Exact formulas for the approximation of connections and curvature

Snorre H. Christiansen

Abstract

First we express the holonomy along a boundary curve as the integral on the domain, of an expression which is linear in the curvature. Then we provide a rigorous justification of the definition of curvature in Regge calculus.

Je t’apporte l’enfant d’une nuit d’Idumée!
Don du poème, Mallarmé.

1 Introduction

We present two results on connections and curvature that aim to relate the continuous and the discrete. Whether nature is one or the other, remains open.

The first result was inspired by the desire to extend the Lattice Gauge Theory initiated by Wilson [11], to a higher order method. While we did not quite achieve this goal, a formula was obtained, that might be of independent interest. It expresses the holonomy around a closed curve as an exact integral which is linear in the curvature. This continues our earlier investigations on LGT [3][4][5][6], which were concerned with convergence analysis, mainly when the gauge field describes electromagnetism, and extending the method to simplicial meshes (rather than the cubical ones that are customary).

The second result is a justification of the definition of curvature in the calculus of Regge [9]. Those provided in [9] and [7] were not found to be completely rigorous. Earlier [1][2], we have related Regge calculus to finite elements and studied linearization. Here we present a result on the non-linear method.

2 Definitions

We present here some some notions on connections and curvature, to fix notations. A standard reference on the subject is [8]. We have mainly used [10] (Appendix C).

Let $V$ be a finite dimensional Euclidean vector space. The space of endomorphisms of $V$ (that is, linear maps $V \to V$) is denoted $\text{End}(V)$. Let $G$ be a closed subgroup of the orthogonal endomorphisms of $V$ and $\mathfrak{g}$ its associated Lie algebra.
Given a function $Q : S \rightarrow G$ one transforms elements of $\Omega^k(S) \otimes V$ as follows:

$$\Phi \mapsto Q\Phi. \quad (2.1)$$

One also transforms connection one-forms $A \in \Omega^1(S) \otimes \mathfrak{g}$ as follows:

$$A \mapsto G_Q(A) = QAQ^{-1} - (DQ)Q^{-1}. \quad (2.2)$$

This formula ensures that we have:

$$\nabla_{G_Q(A)}Q\Phi = Q(\nabla_A\Phi). \quad (2.3)$$

Parallel transport with respect to $A$, along a curve $\gamma : [a, b] \rightarrow S$, from $x$ to $y$, is denoted:

$$\text{PT}_A(\gamma). \quad (2.4)$$

It is defined as the linear map $V \rightarrow V$, which to a vector $u(x) \in V$, associates $u(y) \in V$ in such a way that there is a field $u$, defined on $\gamma$, that satisfies:

$$\nabla_Au(\dot{\gamma}) = 0. \quad (2.5)$$

In the commutative case we have the formula:

$$\text{PT}_A(\gamma) = \exp(-\int_\gamma A). \quad (2.6)$$

If the endpoint $y$ of $\gamma$ is also its origin $x$, one speaks of a holonomy, and we denote it by:

$$\text{Hol}_A(\gamma). \quad (2.7)$$

Parallel transport along a curve $\gamma$ from $x$ to $y$, behaves as follows under gauge transformations:

$$\text{PT}_{G_Q(A)}(\gamma)Q(x) = Q(y)\text{PT}_A(\gamma). \quad (2.8)$$

In particular, around a closed curve from $x$ to $x$, we get:

$$\text{Hol}_{G_Q(A)}(\gamma) = Q(x)\text{Hol}_A(\gamma)Q(x)^{-1}. \quad (2.9)$$

The curvature of $A$ is denoted $\mathcal{F}(A)$:

$$\mathcal{F}(A) = dA + \frac{1}{2}[A, A]. \quad (2.10)$$

We have:

$$\mathcal{F}(G_Q(A)) = Q\mathcal{F}(A)Q^{-1}. \quad (2.11)$$

## 3 Holonomy from curvature

It is well known that the holonomy around a curve, minus the identity, is a good approximation of the integral of the curvature on the surface the curve bounds, in the sense that the difference between the two is smaller by one order of the length of the curve, see e.g. [10] (Appendix C, Proposition 5.1). That is, for small domains $T$:

$$\text{Hol}_A(\partial T) - I = -\int_T \mathcal{F}(A)(x) \, dx + \mathcal{O}((\text{area}(T))^{3/2}). \quad (3.1)$$
This fact is the basis for Lattice Gauge Theory, introduced in [11]. See for instance in [6], how Proposition 2 is used as an ingredient to prove consistency. It turns out that in discretizations, the left hand side has better invariance properties than the right hand side, under discrete gauge transformations. Discrete gauge invariance is a crucial property, linked to charge conservation by Noether’s theorem.

