Fractional Quantum Hall Effect in $n = 0$ Landau Band of Graphene with Chern Number Matrix

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Fully taking into account the honeycomb lattice structure, fractional quantum Hall states of graphene are considered by a pseudopotential projected into the $n = 0$ Landau band. By using chirality as an internal degree of freedom, the Chern number matrices are defined and evaluated numerically. Quantum phase transition induced by changing a range of the interaction is demonstrated that is associated with chirality ferromagnetism. The chirality-unpolarized ground state is consistent with the Halperin 331 state of the bilayer quantum Hall system.

The concept of topological order$^{1-3}$ substantially expands our understanding of phases of matter that cannot be described by the conventional order parameters associated with symmetry breaking. The quantum Hall effect$^4$ is a prominent example of topologically non-trivial quantum phases. The quantized Hall conductance is expressed as the Chern number$^5-8$ associated with the Berry connection.$^9$ Although the integer quantum Hall state can be described by non-interacting electrons, the electron-electron interaction plays a crucial role in the fractional quantum Hall (FQH) phase. The characteristics of these correlated quantum states are well captured by the Laughlin wave function.$^{10}$ In addition, its excitations are quasi-particles with fractional charges and fractional statistics.$^{11}$ This FQH effect is understood as an integer quantum Hall effect of composite fermions as flux charge composites.$^{12}$ It also provides a consistent picture at an even-denominator Landau level (LL) filling.$^{13}$

The internal degrees of freedom, such as spin or layer index, bring further diversity to the FQH phases. The ground state at $\nu = 5/2$ is described by the Moore-Read Pfaffian state$^{15}$ with the excitation obeying non-Abelian statistics$^{16}$ when the interaction is short-range. As for two-component Abelian FQH systems, the Halperin $lmm$ state$^{17}$ is a typical example. This is realized in a bilayer FQH system at $\nu = 1/2, 18-23$ for example, but its appearance depends strongly on the system parameters. For multi-component systems, the symmetric integer matrix$^{K24-27}$ provides classification of the FQH phases, which is discussed in relation to the Chern number matrix.$^{28,29}$

The FQH effect of graphene$^{30-34}$ is also an example of the multi-component FQH systems. The low-energy behavior of electrons in graphene is described by the doubled massless Dirac fermions at $K$ and $K'$ points in the Brillouin zone. These characteristics give rise to the FQH phases peculiar to graphene.$^{35-45}$ Since the $n = 0$ LL is a standard lowest LL of the valley polarized Dirac fermions, the FQH effect of the $n = 0$ LL has been discussed similarly with the SU(2) invariance arising from the valley degree of freedom. The ground states at the $n = 0$ LL filling factor $\nu = 1/3$ and $1/2$ are described by the pseudospin (valley) polarized Laughlin state and pseudospin singlet composite fermion Fermi sea, respectively.$^{36,37}$

In this study, the FQH system for the $n = 0$ Landau band is investigated by fully taking into account the honeycomb lattice structure of the interaction. Short-range electron-electron interaction of the nearest neighbor (NN) and next-nearest neighbor (NNN) is discussed in this paper by constructing the pseudopotential$^{30}$ projected into the $n = 0$ Landau band. The chiral symmetry of the honeycomb lattice plays an important role in the many-body problems as well.$^{47-49}$ The quantum phase transitions associated with the chirality ferromagnetism occur by changing the interaction range. Since the total pseudospin is not conserved because of the lattice effects, the SU(2) symmetry discussed by the continuous approximation is absent. In order to characterize the quantum phases topologically, the Chern number matrices specified by the chiral basis are constructed numerically. The results are also discussed in relation to the conventional bilayer quantum Hall system.

