Euclidean Jordan algebras and generalized Krein parameters of a strongly regular graph

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Abstract

Let $\tau$ be a strongly $(n, p; a, c)$ regular graph, such that $0 < c < p < n - 1$, $A$ its matrix of adjacency and let $V_n$ be the Euclidean space spanned by the powers of $A$ over the reals where the scalar product $\langle x,y \rangle$ is defined by $\langle x|y \rangle = \text{trace}(x \cdot y)$. In this work one proves that $V_n$ is an Euclidean Jordan algebra of rank 3 when one introduces in $V_n$ the usual product of matrices. Working inside the Euclidean Jordan algebra $V_n$ and making use of the properties of $|A|^\tau$ one defines the generalized Krein parameters of the strongly $(n, p; a, c)$ regular graph $\tau$ and finally one presents necessary conditions over the parameters and the spectra of the $\tau$ strongly $(n, p; a, c)$ regular graph.

1 Introduction

Euclidean Jordan algebras are more and more used in various branches of Mathematics. For example, one may cite the application of this theory to statistics [10], to interior point methods [4, 5, 7], combinatorics [1, 2, 11]. Among several readable texts about or which includes Euclidean Jordan algebras, we may refer [3] and [9]. Previous works on strongly regular graphs included in Euclidean Jordan algebras were developed aiming to obtain the minimal coherent algebra which includes a particular set of matrices and then its symmetric homogeneous coherent 3-dimension subalgebras (all of them defining strongly regular graphs, as it is well known) [11].

This report is organized as follows. In the next section one presents the basic results of Euclidean Jordan algebras. In section 3 one presents a brief introduction on strongly regular graphs. In section 4 one associates an Euclidean Jordan Algebra $\mathcal{V}$ of rank three to the matrix of adjacency of a strongly regular graph. In section 5 one defines the power of $|A|^\tau$ inside the Euclidean Jordan algebra $\mathcal{V}$ associated to a strongly regular graph $\tau$ and one obtains his coordinates in the basis $\{I_n, A, E_1\}$ where $A$ is the matrix of adjacency of $\tau$ and $E_1$ is a primitive idempotent of the unique Jordan frame of the Euclidean Jordan algebra $\mathcal{V}$ associated to $A$. In section 6 one defines the generalized Krein parameters of a strongly regular graph. Finally one establishes necessary conditions for the existence of a strongly $(n, p; a, c)$ regular graph.
2 Basic results on Euclidean Jordan algebras

Consider a real finite and \(n\)-dimensional vectorial space \(V\) with a multiplication \(\circ\) such that the map \((x, y) \to x \circ y\) is bilinear. If \(\forall x \in V\) \((x \circ x) \circ x = x \circ (x \circ x)\) one says that \(V\) is a power associative algebra. In a power associative algebra, \(V\), for all \(x \in V\) and for all \(p, q \in \mathbb{N}\) \(x^p \circ x^q = x^{p+q}\). If \(e \in V\) is such that \(\forall x \in V\) \(x \circ e = e \circ x = x\), then \(e\) is called the unit of \(V\).

Let \(V\) be a power associative algebra with unit element \(e\). For all \(x \in V\) let \(k\) be the least natural number such that \(\{e, x, x^2, \ldots, x^k\}\) is linear dependent. Then \(k\) is the rank of \(x\) and one writes \(rank(x) = k\). One defines the rank of \(V\) as being the natural number \(r = rank(V) = \max\{rank(x) : x \in V\}\). An element \(x \in V\) is regular if its rank is equal to the rank of the algebra. Given a regular element, \(x\), in a power associative algebra, \(V\), with unit element, \(e\), and rank \(r\), since the set \(\{e, x, x^2, \ldots, x^r\}\) is linear independent, one may conclude that there are \(r\) unique real numbers, \(a_1(x), a_2(x), \ldots, a_r(x)\), such that

\[
x^r - a_1(x)x^{r-1} + \ldots + (-1)^r a_r(x)e = 0,
\]

where 0 is the null vector of \(V\). Taking in account (1), the polynomial

\[
p(x, \lambda) = \lambda^r - a_1(x)\lambda^{r-1} + \ldots + (-1)^r a_r(x)
\]

is called the characteristic polynomial of \(x\), where the coefficients \(a_i(x)\) are polynomial functions in the coordinates of \(x\) in a fixed basis of \(V\). The definition of the characteristic polynomial may be extended to any element of \(V\). Indeed, since the set of regular elements of \(V\) is a dense set in \(V\) [3], if \(x \in V\) then there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) of regular elements of \(V\) converging to \(x\). Defining \(\lim_{n \to \infty} a_i(x_n) = a_i(\lim_{n \to \infty} x_n) = a_i(x)\), one obtains the characteristic polynomial of a non regular element as being the polynomial (2).

Let \(x\) be a regular element of \(V\). If one considers the real subalgebra of \(V\), \(\mathbb{R}[x]\), spanned by \(\{e, x, x^2, \ldots, x^{r-1}\}\), then the restriction of the application \(L(x)\), such that \(L(x)y = x \circ y\), to \(\mathbb{R}[x]\) denoted by \(L_0(x)\) has the matrix representation, in the basis \(B = \{e, x, \ldots, x^{r-1}\}\), given by

\[
M_{L_0(x)} = \\
\begin{bmatrix}
0 & 0 & \cdots & 0 & (-1)^{r-1} a_r(x) \\
1 & 0 & \cdots & 0 & (-1)^{r-2} a_{r-1}(x) \\
0 & 1 & \cdots & 0 & (-1)^{r-3} a_{r-2}(x) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a_1(x)
\end{bmatrix},
\]

whose characteristic polynomial is the polynomial (2).

From now on, \(p(x, \lambda)\) stands for the characteristic polynomial of an element \(x\) of a power associative algebra. Easily one can see that when \(x\) is a regular element, \(a_r(x)\) is equal to determinant of the matrix \(M_{L_0(x)}\) and \(a_1(x)\) is equal to the trace of the matrix \(M_{L_0(x)}\). For this reason, in a real power associative algebra, \(V\), with unit element, one defines for each element \(x\) of \(V\) the trace and the determinant of \(x\) and one denotes them, respectively, by \(\text{tr}(x)\) and \(\text{det}(x)\),
as being, respectively, the coefficients $a_1(x)$ and $a_r(x)$ of the polynomial $2$. As an example of real power associative algebras with unit element one may refer the real Jordan algebras with unit element which one describes next.

**Definition 2.1** Let $V$ be a real finite dimensional vector, with the operation of multiplication of vectors, $\circ$, determined by the bilinear application $(x, y) \rightarrow x \circ y$. One says that $V$ is a Jordan algebra if

1) $x \circ y = y \circ x$;
2) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$.

From now on, when one refers to a Jordan algebra, $V$, one admits that $V$ is a real finite dimensional algebra and with unit element $e$. Since Koecher [9] proves that if $V$ is a Jordan algebra then $V$ is a power associative algebra, then to each element $x \in V$ one may associate the respective characteristic polynomial.

Thus, if $r$ is the characteristic of $V$ and the characteristic polynomial of $x$ is the polynomial $2$, then one defines the determinant and the trace of $x$, which are denoted, respectively, $\det(x)$ and $\text{tr}(x)$, by the equalities $\text{tr}(x) = a_1(x)$ and $\det(x) = a_r(x)$. Given a Jordan algebra one says that the element $x$ is invertible if $\exists y \in \mathbb{R}[x]$ such that $x \circ y = e$, and then $y$ is designated the inverse of $x$ and denoted by $x^{-1}$.

