THE TANGENT AND SECANT VARIETY TO A GENERIC PROJECTIVE K3 SURFACE

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Abstract. In this paper we present a way of computing the degree of the secant and tangent variety of the generic projective K3 surface, under the assumption that the divisor giving the embedding in the projective space is 2-very ample. This method exploits a deep link between these varieties and the Hilbert scheme 0-dimensional subschemes of length 2 of the surface, as well as the the structure of irreducible holomorphic symplectic manifold on this last.

Introduction

In this paper we want to study the link between the secant variety of a smooth projective surface $S$ and the Hilbert scheme of 0-dimensional subschemes of length 2 of $S$. In particular, when the embedding of $S$ in the projective space $\mathbb{P}^n$ is given by a 2-very ample divisor, then (cf. [7]) we can identify the Hilbert scheme $\text{Hilb}^2 S$ with the subvariety of the Grassmannian $G(1, n)$ parametrising all the secant lines to $S$.

Our method to compute the degree of the secant variety $\text{Sec} S$ is particularly interesting when $S$ is a $K3$ surface: in this case in fact, the corresponding Hilbert scheme $\text{Hilb}^2 S$ has a richer structure since it is an irreducible holomorphic symplectic manifold. This fact will allow us to compute the degree of $\text{Sec} S$ explicitly in Theorem 6.1 when $S$ is generic, meaning that its Picard group is generated only by the hyperplane class $h$.

The Hilbert scheme shows up also when we want to determine the degree of the tangent variety $\text{Tan} S$ of a surface, and also in this case the situation where $S$ is a generic projective $K3$ surface is particularly interesting. The degree of $\text{Tan} S$ will be determined explicitly in Theorem 6.4 in this last case.

The origin of this paper can be traced back to [5]. Here the authors determine the automorphism group of the Hilbert scheme of two points on a generic projective $K3$ surface, showing that there are at most two automorphisms and giving a characterisation of the cases when there is a non-trivial automorphism. There is no geometric description of this extra automorphism, and the present paper was written while trying to achieve such description. We shall then say that these results can be obtained also in other more classical ways, for example as an application of the double point formula (see e.g. [9, Thm. 9.3] or [6, §0]), but in our exposition

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we want to emphasize and exploit the link between the enumerative properties of a $K3$ surface $S$ and the symplectic geometry of its Hilbert scheme $\text{Hilb}^2 S$.

The structure of the paper is as follows. In Section 1 we recall the construction of the secant variety and of the Hilbert scheme, giving in the last part the connection between them. In Section 2 and Section 3 we give some general results on the secant and tangent variety to a projective surface: in particular, we want to determine the class of the varieties parametrising the secant lines and the tangent planes to $S$ in the corresponding Grassmannians, giving an explicit enumerative meaning of all the coefficients involved. From Section 4 we restrict our attention to the case of $K3$ surfaces. In particular, when the $K3$ surface is generic we show how we can fully characterize the $k$-very ampleness of the hyperplane class in terms of its self-intersection. In Section 5 we recall briefly the facts we need on the Hilbert schemes of points on $K3$ surfaces, and finally in Section 6 we determine the degree of $\text{Sec} S$ and $\text{Tan} S$.

1. THE SECANT VARIETY AND THE HILBERT SCHEME

In this Section we recall briefly the definition and the construction of the secant variety $\text{Sec} S$ to a smooth projective surface $S$ as well as of the Hilbert scheme $\text{Hilb}^2 S$ parametrising the 0-dimensional subschemes of length 2 of $S$. Our main references are [8, §1(a)] and [2, §6] respectively.

We introduce here the notation we will use throughout all the paper for the Grassmannians: $\text{Grass}(k, n)$ will denote the Grassmannian parametrising the $k$-dimensional subspaces of a complex $n$-dimensional vector space, while $\mathbb{G}(k, n)$ will denote the Grassmannian parametrising the $k$-dimensional subspaces of a complex $n$-dimensional projective space. Hence $\mathbb{G}(k, n) = \text{Grass}(k + 1, n + 1)$.

1.1. The secant variety. Let $S \subseteq \mathbb{P}^n$ be a smooth projective surface, which is not contained in any hyperplane. Consider the map

$$f: (S \times S) \setminus \Delta_S \rightarrow \mathbb{G}(1, n) \quad (P, Q) \mapsto \text{line } \langle P, Q \rangle,$$

where $\Delta_S$ is the diagonal of $S \times S$. Let $\Gamma(S)$ be the closure of the graph of $f$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{G}(1, n)$, and $\Sigma(S)$ be the image of $\Gamma(S)$ in $\mathbb{G}(1, n)$ under the projection on the last factor. Then $\Sigma(S)$ is the subset of the Grassmannian which parametrises the lines which are secant to $S$.

In order to give the definition of the secant variety, we consider the incidence variety

$$I = \{(x, l) \in \mathbb{P}^n \times \mathbb{G}(1, n) \mid x \in l\},$$

and restrict it to the set of secant lines: we let $\Sigma B(S)$ be the inverse image in $I$ of $\Sigma(S)$ under the projection on the second factor. The secant variety, $\text{Sec} S$, of $S$ is the image of $\Sigma B(S)$ in $\mathbb{P}^n$.

We have then the following situation (we denote by $pr$ the projections, and use the subscripts to indicate the factors):

$$\begin{align*}
\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{G}(1, n) & \supset \Gamma(X) \quad \Sigma B(S) \subseteq I \subseteq \mathbb{P}^n \times \mathbb{G}(1, n) \\
S \times S & \xrightarrow{pr_1} \Sigma(S) \xrightarrow{pr_2} \bigcup \text{Sec} S \xrightarrow{pr_3} \mathbb{P}^n \times \mathbb{G}(1, n)
\end{align*}$$

(1)
We recall that

1. \( pr_{12} \) is the blow up of \( S \times S \) along \( \Delta_S \);

2. \( pr_3 \) is finite, and if the generic secant line cuts \( S \) in \( m \) distinct points, then it is \( m(m - 1) : 1 \). In particular:
   - (a) if \( n = 3 \), and \( S \) is a surface of degree \( d \), then \( m = d \),
   - (b) if \( n > 3 \), then by [8, Thm. 1.8] or [11, Cor. 2.7, Cor. 2.8] we have \( m = 2 \), and so \( pr_3 \) is \( 2 : 1 \);

3. \( pr_2 \) is a \( \mathbb{P}^1 \)-bundle;

4. the fibre \( pr_1^{-1}(x) \) represents all the secants passing through \( x \in \text{Sec} \ S \).

1.2. The Hilbert scheme. Let \( S \) be a projective surface. The (second) symmetric product \( S^{(2)} \) of \( S \) is the quotient of \( S \times S \) by the involution exchanging the two factors: it is the variety representing the effective 0-dimensional cycles on \( S \), and is singular along the image of the diagonal. Let \( \varepsilon : \text{Hilb}^2 S \to S^{(2)} \) be the blow up of the singular locus. We obtain then a smooth variety, whose points represents the 0-dimensional subschemes of length 2 of \( S \). The morphism \( \varepsilon \) is called the Hilbert–Chow morphism.

