STABILITY, FINITE-TIME STABILITY AND PASSIVITY CRITERIA FOR DISCRETE-TIME DELAYED NEURAL NETWORKS

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Abstract. In this paper, we present the problem of stability, finite-time stability and passivity for discrete-time neural networks (DNNs) with variable delays. For the purposes of stability analysis, an augmented Lyapunov-Krasovskii functional (LKF) with single and double summation terms and several augmented vectors is proposed by decomposing the time-delay interval into two non-equidistant subintervals. Then, by using the Wirtinger-based inequality, reciprocally and extended reciprocally convex combination lemmas, tight estimations for sum terms in the forward difference of LKF are given. In order to relax the existing results, several zero equalities are introduced and stability criteria are proposed in terms of linear matrix inequalities (LMIs). The main objective for the finite-time stability and passivity analysis is how to effectively evaluate the finite-time passivity conditions for DNNs. To achieve this, some weighted summation inequalities are proposed for application to a finite-sum term appearing in the forward difference of LKF, which helps to ensure that the considered delayed DNN is passive. The derived passivity criteria are presented in terms of linear matrix inequalities. Some numerical examples are presented to illustrate the proposed methodology.

Key words: Stability, finite-time stability, finite-time passivity, neural networks, time delay, Lyapunov-Krasovskii functional

1. INTRODUCTION

During the past few decades, neural networks (NNs) have received great attention because of their wide applications in various fields such as image processing, signal
processing, pattern recognition, associative memory, parallel computation, optimization, error diagnosis and so on [1, 2]. One of the most important questions in theoretical analysis of NNs is dynamical behaviours of the NNs, such as their stability [3, 4], periodic oscillatory, and chaos. It is well known that a time delay is inherent in various systems, including NNs, owing to the finite speed of signal transmission and conversion rate of the processors. Delays in a system may cause oscillation and divergence and further degrade the performance [5-10]. Since most systems use a digital processor to acquire information from computers at discrete instants of time, it is essential to formulate discrete-time neural networks (DNNs) that are an analogue of continuous ones [11-18]. In order to improve results regarding this problem, various techniques have been applied to the delay-dependent category, such as augmented Lyapunov-Krasovskii (LK) functional [13, 19-22], free-weighting matrix method [18, 23], summation inequality method [16, 24-27], delay-partitioning method [5, 28, 29] and reciprocally convex approach [20, 30, 31].

The passivity is part of a broader general theory of dissipativeness, which postulates that the energy dissipated inside a dynamic system is less than the energy supplied from an external source and is often linked to stability problems. In particular, the main idea behind the passivity theory (PT) is that the passive property of a system can keep the system internally stable. PT has been established in various control problems, including $H_{\infty}$ control and strict output (or input)-positive realness, so it provides an effective tool for analyzing the dynamic behaviours of a nonlinear system, namely, stability [32], chaos control and synchronization [33], [34], signal processing [35], and complexity [36]. Passivity has been analyzed for various systems, specifically NNs and chaotic, linear, and switched systems [37-42].

Generally, existing results on the passivity problem for DNNs have been from studies considering an infinite-time interval. However, the dynamic properties of the system have been studied over a fixed short time in many practical applications, such as biochemical reaction systems, communication network systems, and other engineering systems. The finite-time stability (FTS) approach was introduced by Dorato in [43]. A very few FTS problems have been studied for discrete-time cases in the literature [44-47]. In addition, Mathiyalagan et al. [48] examined the robust finite-time passivity (FTP) problem of DNNs with time delay by using the concept of finite-time boundedness. However, this approach might have produced conservative results, which has motivated authors in [16] to improve the FTP criteria of DNNs using new weighted summation inequalities.

After the computational complexity became one of the crucial aspects of a research in the area of the system stability, the direct bounding method based on summation inequalities once again becomes the most popular method [17, 19, 49, 50]. Very recently, various types of the Wirtinger-based summation inequalities, tighter than the Jensen-based summation inequality, have been proposed for discrete-time linear time-delay systems [25, 51, 52] and have also been used for the study of the discrete-time DNNs [39]. As one of useful methods to deal with the stability of delayed systems, the reciprocally convex approach was developed in [30] and has been extensively used to study the dynamical behaviours of time-delay systems [32, 53-55].

The goal of this paper is to present the main authors’ results in the stability [56], finite-time stability and finite-time passivity analysis [16] of discrete-time neural networks with interval time-varying delay.
In order to reduce the conservativeness of proposed stability criteria, delay-decomposition method (DDM) is used in [56]. DDM is based on discrete LKF with a free parameter $\alpha$ that divides the summation discrete interval $[k-h_2, k-h_1]$ into two asymmetric discrete subintervals $[k-h_2, k-\alpha-1]$ and $[k-\alpha, k-h_1]$. In this way, a greater degree of freedom is enabled in estimating the stability of the DDN, which leads to a smaller conservative stability criterion. Further, the convenient summations inequalities, extended reciprocally convex combination lemma and zero equalities (ZE) have been used for calculation difference of KLF. As a result of applying the mentioned techniques, some less conservative results are derived in [56]. As a result of applying the mentioned techniques, it is shown that the derived results are less conservative then the existing ones [24, 31, 57, 58].

Using the concepts of FTS and FTP, the researchers in [16] are focused on developing a new method to analyse the FTP of delayed DNNs. To reach this goal, a new definition of EP for DNNs is introduced and new finite-sum inequalities with exponential functions are proposed. A suitable Lyapunov–Krasovskii functional (LKF) with single, double, and triple sums is proposed. To obtain less conservative results, the proposed weighted summation inequalities and a reciprocal convex combination approach are used in [16].

Finally, the numerical examples are presented to illustrate the effectiveness of the proposed results and their improvement over the existing literature.

Section 2 presents the problem formulation and mathematical preliminaries. In Section 3, we present the stability, finite-time stability and finite-time passivity of DDNs. Section 4 presents a numerical example to validate the proposed results.

**Notations.** Throughout the paper, $\mathbb{N}^+$ denotes the set of positive integers, $\mathbb{R}^+$ denotes the $n$-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ the set of all $n\times m$ real matrices. For the positive integers $a$ and $b$ ($b > a$), $[a,b]$ denotes the set of all positive integers $z$ satisfying $a \leq z \leq b$. $I_n$ and $0_{n\times m}$ denote the $n \times n$ identity matrix and $n \times m$ zero matrix, respectively. $X^T$ denotes the matrix transpose of $X$ and $^*$ represent the elements below the main diagonal of a symmetric matrix. $\text{diag}(a,b,\ldots,z)$ denotes the block-diagonal matrix with elements $a,b,\ldots,z$ in the diagonal entries and $\text{Sym}(X) = X + X^T$. For any symmetric matrix $X \in \mathbb{R}^{n\times n}$ the notation $X > 0$ ($X \geq 0$) means that $X$ is a positive definite (positive semi-definite) matrix and $X < 0$ ($X \leq 0$) means that $X$ is a negative definite (negative semi-definite) matrix. For the matrices $A_i \in \mathbb{R}^{m\times m}$, $i=1,2,\ldots,l$, $\text{Col}[A_1,A_2,\ldots,A_l]$ denotes the column block matrix $[A_1^T A_2^T \cdots A_l^T]^T$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. **PRELIMINARIES**

Consider the discrete-time neural networks (DNNs) with an interval time-varying delay of the form [16, 56]:

$$
\begin{align*}
    x(k+1) &= Cx(k) + A f(x(k)) + A_{d_k} f(x(k-h(k)))+u(t), \quad k \geq 0 \\
    y(k) &= f(x(k)), \\
    x(j) &= \phi(j), \quad j \in \{-h_2,-h_2+1,\cdots,-1,0\}
\end{align*}
$$

(1)

where $k \in \mathbb{N}^+$, $x(k) \in \mathbb{R}^n$ is the state vector, $y(k)$ is the output vector, $u(k)$ is the exogenous disturbance input vector, $C, A, A_{d_k} \in \mathbb{R}^{n\times n}$ are the known real constant matrices,
denotes nonlinear activation functions with \( f_i(0) = 0 \), and \( h(k) \) is the time-varying delay satisfying
\[
0 < h_k \leq h(k) \leq h_{k_2},
\]
where \( h_1 \) and \( h_2 \) are the known positive constants. In addition, \( \phi(j) \) denotes a vector-valued initial function that satisfies
\[
\sup_{j \in [-h_0, -h_1 + 1, \ldots, -1]} \left( \phi(j + 1) - \phi(j) \right)^T (\phi(j + 1) - \phi(j)) \leq \delta
\]
where \( \delta \) is a positive constant.

In this paper, we make the following assumption.

**Assumption 1.** [48] For any \( s_1, s_2 \in \mathbb{R}, s_1 \neq s_2 \), the continuous and bounded activation functions \( f_i(\cdot) \) satisfy
\[
k_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq k_i^+, \quad i = 1, 2, \ldots, n
\]
where \( k_i^- \) and \( k_i^+ \) are known constants. For \( s_2 = 0 \) we have \( f(0) = 0 \) and
\[
k_i^- \leq \frac{f_i(s_1)}{s_1} \leq k_i^+, \quad i = 1, 2, \ldots, n
\]
for all \( s_1 \neq 0 \).

The following lemmas will be used in the sequel to establish the main results.

**Lemma 1.** (Jensen’s inequality [59]) For any positive definite symmetric matrix \( R \in \mathbb{R}^{n \times n} \), two positive integers \( a \) and \( b > a \), the sum term \( \sum_{j=a}^{b-1} x^T(j)Rx(j) \) is estimated as
\[
(b - a) \sum_{j=a}^{b-1} x^T(j)Rx(j) \geq \sum_{j=a}^{b-1} x^T(j) R \sum_{j=a}^{b-1} x(j)
\]
\[
(b - a) \sum_{k=a}^{b-1} \eta^T(i) R \eta(i) \geq \begin{bmatrix} \theta_1^T \theta_2^T \end{bmatrix} R \begin{bmatrix} 0 & 3 \rho(a, b) R \end{bmatrix} \begin{bmatrix} \theta_1^T \\ \theta_2^T \end{bmatrix} \geq \begin{bmatrix} \theta_1^T \theta_2^T \end{bmatrix} R \begin{bmatrix} 0 & \rho(a, b) R \end{bmatrix} \begin{bmatrix} \theta_1^T \\ \theta_2^T \end{bmatrix}
\]
\[
(7)
\]
where
\[
\eta(k) = x(k + 1) - x(k), \quad \theta_1 = x(b) - x(a), \quad \theta_2 = x(b) + x(a) - \frac{2}{b - a + 1} \sum_{i=a}^{b} x(i)
\]
\[
\rho(a, b) = \begin{cases} (b - a + 1)/(b - a - 1), & b - a \neq 1 \\ 1, & b - a = 1 \end{cases}
\]
\[
(8)
\]
The reciprocally convex combination lemma (RCCL) plays an important role in the estimation of the forward difference of LKF.

**Lemma 3.** (Reciprocally convex combination lemma [30, 31, 60]) For a real scalar \( \delta \in (0,1) \), a symmetric matrix \( R > 0 \), and any matrix \( S \) satisfying
\[
\begin{bmatrix}
R & S \\
* & R
\end{bmatrix} \succeq 0
\]
the following inequality holds
\[
\begin{bmatrix}
\frac{1}{\delta} R & 0 \\
* & \frac{1}{1-\delta} R^T
\end{bmatrix} \succeq \begin{bmatrix}
R & S \\
* & R
\end{bmatrix} \succeq \begin{bmatrix}
R + (1-\delta)T_1 & S \\
* & R + \delta T_2
\end{bmatrix}
\]

An extended reciprocally convex combination lemma (ERCC), which estimate the sum terms in the forward difference of LKF tightly than RCCL is presented as follows.

