EVOLUTION OF DISPERsal IN ADVECTIVE HOMOGENEOUS ENVIRONMENTS

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Abstract. The effects of weak and strong advection on the dynamics of reaction-diffusion models have long been investigated. In contrast, the role of intermediate advection still remains poorly understood. This paper is devoted to studying a two-species competition model in a one-dimensional advective homogeneous environment, where the two species are identical except their diffusion rates and advection rates. Zhou (P. Zhou, On a Lotka-Volterra competition system: diffusion vs advection, Calc. Var. Partial Differential Equations, 55 (2016), Art. 137, 29 pp) considered the system under the no-flux boundary conditions. It is pointed out that, in this paper, we focus on the case where the upstream end has the Neumann boundary condition and the downstream end has the hostile condition. By employing a new approach, we firstly determine necessary and sufficient conditions for the persistence of the corresponding single species model, in forms of the critical diffusion rate and critical advection rate. Furthermore, for the two-species model, we find that (i) the strategy of slower diffusion together with faster advection is always favorable; (ii) two species will also coexist when the faster advection with appropriate faster diffusion.

1. Introduction. All species disperse to some extent, in part because resources become limited locally as population grows. The persistence and population dynamics of a species are strongly influenced by dispersal. Therefore, evolutionary biologists and ecologists have been fascinated by the question of why individuals disperse for many decades. So far, they have proposed many kinds of models to represent dispersal patterns, investigate dispersal processes, elucidate the consequences of dispersal for population and communities, and then explain dispersal evolution.

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Among these models are ones described by reaction-diffusion equations, which can describe the uneven distributions of concentrations of individuals across an area. One of the most successful examples is the two-species Lotka-Volterra competition-diffusion system (see, for example, the books [1, 14] and some latest advances in [5, 9, 11]). One well-known and widely accepted result on the evolution of dispersal is due to Hastings [4]. He found that as long as the organisms disperse by random diffusion only, slow diffusion rate will be selected provided that the environment is spatially heterogeneous but constant in time (slower diffuser wins).

As a further development of traditional reaction-diffusion models, recently there is a growing interest in the study of population dynamics in advective environments, where due to certain external environmental force (such as wind, water flow, gravity, etc), there appears a new advection term in the modeling system. Most obviously, it occurs in rivers where individuals are transported downstream by the water flow. In [9], Lou and Lutscher prescribed various boundary conditions at the upstream and downstream ends motivated by different ecological scenarios. As long as advection terms incorporated in classical Lotka-Volterra competition-diffusion systems, the study of the persistence for the corresponding single equation seems nontrivial, and the global dynamics of the resulting systems is very challenging.

A typical example from river ecosystems (see [17]) is given by the following logistic type reaction-diffusion-advection equation

\[
\begin{cases}
  u_t = d_1 u_{xx} - \alpha_1 u_x + u(r - u), & 0 < x < L, t > 0, \\
  du_x(0, t) - \alpha u(0, t) = 0, & t > 0, \\
  u(L, t) = 0, & t > 0, \\
  u(x, 0) = u_0(x) \geq 0, & 0 < x < L,
\end{cases}
\]

where \(u(x, t)\) denotes the population density of an aquatic species at location \(x\) and time \(t > 0\), \(d > 0\) is the random movement rate due to water turbulence or self-propelling, \(\alpha > 0\) measures the velocity of advective transport caused by river flow, \(L > 0\) signifies the length of the river, and \(r > 0\) accounts for the intrinsic growth rate.

The single species problem (1) with different types of boundary conditions has been extensively studied; see Dirichlet type in [17], Neumann type in [20] and Robin type in [12, 22].

In this paper, we try to study the equation (1) with special scenarios, and then investigate complexity of competitive consequence when a new or invasive species is introduced into such advective environments. To be precise, the system to be investigated is as follows.

\[
\begin{cases}
  u_t = d_1 u_{xx} - \alpha_1 u_x + u(r - u - v), & 0 < x < L, t > 0, \\
  v_t = d_2 v_{xx} - \alpha_2 v_x + v(r - u - v), & 0 < x < L, t > 0, \\
  u_x(0, t) = v_x(0, t) = 0, & t > 0, \\
  u(L, t) = v(L, t) = 0, & t > 0, \\
  u(x, 0) = u_0(x) \geq 0, & 0 < x < L, \\
  v(x, 0) = v_0(x) \geq 0, & 0 < x < L,
\end{cases}
\]

where \(u(x, t)\) and \(v(x, t)\) represent the population densities of the two competing species at location \(x\) and time \(t > 0\), respectively; \(d_1, d_2 > 0\) denote the random diffusion rates of the two species; \(\alpha_1, \alpha_2 > 0\) stand for the effective advection rates; and \(r > 0\) signifies the intrinsic growth rate for both species, which indicates that the environment is spatially homogeneous. The upstream end \((x = 0)\) is the Neumann boundary condition and the downstream end \((x = L)\) is the hostile
condition. Without loss of generality, we assume that $\alpha_1 < \alpha_2$ as the results for the cases where $\alpha_1 > \alpha_2$ can be stated symmetrically. We note here that Tang and Chen [18] concerned the special case $\alpha_1 = \alpha_2$ with more general boundary conditions.

To our knowledge, currently there seems very few work attempting to understand the population dynamics of system (2) with the special scenarios where the upstream end is the free-flow condition (Neumann type) and the downstream end is the hostile condition (Dirichlet type). Biologically, we consider the special scenarios that the river flows from a lake to a sea. The study of system (2) will enable us to better understand the effect of boundary conditions, diffusion rates and advection rates on the outcome of competition.

1.1. Motivation and related work. In the last few years, various special cases and variants of system (2) have been qualitatively investigated.

We begin with the closed environment, that is, both the upstream end and the downstream end are no-flux boundary conditions which means that no individuals will pass through the boundary. If $d_1 = d_2$ and $\alpha_1 \neq \alpha_2$, Lou, Xiao and Zhou [10] have confirmed that weak advection is more beneficial for species to exclude its competitor; while if $d_1 \neq d_2$ and $\alpha_1 = \alpha_2$, it was proved by Lou and Zhou [12] that the competitor with faster diffusion rate would displace the slower one, that is, faster diffusion will evolve, in sharp contrast to the well-known “slower diffuser wins” in non-advective case [2, 4]. The general case $d_1 \neq d_2$ and $\alpha_1 \neq \alpha_2$ was later investigated by Zhou [24], where, among other things, two observations were found: (i) the strategy of faster diffusion together with slower advection is always favorable, which can be seen as a generation of [10] and [12]; (ii) two species will coexist when the faster advection with appropriate faster diffusion.

For the other scenarios, Lou [12] and Zhou [27] introduce a net loss of individuals at the downstream end to describe the different scenarios. Mathematically, as the net loss of individuals varies from zero to infinity, it may yields different types of boundary conditions, including the standard Neumann, Robin and Dirichlet types.

However, it seems few work focusing on the special scenarios where the upstream end is the Neumann boundary condition. We will address on the special scenarios in this paper.

1.2. Main results. In fact, system (2) generates a monotone dynamical system and the qualitative properties of its steady states almost determine the potential population dynamics. For monotone dynamical systems, there are some well known results. To mention a few, see [6, Proposition 9.1 and Theorem 9.2] and [16]. In particular, we can see that

Proposition 1. If system (2) has no coexistence steady state, then one of the steady states $(\hat{u}, 0)$, $(0, \hat{v})$ and $(0, 0)$ is globally asymptotically stable and the others are linearly unstable.

