Orthant Based Proximal Stochastic Gradient Method for $\ell_1$-Regularized Optimization

Tianyi Chen$^1$, Tianyu Ding$^2$, Bo Ji$^3$, Guanyi Wang$^4$, Yixin Shi$^1$, Sheng Yi$^1$, Xiao Tu$^1$, and Zhihui Zhu$^5$

$^1$ Microsoft, Redmond, USA
$^2$ Johns Hopkins University, Baltimore, USA
$^3$ Zhejiang University, China
$^4$ Georgia Institute of Technology, USA
$^5$ University of Denver, USA

Abstract. Sparsity-inducing regularization problems are ubiquitous in machine learning applications, ranging from feature selection to model compression. In this paper, we present a novel stochastic method – Orthant Based Proximal Stochastic Gradient Method (OBProx-SG) – to solve perhaps the most popular instance, i.e., the $\ell_1$-regularized problem. The OBProx-SG method contains two steps: (i) a proximal stochastic gradient step to predict a support cover of the solution; and (ii) an orthant step to aggressively enhance the sparsity level via orthant face projection. Compared to the state-of-the-art methods, e.g., Prox-SG, RDA and Prox-SVRG, the OBProx-SG not only converges to the global optimal solutions (in convex scenario) or the stationary points (in non-convex scenario), but also promotes the sparsity of the solutions substantially. Particularly, on a large number of convex problems, OBProx-SG outperforms the existing methods comprehensively in the aspect of sparsity exploration and objective values. Moreover, the experiments on non-convex deep neural networks, e.g., MobileNetV1 and ResNet18, further demonstrate its superiority by achieving the solutions of much higher sparsity without sacrificing generalization accuracy.

Keywords: Sparse Optimization · Stochastic Learning · Orthant Prediction.

1 Introduction

Plentiful tasks in machine learning and deep learning require formulating and solving particular optimization problems [39], of which the solutions may not be unique. From the perspective of the application, people are usually interested in a subset of the solutions with certain properties. A common practice to address the issue is to augment the objective function by adding a regularization term [24]. One of the best known examples is the sparsity-inducing regularization, which encourages highly sparse solutions (including many zero elements). Besides, such regularization typically has shrinkage effects to reduce the magnitude of the solutions [23]. Among the various ways of introducing sparsity, the $\ell_1$-regularization is perhaps the most popular choice. Its utility has been demonstrated ranging from improving the interpretation and accuracy of model estimation [21,22] to compressing heavy model for efficient inference [7,12].
In this paper, we propose and analyze a novel efficient stochastic method to solve
the following large-scale $\ell_1$-regularization problem:

$$\minimize_{x \in \mathbb{R}^n} \{ F(x) \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} f_i(x) + \lambda \|x\|_1 \},$$

(1)

where $\lambda > 0$ is a weighting term to control the level of sparsity in the solutions, and
$f(x)$ is the raw objective function. We pay special interests to the $f(x)$ as the average
of numerous $N$ continuously differentiable instance functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, such as
the loss functions measuring the deviation from the observations in various data fitting
problems. A larger $\lambda$ typically results in a higher sparsity while sacrifices more on the
bias of model estimation, hence $\lambda$ needs to be carefully fine-tuned to achieve both low
$f(x)$ and high sparse solutions. Above formulation is widely appeared in many con-
texts, including convex optimization, e.g., LASSO, logistic regression and elastic-net
formulations [23,33], and non-convex problems such as deep neural networks [30,31].

Problem (1) has been well studied in deterministic optimization with various meth-
ods that capable of returning solutions with both low objective value and high sparsity
under proper $\lambda$. Proximal methods are classical approaches to solve the structured non-
smooth optimization problems with the formulation (1), including the popular proximal
gradient method (Prox-FG) and its variants, e.g., ISTA and FISTA [2], in which only
the first-order derivative information is used. They have been proved to be quite useful
in practice because of their simplicity. Meanwhile, first-order methods are limited due
to the local convergence rate and lack of robustness on ill-conditioned problems, which
can often be overcome by employing the second-order derivative information as is used
in proximal-Newton methods [18,29]. However, when $N$ is enormous, a straightforward
computation of the full gradients or Hessians could be prohibitive because of the
costly evaluations over all $N$ instances. Thus, in modern large-scale machine learning
applications, it is inevitable to use stochastic methods that operate on a small subset of
above summation to economize the computational cost at every iteration.

Nevertheless, in stochastic optimization, the studies of $\ell_1$-regularization (1) become
somewhat limited. In particular, the existing state-of-the-art stochastic algorithms rarely
achieve both fast convergence and highly sparse solutions simultaneously due to the
stochastic nature [26]. Proximal stochastic gradient method (Prox-SG) [10] is a natu-
ral extension of Prox-FG by using a mini-batch to estimate the full gradient. However,
there are two potential drawbacks of Prox-SG: (i) the lack of exploitation on the certain
problem structure, e.g., the $\ell_1$ regularization (1); (ii) the slower convergence rate than
Prox-FG due to the variance introduced by random sampling. To exploit the regulariza-
structure more effectively (produce sparser solutions), regularized dual-averaging
method (RDA) [26] is proposed by extending the simple dual averaging scheme in [20].
The key advantages of RDA are to utilize the averaged accumulated gradients of $f(x)$
and an aggressive coefficient of the proximal function to achieve a more aggressive trun-
cation mechanism than Prox-SG. As a result, in convex setting, RDA usually generates
much sparser solutions than that by Prox-SG in solving (1) but typically has slower
convergence. On the other hand, to reduce the variance brought by the stochastic ap-
proximation, proximal stochastic variance-reduced gradient method (Prox-SVRG) [27]
Orthant Based Proximal Stochastic Gradient Method for \( \ell_1 \)-Regularized Optimization

is developed based on the well-known variance reduction technique SVRG developed in [15]. Prox-SVRG has both capabilities of decent convergence rate and sparsity exploitation in convex setting, while its per iteration cost is much higher than other approaches due to the calculation of full gradient for achieving the variance reduction.

The above mentioned Prox-SG, RDA and Prox-SVRG are valuable state-of-the-art stochastic algorithms with apparent strength and weakness. RDA and Prox-SVRG are derived from proximal gradient methods, and make use of different averaging techniques across all instances to effectively exploit the problem structure. Although they explore sparsity well in convex setting, the mechanisms may not perform as well as desired in non-convex formulations [8]. Moreover, observing that the proximal mapping operator is applicable for any non-smooth penalty function, this generic operator may not be sufficiently insightful if the regularizer satisfies extra properties. In particular, the non-smooth \( \ell_1 \)-regularized problems of the form \( \mathbf{1} \) degenerate to a smooth reduced space problem if zero elements in the solution are correctly identified.

