A CRITERION FOR THE EXISTENCE OF LOGARITHMIC
CONNECTIONS ON CURVES OVER A PERFECT FIELD

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Abstract. Let $k$ be a perfect field, and $X$ an irreducible smooth projective curve over $k$. We give a criterion for a vector bundle over $X$ to admit a logarithmic connection singular over a finite subset of $X$ with given residues, where residues are assumed to be rigid.

1. Introduction and statements of the results

Let $X$ be a compact Riemann surface. A famous theorem due to Atiyah [3] and Weil [11], which is known together as the Atiyah-Weil criterion, says that a holomorphic vector bundle over a $X$ admits a holomorphic connection if and only if the degree of each indecomposable component of the holomorphic vector bundle is zero (see [5] for an exposition of the Atiyah-Weil criterion). In [6] and [8], the Atiyah-Weil criterion has been generalised for the smooth projective curve over infinite perfect field, and perfect field, respectively. In [4], a criterion for the existence of a logarithmic connection with prescribed residues has been established, and hence generalising the Atiyah-Weil criterion in logarithmic set up. More precisely, let $S = \{x_1, \ldots, x_m\}$ be a subset of $X$ such that $x_i \neq x_j$ for all $i \neq j$, and let $E$ be a holomorphic vector bundle over $X$. Fix a rigid endomorphism $A(x) \in \text{End}(E(x))$ for every $x \in S$, where $E(x)$ denote the fibre of $E$ over $x \in S$. Then, we have the following.

Theorem 1.1. [4, Theorem 1.3] The vector bundle $E$ admits a logarithmic connection singular over $S$ with residues $A(x)$ at every $x \in S$ if and only if for every direct summand $F \subset E$,

$$\deg F + \sum_{x \in S} \text{tr}(A(x)\vert_{F(x)}) = 0,$$  \hspace{1cm} (1.1)

where $F(x)$ denote the fibre of $F$ over $x \in S$.

The proof in [4, Theorem 1.3] will work for vector bundles on a smooth projective curve defined over an algebraically closed field of characteristic zero.

Motivated by the above discussion, we have problems related to existence of logarithmic connections when the curve is over an algebraically closed field of characteristic $p > 0$, a prime number. Also, what will be the suitable criterion when the base field $k$ fails to be an algebraically closed field.

In case $k$ is an algebraically closed field of characteristic $p > 0$, and $X$ is an irreducible smooth projective curve over $k$, we prove the following (see section [3] Theorem [3.1]).

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Theorem 1.2. Let $E$ be an algebraic vector bundle on $X$ defined over algebraically closed field $k$ of characteristic $p > 0$. Then $E$ admits a logarithmic connection singular over $S$ with residue $A(x)$ for every $x \in S$ if and only if every indecomposable component $F$ of $E$ satisfies the following condition
\[ \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) \equiv 0 \pmod{p}, \] (1.2)
that is, the number \( \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) \in k \) is a multiple of $p$.

By the abuse of notation, we denote the image of \( \deg E \in \mathbb{Z} \) under the morphism
\[ \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \hookrightarrow k \]
by \( \deg E \) itself, and this is used throughout the paper.

We also show the following result (see section 4, Theorem 4.1), and this will generalise [4, Theorem 1.1] in the logarithmic framework.

Theorem 1.3. Let $k$ be a perfect field of characteristic $p$. Let $E$ be a vector bundle on an irreducible smooth projective curve $X$ over $k$. Then, we have

1. Assume that $p > 0$, and suppose that rank of each indecomposable components of $E$ is not divisible by $p$. Then $E$ admits a logarithmic connection singular over $S$ with residue $A(x)$ for every $x \in S$ if and only if for every indecomposable component $F$ of $E$ satisfies
\[ \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) \equiv 0 \pmod{p}, \] (1.3)

2. If for every indecomposable component $F$ of $E$ satisfies
\[ \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) = 0, \] (1.4)
then $E$ admits a logarithmic connection singular over $S$ with residue $A(x)$ for every $x \in S$.

