Preservation of dynamics in coupled cavity system using second order nonlinearity

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We introduce a harmonic time-dependent second-order nonlinear process in one of the cavities in a coupled-cavity array with qubits and observe its effect in generation and preservation of quantum states. Von Neumann entropy and mutual information of the subsystems are used to interpret and measure entanglement between the subsystems. Using a time dependent second order process we could preserve the dynamics of the system. By treating mutual information as an order parameter for synchronisation, we observed that a similar evolution of population should not be interpreted as synchronisation or entanglement.

Keywords: Coupled cavities, second order nonlinearity, entanglement, entropy

I. INTRODUCTION

A two-level quantum system (qubit [1]) is the simplest model of matter one can think. Jaynes-Cummings model describes the interaction of quantized electromagnetic radiation with a single two-level system in the rotating wave approximation [2]. Work by R.H. Dicke [3] has also made significant impact on the extension of Jaynes-Cummings model to multilevel system. Over the last five decades of research, Jaynes Cummings model has undergone various modifications and has covered different aspects of light-matter physics [4–9].

Quantum computation [10] and information theory [11] has built an exponential demand on the need for quantum systems that can showcase well-defined loophole free quantum signatures. Effective protocols and designs for the same have been reported in many works [12–20]. The difficulties in preserving quantum states, such as the entangled state, has made the topic an active area of research since 1960s. Entanglement generation and preservation being top in the checklist, have been addressed in different cavity architectures [21–34]. These “spooky” states [35, 36] in quantum mechanics have accelerated the development of quantum information and communication [37–40].

Various measures have been proposed to quantify the entanglement between the systems, such as the entropies, concurrence etc. [41–47]. On the other hand features, such as the synchronization between the coupled oscillators [48, 49] and mutual information [50–52] are also being used to estimate correlations between the systems. The motivation to inspect the possibility of defining an entanglement or correlation measure between subsystems can be seen in many works [53–58]. In information theory, mutual information is treated a measure of entanglement [59]. If there is no mutual information between the systems, knowledge about one does not reveal any information about the other, which is not the case with entangled state. As the number of subsystems increases, quantifying the distribution of entanglement between a given pair needs a mutual information measure. Since this cannot reveal the entire information about entanglement, one could also has to consider von Neumann entropy to take into account of entanglement of individual subsystems.

Quantum descriptions of nonlinear processes improved the realisation of versatile quantum systems. The theoretical modelling and experimental realization of such systems has been reported [60]. One such candidate for realizing nonlinear phenomenons are the optomechanical systems. These systems are well studied [61–63], leading to the advancement in experimental quantum optics.

Second order nonlinear interactions are of extreme importance in quantum entanglement [64, 65] and phenomenons such as optical squeezing [66]. Here we study the effects of a degenerate second order nonlinear process (degenerate spontaneous parametric down-conversion [67, 68] on the dynamics of quantum states and its role in generating entanglement between coupled oscillators. Further, the influence of a time-dependent nonlinear coupling is also investigated. We study the emergence of entanglement between the subsystem and its relation with mutual information. We measure both mutual information between subsystems and von Neumann entropy to quantify the amount of entanglement.

II. TWO COUPLED CAVITIES WITH TWO-LEVEL SYSTEM

Initially we consider two coupled optical cavity system with a cavity mode frequency $\omega_c$, and a two-level quantum system (qubit) in both of them. By taking $\hbar = 1$ the system can be described by the free Hamiltonian and the interaction Hamiltonian respectively as,

$$\hat{H}_0 = \sum_{i=1}^{2} \frac{1}{2} \omega_{ci} \sigma_z^{(i)} + \omega_{ci} a_i^{\dagger} a_i$$  \hspace{1cm} (1)
where

\[ H_I = \sum_{i=1}^{2} \lambda_i \left( a_i^\dagger \sigma_-^{(i)} + a_i \sigma_+^{(i)} \right) + J \left( a_1^\dagger a_2 + a_1 a_2^\dagger \right). \] (2)

