On a Stochastic Representation Theorem for
Meyer-measurable Processes

Peter Bank\textsuperscript{a} and David Besslich\textsuperscript{a}

\textsuperscript{a}Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany.
E-mail: bank@math.tu-berlin.de; besslich@math.tu-berlin.de

Mathematical Subject Classification (2010): 60G07, 60G40, 60H30, 93E20

Keywords: Stochastic representation theorem, Meyer-σ-fields, Divided stopping times, Optimal stochastic control, Optimal stopping.

Abstract. In this paper we study a representation problem first considered in a simpler version by Bank and El Karoui \cite{2004}. A key ingredient to this problem is a random measure $\mu$ on the time axis which in the present paper is allowed to have atoms. Such atoms turn out to not only pose serious technical challenges in the proof of the representation theorem, but actually have significant meaning in its applications, for instance, in irreversible investment problems. These applications also suggest to study the problem for processes which are measurable with respect to a Meyer-σ-field that lies between the predictable and the optional σ-field. Technically, our proof amounts to a delicate analysis of optimal stopping problems and the corresponding optimal divided stopping times.

1. Introduction

In this paper we study a stochastic representation problem that was first considered in a simpler framework by Bank and El Karoui \cite{2004}. Specifically, we consider a Meyer-σ-field $\Lambda$ such as the predictable or optional σ-field and, under weak regularity assumptions, we construct a $\Lambda$-measurable process $L$ such that a given $\Lambda$-measurable process $X$ can be written as

$$X_S = \mathbb{E}\left[ \int_{[S, \infty)} g_t \left( \sup_{t \in [S, \ell]} L_v \right) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right]$$

at every $\Lambda$-stopping time $S$. Representations of this nature have proven useful in various stochastic optimal control problems, for instance, irreversible investment \cite{Riedel2011}, dynamic allocation via Gittins indices \cite{ElKaroui1994, Bank2007}, and optimal stopping \cite{Bank2003}. In Bank and El Karoui \cite{2004}, stochastic representations like (1) are proven for optional processes $X$ and atomless optional random measures $\mu$ with full support. Our main result, Theorem 2.9, generalizes their result in several ways.

Most notably, we solve the representation problem for measures $\mu$ with atoms. Such atoms not only pose considerable technical challenges for the representation problem (1), but also convey significant meaning in its applications. For instance, in an application of this representation problem to a novel version of the singular stochastic control problem of irreversible investment with inventory risk (see Bank and Besslich...
2018b] and, e.g., Riedel and Su [2011], Chiarolla and Ferrari [2014] for earlier versions), \( \mu \) and \( g \) are used to measure the incurred risk and the atoms of \( \mu \) reflect times of particular importance for the risk assessment; the process \( X \) describes the reward per additional investment unit. As proven in the companion paper Bank and Besslich [2018b], it then turns out that \( (\sup_{t \in [0, T]} L_t)_{t \geq 0} \) yields an optimal investment strategy. At any atom of \( \mu \), the optimal control has to trade off an improvement in the impending risk assessment against any reward from additional investment. How exactly this comes down also depends crucially on what information is available to the controller in this moment. This can be modelled by a Meyer-\( \sigma \)-field interpolating between “reactive” predictable controls and “proactive” optional ones. We refer to Lenglart [1980] for a detailed exposition on Meyer-\( \sigma \)-fields. To account for the full variety of such information dynamics, we solve (1) for an arbitrary Meyer-\( \sigma \)-field \( \Lambda \) instead of merely the optional \( \sigma \)-field. Another extension over Bank and El Karoui [2004] is that we can choose any random time horizon \( \hat{T} \) in (1), not just predictable stopping times. This is possible as in our result the assumption of full support on \( \mu \) for an arbitrary Meyer-exposition on Meyer-\( \sigma \)-fields. Another extension over Bank and El Karoui [2004] is based on the properties of optimal stopping times for the paths of \( \mu \) which, however, do not always obtain.

The technical challenges in establishing the representation (1) arise first due to the fact that the original construction of \( L \) in Bank and El Karoui [2004] is based on the properties of optimal stopping times for the family of auxiliary stopping problems

\[
Y^\ell_S = \esssup_{T \in S^\Lambda((S, \infty))} \mathbb{E} \left[ X_T + \int_{(S, T)} g(t) \mu(dt) \mid \mathcal{F}^\Lambda_S \right], \quad \ell \in \mathbb{R}, \ S \text{-}\Lambda\text{-stopping time.} \tag{2}
\]

When \( \mu \) has atoms, the running costs can exhibit upward and downward jumps and, so, optimal stopping times for (2) may exist only in the relaxed form of divided stopping times (or temps divisés) as introduced by El Karoui [1981]. Such divided stopping times are quadruples consisting of a stopping time and three disjoint sets decomposing the probability space. The analysis how these tuples depend on \( \ell \) requires a considerable more refined analysis than was necessary in Bank and El Karoui [2004]. Conversely, we show in Bank and Besslich [2018a] (see also Section 3.2), that a solution to (1) also allows us to solve such generalized stopping problems and thus offers an alternative to the usual approach via Snell envelopes as pursued in El Karoui [1981].

While conceptually very versatile for modelling information flows and technically convenient to unify the treatment of predictable and optional settings, the consideration of Meyer-\( \sigma \)-fields adds mathematical challenges of its own. For instance, the level passage times of a Meyer-measurable process may not necessarily be Meyer stopping times as is well-known for predictable processes. Second, for optional processes, right-upper-semi-continuity in expectation implies pathwise right-upper-semi-continuity; see Bismut and Skalli [1977]. For \( \Lambda \)-measurable processes however, this implication does not hold true in general. Such subtleties can be disregarded when \( \mu \) does not have atoms as in Bank and El Karoui [2004], but they become crucial for (1) and its applications both technically and conceptually when atoms are present.

This paper is organized as follows. Section 2 introduces the framework and the main result. In Section 3 we sketch some applications of the main theorem, first in irreversible investment and then in optimal stopping over divided stopping times; afterwards we state several open questions. In Section 4 we prove maximality and we prove existence of a solution \( L \) to (1). The technical proofs of auxiliary results are deferred to the appendix.

2. The stochastic representation problem

Let us fix throughout a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) where the filtration and \( \mathcal{F} := (\mathcal{F}_t)_{t \geq 0} \) satisfies the usual conditions of right-continuity and completeness and where \( \mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \subset \mathcal{F} \). Furthermore,
let Λ be a ℱ-complete Meyer-σ-field which contains the predictable-σ-field with respect to ℱ and which itself is contained in the optional-σ-field with respect to ℱ. We use the concept of Meyer-σ-fields as a flexible tool to model different information flows in optimal control problems (see our companion paper Bank and Besslich [2018b]). Moreover, Meyer σ-fields also allow us to prove our main result simultaneously for the predictable and the optional-σ-field, which are both special cases of Meyer-σ-fields. The theory of Meyer-σ-fields was initiated in Lenglart [1980]. We review and expand some of this material in the companion paper Bank and Besslich [2018a]. Let us recall here only the key definitions, which are particularly relevant in this paper. Upon first reading, the reader is invited to think of Λ as the optional-σ-field. In this case, Λ-stopping times S ∈ SA are just classical stopping times and the reader can skip directly to Section 2.2.

2.1. Meyer-σ-fields

The first theorem characterizes complete Meyer-σ-fields and, for the purpose of this paper, this notion can be thought of as a definition in our setting:

**Theorem 2.1** (Lenglart [1980], Theorem 5, p.509). A σ-field on Ω × [0, ∞) generated by càdlàg processes is a ℱ-complete Meyer-σ-field (Lenglart [1980], Definition 2, p.502) if and only if it lies between the predictable and the optional σ-field of a filtration satisfying the usual conditions.

For Meyer-σ-fields one can also define Meyer-projections, which are generalization of optional and predictable projections (see Bank and Besslich [2018a], Theorem 2.14). Uniqueness up to indistinguishability of the projection follows as usual from a suitable section theorem. For stating this theorem we have to use a generalized notion of stopping times:

**Definition 2.2** (Following Lenglart [1980], Definition 1, p.502). A mapping S : Ω → [0, ∞) is a Λ-stopping time, if [S,∞] := {(ω,t) ∈ Ω × [0, ∞) | S(ω) ≤ t} ∈ Λ. The set of all Λ-stopping times is denoted by SA. Additionally, we associate to each mapping S : Ω → [0, ∞] the σ-field FS := σ(ZS | Z : Ω × [0, ∞) → ℜ Λ-measurable).

Having introduced the concept of Λ-stopping times, we can now state the Meyer Section Theorem, which is the Meyer-σ-field extension of the powerful Optional and Predictable Section Theorems:

**Theorem 2.3** (Meyer Section Theorem, Lenglart [1980], Theorem 1, p.506). Let B be an element of Λ. For every ε > 0, there exists S ∈ SA such that B contains the graph of S, i.e. B ⊃ {(ω, S(ω)) | S(ω) < ∞}, and such that ℙ(S < ∞) > ℙ(π(B)) − ε, where π(B) := {ω ∈ Ω | (ω, t) ∈ B for some t ∈ [0, ∞)) denotes the projection of B onto Ω.

An important consequence is the following corollary:

**Corollary 2.4** (Lenglart [1980], Corollary, p.507). If Z and Z′ are two Λ-measurable processes, such that for each bounded T ∈ SA we have ZT ≤ Z′T a.s. (resp. ZT = Z′T a.s.), then the set {Z > Z′} is evanescent (resp. Z and Z′ are indistinguishable).

Let us conclude this section by introducing the following notation for various sets of stopping times:

SA ([0, ∞)) := {T ∈ SA | T < ∞ ℜ-a.s.} and, for a given S ∈ SA, we shall furthermore make frequent use of SA ([S, ∞)) := {T ∈ SA | T ≥ S ℜ-a.s.} as well as SA ((S, ∞)) := {T ∈ SA | T > S ℜ-a.s. on {S < ∞}}. Analogously, we define for R ∈ SA the sets SA , [S, R]), SA ([S, R]). Moreover, we also define ∥S,T∥ := {(ω, t) ∈ Ω × [0, ∞) | S(ω) ≤ t ≤ T(ω)} for S, T ∈ SO, which are stochastic intervals, and, analogously, ∥S,T∥, ∥S,T∥ and ∥S,T∥. Observe that the stochastic intervals defined in this way are always subsets of Ω × [0, ∞), even if the considered stopping times attain the value ∞ for some ω ∈ Ω. Finally, we use the convention that inf ∅ = ∞, sup ∅ = −∞, ∞ · 0 = 0, 0 = ∞, and we restrict a random time T to a set A by

\[ T_A = \begin{cases} & T \quad \text{on } A, \\ & \infty \quad \text{on } A^c. \end{cases} \]
2.2. The main theorem – statement and discussion

Let us now set the stage for our main result. Apart from the Meyer-σ-field \( \Lambda \), our representation problem needs a random Borel-measure \( \mu \) on \([0, \infty)\) and a random field \( g : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R} \) as input. These are assumed to satisfy the following conditions:

**Assumption 2.5.**  
(i) \( \mu \) is a random Borel-measure on \([0, \infty)\) with \( \mu(\{\infty\}) := 0 \).
(ii) The random field \( g : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R} \) satisfies:
   (a) For each \( \omega \in \Omega, \ t \in [0, \infty) \), the function \( g_t(\omega, \cdot) : \mathbb{R} \to \mathbb{R} \) is continuous and strictly increasing from \(-\infty\) to \( \infty \).
   (b) For each \( \ell \in \mathbb{R} \), the process \( g(\cdot, \ell) : \Omega \times [0, \infty) \to \mathbb{R} \) is \( \mathcal{F} \otimes \mathcal{B}([0, \infty)) \)-measurable with \( g(\cdot, \ell) \in L^1(\mathbb{P} \otimes \mu) \).

Furthermore the \( \Lambda \)-measurable process \( X \) with \( X_\infty = 0 \) to be represented should be uniformly integrable as proposed in El Karoui [1981], Proposition 2.26, p.123:

**Definition 2.6.** A process \( X \) is of class(\( D^\Lambda \)) if, with \( X_\infty := 0 \), \( \{X_T \mid T \in \mathcal{S}^\Lambda \} \) is uniformly integrable, i.e. if we have that \( \lim_{T \to \infty} \sup_{T \in \mathcal{S}^\Lambda} \mathbb{E}[(X_T 1_{\{|X_T| > r\}})] = 0 \).

In addition, the process to be represented should exhibit certain regularity properties, specified next:

**Definition 2.7.** Let \( X \) be a \( \Lambda \)-measurable process of class(\( D^\Lambda \)) with \( X_\infty = 0 \).
(a) The process \( X \) is left-upper-semicontinuous in expectation at every \( S \in \mathcal{S}^\Lambda \) in the sense that for any non-decreasing sequence \( (S_n)_{n \in \mathbb{N}} \subset \mathcal{S}^\Lambda \) with \( S_n < S \) on \( \{S > 0\} \) and \( \lim_{n \to \infty} S_n = S \) we have \( \mathbb{E}[X_S] \geq \limsup_{n \to \infty} \mathbb{E}[X_{S_n}] \).
(b) The process \( X \) is \( \Lambda \)-right-upper-semicontinuous in expectation at every \( S \in \mathcal{S}^\Lambda \) in the sense that for \( S \in \mathcal{S}^\Lambda \) and any sequence \( (S_n)_{n \in \mathbb{N}} \subset \mathcal{S}^\Lambda ([S, \infty]) \) satisfying \( \lim_{n \to \infty} \mu([S, S_n]) = 0 \) almost surely, we have \( \mathbb{E}[X_S] \geq \limsup_{n \to \infty} \mathbb{E}[X_{S_n}] \).
(c) The process \( X \) is \( \Lambda \)-upper-semicontinuous in expectation if it satisfies both (a) and (b).

**Remark 2.8.** (a) For \( \mu \) with no atoms and full support, we see that \( \Lambda \)-right-upper-semi-continuity in expectation is equivalent to the property that, for all \( S \in \mathcal{S}^\Lambda \) and every sequence \( S_n \in \mathcal{S}^\Lambda ([S, \infty]) \) which converges to \( S \) from above, we have that \( \mathbb{E}[X_S] \geq \limsup_{n \to \infty} \mathbb{E}[X_{S_n}] \). In the optional case \( \Lambda = \mathcal{O} \), our notion of right-upper-semi-continuity thus boils down to the classical condition of right-upper-semi-continuity in expectation in all \( S \in \mathcal{S}^\mathcal{O} \) (cf. El Karoui [1981], Proposition 2.42, p.141-142), which is well known to be equivalent to pathwise right-upper-semi-continuity (cf. Bismut and Skalli [1977], Theorem II.1.1, p.305).
(b) Notice that in our notions of \( \Lambda \)-upper-semi-continuity we only require to approximate with \( \Lambda \)-stopping times. In order to deduce path properties of \( X \), we extend in Bank and Besslich [2018a], Lemma 3.4, some results of Bismut and Skalli [1977] who confine themselves to the optional case \( \Lambda = \mathcal{O} \).
(c) Over periods between stopping times \( S, T \in \mathcal{S}^\Lambda \) with \( \mu([S, T)) = 0 \) a.s., \( \Lambda \)-upper-semi-continuity amounts to \( X \) being a \( \Lambda \)-supermartingale. This is inline with the requirements for the representations obtained in Föllmer and Knispel [2006] or El Karoui and Meziou [2008].

We are now in a position to state and discuss the main result of this paper:

**Theorem 2.9.** Suppose a given random field \( g \) and a given random measure \( \mu \) satisfy Assumption 2.5 and let \( X \) be a \( \Lambda \)-measurable process of class(D\( ^\Lambda \)) and satisfies \( X_S = 0 \) for \( S \in \mathcal{S}^\Lambda \) with \( \mu([S, \infty)) = 0 \) almost surely.

Then \( X \) admits a representation of the form
\[
X_S = \mathbb{E}\left[ \int_{[S, \infty)} g_t \left( \sup_{v \in [S, t]} L_v \right) \mu(dt) \mid \mathcal{F}_S^\Lambda \right], \quad S \in \mathcal{S}^\Lambda,
\]
for the unique (up to indistinguishability) \( \Lambda \)-measurable process \( L \) such that
\[
L_S = \text{ess inf}_{T \in \mathcal{S}^\Lambda([S, \infty))} f_{S, T}, \quad S \in \mathcal{S}^\Lambda([0, \infty)),
\]
where for $S \in \mathcal{S}^\Lambda([0,\infty))$ and $T \in \mathcal{S}^\Lambda((S,\infty))$, $\ell_{S,T}$ is the unique (up to a $\mathbb{P}$-null set) $\mathcal{F}_{S,T}^\Lambda$-measurable random variable such that
\[
\mathbb{E} \left[ X_S - X_T \mid \mathcal{F}_{S,T}^\Lambda \right] = \mathbb{E} \left[ \int_{[S,T]} g_t(\ell_{S,T}) \mu(dt) \mid \mathcal{F}_{S,T}^\Lambda \right] \quad \text{on} \quad \{ \mathbb{P}(\mu([S,T])) > 0 \mid \mathcal{F}_{S,T}^\Lambda > 0 \}
\]
and $\ell_{S,T} = \infty$ on $\{ \mathbb{P}(\mu([S,T])) > 0 \mid \mathcal{F}_{S,T}^\Lambda = 0 \}$. Furthermore, this process $L$ satisfies
\[
\sup_{S \in \mathcal{S}^\Lambda} \mathbb{E} \left[ \int_{[S,\infty)} \left| g_t \left( \sup_{v \in [S,T]} L_v \right) \right| \mu(dt) \right] < \infty. \tag{6}
\]

The process $L$ is maximal in the sense that $\hat{L}_S \leq L_S$ for any $S \in \mathcal{S}^\Lambda([0,\infty))$ for every other $\Lambda$-measurable process $\hat{L}$ satisfying, mutatis mutandis, (4) and (6). In addition, $L$ is uniquely determined in the atoms of $\mu$ in the sense that, for any $S \in \mathcal{S}^\Lambda$, we have $\hat{L}_S = L_S$ a.s. on $\{ \mathbb{P}(\mu(S)) > 0 \mid \mathcal{F}_{S,T}^\Lambda > 0 \}$.

The most important difference technically and conceptually with the result from Bank and El Karoui [2004] is that we now allow the measure $\mu$ to have atoms. In applications, such atoms mark points in time of particular significance to the problem at hand and it becomes important what is known about and expected for the considered processes at those crucial moments. Thus, one has to specify carefully the information flow under consideration. This can be accomplished in a flexible, yet rigorous manner by using Meyer-$\sigma$-fields and indeed the solution $L$ constructed by our representation theorem depends on the considered Meyer-$\sigma$-field. In particular, a predictable process $X$ can have different representations $L^\Lambda$ as we let $\Lambda$ interpolate between the predictable and the optional $\sigma$-field. In Bank and Besslich [2018b] (see the short summary in Section 3.1), we illustrate in a control theoretic setup how the particular choice of Meyer-$\sigma$-field $\Lambda$ affects $L$ as $\Lambda$ is interpolating between the two extreme cases. In this application, the atoms of $\mu$ lead to the optimality of ladlag controls whose jumps reflect the different information levels corresponding to these atoms. The optimality of ladlag policies is closely related to the notion of divided stopping times that are introduced by El Karoui [1981] to describe a most general solution of optimal stopping problems; see Section 3.2 for more details.

Another insight of our Theorem 2.9 that goes beyond the earlier representation theorem of Bank and El Karoui [2004] is that the construction of a maximal solution and its characterization via (5). Indeed, the latter characterization is obtained in Bank and El Karoui [2004] only under the additional assumption that $L$ has right-upper-continuous paths, a property which does not hold true in general. Our result, by contrast, shows that (5) in fact always holds true for the maximal solution. Indeed, maximality of a solution is, arguably, the appropriate notion here as obviously any solution $L$ can be modified downwards on a $\mu$-nullset without significantly affecting its running suprema $(\sup_{v \in [S,T]} L_v)_{t \in [0,\infty)}$, $S \in \mathcal{S}^\Lambda$ (and thus its representation property). This reasoning of course does not apply for the atoms of $\mu$ at which we in fact establish uniqueness of the solution to the representation problem.

A final improvement due to our result is that we do not require $\mu$ to have full support as assumed in Bank and El Karoui [2004]. In particular, we can readily embed discrete-time frameworks and can take any random time $T$ as time horizon. Not insisting on full support allows us in the optimal control application of Section 3.1 to specify the measure $\mu$ by a Poisson process. More importantly, our representation theorem unifies the one of Bank and El Karoui [2004] with the representation theorem of El Karoui and Föllmer [2005], El Karoui and Meziou [2008], Föllmer and Knispel [2006]; The former is concerned with the special case $\mu(dt) = dt$ (see El Karoui and Föllmer [2005], Theorem 5.2, p.15), while the latter two consider $\mu(dt) = \text{Dirac}_T(dt)$ for a fixed stopping time $T > 0$ (see Föllmer and Knispel [2006], Corollary 3.1, p.7 and El Karoui and Meziou [2008], Theorem 2.7, p.653).
work with the relaxed notion of divided stopping times when studying the structure of the considered family of stopping problems. Furthermore, the consideration of Meyer-σ-fields rather than just the optional one poses technical challenges of its own. For instance, passage times of Λ-measurable processes might not be Λ-stopping times. Moreover, Λ-µ-upper-semi-continuity does not imply pathwise upper-semicontinuity even when µ has full support. This makes some estimates more difficult to argue.

3. Applications of the extended representation theorem and open questions

Let us give a short outline of the new applications in irreversible investment and optimal stopping that are mentioned in the introduction and in the discussion after Theorem 2.9.

3.1. Irreversible investment with inventory risk

In optimal control one has to specify what the controller knows when about exogenous shocks and how and when she can act on this information. Meyer-σ-fields can be used to model the precise information flow in such situations. Given such an information setup, the controller has to trade off revenue from investment against penalizations for inventory risk. When searching for an optimal control, one obtains a representation problem as in (4) where X represents the reward from investment, µ is a “risk clock” and \( g = \partial_\mu \rho_t \) measures the marginal risk effect, where \( \rho(c) \) is the risk assessment when having invested \( c \). The solution \( L \) to (4) and (6) then gives us an optimal control starting in \( c_0 \) via \( c_0 \lor \sup_{v \in [0, t]} L_v \). In particular, this optimal control might be just \( \delta \), which means that at special time points an investor could make a precautionary investment before and a reactive investment after this moment. For more details, we refer to our companion paper Bank and Besslich [2018b]. This also provides a first nontrivial explicit solution to our general representation problem (4) where \( X \) is a compound Poisson process and \( \mu \) is the counting measure of its jumps.

