Experimental study of generic billiards with microwave resonators

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In this work we study the eigenstates and the energy spectra of a generic billiard system with the use of microwave resonators. This is possible due to the exact correspondence between the Schrödinger equation and the electric field equations of the lowest modes in thin microwave resonators. We obtain a good agreement between the numerical (exact) and experimental eigenstates, while the short range experimental spectral statistics show the expected Brody-like behaviour in this energy range, as opposed to the Berry-Robnik picture which is valid only in the semiclassical region of sufficiently small effective Planck's constant.

§1. Introduction

The subject of quantum chaos deals with the quantum mechanical properties of classically chaotic systems, namely, it tries to relate the classical behaviour of Hamiltonian systems to the quantal objects such as wavefunctions and energy spectra of their corresponding quantum mechanical counterparts.

Due to the formal similarity of the Schrödinger equation to the other wave equations of physics the concept of quantum chaos can be generalized to the one of wave chaos. In these systems we may also observe correspondences between the wave mechanics and a suitably defined 'classical' mechanics (ray dynamics), similar to those of the Hamiltonian systems. For example, in the case of electromagnetic waves the appropriate 'classical' picture is the ray optics. Furthermore, as will be shown later, the Schrödinger equation of a two-dimensional billiard system, which is a system of a free point particle elastically bouncing off the boundaries of a given domain, and the wave equation describing the electric field of the low frequency modes in a thin microwave resonator of the same planar shape are in a one-to-one correspondence1). This enables us to experimentally study the wavefunctions and the spectra of a quantal billiard system through the electric field properties of its microwave counterpart. A rich variety of microwave billiards have been studied in the past decade. For a review and further references see the recent book by Stöckmann (2), and the article by Richter (3).

In this work we experimentally studied the quantum properties of a family of billiards with analytical boundaries, introduced by Robnik (4). This work is directly related and complementary to the work done by Veble et al. (5), where the quantum
mechanics of the same system is studied numerically and theoretically. In a similar work Rehfeld et al. studied the spectra of these billiards in superconducting microwave cavities for different shape parameters $\lambda^6)$. When the shape parameter $\lambda$ of the family is varied from 0 to 0.5, this system undergoes a transition from a fully integrable to a fully ergodic system. In the transition region the system is of the mixed type, where the energy shell of the phase space is split into regions of regular, quasiperiodic motion, and those of chaotic motion, where almost each trajectory explores the whole appropriate chaotic region, in the sense that after a sufficiently long time it reaches any point on the chaotic component with arbitrary accuracy. This type of motion may be called generic, as almost any randomly picked Hamiltonian will exhibit this type of motion.

The wavefunctions and spectra of fully integrable and fully chaotic systems show remarkable differences. In the semiclassical limit when $\hbar$ tends to 0 the wavefunctions of the former appear regular with their nodal lines forming a rather regular grid, while those of the latter appear disordered with their amplitude being Gaussian distributed. On the other hand, the energy levels of classically integrable systems show no correlations in this limit and their level spacing distribution is Poissonian$^7$-$^9)$, while the energy levels of classically chaotic systems behave statistically as eigenvalues of random matrices, with the matrices belonging to a Gaussian orthogonal ensemble (GOE) if the system possesses the time reversal symmetry (or some other antiunitary symmetry) and a unitary one (GUE) in the systems where this symmetry is broken$^{10)$-$^{14})$.

The semiclassical picture of a generic, mixed type system can be considered, according to the principle of semiclassical uniform condensation (PUSC$^{14)$,$^{15}$)), as composed of the independent contributions of the chaotic and regular phase space components of its classical counterpart. In the limit $\hbar \to 0$ almost each state can be classified as a regular or irregular, depending on whether its Wigner function, which is the quantum analogue of the classical phase space density, is supported by a chaotic component or an invariant torus in the phase space. Although the PUSC has not yet been proven, there exists strong numerical evidence in its support$^{16)$-$^{21}$,$^{14)$}.