The existence(nonexistence) and uniqueness of the semi-trivial steady states $(\hat{u}, 0)$ and $(0, \hat{v})$ are given in Lemma 2.2. According to the above observations, it is vital to justify whether there are coexistence (positive) steady states for system (2). Lemma 3.1 tells us that system (2) has no coexistence (positive) steady states when $\alpha_1 \leq \alpha_2$, $d_1 \geq d_2$ and $|d_1 - d_2| + |\alpha_1 - \alpha_2| \neq 0$. This suggests that linear stability totally determines the globally asymptotical stability.

The main result of this paper is stated below.
Theorem 1.1. Assume that \( 0 < \alpha_1 < \alpha_2 \). Given \( r, L > 0 \), then the following statements hold:

(i): If \( d_1 \geq d_1^* \), then we have
   (i.1): for \( d_2 \geq d_2^* \), \((0,0)\) is g.a.s;
   (i.2): for \( d_2 < d_2^* \), \((0,\hat{v})\) is g.a.s;

(ii): If \( d_1 < d_1^* \), then there exist two small positive number \( \epsilon \) and \( \delta \) such that
   (ii.1): for \( d_2 \in (0,d_1 + \epsilon) \), \((0,\hat{v})\) is g.a.s;
   (ii.2): for \( d_2 \in (d_2^* - \delta,\infty) \), \((\hat{u},0)\) is g.a.s;
   (ii.3): for some \( d_2 \in [d_1 + \epsilon,d_2^* - \delta] \), system (2) admits a co-existence steady state,

where \( d_1^* \) and \( d_2^* \) are given in Remark 3, and by “g.a.s”, we mean that the steady state is globally asymptotically stable among all non-negative and nontrivial initial conditions.

Remark 1. The results of Theorem 1.1 are in sharp contrast to those of Theorem 1.2 in [24], where the boundary conditions are no-flux. This suggests us that the boundary conditions play an important role in the outcome of competition.

Throughout this paper, we straighten out some terminologies for convenience.

•: It is pointed out that (2) only admits three types of nonnegative steady state solutions as follows:
   -: \((u,v) = (0,0)\) is called a trivial steady state;
   -: \((u,v) = (\hat{u},0)\) or \((u,v) = (0,\hat{v})\) is called a semi-trivial steady state, where \( \hat{u} > 0 \) and \( \hat{v} > 0 \);
   -: \( u > 0 \), \( v > 0 \), and we call \((u,v)\) a coexistence/positive steady state.

The rest of this paper is organized as follows. In Section 2, we establish necessary and sufficient conditions for the persistence of the corresponding single specie model. Section 3 is devoted to show that system (2) has no coexistence steady state under the appropriate conditions and prove Theorem 1.1. In Section 4, we give a short discussion. Though the presentation is kind of parallel to those of [12, 24], we have to modify most of the arguments, especially that for the non-existence of positive steady states.

2. Persistence for the single equation. In this section, we study the following model of single specie

\[
\begin{cases}
  u_t = du_{xx} - \alpha u_x + u(r - u), & x \in (0,L), t > 0, \\
  u_x(0,t) = 0, & t > 0,
  u(L,t) = 0, & t > 0,
  u(x,0) = u_0(x) \geq 0, & 0 < x < L,
\end{cases}
\]

where, \( d, \alpha, r, L > 0 \) are constants. We focus our attention on the steady state of (3), i.e.

\[
\begin{cases}
  du_{xx} - \alpha u_x + u(r - u) = 0, & x \in (0,L), \\
  u_x(0) = u(L) = 0.
\end{cases}
\]

This in turn leads to the study of the following linear eigenvalue problem

\[
\begin{cases}
  d\phi_{xx} - \alpha \phi_x + m(x)\phi + \lambda \phi = 0, & x \in (0,L), \\
  \phi_x(0) = \phi(L) = 0,
\end{cases}
\]

where \( m \in C^1([0,L]) \). In view of the Krein-Rutman Theorem [7], problem (5) admits a unique principal eigen-pair, denoted by \((\lambda_1,\phi_1)\), which satisfy \( ||\phi_1||_{L^\infty([0,L])} = \).
1 and \( \phi_1(x) > 0 \) on \([0, L]\). Sometimes, we also use \((\lambda_1(d, \alpha, m), \phi_1(d, \alpha, m))\) to stress the dependence on parameters. Moreover, by the variational approach, \( \lambda_1(d, \alpha, m) \) can be characterized by
\[
\lambda_1(d, \alpha, m) = \inf_{\varphi \in S} \frac{\int_0^L (d\varphi'^2 e^{\frac{2}{z} x} - m\varphi'^2 e^{\frac{z}{z} x}) dx - \alpha \varphi'^2(0)}{\int_0^L e^{\frac{2}{z} x} \varphi'^2 dx},
\] (6)
where \( S := \{ \varphi \in H^1(0, L) \mid \varphi \neq 0 \text{ and } \varphi(L) = 0 \} \).

Firstly, we establish the monotonic dependence for the principal eigenvalue \( \lambda_1(d, \alpha, m) \) on parameters \( d \) or \( \alpha \), which plays a significant role in later analysis.

**Lemma 2.1.** Fix \( m \in C^1([0, L]) \) and \( L > 0 \). Then we have
\[
\frac{\partial \lambda_1(d, \alpha, m)}{\partial d} = \frac{\int_0^L (\varphi_1 e^{-\frac{z}{z} x})_x \varphi_1 dx}{\int_0^L \varphi_1^2 e^{-\frac{2}{z} x} dx},
\] (7)
and
\[
\frac{\partial \lambda_1(d, \alpha, m)}{\partial \alpha} = \frac{\int_0^L e^{-\frac{z}{z} x} \varphi_1 \varphi_1 dx}{\int_0^L \varphi_1^2 e^{-\frac{2}{z} x} dx}.
\] (8)

Furthermore, we have the following results for \( \lambda_1(d, \alpha, m) \) in problem (5).

(i): If \( m_x(x) \leq 0 \) on \((0, L)\), then
\[
\frac{\partial \lambda_1(d, \alpha, m)}{\partial d} > 0 \text{ and } \frac{\partial \lambda_1(d, \alpha, m)}{\partial \alpha} < 0.
\]

Specially, if \( m(x) = m_0 \), where \( m_0 \) is a positive constant, then the results still hold.

(ii): If \( m(0) > 0 \) and \( m_x(x) \geq 0 \) in \((0, L)\), then we have the following statements:

(i.i.1): for any given \( \alpha > 0 \), taking \( \lambda_1(d, \alpha, m) \) as a function of \( d \), then it has at most one positive root. Moreover, if \( \lambda_1(d^*, \alpha, m) = 0 \) for some \( d^* > 0 \), then \( \frac{\partial \lambda_1(d, \alpha, m)}{\partial d} \bigg|_{d=d^*} > 0 \);

(i.i.2): for any given \( d > 0 \), taking \( \lambda_1(d, \alpha, m) \) as a function of \( \alpha \), then it has at most one positive root. Moreover, if \( \lambda_1(d, \alpha^*, m) = 0 \) for some \( \alpha^* > 0 \), then \( \frac{\partial \lambda_1(d, \alpha, m)}{\partial \alpha} \bigg|_{\alpha=\alpha^*} < 0 \).

