Kinetic theory based segregation model for density-bidisperse dense granular flows

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Individual constituent balance equations are often used to derive expressions for species specific segregation velocities in flows of dense granular mixtures. We propose an expression for the interspecies momentum exchange in density bidisperse granular flows as an extension of ideas from kinetic theory, expanding its range from short-duration binary collisions to the multiple enduring contacts characteristic of dense shear flows, and incorporating the effects of particle friction, concentration ratio, and local flow conditions. The segregation velocity derived from the momentum balance equation with the new interspecies drag model matches results from DEM simulations and accurately predicts the downward and upward segregation velocities of heavy and light particles through the depth of the flowing layer for different density ratios and constituent concentrations for both confined shear and free surface heap flows.

I. INTRODUCTION

When sheared, dense granular materials tend to segregate by particle size, density, shape or friction coefficient, which can be problematic in many industries due to its impact on product quality and uniformity [1–8]. As a result, the segregation behavior of sheared granular mixtures is of interest in a wide variety of fields from both fundamental and applied standpoints.

We focus on bidisperse granular mixtures of particles having the same size but different densities such that segregation is driven by a “buoyant force” mechanism [9, 10]. In this paper, we propose a model for the interspecies drag between heavy and light particles in density bidisperse granular flows and use the model to derive an expression for the segregation velocity of each species by considering the balance between the net buoyant force and the interspecies drag force.

To model the segregation process, we assume that two different particle species can be treated as interpenetrable continua such that each constituent has its own partial pressure, consistent with previous studies [1, 3, 4]. The imbalance between a species’ partial pressure and its body force results in segregation of the two species. A linear relation between the drag force on a particle and its segregation velocity has been used in several gravity-driven segregation models for size or density segregation [1, 4, 11–14] based on either the similarity of kinetic sieving in granular segregation with fluid percolation through a porous material or an analogy to the drag force on a sphere settling in a fluid. The linear drag model has been combined with the constituent mass and momentum balances to derive a general multicomponent theory for segregation [15]. However, Weinhart et al. [16] demonstrated that the coefficient fitted from the linear drag model varies with time, indicating that a simple linear drag law fails to describe the segregation behavior. Recent DEM simulations also suggest that other factors, such as the local pressure [17, 18] and the species concentration ratio [19, 20], affect the segregation velocity, which again raises questions concerning the validity of a linear drag model.

As an alternative to the linear drag model, it is natural to consider the Kinetic Theory of Granular Flow (KTGF) to model the momentum transfer between two segregating species. The KTGF provides a drag model connecting stresses and velocities in a granular mixture while accounting for complex particle interactions [21–27], thereby yielding a drag model that takes particle density and diameter into consideration at the particle level. However, there are two major problems with using the KTGF approach to model dense flows. First, the stress generation mechanism in the KTGF is based on short-duration, binary collisions typical of dilute particle flows as opposed to multiple, enduring contacts typical of the dense granular flows considered here. Second, the KTGF approach does not include physics known to be critical to segregation in dense granular flows such as the dependence of segregation on the local pressure [17]. Nevertheless, a potential path toward a robust drag model for dense flows lies in modifying the KTGF drag model based on empirical results for density segregation in dense granular flows.

In this paper, we propose an expression for the density segregation velocity based on mixture theory [1] with a drag model based on KTGF but extended to the dense regime by introducing appropriate coefficients. The proposed drag
model is described in Section II and an expression for the segregation velocity in density bidisperse granular flow is derived by combining the drag and the equilibrium momentum balance equation. The effects of the local inertial number, particle friction, and relative species concentration proposed in the model are tested separately by DEM simulations in Section III. In Section IV, results of the segregation model are compared with DEM simulations for flows under more general conditions. Conclusions are given in Section V.

II. KTGF BASED DRAG MODEL AND SEGREGATION VELOCITY

For dense granular flows, the total solids volume fraction, \( \phi_{\text{solid}} = \sum \phi_i \), where \( \phi_i \) represents the local volume fraction of each solid species (here \( i = h \) for heavy particles and \( i = l \) for light particles), is around 0.6. The local constituent concentration is \( c_i = \phi_i/\phi_{\text{solid}} \) and the solid density is \( \rho_{\text{solid}} = \sum \rho_i c_i \), where \( \rho_i \) is the particle material density. The gradient of the lithostatic pressure \( P \) is

\[
\frac{\partial P}{\partial z} = -\rho_{\text{solid}} \phi_{\text{solid}} g,
\]

where \( z \) is the vertical coordinate (assuming negative \( z \) is the gravity direction) and \( g \) is the acceleration due to gravity.

The partial pressure \( P_i \) is determined by partitioning the lithostatic pressure induced by gravity among the two constituents such that \( P = \sum P_i \). For particles that only differ in density, we assume that the proportion of the hydrostatic load carried by each species equals its concentration \( c_i \) regardless of the density difference as proposed by Marks et al. [14] and Tunuguntla et al. [4]. The momentum balance for each species can be written as [4]

\[
\frac{\partial}{\partial t} (\rho_i \phi_i u_i) + \nabla \cdot (\rho_i \phi_i u_i \otimes u_i) = -c_i \nabla P + \rho_i \phi_i g + \beta_i,
\]

where \( \otimes \) is the dyadic product, \( t \) is time, \( u_i \) is the vector velocity, and \( \beta_i \) is the interspecies drag vector. (Eq. (2) is similar to Eq. (2.5) in [4], noting that there the intrinsic density \( \rho^* \) is the bulk density of particles in a non-mixed state, which is equal to \( \rho_i \phi_{\text{solid}} \), and that \( \rho^* = c^i \rho^* \) is equivalent to \( \rho_i \phi_i \) used here.) Assuming that the vertical acceleration terms are negligible [4, 28], which is reasonable for the relatively slow segregation in a typical granular flowing layer, the momentum conservation equation in the \( z \)-direction can be simplified using Eq. (1) to

\[
(\rho_{\text{solid}} - \rho_i) \phi_i g + \beta_i = 0.
\]

Equation (3) is analogous to a simple force balance between the the net buoyant force \((\rho_{\text{solid}} - \rho_i) \phi_i g\) and the interspecies drag force \( \beta_i \).

