Notes on N=2 σ-models*

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These lectures, given at the 1992 Trieste Spring School, are devoted to some selected topics in N=2 σ-models on Calabi-Yau manifolds and the associated N=2 superconformal field theories. The first lecture is devoted to the “special geometry” of the moduli space of $c = 9$ N=2 superconformal field theories. An important role is played by the extended chiral algebra which appears in theories with integer $U(1)$ charges. The second lecture is devoted to the σ-model approach. The main focus is an explication of a calculation of Aspinwall and Morrison.

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These lectures are devoted to some selected topics in N=2 \(\sigma\)-models on Calabi-Yau manifolds and the associated N=2 superconformal field theories. The first lecture is devoted to the “special geometry” of the moduli space of \(c = 9\) N=2 superconformal field theories. An important role is played by the extended chiral algebra which appears in theories with integer \(U(1)\) charges. The second lecture is devoted to the \(\sigma\)-model approach. The main focus is an explication of a calculation of Aspinwall and Morrison.

1. Introduction

The subject of N=2 \(\sigma\)-models and N=2 superconformal field theory is simply too vast to be adequately treated in these brief lecture notes. A truly comprehensive review would be a book in itself. Instead of trying to be comprehensive, I will focus on two topics which I hope will lead the student to a better appreciation of this rich subject. At the same time, I hope my slightly unconventional treatment of these topics will be of some interest to the “expert”. Necessarily, I will leave out much that should be said. Luckily, two of the other lectures in this volume, those by Cecotti and Candelas, will fill in some of the gaps. Also, there are some truly excellent reviews already existing [1, 2, 3].

Supersymmetric \(\sigma\)-models on Calabi-Yau manifolds, aside from any intrinsic mathematical interest, are usually considered as backgrounds for “compactified” string theories. If one wants to get a fermionic string theory with four noncompact dimensions \((c = 6)\), one replaces six of the ten flat dimensions with a \(c = 9\) superconformal field theory. So, for string theory, we are really interested in the conformally invariant theory, \textit{i.e.} the Calabi-Yau \(\sigma\)-model at its infrared fixed point.

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Actually, fixed “point” is a misnomer. Even at the fixed point, there generally exist
some number of exactly marginal couplings, so the fixed point set is actually a finite-
dimensional variety, “the moduli space of the conformal field theory”

Except, perhaps, at some exceptional points in the moduli space, we don’t really know
how to construct the conformal theory directly. There are lots of different approaches one
can take to overcome this problem. They tend to fall into two broad classes. One is to
try to do what one can, exploiting the rich set of symmetries imposed by the superconfor-
mal invariance to prove certain things about the CFT. This turns out to be surprisingly
powerful. The other alternative is to study the nonconformal $\sigma$-model which flows to the
conformal theory in the infrared. Clearly, if we can find renormalization group-invariant
quantities, we can study them in the massive theory (where they are generally easier to
calculate), and thereby learn something about the CFT. The most successful approach is,
of course, to combine it both techniques: use conformal methods to uncover symmetries of
the theory which make it easier to calculate quantities of interest in the noncritical theory.

One of the most striking examples of this is mirror symmetry (the subject of Candelas’
lectures). There, one finds a nontrivial automorphism of the conformal field theory which
allows one to relate quantities which have very different $\sigma$-model interpretations. This
opens up to direct $\sigma$-model calculation quantities which previously would have been very
difficult to calculate. I will leave the details for Candelas to unfold, but I will at least lay
the groundwork for some of his calculations.

A word about supersymmetry: the massive theories we are considering have N=2
supersymmetry. At the conformal point, this symmetry is extended to N=2 supercon-
fomal symmetry. Actually, one has independent left- and right-moving superconformal
symmetries, so one often says that such theories possess (2,2) superconformal symmetry.
In order to couple one of these theories to heterotic string theory, it must possess at least
(1,0) supersymmetry. In order to perform a chiral GSO projection, we need an operator
$(-1)^F_L$ which anticommutes with the left-moving supercurrent, and squares to 1. In a
theory with (2,0) supersymmetry (and integer U(1) charges) there is a natural candidate
for this operator, namely let $F_L = Q = \oint J$, the conserved charge associated to the $U(1)$
current $J$ in the N=2 superconformal algebra. With this definition, the theory naturally
has spacetime supersymmetry because the states of the Ramond sector are related to
those of the NS sector by spectral flow.

We could study models with just (2,0) supersymmetry. These are very interesting
from the point of view of the phenomenology of the resulting compactified string theories.
They also pose some outstanding theoretical challenges. However, unlike the (2,2) models we will study, there is no natural way to consider them as the IR limits of massive “(2,0)” theories. At present, our only tool for studying them is to try to construct them directly at the conformal point. This is hard, and as a consequence, little is known about such theories in general. For the purposes of these notes, I will stick to the more frequently trodden path of (2,2) supersymmetry.

Section 2 of these notes is devoted to studying the moduli space of these (2,2) superconformal field theories. In keeping with my description above of the various “approaches” to this subject, we will hew to the superconformal approach, and study directly the properties of the superconformal theory. I will rely heavily on the calculations of [5], who have done the hard work of calculating the relevant superconformal correlators. Our main focus will be on the intrinsic geometry of the moduli space, but the main result – applicable in string theory – is a determination of the Zamolodchikov metrics for various fields (i.e. the kinetic terms in the effective spacetime field theory).

In section 3, I will do an about-face, and take the approach of trying to find RG-invariant quantities to study in the nonconformal σ-model. For concreteness, we will focus on the cubic couplings which entered into the formulæ of section 2. These are also of “phenomenological” interest, for they determine the superpotential of the effective spacetime theory.

2. N=2 superconformal symmetry and special geometry

One approach to this subject is to concentrate on the properties of the theory that follow from the N=2 superconformal symmetry. For an excellent review of the subject in this spirit, see [1]. Here we will unravel the geometry of the moduli space of (2,2) superconformal theories which follow from (2,2) superconformal invariance.

The N=2 superconformal algebra is

\[
\begin{align*}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T}{z-w} \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} \\
G^+(z)G^-(w) &= \frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w) + \frac{1}{2} \partial_w J}{z-w} \\
T(z)G^\pm(w) &= \frac{3}{2} \frac{G^\pm(w)}{(z-w)^2} + \frac{\partial_w G^\pm}{z-w} \\
J(z)G^\pm(w) &= \pm \frac{G^\pm(w)}{z-w}
\end{align*}
\]
Primary states of the theory can be classified by their conformal weight $\Delta$ and their $U(1)$ charge $Q$. Unitarity of the theory requires

$$\Delta \geq \frac{1}{2}|Q|$$  \hspace{1cm} (2.2)

A primary state $|\phi\rangle$ which violated this inequality would lead to a negative norm state $G_{-1/2}^{\pm}|\phi\rangle$. There are a similar set of inequalities which prevent the occurrence of negative-norm states at higher levels. Together, these inequalities etch out a convex polygon in the $(\Delta, Q)$-plane whose vertices lie on the envelope

$$\Delta = \frac{3}{2c}Q^2$$

Fields which saturate the bound (2.2) are called chiral primaries \[6\]. They have a null state at the first level $G_{-1/2}^{\pm}|\phi\rangle = 0$. Putting together left- and right-movers, we have various possibilities, which we label as follows:

- chiral primary fields have $Q = \Delta/2 \bar{Q} = \bar{\Delta}/2$,
- twisted-chiral primaries have $Q = -\Delta/2 \bar{Q} = \bar{\Delta}/2$.

