Coupling Load-Following Stability with OPF

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Abstract—In this paper, the optimal power flow (OPF) problem is augmented to account for the costs associated with the load-following stability of a power network. Load-following stability costs are expressed through the linear quadratic regulator (LQR). The power network is described by a set of nonlinear differential algebraic equations (DAEs). By linearizing the DAEs around a known equilibrium, a linearized OPF that accounts for steady-state operational constraints is formulated first. This linearized OPF is then augmented by a set of linear matrix inequalities that are algebraically equivalent to the implementation of an LQR controller. The resulting formulation, termed LQR-OPF, is a semidefinite program which furnishes optimal steady-state setpoints and an optimal feedback law to steer the system to a new steady state with minimum stability costs. Numerical tests demonstrate that the setpoints computed by LQR-OPF result in lower overall costs and frequency deviations compared to the setpoints of a scheme where OPF and load-following stability are considered separately.

Index Terms—Optimal power flow, load-following control, linear quadratic regulator, semidefinite programming.

I. INTRODUCTION

Capacity expansion and generation planning, economic dispatch, frequency regulation, and automatic generation control (AGC) are decision making problems in power networks that are solved over different time horizons. These problems range from decades in planning to several seconds in transient control. Although decisions made in shorter time periods may negatively affect performance over longer time periods, these problems have traditionally been treated separately. For example, optimization is used for optimal power flow (OPF) and economic dispatch while feedback control theory is used for frequency regulation.

This paper aims to integrate two crucial power network problems with different time-scales. The first problem is the steady-state OPF, whose decisions are updated every few minutes (e.g., 5 minutes for real-time market balancing). The second is the problem of load-following stability that spans the time-scale of several seconds to one minute.

For a forecasted load level, optimal steady-state setpoints that also minimize load-following stability costs are sought. Steady-state costs account for generator power outputs. Stability costs account for the required control action to drive the deviation of frequency and voltage signals from their optimal OPF setpoints to zero via the linear quadratic regulator (LQR). The proposed formulation also provides, as an output, a feedback law to guide the system dynamics to optimal OPF setpoints.

A. Literature Review

Recent research efforts, organized below in two categories, have showcased the economic and technical merits of jointly tackling steady-state and control problems in power networks.

The first category focuses on frequency regulation with a view towards the economic dispatch [1]–[8]. These works design controllers based on feedback from frequency measurements and guarantee the stability of system dynamics—characterized by the swing equation—while ensuring that system states converge to steady-state values optimal for some form of the economic dispatch problem.

The designs of these feedback control laws may come from averaging-based controllers as in [1]–[3]. By leveraging a continuous-time version of a dual algorithm, generalized versions of the aforementioned controllers are developed in [4], including the classical AGC as a special case. Similar control designs come from interpretation of network dynamics as iterations of a primal-dual algorithm that solves an OPF. For example, [5] introduces a primary frequency control that minimizes a reverse-engineered load disutility function, and [6] introduces a modified AGC that solves an economic dispatch with limited operational constraints.

A frequency control law that further solves economic dispatch with nonlinear power flows in tree networks is devised in [7] by borrowing virtual dynamics from the KKT conditions. A novel method to tackle a load-side control problem including linear equality or inequality constraints for arbitrary network topologies is presented in [8]. The chief attractive feature of [1]–[8] is that they afford decentralized or distributed implementations. Moreover, works such as [4] and [7] develop controllers accounting for nonlinear power flow equations.

The second category focuses on OPF variations with enhanced stability measures. The goal is to obtain optimal steady-state setpoints less vulnerable to disturbances, rather than seeking a stabilizing control law. This goal is achieved by incorporating additional constraints in OPF that account for the stability of the steady-state optimal point.

In transient stability, a classical reference is [9] where system differential equations are converted to algebraic ones and added to the OPF. Based on trajectory sensitivity analysis, [10] incorporates in the OPF a set of automatically generated transient-stability-aware linear constraints. Leveraging the equal-area criterion, a robust transient stability-constrained OPF with dynamical and uncertain loads is presented in [11].

