Analytical solution to the problem of diffusion of the light component of the binary gas mixture in a plane channel

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Abstract. Within the framework of the kinetic approach, an analytical solution to the problem of diffusion of the light component of a binary mixture in a flat channel with infinite parallel walls is constructed. It is assumed that the mass of light component molecules and their concentration is much less than the mass of molecules and the concentration of heavy components. The flow rate of the heavy component is assumed to be zero. The change in the state of a light gas component is described on the basis of the BGK (Bhatnagar, Gross, Kruk) model of the Boltzmann kinetic equation. The diffuse reflection model is used as a boundary condition on the channel walls. The mass velocity profile of the light gas component is constructed. The flow rate of the light gas component per unit channel width is calculated. A comparison with similar results presented in open sources was done.

1. Introduction
In boundary layer theory [1] often there is a case of binary gas, the mass of a molecule of one of the components which is much less than the mass of the molecule of the other component. If the partial density of the light component is significantly less than the heavy one, the motion of the heavy component can be neglected, considering it as stationary. For the half-space case (i.e. the case when a binary mixture fills a half-space bounded by a solid flat surface lying in the plane of a rectangular Cartesian coordinate system), an analytical solution to this problem is obtained in [2]. No numerical analysis of the final expressions describing the diffusion process was performed in [2]. Using the method of discrete ordinates solution of the problem of flow of a binary mixture in the channel, the thickness of which is comparable to the average free path length of gas molecules, constructed in [3]. In [4] the flow of binary mixtures of neon-argon, helium-xenon in the presence of pressure, temperature and concentration gradients was considered. In the first case, the mass of the molecules of both components are approximately equal, in the second – the mass of the molecule of one component is much greater than the mass of the molecule of the other. In the article [3] the problem of the Couette for the mixture of helium-argon in the channel, the walls of which are made of different materials. A study of the gas mixture flow in ultrathin capillaries, the size of which is comparable to the radius of action of surface forces, was carried out in [5]. The aim of the present work was to construct an
analytical solution to the problem of diffusion of a light component of a binary mixture relatively heavier in a channel with infinite parallel walls.

Currently, there are two main methods to build exact analytical solutions to boundary value problems of the kinetic theory of gas and plasma using linearized model kinetic equations. This is the Case method (the method of decomposition of the solution by eigenvular generalized functions of the characteristic equation corresponding to the given kinetic equation) and the Wiener-Hopf method (the method of reducing the kinetic equation to unilateral equation of the convolution type with an integral taken along the real positive half-axis). The main advantage of the Case method is that it allows to obtain in an explicit form the distribution function of gas molecules by coordinates and velocities, and, as a consequence, to calculate any macroparameters characterizing the state of the gas. In this paper, a generalization of the Case method to the case of problems with bounded geometry presented in [6] is used as the main method. The general solution of the inhomogeneous integro differential equation, to which the model kinetic equation is reduced, is constructed in the space of generalized functions. The substitution of boundary conditions in the general solution leads to a system of two coupled singular integral equations with a Cauchy-type kernel, which after the transformation are reduced to the Riemann boundary value problem on a real positive half-axis. The use of Sohotsky’s formulas for finding coefficients in the decomposition of the solution of the problem by eigenvectors of the continuous spectrum leads to the Fredholm integral equation of the second kind. The solution of the latter is sought in the form of Neumann series.

2. Materials and methods
Let us consider the diffusion of the light component of a binary gas mixture with respect to heavy in the channel with its walls located in planes $x' = \pm d$ in the rectangular Cartesian coordinate system. Let us suppose that the mixture consists of gas $a$ and $b$ (respectively light and heavy component) with numerical densities (concentrations) $n_a$ and $n_b$ and mass densities $\rho_a = m_an_a$ and $\rho_b = m_bn_b$. Here $m_a$ and $m_b$ are masses of gas molecules $a$ and $b$. We assume that a temperature is constant equal to $T$. We assume that the $Oy'$ axis is directed along the concentration gradient of light component. If the partial density of the light component is substantially less than heavy, then we can neglected by movement of the heavy component and considering it as motionless. Under these assumptions the BGK-equation for the light component of gas mixture is written as:

$$v_x \frac{\partial f}{\partial x'} + v_y \frac{\partial f}{\partial y'} = \nu_a (f_{eq} - f) + \nu_{ab} (f_0 - f), \quad (1)$$

