MINI-FLASH CRASHES, MODEL RISK, AND OPTIMAL EXECUTION

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ABSTRACT. Oft-cited causes of mini-flash crashes include human errors, endogenous feedback loops, the nature of modern liquidity provision, fundamental value shocks, and market fragmentation. We develop a mathematical model which captures aspects of the first three explanations. Empirical features of recent mini-flash crashes are present in our framework. For example, there are periods when no such events will occur. If they do, even just before their onset, market participants may not know with certainty that a disruption will unfold. Our mini-flash crashes can materialize in both low and high trading volume environments and may be accompanied by a partial synchronization in order submission.

Instead of adopting a classically-inspired equilibrium approach, we borrow ideas from the optimal execution literature. Each of our agents begins with beliefs about how his own trades impact prices and how prices would move in his absence. They, along with other market participants, then submit orders which are executed at a common venue. Naturally, this leads us to explicitly distinguish between how prices actually evolve and our agents’ opinions. In particular, every agent’s beliefs will be expressly incorrect.

As far as we know, this setup suggests both a new paradigm for modeling heterogeneous agent systems and a novel blueprint for understanding model misspecification risks in the context of optimal execution.

1. OVERVIEW

Amidst the violent market disruption on May 6, 2010, the infamous Flash Crash,

“Over 20,000 trades across more than 300 securities were executed at prices more than 60% away from their values just moments before. Moreover, many of these trades were executed at prices of a penny or less, or as high as $100,000, before prices of those securities returned to their ‘pre-crash’ levels” ([3]).

Today, this particular event remains so memorable due to its remarkable scale.

In fact, lesser versions of the Flash Crash, or mini-flash crashes, happen quite often. Anecdotal evidence suggests that there may be over a dozen every day ([58]). A rigorous empirical analysis uncovered “18,520 crashes and spikes with durations less than 1,500
A popular definition characterizes a mini-flash crash as an event in which the price of some security changes at least 0.8% and ticks ten times consecutively in a single direction ([74]). Price swings need not be so mild, though. Johnson et al. noted that “both crashes and spikes are typically more than 30 standard deviations larger than the average price movement either side of an event” ([74]). The SEC also recently described several 99% plunges as “mini-flash crashes” ([13]). Price surges can also fit these requirements. For example, the share price of Kraft Foods underwent a mini-flash crash on October 3, 2012 when it rocketed up 28% in less than a minute ([88]).

Now, it is true that the most irregular trades executed during some mini-flash crashes are eventually nullified and removed from the consolidated tape. For instance, when shares of the network security firm Qualys, Inc. jumped from $10 to $0.0001 and back during a 300ms period on April 25, 2013, all trades below $10.15 were ultimately canceled ([7]). The idea is that these transactions were clearly erroneous, that is, there was “an obvious error in [a] term, such as price, number of shares or other unit of trading, or identification of the security” ([10]).

Regardless of whether they are reflected in final data feeds, why do such phenomena occur?

Several answers have been proposed. Roughly, most point to human errors, endogenous feedback loops, the nature of modern liquidity provision, fundamental value shocks, or market fragmentation. These ideas can be viewed as different ways to rationalize how an extreme (local or global) dislocation in supply and demand can arise in modern markets. We will thoroughly review them all in Subsection 2.1.

One of our contributions in the present paper is the development of a model which captures aspects of the first three theories. The remaining explanations are plausible sources of a subset of mini-flash crashes, and we discuss their relationship to our framework in Subsection 2.1.

Our model also appears to exhibit features of historical mini-flash crashes. For instance, there are periods in which extreme price moves will not manifest. If they do, accompanying trade volumes can be high or low. Some market participants may partially synchronize their trading during a mini-flash crash. Our agents may not know that a mini-flash crash is about to begin even just before its onset.

Our results seem to be aligned with intuitive expectations as well. For example, our mini-flash crashes can begin if some of our agents are too uncertain about their initial beliefs, inaccurate in their understanding of price dynamics, slow to update their models and objectives, or willing to take on risk.

Subsection 2.1 contains details on where to find the proofs and figures corresponding to these claims.

We construct our model beginning with a finite population of agents trading in a single risky asset, each of whom must decide how to act based upon his own preferences, beliefs, and observations. Our specifications are drawn from ideas in the price impact and optimal execution literature and are given in Subsections 4.2 - 4.4.

We imagine that our agents’ orders are submitted to a single venue, where they are executed together with trades from other (unmodeled) market participants. This naturally compels us to make an explicit distinction between how the risky asset’s price actually evolves and our agents’ beliefs about its future evolution (see Section 5).

Since we view our agents as simultaneously solving their own optimal execution problems, we avoid certain strong assumptions that would have been implicitly needed, if we had used a classical equilibrium-based approach instead. An additional consequence
is that we precisely describe the errors in our agents’ beliefs. Potentially, each agent could be wrong both about how his trades affect prices and how prices would move in his absence. By appealing to theoretical and practical considerations, we argue away the consistency issues that one might feel would arise.

To the best of our knowledge, this general setup appears to be a new paradigm for modeling heterogeneous agent systems in the contexts of optimal execution and mini-flash crashes.

Additionally, we feel that our general framework could be viewed as a novel method for understanding, to some extent, model misspecification risks and Knightian uncertainty in optimal trading. The basic point is that “all models are wrong” and some (most) risks may be “unquantifiable” ([37], [79]). Existing techniques for managing these unknowns often involve position limits, sensitivity analysis, Bayesian model averaging, the worst-case framework, and interpolations between the worst-case and classical setups. We illustrate how employing our abstract process, potentially in conjunction with these standard methods, may give a more robust perspective.

Our in-depth discussion of these contributions and connections to previous literature on optimal execution and model misspecification appears in Subsection 2.2.

We are ready to begin presenting our work in detail. We highlight key background material and our paper’s contributions in relation to it in Section 2. Our definition of mini-flash crashes is given and discussed in Section 3. Our agents and their beliefs are described in Section 4. We characterize the correct dynamics of the risky asset’s price in Section 5. General results on what unfolds when our agents act as prescribed by Section 4 but prices actually move as in Section 5 are given in Section 6. Using the material in Section 6, a broad particular case of our model is investigated theoretically and numerically in Section 7. Our longer proofs are contained in Appendices A - C.

2. Background & Contributions

In Section 2, we clarify our contributions and explain how they fit into the current literature. Subsections 2.1 and 2.2 contain the relevant discussions for our work on mini-flash crashes and model risk in optimal execution, respectively.

2.1. Mini-Flash Crashes. We already mentioned that existing theories on the causes of mini-flash crashes could be viewed as falling into one of five categories (see Section 1). Here are further details.

i) Human errors (and, relatedly, improper risk management) are among the most commonly cited causes of mini-flash crashes ([72], [82], [13]). The SEC claims that the majority of mini-flash crashes originate from such sources, in fact ([82]). When we read about fat finger trades, rogue algorithms, or glitches in the media, typically human errors are indirectly responsible. For example, due to a bug in the systems at the Tokyo Stock Exchange and a typo in a trade submitted by Mizuho Securities, the share price of the recruitment agency J-Com fell in minutes from ¥672,000 to ¥572,000 on December 8, 2005 ([2]).

ii) Mini-flash crashes may be caused by the rapid, endogenous formation of positive feedback loops ([6], [74], [59], [72], [70], [73]). As Johnson et al. put it, “Crowds of agents frequently converge on the same strategy and hence simultaneously flood the market with the same type of order, thereby generating the frequent extreme price-change events” ([74]). A separate empirical study on the Flash Crash of May 6, 2010, specifically, determined that at its peak, “95% of the trading was due to endogenous triggering effects” ([59]).
iii) The nature of liquidity provision in modern markets is thought to cause some mini-flash crashes ([78], [52], [50], [67], [75], [66], [53]). Today, the majority of liquidity is provided by participants that are free from formal market-making obligations ([50]). In particular, they can instantly vanish, effectively taking one or both sides of the order book at some venue with them. A mini-flash crash can arise either directly as bid-ask spreads blow out or indirectly when a market order (of any size) tears through a nearly empty collection of limit orders. Such a phenomenon has been called *fleeting liquidity* and may have contributed to the occurrence of 38% of mini-flash crashes from 2006 - 2011 ([67]).

This proposed explanation is deeply intertwined with a crucial empirical observation: Mini-flash crashes occur in both high and low trading volume regimes. For instance, the trading volume during the 30s mini-flash crash of “WisdomTree LargeCap” Growth Fund on November 27, 2012 was nearly eight times the average daily trading volume for this security ([13]). The empirical study by Florescu et al. offers extensive evidence that mini-flash crashes often occur during low trading volume periods as well.

Why modern liquidity providers might wish to briefly disappear at times is a separate issue. Broadly, the idea is that liquidity providers choose to pull back when they fear they will be adversely selected. Some suggest that the clearly erroneous trade regulations might discourage the submission of market-stabilizing orders in the midst of a mini-flash crash ([50]). Gayduk and Nadtochiy propose a mechanistic theory: As trading frequencies increase, the very design of auction-style exchanges might ensure that markets become fragile and participants stop offering liquidity ([66]). Adverse selection fears are also stoked by genuine order flow toxicity, delays in consolidated quote feeds like the Security Information Processor, or activities by spoofers and other market manipulators ([5], [52], [53], [4]).

iv) Shocks to perceived fundamental values may lead to mini-flash crashes in some, albeit not most, cases ([75], [98]). We reiterate that these shocks must be perceived only: They may have no factual basis. For instance, a tweet sent on April 23, 2013 after a successful hack on the AP’s Twitter account falsely claimed that President Obama was injured in a series of explosions at the White House. Within two minutes, $136 billion was erased from the S&P 500 Index ([77]).

v) Market fragmentation itself, as well as the current regulations concerning this issue, may give rise to some mini-flash crashes ([50], [67], [45]). In present-day markets, a particular security might be traded at a number of venues, and liquidity need not be uniformly distributed. This injects sophisticated considerations into the problem of optimal execution: How does one route an order to achieve the best possible price? The SEC introduced Rule 611, as well as various exceptions including intermarket sweep orders (ISOs), in an attempt to ensure that traders would receive the most favorable prices available across all venues ([12]). Some argue that, inadvertently, this regulation may have made matters worse. For instance, Dick posits a scenario in which a trader receives an inferior execution because Rule 611 only protects quotes at the top of the book ([50]). In their empirical analysis, Golub et al. find that most mini-flash crashes are initiated by aggressive ISO-submission ([67]).

Aspects of (i), (ii), and (iii) are reflected in our work. For example, the human error theory arises in each of the following ways:

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1 Since market fragmentation may also contribute to local liquidity shortages (see (v)), it may be connected with this observation as well.
a) Every agent believes that a mini-flash crash is a null event (see Remark 4.9). On the contrary, there are cases in which one will occur almost surely (see Lemmas 7.8 and 7.11).
b) Every agent thinks that his trades affect prices through specific temporary and permanent price impact coefficients (see Subsection 4.2). His estimates for these parameters might be wrong (see Section 5).
c) Every agent’s trades may also indirectly impact prices by inducing others to make different decisions than they would otherwise (see Subsection 4.5 and Section 5). This potential effect is not modeled by our agents (see Subsection 4.2). More generally, even if we have a single agent in our setup trading with other unspecified market participants, the parameters in his fundamental value model might be inaccurate (see Subsection 4.2 and Section 5).
d) No agent revises the general class of his beliefs, admissible strategies, or objectives on our time horizon (see Subsections 4.2 - Subsection 4.4). In some cases, a mini-flash crash will not occur if this period is fairly short but will if it is too long (see Lemmas 6.11 and 7.3).
e) Every agent is averse to his position’s apparent volatility risks (see Subsection 4.4). In some cases, there will be no mini-flash crash when our agents are sufficiently averse to these risks; otherwise, there will be one (see Lemmas 6.11 and 7.3).
f) Every agent has the opportunity to update the drift parameter in his price model based upon his observations (see Subsection 4.2). In some cases, a mini-flash crash will unfold because our agents are too easily persuaded to revise their priors (see Lemmas 6.11 and 7.3).
g) Every agent has a model for how prices are affected by the temporary impact of trades (see Subsection 4.2). In some cases, there will be a mini-flash crash if our agents sufficiently underestimate the role of aggregate temporary impact. No such disturbance will occur otherwise. Our agents may be more prone to induce mini-flash crashes in this way when there are many of them (see Lemmas 6.11 and 7.3).

Notice that some of our agents’ human errors directly cause mini-flash crashes, though not all (see Lemma 6.11). We highlight this observation in Figures 1 - 3. Implicitly, the occasional absence of mini-flash crashes also agrees with (i). Despite the regularity of these disruptions on a market-wide basis, individual securities may rarely experience such an event. Similarly, traders’ models and strategies do roughly achieve their intended goals much of the time, which we observe as well (see Lemma 6.11).

Several key ideas from the endogenous feedback loop theory are present in our paper. For example, if a mini-flash crash does occur, it almost surely does so because of “endogenous triggering effects.” Specifically, our mini-flash crashes arise when some of our agents buy or sell at faster and faster rates, which they only do because they started trading more rapidly in the first place (see Section 5 and Lemma 6.6). As predicted by this theory, some of our agents also “converge on the same strategy” during mini-flash crashes: In certain cases, the agents driving these events all buy or sell together with the same (exploding) growth rate (see Lemmas 7.8 and 7.11). Figures 5, 8, and 11 graphically illustrate this partial synchronization.

We do not explicitly model liquidity providers in our framework, as we view our agents as submitting market orders to a single venue (see Section 5). We still view our paper as reflecting (iii), at least in some sense, since our mini-flash crashes can be accompanied by both high and low trading volumes (see Corollary 4.14 and Lemmas 7.8 and 7.11). Visualizations of this point are provided in Figures 4, 6, 7, 9, 10, and 12.
The fundamental value shock theory is beyond the scope of our work. To study it, we could extend our model, say, by including a jump term in our specification of the actual price dynamics (see Section 5). Provided these jumps were almost surely finite, we suspect that they would not induce a mini-flash crash in the sense of our definition (see Section 3). This point is left for future work.

For the sake of tractability, we chose to model our agents as trading at a single venue (see Section 5). This puts the market fragmentation theory also beyond the scope of our paper. Especially since routing decisions are inextricably linked with optimal execution problems in practice, we hope to return to this topic in the future ([1]).

2.2. Model Risk & Optimal Execution. Problems in which agents make their decisions based upon misspecified models are well-studied in the economics and behavioral finance communities ([35], [38], [63], [25], [57], [20], [91], [56]). To the best of our knowledge, such an approach has not been directly pursued in the financial mathematics literature on optimal execution.

Much of the previous work in this area assumes that agents have complete and correct knowledge of all model parameters ([34], [15], [16], [18], [85], [64], [86], [32]). Others consider the possibility that their agents’ models have the correct form; however, the agents must gradually learn the values of certain unobserved features ([17], [31], [43], [54], [62], [48]).

It is understood that anyone using the resulting strategies would be highly exposed to model (misspecification) risks. The concern is partially mitigated since methods including position limits, sensitivity analysis, Bayesian model averaging, the worst-case framework, and interpolations between the worst-case and classical setups may help to manage these issues. Agents that explicitly take model risks into account, say, by using one of these techniques, are typically called ambiguity averse ([41], [42], [44]).

The idea with a control like a position limit is that although a model or strategy may never be perfect, their errors cannot cause ruinous damage. In practice, there are many related risk limits. Key differences among these variants tend to lie in what, specifically, is being limited in size and how its size limit is implemented ([84]). For example, the sizes of single positions, sector positions, market bets, market capitalization bets, and leverage might all be limited. The limits themselves might be inflexible constraints or appear as penalty functions.

Sensitivity analyses attempt to precisely measure how aspects of a strategy or its performance would change, if model assumptions are varied. Here, a model’s parameters or probabilistic structure are often modified ([61], [60], [65]). If a strategy and its performance are found to be sufficiently stable, one might be somewhat assured that model risks are contained.

Agents using the final three techniques above can be viewed as having several candidate models (not one). Alternatively, they can occasionally be interpreted as trying to reduce their exposure to Knightian (or unquantifiable) uncertainty ([79]). Possibly after assigning seemingly appropriate weights, such agents simultaneously measure the performance of their admissible strategies under all candidate models ([93], [49], [41]).

Despite the protection afforded by these methods, they do not offer complete inoculation against model risk.

Due to practical flaws in design or implementation, position limits may not always avert disaster. For example, the SEC found that in some cases, Merrill Lynch’s controls

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2Such agents appear throughout the literature, not just in the financial mathematics strand on optimal execution. The seminal book by Hansen & Sargent offers a comprehensive discussion ([69]). There have been many more recent developments as well (see [51], [90], [36], [33], [29], [27], [28], [30], [14], and the references therein).
allowed single orders to be placed with sizes that were over fifty times larger than a
security’s average daily trading volume ([13]). The SEC further claimed that these
allegedly ineffective limits contributed to the onset of several mini-flash crashes, and
Merrill Lynch was fined $12.5 million.

Certain types of sensitivity analysis, e.g., differentiating a strategy or its performance
with respect to some parameter, may have shortcomings. For instance, they may be
most useful when an agent’s model is a slight perturbation of the actual price dynamics.
Efficacy might be further lowered, if optimal strategies in one regime are only compared
against optimal strategies in another (rather than studying how a single proposed strategy
would perform under a new framework).

