The two-dimensional electron gas in a bilayer graphene in the Bernal stacking supports a variety of uniform broken-symmetry ground states in Landau level $N = 0$ at integer filling factors $\nu \in [-3, 4]$. When an electric potential difference (or bias) is applied between the layers at filling factors $\nu = -1, 3$, the ground state evolves from an interlayer coherent state at small bias to a state with orbital coherence at higher bias where electric dipoles associated with the orbital pseudospins order spontaneously in the plane of the layers. In this paper, we show that by further increasing the bias at these two filling factors, the two-dimensional electron gas goes first through a Skyrmion crystal state and then into an helical state where the pseudospins rotate in space. The pseudospin textures in both the Skyrmion and helical states are due to the presence of a Dzyaloshinskii-Moriya interaction in the effective pseudospin Hamiltonian when orbital coherence is present in the ground state. We study in detail the electronic structure of the helical and Skyrmion crystal states as well as their collective excitations and then compute their electromagnetic absorption.

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I. INTRODUCTION

Experiments\cite{1} have shown that, when placed in a perpendicular magnetic field, a graphene bilayer supports a set of Landau levels with energies given by $E_N = \pm h\omega_c \sqrt{N(|N|+1)}$ where $N = 0, \pm 1, \pm 2, \ldots$. All levels except $N = 0$ are fourfold degenerate in addition to the usual degeneracy related to the guiding center coordinate. An electronic state is thus specified by its Landau level index $N$, its guiding center index $X$ (in the Landau gauge), its spin index $\sigma = \pm 1$ and its valley index $\xi = \pm K$. In $N = 0$, we must add an additional orbital index because states with both $n = 0$ and $n = 1$ Landau level character have zero kinetic energy. The eightfold degeneracy of the $N = 0$ Landau level is revealed experimentally by a jump in the quantized Hall conductivity from $-4 (e^2/h)$ to $+4 (e^2/h)$ when the charge density is tuned across neutrality in moderately disordered samples\cite{2}.

Some of us\cite{2} have shown that the close proximity between the two graphene layers in bilayer graphene leads to spontaneous interlayer coherence in $N = 0$ when Coulomb interaction is taken into account. In a pseudospin language where the spin, layer and orbital degrees of freedom are each mapped to a spin, layer and orbital degrees of freedom are each mapped to a $S_i = 1/2$ pseudospin ($i = \sigma, \xi, n$), electron-electron interactions at integer filling factors completely lift the degeneracy of the bilayer octet producing spontaneously broken-symmetry ground states with spin, valley and orbital polarizations. In consequence, quantum Hall plateaux should occur at all integer values of the filling factors from $\nu = -3$ to $\nu = 4$ in Landau level $N = 0$. The existence of these additional plateaux has recently been confirmed experimentally in suspended bilayer graphene samples and bilayer graphene on SiO$_2$/Si substrates\cite{2}.

The possibility to study novel broken-symmetry (BS) states is always exciting and the new BS states in bilayer graphene are no exceptions. For example, the interlayer-coherent state at $\nu = -3$ (and $\nu = 1$ if the two-dimensional electron gas (2DEG) can be considered as fully spin polarized) is a quantum Hall layer-pseudospin ferromagnet with pseudospin wave dispersion $\omega \sim q^2$ that contrasts with the usual linear dispersion found in interlayer-coherent states in semiconductor double-quantum-well systems. This unusual dispersion can be related\cite{3} to a vanishing of the counterflow superfluid density. At filling factor $\nu = -1, 3$, the interlayer-coherent state has the usual linear dispersion.\cite{4} If a positive electric potential difference $\Delta_B$ (which we refer to, in this paper, as the bias potential) is applied between the layers, the charge is progressively transferred into the bottom layer. At $\nu = -3, 1$ the ground state with all the charge in one layer supports an orbital pseudospin mode that can be viewed as an *intralayer* cyclotron resonance.\cite{5} This mode is gapped due to the finite bias and should be detectable in microwave absorption experiments.

The broken-symmetry states related to the orbital degree of freedom are especially interesting. For a spin-polarized 2DEG, they occur at filling factors $\nu = -1, 3$ and above a certain critical bias $\Delta_B$ where one of the layer is completely filled. The homogeneous orbital broken-symmetry states have a finite density of electric dipoles that collectively order in the $x-y$ plane.\cite{6} The orbital pseudospin mode corresponding to the collective motion of these dipoles is gapless despite the finite bias. It is a Goldstone mode due to the breaking of the $U(1)$ symmetry of the pseudospins in the $x-y$ plane.

It was shown recently\cite{7} that there exists a Dzyaloshinskii-Moriya (DM) interaction between the orbital pseudospins at $\nu = -1, 3$ that causes the orbital pseudospin mode to soften at a finite wave vector $q$ as the bias $\Delta_B$ is increased. It was conjectured that, above a critical bias, the ground state should be some kind of helical state. In the present work, we find a more complex scenario. Working in the Hartree-Fock approximation, we find that, as the bias...
is increased, the ground state follows the sequence of transitions: uniform state - Skyrmion crystal - helical state - Skyrmion crystal - uniform state. The phase diagram is symmetrical about the bias $\Delta_B^{(1)}/2$ where the charge is equally distributed in both orbitals $n = 0$ and $n = 1$. Interestingly, our phase diagram looks very similar to that found recently in a thin film of the helical magnet Fe$_{0.5}$Co$_{0.5}$Si where a Skyrmion crystal as well as an helical and a ferromagnetic uniform phases have all been observed using Lorentz transmission electron microscopy. In this system, the phase transitions are induced by a transverse magnetic field. In the bilayer, the role of the magnetic field is played by the bias. The effective pseudospin Hamiltonian of the orbital states in a graphene bilayer is quite complex. Since charge and pseudospin are coupled, a pseudospin texture such as that found in the orbital Skyrmion crystal produces a charge density in real space and so the direct (or Hartree) Coulomb interaction must be considered together with the other competing interactions mentioned above.

In this work, we study in detail the helical and Skyrmion crystal states. We derive their electronic properties as well as their collective excitations and compute their electromagnetic absorption. We show that the effective pseudospin model that describes these states involves nonlocal as well as anisotropic exchange interactions between the pseudospins. These Coulomb exchange interaction tend to align the orbital pseudospins while the DM term favors a rotation of the pseudospins in space. The resulting ground states result from a balance between these competing forces just as in helical magnets such as MnSi and Fe$_{1-x}$Co$_x$Si. One major difference in the graphene bilayer is that the DM interaction in the orbitally coherent state is of an entirely different origin than the spin-orbit interaction at work in usual helical magnets.

Our paper is organized in the following way. In Sec. II, we study the non-interacting states of the graphene bilayer within a two-band low-energy model. In Sec. III, we discuss the validity of our approximation of assuming complete spin polarization. This discussion is needed since increasing the bias pushes down (up) half of the spin down (up) levels. Some levels of opposite spin cross at finite bias and this can introduce new phases in the phase diagram or make some phases disappear. In Sec. IV, we summarize the Hartree-Fock approximation as well as the generalized random-phase approximation (GRPA) which we use to study the collective excitations. In Sec. V, we study the electronic properties of the different phases as well as their collective excitations. We discuss their electromagnetic absorption in Sec. VI and conclude in Sec. VII.

II. EFFECTIVE TWO-BAND MODEL

We consider a graphene bilayer in Bernal stacking as represented in Fig. 1. The bilayer is placed in an external transverse electric field in order to allow an electrical potential difference $\Delta_B$ between the layers and in a transverse magnetic field $B$. We denote the two basis atoms in the top layer by $A_1$ and $B_1$ and those of the bottom layer by $A_2$ and $B_2$ with atoms $A_1$ sitting directly above atoms $B_2$ as shown in the figure. The band structure of this system is calculated in the tight-binding approximation with the hopping parameters: $\gamma_0 = 3.12$ eV, $\gamma_1 = 0.39$ eV, $\gamma_4 = 0.12$ eV, $\Delta = 0.0156$ eV, taken from Ref. [10]. The parameter $\gamma_0$ is the intralayer hopping term between nearest-neighbor carbon atoms, $\gamma_1$ is the interlayer hopping between carbon atoms that are part of a dimer, $\gamma_4$ is an interlayer hopping term between two carbons atoms that are not part of a dimer ($A_1 - A_2$ and $B_1 - B_2$) and $\Delta$ represents the difference in the crystal field experienced by the inequivalent atoms $A_i$ and $B_i$ in the same plane. We neglect the trigonal warping term $\gamma_3$ (the $A_2 - B_1$ hopping), a correct approximation at sufficiently high magnetic field.

![FIG. 1: (Color online) Lattice structure of the bilayer graphene in the Bernal stacking. The spheres represent carbon atoms.](image)

The electronic dispersion consists of four bands. In the absence of bias and magnetic field, two of these bands meet at the six valley points $K, K'$ of the reciprocal lattice. Here, we take as the two non-equivalent points $K = (-4\pi/3a_0, 0)$ and $K' = -(4\pi/3a_0, 0)$ where $a_0 = 2.46$ Å is the lattice parameter of graphene. The dispersion of the two low-energy bands, for small wavevector $k$ measured from either $K$ or $K'$, is given by $E_{\xi\mathbf{k}}(k) = \pm \hbar^2k^2/2m^*$ with valley index $\xi = \pm 1$. The effective electronic mass is defined by $m^* = 2\hbar^2\gamma_1/3\gamma_0^2a_0^2 = 0.054m_0$ where $m_0$ is the bare electronic mass. The two high-energy bands are separated from the two low-energy bands by a gap of order $\gamma_1$.

