BPS Wilson loops in $\mathcal{N} = 4$ SYM: Examples on hyperbolic submanifolds of space-time

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Abstract

In this paper we present a family of supersymmetric Wilson loops of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in Minkowski space. Our examples focus on curves restricted to hyperbolic submanifolds, $\mathbb{H}_3$ and $\mathbb{H}_2$, of space-time. Generically they preserve two supercharges, but in special cases more, including a case which has not been discussed before, of the hyperbolic line, conformal to the straight line and circle, which is half-BPS. We discuss some general properties of these Wilson loops and their string duals and study special examples in more detail. Generically the string duals propagate on a complexification of $AdS_5 \times S^5$ and in some specific examples the compact sphere is effectively replaced by a de-Sitter space.
1 Introduction and summary

The spectrum of operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) which preserve some supersymmetry is very rich. The list of operators which are half-BPS is still not complete,\(^{1}\) and of course much less is known about operators preserving fewer supercharges. The dynamics of a supersymmetric operator is much more constrained than that of a generic object and the more supercharges it preserves, the more constrained it is. Therefore an interesting endeavor is to find operators preserving relatively few supercharges, whose dynamics are rather rich, but may still be under control.

An example of such operators are the BPS Wilson loops constructed in [2, 3, 4]. For an arbitrary curve on an $S^3$ submanifold of $\mathbb{R}^4$ a prescription is given for choosing extra scalar coupling such that the resulting Wilson loop preserves at least two supercharges. When the loop is on $S^2$ it preserves four and in some special examples, eight, while for the circle it preserves sixteen.

An earlier construction of Zarembo [5] does much the same with arbitrary curves on linear subspaces of $\mathbb{R}^4$. In that case the expectation value of all the loops is unity, even if they preserve only a single supercharge [6, 7, 8]. The interesting thing about the Wilson loops on $S^3$ is that their expectation value is not so simple. In the case of

\[^{1}\text{For recent progress see [1].}\]
the circle it is given by a zero dimensional matrix model \[9, 10\] as was recently proven by Pestun \[11\].

More remarkably, in \[3, 4\] some evidence was presented that this result extends to arbitrary loops on \(S^2\), preserving four supercharges. In that case the propagators are not constant in the Feynman gauge, but the final result seems to be the same matrix model, with a modified coupling. This result can be motivated, as an intermediate step, by a perturbative calculation in 2-dimensional bosonic Yang-Mills (YM) theory, an observation which was confirmed at the two-loop order (for single loops) in \[12, 13, 14\].

In this paper we study another family of supersymmetric Wilson loops, this time in Minkowski space \(\mathbb{R}^{3,1}\). Working with indefinite metric expands the range of possibilities. For one, there are Euclidean two-spheres in Minkowski space, so the previously mentioned results would apply there too, but there are no three-spheres. Instead the maximally symmetric hypersurfaces are the light-cone, the Lorentzian hyperboloid (de-Sitter space) and the Euclidean hyperboloid \(\mathbb{H}_3\) (Euclidean \(AdS_3\)).

Light-like Wilson loops, even without coupling to the scalar fields, are supersymmetric. More generally, we expect constructions like that of Zarembo \[5\] would work also on Minkowski space, as was recently utilized in \[15\].

In this paper we study the case of \(\mathbb{H}_3\). For a general curve constrained to this submanifold we find a prescription to choose the scalar couplings such that the resulting loop is BPS. We then focus on specific subclasses of examples preserving more supersymmetry, in particular \(\mathbb{H}_2\) and special loops along curves of constant curvature. We find results mostly in parallel with those on \(S^3\) and its submanifolds. Yet there are some differences that are not so trivial, specifically in understanding the string-theory dual — where an analytic continuation has to be performed on an \(S^2\) subspace of \(S^5\) which yields a two-dimensional de-Sitter space \(dS_2\).

One particular reason to study Wilson loops in Minkowski space is the connection between Wilson loops and scattering amplitudes. A great deal of evidence shows that Wilson loops made out of light-like segments calculate maximum helicity violating gluon scattering amplitudes \[16, 17, 18, 19, 20\] (for a review, see \[21\]). It is therefore also natural to try to find supersymmetric Wilson loops in Minkowski space.

We present the construction of the Wilson loops on \(\mathbb{H}_3\) in Section 2 and prove that they are supersymmetric. Then we discuss the restriction to \(\mathbb{H}_2\), show how the connection to 2-dimensional Yang-Mills shows up again at the leading order in perturbation theory. This connection suggests once more that the expectation value of these Wilson loops is given by a Gaussian matrix model, at least for compact curves.\(^2\) We then

\(^2\)Unlike the sphere the Hyperboloid is not compact, so one can construct Wilson loop operators with an infinite extent, some examples of which are presented in Section 3.
discuss some general properties of the string duals of these Wilson loops.

The construction of BPS Wilson loops in $\mathcal{N} = 4$ SYM always involves extra couplings to the scalar fields. For curves on $S^3$ these couplings can be all real, but for our construction here they will generically be complex. As the scalar couplings translate in the string dual to positions on $S^5$, we are forced to consider the embedding of the string into a complexification of $AdS_5 \times S^5$. This is discussed in Section 2.5.

In Section 3 we present some special examples. The first is the hyperbolic line, which is $1/2$-BPS, like the line and circle. Indeed it is related to them by a conformal transformation. After that, we present three other examples of circles, other hyperbolic lines and cusps which are all $1/4$ BPS and in all cases we can find the corresponding string solutions. Two of the examples require a complexification of the target space, so the string can be viewed as propagating in a de-Sitter space rather than on an $S^2$. In the one case where the loop is compact, the $1/4$ BPS circle, we can interpolate the gauge theory result to strong coupling by summing over ladder diagrams and find complete agreement.

Some of the results contained in the paper are a summary of the diploma thesis [22].

2 Generalities

2.1 Construction

We consider Wilson loops confined to an $H_3$ subspace of Minkowski space $\mathbb{R}^{3,1}$ given by

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1, \quad x_0 > 0. \quad (2.1)$$

This is a Euclidean manifold contained within the future light-cone.

Our construction is based on a Wick-rotation of the Wilson loops on $S^3$ [[2] [4]] where an important ingredient were the invariant one-forms. They are Wick-rotated to

$$\omega_1 = x^0 dx^1 - x^1 dx^0 + i(x^2 dx^3 - x^3 dx^2),$$
$$\omega_2 = x^0 dx^2 - x^2 dx^0 + i(x^3 dx^1 - x^1 dx^3),$$
$$\omega_3 = x^0 dx^3 - x^3 dx^0 + i(x^1 dx^2 - x^2 dx^1). \quad (2.2)$$

We allow the Wilson loops to follow an arbitrary path on this manifold and in order to preserve supersymmetry will couple them to three of the real scalar fields, which we take to be $\Phi^1, \Phi^2$ and $\Phi^3$. The coupling can be expressed with the use of the one-forms
in terms of the modified connection \((A = A_\mu dx^\mu\) is the gauge connection)

\[
\tilde{A} = A - i \omega_i \Phi^i,
\]

as

\[
W = \frac{1}{N} \text{Tr} \mathcal{P} \exp \int i \tilde{A}.
\]

### 2.2 Supersymmetry

To check that these Wilson loops are indeed BPS objects, consider the supersymmetry variation which is proportional to

\[
\delta_i W \propto (\gamma_\mu dx^\mu - i \rho^i \gamma^5 \omega_i) \epsilon(x).
\]

Here \(\gamma_\mu\) are the usual Dirac matrices, \(\rho^i\) are three of the gamma matrices of \(SO(6)\) and we take them to commute with \(\gamma_\mu\). \(\epsilon(x)\) is the conformal Killing spinor in flat space given by two constant spinors as

\[
\epsilon(x) = \epsilon_0 + x^\mu \gamma_\mu \epsilon_1.
\]

\(\epsilon_0\) is the parameter for the action of the Poincaré supercharges \((Q's)\) and \(\epsilon_1\) that for the action of the superconformal charges \((S's)\).

