INTERSECTION SPACE CONSTRUCTIBLE COMPLEXES

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Abstract. We present an obstruction theoretic inductive construction of intersection space pairs, which generalizes Banagl’s construction of intersection spaces for arbitrary depth stratifications. We construct intersection space pairs for pseudomanifolds with compatible trivial structures at the link fibrations; this includes the case of toric varieties. We define intersection space complexes in an axiomatic way, similar to Goresky-McPherson axioms for intersection cohomology. We prove that if the intersection space exists, then the pseudomanifold has an intersection space complex whose hypercohomology recovers the cohomology of the intersection space pair. We characterize existence and uniqueness of intersection space complexes in terms of the derived category of constructible complexes. We show that intersection space complexes of algebraic varieties lift to the derived category of Mixed Hodge Modules, endowing intersection space cohomology with a Mixed Hodge Structure. We find classes of examples admitting intersection space complexes, and counterexamples not admitting them; they are in particular the first examples known not admitting Banagl intersection spaces. We prove that the (shifted) Verdier dual of an intersection space complex is an intersection space complex. We prove a generic Poincare duality theorem for intersection space complexes.

1. Introduction

Intersection spaces have been recently introduced by Banagl as a Poincare duality homology theory for topological pseudomanifolds which is an alternative to Goreski and McPherson intersection homology. When they are available they present the advantages of being spatial modifications of the given topological pseudomanifold, to which one can later apply algebraic topology functors in order to obtain invariants. In this sense, if one applies (reduced) singular cohomology one obtains a homology theory with internal cup products and a Poincare duality is satisfied between the homology theories corresponding to complementary perversities. Moreover, one can apply many other functors leading to richer invariants. The idea of intersection spaces was sketched for the first time in [2], and was fully developed for spaces with isolated singularities in [3].

In [3] Banagl carefully analyzed the case of quintic 3-folds with ordinary double points appearing in the conifold transition and noticed that, in the same way that intersection cohomology gives the cohomology of a small resolution, cohomology of intersection spaces gives the cohomology of a smoothing in this case. This fitted with predictions motivated by string theory (see Banagl papers for full explanations).

This motivated further work by Banagl, Maxim and Budur ([8], [9], [5], [16]) in which the relation between the cohomology of intersection spaces for the middle perversity and the Milnor fibre of a hypersurface X with isolated singularities is analyzed. The latest evolution of the

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results of these papers, contained in [5], is the construction of a perverse sheaf in $X$ whose hypercohomology computes the cohomology of the intersection space of $X$ in all degrees except for the top degree. Such a perverse sheaf is a modification of the nearby cycle complex and, in fact, when the monodromy is semi-simple in the eigenvalue 1, the middle perversity intersection space perverse sheaf is a direct summand of the nearby cycle complex.

The results described up to now are valid for only isolated singularities (sometimes even assuming that they are hypersurface singularities). In [3] Banagl generalizes the construction for the case of topological pseudomanifolds with two strata and trivial link fibration and sketches a method for more general class of non-isolated singularities. In [6] intersection spaces are constructed for the case of two strata assuming non-trivial conditions on the fibration by links. The only case in which intersection spaces are constructed for a topological pseudomanifold with more than two strata is in [4]. There, the depth 1 strata are circles or intervals, and the depth 2 strata are isolated singularities; in this case, it is the topology of the strata which is very restrictive.

In [8] the following open questions are proposed: Is there a sheaf theoretic approach to intersection spaces, similar to the one of Goreski and McPherson [13] for intersection cohomology?. Up to which kind of singularities the intersection space constructions can be extended? Is intersection space cohomology of algebraic varieties endowed with a Hodge structure?

The papers [9], [5] are contributions to the first and third question for the case of isolated singularities. The paper [7] is also a contribution towards the third question.

In a recent paper [11], Genske takes a new viewpoint: instead of only giving up the topological construction and focusing in producing a complex of sheaves at the original space, he constructs a complex of vector spaces which is related with the complex computing the (co)homology of the original space $X$, but that satisfies Poincaré duality. The construction is valid for any analytic variety (Poincaré duality is satisfied in the compact case). His construction is a bit further to original Banagl ideas than ours, since his procedure is to make a modification which is global in a neighbourhood of the singular set, instead of stratifying it conveniently and making a fibrewise construction.

Finally, in order to finish our review of existing results, let us mention the rational Poincaré spaces approach developed in [14].

The present paper is a contribution to the three questions formulated above for the general singularity case. Before explaining our results in detail in the next Section, let us enumerate them in a very condensed way:

We realize that, for constructing Banagl’s intersection spaces in a general pseudomanifold $X$, one needs to adopt the viewpoint of pairs of spaces and associate an intersection space pair, which is a spatial modification of the pair $(X, \text{Sing}(X))$. We find a procedure which runs inductively on the codimension of the strata and, if it is not obstructed, produces the intersection space pair. We show that, if the link fibrations of the pseudomanifold are trivial and the trivializations of these fibrations verify some compatibility conditions, the intersection space pair exists. This includes the case of toric varieties.

We prove that, if an intersection space pair exists for a topological pseudomanifold and a given perversity, then there exist a constructible complex of sheaves in our original space $X$ that satisfies a set of properties of the same kind that those that characterize intersection cohomology complexes in [13]; we call this complex an intersection space complex for the given perversity. Its hypercohomology recovers the reduced cohomology of the intersection space in the case of isolated singularities. In the case of depth 1 topological pseudomanifold, it recovers the cohomology of the intersection space relative to the singular stratum (like in [3]), which is the one that satisfies Poincaré duality for complementary perversities. For depth 2 and higher, if the dimension of the strata is sufficiently high, the intersection space construction is intrinsically
a construction of pairs of spaces, as we will see below; the hypercohomology of our intersection space complex computes the rational cohomology of the pair of spaces.

Next, we leave the realm of topology and shift to a sheaf theoretic viewpoint studying under which conditions intersection space complexes exist. We find obstructions for existence and uniqueness of intersection space complexes and give spaces parametrizing the possible intersection space complexes in case that the obstruction for existence vanish. Both of these obstructions vanish in the case of isolated singularities and the obstruction for existence also vanish in the case considered in [4], as one should expect. If one assumes that the topological pseudomanifold is an algebraic variety, we show how to carry our constructions in the category of mixed Hodge modules, yielding a polarizable mixed Hodge structure in the hypercohomology of the intersection space complex, and hence in the cohomology of intersection spaces when they exist.

We turn to analyze classes of topological pseudomanifolds in which we can prove the existence of intersection space complexes. We show that they exist for any perversity when the successive link fibrations are trivial (without the compatibility conditions needed to construct the intersection space pair). We also prove the existence if the homological dimension of the strata is at most 1. This includes the case treated in [4]. On the other hand, building on the obstructions for existence, we produce the first examples of topological pseudomanifolds such that intersection space complexes do not exist for given perversities. As a consequence, Banagl intersection spaces can not exist either. One of the examples is a normal algebraic variety whose stratification has depth 1 and whose transversal singularity is an ordinary double point of dimension 3 (those appearing in the conifold transition examples); the perversity used is the middle one.

Finally, we turn to duality questions. We show that the Verdier dual of an intersection space complex with a given perversity is an intersection space complex with the complementary perversity. The proof resembles the one given in [13] for intersection cohomology complexes. However, since (unlike intersection cohomology complexes) intersection space complexes are not unique, this does not yield self dual sheaves for the middle perversity on algebraic varieties. In the case of depth 1 stratifications, we prove that generic choices of the intersection space complex yield the same Betti numbers in hypercohomology and we obtain Poincaré duality at the level of generic Betti numbers for complementary perversities.

Some open questions and further directions are hinted at the end of next section.

2. Main results

This section is the guide to the paper. Here we describe in detail the content of the paper section by section, with cross-references to the main results. The reader may jump to the corresponding sections for a full exposition.

2.1. Topological constructions. The paper starts with a topological construction of pairs of intersection spaces in Section 3.

Banagl construction of the intersection space [3] for a d-dimensional topological pseudomanifold X with isolated singularities for a given perversity \( \bar{p} \) runs as follows. Let \( \Sigma = \{p_1, ..., p_r\} \) be the singular set. Around \( p_i \), consider a conical neighbourhood \( B_i \). Let \( L_i \) denote the link \( \partial B_i \). Let \( \bar{q} \) be the complementary perversity of \( \bar{p} \). Consider a homological truncation

\[
(L_i)_{\leq \bar{q}(d)} \rightarrow L_i.
\]

This is a mapping of spaces inducing isomorphisms in homology in degrees up to \( \bar{q}(d) \) and such that \( H_i((L_i)_{\leq \bar{q}(m)}) \) vanish for \( i > \bar{q}(d) \). Assume for simplicity that the truncation map is an inclusion. Construct the intersection space replacing each of the \( B_i \)'s by the cone over \( (L_i)_{\leq \bar{q}(d)} \), call the resulting space \( Z \). The vertices of the cone are called \( \Sigma = \{p_1, ..., p_r\} \) as well. The intersection space is the result of attaching the cone over \( \Sigma \) to \( Z \). If the truncation map is not
an inclusion, one may force this using an appropriate homotopy model for it. The intersection space homology is the reduced homology of the intersection space.

The construction for the case of topological pseudomanifolds $X = X_d \supset X_{d-m}$ of dimension $d$ with a single singular stratum of codimension $m$ contained in $\mathbb{R}$ and $[0]$ is the following generalization. Let $T$ be a tubular neighbourhood of the singular set $X_{d-m}$. Consider the locally trivial fibration $T \to X_{d-m}$, and let $\partial T \to X_{d-m}$ be the associated fibration of links. Consider a fibrewise homology truncation

$$\partial T_{\leq \bar{q}(m)} \to \partial T.$$ 

This is a morphism of locally trivial fibrations which is a $\bar{q}(m)$-homology truncation at each fibre. Remove $T$ from $X$ and replace it by the fibrewise cone over $\partial T_{\leq \bar{q}(m)}$ (see Definition 3.3); call the resulting space $Z$. As before $\Sigma$ is a subspace of $Z$. The intersection space is the result of attaching the cone over $\Sigma$ to $Z$ and the intersection space homology is the reduced homology of the intersection space.

Notice that the intersection space homology coincides with the relative homology $H_*(Z, \Sigma)$. This observation is the starting point of our construction for more than two strata and of our constructible complex approach to intersection space homology. Consider a topological pseudomanifold with stratification

$$X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

and a suitable system of tubular neighbourhoods of the strata (the conical structure of Definition 3.5). A great variety of topological pseudomanifolds can be endowed with this structure (see Remark 3.14).

Our topological construction of intersection spaces consist in modifying the space $X$ inductively, going each step deeper in the codimension of the strata, by taking successive fibrewise homology truncations. In doing so, one necessarily modifies the singular set $X_{d-2}$ and, as a result, the singular set is not going to be contained in the modified space $Z$. Instead, one obtains a modification $Y$ of $X_{d-2}$ contained in $Z$. One obtains a pair $(Z, Y)$, and it is the homology of this pair our definition of intersection space homology. An important feature of the construction is that one needs to adopt the viewpoint of pairs of spaces right from the beginning if one wants to have a chance of proving duality results; this incarnates in the need of taking homology truncations of pairs of spaces. This new feature appears from the depth 2 strata and hence it did not appear in Banagl constructions explained above. It also may happen that, if the singular stratum, is of small dimension in comparison with the perversity, the situation does not appear at all.

As in Banagl construction, the homology truncations need not be inclusions. This forces us to work with an adequate homotopy model for $X$.

It is important to record for future reference that, at the $k$-th inductive step of the construction, one obtains a pair of spaces $(I^p_k X, I^d_k (X_{d-k}))$ which contain $X_{d-k-1}$ and verify

1. the pair $(I^p_k X \setminus X_{d-k-1}, I^d_k (X_{d-k}) \setminus X_{d-k-1})$ is an intersection space pair of $X \setminus X_{d-k-1}$.
2. there is a system of tubular neighbourhoods $T_{k-1}$ of $X_{d-k-1} \setminus X_{d-k-2}$ in $I^p_k X \setminus X_{d-k-2}$ such that we have locally trivial fibrations of pairs

$$\partial T_{k-1} \cap (I^p_k X \setminus X_{d-k-2}, I^d_k (X_{d-k}) \setminus X_{d-k-2}) \to X_{d-k-1} \setminus X_{d-k-2},$$

being the first the fibrewise cone over the second (see Definition 3.3).

The second locally trivial fibration is called the fibration of link pairs at the $k$-th step of the construction.

The construction follows the scheme of obstruction theory: it is inductive and, at each step, choices are made. The next step may be obstructed and this may depend on the previous choices.
The obstruction consists in the impossibility of constructing a fibrewise homology truncation of the fibration of link pairs at the $k$-th step of the construction.

When there is a set of choices so that the process terminates, we say that an intersection space pair exists. It needs not be unique.

In the case where the conical structure is trivial, that is, if the link fibrations are trivial and the trivializations are compatible with each other (see Definition 3.13), the intersection space pair exists (see Theorem 3.30). This includes the case of arbitrary toric varieties.

2.2. From topology to constructible complexes. In the rest of the paper, we investigate the existence and uniqueness of intersection space pairs and their duality properties by sheaf theoretic methods. For this, we associate to each intersection space pair an element in the derived category of constructible complexes whose hypercohomology computes the rational cohomology of the intersection space pair.

To get this, we need to construct a sequence of intersection space pairs which modify the pair $(X_d, X_{d-2})$ in increasingly smaller neighbourhoods of the strata of $X$. This is done in Section 4.

In Section 5, we exploit the sequence of intersection spaces to derive a constructible complex $IS$ (see Definition 5.15) and prove, in Theorem 5.16, that the hypercohomology of $IS$ recovers the cohomology of the intersection space pair. Finally, in Theorem 5.18, we prove that $IS$ satisfies a set of properties in the same spirit that those that characterize intersection cohomology complexes in [13]. This is the basis for the axiomatic treatment of the next section.

2.3. A derived category approach to intersection space (co)homology. In Section 6, we take an axiomatic approach to intersection space complexes in the same way as Goresky-McPherson approach to intersection cohomology in [13, section 3.3]. We define two sets of properties in the derived category of cohomologically constructible sheaves on $X$. The first set are the properties of the intersection cohomology sheaf composed with a shift. The second set of properties are inspired by Theorem 5.18.

We will call a complex of sheaves verifying the second set of properties intersection space complex of $X$ (Definition 6.2). Theorem 5.18 implies that if there exist an intersection space pair of $X$ (see Definition 3.27), then there exist an intersection space complex of $X$ whose hypercohomology coincides with the cohomology of the intersection space pair. Moreover, we compare the support and cosupport properties of intersection cohomology complexes and intersection space complexes. From the comparison, one sees that intersection space complexes, except possibly in the case of isolated singularities, can not be perverse sheaves.

At this point, we investigate in a purely sheaf theoretical way in which conditions an intersection space complex may exist. For $k = 2, ..., d$, we define $U_k := X \setminus X_{d-k}$ and we denote the canonical inclusions by $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-(k+1)} \to U_{k+1}$.

In Theorem 7.3, we give necessary and sufficient conditions for the existence of intersection space complexes. As in the topological setting, the construction proceeds inductively on each time deeper strata. At the $(k+1)$-th step, we have constructed an intersection space complex $IS_k$ on $X \setminus X_{d-k-1}$ such that the complex $j_{k+1}^* i_{k+1}^* IS_k$ on $X_{d-k-1} \setminus X_{d-k-2}$ is cohomologically locally constant. Comparing with the topological construction these local systems are the cohomology local systems of the fibration of link pairs at the $(k+1)$-th step of the construction. One can consider the natural triangle in the derived category:

$$\tau_{\leq q(k+1)} j_{k+1}^* i_{k+1}^* IS_k \to j_{k+1}^* i_{k+1}^* IS_k \to \tau_{> q(k-1)} j_{k+1}^* i_{k+1}^* IS_k \xrightarrow{[1]}.$$ 

The obstruction to perform the next step in the construction is the obstruction to split the triangle in the derived category. This is the constructible sheaf counterpart of the possibility to construct fibrewise homology truncations in the topological world. In Theorem 7.3, we study the parameter spaces classifying the possible intersection space complexes in step $k+1$ having fixed the step $k$ (they are not unique in general). In the same theorem, for the sake of comparison we...
provide a proof of the existence and uniqueness of intersection cohomology complexes using the same kind of techniques.

There are extension groups controlling the obstructions for existence and uniqueness at each step. They are recorded in Corollary 7.6.

If \( X \) is an algebraic variety, we can lift our construction of intersection space complexes to the category of mixed Hodge modules over \( X \). This is Theorem 8.1 and, as a corollary, one obtains a mixed Hodge structure in the cohomology of intersection spaces. The obstructions for existence and uniqueness are the same kind as extension groups, but taken in the category of mixed Hodge modules (see Corollary 8.2). Using the same techniques, we show that, for an arbitrary perversity, the intersection cohomology complexes are mixed Hodge modules (Theorem 8.3). This puts a mixed Hodge structure on intersection cohomology with arbitrary perversity. We believe that this should be well known, but we provide a proof since it is a simple consequence of our ideas.

2.4. Classes of spaces admitting intersection space complexes and counterexamples.

From the previous section, it is clear that the spaces admitting intersection space complexes need to be special. In this Section 9, we find two sufficient conditions for this, yielding an ample class of (yet special) examples.

By the results explained previously it is clear that pseudomanifolds admitting a trivial conical structure (see Definition 3.13), the intersection space complex exists by the combination of Theorems 3.30 and 5.18.

The point is that if one only needs the existence of intersection space complexes one can relax the triviality properties. In Theorem 9.8 we prove that if the link fibrations are trivial in the sense of Definition 3.10 then the intersection space complex exists. This leaves out the hypothesis on the compatibility of trivializations.

The next class of examples admitting intersection space complexes are pseudomanifolds whose strata are of “cohomological dimension at most 1 for local systems” in the sense of Definition 9.9. This is proved in Theorem 9.10. This includes the case studied by Banagl in [4] and the case of complex analytic varieties with critical set of dimension 1 which are sufficiently singular, in the sense that there are no positive dimensional compact strata (Corollary 9.11).

From our previous results, it is clear that a necessary condition for the existence of an intersection space is the existence of an intersection space complex. We find a few examples (see Example 9.15, Example 9.16 and Example 9.17) not admitting intersection space complex. The last example is an algebraic variety with two strata and the perversity is the middle one. So, one should not expect that algebraicity helps in the existence of intersection spaces. The idea to produce the examples is to observe that, if an space admits intersection space complexes, certain differentials in the Leray spectral sequence of the fibrations of links (which in our sheaf theoretic treatment is a local to global spectral sequence) have to vanish (Proposition 9.13 and Corollary 9.14).

2.5. Duality.

In Section 10, we prove that, if \( \bar{p} \) and \( \bar{q} \) are complementary perversities and \( \text{IS}_{\bar{p}} \) is an intersection space complex for perversity \( \bar{p} \), then its Verdier dual is an intersection space complex for perversity \( \bar{q} \) (Theorem 10.1). So, the Verdier duality functor exchanges the sets of intersection space complexes for complementary perversities. The proof follows the axiomatic treatment of [13] for intersection cohomology complexes.

A surprising consequence is that the existence of intersection space complexes is equivalent for complementary perversities (Corollary 10.2).

Then, we move to the case of depth 1 stratifications and prove, in Proposition 10.3 that, for generic choices of the intersection space complexes, the Betti numbers are always the same (they are minimal). Then, in Theorem 10.6 we show that the Betti numbers symmetry predicted by
Poincaré duality for complementary perversities is satisfied for generic intersection space Betti numbers.

2.6. Open questions. Here is a list of natural questions for further study:

1. We conjecture that the intersection space complexes associated via Definition 5.15 to the intersection space pairs constructed for pseudomanifolds with trivial conical structure in Theorem 3.30 are self Verdier dual when the strata are of even codimension and the perversity is the middle one.

2. Assume that the intersection space complex exist. Does there exist an associated rational homotopy intersection space? Are there further restrictions than the existence of the intersection space complex?

3. If the intersection space complex exist, can one define on its hypercohomology a natural internal cup product? Can one find a product turning its space of sections into a differential graded algebra inducing a cup product? This would lead to a definition of “intersection space rational homotopy type”.

4. Toric varieties have intersection space pairs. Compute their Betti numbers in terms of the combinatorics of the fan.

5. Generalize the generic Poincare duality Theorem 10.6 to the case of arbitrary depth stratifications.

6. This is a suggestion of Banagl: relate our sheaf theoretical methods with the characteristic class obstructing Poincare duality discussed in [6]. If the characteristic class vanishes, is the intersection space complex self-dual in the case of pseudomanifolds with trivial conical structure, even codimensional strata and middle perversity?

3. A topological construction of intersection spaces

3.1. Topological preliminaries. First, we give some basic definitions about \( t \)-uples of spaces in order to fix notation.

**Definition 3.1.**

1. A \( t \)-uple of spaces is a set of topological spaces \((Z_1, \ldots, Z_t)\).

2. A morphism from a \( t \)-uple of spaces into a space \((Z_1, \ldots, Z_t) \to Z\) is a set of morphisms \(\varphi_i : Z_i \to Z\).

3. A morphism between \( t \)-uples of spaces \((Z_1, \ldots, Z_t) \to (Z'_1, \ldots, Z'_t)\) is a set of morphisms \(\varphi_i : Z_i \to Z'_i\).

4. The mapping cylinder of a morphism \(\varphi = (\varphi_1, \ldots, \varphi_t) : (Z_1, \ldots, Z_t) \to Z\), \(\text{cyl}(\varphi)\), is the union of the \( t \)-uple \((Z_1, \ldots, Z_t) \times [0, 1]\) with \((Z, \text{Im}(\varphi_2), \ldots, \text{Im}(\varphi_t))\) with the equivalence relation \(\sim\) such that for \(i = 1, \ldots, t\) and for every \(x \in Z_i\), we have \((x, 1) \sim \varphi_i(x)\).

**Remark 3.2.** Remember that the mapping cylinder of a morphism of spaces \(f : X \to Y\) is \(\text{cyl}(f) := (X \times [0, 1] \sqcup Y)/\sim\) where \(\sim\) is the equivalence relation such that for every \(x \in X\), \((x, 1) \sim f(x)\).

**Definition 3.3.** Let \(\sigma : (Z_1, \ldots, Z_t) \to B\) be a locally trivial fibration of \( t \)-uples of spaces. The cone of \(\sigma\) over the base \(B\) is the locally trivial fibration

\[\pi : \text{cyl}(\sigma) \to B,\]

where \(\text{cyl}(\sigma)\) is the mapping cylinder of \(\sigma\), \(\pi(x, t) := \sigma(x)\) for \((x, t) \in (Z_1, \ldots, Z_t) \times [0, 1]\) and \(\pi(b) := b\) for \(b \in B\) (the definition of \(\pi\) is compatible with the identifications made to construct \(\text{cyl}(\sigma)\)). The cone over a fibration has a canonical vertex section

\[s : B \to \text{cyl}(\sigma)\]

sending any \(b \in B\) to the vertex of the cone \((\text{cyl}(\sigma))_b\).
The following figure shows schematically $cyl(\sigma)$ when the fiber of $\sigma$ is the pair $(T, \Sigma)$ where $T$ is a torus and $\Sigma$ is isomorphic to $S^1$, and the base $B$ is a circle. The fibre $\Sigma$ is depicted into the torus $T$ as a discontinuous circle.

![Figure 1](image)

**Figure 1**

We use a definition of topological pseudomanifold similar to [1, Definition 4.1.1].

**Definition 3.4.** A topological pseudomanifold is a paracompact Hausdorff topological space with a filtration by closed subspaces

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

such that

- Each pair $(X_i, X_{i-1})$ is a locally finite relative CW-complex.
- Every non-empty $X_{d-k} \setminus X_{d-k-1}$ is a topological manifold of dimension $d-k$ called pure stratum of $X$.
- $X \setminus X_{d-2}$ is dense in $X$.
- **Local normal triviality.** For each point $x \in X_{d-k} \setminus X_{d-k-1}$, there exists an open neighborhood $U$ of $x$ in $X$, a compact topological pseudomanifold $L$ of dimension $k-1$ with stratification

$$L = L_{k-1} \supset L_{k-3} \supset ... \supset L_0 \supset L_{-1} = \emptyset$$

and a homeomorphism

$$\varphi : U \xrightarrow{\cong} \mathbb{R}^{d-k} \times c^0(L),$$

where $c^0(L)$ is the open cone of $L$, such that it preserve the strata, that is, $\varphi(U \cap X_{d-r}) = \mathbb{R}^{d-k} \times c^0(L_{r-1})$.

$L$ is called the link of $X$ over the point $x$.

The following new notion is important in our constructions.

