A STOCHASTIC INTEGRAL OF OPERATOR-VALUED FUNCTIONS

VOLODYMYR TESKO

To Professor M. L. Gorbachuk on the occasion of his 70th birthday.

ABSTRACT. In this note we define and study a Hilbert space-valued stochastic integral of operator-valued functions with respect to Hilbert space-valued measures. We show that this integral generalizes the classical Itô stochastic integral of adapted processes with respect to normal martingales and the Itô integral in a Fock space.

1. Introduction

Here and subsequently, we fix a real number \( T > 0 \). Let \( \mathcal{H} \) be a complex Hilbert space, \( M \) be a fixed vector from \( \mathcal{H} \) and \([0, T] \ni t \mapsto E_t\) be a resolution of identity in \( \mathcal{H} \).

Consider the \( \mathcal{H} \)-valued function (abstract martingale)
\[
[0, T] \ni t \mapsto M_t := E_t M \in \mathcal{H}.
\]

In this paper we construct and study an integral
\[
(1) \quad \int_{[0,T]} A(t) dM_t
\]
for a certain class of operator-valued functions \([0, T] \ni t \mapsto A(t)\) whose values are linear operators in the space \( \mathcal{H} \). We define such an integral as an element of the Hilbert space \( \mathcal{H} \) and call it a Hilbert space-valued stochastic integral (or \( \mathcal{H} \)-stochastic integral). By analogy with the classical integration theory we first define integral (1) for a certain class of simple operator-valued functions and then extend this definition to a wider class.

We illustrate our abstract constructions with a few examples. Thus, we show that the classical Itô stochastic integral is a particular case of the \( \mathcal{H} \)-stochastic integral. Namely, let \( \mathcal{H} := L^2(\Omega, \mathcal{A}, P) \) be a space of square integrable functions on a complete probability space \((\Omega, \mathcal{A}, P)\), \( \{A_t\}_{t \in [0,T]} \) be a filtration satisfying the usual conditions and \( \{N_t\}_{t \in [0,T]} \) be a normal martingale on \((\Omega, \mathcal{A}, P)\) with respect to \( \{A_t\}_{t \in [0,T]} \), i.e.,
\[
\{N_t\}_{t \in [0,T]} \quad \text{and} \quad \{N_t^2 - t\}_{t \in [0,T]}
\]
are martingales for \( \{A_t\}_{t \in [0,T]} \). It follows from the properties of martingales that
\[
N_t = E[N_T|A_t], \quad t \in [0,T],
\]
where \( E[\cdot|A_t] \) is a conditional expectation with respect to the \( \sigma \)-algebra \( A_t \). It is well known that \( E[\cdot|A_t] \) is the orthogonal projector in the space \( L^2(\Omega, A, P) \) onto its subspace \( L^2(\Omega, A_t, P) \) and, moreover, the corresponding projector-valued function \( \mathbb{R}_+ \ni t \mapsto E_t := E[\cdot|A_t] \) is a resolution of identity in \( L^2(\Omega, A, P) \), see e.g. [13] [3] [4] [12] [7]. In this way the normal martingale \( \{N_t\}_{t \in [0,T]} \) can be interpreted as an abstract martingale, i.e.,
\[
[0, T] \ni t \mapsto N_t = E[N_T|A_t] = E_t N_T \in \mathcal{H}.
\]

Hence, in the space \( L^2(\Omega, A, P) \) we can construct the \( \mathcal{H} \)-stochastic integral with respect to the normal martingale \( N_t \). Let \( F \in L^2([0,T] \times \Omega, dt \times P) \) be a square integrable

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stochastic process adapted to the filtration \( \{ A_t \}_{t \in [0,T]} \). We consider the operator-valued function \( [0,T] \ni t \mapsto A_F(t) \) whose values are operators \( A_F(t) \) of multiplication by the function \( F(t) = F(t, \cdot) \in L^2(\Omega, A, P) \) in the space \( L^2(\Omega, A, P) \),

\[
L^2(\Omega, A, P) \ni \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, A, P).
\]

In this paper we prove that the \( H \)-stochastic integral of \( [0,T] \ni t \mapsto A_F(t) \) coincides with the classical Itô stochastic integral \( \int_{[0,T]} F(t) \, dN_t \) of \( F \). That is,

\[
\int_{[0,T]} A_F(t) \, dN_t = \int_{[0,T]} F(t) \, dN_t.
\]

In the last part of this note we show that the Itô integral in a Fock space is the \( H \)-stochastic integral and establish a connection of such an integral with the classical Itô stochastic integral. The corresponding results are given without proofs (the proofs will be given in a forthcoming publication). Note that the Itô integral in a Fock space is a useful tool in the quantum stochastic calculus, see e.g. [2] for more details.

