Bayesian Approach of Model Updating of Structural Parameters with Simulated Modal Data

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Abstract. The present study investigates the effectiveness of Bayesian approach of model updating of structure, using simulated modal data sets as evidence. The importance of the prediction of error implemented in the proposed Bayesian framework to assess its capability in sampling posterior values with improved accuracy has been highlighted. The natural frequencies and mode shapes of the structure are taken only as the evidence. The study highlighted the importance of the likelihood functions corresponding to the prediction of error between the experimental and predicted data. The study also implemented the likelihood functions corresponding to the prediction of error variances of natural frequencies only in sampling posterior values with higher accuracy. The efficacy of the proposed approach in dealing with the issue of incomplete data sets, that makes the problem typically ill-conditioned, is duly addressed. The present work also implemented model reduction technique to synthesise the mode shape ordinates at unmeasured degrees of freedom (DOF) from the reduced-order model. In addition, the efficiency of the proposed model reduction algorithm is explored by adding noises of varying percentages to the measured mode shape values. The proposed methodology is illustrated numerically to update the stiffness parameters of an eight-storey shear building model, considering simulated datasets contaminated by Gaussian error as evidence. Results with both incomplete and complete data sets using the proposed Markov Chain Monte Carlo approach are studied. The accuracy on the synthesis of mode shapes from unmeasured DOF using model reduction algorithm is also studied.

1. Introduction
Bayesian inference is widely used to reduce both computational burdens and increased experimental investigations for updating model parameters in a probabilistic framework [1-4]. The Bayesian inference of model parameter updating using data sensitivity [5] and employing the best achievable eigenvectors [6] is found to be appealing, provided the damage locations of the structure are known. The applications of techniques such as transitional Markov Chain Monte Carlo (MCMC) method and Gibbs Sampling, derived from the basic MCMC technique, are also notable in this regard [10, 13-14]. The Gibbs sampling approach becomes computationally expensive and inefficient, especially when the parameter values depend on each other. The inverse problems in system identification require the solution of a family of plausible values of model parameters rather than the search for a single optimal parameter vector [10-14]. The MCMC simulation approach using Metropolis Hastings (MH) algorithm enables the propagation through the parameter space and converges to the globally optimum region [7-9]. In this regard, the re-analyses of full models can be suppressed by introducing model-reduction techniques such as Guyan’s reduction, substructure coupling and component mode synthesis [11-12]. The model reduction between the measured and the synthesised modal quantities such as mode shapes is fruitful in comparison to the direct mode matching for uncertainty quantification of model parameters [15].
The present study investigates the efficacy of Bayesian approach of updating model parameters in determining their posterior distribution using MCMC technique, based on the estimates of likelihood functions of the evidences conditioned on the concerned parameters. The prediction of the likelihood function of the evidences is implemented through Gaussian pdfs, to assess its effectiveness in determining posterior values by the proposed Bayesian framework. The model reduction technique using Guyan’s reduction is used to determine the unknown responses in terms of the known responses, i.e. at the measured DOFs. This study is performed considering the number of modes equivalent to the number of measured DOFs. The algorithm for model reduction is tested for incomplete measurements by considering two different sensor placement combinations. The sensor placement is considered by reducing the number of measured DOFs. The accuracy of determination of unknown mode values using model reduction algorithm is also studied. The proposed techniques are illustrated numerically by considering an eight-storey shear building model to study the accuracy of estimating model parameters and associated uncertainties considering both complete and incomplete modal data sets.

