ostroy [12] studied a model in which packages of items are sold and the sellers may use nonlinear (non-anonymous) pricing functions (see also Parkes and Ungar [68]). Pricing functions assign prices to packages (instead of prices for individual items) that may depend on the package type only, or on the package and buyer (or seller) identity, or on the identity of all parties, i.e., the package, buyer and seller (see Bikhchandani and Ostroy [12]). A competitive equilibrium arises, if there are package prices and package allocations so that the allocation maximizes the overall quasi-linear utility of every buyer and seller, respectively.

One can incorporate this model into the current framework by defining an appropriate game \( G^{\text{max}}(0) \) as follows. The resource set \( E \) is constructed according to the qualitatively different packages traded on the market. Packages of the same type correspond to a resource (yielding a price function per type) but packages with dependencies on the buyers/sellers would correspond to individual resources.

Buyers \( b \in B \) have as strategy space \( X_b \subseteq \mathbb{Z}_+^m \), where \( x_{bj} \) is the number of packages of type \( j \) (or from seller \( s \) if the seller identity matters) while the strategy space of every seller \( s \in A \) is given by some set \( X_s \subseteq \mathbb{Z}_+^p \), where \( x_{sj} \) represents the number of packages of type \( j \) produced by seller \( s \). By assigning resource consumption functions \( g_b(x_b) = x_b \in \mathbb{Z}_+^m, b \in B \) for buyers and \( g_s(x_s) = -x_s \in \mathbb{Z}_+^m, s \in A \) for sellers, respectively, a competitive equilibrium then corresponds to a pair \( (x^*, \lambda) \in X \times \mathbb{R}_+^m \) that weakly enforces 0 for \( G^{\text{max}}(0) \) with market prices. Note that the condition \( \ell(x) \leq 0 \) ensures that supply exceeds demand and the dual variable \( \lambda \) corresponds to the market clearing equilibrium prices (which may be nonlinear in terms of item prices). In this construction, the level of price differentiation depends the constructed set \( E \), that is, \( E \) might model the number of anonymous package types (leading to anonymous package prices), or packages prices depending on the buyer and seller (leading to an increased number \(|E|\)). The LP characterization of Bikhchandani and Ostroy [12] regarding the existence of competitive equilibria can be deduced from Theorem 3.11, because the game \( G^{\text{max}}(0) \) exhibits a finite strategy space for every player (buyer or seller) and besides the demand-supply condition the utilities are separable over players. In fact, looking at Theorem 3.11, one can generalize the characterization of Bikhchandani and Ostroy [12] along several directions. One direction is to allow that players may be buyers and sellers at the same time. In Section 8, we consider so-called trading networks that exhibit this property and the reduction we present is very similar to the one sketched here.

7.3 Item Multiplicity, Additive Linear Valuations with Externalities and Polymatroids

Now we turn to a multi-item model that allows for several items of the same type and some degree of externalities of allocations. There is a finite set \( E = \{1, \ldots, m\} \) of item types and every item may be available at a certain multiplicity. Assume further that \( X_i \subset \{0,1\}^m, i \in N \). This implies that every player wants to receive at most one item per type - however \( X_i \) may still carry some combinatorial restrictions for feasible item sets for player \( i \in N \). Suppose that valuations of players are additive over items, that is,

\[
v_i(\ell(x), x_i) := \sum_{j \in E} v_{ij}(\ell_j(x))x_{ij},
\]
where \( v_{ij} : \mathbb{Z}_+ \rightarrow \mathbb{R}_+ \) is the nonnegative value player \( i \) gets from receiving an item of type \( j \) assuming that item \( j \) is sold to \( \ell_j(x) \) many players. This formulation is not directly comparable to the one before. On the one hand side, additivity of valuations over items is less general. On the other hand, several items of the same type can be sold and we allow for a functional dependency of the valuations with respect to the load \( \ell_j(x) \). Such dependency may be interesting for situations, where the value \( v_{ij}(\cdot) \) of receiving item type \( j \) drops as other players also receive the same type – this is referred to as a setting with negative externalities.