To describe the low-energy ($E << \gamma_1$) excitations of the tight-binding model, we assume complete spin polar-
ization and use an effective two-band model, where the Hamiltonian is given by

\[ H^0_{\xi\mathbf{K}} = \left( -\xi \frac{\Delta^R_n}{2} + (\zeta + \xi \beta \Delta_B) a a^\dagger + \hbar \omega^e_n a^2 \right) \frac{\hbar \omega^e_n}{(a^\dagger)^2} - \xi \frac{\Delta^R_n}{2} + (\zeta - \xi \beta \Delta_B) a^\dagger a \]  

in the basis \((A_2, B_1)\) for \(H^0_{\mathbf{K}}\) and \((B_1, A_2)\) for \(H^0_{-\mathbf{K}}\). In Eq. (1), \(a, a^\dagger\) are the ladder operators and we have defined the parameters

\[ \zeta = 2 \text{sgn}(\gamma_0 \gamma_4) \sqrt{\beta \Delta_4 \gamma_1 + \beta \Delta}, \]

where \(\text{sgn}\) denotes the sign function and

\[ \beta = \frac{\hbar \omega^e_n}{\gamma_1} = 8.86 \times 10^{-3} B, \]

\[ \beta_4 = \left( \frac{\gamma_4}{\gamma_0} \right)^2 \frac{\hbar \omega^e_n}{\gamma_1} = 1.31 \times 10^{-5} B, \]

(with \(B\) in Tesla). The effective cyclotron frequency is defined by \(\omega^e_c = eB/m^*c\).

When \(\gamma_4 = \Delta = \Delta_B = 0\), the Landau level energies in each valley are given by

\[ E^0_{\xi\mathbf{K}} = \pm \sqrt{|N|(|N|+1)} \hbar \omega^e_n, \]

with \(N = 0, \pm 1, \pm 2, \ldots\) where \(N\) is the Landau level. If the Zeeman coupling is taken as zero, all Landau levels are thus four time degenerate (including spin and valley degrees of freedom) with the exception of \(N = 0\) that is eight times degenerate as shown below. With finite values of \(\gamma_4, \Delta\) or \(\Delta_B\), the degeneracy is lifted and we find for the states in \(N = 0\) the following spinors and energies (we use the Landau gauge with \(A = (0, Bx, 0)\)):

\[ \begin{pmatrix} 0 \\ h_{0,x,\sigma}(r) \end{pmatrix}, \quad E^0_{\xi\mathbf{K},0,x,\sigma} = \frac{1}{2} \xi \Delta_B - \sigma \Delta_z, \]

\[ \begin{pmatrix} 0 \\ h_{1,x,\sigma}(r) \end{pmatrix}, \quad E^0_{\xi\mathbf{K},1,x,\sigma} = \frac{1}{2} \xi \Delta_B - \sigma \Delta_z - \xi \beta \Delta_B + \zeta, \]

where we have added a Zeeman coupling \(\Delta_z = \mu_B B/2 = 0.58 \times 10^{-4} B\) eV (with \(B\) in Tesla) for more generality. In the absence of couplings and with \(\zeta = 0\), the \(N = 0\) Landau level has an extra orbital degeneracy since the two spinors above are then degenerate. The wave functions \(h_{n,x}(r) = e^{-iKx/\ell} \phi_n(x - X) / \sqrt{L_x}\) are the eigenstates of an electron with guiding center \(X\) in the Landau gauge, and \(\phi_n(x)\) is the wave function of the one-dimensional harmonic oscillator. Note that with our choice of normalization for the functions \(\phi_n(x)\), the action of the ladder operators on the states \(\phi_n(x)\) is given by \(a^\dagger \phi_n(x) = i \sqrt{n+1} \phi_{n+1}(x)\) and \(a \phi_n(x) = -i \sqrt{n} \phi_{n-1}(x)\).

We only consider Landau level \(N = 0\) in our work so that, from now on, we will drop the index \(N\). The index \(n\) will always refer to the orbital states \(n = 0, 1\) i.e. to \(\phi_0(x)\) and \(\phi_1(x)\). Note that it is clear from Eq. (6) that the valley \(\mathbf{K}(\mathbf{K}')\) eigenstates are localized in the top(bottom) layer. For \(N = 0\), the layer index is thus equivalent to the valley index.

### III. Quantum Hall Ferromagnets

To describe the electronic phases in Landau level \(N = 0\), we use a pseudospin language where we associate to the layer \(\xi = \pm \mathbf{K}\) (or valley) and orbitals \(n = 0, 1\) a spin-half pseudospin. When Coulomb interaction is considered in the Hartree-Fock approximation, the ground states at zero bias and \(\zeta = 0\) obey a set of Hund’s rules in which spin polarization is maximized first, then layer polarization to the greatest extent possible, and finally orbital polarization to the extent allowed by the first two rules.

In the pseudospin language, the ground states are thus various types of quantum Hall ferromagnets (QHF’s).

At zero bias and with \(\zeta = 0\), interlayer coherence is present in the ground state at all integer filling factors but there is no orbital or spin coherence. Interlayer coherence occurs naturally because of the proximity of the two layers (the interlayer separation \(d = 3.34\) Å is very small in bilayer graphene) but disappears quickly when a finite bias \(\Delta_B\) is applied. Adding a finite bias \(\Delta_B\) enriches considerably the phase diagram with new states such as orbital and spin QHF’s.

In previous works\cite{34}, it was found that above a critical bias \(\Delta_B(\nu)\), the ground state is an orbital QHF with Ising character (i.e. no orbital coherence) for \(\nu = -3\) and a ground state with an \(x - y\) character (or orbital coherence) for \(\nu = -1\). The dispersion relation of the orbital pseudospin mode is gapped at \(\omega(q = 0) = \beta \Delta_B\) in the former case and gapless with an highly anisotropic dispersion at small wave vector in the latter case. To get these results, it was assumed that complete spin polarization holds even at finite bias and the correction \(\zeta\) was not considered in the single-particle energies. When these two assumptions are relaxed, the phase diagram at \(\Delta_B \neq 0\) is modified. An exhaustive study of the phase diagram for uniform states at all integer filling factors \(\nu \in [-3, 4]\) will be presented elsewhere. We want to mention here, however, some changes that occur in the phase diagram.

First of all, we remark that orbital coherence is driven by the \(\Delta_B\) term in Eq. (5). Together with the term \(\zeta\), they lift the degeneracy between levels \(n = 0\) and \(n = 1\). The ordering of the non-interacting levels is then as illustrated in Fig. 4.

When the correction \(\zeta\) is neglected, the orbital splitting \(\Delta_B \ll \Delta_B\). For positive bias, the \(n = 0\) orbital state in the bottom(top) layer is lower(higher) in energy than the \(n = 1\) orbital state. The orbital energy splitting has thus different signs in the \(\mathbf{K}\) and \(\mathbf{K}'\) valleys. No orbital coherence is possible at \(\nu = -3\) because level \(n = 1\) is always above level \(n = 0\) at all bias. In that case, the ground state has all electrons in valley \(\mathbf{K}'\) with \(n = 0\) because the Coulomb exchange energy is more negative in \(n = 0\) than in \(n = 1\). The situation is different for \(\nu = -1\). In that case, the valley \(\mathbf{K}'\) is filled above a critical bias and the remaining electrons occupy valley \(\mathbf{K}\) where level \(n = 1\) is now below \(n = 0\). Now, because of the better exchange interaction in \(n = 0\), the eigenstates
in \( K \) are bonding and anti-bonding states of \( n = 0 \) and \( n = 1 \) with the electrons mostly in \( n = 0 \) at low bias and mostly in \( n = 1 \) at large bias. This produces an orbital coherent state.

If \( \zeta \neq 0 \), we see from Eq. (6) that the only effect is to push the orbital coherent states to higher bias. In itself, this is not dramatic provided our effective two-band model is still valid at the new critical bias. But, if the bias is increased, the spin degree of freedom must be considered. At zero bias, the four spin up states are separated in energy from the four spin down states by an exchange-enhanced Zeeman gap of order \( c^2/\kappa \ell \) where \( \kappa \) is the dielectric constant of the substrate and \( \ell = \sqrt{\hbar c/eB} \) is the magnetic length. With bias, two levels with spin up(down) are shifted upward(downward) in energy. When levels with different spins cross, states with spin coherence \((x-y\text{ spin QHF})\) become possible and they may replace the orbital state in the phase diagram. Whether this is the case or not must be decided by numerical calculations. For \( \gamma_4 = 0.12 \), a numerical calculation shows that the orbital coherent state is absent for \( \nu = -1 \). We remark, however, that the exact value of \( \gamma_4 \) is not precisely known and than orbital coherence does occur at smaller values of this parameter.

Fortunately, the phase diagram for the eight-level system is very rich and numerical calculations show that orbital coherence with no interlayer or spin coherences occurs in some range of bias but at the higher filling factors \( \nu = 1 \) and \( \nu = 3 \) and even when \( \gamma_4 \) is as large as \( \gamma_4 = 0.12 \) and spin mixing is considered. Since we are confident that orbital coherent states do occur in the phase diagram in bilayer graphene, we will study, in this paper, the simplest case with \( \nu = -1, \gamma_4 = 0 \) and complete spin polarization. Our results should apply with some minor changes to the orbital states at \( \nu = 1 \) and \( \nu = 3 \).