Here, \( \lambda_i \) corresponds to the coupling between the cavity mode and the qubit in the \( i \) th cavity, \( J \) is the cavity-cavity hopping factor. The operators \( a_i \) (\( a_i^\dagger \)) is the annihilation (creation) operator of the \( i \) th cavity mode, \( \sigma_+^{(i)} \) is the population inversion operator, \( \sigma_-^{(i)} \) (\( \sigma_+^{(i)} \)) is the raising (lowering) operator for the qubit in the \( i \)th cavity (\( i=1, 2 \) for first and second cavity respectively). The dynamics of these coupled cavities have been investigated extensively in the literature. A general state of the form

\[ |\psi(t)\rangle = |\text{qubit}_1, \text{field}_1, \text{qubit}_2, \text{field}_2\rangle, \]

where \( q_s \) and \( f_s \) are the time-dependent coefficients of qubits and fields respectively. At resonance the following coupled dynamical equations can be obtained from the corresponding Schrödinger equation,

\[ i \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle \] (4)

as,

\[ i \frac{\partial q_1(t)}{\partial t} = \lambda f_1(t), \] (5)

\[ i \frac{\partial q_2(t)}{\partial t} = \lambda f_2(t), \] (7)

\[ i \frac{\partial f_1(t)}{\partial t} = \lambda q_1(t) + J f_2(t), \] (6)

\[ i \frac{\partial f_2(t)}{\partial t} = \lambda q_2(t) + J f_1(t). \] (8)

We have taken \( \lambda_1 = \lambda_2 = \lambda \) and \( \omega_{ai} = \omega_{ci} = \omega \). These equations can be solved using Laplace transform method and the Laplace transforms are given in Eqs. 9 to 12.

\[ Q_1(s) = \frac{-iJ\lambda^2 q_2(0) + J\lambda f_2(0)s + i\lambda f_1(0) (\lambda^2 + s^2) - q_1(0)s (J^2 + \lambda^2 + s^2)}{J^2 s^2 + \lambda^4 + 2\lambda^2 s^2 + s^4}, \] (9)

\[ Q_2(s) = \frac{-iJ\lambda^2 q_1(0) + J\lambda f_1(0)s + i\lambda f_2(0) (\lambda^2 + s^2) - q_2(0)s (J^2 + \lambda^2 + s^2)}{J^2 s^2 + \lambda^4 + 2\lambda^2 s^2 + s^4}, \] (11)

\[ F_1(s) = \frac{J\lambda q_2(0)s + \lambda f_2(0)s + i\lambda q_1(0) (\lambda^2 + s^2) - f_1(0)s (J^2 + \lambda^2 + s^2)}{J^2 s^2 + \lambda^4 + 2\lambda^2 s^2 + s^4}, \] (10)

\[ F_2(s) = \frac{J\lambda q_1(0)s + \lambda f_1(0)s + i\lambda q_2(0) (\lambda^2 + s^2) - f_2(0)s (J^2 + \lambda^2 + s^2)}{J^2 s^2 + \lambda^4 + 2\lambda^2 s^2 + s^4}. \] (12)

Now by choosing appropriate initial conditions we can obtain solutions to the Eqs. 5 to 8 by taking the inverse Laplace transform of Eqs. 9 to 12. For instance, with \( |\psi(0)\rangle = |1000\rangle \), we have \( q_1(0) = q_2(0) = f_2(0) = 0 \) and the corresponding time evolution of probabilities becomes,

\[ |q_1(t)|^2 = \left| \frac{(\xi_+^2 + 2\lambda^2) \cosh \left( \frac{t\xi_+}{\sqrt{J^2}} \right) - (\xi_-^2 + 2\lambda^2) \cosh \left( \frac{t\xi_-}{\sqrt{J^2}} \right)}{4(J^4 + 4\lambda^2 J^2)} \right|^2, \] (13)