In the realm of small-signal stability, distance from rotor instability is guaranteed in [12] by providing a set of stressed load conditions as a supplement to OPF. The spectral abscissa, that is, the largest real part of system state matrix eigenvalues, is upper bounded by a negative number in [13], yielding a non-smooth OPF. A simpler optimization problem is pursued in [14] using the pseudo-spectral abscissa as stability measure.
B. Paper contributions and organization

The paper contributions are as follows:

- An OPF framework is developed that solves for optimal steady-state setpoints while providing an optimal feedback law to drive the system to the optimal setpoints. Optimality of load-following stability is appraised by a classical LQR that measures the deviation of time-varying system states and controls from their corresponding optimal steady state. LQR has previously been used in the context of megawatt-frequency control [15] and control of oscillatory dynamics [16]. The proposed framework, in distinction, accounts for LQR costs from within the steady-state time-scale.

- The proposed formulation, LQR-OPF, produces optimal steady-state setpoints with minimum stability costs. Using an LQR controller for load following, the setpoints computed by LQR-OPF result in lower overall costs and frequency fluctuations compared to the setpoints of a scheme where OPF and load-following stability are decoupled.

- The proposed framework allows time-varying stability costs to be dependent on steady-state variables. This dependence enables subsequent regulation pricing schemes similar to [17] where, as an example, the cost of frequency regulation is made dependent on steady-state variables. This dependence on the load conditions is made explicit. Suppose now let \( F \) denote optimization variables, and \( (x^e, a^s, u^e) \) represent generator algebraic equations as well as the network power flow equations. Vector \( d \) collects all the network loads as well as leading zero entries from their corresponding optimal steady state.

In order to obtain a stabilizing feedback law, nonlinear DAEs are linearized around a known equilibrium. This approach is different than trajectory sensitivity analysis where the linearization is carried out around a known trajectory. Trajectory sensitivity analyses aim to quantify the effects of parameter changes or inaccuracies on system behavior [18], and mitigate the computational burden during time-domain simulations [19].

The paper is organized as follows. The power system model is laid out in Section II, followed by a description of a generalized OPF. System linearization is pursued in Section III. The proposed formulation coupling OPF and load-following stability is detailed in Section IV. Specific generator models, power flow equations, and connections to the standard OPF are provided in Section V. Section VI numerically verifies the merits of the proposed method. Section VII provides pointers for integrating more power system applications into our proposed framework as future work.

II. POWER SYSTEM MODEL

Consider a power network with \( N \) buses where \( \mathcal{N} := \{1, \ldots, N\} \) is the set of nodes. Define the partition \( \mathcal{N} = \mathcal{G} \cup \mathcal{L} \) where \( \mathcal{G} = \{1, \ldots, G\} \) collects \( G \) buses that contain generators (and possibly also loads) and \( \mathcal{L} = \{G + 1, \ldots, G + L\} \) collects the remaining \( L \) load-only buses. Notice that \( N = G + L \). For a generator \( i \in \mathcal{G} \) with \( n_s \) states and \( n_c \) control inputs, denote by \( x_i(t) \in \mathbb{R}^{n_s} \) the time-varying vector of state variables, and denote by \( u_i(t) \in \mathbb{R}^{n_c} \) the time-varying control inputs. For example, adopting a fourth-order model yields \( n_s = 4 \) and \( n_c = 2 \) with \( x_i(t) = (\delta_i(t), \omega_i(t), e_i(t), m_i(t)) \) and \( u_i(t) = (r_i(t), f_i(t)) \) where \( \delta_i(t), \omega_i(t), e_i(t), m_i(t), r_i(t), f_i(t) \) respectively denote the generator internal phase angle, rotor electrical velocity, internal EMF, mechanical power input, reference power setting, and internal field voltage. Further details are given in Section V-A.

