1 Introduction

In this review, we consider Euclidean field theory as a formulation of quantum field theory which lives in some Euclidean space, and is expressed in probabilistic terms. Methods arising from Euclidean field theory have been introduced in a very successful way in the study of the concrete models of Constructive Quantum Field Theory.

Euclidean field theory was initiated by Schwinger [1] and Nakano [2], who proposed to study the vacuum expectation values of field products analytically continued into the Euclidean region (Schwinger functions), where the first three (spatial) coordinates of a world point are real and the last one (time) is purely imaginary (Schwinger points). The possibility of introducing Schwinger functions, and their invariance under the Euclidean group are immediate consequences of the by now classic formulation of quantum field theory in terms of vacuum expectation values given by Wightman [3]. The convenience of dealing with the Euclidean group, with its positive definite scalar product, instead of the Lorentz group is evident, and has been exploited by several authors, in different contexts.

The next step was made by Symanzik [4], who realized that Schwinger functions for Boson fields have a remarkable positivity property, allowing to introduce Euclidean fields on their own sake. Symanzik also pointed out an analogy between Euclidean field theory and classical statistical mechanics, at least for some interactions [4].

This analogy was successfully extended, with a different interpretation, to all Boson interaction by Guerra, Rosen and Simon [6], with the purpose

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of using rigorous results of modern statistical mechanics for the study of constructive quantum field theory, inside the program advocated by Wightman \cite{7}, and further pursued by Glimm and Jaffe (see \cite{8} for an overall presentation).

The most dramatic advance of Euclidean theory was responsibility of Nelson \cite{9} \cite{10}. He was able to isolate a crucial property of Euclidean fields (the Markov property) and gave a set of conditions for Euclidean fields, which allow to derive all properties of relativistic quantum fields satisfying Wightman axioms. Nelson theory is very deep and rich of new ideas. After so many years from the basic papers, we still lack a complete understanding of the radical departure from the conventional theory afforded by Nelson ideas, especially about their possible further developments.

By using Nelson scheme, in particular a very peculiar symmetry property, it was very easy to prove \cite{11} the convergence of the ground state energy density, and the Van Hove phenomenon in the infinite volume limit for two dimensional Boson theories. A subsequent analysis \cite{12} gave other properties of the infinite volume limit of the theory, and allowed a remarkable simplification in the proof of a very important regularity property for fields, previously established by Glimm and Jaffe.

Since then all work on constructive quantum field theory has exploited in different ways ideas coming from Euclidean field theory. Moreover, a very important reconstruction theorem has been established by Osterwalder and Schrader \cite{13}, allowing reconstruction of relativistic quantum fields from the Euclidean Schwinger functions, and avoiding the previously mentioned Nelson reconstruction theorem, which is technically more difficult to handle.

This paper is intended to be an introduction to the general structure of Euclidean quantum field theory, and to some of the applications to constructive quantum field theory. Our purpose is to show that, fifty years after its introduction, the Euclidean theory is still interesting, both from the point of view of technical application and physical interpretation.

The paper is organized as follows. In Section 2, by considering simple systems made of a single spinless relativistic particle, we introduce the relevant structures in both Euclidean and Minkowski world. In particular, a kind of (pre)Markov property is introduced already at the one particle level.

Next Section 3 contains the description of the procedure of second quantization on the one particle structure. The free Markov field is introduced, and its crucial Markov property explained. By following Nelson, we use probabilistic concepts and methods, whose relevance for constructive quantum field theory became immediately more and more apparent. The very structure of classical statistical mechanics for Euclidean fields is firmly based on these probabilistic methods. In Section 4 we introduce the interaction and we
show the connection between the Markov theory and the Hamiltonian theory, for two-dimensional space-cutoff interacting scalar fields. In particular, we present the Feynman-Kac-Nelson formula that gives an explicit expression of the semigroup generated by the space-cutoff Hamiltonian in $\Phi_\Omega$ space. We deal also with some applications to constructive quantum field theory. Section 5 is dedicated to a short discussion about the physical interpretation of the theory. In particular we discuss the Osterwalder-Schrader reconstruction theorem on Euclidean Schwinger functions, and the Nelson reconstruction theorem on Euclidean fields. For the sake of completeness, we sketch the main ideas of a proposal, advanced in [14], according to which the Euclidean field theory can be interpreted as a stochastic field theory in the physical Minkowski space-time.

Our treatment will be as simple as possible, by relying on the basic structural properties, and by describing methods of presumably very long lasting power. The emphasis given to probabilistic methods, and to the statistical mechanics analogy, is a result of the historical development. Our opinion is that not all possibility of Euclidean field theory have been fully exploited yet, both from a technical and physical point of view.