This observation has motivated the exploitation of orthant based methods, a class of deterministic second-order methods that utilizes the particular structure within the \( \ell_1 \)-regularized problem \( \mathbf{1} \). During the optimization, they predict a sequence of orthant faces, and minimize smooth quadratic approximations to \( \mathbf{1} \) on those orthant faces until a solution is found \( \mathbf{1} \mathbf{5} \mathbf{16} \). Such a process normally equips with second-order techniques to yield superlinear convergence towards the optimum, and introduces sparsity by Euclidean projection onto the constructed orthant faces. Orthant based methods have been demonstrated competitiveness in deterministic optimization to proximal methods \( \mathbf{1} \mathbf{5} \mathbf{16} \). In contrast, related prior work in stochastic settings is very rare, perhaps caused by the expensive and non-reliable orthant face selection under randomness.

**Our Contributions.** In this paper, we propose an Orthant Based Proximal Stochastic Gradient Method (OBProx-SG) by capitalizing on the advantages of orthant based methods and Prox-SG, while avoiding their disadvantages. Our OBProx-SG is efficient, promotes sparsity more productively than others, and converges well in both practice and theory. Specifically, we have the following contributions.

- We provide a novel stochastic algorithmic framework that utilizes Prox-SG Step and reduced space Orthant Step to effectively solve problem \( \mathbf{1} \). Comparing with the existing stochastic algorithms, it exploits the sparsity significantly better by combining the moderate truncation mechanism of Prox-SG and an aggressive orthant face projection under the control of a switching mechanism. The switching mechanism is specifically established in the stochastic setting, which is simple but efficient, and performs quite well in practice. Moreover, we present the convergence characteristics under both convex and non-convex formulations, and provide analytic and empirical results to suggest the strategy of the inherent switching hyperparameter selection.
- We carefully design the Orthant Step for stochastic optimization in the following aspects: (i) it utilizes the sign of the previous iterate to select an orthant face, which is more efficient compared with other strategies that involve computations of (sub)-gradient in the deterministic orthant based algorithms \( \mathbf{1} \mathbf{16} \); (ii) instead of optimizing with second-order methods, only the first-order derivative information is used to make progress on the constructed orthant face.
Experiments on both convex (logistic regression) and non-convex (deep neural networks) problems show that OBProx-SG usually outperforms the other state-of-the-art methods comprehensively in terms of the sparsity of the solution, final objective value, and runtime. Particularly, in the popular deep learning applications, without sacrificing generalization performance, the solutions computed by OBProx-SG usually possess multiple-times higher sparsity than those searched by the competitors.

2 The OBProx-SG Method

To begin, we summarize the proposed Orthant Based Proximal Stochastic Gradient Method (OBProx-SG) in Algorithm 1. In a very high level, it proceeds one of the two subroutines at each time, so called Prox-SG Step (Algorithm 2) and Orthant Step (Algorithm 3). There exist two switching parameters \( N_P \) and \( N_O \) that control how long we are sticking to each step and when to switch to the other. We will see that the switching mechanism (choices of \( N_P \) and \( N_O \)) is closely related to the convergence of OBProx-SG and the sparsity promotions. But we defer the detailed discussion till the end of this section, while first focus our attention on the Prox-SG Step and Orthant Step.

Algorithm 1 Outline of OBProx-SG for solving (1).

1. **Input:** \( x_0 \in \mathbb{R}^n, \alpha_0 \in (0, 1), \) and \( \{N_P, N_O\} \subset \mathbb{Z}^+ \).
2. **for** \( k = 0, 1, 2, \ldots \) **do**
3. **Switch** Prox-SG Step or Orthant Step by Algorithm 4.
4. **if** Prox-SG Step is selected **then**
5. Compute the Prox-SG Step update:
\[
x_{k+1} \leftarrow \text{Prox-SG}(x_k, \alpha_k) \text{ by Algorithm 2}.
\]
6. **else if** orthant step is selected **then**
7. Compute the Orthant Step update:
\[
x_{k+1} \leftarrow \text{Orthant}(x_k, \alpha_k) \text{ by Algorithm 3}.
\]
8. Update \( \alpha_{k+1} \) given \( \alpha_k \) according to some rule.

**Prox-SG Step.** In Prox-SG step, the algorithm performs one iteration of standard proximal stochastic gradient step to approach a solution of (1). Particularly, at \( k \)-th iteration, we sample a mini-batch \( B_k \) to make an unbiased estimate of the full gradient of \( f \) (line 2 Algorithm 2). Then we utilize the following proximal mapping to yield next iterate as

\[
x_{k+1} = \text{Prox}_{\alpha_k} \lambda \| \cdot \|_1 (x_k - \alpha_k \nabla f_{B_k}(x_k))
= \argmin_{x \in \mathbb{R}^n} \frac{1}{2\alpha_k} \| x - (x_k - \alpha_k \nabla f_{B_k}(x_k)) \|_2^2 + \lambda \| x \|_1.
\] (3)
Orthant Based Proximal Stochastic Gradient Method for $\ell_1$-Regularized Optimization

Algorithm 2 Prox-SG Step.
1: **Input:** Current iterate $x_k$, and step size $\alpha_k$.
2: Compute the stochastic gradient of $f$ on $B_k$
\[ \nabla f_{B_k}(x_k) \leftarrow \frac{1}{|B_k|} \sum_{i \in B_k} \nabla f_i(x_k). \]
\[ (2) \]
3: Compute $x_{k+1} \leftarrow \text{Prox}_{\alpha_k \lambda \| \cdot \|_1} \left( x_k - \alpha_k \nabla f_{B_k}(x_k) \right)$.
4: Return $x_{k+1}$.

Algorithm 3 Orthant Step.
1: **Input:** Current iterate $x_k$, and step size $\alpha_k$.
2: Compute the stochastic gradient of $\tilde{F}$ on $B_k$
\[ \nabla \tilde{F}_{B_k}(x_k) \leftarrow \frac{1}{|B_k|} \sum_{i \in B_k} \nabla \tilde{F}_i(x_k) \]
\[ (5) \]
3: Compute $x_{k+1} \leftarrow \text{Proj}_{O_k} \left( x_k - \alpha_k \nabla \tilde{F}_{B_k}(x_k) \right)$.
4: Return $x_{k+1}$.