Adjoints of these fields (“antichiral” and “twisted antichiral”) have opposite charge assignments. Unitarity implies the largest allowed charge for (twisted) chiral field is $|Q_{\text{max}}| = c/3$. We demand that there be a holomorphic field with this charge $\epsilon^{\pm}(z)$ $Q = \pm c/3$, $\Delta = c/6$, $\bar{Q} = \bar{\Delta} = 0$. Clearly, $c$ must be a multiple of 3, so that the $\epsilon^{\pm}(z)$ are mutually local. Also, all of the other $U(1)$ charges in the theory must be integral, so that $\epsilon^{\pm}(z)$ are local with respect to those fields. When embedded in a string theory, a (2,2) superconformal theory of this sort leads to a string theory with spacetime supersymmetry \[1,4\]. The existence of extra holomorphic fields means our theory has an extended chiral algebra. For $c = 6$, this is simply the N=4 superconformal algebra, with $\epsilon^{\pm}, J$ forming the $SU(2)$ Kač-Moody algebra contained in the N=4 algebra. For $c = 9$, the extended algebra one gets \[7\] has no commonly accepted name. It contains two non-commuting N=2 algebras, the usual one with supercurrents $G^{\pm}(z)$, and a second one with supercurrents $\epsilon^{\pm}(z)$, $U(1)$ current $\bar{J} = \frac{1}{3}J$ and stress tensor $\bar{T} = \frac{1}{6} :J^2:$.

We will consider $c = 9$ for definiteness. We can organize \[6,7\] the chiral fields of the model into those which are chiral both with respect to both of the N=2’s in the extended chiral algebra, and those which are descendent with respect to the “exotic” N=2 generated by $\epsilon^{\pm}$. 

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The chiral primary $h$ with $Q = \bar{Q} = 1$ has the property that $|h\rangle$ is annihilated by $\epsilon^\pm_{-1/2}$ and $\bar{\epsilon}^\pm_{-1/2}$, so it is a chiral primary with respect to both $N=2$'s. Its adjoint, $h^\dagger$ has $Q = \bar{Q} = -1$ and is antichiral with respect to both $N=2$'s. The twisted chiral primary $b$ with $Q = -1$, $\bar{Q} = 1$ has $|b\rangle$ annihilated by $\epsilon^-_{-1/2}$ and $\bar{\epsilon}^\pm_{-1/2}$ and so is twisted-chiral with respect to both $N=2$'s. $b^\dagger$ has $Q = 1$, $\bar{Q} = -1$.

The rest of the chiral and twisted-chiral primaries (and their adjoints) are obtained by acting on these with $\epsilon^\pm_{-1/2}$ and $\bar{\epsilon}^\pm_{-1/2}$. For instance, we have the chiral primary $\tilde{h} = (\epsilon^+_{-1/2} \bar{\epsilon}^+_{-1/2} h^\dagger)$ which is the primary field associated to the state $|\tilde{h}\rangle = \epsilon^+_{-1/2} \bar{\epsilon}^+_{-1/2} |h\rangle$ which has $Q = \bar{Q} = 2$. In the tables below I have listed the chiral primaries and twisted chiral primaries and their charges.

| \( Q \rightarrow \) | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\tilde{Q}$ | 0 | $\mathbb{I}$ | $\epsilon^+$ | $\bar{\epsilon}^+$ |
| $\downarrow$ | 1 | $h$ | $(\epsilon^+_{-1/2} b^\dagger)$ | $\epsilon^+ \bar{\epsilon}^+$ |
| 2 | $(\epsilon^+_{-1/2} b^\dagger)$ | $(\epsilon^+_{-1/2} \bar{\epsilon}^+_{-1/2} h^\dagger)$ | $\epsilon^+ \bar{\epsilon}^+$ |
| 3 | $\bar{\epsilon}^+$ | $\epsilon^+ \bar{\epsilon}^+$ |

**Chiral Primaries**

| \( Q \rightarrow \) | 0 | -1 | -2 | -3 |
|---|---|---|---|---|
| $\tilde{Q}$ | 0 | $\mathbb{I}$ | $\epsilon^-$ | $\bar{\epsilon}^-$ |
| $\downarrow$ | 1 | $b$ | $(\epsilon^-_{-1/2} h^\dagger)$ | $\epsilon^- \bar{\epsilon}^-$ |
| 2 | $(\epsilon^-_{-1/2} b^\dagger)$ | $(\epsilon^-_{-1/2} \bar{\epsilon}^+_{-1/2} h^\dagger)$ | $\epsilon^- \bar{\epsilon}^-$ |
| 3 | $\bar{\epsilon}^+$ | $\epsilon^- \bar{\epsilon}^-$ |

**Twisted Chiral Primaries**

Exactly marginal operators are easy to construct in $N=2$ theories. $C = (G^+_{-1/2} \bar{G}^-_{-1/2} h)$, and $R = (G^-_{-1/2} \bar{G}^+_{-1/2} b)$ are neutral, dimension (1,1) Virasoro primaries. The primary fields $C, R$ (and their adjoints $C^\dagger, R^\dagger$) transform into total derivatives under supersymmetry. So $\int C$ and $\int R$ are superconformally-invariant operators. We would like to contemplate adding them to the action

$$S(\tau, t) = S_0 + \tau^a \int C_a + \bar{\tau}^a \int (C^\dagger)_{\bar{a}} + t^m \int R_m + \bar{t}^m \int (R^\dagger)_{\bar{m}}$$

and thereby defining the deformed theory, whose partition function $Z(\tau, t) = \int e^{-S(\tau, t)}$, say, can be calculated as a power series expansion in the $\tau$'s and $t$'s.

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1 In the context of $N=2$ $\sigma$-models to be discussed in the next section, these are, respectively, the moduli corresponding to deformations of the complex structure and the Kähler class of the Calabi-Yau manifold.
Although (2.3) does indeed define a family of new (2,2) superconformal theories parametrized by the moduli $\tau, t$, the idea of constructing the deformed theory as a power series is, unfortunately, a complete fake.

The problem is simple. Each term in the power series expansion is a correlation function of a product of $C$'s $C^\dagger$'s, $R$'s and $R^\dagger$'s, each integrated over the world-sheet. But in general, there are singularities when these operators collide. These singularities render the integrals ambiguous, and could potentially even ruin the superconformal invariance of the theory [8]. One prescription for evading these ambiguities (and showing that (2,2) superconformal invariance is indeed preserved) is to embed the theory in a string theory. Then the marginal operators in question are the “internal” parts of vertex operators for spacetime fields. We can then consider the correlation function of vertex operators in a regime in momentum space where the integrals converge, and define the full correlation function by analytic continuation in the momenta. This provides a physical way to understand (and subtract off) the singularities of the integrand – they are the residues of on-shell poles due to intermediate massless string states. This program was carried out to fourth order by Dixon, Kaplunovsky and Louis [5]. The calculation is quite tedious, and probably too unwieldy to carry out to higher order. Still, as we shall see, it is enough to recover the local geometry of the moduli space. Indeed, what I am going to do in the rest of this section is “borrow” their results. All of the formulæ which follow appear (perhaps somewhat disguised) in their paper.