2 One particle systems

A system made of only one relativistic scalar particle, of mass $m > 0$, has a quantum state space represented by the positive frequency solutions of the Klein-Gordon equation. In momentum space, with points $p_\mu, \mu = 0, 1, 2, 3$, let us introduce the upper mass hyperboloid, characterized by the constraints $p^2 = p_0^2 - \sum_{i=1}^{3} p_i^2 = m^2$, $p_0 \geq m$, and the relativistic invariant measure on it, formally given by $d\mu(p) = \theta(p_0)\delta(p^2 - m^2)dp$, where $\theta$ is the step function $\theta(x) = 1$ if $x \geq 0$, and $\theta(x) = 0$ otherwise, and $dp$ is the four-dimensional Lebesgue measure. The Hilbert space of quantum states $F$ is given by the square integrable functions on the mass hyperboloid equipped with the invariant measure $d\mu(p)$. Since in some reference frame the mass hyperboloid is uniquely characterized by the space values of the momentum $p$, with the energy given by $p_0 \equiv \omega(p) = \sqrt{p^2 + m^2}$, the Hilbert space $F$ of the states is in fact made of those complex valued tempered distributions $f$ in the configuration space $R^3$, whose Fourier transforms $\hat{f}(p)$ are square integrable functions in momentum space with respect to the image of the relativistic invariant measure $d\p /2\omega(p)$, where $d\p$ is the Lebesgue measure in momentum space. The scalar product on $F$ is defined by

$$\langle f, g \rangle_F = (2\pi)^3 \int \hat{f}^*(p)\hat{g}(p)\frac{d\p}{2\omega(p)}.$$
where we have normalized the Fourier transform in such a way that

\[ f(x) = \int \exp(i p \cdot x) \tilde{f}(p) dp, \]

\[ \tilde{f}(p) = (2\pi)^{-3} \int \exp(-i p \cdot x) \tilde{f}(x) dx, \]

\[ \int \exp(i p \cdot x) dp = (2\pi)^3 \delta(x). \]

The scalar product on \( F \) can be expressed also in the form

\[ \langle f, g \rangle_F = \int \int f(x')^* W(x' - x) g(x) \, dx' \, dx, \]

where we have introduced the two point Wightman function at fixed time, defined by

\[ W(x' - x) = (2\pi)^{-3} \int \exp(i p \cdot (x' - x)) \frac{dp}{2\omega(p)}. \]

A unitary irreducible representation of the Poincaré group can be defined on \( F \) in the obvious way. In particular, the generators of space translations are given by multiplication by the components of \( p \) in momentum space, and the generator of time translations (the energy of the particle) is given by \( \omega(p) \).

For the scalar product of time evolved wave functions we can write

\[ \langle \exp(-it')f, \exp(-it)g \rangle_F = \int \int f(x')^* W(t' - t, x' - x) g(x) \, dx' \, dx, \]

where we have introduced the two point Wightman function, defined by

\[ W(t' - t, x' - x) = (2\pi)^{-3} \int \exp(-i(t - t')) \exp(i p \cdot (x' - x)) \frac{dp}{2\omega(p)}. \]

To the physical single particle system living in Minkowski space-time we associate a kind of mathematical image, living in Euclidean space, from which all properties of the physical system can be easily derived. We start from the two point Schwinger function

\[ S(x) = \frac{1}{(2\pi)^4} \int \frac{\exp(i p \cdot x)}{\sqrt{p^2 + m^2}} \, dp, \]

which is the analytic continuation of the previously given two point Wightman function into the Schwinger points. Here \( x, p \in \mathbb{R}^4 \), and \( p \cdot x = \sum_{i=1}^{4} x_i p_i \).
While $dp$ and $dx$ are the Lebesgue measures in the $R^4$ momentum and configuration spaces respectively. The function $S(x)$ is positive and analytic for $x \neq 0$, decreases as $\exp (-m \|x\|)$ as $x \to \infty$, and satisfies the equation

$$(-\Delta + m^2)S(x) = \delta(x),$$

where $\Delta = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in four dimensions.

The mathematical image we are looking for is described by the Hilbert space $N$ of those tempered distributions in four-dimensional configuration space $R^4$, whose Fourier transforms are square integrable with respect to the measure $dp/\sqrt{p^2 + m^2}$. The scalar product on $N$ is defined by

$$\langle f, g \rangle_N = (2\pi)^4 \int \tilde{f}^*(p) \tilde{g}(p) \frac{dp}{\sqrt{p^2 + m^2}}.$$

Four dimensional Fourier transform are normalized as follows

$$f(x) = \int \exp (ip.x) \tilde{f}(p) \, dp,$$

$$\tilde{f}(p) = (2\pi)^{-4} \int \exp (-ip.x) \tilde{f}(x) \, dx,$$

$$\int \exp (ip.x) dp = (2\pi)^4 \delta(x).$$

We write also

$$\langle f, g \rangle_N = \int \int f^*(x) S(x-y) g(y) \, dx \, dy = \langle f, (-\Delta + m^2)^{-1} g \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the ordinary Lebesgue product defined on Fourier transforms and $(-\Delta + m^2)^{-1}$ is given by multiplication by $(p^2 + m^2)^{-1}$ in momentum space. The Schwinger function $S(x-y)$ is formally the kernel of the operator $(-\Delta + m^2)^{-1}$. The Hilbert space $N$ is the carrier space of a unitary (non-irreducible) representation of the four-dimensional Euclidean group $E(4)$. In fact, let $(a, R)$ be an element of $E(4)$

$$(a, R) : R^4 \to R^4$$

$$x \to Rx + a,$$

where $a \in R^4$, and $R$ is an orthogonal matrix, $RR^T = R^TR = 1_4$. Then the transformation $u(a, R)$ defined by

$$u(a, R) : N \to N$$

$$f(x) \to (u(a, R)f)(x) = f(R^{-1}(x-a)).$$
provides the representation. In particular, we consider the reflection \( r_0 \) with respect to the hyperplane \( x_4 = 0 \), and the translations \( u(t) \) in the \( x_4 \) direction. Then we have \( r_0 u(t) r_0 = u(-t) \), and analogously for other hyperplanes.