To determine constituent velocities from Eq. (3), \( \beta_i \) must be expressed as a function of the velocity difference between the two species, and the KTGF offers a means to do this. Changes to the KTGF are needed, however, because it was developed to model rapid flow of dilute granular materials based on instantaneous binary collisions [28, 30]. This is not the case in the dense flow regime where enduring contacts dominate and many adjacent particles are part of contact force chains [31]. Correlations in motion and force due to the dense contact network reduce the collisional energy dissipation [32]. Substantial effort has been expended to extend the applicability of the KTGF to the dense regime, mainly by modifying the expression for the collisional energy dissipation rate to account for the effects of sustained contacts [33, 34]. In fact, an augmented KTGF approach has been proposed for segregation problems, but it is limited to small size and density differences [35, 36]. In addition, several stand-alone drag models based on the KTGF have been proposed for dilute granular flows [21, 27].

Here we follow an approach similar to one of these models [27] by assuming that the long-duration collisional drag follows similar physics to the short-duration collisional drag, but with a much longer contact time. This can be represented by correction coefficients modifying the well-known solid-solid drag model proposed by Syamlal [22] that has been successfully used for fluidized bed simulations [40, 41]. The Syamlal expression for the local drag \( \mathbf{\beta}_i \) between two constituents \( i \) and \( j \) is

\[
\mathbf{\beta}_i = -\left[ \frac{3(1 + e)}{2\pi(\rho_i d_i^3 + \rho_j d_j^3)} \left( \frac{\pi}{2} + \frac{\mu \pi^2}{8} \right) \right] g_{ij} \rho_i \phi_i \phi_j (d_i + d_j)^2 |u_i - u_j| (u_i - u_j),
\]

where \( g_{ij} \) is the radial distribution function for a collision between particle \( i \) and \( j \) [43], \( \mu \) is the friction coefficient, and \( e \) is the restitution coefficient. For density segregation, \( d_i = d_j \), and \( g_{ij} \) becomes a constant if we assume constant total solid volume fraction \( \phi_{\text{solid}} \). The bulk velocity in the vertical direction of the combined constituents
is $w_{\text{bulk}} = c_i w_i + c_j w_j$ based on volume conservation. The segregation, or percolation velocity $w_{p,i}$, for species $i$ is defined as the velocity difference between the constituent velocity and the bulk velocity in the $z$ direction:

$$w_{p,i} = w_i - w_{\text{bulk}} = c_j (w_i - w_j).$$  

(5)

Using Eq. (5) in Eq. (4) for bidisperse granular mixtures of heavy and light particles having the same diameter $d$, the interspecies drag force in the segregation ($z$) direction is

$$\beta_i = -\left[3(1 + c)\left(1 + \frac{\mu_\tau}{4}\right) g_{ij}\right] \left[\frac{\rho_h \rho_i \phi_i}{\rho_l + \rho_h} \phi_j \phi_{\text{solid}}^2 \frac{|w_{p,i}|}{d} w_{p,i}\right].$$  

(6)

Based on Eq. (6), we propose a semi-empirical interspecies drag model of the form

$$\beta_i = \frac{\text{KTGF}}{\text{enduring contacts}} \left[\frac{\rho_h \rho_i \phi_i}{\rho_l + \rho_h} \phi_j \phi_{\text{solid}}^2 \frac{|w_{p,i}|}{d} w_{p,i}\right] \frac{1}{I^2} \sqrt{\frac{\phi_h}{\phi_l}}.$$  

(7)

Equation (7) has four multiplicative terms. Consider them from left to right. $B(\mu)$ replaces the first term in square brackets in Eq. (6) to account for enduring contacts in dense flows. In dense granular mixtures short-duration collisions are unlikely to play as important of a role as in the dilute flows upon which Eq. (6) for KTGF is based. As a result, we assume $B(\mu)$ is independent of $\epsilon$, which is proven in Section III.D. The second term is inherited from the KTGF and reflects the momentum transfer between two particles with different mass, taking into consideration density and diameter at the particle level. The third term accounts for the local flow conditions, since Eq. (6) does not explicitly account for the presence of shear. As such, Eq. (6) indicates that the two particle species would segregate even without the presence of shear, which, of course, does not occur. There are several ways to rationalize this. For example, Gera et al. extended the KTGF drag model to dense fluidized beds by adding a “hindrance effect” term related to the friction and pressure, such that the momentum exchange has to exceed a threshold to drive the segregation. Here we take a heuristic approach by introducing the inertial number $I$ from the $\mu(I)$ rheology to account for local flow conditions, such that

$$I = \dot{\gamma} d \sqrt{\frac{\rho_{\text{solid}}}{P}},$$  

(8)

where the bulk density is $\rho_{\text{bulk}} = \rho_{\text{solid}} \phi_{\text{solid}}$ and $\dot{\gamma}$ is the local shear rate. A small value of $I$ (small $\dot{\gamma}$ and/or large $P$) corresponds to the quasistatic regime where granular flows behave more like deforming solids. Conversely, a large value of $I$ (large $\dot{\gamma}$ and/or small $P$) corresponds to the collisional regime where granular flows behave more like fluids. It is reasonable to expect that the interspecies drag increases with pressure, so the inertial number appears as $1/I^2$ in Eq. (7). Finally, the fourth term in Eq. (7) is an empirical modification to account for the nonlinear dependence of segregation on particle concentration (i.e., heavy particles among many light particles segregate faster than light particles among many heavy particles), as it increases the drag with increasing $\phi_h/\phi_l$.

