Fuel Minimisation for a Vehicle Equipped with a Flywheel and Battery on a Three-Dimensional Track

Mehdi Imani Masouleh and David J. N. Limebeer

Abstract—An optimal control based methodology is proposed for minimising the combustible fuel consumption of a hybrid vehicle equipped with an internal combustion engine, a high-speed flywheel and a battery. The three-dimensionality of the road is recognised by the optimal control calculations. Fuel efficiency is achieved by optimally exploiting the primary and secondary energy sources and controlling the vehicle so that the fuel consumption is minimised for a given, but arbitrary three-dimensional route. A time-of-arrival constraint rather than a driving cycle is used. The benefits of using multiple auxiliary storage systems are demonstrated and a lower-bound estimate of the fuel consumption is presented.

I. INTRODUCTION

The low efficiency and the air-pollution side effects associated with internal combustion engine (ICE) usage are well known. As is now widely appreciated, several of the problems endemic to ICES can be mitigated using a variety of secondary energy storage technologies such as fuel cells, lithium-based battery systems, high-speed flywheels and supercapacitors. In broad terms, fuel cells and lithium-based battery systems have good energy storage properties, while high-speed flywheels and supercapacitors can be utilised for their high power density characteristics [1], [2]. As a result of their poor power density properties, battery packs in commercial electric vehicles tend to be oversized. Other battery-related issues include long charging times and shortened life expectancy, especially when they are cycled at high charge and discharge rates. Supercapacitors and flywheel-based systems are an ideal complement to batteries, because of their high power density characteristics under both charging and discharging. The fundamentals and applications of fuel cells, including the main reactions, are reviewed in [3]. A comprehensive review of lithium-ion battery technologies is given in [4]. The physical structure of some flywheel-based systems is reviewed in a vehicle context in [5]. Hybrid energy storage systems (HESS) based on batteries and supercapacitors are reviewed in [6] and the references therein.

Aside from the technologies themselves, energy storage modelling as well as their optimal deployment are important issues. The modelling and control of hybrid electric vehicles (HEVs) are reviewed in [7], where various powertrain topologies and control strategies are discussed. It is pointed out that global optimization can be used as a ‘what’s possible’ benchmark for evaluating energy management strategies. Another good survey paper reviewing 180 papers on the optimal energy management of HEVs and plug-in HEVs can be found in [8]. Power management control strategies can be divided into offline and online methodologies. Online management strategies include methods such as look-up tables, state machines, thermostat control, Equivalent Consumption Minimisation Strategies (ECMS) [9]–[11], Neural networks [12], particle swarm optimisation [13], model predictive control (MPC) [14] and fuzzy control [15]. Offline strategies that often focus on global optimisation include dynamic programming (DP) [10], [16], linear programming [17], nonlinear programming [18], Stochastic DP [19]–[21] and genetic algorithms [22]. An evolutionary algorithm was applied in [23] over a sliding window to allow real-time power management and a data-driven reinforcement learning algorithm was proposed in [24]. ECMS does not guarantee charge sustainability and hence an Adaptive ECMS (A-ECMS) algorithm was introduced in [25] to update the equivalence factor ‘on-the-fly’ on the basis of past and predicted driving conditions.

The majority of the work in the literature uses driving cycles as benchmarks for performance evaluation. As was recognised in [26], optimising vehicle parameters over one drive cycle does not necessarily mean that the vehicle will perform well on other drive cycles. In [26], a method was proposed for optimising an HEV over a range of drive cycles with different levels of driving aggressiveness and traffic conditions, in order to reduce the fuel economy variability with respect to drive cycle changes. While seeking to address the limitations associated with single driving cycle usage, this method is still restricted by the driving cycle combination used and there were cases where a single driving cycle resulted in a lower fuel variability compared with the proposed multi-cycle method.

The (combustible) fuel used by any vehicle will depend on the vehicle’s speed, with higher average speeds typically resulting in a higher fuel usage. A contribution of this work is proposing a method of minimising fuel consumption over different driving conditions without using drive cycles. The idea is to simultaneously optimise the powertrains energy deployment and driving strategy over a given (but arbitrary) route. Key in this procedure is the selection of a time-of-arrival constraint, which acts as an ‘aggressiveness’ surrogate. A short travel time corresponds to aggressive driving and a generally higher fuel consumption. An optimal control algorithm then seeks to minimise the fuel consumption, while ensuring a ‘just-in-time’ arrival. The optimal control calculation makes use of a realistic vehicle model, with three degrees of freedom, and a non-linear tire model. It is shown that flywheels are an excellent means of reducing fuel consumption in manoeuvres where high levels of braking are involved, such as extra-urban driving on fast roads.

Another distinguishing feature of this work over the major-
ity of the existing literature is that unlike rule-based methods, system dynamics and non-linear constraints can be included explicitly as part of the optimal control problem and optimal rather than near optimal results are obtained. A pseudo-spectral method will be used to solve the optimal control problem, which makes use of first- and second-order gradient information for fast convergence. Compared to DP/SDP, models of much higher complexity can be solved using this approach.

In this paper, we will take the roads three-dimensional geometric profile into consideration. In mechanical systems with comparable potential and kinetic energies, the optimal control strategy is strongly dependent on trade-offs between the two energy sources, as was demonstrated by the famous ‘Minimum Time to Climb’ problem associated with jet-powered aircraft [27]. In Section II a simple motivating example is given that highlights the importance of changes in the road gradient. In Section III the vehicle and track models employed in this study are described. In Section V the optimal control problem is cast in a standard form and the numerical method used to solve it is described. Numerical results are presented and discussed in Section VI. The conclusions are given in Section VII and the vehicle parameters used in the study are provided in the Appendix.