Another way to define the Hilbert scheme \( \text{Hilb}^2 S \) is to blow up first the fixed locus of the involution, i.e. the diagonal \( \Delta_S \), and then take the quotient of the blown up variety by the induced involution. This leads to the commutative square

\[
\begin{array}{ccc}
S \times S & \xrightarrow{\rho} & S \times S \\
\downarrow & & \downarrow \pi \\
\text{Hilb}^2 S & \xrightarrow{\varepsilon} & S^{(2)}
\end{array}
\]

and we recall that the action induced on \( \widetilde{S} \times S \) by the one on \( S \times S \) acts as the identity on the exceptional divisor.

**Remark 1.1.** We can define \( \text{Hilb}^k S \) as the variety parametrising the 0-dimensional subschemes of \( S \) of length \( k \). The first construction we recalled for \( \text{Hilb}^2 S \) generalises to the Hilbert schemes \( \text{Hilb}^k S \), while generalizing the second for \( k > 2 \) we find only an open part of \( \text{Hilb}^k S \), whose complement has codimension at least 3.

1.3. \( k \)-very ampleness. We address now to the concept of \( k \)-very ampleness (cf. [4, §2] and [7]). Let \( S \) be a surface, and \( h \) be a divisor on it. Fix a 0-dimensional subscheme \( Z \) of \( S \) of length \( k + 1 \), defined by the ideal sheaf \( \mathcal{I}_Z \), and consider the exact sequence

\[
0 \to \mathcal{I}_Z \otimes \mathcal{O}_S(h) \to \mathcal{O}_S(h) \to \mathcal{O}_Z \otimes \mathcal{O}_S(h) \to 0.
\]

This sequence induces the long cohomology sequence

\[
0 \to H^0(S, \mathcal{I}_Z \otimes \mathcal{O}_S(h)) \to H^0(S, \mathcal{O}_S(h)) \xrightarrow{\text{res}_Z} H^0(Z, \mathcal{O}_Z \otimes \mathcal{O}_S(h)) \to \ldots,
\]

and we say (cf. [7]) that \( h \) is \( k \)-very ample if the restriction map \( \text{res}_Z \) in (3) is onto for every 0-dimensional subscheme of length \( k + 1 \).

It is immediate to see that 0-very ampleness is equivalent to global generation, and that 1-very ampleness is equivalent to very ampleness. Moreover, any \((k - 1)\)-very ample divisor \( h \) induces a map

\[
\varphi_{k-1} : \text{Hilb}^k S \to \text{Grass}(k, H^0(S, \mathcal{O}_S(h))^*),
\]
associating to any 0-dimensional subscheme of length \( k \) of \( S \) the point representing the \( k \)-dimensional subspace \( H^0(Z, \mathcal{O}_Z \otimes O_S(h))^* \) in \( H^0(S, \mathcal{O}_S(h))^* \). The answer to the question whether this map is an embedding is given in the following Theorem.

**Theorem 1.2 (\cite{7} Main Theorem).** The map \( \varphi_{k-1} \) defined in (4) is an embedding if and only if \( h \) is \( k \)-very ample.

1.4. The link. We begin to see the link between the Hilbert scheme and the secant variety since the diagrams (1) and (2) overlap

\[
\Gamma(S) = \widetilde{S \times S} \xrightarrow{\eta=pr_{12}} S \times S
\]

\[
\text{Hilb}^2 S \xrightarrow{pr_3} \Sigma(S) \subseteq G(1, n).
\]

As observed, the action on \( S \times S \) is the one exchanging the two factors, and fixes the diagonal \( \Delta_S \). The induced action on \( \widetilde{S \times S} \) coincides with this one outside the exceptional divisor, and fixes it pointwise. This implies that the morphism \( pr_3 : \widetilde{S \times S} \to \Sigma(S) \) is constant on the orbits of the action, and so we have a morphism \( \varphi \) making the following diagram commute:

\[
\Gamma(S) = \widetilde{S \times S} \xrightarrow{pr_3} \Sigma(S).
\]

We now want to compare this map with the map \( \varphi_1 \) defined in (4).

**Lemma 1.3.** Let \( h \) be a very ample divisor on the surface \( S \). Then the maps \( \varphi_1 \) of (4) and \( \varphi \) of (5) coincide.

**Proof.** Let \( Z \) be a 0-dimensional subscheme of \( S \) of length 2, defined by the ideal sheaf \( \mathcal{I}_Z \). Since \( \mathcal{O}_S(h) \) is a very ample line bundle on \( S \), we have the short exact sequence

\[
0 \to H^0(S, \mathcal{I}_Z \otimes \mathcal{O}_S(h)) \to H^0(S, \mathcal{O}_S(h)) \to H^0(S, \mathcal{O}_Z \otimes \mathcal{O}_S(h)) \to 0,
\]

whose dual

\[
0 \to H^0(Z, \mathcal{O}_Z \otimes \mathcal{O}_S(h))^* \to H^0(S, \mathcal{O}_S(h))^* \to H^0(S, \mathcal{I}_Z \otimes \mathcal{O}_S(h))^* \to 0
\]

shows that \( H^0(Z, \mathcal{O}_Z \otimes \mathcal{O}_S(h))^* = \text{Ann} H^0(S, \mathcal{I}_Z \otimes \mathcal{O}_S(h)) \). But then

\[
\varphi_1(Z) = \text{line } \mathbb{P}(H^0(Z, \mathcal{O}_Z \otimes \mathcal{O}_S(h))^*) \text{ in } \mathbb{P}(H^0(S, \mathcal{O}_S(h))^*)
\]

\[
= \text{line through } Z \text{ in } \mathbb{P}(H^0(S, \mathcal{O}_S(h))^*)
\]

\[
= \varphi(Z).
\]

\( \square \)

**Corollary 1.4.** Let \( h \) be a very ample divisor on the surface \( S \). Then the morphism \( \varphi \) of (5) is an embedding if and only if \( h \) is 2-very ample.

**Proof.** In fact this is true for \( \varphi_1 \) of (4) by Theorem 1.2. \( \square \)
Remark 1.5. Let $S$ be a surface embedded in $\mathbb{P}^n$ by means of the very ample divisor $h$. The map $\varphi_{k-1}$ defined in (4) associates to any 0-dimensional subscheme $Z$ of length $k$ of $S$ the point representing the linear subspace of $\mathbb{P}^n$ spanned by $Z$, i.e. the unique $(k-1)$-dimensional linear subspace of $\mathbb{P}^n$ containing $Z$.

Thanks to this geometric description it is now easy to see that if the embedding of $S$ in $\mathbb{P}^n$ is given by a 2-very ample line bundle, then (the image of) $S$ contains no lines.

Proposition 1.6. Let $S$ be a surface and $h$ a very ample divisor on it. If there exists a divisor $l$ such that $l \simeq \mathbb{P}^1$ and $\deg \mathcal{O}_S(h)^l = 1$, then $h$ is not 2-very ample.