The ultimate goal of this paper is to propose an accurate model for the segregation velocities of the light and heavy particle species in density bidisperse granular flows. Such an expression can be readily derived by substituting the expression for the interspecies drag $\beta_i$ from Eq. (7) into the equilibrium momentum balance of Eq. (3). For light particles,

$$\left(\rho_{\text{solid}} - \rho_l\right) g \phi_l - \frac{B(\mu)}{\rho_l + \rho_h} \phi_h \phi_{\text{solid}}^2 \left(\frac{|w_{p,l}|}{I}\right)^2 \sqrt{\frac{\phi_h}{\phi_l}} = 0,$$

(9)

such that $w_{p,l}$ takes the form

$$w_{p,l} = \left[\frac{gd}{B(\mu) \phi_{\text{solid}}} \left(R_p - \frac{1}{R_p}\right) \sqrt{\frac{\epsilon_l}{\epsilon_h}} \right]^{1/2} (1 - c_l)I,$$

(10)

noting that $\phi_h/\phi_l = c_h/c_l$ and that $R_p = \rho_h/\rho_l$ is the density ratio. For heavy particles, a similar approach yields

$$w_{p,h} = \left[\frac{gd}{B(\mu) \phi_{\text{solid}}} \left(R_p - \frac{1}{R_p}\right) \sqrt{\frac{\epsilon_l}{\epsilon_h}} \right]^{1/2} (1 - c_h)I.$$  

(11)
Equations (10) and (11) indicate that \( w_{p,i} \propto \sqrt{R_p - \frac{1}{R_p}} \), which is similar to the empirical relation \( w_{p,i} \propto (\sqrt{R_p} - \frac{1}{\sqrt{R_p}}) \) observed in monodisperse systems with several heavy intruder particles [18]. Equations (10) and (11) also resemble the general form of the segregation velocity given by Fry et al. [17]

\[
w_{p,i} = \sqrt{gdf(R_p)(1 - \beta_i)}I,
\]

by making

\[
f(R_p) = \left[ \frac{1}{B(\mu)\rho_{\text{solid}}(R_p - \frac{1}{R_p})} \right]^{1/2}.
\]

These similarities suggest that the drag model proposed in Eq. (7) is reasonable and consistent with previous research.

### III. CONFIRMING THE MODEL

The challenge now is to confirm the interspecies drag model in Eq. (7) and validate the resulting segregation model in Eqs. (10) and (11). This is accomplished by performing DEM simulations under a variety of conditions. Direct measurement of \( \beta_i \) from DEM simulations is difficult since both the segregation driving forces and the drag forces result from interparticle collisional forces; it is hard to distinguish them at the particle level. A better way to determine \( \beta_i \) is to utilize momentum conservation at force equilibrium, Eq. (9), in which \( \beta_i \) equals \(- (\rho_{\text{solid}} - \rho_i)\phi g\), a quantity that is easily measured locally in a granular flow. This is equivalent to validating the segregation model in Eqs. (10) and (11), noting that all of the variables in these equations except \( B(\mu) \) are known \((R_p, \rho_{\text{solid}}, d)\) or can be calculated from the simulation \((w_{p,i}, I)\). Hence, the problem comprises finding \( B(\mu) \) under a variety of local flow conditions, \( I \), and local concentration ratios, \( c_h/c_l \).

In the DEM simulations a density bidisperse mixture of \( d = 4 \text{ mm} \) spherical particles with collision time \( t_c = 1.25 \times 10^{-4} \) is sheared between top and bottom horizontal frictional planes in a domain with periodic boundary conditions in the streamwise \((x)\) and spanwise \((y)\) directions as shown in Fig. 1. The massive planar top wall moves horizontally with constant velocity \( U \) and is free to move vertically. The position of the top wall is determined by the wall weight and contact forces from the top layer of particles. The distance between the top and bottom walls \( h \) remains relatively constant (fluctuating by \( \pm 2\% \)) after an initial rapid dilation of the particles at flow onset. The domain extends about 70\( d \), or 0.28 m, in the streamwise direction and 10\( d \), or 0.04 m, in the spanwise direction. The weight of the massive planar top wall is based on the configuration of the system so that the overburden pressure applied by the wall to the system is \( P_{\text{wall}} = 0.05\rho_{\text{bulk}}gh \) for all cases in this study. There are 36864 particles in the system, differentiated by their densities \((\rho_h \text{ for heavy particles and } \rho_l \text{ for light particles having density ratio } R_p = \rho_h/\rho_l)\). The initial dense packing is achieved by placing particles in a grid pattern and letting them settle under gravity. The depth of the flow \( h \) in the vertical \( z \)-direction is approximately 50\( d \) or 0.2 m, and the volume fraction ratio \((\phi_h/\phi_l)\) or, equivalently, the concentration ratio \((c_h/c_l)\), is approximately uniform across the domain. Varying \( c_h/c_l \) or \( \rho_h/\rho_l \) in the simulations causes the segregation velocity to vary. Over 200 simulations are performed over the wide range of flow conditions and particle properties listed in Table 1.

To achieve specific shear rate profiles, the corresponding velocity profile is imposed on the particle by applying a streamwise stabilizing force on each particle \( k \) at every time step according to

\[
F_{\text{stabilize},k} = K_s[u(z_k) - u_k],
\]

where \( u(z) \) is the imposed velocity profile, \( u_k \) is the particle’s streamwise velocity, \( z_k \) is the particle’s vertical position, and \( K_s \) is a gain parameter. The details of the approach for the imposed velocity profile are provided elsewhere [17]. Three different profiles are considered: \( u = Uz/h, Ue^{2.3(z/h-1)}, \) and \( Uz^2/h^2 \). The first and second profiles correspond to those for uniform shear [17] and free surface flow down a heap [48], respectively. The third profile reflects a situation that sometimes occurs in shear flows where the shear rate vanishes at the bottom wall due to the rough boundary condition [49].

The simulation domain is divided into 20 horizontal layers for averaging purposes; each layer is 2.5\( d \) in the \( z \)-direction. The streamwise mean velocity profile for a density segregation simulation 0.1 s after flow onset is shown in Fig. 2 for each of the three imposed velocity profiles. Due to the stabilizing force, the mean velocity matches the imposed profile, with a shear rate that is either uniform \((\dot{\gamma} = U/h)\) or decreases through the depth of the particle bed \((\dot{\gamma} = 2Uz/h^2, 2.3Ue^{2.3(z/h-1)}/h)\). The different symbols in Fig. 2 represent heavy and light particles, which have similar streamwise velocities. Slight deviations from the imposed velocity profiles occur within 5\( d \) from the top and
FIG. 1: Schematic of DEM simulation setup.