Now we introduce a local structure on \( N \) by considering, for any closed region \( A \) of \( R^4 \), the subspace \( N_A \) of \( N \) made by distributions in \( N \) with support on \( A \). We call \( e_A \) the orthogonal projection on \( N_A \). It is obvious that if \( A \subset B \) then \( N_A \subset N_B \) and \( e_A e_B = e_B e_A = e_A \). A kind of (pre)Markov property for one particle systems is introduced as follows. Consider a closed three dimensional piece-wise smooth manifold \( \sigma \), which divides \( R^4 \) in two closed regions \( A \) and \( B \), having \( \sigma \) in common. Therefore \( \sigma \subset A, \sigma \subset B, A \cap B = \sigma, A \cup B = R^4 \). Let \( N_A, N_B, N_\sigma \), and \( e_A, e_B, e_\sigma \) be the associated subspaces and projections respectively. Then \( N_\sigma \subset N_A, N_\sigma \subset N_B \), and \( e_\sigma e_A = e_A e_\sigma = e_\sigma, e_\sigma e_B = e_B e_\sigma = e_\sigma \). It is very simple to prove the following.

**Theorem 1.** Let \( e_A, e_B, e_\sigma \) be defined as above, then \( e_A e_B = e_B e_A = e_\sigma \).

Clearly, it is enough to show that for any \( f \in N \) we have \( e_A e_B f \in N_\sigma \). In that case \( e_\sigma e_A e_B f = e_A e_B f \), from which the theorem easily follows. Since \( e_A e_B f \) has support on \( A \), we must show that for any \( C^0 \) function \( g \) with support on \( A_\sigma \) we have \( \langle g, e_A e_B f \rangle = 0 \). Then \( e_A e_B f \) has support on \( \sigma \), and the proof is complete. Now we have

\[
\langle g, e_A e_B f \rangle = \langle (-\Delta + m^2) g, e_A e_B f \rangle_N = \langle e_A (-\Delta + m^2) g, e_B f \rangle_N = \langle (-\Delta + m^2) g, e_B f \rangle_N = \langle g, e_B f \rangle = 0,
\]

where we have used the definition of \( \langle \cdot, \cdot \rangle_N \) in terms of \( \langle \cdot, \cdot \rangle \), the fact that \( e_A (-\Delta + m^2) g = (-\Delta + m^2) g \), and analogously for other hyperplanes. This ends the proof of the (pre)Markov property for one particle systems.

A very important role in the theory is played by subspaces of \( N \) associated to hyperplanes in \( R^4 \). To fix ideas, consider the hyperplane \( x_4 = 0 \), and the associated subspace \( N_0 \). A tempered distribution in \( N \) with support on \( x_4 = 0 \) has necessarily the form \( (f \otimes \delta_0)(x) = f(x) \delta(x_4) \), with \( f \in F \). By using the basic magic formula, for \( x \geq 0 \) and \( M > 0 \),

\[
\int_{-\infty}^{+\infty} \frac{\exp(ipx)}{p^2 + M^2} dp = \frac{\pi}{M} \exp(-Mx),
\]

it is immediate to verify that \( \| f \otimes \delta_0 \|_N = \| f \|_F \). Therefore, we have an isomorphic and isometric identification of the two Hilbert spaces \( F \) and \( N_0 \).
Obviously, similar considerations hold for any hyperplane. In particular, we consider the hyperplanes \( x_4 = t \), and the associated subspaces \( N_t \). Let us introduce injection operators \( j_t \) defined by

\[
\begin{align*}
    j_t : F &\to N \\
    f &\mapsto f \otimes \delta_t,
\end{align*}
\]

where \( f \) is a generic element of \( F \), with values \( f(x) \), and \((f \otimes \delta_t)(x) = f(x) \delta(x_4 - t)\). It is immediate to verify the following properties for \( j_t \) and its adjoint \( j_t^* \). The range of \( j_t \) is \( N_t \). Moreover, \( j_t \) is an isometry, so that \( j_t^* j_t = 1_F \), \( j_t j_t^* = e_t \), where \( 1_F \) is the identity on \( F \), and \( e_t \) is the projection on \( N_t \). Moreover, \( e_t j_t = j_t \) and \( j_t^* = j_t e_t \).

If we introduce translations \( u(t) \) along the \( x_4 \) direction and the reflection \( r_0 \) with respect to \( x_4 = 0 \), then we also the covariance property \( u(t) j_s = j_{t+s} \), and the reflexivity property \( r_0 j_0 = j_0, j_0^* r_0 = j_0^* \). The reflexivity property is very important. It tells us that \( r_0 \) leaves \( N_0 \) pointwise invariant, and it is an immediate consequence of the fact that \( \delta(x_4) = \delta(-x_4) \).

Therefore, if we start from \( N \) we can obtain \( F \), by taking the projection \( j_s \) with respect to some hyperplane \( \pi \), in particular \( x_4 = 0 \). It is also obvious that we can induce on \( F \) a representation of \( E(3) \) by taking those element of \( E(4) \) that leave \( \pi \) invariant.

Let us now see how we can define the Hamiltonian on \( F \) starting from properties of \( N \). Since we are considering the simple case of the one particle system, we could just perform the following construction explicitly by hands, through a simple application of the basic magic formula given before. But we prefer to follow a route that emphasizes Markov property and can be immediately generalized to more complicated cases.

Let us introduce the operator \( p(t) \) on \( F \) defined by the dilation \( p(t) = j_0 u(t) j_0, t \geq 0 \). Then we prove the following.