A real Jordan algebra $V$ is Euclidean if there is a scalar product $\langle \cdot, \cdot \rangle$ such that $\langle u \circ v, w \rangle = \langle v, u \circ w \rangle$. Additionally, two elements $c, d \in V$ are orthogonal, relatively to the algebra $V$, if $c \circ d = 0$. On the other hand, assuming that $e$ is the unit element of $V$, $c \in V$ is an idempotent if $c^2 = c$, and one says that $\{c_1, c_2, \ldots, c_k\}$ is a complete system of orthogonal idempotents if

1. $c_i^2 = c_i \forall i \in \{1, \ldots, k\}$,
2. $c_i \circ c_j = 0 \forall i \neq j$,
3. $c_1 + c_2 + \cdots + c_k = e$.

An idempotent $c$ is primitive if it is not the sum of two other idempotents. One says that $\{c_1, c_2, \ldots, c_k\}$ is a complete system of orthogonal primitive idempotents or a Jordan frame, if $\{c_1, c_2, \ldots, c_k\}$ is a complete system of orthogonal idempotents such that each idempotent is primitive.

**Theorem 2.1** ([3], p.43) Let $V$ be a Euclidean Jordan algebra with unit element $e$. For $x \in V$ there exist unique real numbers $k$ unique real numbers, $\lambda_1, \lambda_2, \ldots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents, $\{c_1, c_2, \ldots, c_k\}$, such that

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_k c_k.$$  \(3\)

Additionally, $c_j \in \mathbb{R}[x]$, for $j = 1, \ldots, k$.

\[^1\text{Subalgebra of } V \text{ generated by } e \text{ and } x.\]
The numbers $\lambda_j$ of (3) are called the eigenvalues of $x$ and $x = \sum_{i=1}^{k} \lambda_i c_i$ is the spectral decomposition of $x$.
If $V$ is a Euclidean Jordan algebra with unit element $e$ and characteristic $r$, and $c$ is a primitive idempotent of $V$, then $\text{tr}(c) = 1$ and, therefore, one may conclude that $\text{tr}(e) = r$.

**Theorem 2.2** ([3], p.44) Let $V$ be an Euclidean Jordan algebra with characteristic $r$ and unit element $e$. Then, for each $x \in V$, there is a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that

$$x = \sum_{j=1}^{r} \lambda_j c_j.$$  

The $\lambda_i$s, together with their multiplicities, are uniquely determined by $x$. Additionally one verifies that

$$\text{tr}(x) = \sum_{j=1}^{r} \lambda_j.$$  

### 3 Brief introduction to strongly regular graphs

A graph $G$, is a pair of sets $(V(G), E(G))$, where $V(G)$ denotes the nonempty set of vertices and $E(G)$ the set of edges. An element of $E(G)$, which the endpoints are the vertices $i$ and $j$, is denoted by $ij$ and, in such case, we say that the vertex $i$ is adjacent to the vertex $j$. If $v \in V(G)$, then we call neighborhood of $v$ the vertex set denoted by $N_G(v) = \{w : vw \in E(G)\}$. The complement of $G$, denoted by $\bar{G}$, is such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{ij : i, j \in V(G) \land ij \notin E(G)\}$. A graph of order $n$ in which all pairs of vertices are adjacent is called a complete graph and denoted by $K_n$. On the other hand, when there are no pair of adjacent vertices the graph is called a null graph and denoted by $N_n$. In this paper we assume that $G$ is of order $n > 1$, i.e., $|V(G)| = n > 1$ and we deal only with simple graphs, that is, graphs with nor loops (edges with both ends in the same vertex) neither parallel edges (more than one edge between the same pair of vertices). Consider the set of vertices of the graph $G$, $X = \{x_1, x_2, \ldots, x_k\}$, and suppose that $x_i x_{i+1} \in E(G) \ \forall i \in \{1, \ldots, k-1\}$. If the vertices of $X$ are all distinct then we say that $X$ induces a path (and if they are all distinct but $x_1$ and $x_k$, which are equal, then we say that $X$ induces a cycle). If there is no $i \in \{1, \ldots, k-2\}$ and $j > 1$ such that $x_i x_{i+j} \in E(G)$, then $X$ is denoted by $P_k$ ($C_k$) and we say that it is a chordless path (cycle). The adjacency matrix of the graph $G$ is the matrix $A_G = (a_{ij})_{n \times n}$ such that

$$a_{ij} = \begin{cases} 1 & \text{, if } ij \in E(G) \\ 0 & \text{, otherwise.} \end{cases}$$

The number of neighbors of $v \in V(G)$ will be denoted by $d_G(v)$ and called degree of $v$. If $G$ is such that $\forall v \in V(G) \quad d_G(v) = p$ then we say that $G$ is $p$-regular. A graph $G$ is called strongly regular if it is regular, not complete,
not null and, given any two distinct vertices $i, j \in V(G)$, the number of vertices which are neighbors of both $i$ and $j$ depends on whether $i$ and $j$ are adjacent or not. When $G$ is a strongly regular graph of order $n$ which is $k$-regular and any pair of adjacent vertices have $p$ common neighbors and any two distinct non-adjacent vertices have $q$ common neighbors, then we say that $G$ is a $(n, p; a, c)$-strongly regular graph. The chordless cycle on five vertices $C_5$ is an example of a strongly regular graph which is $(5, 2; 0, 1)$-strongly regular. It is well known (see for instance [6]) that a graph $G$ which is not complete and not null is strongly regular iff $A^2_G$ is a linear combination of $A_G$, $I_n$ and $J_n$, that is, $\exists \tau_1, \tau_2, \tau_3 \in \mathbb{R}$ such that

$$A^2_G = \tau_1 I_n + \tau_2 A_G + \tau_3 J_n, \quad (4)$$

where $J_n$ denotes the all ones square matrix of order $n$. Therefore, since $A_G = J_n - I_n - A_G$ we may conclude that a graph is strongly regular if and only if its complement is also strongly regular. From now on one will use $A$ for representing the matrix of adjacency of a strongly regular graph.

Let $\tau$ be a strongly $(n, p; a, c)$ regular graph such that $0 < c < p < n - 1$, $A$ his matrix of adjacency and $V_n$ be the Euclidean space over the reals of the linear combinations of the powers of $A$ with exponent in $\mathbb{N}_0$ where the sum of vectors is the usual sum of matrices, the product of a vector by a scalar is the product of matrix by a real number and the scalar product is $\cdot | \cdot$ defined by $x | y = \text{trace}(x \cdot y)$.

4 Euclidean Jordan algebra associated to the matrix of adjacency of a strongly regular graph

Since the powers of $A^n$ with $n$ a natural number commute among each other then $x \cdot y = y \cdot x$, $\forall x, y \in V_n$. For the same reason and by the fact that the product of matrices his associative, one concludes that

i) $x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y), \forall x, y \in V_n,$

ii) $\langle L(x)y, z \rangle = \langle y, L(x)z \rangle, \forall x, y, z \in V_n.$

Thus, the structure $(V_n, \cdot)$ where $\cdot$ is the usual product of matrices is an Euclidean Jordan algebra.