2. LOGARITHMIC CONNECTION AND RESIDUES OVER A FIELD

Let $k$ be a field. Let $X$ be an irreducible smooth projective curve over $k$. Let
\[ S := \{x_1, \ldots, x_m\} \]
be a finite subset of closed points of $X$ such that $x_i \neq x_j$ for $i \neq j$, and let
\[ Z := x_1 + \ldots + x_m \]
denote the reduce effective divisor associated with $S$. Let $\Omega^1_X$ denote the cotangent bundle of $X$. Let $E$ be a vector bundle over $X$. A logarithmic connection on $E$ singular over $S$ is a $k$-linear map
\[ D : E \rightarrow E \otimes \Omega^1_X \otimes \mathcal{O}_X(Z) \] (2.1)
which satisfies the Leibniz identity
\[ D(fs) = fD(s) + df \otimes s, \] (2.2)
where $f$ is a local section of $\mathcal{O}_X$ and $s$ is a local section of $E$. 
We shall give an equivalent definition of a logarithmic connection in terms of splitting of logarithmic Atiyah exact sequence.

Let $\text{Diff}^1_X(E, E)$ be the vector bundle over $X$ whose sections over any open subset $U \subset X$ are the differential operators on $E|_U$ of order at most one. Let

$$\sigma_1 : \text{Diff}^1_X(E, E) \to TX \otimes \text{End}(E)$$

be the symbol operator, where $TX$ be the tangent bundle of $X$. The symbol operator $\sigma_1$ is surjective. Consider the symbol exact sequence

$$0 \to \text{End}(E) \overset{i}{\to} \text{Diff}^1_X(E, E) \overset{\sigma_1}{\to} TX \otimes \text{End}(E) \to 0. \quad (2.3)$$

Let $\mathcal{O}_X(-Z) \subset \mathcal{O}_X$ be the line bundle associated to the divisor $-Z$. Then

$$\mathcal{O}_X(-Z) \otimes TX \otimes \text{End}(E) \subset TX \otimes \text{End}(E).$$

Define a vector bundle on $X$ as follows.

$$\text{At}(E)(-\log Z) := \sigma_1^{-1}(1_E \otimes TX \otimes \mathcal{O}_X(-Z)).$$

From the symbol exact sequence (2.3), we get a short exact sequence

$$0 \to \text{End}(E) \overset{i}{\to} \text{At}(E)(-\log Z) \overset{\tilde{\sigma}_1}{\to} TX(-\log Z) \to 0,$$

called the logarithmic Atiyah exact sequence, where $\tilde{\sigma}_1$ is the restriction of $\sigma_1$ and $TX(-\log Z) := TX \otimes \mathcal{O}_X(-Z)$.

Now, $E$ admits a logarithmic connection singular over $S$ if and only if the logarithmic Atiyah exact sequence (2.4) splits algebraically, that is, there exists an $\mathcal{O}_X$-linear homomorphism

$$\alpha : TX(-\log Z) \to \text{At}(E)(-\log Z) \quad (2.5)$$

such that

$$\tilde{\sigma}_1 \circ \alpha = 1_{TX(-\log Z)}.$$

Next we will define residue of a logarithmic connection in $E$ at $x \in S$. Fix $x \in S$, and let $U$ be an open subset of $X$ such that $U \cap S = \{x\}$. Let $s$ be an algebraic section of $\text{At}(E)(-\log Z)$ over $U$. Then $\tilde{\sigma}_1(s)$ will be a section of $TX(-\log Z)$ over $U$. Now, evaluating $\tilde{\sigma}_1(s)$ at $x$, we get

$$\tilde{\sigma}_1(s)(x) = \tilde{\sigma}_1_x(s(x)) = 0.$$

Since $\text{Ker}(\tilde{\sigma}_1) = \text{End}(E)$, we get a $k$-linear map

$$j_x : \text{At}(E)(-\log Z)(x) \to \text{End}(E)(x) = \text{End}(E(x)). \quad (2.6)$$

Note that for any $x \in S$, the fiber $\Omega^1_X \otimes \mathcal{O}_X(Z)(x)$ is canonically identified with $k$ by sending a rational 1-form to its residue at $x$.

Given a logarithmic connection $D$ on $E$ singular over $S$, we get a unique algebraic splitting $\alpha$ of (2.4) corresponding to $D$, and thus we define the residue of $D$ at $x \in S$ by

$$\text{Res}(D, x) := j_x(\alpha_x(1)) \in \text{End}(E(x)), \quad (2.7)$$

where $1 \in k$ and the fiber $TX(-\log Z)(x)$ is canonically identified with $k$ as described above.

