ABSTRACT

We argue that all Einstein-Maxwell or Einstein-Proca solutions to general relativity may be used to construct a large class of solutions (involving torsion and non-metricity) to theories of non-Riemannian gravitation that have been recently discussed in the literature.
1. Introduction

Non-Riemannian geometries feature in a number of theoretical descriptions of the interactions between fields and gravitation. Since the early pioneering work by Weyl, Cartan, Schrödinger and others such geometries have often provided a succinct and elegant guide towards the search for unification of the forces of nature [1]. In recent times interactions with supergravity have been encoded into torsion fields induced by spinors and dilatonic interactions from low energy effective string theories have been encoded into connections that are not metric-compatible [2], [3], [4], [5]. However theories in which the non-Riemannian geometrical fields are dynamical in the absence of matter are more elusive to interpret. It has been suggested that they may play an important role in certain astrophysical contexts [6]. Part of the difficulty in interpreting such fields is that there is little experimental guidance available for the construction of a viable theory that can compete effectively with general relativity in domains that are currently accessible to observation. In such circumstances one must be guided by the classical solutions admitted by theoretical models that admit dynamical non-Riemannian structures [7], [8], [9], [10], [11]. A number of recent papers have pursued this approach and have found static spherically symmetric solutions to particular models [12], [13]. In [6] it was pointed out that in a particularly simple model all Einstein-Maxwell solutions to general relativity could be used to generate dynamic non-Riemannian geometries and a tentative interpretation was offered for the matter couplings in such a model. Particular solutions have also been found to more complex models, provided the coupling constants in the action are correlated [12], [13], [14], [15]. It is the purpose of this note to point out that, if such correlations are maintained then solutions may be generated from all Einstein-Maxwell solutions of general relativity. Furthermore the correlations may be discarded and solutions generated from all Einstein-Proca solutions of general relativity with or without the inclusion of a cosmological term in the action.

2. Non-Riemannian Geometry

To establish notation (which follows [16]) we briefly recall some basic definitions. A non-Riemannian geometry is specified by a metric tensor field $g$ and a linear connection $\nabla$ on the bundle of linear frames over a manifold. In a local coframe $\{e^a\}$ with dual frame $\{X^b\}$ such that $e^a(X_b) = \delta^a_b$, the connection forms $\omega^a_b$ satisfy $\omega^c_b(X_a) \equiv e^c(\nabla_X X_a)$. The connection is not metric compatible when the tensor field $S = \nabla g$ is non zero. In the following we use local orthonormal frames so $g = \eta_{ab} e^a \otimes e^b$, ($\eta_{ab} = \text{diag}(-1, 1, 1, 1, \ldots)$), non-metricity 1-forms $Q_{ab} \equiv \eta_{ab} e^a \otimes e^b$, torsion 2-forms $T^a \equiv d e^a + \omega^a_b \wedge e^b$ and curvature 2-forms $R^a_b \equiv d \omega^a_b + \omega^a_c \wedge \omega^c_b$. The general curvature scalar $R$ associated with the connection is defined by $R \ast 1 \equiv R^a_b \wedge * (a_b \wedge e^b)$ in terms of the Hodge operator for the metric.

3. Field Equations for Non-Riemannian Gravity

We begin by analysing the theory derived from the action functional

$$\mathcal{A}[e, \omega] = \kappa R \ast 1 + \frac{\alpha}{2} d Q \wedge * d Q + \frac{\beta}{2} Q \wedge * Q + \frac{\gamma}{2} T \wedge * T \quad (1)$$

