A Review of Dynamic NURBS Approach

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Abstract. Dynamic NURBS, also called D-NURBS, is a known dynamic version of the nonuniform rational B-spline (NURBS) which integrates free-form shape representation and a physically-based model in a unified framework. More recently, computer aided design (CAD) and finite element (FEM) community realized the need to unify CAD and FEM descriptions which motivates a review of D-NURBS concepts. Therefore, in this paper we describe D-NURBS theory in the context of 1D shape deformations. We start with a revision of NURBS for parametric representation of curve spaces. Then, the Lagrangian mechanics is introduced in order to complete the theoretical background. Next, the D-NURBS framework for 1D curve spaces is presented as well as some details about constraints and numerical implementations. In the experimental results, we focus on parameters choice and computational cost.

1 Introduction

In the context of animation of soft objects every engine is composed by three linked parts: the geometric model, dynamic model and rendering module. The former can be realized in the context of parametric frameworks like nonuniform rational B-spline (NURBS) [Piegl and Tiller 1997, Farin 1997]. The dynamic model needs physic models that incorporate dynamic quantities like velocity, mass and force distributions, into an evolution equation that governs the shape deformation [Erleben et al. 2005]. The latter includes global/local illumination techniques to generate the scene with the desired realism [?]. In this work we focus only on the first two components.

Non-uniform Rational B-spline (NURBS) is a mathematical framework commonly used for generating and representing curves, surfaces and volumes [Piegl and Tiller 1997]. It offers an unified mathematical basis to describe analytic and free-form shapes with great flexibility and precision. NURBS became a standard for CAD (Computer Aided Design) systems due to its excellent mathematical, numeric and algorithmic properties. NURBS are built from the B-spline function basis and a NURBS curve is a composition of NURBS functions, a set of control points \( \{p_1, p_2, \cdots, p_n\} \subset \mathbb{R}^3 \) and a weight vector \((w_1, w_2, \cdots, w_n)\). The control points and the weights compose the degrees of freedom of the NURBS curve.

For computer graphics applications, the dynamic model in general is based on classical mechanics which is concerned with physical laws to describe the behavior of a macroscopic system under the action of forces [Deusen et al. 2004]. For instance, when considering a particle in the 3D space under the action of gravity, we can take its position vector along the time \( t \), which in cartesian coordinates is given by \( (x(t), y(t), z(t)) \), and use the Newton’s laws to get the governing equation written in terms of the cartesian coordinates and the time \( t \). In a more general situation, the instantaneous configuration of a system may be described by the values of \( n \) generalized coordinates \( (p_1, p_2, \cdots, p_n) \). So, we need a methodology to write the evolution equation of the system in terms of the generalized coordinates.

The Lagrangian formulation of mechanics is a framework to address this issue [Goldstein 1981]. It is a variational formulation of mechanics based on the integral Hamilton’s Principle which states that the motion of the system between times \( t_1 \) and \( t_2 \) is derivable from the solution of a variational problem [Goldstein 1981].
The corresponding Lagrange’s equations allow to write the evolution of the system in term of the generalized coordinates. That is what we need to link the geometric model of NURBS and the dynamic model: we can use the control points and the weights as generalized coordinates to describe the physical system. Therefore, we get an approach that integrates shape representation and a dynamic model in a unified framework called D-NURBS in the literature [Terzopoulos and Qin 1994, Qin and Terzopoulos 1996].

Continuous systems, like an elastic curve, have infinite degrees of freedom which difficult its description for both the geometric and dynamic aspects. In mathematical terms, we are dealing with infinite basis functions, may be uncountable. One possibility to simplify the problem is to consider finite dimensional representation with enough flexibility in order to represent the solution with the desired precision. In the context of mechanical systems the Finite Element Method (FEM) is the traditional way to perform this task. However, as pointed out in [Cottrell et al. 2009], NURBS framework can be also considered. That is way geometric modeling and FEM community realized the need to unify CAD and FEM descriptions which motivates our review of NURBS and D-NURBS concepts.

So, in section 2 we start with an objective review of B-splines functions in order to set up the background for NURBS development. Next, in section 3 we describe the Lagrangian mechanics framework in the presence of constraints and a generalized potential for dissipation forces. Then, section 4 considers the D-NURBS model following the presentation given in [Terzopoulos and Qin 1994]. We present details of the evolution equation generation, constraints introduction and numerical aspects. For simplicity, we focus on curve spaces but the theory can be straightforward generalize for surfaces and volumes. In the experimental results (section 5), we consider a set up for a linear mass distribution with fixed endpoints. We discuss the influence of parameters choice, effects of NURBS weights and computational cost. The conclusions and further works are presented in section 6. The appendices A and B give some details about specific terms of the D-NURBS governing equation.

2 NURBS: Nonuniform Rational B-spline

The spline framework is the starting point for NURBS development. A polynomial spline of order \( k \) (degree \( k - 1 \)) is a piecewise polynomial function of order \( k \) with continuity of derivative of order \( k - 2 \) at the common joints between segments, which are called patches [Rogers and Adams 1976, Persiano 1996].

