Classification of simple weight modules with finite-dimensional weight spaces over the Schrödinger algebra

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Abstract
We classify simple weight modules with finite-dimensional weight spaces over the (centrally extended complex) Schrödinger algebra in (1 + 1)-dimensional space-time. Our arguments use the description of lowest weight modules by Dobrev, Doebner and Mrugalla; Mathieu’s twisting functors and results of Wu and Zhu on dimensions of weight spaces in dense modules.

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1. Introduction and preliminaries

The Schrödinger Lie group is the group of symmetries of the free particle Schrödinger equation. The (centrally extended) Lie algebra $\mathcal{S}$ of this group in the case of (1+1)-dimensional space-time is called the Schrödinger algebra, see [1, 2]. The algebra $\mathcal{S}$ has basis $\{f, q, h, z, p, e\}$ and the Lie bracket is given as follows:

\[
\begin{align*}
[h, e] &= 2e, & [h, p] &= p, & [h, f] &= -2f, \\
[e, q] &= p, & [e, p] &= 0, & [e, f] &= h, \\
[p, f] &= -q, & [f, q] &= 0, & [h, q] &= -q, \\
[p, q] &= z, & [z, \mathcal{S}] &= 0.
\end{align*}
\]

(1)

From this we see that $e$, $f$ and $h$ generate an $\mathfrak{sl}_2$-subalgebra of $\mathcal{S}$.

In this paper we study so-called weight modules over $\mathcal{S}$, that is $\mathcal{S}$-modules which are diagonalizable over the Cartan subalgebra $\mathfrak{h}$ of $\mathcal{S}$ spanned by $h$ and $z$. Put differently, these are modules which may be written as a sum of simultaneous eigenspaces for $h$ and $z$, so-called weight spaces. The goal
of this paper is to classify, up to isomorphism, all simple weight $S$-modules with finite-dimensional weight spaces.

As usual, $S$-modules are the same as modules over the universal enveloping algebra $U$ of $S$. By the Poincaré-Birkhoff-Witt theorem (see, e.g., [3, p. 156-160]), $U$ is a unital and associative algebra with basis

$$\{q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}\}_{i_1, \ldots, i_6 \in \mathbb{N}},$$

where $\mathbb{N}$ denotes the set of nonnegative integers.

Schur’s lemma says that the central element $z$ acts as a scalar on each simple $S$-module $V$, which is called the central change of $V$. Hence the weight spaces of a simple module are, in fact, precisely the eigenspaces of $h$. For a module $V$ on which $z$ acts as a scalar we denote by $\text{supp}(V)$ the set of $h$-eigenvalues on $M$ and we will call these eigenvalues weights. All $h$-eigenvectors will be called weight vectors. For a weight $\lambda$ we denote by $V_\lambda$ the corresponding weight space.

The weights of a weight module $V$ are weakly ordered as follows: $\lambda_1 \leq \lambda_2$ if and only if $\lambda_2 - \lambda_1$ is a nonnegative integer. If there is a maximal element, $\lambda$, in $\text{supp}(V)$, then $\lambda$ is called a highest weight of $V$, and any $v \in V_\lambda$ is called a highest weight vector. If $V$ is generated by a highest weight vector, then it is called a highest weight module. Lowest weights, lowest weight vectors and lowest weight modules are defined analogously. We have the following standard fact (cf. [4, Lemma 1.15]).

**Lemma 1.** Let $V$ be a weight $U$-module with $\lambda \in \text{supp}(V)$. Then the following hold:

$$f : V_\lambda \to V_{\lambda-2}, \quad q : V_\lambda \to V_{\lambda-1}, \quad h : V_\lambda \to V_\lambda, \quad z : V_\lambda \to V_\lambda, \quad p : V_\lambda \to V_{\lambda+1}, \quad e : V_\lambda \to V_{\lambda+2}.$$

Those $S$-modules on which $p$ and $q$ (and therefore also $z$) act as 0 are $\mathfrak{sl}_2$-modules, and vice versa, so a classification of these modules is well-known, see, e.g., [4, p. 72].

In the next section we recall known partial results from [1, 2] on classification of simple weight $U$-modules with finite-dimensional weight spaces. In Section 3 we formulate our main result, Theorem 5. In Section 4 we discuss our main tool, namely Mathieu’s twisting functors from [5]. Finally, Theorem 5 is proved in Section 5.
2. Known partial results

A classification and explicit description of all simple lowest weight $S$-modules (on which either $p$ or $q$, or both, act nonzero) was given by Dobrev et al. in [1]. Classification and explicit description of simple highest weight $S$-modules follows easily. As usual, simple highest weight modules are unique simple quotients of Verma modules, that is $S$-modules induced from one-dimensional modules over the subalgebra spanned by $f$, $q$, $h$ and $z$, on which both $f$ and $q$ act as 0. Dobrev et al. explicitly compute maximal submodules in Verma modules. Here we recall their result in a slightly different version which is necessary for our arguments. In the following proposition, as well as the rest of this paper, we for a set, $X$, we denote by $X^*$ the set $X \setminus \{0\}$.

**Proposition 2.** For $\lambda \in \mathbb{C}\backslash(-\frac{1}{2}+\mathbb{N})$ and $c \in \mathbb{C}^*$, let $M(\lambda, c)$ be the $U$-module with basis $\{v_{i,j}\}_{i,j\in \mathbb{N}}$ on which the action of $U$ is given by

\[
qv_{i,j} = v_{i+1,j}; \\
f_{i,j} = v_{i,j+1}; \\
zv_{i,j} = cv_{i,j}; \\
hv_{i,j} = (\lambda - i - 2j)v_{i,j}; \\
pv_{i,j} = -jv_{i+1,j-1} + cv_{i-1,j}; \\
ev_{i,j} = j(\lambda + 1 - i - j)v_{i,j-1} + \frac{1}{2}ci(i - 1)v_{i-2,j}.
\]

