Non-Abelian W-representation for GKM

A. Mironov\textsuperscript{a,b,c,1}, V. Mishnyakov\textsuperscript{d,a,b,2}, A. Morozov\textsuperscript{d,b,c,3}

\textsuperscript{a} Lebedev Physics Institute, Moscow 119991, Russia
\textsuperscript{b} ITEP, Moscow 117218, Russia
\textsuperscript{c} Institute for Information Transmission Problems, Moscow 127994, Russia
\textsuperscript{d} MIPT, Dolgoprudny, 141701, Russia

Abstract

W-representation is a miraculous possibility to define a non-perturbative (exact) partition function as an exponential action of somehow integrated Ward identities on unity. It is well known for numerous eigenvalue matrix models, when the relevant operators are of a kind of \( W \)-operators: for the Hermitian matrix model with the Virasoro constraints, it is a \( W_3 \)-like operator, and so on. We extend this statement to the monomial generalized Kontsevich models (GKM), where the new feature is appearance of an ordered \( P \)-exponential for the set of non-commuting operators of different gradings.

1 Introduction. Hermitian model and the idea of W-representation

Partition function of matrix models \[1\] usually satisfies an exhaustive set of Virasoro and \( W \)-constraints, which are, however, not so easy to solve. For example, for the Hermitian matrix model with the partition function \( Z_N\{p\} \) where \( N \) is the size of matrix, the Ward identities read \[2\]

\[ \hat{L}_n Z_N\{p\} = 0, \quad n \geq -1 \] (1)

and the operators

\[ \hat{L}_n := \sum_k (k+n)p_k \frac{\partial}{\partial p_{k+n}} + \sum_a (a-a) \frac{\partial^2}{\partial p_a \partial p_{n-a}} + 2Nn \frac{\partial}{\partial p_n} + N^2 \delta_{n,0} + N p_1 \delta_{n+1,0} - (n+2) \frac{\partial}{\partial p_{n+2}} \] (2)

form a Borel subalgebra of the Virasoro algebra. The underlined term breaks the grading, the grading of \( p_k \) being \( k \). Such a choice of this term corresponds to choice of the Gaussian phase. In this phase, this system of equations has a unique solution \[3\-4\], which is given by \[5\] (see \[6\-8\] for early precursors)

\[ Z\{p\} = e^{\hat{O}_2} \cdot 1 \] (3)

where

\[ \hat{O}_2 := \frac{1}{2} \sum_{k,n} (k+n)p_{n+2} \frac{\partial}{\partial p_{k+n}} + \frac{1}{2} \sum_{a,b} abp_{a+b+2} \frac{\partial^2}{\partial p_a \partial p_b} + N \sum_{k=3} (k-2)p_k \frac{\partial}{\partial p_{k-2}} + \frac{N^2}{2} + \frac{N^2}{2} p_2 \] (4)

As explained in \[9\] (see also \[10\]), this representation can be deduced from the fact that the Virasoro constraints \[2\] can be all encoded in a single equation

\[ \sum_{n \geq 1} p_n \hat{L}_{n-2} Z_N\{p\} = \left( 2\hat{O}_2 - \sum_{n \geq 1} np_n \frac{\partial}{\partial p_n} \right) Z\{p\} = 0 \] (5)
that has a unique solution. The operators commute in the simple way:
\[
[\hat{l}_0, \hat{O}_2] = 2\hat{O}_2
\]  
(6)
and
\[
\hat{l}_0 \cdot 1 = 0
\]  
(7)
i.e. \(\hat{l}_0\) is the grading operator, and the grading of \(\hat{O}_2\) is 2. This is the main point: we combined Virasoro constraints in such a way that the grading-breaking piece is converted into grading operator \(\hat{l}_0\). Now introducing the grading parameter \(x\) via the rescaling \(p_k \rightarrow x^k p_k\), one comes to the equation
\[
(-x \frac{d}{dx} + 2x^2\hat{O}_2) Z = 0
\]  
(8)
with an obvious solution
\[
Z \sim \exp\left(\hat{O}_2 x^2\right)
\]  
(9)
Since the solution is unique, one establishes that (9) provides a representation of the Hermitian matrix model partition function. With our operators we do not need \(x\), and (5) just has (3) as an obvious solution.

Since [5], there were many more examples of \(W\)-representations for many different models [12], see [9] for a recent summary and for an evidence for unambiguity of solutions. However, there remains an important exception from the general list: the monomial Kontsevich models [13,14] beyond the simplest cubic example [11]. The goal of this letter is to fill the gap and provide a simple description of what happens to \(W\)-representation for the generalized Kontsevich model (GKM).

The partition function of the monomial GKM is given by the matrix integral over \(N \times N\) Hermitian matrix \(X\),
\[
Z_r(M) := \mathcal{N}_r \cdot \int dX e^{-\frac{\text{Tr} X^{r+1}}{r+1} + \text{Tr} M^r X}
\]  
(10)
and depends on the external matrix \(M\). At large \(M\) (so called Kontsevich phase [15]), this partition function is understood as a power series in time-variables \(p_k := \text{Tr} M^{-k}\), the coefficients of this power series being independent of the size of matrix \(N\). Hence, the notation \(Z_r(M) = Z_r(p)\). This partition function does not depend on \(p_k\)-variables, and is normalized so that \(\lim_{M \to \infty} Z_r(M) = Z_r\{0\} = 1\).

The Ward identities of this matrix model are described by constraints from the \(W^{(r)}\)-algebra and become rather involved at large \(r\). In the next section, we consider the simplest case of \(r = 2\), when they form a Borel subalgebra of the Virasoro algebra. In section 4, we consider the first non-trivial case of \(r = 3\), when the \(W\)-algebra constraints emerge for the first time, and, in section 5, we consider the generic \(r\) case.

2 W-representation in cubic case

We start with the partition function [10] with \(r = 2\) [16]. In this case, the partition function satisfies the Virasoro constraints [17,19],
\[
2\hat{L}_n Z_2(p) = 0, \quad n \geq -1
\]  
(11)
\[
2\hat{L}_n := \frac{1}{2} \sum_{k \in \text{odd}} (k + 2n)p_k \frac{\partial}{\partial p_{k+2n}} + \frac{1}{4} \sum_{a+b=2n} ab \frac{\partial^2}{\partial p_a \partial p_b} + \frac{p_k^2}{4} \delta_{n,-1} + \frac{1}{16} \delta_{n,0} - (2n + 3) \frac{\partial}{\partial p_{2n+3}}
\]  
(12)
Here the sums over \(k\) and \(a\) run over odd numbers. These constraints can be encoded in a single equation that has a unique solution,
\[
\sum_{n=1}^{p_{2n-1}} 2\hat{L}_{n-2} Z_2(p) = 0
\]  
(13)
This equation contains the terms of gradings 0 and 3. The zero grading term comes from the last (underlined) term in (12) and is

\[ \hat{l}_0 = \sum_{k \in \text{odd}} kp_k \frac{\partial}{\partial p_k} \]  

so that (13) takes the form

\[ (\hat{l}_0 - 3\hat{O}_3) Z_2\{p\} = 0 \]  

with the operators of grading 3 being

\[ \hat{O}_3 := \frac{1}{6} \sum_{k,l\in\text{odd}} (k + l - 3)p_k p_l \frac{\partial}{\partial p_{k+l-3}} + \frac{1}{12} \sum_{k,l\in\text{odd}} (k - l - 3)p_k \frac{\partial^2}{\partial p_{k-l-3} \partial p_l} + \frac{1}{48} p_3 + \frac{1}{12} p_1^3 \]  

The commutation relation is

\[ [\hat{l}_0, \hat{O}_3] = 3\hat{O}_3 \]  

Introducing the grading parameter \( x \) via the rescaling \( p_k \rightarrow x^k p_k \), we come to the equation

\[ (-x \frac{d}{dx} + 3x^3\hat{O}_3) Z_2\{p\} = 0 \]  

Its solution is exponential,

\[ Z_2\{p\} = \exp \left( x^3\hat{O}_3 \right) \cdot 1 \]  

which is nothing but the standard \( W \)-representation [9,11].

