Performance Analysis of Joint-Sparse Recovery from Multiple Measurement Vectors with Prior Information via Convex Optimization

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Abstract

We address the problem of compressed sensing with multiple measurement vectors associated with prior information in order to better reconstruct an original sparse signal. This problem is modeled via convex optimization with $\ell_2,1-\ell_2,1$ minimization. We establish bounds on the number of measurements required for successful recovery. Our bounds and geometrical interpretations reveal that if the prior information can decrease the statistical dimension and make it lower than that under the case without prior information, $\ell_2,1-\ell_2,1$ minimization improves the recovery performance dramatically. All our findings are further verified via simulations.

Index Terms—Convex optimization, Multiple measurement vectors, Sparsity, Statistical dimension

1. Introduction

1.1. Background and Problem Definition

Compressive sensing (CS) [1, 2, 3] of sparse signals in achieving simultaneous data acquisition and compression has been extensively studied in the past few years. In this paper, we focus on multiple measurement vectors (MMVs) that are sensing results with respect to observed signals. MMVs generally exhibit the applicability especially in the areas of wireless sensor networks and wearable sensors [4, 5, 6].

Let $S = [s_1, s_2, ..., s_l] \in \mathbb{R}^{n \times l}$ be the matrix of $l (>1)$ original signals to be sensed by a sensing matrix $\Phi \in \mathbb{R}^{m \times n}$ ($m < n$) and let the matrix of measurement vectors be $Y = [y_1, y_2, ..., y_l] \in \mathbb{R}^{m \times l}$, where $y_i = \Phi s_i$, $i = 1, 2, ..., l$. Suppose there exists an orthonormal basis $\Psi$ such that $s_i = \Psi x_i$ and $X_0 = [x_1, x_2, ..., x_l] \in \mathbb{R}^{n \times l}$ is $k$-joint sparse. In other words, all $x_i$’s share the common support. Given $A = \Phi \Psi$, recovery from MMVs can be efficiently solved via convex optimization as:

$$(\text{Mconvex}) \min_X f(X) \text{ s.t. } Y = AX,$$

where $f(\cdot)$ denotes a convex function. We call the problem (Mconvex) succeeds if it has a unique optimal solution and is ground truth $X_0$. In this paper, the convex function is chosen as $f(X) = \|X\|_{2,1}$ to enhance the joint-sparsity of $X$:

$$(\text{ML1}) \min_X \|X\|_{2,1} \text{ s.t. } Y = AX.$$

So far, there is very limited literature about MMVs with prior information via convex optimization. In fact, we can have some prior knowledge about the ground truth $X_0$ in, for example, the problem of distributed compressive video sensing (DCVS) [7]. In DCVS, we usually adopt higher/lower measurement rates to sample and transmit key/non-key frames at encoder, and then we treat these reconstructed key frames as the prior information for better recovery of the non-key frames at decoder. Mota et al. [8] first propose the analysis of single measurement vector (SMV) with prior information via convex optimization. They show that the performance can be improved provided good prior information can be available. In [9], we characterize when problem (ML1) succeeds and derive the phase transition of success rate inspired by the framework of conic geometry [10].

In this paper, we further extend the problem (ML1) to (ML1P) plus prior information as:

$$(\text{ML1P}) \min_X \|X\|_{2,1} + \lambda \|X - W\|_{2,1} \text{ s.t. } Y = AX,$$

where $W$ is prior information associated with ground truth $X_0$. The goal here is to provide theoretical but practical bound of the probability of successful recovery and analyze the relationship between prior information and performance.

1.2. Contributions of This Paper

We summarize the contributions of our works here.

- Based on conic geometry, the phase transition of success rate in (ML1P) is derived and is consistent with the empirical results. This study indeed provides the useful insights into how to solve the problem of MMVs with prior information.

- What prior information is “good” can be concluded by our theoretical analysis. For example, instead of giving the rough conclusion such as $\|X_0 - W\|_{2,1}$ being close to 0, we clearly show how the supports of $X - W$ and the signs of $X - W$ affect the performance.

1.3. Notations

For a matrix $H$, we denote its transpose by $H^T$; its $i^{th}$ row by $h_i$; its $j^{th}$ column by $h_j$; and the $i^{th}$ entry of $j^{th}$ column by
which means that \( \text{lem} \) (\( M_{\text{convex}} \)).

is also feasible. If \( (M_{\text{convex}}) \), for any matrix \( A \), and let \( \{0_{n \times 1}\} \) denote the \( \ell_p \)-norm and Frobenius norm, respectively. The \( \ell_p,q \)-norm of a matrix is defined as \( \|X\|_{p,q} = \|\|x^i\|_p\|_q \). The null space of matrix \( A \in \mathbb{R}^{m \times n} \) is defined as \( \text{null}(A,l) = \{ Z \in \mathbb{R}^{n \times l} : AZ = 0_{m \times l} \} \). Let \( \mathbb{E} \) denote the expected value and let \( \mathbb{B} = \{ x : \|x\|_2 \leq 1, x \in \mathbb{R}^n \} \) denote closed unit ball. The dot product of two matrices is \( \langle X,Y \rangle = \text{tr}(X^TY) \).

