REMARK ON SERRE $C^\ast$-ALGEBRAS

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Abstract. We study non-commutative algebraic geometry of Artin, Serre and Tate in terms of the operator algebras. Namely, we define the Serre $C^\ast$-algebra $A_X$ of a projective variety $X$ as the norm-closure of a representation of the twisted homogeneous coordinate ring of $X$ by the linear operators on a Hilbert space $H$. It is proved that $X$ is homeomorphic to the space of all irreducible representations of the crossed product of $A_X$ by an automorphism of $A_X$. The case of rational elliptic curves $X$ is considered in detail.

1. Introduction

Since the rings of polynomials are commutative, it is recognized that algebraic geometry must be based on commutative algebra. Yet in 1950’s Serre [9] proved that some truly non-commutative rings $B$ satisfy an analog of the fundamental duality between polynomial rings and varieties:

$$\text{Coh}(X) \cong \text{Mod}(B) / \text{Tors},$$

where $X$ is a projective variety, $\text{Coh}(X)$ a category of the quasi-coherent sheaves on $X$, $\text{Mod}(B)$ a category of the finitely generated graded modules over $B$ and $\text{Tors}$ is the torsion [Stafford & van den Bergh 2001] [12, p. 172]. The rings $B$ satisfying (1.1) are called twisted homogeneous coordinate rings, see Section 2 for an exact definition. For the sake of clarity, consider the following analogy. If $X$ is a compact Hausdorff space and $C(X)$ the commutative algebra of continuous functions from $X$ to $\mathbb{C}$ then by the Gelfand Duality the topology of $X$ is determined by the algebra $C(X)$. In terms of the K-theory this can be written as $K^\text{top}_0(X) \cong K^\text{alg}_0(C(X))$. Taking the two-by-two matrices with entries in $C(X)$, one gets an algebra $C(X) \otimes M_2(\mathbb{C})$; in view of stability of the K-theory under tensor products, it holds $K^\text{top}_0(X) \cong K^\text{alg}_0(C(X)) \cong K^\text{alg}_0(C(X) \otimes M_2(\mathbb{C}))$. In other words, the topology of $X$ is defined by algebra $C(X) \otimes M_2(\mathbb{C})$, which is no longer a commutative algebra. In algebraic geometry, one replaces $X$ by a projective variety, $C(X)$ by its coordinate ring, $C(X) \otimes M_2(\mathbb{C})$ by a twisted homogeneous coordinate ring of $X$ and $K^\text{top}_0(X)$ by a category of the quasi-coherent sheaves on $X$. The simplest concrete example of $B$ is as follows.

Example 1.1. ([12, p. 173]) Let $k$ be a field and $U_\infty(k)$ the algebra of polynomials over $k$ in two non-commuting variables $x_1$ and $x_2$, and a quadratic relation $x_1^2 - x_2 x_1 - x_2^2 = 0$; let $\mathbb{P}^1(k)$ be the projective line over $k$. Then $B = U_\infty(k)$ and $X = \mathbb{P}^1(k)$ satisfy (1.1).

In general, there exists a canonical non-commutative ring $B$, attached to the projective variety $X$ and an automorphism $\alpha : X \rightarrow X$; we refer the reader to...
[Stafford & van den Bergh 2001] [12, pp.180-182]. To give an idea, let $X = \text{Spec} (R)$ for a commutative graded ring $R$. One considers the ring $B := R[t, t^{-1}; \alpha]$ of skew Laurent polynomials defined by the commutation relation

$$b^\alpha t = tb,$$

(1.2)

for all $b \in R$, where $b^\alpha \in R$ is the image of $b$ under automorphism $\alpha$; then $B$ satisfies equation (1.1), see lemmas 2.2 and 3.1. The ring $B$ is non-commutative, unless $\alpha$ is the trivial automorphism of $X$.