**Definition 3.5.** Let $(X, Y)$ be a pair of topological spaces and let

$$X_{d-k} \supset X_{d-k-1} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

be a topological pseudomanifold such that $X_{d-k}$ is a subspace of $Y$. We say that the pair $(X, Y)$ has a conical structure with respect to the stratified subspace if for every $r \geq k$ there exits an open neighbourhood $TX_{d-r}$ of $X_{d-r} \setminus X_{d-r-1}$ in $X \setminus X_{d-r-1}$, with the following properties:
(1) Let $\overline{TX_{d-r}}$ be the closure of $TX_{d-r}$ in $X$. There is a locally trivial fibration of $2(r-k+1)$-uples of spaces

$$
(\overline{TX_{d-r}} \setminus X_{d-r-1}) \cap (X, Y, TX_{d-k}, X_{d-k}, TX_{d-k-1}, X_{d-k-1}, \ldots, TX_{d-r+1}, X_{d-r+1})
$$

such that its restriction to the boundary

$$
(\partial \overline{TX_{d-r}} \setminus X_{d-r-1}) \cap (X, Y, TX_{d-k}, X_{d-k}, TX_{d-k-1}, X_{d-k-1}, \ldots, TX_{d-r+1}, X_{d-r+1})
$$

is a locally trivial fibration.

(2) The fibration $\sigma_{d-r}$ is the cone of $\sigma_{d-r}$ over the base $X_{d-r} \setminus X_{d-r-1}$.

(3) Let $k \leq r_1 < r_2 \leq d$ and consider the isomorphism induced by property (2)

$$
TX_{d-r_1} \cap (\overline{TX_{d-r_2}} \setminus X_{d-r_2-1}) \cong (TX_{d-r_1} \cap (\partial \overline{TX_{d-r_2}} \setminus X_{d-r_2-1})) \times [0, 1]/\sim
$$

where $\sim$ is the equivalence relation of the mapping cylinder.

If we remove $X_{d-r_1-1}$ in both parts of this isomorphism, we obtain an isomorphism

$$
\phi_{r_1,r_2} : (TX_{d-r_1} \setminus X_{d-r_1-1}) \cap TX_{d-r_2} \cong ((TX_{d-r_1} \setminus X_{d-r_1-1}) \cap \partial TX_{d-r_2}) \times [0, 1).
$$

Note that since $X_{d-r_2}$ is contained in $X_{d-r_1-1}$, the vertex section of the cone is not included in the previous spaces.

With this notation, we have the equality

$$(\sigma_{d-r_1})_!(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap TX_{d-r_2} = \phi_{r_1,r_2}^{-1} \circ ((\sigma_{d-r_1})_!(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap \partial TX_{d-r_2}) \circ \phi_{r_1,r_2},$$

that is, the fibration $\sigma_{d-r_1}$ in the intersection $(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap TX_{d-r_2}$ is determined by its restriction to $(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap \partial TX_{d-r_2}$.

Figure 1 illustrates this property.

(4) Let $k \leq r_1 < r_2 \leq d$. If $\partial \overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}) \neq \emptyset$, then we have the following equality of $2(r_1 - k + 1)$-uples

$$
(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap (X, Y, TX_{d-k}, X_{d-k}, TX_{d-k-1}, X_{d-k-1}, \ldots, TX_{d-r_1+1}, X_{d-r_1+1})
$$

and, in this space, we have

$$
\sigma_{d-r_1} \circ \sigma_{d-r_2} = \sigma_{d-r_2}.
$$

Figure 2 illustrates this property.

**Notation 3.6.** Consider a tuple $(Y_1, \ldots, Y_l)$ of subspaces whose components are a subset of the components of the tuple $(X, Y, TX_{d-k}, X_{d-k}, TX_{d-k-1}, X_{d-k-1}, \ldots, TX_{d-r_1+1}, X_{d-r_1+1})$ considered in the previous definition. The fibre of the fibration $\sigma_{d-r}$ restricted to the tuple $(Y_1, \ldots, Y_l)$ is called the fibre of the link bundle of $(Y_1, \ldots, Y_l)$ over $X_{d-r}$.

**Remark 3.7.** The fibration $\sigma_{d-r}$ is the fibration of links of $X_{d-r} \setminus X_{d-r-1}$ and the fibration $\sigma_{d-r}$ is the fibration associated to a tubular neighborhood.

The fact that these fibrations of $2(r_k + 1)$-uples are locally trivial yields that the intersection of the link $L_x$ of over the point $x \in X_{d-r} \setminus X_{d-r-1}$ with the open neighbourhoods $TX_{d-k}, TX_{d-k-1}, \ldots, TX_{d-r_1+1}$ only depends on the connected component of $X_{d-r} \setminus X_{d-r-1}$ containing $x$. 
Let $k \leq r_1 < r_2 \leq d$. The following figure shows how the open neighbourhoods $TX_{d-r}$ intersect each other.

The following figures show how are the morphisms $\sigma_{d-r_1}$ and $\sigma_{d-r_2}$ in $(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap TX_{d-r_2}$ because of Properties (3) and (4) of Definition 3.5.
Remark 3.8. Using properties (3) and (4) of Definition 3.5 we can also deduce that

\[(TX_{d-r_1} \setminus X_{d-r_1-1}) \cap (X,Y,TX_{d-k},X_{d-k},TX_{d-k-1},X_{d-k-1},\ldots,TX_{d-r_1+1},X_{d-r_1+1}) \cap TX_{d-r_2} = \sigma_d^{-1} (TX_{d-r_2} \cap (X_{d-r_1} \setminus X_{d-r_1-1}))\]

and, in this space, we have

\[\sigma_{d-r_2} \circ \sigma_{d-r_1} = \sigma_{d-r_2}.\]

Notation 3.9. Along this chapter, we will use the superindex \(\partial\) to denote the fibrations of boundaries of suitable tubular neighborhoods.

Definition 3.10. We say that a conical structure verifies the \(r\)-th triviality property \((T_r)\) if the locally trivial fibration \(\sigma_{d-r}^\partial\) is trivial, that is, the following two properties hold for every connected component \(S_{d-r}\) of \(X_{d-r} \setminus X_{d-r-1}\).

1. There exist an isomorphism

\[(\sigma_{d-r}^\partial)^{-1}(S_{d-r}) \cong L \times S_{d-r},\]

where \(L\) denotes the \(2(r-k+1)\)-uple of the links of \(S_{d-r}\) in

\[(X,Y,TX_{d-k},X_{d-k},TX_{d-k-1},X_{d-k-1},\ldots,TX_{d-r+1},X_{d-r+1}).\]

2. Under the identification given by property (1), \(\sigma_{d-r}^\partial\) restricted to \(L \times S_{d-r}\) is the canonical projection over \(S_{d-r}\).

Definition 3.11. Let \((X,Y)\) be a pair of spaces with a conical structure as in Definition 3.5 which verifies the \(r\)-th triviality property \((T_r)\) for any \(r\). Fix a trivialization

\[(\sigma_{d-r}^\partial)^{-1}(S_{d-r}) \cong L \times S_{d-r}\]

over each connected component of each stratum. The set of all trivializations is called a system of trivializations for the conical structure.

Let \((X,Y)\) be a pair of spaces with a conical structure as in Definition 3.5. Fix a system of trivializations for the conical structure.

Let \(k \leq r_1 < r_2 \leq d\) verifying that there exist connected components \(S_{d-r_1}\) and \(S_{d-r_2}\) of \(X_{d-r_1} \setminus X_{d-r_1-1}\) and \(X_{d-r_2} \setminus X_{d-r_2-1}\), respectively, such that \((\sigma_{d-r_2}^\partial)^{-1}(S_{d-r_2}) \cap S_{d-r_1} \neq \emptyset\), or what is the same, that the closure \(S_{d-r_1}\) contains \(S_{d-r_2}\).
We say that the system of trivializations is where all the morphisms except $(S_d - r_2) \cap \sigma_{d - r_1}^{-1}(S_{d - r_1})$ and $\sigma_{d - r_1}^{-1}((\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1})$ are $2(k - r + 1)$-uplas. To simplify the notation, along the following reasoning we denote by $(\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap \sigma_{d - r_1}^{-1}(S_{d - r_1})$ and $\sigma_{d - r_1}^{-1}((\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1})$ the first components of these $2(k - r + 1)$-uplas.

Consider also the space $(\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1}$.

Since we have fixed a system of trivializations, we have isomorphisms

\[ L_{d - r_2}^S \times S_{d - r_2} \cong (\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1} \]

and

\[ L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})} \times S_{d - r_2} \cong (\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap \sigma_{d - r_1}^{-1}(S_{d - r_1}) \]

where $L_{d - r_2}^S$ denotes the fibre of the link bundle of $S_{d - r_2}$ over $S_{d - r_2}$ and $L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})}$ denotes the fibre of the link bundle of $\sigma_{d - r_1}^{-1}(S_{d - r_1})$ over $S_{d - r_2}$. Moreover, by the property (4) of Definition 3.5 we have an equality

\[ \sigma_{d - r_1}^{-1}(S_{d - r_1}) \cap (\sigma_{d - r_2})^{-1}(S_{d - r_2}) = \sigma_{d - r_1}^{-1}((\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1}) \]

and, again using the fixed system of trivializations, we obtain an isomorphism

\[ \sigma_{d - r_1}^{-1}((\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1}) \cong c(L_{d - r_2}^X) \times ((\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1}) \]

where $L_{d - r_2}^X$ is the fibre of the link bundle of $X$ over $S_{d - r_1}$.

Combining the previous isomorphisms, we have

\[ L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})} \times S_{d - r_2} \cong c(L_{d - r_2}^X) \times ((\sigma_{d - r_2})^{-1}(S_{d - r_2}) \cap S_{d - r_1}) \cong c(L_{d - r_2}^X) \times L_{d - r_2}^{S_{d - r_1}} \times S_{d - r_2} \]

Let us denote by $\gamma$ the isomorphism

\[ \gamma : c(L_{d - r_2}^X) \times L_{d - r_2}^{S_{d - r_1}} \times S_{d - r_2} \rightarrow L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})} \times S_{d - r_2}. \]

Since under the equivalences given by the trivializations the morphisms $\sigma_{d - r_1}$ and $\sigma_{d - r_2}$ are the canonical projections, using the property (4) of Definition 3.5 the diagram

\[ \begin{array}{ccc}
L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})} \times S_{d - r_2} & \xrightarrow{\gamma} & L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})} \times S_{d - r_2} \\
(\sigma_{d - r_2})^{-1}(S_{d - r_2}) \times S_{d - r_2} & \xrightarrow{\gamma} & (\sigma_{d - r_2})^{-1}(S_{d - r_2}) \times S_{d - r_2} \\
& \downarrow & \\
& S_{d - r_2} &
\end{array} \]

where all the morphisms except $\gamma$ are the canonical projections, is commutative.

So $\gamma$ verifies the following condition:

\[ \gamma : c(L_{d - r_2}^X) \times L_{d - r_2}^{S_{d - r_1}} \times S_{d - r_2} \rightarrow L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})} \times S_{d - r_2} \]

\[ (x, y, z) \rightarrow (\gamma_1(x, y, z), z) \]

**Definition 3.12.** We say that the system of trivializations is compatible if for any two connected components $S_{d - r_1}$ and $S_{d - r_2}$ as above, the map if $\gamma_1$ does not depend on $z$, that is, if there exist an isomorphism $\beta : c(L_{d - r_2}^X) \times L_{d - r_2}^{S_{d - r_1}} \rightarrow L_{d - r_2}^{\sigma_{d - r_1}^{-1}(S_{d - r_1})}$ such that $\gamma = (\beta, Id_{S_{d - r_2}})$.

**Definition 3.13.** We say that the conical structure is trivial if it verifies the $r$-th triviality property $(T_r)$ for any $r$ and there exists a compatible system of trivializations.
Remark 3.14. For a great variety of topological pseudomanifolds

\[ X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset, \]

the pair \( (X, X_{d-2}) \) has a conical structure with respect to the stratification

\[ X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset. \]

Whitney stratifications, for instance, verify this property (see [12]). We consider every topological pseudomanifold which appear in sections 3, 4 and 5 holds this. Moreover, we fix such a conical structure and denote the relevant neighbourhoods \( TX_{d-r} \) for \( r \) varying.

From Section 6, it is not necessary to do this assumption

Remark 3.15. Toric varieties with its canonical stratification are topological pseudomanifolds which have a trivial conical structure with respect to the stratification. This is derived easily via the torus action.

3.2. An inductive construction of intersection spaces. Given a topological pseudomanifold we define an inductive procedure on the depth of the strata. The procedure depends on choices made at each inductive step, and may be obstructed for a given set of choices or carried until the deepest stratum. If for a given set of choices can be carried until the end, it produces a pair of spaces which generalizes Banagl intersection spaces.

Definition 3.16. A perversity is a map \( \bar{p} : \mathbb{Z}_{\geq 2} \rightarrow \mathbb{Z}_{\geq 0} \) such that \( \bar{p}(2) = 0 \) and \( \bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1 \).

Some special perversities are

- The zero perversity, \( \bar{0}(k) = 0 \).
- The total perversity, \( \bar{t}(k) = k - 2 \).
- The lower middle perversity, \( \bar{m}(k) = \lfloor \frac{k}{2} \rfloor - 1 \).
- The upper middle perversity, \( \bar{n}(k) = \lceil \frac{k}{2} \rceil - 1 \).

Given a perversity \( \bar{p} \), the complementary perversity is \( \bar{q} \). It is usually denoted by \( \bar{q} \).

The lower and the upper middle perversities are complementary.

For our construction we need the notion of fibrewise homology truncation of fibrations of pairs of locally finite \( CW \)-complexes:

Definition 3.17. Let \( \sigma : (X, Y) \rightarrow B \) be a locally trivial fibration. We say that \( \sigma \) admits a fibrewise rational \( q \)-homology truncation if there exits a morphism of pairs of spaces

\[ \phi : (X_{\leq q}, Y_{\leq q}) \rightarrow (X, Y) \]

such that \( \sigma \circ \phi \) is a locally trivial fibration and, for any \( b \in B \),

1. the homomorphism in homology of fibres

\[ H_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q}) \rightarrow H_i(X_b, Y_b; \mathbb{Q}) \]

is an isomorphism if \( i \leq q \).
2. the homology group \( H_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q}) \) vanishes if \( i > q \).

Notation 3.18. Given a pair of spaces \( (X, Y) \) we denote by \( (X, Y)_{\leq q} \) the pair \( (X_{\leq q}, Y_{\leq q}) \) appearing in the definition above.

The following figure shows a fibrewise rational 1-homology truncation of the fibration \( \sigma \) in Figure 1. The fiber of the resulting fibration becomes a pointed circle.
**Definition 3.19.** Given a pair of spaces \((X, Y)\) with a conical structure as in Definition 3.5, a fibrewise rational \(q\)-homology truncation of \(\sigma_{d-r}^0\)

\[
\left(\partial TX_{d-r} \setminus X_{d-r-1} \cap (X, Y)\right)_{\leq q}
\]

is compatible with the conical structure if, for every \(r' > r\),

\[
\sigma_{d-r'} \circ (\sigma_{d-r}^0)_{\leq q} : (\sigma_{d-r}^0)^{-1}(TX_{d-r'} \cap (X_{d-r} \setminus X_{d-r-1})) \to X_{d-r'} \setminus X_{d-r'-1}
\]

is the cone of

\[
\sigma_{d-r'}^0 \circ (\sigma_{d-r}^0)_{\leq q} : (\sigma_{d-r}^0)^{-1}(\partial TX_{d-r'} \cap (X_{d-r} \setminus X_{d-r-1})) \to X_{d-r'} \setminus X_{d-r'-1}
\]

**Proposition 3.20.** Given a pair of spaces \((X, Y)\) with a conical structure as in Definition 3.5, if there exist a fibrewise rational \(q\)-homology truncation of \(\sigma_{d-r}^0\), then there exist a fibrewise rational \(q\)-homology truncation of \(\sigma_{d-r}^0\) compatible with the conical structure.

**Proof.** Let us consider a fibrewise rational \(q\)-homology truncation of \(\sigma_{d-r}^0\), \((\sigma_{d-r}^0)_{\leq q}\).

Then, the restriction of \((\sigma_{d-r}^0)_{\leq q}\) to

\[
(\sigma_{d-r}^0)_{\leq q}^{-1}(X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r' > r} TX_{d-r'}
\]

is a fibrewise rational \(q\)-homology truncation of the restriction of \(\sigma_{d-r}^0\) to

\[
(\sigma_{d-r}^0)^{-1}(X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r' > r} TX_{d-r'}
\]

Using the property (2) of Definition 3.5, we can extend the previous restriction to a fibrewise rational \(q\)-homology truncation over

\[
(X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r'' > r'''} TX_{d-r'''}
\]
for any $r' > r$ inductively. Moreover, using the property (3) of Definition 3.5, we can check that these extensions are compatible with the conical structure. So, when $r' = d$, we obtain a fibrewise rational $q$-homology truncation of $\sigma^0_{d-r}$ compatible with the conical structure. □

**Definition 3.21.** Given a pair of spaces $(X, Y)$ with a conical structure verifying the triviality property $(T_r)$ for any $r$ (see Definition 3.10), choose a system of trivializations as in Definition 3.11. A fibrewise rational $q$-homology truncation of $\sigma^0_{d-r}$

\[
(\partial TX_{d-r} \setminus X_{d-r-1} \cap (X, Y))_{\leq q} \\
\begin{array}{c}
(\sigma^0_{d-r})_{\leq q} \\
\sigma^0_{d-r}
\end{array} \\
\begin{array}{c}
X_{d-r} \setminus X_{d-r-1} \\
\partial TX_{d-r} \setminus X_{d-r-1} \cap (X, Y)
\end{array}
\]

is compatible with the trivialization if the following conditions hold.

1. Given a connected component $S_{d-r}$ of $X_{d-r} \setminus X_{d-r-1}$, if $L_{d-r} = (L^X_{d-r}, L^Y_{d-r})$ denotes the fibre of the link bundle of $(X, Y)$ over $S_{d-r}$ and

\[
(\sigma^0_{d-r})^{-1}(S_{d-r}) \cong L_{d-r} \times S_{d-r}
\]

is the isomorphism induced by the system of trivializations, then there exist a pair of spaces $(L_{d-r})_{\leq q} := ((L^X_{d-r})_{\leq q}, (L^Y_{d-r})_{\leq q})$ such that the group $H((L^X_{d-r})_{\leq q}, (L^Y_{d-r})_{\leq q}; \mathbb{Q})$ vanishes if $i > q$, we have an isomorphism

\[
(\sigma^0_{d-r})^{-1}(S_{d-r}) \cong (L_{d-r})_{\leq q} \times S_{d-r}
\]

and, under these identifications, $((\sigma^0_{d-r})_{\leq q})(L_{d-r})_{\leq q} \times S_{d-r} = (\phi_1, Id_{S_{d-r}})$ where $\phi_1 : (L_{d-r})_{\leq q} \to L_{d-r}$ is a morphism such that

\[
H_i(\phi_1) : H_i((L^X_{d-r})_{\leq q}, (L^Y_{d-r})_{\leq q}; \mathbb{Q}) \to H_i(L^X_{d-r}, L^Y_{d-r}; \mathbb{Q})
\]

is an isomorphism if $i \leq q$.

2. Given $r' > r$ and a connected component $S_{d-r'}$ of $X_{d-r'} \setminus X_{d-r'-1}$ such that

\[
(\sigma^0_{d-r'})^{-1}(S_{d-r'}) \cap S_{d-r} \neq \emptyset,
\]

let $L^S_{d-r'}$ and $\sigma^0_{d-r'}^{-1}(S_{d-r'})$ denote the fibres of the link bundles of $S_{d-r}$ and $\sigma^0_{d-r'}(S_{d-r})$ over $S_{d-r'}$, respectively. Moreover, let

\[
\gamma : c(L^X_{d-r}) \times L^S_{d-r'} \times S_{d-r'} \cong L^\sigma_{d-r'}^{-1}(S_{d-r}) \times S_{d-r'}
\]

be the isomorphism defined in Equation [1], in the discussion preceding Definition 3.11. Then, the image of the composition

\[
c((L^X_{d-r})_{\leq q}) \times L^S_{d-r'} \times S_{d-r'} \xrightarrow{(c(\phi_1), Id)} c(L^X_{d-r}) \times L^\sigma_{d-r'}^{-1}(S_{d-r}) \times S_{d-r'}
\]

\[
\gamma
\]

\[
L^\sigma_{d-r'}^{-1}(S_{d-r}) \times S_{d-r'}
\]

is equal to $A \times S_{d-r'}$ for some subset $A \subset L^\sigma_{d-r'}^{-1}(S_{d-r})$.

**Remark 3.22.** If the conical structure is trivial (see Definition 3.13), the condition (1) of the previous definition implies the condition (2).
Remark 3.23. If a fibrewise rational $q$-homology truncation of $\sigma^\partial_{d-r}$ is compatible with the trivialization, then it is also compatible with the conical structure.

The initial step of the induction.
Let $X$ be a topological pseudomanifold such that the pair $(X, X_{d-2})$ has a conical structure with respect to the stratification, we consider the open neighbourhoods $TX_{d-r}$ fixed in Remark 3.14.

Let $m$ be the minimum such that $X_{d-m} \setminus X_{d-m-1} \neq \emptyset$. If the fibration

$$\sigma^\partial_{d-m} : \partial TX_{d-m} \setminus X_{d-m-1} \to X_{d-m} \setminus X_{d-m-1}$$

predicted in Definition 3.5 does not admit a fibrewise rational $\bar{q}(m)$-homology truncation, then the intersection space does not exist. Otherwise we choose a fibrewise rational $\bar{q}(m)$-homology truncation compatible with the conical structure (see Proposition 3.20).

\[
\begin{aligned}
&\left(\partial TX_{d-m} \setminus X_{d-m-1}\right)_{\leq \bar{q}(m)} \\
\phi_d^\partial_{d-m} : (\sigma^\partial_{d-m})_{\leq \bar{q}(m)} &\to X_{d-m} \setminus X_{d-m-1} \\
\sigma^\partial_{d-m} : X_{d-m} \setminus X_{d-m-1} &\to X_{d-m} \setminus X_{d-m-1} \\
\partial TX_{d-m} \setminus X_{d-m-1} &\to X_{d-m} \setminus X_{d-m-1}
\end{aligned}
\]

We construct a new space $X'_m$, a homotopy equivalence $\pi_m : X'_m \to X$ with contractible fibres and a subspace $T^p_m X \hookrightarrow X'_m$ as follows.

Define the map

$$\left(\sigma_{d-m}\right)_{\leq \bar{q}(m)} : \text{cyl}((\sigma^\partial_{d-m})_{\leq \bar{q}(m)}) \to X_{d-m} \setminus X_{d-m-1}$$

to be the cone of the fibration $(\sigma^\partial_{d-m})_{\leq \bar{q}(m)}$ over $X_{d-m} \setminus X_{d-m-1}$. By property (2) of Definition 3.5 there exists a fibre bundle morphism

$$\phi_d^\partial_{d-m} : \text{cyl}((\sigma^\partial_{d-m})_{\leq \bar{q}(m)}) \to TX_{d-m} \setminus X_{d-m-1}$$

ever the base $X_{d-m} \setminus X_{d-m-1}$ which preserves the vertex sections. Let

$$\theta_d^\partial_{d-m} : \text{cyl}((\sigma^\partial_{d-m})_{\leq \bar{q}(m)}) \to X \setminus X_{d-m-1}$$

be the composition of the fibre bundle morphism $\phi^\partial_{d-m}$ with the natural inclusion of the closed subset $TX_{d-m} \setminus X_{d-m-1}$ into $X \setminus X_{d-m-1}$. Let $\text{cyl}(\theta_d^\partial_{d-m})$ be the mapping cylinder of $\theta_d^\partial_{d-m}$. It is by definition the union $\text{cyl}((\sigma^\partial_{d-m})_{\leq \bar{q}(m)}) \times [0,1] \coprod (X \setminus X_{d-m-1})$ under the usual equivalence relation. Denote by

$$s_d^\partial_{d-m} : X_{d-m} \setminus X_{d-m-1} \to \text{cyl}((\sigma^\partial_{d-m})_{\leq \bar{q}(m)})$$

the vertex section. We define $Z_m$ to be the result of quotienting $\text{cyl}(\theta_d^\partial_{d-m})$ by the equivalence relation which identifies, for any $x \in X_{d-m} \setminus X_{d-m-1}$, the subspace $s_d^\partial_{d-m}(x) \times [0,1]$ to a point.

In order to follow in an easier way our further constructions, observe at this point that the mapping cylinder $\text{cyl}(\phi^\partial_{d-m})$ of the vertical map $\phi^\partial_{d-m}$ of diagram (2) above is a subspace both of $\text{cyl}(\theta_d^\partial_{d-m})$ and of $Z_m$.

