F-Theories On Double Sextics and Effective String Theories

Christos Kokorelis
Hanover Court, Wellington Road
BN2 3AZ, Brighton, U.K

ABSTRACT
We construct new F-theory vacua in 8-dimensions. They are coming by projective realizations of F-theory on $K_3$ surfaces admitting double covers onto $P^2$, branched along a plane sextic curve, the so called double sextics. The new vacua are associated with singular $K_3$ surfaces. In this way the stable picture of the heterotic string is mapped at the triple points of the sextic. We argue that this formulation incorporates naturally the $Sp(4, \mathbb{Z})$ invariance that the extrapolating four dimensional vector multiplet sector of all heterotic vacua may possess. In addition, we describe the way that the 4D $g=2$ description of (0,2) moduli dependence of $N=1$ gauge coupling constants may be connected to Riemann surfaces, with natural $Sp(4, \mathbb{Z})$ duality invariance. Here we recover a novel way to break space-time supersymmetry and fix the moduli parameters in the presence of Wilson lines. In the context of arithmetic of torsion points on elliptic curves, we describe in detail, the derivation of the elliptic fibrations in Weierstrass form. We also consider the heterotic duals to compactifications of F-theory in four dimensions belonging to isomorphic classes of elliptic curves with points-cusps of order two. For the latter theories, we calculate the $\mathcal{N}=2$ 4D heterotic prepotential $f_{TTT}$ corresponding to $\Gamma_0(2)_T \times \Gamma_0(2)_U$ classical perturbative duality group and their conjugate modular theories.
1 Introduction and Motivation

At the current state of the art the five perturbative string theories (PST’s) are connected among themselves through the various dualities. In addition, they are regarded that they originate from compactifications of the same underlying higher dimensional theory, M- of F-theory. In this way, different choice of formulations of unified theories at different directions in the moduli space are being used to derive novel features than could not be seen, easily, by the use of another unified theory. Thus, it appears that problems like that of the correct prediction of Newton constant may be solved in four dimensional compactifications, if M-theory is used [2]. On the other hand, F-theory compactifications [7, 8] are used to examine the appearance of non-perturbative gauge symmetries for the heterotic string. The common principle in the examination of consistency of the compactifications involved is that they all correctly reproduce, via duality, the weak coupling expansion of the heterotic string in four dimensions. On the other hand, 4D $\mathcal{N} = 1$ perturbative heterotic string vacua and their eleven dimensional compactification extensions in four dimensions are considered to be phenomenologically promising and furthermore possess a lagrangian description contrary to the existing F-theory formulation.

However, the $\mathcal{N} = 1$ 4D effective supergravity vacua of its low energy modes are defined in terms of three quantities, namely the Kähler potential $K$, the superpotential $W$, and the gauge kinetic function $f$, which appear as too many. Instead we would prefer to have a theory that everything could be defined in terms of only one quantity. Is there such a theory? The answer is yes. In 4D $\mathcal{N} = 2$ effective low energy supergravity theories, the vector multiplet sector of the theory, in its Coulomb phase, is defined in terms of one quantity, the holomorphic prepotential $f$.

However, there is a general weakness in the way we take into account quantum corrections. That is the quantum theory when loop corrections are taken, or not, into account is not defined so that it is manifestly $Sp(4, Z)$ invariant. Let us explain this in more detail. Take for example the four dimensional compactifications of the heterotic string on a $K_3 \times T^2$. The 4D theory has $\mathcal{N} = 2$ supersymmetry and originates from further toroidally compactified the $\mathcal{N} = 1$ $D = 6$ heterotic vacua which in turn are dual to F-theory compactified on a Calabi-Yau 3-fold on an Hirzebruch $F_n$ surface base. The vector multiplet effective action, all the couplings in the effective lagrangian, is fixed completely, in perturbation theory, by the knowledge of the holomorphic prepotential $F$. In turn, the Kähler potential is defined via the use of the special geometry by the holomorphic symplectic vectors

\footnote{The equations which determine directly the one loop correction to the prepotential $F$ of any $\mathcal{N} = 2$ $D = 4$ heterotic vacuum for rank three (S-T), and four (S-T-U) models were given in [3].}
\[ \Omega = (X^I(M^I), F_I(X)), \] 

dependent on the moduli fields \( M^I \), as

\[
K = - \log(-i\Omega^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Omega) = \log(i\bar{X}^IF_I - iX^IF^I), \quad I = 0, \ldots, n. \tag{1}
\]

The target-space duality transformations act on the space of those vectors as \( Sp(2n+2, \mathbb{R}) \) transformations on the period vector \( \Omega \). Alternatively \( K \) can be expressed in terms of the \( T, U \) neutral moduli and the Wilson lines \( B, C \) of the \( T^2 \) torus as

\[
K = \log[(T + \bar{T})(U + \bar{U}) - (B + \bar{C})(\bar{B} + C)] = \det(M - M^\dagger), \tag{2}
\]

where

\[
M = \begin{pmatrix} T & B \\ -C & U \end{pmatrix}. \tag{3}
\]

In the latter case the effective theory of light modes is invariant, if we ignore the gravitational sector contribution of the dilaton and graviphoton, due to the presence of the discrete shifts in the theta angle at the quantum level, under the target space modular group \( Sp(2r; \mathbb{Z}) \). Here \( r \) is the number of moduli. In our case \( r = 2 \) and \( Sp(2r; \mathbb{Z}) \) acts as

\[
M \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} M, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(4; \mathbb{Z}). \tag{4}
\]

However, the calculation of the quantum corrections e.g the one loop corrections to the gauge or gravitational couplings, the \( \mathcal{N} = 2 \) prepotential, are determined in terms of the basis for modular forms for the group \( SL(2, \mathbb{Z}) \) and the \( SL(2, \mathbb{Z}) \) \( j \)-invariant and not those associated with \( Sp(4, \mathbb{Z}) \).

The question that now arises is if the theories involved in the compactifications of the five perturbative string theories, including M, F-theories, with quantum effects may be defined so that they are manifestly \( Sp(4, \mathbb{Z}) \) invariant? Putting the question differently, can we express the quantum corrections in terms of \( Sp(4, \mathbb{Z}) \) entities like the basis of modular forms for the group \( SL(2, \mathbb{Z}) \) and the \( SL(2, \mathbb{Z}) \) \( j \)-invariant and not those associated with \( Sp(4, \mathbb{Z}) \)?

Previously attempts of description of quantum effects where the ring of modular forms for the \( Sp(4, \mathbb{Z}) \) is used are evident in some works. In the moduli dependence of the one loop corrections to the gauge coupling constants in 4D \( \mathcal{N} = 1 \) orbifold compactifications of the heterotic string was translated from the language of the basis of modular forms for the \( SL(2, \mathbb{Z}) \) on the basis of

\[ \text{2}\] \[ \text{The latter entities are made of the usual modular forms for the } SL(2, \mathbb{Z}) \text{ modular group } E_4, E_6 \text{ and the cusp forms } C_{10}, C_{12}. \]
modular forms for $Sp(4, Z)$. In [11] the calculation of the degeneracy of $N = 4$ BPS states was defined in terms of genus 2 theta functions. However, an apparent question remains on those approaches. Namely, which is the Riemann surface where the vector moduli of the $SO(3, 2) \mathbb{T}^2$ torus live? The latter question is meaningful both in the context of heterotic string [10] and in the context of F-theory.

