KINETICALLY CONSTRAINED MODELS WITH RANDOM CONSTRAINTS

ASSAF SHAPIRA

Abstract. We study two kinetically constrained models in a quenched random environment. The first model is a mixed threshold Fredrickson-Andersen model on $\mathbb{Z}^2$, where the update threshold is either 1 or 2. The second is a mixture of the Fredrickson-Andersen 1-spin facilitated constraint and the North-East constraint in $\mathbb{Z}^2$. We compare three time scales related to these models – the bootstrap percolation time for emptying the origin, the relaxation time of the kinetically constrained model, and the time for emptying the origin of the kinetically constrained model – and understand the effect of the random environment on each of them.

1. Introduction

Kinetically constrained models (KCMs) are a family of interacting particle systems introduced by physicists in order to study glassy and granular materials [14, 25]. These are reversible Markov processes on the state space $\{0, 1\}^V$, where $V$ is the set of vertices of some graph. The equilibrium measure of these processes is a product measure of i.i.d. Bernoulli random variables, and their nontrivial behavior is due to kinetic constraints – the state of each site is resampled at rate 1, but only when a certain local constraint is satisfied. This condition expresses the fact that sites are blocked when there are not enough empty sites in their vicinity. One example of such a constraint is that of the Fredrickson-Andersen $j$-spin facilitated model on $\mathbb{Z}^d$, in which an update is only possible if at least $j$ nearest neighbors are empty [13]. We will refer to this constraint as the FA$j$ constraint. Another example is the North-East constraint: the underlying graph is $\mathbb{Z}^2$, and an update is possible only if both the site above and the site to the right are empty [21]. These constraints result in the lengthening of the time scales describing the dynamics as the density of empty sites $q$ tends to 0. This happens since sites belonging to large occupied regions could only be changed when empty sites penetrate from the outside. The main difficulty in the analysis of KCMs is that they are not attractive, which prevents us from using tools such as monotone coupling and censoring often used in the study of Glauber dynamics. For this reason spectral analysis and inequalities related to the spectral gap are essential for the study of time scales in these models. See [23] for more details.

A closely related family of models are the bootstrap percolation models, which are, unlike KCMs, monotone deterministic processes in discrete time. The state space of the bootstrap percolation is the same as that of the KCM, and they share the same family of constraints; but in the bootstrap
percolation occupied sites become empty (deterministically) whenever the constraint is satisfied, and empty sites can never be filled. The initial conditions of the bootstrap percolation are random i.i.d Bernoulli random variables with parameter $1 - q$, i.e., they are chosen according to the equilibrium measure of the KCM. In this paper we will refer to the bootstrap percolation that corresponds to certain constraint by their KCM name, so, for example, the $j$-neighbor bootstrap percolation will be referred to as the bootstrap percolation with the FA$j$f constraint.

In the examples given previously of the FA$j$f model and the North-East model the constraints are translation invariant. Universality results for general homogeneous models have been studied recently for the bootstrap percolation in a series of works that provide a good understanding of their behavior [5, 8, 10, 17]. Inspired by the tools developed for the bootstrap percolation, universality results on the KCMs could also be obtained for systems with general translation invariant constraints [24, 23]. Another type of models vastly studied for the bootstrap percolation are models in a random environment, such as the bootstrap percolation on the polluted lattice [16, 15], Galton-Watson trees [9], random regular graphs [7, 19] and the Erdős-Rényi graph [20]. KCMs in random environments have also been studied in the physics literature, see [26, 27].

In this paper we will consider two models on the two dimensional lattice with random constraints. We will focus on the divergence of time scales when the equilibrium density of empty sites $q$ is small.

The time scale that is commonly considered in KCMs is the relaxation time, i.e., the inverse of the spectral gap. This time scale determines the slowest possible relaxation of correlation between observables, but for homogeneous system it often coincides with typical time scales of the system (see, e.g., [22]). However, when the system is not homogeneous it will in general not describe actual time scales that we observe. We will see in this paper that very unlikely configurations of the disorder that appear far away determine the relaxation time, even though the observed local behavior is not likely to be effected by these remote regions.

Another time scale that is natural to look at is the first time at which the origin (or any arbitrary vertex) is empty. In the bootstrap percolation literature, this is indeed the time most commonly studied. This time could be observed physically, and we will see that it is not significantly effected by the “bad” regions far away from the origin.

In this paper we compare the three time scales – the time it takes for the origin to be emptied with the bootstrap percolation, the relaxation time for the KCM, and the first time the origin is empty in the KCM. We will first analyze these time scales in a mixed threshold FA model on the two dimensional lattice. The second model we consider is a mixed KCM, in which vertices have either the FA1f constraint, or the North-East constraint. This will be an example of a model in which the relaxation time is infinite, but the time at which the origin becomes empty is almost surely finite.
2. Mixed threshold Fredrickson-Andersen model

2.1. Model and notation. In this section we will treat two models – the mixed threshold bootstrap percolation on $\mathbb{Z}^2$ and the mixed threshold Fredrickson-Andersen model on $\mathbb{Z}^2$. Both models will live on the same random environment, that will determine the threshold at each vertex of $\mathbb{Z}^2$. It will be denoted $\omega$, and the threshold at a vertex $x$ will be $\omega_x \in \{1, 2\}$. This environment is chosen according to a measure $\nu$, which depends on a parameter $\pi \in (0, 1)$. $\nu$ will be a product measure – for each vertex $x \in \mathbb{Z}^2$ $\omega_x$ equals 1 with probability $\pi$ and 2 with probability $1 - \pi$, independently from the other vertices. Sites with threshold 1 will be called easy, and sites with threshold 2 difficult.

Both the bootstrap percolation and the FA dynamics are defined on the state space $\Omega = \{0, 1\}^{\mathbb{Z}^2}$. For a configuration $\eta \in \Omega$, we say that a site $x$ is empty if $\eta_x = 0$ and occupied if $\eta_x = 1$. For $\eta \in \Omega$ and $x \in \mathbb{Z}^2$ define the constraint

$$ c_x(\eta) = \begin{cases} 1 & \sum_{y \sim x} (1 - \eta_y) \geq \omega_x \\ 0 & \text{otherwise} \end{cases} $$

(2.1)

We can now define the bootstrap percolation with these constraints – it is a deterministic dynamics in discrete time, where at each time step $t$ empty vertices stay empty, and an occupied vertex $x$ becomes empty if the constraint is satisfied, namely $c_x(\eta(t-1)) = 1$. The initial conditions for the bootstrap percolation are random, depending on a parameter $q \in (0, 1)$. They are chosen according to the measure $\mu$, defined as a product of independent Bernoulli measures:

$$ \mu = \bigotimes_{x \in \mathbb{Z}^2} \mu_x, \quad \mu_x \sim \text{Ber}(1 - q). $$

The Fredrickson-Andersen model is a continuous time Markov process on $\Omega$. It is reversible with respect to the equilibrium measure $\mu$ defined above, and its generator $\mathcal{L}$ is defined by

$$ \mathcal{L}f = \sum_x c_x(\mu_x f - f) $$

(2.2)

for any local function $f$. We will denote by $\mathcal{D}$ the Dirichlet form associated with $\mathcal{L}$. Probabilities and expected values with respect to this process starting at $\eta$ will be denoted by $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$. When starting from equilibrium we will use $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$.

Finally, for any event $A \subseteq \Omega$, we define the hitting time

$$ \tau_A = \inf \{ t : \eta(t) \in A \}. $$
The hitting time is defined for both the KCM and the bootstrap percolation. For the time it takes to empty the origin we will use the notation

\[ \tau_0 = \tau_{\{\eta_0 = 0\}}. \]

2.2. Results. The first result concerns the bootstrap percolation. It will say that for small values of \( q \), \( \tau_0 \) scales as \( \frac{1}{\sqrt{q}} \). To avoid confusion we stress that \( \mu \) and \( P_\mu \) depend on \( q \), even though this dependence is not expressed explicitly in the notation.

