Finite volume treatment of $\pi\pi$ scattering in the $\rho$ channel

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We make a theoretical study of $\pi\pi$ scattering with quantum numbers $J^{PC} = 1^{−−}$ in a finite box. To calculate physical observables for infinite volume from lattice QCD, the finite box dependence of the potentials is not usually considered. We quantify such effects by means of two different approaches for vector-isovector $\pi\pi$ scattering based on Unitarized Chiral Perturbation Theory results: the Inverse Amplitude Method and another one based on the N/D method. We take into account finite box effects stemming from higher orders through loops in the crossed $t$, $u$–channels as well as from the renormalization of the coupling constants. The main conclusion is that for $\pi\pi$ phase shifts in the isovector channel one can safely apply Lüscher based methods for finite box sizes of $L$ greater than $2m_{\pi}^{-1}$.

I. INTRODUCTION

Along years significant efforts have been devoted to determining the hadron spectrum with lattice QCD calculations [1–20]. We refer to Ref. [21] for a recent review on the different methods and results. A crucial feature of the lattice QCD studies is that one must extrapolate from the finite box used in the lattice calculations to infinite space in order to obtain physical observables. To this aim, one of the tools most widely used is the Lüscher’s approach [22, 23]. At the practical level, the latter method was recently rederived in a simpler way in Ref. [24] by employing general expressions for two-body scattering from Ref. [25]. It is worth stressing that the method of Ref. [24] keeps the relativistically covariant two-body propagator. This method has been applied in Refs. [26–31] to calculate physical observables from possible calculations of different hadronic processes in a finite box as well as associated errors.

However, a common feature of Refs. [22–24] is that they do not take into account the finite size dependence of potentials from which the scattering in infinite volume is obtained. This simplification is based on the result of Ref. [22] that the volume dependence of the potentials is “exponentially suppressed” with the box volume. In Ref. [30] the effects of the finite size in the isoscalar and isotensor $\pi\pi$ $S$-wave scattering channels were studied quantitatively. An estimation was given of the minimum box size of possible lattice calculations so that the errors coming from these finite size effects were negligible.

Following a similar procedure as in Ref. [30], the main aim of the present work is to analyze the effects of finite volume for $1^{−−}$ $\pi\pi$ scattering in order to extract physical observables for infinite volume from lattice QCD. This channel is of special relevance since it has the quantum numbers of the resonance $\rho (770)$ whose study in lattice QCD has attracted special attention in the last decade. Several works have used Lüscher’s method to extract physical observables in infinite volume from actual lattice QCD data [4, 32–38]. The method of Ref. [24] was applied in Ref. [39] in order to theoretically illustrate the improvement of using the method of Ref. [24] for determining $\pi\pi$ scattering in the $\rho$ channel from finite box calculations over the standard Lüscher’s method. In Ref. [39] the potential for the $\pi\pi$ scattering in $I = 1$ and $P$-wave was obtained from Ref. [40] which considered as dynamical input tree level diagrams from lowest order Chiral Perturbation Theory (ChPT) with explicit exchange of vector resonances. In Ref. [39] the potential is not modified in the finite box and, hence, the finite volume effects come only from the discretization in the box of the unitarization loops. In the present work we use the amplitudes from the Inverse Amplitude Method (IAM) and another method based on an approximate algebraic solution to the N/D method that we refer as the N/D$_A$ in the following. In both cases the amplitudes include loops in the $t$– and $u$– channels, which indeed

1 The potentials are analytical functions without the elastic two-body unitarity cut.
get modified in the finite volume. This provides two different volume dependent amplitudes for the isovector $\pi\pi$ $P$-wave scattering. We can then study the impact of volume dependence from these amplitudes in the process of extracting physical observables in the infinite volume, like phase shifts, from scattering energy levels obtained in the finite box. The error made by neglecting possible volume dependence of the potential in the direct Lüscher’s analysis of the data is discussed.

II. FORMALISM

For the study of the finite volume effects in $\pi\pi$ interactions in the $\rho$ channel, we will consider two approaches, the $N/D_A$ method and the IAM, whose dynamical input is obtained from the $SU(2)$ chiral amplitudes at $\mathcal{O}(p^4)$.

