The Early-Time Evolution of the Cosmological Perturbations in $f(R)$ Gravity

Je-An Gu$^{1,8}$, Tse-Chun Wang$^2$, Yen-Ting Wu$^{1,2,4}$, Pisin Chen$^{1,2,3,4}$, and W-Y. Pauchy Hwang$^{2,3,5,6}$

$^1$Leung Center for Cosmology and Particle Astrophysics, Department of Physics, and $^3$Graduate Institute of Astrophysics, National Taiwan University, Taipei 10617, Taiwan, R.O.C.
$^4$Kavli Institute for Particle Astrophysics and Cosmology, SLAC National Accelerator Laboratory, Menlo Park, CA 94025, U.S.A.
$^5$Asia Pacific Organization for Cosmology and Particle Astrophysics, and $^6$Center for Theoretical Physics, National Taiwan University, Taipei 10617, Taiwan, R.O.C.

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Abstract

We investigate the evolution of the linear cosmological perturbations in $f(R)$ gravity, an alternative to dark energy for explaining the late-time cosmic acceleration. We numerically calculate the early-time evolution with an approximation we contrive to solve a problem that commonly appears when one solves the full evolution equations. With the approximate evolution equations we can fairly assess the effect of the gravity modification on the early-time evolution, thereby examining the validity of the general-relativity (GR) approximation that is widely used for the early universe. In particular, we compare the CMB photon density perturbation and the matter density perturbation obtained respectively by our approximation and the conventional GR approximation. We find that the effect of the gravity modification at early times in $f(R)$ gravity may not be negligible. We conclude that to be self-consistent, in the $f(R)$ theory one should employ the approximation presented in this paper instead of that of GR in the treatment of the early-time evolution.

*Electronic address: jagu@ntu.edu.tw
†Electronic address: r98222067@ntu.edu.tw
‡Electronic address: r97222061@ntu.edu.tw
§Electronic address: pisinchen@phys.ntu.edu.tw
¶Electronic address: wyhwang@phys.ntu.edu.tw
I. INTRODUCTION

The accelerating expansion of the present universe can be explained by an energy source of anti-gravity, generally termed dark energy, or alternatively by the large-scale, low-energy modification of the gravity theory. In this paper we focus on the $f(R)$ theory of modified gravity (for a review, see [1, 2]) with the gravity action

$$ S_g = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} [R + f(R)]. \quad (1) $$

In this theory the deviation from general relativity (GR) is represented by a function of the Ricci scalar, $f(R)$, within the gravity action.$^1$

The gravitational field equations obtained from the above action are

$$ (1 + f_R)R_{\mu\nu} - \frac{1}{2}(R + f)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)f_R = 8\pi G_N T_{\mu\nu}, \quad (2) $$

where $\Box$ is the d’Alembertian. We use the notation, $f_R \equiv df/dR$ and $f_{RR} \equiv d^2f/dR^2$, in this paper.

For the background expansion of the universe we consider a homogeneous and isotropic space-time described by the flat Robertson-Walker metric:

$$ ds^2 = a^2(\tau) \left\{ -d\tau^2 + \vec{x}^2 \right\}, \quad (3) $$

where $a$ is the scale factor and $\tau$ is the conformal time. With this metric the above gravitational field equations lead to

$$ H^2 + H^2 f_R + \frac{f}{6} + \frac{H}{a} \dot{f}_R - \frac{1}{6} R f_R = \frac{8\pi G_N}{3} \rho, \quad (4) $$

$$ \left( H^2 - \frac{R}{3} \right) + f_R \left( \frac{R}{6} + H^2 \right) - \frac{f}{2} - \frac{\ddot{f}_R}{a^2} - \frac{H}{a} \dot{f}_R = 8\pi G_N P, \quad (5) $$

where the overhead dot denotes the derivative w.r.t. the conformal time $\tau$, the Hubble expansion rate $H = \dot{a}/a^2$, and $\rho$ and $P$ are the average energy density and the average pressure of the universe, for which we will consider matter and radiation.