The following figures show $\text{cyl}(\theta_d^\partial_{d-m})$ and $Z_m$. 

\[
\begin{aligned}
&\text{cyl}(\theta_d^\partial_{d-m}) \\
\coprod \text{cyl}(\theta_d^{\partial \partial}_{d-m}) &\to X \setminus X_{d-m-1} \\
\partial TX_{d-m} \setminus X_{d-m-1} &\to X_{d-m} \setminus X_{d-m-1}
\end{aligned}
\]
The equivalence relation collapses the vertical line over the origin of the horizontal plane in Figure 6, therefore the yellow line in Figure 7 becomes diagonal.

We have a natural projection map \( \pi_m : Z_m \to X \setminus X_{d-m-1} \) which is a homotopy equivalence whose fibres are contractible and has a natural section denoted by \( \alpha_m \). \( \alpha_m \) is a closed inclusion of \( X \setminus X_{d-m-1} \) into \( Z_m \). In the previous figure, \( \pi_m \) is the projection onto the horizontal plane and \( \alpha_m \) is the inclusion of the horizontal plane in the rest of the picture.

Define \( X_m \) as the set \( Z_m \cup X_{d-m-1} \). The projection map extends to a projection \( \pi_m : X'_m \to X \).

Consider in \( X'_m \) the topology spanned by all the open subsets of \( Z_m \) and the collection of subsets of the form \( \pi_m^{-1}(U) \) for any open subset \( U \) of \( X \).

This projection is also a homotopy equivalence whose fibres are contractible and such that the natural section \( \alpha_m \) extends to it giving a closed inclusion of \( X \) into \( X'_m \).

Define the step \( m \) intersection space \( I_{\bar{p}m}X \) to be the subspace of \( X'_m \) given by

\[
I_{\bar{p}m}X := \text{cyl}(\{\sigma_{d-m}^0 \leq \bar{q}(m)\}) \cup \text{cyl}(\phi_{d-m}^0) \cup (X \setminus TX_{d-m}),
\]

with the restricted topology.

The following figure shows \( I_{\bar{p}m}X \).

**Remark 3.24.** Note that we have the equality \( \text{cyl}(\phi_{d-m}^0) = \pi_m^{-1}(\partial TX_{d-m} \setminus X_{d-m-1}) \).

With the above definitions we have the following chains of inclusions

\[
X'_m \supset X \supset X_{d-m} \supset ... \supset X_0,
\]

\[
X'_m \supset I_{\bar{p}m}X \supset X_{d-m} \supset ... \supset X_0,
\]

where \( X \) is embedded in \( X'_m \) via the section \( \alpha_m \).

An immediate consequence of our construction is
Lemma 3.25. The pairs \((X'_r, X_{d-r})\) and \((I'_{m}X, X_{d-m})\) have a conical structure with respect to the stratified subspace \(X_{d-m-1} \supset \ldots \supset X_0\), given by the following open neighbourhoods of \(X_{d-r} \setminus X_{d-r-1}\) : \(\pi_{m-1}^{-1}(TX_{d-r})\) is a neighbourhood in \(X'_m \setminus X_{d-r-1}\) and \(\pi_{m-1}^{-1}(TX_{d-r}) \cap I'_{m}X\) is a neighbourhood in \(I'_{m}X \setminus X_{d-r-1}\).

The inductive step.

At this point we are ready to set up the inductive step of the construction of intersection spaces. The inductive step is different in nature to the initial step in the following sense. The necessary condition to be able to carry the initial step is that a link fibration admits a fibrewise rational \(q\)-homology truncation. In the inductive step, this condition is replaced by the fact that a fibration of link pairs admits a fibrewise rational \(q\)-homology truncation. The smaller space in the pair is constructed by iterated modifications of \(X_{d-2} = X_{d-m}\). Define

\[ I'_{m}(X_{d-2}) := X_{d-2}. \]

We assume by induction that, for \(k \geq m\), we have constructed

(i) a space \(X'_k\) and a projection

\[ \pi_k : X'_k \to X \]

which is a homotopy equivalence with contractible fibres, together with a section \(\alpha_k\) providing a closed inclusion of \(X\) into \(X'_k\).

(ii) subspaces \(I'_{k}(X_{d-2}) \subset I'_{k}X \subset X'_k\) such that, embedding \(X\) into \(X'_k\) via \(\alpha_k\), we have the topological pseudomanifold

\[ X_0 \subset X_1 \subset \ldots \subset X_{d-k-1} \]

embedded into \(I'_{k}(X_{d-2})\),

(iii) the pairs \((X'_k, I'_{k}X), (I'_{k}X, I'_{k}(X_{d-2}))\) have respective conical structures with respect to the stratified subspace described in the previous point. The open neighbourhoods of \(X_{d-r} \setminus X_{d-r-1}\) appearing in these structures are \(\pi_{k-1}^{-1}(TX_{d-r})\) in \(X'_k \setminus X_{d-k-1}\) and \(\pi_{k-1}^{-1}(TX_{d-r}) \cap I'_{k}X\) in \(I'_{k}X \setminus X_{d-k-1}\) respectively.

If \(X_{d-k-1} \setminus X_{d-k-2}\) is empty we define \(X'_{k+1} := X'_k, \pi_{k+1} := \pi_k, \alpha_{k+1} := \alpha_k, I'_{k+1} := I'_{k}X, \) and \(I'_{k+1}(X_{d-2}) := I'_{k}(X_{d-2})\). It is clear that the required conditions are satisfied.

If \(X_{d-k-1} \setminus X_{d-k-2}\) is not empty, since the pair \((I'_{k}X, I'_{k}(X_{d-2}))\) has a conical structure with respect to the stratified subspace

\[ X_0 \subset X_1 \subset \ldots \subset X_{d-k-1}, \]

we have a locally trivial fibration of pairs

\[ \sigma^0_{d-k-1} : (\partial \pi_k^{-1}(TX_{d-k-1}) \setminus X_{d-k-2}) \cap (I'_{k}X, I'_{k}(X_{d-2})) \to X_{d-k-1} \setminus X_{d-k-2}. \]

If the fibration does not admit a fibrewise rational \(\bar{q}(k+1)\)-homology truncation, then the intersection space construction cannot be completed with the previous choices.

Otherwise we choose a fibrewise rational \(\bar{q}(k+1)\)-homology truncation compatible with the conical structure (see Proposition 3.20).
We construct now a homotopy equivalence \( \pi_{k+1} : X'_{k+1} \to X \) with contractible fibres and a pair of subspaces \((I^p_{k+1}X, I^p_{k+1}(X_{d-2})) \hookrightarrow X'_{k+1}\) as follows.

Let

\[
(\sigma_{d-k-2})_{\leq \bar{q}(k+1)} : cyl((\sigma_{d-k-1})_{\leq \bar{q}(k+1)}) \to X_{d-k-1} \setminus X_{d-k-2}
\]

be the cone of the fibration of pairs \((\sigma_{d-k-1})_{\leq \bar{q}(k+1)}\) over \(X_{d-k-1} \setminus X_{d-k-2}\). Recall that, according with Definition 3.3, \(cyl((\sigma_{d-k-1})_{\leq \bar{q}(k+1)})\) is a pair of spaces.

By property (2) of Definition 3.5 there exists a morphism of fibre bundles of pairs

\[
\phi_{d-k-1} : cyl((\sigma_{d-k-1})_{\leq \bar{q}(k+1)}) \to (\pi_k^{-1}(TX_{d-k-1}) \setminus X_{d-k-2}) \cap (I^p_{k}X, I^p_{k}(X_{d-2}))
\]

over the base \(X_{d-k-1} \setminus X_{d-k-2}\) which preserves the vertex sections.

Let

\[
\theta_{d-k-1} : cyl((\sigma_{d-k-1})_{\leq \bar{q}(k+1)}) \to X'_{k} \setminus X_{d-k-2}
\]

be the composition of the fibre bundle morphism \(\phi_{d-k-1}\) with the natural inclusion

\[
(\pi_k^{-1}(TX_{d-k-1}) \setminus X_{d-k-2}) \cap (I^p_{k}X, I^p_{k}(X_{d-2})) \to X'_{k} \setminus X_{d-k-2}.
\]

Let \(cyl(\theta_{d-k-1})\) be the mapping cylinder of \(\theta_{d-k-1}\). It is by definition the union of the pair \(cyl((\sigma_{d-k-1})_{\leq \bar{q}(k+1)}) \times [0, 1] \) with the pair \((X'_{k} \setminus X_{d-k-2}, Im(\theta_{d-k-1}))\) with the usual equivalence relation (where \(\phi_{d-k-1}\) denotes the second component of the fibre bundle of pairs \(\phi_{d-k-1}\)).

Denote by

\[
s_{d-k-1} : X_{d-k-1} \setminus X_{d-k-2} \to cyl((\sigma_{d-k-1})_{\leq \bar{q}(k+1)})
\]

the vertex section. We define \(Z_{k+1}\) to be the pair of spaces which results of quotienting \(cyl(\theta_{d-k-1})\) by the equivalence relation which identifies, for any \(x \in X_{d-k-1} \setminus X_{d-k-2}\), the subspace \(s_{d-k-1}(x) \times [0, 1]\) to a point.

We denote the spaces forming the pair \(Z_{k+1}\) by \(Z_{k+1}^1, Z_{k+1}^2\). We have a natural projection map \(\rho_{k+1} : Z_{k+1}^1 \to X'_{k} \setminus X_{d-k-2}\) which is a homotopy equivalence whose fibres are contractible, and has a natural section denoted by \(\beta_{k+1}\). The composition

\[
\pi_{k+1} := \pi_k|X'_{k} \setminus X_{d-k-2} \circ \rho_{k+1} : Z_{k+1}^1 \to X \setminus X_{d-k-2}
\]

is a homotopy equivalence with contractible fibres, and has a section \(\alpha_{k+1} := \beta_{k+1} \circ \alpha_k|X \setminus X_{d-k-2}\) providing a closed inclusion of \(X \setminus X_{d-k-2}\) into \(Z_{k+1}^1\).

Define \(X'_{k+1}\) as the set \(Z_{k+1}^1 \cup X_{d-k-2}\). The projection maps \(\rho_{k+1}\) and \(\pi_{k+1}\) extends to projections

\[
(3) \quad \rho_{k+1} : X'_{k+1} \to X'_{k}
\]

\[
(4) \quad \pi_{k+1} : X'_{k+1} \to X.
\]

Consider the topology in \(X'_{k+1}\) spanned by the all the open subsets of \(Z_{k+1}\) and the collection of subsets of the form \(\pi_{k+1}^{-1}(U)\) for any open subset \(U\) of \(X\). With this topology the projections are also homotopy equivalences whose fibres are contractible, and such that the natural sections \(\beta_{k+1}\) and \(\alpha_{k+1}\) extend to them as closed inclusions.

Define the step \(k+1\) intersection space pair to be the pair of subspaces of \(X'_{k+1}\) given by

\[
(I_{k+1}^pX, I_{k+1}^p(X_{d-2})) := cyl((\sigma_{d-k-1})_{\leq \bar{q}(k)}) \times \{0\} \cup cyl(\phi_{d-k-1}) \cup (I^p_{k}X, I^p_{k}(X_{d-2})) \setminus \pi_k^{-1}(TX_{d-k-1}),
\]

with the restricted topology.

**Remark 3.26.** Note that we have the equality \(cyl(\phi_{d-k-1}) = \pi_{k+1}^{-1}(\partial TX_{d-k-1} \setminus X_{d-k-2})\).

With the definitions above, and using that the homology truncation is compatible with the conical structure, it is easy to check that conditions (i)-(iii) are satisfied replacing \(k\) by \(k+1\) and the induction step is complete.
Definition 3.27. Given a topological pseudomanifold \( X_d \supset \ldots \supset X_0 \) such that the pair \((X_d, X_{d-2})\) has a conical structure with respect to the stratification (see Definition 3.21 and Remark 3.14), we say that it has an intersection space pair if there exist successive choices of suitable fibrewise homology truncations so that the construction above can be carried up to \( k = d \). In that case the pair 

\[
(I_\bar{p}^d X, I_\bar{p}^d (X_{d-2})) = (I_\bar{p}_d^d X, I_\bar{p}_d^d (X_{d-2}))
\]

is called an intersection space pair associated with the stratification.

Definition 3.28. We denote \( X' := X'_d \). The homotopy model of \( X \) is the homotopy equivalence \( \pi_d \) which we denote \( \pi : X' \rightarrow X \). The section \( \alpha_d \) is denoted by \( \alpha : X \rightarrow X' \) and provides a closed inclusion of \( X \) into \( X' \).

Remark 3.29. If the intersection space pair exists it does not have to be unique up to homotopy. The different choices of fibrewise homology truncations may yield different choices of intersection spaces. The construction of intersection spaces follow the scheme of obstruction theory in algebraic topology: previous choices of fibrewise homology truncation may affect the possibility of finishing the construction in the subsequent steps.

3.3. Intersection spaces for pseudomanifolds having trivial conical structures. Let \( X \) be a topological pseudomanifold with a trivial conical structure (see Definition 3.13). Fix a compatible system of trivializations (see Definitions 3.11 and 3.12). We carry the inductive construction of the intersection space pair as above, but we add the following property to the properties (i)-(iii) which are checked along the induction:

(iv) the conical structures of the pairs \((X'_k, I_k^p X)\), \((I_k^p X, I_k^p (X_{d-2}))\) with respect to

\[
X_0 \subset X_1 \subset \ldots \subset X_{d-k-1}
\]

are trivial, and a compatible system of trivializations is inherited from the inductive construction.

At the initial step of the construction we have a topological pseudomanifold

\[
X \supset X_{d-m} \supset \ldots \supset X_0
\]

with a trivial conical structure and a compatible set of trivializations (as before the codimension of the first non-open stratum is \( m \)).

The compatible system of trivializations gives us a fixed trivialization of the fibration

\[
\sigma_{d-m}^p : \partial TX_{d-m} \setminus X_{d-m-1} \rightarrow X_{d-m} \setminus X_{d-m-1}.
\]

Choose a rational \( \bar{q}^p(m) \)-homology truncation of the fibre. This is always possible and elementary. Now, using the trivialization, the rational \( \bar{q}(m) \)-homology truncation of the fibre propagates to a fibrewise \( \bar{q}(m) \)-homology truncation of the fibration above. This is the truncation chosen at the initial step.

Now, using the compatibility of our system of trivializations, it is easy to show that the pairs \((X'_k, I_k^p X)\), \((I_m^p X, I_m^p (X_{d-2}))\) satisfy the required properties (i)-(iv). The compatible systems of trivializations required in property (iv) are inherited, by construction, by the compatible system of trivializations used at the beginning.

The inductive step of the construction is carried in the same way: the fixed trivializations propagate rational homology truncations of the corresponding fibrations of pairs of links.

We have proven:

Theorem 3.30. If \( X \) is a topological pseudomanifold with a trivial conical structure (see Definition 3.13) then there exist an intersection space pair associated with it for every perversity.

Corollary 3.31. Let \( X \) be a toric variety. Then, \( X \) has an intersection space pair for every perversity.
4. A sequence of Intersection Space pairs

Our aim is to associate with any choice of intersection space pair, a constructible complex on the original topological pseudomanifold $X$, whose hypercohomology coincides with the hypercohomology of the intersection space $IPX$. In order to do so, we define an increasing sequence of modified intersection space pairs, all of them included in the homotopy model $X'$. We provide precise definitions of the sequence, but leave many of the straightforward checking to the reader.

4.1. Systems of neighborhoods. Given a topological pseudomanifold

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

such that the pair $(X, X_{d-2})$ has a conical structure with respect to the stratification

$$X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

(see Remark 3.14), we denote the relevant neighbourhoods by $T^rX_{d-r}$, for $r$ varying.

Property (2) of Definition 3.5 states that the fibration $\sigma_{d-r}$ is the cone of the fibration $\sigma_{d-r}^0$ over the base $X_{d-r} \setminus X_{d-r-1}$. This means precisely that $\partial TX_{d-r} \setminus X_{d-r-1}$ is equal to the product

$$\partial TX_{d-r} \setminus X_{d-r-1} \cong [0, 1],$$

modulo the equivalence relation which identifies $(x, 1)$ and $(y, 1)$ if $\sigma_{d-r}^0(x) = \sigma_{d-r}^0(y)$.

For any $r \in 2, ..., d$ and any $n \in \mathbb{N}$ we define the open neighborhood $T^n X_{d-r}$ to be the quotient of

$$\partial TX_{d-r} \setminus X_{d-r-1} \times (1 - 1/(n + 1), 1]$$

under the same equivalence relation.

The open subsets $T^n X_{d-r}$ for $n$ varying, form a system of tubular neighborhoods of $X_{d-r} \setminus X_{d-r-1}$ in $X \setminus X_{d-r-1}$, whose intersection is the stratum $X_{d-r} \setminus X_{d-r-1}$. Moreover, for any fixed $n$ the collection of neighborhoods $T^n X_{d-r}$, for $r$ varying, give a conical structure to $(X, X_{d-2})$ with respect to the topological pseudomanifold $X_{d-2} \supset ... \supset X_0$.

4.2. The sequence of intersection space pairs. Suppose that there exists successive choices of suitable fibrewise homology truncations so that the construction of intersection space pairs described in the previous section can be carried up to $k = d$. Fix such a choice.

Fix $n \in \mathbb{N}$. Following the inductive construction of the previous section we produce a sequence of pairs

$$(IP_k X, IP_k (X_{d-2}))$$

for $k = 2, ..., r$ as follows.

Let $m$ be the minimum such that $X_{d-m} \setminus X_{d-m-1} \neq \emptyset$. Define

$$K_m^n(X) := \pi_{m-1}^{-1}(TX_{d-m} \setminus T^n X_{d-m}) \subset X_m',$$

$$IP_m^n(X) := IP_m(X) \cup K_m^n(X),$$

$$C_m^n(X_{d-2}) := \emptyset,$$

$$IP_m^n(X_{d-2}) := IP_m^n(X_{d-2}) = X_{d-2}.$$  

Remark 4.1. Note that $\pi_{m-1}^{-1}(TX_{d-m} \setminus T^n X_{d-m}) \setminus X_{d-m-1} \cong \text{cyl}(\sigma_{d-m}^0) \times [0, 1 - 1/(n + 1)]$ (see Remark 3.24).

The following figure shows the previous modification in Figure 8. $K_m^n(X)$ is the union of blue set and brown set.
The following figure shows $TX_{d-m} \setminus T^n X_{d-m}$ in more dimensions than in the previous figure. $K_m^n(X)$ is the preimage of the blue set by the morphism $\pi_m$.

Assume that $K_k^n(X)$, $C_k^n(X_{d-2})$, $I_k^{p,n}(X)$ and $I_k^{p,n}(X_{d-2})$ have been defined. Recall that $\rho_{k+1}$ is the projection defined in Equation (3). Define

$$ K_{k+1}^n(X) := \rho_{k+1}^{-1} \left( (I_k^p X \cap \pi_k^{-1}(TX_{d-(k+1)})) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}) \right) \cup (K_k^n(X) \setminus \pi_k^{-1}(T^n X_{d-(k+1)})) \cup (K_k^n(X) \setminus \pi_k^{-1}(T^n X_{d-(k+1)})) \cup (C_k^n(X_{d-2}) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))$$

$$ I_{k+1}^{p,n}(X_{d-2}) := I_{k+1}^p(X_{d-2}) \cup C_{k+1}^n(X_{d-2}) $$

The following figure illustrates the second induction step. Recall that the codimension of the biggest non-open stratum is $m$. The figure shows $(TX_{d-m} \setminus T^n X_{d-m-1}) \setminus T^n X_{d-m}$ in blue and green and $(TX_{d-m-1} \setminus T^n X_{d-m-1}) \setminus (TX_{d-m} \setminus T^n X_{d-m})$ in yellow. $K_{m+1}^n(X)$ is the union of

- the preimage of the blue and green set by $\pi_{m+1}$
• the preimage of the yellow set by \( \pi_{m+1} \) intersected with the preimage of \( I_m^p X \) by \( \rho_{m+1} \) 
\( C_{m+1}^n(X_{d-2}) \) is the union of
  • the preimage of the blue and green set by \( \pi_{m+1} \)
  • the preimage of the yellow set by \( \pi_{m+1} \) intersected with the preimage of \( I_m^\bar{p}_m(X_{d-2}) \) by \( \rho_{m+1} \)

**Figure 11**

Iterate the construction until \( k = d \) and define

\[
(I_{\bar{p},n}^p X, I_{\bar{p},n}^p(X_{d-2})) := (I_{d}^p X, I_{d}^p(X_{d-2})),
\]

which is a pair of subsets of \( X' \).

Since the closed subsets \( K^n_k(X), C^n_k(X_{d-2}) \) are increasingly larger when \( n \) increases we have constructed a sequence of pairs of closed subsets

\[
(I_{\bar{p},n}^p X, I_{\bar{p},n}^p(X_{d-2})) \subset \ldots \subset (I_{\bar{p},n}^p X, I_{\bar{p},n}^p(X_{d-2})) \subset (I_{\bar{p},n+1}^p X, I_{\bar{p},n+1}^p(X_{d-2})) \subset \ldots
\]

An easy inspection on the construction shows:

**Proposition 4.2.** The previous construction has the following properties.

1. The inclusions \( I_{\bar{p},n}^p X \subset I_{\bar{p},n+1}^p X \) and \( I_{\bar{p},n}^p(X_{d-2}) \subset I_{\bar{p},n+1}^p(X_{d-2}) \) are strong deformation retracts for any \( n \in \mathbb{N} \). If we denote the inclusions by \( \nu_{n}^{X_{d-2}} : I_{\bar{p},n}^p(X_{d-2}) \to I_{\bar{p},n}^p X \), \( i_{X_{d-2}}^{n} : I_{\bar{p},n+1}^p(X_{d-2}) \to I_{\bar{p},n+1}^p X \) and \( i_{X_{d-2}}^{n} : I_{\bar{p},n}^p(X_{d-2}) \to I_{\bar{p},n+1}^p(X_{d-2}) \) and the retractions by \( r_{X_{d-2}}^{n} : I_{\bar{p},n+1}^p X \to I_{\bar{p},n}^p X \) and \( r_{X_{d-2}}^{n} : I_{\bar{p},n+1}^p(X_{d-2}) \to I_{\bar{p},n}^p(X_{d-2}) \), we have commutative diagrams

   \[
   \begin{array}{ccc}
   I_{\bar{p},n}^p(X_{d-2}) & \xrightarrow{\nu_{X_{d-2}}^{n}} & I_{\bar{p},n}^p X \\
   I_{\bar{p},n+1}^p(X_{d-2}) & \xrightarrow{r_{X_{d-2}}^{n+1}} & I_{\bar{p},n+1}^p X \\
   \end{array}
   \]

2. We have the equality \( I_{\bar{p},n}^p(X_{d-2}) = I_{\bar{p},n+1}^p(X_{d-2}) \cap I_{\bar{p},n}^p(X) \).
3. For any \( x \in X \setminus X_{d-2} \), there exist a small contractible neighbourhood \( U_x \) of \( x \) in \( X \) and a natural number \( n_0 \) such that, for every \( n > n_0 \), \( \pi^{-1}(U_x) \) is contained in \( I_{\bar{p},n}^p X \) and \( \pi^{-1}(U_x) \cap I_{\bar{p},n}^p(X_{d-2}) = \emptyset \).
(4) For any $x \in X_{d-r} \setminus X_{d-r-1}$, there exists a small contractible neighbourhood $U_x$ of $x$ in $X$ and a natural number $n_0$ such that, for any $n > n_0$, the diagram restricts to the diagram

\[
\begin{align*}
I^\nu n(X_{d-2}) \cap \pi^{-1}(U_x) \xrightarrow{\nu^n} I^\nu n X \cap \pi^{-1}(U_x) \\
r^n_{X_{d-2}} \bigg| r^n_{X_{d-2}} & = I^\nu n(X_{d-2}) \cap \pi^{-1}(U_x) \\
r^n_{X_{d-2}} \bigg| r^n_{X_{d-2}} & = I^\nu n(X_{d-2}) \cap \pi^{-1}(U_x)
\end{align*}
\]

and we have the equalities $r^n_{X_{d-2}}(I^\nu n+1(X_{d-2}) \cap \pi^{-1}(U_x)) = I^\nu n(X_{d-2}) \cap \pi^{-1}(U_x)$ and $r^n_{X}(I^\nu n+1(X) \cap \pi^{-1}(U_x)) = I^\nu n(X) \cap \pi^{-1}(U_x)$.