**Example 1.2.** The ring $B = U_\infty (k)$ in Example 1.1 corresponds to the automorphism $\alpha(u) = u + 1$ of the projective line $\mathbb{P}^1 (k)$. Indeed, $u = x_2 x_1^{-1} = x_1^{-1} x_2$ and, therefore, $\alpha$ maps $x_2$ to $x_1 + x_2$; if one substitutes in (1.2) $t = x_1, b = x_2$ and $b^\alpha = x_1 + x_2$, then one gets the defining relation $x_1 x_2 - x_2 x_1 - x_1^2 = 0$ for the algebra $U_\infty (k)$.

In what follows, we consider infinite-dimensional representations of $B$ by bounded linear operators on a Hilbert space $\mathcal{H}$. The idea goes back to Sklyanin, who asked about such representations for the twisted homogeneous coordinate ring of an elliptic curve, see remark in brackets to the last paragraph of [Sklyanin 1982] [11, Section 3].

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a ring of skew Laurent polynomials $R[t, t^{-1}; \alpha]$ described by formula (1.2), we shall consider a homomorphism

$$\rho : R[t, t^{-1}; \alpha] \rightarrow \mathcal{B}(\mathcal{H}).$$

(1.3)

Recall that algebra $\mathcal{B}(\mathcal{H})$ is endowed with a $*$-involution; such an involution is the adjoint with respect to the scalar product on the Hilbert space $\mathcal{H}$.

**Definition 1.3.** We shall call representation (1.3) $*$-coherent if:

(i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(t) = \rho(t^{-1})$;

(ii) for all $b \in R$ it holds $\rho^* (b)^\alpha (\rho) = \rho^* (b^\alpha)$, where $\alpha(\rho)$ is an automorphism of $\rho(R)$ induced by $\alpha$.

**Example 1.4.** The ring $U_\infty (k)$ in examples 1.1 and 1.2 has no $*$-coherent representations. Indeed, involution acts on the generators of $U_\infty (k)$ by formula $x_1^* = x_2$; the latter does not preserve the defining relation $x_1 x_2 - x_2 x_1 - x_1^2 = 0$.

Whenever $B = R[t, t^{-1}; \alpha]$ admits a $*$-coherent representation, $\rho(B)$ is a $*$-algebra; the norm-closure of $\rho(B)$ yields a $C^*$-algebra [7, Chapter 3]. We shall refer to such as the *Serre $C^*$-algebra* of the projective variety $X$; it will be denoted by $\mathcal{A}_X$. In this case $\alpha$ induces the Frobenius map, see remark 4.2.

**Example 1.5.** When $X = \mathcal{E}(\mathbb{C})$ a non-singular elliptic curve over the complex numbers, then $B = R[t, t^{-1}; \alpha]$ is the Sklyanin algebra [11]. There exists a $*$-coherent representation of $B$; the resulting Serre $C^*$-algebra $\mathcal{A}_X \cong \mathcal{A}_\theta$, where $\mathcal{A}_\theta$ is the noncommutative torus [7, Section 1.1].

Recall that if $B$ is commutative, then $X \cong \text{Spec} (B)$, where $\text{Spec} (B)$ is the set of all prime ideals of $B$. An objective of our paper is to find a similar formula (if any) for the Serre $C^*$-algebra $\mathcal{A}_X$.

To get the formula, consider continuous homomorphism $\alpha : G \rightarrow \text{Aut} (\mathcal{A})$, where $G$ is a locally compact group, $\mathcal{A}$ is a $C^*$-algebra and $\text{Aut} (\mathcal{A})$ the group of
automorphisms of $A$. The triple $(A, G, \alpha)$ defines a crossed product $C^*$-algebra denoted by $A \rtimes G$; we refer the reader to [Williams 2007] [14, pp. 47-54] for the details. Let $G = \mathbb{Z}$ and let $\hat{\mathbb{Z}} \cong S^1$ be its Pontryagin dual. We shall write $\text{Irred } A$ for the space of all irreducible representations of $A$ endowed with the hull-kernel (Jacobson) topology. Our main result can be stated as follows.

**Theorem 1.6.** The spaces $\text{Irred } (A \rtimes \hat{\alpha} \hat{\mathbb{Z}})$ and $X$ are homeomorphic, where $\hat{\alpha}$ is an automorphism of the Serre $C^*$-algebra $A \rtimes G$.