These chiral amplitudes are discussed in Subsec. II A, whereas the amplitudes involving $\rho$ exchanges are considered in Subsec. II B. In finite volume, since the loop integrals are replaced with sums over discrete momenta, these amplitudes are volume dependent. The finite volume effects on the amplitudes are treated in Subsec. II C. In Subsecs. II D and II E we discuss the $N/D_A$ and the IAM methods, respectively, and show how they make use of the one-loop perturbative chiral amplitudes discussed before. The amplitudes depend on some free parameters, that are fixed in Subsec. II F by reproducing several sets of data.

A. Chiral amplitudes at $\mathcal{O}(p^4)$

Let us denote the different $\pi\pi$ partial waves of isospin $I$ and angular momentum $J$ by $T^{IJ}(s)$. These amplitudes are normalized such that:

$$T^{IJ}(s) = -\frac{8\pi\sqrt{s}}{p} \frac{1}{\cot \delta^{IJ}(s) - i},$$

where $p = \sqrt{s}/4 - m_\pi^2$ is the CM momentum of each pion (with mass $m_\pi$), and $\delta^{IJ}(s)$ is the partial wave phase shift. These amplitudes can be calculated perturbatively as the projection into angular momentum $J$ wave of the different isospin amplitudes $A^{I}(s,t,u)$, that are computed in $SU(2)$ ChPT [42]. Here, $s$, $t$, and $u$ are the usual Mandelstam variables. We denote by $A^{I}_{2n}(s,t,u)$ the $\mathcal{O}(p^{2n})$ contribution to this amplitude, and by $T^{IJ}_{2n}(s)$ its projection into the partial wave with angular momentum $J$. Both the IAM and the $N/D_A$ methods, as we shall see below, make use of the amplitudes $T^{IJ}_{2n}(s)$ to calculate the partial waves $T^{IJ}(s)$ in a non-perturbative manner, in the sense that they resum the unitarity cut. (In what follows, the superscript $IJ$ is dropped to simplify notation.)

The amplitude $A^4_4$ can be written in a generic way, for the different isospin channels, as:

$$A^4_4(s,t,u) = P_L + P_H H(n^2) + P_{G,s} G(s) + P_{G,t} G(t) + P_{G,u} G(u).$$

(2)

The functions $P_X$ above are polynomials of the Mandelstam variables. In particular, the low energy constants (LECs) $\tilde{l}_i$ appear just in the term $P_L$.\footnote{We work here with the finite and scale-independent LECs, $\tilde{l}_i$. In the case of $\pi\pi$ scattering, only the LECs with $i = 1, \ldots, 4$ are involved.} In Eq. (2), $H$ and $G(P^2)$ are the one– and two–point one–loop functions, respectively, given by:

$$G(P^2) = \int \frac{d^3q}{(2\pi)^3} I(q, P),$$

(3)

$$I(q, P) = \frac{(\omega_q + \omega_{\bar{P} - q})/(2\omega_q \omega_{\bar{P} - q})}{(P_0 - \omega_q - \omega_{\bar{P} - q})(P_0 + \omega_q + \omega_{\bar{P} - q})},$$

(4)

$$H = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q},$$

(5)

where $P$ is the four–momentum entering the loop. Whence, $G(s)$, $G(t)$ and $G(u)$ in Eq. (2) stem from the $s$–, $t$– and $u$–channel loops (3) with $P^2 = s$, $t$ and $u$, respectively. These functions are regularized through dimensional regularization. After the divergences and scale dependencies are absorbed in the LECs [42], the loop function, denoted now $G^D$, reads:

$$G^D(P^2) = \frac{1}{16\pi^2} \left( -1 + \sigma(P^2) \log \frac{1 + \sigma(P^2)}{1 - \sigma(P^2)} \right),$$

(6)

with $\sigma(P^2) = \sqrt{1 - 4m_\pi^2/P^2}$. On the other hand, because of the regularization approach followed, we have $H^D = 0$.

B. Chiral amplitudes and the $\rho$ meson

From a broader point of view, one can include, together with the $\mathcal{O}(p^4)$ amplitudes explained so far, the contribu-
tion coming from Resonance ChPT \cite{43,44}, which start to contribute at one-loop order as well. For the problem discussed here, the relevant Lagrangian is the one that incorporates exchanges of the $\rho$ meson field. This term involves, in our case, two free parameters, the bare $\rho$ mass, $M_\rho$, and the coupling constant $G_V$. This contribution is relevant in the $N/D_A$ method for the $I = 1, J = 1$ channel, since the physical $\rho$ resonance is generated by dressing the bare $\rho$ pole with $\pi\pi$ rescattering \cite{25}. We also calculate the $\rho$ exchanges in the $t$- and $u$-channels for the $I = 0, 2$ $\pi\pi$ $S$-wave channels which are also studied here for consistency. These exchanges cancel partially with the crossed channel $\pi\pi$ loops \cite{25}.