II. AN ILLUSTRATIVE EXAMPLE

Imagine a path-constrained point mass (bead) that is required to reach a given destination within a pre-specified time. Suppose the trajectory’s starting point is (0,0) in absolute Cartesian coordinates and that the path is parabolic and defined by $y = ax^2 + bx$. If the destination point $(x_f, y_f)$ is given, there holds $y_f = ax_f^2 + bx_f$ and so the path is specified by a single free parameter. We would like to minimise the energy supplied to the bead in order that it arrives at the destination ‘just in time’. If the effects of air resistance and friction forces are neglected, the bead’s height is indicative of its speed. If the bead has unity mass, conservation of energy dictates

$$\frac{1}{2}v^2 = E_{fuel} - gy. \quad (1)$$

The bead’s speed is $v$, its instantaneous height above the origin is $y$ and $E_{fuel}$ is the amount of external energy supplied. The speed of the bead is thus

$$v = \sqrt{2E_{fuel} - 2gy}. \quad (2)$$

An infinitesimal path segment can be described as

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + (2ax + b)^2}. \quad (3)$$

Using (2) and (3) the manoeuvre time is given by

$$T = \int_0^{x_f} \frac{1}{v} ds = \int_0^{x_f} \frac{dx}{\sqrt{2E_{fuel} - 2gy(ax^2 + bx)}}. \quad (4)$$

The thrust programme that minimises the total external energy supplied is the optimal control problem of minimising

$$E_{fuel} = \int_{s_0}^{s_f} F(s) ds = \int_{0}^{t_f} F(t)v(t) dt \quad (6)$$

subject to state dynamics

$$\begin{align*}
\dot{x} &= \frac{v}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \\
\dot{v} &= F - \frac{g\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}
\end{align*} \quad (7)$$

where $F$ is the externally applied force. The boundary conditions are

$$\begin{align*}
x(t_0) &= 0 & x(t_f) &= 5 \\
v(t_0) &= 0 & v(t_f) &= \text{free}
\end{align*} \quad (8)$$

The control Hamiltonian of this system is

$$\mathcal{H} = \lambda_1 \dot{x} + \lambda_2 \dot{v} + Fv. \quad (9)$$

Since $v(t_f)$ is free, $\lambda_2(t_f) = 0$. For the quadratic path the control Hamiltonian becomes

$$\mathcal{H} = F(\lambda_2 + v) + \frac{\lambda_1 v}{\sqrt{1 + (2ax + b)^2}} + \frac{\lambda_2 g(2ax + b)}{\sqrt{1 + (2ax + b)^2}} \quad (10)$$

with co-state equations

$$\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial x} = \left(-F - \frac{\partial \mathcal{H}}{\partial v}\right) \quad (11)$$

where

$$\frac{\partial \mathcal{H}}{\partial v} = \frac{\lambda_2 g (2ax + b)}{(1 + (2ax + b)^2)^{3/2}} \quad (12)$$

Since the drive force is bounded, Pontryagin’s minimum principle determines that

$$\begin{align*}
F &= F_{max} & \text{if } \lambda_2 < -v \\
F &= 0 & \text{if } \lambda_2 > -v \\
F &= \text{singular} & \text{if } \lambda_2 = -v
\end{align*} \quad (13)$$

In a manner reminiscent of simple variants of the Goddard rocket problem, the optimal thrust programme turns out to be impulsive [28], with all the external energy injected into the system at the beginning of the manoeuvre. To illustrate this the described optimal control problem was solved with GPOPS in the case that $y_f = -1$ and $a = 0.1$; the states, control and co-states are shown in Fig. I.

Nine cases are considered that highlight the significance of interactions between the kinetic and potential energy of the bead. These paths correspond to three values for the $a$ parameter and three values for the terminal point; the resulting paths are shown in Fig. 2.
Fig. 1. Solution of the optimal control problem for \( a = 0.1 \) and \( y_f = -1 \).

Fig. 2. Quadratic paths taken by the bead from the starting point \((0, 0)\), to the terminal points \((5, -1)\), \((5, 0)\) and \((5, 1)\).

For each of the nine cases, Table I shows the path lengths \( L \), the zero-fuel times \( T_0 \) (in the cases that a zero-fuel journey is possible), and the external energy \( E_{\text{fuel}} \) required to complete the journey under a 1 s time-of-arrival constraint.

| Case | \( y_f \) | \( a \) | \( b \) | \( L(m) \) | \( T_0(s) \) | \( E_{\text{fuel}}(J) \) |
|------|----------|------|------|------|------|------|
| 1    | -1       | 0.1  | -0.7 | 5.29 | 1.57 | 6.02 |
| 2    | -1       | 0    | -0.2 | 5.10 | 2.30 | 8.56 |
| 3    | -1       | -0.1 | 0.3  | 5.29 | -   | 13.51|
| 4    | -1       | 0.1  | -0.5 | 5.20 | 2.38 | 9.70 |
| 5    | 0        | 0    | 0    | 5.00 | -   | 12.50|
| 6    | 0        | -0.1 | 0.5  | 5.20 | -   | 17.73|
| 7    | +1       | 0.1  | -0.3 | 5.29 | -   | 15.83|
| 8    | +1       | 0    | -0.2 | 5.10 | -   | 18.37|
| 9    | +1       | -0.1 | 0.7  | 5.29 | -   | 23.31|

The second column in Table I shows the elevation of the terminal point. To calculate the arrival time, (5) is solved numerically by running a bisection on \( E_{\text{fuel}} \) until the journey time constraint is met.