**Remark 1.7.** Since $A \rtimes G$ is usually a simple $C^*$-algebra, its spectrum $\text{Spec } (A \rtimes G)$ is trivial. Thus $X \cong \text{Irred } (A \rtimes \hat{\alpha} \hat{\mathbb{Z}})$ can be viewed as a non-commutative analog of the well-known isomorphism $X \cong \text{Spec } (B)$.

The article is organized as follows. The preliminary facts are reviewed in Section 2. Theorem 1.6 is proved in Section 3. We consider an application of theorem 1.6 in Section 4.

## 2. Twisted homogeneous coordinate rings

Let $X$ be a projective scheme over a field $k$, and let $L$ be the invertible sheaf $\mathcal{O}_X(1)$ of linear forms on $X$. Recall that the homogeneous coordinate ring of $X$ is a graded $k$-algebra, which is isomorphic to the algebra

$$ B(X, L) = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n}). \quad (2.1) $$

Denote by $\text{Coh}$ the category of quasi-coherent sheaves on a scheme $X$ and by $\text{Mod}$ the category of graded left modules over a graded ring $B$. If $M = \bigoplus M_n$ and $M_n = 0$ for $n >> 0$, then the graded module $M$ is called right bounded. The direct limit $M = \lim M_{\alpha}$ is called a torsion, if each $M_{\alpha}$ is a right bounded graded module. Denote by $\text{Tors}$ the full subcategory of $\text{Mod}$ of the torsion modules. The following result is basic about the graded ring $B = B(X, L)$.

**Lemma 2.1.** ([Serre 1955] [9]) $\text{Mod } (B) / \text{Tors} \cong \text{Coh } (X)$.

Let $\alpha$ be an automorphism of $X$. The pullback of sheaf $\mathcal{L}$ along $\alpha$ will be denoted by $\mathcal{L}^{\alpha}$, i.e. $\mathcal{L}^{\alpha}(U) := \mathcal{L}(\alpha U)$ for every $U \subset X$. We shall set

$$ B(X, \mathcal{L}, \alpha) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L} \otimes \mathcal{L}^{\alpha} \otimes \cdots \otimes \mathcal{L}^{\alpha^n}). \quad (2.2) $$

The multiplication of sections is defined by the rule

$$ ab = a \otimes b^{\alpha^n}, \quad (2.3) $$

whenever $a \in B_m$ and $b \in B_n$.

Given a pair $(X, \alpha)$ consisting of a Noetherian scheme $X$ and an automorphism $\alpha$ of $X$, an invertible sheaf $\mathcal{L}$ on $X$ is called $\alpha$-ample, if for every coherent sheaf $\mathcal{F}$ on $X$, the cohomology group $H^q(X, \mathcal{L} \otimes \mathcal{L}^{\alpha} \otimes \cdots \otimes \mathcal{L}^{\alpha^n} \otimes \mathcal{F})$ vanishes for $q > 0$ and $n >> 0$. Notice, that if $\alpha$ is trivial, this definition is equivalent to the usual definition of ample invertible sheaf [Serre 1955] [9]. A non-commutative generalization of the Serre theorem is as follows.
Lemma 2.2. ([Artin & van den Bergh 1990] [1]) Let $\alpha : X \to X$ be an automorphism of a projective scheme $X$ over $k$ and let $\mathcal{L}$ be a $\alpha$-ample invertible sheaf on $X$. If $B(X, \mathcal{L}, \alpha)$ is the ring (2.2), then
\[ \text{Mod} \ (B(X, \mathcal{L}, \alpha)) / \text{Tors} \cong \text{Coh} \ (X). \] (2.4)

Remark 2.3. The necessary and sufficient conditions for an invertible sheaf to be $\alpha$-ample have been established by [Keeler 2000] [6].