For a given expansion history $a(\tau)$, as well as given $\rho(a)$ and $P(a)$, Eq. (4) becomes a second-order differential equation of $f(\tau)$ or $f(R)$. The functions $f(R)$ that satisfy this equation can generate the required expansion history. On the other hand, the dark energy

$^1$ We consider the metric formalism of $f(R)$ gravity and use the natural units where $c = \hbar = 1$. 
models can also generate the required expansion history by choosing an appropriate dark
energy density $\rho_{\text{de}}(a)$ and pressure $P_{\text{de}}(a)$. Consequently, measurements of cosmic expa-
sion alone cannot distinguish $f(R)$ gravity from dark energy, and additional independent
measurements such as the cosmic structures are indispensable.

For the cosmic structure formation in $f(R)$ gravity, people studied the evolution of the
cosmological perturbations [3–5]. While the evolution at late times has been widely studied
[6–8], the evolution at early times is typically treated with a simple approximation, the GR
approximation, where the deviation from GR is ignored. (For a treatment different from the
GR approximation for the early times, see [5] where the evolution from the early times to
the present is studied.)

In this paper we take into account and carefully investigate the effect of the gravity
modification in $f(R)$ gravity on the early-time evolution of the linear perturbations. When
numerically solving the full evolution equations in $f(R)$ gravity for the early times, one
is usually confronted with a tight-coupling issue. To solve this issue we contrive a better
approximation, with which we can fairly assess the effect of the gravity modification at early
times on the evolution. With those at hand, we then examine the validity of the conventional
GR approximation. In particular, we will compare the density perturbations of the CMB
photons and matter obtained respectively by our approximation and the conventional GR
approximation. We will show that the effect of the gravity modification at early times in $f(R)$
gravity may not be negligible. Accordingly, for the early-time evolution of the perturbations
in $f(R)$ gravity, the GR approximation is problematic, and a better treatment is necessary.

II. COSMOLOGICAL PERTURBATIONS IN $f(R)$ GRAVITY

For the cosmological perturbations in the early universe we analyze the evolution equa-
tions of the linear perturbations in the Fourier space and in the synchronous gauge [9]. For
the metric perturbations we consider the scalar modes, $h(\vec{k}, \tau)$ and $\eta(\vec{k}, \tau)$, defined by the
line element:

$$ds^2 = a^2(\tau) \left\{ -d\tau^2 + [\delta_{ij} + h_{ij}(\vec{x}, \tau)] dx^i dx^j \right\},$$

and the Fourier integral:

$$h_{ij}(\vec{x}, \tau) = \int d^3k e^{i\vec{k}\cdot\vec{x}} \left[ \dot{k}_i \dot{k}_j h(\vec{k}, \tau) + \left( \dot{k}_i \dot{k}_j - \frac{1}{3} \delta_{ij} \right) 6\eta(\vec{k}, \tau) \right],$$
where $\vec{k} = k\hat{k}$ and $k$ is the comoving wave number. With regard to the energy part, we consider the stress-energy perturbations of cold dark matter (CDM), baryons, photons and massless neutrinos. For each of the particle species,

$$\delta(\vec{k}, \tau) \equiv \delta \rho(\vec{k}, \tau)/\rho, \quad (8)$$

$$\theta(\vec{k}, \tau) \equiv ik^j \delta T^0_j(\vec{k}, \tau)/(\rho + P), \quad (9)$$

$$\sigma(\vec{k}, \tau) \equiv -(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\Sigma^i_j(\vec{k}, \tau)/(\rho + P), \quad (10)$$

where $\Sigma^i_j \equiv T^i_j - \delta^i_j T^k_k/3$.

For organizing the evolution equations of the above perturbed quantities, we introduce two new dynamical variables:

$$q \equiv \dot{h} + 6\dot{\eta}, \quad (11)$$

$$\chi \equiv f_{RR}\delta R_N, \quad (12)$$

where $\delta R_N$ is defined as the perturbation of the Ricci scalar in the conformal Newtonian gauge, and its relation to the metric perturbations in the synchronous gauge is:

$$\delta R_N = -\frac{6}{a^2}\ddot{\eta} - \frac{18H}{a}\dot{\eta} - \frac{4k^2}{a^2}\eta + \frac{1}{a^2}\ddot{q} + 3\left(\frac{\dddot{H}ak^2}{a^2} + \frac{3H\dddot{H}}{k^2} + \frac{H}{a}\right)q. \quad (13)$$

**A. Evolution Equations**

Since the evolution equations of the stress-energy perturbations in $f(R)$ gravity are given by the Boltzmann equations with the same form as those in GR \cite{9},\footnote{The Boltzmann equations describe the microscopic physics and therefore the form of the equations is independent of the gravity theories.} here we will simply present the evolution equations of the metric perturbations, $q$, $\eta$, and $\chi$. (The information about $h$ can be derived from that about $q$ and $\eta$.)