Here $f_{eq} = n(y') (\beta/\pi)^{3/2} \exp[-\beta(v - u(x'))^2]$ – locally equilibrium distribution function, $u(x')$ – the mass average velocity of the lighter component, $f_0 = n(y') (\beta/\pi)^{3/2} \exp(-\beta u^2)$, $\nu_a$ – collisions frequency of light molecules among themselves, $\nu_{ab}$ – collisions frequency of light molecules with heavy

$$\nu_{ab} = \nu_a \left( \frac{d_{ab}}{d_a} \right)^2 \sqrt{\frac{m_a + m_b}{m_b}} \frac{n_b}{n_a}, \quad \nu_a = \frac{\rho_a}{\mu_a} \Omega(\omega_a, \alpha_a),$$

$$d_{ab} = \frac{d_a + d_b}{2} \Omega(\omega_a, \alpha_a) = \frac{5(\alpha_a + 1)(\alpha_a + 2)}{\alpha_a(7 - \omega_a)(5 - \omega_a)}.$$
(VHS). In this case

$$\Omega(\omega_a, 1) = \Omega(\omega_a) = \frac{30}{(7 - \omega_a)(5 - \omega_a)}.$$ 

If $\beta u^2(x') << 1$, we can linearize the distribution function and write it as:

$$f = f_0(y', v)(1 + h(x', \mathbf{v})).$$

Then

$$\frac{\partial f}{\partial x'} = f_0(y', v) \frac{\partial h}{\partial x'},$$

$$\frac{\partial f}{\partial y'} = \frac{dn}{dy'}(\beta/\pi)^{3/2} \exp(-\beta v'^2) = \frac{1}{n} \frac{dn}{dy'} f_0(y', v) = f_0(y', v) \frac{d\ln n}{dy'},$$

$$f_{eq} = n(y') (\beta/\pi)^{3/2} \exp[-\beta v] (1 + 2 \beta v_y u_y(x')) = f_0(y', v) (1 + 2 \beta v_y u_y(x')).$$

Substituting the obtained results in (1), we arrive at the equation

$$v_x f_0(y', v) \frac{\partial h}{\partial x'} + v_y f_0(y', v) \frac{d\ln n}{dy'} =$$

$$= \nu_a f_0(y', v) (1 + 2 \beta v_y u_y(x') - 1 - h(x', \mathbf{v})) + \nu_{ab} f_0(y', v) (1 - 1 - h(x', \mathbf{v})),$$

or

$$v_x \frac{\partial h}{\partial x'} + v_y \frac{d\ln n}{dy'} = 2 \nu_a \beta v_y u_y(x') - \nu h(x', \mathbf{v}),$$

or

$$\nu = \nu_a + \nu_{ab}.$$ 

Let us introduce the notation: $C_x = \beta^{-1/2} v_x, C_y = \beta^{-1/2} v_y, U(x) = \beta^{-1/2} u_y(x), x = x' \beta^{1/2} \nu, y = y' \beta^{1/2} \nu, G_n = d\ln n/dy, c = \nu_a/\nu$. In the accepted notations, we rewrite the equation (3) as

$$C_x \frac{\partial h}{\partial x} + C_y G_n = 2c C_y U(x) - h(x, \mathbf{v}).$$

(4)

The solution of (4) is sought in the form

$$h(x, \mathbf{C}) = \frac{C_y G_n}{1 - c} [Z(x, C_x) - 1].$$

(5)

Substituting (5) in (4) and considering that from the statistical the meaning of the distribution function

$$U(x) = \pi^{-3/2} \int \exp(-C^2) C_y h(x, \mathbf{C}) d^3C = \frac{G_n}{2(1 - c)} \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\tau^2) Z(x, \tau) d\tau - 1 \right],$$

we arrive to the equation (\mu = C_x)

$$\frac{\mu}{1 - c} \frac{\partial Z}{\partial x} + 1 = \frac{c}{1 - c} \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\tau^2) Z(x, \tau) d\tau - 1 \right] - \frac{1}{1 - c} [Z(x, \mu) - 1],$$

or, after reduction of similar terms to the homogeneous equation $c \in (0, 1)$

$$\mu \frac{\partial Z}{\partial x} + Z(x, \mu) = \frac{c}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\tau^2) Z(x, \tau) d\tau. ($$
As a boundary condition on the walls of the channel, we use diffuse reflection model. In this case, the boundary conditions for function $Z(x, \mu)$ on the channel walls taking into account (2) and (5), we written in the form

$$Z(\pm d, \mp \mu) = 1, \quad \mu > 0. \tag{8}$$

Thus, the solution of the considered problem is reduced to the solution equations (7) with boundary conditions (8).

