Optical stabilization of voltage fluctuations in half-Josephson lasers

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Abstract – A recently proposed device, dubbed half-Josephson laser, provides a phase-lock between the optical phase and the superconducting phase difference between the leads of the device. In this paper we propose to utilize this phase-lock for the stabilization of voltage fluctuations, by two optical feedback schemes. The first scheme involves a single half-Josephson laser and allows to significantly decrease the diffusion coefficient of the superconducting phase difference. The second scheme involves a stable optical source and a fluctuating half-Josephson laser and permits quenching of the diffusion of the relative phase of the lasers. This opens up perspectives of the optical control of the superconducting phase and voltage fluctuations.

In the past years technological developments have led to the realization of novel hybrid devices that combine superconductors and semiconductors [1]. These hybrid devices form a basis to study materials [2], realize new functionalities, like in a supercurrent transistor [3], and investigate exotic states of matter, like topological superconductivity [4]. As a part of this development, superconducting optoelectronic devices have been realized and proposed, where the interaction between light and superconductor-semiconductor structures is important. Enhanced emission of light-emitting diodes (LED) has been demonstrated [5]. Interesting theoretical proposals include the Josephson LED [6], the half-Josephson laser [7,8] (HJL) and devices useful for quantum information purposes [9].

In this letter, we explore some intriguing applications which exploit the most important property of the HJL: a phase-lock between optical phase and superconducting phase difference. The HJL consists of a biased Josephson junction containing a structure capable of emitting light by electron-hole recombination. The light is emitted in an optical resonator. Importantly, the eigenstates associated with the light emission also couples to the superconducting leads. As a result of this, the optical phase is proven to be locked to the superconducting phase difference. In ref. [7], a HJL model based on a single quantum emitter was studied, whereas a model for many quantum emitters was considered in ref. [8]. In the latter, the coherence time of the optical phase was found to be exponentially long. Decoherence is caused only by switchings between stable states of radiation, corresponding to two locked values of the optical phase. The phase-lock thus quenches the phase diffusion which is inevitable in usual lasers.

In previous studies, the superconducting phase difference was assumed to be fixed. For any physical realization of the HJL, however, the superconducting phase difference is expected to diffuse. The diffusion coefficient is proportional to the zero frequency noise of the bias voltage. As a consequence of the phase-lock, also the optical phase will be subjected to this diffusion, thus limiting the coherence time of the Josephson laser. In this letter, we describe two optical feedback schemes, that exploit the phase-lock in the HJL, to stabilize the fluctuations of the bias voltage.

The first scheme involves a single HJL and significantly decreases the diffusion coefficient of the superconducting phase difference. The second scheme involves the optical locking of a fluctuating HJL to a stable optical source. This can be exploited to control the superconducting phase of the HJL and create voltage pulses, by changing the optical phase in time. The optical phase is changed by changing the optical path lengths of the laser beams.

The half-Josephson laser. – We start with a brief overview of the dynamics of the HJL, as it is described in [8] in more detail, and we will specify the feedback scheme later.

The HJL is driven by a dipole moment that oscillates with half the Josephson frequency, \( \omega_J/2 \), corresponding to the average bias voltage, and is composed of individual...
contributions of a large number of quantum emitters. As in the case of any laser, the dipole moment saturates with increasing field strength in the resonant mode. The dipole moment originates from the coupling of the quantum emitters to the superconducting leads. The dipole moment fluctuates due to quantum noise in the optical field, as in usual lasers [10], and also due to spontaneous switchings of the quantum emitters between their eigenstates. The latter fluctuations can be seen as a renormalization of the quantum noise.

In [8], we derive the semiclassical equations of motion for the optical field in the HJL, which do not depend on microscopic details, and provide the basis of the phenomenological description of the HJL. Under assumption of weak coupling of the quantum emitters to the superconductivity and the optical field, the equations of motion are given by

\[
\frac{d}{dt}[\tilde{\xi}]=\frac{\Gamma}{2}\tilde{b} - A\tilde{b}\sin(2\Phi_\Delta - \Phi_\Delta + \xi_\tilde{b}(t)), \quad \Phi_\Delta = -\omega - A\cos(2\Phi_\Delta - \Phi_\Delta) - \Omega''|b|^2 + \xi_\phi(t), \quad \Phi_\Delta = \frac{2e}{h}\tilde{\nu}(t). \tag{1}
\]