FIG. 2: Instantaneous streamwise velocity $u_i$ of each constituent for mixed light (+) and heavy (○) particles averaged in each horizontal layer 0.1 s after shear onset. Dashed curves represent the velocity profiles imposed using the forcing specified in Eq. (14). ($d_h = d_l = 4$ mm, $c_h = c_l = 0.5$, $\mu = 0.2$, $e = 0.9$, and $R_{\rho} = 4$.)

bottom boundaries due to particle ordering adjacent to the flat walls [49, 50]. Here, we focus on the flow away from the walls to avoid any artifacts related to these bounding walls. Companion simulations with bumpy walls (particles attached randomly on flat walls [51]) show little difference with simulations using flat walls for the particles between $0.3 \leq z/h \leq 0.7$.

To characterize the evolution of the segregation, we measure the average center of mass height for each species relative to the mean height of all particles, which is calculated as

$$\bar{z}_i = \frac{1}{N_i} \sum_{k \in i} z_k - \frac{1}{N} \sum_{k=1}^{N} z_k,$$  \hspace{1cm} (15)$$

where $N_i$ and $N$ are the number of particles of species $i$ and the total number of particles in the horizontal averaging.
TABLE I: Simulation parameter ranges.

| Parameter | Value |
|-----------|-------|
| $d$       | 4 mm  |
| $R_\rho$  | $1 - 10$ |
| $\mu$     | $0 - 0.6$ |
| $c_h$, $c_l$ | $0.1 - 0.9$ |
| $\epsilon$ | $0.2 - 0.9$ |
| $U$       | $1 - 5 \text{ m/s}$ |
| $u$       | $U_z/h$, $U z^2/h^2$, $U e^{2.3(z/h-1)}$ |
| $\dot{\gamma}$ | $U/h$, $2U z/h^2$, $2.3U e^{2.3(z/h-1)/h}$ |

FIG. 3: Determining the segregation velocity. (a) Average center of mass displacement $\Delta \bar{z}/d$ time series at different vertical positions under uniform shear ($\dot{\gamma} = U/h = 25 \text{ s}^{-1}$ and $R_\rho = 8$). (b) Segregation velocity profiles for light particles $w_{p,l}$ ($\times$) and heavy particles $w_{p,h}$ ($\circ$) are calculated as the average rate of change of $\Delta \bar{z}$ over the first 1 s of the simulation. Both shear rate $\dot{\gamma}$ and density ratio $R_\rho$ affect the segregation velocity profile. Error bars represent the uncertainty in determining $\Delta \bar{z}$ and $w_{p,h}$; $c_h = c_l = 0.5$, $\mu = 0.2$, $\epsilon = 0.9$.
the error bars at time $t$ slightly less than 1 s in Fig. 3(a).

By considering many thin horizontal layers, the local segregation velocity profile can be measured, as shown in Fig. 3(b) for three sample cases of uniform shear flow. The segregation velocities increase with both increasing density ratio and shear rate. For each case, segregation velocity decreases from top to bottom even though the shear rate and density ratio do not vary with depth. This is a consequence of the overburden pressure $P$ increasing with depth, consistent with previous results [17]. Note that the sum of segregation velocities of heavy and light particles should equal zero due to volume conservation for $c_h = c_l = 0.5$. Figure 3(b) indeed demonstrates that the segregation velocities are nearly equal (and, of course, opposite). The slight difference between $w_{p,l}$ and $-w_{p,h}$ is likely caused by the small amount of segregation during the initial filling process before the start of the simulation. The error bars represent uncertainties in the measurement of the slope in Fig. 3(a).

From the momentum balance equation, Eq. (2), the convection term in the $z$-direction $\frac{\partial}{\partial z}(\rho_i \phi_i w_i^2)$, which can be estimated from the simulation results, is three orders of magnitude less than the net buoyant force $-c_i \frac{\partial P}{\partial z} - \rho_i \phi_i g$ (noting that the constituent velocity $w_i$ is equal to the segregation velocity $w_{p,i}$ since the bulk velocity in the $z$-direction is zero). Likewise, the unsteady term $\frac{\partial}{\partial t}(\rho_i \phi_i \dot{w}_i)$ is also negligible. This confirms that the system is in steady state during the sampling window and that the segregation velocity results from the balance between the net buoyant force and the interspecies drag as indicated by the simplified momentum expression in Eq. (3).

A. Inertial number dependence

The inertial number dependence assumed in the proposed drag model, Eq. (7), requires that the segregation velocity is linear in $I$ [Eqs. (10) and (11)]. Therefore, to verify the drag model, segregation velocities of light particles $w_{p,l}$ at different depths are plotted versus $I$ for different density ratios $R_p$ in Fig. 4 for flows with nonlinear streamwise velocity profiles. Similar results are also found for the segregation velocity of heavy particles $w_{p,h}$ since $w_{p,h} \approx w_{p,l}$ for flows with $c_h = c_l = 0.5$. The linear relation between $w_{p,i}$ and $I$ in Fig. 4 for varying $\gamma$ and $P$ (expressed in terms of varying $I$) confirms the assumed dependence on $I$ and is consistent with results from previous studies [17, 18]. That is, the segregation velocity depends linearly on $I$ for each density ratio. The error bars are somewhat large, consistent with the difficulty in accurately measuring $w_{p,i}$ as indicated in Fig. 3. Nevertheless, a line can be fit through the data at each density ratio. Furthermore, the fitting lines are quite similar for both nonlinear velocity profiles.

