NEW STRONG DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF MEROMORPHIC MULTIVALENT QUASI-CONVEX FUNCTIONS

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ABSTRACT. New strong differential subordination and superordination results are obtained for meromorphic multivalent quasi-convex functions in the punctured unit disk by investigating appropriate classes of admissible functions. Strong differential sandwich results are also obtained.

1. INTRODUCTION AND PRELIMINARIES

Let Σp denote the class of all functions f of the form:

\[ f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \]

which are analytic in the punctured unit disk \( U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \).

A function \( f \in \Sigma_p \) is meromorphic multivalent starlike if \( f(z) \neq 0 \) and

\[ -\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U^*). \]

Similarly, \( f \in \Sigma_p \) is meromorphic multivalent convex if \( f'(z) \neq 0 \) and

\[ -\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U^*). \]

Moreover, a function \( f \in \Sigma_p \) is called meromorphic multivalent quasi-convex function if there exists a meromorphic multivalent convex function \( g \) such that \( g'(z) \neq 0 \)

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and
\[-\text{Re} \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > 0 \quad (z \in U^*).\]

Let \(\mathcal{H}(U)\) be the class of analytic functions in the open unit disk 
\(U = \{ z \in \mathbb{C} : |z| < 1 \}\). For a positive integer \(n\) and \(a \in \mathbb{C}\), let \(\mathcal{H}[a, n]\) be the subclass of \(\mathcal{H}(U)\) consisting of functions of the form:
\[f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\]
with \(\mathcal{H} = \mathcal{H}[1, 1]\).

Let \(f\) and \(g\) be members of \(\mathcal{H}(U)\). The function \(f\) is said to be subordinate to \(g\), or (equivalently) \(g\) is said to be superordinate to \(f\), if there exists a Schwarz function \(w\) which is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1 (z \in U)\) such that \(f(z) = g(w(z))\). In such a case, we write \(f \prec g\) or \(f(z) \prec g(z)\), \(z \in U\). Furthermore, if the function \(g\) is univalent in \(U\), then we have the following equivalent (see [5]):
\[f(z) \prec g(z) \quad \text{iff} \quad f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).\]

Let \(G(z, \zeta)\) be analytic in \(U \times \bar{U}\) and let \(f(z)\) be analytic and univalent in \(U\). Then the function \(G(z, \zeta)\) is said to be strongly subordinate to \(f(z)\) or \(f(z)\) is said to be strongly superordinate to \(G(z, \zeta)\), written as \(G(z, \zeta) \prec\prec f(z)\), if for \(\zeta \in \bar{U} = \{ z \in \mathbb{C} : |z| \leq 1 \}\), \(G(z, \zeta)\) as a function of \(z\) is subordinate to \(f(z)\). We note that
\[G(z, \zeta) \prec\prec f(z) \quad \text{iff} \quad G(0, \zeta) = f(0) \quad \text{and} \quad G(U \times \bar{U}) \subset f(U).\]

**Definition 1.1.** [6] Let \(\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}\) and let \(h\) be a univalent function in \(U\). If \(F\) is analytic in \(U\) and satisfies the following (second-order) strong differential subordination:
\[(1.1) \quad \phi \left( F(z), zF'(z), z^2 F''(z); z, \zeta \right) \prec\prec h(z),\]
then \(F\) is called a solution of the strong differential subordination (1.1). The univalent function \(q\) is called a dominant of the solutions of the strong differential subordination or more simply a dominant if \(F(z) \prec q(z)\) for all \(F\) satisfying (1.1). A dominant \(\bar{q}\) that satisfies \(\bar{q}(z) \prec q(z)\) for all dominants \(q\) of (1.1) is said to be the best dominant.

**Definition 1.2.** [7] Let \(\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}\) and let \(h\) be analytic function in \(U\). If \(F\) and \(\phi (F(z), zF'(z), z^2 F''(z); z, \zeta)\) are univalent in \(U\) for \(\zeta \in \bar{U}\) and satisfy the following (second-order) strong differential superordination:
\[(1.2) \quad h(z) \prec\prec \phi \left( F(z), zF'(z), z^2 F''(z); z, \zeta \right),\]
then \(F\) is called a solution of the strong differential superordination (1.2). An analytic function \(q\) is called a subordinant of the solutions of the strong differential superordination or more simply a subordinant if \(q(z) \prec F(z)\) for all \(F\) satisfying (1.2). A univalent subordinant \(\bar{q}\) that satisfies \(q(z) \prec \bar{q}(z)\) for all subordinants \(q\) of (1.2) is said to be the best subordinant.
Definition 1.3. [6] Denote by $Q$ the set consisting of all functions $q$ that are analytic and injective on $\bar{U} \setminus E(q)$, where

$$E(q) = \left\{ \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty \right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(q)$.