Let us begin by introducing the projected fermion operators into the $n = 0$ Landau band. Here, we assume that the system is always spin-polarized. The kinetic Hamiltonian is written as

$$H_{\text{kin}} = t \sum_{\langle ij \rangle} e^{\phi_i} c_j^\dagger c_j = e^{\phi_i} h_{\text{kin}} c,$$

which describes hopping between the NN pairs of sites with strength $t$. Here, $e^{\phi_i} = (e^{i \phi_{i}} , e^{i \phi_{i}}) , c_{\mu(i)}^\dagger = (c_{\mu(i)}^\dagger , \cdots , c_{N_{\text{cell}}(i)}^\dagger)$ and $c_{\mu(i)}^\dagger$ creates a fermion at the sublattice $\bullet(i)$ for unit cell $i$. The Peierls phase $\phi_i$ is determined such that the sum of the phases around an elementary hexagon is equal to the magnetic flux $2\pi \phi$ in units of the flux quantum $\phi_0 = h/e$. In the calculation, the string gauge$^{50,51}$ is employed, which enables us to realize the minimum magnetic fluxes that are consistent with the lattice periodicity. When the system is put on the $N_s \times N_s$ unit cells with a periodic boundary condition, the magnetic field can be provided as $\phi = N_{\text{cell}} N_s / q (N_{\text{cell}} = 1, 2, \cdots, N_{\text{cell}})$, where $N_{\text{cell}} = N_s N_s$. Here, $N_s$ corresponds to the total magnetic flux. The lattice model with $\phi = p/q (p, q : \text{relatively prime})$ has $2q$ single-electron bands, where $2$ comes from the sublattice degree of freedom. The number of states per band is obtained as $N_s N_s / q$. For the weak magnetic field ($\phi \ll 1$), $2p$ bands flow into each other around the zero energy, which form the $n = 0$ LL in the large $q$ limit. Thus, in this paper, “the $n = 0$ Landau band” is defined as a group of these bands, where there are $(N_s N_s / q) \times 2p = 2 N_{\phi}$ one-body states.
Since the honeycomb lattice is bipartite, the Hamiltonian $H_{\text{kin}}$ has chiral symmetry. The $2N_{\text{cell}} \times 2N_{\text{cell}}$ matrix $\Gamma = \text{diag}(\Gamma_{\text{kin}}, -\Gamma_{\text{kin}})$ anticommutes with the Hamiltonian as $[h_{\text{kin}}, \Gamma] = 0$ and $\Gamma^2 = I_{2N_{\text{cell}}}$. If $\psi_k$ is the eigenvector of $h_{\text{kin}}$ with the energy $\epsilon_k$, the chiral symmetry guarantees that $\Gamma \psi_k$ is identical to the one with $-\epsilon_k$. Thus, the chiral operator $\Gamma$ can be diagonalized within the one-body states of the $n = 0$ Landau band. Then, the chiral basis can be defined as $\psi = (\psi_+, \psi_-)$, $\psi_k = (\psi_{1,k}, \cdots, \psi_{N,k})$, and $\Gamma \psi_k = \pm \psi_k$. Note that $\psi_{k,-}$ is localized on the sublattice $\bullet$ so that the multiplet can be expressed as $\psi = (\begin{pmatrix} 0 \\ -\psi \end{pmatrix})$, where $\psi_{k,-}$ is a proper $N_{\text{cell}} \times N_{\text{g}}$ matrix.

Next, let us consider the two-body interactions written as

$$H_{\text{int}} = \sum_{\sigma \sigma', \nu} \sum_{\langle ij \rangle} V_{ij}^{\nu} n_{i,\sigma} n_{j,\sigma'} + \sum_{\nu} V_{ij}^{\nu} n_{i,\sigma} n_{j,\sigma},$$

where $n_{\nu}(\sigma) = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}_{\nu}(\sigma) C_{\nu}(\sigma)$ and $V_{ij}^{\nu}$ is the strength of the electron-electron interaction. In order to construct the pseudopotential, the projected creation-annihilation operators are defined as $\hat{c}^\dagger = \hat{c}^\dagger P_{\nu}$, where $P_{\nu} = \psi_{k,\nu} \psi_{k,\nu}^\dagger$. This expression is simplified by writing $\chi_{\nu}(\sigma) = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}_{\nu}(\sigma) P_{\nu} = \psi_{k,\nu} \psi_{k,\nu}^\dagger$. Note that these projected operators no longer satisfy the canonical anticommutation relations ($\{ \hat{c}_{i}, \hat{c}_{j}^\dagger \} = \delta_{ij}$). By taking into account the ordering of fermions, the replacement of $\hat{c}^\dagger$, $\hat{c}$ with $\hat{d}^\dagger$, $\hat{d}$ causes the Hamiltonian to be projected into the $n = 0$ Landau band. Then, the projected Hamiltonian can be defined as