One now introduces some notation. Let $r$ and $s$ be the real numbers defined by

$$r = \frac{a - c + \sqrt{(a - c)^2 + 4 \ast (p - c)}}{2},$$

$$s = \frac{a - c - \sqrt{(a - c)^2 + 4 \ast (p - c)}}{2}.$$

Since the linear operator $L(A)$ is a symmetric linear operator from $V_n$ to $V_n$ with only three distinct eigenvalues, namely $p, r$ and $s$ one concludes that $S =$
\{E_1, E_2, E_3\} is the unique complete system of orthogonal idempotents of \(V_n\) associated to \(A\), where

\[
E_1 = \frac{A^2 - (r + s)A + rsI_n}{(p - r)(p - s)} = \frac{J}{n},
\]

\[
E_2 = \frac{A^2 - (p + s)A + psI_n}{(r - s)(r - p)},
\]

\[
E_3 = \frac{A^2 - (p + r)A + prI_n}{(s - r)(s - p)}
\]

and \(J_n\) is the matrix of ones. In the following one shows that \(S\) is a Jordan frame of the Euclidean Jordan algebra \(V_n\) that is a basis of \(V_n\) with the aim of proving that \(V_n\) is an Euclidean Jordan algebra with rank three. Since the spectral decomposition of \(A\) is \(A = pE_1 + rE_2 + sE_3\) one concludes that

\[
A^k = p^kE_1 + r^kE_2 + s^kE_3, \forall k \in \mathbb{N}_0
\]

and so the set \(S\) spans \(V_n\) and since \(\{E_1, E_2, E_3\}\) is a linear independent set of \(V_n\) then \(\langle E_1, E_2, E_3 \rangle\) is a basis of \(V_n\). Therefore one may affirm that the dimension of \(V_n\) is three and that \(S\) is a Jordan frame of \(V_n\). Now let us prove that rank(\(A\)) = 3. Since \(A\) is the matrix of adjacency \(\tau\) then

\[
(A - pI_n)(A^2 - (s + r)A + srI_n) = 0.
\]

So, after some calculations it follows that

\[
A^3 = A^2(s + r + p) - (sr + (s + r)p)A + psrI_n.
\]

Thus, one may conclude that \(\{I_n, A, A^2, A^3\}\) is a linear dependent set. We will now prove that \(\{I_n, A, A^2\}\) is a free set of \(V_n\). Let \(\alpha, \beta\) and \(\gamma\) be real numbers. Since

\[
\begin{vmatrix}
1 & p & p^2 \\
1 & r & r^2 \\
1 & s & s^2
\end{vmatrix} = (r - p)(s - p)(s - r) \neq 0
\]

and

\[
\alpha I_n + \beta A + \gamma A^2 = 0
\]

\(\uparrow\)

\[
\alpha E_1 + \alpha E_2 + \alpha E_3 + \beta(pE_1 + rE_2 + sE_3) + \gamma(p^2E_1 + r^2E_2 + s^2E_3) = 0
\]

\(\uparrow\)

\[
(\alpha + p\beta + p^2\gamma)E_1 + (\alpha + r\beta + r^2\gamma)E_2 + (\alpha + s\beta + s^2\gamma) = 0
\]

\(\uparrow\)

\[
\begin{cases}
\alpha + p\beta + p^2\gamma = 0 \\
\alpha + r\beta + r^2\gamma = 0 \\
\alpha + s\beta + s^2\gamma = 0
\end{cases}
\]

\(\uparrow\)

\[
\alpha = 0 \land \beta = 0 \land \gamma = 0
\]

then \(\{I_n, A, A^2\}\) is a free set of \(V_n\), and therefore the least natural number \(k\) such that the set \(\{I_n, A, A^2, \cdots, A^k\}\) is linear dependent is three and so rank(\(A\)) = 3. Since the dimension of \(V_n\) is three it follows that rank(\(V_n\)) = 3.
5 Properties of the Euclidean Jordan algebra associated to a strongly regular graph

Let $\tau$ be a strongly regular graph, $\mathcal{V}_n$ the Euclidean Jordan algebra associated to $\tau$ and $S = \{E_1, E_2, E_3\}$ the unique Jordan frame associated to the matrix of adjacency of $\tau$. Making use of the spectral decomposition of $I_n$ and of $A$ one concludes that the set $\{I_n, A, E_1\}$ is a linear independent set of $\mathcal{V}_n$. Since $\mathcal{V}_n$ is 3-dimensional then one may conclude that $\{I_n, A, E_1\}$ is a basis of $\mathcal{V}_n$. In this section one considers the basis $\mathcal{B} = \{I_n, A, E_1\}$ and we make use of the scalar product $< x, y > = \text{tr}_{\mathcal{V}_n}(x \cdot y)$, $\forall x, y \in \mathcal{V}_n$ where now the notation $\text{tr}_{\mathcal{V}_n}(u)$ represents the trace of the vector $u$ in the sense of the Euclidean Jordan algebra $\mathcal{V}_n$. In first place, one defines the power of $|A|^x$ for any real number $x$ and next one calculates the coordinates of $|A|^x$ in the basis $\mathcal{B}$. Since $S$ is an Jordan frame of $\mathcal{V}_n$ it follows that $\text{tr}_{\mathcal{V}_n}(E_i) = 1$, $\forall i = 1, \cdots, 3$. Let $x$ be a real number. One defines $|A|^x$ by the equality

$$|A|^x = p^xE_1 + r^xE_2 + |s|^xE_3.$$  

Now one determines the coordinates of $|A|^x$ in the basis $\mathcal{B}$. So let $\alpha_x, \beta_x$ and $\gamma_x$ be real numbers such that

$$|A|^x = \alpha_xI_n + \beta_xA + \gamma_xE_1.$$  

Using the spectral decomposition of $|A|^x$ it follows that

$$p^xE_1 + r^xE_2 + |s|^xE_3 = \alpha_xI_n + \beta_xA + \gamma_xE_1.$$  

Now let us consider in $\mathcal{V}_n$ the scalar product $< \bullet, \bullet >$ defined by

$$< x, y > = \text{tr}_{\mathcal{V}_n}(xy), \forall x, y \in \mathcal{V}_n.$$  

Multiplying both members of (6) successively by $E_1, E_2$ and $E_3$ and applying $\text{tr}_{\mathcal{V}_n}$ to both members of (6) one obtains the system

$$\begin{cases}
  r^x &= 1\alpha_x + r\beta_x + 0\gamma_x \\
  |s|^x &= 1\alpha_x + s\beta_x + 0\gamma_x \\
  p^x &= 1\alpha_x + p\beta_x + \gamma_x
\end{cases}.$$  

Since

$$\begin{vmatrix}
  1 & r & 0 \\
  1 & s & 0 \\
  1 & p & 1
\end{vmatrix} = s - r \neq 0$$

then one may conclude that the system (7) is a Cramer system. Therefore solving (7) one obtains:

$$\alpha_x = \frac{(p-c)(r^x-1+|s|^x)}{r-s}, \quad \beta_x = -\frac{|s|^x-r^x}{r-s}, \quad \gamma_x = p^x - r^x + (p-r)\frac{|s|^x-r^x}{r-s}.$$
Finally, one may write
\[ |A|^x = \left( \frac{(p-c)(r^{x-1} + |s|^{x-1})}{r-s} I_n - \frac{|s|^x - r^x}{r-s} A + (p^x - r^x + (p-r)\frac{|s|^x - r^x}{r-s})E_1. \]