Therefore, the spline space is a functional space composed by piecewise polynomial functions with the property already stated. A fundamental element in the spline theory is the knot vector which defines the end points of the patches of the spline function. Given the order \( k \) and a knot vector \( \nu = (u_0, u_1, \cdots, u_n) \), we can denote the space of polynomial splines of order \( k \) with domain in the range \([u_0, u_n]\) as \( S^k(u_0, u_1, \cdots, u_n) \). The Figure 1 shows some elements of this set when \( k = 1, 2, 3 \).

We can show that the \( S^k(u_0, u_1, \cdots, u_n) \) is a vector space of dimension \( n - k + 1 \) [Persiano 1996]. The main point in the spline theory is to construct a basis for this space. From the functional analysis viewpoint the space properties are invariant respect to the basis choice. However, for computer graphics aspects it is important that every tool and algorithm generated has an intuitive geometric and visual interpretation with local control of the target objects. The B-spline basis attend these requirements.

Following traditional texts in this area [Farin 1997, Rogers and Adams 1976] we perform a recursive definition of the B-spline basis. So, let us consider the \( S^1(u_0, u_1, \cdots, u_n) \); that means, the space of piecewise polynomial functions of order \( k = 1 \) (degree 0) which are just piecewise constant functions, like the one presented on Figure 1 for \( k = 1 \).

A basis for this space is in fact the first B-spline basis in our recursive scheme, which is defined as follows:

\[
B_{i,1}(u) = \begin{cases} 
1, & \text{if } u_i \leq u < u_{i+1} \\
0, & \text{otherwise}
\end{cases}
\]  

(1)

for \( i = 0, 1, \cdots, n - 1 \).

Now, let us consider the space \( S^2(u_0, u_1, \cdots, u_n) \); that is, the space of piecewise polynomial functions of order \( k = 2 \) (degree \( k - 1 = 1 \)) which are just piecewise linear functions with continuity of derivative of order
Figure 1: Polynomial spline examples with knot vector $v = (t_0, t_1, t_2, t_3, t_4, t_5)$.

$k - 2 = 0$ (see Figure 1). We are supposing that $u_0 < u_1 < \cdots < u_n$. We already know that this is a vector space with dimension $n - k + 1 = n - 1$. Besides, when integrating polynomial functions of order $k$ we get again polynomial functions but with order $k + 1$. Also, we want that the support of the functions $B_{i,2}$ would be as small as possible (for local geometric control) and that they have continuity of derivative of order $k - 2 = 0$ at the common joints between patches. The following functions fulfill these requirements:

$$B_{i,2}(u) = \int_{-\infty}^{u} \left( \frac{B_{i,1}(s)}{u_{i+1} - u_i} - \frac{B_{i+1,1}(s)}{u_{i+2} - u_{i+1}} \right) ds, \quad i = 0, 1, \cdots, n - 2.$$  \hfill (2)

By repeating the above arguments, we can show that the following recursive scheme will generate a basis $B^k = \{B_{i,k}, \quad i = 0, 1, \cdots, n - k - 1\}$ for splines $f : [u_0, u_n] \rightarrow \mathbb{R}$ such that $f(u_0) = f(u_n) = 0$:

$$B_{i,k}(u) = \frac{(u - u_i) B_{i,k-1}(u)}{u_{i+k-1} - u_i} + \frac{(u_{i+k} - u) B_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}, \quad i = 0, 1, \cdots, n - k \hfill (3)$$

where $k = 2, 3, \cdots$ and the $B_{i,1}(u)$ is given by expression (1). The Figure 2 pictures the obtained basis for $k = 2$.

So, in the above development, the span of $B^k$ is in fact a subspace of $S^k(u_0, u_1, \cdots, u_n)$ once $B^k$ can only generate functions with support in the interval $(u_0, u_n)$ as we already observed above. However, we can cover all the spline space by considering more general knot vectors. In fact, the knot vector has a significant influence in the spline basis generated. In general, it is used three types of knot vectors: uniform, open uniform (or just open) and nonuniform.

Uniform knot vectors satisfies $u_{i+1} - u_i = \Delta u = \text{const.}$, for $i = 1, 2, \cdots, n$. Uniform knot vectors yield periodic uniform basis functions, like the one presented in Figure 3 that means:

$$B_{i,k}(u) = B_{i-1,k}(u - \Delta u) = B_{i+1,k}(u + \Delta u).$$

An open uniform knot vector has also the property $u_{i+1} - u_i = \Delta u$ for internal knots but it has multiplicity of knot values at the ends equal to the order $k$ of the B-spline functions. For instance:
Figure 2: B-spline of order $k = 2$ with knot vector $v = (u_0, u_1, u_2, u_3, u_4, u_5)$.

$$v = (0, 0, 1, 2, 3, 4, 5, 5), \text{ if } k = 2,$$

$$v = (0, 0, 0, 1, 0.2, 0.3, 0.4, 0.5, 0.5, 0.5), \text{ if } k = 3.$$

These kind of knot vectors may yield more general B-spline basis $B^k$ that can generate functions that are not null at the ends of the knot vector, as we can visualize in Figure[4]

Finally, nonuniform knot vectors may have either unequally spaced ($u_{i+1} - u_i = \Delta u_i$) and/or multiple knot values at the ends or even for the internal knots.