3 \ W-representation in quartic case

Now we consider the \( r = 3 \) case. This is the first truly non-trivial case. We have now a combination of Virasoro and \( W \)-constraints [20,23]

\[ 3\hat{L}_n Z_3\{p\} = 0, \quad n \geq -1 \]  

\[ 3\hat{W}^{(3)}_n Z_3\{p\} = 0, \quad n \geq -2 \]  

\[ 3\hat{L}_n := \frac{1}{3} \sum_{k} (k + 3n)p_k \frac{\partial}{\partial p_{k+3n}} + \frac{1}{6} \sum_{a,b,c=1 \atop a+b+c=3n} abP_k \frac{\partial^2}{\partial p_a p_b} + \frac{p_1 p_2}{3} \delta_{n,-1} + \frac{1}{9} \delta_{n,0} - (3n+4) \frac{\partial}{\partial p_{3n+4}} \]  

\[ 3\hat{W}^{(3)}_n := \frac{1}{9} \sum_{k,l=1} (k + l + 3n)P_k P_l \frac{\partial}{\partial p_{k+l+3n}} + \frac{1}{9} \sum_{a,b,c=1 \atop a+b+c=3n} abP_k \frac{\partial^2}{\partial p_a p_b} + \frac{1}{27} \sum_{a,b,c=1 \atop a+b+c=3n} abc \frac{\partial^3}{\partial p_a p_b p_c} + \frac{1}{27} \sum_{a,b,c=1 \atop a+b+c=3n} P_a P_b P_c \]  

where \( P_k := p_k - 3 \cdot \delta_{k,4} \), and \( a, b, c, k, l \) in the sums are not divisible by 3. They can be combined into a single equation that unambiguously determines the partition function

\[ \sum_{n=1}^{p_{3n-1}} \cdot 3\hat{W}^{(3)}_{n-3} Z_3\{p\} + c \sum_{n=1}^{p_{3n-2}} 3\hat{L}_{n-2} Z_3\{p\} = 0 \]  

where the parameter \( c \) can be chosen rather arbitrarily (only non-negative rational \( c \) can give rise to additional superfluous solutions of this equation) [9]. At the l.h.s. of this equation, there are operators of gradings 0, 4 and 8.
For the special choice of \( c = -1 \), the coefficients in front of the sum \( \sum_{n} (3n - 1) p_{3n - 1} \frac{\partial}{\partial p_{3n - 1}} \), coming from the first term in (22), and in front of the sum \( \sum_{n} (3n - 2) p_{3n - 2} \frac{\partial}{\partial p_{3n - 2}} \), coming from the second term, are equal to each other, so that the zero grading operator is nothing but \( \hat{l}_0 \)

\[
\hat{l}_0 = \sum_{k} kp_{k} \frac{\partial}{\partial p_{k}} \tag{23}
\]

with \( \text{k not divisible by 3} \). With this choice, (22) looks like

\[
\left( \hat{l}_0 - 4\hat{O}_4 - 8\hat{O}_8 \right) Z_3\{p\} = 0
\]

where the operators of gradings 4 and 8 are

\[
\hat{O}_4 := \frac{1}{12} \sum_{n=1}^{p_{3n-1}} \left( 2 \sum_{k} (k + 3n - 5) p_{k} \frac{\partial}{\partial p_{k+3n-5}} + \sum_{a+b=3n-5} ab \frac{\partial^2}{\partial p_{a} \partial p_{b}} \right) \\
+ \frac{1}{24} \sum_{n=1}^{p_{3n-2}} \left( 2 \sum_{k} (k + 3n - 6) p_{k} \frac{\partial}{\partial p_{k+3n-6}} + \sum_{a+b=3n-6} ab \frac{\partial^2}{\partial p_{a} \partial p_{b}} \right) + \frac{p_4}{36} + \frac{p_1^2 p_2}{6} \tag{25}
\]

and

\[
\hat{O}_8 := -\frac{1}{8 \cdot 2^7} \left\{ \sum_{n=1}^{p_{3n-1}} \left( 3 \sum_{k,l} (k + l + 3n - 9) p_{k} p_{l} \frac{\partial}{\partial p_{k+l+3n-9}} + 3 \sum_{k=1}^{p_{3n-1}} \frac{\partial}{\partial p_{k}} \sum_{a+b=3n-9} abp_{k} \frac{\partial^2}{\partial p_{a} \partial p_{b}} \right) \\
+ \sum_{a,b,c=1}^{p_{3n-9}} abc \frac{\partial^3}{\partial p_{a} \partial p_{b} \partial p_{c}} \right\} + p_1^3 p_5 + p_2^3 + 3p_1^2 p_2 p_4 \tag{26}
\]

and the sums over \( k, l, a, b, c \) run over positive integers not divisible by 3.

The commutation relations are

\[
[l_0, \hat{O}_4] = 4\hat{O}_4, \quad [l_0, \hat{O}_8] = 8\hat{O}_8
\]

Introducing the grading parameter \( x \) via the rescaling \( p_k \to x^k p_k \), we come to the equation

\[
\left( -x \frac{d}{dx} + 4x^4\hat{O}_4 + 8x^8\hat{O}_8 \right) Z_3\{p\} = 0
\]

Its solution is going to be an ordered exponential

\[
Z_3\{p\} = P \exp \left( \int x \left( 4x^4\hat{O}_4 + 8x^8\hat{O}_8 \right) \frac{dx'}{x'} \right) \cdot 1 = \\
= 1 + \int x \left( 4x^4\hat{O}_4 + 8x^8\hat{O}_8 \right) \frac{dx'}{x'} \cdot 1 + \int x \left( 4x^4\hat{O}_4 + 8x^8\hat{O}_8 \right) \frac{dx'}{x} \int x' \left( 4x'^4\hat{O}_4 + 8x'^8\hat{O}_8 \right) \frac{dx''}{x''} \cdot 1 + \ldots = \\
= \left( 1 + x^4\hat{O}_4 + x^8 \left( \frac{1}{2}\hat{O}_4 + \hat{O}_8 \right) \right) + x^{12} \left( \frac{1}{6}\hat{O}_4^3 + \frac{1}{3}\hat{O}_4 \hat{O}_8 + \frac{2}{3} \hat{O}_8 \hat{O}_4 \right) + \ldots \tag{30}
\]

\[\text{In the case of arbitrary } c \text{ in (22), one has to consider two different gradings, } p_{3k-1} \to x^{3k-1} p_{3k-1} \text{ and } p_{3k-2} \to (x\alpha)^{3k-2} p_{3k-2} \text{ which leads to the equation} \]

\[
\left( -x \frac{d}{dx} + (1+c)\alpha \frac{d}{dx} + 4x^4\hat{O}_4(\alpha, c) + 8x^8\hat{O}_8(\alpha) \right) Z_3\{p\} = 0
\]

with the operators \( \hat{O}_{4,8} \) depending on \( \alpha \) and \( \hat{O}_4 \) on the constant \( c \). Another possibility is to define yet another operator of zero grading, \( \hat{O}_0 := \sum_{k} (3k - 2) p_{3k-2} \frac{\partial}{\partial p_{3k-2}} \) and deal with the equation

\[
\left( -x \frac{d}{dx} + (1+c)\hat{O}_0 + 4x^4\hat{O}_4(c) + 8x^8\hat{O}_8 \right) Z_3\{p\} = 0
\]