Safeguards provided by Bayesian model averaging, the worst-case framework, and
interpolations between the worst-case and classical setups may be weakened if the agent’s

1) candidate models all poorly represent the actual price dynamics,
2) designated model weights are assigned inappropriately,
3) or conceptions about future outcomes and their payoffs are mistaken.

Now, in much of the optimal execution literature with ambiguity averse agents, the
sources of uncertainty are driven by Brownian motions, Poisson processes, or Poisson
random measures, and the agent’s candidate models are characterized by a suitable class
of equivalent measures ([41], [42], [44]). Even when candidate models are allowed to be
mutually singular, e.g., in the separate body of work on option pricing under volatility
uncertainty ([81], [21]), they are philosophically similar, say, in the sense that they might
have the same general form but differ in their parameter specifications. These points
heighten the possibility of (i).

Weights corresponding to candidate models are typically determined in a Bayesian
fashion or according to a chosen notion of distance from the candidate to the agent’s
reference model. (ii) may then arise, if either the agent’s priors or belief metrics are not
reflective of the actual price dynamics.

Historical examples of (iii) are abundant. Though not in the context of optimal
execution, Taleb recounts an especially striking anecdote involving a casino ([95]). This
organization put on a show which included a tiger. The firm insured against a variety of
incidents but did not envision that the creature would attack its star performer. When
this tragically occurred, the casino lost $100 million and suffered one of its largest losses
ever.

While model risks in optimal execution can never be entirely eliminated, these obser-
vations suggest that a new paradigm for managing them could be helpful. We hope that
our general procedure, possibly applied together with existing techniques, might be such
a paradigm. To clarify what we are introducing in the context of optimal execution, note
the following alternative interpretation of our setup:

We have a single agent trading in a risky asset over a finite time horizon.
In part, he has the beliefs and objectives described in Subsections 4.2
- 4.4; however, he is also concerned about model misspecification risks.
Before he begins trading, he wishes to have a more robust understanding
of how the strategy he derives in Subsection 4.5 might perform. To get
this, he imagines a new plausible way that the price might evolve and
tests his strategy’s performance in this scenario. He first hypothesizes
that there might be other market participants who stumbled upon the
basics of his strategy and might be planning to use these ideas. He is
then led to consider the possibility that the actual price dynamics are as
given in Section 5. He studies what might unfold in Sections 6 - 7. He
concludes that since it seems like his original strategy might lead to mini-flash crashes and devastating losses at times, he would like to reconsider his trading plans.

In short, instead of emphasizing mathematical similarity when checking his strategy’s performance in additional models, we could view our agent as emphasizing his human similarity with other market participants. By doing so, he seems to test his ideas in alternative settings which are both plausible and yield a different perspective on his model risks (compared to the insights offered by more traditional approaches). Observe that the idea that our agents, real or fictitious, individually solve their optimal execution problems and are not in a classical equilibrium state is crucial here.

The concept that strategy replication among market participants may significantly affect the future and, hence, may bring unforeseen risks is not limited to the context of optimal execution. For example, there is growing concern that the dramatic rise in index investing may have unanticipated, detrimental effects on the broader economy ([39], [55], [22], [23]).

Also, when we consider our setup from this new viewpoint, the consistency issues which may arise in conjunction with some of the human errors in our framework are not worrisome (see (a) - (g) in Subsection 2.1). After all, our agent is deliberately falsifying his beliefs to better discern his model risk exposure.

Even if we retain our initial finite population system view, we feel that these concerns may not be significant. While we could directly attempt to fit our framework into one of the modern equilibrium notions in the model misspecification literature, there may not be a need to do so: It appears that there are simple, practically-oriented reasons why the apparent issues might naturally come about in our setting.

First, we think of our agents as having the opportunity to witness just a single realization of the price, meaning that each agent believes that null events (including the price path itself) must occur. This seems to reflect the non-stationarity of markets: Models, parameters, and strategies which work quite well in one period may fail in the next.

Second, in practice, agents do not necessarily notice all of the ways in which their models are wrong. If they do, they may not want or be able to fix them. These behavioral arguments seem especially valid over the short time horizons that we consider and are supported by general observations from both the philosophy of science and psychology communities ([46], [19], [68], [94], [83]). For a specific example verifying these claims, recall the circumstances which engulfed Knight Capital on August 1, 2012: A bug arose in a critical piece of software. It has been alleged that the firm did not detect the glitch themselves; rather, they only became aware of it after being notified by the New York Stock Exchange. Supposedly, it then took 30 to 45 minutes for the firm to implement corrections, leading to a $440 million loss and Knight Capital’s subsequent acquisition by Getco LLC ([87]).

3. Mini-Flash Crashes

In Section 3, we introduce and discuss our definition of mini-flash crashes.

Definition 3.1. We say that a mini-flash crash occurs, if the risky asset’s price tends to either $+\infty$ or $-\infty$ on our time horizon.

For now, while this definition communicates our broad notion, its details are fairly vague.

We say more about our time horizon and price in Sections 4 - 5. The former is finite and deterministic, while the latter is a particular stochastic process.
We precisely describe the sense in which the price explodes and at what kinds of times\(^3\) mini-flash crashes can occur in Sections 6 - 7. Roughly, we analyze the occurrence of mini-flash crashes pathwise. In the cases that we consider, they happen (or not) almost surely; however, their direction is random. If a mini-flash crash unfolds, it does so at a deterministic time; yet, none of our agents have enough information to compute this time or even know that a mini-flash crash is imminent.

We said nothing about the classification of unbounded price oscillations as well. In the scenarios that we investigate, unbounded price oscillations occur with probability zero (see Sections 6 - 7), but this may be an artifact of our technical choices. Such an oscillation appears to reflect the practical duration of mini-flash crashes, and we hope to explore this point in a future work.

Clearly, there are many reasons why mini-flash crashes would be non-existent, if Definition 3.1 were used in practice. For instance, the SEC has instituted the “Limit Up-Limit Down Mechanism” to temporarily suspend trading in individual securities whose prices escape certain upper and lower bounds in specified short periods ([11]). Market-wide circuit breakers might be employed as well, which temporarily halt all trading when the S&P 500 Index declines sufficiently in a single trading day ([11]).

Our intuitive justification for this approximation is four-fold: First, though finite, price swings during a mini-flash crash can be quite extreme and shocking (see Section 1 and Subsection 2.1). Second, in our setting, we roughly view that trading would be suspended just before the occurrence of the event in Definition 3.1: A mini-flash crash is more appropriately understood to be the local behavior of the price near such a disruption. Third, since there are cases in which prices explode almost surely in our framework, Definition 3.1 avoids seemingly more arbitrary cut-offs that might have been necessary, if we included finite disturbances (see Sections 1 and 7). Finally, when we view our setup from the model risk-averse single agent perspective explained in Subsection 2.2, it might be reasonable to hypothesize that our agent would consider the possibility of exploding prices, even as only a limiting case which must be averted.

4. Agents

In Section 4, we describe our agents and their individual optimal execution problems. Important preliminary details are given in Subsection 4.1. We present our agents’ models and beliefs in Subsection 4.2. Each agent’s admissible strategies are characterized in Subsection 4.3. We discuss the agents’ objectives in Subsection 4.4. We prove Lemma 4.8, the main result of Section 4, in Subsection 4.5. Lemma 4.8 prescribes optimal strategies for our agents given their beliefs and preferences. Our agents attempt to trade according to these plans on our time horizon. When he does so, each agent believes that a mini-flash crash will occur with probability zero.

4.1. Preliminaries. We consider a population of \(N\) agents: Agent 1, . . . , Agent \(N\). The intuition underlying our agent’s models suggests that \(N\) should be interpreted as a large number (see Subsection 4.2). Mathematically, its qualitative size does not matter (see Lemma 6.6).

There is a special nonnegative parameter \(\nu_j^2\) associated to Agent \(j\) (see Subsection 4.2).

**Definition 4.1.** If \(\nu_j^2 > 0\), then we call Agent \(j\) an *uncertain* agent. If \(\nu_j^2 = 0\), then we call Agent \(j\) a *certain* agent.

\(^3\)We thank Shige Peng for this observation.
The reason for choosing these particular words will become clear in Subsection 4.2, and the distinction between these two types of agents will be crucial throughout the rest of the paper. For now, we assume that there are \( K \in \{0, \ldots, N\} \) uncertain agents, namely, Agents 1 through \( K \). A critical role is played by the value of \( K \) (see Lemma 6.6).

All agents attempt to trade in a single risky asset over a time horizon \([0, T]\). Our arguments proceed as long as \( T \) is deterministic and finite; however, our rationale is reasonable only when this period is short, say, no more than 1 day (see Subsection 4.2).

### 4.2. Models

Our agents trade continuously by optimally selecting a trading rate from a particular class of admissible strategies. To motivate our specifications of their choices and objectives, we first define their models and beliefs.

All trades submitted at time \( t \) are executed immediately at the price \( S_t^{exc} \). At each time \( t \), every agent observes the correct value of \( S_t^{exc} \). No agent knows the true dynamics of the stochastic process \( S_t^{exc} \), though.

Instead, prior to \( t = 0 \), Agent \( j \) has developed a model \( S_{j,\beta_j}^{exc} \) for \( S_t^{exc} \). \( S_{j,\beta_j}^{exc} \) evolves on a different probability space satisfying the usual conditions. Every agent models \( S_t^{exc} \) on a different probability space, despite the fact that their observations of \( S_t^{exc} \) will be identical. Our point is that Agent \( j \) interprets his observations as a sample path of his individual model for \( S_t^{exc} \), which may be unrelated to the process that Agent \( k \) uses to interpret the same data.

The space \((\Omega_j, \mathcal{F}_j, \{\mathcal{F}_{j,t}\}_{0 \leq t \leq T}, P_j)\) comes equipped with \( W_j \), an \( \mathcal{F}_{j,t} \)-Wiener process under \( P_j \). There is also an \( \mathcal{F}_{j,0} \)-measurable random variable \( \beta_j \), which is independent of \( W_j \) and normally distributed with mean \( \mu_j \) and variance \( \nu_j^2 \) under \( P_j \).

Recalling Definition 4.1, we see that Agent \( j \) is certain if he believes that he knows the correct value of \( \beta_j \) at \( t = 0 \). Otherwise, he is uncertain. Regardless of whether he is certain or uncertain in this sense, we will soon see that Agent \( j \) can always be viewed as certain about many things, e.g., he will not change the form of his models, objectives, or admissible strategies on \([0, T]\).

**Definition 4.2.** Following ([17]), Agent \( j \) defines an \( \mathcal{F}_{j,t} \)-adapted process \( S_{j,t}^{unf} \) by

\[
S_{j,t}^{unf} = S_{j,0} + \beta_j t + W_{j,t}, \quad t \in [0, T].
\]

\( S_{j,t}^{unf} \) is Agent \( j \)'s estimate of the unaffected or fundamental price of the risky asset at time \( t \). The drift term represents the price pressure that Agent \( j \) believes will arise due to the trades of (other) institutional investors. Agent \( j \) approximates the average behavior of uninformed or noise traders using the Brownian term.

After a fashion, Agent \( j \) believes that he can compute \( S_{j,t}^{unf} \) at \( t \) (see Subsection 4.3). Implicitly, he believes that his observations of \( S_{j,t}^{unf} \) will be independent of his trading

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4From a technical perspective, we will soon see that there is no need to introduce the filtration \( \{\mathcal{F}_{j,t}\} \). It would be equivalent to work with \( \{S_{j,t}^{exc}\} \) (see Subsections 4.3 and 4.5). The basic observation is that Agent \( j \) believes he can correctly reformulate his original optimal execution problem with partial information as one with complete information. Keeping the first problem seems to help motivate our setup.

5From this perspective, one might partially connect our work on mini-flash crashes to explanations of longer term financial bubbles based on overconfident investors ([92]).

6Almgren & Lorenz provide further details regarding the interpretation and limitations of (4.2) ([17]). A possible extension of our work could replace (4.2) with one of the more recent models considered in the literature on optimal trading problems with a learning aspect ([17], [54], [43], [64], [86], [62], [48]).
decisions. Agent $j$ knows which deterministic constant $S_{j,0}$ he has selected in (4.2). Unless $\nu_{j}^{2} = 0$, he does not assume that he can determine the realized values of $\beta_{j}$ or $W_{j,t}$. Instead, Agent $j$ will attempt to learn the value of $\beta_{j}$ by computing its expectation conditional on his accumulated observations as time passes (see Subsection 4.5).

Intuitively, Agent $j$’s selection of (4.2) makes the most sense when $N$ is large and $T$ is short. Notice that Agent $j$ makes no attempt to precisely estimate the number of other market participants, nor their individual goals or beliefs. That he believes he cannot improve the predictive accuracy of (4.2) by doing so appears to suggest that the population of traders is of sufficient size. In practice, many securities’ prices must be positive. Together with the fact that real drifts and volatilities are non-constant, (4.2) only seems even potentially plausible over short periods.

We now are ready to discuss $S_{j,\theta_{j}}^{{\text{exc}}}$.

**Definition 4.3.** Let $\theta_{j,t}$ denote Agent $j$’s trading rate at time $t$ (see Subsection 4.3). He defines $S_{j,\theta_{j}}^{{\text{exc}}}$ as the $\mathcal{F}_{j,t}$-adapted process

$$
S_{j,\theta_{j},t}^{{\text{exc}}} = S_{j,t}^{{\text{unf}}} + \eta_{j,\text{per}} \int_{0}^{t} \theta_{j,s} \, ds + \frac{1}{2} \eta_{j,\text{tem}} \theta_{j,t}, \quad t \in [0, T].
$$

(4.3)

Agent $j$ has chosen the deterministic positive constants $\eta_{j,\text{per}}$ and $\eta_{j,\text{tem}}$ in (4.3) prior to time $t = 0$.

There are two primary rationales behind (4.3). First, Agent $j$ could be viewed as taking into account his own effects on the execution price via an Almgren-Chriss reduced-form model ([16], [15], [18]). $\eta_{j,\text{per}}$ would denote Agent $j$’s estimate for his permanent price impact parameter, while he would approximate his temporary price impact parameter with $\eta_{j,\text{tem}}$. There is an alternative explanation in which Agent $j$ believes he submits market orders to a limit order book with certain characteristics. We present further details for both viewpoints in Section 5.

While Agent $j$ can use his prior for $\beta_{j}$, as well as his observations, to improve his estimate for the realized value of $\beta_{j}$, all model parameters in (4.2) and (4.3) are fixed (see Subsection 4.3). He cannot change the form of these models either, e.g., by making $\beta_{j}$ time-dependent in (4.2) or including a transient impact term in (4.3). The idea is that $T$ is short and, in practice, the time scale for developing an appropriate class of models for trading some instrument is often much longer than the time scale for revising parameters to better reflect current market conditions.

### 4.3. Admissible Strategies.

Agent $j$ selects his trading rate $\theta_{j}$ from $\mathcal{A}_{j}$, a class of admissible strategies that we will now define precisely.

Recall that Agent $j$ does not observe the realizations of either $\beta_{j}$ or $W_{j}$. Hence, it would not make sense for Agent $j$’s trading rate to be $\mathcal{F}_{j,t}$-adapted. Agent $j$ does watch $S^{{\text{exc}}}$, though, which he interprets as $S_{j,\theta_{j}}^{{\text{exc}}}$. Working with the filtration generated by $S_{j,\theta_{j}}^{{\text{exc}}}$ is somewhat cumbersome, as Agent $j$ believes that it depends on his choice of trading rate. The key is to notice that when Agent $j$ selects a continuous trading rate adapted to $\mathcal{F}_{j,t}^{{\text{unf}}}$, the filtration generated by $S_{j}^{{\text{unf}}}$, he believes that this filtration describes the same flow of information as his execution price observations. This is advantageous, as Agent $j$ thinks that $\mathcal{F}_{j,t}^{{\text{unf}}}$ is independent of his trading decisions (see Subsection 4.2).

---

7Alternatively, one could argue that there are only a few agents, all of whom are effectively hidden from one another; however, the securities for which our framework seems most reasonable would probably be traded by a large population anyway.

8Certain commodities have traded at negative prices ([9]).
Intuitively, the idea is that at each time $t$, Agent $j$ observes the correct value of $S_{t}^{exc}$. He views this value as the realization of $S_{j,t}^{exc,\theta_{j,t}}$. Using his knowledge of his past trading rate, he determines $S_{j,t}^{unf}$ as in (4.3). Agent $j$ thinks that these steps can be effectively taken all at once due to the (perceived) continuity of each process involved in the calculations.

Agent $j$ also believes his trades suffer from transaction costs due to both temporary and permanent price impact (see (4.3)). It seems reasonable to assume that he would never adopt a strategy that he thought would saddle him with infinite costs. As temporary impact induces a quadratic cost, we specify that he can only choose a strategy that satisfies

$$E_{P_{j}} \left[ \int_{0}^{T} \theta_{j,t}^{2} dt \right] < \infty.$$  \hfill (4.4)

Costs arising from permanent impact do not depend on Agent $j$’s trading rate, if his terminal inventory is deterministic. In fact, for reasons discussed in Subsection 4.4, we specify that Agent $j$’s terminal inventory must be zero, i.e., Agent $j$ solves an optimal liquidation problem.\footnote{Why would Agent $j$ produce the estimate $\eta_{j,per}$? After all, he believes that its value will not affect his trading decisions. Roughly, we feel that Agent $j$ might have such an approximation for business purposes, e.g., he may hope to accurately forecast P&L, even if he believes some components are uncontrollable. Unbeknownst to Agent $j$, $\eta_{j,per}$ is quite important for additional reasons (see Subsection 7.1).}

Formalizing these comments leads to the following definition.

