Graham’s pebbling conjecture on Cartesian product of the middle graphs of even cycles

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Abstract: A pebbling move on a graph \( G \) consists of taking two pebbles off one vertex and placing one on an adjacent vertex. The pebbling number of a graph \( G \), denoted by \( f(G) \), is the least integer \( n \) such that, however \( n \) pebbles are located on the vertices of \( G \), we can move one pebble to any vertex by a sequence of pebbling moves. Let \( M(G) \) be the middle graph of \( G \). For any connected graphs \( G \) and \( H \), Graham conjectured that \( f(G \times H) \leq f(G)f(H) \). In this paper, we give the pebbling number of some graphs and prove that Graham’s conjecture holds for the middle graphs of some even cycles.

Keywords: Graham’s conjecture, even cycles, middle graphs, pebbling number.

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1 Introduction

Pebbling in graphs was first introduced by Chung [2]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. The pebbling number of a vertex \( v \), the target vertex, in a graph \( G \) is the smallest number \( f(G, v) \) with the property that, from every placement of \( f(G, v) \) pebbles on \( G \), it is possible to move one pebble to \( v \) by a sequence of pebbling moves. The pebbling number of a graph \( G \), denoted by \( f(G) \), is the maximum of \( f(G, v) \) over all the vertices of \( G \).

There are some known results regarding the pebbling number (see [2,5,7]). If one pebble is placed on each vertex other than the vertex \( v \), then no pebble can be moved to...
v. Also, if u is at a distance d from v, and \(2^d - 1\) pebbles are placed on u, then no pebble can be moved to v. So it is clear that \(f(G) \geq \max\{|V(G)|, 2^D\}\), where \(D\) is the diameter of graph \(G\). Furthermore, we know that \(f(K_n) = n\) and \(f(P_n) = 2^n - 1\) (see [2]), where \(K_n\) is the complete graph and \(P_n\) is the path, respectively on \(n\) vertices.

The middle graph of a graph \(G\), denoted by \(M(G)\), is obtained from \(G\) by inserting a new vertex into each edge of \(G\), and joining the new vertices by an edge if the two edges they inserted share the same vertex of \(G\).

Given two disjoint graphs \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\), the Cartesian product of them is denoted by \(G_1 \times G_2\). It has vertex set \(V_1 \times V_2 = \{(u_i, v_j) | u_i \in V_1, v_j \in V_2\}\), where \((u_1, v_1)\) is adjacent to \((u_2, v_2)\) if and only if \(u_1 = u_2\) and \((v_1, v_2) \in E_2\), or \((u_1, u_2) \in E_1\) and \(v_1 = v_2\). One may view \(G_1 \times G_2\) as the graph obtained from \(G_1\) by replacing each of its vertices with a copy of \(G_1\), and each of its edges with \(|V_1|\) edges joining corresponding vertices of \(G_1\) in the two copies. Let \(u \in G, v \in H\), then \(u(H)\) and \(v(G)\) are subgraphs of \(G \times H\) with \(V(u(H)) = \{(u, v) | v \in V(H)\}\), \(E(u(H)) = \{(u, v)(u, v') | vv' \in E(H)\}\) and \(V(v(G)) = \{(u, v) | u \in V(G)\}\), \(E(v(G)) = \{(u, v)(u', v) | uu' \in E(G)\}\). It is clear that \(u(H) \cong H\) and \(v(G) \cong G\).

The following conjecture (see [2]), by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

Conjecture (Graham). The pebbling number of \(G \times H\) satisfies \(f(G \times H) \leq f(G)f(H)\).

Ye et al. (see [3]) proved that \(f(M(C_{2n+1}) \times M(C_{2m+1})) \leq f(M(C_{2n+1}))f(M(C_{2m+1}))\) and \(f(M(C_{2n}) \times M(C_{2m+1})) \leq f(M(C_{2n}))f(M(C_{2m+1}))\). In this paper, we will prove that \(f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n}))f(M(C_{2m}))\) for \(m, n \geq 5\) and \(|n - m| \geq 2\).