Here, the optical field is represented by \(b = \tilde{b}\), the expectation value of the photon annihilation operator, with phase \(\Phi_\phi\). The phase \(\Phi_\Delta\) is the superconducting phase difference across the Josephson junction, in a rotating frame of reference. Because of the phase-lock only the phase combination \(2\Phi_\phi - \Phi_\Delta\) occurs at the right-hand side of this equation. In contrast to what is the case now, \(\Phi_\Delta\) was taken constant in ref. [8]. As regards the other coefficients, \(\omega\) is the detuning of the photon frequency with respect to the resonator frequency, \(A\) and \(\Omega''\) are coefficients with a value determined by the dipole moment, \(\tilde{\nu}(t)\) is the time dependent fluctuation of the voltage bias and \(\Gamma\) is the decay rate of the resonator. The quantities \(\xi_\tilde{b}(t), \xi_\phi(t)\) and \(\tilde{\nu}(t)\) are Langevin noise sources, with zero time average and satisfying

\[
\left\langle \xi_\tilde{b}(t)\xi_\tilde{b}(t') \right\rangle = \frac{\Gamma}{4}\delta(t-t') = n_s \left\langle \xi_\phi(t)\xi_\phi(t') \right\rangle, \\
\left\langle \tilde{\nu}(t)\tilde{\nu}(t') \right\rangle = k_B T Z \delta(t-t'), \tag{2}
\]

with \(n_s\) the stationary number of photons in the resonator, \(Z\) the impedance of the junction and \(k_B T\) the thermal energy. The stationary solutions to eq. (1) describe steady-state lasing with a fixed value of the phase combination \(2\Phi_\phi - \Phi_\Delta\). The two phases are indeed locked.

To study noise in the HJL, eq. (1) can be simplified by linearizing it about its stationary value. Surely, fluctuations of the optical field are expected to be small compared to the field itself, in the steady-state operation of the laser. Taking frequency in units of \(\Gamma/2\), the linearized equations in Fourier space are given by

\[
C_1(\phi_\Delta'' - 2\phi_\Delta') - i\nu\phi_\Delta'' = \xi_\tilde{b}, \\
(2 - i\nu)\phi''_\phi - \phi''_\Delta + C_2\phi_\phi = \xi_\phi, \tag{3}
\]

where \(a, \phi_\Delta\) and \(\phi_\Delta\) (and their Fourier transforms) are, respectively, the deviation from \(|b_s|, \Phi_\phi^s\) and \(\Phi_\Delta^s\), the stationary solutions to eq. (1), and \(\nu\) is the dimensionless frequency in a rotating frame of reference. It is taken in units of \(\Gamma/2\). The coefficients are given by \(C_1 = |b_s|(|\omega + \Omega''|b_s|^2|\) and \(C_2 = 2|b_s|^2\Omega''\).

The linearized equations can be further reduced. First, as we are only interested in the time dependence of the phases, we eliminate the term \(C_2\phi_\phi\) in the second line of eq. (3), using the equation in the first line. Second, we concentrate on a small frequency scale, \(\nu \ll 1\), where \(\phi_\phi(t)\) adiabatically follows \(\phi_\Delta(t)\). Hence, eliminating \(\phi_\phi\) from eq. (3) and assuming that the relevant frequencies satisfy \(\nu \ll 1\), we arrive at

\[
C_1C_2[\phi''_\Delta - 2\phi'_\Delta] = i\nu\xi_\tilde{b} + C_2\xi_\phi, \quad -i\nu\phi''_\Delta = \frac{2e}{h}\tilde{\nu}. \tag{4}
\]

The first of these equations describes fluctuations of the combined phase \(\phi_\Delta - 2\phi_\phi\), which is not subjected to drift. These fluctuations manifest only at large frequency scales, or differently stated, at short timescales. Since we concentrate on small frequency scales, corresponding to long timescales, we can assume that the effect of the high-frequency fluctuations on the phases has averaged out to zero. Therefore we assume to have the time averaged value of the phase difference \(\phi_\Delta - 2\phi_\phi\). Consequently, as required, we have \(\phi_\Delta = 2\phi_\phi\). The second of these equations describes diffusion of the phases. Indeed, the variance at low frequencies satisfies \(\langle |\phi''_\Delta|^2 \rangle \sim \langle \nu^2 \rangle / \nu^2\). In the time domain the variances are proportional to time. This is very much like phase diffusion in common lasers.