2. Bayesian approach of Model Updating

In the Bayesian approach, subjective judgements based on instinct, experience or incidental information is systematically incorporated with the measured data to obtain a balanced estimate of model parameters. The primary rule that governs Bayesian model updating is Bayes theorem, which states that given the measured data, the posterior pdf of the model depends on the likelihood function, the prior distribution and the evidence [7]. This can be mathematically expressed as,

$$P(\theta | D) = \frac{P(D | \theta) P(\theta)}{P(D)}$$

(1)

where, $\theta$ is the vector of updating parameters, $D$ is the available data, $P(D | \theta)$ is the likelihood that represents the possibility of the data $D$ when a trust of $\theta$ is taken as true, $P(\theta)$ is the prior pdf that contains the knowledge about $\theta$ before observing the data, $P(\theta | D)$ is the posterior pdf and $P(D)$ is the normalization factor (the evidence given by the data). The related formulation of likelihood function to obtain the posterior statistics of the model parameter are presented in the following.

2.1. Likelihood function

The difference between the measured responses and the predicted outputs of evidences from a model is the prediction error, which can be defined as,

$$e(\theta) = Y - X(\theta)$$

(2)

where, $e(\theta)$ is the prediction error, $Y$ is the measured response and $X(\theta)$ is the predicted output of the evidences from the model. The likelihood function of evidences for the outcome ($Y$) can be expressed as:

$$p(Y | \theta, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{[e(\theta)]^2}{2\sigma^2}\right\} = \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{[Y - X(\theta)]^2}{2\sigma^2}\right\}$$

(3)

The evidences ($D$) in the present study are the modal data of a structure comprising modal frequencies. In case of incomplete modal data, i.e. the number of measured degrees of freedom of mode shape, $n_i$ is less than the actual number of degrees of freedom ($n$). Assuming that the frequency of an energy mode is statistically independent to each other, the likelihood of $D$ is expressed as:

$$p(D | \theta) = \prod_{i=1}^{m} p(\omega_i | \theta, \sigma_{\omega_i}^2)$$

(4)

The unknown parameters in the model updating problem considering natural frequencies as evidence are increased to $[d + m]$ in which $d$ is the extent of the vector $\theta$ and $m$ is the number of modes considered.
2.2. Prior Distribution
In Bayesian deduction, the prior distributions of parameters describing a model are necessary. The prior pdf of the unknown parameters $\theta$, i.e. $P(\theta)$ is assumed to be exponential, and the different components of $\theta$ is assumed to be independent. The likelihood functions of the error variances are assumed as inverse gamma ($IG$) for natural frequencies.

2.3. Posterior Distribution
The posterior distribution of the updating parameters can be now obtained by Bayes theorem using equation (1). It may be noted that apart from model parameters ($\theta$), the updating parameters vector involve the variances of the frequency. Thus, the posterior values of the parameters in equation (1) is redefined as, $\bar{\theta} = \left\{ \theta^T, \{\sigma^2_{\theta_j}\}_{j=1}^m \right\}^T$. In the present study, the MCMC technique is implemented to obtain samples from the posterior distribution.

3. MCMC based Model Parameter Updating
The MCMC allows efficient generation of random samples as a series of a “Markov Chain” according to an arbitrarily given probability distribution. The MH based MCMC approach is used. The proposal distribution, $f$ governs the distribution of the candidate sample and affects the passage of the Markov chain from one state to another. In the present study, Gaussian distribution is used as the proposal distribution to draw samples from the high dimensional posterior distribution. The acceptance criteria ratio of MH algorithm in the following manner,

$$r = \frac{p(\omega | \theta, \sigma^2_{\omega}) p(\sigma^2_{\omega} | \bar{\theta}) p(\bar{\theta})}{p(\omega | \bar{\theta}, \sigma^2_{\omega}) p(\sigma^2_{\omega} | \theta) p(\theta)}$$

(5)

The posterior parameter value $\bar{\theta}$ is then accepted either with probability $\alpha_\alpha$ or rejected with a probability of $1 - \alpha_\alpha$. The initial acceptance ratio is kept low to avoid any bias of updating the parameter value right from the start of the algorithm.