**Assumption 7.5.** For every \( i \in N, j \in E \), the functions \( v_{ij} \) are nonnegative and exhibit negative externalities, that is, \( v_{ij}(z) \geq v_{ij}(z + 1) \) for all \( z \in \mathbb{Z}_+ \).

The model so far does not satisfy the assumption of a maximization game \( G^{\text{max}}(u) \) augmented with prices, as the utility of a player is allowed to depend on the load vector \( \ell(x) \). However, one can easily verify that with Assumption 7.5, the game is in fact a monotone aggregative game as defined in Section 5. With this insight at hand, by Theorem 5.3 it suffices to analyze cases for which \( u \) is enforceable for \( G^{\text{max}}(u) \), which in turn is equivalent to the property that Problem \( P^{\text{min}}(u) \) has zero duality gap and and satisfies \( \ell(x^*) = u \) for a primal-dual optimal pair \((x^*, \lambda)\). We get the following result using Theorem C.1.

**Corollary 7.6.** Let \( X_i = \mathcal{P}_E \subset \{0,1\}^m, i \in N \) be integral polymatroid polyhedra and assume that valuation functions satisfy Assumption 7.5. Then, any supply vector \( u \in \mathbb{Z}_+^m \) for which there is \( x \in X \) with \( \ell(x) = u \) is enforceable.

**Proof.** By Assumption 7.5 and the structure of the valuation functions (see (18)) it follows directly that the game is a monotone aggregative game. Thus, for any supply vector \( u \in \mathbb{Z}_+^m \) with \( \ell(x) = u \) for some \( x \in X \), by Theorem 5.3 it suffices to show that \( u \) is enforceable for \( G^{\text{max}}(u) \). This, however, follows directly from Theorem C.1.

## 8 Application to Trading Networks

A bilateral trading network is represented by a directed multigraph \( G = (N, E) \), where \( N \) is the set of vertices and \( E = \{e_1, \ldots, e_m\} \) the set of edges. Each vertex corresponds to a player and each edge \( e = (s, b) \) represents a bilateral trade that can take place between the pair of incident vertices \( s, b \in N \). For each \( e = (s, b) \in E \), the source vertex \( s \) corresponds to the seller and the sink vertex \( b \) corresponds to the buyer in the trade. For \( i \in N \), let \( \delta^+(i) \) and \( \delta^-(i) \) be the set of outgoing and incoming edges of vertex \( i \in N \) and as usual we denote the set of all edges incident to \( i \) by \( \delta(i) = \delta^+(i) \cup \delta^-(i) \). For a set of edge prices \( \lambda_e \geq 0, e \in E \), we can associate with each possible trade \( e = (s, b) \in E \) a price \( \lambda_e \geq 0 \) with the understanding that the buyer \( b \) pays \( \lambda_e \) to the seller \( s \). An outcome of the market is a set of realized trades \( S \subseteq E \) and a vector of prices \( \lambda \in \mathbb{R}_+^m \). Given an outcome, the quasi-linear utility of a player \( i \in N \) is defined as the sum of the utility gained from trades plus the income minus the cost of trades, respectively. The utility of realized trades is given by a function \( \bar{w}_i : 2^\delta(i) \rightarrow \mathbb{R} \). We extend \( \bar{w}_i \) to \( 2^m \) by taking \( w_i : 2^m \rightarrow \mathbb{R}, S \mapsto \bar{w}_i(S \cap \delta(i)) \). The overall utility for given \( S \subseteq E \) and \( \lambda \in \mathbb{R}_+^m \) is defined as

\[
w_i(S) + \sum_{e \in \delta^+(i) \cap S} \lambda_e - \sum_{e \in \delta^-(i) \cap S} \lambda_e
\]

(19)

For the function \( w_i, i \in N \) we only assume monotonicity on the buyer side, that is, \( w_i(S) \geq w_i(T) \) for all \( T \subseteq S \subseteq \delta^-(i) \). Free disposal at the buyer side is a sufficient condition for this assumption.
The market maker wants to determine a price vector \( \lambda \in \mathbb{R}_+^m \) and a set of realized trades \( S^* \subseteq E \) such that

\[
S^* \in \arg \max_{S \subseteq E} \left\{ w_i(S) + \sum_{e \in \delta^+(i) \cap S} \lambda_e - \sum_{e \in \delta^-(i) \cap S} \lambda_e \right\}
\]

holds for all \( i \in N \).