$$\hat{H}_{\text{int}} = \sum_{\alpha \alpha', \nu \nu'} \sum_{\langle ij \rangle} V_{ij}^{\nu} \hat{c}_{i,\alpha} \hat{c}_{j,\alpha'} + \sum_{\nu} V_{ij}^{\nu} \hat{c}_{i,\alpha} \hat{c}_{j,\alpha},$$

where $A_{\alpha \alpha'}^{\nu \nu'} = \sum_{\nu} \sum_{\nu'} V_{ij}^{\nu} \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}_{\nu}(\sigma) \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}_{\nu'}(\sigma') \psi_{k,\nu}^\dagger \psi_{k,\nu'}$, and $\chi_{\nu}(\sigma) = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}_{\nu}(\sigma) P_{\nu}$. Dependence of the phase transition point $V_2/V_1$ on magnetic flux $\phi$ is shown in Fig. 1 (c) and (d). The transition point denoted as $a-b$ indicates the transition between $\chi_{\text{tot}} = \nu$ and $b$.

![Fig. 1.](image)

Note that the interaction $V_1$ ($V_2$) contributes $V_{ij}^\nu$ ($V_{ij}^\nu$). Figures 1 (a) and (b) show the $V_2/V_1$ dependence of the four-electron energy obtained by the exact diagonalization at $\nu = 1/3$ and $1/2$. Since the exact topological degeneracy due to the center-of-mass translation is given by the Landau gauge in a finite system, we choose a system of $12 \times 12$ size for $\phi = 1/12$ as shown in Fig. 1 (a) at $\nu = 1/3$. For example, the chirality-polarized ground state multiplet is exactly three-fold degenerated. (See the inset in Fig. 1 (a).) As for the system at $\nu = 1/2$, it is compatible to the Dirac cones at $K$ and $K'$ points of graphene, the system size should be $3n \times 3n$ ($n = 1, 2, \cdots$). Then, we choose the magnetic flux as $\phi = 8/(12 \times 12) = 1/18$ using the string gauge. In this case, the topological degeneracy is not exact. However, the energy separation of the entangled states within the ground state multiplet is very small. For example, it is less than 0.003 times the energy gap for the six-fold ground state multiplet with $\chi_{\text{tot}} = \nu$. (See the inset in Fig. 1 (b).) Note that the magnetic length $l_B = \sqrt{\hbar/eB} = \sqrt{S}/2\pi \phi$, where $S = 3\sqrt{3}a^2/2$ is the area of the elementary hexagon with lattice constant $a$, is approximately $2.2a$ and $2.7a$ for $\phi = 1/12$ and $1/18$, respectively. In the case of $V_1 \neq V_2 = 0$, that is, only the bipartite interactions, many-body states with $\chi_{\text{tot}} = \nu$ should be eigenstates with zero eigenvalues. Generally, the system at $\nu < 1$ provides $2 \times N_{\text{cell}}$-fold degenerate ground states (2 comes from the sign of chirality), which are lifted by the infinitesimal $V_2$. Further, an increase in $V_2$ leads the phase transition from $\chi_{\text{tot}} = \nu$ to $0$ since the NNN interactions act between the same sublattices. In Fig. 1 (c) and (d), the ratio $V_2/V_1$ at the phase transition points is plotted against the magnetic flux.
\( \phi \) at \( v = 1/3 \) and \( 1/2 \). The results suggest that the strong short-range interaction compared with the magnetic flux (\( \phi \ll 1 \)) favors the unpolarized chirality unless \( V_2 \) is not vanishing.

