LOCAL CONNECTIVITY OF JULIA SETS OF SOME RATIONAL MAPS WITH SIEGEL DISKS

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Abstract. We prove that a long iteration of rational maps is expanding near boundaries of bounded type Siegel disks. This leads us to extend Petersen’s local connectivity result on the Julia sets of quadratic Siegel polynomials to a general case. A new key feature in the proof is that the puzzles are not used.

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1. Introduction

The local connectivity of the Julia sets of rational maps is one of the central themes in complex dynamics. By constructing expanding metrics, the connected Julia sets of hyperbolic, subhyperbolic and geometrically finite rational maps have been proved to be locally connected (see [DH84], [TY96]). For the maps which do not have expanding metric near the Julia sets, constructing puzzles is the main idea in the study of the local connectivity of the Julia sets. In the 1980s, Yoccoz proved that if the non-renormalizable quadratic polynomials have no irrationally indifferent periodic points, then their Julia sets are locally connected provided they are connected [Hub93]. The so-called Yoccoz puzzle plays a crucial role in the proof. To study the topology of the Julia sets of cubic polynomials, the Branner-Hubbard puzzle was introduced in [BH92]. By developing the puzzle techniques further, various Julia sets without Siegel disks have been proved to be locally connected. See [HJ93], [Lyu97], [LS98], [KSS07], [KL09], [KS09], [Roe10], [CST17] for polynomials, and [Roe08], [QWY12], [DS22], [WYZ23] for some special rational maps (see also [CDKS22]).

Based on the Douady-Ghys surgery on cubic Blaschke products, Petersen proved that the Julia sets of quadratic polynomials with a fixed bounded type Siegel disk are locally connected [Pet96] (see also [Pet98] for the quadratic polynomials with periodic bounded type Siegel disks). Later Yampolsky derived this classical result from a priori bounds of unicritical analytic circle maps [Yam99]. One of the main tools in their proofs is a puzzle structure, which is refereed to as Petersen’s puzzle. Petersen’s puzzle construction can be adapted to polynomials of higher degrees by which J. Yang
proved that the Julia sets of any polynomials (assumed to be connected) are locally connected at the boundary points of bounded type Siegel disks [Yan23]. However, for rational maps, except in rare cases, for instances, the mating examples of Yampolsky and Zakeri [YZ01] and some other examples in [Yan15], there is in general no puzzle structure. Hence the study of the topology of the Julia sets of rational maps is always a great challenge, especially when there is no expanding metric near the Julia sets.

Near boundaries of Siegel disks, the rational maps are far from being expanding. Since the puzzle construction may not exist for general rational maps with Siegel disks, the corresponding Julia sets may have exotic topology in this situation. For instance, when a rational map has both Siegel disks and attracting basins, a priori, the boundary of an immediate attracting basin may spiral around a Siegel disk in a very complicated manner and the Julia set may be non-locally connected.

1.1. **Main results.** The main purpose of this paper is to extend Petersen’s local connectivity result to a class of rational maps with bounded type Siegel disks. In particular, it implies that the exotic topology of the whole Julia sets mentioned above cannot occur when the critical orbits are well controlled, and moreover, it provides an alternative proof of Petersen’s result without using puzzles.

**Main Theorem.** Suppose $f$ is a rational map with Siegel disks such that the Julia set $J(f)$ is connected, and moreover, the forward orbit of every critical point of $f$ satisfies one of the following:

(a) It is finite; or
(b) It lies in an attracting basin; or
(c) It intersects the closure of a bounded type Siegel disk.

Then $J(f)$ is locally connected.

As an immediate application of the Main Theorem, the Julia sets of all cubic Siegel polynomials in the Zakeri curve and in capture domains are locally connected, where the Zakeri curve is the collection of all cubic polynomials with both critical points on the boundary of a fixed Siegel disk of a given bounded type rotation number [Zak99]. As another application of the Main Theorem, we conclude that if the cubic Newton map or the McMullen map $z \mapsto z^n + \lambda/z^n$ with $n \geq 3$ has a bounded type Siegel disk, then the Julia sets are locally connected. For in this case their Julia sets are connected and their critical points lie either in attracting basins or on the boundaries of bounded type Siegel disks. This complements the results of [Roe08, Theorem 4] and [QWY12, Theorem 1.3].

For a rational map $f$ with degree at least two, let $\text{Crit}(f)$ be the set of all critical points of $f$. The *postcritical set* of $f$ is

$$\mathcal{P}(f) := \bigcup_{n \geq 1} f^n(\text{Crit}(f)).$$

Let $\hat{\mathbb{C}}$, $\text{dist}_{\hat{\mathbb{C}}} \cdot \cdot \cdot$ and $\text{diam}_{\hat{\mathbb{C}}} \cdot \cdot \cdot$ denote respectively the Riemann sphere, the distance and diameter with respect to the spherical metric. The following lemma is the key ingredient in the proof of the Main Theorem.

**Main Lemma.** Let $f$ be a rational map with degree at least two and having a fixed bounded type Siegel disk $\Delta$. Suppose $\text{dist}_{\hat{\mathbb{C}}} (\mathcal{P}(f) \setminus \partial \Delta, \partial \Delta) > 0$. Then for any $\varepsilon > 0$ and any Jordan disk $V_0 \subset \hat{\mathbb{C}} \setminus \Delta$ satisfying $\overline{V_0} \cap \mathcal{P}(f) \neq \emptyset$ and $\overline{V_0} \cap \mathcal{P}(f) \subset \partial \Delta$, there exists $N \geq 1$, such that $\text{diam}_{\hat{\mathbb{C}}} (V_n) < \varepsilon$ for all $n \geq N$, where $V_n$ is any connected component of $f^{-n}(V_0)$. 
The condition $\text{dist}_{\hat{C}}(\mathcal{P}(f) \setminus \partial \Delta, \partial \Delta) > 0$ in the Main Lemma is seen to be automatically satisfied if $\hat{\mathcal{P}}(f) \setminus \partial \Delta = \emptyset$. We would like to point out that the contraction property of the composition of a certain sequence of inverse branches near boundaries of bounded type Siegel disks was previously obtained by using complex a priori bounds, see [Yam99] for the case of unicritical analytic circle maps, and [ESY22] for the case of multi-critical analytic circle maps (see also [Yam19]). Our Main Lemma asserts that such contraction holds for all the pullback sequences and moreover, the contraction is actually “uniform”. That is, for given $V_0$, the contraction depends only on the length of the pullback sequence, but not on the choice of the inverse branch for each pullback.

Such uniform contraction is essential for mating Siegel polynomials as well as the proof of local connectivity of the Julia sets with Siegel disks. In the celebrated work of Yampolsky-Zakeri [YZ01] on the mating of quadratic Siegel disks, such uniform contraction property is obtained by using complex a prior bounds together with certain puzzle structure. See [Yan15] for a similar situation. The Main Lemma does not assume the existence of any puzzle structure which may not exist for general rational maps. In fact, a similar result as the Main Lemma can be used to prove the local connectivity of the Julia sets of transcendental entire functions with bounded type Siegel disks, where the puzzles are definitely not available (This will appear in a forthcoming paper, see [WYZZ21] for partial results).

1.2. Sketch of the proofs. Let us first sketch the general idea of the proof of the Main Lemma. Let $f$ be the rational map in the Main Lemma having a fixed bounded type Siegel disk $\Delta$. Note that analytic Blaschke models may not exist for general rational maps. In §2 we construct a quasi-Blaschke model $G$ (which is quasi-regular) for $f$ such that the Siegel disk $\Delta$ (resp. the boundary $\partial \Delta$) of $f$ is replaced by the unit disk $D$ (resp. the unit circle $T$) of $G$. Such a model was first used by Petersen to study the Herman-´Swia¸tek theorem for holomorphic self-homeomorphisms of quasi-circles [Pet04]. The heart of the proof in this paper is to show that a long iteration of $G$ is expanding near the unit circle. Let $\mathcal{P}(G)$ be the corresponding postcritical set of the model map $G$ in $\hat{C} \setminus D$. Since $\partial \Delta \subset \mathcal{P}(f)$, we have $T \subset \mathcal{P}(G)$ (see §2.1). For simplicity, here we only consider the Jordan disk $V_0 \subset \hat{C} \setminus D$ which satisfies that $V_0 \cap \mathcal{P}(G)$ is a non-empty subarc on $T$ and the general case can be reduced to such case (see §4.3). We use $\{V_n\}_{n \geq 0}$ to denote the pullbacks of $V_0$ in $\hat{C} \setminus D$ under $G$. Note that $G^n : V_n \rightarrow V_0$ is conformal and each $V_n$ is a Jordan disk (see Lemma 2.1). For some $N_0$ large enough, $\overline{V}_{N_0} \cap T$ is contained in a small open arc $I_0 \subset T$ (see Lemma 4.7) and the Jordan disk $V_{N_0}$ is contained in the following finite union in $\hat{C} \setminus D$:

$$H_d(I_0) \cup \bigcup_{1 \leq i \leq M_0} W_i,$$

where each $W_i$ is a Jordan disk compactly contained in $\hat{C} \setminus D$ satisfying \#($W_i \cap \mathcal{P}(G)$) ≤ 1 and $H_d(I_0)$ is a half hyperbolic $d$-neighborhood of $I_0$ with $d > 0$ being a given large number (see §2.13 and Figure 2 for the definition of $H_d(I)$ and §4.3 for the way to choose $W_i$’s). By applying the classical Shrinking Lemma (see Lemma 4.6) to each $W_i$, the Main Lemma can be reduced to proving Lemma 4.5: For any $\varepsilon > 0$, if $V_0 = H_d(I_0)$ with $I_0$ small enough, then

$$\text{diam}_{\hat{C}}(V_n) < \varepsilon \quad \text{for all } n \geq 0.$$
To this end, for any small subarc $I_0 \subset \mathbb{T}$, we first give a rule in §3 by which one can dynamically define an infinite sequence of arcs $\{I_n\}_{n \geq 0}$ on $\mathbb{T}$ by choosing two constants $0 < \delta < \eta < 1/2$ suitably, which is called a $(\delta, \eta)$-admissible sequence. For each $n \geq 0$, one of the branches of $G^{-1}$, say $\Phi_n$, is associated to $I_n \to I_{n+1}$. We call $I_{n+1}$ the pullback of $I_n$ but note that sometimes $G(I_{n+1}) \neq I_n$, especially when $I_n$ contains a critical value. The process $I_n \to I_{n+1}$ is called a critical pullback if there is a critical value on $\mathbb{T}$ close to $I_n$ (see §3 for the precise definition). We show that finitely many critical pullbacks will decrease the dynamical length (see (2.8) for the definition) $\sigma(I_n)$ of $I_n$ by a definite amount. In particular, $\sigma(I_n) < 2\sigma(I_m)$ for all $n > m \geq 0$ and $\sigma(I_n) \to 0$ as $n \to \infty$ (see Lemma 3.3).

Let $I_{n_j} \to I_{n_j+1}$, where $j \geq 1$, be the sequence of all critical pullbacks. For a small $s_0 > 0$, let $D_{s_0}$ be the half Euclidean $s_0$-neighborhood of $I_{n_j}$ in $\mathbb{C} \setminus \overline{D}$ which is disjoint with $\mathcal{P}(G)$ (see (3.11) and Figure 8). Denote by $\Omega$ the unique component of $\mathbb{C} \setminus \mathcal{P}(G) \cup \overline{D}$ with $\mathbb{T} \subset \partial \Omega$. The main result in §3 is Proposition 3.7: There exists a number $T_0 \geq 1$ depending only on $G$, such that for any $z \in D_{s_0}^n$, the composition $\Phi_{n_j+T_0} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}$ of the inverses of $G$ either contracts the hyperbolic metric $\rho_\Omega(z)|dz|$ uniformly at $z$, or maps $z$ into a half hyperbolic $d$-neighborhood $H_d(J)$ of an arc $J \subset \mathbb{T}$ containing the base interval $I_{n_j+T_0+1}$, where the dynamical lengths of $J$ and $I_{n_j+T_0+1}$ are comparable.

For the proof of Proposition 3.7, we analyze the combinatoric information of the critical orbit of $G$ on $\mathbb{T}$ and apply the contraction property of $G^{-1}$ (see Lemma 2.9) several times at most of the place in $D_{s_0}^n$, except in a triangle-shaped region where the points will be mapped very close to the base interval $I_{n_j+T_0+1}$ because of the finitely many critical pullbacks. The contraction principle in Lemma 2.9 was first observed by Petersen in [Pet96].

Based on Proposition 3.7 and the property $\sigma(I_n) \to 0$ as $n \to \infty$, for each $(\delta, \eta)$-admissible sequence $\{I_n\}_{n \geq 0}$, we construct an improved $(\delta, \eta)$-admissible sequence $\{(J_n, F_n, d_n)\}_{n \geq 0}$ in §4.1, where

- $J_n$ is an arc of $\mathbb{T}$ containing $I_n$ so that $\sigma(J_n) < 2\sigma(J_0) = 2\sigma(I_0)$ and $\sigma(J_n) \to 0$ as $n \to \infty$; and
- $F_n$ is the union of at most countably many Jordan disks intersecting $H_{d_n}(J_n)$ with $d_n \in [d, d+1]$ and having hyperbolic diameters with respect to $\rho_\Omega(z)|dz|$ less than a constant $K = K(\delta, \eta, d) > 0$.

Such a sequence allows us to capture further the contraction property of a long iteration of the inverse of $G$ near $\mathbb{T}$ as the following (see §4.1):

$$\Phi_{m-1} \circ \cdots \circ \Phi_{n+1} \circ \Phi_n(H_{d_n}(J_n) \cup F_n) \subset H_{d_m}(J_m) \cup F_m \subset D_{s_0}^m,$$

where $m > n \geq 0$. We have $\text{diam}_\infty(H_{d_n}(J_n) \cup F_n) \to 0$ as $n \to \infty$ since $\sigma(J_n) \to 0$ as $n \to \infty$, $d_n \in [d, d+1]$ and the hyperbolic diameter of every component of $F_n$ is bounded above by $K$.

Suppose $\{V_n\}_{n \geq 0}$ is the sequence obtained by pulling back $V_0 = H_d(I_0)$ under $G$. For given $\varepsilon > 0$, by assuming $I_0$ small, if there is a sequence of improved $(\delta, \eta)$-admissible sequences $\{(J_n^m, F_n^m, d_n^m)\}_{n \geq 0}$ such that for any $m \geq 0$,

$$V_n \subset H_{d_n^m}(J_n^m_n) \cup F_n^m \text{ for all } 0 \leq n \leq m,$$

then $\text{diam}_\infty(V_n) < \varepsilon$ for all $n \geq 0$. Otherwise, there exist a maximal $N < +\infty$ and an improved $(\delta, \eta)$-admissible sequence $\{(J_n, F_n, d_n)\}_{n \geq 0}$ such that $V_n \subset H_{d_n}(J_n) \cup F_n$ for all $0 \leq n \leq N$ but not hold for $N + 1$. In this case we say that the pullback $V_N \to V_{N+1}$ is a jump off. There are two types of jump offs. For the first type
jump off, \( V_{N+1} \) is bounded away from \( \mathbb{T} \). In this case, the size of all the subsequent pullbacks will be small by a routine argument. For the second type jump off, \( V_{N+1} \) is still very close to \( \mathbb{T} \) so that \( V_{N+1} \subset H_d(J) \) with \( J \) being arbitrarily small provided that \( I_0 \) is small enough, and most importantly, the hyperbolic diameter of \( V_{N+1} \) is bounded above by some constant, and the pullback \( V_N \to V_{N+1} \) decreases the hyperbolic metric by a definite amount. So if \( I_0 \) is small enough and \( V_N \to V_{N+1} \) is a second type jump off, we can consider the jump off of the sequence \( \{V_k\}_{k \geq N+1} \) with respect to \( J \). This process can thus be repeated. So we will either have a first type jump off at some time or the number of the second type jump offs will be eventually large enough so that the hyperbolic diameter of \( V_n \) is smaller than a given number. This implies Lemma 4.5 and the Main Lemma follows.

The proof of the Main Theorem is presented in §5. There are two basic tools in the proof – the Main Lemma and a criterion of Whyburn. The criterion says that a compact subset \( X \) of \( \hat{\mathbb{C}} \) is locally connected if and only if the boundary of every component of \( \hat{\mathbb{C}} \setminus X \) is locally connected, and moreover, for any \( \varepsilon > 0 \), the number of the components of \( \hat{\mathbb{C}} \setminus X \) whose size is greater than \( \varepsilon \) is finite.

The second condition of the Whyburn’s criterion follows directly from the Main Lemma. Note that the boundaries of bounded type Siegel disks are quasi-circles [Zha11]. To verify the first condition, it suffices to show that the boundaries of immediate attracting basins are locally connected if the map has attracting cycles. The argument here is a bit subtle. As we mentioned at the beginning of the introduction, the boundary of an attracting basin may turn around some Siegel disk in a very complicated manner so that the uniform contraction property may not hold – for a large number of pullbacks are needed to unwind the object first before shrinking it. To overcome this we will show that the homotopy complexity of the internal rays in the immediate attracting basins actually have a uniform upper bound (see Lemma 5.5). This allows us to apply the Main Lemma to deduce that the equipotential curves in the immediate attracting basins converge uniformly. Thus the boundaries of immediate attracting basins are locally connected, the first condition of the Whyburn’s criterion follows and the Main Theorem holds.

**Notations.** We will use the following notations throughout this paper.

- Let \( \mathbb{R} \), \( \mathbb{C} \) and \( \hat{\mathbb{C}} \), respectively, be the real axis, complex plane and Riemann sphere.
- For \( a \in \mathbb{C} \) and \( r > 0 \), denote \( \mathbb{D}_r(a) := \{ z \in \mathbb{C} : |z - a| < r \} \), \( \mathbb{D}_r := \mathbb{D}_r(0) \), \( \mathbb{D} := \mathbb{D}_1 \), \( T_r := \{ z \in \mathbb{C} : |z| = r \} \) and \( T := T_1 \).
- For \( r > 1 \), denote the annulus \( A_r := \{ z \in \mathbb{C} : 1/r < |z| < r \} \).
- If \( X = \hat{\mathbb{C}} \) (resp. \( \mathbb{C} \), or a hyperbolic domain), let \( \text{dist}_X(\cdot, \cdot) \) and \( \text{diam}_X(\cdot) \) be the distance and diameter with respect to the spherical (resp. Euclidean, or hyperbolic) metric. In particular, \( \rho_X(z)|dz| \) is the hyperbolic metric in a hyperbolic domain \( X \).
- Two positive numbers \( a, b \) are said to be \( C \)-comparable if \( b/C \leq a \leq bC \). For two family of positive numbers \( \{a_\lambda\} \) and \( \{b_\lambda\} \), we write \( a_\lambda \leq b_\lambda \) if there exists a constant \( C > 1 \) such that \( a_\lambda \leq Cb_\lambda \) for all \( \lambda \). We write \( a_\lambda \asymp b_\lambda \) if \( b_\lambda \leq a_\lambda \leq b_\lambda \).

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2. Quasi-Blaschke models

In this section we define quasi-Blaschke models for the maps in the Main Lemma and introduce some tools in hyperbolic geometry.

2.1. Quasi-Blaschke models and dynamical lengths. In the following, we fix the rational map \( f \) which has a fixed bounded type Siegel disk \( \Delta \) in the Main Lemma. According to [Zha11], \( \partial \Delta \) is a quasi-circle containing at least one critical point of \( f \). Without loss of generality, we assume that \( \Delta \) is bounded in the complex plane \( \mathbb{C} \). Let

\[
\phi : \hat{\mathbb{C}} \setminus \overline{\Delta} \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}
\]  

(2.1)

be a conformal isomorphism fixing \( \infty \).

For \( z \in \mathbb{C} \) (resp. \( Z \subset \hat{\mathbb{C}} \)), let \( z^* = 1/z \) (resp. \( Z^* = \{ z^* : z \in Z \} \)) be the symmetric image of \( z \) (resp. \( Z \)) about the unit circle \( \mathbb{T} \). Let \( \text{Crit}(f) \) be the set of all critical points and \( \mathcal{P}(f) \) the postcritical set of \( f \). We extend \( \phi \) to a quasiconformal homeomorphism \( \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) by considering the following two cases:

- (Non-capture case) If \( \mathcal{P}(f) \cap \Delta = \emptyset \), let \( \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be any quasiconformal extension of (2.1).
- (Capture case) If \( \mathcal{P}(f) \cap \Delta \neq \emptyset \), then the following set is nonempty and finite:

\[
\mathcal{V} := \left\{ f^n(c) \left| c \in \text{Crit}(f) \text{ and } n \geq 1 \text{ such that } f^{n-1}(c) \notin \Delta \text{ and } f^n(c) \in \Delta \right. \right\}.
\]

Suppose \( \mathcal{V} = \{ b_1, \cdots, b_m \} \), where \( m \geq 1 \). Let \( c_1 \in \text{Crit}(f) \) and \( n_1 \geq 1 \) such that \( f^{n_1}(c_1) = b_1 \in \Delta \) but \( f^{n_1-1}(c_1) \notin \Delta \). We take \( m \) different points \( b_1', \cdots, b_m' \) in \( \hat{\mathbb{C}} \setminus \overline{\Delta} \) and \( m \) positive integers \( k_1, \cdots, k_m \) such that

\[
f^{k_i}(b'_i) = c_1 \quad \text{for all } 1 \leq i \leq m.
\]

(2.2)

Let \( \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a quasiconformal extension of (2.1) such that

\[
\phi(b_i) = (\phi(b'_i))^* \in \mathbb{D} \quad \text{for all } 1 \leq i \leq m.
\]

(2.3)

In both cases, we define a quasi-Blaschke model corresponding to \( f \):

\[
G(z) := \left\{ \begin{array}{ll} 
\phi \circ f \circ \phi^{-1}(z) & \text{if } z \in \hat{\mathbb{C}} \setminus \mathbb{D}, \\
\phi \circ f \circ \phi^{-1}(z^*)^* & \text{if } z \in \mathbb{D}.
\end{array} \right.
\]

(2.4)

By the construction and the assumption \( \text{dist}_\mathbb{C}(\mathcal{P}(f) \setminus \partial \Delta, \partial \Delta) > 0 \) in the Main Lemma, \( G \) has the following properties:

- (i) \( G \) commutes with \( z \mapsto z^* \), i.e., \( G(z^*) = (G(z))^* \);
- (ii) \( G : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \) is conjugate to \( f : \hat{\mathbb{C}} \setminus \Delta \to \hat{\mathbb{C}} \) by the quasiconformal mapping \( \phi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \);
- (iii) \( G : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a quasi-regular map and analytic in \( \hat{\mathbb{C}} \setminus (Q \cup Q^*) \), where \( Q = \phi(f^{-1}(\Delta) \setminus \Delta) \); and
- (iv) \( \text{dist}_\mathbb{C}(\mathcal{P}(G) \setminus \mathbb{T}, \mathbb{T}) > 0 \), where

\[
\tilde{\mathcal{P}}(G) := \bigcup_{n \geq 1, c \in \text{Crit}(f)} G^n(\phi(c) \cup \phi(c^*))
\]

(2.5)

Indeed, the property (iv) holds since by (2.2) and (2.3), we have the forward orbit

\[
(\phi(b_i))^* \xrightarrow{G^1} \phi(c_1) \xrightarrow{G^1} \phi(b_1) \xrightarrow{G^1} (\phi(c_1))^* \xrightarrow{G^1} (\phi(b_1))^* \xrightarrow{G^1} \phi(c_1),
\]
dependence of all the constants on expanding near the unit circle. In the following we fix the model map $G$ since we are mainly interested in the dynamics of $G$ in $\hat{\mathbb{C}} \setminus \mathbb{D}$.

Let $V_0$ be a Jordan disk in $\hat{\mathbb{C}} \setminus \mathbb{D}$. We call $\{V_n\}_{n \geq 0}$ a pullback sequence of $V_0$ if $V_{n+1}$ is a connected component of $G^{-1}(V_n)$ in $\hat{\mathbb{C}} \setminus \mathbb{D}$ for all $n \geq 0$. The following result implies the Main Lemma immediately.

**Main Lemma**. For any $\varepsilon > 0$ and any Jordan disk $V_0 \subset \hat{\mathbb{C}} \setminus \mathbb{D}$ with $V_0 \cap \mathcal{P}(G) \neq \emptyset$ and $V_0 \setminus \mathcal{P}(G) \subset \mathbb{T}$, there exists $N \geq 1$, such that for any pullback sequence $\{V_n\}_{n \geq 0}$ of $V_0$, we have $\text{diam}_\mathbb{C}(V_n) < \varepsilon$ for all $n \geq N$.

This lemma implies that a long iteration of the quasi-Blaschke model $G$ is expanding near the unit circle. In the following we fix the model map $G$ and omit the dependence of all the constants on $G$. The heart of this paper is the proof of the Main Lemma’, which occupies §2 to §4.

The following result is useful when we consider the pullbacks of Jordan disks. For a proof, see [Pil96, Proposition 2.8].