3. Proof of theorem 1.6

We shall split the proof in a series of lemmas, starting with the following

Lemma 3.1. $B(X, \mathcal{L}, \alpha) \cong R[t, t^{-1}; \alpha]$, where $X = \text{Spec} \ (R)$ of the commutative ring $R$.

Proof. Let us write the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \alpha)$ of projective variety $X$ in the following form:
\[ B(X, \mathcal{L}, \alpha) = \bigoplus_{n \geq 0} H^0(X, \mathcal{B}_n), \] (3.1)
where $\mathcal{B}_n = \mathcal{L} \otimes \mathcal{L}^\alpha \otimes \cdots \otimes \mathcal{L}^{\alpha^n}$ and $H^0(X, \mathcal{B}_n)$ is the zero sheaf cohomology of $X$, i.e. the space of sections $\Gamma(X, \mathcal{B}_n)$; compare with formula (3.5) of [Artin & van den Bergh 1990] [1].

If one denotes by $\mathcal{O}$ the structure sheaf of $X$, then
\[ \mathcal{B}_n = \mathcal{O} t^n \] (3.2)
can be interpreted as a free left $\mathcal{O}$-module of rank one with basis $\{t^n\}$ [Artin & van den Bergh 1990] [1], p. 252.

Recall, that spaces $B_i = H^0(X, \mathcal{B}_i)$ have been endowed with the multiplication rule (2.3) between the sections $a \in B_m$ and $b \in B_n$; such a rule translates into the formula:
\[ at^m b^n = ab^{\circ^m} t^{m+n}. \] (3.3)

One can eliminate $a$ and $t^n$ in the both sides of (3.3); this operation gives us the following equation:
\[ t^m b = b^{\circ^m} t^m. \] (3.4)

First notice, that our ring $B(X, \mathcal{L}, \alpha)$ contains a commutative subring $R$, such that $\text{Spec} \ (R) = X$. Indeed, let $m = 0$ in formula (3.4); then $b = b^{\text{Id}}$ and, thus, $\alpha = \text{Id}$. We conclude therefore, that $R = B_0$ is a commutative subring of $B(X, \mathcal{L}, \alpha)$, and $\text{Spec} \ (R) = X$.

Let us show that equations (1.2) and (3.4) are equivalent. First, let us show that (1.2) implies (3.4). Indeed, equation (1.2) can be written as $b^{\circ} = tbt^{-1}$. Then:
\[
\begin{align*}
\alpha^2 &= tb^{t-1} = t^2bt^{-2}, \\
\alpha^3 &= tb^{2t-1} = t^3bt^{-3}, \\
&\vdots \\
\alpha^m &= tb^{(m-1)t-1} = t^mbt^{-m}.
\end{align*}
\] (3.5)

The last line of (3.5) is equivalent to equation (3.4). The converse is evident; one sets $m = 1$ in (3.4) and obtains equation (1.2). Thus, (1.2) and (3.4) are equivalent equations.
It is easy now to establish an isomorphism $B(X,\mathcal{L},\alpha) \cong R[t,t^{-1};\alpha]$. For that, take $b \in R \subset B(X,\mathcal{L},\alpha)$; then $B(X,\mathcal{L},\alpha)$ coincides with the ring of the skew Laurent polynomials $R[t,t^{-1};\alpha]$, since the commutation relation (1.2) is equivalent to equation (3.4). Lemma 3.1 follows.

Remark 3.2. Lemma 3.1 is proved for an affine variety. For such varieties the theory of twisted homogeneous coordinate rings can be extended by replacing an $\mathbb{N}$-grading by the $\mathbb{Z}$-grading.

Lemma 3.3. $A_X \cong C(X) \rtimes_\alpha \mathbb{Z}$, where $C(X)$ is the $C^*$-algebra of all continuous complex-valued functions on $X$ and $\alpha$ is a $*$-coherent automorphism of $X$.

Proof. By definition of the Serre algebra $A_X$, the ring of skew Laurent polynomials $R[t,t^{-1};\alpha]$ is dense in $A_X$; roughly speaking, one has to show that this property defines a crossed product structure on $A_X$. We shall proceed in the following steps.