Then, the inclusions

\[
\begin{align*}
\pi^{-1}(U_x) \cap I^\nu n X & \hookrightarrow \pi^{-1}(U_x) \cap I^\nu n+1 X, \\
\pi^{-1}(U_x) \cap I^\nu n(X_{n-2}) & \hookrightarrow \pi^{-1}(U_x) \cap I^\nu n+1(X_{n-2})
\end{align*}
\]

are strong deformation retracts.

Sketch of the proof. For any $r$, we have an equality

\[
\left(\partial X_{d-r} \setminus T^n X_{d-r}\right) \setminus X_{d-r-1} = \partial T^n X_{d-r} \setminus X_{d-r-1} \times [0, 1 - 1/(n + 1)].
\]

So, there are canonical retractions $\partial T^n X_{d-r} \setminus T^n X_{d-r} \to \partial T^n X_{d-r} \setminus T^n X_{d-r}$. These retractions induce retractions $K^n r(X) \to K^n r(X)$ and $C^n r(X_{d-2}) \to C^n r(X_{d-2})$ which produce the morphisms $r^n_X$ and $r^n_{X_{d-2}}$ respectively.

Let $x \in X \setminus X_{d-2}$. A small ball $U_x$ around of $x$ verifies property (3) if there exist a natural number $n_0$ such that $U_x$ does not intersect $T^{n_0} X_{d-r}$ for any $r$. Moreover, this number $n_0$ exists if and only if $U_x \cap X_{d-2}$ is empty.

Let $x \in X_{d-r} \setminus X_{d-r-1}$. A small neighbourhood of $x$, $U_x$, verifying property (4) can be constructed as follows: let $V_x$ be a ball around $x$ in the stratum $X_{d-r} \setminus X_{d-r-1}$. Take $V_x$ small enough so that there exist a natural number $n_0$ such that $V_x$ does not intersect $T^{n_0} X_{d-k}$ for any $k > r$. Consider the retraction $\sigma_{d-r}$ appearing in Definition (3.5) (1). Define

\[
U_x := \sigma_{d-r}^{-1}(V_x) \cap T^{n_0-1} X_{d-r}.
\]

The following figure shows $U_x$ where $x \in X_{d-m} \setminus X_{d-m-1}$ in Figure 11.
Definition 4.3. Let $x$ be any point of $X$. If $x \in X \setminus X_{d-2}$, a principal neighbourhood of $x$ is a small neighbourhood which verifies Property (3) of Proposition 4.2. If $x \in X_{d-2}$, a principal neighbourhood of $x$ is a small neighbourhood which verifies Property (4) of Proposition 4.2.

Definition 4.4. Let $x \in X_{d-r} \setminus X_{d-r-1}$. A carved principal neighbourhood of $x$ is an open subset $U_x^* \setminus X_{d-r-1}$. A carved principal neighbourhood of $x$ is an open subset $U_x^* \setminus X_{d-r}$ where $U_x$ is a principal neighbourhood of $x$.

Analogously to Property (4) of Proposition 4.2, we have

**Proposition 4.5.** If $U_x^*$ is a carved principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$, there exist $n_0 \in \mathbb{N}$ such that, for every $n > n_0$, the diagram \((7)\) restricts to the diagram

\[
\begin{align*}
I_\nu^n (X_{d-r-2}) \cap \pi^{-1} (U_x^*) & \xrightarrow{\nu^n} I_\nu^n X \cap \pi^{-1} (U_x^*) \\
\rho^\nu_\delta X_{d-r-2} & \quad \bigg| \bigg| \\
I_{\nu}^{n+1} (X_{d-r-2}) \cap \pi^{-1} (U_x^*) & \quad \longrightarrow \quad I_{\nu}^{n+1} X \cap \pi^{-1} (U_x^*)
\end{align*}
\]

For the next propositions recall that $\sigma^\rho_{d-r}$ is the fibration of Definition 3.5. (2).

**Proposition 4.6.** If $U_x$ is a principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$ for any $r > 0$ and $n \in \mathbb{N}$ big enough, the cohomology group

\[
H^i(I_\nu^n X \cap \pi^{-1} (U_x)), I_\nu^n X_{d-r} \cap \pi^{-1} (U_x); \mathbb{Q})
\]

is 0 if $i \leq \bar{q}(r)$ and isomorphic to the $i$-th cohomology group of the pair $(\sigma^\rho_{d-r})^{-1}(x) \subset (I_{\nu}^{r-1} X, I_{\nu}^{r-1} (X_{d-r-2}))$ if $i > \bar{q}(r)$.

**Proposition 4.7.** If $U_x^*$ is a carved principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$ for any $r > 0$ and $n \in \mathbb{N}$ big enough, the cohomology group

\[
H^i(I_\nu^n X \cap \pi^{-1} (U_x)), I_\nu^n X_{d-r} \cap \pi^{-1} (U_x); \mathbb{Q})
\]

is isomorphic to the $i$-th cohomology group of the pair $(\sigma^\rho_{d-r})^{-1}(x) \subset (I_{\nu}^{r-1} X, I_{\nu}^{r-1} (X_{d-r-2}))$ for every $i \in \mathbb{Z}$.

5. Sheafification

5.1. Sheaf of cubical singular cochains. In this section, every topological space is hereditarily paracompact and locally contractible. In particular, the topological pseudomanifold and the intersection spaces of the previous section verify these properties.

In order to produce constructible complexes whose hypercohomology compute the cohomology of intersection spaces we use sheaves of singular cohomology cochains. For technical reasons cubical cochains, as developed by Massey in [15] Chapters 7 and 12, adapt best to our construction. Here we sketch very briefly the main points we need; the reader should check [12] for complete definitions and proofs.

We denote by $(C_\bullet(X, \mathbb{Q}), \partial)$ the complex of cubical chains of a space $X$. The group $C_i(X, \mathbb{Q})$ is defined to be the quotient

\[
C_i(X, \mathbb{Q}) := Q_i(X, \mathbb{Q})/D_i(X, \mathbb{Q}),
\]

where $Q_i(X, \mathbb{Q})$ is the vector space spanned by maps from the $i$-cube to $X$ and $D_i(X, \mathbb{Q})$ is the subspace of degeneratemaps (maps which are constant in one direction of the cube). The differential $\partial$ is defined in the usual way. The functor given by the homology of the complex $(C_\bullet(X, \mathbb{Q}), \partial)$ defines a homology theory with coefficients in $\mathbb{Q}$.

Let $(C^\bullet(X, \mathbb{Q}), \delta)$ be the complex of cubical cochains of $X$. It is by definition the dual of $(C_\bullet(X, \mathbb{Q}), \partial)$, and hence $C^\bullet(X, \mathbb{Q})$ is the subspace of $Hom(Q_i(X, \mathbb{Q}), \mathbb{Q})$ formed by elements vanishing at $D_i(X, \mathbb{Q})$. The functor given by the cohomology of the complex $(C^\bullet(X, \mathbb{Q}), \delta)$ defines a cohomology theory with coefficients in $\mathbb{Q}$.
Let $f : X \to Y$ be a continuous map. We denote by
\[
\begin{align*}
f_{\#} : C_\bullet(X, \mathbb{Q}) &\to C_\bullet(Y, \mathbb{Q}), \\
f^\# : C^\bullet(Y, \mathbb{Q}) &\to C^\bullet(X, \mathbb{Q})
\end{align*}
\]
the associated transformations of complexes of cubical chains and cochains. They form morphisms of complexes
\[
\begin{align*}
f_{\#} : (C_\bullet(X, \mathbb{Q}), \partial) &\to (C_\bullet(Y, \mathbb{Q}), \partial), \\
f^\# : (C^\bullet(Y, \mathbb{Q}), \delta) &\to (C^\bullet(X, \mathbb{Q}), \delta).
\end{align*}
\]

Let $f, g : X \to X$ two continuous maps. If $f$ and $g$ are homotopic, then $f_{\#}$ and $g_{\#}$ are homotopic morphism of complexes, and the same happens for $f^\#$ and $g^\#$. We need for later use an explicit form of a homotopy of complexes between $f_{\#}$ and $g_{\#}$. Let
\[
\rho : C_i(X, \mathbb{Q}) \to C_{i+1}(I \times X, \mathbb{Q})
\]
be the morphism such that, if $\sigma_i$ is a singular $i$-cube in $X$, $\rho(\sigma_i) = Id_I \times \sigma_i$ (the homomorphism $\rho$ takes degenerate cubical chains to degenerate cubical chains). Let $H : I \times X \to Y$ be an homotopy between $f$ and $g$, that is, $H_0 = f$ and $H_1 = g$. A homotopy between the morphism of complexes $f_{\#}$ and $g_{\#}$ is given by $H_{\#} \circ \rho$. The dual morphism of $H_{\#} \circ \rho$ is an homotopy between $f^\#$ and $g^\#$.

**Lemma 5.1.** Let $h : Z \to X$ a continuous map. If $(H_t)_{t \in I}$ is independent of $t \in I$, then for every $\sigma_i \in Q_i(Z, \mathbb{Q})$, $H_{\#} \circ \rho \circ H_{\#}(\sigma_i)$ is degenerate.

Now we produce a sheafification of cubical cochains. This is an adaptation of the sheafification of singular chains appearing in [17].

**Definition 5.2.** For every $i \in \mathbb{Z}_{\geq 0}$, let $C^i$ be the presheaf of vector spaces

\[
U \sim \rightarrow C^i(U, \mathbb{Q})
\]

where the restriction morphisms are the obvious ones.

The sheaf of cubical singular $i$-cochains $C^i_X$ is defined to be the sheafification of $C^i$.

For every $i \in \mathbb{Z}_{\geq 0}$, let $C^i_X(X)$ be the vector subspace of $C^i(X)$ given by the set of cochains $\xi^i \in C^i(X)$ such that there exist an open covering $\{U_j\}_{j \in J}$ of $X$ such that $\xi^i_{|U_j} = 0$ for every $j \in J$. As in [17] one shows that the sheafification is defined by

\[
C^i_X(U) = C^i(U)/C^i_X(U).
\]

At the level of sheaves we have functoriality as well. Let $f : X \to Y$ be a continuous map. Then, $f$ induces a morphism of complexes of sheaves on $Y$

\[
f^\# : C^\bullet_Y \to f_*C^\bullet_X.
\]

As one expects, if $X$ is contractible then

\[
H^i(C^i(X)) \cong \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}
\]

This implies that the complex of sheaves $C^\bullet_X$ is a resolution of the constant sheaf $\mathbb{Q}_X$.

Moreover, for every $i \in \mathbb{Z}_{\geq 0}$, the sheaf $C^i$ is flabby. Indeed, it is enough to prove the restriction morphisms of the presheaf, $C^i(X, \mathbb{Q}) \to C^i(U, \mathbb{Q})$, are surjective for every open subset $U \subset X$. Given $\xi \in C^i(U, \mathbb{Q})$, let $\xi_X \in C^i(X, \mathbb{Q})$ be the linear morphism $C_i(X, \mathbb{Q}) \to \mathbb{Q}$ such that, for every singular $i$-cube $\sigma$ in $X$, we have

\[
\xi_X(\sigma) = \begin{cases} \xi(\sigma) & \text{if } \text{Im}(\sigma) \subset U \\ 0 & \text{if } \text{Im}(\sigma) \not\subset U \end{cases}
\]

Then, $(\xi_X)_{|U} = \xi$. 

Corollary 5.3. For every \( i \in \mathbb{Z}_{\geq 0} \), the \( i \)-th cohomology group \( H^i(X, \mathbb{Q}) \) is isomorphic to \( i \)-th group of cohomology of the complex \( C_X^*(X) \).

5.2. The intersection space constructible complex. Let \( X \) be a topological pseudomanifold with stratification:
\[
X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset
\]
and a conical structure given by the stratification such that there exist a set of choices so that the inductive construction of the intersection space of \( X \) is not obstructed. Let \( X' \) be the homotopy model of \( X \) and \( \pi : X' \to X \) the homotopy equivalence. Let \((I^\bar{p}, n) X, I^\bar{p}, n(X_{d-2})\) with \( n \in \mathbb{N} \) be the associated sequence of intersection space pairs and
\[
j^n : I^\bar{p}, n X \to X'
\]
the canonical inclusions.

In order to lighten the formulas appearing in this section we denote by \( C^n_X \) and \( C^n_{X,d-2} \) the complex of sheaves of cubical singular chains in \( I^\bar{p}, n X \) and \( I^\bar{p}, n(X_{d-2}) \) respectively.

Proposition 5.4. For every \( n \in \mathbb{N} \), there exist a commutative diagram
\[
\begin{array}{ccc}
I^\bar{p}, n+1(X_{d-2}) \cap U & \xrightarrow{\nu^{n+1}} & I^\bar{p}, n X \cap U \\
\downarrow i_X^{n+1} & & \downarrow i_X^n \\
I^\bar{p}, n(X_{d-2}) \cap U & \xrightarrow{\nu^n} & I^\bar{p}, n+1 X \cap U
\end{array}
\]
where all the morphisms are surjective.

Proof. For every open subset \( U \subset X' \), the inclusions of Diagram \([5]\) restrict to a diagram
\[
\begin{array}{ccc}
I^\bar{p}, n(X_{d-2}) \cap U & \xrightarrow{\nu^n} & I^\bar{p}, n X \cap U \\
\downarrow i_X^{n+1} & & \downarrow i_X^n \\
I^\bar{p}, n+1(X_{d-2}) \cap U & \xrightarrow{\nu^n} & I^\bar{p}, n+1 X \cap U
\end{array}
\]
So, we have the following diagram between the cubical cochain groups:
\[
\begin{array}{ccc}
C^i(I^\bar{p}, n+1 X \cap U, \mathbb{Q}) & \xrightarrow{\nu^{n+1}} & C^i(I^\bar{p}, n+1(X_{d-2}) \cap U, \mathbb{Q}) \\
\downarrow i_X^n & & \downarrow i_X^{n+1} \\
C^i(I^\bar{p}, n X \cap U, \mathbb{Q}) & \xrightarrow{\nu^n} & C^i(I^\bar{p}, n(X_{d-2}) \cap U, \mathbb{Q})
\end{array}
\]
The morphisms of this diagram induce the morphisms of the proposition.

Moreover, these morphisms are surjective since every inclusion of topological spaces induces a surjection between the corresponding cubical cochain groups. \(\square\)

Denote by \( K^{n,*} \) the kernel of \( \nu^n \). There is a canonical morphism
\[
i^{n} : K^{n+1,*} \to K^{n,*}.
\]
Remark 5.5. For every \( i \in \mathbb{Z}_{\geq 0} \) and every \( n \in \mathbb{N} \), the \( i \)-th rational cohomology group of the pair \((I^\bar{p}, n X, I^\bar{p}, n(X_{d-2}))\) is isomorphic to \( i \)-th cohomology group of the complex \( K^{n,*} \).

Definition 5.6. Given a pair of natural numbers \( n_1 < n_2 \), we will define
\[
i^{n_1,n_2} := i^{n_1} \circ \ldots \circ i^{n_2-1} : K^{n_2,*} \to K^{n_1,*}
\]
Then, the complexes of sheaves \( \{ K^n_\bullet \}_{n \in \mathbb{N}} \) and the morphisms \( i^{n_1}_{n_2} \) form an inverse system and we can consider the inverse limit

\[
\lim_{\leftarrow n \in \mathbb{N}} K^n_\bullet,
\]

which is a complex of sheaves.

**Lemma 5.7.** \( (\pi_* \lim_{\leftarrow n \in \mathbb{N}} K^n_\bullet)_{X \setminus X_{d-2}} \) is quasi-isomorphic to \( Q_{X \setminus X_{d-2}} \).

**Proof.** Let \( x \in X \setminus X_{d-2} \). We have the following equalities:

\[
(\pi_* \lim_{\leftarrow n \in \mathbb{N}} K^n_\bullet)_x = \lim_{x \in U \text{open}} \pi_*(\lim_{n \in \mathbb{N}} K^n_\bullet(U)) = \lim_{x \in U \text{open}} (\lim_{n \in \mathbb{N}} K^n_\bullet)(\pi^{-1}(U))) = \lim_{x \in U \text{open}} \lim_{n \in \mathbb{N}} (K^n_\bullet(\pi^{-1}(U))).
\]

Let \( U_x \) be a principal neighbourhood of \( x \) (see definition 4.3). There exist \( n_0 \in \mathbb{N} \) such that for every open subset \( U \subseteq U_x \) and for every \( n > n_0 \), we have

\[
\pi^{-1}(U) \cap I^n_{\pi U} = \pi^{-1}(U)
\]

and

\[
\pi^{-1}(U) \cap I^n_{\pi U}(X_{d-2}) = \emptyset.
\]

Consequently, \( K^n_\bullet(\pi^{-1}(U)) = j^n_\pi C^n_\pi(\pi^{-1}(U)) = C^n_\pi(\pi^{-1}(U)) \) where \( C_\pi \) is the sheaf of singular \( i \)-cochains in \( X' \).

Moreover, for every \( n > n_0 \), \( i^n_\pi = i^n_\pi = Id_{C^n_\pi(\pi^{-1}(U))}. \) So, \( \lim_{\leftarrow n \in \mathbb{N}} (K^n_\bullet(\pi^{-1}(U))) = C^n_\pi(\pi^{-1}(U)). \)

Thus, we have shown that \( (\pi_* \lim_{\leftarrow n \in \mathbb{N}} K^n_\bullet)_{X \setminus X_{d-2}} \) is quasi-isomorphic to \( \pi_* Q_{\pi^{-1}(X \setminus X_{d-2})} \), and the later sheaf is quasi-isomorphic to \( Q_{X \setminus X_{d-2}} \) since \( \pi|_{\pi^{-1}(X \setminus X_{d-2})} \) is a homotopy equivalence. \( \square \)

We study now the cohomology of the complex \( \pi_* \lim_{\leftarrow n \in \mathbb{N}} K^n_\bullet \) over each of the deeper strata of \( X \) and over the global sections. With this purpose, we study the cohomology of \( \pi_* \lim_{\leftarrow n \in \mathbb{N}} K^n_\bullet \) in the principal and the carved principal neighbourhoods (see definitions 4.3 and 4.4) and in the total space.

**Proposition 5.8.** Let \( U \) be equal to \( X \) or a principal neighbourhood or a carved principal neighbourhood of some \( x \in X_{d-k} \setminus X_{d-(k+1)}. \) Then,

\[
H^i(\lim_{\leftarrow n \in \mathbb{N}} (K^n_\bullet(\pi^{-1}(U)))) \cong H^i(\lim_{\leftarrow n \in \mathbb{N}} (K^n_\bullet(\pi^{-1}(U))))
\]

Now, we need some preliminary work in order to prove Proposition 5.8.

**Lemma 5.9.** Let \( U \) be equal to \( X \) or a principal neighbourhood or a carved principal neighbourhood of some \( x \in X_{d-k} \setminus X_{d-(k+1)}. \) Then, there exist \( n_0 \in \mathbb{N} \) such that if \( n > n_0 \), there exist a morphism

\[
i^n_\# : K^n_\bullet(\pi^{-1}(U)) \to K^{n+1}_\bullet(\pi^{-1}(U))
\]

such that \( i^n_\#(\pi^{-1}(U)) \circ i^n_\# = Id_{K^n_\bullet(\pi^{-1}(U))} \) and \( r^n_\# \circ i^n_\#(\pi^{-1}(U)) \) is homotopic to the identity.

Moreover, there exists a homotopy \( h^n_U \) between \( r^n_\# \circ i^n_\#(\pi^{-1}(U)) \) and \( Id_{K^{n+1}_\bullet(\pi^{-1}(U))} \) such that \( i^n_\#(\pi^{-1}(U)) \circ h^n_U = 0. \)
Proof. Let $U$ be a principal neighbourhood and let $n_0$ be the natural number of Proposition 4.2 (4). For every $n > n_0$, the diagram (6) induces a diagram

$$ j_{n+1}^*C_{n+1}^X \xrightarrow{\nu_{n+1}^*} \mu_{n+1}^*C_{n+1}^{X_{d-2}}(\pi^{-1}(U)) \\
\begin{array}{ccc}
  r_{X_n^#} & i_{X_n^#} & r_{X_{d-2}^#} \\
  \downarrow & \downarrow & \downarrow \\
  j_n^*C_n^X(\pi^{-1}(U)) & \nu_n^* & r_{X_{d-2}^#} \\
\end{array} \\
\begin{array}{ccc}
  i_{X_n^#} & i_{X_{d-2}^#} & \nu_{n+1}^* \\
\end{array} \xrightarrow{\pi_n^*} \mu_n^*C_{n+1}^{X_{d-2}}(\pi^{-1}(U)) \\
$$

where $i_{X_{d-2}^#} \circ r_{X_{d-2}^#} = \text{Id}_{\nu_n^*C_n^X(\pi^{-1}(U))}$, $i_{X_n^#} \circ i_{X_{d-2}^#} = \text{Id}_{\nu_n^*C_n^X(\pi^{-1}(U))}$ and $r_{X_{d-2}^#} \circ i_{X_{d-2}^#}$ and $r_{X_n^#} \circ i_{X_n^#}$ are homotopic to the identity.

Then, we obtain a canonical morphism

$$ r_{U}^n^# : K^n_*(\pi^{-1}(U)) \rightarrow K^{n+1}_*(\pi^{-1}(U)) $$

such that $i^n_#(\pi^{-1}(U)) \circ r_{U}^n^# = \text{Id}_{K^n_*(\pi^{-1}(U))}$ and $r_{U}^n^# \circ i^n_#(\pi^{-1}(U))$ is homotopic to the identity.

Consider the diagram (6) of Proposition 4.2 (4). There exist homotopies $H_U^n : \pi^{-1}(U) \cap I^{p,n+1}X \times I \rightarrow \pi^{-1}(U) \cap I^{p,n+1}X$ between $i^n_# \circ r^n_#$ and $\text{Id}_{\pi^{-1}(U) \cap I^{p,n+1}X}$ and $H_{X_{d-2}^#} : \pi^{-1}(U) \cap I^{p,n+1}(X_{d-2}) \times I \rightarrow \pi^{-1}(U) \cap I^{p,n+1}(X_{d-2})$ between $i^n_# \circ r^n_#$ and $\text{Id}_{\pi^{-1}(U) \cap I^{p,n+1}(X_{d-2})}$. Moreover, we can suppose that, for every $t \in I$, the restrictions of $H_U^n$ and $H_{X_{d-2}^#}$ to $\pi^{-1}(U) \cap I^{p,n}X$ and $\pi^{-1}(U) \cap I^{p,n}(X_{d-2})$ are the identity respectively.

Following the procedure explained in Section 5.1 the mapping $H_U^n$ induces a homotopy between $i^n_# \circ r^n_#$ and $\text{Id}_{i^n_#C_{n+1}^X(\pi^{-1}(U))}$. Moreover, applying Lemma 5.1 we have that $i^n_# \circ h_{X_n}^n$ is equal to 0.

Similarly the mapping $H_{X_{d-2}^#}$ induce a homotopy $h_{X_{d-2}^#}^n$ between $i^n_# \circ r^n_#$ and $\text{Id}_{i^n_#C_{n+1}^X(\pi^{-1}(U))}$ such that $i^n_# \circ h_{X_{d-2}^#}^n$ is equal to 0.

So, there exist a homotopy $h_{U}^n$ between $i^n_#(\pi^{-1}(U)) \circ r_{U}^n_#$ and $\text{Id}_{K^{n+1}_*(\pi^{-1}(U))}$ such that $i^n_#(\pi^{-1}(U)) \circ h_{U}^n = 0$.

If $U$ is equal to $X$ or a carved principal neighbourhood we can apply the same method using the diagram (6) of Proposition 4.2 or the diagram of Proposition 4.3 respectively.

Remark 5.10. The diagram (5) of Proposition 4.2 is valid for every $n \in \mathbb{N}$. So, in the previous lemma, we can take $n_0 = 0$ if $U$ is the total space.

Remark 5.11. Note that the morphisms $r_{U}^n^#$ do not induce a morphism of complexes of sheaves since they are not defined for every open subset.

Definition 5.12. Given a pair of natural numbers $n_1 < n_2$ such that $n_0 < n_1$, we define

$$ r_{U}^{n_2, n_1} := r_{U}^{n_2-1, n_2} \circ \ldots \circ r_{U}^{n_1, n_2} : K^{n_1}_*(\pi^{-1}(U)) \rightarrow K^{n_2}_*(\pi^{-1}(U)) $$

Remark 5.13. For every $n > n_0$, the inclusions $\pi^{-1}(U) \cap I^{p,n}X \hookrightarrow \pi^{-1}(U) \cap I^{p,n+1}X$ and $\pi^{-1}(U) \cap I^{p,n}X_{d-2} \hookrightarrow \pi^{-1}(U) \cap I^{p,n+1}X_{d-2}$ are homotopy equivalences. So, $i_{X_n}^n(\pi^{-1}(U))$ and $i_{X_{d-2}^#}^n(\pi^{-1}(U))$ are quasi-isomorphisms.