Proof. This is a classical result on $k$-very ampleness, so we refer to [4] for the proof. \hfill \Box

Remark 1.7. Assume that $S$ is embedded in $\mathbb{P}^n$ by means of a $k$-very ample divisor $h$, with $k \geq 2$. This embedding has the property that for any 0-dimensional subscheme $Z$ of $S$ of length $k$, the linear subspace spanned by $Z$ intersects $S$ exactly in $Z$ (a priori, it could have cut on $S$ a subscheme of higher length containing $Z$). In particular:

(1) If $k \geq 2$, then for any 0-dimensional subscheme $Z$ of $S$ of length 2 the secant line spanned by $Z$ has no further intersections with $S$. As a consequence a tangent line meets $S$ only in the tangency point, and so we deduce that for any $P \in S$ we have $T_P S \cap S = \{P\}$.

(2) If $k \geq 3$, then for any pair of distinct 0-dimensional subschemes $Z, Z'$ of $S$ of length 2, the secant lines spanned by $Z$ and $Z'$ are either disjoint or they meet in a point of $S$. In fact, if they intersect away from $S$, then these two lines span a plane which contains a length 4 subscheme of $S$, which contradicts the 3-very ampleness of $h$. In particular, any pair of tangent planes are disjoint, i.e. for $P, Q \in S$ we have that $T_P S \cap T_Q S \neq \emptyset$ if and only if $P = Q$.

2. The secant variety in the Grassmannian

In this Section we want to present a strategy to determine the degree of the secant variety of a smooth surface. Soon after we will use this method when $S$ is a generic projective $K3$ surface.

Let $S$ be a smooth projective surface, embedded in $\mathbb{P}^n$ by means of the very ample divisor $h$: call $d = h^2$ the degree of $S$ in $\mathbb{P}^n$. Through all the Section we assume that $n \geq 5$ and that $h$ is 2-very ample; as a consequence we have $\text{Hilb}^2 S \simeq \Sigma(S) \subseteq G(1, n)$. We want to determine the class of $\Sigma(S)$ in the cohomology ring of $G(1, n)$, so we begin this Section describing the Schubert cycles we are interested in the Grassmannian (our main reference is [10 §1.5]). As a matter of notation, we will denote by $A_k$ a generic $k$-dimensional linear subspace of $\mathbb{P}^n$.

2.1. The variety of secant lines in the Grassmannian. The dimension of $\text{Grass}(2, n + 1)$ is $2(n - 1)$, and the Plücker map embeds it in $\mathbb{P}^{N-1}$ with $N = \binom{n+1}{2} = \frac{n(n+1)}{2}$. The Plücker embedding is induced by the linear system associated to the Schubert cycle $\sigma_{1,0}$, which represents (i.e. its points are in bijection with) all the lines in $\mathbb{P}^n$ which meet a fixed $A_{n-2}$. This means that

$$P = \varphi_{\sigma_{1,0}} : G(1, n) \longrightarrow \mathbb{P}^{\binom{n+2}{2} - 1}, \quad P^*(\mathcal{H}) = \sigma_{1,0},$$
where $P$ is the Plücker embedding and $\mathcal{H}$ is a hyperplane in $\mathbb{P}^n$.

There are three Schubert cycles of codimension 4 in $G(1, n)$:

1. $\sigma_{4,0}$, which represents all the lines in $\mathbb{P}^n$ meeting a fixed $\Lambda_{n-5}$;
2. $\sigma_{3,1}$, which represents all the lines in $\mathbb{P}^n$ contained in a fixed $\Lambda_{n-1}$ and meeting a fixed $\Lambda_{n-4} \subseteq \Lambda_{n-1}$;
3. $\sigma_{2,2}$, which represents all the lines in $\mathbb{P}^n$ contained in a fixed $\Lambda_{n-2}$.

There are three Schubert cycles of dimension 4, i.e. of codimension $(n - 3)$, in $G(1, n)$:

1. $\sigma_{n-1,n-5}$, which represents all the lines in $\mathbb{P}^n$ contained in a fixed $\Lambda_5$ through a fixed point $P \in \Lambda_5$;
2. $\sigma_{n-2,n-4}$, which represents all the lines in $\mathbb{P}^n$ contained in a fixed $\Lambda_4$ and meeting a fixed line $\Lambda_1 \subseteq \Lambda_4$;
3. $\sigma_{n-3,n-3}$, which represents all the lines in $\mathbb{P}^n$ contained in a fixed $\Lambda_3$.

**Remark 2.1.** We have then that $\sigma_{n-3,n-3} \simeq G(1, 3)$, that $\sigma_{n-2,n-4}$ is isomorphic to the Schubert cycle $\sigma_{2,0} \subseteq G(1, 4)$ and finally that $\sigma_{n-1,n-5} \simeq \mathbb{P}^4$.

These Schubert cycles intersect according to Table 1.

| $\cdot$ | $\sigma_{4,0}$ | $\sigma_{3,1}$ | $\sigma_{2,2}$ |
|--------|----------------|----------------|----------------|
| $\sigma_{n-1,n-5}$ | 1 | 0 | 0 |
| $\sigma_{n-2,n-4}$ | 0 | 1 | 0 |
| $\sigma_{n-3,n-3}$ | 0 | 0 | 1 |

Table 1. Intersection table of the Schubert cycles of dimension and of codimension 4 in $G(1, n)$.

Since $h$ is 2-very ample on $S$, we can write

$$\Sigma(S) = \alpha \sigma_{n-1,n-5} + \beta \sigma_{n-2,n-4} + \gamma \sigma_{n-3,n-3},$$

and we want to compute the coefficients $\alpha$, $\beta$ and $\gamma$. Thanks to Table 1 we have that

$$\alpha = \Sigma(S) \cdot \sigma_{4,0}, \quad \beta = \Sigma(S) \cdot \sigma_{3,1}, \quad \gamma = \Sigma(S) \cdot \sigma_{2,2}.$$ 

By the 2-very-ampleness of $h$ (and Remark 1.7), a line cutting $S$ in 2 points can not have further intersections with $S$, and so we can characterize $\Sigma(S) \subseteq G(1, n)$ as

$$\Sigma(S) = \{l \in G(1, n) \mid \text{length}(l \cap S) = 2\}.$$ (7)

### 2.2. Enumerative meaning of the coefficients

In this Section we want to give an enumerative meaning of the coefficients $\alpha$, $\beta$ and $\gamma$ introduced in (6). We will compute two of them in general in terms of numerical properties of the surface, and describe how to determine the last.

We start by determining $\gamma$. Since $\gamma = \Sigma(S) \cdot \sigma_{2,2}$, by (7) it coincides with the number of lines in $\mathbb{P}^n$ which are contained in a given $\Lambda_{n-2}$ and meet $S$ in 2 points. Now, a generic $\Lambda_{n-2}$ cuts $d = h^2$ distinct points on $S$, and so we have at most $\binom{d}{2}$ lines. Since $h$ is 2-very-ample, no three of those points lie on the same line (cf. Remark 1.7), hence this gives a proof of the following Lemma.
Lemma 2.2. The value of the coefficient $\gamma$ in (6) is

$$\gamma = \frac{1}{2} h^2 (h^2 - 1).$$

Now we determine $\beta$. Since $\beta = \Sigma(S) \cdot \sigma_{3,1}$, by (4) it coincides with the number of lines in $\mathbb{P}^n$ which are contained in a given $\Lambda_{n-1}$, meet a given $\Lambda_{n-4} \subseteq \Lambda_{n-1}$ and cut $S$ in 2 points.