The $i-j$ component ($i \neq j$) of the gravitational field equations in Eq. (2) gives

$$\dot{q} = -2aHq + 2k^2\eta - \frac{2k^2\chi}{1 + f_R} - \frac{12\pi G_N a^2}{1 + f_R} \sum_{a} \rho_a \sigma_a (1 + w_a). \quad (14)$$
The linear combinations of the 0–0 and 0–j components give

\[ \dot{\eta} = \left( \frac{a^2 H^2}{k^2 f_R} \right) \left\{ \left[ \frac{k^2 (1 + f_R)}{3aH} - \frac{\dot{H} f_R}{2aH^2} \right] q - \frac{2k^4 (1 + f_R)}{3a^2 H^2} \eta + \left( \frac{k^4}{3a^2 H^2} - \frac{k^2 \dot{H}}{aH^2} \right) \chi \right\} \]

\[ - \frac{8\pi G_N a}{H} \sum_a \rho_a \theta_a (1 + w_a) - \frac{8\pi G_N k^2}{3H^2} \sum_a \rho_a \delta_a \right\}, \tag{15} \]

\[ \dot{\chi} = - \frac{aH}{k^2} \left[ \frac{\dot{f}_R}{2} + \frac{\dot{H} f_R - 4\pi G_N a}{H} \rho_{\text{eff}} (1 + w_{\text{eff}}) \right] q + \dot{f}_R \eta + 2 (1 + f_R) \dot{\eta} + \left( aH - \frac{\dot{f}_R}{1 + f_R} \right) \chi \]

\[ - \frac{8\pi G_N a^2}{k^2} \sum_a \rho_a \theta_a (1 + w_a) - \frac{12\pi G_N a^2}{k^2 (1 + f_R)} \sum_a \rho_a \sigma_a (1 + w_a). \tag{16} \]

In these three evolution equations the subscript \( a \) runs over the particle species including CDM, baryons, photons and massless neutrinos. In Eqs. (15) and (16) the effective energy density \( \rho_{\text{eff}} \) and the effective equation of state \( w_{\text{eff}} \) are defined as

\[ \rho_{\text{eff}} \equiv \frac{1}{8\pi G_N} \left( \frac{1}{2} Rf_R - 3H^2f_R - \frac{f}{2} - \frac{3H}{a} \dot{f}_R \right), \tag{17} \]

\[ w_{\text{eff}} \equiv \frac{P_{\text{eff}}}{\rho_{\text{eff}}} = - \frac{1}{3} - \frac{2}{3} \left( -\frac{1}{2\pi} \dot{\dot{f}}_R - \frac{1}{2} \dot{f} + \frac{H^2}{a} f_R \right) \left( -\frac{H}{a} \dot{f}_R - H^2f_R + \frac{1}{6} Rf_R - \frac{1}{6} f \right). \tag{18} \]

They characterize the effects of the modification of gravity at the background expansion level.

**B. A Problem of Solving the Full Evolution Equations**

One is confronted with a problem when numerically solving the above evolution equations for the early universe, particularly Eq. (15). Here we elucidate the problem.