It makes the whole consideration more involved. In particular, at some peculiar rational values of \( c \), there is a degeneration, which gives rise to additional superfluous solutions to Eq. (22). For instance, at \( c = +2 \), one gets

\[
Z_3\{p\} = 1 + \alpha \cdot p_1 p_2 + \frac{1}{6} p_1^2 p_2 + \frac{1}{36} p_4 + \ldots \tag{28}
\]

and the coefficient \( \alpha \) is not determined from Eq. (22).
The simplest way to generate this expansion is as follows. Let us look for a solution in the form \( Z_4(p) = \sum_k x^{4k} \hat{\Psi}_k \cdot 1 \). Then, (23) is equivalent to the recursion relation

\[
\hat{\Psi}_k = \frac{1}{k} \hat{O}_4 \hat{\Psi}_{k-1} + \frac{2}{k} \hat{O}_8 \hat{\Psi}_{k-2}
\]

with the initial conditions \( \hat{\Psi}_0 = 1, \hat{\Psi}_1 = \hat{O}_4 \).

Note that the recursion relation is consistent with similar relations obtained by J.Zhou \([24]\), though we derive them within a different framework. However, the operators \( \hat{O}_4 \) and \( \hat{O}_8 \) do not commute and, hence, do not lead to a simple exponential \( W \)-representation form of (30) (this is not quite consistent with the conclusion of \([24]\)):

\[
P \exp \left( x^4 \hat{O}_4 + x^8 \hat{O}_8 \right) = \exp \left( x^4 \hat{O}_4 + x^8 \hat{O}_8 - \frac{x^{12}}{6} \left[ \hat{O}_4, \hat{O}_8 \right] - \frac{x^{20}}{60} \left[ \hat{O}_8, \left[ \hat{O}_4, \hat{O}_8 \right] \right] + \ldots \right)
\]

(32)

The series (30) is one of the most effective technical ways to generate the partition function \( Z_3(p) \) as an expansion in powers of \( p_k \)’s (see the associated data and Appendix B, where, as an illustration, we evaluate \( Z_3(p) \) up to \( x^{16} \)).

4 \ W-representation for arbitrary monomial potential

4.1 The case of \( r = 4 \)

Let us briefly sketch the next \( r = 4 \) case. This time we should use the following constraints:

\[
4 \hat{L}_n Z_4(p) = 0, n \geq -1 \\
4 \hat{W}^{(3)}_n Z_4(p) = 0, n \geq -2 \\
4 \hat{W}^{(4)}_n Z_4(p) = 0, n \geq -3
\]

(33)

The corresponding \( W \) algebra can be expressed in terms of bosonic currents:

\[
4 \hat{L}_n = \frac{1}{8} \sum_{n_1+n_2=4n} : J_{n_1} J_{n_2} : + \frac{5}{32} \delta_{n,0}
\]

(34)

\[
4 \hat{W}^{(3)}_n = \frac{1}{48} \sum_{n_1+n_2+n_3=4n} : J_{n_1} J_{n_2} J_{n_3} :
\]

(35)

\[
4 \hat{W}^{(4)}_n = \frac{1}{256} \sum_{n_1+n_2+n_3+n_4=4n} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} : - \frac{1}{128} \sum_{p+q=n} \delta_{n_1+n_2=4p} \delta_{n_3+n_4=4q} \sum_{(n_1)_r (n_2)_r} : J_{n_1} J_{n_2} : - \frac{9}{4096} \delta_{n,0}
\]

(36)

where \((n)_r\) denotes \( n \) modulo \( r \). The currents are:

\[
\begin{cases}
J_{-n} &= p_n - 4 \delta_{n,5} \\
J_{n} &= n \frac{\partial}{\partial p_n}
\end{cases}
\]

(37)

and the normal ordering implies all the derivatives moved to the right. The sums in these expressions run over integers not divisible by 4. The last term in \( 4 \hat{W}^{(4)}_n \) comes from the anomaly. Notice a misprint in \( 4 \hat{W}^{(4)}_n \) of \([25]\) Appendix C).

As usual \([2]\) we consider a peculiar linear combination of these constraints:

\[
c_3 \sum_{n=1} p_{4n-1} \cdot 4 \hat{W}^{(4)}_{n-4} Z_4(p) + c_2 \sum_{n=1} p_{4n-2} \cdot 4 \hat{W}^{(3)}_{n-3} Z_4(p) + c_1 \sum_{n=1} p_{4n-3} \cdot 4 \hat{L}_{n-2} Z_4(p) = 0
\]

(38)
and according to [13] this equation has a unique solution for almost arbitrary constants \( c_i \). For our current purposes they can be chosen so that the zero grading operators combine into \( l_0 = \sum p_k \frac{\partial}{\partial p_k} \). This choice is

\[
c_i = (-1)^i
\]

(39)

It deserves making a brief remark on grading. If we neglect the shift of the fifth time in (37), then all the terms coming from \( \hat{W}^{(4)}_{r,n} \) have the grading 15. The third term in (34) contains at maximum one shift, which means there is also a term of grading 10. The second term contains terms with one or two shifts, which means there are terms of grading 5 and 10. Hence, the zero grading terms come only from the leading term in (36) and, similarly, in (34) and (35). This immediately gives (39).

For this choice (39) the equation for the partition function acquires the form

\[
\left( l_0 - 5 \hat{O}_5 - 10 \hat{O}_{10} - 15 \hat{O}_{15} \right) Z_4(p) = 0
\]

(40)

Then the \( W \)-representation is given by:

\[
Z_4(p) = P \exp \left( x^5 \hat{O}_5 + x^{10} \hat{O}_{10} + x^{15} \hat{O}_{15} \right) \cdot 1
\]

(41)

We illustrate this representation by evaluation of the partition function \( Z_4(p) \) in Appendix B up to order 15. In the 5-th and 10-th order, available at [26], it coincides with the answer in that paper.