It is known that the above subproblem (3) has a closed form solution [2,4]. Denote the trial iterate $\hat{x}_{k+1} := x_k - \alpha_k \nabla f_{B_k}(x_k)$, then $x_{k+1}$ can be computed efficiently as:
\[ [x_{k+1}^i] = \begin{cases} [\hat{x}_{k+1}^i] - \alpha_k \lambda, & \text{if } [\hat{x}_{k+1}^i] > \alpha_k \lambda \\ [\hat{x}_{k+1}^i] + \alpha_k \lambda, & \text{if } [\hat{x}_{k+1}^i] < -\alpha_k \lambda \\ 0, & \text{otherwise} \end{cases} \]
\[ (4) \]

In OBProx-SG, Prox-SG Step generally serves as a globalization mechanism to guarantee convergence and predict a cover of supports (non-zero entries) in the solution. But it alone is insufficient to exploit the sparsity structure because of the relatively moderate truncation mechanism effected in a small projection region, i.e., the trial iterate $\hat{x}_{k+1}$ is projected to zero only if it falls into $[-\alpha_k \lambda, \alpha_k \lambda]$. Our remedy here is to combine it with our Orthant Step, which exhibits an aggressive sparsity promotion mechanism while still remains efficient, as described next.

**Orthant Step.** Since that fundamental to Orthant Step is the manner in which we handle the zero and non-zero elements, we define the following index sets for any $x \in \mathbb{R}^n$:
\[ \mathcal{I}^0(x) := \{ i : [x]^i = 0 \}, \mathcal{I}^+(x) := \{ i : [x]^i > 0 \}, \mathcal{I}^-(x) := \{ i : [x]^i < 0 \}. \]
\[ (6) \]
Furthermore, we denote the non-zero indices of $x$ by $\mathcal{I}^\neq(x) := \mathcal{I}^+(x) \cup \mathcal{I}^-(x)$. To proceed, we define the orthant face $O_k$ that $x_k$ lies in to be
\[ O_k := \{ x \in \mathbb{R}^n : \text{sign}([x]^i) = \text{sign}([x_k]^i) \text{ or } [x]^i = 0, 1 \leq i \leq n \} \]
\[ (7) \]
such that $x \in O_k$ satisfies: (i) $[x]^i_{\mathcal{I}^0(x_k)} = 0$; (ii) for $i \in \mathcal{I}^\neq(x_k)$, $[x]^i$ is either 0 or has the same sign as $[x_k]^i$. 
The key assumption for Orthant Step is that an optimal solution $x^*$ of problem \([1]\) inhabits $O_k$, i.e., $x^* \in O_k$. In other words, the orthant face $O_k$ already covers the support (non-zero entries) of $x^*$. Our goal becomes now minimizing $F(x)$ over $O_k$, i.e., solving the following subproblem:

$$x_{k+1} = \arg\min_{x \in O_k} F(x) = f(x) + \lambda \|x\|_1. \quad (8)$$

By the definition of $O_k$, we know $[x]_{\mathcal{I}^0(x_k)} \equiv 0$ are fixed, and only the entries of $[x]_{\mathcal{I}^0(x_k)}$ are free to move. Hence, (8) is essentially a reduced space optimization problem. Observing that for any $x \in O_k$, $F(x)$ can be written precisely as a smooth function $\tilde{F}(x)$ in the form

$$F(x) = \tilde{F}(x) := f(x) + \lambda \text{sign}(x_k)^T x, \quad (9)$$

therefore (8) is equivalent to the following smooth problem

$$x_{k+1} = \arg\min_{x \in O_k} \tilde{F}(x). \quad (10)$$

A direct way for solving problem (10) would be the projected stochastic gradient descent method, as stated in Algorithm 3. It performs one iteration of stochastic gradient descent method, as stated in Algorithm 3. It performs one iteration of stochastic gradient descent (SGD) step combined with projections onto the orthant face $O_k$. At $k$-th iteration, a mini-batch $\mathcal{B}_k$ is sampled, and is used to approximate the full gradient $\nabla \tilde{F}(x_k)$ by the unbiased estimator $\nabla \tilde{F}_{\mathcal{B}_k}(x_k)$ (line 2, Algorithm 3). The standard SGD update computes a trial point $\hat{x}_{k+1} = x_k - \alpha_k \nabla \tilde{F}_{\mathcal{B}_k}(x_k)$, which is then passed into a projection operator $\text{Proj}_{O_k}(\cdot)$ defined as

$$[\text{Proj}_{O_k}(z)]_i = \begin{cases} [z]_i & \text{if } \text{sign}([z]_i) = \text{sign}([x_k]_i) \\ 0 & \text{otherwise} \end{cases}. \quad (11)$$

Notice that $\text{Proj}_{O_k}(\cdot)$ is an Euclidean projector, and ensures that the trial point $\hat{x}_{k+1}$ is projected back to the current orthant face $O_k$ if it happens to be outside, as illustrated in Figure 1. In the demonstrated example, the next iterate $x_{k+1} = \text{Proj}_{O_k}(\hat{x}_{k+1})$ turns out to be not only a better approximated solution but also sparser compared with $x_k$ since $[x_{k+1}]_2 = 0$ after the projection, which suggests the power of Orthant Step in sparsity promotion. In fact, compared with Prox-SG, the orthant-face projection (11) is a more aggressive sparsity truncation mechanism. Particularly, Orthant Step enjoys a much larger projection region to map a trial iterate to zero comparing with other state-of-the-art stochastic algorithms. Consider the 1D example in Figure 2 where $x_k > 0$, it is clear that the projection region of Orthant Step $(-\infty, \alpha_k \lambda]$ is a superset of that of Prox-SG and Prox-SVRG $[-\alpha_k \lambda, \alpha_k \lambda]$, and it is apparently larger than that of RDA.\(^6\)

In practice, by taking advantage of the fact that (10) is a reduced space problem, i.e., $[x_{k+1}]_{\mathcal{I}^0(x_k)} \equiv 0$, we only need to store a small part of stochastic gradient information $\left[\nabla \tilde{F}(x_k)\right]_{\mathcal{I}^0(x_k)}$, and compute the projection of $\left[\hat{x}_{k+1}\right]_{\mathcal{I}^0(x_k)}$. This makes the whole procedure even more efficient since it is expected that $|\mathcal{I}^0(x_k)| \ll n$.