Lemma 2.3. Let $S \subseteq \mathbb{P}^n$ be a surface embedded by a 2-very ample divisor $h$. Let $C = S \cap \Lambda_{n-1}$ be an irreducible smooth hyperplane section of $S$. The coefficient $\beta$ in (6) coincides with the degree of the secant variety of $C$ in $\Lambda_{n-1}$, and its value is

$$\beta = \frac{1}{2} (h^2 (h^2 - 4) - h \cdot K_S).$$

Proof. For a generic choice of $\Lambda_{n-1} \subseteq \mathbb{P}^n$, the curve $C = S \cap \Lambda_{n-1}$ is an irreducible and smooth curve of genus

$$g(C) = 1 + \frac{1}{2} h \cdot (K_S + h),$$

where $K_S$ is a canonical divisor on $S$, and its degree in $\Lambda_{n-1}$ is $d = h^2$.

A line $l$ contained in $\Lambda_{n-1}$ and meeting $S$ in a length 2 subscheme can be of the following types:

1. $l$ is a secant of $C$;
2. $l$ is a tangent of $C$;
3. $l$ is tangent to $S$ in a point $P$ of $C$, but $l$ is not the tangent of $C$ at $P$.

Assume we are in the third case: then $\Lambda_{n-1}$ contains $l$ and the tangent line $T_P C$ to $C$ at $P$, since $C$ is smooth. Then $\Lambda_{n-1}$ contains the linear subspace generated by these two lines in $\Lambda_{n-1}$, i.e. $T_P S$, and this implies that $C$ is singular at $P$, which is a contradiction. Hence a line $l \subseteq \Lambda_{n-1}$ meeting $S$ in 2 points is of the first or second kind. Then $l$ is contained in the secant variety of $C$ in $\Lambda_{n-1}$.

The secant variety $\text{Sec} C$ is a threefold contained in $\Lambda_{n-1}$, and so $\beta$ can as well be computed as the degree of $\text{Sec} C$ in $\Lambda_{n-1}$. By [8, Thm. 4.3] or [1, Thm. 3.5], the degree of the secant variety to a smooth curve of genus $g$ and degree $d$ is

$$\deg \text{Sec} C = \binom{d-1}{2} - g,$$

which in our case says that

$$\beta = \frac{1}{2} (h^2 (h^2 - 4) - h \cdot K_S).$$

Remark 2.4. The proof of Lemma 2.3 can be adapted to show that the coefficient $\alpha$ in (6) is the degree of the secant variety to $S$ in $\mathbb{P}^n$.

The degree of $\Sigma(S)$ in $\mathbb{P}^{(n+2)(n-1)/2}$ under the Plücker embedding $P$ can easily be computed in terms of the coefficients $\alpha$, $\beta$ and $\gamma$. As the restriction of the hyperplane class to the Grassmannian is the Schubert cycle $\sigma_{1,0}$, this degree coincides with

$$\deg \sigma_{1,0}^4 \cdot \Sigma(S).$$

Using Pieri formula, we can compute that $\sigma_{1,0}^4 = \sigma_{4,0} + 3\sigma_{3,1} + 2\sigma_{2,2}$, and

$$\deg \sigma_{1,0}^4 \cdot \Sigma(S) = \alpha + 3\beta + 2\gamma.$$ (8)
To determine the value of $\alpha$ is then the same as to determine the degree of $\sigma^4_{1,0} \cdot \Sigma(S)$.

3. The tangent variety

In the same spirit we define the secant variety to a surface $S$ embedded in $\mathbb{P}^n$, we can define the tangent variety to $S$ in the following way. We begin with the map

$$
T: \quad S \quad \longrightarrow \quad G(2,n) \\
\quad P \quad \longmapsto \quad T_P S,
$$

and the incidence relation

$$
I' = \{(x,\pi) \in \mathbb{P}^n \times G(2,n) \mid x \in \pi\} \subseteq \mathbb{P}^n \times G(2,n).
$$

We denote $T(S) = \text{im} T \subseteq G(2,n)$, and then define

$$
\text{Tan} S = pr_1(pr_2^{-1}(T(S))) \subseteq \mathbb{P}^n.
$$

Observe that we can describe $\text{Tan} S$ as well as the variety given by the union of all the (embedded) tangent planes to $S$:

$$
\text{Tan} S = \{x \in \mathbb{P}^n \mid x \in T_P S \text{ for some } P \in S\}.
$$

Remark 3.1. Let $S$ be a surface embedded in $\mathbb{P}^n$ by means of the very ample divisor $h$. As a consequence of Remark 1.7, we have that if $h$ is 2-very ample then $T$ is injective, hence it is an embedding of $S$ in $G(2,n)$. In particular, we have $S \simeq T(S)$.

3.1. The variety of tangent planes in the Grassmannian. Assume from now on that the embedding of $S$ in $\mathbb{P}^n$ is induced by a 2-very ample divisor $h$. We want to describe $T(S)$ in terms of the 2-dimensional Schubert cycles of $G(2,n) = \text{Grass}(3, n+1)$.

We give a brief description of the Schubert cycles involved. First of all, in codimension 1 we have the cycle $\sigma_{1,0,0}$, representing the set of all the planes in $\mathbb{P}^n$ intersecting a given $\Lambda_{n-3}$. The map induced by $\sigma_{1,0,0}$ is the Plücker embedding in $\mathbb{P}^{N-1}$ (with $N = \binom{n+1}{3}$).

In codimension 2 we have the cycles

1. $\sigma_{2,0,0}$ parametrizing all the planes of $\mathbb{P}^n$ which intersect a given $\Lambda_{n-4}$;
2. $\sigma_{1,1,0}$ parametrizing all the planes of $\mathbb{P}^n$ which intersect a given $\Lambda_{n-2}$ in (at least) a line;

and finally, in dimension 2 we have the cycles

1. $\sigma_{n-2,n-2,n-4}$ parametrizing all the planes of $\mathbb{P}^n$ which are contained in a given $\Lambda_4$ and contain a given line $\Lambda_1 \subseteq \Lambda_4$;
2. $\sigma_{n-2,n-3,n-3}$ parametrizing all the planes of $\mathbb{P}^n$ which are contained in a given $\Lambda_3$ and pass through a given point $P \in \Lambda_3$.

Remark 3.2. Observe that $\sigma_{n-2,n-2,n-4}$ is isomorphic to the Schubert cycle $\sigma_{2,2,0} \subseteq G(2,4)$, and that $\sigma_{n-2,n-3,n-3} \simeq \mathbb{P}^2$.

The intersection table among these cycles is given in Table 2.