**Lemma 4.** (Extended reciprocally convex combination lemma [31, 57]) For a real scalar \( \delta \in (0,1) \), a symmetric matrix \( R > 0 \), and any matrix \( S \), the following inequality holds
\[
\begin{bmatrix}
\frac{1}{\delta} R & 0 \\
* & \frac{1}{1-\delta} R^T
\end{bmatrix} \succeq \begin{bmatrix}
R + (1-\delta)T_1 & S \\
* & R + \delta T_2
\end{bmatrix}
\]
where \( T_1 = R - SR^{-1}S^T \) and \( T_2 = R - S^T R^{-1}S \).

**Remark 1.** In reference [20], the specific reciprocally convex inequality is proposed. For comparison, we rewrite the result of Theorem 1 in [20] in a block form as
\[
\begin{bmatrix}
\frac{1}{\delta} R_1 & 0 \\
* & \frac{1}{1-\delta} R_2
\end{bmatrix} \succeq \begin{bmatrix}
R_1 + (1-\delta)T_1 & \delta S_1 + (1-\delta)S_2 \\
* & R_2 + \delta T_2
\end{bmatrix}
\]
where \( T_1 = R_1 - Y_1 R_1^{-1}Y_1^T \) and \( T_2 = R_2 - Y_2 R_2^{-1}Y_2^T \). If taking \( R_1 = R_2 = R \) and \( S_1 = S_2 = S \), inequality (12) immediately reduces to (11), which means that ERCC (Lemmas 4) is a special case of Theorem 1 in [20].

**Lemma 5.** ([57]) For a positive definite symmetric matrix \( R \) matrices \( \Upsilon \) and \( \Xi \), the following statements are equivalent
(i) \( \Xi - \Upsilon^T R \Upsilon < 0 \)
(ii) There exists a matrix \( \Psi \) with appropriate dimension such that
\[
\begin{bmatrix}
\Xi & \Upsilon^T \\
* & -R
\end{bmatrix} < 0
\]
To study the finite-time stability of the DNN (1), we introduce the following definitions.

**Definition 1.** [61] The DNN (1) with time varying delay is said to be finite-time stable with respect to \( (\alpha, \beta, N) \), where \( 0 \leq \alpha < \beta \), if
\[
\sup_{j \in [-h_j, -h_{j+1}, \ldots, 0]} x^T(j)x(j) \leq \alpha \quad \Rightarrow \quad x^T(k)x(k) < \beta, \quad \forall k \in \{1, 2, \ldots, N\}
\]
Definition 2. [16] With respect to \((\alpha, \beta, N, \mu)\), where \(0 \leq \alpha < \beta\), \(\gamma \geq 1\) and \(\mu\) is a positive scalar, DNN (1) with time-varying delay is said to be finite-time passive if it is FTS with respect to \((\alpha, \beta, N)\), and under the zero initial condition, output \(y(k)\) satisfies
\[
2 \sum_{j=0}^{k-1} y^T(j)u(j) \geq -\mu \sum_{j=0}^{k-1} u^T(j)u(j), \quad \forall k \in \{1, 2, \ldots, N\}
\]  
(15)

Remark 2. Since the system output \(y(j)\) in (15) is scaled by the exponential function \(\gamma^{-j}\), then the exponential behaviour of discrete systems is embedded in the inequality (15). Therefore, we define the condition (15) as exponential passivity (EP) for discrete-time systems. This EP is analogous to EP for continuous-time systems which are considered in the existing literature. In the case \(\gamma = 1\), the inequality (15) becomes a well-known condition for the passivity of discrete-time systems
\[
2 \sum_{j=0}^{k-1} y^T(j)u(j) \geq -\mu \sum_{j=0}^{k-1} u^T(j)u(j)
\]  
(16)

Based on Definition 2, FTP is expressed as a combination of two properties: the exponential passivity and the finite-time stability.

Before deriving the main results, we introduce three weighted summation inequalities, which play a significant role in obtaining new criteria.

Lemma 6. [16] For given integers \(h_1, h_2, 0 \leq h_1 < h_2\), and \(\gamma > 0\), a vector function \(\omega(k)\), \(k \in \mathbb{Z}^+\), and symmetric matrix \(R > 0\), the following inequalities hold:
\[
\pi \sum_{j=k-h_1}^{k-h_1-1} \gamma^{j-i} \omega^T(j)R\omega(j) \geq \left( \sum_{j=k-h_1}^{k-h_1-1} \omega(j) \right)^T R \left( \sum_{j=k-h_1}^{k-h_1-1} \omega(j) \right)
\]  
(17)

\[
k \sum_{i=-h_1}^{k-h_1} \sum_{j=k-i}^{k-i-1} \gamma^{j-i} \omega^T(j)R\omega(j) \geq \left( \sum_{i=-h_1}^{k-h_1} \sum_{j=k-i}^{k-i-1} \omega(j) \right)^T R \left( \sum_{i=-h_1}^{k-h_1} \sum_{j=k-i}^{k-i-1} \omega(j) \right)
\]  
(18)

\[
\vartheta \sum_{i=k-h_1}^{k-h_1-1} \sum_{j=k-i}^{k-i-1} \gamma^{j-i} \omega^T(j)R\omega(j) \geq \left( \sum_{i=k-h_1}^{k-h_1-1} \sum_{j=k-i}^{k-i-1} \omega(j) \right)^T R \left( \sum_{i=k-h_1}^{k-h_1-1} \sum_{j=k-i}^{k-i-1} \omega(j) \right)
\]  
(19)

where
\[
\pi = \sum_{j=k-h_1}^{k-h_1-1} \gamma^{-(k-j)}, \quad k = \sum_{i=-h_1}^{k-h_1} \sum_{j=k-i}^{k-i-1} \gamma^{-(k-j)} \quad \vartheta = \sum_{i=k-h_1}^{k-h_1-1} \sum_{j=k-i}^{k-i-1} \gamma^{-(k-j)}
\]  
(20)

Proof. Let \(X = \begin{bmatrix} R^{1/2} & Y^{1/2} \\ 0 & 0 \end{bmatrix} \) with \(Y \in \Re^n, \ 0 < R \in \Re^{n \times n} \). Then,
\[
X^TX = \begin{bmatrix} Y^TR^{1/2} & 0 & \sqrt{R}Y^{1/2} \\ R^{1/2} & 0 & 0 \end{bmatrix} \begin{bmatrix} Y^T & R^{1/2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Y^TRY & Y^TR \\ Y^TR & R \end{bmatrix} \geq 0
\]  
(21)


Replacing \( Y \) and \( R \) in (21) by \( \omega(j) \) and \( \gamma^{k-j} R \), respectively and summing from \( k-h_2 \) to \( k-h_1-1 \) gives

\[
\sum_{j=k-h_1}^{k-h_2-1} \begin{bmatrix}
\omega^T(j) \gamma^{k-j} R \omega(j) & \omega^T(j) \\
\omega(j) & \gamma^{(k-j)} R^{-1}
\end{bmatrix} \geq 0
\]  

(22)

Through the Schur complement, we obtain

\[
\sum_{j=k-h_1}^{k-h_2-1} \gamma^{k-j} \omega^T(j) R \omega(j) \geq \pi^{-1} \sum_{j=k-h_1}^{k-h_2-1} \omega(j) \begin{bmatrix}
\begin{pmatrix}
\gamma^{k-j} R^{-1}
\end{pmatrix} & - \omega^T(j)
\end{bmatrix} \begin{bmatrix}
\begin{pmatrix}
\gamma^{k-j} R^{-1}
\end{pmatrix} & - \omega(j)
\end{bmatrix}^{-1}
\]

(23)

where \( \pi = \sum_{j=k-h_1}^{k-h_2-1} \gamma^{-(k-j)} \), which is equivalently represented by \( \pi \) in (20). We then obtain (17).

Using a similar method, we obtain

\[
\sum_{i=k-h_2}^{k-h_1-1} \sum_{j=k-i}^{k-h_1} \gamma^{k-j} \omega^T(j) R \omega(j) \geq \kappa^{-1} \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=k-i}^{k-h_1} \omega(j) \begin{bmatrix}
\begin{pmatrix}
\gamma^{k-j} R^{-1}
\end{pmatrix} & - \omega^T(j)
\end{bmatrix} \begin{bmatrix}
\begin{pmatrix}
\gamma^{k-j} R^{-1}
\end{pmatrix} & - \omega(j)
\end{bmatrix}^{-1}
\]

(25)

where \( \kappa = \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=k-i}^{k-h_1} \gamma^{-(k-j)} \). Thus, we obtain (18). A similar proof can be derived for the inequality (19).

\( \blacksquare \)

**Remark 3.** Jensen’s inequities for the single and double summation represent the special cases \( \gamma = 1 \) of inequalities (17)-(19).

**Lemma 7.** [30] For given positive integers \( n \) and \( m \), a scalar \( \alpha \) in the interval \((0,1)\), a given \( n \times m \) matrix \( R > 0 \), and two matrices \( W_1 \) and \( W_2 \) in \( \mathbb{R}^{m \times m} \), for all vectors \( \xi \) in \( \mathbb{R}^m \) let us define the function \( \vartheta(\alpha, R) \) given by

\[
\vartheta(\alpha, R) = \frac{1}{\alpha} \xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha} \xi^T W_2^T R W_2 \xi
\]  

(26)
Then, if there exists a matrix $X$ in $\mathbb{R}^{n \times n}$ such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$, the following inequality holds:

$$\min_{\alpha \in (0,1)} \theta(\alpha, R) \geq \begin{bmatrix} W_1 \xi^T \\ W_2 \xi^T \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}$$  \hspace{1cm} (27)

**Lemma 8.** [53] For symmetric matrices of appropriate dimensions $R > 0$, $\Omega$, and a matrix $\Gamma$, the following two statements are equivalent:

1. $\Omega - \Gamma R^T < 0$.
2. There exist a matrix of the appropriate dimensions $\Pi$ such that

$$\begin{bmatrix} \Omega + \Gamma \Pi^T + \Pi \Gamma^T \\ -R \end{bmatrix} < 0$$

3. **MAIN RESULTS**

### 3.1. Stability of DNNs with time-varying delay

In this section, we present a criterion on the asymptotical stability for the discrete-time neural networks with interval time-varying delay (1) that was proposed in [56]. The following notations are introduced for later use:

- $\xi(k) = \begin{bmatrix} x^T(k) & x^T(k-h_1) & x^T(k-h_2) & x^T(k-h_3) & x^T(k-h) & x^T(k-\alpha) & v_1^T(k) & v_2^T(k) \end{bmatrix}^T$
- $\zeta_1(k) = \begin{bmatrix} x^T(k) & \sum_{j=k-h_1}^{k-1} x^T(j) & \sum_{j=k-h_2}^{k-1} x^T(j) & \sum_{j=k-h_3}^{k-1} x^T(j) \end{bmatrix}^T$, $\eta(k) = x(k+1) - x(k)$
- $\zeta_2(k) = \begin{bmatrix} x^T(k) & f^T(x(k)) \end{bmatrix}^T$, $\zeta_3(k) = \begin{bmatrix} x^T(k) & \eta^T(x(k)) \end{bmatrix}^T$
- $v_1(k) = \sum_{j=k-h}^{k-1} \frac{x(j)}{h_1+1}$, $v_2(k) = \sum_{j=k-h}^{k-1} \frac{x(j)}{h(h(k)-h_1+1)}$
- $v_3(k) = \sum_{j=k-h}^{k-1} \frac{x(j)}{h_2-h(k)+1}$, $v_4(k) = \sum_{j=k-h}^{k-1} \frac{x(j)}{h(h(k)-h_2+1)}$
- $\alpha_1 = (e_1 - e_2) \xi(k)$, $\alpha_2 = (e_1 + e_2 - 2e_0) \xi(k)$
- $\alpha_3 = (e_1 - e_2) \zeta_1(k)$, $\alpha_4 = (e_1 + e_2 - 2e_0) \zeta_1(k)$
- $e_i = \begin{bmatrix} 0_{n+i-1 \times n}, I_n, 0_{n+i-1 \times n} \end{bmatrix}$, $i = 1, 2, \ldots, 11$
- $e_s = Ce_i + Ae_{in} + A_j e_{i1}$,
- $\rho(a,b) = \begin{cases} b-a+1 & b-a > 1 \\ b-a-1 & 1 \\ 1 & b-a = 1 \end{cases}$
\[ \delta_i = \frac{h(k) - h_i}{h_{i+1}}, \quad h_{i+1} = h_i - h_k \]

\[ \Sigma(h(k), \alpha) = \Sigma_{i_1} + \Sigma_{i_2}(h(k), \alpha) + \Sigma_3 + \Sigma_{i_4} + \Sigma_{i_5} \]

\[ \Sigma_{i_1} = \mathbf{P}^T \mathbf{P}_{i_1} - \mathbf{P}_{i_2}^T \mathbf{P}_{i_1}, \quad \Sigma_{i_2}(h(k), \alpha) = \text{Sym}\{(\mathbf{P}_{i_1} - \mathbf{P}_{i_2})^T \mathbf{P}_{i_1}(h(k), \alpha)\} \]

\[ \mathbf{P}_{i_1} = \begin{bmatrix} e_1 \\ (h_i + 1)e_6 - e_2 \\ -e_3 - e_4 \\ -e_3 \end{bmatrix}, \quad \mathbf{P}_{i_2} = \begin{bmatrix} e_1 \\ (h_i + 1)e_6 - e_1 \\ -e_2 - e_3 \\ -e_2 \end{bmatrix} \]

\[ \Sigma_2 = e_i^T \mathbf{Q}_i e_i + e_2^T (\mathbf{Q}_i - \mathbf{Q}_e) e_2 + e_3^T (\mathbf{Q}_3 - \mathbf{Q}_e) e_3 - e_4^T \mathbf{Q}_4 e_4 \]

\[ \Sigma_3 = \begin{bmatrix} e_i \\ e_{i_1} \end{bmatrix}^T (h_i + 1)(e_i + (h_i - h(k) + 1)e_6 \left(\alpha - h_k + 1\right)e_9) \]

\[ \Sigma_{i_4}(h(k)) = \Sigma_{i_4} - \Sigma_{i_5}, \quad \Sigma_{i_5} = (e_i - e_1) \left(\mathbf{h}_{i_2}^T \mathbf{Z}_i + \mathbf{h}_{i_3}^T \mathbf{Z}_e\right)(e_i - e_1) \]

\[ \Sigma_{i_2} = E_i^T \begin{bmatrix} 0 \\ 3 \rho(0, \mathbf{h}) Z_i \end{bmatrix} E_i, \quad E_i = \begin{bmatrix} e_i - e_2 \\ e_i + e_3 - 2e_6 \end{bmatrix} \]

\[ \Sigma_{i_3}(h(k)) = \mathbf{E}_2^T \begin{bmatrix} \mathbf{Z}_2 \\ S \end{bmatrix} \begin{bmatrix} (1 - \delta_2) T_i \\ 0 \end{bmatrix} \mathbf{E}_2 \]

\[ T_i = \mathbf{Z}_2 - S \mathbf{Z}_2^T S, \quad T_2 = \mathbf{Z}_2 - S^T \mathbf{Z}_2^{-1} S \]

\[ \Sigma_{i_4}(h_i) = \mathbf{E}_2^T \begin{bmatrix} \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} \mathbf{E}_2 \end{bmatrix} + \mathbf{E}_2^T \mathbf{Z}_2 \mathbf{E}_2 \]

\[ \Sigma_{i_5}(h_i) = \mathbf{E}_2^T \begin{bmatrix} \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} \mathbf{E}_2 \end{bmatrix} + \mathbf{E}_2^T \mathbf{Z}_2 \mathbf{E}_2 \]

\[ \mathbf{Z}_2 = \begin{bmatrix} 0 \\ 3 \mathbf{Z}_2 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} e_i - e_1 \\ e_i + e_3 - 2e_6 \end{bmatrix}, \quad \mathbf{E}_3 = \begin{bmatrix} e_i - e_1 \\ e_i + e_3 - 2e_6 \end{bmatrix}, \quad \mathbf{E}_4 = \begin{bmatrix} e_i - e_1 \\ e_i + e_3 - 2e_6 \end{bmatrix} \]

\[ E_n(h(k)) = \begin{bmatrix} (h(k) - h_1 + 1)e_i - e_2 \\ e_i - e_3 \end{bmatrix}, \quad E_n(h(k)) = \begin{bmatrix} (h(k) - h(k) + 1)e_i - e_1 \\ e_i - e_3 \end{bmatrix} \]
\[ E_{46}(h(k)) = \begin{bmatrix} E_4(h(k)) \\ E_6(h(k)) \end{bmatrix}, \quad \Sigma_{53}(\alpha) = (e_i - e_j)^T (\alpha - h_i) Z_\alpha (e_i - e_j) \]

\[ \Sigma_{42}(\alpha) = E_4^T Z_\alpha 3_\rho(h_i, \alpha) Z_\alpha E_4, \quad \Sigma_6 = \Sigma_{41} - \Sigma_{42}(h(k)) \]

\[ K_1 = \text{diag} \begin{bmatrix} k_1^-, k_2^-, \ldots, k_n^- \end{bmatrix}, \quad K_2 = \text{diag} \begin{bmatrix} k_1^+, k_2^+, \ldots, k_n^+ \end{bmatrix} \]

\[ \Sigma_2 = \text{Sym} \left\{ (e_{i_0} - K_1 e_i)^T M_1 (e_{i_0} - K_2 e_i) \right\} + \text{Sym} \left\{ (e_{i_1} - K_1 e_i)^T M_2 (e_{i_1} - K_2 e_i) \right\} \]

\[ \Sigma_{41} = h_i^2 \begin{bmatrix} e_i \\ e_i - e_j \end{bmatrix}^T G \begin{bmatrix} e_i \\ e_i - e_j \end{bmatrix} + h_3 e_i^T H_1 e_i - h_3 e_i^T H_2 e_i - h_5 e_i^T H_3 e_i - h_5 e_i^T (H_1 - H_2) e_i \]

\[ \Sigma_{42}(h(k)) = E_{56}^T (h(k)) G_{12} E_{56}(h(k)) \]

\[ G_i = G + \begin{bmatrix} 0 \\ H_i \end{bmatrix}, \quad i = 1, 2, G_{12} = \begin{bmatrix} G_1 \\ * \\ G_2 \end{bmatrix} \]

**Theorem 1.** [56] For given positive integers \( h_i \) and \( h_i \), system (1) with interval time-varying delay satisfying condition (2) is asymptotically stable, if there exist positive-definite matrices \( P_i \in \mathbb{R}^{n_i \times n_i}, \quad Q_i \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, 2, 3, \quad R_i \in \mathbb{R}^{n_i \times 2n} \), \( Z_i \in \mathbb{R}^{n_i \times n}, \quad i = 1, 2, 3 \), positive definite diagonal matrices \( M_i \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, 2, 3 \) and any matrix \( S \in \mathbb{R}^{2n \times 2n} \) such that the following LMIs hold

\[ \begin{bmatrix} \Phi(h_i, \alpha) \\ E_2^T S \\ * \\ -Z_2 \end{bmatrix} < 0 \quad (29) \]

\[ \begin{bmatrix} \Phi(h_i, \alpha) \\ E_2^T S^T \\ * \\ -Z_2 \end{bmatrix} < 0 \quad (30) \]

where

\[ \Phi(h_i, \alpha) = \Sigma_{41} + \Sigma_{52}(h_i, \alpha) + \Sigma_2 + \Sigma_3 + \Sigma_{41} - \Sigma_{42} - \Sigma_{43}(h_i) + \Sigma_{51}(\alpha) - \Sigma_{52}(\alpha) - \Sigma, \quad i = 1, 2 \]

**Proof.** Construct a LKF for the DNN (1) as follows:

\[ V(k) = \sum_{i=1}^{4} V_i(k) \quad (31) \]

where

\[ V_1(k) = \xi^T_1(k) P \xi_1(k) \]

\[ V_2(k) = \sum_{j=-H-k}^{H-k} x^T(j) Q x(j) + \sum_{j=-H-d}^{H-d} x^T(j) Q x(j) + \sum_{j=-H-d}^{H-d} x^T(j) Q x(j) \]

\[ V_3(k) = \sum_{j=-H-k}^{H-k} \xi^T_2(j) R \xi_2(j) + \sum_{i=-H-k}^{H-k} \sum_{j=-H-d}^{H-d} \xi^T_2(j) R \xi_2(j) \]
\[ V_4(k) = h_1 \sum_{j=0}^{k} \sum_{j=1}^{k} \eta^T(j)Z_i(j) + h_2 \sum_{j=0}^{k} \sum_{j=1}^{k} \eta^T(j)Z_i(j) \]

\[ V_3(k) = (\alpha - h_1) \sum_{j=0}^{k} \sum_{j=1}^{k} \eta^T(j)Z_i(j) \]

The forward difference of LKF (31), \( \Delta V(k) = V(k+1) - V(k) \), along the trajectories of the DNN (1) gives

\[ \Delta V_i(k) = \zeta_i^T(k+1)P_1\zeta_i(k+1) - \zeta_i^T(k)P_1\zeta_i(k) \quad (32) \]

Since

\[ \zeta_1(k) = \text{Col} \left\{ e_1, (h_1 + 1)e_6 - e_1, \right. \]
\[ \left. (h(k) - h_1 + 1)e_j + (h_2 - h(k) + 1)e_9 - e_j, (\alpha - h_1 + 1)e_9 - e_j \right\} \]

\[ \zeta_1(k+1) = \text{Col} \left\{ e_1, (h_1 + 1)e_6 - e_1, \right. \]
\[ \left. (h(k) - h_1 + 1)e_j + (h_2 - h(k) + 1)e_9 - e_j, (\alpha - h_1 + 1)e_9 - e_j \right\} \]

it follows

\[ \Delta V_1(k) = \zeta_i^T(k) \left( \Sigma_{11} + \Sigma_{12}(h(k), \alpha) \right) \xi(k) \quad (35) \]

The forward differences of \( V_2(k) \) and \( V_3(k) \) can be obtained as

\[ \Delta V_2(k) = \xi^T(k) \left\{ e_i^TQ_i e_i + e_i^T(Q_i - Q_j)e_j + e_i^T(Q_j - Q_i)e_j - e_i^TQ_j e_i \right\} \xi(k) \]
\[ = \xi^T(k) \Sigma_2 \xi(k) \quad (36) \]

\[ \Delta V_3(k) = \sum_{j=k-h_1}^{k} \zeta_2^T(j)R_2 \zeta_2(j) - \sum_{j=k-h_1}^{k-1} \zeta_2^T(j)R_2 \zeta_2(j) \]
\[ + \zeta_2^T(k)R_2 \zeta_2(k) - \sum_{j=k-h_1}^{k-1} \zeta_2^T(j)R_2 \zeta_2(j) \]
\[ \leq \xi^T(k) \left[ e_i^T \right]^T(h_1 + 1)R \left[ e_i^T \right] \xi(k) - \xi^T(k) \left[ e_i^T \right]^T R \left[ e_i^T \right] \xi(k) \]
\[ = \xi^T(k) \Sigma_3 \xi(k) \quad (37) \]
\[ \Delta V_4(k) = h_2 \sum_{i=h}^{k-1} \left( \eta^T(k)Z_i \eta(k) - \eta^T(k + i)Z_i \eta(k + i) \right) \]
\[ + h_2 \sum_{i=h}^{k-1} \left( \eta^T(k)Z_i \eta(k) - \eta^T(k + i)Z_i \eta(k + i) \right) \]
\[ = \xi^T(k)(e_i - e_i^T \left( h_i^2Z_i + h_i^2Z_i \right)(e_i - e_i)\xi(k) \]
\[ - h_2 \sum_{i=h}^{k-1} \eta^T(j)Z_i \eta(j) - h_2 \sum_{j<k-h}^{k-1} \eta^T(j)Z_j \eta(j) \]

