Three proofs of the Casas-Alvero conjecture

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Abstract

The Casas-Alvero conjecture claims that a complex univariate polynomial having roots in common with each of its derivatives must be a power of a linear polynomial. Up to now, only partial proofs and numerical evidences have been presented. In this paper we give three different proofs of the conjecture.

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1. Introduction

This paper is concerned with the following question posed by E. Casas-Alvero more than a decade ago.

Casas-Alvero Conjecture. Let $f$ be a monic complex polynomial of degree $n$ in a single variable $z$. Suppose that $\gcd(f, f^{(k)}) \neq 1$ for $k = 1, \ldots, n - 1$, where $f^{(k)}$ denotes the $k$-th derivative of $f$. Then, there exists a constant $a \in \mathbb{C}$ such that $f(z) = (z - a)^n$.

It may be proven that if the conjecture is true over $\mathbb{C}$ then it is true over all fields of characteristic 0. In contrast, the conjecture is not true in prime characteristic.

In [2] the Casas-Alvero conjecture was proven for polynomials of degree $n$ less than or equal to 7. The conjecture has also been proven for infinitely many values of $n$, see [1].
We can rewrite the conjecture in terms of interpolation polynomials on the complex plane $\mathbb{C}$, see [3, 4].

**Theorem 1.1.** Let $z_1, z_2, \ldots, z_n$ be $n$ complex numbers and let $p$ be a monic complex polynomial of degree $n$. Suppose that the polynomial $p$ satisfies

\[
p(z_k) = 0, \quad k = 1, \ldots, n;
p^{(k)}(z_{k+1}) = 0, \quad k = 1, \ldots, n - 1.
\]

Then

\[p(z) = (z - a)^n, \quad a \in \mathbb{C}.
\]

In the sequel any monic polynomial satisfying the conditions of Theorem 1.1 will be called a Casas-Alvero polynomial.

**2. Birkhoff interpolation**

First, we pose the following problems which may be considered as particular cases of Birkhoff or lacunary interpolation problems [5]. They are closely related to the Casas-Alvero conjecture.

**Problem 2.1.** Given $n$ complex numbers $\alpha_1, \ldots, \alpha_n$, find all monic polynomials $p$ that satisfy

\[p^{(k)}(\alpha_{k+1}) = 0, \quad k = 0, 1, \ldots, n - 1.
\]

**Solution.** A monic polynomial $p_n$ of degree $n$ can be expressed in the form

\[p_n(z) = \sum_{j=0}^{n} a_j z^j,
\]

where $a_n = 1$. For $k = 1, \ldots, n$, its $k$-th derivative is given by

\[p_n^{(k)}(z) = \sum_{j=k}^{n} a_j \frac{j!}{(j-k)!} z^{j-k}.
\]

If we evaluate the above expressions at the nodes, we obtain

\[p_n^{(k)}(\alpha_{k+1}) = \sum_{j=k}^{n} a_j \frac{j!}{(j-k)!} \alpha_{k+1}^{j-k} = 0, \quad k = 0, 1, \ldots, n - 1.
\]
We can write the above equations in matrix form as

\[
A \cdot \begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n-1} \\
a_n
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
n!
\end{pmatrix},
\]

where

\[
A = \begin{pmatrix}
1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-2} & \alpha_1^{n-1} & \alpha_1^n \\
0 & 1 & 2\alpha_2 & \cdots & \frac{n-2}{(n-3)!} \alpha_2^{n-3} & \frac{n-1}{(n-2)!} \alpha_2^{n-2} & \frac{n}{(n-1)!} \alpha_2^{n-1} \\
0 & 0 & 2! & \cdots & \frac{(n-2)!}{(n-4)!} \alpha_3^{n-4} & \frac{(n-1)!}{(n-3)!} \alpha_3^{n-3} & \frac{n!}{(n-2)!} \alpha_3^{n-2} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (n-2)! & \frac{(n-1)!}{1!} \alpha_{n-1} & \frac{n!}{2!} \alpha_{n-1}^2 \\
0 & 0 & 0 & \cdots & 0 & (n-1)! & n! \alpha_n \\
0 & 0 & 0 & \cdots & 0 & 0 & n!
\end{pmatrix}.
\]

Clearly the interpolation problem has a solution and it is unique, because the matrix is of full rank. The solution of the system is trivial but laborious. The incidence matrix will have as many rows as the number of different nodes. In the extreme case that there is only one node, the problem is of Hermite type. As in all cases the matrix is triangular, there is always a unique solution.

A straightforward calculation shows that the unique solution \( p_n \) of Problem 2.1 admits the following integral representation

\[
p_n(z) = n! \int_{\alpha_1}^{z} dx_1 \int_{\alpha_2}^{x_1} dx_2 \cdots \int_{\alpha_{n-1}}^{x_{n-2}} dx_{n-1} \int_{\alpha_n}^{x_{n-1}} dx_n.
\]

We can study now the inverse problem.