Take for example the F-theory/heterotic duality in 8-dimensions [7, 22]. Here, compactifications of F-theory on a $K_3$ which admits an elliptic fibration with a section is on the same moduli space $SO(18, 2; \mathbb{Z}) \backslash SO(18, 2)/SO(18) \times SO(2)$, as the $E_8 \times E_8$ heterotic string on a $T^2$ torus. The elliptic fibration with a section, represented by defining a torus with a $\mathbb{P}^1$ base, represents the F theory dual to the $E_8 \times E_8$ heterotic string are given by a two parameter $(\alpha, \beta)$ family with base coordinate $z$, as

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7).$$

Because the torus is a genus one curve the map between the parameters $(\alpha, \beta)$ and the complex structure parameters of the $T^2$ torus, namely $(\tau, \rho)$, is expressed in terms of the $j$-invariant, the modular function for the torus [12]. As a result at the quantum level the basic quantities of the heterotic string are defined in terms of the inherited F-theory $j$-invariant. Note that at the large $\rho$ limit the complex structure $\tau$ of the $T^2$ of the heterotic string is identified with the complex structure of the elliptic fiber (5). It is the presence of the $j$-invariant that signals the lacking $Sp(4, Z)$ invariance which is eminent through all the F-theory formulation.

There is another area however, when the question of the Riemann surface with $Sp(4, Z)$ invariance is meaningful. Take for example $\mathcal{N} = 2$ supersymmetric Yang-Mills. At the Coulomb phase in four dimensions the moduli space of the $r$ vector multiplets, when charged hypermultiplet matter is not present, is invariant under $Sp(4r; Z)$. The $\mathcal{N} = 2$ $SU(r + 1)$ theory is associated with genus $r$ Riemann surfaces, e.g the $\mathcal{N} = 2$ $SU(2)$ theory is described by a genus 1 surface, the $\mathcal{N} = 2$ $SU(3)$ with genus 2 and so on. Uniqueness and universality is lost as for $r > 1$, the theory instead of being simpler, as the gauge group increases and the number of Wilson lines that break the initial “observable” $E_8$ gauge group decreases, it is defined on even higher genus surfaces. Instead we require in this work that we want to describe 8D F-theory realizations in terms of double covers of $K_3$ fibrations that are branched always along a fixed form plane curve, the double sextic. In the latter sense we always work with the universal form sextic curve.

In this work, we will present a Riemann surface that possess $Sp(4, Z)$ invariance that we will argue is connected to the 4D heterotic string. In addition, we will present F-theory solutions in eight dimensions whose six dimensional versions, that we hope to address in a future work, may be connected directly to the $Sp(4, Z)$ Riemann surfaces.

In section 2, we describe the connection of the Mordell-Weyl group to the existence of Weierstrass form of rational elliptic surfaces with a section. We give explicitly the derivation
of the Weierstrass models for the various forms of torsion subgroups. These Weierstrass models as rational elliptic surfaces will be used in the next section to understand the equivalence between $K_3$ surfaces admitting elliptic fibration and maximizing sextics at the degeneration limit of the F-theory/heterotic duality limit. In section 3, we describe the representation of $K_3$ surfaces as double covers onto $\mathbb{P}^2$ branched along a plane sextic and its connection to 8D F-theory/heterotic duality. In addition, we explain the correspondence of the plane sextics to the extremal elliptic fibrations. In section 4, we discuss the Riemann surfaces, appearing in genus two and are connected to the calculations of moduli dependence of the one loop gauge couplings for non-vanishing background fields in $\mathcal{N} = 1$ four dimensional heterotic $(0,2)$ string compactifications, namely the binary sextics. In section 5, we present our results for the $\mathcal{N} = 2$ 4D heterotic theories which exhibit target space modular group $\Gamma_o(2)_T \times \Gamma_o(2)_U$. These theories may come from further compactification on a $K_3$ of 8D F-theory compactifications with $\mathbb{Z}_2$ torsion subgroup or from toroidal compactification of the 6D F-theory on a Calabi-Yau 3-fold over a 2D base.

2 Rational points on Elliptic Curves

In this section we will explain the appearance of the Mordell-Weyl group in 8-dimensional compactifications of F-theory, realised on elliptically fibered $K_3$ surfaces. We will particularly explore, one side of the iceberg, namely representations of $K_3$ surfaces which are realised on genus one curves, those admitting elliptic fibration with a section. We are particularly interested in examining the degeneration limit of the F-theory/heterotic duality where the $K_3$ surface breaks into two rational elliptic surfaces. The other side of the iceberg is the representation of $K_3$ surfaces admitting double covers of $\mathbb{P}^2$ along a plane sextic and their projective realizations and it will be treated in the next section.

Let me start with a few definitions. The Weierstrass form comes from the most general cubic that can be written in $\mathbb{P}^2$ coordinates as

$$F(x, y, w) = c_{yy}y^3 + c_{xy}xy^2 + c_{xyy}x^2y + c_{yyw}yw^2 + c_{yww}ww^2 + c_{xxx}x^3 + c_{xxw}x^2w + c_{xww}xw^2 + c_{www}w^3.$$  \hspace{1cm} (6)

When a number of conditions is applied to (6) it can always be reduced in the general projective\(^3\) Weierstrass form in $\mathbb{P}^2$ which reads

$$y^2w + a_1xyw + a_3yw^2 = x^3 + a_2x^2w + a_4xw^2 + a_6w^3.$$  \hspace{1cm} (7)

These conditions can be summarized as follows:

\(^3\) The subscripts of the coefficients denote the homogeneity of the corresponding term under a change of variables.
• Demanding the curve \((\mathfrak{B})\) to pass through \((x, y, w) = (0, 1, 0)\) we get \(c_{yyy} = 0\).
• Having \((0,1,0)\) as a non-singular point gives us \(c_{yyw} \neq 0\) or \(c_{xyy} \neq 0\).
• Having the curve \(f(x, y, w) = w\) as a tangent line at infinity point \((0,1,0)\) we get one more condition \(c_{xyy} = 0\) and from the previous condition \(c_{yyw} \neq 0\).
• Lastly because a non-singular point \(P\) of \(F\) is a flex or inflection point if the intersection multiplicity of the tangent line of \(F\) at \(P\) is greater or equal to 3, the curve \((\mathfrak{B})\) can have an inflection point at \((0,1,0)\) if \(c_{xxy} = 0\) and \(c_{xxx} \neq 0\).

Under these conditions the cubic takes the form

\[ F = c_{xxy}y^2w + c_{xyy}xyw + c_{yyw}yw^2 + c_{xxx}x^3 + c_{xxw}x^2w + c_{xww}xw^2 + c_{www}w^3. \] (8)

This form is known as Weierstrass form. By setting \(w = 1\) we can put the cubic in the affine Weierstrass form as

\[ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \] (9)

The subscripts in the coefficients of the non-singular cubic indicate the degree of homogeneity under a certain change of variables. Note that \((\mathfrak{B})\) is singular only if \(a_3 = a_4 = 0\).

As usual, compactifications of heterotic string on \(T^2\), in the context of its duality with F-theory on a \(K_3\) surface in 8 dimensions, are realized in terms of an elliptic curve over \(^4\) a field \(k\), namely as a non-singular cubic that is in a Weierstrass form.