Next, we describe logarithmic connections with prescribed residues. For that first notice the following.
Lemma 2.1. For every $x \in S$, the fibre $\text{At}(E)(-\log Z)(x)$ has the canonical decomposition
\[
\text{At}(E)(-\log Z)(x) = \text{End}(E(x)) \oplus k
\] (2.8)

Proof. See [4, Lemma 2.1]. □

Fix $A(x) \in \text{End}(E(x))$ for every $x \in S$. Consider the one dimensional vector space over $k$ generated by the vector $(A(x), 1)$ in $\text{End}(E(x)) \oplus k$, that is,
\[
l_x := k.(A(x), 1) \subset \text{End}(E(x)) \oplus k = \text{At}(E)(-\log Z)(x).
\] (2.9)

Let $\mathcal{A}(E) \rightarrow X$ be the vector bundle that fits in the following short exact sequence
\[
0 \rightarrow \mathcal{A}(E) \rightarrow \text{At}(E)(-\log Z) \rightarrow \bigoplus_{x \in S} \frac{\text{At}(E)(-\log Z)(x)}{l_x} \rightarrow 0.
\] (2.10)

where $l_x$ is constructed above.

From logarithmic Atiyah exact sequence (2.4) we have the short exact sequence
\[
0 \rightarrow \text{End}(E) \otimes \mathcal{O}_X(-Z) \xrightarrow{\iota} \mathcal{A}(E) \xrightarrow{\tilde{\sigma}_1} TX(-\log Z) \rightarrow 0,
\] (2.11)

where $\tilde{\sigma}_1$ is the restriction of $\tilde{\sigma}_1$.

Lemma 2.2. A logarithmic connection in $E$ with given residues $A(x) \in \text{End}(E(x))$, for every $x \in S$ is an algebraic splitting of the short exact sequence (2.11), that is, there exists a morphism $h : TX(-\log Z) \rightarrow \mathcal{A}(E)$ such that $\tilde{\sigma}_1 \circ h = 1_{TX(-\log Z)}$.

We recall the notion of rigid endomorphism. Let $x \in X$. An endomorphism $\beta \in \text{End}(E(x))$ is said to be a rigid if
\[
\beta \circ \phi(x) = \phi(x) \circ \beta
\]
for all $\phi \in H^0(X, \text{End}(E))$. Now onwards we shall assume that the endomorphism $A(x)$ is rigid for every $x \in S$.

Recall that a vector bundle $E$ over $X$ is said to be decomposable if there are vector bundles $F$ and $F'$ such that $\text{rk}(F) > 0$, $\text{rk}(F') > 0$ and
\[
E \cong F \oplus F'.
\]

A vector bundle is called indecomposable if it is not decomposable.

From [21, p.315, Theorem 2], any vector bundle $E$ over $X$ is isomorphic to a unique, up to reordering, direct sum of indecomposable vector bundles, and we call it Krull-Remak-Schmidt decomposition of $E$. Let
\[
E = \bigoplus_{i=1}^n E^i
\]
be the Krull-Remak-Schmidt decomposition of $E$. Since $A(x) \in \text{End}(E(x))$ is rigid for every $x \in S$,
\[
A(x)(E^i(x)) \subset E^i(x).
\]

The restriction of $A(x)$ on $E^i(x)$ is denoted by $A^i(x)$ for every $x \in S$ and for every $i = 1, \ldots, n$. We use the notations as above for the following.
Lemma 2.3. \( E \) admits a logarithmic connection singular over \( S \) with residues \( A(x) \) at every \( x \in S \) if and only if each indecomposable component \( E^i \) admits a logarithmic connection \( D^i \) with residue \( A^i(x) \) at every \( x \in S \).

Proof. Let \( \iota : E^i \to E \) be the inclusion map, and \( q^i : E \to E^i \) the quotient map. Let \( D : E \to E \otimes \Omega^1_X(\log Z) \) be a logarithmic connection singular over \( S \) with residue \( A(x) \) at every \( x \in S \). Consider the composition
\[
(q^i \otimes 1_{\Omega^1_X(\log Z)}) \circ D \circ \iota : E^i \to E^i \otimes \Omega^1_X(\log Z),
\]
which satisfies the Leibniz rule and singular over \( S \) with residue \( A^i(x) \) at every \( x \in S \). Conversely, given logarithmic connection \( D^i \) on \( E^i \) for every \( 1 \leq i \leq n \), singular over \( S \) with residue \( A^i(x) \) at every \( x \in S \). Then \( \bigoplus_{i=1}^n D^i \) gives a logarithmic connection on \( E \) singular over \( S \) with residue \( A(x) \) at every \( x \in S \). \( \square \)

3. Criterion over an algebraically closed field of characteristic \( p > 0 \)

In this section, we assume that \( k \) is an algebraically closed field of characteristic \( p > 0 \), and \( X \) an irreducible smooth projective curve over \( k \).