We remark that in [3, 4] the authors gave a definition of the operator-valued stochastic integral

\[
B := \int_{[0,T]} A(t) \, dE_t
\]

for a family \( \{ A(t) \}_{t \in [0,T]} \) of commuting normal operators in \( \mathcal{H} \). Such an integral was defined using a spectral theory of commuting normal operators. It is clear that for a fixed vector \( M \in \text{Dom}(B) \subset \mathcal{H} \) the formula

\[
\int_{[0,T]} A(t) \, dM_t := \left( \int_{[0,T]} A(t) \, dE_t \right) M
\]

can be regarded as a definition of integral (1). In this way we obtain another definition of integral (1) different from the one we have proposed in this paper.

2. THE CONSTRUCTION OF THE \( H \)-STOCHASTIC INTEGRAL

Let \( \mathcal{H} \) be a complex Hilbert space, \( \mathcal{L}(\mathcal{H}) \) be a space of all bounded linear operators in \( \mathcal{H} \), \( M \neq 0 \) be a fixed vector from \( \mathcal{H} \) and

\[
[0,T] \ni t \mapsto E_t \in \mathcal{L}(\mathcal{H})
\]

be a resolution of identity in \( \mathcal{H} \), that is a right-continuous increasing family of orthogonal projections in \( \mathcal{H} \) such that \( E_T = 1 \). Note that the resolution of identity \( E \) can be regarded as a projector-valued measure \( \mathcal{B}([0,T]) \ni \alpha \mapsto E(\alpha) \in \mathcal{L}(\mathcal{H}) \) on the Borel \( \sigma \)-algebra \( \mathcal{B}([0,T]) \). Namely, for any interval \( (s,t] \subset [0,T] \) we set

\[
E((s,t]) := E_t - E_s, \quad E(\{0\}) := E_0, \quad E(\emptyset) := 0,
\]

and extend this definition to all Borel subsets of \( [0,T] \), see e.g. [6] for more details.

By definition, the \( \mathcal{H} \)-valued function

\[
[0,T] \ni t \mapsto M_t := E_t M \in \mathcal{H}
\]

is an abstract martingale in the Hilbert space \( \mathcal{H} \).

In this section we give a definition of integral (1) for a certain class of operator-valued functions with respect to the abstract martingale \( M_t \). A construction of such an integral is given step-by-step, beginning with the simplest class of operator-valued functions. Let us introduce the required class of simple functions.

For each point \( t \in [0,T] \), we denote by

\[
\mathcal{H}_M(t) := \text{span}\{ M_{s_2} - M_{s_1} \mid (s_1, s_2) \subset (t, T] \} \subset \mathcal{H}
\]

the linear span of the set \( \{ M_{s_2} - M_{s_1} \mid (s_1, s_2) \subset (t, T] \} \) in \( \mathcal{H} \) and by

\[
\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t) \to \mathcal{H})
\]
the set of all linear operators in $\mathcal{H}$ that continuously act from $\mathcal{H}_M(t)$ to $\mathcal{H}$. The increasing family $\mathcal{L}_M = \{\mathcal{L}_M(t)\}_{t \in [0,T]}$ will play here a role of the filtration $\{\mathcal{A}_t\}_{t \in [0,T]}$ in the classical martingale theory.

For a fixed $t \in [0, T)$, a linear operator $A$ in $\mathcal{H}$ will be called $\mathcal{L}_M(t)$-measurable if

(i) $A \in \mathcal{L}_M(t)$ and, for all $s \in [t, T)$,

$$\|A\|_{\mathcal{L}_M(t)} = \|A\|_{\mathcal{L}_M(s)} := \sup \left\{ \frac{\|Ag\|_{\mathcal{H}}}{\|g\|_{\mathcal{H}}} : g \in \mathcal{H}_M(s), g \neq 0 \right\}.$$ 

(ii) $A$ is partially commuting with the resolution of identity $E$. More precisely,

$$AE_s g = E_s Ag, \quad g \in \mathcal{H}_M(t), \quad s \in [t, T].$$

Such a definition of $\mathcal{L}_M(t)$-measurability is motivated by a number of reasons:

- $\mathcal{L}_M(t)$-measurability is a natural generalization of the usual $\mathcal{A}_t$-measurability in classical stochastic calculus, see Lemma 4 (Section 3) for more details;
- in some sense, $\mathcal{L}_M(t)$-measurability (for each $t$) is the minimal restriction on the behavior of a simple operator-valued function $[0, T] \ni t \mapsto A(t)$ that will allow us to obtain an analogue of the Itô isometry property (see inequality (4) below) and to extend the $H$-stochastic integral from a simple class of functions to a wider one.

In what follows, it is convenient for us to call $\mathcal{L}_M(T)$-measurable all linear operators in $\mathcal{H}$. Evidently, if a linear operator $A$ in $\mathcal{H}$ is $\mathcal{L}_M(t)$-measurable for some $t \in [0, T]$ then $A$ is $\mathcal{L}_M(s)$-measurable for all $s \in [t, T]$.