2. CONIC GEOMETRY

We briefly introduce how a convex function can be specified in terms of conic geometry to make this paper self-contained. First, we introduce a cone and measure its size in a sense of statistical dimension. Then, they are connected with optimality condition for the MMVs recovery problem.

Definition 2.1. (Descent cone [10])
The descent cone \( D(f,x) \) of a function \( f : \mathbb{R}^n \to \mathbb{R} \) at a point \( x \in \mathbb{R}^n \), defined as:

\[
D(f,x) := \bigcup_{\tau > 0} \{ u \in \mathbb{R}^n : f(x + \tau u) \leq f(x) \},
\]

is the conical hull of the perturbations that do not increase \( f \) near \( x \).

By the definition of descent cone, the necessary and sufficient condition of the success of problem (ML1) is described and proved in our earlier work [9]. But in this paper, the main problem we are studying is not related to a norm function, so we need to modify the proof slightly to fit the problem (Mconvex) with general convex function as follows.

Lemma 2.2. (Optimality condition for MMVs recovery with general convex function)
The matrix \( X_0 \) is the unique optimal solution to problem (Mconvex) if and only if \( D(f,X_0) \cap \text{null}(A,l) = \{0_{n \times 1}\} \).

Proof: Assume \( X_0 \) is the unique optimal solution to problem (Mconvex). Given a matrix \( Z \in D(f,X_0) \cap \text{null}(A,l) \), we know that \( X_0 + Z \) is a feasible point of problem (Mconvex) and \( f(X_0 + Z) \leq f(X_0) \), which implies that \( X_0 + Z \) is an optimal solution to problem (Mconvex). According to the uniqueness of optimal solution of problem (Mconvex), we have \( E = 0 \), and thus \( D(f,X_0) \cap \text{null}(A,l) = \{0_{n \times 1}\} \).

Conversely, suppose \( D(f,X_0) \cap \text{null}(A,l) = \{0_{n \times 1}\} \). Since we know that \( X_0 \) is a feasible solution of problem (Mconvex), for any matrix \( Z \in \text{null}(A,l) \setminus \{0_{n \times 1}\} \), \( X_0 + Z \) is also feasible. If \( f(X_0 + Z) \leq f(X_0) \), then we have \( Z \in D(f,X_0) \cap \text{null}(A,l) \setminus \{0_{n \times 1}\} = 0 \), which is impossible. Therefore

\[
f(X_0 + Z) > f(X_0) \quad \text{for all } Z \in \text{null}(A,l) \setminus \{0_{n \times 1}\},
\]

which means that \( X_0 \) is the unique optimal solution to problem (Mconvex).

Since linear subspace is also a cone, Lemma 2.2 connects the optimal conditions to the relation that the intersection between the descent cone at \( X_0 \) and matrix null space is singleton (i.e., problem (Mconvex) succeeds).

For a random sensing matrix \( A \), the probability of success for problem (Mconvex) can be related to the “sizes” of two cones in Lemma 2.2. Unfortunately, since a cone may be not linear, there’s no a standard definition to describe the size of a cone. Amelunxen et al. [10] give a way to measure the size of a cone, as described in the following.

Definition 2.3. (Statistical Dimension [10])
The statistical dimension (S.D.) \( \delta(C) \) of a closed convex cone \( C \subset \mathbb{R}^n \) is defined as:

\[
\delta(C) := \mathbb{E} \left[ \left\| \prod (g,C) \right\|_2^2 \right],
\]

where \( g \in \mathbb{R}^n \) is a standard normal vector and \( \prod (\cdot,C) \), denoting the Euclidean projection onto \( C \), is defined as:

\[
\prod (x,C) := \arg \min \left\{ \|x - y\|_2 : y \in C \right\}.
\]

According to the definition of S.D. of a cone, Amelunxen et al. [10] derive the probability that two cones with a random rotation are separated as follows.

Theorem 2.4. (Approximate kinematic formula [10])
Fix a tolerance \( \eta \in (0,1) \). Suppose that \( C_1, C_2 \subset \mathbb{R}^N \) are closed convex cones, but one of them is not a subspace. Draw an orthogonal matrix \( Q \in \mathbb{R}^{n \times n} \) uniformly at random. Then

\[
\delta(C_1) + \delta(C_2) \leq n - a_\eta \sqrt{n} \Rightarrow \mathbb{P}\{C_1 \cap QC_2 = \{0\}\} \geq 1 - \eta,
\]

\[
\delta(C_1) + \delta(C_2) \geq n + 2a_\eta \sqrt{n} \Rightarrow \mathbb{P}\{C_1 \cap QC_2 = \{0\}\} \leq \eta.
\]

The quantity \( a_\eta := 8 \sqrt{\log(4/\eta)} \).