Denote by \( a_i(t) \) the vector of algebraic variables. For load nodes \( i \in \mathcal{L} \), \( a_i(t) = (v_i(t), \theta_i(t)) \), where \( v_i(t) \) and \( \theta_i(t) \) denote the terminal load voltage and phase angle. For generator nodes \( i \in \mathcal{G} \), \( a_i(t) = (p_{g_i}(t), q_{g_i}(t), v_i(t), \theta_i(t)) \), where \( p_{g_i}(t), q_{g_i}(t), v_i(t), \) and \( \theta_i(t) \) respectively denote generator real and reactive power, terminal voltage, and phase angle. For brevity, we drop the dependencies of variables \( x_i, a_i, \) and \( u_i \) on \( t \) and we introduce \( x := \{x_i\}_{i \in \mathcal{G}} \in \mathbb{R}^{n_s, G} \), \( u := \{u_i\}_{i \in \mathcal{G}} \in \mathbb{R}^{n_c, G} \), and \( a := \{a_i\}_{i \in \mathcal{L}} \in \mathbb{R}^{2N+2G} \). Finally, let \( z = \{x, a, u\} \in \mathbb{R}^{n_s+n_c+2G+2N} \). The dynamics of a power system can be captured by a set of nonlinear DAEs

\[
\dot{x} = g(x, a, u),
\]

\[
d = h(x, a),
\]

where \( g : \mathbb{R}^{(n_s+n_c+2)G+2N} \to \mathbb{R}^{n_s, G} \) is given by adopting an appropriate dynamical model of the generator, and \( h : \mathbb{R}^{(n_s+2)G+2N} \to \mathbb{R}^{2G+2N} \) includes generator algebraic equations as well as the network power flow equations. Vector \( d \) collects all the network loads as well as leading zero entries coming from two generator algebraic equations per generator. A particular example of the mapping \( g \) is provided in Section V-A; for the corresponding form of the mapping \( h \) and the vector \( d \) see Section V-B.

Given steady-state load conditions \( d \), the system steady-state operating point is represented by an equilibrium of the DAEs (1). By setting \( \dot{x} = 0 \) and allowing \( x, a, \) and \( u \) to reach steady states \( x^{eq}, a^{eq}, \) and \( u^{eq} \), we arrive at a system of \( n_s + n_c + 2N \) algebraic equations in \( n_s + n_c + 2N \) variables:

\[
0 = g(x^{eq}, a^{eq}, u^{eq}),
\]

\[
d^{eq} = h(x^{eq}, a^{eq}).
\]

Let \( \mathcal{F}(d^{eq}) \) denote the set of solutions to (2), where the dependency on the load conditions is made explicit. Suppose now that \( x^{eq}, a^{eq}, u^{eq} \) are to be jointly optimized so that a certain objective function \( c(x^{eq}, a^{eq}, u^{eq}) \) related is minimized. This leads to a generalized OPF [for clarity, we use \( (x, a, u) \) to denote optimization variables, and \( (x^{eq}, a^{eq}, u^{eq}) \) to generically denote a DAE equilibrium].

\[
\min_{x^*, a^*, u^*} c(x^*, a^*, u^*)
\]

subject to \( 0 = g(x^*, a^*, u^*) \).
where (3d) are the algebraic variable constraints on voltage magnitudes, line flow limits, and line current capacities. Note that the parameter vector \( d^* \) (which includes the constant-power loads) is the input to (3). The term generalized refers to the fact that (3) considers models of generators within the OPF, see e.g., [20] for a recent example. The connection between (3) and the standard OPF is explained in Section V-C. The OPF problem (3) guarantees optimal steady-state operating costs, but does not provide minimal stability costs. Prior to introducing a formulation that bridges stability with OPF, linear approximation of the system dynamics is required which is presented in the next section.

III. LINEAR APPROXIMATION OF SYSTEM DYNAMICS

To obtain an approximate dynamic, we linearize (1) around a known operating point \( z^0 := (x^0, a^0, u^0) \in F(d^0) \). For example, the point \( z^0 \) can be a solution of the load-flow corresponding to an operating point known to the system operator. The motivation behind this selection is to obtain tractable constraints to augment (3) as it allows the stability constraints to take the form of properly formulated LMIs. The derived control law is eventually applied to the nonlinear DAEs (1), rather than the linearization derived in this section.