Such a tuple \((S^*, \lambda)\) constitutes a competitive equilibrium. The main difference to the Walrasian market equilibrium model is that players can simultaneously act as buyers and sellers in different trades.

We will cast this problem in the framework by constructing an equivalent game \( G^{\text{max}}(0) \). For each player \( i \), we have a vector \( x_i \in \{-1, 0, 1\}^m \) with the understanding that

\[
x_{ie} = \begin{cases} 
-1, & \text{if } e \in \delta^+(i) \text{ and trade } e \text{ is realized as buyer} \\
1, & \text{if } e \in \delta^-(i) \text{ and trade } e \text{ is realized as buyer} \\
0, & \text{if } e \notin \delta(i) \text{ or } e \text{ is not realized.}
\end{cases}
\]

We thus define \( X_i = \{x_i \in \{-1, 0, 1\}^m | x_{ie} = 0, e \notin \delta(i)\}, i \in N \). To complete the description of \( G^{\text{max}}(0) \), we assume that the resource consumption functions are given as \( g_i(x_i) = x_i \) for all \( i \in N \) and we define the valuation function of player \( i \in N \) on \( X_i \) by

\[
v_i(0, x_i) := w_i(\{e \in \delta(i) : |x_{ie}| = 1\}).
\]

(20)

With this construction, we have a one-to-one correspondence between \( x_i \in X_i \) and sets \( S_i \subseteq \delta(i) \) via \( E(x_i) := \{e \in \delta(i) : |x_{ie}| = 1\} \). We obtain the following characterizations on the existence of competitive equilibria using the notation \( E(x) := \cup_{i \in N} E(x_i) \).

**Lemma 8.1.** Consider a bilateral trading game and let \( G^{\text{max}}(0) \) be an associated pricing game. Then, the following statements are equivalent.

1. There exists a competitive equilibrium \((E(x^*), \lambda) \in E \times \mathbb{R}_+^m\) for the bilateral trading game.
2. The vector \( u = 0 \) is enforceable via \((x^*, \lambda) \in X \times \mathbb{R}_+^m\) for the game \( G^{\text{max}}(0) \).
3. \( p^{\text{max}}(0) \) has zero duality gap and \( x^* \in X \) is an optimal solution \( x^* \in X \) that satisfies \( \ell(x^*) = 0 \).

**Proof.** By Theorem A.1 we have already that (2)\(\Leftrightarrow\)(3) holds, so we only need to show (1)\(\Leftrightarrow\)(2). For any \( x \in X \) with \( \ell(x) = 0 \), we have

\[
\forall e = (s, b) \in E : e = (s, b) \in E(x_s) \Leftrightarrow e = (s, b) \in E(x_b).
\]

(21)

For \( i \in N \) arbitrary, we get

\[
\begin{align*}
\lambda^* \in \arg \max & \{v_i(0, x_i) - \lambda^T x_i | x_i \in X_i\} \\
\Leftrightarrow & \ x_i^* \in \arg \max \{w_i(E(x_i)) - \lambda^T x_i | x_i \in X_i\} & \text{(By definition of } v_i) \\
\Leftrightarrow & \ E(x_i^*) \in \arg \max \left\{ w_i(E(x_i)) + \sum_{e \in E(x_i) \cap \delta^+(i)} \lambda_e - \sum_{e \in E(x_i) \cap \delta^-(i)} \lambda_e \bigg| E(x_i) \subseteq \delta(i) \right\} \\
\Leftrightarrow & \ E(x^*) \in \arg \max \left\{ w_i(S) + \sum_{e \in \delta^+(i) \cap S} \lambda_e - \sum_{e \in \delta^-(i) \cap S} \lambda_e \bigg| S \subseteq E \right\} & \text{(By def. of } w_i \text{ and } (21))
\end{align*}
\]

\[\square\]
With this characterization, we can use the results obtained so far for the enforceability of 0 for $G^{\text{max}}(0)$. Note that every $X_i, i \in N$ consists of $k_i := 3^{b(i)}$ many points, thus, Theorem 3.11 gives a complete characterization of the existence of competitive equilibria.