Using inequality (7) from Lemma 2 to estimate \( Z_1 \)-dependent summation term yields
\[ h_1 \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) \geq \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} ^T \begin{bmatrix} 0 \\ 3\rho(0,h_1)Z_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \xi^T(k)\Sigma_1 \xi(k) \]

Similarly, by using (7) we estimate \( Z_2 \)-dependent summation term
\[ h_2 \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) = h_2 \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) + h_2 \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) \]
\[ \geq \frac{1}{\delta_2} \xi^T(k) E_2 \begin{bmatrix} \begin{bmatrix} Z_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3Z_1 \end{bmatrix} \end{bmatrix} E_2 \xi(k) + \frac{1}{1-\delta_2} \xi^T(k) E_2 \begin{bmatrix} Z_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3Z_1 \end{bmatrix} E_2 \xi(k) \]
\[ = \xi^T(k) \left( E_2 \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \frac{1}{h_1}Z_2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \frac{1}{h_1}Z_2 \end{bmatrix} E_2 \right) \xi(k) \]

Based on Lemma 4, for any matrix \( S \in \mathbb{R}^{2n\times 2n} \) we have
\[ h_2 \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) \geq \xi^T(k) \left( E_2 \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \frac{1}{h_1}Z_2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \frac{1}{h_1}Z_2 \end{bmatrix} E_2 \right) \xi(k) \]
\[ = \xi^T(k)\Sigma_2(h(k))\xi(k) \]

Then, we can get the upper bound of \( \Delta V_4(k) \) as
\[ \Delta V_4(k) \leq \xi^T(k) \left( \Sigma_1 - \Sigma_2 - \Sigma_2(h(k)) \right) \xi(k) = \xi^T(k)\Sigma_2(h(k))\xi(k) \]

Calculating \( \Delta V_3(k) \) gives
\[ \Delta V_3(k) = \left( \alpha - h_1 \right)^2 \eta^T(k)Z_j \eta(k) - \left( \alpha - h_1 \right)^2 \sum_{i=h}^{k-1} \eta^T(k + i)Z_i \eta(k + i) \]
\[ = \xi^T(k)(e_i - e_i^T \left( \alpha - h_1 \right)^2Z_i(e_i - e_i)\xi(k) - \left( \alpha - h_1 \right)^2 \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) \]

By inequality (7), the second term of \( \Delta V_4(k) \) can be written as
\[ \left( \alpha - h_1 \right) \sum_{j=k-h}^{k-1} \eta^T(j)Z_j \eta(j) \geq \begin{bmatrix} \omega_h \\ \omega_{h} \end{bmatrix} ^T \begin{bmatrix} 0 \\ 3\rho(h_1,\alpha)Z_1 \end{bmatrix} \begin{bmatrix} \omega_h \\ \omega_{h} \end{bmatrix} \]

\[ \text{(44)} \]
Then, we can get the upper bound of $\Delta V_k(k)$ as
\[
\Delta V_k(k) \leq \xi^T(k) \left( \Sigma_{\alpha}(\alpha) \right) \xi(k) = \xi^T(k) \left( \Sigma_{\alpha}(\alpha) - \Sigma_{\alpha}^{(2)}(\alpha) \right) \xi(k)
\]
(45)

Under the assumption on the activation function (4) and (5), for any positive definite diagonal matrices $M_j = \text{diag} \{ m_{j1}, m_{j2}, \ldots, m_{jn} \}, j = 1, 2, 3$, the following inequality holds
\[
0 \geq 2 \sum_{i=1}^{n} m_{ii} \left[ f_i(x_i(k)) - k_i^* x_i(k) \right] \times \left[ f_i(x_i(k)) - k_i^* x_i(k) \right] + 2 \sum_{i=1}^{n} m_{ii} \left[ f_i(x_i(k-h(k))) - k_i^* x_i(k-h(k)) \right] \left[ f_i(x_i(k-h(k))) - k_i^* x_i(k-h(k)) \right]
\]
(46)

Then
\[
0 \geq 2 \sum_{i=1}^{n} \left[ f(x(k)) - K_i x(k) \right] M_i \left[ f(x(k)) - K_i x(k) \right] + 2 \sum_{i=1}^{n} \left[ f(x(k-h(k))) - K_i x(k-h(k)) \right] M_i \left[ f(x(k-h(k))) - K_i x(k-h(k)) \right] + 2 \sum_{i=1}^{n} \left[ f(x(k)) - f(x(k-h(k))) \right] M_i \left[ f(x(k)) - f(x(k-h(k))) \right]
\]
(47)

\[
0 \geq 2 \xi^T(k) \left( e_{10} - K_1 e_1 \right)^T M_1 \left( e_{10} - K_1 e_1 \right) \xi \xi^T(k) + 2 \xi^T(k) \left( e_{11} - K_2 e_1 \right)^T M_2 \left( e_{11} - K_2 e_1 \right) \xi \xi^T(k) + 2 \xi^T(k) \left( e_{11} - K_2 e_1 \right)^T M_3 \left( e_{11} - K_2 e_1 \right) \xi \xi^T(k)
\]
(48)

So, we have
\[
\xi \xi^T(k) \Sigma_{\alpha} \xi(k) \leq 0
\]
(49)

Therefore, by combining (35)-(37), (42), (45) and (49), the forward difference of $V(k)$ is obtained as
\[
\Delta V(k) \leq \xi^T(k) \Sigma(h(k), \alpha) \xi(k)
\]
(50)

Since the matrix $\Sigma(h(k), \alpha)$ is affine with respect to the delay $h(k)$, the condition
\[
\Sigma(h(k), \alpha) < 0 \text{ is satisfied if and only if } \Sigma(h_i, \alpha) < 0, \quad i = 1, 2
\]
(51)

By calculating, we have
\[
\Sigma_{1i}(h_i, \alpha) = \text{Sym} \left\{ \left( \Pi_i - \Pi_j \right)^T P \Pi_i(h_i, \alpha) \right\}
\]
(52)

\[
\Sigma_{2i}(h_i, \alpha) = \text{Sym} \left\{ \left( \Pi_i - \Pi_j \right)^T P \Pi_i(h_i, \alpha) \right\}
\]
\[
\Sigma_{\alpha}(h_i) = \begin{bmatrix}
E_2^T \\
E_3^T
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}_2 & S \\
* & \tilde{Z}_2
\end{bmatrix}
\begin{bmatrix}
E_2 \\
E_3
\end{bmatrix} + E_2^T \tilde{Z}_2 E_2 - E_3^T \tilde{Z}_2 E_3
\] (53)

\[
\Sigma_{\alpha}(h_i) = \begin{bmatrix}
E_2^T \\
E_3^T
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}_2 & S \\
* & \tilde{Z}_2
\end{bmatrix}
\begin{bmatrix}
E_2 \\
E_3
\end{bmatrix} + E_2^T \tilde{Z}_2 E_2 - E_3^T \tilde{Z}_2 E_3
\] (54)

\[
\Sigma(h_i, \alpha) = \Phi(h_i, \alpha) + E_2^T \tilde{Z}_2 E_2 < 0
\] (55)

\[
\Sigma(h_i, \alpha) = \Phi(h_i, \alpha) + E_3^T \tilde{Z}_2 E_3 < 0
\] (56)

Then, by taking Schur complement, it can be seen that (55) and (56) are equivalent to (29) and (30), respectively. Therefore, when (29) and (30) hold, \( \Delta V(k) < 0 \), which shows that system (1) is asymptotically stable.

**Remark 4.** In Theorem 1, the Wirtinger-based summation inequality (7) is applied to summation with the constant lower and upper bound \( \sum_{j=k-h_i}^{k-1} \eta^T(j)Z(j)\eta(j) \). However, in the case of the summation with the time-varying lower or upper bound

\[
h_1 \sum_{j=k-h_i}^{k-1} \eta^T(j)Z(j)\eta(j) + h_2 \sum_{j=k-h_i}^{k-1} \eta^T(j)Z(j)\eta(j)
\] (57)

combination of the Wirtinger-based inequality (7) and the reciprocally convex approach (11) is applied.

By introducing an augmented LKF and zero equations, a further improved stability condition of system (1) can be obtained as follows.

**Theorem 2.** [56] For given positive integers \( h_1 \) and \( h_2 \), system (1) with interval time-varying delay satisfying condition (2) is asymptotically stable, if there exist positive-definite matrices \( P \in \mathbb{R}^{n \times n} \), \( Q_i \in \mathbb{R}^{n \times n} \), \( i = 1, 2, 3 \), \( R \in \mathbb{R}^{n \times n} \), \( Z_i \in \mathbb{R}^{n \times n} \), \( i = 1, 2, 3 \), \( G \in \mathbb{R}^{2n \times 2n} \), positive definite diagonal matrices \( M_i \in \mathbb{R}^{n \times n} \), \( i = 1, 2, 3 \), symmetric matrices \( H_i \in \mathbb{R}^{n \times n} \), \( i = 1, 2 \) and any matrices \( S \in \mathbb{R}^{2n \times 2n} \) and \( X \in \mathbb{R}^{2n \times 2n} \) such that the following LMIs hold

\[
\begin{bmatrix}
G_1 & X \\
* & G_2
\end{bmatrix} > 0
\] (58)

\[
\begin{bmatrix}
\Phi(h_i, \alpha) + \Sigma_{\alpha} - \Sigma_{\alpha}(h_i) & E_2^T S \\
* & -Z_2
\end{bmatrix} < 0
\] (59)
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\[
\begin{bmatrix}
\Phi(h_i, \alpha) + \Sigma_{\alpha_i} - \Sigma_{\alpha_2}(h_i) \\
E^T_i S^T
\end{bmatrix} < 0
\]

(60)

where

\[
\Phi(h_i, \alpha) = \Sigma_{\alpha} + \Sigma_{\alpha_2}(h_i) + \Sigma_{\alpha_3} + \Sigma_{\alpha_4} - \Sigma_{\alpha_2}(h_i) + \Sigma_{\alpha_5}(\alpha) - \Sigma_{\alpha_6}(\alpha) - \Sigma_{\alpha}.
\]

\(i = 1, 2\).