Then,

$$ i^n_#(\pi^{-1}(U)) : K^{n+1}_*(\pi^{-1}(U)) \rightarrow K^{n}_*(\pi^{-1}(U)) $$
is also a quasi-isomorphism and we have an isomorphism
\[
\lim_{\leftarrow n \in \mathbb{N}} H^i(K^n \cdot (\pi^{-1}(U))) \rightarrow H^i(K^{n_0+1} \cdot ((\pi^{-1}(U))))
\]
\[
\{[\xi^i_n]\}_{n \in \mathbb{N}} \rightarrow [\xi^i_{n+1}]
\]

**Notation 5.14.** For every open subset \( V \subset X' \), the elements of \( H^i(\lim_{\leftarrow n \in \mathbb{N}} K^n \cdot (\pi^{-1}(V))) \) are equivalence classes of elements
\[
\{\xi^i_n\}_{n \in \mathbb{N}} \in \text{Ker}(\lim_{n \in \mathbb{N}} \partial^{i+1}_n(V)) \subset \lim_{n \in \mathbb{N}} K^{n,i}(V)
\]
which we are going to denote with \([\{\xi^i_n\}_{n \in \mathbb{N}}]\).

In addition, given \( n \in \mathbb{N} \) and an element \( \xi^i_n \in \text{Ker}(\partial^{i+1}_n(V)) \subset K^{n,i}(V) \), we are going to denote its equivalence class in \( H^i(K^n \cdot (\pi^{-1}(V))) \) with \([\xi^i_n]\).

**Proof of Proposition 5.8.** It is enough to prove that, if \( U \) is equal to \( X \) or a principal neighbourhood or a carved principal neighbourhood of \( x \), then the morphism
\[
\text{Ker}(\lim_{\leftarrow n \in \mathbb{N}} \partial^{i+1}_n(\pi^{-1}(U))) \xrightarrow{\alpha} \lim_{\leftarrow n \in \mathbb{N}} H^i(K^n \cdot (\pi^{-1}(U)))
\]
\[
\{\xi^i_n\}_{n \in \mathbb{N}} \rightarrow [[\xi^i_n]]_{n \in \mathbb{N}}
\]
factorises into a morphism
\[
H^i(\lim_{\leftarrow n \in \mathbb{N}} K^n \cdot (\pi^{-1}(U))) \xrightarrow{\beta} \lim_{\leftarrow n \in \mathbb{N}} H^i(K^n \cdot (\pi^{-1}(U)))
\]
which is an isomorphism.

First, we prove \( \alpha \) factorises. Let us consider an element \( \{\xi^i_n\}_{n \in \mathbb{N}} \in \text{Ker}(\lim_{\leftarrow n \in \mathbb{N}} \partial^{i}_n(\pi^{-1}(U))). \) Then, there exist an element \( \{\delta^{i-1}_n\}_{n \in \mathbb{N}} \in \text{Im}(\lim_{\leftarrow n \in \mathbb{N}} \partial^{i}_n(\pi^{-1}(U))) \) such that
\[
\lim_{n \in \mathbb{N}} \partial^{i}_n(\pi^{-1}(U))\{\delta^{i-1}_n\}_{n \in \mathbb{N}} = \{\xi^i_n\}_{n \in \mathbb{N}}.
\]

So, for every \( n \in \mathbb{N} \), \( \partial^{i}_n(\pi^{-1}(U))\delta^{i-1}_n = \xi^i_n. \) Consequently, \( \alpha([\xi^i_n]) = 0 \) and we conclude that the morphism factorises. Now, we prove that \( \alpha \) and, consequently, \( \beta \) are surjective.

Because of Remark 5.13, it is enough to prove that, for every element \( \xi \in \text{Ker}(\partial^{i+1}_{n_0+1}((\pi^{-1}(U)))) \), there exist an element \( \{\xi^i_n\}_{n \in \mathbb{N}} \in \text{Ker}(\lim_{\leftarrow n \in \mathbb{N}} \partial^{i+1}_n(\pi^{-1}(U))) \) such that \( \xi^i_{n_0+1} = \xi. \)

Let us consider
\[
\xi^i_n = \begin{cases} 
(i^{n,n_0+1}(\pi^{-1}(U)))\xi & \text{if } n < n_0 + 1 \\
\xi & \text{if } n = n_0 + 1 \\
r_{U}^{n+1,n}(\xi) & \text{if } n > n_0 + 1
\end{cases}
\]

Then, for every pair of natural numbers \( n_1 < n_2 \), we have the equality
\[
((i^{n_1,n_2})(\pi^{-1}(U)))\xi^i_{n_2} = \xi^i_{n_1}.
\]

Hence \( \{\xi^i_n\}_{n \in \mathbb{N}} \) belongs to \( \lim_{\leftarrow n \in \mathbb{N}} K^{n,i}(\pi^{-1}(U)). \)

Moreover
- we have the vanishing \( (\partial^{i+1}_{n_0+1}(\pi^{-1}(U)))\xi = 0, \)
- for every pair of natural numbers \( n_1 < n_2 \) we have the equality
  \[
  \partial^i_{n_1} \circ i^{n_1,n_2} = i^{n_1,n_2} \circ \partial^i_{n_2}
  \]
  and,
- if \( n_1 > n_0 \), we have the equality
  \[
  \partial^i_{n_2}(\pi^{-1}(U)) \circ r^{n_1,n_2}_U = r^{n_1,n_2}_U \circ \partial^i_{n_1}(\pi^{-1}(U)).
  \]
Consequently \((\partial_{n+1}^{i+1}(\pi^{-1}(U))) (\xi_n^i) = 0\) for every \(n \in \mathbb{N}\), \(\{\xi_n^i\}_{n \in \mathbb{N}}\) belongs to \(\text{Ker} \lim \leftarrow_{n \in \mathbb{N}} \partial_n^{i+1}(\pi^{-1}(U))\) and \(\alpha\) and \(\beta\) are surjective.

Finally, we prove \(\beta\) is injective, that is, \(\text{Ker}(\alpha) = \text{Im}(\text{lim} \leftarrow_{n \in \mathbb{N}} \partial_n^i(\pi^{-1}(U)))\).

Let \(\{\xi_n^i\}_{n \in \mathbb{N}} \in \text{Ker} \lim \leftarrow_{n \in \mathbb{N}} \partial_n^{i+1}(\pi^{-1}(U))\) such that \(\alpha(\{\xi_n^i\}_{n \in \mathbb{N}}) = \{[\xi_n^i]\}_{n \in \mathbb{N}} = 0\). Then, \(\xi_{n_0+1}^i \in \text{Im}(\partial_{n_0+1}(\pi^{-1}(U)))\). So, there exist \(\delta \in \mathcal{K}^{n_0+1, i-1}(\pi^{-1}(U))\) such that \((\partial_{n_0+1}(\pi^{-1}(U)))(\delta) = \xi_{n_0+1}^i\).

For every \(n \in \mathbb{N}\), we define

\[
\delta_{n}^{i-1} = \begin{cases} 
(i^{n,n_0+1}(\pi^{-1}(U)))(\delta) & \text{if } n < n_0 + 1 \\
\xi_{n_0+1}^i & \text{if } n = n_0 + 1 \\
r_U^{n_0+1,n}(\delta) & \text{if } n > n_0 + 1
\end{cases}
\]

Then, \(\{\delta_{n}^{i-1}\}_{n \in \mathbb{N}} \in \text{lim} \leftarrow_{n \in \mathbb{N}} \mathcal{K}^{n, i-1}(\pi^{-1}(U))\).

Let us denote \(\xi_n^i := (\partial_n^i(\pi^{-1}(U))) (\delta_n^{i-1})\) for every \(n \in \mathbb{N}\). Then we have the equality \(\text{lim} \leftarrow_{n \in \mathbb{N}} \partial_n^i(\pi^{-1}(U))(\{\delta_{n}^{i-1}\}_{n \in \mathbb{N}}) = \{\xi_n^i\}_{n \in \mathbb{N}}\) and \([\xi_n^i]_{n \in \mathbb{N}} = 0\) in \(H^i(\text{lim} \leftarrow_{n \in \mathbb{N}} \mathcal{K}^{n, \bullet}(\pi^{-1}(U)))\).

So, to prove \(\beta\) is injective, it is enough to prove the equality \([\{\xi_n^i\}_{n \in \mathbb{N}}] = [\{\xi_n^i\}_{n \in \mathbb{N}}]\) in \(H^i(\text{lim} \leftarrow_{n \in \mathbb{N}} \mathcal{K}^{n, \bullet}(\pi^{-1}(U)))\).

If \(n < n_0 + 1\), we have

\[
\xi_n^i = \partial_n^i(\pi^{-1}(U))(\delta_n^{i-1}) = \partial_n^i(\pi^{-1}(U))(i^{n,n_0+1}(\pi^{-1}(U)))(\delta) = i^{n,n_0+1}(\pi^{-1}(U))(\partial_{n_0+1}^i(\pi^{-1}(U)))(\delta) = i^{n,n_0+1}(\pi^{-1}(U))(\xi_{n_0+1}^i) = \xi_n^i.
\]

If \(n = n_0 + 1\), we have

\[
\xi_{n_0+1}^i = \partial_{n_0+1}^i(\pi^{-1}(U))(\delta) = \xi_{n_0+1}^i.
\]

If \(n > n_0 + 1\), we have

\[
\xi_n^i = \partial_n^i(\pi^{-1}(U))(\delta_n^{i-1}) = \partial_n^i(\pi^{-1}(U))(r_U^{n_0+1,n}(\delta)) = r_U^{n_0+1,n}(\partial_{n_0+1}^i(\pi^{-1}(U)))(\delta) = r_U^{n_0+1,n}(\xi_{n_0+1}^i) = r_U^{n_0+1,n}(i^{n_0+1,n}(\pi^{-1}(U)))(\xi_n^i).
\]

For every \(n > n_0 + 1\), let \(h_U^n\) be the homotopy defined in Lemma \(5.9\). A simple computation shows that, for every \(n > n_0 + 1\), we have the equality:

\[
\tilde{\xi}_n^i - \xi_n^i = \partial_n^i(\sum_{k=n_0+1}^{n-1} (r_U^k \circ h_U^{k-1})(\xi_k^i) + h_U^{n-1}(\xi_n^i)).
\]

Let

\[
\xi_n^{i-1} = \begin{cases} 
0 & \text{if } n \leq n_0 + 1 \\
\sum_{k=n_0+2}^{n-1} (r_U^k \circ h_U^{k-1})(\xi_k^i) + h_U^{n-1}(\xi_n^i) & \text{if } n > n_0 + 1
\end{cases}
\]

Let us prove \(\{\xi_n^{i-1}\}_{n \in \mathbb{N}} \in \text{lim} \leftarrow_{n \in \mathbb{N}} \mathcal{K}^{n, i-1}(\pi^{-1}(U))\).

Since, for every \(n > n_0 + 1\), \(i^{n,\#}(\pi^{-1}(U)) \circ h_U^n = 0\) and \(i^{n,\#}(\pi^{-1}(U)) \circ r_U^{n,\#} = \text{IdK}_{n,\bullet}(\pi^{-1}(U))\), if \(n_1 > n_0 + 1\), we have the equality

\[
\begin{align*}
&i^{n_1,n_2}(\pi^{-1}(U))\left(\sum_{k=n_0+2}^{n_2-1} (r_U^{k,n_2} \circ h_U^{k-1})(\xi_k^i) + h_U^{n_2-1}(\xi_{n_2}^i)\right) \\
&\quad + h_U^{n_2-1}(\xi_{n_2}^i) = 0.
\end{align*}
\]

and if \(n_1 \leq n_0 + 1\), we have

\[
\begin{align*}
&i^{n_1,n_2}(\pi^{-1}(U))\left(\sum_{k=n_0+2}^{n_2-1} (r_U^{k,n_2} \circ h_U^{k-1})(\xi_k^i) + h_U^{n_2-1}(\xi_{n_2}^i)\right) = 0.
\end{align*}
\]
Then, $\{\epsilon_i^{-1}\}_{n \in \mathbb{N}}$ belongs to $\lim_{\leftarrow n \in \mathbb{N}} (K^{n,i-1}(\pi^{-1}(U)))$ and we have the equality

$$\lim_{n \in \mathbb{N}} \partial^{i}(\pi^{-1}(U))((\epsilon_i^{-1})_{n \in \mathbb{N}}) = (\xi_i^n)_{n \in \mathbb{N}} - (\xi^n_i)_{n \in \mathbb{N}}.$$  

Therefore, $[(\xi^n_i)_{n \in \mathbb{N}}]$ equals $[(\xi^n_i)_{n \in \mathbb{N}}]$ in $H^i(\lim_{\leftarrow n \in \mathbb{N}} (K^{n,i}(\pi^{-1}(U))))$ and we conclude. \( \square \)

**Definition 5.15.** Let us define $[\pi_* \lim_{\leftarrow n \in \mathbb{N}} K^{n,\bullet}].$

**Theorem 5.16.** The hypercohomology of $IS := \pi_* \lim_{\leftarrow n \in \mathbb{N}} K^{n,\bullet}.$

**Proof.** The sheaves $K^{n,i}$ are flabby. Let $U$ be any open subset in $X'.$ Every section in $K^{n,i}(U)$, $\xi$, is also a section of $j^2_n C^{n,i}(U)$. Then, $\xi$ is the equivalence class of a singular cubical $i$-cochain $\xi' \in C^i(U \cap I^{\bar{p}n}X, \mathbb{Q})$. We can extend $\xi'$ by 0 to get a singular cubical $i$-cochain $\xi'_{I^{\bar{p}n}X} \in C^i(I^{\bar{p}n}X, \mathbb{Q})$, that is, for every singular $i$-cube $\sigma$ in $I^{\bar{p}n}X$, we have

$$\xi'_{I^{\bar{p}n}X}(\sigma) = \begin{cases} \xi'(\sigma) & \text{if } \text{Im}(\sigma) \subset U, \\ 0 & \text{if } \text{Im}(\sigma) \not\subset U. \end{cases}$$

The equivalence class of $\xi'_{I^{\bar{p}n}X}$ in the sheaf of singular cubical $i$-cochains is a section $\xi_{X'}$ in $j^2_n C^{n,i}(X')$. It is easy to check that $\xi_{X'}$ is contained in $K^{n,i}(X')$ since it is an extension by 0 of $\xi$.

Now, we prove $\lim_{\leftarrow n \in \mathbb{N}} K^{n,i}$ is also flabby. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a section of $(\lim_{\leftarrow n \in \mathbb{N}} K^{n,i})(U) = \lim_{\leftarrow n \in \mathbb{N}} (K^{n,i}(U)).$ Since $K^{n,i}(U)$ is flabby for every natural $n$, the sections $\xi_n$ extend to 0 to sections $\xi_{X'}$ in $K^{n,i}(X')$. It is easy to check that $i^{n_1,n_2}(X')(\xi_{X'}) = \xi_{X'}$ for every pair of natural numbers $n_1 < n_2$. So, $\{\xi_{X'}\}_{n \in \mathbb{N}}$ is a global section of $\lim_{\leftarrow n \in \mathbb{N}} K^{n,i}$.

Then, all the sheaves of the complex $IS$ are flabby and, consequently, the hypercohomology of $IS$ is equal to the cohomology of the global sections of $IS$. By Proposition 5.8, there is an isomorphism

$$H^i(\Gamma(X, IS)) \cong \lim_{\leftarrow n \in \mathbb{N}} H^i(\Gamma(X', K^{n,\bullet})).$$

Since the cohomology $H^i(\Gamma(X', K^{n,\bullet}))$ is the cohomology of the intersection space pair for every $n \in \mathbb{N}$, we conclude. \( \square \)

Now we prove a set of properties of the complex $IS$, in a similar vein that those satisfied by intersection cocohomology sheaves.

**Definition 5.17.** For $k = 2, \ldots, d$, we define $U_k := X \setminus X_{d-k}$ and we denote the canonical inclusions with $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-(k+1)} \to U_{k+1}$.

**Theorem 5.18.** The complex of sheaves $IS$ satisfies the following properties.

1. $IS|_{U_2}$ is quasi-isomorphic to $\mathbb{Q}|_{U_2}$.
2. The cohomology sheaves $H^i(IS)$ are 0 if $i \notin \{0, 1, \ldots, d\}$
3. For $k = 2, \ldots, d$, the cohomology sheaves $H^i(j_k^! IS|_{U_{k+1}})$ are 0 if $i < \bar{q}(k)$.
4. For $k = 2, \ldots, d$, the usual morphisms between the cohomology sheaves $H^i(j_k^! IS|_{U_{k+1}}) \to H^i(j_{k+1}^* IS|_{U_{k+1}})$ are isomorphisms if $i > \bar{q}(k)$.

**Proof.** (1) is shown in lemma 5.7.

Let $x \in X_{d-k} \setminus X_{d-(k+1)}$ for some $k \in \{2, 3, \ldots, d\}$. Given a complex of sheaves we denote by $H^i$ and $\mathcal{H}^i$ its $i$-th cohomology presheaf and sheaf respectively. We have the obvious chain of equalities:

$$\mathcal{H}^i(\pi_* \lim_{\leftarrow n \in \mathbb{N}} K^{n,\bullet})_x = H^i(\pi_* \lim_{\leftarrow n \in \mathbb{N}} K^{n,\bullet})_x = \lim_{x \in U \text{open}} H^i(\lim_{\leftarrow n \in \mathbb{N}} K^{n,\bullet}(\pi^{-1}(U))) =$$
= \lim_{x \in U \text{ open}} H^i(\lim_{n \in \mathbb{N}} (K^{n,\bullet}(\pi^{-1}(U)))).

Since the principal neighbourhoods form a system of neighborhoods for any point, we can suppose every open subset $U$ appearing in the previous formula is a principal neighbourhood of $x$. Then, applying Proposition 5.8 we have

$$\mathcal{H}^i(\pi_* \lim_{n \in \mathbb{N}} K^{n,\bullet})_x = \lim_{U \text{ principal neighbourhood of } x} \lim_{n \in \mathbb{N}} H^i((K^{n,\bullet}(\pi^{-1}(U)))).$$  

So, applying proposition 4.6 $\mathcal{H}^i(\pi_* \lim_{n \in \mathbb{N}} K^{n,\bullet})_x$ is 0 if $i \leq q(k)$ and equal to $H^i((\sigma^0_{d-r})^{-1}(x); \mathbb{Q})$

if $i > q(k)$.

Hence, we have proven (2) and (3) of the theorem.

Moreover, applying again Proposition 5.8

$$\mathcal{H}^i(j^k_* i_* IS|_{U_k})_x = \lim_{U \text{ principal neighbourhood of } x} H^i(\lim_{n \in \mathbb{N}} \pi_* K^{n,\bullet}(U \setminus X_{d-k})) = \lim_{U \text{ principal neighbourhood of } x} \lim_{n \in \mathbb{N}} H^i(\pi_* K^{n,\bullet}(U \setminus X_{d-k}))$$

and, because of Proposition 4.7,

$$\mathcal{H}^i(j^k_* i_* IS|_{U_k})_x = H^i((\sigma^0_{d-r})^{-1}(x); \mathbb{Q})$$

for every $i \in \mathbb{Z}$, which concludes the proof. \qed

6. AXIOMS OF INTERSECTION SPACE COMPLEXES

From now on, we do not need to assume that our topological pseudomanifold has a conical structure with respect to the stratification, like in Remark 3.14.

Let $X$ be a topological pseudomanifold with the following stratification:

(9) $X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$

Let $U_k := X \setminus X_{d-k}$ and let $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ be the usual inclusions.

Let us denote by $D^b_{cc}(X)$ the bounded derived category of cohomologically constructible sheaves of rational vector spaces on $X$ with the previous stratification. Fix a perversity $\bar{p}$ and let us consider the following sets of properties in this category:

(1) We say that $B^\bullet \in D^b_{cc}(X)$ verifies $[AX1]_{\bar{p}}$ for perversity $\bar{p}$ if:

(a) $B^\bullet_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2},$

(b) the cohomology sheaf $\mathcal{H}^i(B^\bullet)$ is 0 if $i \notin \{0, 1, ..., n\},$

(c) $\mathcal{H}^i(j^k_* B^\bullet_{|U_{k+1}})$ is equal to 0 if $i > \bar{p}(k),$  

(d) the natural morphism $\mathcal{H}^i(j^k_* B^\bullet_{|U_{k+1}}) \to \mathcal{H}^i(j^k_* i_* B^\bullet|_{U_k})$ is an isomorphism if $i \leq \bar{p}(k).$

(2) Let $\bar{q}$ be the complementary perversity of $\bar{p}$. We say that $B^\bullet \in D^b_{cc}(X)$ verifies $[AXS1]_{\bar{q}}$ for perversity $\bar{q}$ if:

(a) $B^\bullet_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2},$

(b) the cohomology sheaf $\mathcal{H}^i(B^\bullet)$ is 0 if $i \notin \{0, 1, ..., n\},$

(c) $\mathcal{H}^i(j^k_* B^\bullet_{|U_{k+1}})$ is equal to 0 if $i \leq \bar{q}(k),$  

(d) the natural morphism $\mathcal{H}^i(j^k_* B^\bullet_{|U_{k+1}}) \to \mathcal{H}^i(j^k_* i_* B^\bullet|_{U_k})$ is an isomorphism if $i > \bar{q}(k).$
Definition 6.5. \( B^\bullet \) verifies \([AX1]_k\) for \( k = 2, \ldots, d \) if and only if \( B^\bullet [d] \) verifies the axioms \([AX1]\) of \([13]\) section 3.3, that is, if \( B^\bullet [d] \) is the intersection cohomology sheaf of \( X \). So, we will denote an object of \( D^b_{c\text{c}}(X) \) verifying \([AX1]_k\) for \( k = 2, \ldots, d \) by \( IC_p[-d] \).

Definition 6.2. An intersection space complex of \( X \) with perversity \( \bar{p} \) and stratification \( \{9\} \) is a complex of sheaves verifying \([AXS1]_k\) for \( k = 2, \ldots, d \).

We denote by \( IS\bar{p} \) a complex of sheaves in \( X \) with these properties.

Remark 6.3. If the stratification of \( X \) induces a conical structure (see Definition 3.5) and there exist an intersection space pair of \( X \) with perversity \( \bar{p} \) in the sense of Definition 3.27, then there exist an intersection space complex of \( X \) (see Theorem 5.18).

In the sequel, we will need equivalent axioms to \([AXS1]_k\). In the following remark, we review the method of \([13]\) section 3.4 to get equivalent axioms to \([AX1]_k\) and \([AXS1]_k\).

Remark 6.4. Using the long exact sequence of cohomology associated to the distinguished triangle
\[
(j_x^! B^\bullet_{\mathbb{U}_{k+1}} \to j_k^! B^\bullet_{\mathbb{U}_{k+1}} \to j_k^* i_{k*} B^\bullet_{\mathbb{U}_{k+1}} \to [1])
\]
and, for every \( x \in X \setminus X_{d-k} \setminus X_{d-k-1} \), the canonical inclusion. Then, applying the property 1.13(15) of \([13]\),
\[
j_x^! B^\bullet = u_x^! j_k^! B^\bullet_{\mathbb{U}_{k+1}} = u_x^* j_k^* B^\bullet_{\mathbb{U}_{k+1}} [k - d]
\]
we deduce \((d_k)\) of \([AXS1]_k\) is equivalent to \((d_1_k)\) and \((d_2_k)\) where \((d_1_k)\) and \((d_2_k)\) are the following properties.

\[(d_1_k)\] \( \mathcal{H}^i(j_k^! B^\bullet_{\mathbb{U}_{k+1}}) = 0 \) if \( i > \bar{p}(k) + 1 \).

\[(d_2_k)\] The canonical morphism \( \mathcal{H}^{\bar{p}(k)+1}(j_k^! B^\bullet_{\mathbb{U}_{k+1}}) \to \mathcal{H}^{\bar{p}(k)+1}(j_k^* B^\bullet_{\mathbb{U}_{k+1}}) \) is the morphism 0.

Moreover, using the property 1.13(15) of \([13]\), these properties are equivalent to:

\[(d_{1k}')\] For every \( x \in X \setminus X_{d-k} \setminus X_{d-k-1} \), \( \mathcal{H}^i(j_x^! B^\bullet_{\mathbb{U}_{k+1}}) = 0 \) if \( i > \bar{q}(k) + 1 + d - k = d - \bar{q}(k) - 1 \).

\[(d_{2k}')\] For every \( x \in X \setminus X_{d-k} \setminus X_{d-k-1} \), the canonical morphism \( \mathcal{H}^{d - \bar{q}(k) - 1}(j_x^! B^\bullet_{\mathbb{U}_{k+1}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^* B^\bullet_{\mathbb{U}_{k+1}}) \) (given by property 1.13(15) of \([13]\)) is the morphism 0.