Furthermore, let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$, and $Q(1) \equiv Q_1$.

Definition 1.4. [9] Let $\Omega$ be a set in $\mathbb{C}$, $q \in Q$, and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; z, \zeta) \notin \Omega$, whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \text{Re}\left\{ \frac{t}{s} + 1 \right\} \geq k\text{Re}\left\{ \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right\},$$

$z \in U$, $\xi \in \partial U \setminus E(q)$, $\zeta \in \bar{U}$, and $k \geq n$.

We simply write $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

Definition 1.5. [8] Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; z, \zeta) \in \Omega$, whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \text{Re}\left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m}\text{Re}\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$, $\xi \in \partial U$, $\zeta \in \bar{U}$, and $m \geq n \geq 1$.

In particular, we write $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

In our investigations, we will need the following lemmas.

Lemma 1.1. [9] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $F \in \mathcal{H}[a, n]$ satisfies

$$\psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right) \in \Omega,$$

then $F(z) \prec q(z)$.

Lemma 1.2. [8] Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $F \in Q(a)$ and $\psi(F(z), zF'(z), z^2F''(z); z, \zeta)$ is univalent in $U$ for $\zeta \in \bar{U}$, then

$$\Omega \subset \left\{ \psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right) : z \in U, \zeta \in \bar{U} \right\}$$

implies $q(z) \prec F(z)$.

In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1–4]. In this present investigation, by making use of the strong differential subordination results and strong differential superordination results of Oros and Oros [8,9], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and superordination properties of meromorphic multivalent quasi-convex functions.
2. Strong Subordination Results

**Definition 2.1.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in Q_1 \cap \mathcal{H} \). The class of admissible functions \( \Phi_{\Omega}[\Omega, q] \) consists of those functions \( \phi: \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C} \) that satisfy the admissibility condition: \( \phi(u, v, w; z, \zeta) \notin \Omega \), whenever

\[
u = q(\xi), \quad v = \frac{kq'q(\xi)}{qq(\xi)}, \quad q(\xi) \neq 0 \text{ and } \Re \left\{ \frac{w + v^2}{v} \right\} \geq k \Re \left\{ \frac{\xi q'q(\xi)}{qq(\xi)} + 1 \right\},
\]

where \( z \in U, \zeta \in \bar{U}, \xi \in \partial U \setminus E(q) \), and \( k \geq 1 \).

**Theorem 2.1.** Let \( \phi \in \Phi_{\Omega}[\Omega, q] \). If \( f \in \Sigma_p \) satisfies

\[
\phi \left( \frac{\frac{(z^p f'(z))'}{g'(z)} - \frac{z (z^p f'(z))^"}{g'(z)} - \frac{z^2 (z^p f'(z))'}{g'(z)} + z (z^p f'(z))'}{z (z^p f'(z))'} \times \left[ 1 - \frac{z (z^p f'(z))'}{g'(z)} \right] - \frac{z^2 (z^p f'(z))^"}{g'(z)} + \frac{z^2 (z^p f'(z))^"}{g'(z)} \left( \frac{z^2 (z^p f'(z))^"}{g'(z)} - 1 \right); z, \zeta \right) \in \Omega,
\]

then

\[
\frac{(z^p f'(z))'}{g'(z)} < q(z).
\]

**Proof.** Let the analytic function \( F \) in \( U \) be defined by

\[
F(z) = \frac{(z^p f'(z))'}{g'(z)}.
\]

After some calculation, we have

\[
\frac{z F'(z)}{F(z)} = \frac{z (z^p f'(z))^"}{(z^p f'(z))'} - \frac{z^2 (z^p f'(z))^"}{g'(z)}.
\]