Throughout this paper, \(G\) will denote a simple connected graph with vertex set \(V(G)\) and edge set \(E(G)\). \(P_n\) and \(C_n\) will denote a path and a cycle with \(n\) vertices, respectively. Given a distribution of pebbles on the vertices of \(G\), define \(p(K)\) to be the number of pebbles on a subgraph \(K\) of \(G\) and \(p(v)\) to be the number of pebbles on a vertex \(v\) of \(G\). Moreover, we let \(\tilde{p}(K)\) and \(\tilde{p}(v)\) denote the numbers of pebbles on \(K\) and \(v\) after some sequence of pebbling moves, respectively.

## 2 Main results

**Definition 2.1** (see [5]) Let \(P_n = v_1v_2 \cdots v_n\) be a path. We say that \(P_n\) has weight \(\sum_{i=1}^{n-1} 2^{i-1}p(v_i)\) with respect to \(v_n\) and this is written as \(\omega_{P_n}(v_n)\).

**Proposition 2.2** (see [5]) Let \(P_n = v_1v_2 \cdots v_n\) be a path. If \(\omega_{P_n}(v_n) \geq k2^{n-1}\), then at least \(k\) pebbles can be moved from \(P_n \setminus v_n\) to \(v_n\).
Corollary 2.3 Let $P_n = v_1v_2 \cdots v_n$ be a path. Let $\omega_{P_n}(v_k) = \sum_{i=1}^{k-1} 2^{i-1}p(v_i) + \sum_{j=k+1}^{n} 2^{n-j}p(v_j)$ for $2 \leq k \leq n - 1$. If $\omega_{P_n}(v_k) \geq t2^{k-1} + 2^{n-k} - 1$ for $\frac{n+1}{2} \leq k \leq n$, $\omega_{P_n}(v_k) \geq 2^{k-1} + t2^{n-k} - 1$ for $1 \leq k < \frac{n+1}{2}$, then at least $t$ pebbles can be moved from $P_n \setminus v_k$ to $v_k$.

Proof. Without loss of generality, we assume that $\frac{n+1}{2} \leq k \leq n$.
If $k = n$, it follows from Proposition 2.2.
If $\frac{n+1}{2} \leq k \leq n - 1$, let $L_1 = v_1v_2 \cdots v_k$, $L_2 = v_kv_{k+1} \cdots v_n$ be two subpaths of $P_n$.
Suppose $\omega_{P_n}(v_k) \geq t2^{k-1} + 2^{n-k} - 1$, then either $\sum_{i=1}^{k-1} 2^{i-1}p(v_i) \geq t2^{k-1}$ or $\sum_{j=k+1}^{n} 2^{n-j}p(v_j) \geq 2^{n-k}$ holds.
Case 1. $\sum_{i=1}^{k-1} 2^{i-1}p(v_i) \geq t2^{k-1}$, by Proposition 2.2 we can move $t$ pebbles from $L_1 \setminus v_k$ to $v_k$.
Case 2. $\sum_{j=k+1}^{n} 2^{n-j}p(v_j) \geq 2^{n-k}$, we may assume that $\sum_{j=k+1}^{n} 2^{n-j}p(v_j) = s2^{n-k} + h$, where $s$ and $h$ are integers satisfying $s \geq 1$ and $0 \leq h < 2^{n-k}$. With $p(v_j)$ pebbles on $v_j$ ($k+1 \leq j \leq n$), we can move $s$ pebbles from $L_2 \setminus v_k$ to $v_k$.
Note that $2^{k-1} \geq 2^{n-k}$ for $k \geq \frac{n+1}{2}$, we have
\[
\sum_{i=1}^{k-1} 2^{i-1}p(v_i) = \omega_{P_n}(v_k) - \sum_{j=k+1}^{n} 2^{n-j}p(v_j) \\
\geq t2^{k-1} + 2^{n-k} - 1 - (s2^{n-k} + h) \\
= (t2^{k-1} - s2^{n-k}) + (2^{n-k} - h) - 1 \\
\geq (t-s)2^{k-1}.
\]
So we can move $t-s$ pebbles from $L_1 \setminus v_k$ to $v_k$ with $p(v_i)$ pebbles on $v_i$ ($1 \leq i \leq k-1$). That is to say we can move $s + (t-s) = t$ pebbles to $v_k$.

