Concave Shape of the Yield Curve and No Arbitrage

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In fixed income sector, the yield curve is probably the most observed indicator by the market for trading and financing purposes. A yield curve plots interest rates across different contract maturities from short end to as long as 30 years. For each currency, the corresponding curve shows the relation between the level of the interest rates (or cost of borrowing) and the time to maturity. For example, the U.S. dollar interest rates paid on U.S. Treasury securities for various maturities are plotted as the US treasury curve. For the same currency, if the swap market is used, we could also plot the swap rates across the tenors which would be called the swap curve.

The shape of the yield curve gives an idea of future interest rate changes and economic activity. There are three main types of yield curve shapes: normal, inverted and flat (or humped). A normal yield curve is one in which longer maturity bonds have a higher yield compared to shorter-term bonds due to the risks associated with time. An inverted yield curve is one in which the shorter-term yields are higher than the longer-term yields, which can be a sign of upcoming recession. In a flat or humped yield curve, the shorter- and longer-term yields are very close to each other, which is also a predictor of an economic transition.

A normal or up-sloped yield curve indicates yields on longer-term bonds may continue to rise, responding to periods of economic expansion. When investors expect longer-maturity bond yields to become even higher in the future, many would temporarily park their funds in shorter-term securities in hopes of purchasing longer-term bonds later for higher yields.

In a rising interest rate environment, it is risky to have investments tied up in longer-term bonds when their value has yet to decline as a result
of higher yields over time. The increasing temporary demand for shorter-term securities pushes their yields even lower, setting in motion a steeper up-sloped normal yield curve.

An inverted or down-sloped yield curve suggests yields on longer-term bonds may continue to fall, corresponding to periods of economic recession. When investors expect longer-maturity bond yields to become even lower in the future, many would purchase longer-maturity bonds to lock in yields before they decrease further. The increasing onset of demand for longer-maturity bonds and the lack of demand for shorter-term securities lead to higher prices but lower yields on longer-maturity bonds, and lower prices but higher yields on shorter-term securities, further inverting a down-sloped yield curve.

A flat yield curve may arise from normal or inverted yield curve, depending on changing economic conditions. When the economy is transitioning from expansion to slower development or even recession, yields on longer-maturity bonds tend to fall and yields on shorter-term securities likely rise, inverting a normal yield curve into a flat yield curve. When the economy is transitioning from recession to recovery and potential expansion, yields on longer-maturity bonds are set to rise and yields on shorter-maturity securities are sure to fall, tilting an inverted yield curve towards a flat yield curve.

According to Salomon Brothers working paper [7], it is commonly known there are three main influences on the treasury yield curve shape:

1. Yield curve shape reflects the market’s expectations of future rate change.

2. Yield curve shape reflects the bond risk premia (expected return differentials across different maturities)

3. Yield curve shape reflects the convexity benefit of bonds of different tenors.

Even the yield curve can be flat, upward or downward (inverted), however, yield curve is generally concave. There is a lack of explanation of the concavity of the yield curve shape from economics theory. We offer in this article an explanation of the concavity shape of the yield curve from trading perspectives.

Our main argument is to construct an investment portfolio consisting fixed income instruments and demonstrate that if the yield curve is not
concave, an arbitrage will emerge. Our results also depend on an assumption that yield curve moves up and down in parallel. This assumption is not precisely true but approximately acceptable in reality.

1 Interest Rates

We first discuss the relations between different notions of interest rates. In the discussion below, certain simplifications will be made. We mainly discuss three types of rates which are often used in industry: zero rates, forward rates and par rates. Correspondingly, there are three types of instruments: zero coupon bonds, forward rate agreements and swaps (or par bonds). In all the calculations, we ignore the day count and business conventions. For swaps we assume the exchange of payment occurs once a year and for forward rate agreements we also assume that the contract expire in integer years and span for one year.