Consider a generic equilibrium point \( z^0 \in F(d^0) \), where \( d^0 \) is a known load vector. That is, the following holds:

\[
0 = g(x^0, a^0, u^0), \quad d^0 = h(x^0, a^0).
\]

Define \( \Delta d^* := d^* - d^0 \) as the step-change difference between \( d^* \), the load for which the generalized OPF (3) is to be solved, and \( d^0 \), the generic load that will be used for linearization. Equations (1a) and (1b) can be linearized around \( (x^0, a^0, u^0) \) by setting \( (x, a, u) = (x^0, a^0, u^0) + (\Delta x, \Delta a, \Delta u) \):

\[
\Delta x = g_x(z^0) \Delta x + g_a(z^0) \Delta a + g_u(z^0) \Delta u,
\]

\[
\Delta d^* = h_x(x^0, a^0) \Delta x + h_a(x^0, a^0) \Delta a,
\]

where the notation \( g_x \) defines the Jacobian with respect to \( x \), and \( g_a, g_u, h_x, \) and \( h_a \) are similarly defined. It is also understood that \( \Delta x, \Delta a, \Delta u \) are functions of time.

Equation (5) represents a set of linear DAEs. Next, we use (5) in order to develop a proper linear dynamical system that represents the system without algebraic constraints. In particular, assuming invertibility of \( h_a(x^0, a^0) \), (5b) can be solved for the algebraic variables as

\[
\Delta a = -h_a^{-1}(x^0, a^0) \left(-\Delta d^* + h_x(x^0, a^0) \Delta x\right).
\]

The assumption on invertibility of \( h_a(x^0, a^0) \) is very mild in the sense that it holds for practical networks and for various operating points; see also [14] and references therein for sufficient conditions in a similar construction. Then, by substituting (6) in (5a), we obtain

\[
\Delta x = A(z^0) \Delta x + B(z^0) \Delta u + g_a(z^0) h_a^{-1}(x^0, a^0) \Delta d^* + g_u(z^0),
\]

where \( A(z^0) = g_x(z^0) - g_a(z^0) h_a^{-1}(x^0, a^0) h_x(x^0, a^0), \) and \( B(z^0) = g_u(z^0) \).
augmenting the linearized OPF in (9), this LQR-based OPF is written as follows

\[
\begin{align*}
    z^* &= \min_{z' \in \mathbb{R}^n} C(z') + \frac{T_{hr}}{2} \int_0^{t_f} \Delta x^T Q \Delta x' + \Delta u^T R \Delta u' dt \\
    \text{subj. to } & \quad \Delta x' = A(z^0) \Delta x' + B(z^0) \Delta u' \\
    & \quad \Delta x'(0) = x(0) - x^* \\
    & \quad (8a), (8b), a^* \in A,
\end{align*}
\]

where \( Q \) and \( R \) are positive definite matrices penalizing state and control actions deviations; \( T_{hr} \) is a scaling factor to compensate for the time-scale of the stability control problem; \( t_f \) is the optimization time-scale for the OPF problem which is typically in minutes. Since \( t_f \) is in minutes, the finite horizon LQR problem can be replaced with an infinite horizon LQR formulation (i.e., \( t_f = \infty \)) as the solution to the Riccati equation reaches steady state [21].

In (12c), if we assume that the system is operating at \( x^0 \), then \( \Delta x(0) = 0 \), yielding \( \Delta x'(0) = x^0 - x^* \).

Regulating state and control actions can be dependent on the operating points, to encourage smaller steady-state variations. For example regulating a generator’s frequency becomes more costly as the real power generation increases [17]. We capture this by considering \( Q^{-1} \) and \( R^{-1} \) to be affine functions of steady-state variable \( z^* \). In particular, assume \( Q^{-1} \) and \( R^{-1} \) to be diagonal matrices as follows

\[
\begin{align*}
    Q(z^*)^{-1} &= Q_0^{-1} + Q_1^{-1} \text{diag}(a^*), \\
    R(z^*)^{-1} &= R_0^{-1} + R_1^{-1} \text{diag}(a^*).
\end{align*}
\]

The corresponding infinite horizon LQR augmenting the linearized OPF (12) can be written as an SDP:

\[
\begin{align*}
    \text{LQR-OPF: } \\
    \min_{S, z^*} c(z^*) + \frac{T_{hr}}{2} \gamma \\
    \text{subj. to } & \quad (8a), (8b), a^* \in A \\
    & \quad [0, 0, 0, \dot{x}^0 - \dot{x}^*] \leq 0 \\
    & \quad [A^T S + S A - B(z^*)^T B] = S - Q(z^*)^{-1} \leq 0 \\
    & \quad S \succeq 0.
\end{align*}
\]

The optimal control law generated from the optimal control problem (12) is \( \Delta u' = K \Delta x' \) where \( K = -R^{-1} B(z^0)^T S^{-1} \) is the optimal state feedback control gain and \( S^* \) is an optimizer of (14). The reader is referred to [21] for the derivation of this control law.

