LOCAL RINGS OF EMBEDDING CODEPTH 3. EXAMPLES

LARS WINThER CHRISTENSEN AND OANA VELICHE

Abstract. A complete local ring of embedding codepth 3 has a minimal free
resolution of length 3 over a regular local ring. Such resolutions carry a differ-
ential graded algebra structure, based on which one can classify local rings of
embedding codepth 3. We give examples of algebra structures that have been
conjectured not to occur.

1. Introduction

A classification of commutative noetherian local rings of embedding codepth \(c \leq 3\)
took off more than a quarter of a century ago. Up to completion, a local ring of
embedding codepth \(c\) is a quotient of regular local ring \(Q\) by an ideal \(I\) of grade
\(c\), and the classification is based on an algebra structure on \(\text{Tor}_Q^*(Q/I, k)\), where
\(k\) is the residue field of \(Q\). The possible isomorphism classes of these algebras
were identified by Weyman [7] and by Avramov, Kustin, and Miller [2]. Significant
restrictions on the invariants that describe these isomorphism classes were recently
worked out by Avramov [1]. Here is a précis that will suffice for our purposes.

Let \(R\) be a commutative noetherian local ring with maximal ideal \(m\) and residue
field \(k = R/m\). Denote by \(e\) the minimal number of generators of \(m\) and by \(d\) the
depth of \(R\). The number \(e\) is called the embedding dimension of \(R\), and \(c = e - d\) is
the embedding codepth. By Cohen’s Structure Theorem the \(m\)-adic completion \(\hat{R}\)
of \(R\) has the form \(\hat{R} = Q/I\), where \(Q\) is a complete regular local ring with the same
embedding dimension and residue field as \(R\); we refer to \(I\) as the Cohen ideal of \(R\).

The projective dimension of \(\hat{R}\) over \(Q\) is \(c\), by the Auslander–Buchsbaum For-
mula. From now on let \(c \leq 3\); the minimal free resolution \(F\) of \(\hat{R}\) over \(Q\) then
carries a differential graded algebra structure. It induces a graded algebra struc-
ture on \(F \otimes_Q k = \text{H}(F \otimes_Q k) = \text{Tor}_Q^*(\hat{R}, k)\), which identifies \(R\) as belonging to
one of six (parametrized) classes, three of which are called \(B\), \(C(c)\), and \(G(r)\) for
\(r \geq 2\). The ring \(R\) is in \(C(c)\) if and only if it is an embedding codepth \(c\) complete
intersection. If \(R\) is Gorenstein but not a complete intersection, then it belongs
to the class \(G(r)\) with \(r = \mu(I)\), the minimal number of generators of the Cohen
ideal. Work of J. Watanabe [6] shows that \(\mu(I)\) is odd and at least 5. Brown [3]
identified rings in \(B\) of type 2, and thus far no other examples of \(B\) rings have been
known. Rings in \(G(r)\) that are not Gorenstein—rings in \(G(3)\) and \(G(2n)\) for \(n \in \mathbb{N}\)
in particular—have also been elusive; in fact, it has been conjectured [1, 3.10] that
every ring in \(G(r)\) would be Gorenstein and, by implication, that the classes \(G(3)\)
and \(G(2n)\) would be empty.

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In this paper we provide examples of some of the sorts of rings that have hitherto dodged detection; the precise statements follow in Theorems I and II below.

**Theorem I.** Let $k$ be a field, set $Q = k[x, y, z]$, and consider these ideals in $Q$:

\[
\begin{align*}
g_1 &= (xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4) \\
g_2 &= g_1 + (x^3y - z^4) \\
g_3 &= g_2 + (x^2z^2) \\
g_4 &= g_3 + (x^3z) .
\end{align*}
\]

Each algebra $Q/g_n$ has embedding codepth 3 and type 2, and $Q/g_n$ is in $G(n + 1)$.

The theorem provides counterexamples to the conjecture mentioned above: The classes $G(2)$, $G(3)$, and $G(4)$ are not empty, and rings in $G(5)$ need not be Gorenstein. The second theorem provides examples of rings in $B$ of type different from 2.

**Theorem II.** Let $k$ be a field, set $Q = k[x, y, z]$, and consider these ideals in $Q$:

\[
\begin{align*}
b_1 &= (x^3, x^2y, yz^2, z^3) \\
b_2 &= b_1 + (xyz) \\
b_3 &= b_2 + (xy^2 - y^3) \\
b_4 &= b_3 + (y^2z) .
\end{align*}
\]

Each algebra $Q/b_n$ has embedding codepth 3 and belongs to $B$. The algebras $Q/b_1$ and $Q/b_2$ have type 1 while $Q/b_3$ and $Q/b_4$ have type 3.

The algebras $Q/b_1$ and $Q/b_2$ have embedding codimension 2, which is the largest possible value for a non-Gorenstein ring of embedding codepth 3 and type 1. The algebras $Q/b_3$ and $Q/b_4$ are artinian; that is, they have embedding codimension 3. An artinian local ring of codepth 3 and type 2 belongs to $B$ only if the minimal number of generators of its Cohen ideal is odd and at least 5; see [1, 3.4]. In that light it appears noteworthy that $b_3$ and $b_4$ are minimally generated by 6 and 7 elements, respectively.