Let the current time be 0, and future annual years be \(1, 2, \cdots, n\). The zeros rates are yield to maturity of the zero coupon bonds maturing at these times. Zero rates and the zero coupon bond prices have the relationship

\[
p_i = \frac{1}{(1 + y_i)^i}
\]

where the \(p_i\) is the zero coupon bond price and \(y_i\) is the yield to maturity. The forward rates for time interval \((i, i+1)\) are the rates specified in the forward rate agreements (FRA) to lock in the interest rates at the future time \(i\) for the one year period. This rate \(f_i\) can be calculated as

\[
f_i = \frac{p_i}{p_{i+1}} - 1 = \frac{(1 + y_{i+1})^{i+1}}{(1 + y_i)^i} - 1
\]

and finally the par rates \(s_i\) are calculated as

\[
s_i = \frac{1 - p_i}{p_1 + p_2 + \cdots + p_i}
\]

It is known that as long as \(f_i > 0\), there is no arbitrage between these trading instruments. Hence in theory, the zero curve can have any shape as long as \(f_i > 0\). For example, the curve might be upward or might be downward. The curve might be convex or might be concave.
But in reality the curves are usually concavely upward shaped. In this article we show that the zero curve and the swap curve has to be concave if the following conditions are true: there is no arbitrage; yield curve moves in parallel. The first assumption is realistic as many trading desks around the world are watching the yield curve and try to take arbitrage opportunities at any time. The second assumption is not realistic, but close to reality. In a rising rates environment, the rates of different tenors all increase in general and in a falling rates environment, rates of different tenors all decrease.

2 Zero Coupon Bonds

Our main results depend on the following crucial well known results:

**Convexity Inequality** The function $f(x), x \in \mathbb{R}$ is a convex function, then by definition it should satisfy the following inequality. For any $\lambda_1 > 0, \lambda_2 > 0$ which $\lambda_1 + \lambda_2 = 1$ and $a, b \in \mathbb{R}$, we should have

$$f(\lambda_1 a + \lambda_2 b) \leq \lambda_1 f(a) + \lambda_2 f(b)$$

We now set our securities. We have three zero coupon bonds, calling them $B_1, B_2, B_3$ corresponds to three maturities $T_1 < T_2 < T_3$ with their yields be $y_1, y_2, y_3$. So far we impose no conditions on these yields as long as the implied forward is positive. We now construct a trading portfolio by purchasing $\lambda_1$ dollar amount of $B_1$, $\lambda_3$ dollar amount of $B_3$ and short $\lambda_2$ dollar amount of $B_2$. We choose quantities $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ by the following rules

$$\lambda_1 + \lambda_3 = \lambda_2$$

and

$$\lambda_1 T_1 + \lambda_3 T_3 = \lambda_2 T_2.$$ 

We notice that by combining the two equations, we have

$$\lambda_1 (T_2 - T_1) = \lambda_3 (T_3 - T_2)$$

In fact by linear algebra, the solutions of $\lambda_i$ is unique up to a scalar

$$\lambda_1 = T_3 - T_2, \lambda_2 = T_3 - T_1, \lambda_3 = T_2 - T_1$$
As a consequence
\[ \frac{\lambda_1}{\lambda_2} + \frac{\lambda_3}{\lambda_2} = 1 \]

We now claim that portfolio we have constructed has zero cost. Zero cost is obvious given the rule that \( \lambda_1 + \lambda_2 + \lambda_3 = 0. \)

**Theorem 1.** If yields \( y_1, y_2, y_3 \) as a function of time to maturity \( T_1, T_2, T_3 \) is convex, i.e.
\[ (T_3 - T_2)y_1 + (T_2 - T_1)y_3 \geq (T_3 - T_1)y_2 \] (4)
the portfolio we constructed admits an arbitrage.