**Corollary 8.2.** Competitive equilibria for bilateral trading networks exist if and only if the following LP admits integral optimal solutions $\alpha$ with $\ell(\alpha) = 0$:

$$\max \sum_{i \in N} v_i^T \alpha_i, \ell(\alpha) \leq 0, \alpha_i \in \Lambda_i \text{ for all } i \in N,$$  \hspace{1cm} (LP$^{\max}(0)$)

where $v_i := (v_i(0, x_i))_{x_i \in X_i}$ and $\ell(\alpha) := \sum_{i \in N} \sum_{j \in \lambda^{+}(i)} x_i^T_{\alpha ij}$.

We obtain the following result as a direct corollary of Theorem 3.15.

**Corollary 8.3.** $LP^{\max}(0)$ can be solved in polynomial time, if there is a polynomial time demand oracle.

In order to obtain existence results, one needs to enforce some assumptions on the valuation functions $w_i, i \in N$. Hatfield et al. [47] introduced the concept of fully substitutable valuations. We omit here the precise definition but it is important to know that this concept is in fact equivalent to the known concept of GS valuations or $M^2$-concave valuations (see Hatfield et al. [48]). Instead of the monotonicity property of valuations (as e.g. in Fujishige and Yang [38]), we assume that valuations are buyer-monotone, that is, for every $i \in N$ and $S \subseteq T \subseteq \delta^{-}(i)$, we have $w_i(S) \leq w_i(T)$. Free disposal for the buyer is a sufficient condition. We obtain the following result.

**Theorem 8.4 (Hatfield et al. [47]).** For fully substitutable buyer-monotone valuations, there exists a competitive equilibrium.

**Proof.** In order to apply our previous results, we need to check whether

$$\max \left\{ v(x) := \sum_{i \in N} v_i(0, x_i) \big| x \in X, \ell(x) \leq 0 \right\}$$

has zero duality gap and admits an optimal solution $x^* \in X$ with $\ell(x^*) = 0$. Again the result of Yokote [87] implies zero duality gap as $\ell(x) \leq 0$ is a laminar system. With the buyer monotonicity, any optimal solution $x^*$ to (22) can be turned into one with $\ell(x^*) = 0$, hence, Theorem A.1 implies Theorem 7.3.

### 9 Application to Congestion Control in Communication Networks

In the domain of network-based TCP congestion control, we are given a directed capacitated graph $G = (V, E, u)$, where $V$ are the nodes, $E$ with $|E| = m$ is the edge set and $u \in \mathbb{R}^{m}_+$ denote the edge capacities. There is a set of players $N = \{1, \ldots, n\}$ and every $i \in N$ is associated with an end-to-end pair $(s_i, t_i) \in V \times V$ and a bandwidth utility function $U_i : \mathbb{R}_+ \to \mathbb{R}_+$ measuring the received benefit from sending net flow from $s_i$ to $t_i$. As in congestion games, a flow for $i \in N$ is a nonnegative vector $x_i \in \mathbb{R}^{|E|}_+$ that lives in the flow polyhedron:

$$X_i = \left\{ x_i \in \mathbb{R}^{m}_+ \bigg| \sum_{j \in \delta^+(v)} x_{ij} - \sum_{j \in \delta^-(v)} x_{ij} = 0, \text{ for all } v \in V \setminus \{s_i, t_i\} \right\},$$
where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering $v$. We assume $X_i \neq \emptyset$ for all $i \in N$ and we denote the net flow reaching $t_i$ by $\text{val}(x_i) := \sum_{j \in \delta^+(s_i)} x_{ij} - \sum_{j \in \delta^-(s_i)} x_{ij}, i \in N$. The goal in price-based congestion control is to determine edge prices $\lambda_{ij}, j \in E$ so that a strategy distribution $x^*$ is induced as an equilibrium respecting the network capacities $u$ and, hence, avoiding congestion. Assuming that resource consumption is linear, that is, $g_i(x_i) = x_i, i \in N$, the equilibrium condition amounts to
\[ x_i^* \in \operatorname{arg\,max}\{U_i(\text{val}(x_i)) - \lambda^\top x_i | x_i \in X_i\} \text{ for all } i \in N. \]

We obtain the following result for concave bandwidth utility functions.

**Theorem 9.1** (Kelly et al. [54]). For concave bandwidth utility functions $U_i, i \in N$, every capacity vector $u \in \mathbb{R}_+^m$ is weakly enforceable with market prices.

**Proof.** With the concavity of $U_i, i \in N$, problem $P_{\text{max}}^\text{max}(u)$ is a convex optimization problem over a polytope and hence satisfies Slater’s constraint qualification conditions for strong duality. Thus, Theorem A.1 implies the result. \hfill \Box

Let us turn to models, where the flow polyeder $X_i$ is intersected with $\mathbb{Z}_+^m$. Most of the previous works in the area of congestion control assume either that there is only a single path per $(s_i, t_i)$ pair or as in Kelly et al. [54], the flow is allowed to be fractional. Allowing a fully fractional distribution of the flow, however, is not possible in some interesting applications - the notion of data packets as indivisible units seems more realistic. The issue of completely fractional routing versus integrality requirements has been explicitly addressed by Orda et al. [65], Harks and Klimm [45] and Wang et al. [84]. Using the TDI and IDP property of network matrices, we obtain the following result for integral flow polytopes.

**Corollary 9.2.** Let the bandwidth utility functions $U_i, i \in N$ be non-decreasing, identical and linear and assume that all players share the same source $s_i = s_i, i \in N$. Then, for integral routing models with strategy spaces $X_i' = X_i \cap \mathbb{Z}_+^m$, every capacity vector $u \in \mathbb{Z}_+^m$ is weakly enforceable with market prices.

**Proof.** For problem $P_{\text{max}}^\text{max}(u)$, we can w.l.o.g. change the instance by introducing a super-sink and connect all $t_i$'s to the sink with large enough integral capacity. This way, we obtain an ordinary $s$-$t$ max-flow problem for which the LP-formulation $LP_{\text{max}}^\text{max}(u)$ is known to be integral. \hfill \Box

**Remark 9.3.** The above proof shows that for a capacity vector $u \in \mathbb{Z}_+^m$, we can compactly represent the enforcing prices/allocation space and efficiently optimize linear functions over it.

While the above result seems to require somewhat restrictive assumptions (linear identical bandwidth utilities and a common source), we show in the following that already for two source-sink pairs with identical linear capped bandwidth utilities, enforceability is not guaranteed, unless $P = NP$. A capped linear function $f : \mathbb{R} \to \mathbb{R}$ has the form $f(x) = ax$, for $x \leq x_{\text{max}}$ and $f(x) = ax_{\text{max}}$ for $x \geq x_{\text{max}}$. This type of function is concave and arises quite naturally as we only require the existence of an upper bound on the requested bandwidth of every player.

**Proposition 9.4.** Unless $P \neq NP$, there is an instance with only two players with different source sink pairs $(s_i, t_i), i \in \{1, 2\}$ and non-decreasing, identical and linear capped bandwidth utilities $U_i, i \in \{1, 2\}$, for which there is a vector $u \in \mathbb{Z}_+^m$ that is not weakly enforceable by market prices.