The data, however, do not show as clear a linear dependence on $I$ when plotted in the same way for uniform shear flows, as illustrated in Fig. 5(a). This is because unlike the other two nonlinear velocity profiles [$u = U z^2/h^2$ and $U e^{2.3(z/h-1)}$], the local inertial number in a uniform shear flow is not equal to zero or nearly zero at the bottom wall. Instead, in the case of the linear velocity profile ($u = U z/h$), the segregation is forced to cease at the fixed bottom wall because of the wall itself. To account for the cessation of density segregation at the bottom wall, we use a modified inertial number for uniform shear flows

$$I^* = \sqrt{I^2 - I_0^2},$$

where $I_0$ is the inertial number at which density segregation effectively ceases. According to Fig. 3(b), segregation ceases at $z/h \approx 0$. Therefore, we assume $I_0$ is the inertial number at that location, such that $I_0 = \gamma d \sqrt{\rho_{bulk}/I_0}$ with $P_0 = \rho_{bulk} g h + P_{wall}$. As shown in Fig. 5(b), using $I^*$ instead of $I$ collapses the data for uniform shear flows such that the dependence of $w_{p,i}$ on $I^*$ is close to linear. Note that for the two nonlinear velocity profiles $I^*$ equals $I$ since $I_0 \approx 0$. From here on, unless otherwise specified, we drop the asterisk on $I$, using the corrected inertial number [Eq. (16)].

Figures 4 and 5(b) show that $w_{p,i}$ depends linearly on $I$ for varying shear profiles, shear rates, and density ratios. This confirms $\beta_i \propto I^{-2}$ in Eq. (7) since equating the buoyant and drag forces gives $w_{p,i} \propto I$ in Eqs. (10) and (11). The linear relation between $w_{p,i}$ and $I$ is also consistent with previous results in free surface flows. Consider, for example, heap flows, in which the flowing layer thickness is about eight particle diameters [48, 52]. For such a thin layer with the local shear rate decreasing exponentially with depth, the overburden, or lithostatic, pressure can be treated as a uniform parameter, and the linear relation between $w_{p,i}$ and $I$ can be reduced to a linear relation between $w_{p,i}$ and $\gamma d$ [53, 54]. On the other hand, for a thick flowing layer with a uniform shear rate, the segregation velocity at different depths in the flowing layer is only affected by the overburden pressure so long as we consider segregation away from the upper and lower bounding walls. The effect of lithostatic pressure has to be included, resulting in a dependence of $w_{p,i}$ on $I$ [17, 18]. We further note that in our previous work [17], we proposed a form for $I^*$ in terms of a critical pressure that is equivalent to Eq. (16). However, it should be noted that at larger inertial numbers ($I > 0.5$) [18], instantaneous binary collisions dominate over enduring contacts, resulting in a rheological change from dense to rapid dilute granular flow. In the latter regime, the linear relation between $w_{p,i}$ and $I$ may no longer be valid.
FIG. 4: Segregation velocities of light particles vs. local inertial number \( I \) at different depths for nonlinear velocity profiles (a) \( u = U z^2/h^2 \) and (b) \( u = U e^{2.3(z/h-1)} \) for simulations with \( U = 2 \) m/s, \( R_p = \rho_h/\rho_l \in \{2, 4, 10\} \), \( c_h = c_l = 0.5 \), \( \mu = 0.2 \), and \( e = 0.9 \). Dashed lines are linear fits for each density ratio, demonstrating a linear dependence of \( w_{p,l} \) on \( I \). Error bars represent the uncertainty in determining \( w_{p,l} \) as indicated in Fig. 3.

FIG. 5: Segregation velocities of light particles \( w_{p,l} \) at different depths vs. (a) inertial number \( I \) and (b) modified inertial number \( I^* = \sqrt{I^2 - I_0^2} \) for simulations with particle properties given in Fig. 4 but for uniform shear rates \( \dot{\gamma} = 5 \) s\(^{-1} \) (○), 15 s\(^{-1} \) (□), and 25 s\(^{-1} \) (○). \( I_0 \) is the cut-off inertial number at which density segregation effectively ceases [17]. The dashed line in both figures is identical and a linear fit to the data in (b). Error bars represent the uncertainty in determining \( w_{p,l} \) as indicated in Fig. 3.

B. Density ratio dependence

As shown in Fig. 4, \( w_p/I \) increases with density ratio \( R_p \). Equations (10) and (11) indicate that \( w_p/I \) is proportional to \( \sqrt{R_p - 1}/R_p \). To test this proposed dependence, \( w_{p,l}/I \), calculated as the slope of the fitting lines in Figs. 4 and 5(b), is plotted versus \( R_p \) in Fig. 6. The dashed curve in Fig. 6 represents the predictions of Eq. (10) with \( B(\mu = 0.2) = 400 \). (We characterize the dependence of \( B \) on \( \mu \) in Section III.D.) Indeed, \( w_{p,l}/I \) is proportional to \( \sqrt{R_p - 1}/R_p \) over the wide range of \( R_p \) tested (\( 1.3 \leq R_p \leq 10 \)) in different flow profiles where \( I \) varies due to changes in both \( P \) and \( \dot{\gamma} \). Good agreement between the curve and the data confirms the proposed dependence of the drag model on particle densities and, equivalently, the functional dependence of \( w_{p,l} \) on \( R_p \) in Eqs. (10) and (11).
A recent study [20] indicates that the segregation in mixtures of heavy and light particles has an underlying asymmetry that depends on the local particle concentration, similar to mixtures where particles differ in size [19]. Specifically, a heavy particle among mostly light particles segregates faster than a light particle among mostly heavy particles. The last term $\sqrt{\phi_h/\phi_l}$, on the right hand side of the drag model in Eq. (7), accounts for the nonlinear dependence of segregation velocity on particle concentration. To confirm this term, we rewrite Eq. (10) using $w_{p,l} = c_h(w_l - w_h)$ from Eq. (5), such that $c_h$ on the l.h.s and $(1 - c_l)$ on the r.h.s cancel and the equation becomes

$$\begin{equation}
(w_l - w_h)^2 = \frac{1}{B(\mu)} \frac{gd}{\phi_{solid}} \left( R_p - \frac{1}{R_p} \right) t^2 \frac{\sqrt{c_l}}{c_h}.
\end{equation}$$