The spectrum is then a superposition of independent contributions stemming from different invariant components of the classical phase space, with the average density of states of each contribution proportional to their relative classical phase space volume (Liouville measure). The corresponding statistical properties of each contribution are that of the GOE or GUE for the chaotic components and Poisson for the regular ones. This is the so called Berry-Robnik$^{22)$ picture.

The experiments performed in this work were, however, still far from the semiclassical limit. Nevertheless, even in this low energy region the wavefunctions show a certain correspondence with the classical mechanics as they are typically related to the shortest periodic orbits of the system. The spectral statistics are also not expected to follow the Berry-Robnik picture in this regime. Although lacking a deeper physical foundation, the so-called Brody picture$^{23)$,$^{24)$} is usually applicable to the classically mixed type systems at these lower energies. It interpolates between the Poisson and the GOE (GUE) statistics by introducing a fractional power law level repulsion between levels$^{16)$,$^{17)$,$^{25)$}. As these results have been confirmed numerically,
we wanted to see whether they can be observed in an experimental setup.

In section 2 we present the theoretical background and some mathematical definitions. In section 3 we show the experimental setup and explain the data interpretation. We compare the experimentally obtained states and spectral statistics with their numerical counterparts in section 4.

§2. Theory and Definitions

The system we studied is a billiard. It consists of a free particle elastically bouncing off the boundaries of a domain. In our case the domain is two-dimensional and defined by conformally mapping the unit circle in the complex plane \( z \) by the quadratic transformation onto the complex plane \( w \) (physical plane), namely

\[
w(z) = z + \lambda z^2, \quad w = x + iy.
\]  

(2.1)

When \( y = 0 \) the \( x \) component ranges within the interval \([-1 + \lambda, 1 + \lambda]\).

The classical dynamics of the billiard systems is on the one hand very simple since the trajectories are just straight lines between consecutive bounces and are independent of the energy, while on the other hand their behaviour can range from both integrable (such as the circular or rectangular billiard) to completely chaotic (see e.g. Bunimovich stadium, Sinai, \( \lambda = 0.5 \) billiards). Another type of one-parameter family of generic billiards are the oval billiards introduced by Benettin and Strelcyn, whose quantum mechanics was recently studied by Makino et al. However, this type of billiards do not have an analytic boundary (there are points of discontinuous second derivative).

In this work we chose the parameter \( \lambda = 0.15 \), where the system is of a mixed type with areas of regular, quasiperiodic motion and chaotic components sharing the phase space. The billiard shape is still convex, so that the Lazutkin’s tori and caustics still exist (the so-called whispering gallery modes), for a review see reference 28). Therefore, the system is indeed very well described by the KAM picture. In figure 1 we show the surface of section (SOS) plot of the main chaotic component of this billiard, where we plot the \( x \) coordinate and the \( x \) component of the momentum unit vector (denoted by \( p_x \)) whenever the trajectory, which started somewhere in the chaotic region, passes the \( y = 0 \) line. The coordinate \( x \) is taken relative to the center of the billiard at \( y = 0 \), so that its range is now \( x \in [-1, 1] \).

The stationary Schrödinger equation for billiards is the Helmholtz equation with the Dirichlet boundary conditions,

\[
(\Delta + k^2)\psi = 0, \quad \psi|_{\partial D} = 0,
\]  

(2.2)

where \( k^2 = 2mE/\hbar^2 \). We are interested in the properties of the eigenstates \( \psi_i \) and the corresponding eigenvalues \( E_i \) of the system.

To study the spectra it is convenient to unfold the energy scale so that the density of levels on the unfolded scale is everywhere equal to one. This removes the density of states which is an individual property of a system, so that the possible universal properties may be observed. If the average number of states up to a given
energy $\bar{N}(E)$ is known, then the mapping

$$x_i = \bar{N}(E_i)$$  \hspace{1cm} (2.3)

produces the unfolded spectrum with the desired (unit) density of states. For the 2D billiard system the average number of states is given by the Weyl formula

$$\bar{N}(E) = \frac{1}{4\pi} \left[ Sk^2 - Lk + K \right],$$  \hspace{1cm} (2.4)

where $S$ is the surface of the billiard, $L$ the length of its boundary and $K$ corrections due to the curvature and corners of the boundary.\(^{29}\)