Superior fuel consumption performance is achieved when the initial part of the journey is downhill, because the early conversion of potential energy into kinetic energy results in higher speeds throughout the journey. Shorter journey paths do not necessarily result in lower fuel consumption (e.g. compare Case 1 and 2). As one would expect, the journeys in Cases 7 to 9 are the most arduous in terms of fuel consumption, due to elevated terminal points. This example demonstrates that the path elevation curvature can have a significant influence on fuel usage and sometimes in a counter-intuitive manner.

III. THE MATHEMATICAL MODEL

A bicycle model of the car is used that has yaw, lateral and longitudinal freedoms, and nonlinear tires. The road is assumed three-dimensional and is represented by a geometric construct called a ‘ribbon’, which is describable in terms of three curvature variables [29].

A. Track Model

A moving coordinate system (called a Darboux frame) is used to describe the track. As shown in Fig. 3 the origin of this moving system is ‘dragged along’ by the car. The independent variable in the track description is \( s \), which is the distance travelled by the car from some starting point, projected onto the track spine. The spine could be, but need not be, the track centre line.

The Darboux frame is described by the orthogonal moving triad \([\hat{t} \hat{n} \hat{m}]\), with \( \hat{m} \) normal to the road surface. The road is represented locally by the \( \hat{t}-\hat{n} \) plane, which ‘travels’ with the car down the road.

The orientation of this local ‘patch’ is described in terms of three Euler angles that are all functions of \( s \) [29]. If the...
track orientation is described by the roll, yaw and pitch angles, respectively \( \phi, \mu \) and \( \theta \), then the track curvatures are given by

\[
\Omega = \begin{bmatrix}
    \Omega_x \\
    \Omega_y \\
    \Omega_z
\end{bmatrix} = \frac{1}{\ell} \begin{bmatrix}
    \dot{\phi} - \sin(\mu)\dot{\theta} \\
    \cos(\phi)\mu + \cos(\mu)\sin(\phi)\dot{\theta} \\
    -\sin(\phi)\mu + \cos(\mu)\cos(\phi)\dot{\theta}
\end{bmatrix}. \tag{14}
\]

The angular velocity of the Darboux frame is given by

\[
\omega = \begin{bmatrix}
    \omega_x \\
    \omega_y \\
    \omega_z
\end{bmatrix}^T = \ell \Omega. \tag{15}
\]

The next kinematic relationships we will require relates to the way in which the car progresses down the road. Suppose that the absolute velocity of the car in its body-fixed coordinate system is \([u \ v \ w]^T\); the longitudinal velocity component \( u \) is determined by the throttle/brakes, while the lateral component \( v \) is determined by the steering, and the vertical component \( w \) is determined by the track characteristics. The car’s geometric centre (the car’s mass centre projected down to the road) is given by \( n = [0 \ 0 \ 0]^T \) in the Darboux frame. The absolute velocity of the car in the Darboux frame is given by

\[
\begin{bmatrix}
    \dot{s} \\
    \dot{n} \\
    0
\end{bmatrix} = n \times \omega + R_z(\xi) v = \begin{bmatrix}
    n_\omega + u \cos \xi - v \sin \xi \\
    v \cos \xi + u \cos \xi \\
    w - n_\omega
\end{bmatrix}, \tag{16}
\]

where

\[
R_z(\xi) = R(e_z, \xi) = \begin{bmatrix}
    \cos \xi & -\sin \xi & 0 \\
    \sin \xi & \cos \xi & 0 \\
    0 & 0 & 1
\end{bmatrix}. \tag{17}
\]

This will be used in the next section to derive the vehicle’s equations of motion. The car’s yaw angle in Darboux frame \( \xi \) expressed in distance domain is deduced from the third row of \( \text{(22)} \) by integrating

\[
\xi' = S_f(s) \bar{\omega}_z - \Omega_z. \tag{23}
\]

### B. Dynamics

The equations describing the dynamics of the car are derived using standard vectorial methods. The absolute velocity of the car’s mass centre (expressed on the vehicle’s coordinate system) can be written as

\[
v_B = v + \bar{\omega} \times h
\]

where \( \times \) denotes the cross product. The Newton-Euler equations for this system are given by

\[
M(\dot{v}_B + \bar{\omega} \times v_B) = F_B + MgR^T e_z \tag{25}
\]

\[
I_B \dot{\bar{\omega}} + \bar{\omega} \times (I_B \bar{\omega}) = M_B, \tag{26}
\]

where the car’s inertia matrix is assumed to be diagonal and is given by \( I_B = \text{diag}(I_x, I_y, I_z) \), with \( F_B = [F_x \ F_y \ F_z]^T \) and \( M_B = [M_x \ M_y \ M_z]^T \) being the external force and moment. The last term in \( \text{(25)} \) is due to the gravitational acceleration of the car’s mass centre and can be expressed as

\[
MgR^T e_z = R_z(\xi) R_y(\phi) R_x(\mu) \begin{bmatrix}
    0 & 0 & Mg
\end{bmatrix}^T
\]

\[
= Mg \begin{bmatrix}
    \sin \xi \sin \phi \cos \mu - \cos \xi \sin \mu \\
    \sin \xi \sin \mu + \cos \xi \sin \phi \cos \mu \\
    \cos \phi \cos \mu
\end{bmatrix}. \tag{27}
\]