Corollary 2.4 Let $P_n = v_1v_2 \cdots v_n$ be a path. Then $f(M(P_n) - \{v_1, v_n\}) = 2^{n-2} + n - 2$.

![Diagram](image1.png)

Figure 1: The graph $M(P_n) - \{v_1, v_n\}$ in Corollary 2.4.

Proof. To get $M(P_n)$, we insert $u_i$ into the edge $v_iv_{i+1}$ and add the edge $u_iu_{i+1}$ for each $i \in \{1, 2, \ldots, n-2\}$. Let $U = u_1u_2 \cdots u_{n-1}$ be a subpath of $M(P_n) - \{v_1, v_n\}$.
It is clear that $f(M(P_n) - \{v_1, v_n\}) \geq 2^{n-2} + n - 2$. If we place one pebble on each of vertices $v_2, \ldots, v_{n-1}$, and place $2^{n-2} - 1$ pebbles on $u_{n-1}$, then we can not move one pebble to $u_1$. So $f(M(P_n) - \{v_1, v_n\}) \geq 2^{n-2} + n - 2$. 

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Now, assume that \(2^{n-2} + n - 2\) pebbles are located at \(V(M(P_n) - \{v_1, v_n\})\).
First, we prove that one pebble can be moved to \(u_k\) (1 \(\leq k \leq n - 1\)).

While \(m \leq k\), we can move \(\lfloor p(v_m)/2 \rfloor\) pebbles from \(v_m\) to \(u_m\). While \(m > k\), we can move \(\lfloor p(v_m)/2 \rfloor\) pebbles from \(v_m\) to \(u_{m-1}\).

\[
\omega_U(u_k) \geq 2^{n-2} + n - 2 - \sum_{i=2}^{n-1} p(v_i) + 2 \sum_{i=2}^{n-1} \lfloor p(v_i)/2 \rfloor \\
\geq 2^{n-2}.
\]

It is clear that \(2^{n-2} \geq 2^{k-1} + 2^{n-k-1} - 1\) for \(1 \leq k \leq n - 1\). By Corollary 2.3, we can move one pebble from \(U \setminus u_k\) to \(u_k\) (1 \(\leq k \leq n - 1\)).

Now we prove that one pebble can be moved to \(v_k\) (2 \(\leq k \leq n - 1\)). Without loss of generality, we assume that \(k \geq \frac{n+1}{2}\).

While \(m < k\), we can move \(\lfloor p(v_m)/2 \rfloor\) pebbles from \(v_m\) to \(u_m\). While \(m > k\), we can move \(\lfloor p(v_m)/2 \rfloor\) pebbles from \(v_m\) to \(u_{m-1}\).

We will prove that after a sequence of pebbling moves above, two pebbles can be moved from \(U\) to \(u_{k-1}\), so that one pebble can be moved from \(u_{k-1}\) to \(v_k\).

We consider the worst case, that is \(p(u_{k-1}) = 0\).

\[
\omega_U(u_{k-1}) \geq 2^{n-2} + n - 2 - \sum_{j=2}^{n-1} p(v_j) + 2 \sum_{j=2}^{n-1} \lfloor p(v_j)/2 \rfloor \\
\geq 2^{n-2} + 1.
\]

It is clear that \(2^{n-2} + 1 \geq 2 \times 2^{(k-1)-1} + 2^{n-(k-1)-1} - 1\) for \(\frac{n-1}{2} \leq k - 1 \leq n - 2\). By Corollary 2.3, we can move two pebbles from \(U \setminus u_{k-1}\) to \(u_{k-1}\) (\(\frac{n-1}{2} \leq k - 1 \leq n - 2\)). So we can move one pebble to \(v_k\) (\(\frac{n+1}{2} \leq k \leq n - 1\)), and we are done.