From the definition of the one-forms \((2.2)\) one can see that they are related to the generators of the Lorentz group in the chiral spinor representation \((\tau^i_L\) are the Pauli matrices acting on the chiral spinors)

\[
\omega_i \tau^i_L = x^\mu dx^\nu \gamma^\mu \gamma^\nu \gamma^5 \gamma_\mu \gamma_\nu \gamma_5 \epsilon_0 = -i \varepsilon^{ijk} \tau^k_L \epsilon^+_1.
\]

The supersymmetry variation \((2.5)\) should vanish at all points on \(\mathbb{H}_3\) and for arbitrary tangent vectors. We note that in a chiral basis the Dirac matrices are represented by Pauli matrices

\[
\gamma^i \epsilon^\pm = \pm i \tau^i \epsilon^\pm, \quad \gamma^0 \epsilon^\pm = i \epsilon^\pm.
\]

If we restrict to chiral spinors, the BPS condition reduces to

\[
\omega_i \left[ -i (\rho^i \epsilon^+_0 - i \tau^i_L \epsilon^+_1) - x^\rho \gamma_\rho (\tau^i_L \epsilon^+_0 - i \rho^i \epsilon^+_1) \right] = 0,
\]

so we get the three independent conditions

\[
\rho^i \epsilon^+_0 = i \tau^i_L \epsilon^+_1, \quad i = 1, 2, 3.
\]

To solve these equations one can eliminate \(\epsilon^+_1\) and get the equations for \(\epsilon^+_0\)

\[
\rho^{ij} \epsilon^+_0 = -\tau^j_L \tau^i_L \epsilon^+_0 = -i \varepsilon^{ijk} \tau^k_L \epsilon^+_0.
\]
This is a set of three equations, but only two are independent.

Now we notice that \( \rho^{ij} \) are the generators of a subgroup of the \( SU(4) \) R-symmetry group, which we label \( SU(2)_A \). The indices for the 4 of \( SU(4) \) are split into a pair \( \dot{a}a \) for \( SU(2)_A \times SU(2)_B \) respectively, where the second group acts on the three remaining scalar fields. \( \rho^{ij} \) can then be represented by pauli matrices \( \rho^{ij} = i \epsilon^{ijk} \tau^k_A \). This leads to the three equations

\[
(\tau^i_A + \tau^i_L) \epsilon^+_{0\dot{a}a} = 0 , \quad i = 1, 2, 3 . \tag{2.12}
\]

This means that \( \epsilon^+_0 \) should be a singlet of the diagonal sum of the \( SU(2)_A \) and \( SU(2)_L \) groups. We can express it in terms of an arbitrary 2-component spinor \( \epsilon_a \) as

\[
\epsilon^+_{0\dot{a}a} = (\delta^a_1 \delta^2_{\dot{a}} - \delta^a_2 \delta^1_{\dot{a}}) \epsilon_a = i(\tau^2)^{\dot{a}}_a \epsilon_a . \tag{2.13}
\]

Now using (2.10) we derive\(^3\)

\[
\epsilon^+_{1\dot{a}a} = -i \epsilon_{\alpha \beta} \epsilon^{ab} (\tau^1_L)_{\dot{b}} \epsilon (\tau^1_L)^{\beta \gamma} \epsilon^{+\gamma}_{0\dot{a}b} = (\tau^2)^{\dot{a}}_a \epsilon^{ab} \epsilon_{\dot{b}} , \tag{2.14}
\]

From this we finally see that the general Wilson loop on \( \mathbb{H}_3 \) of the type (2.4) will be invariant under the two supercharges

\[
Q^a = i(\tau^2)^{\dot{a}}_a Q^a + (\tau^2)^{\dot{a}}_a \epsilon^{ab} S^a_{\dot{b}} . \tag{2.15}
\]

### 2.3 Restriction to \( \mathbb{H}_2 \)

A simple restriction on the possible curves is given by setting \( x_3 = 0 \), resulting in an \( \mathbb{H}_2 \) subspace. In this case the scalar couplings are given by

\[
\omega_i = (x^0 dx^1 - x^1 dx^0, x^0 dx^2 - x^2 dx^0, i(x^1 dx^2 - x^2 dx^1)) . \tag{2.16}
\]

This is clearly the analog of the \( S^2 \) loops discussed in \( \cite{2,3,4} \).

A simple calculation shows that for a general path on this space these scalar couplings will guarantee that the loop preserves four supercharges. In addition to the chiral supercharges, these loops also preserve two anti-chiral ones. In the case of \( S^2 \) it was shown \( \cite{3,4} \) that at leading order in perturbation theory the calculation of the supersymmetric Wilson loops is related to that of loops in 2-dimensional Yang-Mills. In special cases this was checked also to 2-loop order \( \cite{13,14} \), and in more restricted cases there are tests that go even beyond that \( \cite{12,23} \). We show in the next subsection that this result extends, at least at leading order, to the loops on \( \mathbb{H}_2 \).

\(^3\)Writing this requires lowering the indices of \( \epsilon_1^+ \), which corresponds to a specific representation of the \( \rho^i \) matrices.
Note that by restricting \( x_0 \) to be a constant one would end with an \( S^2 \subset H_3 \). Unlike the case of \( H_2 \), a general curve on this subspace does not preserve extra supersymmetry, so we will not consider it in detail.

In Section 3 we present special examples of Wilson loops that preserve more than these 4 supercharges. They will all be constrained within this subspace.

### 2.4 Perturbative calculation: 2-dimensional Yang-Mills on \( H_2 \) and the matrix model

We can write \( H_2 \) embedded in \( \mathbb{R}^{2,1} \) in terms of the complex coordinates \( \zeta \) and \( \bar{\zeta} \) in the unit disc as

\[
x_0 = \frac{1 + \zeta \bar{\zeta}}{1 - \zeta \bar{\zeta}}, \quad x_1 = \frac{\zeta + \bar{\zeta}}{1 - \zeta \bar{\zeta}}, \quad x_2 = -i \frac{\zeta - \bar{\zeta}}{1 - \zeta \bar{\zeta}}.
\]  

(2.17)

This gives the metric on the Poincaré disc

\[
ds^2 = \frac{4 d\zeta d\bar{\zeta}}{(1 - \zeta \bar{\zeta})^2}.
\]  

(2.18)

Now we consider the perturbative expansion of the Wilson loop and write down the propagator between two points along the curve, one at \( x \) represented by \((\zeta, \bar{\zeta})\) and one at \( x' \) represented by \((\eta, \bar{\eta})\). Working in the Feynman gauge the effective propagator combining the gauge fields and scalars is

\[
\langle (iA^a_\mu dx^\mu + \Phi^a_\omega \omega_i)(iA^b_\mu dx'^\mu + \Phi^b_\omega \omega'_i) \rangle = -\frac{g_{4d}^2 \delta^{ab}}{4\pi^2} \frac{dx \cdot dx' - \omega_i \omega'_i}{(x - x')^2}
\]

\[=-\frac{g_{4d}^2 \delta^{ab}}{4\pi^2(1 - \zeta \bar{\zeta})(1 - \eta \bar{\eta})} \left[ \frac{\bar{\zeta} - \bar{\eta} d\zeta d\eta + \frac{\zeta - \eta}{\zeta - \bar{\eta}} d\bar{\zeta} d\bar{\eta}}{\zeta - \bar{\eta}} \right].
\]

(2.19)

We note, that as in the case of the loops on \( S^2 \) [3, 4], this is a propagator for a gauge field in two dimensions. Consider YM\(_2\) on \( H_2 \) in the generalized Feynman gauge

\[
L = \frac{1}{\sqrt{2g}} \left[ \frac{1}{4} (F^a_{\alpha \beta})^2 + \frac{1}{2\zeta} (\nabla^\alpha A_{\alpha \alpha})^2 + \partial_\alpha b^a (D^c_c)^a \right],
\]

(2.20)

where

\[
F^a_{\alpha \beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + f^{abc} A^b_\alpha A^c_\beta, \quad (D_\alpha c)^a = \partial_\alpha c^a + f^{abc} A^b_\alpha c^c,
\]

(2.21)

and \( \nabla^\alpha \) is the covariant derivative with respect to the metric (2.18). Setting \( \xi = -1 \) the gauge term becomes

\[
\sqrt{g} L = -\frac{(1 - \zeta \bar{\zeta})^2}{2g_{4d}^2} \left[ (\partial_\zeta A_\zeta)^2 + (\partial_\bar{\zeta} A_{\bar{\zeta}})^2 \right].
\]  

(2.22)
It is easy to check that the propagators

\[
\Delta^{ab}_{\zeta\zeta}(\zeta, \bar{\zeta}; \eta, \bar{\eta}) = \frac{g_2^2 \delta^{ab}}{\pi(1 - \zeta \bar{\zeta})(1 - \eta \bar{\eta})} \frac{\zeta - \bar{\eta}}{\zeta - \eta},
\]

satisfy

\[
\frac{1}{2g_2^2} \partial_{\zeta} \left[(1 - \zeta \bar{\zeta})^2 \partial_{\zeta} \Delta^{ab}_{\zeta\zeta}(\zeta, \bar{\zeta}; \eta, \bar{\eta})\right] = \delta^{ab} \delta^2(\zeta - \eta),
\]

and likewise for \(\Delta^{ab}_{\bar{\zeta}\bar{\zeta}}\).