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In preparation for the proofs, we recall a few definitions and facts. Let $Q$ be a regular local ring with residue field $k$ and let $I$ be an ideal in $Q$ of grade 3. The quotient ring $R = Q/I$ has codepth 3 and its minimal free resolution over $Q$ has the form

\[
F = 0 \rightarrow Q^n \rightarrow Q^m \rightarrow Q^{l+1} \rightarrow Q \rightarrow 0 ,
\]

where $n$ is the type of $R$ and one has $m = n + l$ and $l + 1 = \mu(I)$. It has a structure of a graded-commutative differential graded algebra; this was proved by Buchsbaum and Eisenbud [4, 1.3]. While this structure is not unique, the induced graded-commutative algebra structure on $A = H(F \otimes_Q k)$ is unique up to isomorphism. Given bases

\[
\begin{align*}
e_1, \ldots, e_{l+1} & \text{ for } A_1, \\
f_1, \ldots, f_m & \text{ for } A_2, \text{ and} \\
g_1, \ldots, g_n & \text{ for } A_3
\end{align*}
\]
graded-commutativity yields

\begin{align}
\mathbf{e}_i \mathbf{e}_j &= \mathbf{e}_j \mathbf{e}_i \quad \text{and} \quad (1.2) \quad \mathbf{e}_i^2 &= 0 \quad \text{for all} \quad 1 \leq i, j \leq l + 1 ; \\
\mathbf{e}_i \mathbf{f}_j &= \mathbf{f}_j \mathbf{e}_i \quad \text{for all} \quad 1 \leq i \leq l + 1 \quad \text{and all} \quad 1 \leq j \leq m .
\end{align}

We recall from [2, 2.1] the definitions of the classes \( \mathbf{G}(r) \) with \( r \geq 2^* \) and \( \mathbf{B} \).

The ring \( R \) belongs to \( \mathbf{G}(r) \) if there is a basis (1.1) for \( A_{\geq 1} \) such that one has

\begin{align}
\mathbf{e}_i \mathbf{f}_j &= \mathbf{g}_1 \quad \text{for all} \quad 1 \leq i \leq r
\end{align}

and all other products of basis elements not fixed by (1.3) via (1.2) are zero.

The ring \( R \) belongs to \( \mathbf{B} \) if there is a basis (1.1) for \( A_{\geq 1} \) such that one has

\begin{align}
(1.4) \quad \mathbf{e}_1 \mathbf{e}_2 &= \mathbf{f}_3 \quad \text{and} \quad \mathbf{e}_1 \mathbf{f}_1 &= \mathbf{g}_1 \\
\mathbf{e}_2 \mathbf{f}_2 &= \mathbf{g}_1
\end{align}

and all other products of basis elements not fixed by (1.4) via (1.2) are zero.

The proofs of both theorems use the fact that the graded algebra \( A \) is isomorphic to the Koszul homology algebra over \( R \). We fix notation for the Koszul complex. Let \( R \) be of embedding dimension 3 and let \( m = (x, y, z) \) be its maximal ideal. We denote by \( K^R \) the Koszul complex over the canonical homomorphism \( \pi : R^3 \rightarrow m \).

It is the exterior algebra of the rank 3 free \( R \)-module with basis \( \varepsilon_x, \varepsilon_y, \varepsilon_z \), endowed with the differential induced by \( \pi \). For brevity we set

\[ \varepsilon_{xy} = \varepsilon_x \wedge \varepsilon_y, \quad \varepsilon_{xz} = \varepsilon_x \wedge \varepsilon_z, \quad \varepsilon_{yz} = \varepsilon_y \wedge \varepsilon_z \quad \text{and} \quad \varepsilon_{xyz} = \varepsilon_x \wedge \varepsilon_y \wedge \varepsilon_z . \]

The differential is then given by,

\begin{align}
\partial(\varepsilon_x) &= x & \partial(\varepsilon_{xy}) &= x\varepsilon_y - y\varepsilon_x \\
\partial(\varepsilon_y) &= y & \partial(\varepsilon_{xz}) &= x\varepsilon_z - z\varepsilon_x & \partial(\varepsilon_{xyz}) &= x\varepsilon_{yz} - y\varepsilon_{xz} + z\varepsilon_{xy} \\
\partial(\varepsilon_z) &= z & \partial(\varepsilon_{yz}) &= y\varepsilon_z - z\varepsilon_y
\end{align}

and it makes \( K^R \) into a graded-commutative differential graded algebra. The induced algebra structure on \( H(K^R) \) is graded-commutative, and there is an isomorphism of graded algebras \( A \cong H(K^R) \); see [1, (1.7.1)].

Note that the homology module \( H_3(K^R) \) is isomorphic, as a \( k \)-vector space, to the socle of \( R \), that is, the ideal \((0 : m)\). To be precise, if \( s_1, \ldots, s_n \) is a basis for the socle of \( R \), then the homology classes of the cycles \( s_1 \varepsilon_{xy}, \ldots, s_n \varepsilon_{xyz} \) in \( K^R_3 \) form a basis for \( H_3(K^R) \). From the isomorphism \( A \cong H(K^R) \) one gets, in particular, \( \sum_{i=0}^3 (-1)^i \text{rank}_k H_i(K^R) = 0 \), as the ranks of the Koszul homology modules equal the ranks of the free modules in \( F \). To sum up one has,

\begin{align}
\text{rank}_k H_1(K^R) &= l + 1 = \mu(I) \\
\text{rank}_k H_2(K^R) &= l + \text{rank}_k H_3(K^R) \\
\text{rank}_k H_3(K^R) &= \text{rank}_k(0 : m) .
\end{align}

Theorem I is proved in Sections 2–4 and Theorem II in Sections 5–7. For each quotient algebra \( Q / \mathfrak{q}_n \) and \( Q / \mathfrak{b}_n \) we shall verify that the Koszul homology algebra has the desired multiplicative structure as described in (1.3) and (1.4), and we shall determine the type of the quotient algebra. As described above, the latter means determining the socle rank, as each of these algebras has depth 0.