**Proof.** Now we assume yields move by the same amount \( a \) and time moves forward by \( t \), therefore our portfolios new value becomes
\[ P(a, t) = \lambda_1 e^{-a(T_1-t)+y_1 t} + \lambda_3 e^{-a(T_3-t)+y_3 t} - \lambda_2 e^{-a(T_2-t)+y_2 t} \]
We want to show that this quantity is positive i.e.
\[ \frac{\lambda_1}{\lambda_2} e^{-a(T_1-t)+y_1 t} + \frac{\lambda_3}{\lambda_2} e^{-a(T_3-t)+y_3 t} \geq e^{-a(T_2-t)+y_2 t} \]
By the convexity inequality we have
\[ \frac{\lambda_1}{\lambda_2} e^{-a(T_1-t)+y_1 t} + \frac{\lambda_3}{\lambda_2} e^{-a(T_3-t)+y_3 t} \geq e^{-a(T_2-t)+\frac{\lambda_1}{\lambda_2}y_1 t + \frac{\lambda_3}{\lambda_2} y_3 t} \]
But if the yield \( y_i \) are convex, by definition we have
\[ \frac{\lambda_1}{\lambda_2} y_1 + \frac{\lambda_3}{\lambda_2} y_3 \geq y_2 \]
therefore
\[ \lambda_1 e^{-a(T_1-t)+y_1 t} + \lambda_3 e^{-a(T_3-t)+y_3 t} \geq \lambda_2 e^{-a(T_2-t)+y_2 t} \]
it true. \( \square \)

We have completed our argument that arbitrage exists by construction a zero cost portfolio consisting of three zero coupon bonds. The argument is valid for any three maturities as long as corresponding yields are convex and the yields move by the same amount. The entire argument is based on the convexity inequality. We have proved so far:
1. Under parallel movement in yields, we can construct zero cost portfolio and achieve positive profit instantaneously.

2. If Yields are convex with respect to time, we construct zero cost portfolio and achieve positive profit at any future time.

3 Nonparallel Movement

We now extend the results in the previous sections to nonparallel movement. We now try to refine the argument above. For now we assume that yields \( y_i \) are different, the movement \( a_i \) can also be different. We first check the instantaneous result. After the yield movement the portfolio value becomes

\[
P(a_1, a_2, a_3) = \lambda_1 e^{-a_1 T_1} + \lambda_3 e^{-a_3 T_3} - \lambda_2 e^{-a_2 T_2}
\]

Once again, we require \( \lambda_1 + \lambda_3 = \lambda_2 \) thus the bond portfolio is a zero cost portfolio. Secondly, we investigate the constraints on \( a_i \) so that the portfolio value is always positive. By the convexity inequality

\[
\frac{\lambda_1}{\lambda_2} e^{-a_1 T_1} + \frac{\lambda_3}{\lambda_2} e^{-a_3 T_3} \geq e^{-\frac{\lambda_1}{\lambda_2} a_1 T_1 - \frac{\lambda_3}{\lambda_2} a_3 T_3}
\]

As a consequence, we would need to set

\[-\frac{\lambda_1}{\lambda_2} a_1 T_1 - \frac{\lambda_3}{\lambda_2} a_3 T_3 \geq -a_2 T_2\]

which is equivalent to

\[
\frac{\lambda_1}{\lambda_2} a_1 T_1 + \frac{\lambda_3}{\lambda_2} a_3 T_3 \leq a_2 T_2
\]

If we require that ratios between \( a_1, a_2, a_3 \) are all fixed, the above equality implies that

\[
\frac{\lambda_1}{\lambda_2} T_1 + \frac{\lambda_3}{\lambda_2} T_3 = T_2
\]

As a consequence, we have \( \lambda_1 = a_3 T_3 - a_2 T_2, \lambda_2 = a_3 T_3 - a_1 T_1, \lambda_3 = a_2 T_2 - a_1 T_1 \).
Secondly, we investigate the requirement on $a_i$ to achieve positive portfolio value at any future time. For this purpose, let time march forward by amount of $t$. Therefore the new portfolio becomes

$$\lambda_1 e^{-a_1(T_1-t)+y_1 t} + \lambda_3 e^{-a_3(T_3-t)+y_3 t} - \lambda_2 e^{-a_2(T_2-t)+y_2 t}$$

and we hope to have

$$\frac{\lambda_1}{\lambda_2} e^{-a_1(T_1-t)+y_1 t} + \frac{\lambda_3}{\lambda_2} e^{-a_3(T_3-t)+y_3 t} \geq e^{-a_2(T_2-t)+y_2 t}$$

