HARNACK INEQUALITY FOR FRACTIONAL LAPLACIAN-TYPE OPERATORS ON HYPERBOLIC SPACES

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Abstract. The Krylov–Safonov theory for fully nonlinear nonlocal operators on hyperbolic spaces of dimension three is established. Since the operators on hyperbolic spaces exhibit qualitatively different behavior than those on manifolds with nonnegative curvature, new scale functions are introduced which take the effect of negative curvature into account.

The regularity theory in this work provides unified regularity results for fractional-order and second-order operators in the sense that the regularity estimates stay uniform as the fractional-order approaches 2. In the unified regularity theory, the asymptotic behavior of the normalizing constant for the fractional Laplacian plays a fundamental role. The dimension restriction has been imposed to compute the explicit value of this constant by using the Fourier analysis on hyperbolic spaces.

1. Introduction

In a celebrated series of papers [3, 4, 5] by Caffarelli and Silvestre, the regularity theories such as Krylov–Safonov, Cordes–Nirenberg, and Evans–Krylov theory are established for fractional-order operators on Euclidean spaces. The most important feature is that the constants in regularity estimates do not blow up and remain uniform as the order of operator converges to 2. It means that the regularity theories for fractional-order and second-order operators are unified. It has also been extended to the parabolic cases [24, 25, 26].

On the other hand, the regularity theory for the local operators on Riemannian manifolds has been studied. In particular, the studies on the Harnack inequalities, initiated by Yau [43] and Cheng–Yau [7], have been extended to second-order divergence and non-divergence form operators. These are extensions of the De Giorgi–Nash–Moser [31] and Krylov–Safonov Harnack inequalities [2, 21, 42], respectively. See also [22, 23] for the parabolic Harnack inequalities.

The natural extension is to obtain the regularity results for fractional-order operators on Riemannian manifolds. Indeed, the Harnack inequality for nonlocal operators on metric measure spaces with volume doubling property, which includes Riemannian manifolds with nonnegative curvature, has been studied via the Dirichlet form theory [6]. However, this approach does not provide the unified regularity theory and is not appropriate for non-divergence form operators. For nonlocal non-divergence form operators, the Krylov–Safonov Harnack inequality on Riemannian manifolds with nonnegative curvature has recently been established by the authors [19]. The result in [19] unifies the Krylov–Safonov Harnack inequalities for local and nonlocal operators on manifolds with nonnegative curvature as in the works of Caffarelli and Silvestre.
In this paper, we continue to pursue the unified regularity theory for fractional-order and standard local nonlinear operators on 3-dimensional hyperbolic space $\mathbb{H}^3$ that has a nonnegative curvature. The study on 3-dimensional hyperbolic spaces is geometrically meaningful especially in general relativity. In this work, the dimension restriction has been imposed to compute the explicit value of normalizing constant for the fractional Laplacian, which is crucial for the unified regularity theory as in [3, 4, 5] due to its asymptotic behavior as $\gamma$ approaches 1.

The fractional Laplacian $-(\Delta_{\mathbb{H}^3})^\gamma$, $\gamma \in (0, 1)$, on hyperbolic spaces $\mathbb{H}^n$ is defined recently in [1] by Banica, González, and Sáez to study extension problem, and the explicit form of its kernel up to normalizing constant is provided. To the best of authors’ knowledge, this normalizing constant is not known. It has been observed [20] that finding the constant and studying its asymptotic behavior are the most important steps when a scaling is not available. In this regard, the first part of the paper is devoted to finding the explicit value of the constant. Indeed, this can be done on hyperbolic spaces with curvature $-\tau^{-2}$.

**Theorem 1.1.** Let $\mathbb{H}^3_\tau$ be the 3-dimensional hyperbolic space with curvature $-\tau^{-2}$. The fractional Laplacian $-(\Delta_{\mathbb{H}^3})^\gamma$ is given by

\[
-(\Delta_{\mathbb{H}^3})^\gamma u(x) = \text{p.v.} \int_{\mathbb{H}^3_\tau} (u(x') - u(x)) K_{\gamma, \tau}(d_{\mathbb{H}^3}(x', x)) \, d\mu_{\mathbb{H}^3_\tau}(x'), \quad x \in \mathbb{H}^3.
\]

Here, the kernel $K_{\gamma, \tau}$ has an explicit form

\[
K_{\gamma, \tau}(\rho) = C(n, \gamma) \frac{1}{\sinh \rho/\tau} (\rho/\tau)^{-1/2-\gamma} \frac{2K_{-3/2-\gamma}(\rho/\tau)}{\Gamma(3/2 + \gamma)(2\tau)^{3/2+\gamma}},
\]

where $K_\nu$ denotes the modified Bessel function of the second kind and

\[
C(n, \gamma) = \frac{2^{2\gamma} \Gamma\left(\frac{n}{2} + \gamma\right)}{\pi^{n/2} \Gamma(-\gamma)}
\]

is the normalizing constant for the fractional Laplacian on Euclidean spaces $\mathbb{R}^n$.

**Remark 1.2.** (i) The novelty of Theorem 1.1 is that the normalizing constant has the same asymptotic behavior with $1 - \gamma$ as $\gamma \to 1$, which is crucial in the upcoming regularity theory, as in [3, 4, 5].

(ii) It is natural to expect that $K_{\gamma, \tau}$ converges to the kernel of the fractional Laplacian on Euclidean space as curvature $-\tau^{-2}$ approaches zero. Indeed, it follows from Lemma A.3 that

\[
K_{\gamma, \tau}(\rho) \to C(3, \gamma) \rho^{-3-2\gamma},
\]

as $\tau \to \infty$.

(iii) The kernel $K_{\gamma, \tau}$ decays exponentially as $\rho \to \infty$, which is different from the behavior of the kernel on manifolds with nonnegative curvature. The difference comes from the exponential growth of the volume of balls in hyperbolic spaces, and this is why the regularity theories on manifolds with negative and nonnegative curvatures are dealt with separately.

(iv) In Theorem 1.1, the only 3-dimensional case is considered due to the difficulty in some computation of integrals in, e.g., Section 4 and Section 5. Such a restriction on dimensions is common on hyperbolic spaces. However, the harmonic analysis in Section 3 is provided for general dimensions. Moreover, most of the arguments in Section 6–Section 9
work in general dimensions with minor modifications if Theorem 1.1 is true for general dimensions.

The proof of Theorem 1.1 is not as simple as in the case of Euclidean spaces. In the Euclidean case, it follows from

\[
(-\Delta)^\gamma u(\xi) = -\frac{1}{2} C(n, \gamma) \int_{\mathbb{R}^n} \frac{u(\cdot + y)(\xi) + u(\cdot - y)(\xi) - 2u(\xi)}{|y|^{n+2\gamma}} \, dy
\]

\[
= -\frac{1}{2} C(n, \gamma) \left( \int_{\mathbb{R}^n} \frac{e^{\xi y} + e^{-\xi y} - 2}{|y|^{n+2\gamma}} \, dy \right) \hat{u}(\xi)
\]

\[
= C(n, \gamma) \left( \int_{\mathbb{R}^n} \frac{1 - \cos(\xi \cdot y)}{|y|^{n+2\gamma}} \, dy \right) \hat{u}(\xi)
\]

\[
= C(n, \gamma) \left( \int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^{n+2\gamma}} \, dy \right) |\xi|^{2\gamma} \hat{u}(\xi)
\]

and the definition \((-\Delta)^\gamma u(\xi) = |\xi|^{2\gamma} \hat{u}(\xi)\) that

\[
C(n, \gamma) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^{n+2\gamma}} \, dy \right)^{-1} = \frac{2^{2\gamma} \Gamma(n/(2\gamma) + \gamma)}{\pi^{n/2} \Gamma(-\gamma)}.
\]

The computation (1.3) relies heavily on the group structure of \((\mathbb{R}^n, +)\) that \(\mathbb{H}_0^n\) does not have. Instead, the gyrogroup structure of hyperbolic spaces, discovered by Ungar [35] and developed as a tool for the formulation of special relativity as an alternative to the use of Lorentz transformations to represent compositions of velocities, can be used to perform a similar computation as in (1.3). See also [39].

The aim of the next part of the paper is to establish the Alexandrov–Bakelman–Pucci (or ABP for short) estimates, Krylov–Safonov Harnack inequality, and Hölder estimates for solutions of nonlinear nonlocal elliptic equations on 3-dimensional hyperbolic space, which are robust in the sense that the estimates remain uniform as \(\gamma\) approaches 1. Modeled on the fractional Laplacian (1.1) with \(\tau = 1\), nonlinear operators of the fractional Laplacian-type can be defined in the standard way. For a class \(\mathcal{L}_0\) of linear operators of the form

\[
Lu(x) = \text{p.v.} \int_{\mathbb{H}^n} (u(x') - u(x)) \mathcal{K}(d_{\mathbb{H}^n}(x', x)) \, d\mu_{\mathbb{H}^n}(x'), \quad x \in \mathbb{H}^n,
\]

with kernels \(\mathcal{K}\) satisfying

\[
\lambda \mathcal{K}_\gamma \leq \mathcal{K} \leq \Lambda \mathcal{K}_\gamma, \quad 0 < \lambda \leq \Lambda,
\]

the maximal and minimal operators are defined by

\[
\mathcal{M}^+ u(x) := \mathcal{M}^+_{\mathcal{L}_0} u(x) := \sup_{L \in \mathcal{L}_0} Lu(x) \quad \text{and} \quad \mathcal{M}^- u(x) := \mathcal{M}^-_{\mathcal{L}_0} u(x) := \inf_{L \in \mathcal{L}_0} Lu(x),
\]

respectively. It is easy to see that these extremal operators are well-defined at \(x \in \mathbb{H}^n\) for any bounded function \(u\) that is \(C^2\) near \(x\) (see (5.2)). An operator \(\mathcal{I}\) is said to be elliptic with respect to \(\mathcal{L}_0\) if

\[
\mathcal{M}^-_{\mathcal{L}_0} (u - v)(x) \leq \mathcal{I}(u, x) - \mathcal{I}(v, x) \leq \mathcal{M}^+_{\mathcal{L}_0} (u - v)(x)
\]

for every point \(x \in \mathbb{H}^n\) and for all bounded functions \(u\) and \(v\) which are \(C^2\) near \(x\).

The first step towards the Krylov–Safonov Harnack inequality and Hölder estimate is showing the ABP-type estimate. The following theorem provides an estimate on the distribution function of supersolutions to nonlocal nonlinear operators. Throughout the paper, the functions \(\frac{\sinht}{t}\) and \(t \coth t\) are denoted by \(\mathcal{S}(t)\) and \(\mathcal{H}(t)\), respectively, as in [42]. Moreover, we
introduce a function $T(t) = t / \tanh^{-1}(1/2 \tanh(t))$. To focus on the limiting behavior as $\gamma \to 1$, we set $\tau = 1$ from now on.

**Theorem 1.3** (ABP-type estimate). Let $n = 3$, $\gamma_0 \in (0, 1)$, and assume $\gamma \in [\gamma_0, 1)$. Let $u \in C^2(B_{5R}) \cap L^\infty(\mathbb{H}^n)$ be a function on $\mathbb{H}^n$ satisfying $\mathcal{M}^- u \leq f$ in $B_{5R}$, $u \geq 0$ in $\mathbb{H}^n \setminus B_{5R}$, and $\inf_{B_{2R}} u \leq 1$. Let $\mathcal{C}$ be a contact set defined by (6.1), then there is a finite collection $\{Q_\alpha \}$ of dyadic cubes, with $\text{diam}(Q_\alpha) \leq r_0$, such that $Q_\alpha \cap \mathcal{C} \neq \emptyset$, $\mathcal{C} \subset \cup_{j} \overline{Q_\alpha}$, and

$$|B_R| \leq \sum_j c F^n |Q_\alpha^n|,$$

where $r_0$ is given by (5.7), and $c$ and $F$ are given by

$$F = S(7R) \left( n \mathcal{H}(7R) + \frac{R^2}{\mathcal{I}_0(R)} \max f \right),$$

and

$$c = C \cosh^{n-1}(C T(r_0)^2 r_0 F) (C T(r_0)^2 F)^{(n-1) \log \cosh(C T(r_0)^2 r_0 F)} T(r_0)^{2n}.$$

The universal constant $C > 0$ depends only on $n$, $\lambda$, $\Lambda$, and $\gamma_0$.

The scale function $\mathcal{I}_0(R)$ in Theorem 1.3 will be introduced in Section 5. The function $\frac{R^2}{\mathcal{I}_0(R)}$ corresponds to $\frac{1}{2\gamma} R^{2\gamma}$ in the case of Euclidean spaces [3] or manifolds with nonnegative curvature [19]. However, this function exhibits qualitatively different behavior in the case of hyperbolic spaces, compared to the simple scale function $\frac{1}{2\gamma} R^{2\gamma}$. This is because the kernels $K_\gamma$ decay exponentially as $\rho \to \infty$ while those in the case of manifolds with nonnegative curvature decay polynomially.

It will be proved in Lemma 5.6 and Lemma 5.5 that $r_0 \to 0$ and $\mathcal{I}_0(R) \to 6$ as $\gamma \to 1$. Since $\lim_{t \to 0} T(t) = 2$, the dependence of $c$ in (1.5) on $R$ disappears in the limit $\gamma \to 1$. As a consequence, the Riemann sum in (1.4) converges to the integral

$$C \int_{\mathcal{C}} S^n(7R) \left( n \mathcal{H}(7R) + R^2 f(x) \right)^n d\mu_{\mathbb{H}^n}(x),$$

with $C$ independent of $R$, which appears in [42, Theorem 1.2]. This implies that Theorem 1.3 recovers the ABP estimate for second-order equations on hyperbolic space as a limit. Moreover, Theorem 1.3 provides a new result even for second-order operators because it covers fully nonlinear operators. However, this paper provides the results for the case of dimension three only.

Finally, the Krylov–Safonov Harnack inequality and Hölder estimates for solutions of nonlocal nonlinear equations on 3-dimensional hyperbolic space are established. Similarly as in Theorem 1.3, we consider the case $\tau = 1$.

**Theorem 1.4** (Harnack inequality). Let $n = 3$, $\gamma_0 \in (0, 1)$ and assume $\gamma \in [\gamma_0, 1)$. If a nonnegative function $u \in C^2(B_{7R}) \cap L^\infty(\mathbb{H}^3)$ satisfies

$$\mathcal{M}^- u \leq C_0 \quad \text{and} \quad \mathcal{M}^+ u \geq -C_0 \quad \text{in } B_{7R},$$

then

$$\sup_{B_{3j_1 R}} u \leq C \left( \inf_{B_{3j_1 R}} u + C_0 \mathcal{I}_0(7R) \right)$$

for some universal constants $\delta_1 \in (0, 1)$ and $C > 0$ depending only on $n$, $\lambda$, $\Lambda$, $R$, and $\gamma_0$. 

Let us denote by $\| \cdot \|^\gamma$ the non-dimensional norm in the following theorem.

**Theorem 1.5** (Hölder estimates). Let $n = 3$, $\gamma_0 \in (0,1)$ and assume $\gamma \in [\gamma_0, 1)$. If $u \in C^2(B_{7R}) \cap L^\infty(\mathbb{H}^3)$ satisfies \((1.6)\), then

$$\|u\|_{C^\gamma(B_{7R})} \leq C \left( \|u\|_{L^\infty(\mathbb{H}^3)} + C_0 \frac{(7R)^2}{I_0(7R)} \right)$$

for some universal constants $\alpha \in (0,1)$ and $C > 0$ depending only on $n$, $\lambda$, $\Lambda$, $R$, and $\gamma_0$.

Since $I_0(R) \to 6$ as $\gamma \to 1$ and the universal constants in Theorem 1.4 and Theorem 1.5 do not depend on $\gamma$, Theorem 1.4 and Theorem 1.5 recover the classical results for second-order equations on hyperbolic space as limits.

**Corollary 1.6.** (i) If a nonnegative function $u \in C^2(B_{7R}) \cap L^\infty(\mathbb{H}^3)$ satisfies \((1.7)\)

$$\mathcal{M}^-(D^2u) \leq C_0 \quad \text{and} \quad \mathcal{M}^+(D^2u) \geq -C_0 \quad \text{in} \ B_{7R},$$

then

$$\sup_{B_{3R}} u \leq C \left( \inf_{B_{3R}} u + C_0 R^2 \right).$$

(ii) If $u \in C^2(B_{7R}) \cap L^\infty(\mathbb{H}^3)$ satisfies \((1.7)\), then

$$\|u\|_{C^\gamma(\mathbb{H}^3)} \leq C \left( \|u\|_{L^\infty(\mathbb{H}^3)} + C_0 R^2 \right).$$

The universal constants $\delta, \alpha \in (0,1)$, and $C > 0$ depend only on $n$, $\lambda$, $\Lambda$, and $R$.

The main difficulties in establishing regularity results, Theorem 1.3, Theorem 1.4, and Theorem 1.5, arise from the effect of negative curvature. The volume of ball in hyperbolic spaces behaves like that in Euclidean spaces when a radius is small, while it grows exponentially as a radius gets bigger. Due to this non-homogeneity of volume, the scaling property does not hold, making the standard arguments for regularity results break. To overcome this difficulty, we introduce new scale functions that take non-homogeneity into account and provide some monotonicity properties (see Section 5).

Another difficulty arising from the non-homogeneity of volumes lies in the dyadic ring argument in ABP estimates. In the ABP estimates, we find a dyadic ring around a given contact point in which a supersolution is quadratically close to a tangent paraboloid in a large portion of the ring. However, the standard dyadic rings $B_{2^{-k}r} \setminus B_{2^{-(k+1)}r}$ no longer work in the framework of hyperbolic spaces. It leads to introducing a hyperbolic dyadic ring whose radii are determined by volumes of balls (see Section 6). The hyperbolic dyadic ring turns out to be the natural “dyadic” ring in the hyperbolic geometry.

For the construction of a barrier function, it is now standard to use the distance function. However, computation is significantly different from the standard one because of the hyperbolic structure. We observe in Section 7 how the negative curvature of hyperbolic spaces affects the computations.

This paper is organized as follows. In Section 2, we recall several models for hyperbolic spaces and Fourier analysis on these spaces. Using the Fourier transform on hyperbolic spaces, the fractional Laplacian is defined as in [1]. Section 3 is devoted to the study of the gyrogroup structure of hyperbolic spaces and some basic harmonic analysis. With these tools at hand, we prove Theorem 1.1 in Section 4. In Section 5, new scale functions are introduced and some monotonicity properties are studied. The regularity theory begins with ABP-type estimates in Section 6. In this section Theorem 1.3 is proved. The next step is the construction of
a barrier function, and this is presented in Section 7. This barrier function, together with the ABP estimates, is used to obtain the so-called $L^p$-estimates in Section 8. Theorem 1.4 and Theorem 1.5 are proved in Section 9 and Section 10, respectively. In Appendix A, some properties of special functions are collected.

2. Preliminaries

In this section, we recall several models for the $n$-dimensional hyperbolic spaces (see, e.g., [34, 30, 15]). Moreover, the fractional Laplacian is defined as in [1] by using the Helgason Fourier transform on these spaces [14, 17, 33]. See also [15, 10]. Furthermore, some well-known properties on hyperbolic spaces are collected.

Let us first recall the hyperboloid model

$$\mathbb{H}_\tau^n = \{ (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : x_0^2 - x_1^2 - \cdots - x_n^2 = \tau^2, x_0 > 0 \},$$

with the metric induced by Lorentzian metric $-dx_0^2 + dx_1^2 - \cdots - dx_n^2$ in $\mathbb{R}^{n+1}$. The space $\mathbb{H}_\tau^n$ has a constant curvature $-\tau^{-2} < 0$. The internal product induced by the Lorentzian metric is denoted by

$$[x, x'] = x_0 x'_0 - x_1 x'_1 - \cdots - x_n x'_n,$$

and the distance between two points $x$ and $x'$ is given by

$$d_{\mathbb{H}_\tau^n}(x, x') = \tau \cosh^{-1} \left( \frac{|x, x'|}{\tau^2} \right).$$

Using the polar coordinates, $\mathbb{H}_\tau^n$ can also be realized as

$$\mathbb{H}_\tau^n = \{ x = \tau(cosh r, \sinh r \omega) \in \mathbb{R}^{n+1} : r \geq 0, \omega \in S^{n-1} \}.$$

Then, the metric and volume element are given by $ds^2 = \tau^2(dr^2 + \sinh^2 r d\omega^2)$ and $d\mu_{\mathbb{H}_\tau^n} = \tau^n \sinh^{n-1} r dr d\omega$, respectively.