Proof. We firstly establish (7) and (8). Since the proofs are similar, we will only show
\[
\frac{\partial \lambda_1(d, \alpha, m)}{\partial d} = \frac{\int_0^L (\varphi_1 e^{-\frac{z}{z} x})_x \varphi_1 dx}{\int_0^L \varphi_1^2 e^{-\frac{2}{z} x} dx}.
\]

Let \( \frac{\partial}{\partial d} \) denotes differentiation with respect to \( d \). Differentiating the equation which the principal eigen-pair \((\lambda_1(d, \alpha, m), \phi_1(d, \alpha, m))\) satisfies, one obtains
\[
\begin{aligned}
& d\phi'_{xx} - \alpha \phi'_{x} + m(x) \phi'_{x} + \lambda_1 \phi'_{x} + \phi_{xx} + \lambda_1 \phi_{x} = 0, \quad x \in (0, L), \\
& \phi'_{x}(0) = 0, \quad \phi_{x}(L) = 0,
\end{aligned}
\] (9)
where we denote \((\lambda_1(d, \alpha, m), \phi_1(d, \alpha, m))\) by \((\lambda_1, \phi_1)\) for simplicity. Since \( d\phi_{xx} - \alpha \phi_{x} = d((\varphi_{x}^2 e^{-\frac{2}{z} x}) e^{\frac{z}{z} x}) , \) one can rewrite the equation of \( \phi_1 \) and (9) as
\[
\begin{aligned}
& d((\varphi_{x}^2 e^{-\frac{2}{z} x}) e^{\frac{z}{z} x})_x + m(x) \phi_{x} + \lambda_1 \phi_{x} = 0, \quad x \in (0, L), \\
& \phi_{x}(0) = 0, \quad \phi_{x}(L) = 0,
\end{aligned}
\] (10)
and
\[
\begin{aligned}
& d((\varphi_{x}^2 e^{-\frac{2}{z} x}) e^{\frac{z}{z} x})_x + m(x) \phi_{x} + \lambda_1 \phi_{x} + \phi_{xx} + \lambda_1 \phi_{x} = 0, \quad x \in (0, L), \\
& \phi_{x}'(0) = 0, \quad \phi_{x}'(L) = 0.
\end{aligned}
\] (11)
Multiplying the first equation in (10) by $e^{-\frac{2}{\lambda}x}\phi_1'$, then integrating on $(0, L)$, we see

$$
\int_0^L d((e^{-\frac{2}{\lambda}x}\phi_1)x e^{-\frac{2}{\lambda}x}) dx + \int_0^L m(x)\phi_1\phi_1'e^{-\frac{2}{\lambda}x}dx + \int_0^L \lambda_1\phi_1\phi_1'e^{-\frac{2}{\lambda}x}dx = 0.
$$

(12)

Similarly, multiplying the first equation in (11) by $e^{-\frac{2}{\lambda}x}\phi_1$, then integrating on $(0, L)$, we obtain

$$
\int_0^L d((e^{-\frac{2}{\lambda}x}\phi_1)'x e^{-\frac{2}{\lambda}x}) dx + \int_0^L m(x)\phi_1\phi_1'e^{-\frac{2}{\lambda}x}dx + \int_0^L \lambda_1\phi_1\phi_1'e^{-\frac{2}{\lambda}x}dx = 0.
$$

(13)

Subtracting (12) from (13) and integrating by parts, one obtains

$$
\lambda_1\int_0^L \phi_1' e^{-\frac{2}{\lambda}x}dx = -\int_0^L \phi_1\phi_1'' e^{-\frac{2}{\lambda}x}dx + d(e^{-\frac{2}{\lambda}x}\phi_1)x \phi_1'(0) - d(e^{-\frac{2}{\lambda}x}\phi_1)'x \phi_1(0)
$$

$$
= \int_0^L \phi_1\phi_1''(x) e^{-\frac{2}{\lambda}x}dx + \phi_1\phi_1'(0) - \phi_1\phi_1'(L) e^{-\frac{2}{\lambda}x}dx.
$$

which establishes (7).

Recalling that $\phi_1(x) > 0$ on $[0, L]$ and the boundary condition $\phi_1(L) = 0$, one immediately obtains that

$$
\phi_1(x) < 0. \text{   for } x \in [L - \epsilon_0, L].
$$

(14)

This together with $\phi_1(L) = 0$ and the uniqueness of solutions of ODEs, one further attains that

$$
\phi_1(x) < 0, \text{   for } x \in [L - \epsilon_0, L].
$$

(15)

Next, we consider case (i): $m_x(x) \leq 0$ on $(0, L)$. Applying the strong maximum principle [3] to problem (15), one immediately gets

$$
\phi_1(x) < 0, \text{   for } x \in [0, L - \epsilon_0],
$$

which suggests that

$$
\phi_1(x) < 0, \text{   for } x \in [0, L - \epsilon_0].
$$

Based on the arbitrariness of $\epsilon_1$ and (14), we obtain that

$$
\phi_1(x) < 0, \text{   on } (0, L).
$$

(16)

From (7), (16) and $\phi_1(x) > 0$ on $[0, L]$, it follows that

$$
\frac{\partial \lambda_1(d, \alpha, m)}{\partial d} > 0 \text{ and } \frac{\partial \lambda_1(d, \alpha, m)}{\partial \alpha} < 0.
$$
Finally, we concern case (ii): $m(0) > 0$ and $m_x(x) \geq 0$ in $(0, L)$. Since the proofs of statements (ii.1) and (ii.2) are similar, we only prove statement (ii.1). Given $\alpha > 0$ and $m \in C^1(0, L)$, for simplicity, we denote $\lambda_1(d, \alpha, m)$ by $\lambda_1(d)$ stressing the dependence on $d$ in the following proof of this lemma. It is clear that this lemma comes from the following assertion:

Claim. If $\lambda_1(d^*) = 0$ for some $d^* > 0$, then $\frac{\partial \lambda_1(d^*)}{\partial d} > 0$.

Denote $\phi_1(d^*, \alpha, m)$ by $\phi^*$. By (7) and $\phi^*(x) > 0$ on $[0, L)$, it suffices to show that

$$\phi^*_x(x) < 0 \text{ in } (0, L).$$

Clearly, $\phi^*$ satisfies

$$\begin{cases} d^* \phi^*_{xx} - \alpha \phi^*_x + m(x) \phi^* = 0, & x \in (0, L), \\
\phi^*_x(0) = 0, & \phi^*(L) = 0. \end{cases}$$

From $m(0) > 0$, $\phi^*(0) > 0$, $\phi^*_x(0) = 0$ and the uniqueness of solutions of ODEs, it follows that

$$\phi^*_x(0) < 0,$$

which together with $\phi^*_x(0) = 0$ infers that there exists some small $\varepsilon_2 > 0$, such that

$$\phi^*_x(x) < 0 \text{ on } (0, \varepsilon_2).$$

We use an indirect argument to verify (17) and suppose that there exists some point $x^* \in (0, L)$ such that

$$\phi^*_x(x^*) = 0.$$

This together with (18) and $\phi^*_x(0) = 0$, suggests that $w^*$ achieves a negative local minimum at some point $x_{min}$ in $(0, x^*)$, where $w^* = \frac{\phi^*}{\phi}$ satisfies

$$\begin{cases} d^* w^*_{xx} + (2d^* w^* - \alpha)w^*_x + m(x) = 0, & x \in (0, x^*), \\
w^*(0) = w^*(x^*) = 0. \end{cases}$$

Again by the strong maximum principle [3], we know that

$$w^*(x) > 0 \text{ on } (0, x^*),$$

which contradicts $w^*(x_{min}) < 0$. This completes the proof. $\square$

Remark 2. From the variational approach (6), it follows that $\lambda_1(d, \alpha, m)$ is strictly decreasing on the function $m(x)$ in the $L^\infty$ sense, that is, if $m_1(x) \leq m_2(x)$ in $[0, L]$, then $\lambda_1(d, \alpha, m_1) > \lambda_1(d, \alpha, m_2)$.