(i) Recall that $R[t,t^{-1};\alpha]$ consists of the finite sums

$$\sum b_k t^k, \quad b_k \in R,$$

subject to the commutation relation

$$b^\alpha_k t = tb_k.$$  (3.7)

Thanks to a $*$-coherent representation, there is also an involution on $R[t,t^{-1};\alpha]$, subject to the following rules:

\begin{align*}
(i) \quad & t^* = t^{-1}, \\
(ii) \quad & (b^\alpha_k)^* = (b^\alpha_k)^*.
\end{align*}  (3.8)

(ii) Following Williams 2007 [14, p.47], we shall consider the set $C_c(\mathbb{Z}, R)$ of continuous functions from $\mathbb{Z}$ to $R$ having a compact support; then the formal sums (3.6) can be viewed as elements of $C_c(\mathbb{Z}, R)$ via the identification

$$k \mapsto b_k.$$  (3.9)

It can be verified that multiplication operation of the formal sums (3.6) translates into a convolution product of functions $f, g \in C_c(\mathbb{Z}, R)$ given by the formula:

$$(fg)(k) = \sum_{l \in \mathbb{Z}} f(l)t^l g(k-l)t^{-l},$$  (3.10)

while involution (3.8) translates into an involution on $C_c(\mathbb{Z}, R)$ given by the formula:

$$f^*(k) = t^k f^*(-k)t^{-k}.$$  (3.11)

The multiplication given by convolution product (3.10) and involution (3.11) turn $C_c(\mathbb{Z}, R)$ into a $*$-algebra, which is isomorphic to the algebra $R[t,t^{-1};\alpha]$.

(iii) There exists the standard construction of a norm on $C_c(\mathbb{Z}, R)$; we omit it here referring the reader to Williams 2007 [14], Section 2.3. The completion of $C_c(\mathbb{Z}, R)$ in that norm defines a crossed product $C^*$-algebra $R \rtimes_\alpha \mathbb{Z}$ [Williams 2007 [14], Lemma 2.27.

(iv) Since $R$ is a commutative $C^*$-algebra and $X = \text{Spec} \, (R)$, one concludes that $R \cong C(X)$. Thus, one obtains $A_X = C(X) \rtimes_\alpha \mathbb{Z}$.

Lemma 3.3 follows.
Remark 3.4. It is easy to see, that (3.7) and (3.8i) imply (3.8ii); in other words, if involution does not commute with automorphism $\alpha$, representation $\rho$ cannot be unitary, i.e. $\rho^*(t) \neq \rho(t^{-1})$.

Lemma 3.5. There exists $\hat{\alpha} \in \text{Aut}(A_X)$, such that:

$$X \cong \text{Irred}(A_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}).$$

(3.12)

Proof. Formula (3.12) is an implication of the Takai duality for the crossed products [Williams 2007] [14, Section 7.1]; let us briefly review this construction.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system with $G$ locally compact abelian group; let $\hat{G}$ be the dual of $G$. For each $\gamma \in \hat{G}$, one can define a map $\hat{\alpha}_\gamma : C_{c}(G, A) \to C_{c}(G, A)$ given by the formula:

$$\hat{\alpha}_\gamma(f)(s) = \gamma(s)f(s), \quad \forall s \in G.$$  

(3.13)

In fact, $\hat{\alpha}_\gamma$ is a $*$-homomorphism, since it respects the convolution product and involution on $C_{c}(G, A)$ [Williams 2007] [14]. Because the crossed product $A \rtimes_{\alpha} G$ is the closure of $C_{c}(G, A)$, one gets an extension of $\hat{\alpha}_\gamma$ to an element of $\text{Aut}(A \rtimes_{\alpha} G)$ and, therefore, a homomorphism:

$$\hat{\alpha} : \hat{G} \to \text{Aut}(A \rtimes_{\alpha} G).$$

(3.14)

Recall that the Takai duality asserts that:

$$(A \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes K(L^2(G)),$$

(3.15)

where $K(L^2(G))$ is the algebra of compact operators on the Hilbert space $L^2(G)$.