Theorem 3.1. Let \( E \) be an algebraic vector bundle on \( X \) defined over the algebraically closed field \( k \) of characteristic \( p > 0 \). Then \( E \) admits a logarithmic connection singular over \( S \) with residue \( A(x) \) for every \( x \in S \) if and only if every indecomposable component \( F \) of \( E \) satisfies the following condition
\[
\deg F + \sum_{x \in S} \text{tr}(A(x) \mid_{F(x)}) \equiv 0 \pmod{p}, \quad (3.1)
\]
that is, the number \( \deg F + \sum_{x \in S} \text{tr}(A(x) \mid_{F(x)}) \in k \) is a multiple of \( p \).

Proof. In view of Lemma 2.3, it is enough to prove the theorem for indecomposable vector bundles. Without loss of generality assume that \( E \) is an indecomposable vector bundle. Consider the short exact sequence (2.11) associated with \( E \) and its extension class (called logarithmic Atiyah class)
\[
\phi^A_E \in H^1(X, \text{End}(E) \otimes \Omega^1_X). \quad (3.2)
\]
By Serre duality, the logarithmic Atiyah class \( \phi^A_E \) corresponds to an element
\[
\widetilde{\phi}^A_E \in H^0(X, \text{End}(E))^*. \quad (3.3)
\]
Now, we shall construct a \( k \)-linear morphism
\[
\delta_x : \text{End}(E(x)) \to H^1(X, \text{End}(E) \otimes \Omega^1_X) \quad (3.4)
\]
of vector spaces for every \( x \in S \). In fact, this morphism \( \delta_x \) can be constructed for any \( x \in X \). Consider the short exact sequence
\[
0 \to \text{End}(E) \otimes \Omega^1_X \to \mathcal{O}_X(x) \otimes \text{End}(E) \otimes \Omega^1_X \xrightarrow{\text{Res}_x} \text{End}(E(x)) \to 0, \quad (3.5)
\]
where the last map is given by sending logarithmic 1-form with values in \( \text{End}(E) \) to its residue at \( x \) with values in \( \text{End}(E(x)) \). The morphism \( \delta_x \) in (3.4) is the coboundary operator in the long exact sequence of cohomology groups induced from the short exact sequence (3.5). From Serre duality, and from \( \delta_x \), we get an induced morphism

\[
\tilde{\delta}_x : \text{End}(E(x)) \to H^0(X, \text{End}(E))^*, \tag{3.6}
\]

such that

\[
\tilde{\delta}_x(\beta)(\gamma) = \text{tr}(\beta \circ \gamma(x)).
\]

Now, consider the Atiyah exact sequence (see [3] and [7, Proposition 4.2] for the proof)

\[
0 \to \text{End}(E) \xrightarrow{j} \text{At}(E) \xrightarrow{\sigma_1} TX \to 0, \tag{3.7}
\]

and let \( \sigma(E) \in H^1(X, \text{End}(E) \otimes \Omega^1_X) \) denote the extension class, known as Atiyah class. Again using Serre duality, we have \( \sigma(E) \in H^0(X, \text{End}(E))^* \). We claim that

\[
\phi^E_1 = \sigma(E) + \sum_{x \in S} \tilde{\delta}_x(A(x)) \tag{3.8}
\]

Since \( k \) is an algebraically closed field and \( E \) is indecomposable, any element \( \theta \in H^0(X, \text{End}(E)) \) is of the form

\[
\theta = \nu 1_E + N,
\]

where \( \nu \in k \) and \( N \) is a nilpotent endomorphism of \( E \). To prove (3.8), it is enough to verify the formula (3.8) for \( 1_E \) and \( N \) separately. From [3, Proposition 18(ii)], it is know that \( \sigma(E)(N) = 0 \). Next, since \( N \in \text{End}(E) \) is nilpotent, \( N_x \in \text{End}(E(x)) \) is nilpotent. Since \( A(x) \) is a rigid endomorphism, \( N(x) \) commutes with \( N(x) \) and hence \( A(x) \circ N(x) \) is nilpotent. Thus,

\[
\tilde{\delta}_x(A(x))(N) = \text{tr}(A(x) \circ N(x)) = 0.
\]

Let \( E = E_0 \subset E_1 \subset \ldots \subset E_l = E \) be a flag of subbundles of \( E \) such that

\[
A(x)(E_i(x)) \subset E_{i+1}(x),
\]

for every \( x \in S \) and for every \( 1 \leq i \leq l \). Let \( \text{End}(E)^0 \) be the \( \mathcal{O}_X \)-submodule of \( \text{End}(E) \) consisting of endomorphisms which preserves the flag \( E_\bullet \). Then \( \text{End}(E)^0 \) is a subbundle of \( \text{End}(E) \). The inclusion morphism \( \iota : \text{End}(E)^0 \hookrightarrow \text{End}(E) \), induces a morphism

\[
\iota^* : H^1(X, \text{End}(E)^0 \otimes \Omega^1_X) \to H^1(X, \text{End}(E) \otimes \Omega^1_X).
\]

It can be shown that the logarithmic Atiyah class \( \phi^E_1 \) is in the image of \( \iota^* \).