Let us also consider the Poincaré ball model $\mathbb{B}_t^n = \{ y \in \mathbb{R}^n : |y| < t \}$ with metric

$$(2.1) \quad ds^2 = \frac{4b^2}{(t^2 - |y|^2)^2} dy^2.$$

The volume measure is given by

$$(2.2) \quad d\mu_{\mathbb{B}_t^n}(y) = \left( \frac{2b}{t^2 - |y|^2} \right)^n dy.$$

If $b/t = \tau$, then the map defined by

$$(2.3) \quad \phi : (x_0, x_1, \cdots x_n) \in \mathbb{H}_\tau^n \mapsto \frac{t}{\tau + x_0}(x_1, \cdots, x_n) \in \mathbb{B}_t^n,$$

or equivalently, by

$$\phi : \tau(cosh r, \sinh r \omega) \in \mathbb{H}_\tau^n \mapsto \frac{\sinh \frac{r}{\tau}}{\cosh \frac{r}{\tau}} \omega \in \mathbb{B}_t^n,$$

is an isometry, and its inverse is given by

$$\phi^{-1} : y \in \mathbb{B}_t^n \mapsto \left( \frac{t^2 + |y|^2}{t^2 - |y|^2}, \frac{2\tau t y_1}{t^2 - |y|^2}, \cdots, \frac{2\tau t y_n}{t^2 - |y|^2} \right) \in \mathbb{H}_\tau^n.$$
with the singular kernel which coincides with the definition in \[(2.6)\]

Note that all the formulas in this section can be written in terms of \[(2.6)\]

where \(\tau\) is the eigenfunction with eigenvalue \(-\frac{2}{t}\). It is well known that the following inversion formula holds:

\[(2.6)\] 

where

\[(2.5)\]

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\[(2.6)\]

where

\[(2.5)\]

is the Harish-Chandra coefficient. Moreover, the Plancherel formula holds:

\[(2.7)\]

Note that all the formulas in this section can be written in terms of \(|\mathbb{H}^n|\) by using isometry \[(2.3)\]. Furthermore, it is well known \([10]\) that \[(2.4)\], \[(2.6)\], and \[(2.7)\] converge to the standard Fourier transform, inversion formula, and Plancherel formula on Euclidean spaces as \(t \to \infty\).

We define the fractional Laplacian \((-\Delta_{|\mathbb{B}^n|})^\gamma\) by

\[(2.8)\]

which coincides with the definition in \([1]\) when \(a = t/2 = 1\). Similarly as in \([1]\), it has an integral representation

\[\left(-\Delta_{|\mathbb{B}^n|}\right)^\gamma u(x) = p.v. \int_{|\mathbb{B}^n|} (u(x') - u(x)) K_{\gamma,t/2} (d_{|\mathbb{B}^n|} (x', x)) d\mu_{|\mathbb{B}^n|} (x'),\]

with the singular kernel

\[(2.9)\]
where
\[ k_{\lambda,t/2}(x,x') = k_{\lambda,t/2}(d_{2B_n}(x,x')) = -\int_{S^{n-1}} e_{\lambda,\xi,t}(x) e_{-\lambda,\xi,t}(x') \frac{d\sigma(\xi)}{(t/2)^{n-1}|c(\lambda t)|^2}. \]

The kernel (2.9) with the explicit value of constant will be investigated in Section 4.

Let us now state some well-known results on hyperbolic space \( \mathbb{H}^n \). The first one is the volume doubling property that will be used frequently throughout the paper.

**Lemma 2.1.** For any \( B_r \subset B_R \subset \mathbb{H}^n \),
\begin{equation}
\left( \frac{R}{r} \right)^n \leq \frac{|B_R|}{|B_r|} \leq D \left( \frac{R}{r} \right)^{\log_2 D},
\end{equation}
where \( D = 2^n \cosh^{n-1}(2R) \).

**Lemma 2.1** is a direct consequence of the Bishop–Gromov inequality. See [40, Theorem 18.8 and Corollary 18.11] for the first inequality and the second inequality with \( R = 2r \) in (2.10), respectively. For the full inequality, we find \( k \in \mathbb{N} \) such that \( R \in [2^{k-1}r, 2^k r) \) and then iterate the inequality.

The next result is the bound of Hessian of the squared distance. See [9, Lemma 3.12] for instance.

**Lemma 2.2.** Fix \( y \in \mathbb{H}^n \) and consider the distance function \( d_{\mathbb{H}^n}(\cdot, y) \). Then,
\[ D^2(d_{\mathbb{H}^n}^2(x,y)/2)(\xi,\xi) \leq \mathcal{H}(d_{\mathbb{H}^n}(x,y))|\xi|^2 \]
for all \( \xi \in T_x \mathbb{H}^n \).

Let us close this section with the following generalization of Euclidean dyadic cubes that will be used in the decomposition of the contact set and in the Calderón–Zygmund technique.

**Theorem 2.3 (Christ [8]).** There is a countable collection \( \{Q^j_\alpha \subset \mathbb{H}^n : j \in \mathbb{Z}, \alpha \in I_j \} \) of open sets and constants \( c_1, c_2 > 0 \) (with \( 2c_1 \leq c_2 \)) and \( \delta_0 \in (0,1) \), depending only on \( n \), such that
\[(i) \ \ |\mathbb{H}^n \setminus \bigcup \alpha Q^j_\alpha| = 0 \text{ for each } j \in \mathbb{Z},
(ii) \ if \ i \geq j, \ then \ either \ Q^j_\beta \subset Q^i_\alpha \ or \ Q^j_\alpha \cap Q^i_\beta = \emptyset,
(iii) \ for \ each \ (j,\alpha) \ and \ each \ i < j, \ there \ is \ a \ unique \ \beta \ such \ that \ Q^j_\alpha \subset Q^i_\beta,
(iv) \ diam(Q^j_\alpha) \leq c_2\delta_0^j, \ and
(v) \ each \ Q^j_\alpha \ contains \ some \ ball \ B(z^j_\alpha, c_1\delta_0^j). \]

The original statement of [8, Theorem 11] consists of six properties. As mentioned in [8], the first five properties concern only the quasi-metric space structure and the last property requires the space to be of homogeneous type. Since the hyperbolic spaces are not of homogeneous type, the last property—which is not needed in this work—cannot be included.

## 3. The Möbius Gyrogroup

In this section, the gyrogroup structure of the Poincaré ball model \( \mathbb{B}^n_+ \) is studied as a tool for analyzing kernels for the fractional Laplacian. After first discovered in 1988 by Ungar [35], it has been extensively studied in the context of abstract algebra, non-Euclidean geometry, mathematical physics, and quantum information and computation [10, 11, 12, 36, 37, 38].
Let us first recall the Clifford algebra $\mathcal{C}l_{0,n}$ over $\mathbb{R}^n$ before we define the gyropseudgroup addition. The Clifford algebra $\mathcal{C}l_{0,n}$ is the associative real algebra generated by $\mathbb{R}^n$ and $\mathbb{R}$ subject to the relation
\begin{equation}
3.1 \quad x^2 = -|x|^2, \quad x \in \mathbb{R}^n,
\end{equation}
where $|\cdot|$ denotes the standard Euclidean norm. For example, $\mathcal{C}l_{0,1}$ and $\mathcal{C}l_{0,2}$ are isomorphic to the field of complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$, respectively. In the Clifford algebra, there is a unique anti-automorphism satisfying $\bar{x} = -x$ and $\bar{xy} = y\bar{x}$ for all $x, y \in \mathbb{R}^n$. Clearly, it holds that $x\bar{x} = |x|^2 = \bar{x}x$, $|xy|^2 = |x|^2|y|^2$, and $|x| = |\bar{x}|$. Moreover, it follows from the relation (3.1) that
\begin{equation}
3.2 \quad x\bar{y} = \frac{1}{2}(x\bar{y} + y\bar{x}) + \frac{1}{2}(x\bar{y} - y\bar{x}) = \langle x, y \rangle y + \frac{1}{2}(x\bar{y} - y\bar{x}),
\end{equation}
where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$.

The group of all conformal orientation preserving transformations of $\mathbb{R}^n$ consists of the mappings $K\varphi_y$, where $K \in SO(n)$ and $\varphi_x$ is the Möbius transformation on $\mathbb{B}^n_t$. It is known [37, 38] that $\varphi_x$ is given by
\begin{equation}
\varphi_x(y) = (x + y) \left( 1 + \frac{x\bar{y}}{t^2} \right)^{-1} = \frac{1 + \frac{2}{t^2}\langle x, y \rangle + \frac{1}{t^2}|y|^2}{1 + \frac{2}{t^2}\langle x, y \rangle + \frac{1}{t^2}|x|^2|y|^2} x + (1 - \frac{1}{t^2}|x|^2) y.
\end{equation}

By defining the Möbius addition by
\begin{equation}
x \oplus y = \varphi_x(y),
\end{equation}
the tuple $(\mathbb{B}^n_t, \oplus)$ forms a gyrogroup, that is, the following properties hold:

(i) (Left identity) $0 \oplus x = x$ for all $x \in \mathbb{B}^n_t$.

(ii) (Left inverse) For each $x \in \mathbb{B}^n_t$, there is $\ominus x = -x$ such that $(\ominus x) \oplus x = 0$.

(iii) (Gyroassociative) $x \oplus (y \oplus z) = (x \oplus y) \oplus \text{gyr}[x, y]z$ for all $x, y, z \in \mathbb{B}^n_t$, where
\begin{equation}
\text{gyr}[x, y]z = \left( 1 + \frac{x\bar{y}}{t^2} \right) z \left( 1 + \frac{x\bar{y}}{t^2} \right)^{-1}
\end{equation}
is the gyration operator.

(iv) (Left loop property) $\text{gyr}[x, y] = \text{gyr}[x \oplus y, y]$ for all $x, y \in \mathbb{B}^n_t$.

The gyrogroup $(\mathbb{B}^n_t, \oplus)$ is gyrocommutative, i.e., $x \oplus y = \text{gyr}[x, y](y \ominus x)$ for all $x, y \in \mathbb{B}^n_t$. Let us define the second operation $\boxplus$ so-called coaddition by
\begin{equation}
x \boxplus y = x \oplus \text{gyr}[x, y]y.
\end{equation}

It can be easily checked that the tuple $(\mathbb{B}^n_t, \boxplus)$ forms a gyrogroup with the left identity $0$ and left inverse $\boxminus x = -x$. We will use the abbreviation $x \ominus y = x \oplus (\ominus y)$ and $x \boxminus y = x \boxplus (\boxminus y)$. Then we have
\begin{equation}
3.3 \quad x \boxplus y = x \ominus \text{gyr}[x, y]y.
\end{equation}

See [37, 38] for more properties of gyrogroups $(\mathbb{B}^n_t, \oplus)$ and $(\mathbb{B}^n_t, \boxplus)$. Among many properties, the following cancellation laws will be used in Section 4.

**Theorem 3.1.** [37, Theorem 2.15] Let $a, b \in \mathbb{B}^n_t$. Then $x = \ominus a \oplus b$ is the unique solution of the equation $a \oplus x = b$, and $x = b \boxminus a$ is the unique solution of the equation $x \oplus a = b$.

In the rest of this section, we perform some computations on the eigenfunctions $e_{\lambda,\xi}^t$ and the measures $\mu_{\mathbb{B}^n_t}$ for future references.
Lemma 3.2. For $\lambda \in \mathbb{R}$, $\xi \in S^{n-1}$, and for $y, z \in \mathbb{R}^n$,

$$e_{-\lambda, \xi, t}(z \sqcap y) = \left( \frac{|\xi - \frac{z}{t}|^2 \left(1 - \frac{|y|^2}{t^2}\right)|1 + \frac{y}{t}|^2}{\left(1 - \frac{|z|^2|y|^2}{t^4}\right)\xi - \left(1 - \frac{|y|^2}{t^2}\right)\frac{y}{t} + \left(1 - \frac{|z|^2}{t^2}\right)\frac{y}{t}} \right)^{n-1} e_{-\lambda, \xi, t}(z).$$

Proof. We use (3.2) and the formula

$$1 - \frac{|a \oplus b|^2}{t^2} = \left(1 - \frac{|a|^2}{t^2}\right)\left(1 - \frac{|b|^2}{t^2}\right)\left|1 + \frac{ab}{t^2}\right|^{-2},$$

which is given in [10, Equation (16)], to obtain

$$1 - \frac{|z \sqcap y|^2}{t^2} = 1 - \frac{|z \oplus (-\text{gyr}[z, y])|^2}{t^2}$$

$$= \left(1 - \frac{|z|^2}{t^2}\right)\left(1 - \frac{|y|^2}{t^2}\right)\left|1 - \frac{\text{gyr}[z, y]}{t^2}\right|^{-2}.$$ (3.3)

On the other hand, using (3.2) and the definition of gyration, we have

$$\left|\xi - \frac{z \oplus \text{gyr}[z, y]}{t}\right|^2 = \left|\xi - \frac{\text{gyr}[z, y]}{t}\right|^2$$

$$= \left|\xi - \frac{1}{t}(z - \text{gyr}[z, y])\left(1 - \frac{\text{gyr}[z, y]}{t^2}\right)^{-2}\right|^2$$

$$= \left|\xi \left(1 - \frac{\text{gyr}[z, y]}{t^2}\right) - \frac{1}{t}(z - \text{gyr}[z, y])\left|1 - \frac{\text{gyr}[z, y]}{t^2}\right|^{-2}\right|^2.$$ (3.4)

A further computation shows that

$$\left|\xi \left(1 - \frac{\text{gyr}[z, y]}{t^2}\right) - \frac{1}{t}(z - \text{gyr}[z, y])\right|^2$$

$$= \left|\xi - \frac{z}{t} + \frac{1}{t}\left(1 - \frac{\xi z}{t}\right)\left(1 + \frac{zy}{t^2}\right)\left|1 + \frac{zy}{t^2}\right|^{-2}\right|^2$$

$$= \left|\left(1 - \frac{|z|^2|y|^2}{t^4}\right)\xi - \left(1 - \frac{|y|^2}{t^2}\right)\frac{y}{t} + \left(1 - \frac{|z|^2}{t^2}\right)\frac{y}{t}\right|^2$$

$$= \left(1 - \frac{|z|^2|y|^2}{t^4}\right)\xi - \left(1 - \frac{|y|^2}{t^2}\right)\frac{y}{t} + \left(1 - \frac{|z|^2}{t^2}\right)\frac{y}{t}\right|^2$$

Combining (3.3), (3.4), and (3.5), and recalling the definition of eigenfunction (2.5), the lemma is proved. \(\square\)

Lemma 3.3. For $y, z \in \mathbb{R}^n$,

$$d\mu_{\mathbb{R}^n}(z \sqcap y) = \left(\frac{1 - \frac{|z|^2|y|^2}{t^4}}{|1 + \frac{zy}{t^2}|^2}\right)^{n-1} d\mu_{\mathbb{R}^n}(z).$$

Let us prove the following two lemmas in order to prove Lemma 3.3. The first one is a computation of (3.2).
Lemma 3.4. For \( y, z \in \mathbb{B}^n \),
\[
z \boxdot y = \left( 1 - \frac{|z|^2|y|^2}{t^4} \right)^{-1} \left( 1 - \frac{|y|^2}{t^2} \right) z - \left( 1 - \frac{|z|^2}{t^2} \right) y.
\]

Proof. A straightforward computation shows that
\[
z \boxdot y = z \odot \text{gyr}|z, y| y = z \odot \left( 1 + \frac{z \bar{y} y}{t^2} \right) y \left( 1 + \frac{\bar{y} y}{t^2} \right)^{-1}
\]
\[
= \left( z - \left( 1 + \frac{z \bar{y} y}{t^2} \right) y \left( 1 + \frac{\bar{y} y}{t^2} \right)^{-1} \right) \left( 1 - \frac{1}{t^2} \bar{z} \left( 1 + \frac{z \bar{y} y}{t^2} \right) y \left( 1 + \frac{\bar{y} y}{t^2} \right)^{-1} \right)
\]
\[
= \left( z + \frac{|z|^2}{t^2} y - y - \frac{|y|^2}{t^2} \bar{z} \right) \left( 1 + \frac{\bar{y} y}{t^2} \right)^{-1} \left( 1 - \frac{|z|^2|y|^2}{t^4} \right)^{-1} \left( 1 + \frac{\bar{y} y}{t^2} \right)
\]
\[
= \left( \left( 1 - \frac{|y|^2}{t^2} \right) z - \left( 1 - \frac{|z|^2}{t^2} \right) y \right) \left( 1 - \frac{|y|^2}{t^2} \right)^{-1}.
\]

\[\square\]

The next lemma provides the Jacobian determinant of \( z \boxdot y \).

Lemma 3.5. Let \( J \) be the Jacobian matrix \( \frac{\partial(z \boxdot y)}{\partial z} \). Then,
\[
|J| = \left( 1 - \frac{|z|^2|y|^2}{t^4} \right)^{-n} \left( 1 - \frac{|y|^2}{t^2} \right)^n \left| 1 + \frac{\bar{y} y}{t^2} \right|^2.
\]

Proof. Let \( A = 1 - \frac{|z|^2|y|^2}{t^4} \). Using Lemma 3.4, the \((i, j)\)-th entry, for \( i, j = 1, \ldots, n \), is obtained as
\[
\frac{\partial(z \boxdot y)_i}{\partial z_j} = \frac{1}{A^2} \left( 1 - \frac{|y|^2}{t^2} \right) \left( A \delta_{ij} + 2 \frac{z_j y_i}{t^2} + 2 \frac{|y|^2 z_i z_j}{t^4} \right).
\]

We use the technique in [41, Lemma 5] to compute the Jacobian determinant \( |J| \). That is, we change \(|J|\) into
\[
|J| = \frac{1}{A^{2n}} \left( 1 - \frac{|y|^2}{t^2} \right)^n \begin{vmatrix}
1 & 2z_1 & \cdots & 2z_n \\
0 & A + 2 \frac{z_1 y_i}{t^2} + 2 \frac{|y|^2 z_1^2}{t^4} & \cdots & 2 \frac{z_n y_i}{t^2} + 2 \frac{|y|^2 z_1 z_n}{t^4} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 2 \frac{z_n y_i}{t^2} + 2 \frac{|y|^2 z_n z_i}{t^4} & \cdots & A + 2 \frac{z_1 y_i}{t^2} + 2 \frac{|y|^2 z_1 z_n}{t^4}
\end{vmatrix}
\]
by adding zeroth row and zeroth column. For \( i = 1, \ldots, n \), we multiply the first row of the above matrix by \(- \frac{y_i}{t^2} - \frac{|y|^2 z_i}{t^4}\) and add it to the \( i \)-th row. Then we obtain
\[
|J| = \frac{1}{A^{2n}} \left( 1 - \frac{|y|^2}{t^2} \right)^n \begin{vmatrix}
1 & 2z_1 & 2z_2 & \cdots & 2z_n \\
- \frac{y_1}{t^2} - \frac{|y|^2 z_1}{t^4} & A & 0 & \cdots & 0 \\
- \frac{y_2}{t^2} - \frac{|y|^2 z_2}{t^4} & 0 & A & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- \frac{y_n}{t^2} - \frac{|y|^2 z_n}{t^4} & 0 & 0 & \cdots & A
\end{vmatrix}.
\]
By multiplying the $j$-th column by $\frac{1}{A}(\frac{y_j}{t^2} + |y_j|^2 \frac{x_j}{t^2})$, $j = 1, \ldots, n$, and adding it to the first column, we arrive at

$$|J| = \frac{1}{A^{2n}} \left(1 - \frac{|y|^2}{t^2}\right)^n \begin{vmatrix} 1 + \frac{|y|^2}{t^2} & 2z_1 & \cdots & 2z_n \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{vmatrix} = \frac{1}{A^{n+1}} \left(1 - \frac{|y|^2}{t^2}\right)^n \left|1 + \frac{\bar{z}y}{t^2}\right|^2.$$

We are now ready to prove Lemma 3.3.