* One does not define \( \mathbf{G}(1) \) because it would overlap with another class called \( \mathbf{H}(0, 1) \).
2. Proof that $Q/\mathfrak{g}_1$ is a type 2 algebra in $G(2)$

The ideal $\mathfrak{g}_1 = (x^2y, xyz, yz^2, x^4 - y^3z, xz^3 - y^4)$ in $Q = k[x, y, z]$ is generated by homogeneous elements, so $R = Q/\mathfrak{g}_1$ is a graded $k$-algebra. For $n \geq 0$ denote by $R_n$ the subspace of $R$ of homogeneous polynomials of degree $n$. It is simple to verify that the elements in the second column below form bases for the subspaces $R_n$; for convenience, the third column lists the relations among non-zero monomials.

\[
\begin{array}{ccc}
R_0 & 1 \\
R_1 & x, y, z \\
R_2 & x^2, xy, xz, y^2, yz, z^2 \\
R_3 & x^3, x^2y, x^2z, xz^2, y^3, y^2z, z^3 \\
R_4 & x^4, x^3y, x^3z, x^2z^2, xz^3, z^4 \\
R_5 & x^4y, x^3z^2, z^5 \\
R_{n \geq 6} & z^n \\
\end{array}
\]

(2.1)

Set $A = H(K^R)$; we shall verify that $A$ has the multiplicative structure described in (1.3) with $r = 2$, and that $R$ has socle rank 2.

A basis for $A_3$. From (2.1) it is straightforward to verify that the socle of $R$ is generated by $x^4y$ and $x^3z^2$, so it has rank 2 and the homology classes of the cycles

\[
g_1 = x^4y \varepsilon_{xyz} \quad \text{and} \quad g_2 = x^3z^2 \varepsilon_{xyz}
\]

form a basis for $A_3$. As there are no non-zero boundaries in $K^R_3$, the homology classes $g_1$ and $g_2$ contain only $g_1$ and $g_2$, respectively, and the bar merely signals that we consider the cycles as elements in $A$ rather than $K^R$.

As the ideal $\mathfrak{g}_1$ is minimally generated by 5 elements, one has rank$_k A_1 = 5$, and hence rank$_k A_2 = 6$ by (1.5).

A basis for $A_2$. It is elementary to verify that the next elements in $K^R_2$ are cycles.

\[
\begin{align*}
f_1 &= -yz \varepsilon_{xy} + y^2 \varepsilon_{xz} \\
f_2 &= yz \varepsilon_{xz} \\
f_3 &= x^4 \varepsilon_{xy} \\
f_4 &= xz^3 \varepsilon_{xy} \\
f_5 &= (x^3y - z^4) \varepsilon_{xy} \\
f_6 &= x^3z^2 \varepsilon_{xz}
\end{align*}
\]

(2.3)

To see that their homology classes form a basis for $A_2$ it suffices, as $A_2$ is a $k$-vector space of rank 6, to verify that they are linearly independent modulo boundaries. A boundary in $K^R_2$ has the form

\[
\partial(a \varepsilon_{xyz}) = ax \varepsilon_{yz} - ay \varepsilon_{xz} + az \varepsilon_{xy},
\]

for some $a$ in $R$. As the differential is graded, one needs to verify that $f_1$ and $f_2$ are linearly independent modulo boundaries, that $f_3$, $f_4$, and $f_5$ are linearly independent modulo boundaries, and that $f_6$ is not a boundary.

If $u$ and $v$ are elements in $k$ such that $uf_1 + vf_2 = -uyz \varepsilon_{xy} + (uy^2 + vyz) \varepsilon_{xz}$ is a boundary, then it has the form (2.4) for some $a$ in $R_1$. In particular, one has $ax = 0$, and that forces $a = 0$; see (2.1). With this one has $uyz = 0$ and $uy^2 + vyz = 0$, whence $u = 0 = v$. Thus, $f_1$ and $f_2$ are linearly independent modulo boundaries.
If \( u, v, \) and \( w \) in \( k \) are such that \( uf_3 + vf_4 + wf_5 = (wx^3y + ux^3z + vxz^3 - wz^3)e_{xy} \) is a boundary, then it has the form (2.4) for some element
\[
a = a_1x^3 + a_2x^2y + a_3x^2z + a_4xz^2 + a_5y^2 + a_6y^2z + a_7z^3
\]
in \( R_3 \). From \( az = wx^3y + ux^3z + vxz^3 - wz^3 \) one gets \( w = 0, u = a_1, \) and \( v = a_4; \) see (2.1). From \( ax = 0 \) one gets \( a_1 = 0 = a_4 \), that is, \( u = 0 = v \).

Finally, \( f_6 \) is not a boundary as no element \( a \) in \( R_4 \) satisfies \(-ay = x^3z^2\).

**A basis for** \( A_1 \). The following elements are cycles in \( K_1^R \).
\[
e_1 = x^3\varepsilon_x - y^2z\varepsilon_y
\]
\[
e_2 = z^3\varepsilon_x - y^3\varepsilon_y
\]
\[
e_3 = yz\varepsilon_x
\]
\[
e_4 = z^2\varepsilon_y
\]
\[
e_5 = y^2\varepsilon_x
\]
(2.5)

The vector space \( A_1 \) has rank 5, so as above the task is to show that \( e_1, \ldots, e_5 \) are linearly independent modulo boundaries. A boundary in \( K_1^R \) has the form
\[
\partial(az\varepsilon_x + b\varepsilon_y + c\varepsilon_{yz}) = -(ay + bz)\varepsilon_x + (ax - cz)\varepsilon_y + (bx + cy)\varepsilon_z,
\]
for \( a, b, \) and \( c \) in \( R \). As above, the fact that the differential is graded allows us to treat cycles with coefficients in \( R_2 \) and \( R_3 \) independently.

If \( u \) and \( v \) are elements in \( k \) such that \( u e_1 + v e_2 = (ux^3 + vz^3)e_x - (uy^2z + vy^3)e_y \) is a boundary in \( K_1^R \), then it has the form (2.6) for elements
\[
a = a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2,
\]
\[
b = b_1x^2 + b_2xy + b_3xz + b_4y^2 + b_5yz + b_6z^2, \quad \text{and}
\]
\[
c = c_1x^2 + c_2xy + c_3xz + c_4y^2 + c_5yz + c_6z^2
\]
in \( R_2 \). The equality \( cz - ax = uy^2z + vy^3 \) yields \( v = 0 \) and \( u = c_4 \), and from \( bx + cy = 0 \) one gets \( c_6 = 0 \).