We reorganize Eq. (15) as follows.

\[ \dot{\eta} = \left( \frac{a^2 H^2}{k^2 f_R} \right) \left\{ \left[ \frac{k^2}{3aH} q - \frac{2k^4}{3a^2 H^2} \eta - \frac{8\pi G_N a}{H} \sum_a \rho_a \theta_a (1 + w_a) - \frac{8\pi G_N k^2}{3H^2} \sum_a \rho_a \delta_a \right] \right\} \]

\[ + \left\{ \left[ \frac{k^2}{3aH} f_R - \frac{\dot{H}}{2aH^2} \dot{f}_R \right] q - \frac{2k^4}{3a^2 H^2} \dot{f}_R \eta + \left( \frac{k^4}{3a^2 H^2} - \frac{k^2 \dot{H}}{aH^2} \right) \chi \right\} \right\} / \dot{f}_R. \tag{19} \]

This equation can be read as

\[ \dot{\eta} = \left[ \text{GR-terms + } f\text{-terms} \right] / \dot{f}_R, \tag{20} \]
where the “$f$-terms” denote the terms proportional to the derivatives of $f$ (including $\chi$), and the “GR-terms” are the other terms that also appear in the GR equations. When the $f(R)$ theory is very close to GR (e.g., at early times), $\dot{f}_R$ and the $f$-terms are much smaller than the GR-terms. In this case, to correctly obtain $\dot{\eta}$ via the above equation, the summation of the GR-terms should be as small as $\dot{f}_R\dot{\eta}$. However, the error in calculating each GR term in the perturbation theory, i.e., due to ignoring high-order perturbations, can be much larger than $\dot{f}_R\dot{\eta}$, thereby making the calculation of $\dot{\eta}$ in Eq. (15) incorrect.

This problem is analogous to the issue caused by the tight coupling between photons and baryons (before the decoupling around $a \sim 10^{-3}$) that renders the evolution equations of the perturbations in the standard cosmology difficult to solve. A simple but rough solution is to invoke the approximation where baryons and photons behave like a single coupled fluid with $\dot{\theta}_\gamma = \dot{\theta}_b$. Instead, cosmologists invoke a more accurate approximation, termed “tight-coupling approximation”, to account for the slip between the photon and baryon fluids. In our case for $f(R)$ gravity at early times, the largeness of the factor $1/\dot{f}_R$ leads to the tight coupling among the GR-terms, which makes the calculation of $\dot{\eta}$ in Eq. (15) incorrect. Similarly, a simple but rough solution is to use the GR approximation in Eq. (15) such that the summation of the GR-terms vanishes and the $f$-terms and $\dot{f}_R\dot{\eta}$ are ignored. Nevertheless, to have better accuracy we contrive an approximation to account for the modification of gravity in $f(R)$ gravity.

The tightness of the coupling can be characterized by $aH/\dot{f}_R$ (for the super-horizon modes) or by $(k^2/aH)/\dot{f}_R$ (for the sub-horizon modes), where the largeness of which suggests the tightness. For demonstration we will show in Sec. III the evolution of $aH/\dot{f}_R$ in a $f(R)$ model, where $aH/\dot{f}_R \sim 10^{11}$ when $a \sim 10^{-3}$ and $\sim 10^7$ when $a \sim 10^{-2}$.

We know that the CMB observational results are consistent with GR with high precision. Therefore the allowed deviation from GR at early times must be small. Accordingly, in the viable $f(R)$ models of the late-time cosmic acceleration, $f_R$ and its derivatives should be tiny at early times, which leads to the tight-coupling issue.
C. “Tight-Coupling” Approximation

For the early-time evolution we construct a new approximation to solve the problem discussed above. In dealing with Eq. (15) we decompose \( \eta \) into two parts,

\[
\eta = \eta^{(0)} + \eta^{(1)},
\]

where \( \eta^{(0)} \) and \( \eta^{(1)} \) are designed to be comparable respectively to the GR-terms and \( f \)-terms normalized by \( H^2 \), so that Eq. (15) can also be divided into two parts which respectively lead to the evolution equations of \( \eta^{(0)} \) and \( \eta^{(1)} \) with no tight-coupling issue. For this purpose, we set

\[
\dot{\eta}^{(0)} = \frac{4\pi G_N a^2}{k^2} \sum_a \rho_a \theta_a (1 + w_a) - \frac{2\pi G_N a^2}{(1 + f_R) k^2} \rho_{\text{eff}} (1 + w_{\text{eff}}) q ,
\]

and then derive the evolution equation of \( \eta^{(1)} \):

\[
\dot{\eta}^{(1)} = \frac{1}{1 + f_R} \left[ \frac{1}{2} \dot{\chi} + \left( \frac{aH}{4k^2} \dot{f}_R + \frac{a\dot{H}}{2k^2} f_R \right) q - f_R \dot{\eta}^{(0)} - \frac{1}{2} f_R (\eta^{(0)} + \eta^{(1)}) ight.
\]

\[
- \frac{1}{2} \left( aH - \frac{\dot{f}_R}{1 + f_R} \right) \chi + \frac{6\pi G_N a^2 f_R}{k^2(1 + f_R)} \sum_a \rho_a \sigma_a (1 + w_a) \right].
\]