**Proof of Lemma 3.3.** By using (2.2), (3.3), and Lemma 3.5, we have

$$d\mu_{B^*_t}(z \boxdot y) = \left(1 - \frac{|z \boxdot y|^2}{t^2}\right)^{-n} d(z \boxdot y)$$

(3.6)

$$= \left(1 - \frac{|z|^2}{t^2}\right)^{-n} \left|1 - \frac{\bar{z}gy[z,y]/y}{t^2}\right|^{2n} \left(1 - \frac{|z|^2|y|^2}{t^4}\right)^{-(n+1)} \left|1 + \frac{\bar{z}y}{t^2}\right|^2 dz.$$

Therefore, the desired result follows from

$$\left|1 - \frac{\bar{z}gy[z,y]/y}{t^2}\right| = \left|1 - \frac{|z|^2|y|^2}{t^4}\right|^{1 - \frac{\bar{z}y}{t^2}}$$

and (3.6). □

4. Normalizing constant

In this section, we prove Theorem 1.1. Following the proof of [1, Lemma 2.1], one can easily obtain that

$$k_{\lambda,t/2}(\rho) = \frac{(-1)^n}{(2\pi)^{n+1}} \left(\frac{2}{t}\right)^{\frac{n-1}{2}} \left(\frac{\partial_\rho}{\sinh(2\rho/t)}\right)^{\frac{n-1}{2}} \cos(\lambda \rho)$$

(4.1)

for $n = 3$. Indeed, (4.1) holds for any $n \geq 3$ odd, and a similar formula can be obtained for $n \geq 2$ even. However, let us focus on the case $n = 3$ in the sequel. As in [1, Theorem 2.4], we also have

$$K_{\gamma,t/2}(\rho) = \alpha_\gamma \left(\frac{\partial_\rho}{\sinh(2\rho/t)}\right)^{(n-1)/2} \left(\frac{n-1}{t} \rho\right)^{-1/2-\gamma} \left(\frac{n-1}{t} \rho\right)^{-\gamma} K_{-1/2-\gamma} \left(\frac{n-1}{t} \rho\right)$$

(4.2)

for some constant $\alpha_\gamma$. The rest part of this section is devoted to finding the constant $\alpha_\gamma$ in 3-dimensional case.

The idea is to follow the lines (1.3) as in the case of Euclidean spaces, in light of the gyrogroup structure of $B^n_t$. By utilizing Theorem 3.1, Lemma 3.2, and Lemma 3.3, we have

$$f(\overline{(\oplus y)}) (\lambda, \xi) = \int_{B^n_t} f(x)(x) e_{-\lambda,\xi,t}(x) d\mu_{B^n_t}(x)$$

(4.3)

$$= \int_{B^n_t} f(z) e_{-\lambda,\xi,t}(z \boxdot y) d\mu_{B^n_t}(z \boxdot y)$$

$$= \int_{B^n_t} f(z) e_{-\lambda,\xi,t}(z) E(\lambda, \xi, y, z) d\mu_{B^n_t}(z)$$
and
\[ f(\cdot \ominus y)(\lambda, \xi) = \int_{\mathbb{B}_t^n} f(z)e^{-\lambda, \xi, t}(z)E(\lambda, \xi, -y, z) \, d\mu_{\mathbb{B}_t^n}(z), \tag{4.4} \]
where
\[ E(\lambda, \xi, y, z) = \left( \frac{|\xi - \frac{y}{t}|^2}{\left(1 - \frac{|y|^2}{t^2}\right)\xi - \left(1 - \frac{|y|^2}{t^2}\right)\frac{y}{t} + \left(1 - \frac{|y|^2}{t^2}\right)\frac{y}{t}^2} \right)^{\frac{n-1}{2}} (1 - \frac{|y|^2}{t^2})^{-\frac{n-1}{2}}. \]

It is to be noted that \( E \) in (4.3) and (4.4) are not independent of \( z \), which implies that the second equality in (1.3) already fails to hold in the framework of hyperbolic spaces. We have obtained
\[ (-\Delta_{\mathbb{B}_t^n})^{\gamma} f(\lambda, \xi) \tag{4.5} \]
\[ = -\frac{1}{2} \int_{\mathbb{B}_t^n} (f(\cdot \ominus y)(\lambda, \xi) + f(\cdot \ominus y)(\lambda, \xi) - 2f(\lambda, \xi)) K_{\gamma, t/2}(d) \, d\mu_{\mathbb{B}_t^n}(y) \]
\[ = -\frac{1}{2} \int_{\mathbb{B}_t^n} f(z)e_{-\lambda, \xi, t}(z) \int_{\mathbb{B}_t^n} E(\lambda, \xi, y, z) + E(\lambda, \xi, -y, z) - 2) K_{\gamma, t/2}(d) \, d\mu_{\mathbb{B}_t^n}(y) \, d\mu_{\mathbb{B}_t^n}(z), \]
where \( d \) denotes the distance \( d_{\mathbb{B}_t^n}(0, y) \). However, even though \( E \) itself is not independent of \( z \), it is plausible to guess that the inner integral in the right-hand side of (4.5) is independent of \( z \). Indeed, we have the following result.

**Lemma 4.1.** For any \( \lambda \in \mathbb{R}, \xi \in \mathbb{S}^2 \), and any \( z \in \mathbb{B}_t^n \),
\[ -\frac{1}{2} \int_{\mathbb{B}_t^n} (E(\xi, y, z) + E(\xi, -y, z) - 2) K_{\gamma, t/2}(d_{\mathbb{B}_t^n}(0, y)) \, d\mu_{\mathbb{B}_t^n}(y) \tag{4.6} \]
\[ = \alpha_\gamma 2^{-3/2-3\gamma} \pi^{3/2} t^{2+2\gamma} \Gamma(-\gamma) \left( \lambda^2 + \frac{4}{t^2} \right)^\gamma. \]

Lemma 4.1 actually finishes the proof of Theorem 1.1. Indeed, it follows from (2.8), (4.5), and Lemma 4.1, that
\[ \alpha_\gamma = \frac{2^{3/2+3\gamma}}{\pi^{3/2} t^{2+2\gamma} \Gamma(-\gamma)}. \tag{4.7} \]
Therefore, we obtain
\[ K_{\gamma, t/2}(\rho) = C(3, \gamma) \frac{2/t}{\sinh(2\rho/t)} \rho^{-1/2-\gamma} 2K_{-3/2-\gamma}(2\rho/t) \frac{2K_{-3/2-\gamma}(2\rho/t)}{\Gamma(3/2 + \gamma) t^{3/2+\gamma}} \]
from (4.2), (4.7), and (A.2). Recalling that \( a = t/2 \), we arrive at (1.2).

From now on, let us focus on proving Lemma 4.1. The proof of Lemma 4.1 consists of two parts: we will show that the imaginary part of the left-hand side of (4.6) is zero in Lemma 4.2, and that the real part becomes the right-hand side of (4.6), independently of \( \xi \) and \( z \).

**Lemma 4.2.** Let \( n = 3 \). For any \( \lambda \in \mathbb{R}, \xi \in \mathbb{S}^{n-1} \), and any \( z \in \mathbb{B}_t^n \),
\[ \text{Im} \left( \int_{\mathbb{B}_t^n} (E(\lambda, \xi, y, z) + E(\lambda, \xi, -y, z) - 2) K_{\gamma, t/2}(d_{\mathbb{B}_t^n}(0, y)) \, d\mu_{\mathbb{B}_t^n}(y) \right) = 0. \tag{4.8} \]
Proof. For any $T \in O(n)$, we have $E(\lambda, T\xi, y, Tz) = E(\lambda, \xi, T^{-1}y, z)$, and hence
\[
\int_{\mathbb{R}^n} \left( E(\lambda, T\xi, y, Tz) + E(\lambda, T\xi, -y, Tz) - 2\right) K_{\gamma,t/2}(d_{\mathbb{R}^n}(0, y)) \, d\mu_{\mathbb{R}^n}(y)
\]
\[
= \int_{\mathbb{R}^n} \left( E(\lambda, \xi, T^{-1}y, z) + E(\lambda, \xi, -T^{-1}y, z) - 2\right) K_{\gamma,t/2}(d_{\mathbb{R}^n}(0, y)) \, d\mu_{\mathbb{R}^n}(y)
\]
\[
= \int_{\mathbb{R}^n} \left( E(\lambda, \xi, y, z) + E(\lambda, \xi, -y, z) - 2\right) K_{\gamma,t/2}(d_{\mathbb{R}^n}(0, y)) \, d\mu_{\mathbb{R}^n}(y).
\]
Thus, by taking $T \in O(n)$ that maps $\xi$ to $e_1$ and $z$ to a point in $e_1 e_2$-plane, we may assume that $\xi = e_1$ and $z = (z_1, z_2, 0, \cdots 0)$. In fact, we will prove
\[
\int_{\mathbb{S}^{n-1}} \text{Im} \, E(\lambda, \xi, r\omega, z) \, d\omega = 0,
\]
from which (4.9) follows.

For notational convenience, let us write
\[
\text{Im} \, E(\lambda, e_1, y, z) = \left( \frac{A^2 BD}{|Ae_1 - B\bar{z} + Cy|^2 |e_1 + \bar{z}y|^2} \right)^{\frac{n-1}{2}} \sin \left( -\frac{\lambda t}{2} \log \frac{BD |e_1 + \bar{z}y|^2}{|Ae_1 - B\bar{z} + Cy|^2} \right),
\]
where $y = r(\omega_1 \cos \theta, \omega_1 \sin \theta, \omega') \in \partial B_r$ with $(\omega_1, \omega') \in \mathbb{S}^{n-2}$, $\omega_1 \in \mathbb{R}$, and $\omega' \in \mathbb{R}^{n-2}$. Here, we have set
\[
A = 1 - \frac{|z|^2 r^2}{t^4}, \quad B = 1 - \frac{r^2}{t^2}, \quad C = 1 - \frac{|z|^2}{t^2}, \quad \text{and} \quad D = \left| e_1 - \frac{z}{t} \right|^2.
\]
To prove (4.9), let us first fix $(\omega_1, \omega')$ and set
\[
I := \int_{\theta_0}^{2\pi + \theta_0} \text{Im} \, E(\lambda, e_1, r(\omega_1 \cos \theta, \omega_1 \sin \theta, \omega'), z) \, d\theta,
\]
where $\theta_0 \geq 0$ is the smallest angle such that the function
\[
\Theta(\theta) := \frac{BD |e_1 + \bar{z}y(\omega_1 \cos \theta, \omega_1 \sin \theta, \omega')|^2}{|Ae_1 - B\bar{z} + Cy(\omega_1 \cos \theta, \omega_1 \sin \theta, \omega')|^2}
\]
attains its minimum. Let $\theta_1 > \theta_0$ be the smallest angle such that $\Theta$ attains its maximum, and $\theta_2 > \theta_1$ the smallest angle such that $\Theta$ attains its minimum. Then, we have $\theta_2 = \theta_0 + 2\pi$, and hence,
\[
(4.10) \quad I = \int_{\theta_0}^{\theta_1} \text{Im} \, E(\lambda, e_1, y, z) \, d\theta + \int_{\theta_1}^{\theta_2} \text{Im} \, E(\lambda, e_1, y, z) \, d\theta = : I_0 + I_1.
\]
For $I_i$, $i = 0, 1$, let us consider the change of variables $s = \Theta(\theta)$, $\theta_i \leq \theta \leq \theta_{i+1}$. We have
\[
s = \Theta(\theta) = \frac{X_1 \omega_1 \cos \theta + Y_1 \omega_1 \sin \theta + Z_1}{X_2 \omega_1 \cos \theta + Y_2 \omega_1 \sin \theta + Z_2},
\]
or equivalently,
\[
X \omega_1 \cos \theta + Y \omega_1 \sin \theta = Z,
\]
Thus, we have
\[ X := X_1 - X_2s := 2BD\frac{r_{z_1}}{t^2} - (2AC\frac{r_{z_1}}{t} - 2BC\frac{r_{z_1}}{t^2}) s, \]
\[ Y := Y_1 - Y_2s := 2BD\frac{r_{z_2}}{t^2} - (-2BC\frac{r_{z_2}}{t^2}) s, \]
and
\[ Z := Z_2s - Z_1 := \left(A^2 - 2AB\frac{z_1}{t} + B^2\frac{|z|^2}{t^2} + C^2\frac{r_{z_2}}{t^2}\right) s - \left(BD + \frac{r^2|z|^2}{t^4}BD\right). \]

Let \( W = \sqrt{X^2 + Y^2} \), and let \( \tilde{\theta} \) be the angle such that \( \cos \tilde{\theta} = \frac{X}{W} \) and \( \sin \tilde{\theta} = \frac{Y}{W} \), then
\[ \theta = \tilde{\theta} + 2k\pi \pm \arccos \left(\frac{Z}{W\omega_1}\right) \text{ for some } k \in \mathbb{Z}, \]
where the principal branch for \( \arccos \) is taken to be \([0, \pi]\). Note that \( s = \Theta(\theta) \) increases on \([\theta_0, \theta_1]\) and decreases on \([\theta_1, \theta_2]\). Since \( Z_2 > 0 \), \( \arccos \frac{Z}{W\omega_1} \) decreases on \([\theta_0, \theta_1]\) and increases on \([\theta_1, \theta_2]\). Thus, we have the minus sign in (4.11) on \([\theta_0, \theta_1]\) and the plus sign on \([\theta_1, \theta_2]\). Let us consider the case \( \theta \in [\theta_0, \theta_1] \), then
\[ \cos \theta = \frac{XZ + Y\sqrt{W^2\omega_1^2 - Z^2}}{W^2\omega_1} \text{ and } \sin \theta = \frac{YZ - X\sqrt{W^2\omega_1^2 - Z^2}}{W^2\omega_1}. \]

Using (4.12), we see that
\[ X_1\omega_1 \cos \theta + Y_1\omega_1 \sin \theta + Z_1 = (X_2\omega_1 \cos \theta + Y_2\omega_1 \sin \theta + Z_2)s = \frac{U - V\sqrt{W^2\omega_1^2 - Z^2}}{W^2}s, \]
where
\[ U = -(X_1X_2 + Y_1Y_2)Z_1 + (X_1^2 + Y_1^2)Z_2 + ((X_2^2 + Y_2^2)Z_1 - (X_1X_2 + Y_1Y_2)Z_2) s, \]
and
\[ V = X_1Y_2 - Y_1X_2. \]

Thus, we have
\[ \left|e_1 + \frac{r}{t^2} \tilde{z}(\omega_1 \cos \theta, \omega_1 \sin \theta, \omega)\right|^2 = \frac{(X_1\omega_1 \cos \theta + Y_1\omega_1 \sin \theta + Z_1)}{\frac{BD}{U - V\sqrt{W^2\omega_1^2 - Z^2}}}. \]

Moreover, the equality
\[ (Y_1Z_2 - Y_2Z_1)\omega_1 \cos \theta + (Z_1X_2 - Z_2X_1)\omega_1 \sin \theta - (X_1Y_2 - X_2Y_1)\omega_1^2 \]
\[ = ((Y_1Z_2 - Y_2Z_1)Y - (Z_1X_2 - Z_2X_1)X)\sqrt{W^2\omega_1^2 - Z^2} \]
\[ + (Y_1Z_2 - Y_2Z_1)XZ + (Z_1X_2 - Z_2X_1)YZ - (X_1Y_2 - X_2Y_1)W^2\omega_1^2 \]
\[ = ((Y_1Z_2 - Y_2Z_1)Y - (Z_1X_2 - Z_2X_1)X)\sqrt{W^2\omega_1^2 - Z^2} - \frac{(X_1Y_2 - X_2Y_1)(W^2\omega_1^2 - Z^2)}{W^2} \]
\[ = \frac{U - V\sqrt{W^2\omega_1^2 - Z^2}}{W^2}\sqrt{W^2\omega_1^2 - Z^2}, \]
together with (4.13), yields that

\[
\begin{align*}
d\theta &= \frac{(X_2 \omega_1 \cos \theta + Y_2 \omega_1 \sin \theta + Z_2)^2}{(Y_1 Z_2 - Y_2 Z_1) \omega_1 \cos \theta + (Z_1 X_2 - Z_2 X_1) \omega_1 \sin \theta - (X_1 Y_2 - X_2 Y_1) \omega_1^2} ds \\
&= \frac{U - V \sqrt{W^2 \omega_1^2 - Z^2}}{W} \frac{1}{\sqrt{W^2 \omega_1^2 - Z^2}} ds.
\end{align*}
\]

(4.15)

Therefore, combining (4.10), (4.14), and (4.15), and performing the same computation on \([\theta_1, \theta_2]\), we obtain

\[
I = \int_{s_0}^{s_1} (ABD)^{n-1}s^{-\frac{n-1}{2}} \left( \frac{W^2}{U - V \sqrt{W^2 \omega_1^2 - Z^2}} \right)^{n-2} \sin \left( \frac{\lambda t}{2} \log s \right) ds
\]

(4.16)

\[
- \int_{s_1}^{s_2} (ABD)^{n-1}s^{-\frac{n-1}{2}} \left( \frac{W^2}{U + V \sqrt{W^2 \omega_1^2 - Z^2}} \right)^{n-2} \sin \left( \frac{\lambda t}{2} \log s \right) ds,
\]

where \(s_i = \Theta(\theta_i), i = 0, 1, 2\). Since \(n = 3\), we have \(\omega_1 = \sin \varphi\) and \(\omega' = \cos \varphi\), and hence

\[
\int_{S^2} \text{Im} E(e_1, y, z) d\omega
\]

\[
= \int_0^\pi \int_{s_0}^{s_1} (ABD)^2s^{-1} \frac{2UW^2}{\sqrt{W^2 \sin^2 \varphi - Z^2} U^2 - V^2(W^2 \sin^2 \varphi - Z^2)} \sin \left( \frac{\lambda t}{2} \log s \right) ds \sin \varphi d\varphi
\]

\[
= \int_0^\pi \int_{s_0}^{s_1} (ABD)^2s^{-1} \frac{UW^2}{\sqrt{W^2 \sin^2 \varphi - Z^2} U^2 - V^2(W^2 \sin^2 \varphi - Z^2)} \sin \left( \frac{\lambda t}{2} \log s \right) ds \sin \varphi d\varphi.
\]

Note that \(s_0\) and \(s_1\) are solutions of \(W^2 \sin^2 \varphi - Z^2 = 0\). Since

\[
W^2 - Z^2 = -A^2B^2D^2 \left( s - \frac{t - r}{t + r} \right) \left( s - \frac{t + r}{t - r} \right),
\]

(4.17)

Fubini’s theorem and change of variables \(v = \sin \varphi\) lead us to

\[
\int_{S^2} \text{Im} E(e_1, y, z) d\omega
\]

\[
= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \int_{v_0}^1 \frac{4(ABD)^2UW^2v}{(U^2 - V^2(W^2v^2 - Z^2))\sqrt{W^2v^2 - Z^2} \sqrt{1 - v^2}} dv \right) \frac{1}{s} \sin \left( \frac{\lambda t}{2} \log s \right) ds
\]

\[
= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \int_{v_0}^1 \frac{4(ABD)^2Uv}{V^2W(U^2v^2 + v_0^2 - v^2)\sqrt{v^2 - v_0^2} \sqrt{1 - v^2}} dv \right) \frac{1}{s} \sin \left( \frac{\lambda t}{2} \log s \right) ds,
\]

where \(v_0 = \frac{|Z|}{W}\). Note that \(R^2 := \frac{U^2}{V^2W^2} + v_0^2 \geq 1\) since \(Z^2 + v_0^2 \geq Z^2 \geq X^2 + Y^2 \geq W^2\). Moreover, since

\[
\frac{d}{dv} \left( - \frac{VW}{U\sqrt{R^2 - 1}} \arctan \left( \frac{U}{VW\sqrt{R^2 - 1}} \sqrt{1 - v^2} \right) \right) = \frac{v}{(R^2 - v^2)\sqrt{(v^2 - v_0^2)(1 - v^2)}}
\]

we obtain

\[
\int_{v_0}^1 \frac{v}{(U^2v^2 + v_0^2 - v^2)\sqrt{v^2 - v_0^2} \sqrt{1 - v^2}} dv = \frac{\pi V^2W^2}{2U \sqrt{U^2 - V^2(W^2 - Z^2)}}.
\]
Therefore, we have
\[ \int_{S^2} \text{Im } E(e_1, y, z) \, d\omega = \int_{\frac{1-\pi}{4\pi}}^{\frac{1+\pi}{4\pi}} \frac{2\pi (ABD)^2 W}{\sqrt{U^2 - V^2(W^2 - Z^2)}} s^{-1} \sin \left( \frac{\lambda t}{2} \log s \right) \, ds. \]

A simple computation shows that
\[ U^2 - V^2(W^2 - Z^2) = ((Z_1X_2 - Z_2X_1)^2 + (Y_1Z_2 - Y_2Z_1)^2 - (X_1Y_2 - X_2Y_1)^2) W^2. \]

Furthermore, we have
\[
Z_1X_2 - Z_2X_1 = -2ABD \left( \frac{C}{t} \left( 1 + \frac{r^2 \norm{z}^2}{t^4} \right) - 2 \frac{r z_1}{t^2} \left( A - B \frac{z_1}{t} \right) \right),
\]
\[
Y_1Z_2 - Y_2Z_1 = 4ABD \frac{r z_2}{t^2} \left( A - B \frac{z_1}{t} \right), \quad \text{and}
\]
\[
X_1Y_2 - X_2Y_1 = -4ABCD \frac{r^2 z_2}{t^3},
\]

which yield that
\[ U^2 - V^2(W^2 - Z^2) = 4(ABD)^4 W^2 \left( \frac{rt}{t^2 - r^2} \right)^2. \]

Finally, the change of variables \( u = \frac{t}{2} \log s \) yields that
\[
\int_{S^2} \text{Im } E(e_1, y, z) \, d\omega = \pi \int_{\frac{1-\pi}{4\pi}}^{\frac{1+\pi}{4\pi}} \left( \frac{t}{r} - \frac{r}{t} \right) s^{-1} \sin \left( \frac{\lambda t}{2} \log s \right) \, ds
\]
\[ = \frac{2\pi}{t} \left( \frac{t}{r} - \frac{r}{t} \right) \int_{-d_{S^2}(0, y)}^{d_{S^2}(0, y)} \sin(\lambda u) \, du = 0,
\]

which proves (4.9).