**Lemma 2.1.** If a Jordan disk $V \subset \hat{\mathbb{C}} \setminus \mathbb{D}$ contains no critical values, or its closure contains at most one critical value, then every component $U$ of $G^{-1}(V)$ is a Jordan disk. Moreover, $G : U \to V$ is a homeomorphism in the first case.

Let $\mathbb{R}$ be the real axis. An orientation-preserving homeomorphism $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is called a quasi-symmetric map if there exists $k > 1$ such that

$$
\frac{k^{-1}}{g(x + t) - g(x)} \leq g(x) - g(x - t) \leq k
$$

for all $x \in \mathbb{R}$ and all $t > 0$. Let $g : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism of the unit circle $\mathbb{T}$ and $\tilde{g} : \mathbb{R} \to \mathbb{R}$ a lift of $g$ under the covering map $x \mapsto e^{2\pi i x}$. The map $g : \mathbb{T} \to \mathbb{T}$ is called quasi-symmetric if (2.7) holds for $\tilde{g}$.

Let $\Delta$ be the fixed bounded type Siegel disk of $f$ with rotation number $\alpha$. Since $\partial \Delta$ is a quasi-circle, there exists a quasiconformal mapping $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\psi(\partial \Delta) = \mathbb{T}$ such that

$$
\psi \circ f \circ \psi^{-1}(\zeta) = R_\alpha(\zeta) := e^{2\pi i \alpha} \zeta \quad \text{for all } \zeta \in \mathbb{T}.
$$

The restriction $G = \phi \circ f \circ \phi^{-1} : \mathbb{T} \to \mathbb{T}$ of the quasi-regular map defined in (2.4) has rotation number $\alpha$. Therefore,

$$
h := \psi \circ \phi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}
$$

is a quasiconformal mapping satisfying

$$
h \circ G \circ h^{-1}(\zeta) = R_\alpha(\zeta) \quad \text{for all } \zeta \in \mathbb{T}.
$$

In particular, $h : \mathbb{T} \to \mathbb{T}$ is a quasi-symmetric map conjugating $G$ to the rigid rotation $R_\alpha$. Note that $h$ is unique up to a post-composite rigid rotation of $\mathbb{T}$. 
**Definition** (Dynamical length). For any arc $I \subset \mathbb{T}$, we use $|I|$ to denote the Euclidean length of $I$. Define
\[
\sigma(I) := |h(I)|. \tag{2.8}
\]
We call $\sigma(I)$ the dynamical length of $I$. Clearly, $\sigma(\cdot)$ is $G$-invariant in the sense that $\sigma(G(I)) = \sigma(I)$.

Since both $h : \mathbb{T} \to \mathbb{T}$ and $h^{-1} : \mathbb{T} \to \mathbb{T}$ are quasi-symmetric maps, the following result follows from (2.7) immediately.

**Lemma 2.2.** We have $\sigma(I) \asymp \sigma(J)$ if and only if $|I| \asymp |J|$, where $I$ and $J$ are disjoint and adjacent subarcs of $\mathbb{T}$.

In this paper, the anticlockwise direction of the unit circle is regarded as the positive direction. This induces an orientation for any subarc $I \subset \mathbb{T}$. If $I$ is a subarc of $\mathbb{T}$ with $\mathbb{T} = [a, b] \neq \mathbb{T}$, we say that $a$ is on the left of $b$ (or $b$ is on the right of $a$). If $I = \mathbb{T} \setminus \{a\}$, we denote $I = (a^-, a^+)$ for convenience. For any two different $a, b \in \mathbb{T}$, by definition we have $(a, b) \cup (b, a) \cup \{a, b\} = \mathbb{T}$.

### 2.2. First return properties

We first give a brief account of the combinatorics of the closest returns and the associated dynamical partitions of the unit circle under the irrational rotation $R_\alpha(\zeta) = e^{2\pi i \alpha} \zeta$. One may refer to [Mil06, Appendix C] for the details. For $n \geq 1$, let $p_n/q_n := [0; a_1, \cdots, a_n]$ be the $n$-th approximation of the continued fraction expansion of $\alpha \in (0, 1)$, where $p_n$ and $q_n$ are coprime positive integers. Denote $p_0 = 0$ and $q_0 = 1$. Note that $p_1 = 1$ and $q_1 = a_1$. It is well known that for all $n \geq 2$,
\[
\begin{align*}
p_n &= a_np_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_nq_{n-1} + q_{n-2}. \tag{2.9}
\end{align*}
\]

Denote $\lambda = e^{2\pi i \alpha}$. We say that the sequence $\lambda^1, \lambda^2, \lambda^3, \cdots$ has a close return to $\lambda^0 = 1$ at the time $q \geq 1$ if $\lambda^q$ is closer to 1 than any of its predecessors:
\[
|\lambda^q - 1| < |\lambda^k - 1| \text{ for } k = 1, 2, 3, \cdots, q - 1.
\]

Actually the collection of all close return times are exactly the $q_n$’s: $1 = q_0 < q_1 < q_2 < \cdots$. Let $d_n$ be the Euclidean length of the shortest arc in $\mathbb{T}$ connecting 1 with $\lambda^{q_n}$. Then $2\pi \min\{\alpha, 1 - \alpha\} = d_0 > d_1 > d_2 > \cdots > 0$ and for all $n \geq 2$,
\[
\begin{align*}
d_n &= d_{n-2} - a_nh_{n-1} \quad \text{and} \quad a_n < \frac{d_{n-2}}{d_{n-1}} \leq a_n + 1. \tag{2.10}
\end{align*}
\]

If $0 < \alpha < 1/2$, then $\lambda^{q_n}$ lies in the open arc $(\lambda, 1) \subset \mathbb{T}$ and $\lambda^{q_1} \in (1, \lambda) \subset \mathbb{T}$. Moreover, we have $\lambda^{q_n} \in (\lambda^{q_n-2}, 1)$ if $n$ is odd and $\lambda^{q_n} \in (1, \lambda^{q_n-2})$ if $n$ is even, where $n \geq 3$. In the closed arc $[\lambda^{q_n-2}, \lambda^{q_n-1}] \subset \mathbb{T}$ containing 1, the points with the form $\lambda^k : 0 \leq k \leq q_n$ are (listed by the positive direction on $\mathbb{T}$):
\[
\lambda^{q_n-1}, \lambda^{q_n-2+q_{n-1}}, \lambda^{q_n-2+2q_{n-1}}, \cdots, \lambda^{q_n} = \lambda^{q_n-2+a_nh_{q_{n-1}}-1}, 1, \lambda^{q_{n-1}}.
\]

If $1/2 < \alpha < 1$, then $\lambda^{q_n} \in (1, \lambda)$, $\lambda^{q_1} \in (\lambda, 1)$ and the above statements are still true by permuting the parity of $n$.

In the following, every subarc in $\mathbb{T}$ is assumed to be open and nonempty unless otherwise specified. Let $I \subset \mathbb{T}$ be an arc. For any $x \in I$, we denote
\[
N(x, I) := \min\{n \geq 1 : G^n(x) \in I\}.
\]

Define
\[
N(I) := \max_{x \in I} N(x, I) \quad \text{and} \quad n(I) := \min_{x \in I} N(x, I).
\]

**Lemma 2.3.** We have $N(I) \asymp n(I)$ for any subarc $I \subset \mathbb{T}$.
Proof. We assume that \(|I|\) is small since otherwise, the conclusion is obvious. For any \(x \in I = (a, b)\), let \(n = N(x, I) \geq 1\) be the first return time of \(x\). Without loss of generality, we assume that \(G^n(x) \in (a, x)\). Then there exists an integer \(k = k(x) \geq 2\) such that

\begin{enumerate}[(i)]
  \item \((a, x) \subset (G^{q_k-2}(x), x)\) and \((x, b) \subset (x, G^{q_k-1}(x))\); and
  \item \(n = q_{k-2} + jq_{k-1}\) with \(1 \leq j \leq a_k\).
\end{enumerate}

Let \(\sigma(I)\) be the dynamical length of \(I \subset \mathbb{T}\) defined in (2.8). Since \(\alpha\) is of bounded type, by (2.10) we have \(\sigma(I) \asymp |(1, R_{\alpha}^{q_1}(1))|\). Hence

\[
|(1, R_{\alpha}^{q_k}(x))| \asymp |(1, R_{\alpha}^{q_k}(1))| \quad \text{for } x, y \in I.
\]

Therefore, there exists an integer \(\ell_0 = \ell_0(\alpha) \geq 1\) such that \(|k(x) - k(y)| \leq \ell_0\) for any \(x, y \in I\). This, together with (ii), (2.9) and the fact that \(\alpha\) is of bounded type, implies the lemma.

Lemma 2.4. We have \(\sigma(I) \asymp \sigma(J)\) if and only if \(N(I) \asymp N(J)\), where \(I, J\) are subarcs of \(\mathbb{T}\).

Proof. We only prove the necessity since the sufficiency is completely similar. From the proof of Lemma 2.3 it follows that there exist two integers \(k_1, k_2 \geq 2\) such that

\[
\sigma(I) \asymp |(1, R_{\alpha}^{q_{k_1}}(1))|, \quad N(I) \asymp q_{k_1}, \quad \text{and}
\]

\[
\sigma(J) \asymp |(1, R_{\alpha}^{q_{k_2}}(1))|, \quad N(J) \asymp q_{k_2}.
\]

Thus if \(\sigma(I) \asymp \sigma(J)\), then \(|(1, R_{\alpha}^{q_{k_1}}(1))| \asymp |(1, R_{\alpha}^{q_{k_2}}(1))|\). By the fact that \(\alpha\) is of bounded type, this is equivalent to that \(q_{k_1} \asymp q_{k_2}\). Thus \(N(I) \asymp N(J)\).

Lemma 2.5. For any arc \(I \subset \mathbb{T}\) and \(x \in I\), if \(G^{n_1}(x)\) and \(G^{n_2}(x)\) are the first and the second returns of \(x\) to \(I\), i.e.,

\[
n_1 = N(x, I) \quad \text{and} \quad n_2 = n_1 + N(G^{n_1}(x), I),
\]

then \(\sigma(I) \asymp \sigma(J)\), where \(J\) is the shortest arc between two points of \(x, G^{n_1}(x)\) and \(G^{n_2}(x)\) with respect to the dynamical length.

Proof. By Lemmas 2.3 and 2.4 the dynamical lengths of \((x, G^{n_1}(x))\) (or \((G^{n_1}(x), x)\)) and \((G^{n_1}(x), G^{n_2}(x))\) (or \((G^{n_2}(x), G^{n_1}(x))\)) are comparable with \(\sigma(I)\). Without loss of generality, we assume that \((x, G^{n_1}(x)) \subset I\), i.e., \(G^{n_1}(x)\) is on the right of \(x\). There are following two cases:

**Case I:** Suppose \(I' := (x, G^{n_2}(x)) \subset I\). If \(G^{n_1}(x) \in I'\), then

\[
\sigma((x, G^{n_1}(x))) \asymp \sigma((G^{n_1}(x), G^{n_2}(x))) \asymp \sigma(I).
\]

Otherwise, \(G^{n_2}(x) \in (x, G^{n_1}(x))\). Then for any sufficiently small \(\varepsilon > 0\), we have

\[
n_2 = N(x, I'_\varepsilon), \quad \text{where } I'_\varepsilon := (x - \varepsilon, G^{n_2}(x) + \varepsilon).
\]

Note that \(n_1 = N(x, I)\) and \(n_2 - n_1 = N(G^{n_1}(x), I)\). By Lemma 2.3, we have \(n_1 \asymp n_2 - n_1\) and hence \(n_1 \asymp n_2\). By Lemmas 2.3 and 2.4 and letting \(\varepsilon \to 0\), we conclude that \(\sigma(I) \asymp \sigma(I')\). In the first paragraph, we know that \(\sigma(I) \asymp \sigma(I')\). Therefore, \(\sigma(I) \asymp \sigma(I') = \sigma(I'')\).

**Case II:** Suppose \(I' := (G^{n_2}(x), x) \subset I\). Then one can define \(I'_\varepsilon := (G^{n_2}(x) - \varepsilon, x + \varepsilon)\) as in Case I and the rest argument is completely the same. \(\square\)
2.3. Half hyperbolic neighborhoods. We have assumed that the bounded type Siegel disk $\Delta$ is bounded in $\mathbb{C}$. Without loss of generality, in the following we assume further that $f(\infty) \in \Delta$ (hence $G(\infty) \in \mathbb{D}$). Then for any connected set $V_0$ in $\hat{\mathbb{C}} \setminus \mathbb{D}$, every component of $G^{-1}(V_0)$ in $\hat{\mathbb{C}} \setminus \mathbb{D}$ is a bounded set in $\mathbb{C} \setminus \mathbb{D}$. Let $\mathcal{P}(G)$ be defined in (2.5). There exists a constant $r_0 > 1$ such that

$$\left( \mathcal{P}(G) \setminus \mathcal{T} \right) \cap \overline{A}_{r_0} = \emptyset,$$

(2.11)

where $A_{r_0} = \{ z \in \mathbb{C} : 1/r_0 < |z| < r_0 \}$. Let $I = (a, b) \subset \mathcal{T}$ be an open arc with $I \neq \mathcal{T}$. Then $\mathcal{P}(G) \setminus I$ is a compact subset of $\mathbb{C} \setminus \mathbb{D}$. We denote

$$\Omega_I := \text{The component of } \mathbb{C} \setminus (\mathcal{P}(G) \setminus I) \text{ containing } I.$$ (2.12)

Then $\Omega_I$ is a domain which is symmetric about $\mathcal{T}$.

**Definition (Half hyperbolic neighborhoods).** For any given $d > 0$, let

$$H_d(I) := \{ z \in \Omega_I : \text{dist}_{\Omega_I}(z, I) < d \text{ and } |z| > 1 \}$$

(2.13)

be a half hyperbolic neighborhood of $I$ in $\hat{\mathbb{C}} \setminus \mathbb{D}$. See Figure 1.

Let $\text{Crit}(f)$ be the set of all critical points of $f$ and we denote $\overline{\text{Crit}}(f) := \{ c \in \text{Crit}(f) : \exists n \geq 0 \text{ such that } f^n(c) \in \partial\Delta \}$. Note that $\text{Crit}(f) \subset \hat{\mathbb{C}} \setminus \Delta$. Denote

$$\text{Crit}(G) := \phi(\text{Crit}(f)) \text{ and } \overline{\text{Crit}}(G) := \phi(\overline{\text{Crit}}(f)).$$

We denote

$$\mathcal{CV} := \{ G(c) : c \in \text{Crit}(G) \cap \mathcal{T} \}, \text{ and }$$

$$\overline{\mathcal{CV}} := \left\{ G^n(c) \left| c \in \overline{\text{Crit}}(G) \text{ and } n > 1 \text{ is minimal such that } G^n(c) \in \mathcal{T} \right. \right\}.$$ (2.14)

Then $\mathcal{CV} \subset \overline{\mathcal{CV}} \subset \mathcal{T}$ and $\overline{\mathcal{CV}}$ is a finite set.

Let $r_0 > 1$ be the constant introduced in (2.11). Then there exists $0 < \tilde{r}_0 \leq \min \left\{ \frac{1}{d}, r_0 - 1 \right\}$ such that every subarc $I \subset \mathcal{T}$ satisfies:

if $\min \{|I|, |\sigma(I)|\} < \tilde{r}_0$, then $\text{dist}_{\mathcal{P}(G)}(I \cap G^n(\mathcal{CV})) \leq 1$ for every $n \geq 0$. (2.15)

**Lemma 2.6.** For any $d > 0$, there exists $r_1 = r_1(d) \in (0, \tilde{r}_0]$ such that if $I \subset \mathcal{T}$ is an arc satisfying $\min \{|I|, |\sigma(I)|\} < r_1$, then $H_{\tilde{d}}(I)$ is a bounded Jordan disk in $\mathbb{C} \setminus \mathbb{D}$ for all $\tilde{d} \in (0, d + 1]$. 

![Figure 1: A half hyperbolic neighborhood $H_d(I)$ of the open interval $I = (a, b)$.](image-url)
Proof. Let $d > 0$ and $I \subset \mathbb{T}$ be an arc with $T \neq \mathbb{T}$. Consider the hyperbolic domain
\[ \tilde{\Omega}_I := \hat{\mathbb{C}} \setminus (\mathbb{T} \setminus I) \] (2.16)
and denote
\[ \tilde{H}_d(I) := \{ z \in \tilde{\Omega}_I : \text{dist}_{\tilde{\Omega}_I}(z, I) < d \}. \] (2.17)
Then the boundary $\partial \tilde{H}_d(I)$ consists of two subarcs of Euclidean circles which are symmetric to $\mathbb{T}$ (see [MS93], and the angle $\beta = \beta(d) \in (0, \pi)$ between $\partial \tilde{H}_d(I)$ and $\mathbb{T} \setminus I$ depends only on $d$. In particular,
\[ \tilde{H}^3(I) := \tilde{H}_d(I), \quad \text{where } \log \cot \left( \frac{\beta}{2} \right) = d. \] (2.18)
Hence there exists a small $r_1 = r_1(d) > 0$ such that if $|I| < r_1$ or $|\sigma(I)| < r_1$, then the Euclidean diameter of $\tilde{H}_{d+1}(I)$ is small and
\[ \tilde{H}_{d+1}(I) \subset A_{r_0} \setminus (\mathbb{T} \setminus I), \] (2.19)
where $r_0 > 1$ is the number in (2.11). Since $\Omega_I \subset \tilde{\Omega}_I$, we have
\[ H_d(I) \subset \tilde{H}_d(I) \setminus \mathbb{D} \quad \text{for all } d \in (0, d + 1). \] (2.20)

If $\Omega_I = \tilde{\Omega}_I$, then the lemma holds immediately. Without loss of generality, we assume that $\Omega_I \subset \tilde{\Omega}_I$. Note that $\Omega_I$ is a hyperbolic domain which is symmetric about the unit circle. The arc $I \subset \mathbb{T}$ is a geodesic in $\Omega_I$ with respect to the hyperbolic metric $\rho_{\Omega_I}(z)|dz|$. Consider the holomorphic universal covering $\pi : \mathbb{D} \to \Omega_I$. The preimages of $I$ under $\pi$ consist of countably many pairwise disjoint Euclidean arcs $\{L_i : i \in \mathbb{N}\}$ in $\mathbb{D}$ which are orthogonal to the unit circle. The $(d + 1)$-neighborhood
\[ S_{d+1}(L_i) := \{ w \in \mathbb{D} : \text{dist}_{\mathbb{D}}(w, L_i) < d + 1 \} \]
of each $L_i$ with respect to the hyperbolic metric in $\mathbb{D}$ is a simply connected domain. By (2.19) and (2.20), for any different $i$ and $j$, $S_{d+1}(L_i)$ and $S_{d+1}(L_j)$ lie in their respective fundamental domains and $S_{d+1}(L_i) \cap S_{d+1}(L_j) = \emptyset$. This implies that the restriction of $\pi$ in an open neighborhood of $S_{d+1}(L_i) \subset \mathbb{D}$ is conformal and $\pi(S_d(L_i)) = H_d(I) \cup (H_d(I))^*$ is a bounded Jordan disk for all $d \in (0, d + 1)$. In particular, $H_d(I)$ is a bounded Jordan disk in $\mathbb{C} \setminus \mathbb{D}$ for all $d \in (0, d + 1)$. \hfill \Box

Lemma 2.7. For any $d > 0$, there exist $C'_0 = C'_0(d) > 0$ and $r_2 = r_2(d) \in (0, r_1]$ such that if $I \subset \mathbb{T}$ is an arc satisfying $\min\{|I|, |\sigma(I)|\} < r_2$, then

(a) $H_d(I) \subset \tilde{H}_d(I) \setminus \mathbb{D} \subset H_d'(I)$, where $d' = d + C'_0 |I| < d + 1$;

(b) If $I \cap \bigcup_{n=0}^{d-1} G^d(CV) = \emptyset$ for some $n \geq 1$, let $J \subset \mathbb{T}$ be the arc satisfying $G^n(J) = I$ and $Q \subset \mathbb{C} \setminus \mathbb{D}$ be the component of $G^{-n}(H_d(I))$ satisfying $\partial Q \cap \mathbb{T} = J$. Then $Q \subset H_d'(J)$.

Proof. (a) Given $d > 0$. By Lemma 2.6, we first assume that $\min\{|I|, |\sigma(I)|\} < r_1$ such that $H_d(I)$ is a Jordan disk for all $d \in (0, d + 1)$.

Without loss of generality, we assume that $I = \{ e^{i\theta} : |\theta| < \frac{|I|}{2} \}$. Denote $\mathbb{H}_+ = \{ w \in \mathbb{C} : \text{Re } w > 0 \}$ and let $\tilde{\Omega}_I = \tilde{\mathbb{C}} \setminus (\mathbb{T} \setminus I)$ be defined in (2.16). Let
\[ \varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1 : \tilde{\Omega}_I \to \mathbb{H}_+ \] (2.21)
be the composition of the following three conformal maps (see Figure 2), where
\[ \zeta = \varphi_1(z) = \frac{i}{\tan\left(\frac{|I|}{4}\right)} \frac{1-z}{1+z} : \mathcal{D}_I \to \mathbb{C} \setminus \mathbb{R} \setminus (-1, 1), \]
\[ \xi = \varphi_2(\zeta) = \frac{1+\zeta}{1-\zeta} : \mathbb{C} \setminus \mathbb{R} \setminus (-1, 1) \to \mathbb{C} \setminus (-\infty, 0], \text{ and} \]
\[ w = \varphi_3(\xi) = \sqrt{\xi} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{H}_+. \]

Then we have \( \varphi(I) = \varphi_3 \circ \varphi_2((-1, 1)) = \varphi_3(\mathbb{R}_+) = \mathbb{R}_+, \) where \( \mathbb{R}_+ := (0, +\infty). \) Let \( \tilde{H}_d(I) \) be defined in (2.17). A direct calculation shows that
\[ \varphi(\tilde{H}_d(I)) = \{ w \in \mathbb{H}_+ : \text{dist}_{\mathbb{H}_+}(w, \mathbb{R}_+) < d \} = \{ re^{i\theta} : r > 0 \text{ and } |\theta| < \frac{\pi}{2} - \frac{\beta}{2} \}, \]
where \( \beta \in (0, \pi) \) satisfies \( d = \log \cot\left(\frac{\beta}{4}\right). \) See (2.18).

Let \( r_0 > 1 \) be the number introduced in (2.11). Define
\[ \hat{\Omega}_I := \mathbb{A}(r_0) \setminus ((\mathbb{T} \setminus I) \cup (-r_0, -1/r_0)). \]

Let \( \tilde{P}(G) \) be the set defined in (2.5). By (2.19), \( \hat{\Omega}_I \) is a simply connected domain satisfying
\[ \hat{\Omega}_I \cap (\tilde{P}(G) \setminus \mathbb{T}) = \emptyset \quad \text{and} \quad H_d(I) \subset \tilde{H}_d(I) \subset \hat{\Omega}_I \subset \Omega_I \subset e\Omega_I. \quad (2.22) \]
For a subset $Z$ of $\mathbb{C}$, we denote $-Z := \{ -z : z \in Z \}$. Let $D_r(a)$ be the Euclidean disk centered at $a \in \mathbb{C}$ with radius $r > 0$ and we denote $D_r := D_r(0)$. By a direct calculation, we have

$$\varphi_1(\Omega_I) = \mathbb{C} \setminus \left( (\mathbb{R} \setminus (-1, 1)) \cup \overline{D} \cup (-\overline{D}) \cup L \cup (-L) \right),$$

where $D = D_{\frac{r_0 + 1}{2}}(\frac{r_0}{2} + 1) i$ and $L = [R, +\infty) i$ with

$$R = \frac{1}{\tan(\frac{\pi}{4})} r_0 + 1 \quad \text{and} \quad r = \frac{1}{\tan(\frac{\pi}{4})} r_0 - 1.$$ 

Therefore, there exist constants $C_1, C_2 > 0$ such that

$$\varphi_3 \circ \varphi_2(\overline{D} \cup (-\overline{D}) \cup L \cup (-L)) \subset \varphi_3(D_{C_1}(1) \setminus \mathbb{R}) \subset D_{C_2}(i) \cup D_{C_2}(1) \setminus (-1).$$

Decreasing the length of $I$ if necessary, there exists $C_3 > 0$ such that

$$\varphi(\Omega_I) \supset U_0 \cup U_1 \cup U_\infty,$$ 

where $U_0 := D_{\frac{1}{2}} \cap H_+, U_\infty := H_+ \setminus \overline{D}_2$ and

$$U_1 := \{ re^{i\theta} : r > 0 \text{ and } |\theta| < \frac{\pi}{2} - C_3 |I| \} \quad \text{with } C_3 |I| < \frac{\beta}{4}.$$

Note that $d = \log \cot(\frac{\beta}{4})$ with $\beta \in (0, \pi)$. By (2.23), if $|I|$ is small, then for every $z \in \partial H_d(I) \setminus \partial I$ and $w = \varphi(z)$, we have

$$\text{dist}_{\tilde{\Omega}_I}(z, I) \leq \text{dist}_{U_1}(w, \mathbb{R}_+) = \log \cot \left( \frac{\beta}{4} \right),$$

where

$$\hat{\beta} = 2 \left( \frac{\pi}{2} - \frac{\pi}{2} - C_3 |I| \cdot \left( \frac{\pi}{2} - \frac{\beta}{2} \right) \right) = \beta - \eta_0 |I| \quad \text{with } \eta_0 := \frac{2C_3(\pi - \beta)}{\pi - 2C_3 |I|}.$$ 

If $|I|$ is small, then there exists $C'_0 = C'_0(d) > 0$ such that for all $z \in \tilde{H}_d(I)$,

$$\text{dist}_{\tilde{\Omega}_I}(z, I) \leq d + \log \frac{1 + \tan \left( \frac{\eta_0 |I|}{2} \tan \left( \frac{\beta}{4} \right) \right)}{1 - \tan \left( \frac{\eta_0 |I|}{4} \right) / \tan \left( \frac{\beta}{4} \right)} < d + C'_0 |I|. \quad (2.24)$$

Decreasing the length of $I$ if necessary, we assume that $C'_0 |I| < 1$ and hence

$$d' := d + C'_0 |I| < d + 1.$$ 

Denote

$$\tilde{H}'_d(I) := \{ z \in \tilde{\Omega}_I : \text{dist}_{\tilde{\Omega}_I}(z, I) < d' \}.$$ 

By (2.22) and (2.24), we have

$$H_d(I) \subset \tilde{H}_d(I) \setminus \overline{B} \subset \tilde{H}'_d(I) \setminus \overline{B} \subset H_d(I). \quad (2.25)$$

(b) Suppose $I \cap \left( \bigcup_{k=0}^{n-1} G^k(C\mathbb{V}) \right) = \emptyset$ for some $n \geq 1$. Let $\tilde{Q}$ be the connected component of $G^{-n}(\tilde{\Omega}_I)$ containing $J$, where $J \subset \mathbb{T}$ is the unique arc satisfying $G^n(J) = I$. We claim that $\tilde{Q}$ is simply connected and that $G^n : \tilde{Q} \to \tilde{\Omega}_I$ is a conformal map.