**Proof.** Having capped bandwidth utilities implies that there are only finitely many strategies per player. Thus, we can use the LP characterization result of Theorem 3.15: It remains to prove that
the master problem $P_{\text{max}}(u)$ is NP-hard and that the demand problem is polynomial time solvable. The demand problem amounts to
\[
\max \{ \text{val}(x_i) - \lambda^\top x_i | x_i \in X_i \},
\]
which is just a max flow problem. For the master problem, it is not hard to see that we can reduce from the two-directed disjoint path problem. For an instance of two-directed disjoint path, we associate the given two source-sink pairs naturally with those of two players $\{1, 2\}$ and assume $u = 1$ and $U(\text{val}(x_i)) = \text{val}(x_i), i \in \{1, 2\}$ with a cap at any value larger equal 1. This way, due to the integrality of the flows, there is a solution to the disjoint path problem iff the objective value of the master problem is 2.

10 Conclusions and Extensions

We introduced a generic resource allocation problem and studied the question of enforceability of certain load vectors $u$ via (anonymous) pricing of resources. We derived a characterization of enforceable load vectors via studying the duality gap of an associated optimization problem. We further derived a characterization connecting enforceability for arbitrary non-convex settings to enforceability of a convex model. Using this general result, we studied consequences of known structural results in the area of linear integer optimization, polyhedral combinatorics and discrete convexity for several application cases.

Understanding duality gaps of optimization problems is an active research area, see for instance the progress on duality for nonlinear mixed integer programming (cf. Baes et al. [5]). Thus, our general characterization yields the opportunity to translate progress in this field to economic situations mentioned in the applications.

For our general model we assumed that the strategy spaces are subsets $X_i \in \mathbb{R}^m, i \in N$. This assumption is not necessary for proving our main result. We could have chosen $X_i$ as a Banach space and the results would have gone through. In fact, in the area of dynamic traffic assignments (cf. Friesz et al [37]), the flow trajectories live in function spaces, thus, offering the possibility that our characterization on Banach spaces yields the existence of (time varying) tolls for these applications too.

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A Utility Maximization Problems

We turn to utility maximization problems and define the following analogous problem:

$$\max \left\{ v(x) \mid \ell_j(x) \leq u_j, j \in E, x_i \in X_i, i = 1, \ldots, n \right\}, \quad (P_{\text{max}}(u))$$

where the objective function is defined as $v(x) := \sum_{i \in N} v_i(u, x_i)$.

The Lagrangian function for problem $P_{\text{max}}(u)$ becomes $L(x, \lambda) := v(x) - \lambda^T(\ell(x) - u)$, $\lambda \in \mathbb{R}^m_+$, and we can define the Lagrangian-dual as:

$$\mu: \mathbb{R}^m_+ \to \mathbb{R}, \quad \mu(\lambda) = \sup_{x \in X} L(x, \lambda) = \sup_{x \in X} \{v(x) - \lambda^T(\ell(x) - u)\}.$$ We assume that $\mu(\lambda) = \infty$, if $L(x, \lambda)$ is not bounded from above on $X$. The dual problem is defined as:

$$\inf_{\lambda \geq 0} \mu(\lambda), \quad (D_{\text{max}}(u))$$

We obtain the following analogous results to the minimization case.

**Theorem A.1.** Consider a game of type $G_{\text{max}}(u)$. Then, the following statements hold:

1. A supply vector $u \in \mathbb{R}^m$ is enforceable via $(x^*, \lambda^*)$ if and only if $(x^*, \lambda^*)$ has zero duality gap for $P_{\text{max}}(u)$ and $x^*$ satisfies (1) with equality.

2. A supply vector $u \in \mathbb{R}^m$ is weakly enforceable with market clearing prices if and only if $(x^*, \lambda^*)$ has zero duality gap for $P_{\text{max}}(u)$.

3. A supply vector $u \in \mathbb{R}^m$ is uniquely enforceable via $(x^*, \lambda^*)$ if and only if $(x^*, \lambda^*)$ has zero duality gap for $P_{\text{max}}(u)$ and $x^*$ is a unique optimal solution for $P_{\text{max}}(u)$ satisfying (1) with equality.