In concluding this section, we remark that the phase diagram for filling factors \( \nu = -1 \) has been studied in some details before but only the homogeneous states have been considered. Non-uniform states such as Skyrmion crystals with valley or orbital pseudospin textures have also been considered but not integer filling factors only. In this work, our focus is on the non-uniform states at precisely \( \nu = -1 \).

IV. FORMALISM

In the rest of this paper, we concentrate on the study of the orbital coherent states at \( \nu = -1 \), assuming \( \zeta = 0 \) and complete spin polarization. Our calculations are all done at zero temperature.

A. Hartree-Fock Hamiltonian

With the restrictions outlined above, the analysis is reduced to that of a two-level system in valley \( K \) since the two filled levels in valley \( K' \) can be considered as inert. The non-interacting Hamiltonian is given by Eq. (1) while the many-body Hamiltonian in the Hartree-Fock approximation is

\[
H_{HF} = N_\varphi E_n \rho_{n,n} (0) + N_\varphi \sum_q H_{n_1,n_2,n_3,n_4} (q) \langle \rho_{n_1,n_2} (-q) \rangle \rho_{n_3,n_4} (q) - N_\varphi \sum_q X_{n_1,n_4,n_3,n_2} (q) \langle \rho_{n_1,n_2} (-q) \rangle \rho_{n_3,n_4} (q),
\]

where \( n_i = 0, 1 \) is the orbital index and \( N_\varphi = S/2\pi \ell^2 \) is the Landau level degeneracy. The single-particle energies can be simplified to \( E_q = -n\beta \Delta_B \). In deriving Eq. (7), we have taken into account a neutralizing positive background so that the \( q = 0 \) contribution is absent in the Hartree term. This convention is indicated by the bar over the summation.

The density operators in Eq. (4), are defined by

\[
\rho_{n_1,n_2} (q) = \frac{1}{N_\varphi} \sum_{X_1,X_2} e^{-\frac{i}{\hbar} \mathbf{q} \cdot (X_1+X_2)} \times c_{X_1,n_1}^\dagger c_{X_2,n_2}^\dagger \delta_{X_1,X_2+q} e^{\varphi q^2},
\]

where \( c_{X,n}^\dagger (c_{X,n}) \) creates(destroys) an electron in state \((X,n)\) in the Landau gauge. The Hartree and Fock interactions are given by

\[
H_{n_1,n_2,n_3,n_4} (q) = \left( \frac{e^2}{\kappa \ell} \right) \frac{1}{q^2} K_{n_1,n_2} (q) K_{n_3,n_4} (-q),
\]

\[
X_{n_1,n_2,n_3,n_4} (q) = \int \frac{dp^2}{2\pi} H_{n_1,n_2,n_3,n_4} (p) e^{iq \cdot p \ell^2}.
\]
where $\kappa$ is the effective dielectric constant at the position of the graphene layers. The Coulomb energy $e^2/\kappa \ell = 1.3 \sqrt{B}$ meV with $B$ in Tesla and we take $\kappa = 5$.

The form factors which appear in $H$ and $X$ are given by

$$K_{0,0}(q) = \exp\left(-\frac{q^2 \ell^2}{4}\right), \quad (11)$$

$$K_{1,1}(q) = \exp\left(-\frac{q^2 \ell^2}{4}\right) \left(1 - \frac{q^2 \ell^2}{2}\right),$$

$$K_{1,0}(q) = \left(\frac{iq_\parallel + iy_\parallel}{\sqrt{2}}\right) \exp\left(-\frac{q^2 \ell^2}{4}\right),$$

$$K_{0,1}(q) = \left(\frac{-iq_\parallel + iy_\parallel}{\sqrt{2}}\right) \exp\left(-\frac{q^2 \ell^2}{4}\right).$$

They capture the character of the two different orbital states and play an important role in the physics of the orbital phase. Detailed expressions for the Hartree and Fock interactions $H$ and $X$ are given in Appendix A of Ref. [3].

**B. Order parameters in the coherent phases**

The order parameters of the orbital phases are obtained from the single-particle Green’s function

$$G_{n,n'}(X, X', \tau) = -\left\langle T_\tau c_n(X) c^\dagger_{n',X'}(0)\right\rangle, \quad (12)$$

where $T_\tau$ is the imaginary time ordering operator. We define the Fourier transform of the single-particle Green’s function as

$$G_{n,n'}(q, \tau) = \frac{1}{N_g} \sum_{X,X'} e^{-i \frac{2\pi}{L_g} (X + X')} \times \delta_{X,X'-q_\parallel \ell^2} G_{n,n'}(X, X', \tau) \quad \text{(13)}$$

so that the order parameters of the coherent phases are related to the Green’s function by

$$\langle \rho_{n,n'}(q) \rangle = G_{n',n}(q, \tau = 0^-). \quad (14)$$

The equation of motion for the Green’s function in the Matsubara formalism and in the Hartree-Fock approximation is given by

$$\sum_{m=0,1} \sum_{q'} T^\dagger_{n,m}(q, q') \gamma_{q,q'} G_{m,n'}(q, i\omega_n) = \delta_{n,n'} \delta_{q,0}, \quad (15)$$

and the phase factor

$$\gamma_{q,q'} = e^{-iq \times q' \ell^2/2}. \quad (17)$$

The Hartree and Fock potentials are defined by

$$U^H_{n_3,n_4}(q) = \sum_{n_1,n_2} H_{n_1,n_2,n_3,n_4} (-q) \langle \rho_{n_1,n_2}(q) \rangle, \quad (18)$$

$$U^F_{n_3,n_4}(q) = \sum_{n_1,n_2} X_{n_1,n_4,n_3,n_2} (-q) \langle \rho_{n_1,n_2}(q) \rangle. \quad (19)$$

The self-consistent Eq. (15) must be solved numerically in an iterative way in order to get the order parameters in the different orbital phases. Once this is done, the Hartree-Fock energy is obtained from

$$\frac{E_{HF}}{N} = -\beta \Delta_B \langle \rho_{1,1}(0) \rangle \quad (20)$$

$$+ \frac{1}{2} \sum_q \sum_{n_1,...,n_4} H_{n_1,n_2,n_3,n_4}(q) \langle \rho_{n_1,n_2}(-q) \rangle \langle \rho_{n_3,n_4}(q) \rangle$$

$$- \frac{1}{2} \sum_q \sum_{n_1,...n_4} X_{n_1,n_4,n_3,n_2}(q) \langle \rho_{n_1,n_2}(-q) \rangle \langle \rho_{n_3,n_4}(q) \rangle.$$
we must use the true density \( n(\mathbf{r}) \) which is the Fourier transform of

\[
    n(\mathbf{q}) = N_\varphi \sum_{n_1, n_2} \rho_{n_1, n_2}(\mathbf{q}) K_{n_1, n_2}(-\mathbf{q}).
\]

(25)

This coupling can be written as

\[
    H = -eN_\varphi \int d\mathbf{r} \overline{\varrho}_\mathbf{n}(\mathbf{r}) \phi(\mathbf{r})
    + \sqrt{2e} N_\varphi (\rho_x(\mathbf{q} = 0) E_x - \rho_y(\mathbf{q} = 0) E_y),
\]

where we have defined \( \overline{\varrho}_\mathbf{n}(\mathbf{q}) = \exp(-q^2\varepsilon^2/4) \rho_i(\mathbf{q}) \) with \( i = n, x, y, z \). In the three states studied in this paper (uniform, helical, and Skyrme crystal), the guiding-center density \( \langle \rho_n(\mathbf{r}) \rangle \) is uniform and so \( \langle \rho_n(\mathbf{q}) \rangle = \delta_{\mathbf{q}, 0} \). For these states, the coupling with the electric field is simply given by

\[
    H_E = -N_\varphi \mathbf{d}(0) \cdot \mathbf{E},
\]

(27)

where we have defined the electric dipole operator

\[
    \mathbf{d}(\mathbf{q}) = -e\sqrt{2e} \frac{\varepsilon^2}{4} (\overline{\varrho}_n(\mathbf{q}), -\overline{\varrho}_y(\mathbf{q})).
\]

(28)

The fact that orbital coherence leads to a finite density of electric dipoles in the plane of the layers was first shown in Ref. [8].

Note that for the states with \( \langle \rho_n(\mathbf{q}) \rangle = \delta_{\mathbf{q}, 0} \), the sum rules of Eq. (21) are, in pseudospin language, equivalent to

\[
    \sum_{\mathbf{q}} |\langle \rho_x(\mathbf{q}) \rangle|^2 + |\langle \rho_y(\mathbf{q}) \rangle|^2 + |\langle \rho_z(\mathbf{q}) \rangle|^2 = \frac{1}{4}
\]

(29)

which is an average normalization condition for the pseudospin field vector i.e. \( \int d\mathbf{r} |\langle \rho(\mathbf{r}) \rangle|^2 = 1/4S \) where \( S \) is the area of the 2DEG. The modulus of the pseudospin field is, in general, not constant in space.