Since the pseudopotential is constructed on the basis of the honeycomb lattice model, only the \( z \)-component of the pseudospin, \( \hat{A}_{\text{tot}} \), is conserved. The SU(2) symmetry arising from the chirality is absent in contrast to the cases in the continuum limit.\(^{36,37} \) Then, in order to investigate the internal topological structure of the many-body states, we evaluate the Chern number matrices associated with the chirality. First, we investigate the twisted boundary condition,\(^{31} \) \( c_{n+1,n} = e^{i\theta_n} c_{n+1,n} \), and \( c_{n+1,n} = e^{-i\theta_n} c_{n+1,n} \), where \( n_1 \) and \( n_2 \) are the labels of the unit cell for the \( x \) and \( y \) directions. Since the chiral symmetry remains in the kinetic Hamiltonian \( H_{\text{kin}}(\theta^1, \theta^2) \), the connection between chirality and sublattice is preserved.

Let us consider each projected fermion operator \( \tilde{c}_\bullet \) and \( \tilde{c}_\circ \) with different twisted boundary conditions,

\[
\tilde{c}_{\bullet}(\theta) = \tilde{c}_{\bullet}(\theta), \quad \tilde{c}_{\circ}(\theta) = \tilde{c}_{\circ}(\theta),
\]

where \( \theta_{\bullet}(\theta) = \theta_{\bullet}(\theta) \) and \( \theta_{\circ}(\theta) = \theta_{\circ}(\theta) \). The projected Hamiltonian \( \tilde{H}_{\text{kin}}(\theta^1, \theta^2) \) can be defined by replacing \( \tilde{c}_{\bullet}(\theta) \) with \( \tilde{c}_{\bullet}(\theta) \) in Eq. (3). Note that this Hamiltonian can be written by the fermion operators \( d_{\bullet}(\theta) = \tilde{c}_{\bullet}(\theta) \), which satisfy the canonical anticommutation relations \( [d_{\bullet}(\theta), d_{\bullet}(\theta')] = [d_{\circ}(\theta), d_{\circ}(\theta')] = 0 \) for any \( \theta \).

Now, let us further define the non-Abelian Berry connection and curvature\(^{3,9} \) of the \( m \)-fold ground state multiplet \( \Phi = (\{G_1\}, \ldots, \{G_m\}) \) by selecting the sublattices in each direction \( x, y \) as

\[
A_{\sigma,\sigma'} = \Theta_{\sigma'} \frac{d}{d\theta_{\sigma'}}, \quad d = \sum_{\sigma,\sigma'} dd_{\sigma}^\dagger \frac{d}{d\theta_{\sigma}},
\]

\[
F_{\sigma,\sigma'} = dA_{\sigma,\sigma'} + A_{\sigma,\sigma'}^2,
\]

where \( \sigma = \bullet, \circ \), and the two parameters except for \( \theta_{\bullet} \) and \( \theta_{\circ} \) are fixed to 0. Then, the Chern number matrix is defined as

\[
C = \begin{pmatrix} C_{\bullet,\bullet} & C_{\bullet,\circ} \\ C_{\circ,\bullet} & C_{\circ,\circ} \end{pmatrix}, \quad C_{\sigma,\sigma'} = \frac{1}{2\pi i} \int \text{Tr} F_{\sigma,\sigma'},
\]

The element is evaluated as\(^{51,53} \)

\[
\begin{align*}
C_{\bullet,\bullet} &= \frac{1}{2\pi i} \sum_{\sigma} \int \theta_{\bullet} \frac{d}{d\theta_{\bullet}}, \\
C_{\bullet,\circ} &= \frac{1}{2\pi i} \sum_{\sigma} \int \theta_{\circ} \frac{d}{d\theta_{\circ}}, \\
C_{\circ,\bullet} &= \frac{1}{2\pi i} \sum_{\sigma} \int \theta_{\bullet} \frac{d}{d\theta_{\bullet}}, \\
C_{\circ,\circ} &= \frac{1}{2\pi i} \sum_{\sigma} \int \theta_{\circ} \frac{d}{d\theta_{\circ}}.
\end{align*}
\]

In order to construct the ground state multiplet \( \Phi(\theta) \) numerically, we numerically evaluate \( C(\theta) \) for \( N_\text{el} \)-electron states classified by the total chirality \( \chi_{\text{tot}} \) as defined as \( \Phi(\theta) = \{\psi_{1}(\theta), \ldots, \psi_{N_\text{el}}(\theta)\} \).