We seek a solution of (7) as a decomposition of the spectral parameter $\eta$

$$Z_\eta(x, \mu) = \exp\left(-\frac{x}{\eta}\right) F(\eta, \mu). \tag{9}$$

Substituting (9) into (7) and introducing the normalization integral

$$\int_{-\infty}^{+\infty} \exp(-\tau^2) F(\eta, \tau) d\tau = 1, \tag{10}$$

we arrive at the characteristic equation

$$(\eta - \mu) F(\eta, \tau) = \frac{1}{\sqrt{\pi}} c\eta. \tag{11}$$

Solution of (11) in the space of generalized functions has the form

$$F(\eta, \mu) = \frac{1}{\sqrt{\pi}} c\eta \text{P} \frac{1}{\eta - \mu} + g(\eta) \delta(\eta - \mu). \tag{12}$$

Here $\text{P}(1/z)$ and $\delta(z)$ are the distribution in the sense of a principal value of the integral of $1/z$ and the Dirac delta function, respectively.

Substituting (12) for the normalization integral (10), we find an explicit form of the function $g(\eta)$.

$$\frac{1}{\sqrt{\pi}} c\eta \int_{-\infty}^{+\infty} \frac{\exp(-\tau^2)}{\tau - \mu} d\tau + g(\eta) \exp(-\eta^2) = 1.$$ 

Thus,

$$g(\eta) = \exp(\eta^2) \lambda(\eta),$$

where

$$\lambda(z) = 1 + \frac{1}{\sqrt{\pi}} cz \int_{-\infty}^{\infty} \frac{\exp(-\mu^2)}{\mu - z} d\mu \tag{13}$$

is the dispersion function.

On the real positive half-axis

$$\lambda(\mu) = 1 - 2c\mu \exp(-\mu^2) \int_{0}^{\mu} \exp(u^2) du. \tag{14}$$

In the complex plane the dispersion function is calculated by the formula

$$\lambda(z) = 1 - 2cz \exp(-z^2) \int_{0}^{z} \exp(u^2) du \pm ic\sqrt{\pi}z \exp(-z^2), \quad \pm Im(z) > 0.$$
Substituting (13) into (12), we find the distinct form of the function $F(\eta, \mu)$

$$F(\eta, \mu) = \frac{1}{\sqrt{\pi}} c \eta P \frac{1}{\eta - \mu} + \exp(\eta^2)\lambda(\eta) \delta(\eta - \mu).$$

(15)

Let us estimate the behavior of $\lambda(z)$ in the vicinity of the infinitely remote points. Decomposing (13) into asymptotic series, we obtain

$$\lambda(z) = 1 - \frac{1}{\sqrt{\pi}} c \int_{-\infty}^{\infty} \frac{\exp(-\mu^2)}{1 - \mu/z} d\mu = 1 - c \left[ 1 + \left( \frac{1}{2z^2} + \frac{3}{4z^4} + \ldots \right) \right], \quad |z| \to \infty.$$ 

(16)

From the decomposition (16) it is seen that $\lambda(\infty) = 1 - c > 0$. Imaginary part of the function $\lambda^+(\mu)$ for all $\mu > 0$ is positive, and for all $\mu < 0$ – is negative. The real part of this function generally has four roots $\pm \tau_0(c)$ and $\pm \tau_1(c)$. Denote by $\theta(\mu)$ the main value of the function argument $\lambda^+(\mu)$, fixed at zero by the condition $\theta(0) = 0$. The value of $\theta(\mu)$ is defined by the equality

$$\theta(\mu) = \frac{\pi}{2} - \arctg \frac{\lambda(\mu)}{\sqrt{\pi c \mu \exp(-\mu^2)}}.$$ 

(17)

The increment of the main value $\theta(\mu) = \arg\lambda^+(\mu)$ is zero, both on the positive half-axis and the entire axis. Hence, as follows from the argument principle, the function $\lambda(z)$ has no zeros in the complex plane with a cut along the real numerical line. Thus, the discrete spectrum in the considered problem is an empty set.