Let us substitute $A = C_0(X)$ and $G = \mathbb{Z}$ in (3.15); one gets the following isomorphism:

$$(C_0(X) \rtimes_{\alpha} \mathbb{Z}) \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \cong C_0(X) \otimes K(L^2(\mathbb{Z})).$$

(3.16)

Lemma 3.3 says that $C_0(X) \rtimes_{\alpha} \mathbb{Z} \cong A_X$; therefore one arrives at the following isomorphism:

$$A_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \cong C_0(X) \otimes K(L^2(\mathbb{Z})).$$

(3.17)

Consider the set of all irreducible representations of the $C^*$-algebras in (3.17); then one gets the following equality of representations:

$$\text{Irred}(A_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) = \text{Irred}(C_0(X) \otimes K(L^2(\mathbb{Z}))).$$

(3.18)

Let $\pi$ be a representation of the tensor product $C_0(X) \otimes K(L^2(\mathbb{Z}))$ on the Hilbert space $\mathcal{H} \otimes L^2(\mathbb{Z})$; then $\pi = \varphi \otimes \psi$, where $\varphi : C_0(X) \to B(\mathcal{H})$ and $\psi : K \to B(L^2(\mathbb{Z}))$. It is known, that the only irreducible representation of the algebra of compact operators is the identity representation. Thus, one gets:

$$\text{Irred}(C_0(X) \otimes K(L^2(\mathbb{Z}))) = \text{Irred}(C_0(X)) \otimes \{pt\} = \text{Irred}(C_0(X)).$$

(3.19)

Further, the $C^*$-algebra $C_0(X)$ is commutative, hence the following equations are true:

$$\text{Irred}(C_0(X)) = \text{Spec}(C_0(X)) = X.$$  

(3.20)

Putting together (3.18) – (3.20), one obtains:

$$\text{Irred}(A_X \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}) \cong X.$$  

(3.21)

The conclusion of lemma 3.5 follows from (3.21). □

Theorem 1.6 follows from lemma 3.5.
4. Example

Recall that noncommutative torus is the universal \( C^* \)-algebra \( A_\theta \) generated by unitaries \( u \) and \( v \) satisfying the commutation relation \( vu = e^{2\pi i \theta} uv \), where \( \theta \) is an irrational constant [7, Section 1.1]. The K-theory of \( A_\theta \) is Bott periodic with \( K_0 (A_\theta) = K_1 (A_\theta) \cong \mathbb{Z}^2 \); the range of trace on projections of \( A_\theta \otimes K \) is a subset \( \Lambda = \mathbb{Z} + \mathbb{Z} \theta \) of the real line called a pseudo-lattice. The torus \( A_\theta \) is said to have real multiplication, if \( \theta \) is a quadratic irrationality; in this case the endomorphism ring of pseudo-lattice \( \Lambda \) is bigger than \( \mathbb{Z} \) – hence the name. The noncommutative torus with real multiplication will be written as \( A_{RM} \). We denote by \( (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}) \) be the minimal period of continued fraction of the irrational quadratic \( \theta \). Consider the matrix

\[
A := \begin{pmatrix}
a_1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
a_2 & 1 \\
1 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
a_n & 1 \\
1 & 0
\end{pmatrix};
\]

(4.1)

the matrix is an invariant of torus \( A_{RM} \). A specialization of theorem 1.6 to \( A_{RM} \) gives us the following result.

**Proposition 4.1.** \( \text{Irred} \left( A_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \right) \cong \mathcal{E}(K) \), where \( \mathcal{E}(K) \) is a non-singular elliptic curve defined over an algebraic number field \( K \).

**Proof.** We shall view the crossed product \( A_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \) as a \( C^* \)-dynamical system \( (A_{RM}, \hat{\mathbb{Z}}, \hat{\alpha}) \), see [Williams 2007] [14] for the details. Recall that the irreducible representations of \( C^* \)-dynamical system \( (A_{RM}, \hat{\mathbb{Z}}, \hat{\alpha}) \) are in the one-to-one correspondence with the minimal sets of the dynamical system (i.e. closed \( \hat{\alpha} \)-invariant sub-\( C^* \)-algebras of \( A_{RM} \) not containing a smaller object with the same property).