Even after eliminating the divergences, one is faced with an ineluctable fact: the moduli space is a curved manifold. The marginal operators we have written down represent infinitesimal perturbations i.e. they are tangent vectors to the moduli space. To perturb the theory to higher order, we need to take covariant derivatives, rather than ordinary derivatives. The procedure used by DKL amounts to a definition of the connection one is supposed to use to define these covariant derivatives. Other ways of subtracting the short-distance singularities of the correlation-functions lead, in principle, to different definitions of the connection [9]. Clearly, if our ultimate interest is in embedding these theories in a string theory, the DKL definition is a good one to follow.

It is a simple fact of life that on a curved manifold, covariant derivatives do not commute, and the result of defining the theory as a function of the moduli by “parallel
transport” is path-dependent.

Since the moduli space is curved, when we parallel transport the theory around a closed loop, we return to an isomorphic, but not quite identical conformal field theory. In general, the states at each level return to themselves permuted by a $U(n)$ transformation. This phenomenon, in the much simpler context of nonrelativistic quantum mechanics is known as Berry’s Phase (or its nonabelian generalization \[10\]). In particular, the chiral primaries $b$ and $h$ come back mapped nontrivially among themselves. Even the generators of the $c = 9$ extended chiral algebra are affected, but there the effect is rather tightly constrained, since they must satisfy the same algebra as before. The result is that the N=2 supercurrents $G^\pm$ and $\epsilon^\pm$ (and the corresponding barred fields) come back to themselves multiplied by phases

\[
G^\pm(z) \to e^{\pm i\theta} G^\pm(z), \quad \bar{G}^\pm(\bar{z}) \to e^{\pm i\tilde{\theta}} \bar{G}^\pm(\bar{z})
\]

\[
\epsilon^\pm(z) \to e^{\mp i\theta} \epsilon^\pm(z), \quad \bar{\epsilon}^\pm(\bar{z}) \to e^{\mp i\tilde{\theta}} \bar{\epsilon}^\pm(\bar{z})
\]

The stress tensor $T$ and $U(1)$ current $J$ (and their barred counterparts) must come back to themselves, with no phases because of the central terms in the algebra \[2.1\].

When discussing a family of theories, then, we should think of $G^+$ not as a fixed operator, but as an operator-valued section of a line bundle $L$. The Berry phase that we pick up is simply a manifestation of the curvature of $L$. $G^-$ is an operator-valued section of the dual line bundle $L^{-1}$. Similarly, $\bar{G}^\pm$ are operator-valued sections of line bundle $\tilde{L}, \tilde{L}^{-1}$ and $\epsilon^\pm$ are sections of $L^{-3}, L^3$ (the third tensor powers of $L^{-1}, L$), etc.

We have already identified the marginal operators $C$ and $R$ as operator-valued sections of $T$, the (holomorphic) tangent bundle to the moduli space. Therefore we learn that the chiral primaries $h$ and $b$ are, respectively, sections of $T \otimes L \otimes \tilde{L}$ and $T \otimes L^{-1} \otimes \tilde{L}$. In particular, the precise mapping between the chiral primaries $b, h$ and the moduli $C, R$ for a family of theories depends on a choice of section $s \in \Gamma(L)$ and $\tilde{s} \in \Gamma(\tilde{L})$. To jump slightly ahead, this means that the Zamolodchikov metric for the chiral primaries differs from that of the moduli (c.f. \[3\], eqn. 3.36)

\[
\mathcal{G}_{ab} = \langle h_a \bar{(h^\dagger)} b(0) \rangle = \|s\|^2 \|\tilde{s}\|^2 g_{ab}
\]

\[2\] Of course, these complications don’t arise for a 1-parameter family of perturbations. One also doesn’t notice them at low orders in the perturbation expansion. Finally, in the particular context in which we are working, holomorphic covariant derivatives, $\nabla_a$ and $\nabla_m$ all commute. This can fool one into the false impression that the moduli space is flat.
where

\[ g_{ab} = \langle C_a(1)(C^\dagger)_b(0) \rangle \]

(with a similar formula for \( b \)'s and \( R \)'s).

Let me repeat that we’re not saying anything exotic here. We started out with states \(| C_a \rangle \) and \(| h_a \rangle \) related by \(| C_a \rangle = C^+_1 G^+_1 | h_a \rangle \). We parallel transport them around a closed loop in moduli space, and they come back rotated by a \( U(n) \) matrices, \(| C_a \rangle \rightarrow U_a^b | C_b \rangle, | h_a \rangle \rightarrow U'_a^b | h_b \rangle \). Superconformal symmetry dictates that the two \( U(n) \) matrices must be related: \( U_a^b = e^{i(\theta + \tilde{\theta})} U'_a^b \) and that, moreover, the Berry phases for all of the states in the superconformal module built on \(| h_a \rangle \) must be similarly related.

To understand the Berry’s phase in this sector of the theory, we need to find the metric on moduli space, and the fiber metric on the line bundle \( L \) and \( \tilde{L} \).

The trick to constructing these objects is to realize that the same moduli space parametrizes the topological field theories obtained by twisting this \( N=2 \) superconformal theory\([11]\). The topological theory is obtained by “improving” the stress tensor \( T \rightarrow T \pm \frac{1}{2} \partial \bar{z} J \) (and similarly for \( \bar{T} \)) so that one of the supercurrents \( G^\pm \) becomes dimension 1, while the other becomes dimension 2. The integral of the dimension 1 supercurrent then becomes a nilpotent global charge \( Q \), and we define the physical states of the topological theory to be those in the cohomology of \( Q \). There are two independent twistings of relevance: the “A” model, in which \( Q = \oint G^- \) and \( \bar{Q} = \oint \bar{G}^+ \) are the nilpotent charges which define the theory, and the “B” model, in which \( Q = \oint G^+ \) and \( \bar{Q} = \oint \bar{G}^- \) are the nilpotent supercharges \([12]\). In each case, the physical states of the topological theory are in 1-1 correspondence with the twisted chiral or chiral primaries listed above.

In the topological theory, charge is violated by 3 units on the sphere. So the two-point function provides a quadratic form on the (twisted) chiral ring

\[ \eta(\phi_i, \phi_j) = \langle \phi_i(z, \bar{z}) \phi_j(0) \rangle \]

where, in order to get a nonzero 2-point function, the total charge must add up to \( \bar{Q} = 3 \) and \( Q = \pm 3 \). This quadratic form is symmetric or skew-symmetric\(^3\), depending on whether the

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\(^3\) The skew-symmetric part of this quadratic form for the twisted chiral ring (i.e. the quadratic form restricted to the skew-diagonal part of the above table of twisted chiral primaries) is the conformal field theory version of the quadratic form studied in the “variation of Hodge structures” approach to the moduli space of Calabi-Yau manifolds\([13,14]\).
total charge \((Q + \bar{Q})\) of \(\phi_i\) is even or odd. (By topological invariance, the 2-point functions is actually independent of \(z, \bar{z}\).)