According to Eq. (17), $(w_l - w_h)^2$ depends linearly on $\sqrt{c_l/c_h}$. To test this dependence, we keep the particle densities and other simulation parameters constant, but vary $c_h$ from 0.1 to 0.9 in steps of 0.1 for uniform shear flow. Thus, $c_l/c_h$ varies from 1/9 to 9. The particle concentration is limited to the range 0.1 to 0.9 because $(w_l - w_h)^2$ is close to zero for $c_h > 0.9$, making the segregation velocity too small to accurately determine. Figure 7 plots the square of the normalized velocity difference between the two species $\Delta \hat{w}$, as a function of $\sqrt{c_l/c_h}$ for uniform shear flows. Note that $(w_{p,l} - w_{p,h})/I$ is calculated using the slope of the fitting line through the data at different depths like that in Fig. 5(b). This avoids propagating the random error evident in these figures as the deviation of the data points for individual simulations from the fitting line. By varying only the concentration ratio in Fig. 7, it is evident that $\Delta \hat{w}^2$ depends linearly on $\sqrt{c_l/c_h}$. As expected, larger uncertainties exist for small $\sqrt{c_l/c_h}$ due to the difficulty in measuring small values of the segregation velocity. Nonetheless, Eq. 7 confirms the linear relation between $\Delta \hat{w}^2$ and $\sqrt{c_l/c_h}$ used in Eqs. (10) and (11), and thus the proposed form of the drag model [Eq. (7)].

D. Friction and restitution dependence

The functional form for $B(\mu)$ characterizing the dependence of the drag on particle friction, in Eqs. (7), (10), and (11), is difficult to derive analytically due to the complex particle interactions typical of dense granular flows. Instead, DEM simulations under a variety of flow conditions are used to determine $B(\mu)$ and demonstrate that it is independent of the restitution coefficient $e$.

Consider first $e$, which characterizes energy dissipation in short-duration collisions. We vary $e$ from 0.1 to 0.9 for uniform shear flow while keeping all other parameters unchanged, such that the damping coefficient of the linear-spring-dashpot collision model used in the DEM simulations varies accordingly. The segregation velocities for both light and heavy particles at $z/h = 0.5$ for the case with $c_h = c_l = 0.5$ and $R_p = 8$ are plotted versus $e$ in Fig. 8(a) for...
FIG. 7: Square of the normalized velocity difference \( \Delta \hat{\omega}^2 \) versus square root of concentration ratio \( \sqrt{c_l/c_h} \) for simulations with \( R_\rho = 4, \mu = 0.2, e = 0.9, \) and \( \dot{\gamma} = U/h = 25 \text{ s}^{-1} \). \( c_h \) varies from 0.1 to 0.9 in steps of 0.1 and \( c_l = 1 - c_h \). The dashed line is a linear fit to the data forced through zero.

FIG. 8: Dependence of segregation velocities \( w_{p,i} \) on contact parameters. (a) Light (×) and heavy (○) particle segregation velocities are nearly independent of restitution coefficient \( e \) at \( z/h = 0.5 \) with \( \mu = 0 \) (black), 0.1 (red), and 0.4 (blue). (b) Particle segregation velocity decreases as \( \mu \) increases for \( z/h = 0.3 \) (red), 0.5 (blue), 0.7 (black) with \( e = 0.9 \). \( R_\rho = 8, c_h = c_l = 0.5, \) and \( \dot{\gamma} = U/h = 25 \text{ s}^{-1} \).

three different friction coefficients \( \mu \). Although there is some scatter in the data, it is clear that \( w_{p,i} \) is independent of \( e \), regardless of \( \mu \), as is reasonable to expect for dense granular mixtures in which short-duration collisions are unlikely to play as important of a role as in dilute flows.

For dense granular mixtures, enduring contacts are dominant, and stresses are generated by long-duration sliding and rolling contacts. Unlike \( e \), the surface friction coefficient \( \mu \) has significant impact on \( w_{p,i} \) as shown in Fig. 8(b). The particle segregation velocity generally decreases as \( \mu \) increases at different bed depths. Note that \( w_{p,i} \) measured from DEM simulations is affected by not only \( \mu \) but also the \( \mu \) dependent initial packing, which adds more uncertainties resulting in scatter in the data. Nevertheless, the overall trend of decreasing segregation velocity with increasing friction remains evident.

In this context, \( B(\mu) \) is determined through a series of DEM simulations for uniform shear flow with different density ratios \( R_\rho \) and friction coefficients \( \mu \). Since the dependence of drag on the other parameters has been verified, it is possible to determine the functional form for \( B(\mu) \) from Eq. 10.
FIG. 9: (a) $\frac{gd}{\phi_{solid}}(R_\rho - \frac{1}{\rho_c})\sqrt{\frac{2}{c_h}}(1 - c_i)^2$ vs. $(w_{p,i}/I)^2$ for uniform shear flows with $R_\rho \in \{2, 4, 6, 8, 10\}$ and three different $\mu$ values. $B(\mu)$ is determined by the fitted slope of the dashed lines, which are forced through zero ($c_h = c_i = 0.5$, $\gamma = 25$, and $e = 0.9$). (b) $B(\mu)$ vs. $\mu$ based on 135 simulations. Different symbols correspond to $e = 0.2$ (x), $0.8$ ( ), and $0.9$ ( ) for uniform shear flows.

$$B(\mu) = \frac{gd}{\phi_{solid}}(R_\rho - \frac{1}{\rho_c})\sqrt{\frac{2}{c_h}}(1 - c_i)^2 \sqrt{\frac{(w_{p,i}/I)^2}{1 - \mu^2}}.$$  \hspace{1cm} \text{(18)}$$

In applying Eq. \text{(18)}, the slope of the fitting line in Fig. 5(b) is again used for $w_{p,i}/I$. We estimate $B(\mu)$ by plotting $\frac{gd}{\phi_{solid}}(R_\rho - \frac{1}{\rho_c})\sqrt{\frac{2}{c_h}}(1 - c_i)^2$ versus $(w_{p,i}/I)^2$ over a range of density ratios for three values of $\mu$ in Fig. 9(a). The slope of the data at each $\mu$ gives the corresponding value of $B(\mu)$.