The proof of Lemma 4.2 for 2-dimensional case is simpler. Indeed, it follows from (4.16) and (4.17) that
\[
I = \int_{\frac{1-\pi}{4\pi}}^{\frac{1+\pi}{4\pi}} ABD s^{-1/2} \sin \left( \frac{\lambda t}{2} \log s \right) \, ds = \int_{-\log \frac{1-\pi}{1+\pi}}^{\log \frac{1+\pi}{1+\pi}} ABD e^{u/2} \sin \left( \frac{\lambda t}{2} u \right) \, du = 0.
\]

The authors believe that Lemma 4.2 holds true for any dimension \( n \), but could not find how to compute the integral (4.16). We leave it for future discussion.

Let us next prove Lemma 4.1.

Proof of Lemma 4.1. The same computation as in Lemma 4.2 shows that
\[
\int_{S^2} \text{Re } E(\lambda, \xi, y, z) \, d\omega = \frac{2\pi}{t} \left( \frac{t}{r} - \frac{r}{t} \right) \int_{-d_{S^2}(0, y)}^{d_{S^2}(0, y)} \cos(\lambda u) \, du
\]
\[ = \frac{4\pi}{\lambda t} \left( \frac{t}{r} - \frac{r}{t} \right) \sin \left( \lambda d_{S^2}(0, y) \right),
\]
where $y = r\omega$. Thus, we have

\[
I := -\frac{1}{2} \int_0^t \int_{S^2} \Re \left( E(\lambda, \xi, r\omega, z) + E(\lambda, \xi, -r\omega, z) - 2 \right) K_{\gamma,t/2} \left( d_{B_t^\nu}(0, y) \right) \frac{r^2}{(1 - \frac{r^2}{t^2})^3} \, d\omega \, dr
\]

\[
= 4\pi \int_0^t \left( 1 - \frac{1}{\lambda t} \left( \frac{t}{r} - \frac{r}{t} \right) \sin \left( \lambda d_{B_t^\nu}(0, y) \right) \right) K_{\gamma,t/2} \left( d_{B_t^\nu}(0, y) \right) \frac{r^2}{(1 - \frac{r^2}{t^2})^3} \, dr.
\]

Using the change of variables

\[
\rho = \cosh^{-1} \left( \frac{t^2 + r^2}{t^2 - r^2} \right), \quad r^2 = t^2 \cosh \rho - 1, \quad \frac{dr}{\cosh \rho + 1} = \frac{t}{\cosh \rho + 1} \, d\rho,
\]

we obtain

\[
(4.18) \quad I = \frac{\pi t^3}{2} \int_0^\infty \left( 1 - \frac{2 \sin(\lambda t \rho/2)}{\lambda t \sin \rho} \right) K_{\gamma,t/2}(t \rho/2) \sin^2 \rho \, d\rho.
\]

Recall that the kernel $K_{\gamma,t/2}$ is given by

\[
K_{\gamma,t/2}(\rho) = \alpha_{2\gamma} \frac{1}{\sin(2\rho/t)} \partial_\rho \left( (2\rho/t)^{-1/2-\gamma} K_{-1/2-\gamma}(2\rho/t) \right)
\]

\[
= (-\alpha_{2\gamma}) \frac{2/t}{\sin(2\rho/t)} (2\rho/t)^{-1/2-\gamma} K_{-3/2-\gamma}(2\rho/t),
\]

where we have used (A.2). By plugging (4.19) into (4.18), we have

\[
I = (-\alpha_{2\gamma}) \pi t^2 \int_0^\infty \left( \sin \rho - \frac{2}{\lambda t} \sin \left( \frac{\lambda t}{2} \rho \right) \right) \rho^{-1/2-\gamma} K_{-3/2-\gamma}(\rho).
\]

Let $S^k_\nu$ and $L^k_\nu$ be indefinite integrals given in (A.3). Then, for $\nu = 3/2 + \gamma$,

\[
I = (-\alpha_{2\gamma}) \pi t^2 \left[ S^1_\nu(\rho) - \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \left( \frac{\lambda t}{2} \right)^{2k} L^{2k+2}_\nu(\rho) \right]_0^\infty.
\]

Lemma A.1, Lemma A.2, and Lemma A.3 show that

\[
S^1_\nu - \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \left( \frac{\lambda t}{2} \right)^{2k} L^{2k+2}_\nu \to 0
\]

as $\rho \to 0$, and

\[
S^1_\nu - \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \left( \frac{\lambda t}{2} \right)^{2k} L^{2k+2}_\nu \to -\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \left( \frac{\lambda t}{2} \right)^{2k} \sqrt{\pi(2k+2)!} \frac{2k}{2k+1(2k+1)!} 2^{k-\nu} \Gamma(k-\gamma)
\]
as \( \rho \to \infty \). Therefore, we arrive at

\[
I = (-\alpha_{\gamma})2^{-\nu} \pi^{3/2} t^2 \left[ -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\lambda t}{2} \right)^{2k}(k - 1 - \gamma)(k - 2 - \gamma) \cdots (-\gamma) \Gamma(-\gamma) \right]
\]

\[
= (-\alpha_{\gamma})2^{-\nu} \pi^{3/2} t^2 |\Gamma(-\gamma)| \sum_{k=0}^{\infty} \frac{\gamma(\gamma - 1) \cdots (\gamma - k + 1)}{k!} \left( \frac{\lambda t}{2} \right)^{2k}
\]

\[
= (-\alpha_{\gamma})2^{-\nu} \pi^{3/2} t^2 |\Gamma(-\gamma)| \left( 1 + \frac{\lambda^2 t^2}{4} \right)^{\gamma}
\]

\[
= (-\alpha_{\gamma})2^{-\nu} \pi^{3/2} t^2 |\Gamma(-\gamma)| \left( \lambda^2 + \frac{t^2}{\lambda^2} \right)^{\gamma},
\]

where we have used the binomial theorem for fractional exponent \( \gamma \) in the third equality. \( \square \)

5. Scale functions

Having the explicit form of kernels (1.2) at hand, we will proceed to the regularity theory for fractional Laplacian-type operators on hyperbolic spaces. For this purpose, some scale functions that are appropriate in the framework of hyperbolic spaces are introduced in this section.

As mentioned in Section 1, it is of interest to study regularity estimates whose constants do not blow up and stay uniform as the order of operator approaches 2. To simplify computations and focus on the limit \( \gamma \to 1 \), let us consider the the case \( \tau = t/2 = 1 \) and write \( K_{\gamma,1} = K_{\gamma} \).

Let us define the scale functions

\[
I_0(R) := I_{0,\gamma}(R) := \int_{B_R(x)} d^2_{\mathbb{H}}(z, x) K_{\gamma}(d_{\mathbb{H}}(z, x)) \, d\mu(z),
\]

\[
I_{\infty}(R) := I_{\infty,\gamma}(R) := \int_{\mathbb{H}^n \setminus B_R(x)} R^2 K_{\gamma}(d_{\mathbb{H}}(z, x)) \, d\mu(z),
\]

and \( I(R) := I_{\gamma}(R) := I_{0,\gamma}(R) + I_{\infty,\gamma}(R) \). It follows from Theorem 1.1 and Lemma A.3 that \( \rho^2 K_{\gamma}(\rho) \sinh^2 \rho \sim \rho^{1-2\gamma} \) as \( \rho \to 0 \) and \( K_{\gamma}(\rho) \sinh^2 \rho \sim \rho^{-1-\gamma} \) as \( \rho \to \infty \), which guarantee the well-definedness of integrals in (5.1). Moreover, we observe that

\[
|Lu(x)| \leq A \|u\|_{C^2(B_R(x))} I_0(R) + 2A \|u\|_{L^\infty(\mathbb{H}^n)} I_{\infty}(R) < +\infty
\]

for any \( L \in \mathcal{L}_0 \) and any function \( u \in C^2(B_R(x)) \cap L^\infty(\mathbb{H}^n) \).

In 3-dimensional case, the scale functions (5.1) are represented explicitly by using Bessel functions.

**Lemma 5.1.** Let \( n = 3 \). Then,

\[
I_0(R) = \frac{2^\gamma}{(1 - \gamma) \Gamma(-\gamma)} R^{3-\gamma} \left( I_{1/2} K_{3/2+\gamma} + I_{3/2} K_{1/2+\gamma} \right)
\]

\[
- \frac{2^\gamma}{(1 - \gamma)(2 - \gamma) \Gamma(-\gamma)} R^{3-\gamma} \left( I_{3/2} K_{1/2+\gamma} + I_{5/2} K_{-1/2+\gamma} \right)
\]

and

\[
I_{\infty}(R) = \frac{2^\gamma}{\Gamma(-\gamma)} R^{3-\gamma} \left( I_{1/2} K_{3/2+\gamma} + I_{3/2} K_{1/2+\gamma} \right),
\]

where \( I_\nu \) and \( K_\nu \) denote the modified Bessel function of the first and second kind, respectively.
Proof. The proof of (5.4) is very similar to that of (5.3), so let us prove (5.3) only. Let $S^k_\nu$ and $C^k_\nu$ be functions defined by (A.3), then

$$I_0(R) = |S^2| \int_0^R \rho^2 K_\nu(\rho) \sinh^2 \rho \, d\rho$$

$$= \frac{C(3, \gamma)|S^2|}{\Gamma(3/2 + \gamma)2^{1/2+\gamma}} \int_0^R \rho^{3/2-\gamma} K_{3/2-\gamma}(\rho) \sinh \rho \, d\rho$$

$$= \frac{2^{3/2+\gamma}}{\pi^{1/2}\Gamma(-\gamma)} \left( S^3_{3/2+\gamma}(R) - S^3_{3/2+\gamma}(0) \right),$$

where $\nu = 3/2 + \gamma$. By using Lemma A.1 and the recurrence relations (A.1), we obtain

$$S^3_{3/2+\gamma}(R) = \frac{2^{-3/2}\pi^{1/2}}{(1-\gamma)} R^{3-\gamma} \left( I_{1/2}K_{3/2+\gamma} + I_{5/2}K_{1/2+\gamma} \right)$$

$$- \frac{2^{-3/2}\pi^{1/2}}{(1-\gamma)(2-\gamma)} R^{3-\gamma} \left( I_{3/2}K_{1/2+\gamma} + I_{5/2}K_{-1/2+\gamma} \right),$$

from which (5.3) follows. \[ \square \]

The scale functions $I_0$ and $I_\infty$ correspond to $\frac{C(3, \gamma)|S^2|}{2^{-2-\gamma}} R^{2-2\gamma}$ and $\frac{C(3, \gamma)|S^2|}{2^{1-\gamma}} R^{2-2\gamma}$, respectively, in the case of Euclidean spaces. While $R^{2-2\gamma}$ has a scaling property, the scale functions $I_0$ and $I_\infty$ do not have it. This is one of the main difficulties in the upcoming regularity theory. However, we will observe that some monotonicity properties play a similar role in the following sections. For example, $I_0(R)$ is increasing and $I_\infty$ is decreasing by definition. Moreover, the following lemma provides another monotonicity.

**Lemma 5.2.** The function $I_0(R)/R^{2-\gamma}$ is decreasing in $R$.

Proof. Let us define the function

$$F(R) := \frac{(1-\gamma)|\Gamma(-\gamma)|I_0(R)}{2^{\gamma} R^{2-\gamma}}$$

$$= R \left( I_{1/2}K_{3/2+\gamma} + \frac{1-\gamma}{2-\gamma} I_{3/2}K_{1/2+\gamma} - \frac{1}{2-\gamma} I_{5/2}K_{-1/2+\gamma} \right).$$

Using the relations (A.1) and (A.2), a straightforward calculation yields

$$F'(R) = -\gamma I_{1/2}K_{3/2+\gamma} - (1-\gamma) I_{3/2}K_{1/2+\gamma} + I_{5/2}K_{-1/2+\gamma}.$$

Thus, we conclude that

$$F'(R) \leq (-\gamma - (1-\gamma) + 1) I_{5/2}K_{-1/2+\gamma} = 0,$$

by using the facts that $I_\nu$ is decreasing and $K_\nu$ is increasing with respect to $\nu$ when $\nu > 0$. \[ \square \]

An obvious consequence of Lemma 5.2 is the following.

**Corollary 5.3.** The function $I_0(R)/R^2$ is decreasing in $R$.

The following lemma shows a relation between two scale functions $I_0$ and $I_\infty$. Let us mention that the function $H(t) = t \coth t$ naturally appears when the negative curvature is involved as in Lemma 2.2.
Lemma 5.4. For all $R > 0$,

$$I_\infty(R) \leq \frac{1 - \gamma}{\gamma} \mathcal{H}(R) I_0(R).$$

Proof. Let us define the function

$$F = \frac{\gamma |\Gamma(-\gamma)|}{2\gamma} \left( I_\infty - \frac{1 - \gamma}{\gamma} \mathcal{H} I_0 \right)$$

and show that $R^{-2 + \gamma} F' \leq 0$. Using Lemma 5.1 and the relations (A.1) and (A.2), we have

$$R^{-2 + \gamma} F' = 2(1 - \gamma)I_{1/2}K_{3/2 + \gamma} + 2I_{3/2}K_{1/2 + \gamma} - 2(1 - \gamma)\mathcal{H}I_{1/2}K_{3/2 + \gamma}$$

$$- \left( \mathcal{H} - \frac{R^2}{\sinh R} \right) \left( I_{1/2}K_{3/2 + \gamma} + \frac{1 - \gamma}{2 - \gamma} I_{3/2}K_{1/2 + \gamma} - \frac{1}{2 - \gamma} I_{5/2}K_{1/2 + \gamma} \right).$$

Using $\mathcal{H}I_{1/2} = RI_{1/2}$ and (A.1) again, we obtain

$$\mathcal{H}I_{1/2}K_{3/2 + \gamma} = RI_{3/2}K_{1/2 + \gamma} + (1 + 2\gamma)I_{3/2}K_{1/2 + \gamma} + I_{1/2}K_{3/2 + \gamma}.$$

Thus, in light of $\mathcal{H} - \frac{R^2}{\sinh R} \geq 0$ and $\gamma \in (0, 1)$, we have

$$R^{-2 + \gamma} F' = (2 - 2(1 - \gamma)(1 + 2\gamma))I_{3/2}K_{1/2 + \gamma} - 2(1 - \gamma)RI_{3/2}K_{1/2 + \gamma}$$

$$- \left( \mathcal{H} - \frac{R^2}{\sinh R} \right) \left( I_{1/2}K_{3/2 + \gamma} + \frac{1 - \gamma}{2 - \gamma} I_{3/2}K_{1/2 + \gamma} - \frac{1}{2 - \gamma} I_{5/2}K_{1/2 + \gamma} \right)$$

$$\leq - 2\gamma(1 - 2\gamma)I_{3/2}K_{1/2 + \gamma} - \left( \mathcal{H} - \frac{R^2}{\sinh R} \right) \left( I_{1/2}K_{3/2 + \gamma} - I_{5/2}K_{1/2 + \gamma} \right).$$

Moreover, since $I_{1/2}K_{3/2 + \gamma} - I_{5/2}K_{1/2 + \gamma} \geq 0$, we use

$$\mathcal{H} - \frac{R^2}{\sinh^2 R} \geq \frac{R^2}{\sqrt{R^2 + 1} + 1},$$

to obtain

$$\frac{R^{-2 + \gamma} F'}{I_{3/2}K_{1/2 + \gamma}} \leq - 2\gamma(1 - 2\gamma) - \frac{R^2}{\sqrt{R^2 + 1} + 1} \left( \frac{I_{1/2}K_{3/2 + \gamma} - I_{5/2}K_{1/2 + \gamma}}{I_{3/2}K_{1/2 + \gamma}} \right).$$

We now use bounds for ratios of modified Bessel functions

$$\frac{I_{\nu + 1/2}(R)}{I_{\nu - 1/2}(R)} < \frac{R}{\sqrt{R^2 + \nu^2 + \nu}}, \quad \nu \geq 0,$$

and

$$\frac{K_{\nu}(R)}{K_{\nu + 1}(R)} \leq \frac{R}{\sqrt{R^2 + (\nu - 1/2)^2 + \nu + 1/2}}, \quad \nu \geq 1/2.$$
See [32, Theorem 3 and 6] for the proofs of these bounds. By using (5.5)–(5.6) and monotonicty of \( K_\nu \) with respect to \( \nu \), we have

\[
\frac{I_{1/2}K_{3/2+\gamma} - I_{5/2}K_{-1/2+\gamma}}{I_{3/2}K_{1/2+\gamma}} \geq \frac{(\sqrt{R^2 + 1} + 1)(\sqrt{R^2 + \gamma^2 + 1 + \gamma})}{\sqrt{R^2 + 4} - 1} \geq \frac{(\sqrt{R^2 + 1} + 1)(\sqrt{R^2 + \gamma^2 + 1 + \gamma})}{\sqrt{R^2 + 1} + 1} \geq (1 + \gamma)\frac{\sqrt{R^2 + 1} + 1}{R^2}.
\]

Therefore, we conclude that

\[
\frac{R^{-2+\gamma}F'}{I_{3/2}K_{1/2+\gamma}} \leq -2\gamma(1 - 2\gamma) - (1 + \gamma) = -(1 + 4\gamma)(1 - \gamma) \leq 0.
\]

\[\Box\]

Since the limit behavior of the scale functions as \( \gamma \to 0 \) are of interest in the unified regularity theory, we also prepare the following lemma.

**Lemma 5.5.** For any fixed \( R > 0 \),

\[
\lim_{\gamma \to 1^-} I_0(R) = 6 \quad \text{and} \quad \lim_{\gamma \to 1^+} I_\infty(R) = 0.
\]

**Proof.** The modified Bessel functions \( I_\nu \) and \( K_\nu \) are elementary functions when \( \nu \) is a half-integer. For examples, we have

\[
I_{1/2} = \left(\frac{2}{\pi R}\right)^{1/2} \sinh R, \quad K_{1/2} = \left(\frac{\pi}{2R}\right)^{1/2} e^{-R},
\]

\[
I_{3/2} = \left(\frac{2}{\pi R}\right)^{1/2} \left( -\frac{\sinh R}{R} + \cosh R \right), \quad K_{3/2} = \left(\frac{\pi}{2R}\right)^{1/2} \left( 1 + \frac{1}{R} \right) e^{-R},
\]

\[
I_{5/2} = \left(\frac{2}{\pi R}\right)^{1/2} \left( \left( 1 + \frac{3}{R^2} \right) \sinh R - \frac{3}{R} \cosh R \right), \quad K_{5/2} = \left(\frac{\pi}{2R}\right)^{1/2} \left( 1 + \frac{3}{R} + \frac{3}{R^2} \right) e^{-R}.
\]

Thus, by using

\[
\lim_{\gamma \to 1^-} (1 - \gamma) |\Gamma(-\gamma)| = 1 \quad \text{and} \quad \lim_{\gamma \to 0} \gamma |\Gamma(-\gamma)| = 1,
\]

a straightforward computation proves the lemma. \[\Box\]

In the work of Caffarelli and Silvestre [3], the quantity \( r_0 = \rho_0 2^{1/(2 - 2\gamma)} R \), which is characterized by the relation \((r_0/\rho_0)^{2-2\gamma} = R^{2-2\gamma}/2\), plays a fundamental role. The most important feature of this quantity is that it converges to 0 as \( \gamma \to 0 \).