**Definition 4.4.** Let $A_{j}$ be the space of $F_{j,t}$-adapted processes $\theta_{j}$ such that $\theta_{j,\cdot}(\omega)$ is continuous on $[0,T]$ for $P_{j}$-almost every $\omega \in \Omega_{j}$, (4.4) holds, and

$$x_{j} + \int_{0}^{T} \theta_{j,t} dt = 0 \quad P_{j} - \text{a.s.}$$  \hfill (4.5)

For any $\theta_{j} \in A_{j}$, we define the process $X_{j}^{\theta_{j}}$ by

$$X_{j}^{\theta_{j},t} = x_{j} + \int_{0}^{t} \theta_{j,s} ds.$$  \hfill (4.6)

$X_{j}^{\theta_{j}}$ is our notation for Agent $j$’s inventory: When he trades according to $\theta_{j}$, Agent $j$ holds $X_{j,t}^{\theta_{j}}$ shares of the risky asset at time $t$. In particular, we could also write (4.5) as

$$X_{j,T}^{\theta_{j}} = 0 \quad P_{j} - \text{a.s.}.$$  

In agreement with our intuition, the process $X_{j}^{\theta_{j}}$ is $F_{j,t}$-adapted due to our construction of $A_{j}$.

**4.4. Objective Functions.** Agent $j$ would like to trade such that, on average, his realized trading revenue will be as high as possible. He is also concerned about the various risks he might encounter while trading and hopes to take some of these into account. Since Agent $j$ proxies $S^{exc}$ with $S_{j,t}^{exc,\theta_{j}}$, he believes that the expected revenue corresponding to $\theta_{j} \in A_{j}$ is given by

$$E_{P_{j}} \left[ - \int_{0}^{T} \theta_{j,t} S_{j,t}^{exc,\theta_{j}} dt \right].$$  \hfill (4.7)

Now Agent $j$ must consider how to manage several risks. First, there are volatility risks associated with delayed liquidation. Since he uses (4.2) and (4.3), he believes that
these can be quantified by

\[
E^P_j \left[ -\frac{\kappa_j}{2} \int_0^T \left( X_{j,t}^{\theta_j} \right)^2 \, dt \right].
\] (4.8)

In (4.8), Agent \( j \) selects the deterministic risk aversion parameter \( \kappa_j > 0 \) based upon his appetite. On the other hand, Agent \( j \) presumably believes that as he observes the execution price’s path, he can better estimate \( \beta_j \)'s realized value. He might then think that he is more likely to regret earlier trades than later trades. A simple, though admittedly ad-hoc, way that Agent \( j \) could adjust for this risk is to artificially lower \( \kappa_j \). Similarly, it might be possible for him to partially account for his other risks including those arising from model misspecification with such an approach. In fact, in a slightly different setting, Jaimungal et al. show the equivalence between certain forms of ambiguity aversion and quadratic inventory penalties ([41]).

This discussion suggests the following objective for Agent \( j \).

**Definition 4.5.** Agent \( j \)'s objective is to maximize

\[
E^P_j \left[ -\int_0^T \theta_{j,t} S_{j,t} \, dt - \frac{\kappa_j}{2} \int_0^T \left( X_{j,t}^{\theta_j} \right)^2 \, dt \right]
\] (4.9)

over \( \theta_j \in \mathcal{A}_j \).

As mentioned in Subsection 4.3, (4.9) motivates our requirement that Agent \( j \) must liquidate by \( T \) (see (4.5)). Although the permanent impact term in (4.9) disappears, regardless of the deterministic value of \( X_{j,T}^{\theta_j} \), non-zero values of \( X_{j,T}^{\theta_j} \) could perversely incentivize Agent \( j \) via (4.8). For example, if Agent \( j \) started with a large inventory and needed a larger terminal inventory, he might be inclined to pay unnecessary round-trip (sell early/buy later) costs induced by temporary impact.

In Lemma 4.8’s proof, we see that (4.9) can be equivalently formulated as the following complete information problem: Maximize

\[
E^P_j \left[ \int_0^T X_{j,t}^{\theta_j} \, dt - \frac{\eta_{j,\text{tem}}}{2} \int_0^T \theta_{j,t}^2 \, dt - \frac{\kappa_j}{2} \int_0^T \left( X_{j,t}^{\theta_j} \right)^2 \, dt \right]
\]

over \( \theta_j \in \mathcal{A}_j \). It can also be thought of as an optimal tracking problem, in which Agent \( j \) must minimize

\[
\frac{1}{2} \int_0^T \left( X_{j,t}^{\theta_j} - \frac{E^P_j \left[ \beta_j \left| \mathcal{F}^{\text{unf}}_{t,t} \right. \right]}{\kappa_j} \right)^2 \, dt + \frac{\eta_{j,\text{tem}}}{2\kappa_j} \int_0^T \theta_{j,t}^2 \, dt
\]

Variants of both the former and the latter have been previously investigated, although not, to the best of our knowledge, with our intentions ([64], [24]).

4.5. **Results.** Before proving Lemma 4.8, we introduce the following notation. It will be useful throughout the paper.

**Definition 4.6.** We define \( \tau_j (\cdot) \) by

\[
\tau_j (t) \triangleq \sqrt{\frac{\kappa_j}{\eta_{j,\text{tem}}}} (T - t), \quad t \in [0, T].
\]

**Remark 4.7.** For our purposes, the key point in Definition 4.6 is that \( \tau_j \) strictly decreases to 0 as \( t \uparrow T \).
Lemma 4.8. \((4.9)\) has a unique\(^{10}\) solution \(\theta^*_j \in A_j\). When \(\omega \in \Omega_j\) is chosen such that \(W_j, (\omega)\) is continuous on \([0, T]\), \(X_{j,t}^\theta(\omega)\) satisfies the linear ODE
\[
\theta^*_{j,t}(\omega) = -\sqrt{\frac{\kappa_j}{\eta_{j,\text{tem}}}} \coth(\tau_j(t)) X_{j,t}^\theta(\omega) \\
+ \frac{\tanh(\tau_j(t)/2) \left[ \mu_j + \nu_j^2 \left( S_{j,t}^{\text{unf}}(\omega) - S_{j,0} \right) \right]}{\sqrt{\eta_{j,\text{tem}} \kappa_j} (1 + \nu_j^2 t)}, \quad t \in (0, T)
\]
(4.10)

Remark 4.9. In conjunction with Subsections 4.2 - 4.3, Lemma 4.8 implies that Agent \(j\) believes that a mini-flash crash is a null event. More precisely, under his setup, \(S_{j,\theta_j}^{\text{exc}}\) will remain finite on \([0, T]\) \(P_j\)-a.s. Of course, he believes that this will be true of \(X_{j,t}^\theta\) and \(\theta^*_{j,t}\) as well.

Remark 4.10. Agent \(j\) believes that (4.10) characterizes his optimal trading rate almost surely. Therefore, it seems reasonable to view that he would always attempt to implement this strategy: He thinks it is nearly impossible for this approach to be flawed, after all.

Remark 4.11. The first term in (4.10) arises from our constraint that Agent \(j\) must liquidate by the terminal time (see (4.5)). In fact, the weighting factor
\[
-\sqrt{\frac{\kappa_j}{\eta_{j,\text{tem}}}} \coth(\tau_j(t))
\]
tends to \(-\infty\) as \(t \uparrow T\). Intuitively, the reason that Agent \(j\) believes that \(X_{j,t}^\theta\) and \(\theta^*_{j,t}\) remain finite as \(t \uparrow T\) is that \(X_{j,t}^\theta\) tends very rapidly to zero.

Agent \(j\) thinks that he learns about \(\beta_j\)'s realized value over time, which is captured by the second term in (4.10) since
\[
E^{P_j} \left[ \beta_j \mid \mathcal{F}_{j,t}^{\text{unf}} \right] = \frac{\mu_j + \nu_j^2 \left( S_{j,t}^{\text{unf}} - S_{j,0} \right)}{1 + \nu_j^2 t} P_j - \text{a.s.} \quad (4.11)
\]
(\([80]\)). The factor
\[
\frac{\tanh(\tau_j(t)/2)}{\sqrt{\eta_{j,\text{tem}} \kappa_j}}
\]
is bounded by \(1/\sqrt{\eta_{j,\text{tem}} \kappa_j}\) and tends to zero as \(t \uparrow T\).

The second term may either dampen or amplify the effects of the first. Agent \(j\) believes that the weighting factors reflect that his need to liquidate must eventually overwhelm his desire to profit by trading in the direction of the risky asset’s drift.

Remark 4.12. As anticipated, Agent \(j\)'s permanent impact parameter estimate \(\eta_{j,\text{per}}\) is absent in (4.10) (see Subsection 4.3).

Remark 4.13. Lemma 4.8’s proof has five steps.

First, we introduce an auxiliary problem in which Agent \(j\) can select a trading rate from a larger class of admissible strategies. Our original formulation did not consider these, as they may not be aligned with the intuition underlying our framework. For instance, some of them suggest that Agent \(j\) could peak into the future or that he might knowingly select a trading rate that would cause \(S_{j,\theta_j}^{\text{exc}}\) to explode.

\(^{10}\)Here, uniqueness holds up to \(dP_j \otimes dt\)-a.s. equality on \(\Omega_j \times [0, T]\).
We then show that Agent $j$ does not believe that he can benefit from the auxiliary problem’s new informational structure. This part of the argument uses (4.11), as well as the Vitali convergence theorem and uniform integrability.

The third step is to find a suitable complete information equivalent for Agent $j$’s auxiliary problem. We have effectively discussed this in Subsection 4.4. The idea is to use integration by parts and an innovation process.

Next, we use a result from Bank et al. to determine the unique solution to our auxiliary problem ([24]). The introduction of our auxiliary problem was motivated by this step. Admittedly, this means we use a tool which seems to be far more powerful than our problem demands. For instance, the result of Bank et al. applies to a broad class of optimal tracking problems with a non-Markovian target, while (4.11) suggests that we track a Markovian one (see Subsection 4.4). We still adopt this approach, as it may allow us to extend our work in the future.

We conclude by demonstrating that the trading rate identified in the previous step is actually in our original set of admissible strategies. Much of the work to prove that the trading rate is adapted to $\{F_{j,t}^{unf}\}$ comes from our second step, while the remaining ideas are taken care of by Bank et al. ([24]).

Proof. See Appendix A.1. □

**Corollary 4.14.** If $\nu_j^2 = 0$, then $X_{j,t}^{\theta^*_j}$ does not depend on $S_j^{unf}$. In particular, it is deterministic and satisfies the linear ODE

$$
\begin{align*}
\theta_{j,t}^{\theta^*_j} &= -\sqrt{\kappa_j} \coth (\tau_j(t)) X_{j,t}^{\theta^*_j} + \frac{\mu_j \tanh(\tau_j(t)/2)}{\sqrt{\eta_j,tem} \kappa_j}, \quad t \in (0, T) \\
X_{j,0}^{\theta^*_j} &= x_j.
\end{align*}
$$

(4.12)

**Remark 4.15.** Corollary 4.14 confirms that there are significant differences between our certain and uncertain agents, as expected: If Agent $j$ feels completely certain of $\beta_j$’s realized value, he would not glean profitable information and modify his trades based upon his observations of the execution price (see Subsections 4.3 - 4.4). Mathematically, it is also especially evident from (A.5) and (A.6).

**Proof.** This is immediate from Lemma 4.8. □

5. **Execution Price**

In Section 5, we specify how $S^{exc}$ actually evolves. While each agent observes the same realized path of this process, in general, no agent knows the correct dynamics.\(^{11}\)

An agent’s trading decisions are entirely determined by his beliefs, preferences, and observations of a single realized path of $S^{exc}$ (see Lemma 4.8).

Let

$$
\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \left\{\tilde{\mathcal{F}}_t\right\}_{0 \leq t \leq T}, \tilde{P}\right)
$$

be a filtered probability space satisfying the usual conditions. The space is equipped with an $\tilde{\mathcal{F}}_t$-Wiener process under $\tilde{P}$, which we denote by $\tilde{W}$. We also have the following deterministic real constants:

$$
\tilde{\beta}, \quad S_0, \quad \tilde{\eta}_{1,per}, \ldots, \tilde{\eta}_{N,per}, \quad \text{and} \quad \tilde{\eta}_{1,tem}, \ldots, \tilde{\eta}_{N,tem}.
$$

$\tilde{\beta}$ can be arbitrary; however, the remaining constants are strictly positive.

\(^{11}\)There is a single trivial case where this is not true. If $N = 1, \tilde{\beta} = \beta, \nu_1^2 = 0, \tilde{\eta}_{1,tem} = \eta_{1,tem}$, and $\tilde{\eta}_{1,per} = \eta_{1,per}$, our lone agent’s model would be exactly right.
Definition 5.1. The true execution price $S_{exc}$ under $\tilde{P}$ is the $\tilde{F}_t$-adapted process

$$S_t^{exc} = S_0 + \tilde{\beta} t + \sum_{i=1}^{N} \tilde{\eta}_{i,per} (X_{i,t}^{\theta_i} - x_i) + \frac{1}{2} \sum_{i=1}^{N} \tilde{\eta}_{i,tem} \theta^*_{i,t} + \tilde{W}_t, \quad t \in [0, T]. \quad (5.1)$$

(5.1) can be viewed as a multi-agent extension of the Almgren-Chriss model ([16], [15], [18]). Models of this form, particularly when the $\tilde{\eta}_{j,tem}$'s ($\tilde{\eta}_{j,per}$'s) are all identical, have been applied in the context of predatory trading ([40]).

From this perspective, $\tilde{\eta}_{j,per}$ and $\tilde{\eta}_{j,tem}$ are the correct values of Agent $j$'s permanent and temporary price impact parameters, respectively. We allow these quantities to have arbitrary relationships to Agent $j$'s corresponding estimates $\eta_{j,per}$ and $\eta_{j,tem}$. For instance, Agent $j$ might underestimate his permanent impact ($\eta_{j,per} < \tilde{\eta}_{j,per}$) but perfectly estimate his temporary impact ($\eta_{j,tem} = \tilde{\eta}_{j,tem}$). Similarly, Agent $j$'s prior $\beta_j$ for the correct drift $\tilde{\beta}$ may be accurate or severely mistaken.

Comparing our descriptions of $S_{exc}^{\theta_j}$ in (4.3) and $S^{exc}$ in (5.1), we see that Agent $j$ proxies each term in (5.1) as follows:

$$\eta_{j,per} (X_{j,t}^{\theta_j} - x_j) \leftrightarrow \tilde{\eta}_{j,per} (X_{j,t}^{\theta_j} - x_j)$$
$$\frac{1}{2} \eta_{j,tem} \theta^*_{j,t} \leftrightarrow \frac{1}{2} \tilde{\eta}_{j,tem} \theta^*_{j,t}$$

$$S_{j,0} + \beta_j t + W_{j,t} \leftrightarrow S_0 + \tilde{\beta} t + \sum_{i \neq j} \tilde{\eta}_{i,per} (X_{i,t}^{\theta_i} - x_i) + \frac{1}{2} \sum_{i \neq j} \tilde{\eta}_{i,tem} \theta^*_{i,t} + \tilde{W}_t.$$

Heuristically, we could also interpret (5.1) through the lens of a single order book. This connection was observed by Kallsen & Muhle-Karbe ([76]). The process

$$S_0 + \tilde{\beta} t + \tilde{W}_t$$

would be viewed as the fundamental price, while the sum of the fundamental price and the permanent impact terms

$$S_0 + \tilde{\beta} t + \tilde{W}_t + \sum_{i=1}^{N} \tilde{\eta}_{i,per} (X_{i,t}^{\theta_i^*} - x_i)$$

would be the reference price. We would set the $\tilde{\eta}_{j,tem}$'s to a single value, and do the same for the $\tilde{\eta}_{j,per}$'s. Our agents would submit market orders, and only the net agent order flow would be executed in the book (remaining orders would be matched together). All agents would receive the same average execution price at each time. The bid-ask spread would be infinitesimally small, while the book would be infinitely resilient and block-shaped with height $1/\tilde{\eta}_{j,tem}$. That is, agents would trade in an Obizhaeva-Wang book which instantly recovers to the reference price after each execution (no transient impact) ([85])

6. General Results

When our agents implement the strategies that they believe are optimal (see Lemma 4.8) but $S^{exc}$ has the dynamics in (5.1), what happens? The goal of Section 6 is to offer some general answers to this question.

To simplify our presentation, we begin by introducing and analyzing additional notation (see Definition 6.2 and Lemma 6.4). We find that our agents’ inventories and trading rates evolve according to a particular ODE system with stochastic coefficients (see Lemma 6.6). Under certain conditions, the system can have a singular point (see Lemma 6.7). For convenience, we study what unfolds when this singular point is of the first kind (see Lemma 6.10). We also examine the case in which there is no singular point (Lemma 6.11). Due to tractability issues, in order to determine whether or not
Mini-flash crashes arise, we consider a particular, though broad, class of examples (see Remark 6.13). While we present these findings in Section 7, we provide a high-level summary of them in Theorem 6.14. In particular, we see that in some cases, mini-flash crashes occur $\tilde{P}$-a.s. at the first singular point of our system. Again, our agents still believe that a mini-flash crash is a null event.