D. Collective modes

To study the collective excitations, we compute the two-particle Green’s function

\[
    \chi_{n_1, n_2, n_3, n_4}(\mathbf{q}, \mathbf{q}'; \tau)
    = -N_\varphi \langle T_{\tau} \rho_{n_1, n_2}(\mathbf{q}, \tau) \rho_{n_3, n_4}(\mathbf{q}', 0) \rangle
    + N_\varphi \langle \rho_{n_1, n_2}(\mathbf{q}) \rangle \langle \rho_{n_3, n_4}(\mathbf{q}') \rangle
\]

(30)

in the generalized random-phase approximation (GRPA). In this approximation, \( \chi_{n_1, n_2, n_3, n_4}(\mathbf{q}, \mathbf{q}'; \tau) \) is the solution of the equation

\[
    \chi_{n_1, n_2, n_3, n_4}(\mathbf{q}, \mathbf{q}'; \tau)
    = \chi_{n_1, n_2, n_3, n_4}^0(\mathbf{q}, \mathbf{q}'; i\Omega_n)
    \times H_{n_5, n_6, n_7, n_8}(\mathbf{q}''') \chi_{n_7, n_8, n_9, n_4}(\mathbf{q}'', \mathbf{q}'; i\Omega_n)
    - \frac{1}{4\hbar^2} \sum_{n_5, n_6, n_7, n_8} \chi_{n_5, n_6, n_7, n_8}(\mathbf{q}''') \chi_{n_7, n_8, n_9, n_4}(\mathbf{q}'', \mathbf{q}'; i\Omega_n)
    \times X_{n_5, n_6, n_7, n_8}(\mathbf{q}''') \chi_{n_7, n_8, n_9, n_4}(\mathbf{q}'', \mathbf{q}'; i\Omega_n)
\]

where \( \Omega_n \) is a bosonic Matsubara frequency and the Hartree-Fock two-particle Green’s function \( \chi_{n_1, n_2, n_3, n_4}^0(\mathbf{q}, \mathbf{q}'; i\Omega_n) \) is given by

\[
    [i\Omega_n - (E_{n_2} - E_{n_1})] \chi_{n_1, n_2, n_3, n_4}^0(\mathbf{q}, \mathbf{q}'; \Omega_n)
    = \hbar \langle \rho_{n_1, n_4}(\mathbf{q} - \mathbf{q}') \rangle \delta_{\mathbf{q}, \mathbf{q}'} \gamma^{*}_{\mathbf{q}'} \mathbf{q}'
    - \frac{\hbar}{4\hbar^2} \sum_{n_5, n_6} U_{n_5, n_6}^H(\mathbf{q} - \mathbf{q}'') \gamma_{\mathbf{q}, \mathbf{q}'} \chi_{n_5, n_6, n_2, n_3, n_4}(\mathbf{q}'', \mathbf{q}', i\Omega_n)
    + \sum_{n_5, n_6} U_{n_5, n_6}^H(\mathbf{q} - \mathbf{q}'') \gamma_{\mathbf{q}, \mathbf{q}'} \chi_{n_5, n_6, n_2, n_3, n_4}(\mathbf{q}'', \mathbf{q}', i\Omega_n)
    + \sum_{n_5, n_6} U_{n_5, n_6}^F(\mathbf{q} - \mathbf{q}'') \gamma_{\mathbf{q}, \mathbf{q}'} \chi_{n_5, n_6, n_2, n_3, n_4}(\mathbf{q}'', \mathbf{q}', i\Omega_n)
    - \sum_{n_5, n_6} U_{n_5, n_6}^F(\mathbf{q} - \mathbf{q}'') \gamma_{\mathbf{q}, \mathbf{q}'} \chi_{n_5, n_6, n_2, n_3, n_4}(\mathbf{q}'', \mathbf{q}', i\Omega_n)
\]

(31)

(32)

Note that the response functions depend only on the order parameters \( \langle \rho_{n, m}(\mathbf{q}) \rangle \) computed in the HFA. Eqs. (31,32) can be solved numerically by writing them in a matrix form. The collective excitations are then given by the poles of the retarded Green’s function \( \chi_{n_1, n_2, n_3, n_4}^R(\mathbf{q}, \mathbf{q}', \omega) \) which is obtained by the analytic continuation \( i\Omega_n \to \omega + i\delta \) of the corresponding two-particle Green’s function.

V. PHASE DIAGRAM FOR THE ORBITAL PHASES AT \( \nu = -1 \)

We define an average pseudospin field by

\[
    \mathbf{p}(\mathbf{q}) = \sqrt{2e} \varepsilon (\langle \rho_x(\mathbf{q}) \rangle, \langle \rho_y(\mathbf{q}) \rangle, \langle \rho_z(\mathbf{q}) \rangle),
\]

(33)

so that the dipole field can be related to the parallel component of this vector by

\[
    \mathbf{d}(\mathbf{q}) = e^{-z^2/2} \mathbf{p}_\parallel(\mathbf{q}).
\]

(34)

After a lengthy calculation, we can put the Hartree-Fock energy per electron given by Eq. (20) in the pseudospin
In this equation, the bias 
\[ \overline{\Delta}_B = \Delta_B / (e^2/\kappa \ell) \], \( \alpha = \sqrt{2} \kappa \ell \), \( I \) is the \( 2 \times 2 \) unit matrix and \( \Lambda (\mathbf{q}) \) is given by

\[
\Lambda (\mathbf{q}) = \begin{pmatrix} \cos (2 \varphi_{\mathbf{q}}) & \sin (2 \varphi_{\mathbf{q}}) \\ \sin (2 \varphi_{\mathbf{q}}) & -\cos (2 \varphi_{\mathbf{q}}) \end{pmatrix},
\]

(36)

where \( \varphi_{\mathbf{q}} \) is the angle between the wave vector \( \mathbf{q} \) and the \( x \) axis. The bias \( \overline{\Delta}_B /2 \) is defined as the bias for which the charge is equally distributed between the two levels \( n = 0, 1 \) in the uniform phase. Finally, the pseudospin interactions are given by

\[ a (\mathbf{q}) = 2 \left( H_{0,1,1,0} (\mathbf{q}) - X_{1,1,0,0} (\mathbf{q}) \right) \]

(37)

\[ = q \ell e^{-\frac{2q^2 \ell^2}{4}} \left( -2 \int_0^\infty dy \left( 1 - \frac{y^2}{2} \right) e^{-y^2/2} J_0 (q \ell y) \right), \]

\[ b (\mathbf{q}) = 2 e^{2i \varphi_{\mathbf{q}}} \left( H_{1,0,0,1} (\mathbf{q}) - X_{1,0,1,0} (\mathbf{q}) \right) \]

(38)

\[ = q \ell e^{-\frac{2q^2 \ell^2}{4}} - \int_0^\infty dy y^2 e^{-y^2/2} J_2 (q \ell y), \]

\[ c (\mathbf{q}) = H_{0,0,0,0} (\mathbf{q}) - X_{0,0,0,0} (\mathbf{q}) + H_{1,1,1,1} (\mathbf{q}) - X_{1,1,1,1} (\mathbf{q}) - 2 (H_{1,1,0,0} (\mathbf{q}) - X_{0,1,1,0} (\mathbf{q})) \]

(39)

\[ = \frac{q^3 \ell^3 e^{-\frac{2q^2 \ell^2}{4}}}{4} \left( - \int_0^\infty dy \left( 2 - 2y^2 + \frac{y^4}{4} \right) e^{-y^2/2} J_0 (q \ell y) \right), \]

\[ d (\mathbf{q}) = -4 i e^{i \varphi_{\mathbf{q}}} \left( H_{1,0,0,0} (\mathbf{q}) - X_{1,0,0,0} (\mathbf{q}) \right) -4 i e^{i \varphi_{\mathbf{q}}} \left( H_{1,1,1,0} (\mathbf{q}) - X_{1,1,1,0} (\mathbf{q}) \right) \]

(40)

\[ = \frac{4 q^2 \ell^2 e^{-\frac{2q^2 \ell^2}{4}}}{\sqrt{2}} \left( - \frac{1}{\sqrt{2}} \int_0^\infty dy y e^{-y^2/2} J_1 (q \ell y) \right). \]

These interactions are plotted in Fig. 3. Their values at \( q = 0 \) are \( a (0) = -\sqrt{2} b (0) = 0, c (0) = -\frac{4}{3} \sqrt{2} d (0) = 0. \)

Eq. (35) is the energy functional of an effective helical pseudospin ferromagnet. Apart from the constant terms, there are four distinct contributions to the total energy:

1. The term \( \beta \left( \overline{\Delta}_B - \overline{\Delta}_B^{(1)}/2 \right) p_z (0) \) is an effective Zeeman coupling that changes sign at the bias \( \overline{\Delta}_B^{(1)}/2 \). For \( \overline{\Delta}_B < \overline{\Delta}_B^{(1)}/2 \), the pseudospins order along the positive \( z \) axis. They order along \( -z \) for \( \overline{\Delta}_B > \overline{\Delta}_B^{(1)}/2 \).

2. The terms involving \( a (\mathbf{q}) \) and \( c (\mathbf{q}) \) are non local Heisenberg exchange interactions between the pseudospins.

3. The term with \( b (\mathbf{q}) \Lambda (\mathbf{q}) \) is a dipolar interaction. In the small wave vector limit, only the Hartree term \( q \ell e^{-\frac{2q^2 \ell^2}{4}} \) in Eq. (38) contributes significantly to \( b (\mathbf{q}) \). To first order in \( q \ell, b (\mathbf{q}) \rightarrow q \ell \). The dipolar term can be related to the Fourier transform of the dipole-dipole electrostatic interaction.