As was discussed in the \(S^2\) case in [4], these propagators use the Wu-Mandelstam-Leibbrandt prescription for dealing with singularities in the Euclidean light cone propagator [24, 25, 26]. This prescription leads to different results than one would get by a Wick-rotation from Lorentzian signature. This difference was stressed in [27] and resolved in [28], where the authors realized that this prescription gives the same result as one gets by doing the instanton expansion of [29] and focusing on the zero-instanton sector only.

Now consider an arbitrary Wilson loop of 2-dimensional YM on \(\mathbb{H}_2\)

\[
W_{2d} = \frac{1}{N} \text{Tr} \mathcal{P} \exp \int i(A_\zeta \, d\zeta + A_{\bar{\zeta}} \, d\bar{\zeta}).
\]

At leading order in perturbation theory, using the propagators (2.23) this loop will agree precisely with the expression for the Wilson loop in four dimensions (2.19) under the identification of the couplings

\[
g_2^2 = \frac{g_4^2}{4\pi}\]

(2.26)

We conclude that as in the case of the BPS loops on \(S^2\), the Wilson loops on \(\mathbb{H}_2\) seem to be given by this perturbative calculation in two dimensions, at least at the one-loop level. In the case of \(S^2\) the relation between the couplings was such that \(g_2^2\) was negative, but here it is positive.

If the loop is compact and encloses a region of area \(A\), it is easy to evaluate the result of the YM$_2$ calculation. Following [27] it is possible to use the invariance under area-preserving diffeomorphisms and do the explicit calculation in the case of the circle. At leading order we get

\[
\langle W \rangle_{2d} = 1 - g_2^2 N \frac{A(A + 4\pi)}{8\pi}.
\]

(2.27)
The full perturbative series can be expressed in terms of Laguerre polynomials \[27, 10\]

\[
\langle W \rangle^2_{d} = \frac{1}{N} L_{N-1}^1 \left( \frac{g_2^2 A(A + 4\pi)}{4\pi} \right) \exp \left( -\frac{g_2^2 A(A + 4\pi)}{8\pi} \right).
\]

(2.28)

Assuming this result is exact in the four dimensional theory too and concentrating on the large \( N \) limit, the result reduces to a Bessel function

\[
\langle W \rangle_{\text{planar}} = \frac{4\pi}{\sqrt{g_2^2 N A(A + 4\pi)}} J_1 \left( \frac{\sqrt{g_2^2 N A(A + 4\pi)}}{2\pi} \right).
\]

(2.29)

In the case of the loops on \( S^2 \) there were some very interesting 2-loop calculations by Bassetto et al. and Young \[13, 14\]. It would be interesting to repeat those checks for this case.

This result is identical to that of the 1/2 BPS circle \[9, 10, 11\] under the replacement \( g_2^2 N \rightarrow -g_2^2 N A(A + 4\pi)/4\pi^2 \). This expression is also equal to that of the Wilson loop in the Gaussian matrix model. One difference with respect to the circle is the change in sign on the coupling, which has the effect of replacing the modified Bessel function \( I_1 \) with the regular one \( J_1 \). This function has oscillatory behavior at large \( \lambda \), where the Wilson loop is described by a string, suggesting that the string dual should be Lorentzian.

Another difference from the circle, or more generally the loops on \( S^2 \), is that on \( \mathbb{H}_2 \) there are non compact loops that go off to infinity. Such loops are interesting since they asymptote to the light-cone and are therefore similar to light-like cusped Wilson loops that have been used to calculate scattering amplitudes. In these cases it is unclear how the reduction to YM\(_2\) would work. It is still true that the effective propagator is the same as in the lower dimensional theory, but a non-compact curve is not completely gauge invariant. Large gauge transformations will change its expectation value and therefore one cannot rely on the gauge invariance of the two-dimensional theory and perform the calculation in the light-cone gauge. It is therefore left as a question whether the non compact loops are also captured beyond the leading level by two-dimensional YM in some gauge.

## 2.5 String theory description: Complexification and de-Sitter space

We now discuss some of the basic properties of the string theory duals \[30, 31\] of the BPS Wilson loops on \( \mathbb{H}_3 \). The issue we would like to address is that for these loops the scalar couplings are in general imaginary, hence the strings describing the loops will
live on a complexified $S^5$. The most general Wilson loop we constructed has complex
couplings to three scalars, so only three complex directions inside the complexified $S^5$
are turned on.

A general requirement for a Wilson loop to be BPS is that the strength of coupling
to the scalars is the same as that to the gauge fields. Viewing the gauge theory as
dimensionally reduced from $\mathcal{N} = 1$ in ten dimensions, the loop in ten dimensions has
to be light-like.

Therefore one usually considers a real coupling for a space-like Wilson loop, imaginary
coupling for a time-like one and no scalar couplings at all for a light-like loop in
4-dimensions. These loops then have a natural description in the $AdS_5 \times S^5$ dual of
the gauge theory.

But in general one may consider complex scalar couplings, where now the BPS-
condition is that they are light-like in a complexification of (six of the coordinates of)
the ten-dimensional space. Indeed in our construction (2.2) the scalar couplings are
complex.

In such situations the dual $AdS$ interpretation is rather subtle, and requires also
complexification. A related example is in the study of charged local operators in the
Euclidean gauge theory. The charged local operators (like the BMN ground state $\text{Tr} Z^J$
[32]) can be described semiclassically in the Lorentzian theory by particle trajectories
or giant gravitons. In the Euclidean theory, there is no real time and therefore one
cannot describe a real process of propagating from the boundary of $AdS_5$ into the bulk.
This can be resolved by considering a tunneling picture, or a complexification of the
space, which essentially Wick-rotates one of the directions on $S^5$ (see the discussion in
[33]).

In the case of the Wilson loops on $S^3$ coupled to three scalars, the dual string is on
an $H_4 \times S^2$ subspace of $AdS_5 \times S^5$. This subspace can be represented in terms of the
coordinates
\[
 ds^2 = \frac{1}{z^2} \left( dz^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \right) + dy_1^2 + dy_2^2 + dy_3^2 ,
\]
subject to the constraints
\[
 z^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 , \quad y_1^2 + y_2^2 + y_3^2 = 1 .
\]
The first condition is the extension into the bulk of the condition of being on $S^3$ on
the boundary, while the second is that of $S^2 \subset \mathbb{R}^3$.

We want to modify this setup for the case at hand, of the Wilson loops on $H_3$. Now
some of the scalar couplings may be complex, and hence the $y_i$ coordinates too. We
consider in detail only loops on $H_2$, for which only one of the three scalar couplings
becomes imaginary while the others remain real. Wick-rotating \( y_3 = iy_0 \) and \( x_4 = ix_0 \) and setting \( x_3 = 0 \) we get
\[
\begin{align*}
 ds^2 &= \frac{1}{z^2} \left( dx^2 - dx_0^2 + dx_1^2 + dx_2^2 \right) - dy_0^2 + dy_1^2 + dy_2^2 , \\
 x_0^2 - z^2 - x_1^2 - x_2^2 &= 1 , \\
 -y_0^2 + y_1^2 + y_2^2 &= 1 .
\end{align*}
\] (2.32)

Now the \( y_i \) coordinates parameterize de-Sitter space \( dS_2 \) with Lorentzian signature. Any non-trivial string embedding into this space will be of a string with Lorentzian world-sheet and hence the string should propagate also on a Lorentzian submanifold of \( AdS_4 \). Alas, the extra condition on the \( x_i \) and \( z \) coordinates imply that the string moves in an \( H^3 \) subspace of \( AdS_4 \), which is a Euclidean submanifold.