First, observe that our assumptions in Section 5 do not affect our certain agents’ trading decisions (see Corollary 4.14). It remains to characterize our uncertain agents’ strategies.

We will have an even mix of deterministic and stochastic maps. In what follows, we always explicitly indicate $\omega$-dependence to distinguish between the two. Our equations are solved pathwise, so we do not encounter probabilistic concerns.

**Notation 6.1.** Fix $\omega \in \tilde{\Omega}$ such that $\tilde{W}(\omega)$ has a continuous path.

**Definition 6.2.** Define the maps 

\[
\Phi_i : [0, T] \rightarrow \mathbb{R} \\
A : [0, T] \rightarrow M_K(\mathbb{R}) \\
B : [0, T) \rightarrow M_K(\mathbb{R}) \\
C(\cdot, \omega) : [0, T] \rightarrow \mathbb{R}^K
\]

by

\[
\Phi_i(t) \triangleq \frac{\tanh (\tau_i(t)/2) \nu_i^2}{\sqrt{\eta_{i,\text{tem}}(1 + \nu_i^2 t)}} \\
A_{ik}(t) \triangleq \begin{cases} 
1 - \frac{1}{2} (\tilde{\eta}_{i,\text{tem}} - \eta_{i,\text{tem}}) \Phi_i(t) & \text{if } i = k \\
-\frac{1}{2} \tilde{\eta}_{k,\text{tem}} \Phi_i(t) & \text{if } i \neq k
\end{cases} \\
B_{ik}(t) \triangleq \begin{cases} 
(\tilde{\eta}_{i,\text{per}} - \eta_{i,\text{per}}) \Phi_i(t) - \frac{\kappa_i}{\eta_{i,\text{tem}}} \coth (\tau_i(t)) & \text{if } i = k \\
\tilde{\eta}_{k,\text{per}} \Phi_i(t) & \text{if } i \neq k
\end{cases} \\
C_i(t, \omega) \triangleq \Phi_i(t) \left[ \frac{\mu_i}{\nu_i^2} + (S_0 - S_{i,0}) + \tilde{\beta} t - \sum_{k \leq K, k \neq i} \tilde{\eta}_{k,\text{per}} x_k - x_i (\tilde{\eta}_{i,\text{per}} - \eta_{i,\text{per}}) \\
+ \sum_{k > K} \tilde{\eta}_{k,\text{per}} \left( X_{k,t}^\alpha - x_k \right) + \frac{1}{2} \sum_{k > K} \tilde{\eta}_{k,\text{tem}} \theta_{k,t}^\alpha + \tilde{W}_i(\omega) \right].
\]

Here, $i \in \{1, \ldots, K\}$.

**Remark 6.3.** Observe that we can now write the dynamics in (4.10) as

\[
\theta_{j,t}^\alpha(\omega) = -\sqrt{\frac{\kappa_j}{\eta_{j,\text{tem}}}} \coth (\tau_j(t)) X^\alpha_{j,t}(\omega) + \Phi_j(t) \left[ \frac{\mu_j}{\nu_j^2} + (S^{\text{unf}}_{j,t}(\omega) - S_{j,0}) \right]
\]

when Agent $j$ is uncertain.

We frequently reference various easy properties of the functions in Definition 6.2. We collect these below for convenience.

**Lemma 6.4.** Fix $j \in \{1, \ldots, K\}$. We have the following:

i) $\Phi_j$ is a strictly decreasing nonnegative function on $[0, T]$ with $\Phi_j(T) = 0$.

ii) The entries of $A$ are analytic on $[0, T]$ and $A(T) = I_K$. 
iii) If \( \det A \) has a root on \([0, T]\), we can find the smallest one which we denote by \( t_e \).

In this case, \( t_e < T \) and the zero of \( \det A \) at \( t_e \) is of finite multiplicity.

iv) The entries of \( B \) are analytic on \([0, T)\) but
\[
\lim_{t \uparrow T} B_{jj}(t) = -\infty.
\]

v) \( C(\cdot, \omega) \)'s entries are continuous on \([0, T]\).

**Proof.** Parts (i) and (ii) are clear. After recalling that the zeros of an analytic function are isolated and of finite multiplicity, we get (iii) from (ii). The singularity in \( B \) at \( T \) arises from the \( \text{coth} \) term, yielding (iv). Corollary 4.14 and our choice of \( \omega \) give (v). □

**Definition 6.5.** When \( \det A \) has a root on \([0, T]\), we let \( t_e \) denote the smallest one (see Lemma 6.4).

Up to some deterministic time, the uncertain agents’ inventories evolve according to a particular first order linear ODE system when \( \omega \) is fixed. Lemma 6.6 makes this precise when this time is positive. We leave the investigation and interpretation of the case when it is zero for future work. Also, note that our agents effectively assume that this time is \( T \) almost surely.

**Lemma 6.6.** Suppose that \( \det A \) has a root on \([0, T]\). If \( t_e > 0 \), then \( S^{\text{exc}}(\omega) \), the \( X_t^\theta(\omega) \)'s and the \( \theta_t^\omega(\omega) \)'s are all uniquely defined and continuous on \([0, t_e)\). Moreover, letting the \( u \)-superscript signify restriction to the uncertain agents, the uncertain agents’ strategies are characterized by
\[
A(t) \theta_t^{u, \ast}(\omega) = B(t) X_t^{u, \ast}(\omega) + C(t, \omega), \quad t \in (0, t_e)
\]
(6.1)

When \( \det A \) does not have a root on \([0, T]\), the same statements hold after replacing \( t_e \) with \( T \).

**Proof.** See Appendix B.1. □

Lemma 6.6 does not address the behavior of our uncertain agents’ inventories and trading rates as \( t \uparrow t_e \) or \( t \uparrow T \). The difficulties are that \( A \) is non-invertible at \( t_e \), while \( B \)'s entries explode at \( T \) (see Lemma 6.4).

The approach for resolving these issues is well-established (see Chapter 6 of [47]). We sketch the key points when \( \det A \) has a root on \([0, T]\) and \( t_e > 0 \). Analyzing the effects of \( B \)'s explosion at \( T \) is similar (see Lemma 6.11).

We begin by considering the homogeneous equation corresponding to (6.1):
\[
A(t) X_t^u(\omega) = B(t) X_t^u(\omega) + C(t, \omega), \quad t \in (0, t_e)
\]
(6.2)

We change notation to emphasize that (6.2) no longer describes the uncertain agents’ optimal strategies. We next write (6.2) in a more convenient form.

**Lemma 6.7.** Suppose that \( \det A \) has a root on \([0, T]\) and \( t_e > 0 \). Near \( t_e \), the solution of (6.2) satisfies
\[
(t - t_e)^{m+1} X_t^u(\omega) = D(t) X_t^u(\omega).
\]
(6.3)

In (6.3), \( m \) is a nonnegative integer such that the multiplicity of the zero of \( \det A \) at \( t_e \) is \((m + 1)\). \( D \) is a particular analytic map for which \( D(t_e) \) has rank 0 or 1 (see (B.5)).

\[^{12}\text{For instance, } \theta_t^{u, \ast}(\omega) \text{ denotes the first } K \text{-entries of } \theta_t^\omega(\omega).\]
Proof. See Appendix B.2.

Definition 6.8. If det $A$ has a root on $[0, T]$ and $t_e > 0$, we let $m$, $D$, and $f$ be defined as in Lemma 6.7’s proof (see (B.4) and (B.5)). Also, $D(t_e)$ has at most one non-zero eigenvalue (see Lemma 6.7’s proof), which we denote by $\lambda$.

Remark 6.9. Unless $D(t_e) = 0$, $D(t_e)$ has rank 1 (see Lemma 6.7’s proof). Hence, we can find $v, \hat{v} \in \mathbb{R}^K$ such that

$$v\hat{v}^T = D(t_e) \quad \text{and} \quad \hat{v}^Tv = \lambda.$$ 

Moreover, $v$ is an eigenvector of $D(t_e)$ corresponding to $\lambda$. While $v$ and $\hat{v}$ are not unique, algorithms are available to compute an example of such a pair ([89]). In future work, we may use this decomposition to investigate the occurrence of mini-flash crashes in broader cases than those considered in Section 7.

Suppose that $D(t_e) \neq 0$. Since $t_e < T$, the coefficients of (6.2) are analytic in a neighborhood of $t_e$ (see Lemma 6.4). It follows that (6.2) has a singular point of the first kind at $t_e$ when $m = 0$ in Lemma 6.7 (see Chapter 6 of ([47])). Otherwise, the singular point is of the second kind. In the former case, the fundamental solution of (6.2) near $t_e$ is the product of a certain analytic function with a matrix exponential.

The analysis of solution behavior when there is a singular point of the second kind at $t_e$ is significantly more difficult. For instance, while we may be able to find a formal series solution for (6.2) near $t_e$, it may converge at just one point. We do not consider scenarios with such singularities in the present work, as our examples in Section 7 do not exhibit them (see Lemma 7.3).

As soon as we have the fundamental solution near $t_e$, we use variation of parameters to solve (6.1). This gives our uncertain agents’ optimal inventories. We immediately get their optimal trading rates by differentiating and the corresponding execution price by plugging all agents’ strategies into (5.1).

This discussion is made precise in the next result.

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13 We adopt the nomenclature from Coddington & Carlson ([47]); however, other sources refer to such points as regular and irregular singular points, respectively ([71]). There are nonequivalent definitions of these terms too.

14 See the books by Wasow ([97]) and Ilyashenko & Yakovenko ([71]) for detailed discussions on these issues.
Lemma 6.10. Suppose that $\det A$ has a root on $[0, T]$, $t_e > 0$, and $m = 0$. If $\lambda \notin \mathbb{Z}$, then for some small $\rho > 0$,

$$X_t^{u,*} (\omega) = P(t) \left[ \sum_{j=1}^{K-1} \left( y_j (\omega) - \int_{t-e-\rho}^{t} \frac{F_j (s, \omega)}{s - \rho} ds \right) v_j + |t - e| \lambda \left( y_K (\omega) - \int_{t-e-\rho}^{t} \frac{F_K (s, \omega)}{s - \rho} ds \right) v_K \right]$$

for $t \in (t_e - \rho, t_e)$. Here,

- $\{v_1, \ldots, v_K\}$ is an eigenbasis for $D(t_e)$ ($v_K$ corresponds to $\lambda$);
- $P$ is a (non-singular-)matrix-valued analytic function on $[t_e - \rho, t_e]$ such that $P(t_e) = I_K$ (see (B.6));
- $\{y_1 (\omega), \ldots, y_K (\omega)\}$ are constants (see (B.9));
- and $\{F_1 (\cdot, \omega), \ldots, F_K (\cdot, \omega)\}$ are continuous real-valued functions on $[t_e - \rho, t_e]$ (see (B.9)).

We get $\theta^{u,*} (\omega)$ and $S^{exc}(\omega)$ on $(t_e - \rho, t_e)$ by differentiating (6.4) and by substituting $X_t^{\theta^*} (\omega)$ and $\theta^* (\omega)$ into (5.1), respectively.\(^{16}\)

Proof. See Appendix B.3.

Lemma 6.11. Suppose that $\det A$ does not have a root on $[0, T]$. Then $S^{exc}(\omega)$, the $X_t^{\theta^*} (\omega)$'s and the $\theta^* (\omega)$'s are all uniquely defined and continuous on $[0, T]$. Moreover,

$$\lim_{t \uparrow T} X_t^{\theta^*} (\omega) = 0.$$  \(\text{(6.5)}\)

Remark 6.12. Each agent believes that his terminal inventory will be zero almost surely (see (4.5)). Lemma 6.11 specifies conditions under which the agents are effectively correct in this regard.

Proof. See Appendix B.4. \(\square\)

Remark 6.13. For general parameter choices, using Lemma 6.10 to investigate the occurrence of mini-flash crashes may be difficult. Here are the key challenges:

- a) Transparent conditions governing the existence of a root of $\det A$ on $[0, T]$ are not immediate.\(^{17}\)
- b) It is not yet obvious when, if ever, the multiplicity of $\det A$'s root at $t_e$ will be 1.
- c) It is unclear that we can ensure that $\lambda \notin \mathbb{Z}$, even after a perturbation of our parameters.

\(^{15}\)In Section 7, we can always slightly perturb our parameters, if necessary, to ensure that $\lambda \notin \mathbb{Z}$ (see Lemma 7.6).

Generally, the $\lambda \in \mathbb{Z}$ case may or may not be more difficult to avoid. We leave this point for future work. In principle, there could be three additional scenarios to consider: $D(t_e) = 0$, $D(t_e)$ is a non-zero nilpotent matrix, and $\lambda \neq 0$.

When $D(t_e) = 0$, the matrix exponential in the fundamental solution of (6.2) can be dropped (see Sections 2.3 and 5.6 of [47]). If $D(t_e)$ is a non-zero nilpotent matrix, the series representation of the matrix exponential terminates after $(K - 1)$ terms, maybe fewer, as the degree of $D(t_e)$ is no higher than $K$.

In the last case, the argument is less transparent. A crucial recursion used in the determination of $P$ is no longer valid, necessitating an intricate change of variables (see Chapter 6 of [47]). Consequently, the fundamental solution of (6.2) is that in (B.6) but with $D(t_e)$ replaced by a more opaque matrix, which significantly complicates further analysis.

\(^{16}\)Recall that the certain agents' inventories and trading rates were found in Corollary 4.14.

\(^{17}\)Once we have such conditions, easily checking whether or not $t_e > 0$ would be presumably trivial but could be troublesome as well.
d) More analysis of (6.4) is needed to characterize potential explosions in the coordinates of \( X^u,\theta^* \) and \( \theta^u,\star \) as \( t \uparrow t_e \).
e) Determining whether or not \( S^{exc} \) explodes requires a more thorough study of (5.1) and (6.4).

Resolving (a) and (b) is rather intractable, unless \( K \) is small or our uncertain agents are fairly similar (see Definition 6.2). Completing the studies suggested by (c) and (d) requires further knowledge of \( \lambda \) and the eigenbasis \( \{v_1, \ldots, v_K\} \) of \( D(t_e) \). Even then, the \( y_j \)'s and the \( F_j \)'s in (B.9) may be quite opaque and pose obstacles. These observations further restrict the size of \( K \) or the differences among our agents. After all of these restrictions, finishing (e) may still not be straightforward, as in principle, the coordinates of \( X^u,\theta^* \) or \( \theta^u,\star \) might explode at the same rates but in opposite directions.

Hence, we investigate mini-flash crashes only in the context of a particularly tractable class of examples (see Section 7). We offer a rough summary of our mathematical findings in Theorem 6.14; however, the details and practical connections to mini-flash crashes are in Section 7.

We leave the study of other scenarios for future work. For instance, it would be interesting to know whether or not we could observe unbounded price oscillations near \( t_e \) or mini-flash crashes in the absence of synchronized trading (see Section 3).

**Theorem 6.14.** When our agents are as characterized in Section 4 but the risky asset’s price evolves as in Section 5, at least three cases emerge (see Lemmas 6.11, 7.3, 7.8, and 7.11 for precise statements): There are broad sufficient conditions on our deterministic parameters such that

i) \( S^{exc} \), the \( X^u,\theta^* \)'s and the \( \theta^* \)’s are all uniquely defined and continuous on \([0,T]\)
   and
   \[ \lim_{t \uparrow T} X^\theta_t = 0 \quad \tilde{P} - a.s.; \]

ii) \( S^{exc} \), the \( X^u,\theta^* \)'s, and the \( \theta^u,\star \)'s explode as \( t \uparrow t_e \) \( \tilde{P} \)-a.s.;

iii) or all coordinates of \( X^\theta \) have a finite limit but \( S^{exc} \) and the \( \theta^u,\star \)'s explode as \( t \uparrow t_e \) \( \tilde{P} \)-a.s.

In scenarios (ii) and (iii), all explosions occur in the same random direction: \( +\infty \) or \( -\infty \). The \( \tilde{P} \)-probability of infinite spikes (crashes) tends to either 0 or 1 as \( t \uparrow t_e \); however, it is positive for any fixed \( t < t_e \).

While our conditions are deterministic, no agent knows the critical parameters in these calculations. In particular, our agents believe that a mini-flash crash is a null event.

**Proof.** The result follows immediately from Lemmas 6.11, 7.3, 7.6, 7.8, and 7.11.

\[ \square \]

**7. Semi-Symmetric Uncertain Agents**

In Section 7, we thoroughly analyze a broad but tractable class of scenarios. This will enable us to both theoretically and numerically investigate the occurrence of mini-flash crashes.

Based on Remark 6.13, we specify that our uncertain agents’ parameters are identical, except for their initial inventories \( x_i \), means of their initial drift priors \( \mu_j \), and their initial estimates for the fundamental price \( S_{j,0} \). Such agents are nearly symmetric, so we call them semi-symmetric.

**Definition 7.1.** We say that our uncertain agents are semi-symmetric when there are positive constants

\[ \tilde{\eta}_{tem}, \quad \eta_{tem}, \quad \tilde{\eta}_{per}, \quad \eta_{per}, \quad \nu^2, \quad \text{and} \quad \kappa \]
such that for each $i \in \{1, \ldots, K\}$

$$\tilde{\eta}_{i,\text{tem}} = \bar{\eta}_{\text{tem}}, \quad \eta_{i,\text{tem}} = \eta_{\text{tem}}, \quad \tilde{\eta}_{\text{per}} = \bar{\eta}_{\text{per}},$$

$$\eta_{i,\text{per}} = \eta_{\text{per}}, \quad \nu_i^2 = \nu^2, \quad \kappa_i = \kappa.$$

Definition 7.1 implies that the diagonal entries of $A$ are identical, as are the off-diagonal entries. The same is true for $B$ (see Definition 6.2). Such a simplification considerably reduces the difficulties in computing $\det A$, $\lambda$, and an eigenbasis for $D (t_e)$ (see (C.2) and Lemma 7.6). The $x_j$'s, $\mu_j$'s, and $S_{j,0}$'s only enter in $C$, which also has a nice structure (see (C.12)).