Solving Eq. (23) requires the information about \( \chi \) and \( \dot{\chi} \). Instead of using Eq. (16), to solve the problem we invoke the following approximation for \( \chi \):

\[
\chi \equiv f_{RR} \delta R_N \approx \chi_{\text{approx}} = -8\pi G_N f_{RR} \delta T_N = 8\pi G_N f_{RR} \sum_a \rho_a (1 - 3w_a) \delta_{N,a} ,
\]

that is, \( \delta R_N \approx -8\pi G_N \delta T_N \), where \( \delta T_N \) is the perturbation of the trace of the stress-energy tensor in the conformal Newtonian gauge, \( \delta T_N \equiv (\delta T_{\mu\nu}^\text{\&})_{N} \), and \( \delta_{N,a} \) is the density perturbation of the \( a \)-th fluid in the conformal Newtonian gauge [9].

With regard to \( \dot{\chi} \), we derive its relation to other perturbed quantities from the time derivative of Eq. (24):

\[
\dot{\chi} \approx \dot{\chi}_{\text{approx}} \equiv 8\pi G_N \sum_a \rho_a \left[ \dot{f}_{RR} - 3aH(1 + w_a)f_{RR} \right] (1 - 3w_a) \delta_{N,a} \n
\]

\[
+ 8\pi G_N f_{RR} \sum_a \rho_a (1 - 3w_a) \delta_{N,a}^{(0)} ,
\]

where

\[
\delta_{N,a}^{(0)} \equiv (1 + w_a) \left( -\theta_a - \frac{q}{2} + 3\dot{\eta}^{(0)} - \frac{3aH}{2k^2} \frac{q}{q} - \frac{3a^2 H^2}{2k^2} + 3a\dot{H} \right) ,
\]

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That is, we neglect $\eta^{(1)}$ when calculating $\dot{\delta}_{N,a}$ in the above $\chi$ relation, as the second approximation. This approximation and that in Eq. (24) are the two approximations we make in our treatment of the early-time evolution.

To calculate the early-time evolution of the perturbations in $f(R)$ gravity, we solve the set of the coupled evolution equations and relations including Eqs. (14), (21), (22), (23), (24), (25), and the Boltzmann equations. No tight coupling appears in this set of equations.

D. GR Approximation vs. Tight-Coupling Approximation

Conventionally people take the GR approximation [6, 8], where the early universe is described by the ΛCDM model, to solve the early-time evolution equations of the perturbations in $f(R)$ gravity, thereby giving an initial condition for the late-time evolution equations under the matter-domination approximation (and maybe other approximations). In many cases people solve the late-time approximate evolution equations from an initial time between $a = 0.01$ and $a = 0.03$. That is, it is widely believed that the GR approximation is valid to a high precision at least before $a = 0.01$ for most viable $f(R)$ models.

In the conventional method the effects of the modification of gravity at early times are neglected and therefore can hardly be assessed. On the contrary, our approximation takes into account the effect of the gravity modification in $f(R)$ gravity. Our approximate equations in Sec. II C go back to the evolution equations in GR when $f_R$ and $f_{RR}$ go to zero, i.e., when the effects of the gravity modification are eliminated. Therefore, the GR approximation is a limiting case of our approximation and also a rougher approximation than ours. With our approximation we can assess the effect of the gravity modification on the early-time evolution, thereby examining the validity of the GR approximation.