In fact, let $\tilde{Q}_1^+ \subset \tilde{\Omega}_I \setminus \overline{B}$ be the unique component of $G^{-1}(\tilde{\Omega}_I \setminus \overline{B})$ such that $J_1 \subset \partial \tilde{Q}_1^+$, where $J_1 := G^{n-1}(J)$. Since $(\tilde{\Omega}_I \setminus \overline{B}) \cap \mathbb{V}(G) = \emptyset$ and $\tilde{\Omega}_I \setminus \overline{B}$ is simply connected, it follows that $G : \tilde{Q}_1^+ \to \tilde{\Omega}_I \setminus \overline{B}$ is conformal. Let $\tilde{Q}_1^- \subset \overline{B}$ be the unique component of $G^{-1}(\tilde{\Omega}_I \cap \overline{B})$ such that $J_1 \subset \partial \tilde{Q}_1^-$. Then similarly, $G : \tilde{Q}_1^- \to \tilde{\Omega}_I \cap \overline{B}$ is also conformal. Let $\tilde{Q}_1$ be the component of $G^{-1}(\tilde{\Omega}_I)$ containing $J_1$. We have $\tilde{Q}_1 \supset \tilde{Q}_1^+ \cup \tilde{Q}_1^- \cup J_1$. By (2.15), we have $\mathcal{H}(I \cap C\mathbb{V}) \leq 1$. By Riemann-Hurwitz’s formula, $\tilde{Q}_1$ is simply
connected and contains at most one critical point (without counting multiplicity). If \(c \in \hat{Q}_1 \setminus \hat{T}\) is a critical point, then its symmetric image \(c' = 1/\tau\) about \(\hat{T}\) is also a critical point which is contained in \(\hat{Q}_1\). This is a contradiction. Since \(I \cap CV = \emptyset\), it follows that \(Q_1\) contains no critical point and \(G : \hat{Q}_1 \to \hat{\Omega}_I\) is a homeomorphism. In particular, we have \(\hat{Q}_1 = \hat{Q}_1^+ \cup \hat{Q}_1^- \cup J_1\) and hence \(G : \hat{Q}_1 \to \hat{\Omega}_I\) is conformal. \(\text{By} (2.15), \|i(I \cap G^k(CV))\| \leq 1\) for every \(0 \leq k \leq n - 1\). Since \((\hat{Q}_1 \setminus \hat{T}) \cap \hat{\mathcal{P}}(G) = \emptyset\), it follows that \(G^n : \hat{Q} \to \hat{\Omega}_I\) is conformal by an inductive argument.

Let \(Q'\) be the component of \(G^{-n} (\hat{H}_d(I)) \) containing \(J\). Since \(\hat{H}_d(I) \subset \hat{\Omega}_I\), we have \(Q' \subset \hat{Q}\). Note that \(G^n : \hat{Q} \to \hat{\Omega}_I\) is conformal. By Schwarz-Pick Lemma, we have

\[
Q' = \{z \in \hat{Q} : \text{dist}_{\hat{\mathcal{P}}}(z, J) < d'\}.
\]

By \(2.25\), we have \(H_d(I) \subset \hat{H}_d(I) \setminus \overline{D}\). Then for the component \(Q \subset \hat{C} \setminus \overline{D}\) of \(G^{-n}(H_d(I))\) satisfying \(\partial Q \cap \hat{T} = J\), we have \(Q \subset Q'\). Since \(\hat{Q} \subset \Omega_J\), where \(\Omega_J\) is defined similarly as \(\Omega_I\) in \(2.12\), it follows that \(Q' \subset H_d(I)\) and hence \(Q \subset H_d(J)\).

Based on the proof of Lemma 2.7 (see also Lemma 2.6), we show that the angle between \(\partial H_d(I) \setminus I\) and \(\hat{T} \setminus I\) is well-defined and equal to the angle between \(\partial \hat{H}_d(I)\) and \(\hat{T} \setminus I\). Indeed, we verify this by comparing the hyperbolic metrics of \(\hat{\Omega}_I\) with \(\hat{\Omega}_I\) near the end points of \(I\). For \(U_0 = D_{1/2} \cap \mathbb{H}^+_I\), the following map is conformal:

\[
\psi(w) = \frac{1}{w} \left(\frac{2w - i}{2w + i}\right)^2 : U_0 \to \mathbb{H}^+_I.
\]

Note that \(\rho_{\mathbb{H}^+_I}(w) = \frac{1}{1 - w^2}\). A direct calculation shows that for every \(w = x + yi \in D_{1/2} \cap \mathbb{H}^+_I\), we have

\[
\frac{\rho_{\hat{H}_I}(w)}{\rho_{\hat{H}_I}(w)} = \frac{x|\psi(w)|}{\operatorname{Re} \psi(w)} = \frac{4x \cdot |2w - i|}{\operatorname{Re} \left(\frac{2w - i}{2w + i}\right)} \frac{4w}{\operatorname{Im} \left(\frac{2w - i}{2w + i}\right)}
\]

\[
= \left(1 + \frac{x^2}{1 - y^2 - x^2}\right) \left(1 + \frac{x^2}{(1 + y)^2}\right) \left(1 + \frac{x^2}{(2 - y)^2}\right) = 1 + O(x^2).
\]

For \(U_\infty = \mathbb{H}^+_I \setminus \overline{D}_2\) and \(w \in \mathbb{H}^+_I \setminus \overline{D}_4\), we have

\[
\frac{\rho_{\hat{H}_I}(w)}{\rho_{\hat{H}_I}(w)} = \frac{\rho_{\hat{H}_I}(w)}{\rho_{\hat{H}_I}(w)} = 1 + O(\left(\operatorname{Re} \frac{1}{w}\right)^2).
\]

By \(2.21\), there exists \(C_1 \in (0, 1]\) such that for any \(z \in \hat{\Omega}_I\) satisfying \(\operatorname{dist}_\mathbb{C}(z, \partial I) = \kappa |I|\) with \(\kappa \in (0, C_1]\), then \(z \in \Omega_I \subset \Omega_I \subset \hat{\Omega}_I\) and \(w = \varphi(z) \in D_{1/2} \cap \mathbb{H}^+_I\) or \(w \in \mathbb{H}^+_I \setminus \overline{D}_4\).

By \(2.23\), the hyperbolic metrics satisfy

\[
1 \leq \frac{\rho_{\hat{H}_I}(z)}{\rho_{\hat{H}_I}(z)} \leq \frac{\rho_{\hat{H}_I}(z)}{\rho_{\hat{H}_I}(z)} = \frac{\rho_{\mathbb{H}^+_I}(w)}{\rho_{\mathbb{H}^+_I}(w)} = 1 + O(\kappa).
\]

This implies that the angle \(\beta = \beta(d) \in (0, \pi)\) between \(\partial H_d(I) \setminus I\) and \(\hat{T} \setminus I\) is well-defined and equal to the angle between \(\partial \hat{H}_d(I)\) and \(\hat{T} \setminus I\). Hence we have (compare \(2.18\) and see Figure 1) \(H^\beta(I) := H_d(I)\), where \(\log \cot \left(\frac{\beta}{2}\right) = d\).
For two different points $z_1, z_2 \in \mathbb{C}$, we use $[z_1, z_2]$ to denote the segment in $\mathbb{C}$ connecting $z_1$ with $z_2$ (This notation is used to distinguish from arcs on the circle). For $0 < \omega < 1$ and an arc $I \subset \mathbb{T}$, we denote

$$S_\omega(I) := \{ z \in \mathbb{C} : 1 < |z| < 1 + \omega|I| \text{ and } |0, z| \cap I \neq \emptyset \}.$$  

**Lemma 2.8.** For any given $n \geq 1$, there exist $\tilde{r}_2 = \tilde{r}_2(n) > 0$, $0 < \omega = \omega(n) < 1$ and $0 < \beta_0 = \beta_0(n) < \pi$ such that for any $n - 1$ points $x_1 < x_2 < \cdots < x_{n-1}$ in $I = (a, b) = (x_0, x_n) \subset \mathbb{T}$ with $|I| < \tilde{r}_2$, the following two statements hold:

- $H^{\beta_0}\big((x_i, x_{i+1})\big)$ is a Jordan disk for all $0 \leq i \leq n - 1$; and
- $S_\omega(I) \cap \mathbb{T} \subset \bigcup_{i=0}^{n-1} H^{\beta_0}\big((x_i, x_{i+1})\big)$.

This lemma implies that $\tilde{r}_2$, $\omega$, and $\beta_0$ depend on the number (i.e., $n + 1$) of the points $x_0, x_1, \cdots, x_n$ but not on their position.

**Proof.** For any $\beta \in (0, \pi)$, by [2.18] and [2.27], we write $H_d(I) = H^{\beta}(I)$ and $\widetilde{H}_d(I) = \widetilde{H}^{\beta}(I)$ for the subarc $I \subset \mathbb{T}$ with small width, where

$$d = \tilde{d} = \log \cot \left(\frac{\beta}{2}\right).$$

For given $\beta_0 \in (0, \pi)$, by Lemmas 2.6 and 2.7(a), there exists a small $r_2 = r_2(\beta_0) > 0$ such that if $|I| < r_2$, then

- $\widetilde{H}^{\beta}(I)$ and $H^{\beta}(I)$ are Jordan disks, for any $\beta \in [\beta_0, \pi]$;
- $\widetilde{H}_d(I)$ and $H_d(I)$ are Jordan disks, for any $d \in (0, d_0 + 1]$, where $d_0 = \log \cot \left(\frac{\beta_0}{2}\right)$;
- $\widetilde{H}_d(I) \setminus \overline{D} \subset H_d(I)$, where $d \in (0, d_0]$ and $d' = d + C_0'|I| < d + 1$.

Therefore, $\widetilde{H}^{\beta}(I) \setminus \overline{D} \subset H^{\beta'}(I)$, where $\beta' \in (0, \pi)$ satisfies $\log \cot \left(\frac{\beta'}{2}\right) = d'$. Without loss of generality we assume that $r_2 > 0$ is small such that for any $d \in (0, d_0]$ and any arc $I \subset \mathbb{T}$ with $|I| < r_2$,

$$4 \arctan \left(e^{-(d+C_0'|I|)}\right) > 3 \arctan \left(e^{-d}\right).$$

This implies that $\beta' > \frac{3}{2}\beta$. Therefore, we have

$$\widetilde{H}^{\beta}(I) \setminus \overline{D} \subset H^{\beta'}(I) \subset H^{\frac{3\beta}{2}}(I).$$

If $n = 1$, we take $\beta_1 = \frac{\pi}{8}$. There exist small $r'_1 > 0$ and $0 < \omega_1 < 1$ such that if $|I| < r'_1$, then $H^{\beta_1}(I)$ is a Jordan disk and

$$S_{\omega_1}(I) \cap \mathbb{T} \subset \widetilde{H}^{\frac{3\beta_1}{4}}(I) \setminus \overline{D} \subset H^{\beta_1}(I).$$

Inductively, we assume that for every $1 \leq k \leq n - 1$ with $n \geq 2$, there exist $r'_k > 0$, $0 < \omega_k < 1$ and $0 < \beta_k < \pi$ such that for any $k - 1$ points $x'_1 < x'_2 < \cdots < x'_{k-1}$ in $I = (a, b) = (x'_0, x'_k) \subset \mathbb{T}$ with $|I| < r'_k$, the following two statements hold:

- $H^{\beta_k}\big((x'_i, x'_{i+1})\big)$ is a Jordan disk for all $0 \leq i \leq k - 1$; and
- $S_{\omega_k}(I) \cap \mathbb{T} \subset \bigcup_{i=0}^{k-1} H^{\beta_k}\big((x'_i, x'_{i+1})\big)$.

Note that any $n - 1$ points $x_1 < x_2 < \cdots < x_{n-1}$ in $I = (a, b) = (x_0, x_n) \subset \mathbb{T}$ divides $I$ into $n$ subarcs and there exists one subarc $I_j = (x_j, x_{j+1})$ with $0 \leq j \leq n - 1$ satisfying $|I_j| \geq |I|/n$. Hence there exist positive numbers

$$r'_n \leq \min_{1 \leq k \leq n-1} \{r'_k\}, \quad \omega_n \leq \min_{1 \leq k \leq n-1} \{\omega_k\} \quad \text{and} \quad \beta_n \leq \min_{1 \leq k \leq n-1} \{\beta_k\}$$

depending on $n$ such that for any $n - 1$ points $x_1 < x_2 < \cdots < x_{n-1}$ in $I$ with $|I| < r'_n$, then

- $H^{\beta_n}\big((x_i, x_{i+1})\big)$ is a Jordan disk for all $0 \leq i \leq n - 1$; and
Lemma 2.8 will be used in the proof of Proposition 3.7 and the corresponding number \( n \geq 1 \) there is an integer determined by \( G \).

2.4. Contraction of \( G^{-1} \). Let \( z_0, z_1, z_2 \in \mathbb{C} \setminus \mathbb{D} \) be three different points. In this paper we assume that the angle \( \angle z_1z_0z_2 \) is measured in the logarithmic plane of \( G \). Specifically, when we write \( \beta = \angle z_1z_0z_2 \in [0, 2\pi] \), it means that there exists \( x \in \mathbb{R} \) such that the half-strip \( \{ w \in \mathbb{C} : x < \Re w < x + 1 \) and \( \Im w \leq 0 \} \) contains \( z'_0, z'_1, z'_2 \) with \( z_i = \text{Exp}(z'_i) \) for \( 1 \leq i \leq 3 \) and

\[
\beta = \arg \left( \frac{z'_2 - z'_0}{z'_1 - z'_0} \right) \in [0, 2\pi). \tag{2.28}
\]

Let \( \mathcal{P}(G) = \phi(\mathcal{P}(f) \setminus \Delta) \) be the postcritical set defined in (2.6). Note that \( \mathcal{P}(G) \subset \hat{\mathcal{P}}(G) \). By (2.11), the following set is non-empty:

\[
\Omega := \text{The unique component of } \hat{\mathcal{C}} \setminus (\mathcal{P}(G) \cup \mathbb{D}) \text{ with } T \subset \partial \Omega. \tag{2.29}
\]

In this subsection, we specify the places where \( G^{-1} \) contracts the hyperbolic metric \( \rho_{\Omega}(z)|dz| \) in \( \Omega \) strictly. By the definition of \( H_d(I) \) in (2.13), we have \( H_d(I) \subset \Omega \). Let \( CV \) and \( \tilde{r}_0 > 0 \) be defined in (2.14) and (2.15) respectively. The principle of the following result is observed essentially in [Pet96, Lemma 1.11].

Lemma 2.9. Let \( V \) be a Jordan disk in \( \Omega \) such that \( \overline{T} = \overline{V} \cap \mathbb{T} \), where \( I \) is an arc with \( \sigma(I) < \tilde{r}_0 \), \( \partial I \cap CV = \{ v \} \) and the angle between \( \partial V \setminus I \) and \( T \setminus I \) at \( v \) is well-defined and positive. Then there exists \( 0 < \mu = \mu(V) < 1 \) such that for every component \( U \) of \( G^{-1}(V) \cap \Omega \) with \( \tilde{z}(U \cap \mathbb{T}) = 1 \) and for any \( z \in U \), we have

\[
\rho_{\Omega}(z) \leq \mu \rho_{\Omega}(G(z))|G'(z)|. \tag{2.30}
\]

In particular, if \( V = H_d(I) \) (resp. \( V = H^\beta(I) \)) is simply connected for some \( d > 0 \) (resp. for some \( 0 < \beta < \pi \)), then \( \mu \) depends only on \( d \) (resp. \( \beta \)).

Proof. Since \( \sigma(I) < \tilde{r}_0 \), (2.15) shows that \( V \cap CV = \emptyset \) and \( \tilde{z}(\partial V \cap CV) = 1 \). By Lemma 2.1 every component \( U \) of \( G^{-1}(V) \) in \( \hat{\mathcal{C}} \setminus \mathbb{D} \) is a Jordan disk and \( G : U \rightarrow V \) is conformal. Let \( c \in \text{Crit}(G) \cap \mathbb{T} \) be the unique critical point such that \( G(c) = v \), where \( G \) has local degree \( 2\ell - 1 \) at \( c \) with \( \ell \geq 2 \). Without loss of generality, we assume that \( I = (a, v) \), i.e., the critical value \( v \) is the right end point of \( I \). There exist \( \ell \) components of \( G^{-1}(V) \) in \( \hat{\mathcal{C}} \setminus \mathbb{D} \) which contain \( c \) as their boundary point. We label them by \( U_1, \cdots, U_\ell \) clockwise such that \( U_i \cap \mathbb{T} = [a',c] \) with \( G(a') = a \) and \( U_i \cap \mathbb{T} = \{ c \} \) for \( 1 \leq i \leq \ell - 1 \). See Figure 3.

Let \( U := U_i \) for some \( 1 \leq i \leq \ell - 1 \). Since \( G \) is a quasi-regular map satisfying \( G(T) = T \), there exists a small Jordan disk \( W \) containing \( c \), such that \( G \) can be written as \( G = g \circ \varphi \), where

- \( \varphi : W \rightarrow \varphi(W) \) is quasiconformal and \( g : \varphi(W) \rightarrow G(W) \) is a proper holomorphic map having exactly one critical value \( v \); and
- \( W \cap \mathbb{T} \) is an arc and \( \varphi(W \cap \mathbb{T}) \subset \mathbb{T} \).
Note that the angle $\beta > 0$ between $\partial V \setminus I$ and $\mathbb{T} \setminus I$ at $v$ is well-defined, i.e., as $z \in \partial V \setminus I$ and $z \to v$, the limit of $\angle avz$ exists and equals to $\pi - \beta$. Since the bounded turning property is preserved by quasiconformal mappings (see [LV73, p. 100]), there exists a small number $0 < \beta' < \pi/3$ depending only on $V$ (hence also on $\beta$), such that for all $z \in U \cap W$, we have

$$\angle a'cz \in (\beta', \pi - \beta').$$  \hfill (2.30)

Let $\Omega'$ be the component of $\mathbb{C} \setminus (\mathbb{T} \cup G^{-1}(\mathcal{P}(G) \cup \mathbb{T}))$ containing $U$. Then $\Omega'$ is a proper subset of $\Omega$ and $G : \Omega' \to \Omega$ is a holomorphic covering map (see (2.1)). By Schwarz-Pick Lemma,

$$\rho_{\Omega}(G(z))|G'(z)| = \rho_{\Omega'}(z), \quad \text{for all } z \in \Omega'.$$

Except a small neighborhood of $c$, the rest part of $U$ is compactly contained in $\Omega$. To prove this lemma, it suffices to prove that for all $z \in U \cap W$, we have

$$\frac{\rho_{\Omega'}(z)}{\rho_{\Omega}(z)} \geq \frac{1}{\mu'} > 1,$$

where $0 < \mu' < 1$ is a number depending only on $\beta'$. Based on (2.26) and (2.30), by changing coordinates, we may assume that $\Omega = \{z \in \mathbb{C} : \text{Re} \ z > 0\}$ and $\Omega' = \{z \in \mathbb{C} : |\arg z| < (\pi - \beta')/2\}$. Then a direct calculation shows that one can take

$$\mu' := (\pi - \beta')/\pi \in (0, 1).$$

In particular, suppose $V = H_d(I) = H_{\beta}(I)$ for some $d > 0$ and $0 < \beta < \pi$. Then the angle between $\partial V \setminus I$ and $\mathbb{T} \setminus I$ at $v$ is $\beta$ and (2.30) holds for all $z \in U$ for a constants $\beta' > 0$ depending only on $\beta$ (or $d$). The proof is complete. \hfill \Box

Let $\text{Crit}(G) \cap \mathbb{T}$ be the set of critical points of $G$ on $\mathbb{T}$. Counting without multiplicity, we denote

$$t_0 := \sharp(\text{Crit}(G) \cap \mathbb{T}) = \sharp CV \geq 1.$$  \hfill (2.31)

For each $c_j = e^{i\theta_j} \in \text{Crit}(G) \cap \mathbb{T}$ with $\theta_j \in \mathbb{R}$, where $1 \leq j \leq t_0$, there exist finitely many connected components $W_{j,1}, \cdots, W_{j,t(j)}$ of $G^{-1}(\mathbb{D}) \setminus \mathbb{T}$ attaching to $c_j$. Similar to the proof of (2.30), we conclude that there exist two small numbers $\hat{r} > 0$ and $\hat{\beta} \in (0, \frac{\pi}{2})$ such that

$$\left( \bigcup_{i=1}^{t(j)} W_{j,i} \right) \cap \{z \in \mathbb{C} : |z - c_j| < \hat{r} \}$$

$$\subset \{ z = c_j + re^{i\theta} : 0 < r < \hat{r} \text{ and } |\theta - \theta_j| < \frac{\pi}{2} - \hat{\beta} \}. $$
In the rest of this paper, we always assume that $d > 1$ is large (based on Lemma 2.8) and the number $d$ will be fixed after Proposition 3.7 such that

$$H_d(I) = H^d(I) \quad \text{with} \quad 0 < \beta = \beta(d) \leq \frac{\delta}{\ell},$$

(2.32)

where $I$ is any subarc of $T$ satisfying $|I| < r_2$ and $r_2 := r_2(d) > 0$ is the number introduced in Lemma 2.7.

3. Admissible sequences and contraction regions

For an arc $I \subset T$ and a number $\kappa > 0$, we use $\kappa I$ to denote the arc in $T$ which has dynamical length $\kappa \sigma(I)$ and has the same middle point as $I$ (with respect to the dynamical length). Let $\mathcal{C}V$ and $\mathcal{C}V'$ be the finite subsets of $T$ defined in (2.14). Let $I = (a, b) \subset T$ be an arc with $\sigma(I) < r_2/2$. Then $2((2T) \cap \mathcal{C}V) \leq 1$ and $H_d(2I)$ is a Jordan disk for all $\tilde{d} \in (0, d + 1]$ by Lemma 2.6.

3.1. Non-critical predecessors. Suppose $I \cap \mathcal{C}V = \emptyset$. There exists a unique arc $J \subset T$ such that $G(J) = I$. We call $J$ a non-critical predecessor of $I$ and $I \to J$ a non-critical pullback. The pullback $I \to J$ induces a branch $\Phi$ of $G^{-1}$ such that $\Phi(I) = J$. We say that $\Phi$ is the inverse branch of $G$ associated to the pullback $I \to J$.

For the inverse branch $\Phi$, there exist a pair of critical points $c^-, c^+ \in T$ and a pair of critical values $v^-, v^+ \in \mathcal{C}V$ such that

- $I \subset (v^-, v^+) \quad \text{and} \quad J \subset (c^-, c^+);$
- $(c^-, c^+)$ does not contain any critical point of $G$; and
- $\Phi((v^-, v^+)) = (c^-, c^+).

We call $v^-$ and $v^+$ the two singular points associated to $\Phi$. In the following we shall see that this will not cause ambiguity when $T$ contains exactly one critical point of $G$.