If \( u, v, \) and \( w \) are elements in \( k \) such that \( u e_3 + v e_4 + w e_5 = (uyz + wy^2)\varepsilon_x + vz^2\varepsilon_y \) is a boundary, then it has the form (2.6) for \( a = a_1x + a_2y + a_3z, b = b_1x + b_2y + b_3z, \) and \( c = c_1x + c_2y + c_3z \) in \( R_1 \). This yields equations:
\[
-a_1xy - a_2y^2 - (a_3 + b_2)yz - b_1xz - b_3z^2 = uyz + wy^2,
\]
\[
a_1x^2 + a_2xy + (a_3 - c_1)xz - c_2y^2 - c_3z^2 = vz^2, \quad \text{and}
\]
\[
b_1x^2 + (b_2 + c_1)xy + b_3xz + c_2y^2 + c_3yz = 0.
\]

From the last equation one gets, in particular, \( c_3 = 0 \) and \( b_2 + c_1 = 0 \). The second one now yields \( v = 0, a_2 = 0, \) and \( a_3 = c_1 \). With these equalities, the first equation yields \( w = 0 \) and \( u = 0 \).

**The product** \( A_1 \cdot A_1 \). To determine the multiplication table \( A_1 \times A_1 \) it is by (1.2) sufficient to compute the products \( e_i e_j \) for \( 1 \leq i < j \leq 5 \). The product \( e_3 e_5 \) is zero by graded-commutativity of \( K^R \), and the following products are zero because the
coefficients vanish in $R$; cf. $(2.1)$.

$$
e_1e_2 = (-x^3y^3 + y^2z^4)x_{xy} \quad e_3e_4 = y^3e_{xy}$$
$$e_1e_3 = y^3z^2e_{xy} \quad e_4e_5 = -y^2z^2e_{xy}$$
$$e_2e_5 = y^5e_{xy}$$

Finally, the computations

$$e_1e_4 = x^3z^2e_{xy} = \partial(x^3z^3_3)$$
$$e_1e_5 = y^4ze_{xy} = xz^4e_{xy} = \partial(xz^3_3e_{xyz})$$
$$e_2e_3 = y^3ze_{xy} = xz^4e_{xy} = \partial(xz^3_3e_{xyz})$$
$$e_2e_4 = z^5e_{xy} = \partial((z^4 - x^3y)e_{xyz})$$

show that also the products $e_1e_4, e_1e_5, e_2e_3,$ and $e_2e_4$ in homology are zero. Thus, one has $A_1 \cdot A_1 = 0$.

**The product** $A_1 \cdot A_2$. Among the products $e_if_j$ for $1 \leq i \leq 5$ and $1 \leq j \leq 6$ several are zero by graded-commutativity of $K^R$:

$$e_3f_j \text{ and } e_5f_j \text{ for } 1 \leq j \leq 6$$
$$e_1f_3, e_1f_4, e_1f_5, e_2f_3, e_2f_4, e_2f_5, e_4f_3, e_4f_4, \text{ and } e_4f_5.$$ 

The following products are zero because the coefficients vanish in $R$.

$$e_1f_2 = y^3z^2e_{xyz} \quad e_4f_1 = -y^2z^2e_{xyz}$$
$$e_1f_6 = x^3y^2z^3e_{xyz} \quad e_4f_2 = -yz^3e_{xyz}$$
$$e_2f_1 = y^5e_{xyz} \quad e_4f_6 = -x^3z^4e_{xyz}$$
$$e_2f_5 = x^3y^3z^2e_{xyz}$$

This leaves two products to compute, namely $e_1f_1 = y^4ze_{xyz} = e_2f_2$.

The computations above show that in terms of the $k$-basis $ar{e}_1, \ldots, \bar{e}_5, \bar{f}_1, \ldots, \bar{f}_6$, $\bar{g}_1, \bar{g}_2$ for $A_{\geq 1}$ the non-zero products of basis vectors are

$$\bar{e}_1\bar{f}_1 = \bar{e}_2\bar{f}_2 = \bar{g}_1,$$  

whence $R$ belongs to $G(2)$.

### 3. Proof that $Q/\mathfrak{g}_2$ is a type 2 algebra in $G(3)$

Set $R = Q/\mathfrak{g}_2$; as one has $\mathfrak{g}_2 = g_1 + (x^3y - z^4)$ it follows from $(2.1)$ that the elements listed below form bases for the subspaces $R_i$. As in $(2.1)$ the third column records the relations among non-zero monomials.

| $R_0$ | $1$ |
| $R_1$ | $x, y, z$ |
| $R_2$ | $x^2, xy, xz, y^2, yz, z^2$ |
| $R_3$ | $x^3, x^2y, x^2z, y^3, y^2z, z^3$ |
| $R_4$ | $x^4, x^3y, x^3z, x^2z^2, xz^3$ |
| $R_5$ | $x^4y, x^3z^2, xz^4 = y^4z = x^4y$ |

Set $A = H(K^R)$; we shall verify that $A$ has the multiplicative structure described in $(1.3)$ with $r = 3$, and that $R$ has socle rank 2.
The next remark will also be used in later sections; loosely speaking, it allows us to recycle the computations from Section 2 in the analysis of $A$.

(3.2) **Remark.** Let $a \subseteq b$ be ideals in $Q$. The canonical epimorphism $Q/a \to Q/b$ yields a morphism of complexes $K^{Q/a} \to K^{Q/b}$. It maps cycles to cycles and boundaries to boundaries. To be explicit, let $E = \{1, \varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{xyz}\}$ be the standard basis for either Koszul complex; if $\sum_{e \in E}(q_e + a)e$ is a cycle (boundary) in $K^{Q/a}$, then $\sum_{e \in E}(q_e + b)e$ is a cycle (boundary) in $K^{Q/b}$. By habitual abuse of notation, we write $x, y,$ and $z$ for the cosets of $x, y,$ and $z$ in any quotient algebra of $Q$, and as such we make no notational distinction between an element in $K^{Q/a}$ and its image in $K^{Q/b}$.