We can write

$$
T(S) = \alpha' \sigma_{n-2,n-2,n-4} + \beta' \sigma_{n-2,n-3,n-3},
$$

and we want now to determine the values of $\alpha'$ and $\beta'$. 


3.2. Enumerative meaning of the coefficients. As we have the description

\[ T(S) = \{ \pi \in G(2, n) \mid \pi \text{ is tangent to } S \} , \]  

we can give an enumerative meaning to the coefficients \( a' \) and \( \beta' \) in (9).

We start with \( \beta' = T(S) \cdot \sigma_{1,1,0} \), which by (10) corresponds to the number of tangent planes to \( S \) intersecting an \((n - 2)\)-dimensional linear subspace in at least one line.

Let \( \Lambda_{n-2} = \Lambda_{n-1} \cap \Lambda'_{n-1} \), and call \( C = S \cap \Lambda_{n-1} \) and \( C' = S \cap \Lambda'_{n-1} \). For a generic choice of \( \Lambda_{n-1}, \Lambda'_{n-1} \) we have that \( C \) and \( C' \) are smooth curves meeting transversely in \( d = h^2 \) distinct points. Let \( P \) be one of such points: then the lines \( T_P S \cap \Lambda_{n-1} \) and \( T_P S \cap \Lambda'_{n-1} \) are distinct and meet only at \( P \). So \( T_P S \cap \Lambda_{n-1} \cap \Lambda'_{n-1} = \{ P \} \), and this means that no line contained in \( T_P S \) can be contained in \( \Lambda_{n-2} \). As a consequence, if we have a line contained in \( T_P S \cap \Lambda_{n-2} \), we can assume that \( P \in S \setminus \Lambda_{n-2} \).

**Lemma 3.3.** The point \( P \in S \setminus \Lambda_{n-2} \) is such that \( T_P S \cap \Lambda_{n-2} \) is a line if and only if there is a curve \( \Gamma \) in the pencil generated by \( C \) and \( C' \) which is singular at \( P \).

**Proof.** Let \( P \) be a point such that \( T_P S \cap \Lambda_{n-2} \) contains a line \( l \). Since \( P \notin \Lambda_{n-2} \), there exists a unique hyperplane \( \mathcal{H} \) in the pencil of hyperplanes through \( \Lambda_{n-2} \) passing through \( P \). Let \( \Gamma = S \cap \mathcal{H} \), then \( \Gamma \) is a curve in the pencil generated by \( C \) and \( C' \), and since \( l \subseteq \mathcal{H} \) and \( P \in \mathcal{H} \) we deduce that \( T_P S \subseteq \mathcal{H} \) which implies that \( \Gamma \) is singular at \( P \).

Viceversa, let \( \Gamma \) be a curve in the pencil generated by \( C \) and \( C' \) which is singular at \( P \). Then \( \Gamma \) is the intersection of \( S \) with a hyperplane \( \mathcal{H} \) in the pencil of hyperplanes through \( \Lambda_{n-2} \), and \( T_P S \subseteq \mathcal{H} \) since \( \Gamma \) is singular at \( P \). But then \( T_P S \cap \Lambda_{n-2} = T_P S \cap \mathcal{H} \cap \Lambda_{n-1} = T_P S \cap \Lambda_{n-1} \) is a line by Grassmann formula. \( \square \)

We can then conclude that

\[ \beta' = \sum_{C \in \text{Pencil in } |h|} \#(\text{Sing } C) , \]

which is a number we can determine.

**Lemma 3.4.** The value of \( \beta' \) in (9) is

\[ \beta' = \chi_{\text{top}}(S) + h \cdot (2K_S + 3h) . \]

**Proof.** The surface \( S \) is embedded in \( \mathbb{P}^n \) by means of the 2-very ample divisor \( h \), hence for a pencil of curves in \( |h| \) we have that the generic curve \( C \) is smooth of genus \( 1 + \frac{1}{2} h \cdot (K_S + h) \) and its topological Euler characteristic is

\[ \chi_{\text{top}}(C) = -h \cdot (K_S + h) . \]

The generic singular curve has only one node, and so its topological Euler characteristic is \(-h \cdot (K_S + h) + 1\). Finally, the base points of the generic pencil are \( d = h^2 \) distinct points. We blow up \( S \) in these points to find a surface \( \tilde{S} \) having a fibration

\[ \begin{array}{c|c|c} \sigma_{n-2,n-2,n-4} & 1 & 0 \\ \sigma_{n-2,n-3,n-3} & 0 & 1 \end{array} \]

**Table 2.** Intersection table of the Schubert cycles in \( G(2, n) \).
\( \hat{S} \to \mathbb{P}^1 \) induced by the pencil. In this setting \( \beta' \) corresponds to the number of singular fibres, i.e. to the degree of the discriminant locus of the fibration: we can compute this degree by means of topological methods. Call \( F \) the generic (smooth) fibre of the fibration, \( \Delta \subseteq \mathbb{P}^1 \) the discriminant locus and \( F_{\text{Sing}} \) the singular fibre over the points of \( \Delta \). Choosing the pencil generically, \( \Delta \) consists of \( \beta' \) distinct points and \( F_{\text{Sing}} \) has only one node. Since \( \hat{S} \) is obtained from the surface \( S \) after the blow up of \( d \) points, we have that \( \chi_{\text{top}}(\hat{S}) = \chi_{\text{top}}(S) + d \). But then:

\[
\chi_{\text{top}}(S) + d = \chi_{\text{top}}(\hat{S}) = \chi_{\text{top}}(\mathbb{P}^1 \setminus \Delta) \cdot \chi_{\text{top}}(F) + \chi_{\text{top}}(\Delta) \cdot \chi_{\text{top}}(F_{\text{Sing}}) = (2 - \beta')(-h \cdot (K_S + h)) + \beta'(-h \cdot (K_S + h) + 1),
\]

and this allows us to conclude that \( \beta' = \chi_{\text{top}}(S) + h^2 + 2h \cdot (K_S + h) \) hence to prove the Lemma. \( \square \)

Now we focus on \( \alpha' = \tau(S) \cdot \sigma_{2,0,0} \). By (11) its value corresponds to the number of tangent planes to \( S \) meeting a given \((n - 4)\)-dimensional linear subspace \( \Lambda_{n-4} \) of \( \mathbb{P}^n \). Such a \( \Lambda_{n-4} \) is the intersection of 4 hyperplanes, say \( \Lambda_{n-1}, \Lambda_{n-1}', \Lambda_{n-1}'', \Lambda_{n-1}''' \); it is not restrictive to assume that each of them cuts \( S \) in a smooth curve and that \( \Lambda_{n-1} \cap S = \emptyset \).

**Remark 3.5.** Under the assumption that \( C = S \cap \Lambda_{n-1} \) is smooth, we have that \( \Lambda_{n-1} \) intersects any tangent space \( T_P S \) in a line. In fact, by Grassmann formula we have that \( \dim(\Lambda_{n-1} \cap T_P S) \geq 1 \) and finally that \( \dim(\Lambda_{n-1} \cap T_P S) = 2 \) if and only if \( T_P S \subseteq \Lambda_{n-1} \) which happens if and only if \( C \) is singular at \( P \).