The car’s equations of motion can now be assembled from \( \text{(25), (26) and (27)} \) as follows:

\[
\dot{v} = (v + h \bar{\omega}) \bar{\omega} - n_\omega \bar{\omega}_y + h \bar{\omega}_y + g(\sin \xi \sin \phi \cos \mu - \cos \xi \sin \mu) + F_x / M \tag{28}
\]

\[
\dot{\bar{\omega}} = n_\omega \bar{\omega}_x - (u - h \bar{\omega}_y) \bar{\omega}_z - h \bar{\omega}_z + g(\sin \xi \sin \mu + \cos \xi \sin \phi \cos \mu) + F_y / M \tag{29}
\]

\[
\dot{\bar{\omega}}_z = ((I_x - I_y)\bar{\omega}_x \bar{\omega}_y + M_z) / I_z, \tag{30}
\]

in which \( F_x, F_y \) are the resultant longitudinal and lateral forces, and \( M_z \) is the z-axis tire moment acting on the car. These quantities are given by

\[
F_x = F_{fz} \cos \delta - F_{fz} \sin \delta + F_{rx} + F_{ax}
+ (F_{fz} \cos \delta + F_{fz} \cos \mu) C_r \tag{31}
\]

\[
F_y = F_{fy} \cos \delta + F_{fy} \sin \theta F_{fy} \sin \delta C_r \tag{32}
\]

\[
M_z = a(F_{fy} \cos \delta + F_{fy} \sin \delta) - bF_{tr} \tag{33}
\]

The tire force system is illustrated in Fig 4, and is discussed in Section III-D. The coefficient \( C_r \) is the tire rolling resistance coefficient and the last two terms in \( \text{(31) and (32)} \) represent the rolling resistance forces. The aerodynamic drag force \( F_{ax} \) acts in negative x-axis direction and is given by

\[
F_{ax} = -\frac{1}{2} C_D \rho A u^2 \tag{34}
\]
The tire forces have normal, longitudinal and lateral components that act on the vehicle chassis at the tire ground contact points and react on the inertial frame. The rear-wheel tire force is expressed in the vehicle’s body-fixed reference frame, while the front tire force is expressed in a steered reference frame; refer again to Fig. 4. We make use of the well-known Magic Formula tire model [30], where these forces are a function of the normal load and the tire’s longitudinal slip coefficient \( \kappa \) and a lateral slip angle \( \alpha \). The tire equations were also described in details in the Appendix of [31]. The same tire parameters are used in this work except that the peak longitudinal and lateral friction coefficients have been scaled down by 30\%. Following standard conventions we use

\[
\kappa = -\left(1 + \frac{R\omega}{u_w}\right)
\]

and

\[
\tan \alpha = -\frac{v_w}{u_w},
\]

where \( R \) is the wheel radius and \( \omega \) the wheel’s spin velocity. The quantities \( u_w \) and \( v_w \) are the absolute velocity components of the wheel centre in a wheel-fixed coordinate system. The front and rear tire lateral slip angles are given by

\[
\alpha_f = \arctan \left( \frac{v - \psi b}{u} \right),
\]

and

\[
\alpha_r = \arctan \left( \frac{\cos \delta (\psi a + v) - \sin \delta u}{\cos \delta u + \sin \delta (\psi a + v)} \right).
\]

### E. Battery Model

We will use a simple ‘voltage behind output resistance’ model for the (lithium-ion) battery, as shown in Fig. 5. The terminal voltage is

\[
V_b = V_{OC} - I_b R_b,
\]

where \( V_{OC} \) is the battery open-circuit voltage, \( I_b \) is the battery current and \( R_b \) is the internal resistor. In this convention positive \( I_b \) corresponds to discharging the battery, while negative \( I_b \) corresponds to charging. The unloaded battery voltage \( V_{OC} \) has a dependency on the state of the charge \( \text{SoC} \) as follows

\[
V_{OC} = V_{OC}^{\text{min}} + (V_{OC}^{\text{max}} - V_{OC}^{\text{min}}) \text{SoC},
\]

where the \( \text{SoC} \) is defined as

\[
\text{SoC} = \frac{Q_b}{Q_b^{\text{max}}},
\]
The maximum battery charge $Q_{b}^{\text{max}}$ is given by
\begin{equation}
Q_{b}^{\text{max}} = \frac{2E_{b}^{\text{max}}}{V_{OC}^{\text{max}} + V_{OC}^{\text{min}}} \tag{47}
\end{equation}
where $E_{b}^{\text{max}}$ represents the maximum energy storage capacity of the battery. Using (44) and (45) the power delivered, or drawn from the battery is
\begin{equation}
P_b = (V_{OC}^{\text{min}} + (V_{OC}^{\text{max}} - V_{OC}^{\text{min}})SoC - I_b R_b)J_b. \tag{48}
\end{equation}
Again negative $P_b$ implies charging and positive $P_b$ implies discharging.