**Theorem 2.1.** Consider the bootstrap percolation with the mixed FA constraint. Then \( \nu \)-almost surely

\[
\lim_{q \to 0} \mu \left[ \tau_0 \geq \frac{a}{\sqrt{q}} \right] \xrightarrow{a \to \infty} 0, \tag{2.3}
\]

\[
\lim_{q \to 0} \mu \left[ \tau_0 \leq \frac{a}{\sqrt{q}} \right] \xrightarrow{a \to 0} 0. \tag{2.4}
\]

For the KCM we have an exponential divergence of the relaxation time, but a power law behavior of \( \tau_0 \).

**Theorem 2.2.** Consider the KCM with the mixed FA constraint.

1. There exist a constant \( c > 0 \) (that does not depend on \( \pi, q \)) such that \( \nu \)-almost surely the relaxation time of the dynamics is at least \( e^{c/q} \).

2. \( \nu \)-almost surely there exist \( \alpha \) and \( \overline{\alpha} \) (which may depend on \( \omega \)) such that

\[
P_\mu \left[ \tau_0 \geq q^{-\alpha} \right] \xrightarrow{q \to 0} 0, \tag{2.5}
\]

\[
P_\mu \left[ \tau_0 \leq q^{-\overline{\alpha}} \right] \xrightarrow{q \to 0} 0. \tag{2.6}
\]

Moreover, \( E_\mu [\tau_0] \geq q^{-\alpha} \) for \( q \) small enough.

**Remark 2.3.** We will see that the two exponents \( \alpha \) and \( \overline{\alpha} \) cannot be deterministic – there is \( \alpha_0 \in \mathbb{R} \) such that \( \nu (\overline{\alpha} < \alpha_0) > 0 \) but \( \nu (\alpha < \alpha_0) < 1 \).

**Remark 2.4.** In these two theorems we see that while \( \tau_0 \) for the bootstrap percolation behaves like \( q^{-1/2} \), its scaling for the FA is random. In the proof we will see in details the reason for this difference, but we could already try to describe it heuristically. The bootstrap percolation is dominated by the sites far away from the origin, and once these sites are emptied the origin will be emptied as well. The influence of the environment far away becomes deterministic by a law of large numbers, so we do not see the randomness of \( \omega \) in the exponent. To the contrary, in the FA dynamics even when sites far away are empty, one must empty many sites in a close neighborhood of the origin simultaneously before the origin could be emptied. Therefore, in order to empty the
origin we must overcome a large energy barrier, which makes \( \tau_0 \) bigger. This effect depends on the structure close to the origin, so it feels the randomness of the environment.

For simplicity, we have chosen to focus on the two dimensional case. However, a more general result can also be obtained. In the next two theorems we will consider the bootstrap percolation and KCM on \( \mathbb{Z}^d \). The thresholds \( \{\omega_x\}_{i \in \mathbb{Z}^2} \) are i.i.d., according to a law that we denote by \( \nu \). We will also assume that the probability that the threshold is 1 is nonzero, and that the probability that the threshold is more than \( d \) is zero.

**Theorem 2.5.** For the bootstrap percolation model described above, \( \nu \)-almost surely

\[
\lim_{q \to 0} \mu \left[ \tau_0 \geq aq^{-1/a} \right] \overset{a \to \infty}{\longrightarrow} 0, \tag{2.7}
\]

\[
\lim_{q \to 0} \mu \left[ \tau_0 \leq aq^{-1/a} \right] \overset{a \to 0}{\longrightarrow} 0. \tag{2.8}
\]

**Theorem 2.6.** For the KCM described above, \( \nu \)-almost surely there exist \( \alpha \) and \( \overline{\alpha} \) (which may depend on \( \omega \)) such that

\[
P_\mu \left[ \tau_0 \geq q^{-\alpha} \right] \overset{q \to 0}{\longrightarrow} 0, \tag{2.9}
\]

\[
P_\mu \left[ \tau_0 \leq q^{-\overline{\alpha}} \right] \overset{q \to 0}{\longrightarrow} 0. \tag{2.10}
\]

3. **Mixed north-east and FA1f KCM**

3.1. **Model and notation.** In this section we will consider again a kinetically constrained dynamics in an environment with mixed constraints. This time, however, the two constraints we will have are FA1f and north-east. That is, using the same \( \omega \) and \( \nu \) as before, for \( x \) such that \( \omega_x = 1 \)

\[
c_x(\eta) = \begin{cases} 1 & \sum_{y \sim x} (1 - \eta_y) \geq 1 \\ 0 & \text{otherwise} \end{cases},
\]

and when \( \omega_x = 2 \)

\[
c_x(\eta) = \begin{cases} 1 & \eta_x + e_1 = 0 \text{ and } \eta_x + e_2 = 0 \\ 0 & \text{otherwise} \end{cases}.
\]

For the same \( \mu \), we can define \( \mathcal{L} \) as in [2, 2]. Note that \( c_x \) (and therefore \( \mathcal{L} \)) are not the same as those of the previous section, even though we use the same letters to describe them. Again, the hitting time of a set \( A \) will be denoted by \( \tau_A \), and \( \tau_0 = \tau_{\{\eta_0 = 0\}} \).
We restrict ourselves to the case where \( \pi \) is greater than the critical probability for the Bernoulli site percolation on \( \mathbb{Z}^2 \), denoted by \( p^{\text{SP}} \). The critical probability for the oriented percolation on \( \mathbb{Z}^2 \) will be denoted by \( p^{\text{OP}} \).

Remark 3.1. Our choice of regime, where easy sites percolate, guarantees that all sites are emptiable for the bootstrap percolation. The infinite cluster \( C \) of easy sites is emptiable since it must contain an empty site somewhere. The connected components of \( \mathbb{Z}^2 \setminus C \) are finite, and have an emptiable boundary, so each of them will also be emptied eventually.

This choice, however, is not the only one for which all sites are emptiable. For any fixed environment \( \omega \) there is a critical value \( q_c \) such that above \( q_c \) all sites are emptiable and below \( q_c \) some sites remain occupied forever. For \( \pi > p^{\text{SP}} \) we already know that \( \nu \)-almost surely \( q_c = 0 \). In fact, the same argument gives a slightly better result by allowing sites to be difficult if they are also empty. This implies that \( q_c \leq 1 - \frac{1 - p^{\text{SP}}}{1 - \pi} \). On the other hand, if there is an infinite up-right path of difficult sites that are all occupied, this path could never be emptied. This will imply that \( q_c \geq 1 - \frac{p^{\text{OP}}}{1 - \pi} \).

3.2. Results. We will see for this model that it is possible to have an infinite relaxation time, and still the tail of the distribution of \( \tau_0 \) decays exponentially, with a rate that scales polynomially with \( q \).

**Theorem 3.2.** Consider the kinetically constrained model described above, with \( \pi > p^{\text{SP}} \) and \( q \leq q^{\text{OP}} \).

1. \( \nu \)-almost surely the spectral gap is 0, i.e., the relaxation time is infinite.
2. There exist two positive constants \( c, C \) depending on \( \pi \) and a \( \nu \)-random variable \( \tau \) such that
   (a) \( \mathbb{P}_\mu (\tau_0 \geq t) \leq e^{-t/c} \) for all \( t > 0 \),
   (b) \( \nu (\tau \geq t) \leq C t^{\frac{1}{1 + \pi}} \) for \( t \) large enough.