(8)

Now let suppose that \( r, |s| \) and \( p \) are distinct numbers. Since
\[ |A|^x = p^x E_1 + |r|^x E_2 + |s|^x E_3 \]
then rewriting (8) one obtains
\[ p^x E_1 + r^x E_2 + |s|^x E_3 = \frac{(p-c)(r^{x-1} + |s|^{x-1})}{r-s} I_n - \frac{|s|^x - r^x}{r-s} A + (p^x - r^x + (p-r)\frac{|s|^x - r^x}{r-s})E_1. \]

(9)

Working in the real vector space \( \mathcal{F} \) of real functions with the usual operations of addition of functions and scalar multiplication of a function by a real and equalizing the coefficients of \( p^x, r^x \) and of \( |s|^x \) in both members of (9) one obtains:
\[
E_1 = \frac{J}{n}, \\
E_2 = \frac{|s|}{r-s} I_n + \frac{1}{r-s} A + \frac{s-p}{r-s} J, \\
E_3 = \frac{r}{r-s} I_n - \frac{1}{r-s} A + \frac{p-r}{r-s} J.
\]

Using the basis \( \{I_n, A, J_n - A - I_n\} \) of \( \mathcal{V}_n \) it follows that
\[
E_1 = \frac{r-s}{n(r-s)} I_n + \frac{r-s}{n(r-s)} A + \frac{r-s}{n(r-s)} (J_n - A - I_n), \\
E_2 = \frac{|s|}{r-s} I_n + \frac{n+s-p}{n(r-s)} A + \frac{s-p}{n(r-s)} (J_n - A - I_n), \\
E_3 = \frac{r n + p - r}{n(r-s)} I_n - \frac{n+p-r}{n(r-s)} A + \frac{p-r}{n(r-s)} (J_n - A - I_n).
\]

(10)

One now suppose that \( r, s \) and \( p \) are distinct real numbers and that \( r = |s| \). Let \( \epsilon \) be a sufficient small real number such that the set \( \{(p+\epsilon)^x, (r+\epsilon)^x, |s+\epsilon|^x\} \) is a linear independent set of the real vector space \( \mathcal{F} \). Let now work in the Euclidean Jordan Algebra generated by the powers of the matrix \( A+\epsilon I_n \). Then proceeding in the same way as when one has deduced (9) with \( p, r \) and \( |s| \) distinct one obtains:
\[
(p+\epsilon)^x E_1 + (r+\epsilon)^x E_2 + |s+\epsilon|^x E_3 = \\
|s+\epsilon|(r+\epsilon)\frac{(r+\epsilon)^{x-1} + |s+\epsilon|^{x-1}}{r-s} I_n - \frac{|s+\epsilon|(r+\epsilon)^x}{r-s} (A + \epsilon I) + \\
+ (p+\epsilon)^x - (r+\epsilon)^x + (p-r)\frac{|s+\epsilon|(r+\epsilon)^x}{r-s})E_1.
\]

(11)
Now working in the real vector space $\mathcal{F}$ and equalizing the coefficients of $(p + \epsilon)x, (r + \epsilon)x$ and $|s + \epsilon|x$ in both members of (11) one obtains:

\[
E_1 = \frac{J}{n}, \\
E_2 = \frac{|s + \epsilon|}{r - s} A_n + \frac{1}{r - s} (A + \epsilon I_n) + \frac{s - p}{(r - s)} J_n, \\
E_3 = \frac{r + \epsilon}{r - s} A_n - \frac{1}{r - s} (A + \epsilon I_n) + \frac{p - r}{(r - s)} J_n.
\]

Using the basis $\{I_n, A, J_n - A - I_n\}$ of $\mathcal{V}_n$ it follows that

\[
E_1 = \frac{J}{n}, \\
E_2 = \frac{|s + \epsilon|}{n(r - s)} A_n + \frac{n + s - p}{n(r - s)} A + \frac{ne}{n(r - s)} I_n + \frac{s - p}{n(r - s)} (J_n - A - I_n), \\
E_3 = \frac{(r + \epsilon)n + p - r}{n(r - s)} I_n + \frac{-n + p - r}{n(r - s)} A + \frac{ne}{n(r - s)} I_n + \frac{p - r}{n(r - s)} (J_n - A - I_n).
\]

Then making $\epsilon$ converge to zero one obtains:

\[
E_1 = \frac{J}{n}, \\
E_2 = \frac{|s|}{n(r - s)} A_n + \frac{n + s - p}{n(r - s)} A + \frac{s - p}{n(r - s)} (J_n - A - I_n), \\
E_3 = \frac{rn + p - r}{n(r - s)} I_n + \frac{-n + p - r}{n(r - s)} A + \frac{p - r}{n(r - s)} (J_n - A - I_n).
\]

(12)

Now one will establish necessary conditions for the existence of a strongly $(n, p; a, c)$ regular graph $\tau$ working inside the Euclidean Jordan algebra $\mathcal{V}_n$ associated to $\tau$ and using the theorem 6.1.

6 Generalized Krein parameters of a strongly regular graph

In this section one will define the generalized Krein parameters of the strongly regular graph $\tau$, establish some properties of these parameters and finally one will deduce necessary conditions for the existence of a strongly $(n, p; a, c)$ regular graph. But first one presents the theorems 6.1, 6.2, the lemma 6.1 and some notation.

Theorem 6.1 \((12), p.439\) Let $A$ be a symmetric matrix of order $n$ with eigenvalues

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.
\]

Suppose $N$ is an $m \times n$ real matrix such that $NN^T = I_m$, so $m \leq n$. Let $B = NAN^T$, and let

\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m
\]
be the eigenvalues of $B$. Then the eigenvalues of $B$ interlace those of $A$, in the sense that

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$$

for $i = 1, 2, \cdots, m$.

From (12) it follows that

$$E_1 = \frac{(r-s)I_n + (r-s)A + J_n - I_n}{n(r-s)},$$
$$E_2 = \frac{|s_n + p| I_n + n(r-s)A + J_n - I_n}{n(r-s)},$$
$$E_3 = \frac{r_n + p}{n(r-s)} I_n + \frac{n(r-s)}{n(r-s)} A + \frac{J_n - I_n}{n(r-s)},$$
$$E_1 + E_2 = \frac{|s_n + p| I_n + n(r-s)A + J_n - I_n}{n(r-s)},$$
$$E_1 + E_3 = \frac{r_n + p}{n(r-s)} I_n + \frac{n(r-s)}{n(r-s)} A + \frac{J_n - I_n}{n(r-s)},$$
$$E_2 + E_3 = \frac{n-1}{n} I_n - \frac{1}{n} A - \frac{1}{n} (J_n - A - I_n).$$

Now one will introduce some notation.

**Definition 6.1** let $A$ and $B$ be matrices of $\mathcal{M}_{m \times n}(\mathbb{R})$ and $\mathcal{M}_{p \times q}(\mathbb{R})$. Then one defines the Kronecker product of $A$ and $B$ as being the matrix $A \otimes B$ such that

$$A \otimes B = \left[ \begin{array}{llll}
  a_{11}B & a_{12}B & \cdots & a_{1n}B \\
  a_{21}B & a_{22}B & \cdots & a_{2n}B \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{array} \right].$$

**Definition 6.2** Let $A \in B$ be matrices of $\mathcal{M}_n(\mathbb{R})$. Then one defines the componentwise product of $A$ and $B$ as being the matrix $A \circ B$ such that $(A \circ B)_{ij} = a_{ij}^b_{ij}$, $\forall i, j \in \{1, \cdots, n\}$.