\(^6\) The deduction for the projection regions of different methods can be found in Appendix A.
Orthant Based Proximal Stochastic Gradient Method for \(\ell_1\)-Regularized Optimization

\[
\hat{x}_{k+1} = \frac{\nabla \tilde{F}_B(x_k)}{-\alpha_k} + \hat{x}_k
\]

\[
\{x \in \mathbb{R}^3 : [x]_1 \geq 0, [x]_2 \geq 0, [x]_3 = 0\}
\]

Fig. 1: Illustration of Orthant Step with projection in (11), where \(O_k = \{x \in \mathbb{R}^3 : [x]_1 \geq 0, [x]_2 \geq 0, [x]_3 = 0\}\).

We emphasize that Orthant Step is one of the keys to the success of our proposed OBProx-SG method in terms of sparsity exploration. It is originated from the orthant based methods in deterministic optimization, which normally utilize second-order information. When borrowing the idea, we make the selection of orthant face \(O_k\) more effective by looking at the sign of the previous iterate (see (7)). Then, we make use of a stochastic version of the projected gradient method in solving the subproblem (10) to introduce sparsity aggressively. As a result, Orthant Step always makes rapid progress to the optimality, while at the same time promotes sparse solutions dedicatedly.

Switching Mechanism. To complete the discussion of the OBProx-SG framework, we now explain how we select Prox-SG or Orthant Step at each iteration, which is crucial in generating accurate solutions with high sparsity. A popular switching mechanism for deterministic multi-routine optimization algorithms utilizes the optimality metric of each routine (typically the norm of (sub)gradient) [5,6]. However, in stochastic learning, this approach does not work well in general due to the additional computation cost of optimality metric and the randomness that may deteriorate the progress of sparsity exploration as numerically illustrated in Appendix C.

To address this issue, we specifically establish a simple but efficient switching mechanism consisting of two hyperparameters \(N_P\) and \(N_O\), which performs quite well in both practice and theory. As stated in Algorithm 4, \(N_P\) (\(N_O\)) controls how many consecutive iterations we would like to spend doing Prox-SG Step (Orthant Step), and then switch to the other. OBProx-SG is highly flexible to different choices of \(N_P\) and \(N_O\). For example, an alternating strategy between one Prox-SG Step and one Orthant Step.
Algorithm 4 Switching Mechanism.

1: **Input:** \( k, N_P, N_O \).
2: if \( \text{mod}(k, N_P + N_O) < N_P \) then
3:   **Return** Prox-SG Step is selected.
4: else
5:   **Return** Orthant Step is selected.

This corresponds to setting \( N_P = N_O = 1 \), and a strategy of first doing several Prox-SG Steps then followed by Orthant Step all the time corresponds to setting \( N_P < \infty, N_O = \infty \). A larger \( N_P \) helps to predict a better orthant face \( O_k \) which hopefully covers the support of \( x^* \), and a larger \( N_O \) helps to explore more sparsity within \( O_k \).

As we will see in Section 3, convergence of Algorithm \( 1 \) requires that either doing Prox-SG Step infinitely many times \( (N_P \leq \infty, N_O < \infty) \), or doing finitely many Prox-SG Steps followed by infinitely many Orthant Steps \( (N_P < \infty, N_O = \infty) \) given the support of \( x^* \) has been covered by some \( O_k \). In practice, without knowing \( x^* \) ahead of time, we can always start by employing Prox-SG Step \( N_P \) iterations, followed by running Orthant Step \( N_O \) iterations, then repeat until convergence. Meanwhile, experiments in Section 4 show that first performing Prox-SG Step sufficiently many times then followed by running Orthant Step all the time usually produces even slightly better solutions. Moreover, for the latter case, a bound for \( N_P \) is provided in Section 3. For simplicity, we refer the OBProx-SG under \( (N_P < \infty, N_O = \infty) \) as OBProx-SG+ throughout the remainder of this paper.

We end this section by giving empirical suggestions of setting \( N_P \) and \( N_O \). Overall, in order to obtain accurate solutions of high sparsity, we highly recommend to start OBProx-SG with Prox-SG Step and ends with Orthant Step. Practically, employing finitely many Prox-SG Steps followed by sticking on Orthant Steps \( (N_P < \infty, N_O = \infty) \) until the termination, is more preferable because of its attractive property regarding maintaining the progress of sparsity exploration. In this case, although the theoretical upper bound of \( N_P \) is difficult to be measured, we suggest to keep running Prox-SG Step until reaching some acceptable evaluation metrics e.g., objectives or validation accuracy, then switch Orthant Step to promote sparsity.

3 Convergence Analysis

In this section, we give a convergence analysis of our proposed OBProx-SG and OBProx-SG+, referred as OBProx-SG(+) for simplicity, corresponding to convex and non-convex problems. Towards that end, we first make the following assumption.

**Assumption 1** The function \( f : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable, and bounded below on the compact level set \( \mathcal{L} := \{ x \in \mathbb{R}^n : F(x) \leq F(x_0) \} \), where \( x_0 \) is the initialization of Algorithm \( 1 \). Moreover, the stochastic gradient \( \nabla f_{B_k} \) and \( \nabla \tilde{F}_{B_k} \) evaluated on the mini-batch \( B_k \) are Lipschitz continuous on the level set \( \mathcal{L} \) with a shared Lipschitz constant \( L \) for all \( B_k \). The gradient \( \nabla \tilde{F}_{B_k}(x) \) is uniformly bounded over \( \mathcal{L} \), i.e., there exists a \( M < \infty \) such that \( \| \nabla \tilde{F}_{B_k}(x) \|_2 \leq M \) holds for any \( x \in \mathbb{R}^n \).
We mention that many of the terms in Assumption 1 often appear in numerical optimization literatures \cite{27,28}. Let $x^*$ be an optimal solution of problem (1), $F^*$ be the minimum, and $\{x_k\}_{k=0}^\infty$ be the iterates generated by Algorithm 1. We then define

$$S_P = \{k \in \mathbb{Z}^{\geq 0} : \text{Prox-SG Step is selected at } k\text{-th iteration}\}$$

$$S_O = \{k \in \mathbb{Z}^{\geq 0} : \text{Orthant Step is selected at } k\text{-th iteration}\}$$