**Problem 2.2.** Given \( n \) complex numbers \( z_1, z_2, \ldots, z_n \), let

\[
\hat{z} = (z_1, \ldots, z_n) \quad \text{and} \quad p(z) = \prod_{k=1}^{n} (z - z_k).
\]

Find all the vectors of complex numbers

\[
\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)
\]
that satisfy
\[ p^{(k)}(\alpha_{k+1}) = 0, \quad k = 0, 1, \ldots, n - 1. \]

**Solution.** The problem has a solution. We begin with the last equation of the system given by Problem 2.1 to obtain \( \alpha_n \). Then we solve the \((n - 1)\)-th equation to get the two values of \( \alpha_{n-1} \) and we continue until considering the first equation of degree \( n \) which allows us to obtain the \( n \) values of \( \alpha_1 \).

The problem has a solution but the solution is not unique. The problem may have until \( n! \) different solutions. Of course, it is possible to approximate the solutions by numerical methods, but it should be noticed that this problem is invariant under permutation of the symmetric group of the \( n \) components of \( \hat{z} \). If we change the order of the roots in \( \hat{z} \), the polynomial \( p \) does not change and the solutions of the problem are the same ones as before. The number of solutions is just equal to the number of permutations of \( \hat{z} \).

We can make Problem 2.2 more complicated considering additional conditions like requiring the nodes of interpolation \( \alpha \) to be equal to the zeros of the polynomial \( p \). Thus, the polynomial \( p \) and its derivatives will have common zeros.

3. **The first proof**

There are several different ways to tackle the proof. We have chosen to consider it as a problem of the same type as that of Problem 2.2. That is, assuming that the roots of the polynomial are known, find the nodes of interpolation. Subsequently, impose the condition that the roots coincide with the nodes. Then this problem is invariant under permutations of the zeros of the polynomial as in Problem 2.2 But if we permute the zeros of the polynomial, the interpolation nodes are automatically interchanged. Thus the polynomial and its derivatives up to order \( n - 1 \) must take the value zero at all the zeros of the polynomial. Therefore, all the zeros are of multiplicity \( n \). But this is possible only if there is a single zero of multiplicity \( n \). Thus, the Casas-Alvero conjecture is proven.
4. A second proof by induction

Taking account of the representation (2), for \( n = 2 \), the Casas-Alvero polynomial is

\[
p_2(z) = 2! \int_\alpha^z dx_1 \int_\alpha^{x_1} dx_2 = 2! \int_\alpha^z dx_1 \int_\alpha^{x_1} dx_2
\]

\[
= (z - \alpha_2)^2 - (\alpha_1 - \alpha_2)^2 = (z - \alpha_2)^2 = (z - \alpha_1)^2.
\]

Now suppose that, for \( k = 1, \ldots, n - 1 \), any Casas-Alvero polynomial is of the form

\[
p_k(z) = (z - \alpha_1)^k.
\]

Then

\[
p_n(z) = n! \int_\alpha^z \int_\alpha^{x_1} \int_\alpha^{x_2} \cdots \int_\alpha^{x_{n-2}} \int_\alpha^{x_{n-1}} dx_n.
\]

Notice that \( p_n(z) \) takes the value zero at \( z = \alpha_k \) from \( k = 1 \) to \( n \). So, for \( k = 1, \ldots, n \), we have

\[
p_n(z) = n! \int_\alpha^z \int_\alpha^{x_1} \int_\alpha^{x_2} \cdots \int_\alpha^{x_{n-2}} \int_\alpha^{x_{n-1}} dx_n.
\]

Therefore

\[
p_n(z) = n! \int_\alpha^z dx_1 \int_\alpha^{x_1} dx_2 \cdots \int_\alpha^{x_{n-2}} (x_{n-1} - \alpha_n) dx_{n-1}
\]

\[
= n! \int_\alpha^z dx_1 \int_\alpha^{x_1} dx_2 \cdots \int_\alpha^{x_{n-2}} x_{n-1} dx_{n-1} - n\alpha_n p_{n-1}(z),
\]

for \( k = 1, \ldots, n \), where \( p_{n-1} \) is a Casas-Alvero polynomial of degree \( n - 1 \). Then

\[
p_{n-1}(z) = (z - \alpha_1)^{n-1}
\]

and

\[
\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1}.
\]

Therefore

\[
p_n(z) = n! \int_\alpha^z dx_1 \int_\alpha^{x_1} dx_2 \cdots \int_\alpha^{x_{n-3}} \left( \frac{1}{2} (x_{n-2}^2 - \alpha_1^2) \right) dx_{n-2} - n\alpha_n p_{n-1}(z)
\]

\[
= n! \int_\alpha^z dx_1 \int_\alpha^{x_1} dx_2 \cdots \int_\alpha^{x_{n-3}} \frac{1}{2} x_{n-2}^2 dx_{n-2}
\]

\[
- \frac{n(n - 1)}{2} \alpha_1^2 p_{n-2}(z) - n\alpha_n p_{n-1}(z),
\]
where \( p_{n-2}(z) = (z - \alpha_1)^{n-2} \).