**Proof.** Construct the following LKF candidate

\[
\tilde{V}(k) = V(k) + V_6(k)
\]

where

\[
V_6(k) = h_2 \sum_{j=0}^{k-1} \sum_{j+k} \zeta^T_5(j) G \tilde{\xi}_5(j)
\]

(62)

The forward difference of (62) along the trajectories of DNN (1) gives

\[
\Delta V_6(k) = h_2 \sum_{j=0}^{k-1} \zeta^T_5(j) G \tilde{\xi}_5(j) - h_2 \sum_{j=0}^{k-1} \zeta^T_5(j) G \tilde{\xi}_5(j) - h_2 \sum_{j=0}^{k-1} \zeta^T_5(j) G \tilde{\xi}_5(j)
\]

(63)

For any symmetric matrices \(H_i\), \(i = 1, 2\) we have

\[
\sum_{j=k}^{k+1} [x^T(j+1) H_j x(j+1) - x^T(j) H_j x(j)]
\]

\[
= x^T(k-l_j) H_j x(k-l_j) - x^T(k-l_j) H_j x(k-l_j)
\]

(64)

Since

\[
x^T(j+1) H_j x(j+1) - x^T(j) H_j x(j) = \left[ x(j)^T \begin{bmatrix} 0 & H_j \end{bmatrix} \right] \left[ \begin{bmatrix} x(j) \\ \eta(j) \end{bmatrix} \right] = \zeta^T_5(j) \begin{bmatrix} 0 & H_j \end{bmatrix} \tilde{\xi}_5(j)
\]

then the following zero equation holds

\[
EQ(l_j, H_j) = x^T(k-l_j) H_j x(k-l_j) - x^T(k-l_j) H_j x(k-l_j)
\]

\[
- \sum_{j=k}^{k+1} \zeta^T_5(k) \begin{bmatrix} 0 & H_j \end{bmatrix} \tilde{\xi}_5(j) = 0
\]

(65)

Now, from (66) we get the following two zero equations

\[
EQ(h(k), H_j) = x^T(k-h(k)) H_j x(k-h(k)) - x^T(k-h(k)) H_j x(k-h(k))
\]

\[
- \sum_{j=k}^{k-1} \zeta^T_5(k) \begin{bmatrix} 0 & H_j \end{bmatrix} \tilde{\xi}_5(j) = 0
\]

(67)

\[
EQ(h(k), H_j) = x^T(k-h(k)) H_j x(k-h(k)) - x^T(k-h(k)) H_j x(k-h(k))
\]

\[
- \sum_{j=k}^{k-1} \zeta^T_5(k) \begin{bmatrix} 0 & H_j \end{bmatrix} \tilde{\xi}_5(j) = 0
\]

(68)
Adding the inequality (67) and (68) to (63) implies
\[
\Delta V_k(k) = h_2^2 \xi^T(k) G \xi(k) + h_2 x^T(k) \mathcal{H}_2 x(k) + h_2 x^T(k) \mathcal{H}_1 x(k - h_1) - \sum_{j=h}^{\delta_h - 1} \xi^T(j) G \xi(j) - h_2 \sum_{j=h}^{\delta_h - 1} \xi^T(j) G \xi(j)
\]
(69)

Applying Jensen inequality (6), the \(\Delta V_k(k)\) is estimated as
\[
\Delta V_k(k) \leq \xi^T(k) \left( h_2^2 \left[ e_h \right]^T G \left[ e_h \right] + \frac{1}{\delta_h} \sum_{j=h}^{\delta_h - 1} \xi^T(j) G \xi(j) \right) \xi(k) - \frac{1}{\delta_h} \sum_{j=h}^{\delta_h - 1} \xi^T(j) G \xi(j)
\]
(70)

Using Lemma 3, for any matrix \(X \in \mathbb{R}^{2n \times 2n}\) we have
\[
\Delta V_k(k) \leq \xi^T(k) \left( \sum_{\alpha_1} - E^\alpha_{\alpha_1}(h(k)) \left[ G_1 X G_2 \right] E^\alpha_{\alpha_1}(h(k)) \right) \xi(k)
\]
\[
= \xi^T(k) \left( \sum_{\alpha_1} - E^\alpha_{\alpha_1}(h(k)) G_1 G_2 E^\alpha_{\alpha_1}(h(k)) \right) \xi(k)
\]
\[
= \xi^T(k) \left( \sum_{\alpha_1} - E^\alpha_{\alpha_1}(h(k)) \right) \xi(k) = \xi^T(k) \sum_{\alpha_1} (h(k)) \xi(k)
\]
(71)

Finally
\[
\Delta \tilde{V}(k) \leq \xi^T(k) \left[ \sum_{\alpha_1} (h(k)), \alpha \right] + \xi^T(k) \left[ \sum_{\alpha_1} - \sum_{\alpha_2} (h(k)) \right] \xi(k)
\]
(72)

Based on Lemma 5, the equivalent condition of \(\sum_{\alpha_1} (h(k)), \alpha \) + \(\sum_{\alpha_1} - \sum_{\alpha_2} (h(k)) < 0\) is that exists a matrix \(\Psi\) such that
\[
\Theta(h(k), \alpha) = \left[ \begin{array}{c} \sum_{\alpha_1} (h(k)), \alpha + \sum_{\alpha_1} \left[ E^\alpha_{\alpha_1}(h(k)) \right] \Psi^T \left[ E^\alpha_{\alpha_1}(h(k)) \right] \left[ E^\alpha_{\alpha_1}(h(k)) \right] \right] \xi(k) < 0
\]
(73)

Since the matrix \(\Theta(h(k), \alpha)\) is affine with respect to the delay \(h(k)\), the condition \(\Theta(h(k), \alpha) < 0\) is satisfied if and only if
\[
\Theta(h_i, \alpha) < 0, \quad i = 1, 2
\]
(74)

Based on Lemma 5, (74) is equivalent to
\[
\sum_{\alpha_1} - \sum_{\sum_{\alpha_2} (h(k))} < 0, \quad i = 1, 2
\]
\[
\sum_{\alpha_1} - \sum_{\sum_{\alpha_2} (h(k))} < 0
\]
(75)

i.e.
\[
\Phi(h_i, \alpha) + E^\alpha_{\alpha_1} \left[ Z_{\alpha_2} - S \right] E_{\alpha_2} + \sum_{\alpha_1} - \sum_{\sum_{\alpha_2} (h(k))} < 0
\]
\[
\Phi(h_i, \alpha) + E^\alpha_{\alpha_1} \left[ Z_{\alpha_2} - S \right] E_{\alpha_2} + \sum_{\alpha_1} - \sum_{\sum_{\alpha_2} (h(k))} < 0
\]
(76)

\[
\Phi(h_i, \alpha) + E^\alpha_{\alpha_1} \left[ Z_{\alpha_2} - S \right] E_{\alpha_2} + \sum_{\alpha_1} - \sum_{\sum_{\alpha_2} (h(k))} < 0
\]
(77)
Then, by taking the Schur complement, it can be seen that (76) and (77) are equivalent to (59) and (60), respectively. Therefore, when (59) and (60) hold, \( \Delta \dot{V}(k) < 0 \), which shows that system (1) is asymptotically stable.

**Remark 5.** In order to improve the stability criterion proposed in Theorem 1, in the proof of Theorem 2 a new double summation term \( V_6(k) \) (with free matrix \( G \)) is introduced in LKF and two zero equalities (with symmetric matrices \( H_1 \) and \( H_2 \)) are introduced in \( \Delta V_6(k) \). In this way, the conservatism of the Theorem 2 is reduced.

**Remark 6.** In order to reduce the number of decision variables, Lemma 5 is used twice. First, by using the lemma, non-affine term

\[
\Sigma(h(k), \alpha) + \Sigma_{6i} - E_{6i}(h(k))G_{12}E_{6i}(h(k)) < 0
\]

with respect to \( h(k) \) is transformed into the affine term \( \Theta(h(k), \alpha) \). On that occasion, additional matrix \( \Psi \in \mathbb{R}^{11 \times 2n} \) is introduced in (73) and the number of decision variables have increased significantly (for \( 22n^2 \)). Second, if we apply Lemma 5 on (74), then the matrix \( \Psi \in \mathbb{R}^{11 \times 2n} \) can be eliminated from \( \Theta(h, \alpha) \). In this way, the significantly reduction of decision variables was performed.

**Remark 7.** The proposed stability criteria depend on the lower and upper bounds of time-delay, \( h_1 \) and \( h_2 \). In order to compare proposed results with existing ones, we calculate a maximum allowable upper bound (MAUB) of the time-delay, \( h_{\text{max}} = h_{2\text{max}} \) such that the concerned system is asymptotically stable for any delay size less than the MAUB. Note that a criterion that gives a lower value of MAUB is less conservative with respect to other criteria.

### 3.2. FTS and FTP for DNNs with time-varying delay

In this section, we present FTS and FTP for DNN (1) with time-varying delay using the weighted summation inequalities (17)-(19) [16].

**Theorem 3.** [16] Under Assumption 1, DNN (1) is finite time stable with respect to \( (\alpha, \beta, N) \), where \( 0 \leq \alpha < \beta \) if there exist a scalar \( \gamma > 1 \), positive scalars \( \lambda_i \), \( i = 1, 2, \ldots, 9 \), positive definite matrices \( P, Q_1, Q_2, Q_3, R, Z_1, Z_2, Z_3 \), positive definite diagonal matrices \( \lambda_1 \) and \( \lambda_2 \), matrices \( U, V \) and \( \Omega = [\Omega_1 \Omega_2] \), such that the following LMIs hold:

\[
\begin{bmatrix}
\Sigma + \Gamma \Omega^T + \Omega \Gamma^T & \Omega_1 & \Omega_2 \\
* & -Z_2 & -U \\
* & * & -Z_2
\end{bmatrix} < 0, \quad i = 1, 2
\]

\[
\begin{bmatrix}
Z_1 & V \\
* & Z_2
\end{bmatrix} > 0, \quad \begin{bmatrix}
Z_2 & U \\
* & Z_2
\end{bmatrix} > 0,
\]

\[
\lambda_1 I < P < \lambda_2 I, \quad Q_1 < \lambda_1 I, \quad Q_2 < \lambda_2 I, \quad Q_3 < \lambda_3 I,
\]

\[
R < \lambda_4 I, \quad Z_1 < \lambda_4 I, \quad Z_2 < \lambda_4 I, \quad Z_3 < \lambda_4 I
\]

\[
\gamma n \left[ \alpha (\lambda_2 + \varepsilon_2 \lambda_3 + \varepsilon_3 \lambda_4 + (\varepsilon_2 + \varepsilon_3) \lambda_5) + (\varepsilon_2 + \varepsilon_3) \dot{\lambda}_2 \right] - \beta \lambda_1 < 0.
\]
where

\[ \Sigma = \Theta \Sigma_{\Theta} \Theta \]

\[ \Sigma_{11} = C^T P C - \gamma P + Q_1 + Q_2 + (h_2 - h_1 + 1)Q_3 + (C - I)^T Z_{121} (C - I) - F_1 A_1, \]

\[ \Sigma_{15} = C^T P A + (C - I)^T Z_{121} A + F_2 A_1, \]

\[ \Sigma_{16} = C^T P A_j + (C - I)^T Z_{121} A_j, \quad \Sigma_{22} = -\gamma h_2 Q_2 - Z_1 - h_2^2 Z_3, \quad \Sigma_{23} = Z_1 - V^T, \]

\[ \Sigma_{24} = V^T, \quad \Sigma_{27} = h_2 Z_1, \quad \Sigma_{28} = h_2 Z_3, \quad \Sigma_{33} = -\gamma h_2 Q_2 - 2Z_1 + V + V^T - F_1 A_2, \quad (83) \]

\[ \Sigma_{41} = Z_1 - V^T, \quad \Sigma_{16} = F_2 A_2, \quad \Sigma_{44} = -\gamma h_2 Q_2 - Z_1, \]

\[ \Sigma_{33} = A^T P A + (h_2 - h_1 + 1)R + A^T Z_{121} A - A_1, \]

\[ \Sigma_{66} = A^T P A_j + A^T Z_{121} A_j, \quad \Sigma_{66} = -\gamma R + A_j^T PA_j + A_j^T Z_{121} A_j - A_2, \]

\[ \Sigma_{77} = -Z_3, \quad \Sigma_{78} = -Z_3, \quad \Sigma_{88} = -Z_3. \]