(12)

to indicate which iteration conducts Prox-SG Step (Orthant Step). Next, we define the gradient mapping and its estimator on mini-batch $B$ as follows

$$G_\eta(x) = \frac{1}{\eta} (x - \text{Prox}_{\eta\lambda\|\cdot\|_1} (x - \eta \nabla f(x)))$$

and

$$G_\eta,B(x) = \frac{1}{\eta} (x - \text{Prox}_{\eta\lambda\|\cdot\|_1} (x - \eta \nabla f_B(x))).$$

(13)

Here we define the noise $e(x)$ be the difference between $G_\eta(x)$ and $G_\eta,B(x)$ with zero-mean due to the random sampling of $B$, i.e., $\mathbb{E}_{B}[e(x)] = 0$. $\tilde{x}$ is so-called a stationary point of $F(x)$ if $G_\eta(\tilde{x}) = 0$. Additionally, establishing some convergence results require the below constant to measure the least magnitude of non-zero entries in $x^*$:

$$0 < \delta = \frac{1}{2} \min_{i \in I^{\neq 0}(x^*)} |[x^*]_i|.\quad(15)$$

In our analysis, we consider the following two commonly used step size $\alpha_k$ selections, i.e., either as constant

$$\alpha_k \equiv \alpha \in (0, 1/L], \quad \forall k = 0, 1, 2 \cdots; \quad(16)$$
or decaying per iteration and satisfying

$$\sum_{k=0}^{\infty} \alpha_k = \infty, \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty.\quad(17)$$

3.1 Convex analysis

Now we state the main theorem of OBProx-SG(+), in the convex case.

**Theorem 1.** Suppose $f$ is a convex function, and the step size $\{\alpha_k\}$ is specified as either (16) or (17). Then $\{F(x_k)\}$ obtained by Algorithm 1 converges to optimum in expectation such that

$$\lim_{k \to \infty} \mathbb{E}[F(x_k)] = F^*, \quad(18)$$

if one of the following two situations is satisfied:

(i) $N_P < \infty$ and $N_O < \infty$;

(ii) $N_P < \infty$ and $N_O = \infty$, i.e., $|S_P| = N_P, |S_O| = \infty$. Additionally, the last Prox-SG Step yields $\|x_{N_P+1} - x^*\|_2 < \delta$, and step size of Orthant Steps satisfies

$$\alpha_k \in \left(0, \min \left\{ \frac{2}{L}, \frac{2\delta}{M} \right\} \right), \quad \forall k \in S_O.\quad(19)$$
Moreover, if problem \((1)\) has a unique solution \(x^*\), then we have
\[
\lim_{k \to \infty} \mathbb{E}[x_k] = x^*.
\] (20)

Theorem \([1]\)(i) implies that if OBProx-SG employs Prox-SG Step and Orthant Step alternatively, then the objective values \(\{F(x_k)\}\) and iterates \(\{x_k\}\) converge to optimum in expectation, which is mainly derived from the convergence properties of Prox-SG to optimality. However, in practice, we may hesitate to repeatedly switch back to Prox-SG Step since most likely it is going to ruin the sparsity from the previous iterates by Orthant Step due to the stochastic nature.

It is worth asking that if the convergence is still guaranteed by doing only finitely many Prox-SG Steps and then keeping doing Orthant Steps. In line with this idea, Theorem \([1]\)(ii) states the convergence is still ensured if the last iterate yielded by Prox-SG Step locates close enough to \(x^*\), i.e.,
\[
\|x_{N^p+1} - x^*\|_2 < \delta.
\]
We will see in appendix that it further indicates \(x^*\) inhabits the orthant faces \(\{O_k\} \in S\) of all subsequent iterates updated by Orthant Steps. Consequently, the convergence is then naturally followed by the property of Project Stochastic Gradient Method. Although the assumption \(\|x_{N^p+1} - x^*\|_2 < \delta\) is hard to be verified in practice, setting \(N^p\) to be large enough and \(N_O = \infty\) usually performs quite well, as we will see in Section 4. To end this part, we present an upper bound of \(N^p\) via the probabilistic characterization to reveal that if the step size is sufficiently small, and the mini-batch size is large enough, then after \(N^p\) Prox-SG Steps, OBProx-SG computes iterate \(x_{N^p+1}\) sufficiently close to \(x^*\) with high probability. This result is established under a popular Polyak-ojasiewicz (PL) condition for non-smooth problem \([19]\), i.e., there exists a \(\mu > 0\) such that for any \(x \in \mathbb{R}^n\) and \(\eta > 0\),
\[
\|G_{\eta}(x)\|_2^2 \geq 2\mu(F(x) - F^*).
\] (21)

And based on the Assumption \([1]\) for any \(x\) in the compact level set \(L\), there exists an upper bound \(\sigma^2 > 0\) for the variance of random noise \(e(x)\) generated from the one-point mini-batch, i.e., \(B = \{i\}, i = 1, \ldots, N\).

**Theorem 2.** Suppose \(f\) is convex and satisfies the PL condition \((21)\). There exists some constants \(C > 0, \frac{1}{\gamma L} > \gamma > 0\) such that for any constant \(\tau \in (0, 1)\), if \(\alpha_k\) satisfies
\[
\alpha_k \equiv \alpha < \min \left\{ \frac{8\gamma \mu \tau \delta^2}{(2L\gamma - 1)C}, \frac{1}{2\mu}, \frac{1}{L} \right\},
\] (22)
and the mini-batch size \(|B_k|\) satisfies
\[
|B_k| > \frac{8\gamma \mu \sigma^2}{8\gamma \mu \tau \delta^2 - (2L\gamma - 1)C\alpha}, \quad \forall k = 0, 1, 2, \ldots,
\] (23)
then
\[
\mathbb{P}(\|x_{N^p+1} - x^*\|_2 \leq \delta) \geq 1 - \tau
\] (24)
for any \(N^p \geq K\) where
\[
K := \left\lfloor \log (\text{poly}(\tau \delta^2, 1/|B_k|, \alpha)/(F(x_0) - F^*))) \right\rfloor \log (1 - 2\mu \alpha)
\] (25)
and \(\text{poly}(\tau \delta^2, 1/|B_k|, \alpha)\) represents some polynomial of \(\tau \delta^2, 1/|B_k|\) and \(\alpha\).
In words, Theorem 2 implies that after sufficient number of iterations, with high probability Prox-SG produces an iterate $x_{N_P+1}$ that is $\delta$-close to $x^*$. However, we note that it does not guarantee $x_{N_P+1}$ as sparse as $x^*$; as we explained before, due to the limited projection region and randomness, $x_{N_P+1}$ may still have a large number of non-zero elements, though many of them could be small. As will be demonstrated in Section 4, the following Orthant Steps will significantly promote the sparsity of the solution.