In order to satisfy the requirement of Theorem 2.4, both \( \Phi \) and \( \Psi \) can be easily selected such that \( A = \Phi \Psi \) is a Gaussian random matrix [11]. In compressive sensing, \( \Phi \) and \( \Psi \) are conventionally used to set as a Gaussian random matrix and orthonormal basis, respectively, so that \( A = \Phi \Psi \) is also a Gaussian random matrix [11]. Let \( C_1 = D(f,X_0) \) and let \( QC_2 = \text{null}(A,l) \) with a random matrix \( A = \Phi \Psi [11] \). The probability of intersection given in Theorem 2.4 can be reformulated as the probability of existence of unique optimal solution by Lemma 2.2, i.e.,

\[
\mathbb{P}\{C_1 \cap QC_2 = \{0\}\} = \mathbb{P}(D(f,X_0) \cap \text{null}(A,l) = \{0_{n \times 1}\}) = \mathbb{P}(M_{\text{convex}} \text{ succeeds}).
\]

Since the nullity of \( A \) is \( n - m \) almost surely, the dimension of \( C_2 \) is \( \delta(\text{null}(A,l)) = \dim(\text{null}(A,l)) = (n - m)||l. \) Then, the probability that (Mconvex) succeeds can be estimated by Theorem 2.5, which was derived in our earlier work [9].
Theorem 2.5. (Phase transitions in MMVs recovery) Fix a tolerance $\eta \in (0, 1)$. Let $X_0 \in \mathbb{R}^{n \times l}$ be a fixed matrix. Suppose $A \in \mathbb{R}^{m \times n}$ has independent standard normal entries and $Y = AX_0$. Then

$$m \geq \frac{\delta(D(f, X_0))}{\sqrt{n}} \Rightarrow \mathbb{P}(D_{\text{convex}} \text{ succeeds}) \geq 1 - \eta,$$

$m \leq \frac{\delta(D(f, X_0))}{\sqrt{n}} \Rightarrow \mathbb{P}(D_{\text{convex}} \text{ succeeds}) \leq \eta,$

where the quantity $a_\eta := 8 \sqrt{\log(4/\eta)}$.

3. ESTIMATION OF S.D. IN (ML1P)

In Theorem 2.5, $\delta(D(f, X_0))$ plays an important role to estimate the probability that (Mconvex) succeeds. However, calculating the exact value of S.D. of a cone is still open. In this section, we provide the bounds of S.D. of descent cone at the point $X_0$ associated with convex function $\zeta_W(X) = \|X\|_{2,1} + \lambda \|X - W\|_{2,1}$ in problem (ML1P), where function $\zeta_W$ is called $\ell_{2,1}$-norm with prior information.

Theorem 3.1. (Error bound of S.D. in (ML1P))

Let $\partial \zeta_W$ be subdifferential of $\zeta_W$. Suppose $\partial \zeta_W(X)$ is nonempty and compact, and does not contain the origin. Then, we have

$$\inf_{\tau \geq 0} F(\tau) - \xi(X) \leq \delta(D(\zeta_W, X)) \leq \inf_{\tau \geq 0} F(\tau),$$

where $\xi(X) = \frac{2\|X\|_p \sup \|S\|_p : S \in \partial \zeta_W(X)}{(\partial \zeta_W(X), X)}$.

$$F(\tau) := F(\tau, X) = \mathbb{E} \left[ \text{dist}^2(G, \tau \cdot \partial \zeta_W(X)) \right] \text{for } \tau \geq 0,$$

and $G \in \mathbb{R}^{n \times l}$ is a Gaussian random matrix.

Moreover, for $k$-joint sparse matrix $X_0 \in \mathbb{R}^{n \times l}$, we have

$$\inf_{\tau \geq 0} F(\tau) - 2(1 + \lambda) \sqrt{n} \left( \frac{1}{1 - \lambda} \right) \leq \delta(D(\zeta_W, X_0)) \leq \inf_{\tau \geq 0} F(\tau).$$

Proof.

For any given matrix $X$, we have

$$\delta(D(\zeta_W, X)) = \mathbb{E} \left[ \text{dist}^2(G, D(\zeta_W, X)) \right],$$

where the distance function is $\text{dist}(G, C^0) = \| \prod_C (G, C) \|_F$ for a fixed cone $C$. According to Corollary 23.7.1 in [12], the polar cone can be rewrite as $D(\zeta_W, X) = \bigcup_{\tau \geq 0} \tau \cdot \partial \zeta_W(X)$, thus

$$\mathbb{E} \left[ \text{dist}^2(G, D(\zeta_W, X)) \right] = \mathbb{E} \left[ \inf_{\tau \geq 0} F_G(\tau) \right],$$

(1)

where

$$F_G(\tau) := F_G(\tau, X) = \text{dist}^2(U, \tau \partial \zeta_W(X)) \text{ for } \tau \geq 0.$$

For the upper bound of $\delta(D(\zeta_W, X))$, since

$$\mathbb{E} \left[ \inf_{\tau \geq 0} F_G(\tau) \right] \leq \inf_{\tau \geq 0} \mathbb{E} [F_G(\tau)] = \inf_{\tau \geq 0} F(\tau),$$

the result follows.