**A basis for $A_{x=1}$**. From (3.1) it is straightforward to verify that the socle of $R$ is $R_5$, so it has rank 2, and the homology classes of the cycles $g_1$ and $g_2$ from (2.2) form a basis for $A_3$. The ideal $g_2$ is minimally generated by 6 elements, so one has $R \subseteq 6$, and rank $A_2 = 7$; see (1.5). Proceeding as in Section 2 it is straightforward to verify that the homology classes $e_1, \ldots, e_5$ from (2.5) together with the homology class of the cycle

$$e_6 = x^2y\varepsilon_x - z^3\varepsilon_z$$

form a basis for $A_1$. Similarly, one verifies that the homology classes of $f_1, f_2, f_4,$ and $f_6$ from (2.3) together with those of the cycles

$$f_3 = x^3\varepsilon_{xy} - y^3\varepsilon_{xz} + y^3\varepsilon_{yz},$$

$$f_5 = z^4\varepsilon_{xz} - y^3 z^3 \varepsilon_{yz},$$

and

$$f_7 = x^2 y \varepsilon_{xy} + z^3 \varepsilon_{yz}$$

make up a basis for $A_2$.

**The product $A_1 \cdot A_1$**. It follows from the computations in the previous section that one has $e_i e_j = 0$ for $1 \leq i, j \leq 5$. To complete the multiplication table $A_1 \times A_1$ it is by (1.2) sufficient to compute the products $e_i e_6$ for $1 \leq i \leq 5$. The products $e_1 e_6$ and $e_2 e_6$ are zero as one has $R_3 \cdot R_3 = 0$. The remaining products involving $e_6$,

$$e_3 e_6 = -y z^4 \varepsilon_{xz},$$

$$e_4 e_6 = -x^2 y z^2 \varepsilon_{xy} - z^5 \varepsilon_{yz},$$

and

$$e_5 e_6 = -y^2 z^3 \varepsilon_{xz}$$

are zero because the coefficients vanish in $R$. Thus, one has $A_1 \cdot A_1 = 0$.

**The product $A_1 \cdot A_2$**. It follows from the computations in the previous section that the only non-zero products $e_i f_j$ for $1 \leq i \leq 5$ and $j \in \{1, 2, 4, 6\}$ are the ones listed in (2.7). The products $e_i f_5$ for $1 \leq i \leq 6$ are zero as one has $R_{\geq 2} \cdot R_4 = 0$; similarly, the products

$$e_1 f_3, e_1 f_7, e_2 f_3, e_2 f_7, e_5 f_3, e_6 f_4,$$

and $e_6 f_7$

are zero as one has $R_3 \cdot R_{\geq 3} = 0$. Among the remaining products $e_4 f_7$ and $e_6 f_2$ are zero by graded-commutativity, and the following are zero because the coefficients
vanish in $R$.

$$
\begin{align*}
e_3f_7 &= y^2z^3e_{xyz} \\
e_5f_7 &= y^2z^3e_{xyz} \\
e_4f_3 &= y^2e_{xyz} \\
e_5f_3 &= y^5e_{xyz}
\end{align*}
$$

The one remaining product is $e_3f_3 = y^4e_{xyz} = g_1$.

In terms of the basis $\bar{e}_1, \ldots, \bar{e}_6, \bar{f}_1, \ldots, \bar{f}_7, \bar{g}_1, \bar{g}_2$ for $A_{\geq 1}$ the only non-zero products of basis vectors are

$$
\bar{e}_i\bar{f}_i = \bar{g}_1 \quad \text{for } 1 \leq i \leq 3,
$$

whence $R$ belongs to $G(3)$.

4. Proof that $Q/g_3$ and $Q/g_4$ are type 2 algebras in $G(4)$ and $G(5)$

The arguments that show that $Q/g_3$ and $Q/g_4$ are $G$ algebras follow the argument in Section 3 closely; we summarize them below.

(4.1) The quotient by $g_3$. Set $R = Q/g_3$; as one has $g_3 = g_2 + (x^2z^2)$ it follows from (3.1) that the elements listed in the second column below form bases for the subspaces $R_n$.

| $R_0$ | 1 |
| $R_1$ | $x, y, z$ |
| $R_2$ | $x^2, xy, xz, y^2, yz, z^2$ |
| $R_3$ | $x^3, x^2y, x^2z, xz^2, y^3, y^2z, z^3$ |
| $R_4$ | $x^4, x^3y, x^3z, xz^3, y^4, y^3z = x^4, z^4 = x^3y$ |
| $R_5$ | $x^4y$ |

It is straightforward to verify that the socle of $R$ is generated by the elements $x^4y$ and $x^3z$, so it has rank 2. Set $A = H(K^R)$; the homology classes of cycles

$$
\quad g_1 = x^4y e_{xyz} \quad \text{and} \quad g_2 = x^3z e_{xyz}
$$

form a basis for $A_3$. One readily verifies that the homology classes of $e_1, \ldots, e_6$ and $f_1, f_2, f_3, f_5, f_7$ from Section 3, see also (3.2), together with those of the cycles

$$
\begin{align*}
e_7 &= x^2e_x, & f_4 &= y^3e_{xy} - xz^2e_{xz}, & f_6 &= xz^2e_{xy}, & f_8 &= x^3z e_{xz}
\end{align*}
$$

form bases for $A_1$ and $A_2$.

The products $e_1e_7$, $e_2e_7$, and $e_6e_7$ vanish as one has $R_3 \cdot R_3 = 0$, while $e_3e_7$ and $e_5e_7$ are zero by graded-commutativity. Finally, one has $e_4e_7 = -x^4e_{xy} = -\partial(xz^3e_{xyz})$, so also the product $\bar{e}_4\bar{e}_7$ is zero. Together with the computations from the previous section this shows that $A_1 \cdot A_1$ is zero.