3.2. An optimal stopping problem over divided stopping times

Suppose \( X \) can be represented as in (4) for some given \( \mu \) and \( g \) as in Assumption 2.5. Then the process \( L \) can be viewed as a universal stopping signal for the parametrized optimal stopping problems

\[
\sup_{\tau \in \mathcal{S}^{\Lambda, \text{div}}} \mathbb{E} \left[ X_\tau + \int_{[0, \tau]} g_t(\ell) \mu(\mathrm{d}t) \right], \quad \ell \in \mathbb{R}.
\]

Indeed, an optimal divided stopping time (cf. El Karoui [1981], Definition 2.37, p.136 or Bank and Besslich [2018a], Definition 2.36) for parameter \( \ell \) is given by \( \tau_\ell := (T_\ell, \emptyset, H_\ell, (H_\ell)_c) \), where \( H_\ell := \{ L_{T_\ell} \geq \ell \} \) and \( T_\ell := \inf \left\{ t \geq 0 \mid \sup_{v \in [0, t]} L_v \geq \ell \right\} \). Thus the same representing process \( L \) indeed works simultaneously for all stopping problems parametrized by \( \ell \in \mathbb{R} \). Hence, the universal signal concept first observed for classical stopping problems in Bank and Föllmer [2003], Theorem 2, p.6, who have to assume upper-right-continuity of \( L \), though. Without this regularity property on \( L \), optimal stopping times attaining the value in (7) can no longer be expected. But, as stated above, we still can describe optimal divided stopping times and thus provide an optimal stopping characterization alternative to the Snell-envelope approach of El Karoui [1981], Theorem 2.39, p.138. For more details see Bank and Besslich [2018a], Chapter 4.

Let us observe also that the above result can be viewed as a converse to the construction of the maximal solution \( L \) that we carry out in Section 4.2.

3.3. Some open questions

Let us mention several open questions, which might be interesting for future research.
3.3.1. Regularity properties of $L$

In applications, it is often of interest to know about the regularity of the solution $L$ to the representation problem; see Ferrari [2015], Proposition 3.4, p.9, for applications in singular control problems or Bank and Küchler [2007], Theorem 1, p.1360-1361 for an application in dynamic allocation problems going back to Gittins’s multi-armed bandit problems. In the optional case $\Lambda = \mathcal{O}$, we have the following result:

**Lemma 3.1.** In the setting of Theorem 2.9, suppose we have $\Lambda = \mathcal{O}$ and assume $\mu$ does not admit atoms. Then $L$ satisfying (4), (5) and (6) is pathwise right-upper-semicontinuous.

**Proof.** Fix $S \in \mathcal{S}^\Lambda$ and consider $(S_n)_{n \in \mathbb{N}}$ with $S_n \geq S$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} S_n = S$. From Bank and El Karoui [2004], Lemma 4.1, p.1041, we obtain that $\tilde{L}$ defined by $\tilde{L}_t := \limsup_{s \downarrow t} L_s := \lim_{s \downarrow t} \sup_{[t,t+\epsilon]} L_v$ also solves (4) with (6). As $\tilde{L}$ is $\mathcal{F}$-progressively measurable (see Dellacherie and Meyer [1978], Theorem 33, p.103) we obtain by the same arguments as in Bank and El Karoui [2004], Proof of Theorem 1, p.1042, that $\limsup_{n \to \infty} L_{S_n} \leq \tilde{L}_S \leq \text{ess inf}_{T \in \mathcal{S}^\Lambda([S,\infty))} \ell_{S,T} = \tilde{L}_S$. Meyer’s optional Section Theorem allows us to conclude the claimed pathwise right-upper-semicontinuity.

For general Meyer-$\sigma$-fields $\Lambda$, the argument proving the previous lemma breaks down. This is due to the fact that even when the above $\tilde{L}$ has the representation property (4) (as will be the case, e.g., for atomless $\mu$), its evaluation at time $S \in \mathcal{S}^\Lambda$ will be $\mathcal{F}_S$-measurable, but not $\mathcal{F}_S^\Lambda$-measurable in general, and we thus cannot estimate it against the $\mathcal{F}_S^\Lambda$-measurable $\text{ess inf}_{T \in \mathcal{S}^\Lambda([S,\infty))} \ell_{S,T}$. This seemingly technical point corresponds in fact to an observation which is natural in the singular control problem mentioned in Section 3.1: in the case of a right-upper-semicontinuous $L$, there will be no reactive investment; for $\Lambda = \mathcal{P}$, though, this reactive control turns out to be the only way the controller will intervene. We refer to Bank and Besslich [2018a], Chapter 4 for the details in an explicit example. These considerations lead us to ask the following:

**Open question:** Is there a natural extension of Lemma 3.1? Is $\{L < ^L\tilde{L}\}$ an evanescent set?

3.3.2. Necessary conditions for the existence of $L$

In Theorem 2.9 we have seen sufficient conditions for the existence of a solution $L$ to (4). Concerning necessary conditions, one can prove that to obtain a process $L$ satisfying (4) and (6) a $\Lambda$-measurable process of class$(\mathcal{D}^\Lambda)$ with $X_\infty = 0$ has to be $\Lambda\mu$-right-upper-semicontinuous (for a proof see Besslich [2019], Proposition 1.10, p.15) and that it has to satisfy $X_S = 0$ for $S \in \mathcal{S}^\Lambda$ with $\mu([S,\infty)) = 0$ almost surely (for a proof see Besslich [2019], Proposition 1.15, p.19). On the other hand, Besslich [2019], p.18-19, shows that a process $X$ which satisfies the previous two necessary conditions may not have a representation as in (4). Moreover, it is shown there that there are processes which can be represented, but which are not left-upper-semicontinuous in expectation. This leads to the following natural question:

**Question:** What are the necessary conditions to ensure that a $\Lambda$-measurable process of class$(\mathcal{D}^\Lambda)$ can be represented as in (6)? Moreover, how can we weaken the left-semi-continuity requirement on $X$ and still get a representation?

3.3.3. Continuous dependence on $\Lambda$

In Bank and Besslich [2018a], Chapter 4, one can see that for the family of Meyer-$\sigma$-fields $\Lambda^n$, $\eta \geq 0$, considered there, the solution $L^{\Lambda^n}$ to (4) converges to $L^{\mathcal{P}}$ (resp. $L^{\mathcal{O}}$), when “$\Lambda^n$ approaches $\mathcal{P}$” (resp. $\mathcal{O}$) in a suitable way. As the dependence on $\Lambda$ of a solution $L$ to (4) is rather intricate in general this suggests the following:

**Open question:** Which notion of convergence for Meyer-$\sigma$-fields ensures continuity of $\Lambda \mapsto L^{\Lambda}$? And what notion of convergence for $X$ ensures continuous dependence of $\Lambda$ on $X$?

4. Proof of the representation theorem

We start the proof of Theorem 2.9, by establishing in Section 4.1 the maximality of a given solution $L$ to (4) with (5) and (6). The construction of such a solution is carried out in Section 4.2, where we relegate the proofs of auxiliary results to Appendix A.
4.1. Proof of maximality and uniqueness at atoms

To prove the maximality claim of Theorem 2.9, let \( L \) be a \( \Lambda \)-measurable process with (4), (5) and (6) and consider another \( \Lambda \)-measurable process \( \tilde{L} \) satisfying, mutatis mutandis, (4) and (6). By the same arguments as in Bank and El Karoui [2004], Proof of Theorem 1, p.1042, one can see that \( \tilde{L}_S \leq \text{ess inf}_{T \in \mathcal{S}^\lambda(\{S,\infty\})} \ell_{S,T} = L_S \) for all \( S \in \mathcal{S}^\lambda \). Hence, \( L \) with (5) is indeed maximal. To obtain the claimed uniqueness at atoms of \( \mu \), use the representation property (4), strict monotonicity of \( \ell \) and the maximality of \( L \) to deduce

\[
X_S = \mathbb{E}[g_S(\tilde{L}_S)\mu(\{S\})|\mathcal{F}_S^\lambda] + \mathbb{E}\left[ \int_{(S,\infty)} g_t \left( \sup_{v \in [S,t]} \tilde{L}_v \right) \mu(dt) \bigg| \mathcal{F}_S^\lambda \right] \leq \mathbb{E}(g_S(\tilde{L}_S) - g_S(L_S))\mu(\{S\})|\mathcal{F}_S^\lambda] + X_S \leq X_S.
\]

Therefore, \( \mathbb{E}(g_S(\tilde{L}_S) - g_S(L_S))\mu(\{S\})|\mathcal{F}_S^\lambda] = 0 \) which implies \( (g_S(\tilde{L}_S) - g_S(L_S))\mu(\{S\}) = 0 \) (see for example Proposition A.5). By strict monotonicity of \( \ell \) we obtain that \( \{L_S = \tilde{L}_S \} \subset \{\mu(\{S\}) > 0 \} \) almost surely. As \( \{L_S = \tilde{L}_S \} \in \mathcal{F}_S^\lambda \) we get \( \{L_S = \tilde{L}_S \} \subset \{\mathbb{P}(\mu(\{S\}) > 0)|\mathcal{F}_S^\lambda) > 0 \} \) a.s. (see again Proposition A.5).

4.2. Construction of a solution to the representation problem

In this section, we construct a solution to the representation problem (4) satisfying (5) and (6) as stated in Theorem 2.9. As in Bank and El Karoui [2004], the idea is to introduce suitable stopping problems which can be analyzed using the general results of El Karoui [1981]. El Karoui [1981] also uses the theory of Meyer-\( \sigma \)-fields, developed by Lenglart [1980], and introduces stopping problems for those \( \sigma \)-fields. One key tool to describe optimality results and to get an intuition about those stopping problems is the Snell envelope. We also use this concept and more precisely we construct, as in Bank and El Karoui [2004], a regular version of the family of Snell envelopes \( \{Y^t\}_{t \in \mathbb{R}} \) given by

\[
Y^t_S = \text{ess sup}_{T \in \mathcal{S}^\lambda(\{S,\infty\})} \mathbb{E}\left[ X_T + \int_{(S,T]} g_t(\ell)\mu(dt) \bigg| \mathcal{F}_S^\lambda \right], \quad S \in \mathcal{S}^\lambda.
\]

This regular version will allow us to show that a solution \( L \) to the representation problem (4) is given by

\[
L_t(\omega) := \sup \{ \ell \in \mathbb{R} \mid Y^\ell_S(\omega) = X_\ell(\omega) \}, \quad (\omega, t) \in \Omega \times [0, \infty),
\]

i.e. at any time \( S \in \mathcal{S}^\lambda \), \( L_S \) is the maximal value \( \ell \) for which the optimal stopping problem introduced by \( Y^\ell_S \) is solved by stopping immediately at time \( S \).

One key problem in our work compared to the approach of Bank and El Karoui [2004] is that, for fixed \( \ell \), we do not get in general a stopping time which solves the optimal stopping problem introduced by \( Y^\ell \). This is due to the atoms of \( \mu \) and because the process \( X \) is not pathwise right-upper-semicontinuous in general. Therefore we use the so-called “temps divisés” or divided stopping times as discussed more precisely in El Karoui [1981], p. 136-140; see also Bank and Besslich [2018a]. These will give us solutions to a suitable relaxation of the above optimal stopping problem.

This section is organized as follows. We start with a convenient normalization of \( g \) in Section 4.2.1. Afterwards we construct in Section 4.2.2 the mentioned processes \( \{Y^t\}_{t \in \mathbb{R}} \) and their aggregation \( Y \). In Section 4.2.3, we define \( L \) and prove some auxiliary properties, especially that \( L \) satisfies (5), which we will use in Section 4.2.4 to finally show that \( L \) solves the representation problem (4) with integrability condition (6).

4.2.1. A convenient normalization

We will assume henceforth that \( g \) is normalized in the sense that

\[
g_t(\omega, 0) = 0, \quad (\omega, t) \in \Omega \times [0, \infty).
\]

This is without loss of generality as one could consider the auxiliary processes \( \tilde{g} \) and \( \tilde{X} \) given by \( \tilde{g}(\ell) := g(\ell) - g(0), \ell \in \mathbb{R}, \tilde{X} := X - \lambda(\int_0^{\infty} g_t(0)\mu(dt)) \); it is readily checked that \( \tilde{X} \) and \( \tilde{g} \) inherit the required regularities from \( X \) and \( g \), and that a process \( L \) representing \( \tilde{X} \) with \( \tilde{g} \) will also represent \( X \) with \( g \).
4.2.2. A family of optimal stopping problems

This section introduces some auxiliary stopping problems and provides a suitably regular choice of the corresponding Snell-envelopes \((Y^\ell)_{\ell \in \mathbb{R}}\). This regularity will be crucial for the construction of the maximal solution \(L\) to our representation problem (4) in Lemma 4.2. For ease of exposition, the proof of the following rather technical result is deferred to the Appendix A.2.

**Lemma 4.1.** There is a jointly measurable mapping \(Y : \Omega \times [0, \infty] \times \mathbb{R} \to \mathbb{R}, (\omega, t, \ell) \mapsto Y^\ell_\omega(t)\) with the following properties:

(i) For each \(\ell \in \mathbb{R}\), the process \(Y^\ell : \Omega \times [0, \infty) \to \mathbb{R}, (\omega, t) \mapsto Y^\ell_\omega(t)\) is \(\Lambda\)-measurable, l\'adl\'ag and of class(D\(^A\)) with

\[
Y^\ell_S = \text{ess sup}_{T \in \mathcal{S}^A_{\Lambda}(S, \infty)} E \left[ X_T + \int_{[S, T)} g_t(\ell) \mu(\text{d}t) \right| \mathcal{F}^A_S], \quad S \in \mathcal{S}^A. \tag{9}
\]

(ii) Define the l\'adl\'ag \(\Lambda\)-measurable processes \(E^\ell\) of class(D\(^A\)), \(\ell \in \mathbb{R}\), by

\[
E^\ell := \Lambda \left( \int_{[0, \cdot)} g_t(\ell) \mu(\text{d}t) \right) + \Lambda \left( \int_{[0, \infty)} |g_t(\ell)| \mu(\text{d}t) \right) + M^X + 1 \tag{10}\]

and

\[
E^\ell_\infty := \int_{[0, \infty)} g_t(\ell) \mu(\text{d}t) + \int_{[0, \infty)} |g_t(\ell)| \mu(\text{d}t) + M^X_\infty + 1 \tag{11}
\]

with the \(\Lambda\)-martingale \(M^X\) such that \(-M^X \leq X \leq M^X\) as in Lemma A.1. Then there is a version of the stochastic field \((E^\ell)_{\ell \in \mathbb{R}}\) such that, for all \(\omega \in \Omega\), \(E_\omega(\cdot)\) has the following properties:

(a) Locally uniform continuity in \(\ell \in \mathbb{R}\), i.e.

\[
\lim_{\delta \downarrow 0} \sup_{\ell', \ell'' \in \mathbb{R}} \sup_{T \in [0, \infty]} |E^\ell(\omega) - E^{\ell'}(\omega)| = 0 \quad \text{for all compact sets } C \subset \mathbb{R}. \tag{12}
\]

(b) L\'adl\'ag paths for \(\ell \in \mathbb{R}\), i.e. for any \(\ell \in \mathbb{R}\) the mapping \(t \mapsto E^\ell_\omega(t)\) is real valued and l\'adl\'ag. In particular the paths \(t \mapsto E^\ell_{t+}(\omega)\) and \(t \mapsto E^\ell_{t-}(\omega)\) are bounded on compact intervals.

(c) \(Y^\ell_\omega(\omega) + E^\ell_\omega(\omega) \geq 1\) for all \(\omega \in \Omega\) and all \(t \in [0, \infty]\).

(iii) For any \(\ell \in \mathbb{R}\) and \(S \in \mathcal{S}^A\), the family of \(\mathcal{F}^A_{\Lambda}\)-stopping times

\[
T^\ell_{S, t}(\omega) := \inf \{ t \in [S(\omega), \infty) \mid X_t(\omega) \geq \lambda Y^\ell_t(\omega) - (1 - \lambda) E^\ell_t(\omega) \}, \quad \lambda \in [0, 1), \tag{13}
\]

is non-decreasing in \(\lambda\) for all \(\omega \in \Omega\) with limit \(\lim_{\lambda \uparrow 1} T^\lambda_{S, t} := T_{S, t}\). Moreover, we have on all of \(\Omega\) that

\[
T^\ell_{S, t} = \min \{ t \in [S, \infty) \mid Y^\ell_t = X_t \text{ or } Y^\ell_{t-} = *X_t \text{ or } Y^\ell_{t+} = X^*_t \}, \tag{14}
\]

and, for every \(\omega \in \Omega\), the mapping \(\ell \mapsto T^\ell_{S, t}(\omega)\) is non-decreasing.

(iv) The following inclusions hold for any \(S \in \mathcal{S}^A\), \(\ell \in \mathbb{R}\):

\[
H^-_{S, t} := \{ T^-_{S, t} < T^\ell_{S, t} \text{ for all } \lambda \in [0, 1) \} \subseteq \{ Y^\ell_{T^\lambda_{S, t-}} = X_{T^\lambda_{S, t}} \}, \tag{15}
\]

\[
H^-_{S, t} := \{ Y^\ell_{T^-_{S, t}} = X_{T^-_{S, t}} \} \cap \left( H^-_{S, t} \right)^c \subseteq \{ Y^\ell_{T^\lambda_{S, t-}} = X_{T^\lambda_{S, t}} \}, \tag{16}
\]

\[
H^+_{S, t} := \{ Y^\ell_{T^+_{S, t}} > X_{T^+_{S, t}} \} \cap \left( H^+_{S, t} \right)^c \subseteq \{ Y^\ell_{T^\lambda_{S, t+}} = X_{T^\lambda_{S, t}} \}. \tag{17}
\]
We have $H_{S,t}^- \in F^\Lambda_{S,t}$ and $H_{S,t}^+, H_{S,t}^\tau \in F^\Lambda_{S,t}$, making the restrictions (compare definition (3)) $(T_{S,t})_H, H_{S,t}^\tau, (T_{S,t})_\Lambda$ and $(T_{S,t})_H$ an $F^A_-$-predictable stopping time, a $\Lambda$-stopping time, and an $F^A_+$-stopping time, respectively. In particular, for $S \in S^\Lambda, \ell \in \mathbb{R}$, the quadruple

$$\tau_{S,t} := (T_{S,t}, H_{S,t}^-, H_{S,t}^\tau, H_{S,t}^+)$$

is a divided stopping time (see Bank and Besslich [2018a], Definition 2.36).

Moreover, we have up to a $\mathcal{P}$-null set

$$H_{S,t}^- \cup H_{S,t} = \{Y^t_{S,t} = X_{T_{S,t}}\}, \quad H_{S,t}^+ = \{Y^t_{S,t} > X_{T_{S,t}}\}.$$  \hfill (19)

(v) For $S \in S^\Lambda$, the mapping $\ell \mapsto \tau_{S,t} \in S^\Lambda, \text{div}$ with $\tau_{S,t}$ from (18) is increasing in the sense that $[S, \tau_{S,t}) \subset [S, \tau_{S,t'}]$ for all $\ell, \ell' \in \mathbb{R}$ with $\ell \leq \ell'$, where for a divided stopping time $\tau = (T, H^-, H^+)$ we define $[S, \tau)$ for $\omega \in \Omega$ by

$$[S, \tau)(\omega) := \begin{cases} \{t \in [0, \infty) \mid S(\omega) \leq t < T(\omega)\} & \text{if } \omega \in H^- \cup H, \\ \{t \in [0, \infty) \mid S(\omega) \leq t \leq T(\omega)\} & \text{if } \omega \in H^+. \end{cases}$$  \hfill (20)

Furthermore, for $\ell \in \mathbb{R}$, the divided stopping time $\tau_{S,t}$ attains the value of the optimal stopping problem in (9), i.e. almost surely

$$Y^\ell_S = \mathbb{E} \left[ X_{\tau_{S,t}} + \int_{S, \tau_{S,t}} g_t(\ell) \mu(\mathfrak{d}t) \middle| F^\Lambda_S \right],$$

where for a divided stopping time $\tau = (T, H^-, H^+)$ we let $X_\tau := X_T \mathbb{1}_{H^-} + X_T \mathbb{1}_{H^+} + X_\tau^\prime \mathbb{1}_{H^+}$.

(vi) For $S \in S^\Lambda$ and $\ell, \ell' \in \mathbb{R}$, we have almost surely that

$$Y^\ell_S \geq \mathbb{E} \left[ X_{\tau_{S,t}} + \int_{S, \tau_{S,t}} g_t(\ell) \mu(\mathfrak{d}t) \middle| F^\Lambda_S \right].$$

Moreover, there is a version of the $\Lambda$-projections \(\Lambda \left( \int_{[0, \infty)} g_t(\ell) \mu(\mathfrak{d}t) \right) \), $\ell \in \mathbb{R}$, such that on all of $\Omega$, we have

$$\lim_{\delta \downarrow 0} \sup_{\ell' \in \mathbb{R}} \sup_{C \subset [0, \infty]} \sup_{s \in [0, \infty]} \left| \Lambda \left( \int_{[0, \infty)} g_t(\ell) \mu(\mathfrak{d}t) \right) - \Lambda \left( \int_{[0, \infty)} g_t(\ell') \mu(\mathfrak{d}t) \right) \right| = 0, \quad C \subset \mathbb{R} \text{ compact},$$

and such that

$$Y^\ell_S \geq Y^\ell_S \geq Y^\ell_s \geq Y^\ell_s + \Lambda \left( \int_{[0, \infty)} g_t(\ell) \mu(\mathfrak{d}t) \right) - \Lambda \left( \int_{[0, \infty)} g_t(\ell') \mu(\mathfrak{d}t) \right), \quad s \in [0, \infty), \ell \leq \ell'.$$  \hfill (24)

(vii) For fixed $\omega, s \in \Omega \times [0, \infty]$, the mapping $\ell \mapsto Y^\ell_s(\omega)$ is continuous and non-decreasing. Furthermore, we have $X = \inf_{\ell \in \mathbb{R}} Y^\ell$ up to indistinguishability.

(viii) For $S \in S^\Lambda$, we have on $\{ \mathbb{P}(\mu([S, \infty)) > 0) \mid F^\Lambda_S \} = 0\}$ that $X_S = Y^\ell_S, T_{S,t} = S$ for all $\ell \in \mathbb{R}$ almost surely.

4.2.3. Construction of the solution

With the help of the stochastic field $Y = (Y^\ell_t)_{\ell \in \mathbb{R}, t \geq 0}$ we now construct the process $L$ which will turn out to be the solution to our stochastic representation problem. At any time $t \geq 0$, it is defined in the same way as in Bank and El Karoui [2004], Lemma 4.13, p.1051, namely as the threshold value $\ell \in \mathbb{R}$ up to which one would immediately stop in the optimal stopping problems corresponding to the Snell envelopes $(Y^\ell_t)_{\ell \in \mathbb{R}}$: 
Lemma 4.2. For $Y = (Y^\ell)_{\ell \in \mathbb{R}}$ as in Lemma 4.1, the process $L$ defined by

$$L_t(\omega) := \sup \{ \ell \in \mathbb{R} \mid Y^\ell_t(\omega) = X_t(\omega) \}, \quad (\omega, t) \in \Omega \times [0, \infty),$$

and $L_\infty(\omega) := \infty$, $\omega \in \Omega$, is $\Lambda$-measurable. Furthermore, we have for $S \in \mathcal{S}^{\Lambda}$ that

$$\mathbb{P}(\{L_S = \infty\} \cap \{\mathbb{P}(\mu([S, \infty)) > 0 \mid \mathcal{F}^S_\infty) > 0\}) = 0, \quad \mathbb{P}(\{L_S = -\infty\} \cap \{\mathbb{P}(\mu(\{S\}) > 0 \mid \mathcal{F}^S_\infty) > 0\}) = 0.$$

For the proof of this lemma and of Lemma 4.3 and Lemma 4.4 below we refer to the Appendices A.3, A.4 and A.5, respectively.

Next, we see that the process $L$ constructed in Lemma 4.2 is the essential infimum over the family of random variables $\ell_A$ and $\ell_A$ respectively.