The LQR-OPF problem (14) is an SDP which solves jointly for the new steady-state variables \( z^* \) and the matrix \( S \) while guaranteeing the stability of the linearized system in (7). The two problems of OPF and stability control are coupled in (14) through the LQR weight matrices—with \( Q^{-1} \) and \( R^{-1} \) being affine functions—as well as through the term \( x^* - x^0 \) in LMI (14c). It is worth noting that \( \gamma \) in (14) is equal to the integral in the objective of (12) with \( t_f = \infty \).

The optimal steady-state solution \( z^* \) of the LQR-OPF problem (14) satisfies the linearized steady-state equations (8).
V. GENERATOR MODEL, POWER FLOWS, AND OPF

The developments in this paper and the proposed LQR formulation are applicable to any generator dynamical model with control inputs. However, as an example, we present a specific form of mappings $g$ and $h$ in this section which will be used for the numerical tests of Section VI.

A. Generator model

The fourth order model of the synchronous generator internal dynamics for node $i \in G$ can be written as

$$\dot{\delta}_i = \omega_i - \omega_s$$

$$\dot{\omega}_i = \frac{1}{M_i} [m_i - D_i (\omega_i - \omega_s) - p_{gi}]$$

$$\dot{e}_i = \frac{1}{\tau_{di}} [\frac{x_{di}}{x'_{di}} e_i + \frac{x_{di}}{x'_{di}} d_i \cos(\delta_i - \theta_i) + f_i]$$

$$\dot{m}_i = \frac{1}{\tau_{ei}} \left[ r_i (t) - \frac{1}{R} (\omega_i - \omega_s) - m(t) \right]$$

where $M_i$ is the rotor’s inertia constant ($\text{pu} \times \text{sec}^2$), $D_i$ is the damping coefficient ($\text{pu} \times \text{sec}$), $\tau_{di}$ is the direct-axis open-circuit time constant ($\text{sec}$), $x_{di}$ and $x_{qi}$ are respectively the direct- and quadrature-axis synchronous reactances, and $x'_{di}$ is the direct-axis transient reactance ($\text{pu}$). Equation (15d) is a simplistic model of a prime-mover generator with $\tau_{ei}$ as the charging time ($\text{sec}$) and a speed-governing mechanism with regulation constant $R$. The mapping $g$ defined in (1a) is given by concatenating (15) for $i \in G$.

The following algebraic equations relate the generator real and reactive power output with generator voltage, internal EMF, and internal angle and must hold at any time instant for generator nodes $i \in G$:

$$0 = -p_{gi} + \frac{e_i v_i}{x_{di}} \sin(\delta_i - \theta_i)$$

$$+ \frac{x_{qi}}{x_{di}} q_i v_i^2 \sin[2(\delta_i - \theta_i)]$$

$$= -q_{gi} - \frac{e_i v_i}{x_{di}} \cos(\delta_i - \theta_i) + \frac{x'_{di}}{x_{di}} q_i v_i^2$$

$$+ \frac{x_{qi}}{x_{di}} q_i v_i^2 \cos[2(\delta_i - \theta_i)].$$

B. Power flow equations

Let $Y = \mathbf{g} + \mathbf{j}B$ denote the network bus admittance matrix based on the $\pi$-model of transmission lines. Notice that $Y$ may include transformers, tap-changing voltage regulators, and phase shifters [25]. The power flow equations are

$$-p_i = -p_{gi} + G_{i,i} v_i^2 + \sum_{j \in N_i} [G_{i,j} v_j v_i \cos \theta_{ij} + B_{i,j} v_j v_i \sin \theta_{ij}], i \in G.$$ (17a)

$$-q_i = -q_{gi} - B_{i,i} v_i^2 + \sum_{j \in N_i} [G_{i,j} v_j v_i \sin \theta_{ij} - B_{i,j} v_j v_i \cos \theta_{ij}], i \in G.$$ (17b)