For $\lambda \in -\frac{1}{2} + \mathbb{N}$ and $c \in \mathbb{C}^*$, denote by $N(\lambda, c)$ the $U$-module of $M(\lambda, c)$ with basis $\{v_{i,j}\}_{i,j\in \mathbb{N}, j\leq \lambda + \frac{1}{2}}$ on which the action of $U$ is given by

\[
qv_{i,j} = v_{i+1,j}; \\
f_{i,j} = \begin{cases} v_{i,j+1}, & \text{if } j < \lambda + \frac{1}{2}; \\ -\sum_{s=0}^{\lambda + \frac{1}{2}} \frac{1}{(2c)^{\lambda + \frac{1}{2} - s}}(\alpha + \frac{1}{2})v_{i+2\lambda+3-2s,s}, & \text{if } j = \lambda + \frac{1}{2}; \end{cases} \\
zv_{i,j} = cv_{i,j}; \\
hv_{i,j} = (\lambda - i - 2j)v_{i,j}; \\
pv_{i,j} = -jv_{i+1,j-1} + cv_{i-1,j}; \\
ev_{i,j} = j(\lambda + 1 - i - j)v_{i,j-1} + \frac{1}{2}ci(i - 1)v_{i-2,j}.
\]

Then every simple highest weight $U$-module with finite-dimensional weight spaces on which either $p$ or $q$ or both act nonzero is isomorphic to precisely one module of the form $M(\lambda, c)$ or $N(\lambda, c)$.
Proof. This result follows easily from [1, Theorem 1]. Consider the Lie algebra $S(1)$ defined in [1]. It is readily checked that the map $\psi : S(1) \rightarrow S$ given by

$$
D \mapsto -h, \quad P_t \mapsto -e, \quad P_x \mapsto p, \quad K \mapsto f, \quad G \mapsto -q, \quad m \mapsto -z
$$

extends to a Lie algebra isomorphism (here $m$ is identified with the scalar with which it acts on the modules in question). Now the statement follows by pushing forward [1, Theorem 1] along $\psi$.

The simple weight $U$-modules with finite-dimensional weight spaces which now remain to be classified are those on which either $p$ or $q$ or both act nonzero and, furthermore, which have neither a highest nor a lowest weight. Let us denote the class of such modules by $\mathcal{N}$.

Theorem 4 below is due to Wu and Zhu, see [2]. It describes dimensions of weight space for modules in $\mathcal{N}$. The proof, though omitted here, requires the following lemma, which we will use.

Lemma 3. (a) For $s \in \{p, q, e, f\}$ the adjoint action of $s$ on $U$ is locally nilpotent.

(b) Let $V$ be a $U$-module, $v \in V$, $u, s \in U$ and $n, m \in \mathbb{N}$. Assume that $\text{ad}_n^s u = 0$. If $s^m v = 0$, then $s^{nm} uv = 0$.

(c) Let $s \in \{p, q, e, f\}$ and $V$ be a simple $U$-module. If the kernel of $s$ on $V$ is nonzero, then $s$ acts on $V$ locally nilpotently.

Proof. For $m \in \mathbb{N}$ let $U_m$ denote the linear span in $U$ of all standard monomials $q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}$ such that $i_1 + \cdots + i_6 \leq m$. Then $U_m$ is finite-dimensional and stable under the adjoint action of $s$. Moreover, $U_m$ is also stable under the adjoint action of $h$. At the same time, the adjoint action of $h$ on $U_m$ is certainly diagonalizable (as all monomials are eigenvectors for this action). Since $U_m$ is finite-dimensional, the adjoint action of $h$ on $U_m$ has only finitely many eigenvalues. Now claim (a) follows from Lemma [1].

To prove claim (b), let $X$ denote the linear subspace of $U$ spanned by all elements of the form $s^i u s^j$, $i, j \in \mathbb{N}$. The identity $\text{ad}_s^n u = 0$ can be written as $s^n u = u' s$ for some $u' \in X$ which implies that for any $x \in X$ there is $x' \in X$ such that $s^n x = x' s$. By induction, it follows that for any $x \in X$ there is $x' \in X$ such that $s^{nm} x = x' s^m$. Therefore there is $u' \in X$ such that $s^{nm} u v = u' s^m v = 0$, proving claim (b).
As a simple module is generated by any nonzero element, claim (c) follows from claims (a) and (b).

**Theorem 4.** Let \( L \in \mathcal{N} \) and \( \lambda \in \text{supp}(L) \). Then \( \text{supp}(L) = \lambda + \mathbb{Z} \) and \( \dim(L_{\lambda+i}) = \dim(L_{\lambda+j}) \) for all \( i, j \in \mathbb{Z} \).

3. Main result

The following theorem is the main result of this paper.

**Theorem 5.** For \( \lambda \in -\frac{1}{2} + \mathbb{N} \), \( c \in \mathbb{C}^* \) and \( x \in \mathbb{C} \) with \( x \neq 0 \) and \( 0 \leq \text{Re}(x) < 1 \), let \( B_x^{(q)}(N(\lambda, c)) \) be the vector space with basis \( \{ v_{i,j} \}_{i,j \in \mathbb{Z}, 0 \leq j \leq \lambda + \frac{1}{2}} \). Then, setting

\[
\begin{align*}
q v_{i,j} &= v_{i+1,j}; \\
f v_{i,j} &= \begin{cases} v_{i,j+1}, & \text{if } j < \lambda + \frac{1}{2}; \\
-\sum_{s=0}^{\lambda + \frac{1}{2}} \frac{1}{(2c)^{\lambda + \frac{1}{2}-s}} (\lambda + \frac{1}{2}) v_{i+2\lambda+3-2s,s}, & \text{if } j = \lambda + \frac{1}{2}; \\
z v_{i,j} &= cv_{i,j}; \\
h v_{i,j} &= (\lambda - (i + x) - 2j)v_{i,j}; \\
p v_{i,j} &= -j v_{i+1,j-1} + c(i + x)v_{i-1,j}; \\
e v_{i,j} &= j (\lambda + 1 - (i + x) - j)v_{i,j-1} + \frac{1}{2}c(i + x)((i + x) - 1)v_{i-2,j};
\end{cases}
\]

(3)

defines on \( B_x^{(q)}(N(\lambda, c)) \) the structure of a \( U \)-module. Moreover, every module in \( \mathcal{N} \) is isomorphic to precisely one module of this form.