To resolve this issue we propose to Wick-rotate instead of \( x_0 \), both \( x_1 \) and \( x_2 \) (which is very similar to rotating \( x_0 \) and \( z \)). This gives
\[
\begin{align*}
 ds^2 &= \frac{1}{z^2} \left( dz^2 + dx_0^2 - dx_1^2 - dx_2^2 \right) - dy_0^2 + dy_1^2 + dy_2^2 , \\
 x_0^2 + z^2 - x_1^2 - x_2^2 &= 1 , \\
 -y_0^2 + y_1^2 + y_2^2 &= 1 .
\end{align*}
\] (2.33)

Now \( x_i \) and \( z \) parametrize the three-dimensional de-Sitter space \( dS_3 \). To see that define the embedding coordinates
\[
X_0 = \frac{x_0}{z} , \quad X_1 = \frac{x_1}{z} , \quad X_2 = \frac{x_2}{z} , \quad X_3 = \frac{1}{z} .
\] (2.34)

The metric (2.33) reduces to the flat metric for these coordinates with signature \((+,-,-,-)\) and the constraint
\[
X_0^2 - X_1^2 - X_2^2 - X_3^2 = -1 .
\] (2.35)

In this space we can embed strings with Lorentzian world-sheets.

In attempts to quantize gravity on de-Sitter space its similarity to \( AdS \) space is often employed (see e.g. [34, 35, 36]). This similarity is realized here by the fact that the target space of our \( \sigma \)-model seems to be automatically continued from \( AdS_3 \times S^2 \) to \( dS_3 \times dS_2 \). We will refrain, however, from trying to interpret our results as a holographic realization of de-Sitter space and view it rather as an analytical continuation.

Yet, for practical purposes we are dealing with de-Sitter space which raises some of the usual issues associated to that space, like the choice of global structure. Global de-Sitter space has two conformal boundaries, one at asymptotic past and one at asymptotic future. An alternative to that exists, where antipodal points are identified, giving \( dS/\mathbb{Z}_2 \) [37, 38]. Taking this choice in our case has the effect of limiting the range of \( y_0 \) to \( \mathbb{R}_+ \) and likewise for \( X_0 \). Since we got these spaces by a Wick-rotation, it is not entirely clear what the range of the coordinates is. While we do not have strong arguments against global de-Sitter space, it seems in the examples we study below that the interpretation in terms of the orbifold \( dS/\mathbb{Z}_2 \) is more natural.

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Figure 1: The hyperbolic line $x_2 = 0$ represented on the hyperboloid $x_0^2 - x_1^2 - x_2^2 = 1$ and its stereographic projection onto the Poincaré disc.

3 Examples

3.1 1/2 BPS hyperbolic line

Consider the hyperbolic line

$$x^\mu = (\cosh t, \sinh t, 0, 0), \quad -\infty < t < \infty.$$  \hspace{1cm} (3.1)

We can immediately read from (2.2) that the three scalar couplings will be given by

$$\omega_i = (1, 0, 0) \, dt,$$  \hspace{1cm} (3.2)

so it has constant coupling to a single scalar $\Phi^1$. Checking the supersymmetry variation of this loop one finds that it is annihilated by half the supercharges, just like the straight line or circle.

One may be quite surprised by this, it is generally assumed that the only half-BPS Wilson loops are the straight line and the circle, which are related to each other by a conformal transformation. That is indeed true in Euclidean $\mathbb{R}^4$, but on Minkowski space there are some more possibilities: The space-like hyperbolic line like (3.1) is clearly a boost of the line in the $x^1$ direction and likewise there is the time-like hyperbolic line

$$x^\mu = (\sinh t, \cosh t, 0, 0).$$  \hspace{1cm} (3.3)

This curve is conformal to a time-like straight line and with a constant (imaginary) scalar coupling will also be 1/2 BPS.
Figure 2: A representation of the minimal surface for the hyperbolic line. The vertical direction represents the “AdS direction” \( z = \sqrt{x_0^2 - x_1^2 - 1} \). The blue line is at the boundary and the thin lines at constant \( z \) are a natural choice for a cutoff on the world-sheet.

In fact this second line belongs to another family of BPS loops that may be constructed on a 3-dimensional de-Sitter subspace of Minkowski space. We will not study them in this paper.

Using the representation (2.17), the Wilson loop is given by \( \zeta = \tanh \frac{t}{2} \). Then expanding the loop to second order and using (2.19) we find

\[
\langle W \rangle = 1 - g_{\text{id}}^2 N \frac{1}{2} 2 \frac{\pi^2}{4} \int dt_1 dt_2 \frac{2\dot{\zeta}(t_1)\dot{\zeta}(t_2)}{(1 - \zeta(t_1)^2)(1 - \zeta(t_2)^2)} + \cdots
\]

As for the case of the circle [9], the integrand, which is the combined propagator, is a constant \((1/2)\) thus

\[
\langle W \rangle = 1 - \lambda \frac{1}{32\pi^2} \int dt_1 dt_2 + \cdots = 1 - \lambda \frac{1}{32\pi^2} (2T)^2 + \cdots
\]

where \( T \) is a cutoff on the parameter \( t \), corresponding to taking the two endpoints

\[
(x_0, x_1) = (\cosh T, - \sinh T), \quad (x'_0, x'_1) = (\cosh T, \sinh T).
\]

Unlike the circle, this Wilson loop suffers from infra-red divergences, which occur also in some of the other examples of Wilson loops listed below. These divergences have to do with the fact that the curves are not compact and their physical meaning will be studied elsewhere.

We turn now to finding the string dual to this Wilson loop. The loop only couples to a single scalar \( \Phi^1 \) and has a real coupling. Therefore we do not need to consider the embedding of the string into a complexified \( AdS_5 \times S^5 \). Rather the string is localized
at a constant point $y_2 = y_3 = 0$ on $S^5$. For the $AdS_5$ solution we consider an $AdS_3$ subspace with metric
\[ ds^2 = \frac{1}{z^2}(dz^2 - dx_0^2 + dx_1^2). \] (3.7)

The string surface should end along the curve $x_0^2 - x_1^2 = 1$ on the boundary at $z = 0$. From symmetry considerations (and since it is conformal to the straight line and to the circle) we know that the string will span an $H_2$ subspace of target space which is given by the equation
\[ z^2 = x_0^2 - x_1^2 - 1. \] (3.8)

This can be verified, of course, by checking the equations of motion. The simplest way to write them is to make a change of coordinates
\[ x_0 = e^w \cosh \alpha \cosh t, \quad x_1 = e^w \cosh \alpha \sinh t, \quad z = e^w \sinh \alpha, \] (3.9)
which gives the metric
\[ ds^2 = \frac{1}{\sinh^2 \alpha} (-dw^2 + d\alpha^2 + \cosh^2 \alpha dt^2). \] (3.10)

Now we take the world-sheet coordinates $\sigma$ and $\tau$ and the ansatz
\[ w = w(\sigma), \quad \alpha = \alpha(\sigma), \quad t = \tau. \] (3.11)

This ansatz is consistent and leads to the action
\[ S = \sqrt{\lambda} \int d\sigma d\tau \frac{1}{\sinh^2 \alpha} (-w'^2 + \alpha'^2 + \cosh^2 \alpha \ dt^2). \] (3.12)

Clearly $w$ is cyclic and setting it to a constant will solve the equations of motion.

The Virasoro constraint is
\[ \alpha'^2 = \cosh^2 \alpha, \] (3.13)
and is solved (up to a trivial shift of $\sigma$) by
\[ \tanh \alpha = \sin \sigma, \quad 0 \leq \sigma < \frac{\pi}{2}. \] (3.14)

It is easy to check that this also solves the equation of motion of $\alpha$, and indeed is a parametrization of the surface $z^2 = x_0^2 - x_1^2 - 1$.

An alternative way of finding this solution (as was done for the circle in [39]) is starting with the solution for the straight line along the $x_1$ direction at $x_0 = 0$, which will simply span the $z$ direction and act on it with the $AdS_5$ isometry dual to the boost in the $x_0$ direction
\[ x_0 \to \frac{1 + x_1^2 + z^2}{1 - x_1^2 - z^2}, \quad x_1 \to \frac{2x_1}{1 - x_1^2 - z^2}, \quad z \to \frac{2z}{1 - x_1^2 - z^2}. \] (3.15)
This clearly is the same surface satisfying the constraint (3.8) parametrized in a different way than above.

Yet another alternative to finding this solution are the techniques presented in [15].