It is well known that the exchange of resonances has a large impact on the values of the LECs. The contribution of the $\rho$ resonance to these constants reads \cite{43}:

$$
\begin{align*}
\bar{\ell}^0_p &= -96\pi^2G^2 \frac{\bar{G}_I}{M^2} , \\
\bar{\ell}^2_p &= 48\pi^2G^2 \frac{\bar{G}_I}{M^2} , \\
\bar{\ell}^0_p &= \bar{\ell}^4_p = 0.
\end{align*}
$$

Since these contributions are already accounted for by the $\rho$-exchange amplitudes, we subtract them from the LECs appearing in the amplitudes to avoid double counting. This is the case when applying the unitarization scheme based on the $N/D$ method, but not in the IAM. In this latter method the physical $\rho$ is generated from the dynamics through the ChPT $O(p^4)$ LECs $\bar{\ell}_i$ \cite{45}. This makes a difference between the $N/D_A$ method and the IAM, that will show up in the different finite volume effects predicted by each approach. This point is not a shortcoming of our study, but rather it will allow us to confront both methods, so that the differences between both are considered as an estimate for uncertainties.

### C. Finite volume effects

Let us now discuss the modifications to the amplitudes explained so far when one considers the interactions in a finite cubic box of edge $L$ and with periodic boundary conditions. The chiral amplitude $A_4(s, t, u)$ receives contribution from loop integrals that, in the finite volume case, are replaced with sums over the allowed (quantized) momenta, giving rise to $\tilde{A}_4(s, t, u)$, which is the finite volume version of the former. Namely, these contribution arise from $s$-, $t$- and $u$-channel loops, and from tadpole diagrams as well. Note also that we write the amplitudes in terms of the physical pion mass $m_\pi$ and decay constant $f_\pi$, and hence the $O(p^4)$ contributions to them (tadpole loop-functions) are included as $O(p^4)$ terms in the amplitudes $A_4$ and $\tilde{A}_4$. The $O(p^4)$ contribution to the partial wave in the finite volume, that we denote by $\tilde{T}_4$, is calculated as $T_4(s)$ with the replacement $A_4 \to \tilde{A}_4(s, t, u)$. This one is calculated from Eq. (2), but replacing the loop functions in Eqs. (3) and (5) with their finite volume counterparts, $\tilde{G}^D$ and $\tilde{H}^D$. Following the procedure in Ref. \cite{27}, the finite volume loop functions are obtained from the infinite volume ones as:

$$
\tilde{G}^D(P) = G^D(P^2) + \lim_{q_{\text{max}} \to \infty} \left[ \frac{1}{L^3} \sum_{\tilde{q}_i} \frac{1}{q_i} \int_{q < q_{\text{max}}} \frac{d^3 \tilde{q}}{(2\pi)^3} I(\tilde{q}, P) \right], \quad (8)
$$

$$
\tilde{H}^D = H^D + \lim_{q_{\text{max}} \to \infty} \left[ \frac{1}{L^3} \sum_{\tilde{q}_i} \frac{1}{2\omega_{\tilde{q}}} \int_{q < q_{\text{max}}} \frac{d^3 \tilde{q}}{(2\pi)^3} \frac{1}{2\omega_{\tilde{q}}} \right], \quad (9)
$$

where $I(\tilde{q}, P)$ the integrand of Eq. (3), defined in Eq. (4). Since the box breaks Lorentz symmetry, the reference frame is fixed to the center of mass frame of the initial pions. For this reason we have used $P$ as the argument of $G^D$ in Eq. (8) instead of $P^2$. For the $s$-channel loop case, where $\vec{P} = 0$ so that $(P^0)^2 = P^2 = s$, we obtain $G^D(P)$ as:

$$
\tilde{G}^D(s) = G^D(s) + \lim_{q_{\text{max}} \to \infty} \left[ \frac{1}{L^3} \sum_{\tilde{q}_i} \frac{1}{q_i} \int_{q < q_{\text{max}}} \frac{d^3 \tilde{q}}{(2\pi)^3} I(\tilde{q}, s) \right], \quad (10)
$$

and:

$$
I(\tilde{q}, s) = \frac{1}{\omega(\tilde{q})} \frac{1}{s - 4\omega(\tilde{q})^2}. \quad (11)
$$