We define such quantity in a similar way in our framework. Since the scale function \( I_0 \) is strictly increasing, its inverse exists. Thus, for given \( R > 0 \), we define \( r_0 \in (0, R) \) by

\[
(5.7) \quad r_0 = \rho_0 I_0^{-1}(I_0(R)/2)
\]

for some universal constant \( \rho_0 \in (0, 1) \) that will be determined later. Let us close the section with the following lemma.

**Lemma 5.6.** For fixed \( R > 0 \), \( \lim_{\gamma \to 1} r_0 = 0 \).
Proof. Let us fix $R > 0$ and suppose that $\lim_{\gamma \to 1} r_0 \neq 0$. Then, since $\tilde{r} := \lim \sup_{\gamma \to 1} r_0 \neq 0$, there is a sequence $\gamma_k \to 1$ such that $r_{0,k} := r_0 \mathcal{I}_{0,\gamma_k}^{-1}(\mathcal{I}_{0,\gamma_k}(R)/2) \in (0, R)$ converges to $\tilde{r}$. We have

\begin{equation}
\mathcal{I}_{0,\gamma_k}(r_{0,k}/p_0) = \frac{1}{2} \mathcal{I}_{0,\gamma_k}(R).
\end{equation}

By Lemma 5.5 and continuity of $\mathcal{I}_0$, the left-hand side of (5.8) converges to 6, whereas the right-hand side of (5.8) converges to 3, which is a contradiction. \qed

6. Discrete ABP-type estimates

In this section, we provide the proof of Theorem 1.3. Throughout the section, $u$ is assumed to be a supersolution given in Theorem 1.3. On Riemannian manifolds, the distance squared function in construction of envelope was suggested by Cabré in [2], and has been used by many in [21, 42, 22, 23]. More precisely, for each $y \in B_{R}$, there is a unique paraboloid

\[ P_y(z) = c_y - \frac{1}{2R^2} d_{\mathbb{H}^n}(z, y) \]

that touches $u$ from below, with a contact point $x \in B_{\frac{R}{5}}$. The envelope $\Gamma$ of $u$ is defined by

\[ \Gamma(z) = \sup_{y \in B_R} P_y(z), \]

and the contact set is given by

\begin{equation}
\mathcal{C} = \{ x \in B_{5R} : u(x) = \Gamma(x) \}.
\end{equation}

The first step toward to ABP-type estimates for nonlocal operators is to find a ring around a given contact point in which supersolution $u$ is quadratically close to the paraboloid in a large portion of the ring. In Euclidean spaces [3], or more generally in Riemannian manifolds with nonnegative curvature [19], the standard dyadic rings $B_{2^{-k}r_0} \setminus B_{2^{-(k+1)}r_0}$ are used. However, these are not appropriate within the framework of hyperbolic spaces due to the lack of homogeneity of the volume of balls. We thus define $r_k$ recursively by

\begin{equation}
\frac{|B_{r_k}|}{|B_{r_{k-1}}|} = 2^{-n}, \quad k = 1, 2, \ldots,
\end{equation}

and a hyperbolic dyadic ring by $R_k = R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$. Note that we have from Lemma 2.1 that $|B_{r_k}|/|B_{r_{k-1}}| \leq (r_k/r_{k-1})^n$, and hence $r_{k+1} \geq r_k/2$. Using the hyperbolic dyadic rings, we will prove a series of lemmas to deduce Theorem 1.3. For notational convenience, we shall write

\[ F(x) := \Lambda \mathcal{H}(\tilde{r}R) + \frac{R^2}{\mathcal{I}_0(R)} f(x) \]

throughout the section.

**Lemma 6.1.** Let $u$ be a supersolution given in Theorem 1.3. Then, there exists a universal constant $C_0 > 0$, independent of $\gamma$ and $R$, such that for each $x \in \mathcal{C}$ and $M_0 > 0$, there is an integer $k \geq 0$ satisfying

\begin{equation}
|G_k| \leq \frac{C_0}{M_0} F(x) |R_k|,
\end{equation}

where $G_k = R_k \cap \{ u > P_y + M_0(r_k/R)^2 \}$. 
Proof. Let \( x \in C \) and \( M_0 > 0 \) be given. Since \( x \) minimizes the function \( u + \frac{1}{2R^2} d_{\mathbb{H}^n}(\cdot, y) \), we have \( \mathcal{M}^{-}\left(u\right) \geq I_1 + I_2 + I_3 \), where

\[
I_1 = \lambda \int_{B^{2}\left(x\right) \cup B^{5}R} \delta \left(u + \frac{1}{2R^2} d_{\mathbb{H}^n}(\cdot, y), x, z\right) K_{\gamma}(d_{\mathbb{H}^n}(z, x)) \, d\mu(z),
\]

\[
I_2 = -\Lambda \int_{B^{2}\left(x\right) \cup B^{5}R} \delta^{+} \left(\frac{1}{2R^2} d_{\mathbb{H}^n}(\cdot, y), x, z\right) K_{\gamma}(d_{\mathbb{H}^n}(z, x)) \, d\mu(z),
\]

\[
I_3 = -\Lambda \int_{\mathbb{H}^n \setminus (B^{2}\left(x\right) \cup B^{5}R)} \delta^{-}(u, x, z) K_{\gamma}(d_{\mathbb{H}^n}(z, x)) \, d\mu(z),
\]

and \( \delta(v, x, z) = (v(z) + v(\exp_{x}(-\exp_{x}^{-1} z)) - 2v(x))/2 \) is the second order incremental quotients. By the mean value theorem for integrals and Lemma 2.2, we obtain

\[
I_2 \geq -C\Lambda \mathcal{H}(7R) \frac{I_{0}(R)}{R^2}.
\]

Since \( u(x) \leq u(x) + \frac{1}{2R^2} d_{\mathbb{H}^n}(x, y) \leq \inf_{B^{2}\left(x\right)} u + \frac{1}{2R^2} d_{\mathbb{H}^n}(x, y) \leq 11/2 < 6 \) and \( u \geq 0 \) in \( \mathbb{H}^n \setminus B_{5}R \), we also have

\[
I_3 \geq -C\Lambda \frac{I_{\infty}(R)}{R^2}.
\]

Let us now focus on \( I_1 \). Assume that (6.3) does not hold for all \( k \). Then, since

\[
\delta \left(u + \frac{1}{2R^2} d_{\mathbb{H}^n}(\cdot, y), x, z\right) \geq M_0 \left(\frac{r_{k}}{R}\right)^{2} \quad \text{on} \quad G_{k}
\]

and \( K \) is decreasing (which can be easily checked by calculating \( K' \) and using (A.2)), we have

\[
I_1 \geq \lambda M_0 \sum_{k=1}^{\infty} \int_{G_{k}} \left(\frac{r_{k}}{R}\right)^{2} K_{\gamma}(d_{\mathbb{H}^n}(z, x)) \, d\mu(z) \geq \lambda C_0 \frac{F(x)}{R^2} \sum_{k=1}^{\infty} r_{k}^{2} K_{\gamma}(r_{k}) |R_{k}|.
\]

Since \( r_{k+1} \geq r_{k}/2 \) and \( |R_{k}| = 2^{n} |R_{k+1}| \), we obtain

\[
I_1 \geq \lambda C_0 \frac{F(x)}{R^2} \sum_{k=0}^{\infty} r_{k+1}^{2} K_{\gamma}(r_{k+1}) |R_{k+1}|
\]

\[
\geq \lambda^{2} \frac{2^{-2-n} C_{0}}{R^2} \sum_{k=0}^{\infty} r_{k}^{2} K_{\gamma}(r_{k+1}) |R_{k}|
\]

\[
\geq \lambda^{2} \frac{2^{-2-n} C_{0}}{R^2} \sum_{k=0}^{\infty} \int_{R_{k}} d_{\mathbb{H}^n}^{2}(z, x) K_{\gamma}(d_{\mathbb{H}^n}(z, x)) \, d\mu(z)
\]

\[
= \lambda^{2} \frac{2^{-2-n} C_{0}}{R^2} \frac{F(x)}{R^2} I_{0}(r_{0}).
\]

Furthermore, by using Corollary 5.3 we have

\[
I_1 \geq \lambda^{2} \frac{2^{-2-n} C_{0}}{R^2} \frac{F(x)}{R^2} r_{0}^{2} I_{0}(r_{0}/r_{0}) = \lambda^{2} \frac{2^{-3-n} C_{0}}{R^2} \frac{F(x)}{R^2} r_{0}^{2} I_{0}(r_{0}).
\]

Combining (6.4), (6.5), and (6.6), and then using (5.7) and Lemma 5.4,

\[
f(x) \geq \lambda^{2} \frac{2^{-3-n} C_{0}}{R^2} \frac{F(x)}{R^2} r_{0}^{2} I_{0}(R) - C \left(1 + \gamma_{0}^{-1}\right) \lambda \mathcal{H}(7R) \frac{I_{0}(R)}{R^2}.
\]

By taking \( C_{0} \) sufficiently large, we arrive at a contradiction. \( \square \)
The next lemma shows that the function $\Gamma - P_y$ is $c$-convex with an appropriate function $c$. See [13, 28] for the definition of $c$-convex function. The proof is exactly the same with that of [19, Lemma 3.4] except for the Hessian bound of distance squared function. That is, we use Lemma 2.2 instead of [19, Lemma 2.1].

**Lemma 6.2.** Let $x \in \mathcal{C}$, $z \in \mathbb{H}^n$, and $y \in B_R$ be a vertex point of a paraboloid $P_y$. Then,

$$(\Gamma - P_y)(z) \leq (1 - t)(\Gamma - P_y)(z_1) + t(\Gamma - P_y)(z_2) + \frac{1}{2R^2} t(1-t)\mathcal{H}(d_{\mathbb{H}^n}(y, z) + |\xi|)|\xi|^2$$

for all $t \in (0, 1)$, where $z_1 = \exp_z(t\xi)$ and $z_2 = \exp_z((1-t)(-\xi))$.

Using Lemma 6.2, we show that the envelope is captured in a small ball near a contact point by two paraboloids that are quadratically close to each other.

**Lemma 6.3.** Under the setting of Lemma 6.1, there is a universal constant $\varepsilon_0 \in (0, 1)$ such that if

$$|\{z \in R_k : \Gamma(z) > P_y(z) + h\}| \leq \varepsilon_0 |R_k|,$$

then

$$\Gamma \leq P_y + h + CH(7R)\left(\frac{R_k}{R}\right)^2$$

in $B_{\tilde{r}_{k+1}}(x)$, where $\tilde{r}_{k+1} = \tanh^{-1}\left(\frac{1}{2} \tanh r_{k+1}\right)$.

**Proof.** Let us fix $z \in B_{\tilde{r}_{k+1}}(x)$ and set $D = \{z \in R_k : \Gamma(z) \leq P_y(z) + h\}$. For $w \in R_k$, let us consider a geodesic $c : \mathbb{R} \to \mathbb{H}^n$ passing through $w$ and $z$. Then, $c(\mathbb{R}) \cap R_k$ consists of two connected components $c(t_1, t_2)$ and $c(t_3, t_4)$, where $t_1 < t_2 < t_3 < t_4$ satisfy $t_4 - t_3 = t_2 - t_1$. We may assume that $w = c(t) \in c(t_1, t_2)$. We define a map $\varphi_z : R_k \to R_k$ by $\varphi_z(w) = c(-t + t_1 + t_4)$, which is clearly one-to-one and onto.

Among all the geodesics passing through the point $z$, let us consider geodesics $c_\perp$ that are perpendicular to the geodesic joining $x$ and $z$. Then $\cup c_\perp$ divides $R_k$ into two regions: let $A_1$ be the smaller one and $A_2$ the bigger one. We have

$$(6.7) \quad |E| \leq |\varphi_z(E)| \quad \text{for any Borel set } E \subset A_1.$$

Indeed, we may assume that $z = 0 = (1, 0) \in \mathbb{H}^n$ by using a global isometry. Then the map $\varphi := \varphi_z$ can be represented by

$$\varphi(w) = (\cosh(r + C_\theta), \sinh(r + C_\theta)(-\theta)), \quad w = (\cosh r, \sinh r \theta),$$

where $C_\theta = d_{\mathbb{H}^n}(\varphi(w^*), 0) - d_{\mathbb{H}^n}(w^*, 0)$, with $w^*$, the intersection point of $\partial B_{\tilde{r}_{k+1}}(x)$ and the geodesic segment joining 0 and $w$. Note that $\varphi$ is a smooth map because it is a composition of smooth maps. Clearly, $C_\theta \geq 0$ if and only if $w \in A_1$. Thus, we obtain

$$|E| = \iint 1_E(w) \sinh^{n-1} r \, dr \, d\theta$$

$$\leq \iint 1_{\varphi(E)}(\varphi(w)) \sinh^{n-1}(r + C_\theta) \, dr \, d\theta$$

$$= \iint 1_{\varphi(E)}(\cosh \tilde{r}, \sinh \tilde{r}(-\theta)) \sinh^{n-1} \tilde{r} \, d\tilde{r} \, d\tilde{\theta}$$

$$= \iint 1_{\varphi(E)}(\cosh \tilde{r}, \sinh \tilde{r} \tilde{\theta}) \sinh^{n-1} \tilde{r} \, d\tilde{r} \, d\tilde{\theta} = |\varphi(E)|,$$

where we have used change of variables $\tilde{r} = r + C_\theta$ and $\tilde{\theta} = -\theta$. This proves (6.7).
We next claim that

\begin{equation}
|R_k| \leq C|A_1| \quad \text{with } C > 0 \text{ a universal constant.}
\end{equation}

Let us first deduce the lemma assuming that (6.8) is true. If we show that \(\varphi_\varepsilon(A_1 \cap D) \cap D \neq \emptyset\), then there are points \(w_i \in A_i \cap D\), \(i = 1, 2\), such that \(\varphi_\varepsilon(w_1) = w_2\). Since \(\Gamma(w_i) \leq P_y(w_i) + h\), for \(i = 1, 2\), the desired result follows from Lemma 6.2. Assume to the contrary that \(\varphi(A_1 \cap D) \cap D = \emptyset\). By (6.8), we have

\[ |A_1 \cap D| \leq |R_k \cap D| \leq \varepsilon_0 |R_k| \leq C\varepsilon_0 |A_1|. \]

By taking \(\varepsilon_0 = (2C)^{-1}\), we obtain \(|A_1 \cap D| > |A_1|/2\). Since \(\varphi_\varepsilon(A_1 \cap D) \subset A_2 \cap D^c\), it follows that

\[ \frac{1}{2}|A_1| < |A_1 \cap D| \leq |\varphi_\varepsilon(A_1 \cap D)| \leq |A_2 \cap D| \leq |R_k \cap D| \leq \frac{1}{2}|A_1|, \]

which is a contradiction.

From now on, we focus on the proof of (6.8). To this end, it is convenient to use the Poincaré ball model \(\mathbb{B}_1^3\) with \(\tau = b = t = 1\). Let \(\tilde{A}_1 = \phi(A_1)\) and \(\tilde{R}_k = \phi(R_k)\), where \(\phi\) is the isometry given by (2.3). Since we are concerned with volumes, we may assume \(\phi(z) = |\phi(z)| e_1\) so that \(\tilde{A}_1\) is rotationally symmetric with respect to \(x_1\)-axis. Let \(\rho_k\) be such that \(r_k = d_{\mathbb{B}_1^3}(0, \rho_k e_1)\). We observe that

\begin{equation}
\{y \in \mathbb{B}_1^3 : \rho_{k+1} < |y| < \rho_k, \ e_1 \cdot y/|y| > 1/2\} \subset \tilde{A}_1.
\end{equation}

Indeed, if we define \(\tilde{A}_1'\) in the same way as \(\tilde{A}_1\) with \(z' \in \partial B_{\tilde{r}_{k+1}}(x)\) instead of \(z \in B_{\tilde{r}_{k+1}}(x)\), then \(\tilde{A}_1 \supset \tilde{A}_1'\). Moreover, any geodesic that is perpendicular to \(x_1\)-axis and passes through \(\tilde{r}_{k+1} e_1\) is contained in the sphere

\begin{equation}
(x_1 - \frac{1 + \tilde{r}_{k+1}^2}{2\tilde{r}_{k+1}})^2 + x_2^2 + x_3^2 = \left( \frac{1 - \tilde{r}_{k+1}^2}{2\tilde{r}_{k+1}} \right)^2.
\end{equation}

The \(x_1\)-coordinate of the intersection of the spheres (6.10) and \(x_1^2 + x_2^2 + x_3^2 = \rho_{k+1}^2\) is given by

\[ \frac{\tilde{r}_{k+1}}{1 + \tilde{r}_{k+1}^2}(1 + \rho_{k+1}^2) = \tanh \tilde{r}_{k+1} \frac{\rho_{k+1}}{\tanh \tilde{r}_{k+1}} = \frac{1}{2}\rho_{k+1}, \]

where we used

\[ d_{\mathbb{B}_1^3}(0, \rho k e_1) = \cosh^{-1} \frac{1 + \rho^2}{1 - \rho^2} = \tanh^{-1} \frac{2\rho}{1 + \rho^2} \]

in the first equality. Note that the radius \(\tilde{r}_{k+1} = \tanh^{-1}(\frac{1}{2}\tanh r_{k+1})\) is chosen so that the last equality holds. Therefore, (6.9) holds.

We now compute

\[ |A_1| = |\tilde{A}_1| \geq \int_0^{2\pi} \int_{\rho_{k+1}}^{\rho_k} \int_0^{\pi/2} \left( \frac{2}{1 - \rho^2} \right)^3 \rho^2 \sin \varphi \, d\varphi \, d\rho \, d\theta = \pi \int_{\rho_{k+1}}^{\rho_k} \left( \frac{2}{1 - \rho^2} \right)^3 \rho^2 \, d\rho. \]

Since

\[ |R_k| = |\tilde{R}_k| = 4\pi \int_{\rho_{k+1}}^{\rho_k} \left( \frac{2}{1 - \rho^2} \right)^3 \rho^2 \, d\rho, \]

(6.8) is proved with \(C = 4\).
We define \( \phi : \mathbb{H}^n \to B_R \) by a map assigning each point \( x \in \mathbb{H}^n \) a vertex point \( y \) of the paraboloid \( P_y \), where \( P_y \) is a paraboloid such that \( \Gamma(x) = P_y(x) \), which is not necessarily unique. Then, the flatness of \( \Gamma \) obtained in Lemma 6.3 allows us to control the image of \( \phi \), which can be understood as the image of gradient of \( \Gamma \).

**Lemma 6.4.** Under the setting of Lemma 6.1, let \( x \in C \) and let \( k \) be such that (6.3) holds, and let \( \varepsilon_0 \) be the constant in Lemma 6.3. Then,

\[
|\{z \in R_k : u(z) > P_y(z) + CF(x)(r_k/R)^2\}| \leq \varepsilon_0 |R_k|
\]

and

\[
\phi \left( \frac{B(x, r_{k+1}/2)}{B(y, CS(\gamma)T(r_{k+1})F(x)r_k)} \right) \subset B(y, CS(\gamma)T(r_{k+1})F(x)r_k),
\]

where \( T(r) = r / \tanh^{-1}(\frac{1}{2} \tanh r) \) and \( C > 0 \) is a universal constant depending only on \( n, \lambda, L, \) and \( \gamma_0 \).

**Proof.** By taking \( M_0 = C_0 F(x)/\varepsilon_0 \) in Lemma 6.1, we obtain (6.11). Moreover, by Lemma 6.3 we have

\[
P_y \leq \Gamma \leq P_y + CF(x) \left( \frac{r_k}{R} \right)^2
\]

in \( B_{r_{k+1}}(x) \), with a universal constant \( C > 0 \).