The classical action for this solution diverges and requires regularization. The area is

\[ S_{\text{cl.}} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \frac{1}{\sin^2 \sigma} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \left[ \frac{1}{z_{\text{min}}} - \frac{1}{z_{\text{max}}} \right]. \] (3.16)

\( z_{\text{min}} \) is a cutoff near the boundary of space and is usually discarded. \( z_{\text{max}} \) is the maximal value of \( z \) on the world-sheet and if we take \( z_{\text{max}} \to \infty \), we get that the classical action vanishes.

Another possible prescription is to impose a cutoff at fixed \( x_0 \). Then ignoring the divergence from \( z_{\text{min}} \) in (3.16) we have

\[ S_{\text{cl.}} = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau \frac{\cosh \tau}{\sqrt{x_0^2 - \cosh^2 \tau}} = -\frac{\sqrt{\lambda}}{2}, \] (3.17)

which is half of the result for the circle. Thus depending on the choice of cutoff we have the same result as for half the circle or the line. This is not surprising, since the hyperbolic line is conformal to a segment on the line or circle.

Since we are dealing with a supersymmetric Wilson loop it seems more natural, though, to follow the prescription of [40], adding a total derivative term

\[ S_{\text{cl.}} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left( \frac{1}{\sin^2 \tau} + \left( z'/z \right)' \right) = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \frac{1}{\cos^2 \sigma} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau z_{\text{max}} \] (3.18)

This expression does not diverge in the UV, near \( z \sim 0 \), but it does diverge in the IR, for large \( z \). Again, a possible regularization prescription is to integrate the world-sheet up to a cutoff \( x_0 = \cosh T \). This gives

\[ \frac{\sqrt{\lambda}}{2\pi} \int d\tau \frac{\sqrt{x_0^2 - \cosh^2 \tau}}{\cosh \tau} = \frac{\sqrt{\lambda}}{2} (\cosh T - 1). \] (3.19)

This is proportional to the area on \( \mathbb{H}_2 \) of half a circle of radius \( \zeta = \tanh \frac{T}{2} \), which also appears in the gauge theory result (2.29). Note though, that the result in (2.29) corresponded to closed curves, while the hyperbolic line is not. Furthermore the extrapolation of (2.29) to large \( \lambda \) gives an oscillating function, while the result we find in the \( AdS \) calculation is real.

Another alternative is to simply keep a fixed IR cutoff \( z_{\text{max}} \) which gives

\[ \frac{\sqrt{\lambda}}{2\pi} 2Tz_{\text{max}}. \] (3.20)
Note that the string world-sheet is open at future infinity (as is the dual Wilson loop operator), so there are different ways to add total derivatives which contribute differently in the infra-red. Furthermore, the string solution itself depends on a choice of behavior at future infinity, not only at the boundary at $z \to 0$. The solution (3.8) should be the only one preserving 1/2 of the supercharges, though these supercharges are probably broken by the cutoff $T$. It is not altogether obvious how to match regularization prescriptions in string theory and the gauge theory that break the supersymmetry in a similar way.

Some issues related to this, and to the existence of other solutions with different behavior at future infinity will be presented elsewhere.

Since this loop is conformal to the line, all the known descriptions of that Wilson loop could be adapted to this case. They include the D3-brane and D5-brane descriptions appropriate for loops in representations of dimension of order $N$ \cite{30, 41, 42, 43, 44, 45}, or the full back-reacted “bubbling geometries” appropriate for loops in representations of dimension of order $N^2$ \cite{46, 47, 48, 49}. Likewise it is possible to calculate the correlation function of this Wilson loop with chiral primary operators in all the different pictures \cite{39, 50, 51}.

### 3.2 1/4 BPS hyperbolic cusp

A geodesic on $\mathbb{H}_2$, as mentioned before, is 1/2-BPS and is the analog of a great circle on $S^2$. Any two such lines will share 1/4 of the supercharges. Of course on $S^2$ any two great circles will cross, while on $\mathbb{H}_2$ there are non-intersecting geodesics. Non-intersecting circles exist on $S^3$, for example the Hopf-fibers, which indeed were an interesting example studied in \cite{2, 4}.

But the intersecting circles were also interesting, since one can then make a closed loop out of two half-circles and it is 1/4-BPS, which is the “longitudes” example in \cite{2, 4} (see also \cite{13, 14}). Now instead we can consider two hyperbolic rays meeting at a point

$$x^\mu = \begin{cases} 
(cosh t, \sinh t, 0, 0), & t < 0, \\
(cosh t, -\cos \delta \sinh t, \sin \delta \sinh t, 0), & t > 0.
\end{cases} \quad (3.21)$$

The scalar couplings are

$$\omega_i = \begin{cases} 
(1, 0, 0) dt, & t < 0, \\
(-\cos \delta, \sin \delta, 0) dt, & t > 0.
\end{cases} \quad (3.22)$$

As mentioned above, a hyperbolic line is conformal to a straight line, and thus this “hyperbolic-cusp” is conformal to a cusp in the $(x^1, x^2)$ plane. Explicitly (3.21) can be
Figure 3: A cusp made of two hyperbolic rays and its projection to the Poincaré disc.

written in terms of the complex coordinates on the unit disc through (2.17) as

\[ \zeta = \begin{cases} \tanh \frac{t}{2}, & t < 0, \\ e^{i(\pi-\delta)} \tanh \frac{t}{2}, & t > 0. \end{cases} \tag{3.23} \]

The Wilson loop made from this cusp in the \((x_1, x_2)\) plane and the above scalar couplings will be also BPS, it is in the class of operators constructed in [5]. The string solution describing this loop was written down in [4] and it can be mapped then to the desired configuration by the \(AdS_5\) isometry which extends the conformal transformation (2.17) to the bulk.

We repeat here the calculation of the string surface, using the Polyakov action and the conformal gauge rather than the Nambu-Goto action as in [4].

Starting with \(AdS_4 \times S^1\) with metric

\[ ds^2 = \frac{1}{z^2} \left( dz^2 - dx_0^2 + dx_1^2 + dx_2^2 \right) + d\varphi^2, \tag{3.24} \]

we change coordinates to

\[ x_0 = e^w \coth \mu, \quad x_1 + ix_2 = r e^{i\phi} = \frac{e^{w+i\phi} \cos \nu}{\sinh \mu}, \quad z = \frac{e^w \sin \nu}{\sinh \mu}, \tag{3.25} \]

so the metric becomes

\[ ds^2 = \frac{1}{\sin^2 \nu} \left( d\mu^2 - \sinh^2 \mu dw^2 + dv^2 + \cos^2 \nu \, d\phi^2 \right) + d\varphi^2. \tag{3.26} \]

The boundary conditions for the string are at \(w = 0\) and it is a consistent ansatz to set \(w = 0\) along the entire world-sheet. This corresponds to the fact that the string is contained within an \(H_3 \times S^1\) subspace of \(AdS_5 \times S^5\) given by \(x_0^2 - r^2 - z^2 = 1\).
We take now the ansatz
\begin{align}
  w &= 0, \quad \mu = \mu(\tau), \quad \nu = \nu(\sigma), \quad \phi = \phi(\sigma), \quad \varphi = \varphi(\sigma).
\end{align}

Using dot for \( \partial_{\tau} \) and prime for \( \partial_{\sigma} \), the action for a Euclidean string world-sheet is
\begin{align}
  S &= \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left( \frac{1}{\sin^2 \nu} (\mu^2 + \nu'^2 + \cos^2 \nu \phi'^2) + \varphi'^2 \right).
\end{align}

Since the only \( \tau \) dependence is in \( \dot{\mu} \), it has to be a constant \( p \). Beyond that there are two obvious conserved quantities
\begin{align}
  E &= \cot^2 \nu \phi', \quad J = \varphi',
\end{align}
where we chose the names since one is related to motion on \( AdS \) and the other on the sphere, but not too much should be read into that choice of symbols.

Lastly there is the Virasoro constraint
\begin{align}
  p^2 &= \nu'^2 + \sin^2 \nu \left( E^2 \tan^2 \nu + J^2 \right).
\end{align}

These sets of equations can be solved for general \( E, J \) and \( p \), but they are particularly simple in the BPS case, when \( E^2 = J^2 = 1 - p^2 \), which will turn out to be the relevant case for us (we take them all positive). We find
\begin{align}
  \nu^2 &= 1 - \frac{1 - p^2}{\cos^2 \nu}.
\end{align}
\( \nu \) varies from zero at the boundary of \( AdS \) to a maximal value \( \sin \nu = p \), at which point \( \nu' = 0 \) and then it turns back towards the boundary.