**Lemma 3.6.** Let \( S \subseteq \mathbb{P}^n \) be a surface embedded by a 3-very ample divisor \( h \). The number \( \alpha' \) in (9) is the degree of \( \text{Tan} S \) in \( \mathbb{P}^n \), and it coincide with the number of tangent lines to \( S \) which intersect a given \( \Lambda_{n-4} \subseteq \mathbb{P}^n \).

**Proof.** For \( Q \in S \), we will call \( l_Q = T_Q S \cap \Lambda_{n-1} \) (and analogously define \( l_Q' \) and so on). Let \( \{P\} = l_Q \cap l_Q' \), we have the following possibilities:

1. \( P = Q \) is one of the \( d \) points where \( \Lambda_{n-1} \cap \Lambda_{n-1}' \) meets \( S \).
2. \( Q \) is one of the finite number of points which determine a tangent space \( T_Q S \) where \( l_Q = l_Q' \). It is easy to see that this number is finite since each such \( Q \) determines a tangent plane meeting a \( \Lambda_{n-2} \) in at least a line, and we know that there are at most \( \beta' \) such planes.
3. \( l_Q \neq l_Q' \) and \( P \neq Q \). This is the generic situation, and we observe that in this case the point \( P \) determines uniquely three data: the tangent plane it belongs to (here is where the 3-very ampleness come into play, in view of Remark [17]), a tangent line in this tangent plane (the line through \( P \) and \( Q \)), and a length 2 non-reduced subscheme of \( S \) (obtained as the intersection of that tangent line with \( S \)).

This shows that the intersection of \( \Lambda_{n-1} \cap \Lambda_{n-1}' \) with \( \text{Tan} S \) is a surface, which has the property that all but a finite number of its points determine uniquely a tangent plane, a tangent line and a non-reduced length 2 subscheme. When we intersect again \( \text{Tan} S \) with \( \Lambda_{n-1}'' \) and \( \Lambda_{n-1}''' \) we get on this surface a finite number of distinct points (this number equals the degree the surface in \( \Lambda_{n-2} \)) and so we have that the number of tangent planes to \( S \) which intersect a given \( \Lambda_{n-4} \) is the same as the degree of \( \text{Tan} S \) as well as the number of tangent lines to \( S \) which intersect that given \( \Lambda_{n-4} \). \( \square \)
3.3. Reduction to the Grassmannian of lines. Lemma 3.6 allows us to compute the number $\alpha' = \mathcal{T}(S) \cdot \sigma_{2,0,0}$ in the Grassmannian $G(1,n)$ rather than in $G(2,n)$. In fact, the set of lines intersecting a given $\Lambda_{n-4}$ is described in $G(1,n)$ by the codimension 3 Schubert cycle $\sigma_{3,0}$, while the set of non-reduced subschemes is the image of the exceptional divisor $E$ of the Hilbert–Chow morphism under the embedding $\varphi : \text{Hilb}^2 S \hookrightarrow G(1,n)$. Hence, calling $X$ the image of $E$ in $G(1,n)$, we have

$$\alpha' = X \cdot \sigma_{3,0}. \quad (11)$$

In the Grassmannian $G(1,n)$ we have the codimension 3 cycles

(1) $\sigma_{3,0}$ representing all the lines in $\mathbb{P}^n$ meeting a given $\Lambda_{n-4}$.

(2) $\sigma_{2,1}$ representing all the lines in $\mathbb{P}^n$ contained in a $\Lambda_{n-1}$ and meeting a given $\Lambda_{n-3} \subseteq \Lambda_{n-1}$;

and the 3-dimensional cycles

(1) $\sigma_{n-1,n-4}$ representing all the lines in $\mathbb{P}^n$ contained in a given $\Lambda_{4}$ and through a fixed point $P \in \Lambda_{4}$,

(2) $\sigma_{n-2,n-3}$ representing all the lines in $\mathbb{P}^n$ contained in a given $\Lambda_{3}$ and meeting a fixed line $\Lambda_{1} \subseteq \Lambda_{3}$.

We can then write

$$X = \alpha' \sigma_{n-1,n-4} + \beta'' \sigma_{n-2,n-3}, \quad (12)$$

and we observe that the notation $\alpha'$ in this Section is coherent to the one used in \footnote{1} thanks to Lemma 3.6 and (11).

Lemma 3.7. Let $S \subseteq \mathbb{P}^n$ be a surface embedded by a 3-very ample divisor $h$, and let $C = S \cap \Lambda_{n-1}$ be an irreducible smooth hyperplane section of $S$. The coefficient $\beta''$ in (12) coincides with the degree of the tangent variety of $C$ in $\Lambda_{n-1}$, and its value is

$$\beta'' = h \cdot (K_S + 3h).$$

Proof. Observe that $\beta'' = X \cdot \sigma_{2,1}$ is the number of tangent lines to $S$ which are contained in a given $\Lambda_{n-1}$ and meet a fixed $\Lambda_{n-3} \subseteq \Lambda_{n-1}$. Choosing $\Lambda_{n-1}$ such that the curve $C = S \cap \Lambda_{n-1}$ is smooth, it is easy to see that $\beta''$ is the degree of the tangent variety to the curve $C$ in $\Lambda_{n-1}$: as in the proof of Lemma 2.3, the smoothness of $C$ implies that a tangent line to $S$ contained in $\Lambda_{n-1}$ must be a tangent line to $S \cap \Lambda_{n-1}$.

By [1, Prop. 3.3], the degree of the tangent variety of a smooth curve of degree $d$ and genus $g$ in $\mathbb{P}^n$ is $2d+2g-2$: in our case we have $d = h^2$ and $2g-2 = h \cdot (K_S+h)$, and so we get

$$\beta'' = 2h^2 + h \cdot (K_S + h). \quad \Box$$

Remark 3.8. We have a linear constraint among $\alpha'$ and $\beta''$: using Pieri formula we see that $\sigma_{1,0}^3 = \sigma_{3,0} + 2\sigma_{2,1}$, and so we see that $X \cdot \sigma_{1,0}^3 = \alpha' + 2\beta''$ is the degree of $X$ under the embedding induced by $i$. 
4. Embedding of $K3$'s in $\mathbb{P}^n$

From now on, we let $S$ be a $K3$ surface with $\text{Pic} \ S = \mathbb{Z} \cdot h$, where $h$ is a very ample divisor on $S$ and $h^2 = 2t$ for some $t \geq 2$. Then we have an embedding $\varphi_{|h|} : S \hookrightarrow \mathbb{P}^{t+1} = |h|^*$. In this Section we want to study the the $k$-very ampleness of $h$, and in particular to determine the integer $k$ such that $h$ is $k$-very ample but not $(k+1)$-very ample.

The main tool is the following result by Knutsen, which improves general result of Reider (see [14] and [3]) in the case of $K3$ surfaces.