The battery charge can be modelled by the dynamic equation
\begin{equation}
\dot{Q}_b = -I_b. \tag{49}
\end{equation}

The power transmission between the battery and the rear wheel requires an electric motor, which is assumed to have an efficiency factor $\mu_{em}$. The electric motor output power is thus
\begin{equation}
P_{em} = \mu_{em}^{\text{sign}(P_f)} P_f. \tag{50}
\end{equation}

F. Engine Map

The engine used in the work presented here is the 1.5L Prius engine with maximum power output of 43 kW. The fuel consumption map (see Fig.[9]) for this engine was obtained from the ADVISOR software [32]. One measure of fuel efficiency is brake specific fuel consumption (BSFC), which represents the fuel mass needed to release one unit of energy. A BSFC map can be calculated from the fuel consumption map using
\begin{equation}
BSFC = \frac{\bar{\eta}_f(\omega_e, T_e)}{\omega_e T_e}, \tag{51}
\end{equation}
where $\bar{\eta}_f$ is the fuel-mass consumption rate and $T_e$ and $\omega_e$ are the engine torque and rotational speed respectively. We will use the BSFC to evaluate the efficiency performance of the engine. The internal combustion engine power is represented by $P_{ICE}$ and is given by
\begin{equation}
P_{ICE} = T_e \omega_e. \tag{52}
\end{equation}

In the optimal control calculations a quadratic multivariate polynomial was fitted to the engine map using a Linear Least Squares algorithm to speed up the fuel consumption calculation. The polynomial captures the shape of the map quite well and has an average absolute error of 2.6% over the entire engine operating range.

G. Flywheel

A high-speed flywheel is included in the drivetrain to provide a high-power energy re-deployment capability, which complements the low-power high-energy storage capability of the battery. While batteries can store energy for a relatively long time due to their low inherent losses, flywheel storage systems suffer from high losses especially when running at high speeds. A basic flywheel storage system can be represented by a spinning inertia with kinetic energy
\begin{equation}
E_{fly} = \frac{1}{2} J_f \omega_f^2, \tag{53}
\end{equation}

where $J_f$ is the moment of inertia and $\omega_f$ is the flywheel’s angular velocity. The dominant losses in the flywheel come from the friction in the bearings. These losses are modelled using the empirical relationship given on pg. 147 of [33]
\begin{equation}
P_{\text{loss}} = 2 \times 10^{-7} \omega_f^2 + 0.0151 \omega_f + 4.0577, \tag{54}
\end{equation}
where $\omega_f$ is given in rpm. The flywheel dynamics are described by
\begin{equation}
\dot{E}_{fly} = -P_{fly} - P_{\text{loss}}. \tag{55}
\end{equation}

There are also losses in the continuous variable transmission (CVT) that can be lumped into an efficiency factor $\mu_{CVT}$.

H. Power Transmission

The power transmitted to the rear wheel can be modelled by the constraint
\begin{equation}
P_{ICE} + P_{em} + \mu_{CVT}^{\text{sign}(P_{fly})} P_{fly} - (F_{rx} + F_{rlx})u \geq 0, \tag{56}
\end{equation}
which ensures that the power delivered to the back wheels never exceeds the combined power delivery capability of the internal combustion engine, the battery and the flywheel. If the rear-wheel tire force is positive, the vehicle is being driven, and the sum of powers delivered by the engine, the flywheel CVT and electric motor will match the mechanical power delivered to the rear wheel. Under braking, rear-wheel tire force is negative, and the mechanical power at the back wheels is used to charge the battery and/or the flywheel, or else it is dissipated as heat.

At $P = 0$, $\mu_{CVT}^{\text{sign}(P_{fly})}$ is discontinuous and hence must be approximated using a smooth function. The approximation used in this study is
\begin{equation}
\mu^{\text{sign}(P)} \approx 0.5 \mu (1 + \tanh(\varphi P)) + \frac{0.5}{\mu} (1 + \tanh(-\varphi P)), \tag{57}
\end{equation}
in which $\varphi$ is a constant. As $\varphi$ is increased, the approximation [57] approaches $\mu^{\text{sign}(P)}$. 

![Fig. 6. Engine fuel consumption map. The consumption rate in g/s is shown on the contours/level sets. The black dashed line is the maximum engine torque available as a function of speed.](image-url)
IV. OPTIMAL CONTROL

The minimum fuel problem can be formulated as an optimal control problem that is now described.

The system is described by a set of equations of the form

\[ x'(s) = f(x(s), u(s)), \]

in which the state-vector is given by

\[ x = [n \ \xi \ \nu \ \omega \ E_{fly} \ Q_y]^T. \]

The associated differential equations are given by (21), (23), (28), (29), (30), (39) and (55), respectively. The control vector is given by

\[ u = [\delta \ k_f \ k_r \ F_{fz} \ F_{rz} \ \omega_e \ T_e \ P_{fly} \ I_b]^T. \]

The problem is also subject to the path constraints (38), (39), (56) and bounds on the states and controls. There is also a constraint on maximum engine torque available as shown in (56).

The arrival time constraint can be written as an integral constraint as below

\[ \int_{s_0}^{s_f} S_f(s) ds \leq T, \]

where \( T \) is the arrival time.

The minimum-fuel performance index to be minimised is given by

\[ J = \int_{s_0}^{s_f} S_f(s) \dot{m}_f ds. \]

In practice, however, to avoid jerky controls and singular arcs, we actually control \( \dot{u} \) and minimise

\[ J_{mod} = \int_{s_0}^{s_f} S_f(s) (\dot{m}_f + \dot{u}^T R \dot{u}) ds \]

in which \( R \) is an appropriate weighting matrix. We also impose slew-rate limits on controls by placing hard constraints on \( |\dot{u}| \).