\[ |q_2(t)|^2 = \left| \frac{\xi_+ \sinh \left( \frac{t\xi_+}{\sqrt{J^2}} \right) - \xi_- \sinh \left( \frac{t\xi_-}{\sqrt{J^2}} \right)}{2(J^2 + 4\lambda^2)} \right|^2, \] (14)

\[ |f_1(t)|^2 = \left| \frac{\lambda^2 \left[ \sinh^2 \left( \frac{t\xi_+}{\sqrt{J^2}} \right) + \sinh^2 \left( \frac{t\xi_-}{\sqrt{J^2}} \right) \right] - \xi_- \xi_+ \sinh \left( \frac{t\xi_+}{\sqrt{J^2}} \right) \sinh \left( \frac{t\xi_-}{\sqrt{J^2}} \right)}{J^2 + 4\lambda^2} \right|^2, \] (15)

\[ |f_2(t)|^2 = \left| \frac{\lambda^2 \left[ \cosh \left( \frac{t\xi_+}{\sqrt{J^2}} \right) - \cosh \left( \frac{t\xi_-}{\sqrt{J^2}} \right) \right]}{J^2 + 4\lambda^2} \right|^2, \] (16)

where \( \xi_{\pm} = \sqrt{-J^2 - 2\lambda^2 \pm \sqrt{J^4 + 4J^2\lambda^2}} \). The evolution of the qubits and fields can be viewed as a quantum state
transfer between the cavities. One could also control the
dynamics by varying the coupling parameters and choice of
initial conditions. It has to be noted that we have not
considered any decay parameters for cavity and qubits.
For the experimental realization one has to account for
the environment by taking the proper Lindblad operators
decay operators).

III. COUPLED CAVITIES WITH TWO
PHOTON PROCESS

We now study the effect of second order process is
introduced by incorporating a χ(2) medium in the first
cavity. This mechanism is used for producing entangled
pairs of photons (known as spontaneous parametric down
conversion, SPDC). Here a ω_b bosonic mode can be
converted into two ω_c cavity modes by means of degenerate
SPDC. The cavities are allowed to interact each other
through photon hopping coupling, J and with the two-
level system with a coupling, λ. Figure 1 illustrates the
coupled cavity system.

![Cavity Diagram](image)

FIG. 1. Two coupled cavities with a two-level quantum sys-
tem in each and a χ(2) nonlinear medium in the first cavity.

The dynamics of this system can be studied by adding an
additional term $\hat{H}_k$ to the previous Hamiltonian as,

$$\hat{H}_k = ik \left[ (a_1^\dagger)^2 b - (a_1^\dagger)^2 b^\dagger \right] + \omega_b b^\dagger b. \quad (17)$$

Here, $k$, ($k = k_0/2$) is the nonlinear coupling between
the cavity mode and the $\omega_1$ mode. The operator $b$ ($b^\dagger$)
is the annihilation (creation) operator of the bosonic mode. For
simplicity, we study the system without any detuning or
decay. By representing $\omega_b$ bosonic mode as field $b$, cavity
excitations as field $a_1$ and qubit excitations as qubit $\chi$, the
general state may be written as,

$$|\psi\rangle = |\text{qubit}_1, \text{field}_1, \text{field}_b, \text{qubit}_2, \text{field}_2\rangle \quad (18)$$

For a maximum of single excitation in $\omega_b$ mode, the gen-
eral state now takes the form,

$$|\psi(t)\rangle = \chi(t)|00100\rangle + a(t)|11000\rangle + b(t)|10010\rangle \quad (19)
+ c(t)|10001\rangle + d(t)|01001\rangle + e(t)|01010\rangle
+ f(t)|00011\rangle + g(t)|02000\rangle + h(t)|00002\rangle .$$

The dynamics can be studied by solving Schrödinger
equation of the system for a given initial state. Since the
analytical solutions are lengthy, we study the dynamics
numerically [69]. We start with a single excited state of
$\omega_b$ frequency which could undergo degenerate SPDC
due to the nonlinear coupling factor $k$, which will result
in two $\omega_c$ photons. It is observed that the population
inversion of qubits $Q_1$ and $Q_2$ goes from $-1$ (ground
state) to 0.0. It means, individual qubits are mostly in
the superposition of ground and excited states. This
superposition indicates a possible entanglement between
the subsystems. From the population inversion there
appears a correlation between the qubits such that
both evolves almost identically. A change in value of
$k$, changes the periodicity and this raise the question of
synchronization between the qubits. To check whether
this is due to SPDC, we repeat the simulation for a
different initial state and $k$.