The products $e_7f_j$ vanish for $3 \leq j \leq 8$ as one has $R_3 \cdot R_{\geq 3} = 0$, and for the same reason any one of the elements $e_1, e_2$, and $e_6$ multiplied by either $f_4$ or $f_6$ is zero. All products $e_if_8$ vanish as one has $R_{\geq 2} \cdot R_4 = 0$. Among the remaining products involving $e_7$, $f_4$, or $f_6$ all but one are zero by graded-commutativity, the non-zero product is $e_4f_4 = xz^4e_{xyz} = g_1$. It follows that in terms of the $k$-basis $\bar{e}_1, \ldots, \bar{e}_7$, $\bar{f}_1, \ldots, \bar{f}_8$, $\bar{g}_1, \bar{g}_2$ for $A_{\geq 1}$ the only non-zero products of basis-elements are

$$
\bar{e}_i\bar{f}_i = \bar{g}_1 \quad \text{for } 1 \leq i \leq 4,
$$

whence $R$ belongs to $G(4)$. 
(4.2) **The quotient by \( g_4 \).** Set \( R = Q/g_4 \); as one has \( g_4 = g_3 + (x^3z) \) it follows from (4.1) that the elements listed in the second column below form bases for the subspaces \( R_n \).

\[
\begin{array}{c|c}
R_0 & 1 \\
R_1 & x, y, z \\
R_2 & x^2, xy, xz, y^2, yz, z^2 \\
R_3 & x^2z, x^2y, x^2y^2, x^2y^2z, x^2y^3, y^2z \\
R_4 & x^4, x^3y, x^3z, y^4 = xz^3, y^3z = x^4, z^4 = x^3y \\
R_5 & x^4y, x^3y, x^2z, y^4z = x^4y \\
\end{array}
\]

It is straightforward to check that the socle of \( R \) is generated by the elements \( x^4y \) and \( x^2z \), so it has rank 2. Set \( A = \text{H}(K^R) \); the homology classes of the cycles

\[
g_1 = x^4y \varepsilon_{xyz} \quad \text{and} \quad g_2 = x^2z \varepsilon_{xyz}
\]

form a basis for \( A_3 \). One readily verifies that the homology classes of \( e_1, \ldots, e_7 \) and \( f_1, f_2, f_3, f_4, f_6, f_7 \) from (4.1), see also (3.2), together with those of

\[
e_8 = x^2 \varepsilon_x, \quad f_5 = -x^3 \varepsilon_{xz} + y^2 \varepsilon_{yz}, \quad f_8 = x^2 \varepsilon_{xy}, \quad \text{and} \quad f_9 = x^2 \varepsilon_{xz}
\]

form bases for \( A_1 \) and \( A_2 \).

All the products \( e_i e_8 \) are zero as \( x^2z \) is in the socle of \( R \). Together with the computations from (4.1) this shows that \( A_1 \cdot A_1 \) is zero.

All products involving \( f_3, e_i f_8, \) and \( e_i f_9 \) vanish as \( x^2z \) is in the socle of \( R \). Any one of the elements \( e_1, e_5, e_6, \) and \( e_7 \) multiplied by \( f_5 \) is zero for as one has \( R_3 \cdot R_3 = 0 \). The remaining products involving \( f_5 \) are

\[
e_3 f_5 = y^3 z^2 \varepsilon_{xyz} = 0, \quad e_4 f_5 = x^3 z^2 \varepsilon_{xyz} = 0, \quad \text{and} \quad e_5 f_5 = y^4 \varepsilon_{xyz} = g_1 .
\]

It follows that in terms of the \( k \)-basis \( \bar{e}_1, \ldots, \bar{e}_8, \bar{f}_1, \ldots, \bar{f}_9, \bar{g}_1, \bar{g}_2 \) for \( A_1 \) the only non-zero products of basis vectors are

\[
\bar{e}_i \bar{f}_j = \bar{g}_1 \quad \text{for } 1 \leq i \leq 5,
\]

whence \( R \) belongs to \( B(5) \).

5. **Proof that \( Q/b_1 \) is a type 1 algebra in \( B \)**

The ideal \( b_1 = (x^3, x^2y, y^2z, z^3) \) in \( Q = k[x, y, z] \) is generated by homogeneous elements, so \( R = Q/b_1 \) is a graded \( k \)-algebra. It is straightforward to verify that the elements listed below form bases for the subspaces \( R_n \).

\[
\begin{array}{c|c}
R_0 & 1 \\
R_1 & x, y, z \\
R_2 & x^2, xy, xz, y^2, yz, z^2 \\
R_3 & x^2z, x^2y, x^2yz, x^2y^2z, x^2y^3z, y^2z \\
R_4 & x^4z, x^3y^2z, y^4, y^3z \\
R_{n\geq5} & xy^{n-1}, xy^{n-2}z, y^n, y^{n-1}z \\
\end{array}
\]

Set \( A = \text{H}(K^R) \); we shall verify that \( A \) has the multiplicative structure described in (1.4) and that \( R \) has socle rank 1.