Thus, we seek a solution of (7) in the form

$$Z(x, \mu) = \int_{-\infty}^{+\infty} \exp\left(-\frac{x}{\eta}\right) F(\eta, \mu)a(\eta) \, d\eta.$$ 

(18)

Here $a(\eta)$ is an unknown function to be further defined. Substituting the decomposition (18) into boundary conditions (8) and taking into account (15) and (13), we arrive at a system of coupled integral equations

$$\frac{1}{\sqrt{\pi}} c \int_{-\infty}^{+\infty} \frac{\eta b(\eta, d)}{\eta - \mu} \, d\eta + \exp(\mu^2)b(\mu, d)\lambda(\mu) = 1, \quad \mu < 0,$$

(19)

$$\frac{1}{\sqrt{\pi}} c \int_{-\infty}^{+\infty} \frac{\eta b(\eta, -d)}{\eta - \mu} \, d\eta + \exp(\mu^2)b(\mu, -d)\lambda(\mu) = 1, \quad \mu > 0.$$ 

(20)

Here we have introduced the notation

$$b(\eta, x) = \exp\left(-\frac{x}{\eta}\right) a(\eta).$$ 

(21)

Replace in (19) $\mu$ with $-\mu$ and present the obtained integral as the sum of two

$$\frac{1}{\sqrt{\pi}} c \int_{-\infty}^{0} \frac{\eta b(\eta, d)}{\eta + \mu} \, d\eta + \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(\eta, d)}{\eta + \mu} \, d\eta + \exp(\mu^2)b(-\mu, d)\lambda(\mu) = 1, \quad \mu > 0.$$ 

(22)
By writing (22) we took into account that the function $\lambda(\mu)$ is even on a real numeric line. The first integral in the obtained equation is singular, and the second is regular. Replace the integration variable $\eta$ with $-\eta$ in the first integral and rewrite it as

$$
\int_{-\infty}^{0} \frac{\eta b(\eta, d)}{\eta + \mu} d\eta = \int_{+\infty}^{0} \frac{-\eta b(-\eta, d)}{-\eta + \mu} d\eta = \int_{0}^{+\infty} \frac{\eta b(-\eta, d)}{\eta - \mu} d\eta.
$$

Taking into account the performed transformations, the equation (19) takes the form

$$
\frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(-\eta, d)}{\eta - \mu} d\eta + \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(\eta, d)}{\eta + \mu} d\eta + \exp(\mu^2) b(-\mu, d)\lambda(\mu) = 1, \quad \mu > 0. \quad (23)
$$

Similarly, we transform the integral in (19). In the end, the equation (19) can be written in the form

$$
\frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(\eta, -d)}{\eta - \mu} d\eta + \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(-\eta, -d)}{\eta + \mu} d\eta + \exp(\mu^2) b(\mu, -d)\lambda(\mu) = 1, \quad \mu > 0. \quad (24)
$$

Let us transform the resulting equations. Folding (23) and (24), taking into account (21) we come to the equation

$$
\frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta [a(\eta) + a(-\eta)]}{\eta - \mu} \exp(d/\eta) d\eta + \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta [a(\eta) + a(-\eta)]}{\eta + \mu} \exp(-d/\eta) d\eta + \exp(\mu^2) [a(\eta) + a(-\eta)] \exp(d/\eta)\lambda(\mu) = 2, \quad \mu > 0. \quad (25)
$$

Similarly, subtracting (23) from (24), we obtain

$$
\frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta [a(\eta) - a(-\eta)]}{\eta - \mu} \exp(d/\eta) d\eta + \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta [a(\eta) - a(-\eta)]}{\eta + \mu} \exp(-d/\eta) d\eta + \exp(\mu^2) [a(\eta) - a(-\eta)] \exp(d/\eta)\lambda(\mu) = 0, \quad \mu > 0. \quad (26)
$$

It is easy to see that (26) for $a(\eta) = a(-\eta)$ refers to the identity. In this case (25) can be rewritten as

$$
\frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(\eta, -d)}{\eta - \mu} d\eta + \exp(\mu^2) b(\mu, -d)\lambda(\mu) = f(\mu), \quad \mu > 0, \quad (27)
$$

where

$$
f(\mu) = 1 - \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(\eta, d)}{\eta + \mu} d\eta. \quad (28)
$$

We seek a solution of (27) using the methods of boundary value problems of the theory of functions of complex variable. For this purpose, we introduce an auxiliary function given by the Cauchy type integral

$$
N(z) = \frac{1}{\sqrt{\pi}} c \int_{0}^{+\infty} \frac{\eta b(\eta, -d)}{\eta - z} d\eta. \quad (29)
$$
On the real positive numerical axis, the function \( N(z) \) has a cut. On the upper and lower banks of the cut are fair equalities

\[
N^+(\mu) - N^-(\mu) = 2\sqrt{\pi}icb(\mu, -d), \quad 0 < \mu < +\infty,
\]

\[
N^+(\mu) + N^-(\mu) = \frac{2}{\sqrt{\pi}}c \int_0^{+\infty} \frac{b(\eta, -d)}{\eta - \mu} d\eta,
\]