Moreover, for a nilpotent endomorphism \( N \), the subbundles \( \text{Ker}(N^j) \) of \( E \) for \( j \geq 1 \), form a flag of \( E \) such that

\[
A(x)(\text{Ker}(N^j)(x)) \subset \text{Ker}(N^j)(x),
\]

because \( A(x) \circ N^j(x) = N^j(x) \circ A(x) \) for every \( j \geq 1 \). We take \( \text{End}(E)^0 \) to be associated with the flag \( \text{Ker}(N^j), j \geq 1 \). Then, from above observation \( \phi^E_1(N) = 0 \). Thus, (3.8) satisfies for \( N \).
Consider the following composition of morphisms
\[ H^0(X, \text{End}(E)) \otimes H^1(X, \text{End}(E) \otimes \Omega^1_X) \xrightarrow{m} H^1(X, \text{End}(E) \otimes \text{End}(E) \otimes \Omega^1_X) \xrightarrow{\text{trace}} H^1(X, \Omega^1_X) = k, \]
and denote it by \( \Psi \), where the first morphism is the cup product, second morphism \( m^* \) is induced from \( m : \text{End}(E) \otimes \text{End}(E) \rightarrow \text{End}(E) \) which is composition of endomorphisms of \( E \).

Under this composition \( \Psi \), the image of \( 1 \otimes \phi^A_E \) coincides with \( \deg E + \sum_{x \in S} \text{tr}(A(x)) \in k \), that is,
\[ \tilde{\phi}^A_E(1_E) = \deg E + \sum_{x \in S} \text{tr}(A(x)). \]

Also, under the same morphism \( \Psi \), the image of \( \text{at}(E) \) coincides with \( \deg E \) (see [6, Proposition 3.1]), and hence
\[ \tilde{\text{at}}(E)(1_E) = \deg E. \]

As observed above, \( \tilde{\delta}_x(A(x))(1_E) = \text{tr}(A(x) \circ 1_E) = \text{tr}(A(x)) \). Thus, (3.8) verifies for \( 1_E \), and hence completes the proof of the claim.

Thus, \( E \) admits a logarithmic connection singular over \( S \) with given residue \( A(x) \) for every \( x \in S \) if and only if the obstruction class \( \phi^A_E \) vanishes which is equivalent to \( \tilde{\phi}^A_E(1_E) = 0 \), that is, \( \deg E + \sum_{x \in S} \text{tr}(A(x)) \) vanishes in \( k \), which is nothing but
\[ \deg E + \sum_{x \in S} \text{tr}(A(x)) \equiv 0 \pmod{p}. \]

This completes the proof of the theorem. \( \square \)

**Remark 3.2.** In the proof of the Theorem 3.1, we observed that \( \tilde{\phi}^A_E(1_E) \in k \) corresponds to \( \deg E + \sum_{x \in S} \text{tr}(A(x)) \in k \). In fact, this is true for every field \( k \). Therefore, if a vector bundle \( E \) on an irreducible smooth projective curve \( X \) defined over any field \( k \) admits a logarithmic connection singular over \( S \) with residue \( A(x) \) for every \( x \in S \), then the number
\[ \deg E + \sum_{x \in S} \text{tr}(A(x)) \]

is a multiple of the characteristic of \( k \).

### 4. Criterion over a perfect field

In this section, we prove an analogous result, when the base field fails to be algebraically closed. In this section, we will assume that \( k \) is a perfect field of arbitrary characteristic, and \( X \) is an irreducible smooth projective curve defined over \( k \).

**Theorem 4.1.** Let \( k \) be a perfect field of characteristic \( p \). Let \( E \) be a vector bundle on an irreducible smooth projective curve \( X \) over \( k \). Then, we have
Assume that $p > 0$, and suppose that rank of each indecomposable com-
ponents of $E$ is not divisible by $p$. Then $E$ admits a logarithmic connection
singular over $S$ with residue $A(x)$ for every $x \in S$ if and only if for every
indecomposable component $F$ of $E$ satisfies
\[ \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) \equiv 0 \pmod{p}, \] (4.1)

If for every indecomposable component $F$ of $E$ satisfies
\[ \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) = 0, \] (4.2)
then $E$ admits a logarithmic connection singular over $S$ with residue $A(x)$ for
every $x \in S$.