Next we aim to estimate the lower bound of $\delta(D(\zeta_W, X))$. By the fact that $F_G(\tau)$ is convex on $\tau \geq 0$ and continuous differentiable on $\tau > 0$ (Lemma C.1 in [10]), we have

$$F_G(\tau) \geq F_G(\tau_0) + F'_G(\tau_0)(\tau - \tau_0),$$

(2)

for any $\tau$ and $\tau_0$.

Let $\tau^*$ and $\tau^*_G$ be the minimizer of $F(\tau)$ and $F_G(\tau)$, respectively. Since $F(\tau)$ is strictly convex on $\tau \geq 0$ and differentiable on $\tau > 0$ (Lemma C.2 in [10]) the minimizer $\tau^*$ of $F(\tau)$ is unique, that is,

$$\tau^* = \arg\min_{\tau \geq 0} F(\tau).$$

Then, Eq. (2) can be written as

$$F_G(\tau^*_G) \geq F_G(\tau^*) + F'_G(\tau^*)(\tau^*_G - \tau^*).$$

($F'_G(\tau^*)$ is the right derivative provided $\tau^* > 0$). Then the expected value of $\inf_{\tau \geq 0} F_G(\tau)$ in Eq. (1) corresponding to $G$ becomes

$$\begin{align*}
\mathbb{E} \left[ \inf_{\tau \geq 0} F_G(\tau) \right] &= \mathbb{E} [F_G(\tau^*_G)] \\
&\geq \mathbb{E} [F_G(\tau^*)] + \mathbb{E} [F'_G(\tau^*)(\tau^*_G - \tau^*)] \\
&= F(\tau^*) + \mathbb{E} \left[ (\tau^*_G - \tau^*) \cdot (F'_G(\tau^*) - \mathbb{E} [F'_G(\tau^*)]) \right] \\
&\geq \mathbb{E} \left[ (\tau^*_G - \tau^*) \cdot (F'_G(\tau^*) - \mathbb{E} [F'_G(\tau^*)]) \right].
\end{align*}$$

We can see that $\mathbb{E}[\tau^*_G - \tau^*] \cdot F'(\tau^*) \geq 0$ since $F'(\tau^*) > 0$ if $\tau^* > 0$ and $F'(\tau^*) \geq 0$ if $\tau^* = 0$ (because $\tau^*$ minimize $F(\tau)$). Therefore,

$$\delta(D(\zeta_W, X)) \geq \inf_{\tau \geq 0} F(\tau) - (\mathbb{E}[\tau^*_G] \cdot \mathbb{E}[F'_G(\tau^*)])^{1/2} + \mathbb{E}[\tau^*_G - \tau^*] \cdot F'(\tau^*).$$

(3)

Next, to compute the variance of $\tau^*_G$, we need to devise a consistent method for selecting a minimizer $\tau_U$ of $F_U$. Introduce the closed convex cone $C := \text{cone}(\partial \zeta_W(X))$, and notice that

$$\inf_{\tau \geq 0} F_U(\tau) = \inf_{\tau \geq 0} \text{dist}^2(U, \tau \cdot \partial \zeta_W(X)) = \text{dist}^2(U, C).$$

In other words, the minimum distance to one of the sets $\tau \partial \zeta_W(X)$ is attained at the point $\prod_C (U) := \arg\min_{U \in C} \|U - C\|_F : C \in C$. As such, it is natural to pick a minimizer $\tau_U$ of $F_U$ according to the rule

$$\tau_U := \inf \{ \tau \geq 0 : \prod_C (U) \in \tau \partial \zeta_W(X) \} = \frac{\prod_C (U), X}{(\partial \zeta_W(X), X)}.$$
In light of Eq. (4), we have
\begin{align*}
|\tau U - \tau V| &= \frac{1}{\partial \zeta W(X)} \langle \Pi c(U) - \Pi c(V), X \rangle \\
&\leq \frac{1}{\partial \zeta W(X)} \| \Pi c(U) - \Pi c(V) \|_F \\
&\leq \frac{1}{\partial \zeta W(X)} \| U - V \|_F.
\end{align*}

We have used the fact (B.3) in [10] that the projection onto a closed convex set is nonexpansive. By the relation between \( \text{Var}(\tau \gamma) \) and Lipschitz constant \( \frac{1}{\partial \zeta W(X)} \), we have
\begin{equation}
\text{Var}(\tau \gamma)^{1/2} \leq \frac{\| X \|_F}{\partial \zeta W(X)}, \quad (5)
\end{equation}