Further computations show that

\[
\frac{z^2 F''(z)}{F(z)} + \frac{z F'(z)}{F(z)} \left( \frac{z F'(z)}{F(z)} \right)^2 = z \left[ \frac{z (z^p f'(z))^"}{(z^p f'(z))'} - \frac{z^2 (z^p f'(z))^"}{g'(z)} \right]'
\]

\[
= \frac{z^2 (z^p f'(z))^"''}{(z^p f'(z))'} + \frac{z (z^p f'(z))^"'}{(z^p f'(z))'} \left( 1 - \frac{z (z^p f'(z))^"'}{(z^p f'(z))'} \right) - \frac{z^2 (z^p f'(z))^"}{g'(z)}
\]

\[
+ \frac{z^2 (z^p f'(z))^"}{g'(z)} \left( \frac{z^2 (z^p f'(z))^"}{g'(z)} - 1 \right).
\]

Define the transforms from \( \mathbb{C}^3 \) to \( \mathbb{C} \) by

\[
u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t + s) - s^2}{r^2}.
\]

Let

\[
\psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi \left( \frac{s}{r}, \frac{r(t + s) - s^2}{r^2}; z, \zeta \right).
\]
The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.5) that

\begin{equation}
\psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right)
= \phi \left( -\frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)} - \frac{z g''(z)}{g'(z)}, \frac{z^2 (zp f'(z))'''}{g'(z)} + \frac{z (zp f'(z))''}{g'(z)} \right)
\times \left( 1 - \frac{z (zp f'(z))''}{g'(z)} - \frac{z^2 g''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \right) ; z, \zeta \right).
\end{equation}

Therefore, (2.1) becomes \( \psi(F(z), zF'(z), z^2F''(z); z, \zeta) \in \Omega \).

To complete the proof, we next show that the admissibility condition for \( \phi \in \Phi_\kappa[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.4. Note that

\[ \frac{t}{s} + 1 = \frac{w + v^2}{v}. \]

Hence \( \psi \in \Psi[\Omega, q] \). By Lemma 1.1, \( F(z) \prec q(z) \) or equivalently

\[ -\frac{(zp f'(z))'}{g'(z)} \prec q(z). \]

We consider the special situation when \( \Omega \neq \mathbb{C} \) is a simply connected domain. In this case \( \Omega = h(U) \), for some conformal mapping \( h \) of \( U \) onto \( \Omega \) and the class \( \Phi_\kappa[h(U), q] \) is written as \( \Phi_\kappa[h, q] \). The following result is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** Let \( \phi \in \Phi_\kappa[h, q] \). If \( f \in \Sigma_p \) satisfies

\begin{equation}
\phi \left( -\frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)} - \frac{z g''(z)}{g'(z)}, \frac{z^2 (zp f'(z))'''}{g'(z)} + \frac{z (zp f'(z))''}{g'(z)} \right)
\times \left( 1 - \frac{z (zp f'(z))''}{g'(z)} - \frac{z^2 g''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \right) ; z, \zeta \prec h(z),
\end{equation}

then

\[ -\frac{(zp f'(z))'}{g'(z)} \prec q(z). \]

By taking \( \phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}, \beta, \gamma \in \mathbb{C}, \) in Theorem 2.2, we state the following corollary.

**Corollary 2.1.** Let \( \beta, \gamma \in \mathbb{C} \) and let \( h \) be convex in \( U \) with \( h(0) = 1 \) and \( \text{Re} \{\beta h(z) + \gamma\} > 0 \). If \( f \in \Sigma_p \) satisfies

\[ -\frac{(zp f'(z))'}{g'(z)} + \frac{z (zp f'(z))''}{g'(z)} \prec h(z), \]

then

\[ -\frac{(zp f'(z))'}{g'(z)} \prec q(z). \]
The next result is an extension of Theorem 2.1 to the case where the behavior of \( q \) on \( \partial U \) is not known.

**Corollary 2.2.** Let \( \Omega \subset \mathbb{C} \) and \( q \) be univalent in \( U \) with \( q(0) = 1 \). Let \( \phi \in \Phi_{\mathcal{H}}[h, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If \( f \in \Sigma_\rho \) satisfies

\[
\phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z g''(z)}{g'(z)} \right) - \frac{z^2 (zp f'(z))''}{g'(z)} + z \frac{1}{(zp f'(z))''} \leq q(z),
\]

then

\[
\frac{z (zp f'(z))'}{g'(z)} < q(z).
\]

**Proof.** Theorem 2.1 yields \( -\frac{(zp f'(z))'}{g'(z)} < q_\rho(z) \). The result is now deduced from the fact that \( q_\rho(z) < q(z) \). \( \square \)