4. Some tools

In this section we will present some tools that will help us analyze the kinetically constrained models that we have introduced. We will start by considering a general state space \( \Omega \), and any Markov process on \( \Omega \) that is reversible with respect to a certain measure \( \mu \). We denote its generator by \( \mathcal{L} \) and the associated Dirichlet form by \( \mathcal{D} \). We will consider, for some event \( A \), its hitting time \( \tau_A \). With some abuse of notation, we use \( \tau_A \) also for the \( \mu \)-random variable giving for every state \( \eta \in \Omega \) the expected hitting time at \( A \) starting from that state:

\[ \tau_A (\eta) = \mathbb{E}_\eta (\tau_A) \].
\( \tau_A(\eta) \) satisfies the following Poisson problem:

\[
\mathcal{L}\tau_A = -1 \text{ on } A^c, \\
\tau_A = 0 \text{ on } A. 
\] (4.1)

By multiplying both sides of the equation by \( \tau_A \) and integrating with respect to \( \mu \), we obtain

**Corollary 4.1.** \( \mu(\tau_A) = D\tau_A \).

Rewriting this corollary as \( \mu(\tau_A) = \frac{\mu(\tau_A)^2}{D\tau_A} \), it resembles a variational principle introduced in [4] that will be useful in the following. In order to formulate it we will need to introduce some notation.

**Definition 4.2.** For an event \( A \subseteq \Omega \), \( V_A \) is the set of all functions in the domain of \( \mathcal{L} \) that vanish on the event \( A \). Note that, in particular, \( \tau_A \in V_A \).

**Definition 4.3.** For an event \( A \subseteq \Omega \),

\[
\mathcal{T}_A = \sup_{0 \neq f \in V_A} \frac{\mu(f^2)}{Df}.
\]

The following proposition is given in the first equation of the proof of Theorem 2 in [4]:

**Proposition 4.4.** \( P_\mu[\tau_A > t] \leq e^{-t/\tau_A} \).

**Remark 4.5.** In particular, Proposition 4.4 implies that \( \mu(\tau_A) \leq \bar{\tau}_A \). This, however, could be derived much more simply from Corollary 4.1—

\[
\mu(\tau_A)^2 \leq \mu(\tau^2_A) \leq \mathcal{T}_A D\tau_A = \mathcal{T}_A \mu(\tau_A).
\]

Note that whenever \( \tau_A \) is not constant on \( A^c \) this inequality is strict. Thus on one hand Proposition 4.4 gives an exponential decay of \( P_\mu[\tau_A > t] \), which is stronger than the information on the expected value we can obtain from the Poisson problem in equation (4.1). On the other hand, \( \mathcal{T}_A \) could be longer than the actual expectation of \( \tau_A \).

In order to bound the hitting time from below we will formulate a variational principle that will characterize \( \tau_A \).

**Definition 4.6.** For \( f \in V_A \), let

\[
\mathcal{T}f = 2\mu(f) - Df.
\]

**Proposition 4.7.** \( \tau_A \) maximizes \( \mathcal{T} \) in \( V_A \). Moreover, \( \mu(\tau_A) = \sup_{f \in V_A} \mathcal{T}f \).
Proof. Consider \( f \in V_A \), and let \( \delta = f - \tau_A \). Using the self-adjointness of \( \mathcal{L} \), equation (4.1), and the fact that \( \delta \in V_A \) we obtain

\[
\mathcal{T} f = \mathcal{T} (\tau_A + \delta) \\
= 2 \mu (\tau_A) + 2 \mu (\delta) - \mathcal{D} \tau_A - \mathcal{D} \delta + 2 \mu (\delta \mathcal{L} \tau) \\
= \mathcal{T} \tau_A - \mathcal{D} \delta.
\]

By the positivity of the Dirichlet form, \( \mathcal{T} \) is indeed maximized by \( \tau_A \). Finally, by Corollary 4.1,

\[
\sup_{f \in V_A} \mathcal{T} f = \mathcal{T} \tau_A = 2 \mu (\tau_A) - \mathcal{D} \tau_A = \mu (\tau_A).
\]

\[\square\]

As an immediate consequence we can deduce the monotonicity of the expected hitting time:

**Corollary 4.8.** Let \( \mathcal{D} \) and \( \mathcal{D}' \) be the Dirichlet forms of two reversible Markov processes defined on the same space, such that both share the same equilibrium measure \( \mu \). We denote the expectations with respect to these processes starting at equilibrium by \( \mathbb{E}_\mu \) and \( \mathbb{E}'_\mu \). Assume that the domain of \( \mathcal{D} \) is contained in the domain of \( \mathcal{D}' \), and that for every \( f \in \text{Dom} \mathcal{D} \)

\[
\mathcal{D} f \leq \mathcal{D}' f.
\]

Then, for an event \( A \subseteq \Omega \),

\[
\mathbb{E}_\mu \tau_A \leq \mathbb{E}'_\mu \tau_A.
\]

We will now restrict ourselves to kinetically constrained models. Fix a graph \( G \) and take \( \Omega = \{0,1\}^G \). For every vertex \( x \in G \) and a state \( \eta \in \Omega \) we define a constraint \( c_x (\eta) \in \{0,1\} \). The constraint does not depend on the value at \( x \), and is non-increasing in \( \eta \). The equilibrium measure \( \mu \) is a product measure. The generator of this process, operating on a local function \( f \), is given by

\[
\mathcal{L} f = \sum_x c_x (\mu_x f - f)
\]

and its Dirichlet form by

\[
\mathcal{D} f = \mu \left( \sum_x c_x \text{Var}_x f \right).
\]

Fix a subgraph \( H \) of \( G \), and denote the complement of \( H \) in \( G \) by \( H^c \).

We will compare the dynamics of this KCM to the dynamics restricted to \( H \), with boundary conditions that are the most constrained ones.

**Definition 4.9.** The restricted dynamics on \( H \) is the KCM defined by the constraints

\[
c^H_x (\eta) = c_x (\eta^H),
\]
where, for $\eta \in \{0, 1\}^H$, $\eta^H$ is the configuration given by

$$\eta^H(x) = \begin{cases} 
\eta_x & x \in H \\
1 & x \in H^c.
\end{cases}$$

We will denote the corresponding generator by $\mathcal{L}_H$ and its Dirichlet form by $\mathcal{D}_H$.

Claim 4.10. For any $f$ in the domain of $\mathcal{L}$

$$\mathcal{D}f \geq \mu_{H^c} \mathcal{D}_H f.$$  

Proof. $c^H_x \leq c_x$ and $\text{Var}_x f$ is positive, therefore

$$\mathcal{D}f = \mu \left( \sum_x c_x \text{Var}_x f \right) \geq \mu \left( \sum_{x \in H} c^H_x \text{Var}_x f \right).$$

$\square$

The next claim will allow us to relate the spectral gap of the restricted dynamics to the variational principles discussed earlier.

Claim 4.11. Let $\gamma_H$ be the spectral gap of $\mathcal{L}_H$, and fix an event $A$ that depends only on the occupation of the vertices of $H$. Then for all $f \in V_A$

1. $\mathcal{D}f \geq \mu(A) \gamma_H (\mu f)^2$,
2. $\mathcal{D}f \geq \frac{\mu(A)}{1 + \mu(A)} \gamma_H \mu (f^2)$

Proof. First, note that $\mu_H(A) \leq \mu_H(f = 0) \leq \mu_H(|f - \mu_H f| \geq \mu_H f)$. Therefore, by Chebyshev inequality and the fact that $\mu(A) = \mu_H(A)$,

$$\mu(A) \leq \frac{\text{Var}_H f}{(\mu_H f)^2}.$$  

(4.2)

Then, Claim 4.10 implies

$$\mathcal{D}f \geq \mu_{H^c} \mathcal{D}_H f \geq \gamma_H \mu_{H^c} \text{Var}_H f \geq \mu(A) \gamma_H \mu_{H^c} (\mu_H f)^2 \geq \mu(A) \gamma_H (\mu f)^2$$

by Jensen inequality. For the second part, we use inequality 4.2

$$\text{Var}_H f \geq \mu(A) \left( \mu_H (f^2) - \text{Var}_H f \right),$$

which implies

$$\text{Var}_H f \geq \frac{\mu(A)}{1 + \mu(A)} \mu_H (f^2).$$

The result then follows by applying Claim 4.10. $\square$
5. Proof of the results

5.1. Mixed threshold bootstrap percolation on $\mathbb{Z}^2$.

5.1.1. Proof of equation (2.3). For the upper bound we will find a specific mechanism in which a cluster of empty sites could grow until it reaches the origin.