### 4.2 Towards arbitrary \( r \)

Partition function \( Z_r \) in the GKM [10] with potential \( x^{r+1} \) does not depend on \( p_{nr} \) and satisfies the whole set of \( W \)-constraints of the orders ranging from 2 (Virasoro) to \( r \) [13],

\[
r \hat{W}^{(k)}_r Z_r\{p\} = 0, \quad k = 2, \ldots, r, \quad n \geq 1 - k
\]

(42)

and the \( W \)-generators are defined in [25] and in Appendix B, the first two being

\[
r \hat{L}_n = \frac{1}{2r} \sum_{n_1+n_2=r} : J_{n_1} J_{n_2} : + \frac{r^2 - 1}{24r} \delta_{n,0}
\]

(43)

\[
r \hat{W}^{(3)}_n = \frac{1}{3r^2} \sum_{n_1+n_2+n_3=r} : J_{n_1} J_{n_2} J_{n_3} :
\]

(44)

They can be expressed through the \( \eta_r \)-twisted scalar fields [25], and higher constraints are rather involved, e.g. (see Appendix B)

\[
r \hat{W}^{(4)}_n = \frac{1}{4r^3} \sum_{n_1+n_2+n_3+n_4=r} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} : - \frac{1}{8r^2} \sum_{p+q=n}^{p+q=n} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} : + \frac{(r^2 - 1)(r - 6)}{48r^2} \sum_{n_1+n_2=r} : J_{n_1} J_{n_2} :
\]

\[
- \frac{1}{8r^3} \sum_{n_1+n_2=r} \left( \langle n_1 \rangle_r + \langle n_2 \rangle_r \right) : J_{n_1} J_{n_2} : - \frac{(r^2 - 1)(r - 2)(r - 3)(5r + 7)}{5760r^3} \delta_{n,0}
\]

(45)

and

\[
\hat{r} \hat{W}^{(5)}_n = \frac{1}{5r^4} \sum_{n_1+n_2+n_3+n_4+n_5=r} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} J_{n_5} : - \frac{1}{6r^3} \sum_{p+q=n}^{p+q=n} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} J_{n_5} :
\]

\[
+ \frac{(r^2 - 1)(r - 12)}{72r^3} \sum_{n_1+n_2+n_3=r} : J_{n_1} J_{n_2} J_{n_3} : - \frac{1}{6r^4} \sum_{n_1+n_2+n_3=r} \left( \langle n_1 \rangle_r + \langle n_2 \rangle_r + \langle n_3 \rangle_r \right) : J_{n_1} J_{n_2} J_{n_3} :
\]

(46)
In these formulas,

\[
\begin{align*}
J_{-n} &= p_n - r\mu\delta_{n,r+1} \\
J_n &= n \frac{\partial}{\partial p_n}
\end{align*}
\]  

(47)

and the sums run over integers not divisible by \( r \). We denote \( (n)_r := (n)_r \left( r - (n)_r \right) \) where, as before, \( (n)_r \) is the value of \( n \) modulo \( r \). At the moment, we introduce a parameter \( \mu \) in the term violating the grading in order to control the grading easier. We will ultimately put \( \mu = 1 \).

The leading term of the \( W \)-generator is

\[
r\hat{W}^{(k)}_n := \frac{1}{r^{k-1}} \sum_{k_i = 1, k_i \notin r\mathbb{Z}} \left( \sum_j k_j + rn \right) \cdot \prod_{i=1}^{r-1} (p_{k_i} - r \cdot \delta_{k_i,r+1}) \cdot \frac{\partial}{\partial p_{\sum_j k_j + rn}} + \ldots
\]  

(48)

These \( W \)-constraints can be combined into a single equation

\[
\sum_{i=1}^{r-1} c_i \frac{\partial}{\partial p_{nr-i}} \cdot r\hat{W}^{(i+1)}_{n-i-1} \cdot Z_r \{ p \} = 0
\]  

(49)

again with the nearly arbitrary constants \( c_i \). This equation is a sum of operators of gradings \{ \( (r+1)i \) \}, \( i = 0..r - 1 \), which is given by the expansion of the operators

\[
r\hat{W}^{(i+1)}_n = \sum_{j=0}^{i} \mu^j \cdot r\hat{W}^{(i+1)}_{n,j}
\]  

(50)

into operators of definite gradings: \( r\hat{W}^{(k)}_{n,j} \) has grading \( rn - j(r + 1) \).

Again the constants \( c_i \) can be adjusted so that all the zero grading operators, i.e. all the terms in

\[
\sum_{n=1}^{r-1} \frac{\partial}{\partial p_{nr-i}}
\]

with all \( i = 1, \ldots, r \), come with the unit coefficients and combine into the grading operator

\[
\hat{l}_0 = \sum_{i=1}^{r-1} \sum_{n=1}^{\infty} p_{nr-i} \frac{\partial}{\partial p_{nr-i}}
\]  

(51)

This is the choice \( c_i = (-1)^i \). Then, introducing the grading parameter \( x \) via the rescaling \( p_k \to x^kp_k \), we come to the equation

\[
\sum_{i=1}^{r-1} (-1)^i \sum_n p_{nr-r+i} \cdot r\hat{W}^{(i+1)}_{n-i-1} \cdot Z_r \{ p \} = \sum_{j=0}^{r-1} \sum_{n>0,n \notin r\mathbb{Z}} p_{nr-r+i} \cdot r\hat{W}^{(i+1)}_{n-i-1,j} \cdot Z_r \{ p \} = 0
\]  

(52)

As we explained, the term with \( j = 0 \) in this sum reproduces the operator \( \hat{l}_0 \), and we finally come to the equation (we put \( \mu = 1 \) here)

\[
\left( -x \frac{d}{dx} + \sum_{i=1}^{r-1} (r+1)i x^{(r+1)i} \cdot \hat{O}_{(r+1)i} \right) Z_r \{ p \} = 0 \\
\left[ \hat{l}_0, \hat{O}_{(r+1)i} \right] = (r+1)i \cdot \hat{O}_{(r+1)i}
\]  

(53)

The solution to this equation is the iterated integral

\[
Z_r \{ p \} = 1 + \int_0^1 \hat{A}(t) \frac{dt}{t} + \int_0^1 \hat{A}(t) \frac{dt}{t} \int_0^t \hat{A}(t') \frac{dt'}{t'} + \int_0^1 \hat{A}(t) \frac{dt}{t} \int_0^t \hat{A}(t') \frac{dt'}{t'} \int_0^{t'} \hat{A}(t'') \frac{dt''}{t''} + \ldots
\]  

(54)
with $\hat{A}(t) := \sum_{k=2}^{r} k t^k \hat{O}_{(r+1)k}$, $t := x^{r+1}$, i.e. the series

$$Z_r\{p\} = \sum_{s=1}^{\infty} \sum_{i_1, \ldots, i_s=1}^{r-1} \frac{i_1 \ldots i_s \cdot \hat{O}_{(r+1)i_1} \cdots \hat{O}_{(r+1)i_s}}{(i_1 + \cdots + i_s) \cdots (i_{s-1} + i_s)i_s}$$

(55)

where some $i_k$ can be the same. The coefficient is the repeated integral

$$\int_0^1 t^{i_1} \frac{dt}{t} \int t^{i_2} \frac{dt'}{t'} \int t^{i_3} \frac{dt''}{t''} \cdots$$

(56)

In the commuting case, the coefficients would sum up just to

$$\sum_{\{k_a\} a=1}^{r-1} \frac{1}{k_a!} \hat{O}^{k_a}_{(r+1)i_a} = \exp \left( \sum_{a=1}^{r-1} \hat{O}_{(r+1)i_a} \right)$$

(57)

but, in the generalized Kontsevich model, $\hat{O}$’s do not commute. Still (55) is a very explicit and practical expression, we give some examples of its application in Appendix B.