If we continue the process of calculating the iterated integral, we arrived at

\[
p_n(z) = z^n - \alpha_1^n - \sum_{k=1}^{n-2} \binom{n}{k} \alpha_1^{n-k}(z - \alpha_1)^k - n\alpha_n p_{n-1}(z)
\]

\[
= z^n - \sum_{k=0}^{n-1} \binom{n}{k} \alpha_1^{n-k}(z - \alpha_1)^k + n(\alpha_1 - \alpha_n)(z - \alpha_1)^{n-1}
\]

\[
= z^n + (z - \alpha_1)^n - \sum_{k=0}^{n} \binom{n}{k} \alpha_1^{n-k}(z - \alpha_1)^k + n(\alpha_1 - \alpha_n)(z - \alpha_1)^{n-1},
\]

since the number \( \alpha_n \) is a root of the polynomial \( p_n \). Actually

\[
p_n(\alpha_n) = (\alpha_n - \alpha_1)^n + n(\alpha_1 - \alpha_n)(\alpha_n - \alpha_1)^{n-1}
\]

\[
= -(n - 1)(\alpha_n - \alpha_1)^n = 0.
\]

Then, \( \alpha_n = \alpha_1 \) and \( p_n(z) = (z - \alpha_1)^n \), as we wanted to prove.

5. The third proof

Although we think that the previous two proofs are correct, we encourage the reader to continue reading the article.

Lemma 5.1. Every Casas-Alvero polynomial has a root equal to the geometric center of gravity of its roots.

Proof. Let \( p_n \) be a monic polynomial of degree \( n \) expressed in the form \([\boxed{1}]\). Then

\[
p_n^{(n-1)}(z) = n! z + (n - 1)! a_{n-1}.
\]

Evaluating at the zero \( z_n \) of the polynomial \( p_n^{(n-1)} \), we obtain

\[
p_n^{(n-1)}(z_n) = n! z_n + (n - 1)! a_{n-1} = 0,
\]

or, equivalently,

\[
z_n = -\frac{a_{n-1}}{n} = \frac{1}{n} \sum_{j=1}^{n} z_j.
\]
Now, let \( p_n \) be a monic polynomial of degree \( n \). The polynomial \( p_n \) may be written in the form

\[
p_n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k c_k z^{n-k}, \tag{3}
\]

where

\[
c_0 = 1, \quad c_1 = \binom{n}{1}^{-1} \sum_{j=1}^{n} z_j, \]

\[
c_k = \binom{n}{k}^{-1} \sum_{j_1+\ldots+j_k=k} \prod_{i=1}^{n} z_i^{j_i}, \quad k = 2, \ldots, n.
\]

Taking derivatives in (3), we obtain

\[
p'_n(z) = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k c_k (n-k) z^{n-k-1}
\]

\[
= n \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k c_k \frac{n-k}{n} z^{n-k-1}
\]

\[
= n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k c_k z^{n-k-1}.
\]

The above calculation indicates that the zeros of the derivative of a polynomial of degree \( n \) and the zeros of this polynomial have in common the following quantities: the average of their zeros, the mean double product of their zeros, and so forth until the average \((n-1)\) product of their zeros.

If \( p_n \) is a Casas-Alvero polynomial, then, for \( k = 1, \ldots, n-1 \), the polynomial \( p_n^{(k)} \) has the same coefficients \( c_j, j = 1, \ldots, n-k, \) as \( p_n \).

For \( k = n-1 \), the zero shared by the polynomial and its derivative of order \( n-1 \) is precisely \( c_1 \) and this determines the coefficient of degree \( n-1 \) of the polynomial \( p_n \).

For \( k = n-2 \), which is the zero shared by the polynomial and the derivative of order \( n-2 \)? The two zeros determine the same value for \( c_2 \).

Therefore, whatever zero \( p_n \) shares with its derivative we obtain same result for the polynomial. But this reasoning is valid for all the derivatives of \( p_n \). Then all the zeros of the successive derivatives of \( p_n \) must be zeros of \( p_n \), therefore all the zeros must be equal.
6. Concluding remarks

We conclude that the Casas-Alvero conjecture is true. In connection with the work carried out in this paper, we are currently studying an optimization problem in which symmetries play an important role. The results related to this problem will appear elsewhere.

In our opinion solving a math problem is not to close the door but rather open a window to new challenges. For this reason, we propose a degenerated Birkhoff interpolation problem in which the number of equations is greater than the number of unknowns. Among the different possibilities we have chosen the problem stated below. Find the conditions to be met by the interpolation nodes and their values for the following problem to be solvable.

**Problem 6.1.** Given $2n$ complex numbers $\alpha_1, \ldots, \alpha_n$ and $c_0, \ldots, c_{n-1}$, find all the polynomials $p$ of degree $n$ that satisfy

$$p(\alpha_{k+1}) = p^{(k)}(\alpha_{k+1}) = c_k, \quad k = 0, 1, \ldots, n - 1.$$ 

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