Demanding that the connection be compatible both with the metric (of the tangent bundle and of the line bundles \(L, \bar{L}\)) and with this quadratic form is extremely restrictive. It implies a set of equations which allows us to solve for the metrics. How exactly one phrases these equations is somewhat a matter of taste. They appear in the theory of variations of Hodge structures \([13]\), and in the Calabi-Yau context are quite elegantly discussed in \([14]\). Slightly repackaged in the context of massive deformations of \(N=2\) theories, they go by the name of “topological-antitopological fusion” equations \([16]\), which are discussed by Cecotti in his lectures. Alternatively, one can simply calculate the relevant 4-point functions in the superconformal field theory \([5]\).

However one arrives at it, the result is simple to state. The line bundles \(L, \bar{L}\) have curvature tensors \(F, \bar{F}\) which are both of type \((1,1)\). What is more, the mixed components of these curvature tensors vanish

\[
F_{a\bar{m}} = F_{m\bar{a}} = \bar{F}_{a\bar{m}} = \bar{F}_{m\bar{a}} = 0 \quad (2.6)
\]

and the remaining components satisfy

\[
F_{a\bar{b}} = \bar{F}_{a\bar{b}}, \quad F_{m\bar{n}} = -\bar{F}_{m\bar{n}} \quad (2.7)
\]

The line bundles \(L, \bar{L}\) can actually be given the structure of holomorphic line bundles, and the DKL definition corresponds to choosing a holomorphic connection on \(L, \bar{L}\).

The metric on the moduli space also takes a block-diagonal form:

\[
g_{a\bar{b}} = 3\bar{F}_{a\bar{b}}, \quad g_{m\bar{n}} = 3\bar{F}_{m\bar{n}} \quad (2.8)
\]

This means the moduli space is Kähler since \(\bar{F}\) is a closed 2-form, and \((2.6), (2.7)\) imply that the mixed components of Christoffel connection, like \(\Gamma^a_{mb}\), and the mixed components of the curvature vanish. This is not quite enough to prove that the moduli space is a product manifold, \(\text{cal}M \times \text{cal}M'\), parametrized by the \(\tau\)'s and \(t\)'s respectively. In many cases of interest, it appears that the moduli space is the quotient of such a product manifold by the action of a discrete group which acts nontrivially on both factors. We will ignore this

\[\footnote{For a discussion of the topological-antitopological fusion equations as compatibility equations, see \([15]\).} \]
subtlety, assuming that we can always go to a suitable covering space which is a product $\text{cal} M \times \text{cal} M'$.

There is one more condition which restricts the geometry of the moduli space, but to state it, we must pause to introduce the 3-point functions (the “(twisted) chiral ring” [3], or the “Yukawa couplings” of the low-energy effective field theory [3])

$$W_{abc}(s \otimes \tilde{s} \otimes 6) = \langle \tilde{h}_a(0) h_b(1) h_c(-1) \rangle$$

$$W'_{kmn}((s^{-1}) \otimes \tilde{s} \otimes 6) = \langle \tilde{b}_k(0) b_m(1) b_n(-1) \rangle$$

where $\tilde{h}_a = (\epsilon_{1/2} - \bar{\epsilon}_{1/2} h_a)$, and $\tilde{b}_m = (\epsilon_{+1/2} - \bar{\epsilon}_{1/2} b_m)$. The 3-point functions depend explicitly on the sections $s, \tilde{s}$ which relate the tangent vectors to the moduli space to the chiral primaries $h, b$ (with some extra powers of $s, \tilde{s}$ thrown in to define the $\epsilon$’s. c.f. (2.4)).

$W_{abc}$ and $W'_{kmn}$ don’t look symmetric in their indices, but by the Ward identities associated to the extended $c = 9$ chiral algebra, they actually are symmetric. So $W$ is a section of $S^3(T^*) \otimes L^{-6} \otimes \tilde{L}^{-6}$ and $W'$ is a section of $S^3(T^*) \otimes L^6 \otimes \tilde{L}^{-6}$. They are actually holomorphic sections:

$$\nabla \bar{m} W_{abc} = \nabla \bar{d} W_{abc} = 0 \quad (2.9a)$$

$$\nabla \bar{m} W'_{kmn} = \nabla \bar{a} W'_{kmn} = 0 \quad (2.9b)$$

The proof of this is an easy application of the superconformal Ward identities [17]. Because of the Kähler geometry, the Christoffel terms vanish and

$$\nabla \bar{m} W_{abc} = \partial \bar{m} W_{abc} = \langle \tilde{h}_a(0) h_b(1) h_c(-1) \rangle \int d^2 w R^\dagger_m(w, \bar{w})$$

Write

$$\int d^2 w R^\dagger(w, \bar{w}) = - \int d^2 w \oint d\bar{y} \frac{\bar{y} - \bar{u}}{\bar{w} - \bar{u}} G^+(\bar{y}) \left( G_{-1/2} b^\dagger \right)(w, \bar{w})$$

where the contour surrounds $w$. The Ward identity says the correlation function is actually independent of $\bar{u}$. Setting $\bar{u} = 0$, one can deform the contour so that it closes around the

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5 The Ward identity in question simply says that

$$\langle \oint d z (z - w) / w \epsilon^-(z) \tilde{h}_a(0) h_b(1) h_c(-1) \rangle,$$

where the contour surrounds the origin, is independent of $w$. Doing the same contour-deformation trick discussed below for both $\epsilon^-$ and $\bar{\epsilon}^-$, we conclude that $W_{abc}$ is independent of which operator has the “tilde” on it.
other vertex operators. One gets no contribution from closing the contour around $\pm 1$, since $G_{-1/2}^{-1} h = 0$. The contribution from 0 vanishes as well, since $G^+$ has only a single pole with the operator at the origin, and there is an explicit factor of $\bar{y}$ in the integrand.

For $\langle \tilde{h}_a(0) h_b(1) h_c(-1) \int d^2 w C_\alpha^d(w, \bar{w}) \rangle$, the argument goes almost the same way, except that we have to worry a little more about short-distance singularities. However, because we can deform both the $G^+$ and the $\bar{G}^+$ contours, we can show that the residues actually vanish.

We would like to use identical reasoning to conclude that the mixed holomorphic covariant derivatives also vanish

$$\nabla_m W_{abc} = \nabla_a W'_{kmn} = 0 \quad (2.9c)$$

It is certainly true that $(L \otimes \tilde{L})|_{calM'}$ is flat, but it is not necessarily holomorphically trivial. If it is not, we can find locally covariantly constant sections which have global monodromies. This will mean that $W$ will also have global monodromy: As we go around a closed loop in $calM'$, $W$ will pick up a phase. This phase cancels in the expression below for the curvature of the moduli space. Modulo this subtlety, we simply need to establish

$$\langle \tilde{h}_a(0) h_b(1) h_c(-1) \int d^2 w R_{a\bar{b}c\bar{d}} \rangle = 0$$

which again follows from the superconformal Ward identities [17].

The final condition that the curvature tensor of the metric $g$ must satisfy is

$$R_{a\bar{b}c\bar{d}} = g_{a\bar{c}}g_{b\bar{d}} + g_{a\bar{d}}g_{b\bar{c}} - \|s\|^{-12}\|\tilde{s}\|^{-12} W(s^6 \otimes \tilde{s}^6)_{a\bar{b}c\bar{d}} \bar{W}(s^6 \otimes \tilde{s}^6)_{\bar{f}c\bar{d}\bar{g}^{e\bar{f}}} \quad (2.10)$$

(The other components of $R$ must vanish by the above conditions.)