A family $\{A(t)\}_{t \in [0, T]}$ of linear operators in $\mathcal{H}$ will be called a simple $\mathcal{L}_M$-adapted operator-valued function on $[0, T]$ if, for each $t \in [0, T]$, the operator $A(t)$ is $\mathcal{L}_M(t)$-measurable and there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that

$$A(t) = \sum_{k=0}^{n-1} A_k \chi_{(t_k, t_{k+1}]}(t), \quad t \in [0, T],$$

where $\chi_{\cdot}(\cdot)$ is the characteristic function of the Borel set $\alpha \in \mathcal{B}([0, T])$.

Let $S = S(M)$ denote the space of all simple $\mathcal{L}_M$-adapted operator-valued functions on $[0, T]$. For a function $A \in S$ with representation (2) we define an $H$-stochastic integral of $A$ with respect to the abstract martingale $M_t$ through the formula

$$\int_{[0, T]} A(t) \, dM_t := \sum_{k=0}^{n-1} A_k (M_{t_{k+1}} - M_{t_k}) \in \mathcal{H}.$$ 

We can show that this definition does not depend on the choice of representation of the simple function $A$ in the space $S$.

In the space $S$ we introduce a quasinorm by setting

$$\|A\|_{S_2} := \left( \int_{[0, T]} \|A(t)\|^2_{\mathcal{L}_M(t)} \, d\mu(t) \right)^{\frac{1}{2}} := \left( \sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_M(t_k)} \mu((t_k, t_{k+1}]) \right)^{\frac{1}{2}}$$

for each $A \in S$ with representation (2). Here the measure $\mu$ is defined by the formula

$$\mathcal{B}([0, T]) \ni \alpha \mapsto \mu(\alpha) := \|M(\alpha)\|^2_M = (E(\alpha)M, M)_{\mathcal{H}} \in \mathbb{R}_+,$$

where $M(\alpha) := E(\alpha)M$ for all $\alpha \in \mathcal{B}([0, T])$, in particular,

$$M((t_k, t_{k+1}]) := E((t_k, t_{k+1}))M = M_{t_{k+1}} - M_{t_k}, \quad (t_k, t_{k+1}] \subset [0, T].$$

The following statement is fundamental.

**Theorem 1.** Let $A, B \in S$ and $a, b \in \mathbb{C}$. Then

$$\int_{[0, T]} (aA(t) + bB(t)) \, dM_t = a \int_{[0, T]} A(t) \, dM_t + b \int_{[0, T]} B(t) \, dM_t$$

for all $A, B \in S$. More generally, for a fixed $\alpha \in \mathcal{B}([0, T])$ and $A, B, C \in S$ with representation (2) and $(\alpha, \beta) \in \mathcal{B}([0, T]) \times \mathcal{B}([0, T])$ we have

$$\int_{[0, T]} (aA(t) + bB(t)) \chi_{(\alpha, \beta]}(\cdot) \, dM_t = a \int_{[0, T]} A(t) \chi_{(\alpha, \beta]}(\cdot) \, dM_t + b \int_{[0, T]} B(t) \chi_{(\alpha, \beta]}(\cdot) \, dM_t.$$
and
\begin{align*}
\left\| \int_{[0,T]} A(t) \, dM_t \right\|_{\mathcal{H}}^2 &\leq \int_{[0,T]} \|A(t)\|_{L_{\mathcal{M}}(t)}^2 \, d\mu(t).
\end{align*}
\hfill (4)

**Proof.** The first assertion is trivial.

Let us check inequality (1). Using (i), (ii) and properties of the resolution of identity $E$, for $A \in \mathcal{S}$ with representation (2), we obtain
\begin{align*}
\left\| \int_{[0,T]} A(t) \, dM_t \right\|_{\mathcal{H}}^2 &= \left( \int_{[0,T]} A(t) \, dM_t, \int_{[0,T]} A(t) \, dM_t \right)_{\mathcal{H}} \\
&= \sum_{k,m=0}^{n-1} (A_k M(\Delta_k), A_m M(\Delta_m))_{\mathcal{H}} \\
&= \sum_{k,m=0}^{n-1} (A_k E(\Delta_k)M, A_m E(\Delta_m)M)_{\mathcal{H}} \\
&= \sum_{k=0}^{n-1} (A_k E(\Delta_k)M, A_k E(\Delta_k)M)_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k M(\Delta_k)\|_{\mathcal{H}}^2 \\
&\leq \sum_{k=0}^{n-1} \|A_k\|^2_{L_{\mathcal{M}}(t_k)} \|M(\Delta_k)\|_{\mathcal{H}}^2 = \sum_{k=0}^{n-1} \|A_k\|^2_{L_{\mathcal{M}}(t_k)} d\mu(\Delta_k) \\
&= \int_{[0,T]} \|A(t)\|^2_{L_{\mathcal{M}}(t)} \, d\mu(t),
\end{align*}

where $\Delta_k := (t_k, t_{k+1}]$ for all $k \in \{0, \ldots, n - 1\}$. \hfill \Box