III. RESULTS

We compare the early-time evolution of the cosmological perturbations obtained respectively by our approximation and the GR approximation. We modify the CMBFAST code [11] to numerically solve our approximate early-time evolution equations of the cosmological perturbations in $f(R)$ gravity, while we use CMBFAST to obtain the early-time evolution under the GR approximation.
FIG. 1: The evolution of $-\dot{f}_R/aH$, $-f/H_0^2$, $-f_R$ and $m \equiv R f_{RR}/(1 + f_R)$ for the designer $f(R)$ model with $w_{\text{eff}} = -1$ and the initial condition: $f_R(a_i) = -1.3923 \times 10^{-39}$ at $a_i = 10^{-8}$.

For the purpose of demonstration, we consider a designer $f(R)$ model $^{12}$ with $w_{\text{eff}} = -1$ and the initial condition: $f_R(a_i) = -1.3923 \times 10^{-39}$ at $a_i = 10^{-8}$. This model is consistent with the observational results about the cosmic structures $^{13}$. With regard to the other cosmological parameters, we use the values suggested by the Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP7) observations $^{14}$: The effective number of neutrino species $N_{\text{eff}} = 4.34$, the mass fraction of helium $Y_{\text{He}} = 3.26$, the Hubble constant $H_0 = 73.8 \text{ km/s/Mpc}$, the baryon density fraction $\Omega_b = 0.0455$, the cold dark matter $\Omega_c = 0.226$, the effective dark energy $\Omega_{\text{eff}0} = 0.728$, and the matter-radiation equality time $z_{\text{eq}} = 4828$.

Figure 1 shows the evolution of several $f$-related quantities for the designer $f(R)$ model under consideration, including $-f/H_0^2$ and the derivatives: $-\dot{f}_R/aH$ (introduced in Sec. II B), $-f_R$ and $m \equiv R f_{RR}/(1 + f_R)$ $^{10}$. The quantity $m$ is conventionally used to characterize the deviation from GR $^{6–8}$. In this model the derivatives of $f$ grow with time from tiny values at early times to the order of unity at present, and accordingly $f$ is nearly a

$^{3}$ A designer $f(R)$ model with the effective equation of state $w_{\text{eff}}$ gives the same expansion history as that of a dark energy model with $w_{\text{DE}} = w_{\text{eff}}$. We invoke the code developed by Wei-Ting Lin to numerically calculate $f(R)$ and its derivatives for given $w_{\text{eff}}$, $f_R(a_i)$, and the values of other cosmological parameters.
constant at early times and slightly changes in the recent epoch around the value $-2\Lambda$.

We present the evolution of two Fourier modes, $k = 0.1\, \text{Mpc}^{-1}$ and $k = 0.01\, \text{Mpc}^{-1}$. In Fig. 2 we present the CMB photon density perturbation $\Theta_0$ in the synchronous gauge and its fractional difference between our approximation and the GR approximation, $|\Theta_0(\text{ours}) - \Theta_0(\text{GR})| / [|\Theta_0(\text{ours})| + |\Theta_0(\text{GR})|]$. In Fig. 3 we present the matter density perturbation $\delta$ in the conformal Newtonian gauge and its fractional difference between two approximations, $|\delta(\text{ours}) - \delta(\text{GR})| / [|\delta(\text{ours})| + |\delta(\text{GR})|]$. The gauge choice for presenting $\delta$ is made for connecting to the late-time evolution of the matter density perturbation that has been widely studied in the conformal Newtonian gauge [6–8].

In addition, we present in these two figures two relevant quantities: $c_\chi$, the fractional difference between $\chi(\text{approx})$ and $\chi$, and $c_m \equiv (aH/k)^2 m$. The fractional difference $c_\chi$ gives a criterion for the validity of our approximation, i.e., the smallness of it indicates the validity of the approximation. The quantity $c_m$ is conventionally used to give a criterion for the validity of the sub-horizon approximation in $f(R)$ gravity. One may use $c_m$ to determine the starting time of invoking the late-time, matter-dominated, sub-horizon approximate evolution equations. This starting time will also be the ending time of invoking the GR approximation if the initial condition of the late-time evolution is given from solving the early-time evolution equations with the GR approximation. While plotting $\Theta_0(\text{GR})$ and $\delta(\text{GR})$ from an early time $a = 10^{-5}$ to a late time $a = 0.1$, we plot $\Theta_0(\text{ours})$ and $\delta(\text{ours})$ till the time when $c_\chi = 0.1$ (so as to the fractional difference), before which our approximation is valid in assessing the effect of the modification of gravity in $f(R)$ gravity.