**Theorem 2.** The operator \( p(t) \) is bounded and selfadjoint. The family \( \{p(t)\} \), for \( t \geq 0 \), is a norm-continuous semigroup.

**Proof.** Boundedness and continuity are obvious. Selfadjointness is a consequence of reflexivity. In fact

\[
p^*(t) = j_0^* u(-t) j_0 = j_0^* r_0 u(t) r_0 j_0 = j_0^* u(t) j_0 = p(t).
\]

The semigroup property is a consequence of the Markov property. In fact, let us introduce \( N_+, N_0, N_- \) as subspaces of \( N \) made by distributions with support in the regions \( x_4 \geq 0, x_4 = 0, x_4 \leq 0 \), respectively, and call \( e_+, e_0, \)
\( e_- \) the respective projections. By Markov property we have \( e_0 = e_- e_+ \). Now write for \( s, t \geq 0 \)

\[
p(t)p(s) = j_0^* u(t) j_0 j_0^* u(s) j_0 = j_0^* u(t) e_0 u(s) j_0.
\]

If \( e_0 \) could be cancelled, then the semigroup property would follow from the group property of the translations \( u(t) u(s) = u(t + s) \) (a miracle of the dilations!). For this, consider the matrix element

\[
\langle f, p(t)p(s)g \rangle_F = \langle u(-t) j_0 f, e_0 u(s) j_0 g \rangle_N,
\]

recall \( e_0 = e_- e_+ \), and use \( u(s) j_0 g \in N_+ \), and \( u(-t) j_0 f \in N_- \).

Let us call \( h \) the generator of \( p(t) \), so that \( p(t) = \exp(-th) \), for \( t \geq 0 \). By definition, \( h \) is the Hamiltonian of the physical system. A simple explicit calculation shows that \( h \) is just the energy \( \omega \) introduced before. Starting from the representation of the Euclidean group \( E(3) \) already given and from the Hamiltonian, we immediately get a representation of the full Poincaré group on \( F \). Therefore, all physical properties of the one particle system have been reconstructed from its Euclidean image on the Hilbert space \( N \).

As a last remark of this section, let us note that we can consider the real Hilbert spaces \( N_r \) and \( F_r \), made of real elements (in configuration space) in \( N \) and \( F \). The operators \( u(a, t), u(t), r_0, j_\pi, j^*_\pi, e_A \) are all reality preserving, i.e. they map real spaces into real spaces.

This exhausts our discussion about the one particle system. For more details we refer to [6] and [15]. We have introduced the Euclidean image, discussed its main properties, and shown how we can derive all properties of the physical system from its Euclidean image. In the next sections, we will show how this kind of construction carries through the second quantized case and the interacting case.

## 3 Second quantization and free fields

We begin this section with a short review about the procedure of second quantization based on probabilistic methods, by following mainly Nelson [10], see also [6] and [15]. Probabilistic methods are particularly useful in the frame of the Euclidean theory.

Let \( \mathcal{H} \) be a real Hilbert space, with symmetric scalar product \( \langle \cdot, \cdot \rangle \). Let \( \phi(u) \) the elements of a family of centered Gaussian random variables indexed by \( u \in \mathcal{H} \), uniquely defined by the expectation values \( E(\phi(u)) = 0 \), \( E(\phi(u)\phi(v)) = \langle u, v \rangle \). Since \( \phi \) is Gaussian we also have

\[
E(\exp(\lambda \phi(u))) = \exp(\frac{1}{2} \lambda^2 \langle u, u \rangle),
\]

whereas for the second quantized case
\[
E(\phi(u_1)\phi(u_2)\ldots\phi(u_n)) = [u_1u_2\ldots u_n].
\]

Here \([\ldots]\) is the Hafnian of elements \([u_iu_j] = \langle u_i, u_j \rangle\), defined to be zero for odd \(n\), and for even \(n\) by the recursive formula

\[
[u_1u_2\ldots u_n] = \sum_{i=2}^{n} [u_1u_i][u_1u_2\ldots u_{n}]',
\]

where in \([\ldots]'\) the terms \(u_1\) and \(u_i\) are suppressed. Hafnians, from the Latin name of Copenhagen, the first seat of the theoretical group of CERN, were introduced in quantum field theory by Caianiello [16], as a useful tool when dealing with Bose statistics.

Let \((Q, \Sigma, \mu)\) be the underlying probability space where \(\phi\) are defined as random variables. Here \(Q\) is a compact space, \(\Sigma\) a \(\sigma\)-algebra of subsets of \(Q\), and \(\mu\) a regular, countable additive probability measure on \(\Sigma\), normalized to \(\mu(Q) = \int_Q d\mu = 1\).

The fields \(\phi(u)\) are represented by measurable functions on \(Q\). The probability space is uniquely defined, but for trivial isomorphisms, if we assume that \(\Sigma\) is the smallest \(\sigma\)-algebra with respect to which all fields \(\phi(u)\), with \(u \in \mathcal{H}\), are measurable. Since \(\phi(u)\) are Gaussian, then they are represented by \(L^p(Q, \Sigma, \mu)\) functions, for any \(p\) with \(1 \leq p < \infty\), and the expectations will be given by

\[
E(\phi(u_1)\phi(u_2)\ldots\phi(u_n)) = \int_Q \phi(u_1)\phi(u_2)\ldots\phi(u_n) \, d\mu,
\]

where, by a mild abuse of notation, the \(\phi(u_i)\) in the right hand side denote the \(Q\) space functions which represent the random variables \(\phi(u_i)\). We call the complex Hilbert space \(\mathcal{F} = \Gamma(\mathcal{H}) = L^2(Q, \Sigma, \mu)\) the \(\Phi\omega\kappa\) space constructed on \(\mathcal{H}\), and the function \(\Omega_0 \equiv 1\) on \(Q\) the \(\Phi\omega\kappa\) vacuum.