\[ \Gamma_1 = \begin{bmatrix} h_2 I_n & 0_{n \times n} & -I_n & 0_n \end{bmatrix}^T, \quad \Gamma_2 = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & -I_n & 0_n \end{bmatrix}^T, \]

\[ Z_{12} = \pi h_1 Z_1 + \kappa \tau Z_2, \quad Z_{123} = Z_{12} + \theta \tau Z_3, \]

\[ \hat{f} = \max_{l \leq i \leq n} \left[ k_i^+, k_i^- \right] \]

\[ \Lambda_i = \text{diag} \{ \lambda_1, \lambda_2, \ldots, \lambda_m \}, \quad i = 1, 2 \]

\[ F_1 = \text{diag} \{ k_1^+, k_2^+, \ldots, k_m^+ \}, \]

\[ F_2 = \text{diag} \left\{ \frac{k_1^- + k_1^+}{2}, \frac{k_2^- + k_2^+}{2}, \ldots, \frac{k_m^- + k_m^+}{2} \right\} \]

\[ h_{11} = h_1 - h_2, \]

\[ r_1 = \sum_{l = h_1}^{h_1 - 1} \sum_{i = l}^{l} 1 = \frac{(h_2 - h_1)(h_2 + h_1 + 1)}{2}, \quad r_2 = \sum_{l = h_1}^{h_1 - 1} \sum_{i = l}^{l} 1 = \frac{(h_2 - h_1)(h_2 - h_1 + 1)}{2} \quad (87) \]

\[ \varepsilon_1 = \sum_{j = h_1}^{h_1} \gamma^{j-1} \quad \gamma = 1 \]

\[ \varepsilon_2 = \sum_{j = h_1}^{h_1} \gamma^{j-1} \quad \gamma = 1 \]

\[ \varepsilon_3 = \sum_{i = h_1}^{h_1 + 1} \sum_{j = i}^{i} 1 \gamma^{j-1} \quad \gamma = 1 \]

\[ \varepsilon_4 = \sum_{i = h_1}^{h_1 + 1} \sum_{j = i}^{i} 1 \gamma^{j-1} \quad \gamma = 1 \]
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\[ \varepsilon_5 = \sum_{l=-h}^{1} \sum_{i=l}^{1} \sum_{j=i}^{1} \gamma^{i-j-1} \]  
\[ \varepsilon_6 = \sum_{l=-h}^{1} \sum_{i=l}^{1} \sum_{j=i}^{1} \gamma^{i-j-1} \]  

**Proof.** Define the following LKF for DNN (1): 

\[ V(k) = \sum_{i=1}^{6} V_i(k) \]  

with 

\[ V_1(k) = x^T(k)Px(k) \]  
\[ V_2(k) = \sum_{j=k-h}^{1} \gamma^{k-j-1}x^T(j)Q_1x(j) + \sum_{j=k-h}^{1} \gamma^{k-j-1}x^T(j)Q_2x(j) \]  
\[ V_3(k) = \sum_{j=k-h}^{1} \gamma^{k-j-1}x^T(j)Q_3x(j) + \sum_{i=h}^{1} \sum_{j=k-h}^{1} \gamma^{k-j-1}x^T(j)Q_3x(j) \]  
\[ V_4(k) = \sum_{j=k-h}^{1} \gamma^{k-j-1}f^T(x(j))Rf(x(j)) + \sum_{i=h}^{1} \sum_{j=k-h}^{1} \gamma^{k-j-1}f^T(x(j))Rf(x(j)) \]  
\[ V_5(k) = \kappa \sum_{i=h}^{1} \sum_{j=k-h}^{1} \gamma^{k-j-1} \eta^T(j)Z_1\eta(j) + \kappa \sum_{i=h}^{1} \sum_{j=k-h}^{1} \gamma^{k-j-1} \eta^T(j)Z_2\eta(j) \]  
\[ V_6(k) = \theta \sum_{i=h}^{1} \sum_{j=k-h}^{1} \sum_{i=i-1}^{1} \gamma^{k-j-1} \eta^T(j)Z_3\eta(j) \]  

where \( \eta(k) = x(k+1) - x(k) \), and \( \pi, \kappa \) and \( \theta \) are defined in (20). Calculating the difference of LKF (31), \( \Delta V(k) = V(k+1) - V(k) \), along the trajectories of DNN (1) gives 

\[ \Delta V_1(k) = (\gamma - 1)V_1(k) + x^T(k)(C^TPC - \gamma P)x(k) + 2x^T(k)C^TPAf(x(k)) \]  
\[ + 2x^T(k)C^TPA_d f(x(k - h(k))) \]  
\[ + f^T(x(k))A^TPAf(x(k)) + 2f^T(x(k))A^TPA_d f(x(k - h(k))) \]  
\[ + f^T(x(k - h(k)))A_d^TPA_d f(x(k - h(k))) \]  
\[ \Delta V_2(k) = (\gamma - 1)V_2(k) + x^T(k)(Q_1 + Q_2)x(k) - \gamma^h x^T(k-h_1)Q_1x(k-h_1) \]  
\[ - \gamma^h x^T(k-h_2)Q_2x(k-h_2) \]  

\[ \Delta V_3(k) = (\gamma - 1)V_3(k) + x^T(k)Q_3x(k) - \gamma x^T(k-h_1)Q_3x(k-h_1) \]  
\[ + \gamma x^T(k-h_2)Q_3x(k-h_2) \]  
\[ \Delta V_4(k) = (\gamma - 1)V_4(k) + x^T(k)Rf(x(k)) - \gamma^h x^T(k-h_1)Rf(x(k-h_1)) \]  
\[ - \gamma^h x^T(k-h_2)Rf(x(k-h_2)) \]  
\[ \Delta V_5(k) = (\gamma - 1)V_5(k) + x^T(k)Z_1\eta(k) - \gamma x^T(k-h_1)Z_1\eta(k-h_1) \]  
\[ + \gamma x^T(k-h_2)Z_1\eta(k-h_2) \]  
\[ \Delta V_6(k) = (\gamma - 1)V_6(k) + x^T(k)Z_2\eta(k) - \gamma x^T(k-h_1)Z_2\eta(k-h_1) \]  
\[ + \gamma x^T(k-h_2)Z_2\eta(k-h_2) \]  

(91)  
(92)  
(93)  
(94)  
(95)
\[ \Delta V_j(k) = \sum_{j=k+1}^{k+1} \gamma^{k-j} x^T(j) Q_k x(j) - \sum_{j=0}^{k} \gamma^{k-j} x^T(j) Q_k x(j) \]

\[ + \sum_{j=0}^{k} \sum_{j=0}^{k} \gamma^{k-j} x^T(j) Q_k x(j) - \sum_{j=0}^{k} \sum_{j=0}^{k} \gamma^{k-j} x^T(j) Q_k x(j) \]

(96)

\[ \leq (\gamma - 1) V_j(k) + (h_k - h_{k+1}) x^T(k) Q_k x(k) - \gamma^h x^T(k - h(k)) Q_k x(k - h(k)) \]

\[ \Delta V_j(k) \leq (\gamma - 1) V_j(k) + (h_k - h_{k+1}) f^T(x(k)) R_f(x(k)) \]

(97)

\[ - \gamma^h f^T(x(k - h(k))) R_f(x(k - h(k))) \]

\[ \Delta V_j(k) = (\gamma - 1) V_j(k) + \eta^T(k) Z_j \eta(k) - \pi \sum_{j=k-h}^{k-h-1} \gamma^{k-j} \eta^T(j) Z_j \eta(j) \]

(98)

\[ - \kappa \sum_{j=k-h}^{k-h-1} \sum_{j=1}^{j-k-h-1} \gamma^{k-j} \eta^T(j) Z_j \eta(j) \]

\[ \Delta V_j(k) = (\gamma - 1) V_j(k) + (\gamma - 1) \eta^T(k) Z_j \eta(k) - \delta \sum_{j=k-h}^{k-h-1} \sum_{j=1}^{j-k-h-1} \gamma^{k-j} \eta^T(i) Z_j \eta(i) \]

(99)

where \( \tau_1 \), \( \tau_2 \) and \( Z_{12} \) are defined in (84) and (87).

Based on Lemma 1, we have

\[ \pi \sum_{j=k-h}^{k-h-1} \gamma^{k-j} \eta^T(j) Z_j \eta(j) = \pi \sum_{j=k-h}^{k-h-1} \gamma^{k-j} \eta^T(j) Z_j \eta(j) + \pi \sum_{j=k-h}^{k-h-1} \gamma^{k-j} \eta^T(j) Z_j \eta(j) \]

\[ \geq \frac{1}{\pi_1(k)} \left( \sum_{j=k-h}^{k-h-1} \eta(j) \right)^T Z_j \left( \sum_{j=k-h}^{k-h-1} \eta(j) \right) + \frac{1}{\pi_2(k)} \left( \sum_{j=k-h}^{k-h-1} \eta(j) \right)^T Z_j \left( \sum_{j=k-h}^{k-h-1} \eta(j) \right) \]

(100)

where

\[ \pi_1(k) = \sum_{j=k-h}^{k-h-1} \gamma^{-(k-j)} = \begin{cases} h_k - h_{k+1}, & \gamma = 1 \\ \gamma - 1, & \gamma \neq 1 \end{cases} \]

\[ \pi_2(k) = \sum_{j=k-h}^{k-h-1} \gamma^{-(k-j)} = \begin{cases} h_k - h_{k+1}, & \gamma = 1 \\ \gamma - 1, & \gamma \neq 1 \end{cases} \]

(101)

\[ \frac{\pi_1(k)}{\pi} + \frac{\pi_2(k)}{\pi} = 1, \]

For \( \begin{bmatrix} Z_1 & V \\ * & Z_1 \end{bmatrix} > 0 \), using Lemma 2 we obtain
Taking the assumption regarding the activation functions (4) into account, we can obtain

\[
\begin{bmatrix}
    x(k) \\
f(x(k))
\end{bmatrix}
\begin{bmatrix}
    F_1\Lambda_1 & -F_2\Lambda_1 \\
    -F_2\Lambda_1 & \Lambda_1
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
f(x(k))
\end{bmatrix} \leq 0
\]
where $\Lambda_i$ and $F_i$, $i = 1, 2$ are defined by (86).

By combining the previous inequalities (32)-(99), (102), (103), and (105)-(107) we have

$$\Delta V(k) - (\gamma - 1) V(k) \leq \xi^T(k) \left( \Sigma - \Gamma(h(k)) \right) \xi(k)$$

where $\Sigma$ is given in (83). If

$$\Sigma - \Gamma(h(k)) \left[ \begin{array}{cc} Z_2 & U \\ * & Z_2 \end{array} \right] \Gamma^T(h(k)) < 0, \quad \forall h(k) \in [h_1, h_2]$$

then

$$\Delta V(k) - (\gamma - 1) V(k) < 0$$

From (109), based on Lemma 3, there exist a matrix $\Omega$ such that

$$\Phi(h(k)) = \left[ \begin{array}{cc} \Sigma + \Gamma(h(k)) \Omega^T + \Omega \Gamma^T(h(k)) & \Omega_1 \\ * & -Z_2 \\ * & -Z_2 \end{array} \right] < 0, \quad \forall h(k) \in [h_1, h_2]$$

(111)

Since $\Phi(h(k))$ is affine with respect to $h(k)$, it is necessary and sufficient to ensure that inequality (111) holds at the vertices of the interval $[h_1, h_2]$, i.e. $\Phi(h_1) < 0$ and $\Phi(h_2) < 0$, where $\Gamma(h_1) = \Gamma_1$ and $\Gamma(h_2) = \Gamma_2$. Thus, the inequality (111) holds if and only if (79) holds as well.