4. The fourth term is a Dzyaloshinskii-Moriya (DM) interaction between the pseudospins. With the redefinition \( p_y \rightarrow -p_z \) and \( p_z \rightarrow p_y \), it takes the more conventional DM form which is the Fourier transform of \( D \int d \mathbf{r} (\mathbf{p} \cdot (\nabla \times \mathbf{p})) \) where \( D \) is a constant. In our case, the DM constant \( D \) becomes a non local function \( d (\mathbf{r} - \mathbf{r}') \). It is interesting to remark that a DM occurs in our model although we
are not dealing with real spins or spin-orbit interaction. The physical origin of this term was discussed in Ref. 3. From a microscopic point of view, $d(q)$ contains only the interactions $H_{n_1,n_2,n_3,n_4}(q)$ and $X_{n_1,n_2,n_3,n_4}(q)$ that do not conserve the orbital index (for example: $X_{1,0,0,0}(q)$). These terms arise in our system because we are dealing with two different orbitals $h_0(r)$ and $h_1(r)$. On the contrary, the other effective interactions $a(q), b(q)$ and $c(q)$ consist of terms that conserve the orbital index.

The exchange interactions tends to align the pseudospins together while the DM term favors a rotation of the pseudospins in real space. This type of competition is usually present in helical magnets such as MnSi and Fe$_{1-x}$Co$_x$Si. In fact, our numerical calculation gives a phase diagram which is similar to that found recently in the helical magnet Fe$_{0.5}$Co$_{0.5}$Si where Skyrmion crystal and an helical phase have been observed using Lorentz transmission electron microscopy.

Because the instability of the uniform phase occurs at $q_y l \approx -2$ (see below), it is not useful to derive a long-wavelength approximation of Eq. (35). The calculation of the optimal wave vector $q_l$ for the helical state must be done numerically. To obtain the phase diagram for the orbital phases, we evaluate the Hartree-Fock energy for (once again, $\langle \rho_n(q) \rangle = \delta_{q,0}$ in all three cases):

1. The uniform phase (UP) defined by $p(q) = p_0 \delta_{q,0}$
2. The Skrymion crystal phase (SKP) defined by the set of order parameters $\{p(G)\}$ where $G$ is a reciprocal lattice vector of the crystal. The triangular crystal has lower energy than the square or rectangular lattices. (We have not tried other lattice types however.)
3. The helical phase (HP) defined the order parameters $\{p(q)\}$ where $q = n q_0$ where $n = 0, \pm 1, \pm 2, ...$

Within this set of states, we find the following ordering:

| $\Delta_B$ | UP | SKP | HP | HP |
|-----------|----|-----|----|----|
| $< 0.06$  |    |     |    |    |
| $0.06 < \Delta_B < 0.52$ |    |     |    |    |
| $0.52 < \Delta_B < 3.02$ |    |     |    |    |
| $2.98 < \Delta_B < 3.44$ |    |     |    |    |
| $3.44 < \Delta_B < 3.536$ |    |     |    |    |
| $> 3.536$ |    |     |    |    |

Above $\Delta_B = \Delta_B^{(1)} = 3.536$, all electrons are in level $n = 1$ and there is no orbital coherence. We remark that the phase diagram is symmetrical with respect to the bias $\Delta_B^{(1)}/2$ (see Eq. (53) below). When the layer index is added to the picture, the ground state is an interlayer coherent phase with no orbital coherence for $\Delta_B < 0.0022$. Also, although the guiding-center density $\langle \rho_1(r) \rangle$ is uniform in space, the real density $n(r)$ is not. For $\langle \rho_1(r) \rangle$ to be constant in space, we must have $\langle \rho_{1,0}(q) \rangle = -\langle \rho_{1,1}(q) \rangle$ for $q \neq 0$ in the Skyrmion and helical phases.

### A. Uniform phase

The uniform phase has all orbital pseudospins pointing in the same direction in space. Its energy is given by

$$E_{HF}/N = \left( \frac{\epsilon^2}{\kappa l} \right)$$

$$= -\frac{11}{32} \sqrt{\frac{\pi}{2}} \frac{\beta}{2} \Delta_B$$

$$+ \frac{1}{\alpha} \left( \Delta_B - \frac{1}{2} \Delta_B^{(1)} \right) p_z(0)$$

$$- \frac{1}{2\alpha^2} \sqrt{\frac{\pi}{2}} \left( p_x(0) \cdot p_y(0) + \frac{3}{4} p_z(0) p_z(0) \right).$$

The Hartree-Fock solution for the order parameters is

$$p_z(0) = \frac{1}{2} - \frac{\Delta_B}{\Delta_B^{(1)}},$$

$$|p_{\parallel}(0)| = \sqrt{\frac{1}{4} - (p_z(0))^2}.$$

At the critical field

$$\Delta_B^{(1)} = \frac{1}{4\beta} \sqrt{\frac{\pi}{2}},$$

all electrons are pushed into level $n = 1$ and the orbital coherence is lost. For $B = 10$ T, we find that $\Delta_B^{(1)} = 3.536$. From Eq. (42), we have the symmetry

$$|p_{\parallel}(0)| = \left| p_{\parallel}(0) \right| \Delta_B^{(1)} \Delta_B,$$

$$p_z(0) \Delta_B = -p_z(0) \Delta_B^{(1)} \Delta_B.$$

The band structure consists of two dispersionless bands with energy

$$E_+ = -\frac{1}{2} \sqrt{\frac{\pi}{2}} \left( \frac{\epsilon^2}{\kappa l} \right),$$

$$E_- = -\sqrt{\frac{\pi}{2}} \left( \frac{\epsilon^2}{\kappa l} \right).$$

The energy in the middle of these two bands is

$$\Delta_B^{(\ast)} = \frac{1}{2} (E_+ + E_-) = -\frac{3}{4} \sqrt{\frac{\pi}{2}} \left( \frac{\epsilon^2}{\kappa l} \right).$$

We obviously have the symmetry

$$E_+ = 2 \Delta_B^{(\ast)} - E_-. $$

As can be seen from Eq. (41), the energy of the UP is independent of the orientation of $p_{\parallel}(0)$ in the $x-y$ plane. The UP is a quantum Hall orbital pseudomagnet. It follows that this phase supports a gapless $x-y$ orbital pseudospin wave excitation, a Goldstone mode related to the breaking of the rotation symmetry about the $z$ axis.
The pseudospins remain parallel during their motion in this mode (for \( q = 0 \)) and so the DM term does not open a gap as it does for the corresponding mode in the helical state (see below). The contribution of the DM is strongest at larger bias and shorter wavelength. The mode’s dispersion relation is plotted in Fig. 1 for different values of the bias. It is highly anisotropic with an unusual square root dispersion in the \( q_y \) direction if the dipoles are oriented in the direction \(-\hat{x}\). The DM instability of the UP occurs at \( \Delta B_{\text{DM}} = 0.37 \) (as well as at \( \Delta B_{\text{DM}} = 0.37 \) if the bias is decreased from \( \Delta B_{\text{DM}} \)) and at finite wave vector \( q_y \ell \approx -2 \). This suggests a transition to a ground state where the orbital pseudospin field is no longer uniform as in the HP. In reality, we find that this instability is preempted by a transition to a Skyrmion crystal at \( \Delta B = 0.06 \) as we discussed in the previous section.

\[
\text{FIG. 4: (Color online) Dispersion relation of the orbital pseudospin mode in the uniform phase for different values of the bias. For (1) to (3) respectively: } \Delta B / (e^2/\kappa\ell) = 0.1, 0.3, 0.37.
\]

B. Helical phase

The exact helical phase (HP) given by the solution of the Hartree-Fock equations is too complex to study analytically and so we give the numerical solution below. However it is instructive, in order to understand the instability of the UP, to look at the energy of a simple helical phase (SHP) with only one wave vector:

\[
\mathbf{p}_{\text{SHP}} (\mathbf{r}) = \frac{\alpha}{2S} p_0 [\mathbf{e}_1 \cos (\mathbf{q} \cdot \mathbf{r}) + \mathbf{e}_2 \sin (\mathbf{q} \cdot \mathbf{r})] + \frac{\alpha}{2S} \eta \hat{z}.
\]

where \( \mathbf{e}_1, \mathbf{e}_2 \) are orthogonal unit vectors and \( \alpha \) is a normalization factor. The last term with \( \eta \) describes the uniform polarization due to the bias. It makes the magnitude of \( \mathbf{p}_{\text{SHP}} \) position-dependent. It is however possible to impose an average normalization condition. For simplicity, we will choose bias \( \Delta B_{\text{HP}} / 2 \) where \( \beta = 0 \) and \( \alpha = 1 \). We define the unit vectors:

\[
\begin{align*}
\mathbf{e}_1 &= \sin (\theta_1) \cos (\varphi_1) \hat{x} + \sin (\theta_1) \sin (\varphi_1) \hat{y} + \cos (\theta_1) \hat{z}, \\
\mathbf{e}_2 &= \sin (\theta_2) \cos (\varphi_2) \hat{x} + \sin (\theta_2) \sin (\varphi_2) \hat{y} + \cos (\theta_2) \hat{z}, \\
\mathbf{q} &= \cos (\varphi_q) \hat{x} + \sin (\varphi_q) \hat{y}.
\end{align*}
\]