Lemma 4.3. For $L$ as in the previous Lemma 4.2 we have

$$L_S = \operatorname{ess inf}_{T \in \mathcal{S}^{\Lambda}((S, \infty))} \ell_{S,T}, \quad S \in \mathcal{S}^{\Lambda}([0, \infty)), \quad (25)$$

where $\ell_{S,T}$ is defined in Theorem 2.9. Moreover, with $Y$ from Lemma 4.1 we have that

$$X_S = Y^L_S \quad \text{almost surely for any } S \in \mathcal{S}^{\Lambda}([0, \infty)). \quad (26)$$

Next, we clarify how $L$ is related to the stopping times $T_{S,\ell}$ ($\ell \in \mathbb{R}$, $S \in \mathcal{S}^{\Lambda}$) constructed in Lemma 4.1. Our result reveals that, as one can also see in Section 3.2, $L$ can be seen as a universal stopping signal for the optimal stopping problems corresponding to $Y$.

Lemma 4.4. For every $S \in \mathcal{S}^{\Lambda}$, there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that with

$$\bar{\Omega}_S^N := \{ (\omega, t, \ell) \in \Omega \times [0, \infty) \times \mathbb{R} \mid \omega \in \mathcal{N}^c, \ S(\omega) \leq t \}$$

the stopping times $(T_{S,\ell})_{\ell \in \mathbb{R}}$ from Lemma 4.1 (iii) and the process $L$ from Lemma 4.2 are related by the inclusions

$$A := \left\{ (\omega, t, \ell) \in \bar{\Omega}_S^N \mid \sup_{v \in [S(\omega), t]} L_v(\omega) < \ell \right\} \subset B := \left\{ (\omega, t, \ell) \in \bar{\Omega}_S^N \mid t \leq T_{S,\ell}(\omega) \right\}, \quad (27)$$

$$C := \left\{ (\omega, t, \ell) \in \bar{\Omega}_S^N \mid \sup_{v \in [S(\omega), t]} L_v(\omega) \leq \ell \right\} \quad \text{and}$$

$$\tilde{A} := A \cap \left\{ (\omega, t, \ell) \in \bar{\Omega}_S^N \mid X_{T_{S,\ell}(\omega)}(\omega) = Y^L_{T_{S,\ell}(\omega)}(\omega) \right\} \subset \tilde{B} := \left\{ (\omega, t, \ell) \in \bar{\Omega}_S^N \mid t < T_{S,\ell}(\omega) \right\}, \quad (28)$$

$$\tilde{C} := \left\{ (\omega, t, \ell) \in \bar{\Omega}_S^N \mid \sup_{v \in [S(\omega), t]} L_v(\omega) \leq \ell \right\}. \quad (29)$$

4.2.4. Synthesis

We follow the blueprint of the proof from Bank and El Karoui [2004] and we prove in four steps that the process $L$ of Lemma 4.2 satisfies (4) for any $S \in \mathcal{S}^{\Lambda}$. Along the way we will also establish the integrability property (6). Our approach is to first use a disintegration formula for the monotone field $\ell \mapsto Y^L_S$ to show that for any $S \in \mathcal{S}^{\Lambda}$ and all $\ell \in \mathbb{Q}$ we have

$$X_S = \mathbb{E}\left[ X_{T_{S,\ell}} + \int_{[S, T_{S,\ell})} g_t \left( \sup_{v \in [S, t]} L(v) \right) \mu(dt) \mid \mathcal{F}^S_\infty \right]. \quad (30)$$

Afterwards, we let $\ell$ tend to infinity and argue that, in this regime, the $X_{T_{S,\ell}}$-term in the preceding expectation vanishes while the integral converges to an integral over all of $[S, \infty)$. This then establishes the desired representation (4). For ease of exposition of the proof for our main result Theorem 2.9, all the proofs of the following results are deferred to Appendices A.6, A.7 and A.8.

We start with the following disintegration formula:
Proposition 4.5. For every $S \in S^\Lambda$, the nonnegative random Borel-measure $Y_S(d\ell)$ associated with the non-decreasing continuous random mapping $\ell \mapsto Y^S_\ell$ (see Lemma 4.1 (vii)) can be disintegrated in the form
\[
\int_\mathbb{R} \phi(\ell) Y_S(d\ell) = \mathbb{E} \left[ \int_{[S,\infty)} \left\{ \int_\mathbb{R} \phi(\ell) 1_{[S,\tau_S,\ell]}(t) g_t(d\ell) \right\} \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right] \tag{32}
\]
for any nonnegative, $\mathcal{F}_S^\Lambda \otimes \mathcal{B}(\mathbb{R})$-measurable $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$. Here $\tau_{S,\ell}$ is the divided stopping time from (18) and $[S, \tau_{S,\ell})$ is given by (20).

The following lemma establishes (31):

Lemma 4.6. For any $\ell \in \mathbb{R}$, we have
\[
\sup_{S \in S^\Lambda} \mathbb{E} \left[ \int_{[S,\tau_{S,\ell})} \left( \sup_{v \in [S,t]} L_v \right) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right] < \infty \tag{33}
\]
with $\tau_{S,\ell}$ from (18); moreover,
\[
X_S = \mathbb{E} \left[ X_{\tau_{S,\ell}} + \int_{[S,\tau_{S,\ell})} \left( \sup_{v \in [S,t]} L_v \right) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right], \quad S \in S^\Lambda.
\]

As a last preparatory step, the following lemma will allow us to let $\ell$ converge to infinity in (31):

Lemma 4.7. For any $S \in S^\Lambda$ and $T_{S,\infty} := \lim_{Q \uparrow \infty} T_{S,\ell}$, the following assertions hold true:
(i) We have
\[
\mu([T_{S,\infty}, \infty)) = 0, \quad \mathbb{P}\text{-almost surely.} \tag{34}
\]
(ii) Additionally, we have with $\Gamma := \bigcap_{n=1}^\infty \{T_{S,n} < T_{S,\infty}\}$ that
\[
\mathbb{P}(\Gamma \cap \{\mu([T_{S,\infty}, \infty)) > 0\}) = 0, \quad \mathbb{P}(\Gamma^c \cap \{\mu([T_{S,\infty}, \infty)) > 0\} \cap \{X_{T_{S,\infty}} = Y^\ell_{T_{S,\infty}} \text{ for all } \ell\}) = 0, \tag{35}
\]
and almost surely we get the pointwise limit
\[
\lim_{n \to \infty} \mathbb{1}_{(H_{S,n} \cup H_{S,\infty}) \cap \Gamma^c} = \mathbb{1}_{\Gamma^c \cap \{X_{T_{S,\infty}} = Y^\ell_{T_{S,\infty}} \text{ for all } \ell\}}. \tag{36}
\]
(iii) Finally, we have
\[
\lim_{Q \uparrow \infty} \mathbb{E} \left[ X_{\tau_{S,\ell}} \bigg| \mathcal{F}_S^\Lambda \right] = 0. \tag{37}
\]

Now we can put all pieces together to finally prove Theorem 2.9:

Proof of Theorem 2.9:

Fix $S \in S^\Lambda$. Lemma 4.6 yields for rational $\ell \geq 0$ that
\[
X_S - \mathbb{E} \left[ X_{\tau_{S,\ell}} \bigg| \mathcal{F}_S^\Lambda \right] = \mathbb{E} \left[ \int_{[S,\tau_{S,\ell})} g_t \left( \sup_{v \in [S,t]} L_v \right) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right] \tag{38}
\]
\[
+ \mathbb{E} \left[ \int_{[S,\tau_{S,\ell}) \setminus [S,\tau_{S,0})} g_t \left( \sup_{v \in [S,t]} L_v \right) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right].
\]

The first summand on the right-hand side in (38) is uniformly integrable in $S \in S^\Lambda$ by (33). Next, we can see that the integrand of the second summand is positive. Indeed, for $t > T_{S,0}$ this follows from $B^c \subset A^c$ (see (27)) and for $t = T_{S,0}$ and $\omega \in H_{S,0}$ we have $X_{T_{S,0}} = Y^0_{T_{S,0}}$ (see (19)) we obtain the result by $B^c \subset A^c$ (see (29)). Having established a non-negative integrand we can use monotone convergence for $\ell \uparrow \infty$ along the rationals on the right-hand side. As also $[S, \tau_{S,\ell}) \setminus [S, \tau_{S,\infty}) = [S, \infty)$ up to a $\mathbb{P} \otimes \mu$-nullset (see Lemma 4.7 (ii)) the right-hand side in (38) converges to the right-hand side of (4). Additionally, the left hand side converges to $X_S$ by (37), which establishes (4). Integrability (6) follows by (33) and the class$(D^\Lambda)$ property of $X$. Hence, $L$ satisfies (4), (5) and (6). In Section 4.1 we have already proven that such a solution is also maximal and uniqueness of such a maximal solution follows by Corollary 2.4.
Appendix A: Proofs for the auxiliary results in our existence argument

A.1. Preliminary path regularity results

In this section we state three preparatory results, two concerning the path properties of the process $X$ considered in Theorem 2.9 and one about the regularity of $\Lambda$-projections of random fields.

First we adapt Bank and El Karoui [2004], Lemma 4.11, p.1050, for our framework with $\Lambda$-measurable processes. The proof and the changed statements are mainly based on Bismut and Skalli [1977], Theorem II.1, p.305, and Dellacherie and Lenglart [1982], Theorem 6, p.303.

**Lemma A.1.** Any $\Lambda$-measurable process $X$ of class($D^\Lambda$) with $X_\infty = 0$ which is left-upper-semicontinuous in expectation at every $S \in \mathcal{S}^P$ has the following properties:

(i) $X$ is pathwise bounded from above and below by a positive $\Lambda$-martingale of class($D^\Lambda$), i.e. there is a positive $\Lambda$-martingale $M^X : \Omega \times [0, \infty) \rightarrow [0, \infty)$ (see e.g. Bank and Besslich [2018a], Definition 2.25) such that $-M^X_t(\omega) \leq X_t(\omega) \leq M^X_t(\omega)$ for $(\omega, t) \in \Omega \times [0, \infty]$.

(ii) We have that $^*X \leq P X$ and $^*X_\infty \leq P X_\infty = 0$ up to an evanescent set.

**Proof.** Part (i) follows as in the proof of Bank and El Karoui [2004], Lemma 4.11, p.1050, with the help of Bank and Besslich [2018a], Theorem 2.29, and Bank and Besslich [2018a], Proposition 2.31. Part (ii) follows by applying Bank and Besslich [2018a], Lemma 3.4 (ii). $\square$

For the sake of completeness let us note a quite immediate conditional version of $\Lambda$-$\mu$-right-upper-semicontinuity in expectation:

**Proposition A.2.** Assume we have a process $X$ of class($D^\Lambda$) with $X_\infty = 0$ which is $\Lambda$-$\mu$-right-upper-semicontinuous in expectation in all $S \in \mathcal{S}^\Lambda$. Then we have for any $S \in \mathcal{S}^\Lambda$ and any sequence $(S_n)_{n \in \mathbb{N}} \subset \mathcal{S}^\Lambda ([S, \infty])$ such that $\mu([S, S_n])$ vanishes almost surely, that $X_S \geq \lim_{n \rightarrow \infty} \mathbb{E} [X_{S_n} | \mathcal{F}_S^\Lambda]$ whenever the latter limes exists almost surely. $\square$

The next result is a special case of Bank and Besslich [2018a], Lemma 3.9.

**Lemma A.3.** The $\Lambda$-projections of the constant processes $h^\ell := \int_{(0, \infty)} g_s(\ell) \mu(\mathrm{ds})$, $\ell \in \mathbb{R}$, can be chosen such that for all $\omega \in \Omega$ we have

$$\lim_{\delta \downarrow 0} \sup_{\ell, \ell' \in \mathbb{R}} \sup_{|t| \leq \delta, \ell \leq S} \left| \Lambda(h^\ell)_t(\omega) - \Lambda(h^{\ell'})_t(\omega) \right| = 0 \quad \text{for any compact set } C \subset \mathbb{R}. \quad \square$$

A.2. Proof of Lemma 4.1

We start by constructing in Proposition A.4 below processes $\hat{Y}^\ell$, $\ell \in \mathbb{R}$, which fulfill the conditions (i)-(vi) of Lemma 4.1 for fixed $\ell$. The random field $Y$ will then be constructed as a limit of the processes $\hat{Y}^\ell$. The process $\hat{Y}^\ell$ for fixed $\ell$ is constructed using the optimal stopping results of El Karoui [1981]. Specifically, we construct $\hat{Y}^\ell$ as a Snell-envelope and the desired properties are established with the help of divided stopping times (see Bank and Besslich [2018a], Definition 2.36).

A.2.1. Snell envelope construction for fixed parameter $\ell$

**Proposition A.4.** (i) For each $\ell \in \mathbb{R}$, there is a $\Lambda$-measurable, lâ quant process $\hat{Y}^\ell : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ of class($D^\Lambda$) with $\hat{Y}^\ell_\infty = 0$, unique up to indistinguishability, such that for all $S \in \mathcal{S}^\Lambda$ we have (9) with $Y$ replaced by $\hat{Y}$. Moreover, for $\ell \leq \ell'$, we have $\mathbb{P} \left( \hat{Y}^\ell_t \leq \hat{Y}^{\ell'}_t \text{ for all } t \geq 0 \right) = 1$.

(ii) There exists a version of the stochastic field ($E^\ell$)$_{\ell \in \mathbb{R}}$ defined in (10) and (11) satisfying for all $\omega \in \Omega$ the properties (a)-(c) stated in Lemma 4.1 (ii).

(iii) For $\ell \in \mathbb{R}$ and $S \in \mathcal{S}^\Lambda$, define $\hat{T}^\lambda_{S, \ell}$ as in (13) with $Y$ replaced by $\hat{Y}$. This family of $\mathcal{F}^\Lambda_{\hat{T}^\lambda_{S, \ell}}$-stopping times is non-decreasing in $\lambda$ for all $\omega \in \Omega$ with limit $\hat{T}^\lambda_{S, \ell} := \lim_{\lambda \downarrow 1} \hat{T}^\lambda_{S, \ell}$.}

imsart-aichp ver. 2014/10/16 file: output.tex date: April 6, 2020
(iv) For any $S \in \mathcal{S}^\Lambda$, $\ell \in \mathbb{R}$, Lemma 4.1 (iv) holds with $(Y^\ell, T_{S,\ell})$ replaced by $(\hat{Y}^\ell, \hat{T}_{S,\ell})$; we thus can define the divided stopping time $\hat{\tau}_{S,\ell}$ analogously to (18).

(v) For $S \in \mathcal{S}^\Lambda$, the mapping $\ell \mapsto \hat{\tau}_{S,\ell} \in \mathcal{S}^{\Lambda, \text{div}}$ is almost surely increasing in the sense that for all $\ell, \ell' \in \mathbb{R}$ with $\ell \leq \ell'$ we have almost surely that $[S, \hat{\tau}_{S,\ell}] \subset [S, \hat{\tau}_{S,\ell'}]$ with the definition of $[S, \tau]$ given in (20). Furthermore for $\ell \in \mathbb{R}$ the divided stopping time $\hat{\tau}_{S,\ell}$ attains the value of the optimal stopping problem in (9) for $\hat{Y}^\ell$, i.e. the analogue of equation (21) holds true.

(vi) For $S \in \mathcal{S}^\Lambda$ fixed, we have almost surely the analogue of (22) for $(\hat{Y}^\ell, \hat{T}_{S,\ell})$. Moreover, there is a version of the $\Lambda$-projections \( (\Lambda \left( \int_{[0,\infty)} g_t(\ell, \mu(\d t)) \psi \right)_s \big|_{s \in [0,\infty)}) \) \(, \ell \in \mathbb{R}\), such that on all of $\Omega$, we have (23) and, for $\ell, \ell' \in \mathbb{R}$ with $\ell \leq \ell'$, we have the analogue of (24).

Proof of Part (i)-(iv): (i), (iii) and (iv) follow readily from the construction of Snell envelopes and results on divided stopping times as developed by El Karoui [1981]; see Bank and Besslich [2018a], Chapter 2 for a summary. For (ii), we refer to Lemma A.1 and Lemma A.3.

Proof of Part (v): To prove the monotonicity, fix $\ell \leq \ell'$ and $(\omega, t) \in \Omega \times [0, \infty)$ such that $\hat{Y}^\ell_s(\omega) \leq \hat{Y}^{\ell'}_s(\omega)$ for all $s \in [0, \infty)$. If $T_{S,\ell}(\omega) < T_{S,\ell'}(\omega)$ or $T_{S,\ell}(\omega) = T_{S,\ell'}(\omega)$ and $\omega \notin \hat{H}_{S,\ell}^+$ there is nothing to show. Therefore let $t = \hat{T}_{S,\ell}(\omega) = \hat{T}_{S,\ell'}(\omega)$ and assume $\omega \in \hat{H}_{S,\ell}^+ = \{X_{T_{S,\ell}} < \hat{Y}^\ell_{T_{S,\ell}}\} \cap (\hat{H}_{S,\ell'})^c$. We now have to show $\omega \in \hat{H}_{S,\ell'}^+.$ By $\hat{Y}^\ell_t(\omega) \leq \hat{Y}^{\ell'}_t(\omega)$ and $\hat{T}_{S,\ell}(\omega) = t = \hat{T}_{S,\ell'}(\omega)$ we get
\begin{equation}
\omega \in \{X_{T_{S,\ell'}} < \hat{Y}^{\ell'}_{T_{S,\ell'}}\}.
\end{equation}

On the other hand, as $\omega \in (\hat{H}_{S,\ell'})^c$, there exists $\lambda \in [0, 1)$ such that $\hat{T}_{S,\ell}(\omega) = \hat{T}_{S,\ell'}(\omega) = t = \hat{T}_{S,\ell'}(\omega).$ In particular, for all $s \in [S(\omega), \hat{T}_{S,\ell}(\omega))$ we obtain with $Y^\ell_s(\omega) + E^s(\omega) \geq 1$ by the definition of $\hat{T}_{S,\ell}$ that
\begin{equation}
X_s(\omega) < Y^\ell_s(\omega) - (1 - \lambda)(E^s(\omega) + Y^\ell_s(\omega)) \leq Y^\ell_s(\omega) - (1 - \lambda)
\end{equation}
and therefore, recalling $\hat{T}_{S,\ell}(\omega) = t = \hat{T}_{S,\ell'}(\omega),$ \begin{equation}
* X_{T_{S,\ell}}(\omega) = * X_{T_{S,\ell'}}(\omega) < \hat{Y}^\ell_{T_{S,\ell}}(\omega) \leq \hat{Y}^{\ell'}_{T_{S,\ell'}}(\omega).
\end{equation}
Hence $\omega \in \{X_{T_{S,\ell'}} < \hat{Y}^{\ell'}_{T_{S,\ell'}}\} \subset (\hat{H}_{S,\ell'})^c$ by (iv). Together with (39) we get $\omega \in \hat{H}_{S,\ell'}^+.$ This establishes the claimed monotonicity of $\ell \mapsto \hat{\tau}_{S,\ell}$. The optimality of $\hat{\tau}_{S,\ell}$ follows by El Karoui [1981], Proposition 2.35 and 2.36, p.133 and 135 and optional sampling for divided stopping times (see El Karoui [1981], Lemma 2.38, p.137).

Proof of Part (vi): For $\ell \in \mathbb{R}$ we define $Z^\ell := X + E^\ell$. Then we obtain by Bank and Besslich [2018a], Proposition 2.31, that we can write $Z^\ell$’s $\Lambda$-Snell envelope $\hat{Z}^\ell = M^\ell - A^\ell - B^\ell$, where $M^\ell$ is a $\Lambda$-martingale and $A^\ell, B^\ell$ are positive $\Lambda$-measurable and non-$\Lambda$-measurable processes. Hence, by optional sampling for divided stopping times
\begin{equation}
\hat{Z}^\ell_S = M^\ell_S - A^\ell_S - B^\ell_S = \mathbb{E} \left[ M^\Lambda_{\hat{T}_{S,\ell'}} \big| \mathcal{F}^\Lambda_S \right] - A^\Lambda_S - B^\Lambda_S \geq \mathbb{E} \left[ \hat{Z}^\Lambda_{\hat{T}_{S,\ell'}} \big| \mathcal{F}^\Lambda_S \right] \geq \mathbb{E} \left[ Z^\Lambda_{\hat{T}_{S,\ell'}} \big| \mathcal{F}^\Lambda_S \right],
\end{equation}
where we have used that on $\hat{H}_{S,\ell}$ we have $\hat{T}_{S,\ell'} > S$ and therefore also $\hat{B}^\Lambda_{\hat{T}_{S,\ell'}} \geq \hat{B}^\Lambda_S.$ By Bank and Besslich [2018a], Lemma 2.38, the inequality (40) is equivalent to (22) for $(\hat{Y}, \hat{\tau}_{S,\ell})$ replacing $(Y, \tau_{S,\ell})$. Finally, (24) follows as in the proof of Bank and El Karoui [2004], Lemma 4.12, p.1050-1051.

A.2.2. Choosing a regular version in the parameter $\ell$

With the help of Proposition A.4, we can now construct the random field $Y$ posited in Lemma 4.1.

Construction of $Y$: First we can choose for all $q \in \mathbb{Q}$ the processes $(\hat{Y}^q)_{q \in \mathbb{Q}}$ from Proposition A.4 such that (24) for $Y$ replaced by $\hat{Y}$ follows as in the proof of Bank and El Karoui [2004], Lemma 4.12, p.1050-1051.

imsart-aighp ver. 2014/10/16 file: output.tex date: April 6, 2020
sup_{q \geq q < t} \tilde{Y}_t^q(\omega). As in the proof of Bank and El Karoui [2004], Lemma 4.12, p.1050-1051, one can show now that $Y_t^\ell$ and $Y_t^t$ are indistinguishable for all $\ell \in \mathbb{R}$.

**Proof of Part (i), (ii) and the last part of (v):** Result (i) and the property in (v) that the divided stopping time attains the value of the optimal stopping problem are stated for fixed $\ell \in \mathbb{R}$ and thus follow directly as $\tilde{Y}_t^\ell$ is indistinguishable from $Y_t^\ell$. For (ii) we get a process $E$ by Proposition A.4, (ii). For the rest of (ii) fix $\omega \in \Omega, t \in [0, \infty)$ and obtain a sequence $(q_n) \in \mathbb{N} \subset \mathbb{Q}$ converging strictly from below to $\ell$ such that $\lim_{n \to \infty} \tilde{Y}_t^{q_n}(\omega) = Y_t^\ell(\omega)$ as $\tilde{Y}_t^{q_n}(\omega) + E_t^{q_n}(\omega) \geq 1$ for any $n \in \mathbb{N}$ this leads by (12) to $Y_t^\ell(\omega) + E_t^\ell(\omega) \geq 1$.

**Proof of (vi) and monotonicity and continuity of $\ell \mapsto Y_t^\ell$:** By taking again non-increasing rational limits in (24) for $Y_t$, we see that $Y_t^\ell$ satisfies (24) and $Y_t^\ell(\omega) \geq X_\ell(\omega)$ for all $\ell, \ell' \in \mathbb{R}$ and any $(\omega, t) \in \Omega \times [0, \infty]$. Hence by (24) and the convergence property of the $\Lambda$-projections we obtain the continuity and monotonicity of the mapping $\ell \mapsto Y_t^\ell$ for any $(\omega, t) \in \Omega \times [0, \infty]$.