**Theorem 4.1** ([12, Thm. 1.1]). Let $L$ be a nef and big divisor on a $K3$ surface and $k \geq 0$ an integer. Then the following are equivalent:

(i) $L$ is $k$-very ample;

(ii) $L^2 \geq 4k$ and there is no effective divisor $D$ satisfying

\[
\begin{cases}
2D^2 \leq LD \leq D^2 + k + 1 \leq 2k + 2 \\
2D^2 = LD \iff L \sim 2D \text{ and } L^2 \leq 4k + 4 \\
D^2 = k + 1 \iff L \sim 2D \text{ and } L^2 = 4k + 4.
\end{cases}
\]

In our situation it is easy to see that the generator $h$ of Pic $S$ is very ample for $t \geq 3$ (e.g. by using results in [15]), here we address the question of finding a bound for its $k$-very ampleness.

**Proposition 4.2.** The divisor $h$ is $\left[\frac{t}{2}\right]$-very ample, but not $(\left[\frac{t}{2}\right]+1)$-very ample.

**Proof.** If $h$ is $k$-very ample, then by Theorem 4.1 we conclude that

\[h^2 \geq 4k \leadsto 2t \geq 4k \leadsto k \leq \frac{t}{2}.\]

So $h$ is not $k$-very ample for $k \geq \left[\frac{t}{2}\right]+1$. To show that $h$ is $k$-very ample for $k \leq \left[\frac{t}{2}\right]$ we only have to show that there is no divisor $D = ah$, with $a > 0$, satisfying (13).

In particular, we will show that the inequality $2D^2 \leq 2k + 2$ is never satisfied. In fact, since $2k + 2 \leq t + 2$, from $2D^2 = 4ta^2 \leq 2k + 2 \leq t + 2$ we deduce that

\[a^2 \leq \frac{t + 2}{4t} = \frac{1}{4} + \frac{1}{2t} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2},\]

and so that $a = 0$. \hfill $\square$

5. Riemann–Roch and intersection form on $\text{Hilb}^2 K3$

Let $S$ be a $K3$ surface, then $\text{Hilb}^2 S$ is an irreducible holomorphic symplectic manifold. Since the foundational paper [2], this kind of manifolds have been intensively studied, and they are known to enjoy many interesting properties. One of the main tools for dealing with an irreducible holomorphic symplectic manifold $X$ is the Beauville–Bogomolov–Fujiki quadratic form $q_X$ defined on $H^2(X, \mathbb{Z})$: one of its properties is that there exists a positive rational constant $c_X$ such that for any class $D \in \text{NS}(X)$ we have the equality $D^\dim X = c_X q_X(\alpha)^\frac{\dim X}{2}$.

In particular, for $\text{Hilb}^2 S$ we have that for any $D \in \text{NS}(\text{Hilb}^2 S)$ the following holds:

\[D^4 = 3(D, D)^2,\]

where $(\_, \_)$ is the symmetric bilinear form associated to $q_X$.

It follows from O’Grady’s paper [13] the following explicit formulation for the Riemann–Roch theorem on $\text{Hilb}^2 S$ in terms of the Beauville–Bogomolov–Fujiki bilinear form.
Lemma 5.1. Let $D$ be any divisor on $\text{Hilb}^2 S$: then
\begin{equation}
\chi(\text{Hilb}^2 S, D) = \frac{1}{8}((D, D) + 4)((D, D) + 6).
\end{equation}

Proof. Since the symplectic form on $\text{Hilb}^2 S$ induces an isomorphism between the tangent and cotangent bundle, we deduce that the odd Chern classes of $\text{Hilb}^2 S$ vanish. So the Hirzebruch–Riemann–Roch specializes to
\begin{equation}
\chi(\text{Hilb}^2 S, D) = \int_{\text{Hilb}^2 S} \text{ch}(D) \cdot Td(\text{Hilb}^2 S) = 
\frac{1}{2\pi}c_1(D)^2 + \frac{1}{2\pi}c_2(\text{Hilb}^2 S) \cdot c_1(D)^2 + 
\frac{1}{12\pi}(3c_2(\text{Hilb}^2 S)^2 - c_4(\text{Hilb}^2 S)).
\end{equation}
The first addend is equal to $\frac{1}{8}(D, D)^2$ thanks to (14). The third addend equals $\chi(\text{Hilb}^2 S, \mathcal{O}_{\text{Hilb}^2 S})$ and so its value is 3. So we are left only to the computation of the second addend: combining [13, Formula 3.0.12] with [13, Formula 3.0.32] one easily gets that
\begin{equation}
c_2(\text{Hilb}^2 S) \cdot \alpha \cdot \beta = 30(\alpha, \beta), \quad \forall \alpha, \beta \in H^2(\text{Hilb}^2 S, \mathbb{Z}),
\end{equation}
and so $\frac{1}{30}c_2(\text{Hilb}^2 S) \cdot c_1(D)^2 = \frac{5}{4}(D, D)$. It follows that
\begin{equation}
\chi(\text{Hilb}^2 S, D) = \frac{1}{8}(D, D)^2 + \frac{5}{4}(D, D) + 3 = 
\frac{1}{8}((D, D) + 4)((D, D) + 6).
\end{equation}

We consider now the case where the $K_3$ surface $S$ is generic, with $\text{Pic} S = \mathbb{Z} \cdot h$ and $h^2 = 2t \geq 4$. On the Hilbert scheme $\text{Hilb}^2 S$ we have that the Néron–Severi group $\text{NS}(\text{Hilb}^2 S)$ is generated by two classes: the class $H$ induced by $h$ and the class $\delta$ whose double is the exceptional divisor $E$ of the Hilbert–Chow morphism (cf. [2, Prop. 6]). The Beauville–Bogomolov–Fujiki form is described in the basis $H, \delta$ by the Gram matrix
\begin{equation}
\begin{pmatrix}
2t & 0 \\
0 & -2
\end{pmatrix}.
\end{equation}

As an application, we can prove that for $t \geq 4$ the divisor $H - \delta$ is very-ample.

Proposition 5.2. Let $S$ be a generic $K_3$ surface, with $\text{Pic} S = \mathbb{Z} \cdot h$. If $h^2 \geq 8$, then the divisor $H - \delta$ on $\text{Hilb}^2 S$ is very ample.

Proof. For $t = \frac{1}{2}h^2 \geq 4$, we have by Proposition 4.2 that $h$ is 2-very ample, and so we get an embedding
\begin{equation}
\text{Hilb}^2 S \stackrel{\varphi_1}{\hookrightarrow} \text{Grass}(2, H^0(S, h)^*) \rightarrow \mathbb{P}^{N-1},
\end{equation}
where the first arrow is the immersion [11]. Since $h^0(S, h) = t + 2$, we have $N = \frac{(t+2)(t+1)}{2}$.