From (110), it follows

$$V(k) < \gamma V(k-1) < \gamma^2 V(k-2) < \cdots < \gamma^k V(0), \quad k = 1, 2, 3, \ldots$$

(112)

From (31) and (112), we obtain

$$V(0) \leq \alpha \lambda_{\max}(P) + \alpha \epsilon_1 \lambda_{\max}(Q_1) + \alpha \epsilon_2 \lambda_{\max}(Q_2) + \alpha \epsilon_2 \lambda_{\max}(Q_1) + \alpha \epsilon_3 \lambda_{\max}(Q_2) + \alpha \epsilon_3 \lambda_{\max}(Q_1)$$

$$+ \tilde{f}^T \epsilon_1 \lambda_{\max}(R) + \tilde{f}^T \epsilon_2 \lambda_{\max}(R) + \alpha \delta c_1 \lambda_{\max}(Z_1) + \alpha \delta c_2 \lambda_{\max}(Z_2) + \alpha \delta c_3 \lambda_{\max}(Z_3)$$

(113)

where $\epsilon_i$, $i = 1, 2, \ldots, 6$ and $\tilde{f}^T$ are defined in Theorem 3. On the other hand, we have

$$V(k) \geq \lambda_{\min}(P) x^T(k) x(k)$$

(114)

From (112), (113) and (114), for $\gamma > 1$, we can obtain

$$\lambda_{\min}(P) x^T(k) x(k) \leq \gamma^k \left[ \alpha \lambda_{\max}(P) + \alpha \epsilon_1 \lambda_{\max}(Q_1) + \alpha \epsilon_2 \lambda_{\max}(Q_2) + \alpha \epsilon_3 \lambda_{\max}(Q_1) + \alpha \delta c_1 \lambda_{\max}(Z_1) + \alpha \delta c_2 \lambda_{\max}(Z_2) + \alpha \delta c_3 \lambda_{\max}(Z_3) \right]$$

(115)
If
\[
\frac{\gamma^N}{\lambda_{\text{min}}(P)} + \alpha \varepsilon_1 \lambda_{\text{max}}(Q_1) + \alpha \varepsilon_2 \lambda_{\text{max}}(Q_2) + \alpha(e_2 + \varepsilon_1) \lambda_{\text{max}}(Q_3) + f^2(e_2 + \varepsilon_1) \lambda_{\text{max}}(R) + \gamma \delta_2 \lambda_{\text{max}}(Z_1) + \delta_3 \lambda_{\text{max}}(Z_2) + \mu \lambda_{\text{max}}(Z_3) < \beta
\]
then we obtain \( x^T(k)x(k) < \beta, \forall k \in \{1, 2, \ldots, N\} \), and the system (1) is finite-time stable with respect to \((\alpha, \beta, N)\).

Let
\[
\lambda_1 I < \lambda_{\text{max}}(P) < \lambda_2 I, \quad \lambda_{\text{max}}(Q_1) < \lambda_3, \quad \lambda_{\text{max}}(Q_2) < \lambda_4, \quad \lambda_{\text{max}}(Q_3) < \lambda_5, \quad \lambda_{\text{max}}(R) < \lambda_6, \quad \lambda_{\text{max}}(Z_1) < \lambda_7, \quad \lambda_{\text{max}}(Z_2) < \lambda_8, \quad \lambda_{\text{max}}(Z_3) < \lambda_9
\]
(117)

Then, the conditions (81) and (82) follow from (116) and (117).

Now, we focus on the finite-time passivity of DNN (1) with time-varying delay.

**Theorem 4.** [16] Under Assumption 1, DNN (1) is finite time passive with respect to \((\alpha, \beta, N, \mu)\), where \(0 \leq \alpha < \beta, \mu \) is a given positive scalar if there exist a scalar \(\gamma > 1\), positive scalars \(\lambda_i, i = 1, 2, \ldots, 9\), positive definite matrices \(P, Q_1, Q_2, R, Z_1, Z_2,\) and \(Z_3\), positive definite diagonal matrices \(\Lambda_1\) and \(\Lambda_2\), matrices \(U, V\) and \(\Omega = [\Omega_1 \Omega_2]\) such that the LMIs (79)-(82) hold and:
\[
\begin{bmatrix}
S + \Gamma \Omega^T + \Omega \Gamma^T + \Omega_1 + \Omega_2 & * & -Z_2 & -U \\
* & * & * & -Z_2
\end{bmatrix} < 0, \quad i = 1, 2
\]
(118)

where
\[
\hat{\Sigma} = \begin{bmatrix}
\Sigma & \gamma I \\
\gamma^T & -\mu I_n
\end{bmatrix}, \quad \gamma = \begin{bmatrix}0_{4 \times n} \end{bmatrix} \in \mathbb{R}^{4 \times n},
\]
\[
\hat{\Gamma}_1 = \begin{bmatrix}
\Gamma_1 \\
0_{n \times 2n}
\end{bmatrix}, \quad \hat{\Gamma}_2 = \begin{bmatrix}
\Gamma_2 \\
0_{n \times 2n}
\end{bmatrix}
\]
(119)

in which \(\Sigma, \Gamma_1\) and \(\Gamma_2\) are defined as in (83) and (84).

**Proof.** To show the passivity, we chose LKF (31). Then, from the proof of Theorem 3, we can write
\[
\Delta V(k) - (\gamma - 1)V(k) - 2y^T(k)u(k) - \mu \dot{y}^T(k)u(k)
\]
\[
\leq \xi^T(k) \begin{bmatrix}
\Sigma - \Gamma(h(k)) & Z_2 & U \\
Z_2 & * & \Gamma(h(k))
\end{bmatrix} \xi(k) - 2f^T(x(k))u(k) - \mu \dot{y}^T(k)u(k)
\]
\[
= \xi^T(k) \begin{bmatrix}
\hat{S} - \hat{\Gamma}(h(k)) & Z_2 & U \\
Z_2 & * & \hat{\Gamma}(h(k))
\end{bmatrix} \xi(k)
\]
(121)
where
\[ \hat{\xi}(k) = \left[ \xi^T(k) \ u^T(k) \right]^T, \quad \hat{\Gamma}(h(k)) = \begin{bmatrix} \Gamma(h(k)) \\ 0_{m \times 2n} \end{bmatrix} \] (122)
and \( \hat{\Sigma} \) is defined by (119). If
\[ \hat{\Sigma} - \hat{\Gamma}(h(k)) \begin{bmatrix} Z_2 & U \\ * & Z_2 \end{bmatrix} \hat{\Gamma}^T(h(k)) < 0, \quad \forall h(k) \in [h_1, h_2] \] (123)
then
\[ \Delta V(k) - (\gamma - 1) V(k) - 2 y^T(k) u(k) - \mu a^T(k) u(k) < 0 \] (124)
Based on Lemma 3, then from (123) we obtain
\[ \hat{\Phi}(h(k)) = \begin{bmatrix} \hat{\Sigma} + \hat{\Gamma}(h(k)) \Omega^T + \Omega \hat{\Gamma}^T(h(k)) & \Omega_1 \\ * & -Z_2 \\ * & * \end{bmatrix} < 0, \quad \forall h(k) \in [h_1, h_2] \] (125)
Since \( \hat{\Phi}(h(k)) \) is affine with respect to \( h(k) \), it is necessary and sufficient to ensure that inequality (125) holds at the vertices of the interval \([h_1, h_2] \), i.e. \( \hat{\Phi}(h_1) < 0 \) and \( \hat{\Phi}(h_2) < 0 \), where \( \hat{\Gamma}(h_1) = \hat{\Gamma}_1 \) and \( \hat{\Gamma}(h_2) = \hat{\Gamma}_2 \). Thus, the inequality (125) holds if and only if (118) holds as well.

From (124), we get
\[ 0 < V(k) < \gamma V(0) + 2 \sum_{j=0}^{k-1} \gamma^{k-j-1} y^T(j) u(j) + \mu \sum_{j=0}^{k-1} \gamma^{k-j-1} u^T(j) u(j) \] (126)
Under the zero initial condition, for \( k \in \{1, 2, \ldots, N\} \) one has
\[ 2 \sum_{j=0}^{k-1} \gamma^{k-j-1} y^T(j) u(j) + \mu \sum_{j=0}^{k-1} \gamma^{k-j-1} u^T(j) u(j) \geq 0 \] (127)
i.e.
\[ 2 \sum_{j=0}^{k-1} \gamma^{k-j-1} y^T(j) u(j) \geq -\mu \sum_{j=0}^{k-1} \gamma^{k-j-1} u^T(j) u(j) \] (128)
Thus, the system (1) is finite-time passive.

**Remark 8.** In [16], the properties of exponential passivity (Definition 2) and finite-time passivity (Theorem 4) for DNNs with time-varying delay are introduced for the first time, in which FTP is based on the concept of FTS (Theorem 3). Since FTS in one part uses the property of exponential stability (see inequality (110)), then in the proof of Theorem 4, the condition (15) has appeared as expected. Unlike the concept of FTP, a well-known concept for the passivity based on Lyapunov asymptotic stability has already been reported in the literature. Note that a sufficient condition for the passivity of DNNs can be obtained when \( \gamma = 1 \) in Theorem 4.

**Remark 9.** Using Lemma 3, inequalities (79) and (118) can be transformed in the following equivalent forms
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\[ \Sigma < \Gamma_1 \begin{bmatrix} Z_2 & U \\ U^T & Z_2 \end{bmatrix} \Gamma_1^T = \begin{bmatrix} h_2^2Z_2 & 0_{nx5n} & -h_2Z_2 & -h_2U \\ * & 0_{5nx5n} & 0_{5nxn} & 0_{5nxn} \\ * & * & Z_2 & U \\ * & * & * & Z_2 \end{bmatrix} = \Xi_1 \] (129)

\[ \Sigma < \Gamma_2 \begin{bmatrix} Z_2 & U \\ U^T & Z_2 \end{bmatrix} \Gamma_2^T = \begin{bmatrix} h_2^2Z_2 & 0_{nx5n} & -h_2U^T & -h_2Z_2 \\ * & 0_{5nx5n} & 0_{5nxn} & 0_{5nxn} \\ * & * & Z_2 & U \\ * & * & * & Z_2 \end{bmatrix} = \Xi_2 \] (130)

\[ \hat{\Sigma} < \hat{\Gamma}_1 \begin{bmatrix} Z_2 & U \\ Z_2 & U \end{bmatrix} \hat{\Gamma}_1^T = \begin{bmatrix} \Xi_1 \\ 0_{5nxn} \\ 0_n \end{bmatrix} \] (131)

\[ \hat{\Sigma} < \hat{\Gamma}_2 \begin{bmatrix} Z_2 & U \\ Z_2 & U \end{bmatrix} \hat{\Gamma}_2^T = \begin{bmatrix} \Xi_2 \\ 0_{5nxn} \\ 0_n \end{bmatrix} \] (132)

The above inequalities will be used in numerical calculations.

**Remark 10.** The proposed stability criteria depend on the parameters \( \gamma, \alpha, \beta, N \) as well as time delay \( h_1 \) and \( h_2 \). Consider the following cases.

- \( \beta_{\text{min}} \) is the minimum allowable lower bound (MALB) of \( \beta \) such that the concerned DNN is FTS (FTP) for any \( \beta \geq \beta_{\text{min}} \).
- \( h_{\text{max}} = h_{2\text{max}} \) is the maximum allowable upper bound (MAUB) of the variable delay \( h(k) \) such that the concerned DNN is FTS (FTP) for any value when the delay \( \leq h_{\text{max}} \).

Noting that a criterion that gives a lower MALB value or a higher MAUB value is less conservative with respect to the other criteria.