3.2 Non-convex analysis

In this section, we study the convergence properties of OBProx-SG for non-smooth non-convex problems, where the gradient mapping $G_\eta(x)$ defined as (13) is typically used as the convergence criterion. We now present the main theorem for OBProx-SG corresponding to non-convex formulations as follows.

**Theorem 3.** Suppose $f$ is a non-convex function, and the step size $\{\alpha_k\}$ is specified as either (16) or (17). Then $\{x_k\}$ converges to some stationary point in expectation:

$$\lim_{k \to \infty} \mathbb{E} \left[ \|G_{\alpha_k, B_k}(x_k)\|_2 \right] = 0$$

(26)

if one of the following two situations is satisfied:

(i) $N_P < \infty$ and $N_O < \infty$;

(ii) $N_P < \infty$ and $N_O = \infty$, i.e., $|S_P| = N_P, |S_O| = \infty$. Additionally, $f$ is locally convex on $\mathcal{X} = \{x : \|x - \hat{x}\|_2 \leq \delta\}$, where $\hat{x}$ is some stationary point of $F$, the last Prox-SG Step yields $x_{N_P+1} \in \mathcal{X}$, and step size $\alpha_k$ further satisfies (19).

As drawn in Theorem 3 for non-convex problems, the gradient mapping sequence $\{G_{\alpha_k, B_k}(x_k)\}$ converges to zero in expectation. In other words, the iterate $\{x_k\}$ converges to some stationary point in the sense of vanishing gradient mapping. Note that the local convexity-type assumption in case (ii) appears in many non-convex problem analysis, such as: tensor decomposition [11] and one-hidden-layer neural networks [32]. Moreover, under a Polyak-ojasiewicz(PL) condition for non-smooth problem as stated in [21], all stationary points are minimizers of problem (1). Consequently, the convergence to global optimum guarantee is attained as stated in Corollary 1.

**Corollary 1.** If Polyak-ojasiewicz(PL) condition holds, then under the assumption of Theorem 3, OBProx-SG computes iterate sequence which converges to global minimum of problem (1) in expectation, i.e., $\lim_{k \to \infty} \mathbb{E} [F_{B_k}(x_k)] = F^*$.

4 Numerical Experiments

In this section, we consider solving $\ell_1$-regularized classification tasks with both convex and non-convex approaches. In Section 4.1, we focus on logistic regression (convex), and compare OBProx-SG with other state-of-the-art methods including Prox-SG, RDA and Prox-SVRG on numerous datasets. Three evaluation metrics are used for comparison: (i) final objective function value, (ii) density of the solution (percentage of nonzero entries), and (iii) runtime. Next, in Section 4.2 we apply OBProx-SG to deep neural network (non-convex) with popular architectures designed for classification tasks to further demonstrate its effectiveness and superiority. For these extended non-convex experiments, we also evaluate the generalization performance on unseen test data.
### Table 1: Summary of datasets.

| Dataset | N   | n   | Attribute       | Dataset     | N   | n   | Attribute       |
|---------|-----|-----|-----------------|-------------|-----|-----|-----------------|
| a9a     | 32561 | 123 | binary [0, 1]   | real-sim    | 72309 | 20958 | real [0, 1]     |
| higgs   | 11000000 | 28  | real [-3, 41]   | rcv1        | 20242 | 47236 | real [0, 1]     |
| ijcnn1  | 49990 | 22  | real [-1, 1]    | susy        | 5000000 | 18 | real [-17, 34]  |
| kdda    | 8407752 | 20216830 | real [-1, 4] | url_combined | 2396130 | 3231961 | real [-4, 9] |
| news20  | 19996 | 1355191 | unit-length    | w8a         | 49749 | 300  | binary [0, 1]   |

#### 4.1 Convex setting: logistic regression

We first focus on the convex $\ell_1$-regularized logistic regression problem with the form

$$
\min_{(x; b) \in \mathbb{R}^{n+1}} \frac{1}{N} \sum_{i=1}^{N} \log(1 + e^{-l_i(x^T d_i + b)}) + \lambda \|x\|_1,
$$

for binary classification, where $N$ is the number of samples, $n$ is the feature size of each sample, $b$ is the bias, $d_i \in \mathbb{R}^n$ is the vector representation of the $i$-th sample, $l_i \in \{-1, 1\}$ is the label of the $i$-th sample, and $\lambda$ is the regularization parameter. We set $\lambda = 1/N$ throughout the convex experiments, and test problem (27) on 10 publicly available large-scale datasets from LIBSVM repository

http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/

as summarized in Table 1.

We train the models with a maximum number of epochs as 30. Here one epoch means we partition $\{1, \cdots, N\}$ uniformly at random into a set of mini-batches. The mini-batch size $|B|$ for all the convex experiments is set to be $\min\{256, \lceil 0.01 N \rceil\}$ similarly to [28]. The step size $\alpha_k$ for Prox-SG, Prox-SVRG and OBProx-SG is initially set to be 1.0, and decays every epoch with a factor 0.995. For RDA, we fine tune its hyperparameter $\gamma$ per dataset to reach the best results. The switching between Prox-SG Step and Orthant Step plays a crucial role in OBProx-SG. Following Theorem 1(i), we set $N_P = N_O = 5N/|B|$ in Algorithm 1, namely first train the models 5 epochs by Prox-SG Step, followed by performing Orthant Step 5 epochs, and repeat such routine until the maximum number of epochs is reached. Inspired by Theorem 1(ii), we also test OBProx-SG+ with $N_P = 15N/|B|$, $N_O = \infty$ such that after 15 epochs of Prox-SG Steps we stick to Orthant Step till the end. Experiments are conducted on a 64-bit machine with an 3.70GHz Intel Core i7 CPU and 32 GB of main memory.

We compare the performance of OBProx-SG(+) with other methods on the datasets in Table 1 and report the final objective value $F$ and $f$ (Table 2), density (percentage of non-zero entries) in the solution (Table 3) and runtime (Figure 3). For ease of comparison, we mark the best result as bold in the tables.