There are many measures of statistical properties of the unfolded spectra. We will mainly deal with the (normalized to unity) distribution $P(S)$ of the level spacings between the consecutive levels. Due to the unfolding of the spectra,

$$\int_0^\infty SP(S) dS = 1$$  \hspace{1cm} (2.5)

holds. It is conjectured that, after removing the symmetries of the system, for fully integrable systems the distribution of levels is uncorrelated, Poissonian\(^{7} - 9\), leading to the distribution

$$P_{\text{integrable}}(S) = \exp(-S),$$

while the quantal levels of classically fully chaotic systems are believed to behave as eigenvalues of random matrices with matrix elements being statistically independent and Gaussian distributed, again after the discrete geometric symmetries of the system are removed. If the distribution of matrix elements is invariant under orthogonal transformations we speak of Gaussian orthogonal ensembles (GOE) of matrices, while Gaussian unitary ensembles (GUE) are invariant under general unitary transformations. The GOE ensembles apply in the systems with time reversal symmetry (such as our billiard), and the level spacing distribution is in this case well approximated by the Wigner distribution

$$P_{\text{GOE}}(S) = \frac{\pi}{2} S \exp \left( -\frac{\pi}{4} S^2 \right)$$

exhibiting a linear repulsion between levels.

Our system is neither fully integrable nor fully chaotic. The picture of Berry and Robnik\(^{22}\) states that in the semiclassical limit $\hbar \to 0$ each of the invariant components, a chaotic component or a regular region in phase space, contributes an independent level sequence to the total spectrum, with the statistical weight of individual contributions being proportional to the phase space volume (relative Liouville measure) of the corresponding component. The regular components thus contribute Poissonian sequences while the contributions from the chaotic components are GOE(GUE)-like. When only one major chaotic component is present (as is the case with our billiard) the distribution $P(S)$ becomes

$$P_{\text{BR}}(S, \rho_1) = \rho_1^2 \exp(-\rho_1 S) \erfc \left( \frac{1}{2} \sqrt{\pi} \rho_2 S \right) +$$

$$+ \left( 2\rho_1 \rho_2 + \frac{1}{2} \pi \rho_2^3 S \right) \exp \left( -\rho_1 S - \frac{1}{4} \pi \rho_2^2 S^2 \right),$$

where $\rho_1$ and $\rho_2$ are the number of states in the chaotic and regular components, respectively.
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where \( \rho_1 \) is the relative measure of the regular and \( \rho_2 \) of irregular regions in phase space, with \( \rho_1 + \rho_2 = 1 \). It goes to Poisson for \( \rho_1 = 1 \) and to Wigner for \( \rho_1 = 0 \). It is interesting to note that \( P_{\text{BR}}(S = 0) = 1 - \rho_2^2 \) and is different from zero when \( S = 0 \) and \( \rho_2 < 1 \) (\( \rho_1 > 0 \)).

However, although confirmed in the semiclassical region by numerical computations, the Berry–Robnik picture does not apply in the lower energy regions such as those accessible by our experiments for qualitatively well understood and known reasons: the dynamical localization of chaotic eigenstates and their deviation from uniform extendedness. Remarkably, a distribution introduced by Brody\(^{23}\) successfully describes the behaviour of level spacings in this regime (for not too large spacings, say \( S \leq 1 \)) by assuming a fractional power law level repulsion between levels, although so far it has yet not been attributed a deeper physical foundation. This distribution reads as

\[
P_{\text{Brody}}(S, \beta) = aS^\beta \exp(-bS^{\beta+1}), \quad a = (\beta + 1)b, \quad b = \left\{ \Gamma \left( \frac{\beta + 2}{\beta + 1} \right) \right\}^{\beta+1}
\]

and becomes Poisson for \( \beta = 0 \) and Wigner for \( \beta = 1 \). Unlike the Berry-Robnik distribution, it becomes \( P_{\text{Brody}}(S = 0) = 0 \) for all \( \beta \neq 0 \).