The idea of the proof is the following: Consider the localization \( U^{(f)} \) of \( U \) with respect to \( f \), which is equipped with a family of Mathieu’s twisting functors as in [5]. We will show that every module in \( \mathcal{N} \) can be “twisted” into a module which has a highest weight subquotient. Reversing the procedure we get that every module in \( \mathcal{N} \) is (up to isomorphism) a twisted highest weight module. At this point, however, it is not clear how to handle possible redundancy issues. We will show that on all module in \( \mathcal{N} \) the action of \( z \) is nonzero. This information will allow us to repeat the entire procedure, with the difference that we now localize with respect to \( q \) instead. This time redundancy is easily dealt with, so that the classification can be completed.

We may now state the promised classification.

**Corollary 6.** Every simple weight \( U \)-module with finite-dimensional weight spaces is either:
• A dense \( \mathfrak{sl}_2 \)-module, see [4, p. 72].
• A highest weight \( \mathcal{U} \)-module, see Proposition 2.
• A lowest weight \( \mathcal{U} \)-module, see [1, Theorem 1].
• Of the form \( B_z^{(2)}(N(\lambda, c)) \), as defined in Theorem 5.

4. Mathieu’s twisting functors

In this section we give all details on Mathieu’s twisting functors from [5], our main technical tool.

Let \( u \in \{q, f\} \). Then, by Lemma 3, \( \text{ad}_u \) is locally nilpotent on \( \mathcal{U} \). Now by Lemma 4.2 in [5], \( q \) and \( f \) satisfy Ore’s localizability conditions and hence we can consider the corresponding localization \( \mathcal{U}(u) \). We have \( U \subset \mathcal{U}(u) \) is a ring extension in which \( u \) is invertible, moreover, every element of \( \mathcal{U}(u) \) can be written on the form \( u^{-n}s \), for some \( s \in U \) and \( n \in \mathbb{N} \). For \( \mathcal{U}(u) \) we have the following analogue to the Poincaré-Birkhoff-Witt theorem.

**Proposition 7.** The set \( \{q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}\}_{i_1 \in \mathbb{Z} \text{ and } i_2, \ldots, i_6 \in \mathbb{N}} \) is a basis for \( \mathcal{U}(q) \), and \( \{q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}\}_{i_2 \in \mathbb{Z} \text{ and } i_1, i_3, \ldots, i_6 \in \mathbb{N}} \) is a basis for \( \mathcal{U}(f) \).

**Proof.** Recall that, by the Poincaré-Birkhoff-Witt theorem,

\[
\{q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}\}_{i_1, \ldots, i_6 \in \mathbb{N}}
\]

is a basis for \( U \). By the definition of localization,

\[
\{q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}\}_{i_1 \in \mathbb{Z} \text{ and } i_2, \ldots, i_6 \in \mathbb{N}}
\]

spans \( \mathcal{U}(q) \). Next, assume that we have linearly dependent different elements

\[
u_1, \ldots, u_n \in \{q^{i_1} f^{i_2} p^{i_3} e^{i_4} h^{i_5} z^{i_6}\}_{i_1 \in \mathbb{Z} \text{ and } i_2, \ldots, i_6 \in \mathbb{N}}
\]

so that \( a_1 u_1 + \cdots + a_n u_n = 0 \), for some \( 0 \neq a_1, \ldots, a_n \in \mathbb{C} \). Then for some \( m \in \mathbb{N} \), the elements \( q^m u_1, \ldots, q^m u_n \) are different basis elements of \( U \) and \( a_1 q^m u_1 + \cdots + a_n q^m u_n = 0 \), a contradiction.

Since \( q \) and \( f \) commute, a similar argument works for \( \mathcal{U}(f) \) as well. \( \square \)
Proposition 8. For $x \in \mathbb{C}$ the assignment
\[
\Theta_x^{(q)}(q^{\pm 1}) = q^{\pm 1}, \quad \Theta_x^{(q)}(f) = f,
\]
\[
\Theta_x^{(q)}(z) = z, \quad \Theta_x^{(q)}(h) = h - x,
\]
\[
\Theta_x^{(q)}(p) = p + xq^{-1}z, \quad \Theta_x^{(q)}(e) = e + xq^{-1}p + \frac{1}{2}x(x - 1)q^{-2}z
\]
extends uniquely to an automorphism $\Theta_x^{(q)} : U^{(q)} \rightarrow U^{(q)}$ and the assignment
\[
\Theta_x^{(f)}(q) = q, \quad \Theta_x^{(f)}(f^{\pm 1}) = f^{\pm 1},
\]
\[
\Theta_x^{(f)}(z) = z, \quad \Theta_x^{(f)}(h) = h - 2x,
\]
\[
\Theta_x^{(f)}(p) = p - xq^{-1}, \quad \Theta_x^{(f)}(e) = e + x(h - 1 - x)f^{-1}
\]
extends uniquely to an automorphism $\Theta_x^{(f)} : U^{(f)} \rightarrow U^{(f)}$.

PROOF. The uniqueness is clear as $U^{(q)}$ is generated by $\{q^{\pm 1}, f, z, h, p, e\}$ and $U^{(f)}$ is generated by $\{q, f^{\pm 1}, z, h, p, e\}$. Hence we prove existence.