$$-p_i = G_{i,i} v_i^2 + \sum_{j \in N_i} [G_{i,j} v_j v_i \cos \theta_{ij} + B_{i,j} v_j v_i \sin \theta_{ij}], i \in L.$$ (17c)

$$-q_i = -B_{i,i} v_i^2 + \sum_{j \in N_i} [G_{i,j} v_j v_i \sin \theta_{ij} - B_{i,j} v_j v_i \cos \theta_{ij}], i \in L.$$ (17d)

where $\theta_{ij} := \theta_i - \theta_j; p_i := p_i(t)$ and $q_i := q_i(t)$ are respectively the real and reactive power demands at node $i$ modeled as a time-varying constant-power load, i.e., $p_i$ and $q_i$ are not functions of $v_i$. The mapping $h$ in (1b) is given by concatenating (16) for $i \in G$ and (17) for $i \in N$. By defining $p_{gi} = \{p_i\}_{i \in G}, q_{gi} = \{q_i\}_{i \in G}$, $p_{li} = \{p_i\}_{i \in L}, q_{li} = \{q_i\}_{i \in L}$, we obtain vector $d = \{0_{2G}, -p_{gi}, -q_{gi}, -p_{li}, -q_{li}\}$.

C. Optimal power flow

The standard OPF problem typically only considers the algebraic variable $x^a$ and is given as

$$\min c(a^a) \quad \text{subj. to} \quad (17) \quad \text{and} \quad a^a \in A.$$ (18)

If the cost function $c(z^a)$ of the generalized OPF in (3) is only a function of algebraic variables $a^a$, (3) can be solved by solving the standard OPF problem (18) to obtain the optimal $a^a$. The variables $x^a$ and $u^a$ can then be found by plugging in $a^a$ in equations (16) and (15) while setting $\dot{x} = 0$.

VI. NUMERICAL SIMULATIONS

This section provides a numerical assessment of the advantages of the LQR-OPF in comparison to a method where OPF and load-following stability problems are treated separately. Prior to analyzing the case studies, we first describe the general simulation workflow as outlined in Fig. 2.

The decoupled approach, one where OPF and stability are solved separately, is considered on the left-hand side of Fig. 2. Initially, the system is in steady-state and operates at $z^0$. For a forecasted load demand, $d^* = d^0 + \Delta d^*$, the OPF (18) is solved yielding optimal steady-state setpoints $a^a$, including generator real and reactive powers ($p_{gi}^*, q_{gi}^*$) and generator voltage magnitudes and phases ($v_i^*, \theta_i^*$). For OPF, it holds that $a^a_{\text{eq}} = a^a$. The algebraic variables are then utilized to solve (15) upon setting $\dot{x} = 0$ together with (16) to obtain steady-state setpoints of generator states $x^a_{\text{eq}}$ and control inputs $u^a_{\text{eq}}$.

The next steady-state equilibrium is then simply represented as $z^0 = (x^0a^0, u^a_{\text{eq}})$. The DAEs (1a)–(1b), upon being subject to $d^*$, are kicked out of the initial equilibrium $z^0$. To steer the DAEs (1a)–(1b) to the next desired equilibrium point $z^a_{\text{eq}}$, an LQR control is performed. Dynamic performance as well as costs of steady-state and stability are evaluated. The standard OPF (18) is solved using MATPOWER’s runopf.m.

The proposed methodology is considered on the right-hand side of Fig. 2 where the OPF block is replaced by LQR-OPF (14) followed by a load-flow. LQR-OPF obtains optimal generator voltage setpoints $\{v_{ij}^q\}_{i \in G}$, real power setpoints $\{p_{gi}^r\}_{i \in G \setminus \{\text{slack}\}}$, and $\theta_{\text{slack}}^r$, while accounting for stability costs that drive the DAEs (1a)–(1b) to the next desired equilibrium. These obtained setpoints are then input to a standard load-flow (performed by MATPOWER’s runpf.m) by setting the following for $i \in G$: $v_{ij}^q = v_{ij}^r$, $p_{gi}^r = p_{gi}^q$, for nonslack $i$, and $\theta_{\text{slack}}^r = \theta_{\text{slack}}^q$. This process yields the remaining algebraic variables in $a^a_{\text{eq}}$. Similar to the previous approach, the DAEs (15) and (16) are solved after setting $\dot{x} = 0$ yielding optimal steady-state setpoints of states $x^a_{\text{eq}}$ and controls $u^a_{\text{eq}}$.