Set

$$k := \inf_{\phi \in \mathcal{S}} \int_0^L \frac{\phi^2}{\phi^2dx},$$

where $\mathcal{S} := \{\varphi \in H^1(0, L) | \varphi \neq 0 \text{ and } \varphi(L) = 0\}$ is defined in (6). Obviously, $0 < k < \infty$.

Lemma 2.2. Fix $r, L > 0$. The following statements are true:

(i): if $0 < d \leq \frac{r}{k}$, then for any $\alpha > 0$, problem (3) admits a unique positive steady state;

(ii): if $d > \frac{r}{k}$, then there exists $\alpha^* > 0$ such that

(ii.1): if $0 < \alpha \leq \alpha^*$, then problem (3) does not admit any positive steady state;

(ii.2): while if $\alpha > \alpha^*$, then problem (3) admits a unique positive steady state.
(iii): given $\alpha > 0$, then there exists $d^* > 0$ such that

(iii.1): if $d \geq d^*$, then problem (3) does not admit any positive steady state;

(iii.2): while if $0 < d < d^*$, then problem (3) admits a unique positive steady state.

Proof. Since the nonlinear reaction term of problem (3) is of the logistic type, it is well known that the existence of a positive steady state for problem (3) is equivalent to that $u = 0$ is linearly unstable (i.e. $\lambda_1(d, \alpha, r) < 0$) [1]. Moreover, we omit the detail on the proof of the uniqueness of positive steady states as it is standard (for example, we refer to the proof of [10, Lemma 2.1]. We firstly estimate the principal eigenvalue $\lambda_1(d, 0, r)$. From (i) in Lemma 2.1 and the definition of $k$, it follows that

$$\lambda_1(d, 0, r) < 0 \text{ when } d < \frac{r}{k} \text{ and } \lambda_1(d, 0, r) > 0 \text{ when } d > \frac{r}{k}. \quad (20)$$

Specially, $\lambda_1\left(\frac{r}{k}, 0, r\right) = 0$.

Now, we consider case (i): $0 < d \leq \frac{r}{k}$. Based on (20) and (i) in Lemma 2.1, for any $\alpha > 0$, one obtains

$$\lambda_1(d, \alpha, r) < \lambda_1(d, 0, r) \leq 0,$$

which proves (i).

Then, we concern case (ii): $d > \frac{r}{k}$. Recall (i) in Lemma 2.1. We obtain that $\lambda_1(d, \alpha, r)$ is strictly decreasing function on parameter $\alpha$. Since $\lambda_1(d, 0, r) > 0$ for $d > \frac{r}{k}$, it suffices to show that $\lim_{\alpha \to \infty} \lambda_1(d, \alpha, r) < 0$. Indeed, $\lim_{\alpha \to \infty} \lambda_1(d, \alpha, r) = -r$ due to Theorem 1.2 in [15]. This completes the proof.

Finally, we focus on case (iii). From (i) in Lemma 2.1, it follows that $\lambda_1(d, \alpha, r)$ is strictly increasing function on parameter $d$. It suffices to show that

$$\lim_{d \to 0} \lambda_1(d, \alpha, r) < 0 \text{ and } \lim_{d \to +\infty} \lambda_1(d, \alpha, r) > 0.$$

Clearly, $\lim_{d \to 0} \lambda_1(d, \alpha, r) < 0$ due to case (i). On the other hand, $\lim_{d \to +\infty} \lambda_1(d, \alpha, r) > 0$ comes from the following claim.

Claim. $\lim_{d \to +\infty} \lambda_1(d, \alpha, r) = +\infty$.

By an indirect argument, and together with the monotonicity of $\lambda_1(d, \alpha, r)$ with respect to $d$, we suppose that there exist some constant $M$ and $d_n > 0$ satisfying $d_n \to +\infty$ as $n \to \infty$, such that $\lambda_1(d_n, \alpha, r) \leq M$ as $n \to \infty$. For simplicity, we denote $\lambda_1(d_n, \alpha, r)$ and $\phi_1(d_n, \alpha, r)$ by $\lambda_n$ and $\phi_n$, which satisfy

$$\begin{cases}
    d_n \phi_{n,xx} - \alpha \phi_{n,x} + r \phi_n + \lambda_n \phi_n = 0, \quad x \in (0, L), \\
    \phi_{n,x}(0) = \phi_n(L) = 0,
\end{cases} \quad (21)$$

where $\|\phi_n\|_{L^\infty((0, L))} = 1$. By the elliptic regularity [3] and the standard Sobolev imbedding theorem, one can deduce from equation (21) that $\phi_n$, passing to a subsequence if necessary, converges to some function $\phi^*$ in the topology of $C^1([0, L])$ as $n \to \infty$, where $\phi^*$ satisfies (in the weak form)

$$\begin{cases}
    \phi^*_{xx} = 0, \quad 0 < x < L, \\
    \phi^*_x(0) = \phi^*(L) = 0,
\end{cases}$$

which derives that $\phi^* \equiv 0$ on $[0, L]$. This contradicts $\|\phi^*\|_{L^\infty([0, L])} = 1$ due to the convergence of $\phi_n$. Then, $\lim_{d \to +\infty} \lambda_1(d, \alpha, r) = +\infty$ which completes the proof. \qed
Remark 3. Given $\alpha_1, \alpha_2 > 0$, there exist $d_1^* > 0$ and $d_2^* > 0$ such that

(i): if $d_1 \geq d_1^*$ and $d_2 \geq d_2^*$, then problem (2) does not admit any semi-trivial steady state;
(ii): if $d_1 \geq d_1^*$ and $d_2 < d_2^*$, then problem (2) admits a unique semi-trivial steady state $(\hat{u}, \hat{v})$;
(iii): if $d_1 \leq d_1^*$ and $d_2 \geq d_2^*$, then problem (2) admits a unique semi-trivial steady state $(\hat{u}, 0)$;
(iv): if $d_1 < d_1^*$ and $d_2 < d_2^*$, then problem (2) admits two semi-trivial steady state $(\hat{u}, 0)$ and $(0, \hat{v})$.

Finally, we provide some properties about the positive steady state of problem (3) (if it exists) to end this section.

Lemma 2.3. If problem (4) admits a positive solution $u$, then $u$ satisfies

$u(0) < r$ and $u_x(x) < 0$ on $(0, L]$.