Let $u \in \{q, f\}$ and assume that $x \in \mathbb{N}$. We claim that in this case Formulae (4) and (5) correspond to restriction (to generators) of the conjugation automorphism $a \mapsto u^{-x}au^{x}$ of $U^{(u)}$. To prove this we proceed by induction on $x$. The base $x = 0$ is immediate. Let us check the induction step from $x = k$ to $x = k + 1$: For $u = q$ and $s = h$ we have
\[
q^{-1}\Theta_k^{(q)}(h)q = q^{-1}(h - k)q = q^{-1}(hq - kq) = q^{-1}(qh - q - kq) = h - (k + 1) = \Theta_{k+1}^{(q)}(h).
\]

For $u = q$ and $s = p$ we have
\[
q^{-1}\Theta_k^{(q)}(p)q = q^{-1}(p + kq^{-1}z)q = q^{-1}px + kq^{-1}z = q^{-1}(pq + z) + kq^{-1}z = p + q^{-1}z + kq^{-1}z = p + (k + 1)q^{-1}z = \Theta_{k+1}^{(q)}(p).
\]

For $u = q$ and $s = e$ we have
\[
q^{-1}\Theta_k^{(q)}(e)q = q^{-1}(e + kq^{-1}p + \frac{1}{2}k(k - 1)q^{-2}z)q = q^{-1}(eq + kq^{-1}pq + \frac{1}{2}k(k - 1)q^{-1}z) = q^{-1}(qe + p + kq^{-1}(qp + z)) + k(k - 1)q^{-2}z = e + q^{-1}p + kq^{-1}p + kq^{-2}z + \frac{1}{2}k(k - 1)q^{-2}z = e + (k + 1)q^{-1}p + \frac{1}{2}(k + 1)(k + 1 - 1)q^{-2}z = \Theta_{k+1}^{(q)}(e).
\]
All other cases for \( u = q \) are obvious. For \( u = f \) and \( s = h \) we have

\[
f^{-1}\Theta_k^{(f)}(h)f = f^{-1}(h-2k)f = f^{-1}(hf - 2kf) = f^{-1}(f h - 2f - 2kf) = h - 2(k + 1) = \Theta_{k+1}^{(f)}(h).
\]

For \( u = f \) and \( s = p \) we have

\[
f^{-1}\Theta_k^{(f)}(p)f = f^{-1}(p - qf - 1)f = f^{-1}pf - kf^{-1}qf^{-1}f = f^{-1}(fp - q) - kf^{-1}q = p - (k + 1)qf^{-1} = \Theta_{k+1}^{(f)}(p).
\]

For \( u = f \) and \( s = e \) we have

\[
f^{-1}\Theta_k^{(f)}(e)f = f^{-1}(e + k(h - 1) - k)f^{-1}f = f^{-1}(ef + k(h - 1 - k)) = f^{-1}(fe + h + k(h - 1 - k)) = e + f^{-1}(k + 1)(h - k)ff^{-1} = e + f^{-1}f(k + 1)(h - 2 - k)ff^{-1} = e + (k + 1)(h - 1 - (k + 1))ff^{-1} = \Theta_{k+1}^{(f)}(e).
\]

To show that \( \Theta_x^{(u)} \), given by extending Formulae 4 and 5 linearly and multiplicatively, defines an endomorphism of \( U^{(u)} \) for general \( x \in \mathbb{C} \), it suffices to show that each \( \Theta_x^{(u)} \) preserves Lie brackets of the generators of \( U^{(u)} \), i.e. that \( \Theta_x^{(u)} ([s_1, s_2]) = \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2) - \Theta_x^{(u)}(s_2)\Theta_x^{(u)}(s_1) \) for all \( s_1, s_2 \in \{q, f, z, h, p, e, u^{-1}\} \). Let \( s_1, s_2 \in \{q, f, z, h, p, e, u^{-1}\} \). Using Formulae 4 and 5 we can write

\[
Q_{s_1, s_2} := \Theta_x^{(u)}([s_1, s_2]) - \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2) - \Theta_x^{(u)}(s_2)\Theta_x^{(u)}(s_1)
\]

in the form \( \sum_{i \in F} g_i(x)b_i \) where \( F \) is a finite subset of the standard basis of \( U^{(u)} \) and each \( g_i(x) \) is a polynomial in \( x \). From the above, we have \( g_i(x) = 0 \) for all \( x \in \mathbb{N} \). Hence all \( g_i = 0 \) and \( Q_{s_1, s_2} = 0 \).

Finally, from the next proposition we have that \( \Theta_x^{(u)} \circ \Theta_{-x}^{(u)} = \Theta_0^{(u)} = \text{Id} \), implying that \( \Theta_x^{(u)} \) is invertible and thus an automorphism.

**Proposition 9.** For all \( x, y \in \mathbb{C} \) and \( u \in \{q, f\} \) we have \( \Theta_x^{(u)} \circ \Theta_y^{(u)} = \Theta_{x+y}^{(u)} \).

**Proof.** This is a consequence of (4) and (5) applied to the generators of \( U^{(u)} \). The only non-immediate case is to check that \( \Theta_x^{(u)} \circ \Theta_y^{(u)}(e) = \Theta_{x+y}^{(u)}(e) \).
This is done by the following direct calculation for \( u = q \) and \( u = f \), respectively:

\[
\Theta^{(q)}_x \circ \Theta^{(q)}_y (e) = \Theta^{(q)}_x (e + yq^{-1}p + \frac{1}{2}y(y - 1)q^{-2}z) \\
= e + xq^{-1}p + \frac{1}{2}x(x - 1)q^{-2}z + yq^{-1}(p + xq^{-1}z) + \frac{1}{2}y(y - 1)q^{-2}z \\
= e + (x + y)q^{-1}p + \frac{1}{2}(x + y)(x + y - 1)q^{-2}z \\
= \Theta^{(q)}_{x+y} (e),
\]

and

\[
\Theta^{(f)}_x \circ \Theta^{(f)}_y (e) = \Theta^{(f)}_x (e + y(h - 1 - y)f^{-1}) \\
= e + x(h - 1 - x)f^{-1} + y(h - 2x - y - 1)f^{-1} \\
= e + (x + y)(h - 1 - x - y)f^{-1} \\
= \Theta^{(f)}_{x+y} (e).
\]

Now we can define Mathieu’s twisting functors from \([3]\) in our situation. Denote by \( U\)-Mod the category of all \( U\)-modules and their morphisms. Let \( u \in \{q, f\} \) and \( x \in \mathbb{C} \). Then the twisting functor

\[
B^{(u)}_x : U\text{-Mod} \to U\text{-Mod}
\]

is defined as composition of the following three functors:

(i) the induction functor \( \text{Ind}^{(u)}_U := U^{(u)} \otimes_U - \),

(ii) twisting the \( U^{(u)}\)-action by \( \Theta^{(u)}_x \),

(iii) the restriction functor \( \text{Res}^{(u)}_U \).