Suppose $I \cap \left( \bigcup_{k=0}^{n-1} G^k(\mathcal{C}V) \right) = \emptyset$ for some $n \geq 1$. Let $J \subset T$ be the unique arc such that $G^n(J) = I$. Then $G^k(J)$ is a non-critical predecessor of $G^{k+1}(J)$ for all $0 \leq k \leq n - 1$. By Lemma 2.7(b), for every $0 \leq k \leq n - 1$, the unique component $Q \subset \widehat{C} \setminus \widehat{B}$ of $G^{-(n-k)}(H_d(I))$ with $\partial Q \cap T = G^k(J)$ satisfies $Q \subset H_{d'}(G^k(J))$, where $d' = d + C_0|I| < d + 1$.

3.2. Critical predecessors. Let $0 < \delta < \eta < \frac{1}{2}$ be two numbers which will be specified later.

Case (1). Suppose $\mathcal{C}V \cap I \neq \emptyset$. Then $\mathcal{C}V \cap ((1+2\delta)I \setminus I) = \emptyset$ since $\mathcal{C}V(2T) \cap \mathcal{C}V \leq 1$. Assume that $I$ contains a critical value $v = G(c)$ for some $c \in \text{Crit}(G) \cap T$ with local degree $2\ell - 1$, where $\ell \geq 2$.

Let $a', b' \in T$ such that $G(a') = a$ and $G(b') = b$. For any $\tilde{d} \in (0, d + 1]$, there are exactly $\ell$ components of $G^{-1}(H_{d'}(I))$ in $\widehat{C} \setminus \widehat{B}$, say $Q_1, \ldots, Q_{\ell}$, which attach to $c$ clockwise such that $\overline{Q_i} \cap T = [c, b']$ and $\overline{Q_i} \cap T = [a', c]$.

For each $1 \leq i \leq \ell$, we associate an arc $J \subset T$ to $I$ for each branch of $G^{-1}$ sending $H_{d'}(I)$ to $Q_i$. In particular, if we consider $Q_1$, then we take $J$ such that $(c, b') \subset J$. If we consider $Q_{\ell}$, then we take $J$ such that $(a', c) \subset J$. If we consider $Q_i$ with $2 \leq i \leq \ell - 1$, then we take $J$ such that $(c, b') \subset J$ or $(a', c) \subset J$. In all situations, $c$ is an end point of $J$. To fix the idea, we only consider the situation $(c, b') \subset J$ and the rest cases can be treated in the same way. Then we have following two types (see Figure 4):
(i) If \( CV \cap I \neq \emptyset \) and \( \sigma((v, b)) < (1 - \eta)\sigma(I) \), we take \( J = (c, x) \) such that \( \sigma(J) = (1 - \eta/3)\sigma(I) \); and

(ii) If \( CV \cap I \neq \emptyset \) and \( \sigma((v, b)) \geq (1 - \eta)\sigma(I) \), we take \( J = (c, x) \) such that \( \sigma(J) = (1 + \delta)\sigma(I) \).

**Case (2).** Suppose \( CV \cap I = \emptyset \) and \( CV \cap ((1 + 2\delta)I \setminus I) \neq \emptyset \). Note that in this case one may consider the non-critical predecessor of \( I \) as in \( \S 3.1 \). However, we can also define critical predecessors as following.

Let \( v = G(c) \) be the unique critical value in \( CV \cap ((1 + 2\delta)I \setminus I) \) for \( c \in \text{Crit}(G) \cap \mathbb{T} \) of local degree \( 2\ell - 1 \) with \( \ell \geq 2 \). To fix the idea, we only consider the situation that \( v \) is on the right of \( I \). Denote \( \tilde{I} := (1 + 2\delta)I = (\tilde{a}, \tilde{b}) \) and let \( \tilde{a}', \tilde{b}' \in \mathbb{T} \) be two points such that \( G(\tilde{a}') = \tilde{a} \) and \( G(\tilde{b}') = \tilde{b} \). For any \( \tilde{d} \in (0, d + 1] \), there are \( \ell \) components \( \tilde{Q}_1, \ldots, \tilde{Q}_\ell \) of \( G^{-1}(H_{d}(\tilde{I})) \) in \( \tilde{C} \setminus \tilde{D} \) which attach to \( c \) clockwise.

We use \( Q_i \) to denote the component of \( G^{-1}(H_{d}(I)) \) which is contained in \( \tilde{Q}_i \) for every \( 1 \leq i \leq \ell \). Note that \( Q_1, \ldots, Q_{\ell-1} \) do not attach to the unit circle. For each \( 1 \leq i \leq \ell - 1 \), we associate an arc \( J \) to \( I \) such that \( c \) is an end point of \( J \). Specifically, we have the following third type (see Figure 5):

(iii) If \( CV \cap ((1 + 2\delta)I \setminus I) \neq \emptyset \), we take \( J = (c, x) \) such that \( \sigma(J) = \sigma(I)/2 \).

**Figure 4:** A sketch of critical pullbacks under \( G \) of types (i) and (ii). From the arc \( I = (a, b) \) we obtain a new arc \( J = (c, x) \).

**Figure 5:** A sketch of critical pullbacks \( Q_1, \ldots, Q_{\ell-1} \) under \( G \) of type (iii), which are colored yellow. A non-critical pullback \( Q_\ell \) is colored cyan. From the arc \( I = (a, b) \) we obtain a new arc \( J = (c, x) \).

The arcs \( J \)'s defined in above two cases are called the **critical predecessors** of \( I \) of type (i), (ii) and (iii) respectively. We call \( I \to J \) a **critical pullback**.

If we consider the situation that \((a', c) \subset J \) in Case (1), or the situation that \( v \) is on the left of \( I \) in Case (2), then we label the components \( Q_1, \ldots, Q_\ell \) of \( G^{-1}(H_{d}(I)) \) such that they locate around \( c \) anticlockwise.
In Cases (1) and (2), $G^{-1}$ induces $\ell$ branches $\{\Phi^{(i)} : 1 \leq i \leq \ell\}$ of $G^{-1}$ near $v$ such that $\Phi^{(i)}(H_d(I)) = Q_i$. We say that $\{\Phi^{(i)} : 1 \leq i \leq \ell - 1\}$ are the inverse branches of $G$ associated to the pullback $I \to J$.

For the inverse branch $\Phi^{(i)}$, there exist a pair of critical points $c^-, c^+ \in \mathbb{T}$ and a pair of critical values $v^-, v^+ \in CV$ such that

- $J \subset (c^-, c^+)$. In particular, if $J = (c, x)$, then $c^- = c$ and $v^- = v$, and if $J = (x, c)$, then $c^+ = c$ and $v^+ = v$;
- $(c^-, c^+)$ does not contain any critical point of $G$; and
- $\Phi^{(i)}((v^-, v^+)) = (c^-, c^+)$. We call $v^-$ and $v^+$ the two singular points associated to $\Phi^{(i)}$.

For every $\Phi^{(i)}$ with $2 \leq i \leq \ell - 1$, we call the critical value $v^0 = v = G(c^0) \in (1+2\delta)I$ with $c^0 = c$ the singular point associated to $\Phi^{(i)}$.

Let $\Omega$ be the domain defined in (2.29). Let $J$ be a critical predecessor of $I$ and $Q_1, \ldots, Q_{\ell-1}$ be the components of $G^{-1}(H_d(I))$ of type (i), (ii) or (iii) obtained above.

**Lemma 3.1.** There exist $r_3 = r_3(\delta, \eta, d) \in (0, r_2]$ and $K_0 = K_0(\delta, \eta, d) > 0$ such that if $\gamma(I) < r_3$, then there exists a Jordan disk $B_i \subset \Omega$ with $\operatorname{diam}_\Omega(B_i) < K_0$ such that $Q_i \subset H_d(J) \cup B_i$ for every $1 \leq i \leq \ell - 1$.

**Proof.** In a small neighborhood $W$ of $c$, the quasi-regular map $G$ can be written as $G = g \circ \varphi$, where $\varphi : W \to \varphi(W)$ is quasiconformal and $g : \varphi(W) \to G(W)$ is a proper holomorphic map having exactly one critical value $v$. Without loss of generality, we assume that

- $W, \varphi(W)$ and $G(W)$ are Jordan disks;
- $W \cap \mathbb{T}$ is an arc and $\varphi(W \cap \mathbb{T}) \subset \mathbb{T}$; and
- $H_{d+1}(2I) \subset G(W)$.

We only consider the situation that $Q_i$, where $1 \leq i \leq \ell - 1$, are obtained from Case (1) (i.e., type (i) or (ii)) since the proof for Case (2) (i.e., type (iii)) is completely similar. For the above decomposition $G = g \circ \varphi$, let $\tilde{c}, \tilde{b}, \tilde{x} \in \mathbb{T}$ and $\tilde{Q}_i \subset \mathbb{C} \setminus \overline{D}$, respectively, be the preimages of $v, b, G(x)$ and $H_d(I)$ under the holomorphic map $g$ such that $\tilde{Q}_i = \varphi(\tilde{Q}_i)$. See Figure 6.

![Figure 6: A sketch of the decomposition $G = g \circ \varphi$, where $g$ is holomorphic and $\varphi$ is quasiconformal. We show that every $Q_i$ is contained in the union of $H_d(J)$ and a Jordan disk $B_i$ whose hyperbolic diameter has a uniform upper bound.](image)

By Lemma 2.2, $|I| = |(a, b)|$ and $|(b, G(x))|$ are $C_0$-comparable, where $C_0 = C_0(\delta, \eta) > 0$ is a constant depending only on $\delta$ and $\eta$. Since $g$ is holomorphic, it follows
that \( \text{diam}_C(Q_i) \) and \(|(c, x)|\) are \( C_1 \)-comparable for a constant \( C_1 = C_1(\delta, \eta, d) > 0 \). For three different points \( z_0, z_1, z_2 \in C \setminus \mathbb{D} \), let \( \angle z_1 z_0 z_2 \) be the angle measured in the logarithmic plane of \( G \) (see (2.23)). Then there exist \( C_2 = C_2(\delta, \eta, d) > 0 \) and two small numbers \( \beta_1 = \beta_1(\delta, \eta) > 0 \), \( \beta_2 = \beta_2(\delta, \eta, d) > 0 \) such that for every \( 1 \leq i \leq \ell - 1 \), we have

\[
\bar{Q}_i \subset \left\{ z \in C \setminus \overline{\mathbb{D}} : \text{dist}_C(z, \bar{c}) < C_2((\bar{c}, \bar{x})), \quad \angle z \bar{c} x \in (0, \pi - \beta_1) \text{ and } \angle z b x \in (\beta_2, \pi) \right\}.
\]

Since \( H_{d+1}(2I) \subset G(W) \), without loss of generality, we assume that \( \{ z \in C \setminus \overline{\mathbb{D}} : \text{dist}_C(z, \bar{c}) \leq C_2((\bar{c}, \bar{x})) \} \subset \varphi(W) \).

Since \( \varphi : W \to \varphi(W) \) is quasiconformal, it follows that \( \text{diam}_C(Q_i) \) and \(|J| = \|(c, x)\)| are \( C_3 \)-comparable for a constant \( C_3 = C_3(\delta, \eta, d) > 0 \). Hence there exist \( C_4 = C_4(\delta, \eta, d) > 0 \) and two smaller numbers \( \beta_1 = \beta_1(\delta, \eta) > 0 \), \( \beta_2 = \beta_2(\delta, \eta, d) > 0 \) such that for every \( 1 \leq i \leq \ell - 1 \), we have

\[
Q_i \subset S := \left\{ z \in C \setminus \overline{\mathbb{D}} : \text{dist}_C(z, c) < C_4((c, x)), \quad \angle z c r \in (0, \pi - \beta_1) \text{ and } \angle z b x \in (\beta_2, \pi) \right\}.
\]

Let \( \beta = \beta(d) \in (0, \frac{\pi}{2}) \) be the angle formed by \( \partial H_d(J) \setminus J \) and \( T \setminus J \) (see (2.32)). Note that \( \beta_2 = \beta_2(\delta, \eta, d) > 0 \) does not depend on \( d > 0 \). Therefore, increasing \( d > 0 \) and decreasing \( \sigma(I) \) if necessary, we assume that \( 0 < \beta < \beta_1/2 \). In particular, there exists a small \( r_3 = r_3(\delta, \eta, d) \in (0, r_2) \) such that if \( \sigma(I) < r_3 \), then \( H_{d+1}(2I) \subset G(W) \) is still satisfied

\[
S \subset \{ z \in C : 1 < |z| < (r_0 - 1)/2 \}, \tag{3.1}
\]

where \( r_0 > 1 \) is introduced in (2.11). Note that \(|(b', x)|\) and \(|J|\) are \( C_5 \)-comparable, where \( C_5 = C_5(\delta, \eta) > 0 \). It follows that for every \( 1 \leq i \leq \ell - 1 \), \( S \setminus (H_d(J) \cap Q_i) \) is contained in a Jordan disk \( B_i \) whose Euclidean diameter is \( C_6 \)-comparable to its Euclidean distance to \( T \), where \( C_6 = C_6(\delta, \eta, d) > 0 \). Hence \( Q_i \setminus H_d(J) \cup B_i \) and by (3.1), we conclude that there exists \( K_0 = K_0(\delta, \eta, d) > 0 \) such that \( \text{diam}_{Q_i}(B_i) < K_0 \) for every \( 1 \leq i \leq \ell - 1 \).

3.3. \( (\delta, \eta) \)-admissible sequence \( \{I_n\} \). Let us start with a small arc \( I_0 \) with \( \sigma(I_0) < r_2/2 \).

Applying the rules in (3.1) and (3.2) inductively, we obtain a sequence of arcs \( \{I_n : 0 \leq n < N'\} \) so that \( I_{n+1} \) is a predecessor of \( I_n \) for all \( 0 \leq n < N' - 1 \), where \( N' = +\infty \) if \( \sigma(I_n) = \infty \) or is finite. The sequence \( \{I_n : 0 \leq n < N'\} \) is called a \( (\delta, \eta) \)-admissible sequence. A priori, there is a possibility that \( \sigma(I_n) \) is large for some \( n \), for instance, \( I_n \) contains two critical values, and thus \( I_{n+1} \) may not be defined and the number \( N' \) is finite.

In the following we prove that, by choosing \( 0 < \delta \ll \eta < \frac{1}{2} \) appropriately, the predecessors can be defined infinitely many times provided that \( I_0 \) is small enough. The basic idea behind the proof is that the critical predecessors of types (i) and (iii) appear more often than the critical predecessors of type (ii). Let

\[
0 \leq n_1 < n_2 < \cdots < n_j < \cdots \tag{3.2}
\]

be the sequence of all the integers so that \( I_{n_{j+1}} \) is a critical predecessor of \( I_{n_j} \). As we mentioned above, the sequence (3.2) may be finite. Let \( t_0 \geq 1 \) be the number of critical points (without counting multiplicity) of \( G \) on \( T \) (see (2.31)).

**Lemma 3.2.** If \( 0 < \delta < \eta < \frac{1}{2} \) are small enough, then for any \( (\delta, \eta) \)-admissible sequence \( \{I_n : 0 \leq n < N'\} \) containing \( I_{n_{j+2t_0}+1} \) for some \( j \geq 1 \), there exists \( 0 \leq i \leq 2t_0 \), such that \( I_{n_{j+i+1}} \) is a critical predecessor of \( I_{n_{j+i}} \) of type (i) or (iii).
Proof. We prove this lemma by contradiction. Assume that $I_{n_{j+i}}$ is a type (ii) critical predecessor of $I_{n_j+i}$ containing the critical value $v_i \in CV$ for all $0 \leq i \leq 2t_0$. Then from $0 < \delta < \eta < \frac{1}{2}$ we have $G^{n_j-i-n_{j+i}}(I_{n_{j+i}}) \subset (1 + 4\eta)I_{n_{j+i}}$ for every $1 \leq i \leq 2t_0$. Thus $G^{n_j-i-n_{j}}(I_{n_{j+i}}) \subset (1 + 4\eta)^{2t_0}I_{n_j}$ for $0 \leq i \leq 2t_0$ and

$$G^{n_j-i-n_{j}}(v_i) \in (1 + 4\eta)^{2t_0}I_{n_j},$$

(3.3)

Let $\lambda_0 > 1$ be the constant guaranteed by Lemma 2.5. We claim that one can take $0 < \delta < \eta < \frac{1}{2}$ small enough such that

$$(1 + 4\eta)^{2t_0} < 1 + \frac{1}{2\lambda_0}$$

(4.4)

and

$$G^k(v) \notin \left(1 - \frac{1}{2\lambda_0}\right)I_{n_j}$$

for any $0 \leq k \leq n_j + 2t_0 - n_j$ and any $v \in CV$. In fact, we get (4.5) by contradiction. Since otherwise, from $G^k(v) \in \left(1 - \frac{1}{2\lambda_0}\right)I_{n_j}$ and that $0 < \delta < \eta < 1/2$ are small, it follows that $v \in (1 - 2\eta)n_j + k$. But this implies that $G_{n_j+i}$ is a critical predecessor of $I_{n_j+i}$ of type (i). This contradicts the assumption and the claim holds.

Note that $G$ has exactly $t_0$ critical points on $T$. By (3.3), there exist a critical value $v \in CV$ and three integers $j \leq j_0 < j_1 < j_2 \leq j + 2t_0$ such that $v = v_{j_i}$ for every $0 \leq i \leq 2$ and

$$G^{n_j-i-n_{j}}(v) \in (1 + 4\eta)^{2t_0}I_{n_j}.$$

Denote $k_0 := n_{j_0} - n_j$. There exist two integers $k_1, k_2 \in [k_0 + 1, n_{j_2} - n_j]$ such that $G^{k_1}(v)$ and $G^{k_2}(v)$ are the first and second returns of $G^{k_0}(v)$ to $(1 + 4\eta)^{2t_0}I_{n_j}$ respectively. By Lemma 2.5 and (4.4), there exists an $0 \leq l \leq 2$ such that $G^{k_l}(v) \in \kappa l I_{n_j}$, where

$$\kappa \leq \left(1 - \frac{2}{\lambda_0}\right)\left(1 + 4\eta\right)^{2t_0} \left(1 - \frac{2}{\lambda_0}\right)\left(1 + \frac{1}{2\lambda_0}\right) < 1 - \frac{1}{2\lambda_0}.$$

This contradicts (4.5). The proof is complete. \hfill \Box

Lemma 3.3. There exist $0 < \delta < \eta < \frac{1}{2}$ such that if $\sigma(I_0) < r_2/4$, then every $(\delta, \eta)$-admissible sequence $\{I_n : 0 \leq n < N\}$ is infinite, i.e., $N' = +\infty$. In particular,

(a) There exists $T_\delta = T_\delta(\eta) > 0$ such that $\sigma(I_s) < \frac{\eta}{2} \sigma(I_s)$ for any $0 \leq s < t < \infty$

if $\{I_n : s \leq n \leq t\}$ contains at least $T_\delta$ critical predecessors; and

(b) $\sigma(I_t) < 2\sigma(I_s)$ for any $0 \leq s < t < \infty$ and $\sigma(I_n) \to 0$ as $n \to \infty$.

Proof. Let $0 < \delta < \eta < \frac{1}{2}$ be small enough such that Lemma 3.2 holds. Moreover, for the chosen small $\eta > 0$, we let $\delta \in (0, \eta)$ smaller such that

$$(1 + \delta)^{2t_0} < \frac{5}{4} \text{ and } (1 + \delta)^{2t_0}(1 - \frac{2}{\eta}) < 1 - \frac{\eta}{4}.$$ 

For any $I_{n_j}$ with $\sigma(I_{n_j}) < r_2/4$ for some $j \geq 1$, by Lemma 3.2 we have

- $\sigma(I_k) = \sigma(I_{n_j+i}) \leq (1 + \delta)\sigma(I_{n_j})$, for each $n_j + 1 \leq k \leq n_j + 1$;
- $\sigma(I_k) \leq (1 + \delta)^{2t_0}\sigma(I_{n_j}) < \frac{5}{4}\sigma(I_{n_j})$, for each $n_j + 1 \leq k \leq n_j + 2t_0$; and
- $\sigma(I_k) = \sigma(I_{n_j+i+2t_0}) \leq (1 + \delta)^{2t_0}(1 - \frac{\eta}{4})\sigma(I_{n_j}) < (1 - \frac{\eta}{4})\sigma(I_{n_j})$, for each $n_j + 2t_0 + 1 \leq k \leq n_j + 2t_0 + 1$.

Therefore, we have

$$\sigma(I_k) < \frac{5}{4}\sigma(I_{n_j}) \text{ for } n_j \leq k \leq n_j + 2t_0 + 1 \text{ and }$$

$$\sigma(I_{n_j + 2t_0 + 1}) < (1 - \frac{\eta}{4})\sigma(I_{n_j}).$$

(3.6)
This implies that every \((\delta, \eta)\)-admissible sequence \(\{I_n : 0 \leq n < N'\}\) is infinite.

Note that \(\sigma(I_k) = \sigma(I_0)\) for each \(0 \leq k \leq n_1\). By (3.6), we have \(\sigma(I_r) < \frac{1}{2} \sigma(I_0) < 2\sigma(I_s)\) for any \(0 \leq s < t < \infty\). Let \(k_0 = k_0(\eta) \geq 1\) be the minimal integer which is larger than \(\log \frac{1}{\log \frac{1}{4}}\). Then \((1 - \frac{3}{4})^{k_0} < \frac{1}{4}\). Note that (3.2) is an infinite sequence since \(G|_T\) is conjugate to the irrational rotation \(R_\alpha(\zeta) = e^{2\pi i \alpha \zeta}\). Set

\[T_1 := (2t_0 + 1)k_0.\]

Still by (3.6), if \(\{I_n : s \leq n \leq t\}\) contains at least \(T_1\) critical predecessors, then

\[\sigma(I_r) < \frac{1}{4}(1 - \frac{3}{4})^{k_0} \sigma(I_s) < \frac{1}{2} \sigma(I_s).\]

Hence \(\sigma(I_n) \to 0\) as \(n \to \infty\). The proof is complete. \(\square\)

From now on, when we talk about a \((\delta, \eta)\)-admissible sequence \(\{I_n\}\), we always assume that \(\sigma(I_0) < \frac{\eta}{2}/4\) and that \(0 < \delta \ll \eta < \frac{1}{2}\) are taken appropriately so that Lemma 3.3 holds.

For a \((\delta, \eta)\)-admissible sequence \(\{I_n\}_{n \geq 0}\) and \(0 \leq k_1 \leq k_2\), we denote

\[\Theta(\{I_n\}, k_1, k_2) := \#\{\text{critical predecessors in } \{I_n : k_1 \leq n \leq k_2\}\}.\]

For \(j \geq 1\), let \(n_j\) be the critical (value) position of \(\{I_n\}_{n \geq 0}\) defined in (3.2).

**Lemma 3.4.** Let \(\{I_n\}_{n \geq 0}\) and \(\{J_n\}_{n \geq 0}\) be two \((\delta, \eta)\)-admissible sequences with \(I_n \subset J_n\) for all \(n \geq 0\). Suppose \(\sigma(I_0) \leq C_1\sigma(I_0)\) for some \(C_1 = C_1(\delta, \eta) \geq 1\). Then for every \(j \geq 1\), there exists \(C'_1 = C'_1(\delta, \eta, j) \geq 1\) such that

\[\Theta(\{J_n\}, 0, n_j) \leq C'_1.\]

**Proof.** By the definition of \(n_j\), we have \(\sigma(I_{n_1}) = \sigma(I_0)\). Let \(i_0\) be the minimal integer such that \(i_0 \geq \frac{\log C_1}{\log 2} + 1\). We claim that

\[\Theta(\{J_n\}, 0, n_1) \leq i_0 T_1.\]

(3.7)

Indeed, otherwise by Lemma 3.3 we have

\[\sigma(J_{n_1}) < 2 \left(\frac{1}{4}\right)^{i_0} \sigma(J_0) \leq \frac{1}{C_1} \sigma(J_0) \leq \sigma(J_0) = \sigma(I_0).\]

This is a contradiction since \(I_{n_1} \subset J_{n_1}\). Hence (3.7) holds. By Lemma 3.3(b), we have \(\sigma(J_{n_1+1}) < 2\sigma(J_0) \leq 2C_1\sigma(I_0)\). By the definition of critical predecessors, we have \(\sigma(I_{n_1+1}) \geq \frac{1}{2} \sigma(I_{n_1})\). Hence

\[\sigma(J_{n_1+1}) < 2C_1 \sigma(I_0) = 2C_1 \sigma(I_{n_1}) \leq 2C_2 \sigma(I_{n_1+1}), \quad \text{where } C_2 := 4C_1.\]

(3.8)

Note that \(\sigma(I_{n_1+1}) = \sigma(I_{n_2}) \leq 2\sigma(I_{n_2+1})\). Let \(i_1\) be the minimal integer such that \(i_1 \geq \frac{\log C_1}{\log 2} + 1\). Then by Lemma 3.3 and (3.8) we have

\[\Theta(\{J_n\}, n_1 + 1, n_2) \leq i_1 T_1\]

and

\[\sigma(J_{n_2+1}) < 2\sigma(J_{n_1+1}) \leq 2C_2 \sigma(I_{n_1+1}) \leq C_3 \sigma(I_{n_2+1}),\]

where \(C_3 := 4C_2 = 4^2C_1\).