**Proof of (iii), (iv) and the rest of (v):** First of all one can adapt the proof of El Karoui [1981], Proposition 2.35, p.133, to show the inclusions (15), (16), (17). For (14), we have that $\geq \ell$ monotonicity of the mapping $\Omega \times$ stopping time attains the value of the optimal stopping problem are stated for fixed limits in (24) for $\tilde{Y}_t^\ell$. Since $\tilde{Y}_t^\ell(\omega) + E_t^\ell(\omega) \leq 1$ and $\tilde{Y}_t^\ell(\omega)$ for any $0 \leq \lambda < 1, \inf U(\ell) \geq \inf \{ t \in \{ S(\omega), \infty \} | X_t(\omega) \geq \lambda Y_t^\ell(\omega) - (1 - \lambda)E_t^\ell(\omega) \} = T_{S,\ell}^\lambda$.

Proof of the rest of Part (vii): It remains to prove that $Y_{s,\infty}^\ell := \lim_{t \to \infty} = \inf_{t \in \mathbb{R}} Y_t^\ell = X_S$ almost surely for $S \in S^\Lambda$. By $X_S \leq Y_t^\ell$ for all $\omega \in \Omega, t \in [0, \infty]$ and $\ell \leq \ell', \ell' \in \mathbb{R}$ can be used.

**Proof of (vi) and monotonicity and continuity of $\ell \mapsto Y_t^\ell$:** By taking again non-increasing rational limits in (24) for $Y_t$, we see that $Y_t^\ell$ satisfies (24) and $Y_t^\ell(\omega) \geq X_\ell(\omega)$ for all $\ell, \ell' \in \mathbb{R}$ and any $(\omega, t) \in \Omega \times [0, \infty]$. Hence by (24) and the convergence property of the $\Lambda$-projections we obtain the continuity and monotonicity of the mapping $\ell \mapsto Y_t^\ell$ for any $(\omega, t) \in \Omega \times [0, \infty]$.

Proof of (iii), (iv) and the rest of (v): First of all one can adapt the proof of El Karoui [1981], Proposition 2.35, p.133, to show the inclusions (15), (16), (17). For (14), we have that $\geq \ell$ follows from $H_{S,\ell}^\Lambda \cup H_{S,\ell} \cup H_{S,\ell}^+ = \Omega$. Furthermore, we get with the short hand notation

$$U(\ell) := \{ t \in [S(\omega), \infty] | Y_t^\ell(\omega) = X_t(\omega) \text{ or } Y_t^\ell(\omega) = \ast X_t(\omega) \text{ or } Y_t^\ell(\omega) = X_t^\ast(\omega) \}$$

that, for $0 \leq \lambda < 1$,

$$\inf U(\ell) \geq \inf \{ t \in [S(\omega), \infty] | X_t(\omega) \geq \lambda Y_t^\ell(\omega) - (1 - \lambda)E_t^\ell(\omega) \} = T_{S,\ell}^\lambda.$$

Since $T_{S,\ell} = \lim_{\lambda \downarrow 0} T_{S,\ell}^\lambda$ by definition this yields “<” in (14). Finally, due to $H_{S,\ell}^\Lambda \cup H_{S,\ell} \cup H_{S,\ell}^+ = \Omega$ and (15)-(17), we get for any $\omega \in \Omega$ that $T_{S,\ell}(\omega)$ is contained in $U(\ell)$ and hence $\inf U(\ell) = \inf \{ t \in [S(\omega), \infty] | X_t(\omega) \geq \lambda Y_t^\ell(\omega) - (1 - \lambda)E_t^\ell(\omega) \} = T_{S,\ell}^\lambda$.
Since \( g_\ell(\ell) \leq 0 \) for \( \ell < 0 \), because of our normalization (8), this gives
\[
\mathbb{E}[X_S] \leq \mathbb{E}[M^X_S] + \mathbb{E} \left[ \int_{[S,T_S,\ell)} g_\ell(\ell) \mu(d\ell) \right], \quad \ell < 0.
\]

Thus, by Fatou's Lemma, we get for any \( \ell_0 < 0 \) that
\[
-\infty < \limsup_{Q \ni \ell_0 \downarrow -\infty} \mathbb{E} \left[ \int_{[S,T_S,\ell)} g_\ell(\ell) \mu(d\ell) \right] < \mathbb{E} \left[ \int_{[S,T_S,-\infty)} g_\ell(\ell_0) \mu(d\ell) \right] \xrightarrow{\ell_0 \downarrow -\infty} \mathbb{E} \left[ (-\infty) \mathbb{I}_{\mu((S,T_S,-\infty)) > 0} \right].
\]
Hence, \( \mu((S,T_S,-\infty)) = 0 \), \( \mathbb{P} \)-almost surely.

**Proof of (43) and (44):** For \( \ell_0 \leq 0 \) we have again that \( g(\ell_0) \leq 0 \). Hence, we can use (42) in (45) to obtain with Fatou's Lemma
\[
-\infty < \limsup_{Q \ni \ell \downarrow -\infty} \mathbb{E} \left[ g_{T_S,\ell}(\ell) \mu(T_S,\ell) \mathbb{I}_{H^+_S,\ell \cap \Gamma^c} + \left( \int_{[T_S,\ell,-\infty,T_S,\ell)} g_\ell(\ell) \mu(d\ell) + g_{T_S,\ell}(\ell) \mu(T_S,\ell) \mathbb{I}_{H^+_S,\ell} \right) \mathbb{I}_{\Gamma} \right]
\]
\[
\leq \limsup_{Q \ni \ell \downarrow -\infty} \mathbb{E} \left[ g_{T_S,\ell}(\ell) \mu(T_S,\ell) \mathbb{I}_{H^+_S,\ell \cap \Gamma^c} + \left( \int_{[T_S,\ell,-\infty,T_S,\ell)} g_\ell(\ell) \mu(d\ell) \right) \mathbb{I}_{\Gamma} \right]
\]
\[
\leq \mathbb{E} \left[ \left( \mu(T_S,\ell) g_{T_S,\ell}(\ell) \left( \liminf_{Q \ni \ell \downarrow -\infty} \mathbb{I}_{H^+_S,\ell \cap \Gamma^c} + \mathbb{I}_{\Gamma} \right) \right) \right]
\]
\[
\xrightarrow{l_0 \downarrow -\infty} \mathbb{E} \left[ (-\infty) \mathbb{I}_{\mu((T_S,\ell)) > 0} \right].
\]

This allows us to deduce (43). For (44) note that the limes inferior inside the last expectation is actually a limes as
\[
\lim_{Q \ni \ell \downarrow -\infty} \mathbb{I}_{H^+_S,\ell \cap \Gamma^c} = \mathbb{1}_{\{T_S,-\infty < Y_{T_S,-\infty}^\ell \text{ for all } \ell \leq 0\}} \mathbb{P} \text{-a.s.}.
\]

Indeed, this follows because \( H^+_S,\ell = \{X_{T_S,\ell} < Y_{T_S,\ell}^\ell\} \) for all \( \ell \in Q \) up to a \( \mathbb{P} \)-null set (see (19)) and because \( \Gamma^c = \{T_S,\ell = T_S,-\infty \) for some \( \ell < 0\). We note for future use that by the same reasoning almost surely
\[
\lim_{Q \ni \ell \downarrow -\infty} \mathbb{1}_{(H^+_S,\ell \cup H^S,\ell) \cap \Gamma^c} = \mathbb{1}_{\{T_S,-\infty < Y_{T_S,-\infty}^\ell \text{ for some } \ell < 0\}} \mathbb{P} \text{-a.s.}
\]

With (47), estimate (43) gives (44).

**Establishing** \( \mathbb{E}[X_S] = \mathbb{E}[Y_{S,-\infty}^\ell] \): First, \( (T_S,\ell)_{H^-} \) is a predictable stopping time (see Lemma 4.1 (iv)) and therefore we get by Lemma A.1 (ii) that
\[
\mathbb{E} \left[ X_{(T_S,\ell)_{H^-}} \right] \leq \mathbb{E} \left[ X_{(T_S,\ell)_{H^+}} \right] = \mathbb{E} \left[ X_{T_S,\ell} \mathbb{I}_{H^+_S,\ell} \right].
\]

Now we obtain by (41), (49) and \( g(\ell) \leq 0 \) for \( \ell < 0 \) (see (8)) that
\[
\mathbb{E}[X_S] \leq \mathbb{E}[Y_{S,-\infty}^\ell] \xrightarrow{(41)} \limsup_{Q \ni \ell \downarrow -\infty} \mathbb{E} \left[ X_{T_S,\ell} \mathbb{I}_{H^+_{S,\ell}} + X_{T_S,\ell} \mathbb{I}_{H^-_{S,\ell}} + X_{T_S,\ell}^* \mathbb{I}_{H^+_{S,\ell}} + \int_{[S,T_S,\ell)} g_\ell(\ell) \mu(d\ell) \right]
\]
\[
\xrightarrow{(49),(8)} \limsup_{Q \ni \ell \downarrow -\infty} \mathbb{E} \left[ X_{T_S,\ell} \mathbb{I}_{H^+_{S,\ell}} + X_{T_S,\ell}^* \mathbb{I}_{H^+_{S,\ell}} \right].
\]

By Fatou's Lemma it thus follows that
\[
\mathbb{E}[X_S] \xrightarrow{(47),(48)} \mathbb{E} \left[ X_{T_S,-\infty} \mathbb{I}_{\Gamma} + X_{T_S,-\infty} \left( \limsup_{Q \ni \ell \downarrow -\infty} \mathbb{I}_{(H^+_S \cup H^-_{S,\ell}) \cap \Gamma^c} \right) + X_{T_S,-\infty}^* \left( \limsup_{Q \ni \ell \downarrow -\infty} \mathbb{I}_{H^+_{S,\ell} \cap \Gamma^c} \right) \right],
\]
where we have used that $T_{S,t}$ is converging strictly from above to $T_{S,-\infty}$ on $\Gamma$ and therefore $\limsup_{t \to -\infty} X_{T_{S,t}}$ and $\limsup_{t \to -\infty} X_{T_{S,t}}^\circ$ are both less than or equal to $X_{T_{S,-\infty}}^\circ$. By (47), (48) and (50) it remains to show that
\[
\mathbb{E} \left[ X_{T_{S,-\infty}}^\circ 1_E + X_{T_{S,-\infty}}^\circ 1_{E^c} \right] \leq \mathbb{E}[X_S],
\]
where for ease of notation, we put $E$ for $\Gamma \cup \left\{ X_{T_{S,-\infty}} < Y_{T_{S,-\infty}}^\circ \text{ for all } t \leq 0 \right\}$ and observe that $E^c = \left\{ X_{T_{S,-\infty}} = Y_{T_{S,-\infty}}^\circ \text{ for some } t \leq 0 \right\}$ and $\Gamma^c$. For proving (51), we will need the following claim, which will be established in the end.

**Claim:** The restriction $(T_{S,-\infty})_{E^c}$ is a $\Lambda$-stopping time.

To conclude (51), use Bank and Besslich [2018a], Proposition 3.2 (i), to obtain a non-increasing sequence $T_n \in S^\Lambda (\{ (T_{S,-\infty})_{E^c} \})$ with limit $(T_{S,-\infty})_{E^c}$ and $T_n > (T_{S,-\infty})_{E^c}$ on $\{ (T_{S,-\infty})_{E^c} < \infty \}$ such that $X^*_{(T_{S,-\infty})_{E^c}} = \lim_{n \to \infty} X^*_{T_n}$ $P$-almost surely. Due to the claim above $T_n := T_n \wedge (T_{S,-\infty})_{E^c}$ is a sequence of $\Lambda$-stopping times. Now we see that $\lim_{n \to \infty} \mu((S,T_n)) = 0$ a.s., because by (42), (43) and (44), we have $\mu((S,T_{S,-\infty})) = 0$ a.s. and $E \subset \{ \mu((S,T_{S,-\infty})) = 0 \}$. As $T_n$ converges strictly from above to $T_{S,-\infty}$ on $E$ and is equal to $T_{S,-\infty}$ on $E^c$ we get

\[
\lim_{n \to \infty} \mu((S,T_n)) = \mu((S,T_{S,-\infty})) 1_E + \mu((S,T_{S,-\infty})) 1_{E^c} = 0.
\]

Therefore, by $\Lambda$-$\mu$-right-upper-semicontinuity of $X$, and because $X$ is of class(D$^\Lambda$), we get (51) by

\[
\mathbb{E} \left[ X_{T_{S,-\infty}}^\circ 1_E + X_{T_{S,-\infty}}^\circ 1_{E^c} \right] = \mathbb{E} \left[ \lim_{n \to \infty} X_{T_n} \right] = \lim_{n \to \infty} \mathbb{E}[X_{T_n}] \leq \mathbb{E}[X_S].
\]

**Proof of the above claim:** First,

\[
E^c = \left\{ T_{S,-\infty} = T_{S,-n} \text{ and } X_{T_{S,-\infty}} = Y_{T_{S,-\infty}}^\circ \text{ for some } n \in \mathbb{N} \right\} = \bigcup_{n=1}^\infty A^{-n},
\]

for $A^{-n} := \{ T_{S,-\infty} = T_{S,-n} \} \cap (H_{S,-n}^\circ \cup H_{S,-n})$, where the first equality follows by monotonicity of $\ell \mapsto X^\ell$, $\ell \mapsto T_{S,\ell}$ and the second by (19).

Next, we claim that $(T_{S,-n})_{A^{-n}}$ is a $\Lambda$-stopping time, i.e. $[(T_{S,-n})_{A^{-n}}]_{E^c} \in \Lambda$. We start by observing that by Lemma 4.1 (iv) $(T_{S,-n})_{H_{S,-n}^- \cup H_{S,-n}}$ is a $\Lambda$-stopping time for $n \in \mathbb{N}$. Moreover, as $T_{S,-\infty}$ is by Lenglart [1980], Theorem 2, p.503, a stopping time (see Lenglart [1980], Definition 1, p.502) we have due to Lenglart [1980], Corollary 1, p.505, that

\[
\left\{ T_{S,-\infty} = (T_{S,-n})_{(H_{S,-n}^- \cup H_{S,-n})} \right\} \in F_{(T_{S,-n})_{(H_{S,-n}^- \cup H_{S,-n})}}^\Lambda,
\]

By Lenglart [1980], Theorem 4.1 2), p.505, this shows that $A^{-n} \in F_{(T_{S,-n})_{(H_{S,-n}^- \cup H_{S,-n})}}^\Lambda$ and that $(T_{S,-n})_{A^{-n}}$ is a $\Lambda$-stopping time. Finally, we get

\[
[(T_{S,-\infty})_{E^c}, \infty] = \bigcup_{n=1}^\infty \{ (\omega,t) \in \Omega \times [0,\infty) | T_{S,-n}(\omega) \leq t, \omega \in A^{-n} \} = \bigcup_{n=1}^\infty [(T_{S,-n})_{A^{-n}}, \infty] \in \Lambda.
\]

**Proof of (viii):** By Proposition A.5 below we have, up to a $P$-null set,

\[
\check{E} := \{ P \left( \mu([S,\infty)) > 0 \mid F_S^\Lambda \right) = 0 \} \subset \{ \mu([S,\infty)) = 0 \}.
\]

We conclude that $\mu((S,E,\infty)) = 0$ a.s. and, for any $T \in S^\Lambda ([S,E,\infty])$ that $\mu(T,\infty) = 0$ a.s. Therefore, we have by the properties of $X$ that $X_T = 0$ and, also, $Y_{S,E}^\ell = 0 = X_{S,E}$ for all $\ell$. 

imsart-aihp ver. 2014/10/16 file: output.tex date: April 6, 2020
A.3. Proof of Lemma 4.2

We first note that $L$ is $\Lambda$-measurable and that for $S \in \mathcal{S}^\Lambda$, $L_S = \infty$ at most on the set $\{P (\mu([S, \infty)) > 0 \mid \mathcal{F}_{S}^\Lambda) = 0\}$, which can be verified readily as in Bank and El Karoui [2004], Proof of Lemma 4.13, p.1066. Indeed, we have on $\{L_S = \infty\}$ that for any $\ell \in \mathbb{R}$ we have

$$X_S = Y_S^\ell \geq \mathbb{E} \left[ \int_{[S, \infty)} g_t(\ell) \mu(dt) \bigg\vert \mathcal{F}_{S}^\Lambda \right]$$

This shows our claim as the right-hand side converges to $\infty$ on $\{P (\mu([S, \infty)) > 0 \mid \mathcal{F}_{S}^\Lambda) = 0\}$, while the left-hand side yields a finite upper bound.

Next, fix $S \in \mathcal{S}^\Lambda ([0, \infty))$ and assume, by way of contradiction, that

$$P \{\{L_S = -\infty\} \cap \{P (\mu(\{S\}) > 0 \mid \mathcal{F}_{S}^\Lambda) > 0\} > 0.$$  

Then we can use the following claim, to be proven at the end:

**Claim:** There exists a sequence $(T_k)_{k \in \mathbb{N}} \subset \mathcal{S}^\Lambda ([S, \infty])$ and a corresponding non-increasing sequence $(E_k)_{k \in \mathbb{N}} \subset \mathcal{F}_{S}^\Lambda$ with $P(\bigcap_{k=1}^\infty E_k) > 0$ such that, for any $k \in \mathbb{N}$,

$$X_S < \mathbb{E} \left[ X_{T_k} + \int_{[S,T_k)} g_t(-k) \mu(dt) \bigg\vert \mathcal{F}_{S}^\Lambda \right] \quad \text{on} \quad E_k$$

and

$$E_k \subset \{P (\mu(\{S\}) > 0 \mid \mathcal{F}_{S}^\Lambda) > 0\}.$$  

Using this claim and recalling that $g_S(-n) \leq 0$ for all $n \in \mathbb{N}$ (see (8)), we get on $\bigcap_{k=1}^\infty E_k$

$$-\infty < -M_S^X \leq X_S < \mathbb{E} \left[ X_{T_n} + \int_{[S,T_n)} g_t(-n) \mu(dt) \bigg\vert \mathcal{F}_{S}^\Lambda \right] \leq M_S^X + \mathbb{E} \left[ g_S(-n) \mu(\{S\}) \bigg\vert \mathcal{F}_{S}^\Lambda \right], \quad n \in \mathbb{N},$$

with $M^X$ the martingale from Lemma A.1. Observing that $0 > g_S(-n) \downarrow -\infty$ for $n \to \infty$ we deduce from (54) that $P (\mu(\{S\}) > 0 \mid \mathcal{F}_{S}^\Lambda) = 0$ a.s. on $\bigcap_{k=1}^\infty E_k$. This contradicts the properties of the sets $(E_k)_{k \in \mathbb{N}}$ and finishes our proof once the above claim is proven.

**Proof of the above claim:** The family

$$\left\{ \mathbb{E} \left[ X_T + \int_{[S,T)} g_t(-k) \mu(dt) \bigg\vert \mathcal{F}_{S}^\Lambda \right] \ \bigg| \ T \in \mathcal{S}^\Lambda ([S, \infty]) \right\}$$

is upwards directed for any fixed $k \in \mathbb{N}$. Hence there exists by Neveu [1975], Proposition VI-1-1, p.121, for every $k \in \mathbb{N}$ a sequence $(R^k_m)_{m \in \mathbb{N}} \subset \mathcal{S}^\Lambda ([S, \infty])$, such that

$$Y_{S}^{-k} = \lim_{m \to \infty} \mathbb{E} \left[ X_{R^k_m} + \int_{[S,R^k_m)} g_t(-k) \mu(dt) \bigg\vert \mathcal{F}_{S}^\Lambda \right],$$

where the limit on the right hand side is non-decreasing. Note furthermore that

$$\{L_S = -\infty\} = \{X_S < Y_{S}^{-k} \text{ for all } k \in \mathbb{N}\}.$$  

Now we can construct $T_k$ and $E_k$ inductively as follows: For $T_1$ and $E_1$ consider for $m = 1, 2, \ldots$ the sets
\[
\tilde{E}_m^1 := \left\{ X_S < \mathbb{E} \left[ X_{R_m^\infty} + \int_{(S, R_m^\infty)} g_t \left( -1 \right) \mu(\text{d}t) \bigg| \mathcal{F}_S^A \right] \right\} \cap \{ L_S = -\infty \} \cap \{ \mathbb{P} \left( \mu(\{S\}) > 0 \big| \mathcal{F}_S^A \right) > 0 \} \in \mathcal{F}_S^A,
\]
which, up to a $\mathbb{P}$-null set, grow to $\{ L_S = -\infty \} \cap \{ \mathbb{P} \left( \mu(\{S\}) > 0 \big| \mathcal{F}_S^A \right) > 0 \}$, because of (55). Now just choose $m$ large enough to ensure that
\[
\mathbb{P} \left( \tilde{E}_m^1 \right) > \frac{1}{2} \left( \mathbb{P} \left( \{ L_S = -\infty \} \cap \{ \mathbb{P} \left( \mu(\{S\}) > 0 \big| \mathcal{F}_S^A \right) > 0 \} \right) \right) =: \delta,
\]
and set $E_1 := \tilde{E}_m^1$ and $T_1 := R_m^1$.

Now assume $E_k$ and $T_k$ have been constructed already. For the construction of $T_{k+1}$ and $E_{k+1}$, consider the sequence of sets
\[
\tilde{E}_{m+1}^{k+1} := \left\{ X_S < \mathbb{E} \left[ X_{R_{m+1}^\infty} + \int_{(S, R_{m+1}^\infty)} g_t \left( -(k+1) \right) \mu(\text{d}t) \bigg| \mathcal{F}_S^A \right] \right\} \cap E_k,
\]
which, up to a $\mathbb{P}$-null set, grows to $E_k$ again by (55) and because by construction
\[
E_k \subset \{ L_S = -\infty \} \cap \{ \mathbb{P} \left( \mu(\{S\}) > 0 \big| \mathcal{F}_S^A \right) > 0 \}.
\]
For $m$ large enough with $\mathbb{P} \left( \tilde{E}_{m+1}^{k+1} \right) > \delta$, we set $E_{k+1} := \tilde{E}_{m+1}^{k+1}$ and $T_{k+1} := R_{m+1}^{k+1}$.

We conclude that the stopping times $T_k$ and sets $E_k$, $k \in \mathbb{N}$, are as required by (52) and (53) by construction. Additionally, $\mathbb{P} \left( \bigcap_{k=1}^{\infty} E_k \right) \geq 0$ completing the proof of our claim.

A.4. Proof of Lemma 4.3

Fix $S \in \mathcal{S}^A \left( [0, \infty) \right)$. On the set $\{ \mathbb{P} \left( \mu(\{S, \infty\}) > 0 \big| \mathcal{F}_S^A \right) = 0 \}$ we have by Lemma 4.1 (viii) that $X_S = Y_S^\infty$ and $S = T_{S, \ell}$ for all $\ell \in \mathbb{R}$ and hence $L_S = \infty$. In particular $X_S = Y_S^\infty = Y_S^{L_S}$ by definition of $Y_S^\infty$ in Lemma 4.1 (vii). On the other hand, we have by definition of $T_{S, \ell}$ that also $\ell_{S, T} = \infty$ for all $T \in \mathcal{S}^A \left( [S, \infty) \right)$, which shows (25) on $\{ \mathbb{P} \left( \mu(\{S, \infty\}) > 0 \big| \mathcal{F}_S^A \right) = 0 \}$. From now on we focus on the set $\{ \mathbb{P} \left( \mu(\{S, \infty\}) > 0 \big| \mathcal{F}_S^A \right) > 0 \}$, which is by Lemma 4.2 contained in $\{ L_S < \infty \}$ up to a $\mathbb{P}$-nullset.