The equation for \( \nu \) (3.31) integrates to
\begin{align}
  \sin \nu &= p \sin \sigma.
\end{align}
Using the conservation equations
\begin{align}
  \phi' &= \sqrt{1-p^2} \tan^2 \nu = \sqrt{1-p^2} \frac{p^2 \sin^2 \sigma}{1-p^2 \sin^2 \sigma}, \quad \varphi' = \sqrt{1-p^2},
\end{align}
we can integrate \( \phi \) and \( \varphi \)
\begin{align}
  \tan(\phi + \sqrt{1-p^2} \sigma) &= \sqrt{1-p^2} \tan \sigma, \quad \varphi = \sqrt{1-p^2} \sigma.
\end{align}

Going back to the original coordinates (3.25) we have
\begin{align}
  x_0 &= \coth p\tau, \quad x_1 + ix_2 = \frac{e^{-i\sqrt{1-p^2} \sigma} (\cos \sigma + i \sqrt{1-p^2 \sin \sigma})}{\sinh p\tau}, \quad z = \frac{p \sin \sigma}{\sinh p\tau}.
\end{align}
This solution approaches the boundary at $\sigma = 0$, where $\phi = 0$ and at $\sigma = \pi$ where $\phi = (1 - \sqrt{1 - p^2})\pi$. This means that $p$ is related to the opening angle $\delta$ in (3.21) by
\[ p = \frac{1}{\pi} \sqrt{\delta(2\pi - \delta)}. \] (3.36)

We wish to evaluate the action of the classical string solution. Integrating the area gives a divergent result, proportional to the length of the curve $2T$. As discussed in the previous section, there are different possible regularizations. Here we follow [40] and add a total derivative term
\[ S_{\text{cl.}} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left( \frac{1}{\sin^2 \sigma} + \frac{z''z - z'^2 + \dot{z}z - \dot{\dot{z}}^2}{z^2} \right) = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma p \left( \coth p\tau_{\text{min}} - 1 \right). \] (3.37)

The total derivative eliminated the divergence from the small $z$ region, but the integral is still divergent and it is left to specify the boundary of the world-sheet along which the $\sigma$ integral is to be performed.

One regularization is to consider the world-sheet up to a fixed value of $x_0 = \cosh T = \coth p\tau_{\text{min}}$ giving
\[ \frac{\sqrt{\lambda}}{2} p (\cosh T - 1) = \frac{\sqrt{\lambda}}{2\pi} \sqrt{\delta(2\pi - \delta)} (\cosh T - 1). \] (3.38)

This is the analog of (3.19), and again is proportional to the area bounded by the cusp and the cutoff on $x_0$.

An alternative regularization is to restrict the world-sheet to $z \leq z_{\text{max}}$ and $x_0 \leq \cosh T$. Now the range of integration is a bit more complicated. For $0 \leq \sin \sigma \leq z_{\text{max}}/p\sinh T$ we take $\coth p\tau_{\text{min}} = \cosh T$, while in the rest of the interval we need to restrict to $\coth p\tau_{\text{min}} = \sqrt{z_{\text{max}}^2 + p^2 \sin^2 \sigma}/p \sin \sigma$. This gives
\[ S_{\text{cl.}} = \frac{\sqrt{\lambda}}{2\pi} \left[ 2p \cosh T \arcsin \frac{z_{\text{max}}}{p \sinh T} - \pi p + \int d\sigma \frac{\sqrt{z_{\text{max}}^2 + p^2 \sin^2 \sigma}}{\sin \sigma} \right] \] (3.39)
\[ = \frac{\sqrt{\lambda}}{\pi} \left[ p \cosh T \arcsin \frac{z_{\text{max}}}{p \sinh T} - p \arcsin \frac{z_{\text{max}} \cosh T}{\sqrt{p^2 + z_{\text{max}}^2}} + z_{\text{max}} \arccosh \frac{p \cosh T}{\sqrt{p^2 + z_{\text{max}}^2}} \right]. \]

### 3.3 Constant curvature curves: 1/4 BPS circle

A special class of curves on $\mathbb{H}_2$ are those with constant curvature. The analog curves on a sphere are latitudes, while on hyperbolic spaces there are two possibilities: Circles and non-compact lines.
A circle is given by constant $x_0$

$$x = (\cosh v_0, \sinh v_0 \cos t, \sinh v_0 \sin t, 0), \quad 0 \leq t \leq 2\pi.$$  \hspace{1cm} (3.40)

Our prescription (2.2) gives periodic scalar couplings

$$\omega_i = \sinh v_0 ( -\cosh v_0 \sin t, \cosh v_0 \cos t, i \sinh v_0 ) \, dt.$$  \hspace{1cm} (3.41)

This loop preserves 8 supercharges, just like the latitude on $S^2$, as can be easily verified by studying (2.5), (the details can be found in [22]).

In perturbation theory this loop is a lot like the latitude on $S^2$ [12], which in turn is a lot like the usual circle [9]. The combined gauge-field plus scalar propagator (2.19) is a constant

$$\left\langle (iA^a_{\mu} dx^\mu + \Phi^i a \omega_i)(iA^b_{\mu} dx'^{\mu} + \Phi^i b \omega'_i) \right\rangle = -\frac{g^2_{id} \delta^{ab}}{8\pi^2} \sinh^2 v_0 \, dt \, dt'.$$  \hspace{1cm} (3.42)

Likewise the interaction graphs at order $g^2_{id}$ cancel in the Feynman gauge. So one would expect that as in the other cases, the entire perturbative series will be captured by the Gaussian matrix model, which in the planar approximation gives (2.29)

$$\langle W \rangle \simeq \frac{2}{\sqrt{\lambda} \sinh v_0} J_1 \left( \sqrt{\lambda} \sinh v_0 \right).$$  \hspace{1cm} (3.43)

We turn now to finding the string dual of this Wilson loop. Unlike the previous two examples, the line and cusp, in this case the scalar couplings (3.41) are complex. Therefore we will not be able to embed the string dual in $AdS_5 \times S^5$, rather we need to consider a complexification of this space.
Figure 5: A representation of the minimal surface solutions for the circle on the hyperboloid. (a.) The AdS metric is Wick-rotated to $dS_3$ where the string fills half of a $dS_2$ subspace bounded by the blue circle at $z = 0$. The $S^2 \subset S^5$ is Wick-rotated to $dS_2$ where the boundary values are again represented by the blue circle. The surface either covers the part of space above the circle (b.) or below the circle (c.) where the string is represented on the orbifold $dS_2/\mathbb{Z}_2$, so the surface is reflected back up from the $y_0 = 0$ plane.

In the case of the latitude the string solution is given in the metric (2.30) by [52, 12]

\[
\begin{align*}
x_1 + ix_2 &= \frac{e^{i\tau} \tanh \sigma_0}{\cosh \sigma}, & x_3 &= \frac{1}{\cosh \sigma_0}, & z &= \tanh \sigma_0 \tanh \sigma, \\
y_1 + iy_2 &= \frac{e^{i\tau}}{\cosh(\sigma_0 \pm \sigma)}, & y_3 &= \tanh(\sigma_0 \pm \sigma).
\end{align*}
\tag{3.44}
\]

This loop ends at the boundary at $\sigma = 0$ along the latitude at $|x_1 + ix_2| = \tanh \sigma_0$ and $x_3 = 1/\cosh \sigma_0$.

Now we want to consider instead a loop ending along the circle with $|x_1 + ix_2| > 1$, which can be represented by an imaginary $\sigma_0$. We therefore Wick-rotate both $\sigma$ and $\sigma_0$ in the latitude solution into

\[
\begin{align*}
x_1 + ix_2 &= \frac{e^{i\tau} \tan \sigma_0}{\cos \sigma}, & x_0 &= \frac{1}{\cos \sigma_0}, & z &= \tan \sigma_0 \tan \sigma, \\
y_1 + iy_2 &= \frac{e^{i\tau}}{\cos(\sigma_0 \pm \sigma)}, & y_0 &= \tan(\sigma_0 \pm \sigma).
\end{align*}
\tag{3.45}
\]

While the original solution satisfied

\[
\begin{align*}
z^2 + x_1^2 + x_2^2 + x_3^2 &= 1, & y_1^2 + y_2^2 + y_3^2 &= 1,
\end{align*}
\tag{3.46}
\]

this new configuration satisfies

\[
\begin{align*}
z^2 + x_0^2 - x_1^2 - x_2^2 &= 1, & y_1^2 + y_2^2 - y_0^2 &= 1.
\end{align*}
\tag{3.47}
\]
This indeed fits a target space Wick-rotated to the metric (2.33).