**Theorem 2.3.** Let \( h \) and \( q \) be univalent in \( U \) with \( q(0) = 1 \) and set \( q_\rho(z) = q(\rho z) \) and \( h_\rho(z) = h(\rho z) \). Let \( \phi : \mathbb{C} \times U \times \bar{U} \rightarrow \mathbb{C} \) satisfy one of the following conditions:

1. \( \phi \in \Phi_{\mathcal{H}}[h, q_\rho] \) for some \( \rho \in (0, 1) \);
2. there exists \( \rho_0 \in (0, 1) \) such that \( \phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho] \) for all \( \rho \in (\rho_0, 1) \).

If \( f \in \Sigma_\rho \) satisfies (2.7), then

\[
\frac{z (zp f'(z))'}{g'(z)} < q(z).
\]

**Proof.** (1) By applying Theorem 2.1, we obtain \( -\frac{(zp f'(z))'}{g'(z)} < q_\rho(z) \), since \( q_\rho(z) < q(z) \), we deduce

\[
\frac{z (zp f'(z))'}{g'(z)} < q(z).
\]

(2) Let \( F(z) = -\frac{(zp f'(z))'}{g'(z)} \) and \( F_\rho(z) = F(\rho z) \). Then

\[
\phi \left( F_\rho(z), z F'_\rho(z), z^2 F''_\rho(z); \rho z, \zeta \right) = \phi \left( F(\rho z), z F'(\rho z), z^2 F''(\rho z); \rho z, \zeta \right) \in h_\rho(U).
\]

By using Theorem 2.1 and the comment associated with

\[
\phi \left( F(z), z F'(z), z^2 F''(z); w(z), \zeta \right) \in \Omega,
\]

where \( w \) is any function mapping \( U \) into \( U \), with \( w(z) = \rho z \), we obtain \( F_\rho(z) < q_\rho(z) \) for \( \rho \in (\rho_0, 1) \). By letting \( \rho \rightarrow 1^- \), we get \( F(z) < q(z) \). Therefore,

\[
\frac{z (zp f'(z))'}{g'(z)} < q(z).
\]

\( \square \)

The next result gives the best dominant of the strong differential subordination (2.7).
Theorem 2.4. Let \( h \) be univalent in \( U \) and \( \phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C} \). Suppose that the differential equation
\[
(2.8) \quad \phi \left( q(z), \frac{zq'(z)}{q(z)}, \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left( \frac{zq'(z)}{q(z)} \right)^2 ; z, \zeta \right) = h(z)
\]
has a solution \( q \) with \( q(0) = 1 \) and satisfies one of the following conditions:

1. \( q \in Q_1 \) and \( \phi \in \Phi_\Omega[h, q] \);
2. \( q \) is univalent in \( U \) and \( \phi \in \Phi_\Omega[h, q_\rho] \) for some \( \rho \in (0, 1) \);
3. \( q \) is univalent in \( U \) and there exists \( \rho_0 \in (0, 1) \) such that \( \phi \in \Phi_\Omega[h_\rho, q_\rho] \) for all \( \rho \in (\rho_0, 1) \).

If \( f \in \Sigma_\rho \) satisfies (2.7), then
\[
-(z^p f'(z))' < q(z)
\]
and \( q \) is the best dominant.

Proof. By applying Theorem 2.2 and Theorem 2.3, we deduce that \( q \) is a dominant of (2.7). Since \( q \) satisfies (2.8), it is also a solution of (2.7) and therefore \( q \) will be dominated by all dominants. Hence, \( q \) is the best dominant of (2.7). \( \square \)

In the particular case \( q(z) = 1 + Mz, M > 0 \) and in view of Definition 2.1, the class of admissible functions \( \Phi_\Omega[\Omega, q] \) denoted by \( \Phi_\Omega[\Omega, M] \) can be expressed in the following form.

Definition 2.2. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( M > 0 \). The class of admissible function \( \Phi_\Omega[\Omega, M] \) consists of those functions \( \phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C} \) such that
\[
(2.9) \quad \phi \left( 1 + Me^{i\theta}, \frac{kM}{M + e^{i\theta}} + \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left( \frac{kM}{M + e^{-i\theta}} \right)^2 ; z, \zeta \right) \notin \Omega,
\]
whenever \( z \in U, \zeta \in \bar{U}, \theta \in \mathbb{R}, \text{Re} \{Le^{-i\theta}\} \geq k(k - 1)M \), for all \( \theta \) and \( k \geq 1 \).