First, (26) holds because $X_S = Y_S^{L_S}$ on $\{ L_S > -\infty \}$ by continuity of $\ell \mapsto Y_S^\ell$ (Lemma 4.1 (vii)); on $\{ L_S = -\infty \}$, we have by Lemma 4.1 (vii) that $X_S = Y_S^\infty = Y_S^{L_S}$ almost surely. Second, we get for any $T' \in \mathcal{S}^A \left( (S, \infty) \right)$
\[
X_S = Y_S^{L_S} = \text{ess sup}_{T \in \mathcal{S}^A \left( (S, \infty) \right)} \left[ X_T + \int_{(S, T)} g_t(L_S) \mu(\text{d}t) \right| \mathcal{F}_S^A \right] \geq \mathbb{E} \left[ X_T + \int_{(S, T')} g_t(L_S) \mu(\text{d}t) \bigg| \mathcal{F}_S^A \right].
\]
Since $L_S \in \mathcal{F}_S^A$, we obtain $L_S \leq L_{S, T'}$ for any such $T'$, which shows $\leq$ in (25).

For $\geq$ in (25), it suffices to show $K^n \geq \text{ess inf}_{T \in \mathcal{S}^A \left( [S, \infty) \right)} \ell_{S, T}$ for all $n \in \mathbb{N}$, where $K^n$ is the $\mathcal{F}_S^A$-measurable random variable defined by
\[
K^n := \left( L_S + \frac{1}{n} \right) I_{\{ L_S > -\infty \}} - n I_{\{ L_S = -\infty \}} > L_S \quad \text{on} \quad \{ L_S < \infty \}
\]
and $K^n := \infty$ on $\{ L_S = \infty \}$. For each $n \in \mathbb{N}$ there exists by Neveu [1975], Proposition VI-1-1, p.121, a sequence $(T^n_m)_{m \in \mathbb{N}} \subset \mathcal{S}^A \left( [S, \infty) \right)$ with
\[
\mathcal{E}_m^n := \mathbb{E} \left[ X_{T^n_m} - X_S + \int_{(S, T^n_m)} g_t(K^n) \mu(\text{d}t) \bigg| \mathcal{F}_S^A \right] \nearrow Y_S^{K^n} - X_S \quad \text{as} \quad n \uparrow \infty.
\]
since the family of random variables in the essential supremum defining $Y_n^K$ is upwards direct. Since, by definition, $K^n = \infty$ on $\{ L_S = \infty \}$, the inequality is only to be proven on $\{ L_S < \infty \} \subset \{ \mathbb{P}(\mu([S, \infty)) > 0 | \mathcal{F}_S^\Lambda) > 0 \}$, where we also have $\ell_{S, \infty} < \infty$. Now, we have for $m \in \mathbb{N}$ that

$$\{ \mathcal{E}_m^n > 0 \} \subset \{ \mathbb{P}(\mu([S, T_m^n])) > 0 | \mathcal{F}_S^\Lambda) > 0 \},$$

which follows by $\Lambda$-$\mu$-right-upper-semi-continuity in expectation of $X$, Proposition A.2 and Proposition A.5. By definition of $K^n$ and $L$ we have $\{ L_S < \infty \} \subset \{ 0 < \lim_{m \to \infty} \mathcal{E}_m^n \}$ and, due to the monotonicity of $\mathcal{E}_m^n$ in $m$, we also have $\{ \mathcal{E}_m^n > 0 \} \subset \{ \mathcal{E}_{m+1}^n > 0 \}$ for any $m \in \mathbb{N}$. As we have by Lemma 4.2 that

$$\mathbb{P}(\{ L_S = \infty \} \cap \mathbb{P}(\mu([S, \infty)) > 0 | \mathcal{F}_S^\Lambda) > 0 \}) = 0$$

we get

$$\mathbb{P}(\{ \mathbb{P}(\mu([S, \infty)) > 0 | \mathcal{F}_S^\Lambda) > 0 \}) \leq \mathbb{P}(\{ L_S < \infty \}) \leq \lim_{m \to \infty} \mathbb{P}(\{ \mathcal{E}_m^n > 0 \} \subset \{ \mathbb{P}(\mu([S, \infty)) > 0 | \mathcal{F}_S^\Lambda) > 0 \}).$$

and in particular we have up to a $\mathbb{P}$-null set that

$$\mathbb{P} \left( \bigcup_{m \in \mathbb{N}} \{ \mathcal{E}_m^n > 0 \} \right) = \mathbb{P}(\{ \mathbb{P}(\mu([S, \infty)) > 0 | \mathcal{F}_S^\Lambda) > 0 \}). \quad (56)$$

Furthermore, we have $\{ \mathcal{E}_m^n > 0 \} \subset \{ T_m^n > S \}$ and, therefore, we have for any $m \in \mathbb{N}$ that, on $\{ \mathcal{E}_m^n > 0 \}$,

$$\mathbb{E} \left[ \int_{|S,T_m^n|} g_l(\ell_{S,T_m^n}) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right] = \mathbb{E} \left[ X_S - X_{T_m^n} \bigg| \mathcal{F}_S^\Lambda \right] \leq \mathbb{E} \left[ \int_{|S,T_m^n|} g_l(K^n) \mu(dt) \bigg| \mathcal{F}_S^\Lambda \right].$$

As $K^n$ is $\mathcal{F}_S^\Lambda$-measurable this yields for any $m \in \mathbb{N}$ that $K^n \geq \ell_{S,T_m^n} \geq \essinf_{T \in S^\Lambda([S, \infty)) \ell_{S,T}}$ on $\{ \mathcal{E}_m^n > 0 \}$ and by (56) also on $\{ \mathbb{P}(\mu([S, \infty)) > 0 | \mathcal{F}_S^\Lambda) > 0 \}$ almost surely.

A.5. Proof of Lemma 4.4

A.5.1. Preliminary results

We start with the following simple observation:

**Proposition A.5.** Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3$ be nested $\sigma$-fields on $\Omega$. Then for any $\mathcal{G}_3$-measurable random variable $X \geq 0$ and $B \in \mathcal{G}_1$ with $\{ \mathbb{E}[X | \mathcal{G}_2] > 0 \} \subset B$ we have

$$\{ \mathbb{E}[X | \mathcal{G}_2] > 0 \} \subset \{ \mathbb{E}[X | \mathcal{G}_1] > 0 \} \subset B \quad (57)$$

up to a $\mathbb{P}$-null set. In particular, for $A \in \mathcal{G}_3$, $B \in \mathcal{G}_1$ and $\{ \mathbb{P}(A | \mathcal{G}_2) > 0 \} \subset B$ we have $\{ \mathbb{P}(A | \mathcal{G}_1) > 0 \} \subset B$ up to a $\mathbb{P}$-null set.

**Proof.** For the first inclusion in (57), note that

$$\mathbb{E} \left[ \mathbb{E}[X | \mathcal{G}_2] \mathbb{1}_{\{ \mathbb{E}[X | \mathcal{G}_2] = 0 \}} \right] = \mathbb{E} \left[ \mathbb{E}[X | \mathcal{G}_1] \mathbb{1}_{\{ \mathbb{E}[X | \mathcal{G}_1] = 0 \}} \right] = 0,$$

which proves $\{ \mathbb{E}[X | \mathcal{G}_2] > 0 \} \subset \{ \mathbb{E}[X | \mathcal{G}_1] > 0 \}$ up to a $\mathbb{P}$-null set. For the second inclusion in (57), observe that $\{ \mathbb{E}[X | \mathcal{G}_2] > 0 \} \subset B$ and $B \in \mathcal{G}_1 \subset \mathcal{G}_2$ yields

$$\mathbb{E} \left[ \mathbb{E}[X | \mathcal{G}_1] \mathbb{1}_{B^c} \right] = \mathbb{E}[X \mathbb{1}_{B^c}] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] \mathbb{1}_{B^c}] = 0,$$

which implies by $X \geq 0$ that $\mathbb{E}[X | \mathcal{G}_1] \mathbb{1}_{B^c} = 0$ almost surely and therefore $\{ \mathbb{E}[X | \mathcal{G}_1] > 0 \} \subset B$ up to a $\mathbb{P}$-null set.
Next, we need to prove some technical result, which is used in the proof of Lemma 4.4.

**Proposition A.6.** Consider $S \in \mathcal{S}^\Lambda$, $t \in \mathbb{R}$ and let $T \in \mathcal{S}^\Lambda ([S, \infty])$ with $\mathbb{P}(T < \infty) > 0$ be such that $\bar{L}_{S,T} < t$ on $\{T < \infty\}$ where

$$
\bar{L}_{S,T}(\omega) := \begin{cases} 
\sup_{v \in [S(\omega), T(\omega)]} L_v(\omega) & \text{for } T(\omega) \geq S(\omega), \\
-\infty & \text{for } T(\omega) < S(\omega),
\end{cases}
$$

(58)

with $L$ as in Lemma 4.2.

Then, for any $U \in \mathcal{S}^\Lambda ([S, T])$, the set $E^U := \{\mathbb{P}(T < \infty | \mathcal{F}_U^\Lambda) > 0\}$ has strictly positive probability and there exists $R^U \in \mathcal{S}^\Lambda ([U, \infty])$ such that $R^U = \infty$ on $(E^U)^c$ and

$$
R^U > T, \quad \mathbb{P}(\mu((U, R^U)) > 0 | \mathcal{F}_U^\Lambda) > 0, \quad X_U \leq \mathbb{E} \left[ X_{R^U} + \int_{(U, R^U)} g_t(\ell) \mu(\text{d}t) \middle| \mathcal{F}_U^\Lambda \right] \quad \text{on } E^U. \quad (59)
$$

**Proof of Proposition A.6.** Step 1. Constructing a sequence of sets $E^U_m$ exhausting $E^U$: As the family

$$
\left\{ \mathbb{E} \left[ X_R + \int_{[U,R]} g_t(\ell) \mu(\text{d}t) \middle| \mathcal{F}_U^\Lambda \right] \middle| R \in \mathcal{S}^\Lambda ([U, \infty]) \right\}
$$

is upwards directed there exists by Neveu [1975], Proposition VI-1-1, p.121, a sequence of stopping times $(\bar{R}_m^U)_{m \in \mathbb{N}} \subset \mathcal{S}^\Lambda ([U, \infty])$ such that with $Y_U^\ell$ from Lemma 4.1 we have

$$
\mathbb{E} \left[ X_{\bar{R}_m^U} + \int_{[U, \bar{R}_m^U]} g_t(\ell) \mu(\text{d}t) \middle| \mathcal{F}_U^\Lambda \right] \not\supset Y_U^\ell \quad \text{as } m \uparrow \infty.
$$

Let us define for $m \in \mathbb{N}$

$$
E^U_m := \left\{ X_U \leq \mathbb{E} \left[ X_{R_m^U} + \int_{[U, R_m^U]} g_t(\ell) \mu(\text{d}t) \middle| \mathcal{F}_U^\Lambda \right] \right\} \cap E^U \in \mathcal{F}_U^\Lambda. \quad (60)
$$

This gives us a non-decreasing sequence of sets whose union is $E^U$, up to a $\mathbb{P}$-null set. Indeed, as $U \leq T$, we know that $X_U < Y_U^\ell$ at least on the set $\{T < \infty\}$, because by assumption on $T$,

$$
\{T < \infty\} \subset \{\bar{L}_{S,T} < t\} \subset \{\bar{L}_{S,U} < t\} \subset \{X_v < Y_v^\ell \text{ for } v \in [S, U]\}.
$$

Hence, $\{T < \infty\} \subset \{X_U < Y_U^\ell\} \in \mathcal{F}_U^\Lambda$ and, so, by Proposition A.5 we have

$$
\{T < \infty\} \subset E^U \subset \{X_U < Y_U^\ell\}.
$$

Thus, $(E^U_m)_{m \in \mathbb{N}}$ grows to $E^U$ and

$$
0 < \mathbb{P}(T < \infty) \leq \mathbb{P}(E^U) = \mathbb{P} \left( U \cup_{m=1}^\infty E^U_m \right).
$$

Therefore, there exists $M \in \mathbb{N}$ such that $\mathbb{P}(E^U_m) > 0$ for $m \geq M$ and we assume without loss of generality that in fact $\mathbb{P}(E^U_m) > 0$ holds for all $m \in \mathbb{N}$.

**Step 2.** Fix $m \in \mathbb{N}$ and construct $R_m^U$ corresponding to $E^U_m$ such that $R_m^U$ satisfies the desired conditions (59) for $R_m^U$ on $E^U_m$: Define

$$
G^U_m := \left\{ R \in \mathcal{S}^\Lambda ([U, \infty]) \middle| X_U \leq \mathbb{E} \left[ X_R + \int_{[U,R]} g_t(\ell) \mu(\text{d}t) \middle| \mathcal{F}_U^\Lambda \right] \quad \text{on } E^U_m, \quad \mathbb{P}(\mu((U, R)) > 0 | \mathcal{F}_U^\Lambda) > 0 \quad \text{on } E^U_m \text{ and } R = \infty \text{ on } (E^U)^c \right\}.
$$
For $T_1, T_2 \in \mathcal{S}^\Lambda$ we define $T_1 \leq T_2 :\iff T_1(\omega) \leq T_2(\omega)$ for a.e. $\omega \in \Omega$. This defines a partial order (see, e.g., Rudin [1964], 4.20, p.87). By definition of $E^U_m$ the set $\Theta^U_m$ is a partially ordered set, which is nonempty as it contains $(\tilde{R}^U_m)_{E^U_m}$ from above. Indeed, by (60) we just have to show $P(\mu([U, \tilde{R}^U_m])) > 0|F^A_U) > 0$ on $E^U_m$. 

On the set $\Psi := \{P(\mu([U, \tilde{R}^U_m])) > 0|F^A_U) = 0\} \cap E^U_m \in F^A_U$ we have by Proposition A.5 that $\mu([U, \tilde{R}^U_m])) = 0$. Hence we get on $\Psi$ by definition of $E^U_m$ that

$$X_U \leq \mathbb{E} \left[ X_{\tilde{R}^U_m} + \int_{[U, \tilde{R}^U_m]} g_t(\ell) \mu(\text{d}t) \left| F^A_U \right. \right] = \mathbb{E} \left[ X_{\tilde{R}^U_m} \left| F^A_U \right. \right] \leq X_U,$$

which is a contradiction. Here, the first inequality is due to the definition of $E^U_m$, the first equality follows from $\mu([U, \tilde{R}^U_m])) = 0$ and the last inequality is due to the $\Lambda$-right-upper-semicontinuity in expectation of $X$ and Proposition A.2.

Now we have seen that $\Theta^U_m$ is a partially ordered, non-empty set and hence we obtain by the Hausdorff Maximality Theorem (e.g. Rudin [1964], 4.21, p.87), that there exists a maximal totally ordered subset $\tilde{\Theta}^U_m$: For any two elements $\tilde{R}_1, \tilde{R}_2$ of $\Theta^U_m$, we have $\tilde{R}_1 \leq \tilde{R}_2$ or $\tilde{R}_2 \leq \tilde{R}_1$ a.s. and, if we add any element of $\Theta^U_m \setminus \tilde{\Theta}^U_m$, then the resulting set is not totally ordered any more.

Now set $R^U_m := \text{ess sup}_{\tilde{R} \in \Theta^U_m} \tilde{R}$. As the set $\Theta^U_m$ is totally ordered, it is in particular upwards directed and hence, by Neveu [1975], Proposition VI-1-1, p.121, there exists a non-decreasing sequence $(R^U_{m,k})_{k \in \mathbb{N}}$ in $\Theta^U_m$ with $R^U_m = \lim_{k \to \infty} R^U_{m,k}$. It is then immediate that $R^U_m = \infty$ on $(E^U_m)^c$. Observing

$$\left[ R^U_m, \infty \right] = \bigcap_{k=1}^{\infty} \left[ \tilde{R}^U_{m,k}, \infty \right],$$

we see that $R^U_m \in \mathcal{S}^\Lambda([U, \infty))$. For notational simplicity, let us henceforth omit $m$ and $U$ in our notation of $R^U_m, \tilde{R}^U_{m,k}, E^U_m, \Theta^U_m, \tilde{\Theta}^U_m$ and instead just write $R, \tilde{R}_k, E, \Theta, \tilde{\Theta}$. Now define for $k \in \mathbb{N}$

$$I_k := \{ \tilde{R}_k < R \} \in F_{R,-}, \quad I := \{ \tilde{R}_k < R \text{ for all } k \} = \bigcap_{k=1}^{\infty} I_k \in F_{R,-} \tag{61}$$

and observe that $I_k \subset I_{k-1}$ for all $k \in \mathbb{N}$. The sequence $\tilde{R}_k := (\tilde{R}_k)_\Lambda \wedge k, k \in \mathbb{N}$, announces $R_t$ and $\{R_t = 0\} = \emptyset \in F_{R,-}$, which shows by Dellacherie and Meyer [1982], Theorem 71, p.128, that $R_t$ is an $F$-predictable stopping time. As $R \leq R_t$ we also get $I \in F_{R,-}$ and by Lemma A.1 (ii) we have $\limsup_{k \to \infty} X_{\tilde{R}_k} \leq X_R$ on $\{R = \infty\}$. Hence, with Fatou’s Lemma this gives us on $E$ that

$$X_U \leq \limsup_{k \to \infty} \mathbb{E} \left[ X_{\tilde{R}_k} + \int_{[U, \tilde{R}_k]} g_t(\ell) \mu(\text{d}t) \left| F^A_U \right. \right] \tag{62}$$

where we used in the last equality that $F^A_U \subset F_{R_t-}$ holds, which we will prove shortly. Before, note that by (62) we then have $R \in \Theta$ as $P(\mu([U, R])) \geq P(\mu([U, \tilde{R}_k])) > 0|F^A_U) > 0$ already for $k = 1$. Let us now prove $F^A_U \subset F_{R_t-}$. For that let $A \in F^A_U$. By Dellacherie and Meyer [1978], Theorem 56 (c), (56.2), p.118, we have

$$A \cap \{ U < R_t \} \in F_{R_t-} \tag{63}$$

Since $R_t = R$ on $I$ and, by (61), also $R > \tilde{R}_k \geq U$ on $I$ for an arbitrary $k \in \mathbb{N}$, we get $\{ U < R_t \} = (\{ U < \infty \} \cap I^c) \cup I$. Hence, $\{ U < R_t \}^c = \{ U = \infty \} \cap I^c$ and so, in view of (63), it remains to show that...
we have $A \cap \{U = \infty\} \cap I^c \in \mathcal{F}_{R_1}$. Actually we even have $A \cap \{U = \infty\} \cap I^c \in \mathcal{F}_{U}$. This follows by Dellacherie and Meyer [1978], Theorem 56 (e), p.118, provided $A \cap I^c \in \mathcal{F}_{\infty}$; this last assertion holds true though because $\mathcal{F}_{\infty} = \mathcal{F}_{\infty}$.

Now, using (62) and $R \geq R_k$ for all $k$, we get that $R \in \tilde{\Theta}$ by maximality of $\tilde{\Theta}$.

Finally, we claim that $R > T$ on $E$, so that we have found the desired $R = R^U_m$ of Step 2. We argue by way of contradiction and suppose that $\mathbb{P}(\{R \leq T\} \cap E) > 0$. As, by the properties of $T$,

$$E \cap \{R \leq T\} \cap \{T < \infty\} \subset E \cap \{R \leq T\} \cap \{X_R < Y_R^I\} \in \mathcal{F}_R^A$$

we obtain from Proposition A.5 that even

$$E \cap \{R \leq T\} \cap \{\mathbb{P}(T < \infty | \mathcal{F}_R^A) > 0\} \subset E \cap \{R \leq T\} \cap \{X_R < Y_R^I\}.$$

Now we can construct analogously to the set $E$ a set $\Gamma \in \mathcal{F}_R^A$ with $\mathbb{P}(\Gamma) > 0$, $\Gamma \subset E \cap \{R \leq T\} \cap \{X_R < Y_R^I\}$ and a stopping time $R_2 \geq R$ with

$$X_R < E \left[ X_{R_2} + \int_{[R,R_2]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_R^A$$
on $\Gamma$,

which also implies $R_2 > R$ there. We set $\hat{R} := R_{\Gamma} \vee R_2$, which gives us

$$X_R \leq E \left[ X_{\hat{R}} + \int_{[U,\hat{R}]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_R^A$$
on $\Omega$.

This yields on $E$ that

$$X_U \leq E \left[ X_{R_m} + \int_{[U,R_m]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_U^A \leq E \left[ X_{\hat{R}} + \int_{[U,\hat{R}]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_U^A.$$

Hence, $\hat{R} \in \Theta$, but as $\hat{R} \geq R$ and $\hat{R} > R$ on $\Gamma$, $\hat{R}$ would allow us to extend $\tilde{\Theta}$, a contradiction to the maximality of this totally ordered set. It follows that $\mathbb{P}(\{R \leq T\} \cap E) = 0$.

**Step 3. Construct $R^U$ with the help of $R^U_m$ and $E^U_m$, $m \in \mathbb{N}$:** From the previous steps we get a sequence of $\Lambda$-stopping times $(R^U_m)_{m \in \mathbb{N}}$ and an increasing sequence of $\mathcal{F}_U^A$-measurable sets $(E^U_m)_{m \in \mathbb{N}}$ with $R^U_m > T$ on $E^U_m$, $\mathbb{P}(\mu([U,R^U_m])) > 0 | \mathcal{F}_U^A > 0$ on $E^U_m$.

$$X_U \leq E \left[ X_{R^U_m} + \int_{[U,R^U_m]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_U^A \text{ on } E^U_m,$$

$R^U_m = \infty$ on $(E^U_m)^c$ and $0 < \mathbb{P}(E^U) = \mathbb{P}(\bigcup_{m=1}^{\infty} E^U_m)$. Now define $R_U := \bigwedge_{m=1}^{\infty} (R^U_m)_{m \in \mathbb{N}}^{E^U_m \setminus E^U_{m-1}}$. As for any $m \in \mathbb{N}$ we have $E^U_m \setminus E^U_{m-1} \in \mathcal{F}_U^A \subset \mathcal{F}^A_{R^U_m}$, the random variable $R_U$ is a $\Lambda$-stopping time as a countable minimum of $\Lambda$-stopping times. Furthermore, as for any $m \in \mathbb{N}$ we have $R^U_m > T$ on $E^U_m$, we have $R_U > T$ on $E^U$ almost surely and $R_U = \infty$ on $(E^U)^c$ almost surely. Additionally, we have for any $m \in \mathbb{N}$ by $E^U_m \setminus E^U_{m-1} \in \mathcal{F}_U^A$ that

$$X_U \leq E \left[ X_{R^U_m} + \int_{[U,R^U_m]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_U^A \leq E \left[ X_{R^U_m} + \int_{[U,R^U_m]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_U^A \text{ on } E^U_m \setminus E^U_{m-1}.$$

This implies that almost surely

$$X_U \leq E \left[ X_{R^U_m} + \int_{[U,R^U_m]} g_t(\ell) \mu(d\ell) \right] \mathcal{F}_U^A \text{ on } E^U.$$

and analogously we get $\mathbb{P}(\mu([U,R^U_m])) > 0 | \mathcal{F}_U^A > 0$ almost surely on $E^U$, which finally shows that $R^U$ has all the desired properties.
A.5.2. Proof of the inclusions from Lemma 4.4

First, we get from (14) that for any choice of $\mathbb{P}$-null set $N$

$$B \subset \{ (\omega, t, \ell) \in \Omega^N_S \mid Y^\ell_v(\omega) > X_v(\omega) \text{ for all } v \in [S(\omega), t] \} \subset C,$$

and analogously $\tilde{B} \subset \tilde{C}$. To see that $\tilde{A} \subset \tilde{B}$, note first that again for any choice of $\mathbb{P}$-null set $N$

$$A \subset \{ (\omega, t, \ell) \in \tilde{\Omega}^N_S \mid Y^\ell_v(\omega) > X_v(\omega) \text{ for all } v \in [S(\omega), t] \}.$$  \hspace{1cm} (64)

Hence, for $(\omega, t, \ell) \in A$ with $X_{T_{S,\ell}}(\omega) = Y^\ell_{T_{S,\ell}}(\omega)$, i.e. for $(\omega, t, \ell) \in \tilde{A}$, we have $t < T_{S,\ell}(\omega)$, i.e. $(\omega, t, \ell) \in \tilde{B}$. For the proof of $A \subset B$ we need the following auxiliary result, which is proven below:

**Claim 1:** For $\ell \in \mathbb{R}$ we have outside an evanescent set, possibly depending on $\ell$, that

$$[S, \infty] \cap \{ \tilde{L}_{S, \cdot} < \ell \} \subset [S, T_{S, \ell}],$$

where $\tilde{L}_{S, \cdot}$ is defined in (58).