Feedback. -- Our purpose is to stabilize the fluctuations of the bias voltage of the HJL, by implementing a feedback loop involving the measurement of the optical phase. Since \(\phi_\phi\) and \(\phi_\Delta\) are locked, the measuring of the optical phase will give information about the fluctuation of the superconducting phase. This fluctuation is then corrected by a proper voltage feedback signal.

In our setup we use the well-known Pound-Drever-Hall stabilizer [11], which is a very powerful scheme for stabilization of common lasers [12]. Essentially, in this scheme, the laser beam is reflected of a reference cavity with a high quality factor, which acts as a phase memory element. The reflected beam is then superimposed with the incident beam and a beam that leaks from the reference cavity. Its intensity will therefore depend on the phase difference \(\phi_\phi - \phi_\phi\), with \(\phi_\phi\) the time average of \(\phi_\phi\). The intensity is transferred to a voltage error signal, using a photo detector. The proportionality constant is given by \(h_\gamma/2e\), with frequency \(\gamma\). Subsequently, the voltage error signal is amplified, with a factor \(G\), and added to the voltage noise that is already applied to the laser. The error signal will influence the optical phase, thus closing the feedback loop. The total feedback signal is given by

\[
V(t) = -G[(h_\gamma/2e)(\phi_\phi(t) - \phi_\phi(t)) + \phi(t)], \tag{4}
\]

where the sign is
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Fig. 1: (Colour on-line) The feedback scheme used for stabilization of voltage fluctuations, as described in the main text. The blocks represent the elements of the feedback loop. Each element is characterized by a type of conversion of quantities, for instance conversion of phase difference to voltage, in the detector. The Josephson laser contains two kinds of conversions: voltage is converted to the superconducting phase difference, which in turn is converted to the optical phase. Solid (wavy) lines represent an electrical (optical) connection between the elements. Noise sources are indicated near the lines.

chosen for future convenience and $\rho(t)$ is the noise of the amplifier. The Fourier transform of $V(t)$ is given by

$$V(\nu) = -G \left[ \frac{\hbar \gamma}{2e} H(\nu) \phi^0_{\nu} + \phi_{\nu} \right], \quad H(\nu) = 1 - \frac{e^{i\nu t_a} - 1}{i\nu t_a}. \quad (5)$$

The time average of $\phi_b$ was taken from time $t - t_a$ to $t$, with time $t_a$ being proportional to the quality factor of the reference cavity. We require a reference cavity with a high quality factor, compared to that of the HJL, implying $t_a \gg \Gamma^{-1}$. We assumed to have no time delay between signal measurement and feedback at a timescale $t_a$ [13]. Regarding $H(\nu)$ we mention two important limits. For $\nu t_a \ll 1$, $H(\nu) \simeq -i\nu t_a/2$, while for $\nu t_a \gg 1$, $H(\nu) \simeq 1$. Hence, in the first case, the feedback scheme is sensitive to optical phase changes, while in the second case, it is sensitive to the optical phase itself [11].

Equation (4), with $\bar{\nu}_b \rightarrow \bar{\nu}_b + V(\nu)$, describes noise in the HJL, with feedback at small frequencies, $\nu \ll 1$. A schematic of the feedback is given in fig. 1.

Variance and stability. — Using the above equations, we investigate the time-dependent variance of the optical phase. We concentrate on the variance of the difference between the phase at time zero and time $t$

$$\langle [\phi_b(0) - \phi_b(t)]^2 \rangle = 2 \int \langle |\phi^0_{\nu}|^2 \rangle \left[ 1 - e^{-i\nu t} \right] \frac{d\nu}{2\pi}, \quad (6)$$

$$\frac{\hbar}{2e} \phi^0_{\nu} = \frac{G_0}{2i\nu t_a - z} H(\nu) = \frac{\hbar}{4e} \phi^0_{\nu}. \quad (7)$$

Here $z \equiv G_0 t_a$, plays the role of effective amplification coefficient of our feedback circuit. By increasing the amplification coefficient $G$, we can make $z$ as big as we want. We should only make sure that the poles of this expression lie at frequencies much smaller than $\Gamma$. We need to require this for our approximation to be valid. We also do not expect the feedback to work without delay, at frequencies of the order $\Gamma$. This sets the maximum possible value of $z \simeq \Gamma t_a$. With this condition for $z$ satisfied, the assumption to neglect the optical noise sources $\xi^0_{\nu}$ and $\xi^0_{\nu}$ in the calculation of the variance is justified. Previously, we argued that these noise sources are irrelevant, since they manifest themselves at high-frequencies, while it was assumed that only the low frequencies are relevant. The condition $z \lesssim \Gamma t_a$ now guarantees that this assumption is valid so that the high-frequency fluctuations, and with that, also the optical noise sources, are indeed irrelevant.