Proof. Similar to the proofs of (14) in Lemma 2.1, one finds

$u_x(L) < 0$. \hfill (22)

Define $z(x) = \frac{u_x(x)}{u(x)}$ on $[0, L)$, then $z$ satisfies

$\left\{ \begin{array}{ll}
dz_{xx} + (2dz - \alpha)z_x - uz = 0, & x \in (0, L), \\
z(0) = 0, & \lim_{x \to L^-} z(x) = -\infty.
\end{array} \right.$

By the maximum principle [3], one sees

$z(x) < 0$ in $(0, L)$,

which infers that

$u_x(x) < 0$ in $(0, L)$. \hfill (23)

Besides, from $z(x) < 0$ in $(0, L)$ and $z(0) = 0$, it follows that

$z_x(0) \leq 0$.

This together with the uniqueness of solutions of ODEs yields that

$z_x(0) < 0$,

which combined with the boundary condition suggests that

$u_{xx}(0) < 0$.

Estimating the value of the first equation of (4) at $x = 0$, one obtains

$u(0) < r$. \hfill (24)

Thus, (22), (23) and (24) complete the proof.

3. Proof of Theorem 1.1. To prove Theorem 1.1, we firstly establish the non-existence of coexistence steady state for system (2).
3.1. Nonexistence of coexistence steady state. To establish the non-existence of coexistence steady state for system (2), we need some preliminary results. For any coexistence steady state \((u, v)\) (if exists) of system (2), similar to the proof of (14) in Lemma 2.1, one can obtain that there exists some \(\epsilon_3 > 0\), such that
\[ u_x(x) < 0 \text{ and } v_x(x) < 0 \text{ for } x \in [L - \epsilon_3, L]. \]

Define
\[
\hat{T} := \frac{u_x}{u} \text{ and } \hat{S} := \frac{v_x}{v} \text{ for } x \in (0, L).
\]

A directly computation yields that
\[
\begin{cases}
-d_1 \hat{T}_{xx} + (\alpha_1 - 2d_1 \hat{T}) \hat{T}_x + u \hat{T} + v \hat{S} = 0, & 0 < x < L, \\
-d_2 \hat{S}_{xx} + (\alpha_2 - 2d_2 \hat{S}) \hat{S}_x + u \hat{T} + v \hat{S} = 0, & 0 < x < L, \\
\hat{T}(0) = \hat{S}(0) = 0, \\
\lim_{x \to L} \hat{T}(x) = -\infty, \lim_{x \to L} \hat{S}(x) = -\infty.
\end{cases}
\]

\(\hat{T}\) and \(\hat{S}\) have the following properties (see Lemma 3.5 in [12]).

Proposition 2. Let \(\hat{T}\) and \(\hat{S}\) be defined as in (25). Then
\[ -d_1 \hat{T}_x + \alpha_1 \hat{T} - d_1 \hat{T}^2 = -d_2 \hat{S}_x + \alpha_2 \hat{S} - d_2 \hat{S}^2, \text{ for any } x \in (0, L). \] (27)

Furthermore, the following situations for \(\hat{T}\) and \(\hat{S}\) cannot occur:

(1) \(\hat{T}\) (resp. \(\hat{S}\)) achieves a positive local maximum in \((x_1, x_2)\) and \(\hat{S} \geq 0\) (resp. \(\hat{T} \geq 0\)) in \((x_1, x_2)\);

(2) \(\hat{T}\) (resp. \(\hat{S}\)) achieves a negative local minimum in \((x_1, x_2)\) and \(\hat{S} \leq 0\) (resp. \(\hat{T} \leq 0\)) in \((x_1, x_2)\).

where \((x_1, x_2)\) is any interval in \([0, L]\).

Now, we establish the key lemma about the non-existence of coexistence steady state for system (2).

Lemma 3.1. Fix \(r, L > 0\). If \(d_1 \geq d_2, \alpha_1 \leq \alpha_2 \text{ and } |d_1 - d_2| + |\alpha_1 - \alpha_2| \neq 0\), then system (2) does not admit any coexistence steady state.

Proof. By an indirect argument, we suppose that there exists a coexistence steady state for system (2), denoted by \((u^*, v^*)\). By strong maximum principle, one obtains \(u^*, v^* > 0\) on \([0, L]\). Similar to the analysis in Lemma 2.1, one can obtain that there exists some \(\epsilon_4 > 0\), such that
\[ u^*_x(x) < 0 \text{ and } v^*_x(x) < 0 \text{ for } x \in [L - \epsilon_4, L]. \] (28)

For clarification, we prove this Lemma by several claims.

Define
\[ f(x) = d_1 u^*_x - \alpha_1 u^* \text{ and } g(x) = d_2 v^*_x - \alpha_2 v^*, \text{ } x \in [0, L]. \]

Then \(f(0), g(0) < 0\), \(f(L) = d_1 u^*_x(L) < 0\) and \(g(L) = d_2 v^*_x(L) < 0\) due to the boundary conditions and (28).

Claim 1. \(f, g, u^*_x\) and \(v^*_x\) are real analytic. Moreover, all zero points of \(f\) and \(g\) in \([0, L]\) must be isolated.

See Claim 3.1 of Lemma 3.4 [24].

Claim 2. \(u^*(0) + v^*(0) \neq r\).
Arguing indirectly, we assume that \( u^*(0) + v^*(0) = r \). Differentiating the equations of \( u^* \) and \( v^* \), we see
\[
\begin{align*}
d_1 u_{xx}^* - \alpha_1 u_x^* + u^*(r - 2u^* - v^*) - u^* v_x^* &= 0, \quad 0 < x < L, \\
d_2 v_{xx}^* - \alpha_2 v_x^* + v^*(r - u^* - 2v^*) - v^* u_x^* &= 0, \quad 0 < x < L, \\
u_x^*(0) &= v_x^*(0) = 0, \\
u^*(L) &= v^*(L) = 0.
\end{align*}
\]
As before, set \( T := \frac{u^*}{v^*} \) and \( S := \frac{v^*}{u^*} \). Then \( T \) and \( S \) satisfy the following equation
\[
\begin{align*}
d_1 T_{xx} + (2d_1 T - \alpha_1) T_x - u^* T - v^* S &= 0, \quad 0 < x < L, \\
d_2 S_{xx} + (2d_2 S - \alpha_2) S_x - u^* T - v^* S &= 0, \quad 0 < x < L, \\
T(0) = S(0) = 0, \quad \lim_{x \to L} T(x) = -\infty, \quad \lim_{x \to L} S(x) = -\infty.
\end{align*}
\]
Combining the boundary condition \( u_x^*(0) = v_x^*(0) = 0 \) and the equations of \( u^* \) and \( v^* \), one obtains that \( u_{xx}^*(0) = v_{xx}^*(0) = 0 \). This together with the definitions of \( T \) and \( S \) suggests that \( T_x(0) = S_x(0) = 0 \). Then by the uniqueness of solutions of ODEs, one obtains \( T(x) = S(x) = 0 \) on \([0, L]\), which contradicts (28).