One can see that the left hand side (respectively right hand side) of (65) is the section of $A$ (respectively $B$) for fixed $\ell \in \mathbb{R}$. Therefore we obtain by Claim 1 a set $N$ with $\mathbb{P}(N) = 0$ such that, for $\omega \in N^c$ and $t \geq S(\omega)$, we have for all rational $\ell$ that $\tilde{L}_{S,\ell}(\omega) < \ell$ implies $t \leq T_{S,\ell}(\omega)$. Hence, for this choice of $N$,

$$A \cap (\Omega \times [0, \infty) \times \mathbb{Q}) \subset B \cap (\Omega \times [0, \infty) \times \mathbb{Q}).$$

Let us argue that, in fact, even $A \subset B$ holds for this choice of $N$. Fix $(\omega, t, \ell) \in A \subset \tilde{\Omega}^N_S$. Consider $(q_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$ a sequence which increases strictly to $\ell$. Without loss of generality $\tilde{L}_{S,\ell}(\omega) < q_n$ for all $n \in \mathbb{N}$. Therefore $(\omega, t, q_n) \in A$ and, thus, by the choice of $N$, we obtain $(\omega, t, q_n) \in B$, i.e. $t \leq T_{S,q_n}(\omega)$. As the sequence $(T_{S,q_n}(\omega))_{n \in \mathbb{N}}$ is non-decreasing we obtain $t \leq T_{S,q_n}(\omega) \leq T_{S,\ell}(\omega)$, which shows $(\omega, t, \ell) \in B$. Hence, $A \subset B$ is proven once we have established Claim 1.

**Proof of Claim 1:** Assume by way of contradiction that (65) is not true. By Lemma 4.1 (iii), $T_{S,\ell}$ is an $\mathcal{F}^A_{\omega, \ell}$-stopping time and by Lenglart [1980], Theorem 2, p.503, we have $[S, T_{S,\ell}] \in \Lambda$. Hence if the Claim 1 fails for some $\ell \in \mathbb{R}$ then the Meyer Section Theorem (see Theorem 2.3) yields a stopping time $T \in \mathcal{S}^A([S, \infty])$ with $\mathbb{P}(T < \infty) > 0$, such that $\tilde{L}_{S,T} < \ell$ and $T_{S,\ell} < T$ on $\{ T < \infty \}$. As we have $\tilde{L}_{S,T} < \ell = \bigcup_{\ell' \in \mathbb{Q}, \ell' < \ell} \{ \tilde{L}_{S,T} < \ell' \}$, we can assume without loss of generality that we can fix an $\ell' \in \mathbb{Q}$, $\ell' \leq \ell$ such that $\tilde{L}_{S,T} < \ell' \in \{ T < \infty \}$.

Let us prove that the existence of such a $T$ leads to a contradiction. Define $(\tilde{U}_n)_{n \in \mathbb{N}} \subset \mathcal{S}^A([S, \infty])$ with $\tilde{U}_n \geq T_{S,\ell}$ as the non-increasing sequence from Bank and Besslich [2018a], Proposition 3.2 (i), such that

$$\tilde{U}_n \downarrow T_{S,\ell} \text{ on } \{ T_{S,\ell} < \infty \} \quad \text{and} \quad X^\ell_{T_{S,\ell}} = \lim_{n \to \infty} X_{U_n}.$$ \hspace{1cm} (66)

Let $U_n := \tilde{U}_n \wedge T$ for $n \in \mathbb{N}$. Recall, that by assumption $\tilde{L}_{S,T} < \ell$ and $T > T_{S,\ell}$ on $\{ T < \infty \}$, and by (64) this implies $X_{T_{S,\ell}} < Y^\ell_{T_{S,\ell}}$ on $\{ T < \infty \}$. Hence, Proposition A.5 and (17) in conjunction with (19) implies up to $\mathbb{P}$-null sets the inclusions

$$\{ T < \infty \} \subset \left\{ \mathbb{P}\left( T < \infty \mid \mathcal{F}^A_{T_{U_n}} \right) > 0 \right\} \subset \left\{ X_{T_{S,\ell}} < Y^\ell_{T_{S,\ell}} \right\} \subset \left\{ \lim_{n \to \infty} Y^\ell_{U_n} = \lim_{n \to \infty} X_{U_n} \right\}. \hspace{1cm} (67)$$

The desired contradiction will be deduced using the following result, which we will prove at the end:

**Claim 2:** For all $U \in \mathcal{S}^A([S, T])$, we have

$$X_U \leq Y^\ell_U - \mathbb{E} \left[ \int_{[U, T]} (g_t(\ell) - g_t(\ell')) \mu(dt) \mid \mathcal{F}^A_U \right] \quad \text{on} \quad \{ \mathbb{P}(T < \infty \mid \mathcal{F}^A_U) > 0 \}.$$
Now define $\Gamma := \left\{ \mathbb{P}(T < \infty | \mathcal{F}_{T,S,t}^\Lambda) > 0 \right\}$ and $\bar{\Gamma} := \bigcap_{n=1}^{\infty} \{ \mathbb{P}(T < \infty | \mathcal{F}_{U_n}^\Lambda) > 0 \}$, which are both $\mathcal{F}_{T,S,t}^{\Lambda}$-measurable as by Proposition A.5 we have $\{ \mathbb{P}(T < \infty | \mathcal{F}_{U_n}^\Lambda) > 0 \}$ for all $n \in \mathbb{N}$. Another application of Proposition A.5 gives us

$$\{ T < \infty \} \subset \Gamma \subset \bar{\Gamma}.$$  

By choosing $U = U_n$, $n = 1, 2, \ldots$, in Claim 2 and letting $n \uparrow \infty$, we get on $\Gamma$ that

$$X_{T,S,t}^* \leq Y_{T,S,t}^\ell - \liminf_{n \to \infty} \mathbb{E} \left[ \int_{[U_n,T]} (g_t(\ell) - g_t(\ell')) \mu(dt) \left| \mathcal{F}_{U_n}^\Lambda \right. \right].$$

$$= X_{T,S,t}^* - \liminf_{n \to \infty} \mathbb{E} \left[ \int_{[U_n,T]} (g_t(\ell) - g_t(\ell')) \mu(dt) \left| \mathcal{F}_{U_n}^\Lambda \right. \right]. \quad (68)$$

The latter sequence of conditional expectations defines a backward supermartingale, which converges almost surely to a random variable $Z$ with

$$Z \geq \mathbb{E} \left[ \int (g_t(\ell) - g_t(\ell')) \mu(dt) \left| \bigcap_{n=1}^{\infty} \mathcal{F}_{U_n}^\Lambda \right. \right] \geq 0 \quad (69)$$

by Dellacherie and Meyer [1982], Theorem 30, p.24. Since $\bigcap_{n=1}^{\infty} \mathcal{F}_{U_n}^\Lambda = \mathcal{F}_{T,S,t}^{\Lambda}$, we thus infer from (68) that $Z = 0$ on $\Gamma$ and, hence, by (69) and $g_t(\ell) - g_t(\ell') > 0$,

$$\Gamma \subset \{ \mu((T,S,t,T)) > 0 \left| \mathcal{F}_{T,S,t}^{\Lambda} \right. = 0 \right\}$$

up to $\mathbb{P}$-null sets. By Proposition A.5, this implies for any $n \in \mathbb{N}$ that

$$\Gamma \subset \{ \mu((T,S,t,T)) > 0 \left| \mathcal{F}_{T,S,t}^{\Lambda} \right. = 0 \right\} \subset \{ \mu((T,S,t,T)) > 0 \left| \mathcal{F}_{U_n}^{\Lambda} \right. = 0 \right\} \subset \{ \mu((T,S,t,T)) = 0 \}. \quad (70)$$

Let us now show that $\Gamma \subset \{ \mu((T,S,t,T)) = 0 \}$ yields a contradiction. For that we have to use another sequence of $\Lambda$-stopping times given by the following claim whose proof again is deferred until the end:

**Claim 3:** There exists a sequence $(R_m)_{m \in \mathbb{N}} \subset \mathcal{S}^{\Lambda} ([U_1, \infty))$ such that we have for all $n \in \mathbb{N}$

$$Y_{U_n}^\ell = \limsup_{m \to \infty} \mathbb{E} \left[ X_{R_m} + \int_{(T,R_m)} g_t(\ell') \mu(dt) \left| \mathcal{F}_{U_n}^\Lambda \right. \right] \quad \text{on} \quad \Gamma. \quad (71)$$

With $(R_m)_{m \in \mathbb{N}}$ from Claim 3 we get on $\Gamma$ again by (66) and (67) that

$$X_{T,S,t}^* \leq \lim_{n \to \infty} X_{U_n} \leq \lim_{n \to \infty} Y_{U_n}^\ell \leq X_{T,S,t}^* - \liminf_{n \to \infty} \liminf_{m \to \infty} \mathbb{E} \left[ \int_{(T,R_m)} (g_t(\ell) - g_t(\ell')) \mu(dt) \left| \mathcal{F}_{U_n}^\Lambda \right. \right]. \quad (72)$$

Define, for $\bar{n} \in \mathbb{Z}_{\leq -1} := \{-n \mid n = 1, 2, 3, \ldots \}$

$$\bar{Z}_{\bar{n}} := \liminf_{m \to \infty} \mathbb{E} \left[ \int_{(T,R_m)} (g_t(\ell) - g_t(\ell')) \mu(dt) \left| \mathcal{F}_{U_{\bar{n}}}^\Lambda \right. \right].$$

Then this defines again a backward supermartingale by Fatou’s Lemma with

$$\sup_{\bar{n} \in \mathbb{Z}_{\leq -1}} \mathbb{E} \left[ |\bar{Z}_{\bar{n}}| \right] \leq 2 \mathbb{E} \left[ \int_{[0,\infty)} |g_t(\ell)| \mu(dt) \right] < \infty.$$
for all $\tilde{n} \in \mathbb{Z}_{\leq -1}$. Finally we need the following result, which we also prove at the end:

**Claim 4:** We have $\mathbb{P}(\Gamma \cap \{ E[\tilde{Z}_{\tilde{n}} | \mathcal{F}_{T,S,t}^A] > 0 \}) > 0$.

Combining Claim 4 with (72) and (73) gives us with positive probability that $X_{T_S,t}^\alpha \leq X_{T_S,t}^\alpha - \tilde{Z} < X_{T_S,t}^\alpha$, which is the desired contradiction needed to establish Claim 1. It remains to prove Claims 2, 3 and 4.

**Proof of Claim 2:** By Proposition A.6, there is $R^U \in \mathcal{S}^\Lambda([T,\infty])$ such that on $E^U := \{ \mathbb{P}(T < \infty | \mathcal{F}_U^A) > 0 \}$ we have $R^U > T$ and

$$X_U \leq \mathbb{E} \left[ X_{R^U} + \int_{[U,R^U]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right],$$

$$= \mathbb{E} \left[ X_{R^U} + \int_{[U,R^U]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right] - \mathbb{E} \left[ \int_{[U,R^U]} (g_t(\ell) - g_t(\ell')) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right],$$

$$\leq Y_{U^1}^{\ell'} - \mathbb{E} \left[ \int_{[U,T]} (g_t(\ell) - g_t(\ell')) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right].$$

Here, we have used $E^U \in \mathcal{F}_{U^1}^A$, $(U,R^U) \supset [U,T]$ on $E^U$ and $g_t(\ell) - g_t(\ell') > 0$ for $\ell > \ell'$.

**Proof of Claim 3:** As the family

$$\left\{ \mathbb{E} \left[ X_R + \int_{[U,R]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right] \bigg| R \in \mathcal{S}^\Lambda([U_1,\infty]) \right\}$$

is upwards directed there exists by Neveu [1975], Proposition VI-1-1, p.121, a sequence of stopping times $(R_m)_{m \in \mathbb{N}} \subset \mathcal{S}^\Lambda([U_1,\infty])$, such that

$$\mathbb{E} \left[ X_{R_m} + \int_{[U_1,R_m]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right] \searrow Y_{U^1}^{\ell'} \text{ as } m \uparrow \infty.$$  

(74)

By (70), equation (71) holds for $n = 1$. Let us now show that the sequence $(R_m)_{m \in \mathbb{N}}$ from the previous step also fulfills (71) for arbitrary $n$. On $\Gamma$ we have by (70) that $\mu([U_n,U_1]) = 0$ and therefore also

$$Y_{U_n}^{\ell'} = \text{ess sup}_{R \in \mathcal{S}^\Lambda([U_n,\infty])} \mathbb{E} \left[ X_R + \int_{[U_1,R]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right] \bigg| \mathcal{F}_{U_n}^A,$$

$$\leq \mathbb{E} \left[ Y_{U^1}^{\ell'} \bigg| \mathcal{F}_{U_n}^A \right] \bigg( \text{74} \bigg) \lim_{m \to \infty} \mathbb{E} \left[ X_{R_m} + \int_{[U_1,R_m]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right] \leq Y_{U_n}^{\ell'} \text{ on } \Gamma.$$  

(75)

Here, we have used additionally dominated convergence in the fifth step, which is possible as

$$\mathbb{E} \left[ X_{R_m} + \int_{[T,R_m]} g_t(\ell) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right] \leq M_{U^1}^X + \mathbb{E} \left[ \int_{[0,\infty]} |g_t(\ell')| \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right]$$

with $M^X$ the $\Lambda$-martingale of Lemma A.1. Hence, equation (75) gives us the desired identity (71).

**Proof of Claim 4:** We assume by way of contradiction that on $\Gamma$ we have $\mathbb{E} \left[ \tilde{Z}_{\tilde{n}} | \mathcal{F}_{T_S,t}^A \right] = 0$, which leads by Proposition A.5 to $\tilde{Z}_{\tilde{n}} = 0$ on $\Gamma$. With the short hand notation

$$Q_m := \mathbb{E} \left[ \int_{[T,R_m]} (g_t(\ell) - g_t(\ell')) \mu(dt) \bigg| \mathcal{F}_{U^1}^A \right].$$
this is equivalent to
\[
\liminf_{m \to \infty} Q_m = 0 \quad \text{on} \quad \Gamma.
\] (76)

The rest of the proof is structured in the following way:

(i) First, we construct a sequence \((Q_m)_{m \in \mathbb{N}}\) connected to some suitably defined \(\Lambda\)-stopping times \((R_m)_{m \in \mathbb{N}}\) such that \(Q_m = \min_{n=1, \ldots, m} Q_n\) and \(Q_m > 0\) for all \(m \in \mathbb{N}\) on some set \(\Gamma_2 \in \mathcal{F}_{U_1}^\Lambda\) with \(\mathbb{P}(\Gamma_2) > 0\) and \(\Gamma_2 \subset \Gamma\).

(ii) We show that there exists a subsequence \((R_{m_k})_{k \in \mathbb{N}}\) of \((R_m)_{m \in \mathbb{N}}\) such that on \(\Gamma_2\) we have

\[
\begin{align*}
(1) & \quad \lim_{k \to \infty} \mu([U_1, R_{m_k})) = 0, \\
(2) & \quad Y_{U_1}^{\ell'} = \lim_{k \to \infty} \mathbb{E} \left[ X_{R_{m_k}} \mathbbm{1}_{\{T,R_{m_k})} g_t(\ell') \mu(dt) \bigg\rvert \mathcal{F}_{U_1}^\Lambda \right] = \mathbb{E} \left[ X_{R_{m_k}} \mathbbm{1}_{\{T,R_{m_k})} \bigg\rvert \mathcal{F}_{U_1}^\Lambda \right], \\
(3) & \quad \text{We combine the previous points to obtain by} \ \Lambda-\mu\text{-right-upper-semi-continuity in expectation of} \ X \ \text{our desired contradiction.}
\end{align*}
\] (77)

(iii) We combine the previous points to obtain by \(\Lambda-\mu\)-right-upper-semi-continuity in expectation of \(X\) our desired contradiction.

**Construction of \((Q_m)_{m \in \mathbb{N}}\) and \((R_m)_{m \in \mathbb{N}}\):** We define as in the proof of Proposition A.6 the following increasing sequence of sets

\[
E_m := \left\{ X_{U_1} < \mathbb{E} \left[ X_{R_m} \mathbbm{1}_{\{T,R_m\}} g_t(\ell') \mu(dt) \bigg\rvert \mathcal{F}_{U_1}^\Lambda \right] \cap \{ \mathbb{P}(T < \infty \mid \mathcal{F}_{U_1}^\Lambda) > 0 \} \right\},
\]

where \((R_m)_{m \in \mathbb{N}}\) is the sequence of \(\Lambda\)-stopping times constructed in Claim 3. Here we can assume without loss of generality that for \(\Gamma_2 := \Gamma \cap E_1 \in \mathcal{F}_{U_1}^\Lambda\) we have \(\mathbb{P}(\Gamma_2) > 0\). Indeed, as on \(\{T < \infty\}\) we have \(X_{U_1} < Y_{U_1}^{\ell'}\), the convergence property of the sequence \((R_m)_{m \in \mathbb{N}}\) ensures that

\[
\bigcup_{m \in \mathbb{N}} E_m = \{ \mathbb{P}(T < \infty \mid \mathcal{F}_{U_1}^\Lambda) > 0 \} \supset \{ T < \infty \}.
\]

Next, we argue that on \(\Gamma_2\) we have \(Q_m > 0\) for all \(m \in \mathbb{N}\). Indeed, by Proposition A.5 and because \(g_t(\ell) - g_t(\ell') > 0\), the equation \(Q_m = 0\) for some \(m \in \mathbb{N}\) implies that \(\mu((T, R_m)) = 0\). Together with (77) this gives \(\mu([U_1, R_m)) = 0\) on \(\Gamma_2\). Moreover, as \(\Gamma_2 \cap \{Q_m = 0\} \subset E_1 \subset E_m\) we get by \(\Lambda-\mu\)-right-upper-semi-continuity in expectation of \(X\) combined with Proposition A.2 the following contradiction on \(\Gamma_2 \cap \{Q_m = 0\}\):

\[
X_{U_1} < \mathbb{E} \left[ X_{R_m} \mathbbm{1}_{\{T,R_m\}} g_t(\ell') \mu(dt) \bigg\rvert \mathcal{F}_{U_1}^\Lambda \right] = \mathbb{E} \left[ X_{R_m} \bigg\rvert \mathcal{F}_{U_1}^\Lambda \right] \leq X_{U_1}.
\]

Let us now define the sequence \((R_m)_{m \in \mathbb{N}} \subset \mathcal{S}^\Lambda ([U_1, \infty))\) by

\[
R_m := \sum_{p=1}^{m} (R_p)(Q_{p-\min_{k \in \{1, \ldots, m\}} Q_k}) \cap \cap_{p=1}^{m} (Q_{p-\min_{k \in \{1, \ldots, m\}} Q_k}) \in \mathcal{S}^\Lambda ([U_1, \infty)),
\]

which means \(R_m\) is equal to \(R_j\), where \(j\) is the first index for which \(Q_j\) attains value \(\min_{k \in \{1, \ldots, m\}} Q_k\). One can see

\[
Q_m := \mathbb{E} \left[ \int_{(T,R_m)} (g_t(\ell) - g_t(\ell')) \mu(dt) \bigg\rvert \mathcal{F}_{U_1}^\Lambda \right] = \min_{k \in \{1, \ldots, m\}} Q_k
\]

is non-increasing in \(m \in \mathbb{N}\) and hence \(\lim_{m \to \infty} Q_m\) exists. By (76) we have

\[
\Gamma_2 \subset \left\{ \liminf_{m \to \infty} Q_m = 0 \right\} = \left\{ \lim_{m \to \infty} Q_m = 0 \right\} =: E \in \mathcal{F}_{U_1}^\Lambda.
\]
We can assume without loss of generality $P(E) = 1$, because we can replace $U_1$, $(R_m)_{m \in \mathbb{N}}$, $(\overline{R_m})_{m \in \mathbb{N}}$ and $T$ by the again $\Lambda$-stopping times $(U_1)_E$, $(R_m)_E)_{m \in \mathbb{N}}$, $(\overline{R_m})_E)_{m \in \mathbb{N}}$ and $T_E$.