Before calculating the variances, we need to ensure that circuit with the feedback remains stable. The stability is governed by the positions of the poles of the susceptibility functions, defined by eq. (7). Poles below (above) the real axis represent solutions that decay (grow) exponentially with time, while poles at the real axis represent diverging solutions that grow linearly with time. The relatively complex form of $H(\nu)$ prohibits us from finding the poles positions explicitly. Instead, we look for values of the parameter $z$, at which the poles cross the real axis. Thereby we find the boundary of the stability region, since the circuit is stable without feedback, which is at $z = 0$. To find this value of $z$, we need to solve

$$2i\nu t_a - z \left[ 1 - e^{i\nu t_a} - 1 \right] = 0, \quad (8)$$

for real $z$ and $\nu$. It is possible to prove that this equation can be satisfied only for $\nu = 0$. To investigate the crossover at zero real $\nu$, we consider purely imaginary frequencies. Multiplying eq. (8) with $t_a$ and defining $W = i\nu t_a$, we find

$$z = \frac{-2W^2}{e^W - 1 - W}. \quad (9)$$

Stability is achieved when all solutions for $W$ are positive, which is when $z > -4$.

To find the time-dependent variance of $\phi_b$, we evaluate the integral of eq. (6). Again because of the relative complexity of $H(\nu)$, the integral cannot be evaluated analytically at arbitrary $t \gg t_a$. We restrict ourselves to $t / t_a$. For the long timescales ($t \gg t_a$) the integral of eq. (6) is dominated by low frequencies. The integrand is proportional to $1/\nu^2$. The variance is given by

$$\langle [\phi_b(0) - \phi_b(t)]^2 \rangle = \frac{D_0 + D_v}{(1 + z/4)^2} t \equiv Dt, \quad (10)$$

where we have defined the diffusion constant $D$. The answer is proportional to diffusion constants in the absence of feedback, where $D_v = (e^2/\hbar^2)(\bar{v}^2)$ comes from voltage fluctuations at the superconducting leads and $D_v = (e^2/\hbar^2)G^2(\bar{v}^2)$ comes from voltage noise send to the amplifier. We see that feedback reduces the diffusion constant $D$, the reduction being proportional to $z^{-2}$ at big $z$. We should mention that $D_0$ grows with increasing amplification factor $G$, so that the pure increase of $G$ does not reduce $D$. There is an optimum coefficient which depends on the ratio between $\langle \bar{v}^2 \rangle$ and $\langle \bar{g}^2 \rangle$. In addition to a term
which is linear in $t$, there is also a constant contribution to the variance that is, in the limit of $t \gg t_a$, given by

$$\langle (\phi_b(0) - \phi_b(t))^2 \rangle_{\text{const}} \simeq \left[ D_0 + D_\delta \right] \int \frac{1}{[2i\nu t_a - z \, H(\nu)]^2 - (1 + z/4)^2 \nu^2} \frac{d\nu}{2 \pi} - [D_0 + D_\delta] t_a \, f(z),$$

where $f(z) \sim z$ for $z \ll 1$ and $f(z) \simeq 1/(2z)$ for $z \gg 1$. We see that this term is suppressed with increasing $z$, although less efficient than the diffusion constant. This constant value of the variance is reached after a typical timescale of $t \simeq t_a/z$. For $z \gg 1$ the high-frequency contribution is of the order of $zt_a D \gg Dt_a$. Note that this constant contribution is always larger than $(\bar{\nu}^2)$, as a consequence of the condition $z \ll \Gamma t_a$. Phase coherence is preserved at a timescale $t \simeq t_a$, provided that the constant part is smaller than $\pi$.

**Extended feedback scheme.** – One can extend the feedback scheme to achieve a better reduction of the diffusion constant and optimization of the constant contribution to the variance.