For the rest of Section 7, we assume that our uncertain agents are semi-symmetric but place no restrictions on the certain agents. Our theoretical results are contained in Subsection 7.1. In Subsections 7.2 - 7.4, we provide figures for key conclusions on mini-flash crashes (see Subsection 2.1). With these plots, our goal is to highlight the features of our model, not to recreate any specific historical scenario.

### 7.1. Results

**Notation 7.2.** If our uncertain agents are semi-symmetric, the $\tau_j$'s and the $\Phi_j$'s are the same for $j \leq K$ (see Definitions 4.6, 6.2, and 7.1). We denote these functions by $\tau$ and $\Phi$, respectively.

**Lemma 7.3.** Suppose that the uncertain agents are semi-symmetric. Then $\det A$ has a root on $[0, T]$ and $t_e > 0$ if and only if

$$\left( K \tilde{\eta}_{\text{tem}} - \eta_{\text{tem}} \right) \Phi (0) > 2. \quad (7.1)$$

In this case, the zero of $\det A (\cdot)$ at $t_e$ is of multiplicity 1.

**Remark 7.4.** Definitions 4.6 and 6.2 enable us to re-write (7.1) as

$$\frac{\nu^2 \left( K \tilde{\eta}_{\text{tem}} - \eta_{\text{tem}} \right) \tanh \left( \frac{T}{2} \sqrt{\frac{\kappa}{\eta_{\text{tem}}} \kappa} \right)}{\sqrt{\eta_{\text{tem}} \kappa}} > 2. \quad (7.2)$$

By varying our parameters in (7.2) one at a time, (7.1) can be interpreted as discussed in Subsection 2.1:

i) (7.1) holds when $\nu^2$ is high. Since $\nu^2$ is the variance of the uncertain agents’ drift priors, we are led to (f) in Subsection 2.1.

ii) (7.1) holds when $(K \tilde{\eta}_{\text{tem}} - \eta_{\text{tem}})$ is high. A given uncertain agent believes that his own temporary impact parameter is $\eta_{\text{tem}}$, while the actual collective temporary impact parameter induced by the uncertain agents is $K \tilde{\eta}_{\text{tem}}$. Then $(K \tilde{\eta}_{\text{tem}} - \eta_{\text{tem}})$ is large whenever each uncertain agent severely underestimates his own temporary impact or there are many uncertain agents, giving (g) in Subsection 2.1.

iii) (7.1) holds when $T$ is high. Since $[0, T]$ is our time horizon, we get (d) in Subsection 2.1. Note that $T$ must be small enough for our agents’ modeling rationale to hold (see Section 4); however, $T$ need not be too large here, as the value of $\tanh$ reaches 95% of its supremum on $[0, \infty)$ for arguments greater than 1.8.

iv) (7.1) holds when $\kappa$ is low. We conclude (e) in Subsection 2.1, as $\kappa$ measures our uncertain agents’ aversion to volatility risks (see Subsection 4.4). Observe that both the numerator and the denominator of the LHS in (7.1) roughly look like $\sqrt{\kappa}$ for small $\kappa$; however, when $\kappa$ is large, the whole LHS looks like $1/\sqrt{\kappa}$ since $\tanh$ is bounded by 1 on $[0, \infty)$.

**Proof.** See Appendix C.1. \(\square\)
Remark 7.5. As observed in (C.3), when $\det A$ has a root on $[0, T]$, we have
\[
\Phi(t_e) = \frac{2}{K\eta_{tem} - \eta_{tem}}. \tag{7.3}
\]
No agent would think to compute $t_e$ since they all believe that a mini-flash crash is a null event; however, (7.3) makes it especially clear that they could not do so anyway.

Lemma 7.6. Suppose that the uncertain agents are semi-symmetric and (7.1) holds. Then
\[
\lambda = \frac{2\sqrt{K\eta_{tem} \coth(\tau(t_e))} - 2(K\tilde{\eta}_{per} - \eta_{per})}{(K\tilde{\eta}_{tem} - \eta_{tem}) \Phi(t_e)} \tag{7.4}
\]
and the corresponding eigenvector is $v_K = [1, \ldots, 1]^T$. By slightly perturbing $\tilde{\eta}_{per}$ and/or $\eta_{per}$, if necessary, we can ensure that $\lambda \notin \mathbb{Z}$. In this case, $D(t_e)$ is diagonalizable and the remaining vectors in an eigenbasis for $D(t_e)$ (all with the eigenvalue zero) are
\[
v_1 = [-1, 1, 0, \ldots, 0]^T, \quad \ldots, \quad v_{K-1} = [-1, 0, \ldots, 0, 1]^T.
\]

Remark 7.7. With the exceptions of $\tilde{\eta}_{per}$ and $\eta_{per}$, all parameters in (7.4) determine whether or not $\det A$ has a root on $[0, T]$ (see Lemma 7.3). They also fix the value of $t_e$ (see Remark 7.5). Hence, to interpret (7.4), we only consider the roles of $\tilde{\eta}_{per}$ and $\eta_{per}$. These parameters enter (7.4) via
\[
\frac{K\tilde{\eta}_{per} - \eta_{per}}{K\eta_{tem} - \eta_{tem}}. \tag{7.5}
\]

Intuitively, (7.5) can be viewed as the ratio of two terms: The numerator measures how far a given uncertain agent’s estimate of his own permanent impact is from the uncertain agents’ actual collective permanent impact. The denominator, which must be positive due to Lemma 7.3, is the corresponding measure for the temporary impact. One might call (7.5) a mistake ratio.

Since $\Phi(t_e) < 0$ by Lemma 6.4, $\lambda$ is positive only when (7.5) is high enough. We have $\lambda < 0$ when the uncertain agents’ total permanent impact and a single uncertain agent’s estimate of his own permanent impact are too close or when his estimate exceeds the cumulative permanent impact. More precisely,
\[
\{\lambda > 0\} \iff \left\{ \frac{1}{2} \sqrt{K\eta_{tem} \coth(\tau(t_e))} (K\tilde{\eta}_{tem} - \eta_{tem}) < K\tilde{\eta}_{per} - \eta_{per} \right\}
\]
\[
\{\lambda < 0\} \iff \left\{ \frac{1}{2} \sqrt{K\eta_{tem} \coth(\tau(t_e))} (K\tilde{\eta}_{tem} - \eta_{tem}) > K\tilde{\eta}_{per} - \eta_{per} \right\}. \tag{7.6}
\]

Whether a mini-flash crash is accompanied by high or low trading volumes is effectively determined by which inequality in (7.6) holds (see Lemmas 7.8 and 7.11 and Subsections 2.1, 7.3, and 7.4).

Proof. See Appendix C.2. □

Lemma 7.8. Suppose that the uncertain agents are semi-symmetric and (7.1) holds. Assume that $\lambda \notin \mathbb{Z}$ and $\lambda < 0$ (see Lemma 7.6). Let $p$, $y_K(\omega)$, and $F_K(\cdot, \omega)$ be defined as in Lemma 6.10. Then
\[
\left\{ y_K(\omega) > \lim_{t \uparrow t_e} \int_{t - \rho}^t \frac{F_K(s, \omega)}{|s - t_e|^{1+\delta}} ds \right\} \tag{7.7}
\]
\[
\implies \left\{ \lim_{t \uparrow t_e} X_t^{y, \theta_t^*}(\omega) = \lim_{t \uparrow t_e} \theta_t^{y, \theta_t^*}(\omega) = [+\infty, \ldots, +\infty]^T, \quad \lim_{t \uparrow t_e} S_t^{\omega} = +\infty \right\}
\]
and
\[
\begin{aligned}
y_K(\omega) < \lim_{t \uparrow t_e} \int_{t_e - \rho}^{t} \frac{F_K(s, \omega)}{|s - t_e|^{1+\lambda}} ds
\end{aligned}
\tag{7.8}
\]

\[
\Rightarrow \left\{ \lim_{t \uparrow t_e} X_t^{\theta^*} (\omega) = \lim_{t \uparrow t_e} \theta_t^{u,\ast} (\omega) = [-\infty, \ldots, -\infty]^\top, \quad \lim_{t \uparrow t_e} S_t^{exc} (\omega) = -\infty \right\}.
\]

Moreover,

i) The integral limits in (7.7) and (7.8) exist and are finite.

ii) Either (7.7) or (7.8) holds $\tilde{P}$-a.s.

iii) At $t_e - \rho$, the events (7.7) and (7.8) both have positive $\tilde{P}$-probability; however, the $\tilde{P}$-probability of one event tends to 1 (while the other tends to 0) if we let $\rho \downarrow 0$.

**Remark 7.9.** Although we fixed $\omega$ in Notation 6.1, by abuse, we view it as varying for our probabilistic statements in Lemmas 7.8 and 7.11.

**Remark 7.10.** Since $P(t_e) = I_K$ (see Lemma 6.10), (B.9) and Lemma 7.6 imply that $y_K(\omega)$ will be large and positive when the uncertain agents hold significant, similar long positions. $y_K(\omega)$ will be of high magnitude but negative, if the uncertain agents carry substantial, similarly-sized short positions. Hence, a spike in $S^{exc}(\omega)$ is more likely when the uncertain agents are synchronized aggressive buyers, while the odds of a collapse improve when they are synchronized heavy sellers. These effects play the deciding role as $t \uparrow t_e$, as the integral limits in (7.7) and (7.8) are finite.

Still, due to how we can decompose $F_K$ in our case (see (C.15)), large fluctuations in the fundamental price can make the mini-flash crash’s direction unclear until just before $t_e$ (see Figure 9).

**Proof.** See Appendix C.3. □

**Lemma 7.11.** Suppose that the uncertain agents are semi-symmetric and (7.1) holds. Assume that $\lambda \notin \mathbb{Z}$ and $\lambda > 0$ (see Lemma 7.6). Then $\tilde{P}$-a.s.,

\[
\lim_{t \uparrow t_e} X_t^{\theta^*} (\omega)
\]

exists in $\mathbb{R}^N$. If any coordinates of $\theta_t^{u,\ast} (\omega)$ explode, then $S_t^{exc} (\omega)$ and all coordinates of $\theta_t^{u,\ast} (\omega)$ explode in the same direction. For instance, when $\lambda > 1$,

\[
\begin{aligned}
\left\{ \lim_{t \uparrow t_e} \left[ |t - t_e|^{\lambda-1} \int_{t_e - \rho}^{t} \frac{\bar{W}_s (\omega) - \bar{W}_t (\omega)}{|s - t_e|^{1+\lambda}} ds \right] = +\infty \right\}
\end{aligned}
\tag{7.9}
\]

\[
\Rightarrow \left\{ \lim_{t \uparrow t_e} \theta_t^{u,\ast} (\omega) = [+\infty, \ldots, +\infty]^\top, \quad \lim_{t \uparrow t_e} S_t^{exc} (\omega) = +\infty \right\}
\]

and

\[
\begin{aligned}
\left\{ \lim_{t \uparrow t_e} \left[ |t - t_e|^{\lambda-1} \int_{t_e - \rho}^{t} \frac{\bar{W}_s (\omega) - \bar{W}_t (\omega)}{|s - t_e|^{1+\lambda}} ds \right] = -\infty \right\}
\end{aligned}
\tag{7.10}
\]

\[
\Rightarrow \left\{ \lim_{t \uparrow t_e} \theta_t^{u,\ast} (\omega) = [-\infty, \ldots, -\infty]^\top, \quad \lim_{t \uparrow t_e} S_t^{exc} (\omega) = -\infty \right\}.
\]

Moreover,

i) Either (7.9) or (7.10) holds $\tilde{P}$-a.s.

ii) At $t_e - \rho$, the events (7.9) and (7.10) both have positive $\tilde{P}$-probability; however, the $\tilde{P}$-probability of one event tends to 1 (while the other tends to 0) if we let $\rho \downarrow 0$. 
**Remark 7.12.** We make no rigorous statement regarding the $\lambda \in (0, 1)$ case. Most of Lemma 7.11’s proof would still be valid (see Appendix C.3); however, the final estimates are especially convenient when $\lambda > 1$ (see (C.26) - (C.30)). The over-arching purpose of Lemma 7.11 is only to illustrate that mini-flash crashes can occur in low trading volume environments (see Subsection 2.1). Nevertheless, we suspect that mini-flash crashes might unfold when $\lambda \in (0, 1)$, e.g., see Subsection 7.3 and (C.26) - (C.30).

**Proof.** See Appendix C.3. □

### 7.2. Example 1: No mini-flash crash.

Our mini-flash crashes do not always occur (see Lemmas 6.11 and 7.3). In Subsection 7.2, we illustrate this by numerically simulating a scenario in which $\text{det} A$ has no root on $[0, T]$.

By Lemma 7.3 and (C.2), we know that $\text{det} A$ is non-vanishing on $[0, T]$ if and only if

$$ (K\tilde{\eta}_{tem} - \eta_{tem}) \Phi(0) < 2. \tag{7.11} $$

One selection of parameters for which (7.11) is satisfied is

$$
\begin{align*}
N &= 3, & K &= 2, & T &= 1, & S_0 &= 100, \\
\beta &= 1, & \tilde{\eta}_{tem} &= 1, & \eta_{tem} &= 0.75, & \tilde{\eta}_{per} &= 1, \\
\eta_{per} &= 1, & \nu^2 &= 2, & \kappa &= 5, & x_1 &= 2, \\
x_2 &= -2, & \mu_1 &= 15, & \mu_2 &= -10, & S_{1,0} &= 100, \\
S_{2,0} &= 100, & \tilde{\eta}_{3,tem} &= 1, & \eta_{3,tem} &= 1, & \tilde{\eta}_{3,per} &= 1, \\
\mu_3 &= -3, & \nu^3_3 &= 2, & \kappa_3 &= 5, & x_3 &= 2.
\end{align*}
\tag{7.12}
$$

In fact, the LHS of (7.11) then equals 1.1095. Observe that there is no need to specify $\eta_{3,per}$ and $S_{3,0}$ as they are irrelevant (see Corollary 4.14, Definition 6.2, and Lemma 6.6). Again, our purposes are only illustrative here, and we leave the reproduction of a specific practically meaningful scenario for a future work.

Since $K = 2$ and $N = 3$, we have two uncertain agents and one certain agent in the coming plots. We label the corresponding curves with $U_1$, $U_2$, and $C_1$. For example, the label $U1$ will signify a quantity for Agent 1, the first uncertain agent. In Figures 1 and 2, we plot inventories and trading rates. The execution price is depicted in Figure 3.

The diagrams exhibit all of the important qualities that we expect based upon our theoretical results. Here are a few key features:

i) All agents liquidate their positions by the terminal time $T$ (see (6.5) and Figure 1).

ii) $S^{exc}(\omega)$, the $X^{\theta_j}(\omega)$’s and the $\theta_j(\omega)$’s are all continuous on $[0, T]$ (see Lemma 6.11 and Figures 1 - 3).

iii) The uncertain agents’ trading rates appear to exhibit a Brownian component (see Lemma 4.8 and Figure 2).

iv) The certain agent’s trading rate appears to be smooth on $[0, T]$ (see Corollary 4.14 and Figure 2).

v) The agents need not either strictly buy or strictly sell throughout $[0, T]$ (see Subsection 4.3, Lemma 4.8 and Figure 2).

vi) Even so, the agents may decide to strictly buy or strictly sell throughout $[0, T]$ (see Subsection 4.3, Lemma 4.8 and Figure 2).

vii) The uncertain agents’ trading rates do not appear to synchronize (see Figure 2).

### 7.3. Example 2: A mini-flash crash with low trading volume.

Our mini-flash crashes can be accompanied by low trading volumes (see Lemma 7.11). In Subsection 7.3, we visualize this by studying a concrete scenario in which $\text{det} A$ has a root on $[0, T]$; $t_c > 0$; the zero of $\text{det} A$ at $t_c$ is of multiplicity 1; $\lambda \notin \mathbb{Z}$; and $\lambda > 0$. The behavior of the
$X_j^\phi (\omega)$’s is then characterized by Corollary 4.14 and Lemma 7.11. Lemma 7.11 would rigorously describe $S_{\text{exc}}^\phi (\omega)$ and the $\theta_j^* (\omega)$’s as $t \uparrow t_e$, if $\lambda > 1$. To improve the quality of our plots, we consider a situation where $\lambda \in (0, 1)$ instead (see Remark 7.12).
By Lemmas 7.3 and 7.6, we must select parameters such that (7.1) is satisfied and
\[
\lambda = \frac{2}{\left( K\eta_{\text{tem}} - \eta_{\text{tem}} \right) \Phi (t_e)} \left[ \sqrt{\frac{\kappa}{\eta_{\text{tem}}}} \coth (\tau (t_e)) - 2 \left( \frac{K\tilde{\eta}_{\text{per}} - \eta_{\text{per}}}{K\eta_{\text{tem}} - \eta_{\text{tem}}} \right) \right] \tag{7.13}
\]
is a positive non-integer. We can keep most of our choices in (7.12) the same and only make a few revisions:
\[
\eta_{\text{tem}} = 0.5, \quad \tilde{\eta}_{\text{tem}} = 0.2, \quad \eta_{\text{per}} = 0.8, \\
\tilde{\eta}_{\text{per}} = 0.025, \quad \nu^2 = 3, \quad \kappa = 1. \tag{7.14}
\]
As in Subsection 7.2, we do not seek to replicate a particular historical situation. We immediately get (7.1), as its LHS is 4.3302. Using Remark 7.5 and (7.13), we can show that
\[
t_e = 0.2691 \quad \text{and} \quad \lambda = 0.5939.
\]
Again, we have two uncertain agents and one certain agent. We retain the \{U1, U2, C1\}-labeling system from Subsection 7.2. The inventories, trading rates, and execution price are plotted in Figures 4 - 6. To aid our illustration, we truncate the time domains in Figures 5 - 6 to
\[
[0, 0.75 (t_e - 10^{-6})] \quad \text{and} \quad [0, t_e - 10^{-6}]
\]
for the left and right plots, respectively.

