FERMIONS IN GRAVITY AND GAUGE BACKGROUNDS ON A BRANE WORLD

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We solve the fermionic zero modes in gravity and gauge backgrounds on a brane involving a warped geometry, and study the localization of spin 1/2 fermionic field on the brane world. The result is that there exist massless spin 1/2 fermions which can be localized on the bulk with the exponentially decreasing warp factor if including U(1) gauge background. Two special cases of gauge backgrounds on the extra dimensional manifold are discussed.

Keywords: Fermionic zero modes; Localization; General Dirac equation
PACS Nos.: 04.50.+h, 11.25.-w

1. Introduction

It is nowadays widely believed that extra dimensions play an important role in constructing a unified theory of all interactions and provides us with a new solution to hierarchy problem. (for a review see, e.g. Ref. [9]). Ref. [7] gave the first string realization of low scale gravity and braneworld models and pointed out the motivation of TeV strings from the stabilization of mass hierarchy. Modifying the old Kaluza-Klein picture, the recent developments are based on the idea that ordinary matter fields could be confined to a three-dimensional world, corresponding to our apparent Universe, while gravity could live in some higher-dimensional space-time. In the later, Gogberashvili and Randall and Sundrum (RS) reviving the old idea, [11] [12] have recently pointed out that the extra dimension need not even be compact.

Following the brane world models proposed by Randall and Sundrum (RS), a fair amount of activity has been generated involving possible extensions and generalizations, among which, co-dimension two models in six dimensions have been a topic of increasing interest. [14] [15] [16] [17] [18] [19] A useful review on topological defects in higher dimensional models and its relation to braneworlds is available in Ref. [20].

On the other hand, the other local fields except the gravitational field are not always localized on the brane even in the warped geometry. This localization mech-
anism has been recently investigated within the framework of a local field theory. It has been shown that the graviton\cite{13} and the massless scalar field\cite{21} have normalizable zero modes on branes of same types, that the Abelian vector fields are not localized in the RS model in five dimensions but can be localized in some higher-dimensional generalizations of it\cite{22}. Moreover, spin 1/2 and 3/2 fermionic fields are localized on a brane with negative tension\cite{21,23}. Thus, in order to fulfill the localization of fermionic fields on a brane with positive tension, it seems that some additional interactions except the gravity should be introduced in the bulk\cite{24,25}.

Since spin half fields cannot be localized on the brane\cite{8,13,17} in five or six dimensions by gravitational interaction only, it becomes necessary to introduce additional non-gravitational interactions to get spinor fields confined to the brane or string-like defect. The mechanism of localization of spin 1/2 fermions on a brane was first discovered in a flat space-time long ago by Jackiw and Rebbi\cite{26} and recently extended to the case of $AdS_5$ by Grossman and Neubert\cite{23}. More recently, Randjbar-Daemi et al. studied localization of bulk fermions on a brane with inclusion of scalar backgrounds\cite{27} and minimal gauged supergravity\cite{28} in higher dimensions and gave the conditions under which localized chiral fermions can be obtained. Motivated by the inclusion of the bulk scalars\cite{27}, in this letter, we carry out our search for the gauge fields on a 3-brane in six space-time dimensions instead of the real scalar field on this issue. It is shown that spin 1/2 spinor field is confined on the 3-brane without appealing to the additional bulk interactions except the gravity and gauge fields.

This paper is organized as follows: In Sec. 2, we obtain the effective Lagrangian of fermions in gravity and gauge backgrounds. In Sec. 3, we give two ansatz of the background $U(1)$ gauge fields and the conditions under which the gauge fields satisfy the equation of motion for the vector fields. In Sec. 4, we solve the fermionic zero modes and check the localization of bulk fermions on a 3-brane under two simple assumptions for the $U(1)$ gauge fields. In the last section, a brief conclusion is presented.

2. Dirac equation in gravity and gauge backgrounds

We shall consider the six-dimensional generalizations of the RS model with the warped geometry (the $(+,−,−,−,−,−)$ signature will be assumed below):

$$ds^2 = e^{A(r)} \eta_{\mu \nu} dx^\mu dx^\nu - e^{B(r)} a^2 d\theta^2,$$

(1)

where $\eta_{\mu \nu}$ is the ordinary flat Minkowski metric, $a$ is the radius of the circle covered by the coordinate $\theta$. For the two extra spatial dimensions we have introduced polar coordinates $(r, \theta)$ with $0 \leq r < \infty$ and $0 \leq \theta < 2\pi$.