Similar relations for \( \lambda(z) \), defined by equality (13), have the form

\[
\lambda^+(\mu) - \lambda^-(\mu) = 2\sqrt{\pi}ic \exp(-\mu^2), \quad -\infty < \mu < +\infty,
\]

\[
\lambda^+(\mu) + \lambda^-(\mu) = 2\lambda(\mu).
\]

Here, the cut coincides with the entire real number line. Taking into account (30)-(33), we reduce the integral equation (27) to the Riemann boundary value problem on the real positive half-axis

\[
N^+(\mu) \lambda^+(\mu) - N^-(\mu) \lambda^-(\mu) = 2\sqrt{\pi}icf(\mu) \exp(-\mu^2), \quad \mu > 0,
\]

The functions \( N(z) \) and \( \lambda(z) \) have different cuts. To eliminate this feature it is necessary to solve the problem of factorization, that is, to construct a function (let’s call it \( X(z) \)), which has the following properties:

– the function does not vanish at any end point of the complex plane;
– on the real positive axis for it is true equality

\[
\frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \quad \mu > 0,
\]

– function \( X(z) \) is analytic at all other points of the complex plane.

Taking into account the symmetry of the graph of the function \( \lambda(z) \) with respect to the real axis, we can write

\[
\frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda^+(\mu) \exp(i\theta(\mu))}{\lambda^+(\mu) \exp(-i\theta(\mu))} = \exp(2i\theta(\mu)).
\]

Let us logarithm (36)

\[
\ln X^+(\mu) - \ln X^-(\mu) = 2i(\theta(\mu) + \pi k), \quad \mu > 0, \quad 0, \pm 1, \pm 2, \ldots
\]

Thus, we obtain a problem of defining a piecewise analytic function from a given jump. We write its solution using the Sokhotski formula.

\[
\ln X(z) = \frac{1}{\pi} \int_0^{+\infty} \frac{[\theta(\tau) + \pi k] d\tau}{\tau - z}.
\]

The changing of the angle \( \theta(\mu) \) on the real half-axis is zero. Hence, the integral in the right part (38) converges only at \( k = 0 \). Thus, the factorizing function is constructed and has the form

\[
X(z) = \exp[V(z)], \quad V(z) = \frac{1}{\pi} \int_0^{+\infty} \frac{\theta(\tau) d\tau}{\tau - z},
\]

Note that

\[
V(z) = -\frac{\theta(0)}{\pi} \ln z + O_1(z) = O_1(z), \quad z \to 0.
\]
Thus, the function $X(z)$ defined by the equality (39) does not vanish in any end point of the complex plane, including the origin. In order to estimate the behavior of $X(z)$ in the vicinity of an infinitely remote point, we use the decomposition of the function $V(z)$

$$V(z) = -\frac{1}{2\pi} \int_0^\infty \frac{\theta(\tau)}{1 - \tau/z} d\tau = -\frac{1}{2\pi} \int_0^\infty \theta(\tau) \left[ \frac{1}{z} + \frac{\tau}{z^2} + \ldots \right] d\tau = O\left(\frac{1}{z}\right), \quad |z| \to \infty.$$ 

Thus,

$$X(\infty) = 1.$$ 

(40)

Taking into account the solved factorization problem, the boundary value problem (34) is written as

$$N^+\mu X^+\mu - N^-\mu X^-\mu = \frac{X^-\mu}{\lambda^-\mu} 2\sqrt{\pi}ic\mu f(\mu) \exp(-\mu^2), \quad \mu > 0.$$ 

(41)

The jump lines of $N(z)$ and $X(z)$ coincide with the contour of the boundary condition. Therefore, we have reduced singular integral equation (27) to the Riemann boundary value problem. We write its solution using the Sokhotski formula

$$N(z) = \frac{1}{X(z)} c \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{X^{-}\eta}{\lambda^{-}\eta} \eta f(\eta) \exp(-\eta^2) \frac{d\eta}{\eta - z},$$

or, taking into account (28)

$$N(z) = \frac{1}{X(z)} c \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{X^{-}\eta}{\lambda^{-}\eta} \eta \left[ 1 - \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\mu b(\mu, d)}{\mu + \eta} d\mu \right] \exp(-\eta^2) \frac{d\eta}{\eta - z}. \quad (42)$$

Convert the resulting expression. To do this, we use the integral representation of the function $X(z)$

$$X(z) - X(\infty) = \frac{1}{2\pi i} \int_0^{+\infty} \frac{X^+(u) - X^-(u)}{u - z} du.$$ 