By measuring the slopes of different sets of data with $0 \leq \mu \leq 0.6$, we empirically determine the dependence of $B(\mu)$ on $\mu$. The results are plotted in Fig. 9(b) for several values of $e$ for uniform shear flow. For bidisperse mixtures with a small friction coefficient ($\mu < 0.3$), $B(\mu)$ is linearly proportional to $\mu$. For larger values of $\mu$ ($0.3 \leq \mu \leq 0.6$), the relation for $B(\mu)$ remains linear but with a smaller slope. There is slightly more scatter for different values of $e$ at larger $\mu$. Note that $B(\mu = 0.2) = 400$ from Fig. 9(b) is consistent with the value used in Fig. 6 to calculate $w_{p,i}/I$.

The interspecies drag is a combination of normal and tangential forces resulting from long-duration contacts. When $\mu = 0$, the tangential forces vanish. Thus, $B(\mu = 0) \neq 0$ means that interspecies drag of smooth particle systems is entirely due to normal forces. Figure 9(b) shows that $B(\mu)$ increases with $\mu$. That is, increasing friction increases the interspecies drag, thereby reducing the segregation velocity. This is contrary to observations from size segregation, where part of the segregation driving force comes from interparticle friction, such that increasing friction coefficient promotes the segregation of large particles [35]. A possible explanation is that the friction force adds to the segregation driving force in size segregation, which overcomes the resultant increase in the interspecies drag. Alternately, for particles having different densities but the same size, the friction force has no impact on the segregation driving force but only adds to the drag.

Note that Eqs. \text{(10)} and \text{(11)} are not limited to specific velocity profiles even though $B(\mu)$ is empirically determined using the data from uniform shear flows with $c_h = c_i = 0.5$. $B(\mu)$ is merely a coefficient to account for the particle frictional properties. Thus, Eqs. \text{(10)} and \text{(11)} are generally applicable to predict density segregation in any dense segregating granular flow, as shown in the next section.

### IV. SEGREGATION VELOCITY MODEL

Now that the various terms of the segregation velocity model have been validated, it is possible to compare the predictions of Eqs. \text{(10)} and \text{(11)} to the segregation velocity measured in the DEM simulations. To do this, the global values for $d$, $\phi_{solid}$, $B(\mu)$, $c_l$, $c_h$, and $R_\rho$ are used in the model along with local values of $I$ at each horizontal position.
FIG. 10: Predictions of segregation velocity profiles from Eqs. (10) and (11) (dashed curves) compared to DEM simulation data for light (×) and heavy (○) particles under (a) uniform (\(\dot{\gamma} = U/h\)) and (b) varying shear rate profiles \(\dot{\gamma} = 2Uz/h^2, 2.3Ue^{-2.3(z/h-1)/h}\) with different density ratios. \(\mu = 0.2, c_h = c_l = 0.5, \) and \(e = 0.9.\)

FIG. 11: Predictions of segregation velocity profiles from Eqs. (10) and (11) (dashed curves) compared to DEM simulation data under uniform shear for (a) light and (b) heavy particles with heavy particle concentration \(c_h = 0.3, 0.5 \) and 0.7 (\(c_l = 1 - c_h\)). \(\mu = 0.2, R_p = 8, \dot{\gamma} = 25 \text{ s}^{-1}, \) and \(e = 0.9.\)

in the flow. The local inertial number \(I\) is based on the overburden pressure estimated as \(P = P_{\text{wall}} + \rho_{\text{bulk}}g(h-z),\) such that \(I\) can be expressed as a function of vertical position \(z\) according to Eq. (8) (or, equivalently, Eq. (16) for uniform shear flows).

Figure 10(a) compares the particle segregation velocity profiles predicted by Eqs. (10) and (11), with those measured from DEM simulations of uniform shear flows in Fig. 3(b). Since data for uniform shear flows is used in determining \(B(\mu),\) good agreement between the predicted segregation velocities and those from uniform shear flows is expected. However, the segregation velocities for \(z/h < 0.2\) and \(z/h > 0.8\) in the simulations are affected by the segregating particles accumulating near the bounding walls.

A more rigorous test of the model is to consider flows with varying shear rates \(\dot{\gamma} = 2Uz/h^2, 2.3Ue^{-2.3(z/h-1)/h}\) in Fig. 10(b). The close match between the prediction and the data demonstrates that Eqs. (10) and (11) are also effective for flow with non-uniform shear rates, even though \(B(\mu)\) in the model was derived from flow with uniform shear rate \(\dot{\gamma} = U/h.\)

Figure 11 provides a similar comparison where the concentration ratio \(c_h/c_l\) is varied while keeping other parameters constant for uniform shear. In this case, the segregation velocities for heavy and light particles differ from one another when the concentration of the two species is not equal, as expected [20]. Again, the estimated segregation velocities
match the DEM data. Thus, we conclude that the segregation model [Eqs. (10) and (11)] based on the interspecies drag term derived from analogy with KTGF can be used to estimate the segregation velocities of both light and heavy species through the depth of the flowing layer under a variety of flow conditions.

Unlike the confined shear flows discussed to this point, the shear rate and the concentration ratio of many free surface flows vary across the domain. To test the model under these more general conditions, we consider quasi-2d bounded one-sided heap flow in which particles flow in a thin surface layer down a slope much like what occurs when filling a silo. Figure 12 shows segregation velocities from heap flow DEM simulations versus \((c_l/c_h)^{1/4}(1 - c_i)I\), since, according to Eqs. (10) and (11), these two variables are linearly related when all other conditions are equal. The difference between these results and those up to this point is that these results correspond to a wide range of shear rates and concentrations, all occurring simultaneously in the flowing layer of the heap. The solid lines in Fig. 12 are based on the model, the data points represent DEM results, and the dashed lines represent least squares fits through the data points. Overall, the segregation velocities of both bounded heap flows are reasonably well predicted by the model. However, since segregation in heap flows mainly occurs in a thin layer at the free surface, the averages are more uncertain, so the data have substantial scatter, especially for the weakest segregation case with \(R_p = 1.84\) and \(d = 3\) mm in Fig. 12(a). The case with a small particle diameter \(d = 2\) mm but large density ratio \(R_p = 3.3\) has overall larger \(w_{p,i}\) such that the DEM results match the theoretical predictions better, as shown in Fig. 12(b).