In maximization games, the sets $X_i$ usually contain some capacity restrictions, therefore the notion of minimality of vectors $u$ might not be appropriate. Take for instance the example of auctions in Example 1.4. Here, $u = 0$ arises as the unique minimal $u$ leading to trivial conclusions. Perhaps more interesting are scenarios in which the combined valuation function $v(x)$ is in some sense monotonically non-decreasing on $X$. 
Definition A.2 (Upwards closure of $X$, Monotonicity of valuations). $X$ is *upwards-closed* w.r.t. $u \in \mathbb{R}^m$, if $\ell(x) \leq u$ and $\ell(x) \neq u$ for some $x \in X$ implies that there is $x' \in X$ with $x' \geq x$ and $\ell(x') = u$. We say that $X$ is *upwards-closed*, if this property holds for all $u \in \mathbb{R}^m$. The function $v(x)$ is *monotonically non-decreasing* on $X$, if $v(x) \geq v(y)$ for all $x, y \in X$ with $x \geq y$.

We obtain the following result regarding this monotonicity assumption.

**Theorem A.3.** Assume that $v(x)$ is monotonically non-decreasing and $X$ is upwards closed w.r.t. $u \in \mathbb{R}^m$. Then, the supply vector $u \in \mathbb{R}^m$ is enforceable via $(x^*, \lambda^*)$ if and only if $(x^*, \lambda^*)$ has zero duality gap for $P^{\text{max}}(u)$.

**Proof.** By the monotonicity of $v(x)$ and upwards-closedness of $X$ w.r.t. $u$, any optimal solution of $P^{\text{max}}(u)$ can be turned into one that satisfies (1) with equality. \hfill $\square$

We finally get an existence result for convex sets $X_i, i \in N$ and monotone and concave valuations.

**Corollary A.4.** Let $X_i, i \in N$ be nonempty convex sets such that $X$ is upwards-closed w.r.t. $u \in \mathbb{R}^m$. Assume that $v_i, i \in N$ are concave functions, $g_i, i \in N$ are convex functions and $v(x)$ is monotonically non-decreasing and that there exists $x^0 \in \text{relint}(\{x \in X | \ell(x) \leq u\})$. Then, $u$ is enforceable. If $v(x) = \sum_{i \in N} v_i(u, x_i)$ is strictly concave over $X$, then $u$ is uniquely enforceable.

### B Proof of Theorem 4.2

Let us now restate problem $P^{\text{min}}(u)$ in the context of polymatroids.

\[
\begin{align*}
\min \sum_{i \in N} \sum_{j \in E} \pi_{ij}(u)x_{ij} & \quad (P^{\text{min- polymatroid}}(u)) \\
x_{i} & \in B_{f_i}, \ i \in N \\
\ell_{j}(x) & \leq u_{j}, \ j \in E
\end{align*}
\]  

We call $L^{\text{min- polymatroid}}(u)$ the fractional relaxation, where we optimize over $\mathcal{E}B_{f_i}, i \in N$ instead of $B_{f_i}, i \in N$.

**Proof.** We first lift all integral base polyhedra $B_{f_i} \subset \mathbb{Z}^m$ to the higher dimensional space $\tilde{B}_{f_i} \subset \mathbb{Z}^{n-m}$ by introducing $n$ copies $E_i, i \in N$ of the elements $E$ leading to $\tilde{E} := \bigcup_{i \in N} E_i$ with $E_i = \{e_1^i, \ldots, e_m^i\}, i \in N$. The domain of the integral polymatroid function $f_i$ is extended to $\tilde{E}$ as follows

$$
\tilde{f}_i(S) := f_i(E_i \cap S) \text{ for all } S \subseteq \tilde{E}.
$$

This way $\tilde{f}_i(S)$ remains an integral polymatroid rank function on the lifted space $\mathbb{Z}^{n-m}$. Note that for $\tilde{x}_{i} \in \tilde{B}_{f_i}$, we have $\tilde{x}_{i} \in \mathbb{Z}^{n-m}$ and with $f_i(\emptyset) = 0$, we get $x_{ij} = 0$ for all $j \in \tilde{E} \setminus E_i$. By this construction, we get $x_{i} \in B_{f_i} \iff \tilde{x}_{i} \in \tilde{B}_{f_i}$.