One can easily check that (3.45) is a solution of the equations of motion for a string with Lorentzian world-sheet in the metric (2.33)

\[ S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left( \frac{z'^2 + x'^2_0 - x'^2_1}{z^2} - y'^2_0 + y'^2_1 + y'^2_2 - y'^2_1 - y'^2_2 \right. \]

\[ \left. + \Lambda(y'^2_1 + y'^2_2 - y'^2_0 - 1) \right) \]  

(3.48)

On the classical solution this is equal to

\[ S_{cl.} = \sqrt{\lambda} \int d\sigma \left( \frac{1}{\sin^2 \sigma} - \frac{1}{\cos^2(\sigma_0 \pm \sigma)} \right). \]  

(3.49)

We have not specified the range of \( \sigma \) integration. \( \sigma = 0 \) is the boundary of space, where we should impose \( y_0 = \tan \sigma_0 = \sinh v_0 \). There is the usual UV divergence, which can be removed by adding a total derivative \( (z''z - z'^2)/z^2 \). At \( \sigma \to \pi/2 \) the solution reaches \( |x_1 + ix_2| \to \infty \), which is a reasonable end for that part of the solution. For the \( y_0 \) coordinate it seems like we should also allow it to go to infinity, corresponding to \( \sigma = \pi/2 \mp \sigma_0 \). Then we have

\[ S_{cl.} = \sqrt{\lambda} \int d\sigma \left( \frac{1}{\cos^2 \sigma} - \frac{1}{\cos^2(\sigma_0 \pm \sigma)} \right) = \sqrt{\lambda} \left( \tan \sigma \mp \tan(\sigma_0 \pm \sigma) \right). \]  

(3.50)

With this peculiar choice of boundary conditions the divergences at \( \sigma = \pi/2 \) for the first term and from \( \sigma = \pi/2 \mp \sigma_0 \) for the second term cancel and we end up with the contribution from \( \sigma = 0 \)

\[ S_{cl.} = \pm \sqrt{\lambda} \tan \sigma_0 = \pm \sqrt{\lambda} \sinh v_0. \]  

(3.51)

The choice of sign corresponds to two solutions one where \( y_0 \to \infty \) and the other where \( y_0 \to -\infty \). Together we get that the expectation value of this Wilson loop behaves like

\[ \langle W_{\text{circle}} \rangle \sim \sum_{\pm} e^{iS_{cl.}} \sim \cos \left( \sqrt{\lambda} \sinh v_0 \right). \]  

(3.52)

This agrees with the matrix model result (3.43).

It should be possible to find the D3-brane dual to this Wilson loop in analogy to the calculation in [23] as well as its coupling to chiral primary operators [53].

### 3.4 Constant curvature curves: 1/4 BPS hyperbolic line

The other class of constant curvature curves on \( \mathbb{H}_2 \) is that of hyperbolic lines whose curvature does not match that of the underlying space. Such a line is given by

\[ x = (\cosh v_0 \cosh t, \cosh v_0 \sinh t, \sinh v_0, 0), \quad -\infty < t < \infty, \]  

(3.53)
and the scalar couplings are now hyperbolic
\[
\omega_i = \cosh v_0 (\cosh v_0, -\sinh v_0 \sinh t, -i \sinh v_0 \cosh t) \, dt .
\] (3.54)

Like the previous example, this loop also preserves 8 supercharges. And yet again, the combined propagator is a constant
\[
\langle (i A^i_\mu dx^\mu + \Phi^i_\omega_i)(i A^b_\mu dx'^\mu + \Phi^b_\omega_i') \rangle = -\frac{g^2_4 g^{ab}}{8\pi^2} \cosh^2 v_0 \, dt \, dt' .
\] (3.55)

Unlike the previous example, though, it is not easy to sum up ladder diagrams. This Wilson loop is non-compact and the calculation diverges at both ends of the line. As the situation in the simpler case of the 1/2 BPS hyperbolic line is complicated enough, we do not try to resolve the issues associated with these divergences here.

In considering the string dual we take the solution of the latitude on $S^2$ (3.44) and analytically continue $\tau \to i\tau$ and $\sigma_0 \to \sigma_0 - i\pi/2$. This gives
\[
\begin{align*}
x_0 &= \frac{\cosh \tau \coth \sigma_0}{\cosh \sigma} , & x_1 &= \frac{\sinh \tau \coth \sigma_0}{\cosh \sigma} , & x_2 &= \frac{1}{\sinh \sigma_0} , & z &= \coth \sigma_0 \tanh \sigma , \\
y_1 &= \coth(\sigma_0 \pm \sigma) , & y_2 &= -\frac{\sinh \tau}{\sinh(\sigma_0 \pm \sigma)} , & y_0 &= \frac{\cosh \tau}{\sinh(\sigma_0 \pm \sigma)} .
\end{align*}
\] (3.56)

where we have relabeled the coordinates and Wick rotated three of them. This solution is now embedded in a target-space with metric (2.33).

The boundary conditions are satisfied for $\sinh \sigma_0 = 1/\sinh v_0$ and it is easy to check that this is a solution of the equations of motion for a Lorentzian world-sheet. As $\sigma \to \infty$ the world-sheet approaches the curve given by $z = \cosh v_0$, $x_2 = \sinh v_0$.
and $x_0 = x_1$. The world-sheet has to be analytically continued and we find another patch of the solution with

\[
\begin{align*}
    x_0 &= \frac{\sinh \tau \coth \sigma_0}{\sinh \sigma}, & x_1 &= \frac{\cosh \tau \coth \sigma_0}{\sinh \sigma}, & x_2 &= \frac{1}{\sinh \sigma_0}, & z &= \coth \sigma_0 \coth \sigma, \\
    y_1 &= \tanh(\sigma_0 \pm \sigma), & y_2 &= \frac{\cosh \tau}{\cosh(\sigma_0 \pm \sigma)}, & y_0 &= \frac{\sinh \tau}{\cosh(\sigma_0 \pm \sigma)}.
\end{align*}
\]

Finally we need to add a third patch to the world-sheet which is identical to (3.56) only with negative $x_0$ and $y_1$.

The resulting string solution ends along two curves on the boundary. One on the original hyperboloid with $x_0 > 0$ and the other on the second hyperboloid with $x_0 < 0$. This has been forced on us through the analytical continuation of the solution. We note though that perhaps it is legitimate to consider only half of this solution. With the Wick-rotation of the coordinate $z$ the $AdS$ space has turned into $dS$ and away from $x_0 = z = 0$ we can do the same $\mathbb{Z}_2$ identification on this space as we did on the de-Sitter space that replaced $S^2$. Then the solution will be given by one patch like (3.56) and half of (3.57), with $x_0 \geq 0$. 

Figure 7: A representation of the minimal surface solutions for the 1/4 BPS hyperbolic line. (a.) The string ends on two lines on the hyperboloids with positive and negative $x_0$, but one can consider only half of the world-sheet, assuming the string dual lives on $dS_3/\mathbb{Z}_2$. The red dotted lines are the borders between the coordinate patches described in the text. For the correlator of two lines, the string should end also along two hyperbolic lines on $dS_2$ (b.) and the string goes to one side and is reflected from infinity back. If considering just a single line, it should be enough to take only the piece of the string going to the right as in (c.), or to the left.
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References

[1] E. D'Hoker, J. Estes, M. Gutperle, D. Krym, and P. Sorba, “Half-BPS supergravity solutions and superalgebras,” *JHEP* 12 (2008) 047, arXiv:0810.1484.

[2] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “More supersymmetric Wilson loops,” *Phys. Rev.* D76 (2007) 107703, arXiv:0704.2237.

[3] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Wilson loops: From four-dimensional SYM to two-dimensional YM,” *Phys. Rev.* D77 (2008) 047901, arXiv:0707.2699.

[4] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Supersymmetric Wilson loops on $S^3$,” *JHEP* 05 (2008) 017, arXiv:0711.3226.

[5] K. Zarembo, “Supersymmetric Wilson loops,” *Nucl. Phys.* B643 (2002) 157–171, hep-th/0112116.