Let us consider the action of a massless spin 1/2 fermions coupled to gravity

$$S = \int d^6 x \sqrt{-g} \bar{\Psi} \Gamma^A E_A^M D_M \Psi,$$

(2)
where $A = 0, \ldots, 5$ corresponds to the flat tangent six-dimensional Minkowski space, $M = 0, \ldots, 5$ denotes the six-dimensional spacetime index. The corresponding equation of motion takes the form

$$
\Gamma^A E^M_A (\partial_M + \Omega_M + A_M) \Psi(x,r,\theta) = 0,
$$

(3)

where $E^A_M$ is the sechsbein with $E^A_M = (e^{A(r)} \delta^a_\mu, 1, a e^{B(r)})$, $\Omega_M = \frac{1}{4} \Omega^A_B \Gamma_A \Gamma_B$ is the spin connection and $A_M$ is $U(1)$ gauge fields. The RS model is the special case with $M = 0, \ldots, 4$ and $A_M = 0$. The spin connection $\Omega^A_M$ is defined as

$$
\Omega^A_M = \frac{1}{2} E^N_A (\partial_M E^B_N - \partial_N E^B_M) - \frac{1}{2} E^N_B (\partial_M E^A_N - \partial_N E^A_M)
$$

\begin{equation}
- \frac{1}{2} E^{PA} E^{QB} (\partial_P E_{QC} - \partial_Q E_{PC}) E^C_M.
\end{equation}

So the non-vanishing components of $\Omega_M$ are

$$
\Omega_\mu = \frac{1}{4} e^{A(r)} (\partial_\mu + A_\mu) \delta^a_\mu \Gamma_a \Gamma_4, \quad \Omega_5 = \frac{1}{4} a e^{B(r)} \delta^a_\mu \Gamma_5 \Gamma_4,
$$

(4)

where the prime denotes the derivative with respect to $r$. Substituting the non-vanishing components of $\Omega_M$ into the six-dimensional Dirac equation (3) gives

$$
\left\{ e^{-\frac{1}{4} \Gamma^a \delta^\mu_\mu (\partial_\mu + A_\mu)} + \Gamma^4 \left( \partial_r + A_r + A'(r) + B'(r) \right) + \frac{1}{a} \Gamma^5 e^{-\frac{2}{4} (\partial_\theta + A_\theta)} \right\} \Psi = 0.
$$

We denote the Dirac operators on the four-dimensional manifold $M$ and the two extra dimensional manifold $K$ with $D_M$ and $D_K$, respectively:

$$
D_M = e^{-\frac{1}{4} \Gamma^a \delta^\mu_\mu (\partial_\mu + A_\mu)},
$$

(5)

$$
D_K = \Gamma \left\{ \Gamma^4 \left( \partial_r + A_r + A'(r) + B'(r) \right) + \frac{1}{a} \Gamma^5 e^{-\frac{2}{4} (\partial_\theta + A_\theta)} \right\},
$$

(6)

where $\Gamma = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3$ and $D_K$ is a kind of ‘mass’ operator whose operator eigenvalues are fermion masses as seen in four dimensions. Then we have the following commutative relations:

$$
[D_M, D_K] = 0,
$$

and can expand any spinor $\Psi$ in a set of eigenvectors $\phi_m$ of the operator $D_K$

$$
D_K \phi_m = \lambda_m \phi_m.
$$

(7)

Thus, each $\phi_m$ is observed in four dimensions as a fermion of mass $\lambda_m$. All these eigenvalues play a role of the mass of the corresponding four-dimensional excitations. We assume that the energy scales probed by a four-dimensional observer are smaller than the separation, and thus even the first non-zero level is not excited. This implies that we are looking for the solutions of the zero modes of $D_K$

$$
D_K \phi = 0.
$$

(8)
It is just the Dirac equation on the manifold \( K \). For fermionic zero modes, we have the following decomposition

\[
\Psi(x, r, \theta) = \psi(x) \phi(r, \theta),
\]

where \( \phi(r, \theta) \) satisfies Eq. (8). The effective Lagrangian for \( \psi(x) \) is defined as

\[
L_{\text{eff}} = \int drd\theta \sqrt{-g} \bar{\psi} \Gamma^A P^M (\partial_M + \Omega_M + A_M) \psi = \bar{\psi} \Gamma^a \delta^\mu_a (\partial^\mu + A^\mu) \psi(x)
\]

\[
\int dr d\theta e^{-A(r)} \sqrt{-g} \phi^\dagger \phi, \tag{10}
\]

Thus, to have the localization of finite kinetic energy for \( \psi(x) \), the above integral must be finite. This can be achieved if the function \( \phi(r, \theta) \) does not diverge on the extra dimensional manifold \( K \).