A. NUMERICAL OPTIMAL CONTROL

An optimal control problem formulation general enough for our purposes is of Lagrange form. The aim is to determine states \( x(\tau) \in \mathbb{R}^n \), controls \( u(\tau) \in \mathbb{R}^m \) and static parameters \( p(\tau) \in \mathbb{R}^q \) which minimise a cost functional

\[ J = \frac{t_f - t_0}{2} \int_{-1}^{+1} g[x(\tau), u(\tau), \tau, t_0, t_f, p] d\tau \]

subject to the state dynamics,

\[ \frac{dx}{d\tau} = \frac{t_f - t_0}{2} f[x(\tau), u(\tau), \tau, t_0, t_f, p], \]

path constraints

\[ c_{min} \leq c[x(\tau), u(\tau), \tau, t_0, t_f, p] \leq c_{max} \in \mathbb{R}^r, \]

and boundary conditions

\[ b_{min} \leq b[x(-1), x(+1), t_0, t_f, p] \leq b_{max} \in \mathbb{R}^s. \]

The normalised optimisation interval \( \tau \in [-1, 1] \) can be transformed into the general interval \( t \in [t_0, t_f] \) using the affine transformation \( t = (t_f - t_0)\tau/2 + (t_f + t_0)/2 \).

The pseudo-spectral numerical optimal control solver GPOPS-II [34], which is based on the Legendre-Gauss-Radau (LGR) collocation scheme, was used to solve the minimum lap time problem in this paper. In this scheme the state is approximated using a Lagrange polynomial of order \( N \)

\[ x(\tau) \approx X(\tau) = \sum_{i=1}^{N+1} X_i L_i(\tau) \]

where

\[ L_i(\tau) = \prod_{j=1}^{N+1} \frac{\tau - \tau_i}{\tau_j - \tau_i}, i = 1, \ldots, N+1. \]

The state-derivative approximation is thus given by

\[ \dot{x}(\tau) \approx \dot{X}(\tau) = \sum_{i=1}^{N+1} X_i \dot{L}_i(\tau). \]

Collocating the state dynamics at \( N \) LGR points gives

\[ \dot{X}(\tau_j) = \sum_{i=1}^{N+1} X_i \dot{L}_i(\tau_j) = \sum_{i=1}^{N+1} X_i D_{ji}, j = 1, \ldots, N, \]

where \( D_{ji} = \dot{L}_i(\tau_j) \) are the \( N \times (N+1) \) elements of the LGR differentiation matrix. Note that \( \tau_{N+1} \) is a non-collocated point. Using (71) we can discretise (65) and essentially transform the state dynamics given by ordinary differential equations into algebraic constraints.

The optimal control problem can then be approximated by an NLP problem. The cost function of this NLP is obtained by approximating the cost functional (64) using LGR quadrature. The NLP problem can be described by the task of finding \( X_i \)'s, \( U_i \)'s and \( p \) which minimise

\[ J \approx \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i g[X_i, U_i, \tau_i, t_0, t_f, p] d\tau \]

in which the \( w_i \)'s are the quadrature weights [35], subject to the following constraints

\[ \sum_{i=1}^{N+1} X_i D_{ji} = \frac{t_f - t_0}{2} f[X_i, U_i, \tau_i, t_0, t_f, p], j = 1, \ldots, N; \]

\[ c_{min} \leq c[X_i, U_i, \tau_i, t_0, t_f, p] \leq c_{max}, i = 1, \ldots, N; \]

\[ b_{min} \leq b[X_i, X_N, t_0, t_f, p] \leq b_{max}. \]

For clarity of exposition, the description provided here is for a single-interval pseudo-spectral method (global collocation). GPOPS-II uses a mesh and polynomial degree refinement
scheme [36] so that error reduction can be achieved in the presence of non-smooth problem features. The extension to multiple segments is straightforward, with the only requirement being the need to enforce continuity between each mesh interval.

The transcribed NLP problem is typically large but sparse. The IPOPT [37] software library (based on interior point methods) was used to solve the NLP problem. Automatic Differentiation was used to provide IPOPT with accurate and computationally efficient first and second order derivatives [38].

V. RESULTS

In order to illustrate the concepts described, the motor-racing Circuit de Spa-Francorchamps will be used as an exemplar track. This track is approximately 7 km long with an elevation change of approximately 110 m. The path is restricted to the neighbourhood of the track centre line in order to make comparisons easier; this was achieved by constraining the state $n$, which is the perpendicular distance from the car mass centre to the track centre line, to be ‘small’. The three-dimensional track, as well as a two-dimensional projection on to a ground plane are shown in Fig. 7. The route that the car is constrained to take together with the corner distances are shown in Fig. 8. This figure will be useful in analysing the results to be presented later. In all the simulations described here, the vehicle will start at rest from the start-finish line (SF) (in practice the vehicle will start from a low speed in order to keep its reciprocal well defined), and will complete the circuit such that the combustible fuel usage is minimised. The car will come to a standstill at the end of its journey. The vehicle parameters are summarised in Table III given in the Appendix.

The minimum fuel problem was solved for a journey time of 240 s on the two-dimensional and three-dimensional track descriptions. These comparative calculations are used to quantify the effects of three dimensionality. The optimal speed profile of the full hybrid vehicle for both the two and three dimensional tracks, along with the track elevation changes, are depicted in Fig. 9. One can see that the track is on a slight incline when the car starts its journey. This results in the car’s speed at 200 m from the SF line being slightly higher on the 2D track. After the hairpin bend at 400 m the track falls away and the predicted speed on the 3D track exceeds that of the 2D track model. The 3D track speed advantage is then ‘given back’ as the car enters the uphill section between 1100 m and 2400 m. At the start of the incline at 1100 m, one also sees an increase in the fuel consumption rate as shown in Fig. 10.