In order to introduce the concept of second quantization of operators, we must introduce subspaces of \(\mathcal{F}\) with a “fixed number of particles”. Call \(\mathcal{F}_{(0)} = \{\lambda\Omega_0\}\), where \(\lambda\) is any complex number. Define \(\mathcal{F}_{(\leq n)}\) as the subspace of \(\mathcal{F}\) generated by complex linear combinations of monomials of the type \(\phi(u_1)\ldots\phi(u_j)\), with \(u_i \in \mathcal{H}\), and \(j \leq n\). Then \(\mathcal{F}_{(\leq n-1)}\) is a subspace of \(\mathcal{F}_{(\leq n)}\). We define \(\mathcal{F}_{(n)}\), the \(n\) particle subspace, as the orthogonal complement of \(\mathcal{F}_{(\leq n-1)}\) in \(\mathcal{F}_{(\leq n)}\), so that

\[
\mathcal{F}_{(\leq n)} = \mathcal{F}_{(n)} \oplus \mathcal{F}_{(\leq n-1)}.
\]

By construction the \(\mathcal{F}_{(n)}\) are orthogonal, and it is not difficult to verify that

\[
\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_{(n)}.
\]
Let us now introduce Wick normal products by the definition

\[ \phi(u_1) \phi(u_2) \ldots \phi(u_n) := E_{(n)} \phi(u_1) \phi(u_2) \ldots \phi(u_n), \]

where \( E_{(n)} \) is the projection on \( \mathcal{F}_{(n)} \). It is not difficult to prove the usual Wick theorem (see for example [6]), and its inversion given by Caianiello [16].

It is interesting to remark that, in the frame of the second quantization performed with probabilistic methods, it is not necessary to introduce creation and destruction operators as in the usual treatment. However the two procedures are completely equivalent, as shown for example in [15].

Given an operator \( A \) from the real Hilbert space \( \mathcal{H}_1 \) to the real Hilbert space \( \mathcal{H}_2 \), we define its second quantized operator \( \Gamma(A) \) through the following definitions

\[ \Gamma(A) \Omega_{01} = \Omega_{02}, \]
\[ \Gamma(A) : \phi_1(u_1) \phi_1(u_2) \ldots \phi_1(u_n) := \phi_2(Au_1) \phi_2(Au_2) \ldots \phi_2(Au_n) ; \]

where we have introduced the probability spaces \( Q_1 \) and \( Q_2 \), their vacua \( \Omega_{01} \) and \( \Omega_{02} \), and the random variables \( \phi_1 \) and \( \phi_2 \), associated to \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. The following remarkable theorem by Nelson [10] gives a full characterization of \( \Gamma(A) \), very useful in the applications.

**Theorem 3.** Let \( A \) be a contraction from the real Hilbert space \( \mathcal{H}_1 \) to the real Hilbert space \( \mathcal{H}_2 \). Then \( \Gamma(A) \) is an operator from \( L^1_{(1)} \) to \( L^1_{(2)} \) which is positivity preserving, \( \Gamma(A)u \geq 0 \) if \( u \geq 0 \), and such that \( E(\Gamma(A)u) = E(u) \). Moreover, \( \Gamma(A) \) is a contraction from \( L^p_{(1)} \) to \( L^p_{(2)} \) for any \( p, 1 \leq p < \infty \). Finally, \( \Gamma(A) \) is also a contraction from \( L^p_{(1)} \) to \( L^q_{(2)} \), with \( q \geq p \), if \( \| A \|_2^2 \leq (p - 1)/(q - 1) \).

We have indicated with \( L^p_{(1)}, L^p_{(2)} \) the \( L^p \) spaces associated to \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. This is the celebrated Nelson best hypercontractive estimate. For the proof we refer to the original paper [10], see also [15].

This exhausts our short review on the theory of second quantization based on probabilistic methods.

The usual time-zero quantum field \( \bar{\phi}(u) \), \( u \in F_r \), in the \( \Phi\phi\kappa \) representation, can be obtained through second quantization starting from \( F_r \). We call \((\bar{Q}, \bar{\Sigma}, \bar{\mu})\) the underlying probability space, and \( \mathcal{F} = \Gamma(F_r) = L^2(\bar{Q}, \bar{\Sigma}, \bar{\mu}) \) the Hilbert \( \Phi\phi\kappa \) space of the free physical particles.

Now we introduce the free Markov field \( \phi(f) \), \( f \in N_r \), by taking \( N_r \) as the starting point. We call \((Q, \Sigma, \mu)\) the associated probability space. We introduce the Hilbert space \( \mathcal{N} = \Gamma(N_r) = L^2(Q, \Sigma, \mu) \), and the operators
\[ U(a, R) = \Gamma(u(a, R)), \quad R_0 = \Gamma(r_0), \quad U(t) = \Gamma(u(t)), \quad E_A = \Gamma(e_A), \] and so on, for which the previous Nelson theorem holds (take \( \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{N}_r \)).