Let us note that $\tilde{G}^D(P)$ depends solely on $P^2 = s$ in this case. However, for the $t$-channel loop, we have $P^0 = 0$ so that $P^2 = -\vec{P}^2 = t$, and hence the integrand $I(\tilde{q}, P)$ becomes:

$$
I(\tilde{q}, P) = -\frac{1}{2\omega_{\tilde{q}}\omega_{\tilde{q} - \tilde{q}}} \frac{1}{\omega_{\tilde{q}} + \omega_{\tilde{q} - \tilde{q}}}. \quad (12)
$$

Now, contrary to the $s$-channel case, $G(P)$ depends on $P^2 = t$, but also on $\vec{P}$ and its relative orientation respect to the cubic lattice of allowed momenta in the box, $\{\tilde{q}_i\}$. This dependence translates into a dependence on the scattering angle $\theta$, present in $t = -2(s/4 - m_\pi^2)(1 - \cos \theta)$, but also on the azimuthal angle $\phi$, and this also happens with the $u$-channel case. Thus, when making the partial
wave projections in finite volume, we should now also integrate on $\phi$,
\[ \tilde{T}^{IJ}(s) = \frac{1}{4\pi} \int d\phi \int d(\cos \theta) A^I(s, \cos \theta, \phi) P_J(\cos \theta), \]
being $P_J(x)$ the Legendre polynomial of order $J$.

Finally, the function $\tilde{H}^D$ can be computed using the Poisson resummation formula (see e.g. Ref. [46]) and, since $H^D = 0$, we find:
\[ \tilde{H}^D = \frac{m_\pi}{4\pi^2 L} \sum_{0 \neq \vec{n} \in \mathbb{Z}^3} \frac{1}{|\vec{n}|} K_1(|\vec{n}|m_\pi L), \]
being $K_1$ the Bessel function.

The $s$–channel loops are responsible for the right–hand cut (RHC), or unitarity cut, that stems from the loop function $G(s)$ and it is the most important source of $L$–dependence in the amplitude. This $L$–dependence arising from the RHC is the one used by the Lüscher method to obtain phase shifts from the energy levels obtained in a finite volume. In particular, in the version of Ref. [24], given a certain energy level $E$ on a finite box of size $L$, the infinite volume amplitude is given by
\[ T^{-1}(E) = \tilde{G}^D(E) - G^D(E), \]
from where the phase shifts can be straightforwardly extracted. Note that the $T$-matrix so calculated does not depend on any regularization procedure for the loop function $G$ since its inverse is the limit of the piece between brackets in Eq. (8) (for further discussion on Eq. (15), see Ref. [47]). Eq. (15) only takes into account the $L$–dependence coming from the $s$–channel loops. However, the tadpoles and the $t$– and $u$–channel loops, which give rise to the left-hand cut (LHC) when projected into partial waves, give further dependence on $L$ (polarization corrections in the terminology of Ref. [23]) that is neglected in Eq. (15). Its effects are typically disregarded because they are exponentially suppressed [23].