3. Equation of motion for the \( U(1) \) gauge fields

Let us turn to the \( U(1) \) gauge fields. Here we consider the action of the spin 1 vector fields

\[
S_1 = -\frac{1}{4} \int d^5 x \sqrt{-g} g^{MN} g^{RS} F_{MN} F_{RS}, \tag{11}
\]

where the gauge field tensor \( F_{MN} = \partial_M A_N - \partial_N A_M \) as usual with \( A_M \) the gauge fields. In this letter, to simplify the analysis and without loss of generality, it is assumed that the gauge fields \( A_\mu, A_r, A_\theta \) satisfy the following two ansatz: **Ansatz I**: \( A_\mu = A_\mu(x), A_r = A_r(r), A_\theta = A_\theta(\theta) \) and **Ansatz II**: \( A_\mu = A_\mu(x), A_r = A_r(r), A_\theta = A_\theta(r) \), which are the function of the four dimensional spacetime coordinate \( x \) and extra dimensional spacetime coordinate \( r \) and \( \theta \), respectively. Then one may doubt whether the two different ansatz of gauge fields \( A_r \) and \( A_\theta \) satisfy the equation of motion for the vector fields. This property will be reflected mathematically in the following. From the action (11), the equation of motion is given by

\[
\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} g^{RS} F_{NS}) = 0, \tag{12}
\]

which can be expanded as

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} g^{RS} F_{\nu S}) + \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} g^{RS} F_{jS}) = 0. \tag{13}
\]

Note that in this letter \( \mu, \nu, \lambda \) denote the four dimensional spacetime indices, and \( i, j, k \) denote the extra dimensional spacetime indices. To analyze this equation in more detail, we divide the index \( R \) into the following three cases:

**Case I**: \( R = \tau \)
From the background geometry (11), Eq. (13) changes into

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} F_{\nu \lambda}) + \frac{1}{\sqrt{-g}} \partial_{\lambda} (\sqrt{-g} g^{\mu \lambda} F_{\mu \lambda}) = 0,
\]

where the first term on the LHS of (14) is the usual equation of motion in four dimensional spacetime. Due to the assumption of \( A_j = A_j(r, \theta), A_\lambda = A_\lambda(x) \), the gauge field tensor \( F_{j\lambda} = \partial_j A_\lambda(r, \theta) - \partial_\lambda A_j(x) \) naturally equals to zero. Thus, under the requirement that the \( A_{\mu} \) meets the usual equation of motion in four dimensional spacetime, the two different ansatz of gauge fields \( A_r \) and \( A_\theta \) satisfy the equation of motion for the vector fields in six dimensional spacetime.

**Case II: \( R = r \)**

One can find that the equation (12) changes to

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} g^{rr} F_{\nu r}) + \frac{1}{\sqrt{-g}} \partial_{r} (\sqrt{-g} g^{rr} g^{\theta \theta} F_{r r}) = 0.
\]

It is easy to see that \( F_{rr} \) equals to zero, so we only calculate the value of second term on the LHS of (15). Considering that \( F_{r \theta} \) is an anti symmetric tensor, then \( F_{rr} = 0 \), so we only need to consider the case of \( j = \theta \). For the first ansatz \( A_r = A_r(r), A_\theta = A_\theta(\theta) \), the gauge field tensor is

\[
F_{\theta r} = \partial_\theta A_r(r) - \partial_r A_\theta(\theta),
\]

which also equals to zero. This situation is similar to case I. However, for the second ansatz \( A_r = A_r(r), A_\theta = A_\theta(r) \), we have

\[
F_{\theta r} = \partial_\theta A_r(r) - \partial_r A_\theta(r) = -\partial_r A_\theta(r).
\]

Then equation (15) is reduced to

\[
\partial_\theta (\sqrt{-g} g^{\theta \theta} g^{rr} \partial_r A_\theta(r)) = 0,
\]

which is true for any \( A_r(r) \) and \( A_\theta(r) \). So in the case of \( R = r \), both ansatz are allowed.