**Definition 5.1.** A square (that is, a subset of $\mathbb{Z}^2$ of the form $x + [L]^2$) is *good* if it contains at least one easy site in each line and in each column.

**Claim 5.2.** Fix $L$. The probability that a square of side $L$ is good is at least $1 - 2Le^{-\pi L}$.

**Proof.**

\[ \mathbb{P} [\text{easy site in each line}] = \left(1 - (1 - \pi)^L\right)^L \geq 1 - Le^{-\pi L}. \]

The same bound holds for $\mathbb{P} [\text{easy site in each column}]$, and then we conclude by the union bound. \hfill \Box

**Definition 5.3.** The square $[L]^2$ is *excellent* if for every $2 \leq i \leq L$ at least one of the sites in $\{i\} \times [i-1]$ is easy, and at least one of the sites in $[i-1] \times \{i\}$ is easy. For other squares of side $L$ being excellent is defined by translation.

We will use $p_L$ to denote the probability that a square of side $L$ is excellent. Note that $p_L$ depends only on $\pi$ and not on $q$.

The next two claims will show how a cluster of empty sites could propagate. See Figure 5.1.

**Claim 5.4.** Assume that $[L]^2$ is excellent, and that $(1,1)$ is initially empty. Then $[L]^2$ will be entirely emptied by time $L^2$.

**Proof.** This could be done by induction on the size of the empty square – assume that $[l]^2$ is entirely emptied for some $l \leq L$. By the definition of an excellent square, there is an easy site $x \in \{l+1\} \times [l]$. Its neighbor to the left is empty (since it is in $[l]^2$), so at the next time step this site will also be empty. Once $x$ is empty, the two sites $x \pm e_2$ could be emptied, and then the sites $x \pm 2e_2$ and so on, as long as they stay in $\{l+1\} \times [l]$. Thus, at time $l$ all sites in $\{l+1\} \times [l]$ will be empty, and by the same reasoning the sites of $[l] \times \{l+1\}$ will also be empty. Since $(l+1, l+1)$ has two empty neighbors it will be emptied at step $l+1$, and thus $[l+1]^2$ will be emptied. \hfill \Box

**Claim 5.5.** Assume that $[L]^2$ is good, and that it has a neighboring square that is entirely empty by time $T$. Then $[L]^2$ will be entirely empty by time $T + L^2$.

**Proof.** We can empty $[L]^2$ line by line (or column by column, depending on whether its empty neighbor is in the horizontal direction or the vertical one). For each line, we start by emptying the easy site that it contains, and then continue to propagate. \hfill \Box
Definition 5.6. Until the end of the proof of the upper bound, $L$ will be the minimal length for which the probability to be good exceeds $p^{sp} + 0.01$.

Definition 5.7. $C$ will denote the infinite cluster of good boxes of the form $Li + [L]^2$ for $i \in \mathbb{Z}^2$. $C_0$ will denote the cluster of the origin surrounded by a path in $C$, or just the origin if it is in $C$. $\partial C_0$ will be the outer boundary of $C_0$ (namely the boxes out of $C$ that have a neighbor in $C_0$). Note that $C_0$ is finite and that $\partial C_0$ is connected.

Claim 5.8. Assume that at time $T$ one of the boxes on $\partial C_0$ is entirely empty. Then by time $T + T_0$ the origin will be empty, where $T_0 = (|\partial C_0| + |C_0|) L^2$.

Proof. By Claim 5.5 the boundary $\partial C_0$ will be emptied by time $T_0 + L^2 |\partial C_0|$. Then, at each time step at least one site of $C_0$ must be emptied, since no finite region could stay occupied forever. □

Claim 5.9. Assume that a box $Li + [L]^2$ in $C$ is empty at time $T$. Also, assume that the graph distance in $C$ between this box and $\partial C_0$ is $l$. Then by time $T + lL^2 + T_0$ the origin will be empty.

Proof. This is again a direct application of claims 5.5 and 5.8. □
Finally, we will use the following result from percolation theory:

**Claim 5.10.** For \( l \) large enough, the number of boxes in \( C \) that are at graph distance in \( C \) at most \( l \) from \( \partial C_0 \) is greater than \( \theta l^2 \), where \( \theta \) depends only on the probability that a box is good.

**Proof.** By ergodicity the cluster \( C \) has an almost sure positive density, so in particular

\[
\liminf_{l \to \infty} \frac{|C \cap [-l, l]^2|}{|[-l, l]^2|} > 0.
\]

By \([3]\), there exists a positive constant \( \rho \) such that boxes of graph distance \( l \) from the origin must be in the box \([-\frac{1}{\rho} l, \frac{1}{\rho} l]^2\) for \( l \) large enough. Combining these two facts proves the claim. \( \square \)

This claim together with a large deviation estimate yields

**Corollary 5.11.** For \( l \) large enough, the number of excellent boxes in \( C \) that are at graph distance in \( C \) at most \( l \) from \( \partial C_0 \) is greater than \( \theta' l^2 \), where \( \theta' = 0.99 \theta p_L \).

We can now put all the ingredients together and obtain the upper bound. Fix \( c > 0 \), and \( l = \frac{c}{\sqrt{q}} \). By Corollary 5.11, for \( q \) small, there are at least \( \frac{\theta' c^2}{q} \) excellent boxes at distance smaller than \( l \) from \( \partial C_0 \). If one of them contains an empty site at the bottom left corner, the origin will be emptied by time \((l + 1) L^2 + T_0 \). For \( q \) small enough, this time is bounded by \( \frac{2cL^2}{\sqrt{q}} \).

The probability that none of them do is \((1 - q)^{\frac{\theta' c^2}{q}}\), thus it tends to 0 uniformly in \( q \) as \( c \to \infty \).

**5.1.2. Proof of equation (2.4).** The lower bound results from the simple observation, that the root could only be infected by time \( t \) if there is an empty site at distance smaller than \( t \). The probability of that event is \( 1 - (1 - q)^{4t^2} \), and taking \( t = \frac{a}{\sqrt{q}} \) and \( q \) small enough this probability is bounded by \( 1 - e^{-2a} \). This tends to 0 with \( a \) uniformly in \( q \), which finishes the proof.

**5.2. Mixed threshold KCM on \( \mathbb{Z}^2 \).**

5.2.1. **Spectral gap.** The spectral gap of this model is dominated by that of the FA2f model. Fix any \( \gamma \) strictly greater than the gap of FA2f. Then there is a local non-constant function \( f \) such that

\[
\frac{D_{\text{FA2f}} f}{\text{Var } f} \leq \gamma,
\]

where \( D_{\text{FA2f}} \) is the Dirichlet form of the FA2f model.

Since \( f \) is local, it is supported in some square of size \( L \times L \), for \( L \) big enough. \( \nu \)-almost surely it is possible to find a far away square in \( \mathbb{Z}^2 \) of size \( L \times L \) that contains only difficult sites. By translation invariance of the FA2f model we can assume that this is the square in which \( f \) is supported. In this case, \( D f = D_{\text{FA2f}} f \), and this shows that indeed the gap of the model with random threshold is smaller than that of FA2f, which by \([11]\) is bounded by \( e^{-c/q} \).
5.2.2. Proof of equation (2.5). In this part we will use Corollary 4.1 in order to bound \( \tau_0 \) by a path argument. As in the proof of the upper bound for the bootstrap percolation, we will consider the good squares (see Definition 5.1) and their infinite cluster. In fact, by Claim 5.2 by choosing \( L \) big enough we may assume that the box \([L]^2\) is in this cluster. Let us fix this \( L \) until the end of this part. We will also choose an infinite self avoiding path of good boxes starting at the origin and denote it by \( i_0, i_1, i_2, \ldots \). Note that this path depends on \( \omega \) but not on \( \eta \).

On this cluster empty sites will be able to propagate, and the next definition will describe the seed needed in order to start this propagation.