Inductively, for every \(l \geq 2\), we have

\[\Theta(\{J_n\}, n_l - 1 + 1, n_l) \leq i_{l-1} T_1,\]

(3.9)

where \(i_{l-1}\) is the minimal integer such that

\[i_{l-1} \geq \frac{\log C_l}{\log 2} + 1 = \frac{\log(4^{l-1}C_1)}{\log 2} + 1 = 2l - 1 + \frac{\log C_l}{\log 2}.\]
Therefore, we have
\[ i_{l-1} \leq 2l + \frac{\log C_1}{\log 2}. \] (3.10)
Denote \( n_0 := -1 \). Combining (3.7), (3.9) and (3.10), we have
\[ \Theta(\{J_n\}, 0, n_j) = \sum_{l=1}^{j} \Theta(\{J_n\}, n_{l-1} + 1, n_l) \leq T_1 \sum_{l=1}^{j} i_{l-1} \leq C'_1, \]
where \( C'_1 = (\frac{\log C_1}{\log 2} + j + 1)jT_1 \). The proof is complete. \( \square \)

**Definition** (Deviation). Let \( \{I_n\}_{n \geq 0} \) be a \((\delta, \eta)\)-admissible sequence. For two integers \( k \geq 1 \) and \( n \geq 0 \), the *deviation* from \( I_{n+k} \) to \( I_n \) is defined as the dynamical length of the shorter arc in \( \mathbb{T} \) between \( G^k(I_{n+k}) \) and \( I_n \).

By definition, if \( G^k(I_{n+k}) \) intersects \( I_n \), then the deviation from \( I_{n+k} \) to \( I_n \) is zero. In particular, if \( I_{j+1} \) is a non-critical predecessor of \( I_j \) for all \( n \leq j \leq n+k-1 \), then the deviation from \( I_{n+k} \) to \( I_n \) is zero. Note that the deviation is caused by critical predecessors. Since \( \sigma(I_{n+k}) < 2\sigma(I_n) \) for all \( k \geq 1 \), as a consequence of Lemma 3.3, we have the following immediate corollary.

**Corollary 3.5.** Let \( \{I_n\}_{n \geq 0} \) be a \((\delta, \eta)\)-admissible sequence. Then for every \( l \geq 1 \), there is a number \( K(\delta, \eta, l) > 1 \) depending only on \( \delta, \eta \) and \( l \), such that for any \( n \geq 0 \), the deviation from \( I_{n+k} \) to \( I_n \) is at most \( K(\delta, \eta, l)\sigma(I_n) \) provided the number of the critical predecessors between \( n \) and \( n+k \) is not more than \( l \).

3.4. Contraction regions associated to \( \{I_n\} \). Consider a \((\delta, \eta)\)-admissible sequence \( \{I_n\}_{n \geq 0} \), where \( 0 < \delta \ll \eta < \frac{1}{2} \) are two fixed small numbers. For \( n \geq 0 \), let \( \Phi_n \) be the inverse branch of \( G \) associated to the pullback \( I_n \rightarrow I_{n+1} \). If \( I_n \rightarrow I_{n+1} \) is a non-critical pullback (see (3.1)) or a critical pullback with \( \Phi_n = \Phi_n^{(1)} \) (see (3.2)), we use \( v_n^- = G(c_n^-) \) and \( v_n^+ = G(c_n^+) \) to denote the singular points associated to \( \Phi_n \), where \( c_n^- \), \( c_n^+ \in \text{Crit}(G) \cap \mathbb{T} \). Otherwise, let \( v_n^0 = G(c_n^0) \) be the singular point associated to \( \Phi_n \), where \( c_n^0 \in \text{Crit}(G) \cap \mathbb{T} \).

**Definition** (The rays \( L_n^\pm \) associated to \( v_n^\pm \)). Let \( v_n^\pm = G(c_n^\pm) \) be the singular points associated to \( \Phi_n \), and let \( L_n^\pm \) be the rays (in the logarithmic plane of \( G \)) starting from \( v_n^\pm \) which form an angle \( 0 < \beta < \pi/3 \) with \( \mathbb{T} \) (clockwise for \( v_n^+ \) and anticlockwise for \( v_n^- \)), where \( \beta \) will be specified later. By Lemma 2.9, \( \Phi_n \) contracts the hyperbolic metric \( \rho_G(z)|dz| \) strictly in \( V_n^+ \subset \Omega \), where \( V_n^+ \) consists of the points which are below \( L_n^+ \) and in a small neighborhood of \( v_n^+ \) (see Figure 7). In the following we refer \( L_n^\pm \) the rays associated to \( v_n^\pm \).

![Figure 7: The rays \( L_n^\pm \) associated to the singular points \( v_n^\pm \) of the branch \( \Phi_n \) of \( G^{-1} \). The map \( \Phi_n \) contracts the hyperbolic metric \( \rho_G(z)|dz| \) in two domains \( V_n^\pm \) colored dark cyan and yellow respectively.](image-url)
By Lemma 2.9 if the singular point associated to $\Phi_n$ is $v_0^n$, then $\Phi_n$ contracts the hyperbolic metric $\rho_0(z)|dz|$ strictly in $V_0^n \subset \Omega$, where $V_0^n$ consists of the points which are in a small neighborhood of $v_0^n$ and are outside of $\overline{\Omega}$.

By Lemma 3.3 $\sigma(I_n) < 2\sigma(I_0)$ for all $n \geq 0$ and $\sigma(I_n) \to 0$ as $n \to \infty$. Denote $s_0 := r_1/8$, where $r_1 = r_1(d) > 0$ is introduced in Lemma 2.6. For $n \geq 0$, we define

$$D_n^{s_0} := \{ z \in \mathbb{C} \setminus \overline{\Omega} : \text{dist}_{\mathbb{C}}(z, I_n) < s_0 \}. \quad (3.11)$$

Then $D_n^{s_0} \cap \mathcal{P}(G) = \emptyset$ for all $n \geq 0$. Note that $D_n^{s_0}$ is simply connected and $\Phi_n$ is defined in $D_n^{s_0}$ by analytic continuation of $\Phi_n|_{I_n}$. Let $r_3 = r_3(\delta, \eta, d) > 0$ be introduced in Lemma 3.1. Without loss of generality, we assume that $r_3$ is small such that if $\sigma(I_0) < r_3$, then $H_{d+1}(I_n) \subset D_n^{s_0}$ for all $n \geq 0$.

**Lemma 3.6.** For any integer $T' \geq 1$, there exists $r_4 = r_4(\delta, \eta, d, T') \in (0, r_3)$ such that if $\sigma(I_0) < r_4$, then for any $n \geq 0$ and $m \geq n$, if $\{ I_k : n \leq k \leq m+1 \}$ contains at most $T'$ critical predecessors, then

$$\Phi_m \circ \cdots \circ \Phi_{n+1} \circ \Phi_n(H_d(I_n)) \subset D_{m+1}^{s_0}. \quad (3.12)$$

Consequently, for any simply connected domain $D_n$ in $\hat{\mathbb{C}} \setminus (\overline{\Omega} \cup \mathcal{P}(G))$ containing $H_d(I_n)$, $\Phi_m \circ \cdots \circ \Phi_{n+1} \circ \Phi_n$ can be extended analytically from $H_d(I_n)$ to $D_n$.

**Proof.** Let $n_0 \geq n$ be the smallest integer such that $I_{n_0} \to I_{n_1+1}$ is a critical pullback. If $\sigma(I_0)$ is small, then for every $n \leq k \leq n_0 - 1$, by Lemma 2.7(b), $\Phi_k \circ \cdots \circ \Phi_{n+1} \circ \Phi_n(H_d(I_n))$ is defined and contained in $H_d(I_k+1)$ with $d' \in [d, d+1]$, where $H_d(I_k+1) \subset D_{k+1}^{s_0}$. By Lemma 3.1, $\Phi_n \circ \cdots \circ \Phi_{n+1} \circ \Phi_n(H_d(I_n))$ is defined and contained in $H_d(I_{n+1}) \cup B_{n+1}^1$, where $B_{n+1}^1$ is a Jordan disk in $\Omega$ satisfying

$$\bigcap_{i=1}^{n_0} H_d(I_{n+i}) \neq \emptyset \quad \text{and} \quad \text{diam}_\Omega(B_{n+i}^1) < K_0 = K_0(\delta, \eta, d).$$

If $\sigma(I_0)$ is small enough, then $H_d(I_{n+1}) \cup B_{n+1}^1 \subset D_{n+1}^{s_0}$.

Let $T' \geq 1$ be given. Repeating the above process, there exists a small $r_4 = r_4(\delta, \eta, d, T') \in (0, r_3)$ such that if $\sigma(I_0) < r_4$, then for every $n \leq \ell \leq m$, there exists $1 \leq p_\ell \leq T'$ such that

$$\Phi_\ell \circ \cdots \circ \Phi_{n+1} \circ \Phi_n(H_d(I_n)) \subset H_d(I_{\ell+1}) \cup \bigcup_{k=1}^{p_\ell} B_{\ell+1}^k \subset D_{\ell+1}^{s_0}, \quad (3.12)$$

where each $B_{\ell+1}^k$ with $1 \leq k \leq p_\ell$ is a Jordan disk in $\Omega$ satisfying

$$\bigcap_{i=1}^{n_0} H_d(I_{\ell+i}) \neq \emptyset \quad \text{and} \quad \text{diam}_\Omega(B_{\ell+1}^k) < p_\ell K_0 \leq T' K_0, \quad (3.13)$$

and moreover, $\tilde{d} = d$ if $I_\ell \to I_{\ell+1}$ is a critical pullback and $\tilde{d} = d' \in [d, d+1]$ if $I_\ell \to I_{\ell+1}$ is a non-critical pullback. The proof is complete. \hfill $\Box$

Denote $T_0 := t_0 + 1$, where $t_0 = 2(\text{Crit}(G) \cap \mathbb{T}) = 2CV \geq 1$.\quad (3.14)

Let $n_j$ be defined in (3.2). By Lemma 3.6, $\Phi_{n_j+T_0} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}$ is defined in $D_{n_j}^{s_0}$ if $\sigma(I_0) < r_4 := r_4(\delta, \eta, d, T_0) = r_4(\delta, \eta, d)$. The following result is the most important ingredient of the proof of the Main Lemma.

**Proposition 3.7.** There exist $d > 1$, $r_5 = r_5(\delta, \eta, d) \in (0, r_4)$, $C_0 = C_0(\delta, \eta, d) > 1$ and $0 < \mu_0 < 1$, such that if $\sigma(I_0) < r_5$, then for every $j \geq 1$, there exists an arc $J \subset \mathbb{T}$ containing $I_{n_j+T_0+1}$ with $\sigma(J) < C_0(\sigma(I_{n_j+T_0+1})$ such that for any $z \in D_{n_j}^{s_0}$, the preimage $w = \Phi_{n_j+T_0} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(z)$ satisfies one of the following:
(a) $\rho_d(w) < \mu_0 \rho_d(z)|G^{n_j+\tau_0-n_j+1}'(w)|$; or
(b) $w \in H_d(J)$.

**Proof.** By Corollary 3.5, there exists a number $K_1 = K_1(\delta, \eta, T_0) > 1$ depending only on $\delta, \eta$ and $T_0$ such that for any $n_j \leq k \leq n_j+T_0+2$, if $x \in (1+2\delta)I_k$, then

$$G^{k-n_j}(x) \in (2K_1+1)I_{n_j}. \tag{3.15}$$

For $j \leq i \leq j+T_0$, let $v_{n_i} \in CV$ be the unique critical value such that $v_{n_i} \in (1+2\delta)I_{n_i}$. Then $G^{n_i-n_j}(v_{n_i}) \in (2K_1+1)I_{n_j}$ for every $j \leq i \leq j+T_0$.

Without loss of generality, we assume that $v_{n_j} = v_{n_j}$ The proof for $v = v_{n_j}$ is completely the same and that for $v = v_{n_j}$ is obvious. Let $L = L_{n_j}$ be the ray associated to $v := v_{n_j}$ which forms an angle $0 < \beta < \pi/3$ with $T$. Since $\sigma(I_{n_j}) < 2\sigma(I_0)$, decreasing $\sigma(I_0)$ if necessary, there exists $l \geq 5$ such that the closest return $G^l(v)$ is on the same side as $L$, i.e., the angle formed by $L$ and $[v, G^l(v)]$ is $\beta$, and moreover,

$$\sigma((v, G^{n+2}(v))) \leq (K_1 + 2)\sigma(I_{n_j}) < \sigma((v, G^l(v))). \tag{3.16}$$

Since $s_0 = r_1/8$ and $\sigma(I_{n_j}) < 2\sigma(I_0)$, decreasing $\sigma(I_0)$ if necessary, without loss of generality, we assume that $[G^{n-1}(v), G^l(v)]$ is contained in the interior of $\partial D_0 \cap \mathbb{T} = [x_0', x_0]$ and $\sigma([x_0', x_0]) < r_1$. Let $X$ be the subregion of $D_0$ which is below the ray $L$. Let $Y$ be the closed trapezoid in $\mathbb{C} \setminus \mathbb{D}$ which is bounded by the ray $L$, the interval $[G^{n-1}(v), x_0]$, and the two vertical straight segments passing through $G^{n-4}(v)$ and $x_0$, respectively. Let $Z$ denote the closed right triangle in $\mathbb{C} \setminus \mathbb{D}$ bounded by the interval $[v, G^{n-4}(v)]$, the ray $L$, and the vertical segment passing through $G^{n-4}(v)$. See Figure 8

![Figure 8: The interval $I_{n_j}$ and its half neighborhood $D_{n_j}$ with a partition.](image)

**Case 1:** Suppose $z \in Z$. Denote $I_{n_j} = (a, b)$. If $a \in (v, b)$, we denote $J_0 := (v, G^{n-4}(v))$. Otherwise, we denote $J_0 := (a, G^{n-4}(v))$. Then we have $I_{n_j} \subset J_0$. By (3.16), there exists a constant $C_1 = C_1(\delta, \eta, T_0) = C_1(\delta, \eta) \geq 1$ such that

$$\sigma(J_0) \leq C_1 \sigma(I_{n_j}). \tag{3.17}$$

Without loss of generality, we assume that the angle $\beta \in (0, \frac{\pi}{2})$ between the ray $L$ and $T$ is small enough such that the half hyperbolic neighborhood $H_d(J_0)$ is simply connected and $Z \subset H_d(J_0)$. Let $N := n_j+T_0 - n_j + 1$. For $k = 0, 1, \ldots, N$, we consider the $(\delta, \eta)$-admissible sequence of pullbacks of $J_0$ (which is well-defined if
\(\sigma(I_0)\) is small) such that the pullback \(J_k \to J_{k+1}\) is induced by \(I_{n_j+k} \to I_{n_j+k+1}\) and satisfies \(I_{n_j+k} \subset J_k\) for all \(1 \leq k \leq N\).

By Lemma 3.4 and (3.17), the number of critical predecessors in the sequence \(\{J_k : 0 \leq k \leq N\}\) is bounded above by a constant \(C_2 = C_2(\delta, \eta, T_0) = C_2(\delta, \eta) \geq 1\). By a similar arguments to (3.12) and (3.13), there exists \(1 \leq p \leq C_2\) such that for any \(z \in Z\) we have

\[
w = \Phi_{n_j+1} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(z) \in W_p := H_d(J_N) \cup \bigcup_{k=1}^{p} B_N^k,
\]

where each \(B_N^k\) with \(1 \leq k \leq p\) is a Jordan disk in \(\Omega\) satisfying

\[
B_N^k \cap H_d(J_N) \neq \emptyset \quad \text{and} \quad \text{diam}_{\Omega}(B_N^k) < C_2K_0.
\]

This implies that if \(\sigma(I_0)\) is small enough, then there is a constant \(C_3 = C_3(\delta, \eta, d) > 1\) such that \(w \in W_p \subset H_d(J)\) with \(J := C_3J_N\). Hence

\[
\sigma(J) = C_3\sigma(J_N) < 2C_3\sigma(J_0). \tag{3.18}
\]

Since \(I_{n_j+N} \subset J_N\), we have \(I_{n_j+N+1} \subset J\).

By (3.16), there is a number \(C_4 = C_4(\delta, \eta, d) > 1\) such that \(\sigma(J_0) = C_4\sigma(I_{n_j})\). By the definitions of critical and non-critical predecessors, \(\sigma(I_{n_j}) \leq 2^{f_0+1}\sigma(I_{n_j+T_0+1})\). Therefore, there is a constant \(C_0 = C_0(\delta, \eta, d) > 1\) such that \(\sigma(J) < C_0\sigma(I_{n_j+T_0+1})\).

This implies that Part (b) holds for all \(z \in Z\).

**Case 2:** Suppose \(z \in X\). Then by Lemma 2.9, the map \(I_{n_j}\) contracts the hyperbolic metric at \(z\) by a definite amount. In particular, there is some \(\mu_1 = \mu_1(\beta) \in (0, 1)\) which depends only on the angle \(\beta\) formed by \(L\) and \(T\) such that

\[
\rho_\Omega(\Phi_{n_j}(z)) < \mu_1 \rho_\Omega(z) |G'(\Phi_{n_j}(z))|.
\]

Then by Schwarz-Pick Lemma, Part (a) holds for all \(z \in X\) with \(\mu_0 := \mu_1\).

**Case 3:** Suppose \(z \in Y\). For \(n_j + 1 \leq k \leq n_j + T_0 + 1\), if \(v_k = v_k^0\) is the singular point associated to \(I_{n_j+T_0+1}\), then by definition we have \(v_k^0 \in (1+2\delta)I_k\). According to (3.15) and (3.16) we have

\[
G^{k-n_j}(v_k^0) \in (2K_1+1)I_{n_j} \subset (G^{q-1}(v), G^q(v)). \tag{3.19}
\]

We claim that there exists an integer \(k \in [n_j + 1, n_j + T_0]\) such that the arc \([G^q(v), G^{q-4}(v)]\) contains \(G^{k-n_j}(v_k)\), where \(v_k\) is a singular point associated to \(I_{n_j+T_0+1}\). Indeed, for every \(1 \leq i \leq T_0 = t_0 + 1\), the arc \((1+2\delta)I_{n_j+i}\) contains a critical value \(v_{n_j+i} \in CV \subset T\). Note that \(CV = t_0\). By (3.15) and (3.16), there exist two integers \(k', k'' \in [n_j + 1, n_j + T_0]\) with \(k' < k''\) and a critical value \(v' \in CV \cap (1+2\delta)I_{k'} \cap (1+2\delta)I_{k''}\) such that \([G^{q-1}(v), G^q(v)]\) contains \(G^{k'-n_j}(v')\) and \(G^{k''-n_j}(v')\). This implies that \(k'' - k' > q_{l-2}\). Then

\[
G^{k'-n_j+r_{l-2}}(v') \in [G^{q-1+r_{l-2}}(v), G^{q+r_{l-2}}(v)] \subset [G^q(v), G^{q-4}(v)]. \tag{3.20}
\]

and

\[
n_j + 1 \leq k' \leq k' + q_{l-2} \leq k'' \leq n_j + T_0.
\]

Set \(k := k' + q_{l-2} \in [n_j + 1, n_j + T_0]\). Then by (3.20) we have

\[
G^{k-n_j}(v') \in [G^q(v), G^{q-4}(v)] \subset [G^q(v), x_0).
\]

By the definition of \(\tilde{r}_0\) in (2.15) and note that \(\sigma([x_0', x_0]) < r_1 \leq \tilde{r}_0\), we have

\[
\{G^{k-n_j}(\tilde{v}) : \tilde{v} \in CV \} \cap [x_0', x_0] = \{G^{k-n_j}(v')\} \subset [G^q(v), x_0).
\]
It follows that \( v_k := v' \) is a singular point associated to \( \Phi_k \).

Next we shall find some half hyperbolic neighborhoods to cover \( Y \). By the claim above there exists a least integer \( k_1 \in [n_j + 1, n_j + T_0] \) such that
\[
x_1 := G^{k_1-n_j}(v_{k_1}) \in [G^q(v), x_0),
\]
where \( v_{k_1} \) is a singular point associated to \( \Phi_{k_1} \). If \( x_1 \in [G^q(v), G^{q-4}(v)] \), we stop. Otherwise, there exists a least integer \( k_2 \in [k_1 + 1, n_j + T_0] \) such that
\[
x_2 := G^{k_2-n_j}(v_{k_2}) \in [G^q(v), x_1),
\]
where \( v_{k_2} \) is a singular point associated to \( \Phi_{k_2} \). Repeating the procedure, we obtain two finite sequences:
\[
x_m < \cdots < x_2 < x_1 < x_0 \quad \text{and} \quad n_j + 1 \leq k_1 < k_2 < \cdots < k_m \leq n_j + T_0,
\]
where \( x_i := G^{k_i-n_j}(v_{k_i}) \in (G^{q-4}(v), x_{i-1}) \) for all \( 1 \leq i \leq m \) with \( m \geq 1 \) and \( x_m := G^{k_m-n_j}(v_{k_m}) \in [G^q(v), G^{q-4}(v)] \). Moreover, \( v_{k_i} \) is a singular point associated to \( \Phi_{k_i} \) for every \( 1 \leq i \leq m \).

Let \( \Xi := \{x_0, x_1, \ldots, x_m\} \) and \( h \geq 1 \) be the least integer such that \( x_0 \in [G^{qh}(v), G^{qh-2}(v)] \). Without loss of generality, we assume that \( h \) and \( l \) are even and \( l - h \geq 8 \) is also even (decreasing \( l(I_0) \) if necessary). Then the union of all
\[
[G^{q2k}(v), G^{q2k-2}(v)] \quad \text{with} \quad h + 4 \leq 2k \leq l - 4 \quad \text{and} \quad [G^{qh+2}(v), x_0]
\]
(3.21)
covers \( [G^{q-4}(v), x_0] \). Each of these intervals is contained in a smallest interval with end points in \( \Xi \), say \([x_{l_k}, x_{l'_k}]\) with \( m \geq l_k > l'_k \geq 0 \). In particular, the interval \( [G^{qh+2}(v), x_0] \) is contained in some \([x_{l_k}, x_{l'_k}]\) with \( 2k = h + 2 \).

From the fact that the rotation number \( \alpha \) of \( G \) is of bounded type and that the number of the critical points on \( \mathbb{T} \) is bounded, it follows that there is a uniform constant \( N_0 \), which is independent of \( k \), such that each of the intervals in (3.21) contains at most \( N_0 \) points in \( \Xi \). Let \( Y_k \) denote the closed trapezoid in \( \mathbb{C} \setminus \overline{D} \) which is bounded by the ray \( L \), the interval \([x_{l_k}, x_{l'_k}]\), and the two vertical straight segments passing through \( x_{l_k} \) and \( x_{l'_k} \). Then the interval \([x_{l_k}, x_{l'_k}]\) contains at most \( N_0 + 2 \) points in \( \Xi \). See Figure 9.

Figure 9: The trapezoid \( Y_k \) with some marked boundary points.

By Lemma 2.8 there exist \( \overline{r_2} = \overline{r_2}(N_0) > 0 \), \( \omega = \omega(N_0) \in (0, 1) \) and \( \beta_0 = \beta_0(N_0) \in (0, \pi) \) such that for every \( I'_k = (x_{l_k}, x_{l'_k}) \subset \mathbb{T} \) with \( |I'_k| < \overline{r_2} \), the following two statements hold:
- \( H^{\beta_0}(x_i, x_{i-1}) \) is a Jordan disk for all \( l'_k + 1 \leq i \leq l_k \); and
- \( S_\omega(I'_k) \setminus \mathbb{T} \subset \bigcup_{i=l'_k+1}^{l_k} H^{\beta_0}(x_i, x_{i-1}) \).
Decreasing $\sigma(I_0)$ and $s_0$ in (3.11) if necessary, we assume that $|I'_k| < \tilde{r}_2$ for all $k$.

Since the rotation number $\alpha$ is of bounded type, it follows that there exists $C' > 1$ which is independent of $k$ such that the dynamical lengths of $(v, G^{2k}(v))$ and $(G^{2k}(v), G^{2k-2}(v))$ are $C'$-comparable. By Lemma 2.2, this implies that their Euclidean lengths are $C''$-comparable, where $C'' > 1$ is a number which is also independent of $k$. Decreasing the angle $\beta \in (0, \frac{\pi}{2})$ between the ray $L$ and $T$ to a new angle $\beta = \beta(N_0) \in (0, \frac{\pi}{2})$ if necessary, one can guarantee that $Y_k \subset S_n(I'_k)$ and hence $Y_k \subset \bigcup_{i=k}^{m} H^{\beta_0}((x_i, x_{i-1}))$ for all $k$. This implies that

$$Y \subset \bigcup_{i=1}^{m} H^{\beta_0}((x_i, x_{i-1})).$$

By (2.27), we have $H_{d_0}((x_i, x_{i-1})) = H^{\beta_0}((x_i, x_{i-1}))$ for every $1 \leq i \leq m$, where $d_0 = \log \cot(\frac{\beta_0}{2})$. In the following, we fix a large $d \geq d_0$ such that (2.32) also holds.