Since in general \( \Gamma(AB) = \Gamma(A)\Gamma(B) \), then we have immediately the following expression of the Markov property \( E_\sigma = E_A E_B \), where the closed regions \( A, B, \sigma \) of the Euclidean space have the same properties explained before during the proof of the (pre)Markov property for one particle systems.

It is obvious that \( E_A \) can also be understood as conditional expectation with respect to the sub-\( \sigma \)-algebra \( \Sigma_A \) generated by the field \( \phi(f) \) with \( f \in \mathcal{N}_r \) and the support of \( f \) on \( A \).

The relation, previously pointed out, between \( \mathcal{N}_t \) subspaces and \( \mathcal{F} \) are also valid for their real parts \( \mathcal{N}_{rt} \) and \( \mathcal{F}_r \). Therefore they carry out through the second quantization procedure. We introduce \( J_t = \Gamma(j_t) \) and \( J_t^* = \Gamma(j_t^*) \), then the following properties hold. \( J_t \) is an isometric injection of \( L^p(\bar{Q}, \bar{\Sigma}, \bar{\mu}) \) into \( L^p(Q, \Sigma, \mu) \), the range of \( J_t \) as an operator \( L^2 \to L^2 \) is obviously \( \mathcal{N}_t = \Gamma(\mathcal{N}_{rt}) \), moreover \( J_t J_t^* = E_t \). The free Hamiltonian \( H_0 \) is given for \( t \geq 0 \) by

\[ J_0^* J_t = \exp(-tH_0) = \Gamma(\exp(-t\omega)). \]

Moreover, we have the covariance property \( U(t) J_0 = J_t \), and the reflexivity \( R_0 J_0 = J_0, \quad J_0^* R_0 = J_0^* \).

These relations allow a very simple expression for the matrix elements of the Hamiltonian semigroup in terms of Markov quantities. In fact, for \( u, v \in \mathcal{F} \) we have

\[ \langle u, \exp(-tH_0)v \rangle = \int_Q (J_t u)^* J_0 v \, d\mu. \]

In the next section we will generalize this representation to the interacting case.

Finally, let us derive the hypercontractive property of the free Hamiltonian semigroup.

Since \( \| \exp(-t\omega) \| \leq \exp(-tm) \), where \( m \) is the mass of the particle, we have immediately by a simple application of Nelson theorem

\[ \| \exp(-tH_0) \|_{p,q} \leq 1, \]

provided \( q - 1 \leq (p - 1) \exp(2tm) \), where \( \| \ldots \|_{p,q} \) denotes the norm of an operator from \( L^p \) to \( L^q \) spaces.

### 4 Interacting fields

The discussion of the previous sections was limited to free fields both in Minkowski and Euclidean spaces. Now we must introduce interaction in order to get nontrivial theories.
Firstly, as a general motivation, we will proceed quite formally, then we will resort to precise statements.

Let us recall that in standard quantum field theory, for scalar self-coupled fields, the time ordered products of quantum fields in Minkowski space-time can be expressed formally through the formula

\[
\frac{\langle T(\phi(x_1)\ldots\phi(x_n)\exp(i\int L \, dx)) \rangle}{\langle T \exp(i\int L \, dx) \rangle},
\]

where \( T \) denotes time ordering, \( \phi \) are free fields in Minkowski space-time, \( L \) is the interaction Lagrangian, and \( \langle \ldots \rangle \) are vacuum averages. As it is very well known, this expression can be put for example at the basis of perturbative expansions, giving rise to terms expressed through Feynman graphs. The appropriate chosen normalization provides automatic cancellation of the vacuum to vacuum graphs.

Now we can introduce a formal analytic continuation to the Schwinger points, as previously done for the one particle system, and obtain the following expression for the analytic continuation of the field time ordered products, now called Schwinger functions,

\[
S(x_1,\ldots,x_n) = \frac{\langle \phi(x_1)\ldots\phi(x_n)\exp U \rangle}{\langle \exp U \rangle}.
\]

Here \( x_1,\ldots,x_n \) denote points in Euclidean space, \( \phi \) are the Euclidean fields introduced before. The chronological time ordering disappears, because the fields \( \phi \) are commutative, and there is no distinguished “time” direction in Euclidean space. The symbol \( \langle \ldots \rangle \) denotes here expectation values represented by \( \int \ldots d\mu \), as explained before, and \( U \) is the Euclidean “action” of the system formally given by the integral on Euclidean space

\[
U = -\int P(\phi(x)) \, dx,
\]

if the field self-interaction is produced by the polynomial \( P \).

Therefore, these formal considerations suggest that the passage from the free Euclidean theory to the fully interacting one is obtained through a change of the free probability measure \( d\mu \) to the interacting measure

\[
\exp U \, d\mu / \int_Q \exp U \, d\mu.
\]

The analogy with classical statistical mechanics is evident. The expression \( \exp U \) acts as Boltzmannfaktor, and \( Z = \int_Q \exp U \, d\mu \) is the partition function.
Our task will be to make these statements precise from a mathematical point of view. We will be obliged to introduce cutoffs, and then be involved in their careful removal.

For the sake of convenience, we make the substantial simplification of considering only two-dimensional theories (one space - one time dimensions in the Minkowski region) for which the well known ultraviolet problem of quantum field theory gives no trouble. There is no difficulty in translating the content of the previous sections to the two-dimensional case.