The next equilibrium is then given by $z^a_{\text{eq}} = (x^0a^0, u^a_{\text{eq}})$. 

\[ \begin{align*}
\dot{\delta}_i &= \omega_i - \omega_s \\
\dot{\omega}_i &= \frac{1}{M_i} [m_i - D_i (\omega_i - \omega_s) - p_{gi}] \\
\dot{e}_i &= \frac{1}{\tau_{di}} [\frac{x_{di}}{x'_{di}} e_i + \frac{x_{di}}{x'_{di}} d_i \cos(\delta_i - \theta_i) + f_i] \\
\dot{m}_i &= \frac{1}{\tau_{ei}} \left[ r_i (t) - \frac{1}{R} (\omega_i - \omega_s) - m(t) \right]
\end{align*} \]
A. Network description

Network steady-state data are obtained from MATPOWER’s case9.m, case14.m, and case39.m [27]. These include bus admittance matrix $Y$, steady-state real and reactive power demands, that is, $p_i^0 := \{p_{ij}^0, p_i^0\}$ and $q_i^0 := \{q_{ij}^0, q_i^0\}$, cost of real power generation $c(\alpha^s) = \sum_i c_i (p_{ij}^0)^2 + c_1 p_{ij}^0 + c_0$, as well as the limits of power generation and voltages in constraint (3d). Machine constants of (15a)–(15c) and (16) are respectively the maximum real and reactive power limits of generator $i \in \mathcal{G}$. The rationale behind choosing (19a) is that angle and frequency instability are usually remedied by generating real power. In this case, increase in steady-state real power generation $p_i^s$ leads to higher cost of frequency regulation. Similarly, the rationale behind choosing (19b) is that voltage stability is typically correlated with reactive power injection. This choice means that an increase in steady-state reactive power generation $q_i^s$ inures a higher cost of voltage regulation.

B. Regulation cost matrices $Q$ and $R$

In accordance with (13), $Q^{-1}$ and $R^{-1}$ are selected to be diagonal with affine entries as follows

$$Q_{\omega_i, \delta_i, m_i}^{-1} = R_{\tau_i}^{-1} = \left(1 - \alpha \frac{p_{\text{g}_{i}}^{\text{max}}}{p_{\text{g}_{i}}^{0}}\right), \quad (19a)$$

$$Q_{\varepsilon_i}^{-1} = R_{\tau_i}^{-1} = \left(1 - \alpha \frac{q_{\text{g}_{i}}^{\text{max}}}{q_{\text{g}_{i}}^{0}}\right), \quad (19b)$$

where $Q_{\omega_i, \delta_i, m_i}$ refers to the diagonal entries of $Q$ corresponding to $\omega_i$, $\delta_i$, and $m_i$. Matrices $Q_{\varepsilon_i}$, $R_{\tau_i}$, and $R_{\tau_i}$ are similarly defined. Parameter $\alpha$ is in the interval $[0, 1]$ that determines the amount of coupling between steady-state quantities and stability costs through matrices $Q$ and $R$. Quantities $p_{\text{g}_{i}}^{\text{max}}$ and $q_{\text{g}_{i}}^{\text{max}}$ are respectively the maximum real and reactive power limits of generator $i \in \mathcal{G}$.

| Case | Method | Steady-state cost ($) | Stability cost ($) | Total cost ($) | Max. freq. dev. (Hz) | Max. volt. dev. (pu) |
|------|--------|-----------------------|-------------------|--------------|---------------------|-------------------|
| 9-bus | LQR-OPF | 6151.7                | 16.1              | 6167.8       | 0.0141              | 0.02              |
|      | OPF    | 6113.6                | 258.7             | 6372.4       | 0.0204              | 0.12              |
| 14-bus | LQR-OPF | 9196.5                | 17.2              | 9213.8       | 0.0039              | 0.03              |
|      | OPF    | 9127.4                | 188.6             | 9316.0       | 0.0123              | 0.03              |
| 39-bus | LQR-OPF | 51569.1               | 2207.4            | 55084.1      | 0.1817              | 0.08              |
|      | OPF    | 51569.1               | 2207.4            | 55084.1      | 0.1400              | 0.11              |