Here, \( N_\text{el} \) is the dimension of the Hilbert space, and

\[
\begin{align*}
\psi_{\bullet}(\theta) &= \left( \sum_{n\in\Phi\bullet} \tilde{d}_{n+1}(\theta) \right) \left( \sum_{n\in\Phi\circ} \tilde{d}_{n+1}(\theta) \right) |0\rangle, \\
\psi_{\circ}(\theta) &= \left( \sum_{n\in\Phi\circ} \tilde{d}_{n+1}(\theta) \right) \left( \sum_{n\in\Phi\bullet} \tilde{d}_{n+1}(\theta) \right) |0\rangle,
\end{align*}
\]

where \( \Phi\bullet \) is one of the possible ways to occupy \( N_\text{el} \) states by \( N_\text{el} \) electrons, and \( N_\text{el} = (N_e \pm \chi_{\text{tot}})/2 \). Then, the ground state multiplet is expressed as \( \Phi = \psi_{\text{tot}}(\theta) \), where \( \psi_{\text{tot}} = (\psi_{G_1}, \ldots, \psi_{G_m}) \), and \( \psi_{G_i} \) is the eigenvector of \( \Phi(\theta) \). We have \( \Phi(\theta) \psi_{\text{tot}} = \psi_{\text{tot}}(\theta) \).

The energy gap of the eight-fold ground state multiplet is always gapped for \( \Phi = 0 \). Since the polarized many-body states occupy only the sublattice \( \bullet \), the staggered sublattice potential written as \( \tilde{H}_{\text{tot}} = -M \hat{P}_{\text{tot}} \hat{c}_{\bullet} + M \hat{P}_{\text{tot}} \hat{c}_{\circ} = -MG(M > 0) \) stabilizes the polarized many-body states. Now, the ground state multiplet is defined as a group of lowest-energy many-body states that is separated from the other excited states. Numerically obtained ground states at \( v = 1/3 \) and \( 1/2 \) are similar to those provided by the pseudopotentials projected into the lowest Landau band.\(^{51} \) At \( v = 1/3 \), the ground state is always three-fold degenerated irrespective of the number of electrons. This three-fold ground state multiplet is gapped in the thermodynamic limit similar to the Ref. 51. In addition, its Chern number \( C_{\bullet,\bullet} = 1 \). This implies that the ground state is the lattice analogue of the Laughlin state. On the other hand, for the \( v = 1/2 \) case, the degeneracy of the ground state has no such universal feature, and there is no sign of a finite energy gap from its scaling. This is also consistent with the composite fermion Fermi sea.\(^{51} \)

Next, let us consider the many-body states with \( \chi_{\text{tot}} = 0 \), which occupy the same number of sublattices \( \bullet \) and \( \circ \). Here, we focus on the \( v = 1/2 \) state. Figure 2(a) and (b) shows the \( \theta_{\bullet} \) dependence of the many-body spectrum at \( V_2/V_1 = 0.3 \) and 1.0, respectively. In Fig. 2(a), the ground state mixes with higher states with the change in boundary conditions, and the ground state multiplet is not well-defined. On the other hand, in Fig. 2(b), eight low-energy states are entangled and do not mix with excited states. In Fig. 2(c), the many-body spectrum with \( \chi_{\text{tot}} = 0 \) for \( N_e = 4 \) and \( \phi = 1/8 \) is plotted as a function of \( V_2/V_1 \). The behavior of the ground state in the spectral flow changes at \( V_2/V_1 = 0.53 \). Note that the eight-fold ground state multiplet is always gapped for \( V_2/V_1 \geq 0.53 \) as long as \( V_1 = 0 \). (For \( V_1 = 0 \), the energy gap of the eight-fold ground state multiplet vanishes, and the two decoupled \( \nu = \frac{1}{2} \) and \( \nu = 1 \) states should be the ground states.) In the inset of the Fig. 2(b), the energy gaps \( \Delta E = E_0 - E_1 \) at \( V_2/V_1 = 1 \) with periodic boundary conditions are plotted as a function of the magnetic flux \( \phi \), where \( E_1 \) is the \( \nu \)-th eigenvalue of \( \tilde{H}_{\text{tot}} \) with \( \chi_{\text{tot}} = 0 \). The result indicates that the scaling low \( \Delta E \propto \phi^2 \) is roughly valid in the wide range of \( \phi \). Since the electron density is obtained as \( \rho = N_e/(2Ne) = \frac{\phi}{2} \), we have \( \Delta E \propto \rho^2 \), which means that the excitations are local.