In view of the Remark 3.2, in the first part (1) of the above Theorem 4.1, one
direction is obvious, that is, if $E$ admits a logarithmic connection singular over $S$
with residue $A(x)$ for every $x \in S$, then
\[ \deg F + \sum_{x \in S} \text{tr}(A(x)|_{F(x)}) \equiv 0 \pmod{p}. \]

The hard part is to prove the converse of it. Also, notice that the converse of the
second part (2) of the Theorem 4.1 is not true. In order to prove above Theorem
4.1 we need the notion of absolutely indecomposable vector bundles over $X$ and
some key properties of finite extensions of perfect fields.

Let $K$ be a finite extension of $k$. Let
\[ X_K = X \times_{\text{spec}(k)} \text{spec}(K) \]
be the curve obtained from base change and let
\[ \pi : X_K \to X \] (4.3)
be the natural projection. Consider the subset $S = \{x_1, \ldots, x_m\}$ of $X$ as above.
Then
\[ \tilde{S} := \pi^{-1}(S) = \{\pi^{-1}(x_1), \ldots, \pi^{-1}(x_m)\} \]
will be a subset consists of distinct closed points of $X_K$. For simplicity we set
$y_j = \pi^{-1}(x_j)$ for every $1 \leq j \leq m$. Let
\[ \tilde{Z} := y_1 + \cdots + y_m \]
be the reduced effective divisor associated with $\tilde{S}$.

Let $F$ be a vector bundle over $X_K$, and let
\[ F := \pi_* F \] (4.4)
be the direct image of the vector bundle $F$ under $\pi$. Then $F$ is a vector bundle over
$X$. Let
\[ d := \deg \pi = [K : k] \]
denote the degree of the extension $K|k$. Then
\[ \deg F = d \cdot \deg F, \] (4.5)
and we get a relation between their ranks
\[ \text{rk}(F) = d \cdot \text{rk}(\mathcal{F}). \] (4.6)

Under above notations we have

**Lemma 4.2.** If the vector bundle \( \mathcal{F} \) over \( X_K \) admits a logarithmic connection singular over \( \tilde{S} \), then the vector bundle \( F \) (defined in (4.4)) over \( X \) admits a logarithmic connection singular over \( S \).

**Proof.** Let
\[ D : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X_K} \otimes O_{X_K}(-\tilde{Z}) = \mathcal{F} \otimes \Omega^1_{X_K} (\log \tilde{Z}) \]
be a logarithmic connection on \( \mathcal{F} \) singular over \( \tilde{S} \).

Let \( U \subset X \) be an open subset, and \( s \in (\pi^{-1}(U)) \). Then \( D(s) \) be a local section of \( \mathcal{F} \otimes \Omega^1_{X_K} (\log \tilde{Z}) \) over \( \pi^{-1}(U) \). This implies that \( D(s) \) is a local section of \( \pi_*(\mathcal{F} \otimes \Omega^1_{X_K} (\log \tilde{Z})) \) over \( U \). Since
\[ \pi^*(\Omega^1_X (\log Z)) = \Omega^1_{X_K} (\log \tilde{Z}), \]
from the projection formula, we get
\[ \pi_*(\mathcal{F} \otimes \Omega^1_{X_K} (\log \tilde{Z})) = \pi_*(\mathcal{F} \otimes \pi^*(\Omega^1_X (\log Z))) = \pi_*(\mathcal{F}) \otimes \Omega^1_X (\log Z) = F \otimes \Omega^1_X (\log Z). \]

Thus, \( D(s) \) gives a local section of \( F \otimes \Omega^1_X (\log Z) \) over \( U \).

We define an operator
\[ \nabla : F \to F \otimes \Omega^1_X (\log Z) \]
by
\[ \nabla(s) := D(s), \]
where \( s \in F(U) \), a section given by \( s \in (\pi_*\mathcal{F})(U) = F(U) \), for every open subset \( U \subset X \).