Inequality (1) enables us to extend the $H$-stochastic integral to operator-valued functions $[0,T] \ni t \mapsto A(t)$ which are not necessarily simple. Namely, denote by $\mathcal{S}_2 = \mathcal{S}_2(M)$ a Banach space associated with the quasinorm $\| \cdot \|_{\mathcal{S}_2}$. For its construction, it is first necessary to pass from $\mathcal{S}$ to the factor space
\[ \mathcal{S} := \mathcal{S}/\{A \in \mathcal{S} \mid \|A\|_{\mathcal{S}_2} = 0 \} \]
and then to take the completion of $\mathcal{S}$. It is not difficult to see that elements of the space $\mathcal{S}_2$ are equivalence classes of operator-valued functions on $[0,T]$ whose values are linear operators in the space $\mathcal{H}$.

An operator-valued function $[0,T] \ni t \mapsto A(t)$ will be called $H$-stochastic integrable with respect to $M_t$ if $A$ belongs to the space $\mathcal{S}_2$. It follows from the definition of the space $\mathcal{S}_2$ that for each $A \in \mathcal{S}_2$ there exists a sequence $(A_n)_{n=0}^{\infty}$ of simple operator-valued functions $A_n \in \mathcal{S}$ such that
\begin{align*}
\int_{[0,T]} \|A(t) - A_n(t)\|_{L_{\mathcal{M}}(t)}^2 \, d\mu(t) &\to 0 \quad \text{as} \quad n \to \infty.
\end{align*}
\hfill (5)

Due to (1), for such a sequence $(A_n)_{n=0}^{\infty}$, the limit
\[ \lim_{n \to \infty} \int_{[0,T]} A_n(t) \, dM_t \]
exists in $\mathcal{H}$ and does not depend on the choice of the sequence $(A_n)_{n=0}^{\infty} \subset \mathcal{S}$ satisfying (5). We denote this limit by
\[ \int_{[0,T]} A(t) \, dM_t := \lim_{n \to \infty} \int_{[0,T]} A_n(t) \, dM_t \]
and call it the $H$-stochastic integral of $A \in \mathcal{S}_2$ with respect to the abstract martingale $M_t$. It is clear that for all $A \in \mathcal{S}_2$ the assertions of Theorem 1 still hold.
Note one simple property of the integral introduced above. Let $U$ be some unitary operator acting from $\mathcal{H}$ onto another complex Hilbert space $\mathcal{K}$. Then $[0, T] \ni t \mapsto G_t := UM_t \in \mathcal{K}$ is an abstract martingale in the space $\mathcal{K}$ because, for any $t \in [0, T]$, $G_t = UM_t = X_tG, \quad X_t := UE_tU^{-1}, \quad G := UM \in \mathcal{K},$ and $X_t$ is a resolution of identity in the space $\mathcal{K}$.

Let an operator-valued function $[0, T] \ni t \mapsto A(t)$ be $H$-stochastic integrable with respect to $M_t$. One can show that the operator-valued function $[0, T] \ni t \mapsto UA(t)U^{-1}$ is $H$-stochastic integrable with respect to $G_t$ and $U \left( \int_{[0, T]} A(t) \, dM_t \right) = \int_{[0, T]} UA(t)U^{-1} \, dG_t \in \mathcal{K}$.

3. The Itô stochastic integral as an $H$-stochastic integral

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\{A_t\}_{t \in [0, T]}$ be a right continuous filtration. Suppose that the $\sigma$-algebra $\mathcal{A}_0$ contains all $P$-null sets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_T$. Moreover, we assume that $\mathcal{A}_0$ is trivial, i.e., every set $\alpha \in \mathcal{A}_0$ has probability 0 or 1.

Let $N = \{N_t\}_{t \in [0, T]}$ be a normal martingale on $(\Omega, \mathcal{A}, P)$ with respect to $\{A_t\}_{t \in [0, T]}$. That is, $N_t \in L^2(\Omega, \mathcal{A}, P)$ for all $t \in [0, T]$ and

$$\mathbb{E}[N_t - N_s | A_a] = 0, \quad \mathbb{E}[(N_t - N_s)^2 | A_a] = t - s$$

for all $s, t \in [0, T]$ such that $s < t$. Without loss of generality one can assume that $N_0 = 0$. Note that there are many examples of normal martingales, — the Brownian motion, the compensated Poisson process, the Azéma martingales and others, see for instance [10] [8] [12].