**Case III: \( R = \theta \)**

Equation (12) becomes

\[
\partial_{\mu} (\sqrt{-g} g^{\mu \nu} g^{\theta \theta} F_{\nu \phi}) + \partial_r (\sqrt{-g} g^{rr} g^{\theta \theta} F_{r \phi}) = 0,
\]

which is reduced to

\[
\partial_r (\sqrt{-g} g^{rr} g^{\theta \theta} F_{r \phi}) = 0
\]

for \( F_{r \phi} = 0 \). For the first ansatz \( A_r = A_r(r) \) and \( A_\theta = A_\theta(\theta) \), the gauge field tensor \( F_{r \phi} = \partial_r A_\theta - \partial_\theta A_r = 0 \). While for the second ansatz \( A_r = A_r(r) \) and \( A_\theta = A_\theta(r) \), one has \( F_{r \theta} = \partial_\theta A_r(r) \). Then the equation of motion reduces to

\[
\partial_r (e^{2A(r)} - \frac{\beta(c)}{r^2} \partial_r A_\theta(r)) = 0.
\]
It suggests that, in order to satisfy the equation of motion, there exists a constraint (21) for \( A_\theta (r) \). In fact, this equation can be simply satisfied. For example, a simple choice is \( A_\theta (r) = C \) with \( C \) a constant.

From the above analysis, it can be concluded that, in order to satisfy the equation of motion for the gauge fields, there should exist one constraint equation (21) for the gauge fields \( A_\theta (r) \) under the second ansatz. While for the first ansatz, one need not any constraint.

4. Fermionic zero modes and localization of fermions

In this section, we solve the fermionic zero modes under the above two ansatz for the gauge fields and discuss the localization of the Dirac fermions in these gauge backgrounds. We have the physical setup in mind such that “local cosmic string” sits at the origin \( r = 0 \) and then ask the question of whether various bulk fermions with spin 1/2 can be localized on the brane with the exponentially decreasing warp factor by means of the gravitational interaction and gauge background. Of course, in due analysis, we will neglect the backreaction on the geometry induced by the existence of the bulk fields.

4.1. Ansatz I: \( A_\mu = A_\mu (x), A_r = A_r (r) \) and \( A_\theta = A_\theta (\theta) \)

For our current ansatz, the Dirac equation (8) is read as

\[
\Gamma \left\{ \Gamma^A \left( \partial_r + A_r(r) + A'_r(r) + B'_r(r) \right) + \frac{1}{a} \Gamma^5 e^{-\frac{\Phi}{2}} (\partial_\theta + A_\theta(\theta)) \right\} \phi = 0. \tag{22}
\]

We are now ready to study the above Dirac equation for 6-dimensional fluctuations, and write it in terms of 4-dimensional effective fields. Since \( \phi \) is a 6-dimensional Weyl spinor we can represent it by

\[
\phi(r, \theta) = \begin{pmatrix} \phi_1^{(4)} \\ \phi_2^{(4)} \end{pmatrix}, \tag{23}
\]

where \( \phi_1^{(4)} \) and \( \phi_2^{(4)} \) are 4-dimensional Dirac spinors. Our choice for the 6-dimensional constant gamma matrices \( \Gamma^A, A = 0, 1, 2, 3, 4, 5 \) are:

\[
\Gamma^A = \begin{pmatrix} 0 & \Sigma^A \\ \Sigma^A & 0 \end{pmatrix}. \]

Here \( \Sigma^0 = \Sigma^0 = \gamma^0 \gamma^0, \Sigma^i = -\Sigma^i = \gamma^0 \gamma^i; \Sigma^4 = -\Sigma^4 = i \gamma^0 \gamma^5; \Sigma^5 = -\Sigma^5 = \gamma^0, \gamma^\mu \) and \( \gamma^5 \) are usual four-dimensional Dirac matrices in the chiral representation:

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\
\gamma^5 &= i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]
with \( \sigma \) the Pauli matrices. \( \Gamma^A \) have the relation \( \Gamma^A \Gamma^B + \Gamma^B \Gamma^A = 2\eta^{AB} I \). Then the Dirac equation \((22)\) can be reduced to
\[
\begin{align*}
&\left\{ ae B \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix} \left( \partial_r + A_r(r) + A'(r) + \frac{B'(r)}{4} \right) \\
+ & \left( i \gamma^5 \gamma^0 \right) \left( \partial_\theta + A_\theta(\theta) \right) \right\} \begin{pmatrix} \phi_1(4) \\ \phi_2(4) \end{pmatrix} = 0. 
\end{align*}
\]
Obviously, the solutions of \( \phi_1^{(4)} \) and \( \phi_2^{(4)} \) are the same. For simplicity and without loss of generality, we consider the solution of \( \phi_1^{(4)} \). Denoting
\[
\phi_1^{(4)} = \begin{pmatrix} \phi_{11}^{(2)} \\ \phi_{12}^{(2)} \end{pmatrix}
\]
with \( \phi_{11}^{(2)} \) and \( \phi_{12}^{(2)} \) the 2-dimensional Dirac spinors, one can obtain the following differential equations
\[
\begin{align*}
\left\{ ae B \sigma^0 \left( \partial_r + A_r(r) + A'(r) + \frac{B'(r)}{4} \right) + iI_{2x2}(\partial_\theta + A_\theta(\theta)) \right\} \phi_{11}^{(2)} &= 0, \\
\left\{ ae B \sigma^0 \left( \partial_r + A_r(r) + A'(r) + \frac{B'(r)}{4} \right) - iI_{2x2}(\partial_\theta + A_\theta(\theta)) \right\} \phi_{12}^{(2)} &= 0.
\end{align*}
\]
Solving the above two equations one can easily get the formalized solutions:
\[
\begin{align*}
\phi_{11}^{(2)}(r, \theta) &= e^{-A(r) - \frac{B(r)}{4} - \int dr A_r(r) + \frac{1}{a} \int dr \left( \frac{B(r)}{4} + C \right)} C r \Theta, \\
\phi_{12}^{(2)}(r, \theta) &= e^{-A(r) - \frac{B(r)}{4} - \int dr A_r(r) - \frac{1}{a} \int dr \left( \frac{B(r)}{4} + C \right)} - e^{-i(n + \frac{1}{2}) \theta - A(r) - \frac{B(r)}{4} - \int dr A_r(r) - \frac{1}{a} \int dr \left( \frac{B(r)}{4} + C \right) - \int dr A_\theta(\theta)} C r \Theta.
\end{align*}
\]
with \( C \) being an integration constant. Because of the sechsbein transformation properties, \( \phi_i^{(4)}(r, \theta) \) has to be antiperiodic\(30\) \( \phi_1^{(4)}(r, \theta) = -\phi_1^{(4)}(r, \theta + 2\pi) \), then we get \( C = i(n + \frac{1}{2}) \) \( (n \in \mathbb{Z}) \). Therefore, in the case of the gauge fields \( A_r = A_r(r) \) and \( A_\theta = A_\theta(\theta) \), by substituting the constant \( C \) into the above expression \((28)\), one can get the fermionic zero modes \( \phi(r, \theta) \)
\[
\phi(r, \theta) = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \otimes \left( e^{-i(n + \frac{1}{2}) \theta - A(r) - \frac{B(r)}{4} - \int dr A_r(r) - \frac{1}{a} \int dr \left( \frac{B(r)}{4} + C \right) - \int dr A_\theta(\theta)} C r \Theta \right),
\]
The effective Lagrangian for \( \psi(x) \) then becomes
\[
\mathcal{L}_{\text{eff}} = \int d\theta d\varphi \sqrt{-g} \bar{\psi} \Gamma^A E^M_A \left( \partial_M - \Omega_M + A_M \right) \psi
= 4a \bar{\psi} \Gamma^a \delta^\mu_\nu \partial_\mu \psi I_1 I_2,
\]
where
\[
I_1 = 2 \int_0^\infty dr e^{-\frac{1}{2} \Lambda(r) - \frac{2n+1}{a} \int dr A_\theta(\theta) + \frac{B(r)}{a} - \frac{1}{a} \int dr \left( \frac{B(r)}{4} + C \right)} \cosh \left( \frac{2n+1}{a} \int dr \left( \frac{B(r)}{4} + C \right) \right),
\]
\[
I_2 = 2 \int_0^\infty dr e^{-\frac{1}{2} \Lambda(r) - \frac{2n+1}{a} \int dr A_\theta(\theta) + \frac{B(r)}{a} - \frac{1}{a} \int dr \left( \frac{B(r)}{4} + C \right)} \sinh \left( \frac{2n+1}{a} \int dr \left( \frac{B(r)}{4} + C \right) \right).
\]
and

\[ I_2 = \int_0^{2\pi} d\theta e^{-2f d\theta A_\theta(\theta)}. \]  