**Claim 3.** \( u^*(0) + v^*(0) > r \).

By contradiction, and together with Claim 2, one attains
\[
u^*(0) + v^*(0) < r. \tag{29}\]
Combining the boundary condition \( u_x^*(0) = v_x^*(0) = 0 \) and the equations of \( u^* \) and \( v^* \), we obtain \( u_{xx}^*(0) < 0 \) and \( v_{xx}^*(0) < 0 \), which derive that there exists some \( \delta > 0 \) such that \( u_x^*(x) < 0 \) and \( v_x^*(x) < 0 \) for \( x \in (0, \delta) \). By the definitions of \( T \) and \( S \), one immediately obtains
\[
T(x) < 0 \quad \text{and} \quad S(x) < 0 \quad \text{for} \quad x \in (0, \delta). \tag{30}\]

**Claim 3.1.** \( T(x) < 0 \) and \( S(x) < 0 \) for \( x \in (0, L) \).

By (28) and (30), it suffices to show that
\[
T(x) < 0 \quad \text{and} \quad S(x) < 0 \quad \text{for} \quad x \in (\delta, L - \epsilon_4).
\]

Also by an indirect argument, without loss of generality, we assume that there exists \( x_1 \in [\delta, L - \epsilon_4) \) such that
\[
T(x_1) = 0, \quad T(x) < 0 \quad \text{for} \quad x \in (0, x_1) \quad \text{and} \quad S(x) < 0 \quad \text{for} \quad x \in (0, x_1).
\]
This together with \( T(0) = 0 \) yields that \( T \) achieves a local minimum in \((0, x_1)\) and \( S(x) < 0 \) in \((0, x_1)\), which is impossible due to (2) in Proposition 2. This verifies Claim 3.1. Based on Claim 3.1 and the definitions of \( T \) and \( S \), one attains
\[
u_x^*(x) < 0 \quad \text{and} \quad v_x^*(x) < 0 \quad \text{in} \quad (0, L). \tag{31}\]
Recall that \( u^* \) and \( v^* \) satisfy
\[
\begin{align*}
d_1 u_{xx}^* - \alpha_1 u_x^* + u^*(r - u^* - v^*) &= 0, \quad 0 < x < L, \\
d_2 v_{xx}^* - \alpha_2 v_x^* + v^*(r - u^* - v^*) &= 0, \quad 0 < x < L, \\
u_x^*(0) &= v_x^*(0) = 0, \\
u^*(L) &= v^*(L) = 0,
\end{align*}
\]
which yields that
\[
\lambda_1(d_1, \alpha_1, r - u^* - v^*) = \lambda_1(d_2, \alpha_2, r - u^* - v^*) = 0. \tag{32}\]
However, by (ii.1) in Lemma 2.1, one obtains
\[
\lambda_1(d_1, \alpha_1, r - u^* - v^*) \geq \lambda_1(d_2, \alpha_1, r - u^* - v^*), \tag{33}\]
due to $d_1 \geq d_2$. Besides, by (ii.2) in Lemma 2.1, one attains
\[ \lambda_1(d_2, \alpha_1, r - u^* - v^*) \geq \lambda_1(d_2, \alpha_2, r - u^* - v^*), \]
due to $\alpha_1 \leq \alpha_2$. Since $|d_1 - d_2| + |\alpha_1 - \alpha_2| \neq 0$, by (33) and (34), we have
\[ \lambda_1(d_1, \alpha_1, r - u^* - v^*) > \lambda_1(d_2, \alpha_2, r - u^* - v^*) \]
which contradicts (32). This proves Claim 3.

By the boundary condition $u_0^*(0) = v_0^*(0) = 0$ and the equations of $u^*$ and $v^*$, one obtains $u_{xx}^*(0) > 0$ and $v_{xx}^*(0) > 0$, which yield that there exists some $\delta_1 > 0$ such that $u_0^*(x) > 0$ and $v_0^*(x) > 0$ for $x \in (0, \delta_1)$. This together with (28) suggests that there exist $x_2, x_3 \in (\delta_1, L - \epsilon_4)$ such that
\[ u_0^*(x) > 0 \text{ in } (0, x_2), \quad v_0^*(x) > 0 \text{ in } (0, x_3) \text{ and } u_0^*(x_2) = v_0^*(x_3) = 0. \]

Without loss of generality, we assume $x_2 \leq x_3$. From (35) and $u_0^*(0) = 0$, it follows that $T$ achieves a positive local maximum in $(0, x_2)$, which contradicts Proposition 2 due to $S(x) > 0$ in $(0, x_2)$. This completes the proof of Lemma 3.1.

3.2. Stability of $(\ddot{u}, 0)$ and $(0, \ddot{v})$. In this subsection, we analyse the linear stability of two semi-trivial steady state $(\ddot{u}, 0)$ and $(0, \ddot{v})$.

**Lemma 3.2.** Assume that $0 < \alpha_1 < \alpha_2$. Then $d_2^* > d_1^* > 0$ and the following statements are true:

(i): if $d_1 \geq d_1^*$ and $0 < d_2 < d_2^*$, then $(0, \ddot{v})$ is linearly stable;

(ii): if $0 < d_1 < d_1^*$ and $d_2 \geq d_2^*$, then $(\ddot{u}, 0)$ is linearly stable;

(iii): if $0 < d_1 < d_1^*$ and $0 < d_2 \leq d_1$, then $(0, \ddot{v})$ is linearly stable and $(\ddot{u}, 0)$ is linearly unstable,

where $d_1^*$ and $d_2^*$ are given in Remark 3.

**Proof.** Firstly, we show that $d_2^* > d_1^*$. Recall the definitions of $d_1^*$ and $d_2^*$. One obtains
\[ \lambda_1(d_1^*, \alpha_1, r) = \lambda_1(d_2^*, \alpha_2, r) = 0. \]
From (i) in Lemma 2.1 and $\alpha_1 < \alpha_2$, it follows that
\[ \lambda_1(d_1^*, \alpha_1, r) > \lambda_1(d_1^*, \alpha_2, r). \]
Again by (i) in Lemma 2.1, (36) and (37), one immediately attains $d_2^* > d_1^*$.

Since the proofs of case (i) and case (ii) are similar, we only prove case (i). It is well known that the linear stability of $(0, \ddot{v})$ is determined by $\lambda_1(d_1, \alpha_1, r - \dot{\ddot{v}})$, which is the principal eigenvalue of the following linear eigenvalue problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
d_1 \phi_{xx} - \alpha_1 \phi_x + (r - \dot{\ddot{v}})\phi + \lambda \phi = 0, \quad x \in (0, L), \\
\phi_x(0) = \phi(L) = 0.
\end{array} \right.
\end{aligned}
\]

By Remark 2, (i) in Lemma 2.1, and $d_1 \geq d_1^*$, one knows that
\[ \lambda_1(d_1, \alpha_1, r - \dot{\ddot{v}}) > \lambda_1(d_1, \alpha_1, r) \geq \lambda_1(d_1^*, \alpha_1, r) = 0, \]
which suggests that $(0, \ddot{v})$ is linearly stable.