In the next proposition we transform this estimate into an exact identity, expressing the holonomy as an integral, which is linear in the curvature. The estimate (3.1) can easily be deduced from the proposed identity. The original motivation was construct a discretely gauge invariant discretization of Yang-Mills action, with higher orders of convergence than classical LGT. In this we have not yet succeeded.

**Proposition 1.** Suppose $T$ is an oriented rectangle. Define, for any $x \in T$, two paths, $\gamma^-(x)$ and $\gamma^+(x)$, as follows. The face is equipped with two coordinates determined by the axes of $T$, compatible with its orientation. The origin of $T$ has coordinates $(0,0)$ and the opposite vertex in $T$ has coordinates $(a,b)$. We put $x = (x_0, x_1)$ and let the paths consists of straight lines joining the following points:

\[
\begin{align*}
\gamma^-(x) : (0,0) &\to (x_0, 0) \to (x_0, x_1), \\
\gamma^+(x) : (x_0, x_1) &\to (x_0, b) \to (0, b) \to (0,0).
\end{align*}
\]

Then we have:

\[
\text{Hol}_A(\partial T) - I = - \int_T PT_A(\gamma^+(x)) F(A)(x) PT_A(\gamma^-(x)) \, dx.
\]

**Proof.**

(i) Remark first that if identify (3.4) holds for a gauge potential $A$ then it holds for any gauge transformation $GQ(A)$ of $A$.

(ii) Given $x \in T$ define a path $\alpha(x)$ consisting of straight lines as follows:

\[
\alpha : (x_0, x_1) \to (0, x_1) \to (0,0).
\]

The path, followed in reverse is denoted $\alpha(x)^{-1}$. Define $Q : f \to G$ as follows:

\[
Q(x) = PT_A(\alpha(x)).
\]

The gauge potential $A' = GQ(A)$ now satisfies, by (2.8):

\[
PT_A(\alpha(x)^{-1}) = \operatorname{id}_V,
\]

therefore, for all $x_0 \in [0,a]$, $x_1 \in [0,b]$:

\[
\begin{align*}
A'_0(x_0, x_1) &= 0, \\
A'_1(0, x_1) &= 0.
\end{align*}
\]

(iii) For $A$ of the above form, the proposition is proved for fixed $b$, differentiating with respect to $a$. More precisely, for any point $(x_0, x_1) \in T$, let $P(x_0, x_1)$ be the parallel transport, according to $A$ along the segment from $(x_0, 0)$ to $(x_0, x_1)$. We have:

\[
\text{Hol}_A(\partial T) = P(a, b).
\]

We have:

\[
\partial_1 P(x) = -A_1(x) P(x).
\]
We deduce:
$$\partial_1 (P(x)^{-1} \partial_0 P(x)) = -P(x)^{-1} \partial_0 A_1(x) P(x).$$
(3.12)

Hence:
$$\partial_0 P(x_0, b) = -\int_0^b P(x_0, b) P(x_0, x_1)^{-1} \partial_0 A_1(x_0, x_1) P(x_0, x_1) \, dx_1.$$  
(3.13)

So that:
$$P(a, b) = I - \int_0^a \int_0^b P(x_0, b) P(x_0, x_1)^{-1} \partial_0 A_1(x_0, x_1) P(x_0, x_1) \, dx_0 \, dx_1.$$  
(3.14)

This can be interpreted as the claimed identity.

Remark 1. In the abelian case one can give a much simpler proof of this identity. We consider the rectangle $T = T(a, b)$ as a function of the upper right corner. Define:

$$H(a, b) = \text{Hol}_A(\partial T(a, b)),$$

$$= \exp(-\int_{\partial T(a, b)} A),$$  
(3.15)

$$= \exp(-\int_{T(a, b)} dA).$$  
(3.16)

From the last expression we deduce:
$$\partial_y H(x, b) = -H(x, b) \int_0^b d A(x, y) \, dy.$$  
(3.18)

So we can write:
$$H(a, b) = I - \int_0^a \int_0^b H(x, b) \, dA(x, y) \, dx \, dy,$$  
(3.19)

From the second to last expression on the other hand, we deduce that for $y \in [0, b]$:

$$P T_A(\gamma^+(x, y)) P T_A(\gamma^-(x, y)) = H(x, b).$$  
(3.20)

so that we have obtained the desired identity.