To prove (6.12), let \( z \in \overline{B}(x, r_{k+1}/2) \) and \( y_s \in \phi(z) \). We need to find an upper bound of \( d_{\mathbb{H}^n}(y_s, y) \). Let \( \xi_1 = \exp_z^{-1} y_s \) and \( \xi_2 = \exp_z^{-1} y \). Let us consider a family of geodesics

\[
c(s, t) = \exp_z(t(\xi_1 + s(\xi_2 - \xi_1))),
\]

and the Jacobi field \( J \) along \( c \). Then, by [16, Equation (1.8b)] (or see, e.g. [18]), we have

\[
|J(1)|_{g(y_s)} \leq S(|\xi_1|)|J'(0)|_{g(z)} \leq S(\gamma)\xi_2 - \xi_1|_{g(z)}.
\]

Considering the curve \( s \to c(s, 1) \), we obtain

\[
d_{\mathbb{H}^n}(y_s, y) \leq \int_0^1 |\partial_s c(s, 1)|_{g(y_s)} \, ds \leq S(\gamma)\exp_z^{-1} y_s - \exp_z^{-1} y|_{g(z)}.
\]

By the Gauss lemma, we know that \( |\exp_z^{-1} y_s - \exp_z^{-1} y|_{g(z)} = R^2|\nabla P_y(z) - \nabla P_y(z)|_{g(z)} \). Thus, it only remains to show that

\[
R^2|\nabla P_y(z) - \nabla P_y(z)|_{g(z)} \leq CT(r_{k+1})F(x)r_k
\]

for some universal constant \( C > 0 \).

To this end, we prove that

\[
\left| \frac{d}{dt} \bigg|_{t=0} (P_{y_s} - P_y)(c(t)) \right| \leq CT(r_{k+1})F(x)r_k \frac{r_k}{R^2}
\]

for all geodesics \( c \), with unit speed, starting from \( c(0) = z \). Suppose that (6.15) does not hold for some \( c \). We may assume that

\[
CT(r_{k+1})F(x)r_k \frac{r_k}{R^2} \leq \left| \frac{d}{dt} \bigg|_{t=0} (P_{y_s} - P_y)(c(t)) \right|
\]

by considering \( \tilde{c}(t) = c(-t) \) instead of \( c(t) \) if necessary. Let \( \varepsilon > 0 \), then there is a \( \delta > 0 \) such that if \( |t| < \delta \), we have

\[
CT(r_{k+1})F(x)r_k \frac{r_k}{R^2} - \varepsilon \leq \frac{(P_{y_s} - P_y)(c(t)) - (P_{y_s} - P_y)(c(0))}{t} \leq \frac{h(t) - h(0)}{t},
\]
where \( h(t) = (\Gamma - P_y)(c(t)) \). Let \( T > 0 \) be the first time such that \( c(T) \in \partial B_{3\tilde{r}_{k+1}/4}(x) \). Let \( N \) be the least integer not smaller than \( T/\delta \), and let \( 0 = t_0 < t_1 < \cdots < t_N = T \) be equally distributed times. Then, \( t_{i+1} - t_i = T/N \leq \delta \). By Lemma 6.2, we have

\[
(6.17) \quad \frac{h(t_i) - h(t_{i-1})}{t_i - t_{i-1}} \leq \frac{h(t_{i+1}) - h(t_i)}{t_{i+1} - t_i} + \frac{H(7R)}{2R^2}(t_{i+1} - t_{i-1}), \quad i = 1, 2, \ldots, N - 1.
\]

Thus, it follows from (6.16) and (6.17) that

\[
(6.18) \quad C T (r_{k+1}) F(x) \frac{r_k}{R^2} - \varepsilon \leq \frac{h(t_{i+1}) - h(t_i)}{T/N} + \frac{H(7R)}{2R^2} T_i, \quad i = 1, 2, \ldots, N - 1.
\]

Summing up (6.18) for \( i = 1, 2, \ldots, N - 1 \), we obtain

\[
N \left( C T (r_{k+1}) F(x) \frac{r_k}{R^2} - \varepsilon \right) \leq \frac{h(t_N) - h(t_0)}{T/N} + \frac{H(7R) 2T N (N-1)}{2R^2}.
\]

Since \( c \) has a unit speed, we have \( \tilde{r}_{k+1}/4 < T < \tilde{r}_{k+1} \), and hence

\[
C T (r_{k+1}) F(x) \frac{r_k}{R^2} - \varepsilon \leq \frac{(\Gamma - P_y)(c(T)) - (\Gamma - P_y)(z)}{\tilde{r}_{k+1}/4} + \frac{H(7R)}{2R^2} \tilde{r}_{k+1} \leq \frac{(\Gamma - P_y)(c(T))}{\tilde{r}_{k+1}/4} + \frac{H(7R)}{2R^2} \tilde{r}_{k+1}.
\]

Recalling that \( T (r_{k+1}) = r_{k+1}/\tilde{r}_{k+1} \) and \( r_{k+1} \geq r_k/2 \), and that \( \varepsilon \) was arbitrary, we have

\[
(6.19) \quad C F(x) \frac{r_k^2}{8R^2} \leq (\Gamma - P_y)(c(T)) + H(7R) \frac{r_k^2}{8R^2}.
\]

Since \( c(T) \in \partial B_{3\tilde{r}_{k+1}/4}(x) \subset B_{2\tilde{r}_{k+1}}(x) \), the inequality (6.19) with sufficiently large constant \( C_1 > 0 \) contradicts to (6.13). Therefore, we have proved (6.14), which finishes the proof. \( \square \)

We are now ready to prove a discrete ABP-type estimate, from which Theorem 1.3 follows.

**Lemma 6.5.** Assume the same assumptions as in Theorem 1.3. There is a finite collection \( \mathcal{D} \) of dyadic cubes \( \{Q^i_{\alpha}\} \), with diameters \( d_j \leq r_0 \), such that the following hold:

(i) Any two different dyadic cubes in \( \mathcal{D} \) do not intersect.

(ii) \( \mathcal{C} \subset \bigcup \overline{Q}^i_{\alpha}. \)

(iii) \( |\phi(Q^i_{\alpha})| \leq C \cosh^{-1}((CT(r_0)^2 r_0 F)(CT(r_0)^2 F)^{(n-1) \log \cosh(CT(r_0)^2 r_0 F)(CT(r_0)^2 F)^n}) |Q^i_{\alpha}|. \)

(iv) \( |B(z_0, 2r_0) \cap \{u \leq \Gamma + C(\sup_{Q^i_{\alpha}} F(x)(r_0/R)^2)\} | \geq \mu |Q^i_{\alpha}|. \)

The constants \( C > 0 \) and \( \mu > 0 \) depend only on \( n, \lambda, \Lambda, \) and \( \gamma_0 \).

**Proof.** Let \( c_1, c_2, \) and \( \delta_0 \) be the constants given in Theorem 2.3, which depend only on \( n \). Let us fix the smallest integer \( N \) such that \( c_2 \delta_0^N \leq r_0 \), then there are finitely many dyadic cubes \( Q^i_{\alpha} \) of generation \( N \) such that \( \overline{Q}^i_{\alpha} \cap \mathcal{C} \neq \emptyset \) and \( \mathcal{C} \subset \bigcup_{\alpha} \overline{Q}^i_{\alpha} \). Whenever a dyadic cube \( Q^i_{\alpha} \) \((j \geq N)\) does not satisfy (iii) and (iv), we consider all of its successors \( Q^{j+1}_{\beta} \subset Q^i_{\alpha} \) instead of \( Q^i_{\alpha} \). Among these successors of \( j + 1 \) generation, we only keep those whose closures intersect \( A \) and discard the rest. We prove that this process must finish in a finite number of steps.

Assume to the contrary that the process produces an infinite sequence of nested dyadic cubes \( \{Q^i_{\alpha}\}_{j=N}^{\infty} \). Then, the intersection of their closures is some contact point \( x \in \mathcal{C} \). By
Lemma 6.4, there is a $k \geq 0$ such that (6.11) and (6.12) hold. Let $j \geq N$ be such that
\[ \delta_0 \tilde{r}_{k+1}/2 \leq c_2 \delta_0^j < \tilde{r}_{k+1}/2 \leq r_0, \]
then it follows from Theorem 2.3 that
\[ B(z_{\alpha}^j, c_1 \delta_0^j) \subset Q_{\alpha}^j \subset B(x, \tilde{r}_{k+1}/2). \]
Thus, it follows from (6.12) and (6.20) that
\[ |\phi(Q_{\alpha}^j)| \leq |\phi(B(x, \tilde{r}_{k+1}/2))| \leq |B(y, CS(7R)T(r_{k+1})F(x)r_k)|. \]
Since $S(7R)F(x) \leq F$ and $r_k \leq 2r_{k+1} = 2T(r_{k+1})\tilde{r}_{k+1} \leq 4T(r_0)c_2 \delta_0^{j-1}$, we have
\[ |\phi(Q_{\alpha}^j)| \leq |B(z_{\alpha}^j, CT(r_0)^2F_c \delta_0^j)|. \]
Therefore, by Lemma 2.1 we obtain
\[ |\phi(Q_{\alpha}^j)| \leq D \left(CT(r_0)^2F\right)^{\log_4 D} |Q_{\alpha}^j| \]
where $D = 2^n \cosh^{n-1}(CT(r_0)^2c_2 \delta_0^jF)$, which shows that $Q_{\alpha}^j$ satisfies (iii).
If $z \in B(x, r_k)$, then $d(z, z_{\alpha}^j) \leq d(z, x) + d(x, z_{\alpha}^j) < r_k + c_2 \delta_0^j \leq 2r_0$, which shows that $B(x, r_k) \subset B(z_{\alpha}^j, 2r_0)$. Thus, by using (6.11), we have
\[ |B(z_{\alpha}^j, 2r_0) \cap \{u \leq \Gamma + C(\sup_{Q_{\alpha}^j} F(x))(r_0/R)^2\}| \geq |R_k \cap \{u \leq P_y + CF(x)(r_0/R)^2\}| \]
\[ \geq (1 - \varepsilon_0)|R_k| \]
\[ = (1 - \varepsilon_0)(2^n - 1)|B_{r_{k+1}}| \]
\[ \geq \mu|Q_{\alpha}^j| \]
for some universal constant $\mu > 0$. This proves that $Q_{\alpha}^j$ also satisfies (iv), which yields a contradiction. Therefore, the process must stop in a finite number of steps. \[ \square \]

7. A Barrier Function

This section is devoted to the construction of a special barrier function, which is a key ingredient together with the ABP-type estimates for the Krylov–Safonov Harnack inequality. It is standard to use distance function to construct a barrier function, but computations are significantly different from the standard argument. We will observe how the negative curvature of hyperbolic spaces comes into play. Let us begin with some inequalities.

Lemma 7.1. Let $\alpha > 0$ and $R_0 > 0$. Then,
\[ (\cosh^{-1}(t \cosh R_0))^{-2\alpha} - R_0^{-2\alpha} \geq -2\alpha R_0^{-2\alpha-2} \mathcal{H}(R_0) (t-1) \]
for all $t > 1/ \cosh R_0$. Moreover, for $\mathcal{H}(t) = t \coth t$,
\[ \frac{(\cosh^{-1}(t \cosh R_0))^{-2\alpha-2}}{A^2 \cosh^2 R_0 - 1} - \frac{R_0^{-2\alpha-2}}{\sinh^2 R_0} \geq -\frac{1}{R_0^{2\alpha}} \left( (2\alpha + 2) \mathcal{H}(R_0) \mathcal{H}(R_0) \frac{t}{R_0^2} - \frac{1}{R_0^2 \sinh R_0} \right), \]
and
\[
(cosh^{-1}(t \cosh R_0))^{-2\alpha-1} t \cosh R_0 (t^2 \cosh^2 R_0 - 1)^{3/2} - R_0^{-2\alpha-1} \frac{\cosh R_0}{\sinh^3 R_0}
\geq -R_0^{2\alpha} (\{2\alpha + 1\} \mathcal{H}(R_0) - R_0^2 + 3 \mathcal{H}^2(R_0)) \mathcal{H}(R_0) \frac{t - 1}{R_0^2 R_0^2 \sinh^2 R_0},
\]
for all \( t > 1/ \cosh R_0 \).

Proof. Since the function
\[
f(t) := (cosh^{-1}(t \cosh(R_0)))^{-2\alpha}, \quad t > \frac{1}{\cosh R_0},
\]
is convex, (7.1) follows from the inequality \( f(t) \geq f(1) + f'(1)(t - 1) \). The inequalities (7.2) and (7.3) can be obtained similarly by considering
\[
g(t) := \frac{(cosh^{-1}(t \cosh R_0))^{-2\alpha-2}}{t^2 \cosh^2 R_0 - 1} \quad \text{and} \quad h(t) := \frac{t \cosh R_0}{(t^2 \cosh^2 R_0 - 1)^{3/2}},
\]
which are also convex functions. \( \square \)

Using Lemma 7.1, we first construct a barrier function when \( \gamma \) is sufficiently close to 1.

Lemma 7.2. Let \( \delta \in (0, 1) \). There are constants \( \alpha > 0 \) and \( \gamma_0 \in (0, 1) \), depending only on \( n \), \( \lambda \), \( \Lambda \), \( \delta \), and \( R \), such that the function
\[
v(x) = \max \left\{ -\left(\frac{\delta}{20}\right)^{-2\alpha}, -\left(\frac{d_{\mathbb{H}^n}(x, 0)}{5R}\right)^{-2\alpha} \right\}
\]
is a supersolution to
\[
\frac{(7R)^2}{\mathcal{L}_0(7R)} \mathcal{M}^+ v(x) + \Lambda \mathcal{H}(7R) \leq 0,
\]
for every \( \gamma_0 < \gamma < 1 \) and \( x \in B_{5R} \setminus B_{\delta R/4} \).

Proof. Fix \( x \) and let \( R_0 := d_{\mathbb{H}^n}(x, 0) \in (\delta R/4, 5R) \). We are going to consider the coordinates centered at \( x \). There is an isometry \( \varphi \in SO(1, n) \) such that \( x = \varphi(0) \) and \( 0 = \varphi(\cosh R_0, \sinh R_0 e_1) \) with \( e_1 \in \mathbb{S}^{n-1} \). Notice that 0 denotes \((1, 0, \cdots, 0) \in \mathbb{H}^n \).

Let \( z \in B_{R_0/2}(x) \), then \( z = \varphi(\cosh r, \sinh r \omega) \) for some \( r \in [0, R_0/2) \) and \( \omega \in \mathbb{S}^{n-1} \). By the hyperbolic law of cosines, we have
\[
d_{\mathbb{H}^n}(z, 0) = d_{\mathbb{H}^n}(\varphi(\cosh r, \sinh r \omega), \varphi(\cosh R_0, \sinh R_0 e_1))
= d_{\mathbb{H}^n}((\cosh r, \sinh r \omega), (\cosh R_0, \sinh R_0 e_1))
= \cosh^{-1}(A - B)
\]
where \( A = \cosh r \cosh R_0 \) and \( B = \sinh r \sinh R_0 \omega_1 \). Similarly, we have
\[
d_{\mathbb{H}^n}(\exp_x (- \exp_x^{-1} z), 0) = \cosh^{-1}(A + B).
\]
Thus, we obtain
\[
\delta(v, x, z) = -(5R)^{2\alpha} \left( \cosh^{-1}(A - B) \right)^{-2\alpha} + \left( \cosh^{-1}(A + B) \right)^{-2\alpha} - 2R_0^{-2\alpha}.
\]
Since \((\cosh^{-1}(\cdot))^{-2\alpha}\) is convex at \(A\), we obtain

\[
\delta \leq - (5R)^{2\alpha} \left( \alpha(2\alpha + 1) \frac{(\cosh^{-1} A)^{-2\alpha-2}}{(A^2 - 1)^{1/2}} + \alpha \frac{A(\cosh^{-1} A)^{-2\alpha-1}}{(A^2 - 1)^{3/2}} \right) B^2 \\
- (5R)^{2\alpha} \left( (\cosh^{-1} A)^{-2\alpha} - R_0^{-2\alpha} \right).
\]

Moreover, by applying Lemma 7.1 with \(t = \cosh r\), we have

\[
\delta \leq \alpha(2\alpha + 1)c_\delta \left[ (2\alpha + 2 + 2\mathcal{H}(R_0)) \mathcal{H}(R_0) \frac{\cosh r - 1}{R_0^2} - 1 \right] \frac{\sinh^2 r}{R_0^2} \omega_1^2 \\
+ \alpha c_\delta \left[ ((2\alpha + 1)\mathcal{H}(R_0) - R_0^2 + 3\mathcal{H}^2(R_0)) \frac{\cosh r - 1}{R_0^2} - 1 \right] \mathcal{H}(R_0) \frac{\sinh^2 r}{R_0^2} \omega_1^2 \\
+ 2\alpha c_\delta \mathcal{H}(R_0) \frac{\cosh r - 1}{R_0^2},
\]

where \(c_\delta = (20/\delta)^{2\alpha}\).

Let us now compute

\[
\mathcal{M}^+ v(x) \leq \int_{B(x, \frac{R_0}{2})} (\Lambda \delta^+(v, x, z) - \lambda \delta^- (v, x, z)) \mathcal{K}_\gamma (d_{\mathbb{H}^n}(z, x)) \, d\mu(z) \\
+ \int_{\mathbb{H}^n \setminus B(x, \frac{R_0}{2})} (\Lambda \delta^+(v, x, z) - \lambda \delta^- (v, x, z)) \mathcal{K}_\gamma (d_{\mathbb{H}^n}(z, x)) \, d\mu(z) =: I_1 + I_2.
\]

We take \(\alpha = \alpha(n, \lambda, \Lambda, R) > 0\) sufficiently large so that

\[
\lambda(2\alpha + 1) \int_{\mathbb{S}^{n-1}} \omega_1^2 \, d\sigma - \Lambda \mathcal{H}(7R) > C_1 \Lambda \mathcal{H}(7R),
\]

for some universal constant \(C_1 > 0\) to be determined later. Then, we have

\[
I_1 \leq \Lambda \alpha c_\delta \left[ (2\alpha + 1)\mathcal{H}(R_0) - R_0^2 + 3\mathcal{H}^2(R_0) \right] \mathcal{H}(R_0) \int_{B_{R_0/2}} \frac{\cosh r - 1}{R_0^2} \frac{\sinh^2 r}{R_0^2} \omega_1^2 \mathcal{K}_\gamma \, d\mu(z) \\
+ \Lambda \alpha (2\alpha + 1)c_\delta (2\alpha + 2 + 2\mathcal{H}(R_0)) \mathcal{H}(R_0) \int_{B_{R_0/2}} \frac{\cosh r - 1}{R_0^2} \frac{\sinh^2 r}{R_0^2} \omega_1^2 \mathcal{K}_\gamma \, d\mu(z) \\
+ \alpha c_\delta \int_{B_{R_0/2}} \left( 2\Lambda \mathcal{H}(R_0) \frac{\cosh r - 1}{R_0^2} - \lambda(2\alpha + 1 + \mathcal{H}(R_0)) \frac{\sinh^2 r}{R_0^2} \omega_1^2 \right) \mathcal{K}_\gamma \, d\mu(z) \\
= \alpha c_\delta (I_{1,1} + I_{1,2} + I_{1,3}).
\]
We use (7.7) to estimate $I_{1,3}$ as follows:

$$
I_{1,3} = \int_{B_{R_0}/2} (4\Lambda H(R_0) - 4\lambda(2\alpha + 1 + H(R_0)) \cosh^2(r/2)\omega^2 R_0^2 \sinh^2 \frac{r}{R_0} K_\gamma \, d\mu(z)
\leq \int_0^{R_0/2} \left(4\Lambda|S^{n-1}|H(R) - 4\lambda(2\alpha + 1) \int_{S^{n-1}} \omega^2 \, d\sigma \right) \sinh^2 \frac{r}{R_0} K_\gamma(r) \sinh^{n-1} r \, dr
\leq -4C_1\Lambda \frac{H(R)}{R^2} \int_0^{R_0/2} |S^{n-1}| \left(\frac{r}{2}\right)^2 K_\gamma(r) \sinh^{n-1} r \, dr
= -C_1\Lambda H(R) \frac{I_0(R_0/2)}{R^2}.
$$

(7.9)

For $I_{1,1}$ and $I_{1,2}$, we observe that $\cosh r - 1 \leq Cr^2$ and $\sinh^2 r \leq Cr^2$ for $r \in [0, R_0/2]$, where $C$ is some constant depending on $R$. Thus, by using Theorem 1.1 we obtain

$$
I_{1,1} + I_{1,2} \leq CA \int_0^{R_0/2} \int_{S^{n-1}} \cosh r - 1 \sinh^2 r \omega^2 \frac{R^2}{R_0^2} K_\gamma(r) \sinh^{n-1} r \, d\sigma \, dr
\leq CA \frac{C(3, \gamma)}{\Gamma(3/2 + \gamma)} \frac{1}{2^1/2 + \gamma} R_0^4 \int_0^{R_0/2} r^{4-\nu} K_{-\nu-1}(r) \sinh r \, dr
\leq CA \frac{C(3, \gamma)}{R_0^4} \frac{1}{\Gamma(3/2 + \gamma)} \frac{1}{2^1/2 + \gamma} \frac{1}{\nu} I_0(R_0/2),
$$

(7.10)

where $\nu = 1/2 + \gamma$ and $S^{n-2}_\nu$ is the function defined in (A.3).