The qualities that we expect based upon Corollary 4.15, Lemma 7.11, and Remark 7.12 are all present. We offered some applicable comments in Subsection 7.2, so we only add a few new observations here.

i) All agents’ inventories approach a finite limit as \( t \uparrow t_e \) (see Lemma 7.11 and Figure 4).

ii) The execution price and the uncertain agents’ trading rates explode as \( t \uparrow t_e \) (see Lemma 7.11, Remark 7.12 and Figures 5 - 6).

iii) The uncertain agents’ trading rates synchronize as \( t \uparrow t_e \) (see Lemma 7.11, Remark 7.12, and Figure 5).

iv) That an explosion in \( S^{\text{exc}} (\omega) \) will occur as well as its direction becomes increasingly obvious as \( t \uparrow t_e \); however, it is not necessarily clear at first (see Lemma 7.11, Remark 7.12, and Figure 6).
7.4. Example 3: A mini-flash crash with high trading volume. Our mini-flash crashes can also be accompanied by high trading volumes (see Lemma 7.8). We illustrate this in Subsection 7.4 by simulating a case in which \( \det A \) has a root on \([0, T]\); \( t_c > 0 \); the zero of \( \det A \) at \( t_c \) is of multiplicity 1; \( \lambda \notin \mathbb{Z} \); and \( \lambda < 0 \). The behaviors of \( S^{\text{exc}}(\omega) \), the \( X_{\theta^*_j}(\omega) \)'s, and the \( \theta^*_j(\omega) \)'s are then described by Corollary 4.14 and Lemma 7.8.

We especially wish to emphasize the stochastic explosion direction and do this in two ways.

First, we choose the same deterministic parameters to create Figures 7 - 12. The difference is that one realization of \( \tilde{W} \) is used in Figures 7 - 9, while another is used in Figures 10 - 12. We denote the corresponding \( \omega \)'s by \( \omega_{up} \) and \( \omega_{dn} \), since there are spikes and crashes in the former and latter plots, respectively.

Second, Figures 7 - 9 themselves suggest that the explosion direction is random. This is particularly true in Figures 8 - 9, since we initially notice that the price rapidly rises as the uncertain agents’ buying rates synchronize. Only moments before the mini-flash crash do we see the price collapsing and the uncertain agents’ aggressively selling together.
Now, we need to choose parameters such that (7.1) is satisfied and
\[
\lambda = \frac{2 \left( \sqrt{\frac{\kappa}{\eta_{tem}}} \coth (\tau (t_e)) - 2 \left( \frac{K\tilde{\eta}_{per} - \eta_{per}}{K\tilde{\eta}_{tem} - \eta_{tem}} \right) \right)}{(K\tilde{\eta}_{tem} - \eta_{tem}) \Phi (t_e)}
\]
is a negative non-integer due to Lemmas 7.3 and 7.8. Compared to Subsection 7.3, we set
\[
\tilde{\eta}_{per} = 0.5, \quad \eta_{per} = 0.5
\]
and keep every other parameter the same. As in Subsections 7.2 - 7.3, we do not have in mind a special historical example here. Since we have only changed \( \tilde{\eta}_{per} \) and \( \eta_{per} \), the values of \((K\tilde{\eta}_{tem} - \eta_{tem}) \Phi (0) \) and \( t_e \) do not differ from Subsection 7.3; however, \( \lambda \) is now negative:
\[
(K\tilde{\eta}_{tem} - \eta_{tem}) \Phi (0) = 4.3302, \quad t_e = 0.2691, \quad \lambda = -0.4531.
\]

The numbers of uncertain and certain agents are still two and one, respectively. We also retain the \( \{U1, U2, C1\} \)-labeling system from Subsections 7.2 - 7.3. Figures 7 and 10 depict the agents’ inventories. We plot the agents’ trading rates in Figures 8 and 11. The execution price appears in Figures 9 and 12. To help with our visualization, the time domains in the left plots in Figures 7 - 9 and Figures 10 - 12 are truncated to \([0, 0.94 (t_e - 10^{-6})]\) and \([0, 0.75 (t_e - 10^{-6})]\), respectively.

Our observations regarding Figures 7 - 12 are in agreement with Corollary 4.14 and Lemma 7.8. We have already made note of many important aspects in Subsections 7.2 - 7.3 and only remark upon the new details.

i) The execution price, as well as the uncertain agents’ inventories and trading rates, all explode in the same direction as \( t \uparrow t_e \) (see Lemma 7.8 and Figures 7 - 12).

ii) The explosions take place at the deterministic time \( t_e \) (see Lemma 7.8 and Figures 7 - 12).

iii) The explosion direction depends on \( \omega \in \tilde{\Omega} \) (see Lemma 7.8 and Figures 7 - 12).

iv) The explosion direction cannot be known with complete certainty before \( t_e \) (see Lemma 7.8 and Figures 7 - 12).

v) The explosion rates in the price and uncertain agents’ trading rates in Subsection 7.3 are slower than in Subsection 7.4 (see Figures 5 - 6, Figures 8 - 9, and Figures 11 - 12). We did not explicitly state this previously; however, this is to be expected since trading rates are integrable in Subsection 7.3 but not in Subsection 7.4.

### Appendix A. Section 4 Proofs

**A.1. Proof of Lemma 4.8.** We now implement the steps outlined in Remark 4.13.

**Step 1:** Denote the usual \( P_j \)-augmentation of \( \{F_{j,t}^{unf}\}_{0 \leq t \leq T} \) by \( \{\tilde{F}_{j,t}^{unf}\}_{0 \leq t \leq T} \). Let \( \tilde{A}_j \) be the space of \( \tilde{F}_{j,t}^{unf} \)-progressively measurable processes \( \theta_j \) such that (4.4) and (4.5) hold. We, again, define the process \( X_{j,\theta}^{\tilde{d}_j} \) by (4.6) for any strategy \( \theta_j \in \tilde{A}_j \). Agent \( j \)'s auxiliary problem is to maximize

\[
E^{P_j} \left[ \int_0^T \theta_{j,t} S_{j,\theta_{j,t}}^{\tilde{d}_j} dt - \frac{\kappa_j}{2} \int_0^T \left( X_{j,\theta_j}^{\tilde{d}_j} \right)^2 dt \right]
\]

over \( \theta_j \in \tilde{A}_j \).
Figure 7. Depiction of the agents' inventories for $\omega_{dn}$ in Subsection 7.4.

Figure 8. Depiction of the agents' trading rates for $\omega_{dn}$ in Subsection 7.4.

Figure 9. Depiction of the execution price for $\omega_{dn}$ in Subsection 7.4.
Figure 10. Depiction of the agents’ inventories for $\omega_{up}$ in Subsection 7.4.

Figure 11. Depiction of the agents’ trading rates for $\omega_{up}$ in Subsection 7.4.

Figure 12. Depiction of the execution price for $\omega_{up}$ in Subsection 7.4.
Step 2: We wish to show that
\[ E^P_j \left[ \beta_j \right| F^\text{unf}_{j,t} \right] = E^P_j \left[ \beta_j \right| F^\text{unf}_{j,t} \right] \quad P_j - \text{a.s.} \]

where \((\Omega_j, \mathcal{F}_j, P_j)\) is \(P_j\)-complete by hypothesis (see Subsection 4.2). Suppose that \(t \in [0, T)\). Letting \(N_j\) be the \(P_j\)-null subsets of \(\Omega_j\) and
\[ F^\text{unf}_{j,t} = \bigcap_{t < u \leq T} F^\text{unf}_{j,u} \]
we have
\[ \tilde{F}^\text{unf}_{j,t} = \sigma \left( F^\text{unf}_{j,t+}, N_j \right) \].

Hence,
\[ E^P_j \left[ \beta_j \right| \tilde{F}^\text{unf}_{j,t} \right] = E^P_j \left[ \beta_j \right| F^\text{unf}_{j,t} \right] \quad P_j - \text{a.s.} \quad (A.2) \]

Since \(F^\text{unf}_{j,t} \subseteq \tilde{F}^\text{unf}_{j,t}\), it suffices to show that
\[ E^P_j \left[ 1_U \beta_j \right] = E^P_j \left[ 1_U E^P_j \left[ \beta_j \right| F^\text{unf}_{j,t} \right] \right] \]
for all \(U \in F^\text{unf}_{j,t+}\). Pick \(U \in F^\text{unf}_{j,t+}\) and any positive decreasing sequence \((\epsilon_n)_{n \geq 1}\) in \((0, T - t)\) tending to 0. By (4.2) in Subsection 4.2 and (4.11) in Step 2,
\[ 1_U E^P_j \left[ \beta_j \left| F^\text{unf}_{j,t+\epsilon_n} \right] \right] \xrightarrow{P_j - \text{a.s.}} 1_U E^P_j \left[ \beta_j \left| F^\text{unf}_{j,t} \right] \right]. \quad (A.3) \]

By the Vitali convergence theorem and the uniform integrability of the collection
\[ \left\{ 1_U E^P_j \left[ \beta_j \right| F^\text{unf}_{j,t+\epsilon_n} \right] \right\}_{n \geq 1}, \]
(A.3) also hold in the sense of \(L^1\)-convergence. This finishes the argument, as
\[ E^P_j \left[ 1_U \beta_j \right] = E^P_j \left[ 1_U E^P_j \left[ \beta_j \right| F^\text{unf}_{j,t+\epsilon_n} \right] \right] \]
for \(n \geq 1\).

Step 3: By (4.3) and (4.6),
\[- \int_0^T \theta_j,t S^{\text{exc}}_{j,\theta_j,t} dt = \int_0^T \theta_j,t S^{\text{unf}}_{j,t} dt - \int_0^T \theta_j,t \left[ \eta_{j, \text{per}} \left( X^{\theta_j}_{j,t} - x_j \right) + \frac{1}{2} \theta_j, \text{tem} \theta_j,t \right] dt \]
for \(\theta_j \in \tilde{A}_j\). Section 7.4 of [80] and (4.2) imply that the process \(\{W_{j,t}\}_{0 \leq t \leq T}\) with
\[ W_{j,t} \triangleq S^{\text{unf}}_{j,t} - S_{j,0} - \int_0^t E^P_j \left[ \beta_j \right| F^\text{unf}_{j,s} \right] ds \]
is an \(F^\text{unf}_{j,t}\)-Wiener process under \(P_j\) and
\[ S^{\text{unf}}_{j,t} = S_{j,0} + \int_0^t E^P_j \left[ \beta_j \right| F^\text{unf}_{j,s} \right] ds + W_{j,t}. \quad (A.4) \]

---

\(^{18}\)The result is clear when \(t = T\), since \(\tilde{F}^\text{unf}_{j,T} = \sigma \left( F^\text{unf}_{j,T}, N_j \right)\).

\(^{19}\)In fact, \(W_j\) is an innovation process, i.e., for each \(t \in [0, T]\), we have \(F^\text{unf}_{j,t} = F^W_{j,t}\). Here, \(\left\{ F^W_{j,t} \right\}_{0 \leq t \leq T} \) is the filtration generated by \(W_j\).
After integrating by parts and recalling (4.6) and (A.4), we get
\[
E^P_j \left[ - \int_0^T \theta_{j,t} S_{j,t}^{unf} \, dt \right] = E^P_j \left[ -X_{j,T}^{\theta_j} S_{j,T}^{unf} + \int_0^T X_{j,t}^{\theta_j} E^P_j \left[ \beta_j \left| F_{j,t}^{unf} \right| \right] \, dt \right] + x_j S_{j,0}.
\]

We also have
\[
E^P_j \left[ - \int_0^T \theta_{j,t} \left( \eta_{j,per} \left( X_{j,t}^{\theta_j} - x_j \right) + \frac{1}{2} \eta_{j,tem} \theta_{j,t} \right) \, dt \right] = E^P_j \left[ - \frac{1}{2} \eta_{j,per} \left( X_{j,T}^{\theta_j} - x_j \right)^2 - \frac{1}{2} \eta_{j,tem} \int_0^T \theta_{j,t}^2 \, dt \right].
\]

Now \(X_{j,T}^{\theta_j} = 0\) \(P_j\)-a.s. by the definition of \(\tilde{A}_j\) in Step 1. Since \(x_j, S_{j,0}\) and \(S_{j,T}^{unf}\) do not depend on Agent \(j\)'s choice of \(\theta_j \in \tilde{A}_j\), Step 2 implies that \(\theta_j^*\) maximizes (A.1) over \(\theta_j \in \tilde{A}_j\) if and only if it maximizes
\[
E^P_j \left[ \int_0^T X_{j,t}^{\theta_j} E^P_j \left[ \beta_j \left| F_{j,t}^{unf} \right| \right] \, dt - \frac{1}{2} \eta_{j,tem} \int_0^T \theta_{j,t}^2 \, dt - \frac{\kappa_j}{2} \int_0^T \left( X_{j,t}^{\theta_j} \right)^2 \, dt \right]. \quad (A.5)
\]

Due to (4.2), (4.11), in Step 2, and Agent \(j\)'s Gaussian prior for \(\beta_j\), the process \(E^P_j \left[ \beta_j \left| F_{j,t}^{unf} \right| \right]\) is \(F_{j,t}^{unf}\)-predictable and in \(L^2 (dP_j \otimes dt)\). Clearly, \(\theta_j^*\) maximizes (A.5) over \(\theta_j \in \tilde{A}_j\) if and only if it minimizes
\[
E^P_j \left[ \frac{1}{2} \int_0^T \left( X_{j,t}^{\theta_j} - \frac{E^P_j \left[ \beta_j \left| F_{j,t}^{unf} \right| \right]}{\kappa_j} \right)^2 \, dt + \eta_{j,tem} \int_0^T \theta_{j,t}^2 \, dt \right]. \quad (A.6)
\]

**Step 4:** After defining
\[
K_j (t, s) \triangleq \sqrt{\frac{\kappa_j}{\eta_{j,tem}}} \left( \sinh (\tau_j (s)) \over \cosh (\tau_j (t)) - 1 \right), \quad 0 \leq t \leq s < T
\]
\[
\hat{\beta}_{j,t} \triangleq E^P_j \left[ \frac{1}{\kappa_j} \left( 1 - \frac{1}{\cosh (\tau_j (t))} \right) \right] \int_t^T E^P_j \left[ \beta_j \left| F_{j,s}^{unf} \right| K_j (t, s) \, ds \right] \left| F_{j,t}^{unf} \right|, \quad t \in [0, T),
\]
we see from Theorem 3.2 of [24] that (A.6) has a unique solution \(\theta_j^* \in \tilde{A}_j\). Moreover, the corresponding optimal inventory process \(X_{j,t}^{\theta_j^*}\) satisfies the linear ODE
\[
dX_{j,t}^{\theta_j^*} = \sqrt{\frac{\kappa_j}{\eta_{j,tem}}} \coth (\tau_j (t)) \left( \hat{\beta}_{j,t} - X_{j,t}^{\theta_j^*} \right) \, dt
\]
\[
X_{j,0}^{\theta_j^*} = x_j \quad (A.8)
\]
\(dP_j \otimes dt\)-a.s. on \(\Omega_j \times [0, T)\).
Using Fubini’s theorem and Steps 2, we get that

\[
E_P^j \left[ \int_t^T E_P^j \left[ \beta_j \left| \bar{F}_{j,s}^{unf} \right| K_j (t, s) \right] ds \right] = E_P^j \left[ \beta_j \left| F_{j,t}^{unf} \right| \right], \quad P_j - \text{a.s.} \quad (A.9)
\]

The tanh half-angle formula together with (4.11) and (A.7) imply that (A.8) can be re-written as

\[
\theta_{j,t}^* = -\sqrt{\frac{\kappa_j}{\eta_{j,tem}}} \coth (\tau_j (t)) X_{j,t}^{\theta_j^*} + \frac{\tanh (\tau_j (t) / 2)}{\sqrt{\eta_{j,tem} \kappa_j}} \left( \mu_j + \nu_j^2 S_{j,t}^{unf} - S_{j,0} \right), \quad t \in (0, T)
\]

\[
X_{j,0}^{\theta_j^*} = x_j. \quad (A.10)
\]

**Step 5:** We know that \( \theta_{j,t}^* \) satisfies (4.4) and (4.5), as all strategies in \( \tilde{A}_j \) have these properties. Now \( W_j, (\omega) \) is continuous on \([0, T]\) for \( P_j \)-almost every \( \omega \in \Omega_j \). When such an \( \omega \) is chosen, (A.10) becomes (4.10). The latter is a first order linear ODE with continuous coefficients, so \( \theta_{j,t}^* (\omega) \) is continuous on \([0, T]\) (e.g., by Chapter 1.2 of [96]).