Corollary 2.3. Let \( \phi \in \Phi_\Omega[\Omega, M] \). If \( f \in \Sigma_\rho \) satisfies
\[
\phi \left( \frac{(z^p f'(z))'}{g'(z)}, \frac{z (z^p f'(z))''}{g'(z)} + \frac{z^2 (z^p f'(z))'''}{g'(z)} - \frac{z g'''(z)}{g'(z)} \left( \frac{z g'''(z)}{g'(z)} - 1 \right) ; z, \zeta \right) \in \Omega,
\]
then
\[
\left| \frac{(z^p f'(z))'}{g'(z)} + 1 \right| < M.
\]

When \( \Omega = q(U) = \{w : |w - 1| < M\} \), the class \( \Phi_\Omega[\Omega, M] \) is simply denoted by \( \Phi_\Omega[M] \), then Corollary 2.3 takes the following form.
Corollary 2.4. Let $\phi \in \Phi_\Omega [M]$. If $f \in \Sigma_p$ satisfies
\[
\left| \phi \left( -\frac{(z^p f'(z))'}{g'(z)} - \frac{z g''(z)}{g'(z)} \right) \right| < M,
\]
then
\[
\frac{|(z^p f'(z))'|}{g'(z)} + 1 < M.
\]

Example 2.1. If $M > 0$ and $f \in \Sigma_p$ satisfies
\[
\left| \frac{z^2 (z^p f'(z))''' - \left( \frac{z g''(z)}{g'(z)} \right)^2}{(z^p f'(z))'} - \frac{z^2 g''(z)}{g'(z)} + \left( \frac{z g''(z)}{g'(z)} \right)^2 \right| < M,
\]
then
\[
\frac{|(z^p f'(z))'|}{g'(z)} + 1 < M.
\]

This implication follows from Corollary 2.4 by taking $\phi(u, v, w; z, \zeta) = w - v + 1$.

Example 2.2. If $M > 0$ and $f \in \Sigma_p$ satisfies
\[
\left| \frac{z (z^p f'(z))''}{(z^p f'(z))'} - \left( \frac{z g''(z)}{g'(z)} + 1 \right) \right| < \frac{M}{M+1},
\]
then
\[
\frac{|(z^p f'(z))'|}{g'(z)} + 1 < M.
\]

This implication follows from Corollary 2.3 by taking $\phi(u, v, w; z, \zeta) = v$ and $\Omega = h(U)$, where $h(z) = \frac{M}{z+1}, M > 0$. To apply Corollary 2.3, we need to show that $\phi \in \Phi_\Omega [\Omega, M]$, that is the admissibility condition (2.9) is satisfied follows from
\[
\left| \phi \left( 1 + Me^\theta, \frac{kM}{M+e^\theta}, \frac{kM + Le^{-\theta}}{M+e^{-\theta}} - \left( \frac{kM}{M+e^{-\theta}} \right)^2 ; z, \zeta \right) \right| = \frac{kM}{M+1} \geq \frac{M}{M+1},
\]
for $z \in U, \zeta \in \bar{U}, \theta \in \mathbb{R}$, and $k \geq 1$.

3. Strong Superordination Results

In this section, we obtain strong differential superordination. For this purpose the class of admissible functions given in the following definition will be required.

Definition 3.1. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}$. The class of admissible functions $\Phi_\Omega [\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; \xi, \zeta) \in \Omega$, whenever
\[
u = q(z), \quad v = \frac{zq'(z)}{mq(z)}, \quad q(z) \neq 0 \quad \text{and} \quad \text{Re} \left\{ \frac{w + v^2}{v} \right\} \leq \frac{1}{m} \text{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},
\]
where \( z \in U, \zeta \in \overline{U}, \zeta \in \partial U, \) and \( m \geq 1. \)