The energy of the SHP is then

\[
\frac{E_{\text{SHP}}}{N} / \left( \frac{e^2}{\kappa\ell} \right) = -\frac{11}{32} \sqrt{\frac{\pi}{2}} - \frac{1}{2} B \Delta B
\]

where all interactions \( a(q), b(q), c(q), d(q) \) are negative. Inspection of Eq. (52) and Fig. 3 shows that the lowest-energy solution is obtained when \( \theta_1 = 0, \theta_2 = \pi/2 \) and \( \varphi_2 = \varphi_q \). The vector \( \mathbf{e}_2 \) is free to take any orientation in the \( x-y \) plane. The plane of polarization of the helix is thus \( \hat{z} - \mathbf{e}_2 \) and its energy is given by

\[
\frac{E_{\text{SHP}}}{N} / \left( \frac{e^2}{\kappa\ell} \right) = -\frac{11}{32} \sqrt{\frac{\pi}{2}} - \frac{1}{2} B \Delta B
\]

In this equation, \( \theta_1, \varphi_1 \) and \( \theta_2, \varphi_2 \) must be chosen to make \( \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \). For values of \( q \) where all interactions \( a(q), b(q), c(q), d(q) \) are negative, inspection of Eq. (52) and Fig. 3 shows that the lowest-energy solution is obtained when \( \theta_1 = 0, \theta_2 = \pi/2 \) and \( \varphi_2 = \varphi_q \). The vector \( \mathbf{e}_2 \) is free to take any orientation in the \( x-y \) plane. The plane of polarization of the helix is thus \( \hat{z} - \mathbf{e}_2 \) and its energy is given by

\[
\frac{E_{\text{SHP}}}{N} / \left( \frac{e^2}{\kappa\ell} \right) = -\frac{11}{32} \sqrt{\frac{\pi}{2}} - \frac{1}{2} B \Delta B
\]

The sum of the four interactions is plotted in Fig. 3. From this figure, we see that the wave vector that minimizes the energy of the helix is \( q_0 \ell \approx 2.3 \). Note that an helix in the \( x-y \) plane (with \( \theta_1 = \theta_2 = \pi/2, \varphi_1 = 0, \varphi_2 = \pi/2 \)) has a higher energy given by

\[
\frac{E_{\text{SHP}}}{N} / \left( \frac{e^2}{\kappa\ell} \right) = -\frac{11}{32} \sqrt{\frac{\pi}{2}} - \frac{1}{2} B \Delta B + \frac{a(q)}{16}.
\]
$b + c$) has its minimum at $q = 0$. We conclude that these interactions do not favor the formation of a non-uniform state. In our system, the DM interaction is responsible for the formation of the helix with a finite wavevector. We remark that our helix has its wave vector $q$ parallel to $e_2$ instead of $\hat{n} = e_1 \times e_2$ as is often the case.

1. Energy

When we solve for the exact helical phase using Eq. (15), we find a multicomponent helix with a finite value of $\eta_z$. We choose $e_2 = \hat{x}$ so that the helix wave vector $q = nq_0\hat{x}$ with $q_0 = 2\pi/\lambda$ and the pseudospin vector $\langle \hat{\rho} \rangle$ rotates clockwise in the $z-x$ plane. We find that the optimal period of the helix is $\lambda/\ell = 2.74 + 0.01(\Delta_B - \Delta_B^{(1)})/2$, close to the value we found above for the SHP, and varies only slightly with bias. Since $\ell (\text{Å}) = 256/\sqrt{B}(\text{T})$, the period of the helix is of the order 220 Å for $B = 10$ T.

The energy of the numerical solution is independent of the direction of the wave vector $q$ in the $x-y$ plane, as expected. As for the SHP, the plane of polarization is given by $(e_1, e_2) = (q_0, \hat{z})$. The energy of the HP is plotted in Fig. 5 along with that of the SHP. The energy of the HP is obviously very close to that of the SHP. We find indeed that the number of Fourier components in the HP is small.

2. Pseudospin pattern in real space

Fig. 6 shows the orbital pseudospin field in real space at different biases. The following symmetry relations hold

$$\langle \rho_x(x) \rangle_{\Delta_B} = \langle \rho_x(-x) \rangle_{\Delta_B^{(1)} - \Delta_B},$$

$$\langle \rho_z(x) \rangle_{\Delta_B} = -\langle \rho_z(-x) \rangle_{\Delta_B^{(1)} - \Delta_B}.$$  

In the numerical solution, $\langle \rho_x (q = 0) \rangle = \langle \rho_y (q = 0) \rangle = 0$ but $\langle \rho_z (q = 0) \rangle \neq 0$ (except at the special bias $\Delta_B^{(1)}/2$). As expected, the helix has a finite value for the term $\eta$ in Eq. (50) because the bias tilts the pseudospin vector away from the $z$ axis.

![Fig. 5](image1.png)

**FIG. 5:** (Color online) Energy of uniform (UP), exact helix (HP), and simple helix (SHP) as a function of bias at $B = 10$ T.

![Fig. 6](image2.png)

**FIG. 6:** (Color online) Density profile for the pseudospin fields $\langle \rho_x(X) \rangle$ and $\langle \rho_z(X) \rangle$ in the helical phase for two different biases.

In the helical phase $\langle \rho_{0,0} (q) \rangle + \langle \rho_{1,1} (q) \rangle = 1$ and so the orbital density $\langle \rho_n (r) \rangle = 1/2\pi\ell^2$ is constant in real space. The densities

$$\langle \rho_{0,0} (r) \rangle = \frac{1}{2} + \langle \rho_z (r) \rangle,$$

$$\langle \rho_{1,1} (r) \rangle = \frac{1}{2} - \langle \rho_z (r) \rangle.$$  

3. Band structure

For a modulation of the pseudospin density in only one direction such as in the helical state, Eq. (15) for the single-particle Green’s function can be written more...
simply as

\[ i\omega_n G_{n,m}(X,\omega_n) - \sum_k \Lambda_{n,k}(X) G_{k,m}(X,\omega_n) = \delta_{n,m}, \]

with \( n, m, k = 0, 1 \) where

\[ G_{n,m}(X,\tau) = -\left\langle T_\tau c_{n,X}(\tau) c_{m,X}^\dagger(0) \right\rangle, \]

and

\[ \Lambda_{n,m}(X) = -\beta \Delta_B \delta_{n,m} \delta_{n,1} + \sum_q \left( U^H(n,m,q) - U^F(n,m,q) \right) e^{iqX}. \]

The order parameters are given by

\[ \langle \rho_{n,m}(q) \rangle = \frac{1}{N_\rho} \delta_{q,0} \sum_X e^{-iqX} \langle \rho_{n,m}(X) \rangle, \]

with

\[ \langle \rho_{n,m}(X) \rangle = G_{m,n}(X,\tau = 0^-) . \]

We can solve formally for the Green’s function to find

\[ G_{n,m}(X,\omega_n) = \frac{A_{n,m}(X)}{i\omega_n + \mu/\hbar - E_+(X)/\hbar} + \frac{B_{n,m}(X)}{i\omega_n + \mu/\hbar - E_-(X)/\hbar}. \]

The band structure consists of two bands with dispersion \( E_{\pm}(X) \) given by

\[ E_{\pm}(X) = \frac{1}{2} |A_{0,0}(X) + \Lambda_{1,1}(X)| \]

\[ \pm \frac{1}{2} \sqrt{|A_{0,0}(X) - \Lambda_{1,1}(X)|^2 + 4 |A_{0,1}(X)|^2}, \]

with

\[ A_{n,m}(X) = \frac{E_+(X) \delta_{n,m} - \Lambda(\pi)(X)}{E_+(X) - E_-(X)}, \]

\[ B_{n,m}(X) = \frac{\Lambda(\pi)(X) - E_-(X) \delta_{n,m}}{E_+(X) - E_-(X)}, \]

and \( \pi = 1 - n \), etc. At \( T = 0 \) K,

\[ \langle \rho_{n,m}(X) \rangle = B_{n,n}(X) . \]

It is easy to show from Eq. (59) that, at \( T = 0 \) K, we have the sum rule

\[ |\langle \bar{\rho}(X) \rangle|^2 = \frac{1}{4}. \]

Moreover, From Eqs. (66,67), we have also

\[ \langle \rho_{0,0}(X) \rangle + \langle \rho_{1,1}(X) \rangle = 1, \]

so that the total guiding-center density is unmodulated in the spiral phase. Note that the real density \( n(r) \) given by Eq. (25) is modulated however. The modulus of the pseudospin vector \( |\langle \bar{\rho}(X) \rangle| = 1/2 \) is constant in space. This is not the case when there is a two-dimensional modulation of the pseudospin texture as we will see in the crystal phase.

The band structure of the HP is shown in Fig. 7 for different values of the bias. The lowest band \( E_-(X) \) is completely filled so that the system is insulating. There is a continuum of electron-hole excitations in the energy range \( E_{eh} \in [0.56, 0.80] (e^2/\kappa \ell) \) which is roughly independent of the bias. The band structure has the symmetry

\[ E_+(X)|\Delta_B = 2\Delta_B^{(s)} - E_-(X)|\Delta_B^{(s)} - \Delta_B, \]

where \( \Delta_B^{(s)} \) was defined previously in Eq. (48).

![FIG. 7: (Color online) Band structure \( E_{\pm}(X) \) in the helical state for different values of the bias and for the uniform phase (straight lines) at \( \Delta_B = 0 \) or \( \Delta_B = \Delta_B^{(1)} \).]