An elliptic curve over a the field of rationals \(Q\) is defined as a non-singular cubic in Weierstrass form with rational coefficients. The elliptic curve over \(Q\) is realized by completing the square in the Weierstrass form followed by a change of variables in \(y\) to get \(y^2 = R(x)\), where \(R(x)\) a cubic in degree 3 with distinct roots. In the latter form, namely \((\mathfrak{B})\) the inflection point is mapped to the point at infinity \(O = (0, 1, 0)\) such that it becomes the line at infinity. For the non-singular cubic curve, the existence of a specified point \(O\), defines a group operation on the curve associating the points on the curve with an abelian group, the Mordell-Weyl group (MW), with \(O\) its identity element. What the MW group theorem says is that the group of rational points of an elliptic curve over \(Q\), \(E(Q)\), is finitely generated. The MW group can be written as \(E(Q) \approx Z^r \oplus \Phi\), where \(\Phi\) is a finite abelian group known as the torsion subgroup. The integer \(r\) is the rank of \(E(Q)\). Given now the affine cubic in \((\mathfrak{B})\) and the definition of the MW group we need to know how it is possible to construct for a given MW group its Weierstrass form. The Weierstrass form for the different choises of the MW group may be associated with the appearance of the non-simply connected gauge groups \(^{19}\) in 8-dimensional compactifications of F-theory on a \(K_3\) surface at its degeneration limit in the next section. We note that the torsion subgroup \(F\) of an elliptic curve when the field of integers \(k\) is equal to the field of rational functions becomes the group of sections.

\(^4\)Note that the affine plane \(k^2 = (x, y)\) has a standard one to one embedding into \(\mathbb{P}_2(k)\). In turn , \(\mathbb{P}_2(k)\) is defined as the quotient of \((x, y, w) \in (k^3 - (0,0,0))\).
Before addressing the Weierstrass construction, let me give first few definitions concerning the group formation on elliptic curves that will help us to understand the procedure.

• Given an initial point \( \mathcal{O} \), we define the group law on non-singular cubics in terms of the identity element fixed at \( \mathcal{O} = (x, y, w) = (0, 1, 0) \). In addition, we define \( R = PQ = P \cdot Q \) as the line element between two different points \( P, Q \) on the curve. The addition law for a point in the curve is defined as the multiplication \( P + Q = \mathcal{O} \cdot PQ \). This operation makes the points on the curve to form an abelian group with \( \mathcal{O} \) as the identity element, \( P + \mathcal{O} = \mathcal{O} \cdot P = P \). Negatives are defined by first setting \( \mathcal{O} \) as the third point on the line tangent at \( \mathcal{O} \). By definition \( -P \) \( \text{def} = \mathcal{O} \cdot P \). This means \( -P \) \( \text{def} = (x^2, -y, -a_1 x - a_3) \). So \( P + (-P) = \mathcal{O} \cdot O = \mathcal{O} \).

Next let us look at the determination of the Weierstrass form of the elliptic fibrations admitting torsion points using the affine form (9). For the study of the Mordell-Weyl group it is enough to examine the behaviour of the elliptic curve (9) at the point \( P=(0,0) \). The latter means \( a_6 = 0 \) and eqn.(9) becomes

\[
y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x.
\]

Now what happens is that \( E(k) \) becomes an abelian group with \( \mathcal{O} = (0, 1, 0) \) the point at infinity as identity element of the group law on the elliptic curve. The condition that the torsion subgroup \( \Phi \cong \mathbb{Z}_2 \) is equivalent to the statement that \( P + P = \mathcal{O} \) (or \( P = -P \)). In other words \( P \) is of order two if the tangent is \( \infty \) at \( P \), that means vertical tangency. Since the point at infinity is non-singular, to study singular points, as we said already we translate the points on the curve at the point \( (0,0) \). That had set \( a_6 = 0 \). Note that eqn. (9) is singular only when \( a_3 = a_4 = 0 \). By taking now differentials in eqn. (9) we deduce that the coefficient of \( dy \) equals zero at \( P \). This means that \( a_3 = 0 \). Now the Weierstrass form of the elliptic curve becomes

\[
y^2 + a_1 xy = x^3 + a_2 x^2 + a_4 x.
\]

In the special case that \( a_1 = 0 \) (12) gives the same Weierstrass equation as that appearing in [20, 19]. In general for an elliptic curve over the field \( k = Q \), \( \Phi \) must be one of the following fifteen groups [20]

\[
\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_3, \ldots \mathbb{Z}_9, \mathbb{Z}_{10}, \mathbb{Z}_{12}, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_4 \oplus \mathbb{Z}_2, \mathbb{Z}_6 \oplus \mathbb{Z}_2, \mathbb{Z}_8 \oplus \mathbb{Z}_2
\]

(13)

For an elliptic curve over the complex numbers \( k = C \) there are in addition four more possibilities. They are listed by Cox and Parry [15] as

\[
\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_6 \oplus \mathbb{Z}_3, \mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_5 \oplus \mathbb{Z}_5.
\]

(14)
In the case where the points of the elliptic surface over $C(t)$, are rationals, the rational elliptic curve can be written as an elliptic surface with a section $S$, a Jacobian. Note that the Weierstrass forms that we examine in this paper are associated with standard compactifications of F-theory in 8 dimensions where the antisymmetric B-field is zero. As the methods discussing other possibilities for the torsion subgroup $\Phi$ are not present in string theory literature we can briefly discuss them here. If we examine higher orders in the torsion subgroup $F$ we need the coordinates of the point $2P$. At the tangent line at a general point $P=(x,y)$ on the cubic (9) the coordinates of the point $2P$ are given by

$$x(2P) = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6},$$

where

$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_3^2 + 4a_6,$$

$$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_2a_4 + a_2a_3^2 - a_4^2.$$  \(16\)

Let us examine its consequences. For the singular point $P=(0,0)$ eqn. (10) gives

$$-P = (0,a_3) \quad 2P = (-a_2,a_1a_2-a_3)$$

Let us see now how torsion points of order three, $\Phi \cong \mathbb{Z}_3$ and higher are embedded in the general equation (9) of the elliptic curve. Let us now make the following isomorphic change of variables in eqn. (11).

$$(x, y) \rightarrow (x' + a_3^{-1}a_4x'),$$

which gives us that

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2.$$  \(19\)

Torsion of order three means $2P + P = O$. In other words $2P = -P$, $-P = (0, a_3)$ and $a_2 = 0$. Thus for

- $\Phi \cong \mathbb{Z}_3$, \quad $y^2 + a_1xy + a_3xy = x^3$.  \(20\)

Let us now return back into eqn. (19). Making another change of variables, to an isomorphic curve, in the form

$$(x, y) \rightarrow \left(\frac{x'}{u^2}, \frac{y'}{u^3}\right),$$

Non-standard compactifications of F-theory where the latter field is turned on where considered in \(24, 25\). More details can be found at \(23\).

Changes of variables in a Weierstrass equation in the form $x \rightarrow u^2x' + r, \ y \rightarrow u^3y' + su^2x' + t$ fix the point $(0, 1, 0)$ and carry the line $w=0$ at the same line. The latter is important since the line $w=0$ is the line at infinity.
we get that the point P=(0,0) remains fixed and (19) is transformed into
\[ y^2 + a_3^{-1}a_1a_2xy + a_3^{-2}a_2^3y = x^3 + a_3^{-2}a_2^3x^2. \]  
(22)
or in more elegant form into
\[ y^2 + (1 - c)xy - by = x^3 - bx^2, \]  
(23)
where \( b = -a_3^{-2}a_2^3 \) and \( c = 1 - a_3^{-1}a_1a_2 \). The Weierstrass form of eqn. (23) is well known in mathematics literature as Tate normal form. Its discriminant is given by
\[ \Delta(b, c) = (1 - c)^4b^3 - (1 - c)^3b^3 - 8(1 - c)^2b^4 + 36(1 - c)b^4 + 16b^5 - 27b^4 \]  
(24)
and its connection to the usual from F-theory considerations discriminant \( \Delta = 4a^4 + 27b^2 \) may be obvious.