**Theorem 3.1.** Let \( \phi \in \Phi_{p}[\Omega, q]. \) If \( f \in \Sigma_{p}, -\frac{(zp f'(z))'}{g'(z)} \in Q_{1}, \) and

\[
\phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)} - \frac{z g''(z) (zp f'(z))''}{g'(z)}, \frac{z^{2} (zp f'(z))'''}{g'(z)} \right) \\
+ \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^{2} g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left( \frac{z g''(z)}{g'(z)} - 1 \right) ; z, \zeta \right) \]

is univalent in \( U, \) then

\[
(3.1)
\]

\[
\Omega \subset \left\{ \phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)} - \frac{z g''(z) (zp f'(z))''}{g'(z)}, \frac{z^{2} (zp f'(z))'''}{g'(z)} \right) + \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^{2} g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left( \frac{z g''(z)}{g'(z)} - 1 \right) ; z, \zeta \right) : z \in U, \zeta \in \overline{U} \}
\]

implies

\[
q(z) \prec -\frac{(zp f'(z))'}{g'(z)}.
\]

**Proof.** Let \( F \) defined by (2.2) and \( \psi (F(z), zF'(z), z^{2}F''(z); z, \zeta) \) defined by (2.6). Since \( \phi \in \Phi_{p}[\Omega, q], \) from (2.6) and (3.1), we have

\[
\Omega \subset \{ \psi (F(z), zF'(z), z^{2}F''(z); z, \zeta) : z \in U, \zeta \in \overline{U} \}.
\]

From (2.5), we see that the admissibility condition for \( \phi \in \Phi_{p}[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.5. Hence \( \psi \in \Psi'[\Omega, q] \) and by Lemma 1.2, \( q(z) \prec F(z) \) or equivalently

\[
q(z) \prec -\frac{(zp f'(z))'}{g'(z)}.
\]

We consider the special situation when \( \Omega \neq \mathbb{C} \) is a simply connected domain. In this case \( \Omega = h(U) \), for some conformal mapping \( h \) of \( U \) onto \( \Omega \) and the class \( \Phi_{p}[h(U), q] \) is written as \( \Phi_{p}[h, q]. \) The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.2.** Let \( \phi \in \Phi_{p}[h, q], q \in \mathcal{H}, \) and \( h \) be analytic in \( U. \) If \( f \in \Sigma_{p}, -\frac{(zp f'(z))'}{g'(z)} \in Q_{1}, \) and

\[
\phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)} - \frac{z g''(z) (zp f'(z))''}{g'(z)}, \frac{z^{2} (zp f'(z))'''}{g'(z)} \right) \\
+ \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^{2} g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left( \frac{z g''(z)}{g'(z)} - 1 \right) ; z, \zeta \right) \]

is univalent in \( U, \) then

\[
(3.2)
\]

\[
\Omega \subset \left\{ \phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)} - \frac{z g''(z) (zp f'(z))''}{g'(z)}, \frac{z^{2} (zp f'(z))'''}{g'(z)} \right) + \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^{2} g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left( \frac{z g''(z)}{g'(z)} - 1 \right) ; z, \zeta \right) : z \in U, \zeta \in \overline{U} \}
\]

implies

\[
q(z) \prec -\frac{(zp f'(z))'}{g'(z)}.
\]
is univalent in $U$, then
\begin{equation}
(3.2)
\end{equation}
\begin{align*}
&h(z) \prec \phi \left( -\frac{(z^p f'(z))^\prime}{g'(z)}, \frac{z (z^p f'(z))''}{g'(z)} - \frac{z g''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))''}{g'(z)} \right) \\
&\quad + \frac{z (z^p f'(z))^n}{(z^p f'(z))'} \left( 1 - \frac{z (z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left( \frac{z g''(z)}{g'(z)} - 1 \right) ; z, \zeta
\end{align*}
implies
\[ q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}. \]

By taking $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 3.2, we state the following corollary.

**Corollary 3.1.** Let $\beta, \gamma \in \mathbb{C}$ and let $h$ be convex in $U$ with $h(0) = 1$. Suppose that the differential equation $q(z) + \frac{z q'(z)}{\beta q(z) + \gamma} = h(z)$ has a univalent solution $q$ that satisfies $q(0) = 1$ and $q(z) \prec h(z)$. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{K} \cap Q_1$, and
\[ -\frac{(z^p f'(z))'}{g'(z)} + \frac{z (z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z) \gamma g'(z) - \beta (z^p f'(z))' \]
is univalent in $U$, then
\[ h(z) \prec -\frac{(z^p f'(z))'}{g'(z)} + \frac{z (z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z) \gamma g'(z) - \beta (z^p f'(z))' \]
implies
\[ q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}. \]

The next result gives the best subordinant of the strong differential superordination (3.2).