This relationship of quantum eigenstates to classical mechanics can be best studied by the use of the Wigner function (see reference 30))

\[
W(q, p) = \frac{1}{(2\pi\hbar)^N} \int d^N X \exp(-i p \cdot X/\hbar) \psi^\dagger(q - X/2)\psi(q + X/2)
\]

where \( N \) is the number of degrees of freedom. It is the quantum analogue of the classical phase space density. It is not positive definite, however it becomes so when \( \hbar \to 0 \). The principle of uniform semiclassical condensation (PUSC, see reference 14)) states that in this limit the Wigner function of any eigenstate condenses uniformly onto an invariant classical object in the phase space, which can be either an invariant torus or a whole chaotic component. If PUSC is accepted, the Berry-Robnik picture of level statistics becomes its direct consequence.

In our work we took the value of the Wigner function on the symmetry line \( y = 0 \) and integrated it over \( p_y \),

\[
\rho_{\text{SOS}}(x, p_x) = \int dp_y W(x, y = 0, p_x, p_y),
\]

in order to compare it with the classical SOS plot. This yields

\[
\rho_{\text{SOS}}(x, p_x) = \frac{1}{2\pi\hbar} \int dX \exp(-ip_x X/\hbar) \psi^\dagger(x - X/2, y = 0)\psi(x + X/2, y = 0).
\]

This 'density' is different from 0 only for the states with even parity. As the Wigner function and thus its projection is not positive definite but exhibits oscillations of the order of the wavelength, we smoothed \( \rho_{\text{SOS}} \) with a Gaussian, chosen narrower than the minimum uncertainty one but still wide enough to smooth out the oscillations.
3. Experimental technique

3.1. Experimental setup

There is an exact correspondence between the electric field of the lowest resonances in thin microwave resonators and two-dimensional quantum billiards. The resonator must be shaped as a thin prism with the shape of its base the same as that of the billiard. If we choose the \(x\) and \(y\) coordinates in the plane of the base and \(z\) perpendicular to it, the wave equation for the \(z\) component of electric field in the lowest \(k_z = 0\) TM modes is

\[(\Delta_{(x,y)} + k^2)E_z = 0, \quad E_z|_{\partial D} = 0\] (3.1)

with the wavenumber \(k = \frac{2\pi \nu}{c}\) (\(\nu\) is the measured frequency and \(c\) the velocity of light). The \(E_z\) component of electric field clearly corresponds to the wavefunction \(\psi\) in equation (2.2). By measuring the electric field in a resonator we may then obtain the amplitude of the corresponding wavefunction of the quantum problem.

The experimental setup enabled us to measure not only the amplitude of the wavefunction across the billiard, but also its sign. This is important in the calculation of the experimental Wigner transforms of the eigenfunctions - see section 2 and subsection 4.2. The arrangement consisted of a lower plate into which the billiard shape was drilled with the depth of 8mm and an upper plate. Both of the plates had a microwave antenna inserted through them and protruding through almost the whole depth of the billiard. The antennae were connected to a vector network analyzer (Wiltron 360B) capable of measuring not only the intensity of the signal but also its phase with respect to the input reference signal. The upper antenna both emitted and received microwaves, whilst the lower one was receiving them only.

The lower plate (with the drilled billiard shape) could be moved with respect to the upper one and thus enabled the upper antenna to reach any point in the billiard. While only the upper antenna would be sufficient to measure the square of the electric field in microwave resonances, the transmission between the antennae gives the information of the relative sign (phase) of the electric field between them. More details of the experiment will be described elsewhere.

We prepared two resonators. The first one was drilled (8mm deep) in a brass plate and had the shape of the whole \(\lambda = 0.15\) billiard, with the scale 127mm per unit of length of the billiard as defined in equation (2.1), and was used for measuring the wavefunctions. The wavefunctions were measured over a quadratic grid with the separation of points by 5mm. The second resonator was used for measuring the spectra. It represented only a half of the billiard and thus selected only the odd parity states of the whole billiard in order to remove the symmetry, so that the possible universal properties of its spectrum can emerge. It was made of aluminium with 365mm per unit of length and drilled 8mm deep. When measuring the spectra we tightly screwed the upper plate to the lower one in order to increase the conductivity between them and thus the quality of the resonator. Thus quality factors of some thousand were obtained. As the determination of sign of the electric field was not necessary in this spectral measurement, we used a single antenna in the upper plate,
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but moved it to six different positions in order not to miss any resonances, which can happen if the antenna lies very close to the nodal line of a resonance.