(31)

In order to localize spin 1/2 fermions in this framework, the integrals (30) and (31) should be finite. By considering the Einstein’s equation without sources, the solutions of the metric functions \( A(r) \) and \( B(r) \) are given by

\[ A(r) = B(r) = -cr, \]  

(32)

where the parameter \( c \) is the combination of the bulk cosmological constant and the Newton constant. Therefore, for such exponential warp factors \( A(r) \) and \( B(r) \), to have localized fermions, it is sufficient if \( I_1 = \int_0^\infty dr e^{2cr} \int dr A(r) \cosh \left( \frac{4n+2}{ac} e^{cr} \right) \) and \( I_2 = \int_0^{2\pi} d\theta e^{-2f d\theta A_\theta(\theta)} \) are finite on \( K \). When the gauge background vanishes, this integral is obviously divergent for the exponentially decreasing warp factor \( c > 0 \) while it is finite for the exponentially increasing warp factor \( c < 0 \). This situation is the same as in the case of the domain wall in the RS framework where for localization of spin 1/2 field additional localization method by Jackiw and Rebbi was introduced. However, if the gauge backgrounds are considered, the integrals \( I_1 \) and \( I_2 \) can be normalizable for not only the exponentially increasing but also the exponentially decreasing warp factor.

4.2. \textbf{Ansatz II:} \( A_\mu = A_\mu(x), A_r = A_r(r) \) and \( A_\theta = A_\theta(r) \)

In this simple assumption, note that \( A_\theta(r) \) should be constrained by equation (21) while \( A_r(r) \) is an arbitrary function. But we shall prove that \( A_\theta(r) \) have no contribution to the effective Lagrangian. The Dirac equation (33) becomes

\[
\bar{\Gamma} \left\{ \Gamma^4 \left( \partial_r + A_r(r) + \frac{B_r(r)}{4} + A_r(r) + \Gamma^5 \gamma^4 a^{-1} e^{-\frac{2}{c} A_\theta(r)} + \frac{1}{a} \Gamma^5 e^{-\frac{2}{c} \partial_\theta} \right) + 1 \frac{1}{a} \Gamma^5 e^{-\frac{2}{c} \partial_\theta} \phi \right\} = 0.
\]

(33)

Denoting

\[ \phi(r, \theta) = \begin{pmatrix} \phi_1^{(4)} \\ \phi_2^{(4)} \end{pmatrix}, \]

(34)

the Dirac equation (33) reduces to

\[
\begin{pmatrix} a e^{\frac{B_r(r)}{4}} \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix} \left( \partial_r + \frac{B_r(r)}{4} + A_r(r) + i a^{-1} e^{-\frac{2}{c} \gamma^5 A_\theta(r)} \right) \\
\end{pmatrix} + \begin{pmatrix} 0 \\ i \gamma^5 \gamma^0 \end{pmatrix} \partial_\theta \begin{pmatrix} \phi_1^{(4)} \\ \phi_2^{(4)} \end{pmatrix} = 0, \]

i.e.

\[
\begin{pmatrix} a e^{\frac{B_r(r)}{4}} \gamma^0 \left( \partial_r + \frac{B_r(r)}{4} + A_r(r) + i a^{-1} e^{-\frac{2}{c} \gamma^5 A_\theta(r)} \right) + i \gamma^5 \gamma^0 \partial_\theta \end{pmatrix} \phi_1^{(4)} = 0,
\]

\[
\begin{pmatrix} a e^{\frac{B_r(r)}{4}} \gamma^0 \left( \partial_r + \frac{B_r(r)}{4} + A_r(r) + i a^{-1} e^{-\frac{2}{c} \gamma^5 A_\theta(r)} \right) + i \gamma^5 \gamma^0 \partial_\theta \end{pmatrix} \phi_2^{(4)} = 0.
\]
and, for the right spinors
\[ \gamma^5 \phi_i^{(4)R}(r, \theta) = \phi_i^{(4)R}(r, \theta) \]