where $\kappa, \alpha, \beta, \gamma$ are real couplings, $Q = \eta^{ab} Q_{ab}$ and $T = i_c T^c$. (Here and below $i_c \equiv i_{X_c}$ in terms if the contraction operator.) For any form $\mathcal{T}$ we denote its variation induced by a coframe variation $\{e^a\}$ and a connection variation $\{\omega^a_b\}$ by $\mathcal{T}_{\epsilon}$ and $\mathcal{T}_{\omega}$ respectively.
Consider orthonormal coframe induced variations of $\Lambda[e, \omega]$. The first term gives
\[
(R^a_b \wedge *(e_a \wedge e^b))' = \dot{e}^c \wedge R^a_b \wedge *(e_a \wedge e^b). \tag{2}
\]
Since $(dQ) = 0$ and $(Q) = 0$ it follows that
\[
\left(\frac{1}{2} dQ \wedge dQ\right)' = \frac{1}{2} \dot{e}^a \wedge (dQ \wedge i_a * dQ - i_a dQ * dQ) \tag{3}
\]
and
\[
\left(\frac{1}{2} Q \wedge Q\right)' = -\frac{1}{2} \dot{e}^a \wedge (Q \wedge i_a * Q + i_a Q * Q). \tag{4}
\]
The coframe variation of the last term may be computed by noting that $(\dot{T}_a)' = D \dot{e}^a$ and
\[
i_b \dot{e}^a = -i_{\dot{X}_b} e^a. \tag{5}
\]
Thus the variational condition $(\Lambda)' = 0$ mod $d$ yields the field equation:
\[
\kappa D * (e_a \wedge e^b) = 2 \delta^b_a (\alpha d * dQ + \beta * Q) + \gamma e_b \wedge i_a * T. \tag{10}
\]
It is instructive to take the trace of this equation and replace it by the equivalent equations:

\[ \alpha d \ast d Q + \beta \ast Q = \frac{\gamma(1 - n)}{2n} \ast T \] (11)

\[ \kappa D \ast (e_a \wedge e^b) = \delta^b_a \frac{(1 - n)}{n} \gamma \ast T + \gamma e^b \wedge i_a \ast T \] (12)

where \( n \) is the dimension of the manifold.

The variational condition \( (\Lambda) \dot{e} = 0 \mod d \) yields the Einstein field equation:

\[ \kappa R^a_b \wedge \ast(e_a \wedge e^b \wedge e_c) + \tau_c[\alpha] + \tau_c[\beta] + \tau_c[\gamma] = 0 \] (13)

where

\[ \tau_c[\alpha] = \frac{\alpha}{2} (d Q \wedge i_c \ast d Q - i_c d Q \wedge \ast d Q) \] (14)

\[ \tau_c[\beta] = -\frac{\beta}{2} (Q \wedge i_c \ast Q + i_c Q \wedge \ast Q) \] (15)

\[ \tau_c[\gamma] = \gamma \{ i_k(T^k \wedge \ast(T \wedge e_c)) - Di_c \ast T - \frac{1}{2}(T \wedge i_c \ast T + i_c T \wedge \ast T) \}. \] (16)

4. Non-Riemannian Solutions Generated from the Einstein-Maxwell System

We look for solutions to these field equations in which the torsion and non-metricity forms are expressed in terms of general 1-forms \( A_1, A_2 \) and \( A_3 \) on spacetime, according to the ansatz:

\[ Q_{ab} = e_a i_b A_1 + e_b i_a A_1 - \frac{2}{n} \eta_{ab} A_1 - \frac{\eta_{ab}}{n} A_2 \] (17)

\[ T^a = \frac{1}{2n} e^a \wedge A_3. \] (18)

In the following we are concerned with spacetime solutions and set \( n = 4 \). This ansatz ensures that a remarkable cancellation occurs in the field equations and is responsible for the particular solutions that have been discussed in [14], [15]. To describe the nature of the solutions that arise we introduce an action functional of vacuum Einstein-Maxwell type:

\[ \Lambda_{E-M}[g, A] = \kappa \mathcal{R} * 1 + \frac{\alpha}{2} d A \wedge \ast d A \] (19)

where \( F = d A \) is the Maxwell 2-form. The traditional field equations of Einstein-Maxwell type follow from (13) by putting \( \beta = \gamma = 0 \) and setting the forms \( \omega^n_b \) to define the metric compatible torsion free Levi-Civita connection. (The coupling \( \alpha \) is negative for the physical Einstein-Maxwell system.) As a result of a somewhat tedious calculation (which has been checked using the programme Manifolds [17] in the symbolic algebra language Maple) the following result emerges.