To calculate the minimal sets of \( (A_{RM}, \hat{\mathbb{Z}}, \hat{\alpha}) \), let \( \theta \) be quadratic irrationality such that \( A_{RM} \cong A_\theta \). It is known that every non-trivial sub-\( C^* \)-algebra of \( A_\theta \) has the form \( A_{n\theta} \) for some positive integer \( n \) [Rieffel 1981] [8, p. 419]. It is easy to deduce that the maximal proper sub-\( C^* \)-algebra of \( A_\theta \) has the form \( A_{p\theta} \), where \( p \) is a prime number. (Indeed, each composite \( n = n_1 n_2 \) cannot be maximal since \( A_{n_1 n_2 \theta} \subset A_{n_1 \theta} \subset A_\theta \) or \( A_{n_1 n_2 \theta} \subset A_{n_2 \theta} \subset A_\theta \), where all inclusions are strict.)

We claim that \( (A_{p\theta}, \hat{\mathbb{Z}}, \hat{\alpha}^{\pi(p)}) \) is the minimal \( C^* \)-dynamical system, where \( \pi(p) \) is certain power of the automorphism \( \hat{\alpha} \). Indeed, the automorphism \( \hat{\alpha} \) of \( A_\theta \) corresponds to multiplication by the fundamental unit, \( \varepsilon \), of pseudo-lattice \( \Lambda = \mathbb{Z} + \mathbb{Z} \theta \). It is known that certain power, \( \pi(p) \), of \( \varepsilon \) coincides with the fundamental unit of pseudo-lattice \( \mathbb{Z} + (p\theta) \mathbb{Z} \), see e.g. [Hasse 1950] [5, p.298]. Thus one gets the minimal \( C^* \)-dynamical system \( (A_{p\theta}, \hat{\mathbb{Z}}, \hat{\alpha}^{\pi(p)}) \), which is defined on the sub-\( C^* \)-algebra \( A_{p\theta} \) of \( A_\theta \). Therefore we have an isomorphism

\[
\text{Irred} \left( A_{RM} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}} \right) \cong \bigcup_{p \in \mathcal{P}} \text{Irred} \left( A_{p\theta} \rtimes_{\hat{\alpha}^{\pi(p)}} \hat{\mathbb{Z}} \right),
\]

(4.2)

where \( \mathcal{P} \) is the set of all (but a finite number) of primes.

To simplify the RHS of (4.2), recall that matrix form of the fundamental unit \( \varepsilon \) of pseudo-lattice \( \Lambda \) coincides with the matrix \( A \), see above. For each prime \( p \in \mathcal{P} \) consider the matrix

\[
L_p = \begin{pmatrix}
tr (A^{\pi(p)}) - p & p \\
tr (A^{\pi(p)}) - p - 1 & p
\end{pmatrix}
\]

(4.3)

where \( tr \) is the trace of matrix. Let us show, that

\[
A_{p\theta} \rtimes_{\hat{\alpha}^{\pi(p)}} \hat{\mathbb{Z}} \cong A_\theta \rtimes_{L_p} \hat{\mathbb{Z}},
\]