**Definition 5.12.** A box in \( \mathbb{Z}^2 \) is essentially empty if it is good and contains an entire line or an entire column of empty sites. This will depend on both \( \omega \) and \( \eta \).

In order to guarantee the presence of an essentially empty box we will fix \( l = q^{-L-1} \), and define the bad event

**Definition 5.13.** \( B = \{ \text{none of the boxes } i_0, \ldots, i_l \text{ is essentially empty} \} \). For fixed \( \omega \) the path \( i_0, i_1, i_2, \ldots \) is fixed, and \( B \) is an event in \( \Omega \).

A simple bound shows that

\[
\mu(B) \leq \left(1 - q^L \right)^l \leq e^{-\frac{1}{q}}.
\]  

(5.1)

We can use this bound in order to bound the hitting time at \( B \):

**Claim 5.14.** There exists \( C > 0 \) such that \( \mathbb{P}_\mu(\tau_B \leq t) \leq C e^{-\frac{1}{q}} t \).

**Proof.** We use the graphical construction of the Markov process. In order to hit \( B \), we must hit it at a certain clock ring taking place in one of the sites of \( \cup_{n=1}^{L} (Li + [L]^2) \). Therefore,

\[
\mathbb{P}(\tau_B \leq t) \leq \mathbb{P} \left[ \text{more than } 2 (2L + 1)^2 lt \text{ rings by time } t \right] + 2 (2L + 1)^2 lt \mu(B)
\leq e^{-(2L+1)^2q^{-L-1}t} + 2 (2L + 1)^2 q^{-L-1} t e^{-\frac{1}{q}} \leq C e^{-\frac{1}{q}} t.
\]

In order to bound \( \tau_0 \) we will study the hitting time of \( A = B \cup \{ \eta(0) = 0 \} \).

**Lemma 5.15.** Fix \( \eta \in \Omega \). Then there exists a path \( \eta_0, \ldots, \eta_N \) of configurations and a sequence of sites \( x_0, \ldots, x_N-1 \) such that

(1) \( \eta_0 = \eta \),
(2) \( \eta_N \in A \),
(3) \( \eta_{n+1} = \eta_i^{x_n} \),
(4) \( c_{x_n}(\eta_i) = 1 \),
(5) \( N \leq 4L^2 l \),
For all \( i \leq N \), \( \eta_i \) differs from \( \eta \) at at most 3L points, contained in at most two neighboring boxes.

Proof. If \( \eta \in A \), we take the path \( \eta \) with \( N = 0 \). Otherwise \( \eta \in B^c \), so there is an essentially empty box in \( i_0, \ldots, i_l \), which contain an empty column (or row). We can then create an empty column (row) next to it and propagate that column (row) as in Figure 5.2. When the path rotates we can rotate this propagating column (row) as show in Figure 5.3. \( \square \)

We can use this path together with Corollary 4.1 in order to bound \( \tau_A \).

**Lemma 5.16.** There exists \( C_L > 0 \) (that may depends on \( L \) but not on \( q \)) such that \( \mu (\tau_A) \leq C_L q^{-5L-2} \).

Proof. Since \( \tau \) vanishes on \( A \), taking the path defined in Lemma 5.15

\[
\tau_A (\eta) = \sum_{i=0}^{N-1} (\tau_A (\eta_i) - \tau_A (\eta_{i+1})).
\]

In the following we use the notation

\[
\nabla_x \tau_A (\eta) = \tau_A (\eta) - \tau_A (\eta^x).
\]
Then, by Cauchy Schwartz inequality,
\[ \mu (\tau_A)^2 \leq \mu (\tau_A^2) = \sum_{\eta} \mu (\eta) \left( \sum_{i=0}^{N-1} \nabla x_i \tau_A (\eta_i) \right)^2 \]
\[ \leq \sum_{\eta} \mu (\eta) N \sum_i c_{x_i} (\eta_i) (\nabla x_i \tau_A (\eta_i))^2 \]
\[ = \sum_{\eta} \mu (\eta) N \sum_i \sum_z c_z (\eta') (\nabla z \tau_A (\eta'))^2 \mathbb{1}_{z=x_i} \mathbb{1}_{\eta'=\eta_i}. \]

By property number 6 of the path, we know that
\[ \mu (\eta') \leq q^{-3L} \mu (\eta'), \] so we obtain
\[ \mu (\tau_A)^2 \leq q^{-3L} N \sum_{\eta'} \mu (\eta') \sum_z c_z (\eta') (\nabla z \tau_A (\eta'))^2 \mathbb{1}_{z=x_i} \sum_{\eta} \mathbb{1}_{\eta'=\eta_i}. \]

Still using property 6, \( \eta \) differs from \( \eta' \) at at most \( 3L \) point, all of them in the box containing \( z \) or in one of the two neighboring boxes. This gives the bound \( \sum_{\eta} \mathbb{1}_{\eta'=\eta'} \leq (3L^2)^{3L} \). Finally, bounding \( \mathbb{1}_{z=x_i} \) by 1,
\[ \mu (\tau_A)^2 \leq q^{-3L} (3L^2)^{3L} N^2 \sum_{\eta'} \mu (\eta') \sum_z c_z (\eta') (\nabla z \tau_A (\eta'))^2 \]
\[ \leq 16 (3L^2)^{3L} L^4 q^{-5L-2} D\tau_A. \]

This concludes the proof of the lemma by Corollary 4.1.

Using this lemma and the bound on \( \tau_B \) in Claim 5.14, we can finish the estimation of the upper bound. By Markov inequality
\[ \mu (\tau_A \geq C_L q^{-5L-3}) \leq q. \]

On the other hand, by Claim 5.14
\[ \mu (\tau_A < C_L q^{-5L-5}) \leq \mu (\tau_0 < C_L q^{-5L-3}) + \mu (\tau_B < C_L q^{-5L-3}) \]
\[ \leq \mu (\tau_0 < C_L q^{-5L-3}) + C'_L e^{-1/q}. \]

Therefore
\[ \mu (\tau_0 \geq C_L q^{-5L-3}) \leq q + C'_L e^{-1/q}, \]
and taking \( \alpha = 5L + 3 \) will suffice.

Concerning Remark 2.3, fix \( L_0 \) such that the probability that \([L_0]^2\) is good exceeds \( p^{SP} \). Then, for \( \alpha_0 = 5L_0 + 3, \nu (\alpha \leq \alpha_0) > 0. \) We will see later that the other inequality holds as well for the same \( \alpha_0. \)

5.2.3. Proof of equation (2.6). A trivial bound could be obtained by taking any \( \alpha < 1, \) since the rate at which the origin becomes empty is always at most \( q. \) We will, however, look for a bound
that will describe better the effect of the disorder, and will allow us to prove Remark 2.3. The basic observation for the estimation of this lower bound is that if $[-L, L]^2$ is initially occupied, and if it contains only difficult sites, then at some point we will need to empty at least $\frac{L}{2}$ sites before the origin could be emptied. This energy barrier forces $\tau_A$ to be greater than $q^{-L/2}$.

In the following we will fix $L$ such that $[-L, L]^2$ contains only difficult sites (so it is not the same $L$ we have used for the upper bound).

**Definition 5.17.** For a rectangle $R$, the span of $R$ are the sites that could be emptied with the bootstrap percolation using only the $0$s of $R$. If the span of $R$ equals $R$ we say that $R$ is internally spanned.

The next fact is a consequence of the fact a set which is stable under the bootstrap percolation must be a rectangle [2].

**Fact 5.18.** The span of a rectangle $R$ is a union of internally spanned rectangles.

**Definition 5.19.** For $x \in \mathbb{Z}^2$, let $G_x$ be the event that the origin is in the span of $[-L, L]^2$ for $\eta$, but not for $\eta^x$. $G_x$ is defined as the intersection of $G_x$ with the event \{$c_x = 1$\}.

First, note that legal flips of sites in the interior of a rectangle or outside the rectangle cannot change its span. Therefore, $G_x = \emptyset$ for $x$ which is not on the inner boundary of $[-L, L]^2$.