Given any solution \((g = g_{E-M}, A = A_{E-M})\) of Einstein-Maxwell type to the field equations generated from the traditional action \( \Lambda_{E-M}[g, A] \) above, then a solution to the field equations (13), (11), (12) generated by the action \( \Lambda[e, \omega] \) is given by \( g = g_{E-M} \) together with the ansatz (17), (18) where the 1-forms \( A_1, A_2, A_3 \) are given by

\[ A_1 = -\frac{2\beta}{3\kappa} A_{E-M} \] (20)
provided the couplings in the action satisfy

\[ 12\beta \gamma - 9\gamma \kappa - 64\beta \kappa = 0 \]  

(23)

Thus a result of [14] arises as a particular solution generated from the class of all vacuum Einstein-Maxwell solutions. Furthermore we next show that the constraint on the couplings in the action can be removed, if necessary, by generating solutions from the class of all vacuum Einstein-Proca solutions.

5. Extended Action and Field Equations

In this section we also broaden the theory by considering the action functional:

\[ \Lambda_1[e, \omega] = \Lambda[e, \omega] + \frac{1}{2} \xi T_c \wedge \ast T^c \]  

(24)

where \( \xi \) is an arbitrary real coupling. The variations of the extra term follow from

\[ \frac{1}{2} \left( T_c \wedge \ast T^c \right)_e = (T^c)_e \wedge \ast T_c + \frac{1}{2} \dot{e}^a \{ T_c \wedge i_a \ast T^c - i_a T_c \wedge \ast T^c \} \]

or

\[ \frac{1}{2} \left( T_c \wedge \ast T^c \right)_e = \dot{e}^c \wedge (D \ast T_c + \frac{1}{2} \{ T_c \wedge i_a \ast T^c - i_a T_c \wedge \ast T^c \}) \]  

(25)

and

\[ \frac{1}{2} \left( T_c \wedge \ast T^c \right)_\omega = (T^c)_\omega \wedge \ast T_c = \omega \dot{e}^b \wedge e^b \wedge \ast T_c. \]  

(26)

Thus \((A_1)_e = 0 \mod d\) yields

\[ \kappa R^a_{b \wedge \ast (e_a \wedge e^b \wedge e_c)} + \tau_c[\alpha] + \tau_c[\beta] + \tau_c[\gamma] + \tau_c[\xi] = 0 \]  

(27)

where

\[ \tau_c[\xi] = \xi (D \ast T_c + \frac{1}{2} \{ T_c \wedge i_a \ast T^c - i_a T_c \wedge \ast T^c \}) \]  

(28)

while \((A_1)_\omega = 0 \mod d\) gives

\[ \kappa D \ast (e_a \wedge e^b) = 2\delta_a^b (\alpha d \ast d Q + \beta \ast Q) + \gamma e^b \wedge i_a \ast T - \xi e^b \wedge \ast T_a \]  

(29)

or equivalently:

\[ \alpha d \ast d Q + \beta \ast Q = \frac{\gamma(1 - n)}{2n} \ast T + \frac{\xi}{2n} e^c \wedge \ast T_c \]  

(30)

\[ \kappa D \ast (e_a \wedge e^b) = \delta_a^b \frac{(1 - n)}{n} \gamma \ast T + \delta_a^b \frac{\xi}{n} e^c \wedge \ast T_c + \gamma e^b \wedge i_a \ast T - \xi e^b \wedge \ast T_a. \]  