D. The $N/D_A$ method

From the exact $N/D$ method [48] it was shown in Ref. [25] that when the LHC is neglected or treated perturbatively (see also Refs. [49–53]), a two-body partial wave amplitude can be written in full generality as:
\[ T^{-1}(s) = V^{-1}(s) - G^S(s), \]
where the loop function $G^S(s)$ is given by a one-subtracted dispersion relation of Eq. (3),
\[ G^S = \frac{1}{16\pi^2} \left( a(\mu) + \log \frac{m^2}{\mu^2} + \sigma(s) \log \frac{1 + \sigma(s)}{1 - \sigma(s)} \right). \]
Here, $a(\mu)$ is a subtraction constant, that depends on the renormalization scale $\mu$, taken as $\mu = 770$ MeV. The kernel $V$ is calculated by matching its chiral expansion, $V(s) = V_2(s) + V_4(s) + \cdots$, with the chiral amplitudes $T_2$ and $T_4$, that is:
\[ T(s) = \frac{V(s)}{1 - V(s)G^S(s)} = V_2 + V_4 + V_2^2 G^S + \cdots \]
\[ = T_2 + T_4 + \cdots, \]
where the ellipsis indicate $O(p^6)$ and higher orders in the expansion. It results then:
\[ V_2(s) = T_2(s), \]
\[ V_4(s) = T_4(s) - T_2(s)^2 G^S(s). \]
Note that the kernel $V_4$ has no RHC, since $T_4(s)$ contains the piece $T_2^2 G^D$ and then this cut cancels in the difference in Eq. (19). Hence, the RHC stems solely from the denominator in Eq. (18). Now, to calculate the amplitude in the finite volume, $\tilde{T}$, we must replace $V$ with $\tilde{V} = V_2 + V_4$. Since the $O(p^2)$ term does not depend on $L$, no replacement is needed on it, whereas we have:
\[ \tilde{V}_4 \rightarrow \tilde{V}_4 = \tilde{T}_4 - T_2 \tilde{G}^S, \]
with $\tilde{G}^S$ given by an analogous expression to Eq. (10) but with $G^D$ replaced by $G^S$. Notice that, according to the discussion above, the volume dependence of the $s$–channel loop does not affect the kernel $\tilde{V}_4$. Hence, the volume dependence in the amplitude $\tilde{T}$ enters through: (i) the kernel $V_4$ which volume dependence originates from tadpoles and $t$– and $u$– loop functions generating LHC, and (ii) denominator $1 - \tilde{V} \tilde{G}^S$, that gives rise to the RHC, through the function $\tilde{G}^S$. This is the most important source of volume dependence. Finally, the energy levels $E$ within a box of edges of size $L$ are given by the poles of the amplitude $\tilde{T}(s)$, $s = E^2$, that corresponds to the solution of the equation:
\[ 1 - \tilde{V} \tilde{G} = 0. \]
E. The Inverse Amplitude Method (IAM)

Now, we consider the elastic IAM [45, 54–57]. This method uses elastic unitarity and ChPT [42] to calculate the inverse of the $\pi\pi$ scattering partial wave $T^{11}$ through a dispersion relation.

In the IAM, one considers an auxiliary function, $W(s) \equiv T_2(s)^2/T(s)$, its analytic structure being a RHC from $4m^2_\pi$ to $\infty$, a LHC from $-\infty$ to $0$, and possible poles coming from zeros of the $T$ function (like Adler zeros). Hence, we can write a dispersion relation for $W$ as:

$$W(s) = W(0) + W'(0)s + \frac{1}{2}W''(0)s^2 + \frac{s^3}{\pi} \int_{RHC} ds' \frac{\text{Im } W(s')}{s^3(s' - s)} + LC(W) + PC.$$ (22)

The RHC integral can be evaluated exactly due to unitarity, whereas the subtraction constants can be evaluated with ChPT, since they involve amplitudes or derivatives of them at $s = 0$, and so $W(0) \simeq T_2(0) - T_4(0)$, $W'(0) \simeq T_3(0) - T_4(0)$, $W''(0) \simeq -T_2''(0)$. The LHC is dominated by the low energy region, due to the three subtractions, and it is also dumped by an extra $1/(s' - s)$ for physical values of $s$. Then, it can be evaluated using ChPT to obtain $LC(W) \simeq -LC(T_4)$. The pole contribution, $PC$, is formally $O(p^6)$ and we neglect it (this causes some technical problems in the subthreshold region around the Adler zeros which can be easily solved, but they do not affect the description of scattering nor resonances, for details see Ref. [58]). Taking into account all the above considerations we arrive at the simple IAM formula,

$$T = \frac{T_2}{T_2 - T_4}.$$ (23)

Hence, the amplitude in the box is obtained as $\bar{T}$ by replacing in Eq. (23) the amplitude $T_4$ with $\bar{T}_4$. The energy levels $E$ are then given by the poles of the amplitude $\bar{T}$, that is, by the solution of the equation $T_2 - \bar{T}_4 = 0$. As stated before, the infinite volume $T$-matrix can be reobtained, neglecting the left cut and tadpoles $L$-dependence, from Eq. (15).