Following the argument of the Ref. 29, the obtained Chern number matrix \( C \) suggests the \( K \)-matrix as \( K = (\mathcal{C}/m)^{-1} \), where \( m \) is the degeneracy. The matrices defined by the eight-
considered in terms of the Chern number matrix. In the conventional bilayer system at $\nu = 1/3$, for example, the Halperin 551 state is one of the candidates of the gapped ground state.\cite{18,54} However, the results at $\nu = 1/3$ are not systematic, and no clear picture is obtained in contrast to the $\nu = 1/2$ case. They will be discussed later elsewhere.

To summarize, we have constructed the Chern number matrix in association with the chiral basis by using the Hamiltonian projected into the $n = 0$ Landau band of the honeycomb lattice. Modifying the interaction range induces the quantum phase transitions associated with the chirality ferromagnetism. When the NN interactions are sufficiently strong, the ground states at $\nu = 1/3$ and 1/2 are chirality-polarized, and consistent with the Laughlin state and the composite fermion Fermi sea, respectively. On the other hand, an increase in the strength of the NNN interaction leads the ground state to chirality-uncpolarize. The obtained Chern number matrix indicates that the unpolarized ground state for large $V_2$ at $\nu = 1/2$ is consistent with the Halperin 331 state.

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Fig. 2. (Color online) Many-body spectrum with $\epsilon_{\text{tot}} = 0$ at $\nu = 1/2$ as a function of (a, b) the twist $\theta'_2$ and (c) the ratio of the interaction $V_2/V_1$. (a, b) The remaining parameters are fixed at $\theta'_2 = \theta'_3 = 0$. The inset in (b) shows the scaled energy gaps as a function of the magnetic flux $\phi$.

\begin{align}
C &= \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \tag{12}
\end{align}

As for a conventional bilayer FQH system with finite interlayer separation $d$, the pseudopotential does not have the SU(2) symmetry arising from the pseudospin. However, in the limit $d \to \infty$ or $d = 0$, the system is exceptionally SU(2) invariant. Therefore, the ground states of the $\nu = 1/2$ case in these two limits are described by the composite fermion Fermi sea. The former is the two decoupled states and the latter is the pseudospin-singlet state.\cite{54} On the other hand, the Halperin 331 state, which does not have the SU(2) symmetry, is realized as the ground state in the intermediate separation.\cite{18-20} The FQH system for the $n = 0$ LL of graphene in the continuum limit corresponds to the conventional bilayer FQH system with $d = 0$, where the Halperin 331 state should not be observed because there is SU(2) symmetry in the system.

In contrast, the NN and NNN interactions in the honeycomb lattice act between the different and same sublattices (chiralities). Therefore, by connecting the parameter $V_1$ and $V_2$ with the interlayer and intralayer interactions, the FQH system of the $n = 0$ Landau band corresponds to the conventional bilayer quantum Hall system with $d \neq 0$; the chirality $\pm 1$ plays the role of the two-layer index. Note that the only $z$-component of the pseudospin is conserved in both systems. Thus, it is natural to expect that the analogue of the Halperin 331 state is realized for the $n = 0$ Landau band when the ratio $V_2/V_1$ is on the order of unity. We confirmed this scenario for the four-electron system in terms of the Chern number matrix as Eq. (12).

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