Note that in defining the 3-point functions, we had to make an arbitrary choice of sections $s, \tilde{s}$. Under a change of section $s \rightarrow f s, \tilde{s} \rightarrow \tilde{f} \tilde{s}$, the 3-point functions transform

$$W_{abc} \rightarrow f^6 \tilde{f}^6 W_{abc}, \quad W'_{kmn} \rightarrow f^{-6} \tilde{f}^{-6} W'_{kmn}$$

But the factors of $\|s\|^{-12}\|\tilde{s}\|^{-12}$ in (2.10) transform in a compensating way. Thus (2.10) is a completely covariant equation for the curvature of the moduli space. This collection of restrictions (2.6)–(2.10) are collectively known as “special geometry” [15, 19, 20, 5, 14, 21].

These formulæ are usually written, locally, in terms of the Kähler potential for the metric $g$, rather than the norm-squared of a section of a line bundle. The correspondence is
easy to establish if one notes that, given a meromorphic section \( \tilde{s} \) of \( \tilde{L} \), then the curvature \( \tilde{F} \) is given by

\[
\tilde{F} = -\partial \bar{\partial} \log \| \tilde{s} \|^2
\]

So the Kähler potential for the metric \( g \) is just

\[
K = -3 \log \| \tilde{s} \|^2
\]

Several comments are in order. First, a crucial role was played the fact that the superconformal generators pick up a Berry phase. This is to be expected on general grounds, but is frequently ignored in discussions of this subject.

Second, the framework we have developed here is tailor-made to address questions about the global properties of the moduli space. Other treatments of special geometry, written in local “special coordinates” are incapable of even detecting these subtleties. To cite two of these subtleties that we have seen in our discussion, the first was that the moduli space is in general not a product manifold. We “solved” this problem by going to a suitable covering space which is a product space \( \mathcal{M} \times \text{calM}' \). The second subtlety is that the “chiral ring” \( W \) has monodromies as one goes around loops in \( \text{calM}' \), whereas most people (myself included) have simply assumed that it is independent of the moduli on \( \text{calM}' \) (and vice versa for the twisted chiral ring \( W' \) and \( \text{calM} \)). This is because the line bundles \( (L \otimes \tilde{L})|_{\text{calM}'} \) and \( (L^{-1} \otimes \tilde{L})|_{\text{calM}} \) are flat, but not necessarily holomorphically trivial.

Third, it is clear that, even ignoring the subtle global questions, it is a bad idea to focus on one set of the moduli and ignore the other in discussing the physics of these theories. The Zamolodchikov metrics for various descendent fields (in particular, the fields \( \tilde{G}_{-1/2}^b, (\tilde{G}_{-1/2}^b) \) which go into defining the vertex operators for the matter fields in the string theory) depend on both sets of moduli \( \mathcal{M} \). The Zamolodchikov metrics of the matter fields

\[
G_{a\bar{b}} = \langle (\tilde{G}_{-1/2}^b h_a)(1)(\tilde{G}_{-1/2}^b h_{\bar{b}}^\dagger)(0) \rangle = \| s \|^2 g_{a\bar{b}}
\]

\[
G_{m\bar{n}} = \langle (\tilde{G}_{-1/2}^b b_m)(1)(\tilde{G}_{-1/2}^b b_{\bar{n}}^\dagger)(0) \rangle = \| s \|^{-2} g_{m\bar{n}}
\]
depend nontrivially on both sets of moduli so the “physics” of these theories does not factorize, even locally.

Fourth, our analysis has shown that the Kähler form of the metric on the moduli space, \( J = \frac{i}{2\pi} (g_{a\bar{b}} d\tau^a \wedge d\bar{\tau}^b + g_{m\bar{n}} dt^m \wedge d\bar{t}^n) = 3c_1(\tilde{L}) \) is three times an integer class. This
result, anticipated in a footnote in [14], and implicit in the results of [3] is a somewhat more restrictive condition than that obtained in most previous analyses.

Fifth, and perhaps most important, our ability to determine the geometry of the moduli space has been reduced to the necessity of calculating the 3-point functions $W_{abc}^\prime$, $W_{kmn}^\prime$, and solving some differential equations. In special coordinates, the latter reduce to relatively tractable equations. The key feature which will make it possible to carry out this program is the equation (2.9), which tell us about the covariant constancy of the $W$’s with respect to certain of the moduli. This is a tremendous advantage in calculating them for certain classes of theories.

3. N=2 sigma model in super-space

The superconformal approach followed in the last section is very general – it applies to essentially any family of $c = 9$ N=2 conformal field theories with the extended chiral algebra. In this section we will turn to a particular class of such theories which arise from N=2 supersymmetric $\sigma$-models on Calabi-Yau manifolds [22,3]. We’ll concentrate on the problem of calculating the 3-point functions which were the crux of the special geometry discussed in the last section.

The $\sigma$-model action can be written in N=2 superspace as

$$S = \int d^2z d^4\theta K(\Phi, \bar{\Phi})$$

(3.1)

where $\Phi^i$ are superfields whose lowest component $\phi^i$ are local complex coordinates on a compact Calabi-Yau manifold $M$. The $\Phi^i$ obey a chiral constraint

$$D_+ \Phi^i = \bar{D}_+ \Phi^i = D_- \bar{\Phi}^i = \bar{D}_- \bar{\Phi}^i = 0$$

(3.2)

where the superderivatives are

$$D_+ = \partial_{\theta^+} + \theta^- \partial_z, \quad D_- = \partial_{\theta^-} + \theta^+ \partial_z$$

$$\bar{D}_+ = \partial_{\bar{\theta}^+} + \bar{\theta}^- \partial_{\bar{z}}, \quad \bar{D}_- = \partial_{\bar{\theta}^-} + \bar{\theta}^+ \partial_{\bar{z}}$$

In components,

$$\Phi^i = \phi^i(y, \bar{y}) + \theta^- \lambda^i(y, \bar{y}) + \bar{\theta}^- \psi^i(y, \bar{y}) + \theta^- \bar{\theta}^- F^i(y, \bar{y})$$
where $y = z + \theta^- \theta^+$. The lowest component is, as already mentioned, a scalar which is a local complex coordinate on $M$. $\lambda^i, \psi^i$ are left- and right-chirality spinors taking values in $T$, the holomorphic tangent bundle of $M$, and $F^i$ is an auxiliary field which allows the supersymmetry algebra to close off-shell. The function $K(\phi, \bar{\phi})$ is called the Kähler potential, and determines the metric on $M$ by

$$g_{ij} = \partial_i \partial_j K$$  \hspace{1cm} (3.3)

The Kähler potential is not a globally-defined function on $M$. Across coordinate patches, $K$ transforms by a Kähler transformation

$$K' = K + f(\Phi) + \bar{f}(\bar{\Phi})$$

The Kähler metric is unaffected by Kähler transformations, and so is globally well-defined. The action, too, is globally defined because we can write (3.1) in terms of the Kähler metric as

$$S = -\int d^2zd\theta^+ d\bar{\theta}^- D_\Phi^i \bar{D}_+ \bar{\Phi}^j g_{ij}(\Phi, \bar{\Phi}) - \int d^2zd\theta^- d\bar{\theta}^+ D_+ \bar{\Phi}^j \bar{D}_- \Phi^i g_{ij}(\Phi, \bar{\Phi})$$  \hspace{1cm} (3.4)

Clearly, the action can be generalized by adding a $\theta$-term. Let $B$ be a real closed 2-form on $M$. Add to the action

$$S_\theta = i \int \phi^* B = i \int B_{ij} d\phi^i \wedge d\phi^j$$

$$= -\int d^2zd\theta^+ d\bar{\theta}^- D_\Phi^i \bar{D}_+ \bar{\Phi}^j B_{ij}(\Phi, \bar{\Phi}) + \int d^2zd\theta^- d\bar{\theta}^+ D_+ \bar{\Phi}^j \bar{D}_- \Phi^i B_{ij}(\Phi, \bar{\Phi})$$  \hspace{1cm} (3.5)

This is (almost) a total derivative. Its integral is only nonzero for topologically nontrivial maps $\phi$. It thus has no effect in $\sigma$-model perturbation theory, though it will be very important when we come to discuss $\sigma$-model instantons.