4 A justification of Regge Calculus

Regge calculus [9] can be defined as follows.

Let $T$ be a simplicial complex, that is, a finite set of of finite non-empty sets. The elements of $T$ are called simplices and are thought of as sets of vertices. For each simplex $T \in T$, its geometric realization is the set:

$$|T| = \{ f : T \to \mathbb{R} : \sum_{x \in T} f(x) = 1, \forall x \in T \, f(x) \geq 0 \}.$$  
(4.1)

A vertex $x \in T$ can be identified with the characteristic function of $\{x\}$ on $T$, which is an element of $|T|$. For $T' \subseteq T$ there is a unique affine map $\Phi_{T T'} : |T'| \to |T|$ which is the identity on vertices of $T'$.
The geometric realization of \( \mathcal{T} \) is:

\[
\prod_{T \in \mathcal{T}} |T| / \sim.
\]  

(4.2)

where the equivalence relation is the smallest satisfying:

\[
\Phi_{T T'}(x) \sim x, \text{ whenever } x \in |T'| \text{ and } T' \subseteq T.
\]  

(4.3)

In particular the maps \( \Phi_{T T'} \) are identified with inclusions.

Suppose that \(|T|\) is an oriented \(n\)-dimensional manifold. In Regge calculus one assigns a real number to each edge. These numbers, interpreted at edge lengths squared, determine a constant metric \(\rho\) on each simplex.

Then, to each codimension 2 simplex \(h\) (called hinge) in \(\mathcal{T}\) one associates a so-called deficit angle \(d_h\) as follows. Compute, for each \(n\)-simplex in \(\mathcal{T}\) containing the hinge, the dihedral angle between the two faces arriving at the hinge. Add these dihedral angles, and subtract this number from \(2\pi\), to get the deficit angle \(d_h\). Let \(a_h\) be the area of the hinge. The action defined by Regge to mimick the Einstein-Hilbert action is:

\[
\rho \mapsto \sum_h d_h a_h.
\]  

(4.4)

Critical point of this action are discrete analogues of Einstein metrics.

One goes even further and asserts that the scalar curvature is a measure on \(|T|\) concentrated to the hinges and given by:

\[
\psi \mapsto \sum_h d_h \int_h \psi,
\]  

(4.5)

where one sums over hinges \(h\), the integrals of \(\psi\) on \(h\) equipped with the induced metric. The difficulty, in order to make sense of this assertion, is that scalar curvature is a non-linear expression of the metric involving second order derivatives. That it should be well defined for a discontinuous metric is miraculous.

Regge \[9\] proposed a justification involving an averaging argument and an appeal to the Gauss-Bonnet theorem. An alternative justification can be found in \[7\] (see in particular §3 and Theorem 3.1). A sequence of smooth metrics approximating the Regge metric is considered, and one wants to obtain the curvature of the limit (as defined by Regge) as the limit of curvatures (as usually defined for smooth metrics). However we don’t think the passage to the limit is valid for any approximating sequence, and \[7\] is vague about which approximating sequences can be used.

Ideally one might want to identify a topology on the space of metrics, with respect to which this amounts to continuity of the curvature map (into the space of measures). We have not identified such a topology. But in this paper we prove that the limiting procedure is valid for the canonical approximating sequence, obtained by smoothing by convolution.

Various results connecting integrals of curvature with holonomies can be found in Appendix C.5 of \[10\]. Based on such considerations we are able to evaluate the curvature of the smoothed Regge metrics. Our arguments can be see to reprove a variant of the Gauss-Bonnet theorem. We hope the reader will share our pleasure in doing so.

Here we concentrate on the two-dimensional case \((n = 2)\).
Let $S$ be a two dimensional Euclidean vector space, whose metric is denoted $g$ and serves as a reference. Half-lines emanating from the origin split the space into $I \in \mathbb{N}$ sectors. The half-lines are indexed by a cyclic variable $i \in \mathbb{Z}/I\mathbb{Z}$. The sector between $i$ and $i + 1$ is indexed by $i + 1/2$.

In this context we consider a Regge metric $\rho$. It is constant in each sector, with value in the sector $i + 1/2$ denoted $\rho_{i+1/2}$. Its pullback to the half-lines separating two sectors is well-defined, that is, the restriction is the same from both sides, when evaluated on vectors parallel to the half-line.