F. Fixing the free parameters

The partial wave amplitudes, calculated in the $N/D_A$ method and in the IAM, depend on four LECs, $\bar{l}_i = 1, \ldots, 4$, and, in the case of the $N/D_A$ method, also on the subtraction constant $a(\mu)$ and the parameters related to the $\rho$ bare field, $M_\rho$ and $G_V$. These parameters are fixed by reproducing scattering data as well as lattice results. For $I = 0$, the phase shifts that we fit contain the very precise data of $K_{e4}$ decays below $\sqrt{s} = 400$ MeV [59–63]. Above that energy, the data of Ref. [64] and the average of different experiments [65–70], as used e.g. in Ref. [25], are taken into account. For $I = 1$ and $J = 1$, we use the data of Refs. [71, 72]. For $I = 2$, the data come from Refs. [73, 74]. The lattice QCD results of Refs. [75–77] on the pion mass dependence of $f_\pi$ and $a_0$, the $I = 2$ $J = 0$ scattering length, are also fitted.

We perform independent fits for the $N/D_A$ method and the IAM to the whole set of data, obtaining the parameters collected in Table I. We note here the general agreement between the determinations of both methods. The LECs also compare well with the typical values given in the literature (see e.g. Ref. [78] for comparisons and references). All the fitted data are shown in Figs. 1–3. The results of the $N/D_A$ are represented with dashed blue lines, whereas those of the IAM are shown with solid red lines. As can be see in these figures both models are compatible within the experimental uncertainties. Of course, the main aim in the present work is the relative change when passing from infinite to finite volume and not the actual values of the phase shifts at infinite volume. Nevertheless, one should take as starting point an approach that can reproduce physical data in a fair way, as this is the case for both the N/D and IAM unitarization methods. Finally, for completeness, in Table II we also give the values for the predicted pole positions and couplings of the $\sigma$ and $\rho$ mesons for both methods, that are compatible within the typical errors for these parameters (see PDG, Ref. [79], and references therein related to the $\sigma$ and $\rho$ mesons).

| Fit  | $\bar{l}_1$ | $\bar{l}_2$ | $\bar{l}_3$ | $\bar{l}_4$ | $a(\mu)$ (MeV) | $M_\rho$ (MeV) | $G_V$ (MeV) |
|------|-------------|-------------|-------------|-------------|----------------|----------------|-------------|
| $N/D_A$ | 1.7 5.9 4.7 4.0 | -1.5 | 811 | 62 |
| IAM   | -0.4 5.6 5.5 4.0 | - | - | - |

TABLE I. Values of the parameters obtained from the fit of the data for the $N/D_A$ and the IAM methods.

III. RESULTS

In Fig. 4 we show the first three energy levels for different values of the cubic box size, $L$, obtained from the poles of the $I = 1$, $J = 1$ scattering amplitudes in the fi-
we show the dashed ones represent the outcome of the solid lines stand for the IAM results, whereas the (blue) finite box as explained in the previous sections. The (red) solid line is our fit with the IAM, whereas the (blue) dashed one shows the fit with the $N/D_A$ method. The data are taken from Refs. [71, 72].

![Graph](image1)

**FIG. 1.** (Color online) Comparison of our scalar $\pi\pi$ phase shifts to experimental data for $I = 0$ (top panel) and $I = 2$ (bottom panel). The (red) solid line is our fit with the IAM, whereas the (blue) dashed one corresponds to the fit with the $N/D_A$ method. The inset in the top panel shows in more detail the low energy $K_{e4}$ decays data. The data for $I = 0$ are from the $K_{e4}$ decay data of Refs. [59–63] and other data from Refs. [64–70]. For $I = 2$ the phase shifts are from Refs. [73, 74].

| Method   | $\sqrt{s}$ (MeV) | $|g_\sigma|$ (GeV) | $\sqrt{s}$ (MeV) | $|g_\rho|$ (GeV) |
|----------|------------------|--------------------|------------------|------------------|
| $N/D_A$  | 758 – 70$\pi$    | 2.4                | 434 – 251$\iota$ | 3.0              |
| IAM      | 750 – 74$\iota$  | 2.5                | 439 – 236$\iota$ | 2.9              |

**TABLE II.** Predicted values of the $\rho$ and $\sigma$ pole positions and couplings for the $N/D_A$ and the IAM methods. The couplings are defined, for both the $S$– and $P$–wave, as $g_R^2 = \lim_{s \to R}(s – s_R^\pi) f(s)$.