Generically, the $\sigma$-model (3.1)+(3.5) is not conformally invariant. The nonrenormalization theorem [23] is, however a powerful restriction on the form of the $\beta$-function. At one-loop, one has a counterterm of the form

$$\int d^2zd\theta^+ d\bar{\theta}^- D_\Phi^i \bar{D}_+ \bar{\Phi}^j R_{ij}(\Phi, \bar{\Phi}) + \int d^2zd\theta^- d\bar{\theta}^+ D_+ \bar{\Phi}^j \bar{D}_- \Phi^i R_{ij}(\Phi, \bar{\Phi})$$  \hspace{1cm} (3.6)

where $R_{ij}$ is the Ricci tensor on $M$. If one uses the Ricci-flat Kähler metric which is known to exist on a Calabi-Yau manifold, this actually vanishes, but for any $g_{ij}$, the cohomology class of the Ricci form

$$\mathcal{R} = i R_{ij} d\phi^i \wedge d\phi^j$$

14
vanishes, which means that one can rewrite (3.6) as the $\int d^4 \theta$ of a globally-defined function on $M$ (a “D term”). The nonrenormalization theorem says that all higher-loop counterterms are D-terms.

Clearly, the D-terms are renormalized in some horribly complicated way along renormalization group flows. It is believed that they are marginally irrelevant, and flow to zero in the IR. However, those terms in the action which cannot be written (globally) as D-terms are protected from renormalization, and are constant along RG flows. On such term has already been alluded to: the cohomology class of the Kähler form

$$J = ig_{ij} d\phi^i \wedge d\phi^j$$

which enters into (3.4). Shifting the Kähler form by an exact form changes (3.4) by a D-term.

Another obvious RG invariant is the cohomology class of the 2-form $B = B_{ij} d\phi^i \wedge d\bar{\phi}^j$. It is conventional to combine the two, and call $\omega = [J + iB] \in H^2(M, \mathbb{C})$ the cohomology class of the (generalized) Kähler form.

These RG invariants characterize the IR fixed point(s) of this $\sigma$-model. Thus we have recovered some of the moduli discussed in the last section.

$$R = \int d\theta^+ d\bar{\theta}^- D_+ \Phi^i \bar{D}_- \Phi^j b_{ij}(\Phi, \bar{\Phi})$$

$$R^\dagger = \int d\theta^- d\bar{\theta}^+ D_- \Phi^i \bar{D}_+ \Phi^j b_{ij}(\Phi, \bar{\Phi})$$

(3.7)

where $b_{ij}$ is a real closed 2-form on $M$. Clearly, $R$ shifts both $g_{ij}$ and $B_{ij}$ by $b_{ij}$, whereas $R^\dagger$ shifts them in opposite direction. Since these are composite operators, they are subject to renormalization. However the cohomology class of $b_{ij}$ is an RG invariant.

Are there any other RG invariants of these $\sigma$-models? One which has been lurking in our formalism is the complex structure of $M$. That is what distinguishes the coordinates $\phi^i$ from $\bar{\phi}^i$ which satisfy different chirality constraints (3.2). Since the complex structure is built into the N=2 superspace formalism, it is a little awkward to do explicit finite variations of it. Nevertheless, an infinitesimal variation of the complex structure is given by

---

6 A historical note: it was once believed that if one could make the one-loop $\beta$-function vanish in this theory, the $\beta$-function would then be zero to all orders. Later, a 4-loop contribution to the $\beta$-function was found [24].
$h_{ij}$, an element of the cohomology group $H^1(M, T)$. We can write down the corresponding marginal operator

$$ C = \int d\theta^- d\bar{\theta}^- D_+ \bar{\Phi}^k \bar{D}_+ \bar{\Phi}^j g_{jk}(\Phi, \bar{\Phi}) h_{ij}(\Phi, \bar{\Phi}) $$

(3.8)

Again, the form of this operator gets renormalized in some complicated way along RG flows, but the cohomology class of $h_{ij}$ is an RG invariant, and characterizes the superconformal fixed points.

We have now identified the moduli of the (2,2) superconformal field theory which is the IR fixed point of this $\sigma$-model as deformations of either the Kähler class, or the complex structure of $M$. These are the only deformations which we can probe away from the conformal point. There may be other marginal deformations which break the (2,2) supersymmetry of the conformal point down to something smaller (like, say (2,0)), but, again, that is beyond the scope of these notes.

Let us now turn to the 3-point functions $W, W'$ which we introduced in the last section. We have already written down the operators $h, b$ as (lowest components of) superfields.

$$ h = D_+ \bar{\Phi}^k \bar{D}_+ \bar{\Phi}^j g_{jk}(\Phi, \bar{\Phi}) h_{ij}(\Phi, \bar{\Phi}), \quad b = D_+ \Phi^i \bar{D}_+ \bar{\Phi}^j b_{ij}(\Phi, \bar{\Phi}) $$

(3.9)

The detailed form of these operators may, as I have said, get renormalized, but they are constrained to remain chiral and twisted-chiral superfields, respectively, i.e. $h$ satisfies $D_+ h = \bar{D}_+ h = 0$ and $b$ satisfies $D_- b = \bar{D}_- b = 0$

We also need the operators $\tilde{h}, \tilde{b}$, which which satisfy the appropriate antichiral and twisted antichiral constraints. They take the form

$$ \tilde{h} = D_- \Phi^i D_- \Phi^k \bar{D}_- \bar{\Phi}^j \bar{D}_- \bar{\Phi}^k' \epsilon_{iji'j'kk'} g^{i'i} h_{i}^i $$

$$ \tilde{b} = D_+ \bar{\Phi}^j D_+ \bar{\Phi}^k \bar{D}_+ \bar{\Phi}^i \bar{D}_+ \bar{\Phi}^i' \epsilon_{i'jkk'} g^{i'i} g^{j'j} b_{i}^i $$

(3.10)

where $\epsilon_{ijk}$ is the holomorphic 3-form on $M$.

The 3-point function $W_{abc}$ is then given to lowest order by the supergraph Fig. 1, and $W'_{kmn}$ by Fig. 2.