4.3 To $W$-representation from matrix Ward identity for GKM

An interesting option would be to start directly from the identity $[13, 17, 21, 27]

$$\left\{ V' \left( \frac{\partial}{\partial L^{tr}} \right) - L \right\} Z_V = 0$$

(58)

for the matrix integral

$$Z_V = \int dX e^{tr(V(X) - V'(M)X)} = \frac{e^{trMV'(M) - V(M)}}{\det V''(M)} \cdot Z_V\{p_k\}$$

(59)

from which we extract the GKM partition function $Z_V$ depending only on negative powers of the matrix variable $M$, $p_k = \text{tr} M^{-k}$. For the monomial potentials $V_r(X) = \frac{X^{r+1}}{r+1}$, this means that $M^r = L$, and $Z_V = Z_r$ turns out to be independent of all $p_{rn}$, see $[13, 14]$ for details. In this case,

$$Z_r = \frac{e^{\frac{1}{r+1} \text{tr} M^{r+1}}}{\det \left( \sum_{i=0}^{r-1} M^i \otimes M^{r-1-i} \right)} \cdot Z_r\{p_k\}$$

(60)

and substitution into (58) gives a sum of terms with $r+1$ different gradings, associated with $r$ derivatives of the exponential. If we multiply the equation by $M$ and take a trace, the gradings (powers of $M^{-1}$) will be $n(r+1)$ with $n = -1, 0, \ldots, (r-1)$. Actually the lowest grading with $n = -1$ does not show up, because

$$M \cdot \left( \frac{e^{-\frac{1}{r+1} \text{tr} M^{r+1}}}{\partial (M^{tr})^r} \right)^r \cdot \partial Z_r\{p_k\} = 0$$

(61)

The most interesting is grading 0, where we get the operator $\hat{J}_0$. Indeed,

$$r \cdot \text{tr} \left\{ M \cdot \left( \frac{e^{-\frac{1}{r+1} \text{tr} M^{r+1}}}{\partial (M^{tr})^r} \right)^{r-1} \cdot \frac{\partial Z_r}{\partial (M^{tr})^r} \right\} =
\\= r \cdot \sum_k \text{tr} \left( M^r \frac{\partial p_k}{\partial (M^{tr})^r} \right) \cdot \frac{\partial Z_r}{\partial p_k} = \sum_k k \cdot \text{tr} M^{-k} \cdot \frac{\partial Z_r}{\partial p_k} = \sum_k kp_k \frac{\partial Z_r}{\partial p_k} = \hat{J}_0 Z_r$$

(62)

There are two other contributions in this grading, which do not contain $p$-derivatives of $Z_r$: one appears when the $L$ derivative acts on $\det V''(M)$ instead of $Z_r$, another one, when two $L$ derivatives act twice on the same exponential. Analysis in other gradings gets more involved and will be addressed elsewhere.
5 Conclusion

In this letter, we resolve a puzzle of the $W$-representation [5] for the monomial generalized Kontsevich models [13] beyond the cubic case [11]. As usual, the deviation from the standard situation appeared very simple but unexpected and implies far-going consequences. It turned out that the $W$-representation is not an ordinary exponential but an ordered $P$-exponential of a linear combination of non-commuting $W$-like operators of different gradings. We remind that, like many other matrix models [11], the GKM partition function is a KP $\tau$-function [13], thus what we observe is a striking appearance of $P$-exponential in the field of integrable systems. This brings the seemingly simple matrix models into a direct contact with Yang-Mills theories, where the $P$-exponentials play the central role: as predicted long ago, the non-Abelian nature has no conflict with integrability.

In the narrower field of matrix models per se, the $W$-representations provide a truly effective method for generating as many terms of the GKM partition function as one needs. This opens new possibilities for study of these very interesting and archetypical models. Some details are still lacking, and we have not yet derived a truly closed expression for arbitrary $r$, this is one of the simplest subjects for the future work.

Acknowledgements

We are indebted to A. Alexandrov for pointing to us the paper [24], which attempted to find a $W$-representation of the GKM partition functions. Despite it overlooked non-commutativity of the relevant operators, which led to an oversimplified anzatz for the $W$-representation, that paper forced us to revisit the problem and overcome our prejudices. Another origin of our paper is our recent activity on a systematic approach to solving Virasoro-like constraints [3][28][29], and we are very grateful to R. Rashkov for collaboration.

This work was supported by the Russian Science Foundation (Grant No.21-12-00400).

Appendix A: General formula for $\hat{W}^{(k)}_n$

In this Appendix, we describe how one can obtain the relevant $W$-operators $\hat{W}^{(k)}_n$ by the normal ordering of a product of currents [25]. The main point is that the spectral curve for the monomial GKM model $Z_r$ (which can be obtained from the corresponding loop equations [19]) is the $r$-sheeted covering of a sphere, which is clear both from the integrable hierarchy point of view (since the system is described by the $r$-th reduction of the KP hierarchy) [13][25], and from the topological recursion point of view [3]. This is why it is natural to define the current to be

$$J(z) := \sum J_n z^{-n/r - 1} \quad (63)$$

with the current modes given by (67). This expression involves the $r$-th root of $z$, and, hence, one has to specify which of the roots is used (the sheet of the covering). We denote choosing the $m$-th root as $z_m$. One can arbitrarily choose the ordering of $z_m$, $m = 1, \ldots, r$, but, for the sake of definiteness, we choose them to be $z_{m+1}^{1/r} = \exp\left(\frac{2\pi i}{r}\right) \cdot z_m^{1/r}$ and denote $z_1 = z$. Note that integer powers of $z$ are the same for all $z_m$, but, at the level of $J(z)$, the arguments are all different, and one can use non-singular operator expansions. At the same time, the final answer contains only integer powers of $z$.

Now the procedure of constructing the $r\hat{W}^{(k)}_n$-operators consists of three steps.

1. The starting point is an auxiliary operator

$$r\hat{W}^{(k)}_{aux}(z) = \frac{(-1)^{k+1}}{r^k} \sum_{1 \leq m_1 < m_2 < \ldots < m_k \leq r} J(z_{m_1}) \ldots J(z_{m_k}) \quad (64)$$

which is very simple and general, but not normally ordered.

2. One has to normally order $r\hat{W}^{(k)}_{aux}(z)$ in such a way that all positive current modes are moved to the right.

3. After normal ordering, one has to omit all the current modes divisible by $r$: $J_{nr} = 0$ in order to finally obtain $r\hat{W}^{(k)}(z)$.

Now note that both $r\hat{W}^{(k)}_{aux}(z)$ and $r\hat{W}^{(k)}(z)$ are single-valued, and, hence, are expanded into integer powers of $z$. Thus, one generates $r\hat{W}^{(k)}_n$ as

$$r\hat{W}^{(k)}_n(z) = \sum_{n} r\hat{W}^{(k)}_n z^{-n - k} \quad (65)$$
Now we demonstrate how this procedure works in a few examples.

\[ W_n^{(2)} : \] In this case, we have
\[
r \hat{W}_n^{(2)}(z) = \sum_{1 \leq m_1 < m_2 \leq r} \left( \frac{1}{r^2} \sum_{1 \leq m_1 < m_2 \leq r} J(z_{m_1}) J(z_{m_2}) = \right.
\]  
\[
\text{step 2} \quad - \frac{1}{r^2} \sum_{1 \leq m_1 < m_2 \leq r} \left( \sum_{n_1, n_2 > 0} [J_{n_1}, J_{n_2}] \frac{\omega_{m_1}^{n_1} \omega_{m_2}^{n_2}}{J_{n_1} J_{n_2}} : \frac{z^{-(n_1 + n_2)/r}}{r^2 - 1} \right)
\]  
\[
\text{step 3} \quad \frac{1}{r^2} \sum_{1 \leq m_1 < m_2 \leq r} \left( \sum_{n_1, n_2 > 0} \frac{\omega_{m_1}^{n_1} \omega_{m_2}^{n_2}}{J_{n_1} J_{n_2}} : \frac{z^{-(n_1 + n_2)/r}}{r^2 - 1} \right)
\]
where we denoted \( \omega_m := \exp \left( \frac{2 \pi m}{r} \right) \) and used that \( [J_{n_1}, J_{n_2}] = n_1 \delta_{n_1, n_2} \), and, hence, the anomaly term is
\[
\frac{1}{r^2} \sum_{1 \leq m_1 < m_2 \leq r} \left( \sum_{n_1, n_2 > 0} \frac{\omega_{m_1}^{n_1} \omega_{m_2}^{n_2}}{J_{n_1} J_{n_2}} : \frac{z^{-(n_1 + n_2)/r}}{r^2 - 1} \right)
\]
Calculating the sum over \( n \) requires a regularization as usual for the anomaly.