**Definition 6.3** Let $B \in \mathcal{M}_p(\mathbb{R})$, The $k$th kronecker power $B^{\otimes k}$ is defined inductively for all positive integers $k$ by

$$B^{\otimes 1} = B,$$
$$B^{\otimes k} = B \otimes B^{\otimes (k-1)}, \forall k \geq 2, k \in \mathbb{N}.$$

Using the definition 6.3 one considers the following notation

$$\forall m, n \in \mathbb{N}, (EF)^{\otimes mn} = E^{\otimes m} \oplus F^{\otimes n}.$$
Using the definition \(\mathbf{6.4}\) one considers that
\[
\forall m, n \in \mathbb{N}, \ (EF)^{\ominus mn} = E^{\ominus m} \circ F^{\ominus n}.
\]

Let \(k\) and \(l\) be natural numbers. From now on, one will use the notation that one presents next.

\[
\forall j \in \{1, \cdots , 3\}, \ E^{\ominus jk} = (E_j)^{\ominus k},
\]
\[
\forall u, v \in \{1, \cdots , 3\} : u < v, \ E^{\ominus ukl} = (E_u)^{\ominus k} \circ (E_v)^{\ominus l},
\]
\[
\forall u, v \in \{1, \cdots , 3\} : u < v, \ E^{\ominus (+uv)k} = (E_u + E_v)^{\ominus k},
\]
\[
\forall j, u, v \in \{1, \cdots , 3\} : u < v, \ E^{\ominus j(+uv)kl} = (E_j(E_u + E_v))^{\ominus k}.}
\]

**Theorem 6.2** Let \(\mathcal{M}_p(\mathbb{R})\) represent the vector space of the real matrices of order \(p\), \(E\) and \(F\) be two idempotents matrices of \(\mathcal{M}_p(\mathbb{R})\). Then \(E^{\oplus m}, F^{\oplus n}\) and \(EF^{\oplus mn}\) are idempotents of \(\mathcal{M}_{p^m}(\mathbb{R})\), \(\mathcal{M}_{p^n}(\mathbb{R})\) and of \(\mathcal{M}_{p^m+n}(\mathbb{R})\) respectively.

**Proof.** One uses induction to prove that \(E^{\oplus m}, F^{\oplus n}\) and \(EF^{\oplus mn}\) are idempotents of \(\mathcal{M}_{p^m}(\mathbb{R}), \mathcal{M}_{p^n}(\mathbb{R})\) and of \(\mathcal{M}_{p^m+n}(\mathbb{R})\).

1. \(\forall m \in \mathbb{N}, \ E^{\ominus m}\) is an idempotent of \(\mathcal{M}_{p^n}(\mathbb{R})\).
   
   i) Let \(k = 2\) and \(E = [e_{ij}]\). Partitioning according to the size of \(E\), \(E^{\ominus 2} = [e_{ij}E]\). The \(i, j\) block of \(E^{\ominus 2}E^{\ominus 2}\) is
   \[
   \sum_{k=1}^{p} e_{ik} e_{kj}E = \sum_{k=1}^{p} e_{ik} e_{kj}E = e_{ij}E.
   \]
   But \(e_{ij}E\) is the \(ij\) block of \(E^{\ominus 2}\). Then one may conclude that
   \[
   E^{\ominus 2}E^{\ominus 2} = E^{\ominus 2}.
   \]

   ii) Let now suppose that \(E^{\ominus (k-1)}E^{\ominus (k-1)} = E^{\ominus (k-1)}\). Partitioning according to the size of \(E^{\ominus (k-1)}\), \(E^{\ominus k} = [e_{ij}E^{\ominus (k-1)}]\). The \(ij\) block of \(E^{\ominus k}E^{\ominus k}\) is
   \[
   \sum_{l=1}^{p} e_{il} E^{\ominus (k-1)} e_{lj}E^{\ominus (k-1)}
   = \sum_{l=1}^{p} e_{il} e_{lj}E^{\ominus (k-1)} E^{\ominus (k-1)}
   = e_{ij} E^{\ominus (k-1)} E^{\ominus (k-1)}
   = e_{ij} E^{\ominus (k-1)}.
   \]
   But \(e_{ij} E^{\ominus (k-1)}\) is the \(ij\) block of \(E^{\ominus k}\), therefore \(E^{\ominus k}E^{\ominus k} = E^{\ominus k}\).

   One proves that \(\forall n \in \mathbb{N}, \ F^{\ominus n}\) is an idempotent matrix of \(\mathcal{M}_{p^n}(\mathbb{R})\) in the same way.

2. \(\forall m, n \in \mathbb{N}, \ EF^{\oplus mn}\) is an idempotent matrix of \(\mathcal{M}_{p^{m+n}}(\mathbb{R})\).

   Since \(EF^{\oplus mn} = E^{\ominus (m-1)} \oplus (EF^{\ominus 1n})\) one will proceed in the following way: first one proves that \(\forall n \in \mathbb{N}, \ EF^{\ominus 1n}\) is an idempotent matrix of \(\mathcal{M}_{p^{n+1}}(\mathbb{R})\) and next one proves that \(\forall m \in \mathbb{N}, \ EF^{\ominus mn}\) is an idempotent matrix of \(\mathcal{M}_{p^{m+n}}(\mathbb{R})\).

   i) \(\forall n \in \mathbb{N}, \ EF^{\ominus 1n}\) is an idempotent matrix of \(\mathcal{M}_{p^{n+1}}(\mathbb{R})\). First one proves that
   \[
   EF^{\ominus 11} EF^{\ominus 11} = EF^{\ominus 11}.
   \]
Partitioning according to the size of $F$, $E \oplus F = [e_{ij}F]$. The $ij$ block of $EF^\oplus 11EF^\oplus 11$ is
\[
\sum_{k=1}^p e_{ik}F e_{kj}F
= \sum_{k=1}^p e_{ik}e_{kj}FF
= \sum_{k=1}^p e_{ik}e_{kj}F
= e_{ij}F.
\]
But since the $ij$ block of $EF^\oplus 11$ is $e_{ij}F$ then $EF^\oplus 11 \oplus EF^\oplus 11 = EF^\oplus 11$. Now let suppose that
\[
EF^\oplus (1(k-1)) EF^\oplus (1(k-1)) = EF^\oplus 1(k-1)
\]
and let prove that $EF^\oplus 1k \oplus EF^\oplus 1k = EF^\oplus 1k$. Partitioning according to the size of $F$, $EF^\oplus 1k = [(EF^\oplus 1(k-1))]_{ij}F$. Since
\[
EF^\oplus 1k = (EF^\oplus 1(k-1)) \oplus F
\]
the $ij$ block of
\[
EF^\oplus 1k EF^\oplus 1k
\]
is
\[
\sum_{l=1}^p (EF^\oplus (1(k-1)))_{il}F(EF^\oplus (1(k-1)))_{lj}F
= \sum_{l=1}^p (EF^\oplus (1(k-1)))_{il}(EF^\oplus (1(k-1)))_{lj}FF.
\]
But since $(EF^\oplus (1(k-1)))_{lj}F$ is the $ij$ block of $(EF^\oplus 1(k-1)) \oplus F$, this is of $EF^\oplus 1k$ then $EF^\oplus 1k EF^\oplus 1k = EF^\oplus 1k$.