3.2. Interpretation of experimental data

The data obtained from the vector network analyzer are the complex elements of the scattering matrix $S$ of the open channels (the two antennae in our case). The information we want to extract from it is the Green’s function of the corresponding quantum system,

$$G(r_i, r_j, E) = \sum_n \frac{\psi_n(r_i)\psi_n^\dagger(r_j)}{E_n - E},$$

which carries all the information about the spectrum and wavefunctions of the system. The crucial theoretical tool in analyzing our experiments is the scattering matrix $S$, which by definition connects the vector $a$ of amplitudes of incoming waves with the vector $b$ of outgoing waves through the simple relation $b = Sa$. It was shown by Stein et al.\(^{33}\) that the scattering matrix between the antennae at positions $r_i$ and $r_j$ is related to the Green’s function of the system by

$$S = (1 + \alpha^\dagger G)^{-1}(1 + \alpha G),$$

where $\alpha$ is a complex coupling parameter that depends on the antenna shape and the frequency of the microwaves. As the frequency variation of the coupling parameter is slow it can be taken as constant close to a single resonance (within a few widths of the resonance).

When at a given value $E$ only a single resonance $n$ contributes to the Green’s function, the scattering matrix is simplified to

$$S = 1 + 2\alpha \tilde{G},$$

where

$$\tilde{G}_{ij} = \frac{\psi_n(r_i)\psi_n^\dagger(r_j)}{E_n - E + \Delta_n - i\Gamma_n}$$

is the modified Green’s function with the resonances shifted by

$$\Delta_n = \Im(\alpha) \sum_i |\psi_n(r_i)|^2$$

in the real and

$$\Gamma_n = \Re(\alpha) \sum_i |\psi_n(r_i)|^2$$

in the imaginary direction of the energy plane. The resonances in the scattering matrix thus appear shifted and broadened with respect to their unperturbed positions.

The measured resonances of the $S$ matrix can then be fitted with the functions of the form $\beta/(E - \gamma)$, where $\beta$ and $\gamma$ are complex parameters. We should remark that a good quantitative agreement with the numerical wavefunctions was obtained only when the full complex information of the measured $S$ matrix was used (and not only its absolute value). Thus the full complex information is needed not only to analyze the transmission data (the determination of sign) but also when fitting the complex reflection data to obtain the wavefunction amplitude.
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| Index | Exact | Experimental |
|-------|-------|--------------|
| 1     | 13.782| 13.805       |
| 2     | 25.208| 25.173       |
| 3     | 38.900| 38.902       |
| 4     | 46.265| 46.291       |
| 5     | 55.052| 54.981       |
| 6     | 67.785| 67.742       |
| 7     | 73.583| 73.518       |
| 8     | 90.826| 90.823       |
| 9     | 94.455| 94.335       |
| 10    | 97.588| 97.581       |
| 11    | 116.537| 116.448    |
| 12    | 117.930| 117.858    |
| 13    | 129.462| 129.402    |
| 14    | 142.690| 142.537    |
| 15    | 145.705| 145.649    |
| 16    | 161.086| 161.102    |
| 17    | 168.146| 168.151    |
| 18    | 170.391| 170.270    |
| 19    | 176.678| 176.523    |
| 20    | 197.059| 196.944    |

Table I. The comparison of the lowest 20 exact and experimental energy eigenvalues.

§4. Results

4.1. Spectrum

The measured spectra of one half of the billiard (the odd states of the whole billiard) were compared to the numerically calculated ones. The difference in the level positions in the units of the mean level spacing is shown in the figure 2. The error is determined by the estimation of the shifts $\Delta_n$ of the resonances, by moving the receiving antenna to other positions. In table I we show the lowest 20 energy eigenvalues $E = k^2$ in comparison with the ‘exact’ (numerical) values.