At last, in order to rewrite the remaining sum in the last line of (68), we use the identity
\[
\frac{1}{2} \left( \sum_{1 \leq m_1 < m_2 \leq r} \omega_{m_1}^{n_1} \omega_{m_2}^{n_2} + \sum_{1 \leq m_1 < m_2 \leq r} \omega_{m_1}^{n_2} \omega_{m_2}^{n_1} \right) = -\frac{r}{2} \delta_{n_1 + n_2, r n}
\]  
so that we finally obtain
\[
r \hat{W}_n^{(2)}(z) = \frac{1}{2r} \sum_{n_1, n_2 \neq r \mathbb{Z}} : \frac{J_{n_1} J_{n_2}}{z^{n+2}} + \frac{r^2 - 1}{24r} \frac{1}{z^2}
\]
which gives rise to (68).

\[ W_n^{(3)} : \] In this case,
\[
r \hat{W}_n^{(3)}(z) = \sum_{1 \leq m_1 < m_2 < m_3 \leq r} \left( \frac{1}{r^3} \sum_{1 \leq m_1 < m_2 < m_3 \leq r} J(z_{m_1}) J(z_{m_2}) J(z_{m_3}) = \right.
\]  
\[
\text{step 2} \quad \frac{1}{r^3} \sum_{1 \leq m_1 < m_2 < m_3 \leq r} \left( \sum_{n_1, n_2, n_3 > 0} [J_{n_1}, J_{n_2}] J_{n_3} : \frac{z^{-(n_1 + n_2 + n_3)/r}}{r^3 - 1} \right)
\]  
\[
\text{step 3} \quad \frac{1}{r^3} \sum_{1 \leq m_1 < m_2 < m_3 \leq r} \left( \sum_{n_1, n_2, n_3 > 0} \frac{\omega_{m_1}^{n_1} \omega_{m_2}^{n_2} \omega_{m_3}^{n_3}}{J_{n_1} J_{n_2} J_{n_3}} : \frac{z^{-(n_1 + n_2 + n_3)/r}}{r^3 - 1} \right)
\]
The terms linear in \( J \) in the second line are omitted at the third step, since they are proportional to \( J_{nr} \). This is evident since the whole expression should be single-valued, and, hence, it depends only on integer powers of \( z \), i.e. on \( J_{nr} \). It can be manifestly seen in the following way: it is a sum of three terms of the form
\[
\frac{1}{r^3} \sum_{1 \leq m_1 < m_2 < m_3 \leq r} \sum_{n_1, n_2, n_3 > 0} [J_{n_1}, J_{n_2}] J_{n_3} \frac{z^{-(n_1 + n_2 + n_3)/r}}{r^3 - 1}
\]
\[
= \frac{1}{r^3} \sum_{1 \leq m_1 < m_2 < m_3 \leq r} \sum_{n_1, n_2, n_3 > 0} \frac{\omega_{m_1}^{n_1} \omega_{m_2}^{n_2} \omega_{m_3}^{n_3}}{J_{n_1} J_{n_2} J_{n_3}} : \frac{z^{-(n_1 + n_2 + n_3)/r}}{r^3 - 1}
\]
In this case, the calculation is very similar, but there is a subtlety. That is, one needs a counterpart of \( c \) turns into the difference of two terms:

\[
\frac{\omega_{m_1} \omega_{m_2}}{(\omega_{m_1} - \omega_{m_2})^2 \omega_{m_3}^{-n_3}} = -\frac{r^2 - 1}{24} (r - 2) \delta_{n_3, r n}
\]

giving rise to the sum

\[
-\frac{(r^2 - 1)(r - 2)}{24 r^2} \sum_n J_{n r^{-z}}^{n - 3}
\]

The last line in (70) is due to the identity

\[
\frac{1}{6} \left( \sum_{1 \leq m_1 < m_2 < m_3 \leq r} \omega_{m_1}^{-n_1} \omega_{m_2}^{-n_2} \omega_{m_3}^{-n_3} + \text{all permutations of } n_1, n_2, n_3 \right) = \frac{r}{3} \delta_{n_1 + n_2 + n_3, r n} \text{ for any } n_1, n_2, n_3 \notin r \mathbb{Z}
\]

\[\hat{W}_n^{(4)}: \] In this case, the calculation is very similar, but there is a subtlety. That is, one needs a counterpart of formulas (68) and (73). Now it, however, has a more subtle structure: the r.h.s. depends not only on \( r \) and on divisibility of the sum \( n_1 + n_2 + n_3 + n_4 \) by \( r \), but also on divisibility of pairs \( n_1 + n_2, \) etc. Indeed, the identity is

\[
\frac{1}{24} \left( \sum_{1 \leq m_1 < m_2 < m_3 < m_4 \leq r} \omega_{m_1}^{-n_1} \omega_{m_2}^{-n_2} \omega_{m_3}^{-n_3} \omega_{m_4}^{-n_4} + \text{all permutations of } n_1, n_2, n_3, n_4 \right) = -\frac{r}{4} \left( 1 - c(n_1, n_2, n_3, n_4) \right) \delta_{n_1 + n_2 + n_3 + n_4, r n} \text{ for any } n_1, n_2, n_3, n_4 \notin r \mathbb{Z}
\]

where the coefficient \( c(n_1, n_2, n_3, n_4) \) is the number of different combinations of \( n_i \)'s with pairwise sums divisible by \( r \). For instance, \( c(1, 1, 3, 3) = 0 \) (no combinations), \( c(3, 4, 4, 5) = 1 \) (1 combination), \( c(3, 3, 5, 5) = 2 \) and \( c(2, 2, 2, 2) = 3 \) at \( r = 8 \). This immediately implies that the normally ordered quartic combination of currents turns into the difference of two terms:

\[
r \hat{W}_n^{(4)}(z) = \frac{1}{4 r^3} \sum_n \frac{1}{z^{n+4}} \left( \sum_{n_1 + n_2 + n_3 + n_4 = r n} J_{n_1} J_{n_2} J_{n_3} J_{n_4} : \right) - \frac{r}{2} \sum_{p+q=n} \sum_{n_1 + n_2 = r p, n_3 + n_4 = r q} J_{n_1} J_{n_2} J_{n_3} J_{n_4} : + \ldots
\]

In order to calculate the terms of the form : \( J J \cdot \) one proceeds similarly to (67) and uses the identity

\[
\sum_{1 \leq m_1 < m_2 < m_3 < m_4 \leq r} \text{Sym}_{\omega_{m_i}} \frac{\omega_{m_1} \omega_{m_2} \omega_{m_3}^{-n_3} \omega_{m_4}^{-n_4}}{(\omega_{m_1} - \omega_{m_2})^2 (\omega_{m_3} - \omega_{m_4})^2} = \left[ r^2 \cdot \frac{r^2 - 1}{12} - \frac{r}{2} \left( (n_1)_r^2 + (n_2)_r^2 - 1 \right) \right] \delta_{n_1 + n_2, r n}
\]

for any \( n_1, n_2 \notin r \mathbb{Z} \)

Here the symmetrization symbol \( \text{Sym}_{\omega_{m_i}} \) means that we sum over all permutations of \( \omega_i \).