For every $n_j \leq k \leq n_j + T_0$, we denote

$$D_{k+1} := \Phi_k \circ \cdots \circ \Phi_{n_j + 1} \circ \Phi_{n_j}(D^{\beta_0}_{n_j}).$$

Then $D_{k+1}$ is a simply connected domain contained in $\Omega$. We have two cases (I) and (II).

(I) Suppose there is a least integer $k'_1 \in [n_j + 1, k_1]$ such that $(G^{q-1}(v), G^q(v))$ contains $x'_1 := G^{k'_1-n_j}(v^+_1)$ or $G^{k'_1-n_j}(v^-_1)$. For every $n_j \leq k \leq k'_1 - 1$, we denote

$$P_{k+1,1} := \Phi_k \circ \cdots \circ \Phi_{n_j + 1} \circ \Phi_{n_j}(P_1),$$

where $P_1 := H_{d_0}((G^{q-1}(v), x_0))$. Let $J'_1 \subset T$ be the arc satisfying $G^{k'_1-n_j}(J'_1) = (G^{q-1}(v), x_0)$. By the definition of $k'_1$, it follows that for every $n_j \leq k \leq k'_1 - 1$, $D_{k+1} \cap T$ and $\partial D_{k+1,1} \cap T$ are proper subarcs of $T$ satisfying

$$\partial D_{k+1} \cap T \supset I_{k+1} \cup G^{k'-1-k}(J'_1) \quad \text{and} \quad \partial P_{k+1,1} \cap T = G^{k'-1-k}(J'_1).$$

For $z \in Y \setminus T \subset P_1$, we have $w'_1 := \Phi_{k+1-1} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(z) \in P_{k+1,1} \setminus T \subset H_{d_0+1}(J'_1)$ by Lemma 2.7(b).

Let $\tilde{J}_1 \subset T$ be the arc containing $J'_1$ such that $G^{k'-n_j}(\tilde{J}_1) = (x'_1, x_0)$. Note that $\Phi_{k'_1}$ is defined in the simply connected domain $D_{k_1} \cup P_{k_1,1}$ by analytic continuation of $\Phi_{k'_1}|_{k'_1}$. Since the left end point of $\tilde{J}_1$ is $v^+_1$ or $v^-_1$, by Lemma 2.9 there exists a universal constant $0 < \mu < 1$ such that

$$\rho_{\Omega}(\Phi_{k'_1}(w'_1)) < \mu \rho_{\Omega}(w'_1) |G'(\Phi_{k'_1}(w'_1))|.$$

Hence Part (a) is proved for $z \in Y$ with $\mu_0 = \mu$ by Schwarz-Pick Lemma.

(II) Suppose for any $k \in [n_j + 1, k_1]$, the arc $(G^{q-1}(v), G^q(v))$ contains neither $G^{k-n_j}(v^+_1)$ nor $G^{k-n_j}(v^-_1)$. For every $n_j \leq k \leq k_1 - 1$, we denote

$$\tilde{P}_{k+1,1} := \Phi_k \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(\tilde{P}_1),$$

where $\tilde{P}_1 := H_{d_0}((x_1, x_0))$. Let $J_1 \subset T$ be the arc satisfying $G^{k_1-n_j}(J_1) = (x_1, x_0)$. Similar to Case (I), for every $n_j \leq k \leq k_1 - 1$, $D_{k+1} \cap T$ and $\partial \tilde{P}_{k+1,1} \cap T$ are proper subarcs of $T$ satisfying

$$\partial D_{k+1} \cap T \supset I_{k+1} \cup G^{k_1-1-k}(J_1) \quad \text{and} \quad \partial \tilde{P}_{k+1,1} \cap T = G^{k_1-1-k}(J_1).$$

For $z \in \tilde{P}_1$, we have $w_1 := \Phi_{k_1-1} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(z) \in \tilde{P}_{k_1,1} \subset H_{d_0+1}(J_1)$ by Lemma 2.7(b).
Note that the left end point of \( J_1 \) is \( v_{k_1} = v_{k_1}^+ \) (it cannot be \( v_{k_1}^0 \) by (3.19) and cannot be \( v_{k_1}^- \) by (3.15) and (3.16)). Still by Lemma 2.9 we have
\[
\rho_{\Omega}(\Phi_{k_1}(w_1)) < \mu \rho_{\Omega}(w_1)|G'(\Phi_{k_1}(w_1))|.
\]
Hence Part (a) is proved for \( z \in \tilde{P}_1 = H_{d_0}(x_1, x_0) \) with \( \mu_0 = \mu \) by Schwarz-Pick Lemma. In particular, if \( m = 1 \), then Part (a) is proved for \( z \in Y \).

In the following we assume that we are in Case (II) and \( m \geq 2 \). We again have two subcases (II-i) and (II-ii).

(II-i) Suppose there is a least integer \( k'_2 \in [k_1 + 1, k_2] \) such that \((G^{n-1}(v), G^n(v))\) contains \( x'_2 := G^{k'_2-n_j}(v^+_k) \) or \( G^{k'_2-n_j}(v^-_k) \). For every \( n_j \leq k \leq k'_2 - 1 \), we denote
\[
P_{k+1,2} := \Phi_k \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(P_2),
\]
where \( P_2 := H_{d_0}((G^{n-i}(v), x_1)) \). Let \( J'_2 \subset T \) be the arc satisfying \( G^{k'_2-n_j}(J'_2) = (G^{n-i}(v), x_1) \). By the definition of \( k'_2 \), it follows that for every \( n_j \leq k \leq k'_2 - 1 \), \( D_{k+1} \cap T \) and \( \partial P_{k+1,2} \cap T \) are proper subarcs of \( T \) satisfying
\[
\partial D_{k+1} \cap T \supset I_{k+1} \cup G^{k'_2-1-k}(J'_2) \quad \text{and} \quad \partial P_{k+1,2} \cap T = G^{k'_2-1-k}(T'_2).
\]
For \( z \in P_2 \), we have \( w_2 := \Phi_{k'_2-1} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(z) \in P_{k'_2,2} \subset H_{d_o}+(J'_2) \) by Lemma 2.7(b).

Let \( \tilde{J}_2 \subset T \) be the arc containing \( J'_2 \) such that \( G^{k'_2-n_j}(\tilde{J}_2) = (x'_2, x_1) \). Similar to Case (I), since the left end point of \( \tilde{J}_2 \) is \( v_{k'_2}^+ \) or \( v_{k'_2}^- \), by Lemma 2.9 we have
\[
\rho_{\Omega}(\Phi_{k'_2}(w_2)) < \mu \rho_{\Omega}(w_2)|G'(\Phi_{k'_2}(w_2))|.
\]
Note that \( Y \subset P_2 \cup \tilde{P}_1 = H_{d_0}((G^{n-i}(v), x_1)) \cup H_{d_0}(x_1, x_0) \). Hence Part (a) is proved for \( z \in Y \) with \( \mu_0 = \mu \) by Schwarz-Pick Lemma.

(II-ii) Suppose for any \( k \in [k_1 + 1, k_2] \), the arc \((G^{n-i}(v), G^n(v))\) contains neither \( G^{k-n_j}(v^+_k) \) nor \( G^{k-n_j}(v^-_k) \). For every \( n_j \leq k \leq k_2 - 1 \), we denote
\[
\tilde{P}_{k+1,2} := \Phi_k \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(\tilde{P}_2),
\]
where \( \tilde{P}_2 := H_{d_2}(x_2, x_1) \). Let \( J_2 \subset T \) be the arc satisfying \( G^{k_2-n_j}(J_2) = (x_2, x_1) \). Similarly, for every \( n_j \leq k \leq k_2 - 1 \), \( D_{k+1} \cap T \) and \( \partial \tilde{P}_{k+1,2} \cap T \) are proper subarcs of \( T \) satisfying
\[
\partial D_{k+1} \cap T \supset I_{k+1} \cup G^{k_2-1-k}(J_2) \quad \text{and} \quad \partial \tilde{P}_{k+1,2} \cap T = G^{k_2-1-k}(T_2).
\]
For \( z \in \tilde{P}_2 \), we have \( w_2 := \Phi_{k_2-1} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j}(z) \in \tilde{P}_{k_2,2} \subset H_{d_0}+(J_2) \) by Lemma 2.7(b).

Note that the left end point of \( J_2 \) is \( v_{k_2} = v_{k_2}^+ \). Still by Lemma 2.9 we have
\[
\rho_{\Omega}(\Phi_{k_2}(w_2)) < \mu \rho_{\Omega}(w_2)|G'(\Phi_{k_2}(w_2))|.
\]
Then Part (a) is proved for \( z \in P_2 \cup \tilde{P}_1 = H_{d_0}(x_2, x_1) \cup H_{d_0}(x_1, x_0) \) with \( \mu_0 = \mu \) by Schwarz-Pick Lemma. In particular, if \( m = 2 \), then Part (a) is proved for \( z \in Y \).

Next assume that we are in Case (II-ii) and \( m \geq 3 \). Inductively, there exists a universal \( 0 < \mu < 1 \) such that Part (a) holds for all \( z \in Y \) with \( \mu_0 = \mu \) since \( Y \) is covered by the union of finitely many half hyperbolic neighborhoods \( H_{d_0}(x_1, x_{i-1}) \), where \( 1 \leq i \leq m \). (see (3.22)). The proof is complete.

In the rest of this paper, we fixed a large number \( d > 1 \) such that Proposition 3.7 holds.
4. Proof of the Main Lemma

4.1. Improved $(\delta, \eta)$-admissible sequences. Let $I \subset \mathbb{T}$ be a small arc and $d \geq 1$ a large number fixed before. Let $\Omega$ be the unique component of $\hat{\mathbb{C}} \setminus (\mathcal{P}(G) \cup \mathbb{D})$ such that $\mathbb{T} \subset \partial \Omega$ (see (2.29)). For $d \in [d, d+1]$, we use $\Sigma_{I,d}$ to denote the class consisting of all the sets $F$ such that

$$F = \bigcup_{k \in \mathbb{K}} B_k,$$

where $\mathbb{K}$ is an at most countable set, and each $B_k$ is a Jordan disk compactly contained in $\Omega$ satisfying $B_k \cap H_d(I) \neq \emptyset$. Define

$$\Upsilon_{\Omega}(H_d(I), F) := \sup_{k \in \mathbb{K}} \{ \text{diam}_\Omega(B_k) \}.$$ 

Let $\{I_n\}_{n \geq 0}$ be a $(\delta, \eta)$-admissible sequence. For $n \geq 0$, let $\Phi_n$ be the inverse branch of $G$ associated to the pullback $I_n \to I_{n+1}$, which is defined in any simply connected domain in $\hat{\mathbb{C}} \setminus (\mathbb{D} \cup \mathcal{P}(G))$ containing $D_n^\circ$ (see (3.11)). Let $r_5 = r_5(\delta, \eta, d) > 0$ be introduced in Proposition 3.7.

**Lemma 4.1.** There exist positive numbers $r_6 \in (0, r_5]$, $K'$, $K$, $T_2$ depending only on $\delta$, $\eta$, $d$, such that for any $(\delta, \eta)$-admissible sequence $\{I_n\}_{n \geq 0}$ with $\sigma(I_0) < r_6$ and any $F_0 \in \Sigma_{I_0,d}$ with $\Upsilon_{\Omega}(H_d(I_0), F_0) < K'$ and $H_d(I_0) \cup F_0 \subset D_0^\circ$, there exist

1. $\ell \geq 1$ and $F_i \in \Sigma_{I_{\ell},d}$ with $d_i \in [d, d+1]$, where $1 \leq i \leq \ell$; and
2. $F \in \Sigma_{I_{T_2},d}$ for a subarc $J \subset \mathbb{T}$ containing $I_{\ell}$ with $\sigma(J) < \sigma(I_0)/2$,

such that

- $\{I_i : 0 \leq i \leq \ell\}$ contains exactly $T_2$ critical predecessors;
- $\Phi_1 \circ \cdots \circ \Phi_{\ell-1} \circ \Phi_0(\mathcal{H}(I_0) \cup F_0) \subset \mathcal{H}(I_{\ell}) \cup F_1 \subset D_1^\circ$ for all $1 \leq i \leq \ell$;
- $\Upsilon_{\Omega}(\mathcal{H}(I_{\ell}), F_i) < K$ for all $1 \leq i \leq \ell - 1$; and
- $H_d(I_{\ell}) \cup F_\ell \subset \mathcal{H}(J) \cup F$ and $\Upsilon_{\Omega}(H_d(J), F) < K'$.

**Proof.** Let $K_0 = K_0(\delta, \eta, d) > 0$ and $T_1 = T_1(\eta) \geq 1$ be introduced in Lemmas 3.1 and 3.3 respectively. For the constants $C_0 = C_0(\delta, \eta, d) > 1$ and $0 < \mu_0 < 1$ introduced in Proposition 3.7, let $k_0 \geq 3$ be the minimal integer such that

$$\left(\frac{1}{2}\right)^{k_0-2} C_0 < \frac{1}{2}.$$ 

Let $T_0 = t_0 + 1$ be introduced in (3.14), where $t_0 = \frac{1}{2} (\text{Crit}(G) \cap \mathbb{T}) \geq 1$. Set

$$K' := \frac{\mu_0 k_0 T_1 K_0}{1 - \mu_0}.$$ 

Let $F_0 \in \Sigma_{I_0,d}$ with $\Upsilon_{\Omega}(\mathcal{H}(I_0), F_0) < K'$. We assume that $\sigma(I_0)$ is small enough such that $H_d(I_0) \cup F_0 \subset D_0^\circ$ (see (3.11)). Let $j := k_0 T_1 + 1$ and $n_j$ be the $j$-th integer such that $I_{n_j+1}$ is a critical predecessor of $I_{n_j}$ (see (3.2)). Based on a similar proof to (3.12) and (3.13) (decreasing $\sigma(I_0)$ if necessary), by Lemmas 2.7(b), 3.1 and 3.3 we obtain $F_i \in \Sigma_{I_{n_j,d}}$ for $1 \leq i \leq n_j$ with $d_i \in [d, d+1]$, such that

- There are exactly $k_0 T_1$ critical predecessors in $\{I_i : 0 \leq i \leq n_j\}$;
- $\sigma(I_{n_j}) < \sigma(I_0)/2^{k_0-1}$;
- $\Phi_1 \circ \cdots \circ \Phi_{n_j} \circ \Phi_0(\mathcal{H}(I_0) \cup F_0) \subset \mathcal{H}(I_{n_j}) \cup F_1 \subset D_{n_j}^\circ$ for all $1 \leq i \leq n_j$; and
- $\Upsilon_{\Omega}(\mathcal{H}(I_{n_j}), F_i) < K' + k_0 T_1 K_0$ for all $1 \leq i \leq n_j$.

Considering the pullbacks from $I_{n_j}$ to $I_\ell$ with $\ell := n_j + T_0 + 1$, by Lemmas 2.7(b), 3.1 and 3.3 we obtain $F_i \in \Sigma_{I_{n_j,d}}$ for $n_j + 1 \leq i \leq \ell$ with $d_i \in [d, d+1]$, such that

- There are exactly $T_2$ critical predecessors in $\{I_i : 0 \leq i \leq \ell\}$, where

$$T_2 := k_0 T_1 + T_0 + 1.$$
\[ \sigma(I_i) < 2\sigma(I_{n_j}) < \sigma(I_0)/2^{k_0-2}; \]
\[ \Phi_{i-1} \circ \cdots \circ \Phi_1 \circ \Phi_0 (H_d(I_0) \cup F_0) \subset H_d(I_i) \cup F_i \subset D^0_i \text{ for all } 1 \leq i \leq \ell; \text{ and} \]
\[ \mathcal{Y}_\Omega(H_d(I_i), F_i) < K \text{ for all } 1 \leq i \leq \ell - 1, \]
where \( K := (K' + k_0 T_1 K_0) + T_0 K_0. \)

By Lemma 3.1, \( d_\ell \) is chosen to be \( d. \)

Let \( D^{0_0}_{n_j} \) be the set defined in (3.11). Without loss of generality, we assume that \( \sigma(I_0) \) is small enough such that
\[ \cdot H_{d_{n_j}}(I_{n_j}) \cup F_{n_j} \subset D^{0_0}_{n_j}, \text{ and} \]
\[ \cdot \text{The shortest geodesic curve connecting any different points } z_1, z_2 \text{ in every Jordan disk of } F_{n_j} \text{ is contained in } D_{n_j}^{0_0}. \]

By the proof of Proposition 3.7, there exists a closed triangle \( Z \subset D^{0_0}_{n_j} \) such that \( \Phi := \Phi_{\ell-1} \circ \cdots \circ \Phi_{n_j+1} \circ \Phi_{n_j} \) contracts the hyperbolic metric \( \rho_d(z)|dz| \) in \( D^{0_0}_{n_j} \setminus Z \) strictly and \( \Phi(Z) \subset H_d(J) \), where \( J \subset \mathbb{T} \) is an arc containing \( I_\ell \) which satisfies
\[ \sigma(J) < C_0 \sigma(I_\ell) < C_0 \sigma(I_0)/2^{k_0-2} < \sigma(I_0)/2. \]

If \( z \in H_{d_{n_j}}(I_{n_j}) \), then there exists \( F'_\ell \in \Sigma_{I_\ell,d} \) such that \( \Phi(z) \in H_d(I_\ell) \cup F'_\ell \), where \( \mathcal{Y}_\Omega(H_d(I_\ell), F'_\ell) < (T_0 + 1)K_0. \)

By the construction of \( J \), increasing the number \( C_3 = C_3(\delta, \eta, d) > 1 \) in (3.18) if necessary, without loss of generality by (4.1) we assume that \( \Phi(H_{d_{n_j}}(I_{n_j})) \subset H_d(I_\ell) \cup F'_\ell \subset H_d(J). \)

Note that \( \mathcal{Y}_\Omega(H_{d_{n_j}}(I_{n_j}), F_{n_j}) < K' + k_0 T_1 K_0 \) and \( F_{n_j} \setminus Z \) can be written as
\[ F_{n_j} \setminus Z = \bigcup_{k \in \mathbb{K}} B_k, \]
where \( \mathbb{K} \neq \emptyset \) is an at most countable set, and each \( B_k \) is a Jordan disk compactly contained in \( \Omega \) satisfying
\[ \overline{B}_k \cap (\partial \Omega \setminus \mathbb{T}) \neq \emptyset \text{ and } \text{diam}_\Omega(B_k) < K' + k_0 T_1 K_0. \]

Since \( \Phi(H_{d_{n_j}}(I_{n_j}) \cup Z) \subset H_d(J) \), we have \( \Phi(B_k) \cap H_d(J) \neq \emptyset \) for all \( k \in \mathbb{K} \). Define \( F := \bigcup_{k \in \mathbb{K}} \Phi(B_k) \). Then \( F \in \Sigma_{I_\ell,d} \) and we have \( \mathcal{Y}_\Omega(H_d(J), F) < \mu_0(K' + k_0 T_1 K_0) = K'. \)

The proof is complete. \[ \square \]

To every \((\delta, \eta)\)-admissible sequence \( \{I_n\}_{n \geq 0} \), we associate a sequence of triples \( \{(J_n, F_n, d_n)\}_{n \geq 0} \) as following. Set \( F_0 = \emptyset \). Let \( \ell \geq 1, F_\ell \in \Sigma_{I_\ell,d_\ell} \) for \( 1 \leq i \leq \ell - 1 \) with \( d_\ell \in [d, d + 1] \), \( J \subset \mathbb{T} \) and \( F \subset \Sigma_{I_\ell,d} \) be guaranteed by Lemma 4.1. Define \( \mathcal{Y}_\Omega(H_d(J_i), F_i) < \mu_0(K' + k_0 T_1 K_0) = K'. \)

Let \( I'_0 := I_\ell \) and we define a new \((\delta, \eta)\)-admissible sequence \( \{I'_n\}_{n \geq 0} \) such that the pullback \( I'_i \to I'_{i+1} \) is induced by \( I_{n+\ell} \to I_{n+\ell+1} \) and \( I_{n+\ell} \subset I'_n \) for all \( n \geq 0 \). Denote \( F'_0 := F_\ell \). Then \( \mathcal{Y}_\Omega(H_d(I'_0), F'_0) < K'. \)
By applying Lemma 4.1 to \( \{I'_n\}_{n \geq 0} \) and \( F'_n \), we obtain \( \ell' \geq 1 \), \( F'_i \in \Sigma_{J',d'} \) for \( 1 \leq i \leq \ell' - 1 \) with \( d'_i \in [d,d+1] \), \( J' \subset T \) and \( F' \in \Sigma_{J',d'} \). Define
\[
J_{\ell+i} := I'_i, \quad F_{\ell+i} := F'_i, \quad d_{\ell+i} := d'_i \quad \text{for} \quad 1 \leq i \leq \ell' - 1; \quad \text{and} \\
J_{\ell+\ell'} := J', \quad F_{\ell+\ell'} := F', \quad d_{\ell+\ell'} := d.
\]
Inductively, by taking \( I''_0 := J_{\ell+\ell'} \), \( F''_0 := F_{\ell+\ell'} \) and repeating the above process, we thus obtain an improved \((\delta,\eta)\)-admissible sequence of triples \( \{(J_n,F_n,d_n)\}_{n \geq 0} \).

**Corollary 4.2.** For any \( \varepsilon > 0 \), there exists \( \tilde{\tau} = \tilde{\tau}(\varepsilon) \in (0,r_0) \) such that for any improved \((\delta,\eta)\)-admissible sequence \( \{(J_n,F_n,d_n)\}_{n \geq 0} \) with \( \sigma(J_0) = \sigma(I_0) < \tilde{\tau} \), we have \( \text{diam}_{\tilde{C}}(H_{d_n}(J_n) \cup F_n) < \varepsilon \) for all \( n \geq 0 \).

**Proof.** For each \( n \geq 0 \), suppose
\[
F_n = \bigcup_{k \in \mathbb{K}_n} B^n_k, 
\]
where \( \mathbb{K}_n \) is an at most countable set and each \( B^n_k \) is a Jordan disk in \( \Omega \) satisfying \( B^n_k \cap H_{d_n}(J_n) \neq \emptyset \). Then
\[
\text{diam}_{\tilde{C}}(H_{d_n}(J_n) \cup F_n) \leq \text{diam}_{\tilde{C}}(H_{d_n}(J_n)) + 2 \sup_{k \in \mathbb{K}_n} \{\text{diam}_{\tilde{C}}(B^n_k)\}.
\]
(4.2)

By Lemma 4.1, we have
\[
\sup_{k \in \mathbb{K}_n} \{\text{diam}_{\tilde{C}}(B^n_k)\} < K.
\]
(4.3)

From the construction of \( \{(J_n,F_n,d_n)\}_{n \geq 0} \) and Lemma 4.1, we have \( d_n \in [d,d+1] \) and \( \sigma(J_n) < 2\sigma(I_0) \) for all \( n \geq 0 \). Hence for given \( \varepsilon > 0 \), by (4.2) and (4.3), if \( \sigma(I_0) \) is small enough, then \( \text{diam}_{\tilde{C}}(H_{d_n}(J_n) \cup F_n) < \varepsilon \) for all \( n \geq 0 \).

4.2. Jumping off preimages. Let \( V_0 \subset H_d(I_0) \) be a Jordan disk and \( \{V_n\}_{n \geq 0} \) be a pullback sequence of \( V_0 \) under \( G \) in \( \tilde{C} \setminus \overline{D} \). By Lemma 2.1, every \( V_n \) is a Jordan disk. For any given arc \( I'_0 \subset T \) with \( \sigma(I'_0) < r_0 \), we consider the family of all the possible \((\delta,\eta)\)-admissible sequences:
\[
\mathcal{I} := \left\{ \tau = \{I_n\}_{n \geq 0} \mid \tau \text{ is a } (\delta,\eta)\text{-admissible sequence} \right\}
\]
and the corresponding improved \((\delta,\eta)\)-admissible sequences \( \{\tau' = \{(J_n,F_n,d_n)\}_{n \geq 0}\} \). Define
\[
n(\tau) := \sup \{n \in \mathbb{N} : V_k \subset H_{d_k}(J_k) \cup F_k \text{ for all } 0 \leq k \leq n\}
\]
and
\[
\chi(I'_0,\{V_n\}_{n \geq 0}) := \sup_{\tau \in \mathcal{I}} n(\tau).
\]

**Lemma 4.3.** For any \( \varepsilon > 0 \), let \( \tilde{\tau} = \tilde{\tau}(\varepsilon) > 0 \) be the number in Corollary 4.2. Then for any pullback sequence \( \{V_n\}_{n \geq 0} \) of a Jordan disk \( V_0 \subset H_d(I_0) \) with \( \sigma(I_0) < \tilde{\tau} \), we have the following two cases:

(a) If \( \chi(I_0,\{V_n\}_{n \geq 0}) = +\infty \), then \( \text{diam}_{\tilde{C}}(V_n) < \varepsilon \) for all \( n \geq 0 \);

(b) If \( \chi(I_0,\{V_n\}_{n \geq 0}) = N < +\infty \), then \( \text{diam}_{\tilde{C}}(V_n) < \varepsilon \) for all \( 0 \leq n \leq N \).
Definition (Jump off from $\mathbb{T}$). If $\chi(I_0, \{V_n\}_{n \geq 0}) = N < +\infty$, we say that $V_{N+1}$ is the first jump off from $\mathbb{T}$ with respect to $I_0$.