(31)
6. Non-Riemannian Solutions Generated from the Einstein-Proca System

To describe the nature of the solutions that arise to these equations with the ansatz above we introduce an action functional of vacuum Einstein-Proca type:

\[ \Lambda[g, A] = \kappa R * 1 + \frac{\alpha}{2} d A \wedge * d A + \frac{\beta - \beta_0}{2} A \wedge * A \] (32)

where \( A \) is the Proca 1-form. The field equations of Einstein-Proca type follow from (13) by replacing \( \beta \) by \( \beta - \beta_0 \) and setting \( \gamma = 0 \) and the forms \( \omega^a{}_b \) to the metric compatible torsion free Levi-Civita ones. (The constant \( \beta - \beta_0 \) is positive for the physical Einstein-Proca system describing a massive vector field coupled to Einsteinian gravity.) Given any solution \( (g = g_{E-P}, A = A_{E-P}) \) of Einstein-Proca type to the field equations generated from the traditional action functional \( \Lambda[g, A] \) above, then a solution to the field equations (27), (30), (31) generated by the action functional \( \Lambda_1[e, \omega] \) is given by \( g = g_{E-M} \) together with the ansatz (17), (18) where the 1-forms \( A_1, A_2, A_3 \) are given by

\[ A_1 = -\frac{2\beta}{3\kappa} A_{E-P} \] (33)
\[ A_2 = A_{E-P} \] (34)
\[ A_3 = -\frac{64\beta}{(9\gamma + 3\xi)} A_{E-P} \] (35)

provided the parameter \( \beta_0 \) determining the mass term in the Einstein-Proca solution is chosen to be

\[ \beta_0 = \frac{3\kappa (\xi + 3\gamma)}{4(\xi + 3\gamma - 16\kappa)}. \] (36)

Thus by choosing any Einstein-Proca solution associated with the appropriate “mass parameter” one can generate a solution to the above theory without constraining any of the coupling constants \( \kappa, \alpha, \beta, \gamma, \xi \).

7. Generalisations

We have briefly considered further generalisations. Firstly if the ansatz (18) is replaced by

\[ T^a = \frac{1}{2n} e^a \wedge A_3 + \frac{1}{2n} * (e^a \wedge A_4) \] (37)

where \( A_4 \) is a 1-form on spacetime, then the above result provides a class of solutions with arbitrary \( A_4 \) but constrained couplings. Thus if any Einstein-Proca solution \( (g_{E-P}, A_{E-P}) \) is used to define the 1-forms \( A_1, A_2, A_3 \) and \( g \) as above then a solution to the theory generated by \( A_1[e, \omega] \) can be generated provided the couplings in the action satisfy the constraint:

\[ \xi = \kappa \] (38)

in addition to (23). In such a solution the 1-form \( A_4 \) remains arbitrary. We note further that one may also include a cosmological constant in all the actions above and generate solutions from either Einstein-Maxwell or Einstein-Proca solutions in the presence of a cosmological constant.
While this work was in progress we became aware of the papers [14] and [15] in which particular solutions to a class of more general actions are presented. The authors study the theory based on the action functional

\[ A_2 = \frac{1}{2\kappa} \left[ a_0 R^{ab} \wedge *(e_a \wedge e_b) - 2\lambda * 1 + a_2 T^{(2)}_a \wedge * T^a \right. \]

\[ -2 \left( c_3 Q^{(3)}_{ab} + c_4 Q^{(4)}_{ab} \right) \wedge e^a \wedge * T^b + \left( b_3 Q^{(3)}_{ab} + b_4 Q^{(4)}_{ab} \right) \wedge * Q^{ab} \]

\[ - \frac{z_4}{32} dQ \wedge * dQ \]

where \( \{ \kappa, a_0, a_2, b_3, b_4, c_3, c_4, z_4 \} \) are couplings and

\[ T^{(2)}_a \equiv \frac{1}{3} e^a \wedge T \]

\[ Q^{(3)}_{ab} \equiv \frac{4}{9} \left( e_{(a \ b_j)} Q - \frac{1}{4} g_{ab} Q \right) \]

\[ Q^{(4)}_{ab} \equiv \frac{1}{4} g_{ab} Q \]

\[ Q \equiv e^a \hat{e}^b \hat{Q}_{ab} \]

\[ \hat{Q}_{ab} \equiv Q_{ab} - \frac{1}{4} g_{ab} Q. \]