By [5] Prop. 3.1], we know that the divisor $H - \delta$ (which is the pull-back of the hyperplane divisor of $\mathbb{P}^{N-1}$ under the composite map) is ample. Hence Riemann–Roch [15] gives that
\begin{equation}
h^0(\text{Hilb}^2 S, H - \delta) = \frac{1}{2}(t+1)(t+2),
\end{equation}

We consider now the case where the $K_3$ surface $S$ is generic, with $\text{Pic} S = \mathbb{Z} \cdot h$ and $h^2 = 2t \geq 4$. On the Hilbert scheme $\text{Hilb}^2 S$ we have that the Néron–Severi group $\text{NS}(\text{Hilb}^2 S)$ is generated by two classes: the class $H$ induced by $h$ and the class $\delta$ whose double is the exceptional divisor $E$ of the Hilbert–Chow morphism (cf. [2, Prop. 6]). The Beauville–Bogomolov–Fujiki form is described in the basis $H, \delta$ by the Gram matrix
\begin{equation}
\begin{pmatrix}
2t & 0 \\
0 & -2
\end{pmatrix}.
\end{equation}

As an application, we can prove that for $t \geq 4$ the divisor $H - \delta$ is very-ample.
and so the composite map in the previous diagram is the morphism induced by the complete linear system $|H - \delta|$. This shows that $H - \delta$ is very-ample.

Now we want to determine the top intersection numbers $H^4$, $H^3\delta$, $H^2\delta^2$, $H\delta^3$ and $\delta^4$, which will be used in the following Section. As observed, since for any $D \in \text{NS}(\text{Hilb}^2 S)$ we have $D^4 = 3(D, D)^2$, we can easily compute that

$$H^4 = 12t^2, \quad \delta^4 = 12, \quad (H + \delta)^4 = 12(t - 1)^2,$$

$$(H - \delta)^4 = 12(t - 1)^2, \quad (H + 2\delta)^4 = 12(t - 4)^2,$$

and so expanding the powers of the binomials we can deduce that

$$H^4 = 12t^2, \quad H^3\delta = 0, \quad H^2\delta^2 = -4t, \quad H\delta^3 = 0, \quad \delta^4 = 12.$$

6. The degree of the secant and tangent variety of the generic $K3$ surface

In this Section we compute the degree of the secant and the tangent variety of a generic $K3$ surface $S$. Assume that $\text{Pic} S = \mathbb{Z} \cdot h$, with $h^2 = 2t$ and $t \geq 4$: by Proposition \ref{prop:k3} this means that $h$ is 2-very ample, and the image of the embedding of $S$ in $\mathbb{P}^{t+1}$ is a surface containing no lines.

We can then compute explicitly many of the coefficients we introduced before. In particular, thanks to Lemma \ref{lem:sigma} and Lemma \ref{lem:sigma4} we have that \ref{eq:sigma} reads as

$$\Sigma(S) = \alpha\sigma_{t,t-4} + (2(t - 2))\sigma_{t-1,t-3} + (t(2t - 1))\sigma_{t-2,t-2}. \quad (16)$$

We will now compute the value of the coefficient $\alpha = \deg \text{Sec} S$.

**Theorem 6.1.** Let $S$ be a $K3$ surface with an ample divisor $h$ such that $\text{Pic} S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 4$. Use $h$ to embed $S$ in $\mathbb{P}^{t+1}$. The degree of the secant variety to $S$ is

$$\deg \text{Sec} S = 2(t - 2)(t - 3).$$

**Proof.** We only need to compute the value of $\alpha$ in \ref{eq:sigma}. Thanks to \ref{eq:sigma4} we know that the embedded variety $\Sigma(S)$ has degree

$$\deg \Sigma(S) \cdot \sigma_{1,0}^4 = \alpha + 2t(5t - 7).$$

On the other hand, by Proposition \ref{prop:sigma4} $\deg^i \sigma_{1,0}^4 = (H - \delta)^4 = 12(t - 1)^2$ and so we can now compute the value of $\alpha$. \hfill \Box

As $\sigma_{1,0}$ is the Schubert cycle representing the lines intersecting a codimension 2 linear subspace, from the proof of the Theorem we deduce the following results.

**Corollary 6.2.** Let $S$ be a $K3$ surface with an ample divisor $h$ such that $\text{Pic} S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 4$. Use $h$ to embed $S$ in $\mathbb{P}^{t+1}$. Then there are $12(t - 1)^2$ lines meeting $S$ in 2 points and four $(t - 1)$-dimensional linear subspaces in general position.

**Corollary 6.3.** Let $S$ be a $K3$ surface with an ample divisor $h$ such that $\text{Pic} S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 4$. Then the degree of $\Sigma(S)$ under the Plücker embedding of $G(1, t + 1)$ is $12(t - 1)^2$.

Focus now on the tangent variety $\text{Tan} S$: by Lemma \ref{lem:tan} we have that \ref{eq:tan} reads as

$$X = \alpha'\sigma_{t,t-3} + (6t)\sigma_{t-1,t-2}. \quad (17)$$
Theorem 6.4. Let $S$ be a K3 surface with an ample divisor $h$ such that $\text{Pic} \, S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 6$. Use $h$ to embed $S$ in $\mathbb{P}^{t+1}$. The degree of the tangent variety to $S$ is \[ \deg \text{Tan} S = 12(t-2). \]

Proof. By Lemma 3.6 we only need to compute the value of $\alpha'$ in (17). By Remark 3.8 we have that $\alpha' + 12t$ is the degree of the embedded exceptional divisor $E$ of the Hilbert–Chow morphism under the natural inclusion (4). As $E = 2\delta$ and this embedding is induced by (the restriction to $E$ of) $\varphi_{H-\delta}$, we can compute this degree also as \[ E \cdot (H - \delta)^3 = 2\delta \cdot (H - \delta)^3 = 24(t-1). \]
But then $\alpha' + 12t = 24(t-1)$ leads to $\alpha' = 12(t-2)$. \hfill \Box

Combining this Theorem with Lemma 3.4 we have that (9) reads as \[ \mathcal{T}(S) = (12(t-2)) \sigma_{t-1,t-1,t-3} + (6(t+4)) \sigma_{t-1,t-2,t-2}. \]

Corollary 6.5. Let $S$ be a K3 surface with an ample divisor $h$ such that $\text{Pic} \, S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 6$. Use $h$ to embed $S$ in $\mathbb{P}^{t+1}$. Then there are 18t tangent planes to $S$ which intersect two $(t-2)$-dimensional linear subspaces in general position.

Proof. This number coincides with \[ \mathcal{T}(S) \cdot \sigma_{1,0,0} = \mathcal{T}(S) \cdot (\sigma_{2,0,0} + \sigma_{1,1,0}) = (12t - 24) + (24 + 6t) = 18t. \] \hfill \Box

Corollary 6.6. Let $S$ be a K3 surface with an ample divisor $h$ such that $\text{Pic} \, S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 6$. Use $h$ to embed $S$ in $\mathbb{P}^{t+1}$. Then the degree of $\mathcal{T}(S)$ under the Plücker embedding of $G(2, t+1)$ is 18t.

Corollary 6.7. Let $S$ be a K3 surface with an ample divisor $h$ such that $\text{Pic} \, S = \mathbb{Z} \cdot h$ and $h^2 = 2t$ with $t \geq 6$. Use $h$ to embed $S$ in $\mathbb{P}^{t+1}$. Then the degree of $E$ under the Plücker embedding of $G(1, t+1)$ (i.e. the degree of the variety parametrising the tangent lines to $S$) is $24(t-1)$.

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