We will denote by $L^2_a([0, T] \times \Omega)$ the set of all functions (equivalence classes), adapted to the filtration $\{A_t\}_{t \in [0, T]}$, from the space

$$L^2([0, T] \times \Omega) := L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{A}, dt \times P)$$

where $dt$ is the Lebesgue measure on $\mathcal{B}([0, T])$.

Let us show that the Itô stochastic integral $\int_{[0, T]} F(t) \, dN_t$ of $F \in L^2_a([0, T] \times \Omega)$ with respect to the normal martingale $N$ can be considered as an $H$-stochastic integral (see e.g. [15] [10] for the definition and properties of the classical Itô integral). To this end, we set $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ and consider, in this space, the resolution of identity $[0, T] \ni t \mapsto E_t := \mathbb{E}[\cdot | A_t] \in \mathcal{L}(\mathcal{H})$ generated by the filtration $\{A_t\}_{t \in [0, T]}$. Let $M := N_T \in L^2(\Omega, \mathcal{A}, P)$, then the corresponding abstract martingale $[0, T] \ni t \mapsto N_t := E_tN_T = \mathbb{E}[N_T | A_t] \in \mathcal{H}$ is our normal martingale. Note also that $\mu([0, t]) = \|N([0, t])\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|N_t\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \mathbb{E}[N_T^2] = \mathbb{E}[N_T^2 | A_0] = t$.

In the context of this section, $\mathcal{L}_M(t)$-measurability is equivalent to the usual $\mathcal{A}_t$-measurability. More precisely, the following result holds.

**Lemma 1.** Let $t \in [0, T)$. For given $F \in L^2(\Omega, \mathcal{A}, P)$ the operator $A_F$ of multiplication by the function $F$ in the space $L^2(\Omega, \mathcal{A}, P)$ is $\mathcal{L}_N(t)$-measurable if and only if the function $F$ is $\mathcal{A}_t$-measurable, i.e., $F = \mathbb{E}[F | A_t]$. Moreover, if $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function then

$$\|A_F\|_{\mathcal{L}_N(t)} = \|A_F\|_{\mathcal{L}_N(s)} = \|F\|_{L^2(\Omega, \mathcal{A}, P)}, \quad s \in [t, T].$$
Proof. Suppose $F \in L^2(\Omega, A, P)$ is an $\mathcal{A}_t$-measurable function. Let us show that the operator $A_F$ is $\mathcal{L}_N(t)$-measurable.

First, we prove that $A_F \in \mathcal{L}_N(t)$. Taking into account that $F$ is an $\mathcal{A}_t$-measurable function, $\{N_t\}_{t \in [0, T]}$ is the normal martingale and the $\sigma$-algebra $\mathcal{A}_0$ is trivial, for any interval $(s_1, s_2) \subset (t, T)$, we obtain

$$
\|A_F (N_{s_2} - N_{s_1})\|_{L^2(\Omega, A, P)}^2 = E[|F(N_{s_2} - N_{s_1})|^2] = E[E[|F|^2 | A_0]] = E[E[|F|^2(s_{s_1})] = E[E[|F|^2 | s_{s_1})]
$$

We can similarly show that

$$
\|A_F G\|_{L^2(\Omega, A, P)}^2 = \|F\|_{L^2(\Omega, A, P)}^2 \|G\|_{L^2(\Omega, A, P)}^2
$$

for all $G \in \mathcal{H}_N(t) = \text{span}\{N_{s_2} - N_{s_1} | (s_1, s_2) \subset (t, T)\}$. Hence $A_F \in \mathcal{L}_N(t)$ and, moreover, equality (7) takes place.

Let us check that $A_F$ is partially commuting with $E$, i.e.,

$$
A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t), \quad s \in [t, T].
$$

Since $F \in L^2(\Omega, A, P)$ is an $\mathcal{A}_t$-measurable function and $FG \in L^2(\Omega, A, P)$, for any $s \in [t, T]$ and any function $G \in \mathcal{H}_N(t)$, we have

$$
A_F E_s G = F E_s G = FE[G|\mathcal{A}_s] = E[FG|\mathcal{A}_s] = E_s A_F G.
$$

Thus, the first part of the lemma is proved.

Let us prove the converse statement of the lemma: if for a given $F \in L^2(\Omega, A, P)$ the operator $A_F$ is $\mathcal{L}_N(t)$-measurable then $F$ is an $\mathcal{A}_t$-measurable function.