ii) $\forall m \in \mathbb{N}$, $EF^m n$ is an idempotent matrix of $\mathcal{M}_{pm + n}(\mathbb{R})$. One has already proved that $EF^m n \oplus EF^m n = EF^m n$. Let now suppose that
\[
EF^\oplus (k-1)n EF^\oplus (k-1)n = EF^\oplus (k-1)n.
\]
and prove that $EF^\oplus kn EF^\oplus kn = EF^\oplus kn$. Since $EF^kn = E \oplus EF^((k-1)n)$ and partitioning according to the size of $EF^\oplus (k-1)n$, $EF^\oplus kn = [e_{ij}EF^\oplus (k-1)n]$ then the $ij$ block of $EF^\oplus kn$ is
\[
\sum_{l=1}^p e_{il}EF^\oplus (k-1)n e_{lj}EF^\oplus (k-1)n
= \sum_{l=1}^p e_{il}e_{lj}EF^\oplus (k-1)n
= e_{ij}EF^\oplus (k-1)n.
\]
Since $e_{ij}EF^\oplus (k-1)n$ is the $ij$ block of $EF^\oplus kn$ then $EF^\oplus kn EF^\oplus kn = EF^\oplus kn$.

Lemma 6.1 Let $E$ and $F$ be idempotents matrices of $\mathcal{M}_p(\mathbb{R})$. Then $E^m n$, $F^m n$, $EF^mn$ are principal submatrices of $E^m n$, $F^m n$, $EF^mn$ respectively.

7 Necessary conditions for the existence of a strongly regular graph

Let $k,l$ be natural numbers. Since
\[
\forall i = 1, \ldots , 3, \forall u, v, j \in \{1, \ldots , 3\} : u < v, E^{oj}k, E^{uv}kl, E^{(uv)k}, E^{j(uv)kl}
\]
are elements of the Euclidean Jordan algebra $\mathcal{V}_n$ then $\forall j = 1, \cdots, 3$ and $\forall u, v \in \{1, \cdots, 3\}$ such that $u < v$ there exists real numbers $q_{ijkl}^j$, $q_{uvkl}^i$, $q_{(uv)kl}^j$ and $q_{j(\cdot \cdot \cdot)}$ for $i = 1, \cdots, 3$ such that

$$E^0_{\cdot \cdot \cdot} = \sum_{i=1}^3 q_{ijkl}^i E_i,$$

$$E^0_{uvkl} = \sum_{i=1}^3 q_{uvkl}^i E_i,$$

$$E^0_{(uv)k} = \sum_{i=1}^3 q_{(uv)k}^i E_i,$$

$$E^0_{j(\cdot \cdot \cdot)kl} = \sum_{i=1}^3 q_{j(\cdot \cdot \cdot)kl}^i E_i.$$ 

By theorems 6.1, 6.2 and lemma 6.1 one may conclude that, for $i = 1, \cdots, 3$ and $k, l \in \mathbb{N}$, one has

$$0 \leq q_{ij}^j \leq 1, \forall j = 1, \cdots, 3,$$

$$0 \leq q_{uv}^i \leq 1, \forall u, v \in \{1, \cdots, 3\} : u < v,$$

$$0 \leq q_{(uv)k}^j \leq 1, \forall u, v \in \{1, \cdots, 3\} : u < v,$$

$$0 \leq q_{j(\cdot \cdot \cdot)kl}^i \leq 1, \forall j = 1, \cdots, 3, \forall u, v \in \{1, \cdots, 3\} : u < v.$$ 

After some calculations and working in the Euclidean Jordan algebra $\mathcal{V}_n$ with the componentwise product of matrices one obtains:

$$q_{11k}^1 = \frac{1}{n^k} + \frac{1}{n^k} p + \frac{1}{n^k}(n-p-1),$$

$$q_{11k}^2 = \frac{1}{n^k} + \frac{1}{n^k} r + \frac{1}{n^k}(-r-1),$$

$$q_{11k}^3 = \frac{1}{n^k} + \frac{1}{n^k} s + \frac{1}{n^k}(-s-1),$$

$$q_{22k}^1 = \frac{(s|n + s - p|}{n(r - s)} p + \frac{(s|n + s - p|}{n(r - s)}(n-p-1),$$

$$q_{22k}^2 = \frac{(s|n + s - p|}{n(r - s)} r + \frac{(s|n + s - p|}{n(r - s)}(-r-1),$$

$$q_{22k}^3 = \frac{(s|n + s - p|}{n(r - s)} s + \frac{(s|n + s - p|}{n(s-1)},$$

$$q_{33k}^1 = \frac{(s|n + p - r|}{n(r - s)} p + \frac{(s|n + p - r|}{n(r - s)}(n-p-1),$$

$$q_{33k}^2 = \frac{(s|n + p - r|}{n(r - s)} r + \frac{(s|n + p - r|}{n(r - s)}(-r-1),$$