**Remark 11.** In [16], the criteria of FTS are defined with respect to the adjustable parameter \( \gamma \) and the new LKF \( V(\psi) \) with power function \( \gamma^{-1} \) is proposed. From the difference of LKF given in (52), \( \Delta V(\psi) < (\gamma-1)V(\psi) \), it follows that the parameter \( \gamma \) provides an exponential convergence information about an upper bound of LKF given in (54), \( V(\psi) < \gamma^\psi V(0) \), \( \psi = 1, 2, 3, \ldots \). On the other hand, parameterized condition (6) with power function \( \gamma^{-1} \) is obtained while analyzing the FTP criterion. This condition can be understood as a condition of exponential passivity with adjustable parameter \( \gamma \). In the case of \( \gamma = 1 \), the condition (6) reduces to the ordinary passivity defined in the existing literature. The parameter \( \gamma \) represent an optimization parameter in the FTP criterion. By solving the corresponding LMIs for FTP, a specific value for \( \gamma \) is obtained and the exponential passivity condition (6) is satisfied.

**Remark 12.** The results proposed in [16] are delay-dependent, but do not depend on the shape of the time delay. Therefore, the delay may be a random variable. In the case of switching systems, the considered FTP problem can also be solved. In this case, it is necessary to modify the Lyapunov functional in accordance with the given switching systems.
4. DEMONSTRATIVE EXAMPLES

In the section, three numerical examples [16, 56] are provided to illustrate the effectiveness of the proposed stability criteria.

Example 1. Consider the DNN (1) with the following parameters [56]
\[
C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \quad A_j = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}
\]
which was used to check the feasible region of stability criteria in [24, 31, 56, 57, 58]. The activation function is in the form of
\[
f(x) = \begin{bmatrix} \tanh(x_j) \\ \tanh(x_z) \end{bmatrix}
\]
and satisfy Assumption 1 with \( k^-_1 = k^-_z = 0 \) and \( k^+_1 = k^+_z = 1 \). By using Theorem 1, the MAUBs of \( h_{2\max} \) are computed with different lower bounds \( h_l \) and the obtained results are given in Tab. 1. The number of decision variables (NDVs) is also given to show computation complexity. From this table, we can see that proposed results are less conservative and/or have less decision variables than [24, 31, 57, 58].

The time-varying delay and the state trajectories of the DNN (1) with parameters (133) for initial value \( \phi(j) = [3 \ 1]^T, j \in \{-22, -21, \ldots, -1, 0\} \) are shown in Fig. 1 and Fig. 2, which shows that the DNN (1) is asymptotically stable for \( 10 \leq h(k) \leq 22 \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\textbf{Method} & \textbf{2} & \textbf{4} & \textbf{6} & \textbf{8} & \textbf{10} & \textbf{15} & \textbf{20} & \textbf{NDVs} \\
\hline
Th. 2 [57] & 19 & 19 & 20 & 21 & 22 & 24 & 27 & \( 17.5n^2 + 8.5n \) \\
Th. 1 [57] & 20 & 20 & 21 & 21 & 22 & 24 & 27 & \( 53.5n^2 + 8.5n \) \\
Th. 1 [31] & 20 & 20 & 21 & 21 & 22 & 24 & - & \( 13.5n^2 + 11.5n \) \\
Th. 3.1 [24] & 19 & 19 & 20 & 20 & 21 & 24 & 27 & \( 11n^2 + 6n \) \\
Cor. 2 [58] & 19 & 20 & 20 & 21 & 21 & 24 & 27 & \( 154n^2 + 6n \) \\
\hline
Theorem 1 & 19 & 20 & 20 & 21 & 22 & 24 & 27 & \( 17n^2 + 9n \) \\
Theorem 2 & 20 & 20 & 21 & 21 & 22 & 24 & 27 & \( 23n^2 + 11n \) \\
\hline
\end{tabular}
\end{table}

Fig. 1 The time-varying delay

Fig. 2 State trajectories of the DNN
Example 2. Consider DNN (1) with the following parameters [31], [56]

\[
C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad A = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.004 \end{bmatrix}, \quad A_t = \begin{bmatrix} -0.01 & 0.01 \\ -0.02 & -0.01 \end{bmatrix}
\]

and activation functions that satisfy Assumption 1 with \( k_i^- = k_i^+ = 0 \) and \( k_i^+ = k_i^+ = 1 \). In this example, the MAUBs of \( h_{\text{max}} \) are computed by using Theorem 1 and 2 for different lower bounds \( h \), and results are shown in Tab. 2. From this table, it can be seen that the proposed stability criteria in Theorems 1 and 2, provide a significantly larger MAUB than the previous result [31], while the numbers of decision variables are slightly higher.

| Method | 2   | 4   | 6   | 8   | 10  | 20  | NDVs         |
|-------|-----|-----|-----|-----|-----|-----|-------------|
| Th. 1 [31] | 99  | 101 | 103 | 105 | 107 | 117 | 13.5n^2 + 11.5n |
| Theorem 1 | 2028 | 2030 | 2032 | 2034 | 2036 | 2046 | 17n^2 + 9n   |
| Theorem 2 | 3140 | 3142 | 3144 | 3146 | 3148 | 3158 | 23n^2 + 11n |

Example 3. Consider DNN (1) with the following parameters [16], [48]

\[
C = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad A = \begin{bmatrix} -0.1 & 0.8 \\ 0.9 & 0.4 \end{bmatrix}, \quad A_t = \begin{bmatrix} -0.2 & -0.9 \\ 0.1 & 0.2 \end{bmatrix}
\]

The activation functions are described by \( f(x(k)) = [\tanh(0.2x_1(k)) \tan(-0.14x_2(k))] \). Therefore, it is easy to see that \( F_1 = \text{diag}(0, 0) \) and \( F_2 = \text{diag}(0.1, -0.07) \).

First, the FTS of DNN (1) is investigated. Let \( h_1 = 1 \) and \( h_2 = 3 \), then by solving LMIs in Theorem 3 for \( \alpha = 1, \beta = 2, N = 30, \delta = 2.5 \) and \( \gamma = 1.01 \) we obtain the feasible solutions. Thus, the system (1) with parameters (133) is finite-time stable with respect to (1, 2, 30). For time-varying delay \( h(k) = 2 + \sin(k\pi/2) \) and the initial condition \([-0.5 \ 0.5]^T \) Fig. 3 shows the state response \( x(k) \) of DNN (1) with the parameters (133) and Fig. 4 depicts the norm \( \|x(k)\| \) of the state vector of the system (1).

Furthermore, by using Theorem 3, MALBs of \( \beta_{\text{min}} \) are calculated for \( \alpha = 1, N = 30, \delta = 2.5, h_1 = 1, \gamma = 1.0001 \) and \( h_2 \in \{3, 4, 5, 8, 10\} \), and results are listed in Tab. 3.

| \( h_2 \) | 3  | 4  | 5  | 8  | 10 | 20  |
|-----------|----|----|----|----|----|-----|
| Theorem 3 | 1.19 | 1.32 | 1.49 | 2.23 | 2.96 | 4.10 |

Based on Fig. 4, we can see that the norm \( x(k) \) does not exceed MALB \( \beta_{\text{min}} \) in Tab. 3, which means that the above system is FTS with respect to (3, \( \beta_{\text{min}}, N \)), \( N \in (5, 10, 20, 40) \).

For different values of \( N \in (50, 100, 150, 200) \) MALBs of \( h_{\text{max}} \) are computed for \( \alpha = 1, \beta = 1000, \delta = 2.5 \) and results are shown in Tab. 4. These results are mutually equal because the considered system is asymptotic stable \( (\gamma \to 1 \Rightarrow \gamma^N \to 1) \), so the inequality (82) becomes insensitive to the parameter \( N \).
Table 4 MAUBs of $h_{\text{max}}$ for a given $N$ when $h_1 = 5$, $\alpha = 1$, $\beta = 10$ and $\gamma = 1.0001$ in Example 3

| $N$ | 50 | 100 | 150 | 200 | 500 |
|-----|----|-----|-----|-----|-----|
| Theorem 3 | 20 | 20 | 20 | 20 | 20 |

Second, we consider FTP of DNN (1) with parameters (133). By applying Theorem 4, we can obtain MALBs of $\beta_{\text{min}}$ for $\alpha = 1$, $\delta = 2.5$, $N = 30$, $h_1 = 1$ and $\mu$ for given $h_2$, which guarantees DNN (1) with parameters (133) is FTP in the sense of Definition 2. In addition, FTP of DNN (1) with parameters (133) is analyzed by using the same method in [48]. The obtained results are addressed in Tab. 5, which demonstrates that this approach provides less conservative results than the approach in [48].

Table 5 MALBs of $\beta_{\text{min}}$ for a given $h_2$ when $\alpha = 1$, $\delta = 2.5$, $N = 30$ and $\gamma = 1.02$ in Example 3.

| $h_2$ | 3 | 4 | 5 | 8 | 10 |
|-------|---|---|---|---|----|
| 4.09  | 4.67 | 5.34 | 7.18 | 9.95 |
| Theorem 4 | 2.15 | 2.38 | 2.67 | 4.02 | 5.38 |
| $\beta_{\text{min}} / \mu$ | 8.90 | 2.82 | 14.22 | 72.30 | 36.19 |

In order to check the conditions of passivity, the following scalar is defined

$$ CP = 2 \sum_{j=0}^{\psi} \gamma^{-j} y^T(j)u(j) + \mu \sum_{j=0}^{\psi} u^T(j)u(j), \quad \forall \psi \in \{1, 2, \ldots, N\} \tag{136} $$

Based on (15), the system will be passive if $CP \geq 0$. By using results of Theorem 4, the values of CP are computed from (136) for $\alpha = 1$, $\delta = 2.5$, $N = 30$, $\gamma = 1.02$ and given $h_2$ and listed in Tab. 6. Since the values of CP are positive, it can be concluded that the passivity condition (15) is satisfied.


Table 6 Checking the passivity condition (15) by using the scalar CP defined in (136) for \( \alpha = 1 \), \( \delta = 2.5 \), \( N = 30 \), \( \gamma = 1.02 \) and given \( h_{2} \) in Example 3.

| \( h_{2} \) | CP       |
|-----------|----------|
| 3         | 69.25    |
| 4         | 23.11    |
| 5         | 109.55   |
| 8         | 549.79   |
| 10        | 276.06   |

**Remark 13.** To reduce the number decision variables, the well-known reciprocally convex combination approach is used in [16]. It is noted that, the addition of slack variables \( U1 \), \( U2 \), \( U3 \) and \( U4 \) in [48] will increase the number of decision variables. The number of decision variables in [48] is \( 15n^{2} / 2 + 11n / 2 + 10 \), while the number of decision variable of Theorem 4 is slightly smaller and amounts to \( 6n^{2} + 6n + 10 \).

**Remark 14.** From Example 3 and Remark 13 it can be concluded that the results in [16] are less conservative then results proposed in [48]. According to the author’s knowledge, no other references, except [48], deal with the problem of FTP for delayed DNNs. Other references in the literature of [16] consider different stability concepts from that used in [16].

5. CONCLUSION

This paper deals with the stability, finite-time stability and passivity problems of delayed DNNs [16, 56]. First, the Lyapunov-Krasovskii functional with single and double summation terms and several augmented vectors is constructed by decomposing the time-delay interval into two non-equidistant subintervals. Using the Wirtinger-based inequality, extended reciprocally convex approach and several zero equalities, stability conditions are developed in the form of linear matrix inequalities. Second, a delay-dependent criterion has been established to ensure that the considered DNN is passive. By constructing the proper LKF and using LMI techniques, sufficient passivity conditions are obtained. Moreover, some weighted summation inequalities are proposed to obtain less conservative results. Finally, several numerical examples are presented to show the effectiveness of the proposed methods.

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