Since $A_F$ is an $\mathcal{L}_N(t)$-measurable operator, we see that for any $s \in [t, T]

$$
A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t),
$$

or, equivalently,

$$
A_F E[G|\mathcal{A}_s] = E[A_F G|\mathcal{A}_s], \quad G \in \mathcal{H}_N(t). \tag{7}
$$

Let $s \in (t, T)$ and $(s_1, s_2) \subset (t, s]$. We take

$$
G := N_{s_2} - N_{s_1} \in \mathcal{H}_N(t).
$$

Evidently, $G$ is an $\mathcal{A}_t$-measurable function and

$$
A_F E[G|\mathcal{A}_s] = A_F G = FG, \quad E[A_F G|\mathcal{A}_s] = E[FG|\mathcal{A}_s] = GE[F|\mathcal{A}_s].
$$

Hence, using (7), we obtain

$$
FG = GE[F|\mathcal{A}_s].
$$

As a result,

$$
F = E[F|\mathcal{A}_s], \quad s \in (t, T).
$$

Since the resolution of identity $[0, T] \ni s \mapsto E_s = E[\cdot|\mathcal{A}_s] \in \mathcal{L}(\mathcal{H})$ is a right-continuous function, the latter equality still holds for $s = t$, and therefore $F$ is an $\mathcal{A}_t$-measurable function. \hfill \Box

As a simple consequence of Lemma 7 we obtain the following result.

**Theorem 2.** Let $F$ belong to $L^2([0, T] \times \Omega)$. The family $\{A_F(t)\}_{t \in [0, T]}$ of the operators $A_F(t)$ of multiplication by $F(t) = F(t, \cdot) \in L^2(\Omega, A, P)$ in the space $L^2(\Omega, A, P)$,

$$
L^2(\Omega, A, P) \ni \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, A, P),
$$

is $H$-stochastic integrable with respect to the normal martingale $N$ (i.e. belongs to $S_2$) if and only if $F$ belongs to the space $L^2_{\mathcal{A}}([0, T] \times \Omega)$. 


The next theorem shows that the Itô stochastic integral with respect to the normal martingale $N$ can be interpreted as an $H$-stochastic integral.

**Theorem 3.** Let $F \in L^2_a([0,T] \times \Omega)$ and $\{A_F(t)\}_{t \in [0,T]}$ be the corresponding family of the operators $A_F(t)$ of multiplication by $F(t)$ in the space $L^2(\Omega, A, P)$. Then

$$\int_{[0,T]} A_F(t) \, dN_t = \int_{[0,T]} F(t) \, dN_t.$$  

**Proof.** Taking into account Theorem 2, Lemma 1 and the definitions of the integrals

$$\int_{[0,T]} A_F(t) \, dN_t \quad \text{and} \quad \int_{[0,T]} F(t) \, dN_t,$$

it is sufficient to prove Theorem 3 for simple functions $F \in L^2_a([0,T] \times \Omega)$. But in this case Theorem 3 is obvious. \hfill $\square$

4. The Itô integral in a Fock space as an $H$-stochastic integral

Let us recall the definition of the Itô integral in a Fock space, see e.g. [2] for more details. We denote by $\mathcal{F}$ the symmetric Fock space over the real separable Hilbert space $L^2([0,T]) := L^2([0,T], dt)$. By definition (see e.g. [3]),

$$\mathcal{F} := \bigoplus_{n=0}^\infty \mathcal{F}_n n!,$$

where $\mathcal{F}_n := \mathbb{C}$ and, for each $n \in \mathbb{N}$, $\mathcal{F}_n := (L^2_a([0,T]))^{\otimes n}$ is an $n$-th symmetric tensor power $\otimes$ of the complex Hilbert space $L^2_a([0,T])$. Thus, the Fock space $\mathcal{F}$ is the complex Hilbert space of sequences $f = (f_n)_{n=0}^\infty$ such that $f_n \in \mathcal{F}_n$ and

$$\|f\|^2_{\mathcal{F}} = \sum_{n=0}^\infty \|f_n\|^2_{\mathcal{F}_n} n! < \infty.$$

We denote by $L^2([0,T]; \mathcal{F})$ the Hilbert space of all $\mathcal{F}$-valued functions

$$[0,T] \ni t \mapsto f(t) \in \mathcal{F}, \quad \|f\|_{L^2([0,T]; \mathcal{F})} := \left( \int_{[0,T]} \|f(t)\|^2_{\mathcal{F}} \, dt \right)^{\frac{1}{2}} < \infty$$

with the corresponding scalar product. A function $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2([0,T]; \mathcal{F})$ is called Itô integrable if, for almost all $t \in [0,T]$,

$$f(t) = (f_0(t), x_{0,t}, f_1(t), \ldots, x_{0,t}^n f_n(t), \ldots).$$

We denote by $L^2_a([0,T]; \mathcal{F})$ the set of all Itô integrable functions.