$$q_{33k}^3 = \frac{(s|n + p - r|}{n(r - s)} s + \frac{(s|n + p - r|}{n(s-1),$$
\[ q_{12kl}^1 = \frac{1}{n^k} \left[ \frac{|s|n + s - p}{n(r - s)} \right]^l + \frac{1}{n^k} \left( \frac{n + s - p}{n(r - s)} \right)^l p + \frac{1}{n^k} \left( \frac{s - p}{n(r - s)} \right)^l (n - p - 1), \]
\[ q_{12kl}^2 = \frac{1}{n^k} \left[ \frac{|s|n + s - p}{n(r - s)} \right]^l + \frac{1}{n^k} \left( \frac{n + s - p}{n(r - s)} \right)^l r + \frac{1}{n^k} \left( \frac{s - p}{n(r - s)} \right)^l (-r - 1), \]
\[ q_{12kl}^3 = \frac{1}{n^k} \left[ \frac{|s|n + s - p}{n(r - s)} \right]^l + \frac{1}{n^k} \left( \frac{n + s - p}{n(r - s)} \right)^l s + \frac{1}{n^k} \left( \frac{s - p}{n(r - s)} \right)^l (-s - 1), \]
\[ q_{13kl}^1 = \frac{1}{n^k} \left[ \frac{rn + p - r}{n(r - s)} \right]^l + \frac{1}{n^k} \left( \frac{n + p - r}{n(r - s)} \right)^l p + \frac{1}{n^k} \left( \frac{p - r}{n(r - s)} \right)^l (n - p - 1), \]
\[ q_{13kl}^2 = \frac{1}{n^k} \left[ \frac{rn + p - r}{n(r - s)} \right]^l + \frac{1}{n^k} \left( \frac{n + p - r}{n(r - s)} \right)^l r + \frac{1}{n^k} \left( \frac{p - r}{n(r - s)} \right)^l (-r - 1), \]
\[ q_{13kl}^3 = \frac{1}{n^k} \left[ \frac{rn + p - r}{n(r - s)} \right]^l + \frac{1}{n^k} \left( \frac{n + p - r}{n(r - s)} \right)^l s + \frac{1}{n^k} \left( \frac{p - r}{n(r - s)} \right)^l (-s - 1), \]
and
\[ q_{23kl}^1 = \frac{(|s|n + s - p)^k (rn + p - r)^l}{(n(r - s))^{k+l}} + \frac{(n + s - p)^k (-n + p - r)^l}{(n(r - s))^{k+l}} p + \frac{(s - p)^k (p - r)^l}{(n(r - s))^{k+l}} (n - p - 1), \]
\[ q_{23kl}^2 = \frac{(|s|n + s - p)^k (rn + p - r)^l}{(n(r - s))^{k+l}} + \frac{(n + s - p)^k (-n + p - r)^l}{(n(r - s))^{k+l}} r + \frac{(s - p)^k (p - r)^l}{(n(r - s))^{k+l}} (-r - 1), \]
\[ q_{23kl}^3 = \frac{(|s|n + s - p)^k (rn + p - r)^l}{(n(r - s))^{k+l}} + \frac{(n + s - p)^k (-n + p - r)^l}{(n(r - s))^{k+l}} s + \frac{(s - p)^k (p - r)^l}{(n(r - s))^{k+l}} (-s - 1), \]
\[ q_{(+12)k}^1 = \left( |s|n + r - p \right)^k + \left( \frac{n + r - p}{n(r - s)} \right)^k p + \left( \frac{r - p}{n(r - s)} \right)^k (n - p - 1), \]
\[ q_{(+12)k}^2 = \left( \frac{|s|n + r - p}{n(r - s)} \right)^k + \left( \frac{n + r - p}{n(r - s)} \right)^k r + \left( \frac{r - p}{n(r - s)} \right)^k (-r - 1), \]
\[ q_{(+12)k}^3 = \left( \frac{|s|n + r - p}{n(r - s)} \right)^k + \left( \frac{n + r - p}{n(r - s)} \right)^k s + \left( \frac{s - p}{n(r - s)} \right)^k (-s - 1), \]
\[ q_{(+13)k}^1 = \left( \frac{r n + p - s}{n(r - s)} \right)^k + \left( \frac{-n + p - s}{n(r - s)} \right)^k p + \left( \frac{p - s}{n(r - s)} \right)^k (n - p - 1), \]
\[ q_{(+13)k}^2 = \left( \frac{r n + p - s}{n(r - s)} \right)^k + \left( \frac{-n + p - s}{n(r - s)} \right)^k r + \left( \frac{p - s}{n(r - s)} \right)^k (-r - 1), \]
\[ q_{(+13)k}^3 = \left( \frac{r n + p - s}{n(r - s)} \right)^k + \left( \frac{-n + p - s}{n(r - s)} \right)^k s + \left( \frac{p - s}{n(r - s)} \right)^k (-s - 1), \]
\[ q_{(+23)k}^1 = \left( \frac{n - 1}{n} \right)^k + (-1)^k \left( \frac{1}{n} \right)^k p + (-1)^k \left( \frac{1}{n} \right)^k (n - p - 1), \]
\[ q_{(+23)k}^2 = \left( \frac{n - 1}{n} \right)^k + (-1)^k \left( \frac{1}{n} \right)^k r + (-1)^k \left( \frac{1}{n} \right)^k (-r - 1), \]
\[ q_{(+23)k}^3 = \left( \frac{n - 1}{n} \right)^k + (-1)^k \left( \frac{1}{n} \right)^k s + (-1)^k \left( \frac{1}{n} \right)^k (-s - 1), \]
\[ q_{3(13)kl}^1 = \left( \frac{r n + p - r}{n(r - s)} \right)^k \left( \frac{r n + p - s}{n(r - s)} \right)^l + \left( \frac{-n + p - r}{n(r - s)} \right)^k \left( \frac{-n + p - s}{n(r - s)} \right)^l \frac{p}{(n(r - s))^{k+l}} + \left( \frac{p - r}{n(r - s)} \right)^k \left( \frac{p - s}{n(r - s)} \right)^l (-n - p - 1), \]
\[ q_{3(13)kl}^2 = \left( \frac{r n + p - r}{n(r - s)} \right)^k \left( \frac{r n + p - s}{n(r - s)} \right)^l + \left( \frac{-n + p - r}{n(r - s)} \right)^k \left( \frac{-n + p - s}{n(r - s)} \right)^l \frac{r}{(n(r - s))^{k+l}} + \left( \frac{p - r}{n(r - s)} \right)^k \left( \frac{p - s}{n(r - s)} \right)^l (-n - r - 1), \]
\[ q_{3(13)kl}^3 = \left( \frac{r n + p - r}{n(r - s)} \right)^k \left( \frac{r n + p - s}{n(r - s)} \right)^l + \left( \frac{-n + p - r}{n(r - s)} \right)^k \left( \frac{-n + p - s}{n(r - s)} \right)^l \frac{s}{(n(r - s))^{k+l}} + \left( \frac{p - r}{n(r - s)} \right)^k \left( \frac{p - s}{n(r - s)} \right)^l (-n - s - 1), \]
\[ q_{2(+13)kl}^1 = \frac{(s|n+s-p)^k(rn+p-s)^l}{(n(r-s))^{k+l}} + \frac{(n+s-p)^k(-n+p-s)^l}{(n(r-s))^{k+l}} p + \frac{(s-p)^k(p-s)^l}{(n(r-s))^{k+l}} (n-p-1), \]
\[ q_{2(+13)kl}^2 = \frac{(s|n+s-p)^k(rn+p-s)^l}{(n(r-s))^{k+l}} + \frac{(n+s-p)^k(-n+p-s)^l}{(n(r-s))^{k+l}} r + \frac{(s-p)^k(p-s)^l}{(n(r-s))^{k+l}} (-r-1), \]
\[ q_{2(+13)kl}^3 = \frac{(s|n+s-p)^k(rn+p-s)^l}{(n(r-s))^{k+l}} + \frac{(n+s-p)^k(-n+p-s)^l}{(n(r-s))^{k+l}} s + \frac{(s-p)^k(p-s)^l}{(n(r-s))^{k+l}} (-s-1). \]

One now makes some observation about this notation.

**Remark 7.1** For \( i = 1, \ldots, 3 \) the parameters \( q_{ij2}^i \), for \( j = 1, \ldots, 3 \), \( q_{uw11}^i \) such that \( u < v \land u, v \in \{1, \ldots, 3\} \) are the parameters of Krein of the strongly \((n, p; a, c)\) regular graph \( \tau \). Next one must observe that \( \forall i = 1, \ldots, 3, \forall j = 1, \ldots, 3, q_{ij1}^i = \delta_{ij} \). From now on, for \( i = 1, \ldots, 3 \) one will call the parameters \( q_{ijk}^i, q_{j(+)uv}^i k \) with \( k \geq 3 \), and the parameters \( q_{uwkl}^i, q_{j(+)uv}^i k \) with \( l + k \geq 3 \) the generalized Krein parameters of the strongly \((n, p; a, c)\) regular graph \( \tau \).

**Remark 7.2** One now presents some consequences of the parameters \( q_{i(+)uv}^i k \) such that \( u < v \land u, v \in \{1, \ldots, 3\} \). Let \( u \) and \( v \) be natural numbers such that \( u < v \land u, v \in \{1, \ldots, 3\} \). Since

\[ (E_u + E_v) \circ (E_u + E_v) = E_u \circ E_u + 2E_u \circ E_v + E_v \circ E_v \]

then for \( i = 1, \ldots, 3 \), \( q_{i(+)uv}^i 2 = q_{iuu}^i + 2q_{iu}^i + q_{iv}^i \). But for \( i = 1, \ldots, 3 \), \( q_{i(+)uv}^i 2 \leq 1 \).