**A basis for \( A_{\geq1} \).** From (5.1) it is straightforward to verify that the socle of \( R \) is generated by \( x^2z^2 \), so it has rank 1 and the homology class of the cycle

\[
g_1 = x^2z^2 \varepsilon_{xyz}
\]
is a basis for $A_3$. The ideal $b_1$ is minimally generated by 4 elements, whence one has \( \text{rank}_k A_1 = 4 = \text{rank}_k A_2 \); see (1.5). Proceeding as in Section 2 it is straightforward to verify that the elements \( e_1, \ldots, e_4 \) and \( f_1, \ldots, f_4 \) listed below are cycles in $K_1^R$ and $K_2^R$ whose homology classes form bases for the $k$-vector spaces $A_1$ and $A_2$.

\[
\begin{align*}
  e_1 &= x^2 e_x & f_1 &= z^2 e_y \\
  e_2 &= z^2 e_z & f_2 &= x^2 e_{xy} \\
  e_3 &= x y e_x & f_3 &= x^2 z^2 e_{xz} \\
  e_4 &= z^2 e_y & f_4 &= x y z e_{xz}
\end{align*}
\]

(5.3)

**The product** $A_1 \cdot A_1$. To determine the multiplication table $A_1 \times A_1$ it is sufficient to compute the products $e_i e_j$ for $1 \leq i < j \leq 4$; see (1.2). The product $e_1 e_3$ is zero by graded-commutativity of $K^R$, and the following products are zero because the coefficients vanish in $R$.

\[
\begin{align*}
  e_2 e_3 &= -x y z^2 e_{xz} \\
  e_2 e_4 &= -z^4 e_{yz} \\
  e_3 e_4 &= x y z^2 e_{xy}
\end{align*}
\]

The remaining products are

\[
\begin{align*}
  e_1 e_2 &= x^2 z^2 e_{xz} = f_3 \\
  e_1 e_4 &= x^2 z^2 e_{xy} = \partial (x^2 z e_{xyz})
\end{align*}
\]

It follows that the product $\bar{e}_1 \bar{e}_2$ in homology is zero, leaving us with a single non-zero product of basis vectors in $A_1$, namely $\bar{e}_1 \bar{e}_2 = \bar{f}_3$.

**The product** $A_1 \cdot A_2$. One has

\[
\begin{align*}
  e_1 f_1 &= x^2 z^2 e_{xyz} = g_1 \\
  e_2 f_2 &= x^2 z^2 e_{xyz} = g_1 \\
  e_3 f_1 &= x y z^2 e_{xyz} = 0 \\
  e_4 f_3 &= -x^2 z^4 e_{xyz} = 0 \\
  e_4 f_4 &= -x y z^3 e_{xyz} = 0
\end{align*}
\]

(5.4)

The remaining products are zero by graded-commutativity.

In terms of the $k$-basis $\bar{e}_1, \ldots, \bar{e}_4, \bar{f}_1, \ldots, \bar{f}_4, \bar{g}_1$ for $A \geq 1$ the only non-zero products of basis vectors are

\[
\begin{align*}
  \bar{e}_1 \bar{e}_2 &= \bar{f}_3 \\
  \bar{e}_1 \bar{f}_1 &= \bar{g}_1 \\
  \bar{e}_2 \bar{f}_2 &= \bar{g}_1
\end{align*}
\]

It follows that $R$ belongs to the class $B$.

6. **Proof that** $Q/b_2$ **is a type 1 algebra in** $B$

Set $R = Q/b_2$; as one has $b_2 = b_1 + (x y z)$ it follows from (5.1) that the elements listed below form bases for the subspaces $R_n$.

\[
\begin{align*}
  R_0 &= 1 \\
  R_1 &= x, y, z \\
  R_2 &= x^2, xy, xz, y^2, yz, z^2 \\
  R_3 &= x^2 z, x y^2, x z^2, y^3, y^2 z \\
  R_4 &= x^2 z, x y^3, y^4, y^3 z \\
  R_{n \geq 5} &= x y^{n-1}, y^n, y^{n-1} z
\end{align*}
\]

(6.1)
Set $A = H(K^R)$; we shall verify that $A$ has the multiplicative structure described in (1.4) and that $R$ has socle rank 1.

**A basis for** $A_{\geq 1}$. From (6.1) it is straightforward to verify that the socle of $R$ is generated by $x^2z^2$, so it has rank 1, and the homology class of the cycle $g_1$ from (5.2) is a basis for $A_3$; cf. (3.2). The ideal $b_2$ is minimally generated by 5 elements, so one has rank$_k A_1 = 5 = $ rank$_k A_2$; see (1.5). Proceeding as in Section 2 it is straightforward to verify that the homology classes of $e_1, \ldots, e_4$ from (5.3) together with the class of the cycle

$$e_5 = yz\varepsilon_x$$

form a basis for $A_1$. Similarly, one verifies that the homology classes of $f_1, f_2, f_3$ from (5.3) together with those of the cycles

$$f_4 = xy\varepsilon_{xz} \text{ and } f_5 = yz\varepsilon_{xz}$$

make up a basis for $A_2$.

**The product** $A_1 \cdot A_1$. It follows from (5.4) that $\bar{e}_1\bar{e}_2 = \bar{f}_3$ is the only non-zero product $\bar{e}_i\bar{e}_j$ for $1 \leq i < j \leq 4$. To complete the multiplication table $A_1 \times A_1$ it is by (1.2) sufficient to compute the products $e_i e_5$ for $1 \leq i \leq 4$. The products $e_1 e_5$ and $e_3 e_5$ are zero by graded-commutativity of $K^R$, and the remaining products involving $e_5$,

$$e_2 e_5 = -yz^3\varepsilon_{xz} \text{ and } e_4 e_5 = -yz^3\varepsilon_{xy},$$

are zero because the coefficients vanish in $R$.

**The product** $A_1 \cdot A_2$. It follows from (5.4) that the only non-zero products $\bar{e}_i\bar{f}_j$ for $1 \leq i \leq 4$ and $1 \leq j \leq 3$ are $\bar{e}_1\bar{f}_1 = \bar{e}_2\bar{f}_2 = \bar{g}_1$. To complete the multiplication table $A_1 \times A_2$ one has to compute the products $e_5 f_j$ for $1 \leq j \leq 5$ and $e_i f_4$ and $e_i f_5$ for $1 \leq i \leq 5$. The next products are zero because the coefficients vanish in $R$,

$$e_4 f_4 = -xyz^2\varepsilon_{xyz} \text{ and } e_4 f_5 = -yz^3\varepsilon_{xyz},$$

and the remaining are zero by graded-commutativity.