Let $f$ belong to the space $L^2_a([0,T]; \mathcal{F})$ of all simple Itô integrable functions. That is, $f$ belongs to $L^2_a([0,T]; \mathcal{F})$ and there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that

$$f(t) = \sum_{k=0}^{n-1} f(k) \otimes (x_{t_k,t_{k+1}})(t) \in \mathcal{F}$$

for almost all $t \in [0,T]$. The Itô integral $\mathbb{I}(f)$ of such a function $f$ is defined by the formula

$$\mathbb{I}(f) := \sum_{k=0}^{n-1} f(k) \diamond (0, x_{t_k,t_{k+1}}, 0, 0, \ldots) \in \mathcal{F},$$

where the symbol $\diamond$ denotes the Wick product in the Fock space $\mathcal{F}$. Let us recall that for given $f = (f_n)_{n=0}^\infty$ and $g = (g_n)_{n=0}^\infty$ from $\mathcal{F}$ the Wick product $f \diamond g$ is defined by

$$f \diamond g := \left( \sum_{m=0}^n f_m \otimes g_{n-m} \right)_{n=0}^\infty,$$
provided that the latter sequence belongs to the Fock space $\mathcal{F}$.

The Itô integral $\mathbb{I}(f)$ of a simple function $f \in L^2_{\text{a.s.}}([0, T]; \mathcal{F})$ has the isometry property
\[
\|\mathbb{I}(f)\|_{\mathcal{F}}^2 = \int_{[0, T]} \|f(t)\|_{\mathcal{F}}^2 \, dt,
\]
see e.g. [2] 1. Hence, extending the mapping
\[
L^2_{\text{a}}([0, T]; \mathcal{F}) \ni L^2_{\text{a.s.}}([0, T]; \mathcal{F}) \ni f \mapsto \mathbb{I}(f) \in \mathcal{F}
\]
by continuity we obtain a definition of the Itô integral for each $f \in L^2_{\text{a}}([0, T]; \mathcal{F})$ (we keep the same notation $\mathbb{I}$ for the extension).

Let us show that the Itô integral $\mathbb{I}(f)$ of $f \in L^2_{\text{a}}([0, T]; \mathcal{F})$ can be considered as an $H$-stochastic integral. To do this we set $H := \mathcal{F}$ and consider in this space the resolution of identity
\[
[0, T] \ni t \mapsto \mathcal{X}_t f := (f_0, \mathcal{X}_{[0, t]} f_1, \ldots, \mathcal{X}_{[0, t]} f_n, \ldots) \in \mathcal{L}(\mathcal{F}), \quad f = (f_n)_{n=0}^{\infty} \in \mathcal{F}.
\]
Let $Z := (0, 1, 0, 0, \ldots) \in \mathcal{F}$ and
\[
[0, T] \ni t \mapsto Z_t := \mathcal{X}_t Z = (0, \mathcal{X}_{[0, t]} 0, 0, 0, \ldots) \in \mathcal{F}
\]
be the corresponding abstract martingale in the Fock space $\mathcal{F}$. Note that now
\[
\mu([0, t]) := \|Z_t\|_{\mathcal{F}}^2 = \|\mathcal{X}_{[0, t]} 0\|_{L^2_{\text{a.s.}}([0, t])}^2 = t, \quad t \in [0, T],
\]
i.e., $\mu$ is the Lebesgue measure on $B([0, T])$.

We have the following analogues of Theorems 2 and 3.

**Theorem 4.** A function $f \in L^2_{\text{a}}([0, T]; \mathcal{F})$ belongs to the space $L^2_{\text{a.s.}}([0, T]; \mathcal{F})$ if and only if the corresponding operator-valued function $[0, T] \ni t \mapsto A_f(t)$ whose values are operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$,
\[
\mathcal{F} \ni \text{Dom}(A_f(t)) \ni g \mapsto A_f(t)g := f(t) \circ g \in \mathcal{F},
\]
belongs to the space $S_2$.

**Theorem 5.** Let $f \in L^2_{\text{a}}([0, T]; \mathcal{F})$ and $\{A_f(t)\}_{t \in [0, T]}$ be the corresponding family of the operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$. Then
\[
\mathbb{I}(f) = \int_{[0, T]} A_f(t) \, dZ_t.
\]

Taking into account Theorem 5 in what follows we will denote the Itô integral $\mathbb{I}(f)$ of $f \in L^2_{\text{a}}([0, T]; \mathcal{F})$ by $\int_{[0, T]} f(t) \, dZ_t$. Note that this integral can be expressed in terms of the Fock space $\mathcal{F}$. Namely, for any $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2_{\text{a}}([0, T]; \mathcal{F})$, we have
\[
\int_{[0, T]} f(t) \, dZ_t = (0, \tilde{f}_1, \ldots, \tilde{f}_n, \ldots) \in \mathcal{F},
\]
where, for each $n \in \mathbb{N}$ and almost all $(t_1, \ldots, t_n) \in [0, T]^n$,
\[
\tilde{f}_n(t_1, \ldots, t_n) := \frac{1}{n} \sum_{k=1}^{n} f_{n-1}(t_k; t_1, \ldots, \hat{t}_k, \ldots, t_n),
\]
i.e., $\tilde{f}_n$ is the symmetrization of $f_{n-1}(t_1, \ldots, t_{n-1})$ with respect to $n$ variables.