We can now look at the derivation of the Weierstrass form for higher than \( \mathbb{Z}_3 \) torsion subgroups. Looking at (22) and repeating the previous procedure that was applied at the \( \mathbb{Z}_2 \) case, namely requiring the tangent at the point P to be tangent, we get
\[ P = (0, 0), \quad 2P = (b, bc), \quad 3P = (c, b - c), \]
\[-P = (0, b), \quad -2P = (b, 0), \quad -3P = (c, c^2). \]  
(25)
By taking subsequently
\[ 3P = -P \] with \( c = 0 \),
\[ \bullet \quad \Phi \cong \mathbb{Z}_4, \quad \Delta = \Delta(b, 0) = b^4 + 16b^5. \]  
(26)
\[ 3P = -2P \] with \( b = c \),
\[ \bullet \quad \Phi \cong \mathbb{Z}_5, \quad \Delta = \Delta(c, c), \]  
(27)
\[ 3P = -3P \] with \( b = c^2 + c \).
\[ \bullet \quad \Phi \cong \mathbb{Z}_6, \quad \Delta = \Delta(c^2 + c, c). \]  
(28)
There are two remaining cases of Weierstrass forms representing rational elliptic surfaces where the torsion is a subgroup of \( \mathbb{Z}_{2n} \times \mathbb{Z}_2 \). In those cases the rational elliptic surface is associated with
\[ \bullet \quad \Phi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad y^2 = x(x - \beta)(x - \gamma), \]  
(29)
with $\beta, \gamma \in \mathbb{Z}$.

- $\Phi \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, $y^2 = x(x+\zeta^2)(x+\lambda^2)$. \hfill (30)

Equation (29) appears in this form after the use of specific theorems while (30) comes by demanding that one of the points $(x_o,0)$, with $x_o = 0, \beta, \gamma$ in (29) must be the double of another number in order for the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to flow at $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. That means that $\beta = \zeta^2, \gamma = \lambda^2$ in (30). The Weierstrass representations of the elliptic fibrations with a section in eqn.'s (12), (20), (27), (28), (29), (30) are all relevant at the degeneration limit of F-theory on the elliptic $K_3$ studied in the next section.

3 Eight-dimensional compactifications of F-theory as $K_3$ fibrations

3.1 Why double covers?

In this section we examine 8 dimensional compactifications of F-theory and examine the heterotic/duality map \cite{22, 7} at the stable degeneration limit \cite{13, 26}. According to this duality map, F-theory on a $K_3$ surface admitting an elliptic fibration with a section is on the same moduli space as the heterotic string on a $T^2$ torus. Take for example the $E_8 \times E_8$ heterotic string. Then at the limit that the $T^2$ torus is large, the $K_3$ surface degenerates into a variety that is made from the union of two intersecting rational elliptic surfaces $S_1, S_2$ intersecting along an elliptic curve $E^{**}$. At this limit the J-invariant of the F-theory elliptic curve $E^{**}$ coincides with the J-invariant of the heterotic elliptic curve \cite{28} representing the $T^2$ torus as

$$y^2 = x^3 + ax + b.$$ \hfill (31)

On the contrary the $K_3$ $S_F$ compactification of F-theory is represented as an elliptic fibration over a $\mathbb{P}^1$ base, namely as a map $\pi : S_F \to \mathbb{P}^1$. This constitutes a form of violation of ”double cover parity” signalling different treatment of the bases of the duality pairs. We demand that the violation of double cover parity, of the F-theory/heterotic duality map can be corrected by considering generating the $K_3$ surface from double covers. The latter happens, in our case, by considering double covers of $K_3$ onto $\mathbb{P}^2$, the so called double sextics. It is known \cite{28} that double covers of $K_3$ onto $\mathbb{P}^2$ are branched along a plane sextic curve. That idea will constitute our main tool in this section. The difference between double covers onto $\mathbb{P}^2$ and $\mathbb{P}^1$ fibrations, between the two dual sides, can intuitively be explained by the following correspondence.

\footnote{the double cover of the complex plane x}
Consider the cubic defined by

$$xy^2 = z(x - z)(x - \lambda z).$$  \tag{32}$$

Then two generic members of the cubic pencil\(^9\) may be described as cubic curves with common flexes at \((0, 1, 0)\) and common tangents at \((0, 1, 0), (1, 0, 1)\) and \((\lambda, 0, 1)\). By blowing up the base points one can obtains the quotient of \(E \times \mathbb{P}^1\) under the involution

$$(y, z) \xrightarrow{\sigma} (-y, -z),$$ \tag{33}

where \(E\) is the elliptic curve connected to \((0, 1, \infty, \lambda)\). The notation \((y, z)\) refers to the elements of \(E \times \mathbb{P}^1\). In turn, two members of the cubic pencil together with its two degenerate fibers calculate four points on \(\mathbb{P}^1\) and thus an elliptic curve \(E'\). As a result the double cover of the rational elliptic fibration along two members has apriori its double cover given by the abelian surface \(E \times E'\). If we can connect in some way the double cover of some form of abelian surface in some form of F-theory compactifications on \(K_3\) as a double cover we are done.

**Maximizing sextics and singular \(K_3\) surfaces**

The way to proceed is to connect the double sextics to singular \(K_3\) surfaces\[^{27}\], e.g \(K_3\) surfaces for which the Picard number \(\rho = h^{1,1}\). Since every singular \(K_3\) surface is the double cover of a Kummer surface thay can be connected to compactifications involving orbifold limits of \(K_3\) and orientifolds. Let us first give some definitions. The lattice associated to the latter has rank \(\rho_{NS} = 20\) and signature \((+1, (-1)^{19})\). In general to singular \(K_3\) surfaces one associates\[^{27}\] the Picard number

$$\rho_{K_3}^{\text{inv}} = 2 + k \sum_{\nu=1}^{k} \mu(E_{\nu}) + rk|\Phi|,$$ \tag{34}

where \(|\Phi|\) is the order of the group of sections \(\Phi\) that is the Mordell-Weyl group and \(r\) its rank. The quantity \(\mu(E_{\nu})\) is associated to the number of components of the set \(E_{\nu}\) of the singular fibers. Similarly, when a curve \(X\) is the double cover of \(Y\) branched along a curve \(C\), then

$$\rho(X) \geq \rho(Y) + \sigma(C).$$ \tag{35}

For a plane sextic curve \(C\) coming as a double cover onto \(\mathbb{P}^2\), \(0 \leq \sigma \leq 19\). So by comparison with (34) we can notice that the index\(^{10}\) \(\sigma(C)\) associated to the sextic has a similar role as

---

\(^9\)Note that the term pencil is defined to be the dimension of the projective space that parametrizes eqn.(32). Namely one in our case.