Take into account that

$$X^+(u) - X^-(u) = X^-\left[ \frac{X^+(u)}{X^-\left( u \right)} - 1 \right] = X^-\left( u \right) \left[ \frac{\lambda^+(u)}{\lambda^-\left( u \right)} - 1 \right] = \frac{X^-\left( u \right)}{\lambda^-\left( u \right)} 2\sqrt{\pi}icu \exp(-u^2).$$ 

Thus, the integral representation of $X(z)$ has the form

$$X(z) = 1 + \frac{1}{\sqrt{\pi}} c \int_0^{+\infty} \frac{X^-\left( u \right) u \exp(-u^2)}{\lambda^-\left( u \right)} \frac{du}{u - z}. \quad (43)$$

Taking into account the constructed integral representation of the function $X(z)$, we can write

$$\frac{1}{X(z)} c \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{X^-\left( \eta \right) \exp(-\eta^2) \eta}{\lambda^-\left( \eta \right) \eta - z} = \frac{1}{X(z)} \left[ X(z) - 1 \right] = 1 - \frac{1}{X(z)}. \quad (44)$$
To convert the second term to (42), we will take advantage of the obvious equality
\[
\frac{1}{\eta - z} \cdot \frac{1}{\mu + \eta} = \frac{1}{\eta - z} - \frac{1}{\eta + \mu}.
\]

Then
\[
\frac{1}{\sqrt{\pi}} c \int_0^{+\infty} X^{-}(\eta) \frac{d\eta}{\lambda^{-}(\eta)} \exp(-\eta^2) \frac{d\eta}{\eta - z} \cdot \frac{1}{\sqrt{\pi}} c \int_0^{+\infty} \frac{\mu b(\mu, d) d\mu}{\mu + \eta} = 0.
\]

Thus, taking into account the obtained results, we rewrite the expression for \( N(z) \) as
\[
N(z) = 1 - \frac{1}{\sqrt{\pi}} c \int_0^{+\infty} \frac{\mu b(\mu, d) d\mu}{\mu + z} - \frac{1}{\sqrt{\pi}} c \int_0^{+\infty} \frac{\mu X(-\mu) b(\mu, d) d\mu}{\mu + \eta}.
\]  
(45)

For the constructed function \( N(z) \) according to Sokhotsky’s formulas we can write
\[
N^+(\eta) - N^-(\eta) = - \left[ \frac{1}{X^+(\eta)} - \frac{1}{X^{-}(\eta)} \right] \left[ 1 - \frac{1}{\sqrt{\pi}} c \int_0^{+\infty} \frac{\mu X(-\mu) b(\mu, d) d\mu}{\mu + \eta} \right].
\]  
(46)

Note that
\[
\frac{1}{X^+(\eta)} - \frac{1}{X^{-}(\eta)} = \frac{1}{X^{-}(\eta)} \left[ \frac{X^{-}(\eta)}{X^+(\eta)} - 1 \right] = \frac{1}{X^{-}(\eta)} \left[ \frac{\lambda^{-}(\eta)}{\lambda^+(\eta)} - 1 \right] = \frac{\lambda^{-}(\eta) - \lambda^+(\eta)}{X^{-}(\eta) \lambda^+(\eta)} = \frac{2\sqrt{\pi}e\eta \exp(-\eta^2)}{X^{-}(\eta) \lambda^+(\eta)} = - \frac{2(1-c)\sqrt{\pi}e\eta \exp(-\eta^2)}{|\lambda^+(\eta)|^2}.
\]

When writing the last equality, we used the formula of factorization of the dispersion function
\[
X^{-}(\eta) X^{-}(\eta) = \frac{\lambda^{-}(\eta)}{\lambda(\infty)} = \frac{\lambda^{-}(\eta)}{1-c}.
\]

Comparing (46) with (30) and taking into account (21), we find the coefficients in the decomposition of the solution of the boundary value problem (7), (8) on eigenvectors of the continuous spectrum
\[
a(\eta) = \frac{(1-c)X(-\eta)\exp(-\eta^2 - d/\eta)}{|\lambda^+(\eta)|^2} \times \left[ 1 - \frac{1}{\sqrt{\pi}} c \int_0^{+\infty} \frac{\mu X(-\mu) a(\mu) \exp(-d/\mu) d\mu}{\mu + \eta} \right].
\]

Let us denote
\[
h(\eta) = \frac{(1-c)X(-\eta)\exp(-\eta^2 - d/\eta)}{|\lambda^+(\eta)|^2},
\]  
(47)

and consider the equation
\[
a(\eta) = h(\eta) \left[ 1 + \lambda \int_0^{+\infty} \frac{\mu X(-\mu) a(\mu) \exp(-d/\mu) d\mu}{\mu + \eta} \right],
\]  
(48)
which at $\lambda = -c/\sqrt{\pi}$ passes into the equation (47).