**Claim 5.20.** Fix $x$ on the inner boundary of $[-L, L]^2$, and let $\eta \in G_x$. Then $x$ and the origin belong to the same internally spanned rectangle.

**Proof.** Recalling Fact 5.18, we consider the internally spanned rectangle containing the origin. If it didn’t contain $x$, the origin would be in the span of $[-L, L]^2$ also for $\eta^x$, contradicting the definition of $G_x$. \[\square\]

**Corollary 5.21.** Fix $x$ on the inner boundary of $[-L, L]^2$. Then $\mu(G_x) \leq \left(\frac{L^2}{L/2}\right)q^{L/2}$.

**Proof.** Assume without loss of generality that $x$ is on the right boundary. Then there must be an internally spanned rectangle in $[-L, L]^2$ whose width is at least $L$. Since all sites of $[-L, L]^2$ are difficult, it cannot contain two consequent columns that are entirely occupied, therefore it must contain at least $\frac{L}{2}$ empty sites. \[\square\]

We can use the same argument as in the proof of Claim 5.14. Defining $G = \cup_x G_x$, this argument will tell us that the hitting time $\tau_G$ is bigger than $Cq^{-L/2+1}$ with probability that tends to 1 as $q \to 0$, for some constant $C$ that depends on $L$. If we start with a configuration for which the origin is not in the span of $[-L, L]^2$, it could only be emptied after $\tau_G$ – at the first instant in which the span of $[-L, L]^2$ includes the origin, $G_x$ must occur for the site that has just been flipped. Since
the probability to start with an entirely occupied \([-L, L]^2\) tends to 1 as \(q \to 0\), equation (2.6) is satisfied for \(\alpha = \frac{L}{2} - 1\).

In order to bound also the expected value of \(\tau_0\) we will use Proposition 4.7. Let us consider the function

\[ f = 1 \text{ is not in the span of } [-L, L]^2. \]

We can bound its Dirichlet form using Corollary 5.21:

\[
\mathcal{D} f = \mu \left( \sum_x c_x \text{Var}_x f \right) \leq \mu \left( \sum_x c_x q^1 G_x \right) \\
\leq q 16L \left( \frac{L^2}{L/2} \right)^{q^{1/2}} = C_L q^{L/2+1}.
\]

The expected value is bounded by the probability that all sites are occupied –

\[
\mu f \geq (1 - q)^{(2L+1)^2}.
\]

Now consider for some \(\lambda \in \mathbb{R}\) the rescaled function \(\overline{f} = \lambda f\).

\[
\mathcal{T} \overline{f} = 2\mu \overline{f} - \mathcal{D} \overline{f} \\
\geq 2\lambda (1 - q)^{(2L+1)^2} - \lambda^2 C_L q^{L/2+1}.
\]

The optimal choice of \(\lambda\) is \(\lambda = \frac{(1-q)^{(2L+1)^2}}{C_L} q^{-L/2-1}\), which yields

\[
\mathcal{T} \overline{f} \geq \frac{(1-q)^{(2L+1)^2}}{C_L} q^{-L/2-1}.
\]

Proposition 4.7 and the fact that \(\overline{f}\) vanishes on \(\{\eta_0 = 0\}\) implies that \(\mu (\tau_0) \geq C_L' q^{-L/2-1}\), and therefore \(\mathbb{E}_\mu (\tau_0) \geq q^{-\alpha_0}\) for \(q\) small enough.

Concerning Remark 2.3 note that for every \(\alpha\)

\[
\nu (\alpha \geq \alpha) \geq \nu (L \geq 2\alpha + 4) = (1 - \pi)^{(4\alpha+9)^2}.
\]

In particular for \(\alpha_0\) defined in the proof of the upper bound \(\nu (\alpha \geq \alpha_0) > 0\).

5.3. **Mixed threshold models on** \(\mathbb{Z}^d\). The argument above, for the case of \(\mathbb{Z}^2\), works also in more general settings, as long as the probability to be easy (i.e., threshold 1) is strictly positive.

The lower bound of the bootstrap percolation is immediate, since only at scale \(q^{-1/d}\) it is possible to find an easy site. The lower bound for the kinetically constrained model could be analyzed similarly to the the two dimensional case using the methods of [6], but in order to keep things simple we could take \(\pi < 1\), which suffices since the rate at which the origin becomes empty is always at most \(q\).
For the upper bound of both the bootstrap percolation and the kinetically constrained model we need to construct a path that will empty the origin. First, note that we may assume that sites have either threshold 1 (easy) or threshold $d$ (difficult) with probabilities $\pi$ and $1 - \pi$, for some $\pi > 0$. In this case the path is described explicitly in [24]. It is constructed for the FA$df$ model, but we will only need to adapt the definitions there in order to take into account the easy sites.

Fix $L$ that will be equal $n$ defined in the beginning of subsection 5.1 of [24], replacing $q$ by $\pi$ and taking $\epsilon$ such that good boxes (see Definition 5.23 that follows) percolate, and the origin belongs to the infinite cluster. We then consider, just as before, an infinite path of good boxes starting at the origin.

**Definition 5.22.** The easy bootstrap percolation will be the threshold $d$ bootstrap percolation defined on $\mathbb{Z}^d$, with initial conditions in which easy sites are empty and difficult sites are filled. A set $V \in \mathbb{Z}^d$ is *easy internally spanned* if it is internally spanned for this process.

**Definition 5.23.** A *good* box $V = x + [L]^d$ is a box for which the event $G_1$ in Definition 5.4 of [24] occurs, replacing “internally spanned” by “easy internally spanned”.

**Definition 5.24.** As *excellent* box is a box that, by adding a single easy site at its corner, will be easy internally spanned. $p_L$ will be the probability that the box $[L]^d$ is excellent. Note that (as for the two dimensional case) $p_L$ is nonzero, and that it does not depend on $q$.

**Definition 5.25.** An *essentially empty* box $V = x + [L]^d$ will correspond to the event $G_2$ in Definition 5.4 of [24] – it is a good box in which the first slice in any direction is empty.

Being good and being excellent are events measurable with respect to the disorder $\omega$, whereas being essentially empty depends also on the configuration of the empty and filled sites. The definition of the bad event $B$ remains the same as in the previous section, and taking $l = q^{-dL^{d-1}}$ its probability has the same decay.

With these definitions, replacing “supergood” by “essentially empty”, the proof in section 5 of [24] shows how to propagate an essentially empty box along the path of good boxes, corresponding to figures 5.2 and 5.3 in the two dimensional case. We may then consider a path as the one of Lemma 5.15. That is, for any configuration $\eta$ there is a path $\eta_0, \ldots, \eta_N$ with flips $x_0, \ldots, x_{N-1}$ such that

1. $\eta_0 = \eta$,
2. $\eta_N \in A$,
3. $\eta_{i+1} \in \eta_i^c$,
4. $c_{x_i}(\eta_i) = 1$,
5. $N \leq cL^dl$ for some $c > 0$. 


(6) For all $i \leq N$, $\eta_i$ differs from $\eta$ at at most $cL^{d-1}$ points, contained in at most two neighboring boxes.

Applying the exact same argument as for the two dimensional case yields the upper bounds.

5.4. Mixed North-East and FA1f.

5.4.1. Spectral gap. This is the same argument as for the previous model – one can always find arbitrarily large regions of difficult sites, so the gap is bounded by that of the north-east model. Since for the parameters that we have chosen the north-east model is not ergodic, it has 0 gap [11].

5.4.2. Hitting time. Let $A$ be the event $\{\eta_0 = 0\}$. Recall Definition 4.3 and let $\tau = \tau_A$.

The exponential tail of $\tau_0$ is a consequence of Proposition 4.4, so we are left with proving that $\nu(\tau \geq t) \leq t^{-c\log q}$ for some constant $c$. We will do that by choosing a subgraph on which we can estimate the gap, and then apply Claim 4.11.