Therefore one may conclude that

\[ \forall i = 1, \ldots, 3, 0 \leq q_{iuu}^i + 2q_{iu}^i + q_{iv}^i \leq 1. \]

**Remark 7.3** Since the generalized Krein parameters of a strongly regular graph \( \tau \) are greater than zero its natural to establish necessary conditions for the existence of a strongly regular graph with these parameters. One will analyze in this work only the generalized Krein parameters of \( \tau \) associated to the eigenvalue \( p \) of \( \tau \). Let \( k \in 2N + 1 \). The parameters \( q_{33k}^1, q_{(+13)k}^1 q_{2(+13)uw}^1 \) with \( v \in 2N + 1 \) and \( q_{3(+13)uw}^1 \) with \( u + v \in 2N + 1 \) permits us to establish easily criteria to see that \( \tau \) doesn't exist a strongly \((n, p; a, c)\) regular graph. Each expression of each generalized parameter interpreted as polynomial in \( n \) give us the information that if \( \tau \) is a strongly \((n, p; a, c)\) regular graph such that coefficient of the power of \( n \) with exponent equal to the degree of this polynomial is negative then one can say that if \( n \) is sufficient large then this parameter is negative. Note that

\[ (n(r-s))^k q_{33k}^1 = (r - p)n^k + \sum_{i=0}^{k-1} \alpha_i n^i (k \in 2N + 1), \]
\[ (n(r-s))^k q_{(+13)k}^1 = (r - p)n^k + \sum_{i=0}^{k-1} \beta_i n^i (k \in 2N + 1), \]
\[ (n(r-s))^u+v q_{2(+13)uw}^1 = (s|u+v - p)n^{u+v} + \sum_{i=0}^{u+v-1} \delta_i n^i (v \in 2N + 1), \]
\[ (n(r-s))^u+v q_{3(+13)uw}^1 = (r^{u+v} - p)n^{u+v} + \sum_{i=0}^{u+v-1} \gamma_i n^i (u + v \in 2N + 1). \]
Theorem 7.1 Let $\tau$ be a strongly $(n,p;a,c)$ regular graph such that $0 < c < p < n - 1$ with adjacency matrix $A$, having eigenvalues $p,r$ and $s$. Then

\[
(n + p - r)^k + (n + p - s)^k + (n + r - s)^k p + (n + p - s)^k (n - p - 1) \geq 0
\]

\[
\forall k \in 2 \mathbb{N} + 1 \quad (q_{33k}^3 \geq 0),
\]

\[
(n + p - s)^k + (n - p - s)^k p + (n - p - s)^k (n - p - 1) \geq 0
\]

\[
\forall k \in 2 \mathbb{N} + 1 \quad (q_{13k}^1 \geq 0),
\]

\[
(n + p - r)^k + (n + p - s)^k + (n + r - s)^k p + (n + p - s)^k (n - p - 1) \geq 0
\]

\[
\forall k, l \in \mathbb{N}, l + k \in 2\mathbb{N} + 1 \quad (q_{3k(13)kl}^1 \geq 0),
\]

\[
(n + s + p)^k + (n + s - p)^k + (n + s - p)^k p + (n + s - p)^k (n - p - 1) \geq 0
\]

\[
\forall k \in \mathbb{N}, l \in 2\mathbb{N} + 1(q_{2k(13)kl}^1 \geq 0).
\]

One presents the lemma 7.1 since the left member of each inequality\footnote{For instance, from the first inequality of lemma 7.1 one may establish the corollary 7.1} of this lemma is a polynomial in $n$ of degree three is more suitable for getting information, like that presented in corollary 7.1 about a strongly regular graph.

Lemma 7.1 Let $\tau$ be a strongly $(n,p;a,c)$ regular graph such that $0 < c < p < n - 1$ with adjacency matrix $A$, having eigenvalues $p,r$ and $s$. Then

\[
(n + p - r)^3 \geq 0
\]

\[
\quad (q_{333}^3 \geq 0),
\]

\[
(n + p - s)^3 + (n - p - s)^3 + (n - p - s)^3 \geq 0
\]

\[
\quad (q_{133}^1 \geq 0),
\]

\[
(n + p - r)^2 + (n - p - s)^2 + (n + p - r)^2 + (n + p - s)^2 \geq 0
\]

\[
\quad (q_{312}^3 \geq 0),
\]

\[
(n + p - s)^2 + (n - p - s)^2 + (n + p - s)^2 \geq 0
\]

\[
\quad (q_{131}^1 \geq 0),
\]

\[
(n + p - r) \cdot (n + p - s)^2 + (n + p - s)^2 \geq 0
\]

\[
\quad (q_{321}^3 \geq 0),
\]

\[
(n + s + p)^2 + (n + s - p)^2 + (n + s - p)^2 \geq 0
\]

\[
\quad (q_{221}^2 \geq 0).
\]

From lemma 7.1 and (13) one obtains the corollary 7.1.
Corollary 7.1 If $\tau$ is a strongly $(n, p; a, c)$ regular graph and $r$ is the positive eigenvalue of its matrix of adjacency then:

1

$$r < p^{\frac{1}{3}}$$

$$\Downarrow$$

$$n < \frac{(p - r)(3r^2 + 3p + \sqrt{r^4 + 18pr^2 + p^2 + 8r^3p + 8pr})}{2(p - r^3)}$$

2

$$r > p^{\frac{1}{3}}$$

$$\Downarrow$$

$$n > \frac{(p - r)(3r^2 + 3p + (p - r)\sqrt{r^4 + 18pr^2 + p^2 + 8r^3p + 8pr})}{2(p - r^3)}$$

Conclusion 7.1 Each necessary condition for the existence of a strongly $(n, p; a, c)$ regular graph of theorem 7.1 allows us to conclude that $n$ can not be too big when the coefficient of the power of $n$ with greatest exponent is negative. One presents the lemma 7.1, since the left member of each inequality of this lemma is a polynomial in $n$ of degree three and therefore is more suitable for getting information, like that presented in corollary 7.1, about a strongly regular graph. Finally one observes that the introduction of the generalized Krein parameters of a strongly $(n, p; a, c)$ regular graph allowed us to conclude that the Krein parameters for $i, u, v \in \{1, \cdots, 3\}$ : $u < v$ must satisfy not only $0 \leq q_{uv}^i \leq 1 \land 0 \leq q_{vv}^i \leq 1$ but also $q_{uv}^i + 2q_{uv}^i + q_{vv}^i \leq 1.$

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3For instance, interpreting [13] as a polynomial in $n$ one has

$$(rn + p - r)^k + (-n + p - r)^k p + (p - r)^k (n - p - 1) = (r^k - p)n^k + \sum_{i=0}^{k-1} \alpha_i n^i$$

and so is natural to conclude that, when $k$ is an odd natural number, if $r^k - p < 0$ and $n$ is big then $(rn + p - r)^k + (-n + p - r)^k p + (p - r)^k (n - p - 1) < 0.$

4For instance, from the first inequality of lemma 7.1 one may establish the corollary 7.1.
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