---

**Fig. 1:** Supergraph contributing to $W_{abc}$

**Fig. 2:** Supergraph contributing to $W'_{kmn}$
I know of no direct proof that there are no higher-loop corrections to these graphs (beyond those which renormalize the operators themselves). But the general arguments lead us to that conclusion. The reason is simple. We proved that $\nabla_{\bar{m}} W_{abc} = \nabla_{\bar{m}} W'_{k'n} = 0$. This means that the 3-point functions depend holomorphically on the generalized Kähler class, i.e. they depend only on the combination $\omega = [J + iB]$. But perturbation theory is completely insensitive to $B$ because $S_\theta$ is locally a total derivative. Thus the lowest-order perturbative result must be exact.

Of course $\sigma$-model instanton corrections do depend on $B$, so $W'_{k'kn}$ does receive instanton corrections [25]. However, we have also shown that $\nabla_m W_{abc} = 0$, so $W_{abc}$ must be completely independent of the generalized Kähler class. Hence it receives no instanton corrections and is entirely given by the lowest order graph Fig. 1. The result is well-known [3],

$$W_{abc} = \int_M \left( h_{\bar{i}}^{(a)} i' h_{\bar{j}}^{(b)} j' h_{\bar{k}}^{(c)} k \epsilon_{i'j'k'} \epsilon_{ijk} \right)$$

where the 6-form in parentheses is integrated over the Calabi-Yau manifold $M$. (One always has this remaining integral to do in background field perturbation theory.)

The perturbative result for $W'_{mnp}$ is similar

$$W'_{mnp} = \int_M \left( b^{(m)}_{\bar{i}i} b^{(n)}_{\bar{j}j} b^{(p)}_{\bar{k}k} \right) \equiv \int_M b^{(m)} \wedge b^{(n)} \wedge b^{(p)} \quad (3.11)$$

This, however, does receive instanton corrections.

The point is that there are nontrivial, finite-action solutions to the equations of motion given by solutions to

$$\partial_{\bar{t}} \Phi^i = D_- D_- \Phi^i + \Gamma_{jk}^i (\Phi, \bar{\Phi}) D_- \Phi^j D_- \Phi^k = 0 \quad (3.12)$$

corresponding to holomorphic curves in $M$.

---

7 Note that in writing this result, we are making an implicit choice for the holomorphic 3-form $\epsilon$ which appeared in (3.10), but has disappeared from this formula. The choice which we are making is to take $\epsilon$ to be a generator of integral cohomology. This makes sense so long as we are holding fixed the complex structure of $M$. The fact that the formula for $W'_{mnp}$ depends on such a choice was transparent in the formalism of section (2), and has been independently emphasized in [26].
In components, these equations (and their barred counterparts) are

\[
\begin{align*}
\partial_{\bar{z}} \phi^i &= \partial_{\bar{z}} \phi^i = \partial_{\bar{z}} \lambda^i = \partial_{\bar{z}} \psi^i = 0 \\
F^i &= \Gamma^i_{jk} \lambda^j \psi^k, \quad \tilde{F}^i = -\Gamma^i_{jk} \bar{\lambda}^j \bar{\psi}^k \\
\partial_z \psi^i + \Gamma^i_{jk} \partial_z \phi^j \psi^k &= R^i_{jkl} \bar{\lambda}^k \lambda^j \psi^l \\
\partial_{\bar{z}} \bar{\lambda}^\bar{i} + \Gamma^\bar{i}_{\bar{j}k} \partial_{\bar{z}} \bar{\phi}^\bar{j} \bar{\lambda}^\bar{k} &= R^\bar{i}_{\bar{j}kl} \bar{\psi}^\bar{l} \psi^\bar{j}
\end{align*}
\] (3.13a)

\[F^i = \Gamma^i_{jk} \lambda^j \psi^k, \quad \tilde{F}^i = -\Gamma^i_{jk} \bar{\lambda}^j \bar{\psi}^k
\] (3.13b)

\[
\begin{align*}
\partial_z \psi^i + \Gamma^i_{jk} \partial_z \phi^j \psi^k &= R^i_{jkl} \bar{\lambda}^k \lambda^j \psi^l \\
\partial_{\bar{z}} \bar{\lambda}^\bar{i} + \Gamma^\bar{i}_{\bar{j}k} \partial_{\bar{z}} \bar{\phi}^\bar{j} \bar{\lambda}^\bar{k} &= R^\bar{i}_{\bar{j}kl} \bar{\psi}^\bar{l} \psi^\bar{j}
\end{align*}
\] (3.13c)

Actually, (3.13c,d), though they exactly stationarize the action, are awkward to solve. Instead one can solve the linearized equations ((3.13c,d) with the R.H.S. set equal to zero), and obtain an approximate stationary point of the action, supplemented by an explicit 4-fermi term. One can further simplify (3.13c,d) by performing a chiral change of variables:

\[
\begin{align*}
\bar{\lambda}^\bar{i} &= g^\bar{i}j \lambda^j, \\
\bar{\psi}^\bar{j} &= g^\bar{i}j \psi^i
\end{align*}
\]

The equations (3.13c,d) then become

\[
\begin{align*}
\partial_{\bar{z}} \lambda^j &= \partial_{\bar{z}} \psi^i = 0 \\
\partial_z \lambda^i &= \partial_z \psi^i
\end{align*}
\] (3.13c’)

The 4-fermi term in the action is

\[
S_4 = \int R^i_{jkl} \lambda^k \lambda^j \psi^l \psi^i
\]

Depending on the number of fermion zero modes in the instanton background, we may have to bring down powers of this term to soak up any “extra” zero modes not absorbed by the operator insertions in the correlation function [25].

To solve (3.13a), let \( C \subset M \) be a holomorphic \( \mathbb{C}P^1 \) in \( M \). We obtain a solution if \( \phi : \Sigma \to C \) is a holomorphic map from the world-sheet \( \Sigma \) (also a \( \mathbb{C}P^1 \)) to \( C \). In appropriate local coordinates on \( M \) (so that, say, \( C \) is described locally as the \( \phi^3 \)-plane), such a map is just a rational function

\[
\phi^3(z) = \frac{\sum_{i=0}^k a_i z^i}{\sum_{k=0}^k b_i z^i}
\] (3.14)

This map has winding number \( k \). Clearly, it is invariant under a common rescaling \( \{a_i, b_i\} \to \{\lambda a_i, \lambda b_i\} \). To obtain a smooth map of winding number \( k \), we should, strictly nonanomalous because \( c_1(M) = 0 \).
speaking, demand that the roots of the two polynomials in the numerator and denominator of (3.14) not coincide. However for computing integrals, it is natural to compactify the space of solutions by relaxing this restriction, allowing all \( \{a_i, b_i\} \) modulo a common rescaling. The compactified “instanton moduli space” is thus isomorphic to \( \mathbb{CP}^{2k+1} \) [27].

To continue with the computation, we need to discuss the fermionic zero modes in the background (3.14). Solving (3.13a,c) is trivial. The \( \lambda \) zero modes are holomorphic, and the \( \psi \)’s antiholomorphic. The only restriction comes from demanding that the zero modes be normalizable. That is to say, the \( \lambda \)'s are sections of certain holomorphic line bundles on \( \Sigma \), and these line bundles have only a finite number of global holomorphic sections. For a holomorphic instanton of winding number \( k \) (3.14), there are \( 2k \) zero modes of \( \lambda^3 \), and \( k \) zero modes each of \( \lambda_1, \lambda_2 \) (with the corresponding number of zero modes of the \( \psi \)'s).