In terms of the $k$-basis $\bar{e}_1, \ldots, \bar{e}_5, \bar{f}_1, \ldots, \bar{f}_5, \bar{g}_1$ for $A_{\geq 1}$ the only non-zero products of basis vectors are the ones listed in (5.4), so $R$ belongs to $B$.

**7. Proof that** $Q/b_3$ **and** $Q/b_4$ **are type 3 algebras in** $B$

The arguments that show that $Q/b_3$ and $Q/b_4$ are $B$ algebras follow the argument in Section 6 closely; we summarize them below.

(7.1) **The quotient by** $b_3$. Set $R = Q/b_3$; as one has $b_3 = b_2 + (xy^2 - y^3)$ it follows from (6.1) that the elements listed below form bases for the subspaces $R_n$.

\[
\begin{align*}
R_0 & \quad 1 \\
R_1 & \quad x, y, z \\
R_2 & \quad x^2, xy, xz, y^2, yz, z^2 \\
R_3 & \quad x^2 z, xy^2, xz^2, y^2 z, y^3 = xy^2 \\
R_4 & \quad x^2 z^2
\end{align*}
\]
It is straightforward to verify that the socle of \( R \) is generated by the elements \( x^2z^2 \), \( xy^2 \), and \( y^2z \), so it has rank 3. Set \( A = \text{H}(K^R) \); the homology classes of the cycles
\[
g_1 = x^2z^2\varepsilon_{xyz}, \quad g_2 = xy^2\varepsilon_{xyz}, \quad \text{and} \quad g_3 = y^2z\varepsilon_{xyz}
\]
form a basis for \( A_3 \). One readily verifies that the homology classes of \( e_1, \ldots, e_6 \) and \( f_1, \ldots, f_5 \) from the previous section, see also (3.2), together with those of the cycles
\[
e_6 = y^2\varepsilon_x - y^2\varepsilon_y, \quad f_6 = xy^2\varepsilon_{xy}, \quad f_7 = y^2z\varepsilon_{xy}, \quad \text{and} \quad f_8 = y^2z\varepsilon_{yz}
\]
form bases for \( A_1 \) and \( A_2 \).

The products \( e_ie_6, e_if_6, e_if_7, \) and \( e_if_8 \) for \( 1 \leq i \leq 5 \) as well as the products \( e_6f_j \) for \( 1 \leq j \leq 8 \) vanish as one has \( y^2R_{\geq 2} = 0 \). It follows that in terms of the \( k \)-basis \( e_1, \ldots, e_6, f_1, \ldots, f_8, g_1, g_2, g_3 \) for \( A_{\geq 1} \) the only non-zero products of basis vectors are the ones listed in (5.4), so \( R \) belongs to \( B \).

(7.2) The quotient by \( b_4 \). Set \( R = Q/b_4 \); as one has \( b_4 = b_4 + (y^2z) \) it follows from (7.1) that the elements listed below form bases for the subspaces \( R_n \).

\[
\begin{array}{l|l}
R_0 & 1 \\
R_1 & x, y, z \\
R_2 & x^2, xy, xz, y^2, yz, z^2 \\
R_3 & x^2z, xy^2, xz^2 \\
R_4 & x^2z^2 \\
\end{array}
\]

It is straightforward to check that the socle of \( R \) is generated by the elements \( x^2z^2 \), \( xy^2 \), and \( y^2z \), so it has rank 3. Set \( A = \text{H}(K^R) \); the homology classes of the cycles
\[
g_1 = x^2z^2\varepsilon_{xyz}, \quad g_2 = xy^2\varepsilon_{xyz}, \quad \text{and} \quad g_3 = y^2z\varepsilon_{xyz}
\]
form a basis for \( A_3 \). One readily verifies that the homology classes of \( e_1, \ldots, e_6 \) and \( f_1, \ldots, f_5 \) from (7.1) together with those of
\[
e_7 = y^2\varepsilon_x \quad f_7 = y^2z\varepsilon_{xy} \quad f_8 = y^2z\varepsilon_{yz} \quad \text{and} \quad f_9 = y^2z\varepsilon_{xz} - y^2z\varepsilon_{yz}
\]
form bases for \( A_1 \) and \( A_2 \).

The products \( e_ie_7 \) for \( 1 \leq i \leq 6 \) and \( e_7f_j \) for \( 1 \leq j \leq 9 \) vanish as \( yz \) is in the socle of \( R \), and for \( 1 \leq i \leq 6 \) the products \( e_if_7 \) and \( e_if_8 \) vanish for the same reason. Finally, all products \( e_if_9 \) vanish as one has \( y^2R_{\geq 2} = 0 \). Thus, in terms of the \( k \)-basis \( e_1, \ldots, e_7, f_1, \ldots, f_9, g_1, g_2, g_3 \) for \( A_{\geq 1} \) the only non-zero products of basis vectors are the ones listed in (5.4), so \( R \) belongs to \( B \).

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REFERENCES

1. Luchezar L. Avramov, A cohomological study of local rings of embedding codepth 3, J. Pure Appl. Algebra 216 (2012), no. 11, 2489–2506. MR2927181
2. Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, Poincaré series of modules over local rings of small embedding codepth or small linking number, J. Algebra 118 (1988), no. 1, 162–204. MR9961334
3. Anne E. Brown, A structure theorem for a class of grade three perfect ideals, J. Algebra 105 (1987), no. 2, 308–327. MR873006
4. David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. 99 (1977), no. 3, 447–485. MR0453723
5. Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.
6. Junzo Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. 50 (1973), 227–232. MR0319985
7. Jerzy Weyman, *On the structure of free resolutions of length 3*, J. Algebra 126 (1989), no. 1, 1–33. MR1023284

Texas Tech University, Lubbock, TX 79409, U.S.A.
E-mail address: lars.v.christensen@ttu.edu
URL: http://www.math.ttu.edu/~lchriste

Northeastern University, Boston, MA 02115, U.S.A.
E-mail address: o.veliche@neu.edu
URL: http://www.math.neu.edu/~veliche