Since $\pi$ is greater than the critical probability for the Bernoulli site percolation, there will be an infinite cluster of easy sites $C$. We denote by $C_0$ the cluster of the origin surrounded by a path in $C$. $\partial C_0$ will be the outer boundary of $C_0$, i.e., the sites in $C$ that have a neighbor in $C_0$. Then, we fix a self avoiding infinite path of easy sites $v_0, v_1, \ldots$ starting with the sites of $\partial C_0$. That is, $v_0, \ldots, v_{|\partial C_0|}$ is a path that encircles $C_0$, and then $v_{|\partial C_0|+1}, \ldots$ continues to infinity. We will denote $V = \{v_i\}_{i \in \mathbb{N}}$. Let $H = V \cup C_0$, and consider the restricted dynamics $L_H$ introduced in Definition 4.9. We split the dynamics in two – for some local function $f$ on $H$

$$L_H f = L_{C_0}^0 f + L_V^V f,$$

$$L_V^V = \sum_{i \in \mathbb{N}} c_{v_i}^H (\mu_{v_i} f - f),$$

$$L_{C_0}^0 = \sum_{x \in C_0} c_x^H (\mu_x f - f).$$

Note that the boundary conditions of the $C_0$ dynamics depend on the state of the vertices in $V$ and vice versa. We will denote by $L_{C_0}^0$ the $C_0$ dynamics with empty boundary conditions and by $L_V^V$ the $V$ dynamics with occupied boundary conditions. All generators come with their Dirichlet forms carrying the same superscript and subscript.

We will bound the gap of $L_H$ using the gaps of $L_V^V, L_{C_0}^0$ and the following block dynamics:

$$L_b^b f = (\mu_V (f) - f) + 1_{\partial C_0 \text{ is empty}} (\mu_C f - f).$$

Denote the spectral gaps of $L_V^V, L_{C_0}^0, L_b^b, L_H$ by $\gamma_V^V, \gamma_{C_0}^0, \gamma_b^b, \gamma_H$.

By Proposition 4.4 of [11]:
Claim 5.26.
\[ \gamma^b = 1 - \sqrt{1 - q|\partial C_0|}, \]
i.e., \( \text{Var} f \leq \frac{1}{1 - \sqrt{1 - q|\partial C_0|}} D^b f \) for any local function \( f \).

Let us now use this gap in order to relate \( \gamma_H \) to \( \gamma^V \) and \( \gamma^C_0 \):

Claim 5.27.
\[ \gamma_H \geq \gamma^b \min \{ \gamma^V_1, \gamma^C_0 \}. \]

Proof. Fix a non-constant local function \( f \).
\[
\begin{align*}
\text{Var} f &\leq \frac{1}{\gamma^b} D^b f = \frac{1}{\gamma^b} [\mu (\text{Var} f) + \mu (1_{\partial C_0 \text{ is empty}} \text{Var} f)] \\
&\leq \frac{1}{\gamma^b} \left[ \frac{1}{\gamma^V_1} \mu (D^V_1 f) + \frac{1}{\gamma^C_0} \mu (1_{\partial C_0 \text{ is empty}} D^C_0 f) \right] \\
&\leq \frac{1}{\gamma^b} \max \left\{ \frac{1}{\gamma^V_1}, \frac{1}{\gamma^C_0} \right\} D_H f.
\end{align*}
\]

We are left with estimating \( \gamma^V_1 \) and \( \gamma^C_0 \).

Claim 5.28. There exists \( C > 0 \) such that \( \gamma^V_1 \geq Cq^3 \).

Proof. The Dirichlet form \( D^V_1 \) is dominated by the Dirichlet form of \( FA1f \) on \( \mathbb{Z}_+ \), and that dynamics has spectral gap which is proportional to \( q^3 \) (see [11]). \( \square \)

For \( \gamma^C_0 \) we will use the bisection method, comparing the gap on a box to that of a smaller box. For \( L \in \mathbb{N} \), let \( L^{\text{NE}}_k \) be the generator of the north-east dynamics in the box \([L]^2\) with empty boundary (for the north east model this is equivalent to putting empty boundary only above and to the right). Denote its gap by \( \gamma^\text{NE}_{[L]^2} \). By monotonicity we can restrict the discussion to this dynamics, i.e.,
\[
\gamma^C_0 \geq \gamma^\text{NE}_{\text{diam} C_0}. \tag{5.2}
\]

We will now bound \( \gamma^\text{NE}_{[L]^2} \) (see also Theorem 6.16 of [11]).

Claim 5.29. \( \gamma^\text{NE}_{[L]^2} \geq e^{3\log q L} \).

Proof. We will prove the result for \( L_k = 2^k \) by induction on \( k \). Then monotonicity will complete the argument for all \( L \). Consider the box \([L_k]^2\), and divide it in two rectangles \(- R_- = [L_{k-1}] \times [L_k] \) and \( R_+ = [L_{k-1} + 1, L_k] \times [L_k] \). We will run the following block dynamics
\[
L^{\text{NE}} f = (\mu_{R_+} f - f) + 1_{\partial_- R_+ \text{ is empty}} (\mu_{R_-} f - f),
\]
where $\partial_- R_+$ is the inner left boundary of $R_+$. Again, by Proposition 4.4 of [11],

$$\text{gap} \left( L^{\text{bNE}} \right) = 1 - \sqrt{1 - \mu \left( 1_{\partial_- R_+ \text{ is empty}} \right)} = 1 - \sqrt{1 - q^{L_k}}.$$ 

Therefore for every local function $f$

$$\text{Var} f \leq \frac{1}{1 - \sqrt{1 - q^{L_k}}} D^{\text{bNE}} f = \frac{1}{1 - \sqrt{1 - q^{L_k}}} \mu \left( \text{Var}_{R_+} f + 1_{\partial_- R_+ \text{ is empty}} \text{Var}_{R_-} f \right) \leq \frac{1}{1 - \sqrt{1 - q^{L_k}}} \mu \left( \frac{1}{\gamma^{\text{NE}}_{R_+}} D^{\text{NE}}_{R_+} f + \frac{1}{\gamma^{\text{NE}}_{R_-}} D^{\text{NE}}_{R_-} f \right),$$

where $\gamma^{\text{NE}}_{R}, D^{\text{NE}}_{R}$ are the spectral gap and Dirichlet form of the north-east dynamics in $R$ with empty boundary conditions for any fixed rectangle $R$. We see that

$$\gamma^{\text{NE}}_{[L_k]^2} \geq \left( 1 - \sqrt{1 - q^{L_k}} \right) \gamma^{\text{NE}}_{[L_{k-1}] \times [L_k]}.$$ 

If we repeat the same argument dividing $[L_{k-1}] \times [L_k]$ into the rectangles $[L_{k-1}] \times [L_{k-1}]$ and $[L_{k-1}] \times [L_{k-1} + 1, L_k]$, we obtain

$$\gamma^{\text{NE}}_{[L_{k-1}] \times L_k} \geq \left( 1 - \sqrt{1 - q^{L_{k-1}}} \right) \gamma^{\text{NE}}_{[L_{k-1}]^2}.$$ 

Hence,

$$\log \gamma^{\text{NE}}_{[L_k]^2} \geq \log \gamma^{\text{NE}}_{[L_{k-1}]^2} + 2^k \log q - \log 4,$$

yielding

$$\log \gamma^{\text{NE}}_{[L_k]^2} \geq \log q \sum_{n=1}^k 2^n - k \log 4$$

which finishes the proof. \hfill \Box

We can now put everything together. Let $L$ be the diameter of $C_0$. By the second part of Claim 4.11

$$\tau \leq \frac{1 + q}{q} \frac{1}{\gamma^r} \leq \frac{1 + q}{q} \frac{1}{ \left( 1 - \sqrt{1 - \gamma^r} \right) \min \{ \gamma^V_{1}, \gamma^C_{C_0} \}} \leq q^{-4L-1}. \tag{5.3}$$

Finally, we will use the sharpness of the phase transition for the site percolation on the dual graph (see [11][12]):
Claim 5.30. There exists a positive constant $c_2$ that depends on $\pi$ such that $\nu(L \geq D) \leq e^{-c_2 D}$ for any $D \in \mathbb{N}$.