On the other hand, using the fact that $v$ is bounded and Lemma 5.4, we obtain

$$
I_2 \leq CA \frac{I_\infty(R_0/2)}{(R_0/2)^2} \leq CA \frac{1}{\gamma} \frac{H(R)}{\gamma} \frac{I_0(R_0/2)}{(R_0/2)^2}.
$$

(7.11)

Thus, (7.6), (7.8), (7.9), (7.10), (7.11), and Corollary 5.3 yield

$$
\frac{(7R)^2}{I_0(7R)} M^+ v(x) \leq CA \frac{(7R)^2}{I_0(7R)} \frac{1}{(R_0/2)^2} \frac{1}{\gamma} S^{n-2}_\nu(R_0/2) + \frac{1}{\gamma} H(R).
$$

(7.12)

Recall from Lemma 5.5 that $I_0(R_0/2) \to 6$ as $\gamma \to 1$. Since $C(3, \gamma) \to 0$ as $\gamma \to 1$, we can make the second and third terms in (7.12) be as small as we want, by choosing $\gamma_0$ close to 1. Therefore, the proof is finished by assuming that we have taken $\alpha$ sufficiently large so that (7.4) holds.

In the following lemma, we construct a barrier function for any $\gamma \in (\gamma_0, 1)$ for given $\gamma_0 \in (0, 1)$.

**Lemma 7.3.** Given $\gamma_0 \in (0, 1)$ and $\delta \in (0, 1)$, there exist universal constants $\alpha > 0$ and $\kappa \in (0, 1/4]$, depending only on $n$, $\lambda$, $\Lambda$, $\delta$, and $\gamma_0$, such that the function

$$
v(x) = \max \left\{ -\left(\frac{\kappa \delta}{20}\right)^{-2\alpha}, -\left(\frac{d_{\delta n}(x, 0)}{5R}\right)^{-2\alpha} \right\}
$$

is a supersolution to

$$
\frac{(7R)^2}{I_0(7R)} M^+ v(x) + \Lambda H(R) \leq 0,
$$

(7.13)

for every $\gamma_0 < \gamma < 1$ and $x \in B_{5R} \setminus \overline{B}_{\delta R/4}$.
Lemma 7.2

Lemma 5.4

Let \( \gamma_1 \) and \( \alpha_1 \) be the \( \gamma_0 \) and \( \alpha \) in Lemma 7.2, respectively. When \( \gamma \in [\gamma_1, 1) \), the desired result holds with \( \alpha_1 \) and \( \kappa = 1/4 \).

Let us now assume \( \gamma \in (\gamma_0, \gamma_1) \). For \( x \) with \( R_0 := d_{\mathbb{H}^n}(x, 0) \in (\delta R/4, 5R) \), we know that \( v \in C^2(B(x, R_0/2)) \) and that \( \delta^+(v, x, z) \) is bounded for \( z \in \mathbb{H}^n \setminus B(x, R_0/2) \). Thus, we have

\[
(7.14) \quad \frac{(7R)^2}{I_0(7R)} M^+ v(x) \leq C \mathcal{H}(7R) - \lambda \int_{\mathbb{H}^n} \delta^-(v, x, z) \mathcal{K}_\gamma(d_{\mathbb{H}^n}(z, x)) \, d\mu(z).
\]

If we take \( \alpha = \max\{\alpha_1, n/2\} \), then the function \( \delta^-(\langle d_{\mathbb{H}^n}(\cdot, 0)/5R\rangle^{-2\alpha}, x, z) \) is not integrable. Therefore, the last integral in (7.14) can be made arbitrarily large, by taking \( \kappa \) small. In particular, we choose \( \kappa \) so that (7.13) holds.

Corollary 7.4. Let \( \delta \in (0, 1) \) and assume \( 0 \leq \gamma_0 \leq \gamma < 1 \). Then, there is a function \( v_\delta \) such that

\[
\begin{cases}
  v_\delta \geq 0 & \text{in } M \setminus B_{5R}, \\
  v_\delta \leq 0 & \text{in } B_{2R}, \\
  \frac{(7R)^2}{I_0(7R)} M^+ v_\delta + \lambda \mathcal{H}(7R) \leq 0 & \text{in } B_{5R} \setminus \overline{B}_{\delta R/4}, \\
  \frac{(7R)^2}{I_0(7R)} M^+ v_\delta \leq C \lambda \mathcal{H}(7R) & \text{in } B_{5R}, \\
  v \geq -C & \text{in } B_{5R},
\end{cases}
\]

for some universal constant \( C > 0 \), depending only on \( n, \lambda, \Lambda, \delta, \) and \( \gamma_0 \).

Proof. Let \( \alpha \) and \( \kappa \) be the constants given in Lemma 7.3, and define a function \( v_\delta(x) = \psi(d_{\mathbb{H}^n}(x, 0)/R^2) \), where \( \psi \) is a smooth and increasing function on \([0, \infty)\) such that

\[
\psi(t) = \left( \frac{3^2}{5^2} \right)^{-\alpha} - \left( \frac{t}{5^2} \right)^{-\alpha} \quad \text{if } t \geq (\kappa \delta)^2.
\]

We already know from Lemma 7.3 that \( \frac{(7R)^2}{I_0(7R)} M^+ v_\delta + \lambda \mathcal{H}(7R) \leq 0 \) in \( B_{5R} \setminus \overline{B}_{\delta R/4} \). Finally, for \( x \in \overline{B}_{\delta R/4} \), we have \( |\delta(v_\delta, x, z)| \leq C \mathcal{H}(7R)d_{\mathbb{H}^n}(x, z)^2/R^2 \) for \( z \in B_R(x) \) and \( |\delta(v_\delta, x, z)| \leq C \) for \( z \in \mathbb{H}^n \setminus B_R(x) \) with a uniform constant \( C > 0 \). Therefore, we conclude \( \frac{(7R)^2}{I_0(7R)} M^+ v_\delta \leq C \lambda \mathcal{H}(7R) \) in \( B_{5R} \), with the help of Lemma 5.4.

8. \( L^p \)-estimate

In this section, we prove the so-called \( L^p \)-estimate, which forms a basis for the proofs of Harnack inequality and Hölder estimates. It connects a pointwise estimate to an estimate in measure.

Lemma 8.1. Assume \( 0 < \gamma_0 \leq \gamma < 1 \), and let \( \delta \in (0, 1) \). If \( u \in C^2(B_{7R}) \) is a nonnegative function on \( \mathbb{H}^n \) satisfying

\[
\frac{(7R)^2}{I_0(7R)} M^ u \leq \varepsilon_\delta \text{ in } B_{7R} \text{ and } \inf_{B_{2R}} u \leq 1,
\]

then

\[
\frac{|\{u \leq M_\delta\} \cap B_{\delta R}|}{|B_{7R}|} \geq \mu_\delta,
\]

where \( \varepsilon_\delta > 0, \mu_\delta \in (0, 1) \), and \( M_\delta > 1 \) are universal constants depending only on \( n, \lambda, \Lambda, \delta, R, \) and \( \gamma_0 \).
Proof. Let \( v_\delta \) be the barrier function constructed in Corollary 7.4 and define \( w = u + v_\delta \). Then \( w \) satisfies \( w \geq 0 \) in \( M \setminus B_{5R} \), \( \inf_{B_{2R}} w \leq 1 \), and \( \mathcal{M}^- w \leq \frac{I_0(7R)}{I_0(7R)} \varepsilon_\delta + \mathcal{M}^+ v_\delta \) in \( B_{5R} \). By applying Theorem 1.3 to \( w \) with its envelope \( \Gamma_w \), we have
\[
|B_R| \leq \sum_j cF^n|Q_j^i|,
\]
where
\[
F = S(7R) \left( \Lambda \mathcal{H}(7R) + \frac{R^2}{I_0(7R)} \left( \frac{I_0(7R)}{(7R)^2} \varepsilon_\delta + \max_{\mathcal{Q}_j^i} \mathcal{M}^+ v_\delta \right) \right) +
\]
and \( c = C \cosh^{n-1}(CT(r_0)^2r_0)(CT(r_0)^2F)(n-1)\log \cosh(CT(r_0)^2r_0F)T(r_0)^2 \). By Corollary 5.3, we obtain
\[
F \leq S(7R) \left( \varepsilon_\delta + \Lambda \mathcal{H}(7R) + \frac{(7R)^2}{I_0(7R)} \max_{\mathcal{Q}_j^i} \mathcal{M}^+ v_\delta \right).
\]
Since \( \Lambda \mathcal{H}(7R) + \frac{(7R)^2}{I_0(7R)} \mathcal{M}^+ v_\delta \leq 0 \) in \( B_{5R} \setminus \mathcal{B}_{7R/4} \) and \( \frac{(7R)^2}{I_0(7R)} \mathcal{M}^+ v_\delta \leq C \Lambda \mathcal{H}(7R) \) in \( B_{5R} \), we have
\[
|B_R| \leq C \varepsilon_\delta \sum_{\mathcal{Q}_j^i \cap \mathcal{B}_{7R/4} = \emptyset} |Q_j^i| + C \sum_{\mathcal{Q}_j^i \cap \mathcal{B}_{7R/4} \neq \emptyset} |Q_j^i|
\]
for some universal constant \( C > 0 \), depending on \( R \). By taking \( \varepsilon_\delta > 0 \) sufficiently small, we have
\[
|B_{7R}| \leq C \sum_{\mathcal{Q}_j^i \cap \mathcal{B}_{7R/4} \neq \emptyset} |Q_j^i|.
\]
By using Lemma 6.5 (iv), we obtain
\[
|B_{7R}| \leq C \sum_{\mathcal{Q}_j^i \cap \mathcal{B}_{7R/4} \neq \emptyset} |B(z_{j}^0, 2r_0) \cap \{ w \leq \Gamma_w + C \}|.
\]
Whenever \( \mathcal{Q}_j^i \cap \mathcal{B}_{7R/4} \neq \emptyset \), the ball \( B(z_{j}^0, 2r_0) \) is contained in \( B_{7R} \) if we have taken \( \rho_0 = \delta/4 \).
Indeed, for \( z \in B(z_{j}^0, 2r_0) \)
\[
d_{H^n}(z, 0) \leq d_{H^n}(z, z_{j}^0) + d_{H^n}(z_{j}^0, z_s) + d_{H^n}(z_s, 0) \leq 2r_0 + r_0 + \delta R/4 < \delta R,
\]
where \( z_s \) is a point in \( \mathcal{Q}_j^i \cap \mathcal{B}_{7R} \). By taking a subcover of \( \{ B(z_{j}^0, 2r_0) \} \) with finite overlapping and using \( v_\delta \geq -C \) in \( B_{5R} \), we arrive at
\[
|B_{7R}| \leq C|\{ u \leq M_\delta \} \cap B_{7R}|
\]
for some \( M_\delta > 1 \). Taking \( \mu_\delta = 1/C \) finishes the proof. \( \square \)

Lemma 8.1, together with the Calderón–Zygmund technique developed in [2], provides the following \( L^2 \)-estimate. As in [2], we fix \( \delta = \frac{2n}{42} \delta_0 \) and \( \delta_0 = \delta_0(1 - \delta_0)/2 \in (0, 1) \). Let \( k_R \) be the integer satisfying
\[
c_2 \delta_0^{k_R-1} < R \leq c_2 \delta_0^{k_R-2},
\]
which is the generation of a dyadic cube whose size is comparable to that of some ball of radius \( R \).
Lemma 8.2. Assume $0 < \gamma_0 \leq \gamma < 1$. Let $\varepsilon_\delta$, $\mu_\delta$, and $M_\delta$ be the constants in Lemma 8.1. Let $u \in C^2(B_{7R})$ be a nonnegative function on $\mathbb{H}^n$ satisfying $\frac{(7R)^2}{T_0(7R)}M^-u \leq \varepsilon_\delta$ in $B_{7R}$ and $\inf_{B_{3R}} u \leq 1$. If $Q_1$ is a dyadic cube of generation $k_R$ such that $\inf_{x \in Q_1} d_{\mathbb{H}^n}(x, 0) \leq \delta_1 R$, then

$$|\{u > M^+_3\} \cap Q_1| \leq (1 - c_\delta)^i|Q_1|,$$

for all $i = 1, 2, \ldots$. As a consequence, we have

$$|\{u > t\} \cap Q_1| \leq C t^{-\varepsilon}|Q_1|, \quad t > 0,$$

for some universal constants $C > 0$ and $\varepsilon > 0$.

Corollary 8.3 (Weak Harnack inequality). Assume $0 < \gamma_0 \leq \gamma < 1$. If $u \in C^2(B_{2R})$ is a nonnegative function satisfying $\mathcal{M}^-u \leq C_0$ in $B_{2R}$, then

$$\left(\int_{B_{2R}} u^p \, d\mu(z)\right)^{1/p} \leq C \left(\inf_{B_{2R}} u + C_0 \frac{R^2}{T_0(R)}\right),$$

where $p > 0$ and $C > 0$ are universal constants depending only on $n$, $\lambda$, $\Lambda$, $R$, and $\gamma_0$.

See, e.g., [2, Theorem 8.1] for the proof of Corollary 8.3.

9. Harnack inequality

The purpose of this section is to prove the Krylov–Safonov Harnack inequality by using Lemma 8.2. A simple scaling argument will provide Theorem 1.4.

Theorem 9.1. Assume $0 < \gamma_0 \leq \gamma < 1$. If a nonnegative function $u \in C^2(B_{7R})$ satisfies

$$\frac{(7R)^2}{T_0(7R)}M^-u \leq \varepsilon_0 \quad \text{and} \quad \frac{(7R)^2}{T_0(7R)}M^+u \geq -\varepsilon_0 \quad \text{in } B_{7R},$$

and $\inf_{B_{3R}} u \leq 1$, then

$$\sup_{B_{3R/4}} u \leq C,$$

where $\varepsilon_0 > 0$ and $C > 0$ are universal constants depending only on $n$, $\lambda$, $\Lambda$, $R$, and $\gamma_0$.

Proof. Let $\varepsilon$ and $\varepsilon_\delta$ be the constants given in Lemma 8.2, and let $t > 0$ be the minimal value such that the following holds:

$$u(x) \leq h_t(x) := t \left(\frac{1}{D} \left(1 - \frac{d_{\mathbb{H}^n}(x, 2R)}{\delta_1 R}\right) \log_2 D\right)^{-1/\varepsilon} \quad \text{for all } x \in B_{3R},$$

where for $D = 2^n \cosh^{n-1}(2\delta_0 R)$. Since $\sup_{B_{3R/4}} u \leq tD^{-\frac{1}{\varepsilon}\log_2 D}$, we can conclude the theorem once we show that $t \leq C$ for some universal constant $C$.

Let $x_0 \in B_{3R}$ be a point such that $u(x_0) = h_t(x_0)$. Let $d = \delta_1 R - d_{\mathbb{H}^n}(x_0, 0)$, $r = d/2$, and $A = \{u > u(x_0)/2\}$, then we have

$$u(x_0) = h_t(x_0) = tD^{1/\varepsilon} \left(\frac{2r}{\delta_1 R}\right)^{-\frac{1}{\varepsilon}\log_2 D}.$$

We apply Lemma 8.2 to $u$ to obtain

$$|A \cap Q_1| \leq C \left(\frac{u(x_0)}{2}\right)^{-\varepsilon}|Q_1| \leq Ct^{-\varepsilon} \frac{1}{D} \left(\frac{r}{R}\right)^{\log_2 D}|Q_1|,$$

where $Q_1$ is the unique dyadic cube of generation $k_R$ that contains the point $x_0$. 

We will show that there is a small constant $\theta > 0$ such that
\begin{equation}
|A^c \cap Q_2| \leq \frac{1}{2}|Q_2|, \tag{9.1}
\end{equation}
where $Q_2 \subset Q_1$ is the dyadic cube of generation $k_{\theta�/14}$ containing the point $x_0$, provided that $t$ is large. However, when $t$ is sufficiently large, we also have
\[ |A \cap Q_2| \leq |A \cap Q_1| \leq \frac{C}{t^\varepsilon} \left( \frac{r}{R} \right)^{\log_2 D} |B(z, c_2 \delta_0^k)| \leq \frac{C}{t^\varepsilon} \left| B(z, c_1 \delta_0^k) \right| \leq \frac{C}{t^\varepsilon} |Q_2| < \frac{1}{2} |Q_2|, \]
where $B(z, c_1 \delta_0^k)$ is a ball contained in $Q_2$. This contradicts to (9.1) and will lead us to a conclusion that $t$ is uniformly bounded.

Let us now focus on proving (9.1). For every $x \in B(x_0, \theta r)$, we have
\[ u(x) \leq h_t(x) \leq t \left( \frac{1}{D} \left( \frac{d - \theta r}{\delta_1 R} \right)^{\log_2 D} \right)^{-1/\varepsilon} = \left( 1 - \frac{\theta}{2} \right)^{-\frac{1}{\varepsilon} \log_2 D} u(x_0). \]
We define a function
\[ v(x) := \left( 1 - \frac{\theta}{2} \right)^{-\frac{1}{\varepsilon} \log_2 D} u(x_0) - u(x). \]
Since we will apply Lemma 8.2, we need a function which is nonnegative on the whole space. Thus, we apply Lemma 8.2 to $w := v^+$ in $B(x_0, \theta r/14)$. For $x \in B(x_0, \theta r/14)$, we have
\begin{align*}
\mathcal{M}^- w(x) &\leq \mathcal{M}^- v(x) + \mathcal{M}^+ v_-(x) \\
&\leq -\mathcal{M}^+ u(x) + \Lambda \int_{\mathbb{R}^n \setminus B(x_0, \theta r)} v^-(z) \mathcal{K}_\gamma(d_{\mathbb{R}^n}(z, x)) \, d\mu(z) \\
&\leq \frac{I_0(7R)}{(7R)^2} \varepsilon_0 + \Lambda \int_{\mathbb{R}^n \setminus B(x_0, \theta r)} (u(z) - (1 - \theta/2)^{-\frac{1}{\varepsilon} \log_2 D} u(x_0))^+ \mathcal{K}_\gamma(d_{\mathbb{R}^n}(z, x)) \, d\mu(z).
\end{align*}
To compute the last integral in (9.2), we introduce another auxiliary function
\[ g_\beta(x) := \beta \left( 1 - \frac{d_{\mathbb{R}^n}(x, 0)^2}{R^2} \right)^+, \]
with the largest number $\beta > 0$ satisfying $u \geq g_\beta$. From the assumption $\inf_{B_{\theta_1 R}} u \leq 1$, we have $(1 - \delta_1^2) \beta \leq 1$. Let $x_1 \in B_R$ be a point where $u(x_1) = g_\beta(x_1)$. Since
\[ \int_{\mathbb{R}^n} \delta^-(u, x_1, z) \mathcal{K}_\gamma(d_{\mathbb{R}^n}(z, x_1)) \, d\mu(z) \leq \int_{\mathbb{R}^n} \delta^-(g_\beta, x_1, z) \mathcal{K}_\gamma(d_{\mathbb{R}^n}(z, x_1)) \, d\mu(z) \leq C \mathcal{H}(7R) \frac{I_0(7R)}{(7R)^2}, \]
we obtain that
\[ \varepsilon_0 \geq \frac{I_0(7R)^2}{\mathcal{I}_0(7R)} \mathcal{M}^- u(x_1) \geq \frac{\lambda}{\mathcal{I}_0(7R)} \int_{\mathbb{R}^n} \delta^+(u, x_1, z) \mathcal{K}_\gamma(d_{\mathbb{R}^n}(z, x_1)) \, d\mu(z) - C\mathcal{H}(7R). \]
Thus, we have

\[
\int_{\mathbb{H}^n} (u(z) - c)^+ \mathcal{K}_\gamma (d_{\mathbb{H}^n}(z, x_1)) \, d\mu(z)
\]

\[
\leq \int_{\mathbb{H}^n} \delta^+ (u, x_1, z) \mathcal{K}_\gamma (d_{\mathbb{H}^n}(z, x_1)) \, d\mu(z) \leq C \mathcal{H}(7R) \frac{I_0(7R)}{(7R)^2},
\]

(9.3)

where \( c := 1/(1 - \delta_1^2) \).