Figure 2 shows that for the Fourier mode with $k = 0.1\, \text{Mpc}^{-1}$ the fractional difference in the CMB photon density perturbation is about 1% around the photon-baryon decoupling time, $z_{\text{dec}} = 1090 (a \sim 10^{-3})$, and reaches as large as 10% around $a = 10^{-2}$. For $k = 0.1\, \text{Mpc}^{-1}$ the fractional difference is about one order of magnitude smaller: $\lesssim 0.1\%$ around the decoupling time; $\sim 1\%$ around $a = 10^{-1.5} \simeq 0.03$. This result indicates that the effect of the gravity modification at early times in the $f(R)$ theory may not be negligible compared to the accuracy of the CMB observations. With regard to the matter density perturbation in Fig. 3 for $k = 0.1\, \text{Mpc}^{-1}$ the fractional difference is about 1% around $a = 10^{-2}$, which is marginally negligible when compared to the current observational accuracy, while for $k = 0.01\, \text{Mpc}^{-1}$ it is smaller: $\lesssim 10^{-3}$ before $a = 10^{-1.5} \simeq 0.03$. 
FIG. 2: The comparison of the CMB photon density perturbations obtained respectively by our approximation $\Theta_0$(ours) and the GR approximation $\Theta_0$(GR). The upper panel is for the case where $k = 0.1 \text{Mpc}^{-1}$, and the lower panel for $k = 0.01 \text{Mpc}^{-1}$. We present the evolution of the fractional difference in $\Theta_0$ between these two approximations, as well as the fractional difference $c_\chi$ between $\chi_{(\text{approx})}$ and $\chi$ as an indicator of the validity of our approximation, and $c_m \equiv (aH/k)^2 m$ (where $m \equiv R f_R R/(1 + f_R)$) that is related to the validity of the sub-horizon approximation.
FIG. 3: The comparison of the matter density perturbations obtained respectively by our approximation $\delta$ (ours) and the GR approximation $\delta$ (GR). The upper and the lower figures are respectively for $k = 0.1 \text{ Mpc}^{-1}$ and $k = 0.01 \text{ Mpc}^{-1}$. We present the evolution of the fractional difference in $\delta$ between these two approximations, as well as the evolution of $c_\chi$ and $c_m$. 
IV. DISCUSSIONS

In this paper we numerically solve the early-time evolution equations of the linear cosmological perturbations in $f(R)$ gravity via an approximation we construct. With our approximation we can fairly assess the effect of the gravity modification in various $f(R)$ models on the early-time evolution of the perturbations, thereby examining the validity of the conventional GR approximation that neglects the deviation from GR. In particular, we obtain the evolution of the density perturbations of the CMB photons and matter, and present the fractional differences in these two quantities between our approximation and the GR approximation. This difference indicates the significance of the effect of gravity modification on the evolution of the cosmological perturbations.

We find that the effect of the gravity modification at early times in $f(R)$ gravity may not be negligible, particularly for the Fourier modes with shorter wavelengths such as $k = 0.1 \text{ Mpc}^{-1}$. Thus for self-consistency’s sake, the GR approximation is problematic, and a better treatment for the early-time evolution is necessary, which our approximation may provide. In our demonstration, even though the deviation from GR looks tiny: $m \simeq -f_R \sim 10^{-11}$ when $a \sim 10^{-3}$ and $\sim 10^{-8}$ when $a \sim 10^{-2}$, the fractional difference in the CMB photon density perturbation can reach 1% at the photon-baryon decoupling time and even 10% around $a = 10^{-2}$, which is significant compared to the accuracy of the CMB observations. That is, even a tiny deviation from GR at early times may induce a significant effect on the cosmological perturbations. This contradicts the conventional thinking. This situation is analogous to the issue about the tight coupling between photons and baryons before decoupling, which one is confronted with when solving the evolution equations of the perturbations in the standard cosmology [9].

As a consequence, the CMB observations may provide a stringent test to the currently viable $f(R)$ models, meanwhile giving tighter constraints on $f(R)$ gravity than expected, and further play an important role in distinguishing $f(R)$ gravity from dark energy.

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