With $|\psi(0)\rangle = (|00100\rangle + |01001\rangle)/\sqrt{2}$, where there
is an entanglement between the subsystems, the system
does not shows any identical evolution as exhibited in
the first case. This highlights the influence of other coupling
parameters and initial state on the system. Still we could
see that there is an approximate identical behaviour ini-
tially. This leads to a conclusion that, an initial entan-
glement does not guarantee an identical evolution for the
subsystem in the future. The population inversion for
two different initial conditions are shown in Fig. 2. In
order to address the question of synchronization, mutual
information between the subsystems has to be studied
[51], which we shall address in the later section. Since
an uncontrolled evolution doesn’t appears to be much
fruitful, we introduce a time dependency on the coupling
$k$.

IV. TIME DEPENDENT COUPLING

Time dependent coupling schemes are addressed in
cavity systems [70]. Here we focus on the effect of time
dependency on the second order process alone such that
the nonlinear coupling factor, $k$ get modified as $k \rightarrow k(t)$.
Time dependence can be of any form. We choose a sinus-
soidal time variation,

$$k(t) = k_0 \left( \frac{1 + \sin(\Omega t)}{2} \right), \quad (20)$$

such that $k(t) \in [0, k_0]$. Here $\Omega$ can vary from 0 to $\infty$ and
when $\Omega = 0$ we get the constant coupling as, $k = k_0/2$. 
Fig. 2. Population inversion $\langle \sigma_z \rangle$, of qubit 1 (red colour) and qubit 2 (blue colour) for $\omega = 10 \times 2\pi$ GHz, $\lambda = 0.01 \omega$, $J = 0.05 \lambda$, with (a) $k_0 = 0.010 \omega$ and initial state $|\psi(0)\rangle = |00100\rangle$ (b) $k_0 = 0.010 \omega$ and initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|00100\rangle + |01001\rangle)$.

* time scale is in nano seconds.

| Sl. No | $\Omega^a$ GHz | $10\% k_0 \rightarrow 0$ | $k = 0$ | $0 \rightarrow 10\% k_0$ |
|--------|----------------|------------------------|----------|------------------------|
| 1      | 0.000000       | –                      | –        | –                      |
| 2      | 0.002222       | 1961                   | 2120     | 2281                   |
| 3      | 0.004444       | 981                    | 1060     | 1140                   |
| 4      | 0.006667       | 654                    | 707      | 760                    |
| 5      | 0.008889       | 491                    | 530      | 570                    |

* unit of $\Omega$ is GHz throughout
* all time is in nano seconds

Table I. Time required to reduce the value of $k$ to reach 0 from its initial value for different values of $\Omega$.

Table I, gives the time at which $k$ becomes zero for a given $\Omega$ in the first cycle. We look for the signature of the time dependence of $k$ over the interval in which the value of $k$ decreases from 10% of the maximum value of $k$ to zero and further increases to 10% of the maximum value of $k$.

The limit $k \rightarrow 0$ makes the evolution less dependent on the second order process and thus get the dynamics depend more on the other coupling factors. It is noted that, the dynamics of the qubits are preserved for a time interval around which $k \rightarrow 0$. Thus, by adjusting the value of $\Omega$ one can choose when to reduce the value of $k$ to be zero and hence we have a control on the dynamics.