We seek a solution of (48) in the form

$$a(\eta) = \sum_{k=0}^{+\infty} \lambda^k a_k(\eta).$$  \hspace{1cm} (49)

Substituting (49) into (48) and equating the coefficients at of equal degrees $\lambda$, we obtain a system of recursive relations for finding $a_k(\eta) \quad (\eta > 0)$

$$a_0(\eta) = h(\eta),$$  \hspace{1cm} (50)

$$a_1(\eta) = h(\eta) \int_{0}^{+\infty} \frac{g(\mu_1)}{\mu_1 + \eta} \, d\mu_1,$$  \hspace{1cm} (51)

$$a_2(\eta) = h(\eta) \int_{0}^{+\infty} \frac{g(\mu_1)}{\mu_1 + \eta} \int_{0}^{+\infty} \frac{g(\mu_2)}{\mu_2 + \mu_1} \, d\mu_2,$$  \hspace{1cm} (52)

$$\ldots$$

$$a_k(\eta) = h(\eta) \prod_{i=1}^{k} \int_{0}^{+\infty} \frac{g(\mu_i)}{\mu_i + \eta} \, d\mu_i,$$  \hspace{1cm} (53)

$$g(\mu) = \mu X(-\mu) \exp(-d/\mu) h(\mu) = \frac{(1 - c)\mu X^2(-\mu) \exp(-\mu^2 - 2d/\mu)}{|\lambda^+(\mu)|^2}. $$  \hspace{1cm} (54)

Finding the explicit form of the $a_k(\eta)$ completes the construction of the distribution function of the molecules of the light component of the binary gas. Substituting (18) into an expression for mass velocity (6) and taking into account the normalization integral (10) we find

$$U(x) = \frac{G_n}{2(1-c)} \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\mu^2)Z(x, \mu) \, d\mu - 1 \right] =$$

$$= \frac{G_n}{2(1-c)} \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp \left( -\frac{x}{\eta} \right) a(\eta) \, d\eta - 1 \right].$$ \hspace{1cm} (56)

Transform the integral in the expression (56)

$$\int_{-\infty}^{+\infty} \exp \left( -\frac{x}{\eta} \right) a(\eta) \, d\eta = \int_{0}^{+\infty} \exp \left( -\frac{x}{\eta} \right) a(\eta) \, d\eta + \int_{0}^{+\infty} \exp \left( -\frac{-x}{\eta} \right) a(\eta) \, d\eta =$$

$$= - \int_{+\infty}^{0} \exp \left( \frac{x}{\eta} \right) a(-\eta) \, d\eta + \int_{0}^{+\infty} \exp \left( -\frac{-x}{\eta} \right) a(\eta) \, d\eta = \int_{0}^{+\infty} \left[ \exp \left( \frac{x}{\eta} \right) + \exp \left( -\frac{-x}{\eta} \right) \right] a(\eta) \, d\eta.$$  

When writing the last equality, we took into account that $a(-\eta) = a(\eta)$. Thus,

$$U(x) = \frac{G_n}{2(1-c)} \left[ \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \left[ \exp \left( \frac{x}{\eta} \right) + \exp \left( -\frac{-x}{\eta} \right) \right] a(\eta) \, d\eta - 1 \right].$$ \hspace{1cm} (57)
Substituting in (57) the decomposition (49) and changing the order of summation and integration, we obtain

\[ U(x) = \frac{G_n}{2(1 - c)} \left[ (1 - c) \sum_{k=0}^{+\infty} \lambda^k J_k(x) - 1 \right]. \tag{58} \]

Here introduced the notation

\[ J_k(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \left[ \exp\left(\frac{x}{\eta}\right) + \exp\left(-\frac{x}{\eta}\right) \right] a_k(\eta) \, d\eta. \tag{59} \]