By the lemma (C.1) in [10],[
\begin{equation}
(\text{Var}(F G(\tau \gamma)))^{1/2} \leq 2 \sup_{S \in \partial \zeta W(X)} \| S \|_F. \quad (6)
\end{equation}

Substitute \( X_0 \) into Eqs. (3), (5), and (6). In Eq. (5), \( \partial \zeta W(X_0) \) can be reformulated by cosine function to \( \| X_0 \|_{2,1} + \lambda \sum_{i=1}^n \| x_i \|_2 \cos(\angle Ox_i w_i) \). It is obvious that the lower bound of \( \partial \zeta W(X_0) \) is \( 1 - \lambda \| X_0 \|_{2,1} \). In Eq. (6), the right hand side \( \sup_{S \in \partial \zeta W(X_0)} \| S \|_F \) will be equal to \( (1 + \lambda) \sqrt{n} \) because the rows of \( \partial X_0 \|_{2,1} \) and \( \partial X_0 - W \|_{2,1} \) are already normalized. We have
\begin{align*}
\delta(D(\zeta W, X_0)) &\geq \inf_{\tau \geq 0} F(\tau) - \frac{2}{\partial \zeta W(X_0)} \| X_0 \|_F \sup_{S \in \partial \zeta W(X_0)} \| S \|_F \\
&\geq \inf_{\tau \geq 0} F(\tau) - \frac{2}{(1 - \lambda) \sqrt{n}} \| X_0 \|_{2,1} \\
&\geq \inf_{\tau \geq 0} F(\tau) - \frac{1}{(1 - \lambda) \sqrt{n}} - \frac{1}{\sqrt{1 + \lambda}}.
\end{align*}

where the last inequality depends on \( -\frac{1}{\partial \zeta W(X_0)} \). We complete the proof.

To calculate the function \( F(\tau) \) in Theorem 3.1, we first compute the subdifferential of both \( \ell_{2,1} \)-norm and \( \zeta W(X) \).

**Lemma 3.2.** (Subdifferential of \( \ell_{2,1} \)-norm [13])

For any \( X, U \in \mathbb{R}^{n \times 1} \), we have
\begin{equation*}
U \in \partial \| X \|_{2,1} \iff u^i \in \partial \| x^i \|_2, \quad 1 \leq i \leq n,
\end{equation*}

where
\begin{equation*}
u^i \in \partial \| x^i \|_2 \iff \begin{cases} u^i = x^i/\| x^i \|_2 & \text{if } x^i \neq 0, \\
\| u^i \|_2 \leq 1 & \text{if } x^i = 0.
\end{cases}
\end{equation*}

The subgradient of \( \ell_{2,1} \)-norm at \( X \) is calculated by row-by-row subgradient of Euclidean norm \( \| x^i \|_2 \), whereas \( \partial \| x^i \|_2 \) consists of the gradient whenever \( x^i \neq 0 \), and \( \partial \| x^i \|_2 = B \) if \( x^i = 0 \). That is, the computation of subgradient of \( \ell_{2,1} \)-norm at \( X \) depends on if a row of \( X \) is zero or not.

Moreover, since the subdifferential of \( \zeta W(X) \) can be calculated separately as \( \partial (\| X \|_{2,1} + \lambda \| X - W \|_{2,1}) = \partial \| X \|_{2,1} + \lambda \partial \| X - W \|_{2,1} \), we calculate the subgradient of \( \zeta W(X) \) according to the indices sets of zero and nonzero rows with respect to \( X \) and \( X - W \). We separate the domain of \( \zeta W(X) \) into four cases, where \( E_1 = \Lambda X \cap \Lambda X - W, E_2 = \Lambda X \cap \Lambda X - W, E_3 = \Lambda X \cap \Lambda X - W, \) and \( E_4 = \Lambda X \cap \Lambda X - W \). Then, we have the following lemma.

**Lemma 3.3.** (Subdifferential of \( \ell_{2,1} \)-norm with prior information)

For any \( X, U \in \mathbb{R}^{n \times 1} \), we have
\begin{equation*}
U \in \partial \zeta W(X) \leftrightarrow u^i \in \partial (\| x^i \|_2 + \lambda \| x^i - w^i \|_2), \quad 1 \leq i \leq n,
\end{equation*}

where
\begin{equation*}
u^i \in \partial (\| x^i \|_2 + \lambda \| x^i - w^i \|_2) \Leftrightarrow \begin{cases} u^i = \frac{x^i}{\| x^i \|_2} + \lambda(\frac{x^i - w^i}{\| x^i - w^i \|_2}), & \text{if } i \in E_1, \\
\| u^i \|_2 + \lambda \| \partial x^i \|_2 \leq 1, & \text{if } i \in E_2, \\
u^i = \alpha^i + \lambda(\frac{\alpha^i - w^i}{\| \alpha^i - w^i \|_2}), & \| \alpha^i \|_2 \leq 1, & \text{if } i \in E_3, \\
u^i = \alpha^i + \lambda \beta^i, & \| \alpha^i \|_2 \leq 1, & \| \beta^i \|_2 \leq 1, & \text{if } i \in E_4.
\end{cases}
\end{equation*}

According to Lemma 3.3, Theorem 3.1 can be rewritten as follows.