With this, one can preserve the dynamics of the system, even though there is no revival of the identical behaviour exhibited in the initial time. We could utilise this to preserve the dynamics of a particular state. Figures 3(a) to 3(f) illustrate the effects of time dependent $k$ on the population inversion. Here with an initial state $|00100\rangle$ and a time dependent $k$, we get a superposed state which tends to remain in that state while $k$ reduces to its minimum. This should not be interpreted as there is no dynamics. Here the dynamics does not alter the superposition considerably. By overlooking the fluctuations, one can say that the system has created a particular superposition. In Fig. 3(b) the population inversion of $Q1$ and $Q2$ remains zero for a longer period. Which means that, they are in a superposition of both excited and ground state. We could also observe that population inversion remains close to -1 in Figs. 3(a) and 3(c) for different time interval. From Fig. 3(c), we see for a small interval ($\sim 1500$ ns to $2000$ ns) the value of $\langle \sigma_z \rangle$ lies between 0 and -0.5. This opens up the possibility of preserving different superposition.

For $|\psi(0)\rangle = (|00100\rangle + |01001\rangle) / \sqrt{2}$, rather than preserving a superposition, the preservation of dynamics is well observed. For any given initial state, we observe the preservation around the same time intervals. This confirms that preservation depends on the temporal behaviour of $k$. This motivate us to investigate the possibility of generating entangled state. Since, in our system, there are five subsystems, it is possible to have different combinations of the entangled state. So we further investigate the amount of entanglement by studying the entropies of the system.

V. MUTUAL INFORMATION AND VON NEUMANN ENTROPY

Entropy is a measure that can account for the lack of information from the works of C.E. Shannon [71]. The works of von Neumann [72], describes the mathematical motivation in formulating the entropy in quantum mechanics. Here we focus on mutual information (also known as mutual entropy) and von Neumann entropy. Mutual information quantifies the amount of information shared between two systems [73]. Given any two continuous random variables, say $x \in X$ and $y \in Y$, the mutual information can be related to classical Shannon entropy as,
FIG. 3. Population inversion $\langle \sigma_z \rangle$, of qubit 1 (red colour) and qubit 2 (blue colour) for $\omega = 10 \times 2\pi$ GHz, $\lambda = 0.01\omega$, $J = 0.05\lambda$ and $k_0 = 0.01\omega$. (a, b and c) $|\psi(0)\rangle = |00100\rangle$ (d, e and f) $|\psi(0)\rangle = 1/\sqrt{2}(|00100\rangle + |01001\rangle)$.

$$I(X : Y) = \sum_{y \in Y} \sum_{x \in X} p(x, y) \log \left( \frac{p(x, y)}{p(x)p(y)} \right)$$

$$I(X : Y) = H(X) + H(Y) - H(X, Y). \quad (21)$$

Where $p(x)$ and $p(y)$ are the marginal probability and $p(x, y)$ is the joint probability. We can find a relation such that, the Shannon entropy a single random variable $H(X)$ and $H(Y)$ is always less than the joint Shannon entropy ($H(X,Y)$) as,

$$H(X,Y) \geq H(X) \text{ or } H(Y) \quad (22)$$

The general quantum analogue of Shannon entropy is the $\alpha$-entropy [74], from which the quantum analogue for mutual information can be written as,

$$I(m : n) = S_\alpha (\rho_m) + S_\alpha (\rho_n) - S_\alpha (\rho_{mn}). \quad (23)$$

where $\rho$ is the density matrix and $m, n$ are the indices for the subsystems. From which one can deduce the inequality,

$$S_\alpha (\rho_{mn}) \geq S_\alpha (\rho_n) \quad (24)$$

The violation of the above inequality (Eq. 24) indicates the presence of entanglement in the system and here we investigate the same for our system. Since there are five subsystems and the respective density matrices are labelled as,

$$\begin{pmatrix}
Qubit 1 & Field 1 & \chi^{(2)\text{mode}} \\
Qubit 2 & Field 2 & \\
\end{pmatrix} =
\begin{pmatrix}
\rho_{q1} & \rho_{f1} & \rho_0 \\
\rho_{q2} & \rho_{f2} & \rho_0 \\
\end{pmatrix} = \rho_m \quad (25)$$