\[\textbf{Definition 2.5 (see [5])}\quad \text{The } t \text{-pebbling number of a graph } G \text{ is the smallest number } f_t(G) \text{ with the property that from every placement of } f_t(G) \text{ pebbles on } G, \text{ it is possible to move } t \text{ pebbles to any vertex } v \text{ by a sequence of pebbling moves.}\]

\[\textbf{Lemma 2.6 (see [6])}\quad \text{If } n \geq 2, \text{ then } f(M(C_{2n})) = 2^{n+1} + 2n - 2.\]

\[\textbf{Corollary 2.7}\quad \text{If } n \geq 2, \text{ then } f_i(M(C_{2n})) \leq t2^{n+1} + 2n - 2.\]

\[\textbf{Proof}. \text{ Let } C_{2n} = v_0v_1 \cdots v_{2n-1}v_0, \text{ M}(C_{2n}) \text{ is obtained from } C_{2n} \text{ by inserting } u_i \text{ into } v_iu_{(i+1)mod(2n)}, \text{ and connecting } u_iu_{(i+1)mod(2n)} \text{ } (0 \leq i \leq 2n - 1).\]

Without loss of generality, we may assume that our target vertex is \(u_0\) or \(v_0\).

\[\text{Case 1. The target vertex is } u_0. \text{ In this case, we use induction on } t.\]

The result is obvious for \(t = 1\) from Lemma 2.6.

Now suppose that \(t2^{n+1} + 2n - 2\) pebbles are located at the vertices of \(M(C_{2n})\).

We consider the worst case, that is \(p(u_0) = 0\).

Let \(A = \{u_0, v_1, u_1, \ldots, v_n, u_n\}, B = \{u_n, v_{n+1}, \ldots, v_{2n-1}, u_{2n-1}, v_0, u_0\}\) and \(G = M(C_{2n})\). Then we have either \(A\) or \(B\) contains more than \(2^n + n\) pebbles.

Note that \(G[A] \cong G[B] \cong M(P_{n+2}) - \{v_1, v_{n+2}\}\), according to Corollary 2.4, with \(2^n + n\) pebbles on \(A\) or \(B\), one pebble can be moved to \(u_0\).
Note that \(2^n + n \leq 2^{n+1}\), the number of remaining pebbles is more than \((t - 1)2^{n+1} + 2n - 2\). So we can move \(t - 1\) pebbles to \(u_0\) with the remaining pebbles by induction, and we are done.

Case 2. The target vertex is \(v_0\).

Let \(A' = \{u_0, v_1, \ldots, v_{n-1}, u_{n-1}\}\), \(B' = \{u_{2n-1}, v_{2n-1}, \ldots, v_{n+1}, u_n\}\).

Suppose that \(t2^{n+1} + 2n - 2\) pebbles are located at the vertices of \(M(C_{2n})\).

We consider the worst case, that is \(p(v_0) = 0\).

By proposition 2.2 while \(p(v_n) \geq t2^{n+1}\), \(t\) pebbles can be moved to \(v_0\).

Now suppose that \(t2^{n+1} - h\) pebbles are located at \(v_n\), without loss of generality, we assume that \(p(A') \geq p(B')\), that is \(p(A') \geq n - 1 + \lceil h/2\rceil\).

Let \(L = v_0u_0u_1\cdots u_{n-1}v_n\) be a subpath of \(G\) with length \(n + 1\) and \(q = \sum_{i=0}^{n-1} p(u_i)\).

While \(q \geq \lceil h/2\rceil\),

\[
\omega_L(v_0) = p(v_n) + \sum_{i=0}^{n-1} 2^{n-i}p(u_i) \geq t2^{n+1} - h + 2q \geq t2^{n+1}.
\]

By Proposition 2.2 \(t\) pebbles can be moved from \(L\setminus v_0\) to \(v_0\).