Let $m_i$ denote the directing vector of half-line $i$ which has unit length, with respect to $\rho_{i+1/2}$. Let $\theta_{i+1/2}$ be the angle between the vectors $m_i$ and $m_{i+1}$ with respect to the metric $\rho_{i+1/2}$. The deficit angle, at the origin, is defined to be:

$$d = 2\pi - \sum_i \theta_{i+1/2}. \quad (4.6)$$

Choose $\phi$, a smooth function on $E$ with compact support in the unit ball and with integral 1, with respect to $g$. For $\epsilon > 0$ define the scaling:

$$\phi_{\epsilon}(x) = \epsilon^{-2}\phi(\epsilon^{-1}x), \quad (4.7)$$

where the factor in front is chosen to preserve the integral. Define the smoothed Regge metrics $\sigma_\epsilon$ by the following convolution product, computed with respect to $g$:

$$\sigma_\epsilon = \phi_\epsilon * \rho. \quad (4.8)$$

We concentrate first on the metric $\sigma = \sigma_1$. We denote by $\nabla$ the Levi-Civita connection, $\kappa$ the scalar curvature and by $\mu$ the volume two-form of $\sigma$. Our goal is to prove that the function $\kappa$ has compact support and:

$$\int_S \kappa \mu = d. \quad (4.9)$$

We do this by evaluating the holonomy (with respect to $\nabla$) along a curve encircling the origin, at sufficient distance, in two different ways.

Let $x \mapsto (e_1(x), e_2(x))$ denote a choice of orthonormal oriented basis (at $x \in S$). Given this frame, denote by $A$ the connection one-form of the Levi-Civita connection of $\sigma$. Thus:

$$A \in \Omega^1(S) \otimes \mathfrak{so}(2). \quad (4.10)$$

The Lie algebra $\mathfrak{so}(2)$ is one-dimensional and spanned by the matrix:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (4.11)$$

**Proposition 2.** Let $T$ be a domain in $S$ with a piecewise smooth boundary curve $\partial T$. We have:

$$\text{Hol}_A(\partial T) = \exp((\int_T \kappa \mu)J). \quad (4.12)$$

**Proof.** We denote by $F$ the curvature tensor, also in the frame $(e_1, e_2)$. We have:

$$F = dA + \frac{1}{2}[A, A] = dA \in \Omega^2(S) \otimes \mathfrak{so}(2). \quad (4.13)$$
By Stokes theorem:

$$\text{Hol}_A(\partial T) = \exp(- \int_{\partial T} A) = \exp(- \int_T F).$$

(4.14)

But we also have (equation (5.13) in [10]):

$$F = -\kappa J \mu.$$

(4.15)

This concludes the proof. □

Next we compute the holonomy by parallel transporting along the curve. This is made easy by the following fact:

**Lemma 3.** In the union of the sectors $i - 1/2$ and $1 + 1/2$, consider the subset $U_i$ of points whose distance to the boundary is strictly larger than 1.

On $U_i$ we have $\nabla m_i = 0$.

**Proof.** For simplicity of notation, we write $m = m_i$. Recall that for any constant vector fields $X, Y$ on $U_i$, we have:

$$2\sigma(\nabla_X m, Y) = X\sigma(m, Y) + m\sigma(X, Y) - Y\sigma(m, X).$$

(4.16)

Here, the vectorfields act on scalar fields as derivations.

Denote by $n$ a vector which is orthogonal to $m$ for the reference metric $g$. We use that $\sigma$ is invariant in the $m$ direction and that $\sigma(m, m)$ is constant, and compute:

$$2\sigma(\nabla_m m, m) = m\sigma(m, m) + m\sigma(m, m) - m\sigma(m, m) = 0,$$

(4.17)

$$2\sigma(\nabla_m m, n) = n\sigma(m, n) + m\sigma(m, n) - n\sigma(m, n) = 0,$$

(4.18)

$$2\sigma(\nabla_n m, m) = n\sigma(m, m) + m\sigma(n, m) - m\sigma(m, n) = 0,$$

(4.19)

$$2\sigma(\nabla_n m, n) = n\sigma(m, n) + m\sigma(n, n) - n\sigma(m, n) = 0.$$

(4.20)

This concludes the proof. □

Let $\alpha_{i+1/2}$ be the angle at the origin of the sector $i + 1/2$, computed with respect to the reference metric $g$. Elementary trigonometry shows that the union of the domains $U_i$ contains the exterior of the ball with radius:

$$r = \max_i 1/\cos(\pi/2 - \alpha_{i+1/2}/2).$$

(4.21)

From the preceding Lemma one gets:

**Corollary 4.** The scalar curvature $\kappa$ is supported in the ball $B_g(0, r)$.

**Proposition 5.** Let $T$ be a domain containing the ball $B_g(0, r)$. We have:

$$\text{Hol}_A(\partial T) = \exp(-\sum_i \theta_{i+1/2})J).$$

(4.22)

**Proof.** For each $i$, define vectors $n_i^+$ and $n_i^-$ such that $(m_i, n_i^\pm)$ is an orthonormal oriented basis with respect to the metric $\rho_{i\pm1/2}$.