\(^{10}\)When we minimally resolve a singularity \(x_n\), \(n\) linearly independent rational curves in the Neron-Severi group appear. Then for a curve \(C\) the index is defined as the sum of the subindices \(n\) of all its the simple singularities.
those of the singular fibers of the singular $K_3$ surfaces. In fact we will see, in the rest of the section, in the cases of interest in 8 dimensional compactifications of F-theory/heterotic duality map, $K_3$ surfaces admitting elliptic fibrations with a section and $K_3$ surfaces admitting double covers on to $\mathbb{P}^2$ coincide. The latter list of $K_3$ fibrations has been worked out in \[14\]. So by considering double covers onto the base manifold on both sides of the 8-dimensional F-theory/heterotic duality map we recover naturality in the treatment of the base.

A sextic is called maximizing if its index is maximal, $\sigma = 19$. A result from (32) it follows that the maximizing sextic $C$ is a singular $K_3$ surface. If the index $\sigma(S) = 19$ the singularities of the sextic are in the form $a_1$, $d_2n$, $e_7$, and $e_8$. In this case the sextic is called supermaximizing and the number of the rational components in this case is less than ten. That can be proved by a simple calculation of the Euler number. We remind that these are singularities of the branch curve and not its double cover. The double cover has as usual A, D, E singularities. This suggests that in the general case $K_3$ surfaces that are acted by involutions not having fixed points have singular fibers with at least ten components. When we have ten components those components are rational. In fact the following table will assist us to our work later.

| fiber-type | singularity-type | $\epsilon$ | $\delta$ | $d$ | order of torsion group | double cover |
|------------|-----------------|------------|----------|-----|-----------------------|-------------|
| $I_n$      | $A_{n-1}$       | $n$        | $n-1$    | $n$ | $\infty$              | $I_{2n}$    |
| $I_n^*$    | $D_{4n}$        | $n+6$      | $n+4$    | $4$ | $4$                   | $I_{2n}$    |
| II         | none            | $2$        | $0$      | $1$ | $1$                   | IV          |
| III        | $A_1$           | $3$        | $1$      | $2$ | $2$                   | $I_o^*$     |
| IV         | $A_2$           | $4$        | $2$      | $3$ | $3$                   | $IV^*$      |
| $II^*$     | $E_6$           | $10$       | $8$      | $1$ | $1$                   | $IV^*$      |
| $III^*$    | $E_7$           | $9$        | $7$      | $2$ | $2$                   | $I_o^*$     |
| $IV^*$     | $E_8$           | $8$        | $6$      | $3$ | $3$                   | IV          |

Here, the Kodaira fiber are placed against their components $\delta$, Euler number $\epsilon$, discriminant and fiber of the double cover.

### 3.2 Elliptic fibrations with involutions

The double cover consists of an elliptic fibration $\pi : Y \rightarrow \mathbb{P}^1$ and an involution $\sigma : z \rightarrow -z$ that respects the fibration. In the general case that $Y$ is an elliptic $K_3$ surface we can distinguish two general cases. The case of isolated fixed points and the case of no isolated fixed points. In the first case the are eight fixed points and the quotient by the involution $X$ is a $K_3$ surface. That is the case that is considered in studying the CHL vacuum in [21]. In the second case there are three possibilities: a) $X$ is a rational surface with the fixed locus appears in two components. The surface $Y$ may be coming from a double sextic, as a union of two cubics. b) $X$ is a rational surface but the fixed locus appears in one component, and c)
X is an Enriques surface with no fixed locus and the elliptic fibration has two double fibers. The point that we follow is to consider elliptic $K_3$ fibrations $\pi : Y \to \mathbb{P}^1$, where $Y$ is an elliptic $K_3$ surface, and examine the effect of the involutions $\sigma$ that respect the fibration. We represent the involution after resolution by $X = Y/\sigma$ where tilde denotes desingularization. We will be interested in singular sextics that can give $K_3$ surfaces after resolution. In fact, there are two possibilities that can be realized for the surface $X$. The surface $X$ can be either a ruled surface or an elliptic surface. Those two distinctions are related to the way that the involution $\sigma$ acts on the $Y$ fibration. We can have $X$ as a rational ruled surface only if each fiber is invariant under the involution and contains fixed points. In fact, by blowing down $X$ to a minimal model $X_o$, containing no exceptional curves of the first kind, that can be the Hirzebrush surface $F_4$, we obtain $Y$, the $K_3$ surface, as a double cover of $X_o$ branched along a 4-section $\Gamma_o = S_\infty + T$. We denoted by $S_\infty$ the minimal section, $S_\infty^2 = -4$ and $T$ is a trisection disjoint from $S_\infty$. One can now observe that the index of the elliptic fibration is the index of $\Gamma_o$ or $T$. In this case the branch locus of the covering will be a transversal divisor hitting each rational fiber exactly four times and it will appear as a sextic. In this case the rational surface $X$ can be represented as a $\mathbb{P}^2$ after a few birational modifications.

An easy way to get an elliptic fibration is to have a singular point $P$ on the sextic and consider the elliptic fibration induced by the lines through it. When done in this way, the elliptic fibration centered at the triple point of a maximizing sextic is extremal. In this case there are more possibilities realized for the rational surface $X$ as the torsion group can be different than unity as it was the case in the standard heterotic/F-theory duality [22, 7, 8]. The computation of the torsion group $\Phi$ for elliptic fibrations of a maximizing sextic can be determined from the relation

$$d(X) = \Pi_\nu d(F_\nu)/|\Phi|^2.$$  \hspace{1cm} (36)$$

Here, $d$ denotes the discriminant while includes the blown up Kodaira singularities and the product runs over the configurations of the singular fibers. Take for example the configuration with $I_9, 3I_1$ fibers. Applying the formula (36) we get that $d(X) = 1$ which must be the case as the Picard lattice for a rational elliptic surface is the whole of $H^2(X, \mathbb{Z})$ and the lattice is unimodular.

The results in the case of double covers are summarized as follows:
### Table 2

| Cubic fibration | order F | $K_3$ fibration |
|------------------|---------|-----------------|
| $II^* 2I_1$      | 1       | $2II^* 2I_2$    |
| $II^* II$        | 1       | $2II^* IV$      |
| $III^* I_2 I_1$  | 2       | $2III^* I_4 I_2$|
| $III^* III$      | 2       | $2III^* I_0^3$  |
| $I_1^* 2I_1$     | 2       | $2I_1^* 2I_2$   |
| $I_9 3I_1$       | 3       | $I_{18} I_2 4I_1, 2I_9 2I_2 2I_1$ |
| $IV^* I_3 I_1$   | 3       | $2IV^* I_6 I_2$ |
| $IV^* IV$        | 3       | $3IV^*$         |
| $I_8 I_2 2I_1$   | 4       | $I_{16} I_4 4I_1, I_{16} 3I_2 2I_1, I_{16} 3I_2 2I_1, 2I_8 I_4 I_2 2I_1, 2I_8 4I_2$ |
| $I_2^* 2I_2$     | 4       | $2I_2^* 2I_4$   |
| $I_1^* I_4 I_1$  | 4       | $2I_1 I_8 I_2$  |
| $2I_5 2I_1$      | 5       | $2I_{10} 4I_1, I_{10} 2I_5 I_2 2I_1, 4I_5 2I_2$ |
| $I_6 I_3 I_2 I_1$| 6       | $I_{12} I_6 2I_2 2I_1, I_{12} I_4 2I_3 2I_1, I_{12} 2I_3 I_2, 3I_6 I_4 2I_1, 3I_6 3I_2, 2I_6 I_4 2I_3 I_2$ |
| $2I_4 2I_2$      | 8       | $2I_8 4I_2, I_8 3I_4 2I_2, 6I_4$ |
| $4I_3$           | 9       | $2I_6 I_3$      |

The first column gives us the Kodaira fibers appearing in the cubic associated with the corresponding Mordell-Weyl group order of the group of sections given in the second column. The third column is the configuration of allowed Kodaira fibers for the associated $K_3$ fibrations.