Our observations are summarized as follows. Table 2 shows that our OBProx-SG(+) performs significantly better than RDA, and is competitive to Prox-SG and Prox-SVRG in terms of the final $F$ and $f$ (round up to 3 decimals), which implies that OBProx-SG(+), Prox-SG and Prox-SVRG can reach comparable convergence results in practice. Besides the convergence, we have a special concern about the sparsity of the solutions. As is demonstrated in Table 3, OBProx-SG(+) is no doubt the best solver. In fact,
OBProx-SG achieves the solutions of highest sparsity (lowest density) on 3 out of 12 datasets, while OBProx-SG+ performs even better, which computes all solutions with the highest sparsity. Apparently, OBProx-SG(+) has strong superiority in promoting sparse solutions while retains almost the same accuracy. Finally, for runtime comparison, we plot the relative runtime of these solvers for each dataset, which is scaled by the maximum runtime consumed by a particular solver on that dataset. Figure 3 indicates that Prox-SG, RDA and OBProx-SG(+) are almost as efficient as each other, while Prox-SVRG takes much more time due to the computation of full gradient.

The above experiments in convex setting demonstrate that the proposed OBProx-SG(+) outperform the other state-of-the-art methods, and have apparent strengths in generating much sparser solutions efficiently and reliably.
4.2 Non-convex setting: deep neural network

We now apply OBProx-SG(+) to the non-convex setting that solves classification tasks by Deep Convolutional Neural Network (CNN) on the benchmark datasets CIFAR10 [17] and Fashion-MNIST [25]. Specifically, we are testing two popular CNN architectures, i.e., MobileNetV1 [14] and ResNet18 [13], both of which have proven successful in many image classification applications. We add an $\ell_1$-regularization term to the raw problem, where $\lambda$ is set to be $10^{-4}$ throughout the non-convex experiments.

We conduct all non-convex experiments for 200 epochs with a mini-batch size of 128 on one GeForce GTX 1080 Ti GPU. The step size $\alpha_k$ in Prox-SG, Prox-SVRG and OBProx-SG(+) is initialized as $0.1$, and decay by a factor $0.1$ per 75 epochs. The $\gamma$ in RDA is fine-tuned to be 20 for CIFAR10 and 30 for Fashion-MNIST in order to achieve the best performance. Similar to convex experiments, we set $N_P = N_O = 5N/|B|$ in OBProx-SG, and set $N_P = 100N/|B|$, $N_O = \infty$ in OBProx-SG+ since running Prox-SG Step 100 epochs already achieves an acceptable validation accuracy.

Based on the experimental results, the conclusions that we made previously in convex setting still hold in the current non-convex case: (i) OBProx-SG(+) performs competitively among the methods with respect to the final objective function values, see Table 4; (ii) OBProx-SG(+) computes much sparser solutions which are significantly better than other methods as shown in Table 5. Particularly, OBProx-SG+ achieves the highest sparse (lowest dense) solutions on all non-convex tests, of which the solutions are 4.24 to 21.86 times sparser than those of Prox-SG, while note that RDA and Prox-SVRG perform not comparable on the sparsity exploration because of the ineffectiveness of variance reduction techniques for deep learning [8]. In addition, we evaluate how well the solutions generalize on unseen test data. Table 6 shows that all the methods reach a comparable testing accuracy except RDA.

Finally, we investigate the sparsity evolution of the iterates to reveal the superiority of Orthant Step on sparsity promotion, where we use OBProx-SG+ as the representative of OBProx-SG(+) for illustration. As shown in Figure 4, OBProx-SG+ produces the highest sparse (lowest dense) solutions compared with other methods. Particularly, at the early $N_P$ iterations, OBProx-SG+ performs merely the same as Prox-SG. However, after the switching to Orthant Step at the 100th epoch, OBProx-SG+ outperforms all the other methods dramatically. It is a strong evidence that because of the construction of orthant face subproblem and the larger projection region, our orthant based technique is more remarkable than the standard proximal gradient step and its variants in terms of

---

Fig. 4: Density evolution. (L): MobileNetV1 on CIFAR10. (R): ResNet18 on Fashion-MNIST
Orthant Based Proximal Stochastic Gradient Method for $\ell_1$-Regularized Optimization

Table 4: Final objective values $F/f$ for tested algorithms on non-convex problems.

| Backbone | Dataset       | Prox-SG       | RDA          | Prox-SVRG     | OBProx-SG | OBProx-SG+ |
|----------|---------------|---------------|--------------|--------------|-----------|------------|
| MobileNetV1 | CIFAR10      | 1.473 / 0.049 | 4.129 / 0.302 | 1.921 / 0.079 | 1.619 / 0.048 | 1.453 / 0.063 |
|          | Fashion-MNIST | 1.314 / 0.089 | 4.901 / 0.197 | 1.645 / 0.103 | 2.119 / 0.089 | 1.310 / 0.099 |
| ResNet18 | CIFAR10       | 0.781 / 0.034 | 1.394 / 0.051 | 0.815 / 0.031 | 0.746 / 0.021 | 0.753 / 0.044 |
|          | Fashion-MNIST | 0.688 / 0.103 | 1.886 / 0.081 | 0.683 / 0.074 | 0.074 / 0.074 | 0.689 / 0.116 |

Table 5: Density (%) for tested algorithms on non-convex problems.

| Backbone | Dataset       | Prox-SG       | RDA          | Prox-SVRG     | OBProx-SG | OBProx-SG+ |
|----------|---------------|---------------|--------------|--------------|-----------|------------|
| MobileNetV1 | CIFAR10      | 14.17         | 74.05        | 92.26        | 9.15      | 2.90       |
|          | Fashion-MNIST | 5.28          | 74.67        | 75.40        | 4.15      | 1.23       |
| ResNet18 | CIFAR10       | 11.60         | 41.01        | 57.92        | 2.72      | 0.88       |
|          | Fashion-MNIST | 6.34          | 42.46        | 35.07        | 5.44      | 0.29       |

Table 6: Testing accuracy (%) for tested algorithms on non-convex problems.

| Backbone | Dataset       | Prox-SG       | RDA          | Prox-SVRG     | OBProx-SG | OBProx-SG+ |
|----------|---------------|---------------|--------------|--------------|-----------|------------|
| MobileNetV1 | CIFAR10      | 90.98         | 81.48        | 87.85        | 90.54     | 90.91      |
|          | Fashion-MNIST | 94.23         | 92.12        | 93.66        | 94.28     | 94.39      |
| ResNet18 | CIFAR10       | 92.43         | 90.54        | 92.28        | 92.54     | 92.75      |
|          | Fashion-MNIST | 94.28         | 93.66        | 94.24        | 94.39     | 93.97      |

the sparsity exploration. As a result, the solutions computed by OBProx-SG generally have a better interpretation under similar generalization performances.