For any \( f \in O_X(U) \), \( d(f \circ \pi) = \pi^* df \in \Omega^1_{X_K} (\log \tilde{Z})(U) \), where \( d : O_X \to \Omega^1_X (\log Z) \) is the universal derivation. The operator \( \nabla \) satisfies the following property (Leibniz rule).
\[ \nabla(f \cdot s) = D((f \circ \pi) \cdot s) = (f \circ \pi)D(s) + d(f \circ \pi) \otimes s = f \nabla(s) + df \otimes \hat{\ast}. \]

Thus, \( \nabla \) is a logarithmic connection on \( F \) singular over \( S \). \( \square \)

**Remark 4.3.** In the above Lemma 4.2, we can choose residue \( B(y) \) for every \( y \in \tilde{S} \), for the logarithmic connection \( D \) on \( \mathcal{F} \) singular over \( \tilde{S} \), such that it maps to the given residue \( A(x) \) for every \( x \in S \), for the induced logarithmic connection \( \nabla \) on \( F \) singular over \( S \), under \( \pi \).
Let $Y$ be a geometrically irreducible projective curve defined over a field $l$. For the following definition of absolutely indecomposable vector bundle, see [1] and [8, Definition 2.2].

A vector bundle $E$ over $Y$ is called **absolutely indecomposable** if there is an algebraic closure $\overline{l}$ of $l$ such that the corresponding vector bundle $E \otimes_l \overline{l}$ over $Y \times_l \overline{l}$ is indecomposable.

In fact, the condition that a vector bundle over $Y$ is absolutely indecomposable does not depend on the choice of the algebraic closure $\overline{l}$ (see the first paragraph on [8, p. n. 86]).

**Proof of Theorem 4.1.** In view of Lemma 2.3, it is enough to show the theorem for indecomposable vector bundles. To prove the first part (1), let $k$ be a perfect field of characteristic $p > 0$, and $E$ be an indecomposable vector bundle over $X$ such that

$$\deg E + \sum_{x \in S} \text{tr}(A(x)) \equiv 0 \pmod{p}, \quad (4.7)$$

where $A(x)$ is given rigid endomorphism on $E(x)$, for every $x \in S$.

Since $k$ is a perfect field, from [1, Theorem 1.8, 4] there is a finite field extension $K$ of $k$ with the following property.

There is an absolutely indecomposable vector bundle $\text{cat} E$ over

$$X_K := X \times_k K$$

such that

$$\pi_* E \cong E, \quad (4.8)$$

where $\pi : X_K \to X$ is the natural projection. The above result is due to A. Tillmann [10] as mentioned in [1].

In view of (4.6), for every $y \in \tilde{S}$, choose endomorphism $B(y) \in E(y) \cong K^{\text{rk}(E)}$ such that

$$d \cdot \text{tr}(B(y)) = \text{tr}(A(x)), \quad (4.9)$$

where $\pi(y) = x$, and $d$ the degree of extension $K|k$.

Note that $\text{tr}(B(y)) \in K$ and $\text{tr}(A(x)) \in k$, so we consider $K$ as a vector space over $k$, and we take trace of $\text{tr}(B(x)) \in K$ to get (4.9).

From (4.7), and (4.5), we have

$$d \cdot \deg E + \sum_{y \in \tilde{S}} d \cdot \text{tr}(B(y)) \equiv 0 \pmod{p}. \quad (4.10)$$

Since $\text{rk}(E)$ is not divisible by $p$, from (4.6), $d$ is coprime to $p$, and hence (4.10) becomes,

$$\deg E + \sum_{y \in \tilde{S}} \text{tr}(B(y)) \equiv 0 \pmod{p}. \quad (4.11)$$

Since $E$ is an absolutely indecomposable vector bundle over $X_K$, by above definition

$$E_K := E \otimes_K K$$

is indecomposable vector bundle over $X_K := X_K \times_K K$, where $K$ is an algebraic closure of $k$. 

Let \( \hat{\pi} : X_\mathbb{K} \to X_K \) be the natural projection. Choose a subset \( \hat{S} = \{\hat{y}_1, \ldots, \hat{y}_m\} \subset X_K \) such that \( \hat{\pi}(\hat{y}_j) = y_j \) for every \( j = 1, \ldots, m \). Notice that \( \deg \mathcal{E}_\mathbb{K} = \deg \mathcal{E} \). From above observation, for every \( \hat{y} \in \hat{S} \), choose endomorphisms \( \overline{B(\hat{y})} \) on \( \mathcal{E}_\mathbb{K}(\hat{y}) \) such that \( \text{tr}(\overline{B(\hat{y})}) = \text{tr}(B(y)) \). Thus, the equation (4.11) becomes
\[
\deg \mathcal{E}_\mathbb{K} + \sum_{\hat{y} \in \hat{S}} \text{tr}(\overline{B(\hat{y})}) \equiv 0 \pmod{p}, \tag{4.12}
\]
and from Theorem 3.1 \( \mathcal{E}_\mathbb{K} \) admits a logarithmic connection singular over \( \hat{S} \) with residue \( \overline{B(\hat{y})} \) for every \( \hat{y} \in \hat{S} \).