Again by applying the convex inequality,

$$\frac{\lambda_1}{\lambda_2} e^{-a_1(T_1-t)+y_1 t} + \frac{\lambda_3}{\lambda_2} e^{-a_3(T_3-t)+y_3 t} \geq e^{-\frac{\lambda_1}{\lambda_2} a_1(T_1-t)+\frac{\lambda_1}{\lambda_2} y_1 t - \frac{\lambda_3}{\lambda_2} a_3(T_3-t)+\frac{\lambda_3}{\lambda_2} y_3 t}$$

For our purpose, we should have three requirements

$$\frac{\lambda_1}{\lambda_2} (y_1 + a_1) + \frac{\lambda_3}{\lambda_2} (y_3 + a_3) \geq y_2 + a_2 \quad (5)$$

The first inequality says that $y_i + a_i$ should be concave which we have examined in the first section. As long as $a_i$ are in this range, the portfolio value is always positive at any future time.

## 4 Swaps

We now turn to the convexity trading strategy using swaps. Swaps are one of the most liquid OTC fixed income instruments. In general, swaps are trading within only as small as 0.25 basis point bid offer spread. Fixed Income desk often uses swaps to express views on interest rate curves and to hedge interest rate risks, in particular the duration risks.

However, swap curve mathematics involves much more than those in zero coupon bond yield curves due to the complexity of bootstrap procedures. This additional complexity makes the arbitrage argument more difficult and lengthier.

An interest rate swap is a contract to exchange certain cash flows at future time. Initially there is an exchange of notional amount and subsequently, there is an exchange of coupon payments, one leg is fixed coupon payment and another leg is floating coupon payment. At maturity there
is another exchange of the notional amount. The floating leg usually has LIBOR as the coupon rate, therefore the floating leg always prices back to be par value. As a consequence, the fixed leg at the inception, also prices back to par value since swap has zero value at inception. As a consequence the swap rates represent the par bond coupon rates.

In real world, the payment frequency for the fixed leg and the floating leg might be different. The fixed leg usually has 6 month as the payment frequency and the floating leg has 3 month as the payment frequency. Also each payment day should adjust for weekend and the holiday. However, in our analysis, we just ignore these technicalities without loss of any generalities.

To establish our results, we use annual payment frequencies at time 1, 2, .... The swap rates correspondingly are $x_1, x_2, \ldots$. Each swap rate $x_n$ corresponds to the fixed leg payment rate of a $n$ year swap. We also use $p_1, p_2, \ldots$ as the discount factors i.e. zero coupon bond prices corresponds to year 1, 2, .... Since we are using annual payment frequency, we can explicitly write down the bootstrap procedure of converting swaps rates to discount factors.

The first year swap rate is simply the compounding rate for the first year hence

$$p_1 = \frac{1}{1 + x_1}$$

Starting from the second year, the conversion is getting complex. The recursive identity can be written in general as

$$p_n = \frac{1 - x_n \sum_{i=1}^{n-1} p_i}{1 + x_n}$$

or equivalently

$$x_n = \frac{1 - p_n}{\sum_{i=1}^{n-1} p_i}$$

Because zero bonds’ prices are decreasing in order to prevent arbitrage, we should impose

$$p_0 = 1, p_n > p_{n+1}, p_n > 0, \text{ for all } n = 1, 2, \ldots,$$

This fact would impose conditions on swap rate itself. We don’t want to explore the necessary and sufficient conditions on swap rates, but the following fundamental fact is very interesting and explains the general shape of swap curve. Let us first introduce more notations. The fixed leg of swap
has one principal payment at the final maturity and all coupon payments in between. The coupon payment has discounted value

\[ x_n(p_1 + p_2 + \cdots + p_n) \]

and professionals always call the sum

\[ P_n = p_1 + p_2 + \cdots + p_n \]

the annuity factor. We now state the first necessary condition on swap rates.