Now, we recall useful definitions to compare the axioms \([AX1]_k\) with \([AXS1]_k\).

Definition 6.5. Let \( B^\bullet \) be a complex of sheaves in a topological space \( X \) and, for every \( x \in X \), let \( j_x : \{x\} \to X \) be the canonical inclusion. Then,

- The local support of \( B^\bullet \) in degree \( m \) is \( \{x \in X | \mathcal{H}^m(j_x^! B^\bullet) \neq 0\} \)

- The local cosupport of \( B^\bullet \) in degree \( m \) is \( \{x \in X | \mathcal{H}^m(j_x^* B^\bullet) \neq 0\} \)
The properties \((c_k)\) of \([AX1]_k\), \((c_k)\) of \([AXS1]_k\) and \((d'_k)\), \((d'_{1,k})\), \((d'_{2,k})\) of Remark 6.4 can be defined in terms of support and cosupport.

Let us consider a complex stratified variety \(X\). Then, the upper middle perversity and the lower middle perversity (see Definition 3.16) are equal over the codimension of the strata of \(X\).

Let \(\bar{m}\) be the middle perversity. The following table, taken from [10], illustrate the conditions of support and cosupport for a complex of sheaves \(IC_{\bar{m}}[-d]\) verifying \([AX1]_k\) with perversity \(\bar{m}\) for \(k = 2, \ldots, d\).

| degree | complex codimension of the strata |
|--------|----------------------------------|
| 8      | \(c\) \(c\) \(c\) \(c\) \(c\) |
| 7      | \(c\) \(c\) \(c\) \(c\) \(c\) |
| 6      | \(c\) \(c\) \(c\) \(c\) \(c\) |
| 5      | \(c\) \(c\) \(c\) \(c\) \(c\) |
| 4      | \(\times\) \(\times\) \(\times\) \(\times\) \(\times\) |
| 3      | \(\times\) \(\times\) \(\times\) \(\times\) \(\times\) |
| 2      | \(\times\) \(\times\) \(\times\) \(\times\) \(\times\) |
| 1      | \(\times\) \(\times\) \(\times\) \(\times\) \(\times\) |
| 0      | \(\times\) \(\times\) \(\times\) \(\times\) \(\times\) |

The symbol \(c\) means the complex can have local cosupport at that place, while the symbol \(\times\) means the complex can have local support at that place.

The following tables illustrate the conditions of support and cosupport for an intersection space complex \(IS_{\bar{m}}\) with perversity \(\bar{m}\).

| degree | complex codimension of the strata |
|--------|----------------------------------|
| 8      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 7      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 6      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 5      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 4      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 3      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 2      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 1      | \(\times\) \(\times\) \(\times\) \(\times\) |
| 0      | \(\times\) \(\times\) \(\times\) \(\times\) |

The symbol \(c\) means the complex can have local cosupport at that place, while the symbol \(\times\) means the complex can have local support at that place. Moreover, the symbol \(\ast\) means the support and the cosupport must verify a special condition given by \((d'_{2,k})\).

Note that in \(U_2\), \((IS_{\bar{m}})_{U_2} \cong (IC_{\bar{m}}[-d])_{U_2} \cong Q_{U_2}\). However, in \(X_{d-2}\), the place at which \(IS_{\bar{m}}\) can have support is exactly the place at which \(IC_{\bar{m}}[-d]\) cannot have support and the place at which \(IS_{\bar{m}}\) can have cosupport is exactly the place at which \(IC_{\bar{m}}[-d]\) cannot have cosupport.

7. A derived category approach to intersection space complexes

In this section, we study necessary and sufficient conditions for the existence of an intersection space complex of \(X\) with a perversity \(\bar{p}\). Unlike intersection cohomology sheaves, intersection
space complexes are not unique. We study the space parametrizing the different choices of intersection space complexes for a fixed perversity.

7.1. Homological algebra review. We need the following lemma, that should be well known, we include a proof for convenience of the reader.

**Lemma 7.1.** Let

\[
A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet
\]

be a distinguished triangle in the category \(D^b_{c,\mathbb{Q}}(X)\).

The following four conditions are equivalent:

1. \(f\) admits a retract
2. \(g\) admits a section
3. There is an isomorphism in the derived category \(\gamma : B^\bullet \cong A^\bullet \oplus C^\bullet\) such that \(f = \gamma^{-1} \circ i_A\) and \(g = p_C \circ \gamma\) where \(i_A : A^\bullet \to A^\bullet \oplus C^\bullet\) is the natural inclusion and \(p_C : A^\bullet \oplus C^\bullet \to C^\bullet\) is the natural projection.
4. \(\phi\) is the morphism 0

**Proof.** First we prove that (1) implies (3). Let \(r\) be a retract of \(f\). Then, \(r \circ f = Id_{A^\bullet}\). So, the morphism induced by \(f\) between the cohomology sheaves \(\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)\) is injective for every \(i \in \mathbb{Z}\). Hence, using the long exact sequence of cohomology

\[
\cdots \to \mathcal{H}^{i-1}(C^\bullet) \xrightarrow{\mathcal{H}^{i-1}(\phi)} \mathcal{H}^i(A^\bullet) \xrightarrow{\mathcal{H}^i(f)} \mathcal{H}^i(B^\bullet) \xrightarrow{\mathcal{H}^i(g)} \mathcal{H}^i(C^\bullet) \xrightarrow{\mathcal{H}^i(\phi)} \mathcal{H}^{i+1}(A^\bullet) \to \cdots
\]

we deduce \(\mathcal{H}^i(\phi) = 0\) for every \(i \in \mathbb{Z}\). Consequently, we have short exact sequences

\[
0 \to \mathcal{H}^i(A^\bullet) \xrightarrow{\mathcal{H}^i(f)} \mathcal{H}^i(B^\bullet) \xrightarrow{\mathcal{H}^i(g)} \mathcal{H}^i(C^\bullet) \to 0
\]

(10)

Now, let us consider the morphism

\[
\gamma = \begin{pmatrix} r \\ g \end{pmatrix} : B^\bullet \to A^\bullet \oplus C^\bullet
\]

The morphism induced by \(\gamma\) between the cohomology sheaves is

\[
\mathcal{H}^i(\gamma) = \begin{pmatrix} \mathcal{H}^i(r) \\ \mathcal{H}^i(g) \end{pmatrix} : \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(A^\bullet) \oplus \mathcal{H}^i(C^\bullet),
\]

which is an isomorphism, since \(\mathcal{H}^i(r)\) is a retract of \(\mathcal{H}^i(f)\) and (10) is exact.

Moreover, since \(f \circ g = 0\), we have \(\gamma \circ f = i_A\) and it is clear that \(p_C \circ \gamma = g\). So, we have proven (1) implies (3).

Now, we prove (2) implies (3). Let \(s\) be a section of \(g\) and let us consider the morphism

\[
\gamma' = (f, s) : A^\bullet \oplus C^\bullet \to B^\bullet
\]

In the same way that in the previous implication, we can show that \(\gamma'\) is a quasi-isomorphism, and that we have \(f = \gamma' \circ i_A\) and \(g \circ \gamma' = p_C\). So, \(\gamma = (\gamma')^{-1}\) is the isomorphism which appears in condition (3).

Moreover, if condition (3) is true, \(p_A : A^\bullet \oplus C^\bullet \to A^\bullet\) denotes the natural projection and \(i_C : C^\bullet \to A^\bullet \oplus C^\bullet\) denotes the natural inclusion, then \(p_A \circ \gamma\) is a retract of \(f\) and \(\gamma^{-1} \circ i_C\) is a section of \(g\). So, (3) implies (1) and (2).
Now, it is enough to prove \((3) \Leftrightarrow (4)\). (3) implies that we have the following isomorphism between distinguished triangles:

\[
\begin{array}{c}
A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet \\
\downarrow{Id_A^\bullet} & & \downarrow{\gamma} & & \downarrow{Id_C^\bullet} \\
A^\bullet & \xrightarrow{i_A} & A^\bullet \oplus C^\bullet & \xrightarrow{p_C} & C^\bullet
\end{array}
\]

\(\phi\) is the morphism 0.

Now, suppose \(\phi = 0\) and let us prove condition (3). By properties of the triangulated categories, we know

\[
\begin{array}{c}
C^\bullet[1] & \xrightarrow{-\phi[1]} & A^\bullet & \xrightarrow{f} & B^\bullet \\
\downarrow{g} & & \downarrow{Id_A^\bullet} & & \downarrow{Id_C^\bullet}
\end{array}
\]

is a distinguished triangle. So, if \(\phi = 0\), there is an isomorphism \(\gamma : B^\bullet \cong A^\bullet \oplus C^\bullet\) which completes the following isomorphism of triangles

\[
\begin{array}{c}
C^\bullet[1] & \xrightarrow{0} & A^\bullet & \xrightarrow{f} & B^\bullet \\
\downarrow{Id_C^\bullet[1]} & & \downarrow{Id_A^\bullet} & & \downarrow{Id_C^\bullet}
\end{array}
\]

\(\gamma\) is the isomorphism which appears in condition (3).

\(\square\)

**Definition 7.2.** The triangle

\[
\begin{array}{c}
A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet \\
\downarrow{\phi} & & \downarrow{Id_A^\bullet} & & \downarrow{Id_C^\bullet}
\end{array}
\]

is said to be split if it verifies the conditions of Lemma 7.1.

### 7.2. Characterization of existence and study of uniqueness

Now, we can state the main theorem in this section.

**Theorem 7.3.** The following holds:

1. [Goreski-McPherson] There exist one object in \(D^b_{cc}(X)\) verifying \([AX1]_k\) for \(k = 2, ..., d\). This object is unique up to isomorphism.

2. Suppose there exist an intersection space complex in \(U_r, IS_{r-1}\), that is, \(IS_{r-1}\) is an object in \(D^b_{cc}(U_r)\) and it verifies \([AXS1]_k\) for \(k = 2, ..., r - 1\). Then, there exist an intersection space complex in \(U_{r+1}, IS_r\), such that \((IS_r)_{U_r} \cong IS_{r-1}\) if and only if the distinguished triangle:

\[
\tau_{\leq \bar{q}(r)}j_{rs}j_{rs}^*i_{rs}IS_{r-1} \xrightarrow{\bar{f}} j_{rs}j_{rs}^*i_{rs}IS_{r-1} \xrightarrow{\tau_{> \bar{q}(r)}j_{rs}j_{rs}^*i_{rs}} IS_r \rightarrow [1]
\]
is split. Moreover, there is a bijection

\[
\left\{ \text{intersection space complexes} \right\}_/\{ \text{isomorphism} \} \leftrightarrow \{ \text{retracts of } f \} / \sim
\]

where \( \sim \) is the equivalence relation such that \( \lambda_1 \sim \lambda_2 \) if and only if there exist isomorphisms \( \alpha : \tau_{\leq \bar{q}(r)} j_{rs}^* i_{rs} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{rs}^* i_{rs} IS_{r-1} \) and \( \beta : i_{rs} IS_{r-1} \to i_{rs} IS_{r-1} \) such that \( \lambda_2 = \alpha \circ \lambda_1 \circ j_{rs}^* \beta \).

**Proof.** 1. is proved in [13, section 3], but we give proof adapted to our needs.

There exist a unique object (up to isomorphism), \( \mathbb{Q}_U \), in \( D^b_{\text{ce}}(U_2) \) verifying (a) and (b) of \([AX1]\)\(_k\). Suppose there exist a unique object (up to isomorphism), \( IC_{\bar{p}}[\mathfrak{r}][-d] \), in \( D^b_{\text{ce}}(U_r) \) verifying \([AX1]\)\(_k\) for \( k = 2, \ldots, r \), and consider the following composition of natural morphisms:

\[
i_{rs} IC_{\bar{p}}[-d] \xrightarrow{\phi_r} j_{rs} j_{rs}^* i_{rs} IC_{\bar{p}}[-d] \xrightarrow{\tau_{\geq \bar{p}(r)} j_{rs} j_{rs}^* i_{rs} IC_{\bar{p}}[-d]} \]

Let us define \( IC_{\bar{p}}[\mathfrak{r}][-d] := \text{cone}(\phi_r)[1] \). Then, there is a distinguished triangle:

\[
IC_{\bar{p}}[\mathfrak{r}][-d] \to i_{rs} IC_{\bar{p}}[-d] \xrightarrow{\phi_r} \tau_{\geq \bar{p}(r)} j_{rs} j_{rs}^* i_{rs} IC_{\bar{p}}[-d] [1]
\]

Using the long exact sequence of cohomology associated to this triangle we can prove that \( IC_r[\mathfrak{r}][-d] \) verifies \([AX1]\)\(_k\) for \( k = 2, \ldots, r \).

Now, suppose there exist another object \( B^* \) in \( D^b_{\text{ce}}(U_{r+1}) \) verifying \([AX1]\)\(_k\) for \( k = 2, \ldots, r \). Then, \( B^* \mid_{U_r} \) verifies \([AX1]\)\(_k\) for \( k = 2, \ldots, r-1 \). So, there exist an isomorphism \( B^* \mid_{U_r} \cong IC_{\bar{p}}[-d] \).

Let \( \varphi : B^* \to i_{rs} IC_{\bar{p}}[-d] \) be the composition of the canonical morphism \( B^* \to i_{rs} B^* \mid_{U_r} \) and an isomorphism \( i_{rs} B^* \mid_{U_r} \cong i_{rs} IC_{\bar{p}}[-d] \) and let \( C^* := \text{cone}(\varphi). \) Since \( i^* \varphi \) is an isomorphism, we have the isomorphism \( i^* C^* \cong 0 \) in the derived category.

Then, the distinguished triangle

\[
i_{rs} i^* C^* \to C^* \to j_{rs} j_{rs}^* C^* [1]
\]

implies there exist an isomorphism \( C^* \cong j_{rs} j_{rs}^* C^*. \)

Moreover, with the long exact sequence of cohomology associated to

\[
j_{rs} j_{rs}^* B^* \xrightarrow{\psi} j_{rs} j_{rs}^* C^* \]

we prove \( H^i(j_{rs} j_{rs}^* C^*) = 0 \) if \( i \leq \bar{p}(r) \) and \( H^i(\psi) : H^i(j_{rs} j_{rs}^* i_{rs} IC_{\bar{p}}[-d]) \to \tau_{\geq \bar{p}(r)} j_{rs} j_{rs}^* C^* \) is an isomorphism if \( i > \bar{p}(r) \). Then, we obtain isomorphisms

\[
C^* \cong \tau_{\geq \bar{p}(r)} j_{rs} j_{rs}^* i_{rs} IC_{\bar{p}}[-d] \quad \text{and} \quad B^* \cong IC_{\bar{p}}[-d].
\]

Repeating this process finitely we obtain \( IC_{\bar{p}}[\mathfrak{r}][-d] \in D^b_{\text{ce}}(X) \) verifying \([AX1]\)\(_k\) for \( k = 2, \ldots, d \) and it is unique up to isomorphism.

Now, we prove 2. Let \( IS_{r-1} \) be an intersection space complex in \( U_r \). We have to prove that there is a bijective map

\[
\left\{ \text{intersection space complexes} \right\}_/\{ \text{isomorphism} \} \leftrightarrow \{ \text{retracts of } f \} / \sim
\]

Let \( \lambda \) be a retract of \( f \) and consider the following composition of morphisms:

\[
i_{rs} IS_{r-1} \xrightarrow{a} j_{rs} j_{rs}^* i_{rs} IS_{r-1} \xrightarrow{\lambda} \tau_{\leq \bar{q}(r)} j_{rs} j_{rs}^* i_{rs} IS_{r-1}
\]

where \( a \) is the canonical morphism.
Let us define \( IS_r := \text{cone}(\varphi_{\lambda})[-1] \). Then, there is a distinguished triangle:

\[
IS_r \xrightarrow{\varphi_{\lambda}} \tau_{\leq q(r)} j_{r+1} i_r IS_{r-1} \xrightarrow{[1]} IS_r
\]

Since \( (j_r j_{r+1} i_r IS_{r-1})_{|U_r} \) equals 0, \( IS_r \) is isomorphic to \( IS_r \) and, using the long exact sequence associated to the triangle, one proves that \( IS_r \) verifies \( \text{[AXS1]} \) for \( k = 2, \ldots, r \).

Let \( \lambda' \) be a different retract of \( f \) such that \( \lambda \sim \lambda' \). We have to prove \( \text{cone}(\varphi_{\lambda'}) \cong \text{cone}(\varphi_{\lambda}) \).

Let \( \alpha : \tau_{\leq q(r)} j_{r+1} i_r IS_{r-1} \rightarrow \tau_{\leq q(r)} j_{r+1} i_r IS_{r-1} \) and \( \beta : i_r IS_{r-1} \rightarrow i_r IS_{r-1} \) be isomorphisms such that \( \lambda' = \alpha \circ \lambda \circ j_{r+1} \beta \).

Since we know the equalities \( \varphi_{\lambda} = \lambda \circ a \) and \( \varphi_{\lambda'} = \lambda' \circ a \), we have to prove the isomorphism \( \text{cone}(\lambda \circ a) \cong \text{cone}(\alpha \circ \lambda \circ j_{r+1} \beta \circ a) \). Moreover, since \( \alpha \) is an isomorphism, \( \text{cone}(\alpha \circ \lambda \circ j_{r+1} \beta \circ a) \) is isomorphic to \( \text{cone}(\lambda \circ j_{r+1} \beta \circ a) \).

Now, consider the following diagrams associated to the octahedral axiom of distinguished triangles.

\[
\begin{array}{cccc}
\lambda a & j_{r+1} i_r IS_{r-1} & \lambda & \tau_{\leq q(r)} j_{r+1} i_r IS_{r-1} \\
\downarrow[1] & \downarrow[1] & \downarrow[1] & \\
\text{cone}(a) & \text{cone}(\lambda) & \text{cone}(\lambda \circ a)
\end{array}
\]

\[
\begin{array}{cccc}
\lambda a & j_{r+1} i_r IS_{r-1} & \lambda a \circ j_{r+1} \beta & \tau_{\leq q(r)} j_{r+1} i_r IS_{r-1} \\
\downarrow[1] & \downarrow[1] & \downarrow[1] & \\
\text{cone}(a) & \text{cone}(\lambda \circ j_{r+1} \beta) & \text{cone}(\lambda \circ j_{r+1} \beta \circ a)
\end{array}
\]

Let \( \phi : \text{cone}(\lambda \circ j_{r+1} \beta) \rightarrow \text{cone}(\lambda) \) be an isomorphism completing the following isomorphism between triangles

\[
\begin{array}{cccc}
j_{r+1} i_r IS_{r-1} & \lambda a \circ j_{r+1} \beta & \tau_{\leq q(r)} j_{r+1} i_r IS_{r-1} & \text{cone}(\lambda \circ j_{r+1} \beta) \\
\downarrow[1] & \downarrow[1] & \downarrow[1] & \\
\text{cone}(a) & \text{cone}(\lambda \circ j_{r+1} \beta) & \text{cone}(\lambda \circ j_{r+1} \beta \circ a)
\end{array}
\]

Then, if \( \rho : \text{cone}(a) \rightarrow \text{cone}(a) \) is an isomorphism completing the triangles isomorphism

\[
\begin{array}{cccc}
i_r IS_{r-1} & j_{r+1} i_r IS_{r-1} & \text{cone}(a) \\
\downarrow[1] & \downarrow[1] & \\
i_r IS_{r-1} & j_{r+1} i_r IS_{r-1} & \text{cone}(a)
\end{array}
\]
the diagram

\[
\begin{array}{ccc}
\text{cone}(\lambda \circ j_r j_r ^\ast \beta) & \xrightarrow{[1]} & \text{cone}(a) \\
\downarrow \phi & & \downarrow \rho \\
\text{cone}(\lambda) & \xrightarrow{[1]} & \text{cone}(a)
\end{array}
\]

is commutative.

Therefore, there exist an isomorphism \(\text{cone}(\lambda \circ j_r j_r ^\ast \beta \circ a) \cong \text{cone}(\lambda \circ a)\), completing the following morphism between triangles

\[
\begin{array}{ccc}
\text{cone}(\lambda \circ j_r j_r ^\ast \beta) & \xrightarrow{[1]} & \text{cone}(\lambda \circ j_r j_r ^\ast \beta \circ a) \\
\downarrow \phi & & \downarrow \rho \\
\text{cone}(\lambda) & \xrightarrow{[1]} & \text{cone}(\lambda \circ a)
\end{array}
\]

Now, suppose there exist an intersection space complex in \(U_{r+1}\), \(IS_r\) such that \((IS_r)\mid_{U_r} \cong IS_{r-1}\). We have to prove that the triangle \(|11|\) is split.

Let \(h : IS_r \to i_{r+1}\) be the composition of the canonical morphism \(IS_r \to i_{r+1}(IS_r)\mid_{U_r}\) and an isomorphism \(i_{r+1}(IS_r)\mid_{U_r} \cong i_{r+1}IS_{r-1}\) and let \(C^\bullet := \text{cone}(h)\). Then, there is a distinguished triangle:

\[
IS_r \xrightarrow{h} i_{r+1}IS_{r-1} \xrightarrow{g} C^\bullet \xrightarrow{[1]}
\]

Note that \(i_r^* h : i_r^* IS_r \to IS_{r-1}\) is an isomorphism. So, \(i_r^* C^\bullet\) is isomorphic to 0 in the derived category. Then, the canonical triangle

\[
i_r i_r^* C^\bullet \to C^\bullet \to j_r j_r ^\ast C^\bullet \xrightarrow{[1]}
\]

implies the isomorphism \(C^\bullet \cong j_r j_r ^\ast C^\bullet\).

Moreover, the long exact sequence of cohomology associated to the triangle

\[
j_r j_r ^\ast IS_r \xrightarrow{j_r j_r ^\ast h} j_r j_r ^\ast i_{r+1}IS_{r-1} \xrightarrow{j_r j_r ^\ast g} j_r j_r ^\ast C^\bullet \xrightarrow{[1]}
\]

implies that

\[H^i(j_r j_r ^\ast C^\bullet) \cong \begin{cases} 0 & \text{if } i > \bar{q}(r) \\ H^i(j_r j_r ^\ast i_{r+1}IS_{r-1}) & \text{if } i \leq \bar{q}(r) \end{cases}\]

Applying the functor \(\tau_{\leq \bar{q}(r)}\) to \(j_r j_r ^\ast g\), we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\tau_{\leq \bar{q}(r)}j_r j_r ^\ast i_{r+1}IS_{r-1} & \xrightarrow{a} & \tau_{\leq \bar{q}(r)}j_r j_r ^\ast C^\bullet \\
\downarrow f & & \downarrow c \\
j_r j_r ^\ast i_{r+1}IS_{r-1} & \xrightarrow{b} & j_r j_r ^\ast C^\bullet
\end{array}
\]

where \(f\) and \(c\) are the canonical morphisms, \(a = \tau_{\leq \bar{q}(r)}j_r j_r ^\ast g\) and \(b = j_r j_r ^\ast g\). Moreover, \(a\) and \(c\) are isomorphisms and the composition \(\lambda = a^{-1} \circ c^{-1} \circ b\) is a retract of \(f\). So, the triangle \(|11|\) is split by Lemma \([11]\).

Let \(IS'_{r+1} \subseteq D^b_{\text{c}r}(U_{r+1})\) be isomorphic to \(IS_r\). Then, we have an isomorphism \((IS'_{r+1})\mid_{U_r} \cong IS_{r-1}\) and \(IS'_{r+1}\) is an intersection space complex in \(U_{r+1}\).

Let \(h' : IS'_{r+1} \to i_{r+1}IS_{r-1}\) be the composition of the canonical restriction morphism and an isomorphism \(i_{r+1}(IS'_{r+1})\mid_{U_r} \cong i_{r+1}IS_{r-1}\), let \(K^\bullet := \text{cone}(h')\) and consider the triangle

\[
IS'_{r+1} \xrightarrow{h'} i_{r+1}IS_{r} \xrightarrow{g'} K^\bullet \xrightarrow{[1]}
\]
Applying the functor $\tau_{\leq q(r)}$ to $j_{r*}j^*_r g'$, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\tau_{\leq q(r)} j_{r*}j^*_r i_{r*} IS_{r-1} & \xrightarrow{a'} & \tau_{\leq q(r)} j_{r*} j^*_r K^* \\
\downarrow h & & \downarrow c' \\
j_{r*}j^*_r i_{r*} IS_{r-1} & \xrightarrow{b'} & j_{r*}j^*_r K^*
\end{array}
\]

where $h$ and $c'$ are the canonical morphisms, $a' = \tau_{\leq q(r)} j_{r*} j^*_r g'$ and $b' = j_{r*} j^*_r g$. Moreover, $a'$ and $c'$ are isomorphisms and $\lambda':= a'^{-1} \circ c'^{-1} \circ b'$ is a retract of $f$.