Consider the short exact sequence (2.11) for the vector bundle \( \mathcal{E} \) over \( X_K \) and for the given endomorphism \( B(y) \in \text{End}(\mathcal{E}(y)) \) for every \( y \in \hat{S} \), that is,
\[
0 \to \text{End}(\mathcal{E}) \otimes \mathcal{O}_{X_K}(-\hat{Z}) \xrightarrow{\pi} \mathcal{A}(\mathcal{E}) \xrightarrow{\xi} TX_K(-\log \hat{Z}) \to 0. \tag{4.13}
\]
Let \( \phi^B_\mathcal{E} \in H^1(X_K, \text{End}(\mathcal{E}) \otimes \Omega^1_{X_K}) \) be the extension class of (4.13).

Changing the base from \( K \) to \( \overline{K} \), and using the base change formula, we get a short exact sequence
\[
0 \to \text{End}(\mathcal{E}) \otimes \mathcal{O}_{X_\mathbb{K}}(-\hat{Z}) \xrightarrow{\pi} \mathcal{A}(\mathcal{E}) \xrightarrow{\xi} TX_\mathbb{K}(-\log \hat{Z}) \otimes_K \overline{K} \to 0. \tag{4.14}
\]
of vector bundles over \( X_\mathbb{K} \). Let \( \hat{\phi}^B_\mathcal{E} \) be the extension class of (4.14), and note that
\[
H^1(X_\mathbb{K}, \text{End}(\mathcal{E}) \otimes \Omega^1_{X_\mathbb{K}} \otimes \overline{K}) = H^1(X_K, \text{End}(\mathcal{E}) \otimes \Omega^1_{X_K}) \otimes \overline{K},
\]
and
\[
\hat{\phi}^B_\mathcal{E} = \phi^B_\mathcal{E} \otimes 1. \tag{4.15}
\]
Next, the short exact sequence (4.14) coincides with the short exact sequence (2.11) associated with the vector bundle \( \mathcal{E}_\mathbb{K} \) over \( X_\mathbb{K} \) with endomorphism \( \overline{B(\hat{y})} \) on \( \mathcal{E}_\mathbb{K}(\hat{y}) \), for every \( \hat{y} \in \hat{S} \), that is,
\[
0 \to \text{End}(\mathcal{E}_\mathbb{K}) \otimes \mathcal{O}_{X_\mathbb{K}}(-\hat{Z}) \xrightarrow{\pi} \mathcal{A}(\mathcal{E}_\mathbb{K}) \xrightarrow{\xi} TX_\mathbb{K}(-\log Z) \to 0.
\]
Therefore, the extension class \( \phi^{B(\hat{y})}_{\mathcal{E}_\mathbb{K}} \) coincides with \( \hat{\phi}^B_\mathcal{E} \).

Since \( \mathcal{E}_\mathbb{K} \) admits a logarithmic connection singular over \( \hat{S} \) with residue \( \overline{B(\hat{y})} \) for every \( \hat{y} \in \hat{S} \), we have
\[
0 = \phi^{B(\hat{y})}_{\mathcal{E}_\mathbb{K}} = \hat{\phi}^B_\mathcal{E}.
\]
Thus, from (4.15),
\[
\phi^B_\mathcal{E} \otimes 1 = 0,
\]
and hence \( \phi^B_\mathcal{E} = 0 \). In view of Lemma 4.2 and Remark 4.3 proof of (1) is complete.

To prove the second part (2), let \( E \) be an indecomposable vector bundle over \( X \) which satisfies (4.2). Using the same technique as above, the vector bundle \( \mathcal{E} \) over \( X_K \) in (4.8) satisfies (4.2). Now, repeating the above argument, we conclude that \( \mathcal{E} \) admits a logarithmic connection singular over \( \hat{S} \) with residue \( B(y) \), for every \( y \in \hat{S} \).
Again, from Lemma \ref{lem:4.2} and Remark \ref{rem:4.3}, proof of the second part (2) of the theorem is complete.

5. Conclusions

The above theorems have been proved over any perfect field and they will be very useful when we study the algebro-geometric invariants for the moduli space of logarithmic connections over a perfect field, with fixed rigid residues.

In case residues are not rigid, finding a suitable criterion for the existence of a logarithmic connection is still an open problem. Our guess is the same criterion should work but we do not know how to prove it.

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