**Theorem 2.** The following limit exits

\[ \lim_{n \to \infty} x_n = x_\infty \geq 0. \]

If the limit \( x_\infty > 0 \), we must have

\[ \lim_{n \to \infty} p_n = 0. \]

**Proof.** In the identity

\[ x_n = \frac{1 - p_n}{\sum_{i=1}^{n} p_i} = \frac{1 - p_n}{P_n} \]

the numerator has limit as \( n \to \infty \) because discount factors are positive and decreasing. The Denominator also has limit because \( P_n \) is positive and increasing. Therefore \( x_n \) must have a finite limit. If the limit is positive, the infinite series \( \sum_{n=1}^{\infty} p_n \) must be finite hence \( p_n \to 0. \)

The financial meaning of this lemma has two folds. First, it shows that the long term swap curve must be asymptotically flat which is indeed the case when we check the market. However, our argument is not based on any economics consideration, but purely based on no arbitrage theory. Secondly, if the long term rates have a positive lower bond, the discount factor has to go to zero. This result is certainly not obvious for swap rates.

## 5 Arbitrage Portfolio Using Swaps

We now introduce the swap curve shifting. If the swap curve changes and the new swap curve for time \( n \) becomes \( x_n + y_n \), the bootstrapped discount curve will also changes. In order to signify the dependence of the changes we
use $p_n(y)$ as the new discount factor. Notice that $y$ here is not necessarily the constant but the entire vector. Correspondingly the annuity factor also changes from $P_n$ to $P_n(y)$. We have to explore the properties of the new discount factors and the old discount factors in order to explore the profit and loss of any trading strategy. These properties are very interesting on their own.

**Lemma 1.** If the changes in swap curve $y_n \geq 0$ for $n > 0$, we must have

$$P_n(y) \leq P_n, \quad n > 0$$

If the changes in swap curve $y_n \leq 0$ for $n > 0$, we have

$$P_n(y) \geq P_n, \quad n > 0$$

**Proof.** We prove the $y_n \geq 0$ case. By the bootstrap identity (7), we see that

$$P_n = \frac{1 - (P_n - P_{n-1})}{x_n}$$

therefore we see that

$$P_n = \frac{1 + P_{n-1}}{1 + x_n}$$

On the other hand it is obvious that

$$P_1(y) = \frac{1}{1 + x_1 + y_1} \leq \frac{1}{1 + x_1} = P_1$$

We can now use induction method. If we already have $P_{n-1}(y) \leq P_{n-1}$, it is clear that

$$P_n(y) = \frac{1 + P_{n-1}(y)}{1 + x_n + y_n} \leq \frac{1 + P_{n-1}(y)}{1 + x_n} \leq \frac{1 + P_{n-1}}{1 + x_n} = P_n$$

Its financial meaning is obvious. If the rates are moving higher, the average (or total) discount factors should be lower. The result is intuitive but certainly not obvious. We cannot stretch it to make the statement that each discount factor is lower i.e.

$$p_n(y) \leq p_n, \quad \text{for all } n > 0$$
but this is not true. For example, if only one year swap rates increase by
10 bps while every other swap rates remain the same, we will see that only
first’s year discount factor is lower, but starting from the second year the
discount factors are actually slightly higher. However, we will prove that
when the movement is parallel, this will be true. But first we need the
following lemma.

**Lemma 2.** If the swap rates change \( y_n = y > 0 \) is constant, for each \( n > 0 \)
we should have

\[
0 < \frac{p_n}{P_n} - \frac{p_n(y)}{P_n(y)} \leq y
\]

\[
0 < \frac{1}{P_n(y)} - \frac{1}{P_n} \leq y
\]

**Proof.** These two inequalities are not obvious at all. We notice by \((\text{7})\), we
have

\[
\frac{1}{P_n} - \frac{p_n}{P_n} = x_n, \quad \frac{1}{P_n(y)} - \frac{p_n(y)}{P_n(y)} = x_n + y
\]

Subtracting, we have

\[
\left[ \frac{p_n}{P_n} - \frac{p_n(y)}{P_n(y)} \right] + \left[ \frac{1}{P_n(y)} - \frac{1}{P_n} \right] = y
\]

We have already proved the second bracket is always positive we just need
to prove the first bracket is also positive. We notice that

\[
\frac{1}{P_1(y)} - \frac{1}{P_1} = y
\]

therefore

\[
\frac{p_1}{P_1} - \frac{p_1(y)}{P_1(y)} = 0
\]