The extended feedback scheme is realized with a frequency dependent amplification factor. The problem we encountered in the previous section is that the response time of the circuit, $t \simeq t_a/z$, became smaller with increasing effective amplification coefficient $\nu$. Since this response time should be much larger than the response time of the HJL, $1/\Gamma$, this restricts the feasible values of $z$. A solution is to increase the amplification factor at low frequencies, keeping it the same at high frequencies.

We modify the feedback voltage signal such that its Fourier transform becomes $\hat{V}(\nu) \rightarrow \hat{F}(\nu) \hat{V}(\nu)$, where $\hat{F}(\nu)$ is the combination of a PI-filter [14] and an extra amplification with a coefficient $r^{-1}$, where $0 < r < 1$,

$$F(\nu) = \frac{1}{r} \frac{1 - i \nu \tau}{1 - i \nu \tau / r}. \tag{12}$$

With this the amplification coefficient remains unchanged at high frequencies, $\nu \gg 1/\tau$, and is increased with a factor $r^{-1}$ at low frequencies, $\nu \ll 1/\tau$. In the frequency interval $r/\tau < \nu < 1/\tau$, $F(\nu)$ works as an integrator of the feedback signal, with $F(\nu) \sim 1/(i \nu \tau)$.

With this, the expression for the optical phase as reaction on the noises is modified to

$$\frac{\dot{h}}{2e} \phi_b^{'} = -\frac{(r - i \nu \tau) \dot{\varphi}_b + (1 - i \nu \tau) G \dot{\varphi}_b}{2i \nu t_a (r - i \nu \tau) - z (1 - i \nu \tau) H(\nu)} + a. \tag{13}$$

The stability analysis is similar to the case of simple feedback. Instability would occur at low frequencies and the stability requires $z > -4r$.

We continue with calculating the variances of the phases. As in the previous section the diffusion constant is determined by low frequencies and is therefore given by eq. (10) with modified value of $z \rightarrow z/r$ and diffusion constant $D_\delta \rightarrow D_\delta/r^2$,

$$D_r = \frac{D_\delta 1 + (z/r)^2 k}{(1 + z/4)^2 r^2}, \quad k \equiv \frac{(g^2)}{(\gamma t_a)^2(\bar{\nu}^2)}. \tag{14}$$

We rewrite the equation in this form since we would like to optimize it with respect to the effective feedback coefficient $z/r$. As we mentioned, the detector noise, $\varphi(t)$, fed to the amplifier, is amplified as well, so that the optimal value is not infinite. It depends on the effective ratio of noises $k$ and is given by $z/r = 1/4k$, so that the optimal value of the diffusion constant becomes $D_{r,\text{opt}} = 16D_\delta k/(16k + 1)$. The reduction of the diffusion constant is significant at sufficiently small $k$.

It is natural to assume that the integration constant $\tau$ is much bigger than $t_a$. Then the time-dependent fluctuation can be expressed as

$$\langle (\phi_b(0) - \phi_b(t))^2 \rangle = D_r t + (D - D_r) \frac{1 - \exp(-\nu t)}{\delta}, \tag{15}$$

where the high-frequency diffusion constant $D$ is given by eq. (10), and $\delta = (4r + z)/|\gamma(4r + z)| \simeq 1/\tau$, for $z \gg 1$, is the reaction frequency of the circuit. The second term on the right-hand side will become constant in time $t \simeq \tau$. This constant replaces the constant term of eq. (11), of the previous case of simple feedback,

$$\langle (\phi_b(0) - \phi_b(t))^2 \rangle_{\text{const}} \simeq \frac{D - D_r}{\delta} = \frac{16\tau(D_\delta + D_\delta)}{(4 + z)(4r + z)} - \frac{16\tau(4 + z)(r^2 + z^2 k)}{(4r + z)^3} \frac{D_\delta}{(16k + 1)} \left[ D_\delta + \frac{D_\delta}{16k + 1} \right], \tag{16}$$

where in the third line we have reduced the expression by inserting the optimal value for the feedback $z/r = 1/4k$. We find low values for the variance compared to eq. (11), when $k \ll 1$ and $z \gg 1$, such that $|\tau z^2 = \tau/(G \gamma t_a) \ll 1$.