(4.4)
where \( L_p \) is an endomorphism of \( A_\theta \) (of degree \( p \)) induced by matrix \( L_p \). Indeed, because \( \text{deg} (L_p) = p \) the endomorphism \( L_p \) maps pseudo-lattice \( \Lambda = \mathbb{Z} + \theta \mathbb{Z} \) to a sub-lattice of index \( p \); any such can be written in the form \( \Lambda_p = \mathbb{Z} + (p\theta)\mathbb{Z} \), see e.g. [Borevich & Shafarevich 1966] [2, p.131]. Notice that pseudo-lattice \( \Lambda_p \) corresponds to the sub-\( C^* \)-algebra \( A_{\rho \theta} \) of algebra \( A_\theta \) and \( L_p \) induces a shift automorphism of \( A_{\rho \theta} \) [Cuntz 1977] [1, Section 2.1]. It is not hard to see, that the shift automorphism coincides with \( \hat{\alpha}^\pi(p) \). Indeed, it is verified directly that \( \text{tr} \ (\hat{\alpha}^\pi(p)) = \text{tr} \ (A^\pi(p)) = \text{tr} \ (L_p) \); thus one gets a bijection between powers of \( \hat{\alpha}^\pi(p) \) and such of \( L_p \). But \( \hat{\alpha}^\pi(p) \) corresponds to the fundamental unit of pseudo-lattice \( \Lambda_p \); therefore the shift automorphism induced by \( L_p \) must coincide with \( \hat{\alpha}^\pi(p) \). The isomorphism (4.4) is proved.

Therefore (4.2) can be simplified to the form

\[
\text{Irred} \ (A_{RM} \rtimes \hat{\alpha} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathbb{P}} \text{Irred} \ (A_{RM} \rtimes L_p \hat{\mathbb{Z}}). \tag{4.5}
\]

To calculate irreducible representations of the crossed product \( C^* \)-algebra \( A_{RM} \rtimes L_p \hat{\mathbb{Z}} \) at the RHS of (4.5), recall that such are in a one-to-one correspondence with the set of invariant measures on a subshift of finite type given by the positive integer matrix (4.3) [Bowen & Franks 1977] [3]; the measures make an abelian group under the addition operation. Such a group is isomorphic to \( \mathbb{Z}^2 / (I - L_p)\mathbb{Z}^2 \), where \( I \) is the identity matrix, see [Bowen & Franks 1977] [3, Theorem 2.2].

Therefore (4.5) can be written in the form

\[
\text{Irred} \ (A_{RM} \rtimes \hat{\alpha} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathbb{P}} \frac{\mathbb{Z}^2}{I - L_p}\mathbb{Z}^2. \tag{4.6}
\]

Let \( \mathcal{E}(K) \) be a non-singular elliptic curve defined over the algebraic number field \( K \); let \( \mathcal{E}(\mathbb{F}_p) \) be the reduction of \( \mathcal{E}(K) \) modulo prime ideal over a “good” prime number \( p \). Recall that \( |\mathcal{E}(\mathbb{F}_p)| = \text{det} \ (I - Fr_p) \), where \( Fr_p \) is an integer two-by-two matrix corresponding to the action of Frobenius endomorphism on the \( \ell \)-adic cohomology of \( \mathcal{E}(K) \), see e.g. [Tate 1974] [13, p.187].

Since \( |\mathbb{Z}^2 / (I - L_p)\mathbb{Z}^2| = \text{det} \ (I - L_p) \), one can identify \( Fr_p \) and \( L_p \); and, therefore, one obtains an isomorphism \( \mathcal{E}(\mathbb{F}_p) \cong \mathbb{Z}^2 / (I - L_p)\mathbb{Z}^2 \). Thus (4.6) can be written in the form

\[
\text{Irred} \ (A_{RM} \rtimes \hat{\alpha} \hat{\mathbb{Z}}) \cong \bigcup_{p \in \mathbb{P}} \mathcal{E}(\mathbb{F}_p). \tag{4.7}
\]

Finally, consider an arithmetic scheme, \( X \), corresponding to \( \mathcal{E}(K) \); the latter fibers over \( \mathbb{Z} \) [Silverman 1994] [10, Example 4.2.2]. It can be immediately seen, that the RHS of (4.7) coincides with the scheme \( X \), where the regular fiber over \( p \) corresponds to \( \mathcal{E}(\mathbb{F}_p) \) \textit{idem}. This argument finishes the proof of proposition 4.1. \( \square \)

\textit{Remark} 4.2. The Frobenius endomorphism \( Fr_p = L_p \) is induced by map \( \alpha \), since \( \alpha \) defines \( \hat{\alpha} \) (i.e. matrix \( A \)) and the latter is linked to \( L_p \) via formula (4.3).

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