**Theorem 3.4.** (Statistical dimension of descent cone of \( \ell_{2,1} \)-norm with prior information)

With the same notations and assumptions as in Theorem 3.1, the S.D. of the descent cone of \( \zeta W \) at the point \( X_0 \) satisfies the inequality
\begin{equation}
\psi_p - \frac{2(1 + \lambda) \sqrt{n}}{(1 - \lambda) \sqrt{k}} \leq \delta(D(\zeta W, X_0)) \leq \psi_p. \quad (7)
\end{equation}

The function \( \psi_p \) is defined as \( \psi_p(E) := \inf_{\tau \geq 0} \{ R_p(\tau, E) \} \), where \( E = \{ |E_1|, |E_2|, |E_3|, |E_4| \} \) and \( R_p = T_1 + T_2 + T_3 + T_4 \) with
\begin{align*}
T_1 &= |E_1| (l + \tau^2 + \tau^2 \lambda^2) + 2\tau^2 \lambda \sum_{i \in E_1} \cos(\angle Ox_i w_i), \\
T_2 &= |E_2| \int_{\tau \lambda}^{\infty} (t - \tau \lambda)^2 \cdot \tau^4 e^{-\frac{t^2}{2\lambda^2}} I_{1/2}(t/\lambda^2) dt, \\
T_3 &= |E_3| \int_{\tau}^{\infty} (t - \tau)^2 \cdot \tau^4 e^{-\frac{t^2}{2\lambda^2}} I_{1/2}(t/\lambda^2) dt, \\
T_4 &= |E_4| \int_{(1 + \lambda)}^{\infty} (t - \tau(1 + \lambda))^2 t^{-1} e^{-t^2/2} dt,
\end{align*}

where \( \Gamma \) is gamma function and
\begin{equation*}
I_0(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \frac{1}{(v+k+1)} \left( \frac{z}{2} \right)^{2k+v} \quad \text{is modified Bessel functions of the first kind.}
\end{equation*}
Proof. First we separate $F_G(\tau)$ as follow:

$$\text{dist}^2(G, \tau \cdot \partial\zeta_W(X_0)) = \sum_{i \in E_1} \left\| g^i - \tau \cdot \left( \frac{x^i_0}{\|x^i_0\|_2} \right) \right\|_2 + \lambda \left\| \frac{x^i_0 - w^i}{\|x^i_0 - w^i\|_2} \right\|_2 (8)$$

By taking the expected value, we have

$$E \left[ \sum_{i \in E_1} \left\| g^i - \tau \gamma^i \right\|_2^2 \right] = E \left[ \sum_{i \in E_1} \sum_{j=1}^l (g^i_j - \tau \gamma^i_j)^2 \right] = E \left[ \sum_{i \in E_1} \sum_{j=1}^l \left( (g^i_j)^2 - 2\tau \gamma^i_j g^i_j + \tau^2 (\gamma^i_j)^2 \right) \right] = \sum_{i \in E_1} \sum_{j=1}^l \left( (g^i_j)^2 - 2\tau \gamma^i_j g^i_j + \tau^2 (\gamma^i_j)^2 \right)$$

$$= \sum_{i \in E_1} \sum_{j=1}^l \left( E \left[ (g^i_j)^2 \right] - 2\tau \gamma^i_j E \left[ g^i_j \right] + \tau^2 (\gamma^i_j)^2 \right) = \sum_{i \in E_1} \sum_{j=1}^l \left( (g^i_j)^2 - 2\tau \gamma^i_j g^i_j + \tau^2 (\gamma^i_j)^2 \right)$$

In Eq. (9), for each $i \in E_2$, let $\vec{\gamma}^i = g^i - \tau \frac{x^i_0}{\|x^i_0\|_2}$, the minimization problem can be written as

$$\inf_{\beta \in B} \left\| \vec{\gamma}^i - \tau \lambda \beta \right\|_2^2$$

Similarly to Eq. (10) and Eq. (11). In Eq. (10), for each $i \in E_3$, let $\vec{\gamma}^i = g^i - \tau \lambda \frac{x^i_0 - w^i}{\|x^i_0 - w^i\|_2}$, the minimization problem can be written as

$$\inf_{\alpha \in B} \left\| \vec{\gamma}^i - \tau \alpha \right\|_2^2$$

which the optimal value is

$$\inf_{\alpha \in B} \left\| \vec{\gamma}^i - \tau \alpha \right\|_2^2$$

and hence Eq. (10) becomes

$$\sum_{i \in E_3} \inf_{\alpha \in B} \left\| \vec{\gamma}^i - \tau \alpha \right\|_2^2 = \sum_{i \in E_3} \left( \left\| \vec{\gamma}^i \right\|_2^2 - \tau \right)^2$$