At last, in order to evaluate the remaining constant anomaly term, one twice uses the summation as in (67), and the identity

\[
\sum_{1 \leq m_1 < m_2 < m_3 < m_4 \leq r} \text{Sym}_{\omega_{m_i}} \frac{\omega_{m_1} \omega_{m_2} \omega_{m_3} \omega_{m_4}}{(\omega_{m_1} - \omega_{m_2})^2 (\omega_{m_3} - \omega_{m_4})^2} = 8 r \cdot \frac{(r^2 - 1)(r - 2)(r - 3)(5 r + 7)}{5760}
\]

in order to ultimately obtain (75).
Similarly, for spin 5 generators, one obtains

$$\frac{1}{120} \left( \sum_{1 \leq m_1 < m_2 < m_3 < m_4 < m_5 \leq r} \prod_{n=1}^{4} \omega_m^{-n_1} \omega_m^{-n_2} \omega_m^{-n_3} \omega_m^{-n_4} \omega_m^{-n_5} + \text{all permutations of } n_1, n_2, n_3, n_4, n_5 \right) =
\frac{r}{5} \left( 1 - \frac{c(n_1, n_2, n_3, n_4, n_5)}{12} \right) \delta_{n_1+n_2+n_3+n_4+n_5, r}$$

for any \( n_1, n_2, n_3, n_4, n_5 \notin r\mathbb{Z} \) \( (78) \) and

$$\hat{W}^{(5)}(z) = \frac{1}{5^r} \sum_{n} \frac{1}{z^{n+5}} \left( \sum_{n_1+n_2+n_3+n_4+n_5 = r} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} J_{n_5} : - \frac{5^r}{6} \sum_{n_1+n_2+n_3+n_4+n_5 = r} : J_{n_1} J_{n_2} J_{n_3} J_{n_4} J_{n_5} : \right) + \ldots (79)$$

The term cubic in currents is obtained from the calculation similar to \( (67) \) with help of the identity

$$\sum_{1 \leq m_1 < m_2 < m_3 < m_4 \leq r} \frac{\text{Sym}_{\omega_{m_1}} \omega_{m_1} \omega_{m_2} \omega_{m_3} \omega_{m_4}}{(\omega_{m_1} - \omega_{m_2})^2} = \left( -r^2 \cdot \frac{r^2 - 1}{6} + 2r \cdot \left[ (n_1)^2 + (n_2)^2 + (n_3)^2 + r(r - n_1)(r - n_2)(r - n_3) \right] - 1 \right) \delta_{n_1+n_2+n_3, r}$$

for any \( n_1, n_2, n_3 \notin r\mathbb{Z} \) \( (80) \)

This finally gives \( (40) \).

One can see that the way to evaluate the \( W \)-generators performed in this subsection is straightforward, but it makes computer calculations rather involved. However, all what one needs is knowledge of the sums

$$\sum_{1 \leq m_1 < \ldots < m_{a+p} \leq r} \text{Sym}_{\omega_{m_1}} \prod_{a=1}^{a} \frac{\omega_{m_{a+1}} \omega_{m_{a+2}} \ldots \omega_{m_{a+p}}} {(\omega_{m_a} - \omega_{m_{a+1}})^2} \prod_{b=1}^{p} \omega_{m_b}$$

\( (81) \)

**Appendix B: Examples of GKM partition functions**

In this Appendix, we present the first orders of expansion of the partition functions \( Z_2 \{ p \}, Z_3 \{ p \} \) and \( Z_4 \{ p \} \) produced by the method described in the paper. They are often needed in applications. In this way, it is easy to generate many more terms: the number here is limited by the length acceptable in a printed version.
\[ Z_3(p) = 1 + x^4 \left( \frac{1}{36} p_4 + \frac{1}{6} p_2 p_2 \right) + x^8 \left( \frac{13}{216} p_6^2 p_2 p_4 + \frac{13}{2592} p_4^2 - \frac{1}{216} p_4^2 + \frac{1}{72} p_1^2 p_5^2 + \frac{1}{27} p_1^2 p_5 + \frac{1}{27} p_1^2 p_7 + \right) + \\
+ x^{12} \left( \frac{325}{279936} p_2^4 - \frac{5}{252} p_2 p_8 + \frac{1}{81} p_1^2 p_3 p_5 + \frac{1}{162} p_1^2 p_3 p_2 - \frac{1}{1296} p_1^2 p_3^2 + \frac{25}{972} p_1 p_3 p_5 - \frac{1}{1296} p_1^2 p_3^2 + \frac{25}{972} p_1 p_3 p_5 - \frac{1}{1296} p_1^2 p_3^2 + \right) + \\
+ x^{16} \left( \frac{12025}{40310784} p_2^4 + \frac{25}{1458} p_2^2 p_8 + \frac{35}{729} p_1 p_3 p_5 + \frac{925}{15552} p_1^2 p_3^2 - \frac{55}{59982} p_1^2 p_3^2 - \frac{1458}{2916} p_1 p_3 p_5 + \frac{7}{324} p_1 p_3 p_5 + \right) - \\
\frac{1}{81} p_5^2 p_2 p_5 p_7 - \frac{1}{1664} p_5^2 p_2 p_5 p_7 - \frac{1}{81} p_5^2 p_2 p_5 p_7 - \frac{1}{1664} p_5^2 p_2 p_5 p_7 - \\
- \frac{10}{243} p_1 p_2 p_3 p_5 - \frac{1}{243} p_1 p_2 p_3 p_5 - \frac{5}{729} p_1 p_2 p_3 p_5 - \frac{10}{729} p_1 p_2 p_3 p_5 - \\
+ \frac{185}{11664} p_4 p_8 p_4 + \frac{1}{1458} p_6^2 p_2 p_4 + \frac{1}{31104} p_8^2 p_2 - \frac{7}{729} p_1 p_1 p_1 p_7 - \frac{10}{729} p_1 p_1 p_1 p_7 - \\
- \frac{37}{1664} p_6^2 p_2 p_4 + \frac{185}{11664} p_4 p_8 p_4 + \frac{37}{64656} p_6^2 p_2 p_4 + \frac{1}{186624} p_4 p_8 p_4 + \frac{1}{93312} p_8^2 - \frac{85}{11664} p_8^2 + O(x^{20}) \]

\[ Z_4(p) = 1 + x^5 \left( \frac{p_5}{32} + \frac{1}{8} p_3 p_1 + \frac{1}{8} p_3 p_1 + \right) + \\
+ x^{10} \left( \frac{1}{128} p_2^2 p_4 + \frac{1}{64} p_2 p_3 p_5 + \frac{1}{32} p_2 p_3 p_5 + \frac{1}{128} p_2 p_3 p_5 + \frac{1}{32} p_2 p_3 p_5 + \right) + \\
+ x^{15} \left( \frac{1}{2048} p_3^2 p_1 p_7 + \frac{1}{32} p_2 p_1 p_7 + \frac{1}{2048} p_3^2 p_1 p_7 + \right) + \\
\frac{5}{64} \frac{p_5}{30} + \frac{1}{3} p_5^2 p_1 p_7 + \frac{1}{3} p_5^2 p_1 p_7 + \frac{1}{3} p_5^2 p_1 p_7 + \frac{1}{3} p_5^2 p_1 p_7 + \\
- \frac{7}{64} p_5^2 p_1 p_7 + \frac{1}{3} p_5^2 p_1 p_7 + \frac{1}{3} p_5^2 p_1 p_7 + \frac{1}{3} p_5^2 p_1 p_7 + \\
- \frac{59}{90} \frac{p_2 p_3 p_1}{10} + \frac{311}{60} \frac{p_2 p_3}{10} - \frac{693 p_1}{60} + O(x^{20}) \]