Since our terminal inventory constraint is deterministic, we observe that

\[
\lim_{t \to T} \theta_{j,t}^* (\omega)
\]

exists and is finite from (28) and (29) in the proof of Theorem 3.2 in [24], as well as (A.9) in Step 4. In particular, we can view the paths of \( \theta_{j,t}^* \) on \([0, T]\) as \( P_j \)-a.s. continuous.\footnote{Alternatively, we could give an argument using singular point theory as in Section 6.}

We conclude by noting that \( \theta_{j,t}^* \) is also \( F_{j,t}^{unf} \)-adapted by (28) and (29) in the proof of Theorem 3.2 in [24], (4.11) in Step 2, and (A.9) in Step 4.

\[
\square
\]
After substituting (5.1) into (B.1), we have
\[ S_{\text{unf},j,t}^\omega - S_{j,0} = (S_0 - S_{j,0}) + \hat{\theta} t + \sum_{i \leq K, i \neq j} \hat{\eta}_{i,\text{per}} X_{i,t}^{\theta^*_i} (\omega) - x_i + \sum_{i > K} \hat{\eta}_{i,\text{per}} X_{i,t}^{\theta^*_i} - x_i \]
\[ + \frac{1}{2} \sum_{i \leq K, i \neq j} \hat{\eta}_{i,\text{tem}} \theta^*_{i,t} (\omega) + \frac{1}{2} \sum_{i > K} \hat{\eta}_{i,\text{tem}} \theta^*_{i,t} \]
\[ + (\hat{\eta}_{j,\text{per}} - \eta_{j,\text{per}}) X_{j,t}^{\theta^*_j} (\omega) - x_j + \frac{1}{2} (\hat{\eta}_{j,\text{tem}} - \eta_{j,\text{tem}}) \theta^*_{j,t} (\omega) + \tilde{W}_t (\omega). \quad (B.2) \]

The quantity on the LHS of (B.2) plays a role in determining Agent j's strategy (see Lemma 4.8). Substituting (B.2) into (4.10) and applying the half-angle formula for \( \tanh (\cdot) \), we get
\[ A_{jj}(t) \theta^*_{j,t} (\omega) - \sum_{i \leq K, i \neq j} A_{ji}(t) \theta^*_{i,t} (\omega) \]
\[ = B_{jj}(t) X_{j,t}^{\theta^*_j} (\omega) + \sum_{i \leq K, i \neq j} B_{ji}(t) X_{i,t}^{\theta^*_i} (\omega) + C_j(t,\omega). \]

It follows that the uncertain agents' strategies are characterized by the ODE system
\[ A(t) \theta^*_{i,t} (\omega) = B(t) X_{i,t}^{\theta^*_i} (\omega) + C(t,\omega) \]
\[ X_{0,\theta^*}^{u,\theta^*} (\omega) = x^u. \quad (B.3) \]

Corollary 4.14, Lemma 6.4 and a standard existence and uniqueness theorem (see Sections 1.1 and 3.1 of [26]) finish the argument. \( \square \)

B.2. Proof of Lemma 6.7. As \( t \uparrow t_e \),
\[ \left\{ A(t) X_t^u (\omega) = B(t) X_t^u (\omega) \right\} \]
\[ \iff \left\{ [\det A(t)] X_t^u (\omega) = [\text{adj} A(t)] B(t) X_t^u (\omega) \right\}. \]

Here, adj denotes the usual adjugate operator.

We can find a non-negative integer \( m \) such that the multiplicity of the zero of \( \det A \) at \( t_e \) is \( m + 1 \) by Lemma 6.4. Hence, there is a unique non-vanishing analytic function \( f \) such that
\[ \det A(t) = (t - t_e)^{m+1} f(t) \quad (B.4) \]
on a small neighborhood of \( t_e \). Note that \( f \) is non-vanishing, as the zeroes of \( \det A \) are isolated and \( \det A(T) = 1 \) (see Lemma 6.4). We then define the analytic (see Lemma 6.4) map \( D \) by
\[ D(t) \triangleq [\text{adj} A(t)] B(t) / f(t) \quad (B.5) \]
and arrive at (6.3).

Since \( \det A(\cdot) \) has a root at \( t_e \), the rank of \( A(t_e) \) is no more than \( K - 1 \). We conclude by observing that \( \text{adj} A(t_e) \) has rank 1 when \( A(t_e) \) has rank \( K - 1 \); otherwise, \( \text{adj} A(t_e) \) must be the zero matrix. The comments about the rank of \( D(t_e) \) immediately follow. \( \square \)
B.3. Proof of Lemma 6.10. \( D(t_e) \neq 0 \) since \( \lambda \neq 0 \). Then (6.2) has a singular point of the first kind at \( t_e \) (see our discussion above). \( \lambda \not\in \mathbb{Z} \) by hypothesis, so Theorem 6.5 of [47] implies that a fundamental solution of (6.2) on \( [t_e - \rho, t_e] \) for some \( \rho > 0 \) is given by

\[
P(t) \left| t - t_e \right|^{D(t_e)}.
\]

In (B.6), \( P(\cdot) \) is an analytic \( M_K(\mathbb{R}) \)-valued function with \( P(t_e) = I_K \). Moreover, \( P(t) \) is invertible for all \( t \in [t_e - \rho, t_e] \) and

\[
\left( P(t) \left| t - t_e \right|^R \right)^{-1} = \left| t - t_e \right|^{-R} \left[ P(t) \right]^{-1}.
\]

The solution of (6.1) satisfies

\[
(t - t_e) \theta_t^{u, *) (\omega) = D(t) X_t^{u, *) (\omega) + \frac{\adj \left[ A(s) \right] C(s, \omega)}{f(s)}
\]

near \( t_e \) (argue as in Lemma 6.7). Since

\[
P(t) \left| t - t_e \right|^{D(t_e)} \rho^{-D(t_e)} \left[ P(t_e - \rho) \right]^{-1}
\]

is also a fundamental solution of (6.3) on \( [t_e - \rho, t_e] \) and equals \( I_K \) at \( t_e - \rho \), we can apply variation of parameters to obtain

\[
X_t^{u, *) (\omega) = P(t) \left| t - t_e \right|^{D(t_e)} \left[ \rho^{-D(t_e)} \left[ P(t_e - \rho) \right]^{-1} \right] \cdot \left[ X_{t_e - \rho}^{(*)} (\omega) \right.
\]

\[
\left. + \int_{t_e - \rho}^t \left( P(t_e - \rho) \rho^{D(t_e)} \left| s - t_e \right|^{-D(t_e)} \left[ P(s) \right]^{-1} \right) \left( \frac{\adj \left[ A(s) \right] C(s, \omega)}{f(s)} \right) ds \right].
\]

We can find an eigenbasis \( \{v_1, \ldots, v_K\} \) for \( D(t_e) \) such that \( v_K \) corresponds to \( \lambda \) (see Lemma 6.7 and Remark 6.9). We then define the continuous real-valued functions \( \{F_1(\cdot, \omega), \ldots, F_K(\cdot, \omega)\} \) on \( [t_e - \rho, t_e] \) and the constants \( \{y_1(\omega), \ldots, y_K(\omega)\} \) as certain eigenbasis coordinates:

\[
\sum_{j=1}^{K} F_j(s, \omega) v_j \triangleq \frac{\left[ P(s) \right]^{-1} \adj \left[ A(s) \right] C(s, \omega)}{f(s)}
\]

\[
\sum_{j=1}^{K} y_j(\omega) v_j \triangleq \rho^{-D(t_e)} \left[ P(t_e - \rho) \right]^{-1} X_{t_e - \rho}^{(*)} (\omega)
\]

Taken with (B.8), these definitions immediately give (6.4) after recalling that for any matrix \( Q \in M_K(\mathbb{R}) \) with eigenvalue \( \gamma \) and corresponding eigenvector \( v \), we have

\[
|t - t_e|^Q v = |t - t_e|^\gamma v.
\]

\[\square\]

\[22\] Any fundamental solution of (6.3) is invertible everywhere, as are matrix exponentials.

\[23\] See Theorem 2.5 of Coddington & Carlson ([47]).

\[24\] See Theorem 2.8 of Coddington & Carlson ([47]).
B.4. Proof of Lemma 6.11. We know that $S_{exc}^\omega (\omega)$, the $X_j^{\omega_t} (\omega)$’s and the $\theta_j^\ast (\omega)$’s are all uniquely defined and continuous on $[0, T)$ (see Lemma 6.6). Corollary 4.14 implies that $X_j^{\omega_t} (\omega)$ and $\theta_j^\ast (\omega)$ are continuous at $T$ for $j > K$ (the certain agents). It also gives us

$$\lim_{t \uparrow T} X_j^{\omega_t} (\omega) = 0$$

for $j > K$. By Definition 5.1, it remains to show that

$$\lim_{t \uparrow T} X_j^{u,\omega_t} (\omega) = 0 \quad \text{and} \quad \lim_{t \uparrow T} \theta_i^{u,\ast} (\omega) \in \mathbb{R}^K.$$  \hspace{1cm} (B.10)

As discussed above, one difficulty is that the diagonal entries of $B$ in (6.1) explode at $T$ (see Lemma 6.4); however, the approach for resolving this issue is similar to that used to analyze solution behavior near $t_e$.

First, we show that (6.2) (after replacing $t_e$ with $T$) has a singular point of the first kind at $T$. Now $\sinh (\tau_j (\cdot))$ has a zero of multiplicity 1 at $T$ since

$$\frac{d \sinh (\tau_j (t))}{dt} \bigg|_{t = T} = - \frac{K_j}{\eta_j,tem} \cosh (\tau_j (t)) \bigg|_{t = T} = - \sqrt{\frac{K_j}{\eta_j,tem}}.$$ 

Hence, there is a unique non-vanishing analytic function $g_j$ such that

$$\sinh (\tau_j (t)) = (t - T) g_j (t) \quad \text{and} \quad g_j (T) = - \sqrt{\frac{K_j}{\eta_j,tem}}$$  \hspace{1cm} (B.11)

on a small neighborhood of $T$. Near $T$, it follows that the entries of $(t - T) B (t)$ are given by

$$(t - T) B_{ik} (t) = \begin{cases} 
(t - T) \left( \eta_{i,per} - \eta_{i,tem} \right) \Phi_i (t) 
& \text{if } i = k \\
- \frac{K_i}{\eta_{i,tem}} \left( \cosh (\tau_i (t)) \right) 
& \text{if } i \neq k
\end{cases}$$  \hspace{1cm} (B.12)

(see Definition 6.2). On this region, the solution of (6.2) satisfies

$$(t - T) X_t^{u} (\omega) = A^{-1} (t - T) B (t) X_t^{u} (\omega).$$  \hspace{1cm} (B.13)

By (B.12) and Lemma 6.4, (B.13) has a singular point of the first kind at $T$.

Second, we find a fundamental solution of (B.13) near $T$. We know that

$$A^{-1} (T) = (t - T) B (t) \bigg|_{t = T} = I_K$$

by (B.11), (B.12), and Lemma 6.4. Theorem 6.5 of [47] implies that a fundamental solution of (B.13) on $[T - \delta, T)$ for some $\delta > 0$ is given by

$$Q (t) | t - T |^{\delta K} = Q (t) | t - T |.$$  \hspace{1cm} (B.14)

In (B.14), $Q$ is an analytic $M_K (\mathbb{R})$-valued function with $Q (T) = I_K$. Also, $Q (t)$ is invertible for all $t \in [T - \delta, T)$.\(^{25}\)

Finally, we use our fundamental solution to solve (6.1) and conclude the proof. Notice that $\tanh (\tau_j (\cdot))$ also has a zero of multiplicity 1 at $T$ since

$$\frac{d \tanh (\tau_j (t))}{dt} \bigg|_{t = T} = - \frac{K_j}{2 \eta_j,tem} \operatorname{sech}^2 (\tau_j (t) / 2) \bigg|_{t = T} = - \frac{1}{2} \sqrt{\frac{K_j}{\eta_j,tem}}.$$ 

\(^{25}\)Any fundamental solution of (B.13) is invertible everywhere.
There is a unique non-vanishing analytic function $h_j$ such that
\[ \tanh (\tau_j (t)/2) = (t - T) h_j (t) \] (B.15)
on a neighborhood of $T$. In particular, the entries of $C(t,\omega)/(t - T)$ near $T$ are given by
\[
\begin{align*}
C_i (t, \omega) &= \left( \frac{h_i (t) \nu_i^2}{\sqrt{\eta_i,tem (1 + \nu_i^2 t)}} \right) \left[ \frac{\mu_i}{\nu_i^2} - (S_0 - S_{i,0}) + \tilde{\theta} t - \sum_{k < K, k \neq i} \tilde{\eta}_{k,per} x_k \\
&\quad - x_i (\tilde{\eta}_{i,per} - \eta_{i,per}) + \sum_{k > K} \tilde{\eta}_{k,per} (X_{k,t}^\theta - x_k) \\
&\quad + \frac{1}{2} \sum_{k > K} \tilde{\eta}_{k,tem} \theta_{k,t} + \tilde{W}_t (\omega) \right]. 
\end{align*}
\] (B.16)

Since
\[ Q(t) |t - T| \delta^{-1} Q^{-1} (T - \delta) \]
is also a fundamental solution of (6.3) on $[T - \delta, T)$\(^{26}\) and equals $I_K$ at $T - \delta$, we can apply variation of parameters\(^{27}\) to obtain
\[ X^{u,\theta^*} (\omega) \]
\[ = Q(t) |t - T| \delta^{-1} Q^{-1} (T - \delta) \cdot \left[ X_{T-\delta}^{\theta^*} (\omega) \\
+ \int_{T-\delta}^t \left( Q(T - \delta) \delta |s - T|^{-1} Q^{-1} (s) \right) A^{-1} (s) C(s, \omega) ds \right]. \] (B.17)

By (B.16), (B.17), and Corollary 4.14, we get (B.10). \(\square\)

**Appendix C. Section 7 Proofs**

C.1. **Proof of Lemma 7.3.** By Definitions 6.2 and 7.1, we see that $A$ is now given by
\[
A_{ik} (t) = \begin{cases} 
1 - \frac{1}{2} (\tilde{\eta}_{tem} - \eta_{tem}) \Phi (t) & \text{if } i = k \\
-\frac{1}{2} \eta_{tem} \Phi (t) & \text{if } i \neq k
\end{cases}. \] (C.1)

A short calculation shows that
\[
\det A (t) = \left[ 1 + \frac{1}{2} \eta_{tem} \Phi (t) \right]^{K-1} \left[ 1 - \frac{1}{2} (K \tilde{\eta}_{tem} - \eta_{tem}) \Phi (t) \right]. \] (C.2)

The first term in (C.2) is always at least 1. The second term is non-zero at 0 but does have a root on $(0, T]$ if and only if (7.1) holds.\(^{28}\) Both of these observations come from Lemma 6.4.

Now, (7.1) implies that $K \tilde{\eta}_{tem} > \eta_{tem}$. Since $t_e$ is a zero of $\det A$, we must have that
\[ 1 - \frac{1}{2} (K \tilde{\eta}_{tem} - \eta_{tem}) \Phi (t_e) = 0. \] (C.3)

---

\(^{26}\)See Theorem 2.5 of Coddington & Carlson ([47]).
\(^{27}\)See Theorem 2.8 of Coddington & Carlson ([47]).
\(^{28}\)In fact, $t_e$ is the unique root of $\det A$ in this case.
Hence, by Lemma 6.4,
\[ \frac{d[\det A(t)]}{dt} \bigg|_{t=t_e} = \left. -\frac{1}{2} \left(K \hat{\eta}_{tem} - \eta_{tem}\right) \left[ 1 + \frac{1}{2} \bar{\eta}_{tem} \Phi(t) \right]^{K-1} \Phi(t) \right|_{t=t_e} > 0. \quad (C.4) \]

C.2. Proof of Lemma 7.6. By (B.4), (C.4), and Lemma 7.3,
\[ f(t_e) = \left. \frac{d[\det A(t)]}{dt} \right|_{t=t_e} = \left. -\frac{1}{2} \left(K \hat{\eta}_{tem} - \eta_{tem}\right) \left[ 1 + \frac{1}{2} \bar{\eta}_{tem} \Phi(t_e) \right]^{K-1} \frac{d}{dt} \Phi(t_e) \right|_{t=t_e}. \quad (C.5) \]

A short calculation shows that \( \text{adj} A(t) \) is given by
\[ \text{adj} A(t)_{ik} = \left( 1 + \frac{1}{2} \bar{\eta}_{tem} \Phi(t) \right)^{K-2} \begin{cases} 1 - \frac{1}{2} \left((K-1) \hat{\eta}_{tem} - \eta_{tem}\right) \Phi(t) & \text{if } i = k \\
1 + \frac{1}{2} \bar{\eta}_{tem} \Phi(t) & \text{if } i \neq k \end{cases}. \quad (C.6) \]

It follows that
\[ \left[ \text{adj} A(t) B(t) \right]_{ik} = \hat{\eta}_{per} \Phi(t) \left( 1 + \frac{1}{2} \bar{\eta}_{tem} \Phi(t) \right)^{K-1} + \left( 1 + \frac{1}{2} \bar{\eta}_{tem} \Phi(t) \right)^{K-2} \left( \eta_{per} \Phi(t) + \sqrt{\frac{\kappa}{\bar{\eta}_{tem}}} \coth(\tau(t)) \right) \cdot \begin{cases} 1 - \frac{1}{2} \left((K-1) \hat{\eta}_{tem} - \eta_{tem}\right) \Phi(t) - 1 & \text{if } i = k \\
-\frac{1}{2} \bar{\eta}_{tem} \Phi(t) & \text{if } i \neq k \end{cases}. \quad (C.7) \]

One can then check that the only potentially non-zero eigenvalue of
\[ D(t_e) = \frac{[\text{adj} A(t_e) B(t_e)]}{f(t_e)} \]
is given by
\[ \lambda = -\frac{2 \left((K \hat{\eta}_{per} - \eta_{per}) \Phi(t_e) - \sqrt{\frac{\kappa}{\bar{\eta}_{tem}}} \coth(\tau(t_e)) \right)}{(K \hat{\eta}_{tem} - \eta_{tem}) \Phi(t_e)} \quad (C.8) \]
with corresponding eigenvector \( v_K \) as above. We get (7.4) from (C.8) after applying (7.3).