Finally, we concern case (iii). Since the proofs are similar, we only prove that $(0, \ddot{v})$ is linearly stable. It suffices to show that $\lambda_1(d_1, \alpha_1, r - \dot{\ddot{v}}) > 0$. If $d_2 = d_1$, by (ii.2) in Lemma 2.1, Lemma 2.3 and $\alpha_1 < \alpha_2$, one finds
\[ 0 = \lambda_1(d_2, \alpha_2, r - \dot{\ddot{v}}) = \lambda_1(d_1, \alpha_2, r - \dot{\ddot{v}}) < \lambda_1(d_1, \alpha_1, r - \dot{\ddot{v}}). \]
If $d_2 < d_1$, from (ii.1) in Lemma 2.1 and Lemma 2.3, it follows that
\[ 0 = \lambda_1(d_2, \alpha_2, r - \dot{\ddot{v}}) < \lambda_1(d_1, \alpha_2, r - \dot{\ddot{v}}). \]
Arguing by contradiction, suppose that \( \lambda_1(d_1, \alpha_1, r - \tilde{v}) \leq 0 \). If \( \lambda_1(d_1, \alpha_1, r - \tilde{v}) = 0 \), then combined (ii.2) in Lemma 2.1, Lemma 2.3 and \( \alpha_1 < \alpha_2 \), one attains

\[
0 = \lambda_1(d_1, \alpha_1, r - \tilde{v}) > \lambda_1(d_1, \alpha_2, r - \tilde{v}),
\]

which contradicts (38). If \( \lambda_1(d_1, \alpha_1, r - \tilde{v}) < 0 \), by (38), \( \alpha_1 < \alpha_2 \) and the continuous dependence of \( \lambda_1(d_1, \alpha, r - \tilde{v}) \) on \( \alpha \), one obtains that there exists \( \alpha_3 \in (\alpha_1, \alpha_2) \) such that

\[
\lambda_1(d_1, \alpha_3, r - \tilde{v}) = 0.
\]

This together with (ii.2) in Lemma 2.1, Lemma 2.3 and \( \alpha_3 < \alpha_2 \), implies that

\[
\lambda_1(d_1, \alpha_2, r - \tilde{v}) < 0,
\]

which contradicts (38). Thus, \( \lambda_1(d_1, \alpha_1, r - \tilde{v}) > 0 \) which completes the proof. \( \square \)

Finally, we prove Theorem 1.1.

Proof. Firstly, we prove the following claim.

Claim. System (2) does not admit any coexistence steady state when \( d_1 \geq d_1^* \) or \( d_2 \geq d_2^* \).

Since the proofs are totally similar, we only consider the case \( d_2 \geq d_2^* \). By an indirect argument, we assume that there exists a coexistence steady state \((\tilde{u}, \tilde{v})\) satisfying

\[
\begin{align*}
d_1 \tilde{u}_{xx} - \alpha_1 \tilde{u}_x + \tilde{u}(r - \tilde{u} - \tilde{v}) &= 0, \quad 0 < x < L, \\
d_2 \tilde{v}_{xx} - \alpha_2 \tilde{v}_x + \tilde{v}(r - \tilde{u} - \tilde{v}) &= 0, \quad 0 < x < L, \\
\tilde{u}_x(0) = \tilde{v}_x(0) &= \tilde{u}(L) = \tilde{v}(L) = 0,
\end{align*}
\]

which suggests that

\[
\lambda_1(d_1, \alpha_1, r - \tilde{u} - \tilde{v}) = \lambda_1(d_2, \alpha_2, r - \tilde{u} - \tilde{v}) = 0. \tag{40}
\]

From Remark 2, it follows that

\[
\lambda_1(d_2, \alpha_2, r - \tilde{u} - \tilde{v}) > \lambda_1(d_2, \alpha_2, r). \tag{41}
\]

Meanwhile, based on Remark 3 and \( d_2 \geq d_2^* \), one finds

\[
\lambda_1(d_2, \alpha_2, r) > 0. \tag{42}
\]

Clearly, (40), (41) and (42) derive a contradiction.

Combining Lemma 3.1, Lemma 3.2, the above Claim and Proposition 1, we know that statement (i) holds and for \( d_1 < d_1^* \), if \( d_2 \in (0, d_1] \), then \((0, \tilde{v})\) is g.a.s.; while if \( d_2 \in [d_2^*, \infty) \), then \((\tilde{u}, 0)\) is g.a.s.

Next, we verify that there exist some small positive \( \epsilon \) and \( \delta \) such that if \( d_1 < d_1^* \), then for \( d_2 \in (d_1, d_1 + \epsilon) \cup (d_2^* - \delta, d_2^*) \), system (2) does not admit any coexistence steady state. Given \( d_1 < d_1^* \), since the proofs are similar, we only present there exists \( \epsilon > 0 \) such that for \( d_2 \in (d_1, d_1 + \epsilon) \), system (2) does not admit any coexistence steady state. Suppose for contradiction that when \( d_2 \rightarrow d_2^* \), there is a coexistence steady state, denoted by \((u^{d_2}, v^{d_2})\). By the elliptic regularity \[3\] and the standard Sobolev imbedding theorem, passing to a subsequence if necessary, one may assume that \((u^{d_2}, v^{d_2}) \rightarrow (u^d, v^d)\) in \( C^2([0, L]) \) as \( d_2 \rightarrow d_2^* \), where \( u^d, v^d \geq 0 \) on \([0, L]\) and satisfy

\[
\begin{align*}
d_1 u_{xx}^d - \alpha_1 u_x^d + u^d(r - u^d - v^d) &= 0, \quad 0 < x < L, \\
d_1 v_{xx}^d - \alpha_2 v_x^d + v^d(r - u^d - v^d) &= 0, \quad 0 < x < L, \\
u_x^d(0) &= v_x^d(0) = u^d(L) = v^d(L) = 0.
\end{align*}
\]
By Lemma 3.1, \( u^d, v^d > 0 \) on \([0, L)\) cannot occur. Then, we need consider three cases: (c1) \( u^d = v^d = 0 \) on \([0, L)\); (c2) \( u^d > 0 \) and \( v^d = 0 \) on \([0, L)\); (c3) \( u^d = 0 \) and \( v^d > 0 \) on \([0, L)\).

For case (c1) \( u^d = v^d = 0 \) on \([0, L)\). Let

\[
\pi^{d_2} = \frac{u^{d_2}}{||u^{d_2}||_\infty([0, L])} \quad \text{and} \quad \pi^{d_2} = \frac{u^{d_2}}{||v^{d_2}||_\infty([0, L])},
\]

Then again by the elliptic regularity [3] and the standard Sobolev imbedding theorem, passing to a subsequence if necessary, we may assume that \((\pi^{d_2}, v^{d_2}) \to (\pi^d, v^d)\) in \(C^2([0, L])\) as \(d_2 \to d_1^+\), and \((\pi^d, v^d)\) satisfies

\[
\begin{align*}
&d_1 \pi^{d_2}_{xx} - \alpha_1 \pi^{d_2}_x + r \pi^d = 0, & 0 < x < L, \\
&d_1 \pi^{d_2}_x - \alpha_2 \pi^{d_2} + r v^d = 0, & 0 < x < L,
\end{align*}
\]

which together with \(||\pi^d||_{L^\infty([0, L])} = ||v^d||_{L^\infty([0, L])} = 1\) yields that

\[
\lambda_1(d_1, \alpha_1, r) = \lambda_1(d_1, \alpha_2, r) = 0.
\]

This contradicts (ii.2) of Lemma 2.1 due to \(\alpha_1 < \alpha_2\).