4. Model Reduction
The model reduction technique evades the full model run for each iteration using dynamic response data. In the present study, the model reduction is implemented by partitioning the mass $M$ and stiffness $K(\theta)$ matrices into matrices based on the master and slave DOFs. The mode shapes corresponding to slave nodes $s_i$ is represented in terms of master nodes for first $m$ modes with $m$ being equal to the number of master nodes as:

$$\Phi_{mn} = t\Phi_{mn}$$

(6)

where, $t \in R^{mn}$ is a transformation parameter. To determine the unknown parameter $t$ in the above, the iterative procedure suggested by Friswell et al. (1995) is applied. The choice of master DOFs depends on the underlying assumption in the Guyan reduction that at the slave coordinates, the inertia forces are insignificant compared with the elastic forces [16]. The present study explores the choice of master DOFs using a different combination of measured DOFs after solving a dynamic eigenvalue problem.

5. Numerical Study
An eight-storied shear building model, as shown in figure 1, is shown to elucidate the proposed model updating technique using synthetic modal response data. The building has uniform nominal stiffness, $k_i$ and mass $m_i$ associated with each story. An updating parameter $\theta_i$ (with $i = 1, 2, ..., 8$) is imposed on each of these stiffness parameters. The multiplication of these two factors represents the actual stiffness of the $i$ th storey i.e. $K_i = k_i \times \theta_i$. The unknown parameter $\theta_i$ are obtained under Bayesian inference. The frequencies of the initial three modes are considered as the mean data $(\omega_m, \Phi_m)$. The modal responses are artificially simulated from a Gaussian distribution with the estimated modal data as mean and 5 %
coefficient of variation i.e. $\omega_j \sim N(\omega_n, 0.05\omega_n)$, where, $j$ represents $j$th sample of measurement. The total number of modal data sets generated in this manner are 15, 30 and 45. The spread of the random walk is adopted as 5% of the current state of the parameter.

For the first case, the modal data were considered to be available for all the DOFs, as shown in figure 1(a). The posterior samples were derived by MH algorithm, and 10000 samples are simulated as the seeds of the Markov Chain. The first 500 samples are discarded as burn-in samples as the initial fluctuations of the parametric values were high. The acceptance ratio at the start is kept low to avoid any bias of updating the parameter value. The posterior distributions of parameters $\theta_1$ and $\theta_2$ are shown in figure 2 in the form of a histogram. The same is observed for the other parameters also. It has been noticed from the histogram plots that the maximum occurrence of the parameters is almost around the actual value of the same and the maximum spread of the parameter value is 8.24%.

For the second case, the modal data were considered to be available for the first case, six measurements of modal data are considered to be available, as shown in figure 1(b). In the next case, only four measurements of modal data are considered to be known, as shown in figure 1(c). The mean and standard deviation of the posterior distribution of parameters are shown in table 1 with the percentage of error in bracket considering 45 sets of simulated incomplete data. It can be noted that the mean values of the posterior distribution of all the parameters are estimated with reasonable accuracy. The maximum percentage of error for posterior values of parameters is 7.82% and 12.11% for six measurements and four measurements respectively.
Table 1. Comparison of actual and predicted parameters.

| Parameters | Actual values | Posterior value of mean and standard deviation of parameters for 45 data sets with percentage of error [shown in bracket] |
|------------|---------------|-----------------------------------------------------------------------------------------------------------------|
|            | Complete measurement | 6 measurement | 4 measurement |
| $\theta_1$ | 0.5 | 0.474, 0.05 [5.10] | 0.535, 0.09 [6.94] | 0.542, 0.11 [8.42] |
| $\theta_2$ | 0.7 | 0.683, 0.08 [2.37] | 0.661, 0.11 [5.54] | 0.644, 0.12 [7.96] |
| $\theta_3$ | 0.2 | 0.194, 0.04 [2.85] | 0.193, 0.07 [3.35] | 0.221, 0.05 [10.50] |
| $\theta_4$ | 0.9 | 0.843, 0.08 [6.29] | 0.843, 0.09 [6.30] | 0.794, 0.11 [11.72] |
| $\theta_5$ | 0.6 | 0.588, 0.06 [2.02] | 0.561, 0.08 [6.47] | 0.626, 0.11 [4.30] |
| $\theta_6$ | 1.0 | 0.945, 0.08 [5.46] | 0.922, 0.11 [7.82] | 0.879, 0.11 [12.11] |
| $\theta_7$ | 0.8 | 0.764, 0.09 [4.46] | 0.784, 0.09 [1.98] | 0.831, 0.13 [3.85] |
| $\theta_8$ | 0.4 | 0.389, 0.06 [2.58] | 0.386, 0.07 [3.58] | 0.379, 0.12 [5.28] |