The coupling gets stronger with frequency and the (absolute) shifts increase (see equation (3.6)). The frequency increase of coupling can be shown for the case of the ideal circular antenna $^{33}$, for which

$$\alpha = 2\pi kR \frac{H_0^{(1)'}(kR)}{H_0^{(1)}(kR)},$$

(4.1)

where $H_0^{(1)}$ is the zero order Hankel function. When the diameter $R$ of the antenna is much smaller than the wavelength (which is true in our experiments),

$$\alpha \approx \frac{2\pi}{\ln(kR)}.$$  

(4.2)

This implies that the (absolute) difference $D_n$ between the experiment and numerics in the units of mean level spacing $\Delta E$ increases with the frequency. When a large relative number of levels start to deviate by more than a few percent of the mean level
spacing the short range level statistics such as $P(S)$ for the actual billiard cannot be studied accurately any more, but have merely a phenomenological value then.

Unfortunately, due to the imperfect geometry of the antennae the theoretically expected values (4.1,4.2) are not realized, but all relations remain intact if we treat $\alpha$ as a phenomenological parameter. This in turn implies of course that $\Delta_n$ (3.6) and $\Gamma_n$ (3.7) are phenomenological parameters too and therefore the determination of the eigenenergies by fitting (3.5) is much less accurate than in the case of ideal geometry.

The accurate range of the experimental spectra can also be estimated independently of the numerical calculation by noting that the shifts and widths of resonances are similar in size (see equations (3.6) and (3.7)). Levels can then be considered accurate as long as the widths of the resonances are not comparable with the mean level spacing.

The cumulative level spacing distribution

$$W(S) = \int_0^S P(S')dS'$$

(4.3)

for the first 180 experimental levels is shown in the left upper part of the figure 3. In the lower left part we plot the same data in a different representation, namely in the form of the deviations of the $U$ function $^{16}$

$$U(W) = \frac{2}{\pi} \arccos \sqrt{1 - W}$$

(4.4)

from the best fitting Berry-Robnik $U$ function. This representation has the nice and convenient property that the statistically expected deviations of $U(W(S))$ are the same for all spacings $S$. The experimental data clearly deviate from the best fitting Berry-Robnik distribution and are fitted by the Brody distribution much better, although the difference in these distributions appears to be only slight. The same plots for the numerically obtained levels are shown in the right part of the figure 3, giving virtually the same results.

The observed discrepancy is of course expected; the Berry-Robnik picture applies only in the sufficiently deep (strict) semiclassical limit where the effective Planck constant is sufficiently small. The deviation that we observe in exact (numerical) and experimental data is attributed to the dynamical localization of the Wigner functions of eigenstates in the phase space, i.e. to the deviations of Wigner functions from the uniform extendedness, as predicted by PUSC (see introduction). These effects have been observed and analyzed by Prosen and Robnik $^{16}$, further studied in references 17), 35), 18). For a recent review see reference 14). A quantitative theory of these (transitional) effects is not yet available.

4.2. Eigenstates

We show the comparison of a selection of odd parity experimental states with their numerical counterparts in figures 5 and 6 respectively, while in figures 7 and 8 the even parity states are shown. We plot the probability density of states in eight contours from 0 to its maximum value. The agreement between the theory is not only
qualitative, but also quantitative as individual contours of most of the plots match nicely. The states with the worst agreement between the theory and experiment are those that lie in the neighbourhood of another eigenstate (within a few typical resonance widths at the given frequency), where the perturbation of the antenna does not only shift the states but also mixes the neighbouring states into linear superpositions of the unperturbed ones. Furthermore, as the perturbation due to the antenna is different at different points, this superposition varies at different measuring points.

Most of the obtained states can be related to the individual shortest periodic orbits of our system shown in figure 4. The two most prominent types of states are those corresponding to the vertical (No. 5, unstable) and horizontal (No. 1, stable) double bounce periodic orbit, which occur regularly in the observed range of states. This correspondence with classical mechanics can be best studied in the phase space.