For a piecewise smooth curve $\gamma$ in $\Omega$, we use $l_\Omega(\gamma)$ to denote the length of $\gamma$ with respect to the hyperbolic metric $\rho_\Omega(z) |dz|$. Let $V$ be a Jordan disk in $\Omega$ with $\overline{V} \subset \Omega$.

For two different points $z_1, z_2 \in V$, let $\Gamma_V(z_1, z_2)$ be the collection of all smooth curves in $V$ connecting $z_1$ with $z_2$.

We define a variant hyperbolic diameter of $V$ in $\Omega$:

$$\overline{\text{diam}}_\Omega(V) := \sup_{z_1, z_2 \in V} \left\{ \inf_{\gamma \in \Gamma_V(z_1, z_2)} \{ l_\Omega(\gamma) \} \right\}.$$

By definition we have $\text{diam}_\Omega(V) \leq \overline{\text{diam}}_\Omega(V)$.

For an annulus $A$ in $\hat{\Omega}$, we use $\text{mod}(A)$ to denote the conformal modulus of $A$.

Let $\mathcal{P}(G)$ be the postcritical set of $G$ defined in (2.6). We denote

$$\widehat{\mathcal{P}}(G) := \{ z \in \mathcal{P}(G) \setminus \mathbb{T} : \exists n \geq 1 \text{ such that } G^n(z) \in \mathbb{T} \}.$$ (4.4)

Let $r_6 > 0$ be the number introduced in Lemma 4.1

Lemma 4.4. Let $M > 0$ and $r' \in (0, r_6]$ be given. Then there exists $r'' = r''(M, r') \in (0, r')$ such that for any pullback sequence $\{V_n\}_{n \geq 0}$ of $V_0 \subset H_d(I_0)$ with $I_0 \subset \mathbb{T}$ and $\sigma(I_0) < r''$, if $V_{N+1}$ is the first jump off from $\mathbb{T}$ with respect to $I_0$, then there exists a Jordan disk $U_{N+1}$ such that one of the following holds:

(a) The first type jump off: $V_{N+1} \subset U_{N+1} \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $\text{mod}(U_{N+1} \setminus \overline{V_{N+1}}) > M$ and $U_{N+1} \cap \mathcal{P}(G) \subset \mathcal{P}(G)$ with $\widehat{\mathcal{P}}(U_{N+1} \cap \mathcal{P}(G)) \leq 1$; or

(b) The second type jump off: $V_{N+1} \subset U_{N+1} \subset H_d(J')$ for an arc $J' \subset \mathbb{T}$ with $\sigma(J') = r'$,

$$\rho_\Omega(w) \leq \mu \rho_\Omega(G(w)) |G'(w)|$$ for $w \in V_{N+1}$ and $\overline{\text{diam}}_\Omega(U_{N+1}) < K,$

where $0 < \mu < 1$ and $K > 0$ are constants depending only on $\delta, \eta$ and $d$.

Proof. Let $\{(J_n, F_n, d_n)\}_{n \geq 0}$ be an improved $(\delta, \eta)$-admissible sequence such that $V_N \subset H_{d_n}(J_N) \cup F_N$. By Corollary 4.2 if $\sigma(I_0) > 0$ is small enough, then $H_{d_n}(J_N) \cup F_N$ can be arbitrarily close to $\mathbb{T}$ and its spherical diameter can be arbitrarily small.

Then one of the following happens:

(i) $v \in (1 + 2\delta)J_N$ for some critical value $v \in CV$; or

(ii) $v \notin (1 + 2\delta)J_N$ for any critical value $v \in CV$.

Suppose Case (i) happens. By the definition of jump off, it follows that the branch of $G^{-1}$ determined by $V_N \to V_{N+1}$, say $\Phi$, is not associated to any $(\delta, \eta)$-critical or non-critical pullback of $J_N$. So $\Phi(v)$ does not belong to $\mathbb{T}$ and $V_{N+1}$ is bounded away from $\mathbb{T}$. In particular, for given $M > 0$, there exists $s_1 = s_1(M) > 0$ such that if $\sigma(I_0) < s_1$, then $\text{diam}_\Omega(V_N)$ is sufficiently small and there exists a Jordan disk $U_{N+1}$ such that Part (a) holds.
Suppose Case (ii) happens. For three different points $z_0, z_1, z_2 \in \mathbb{C} \setminus D$, let $\angle z_1 z_0 z_2$ be the angle measured in the logarithmic plane of $G$ (see \eqref{2.28}). Let $\tilde{V}_{N+1}$ be the component of $G^{-1}(H_{d_J}(J_N) \cup F_N)$ which contains $V_{N+1}$. Let $r' \in (0, r_0]$ be given. We claim that there exist constants $0 < C_1 < C_2 < 1$ and $1 < K_1 = K_1(\delta, \eta, d) < \infty$ such that one of the following is true (see Figure 10):

1. $\text{dist}_C(\tilde{V}_{N+1}, T) > C_1 r'$; or
2. There exist an arc $J' = (x_1, x_2) \subset T$ containing a critical point $c \in T$ with $\sigma(J') = r'$ and constants $\beta \in (0, \pi/2)$, $r_{\text{int}}, r_{\text{out}} \in (0, 2r')$ depending only on $\delta, \eta, d$ with $1 < r_{\text{out}}/r_{\text{int}} < K_1$ such that $\tilde{V}_{N+1} \subset U_{N+1} \subset H_d(J')$, where

$$U_{N+1} = \{ z \in \mathbb{C} | r_{\text{int}} < |z - c| < r_{\text{out}} \text{ and } \angle z c x_2 \in (\beta, \pi - \beta) \}.$$ 

The lemma follows immediately by assuming the claim: if (1) is true, then Part (a) holds, and if (2) is true, then Part (b) holds.

![Figure 10: A sketch of the second type jump off.](image)

Now let us prove the claim. In fact, if $\text{dist}_C(\tilde{V}_{N+1}, T) < r'$ (note that $r' > 0$ is small), there must be a critical point $c \in T$ such that $\tilde{V}_{N+1}$ is contained in a small neighborhood of $c$. This implies that $H_{d_J}(J_N) \cup F_N$ is contained in a small neighborhood of the critical value $v = G(c)$. Without loss of generality, we assume that $v$ is on the right of $J_N = (a, b)$. By \eqref{4.3}, $H_{d_J}(J_N) \cup F_N$ is contained in a simply connected half hyperbolic neighborhood $H_{d_J}((a, v))$ for some $d_1 = d_1(\delta, \eta, d) > 0$, and is thus contained in a cone spanned at $v$ and bounded by $T$ and a ray which forms an angle $\beta_1 = \beta_1(\delta, \eta, d) \in (0, \pi/2)$ with $T$. Note that there exist constants $C_3 = C_3(\delta, \eta, d) > 0$ and $C_4 = C_4(\delta, \eta, d) > 0$ such that

$$\text{diam}_C(H_{d_J}(J_N) \cup F_N) < C_3 |J_N| < C_4 \text{dist}_C(J_N, v).$$

These, together with the fact that $G^{-1}$ is composed by a radical mapping and a quasiconformal mapping, implies that there exist $\beta = \beta(\delta, \eta, d) \in (0, \pi/2)$, $K_1 = K_1(\delta, \eta, d) > 1$ and $0 < r_{\text{int}} < r_{\text{out}}$ with $r_{\text{out}}/r_{\text{int}} < K_1$ such that

$$\tilde{V}_{N+1} \subset U_{N+1} = \{ z | r_{\text{int}} < |z - c| < r_{\text{out}} \text{ and } \angle z c x_2 \in (\beta, \pi - \beta) \}.$$ 

Now the claim follows by taking $0 < C_1 < 1$ small. In fact, let $J'$ be the arc with $c$ being the middle point and $\sigma(J') = r'$. If $\text{dist}_C(\tilde{V}_{N+1}, T) \leq C_1 r'$, then $r_{\text{int}} < C_1 C_5 r'$ for a constant $C_5 = C_5(\delta, \eta, d) > 1$. Since $r_{\text{out}} < K_1 r_{\text{int}}$, we get $0 < C_2 = K_1 C_1 C_5 < 1$ such that $0 < r_{\text{int}} < r_{\text{out}} < C_2 r'$ and $U_{N+1} \subset H_d(J')$. This finishes the proof.

**Lemma 4.5.** For any $\varepsilon > 0$, there exists $r > 0$ such that if $V_0 \subset H_d(I_0)$ for an arc $I_0 \subset \mathcal{T}$ with $\sigma(I_0) < r$, then for any pullback sequence $\{ V_n \}_{n \geq 0}$ of $V_0$, $\text{diam}_C(V_n) < \varepsilon$ for all $n \geq 0$. 


Proof. By the assumption in the Main Lemma, there exists $D_0 \geq 2$ such that for every $z \in \mathbb{T}$, there exists a small Jordan disk $W_0$ containing $z$, such that for any sequence $\{W_n\}_{n \geq 1}$ of pullbacks of $W_0$ in $\hat{C} \setminus \mathbb{D}$, we have $\deg(G^n : W_n \to W_0) \leq D_0$.

For any given $\varepsilon > 0$, there exist $\varepsilon' > 0$ and $M > 0$ depending only on $\varepsilon$ such that for any Jordan disks $U$ and $V$ in $\hat{C} \setminus \mathbb{D}$ with $V \subset U$, we have

$$\text{diam}_{\hat{C} \setminus \mathbb{D}}(V) < \varepsilon' \text{ or mod } (U \setminus V) > M/D_0, \text{ then diam}_{\hat{C}}(V) < \varepsilon.$$  \hspace{1cm} (4.5)

In particular, if diam$_{\Omega}(V) < \varepsilon'$, then diam$_{\hat{C}}(V) < \varepsilon$. For such $\varepsilon' > 0$, let $k_0 \geq 3$ be the smallest integer such that

$$\mu^{k_0 -1}K < \varepsilon', \hspace{1cm} (4.6)$$

where $0 < \mu < 1$ and $K > 0$ are constants guaranteed by Lemma 4.4(b).

Let $r_1' := \tilde{r}$, where $\tilde{r} = \tilde{r}(\varepsilon) > 0$ is the number introduced in Corollary 4.2. For such $r_1'$, by Lemma 4.4 we get $r_2' = r_2'(M, r_1') \in (0, r_1']$. Inductively, we get a sequence of numbers:

$$r_{j+1}' = r_{j+1}'(M, r_j') \in (0, r_j'], \hspace{0.5cm} \text{where } 1 \leq j \leq k_0.$$  \hspace{1cm} \hspace{0.5cm} (4.7)

We claim that $r := r_{k_0+1}'$ is the required number.

In fact, let $V_0 \subset H_d(I_0)$ with $\sigma(I_0) < r_{k_0+1}' \leq r_1'$. If $\chi(I_0, \{V_n\}_{n \geq 0}) = +\infty$, then by Lemma 4.3(a) we have diam$_{\hat{C}}(V_n) < \varepsilon$ for all $n \geq 0$. Otherwise, we have $\chi(I_0, \{V_n\}_{n \geq 0}) = N_1 < +\infty$, i.e., $V_{N_1+1}$ is the first jump off from $\mathbb{T}$ with respect to $I_0$. Then by Lemma 4.3(b) we have

$$\text{diam}_{\hat{C}}(V_n) < \varepsilon \hspace{0.5cm} \text{for all } 0 \leq n \leq N_1.$$  \hspace{1cm} (4.5)

By Lemma 4.4, there exists a Jordan disk $U_{N_1+1}$ such that one of the following holds:

1-i) $U_{N_1+1} \subset U_{N_1+1} \subset \hat{C} \setminus \mathbb{D}$, mod $(U_{N_1+1} \setminus \Omega_{N_1+1}) > M$ and $U_{N_1+1} \cap \mathcal{P}(G) \subset \hat{P}(G)$ with $\sharp(U_{N_1+1} \cap \mathcal{P}(G)) \leq 1$; or

1-ii) $\bar{V}_{N_1+1} \subset U_{N_1+1} \subset H_d(J_1')$ for an arc $J_1' \subset \mathbb{T}$ with $\sigma(J_1') = r_{k_0}'$, and

$$\text{diam}_{\Omega}(U_{N_1+1}) < K.$$  \hspace{1cm} (4.7)

If we are in Case (1-i), then diam$_{\hat{C}}(V_n) < \varepsilon$ for all $n \geq 0$ by (4.5) and (4.7).

Suppose we are in Case (1-ii). Then we consider all the possible improved $(\delta, \eta)$-admissible sequence beginning with $J_1'$. For $n \geq N_1 + 1$, let $W_n$ be the component of $G^{-(n-N_1)}(U_{N_1+1})$ containing $V_n$. Then $G^{n-N_1} : W_n \to U_{N_1+1}$ is conformal by Lemma 2.1. If $\chi(J_1', \{W_n\}_{n \geq N_1+1}) = +\infty$, then by Lemma 4.3(a) we have diam$_{\hat{C}}(W_n) < \varepsilon$ for all $n \geq N_1 + 1$. Combining (4.7), we have diam$_{\hat{C}}(V_n) < \varepsilon$ for all $n \geq 0$. Otherwise, we have $\chi(J_1', \{W_n\}_{n \geq N_1+1}) = N_2 - N_1 - 1 < +\infty$, i.e., $W_{N_2+1}$ is the first jump off from $\mathbb{T}$ with respect to $J_1'$. Then by Lemma 4.3(b) we have

$$\text{diam}_{\hat{C}}(V_n) < \varepsilon \hspace{0.5cm} \text{for all } 0 \leq n \leq N_2.$$  \hspace{1cm} (4.8)

By Lemma 4.4, there exists of the following holds:

2-i) There exists a Jordan disk $U_{N_2+1}$ such that $W_{N_2+1} \subset U_{N_2+1} \subset \hat{C} \setminus \mathbb{D}$, mod $(U_{N_2+1} \setminus \Omega_{N_2+1}) \geq \text{mod } (U_{N_2+1} \setminus \Omega_{N_2+1}) > M$ and $U_{N_2+1} \cap \mathcal{P}(G) \subset \hat{P}(G)$ with $\sharp(U_{N_2+1} \cap \mathcal{P}(G)) \leq 1$; or

2-ii) $W_{N_2+1} \subset H_d(J_2)$ for an arc $J_2 \subset \mathbb{T}$ with $\sigma(J_2) = r_{k_0-1}'$,

$$\rho_0(w) \leq \mu \rho_0(G(w)) |G(w)| \text{ for } w \in W_{N_2+1} \text{ and diam}_{\Omega}(W_{N_2+1}) < \mu K.$$
If we are in Case (2-i), then $\text{diam}_C(V_n) < \varepsilon$ for all $n \geq 0$ by (4.5) and (4.8).

Suppose we are in Case (2-ii). Then we consider the value of $\chi(J'_2, \{W_n\}_{n \geq N_k+1})$ and see whether it is equal to $+\infty$. Let $k_0 \geq 3$ be the integer introduced in (4.6).

Inductively, repeating the above process finitely many times, we have the following two cases:

**Case (I).** There exist an integer $2 \leq i \leq k_0 - 1$, a sequence of arcs $J'_i, \ldots, J'_i$ in $\mathbb{T}$ and a sequence of integers $N_1, \ldots, N_i$ such that

$$\chi(J'_k, \{W_n\}_{n \geq N_k+1}) = N_{k+1} - N_k - 1 \leq +\infty \quad \text{for all} \quad 1 \leq k \leq i - 1,$$

and either

- $\chi(J'_i, \{W_n\}_{n \geq N_i+1}) = +\infty$; or
- $\chi(J'_i, \{W_n\}_{n \geq N_i+1}) = N_{i+1} - N_i - 1 < +\infty$ and there exists a Jordan disk $U_{N_{i+1}} \subset U_{N_{i+1}} \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, mod $(U_{N_{i+1}} \setminus V_{N_{i+1}}) > M$ and $U_{N_{i+1}} \cap \mathcal{P}(G) \subset \mathcal{P}(G)$ with $\mathcal{P}(U_{N_{i+1}} \cap \mathcal{P}(G)) \leq 1$.

In either case we have $\text{diam}_C(V_n) < \varepsilon$ for all $n \geq 0$ by Lemma 4.3 and (4.5).

**Case (II).** There exist a sequence of arcs $J'_i, \ldots, J'_{k_0}$ in $\mathbb{T}$ and a sequence of integers $N_1, \ldots, N_{k_0}$ such that for all $1 \leq i \leq k_0 - 1$,

- $\chi(J'_i, \{W_n\}_{n \geq N_i+1}) = N_{i+1} - N_i - 1 < +\infty$; and
- $W_{N_{i+1}} = H_d(J'_{i+1})$ with $\sigma(J'_{i+1}) = r'_{k_0-i}$.

$$\rho_\Omega(w) \leq \mu \rho_\Omega(G(w)|G'(w)) \quad \text{for} \quad w \in W_{N_{i+1}} \quad \text{and} \quad \widehat{\text{diam}_\Omega(W_{N_{i+1}})} < \mu^i K.$$ 

By (4.6) we have

$$\text{diam}_C(V_{N_{k_0}+1}) \leq \widehat{\text{diam}_\Omega(W_{N_{k_0}+1}) < \mu^{k_0-1} K < \varepsilon'}.$$

By Lemma 4.3 and (4.5), $\text{diam}_C(V_n) < \varepsilon$ for all $n \geq 0$. The proof is complete. \qed

**4.3. Proof of the Main Lemma’.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree at least 2. A sequence $\{V_n\}_{n \geq 0}$ is called a pullback sequence of a Jordan disk $V_0$ under $f$ if $V_{n+1}$ is a connected component of $f^{-n}(V_n)$ for all $n \geq 0$. The following result was proved in [LM97], p. 86 (see also [Mas93], [TY96]).

**Lemma 4.6** ( Shrinking Lemma). Let $D \geq 1$ and $U_0, V_0$ be two Jordan disks in $\hat{\mathbb{C}}$. Suppose $U_0$ is not contained in any rotation domain of $f$ and $V_0$ is compactly contained in $U_0$. Then for any $\varepsilon > 0$, there exists an $N \geq 1$ such that for any pullback sequence $\{U_n\}_{n \geq 0}$ satisfying $\deg(f^n : U_n \to U_0) \leq D$, $\text{diam}_C(V_n) < \varepsilon$ for all $n \geq N$, where $V_n$ is any component of $f^{-n}(V_0)$ contained in $U_n$.

For the map $f$ in the Main Lemma, let $\mathcal{P}(G)$ be the postcritical set of the quasi-Blaschke model $G$ of $f$ defined in (2.6).

**Lemma 4.7**. Let $V_0 \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be a Jordan disk such that $\overline{V_0} \cap \mathcal{P}(G)$ is a subarc on $\mathbb{T}$. Then for any $r > 0$, there exists $N \geq 1$ such that for any pullback sequence $\{V_n\}_{n \geq 0}$ of $V_0$ in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, we have $\sigma(V_n \cap \mathbb{T}) < r$ for all $n \geq N$.

**Proof.** By the assumption, $I_0 = \overline{V_0} \cap \mathcal{P}(G)$ is a proper subarc of $\mathbb{T}$. Note that the forward orbit of any critical point $c_0 \in \text{Crit}(G) \cap \mathbb{T}$ is dense in $\mathbb{T}$ and in particular dense in $I_0$. For any $r > 0$, there exists $N = N(r) \geq 1$ such that

$$\max \left\{ \sigma(I') : I' \text{ is a connected component of } I_0 \setminus \{G^n(c_0) : 1 \leq n \leq N\} \right\} < r. \quad (4.9)$$
Denote $I_n = \nabla_n \cap T$ for $n \geq 1$, where $\{V_n\}_{n \geq 0}$ is any pullback sequence of $V_0$ in $\hat{C} \setminus \overline{D}$. Then $I_n$ is either an arc, a singleton or empty. If $I_n$ is a singleton or empty, then $I_m$ is also for all $m > n$. If $I_n$ is an arc containing a critical value $v = G(c)$ which divides $I_n$ into two subarcs $I'_n$ and $I''_n$, where $c \in \text{Crit}(G) \cap T$, then
\[\sigma(I_{n+1}) \leq \max\{\sigma(I'_n), \sigma(I''_n)\}. \tag{4.10}\]
By (4.9) and (4.10) we have $\sigma(I_n) < r$ for all $n \geq N$.

**Proof of the Main Lemma.** Let $V_0 \subset \hat{C} \setminus \overline{D}$ be a Jordan disk such that $V_0 \cap \mathcal{P}(G) \neq \emptyset$ and $V_0 \cap \mathcal{P}(G) \subset T$. Since $\text{dist}_{\mathbb{C}}(\mathcal{P}(G) \setminus (T, T)) > 0$, we claim that $\hat{C} \setminus (\overline{D} \cup V_0)$ has at most finitely many connected components, say $U_1, \ldots, U_m$, such that
\[U_i \cap (\mathcal{P}(G) \setminus T) \neq \emptyset, \quad \text{for every } 1 \leq i \leq m. \tag{4.11}\]

Indeed, otherwise there are infinitely many $U_i$’s satisfying (4.11) and there exists $z_0 \in T \cap \partial V_0$ such that $\partial V_0$ is not locally connected at $z_0$, which is a contradiction and hence the claim holds. By filling all (except at most one) connected components of $\hat{C} \setminus (\overline{D} \cup V_0)$ which are disjoint with $\mathcal{P}(G)$, we conclude that $V_0$ is contained in a Jordan disk $V_0'$ such that $V_0' \cap \mathcal{P}(G)$ is the union of finitely many subarcs on $T$.

Without loss of generality, we assume that $V_0 = V_0'$.

By considering a homeomorphism defined from $\overline{D}$ onto $V_0$, it is easy to see that $V_0$ can be written as the union of finitely many Jordan disks $V_0^1, \ldots, V_0^\ell$ such that $V_0^i \cap \mathcal{P}(G)$ is a subarc on $T$ for each $1 \leq i \leq \ell$. For any sequence of pullbacks $\{V_n\}_{n \geq 0}$ of $V_0$ under $G$ in $\hat{C} \setminus \overline{D}$, $G^n : V_n \to V_0$ is conformal and $G^n : \nabla_0 \to \nabla_0$ is a homeomorphism by Lemma 2.1. So without loss of generality, we assume that $\ell = 1$ and that $V_0 \subset \hat{C} \setminus \overline{D}$ is a Jordan disk such that $V_0 \cap \mathcal{P}(G)$ is a subarc on $T$.

For given $\varepsilon > 0$, let $r = r(\varepsilon/2) > 0$ be the number guaranteed by Lemma 4.5. By Lemma 4.7 there exists $N_0 = N_0(\varepsilon) \geq 1$ such that for any pullback sequence $\{V_n\}_{n \geq 0}$ of $V_0$ in $\hat{C} \setminus \overline{D}$,
\[\sigma(V_n \cap T) < r/2 \quad \text{for any } n \geq N_0. \]

Let $\varsigma : \overline{D} \to \nabla_{N_0}$ be a homeomorphism. By the uniform continuity of $\varsigma$, the Jordan disk $V_{N_0}$ can be written as the union of finitely many small Jordan disks, such that some of them are covered by $H_d(I_0)$ and the rest of them are $W_1, \ldots, W_{M_0}$, where
- $I_0 \subset T$ is an arc containing $\nabla_{N_0} \cap T$ with $\sigma(I_0) = 3r/4$; and
- Each $W_i$ with $1 \leq i \leq M_0$ is compactly contained in $\hat{C} \setminus \overline{D}$ satisfying $\nabla_i \cap \mathcal{P}(G) \subset \mathcal{P}(G)$ with $\sigma(\nabla_i \cap \mathcal{P}(G)) \leq 1$ and $\mathcal{P}(G)$ is defined in (4.4).

Since there are only finitely many choices of $V_{N_0}$ among all pullback sequences of $V_0$ in $\hat{C} \setminus \overline{D}$, it follows that the number $M_0 \geq 1$ above can be chosen uniformly for all pullback sequences of $V_0$. Note that every $W_i$ is compactly contained in a bigger Jordan disk $\hat{W}_i$, such that for any $n > N_0$, the degree of the restriction of $G^{n-N_0}$ on any connected component $\hat{W}^n_i$ of $G^{-(n-N_0)}(\hat{W}_i)$ in $\hat{C} \setminus \overline{D}$ has uniform upper bound, where $\hat{W}^n_i$ satisfies $G^k(\hat{W}^n_i) \subset \hat{C} \setminus \overline{D}$ for all $1 \leq k \leq n - N_0$. By Lemma 4.6 there exists a uniform $N > N_0$, such that if $n \geq N$, every component of $G^{-(n-N_0)}(\hat{W}_i)$ in $\hat{W}^n_i$ has spherical diameter less than $\varepsilon/(3M_0)$. Combining this with Lemma 4.5 it follows that if $n \geq N$, for any pullback sequence $\{V_n\}_{n \geq 0}$ of $V_0$ in $\hat{C} \setminus \overline{D}$, we have
\[\text{diam}(V_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{3M_0} \cdot M_0 < \varepsilon. \]

The proof is complete. \hfill \Box
5. Proof of the Main Theorem

To prove the local connectivity of a Julia set in $\hat{\mathbb{C}}$, we use the following criterion (see [Whi42, Theorem 4.4, p. 113]).