For the results obtained with the level 2 the difference for both methods are small and compatible with the experimental uncertainties of the phase shifts. For the $N/D_A$ method, indeed, the phase shifts calculated from level 2 and in the infinite volume case are so similar that we only show the latter one. This means that for points coming from the level 2 (or higher) the $L$–dependence of the potential has a minor effect in the resolution of the “inverse problem”. However, if one uses points from level 1 the differences are more important for energies above 700 MeV. This effect is mild for the $N/D_A$, while it is rather strong in the case of the IAM. The reason for this different behavior between both methods is that in the

![Graph](image2)

**FIG. 2.** (Color online) Comparison of our $I = 1$ $J = 1 \pi\pi$ phase shifts to experimental data. The (red) solid line shows our fit with the IAM, whereas the (blue) dashed one shows the fit with the $N/D_A$ method. The data are taken from Refs. [71, 72].
N/D the \( \rho \) meson is included as an explicit pole with fixed bare mass. The physical \( \rho \) generates from the dressing of this bare field due to \( \pi \pi \) rescattering. Therefore, for energies close to the (bare) \( \rho \) mass this \( L \)-independent term dominates the amplitude. This is unlike the IAM case where the \( \rho \) resonance is generated from information codified in the LECs and hence gets much more affected by the \( L \) dependence of the internal dynamics that generates the amplitude.

These results are very illustrative and can be used to select the range of values of \( L \) where the finite box dependence of the potential can be safely neglected to obtain the parameters of the \( \rho(770) \) meson, or phase shifts in this channel. This is represented by the shadowed region in Fig. 4. The horizontal bounds represent the range \( m_\rho \pm \Gamma_\rho \) which would be desired in order to get the resonant shape of the \( \rho \). The left vertical bound represents the limit where the results from level 1 gives acceptable results for both methods. This corresponds approximately to energies below 720 MeV and to an \( L \) value of about 2\( m_\pi^{-1} \). This means that using lattice points from level 1, one can only obtain the same results for both methods for the \( \rho \) phase shifts for energies below 720 MeV. In order to get similar results for both methods for the right part of the \( \rho \) resonance shape one must consider points in the second level for \( L \) values between 2 – 3\( m_\pi^{-1} \).
The $N/D_A$ and IAM models considered in the present work are two sound models rooted in basic properties of strong interactions as unitarity, analyticity and chiral symmetry and have been rigorously and thoroughly tested in many processes in infinite volume. By considering the $L$-dependence that stems from such models we conclude with some confidence that if one uses points obtained from the second level of Fig. 4 in order to extract the $\rho$ meson, then it is safe to neglect such $L$-dependence in the potential and proceed via standard Lüscher method or its version of [24] to solve the inverse problem. If points are used from lattice data from level 1 then the $L$-dependence can certainly be neglected to generate the low part of the $\rho$ resonance but we cannot make the same claim for the upper part of the resonance due to the strong $L$ dependence that we obtain for the IAM method.

We now elaborate on the possible $L$ dependence of the $\rho$ coupling $G_V$. The original type of Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation [81] predicts $G_V = f_\pi/\sqrt{2}$. This relation can also be derived from the high-energy constraint of the pion vector form factor at tree level [44]. However, when the latter study is extended up to the one-loop level [82] it gives rise to the relation $G_V = f_\pi/\sqrt{3}$ that it is considered to be valid in the large $N_C$. This modified KSRF-like relation was also confirmed in other contexts: $\pi\pi$ scattering [83, 84], radiative $\tau$ decays [85], extra-dimension model for $\pi\pi$ scattering [86] and within spectral-function sum rules and semi-local duality [87]. We then consider quite reasonable to take that $G_V$ is proportional to $f_\pi$ and, from the $L$ dependence of the latter, to obtain that of $G_V$. This extra $L$ dependence is now taken into account and reads

$$G_V(L) = G_V(\infty) \left(1 - \frac{\tilde{H}_D(L)}{f_\pi^2} + \frac{m_\pi^2}{16\pi^2 f_\pi^2} \tilde{l}_4 \right) \left(1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2} \tilde{l}_4 \right).$$

One comment is in order here. We should note that the amplitudes involving $\rho$-exchange are already $O(p^4)$, and hence the above modification introduces terms of $O(p^6)$ and higher in the evaluation of $V$($s$) in Eq. (16). However, as argued above, it is of interest in order to identify sources of $L$ dependence in the resulting scattering amplitudes by using the $N/D_A$ method. Next we discuss the results that stem by considering this additional volume dependence of the amplitude. The energy levels and phase shifts obtained now are represented in Fig. 6. The upper panel, Fig. 6(a), shows by the (green) dot-dashed lines the new energy levels obtained. For comparison, we also display the original $N/D_A$ and IAM results, already shown in Fig. 4, by the (blue) dashed and (red) solid lines, respectively. One can observe that