**There exists a subsequence of $(\overline{R_m})_{m \in \mathbb{N}}$ with the desired conditions:** The sequence $(Q_m)_{m \in \mathbb{N}}$ is decreasing to zero and therefore we get by the monotone convergence theorem that $\int_{(T,R_m)} (g_\ell(t) - g_\ell(t')) \mu(dt)$ converges to zero in $L^1(P)$. By possibly passing to a subsequence we can assume that this sequence converges to zero almost surely. As we have $g_\ell(t) - g_\ell(t') > 0$ we also get that on $\Gamma_2$ we have by (70) that

$$\lim_{m \to \infty} \mu((U_1, \overline{R_m})) = 0 \quad \text{a.s..} \quad (78)$$

Next, we get for $m, p \in \mathbb{N}$ and $1 \leq p \leq m$

$$\left\{ Q_p = Q_m \right\} \cap \bigcap_{r=1}^{p-1} \left\{ Q_r > Q_m \right\} \subset \left\{ Q_p = Q_m+1 \right\} \cap \bigcap_{r=1}^{p-1} \left\{ Q_r > Q_m+1 \right\},$$

which implies that

$$\{ \overline{R_m+1} \neq \overline{R_m} \} = \{ Q_{m+1} = Q_{m+1} \} \cap \bigcap_{r=1}^{m} \left\{ Q_r > Q_{m+1} \right\} \subset \{ \overline{R_{m+1}} = R_{m+1} \}. \quad (79)$$

Furthermore we want to remind that $(\overline{R_m})_{m \in \mathbb{N}}$ satisfies by (74)

$$\mathbb{E} \left[ X_{R_m} + \int_{(T,R_m)} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right] \leq \mathbb{E} \left[ X_{R_m} + \int_{(T,R_m)} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right]$$

for any $m, n \in \mathbb{N}$ with $m \leq n$. Hence combing (79) and (80) gives us

$$\mathbb{E} \left[ X_{\overline{R_m}} + \int_{(T,\overline{R_m})} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right] \leq \mathbb{E} \left[ X_{\overline{R_m+1}} + \int_{(T,\overline{R_{m+1}})} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right]. \quad (81)$$

We next show the following result:

**Claim 5:** For fixed $m \in \mathbb{N}$ we have on $\Gamma_2$ that

$$\mathbb{E} \left[ X_{R_m} + \int_{(T,R_m)} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right] \leq \sup_{p \in \mathbb{N}} \mathbb{E} \left[ X_{\overline{R_p}} + \int_{(T,\overline{R_p})} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right]$$

**Proof of Claim 5:** As we have shown $Q_m > 0$ for all $m \in \mathbb{N}$ on $\Gamma_2$ we also have $Q_m > 0$ for all $m \in \mathbb{N}$ on $\Gamma_2$. On the other hand we know that $(Q_m)_{m \in \mathbb{N}}$ decreases to zero on $\Gamma_2$. Fix now $m \in \mathbb{N}$. Then we have for any $m \in \mathbb{N}$ that

$$\Gamma_2 \subset \bigcup_{p=m+1}^{\infty} \left\{ Q_p = Q_p < Q_m \right\}. \quad (82)$$

and actually we can rewrite the right-hand side in (82) as a convenient disjoint union of sets:

$$\Gamma_2 \subset \bigcup_{p=m+1}^{\infty} \left\{ Q_p = Q_p < Q_m \right\} \cap \bigcap_{s=m+1}^{p-1} \left\{ Q_s = Q_m \right\} =: \bigcup_{p=m+1}^{\infty} H^{(p)}.$$

But on $H^{(p)} \in \mathcal{F}^\Lambda_{U_1}$ we have $R_p = R_p$ and hence, by (80),

$$\mathbb{E} \left[ X_{R_m} + \int_{(T,R_m)} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right] \leq \mathbb{E} \left[ X_{\overline{R_p}} + \int_{(T,\overline{R_p})} g_\ell(t) \mu(dt) \bigg| \mathcal{F}^\Lambda_{U_1} \right] \text{ on } H^{(p)},$$
which finishes the proof of Claim 5.

Continuation of the Proof of Claim 4: Now we have by Claim 3 and 5 that on $\Gamma_2$
\[
Y_{U_1}^{\ell} = \lim_{p \to \infty} \mathbb{E} \left[ X_{R_m} + \int_{(T,R_m)} g_t(\ell') \mu(\text{d}t) \bigg| \mathcal{F}_{U_1}^\Lambda \right] \leq \sup_{p \in \mathbb{N}} \mathbb{E} \left[ X_{R_p} + \int_{(T,R_p)} g_t(\ell') \mu(\text{d}t) \bigg| \mathcal{F}_{U_1}^\Lambda \right] 
\]
\[
\overset{(81)}{=} \lim_{p \to \infty} \mathbb{E} \left[ X_{R_p} + \int_{(T,R_p)} g_t(\ell') \mu(\text{d}t) \bigg| \mathcal{F}_{U_1}^\Lambda \right] \overset{(70)}{\leq} Y_{U_1}^{\ell}.
\]

Furthermore, we have by (78)
\[
\lim_{p \to \infty} \mathbb{E} \left[ \int_{(T,R_p)} g_t(\ell') \mu(\text{d}t) \bigg| \mathcal{F}_{U_1}^\Lambda \right] = 0 \quad \text{on} \quad \Gamma_2
\]
and hence also that $\lim_{p \to \infty} \mathbb{E} \left[ X_{R_p} \bigg| \mathcal{F}_{U_1}^\Lambda \right] = Y_{U_1}^{\ell}$ on $\Gamma_2$.

Combining the previous results: On $\Gamma_2$ we have $X_{U_1} < Y_{U_1}^{\ell} = \lim_{p \to \infty} \mathbb{E} \left[ X_{R_p} \bigg| \mathcal{F}_{U_1}^\Lambda \right] \leq X_{U_1}$, which is a contradiction. Here we have used in the first inequality (64), in the first equality (77) and in the second inequality $\Lambda$-$\mu$-right-upper-semicontinuity in expectation of $X$ and Proposition A.2.

A.6. Proof of Proposition 4.5

Our disintegration formula (32) can be proven by following to a large extent the arguments for the analogous result of Bank and El Karoui [2004], Lemma 4.12 (iv), p.1050. In particular, it again suffices to focus on $\phi = 1_{[\ell,\bar{\ell}]}$ with $\ell < \bar{\ell}$ and consider rational partitions $\pi_n = \{ \ell = \ell_0 < \ell_1 < \cdots < \ell_n = \bar{\ell} \}$ of $[\ell, \bar{\ell}]$ whose mesh vanishes as $n \uparrow \infty$. Our divided stopping times $\tau_{S,\ell}$, however, require a more delicate analysis when passing to the limit $n \uparrow \infty$ in the considered integrals. This is taken care of by the following two claims:

Claim 1: For $\text{P}$-a.e. $\omega \in \Omega$ we have
\[
\int_{[S(\omega),\infty)} \int_{[\ell,\bar{\ell}]} \lim_{n \to \infty} \inf \mathbb{1}_{[S,\tau_{S,\ell}(\ell_n(t))]}(\omega, t) g_t(\omega, \text{d}t) \mu(\omega, \text{d}t) \geq \int_{[S(\omega),\infty)} \int_{[\ell,\bar{\ell}]} \mathbb{1}_{[S,\tau_{S,\ell}(t)]}(\omega, t) g_t(\omega, \text{d}t) \mu(\omega, \text{d}t),
\]
where $(\ell_n(t))_{n \in \mathbb{N}}$ are defined by $\ell_n(t) := \max \{ \ell_i \in \pi_n \mid \ell_i \leq t \}$ ($n = 1, 2, \ldots$).

Claim 2: For $\text{P}$-a.e. $\omega \in \Omega$ we have
\[
\int_{[S(\omega),\infty)} \int_{[\ell,\bar{\ell}]} \lim_{n \to \infty} \sup \mathbb{1}_{[S,\tau_{S,\ell}(\ell_n(t))]}(\omega, t) g_t(\omega, \text{d}t) \mu(\omega, \text{d}t) \leq \int_{[S(\omega),\infty)} \int_{[\ell,\bar{\ell}]} \mathbb{1}_{[S,\tau_{S,\ell}(t)]}(\omega, t) g_t(\omega, \text{d}t) \mu(\omega, \text{d}t),
\]
where $(r_n(t))_{n \in \mathbb{N}}$ are defined by $r_n(t) := \min \{ \ell_i \in \pi_n \mid \ell_i > t \}$ ($n = 1, 2, \ldots$).

It remains to prove Claim 1 and 2. Proof of Claim 1: For reasons that will become clear later, we establish (83) only for $\omega \in \bar{\Omega}$, where $\bar{\Omega} \subset \Omega$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for all $q \in \mathbb{Q}$ we have $H_{T_{S,q}} \cap \bar{\Omega} = \{ X_{T_{S,q}} < Y_{T_{S,q}}^q \} \subset \bar{\Omega}$. Notice that such an $\bar{\Omega}$ can be found by (19).

Now fix $\omega \in \bar{\Omega}$. For $(t, \ell) \in [0, \infty) \times [\ell, \bar{\ell}]$ with $t \neq T_{S,\ell}(\omega)$ and $T_{S,\ell}(\omega) = T_{S,\ell}(\omega)$ we have by $\lim_{n \to \infty} \ell_n(t) = \ell$ that $\lim_{n \to \infty} T_{S,\ell_n(t)}(\omega) = T_{S,\ell}(\omega)$ and $\lim_{n \to \infty} \mathbb{1}_{[S,\tau_{S,\ell_n(t)}]}(\omega, t) = \mathbb{1}_{[S,\tau_{S,\ell}]}(\omega, t)$. As for fixed $\omega \in \bar{\Omega}$ the set $\{ \ell \in \mathbb{R} \mid T_{S,\ell}(\omega) < T_{S,\ell}(\omega) \}$ is countable, it is for every $t \in [0, \infty)$ a $g_t(\omega, \text{d}t)$-null set.
Hence, we get
\[
\int_{[S(\omega), \infty)} \liminf_{n \to \infty} \mathbb{1}_{[S, \tau_{S, \ell_n(t)}]}(\omega, t) g_t(\omega, \mathrm{d}t) \mu(\omega, \mathrm{d}t) \\
\geq \int_{[S(\omega), \infty)} \liminf_{n \to \infty} \int_{[\ell, \tilde{\ell}]} \left( \mathbb{1}_{[S, \tau_{S, \ell}]}(\omega, t) \mathbb{1}_{\{T_{S, \ell}(\omega) \neq t\}} \right) g_t(\omega, \mathrm{d}t) \mu(\omega, \mathrm{d}t) \\
+ \int_{[S(\omega), \infty)} \liminf_{n \to \infty} \left( \mathbb{1}_{[S, \tau_{S, \ell_n(t)}]}(\omega, t) \mathbb{1}_{\{T_{S, \ell, n}(\omega) = t\}} \right) g_t(\omega, \mathrm{d}t) \mu(\omega, \mathrm{d}t).
\]
Therefore it remains to show for any fixed \( t \in [S(\omega), \infty) \) that
\[
\liminf_{n \to \infty} \mathbb{1}_{[S, \tau_{S, \ell_n(t)}]}(\omega, t) \geq \mathbb{1}_{[S, \tau_{S, \ell}]}(\omega, t)
\tag{84}
\]
for \( g_t(\omega, \mathrm{d}t) \)-a.e. \( \ell \in J \) with \( J := \{ \ell \in [\ell, \tilde{\ell}] \mid T_{S, \ell}(\omega) = t \} \). As \( \ell \mapsto T_{S, \ell}(\omega) \) is non-decreasing, \( J \) is an interval and since \( g_t(\omega, \mathrm{d}t) \) is an atomless measure, we can focus without loss of generality on the interior of \( J \). If the latter is empty there is nothing to show. Otherwise, fix \( \ell \in \text{int} J \) and observe that there exists some \( N_{\omega, \ell} \in \mathbb{N} \) such that \( \ell_n(\ell) \in \text{int} J \) for \( n \geq N_{\omega, \ell} \) and thus \( T_{S, \ell}(\omega) = T_{S, \ell}^{-}(\omega) = T_{S, \ell}^{+}(\omega) = T_{S, \ell_n(t)}(\omega) = t \) for \( n \geq N_{\omega, \ell} \). This implies that for any fixed \( \ell \), inequality (84) is equivalent to
\[
\liminf_{n \to \infty} \mathbb{1}_{H_{S, \ell_n(t)}^{+}}(\omega) \geq \mathbb{1}_{H_{S, \ell}^{+}}(\omega)
\tag{85}
\]
Now we get by the property of \( \tilde{\Omega} \) and \( (\ell_n(\ell))_{n \in \mathbb{N}} \subset Q \) that
\[
\liminf_{n \to \infty} \mathbb{1}_{H_{S, \ell_n(t)}^{+}}(\omega) = \liminf_{n \to \infty} \mathbb{1}_{\{Y_{T_{S, \ell}}^{\ell_n(\ell)}(t) > X_{T_{S, \ell}}(t)\}}(\omega) = \mathbb{1}_{\bigcup_{m=1}^{\infty} \bigcap_{m=m}^{\infty} \{Y_{T_{S, \ell}}^{\ell_n(\ell)}(t) > X_{T_{S, \ell}}(t)\}}(\omega).
\]
Moreover we have for \( \omega \in \{Y_{T_{S, \ell}}^{\ell_n(\ell)}(t) > X_{T_{S, \ell}}(t)\} \) that there exists by continuity of \( \ell \mapsto Y_{T_{S, \ell}}^{\ell} \) some \( \tilde{N}(\omega) \geq N_{\omega, \ell} \) such that \( Y_{T_{S, \ell}}^{\ell_n(\ell)}(\omega) > X_{T_{S, \ell}}(\omega) \) for \( m \geq \tilde{N}(\omega) \) and so \( \omega \in \bigcap_{m=1}^{\infty} \{Y_{T_{S, \ell}}^{\ell_n(\ell)}(t) > X_{T_{S, \ell}}(t)\} \). As \( \mathbb{1}_{H_{S, \ell}^{+}}(\omega) \leq \mathbb{1}_{\{X_{T_{S, \ell}} < Y_{T_{S, \ell}}^{\ell}\}}(\omega) \) by definition of \( H_{S, \ell}^{+} \), we finally obtain
\[
\liminf_{n \to \infty} \mathbb{1}_{H_{S, \ell_n(t)}^{+}}(\omega) = \mathbb{1}_{\bigcup_{m=1}^{\infty} \bigcap_{m=m}^{\infty} \{Y_{T_{S, \ell}}^{\ell_n(\ell)}(t) > X_{T_{S, \ell}}(t)\}}(\omega) \geq \mathbb{1}_{\{X_{T_{S, \ell}} < Y_{T_{S, \ell}}^{\ell}\}}(\omega) \geq \mathbb{1}_{H_{S, \ell}^{+}}(\omega)
\]
which shows (85) and finishes the proof of Claim 1.

**Proof of Claim 2:** Let \( \Omega \subset \tilde{\Omega} \) with \( P(\Omega) = 1 \) be such that the relation in (19) holds for any \( \omega \in \Omega \) and all \( \ell \in Q \). Analogously to the proof of Claim 1 it suffices to show for any fixed \( t \in [S(\omega), \infty) \) and \( \omega \in \Omega \) that
\[
\limsup_{n \to \infty} \mathbb{1}_{[S, \tau_{S, \ell_n(t)}]}(\omega, t) \leq \mathbb{1}_{[S, \tau_{S, \ell}]}(\omega, t)
\tag{86}
\]
g\(_t(\omega, \mathrm{d}t)\)-a.e. on \( J := \{ \ell \in [\ell, \tilde{\ell}] \mid T_{S, \ell}(\omega) = t \} \). So fix \( t \in [S(\omega), \infty) \). As \( \ell \mapsto T_{S, \ell}(\omega) \) is non-decreasing, \( J \) is an interval and since \( g_t(\omega, \mathrm{d}t) \) is an atomless measure, we can focus without loss of generality on the interior of \( J \) and we can assume that \( \text{int} J \) is non-empty. Now we get for \( \ell \in \text{int} J \) that also \( r_n(\ell) \in \text{int} J \) for \( n \) large enough and thus \( T_{S, \ell}(\omega) = T_{S, \ell}^{-}(\omega) = T_{S, \ell}^{+}(\omega) = T_{S, \ell_n(t)}(\omega) = t \) for sufficiently large \( n \). This implies, analogously to the proof of Claim 1, that for \( \ell \in \text{int} J \), inequality (86) is equivalent to
\[
\limsup_{n \to \infty} \mathbb{1}_{\{X_{T_{S, \ell}} < Y_{T_{S, \ell}}^{\ell_n(\ell)}(t)\}}(\omega) \leq \mathbb{1}_{H_{S, \ell}^{+}}(\omega)
\tag{87}
\]
We claim that \( H_{S, \ell}^{+} = \{X_{T_{S, \ell}} < Y_{T_{S, \ell}}^{\ell}\} \). Indeed, as \( H_{S, \ell}^{+} \subset \{X_{T_{S, \ell}} < Y_{T_{S, \ell}}^{\ell}\} \) is clear we can assume \( \omega \in \{X_{T_{S, \ell}} < Y_{T_{S, \ell}}^{\ell}\} \) and we will show \( \omega \in H_{S, \ell}^{+} \). By continuity and monotonicity of \( \ell \mapsto Y_{T_{S, \ell}}^{\ell}(\omega) \) and \( \ell \in \text{int} J \) there exists \( q \in \text{int} J \cap Q \) with \( q < \ell \) and \( \omega \in \{X_{T_{S, q}} < Y_{T_{S, q}}^{\ell}\} \). As \( \omega \in \Omega \) this implies \( \omega \in H_{S, q}^{+} \). By Lemma 4.1 (vi) the mapping \( \ell \mapsto \tau_{S, \ell} \) is increasing. In particular we get by \( T_{S, \ell}(\omega) = T_{S, q}(\omega) = t \) and for \( \omega \in H_{S, q}^{+} \) that \( \omega \in H_{S, \ell}^{+} \).
Next, we see that the inequality (87) is trivially fulfilled if the left-hand side is zero or if \( \ell = r_n(\ell) \) for sufficiently large \( n \). So, we just have to analyse \( \ell \in J_2 \), where
\[
J_2 := \left\{ \ell \in \text{int} J \mid \omega \in \{ X_{T_S,t} < Y_{T_S,t}^{r_n(\ell)} \} \text{ for infinitely many } n, r_n(\ell) > \ell \text{ for } n \in \mathbb{N} \right\}.
\]

So let us show \( \omega \in \{ X_{T_S,t} < Y_{T_S,t}^{\ell} \} \) for \( \ell \in J_2 \). For that we will use the following claim proven at the end:

**Claim 3:** There exists at most one \( \tilde{\ell} \in J_2 \) with
\[
Y_{T_S,t}^{\ell}(\omega)(\omega) < Y_{T_S,t}^{r_n(\ell)}(\omega) \quad \text{for all } n \in \mathbb{N} \tag{88}
\]
and
\[
X_{T_S,t}(\omega)(\omega) = Y_{T_S,t}^{\ell}(\omega). \tag{89}
\]

As \( g_t(\omega, d\ell) \) is a continuous measure we can now focus by Claim 3 on \( \ell \in J_2 \setminus \{ \tilde{\ell} \} \), which does neither satisfy (88) nor (89). If \( \ell \) does not satisfy (89), we have \( X_{T_S,t}(\omega)(\omega) < Y_{T_S,t}^{\ell}(\omega) \), which is exactly what we want to show. Assume \( \ell \) does not satisfy (88), i.e. \( Y_{T_S,t}(\omega)(\omega) = Y_{T_S,t}^{r_n(\ell)}(\omega) \) for \( n \in \mathbb{N} \) large enough. Then there will be \( \tilde{n} \geq n \) with \( \omega \in \{ X_{T_S,t} < Y_{T_S,t}^{r_n(\ell)} \} \) and by monotonicity of \( r \mapsto Y^r \) we have again \( Y_{T_S,t}^{\ell}(\omega) = Y_{T_S,t}^{r_{\tilde{n}}(\ell)}(\omega) \).

This leads to \( Y_{T_S,t}^{\ell}(\omega) = Y_{T_S,t}^{r_{\tilde{n}}(\ell)}(\omega) > X_{T_S,t}(\omega) \) and hence \( \omega \in \{ X_{T_S,t} < Y_{T_S,t}^{\ell} \} \), which proves Claim 2 once Claim 3 is established.

**Proof of Claim 3:** Assume \( \tilde{\ell} \) fulfills (88) and (89) and \( u \in J_2 \).

Case \( u > \tilde{\ell} \): As \( r \mapsto Y^r(\omega) \) is non-decreasing we get by \( \tilde{\ell} \) satisfying (88) and (89) that
\[
Y_{T_S,u}(\omega)(\omega) > Y_{T_S,t}(\omega)(\omega) = X_{T_S,t}(\omega)(\omega) = X_{T_S,u}(\omega)(\omega),
\]
where we have used \( T_S,\tilde{\ell}(\omega) = t = T_S,u(\omega) \) by \( \tilde{\ell}, u \in J_2 \). Hence \( u \) does not fulfill (89).

Case \( u < \tilde{\ell} \): Again as \( r \mapsto Y^r(\omega) \) is non-decreasing we get that \( X_{T_S,t}(\omega)(\omega) = Y_{T_S,t}(\omega)(\omega) = Y_{T_S,u}(\omega)(\omega) \).

Furthermore we see by \( u < \tilde{\ell} \) that the corresponding sequence \( (r_n(u))_{n \in \mathbb{N}} \) will fulfill \( r_n(u) \leq \tilde{\ell} \) for \( n \) large enough and therefore \( Y_{T_S,u}(\omega)(\omega) = Y_{T_S,t}(\omega)(\omega) = Y_{T_S,u}(\omega)(\omega) \). Therefore \( u \) does not satisfy (89), which proves our claim.

**A.7. Proof of Lemma 4.6**

Let \( S \in \mathcal{S}^h \) and fix \( \ell_0 \in \mathbb{R} \). First we have \( X_S = Y_{S}^{L_S} \) by Lemma 4.3. As we have \( X_S = Y_{S}^{L_S} \) on \( \{ \ell_0 \leq L_S \} \) and as \( \ell \mapsto Y_{S}(\ell) \) is non-decreasing (Lemma 4.1 (vii)), we get
\[
X_S = Y_{S}^{L_S} = Y_{S}^{\ell_0} - \int_{\mathbb{R}} 1_{[L_S\cap [\ell_0, \ell_0)]}(\ell)Y_{S}(d\ell). \tag{90}
\]

Denote by \( I \) the integral on the right hand side of this expression. Due to our disintegration formula (see Proposition 4.5) for the random measure \( Y_{S}(d\ell) \), we can rewrite
\[
I = \mathbb{E} \left[ \int_{[S, \infty]} \left\{ \int_{\mathbb{R}} 1_{[L_S\cap [\ell_0, \ell_0)]}(\ell)1_{[S, r_{\tilde{n}}(\ell)]}(t)g_t(\ell) d\ell \right\} \mu(dt) \right| \mathcal{F}_S^{\ell_0}. \tag{91}
\]

Next, we state a claim, which uses the notation \( L_S,t \) from (58). The claim will be proven at the end.

**Claim:** Let \( \bar{\Omega} := \Omega \setminus \mathcal{N} \), with \( \mathcal{N} \) from Lemma 4.4 and \( \bar{\Omega} \subset \Omega \), \( \mathbb{P}(\bar{\Omega}) = 1 \) such that on \( \bar{\Omega} \) relation (19) holds for all \( \ell \in \Omega \) and \( \ell_0 \). Then we have the following three equations:
(a) For $\omega \in \Omega$, $t \in [0, \infty)$ we have

$$1_{H_{S,t}^+}(\omega)1_{[L_S(\omega),t_0]}(\ell)1_{[S(\omega),T_S,\ell(\omega)]}(t) = 1_{H_{S,t}^+}(\omega)1_{[\bar{L}_S,\ell(\omega),t_0]}(\ell)1_{[S(\omega),T_S,t_0(\omega)]}(t)$$

(92)

for $g_t(\omega, d\ell)$-a.e. $\ell \in \mathbb{R}$.

(b) For $\omega \in \Omega$, $t \in [0, \infty)$ we have

$$1_{H_{S,t}^+}(\omega)1_{[L_S(\omega) \wedge t_0,\ell_0]}(\ell)1_{[S(\omega),T_S,\ell(\omega)]}(t) = 1_{H_{S,t}^+}(\omega)1_{[\bar{L}_S,\ell(\omega),\ell_0]}(\ell)1_{[S(\omega),T_S,t_0(\omega)]}(t)$$

(93)

for $g_t(\omega, d\ell)$-a.e. $\ell \in \mathbb{R}$.