More explicitly, for \( V \in U\)-Mod, then the module \( B^{(u)}_x (V) \) is isomorphic to \( U^{(u)}_x \otimes_U V \), where \( U^{(u)}_x \) is a \( U \)-bimodule obtained as follows: it coincides with the algebra \( U^{(u)} \) as a vector space, the right action is given by multiplication \( p \cdot a := pa \), and the left action is given by multiplication twisted by \( \Theta^{(u)}_x \), that is \( a \cdot p := \Theta^{(u)}_x (a)p \) (here \( p \in U^{(u)} \) and \( a \in U \)).
Lemma 10. Let $V$ be a $U$-module on which $u \in \{q, f\}$ acts bijectively, and let $W$ be a $U^{(u)}$-module. Then

(a) $\text{Ind}^{U^{(u)}}_U(\text{Res}^{U^{(u)}}_U(W)) \cong W$.

(b) $\text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V)) \cong V$.

Proof. Since $u$ acts bijectively on both $V$ and $\text{Res}^{U^{(u)}}_U(W)$, the linear map $v \mapsto 1 \otimes v$ defines an isomorphism from $V$ or $W$ to $\text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V))$ or $\text{Ind}^{U^{(u)}}_U(\text{Res}^{U^{(u)}}_U(W))$, respectively. \hspace{1cm} \square

The next result facilitates the composition and inversion of twisting functors when applied to $\mathcal{N}$.

Proposition 11. Let $x, y \in \mathbb{C}$, $u \in \{q, f\}$. Then we have the following:

(a) $B^{(u)}_x \circ B^{(u)}_y \cong B^{(u)}_{x+y}$.

(b) If $V \in U\text{-Mod}$ and $u$ acts bijectively on $V$, then $B^{(u)}_0(V) \cong V$.

Proof. By definition, for $V \in U\text{-Mod}$ we have natural isomorphisms

\begin{align*}
B^{(u)}_x \circ B^{(u)}_y(V) &= \text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(\text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V))))_x \\
&\cong \text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V))_x \\
&\cong \text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V))_{x+y} \\
&= B^{(u)}_{x+y}(V),
\end{align*}

where the first isomorphism follows from Lemma 10, and the second from Proposition 9. This implies claim (a).

Again by definition, we have

\begin{align*}
B^{(u)}_0(V) &= \text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V))_0 \\
&\cong \text{Res}^{U^{(u)}}_U(\text{Ind}^{U^{(u)}}_U(V)) \\
&\cong V,
\end{align*}

where the second isomorphism follows from Lemma 10, proving claim (b). \hspace{1cm} \square

The modules $B^{(q)}_x(N(\lambda, c))$ are described explicitly in the following proposition.
Proposition 12. For $\lambda \in -\frac{1}{2} + \mathbb{N}$, $x \in \mathbb{C}$ and $c \in \mathbb{C}^*$, the $U$-module $B_x(q)(N(\lambda, c))$ has basis $\{v_{i,j}\}_{i,j \in \mathbb{Z}, 0 \leq j \leq \lambda + \frac{1}{2}}$, where $v_{i,j} = q^{ij}f^{ij} \otimes_U v$, and $v$ is a highest weight vector of $N(\lambda, c)$. The action of $U$ in this basis is given by:

$$
q^{i,j} = v_{i+1,j};
$$

$$
f^{i,j} = \begin{cases} v_{i,j+1}, & \text{if } j < \lambda + \frac{1}{2}; \\
-\sum_{s=0}^{\lambda+\frac{1}{2}} \frac{1}{(2c)^{\lambda+\frac{3}{2}-s}} (\lambda+\frac{3}{2}) v_{i+2\lambda+3-2s,s}, & \text{if } j = \lambda + \frac{1}{2};
\end{cases}
$$

$$
z^{i,j} = cv_{i,j};
$$

$$
h^{i,j} = (\lambda - (i + x) - 2j)v_{i,j};
$$

$$
p^{i,j} = -jv_{i+1,j-1} + c(i + x)v_{i-1,j};
$$

$$
e^{i,j} = j(\lambda + 1 - (i + x) - j)v_{i,j-1} + \frac{1}{2}c(i + x)((i + x) - 1)v_{i-2,j}.
$$

Proof. Using definitions, Proposition 7 and the fact that $z, h, p$ and $e$ all act like scalars on $v$, we get that the vector space $B_x(q)(N(\lambda, c))$ is spanned by $\{v_{i,j}\}_{i,j \in \mathbb{Z}, j \in \mathbb{N}}$. For $j > \lambda + \frac{1}{2}$, however, $v_{i,j} = q^{ij}f^{ij-\lambda-\frac{3}{2}}(fv_{0,\lambda+\frac{1}{2}})$, which is a linear combination of elements $v_{i,j'}$ such that $0 \leq j' < j$. So, by induction, $B_x(q)(N(\lambda, c))$ is spanned by $\{v_{i,j}\}_{i,j \in \mathbb{Z}, 0 \leq j \leq \lambda + \frac{1}{2}}$ as well.

The elements of the latter set are linearly independent because if for some $m \in \mathbb{N}$ and $c_m, \ldots, c_m \in \mathbb{C}$ we would have $\sum_{|i|<m, 0 \leq j \leq \lambda + \frac{1}{2}} c_i v_{i,j} = 0$, then it would follow that

$$
0 = \sum_{|i|<m, 0 \leq j \leq \lambda + \frac{1}{2}} c_i v_{i+m,j} = \sum_{|i|<m, 0 \leq j \leq \lambda + \frac{1}{2}} c_i (1 \otimes q^{i+m} f^{i} v) = 1 \otimes \sum_{|i|<m, 0 \leq j \leq \lambda + \frac{1}{2}} c_i q^{i+m} f^{i} v.
$$

This is impossible as $v_{i+m,j}$ are linearly independent for $|i| < m, 0 \leq j \leq \lambda + \frac{1}{2}$.