Turn our attention to case (c2) \( u^d > 0 \) and \( v^d = 0 \) on \([0, L)\). Again assume

\[
\pi^{d_2} = \frac{u^{d_2}}{||u^{d_2}||_\infty([0, L])},
\]

Similar to the analysis of case (c1), without loss of generality, we may assume that

\((u^{d_2}, \pi^{d_2}) \to (u^d, \pi^d)\) in \(C^2([0, L])\) as \(d_2 \to d_1^+\), and \((u^d, \pi^d)\) satisfies

\[
\begin{align*}
&d_1 u^{d_2}_{xx} - \alpha_1 u^{d_2}_x + u^d(r - u^d) = 0, & 0 < x < L, \\
&d_1 \pi^{d_2}_x - \alpha_2 \pi^{d_2} + \pi^d(r - u^d) = 0, & 0 < x < L,
\end{align*}
\]

which combined with \(||\pi^d||_{L^\infty([0, L])} = 1\) suggests that

\[
\lambda_1(d_1, \alpha_1, r - u^d) = \lambda_1(d_1, \alpha_2, r - u^d) = 0. \tag{43}
\]

By Lemma 2.3, one obtains

\[
u^d(0) < r < u^d_x(x) < 0 \quad \text{on} \quad [0, L]. \tag{44}
\]

Then, (43), (44) and (ii.2) of Lemma 2.1 derives a contradiction.

Now, we consider case (c3) \( u^d = 0 \) and \( v^d > 0 \) on \([0, L)\). Since the proof is similar to that of case (c2), we omit the details.

The rest of the proofs follow the same idea as that of Theorem 1.1 in [24]. We omit the details here.

4. Discussion. In this paper, we mainly study a single species model and a two-species competitive model in a one-dimensional advective homogeneous environment from river ecosystems, where the upstream end is the Neumann boundary condition and the downstream end is the Dirichlet boundary condition.

For the single species model, we develop a new technique to establish necessary and sufficient conditions for the persistence of problem (3), in terms of the critical diffusion rate and critical advection rate. For the two-species competitive model, we assume that the two species have identical diffusive strategy except their diffusion rates and advection rates. By ruling out the coexistence steady state (see Lemma 3.1), we find the amazing phenomenon that the local dynamics totally decides the
global dynamics. More precisely, we find that the strategy of slower advection together with faster diffusion is always superior, see Theorem 1.1.

Although in the current work we have made some progress in understanding the system (2), there are several important problems that are unsolved and deserve further investigation. The first one concerns the case that two species have different interspecific competition abilities, while both interspecific competition coefficients in (2) are normalized to 1. For the non-advective case, He and Ni [5] made a significant breakthrough on the estimate of linear stability of any coexistence steady state and based on this, they finally classified completely all possible long time dynamical behaviors. The second one is to talk about the same topic but in a spatially heterogeneous environment, which clearly is more difficult. See some relevant results in [8, 13, 19, 21, 23, 25, 26] and references therein. We leave these for future work.

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REFERENCES

[1] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Wiley Series in Mathematical and Computational Biology, John Wiley and Sons, Chichester, UK, 2003.
[2] J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, *The evolution of slow dispersal rates: A reaction-diffusion model*, J. Math. Biol., 37 (1998), 61–83.
[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.
[4] A. Hastings, *Can spatial variation alone lead to selection for dispersal?*, Theor. Popul. Biol., 24 (1983), 244–251.
[5] X. He and W.-M. Ni, *Global dynamics of the Lotka-Volterra competition-diffusion system: Diffusion and spatial heterogeneity I*, Comm. Pure. Appl. Math., 69 (2016), 981–1014.
[6] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman Research Notes in Mathematics Series, 247. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1991.
[7] M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Matem. Nauk (N. S.), 3 (1948), 3–95.
[8] Y. Lou, *Some challenging mathematical problems in evolution of dispersal and population dynamics*, Tutorials in Mathematical Biosciences IV, Lecture Notes in Math., 1922, Springer, Berlin, (2008), 171–205.
[9] Y. Lou and F. Lutscher, *Evolution of dispersal in open advective environments*, J. Math. Biol., 69 (2014), 1319–1342.
[10] Y. Lou, D. Xiao and P. Zhou, *Qualitative analysis for a Lotka-Volterra competition system in advective homogeneous environment*, Discrete Contin. Dyn. Syst., 36 (2016), 953–969.
[11] Y. Lou, X.-Q. Zhao and P. Zhou, *Global dynamics of a Lotka-Volterra competition-diffusion-advection system in heterogeneous environments*, J. Math. Pures Appl., 121 (2019), 47–82.
[12] Y. Lou and P. Zhou, *Evolution of dispersal in advective homogeneous environment: The effect of boundary conditions*, J. Differential Equations, 259 (2015), 141–171.
[13] F. Lutscher, M. A. Lewis and E. McCauley, *Effects of heterogeneity on spread and persistence in rivers*, Bull. Math. Biol., 68 (2006), 2129–2160.
[14] W.-M. Ni, *The Mathematics of Diffusion*, CBMS-NSF Regional Conference Series in Applied Mathematics, 82, SIAM, Philadelphia, 2011.
[15] R. Peng and M. Zhou, *Effects of large degenerate advection and boundary conditions on the principal eigenvalue and its eigenfunction of a linear second-order elliptic operator*, Indiana Univ. Math. J., 67 (2018), 2523–2568.
[16] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Math. Surveys Monogr., 41, Amer. Math. Soc., Providence, RI, 1995.
[17] D. C. Speirs and W. S. C. Gurney, *Population persistence in rivers and estuaries*, Ecology, 82 (2001), 1219–1237.
[18] D. Tang and Y. Chen, Global dynamics of a Lotka-Volterra competition-diffusion system in advective homogeneous environments, *J. Differential Equations*, **269** (2020), 1465–1483.

[19] D. Tang and P. Zhou, On a Lotka-Volterra competition-diffusion-advection system: Homogeneity vs heterogeneity, *J. Differential Equations*, **268** (2020), 1570–1599.

[20] O. Vasilyeva and F. Lutscher, Population dynamics in rivers: Analysis of steady states, *Can. Appl. Math. Q.*, **18** (2010), 439–469.

[21] F. Xu and W. Gan, On a Lotka-Volterra type competition model from river ecology, *Nonlinear Anal. Real World Appl.*, **47** (2019), 373–384.

[22] F. Xu, W. Gan and D. Tang, Population dynamics and evolution in river ecosystems, *Nonlinear Anal. Real World Appl.*, **51** (2020), 102983, 16 pp.

[23] X.-Q. Zhao and P. Zhou, On a Lotka-Volterra competition model: The effects of advection and spatial variation, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 73, 25 pp.

[24] P. Zhou, On a Lotka-Volterra competition system: Diffusion vs advection, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 137, 29 pp.

[25] P. Zhou and D. Xiao, Global dynamics of a classical Lotka-Volterra competition-diffusion-advection system, *J. Funct. Anal.*, **275** (2018), 356–380.

[26] P. Zhou and X.-Q. Zhao, Global dynamics of a two species competition model in open stream environments, *J. Dynam. Differential Equations*, **30** (2018), 613–636.

[27] P. Zhou and X.-Q. Zhao, Evolution of passive movement in advective environments: General boundary condition, *J. Differential Equations*, **264** (2018), 4176–4198.

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