While \(q < \lceil h/2\rceil\), then \(\sum_{j=1}^{n-1} p(v_j) \geq n - 1 + \lceil h/2\rceil - q\). So we can move at least \(\lceil \frac{1}{2}(\lceil \frac{n}{2} \rceil + 1 - q) \rceil\) pebbles to the set \(\{u_0, u_1, \ldots, u_{n-2}\}\). Then we have

\[
\omega_L(v_0) = p(v_n) + \sum_{i=0}^{n-1} 2^{n-i}p(u_i) \geq t2^{n+1} - h + 2q + 4 \times \frac{1}{2}(\frac{h}{2} - q) \geq t2^{n+1}.
\]

By Proposition 2.2 \(t\) pebbles can be moved from \(L\setminus v_0\) to \(v_0\). The result follows.

**Theorem 2.8** If \(m, n \geq 5\) and \(|n - m| \geq 2\), then

\[
f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n}))f(M(C_{2m})).
\]

**Proof.** Without loss of generality, we assume that \(n \geq m + 2\) \((m \geq 5)\).

Let \(V(M(C_{2n})) = \{u_1, u_2, \ldots, u_{4n}\}, V(M(C_{2m})) = \{v_1, v_2, \ldots, v_{4m}\}\).

For simplicity, let \(G = M(C_{2n}) \times M(C_{2m})\).

Now assume \((2^{n+1} + 2n - 2)(2^{m+1} + 2m - 2)\) pebbles have been placed arbitrarily at the vertices of \(G\).

We may assume our target vertex is \((u_i, v_j)\), then \((u_i, v_j)\) belongs to both \(u_i(M(C_{2m}))\) and \(v_j(M(C_{2n}))\).

If \(p(u_i(M(C_{2m}))) \geq 2^{m+1} + 2m - 2 \) or \(p(v_j(M(C_{2n}))) \geq 2^{n+1} + 2n - 2\), we can move one pebble to \((u_i, v_j)\) by lemma 2.6.

Suppose that \(p(u_i(M(C_{2m}))) \leq 2^{m+1} + 2m - 3\) and \(p(v_j(M(C_{2n}))) \leq 2^{n+1} + 2n - 3\).

We will prove that if we move as many as possible pebbles from \(u_i(M(C_{2m}))\) to \((u_l, v_j)\) which belongs to \(v_j(M(C_{2n}))\) \((1 \leq l \leq 4n)\), then one pebble can be moved from \(v_j(M(C_{2n}))\) to \((u_i, v_j)\).

We may assume that

\[
p_k = p(u_k(M(C_{2m}))) \leq 2^{m+1} + 2m - 3 \quad (1 \leq k \leq s)
\]
\[
p_k = p(u_k(M(C_{2m}))) \geq 2^{m+1} + 2m - 2 \quad (s + 1 \leq k \leq 4n).
\]

Now we consider the worst case scenario (i.e. the most wasteful distribution of pebbles possible). Therefore we may assume that

\[
p_k = \begin{cases} 
2^{m+1} + 2m - 3 & 1 \leq k \leq s, \\
t_k2^{m+1} + 2m - 2 + (2^{m+1} - 1) & s + 1 \leq k \leq 4n - 1, \\
t_k2^{m+1} + 2m - 2 + R & k = 4n,
\end{cases}
\]

where \(0 \leq R \leq 2^{m+1} - 1\) and \(t_k\) is a positive integer. According to Corollary 2.7, we can move at least \(\sum_{k=s+1}^{4n} t_k\) pebbles to \(v_j(M(C_{2n}))\).