The diffusion constant of the optical phase, $D_r$, can be thus reduced by reducing $r$. Let us address the limit of vanishing noise $\varphi$. In this limit $k \rightarrow 0$ so that the optimal diffusion constant, $D_{r,\text{opt}} \simeq 16D_\delta k/(16k + 1)$. In the limit $r \rightarrow 0$, the transformation $\hat{F}(\nu)$ works as a pure integrator for frequencies $0 < \nu < \tau$, with $\hat{F}(0) \rightarrow \infty$. Thereby we achieve a phase-lock with the phase variance given by the limit of eq. (16) at $r \rightarrow 0$

$$\langle (\phi_b(0) - \phi_b(t))^2 \rangle_{\text{opt}} \simeq \frac{D - D_{r,\text{opt}}}{\delta} = \frac{4r^2}{z} [D_\delta + D_\delta] \ll \pi, \tag{17}$$

To conclude this section, we note that using the extended feedback scheme, which includes the filter $\hat{F}(\nu)$, we can significantly reduce both the diffusion constant and the constant contribution to the variance of the optical phase of the HJL. The time evolution of the variance of the optical phase in both the simple and the extended feedback scheme is shown in fig. 2.
The case of extended feedback (eq. (15), with $\tau = 20 t_a \gg t_a$) is represented by the dashed line and the case of simple feedback (eqs. (6) and (11)) is represented by the solid line. The latter represented by the dashed line and the case of simple feedback $V_{\text{ref}}$ becomes source with that of the HJL. The resulting feedback voltage averaged incoming signal. This can be achieved by measuring the interference of the light from the stable optical source with that of the HJL. The resulting feedback voltage signal becomes $V(t) = -G(t\gamma/2\epsilon)(\phi_0(t) - \phi_{\text{stab}}(t)) + G\gamma\phi_{a}(t)$, with $\phi_{\text{stab}}(t)$ being the phase of the stable source. Using this, the Fourier-transformed phase, $\phi_\nu$, is determined as before (using eq. (4)). With this the Fourier transform of the phase difference, $\phi_- \equiv \phi_0 - \phi_{\text{stab}}$, satisfies

$$\phi_- = \frac{2\epsilon}{G}(G\nu^2 - \nu^2) - 2i\nu\phi_{\text{stab}}, \quad (18)$$

where the typical frequency of the feedback circuit is $y\Gamma$, where dimensionless $y \equiv G\gamma/\Gamma$ must be small to ensure that the feedback occurs at frequencies smaller than the response frequency of the HJL. The stability requires $y > 0$.

**Locking two HJLs.** In this section we study an alternative way to stabilize the voltage fluctuations of a HJL, which is by locking its phase to a stable optical source of close frequency. This source can be another HJL that is not electrically connected to the first one, or another laser. We will show that in this case the feedback quenches the phase diffusion. The fluctuation of the phase remains just finite at big time differences. If the reference source is ideal, this provides infinite decoherence time of the HJL. If the reference source is itself subject to diffusion, the superconducting phase of the HJL follows this diffusion. Still the fluctuation of the phase difference between the phase of the HJL and that of the reference source remains finite.

The locking is modelled using a simple feedback scheme, shown in fig. 3. The scheme is similar to the one in fig. 1. The only difference is that the stabilizer uses as a reference source the external optical signal rather than the time averaged incoming signal. This can be achieved by measuring the interference of the light from the stable optical source with that of the HJL. The resulting feedback voltage signal becomes $V(t) = -G(t\gamma/2\epsilon)(\phi_0(t) - \phi_{\text{stab}}(t)) + G\gamma\phi_{a}(t)$, with $\phi_{\text{stab}}(t)$ being the phase of the stable source. Using this, the Fourier-transformed phase, $\phi_\nu$, is determined as before (using eq. (4)). With this the Fourier transform of the phase difference, $\phi_- \equiv \phi_0 - \phi_{\text{stab}}$, satisfies

$$\phi_- = \frac{2\epsilon}{G}(G\nu^2 - \nu^2) - 2i\nu\phi_{\text{stab}}, \quad (18)$$

where the typical frequency of the feedback circuit is $y\Gamma$, where dimensionless $y \equiv G\gamma/\Gamma$ must be small to ensure that the feedback occurs at frequencies smaller than the response frequency of the HJL. The stability requires $y > 0$.