Let \( P \) be a real polynomial, bounded below and normalized to \( P(0) = 0 \). We introduce approximations \( h \) to the Dirac \( \delta \) function at the origin of the two-dimensional Euclidean space \( \mathbb{R}^2 \), with \( h \in \mathbb{N} \). Let \( h_x \) be the translate of \( h \) by \( x \), with \( x \in \mathbb{R}^2 \). The introduction of \( h \), equivalent to some ultraviolet cutoff, is necessary, because local fields, of the formal type \( \phi(x) \), have no rigorous meaning, and some smearing is necessary.

For some compact region \( \Lambda \) in \( \mathbb{R}^2 \), acting as space cutoff (infrared cutoff), introduce the \( Q \) space function

\[
U^{(h)}_{\Lambda} = - \int_{\Lambda} : P(\phi(h_x)) : \ dx,
\]

where \( dx \) is the Lebesgue measure in \( \mathbb{R}^2 \). It is immediate to verify that \( U^{(h)}_{\Lambda} \) is well defined, bounded below and belongs to \( L^p(Q, \Sigma, \mu) \), for any \( p, 1 \leq p < \infty \). This is the infrared and ultraviolet cutoff action. Notice the presence of the Wick normal products in its definition. They provide a kind of automatic introduction of counter-terms, in the frame of renormalization theory.

The following theorem allows to remove the ultraviolet cutoff.

**Theorem 4.** Let \( h \to \delta \), in the sense that the Fourier transforms \( \tilde{h} \) are uniformly bounded and converge pointwise in momentum space to the Fourier transform of the \( \delta \) function given by \( (2\pi)^{-2} \). Then \( U^{(h)}_{\Lambda} \) is \( L^p \) convergent for any \( p, 1 \leq p < \infty \), as \( h \to \delta \). Call \( U_{\Lambda} \) the \( L^p \) limit, then \( U_{\Lambda}, \exp U_{\Lambda} \in L^p(Q, \Sigma_{\Lambda}, \mu) \), for \( 1 \leq p < \infty \).

The proof uses standard methods of probability theory, and originates from pioneering work of Nelson in [17]. It can be found for example in [6], and [15].

Since \( U_{\Lambda} \) is defined with normal products, and the interaction polynomial \( P \) is normalized to \( P(0) = 0 \), an elementary application of Jensen inequality gives

\[
\int_Q \exp U_{\Lambda} \ d\mu \geq \exp \int_Q U_{\Lambda} \ d\mu = 1.
\]
Therefore, we can rigorously define the new space cutoff measure in $Q$ space

$$d\mu_\Lambda = \exp U_\Lambda d\mu / \int_Q \exp U_\Lambda d\mu.$$ 

The space cutoff interacting Euclidean theory is defined by the same fields on $Q$ space, but with a change in the measure and therefore in the expectation values. The correlations for the interacting fields $\bar{\phi}$ are the cutoff Schwinger functions

$$S_\Lambda(x_1, \ldots, x_n) = \langle \bar{\phi}(x_1) \ldots \bar{\phi}(x_n) \rangle = Z_\Lambda^{-1} \langle \phi(x_1) \ldots \phi(x_n) \exp U_\Lambda \rangle,$$

where the partition function is

$$Z_\Lambda = \langle \exp U_\Lambda \rangle.$$

We see that the analogy with statistical mechanics is complete here. Of course, the introduction of the space cutoff $\Lambda$ destroys translation invariance. The full Euclidean covariant theory must be recovered by taking the infinite volume limit $\Lambda \to \mathbb{R}^2$ on field correlations. For the removal of the space cutoff all methods of statistical mechanics are available. In particular, correlation inequalities of ferromagnetic type can be easily exploited, as shown for example in [6] and [15].

We would like to conclude this section by giving the connection between the space cutoff Euclidean theory and the space cutoff Hamiltonian theory in the physical $\Phi^\kappa$ space.

For $\ell \geq 0$, $t \geq 0$, consider the rectangle in $\mathbb{R}^2$

$$\Lambda(\ell, t) = \{(x_1, x_2) : -\ell / 2 \leq x_1 \leq \ell / 2, 0 \leq x_2 \leq t\},$$

and define the operator in the physical $\Phi^\kappa$ space

$$P_\ell(t) = J_0 e^{U_\Lambda(\ell, t)} J_t,$$

where $J_0$ and $J_t$ are injections relative to the lines $x_2 = 0$ and $x_2 = t$, respectively. Then the following theorem, largely due to Nelson, holds.

**Theorem 5.** The operator $P_\ell(t)$ is bounded and selfadjoint. The family $\{P_\ell(t)\}$, for $\ell$ fixed and $t \geq 0$ is a strongly continuous semigroup. Let $H_\ell$ be its lower bounded selfadjoint generator, so that $P_\ell(t) = \exp(-t H_\ell)$. On the physical $\Phi^\kappa$ space, there is a core $D$ for $H_\ell$ such that on $D$ the following equality holds $H_\ell = H_0 + V_\ell$, where $H_0$ is the free Hamiltonian introduced before and $V_\ell$ is the volume cutoff interaction given by

$$V_\ell = \lim_{\epsilon \to 0} \int_{-\epsilon / 2}^{\epsilon / 2} (\bar{\phi}(h_{x_1})) : dx_1,$$
where $h_{x_1}$ are the translates of approximations to the $\delta$ function at the origin on the $x_1$ space, and the limit is taken in $L^p$, in analogy to what has been explained for the two-dimensional case in the definition of $U_\Lambda$.