For the states of even parity we calculated the smoothed projection of the Wigner function as defined in (2.12). We show them in the figure 9, with the intersections of the relevant periodic orbit with the SOS shown as bullets. The largest values of the Wigner functions do indeed cover the periodic orbits. Although a correspondence to classical mechanics can be established, the observed energy range is still far from the semiclassical limit, where the PUSC applies. In this limit the Wigner functions are expected to be extended across larger invariant objects in phase space such as the whole chaotic components and invariant tori in the regular regions, and not just simple periodic orbits. In our observed region, however, we are still in the localization regime where individual states are supported only by parts of these components, that is by the vicinity of the shortest periodic orbits of the system. This localization is further reflected in the spectral statistics, where the distribution of level spacings is not the limiting Berry-Robnik but rather of the Brody type. The exact Wigner functions for the same states as in figure 9 are shown in figure 10 using exactly the same technique, where quite a nice agreement can be observed.

§5. Conclusion

The equivalence between the two-dimensional Schrödinger equation and the equation for the lowest modes of thin microwave resonators enabled us to experimentally obtain and study the spectra and wavefunctions of billiard systems. We obtained a very good quantitative agreement between the numerically calculated (exact) and experimentally obtained levels and eigenstates. We must stress again that such good agreement was only obtained by using the full complex information of the scattering matrix between the probing antennae.

The properties of obtained spectra and eigenfunctions were in accordance with our expectations for a mixed type system such as ours. Although hints to correspondence with the classical mechanics may be observed in the currently attainable frequency range, the semiclassical limit is still out of reach of the microwave experiments.

In conclusion, we might envision two important improvements of our experimental approach, namely firstly, by a theoretically correct estimation of the complex
coupling parameter $\alpha$ and secondly, by extending the scattering theory\textsuperscript{33}) of subsection 3.2 to a two-channel theory.

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Fig. 1. The surface of section (SOS) plot for the $\lambda = 0.15$ billiard (see text for details).

Fig. 2. The difference $D_n$ between the corresponding experimental and numerical level positions of the odd parity states as a function of the consecutive index in the units of the mean level spacing $\Delta E$, so $D_n = (E_{\text{experimental}}^n - E_{\text{exact}}^n)/\Delta E$.

Fig. 3. In the upper left part we show the cumulative level spacing distribution of the first 180 experimentally obtained odd parity levels, with the best fitting Brody (dashed line) and Berry-Robnik (thin full line) distributions. The limiting Poisson and Wigner distributions are shown dotted. In the lower left part we plot the deviations of the $U(W)$ function, see eq. (4.4), from the best fitting Berry-Robnik $U$ function. The data are shown as a full curve with the thin curves representing the statistically expected deviations, while the dashed curve is the difference between the best fitting Brody and Berry-Robnik $U$ functions. In the right part of the figure the same plots are shown for the corresponding numerically obtained level range.

Fig. 4. The shortest periodic orbits in the $\lambda = 0.15$ billiard. The stable orbits are represented with full lines, while unstable ones with dashed lines.

Fig. 5. A selection of the experimentally obtained odd parity states. We draw the probability density in eight equally spaced contours from zero to the maximum.

Fig. 6. The numerical exact odd states corresponding to the experimental states of figure 5, again with eight equally spaced probability density contours from zero to the maximum (upper part) and the nodal lines (lower part).

Fig. 7. A selection of the experimentally obtained even parity states. We draw the probability density in eight equally spaced contours from zero to the maximum.

Fig. 8. The numerically obtained exact even parity states corresponding to the experimental states of figure 7, with eight equally spaced probability density contours from zero to the maximum (upper part) and the nodal lines (lower part).

Fig. 9. The smoothed projections of the experimental Wigner functions for the even parity states of figure 7. We plot eight equally spaced contours from 0 to the maximum, with the negative value contours shown lighter. In some plots the intersections of the relevant periodic orbit with the SOS are shown as bullets.

Fig. 10. The same plot as in the figure 9, but for the corresponding numerically obtained exact even parity states.
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