We note that the locking can occur even if there is a frequency difference between the stable source and the HJL. We can estimate the frequency difference at which the HJL remains locked to the stable source from eq. (18). For small frequencies, $\nu \ll y$, we find an average lag of phase difference proportional to the time derivative of the phase $\phi_{\text{stab}}(t)$,

$$\langle \phi_- (t) \rangle \approx \frac{2}{y} \frac{d\phi_{\text{stab}}(t)}{dt}. \quad (19)$$

The lag should much be less than $\pi$ for the feedback to remain in the linear regime. Therefore, the phase-lock persists for frequency differences $\Delta \omega \ll y\Gamma$.

From eq. (18) we can infer that the phase difference, $\phi_-$, is not subjected to diffusion. As shown in previous sections, the phase of the stabilized HJL diffuses, which is reflected by the pole at $\nu = 0$. Since in the expression for $\phi_-$, the phase $\phi_{\text{stab}}$ is multiplied with $\nu$, there is no longer a pole at $\nu = 0$, implying that $\phi_- $ is not subjected to drift. Hence, the optical phases of the HJLs are synchronized to each other.

Let us calculate the variance for the phase difference $\phi_-$. For this, we use eqs. (6) and (18), assuming $t \rightarrow \infty$. Furthermore, we assume an ideal stable reference source. We find

$$\langle \phi_-^2 \rangle = \frac{2 D_e + D_b}{y} = 2D_e \frac{1 + y^2 \tilde{q}}{y}, \quad q \equiv \frac{\Gamma^2 (\tilde{q}^2)}{2(\tilde{v}^2)}. \quad (20)$$

In this form, we can easily find the effective feedback coefficient $y$ that optimizes the variance. It is given by $y = 1/\sqrt{q}$. For the phase-lock to persist, the variance should be much less than $\pi$. Figure 3 contains a plot of the variance as a function of $y$, with the optimal value of $y$ indicated.

Fig. 2: Diffusion in the HJL with feedback. The left panel shows the time evolution of the variance of $\phi_0$ for two cases. The case of extended feedback (eq. (15), with $\tau = 20 t_a \gg t_a$) is represented by the dashed line and the case of simple feedback (eqs. (6) and (11)) is represented by the solid line. The latter represented by the dashed line and the case of simple feedback $V_{\text{ref}}$ becomes source with that of the HJL. The resulting feedback voltage averaged incoming signal. This can be achieved by measuring the interference of the light from the stable optical source with that of the HJL. The resulting feedback voltage signal becomes $V(t) = -G(t\gamma/2\epsilon)(\phi_0(t) - \phi_{\text{stab}}(t)) + G\gamma\phi_{a}(t)$, with $\phi_{\text{stab}}(t)$ being the phase of the stable source. Using this, the Fourier-transformed phase, $\phi_\nu$, is determined as before (using eq. (4)). With this the Fourier transform of the phase difference, $\phi_- \equiv \phi_0 - \phi_{\text{stab}}$, satisfies

$$\phi_- = \frac{2\epsilon}{G}(G\nu^2 - \nu^2) - 2i\nu\phi_{\text{stab}}, \quad (18)$$

where the typical frequency of the feedback circuit is $y\Gamma$, where dimensionless $y \equiv G\gamma/\Gamma$ must be small to ensure that the feedback occurs at frequencies smaller than the response frequency of the HJL. The stability requires $y > 0$.
The locking described, yields a new way to control the superconducting phase difference. The optical phase can be easily changed by changing the optical path lengths of the laser beams. This change can be incorporated into the change of delay times $t_d$ of the beams, $\delta \Phi_{\text{stab}} = (\omega J/2) \delta t_d$. Because of the phase-lock $\Phi_b$ changes accordingly and so does the superconducting phase difference between the leads. Owing to the Josephson relation, one can produce voltage pulses by changing the optical path length in time.

Conclusions. – In this paper, we propose stabilization of voltage fluctuations in a half-Josephson laser (HJL), by means of optical feedback. Using a feedback scheme, based on the well-known Pound-Drever-Hall stabilizer, we can significantly decrease the diffusion constant of the phase. The feedback can be further enhanced using the frequency-dependent amplification. In the second feedback scheme, the voltage fluctuations of the HJL are stabilized by locking it to a stable optical source. The variation of the phase difference does not grow with increasing time.

We have shown that with this feedback scheme one can achieve the control of superconducting phase difference by changing optical path lengths. These proposals prove the application potential of the HJL and demonstrate this to be an interesting tool to combine superconductivity and optics.

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