4. Density of states

We compute the density of states (DOS) from the retarded single-particle Green’s function i.e.

\[ g(E,\Delta_B) = -\frac{N_\rho}{\pi} \sum_n \Im \left[ G_{n,n}(q = 0, E) \right], \]

\[ = -\frac{1}{\pi} \sum_n \sum_X \Im \left[ G_{n,n}(X, X, E) \right]. \]

The DOS is represented in Fig. 8 for \( \Delta_B = \Delta_B^{(1)}/2 \). At this bias, the lower energy band is the mirror image (with
respect to a mirror line at the energy $\Delta_B^{(s)}$ of the high energy band. More generally, because of the symmetry of the band structure, the DOS has the corresponding symmetry

$$g(E)|_{\Delta_B} = g\left(2\Delta_B^{(s)} - E\right)|_{\Delta_B}.$$  

(72)

The extrema in the band structure shown in Fig. 7 lead to distinctive van-Hove singularities in the DOS as seen in Fig. 8.

The density of states in the helical state at bias $\Delta_B = \Delta_B^{(1)}/2$.

FIG. 8: Density of states in the helical state at bias $\Delta_B = \Delta_B^{(1)}/2$.

5. Response functions and collective modes

We compute numerically the 16 retarded response functions in the GRPA using Eq. (31). In the non-uniform phases, one must calculate $\chi_{R}^{\rho_{i,j},\rho_{i,j}}(\mathbf{k}, \mathbf{k} + \mathbf{G}, \omega)$ for all reciprocal lattice vector $\mathbf{G}$. Using the transformations given by Eq. (24), we obtain the response function for the pseudospin operators i.e. $\chi_{R}^{\rho_{i,j},\rho_{i,j}}(\mathbf{k}, \mathbf{k} + \mathbf{G}, \omega)$.

We show in Fig. 9 the imaginary part of the response function $\chi_{R}^{\rho_{i,j},\rho_{i,j}}(\mathbf{k}, \mathbf{k}, \omega)$ defined in Eq. (32). This function corresponds to the single-bubble approximation and does not capture the collective modes but only the particle-hole continuum. The continuum appears in the range $E_{eh} \in [0.56, 0.80]$ in accordance with the band structure calculation.

The collective modes can be obtained from the poles of the imaginary part of the full GRPA response functions $\chi_{R}^{\rho_{i,j},\rho_{i,j}}(\mathbf{k}, \mathbf{k}, \omega)$. To get the dispersion relation, we follow the frequencies of these poles as the wave vector is varied in the Brillouin zone. We remark that, in order to capture the electron-hole continuum from $\chi_{R}^{\rho_{i,j},\rho_{i,j}}$ computed in the GRPA, we must sum over all the reciprocal lattice vectors i.e. compute

$$\chi_{R}^{\rho_{i,j},\rho_{i,j}}(\mathbf{k}, \mathbf{k}, \omega) \equiv \sum_{\mathbf{G}} \chi_{R}^{\rho_{i,j},\rho_{i,j}}(\mathbf{k} + \mathbf{G}, \mathbf{k} + \mathbf{G}, \omega).$$  

(73)

This function is shown in Fig. 10 where the electron-hole continuum together with the collective modes are clearly visible. Note that all response functions are coupled in the GRPA equations. Consequently, they all share the same poles. However, the weight of a given pole depends on the nature of the underlying mode and is not the same in all response functions. Electron-hole excitations appear as very localized excitations and are captured in the response functions at finite $\mathbf{G}$.

Fig. 11 shows the dispersion of the first collective modes of the helical phase for $\Delta_B = 1.0$ along the direction of the pseudospin modulation i.e. along $k_x$. (The absence of points in the dispersion of some of the modes is a numerical artefact.) The real-space pattern is periodic with period $\lambda$ along $k_x$ but there is no periodicity in the dispersion in the $k_y$ direction as shown in Fig. 12. We have indicated in Fig. 12 the region of the electron-hole continuum where the collective modes are damped. The higher-energy collective modes are less dispersive and correspond to more localized excitations which eventually vanish in the electron-hole continuum. Since we are dealing with a continuous structure, the number of collective modes is not finite. This is also true.
in the Skyrmion crystal phase that we study in the next section.

The unusual oscillations in the dispersion relation in the $k_y$ direction are due to the phase factor $\gamma_{q,q'} = \exp \left(-i q \cdot q / (\ell^2 / 2) \right) = \exp \left(-i (k + G) \cdot (k + G') / (\ell^2 / 2) \right)$ in Eq. (32). In the spiral phase, $G = (2\pi n / \lambda) \hat{x}$ and $k$ is a vector in the first Brillouin zone. It follows that $\gamma_{q,q'} = \exp \left(-i k_{y'} (G_x - G_x') / (\ell^2 / 2) \right)$. In the direction $k_{y'} = 0$, the phase factor $\gamma_{q,q'}$ is constant and there is no oscillation. Along $k_{y'}$, however, the phase factor is periodic with periods $\lambda^2 / m\pi\ell^2$ in units of $2\pi / \lambda$. In Fig. 12 we can clearly see the periods with $m = 1, 2, 4$ using $\lambda / \ell \approx 2.74$.

The gapless mode of the helical phase is a phonon mode corresponding to a global translation of the density $\rho(r)$ in a space (for $q = 0$) accompanied by an in-phase $z - x$ rotation of all the pseudospins. As can be seen from Eq. (3), this is a global symmetry of the Hamiltonian.

At small wave vector $k$, the dispersion of the phonon mode has $\omega(k_x, k_y = 0) \sim k_x$ (the wave vector of the helix is $q = q_0 \hat{x}$) while $\omega(k_x = 0, k_y) \sim k_{y'}^2$.

The gapless $x - y$ pseudospin mode of the uniform phase acquires a gap in the helical state. Since the bias fixes the $z$ component of the pseudospin in the uniform phase, this mode corresponds to an oscillation of the pseudospins at $\rho_z(r)$ constant when $k = 0$. In the helical phase, the mode that corresponds to this motion is gapped because the DM term sets a preferred plane of rotation for the pseudospins i.e. the direction of $\hat{n}$ and $q$ in Eq. (51) are related. We remark that in a double quantum well systems where a stripe state occurs at $\nu = 1$ in higher Landau levels, both the phonon and pseudospin modes are gapless in the absence of tunneling.

There is no DM term in the Hamiltonian of the stripe state and both motions correspond to a symmetry of the Hamiltonian. A plot (not shown) of the susceptibility...
ties $\chi_{\rho_x,\rho_y}^R(\mathbf{k} \rightarrow \mathbf{0}, \omega)$ shows that the lowest-energy gapped mode in Fig. 12 has a weight in only $\chi_{\rho_x,\rho_y}^R$ and $\gamma_{\rho_x,\rho_y}^R$. It thus seems likely that this mode is the gapped pseudospin $x - y$ mode.

C. Skyrmion crystal phase

The skyrmion crystal phase occurs on both sides of the helical state in the phase diagram. Fig. 13 shows the pseudospin fields defined in the guiding-center representation by Eq. (24) at bias $\Delta_B = 0.2$. From Eq. (44), the parallel component of $\langle \vec{\rho}^\parallel (\mathbf{r}) \rangle$ is directly related to the physical electric dipoles. The crystal at $\Delta_B^\parallel = \Delta_B$ is the electron-hole conjugate of that at $\Delta_B$ i.e.

$$n(\mathbf{r})\big|_{\Delta_B^\parallel = \Delta_B} = 2 - n(\mathbf{r})\big|_{\Delta_B} \tag{74}$$

$$\langle \rho_z(\mathbf{r}) \rangle\big|_{\Delta_B^\parallel = \Delta_B} = - \langle \rho_z(\mathbf{r}) \rangle\big|_{\Delta_B} \tag{75}$$

The pseudospin vorticity in the $x - y$ plane is however the same for both crystals. As in the helical state, the electron-hole conjugation applies to the other properties described in this section.

In the crystal phase, the pseudospin $\langle \vec{\rho}(\mathbf{r}) \rangle$ is not constant in space. In fact, $\langle \vec{\rho}(\mathbf{r}) \rangle$ is a pseudospin density not a unit field. In Fig. 13 the pseudospin density has not been normalized and so $\langle \rho_z(\mathbf{r}) \rangle$ is greater than $1/2$ in some regions of the crystal. The guiding-center density $\langle \rho(\mathbf{r}) \rangle = 1$ is however constant. The density in each orbital is given by Eqs. (74). The difference between the orbital and crystal cases is that $\langle \rho_z(\mathbf{r}) \rangle \in [-1/2, 1/2]$ in the orbital state while $\langle \rho_z(\mathbf{r}) \rangle$ is positive for $\Delta_B < \Delta_B^\parallel /2$ and negative for $\Delta_B > \Delta_B^\parallel /2$ in the crystal phase, in the regions where the crystal is the ground state.

The crystal state is constructed by assuming that the number of vortices is equal to the number of electrons. Since the number of flux quanta is equal to the number of electrons at filling factor $\nu = 1$, we have $\nu = 2\pi n \ell^2$ with $n = 1/\varepsilon a^2$ where $\varepsilon = \sqrt{3}/2$ for a triangular lattice and the lattice constant is given by $a/\ell = \sqrt{2\pi}/\varepsilon$. The lattice spacing is thus controlled by the magnetic field.