Using this claim and equation (5.3)

$$\nu(\tau \geq t) \leq \nu(q^{-4L-1} \geq t) \leq \nu\left(L \geq \frac{\log t}{4 \log \frac{1}{q}} - \frac{1}{4}\right) \leq C t^{c/\log q}.$$ 

6. Conclusions and further questions

We have seen here two simple examples of KCMs in random environments. These examples show that when the environment has some rare remote “bad” regions the relaxation time fails to describe the true observed time scales of the system. Since the dynamics is not attractive, also techniques such as monotone coupling and censoring cannot be applied to these models. In order to overcome this difficulty we considered the hitting time $\tau_0$, which on one hand describes a physically measurable observable, and on the other hand could be studied using variational principles. We formulated some tools based on these variational principles and used them in order to understand the behavior of $\tau_0$ in both models.

For future research, one may try to apply these tools on kinetically constrained models in more types of random environment, such as the polluted lattice, more general mixture of constraints, and models on random graphs.

There are also some questions left open considering the models studied here. For the first model, it is natural to conjecture that $\tau_0$ scales as $q^{-\alpha}$ for some random $\alpha$. We can also look at the $\pi$ dependence of $\alpha$ – we know that when there are not many easy sites this time should become larger, until it reaches the FA2f time when $\pi = 0$. Looking at the proof of Theorem 2.2, we can see that $\tau$ scales like $\frac{\log^{1/2} \pi}{\pi}$, and $\alpha$ scales like $\pi^{-1/2}$. It seems more likely, however, that the actual exponent $\alpha$ behaves like $\frac{1}{\pi}$ – the $\log \frac{1}{q}$ of the lower bound comes up also in the proof bounding the gap of the FA2f model [24], and also there it is conjectured that it does not appear in the true scaling. In fact, if we use the path that we have chosen in order to bound $\tau_0$ also for the bootstrap percolation, we will have the same $\log \frac{1}{q}$ factor, and in this case it is known that it does not appear in the correct scaling. The lower bound of $\pi^{-1/2}$ could be improved, with the price of complicating the proof. In the proof we assume that $[-L, L]^2$ contains only difficult sites. If we require instead that enough lines and columns are entirely difficult (but not necessarily the entire square), we can obtain a bound that scales as $\frac{1}{\pi}$. It is worth noting here that for the bootstrap percolation, by repeating the arguments of [2, 18] with some minor adaptations, we can show that the scaling of the prefactor $a$ with $\pi$ is between $e^{c/\pi}$ and $e^{C/\pi}$ for $c, C > 0$. 
Conjecture 6.1. For the mixed threshold FA model, $\nu$-almost surely the limit $\lim_{q \to 0} \frac{\log \tau_0}{\log \frac{1}{q}}$ exists. Its value $\alpha$ is a random variable whose law depends on $\pi$. Moreover, the law of $\pi \alpha$ converges (in some sense) to a non-trivial law as $\pi$ tends to 0.

The mixed North-East and FA1f model also raises many questions, among them finding the critical probability $q_c$, and characterizing the behavior of both the bootstrap percolation and the KCM in the different parameter regimes.

ACKNOWLEDGMENTS

I would like to thank Cristina Toninelli and Fabio Martinelli for the helpful discussions and useful advice. I acknowledge the support of the ERC Starting Grant 680275 MALIG.

REFERENCES

[1] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.
[2] Michael Aizenman and Joel L Lebowitz. Metastability effects in bootstrap percolation. *Journal of Physics A: Mathematical and General*, 21(19):3801, 1988.
[3] Peter Antal and Agoston Pisztora. On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.*, 24(2):1036–1048, 1996.
[4] Amine Asselah and Paolo Dai Pra. Quasi-stationary measures for conservative dynamics in the infinite lattice. *Ann. Probab.*, 29(4):1733–1754, 2001.
[5] Paul Balister, Béla Bollobás, Michał Przykucki, and Paul Smith. Subcritical $U$-bootstrap percolation models have non-trivial phase transitions. *Trans. Amer. Math. Soc.*, 368(10):7385–7411, 2016.
[6] József Balogh, Béla Bollobás, Hugo Duminil-Copin, and Robert Morris. The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.*, 364(5):2667–2701, 2012.
[7] József Balogh and Boris G Pittel. Bootstrap percolation on the random regular graph. *Random Structures & Algorithms*, 30(1-2):257–286, 2007.
[8] Béla Bollobás, Hugo Duminil-Copin, Robert Morris, and Paul Smith. Universality of two-dimensional critical cellular automata, to appear in *Proc. London Math. Soc.*, 2016.
[9] Béla Bollobás, Karen Gunderson, Cecilia Holmgren, Svante Janson, and Michał Przykucki. Bootstrap percolation on Galton-Watson trees. *Electronic Journal of Probability*, 19, 2014.
[10] Béla Bollobás, Paul Smith, and Andrew Uzzell. Monotone cellular automata in a random environment. *Combin. Probab. Comput.*, 24(4):687–722, 2015.
[11] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli. Kinetically constrained spin models. *Probab. Theory Related Fields*, 140(3-4):459–504, 2008.
[12] Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Comm. Math. Phys.*, 343(2):725–745, 2016.
[13] Glenn H Fredrickson and Hans C Andersen. Kinetic Ising model of the glass transition. *Physical review letters*, 53(13):1244, 1984.
[14] Juan P Garrahan, Peter Sollich, and Cristina Toninelli. *Kinetically constrained models*, volume 150. International Series of Monographs on Physics, 2011.
[15] Janko Gravner and Alexander E Holroyd. Polluted bootstrap percolation with threshold two in all dimensions. *arXiv preprint arXiv:1705.01652*, 2017.

[16] Janko Gravner and Elaine McDonald. Bootstrap percolation in a polluted environment. *Journal of Statistical Physics*, 87(3):915–927, 1997.

[17] Ivailo Hartarsky. U-bootstrap percolation: critical probability, exponential decay and applications. *arXiv preprint arXiv:1806.11405*, 2018.

[18] Alexander E Holroyd. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probability Theory and Related Fields*, 125(2):195–224, 2003.

[19] Svante Janson. On percolation in random graphs with given vertex degrees. *Electronic Journal of Probability*, 14:86–118, 2009.

[20] Svante Janson, Tomasz Łuczak, Tatyana Turova, and Thomas Vallier. Bootstrap percolation on the random graph $G_{n,p}$. *Ann. Appl. Probab.*, 22(5):1989–2047, 2012.

[21] George Kordzakhia and Steven P. Lalley. Ergodicity and mixing properties of the northeast model. *J. Appl. Probab.*, 43(3):782–792, 2006.

[22] Laure Marêché, Fabio Martinelli, and Cristina Toninelli. Exact asymptotics for duarte and supercritical rooted kinetically constrained models. *arXiv preprint arXiv:1807.07519*, 2018.

[23] Fabio Martinelli, Robert Morris, and Cristina Toninelli. Universality results for kinetically constrained spin models in two dimensions. *Communications in Mathematical Physics*, pages 1–49, 2018.

[24] Fabio Martinelli and Cristina Toninelli. Towards a universality picture for the relaxation to equilibrium of kinetically constrained models. *arXiv preprint arXiv:1701.00107*, 2016.

[25] F. Ritort and P. Sollich. Glassy dynamics of kinetically constrained models. *Advances in Physics*, 52(4):219–342, 2003.

[26] M Schulz and E Donth. Glass transition in a modified fredrickson model. *Journal of non-crystalline solids*, 168(1-2):186–194, 1994.

[27] JF Willart, M Tetaert, and M Descamps. Evidence of a dynamical length scale in the f-spin frustrated kineticising model. *Journal of Physics A: Mathematical and General*, 32(48):8429, 1999.