Next, we discuss the expected value of Eq. (13) to (15). For Eq. (13), let $S_{2,i} = \| \vec{\gamma}^i \|_2^2$, for all $i \in E_2$. Since $g^i_j \sim N(0,1)$, $S_{2,i}$ follows the noncentral chi distribution with the same degrees of freedom $l$ and the same mean $\tau$ for all $i \in E_2$, which implies that all $S_{2,i}$ have the same probability density function

$$\rho(S_{2,i} = s,l,\tau) = \frac{s^l \cdot e^{-(s^2 + \tau^2)/2}}{(\tau^s)^{l/2}} \cdot I_{l/2 - 1}(\tau s).$$

By taking the expected value, we have

$$\sum_{i \in E_2} E \left[ (S_{2,i} - \tau \lambda)^2 \right] = \sum_{i \in E_2} \int_0^{\infty} \frac{(t - \tau \lambda)^2 \cdot \rho(t;l,\tau)}{t} dt$$

Similarly, $S_{3,i} = \| \vec{\gamma}^i \|_2^2$ follow the noncentral chi distribution with the same degrees of freedom $l$, the same mean $\tau \lambda$, and the same probability density function $\rho(S_{3,i} = s;l,\tau \lambda)$ for all $i \in E_3$. 

Then, by taking the expected value, Eq. (14) becomes
\[
\sum_{i \in E_3} \mathbb{E} \left[ (S_{3,i} - \tau)^2 \right] = \sum_{i \in E_3} \int_{\tau}^{\infty} (t - \tau)^2 \cdot \rho(t; l, \tau) \, dt = |E_3| \int_{\tau}^{\infty} (t - \tau)^2 \cdot \rho(t; l, \tau) \, dt = T_3.
\]

For Eq. (11), \( S_{4,i} = \| g^i \|_2 \) follow the chi distribution with the same degrees of freedom \( l \), and the same probability density function
\[
\tilde{\rho}(S_{4,i} = s; l) = \frac{2^{1 - \frac{1}{2} s l} s^{-1} e^{-\frac{s^2}{2}}}{\Gamma \left( \frac{l}{2} \right)},
\]
for all \( i \in E_4 \). Then, Eq. (15) can be reformulated as:
\[
\sum_{i \in E_4} \mathbb{E} \left[ \left( \| g^i \|_2 - \tau(1 + \lambda) \right)^2 \right] = \sum_{i \in E_4} \int_{\tau(l + \lambda)}^{\infty} (t - \tau(1 + \lambda))^2 \cdot \tilde{\rho}(t; l) \, dt = |E_4| \int_{\tau(l + \lambda)}^{\infty} (t - \tau(1 + \lambda))^2 \cdot \tilde{\rho}(t; l) \, dt = T_4.
\]

Therefore,
\[
\mathbb{E}[\text{dist}^2(G, \tau \cdot \partial \psi_W(X_0))] = R_p(\tau, E),
\]
and we complete the proof. \( \square \)

Following Theorem 3.4, since \( R_p \) is strictly convex, the infimum value can be computed by finding the root of derivative of \( R_p \). Moreover, if we divide the inequality in Eq. (7) by \( n \), we can see that the error term \( \frac{2(l + \lambda)}{(1 - \lambda) \sqrt{\lambda} \sqrt{n}} \) is inversely proportional to \( n \). That is, the error term is negligible as \( n \) is large enough. We verify Theorem 3.4 in the next section.

4. VERIFICATION

In this section, we verify our theoretical analysis about phase transition in compressive sensing via \( \ell_2.1-\ell_2.1 \) minimization, which were conducted using the CVX package [14]. Based on Theorem 3.4, it’s clear to see that S.D. is highly related to \( \psi_p \), which is dominated by \( E \) and \( \sum_{i \in E_1} \cos(\angle O x^i w^i) \) named cosine term. Hence, our simulations are divided into three categories: (1) Examine how prior information, controlled by \( |E_2| \), improve the performance, (2) Verify how prior information with correct supports but imprecise values, controlled by \( |E_1| \) and cosine term, affect the performance, and (3) Examine how prior information with wrong supports, controlled by \( |E_3| \), affect the performance. All the parameters in the three simulations follow the setting described in the next subsection.