5 Conclusions

We proposed a new Orthant Based Proximal Stochastic Gradient Method (OBProx-SG) for solving $\ell_1$-regularized problem, which combines the advantages of deterministic orthant based methods and proximal stochastic gradient method. In theory, we proved that it converges to some global solution in expectation for convex problems and some stationary point for non-convex formulations. Experiments on both convex and non-convex problems demonstrated that OBProx-SG usually achieves competitive objective values and much sparser solutions compared with state-of-the-arts stochastic solvers.

References

1. Andrew, G., Gao, J.: Scalable training of $l_1$-regularized log-linear models. In: Proceedings of the 24th international conference on Machine learning. pp. 33–40. ACM (2007)
2. Beck, A., Teboulle, M.: A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM journal on imaging sciences 2(1), 183–202 (2009)
3. Bradley, S., Hax, A., Magnanti, T.: Applied mathematical programming (1977)
4. Chen, T.: A Fast Reduced-Space Algorithmic Framework for Sparse Optimization. Ph.D. thesis, Johns Hopkins University (2018)
5. Chen, T., Curtis, F.E., Robinson, D.P.: A reduced-space algorithm for minimizing $\ell_1$-regularized convex functions. SIAM Journal on Optimization 27(3), 1583–1610 (2017)
6. Chen, T., Curtis, F.E., Robinson, D.P.: Farsa for $\ell_1$-regularized convex optimization: local convergence and numerical experience. Optimization Methods and Software (2018)
7. Cheng, Y., Wang, D., Zhou, P., Zhang, T.: A survey of model compression and acceleration for deep neural networks. arXiv preprint arXiv:1710.09282 (2017)
8. Defazio, A., Bottou, L.: On the ineffectiveness of variance reduced optimization for deep learning. In: Advances in Neural Information Processing Systems (2019)
9. Dixit, A.K.: Optimization in economic theory. Oxford University Press on Demand (1990)
10. Duchi, J., Singer, Y.: Efficient online and batch learning using forward backward splitting. Journal of Machine Learning Research 10(Dec), 2899–2934 (2009)
11. Ge, R., Huang, F., Jin, C., Yuan, Y.: Escaping from saddle points: online stochastic gradient for tensor decomposition. In: Conference on Learning Theory. pp. 797–842 (2015)
12. Han, S., Mao, H., Dally, W.J.: Deep compression: Compressing deep neural networks with pruning, trained quantization and huffman coding. arXiv preprint arXiv:1510.00149 (2015)
13. He, K., Zhang, X., Ren, S., Sun, J.: Deep residual learning for image recognition. In: Proceedings of the IEEE conference on computer vision and pattern recognition (2016)
14. Howard, A.G., Zhu, M., Chen, B., Kalenichenko, D., Wang, W., Weyand, T., Andreetto, M., Adam, H.: MobileNets: Efficient convolutional neural networks for mobile vision applications. arXiv preprint arXiv:1704.04861 (2017)
15. Johnson, R., Zhang, T.: Accelerating stochastic gradient descent using predictive variance reduction. In: Advances in neural information processing systems. pp. 315–323 (2013)
16. Keskar, N.S., Nocedal, J., Ozturk, F., Waeber, A.: A second-order method for convex $\ell_1$-regularized optimization with active set prediction. arXiv preprint arXiv:1505.04315 (2015)
17. Krizhevsky, A., Hinton, G.: Learning multiple layers of features from tiny images. Master’s thesis, Department of Computer Science, University of Toronto (2009)
18. Lee, J., Sun, Y., Saunders, M.: Proximal newton-type methods for convex optimization. In: Advances in Neural Information Processing Systems. pp. 836–844 (2012)
19. Li, Z., Li, J.: A simple proximal stochastic gradient method for nonsmooth nonconvex optimization. In: Advances in neural information processing systems (2018)
20. Nesterov, Y.: Primal-dual subgradient methods for convex problems. Mathematical programming (2009)
21. Riezler, S., Vasserman, A.: Incremental feature selection and $\ell_1$ regularization for relaxed maximum-entropy modeling. In: Empirical methods in natural language processing (2004)
22. Sra, S.: Fast projections onto $\ell_1,q$-norm balls for grouped feature selection. In: Joint European Conference on Machine Learning and Knowledge Discovery in Databases (2011)
23. Tibshirani, R.: Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society: Series B (Methodological) 58(1), 267–288 (1996)
24. Tikhonov, N., Arsenin, Y.: Solution of ill-posed problems. Winston and Sons. (1977)
25. Xiao, H., Rasul, K., Vollgraf, R.: Fashion-mnist: a novel image dataset for benchmarking machine learning algorithms (2017)
26. Xiao, L.: Dual averaging methods for regularized stochastic learning and online optimization. Journal of Machine Learning Research 11(Oct), 2543–2596 (2010)
27. Xiao, L., Zhang, T.: A proximal stochastic gradient method with progressive variance reduction. SIAM Journal on Optimization 24(4), 2057–2075 (2014)
28. Yang, M., Milzarek, A., Wen, Z., Zhang, T.: A stochastic extra-step quasi-newton method for nonsmooth nonconvex optimization. arXiv preprint arXiv:1910.09373 (2019)
29. Yuan, G.X., Ho, C.H., Lin, C.J.: An improved glmnet for $l_1$-regularized logistic regression. The Journal of Machine Learning Research 13(1), 1999–2030 (2012)
30. Zaremba, W., Sutskever, I., Vinyals, O.: Recurrent neural network regularization. arXiv preprint arXiv:1409.2329 (2014)
31. Zeiler, M.D., Fergus, R.: Stochastic pooling for regularization of deep convolutional neural networks. arXiv preprint arXiv:1301.3557 (2013)
32. Zhong, K., Song, Z., Jain, P., Bartlett, P.L., Dhillon, I.S.: Recovery guarantees for one-hidden-layer neural networks. In: International Conference on Machine Learning (2017)
33. Zou, H., Hastie, T.: Regularization and variable selection via the elastic net. Journal of the royal statistical society: series B (statistical methodology) (2005)