Taking into account the form of functions \( a(\eta) \) given by equations (50)–(54) we can write

\[ J_0(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \gamma(x, \eta) \, d\eta, \tag{60} \]

\[ J_1(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \gamma(x, \eta) \, d\eta \int_0^{+\infty} \frac{g(\mu_1) \, d\mu_1}{\mu_1 + \eta}, \tag{61} \]

\[ J_2(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \gamma(x, \eta) \, d\eta \int_0^{+\infty} \frac{g(\mu_1) \, d\mu_1}{\mu_1 + \eta} \int_0^{+\infty} \frac{g(\mu_2) \, d\mu_2}{\mu_2 + \mu_1}, \tag{62} \]

\[ J_k(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \gamma(x, \eta) \, d\eta \int_0^{+\infty} \frac{g(\mu_1) \, d\mu_1}{\mu_1 + \eta} \int_0^{+\infty} \frac{g(\mu_2) \, d\mu_2}{\mu_2 + \mu_1} \cdots \int_0^{+\infty} \frac{g(\mu_k) \, d\mu_k}{\mu_k + \mu_{k-1}}, \tag{63} \]

\[ \gamma(x, \eta) = \frac{X(-\eta)}{|\lambda^{-}\eta|^2} \left[ \exp\left(\frac{x - d}{\eta}\right) + \exp\left(-\frac{x + d}{\eta}\right) \right]. \tag{64} \]

The value of the specific mass flow \( J_M \) of the light component in the direction of the axis \( Oy' \) we define as follows

\[ J_M = \frac{1}{2d^2} \int_{-d}^{d} U(x) \, dx. \tag{65} \]

Substituting (58) in the decomposition (65) and changing the order of summation and integration, we obtain

\[ J_M = \frac{G_n}{2(1 - c)d^2} \left[ (1 - c) \sum_{k=0}^{+\infty} \lambda^k K_k - d \right]. \tag{66} \]

Here

\[ K_0 = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \zeta(\eta) \, d\eta, \quad K_1 = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \zeta(\eta) \, d\eta \int_0^{+\infty} \frac{g(\mu_1) \, d\mu_1}{\mu_1 + \eta}, \]

\[ K_2 = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \zeta(\eta) \, d\eta \int_0^{+\infty} \frac{g(\mu_1) \, d\mu_1}{\mu_1 + \eta} \int_0^{+\infty} \frac{g(\mu_2) \, d\mu_2}{\mu_2 + \mu_1}, \]

\[ K_k = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \zeta(\eta) \, d\eta \int_0^{+\infty} \frac{g(\mu_1) \, d\mu_1}{\mu_1 + \eta} \int_0^{+\infty} \frac{g(\mu_2) \, d\mu_2}{\mu_2 + \mu_1} \cdots \int_0^{+\infty} \frac{g(\mu_k) \, d\mu_k}{\mu_k + \mu_{k-1}}, \]

\[ \zeta(\eta) = \frac{\eta X(-\eta)}{|\lambda^{-}\eta|^2} \exp\left(-\eta^2\right) \left[ 1 - \exp\left(-\frac{2d}{\eta}\right) \right]. \]
3. Results and discussion

The values of $J_M/G_n$ for the mixture of He-Xe and the relative concentration of the light component $C = n_a/(n_a + n_b)$ are given in the table

| $2d$ | 0.01 | 0.1 | 0.5 | 1.0 | 2.0 | 5.0 | 10.0 | 100.0 |
|------|------|-----|-----|-----|-----|-----|------|-------|
| $J_M$ (66) | 3.1866 | 1.9082 | 1.0757 | 0.7637 | 0.5013 | 0.2524 | 0.1380 | 0.0150 |
| $C = 0.01$ |       |      |      |      |      |      |      |       |
| $J_M$ (66) | 3.1852 | 1.9133 | 1.0843 | 0.7724 | 0.5088 | 0.2572 | 0.1409 | 0.0153 |
| $C = 0.05$ |       |      |      |      |      |      |      |       |
| $J_M$ (66) | 3.1827 | 1.9200 | 1.0957 | 0.7840 | 0.5192 | 0.2638 | 0.1449 | 0.01576 |
| $C = 0.1$ |       |      |      |      |      |      |      |       |
| $J_M$ [4] | 2.8760 | 1.7652 | 1.0507 | 0.7717 | 0.5236 | 0.2707 | 0.1491 | 0.0162 |

When calculating $J_M$ we assumed that $m_b = 131.30$, $m_a = 4.0026$, $d_b/d_a = 2.226$, $d = (2d'/3)\beta^{1/2} \nu$. As follows from the results given in the table, the approach used in the work leads to results that are in good agreement with the results from [4].

4. Conclusion

In the present work, an analytical solution of the problem of diffusion of the light component of a binary gas mixture in a long flat channel with infinite parallel walls is obtained in the linearized approximation. Expressions describing the mass velocity profile and specific flow of the light component of the mixture, depending on its concentration in the mixture and the channel thickness, are obtained.

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