As Fig. 12 shows, the segregation model we propose in this paper works reasonably well not only for the confined shear flow between two planes from which it is developed, but also for free surface flow in a bounded heap. Previous results indicate that the empirically determined segregation length scale \(h_{c_l}\) for bounded heap flow with inlet concentration \(c_l = c_h = 0.5\). Equations (10) and (11) have this form when \(\hat{S}_D\), the segregation length scale, is

\[
\frac{\hat{S}_D}{d} = \left[\frac{1}{B(\mu)\phi_{solid}}(R_p - 1)\sqrt{\frac{c_l}{c_h}\rho_{bulk}gd}{\rho_{p,i}}}\right]^{1/2}.
\]

(20)

Previous results for free surface flows indicate that the empirically determined segregation length scale \(S_D\) is well approximated by

\[
\frac{S_D}{d} = C_D\ln R_p,
\]

(21)

where \(C_D = 0.081\). The empirical relation and the underlying data are shown in Fig. 13(a).

To compare \(S_D\) and \(\hat{S}_D\) it is necessary to account for the pressure and particle concentration ratio, as well as other flow conditions. The comparison is made by first assuming \(c_l/c_h = 1\), which is the feed concentration ratio in these simulations, such that \(\hat{S}_D\) is no longer concentration dependent. Furthermore, in heap flows, the segregation occurs in a thin flowing layer with a thickness of only a few particle diameters, in which the local shear rate decreases exponentially with depth while the scaled local pressure \(P'/(\rho_{bulk}gd)\) increases linearly from 1 at the surface to \(\delta/d\) at the bottom of the flowing layer at depth \(\delta\), assuming constant \(\rho_{bulk}\). The mean shear rate of the flowing layer equals the local shear rate at a depth of \(0.26\delta\) due to the exponential streamwise velocity profile. This depth corresponds to approximately \(2d\) for \(7d \leq \delta \leq 8.5d\). By assuming the exponentially varying local shear rate is the dominant factor for the depth varying segregation velocity, we expect that the effective pressure is equivalent to the pressure at a depth of about \(2d\) such that \(P'/(\rho_{bulk}gd) = 2\). For comparison, \(P'/(\rho_{bulk}gd) = 1\) and 3 are also considered.

The expression for \(\hat{S}_D\) from Eq. (20) for \(c_h/c_l = 1\) and three different pressures is compared to bounded heap flow results in Fig. 13(b). The data match the model curve well for \(P'/(\rho_{bulk}gh) = 2\). Again, the results in Fig. 13(b) demonstrate the success of the model in replicating a wide range of results even down to the slight curvature evident in the data matching that of the curve corresponding to Eq. (21).
FIG. 12: Segregation velocities from DEM heap flow simulations (data points) \[52\]. (a) $R_\rho = 1.84$, $\mu = 0.2$, $e = 0.9$, and $d = 3$ mm. (b) $R_\rho = 3.33$, $\mu = 0.2$, $e = 0.9$, and $d = 2$ mm. Dashed lines are linear fits to the data points, and solid lines are predictions of the model.

FIG. 13: Dependence of segregation length scale $S_D$ on the density ratio $R_\rho$ from DEM simulations (circles) for segregation in a bounded heap flow \[52\] compared to predictions of (a) the empirical model Eq. (21) and (b) Eq. (20) derived from the segregation velocity model with $c_h/c_l = 1$, $\mu = 0.2$ and different values of $P/(\rho_{\text{bulk}}gd)$.

V. CONCLUSION

In this paper, we have proposed a predictive model, Eqs. (10) and (11), for the segregation velocities of light and heavy particle species in density bidisperse granular flows that is based on a new model for the interspecies drag in segregating dense flows, Eq. (7). The interspecies drag model assumes that the multiple long-duration particle interactions in dense granular flows reflect similar physics to short-duration binary particle interactions typical of dilute granular flows, which have been successfully modeled using the Kinetic Theory of Granular Flows (KTGF) \[29, 30\]. Of course, particle segregation depends on the pressure-shear state, which can be characterized by the inertial number $I$. In particular, $I$ is inversely proportional to the square root of the overburden pressure, which can significantly reduce the segregation velocity \[14\]. In addition, the segregation depends non-linearly on the local particle concentration: heavy particles among many light particles segregate more quickly than light particles among many heavy particles \[20\]. Finally, particles in dense granular flows experience enduring contacts characterized by interparticle friction, which is in contrast to dilute granular flows where particle contacts are short and dominated by the elastic properties of the particles. Hence, the restitution coefficient $e$ has little influence on segregation in the
dense flows considered here.

The advantage of the proposed interspecies drag model for dense granular flows [Eq. (7)] is that it links interspecies drag to segregation velocities and includes the effects of the local flow condition, $I$, local concentration ratio, $c_h/c_l$, and particle properties including the particle densities, sizes, and surface friction, $\mu$. The segregation velocities derived from combining the interspecies drag model with the equilibrium momentum balance equation match the segregation velocities determined from DEM simulations in both confined shear and free surface heap flows. This allows calculation of segregation velocities through the depth of the flowing layer for different density ratios and relative constituent concentrations.

Despite these advances, the modified KTGF drag model proposed here is not without drawbacks. The segregation velocities predicted from Eqs. (10) and (11), while showing the right trends, are only reasonable estimates, as is evident from the scatter in the DEM simulation data from the model predictions in Figs. 10 and 12. This is likely a result of the many variables in the problem, as well as the stochastic nature of the forces on individual particles that drive the segregation and the concurrent collisional diffusion. Nevertheless, the results in Fig. 13 demonstrate a remarkable correspondence between the segregation velocity model and the simulation data, particularly since the data are for a different flow than that from which the model was derived. More research is needed under even more widely varying conditions to refine the model. Furthermore, the model is limited to density bidisperse granular materials. A more challenging problem is to connect the interspecies drag to the segregation velocity for size-bidisperse particles, or, even more difficult, particles that differ in both size and density.

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