**Lemma 5.1 (LC criterion).** A compact subset $X$ in $\hat{\mathbb{C}}$ is locally connected if and only if the following two conditions hold:

(a) The boundary of every component of $\hat{\mathbb{C}} \setminus X$ is locally connected; and
(b) For any given $\varepsilon > 0$, there are only finitely many components of $\hat{\mathbb{C}} \setminus X$ whose spherical diameters are greater than $\varepsilon$.

Let $f$ be a rational map in the Main Theorem. By the assumption, the periodic Fatou components of $f$ can only be attracting basins (including super-attracting) or bounded type Siegel disks and the periodic points in $J(f)$ are all repelling. Moreover, $f$ has at least one cycle of bounded type Siegel disks.

5.1. Local connectivity of the boundaries of attracting basins. In this section, we assume that $f$ has at least one attracting basin. The main goal in this section is to prove the following result:

**Proposition 5.2.** The boundary of every immediate attracting basin of $f$ is locally connected.

Note that $J(f)$ is connected by the assumption. Without loss of generality, we assume that

(i) All periodic Fatou components of $f$ have period one and they consist of $p_0 \geq 1$ fixed Siegel disks $\Delta_p$ with $1 \leq p \leq p_0$ and $q_0 \geq 1$ fixed immediate attracting basins $A_q$ with $1 \leq q \leq q_0$ (Considering an appropriate iteration of $f$ if necessary); and

(ii) $f$ is postcritically finite in the attracting basins (By a standard quasiconformal surgery [BF14, Chapter 4] since every Fatou component of $f$ is simply connected).

By the assumption (ii), we conclude that every immediate attracting basin $A_q$ is super-attracting and contains exactly one critical point $a_q$ (without counting multiplicity). Let $\mathcal{P}(f)$ be the postcritical set of $f$. Then the following set is finite or empty:

$$W_1 := W \cup \mathcal{P}(f).$$

For every $1 \leq q \leq q_0$, there exists a small quasi-disk $B_q$ containing $a_q$ such that $f(B_q) \subset B_q$. Denote

$$W := \hat{\mathbb{C}} \setminus \left( \mathcal{P}(f) \cup \bigcup_{p=1}^{p_0} \Delta_p \cup \bigcup_{q=1}^{q_0} \{a_q\} \right).$$

Then $f^{-1}(W) \subset W$ and $f^{-1}(W_1) \subset W_1$. According to [Zha11], every $\partial \Delta_p$ is a quasi-circle and $\Delta_{p'} \cap \Delta_{p''} = \emptyset$ for any different integers $p', p'' \in [1, p_0]$.

**Lemma 5.3.** There exists $\delta_0 > 0$ such that for any $\varepsilon > 0$, there exists an integer $N \geq 1$, such that for any Jordan disk $V_0$ in $W_1$ with $\text{diam}_\mathbb{C}(V_0) < \delta_0$, we have $\text{diam}_\mathbb{C}(V_n) < \varepsilon$ for all $n \geq N$, where $V_n$ is any component of $f^{-n}(V_0)$.

**Proof.** Note that $W_1$ is a domain whose boundary consists of $p_0 + q_0$ quasi-circles. Then there exist finitely many Jordan disks $\{U_k : 1 \leq k \leq k_0\}$ in $W_1$ and a small number $\delta_0 > 0$ such that
\(W_1 = \bigcup_{k=1}^{k_0} U_k;\)

(ii) For every \(1 \leq k \leq k_0\), \(z(U_k \cap \mathcal{P}(f)) = z(U_k \cap \mathcal{P}_1(f)) \leq 1\) or \(\overline{U}_k \cap \mathcal{P}(f) \subset \partial \Delta_P\) for some \(1 \leq p \leq p_0\) and

(iii) Any Jordan disk \(V_0\) in \(W_1\) with \(\text{diam}_C(V_0) < \delta_0\) is contained in some \(U_k\) with \(1 \leq k \leq k_0\).

Indeed, let \(D\) be a domain in \(\mathbb{C}\) whose boundary consists of \(p_0 + q_0\) Euclidean circles and let \(\varsigma : \overline{D} \to \overline{W}_1\) be a homeomorphism. By the finiteness of \(\mathcal{P}_1(f)\) in \(W_1\), there exists a partition \(D = \bigcup_{k=1}^{k_0} \widetilde{U}_k\) such that (i) and (ii) hold for \(\{U_k = \varsigma(\widetilde{U}_k) : 1 \leq k \leq k_0\}\), where each \(\widetilde{U}_k\) is a Jordan disk. Moreover, \(\{\widetilde{U}_k : 1 \leq k \leq k_0\}\) can be chosen further such that if \(I\) is an arc on \(\partial D\) with the Euclidean length less than a constant \(\nu_0 > 0\), then \(I\) is contained in \(\partial \overline{U}_k \cap \partial D\) for some \(1 \leq k \leq k_0\). Adopting the proof by contradiction which is similar to the Lebesgue number theorem, there exists \(\delta_0 > 0\) such that any Jordan disk \(\widetilde{V}_0\) in \(D\) with \(\text{diam}_C(\widetilde{V}_0) < \delta_0\) is contained in some \(\widetilde{U}_k\) with \(1 \leq k \leq k_0\). Then the existence of \(\delta_0 > 0\) in (iii) follows by the uniform continuity of \(\varsigma^{-1} : \overline{W}_1 \to \overline{D}\).

Let \(\varepsilon > 0\) be given. By Lemma 3.6 and the Main Lemma, there exists \(N \geq 1\) such that for all \(n \geq N\), any component of \(f^{-n}(U_k)\) with \(1 \leq k \leq k_0\) has spherical diameter less than \(\varepsilon\).

Let \(N_+ := \{1, 2, 3, \ldots\}\) be the set of all positive integers. We use the following definition from [Mil06, Appendix E].

**Definition (Orbifolds).** A pair \((S, \nu)\) consisting of a Riemann surface \(S\) and a ramification function \(\nu : S \to N_+\) which takes the value \(\nu(z) = 1\) except on a discrete closed subset is called a **Riemann surface orbifold** (orbifold in short). A point \(z \in S\) is called a **ramified point** if \(\nu(z) \geq 2\).

A map \(h : (\widetilde{S}, \widetilde{\nu}) \to (S, \nu)\) between two orbifolds is called a **branched covering** if \(h : \widetilde{S} \to S\) is a branched covering and \(\nu(h(\zeta)) = \deg_\zeta(h) \widetilde{\nu}(\zeta)\) for all \(\zeta \in \widetilde{S}\), where \(\deg_\zeta(h)\) is the local degree of \(h\) at \(\zeta\). In particular, if \(\widetilde{S}\) is simply connected and \(\widetilde{\nu} \equiv 1\) on \(\widetilde{S}\), then \(h : \widetilde{S} \to (S, \nu)\) is called a **universal covering**.

For the hyperbolic Riemann surface \(W_1\) introduced in (5.1), we define an orbifold \((W_1, \nu)\) as following. If \(z \in W_1 \setminus \mathcal{P}_1(f)\), define \(\nu(z) = 1\). If \(z \in \mathcal{P}_1(f)\), define \(\nu(z)\) as the least common multiple of \(\{\deg_\zeta(f) \nu(\zeta) : \zeta \in f^{-1}(z)\}\). Since \(\mathcal{P}_1(f)\) is a finite set, it is easy to see that \((W_1, \nu)\) is an orbifold.

According to [Mil06, Theorem E.1], the orbifold \((W_1, \nu)\) has a universal covering \(\pi : \mathbb{D} \to (W_1, \nu)\). Let \(W'_1 \subset W_1\) be any connected component of \(f^{-1}(W'_1)\). Note that \(f : W'_1 \to W_1\) is a branched covering between Riemann surfaces and \(\nu(f(\zeta))\) is some integral multiple of \(\nu(\zeta)\) for all \(\zeta \in W'_1\). By [Mil06, §19, pp. 212–213], \(f^{-1}\) lifts to a single-valued holomorphic map \(\widehat{F} : \mathbb{D} \to \mathbb{D}\) such that

\[f \circ \pi \circ \widehat{F}(w) = \pi(w) \quad \text{for all } w \in \mathbb{D}.\]

Since \(f\) has repelling periodic points in \(W_1\), it follows that \(\widehat{F} : \mathbb{D} \to \mathbb{D}\) decreases the hyperbolic metric \(\rho_D(w)|dw| \in \mathbb{D}\). The universal covering \(\pi : \mathbb{D} \to W_1\) induces an orbifold metric \(\sigma(z)|dz|\) in \(W_1\) satisfying

\[\sigma(\pi(w))|\pi'(w)| = \rho_D(w).\]

The following result shows that \(f^{-1}\) contracts the orbifold metric \(\sigma(z)|dz|\) in \(W_1\). See [Mil06, §19, p. 213].
Lemma 5.4. For any compact subset $K$ of $f^{-1}(W_1)$, there exists a number $\lambda = \lambda(K) > 1$ such that for any $z \in K$, if $z$ and $f(z)$ are not ramified points, then
\[
\sigma(f(z))|f^#(z)| \geq \lambda \sigma(z),
\]
where $f^#$ denotes the spherical derivative.

For a given immediate super-attracting basin $A := A_q$, where $1 \leq q \leq q_0$, there exists a conformal map $\psi : \hat{\mathbb{C}} \setminus \bar{D} \to A$ which conjugates $\zeta \mapsto \zeta^u : \hat{\mathbb{C}} \setminus \bar{D} \to \hat{\mathbb{C}} \setminus \bar{D}$ to $f : A \to A$ for some integer $u \geq 2$. For $r > 0$ and $\theta \in \mathbb{R}/\mathbb{Z}$, we denote
\[
R_\theta(r) = E_r(\theta) := \psi(e^{r+2\pi i\theta}).
\]
The images
\[
R_\theta := R_\theta((0, +\infty)) \quad \text{and} \quad E_r := E_r(\mathbb{R}/\mathbb{Z})
\]
are the internal ray of angle $\theta$ and the equipotential curve of potential $r > 0$ in $A$ respectively.

Definition (Ray segments). For every $\theta \in \mathbb{R}/\mathbb{Z}$ and integer $n \geq 1$, the curve
\[
R_{\theta,n} := R_\theta([\frac{1}{e^n}, 1])
\]
is called a ray segment in $A$.

Lemma 5.5. For any $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that for any ray segment $R_{\theta,n}$ with $\theta \in \mathbb{R}/\mathbb{Z}$ and $n \geq 1$, there exists a continuous curve $\tilde{R}_n$ which is homotopic to $R_{\theta,n}$ in $\hat{\mathbb{C}} \setminus \mathcal{P}(f)$ relative to their end points, and moreover, $\tilde{R}_n$ is the union of two continuous curves $\tilde{R}_n^{\text{ess}}$ and $\tilde{R}_n^{\text{end}}$ satisfying
\[
l_{\hat{\mathbb{C}}} (\tilde{R}_n^{\text{ess}}) < C \quad \text{and} \quad \text{diam}_{\hat{\mathbb{C}}} (\tilde{R}_n^{\text{end}}) < \varepsilon,
\]
where $l(\cdot)$ denotes the length with respect to the spherical metric.

Proof. Since the closures of the Siegel disks $\{\Delta_p : 1 \leq p \leq p_0\}$ of $f$ are pairwise disjoint quasi-disks, there exists a quasiconformal mapping $\Psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the restriction $\Psi : \hat{\mathbb{C}} \setminus \bigcup_{p=1}^{p_0} \Delta_p \to \hat{\mathbb{C}} \setminus \bigcup_{p=1}^{p_0} \bar{D}_p$ is conformal, where $\bar{D}_1, \ldots, \bar{D}_{p_0}$ are pairwise disjoint closed spherical disks. Define
\[
F := \Psi \circ f \circ \Psi^{-1}, \quad \mathcal{P}(F) := \Psi(\mathcal{P}(f)) \quad \text{and} \quad \mathcal{P}_1(F) := \Psi(\mathcal{P}_1(f)).
\]
For convenience, we use the same notations in the dynamical plane of $F$ to denote the corresponding objects in the dynamical plane of the quasiregular map $F$ under $\Psi$. For example, we use $R_{\theta,n}$ (not $\Psi(R_{\theta,n})$) to denote the ray segment of $F$ etc. It suffices to prove that the lemma holds in the dynamical plane of $F$ and that $\tilde{R}_n^{\text{ess}}$ lies in the outside of a neighborhood (depending only on the given $\varepsilon$) of $\bigcup_{p=1}^{p_0} \bar{D}_p$ in which $\Psi$ is conformal with bounded distortion.

Without loss of generality, we assume that the corresponding domains $W$ and $W_1$ (see (5.1)) in the dynamical plane of $F$ have the following form:
\[
W = \hat{\mathbb{C}} \setminus \left(\mathcal{P}(F) \cup \bigcup_{p=1}^{p_0} \bar{D}_p \cup \bigcup_{q=1}^{q_0} \bar{B}_r(a_q)\right) \quad \text{and} \quad W_1 = W \cup \mathcal{P}_1(F),
\]
where $F(\bar{B}_r(a_q)) \subset B_{r'}(a_q) = \{z \in \hat{\mathbb{C}} : \text{dist}_{\hat{\mathbb{C}}}(z, a_q) < \delta'\}$. Let $E_1$ be the equipotential curve in $A$ with potential 1. Decreasing the size of $B_r(a_q)$ if necessary, we assume that $E_1 \subset W$. Then there exists a constant $C_0 > 0$ such that
\[
l_\sigma(R_{\theta,1}) < C_0 \quad \text{for all} \ \theta \in \mathbb{R}/\mathbb{Z}, \quad (5.2)
\]
where \( l_{\sigma}(\cdot) \) denotes the length with respect to the orbifold metric \( \sigma(z)|dz| \) in \( W_1 \).

It suffices to prove the lemma for \( \varepsilon > 0 \) which is sufficiently small. For \( 1 \leq p \leq p_0 \), we define

\[
Y_p := \{ z \in W : \text{dist}_{\hat{C}}(z, \partial D_p) < \varepsilon/2 \}.
\]

For two continuous curves \( \gamma_1 \) and \( \gamma_2 \) in \( W \) with the same end points, we denote

\[
\gamma_1 \simeq \gamma_2 \quad \text{in} \quad W,
\]

if \( \gamma_1 \) is homotopic to \( \gamma_2 \) in \( W \) relative to their end points.

We prove the lemma by induction. For \( n = 1 \), we take

\[
\tilde{R}_n = \tilde{R}_{n}^{\text{ess}} := R_{\theta,1} \quad \text{and} \quad \tilde{R}_{n}^{\text{end}} = \emptyset.
\]

Let \( D(E_1) \) be the component of \( \hat{C} \setminus E_1 \) containing the super-attracting fixed point \( a_q \).

We assume that \( \varepsilon > 0 \) is small enough such that \( D(E_1) \cap Y_p = \emptyset \) for all \( 1 \leq p \leq p_0 \).

Suppose there exists \( n \geq 1 \) such that

(i) For any ray segment \( R_{\theta,n} \) with \( \theta \in \mathbb{R}/\mathbb{Z} \), there exists a continuous curve \( \tilde{R}_n \subset W \setminus D(E_1) \) such that \( \tilde{R}_n \simeq R_{\theta,n} \) in \( W \), where \( \tilde{R}_n \) can be written as the union of two continuous curves \( \tilde{R}_n^{\text{ess}} \) and \( \tilde{R}_n^{\text{end}} \) (may be empty);

(ii) \( l_{\sigma}(\tilde{R}_n^{\text{ess}}) < C' \) and \( \tilde{R}_n^{\text{ess}} \cap Y_p = \emptyset \) for all \( 1 \leq p \leq p_0 \), where \( C' = C'(\varepsilon) \in [C_0, +\infty) \) is a number which will be specified later; and

(iii) \( \text{diam}_{\hat{C}}(\tilde{R}_n^{\text{end}}) < \varepsilon \).

Let \( g_n \) be the inverse branch of \( F \) which maps \( R_{u\theta,n} \) to \( R_{\theta}([u^{-\frac{1}{n-1}}, \frac{1}{u}]) \), where \( u = \text{deg}(F : A \to A) \geq 2 \). Then we have

\[
R_{\theta,n+1} = R_{\theta,1} \cup g_n(R_{u\theta,n}).
\]

Note that \( g_n \) can be analytically extended to any continuous curve in \( W \) which is homotopic to \( R_{u\theta,n} \). Therefore, for \( \tilde{R}_n \) obtained in the inductive assumption which
satisfies \( \tilde{R}_n = \tilde{R}_n^{\text{ess}} \cup \tilde{R}_n^{\text{end}} \simeq R_{\theta,n} \) in \( W \); \( g_n(\tilde{R}_n) \) is well-defined and homotopic to \( g_n(R_{\theta,n}) \). Let
\[
\gamma_{n+1}^f := \gamma_{n+1}^f \cup g_n(\tilde{R}_n^{\text{ess}}) \quad \text{and} \quad \gamma_{n+1}'' := g_n(\tilde{R}_n^{\text{end}}).
\]
Then
\[
\gamma_{n+1} := \gamma_{n+1}^f \cup \gamma_{n+1}'' \simeq R_{\theta,n+1} \quad \text{in} \ W.
\]
Since \( \tilde{R}_n \subset W \setminus D(E_1) \), it follows that \( \gamma_{n+1}^f \) and \( \gamma_{n+1}'' \) are continuous curves in \( W \setminus D(E_1) \). See Figure 11. In the following we deform \( \gamma_{n+1} \) in \( W \setminus D(E_1) \) such that the inductive assumptions hold for step \( n+1 \).

By (i), (ii), (5.2) and Lemma 5.4, there exists \( \mu = \mu(\varepsilon) \in (0, 1) \) which is independent of \( n \) such that
\[
l_\sigma(\gamma_{n+1}^f) < C_0 + \mu C'. \tag{5.3}
\]
By (iii), there exists a small number \( \varepsilon' > 0 \) depending only on \( \varepsilon \) such that
\[
\text{diam}_\mathcal{S}(\gamma_{n+1}'' < \varepsilon'. \tag{5.4}
\]
Let \( z_0 = R_\theta(1) \) and \( z_1 = R_\theta(1/u^{n+1}) \) be the two end points of \( \gamma_{n+1} \). We have the following two cases:

(a) \( z_1 \notin Y_p \) for any \( 1 \leq p \leq p_0 \); or
(b) \( z_1 \in Y_p \) for some \( 1 \leq p \leq p_0 \).

Suppose Case (a) holds. Note that \( \varepsilon > 0 \) is assumed to be sufficiently small. By (5.3) and (5.4), there exist a number \( C_1 = C_1(\varepsilon) > 0 \) and a continuous curve \( \tilde{\gamma}_{n+1} \) such that \( \tilde{\gamma}_{n+1} \simeq C_1 \) in \( W \setminus D(E_1) \), \( \tilde{\gamma}_{n+1} \cap Y_p = \emptyset \) for any \( 1 \leq p \leq p_0 \) and
\[
l_\sigma(\tilde{\gamma}_{n+1}) < l_\sigma(\gamma_{n+1}^f) + C_1 < C_0 + \mu C' + C_1.
\]
In this case we define \( \tilde{R}_{n+1} := \tilde{R}_{\text{ess}}^{n+1} \cup \tilde{R}_{\text{end}}^{n+1} \), where
\[
\tilde{R}_{\text{ess}}^{n+1} := \tilde{\gamma}_{n+1} \quad \text{and} \quad \tilde{R}_{\text{end}}^{n+1} := \emptyset.
\]
Then the induction at step \( n+1 \) is finished by setting \( C' := (C_0 + C_1)/(1 - \mu) \).

Suppose Case (b) holds. Let \( z_s \in \partial Y_p \setminus \partial D_p \) be the point in \( W \) such that the segment \( [z_s, z_1] \) is perpendicular to \( \partial Y_p \) (see Figure 12). Note that \( \varepsilon' \) in (5.4) depends on \( \varepsilon \).
Hence there exist a number \( C_2 = C_2(\varepsilon) > 0 \) and a continuous curve \( \tilde{\gamma}_{n+1} \subset W \setminus D(E_1) \) such that \( \tilde{\gamma}_{n+1} \cup [z_s, z_1] \simeq C_1 \) in \( W \), \( \tilde{\gamma}_{n+1} \cap Y_p = \emptyset \) for any \( 1 \leq p \leq p_0 \) and
\[
l_\sigma(\tilde{\gamma}_{n+1}) < C_0 + \mu C' + C_2.
\]
In this case we define \( \tilde{R}_{n+1} := \tilde{R}_{\text{ess}}^{n+1} \cup \tilde{R}_{\text{end}}^{n+1} \), where
\[
\tilde{R}_{\text{ess}}^{n+1} := \tilde{\gamma}_{n+1} \quad \text{and} \quad \tilde{R}_{\text{end}}^{n+1} := [z_s, z_1] \).
\]

Note that \( \text{diam}_\mathcal{S}(\tilde{R}_{\text{end}}^{n+1}) < \varepsilon \). Then the induction at step \( n+1 \) is finished by setting \( C' := (C_0 + C_2)/(1 - \mu) \). Denote \( C_3 := \max\{C_1, C_2\} \).

To sum up, we have proved that for any \( \theta \in \mathbb{R}/\mathbb{Z} \) and any \( n \geq 1 \), there exists a continuous curve \( \tilde{R}_n \) such that \( \tilde{R}_n \simeq R_{\theta,n} \) in \( W \), where \( \tilde{R}_n = \tilde{R}_n^{\text{ess}} \cup \tilde{R}_n^{\text{end}} \) with \( \tilde{R}_n^{\text{ess}} \cap Y_p = \emptyset \) for all \( 1 \leq p \leq p_0 \) and
\[
l_\sigma(\tilde{R}_n^{\text{ess}}) < C' = (C_0 + C_3)/(1 - \mu) \quad \text{and} \quad \text{diam}_{\mathcal{S}}(\tilde{R}_n^{\text{end}}) < \varepsilon.
\]
Note that there exists a constant \( b > 0 \) such that \( \sigma(z) > b \) for all \( z \in W \). Hence we have \( l_\mathcal{S}(\tilde{R}_n^{\text{ess}}) < C := C'/b \). The proof is complete. \( \square \)
Remark. If \( f \) is a polynomial in the Main Theorem, then the Julia set of \( f \) is locally connected by Proposition 5.2 since \( J(f) = \partial A \).

5.2. Proof of the Main Theorem. We give a proof of the Main Theorem in this subsection based on the criterion in Lemma 5.1.
Proof of the Main Theorem. Let $f$ be a rational map in the Main Theorem. Then every periodic Fatou component of $f$ is either a bounded type Siegel disk or an attracting basin. By [Zha11] and Proposition 5.2, the boundaries of all periodic Fatou components of $f$ are locally connected. This implies that the boundaries of all Fatou components of $f$ are locally connected. By Lemma 5.1 it suffices to prove that for every $\varepsilon > 0$ there are at most finitely many Fatou components of $f$ with the spherical diameter $> \varepsilon$.

Iterating $f$ if necessary, we assume that all periodic Fatou components of $f$ have period one. Let $U_0$ be a fixed Fatou component of $f$. If $U_0$ is completely invariant, then $J(f) = \partial U_0$ is locally connected by Proposition 5.2. Suppose $U_0$ is not completely invariant. Then $f^{-1}(U_0) \setminus U_0$ consists of at least one and at most finitely many Fatou components \{U_{1,1}, \ldots, U_{1,i_1}\}, where $i_1 \geq 1$. There exists an integer $k_0 \geq 1$ such that any connected component $U_{k_0+1,j}$ of $f^{-k_0}(U_{1,i})$ with $1 \leq i \leq i_1$ is disjoint with $\mathcal{P}(f)$. Therefore, for any connected component $U_{k+k_0+1,\ell}$ of $f^{-k}(U_{k_0+1,j})$ with $k \geq 1$, the map $f^k : U_{k+k_0+1,\ell} \to U_{k+1,j}$ is conformal.

Let $\delta_0 > 0$ be the constant introduced in Lemma 5.3. Then there exists a uniform constant $M \geq 1$ such that every $U_{k_0+1,j}$ can be covered by $M$ Jordan disks \{D_1, \ldots, D_M\} in $W_1$ whose spherical diameters are less than $\delta_0$. By Lemma 5.3 there exists an integer $N \geq 1$ such that for all $k \geq N$, any connected component of $f^{-k}(D_1)$ has spherical diameter less than $\varepsilon/M$. This implies that if $k \geq N$, then any connected component $U_{k+k_0+1,\ell}$ of $f^{-k}(U_{k_0+1,j})$ satisfies $\text{diam}_R(U_{k+k_0+1,\ell}) < \varepsilon$. Note that $f^{-n}(U_0)$ has only finitely many components for every $1 \leq n \leq N + k_0$ and $f$ has only finitely many fixed Fatou components. Thus there are only finitely many Fatou components of $f$ whose spherical diameters are greater than $\varepsilon$. \hfill \Box

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