Also, in each sector $i + 1/2$, choose a point $p_{i+1/2}$ on the boundary curve, such that:

$$p_{i+1/2} \in U_i \cap U_{i+1}.$$

(4.23)
Let $\gamma_i$ be the portion of the boundary curve from $p_{i-1/2}$ to $p_{i+1/2}$, inside $U_i$.

From Lemma 3 it follows that:

$$\text{PT}_V(\gamma_i) : m_i \mapsto m_i,$$

and then, since parallel transport along $\gamma_i$, is an isometry from the metric $\sigma$ at $p_{i-1/2}$ (which is equal to $\rho_{i-1/2}$) to the metric $\sigma$ at $p_{i+1/2}$ (which is equal to $\rho_{i+1/2}$), it follows that:

$$\text{PT}_V(\gamma_i) : n_i^- \mapsto n_i^+.$$  

(4.24)

The matrix of the identity from the basis $(m_i, n_i^+)$ to the basis $(m_{i+1}, n_{i+1}^-)$ (both of which are orthonormal oriented for $\rho_{i+1/2}$) is:

$$\begin{bmatrix}
\cos \theta_{i+1/2} & \sin \theta_{i+1/2} \\
-\sin \theta_{i+1/2} & \cos \theta_{i+1/2}
\end{bmatrix}.$$  

(4.25)

We write:

$$\text{PT}_V(\partial T) = \text{PT}_V(\gamma_{i-1}) \circ \ldots \circ \text{PT}_V(\gamma_0).$$  

(4.26)

(4.27)

We are now ready to conclude:

**Proposition 6.** We have:

$$\int_S \kappa \mu = d.$$  

(4.29)

*Proof.* From Propositions 2 and 5 we deduce:

$$\int_S \kappa \mu + \sum_i \theta_{i+1/2} \in 2\pi \mathbb{Z}.$$  

(4.30)

Next we consider the following one-parameter family of Regge metrics:

$$[0, 1] \ni s \mapsto \rho(s) = sp + (1 - s)g.$$  

(4.31)

The left hand side in (4.30), evaluated with $\rho$ replaced by $\rho(s)$, varies continuously as a function of $s$, and takes discrete values, so must be constant. Moreover at $s = 0$ one obtains $2\pi$. Therefore the value at $s = 1$ is also $2\pi$.  

We now return to the family of smoothed metrics $\sigma_\epsilon$ defined by (4.8). We let $\kappa_\epsilon$ and $\mu_\epsilon$ denote their respective scalar curvatures and volume forms.

**Proposition 7.** We have:

$$\kappa_\epsilon \mu_\epsilon \rightarrow d\delta,$$

(4.32)

in the sense that for any continuous function $\psi$:

$$\lim_{\epsilon \rightarrow 0} \int \psi \kappa_\epsilon \mu_\epsilon = d\psi(0).$$  

(4.33)
Proof. Let $\Phi_\epsilon : S \rightarrow S$ be the scaling map:

$$\Phi_\epsilon(x) = \epsilon^{-1}x. \quad (4.34)$$

Since:

$$\rho = \epsilon^2 \Phi_\epsilon^* \rho, \quad (4.35)$$

we get:

$$\sigma_\epsilon = \epsilon^2 \Phi_\epsilon^* \sigma. \quad (4.36)$$

It follows that:

$$\kappa_\epsilon = \epsilon^{-2} \Phi_\epsilon^* \kappa, \quad (4.37)$$

$$\mu_\epsilon = \epsilon^2 \Phi_\epsilon^* \mu. \quad (4.38)$$

So that:

$$\kappa_\epsilon \mu_\epsilon = \Phi_\epsilon^* (\kappa \mu). \quad (4.39)$$

Based on this identity, the convergence follows.

Acknowledgements

Stimulating discussions with Tore G. Halvorsen, concerning Lattice Gauge Theory, are gratefully acknowledged.

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