#### 3.3 Examples with Point Like Instantons

We will discuss the case of the creation of an $E_8$ gauge symmetry in the F-theory/heterotic duality realization in eight dimensions when the $K_3$ surface $S_F$ of F-theory compactifications is acted by the involution $\sigma : z \rightarrow -z$. The resulting surface $X$ is an elliptic surface. Now the sextic of the branch locus is made from the union of the two cubics.

At the limit that the heterotic torus is large, the $K_3$ surface breaks up into two rational surfaces and each rational surface is coming by taking the double covers branched at two fibers. Elliptic fibrations generated in this way always have sections. In the case of $E_8 \times E_8$ symmetry on the F-theory side, the observed symmetry group $E_8$ is reproduced by the elements of the vanishing cohomology $H_2(S_i, \mathbb{Z})$, with $i$ either 1 or 2, of the rational elliptic surface $S_i$. In geometrical terms the rational elliptic surface represented as a cubic $C_0$ and a flexed line $L_\infty$ forms a cubic pencil spanned by $C_0$ and $3L_\infty$. This has a $II^*$ and $2I_2$ fibers, an thus generates an unbroken $E_8$ gauge symmetry from point like instantons.
with no local holonomy. Taking the double cover of the sextic formed by the two nodal cubics in the pencil will produce an elliptic fibration of type $2II^*2I_2$. Amazingly enough it appears that the double cover of the $K_3$ fibration can create the whole of $E_8 \times E_8$ gauge symmetry. However, $2II^*2I_2$ is the symmetry of the double cover and does not correspond to the observed heterotic gauge symmetry.

Another elliptic fibration on the same $K_3$ may be build by considering the lines through one of the nodes. Now the intersection of the two cubics\textsuperscript{11} of order nine will give rise to an $I_{18}$, the other node will give us a $I_2$ and there are four tangent lines from the point to the nodal cubic. In total we get the configuration $I_{18}I_24I_1$ with the order of the Mordell-Weyl group to be three. The $Spin(16)/Z_2$ heterotic string comes from the configuration $I^*_42I_1$ of the rational elliptic surface of table 2. The corresponding $K_3$ fibration upon which the involution $\sigma$ is acting is $2II^*IV$. In fact more possibilities are possible [21].

4 Effective string theories from double covers

The question that remains from the previous considerations is if we can find an effective theory that can be formulated in terms of double covers and is defined in lower dimensions, e.g four, that may be defined in the context of F-theory/heterotic duality. The heterotic string theory defined in terms of double covers is considered in this section. We hope to address the $N = 2$ four dimensional compactification of F-theory in terms of double covers in a future work. Let us now consider the usual F-theory compactification, on a Calabi-Yau 3-fold with an $F_n$ base [7, 8] which is on the same moduli space as the heterotic string on a $K_3$ surface. Further compactification on four dimensions on both sides on a $T^2$ torus may produce a $N = 2$ vector multiplet effective theory with e.g an $SL(2, Z)_{T} \times SL(2, Z)_{U}$ duality invariance. If there are Wilson lines on the $T^2$ torus the classical perturbative duality invariance of the vector multiplet effective theory becomes $Sp(4, Z)$. In this case the formulation of the effective theory of light modes is consistently described in terms on the genus two Siegel modular forms which can reproduce consistently the one loop corrections to the gauge coupling constants of $N = 1$ $(0,2)$ orbifold compactifications of the heterotic string[10]. As we said in the introduction it is desirable to have an interpretation of the formulation of the heterotic string compactifications in terms of particular Riemann surfaces with manifest $Sp(4, Z)$ invariance. Here we will do just that. The way that we will proceed in our investigation is to first define the basic elements that define the $N = 1$ $(0,2)$ orbifold compactifications of the heterotic string and then describe the derivation of the Riemann surface possessing the $Sp(4, Z)$ invariance.

For genus $g = 1$ the graded ring of modular forms is generated by the $E_4$, $E_6$ modular

\textsuperscript{11}Remember that a rational elliptic surface is the blow up of $\mathbb{P}^2$ at the nine intersection points of two cubic curves, or the nine basepoints of the cubic pencil.
In the genus two case the graded ring is generated by the Siegel modular forms $E_4, E_6$, and the cusp forms $\chi_{10}, \chi_{12}$ and $\chi_{35}$ where the latter are given by the following expressions

$$I_4 = \sum (\theta_m)^8,$$

$$I_6 = \sum_{\text{syzygous}} \pm (\theta_{m_1} \theta_{m_2} \theta_{m_3})^4,$$

$$I_{10} = -2^{14} \cdot \chi_{10} = \Pi(\theta_m)^2,$$

$$I_{12} = 2^{17} \cdot \chi_{12} = \sum (\theta_{m_1} \theta_{m_2} \cdots \theta_{m_3})^4,$$

$$I_{35} = 2^{39} \cdot \chi_{35} = (\Pi \theta_m) (\sum_{\text{azygous}} \pm (\theta_{m_1} \theta_{m_2} \theta_{m_3})^{20}),$$

where $m_1 = (0,0,0,0), m_2 = (0,0,0,1/2)$ and $m_3 = (0,0,1/2,0)$. The sum in $\chi_{12}$ extends over the fifteen Gorel quadruple a sequence of four even characteristics which form a syzygous sequence. Next consider the hyperelliptic curve for a genus 2g+2 surface in the form

$$y^2 = \Pi_{i=1}^{2g+2} (x - e_i) = P_{2g+2}(x, e_i), \quad e_i \neq e_j, \text{ for } i \neq j,$$

respectively in the genus case as

$$y^2 = \Pi_{i=1}^6 (x - e_i) = P_6(x, e_i), \quad e_i \neq e_j, \text{ for } i \neq j,$$

which represents the double cover of the sphere branched over 2g+2 points, respectively 6 points. Note that $\theta_m = D^{1/8}$, where D is the discriminant of $F_6$. On the curve one can attach a lattice generated by the pair $(I, \Omega)$, where I is a $g \times g$ identity matrix and $\Omega$ is the period matrix. The surface equipped with the pair $(I, \Omega)$ is called a Jacobian. We denote by $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ the canonical homology basis on $F_6$. In addition, we can define by $(\zeta_1, \ldots, \zeta_g)$ as the dual homology basis. Note that the dual homology basis can be defined in terms of $\zeta_i = x_i dz/y$, $i = 1, \cdots, g$. In fact by taking the map from the Jacobian to the complex numbers we define the $\theta$ functions

$$\theta \left( \frac{a}{b} \right) = \sum_{n \in \mathbb{Z}^{g+1}} \exp[(n+a)^T \Omega(n+a) + 2\pi i(n+a) \cdot (z+b)]$$

with the usual build in $Sp(2g, \mathbb{Z})$ invariance. One might now start to feel what we are trying to prove. We will prove that under certain conditions all $N=1 (0,2)$ orbifold compactifications of the heterotic string can be described by certain genus two curves, the binary

---

12 the subscript denotes the modular weight of the respective modular form.
sextics. In particular once the moduli coming from a heterotic string vacum are known, we
can describe the precise form of the Riemann surface that they line on.
However something is still missing from our discussion. The element that we require is that
there is a birational correspondence \([30]\) between the projective varieties associated with
the graded ring of even projective invariants of binary sextics and with the graded ring
of modular forms. In simple terms that means that the projective variety linked with the
graded ring of even projective invariants of binary sextics is a compactification of moduli of
curves of genus two.