By solving the above differential equation similarly as the above case, one can get the solutions
\[ \phi_i^{(4)L}(r, \theta) = e^{-A(r) - \frac{B(r)}{r}} \int dr \left( A(r) - C a - i a^{-1} e^{- \frac{B(r)}{2}} A_b(r) \right) + i C \theta \]
\[ \phi_i^{(4)R}(r, \theta) = e^{-A(r) - \frac{B(r)}{r}} \int dr \left( A(r) - C a - i a^{-1} e^{- \frac{B(r)}{2}} A_b(r) \right) - i C \theta \]

with \( C \) an integration constant. To guarantee the antiperiodicity of \( \phi_i^{(4)}(r, \theta) \), it is easy to get \( C = n + \frac{1}{2} (n \in \mathbb{Z}) \). Then substituting \( C \) into the above equations, the zero mode \( \phi_i^{(4)}(r, \theta) \) on the two extra dimensions takes the following form
\[ \phi_i^{(4)} = \phi_i^{(4)L} + \phi_i^{(4)R} \]

The effective Lagrangian for \( \psi(x) \) then becomes
\[ \mathcal{L}_{\text{eff}} = \int d\theta d\varphi \sqrt{-g} \bar{\Psi} \Gamma^A E_A^M \left( \partial_M - \Omega_M + A_M \right) \Psi \]
\[ = 8 \pi a \bar{\Psi} \Gamma^a \delta^b_0 \partial_{\mu} \psi I_3, \quad \text{where} \]
\[ I_3 = \int_0^\infty dr e^{\frac{1}{2} A(r) - 2 \int dr A(r) + \frac{2 a + 1}{2} \int dr e^{- \frac{B(r)}{2}}. \]

This result shows that, whatever the form of \( A_b(r) \) is, the effective Lagrangian for \( \psi(x) \) has the same form, i.e., \( A_b(r) \) does not affect the effective Lagrangian. By
taking the metric functions $A(r)$ and $B(r)$ in (32), these fermionic zero modes are generically normalizable on the brane with the use of the gauge fields if the integral
\[ I_3 = \int_0^\infty dr e^{\frac{r}{2}} \left( -2 f \frac{dA_r(r)}{dr} - \ln \frac{r}{a} e^{-\frac{r}{2}} \right) \]
does not diverge, and we need not include any other bulk field to localize the bulk fermions.

5. Conclusions

In conclusion, we have studied two issues, those are, finding the solutions of fermionic zero modes with two extra dimensions and investigating the possibility of localizing the spin 1/2 fermionic fields on a brane with the exponentially decreasing warp factor. Localizing the fermionic fields on the brane requires us to introduce other interactions but gravity. In this letter, we include the $U(1)$ gauge fields to study localization of spin 1/2 fermions on a 3-brane in six-dimensional spacetime. Two special ansatz of the gauge fields are presented, and the conditions under which the gauge fields satisfy the equation of motion are obtained. It is worthwhile to stress that, in the case of $A_\mu(x), A_r = A_r(r)$ and $A_\theta = A_\theta(r)$, we obtain the zero modes for chiral fermions and the effective Lagrangian for $\psi(x)$ has the same form whatever the gauge field $A_\theta(r)$ is. It is shown that, the effective Lagrangian for $\psi(x)$ is definitely finite under some assumption of the gauge fields in the extra dimensional manifold, which means that these fermionic zero modes are generically normalizable. And the localization of the bulk fermions on a brane with the exponentially decreasing warp factor is achieved if gauge and gravitational backgrounds are considered.

Moreover, to localize the fermions on the brane or the string-like defect, there are some other backgrounds could be considered besides gauge fields and gravity, for example, vortex background. The localization of the topological Abelian Higgs vortex coupled to fermion can be fund in our another work.

Acknowledgments

It is a pleasure to thank Dr. Zhenhua Zhao for many useful discussions. This work was supported by the National Natural Science Foundation of the People’s Republic of China (No. 10475034 and No. 10705013), the Doctor Education Fund of Educational Department of the People’s Republic of China (No. 20070730055) and the Fundamental Research Fund for Physics and Mathematics of Lanzhou University (No. Lzu07002).

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