We have verified that in accordance with our results the ansatz (17) and (18) enables one to generate solutions to this theory from an Einstein-Proca type system with unconstrained cosmological constant \( \lambda \) and arbitrary couplings \( \{ \kappa, a_0, a_2, b_3, b_4, c_3, c_4, z_4 \} \). Furthermore such an ansatz also enables one to find solutions by solving an Einstein-Maxwell type system (with or without \( \lambda \)) if these couplings satisfy the following relation:

\[ 12 a_0 a_2 b_3 + 24 a_0^2 c_4 + 6 a_0^2 a_2 + 48 a_0 b_4 c_3 + 64 a_0 b_3 b_4 - 32 a_2 b_3 b_4 + 3 a_0 c_4^2 + 18 a_0 c_3 c_4 - 9 a_0 c_3^2 - 4 a_0 a_2 b_4 + 48 a_0 b_3 c_4 + 24 b_4 c_3^2 + 24 b_3 c_4^2 + 32 a_0^2 b_4 = 0. \]

8. Conclusions

We have argued that with the aid of the ansatz (17) and (18) one may construct solutions with non-Riemannian fields to (11), (12), (13) from all solutions of Einstein-Maxwell type in general relativity provided the couplings in the action described by (1) are constrained according to (23). This constraint can be relaxed and the action generalised to (24) provided solutions are then constructed from any solution of Einstein-Proca type in general relativity. In both cases one may generalise all field equations by adding an arbitrary cosmological constant. These general results also apply to the actions discussed recently in [14] and [15] where particular solutions have been recognised. We have also noted that there exists the more general ansatz (37) which generates solutions containing an arbitrary 1-form from this construction, albeit with the further coupling restriction (38).

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REFERENCES

1. H Weyl, Geometrie und Physik, Naturwissenschaften 19 (1931) 49
2. J Scherk, J H Schwarz, Phys. Letts 52B (1974) 347
3. T Dereli, R W Tucker, An Einstein-Hilbert Action for Axi-Dilaton Gravity in 4-Dimensions, Lett. Class. Q. Grav. 12 (1995) L31
4. T Dereli, M Önder, R W Tucker, Solutions for Neutral Axi-Dilaton Gravity in 4-Dimensions, Lett. Class. Q. Grav. 12 (1995) L25
5. T Dereli, R W Tucker, Class. Q. Grav. 11 (1994) 2575
6. R W Tucker, C Wang, Class. Quan. Grav. 12 (1995) 2587
7. F W Hehl, J D McCrea, E W Mielke, Y Ne’eman: “Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance”. Physics Reports, 258 (1995) 1.
8. F W Hehl, E Lord, L L Smalley, Gen. Rel. Grav. 13 (1981) 1037
9. P Baekler, F W Hehl, E W Mielke “Non-Metricity and Torsion” in Proc. of 4th Marcel Grossman Meeting on General Relativity, Part A, Ed. R Ruffini (North Holland 1986) 277
10. V N Ponomariev, Y Obukhov, Gen. Rel. Grav. 14 (1982) 309
11. J D McCrea, Clas. Q. Grav. 9 (1992) 553
12. R Tresguerres, Z. für Physik C 65 (1995) 347
13. R Tresguerres, Phys.Lett. A200 65 (1995) 405
14. Y Obukhov, E J Vlachynsky, W Esser, R Tresguerres, F W Hehl. An exact solution of the metric-affine gauge theory with dilation, shear and spin charges, gr-qc 9604027 (1996)
15. E J Vlachynsky, R Tresguerres, Y Obukhov, F W Hehl. An axially symmetric solution of metric-affine gravity, gr-qc 9604035 (1996)
16. I M Benn, R W Tucker, An Introduction to Spinors and Geometry with Applications in Physics, (Adam Hilger) (1987)
17. R W Tucker, C Wang, Manifolds: A Maple Package for Differential Geometry (1996)