The vehicle’s fuel consumption rate is high initially in order for it to accelerate from rest. As the vehicle approaches the hairpin bend at 400 m, the throttle is eased off at approximately 150 m, with the brake applied at approximately 200 m (see also Fig. 11). The full-lap fuel usage for the 2D case was 321.2 g.
The results presented thus far can be expanded to include real-world influences such as speed limits, enforced stops, changes in the road-surface conditions and wind gusting. One approach to the solution of these problems is to set them up as multi-phase optimal control problems \[39\]. In this formalism each phase can contain new models and/or new model parameters, new inputs, and new constraints. New parameters might include down-graded tyre parameters that particularise degraded road surface conditions, new inputs might include wind-related disturbances, and new constraints might include such things as speed limits and traffic controls. Fig. \[14\] demonstrates how the speed and fuel consumption rate vary when a 35 m/s speed-limit is imposed. At an elapsed distance of 1000 m it is evident that the fuel consumption rate reduces (below the unrestricted speed case) on entry to the speed-restricted section of road, and then increases above the unrestricted speed case in order to make up for the time lost. Similar variations in fuel consumption can be observed on the 5300 m to 6200 m road section.

It will be shown that the flywheel can provide significant fuel savings, especially when the journey times are low, and when the vehicle is required to complete the route aggressively. In these cases there will be many braking regions that will re-generate energy back into the flywheel that can then be quickly re-deployed to accelerate the vehicle. For longer journey times the vehicle can complete the circuit at low speed with little or no braking. These ideas are illustrated in Figs. \[15\] and \[16\] which show the power and energy stored in the flywheel, respectively, for a vehicle with engine and flywheel only. For a journey time of 230 s, the flywheel frequently reaches high levels of stored energy. In contrast, when the journey time is increased to 265 s, the braking regions become less frequent, and lighter, resulting in fewer opportunities to scavenge energy to recharge the flywheel. Flywheel self-discharging power losses are evident as negative gradients on the flywheel energy peaks in Fig. \[16\].

In order to analyse the battery management strategy, the vehicle with ICE and battery is considered in Fig. \[17\] for a journey time of 250 s. The battery state of charge at the start and end of the lap is constrained to be 60%. The battery is discharged in the acceleration regions for power and the re-charged in braking phases. It is evident that unlike the flywheel, the discharge does not happen in an ‘on-off’ fashion. This is because the power losses in the battery are proportional to square of the current. This results in the optimal controller spreading the power delivery over longer time periods in order to maintain high efficiency. In the short charging phases all the available braking power is absorbed. In the regeneration phases, the electric motor losses result in less power than the power available at the wheels being delivered to the battery. These efficiency losses are also evident in the power assist mode when part of battery power is lost on its way to the wheels, and explains why the electric motor power is higher in braking and lower in acceleration.

In a final study, the minimum fuel problem was solved for a number of arrival times for four different energy storage combinations; the results are summarised in Fig. \[18\] As one would expect, at lower journey times, when vehicle has to be driven more aggressively, the fuel consumption increases. Furthermore, at higher speeds the aerodynamic drag losses...
increase. The vehicle with both the flywheel and battery offers maximum advantage at low journey times. However, as the journey time increases, the vehicle with engine and flywheel performs the best as it does not have to carry the additional 100 kg of battery load. With increasing journey times the fuel usage drops significantly and the benefit of using an auxiliary storage system also diminishes, as energy regeneration opportunities decreases. The minimum journey times possible for the vehicle with engine only, engine and battery only, engine and flywheel only, and engine, battery and flywheel were 230.09 s, 225.76 s, 219.67 s and 217.35 s respectively. This highlights the power boost capability of the flywheel in aggressive manoeuvrings.

**Some computational details**

In this study the optimal control solver was initialised with 100 mesh segments. The number of collocation points was allowed to vary between 4 to 10 per mesh segment.
The error tolerance across all meshes was $10^{-3}$ and IPOPT tolerance was set to $10^{-7}$. For the vehicle with both battery and flywheel, when arrival time was set to 240 s, the error tolerance was reached after 18 mesh adaptation iterations. The final mesh had a total of 367 mesh segments and 2861 collocation points. The whole problem took under 2 hours to solve on an 8 core 3.5 GHz computer. For the engine-only case the error tolerance was reached with a similar number of collocations and meshes. However, the total solution time was only 20 minutes as only 10 mesh adaptation iterations were required and each iteration was faster to complete. The solution time is sensitive to the problem set-up. For example, increasing the size of $\varrho$ in (57) slows down convergence as the problem becomes ‘less continuous’. Avoiding the use of look up tables in the cost function speeds up the algorithm significantly. In general, even though the direct pseudo-spectral method shows great robustness, careful attention is required in problem set-up when fast solution speeds are desired.

**Global Optimality**

As with any gradient based optimisation method, the optimal solution obtained from the algorithm might be a local rather than the global minimum. It is therefore important to quantify the sensitivity of the optimal solution to the initial guess. In this study, the problem was initialised using a set of sensible constant values across the entire solution space. A series of cases were then considered to examine whether the solution obtained can be trusted to be a global optimum. In all the cases, the vehicle was equipped with a battery and flywheel and the arrival time was set to 240 s.