![Fig. 6](image-url)
the new energy levels are very similar to the previous ones for \( Lm_\pi \gg 1.7 \). However, the new \( n = 1 \) energy level starts to be different from the original \( N/D_A \) one for \( Lm_\pi \lesssim 1.5 \). Indeed, for smaller values of \( L \) the novel \( n = 1 \) energy level even decreases for decreasing values of \( L \) and tends to values near the bare \( \rho \) mass \((M_\rho = 811 \text{ MeV})\). This is due to the fact that \( G_V \) becomes very small for these small values of \( L \) (for example, \( G_V(L = 1.5m_\pi^{-1}) \) is around 20 MeV, whereas \( G_V(\infty) = 62 \text{ MeV} \)). Thus, the effect of the bare \( \rho \) meson can only be felt for energy values very close to the bare \( \rho \) mass.

Although the \( O(p^6) \) contribution to \( V(s) \) considered here is only partial, it may indicate that the approach is not valid for values of \( Lm_\pi \) in the range 1.0 – 1.5. Let us recall here that these \( L \) values are far from the range \( Lm_\pi \gtrsim 2 \), confirming our previous conclusion that this lower bound can be taken as safe in order to extract reliable \( \pi \pi \) phase shifts and information about the \( \rho \) meson. In the bottom panel, Fig. 6(b), we show by the dot-dashed lines the phase shifts obtained with the new energy levels, together with the original ones, already shown in Fig. 5, resulting from the IAM (solid lines) and the \( N/D_A \) method (dashed lines). For each of these three models, the phase shifts obtained with the energy levels \( n = 1 \) and \( n = 2 \) are shown by the blue and red lines, respectively. As expected, the three approaches are rather similar for the \( n = 2 \) level. The differences between the phase shifts calculated from the \( n = 1 \) energy level obtained with the two \( N/D_A \) results (with and without volume dependence of \( G_V \) ) are large for energy values around \( E = 820 \text{ MeV} \). The deviation of the new calculated phase shifts with respect to the original ones (which were already close to the infinite volume phase shifts) for this range of energies (corresponding to values of \( Lm_\pi \) in the range 1.0 – 1.5) confirms the previous discussion on the large effects that stem from this energy range with low \( L \) and that invalidates the use of Eq. (15). On the other hand, the agreement between the phase shifts calculated with the two versions of the \( N/D_A \) method (with and without volume dependence of the coupling \( G_V \) ) supports the results obtained with energy levels that involve values of the volume \( Lm_\pi \gtrsim 2 \).

IV. SUMMARY

We have made a study of the \( \pi \pi \) scattering in the \( \rho(770) \) channel in a finite box. In particular, we have studied the effect of the exponentially suppressed \( L \)-dependence coming from tadpoles and crossed channel loops that is usually neglected when extracting physical quantities from lattice results in a finite volume via L"uscher approach or the method of Ref. [24]. To do so we have used two realistic models to describe \( \pi \pi \) scattering, the \( N/D_A \) method and the Inverse Amplitude Method (IAM). These approaches describe fairly well the scattering and the \( \rho(770) \) resonance in infinite volume and, when modified to describe the \( \pi \pi \) energy levels in a finite box, they also incorporate the suppressed \( L \)-dependence mentioned above. We obtain for each method the \( \pi \pi \) energy levels in a finite box as a function of the box size \( L \), and study when the L"uscher approach or the method of Ref. [24] can be safely used to obtain the infinite volume phase shifts around the \( \rho(770) \) region. We conclude that if one uses lattice sizes \( L > 2m_\pi^{-1} \), the exponentially suppressed \( L \)-dependence is in fact numerically negligible, and the same results are obtained from the levels derived from either the \( N/D_A \) method or the IAM, which also agree with the infinite volume phase shifts. However, if the high energy part of the \( \rho(770) \) shape is to be reproduced from the first energy level, one needs to use \( L \) values smaller than \( 2m_\pi^{-1} \). Then, the results from the \( N/D_A \) and IAM levels are quite different, which suggests that the neglected \( L \)-dependence in this case might be numerically important also in lattice calculations, and the phase shifts obtained inaccurate.

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