Now the mode shapes are measured for two cases of incomplete measurements at the measured DOFs as manifested in figure 1(b) and figure 1(c). The measured DOFs are taken as master nodes. The iterative model reduction algorithm is implemented to synthesise the mode shape ordinates at the unmeasured or slave DOFs. The efficiency of the model reduction approach is tested by adding noises of varying percentages to the measured mode shape values. For brevity, the mode shape ordinates at the unmeasured DOFs having a maximum percentage of errors are shown in table 2. It can be observed that the modal ordinates at the slave DOFs are estimated quite well using model reduction technique even after applying varied percentages of noise to the modes at the measured DOFs.

Table 2. Comparison of synthesised modes based on model reduction.

| Mode no. | Maximum Percentage of errors of synthesised unknown modes based on model reduction technique |
|----------|------------------------------------------------------------------------------------------|
|          | 6 measurement | 4 measurement |
| No       | 2% | 4% | 6% | 8% | No | 2% | 4% | 6% | 8% |
| Noisy data |     |    |    |    | Noisy data     |    |    |    |    |
| 1        | 0.0001 | 1.5591 | 2.3706 | 3.2613 | 4.5699 | 0.0003 | 2.0746 | 3.9691 | 4.7572 | 5.1347 |
| 2        | 0.0002 | 1.7479 | 1.9867 | 3.9161 | 5.2298 | 0.0001 | 2.1478 | 2.3285 | 4.7768 | 5.0859 |
| 3        | 0.0001 | 1.6918 | 2.0692 | 2.6586 | 3.7282 | 0.0008 | 2.7277 | 3.6517 | 5.2321 | 5.8218 |
| 4        | 0.0008 | 1.0291 | 1.6476 | 2.6935 | 3.5953 | 0.0004 | 2.0541 | 2.9913 | 5.7953 | 6.0451 |
| 5        | 0.0004 | 1.1435 | 2.0115 | 3.7576 | 4.3811 | -- | -- | -- | -- | -- |
| 6        | 0.0007 | 1.6619 | 2.2014 | 3.5871 | 3.5951 | -- | -- | -- | -- | -- |

6. Summary and Conclusions
An effective MCMC technique using MH algorithm based on likelihood estimates of evidences and proper choice of prediction error variance from Bayesian inference is presented to update structural model parameters considering both complete and incomplete data sets. The proposal distribution of the sample for performing a random walk is guided based on physical possibilities of damage to a structure. The algorithm is elucidated to update the stiffness parameters of an eight-storey shear building model with simulated modal data as evidence. As expected, the accuracy for completely known modal data and a higher number of data sets are very high. However, the results of incomplete data sets also estimate the parameters quite well, and the maximum percentage of error of posterior value of parameters is within 12%. The model reduction technique determines mode shape ordinates at the unmeasured DOFs with high accuracy in comparison to the percentage of noise applied to the measured data. Thus, the model reduction approach appears to be a robust choice. It has been generally noted that the MCMC method is quite effective without the additional requirement of any transitional algorithms and sensitivity analysis of data points. However, the degree of accuracy achievable and the iterations required to obtain a converged posterior distribution of the model parameters depend on the type of problems and the interdependency of parameters, which needs to be explored further.
7. References

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