C. Dynamical simulation

A step load increase of 10% in real power with power factor 0.9 is applied at $t = 0$. This implies that $\Delta p_i^s = 0.1 p_i^0$ and $\Delta q_i^s = 0.048 q_i^0$, totaling to a significant load increase of $31.50 + 6.528$ MVA for the 9-bus system, $25.9 + 7.528$ MVA for the 14-bus system, and $625.423 + 6.5808$ MVA for the 39-bus system. This load increase drives the nonlinear dynamics (1) out of the initial equilibrium. By applying LQR control according to the computations in Fig. 2, the dynamics in (1) are steered to arrive at the desired equilibria obtained from the OPF and LQR-OPF.

With selections of $\alpha = 0$ and $T_{\text{iq}} = 1000$, Table I reveals the breakdown of the steady-state, stability, and total costs, as well as maximum frequency and voltage deviations from the optimal equilibrium of the OPF and LQR-OPF approaches when applied to the three networks with the aforementioned load increase. Stability costs reported in Table I are computed as

$$\Delta \omega_i^{\text{eq}} = 1000 \int_0^T (\Delta \omega_i^{\text{eq}}) Q \Delta \omega_i^{\text{eq}} + \Delta u_i^{\text{eq}} R \Delta u_i^{\text{eq}})dt$$

that is, through numerical integration of the trajectories resulting from the simulation of the nonlinear DAEs. The total cost is simply the summation of stability and steady-state costs. Between the two approaches, OPF exhibits lower steady-state cost but higher stability costs. In terms of total costs, the LQR-OPF shows improved performance. The maximum frequency deviation is also much lower for LQR-OPF than the OPF.

Figures 3–6 depict the dynamic performance of the 39-bus system under OPF and LQR-OPF in conjunction with load-following LQR control. Specifically, system frequencies are portrayed in Fig. 3 where it is observed that frequencies resulting from OPF [Fig. 3(a)] undergo higher fluctuations than those resulting from LQR-OPF [Fig. 3(b)].

The deviation of the governor reference power setting $\omega$ from its optimal setting $\omega^{\text{eq}}$ is depicted in Fig. 4, where it...
Deviations of generator angles $\delta$, internal EMF $e$, and internal field voltage $f$ are given in Fig. 5. Generator mechanical power output $m$ and generated power $p_g$ follow the same trend as $r$; thus, their corresponding plots have been omitted for the sake of space, but they are available on the github page. Last but not least, a plot of nodal voltages are provided in Fig. 6.

**D. Effect of coupling**

Here, we study the effect of parameter $\alpha$ which couples steady-state variables to stability costs through (19). When the value of $\alpha$ increases to approach 1, entries of matrices $Q$ and $R$ increase as the values of $p_{gs}^s$ and $q_{gs}^s$ approach their respective maximum. It is depicted in Fig. 7 that as the coupling coefficient $\alpha$ increases, stability costs increase in both OPF and LQR-OPF. However, stability costs of LQR-OPF are significantly lower than the costs incurred by the scheme where OPF and LQR are solved independently.

**VII. SUMMARY AND FUTURE WORK**

An OPF framework is presented that in addition to solving for optimal steady-state setpoints provides an optimal feedback law to perform load-following control. The costs of load-following stability is captured by a classical LQR control that accounts for deviations of system states and controls from their optimal steady-state setpoints. A joint formulation of OPF and load-following control, termed LQR-OPF, is obtained by combining a linearized OPF with an equivalent SDP formulation of the LQR. Numerical tests verify that compared to a scheme where OPF and load-following stability problems are solved separately, LQR-OPF features significantly improved dynamic performance and reduced overall system costs.

The proposed framework is general and allows for seamless incorporation of different power system applications, such as the dynamics of wind turbines and the operation of storage devices—both operating at different time-scales. Recent work, for instance, demonstrates the impact of wind power injection on power system oscillations [28]. We plan to integrate more modern applications into the proposed framework while investigating the regulation and cost benefits of this integration.
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