So, by keeping in mind the analogy with the branched sextic curve coming from double
covering of \(\mathbb{P}^2\) in the previous sections, we conjecture the following theorem :

**Theorem** The projective variety associated with the even projective invariants of binary
sextics, for a particular \((0,2)\) \(N = 1\) heterotic string theory vacuum, represents the Riemann
surface with manifest \(Sp(4,\mathbb{Z})\) invariance that the moduli live. Let us explain in more detail
this issue. The projective invariants \(A, B, C, D\), of the binary sextics have degrees 2,4,6,10.
So if we symbolize by \(\varphi_1, \varphi_2, \ldots, \varphi_6\) the six roots of the sextic \(s_oX^6 + s_1X^5 + \ldots + s_6\) and
we denote their difference \(\varphi_i - \varphi_j\) by \((ij)\) the invariants take the following form

\[
A(s) = s_o^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2 \\
B(s) = s_o^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2 \\
C(s) = s_o^6 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2 \\
D(s) = s_o^{10} \prod_{j<k} (jk)^2.
\]

Because any sextic can be brought in the general form

\[X(X - 1)(X - \lambda_1)(X - \lambda_2)(X - \lambda_3)\]  \hspace{1cm} (42)

we can replace each of the three lambdas in (42) by some theta functions of zero argument, namely

\[
\lambda_1 = (\theta_{1100}\theta_{1000})^2, \quad \lambda_2 = (\theta_{1010}\theta_{1100})^2, \quad \lambda_3 = (\theta_{1001}\theta_{1000})^2,
\]

where

\[
\theta_{g_1g_2h_1h_2}(\tau_1, \epsilon, \tau_2) = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} \frac{d^n}{d\tau_1^n} \theta_{g_1h_1}(\tau_1) \frac{d^n}{d\tau_2^n} \theta_{g_2h_2}(\tau_2) \epsilon^{2n}
\]

\[
(44)
\]

theta functions of genus two. It can be proved that \(\lambda_1, \lambda_2, \lambda_3\) can be expanded in terms of
even powers of \(\epsilon\) when \(\epsilon\) is small. As a result all the variables in eqn. (42) are fixed and
its roots may be calculated. Moreover the invariants \(A, B, C, D\) are expressed in terms of \(\epsilon\)
and \(\lambda\)’s. It is remarkable that the following relations hold

\[
D/A^5 \propto \epsilon^{12}, \quad (B/A^3)^3 \propto j(\tau_1)j(\tau_2)\epsilon^{12}, \quad ((3C - AB))/A^3 \propto (j(\tau_1) - j(\tau_2))\epsilon^{12}.
\]

\[
(45)
\]
or in precise form

\[ I_4 = B, \quad I_6 = \frac{1}{2}(AB - 3C), \quad I_{10} = D, \quad I_{12} = AD, \quad I_{35} = 5^3D^2E. \tag{46} \]

That means that the uniformation parameters of the sextic \((\ref{42})\) are fixed genus two elements therefore possessing manifest \(Sp(4, Z)\) invariance and therefore confirming our theorem. In its non-perturbative form the equation for the sextic \(P_6\) involves the parametric relation \((\ref{40})\). Note that from eqn. \((\ref{43})\) one can notice that the fundamental invariant of the full theory are expressed in terms of products of \(j\)-invariants. What we have not discuss is the expansion of genus two theta functions \((\ref{44})\) in terms of the \(\epsilon\) parameter, our Wilson lines, represents exactly the fact that the space of projective varieties corresponding to the invariants \(A, B, C, D\) has been blown up \([30]\) such that the Jacobian variety of the genus two curve has degenerate to products of elliptic curves. The blow up process is necessary since the projective variety \((\ref{39})\), does not include apriori the Siegel fundamental domain.

The correspondence with the heterotic string comes after identifying \(\tau_1 = T, \tau_2 = U, \epsilon = \text{Wilson line}\). At the points that the discriminant of the projective variety of \((\ref{88})\) degenerates both \(T, U\) and the Wilson line \(A, B\) are involved in a non-trivial relation. That means that at the point where the discriminant of eqn. \((\ref{12})\) vanishes one or more of the moduli may be fixed therefore breaking non-trivially space-time supersymmetry.

Alternatively thinking, eqn. \((\ref{39})\) supplied with the projective invariants \(A, B, C, D\) at the limit of small \(\epsilon\), represents the \((0, 2)\) perturbative expansion of the \(N = 2\) sector of the 4\(\mathcal{D}\) heterotic string vacuum.

Let us now make a comment regarding the 8-dimensional compactifications of F-theory. By taking the equation

\[(\text{branched sextic}) = 0 \tag{47}\]

we may describe the Riemann surface responsible for the 8-dimensional \(K_3\) fibrations as double cover onto \(\mathbb{P}^2\). Those compactifications that are characterized by the vanishing locus have embedded \(Sp(4, Z)\) invariance and may represent how 7-branes degenerate. Of course the real test of our conjecture at this point is to test this result in its four dimensional F-theory counterparts involving compactifications on a three-fold. This task will be performed in a future work.

5 One loop 4D heterotic prepotential for subgroups of \(PSL(2, Z)\)

\(N = 2\) heterotic string theories in four dimensions come from compactification of the ten dimensional heterotic string on the \(K_3 \times T^2\). In the simplest case the effective action of light modes is invariant under the classical modular group is \(SL(2, Z)_T \times SL(2, Z)_U\). In this case the formulation of the six dimensional F-theory compactification, prior to further
compactification of a $T^2$ torus, has been given in [4, 8]. Let us consider the usual F-theory/heterotic duality map in eight dimensions [13]. When compactifying the F-theory side on a $K_3$ surface $S_F$ admitting an elliptic fibration this becomes dual at the heterotic string compactified on a $T^2$ torus. At the degeneration limit the j-invariant of the F-theory elliptic curve becomes identical to the j-invariant of the heterotic elliptic curve. If we further compactify on another $K_3$ surface both dual sides then what effectively happens is that the statement about monodromy of the F-theory elliptic curve translates into a statement about target space duality. So by the use of the Mordell-Weyl group action on the F-theory elliptic curve in the original eight dimensional compactification of F-theory we can control the target space duality group in four dimensions.