If \( u(x_0) \leq c \), then we find an upper bound \( t = u(x_0)(\delta_1 R/d) - \frac{1}{2} \log_2 D \leq c \delta_1^{- \frac{1}{2} \log_2 D} \), which finishes the proof. Otherwise, it follows from (9.2) and (9.3) that

\[
\mathcal{M}^- w(x) \leq \frac{I_0(7R)}{(7R)^2} \varepsilon_0 + \frac{M}{\mathcal{H}(7R)} \frac{I_0(7R)}{(7R)^2} \varepsilon_0 + C M \mathcal{H}(7R) \frac{I_0(7R)}{(7R)^2},
\]

where

\[
M = \sup \left\{ \frac{\mathcal{K}_\gamma (d_{\mathbb{H}^n}(z, x))}{\mathcal{K}_\gamma (d_{\mathbb{H}^n}(z, x_1))} : x \in B(x_0, \theta r/2), x_1 \in B_R, z \in \mathbb{H}^n \setminus B(x_0, \theta r) \right\}.
\]

Let \( d = d_{\mathbb{H}^n}(z, x) \) and \( d_1 = d_{\mathbb{H}^n}(z, x_1) \) for the sake of brevity. If \( d \geq d_1 \), then by [27, Theorem 3.1] we have

\[
\frac{\mathcal{K}_\gamma (d)}{\mathcal{K}_\gamma (d_1)} = \frac{\sinh d_1}{\sinh d} \left( \frac{d_1}{d} \right)^{1/2 + \gamma} \frac{\mathcal{K}_{3/2 + \gamma} (d)}{\mathcal{K}_{3/2 + \gamma} (d_1)} \leq \frac{e^{d_1} \sinh d_1}{d_1} \frac{d}{e^d \sinh d}.
\]

Since the function \( s \mapsto e^s \sinh s/s \) is increasing, we obtain \( \mathcal{K}_\gamma (d)/\mathcal{K}_\gamma (d_1) \leq 1 \). When \( d < d_1 \), by [27, Theorem 3.1] again we have

\[
\frac{\mathcal{K}_\gamma (d)}{\mathcal{K}_\gamma (d_1)} \leq \frac{e^{d_1} \sinh d_1}{e^d \sinh d} \left( \frac{d_1}{d} \right)^{2 + 2\gamma} \leq \frac{e^{d_1} \sinh d_1}{e^d \sinh d} \left( \frac{d_1}{d} \right)^4.
\]

Since \( d_1/d \to 1 \) as \( d \to \infty \), so is \( \mathcal{K}_\gamma (d)/\mathcal{K}_\gamma (d_1) \). Thus, for any cases the ratio \( \mathcal{K}_\gamma (d)/\mathcal{K}_\gamma (d_1) \) is bounded by a universal constant which is independent of \( \gamma \). By using Corollary 5.3, we arrive at

\[
\frac{(\theta r/2)^2}{I_0(\theta r/2)} \mathcal{L}_0^- w \leq C \frac{I_0(R)/R^2}{I_0(\theta r/2)/(\theta r/2)^2} \mathcal{H}(7R) \leq C
\]

in \( B(x_0, 7(\theta r/14)) \).

Let \( Q_2 \subset Q_1 \) be the dyadic cube of generation \( k_{\theta r/14} \) containing the point \( x_0 \). Then by Lemma 8.2, we have

\[
|\{ u < u(x_0)/2 \} \cap Q_2 | = |\{ w > ((1 - \theta/2)^{-\gamma} - 1/2) u(x_0) \} \cap Q_2 |
\]

\[
\leq \frac{C|Q_2|}{((1 - \theta/2)^{-\gamma} - 1/2)^\varepsilon u(x_0)^\varepsilon} \left( \inf_{B(x_0, \delta_1 \theta r/14)} w + C \right)^\varepsilon.
\]
We can make the quantity $(1 - \theta/2)^{-\gamma} - 1/2$ bounded away from $0$ by taking $\theta > 0$ sufficiently small. Recalling that $w(x_0) = ((1 - \theta/2)^{-\gamma} - 1)u(x_0)$, we obtain
\[
|\{u < u(x_0)/2\} \cap Q_2| \leq C|Q_2| \left( ((1 - \theta/2)^{-\gamma} - 1)^\varepsilon + \left( \frac{C}{u(x_0)} \right)^\varepsilon \right).
\]
We choose a constant $\theta > 0$ sufficiently small so that
\[
C \left( (1 - \theta/2)^{-\gamma} - 1 \right)^\varepsilon \leq \frac{1}{4}.
\]
If $t > 0$ is sufficiently large so that $C(C/u(x_0))^{\varepsilon} < 1/4$, then we arrive at (9.1). Therefore, $t$ is uniformly bounded and the desired result follows. \hfill \Box

10. Hölder estimates

In this section, the following Hölder regularity result is proved. Theorem 1.5 follows from simple scaling and covering arguments.

**Lemma 10.1.** Assume $0 < \gamma_0 \leq \gamma < 1$. There is a universal constant $\varepsilon_0$ such that if $u \in C^2(B_7 R)$ is a function such that $|u| \leq \frac{1}{2}$ in $B_7 R$ and
\[
\frac{(7R)^2}{I_0(7R)} M^+ u \geq -\varepsilon_0 \quad \text{and} \quad \frac{(7R)^2}{I_0(7R)} M^- u \leq \varepsilon_0 \quad \text{in} \ B_7 R,
\]
then $u \in C^\alpha$ at $0 \in \mathbb{R}^n$ with an estimate
\[
|u(x) - u(0)| \leq CR^{-\alpha} d_{\mathbb{R}^n}(x, 0)^\alpha,
\]
where $\alpha \in (0, 1)$ and $C > 0$ are universal constants depending only on $n$, $\lambda$, $\Lambda$, $R$, and $\gamma_0$.

**Proof.** Let $R_k := 7 \cdot 4^{-k} R$ and $B_k := B_{R_k}$. It suffices to construct an increasing sequence $\{m_k\}_{k \geq 0}$ and a decreasing sequence $\{M_k\}_{k \geq 0}$ such that $m_k \leq u \leq M_k$ in $B_k$ and $M_k - m_k = 4^{-\alpha k}$. We initially choose $m_0 = -1/2$ and $M_0 = 1/2$ for the case $k = 0$. Let us assume that we have sequences up to $m_k$ and $M_k$ and find $m_{k+1}$ and $M_{k+1}$.

For $x \in B_{2R_{k+1}}$, let $Q_1$ be a dyadic cube of generation $k_{R_{k+1}/7}$. In $Q_1$, either $u > (M_k + m_k)/2$ or $u \leq (M_k + m_k)/2$ in at least half of the points in measure. We assume
\[
(10.1) \quad |\{u > (M_k + m_k)/2\} \cap Q_1| \geq \frac{1}{2}|Q_1|.
\]
A function defined by
\[
v(x) := \frac{u(x) - m_k}{(M_k - m_k)/2}
\]
satisfies $v \geq 0$ in $B_k$ by the induction hypothesis. To apply Lemma 8.2, let us consider a function $w := v^+$, which satisfies
\[
(10.2) \quad |\{w > 1\} \cap Q_1| \geq \frac{1}{2}|Q_1|
\]
by (10.1). Since $\frac{(7R)^2}{I_0(7R)} M^- v \leq 2\varepsilon_0/(M_k - m_k)$ in $B_7 R$, we have
\[
\frac{R_{k+1}^2}{I_0(R_{k+1})} M^- w \leq \frac{R_{k+1}^2}{I_0(R_{k+1})} \left( (M^- v + M^+ v^-) \right)
\]
\[
\leq \frac{2\varepsilon_0}{M_k - m_k} \frac{R_{k+1}^2}{I_0(R_{k+1})} \left( \frac{I_0(7R)}{I_0(7R)} \right)^2 + \frac{R_{k+1}^2}{I_0(R_{k+1})} M^+ v^-
\]
in $B_{3R_{k+1}}$. By Lemma 5.2, we have
\[
\frac{R_{k+1}^2}{I_0(R_{k+1})} \frac{L_0(7R)}{(7R)^2} \leq \left( \frac{R_{k+1}}{7R} \right)^\gamma = 4^{-(k+1)\gamma} < 4^{-k\gamma_0}.
\]
Thus, we obtain
\[
\frac{R_{k+1}^2}{I_0(R_{k+1})} \mathcal{M}^- w \leq 2\varepsilon_0 + \frac{R_{k+1}^2}{I_0(R_{k+1})} \mathcal{M}^+ v^-,
\]
by assuming $\alpha < \gamma_0$.
For $\mathcal{M}^+ v^-$, we use an inequality $v(z) \geq -2((d_{\mathbb{H}^n}(z,0)/R_k)^\alpha - 1)$, $z \in \mathbb{H}^n \setminus B_k$, which follows from the definition of $v$ and the properties of sequences $M_k$ and $m_k$. Then, for any $x_0 \in B_{3R_{k+1}}$, we have
\[
\mathcal{M}^+ v^-(x_0) \leq \lambda \int_{\mathbb{H}^n \setminus B_k} v^-(z) K_\gamma(d_{\mathbb{H}^n}(x_0,z)) \, d\mu(z)
\leq 2\lambda \int_{\mathbb{H}^n \setminus B_k} \left( \left( \frac{d_{\mathbb{H}^n}(z,0)}{R_k} \right)^\alpha - 1 \right) K_\gamma(d_{\mathbb{H}^n}(x_0,z)) \, d\mu(z).
\]
Since $d_{\mathbb{H}^n}(z,0) \leq 4d_{\mathbb{H}^n}(z, x_0)$, we obtain
\[
(10.3)
\frac{R_{k+1}^2}{I_0(R_{k+1})} \mathcal{M}^+ v^- \leq 2\lambda \frac{R_{k+1}^2}{I_0(R_{k+1})} \int_{\mathbb{H}^n \setminus B(x_0, R_{k+1})} \left( \left( \frac{d_{\mathbb{H}^n}(z,0)}{R_k} \right)^\alpha - 1 \right) K_\gamma(d_{\mathbb{H}^n}(x_0,z)) \, d\mu(z).
\]
Let $I$ be the right-hand side of (10.3). By the dominated convergence theorem, we know that $I$ converges to 0 as $\alpha \to 0$ for each $\gamma$. Let $\alpha_\gamma > 0$ be the constant such that $I \leq \varepsilon_0$ whenever $\alpha \leq \alpha_\gamma$. Since $I$ is continuous with respect to $\alpha$ and $\gamma$, $\alpha_\gamma$ is chosen continuously. Thus, the quantity $\min_{\gamma \in [\gamma_0, 1]} \alpha_\gamma$ is positive and depends on $\gamma_0$ (not on $\gamma$). By choosing $\alpha = \min_{\gamma \in [\gamma_0, 1]} \alpha_\gamma$, we obtain
\[
\frac{R_{k+1}^2}{I_0(R_{k+1})} \mathcal{M}^- w \leq 3\varepsilon_0
\]
in $B(x, 7(R_{k+1}/7))$ for $x \in B_{2R_{k+1}}$. Therefore, by Lemma 8.2 and (10.2), we have
\[
\frac{1}{2} |Q_1| \leq |\{ w > 1 \} \cap Q_1| \leq C |Q_1| (w(x) + 3\varepsilon_0)^\epsilon,
\]
or equivalently, $\theta \leq w(x) + 3\varepsilon_0$ for some universal constant $\theta > 0$. By taking $\varepsilon_0 < \theta/6$, we arrive at $w \geq \theta/2$ in $B_{2R_{k+1}}$. Thus, if we set $M_{k+1} = M_k$ and $m_{k+1} = M_k - 4^{-\alpha(k+1)}$, then
\[
M_{k+1} \geq u \geq m_k + \frac{M_k - m_k}{4} \theta = M_k - \left( 1 - \frac{\theta}{4} \right) 4^{-\alpha k} \geq m_{k+1}
\]
in $B_{k+1}$.
When (10.1) does not hold, a similar proof can be made by using $\frac{(7R)^2}{I_0(7R)} \mathcal{M}^+ u \geq -\varepsilon_0$ instead of $\frac{(7R)^2}{I_0(7R)} \mathcal{M}^- u \leq \varepsilon_0$. \hfill $\Box$
Appendix A. Special functions

The equation
\[ \rho^2 \frac{d^2 y}{d \rho^2} + \rho \frac{dy}{d \rho} - (\rho^2 + \nu^2) y = 0 \]

is called the modified Bessel’s equation, and its solutions are given by
\[ I_\nu(\rho) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(\nu + j + 1)} \left( \frac{\rho}{2} \right)^{2j+\nu} \quad \text{and} \quad K_\nu(\rho) = \frac{\pi I_{-\nu}(\rho) - I_\nu(\rho)}{2 \sin \nu \pi}. \]

They are called modified Bessel functions of the first and second kind, respectively. They satisfy the recurrence relations
\[ K_{\nu+1} - K_{\nu-1} = \frac{2\nu}{\rho} K_\nu, \quad I_{\nu-1} - I_{\nu+1} = \frac{2\nu}{\rho} I_\nu, \]

and the following system of first-order differential equations:
\[
\begin{cases}
I'_\nu = I_{\nu-1} + \frac{\nu}{\rho} I_\nu, \\
I'_{\nu+1} = I_{\nu+1} + \frac{\nu}{\rho} I_\nu,
\end{cases}
\quad \text{and} \quad
\begin{cases}
K'_\nu = -K_{\nu-1} - \frac{\nu}{\rho} K_\nu, \\
K'_{\nu+1} = -K_{\nu+1} + \frac{\nu}{\rho} K_\nu.
\end{cases}
\]

Using (A.1) and (A.2), we compute the indefinite integrals
\[ S^k_\nu(\rho) = \int \rho^{k-\nu} K_\nu(\rho) \sinh \rho \, d\rho, \]
\[ C^k_\nu(\rho) = \int \rho^{k-\nu} K_\nu(\rho) \cosh \rho \, d\rho, \]

for \( k = 0, 1, \ldots, \) and \( \nu \neq k + 1/2, \)

and
\[ L^k_\nu(\rho) = \int \rho^{k-\nu} K_\nu(\rho) \, d\rho, \]

for \( k = 0, 2, 4, \ldots, \) and \( \nu \neq k + 1/2. \)

Lemma A.1. For \( k = 0, 1, \ldots, \) and for \( \nu \neq k + 1/2, \) the indefinite integrals \( S^k_\nu \) and \( C^k_\nu \) are given by
\[
S^k_\nu = \sqrt{\frac{\pi}{2}} \sum_{j=0}^{k} \frac{(-1)^j k! (k-j)! \rho^{k+3/2-\nu}}{(k+1-2\nu) \cdots (k+1+j-2\nu)} (K_{\nu-j} I_{(1-j)/2} + K_{\nu-j-1} I_{(1-j)/2})
\]
and
\[
C^k_\nu = \sqrt{\frac{\pi}{2}} \sum_{j=0}^{k} \frac{(-1)^j k! (k-j)! \rho^{k+3/2-\nu}}{(k+1-2\nu) \cdots (k+1+j-2\nu)} (K_{\nu-j} I_{(1-j)/2} + K_{\nu-j-1} I_{(1-j)/2}),
\]

respectively.

Proof. We know from [29, 10.43.3 and 10.39.1] that (A.5) and (A.6) hold true for \( k = 0. \) For \( k \geq 1, \) we use \( S^0_\nu \) to see that
\[
S^k_\nu = \int \rho^k \partial_\rho S^0_\nu \, d\rho = \rho^k S^0_\nu - k \int \rho^{k-1} \rho^{1-\nu} / (1-2\nu) (K_\nu \sinh \rho + K_{\nu-1} \cosh \rho) \, d\rho
\]
\[
= \rho^k S^0_\nu - \frac{k}{1-2\nu} S^k_\nu - \frac{k}{1-2\nu} C^{k-1}_\nu,
\]
which yields
\begin{equation}
A.7 \quad S^{k}_{\nu} = -\frac{k}{k + 1 - 2\nu} C^{k-1}_{\nu-1} + \sqrt{\frac{\pi}{2}} \frac{\rho^{k+3/2-\nu}}{k + 1 - 2\nu} \left(K_{\nu} I_{1/2} + K_{\nu-1} I_{-1/2}\right).
\end{equation}
Similarly, by using $C^{0}_{\nu}$ we obtain
\begin{equation}
A.8 \quad C^{k}_{\nu} = -\frac{k}{k + 1 - 2\nu} S^{k-1}_{\nu-1} + \sqrt{\frac{\pi}{2}} \frac{\rho^{k+3/2-\nu}}{k + 1 - 2\nu} \left(K_{\nu} I_{-1/2} + K_{\nu-1} I_{1/2}\right).
\end{equation}
The lemma follows by solving the recurrence relations (A.7) and (A.8). \hfill \Box

For $L^{k}_{\nu}$, we need the modified Struve functions. The functions
\begin{equation*}
H_{\nu}(\rho) = \sum_{j=0}^{\infty} \frac{(-1)^{j}}{\Gamma(j + 3/2)\Gamma(\nu + j + 3/2)} \left(\frac{\rho}{2}\right)^{2j+\nu+1}
\end{equation*}
and
\begin{equation*}
L_{\nu}(\rho) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + 3/2)\Gamma(\nu + j + 3/2)} \left(\frac{\rho}{2}\right)^{2j+\nu+1}
\end{equation*}
are called the modified Struve functions and solve the modified Struve’s equation
\begin{equation*}
\frac{d^{2}y}{d\rho^{2}} + \frac{1}{\rho} \frac{dy}{d\rho} - \left(1 + \frac{\nu^{2}}{\rho^{2}}\right)y = \frac{(\rho/2)^{\nu-1}}{\sqrt{\pi\Gamma(\nu + 1/2)}}.
\end{equation*}

\begin{lemma}
For $k = 0, 1, \cdots$, and for $\nu \neq k + 1/2$, the indefinite integral $L^{2k}_{\nu}$ is given by
\begin{equation*}
L^{2k}_{\nu} = -\sum_{j=0}^{k-1} \frac{(2k)!(k-j)!}{2j!(2k-2j)!} \rho^{2k-j-\nu} K_{-\nu+1+j}
+ \frac{\pi^{\nu} \Gamma(1/2 - \nu + k)}{2^{\nu+1-k}} (2k)! \rho^{2k} \left(K_{-\nu+k} L_{-\nu+k} + K_{-\nu+k-1} L_{-\nu+k}\right).
\end{equation*}
\end{lemma}
\begin{proof}
By using the integration by parts, we obtain the recurrence relation
\begin{equation}
A.9 \quad L^{2k}_{\nu} = -\int \rho^{2k-1} \partial_{\rho}(\rho^{1-\nu} K_{1-\nu}) \, d\rho = -\rho^{2k-\nu} K_{1-\nu} + (2k - 1)L^{2k-2}_{\nu-1}.
\end{equation}
Therefore, the lemma follows from (A.9) and
\begin{equation}
A.10 \quad L^{0}_{\nu} = \int \rho^{-\nu} K_{-\nu} \, d\rho = \pi^{\nu/2} 2^{-\nu-1} \Gamma(1/2 - \nu) \rho^{1/2} \left(K_{-\nu} L_{-\nu-1} + K_{-\nu-1} L_{-\nu}\right).
\end{equation}
For (A.10), see, e.g., [29, 10.43.2]. \hfill \Box

Let us close the section by collecting some asymptotic behavior of the modified Bessel functions and modified Struve functions.

\begin{lemma}
The asymptotic behavior of the modified Bessel functions are given by
\begin{align*}
I_{\nu}(\rho) &\sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{\rho}{2}\right)^{\nu}, \quad \nu \neq -1, -2, \cdots, \\
K_{\nu}(\rho) &\sim \frac{1}{2} \Gamma(\nu) \left(\frac{\rho}{2}\right)^{-\nu}, \quad \Re \nu > 0,
\end{align*}
\end{lemma}
as $\rho \to 0$, and

$$L_\nu(\rho) \sim e^\rho \sqrt{\frac{\pi}{2\rho}},$$

$$K_\nu(\rho) \sim \sqrt{\frac{\pi}{2\rho}} e^{-\rho},$$

as $\rho \to \infty$. The asymptotic behavior of the modified Struve function $L_\nu$ is given by

$$L_\nu(\rho) \sim \frac{(\rho/2)^{\nu+1}}{\Gamma(3/2)\Gamma(3/2 + \nu)}$$

as $\rho \to 0$, and

$$L_\nu(\rho) \sim e^{s\sqrt{2\pi s}}$$

as $\rho \to \infty$.

For further properties of special functions, the reader may consult the book [29].

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