Recall that \( \Phi(t_e) > 0 \) and \( \hat{\Phi}(t_e) < 0 \) by Lemma 6.4. Since \( t_e, \Phi, \) and \( \tau \) do not depend on \( \hat{\eta}_{per} \) or \( \eta_{per} \), we can ensure that \( \lambda \not\in \mathbb{Z} \) by perturbing the latter parameters. \( D(t_e) \) is then diagonalizable as observed in Lemma 6.10, and \( v_1, \ldots, v_{K-1} \) can be computed using (C.7). \( \square \)
C.3. **Proof of Lemmas 7.8 and 7.11.** Since our uncertain agents are semi-symmetric,

\[ C_i (t, \omega) = \Phi (t) \tilde{W}_t (\omega) \]

\[ + \Phi (t) \left[ \beta t + \sum_{k > K} \tilde{\eta}_{k, \text{per}} \left( X_{k, t}^{i} - x_k \right) + \frac{1}{2} \sum_{k > K} \tilde{\eta}_{k, \text{tem}} \theta_{k, t}^{i} \right] \]

\[ + \Phi (t) \left[ \frac{\mu_i}{\mu^2} + (S_0 - S_{i, 0}) - \sum_{k \leq K, k \neq i} \tilde{\eta}_{\text{per}} x_k - x_i (\tilde{\eta}_{\text{per}} - \eta_{\text{per}}) \right] \tag{C.9} \]

for \( t \leq t_e \) by Definition 6.2. For convenience, we introduce the following deterministic function\(^{29}\) \( c \) and the constants \( c_1, \ldots, c_K \):

\[ c (t) \triangleq \left[ \beta t + \sum_{k > K} \tilde{\eta}_{k, \text{per}} \left( X_{k, t}^{i} - x_k \right) + \frac{1}{2} \sum_{k > K} \tilde{\eta}_{k, \text{tem}} \theta_{k, t}^{i} \right] \]

\[ \sum_{i=1}^{K} c_i v_i \triangleq \left[ \frac{\mu_1}{\mu^2} + (S_0 - S_{1, 0}) - \sum_{k \leq K, k \neq 1} \tilde{\eta}_{\text{per}} x_k - x_1 (\tilde{\eta}_{\text{per}} - \eta_{\text{per}}) \right] \]

\[ \vdots \]

\[ \frac{\mu_K}{\mu^2} + (S_0 - S_{K, 0}) - \sum_{k \leq K, k \neq K} \tilde{\eta}_{\text{per}} x_k - x_K (\tilde{\eta}_{\text{per}} - \eta_{\text{per}}) \]. \tag{C.10} \]

Using (C.9), we get that

\[ C (t, \omega) = \tilde{W}_t (\omega) \Phi (t) v_K + c (t) \Phi (t) v_K + \Phi (t) \sum_{i=1}^{K} c_i v_i \tag{C.11} \]

By (6), \( \{v_1, \ldots, v_K\} \) is an eigenbasis for \( \text{adj} [A (t)] \). Moreover,

\[ \left( 1 + \frac{1}{2} \tilde{\eta}_{\text{tem}} \Phi (t) \right)^{K-2} \left[ 1 - \frac{1}{2} (K \tilde{\eta}_{\text{tem}} - \eta_{\text{tem}}) \Phi (t) \right] \tag{C.12} \]

is the eigenvalue corresponding to each of \( v_1, \ldots, v_{K-1} \), while

\[ \left( 1 + \frac{1}{2} \tilde{\eta}_{\text{tem}} \Phi (t) \right)^{K-1} \tag{C.13} \]

corresponds to \( v_K \).

---

\(^{29}\)The function \( c \) is deterministic by Corollary 4.14.
By (B.9), it follows that
\[
\sum_{j=1}^{K} F_j(t, \omega) v_j
\]
\[
= \left[ P(t) \right]^{-1} \text{adj} \left[ A(t) \right] C(t, \omega)
\]
\[
= \tilde{W}_t(\omega) \left( \Phi(t) \left( 1 + \frac{1}{2} \eta_{\text{item}} \Phi(t) \right) \right)^{-K-1} [P(t)]^{-1} v_K
\]
\[
+ \left( \Phi(t) \left( 1 + \frac{1}{2} \eta_{\text{item}} \Phi(t) \right) \right)^{-K-2} \left[ 1 - \frac{1}{2} (K \eta_{\text{item}} - \eta_{\text{item}}) \Phi(t) \right] \left( c(t) + c_K \right)
\]
\[
\cdot \sum_{i=1}^{K-1} c_i [P(t)]^{-1} v_i.
\]
It follows that we can find analytic deterministic functions $F_{j,1}$ and $F_{j,2}$ such that
\[
F_j(t, \omega) \triangleq \tilde{W}_t(\omega) F_{j,1}(t) + F_{j,2}(t)
\]
for each $j \in \{1, \ldots, K\}$.

Since $P(t_e) = I_K$ (see Lemma 6.10), (C.5) and Remark 7.5 further imply that
\[
F_{j,1}(t_e) = F_{j,2}(t_e) = \cdots = F_{K-1,1}(t_e) = F_{K-1,2}(t_e) = 0
\]
and
\[
F_{K,1}(t_e) = -\frac{\Phi^2(t_e)}{\Phi(t_e)} > 0 \quad \text{and} \quad F_{K,2}(t_e) = -\frac{\Phi^2(t_e)}{\Phi(t_e)} (c(t_e) + c_K).
\]
While $F_{K,1}(t_e) > 0$, determining the sign of $F_{K,2}(t_e)$ is difficult, in general, as it depends upon the sign of $c(t_e) + c_K$ (see (C.10)).

We see from (C.15) and (C.16) that the expression
\[
\frac{F_j(s, \omega)}{|s - t_e|}
\]
is bounded near $t_e$ for each $j < K$ and almost every $\omega \in \tilde{\Omega}$. In particular, the coordinates of both
\[
\sum_{j=1}^{K-1} \left( y_j(\omega) - \int_{t_e-\rho}^{t} \frac{F_j(s, \omega)}{|s-t_e|} ds \right) P(t) v_j
\]
and its time derivative are bounded near $t_e$ for such $\omega$ as well.

Since $P(t_e) = I_K$, the $v_K$-coordinate of $P(t) v_K$ tends to $1$ as $t \uparrow t_e$. For $j < K$, the $v_j$-coordinate of $P(t) v_K$ tends to $0$ as $t \uparrow t_e$. In each situation, we can also obtain

\[\text{Note that } c \text{ is continuously differentiable on } [0, t_e] \text{ by Corollary 4.14.}\]
Lipschitz bounds on the convergence. Due to (6.4) and (C.15), potential explosions in the coordinates of \(X_{t}^{u, \theta^*} (\omega)\) are characterized by

\[
\lim_{t \uparrow t_{e}} \left[ |t - t_{e}|^{\lambda} \left( y_{K} (\omega) - \int_{t_{e} - \rho}^{t} \frac{\tilde{W}_{s} (\omega) F_{K, 1} (s) + F_{K, 2} (s)}{|s - t_{e}|^{1+\lambda}} \, ds \right) \right]. \tag{C.19}
\]

Specifically,

\[
\begin{align*}
\{(C.19) < +\infty \} & \iff \left\{ \lim_{t \uparrow t_{e}} X_{t}^{u, \theta^*} (\omega) \text{ exists in } \mathbb{R}^{K} \right\} \\
\{(C.19) = +\infty \} & \iff \left\{ \lim_{t \uparrow t_{e}} X_{t}^{u, \theta^*} (\omega) = [+\infty, \ldots, +\infty]^{\top} \right\} \\
\{(C.19) = -\infty \} & \iff \left\{ \lim_{t \uparrow t_{e}} X_{t}^{u, \theta^*} (\omega) = [-\infty, \ldots, -\infty]^{\top} \right\}. \tag{C.20}
\end{align*}
\]

To finish the proof, we separately consider the \(\lambda < 0\) and \(\lambda > 0\) cases.

**\(\lambda < 0\) Case.**

Assume that \(\lambda < 0\). It follows that

\[
\lim_{t \uparrow t_{e}} \int_{t_{e} - \rho}^{t} \frac{|\tilde{W}_{s} (\omega) F_{K, 1} (s)|}{|s - t_{e}|^{1+\lambda}} \, ds < \infty \quad \text{and} \quad \lim_{t \uparrow t_{e}} \int_{t_{e} - \rho}^{t} \frac{|F_{K, 2} (s)|}{|s - t_{e}|^{1+\lambda}} \, ds < \infty.
\]

Clearly,

\[
\lim_{t \uparrow t_{e}} |t - t_{e}|^{\lambda} = +\infty,
\]

meaning that

\[
\begin{align*}
\left\{ y_{K} (\omega) - \lim_{t \uparrow t_{e}} \int_{t_{e} - \rho}^{t} \frac{F_{K, 2} (s)}{|s - t_{e}|^{1+\lambda}} \, ds > \lim_{t \uparrow t_{e}} \int_{t_{e} - \rho}^{t} \frac{\tilde{W}_{s} (\omega) F_{K, 1} (s)}{|s - t_{e}|^{1+\lambda}} \, ds \right\} \tag{C.21} \\
\implies \left\{ \lim_{t \uparrow t_{e}} X_{t}^{u, \theta^*} (\omega) = [+\infty, \ldots, +\infty]^{\top} \right\}
\end{align*}
\]

and

\[
\begin{align*}
\left\{ y_{K} (\omega) - \lim_{t \uparrow t_{e}} \int_{t_{e} - \rho}^{t} \frac{F_{K, 2} (s)}{|s - t_{e}|^{1+\lambda}} \, ds < \lim_{t \uparrow t_{e}} \int_{t_{e} - \rho}^{t} \frac{\tilde{W}_{s} (\omega) F_{K, 1} (s)}{|s - t_{e}|^{1+\lambda}} \, ds \right\} \tag{C.22} \\
\implies \left\{ \lim_{t \uparrow t_{e}} X_{t}^{u, \theta^*} (\omega) = [-\infty, \ldots, -\infty]^{\top} \right\}
\end{align*}
\]

Arguing as in our discussion of (C.18), we see that the hypotheses in (C.21) and (C.22) also imply that

\[
\left\{ \lim_{t \uparrow t_{e}} \theta_{t}^{u, \pi} (\omega) = [+\infty, \ldots, +\infty]^{\top} \right\} \quad \text{and} \quad \left\{ \lim_{t \uparrow t_{e}} \theta_{t}^{u, \pi} (\omega) = [-\infty, \ldots, -\infty]^{\top} \right\},
\]

respectively.\(^{31}\) Conditional on \(\tilde{F}_{t_{e} - \rho}\), the RHS of the inequality in (C.21) (and C.22) is deterministic. Since \(F_{K, 1} (t_{e}) > 0\) (see (C.17)), we finish our proof of Lemma 7.8.

**\(\lambda > 0\) Case.**

Assume that \(\lambda > 0\). We can find a constant \(R_{0} (\omega)\) such that

\[
\left| y_{K} (\omega) - \int_{t_{e} - \rho}^{t} \frac{\tilde{W}_{s} (\omega) F_{K, 1} (s) + F_{K, 2} (s)}{|s - t_{e}|^{1+\lambda}} \, ds \right| \leq \frac{R_{0} (\omega)}{|t - t_{e}|^{\lambda}}. \tag{C.23}
\]

\(^{31}\)In particular, the coordinates of \(\theta_{t}^{u, \pi} (\omega)\) will asymptotically explode at the rate \(|t - t_{e}|^{-\lambda-1}\).
Hence, (C.19) is bounded as \( t \uparrow t_e \) and
\[
\lim_{t \uparrow t_e} X_t^{u, \theta^*} (\omega)
\]
exists in \( \mathbb{R}^K \) by our previous comments.

By our discussion surrounding (C.18), we see that explosions in the coordinates of \( \theta_t^{u, \star} (\omega) \) are characterized by
\[
\lim_{t \uparrow t_e} \left[ -\lambda |t - t_e|^{\lambda - 1} \left( y_K (\omega) - \int_{t_e - \rho}^t \frac{\bar{W}_s (\omega) F_{K,1} (s) + F_{K,2} (s)}{|s - t_e|^{1 + \lambda}} ds \right) \right.
\]
\[
\left. - \left( \bar{W}_t (\omega) F_{K,1} (t) + F_{K,2} (t) \right) \right] .
\]
(C.24)

More precisely,
\[
\{(C.24) = +\infty \} \iff \left\{ \lim_{t \uparrow t_e} \theta_t^{u, \star} (\omega) = [+\infty, \ldots, +\infty]^T \right\}
\]
\[
\{(C.24) = -\infty \} \iff \left\{ \lim_{t \uparrow t_e} \theta_t^{u, \star} (\omega) = [-\infty, \ldots, -\infty]^T \right\} .
\]
(C.25)

Suggestively, we first rewrite the expression in (C.24) as
\[
F_{K,2} (t) \left( \lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^t \frac{1}{|s - t_e|^{1 + \lambda}} ds - \frac{1}{|t - t_e|} \right)
\]
\[
+ \lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^t \frac{F_{K,2} (s) - F_{K,2} (t)}{|s - t_e|^{1 + \lambda}} ds
\]
\[
- \lambda |t - t_e|^{\lambda - 1} y_K (\omega)
\]
\[
+ \bar{W}_t (\omega) F_{K,1} (t) \left( \lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^t \frac{1}{|s - t_e|^{1 + \lambda}} ds - \frac{1}{|t - t_e|} \right)
\]
\[
+ \lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^t \frac{\bar{W}_s (\omega) [F_{K,1} (s) - F_{K,1} (t)]}{|s - t_e|^{1 + \lambda}} ds
\]
\[
+ \lambda |t - t_e|^{\lambda - 1} F_{K,1} (t) \int_{t_e - \rho}^t \frac{\bar{W}_s (\omega) - \bar{W}_t (\omega)}{|s - t_e|^{1 + \lambda}} ds
\]
(C.26)

Let \( R_1 \) and \( R_2 \) be the deterministic Lipschitz coefficients for \( F_{K,1} \) and \( F_{K,2} \). The first two lines of (C.26) are deterministic, and we can obtain the following bounds:
\[
\left| F_{K,2} (t) \left( \lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^t \frac{1}{|s - t_e|^{1 + \lambda}} ds - \frac{1}{|t - t_e|} \right) \right|
\]
\[
\leq |F_{K,2} (t)| |t - t_e|^{\lambda - 1}
\]
\[
\rho^\lambda
\]
\[
\left| \lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^t \frac{F_{K,2} (s) - F_{K,2} (t)}{|s - t_e|^{1 + \lambda}} ds \right|
\]
\[
\leq \left( \frac{\lambda R_2}{1 - \lambda} \right) \left( \rho^{1 - \lambda} |t - t_e|^{\lambda - 1} - 1 \right)
\]
(C.27)

In (C.26), the third line is deterministic conditional on \( \bar{F}_{t_e - \rho} \). Lines 4 - 6 of (C.26) are stochastic conditional on \( \bar{F}_{t_e - \rho} \). Letting \( R_3 (\omega) \) be the maximum of \( |\bar{W}_t (\omega)| \) on
of Lemma 7.11. □

Asymptotically, the variance in (C.30) grows like

\[
\frac{\lambda |t - t_e|^{\lambda - 1}}{\rho^{\lambda - 1}} \int_{t_e - \rho}^{t} \frac{1}{|s - t_e|^{1+\lambda}} ds - \frac{1}{|t - t_e|^{1+\lambda}}
\]

(all other terms tend to 0 \(\tilde{P}\)-a.s.). Using integration by parts,

\[
\lambda |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^{t} \frac{\tilde{W}_s(\omega) - \tilde{W}_t(\omega)}{|s - t_e|^{1+\lambda}} ds
\]

also tends to 0 \(\tilde{P}\)-a.s.) as \(t \uparrow t_e\), completing the proof of Lemma 7.11.

\[
\lim_{t \uparrow t_e} \left| |t - t_e|^{\lambda - 1} \int_{t_e - \rho}^{t} \frac{\tilde{W}_s(\omega) - \tilde{W}_t(\omega)}{|s - t_e|^{1+\lambda}} ds \right| = 0
\]

Asymptotically, the variance in (C.30) grows like \(|t - t_e|^{-1}\) as \(t \uparrow t_e\), completing the proof of Lemma 7.11.

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