(c) For $\omega \in \Omega$ and $t = T_{S,t_0}(\omega)$, we have $1_{[L_S,\ell_0]}(\ell)1_{H_{S,t}^+}(\omega) = 1_{[L_S,\ell_0]}(\ell)1_{H_{S,t}^+}(\omega)$ for $g_t(\omega, d\ell)$-a.e. $\ell \in \mathbb{R}$.

Combining (a), (b) and (c) from the above Claim with (91) leads to

$$I = \mathbb{E}\left[ \int_{[S,T_S,\ell_0]} (g_t(\ell_0) - g_t(\bar{L}_S,\ell)) \mu(d\ell) \right] \bigg| \mathcal{F}_S^\Lambda.$$

By (21) and (90), we see that

$$X_S - \mathbb{E}\left[ X_{T_S,\ell_0} \big| \mathcal{F}_S^\Lambda \right] = Y_{t_0}^\ell - I - \mathbb{E}\left[ X_{T_S,\ell_0} \big| \mathcal{F}_S^\Lambda \right] = \mathbb{E}\left[ \int_{[S,T_S,\ell_0]} g_t(\bar{L}_S,\ell) \mu(d\ell) \right] \bigg| \mathcal{F}_S^\Lambda.$$

(94)

Now it stays to show the integrability condition (33). First, we have for $\omega \in \Omega$ and $t < T_{S,t_0}(\omega)$ by Lemma 4.4 that $L_{S,t}(\omega) \leq \ell_0$. On the other hand we get for $\omega \in H_{S,t}^+$ and $t = T_{S,t_0}(\omega)$ that $X_t(\omega) \leq Y_{t_0}^\ell(\omega)$ which shows also in this case $L_t(\omega) < \ell_0$. Hence we have for $t \in [S,\tau_{S,\ell_0}](\omega)$ that $L_t(\omega) \leq \ell_0$, which shows by monotonicity of $\ell \mapsto g_t(\ell)$, $g_t(0) = 0$ (see (8)) and $g(\ell_0) \in L^1$ by Assumption 2.5 (ii) (b) that

$$\mathbb{E}\left[ \int_{[S,T_S,\ell_0]} (g_t(\bar{L}_S,\ell) \lor 0) \mu(d\ell) \right] \leq \mathbb{E}\left[ \int_{0,\infty} g_t(\ell_0 \lor 0) \mu(d\ell) \right] < \infty,$$

which shows that the positive part of $1_{[S,T_S,\ell]}g(\sup_{\nu \in [S,t]} L_{\nu})$ is integrable with an upper bound independent of $S$. Furthermore combining $M^X$ from Lemma A.1 with Bank and Besslich [2018a], Lemma 2.38, and (94) leads to

$$-\infty < \mathbb{E}[X_S - M_S^X] \leq \mathbb{E}\left[ \int_{[S,T_S,\ell_0]} g_t(\bar{L}_S,\ell) \mu(d\ell) \right],$$

which shows that also the negative part of $1_{[S,T_S,\ell]}g(\sup_{\nu \in [S,t]} L_{\nu})$ is integrable. Moreover, by $X$ and $M^X$ of class(D$\Lambda$) we obtain that

$$-\infty < -\left( \sup_{S \in \mathcal{S}^+} \mathbb{E}[\|X_S\|] + \sup_{S \in \mathcal{S}^+} \mathbb{E}[\|M^X_S\|] \right)$$

is a uniform lower bound independent of $S$. This completes the proof of our Lemma once we have proven the above Claim.

**Proof of Part (a) of the above claim:** Fix $\omega \in \Omega$.

“$\geq$” in (92): Assume $t \in [S(\omega),T_{S,t_0}(\omega)]$, $t_0 > \ell > L_{S,t}(\omega) \geq L_S(\omega)$ and $\omega \in H_{S,\ell}^+$. Then we get by $A \subset B$ in Lemma 4.4 (cf. (27)) that $t \leq T_{S,t}(\omega)$. Here we can focus on $t > L_{S,t}(\omega)$ as $\{L_{S,t}(\omega)\}$ is a $g_t(\omega, d\ell)$-null set.

“$\leq$” in (92): Assume $t \in [S(\omega),T_{S,t}(\omega)]$, $t_0 > \ell \geq L_S(\omega)$ and $\omega \in H_{S,\ell}^+$. In the case $t < T_{S,t}(\omega)$ we have by the relation $\bar{B} \subset C$ in Lemma 4.4 (cf. (29) and (30)) that $\ell \geq \bar{L}_S,\ell(\omega)$. For $t = T_{S,t}(\omega)$ we get by $B \subset C$ in Lemma 4.4 (cf. (27) and (28)) that $\sup_{\nu \in [S(\omega),T_{S,t}(\omega)]} L_{\nu}(\omega) \leq \ell$ and $\omega \in H_{S,\ell}^+ \subset \{X_{T_S,\ell} < Y_{T_S,\ell}^\ell\}$ (see (17))
shows by the definition of \( L \) that \( \bar{L}_{S,T,s,t}(\omega) \leq \ell \). This finishes our proof as \( T_{S,t}(\omega) \leq T_{S,t_0}(\omega) \) follows by monotonicity of \( \ell \mapsto T_{S,\ell}(\omega) \).

**Proof of Part (b) of the above claim:** Fix \( \omega \in \tilde{\Omega} \).

“\( \leq \)” in (93): Assume \( L_{S}(\omega) \leq \ell < t_0 \), \( S(\omega) \leq t < T_{S,t}(\omega) \leq T_{S,t_0}(\omega) \) and \( \omega \in H_{\bar{S},\ell} \cup H_{S,\ell} \). From \( \bar{B} \subset \tilde{B} \) in Lemma 4.4 (cf. (29) and (30)) we get \( \bar{L}_{S,t}(\omega) \leq \ell < t_0 \).

“\( \geq \)” in (93): Let \( t \in [S(\omega),T_{S,t_0}(\omega)) \), \( L_{S,t}(\omega) \leq \ell < t_0 \) and \( \omega \in H_{\bar{S},\ell} \cup H_{S,\ell} \). As \( \{L_{S,t}(\omega)\} \) is a \( g_\ell(\omega,d\ell) \)-null set we can focus on \( \bar{L}_{S,t}(\omega) < \ell \). From \( \bar{L}_{S,t}(\omega) < \ell \) we obtain by \( A \subset B \) in Lemma 4.4 (cf. (27)) that \( t < T_{S,t}(\omega) \). Now we have to prove that for fixed \( t \in [S(\omega),T_{S,t_0}(\omega)) \) the set

\[
J := \{ \bar{L}_{S,t}(\omega), t_0 \} \cap \left\{ \ell \in \mathbb{R} \mid \omega \in H_{\bar{S},\ell} \cup H_{S,\ell} \text{ and } T_{S,t}(\omega) = t \right\}
\]

is a \( g_\ell(\omega,d\ell) \)-null set. By Lemma 4.1 (v) the mapping \( \ell \mapsto T_{S,t} \) is increasing and therefore \( J \) is an interval. Assume \( J \) contains more than one point. Then take \( \ell_1, \ell_2 \in J \) and some \( q \in \mathbb{Q} \) with \( \ell_1 < q < \ell_2 \). As \( J \) is an interval also \( q \in J \). From \( \omega \in \tilde{\Omega} \) we get \( q \in H_{\bar{S},t} \cup H_{S,\ell} = \{X_{T_{S,q}} = Y_{T_{S,q}}^q\} \), which implies by \( A \subset \tilde{B} \) in Lemma 4.4 (cf. (29)) that \( t < T_{S,q}(\omega) \), which contradicts \( q \in J \). Hence \( J \) contains at most one point, which shows \( J \) is a \( g_\ell(\omega,d\ell) \)-null set.

**Proof of Part (c) of the above claim:** Fix \( \omega \in \tilde{\Omega} \), \( t = T_{S,t_0}(\omega) \) and \( \ell \in [\bar{L}_{S,T,s,t_0}(\omega),\ell_0) \). We do not have to consider the case \( \ell = \bar{L}_{S,T,s,t_0}(\omega) \) as for fixed \( \omega \) the set \( \{\bar{L}_{S,T,s,t}(\omega)\} \) is a \( g_{\ell_0}(\omega,d\ell) \)-null set. Now we get from \( \bar{L}_{S,T,s,t_0}(\omega) < \ell \) that \( X_{t_0}(\omega) < Y_{t_0}^\ell(\omega) \) for all \( v \in [S(\omega),T_{S,t_0}(\omega)] \). Furthermore \( \bar{L}_{S,T,s,t_0}(\omega) < \ell \) implies by \( A \subset B \) in Lemma 4.4 (cf. (27)) that \( T_{S,t_0}(\omega) \leq T_{S,\ell}(\omega) \) and therefore by monotonicity of \( \ell \mapsto T_{S,\ell}(\omega) \) that \( t = T_{S,t_0}(\omega) = T_{S,\ell}(\omega) \). Hence if \( \ell \in [\bar{L}_{S,T,s,t_0}(\omega),\ell_0) \) we have \( X_{t}(\omega) \leq Y_{t}^\ell(\omega) \) and there exists \( q \in (\bar{L}_{S,T,s,t_0}(\omega),\ell] \cap \mathbb{Q} \) with \( X_{t_0}(\omega) < Y_{t_0}^q(\omega) \). As \( \omega \in \tilde{\Omega} \) this implies \( \omega \in H^2_{\bar{S},q} \) and monotonicity of \( \ell \mapsto T_{S,\ell} \) we get \( \omega \in H_{\bar{S},\ell} \), which proves part (c).

### A.8. Proof of Lemma 4.7

Note first that by Lemma 4.1 (iii) \( \ell \mapsto T_{S,\ell} \) is non-decreasing. Hence \( T_{S,\infty} \) exists as a monotone limit of stopping times. Moreover, by Lemma 4.1 (v) and the definition of the essential supremum, we have

\[
\mathbb{E}[Y_{s}^\ell] = \mathbb{E}[X_{T_{s},\ell}] + \int_{[S,T_{s},\ell]} g_{\ell}(\ell) \mu(d\ell) \geq \mathbb{E}[X_{\infty}] + \int_{[S,\infty]} g_{\ell}(\ell) \mu(d\ell)
\]

or, equivalently, as \( X_{\infty} = 0 \) by assumption,

\[
\mathbb{E}[X_{T_{s},\ell}] \geq \mathbb{E}\left[ \int_{[S,T_{s},\ell]} g_{\ell}(\ell) \mu(d\ell) \right].
\]

Hence, for any \( \mathbb{Q} \ni \ell_0 > 0 \), we can, by monotonicity of \( \ell \mapsto g_{\ell}(\ell) \) and normalization to \( g_{\ell}(0) = 0 \) (see (8)), use monotone convergence to conclude

\[
\mathbb{E}[M_{X}^s] \geq \lim\inf_{\mathbb{Q} \ni \ell \uparrow \infty} \mathbb{E}[X_{T_{s},\ell}] \geq \lim\inf_{\mathbb{Q} \ni \ell \uparrow \infty} \mathbb{E}\left[ \int_{[S,T_{s},\ell]} g_{\ell}(\ell_0) \mu(d\ell) \right] \geq \mathbb{E}\left[ \int_{[T_{S,\infty},\infty]} g_{\ell}(\ell_0) \mu(d\ell) \right] \geq 0,
\]

where \( M^X \geq X \) with \( M^X \) of Lemma A.1 (i) and the first inequality follows with the help of Bank and Besslich [2018a], Lemma 2.38, applied to \( M^X \). For \( \ell_0 \uparrow \infty \), the right-hand side in (95) tends to \( \infty \) on the set \( \{\mu((T_{S,\infty},\infty)) > 0\} \) while the left-hand side yields a finite upper bound. Hence, \( \mathbb{P}(\mu((T_{S,\infty},\infty)) > 0) = 0 \), establishing (34).

Now we want to analyse more precisely the set \( \{\mu((T_{S,\infty})) > 0\} \). By repeating the arguments in (95) and using (34) we obtain

\[
\mathbb{E}[M_{X}^s] \geq \mathbb{E}\left[ \int_{T_{S,\infty}} g_{T_{S,\infty}}(\ell_0) \mu((T_{S,\infty})) \right] \left( 1 + \lim\inf_{\mathbb{Q} \ni \ell \uparrow \infty} \mathbb{P}(X_{T_{s},\ell} = Y_{T_{s},\ell}^\ell) \right).
\]
Hence, again by letting $\ell_0$ tend to $\infty$, we obtain (35) if we can show (36). But actually (36) follows immediately by (19) and monotonicity of $\ell \mapsto Y^\ell$.

Let us now show (37). A repetition of the arguments in (95) using conditional expectations rather than unconditional ones gives us $\liminf_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mid \mathcal{F}^A_S \right] \geq 0$ almost surely. On the other hand, it remains to prove $\limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mid \mathcal{F}^A_S \right] \leq 0$, which will immediately follow from

$$\limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c} \mid \mathcal{F}^A_S \right] \leq 0, \quad \limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c} \mid \mathcal{F}^A_S \right] \leq 0. \quad (96)$$

**Proving the first inequality in (96):** First, $(T_{S,\infty})_T$ is a predictable stopping time with an announcing sequence given by $(T_{S,n})_{(T_{S,n} < T_{S,\infty}) \wedge n}$. By (34) and (35), we get $\mu((T_{S,\infty})_T, \infty) = 0$ almost surely. Hence, we have by assumption on $X$ that $X(T_{S,\infty})_T = 0$. Therefore, we obtain by Fatou’s Lemma and then Lemma A.1 (ii) that

$$\limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c} \mid \mathcal{F}^A_S \right] \leq E \left[ * X_{T,S,\ell} \mid \mathcal{F}^A_S \right] \leq E \left[ P X_{T,S,\ell} \mid \mathcal{F}^A_S \right] = 0.$$

Before proving the second inequality in (96) we need as an intermediate result the following claim:

**Claim:** We have

$$\limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c} \mid \mathcal{F}^A_S \right] \leq E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c = Y^\ell_{T,S,\infty} \text{ for all } \ell} \mid \mathcal{F}^A_S \right].$$

**Proof of the claim:** We define for $k \in \mathbb{N}$ and $\ell \in \mathbb{R}$

$$A_0 := \{ T_{S,p} = T_{S,\infty} \text{ for some } p \leq 0 \}, \quad A_k := \{ T_{S,k-1} < T_{S,k} = T_{S,\infty} \}, \quad A^\ell := \bigcup_{0 \leq k \leq \ell} A_k = \{ T_{S,\ell} = T_{S,\infty} \}$$

such that $\Gamma^c$ is the disjoint union of the sets $(A_k)_{k \in \mathbb{N}}$. Now we get by Fatou’s Lemma and $M^X$ of Lemma A.1 that

$$\limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c} \mid \mathcal{F}^A_S \right] \leq \limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{A^\ell} \mid \mathcal{F}^A_S \right] + \limsup_{Q \ni \ell \uparrow \infty} E \left[ M^X_{T,S,\ell} \mathbb{1}_{\Gamma^c \setminus A^\ell} \mid \mathcal{F}^A_S \right]$$

$$\leq \limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{A^\ell} \mid \mathcal{F}^A_S \right].$$

Using (34) we get for any $\epsilon > 0$ that the predictable stopping time $T_{S,\infty} + \epsilon$ satisfies $\mu((T_{S,\infty} + \epsilon, \infty)) = 0$ almost surely. Hence, by assumptions on $X$ this gives us $X_{T_{S,\infty} + \epsilon} = 0$ and, therefore, $X_{T_{S,\infty}} = 0$. As, by Lemma A.1 (iv), $\bar{T}_\ell := (T_{S,\ell}, H_{S,\ell}^-)$ is an $\mathcal{F}$-predictable stopping time, we obtain by Lemma A.1 (ii) that $X_{\bar{T}_\ell} \leq P X_{\bar{T}_\ell}$. Combining this inequality with $X_{T_{S,\infty}}^* = 0$ yields that for any $\ell \in \mathbb{Q}$ we have

$$E \left[ X_{T,S,\ell} \mathbb{1}_{A^\ell} \mid \mathcal{F}^A_S \right] \leq E \left[ (P X_{T,S,\ell} \mathbb{1}_{H_{S,\ell}^+} + X_{T,S,\ell} \mathbb{1}_{H_{S,\ell}^-}) \mathbb{1}_{A^\ell} \mid \mathcal{F}^A_S \right]. \quad (97)$$

For $k \in \mathbb{N}$, we get by Dellacherie and Meyer [1978], Theorem 56 (c), p.118, that $A_k \in \mathcal{F}_{T_{S,k-}}^A \subset \mathcal{F}_{\bar{T}_k^-}^A$. Hence we see that for any $\ell \in \mathbb{Q}$ and all $k \leq \ell$

$$E \left[ P X_{\bar{T}_\ell} \mathbb{1}_{A_k} \mid \mathcal{F}^A_S \right] = E \left[ X_{\bar{T}_\ell} \mid \mathcal{F}^A_{\bar{T}_\ell^-} \right] \mathbb{1}_{A_k} \mid \mathcal{F}^A_S \right] = E \left[ X_{\bar{T}_\ell} \mathbb{1}_{A_k} \mid \mathcal{F}^A_{\bar{T}_\ell^-} \right] \mid \mathcal{F}^A_S \right] = E \left[ X_{\bar{T}_\ell} \mathbb{1}_{A_k} \mid \mathcal{F}^A_S \right].$$

Plugging this into (97) gives us

$$\limsup_{Q \ni \ell \uparrow \infty} E \left[ X_{T,S,\ell} \mathbb{1}_{\Gamma^c} \mid \mathcal{F}^A_S \right] \leq \limsup_{Q \ni \ell \uparrow \infty} E \left[ \mathbb{1}_{A^\ell} X_{T,S,\ell} \mathbb{1}_{H_{S,\ell}^\pm \cup H_{S,\ell}^-} \mid \mathcal{F}^A_S \right].$$

By Fatou’s Lemma and (36) this finally proves our claim.
Proving the second inequality in (96): The previous claim leads to the second inequality in (96) if we can show
\[ X_{T,S,\infty} 1_{\{X_{T,S,\infty} = Y_{T,S,\infty}\}} = 0. \tag{98} \]
For that we will show that \( \tilde{T} := (T_{S,\infty})_\ell = Y_{T,S,\infty} \) for all \( \ell \in \Gamma_c \) is a \( \Lambda \)-stopping time. If this is true we obtain by (34) and (35) that \( \mu((\tilde{T}, \infty)) = 0 \) almost surely and by assumptions on \( X \) that \( X_{\tilde{T}} = 0 \) establishing (98).

Showing that \( \tilde{T} \) is a \( \Lambda \)-stopping time: First we have
\[ \{X_{T,S,\infty} = Y_{T,S,\infty}^\ell \} \cap \Gamma = \bigcup_{k=0}^\infty B_k \]
with disjoint sets \((B_k)_{k \in \mathbb{N}}\) defined by \( B_0 := \{X_{T,S,0} = Y_{T,S,0}^\ell \} \cap \{T_{S,0} = T_{S,\infty}\} \) and
\[ B_k := \{X_{T,S,k} = Y_{T,S,k}^\ell \} \cap \{T_{S,k-1} < T_{S,k} = T_{S,\infty}\}, \quad k \in \mathbb{N}. \]
Next, we have by Lemma 4.1 (iv), that for every \( k \in \mathbb{N} \), \((T_{S,k})_{H_{T,S,k}}\) is an \( F \)-predictable stopping time and \((T_{S,k})_{H_{T,S,k}}\) is a \( \Lambda \)-stopping time. Hence,
\[ (T_{S,k})_{H_{T,S,k} \cup H_{S,k}} = (T_{S,k})_{H_{T,S,k}} \cap (T_{S,k})_{H_{S,k}} \]
is a \( \Lambda \)-stopping time. Now we define \( T_k := (T_{S,k})_{B_k} \), \( k \in \mathbb{N} \), which is again a \( \Lambda \)-stopping time. Indeed, one can see by (19) that \( B_k \subset \{X_{T,S,k} = Y_{T,S,k}^k\} = H_{T,S,k} \cup H_{S,k} \) up to a \( P \)-null set and we assume without loss of generality that this actually holds true for all \( \omega \in \Omega \). Then we can rewrite \( B_k \) as
\[ B_k = \left\{ X_{(T_{S,k})_{H_{T,S,k} \cup H_{S,k}}} = Y_{(T_{S,k})_{H_{T,S,k} \cup H_{S,k}}}^\ell \ (\text{for all } \ell \in \mathbb{Q}) \right\} \cap (H_{T,S,k} \cup H_{S,k}) \cap \{T_{S,k-1} < T_{S,k} = T_{S,\infty}\}, \]
where we have used that \( \ell \mapsto Y_{\ell} \) is non-decreasing which is why we can restrict to \( \ell \in \mathbb{Q} \). This shows then by the \( \Lambda \)-measurability of \( X \) and \( Y^k \), and by Lenglart [1980], Corollary 1), p.505, that \( B_k \in \mathcal{F}_{(T_{S,k})_{H_{T,S,k} \cup H_{S,k}}} \)
and, hence, that \( T_k \in S^\Lambda \). As for every \( \omega \in \Omega \) there exists at most one \( k \in \mathbb{N} \) with \( T_k(\omega) < \infty \), we have \( \tilde{T} = \bigwedge_{k=1}^\infty T_k \), \( [\tilde{T}, \infty) = \bigcup_{k=1}^\infty [T_k, \infty) \in \Lambda \). This implies that \( \tilde{T} \) is a \( \Lambda \)-stopping time and finishes Step 2.

References

Peter Bank and David Besslich. On Lenglart’s Theory of Meyer-sigma-fields and El Karoui’s Theory of Optimal Stopping. arXiv e-prints, art. arXiv:1810.08485, Oct 2018a.

Peter Bank and David Besslich. Modelling information flows by Meyer-\( \sigma \)-fields in the singular stochastic control problem of irreversible investment. arXiv e-prints, to appear in The Annals of Applied Probability, art. arXiv:1810.08495, Oct 2018b.

Peter Bank and Nicole El Karoui. A stochastic representation theorem with applications to optimization and obstacle problems. Ann. Probab., 32(1B):1030–1067, 2004. ISSN 0091-1798. . URL https://doi.org/10.1214/aop/1079021471.

Peter Bank and Hans Föllmer. American options, multi-armed bandits, and optimal consumption plans: a unifying view. In Paris-Princeton Lectures on Mathematical Finance, 2002, volume 1814 of Lecture Notes in Math., pages 1–42. Springer, Berlin, 2003. . URL https://doi.org/10.1007/978-3-540-44859-4_1.

Peter Bank and Christian Küchler. On Gittins’ index theorem in continuous time. Stochastic Process. Appl., 117(9):1357–1371, 2007. ISSN 0304-4149 . URL https://doi.org/10.1016/j.spa.2007.01.006.

David Besslich. Information flow in stochastic optimal control and a stochastic representation theorem for Meyer-measurable processes. PhD thesis, Technische Universität Berlin, 2019.

Jean-Michel Bismut and Bernard Skalli. Temps d’arrêt optimal, théorie générale des processus et processus de Markov. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 39(4):301–313, 1977. . URL https://doi.org/10.1007/BF01877497.

Maria B. Chiarelli and Giorgio Ferrari. Identifying the free boundary of a stochastic, irreversible investment problem via the Bank–El Karoui representation theorem. SIAM J. Control Optim., 52(2):1048–1070, 2014. ISSN 0363-0129 . URL http://dx.doi.org/10.1137/11085195X.