Now we define the Minkowski sum

$$
\tilde{B}_1 := \sum_{i \in N} \tilde{B}_{f_i} \subset \mathbb{Z}^{n-m},
$$

39
which is again an integral polymatroid base polyhedron. By this construction we can represent all collections of integral base vectors by a single integral polymatroid base polyhedron.

It remains to also handle the capacity constraint \((23)\) (note that this is not a box constraint for polymatroid \(\tilde{B}_1\)). For \(S \subseteq \tilde{E}\), we define \(S_j := \{j \in E \mid \exists i \in N \text{ with } e_i^j \in S\}\) as the union of those original element indices (in \(E\)) for which \(S\) contains at least one copy. With this definition, we define a second polymatroid as follows.

\[
\tilde{B}_2 := \{x \in \mathbb{Z}^{m \cdot n} \mid x(S) \leq h(S) \text{ for all } S \subseteq \tilde{E}, x(\tilde{E}) = h(\tilde{E})\},
\]

where for \(S \subseteq \tilde{E}\) \(h(S) := \sum_{j \in S} u_j\). One can easily verify that \(h\) is an integral polymatroid function.

Now observe that for the sets \(\{e_1^j, \ldots, e_n^j\}, j \in E\) we exactly get the capacity constraint \(x(\{e_1^j, \ldots, e_n^j\}) \leq u_j, j \in E\). Altogether, with the minimality of \(u\), problem \(p_{\text{min-polymatroid}}(u)\) can be reduced to the problem of finding a vector in the intersection of \(\tilde{B}_1\) and \(\tilde{B}_2\) minimizing a linear objective:

\[
\min \left\{ \sum_{i \in N} \sum_{j \in E} \pi_{ij}(u) x_{ij} \middle| x \in \tilde{B}_1 \cap \tilde{B}_2 \right\} \tag{24}
\]

By the fundamental result of Edmonds [29, Thm. (35)], the fractional relaxation \(\tilde{E}B_1 \cap \tilde{E}B_2\) is integral. Note that there are strongly polynomial time algorithms computing an optimal solution to (24) (see Cunningham and Frank [24] and Frank and Tardos [36]).

\[\square\]

### C Utility Maximization on Polymatroids

For the maximization variant, the strategy spaces \(X_i, i \in N\) are usually defined as the vectors of an integral polymatroid polyhedron \(P_{f_i}\). We get the following reformulation of \(p_{\text{max}}(u)\):

\[
\max \sum_{i \in N} \sum_{j \in E} v_{ij}(u) x_{ij} \quad (p_{\text{max-polymatroid}}(u))
\]

\(x_i \in P_{f_i}\) for all \(i \in N\)

\(\ell_j(x) \leq u_j, j \in E\)

The following companion result for maximization problems on polymatroids holds true.

**Theorem C.1.** For polymatroid games, every \(u \in \mathbb{Z}^m\) for which \(p_{\text{max}}(u)\) admits a finite optimal solution is weakly enforceable with market prices. If \(v_{ij}(u) \geq 0\) for all \(i \in N, j \in E\), then, any supply vector \(u \in \mathbb{Z}^m_+\) for which there exists \(x \in X\) with \(\ell(x) = u\) is enforceable.

**Proof.** The proof of the first statement is analogous to the proof of the previous theorem. For the second statement, with \(v_{ij}(u) \geq 0\) it follows that \(v(x)\) is monotonically nondecreasing. Moreover, for any integral \(u \in \mathbb{Z}^m_+\) for which there exists \(x \in X\) with \(\ell(x) = u\) it is known that \(X\) is upwards closed – using polymatroid properties. Thus, with the integrality of the polymatroid-intersection polytope (see the proof of Theorem 4.2) the result follows.  

\[\square\]