4.1. Parameter Setting
For parameter setting, the signal dimension was fixed at \( n = 100 \) and sparsity was set to \( k = 16 \). The number of measurement vectors was \( l \). Since there are no changes with performance when the length of a measurement vector \( m \) is larger than \( \frac{2}{\lambda} \) in all simulations, \( m \) was set to range from 1 to \( \frac{2}{\lambda} \) to focus on the phase transition of performance. In our simulations, we construct a signal matrix \( X_0 \in \mathbb{R}^{n \times l} \) with \( k \) nonzero rows and generate prior information \( W \) with \( k_W \) nonzero rows to satisfy \( w^i = x^i, \forall i \in A_W \subset A_X \).

4.2. Prior Information Controlled by \( |E_2| \)
In the first simulation, \( k_W \) is 4 or 8 and \( l \) is 2 or 5. The following procedure (Step 1 ~ 3) was repeated 100 times for each set of parameters, composed of \( l \) and \( k_W \).

Step 1 Draw a standard normal matrix \( A \in \mathbb{R}^{m \times n} \) and generate \( Y = AX_0 \).

Step 2 Solve problem (ML1P) by CVX to obtain an optimal solution \( X^* \).

Step 3 Declare success if \( \| X^* - X_0 \|_F \leq 10^{-5} \).

As described in Theorem 3.4, \( \delta(D(\psi_W, X_0)) \) depends on \( n, E, l, \) and \( \lambda \). By the definition in Theorem 3.4, \( |E_1| = 12 \) and \( |E_2| = 4 \) in Fig. 1(a) and (c); \( |E_1| = |E_2| = 8 \) in Fig. 1(b) and (d). No matter \( l \) equal to 2 or 5. In Fig. 1, the theoretical curve (in black), indicating \( \delta(D(\psi_W, X_0)) \) derived in Theorem 2.5, is located at the vague region (of separating success and failure) of practical recovery results (in blue). We can observe that the theoretical results (in black) and the practical results (in blue) in Fig. 1(b) are more close to the origin than those in Fig. 1(a) because the \( |E_2| \) in (b) is greater than the \( |E_2| \) in (a), in other words, more correct supports (i.e., larger \( k_W \)) are available. Also we can observe that the practical result (in blue) in Fig. 1(b) is better than Fig. 1(a) as the former is more close to the origin. Similar results can also be observed in Figs. 1(c) and (d) when \( l \) becomes larger. In addition, they show that both the theoretical and practical results will be more close to the origin than those in Figs. 1(a) and (b) due to a larger \( l \) is used. Such phenomena are reasonable because more prior information will be helpful in recovery of sparse signals.

4.3. Prior Information with Correct Supports but Imprecise Values
We discuss how much influence of cosine term on S.D. and performance. This is equivalent to exploring the similarity between \( X_0 \) and \( W \). The parameters were \( l = 5 \) and \( k_W = 8 \). We construct a matrix \( X_0 \in \mathbb{R}^{100 \times 5} \) with \( k = 16 \) nonzero rows and generate prior information \( W \) with \( k_W = 8 \) nonzero rows, where \( A_W \subset A_X \) is chosen. We repeat the procedure
For the last simulation, we verify whether the effect of prior information with wrong supports is correctly predicted by Theorem 3.4. The parameters were set as $l = 5$ and $k_W = 8$.

Prior information with Type 3 was considered here. Next, we choose some $i \in \Lambda_X$ such that $w_i \sim N(0, I_{k_W})$ randomly. The procedure (Step 1 ~ 3) was repeated 100 times for each pair of parameters, $m$ and $k_W$, under four cases of different numbers of wrong supports as the prior information. As shown in Fig. 3, they are $|E_3| = 6$, $|E_4| = 78$ in (a), $|E_3| = 12$, $|E_4| = 72$ in (b), $|E_3| = 18$, $|E_4| = 66$ in (c) and $|E_3| = 24$, $|E_4| = 60$ in (d).

To compare with the case without prior information, the results regarding $\delta(D(\|\cdot\|_{2,1}, X_0))$ are labeled in red line in Fig. 3. In Fig. 3 (a), although $|E_3| = 6$, but it still have 8 correct supports information, overall, S.D. with such $W$ still much lower than red line. In Fig. 3 (b), $|E_3|$ increase to 12, S.D. with such $W$ become almost nothing different then red line. In Fig. 3 (c) and (d), along with the increase of $|E_3|$, the performance degrades and blue line is even greater than red line, in other words, $\ell_{2,1}$-norm minimization without prior information will gives better performance.

4.4. Prior Information with Wrong Supports

For the last simulation, we verify whether the effect of prior information with wrong supports is correctly predicted by Theorem 3.4. The parameters were set as $l = 5$ and $k_W = 8$. In view of the fact that the phase transition analysis in joint-sparse signal recovery with prior information of compressive sensing is relatively unexplored, we have presented a new phase transition analysis based on conic geometry to figure out the effect of prior information for MMVs in this paper. Our studies indeed provide useful insights into the critical problem of selecting prior information to guarantee improvement of signal recovery in the context of compressive sensing.
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