While we refer to [6] and [15] for a full proof, we mention here that boundedness is related to hypercontractivity of the free Hamiltonian, selfadjointness is a consequence of reflexivity, and the semigroup property follows from Markov property. This theorem is remarkable, because it expresses the cutoff interacting Hamiltonian semigroup in an explicit form in the Euclidean theory through probabilistic expectations. In fact we have

$$\langle u, \exp(-tH_\ell)v \rangle = \int_Q (J_t u)^* J_0 v \exp U_\Lambda(\ell, t) \, d\mu.$$

We could call this expression as the Feynman-Kac-Nelson formula, in fact it is nothing but a path integral expressed in stochastic terms, and adapted to the Hamiltonian semigroup.

By comparison with the analogous formula given for the free Hamiltonian semigroup, we see that the introduction of the interaction inserts the Boltzmannfaktor under the integral.

As an immediate consequence of the Feynman-Kac-Nelson formula, together with Euclidean covariance, we have the following astonishing Nelson symmetry

$$\langle \Omega_0, \exp(-tH_\ell)\Omega_0 \rangle = \langle \Omega_0, \exp(-\ell H_\ell)\Omega_0 \rangle,$$

which was at the basis of [11] and [12], and played some role in showing the effectiveness of Euclidean methods in constructive quantum field theory.

It is easy to establish, through simple probabilistic reasoning, that $H_\ell$ has a unique ground state $\Omega_\ell$ of lowest energy $E_\ell$. For a convenient choice of normalization and phase factor, one has $\| \Omega_\ell \|_2 = 1$, and $\Omega_\ell > 0$ almost everywhere on $Q$ space (for Bosonic systems ground states have no nodes in configuration space!). Moreover, $\Omega_\ell \in L^p$, for any $1 \leq p < \infty$. If $\ell > 0$ and the interaction is not trivial, then $\Omega_\ell \neq \Omega_0$, $E_\ell < 0$, and $\| \Omega_\ell \|_1 < 1$. Obviously $\| \exp(-tH_\ell) \|_{2,2} = \exp(-tE_\ell)$.

The general structure of Euclidean field theory, as explained in this section, has been at the basis of all applications in constructive quantum field theory. These applications include the proof of the existence of the infinite volume limit, with the establishment of all Wightman axioms, for two and three-dimensional theories. Moreover, the existence of phase transitions and symmetry breaking has been firmly established. Extensions have been also given to theories involving Fermions, and to gauge field theory. Due to the scope of this review, limited to a description of the general structure of
Euclidean field theory, we can not give a detailed treatment of these applications. Therefore, we refer to recent general reviews on constructive quantum field theory for a complete description of all results, as for example explained by Jaffe in [18]. For recent applications of Euclidean field theory to quantum fields on curved space-time manifolds we refer for example to [19].

5 The physical interpretation of Euclidean field theory

Euclidean field theory has been considered by most researchers as a very useful tool for the study of quantum field theory. In particular, it is quite easy for example to obtain the fully interacting Schwinger functions in the infinite volume limit in two-dimensional space-time. At this point the problem arises of connecting these Schwinger functions with observable physical quantities in Minkowski space-time. A very deep result of Osterwalder and Schrader [13] gives a very natural interpretation of the resulting limiting theory. In fact, the Euclidean theory, as has been shown before, arises from an analytic continuation from the physical Minkowski space-time to the Schwinger points, through a kind of analytic continuation in time, also called Wick rotation, because Wick exploited this trick in the study of the Bethe-Salpeter equation. Therefore, having obtained the Schwinger functions for the full covariant theory, after all cutoff removal, it is very natural to try to reproduce the inverse analytic continuation in order to recover the Wightman functions in Minkowski space-time. Therefore, Osterwalder and Schrader have been able to identify a set of conditions, quite easy to verify, which allow to recover Wightman functions from Schwinger functions. A key role in this reconstruction theorem is played by the so called reflection positivity for Schwinger functions, a property quite easy to verify. In this way a fully satisfactory solution for the physical interpretation of Euclidean field theory is achieved.

From an historical point of view, an alternate route is possible. In fact, at the beginning of the exploitation of Euclidean methods in constructive quantum field theory, Nelson was able to isolate a set of axioms for the Euclidean fields [9], allowing the reconstruction of the physical theory. Of course, Nelson axioms are more difficult to verify, since they involve properties of the Euclidean fields and not only of the Schwinger functions. However, it is still very interesting to investigate whether the Euclidean fields play only an auxiliary role in the construction of the physical content of relativistic theories, or do they have a more fundamental meaning.
From a physical point of view, the following considerations could also lead to further developments along this line. By its very structure, the Euclidean theory contains the fixed time quantum correlations in the vacuum. In elementary quantum mechanics, it is possible to derive all physical content of the theory from the simple knowledge of the ground state wave function, including scattering data. Therefore, at least in principle, it should be possible to derive all physical content of the theory directly from the Euclidean theory, without any analytic continuation.

We conclude this short section on the physical interpretation of the Euclidean theory, with a mention to a quite surprising result \[14\], obtained by submitting classical field theory to the procedure of stochastic quantization in the sense of Nelson \[20\]. The procedure of stochastic quantization associates a stochastic process to each quantum state. In this case, in a fixed reference frame, the procedure of stochastic quantization applied to interacting fields, produces for the ground state a process, in the physical space-time, which has the same correlations as Euclidean field theory. This open the way to a possible interpretation of Euclidean field theory directly in Minkowski space-time. However, a consistent development along this line requires a new formulation of representations of the Poincaré group in the form of measure preserving transformations in the probability space where the Euclidean fields are defined. This difficult task has not been accomplished yet.

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