One can find the reduced density matrix ($\rho_{\text{sub}}$), by taking the partial trace over the rest of the system. Since we are interested in mutual information which takes two subsystems at a time. We must also calculate a partially reduced density matrix, $\rho_{mn}$, where $m$ and $n$ are the subsystems. Then we can estimate the mutual information between subsystem $m$ and $n$. The general expression for $\alpha$-entropy is given as,

$$S_\alpha (\rho_m) = (1 - \alpha)^{-1} \log \text{Tr} (\rho_m^\alpha), \quad (26)$$

Eq. 26 reduces to von Neumann entropy for the limit $\alpha \rightarrow 1$ as,

$$S (\rho_m) = -\text{Tr} (\rho_m \log_2 \rho_m). \quad (27)$$

Thus the mutual information becomes,

$$I(m : n) = S (\rho_m) + S (\rho_n) - S (\rho_{mn}) \quad (28)$$
We can calculate the mutual information and the von Neumann entropy of the system to quantify the entanglement. Since there are five subsystems first we calculate the reduced density matrices of each, \( \rho_m \) and also calculate the reduced density matrix for each pairs, \( \rho_{mn} \). Using which we can compute the von Neumann entropy and mutual information respectively. Here we obtain the total density matrix numerically for each time from the dynamical equation and compute the reduced densities matrices accordingly.

### A. Von Neumann Entropy

Von Neumann entropy is calculated using Eq. 27. In our model we consider only a maximum of one excitation in the 2\( \omega \) mode. Thus the simplest model will need only the following Hilbert space,

\[
\begin{pmatrix}
\text{Qubit-1} \\
\text{Cavity-2} \\
\chi^{(2)}\text{Mode} \\
\text{Qubit-2} \\
\text{Cavity-1}
\end{pmatrix} = \begin{pmatrix}
2 & 2 \\
3 & 2 \\
2 & 3 \\
2 & 3
\end{pmatrix}
\]

We now verify the amount of entanglement produced by measuring the von Neumann entropy. Here we choose \( \Omega = 0.00444 \) and specifically look for the von Neumann entropy during the time period considered in the Table I. The results for \( |\psi(0)\rangle = |00100\rangle \) is tabulated in the Table II.

| \( \rho_m \) | \( \log_2(\text{dim } H_m) \) | \( S_{\text{max}} \) | \( S_{\text{min}} \) | \( \text{Avg. } S \) for \( \Omega = 0.00444 \) |
|-----------|-----------------|--------|--------|------------------|
| Q1        | 1.00000         | 0.99999| 0.00000| 0.99991          |
| C1        | 1.58496         | 1.04188| 0.00000| 1.01221          |
| \chi      | 1.00000         | 0.99999| 0.00000| 0.06086          |
| Q2        | 1.00000         | 0.99999| 0.00000| 0.99999          |
| C2        | 1.58496         | 1.04285| 0.00000| 1.01052          |

\( \rho_m \) - Q-Qubit, C-Cavity, \( \chi \) - Non linear medium  
\( S_{\text{max}} \) - in nano seconds

TABLE II. Von Neumann entropy for \( |\psi(0)\rangle = |00100\rangle \) for \( \Omega = 0.00444 \).

The numerical results shows that there are time when each get maximally entangled with the rest of the system. Since we were looking for the effect of time dependent \( k \) in a particular period in the time range suggested by Table I, we are getting maximum entanglement for the qubits and cavity while the \( \chi^{(2)} \) mode has become more or less pure. The von Neumann entropies for the initial state \( |\psi(0)\rangle = (|00100\rangle + |01001\rangle) / \sqrt{2} \) is tabulated in Table III.

| \( \rho_m \) | \( \log_2(\text{dim } H_m) \) | \( S_{\text{max}} \) | \( S_{\text{min}} \) | \( \text{Avg. } S \) for \( \Omega = 0.00444 \) |
|-----------|-----------------|--------|--------|------------------|
| Q1        | 1.00000         | 0.99999| 0.00000| 0.93277          |
| C1        | 1.58496         | 1.04689| 0.22792| 0.94677          |
| \chi      | 1.00000         | 0.99999| 0.00000| 0.03442          |
| Q2        | 1.00000         | 0.99999| 0.00000| 0.87914          |
| C2        | 1.58496         | 1.04647| 0.15459| 0.89122          |