Let $\alpha : IS_r \rightarrow IS'_r$ be an isomorphism; let $h$ be the composition of the canonical morphism $IS_r \rightarrow i_{r*}(IS'_r|_{U_r})$ and an isomorphism $\gamma : i_{r*}(IS'_r)|_{U_r} \rightarrow i_{r*} IS_{r-1}$; let $h'$ be the composition of the canonical morphism $IS'_r \rightarrow i_{r*}(IS'_r)|_{U_r}$ and an isomorphism $\gamma' : i_{r*}(IS'_r)|_{U_r} \rightarrow i_{r*} IS_{r-1}$. Finally define $\beta := \gamma' \circ i_{r*} \alpha \circ \gamma^{-1}$. Then, there is an isomorphism between triangles:

\[
\begin{array}{ccc}
j_{r*}j^*_r IS_r & \xrightarrow{j_{r*}j^*_r \alpha} & j_{r*}j^*_r i_{r*} IS_{r-1} \\
\downarrow j_{r*}j^*_r h & & \downarrow j_{r*}j^*_r \beta \\
j_{r*}j^*_r IS'_r & \xrightarrow{j_{r*}j^*_r h'} & j_{r*}j^*_r i_{r*} IS_{r-1} \xrightarrow{j_{r*}j^*_r g'} \xrightarrow{j_{r*}j^*_r \delta} j_{r*}j^*_r K^*
\end{array}
\]

where $\alpha$, $\beta$ and $\delta$ are isomorphisms.

Then, the morphisms of diagrams (16) and (17) have the following relations:

\[
a' = \tau_{\leq q(r)} \delta \circ a \circ (\tau_{\leq q(r)} j_{r*} j^*_r \beta)^{-1}
\]

\[
b' = \delta \circ a \circ (j_{r*} j^*_r \beta)^{-1}
\]

\[
c' = \delta \circ c \circ (\tau_{\leq q(r)} h)^{-1}
\]

and, we obtain $\lambda = (\tau_{\leq q(r)} j_{r*} j^*_r \beta)^{-1} \circ \lambda' \circ j_{r*} j^*_r \beta$. So, $\lambda$ is equivalent to $\lambda'$ by the equivalence relation $\sim$.

**Remark 7.4.** There is a unique object in $D^b_{nc}(U_2)$ up to isomorphism verifying (a) and (b) from [AXS1], $Q U_2$.

Now we are going to study the equivalence relation $\sim$, which appears in Theorem 7.3.

Remember that two retracts $\lambda_1$ and $\lambda_2$ of $\tau_{\leq q(r)} j_{r*} j^*_r i_{r*} IS_{r-1} \xrightarrow{f} j_{r*} j^*_r i_{r*} IS_{r-1}$ are equivalent by $\sim$ if and only if there exit isomorphisms

\[
\alpha : \tau_{\leq q(r)} j_{r*} j^*_r i_{r*} IS_{r-1} \rightarrow \tau_{\leq q(r)} j_{r*} j^*_r i_{r*} IS_{r-1}
\]

\[
\beta : i_{r*} IS_{r-1} \rightarrow i_{r*} IS_{r-1}
\]

such that $\lambda_2 = \alpha \circ \lambda_1 \circ j_{r*} j^*_r \beta$.

Let $\alpha : \tau_{\leq q(r)} j_{r*} j^*_r i_{r*} IS_{r-1} \rightarrow \tau_{\leq q(r)} j_{r*} j^*_r i_{r*} IS_{r-1}$ and $\beta : i_{r*} IS_{r-1} \rightarrow i_{r*} IS_{r-1}$ be isomorphisms and let $\lambda$ be a retract of $f$. Then, $\alpha \circ \lambda_1 \circ j_{r*} j^*_r \beta$ is a retract of $f$ if and only if $\alpha \circ \lambda \circ j_{r*} j^*_r \beta \circ f = \text{Id}$, that is, if $\alpha$ is equal to $(\lambda \circ j_{r*} j^*_r \beta \circ f)^{-1}$. So, $\alpha$ is determined by $\beta$ and $\lambda$ and the set of retracts of $f$ which are equivalent to $\lambda$ is determined by the automorphisms $i_{r*} IS_{r-1} \rightarrow i_{r*} IS_{r-1}$.

Since $j_{r*}$ is left-adjoint to $i_{r*}$, the space of automorphisms $\text{Aut}(i_{r*} IS_{r-1})$ is isomorphic to the space of isomorphisms $\text{Iso}(i_{r*} IS_{r-1}, IS_{r-1}) = \text{Aut}(IS_{r-1})$. 

7.3. The space of obstructions to existence and uniqueness. In particular, if $r$ is the dimension of the largest non-trivial stratum, we have $U_r = U_2$ and $\text{Aut}(IS_{r-1})$ is isomorphic to $\text{Aut}(\mathbb{Q}U_2)$, which is the space of homothetic transformations. Moreover, if $\beta \in \text{Aut}(i_r IS_{r-1})$ is a homothetic transformation $\alpha = (\lambda \circ j_r j_r^* \beta \circ f)^{-1}$ is the inverse homothetic transformation. So, if $r$ is the dimension of the largest non-trivial stratum, the equivalence relation $\sim$ is trivial. Consequently, there is a bijective map

$$\{\text{intersection space complexes } IS_r \in D^b_{\text{c}}(U_{r+1})\} / \{\text{isomorphism}\} \longleftrightarrow \{\text{retracts of } f\}$$

Remark 7.5. For any $r$, the triangle $[11]$ induces an exact sequence

$$\ldots \to [\tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}] \to [\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}] \xrightarrow{\partial} [\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}] \to \ldots$$

Moreover, the retracts of $f$ are de elements $\lambda \in [j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}]$ such that $\tilde{f}(\lambda)$ is the identity. So, the space $\{\text{retracts of } f\}$ is modulated by the vector space

$$[\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}]$$

The following is an immediate consequence of the previous construction.

Corollary 7.6. Suppose there exist an intersection space complex $IS_{r-1}$ in $U_r$, that is, $IS_{r-1}$ is an object in $D^b_{\text{c}}(U_r)$ and it verifies $[AXS1]_k$ for $k = 2, \ldots, r-1$.

- The obstruction to existence of intersection space in the next stratum lives in $\text{Ext}^1(\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}) = [\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}]$.

- The obstructions for uniqueness live in the group $\text{Hom}(\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}) = [\tau_{> q(r)} j_r j_r^* i_r IS_{r-1}, \tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1}]$, modulo the equivalence relation described above. The equivalence relation is trivial for the second stratum.

8. A mixed Hodge module structure in intersection space complexes of algebraic varieties

Theorem 8.1. Let $X$ be an algebraic variety. Consider a stratification

$$X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

by algebraic subvarieties, which makes $X$ a topological pseudomanifold. An intersection space complex on $X$ associated with the stratification above admits a lifting to the derived category $D^b_{\text{HM}}(X)$ of mixed Hodge modules on $X$ if and only if the choices of the retractions are morphisms of mixed Hodge modules. Consequently, its hypercohomology groups carry a rational polarizable mixed Hodge structure.

Proof. The proof follows the inductive construction of the intersection space complex considered in the proof of Theorem 7.3. To start with, we notice that $\mathbb{Q}U_2$ is a mixed Hodge module on the algebraic variety $U_2$. Assume, by induction, that the intersection space complex $IS_{r-1}$ defined in $U_r$ is a mixed Hodge module.

In order to construct $IS_r$, we proceed as follows: observe that the triangle

$$\tau_{\leq q(r)} j_r j_r^* i_r IS_{r-1} \to j_r j_r^* i_r IS_{r-1} \to \tau_{> q(r)} j_r j_r^* i_r IS_{r-1} [1]$$

is a triangle of mixed Hodge modules in $U_{r+1}$. This is true because the derived category of mixed Hodge modules is preserved by Grothendieck’s 6 operations, and because the truncation
triangle is a distinguished triangle in the derived category of mixed Hodge modules (use Saito’s anomalous $t$-structure).

By Lemma 7.1, the triangle splits if and only if the connecting morphism equals 0, and if this happens in the derived category of constructible sheaves, then it happens too in the derived category of mixed Hodge modules, since the functor $rat$ is faithful. We consider a retraction $\lambda : j_! j^* r_* I_S \tau_{-1} \to \tau_{\leq q(r)} j_! j^* i_* I_S \tau_{-1}$, which by assumption is a morphism in $D^b MHM(U_{r+1})$. Since $I_S$ is, by definition, the shifted cone $cone(\phi_\lambda)[-1]$, where $\phi_\lambda$ is the composition of $\lambda$ with the canonical morphism $i_* : I_S \tau_{-1} \to j_! j^* i_* I_S \tau_{-1}$ we have that $I_S$ belongs to $D^b MHM(U_{r+1})$.

The obstructions to existence and uniqueness can be lifted to mixed Hodge modules:

**Corollary 8.2.** Let $X$ be an algebraic variety. Consider an stratification

\[ X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset \]

by algebraic subvarieties, which makes $X$ a topological pseudomanifold. Suppose there exist an intersection space complex $I_S_{\tau_{-1}}$ in $U_r$ which belongs to $D^b MHM(U_{r+1})$.

- The obstruction to existence of intersection space in the next stratum lives in
  \[ Ext^1_{D^b MHM(U_{r+1})}(\tau_{>q(r)} j_! j^* i_* I_S \tau_{-1}, \tau_{\leq q(r)} j_! j^* i_* I_S \tau_{-1}) \]

- The obstructions for uniqueness live in the group
  \[ Hom_{D^b MHM(U_{r+1})}(\tau_{>q(r)} j_! j^* i_* I_S \tau_{-1}, \tau_{\leq q(r)} j_! j^* i_* I_S \tau_{-1}), \]

modulo the equivalence relation described above. The equivalence relation is trivial for the second stratum.

A simplification of the proof of Theorem 8.1 yields:

**Theorem 8.3.** Let $X$ be an algebraic variety. Let $\bar{p}$ be any perversity. The intersection cohomology complex associated to it belongs to the derived category of mixed Hodge modules of $X$. Consequently the intersection homology complexes $IH^b_\bar{p}(X, \mathbb{Q})$ carry a canonical polarizable mixed Hodge structure.

**9. Classes of spaces admitting intersection space complexes and counterexamples**

In this section we provide some examples and counterexamples to illustrate our theory. First, we introduce two classes of varieties which admit an intersection space complex for every perversity. The first class depends on the tubular neighbourhoods of the strata: if every stratum admits a trivial tubular neighbourhood, then there exist the intersection space complex for every perversity. The second class depends on the dimension of the strata: if every singular stratum has homological dimension for locally constant sheaves bounded by 1, then there exist the intersection space complex for every perversity. Then, we find concrete examples of pseudomanifolds (including an algebraic variety) not admitting intersection space complexes and, hence, not admitting intersection space pairs. Hence, by Theorem 5.18 $X$ has an intersection space complex. We have shown:
Corollary 9.1. If $X$ has a trivial conical structure, then it has an intersection space complex.

Example 9.2. Toric varieties admit an intersection space pairs and intersection space complex for every perversity.

However, as we prove below, in order to ensure the existence of the intersection space complex we can relax the triviality hypothesis on the stratification: one only needs that the triviality property $(T_r)$ (see 3.10) is satisfied for any stratum. Having a trivial conical structure requires further compatibilities between the trivializations predicted by properties $(T_r)$ (see Definition 3.12).

Definition 9.3. A complex of sheaves is formal and constant if it is quasi-isomorphic to the direct sum of its cohomology sheaves and these cohomology sheaves are constant.

Definition 9.4. If $B^k_{k-1}$ is a complex of sheaves in $U_k$, we say $B^k_{k-1}$ verifies the property $(P_r)$, where $r \geq k$, if $j^*_k i_{k,r+1} B^k_{k-1}$ is formal and constant.

Remark 9.5. If $B^*$ is a formal and constant complex of sheaves in $U_k$ and $(X, X_d-2)$ has a conical structure which verifies the property $(T_r)$ of Definition 3.10 for some $r \geq k$, then $B^*$ verifies the property $(P_r)$.

Proposition 9.6. Given $k \in \{2, ..., d\}$, if there exist an intersection space $IS_{k-1}$ with perversity $\tilde{p}$ in $U_k$, which verifies $(P_k)$, then there exist an intersection space $IS_k$ with perversity $\tilde{p}$ in $U_{k+1}$, such that $(IS_k)_{|U_k}$ is quasi-isomorphic to $IS_{k-1}$.

Proof. $j^*_k i_{k,IS_{k-1}} IS_{k-1}$ is (up to isomorphism in the derived category) a complex of constant sheaves with zero differentials. So, the triangle

$$\tau_{\leq q} j_k^* i_{k,IS_{k-1}} IS_{k-1} \rightarrow j_k^* j_k^* i_{k,IS_{k-1}} IS_{k-1} \rightarrow \tau_{>q} j_k^* j_k^* i_{k,IS_{k-1}} [1]$$

is split for every $q \in \mathbb{Z}$ and, applying Theorem 7.3, we conclude.

Lemma 9.7. Let us suppose that there exist an intersection space complex $IS_{k-1}$ with perversity $\tilde{p}$ in $U_k$. If $(IS_{k-1})_{|U_k}$ verifies the properties $(P_{k-1})$ and $(P_r)$ for a certain $r \geq k$ and $(X, X_d-2)$ has a conical structure which verifies the property $(T_r)$ of Definition 3.10, then $IS_{k-1}$ verifies the property $(P_r)$.

Proof. By Theorem 7.3, $j^*_k i_{k,r+1} IS_{k-1} [1]$ is quasi-isomorphic to the cone of a morphism

$$j^*_k i_{k-1,r+1} (IS_{k-1})_{|U_{k-1}} \rightarrow j^*_k i_{k,r+1} \tau_{\leq q} (IS_{k-1})_{|U_{k-1}}$$

Since $(IS_{k-1})_{|U_{k-1}}$ verifies the properties $(P_{k-1})$ and $(P_r)$, the complexes $j^*_k i_{k-1,r+1} (IS_{k-1})_{|U_{k-1}}$ and $\tau_{\leq q} (IS_{k-1})_{|U_{k-1}}$ are formal and constant. Then, using that $(X, X_d-2)$ has a conical structure verifying the property $(T_r)$, we deduce that the constructible complex $j^*_k i_{k,r+1} \tau_{\leq q} (IS_{k-1})_{|U_{k-1}}$ is also formal and constant. So, $j^*_k i_{k,r+1} IS_{k-1}$ is formal and constant as well and we conclude.

Theorem 9.8. If the pair $(X, X_d-2)$ has a conical structure which verifies the property $(T_r)$ of Definition 3.10 for any $r$, then there exist the intersection space complex of $X$ for every perversity.

Proof. The constant sheaf $\mathbb{Q}_{U_2}$ verifies $(P_r)$ for every $r \geq 2$ such that the pair $(X, X_{d-2})$ has a conical structure with the property $(T_r)$. So, if $(X, X_{d-2})$ has a conical structure which verifies the property $(T_r)$ for any $r$, using Lemma 9.7 and Proposition 9.6 we can construct inductively for every $k$ an intersection space complex with perversity $\tilde{p}$ in $U_k$ which verifies $(P_r)$ for every $r \geq k$.
9.2. Pseudomanifolds with strata of small homological dimension.

**Definition 9.9.** A space $Y$ has homological dimension for locally constant sheaves bounded by $m$ if any locally constant sheaf in $Y$ has no cohomology in degree higher than $m$.

**Theorem 9.10.** Let $X$ be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

such that $X_{d-r} \setminus X_{d-r-1}$ has homological dimension for locally constant sheaves bounded by 1 for any $r$.

Then, there exist the intersection space complex of $X$ for every perversity $\bar{p}$.

Moreover, if $X_{d-r} \setminus X_{d-r-1}$ has homological dimension for locally constant sheaves bounded by 0 for any $r$, the intersection space complex is unique.

**Proof.** To prove the existence it is enough to prove that, for any topological space $Y$ which has homological dimension for locally constant sheaves bounded by 1, any complex of sheaves $B^\bullet$ in $Y$ and any integer $m$, the triangle

$$\tau_{\leq m} B^\bullet \rightarrow B^\bullet \rightarrow \tau_{> m} B^\bullet \xrightarrow{\phi} \tau_{\leq m} B^\bullet [1] \rightarrow \ldots$$

is split.

This triangle is split if and only the morphism

$$\phi \in \text{Ext}^1(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet)$$

is 0.

$\text{Ext}^1(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet)$ is the first hypercohomology group of the complex of sheaves $\text{Hom}^\bullet(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet)$.

Let $E^p,q$ be the local to global spectral sequence of $\text{Hom}^\bullet(\tau_{\leq m} B^\bullet, \tau_{> m} B^\bullet)$. Then,

$$E_2^{p,q} = \mathbb{H}^p(Y, \text{Ext}^q(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet)).$$

Moreover, the fiber of the sheaf $\text{Ext}^q(\tau_{\leq m} B^\bullet, \tau_{> m} B^\bullet)$ in a point $x$ is $[\tau_{> m} B^\bullet_x, \tau_{\leq m} B^\bullet_x]^q$. Since $\tau_{\leq m} B^\bullet_x[q]$ is a complex of injective sheaves, $[\tau_{> m} B^\bullet_x, \tau_{\leq m} B^\bullet_x]^q$ is equal to 0 for every $q \geq 0$. So, the sheaf $\text{Ext}^q(\tau_{\leq m} B^\bullet, \tau_{> m} B^\bullet)$ is 0 for every $q \geq 0$.

Since $Y$ has homological dimension for locally constant sheaves bounded by 1, the group $\mathbb{H}^p(Y, \text{Ext}^q(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet))$ is equal to 0 for every $p > 1$. So, $E_2^{p,q} = 0$ for every $p, q \in \mathbb{Z}$ such that $p + q = 1$.

Hence, $\text{Ext}^1(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet) = 0$, and we conclude the proof of the existence.

Moreover, by Remark 7.5, the retracts of the triangle are modulated by $\text{Ext}^0(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet)$.

With the previous method, we show that, if $Y$ has homological dimension for locally constant sheaves bounded by 0, $\text{Ext}^0(\tau_{> m} B^\bullet, \tau_{\leq m} B^\bullet) = 0$. Then, the retract of the triangle is unique and we conclude.

**Corollary 9.11.** If the strata $X_{d-r} \setminus X_{d-r-1}$ has the homotopy type of a 1-dimensional CW-complex of dimension bounded by 1 for any $r$, then here exist the intersection space complex of $X$ for every perversity $\bar{p}$.

**Example 9.12.** Any algebraic variety with 1 dimensional critical set and the canonical Whitney stratification if each connected component of the critical set is not a stratum as a whole (that happens in the non Whitney-equisingular case) satisfy the hypothesis of the previous corollary, and hence admits an intersection space complex.
9.3. Counterexamples. Now, we illustrate the limits of our theory with a class of varieties which does not admit an intersection space complex for some perversities. With this purpose, the following proposition gives a necessary condition for the splitting of a triangle

\[ \tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \]

**Proposition 9.13.** Let \( X \) be a topological space, let \( B^\bullet \) be a bounded complex of sheaves on \( X \) and let \( E_t^{p,q} \) be the local to global spectral sequence of \( B^\bullet \).

Then, if the canonical triangle

\[ \tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \]

is split, the morphisms \( d_r^{p,q} : E_t^{p,q} \to E_t^{p+r,q-r+1} \) are 0 for every \( r \geq 2 \), \( p \in \mathbb{Z} \) and \( m < q \leq m + r - 1 \).

**Proof.** Let us suppose the triangle is split and let \( \lambda : B^\bullet \to \tau_{\leq m}B^\bullet \) be a retract of the canonical morphism.

Let \( E_t^{p,q} \) be the local to global spectral sequence of \( B^\bullet \), \( E_t^{p,q} \) the local to global spectral sequence of \( \tau_{\leq m}B^\bullet \) and \( \lambda_r^{p,q} : E_t^{p,q} \to E_t^{p,q} \) the morphism induced by \( \lambda \).

For \( r = 2 \),

\[ \lambda_2^{p,q} : \mathbb{H}^p(X, \mathcal{H}^q(B)) \to \mathbb{H}^p(X, \mathcal{H}^q(\tau_{\leq m}B)) \]

is an isomorphism if \( q \leq m \) and \( \mathbb{H}^p(X, \mathcal{H}^q(\tau_{\leq m}B)) = 0 \) if \( q > m \).

Given \( r \geq 2 \), suppose \( \lambda_r^{p,q} : E_t^{p,q} \to E_t^{p,q} \) is an isomorphism for every \( q \leq m \) and \( E_t^{p,q} = 0 \) for every \( q > m \). Then, \( E_{t+1}^{p,q} = 0 \) for every \( q > m \).

Moreover, let us consider the commutative diagram:

\[
\begin{array}{ccc}
E_t^{p,q} & \xrightarrow{\lambda_r^{p,q}} & E_t^{p,q} \\
\downarrow{d_r^{p,q}} & & \downarrow{d_r^{p,q}} \\
E_t^{p+r,q-r+1} & \xrightarrow{\lambda_r^{p+r,q-r+1}} & E_t^{p+r,q-r+1}
\end{array}
\]

If \( q \leq m \), \( \lambda_r^{p,q} \) induces an isomorphism between \( \text{Ker}(d_r^{p,q}) \) and \( \text{Ker}(d_r^{p,q}) \) and \( \lambda_r^{p+r,q-r+1} \) induces an isomorphism between \( \text{Im}(d_r^{p,q}) \) and \( \text{Im}(d_r^{p,q}) \).

If \( q > m \) and \( q-r+1 \leq m \), \( E_t^{p,q} = 0 \) and, since the diagram is commutative, \( d_r^{p,q} = d_r^{p,q} = 0 \). So, we deduce \( d_r^{p,q} = 0 \) for every \( q \leq m + r - 1 \).

Moreover, \( \text{Im}(d_r^{p,q}) = \text{Im}(d_r^{p,q}) = 0 \). Therefore, for every \( q \leq m \), \( \lambda_r^{p,q} \) is an isomorphism and we can finish the proof by induction. \( \square \)

**Corollary 9.14.** Let \( X \) be a topological pseudomanifold with stratification

\[ X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset \]

and let \( k \) be the codimension of \( X_{d-2} \), that is, \( X_{d-2} = X_{d-k} \).

Let \( \bar{p} \) a perversity and \( \bar{q} \) its complementary perversity. If the local to global spectral sequence of \( j_*^k \mathbb{Q}_\mathbb{L}_2 \) has any differential \( d_r^{p,q} : E_t^{p,q} \to E_t^{p+r,q-r+1} \) different from 0 for some \( r \geq 2 \), \( p \in \mathbb{Z} \) and \( \bar{q}(k) < q \leq \bar{q}(k) + r - 1 \), then there does not exist any intersection space complex of \( X \) with perversity \( \bar{p} \).

Now, we construct an example which verify these conditions using the Hopf fibration.

**Example 9.15.** Let \( \rho^3 : S^3 \to S^2 \) be the Hopf fibration and let \( \rho^3 : cyl(\rho^3) \to S^2 \) be the cone of the fibration (see definition 3.3). If \( s : S^2 \to cyl(\rho^3) \) is the vertex section, we consider the space \( X := cyl(\rho^3) \) with the stratification

\[ X \supset s(S^2) \]
Let \( U := X \setminus s(S^2) \) and let \( i : U \to X \) and \( j : s(S^2) \to X \) be the canonical inclusions. Then, since the fiber of \( \rho^\partial \) is \( S^1 \)
\[
\mathcal{H}^i(j_* j^* i_* \mathbb{Q}_U) \begin{cases} 
\mathbb{Q}_{S^2} & \text{if } i = 0, 1 \\
0 & \text{otherwise}
\end{cases}
\]
So, if \( E^{p,q}_r \) is the hypercohomology spectral sequence of \( j_* j^* i_* \mathbb{Q}_U \), \( E^{p,q}_2 \) is
\[
\begin{array}{c|cc}
q & 0 & 0 \\
0 & 1 & 2 \\
\end{array}
\]
where the differential \( d^0_1 \) is different from 0.
Moreover, given any perversity \( \bar{p} \), \( \bar{p}(2) = 0 \). So, applying Corollary 9.14, there does not exist an intersection space complex of \( X \) with stratification \( X \supset s(S^2) \) with any perversity.

Hence, applying Remark 6.3, there does not exist any intersection space pair of \( X \) with the previous stratification.