The pseudospin pattern at each crystal site in Fig. 13 resembles that of a meron (half a Skyrmion) with negative charge. The $z$ component of the pseudospin is up at the center and goes in the $x - y$ plane in-between the crystal sites. The vorticity of the field $\langle \vec{\rho}^\parallel (\mathbf{r}) \rangle$ is positive. Evaluation of the real density $n(\mathbf{r})$ shows that each meron is placed on a uniform background of negative charge $n(\mathbf{r})$ i.e. the density does not go to zero between two crystal sites. Approximatively half of the real density is in the uniform background and half is in the merons. We could thus describe the state at $\Delta_B < \Delta_B^\parallel /2$ as a crystal of charged merons on top of a uniform background of electrons. For $\Delta_B > \Delta_B^\parallel /2$, the vorticity is still positive, but the $z$ component of the pseudospin is down at the center of the vortices and we have positively charged anti-merons (or holes in the electronic density). Note that the electronic density inside each meron is not quantized in our approach since we work with a pseudospin density and the pseudospin modulus changes with position. We could think of the crystal state has a two-dimensional charge-density-wave with an amplitude that can change continuously with bias. We use the term “Skyrmion crystal” in a loose way to refer to that state.

1. Density of states

The density of state is shown in Fig. 14. It is closer in shape to the DOS of the helical phase than to that of a crystal as can be seen by comparing Fig. 14 with the DOS of a Skyrmion crystal at $\nu = 1.2$ and $\Delta_B = 1.28$ shown in Fig. 13 of Ref. 12.

2. Collective modes

The dispersion relation of the collective modes of the skyrmion crystal is shown in Fig. 15. Only the first low-energy modes are shown. The modes become more dense at higher energy until the electron-hole continuum is reached. From Fig. 14 the continuum is in the range $E_{eh} \in [0.47, 0.90]$. The dispersion for $\Delta_B^\parallel - \Delta_B$ (not shown) is exactly the same as that for $\Delta_B$ as expected.

Since our calculation does not include disorder, the lowest-energy mode is a gapless phonon mode as in the helical state. For $\Delta_B = 0.1$, the dispersion is isotropic.
The susceptibilities in the pseudospin Hamiltonian. A plot (not shown) of Skyrmion crystal 14 x with energy gapped mode in Fig. 15 has substantial weights in the path $\Gamma - J - X - \Gamma$ in the irreducible Brillouin zone.

VI. ELECTROMAGNETIC ABSORPTION

The collective modes of the helical and Skyrmion crystal phases can be detected in electromagnetic absorption experiments. With $\kappa = 5$ for graphene on SiO$_2$ substrate, we have $\nu_0 = e^2/2\pi\hbar\kappa = 0.43\sqrt{B}$ THz with $B$ in Tesla. The frequency of the collective modes at $q = 0$ in both the helical and crystal phases are in the range $\nu \in [0.1, 0.6]$ $\nu_0 \approx [0.14, 0.84]$ THz for $B = 10$ T.

Theoretically, the absorption can be related to the current-current correlation function as explained in Ref. 6. We give here only the main results of this calculation.

We write the current operator, projected onto $N = 0$ and valley $K$ as

$$\mathbf{J} = -\frac{e}{c} \frac{\partial H^0_{K'}}{\partial \mathbf{A}^e} \bigg|_{A^e = 0}$$

$$= -\sqrt{2} \beta \Delta_B \frac{e\hbar}{\hbar} N^e (\rho_y (0) \mathbf{x} + \rho_x (0) \mathbf{y}),$$

where $\mathbf{A}^e$ is the vector potential of the external electromagnetic field and $H^0_{K'}$ was defined in Sec. II. We define the current-current correlation function Matsubara Green’s functions as

$$\chi_{J_\alpha, J_\beta} (\tau) = -\frac{1}{A} \langle T_\tau J_\alpha (\tau) J_\beta (0) \rangle$$

$$= \left( \frac{\Delta_B}{\gamma_1} \right)^2 \frac{e^2 \hbar^2}{4\pi m^* e \ell \chi_{\pi, \pi'}} (\tau),$$

where $A$ is the area of the 2DEG and

$$\chi_{\rho_{\alpha}, \rho_{\beta}} (\tau) = -\langle T_\tau \rho_{\alpha} (0, \tau) \rho_{\beta} (0, 0) \rangle$$

with $\alpha, \beta = x, y$. Note that $\mathbf{\pi}, \mathbf{\pi'}$ are defined by $\mathbf{\pi} = y$, $\mathbf{\pi'} = x$.

The electromagnetic absorption for an electric field oriented along the direction $\alpha$ is given by

$$P_{\alpha} (\omega) = -\frac{1}{\hbar} \left[ \frac{\chi_{J_\alpha, J_\alpha} (\omega)}{\omega + i\delta} \right] E_0^2$$

$$= -\frac{1}{2} \left( \frac{e^2}{\ell} \right) \left( \frac{\Delta_B}{\gamma_1} \right)^2 \omega^* E_0^2 \chi_{\pi, \pi'}^R (\omega)$$

$$\times \Im \left[ \frac{\chi_{\rho_{\alpha}, \rho_{\alpha}} (\omega)}{\omega + i\delta} \right],$$

where we have assumed a uniform electric field $\mathbf{E}(t) = E_0 \hat{\alpha} e^{-i\omega t}$ with polarization $\hat{\alpha}$ and taken the analytic

with $\omega \sim k^{1.5}$ at small wave vector as in a Wigner crystal. The pseudospin $x - y$ mode which is usually gapless in a Skyrmion crystal[14] is now gapped because of the DM term in the pseudospin Hamiltonian. A plot (not shown) of the susceptibilities $\chi_{\alpha, \alpha}^R (k \to 0, \omega)$ shows that the lowest-energy gapped mode in Fig. 15 has substantial weights in $\chi_{\rho_x, \rho_x}$ and $\chi_{\rho_y, \rho_y}$. It thus seems likely that this mode is the gapped pseudospin $x - y$ mode.

The energy difference between the crystal and helical phases is quite small, of the order of a few percent. It may be then, that these phases will be very sensitive to disorder as well as thermal and quantum fluctuations. An evaluation of the effect of these perturbations is however beyond the scope of the present paper.
continuation $i\Omega_n \to \omega+i\delta$ of $\chi_{\alpha,\beta} \Omega_\alpha (0,i\Omega_n)$ to get the retarded response function. The response functions $\chi_{\alpha,\beta}^{R} (\omega)$ are calculated in units of $\hbar/e^2/\kappa l$ so that $P_\alpha (\omega)$ is the power absorbed per unit area. In Eq. (79) we have neglected a diamagnetic contribution to the current response which becomes important at low frequencies.

The absorption due to all but the gapless mode in the helical phase is shown in Fig. 16. Comparing with Fig. 12 we see that all the other modes with the exception of the third mode (with frequency near 0.3 $(e^2/\hbar\kappa l)$) are optically active. The lowest gapped mode is the most intense one and its excitation is strongly sensitive to the orientation of the polarization vector of the electromagnetic wave. This is true at all bias voltages. The absorption frequency does not change significantly with bias.

The absorption spectrum for the Skyrmion crystal phase is shown in Fig. 17. In this case, all the modes (except the gapless phonon mode) are equally active in the absorption and the absorption does not seem sensitive to the polarization. There are thus qualitative differences between the absorption in the helical and crystal phases that should help to observe the transition between these two phases. Note that in the UP the orbital pseudospin mode is gapless and does not lead to absorption.

![Graph showing absorption spectrum](image)

**FIG. 16:** Electromagnetic absorption $P_\alpha (\omega)$ in the helical phase for polarization in the $\hat{x}$ and $\hat{y}$ directions at bias $\Delta B/ (e^2/\kappa l) = 1.0$.

**FIG. 17:** Electromagnetic absorption $P_\alpha (\omega)$ in the crystal phase for polarization in the $x$ and $y$ directions at bias $\Delta B/ (e^2/\kappa l) = 0.2$.

The physics of the orbital coherent state is due to a competition between the Coulomb exchange interaction and a Dzyaloshinskii-Moriya interaction between the orbital pseudospins. This competition is responsible for a phase diagram where the ground state evolves from a uniform state with collectively oriented orbital pseudospins at small bias into a Skyrmion crystal state and then into an helical state where the pseudospins rotate in space. If the bias is further increased, the helical states transits into the Skyrmion state again and then back to the uniform state. All three states can be distinguished from their density of states and collective excitations.

As was shown previously, the Goldstone mode due to spontaneous orbital coherence in the uniform phase has the peculiarity of being highly anisotropic. The dispersion is still highly anisotropic in the helical state which is modulated in one direction only but is isotropic in the Skyrmion crystal state. The three phases have one gapless mode and several gapped modes. We have calcu-

**VII. DISCUSSION**

We have studied the phase diagram of the 2DEG in a graphene bilayer in the Bernal stacking at filling factors $\nu = -1, 3$ in Landau level $N = 0$ when orbital coherence is present in one of the layer. Our model uses a tight-binding Hamiltonian that is simplified by working in the two-band model. This simplification is justified since we are interested in the low-energy excitations of the 2DEG. Moreover, we neglected the warping term, an approximation that is valid at sufficiently strong magnetic field. Finally, we assumed complete spin polarization. This last approximation may fail $\nu = -1$ at finite bias when levels with spin up mix with levels with spin down. A more exhaustive study shows that, when this approximation is not made, it is still possible to find states with orbital polarization although at different filling factors and for different ranges of bias than those studied in this paper.
lated that these latter modes lead to absorption in the far-infrared region of the electromagnetic spectrum. In the helical state, the absorption intensity is very sensitive to the orientation of the polarization vector of the electromagnetic wave. This is not the case in the Skyrmion crystal phase.

The helical and Skyrmion crystal phases each support a gapless phonon mode which is accompanied by a motion of the pseudospins. We think that, in the presence of disorder, these Goldstone modes should lead to strong absorption of electromagnetic waves at very small frequencies. This calculation is beyond the scope of this paper and we leave it for further work.

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