Let us determine the action of $U$ on the subspace of $B_0(q)(N(\lambda, c))$ generated by $\{v_{i,j}\}_{i,j \in \mathbb{N}}$. This is clearly a submodule isomorphic to $N(\lambda, c)$ with the action given by (2) (which coincides with evaluation of (3) at $x = 0$).

On the subset $\{v_{i,j}\}_{i,j \in \mathbb{N}}$ of $B_x(q)(N(\lambda, c))$, an element $s \in U$ acts, by definition, as $\Theta_x(q)(s)$ acts on $\{v_{i,j}\}_{i,j \in \mathbb{N}} \subset \text{Ind}_U^{U(q)}(N(\lambda, c))$. This latter action is explicitly described in Proposition 8 and is easily seen to be given by (3). The nontrivial cases $s \in \{h, p, e\}$ are dealt with by direct calculations:

$$
Theta_x(q)(h)v_{i,j} = (h-x)v_{i,j} = (\lambda - i - 2j - x)v_{i,j} = (\lambda - (i + x) - 2j)v_{i,j},
$$
\[
\Theta_{x}^{(q)}(p)v_{i,j} = (p + xq^{-1}z)v_{i,j} = -jv_{i+1,j-1} + (ci + xc)v_{i-1,j} = -jv_{i+1,j-1} + c(i+x)v_{i-1,j},
\]

\[
\Theta_{x}^{(q)}(e)v_{i,j} = (e + xq^{-1}p + \frac{1}{2}x(x-1)q^{-2})v_{i,j} =
\]

\[
= \frac{1}{2}ci(i-1)v_{i-2,j} + j(\lambda + 1 - j - i)v_{i,j-1} + x(-jv_{i,j-1} + c iv_{i-2,j}) + \frac{1}{2}x(x-1)cv_{i-2,j} =
\]

\[
= \frac{1}{2}c(i(i-1) + x(x-1) + 2ix)v_{i-2,j} + j(\lambda + 1 - j - i - x)v_{i,j-1} =
\]

\[
= \frac{1}{2}c(i(x) + (i + x) - 1)v_{i-2,j} + j(\lambda + 1 - j - (i + x))v_{i,j-1}.
\]

Now, note that \(sv_{i,j} = q^i\Theta_{x}^{(q)}(s)v_{0,j}\), from which the action of \(U\) on \(v_{i,j}\) with \(i < 0\) is directly computable. Finally, the action of \(U\) on \(v_{i,j}\) for general \(x \in \mathbb{C}\) and \(i \in \mathbb{Z}\) can now be calculated similarly to when \(i \geq 0\).

5. Proof of main result

The next few results will enable us to relate the modules in \(\mathcal{N}\) to twisted highest weight modules.

Lemma 13. Let \(V\) be a simple weight \(U\)-module with finite-dimensional weight spaces.

(a) If \(e\) acts locally nilpotently on \(V\), then \(V\) is a highest weight module. If \(f\) acts locally nilpotently on \(V\), then \(V\) is a lowest weight module.

(b) Assume, additionally, that \(z\) acts like some \(c \in \mathbb{C}^*\) on \(V\). If \(p\) acts locally nilpotently on \(V\), then \(V\) is a highest weight module. If \(q\) acts locally nilpotently on \(V\), then \(V\) is a lowest weight module.

Proof. We start with claim (a). We will consider this claim for the element \(e\), the other part of the claim is similar. Assume that \(V\) is not a highest weight module. Then \(V\) is not a lowest weight module either, for otherwise \(e\) and \(f\) would both act locally nilpotently on \(V\) and hence \(V\) would be a direct sum of finite-dimensional modules when restricted to \(\mathfrak{sl}_2\). As \(V\) has finite-dimensional weight spaces, this would mean that \(V\) is finite-dimensional, hence highest weight, a contradiction.

Therefore \(V\) is either a dense simple \(\mathfrak{sl}_2\)-module as in [4, Section 3.3] or is in \(\mathcal{N}\). However, \(e\) does not act locally nilpotently on simple dense \(\mathfrak{sl}_2\)-module,
so $V$ is in $\mathcal{N}$. By Theorem 4, we have $\text{supp}(V) = \lambda + \mathbb{Z}$ for some $\lambda \in \mathbb{C}$ and all nonzero weight spaces of $V$ have the same dimension. By [5, Lemma 3.3], $V$ has finite length as an $\mathfrak{sl}_2$-module. The only simple weight $\mathfrak{sl}_2$-modules on which $e$ acts locally nilpotently are highest weight modules, therefore, as an $\mathfrak{sl}_2$-module, $V$ has a finite filtration with subquotients being highest weight modules. Therefore $V$ must have a highest weight, a contradiction. This proves claim (a).

Now we prove claim (b). Again we prove it for the element $p$, the other part is similar. Take any nonzero weight vector $v \in V$ such that $pv = 0$. Let $\lambda$ be the weight of $v$. By claim (a) we may assume that the action of $e$ on $V$ is injective. Since $e$ and $p$ commute, we have that for every $i \in \mathbb{N}$ the element $v_i := e^i v$ is nonzero and satisfies $pv_i = 0$.

Next we observe that the action of $q$ on $V$ is injective for otherwise it would be locally nilpotent and then $q^k v = 0$ for some $k$. The linear span of $v, qv, q^2 v, \ldots, q^{k-1} v$ is then a finite-dimensional space stable under the action of both $q$ and $p$. As $[p, q] = z$ commutes with $p$, by the Kleinecke-Shirokov Theorem it follows that the only eigenvalue of $z$ is zero, which contradicts our assumption on the action of $z$.