\[ Z_5(p) = 1 + x^6 \left( \frac{p_2}{30} + \frac{1}{5} p_1 p_3 p_2 + \frac{1}{10} p_2 p_4 + \frac{p_4}{30} \right) + \\
+ x^{12} \left( \frac{p_3}{1800} + \frac{1}{150} p_1 p_3 p_2 p_2 + \frac{1}{300} p_2 p_4 p_2 + \frac{7}{900} p_5 p_2 + \frac{1}{300} p_2 p_4 p_2 - \frac{1}{100} p_2 p_4 p_2 - \frac{1}{25} p_1 p_7 p_2 - \frac{1}{50} p_1 p_7 p_2 + \frac{1}{50} p_1 p_7 p_2 + \right) + \\
+ \frac{7}{150} p_1 p_3 p_2 p_2 + \frac{3}{50} p_2 p_4 p_2 - \frac{p_4}{300} + \frac{1}{300} p_2 p_4 p_2 + \frac{7}{1800} p_5 p_2 + \frac{7}{300} p_3 p_2 p_2 + \frac{1}{25} p_1 p_7 p_2 + \frac{2}{75} p_1 p_7 p_2 + \frac{p_1 p_1}{75} + \frac{1}{25} p_1 p_1 + \right) + \\
+ O(x^{15}) \]

\[ Z_6(p) = 1 + x^7 \left( \frac{5}{144} p_7 + \frac{1}{12} p_1 p_7 \right) + \frac{1}{12} p_1 p_7 + 1 + O(x^{14}) \]

In general

\[ Z_r(p) = 1 + x^{r+1} \left( \frac{r-1}{24 r^2} p_1 p_2 r + \frac{3^{-4 r}}{2 r} p_1^2 p_2 r - \frac{3^{-4 r}}{2 r} p_1^2 p_2 r - \ldots \right) + \\
+ x^{2 r+2} \left( \frac{r-1}{24 r^2} p_1 p_2 r + \left( \frac{r-1}{24 r^2} p_1 p_2 r - \frac{1}{12} p_1 p_2 r - \frac{1}{12} p_1 p_2 r - \ldots \right) + \\
+ O(x^{3 r+3}) \right) \]

Omitted items depend on selection rules for \( r \), i.e. enter with the Heaviside functions like \( \theta_{r>3} \) in the first bracket.
References

[1] A. Morozov, Phys.Usp.(UFN) 37 (1994) 1; hep-th/9502091, hep-th/0502010
  A. Mironov, Int.J.Mod.Phys. A9 (1994) 4355; Phys.Part.Nucl. 33 (2002) 537; hep-th/9409190

[2] F. David, Mod.Phys.Lett. A5 (1990) 1019
  A. Mironov, A. Morozov, Phys.Lett. B252 (1990) 47-52
  J. Ambjørn, Yu. Makeenko, Mod.Phys.Lett. A5 (1990) 1753
  H. Itoyama, Y. Matsuo, Phys.Lett. 255B (1991) 20

[3] A. Alexandrov, A. Mironov, A. Morozov, Int. J. Mod. Phys. A 19 (2004) 4127, hep-th/0310113

[4] L. Cassia, R. Lodin, M. Zabzine, JHEP 2010 (2020) 126, arXiv:2007.10354

[5] A. Morozov, S. Shakirov, JHEP 0904 (2009) 064, arXiv:0902.2627

[6] A. Givental, math.AG/0008067

[7] A. Alexandrov, A. Mironov, A. Morozov, Physica D235 (2007) 126-167, hep-th/0608228
  A. Alexandrov, A. Mironov, A. Morozov, Theor. Math. Phys. 150 (2007) 153-164, hep-th/0605171

[8] A.Okounkov, Math.Res.Lett. 7 (2000) 447-453;
  V.Bouchard, M.Marino, In: From Hodge Theory to Integrability and tQFT: tt*-geometry, Proceedings of Symposia in Pure Mathematics, AMS (2008), arXiv:0709.1458
  S.Lando, In: Applications of Group Theory to Combinatorics, Koolen, Kwak and Xu, Eds. Taylor & Francis Group, London, 2008, 109-132;
  M.Kazarian, arXiv:0809.3263
  A.Mironov, A.Morozov, JHEP 0902 (2009) 024, arXiv:0807.2843

[9] A. Mironov, V. Mishnyakov, A. Morozov, R. Rashkov, arXiv:2105.09920

[10] L. Cassia, R. Lodin, M. Zabzine, arXiv:2102.05682

[11] A. Alexandrov, Mod.Phys.Lett. A26 (2011) 2193-2199, arXiv:1009.4887

[12] A. Alexandrov, Adv Theor.Math.Phys. 22 (2018) 1347, arXiv:1608.01627
  H. Itoyama, A. Mironov, A. Morozov, JHEP 1706 (2017) 115, arXiv:1704.08648
  A. Mironov, A. Morozov, Phys. Lett. B 771 (2017) 503, arXiv:1705.00976

[13] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, A. Zabrodin, Phys.Lett. B275 (1992) 311, hep-th/9111037
  S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, A. Zabrodin, Nucl.Phys. B380 (1992) 181, hep-th/9201013

[14] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, Nucl.Phys. B397 (1993) 339-378, hep-th/9203043

[15] A. Mironov, A. Morozov, G.W. Semenoff, Int.J.Mod.Phys. A11 (1996) 5031, hep-th/9404005

[16] M. Kontsevich, Commun.Math.Phys. 147 (1992) 1

[17] A. Marshakov, A. Mironov, A. Morozov, Phys.Lett. B274 (1992) 280,

[18] E.Witten, On the Kontsevich model and other models of two-dimensional gravity, in: New York 1991 Proc., Differential geometric methods in theoretical physics, v.1, pp.176-216

[19] A. Alexandrov, A. Mironov, A. Morozov, P. Putrov, Int.J.Mod.Phys. A24 (2009) 4939, arXiv:0811.2825

[20] M. Fukuma, H. Kawai, R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385

[21] A. Mikhailov, Int. J. Mod. Phys. A9 (1994) 873, hep-th/9303129

[22] A. Mironov, S. Pakulyak, Theor. Math. Phys. 95 (1993) 604-625, hep-th/9209100

[23] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, S. Pakuliak, Nucl.Phys. B404 (1993) 717-750, hep-th/9208044
[24] Jian Zhou, arXiv:1305.6991

[25] M. Fukuma, H. Kawai, R. Nakayama, Comm. Math. Phys. 143 (1992) 371-403

[26] A. Mironov, A. Morozov, arXiv:2101.08759

[27] D. Gross, M. Newman, Nucl. Phys. B380 (1992) 168-180

[28] A. Mironov, V. Mishnyakov, A. Morozov, R. Rashkov, JETP Letters 113:11 (2021), arXiv:2104.11550

[29] A. Mironov, V. Mishnyakov, A. Morozov, R. Rashkov, to appear