[6] Z. Guralnik and B. Kulik, “Properties of chiral Wilson loops,” *JHEP* 01 (2004) 065, hep-th/0309118.

[7] Z. Guralnik, S. Kovacs, and B. Kulik, “Less is more: Non-renormalization theorems from lower dimensional superspace,” *Int. J. Mod. Phys.* A20 (2005) 4546–4553, hep-th/0409091.

[8] A. Dymarsky, S. S. Gubser, Z. Guralnik, and J. M. Maldacena, “Calibrated surfaces and supersymmetric Wilson loops,” *JHEP* 09 (2006) 057, hep-th/0604058.

[9] J. K. Erickson, G. W. Semenoff, and K. Zarembo, “Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory,” *Nucl. Phys.* B582 (2000) 155–175, hep-th/0003055.

[10] N. Drukker and D. J. Gross, “An exact prediction of $\mathcal{N} = 4$ SUSYM theory for string theory,” *J. Math. Phys.* 42 (2001) 2896–2914, hep-th/0010274.

[11] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” arXiv:0712.2824.

[12] N. Drukker, “1/4 BPS circular loops, unstable world-sheet instantons and the matrix model,” *JHEP* 09 (2006) 004, hep-th/0605151.
[13] A. Bassetto, L. Griguolo, F. Pucci, and D. Seminara, “Supersymmetric Wilson loops at two loops,” *JHEP* **06** (2008) 083, arXiv:0804.3973.
[14] D. Young, “BPS Wilson loops on $S^2$ at higher loops,” *JHEP* **05** (2008) 077, arXiv:0804.4098.
[15] R. Ishizeki, M. Kruczenski, and A. Tirziu, “New open string solutions in $AdS_5$,” *Phys. Rev.* **D77** (2008) 126018, arXiv:0804.3438.
[16] L. F. Alday and J. M. Maldacena, “Gluon scattering amplitudes at strong coupling,” *JHEP* **06** (2007) 064, arXiv:0705.0303.
[17] J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, “Conformal properties of four-gluon planar amplitudes and Wilson loops,” *Nucl. Phys.* **B795** (2008) 385–408, arXiv:0707.1153.
[18] A. Brandhuber, P. Heslop, and G. Travaglini, “MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills and Wilson loops,” *Nucl. Phys.* **B794** (2008) 231–243, arXiv:0707.1155.
[19] Z. Bern *et al.*, “The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang-Mills theory,” *Phys. Rev.* **D78** (2008) 045007, arXiv:0803.1465.
[20] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, “Hexagon Wilson loop = six-gluon MHV amplitude,” arXiv:0803.1466.
[21] L. F. Alday and R. Roiban, “Scattering Amplitudes, Wilson Loops and the String/Gauge Theory Correspondence,” *Phys. Rept.* **468** (2008) 153–211, arXiv:0807.1889.
[22] V. Branding, “Supersymmetric Wilson loops in the $AdS$/CFT correspondence,”. Diploma thesis, Humboldt University Berlin, 2008; available at http://qft.physik.hu-berlin.de.
[23] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “On the D3-brane description of some 1/4 BPS Wilson loops,” *JHEP* **04** (2007) 008, hep-th/0612168.
[24] T. T. Wu, “Two-dimensional Yang-Mills theory in the leading $1/N$ expansion,” *Phys. Lett.* **B71** (1977) 142.
[25] S. Mandelstam, “Light cone superspace and the ultraviolet finiteness of the $\mathcal{N} = 4$ model,” *Nucl. Phys.* **B213** (1983) 149–168.
[26] G. Leibbrandt, “The light cone gauge in Yang-Mills theory,” *Phys. Rev.* **D29** (1984) 1699.
[27] M. Staudacher and W. Krauth, “Two-dimensional QCD in the Wu-Mandelstam-Leibbrandt prescription,” *Phys. Rev.* **D57** (1998) 2456–2459, hep-th/9709101.
[28] A. Bassetto and L. Griguolo, “Two-dimensional QCD, instanton contributions and the perturbative Wu-Mandelstam-Leibbrandt prescription,” *Phys. Lett.* **B443** (1998) 325–330, hep-th/9806037.
[29] E. Witten, “Two-dimensional gauge theories revisited,” J. Geom. Phys. 9 (1992) 303–368, hep-th/9204083.

[30] S.-J. Rey and J.-T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C22 (2001) 379–394, hep-th/9803001.

[31] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80 (1998) 4859–4862, hep-th/9803002.

[32] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, “Strings in flat space and pp waves from N = 4 Super Yang Mills,” AIP Conf. Proc. 646 (2003) 3–14.

[33] T. Yoneya, “What is holography in the plane-wave limit of AdS5/SYM4 correspondence?,” Prog. Theor. Phys. Suppl. 152 (2004) 108–120, hep-th/0304183.

[34] A. Strominger, “The dS/CFT correspondence,” JHEP 10 (2001) 034, hep-th/0106113.

[35] V. Balasubramanian, J. de Boer, and D. Minic, “Mass, entropy and holography in asymptotically de Sitter spaces,” Phys. Rev. D65 (2002) 123508, hep-th/0110108.

[36] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 05 (2003) 013, astro-ph/0210603.

[37] E. Schrödinger, Expanding universes. University Press, Cambridge [Eng.], 1956.

[38] M. K. Parikh, I. Savonije, and E. P. Verlinde, “Elliptic de Sitter space: dS/Z2,” Phys. Rev. D67 (2003) 064005, arXiv:hep-th/0209120.

[39] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large N limit,” Phys. Rev. D50 (1999) 105023, hep-th/9809188.

[40] N. Drukker, D. J. Gross, and H. Ooguri, “Wilson loops and minimal surfaces,” Phys. Rev. D60 (1999) 125006, hep-th/9904191.

[41] N. Drukker and B. Fiol, “All-genus calculation of Wilson loops using D-branes,” JHEP 02 (2005) 010, hep-th/0501109.

[42] S. Yamaguchi, “Wilson loops of anti-symmetric representation and D5-branes,” JHEP 05 (2006) 037, hep-th/0603208.

[43] J. Gomis and F. Passerini, “Holographic Wilson loops,” JHEP 08 (2006) 074, hep-th/0604007.

[44] S. A. Hartnoll and S. P. Kumar, “Higher rank Wilson loops from a matrix model,” JHEP 08 (2006) 026, hep-th/0605027.

[45] J. Gomis and F. Passerini, “Wilson loops as D3-branes,” JHEP 01 (2007) 097, hep-th/0612022.

[46] S. Yamaguchi, “Bubbling geometries for half BPS Wilson lines,” Int. J. Mod. Phys. A22 (2007) 1353–1374, hep-th/0601089.
[47] O. Lunin, “On gravitational description of Wilson lines,” *JHEP* **06** (2006) 026, [hep-th/0604133](https://arxiv.org/abs/hep-th/0604133).

[48] E. D’Hoker, J. Estes, and M. Gutperle, “Gravity duals of half-BPS Wilson loops,” *JHEP* **06** (2007) 063, [arXiv:0705.1004](https://arxiv.org/abs/0705.1004).

[49] T. Okuda and D. Trancanelli, “Spectral curves, emergent geometry, and bubbling solutions for Wilson loops,” *JHEP* **09** (2008) 050, [arXiv:0806.4191](https://arxiv.org/abs/0806.4191).

[50] S. Giombi, R. Ricci, and D. Trancanelli, “Operator product expansion of higher rank Wilson loops from D-branes and matrix models,” *JHEP* **10** (2006) 045, [hep-th/0608077](https://arxiv.org/abs/hep-th/0608077).

[51] J. Gomis, S. Matsuura, T. Okuda, and D. Trancanelli, “Wilson loop correlators at strong coupling: from matrices to bubbling geometries,” *JHEP* **08** (2008) 068, [arXiv:0807.3330](https://arxiv.org/abs/0807.3330).

[52] N. Drukker and B. Fiol, “On the integrability of Wilson loops in $AdS_5 \times S^5$: Some periodic ansätze,” *JHEP* **01** (2006) 056, [hep-th/0506058](https://arxiv.org/abs/hep-th/0506058).

[53] G. W. Semenoff and D. Young, “Exact 1/4 BPS loop: Chiral primary correlator,” *Phys. Lett.* **B643** (2006) 195–204, [hep-th/0609158](https://arxiv.org/abs/hep-th/0609158).