Finally we claim that $\{q^{2i} v_i \}_{i \in \mathbb{N}}$ is an infinite set of linearly independent elements. We use that $p^{2k} q^{2i} v_i = (\prod_{j=0}^{2k-1} (2i - j)) c^{2k} q^{2(i-k)} v_i$ is zero if and only if $k > i$, where $k \in \mathbb{N}_{>0}$. From this we see that any two elements $q^{2i_1} v_{i_1}$ and $q^{2i_2} v_{i_2}$, where $i_1 \neq i_2$, are distinct, hence follows the first claim. As for the second claim, assume it is false and let $\sum_{i=0}^{k} a_i q^{2i} v_i = 0$ be a nontrivial relation of minimal possible $q$-degree $2k > 0$ (so in particular $a_k \neq 0$). But applying $p^{2k}$ we obtain $(\prod_{j=0}^{2k-1} (2k - j)) c^{2k} a_k v_k = 0$, a contradiction.

As a result, we have infinitely many linearly independent elements of weight $\lambda$, a contradiction. □

**Corollary 14.** Let $L \in \mathcal{N}$. Then both $e$ and $f$ act bijectively on $L$. If in addition $z$ acts nonzero on $L$, then also both $p$ and $q$ act bijectively on $L$.

**Proof.** From Lemma 13 it follows that the actions of $e$, $f$, $p$ and $q$ on $L$ are not locally nilpotent. By Lemma 3 these actions are thus injective. By Theorem 4 these actions restrict to injective actions between finite-dimensional vector spaces of the same dimension. Therefore they all are bijective. □

**Proposition 15.** Let $V$ be a weight $U$-module such that $\text{supp}(V) \subset \lambda + \mathbb{Z}$ for some $\lambda$ and $\sup_{i \in \mathbb{Z}} \dim V_{\lambda+i} < \infty$. Assume that there is some $0 \neq v \in V$ such that $ev = 0$ or $pv = 0$. Then $V$ has a simple highest weight submodule.
Proof. By assumption,

\[ W_e := \{ v \in V | e^n v = 0 \text{ for some } n \in \mathbb{N} \} \neq 0 \]

or

\[ W_p := \{ v \in V | p^n v = 0 \text{ for some } n \in \mathbb{N} \} \neq 0. \]

By Lemma 3, \( W_e \) and \( W_p \) are submodules. Note that \( V \) has finite length (already as an \( \mathfrak{sl}_2 \)-modules by Lemma 3.3). Let \( W \) be a simple submodule of \( W_e \) or \( W_p \). By Lemma 13, \( W \) is a highest weight module.

Lemma 16. Let \( L \in \mathcal{N} \).

(a) There is \( x \in \mathbb{C} \) and \( 0 \neq v \in B_x^{(f)}(L) \) such that \( ev = 0 \).

(b) Assume that \( z \) does not act like \( 0 \) on \( L \). Then there is \( x \in \mathbb{C} \) and \( 0 \neq v \in B_x^{(q)}(L) \) such that \( pv = 0 \).

Proof. Let us first show that, for a non-constant polynomial \( g(x) \) and \( n \times n \)-matrices \( A \) and \( B \), which \( B \) invertible, there is some \( x \in \mathbb{C} \) such that \( A + g(x)B \) is not invertible. It suffices to prove existence of an \( x \) such that \( \det(A + g(x)B) = 0 \). By Leibniz’ determinant formula,

\[
\det(A + g(x)B) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} (A + g(x)B)_{i,\sigma_i}
\]

\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} (A_{i,\sigma_i} + g(x)B_{i,\sigma_i})
\]

\[
= g(x)^n \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} B_{i,\sigma_i} + r(x)
\]

\[
= \det(B)g(x)^n + r(x),
\]

where \( r(x) \) is some polynomial of degree strictly smaller than that of \( g(x)^n \). Since \( \det(B) \neq 0 \), the claim follows from the fundamental theorem of algebra.

To prove claim (a) it suffices to show there is \( x \in \mathbb{C} \) such that \( e \) does not act injectively on \( B_x^{(f)}(L) \). By the definition of \( B_x^{(f)} \) and (5), it is equivalent to show that for some \( x \in \mathbb{C} \), the element \( e + x(h - 1 - x)f^{-1} \) does not act injectively on \( \text{Ind}_{U}^{G}(L) \), which, as follows from Corollary 14, is isomorphic to \( L \) as a vector space. Let \( \lambda \in \text{supp}(L) \), and fix some bases in \( L_{\lambda} \) and...
$L_{\lambda + 2}$. By Theorem 4, these spaces have the same (finite) dimension, say $n$, so $e|_{L_{\lambda}}$ and $f|_{L_{\lambda + 2}}$ are given by $n \times n$-matrices, $E$ and $F$, respectively, with $F$ invertible. Moreover, $h|_{L_{\lambda}} = \lambda$. It now suffices to show that for some $x \in \mathbb{C}$, the matrix $E + x(\lambda - 1 - x)F^{-1}$ is not invertible. This follows from the previous paragraph and proves claim (iii).

Similarly, to prove (b) we have to show, under the assumption that $z$ acts like $c \neq 0$, that there is some $x \in \mathbb{C}$ such that the operator $\Theta_{x}^{(q)}(p) = p - xzq^{-1}$ does not act injectively on $\text{Ind}_{U}^{U_{L}}(L)$. Let the actions $p|_{L_{\lambda}}$ and $q|_{L_{\lambda + 1}}$ be given by some $n \times n$-matrices, $P$ and $Q$, respectively. Then $Q$ is invertible and we need to show that for some $x \in \mathbb{C}$ the matrix $P - xcQ^{-1}$ is not invertible. This follows again from the first part of the proof.

**Proposition 17.** Let $L \in \mathcal{N}$. There are $x \in \mathbb{C}$, $c \in \mathbb{C}^*$ and $\lambda \in -\frac{1}{2} + \mathbb{N}$ such that $L \cong B_{-x}^{(f)}(N)$.

**Proof.** For any $x \in \mathbb{C}$, one sees from Proposition 8 that $B_{x}^{(f)}(L)$ is a weight module, the weight spaces of which have the same, finite, dimension as those of $L$ (all weight spaces of $L$ have the same dimension by Theorem 4). By Lemma 16, there is $x \in \mathbb{C}$ and $0 \neq v \in B_{x}^{(f)}(L)$ such that $ev = 0$. Then, by Proposition 15, there is a simple highest weight submodule $N$ of $B_{x}^{(f)}(L)$. Then the module $B_{x}^{(f)}(N)$ is a submodule of $B_{-x}^{(f)}(B_{x}^{(f)}(L))$, which, by Proposition 11 and Corollary 14, is isomorphic to $L$. But $L$ is simple and hence $L \cong B_{-x}^{(f)}(N)$.