Now if there is \( n \) such that

\[
\frac{p_n}{P_n} - \frac{p_n(y)}{P_n(y)} \geq 0
\]

\[
\frac{p_{n+1}}{P_{n+1}} - \frac{p_{n+1}(y)}{P_{n+1}(y)} < 0
\]

we should have

\[
\frac{1}{P_n(y)} - \frac{1}{P_n} < y < \frac{1}{P_{n+1}(y)} - \frac{1}{P_{n+1}}
\]
Therefore
\[
\left[ \frac{1}{P_{n+1}(y)} - \frac{1}{P_n} \right] - \left[ \frac{1}{P_n(y)} - \frac{1}{P_n} \right] > 0
\]

But reorganizing everything, we have
\[
\frac{p_{n+1}}{P_{n+1}P_n} \geq \frac{p_{n+1}(y)}{P_{n+1}(y)P_n(y)}
\]

But by the condition again
\[
\frac{p_{n+1}}{P_{n+1}P_n} > \frac{p_{n+1}(y)}{P_{n+1}(y)P_n(y)}
\]
\[
> \frac{p_{n+1}}{P_n(y)P_{n+1}}
\]
\[
> \frac{p_{n+1}}{P_nP_{n+1}}
\]

which is a contradiction.

With this lemma, we see that we have the following interesting ordering
\[
\frac{p_n(y)}{P_n(y)} < \frac{p_n}{P_n} < \frac{1}{P_n} < \frac{1}{P_n(y)}
\]

for any constant movement \( y > 0 \).

**Lemma 3.** When the parallel movement \( y > 0 \), the quantity \( p_n(y) < p_n \) for all \( n > 1 \).

**Proof.** Because in lemma 2 we have proved that
\[
\frac{p_n}{P_n} - \frac{p_n(y)}{P_n(y)} \geq 0
\]
then
\[
\frac{p_n(y)}{p_n} < \frac{P_n(y)}{P_n} < 1.
\]

**Lemma 4.** When the parallel movement \( y > 0 \), the quantity \( P_n(y)/P_n \) is monotonically decreasing.
Proof. In order to prove the monotonicity, we just need to show
\[
\frac{P_n(y)}{P_n} - \frac{P_{n+1}(y)}{P_{n+1}} > 0
\]
which is equivalent to
\[
\frac{P_n(y)}{P_{n+1}(y)} > \frac{P_n}{P_{n+1}}
\]
But this is true because
\[
\frac{p_{n+1}(y)}{P_{n+1}(y)} < \frac{p_{n+1}}{P_{n+1}}
\]
by the previous lemma \(^3\). \(\Box\)

Now with all the build-ups in our knowledge, finally we are able to prove the following

**Lemma 5.** When the spread \(y > 0\) and when swap curve increasing, the ratio of two discount factors \(p_n(y)/p_n\) is also decreasing.

Proof. We want to prove that
\[
p_n(y)p_{n-1} \leq p_{n-1}(y)p_n
\]
for all \(n > 1\). But this inequality is equivalent to
\[
p_n(y)(p_{n-1} - p_n) \leq p_n(p_{n-1}(y) - p_n(y))
\]
Due to the fact that
\[
p_n = 1 - x_nP_n, \quad p_{n-1} = 1 - x_{n-1}P_{n-1}
\]
we have
\[
p_{n-1} - p_n = x_nP_n - x_{n-1}P_{n-1} = (x_n - x_{n-1})P_n + x_{n-1}p_n
\]
Similarly,
\[
p_{n-1}(y) - p_n(y) = x_nP_n(y) - x_{n-1}P_{n-1}(y) = (x_n - x_{n-1})P_n(y) + (x_{n-1} + y)p_n(y)
\]
Therefore what we want to prove is equivalent to
\[
-(x_n - x_{n-1}) \left( \frac{p_n}{P_n} - \frac{p_n(y)}{P_n(y)} \right) \leq y \frac{p_n}{P_n} \frac{p_n(y)}{P_n(y)}
\]
which is obvious under our assumptions. \(\Box\)
Finally, we need a technical lemma.

**Lemma 6.** For three integers $n < m < k$, the three points on the plane

$$(P_n, P_n(y)), \quad (P_m, P_m(y)), \quad (P_k, P_k(y))$$

is concave (convex) if and only if $p_n(y)/p_n$ is decreasing (increasing) with respect to $n$.