5. A CONNECTION BETWEEN THE CLASSICAL ITÔ INTEGRAL
AND THE ITÔ INTEGRAL IN THE FOCK SPACE

As before, let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t \in [0, T]}$, $\mathcal{A}_0$ be the trivial $\sigma$-algebra containing all $P$-null sets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_T$. 


Let \( N = \{ N_t \}_{t \in [0,T]} \) be a normal martingale on \((\Omega, \mathcal{A}, P)\) with respect to \( \{ \mathcal{A}_t \}_{t \in [0,T]} \) and \( N_0 = 0 \). It is known that the mapping

\[
\mathcal{F} \ni f = (f_n)_{n=0}^\infty \mapsto I f : = \sum_{n=0}^\infty I_n(f_n) \in L^2(\Omega, \mathcal{A}, P)
\]

is well-defined and isometric. Here \( I_0(f_0) = f_0 \) and, for each \( n \in \mathbb{N} \),

\[
I_n(f_n) := n! \int_0^T \int_0^{t_n} \cdots \left( \int_0^{t_2} f_n(t_1, \ldots, t_n) \, dN_{t_1} \right) \cdots \, dN_{t_{n-1}} \, dN_{t_n}
\]

is an \( n \)-iterated Itô integral with respect to \( N \). We suppose that the normal martingale \( N \) has the chaotic representation property (CRP). In other words, we assume that the mapping \( I : \mathcal{F} \rightarrow L^2(\Omega, \mathcal{A}, P) \) is a unitary. Note that

\[
N_t = \int Z \, dN_t \in L^2(\Omega, \mathcal{A}, P), \quad t \in [0,T],
\]

i.e., \( N \) is the \( I \)-image of the abstract martingale \([0,T] \ni t \mapsto Z_t = (0, \xi_{[0,t]}, 0, 0, \ldots) \in \mathcal{F} \).

The Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP; see e.g. [10, 11].

We note that the spaces \( L^2([0,T] \times \Omega) \) and \( L^2([0,T]; \mathcal{F}) \) can be understood as the tensor products \( L^2([0,T]) \otimes L^2(\Omega, \mathcal{A}, P) \) and \( L^2([0,T]) \otimes \mathcal{F} \), respectively. Therefore,

\[
1 \otimes I : L^2([0,T]; \mathcal{F}) \rightarrow L^2([0,T] \times \Omega)
\]

is a unitary operator.

The next result gives a relationship between the classical Itô integral with respect to the normal martingale with CRP and the Itô integral in the Fock space \( \mathcal{F} \).

**Theorem 6.** We have

\[
L^2_a([0,T] \times \Omega) = (1 \otimes I)L^2_a([0,T]; \mathcal{F})
\]

and, for arbitrary \( f \in L^2_a([0,T]; \mathcal{F}) \),

\[
I \left( \int_{[0,T]} f(t) \, dZ_t \right) = \int_{[0,T]} I f(t) \, dN_t.
\]

Since \( N \) has CRP, for any \( F \in L^2_a([0,T] \times \Omega) \) there exists a uniquely defined vector \( f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2_a([0,T]; \mathcal{F}) \) such that

\[
F(t) = I f(t) = \sum_{n=0}^\infty I_n(f_n(t))
\]

for almost all \( t \in [0,T] \). Hence, using Theorem 6 and equality 8 we obtain

\[
\int_{[0,T]} F(t) \, dN_t = I \left( \int_{[0,T]} f(t) \, dZ_t \right) = \sum_{n=1}^\infty I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P).
\]

It should be noticed that the right hand side of the latter equality was used by Hitsuda [9] and Skorohod [14] to define an extension of the Itô integral. Namely, a function

\[
F(\cdot) = \sum_{n=0}^\infty I_n(f_n(\cdot)) \in L^2([0,T] \times \Omega)
\]

is Hitsuda-Skorohod integrable if and only if

\[
\sum_{n=1}^\infty I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P) \quad \text{or, equivalently,} \quad \sum_{n=1}^\infty \| \hat{f}_n \|^2 \cdot n! < \infty.
\]

The corresponding *Hitsuda-Skorohod integral* \( \mathbb{I}_{HS}(F) \) of \( F \) is defined by the formula

\[
\mathbb{I}_{HS}(F) := \sum_{n=1}^\infty I_n(\hat{f}_n).
\]
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INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3 TERESHCHENKIVSKA, KYIV, 01601, UKRAINE

E-mail address: tesko@imath.kiev.ua

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