Firstly, the initial state and control guesses were chosen constant, and given in terms of the bounds by $x_0 = x_{\min} + (x_{\max} - x_{\min})w$ and $u_0 = u_{\min} + (u_{\max} - u_{\min})w$, with $w$ a weighting between zero and one. The optimal fuel usage for these cases are shown in Table II as ‘Linear x%’. For the 10%, 20% and 80% cases, the solution did not converge (marked as DNC). However, for all the other cases the solutions were essentially identical with the differences being less than the mesh tolerance error of $10^{-3}$.

In an alternative test, a random value of between 20% and 80% of the total bound range, for each state and control, was chosen as the initial guess. This experiment was repeated 10 times and the results are shown as ‘Random n’ in Table II. All the runs converged successfully to the same solution. Finally, the problem was initialised with the lowest charging current and SOC, and then with highest discharge current and SOC.
for the battery. The results are shown as ‘Battery min’ and ‘Battery max’. Identical solutions were obtained once more, demonstrating that the optimal control problem has a good ‘radius of convergence’ when solved using a direct pseudo-spectral method. To ensure that the same minimal cost was not obtained from different trajectories the cases with the highest cost difference were selected (Case Random 1 and Random 2). The state and controls for the two cases where then compared. The worst case difference was found to be in the Engine Torque, $T_e$, as plotted in Fig.19 As can be seen, the trajectories are nearly identical, lending confidence to the idea that the solution obtained is indeed globally optimal.

VI. CONCLUSIONS

We have presented a novel method of evaluating the fuel-consumption performance of hybrid vehicles with multiple secondary energy sources. Rather than utilising standardised driving cycles, we use a specific route and then drive the vehicle as so as to we reach the destination within a given journey time, while controlling the vehicle in order to minimise the combustible fuel consumption. In this framework driving aggressiveness can be systematically controlled by changing the arrival time. Another thrust of this work was to quantify the effectiveness of flywheels and batteries in reducing fuel consumption. Finally, the minimum fuel usage strategy for the hybrid vehicle over a three dimensional route was evaluated and the effects of different combinations of auxiliary energy storage systems were studied.

In future work it might be interesting to consider the effect of route optimisation. Adding traffic information to the model will also make the simulations more realistic. Component sizing of the combustion engine and auxiliary storage system can be formulated easily in the optimal control problem framework by including static parameters.

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Vehicle parameter values

| Symbol | Description | Value |
|--------|-------------|-------|
| $M$    | Vehicle total mass | 1400 kg |
| $M_{car}$ | Vehicle only mass | 1280 kg |
| $M_{bat}$ | Battery and motor mass | 100 kg |
| $M_{fly}$ | Flywheel mass | 20 kg |
| $I_x$ | x moment of inertia | 500 kg m$^2$ |
| $I_y$ | y moment of inertia | 1000 kg m$^2$ |
| $I_z$ | z moment of inertia | 1000 kg m$^2$ |
| $a$ | Mass centre from front axle | 1.35 m |
| $b$ | Mass centre from rear axle | 1.35 m |
| $h$ | Centre of mass height | 0.5 m |
| $C_d$ | Drag coefficient | 0.3 |
| $A$ | Vehicle frontal area | 1.8 m$^2$ |
| $\rho$ | Air density | 1.2 kg/m$^3$ |
| $E_{b_{\text{max}}}$ | Maximum battery energy capacity | 5 M.J |
| $P_{b}$ | Maximum battery power | 25 kW |
| $V_{b_{\text{min}}}$ | Min battery voltage | 240 V |
| $V_{b_{\text{max}}}$ | Max battery voltage | 210 V |
| $R_b$ | Battery internal resistance | 0.5 $\Omega$ |
| $SoC_{\text{min}}$ | Min state of charge | 40% V |
| $SoC_{\text{max}}$ | Max state of charge | 80% V |
| $P_{\text{fly}}$ | Max flywheel power | 60 kW |
| $\mu_{\text{em}}$ | Electric motor efficiency | 85% |
| $E_{fly_{\text{max}}}$ | Max flywheel energy | 400 k.J |
| $J_{fly}$ | Flywheel spinning inertia | 0.02 kg m$^2$ |
| $\eta_{\text{CVT}}$ | CVT efficiency | 85% |
Mehdi Imani Masouleh received the M.Eng. degree in electrical and electronic engineering with first-class honours from the Imperial College London, London, U.K., in 2011. He is currently pursuing the D.Phil. degree in engineering science at the University of Oxford. His current research interests include optimal control, vehicle dynamics, nonlinear stability and hybrid vehicles.

David J N Limebeer received a B.Sc.(Eng) degree from the University of the Witwatersrand in 1974, MSc(Eng) and PhD degrees from the University of Natal in 1977 and 1980, respectively, and the DSc (Eng) from the University of London in 1992. He was a post-doc researcher at the University of Cambridge between 1980 and 1984. He then joined the Electrical and Electronic Engineering Department at Imperial College as a lecturer. He was promoted to Reader in 1989, Professor in 1993, Head of the Control Group in 1996, and Head of Department 1999-2009. In 2009 he moved to Oxford as Professor of Control Engineering and Professorial Fellow at New College Oxford. His research interests include applied and theoretical problems in control systems and engineering dynamics. He is a Fellow of the IEEE (1992), a Fellow of the IET (1994), and a Fellow of the Royal Academy of Engineering (1997).