Let us now consider that our F-theory elliptic curve contains points that are associated to a cyclic subgroup of order 2 that a generator has not been chosen. In this case the F-theory has monodromy $\Gamma_o(2)$ and the $4D$ $N = 2$ theory coming from compactification of the heterotic string compactified on the $K_3 \times T^2$ is invariant under the target space duality group $\Gamma_o(2)_T \times \Gamma_o(2)_U$. The third derivative of the prepotential of the $N = 2$ vector multiplet theory is given by

$$f_{TTT} = \frac{96i}{\pi} \frac{(\Phi_T(T))^2}{\Phi(T)(\Phi(T) - \Phi(\frac{i}{\sqrt{2}}))} \frac{\Phi^\frac{3}{2}(U)}{\Phi_U(U)} \frac{(\Phi(U) - \Phi(\frac{i}{\sqrt{2}}))^\frac{3}{2}}{(\Phi(U) - \Phi(T))}.$$ \hspace{1cm} (48)

At the limit that $T$ drifts towards $U_g = \frac{aU + b}{cU + d}$, where $g$ an $\Gamma_o(2)$ element

$$f_{TTT} \rightarrow -\frac{2i}{\pi} \frac{1}{T - U_g(cU + d)^2},$$ \hspace{1cm} (49)

only when

$$F(U_g) >> F(\frac{i}{\sqrt{2}}).$$ \hspace{1cm} (50)

At this limit the one loop Kähler metric exhibits the usual logarithmic singularity

$$G^{(1)}_{TT} \rightarrow \frac{2}{\pi} \ln |T - U_g|^2 G^{(0)}_{TT}.$$ \hspace{1cm} (51)

When our theory is invariant under the group $\Gamma^o(2)_T \times \Gamma^o(2)_U$, $F_{TTT}$ becomes

$$f_{TTT} = \frac{96i}{\pi} \frac{(\phi_T(T))^2}{\phi(T)(\phi(T) - \phi(i\sqrt{2}))} \frac{\phi^\frac{3}{2}(U)}{\phi_U(U)} \frac{(\phi(U) - \phi(\frac{i}{\sqrt{2}}))^\frac{3}{2}}{(\phi(U) - \phi(T))},$$ \hspace{1cm} (52)

where

$$\phi = \frac{\Delta(z/2)}{\Delta(z)}.$$ \hspace{1cm} (53)
the Hauptmodul for $\Gamma^o(2)$.

**Note Added**

The results of this work have been presented in the author’s talk at SUSY ’98 and its transparencies are available at [http://hepnts1.rl.ac.uk/SUSY98/](http://hepnts1.rl.ac.uk/SUSY98/). We note that exactly the same day appeared in the hep-th archive the work of [18] where singular $K_3$ surfaces are used in the description of the attractor mechanism.

**Acknowledgements**

We would like to thank D. Zagier for useful discussions and B. Mazur for a suggestion. Also we would like to thank the Isaac Newton Institute for Mathematical Sciences at Cambridge for its hospitality and the kind use of its facilities in the context of my Isaac Newton Junior Membership and the Newton Institute Workshop on Computational Results of Arithmetic Geometry. We are especially grateful to U. Persson for sending us a copy of [14] and for important discussions.

**References**

[1] E. Witten, Comments on String Theory Dynamics in Various Dimensions, Nucl. Phys. B443 (1995) 85.

[2] E. Witten, Strong Coupling Expansion of Calabi-Yau Compactification, Nucl. Phys. B471 (1996) 135.

[3] N. Seiberg and E. Witten, Monopole Condensation, And Confinement In $N = 2$ Supersymmetric Yang-Mills Theory, Nucl.Phys. B426 (1994) 19; Erratum-ibid. B430 (1994) 485, [hep-th/9407087]. Monopoles, Duality and Chiral Symmetry Breaking in $N=2$ Supersymmetric QCD Nucl.Phys. B431 (1994) 484, [hep-th/9408099].

[4] Perturbative Couplings of Vector Multiplets in $N = 2$ Heterotic String Vacua, B. de Wit, V. Kaplunovsky, J. Louis, D. Luest Nucl.Phys. B451 (1995) 53, [hep-th/9504006].

[5] Perturbative Prepotential and Monodromies in $N = 2$ Heterotic Superstring, Nucl.Phys. B447 (1995) 35, [hep-th/9504034].

[6] C. Kokorelis, The Master Equation for the Prepotential, [hep-th/9802099].

[7] D. R. Morrison and C. Vafa, Compactifications of F-theory on Calabi-Yau Threefolds I, Nucl. Phys. B473 (1996) 74, [hep-th/9602114].

[8] D. R. Morrison and C. Vafa, Compactifications of F-theory on Calabi-Yau Threefolds II, Nucl.Phys. B476 (1996) 437, [hep-th/9603161].

[9] C. Kokorelis, "The Master Equation for the Prepotential", Nucl. Phys. B542 (1999) 89-111, [hep-th/9802068]. Shorter version of [hep-th/9802099].
[10] P. Mayr and S. Stieberger, Moduli Dependence of One-Loop Gauge couplings in (0, 2) Compactifications, Phys. Lett. B, hep-th/9504123.

[11] R. Dijkgraaf, E. Verlinde and H. Verlinde, Counting Dyons in $N = 4$ String Theory, hep-th/9607026.

[12] G. Cardoso, G. Curio, D. Lust and T. Mohaupt, On the duality between the heterotic string and F-theory in eight dimensions, Phys. Lett. B389 (1996) 479, hep-th/9609111.

[13] R. Friedman, J. Morgan and E. Witten, Vector bundles and F-theory, Commun. Math. Physics. 187 (1997) 679, hep-th/9701162.

[14] U. Persson, Double sextics and singular $K_3$ surfaces, In : Casa-Severo, E. Welters, G. E. Xambo-Descamps, S. (eds) Algebraic geometry. Proceedings, Sitges 1983. (Lect. Notes Math., vol. 1124, pp 262-328 Berlin Heilderberg, New York, Springer 1985.

[15] D. A. Cox and W. R. Perry, Torsion in Elliptic Curves over $k(t)$, Composition Mathematica, Vol. 41, Fasc. 3, (1980) 337-354.

[16] D. S. Kubert, Universal Bounds on the Torsion on Elliptic Curves, Proc. London Math. Soc. (3) 33 (1976) 193-237

[17] R. Miranda and U. Persson, On Extremal Elliptic Surfaces, Math. Z. 193, (1986) 537-558.

[18] G. Moore, Arithmetic and Attractors, hep-th/9807087.

[19] P. Aspinwall and D. Morrison, Non-simply Connected Gauge groups and Rational Points on Elliptic curves, J.High Energy Phys. 07 (1998) 012, hep-th/9805200.

[20] D. Kubert, Universal Bounds on the Torsion of Elliptic Curves, Proc. London. Math. Soc. 33 (1976) 193-237.

[21] C. Kokorelis and U. Persson, In preparation.

[22] C. Vafa, Evidence for F-theory, Nucl. Phys. B469 (1996) 403, hep-th/9602022.

[23] A. Knapp, Elliptic Curves, Mathematical Notes 40, Princeton University Press, Princeton, New jersey, 1992

[24] M. Bershadsky, T. Pantev and V. Sadov, F-theory with Quanitizes Fluxes, hep-th/9805056.

[25] P. Berglund, A. Klemm, P. Mayr and S. Theisen, On Type IIB Vacua With Varying Coupling Constant, hep-th/9805198.

[26] P. S. Aspinwall and David. R. Morrison, Point Like Instantons on $K_3$ Orbifolds, Nucl. Phys. B503 (1997) 533, hep-th/9705104.
[27] T. Shioda and H. Inose, On Singular $K_3$ surfaces, in Complex Analysis and Algebraic Geometry, Cambridge University Press, Cambridge 1977.

[28] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Wiley and Sons, 1978.

[29] E. Witten, Small Instantons in String Theory, Nucl. Phys. B460 (1996) 541, [hep-th/9511030](https://arxiv.org/abs/hep-th/9511030).

[30] J. Igusa, On Siegel Modular Forms of Genus Two, American Journal of Mathematics, Vol. 84 (1962) p 175, ibid, vol. 86 (1964) p. 392.

[31] J. Igusa, Modular Forms and Projective Invariants, American Journal of Mathematics, Vol. 89 (1967) 817.

[32] See for example, H. M. Farkas and I. Kra, Riemann Surfaces, Springer-Verlag, 1992.