\( \rho_m \) - Q-Qubit, C-Cavity, \( \chi \) - Non linear medium  
\( S_{\text{max}} \) - in nano seconds

TABLE III. Von Neumann entropy for \( |\psi(0)\rangle = (|00100\rangle + |01001\rangle) / \sqrt{2} \) for \( \Omega = 0.00444 \).

Mutual information also shows a convincing measure of entanglement, since separable system would have zero mutual information. Fig. 5 and 6 gives the mutual information shared between different subsystems for different initial conditions. As we can see, the entanglement is not strictly bipartite, as there are information shared between others. From the simulations we can see that the mutual information do not reach its maximum possible
FIG. 4. The von Neumann entropy for (a) initial state $|\psi(0)\rangle = |00100\rangle$ and (b) initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|00100\rangle + |01001\rangle)$.

FIG. 5. The mutual information for (a) initial state $|\psi(0)\rangle = |00100\rangle$ and (b) initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|00100\rangle + |01001\rangle)$.

value. This means that the entanglement produced in the system is not bipartite. The average mutual information during the period of interest (as given in Table I) is not maximum. This indicates that an identical behaviour of population inversion should not be interpreted as entanglement. In the first case we could observe that the mutual information and von Neumann entropy of the $\chi^{(2)}$ medium is very low during the period when other subsystems has higher von Neumann entropy. This indicates that the $\chi^{(2)}$ mode has almost decoupled with rest of the subsystems. Another factor that can be deduced from the mutual information is about the synchronization. An almost identical population inversion does not implies a synchronization between the subsystems.
FIG. 6. The mutual information for (a) initial state \( |\psi(0)\rangle = |00100\rangle \) and (b) initial state \( |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|00100\rangle + |01001\rangle) \).

| Pairs   | MI max   | MI min   | Avg. MI (t=981→1140) |
|---------|----------|----------|-----------------------|
| Q1-C1   | 1.75965  | 0.00000  | 1.48266               |
| Q1-C2   | 0.22449  | 0.00000  | 0.14768               |
| Q2-C1   | 0.21794  | 0.00000  | 0.14962               |
| Q2-C2   | 1.75162  | 0.00000  | 1.37360               |
| C1-C2   | 0.99999  | 0.00276  | 0.15041               |
| Q1-\chi | 0.81879  | 0.00000  | 0.00371               |
| Q2-\chi | 0.86259  | 0.00000  | 0.00395               |
| C1-\chi | 1.00038  | 0.00000  | 0.00774               |
| C2-\chi | 0.99999  | 0.00000  | 0.00780               |

* time in nano seconds

TABLE V. Mutual information of partially reduced density matrix pairs with \( |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|00100\rangle + |01001\rangle) \).

VI. CONCLUSION

A coupled cavity system with a second order nonlinear medium in one of the cavity is studied. Numerical analysis of the system for different initial state and coupling factors is performed. In order to have a control over the dynamics a harmonic time dependent nonlinear coupling between the \( \chi^{(2)} \) medium and cavity mode is introduced. With various values for the frequency, \( \Omega \) in the harmonic time dependent coupling, we looked for the dynamical behaviour around the region where the value of \( k \) is reduced to zero. Irrespective of the initial state we observed a preservation of dynamics in the system. Depending upon the value of \( \Omega \), different superposition could be preserved for different time interval. To check for the presence of entanglement we calculated the von Neumann entropy of each subsystems and mutual information between pair of subsystems. From the analysis over a given time limit, we observed that the specific subsystems can be made independent of the rest of the system. From which we conclude that the control can be utilized to preserve entanglement and dynamics in the system. We also confirm that the presence of identical behaviour in the population inversion should not be interpreted as synchronization.

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