Let

\[
\Delta = (2^{n+1} + 2n - 2)(2^{m+1} + 2m - 2) - s(2^{m+1} + 2m - 3) - (4n - s - 1)(2^{m+1} - 1) - (4n - s)(2m - 2) = (2^{n+1} - 2n - 2)(2^{m+1} + 2m - 2) + 2^{m+1} + 4n - 1.
\]

Therefore,

\[
\frac{\Delta}{2^{m+1}} = 2^{n+1} - 2n - 1 + \frac{1}{2^{m+1}} \left[ (2^{n+1} - 2n - 2)(2m - 2) + 4n - 1 \right].
\]

Note that \(\Delta = \left( \sum_{k=s+1}^{4n} t_k \right)^{2^{m+1}} + R\), so \(\sum_{k=s+1}^{4n} t_k > \frac{\Delta}{2^{m+1}} - 1\). It follows that

\[
p(v_j(M(C_{2n}))) \geq \sum_{k=s+1}^{4n} t_k \geq 2^{n+1} - 2n - 2 + \frac{1}{2^{m+1}} \left[ (2^{n+1} - 2n - 2)(2m - 2) + 4n - 1 \right].
\]

To the end, we only need to prove that we can move one pebble from \(v_j(M(C_{2n}))\) to \((u, v_j)\) with \(2^{n+1} - 2n - 2 + \frac{1}{2^{m+1}} \left[ (2^{n+1} - 2n - 2)(2m - 2) + 4n - 1 \right]\) pebbles.

So we only need to prove that

\[
2^{n+1} - 2n - 2 + \frac{1}{2^{m+1}} \left[ (2^{n+1} - 2n - 2)(2m - 2) + 4n - 1 \right] \geq 2^{n+1} + 2n - 2
\]

that is

\[
2^{m+1} < \frac{m - 1}{n} (2^n - 1) - m + 2. \tag{2.1}
\]

For \(n \geq m + 2 \geq 7\), it is clear that the right side of (2.1) is an increasing function of \(n\). So we only need to prove (2.1) under \(n = m + 2\). Substituting \(n = m + 2\) into (2.1), we have

\[
2^{m+1} < \frac{m - 1}{m + 2} (2^{m+2} - 1) - m + 2,
\]

that is

\[
(2m - 8)2^m - m^2 - m + 5 > 0. \tag{2.2}
\]

The left side of (2.2) is an increasing function of \(m\) while \(m \geq 5\). When \(m = 5\), (2.2) holds. This completes the proof.
Remark

In fact, by a similar processing as in the proof of Corollary 2.7 for any $u \in M(C_{2n})$ but $u \not\in C_{2n}$, we can prove that

**Corollary 3.1** If $n \geq 2$, then $f_t(M(C_{2n}), u) \leq 2^{n+1} + 2n - 2 + (t-1)(2^n + n)$.

Then we can prove the following theorem.

**Theorem 3.2** If $(u, v) \not\in C_{2n} \times C_{2m}$, where $C_{2n} \times C_{2m}$ is a subgraph of $M(C_{2n}) \times M(C_{2m})$, then

$$f(M(C_{2n}) \times M(C_{2m}), (u, v)) \leq f(M(C_{2n})) f(M(C_{2m})).$$

**Proof.** If $(u, v) \not\in C_{2n} \times C_{2m}$, then we can get $u(M(C_{2m})) \not\in C_{2n} \times M(C_{2m})$ or $v(M(C_{2n})) \not\in M(C_{2n}) \times C_{2m}$.

Without loss of generality, we assume that $u(M(C_{2m})) \not\in C_{2n} \times M(C_{2m})$.

Let $V(M(C_{2m})) = \{v_1, v_2, \ldots, v_{4m}\}$.

If we move as many as possible pebbles from $v_j(M(C_{2n}))$ to $(u, v_j)$ which belongs to $u(M(C_{2m}))$ ($1 \leq j \leq 4m$), by a similar processing as in the proof of Theorem 2.8 we can prove that the number of pebbles on $u(M(C_{2m}))$ is more than $2^{m+1} + 2m - 2$, so one pebble can be moved from $u(M(C_{2m}))$ to $(u, v)$ with these pebbles.

In this paper, we have shown that while $m, n \geq 5$ and $|m - n| \geq 2$, $f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n})) f(M(C_{2m}))$. The remaining question is open.

**Problem 3.3** $f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n})) f(M(C_{2m}))$, for $m = n$ or $m = n - 1$.

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