Also we know from Proposition 2 that one of the following cases holds:

(i) $pN = 0 = qN$.

(ii) $N \cong M(\lambda, c)$ for some $\lambda \in \mathbb{C} \setminus (-\frac{1}{2} + \mathbb{N})$ and $c \in \mathbb{C}^*$.

(iii) $N \cong N(\lambda, c)$ for some $\lambda \in -\frac{1}{2} + \mathbb{N}$ and $c \in \mathbb{C}^*$.

Now, in the first case, $p$ and $q$ would act on $L \cong B_{-x}^{(f)}(N)$ like $\Theta_{-x}^{(f)}(p)$ and $\Theta_{-x}^{(f)}(q)$, respectively, act on $\text{Ind}_{U}^{U_{L}}(N)$. From Proposition 8 we see that these elements would act like zero, which contradicts $L \in \mathcal{N}$. In the second case, the dimensions of the weight spaces in $N$ are unbounded, so the same is true for $B_{-x}^{(f)}(N) \cong L$, contradicting $L \in \mathcal{N}$ because of Theorem 4. Therefore the third alternative must hold, so that $L \cong B_{-x}^{(f)}(N(\lambda, c))$. □
Corollary 18. On an arbitrary $L \in \mathcal{N}$, $z$ does not act like 0.

Proof. By Proposition 17, we have $L \cong B_{x}^{(f)}(N(\lambda, c))$ for certain $x, c, \lambda \in \mathbb{C}$ where $c \neq 0$ (so that $z$ does not act like 0 on $N(\lambda, c)$). The action of $z$ is not affected by $B_{x}^{(f)}$ which yields the claim. □

Proof of Theorem 5. Let $L \in \mathcal{N}$. By Corollary 18 we know that the action of $z$ on $L$ is nonzero. Now we can apply the same argument as in the proof of Proposition 17, but with $q$ instead of $f$ and $p$ instead of $e$, and with case (i) now ruled out because otherwise $z$ would act like 0 on $N$, and therefore on $B_x^{(q)}(N)$ as well. This gives that $L \cong B_x^{(q)}(N(\lambda, c))$, for some $x \in \mathbb{C}$, $c \in \mathbb{C}^*$ and $\lambda \in -\frac{1}{2} + \mathbb{N}$.

Conversely, let $\lambda \in -\frac{1}{2} + \mathbb{N}$, $c \in \mathbb{C}^*$ and $x \in \mathbb{C}$ be arbitrary. Let also $S \subseteq B_x^{(q)}(N(\lambda, c))$ be a simple (not necessarily proper) submodule. Then, by Proposition 11, $B_x^{(q)}(S) \subseteq B_0^{(q)}(N(\lambda, c))$ is a submodule as well, and is nonzero because of Lemma 10. In addition, $N(\lambda, c)$ is a submodule of $B_0^{(q)}(N(\lambda, c))$. Then $B_x^{(q)}(S) \cap N(\lambda, c) \subseteq N(\lambda, c)$ is yet another submodule. This submodule is nonzero, because the action of $q$ is injective on $B_0^{(q)}(N(\lambda, c))$ and thus $B_x^{(q)}(S)$ has vectors of arbitrarily low weight, some of which must then lie in $N(\lambda, c)$ as well. From simplicity of $N(\lambda, c)$ it follows that $B_x^{(q)}(S) \cap N(\lambda, c) = N(\lambda, c)$ and therefore $B_0^{(q)}(S) = B_x^{(q)}(N(\lambda, c))$.

Then either $S = B_x^{(q)}(N(\lambda, c))$, in which case $B_x^{(q)}(N(\lambda, c))$ is simple, or $S$ is a highest weight module with maximal dimension of weight spaces equal to that of $N(\lambda, c)$. From Proposition 12 we in this latter case get that in fact $S \cong N(\lambda, c)$, so that $B_0^{(q)}(S) \cong B_0^{(q)}(N(\lambda, c))$. All in all, we see that either $B_x^{(q)}(N(\lambda, c))$ is simple, or $B_0^{(q)}(N(\lambda, c)) \cong B_x^{(q)}(N(\lambda, c))$. The latter case is investigated together with possible reducibility as follows.

Assume that $B_x^{(q)}(N(\lambda, c)) \cong B_x^{(q)}(N(\lambda', c'))$. That $c' = c$ is obvious. The isomorphism also implies equality of weight space dimensions, so it follows from Proposition 11 that $\lambda' = \lambda$, and we furthermore get, by Proposition 11, that $B_x^{(q)}(N(\lambda, c)) \cong B_0^{(q)}(N(\lambda, c))$, where $x = x_1 - x_2$. Then

$$Z + \lambda - x = \text{supp}(B_x^{(q)}(N(\lambda, c))) = \text{supp}(B_0^{(q)}(N(\lambda, c))) = Z + \lambda \quad (10)$$

so that $x \in \mathbb{Z}$.

Whenever $x \in \mathbb{Z}$, one conversely easily sees from formulæ 3 that

$$\Phi : B_0^{(q)}(N(\lambda, c)) \to B_x^{(q)}(N(\lambda, c))
\quad v_{i+x,j} \mapsto v_{i,j}$$

16
defines an isomorphism.

Thus $B_0^{(q)}(N(\lambda, c))$ and $B_x^{(q)}(N(\lambda, c))$ are isomorphic if and only if $x \in \mathbb{Z}$, so that in total (via another application of Proposition 11) we have that $B_{x_1}^{(q)}(N(\lambda, c))$ is isomorphic to $B_{x_2}^{(q)}(N(\lambda', c'))$ if and only if $\lambda = \lambda'$, $c' = c$ and $x_1 - x_2 \in \mathbb{Z}$. This completes the proof. \qed

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