**Theorem 3.3.** Let $h$ be analytic in $U$ and $\phi : \mathbb{C}^3 \times U \times \mathbb{C} \to \mathbb{C}$. Suppose that the differential equation
\[ \phi \left( q(z), \frac{z q'(z)}{q(z)}, \frac{z^2 q''(z)}{q(z)}, \frac{z q'(z)}{q(z)} - \left( \frac{z q'(z)}{q(z)} \right)^2 ; z, \zeta \right) = h(z) \]
has a solution $q \in Q_1$. If $\phi \in \Phi_{h, \mathcal{K}}[h, q]$, $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$, and
\begin{align*}
&\phi \left( -\frac{(z^p f'(z))'}{g'(z)}, \frac{z (z^p f'(z))''}{g'(z)} - \frac{z g''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))''}{g'(z)} \right) \\
&\quad + \frac{z (z^p f'(z))''}{(z^p f'(z))'} \left( 1 - \frac{z (z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left( \frac{z g''(z)}{g'(z)} - 1 \right) ; z, \zeta
\end{align*}
implies
\[ q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}. \]
is univalent in $U$, then
\[ h(z) \prec \phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)}, \frac{z^2 (zp f'(z))'''}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)}, \frac{z^2 (zp f'(z))'''}{g'(z)} \right) \]
\[ + \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + z g''(z) \left( \frac{z g''(z)}{g'(z)} - 1 \right); z, \zeta \]
implies
\[ q(z) \prec -\frac{(zp f'(z))'}{g'(z)}. \]
and $q$ is the best subordinant.

**Proof.** The proof is similar to that of Theorem 2.4 and is omitted. \qed

## 4. Sandwich Results

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich theorem.

**Theorem 4.1.** Let $h_1$ and $q_1$ be analytic functions in $U$, $h_2$ be univalent in $U$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi[\Phi[h_2, q_2] \cap \Phi[h_1, q_1]$. If $f \in \Sigma_p$, \( \frac{(zp f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1 \) and
\[ \phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)}, \frac{z (zp f'(z))'''}{g'(z)} \right) \]
\[ + \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + z g''(z) \left( \frac{z g''(z)}{g'(z)} - 1 \right); z, \zeta \]
is univalent in $U$, then
\[ h_1(z) \prec \phi \left( \frac{(zp f'(z))'}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)}, \frac{z (zp f'(z))'''}{g'(z)}, \frac{z (zp f'(z))''}{g'(z)}, \frac{z^2 (zp f'(z))'''}{g'(z)} \right) \]
\[ + \frac{z (zp f'(z))''}{(zp f'(z))'} \left( 1 - \frac{z (zp f'(z))''}{(zp f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + z g''(z) \left( \frac{z g''(z)}{g'(z)} - 1 \right); z, \zeta \]
\[ \prec \prec h_2(z) \]
implies
\[ q_1(z) \prec -\frac{(zp f'(z))'}{g'(z)} \prec q_2(z). \]

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich corollary.

**Corollary 4.1.** Let $\beta, \gamma \in \mathbb{C}$ and let $h_1, h_2$ be convex in $U$ with $h_1(0) = h_2(0) = 1$. Suppose that the differential equations $q_1(z) + \frac{z q_1'(z)}{\beta q_1(z) + \gamma} = h_1(z)$, $q_2(z) + \frac{z q_2'(z)}{\beta q_2(z) + \gamma} = h_2(z)$
have a univalent solutions $q_1$ and $q_2$, respectively, that satisfy $q_1(0) = q_2(0) = 1$ and $q_1(z) \prec h_1(z), q_2(z) \prec h_2(z)$. If $f \in \Sigma_p$, 
\[-\frac{(zp f'(z))'}{g'(z)} + \frac{z(p f'(z))''}{g'(z)}g'(z) - zg''(z) \prec h_1(z) \prec h_2(z)\]
is univalent in $U$, then
\[h_1(z) \prec \prec -\frac{(zp f'(z))'}{g'(z)} + \frac{z(p f'(z))''}{g'(z)}g'(z) - zg''(z) \prec h_2(z)\]
implies
\[q_1(z) \prec -\frac{(zp f'(z))'}{g'(z)} \prec q_2(z).\]

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