For \( k = 1 \), this means that there are a total of 4 zero modes of \( \lambda \), and 4 zero modes of \( \psi \), precisely the number which can be absorbed by the operators in the expression for \( W'_{mnp} \). Thus we can ignore the 4-fermi term in the action, and the computation is completely straightforward. The instanton action is just given by \( S_{\text{inst}} = \int_C \omega \). The operators absorb the fermion zero modes, and the integral over the instanton moduli space can be turned into an integral over three copies of \( \Sigma \), by trading the three instanton moduli for the locations of the three insertion points [25,17].

\[
W'_{mnp}^{(k=1)} = \int_C b^{(m)} \int_C b^{(n)} \int_C b^{(p)} e^{-\int_C \omega}
\]

Note that, as required, this is holomorphic in the Kähler class \( \omega \).

For higher \( k \), things are quite a bit more complicated. There are more fermi zero modes, which means we need to bring down powers of the 4-fermi term from the action. At the same time, the instanton moduli space is higher-dimensional, and we obtain an, in principle very nontrivial, form that we have to integrate over it.

First let’s count zero modes. After the operator insertion have done their work, we still have \( 2k-2 \) zero modes of \( \lambda^3 \) and \( k-1 \) zero modes each of \( \lambda_1, \lambda_2 \) (and the same number of \( \psi \) zero modes) which must be absorbed by bringing down factors of \( S_4 \) from the action. In the geometry we are looking at, the only nonzero components of the Riemann tensor that can absorb the relevant zero modes are \( \int R_{133}^{11} \lambda_1 \lambda^3 \psi_1 \psi^3 \) and \( \int R_{233}^{22} \lambda_2 \lambda^3 \psi_2 \psi^3 \). We

For the cognoscenti, I am assuming that \( C \) is an isolated curve of type \((-1,-1)\). That is, I assume the tangent bundle of \( M \), when restricted to \( C \) splits as \( TM|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \).
need to bring down \((k - 1)\) factors of each from the action to absorb the fermion zero modes.

The insight of Aspinwall and Morrison \[27\] was that the transformation properties of the fermion zero modes as a function of the instanton moduli are simpler in the topological version of this \(\sigma\)-model \[12\], leading to a more tractable calculation. This observation would not be of much use, but for the fact that one can argue that the 3-point function in the “A” model

\[
\tilde{W}_{mnp} = \langle b^{(m)}(0)b^{(n)}(1)b^{(p)}(-1) \rangle
\]

is equal\(^{10}\) to the desired \(W'_{mnp}\). This is a triviality for the tree-level result \((3.11)\); it is far from obvious that it is true for the instanton corrections which are sensitive to the global features of the theory. Still, since these 3-point functions are equal at the conformal point, it makes sense to calculate them in the topological theory, where the description of the fermion zero modes is simpler.

The difference between the topological theory and our original \(\sigma\)-model is that whereas before all of the fermions were spinors on the world-sheet, now \(\lambda^i, \bar{\psi}_\dot{i}\) are scalars, and \(\lambda^i, \bar{\psi}_\dot{i}\) are 1-forms. This changes the counting of zero modes. In the topological theory, there are \(2k + 1\) normalizable zero modes of \(\lambda^3, \bar{\psi}^3\) from \((3.13d)\) and \(k - 1\) zero modes each of \(\lambda_1, \lambda_2, \psi_\dot{1}, \psi_\dot{2}\). The operators in the correlation function now absorb 3 zero modes each of \(\lambda^3, \bar{\psi}^3\) and we still have to bring down \((k - 1)\) factors of \(\int R^{11} \lambda^3 \psi_\dot{1} \bar{\psi}^3\) and \((k - 1)\) factors of \(\int R^{22} \lambda_2 \lambda^3 \psi_\dot{2} \bar{\psi}^3\) to absorb the rest.

The simplification that now arises is that the \(\lambda^3\) zero modes are related by supersymmetry to the bosonic zero modes, that is to say, they transform as tangent vectors to the instanton moduli space\(^{11}\), \(i.e.\) as sections of \(T_{\mathbb{C}P^{2k+1}}\). An explicit examination of the zero modes of \(\lambda_1, \lambda_2\) show that they transform as sections of \(\mathcal{L}\), the tautological line bundle on \(\mathbb{C}P^{2k+1}\). The zero modes of \(\psi\) transform as the complex conjugates of the \(\lambda's\).

Integrating over the fermion zero modes turns \(\int R^{11} \lambda^3 \psi_\dot{1} \bar{\psi}^3\) into a 2-form on \(\mathbb{C}P^{2k+1}\). It is clear that this 2-form in nothing other than \(c_1(\mathcal{L})\), the first Chern class of \(\mathcal{L}\). So

\[
\tilde{W}_{mnp} = \int_{\mathbb{C}P^{2k+1}} b^{(m)} \wedge b^{(n)} \wedge b^{(p)} \wedge c_1(\mathcal{L})^{2k - 2} e^{-S_{inst}}
\]

\(^{10}\) for this choice of holomorphic 3-form. More generally, they are sections of different bundles over the moduli space and are related by \(W'_{mnp} = s^3 \bar{s}^{-3} \tilde{W}_{mnp}\).

\(^{11}\) This is the part that is awkward in the untwisted theory. There the supersymmetry which relates the bosonic zero modes to zero modes of \(\lambda^3\) has a kernel. So the \(\lambda^3\) zero modes transform as sections of a certain subbundle of \(T_{\mathbb{C}P^{2k+1}}\). This complicates the rest of the argument.
Using the fact that $c_1(\mathcal{L}) = -H$, where $H$ is the generator of integral cohomology on $\mathbb{C}P^{2k+1}$, this gives

$$\bar{W}_{mnp} = \int_C b^{(m)} \int_C b^{(n)} \int_C b^{(p)} e^{-k \int_C \omega}$$

which, except for the fact that the instanton action is $k$ times as big, is exactly the result for $k = 1$.

Summing over $k$, we get

$$W'_{mnp} = \int_M b^{(m)} \wedge b^{(n)} \wedge b^{(p)} + \sum_C \sum_{k=0}^{\infty} \int_C b^{(m)} \int_C b^{(n)} \int_C b^{(p)} e^{-k \int_C \omega}$$

Clearly, one can sum this geometric series and obtain

$$W'_{mnp} = \int_M b^{(m)} \wedge b^{(n)} \wedge b^{(p)} + \sum_C \int_C b^{(m)} \int_C b^{(n)} \int_C b^{(p)} \frac{e^{-\int_C \omega}}{1 - e^{-\int_C \omega}}$$

In the simplest case of a 1-dimensional Kähler moduli space, we can let $\omega = t \alpha$, and simply take $b = \alpha$, where $\alpha$ is a generator of $H^2(M, \mathbb{Z})$. Then $\int_C \alpha = n \in \mathbb{Z}^+$, where $n$ is called the degree of the curve $C$. Then

$$W' = \int \alpha^3 + \sum_{n=1}^{\infty} \frac{a_n n^3 e^{-tn}}{1 - e^{-tn}}$$

where $a_n$ is the number of curves of degree $n$. Mirror symmetry [28], which gives one an independent way of calculating $W'$ as the $W$ of some other Calabi-Yau manifold gives one predictions [29] for the $a_n$’s. This is the subject of Candelas’s lectures.

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