The stratification of \( X \) given in the previous example is not natural. Since \( X \) is smooth, the natural stratification of \( X \) has no stratum different from \( X \) and \( \emptyset \). The following example is more natural in the sense that the nontrivial stratum is the singular part of the variety.

**Example 9.16.** Let \( \rho^\partial : S^3 \to S^2 \) be the Hopf fibration and let us consider the locally trivial fibration \( \sigma^\partial : S^3 \times S^2 \to S^2 \).

Moreover, let \( \sigma : cyl(\sigma^\partial) \to S^2 \) be the cone of \( \sigma^\partial \) and \( s : S^2 \to cyl(\sigma^\partial) \) the vertex section. Then, we define \( X := cyl(\sigma^\partial) \) and we consider the stratification \( X \supset s(S^2) \).

Let \( U := X \setminus s(S^2) \) and let \( i : U \to X \) and \( j : s(S^2) \to X \) be the canonical inclusions. Then, since the fiber of \( \sigma^\partial \) is \( S^1 \times S^1 \)
\[
\mathcal{H}^i(j_* j^* i_* \mathbb{Q}_U) \begin{cases} 
\mathbb{Q}_{S^2} & \text{if } i = 0, 2 \\
\mathbb{Q}_{S^2}^2 & \text{if } i = 1 \\
0 & \text{otherwise}
\end{cases}
\]
So, if \( E^{p,q}_r \) is the hypercohomology spectral sequence of \( j_* j^* i_* \mathbb{Q}_U \), \( E^{p,q}_2 \) is
\[
\begin{array}{c|cc}
q & 0 & 0 \\
0 & 1 & 2 \\
\end{array}
\]
where the differentials \( d^0_1 \) and \( d^2_2 \) are different from 0.
Moreover, given any perversity \( \bar{p} \), either \( \bar{p}(3) = 0 \) or \( \bar{p}(3) = 1 \). So, applying Corollary 9.14, there does not exist an intersection space complex of \( X \) with stratification \( X \supset s(S^2) \) with any perversity.

Hence, applying Remark 6.3, there does not exist any intersection space pair of \( X \) with the previous stratification.

A great number of examples can be constructed with this technique. For example, if one wishes to have simply connected link and strata one, can use instead of Hopf fibration the fibration \( \phi : S^7 \to S^4 \) with fibre \( S^3 \).
Now, we give an example of algebraic variety for which the intersection space does not exist for the middle perversity.

**Example 9.17.** Let \( Fr(2,3) \) be the frame bundle over the Grassmannian \( Gr(2,3) \), that is, \( Fr(2,3) := \{ M \in \text{Mat}(3 \times 2, \mathbb{C}) | \text{rk}(M) = 2 \} \) and the canonical bundle

\[
\pi : Fr(2,3) \to Gr(2,3) \cong \mathbb{P}_\mathbb{C}^2
\]

is a \( GL(2,\mathbb{C}) \)-principal bundle with the action

\[
\begin{align*}
GL(2,\mathbb{C}) \times Fr(2,3) &\longrightarrow Fr(2,3) \\
(A,M) &\longmapsto A \cdot M
\end{align*}
\]

Let \( R_1^2 := \{ M \in \text{Mat}(2 \times 2, \mathbb{C}) | \text{rk}(M) \leq 1 \} \) and let us consider the action

\[
\begin{align*}
GL(2,\mathbb{C}) \times R_1^2 &\longrightarrow R_1^2 \\
(A,M) &\longmapsto A \cdot M
\end{align*}
\]

Let \( X := Fr(2,3) \times_{GL(2,\mathbb{C})} R_1^2 \). Since \( \text{Sing}(R_1^2) = \{0\} \), we have the equality

\[
\text{Sing}(X) = Fr(2,3) \times_{GL(2,\mathbb{C})} \{0\} \cong Gr(2,3) \cong \mathbb{P}_\mathbb{C}^2
\]

and the induced fiber bundle

\[
\begin{align*}
Fr(2, n) \times_{GL(2,\mathbb{C})} R_1^2 \setminus \{0\} &\longrightarrow \mathbb{P}_\mathbb{C}^2 \\
(M_1, M_2) &\longmapsto \pi(M_1)
\end{align*}
\]

is the fibration of links over the singularity. The fiber of this morphism is \( R_1^2 \setminus \{0\} \).

Now, let us consider the action

\[
\begin{align*}
GL(2,\mathbb{C}) \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\
(A, (a,b)) &\longmapsto A \cdot \begin{pmatrix} a \\ b \end{pmatrix}
\end{align*}
\]

and let \( Y := Fr(2,3) \times_{GL(2,\mathbb{C})} \mathbb{C}^2 \).

The morphism

\[
\begin{align*}
\mathbb{C}^2 &\longrightarrow R_1^2 \\
(a,b) &\longmapsto \begin{pmatrix} a & a \\ b & b \end{pmatrix}
\end{align*}
\]

is compatible with the actions. So, it induces a morphism \( g : Y \to X \).

Moreover, \( g^{-1}(\text{Sing}(X)) = Fr(2,3) \times_{GL(2,\mathbb{C})} \{0\} \cong Gr(2,3) \cong \mathbb{P}_\mathbb{C}^2 \) and the fiber bundle

\[
\begin{align*}
Fr(2,3) \times_{GL(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\} &\longrightarrow \mathbb{P}_\mathbb{C}^2 \\
(M_1, M_2) &\longmapsto \pi(M_1)
\end{align*}
\]

is the fibration of links. The fiber of this morphism is \( \mathbb{C}^2 \setminus \{0\} \).

In addition,

\[
\begin{align*}
Fr(2,3) \times_{GL(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\} &\longrightarrow Fr(2, n) \times_{GL(2,\mathbb{C})} R_1^2 \setminus \{0\} \\
&M \longmapsto (M,M)
\end{align*}
\]

is the fibration of links over the singularity. The fiber of this morphism is \( R_1^2 \setminus \{0\} \).
is a morphism of fibrations which induces in the fiber the morphism \( f : \mathbb{C}^2 \setminus \{0\} \to R_4^2 \setminus \{0\} \).

Let us denote \( U_X := Fr(2, n) \times_{\text{GL}(2, \mathbb{C})} R_4^2 \setminus \{0\} \) and \( U_Y := Fr(2, n) \times_{\text{GL}(2, \mathbb{C})} \mathbb{C}^2 \setminus \{0\} \). Moreover, let \( j_X : \mathbb{P}^2 \to X \), \( i_X : U_X \to X \), \( j_Y : \mathbb{P}^2 \to Y \) and \( i_Y : U_Y \to Y \) be the canonical inclusions.

The morphism between fibrations \( g \) produces a morphism of complexes

\[
j_Y j_Y^* i_Y^* \mathbb{Q}_{U_Y} \xrightarrow{g} j_X j_X^* i_X^* \mathbb{Q}_{U_X}.
\]

Moreover, \( \mathbb{C}^2 \setminus \{0\} \) is homotopically equivalent to \( S^3 \), \( R_4^2 \setminus \{0\} \) is homotopically equivalent to \( S^3 \times S^2 \) and \( f : \mathbb{C}^2 \setminus \{0\} \to R_4^2 \setminus \{0\} \) induces an isomorphism between the 0-th and the third cohomology groups. Then \( \gamma \) induces an isomorphism between the cohomology sheaves

\[
\mathcal{H}^0(j_Y j_Y^* i_Y^* \mathbb{Q}_{U_Y}) \cong \mathcal{H}^0(j_X j_X^* i_X^* \mathbb{Q}_{U_X})
\]

and

\[
\mathcal{H}^3(j_Y j_Y^* i_Y^* \mathbb{Q}_{U_Y}) \cong \mathcal{H}^3(j_X j_X^* i_X^* \mathbb{Q}_{U_X}).
\]

Let \( E_r^{p,q} \) be the local to global spectral sequence of \( j_X j_X^* i_X^* \mathbb{Q}_{U_X} \), let \( E_r^{p,q} \) be the local to global spectral sequence of hypercohomology of \( j_Y j_Y^* i_Y^* \mathbb{Q}_{U_Y} \), and \( \gamma_r^{p,q} : E_r^{p,q} \to E_r^{p,q} \) the morphism induced by \( \gamma \). Then,

\[
\gamma_r^{p,q} : \mathbb{H}^p(\mathbb{P}^2, \mathcal{H}^q(j_Y j_Y^* i_Y^* \mathbb{Q}_{U_Y})) \to \mathbb{H}^p(\mathbb{P}^2, \mathcal{H}^q(j_X j_X^* i_X^* \mathbb{Q}_{U_X}))
\]

is an isomorphism if \( q = 0, 3 \).

\( E_2^{p,q} \) is

\[
\begin{array}{cccc}
q & 0 & 1 & 2 & 3 \\
3 & \mathbb{Q} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Q} & 0 & 0 & 0 \\
\hline
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

and it does not degenerate in \( r = 2 \). So, \( d_4^{0,3} \) is different from 0 since it is the unique differential different from 0 which can appear.

Moreover, \( E_2^{p,q} \) is

\[
\begin{array}{cccc}
q & 0 & 1 & 2 & 3 \\
6 & \mathbb{Q} & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 \\
3 & \mathbb{Q} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Q} & 0 & 0 & 0 \\
\hline
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

We prove now \( d_4^{0,3} : E_4^{0,3} \to E_4^{r,3-r+1} \) is different from 0 for some \( r \geq 0 \). If \( d_4^{0,3} = 0 \), we have the following isomorphisms

\[
E_4^{0,3} \cong E_3^{0,3} \cong E_2^{0,3} \cong E_2^{0,3} \cong E_4^{0,3}.
\]

Moreover,

\[
E_4^{4,0} \cong E_4^{4,0} \cong E_4^{4,0} \cong E_4^{4,0}.
\]
and the diagram

\[
\begin{array}{ccc}
E_4^{0,3} & \xrightarrow{\alpha} & E_4^{0,3} \\
\downarrow{d_4^{0,3}} & & \downarrow{d_4^{0,3}} \\
E_4^{4,0} & \xrightarrow{\beta} & E_4^{4,0}
\end{array}
\]

is cartesian.

Then, since \(d_4^{0,3}\) is not 0, \(d_4^{0,3}\) is also different from 0.

If \(\bar{p}\) is a perversity such that \(\bar{p}(6) = 2\) and \(\bar{q}\) is the complementary perversity, then we have \(\bar{q}(6) = 2\). This happens for the middle perversity. Consequently, applying Corollary 9.14, there does not exist an intersection space complex of \(X\) with perversity \(\bar{p}\) and stratification \(D\).

Hence, applying Remark 6.3 there does not exist any intersection space pair of \(X\) with perversity \(\bar{p}\).

10. Duality

In this section, we establish the duality properties of the intersection space complexes. First, we study the Verdier dual of the intersection space complex. Next, we give a version of Poincare duality for these complexes.

Let \(X\) be a topological pseudomanifold with the following stratification:

\[X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset\]

Let \(U_k := X \setminus X_{d-k}\) and let \(i_k : U_k \to U_{k+1}\) and \(j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}\) be the natural inclusions.

10.1. Verdier duality.

**Theorem 10.1.** Let \(IS_{\bar{p}}\) be an intersection space complex of \(X\) with perversity \(\bar{p}\) and let \(\bar{q}\) be the complementary perversity of \(\bar{p}\). Then, \(DIS_{\bar{p}}[-d]\), where \(D\) denotes the Verdier dual, is an intersection space complex of \(X\) with perversity \(\bar{q}\).

**Proof.** We have to prove \(DIS_{\bar{p}}[-d]\) verifies \([AXS1]_k\) for perversity \(\bar{q}\) for \(k = 2, \ldots, d\).

(a) We have a chain of isomorphisms

\[(DIS_{\bar{p}}[-d]|_{U_2} \cong DQ_{U_2}[-d] \cong Q_{U_2}.\]

(b) Let \(x \in X\), the group \(\mathcal{H}^i(j_x^*DIS_{\bar{p}}[-d])\) is isomorphic to \(\mathcal{H}^{d-i}(j_x^1IS_{\bar{p}})^v\), which is 0 if \(i \notin \{0, 1, \ldots, d\}\).

(c) Let \(x \in X_{d-k} \setminus X_{d-k+1}\), the group \(\mathcal{H}^i(j_x^*DIS_{\bar{p}}[-d])\) is isomorphic to \(\mathcal{H}^{d-i}(j_x^1IS_{\bar{p}})^v\), which (by property \((d1k)\)) is 0 if \(d-i > d - \bar{p}(k) - 1\), that is, if \(i \leq \bar{p}(k)\).

(d) Let \(x \in X_d \setminus X_{d-1}\), the group \(\mathcal{H}^i(j_x^*DIS_{\bar{p}}[-d])\) is isomorphic to \(\mathcal{H}^{d-i}(j_x^1IS_{\bar{p}})^v\), which (by property \((c_k)\)) is 0 if \(d-i \leq \bar{q}(k)\), that is, if \(i > d - \bar{q}(k) - 1\).

(e) Let \(x \in X_{d-k} \setminus X_{d-k-1}\), the group \(\mathcal{H}^d-\bar{q}(k)-1(j_x^1DIS_{\bar{p}}[-d])\) is isomorphic to \(\mathcal{H}^{d-\bar{q}(k)+1}(j_x^1IS_{\bar{p}})^v\), the group \(\mathcal{H}^{\bar{q}(k)+1}(j_x^1DIS_{\bar{p}}[-d])\) is isomorphic to \(\mathcal{H}^{d-\bar{p}(k)-1}(j_x^1IS_{\bar{p}})^v\) and the canonical morphism

\[\mathcal{H}^{d-\bar{q}(k)-1}(j_x^1DIS_{\bar{p}}[-d]) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^1IS_{\bar{p}})\]

is the dual morphism of \(\mathcal{H}^{d-\bar{p}(k)-1}(j_x^1IS_{\bar{p}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^1IS_{\bar{p}})\), which is the morphism 0 (by property \((d2k)\)).

\[\square\]
In Corollary 7.6 the space of obstructions for existence and uniqueness of intersection spaces are described. Verdier duality \( \mathcal{D} \) interchanges intersection space complexes with complementary perversities. We deduce

**Corollary 10.2.** Let \( X \) be a topological pseudomanifold as above. Let \( \bar{p} \) and \( \bar{q} \) complementary perversities. An intersection space complex for perversity \( \bar{p} \) exists if and only if an intersection space complex for perversity \( \bar{q} \) exists. Verdier duality induces a bijection between the set of intersection space complexes for perversity \( \bar{p} \) and the set of intersection space complexes for perversity \( \bar{q} \).

**10.2. Poincaré duality in the case of 2 strata.** Now, suppose \( X \) has a unique non-trivial stratum. So, the stratification of \( X \) is

\[
X \supset X_{d-k} \supset \emptyset
\]

where \( k \) is the codimension of \( X_{d-k} \).

According with Corollary 7.6, the obstruction for existence of intersection space for perversity \( \bar{p} \) lives in \( \text{Ext}^1(\tau_{\leq q(k)} j_{k}^{*} i_{k}, \mathbb{Q}) \). Assume that the obstruction vanishes so that intersection space complexes exist. In this case the space of intersection space complexes for perversity \( \bar{p} \) is parametrized by the vector space

\[
E_{\bar{p}} := \text{Hom}(\tau_{\leq q(k)} j_{k}^{*} i_{k}, \mathbb{Q}), \tau_{\leq q(k)} j_{k}^{*} i_{k}, \mathbb{Q}).
\]

Corollary 10.2 implies that the obstruction for existence of intersection space for perversity \( \bar{q} \) vanishes and that the space

\[
E_{\bar{q}} := \text{Hom}(\tau_{\leq \bar{p}(k)} j_{k}^{*} i_{k}, \mathbb{Q}), \tau_{\leq \bar{p}(k)} j_{k}^{*} i_{k}, \mathbb{Q}).
\]

of intersection space complexes for perversity \( \bar{q} \) is isomorphic to \( E_{\bar{p}} \).

**Proposition 10.3.** Let \( E_{\bar{p}} \) be the space of all intersection space complexes of \( X \) with perversity \( \bar{p} \) up to isomorphisms.

The dimensions of the vector spaces \( \mathbb{H}^{i}(X, IS_{\bar{p}}) \) with \( IS_{\bar{p}} \in E_{\bar{p}} \) have a minimum and the subset

\[
\{ IS_{\bar{p}} \in E_{\bar{p}} \mid \text{dim}(\mathbb{H}^{i}(X, IS_{\bar{p}})) \text{ is minimum} \} \subset E_{\bar{p}}
\]

is open for every \( i \).

**Proof.** For every intersection space complex \( IS_{\bar{p}} \in E_{\bar{p}} \), we have a triangle

\[
IS_{\bar{p}} \to j_{k}^{*} \mathbb{Q}U_{2} \to \tau_{\leq \bar{q}(k)} j_{k}^{*} i_{k}, \mathbb{Q} \to \mathbb{Q}U_{2}
\]

This triangle induce the long exact sequence of hypercohomology

\[
\cdots \to \mathbb{H}^{i}(X, IS_{\bar{p}}) \to \mathbb{H}^{i}(X, j_{k}^{*} \mathbb{Q}U_{2}) \xrightarrow{\alpha^{i}(IS_{\bar{p}})} \mathbb{H}^{i}(X, \tau_{\leq \bar{q}(k)} j_{k}^{*} i_{k} \mathbb{Q}U_{2}) \to \cdots
\]

So, for every \( i \in \mathbb{Z} \), there is an isomorphism

\[
\mathbb{H}^{i}(X, IS_{\bar{p}}) \cong \text{Ker}(\alpha^{i}(IS_{\bar{p}})) \oplus \text{CoKer}(\alpha^{i-1}(IS_{\bar{p}})).
\]

Moreover,

\[
\text{dim}(\text{CoKer}(\alpha^{i}(IS_{\bar{p}}))) = \text{dim}(\mathbb{H}^{i}(X, \tau_{\leq \bar{q}(k)} j_{k}^{*} i_{k} \mathbb{Q}U_{2})) - \text{dim}(\mathbb{H}^{i}(X, j_{k}^{*} \mathbb{Q}U_{2})) + \text{dim}(\text{Ker}(\alpha^{i}(IS_{\bar{p}})))
\]

So, \( \text{dim}(\mathbb{H}^{i}(X, IS_{\bar{p}})) \) is minimum if and only if \( \text{dim}(\text{Ker}(\alpha^{i}(IS_{\bar{p}}))) \) and \( \text{dim}(\text{Ker}(\alpha^{i-1}(IS_{\bar{p}}))) \) are minimum. The morphism \( \alpha^{i}(IS_{\bar{p}}) \) is the morphisms induced in hypercohomology by the composition

\[
i_{k^{*}} \mathbb{Q}U_{2} \xrightarrow{a} j_{k}^{*} j_{k}^{*} i_{k} \mathbb{Q}U_{2} \xrightarrow{\lambda(IS_{\bar{p}})} \tau_{\leq \bar{q}(k)} j_{k}^{*} j_{k}^{*} i_{k} \mathbb{Q}U_{2}
\]

where \( a \) is the canonical morphisms and \( \lambda(IS_{\bar{p}}) \) is a retraction of the natural truncation morphism \( f : \tau_{\leq \bar{q}(k)} j_{k}^{*} j_{k}^{*} i_{k} \mathbb{Q}U_{2} \to j_{k}^{*} j_{k}^{*} i_{k} \mathbb{Q}U_{2} \).
Let us denote by
\[ a^i : \mathbb{H}^i(X, i_k \mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, j_k \mathbb{Q}_{U_2}), \]
\[ \lambda^i(IS_p) : \mathbb{H}^i(X, j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}), \]
\[ f^i : \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}) \]
the morphisms induced in hypercohomology. Then, we have
\[ \dim(\text{Ker}(\alpha^i(IS_p))) = \dim((\alpha^i)^{-1}(\text{Ker}(\lambda^i(IS_p)))) = \dim(\text{Im}(\alpha^i) \cap \text{Ker}(\lambda^i(IS_p))) \]
Hence, \[ \dim(\text{Ker}(\alpha^i(IS_p))) \] is minimum if and only if \[ \dim(\text{Im}(\alpha^i) \cap \text{Ker}(\lambda^i(IS_p))) \] is minimum.
Since \[ \lambda^i(IS_p) \circ f^i \] is the identity, the homomorphism \[ \lambda^i(IS_p) \] is surjective and we have the equality
\[ \dim(\text{Ker}(\lambda^i(IS_p))) = \dim(\mathbb{H}^i(X, j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2})) - \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2})). \]
So, \[ \dim(\text{Ker}(\lambda^i(IS_p))) \] is independent of \[ IS_p. \] Then, for every \[ IS_p \in E_p, \] there is an isomorphism
\[ \mathbb{H}^i(X, j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}) / \text{Ker}(\lambda^i(IS_p)) \cong \mathbb{Q}^{d_i} \]
where \[ d_i := \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2})). \]
Now, consider the composition of morphisms
\[ \Phi(IS_p) \]
\[ \mathbb{H}^i(X, i_k \mathbb{Q}_{U_2}) \xrightarrow{a^i} \mathbb{H}^i(X, j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}) \xrightarrow{\pi(IS_p)} \mathbb{H}^i(X, j_k \mathbb{i}_k i_k \mathbb{Q}_{U_2}) / \text{Ker}(\lambda^i(IS_p)) \cong \mathbb{Q}^{d_i} \]
where \[ \pi(IS_p) \] is the canonical projection.
Then, \[ \text{Im}(\alpha^i) \cap \text{Ker}(\lambda^i(IS_p)) \] gets the minimum dimension when the morphism \( \Phi(IS_p) \) gets the maximum rank, which happens in an open subset.

**Definition 10.4.** The general \( i \)-th Betti number of the intersection space complexes of \( X \) with perversity \( \bar{p} \) is the minimum of the dimensions of the vector spaces \( \mathbb{H}^i(X, IS_p) \) with \( IS_p \in E_p. \)

**Definition 10.5.** A general intersection space complex of \( X \) with perversity \( \bar{p} \) is an intersection space complex \( IS_p \in E_p \) such that \( \dim(\mathbb{H}^i(X, IS_p)) \) is the general \( i \)-th Betti number for \( i = 0, 1, \ldots, d. \)

**Theorem 10.6.** Let \( \bar{p} \) be a perversity and let \( \bar{q} \) be its complementary perversity. If \( IS_p \) is a general intersection space complex of \( X \) with perversity \( \bar{p} \) and \( IS_q \) is a general intersection space complex of \( X \) with perversity \( \bar{q} \), then, for \( i = 0, 1, \ldots, d, \) there is an isomorphism of \( \mathbb{Q} \)-vector spaces
\[ \mathbb{H}^i(X, IS_p) \cong \mathbb{H}^{d-i}(X, IS_q)^v \]

**Proof.** Given any intersection space complex \( IS_p \) of \( X \) with perversity \( \bar{p} \), we have
\[ \mathbb{H}^i(X, DIS_p[-d])^v \cong \mathbb{H}^{i-d}(X, DIS_p)^v \cong \mathbb{H}^{d-i}(X, IS_p). \]
Applying Theorem 10.1, the complex \( DIS_p[-d] \) is an intersection space complex of \( X \) with perversity \( \bar{q} \). We denote \( IS_q := DIS_p[-d] \).
Suppose \( IS_q \) is not a general intersection space complex of \( X \). Then, there exist another intersection space complex of \( X \) with perversity \( \bar{q}, IS_q' \), such that we have the strict inequality
\[ \sum_{i=0}^d \dim(\mathbb{H}^i(X, IS_q)) < \sum_{i=0}^d \dim(\mathbb{H}^i(X, IS_q)). \]
Consequently we have,
\[ \sum_{i=0}^d \dim(\mathbb{H}^i(X, DIS_q[-d])) = \sum_{i=0}^d \dim(\mathbb{H}^i(X, IS_q')) < \sum_{i=0}^d \dim(\mathbb{H}^i(X, IS_q))^v = \]
\[
= \sum_{i=0}^{d} \dim(\mathbb{H}^{i}(X, D\mathcal{S}_q[-d])) = \sum_{i=0}^{d} \dim(\mathbb{H}^{i}(X, IS_{\bar{q}})).
\]

So, \( IS_{\bar{p}} \) is not a general intersection space complex of \( X \).

We deduce that if \( IS_{\bar{p}} \) is a general intersection space complex, then \( IS_{\bar{q}} \) is also a general intersection space complex. So, there are isomorphisms

\[
\mathbb{H}^{i}(X, IS_{\bar{p}}) \cong \mathbb{H}^{d-i}(X, IS_{\bar{q}})
\]

for some general intersection space complexes \( IS_{\bar{p}} \) and \( IS_{\bar{q}} \).

Since the hypercohomology groups of general intersection space complexes with the same perversity are isomorphic, we conclude. \( \square \)

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