**Proof.** Let three points $n < m < k$ be the maturity of the swaps, The concavity inequality is

$$(P_k - P_n)P_m(y) \geq (P_m - P_n)P_k(y) + (P_k - P_m)P_n(y)$$

which is equivalent to

$$\left( \sum_{i=n+1}^{k} p_i \right) \left( \sum_{j=n+1}^{m} p_j(y) \right) \geq \left( \sum_{i=n+1}^{k} p_i(y) \right) \left( \sum_{j=n+1}^{m} p_j \right)$$

After cancelation, we need to prove

$$\sum_{i=m+1}^{k} p_i \sum_{j=n+1}^{m} p_j(y) \geq \sum_{i=m+1}^{k} p_i(y) \sum_{j=n+1}^{m} p_j$$

But by monotonicity, we know for each $i > j$, we have

$$p_i p_j(y) \geq p_j p_i(y)$$

therefore proved our claim. \(\Box\)

Combining lemma 5 and lemma 6, we have

**Lemma 7.** When the spread $y > 0$ and when swap curve increasing, the three points on the plane

$$(P_n, P_n(y)), \quad (P_m, P_m(y)), \quad (P_k, P_k(y))$$

is concave. When the spread $y <<$ and when swap curve increasing, the three points on the plane

$$(P_n, P_n(y)), \quad (P_m, P_m(y)), \quad (P_k, P_k(y))$$

is convex.
The combination of above lemmas gives the following main result

**Theorem 3.** If the swap curve is upward, plot each swap rate against its duration, the curve should be concave under parallel movement assumption, otherwise we can construct an arbitrage.

**Proof.** We now construct a portfolio consisting with three swaps maturing at time $T_1, T_2, T_3$. The corresponding swap rates are $x_1, x_2, x_3$. The notional of these three swaps are $\lambda_1, \lambda_2, \lambda_3$. In particular we need

$$
\begin{align*}
\lambda_1 &= P_3 - P_2 \\
\lambda_2 &= P_3 - P_1 \\
\lambda_3 &= P_2 - P_1
\end{align*}
$$

According to the condition, we have $x_1 < x_2 < x_3$. We assume by contradiction that $(P_1, x_1), (P_2, x_2), (P_3, x_3)$ is convex, we will construct an arbitrage portfolio. We long the fixed leg in the first and third swap while short the fixed leg in the second swap. After the parallel movement by an amount $y > 0$, our profit and loss comes from two components,

$$L = L_1 + L_2$$

where $L_1$ is the interest accrual and $L_2$ is the mark to market profit and loss. The interest accrual is

$$L_1 = \lambda_1 x_1 + \lambda_3 x_3 - \lambda_2 x_2 > 0$$

by assumption. The $L_2$ component is the market to market profit and loss

$$L_1 = -y \lambda_1 \tilde{P}_1(y) - y \lambda_3 \tilde{P}_3(y) + y \lambda_2 \tilde{P}_2(y)$$

where $\tilde{P}_i(y)$ is the annuity factor from time $t$ the final maturity year $i$. We want to show that if $y > 0$, we have

$$\lambda_1 \tilde{P}_1(y) + \lambda_3 \tilde{P}_3(y) \leq \lambda_2 \tilde{P}_2(y)$$

which is equivalent to $L_2 \geq 0$. First by lemma 5 we see know that

$$(P_1, P_1(y)), (P_2, P_2(y)), (P_3, P_3(y))$$

is concave, given the fact that $P_i(y) - \tilde{P}_i(y)$ is a constant regardless of $i$, we have

$$(P_1, \tilde{P}_1(y)), (P_2, \tilde{P}_2(y)), (P_3, \tilde{P}_3(y))$$

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must be concave. This is equivalent to saying

$$\lambda_1 \tilde{P}_1(y) + \lambda_3 \tilde{P}_3(y) \leq \lambda_2 \tilde{P}_2(y)$$

which proved the theorem.

In summary, technically we have shown that under the parallel movement assumption if the swap curves are increasing, the curve must be of concave shaped.
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