The B-splines generated by open (uniform or nonuniform) knot vectors have important properties [Piegl and Tiller 1997].

1. $B_{i,k} (u) \geq 0 \ \forall u$.

2. $B_{i,k} (u) = 0$ if $u$ is outside the interval $\in [u_i, u_{i+k+1})$.

3. Partition of unity: $\sum_{i=0}^{n-k} B_{i,k} (u) = 1$.

Once defined the basis for the spline space, we can consider curve spaces in $\mathbb{R}^3$ generated through B-splines. So, let us take a set of $p_i, i = 0, 1, 2, \cdots, n - k$ points in $\mathbb{R}^3$ and the vector-valued function given by:

$$c(u) = \sum_{i=0}^{n-k} p_i B_{i,k} (u).$$ (4)

This function defines a curve of class $C^{k-2}$ in $\mathbb{R}^3$, which is called a spline curve. The points $p_i$ are called control points and the corresponding polygon is the defining polygon. Important properties about these curves are:

1. End points interpolation: in the case of open knot vector we have $c(u_0) = p_0$ and $c(u_n) = p_n$.

2. Affine Invariance: If $\psi(r) = Ar + v$ is an affine transformation then $\psi(c(u)) = \sum_{i=0}^{n-k} B_{i,k} (u) \psi(p_i)$.
3. Strong convex hull property: the curve belongs to the convex hull of its control polygon.

A rational B-spline curve is the projection of a polynomial B-spline curve defined in the four-dimensional homogeneous coordinate space back into the three-dimensional physical space [Rogers and Adams 1976]. Therefore, if we represent the control points in the four-dimensional homogeneous coordinate space we obtain:

$$\tilde{p}_i = \left( \frac{w_i p_i}{w_i} \right), \quad i = 0, 1, 2, \ldots, n - k,$$

and applying expression (4) we get a spline curve in the four-dimensional homogeneous space:

$$\tilde{c}(u) = \sum_{i=0}^{n-k} \left( \frac{w_i p_i}{w_i} \right) B_{i,k}(u).$$

By projection in the three-dimensional space we obtain the rational curve:

$$c(u) = \frac{\sum_{i=0}^{n-k} p_i w_i B_{i,k}(u)}{\sum_{j=0}^{n-k} w_j B_{j,k}(u)} = \sum_{i=0}^{n-k} p_i N_{i,k}(u),$$

where $N_{i,k}$ are the rational B-spline functions given by:

$$N_{i,k}(u) = \frac{w_i B_{i,k}(u)}{\sum_{j=0}^{n-k} w_j B_{j,k}(u)}.$$

If the B-splines in expression 3 are generated by nonuniform knot vectors then the functions $R_{i,k}$ in expression 7 are named nonuniform rational B-splines (NURBS) and the curve defined by expression 6 is a NURBS curve.

B-splines can be enriched without modifying the underlying geometry and parameterization through the mechanisms that are called refinements. The most common mechanisms are knot insertion and degree elevation [Farin 1997, Piegl and Tiller 1997].

3 Lagrangian Mechanics

Let us consider a physical system whose instantaneous configuration may be described by the values of $n$ generalized coordinates $(p_1, p_2, \ldots, p_n)$ which can be considered as a point in a $n$ – dimensional Cartesian hyperspace
Figure 4: B-splines functions of order \( k = 2 \) with open uniform knot vector \( v = (0, 0, 1, 2, 3, 4, 5, 5) \).

known as configuration space. As time goes on from a time \( t_1 \) to a time \( t_2 \), the system changes its configuration due to internal and external forces. Therefore, the evolution of the system can be seen as a continuous path, or curve, \( p(t) \) in the configuration space, parameterized through the time \( t \).

The Hamilton’s Principle gives a methodology to write the evolution equation of the system in terms of the generalized coordinates and time \( t \). It states that if for a mechanical systems with kinetic energy \( T = T(p) \), where \( p = dp/dt \), all force fields are derivable from a scalar potential \( V = V(p, t) \) then the motion of the system from time \( t_1 \) to time \( t_2 \) is such that the line integral:

\[
I = \int_{t_1}^{t_2} L(p, \dot{p}, t) \, dt,
\]

where \( L(p, \dot{p}, t) = T(\dot{p}) - V(p, t) \), has a stationary value for the correct path of the motion [Goldstein 1981].

The function \( L \) is named the Lagrangian of the system and we can apply traditional techniques of the variational calculus to show that the correct path must satisfies:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}_i} \right) - \frac{\partial L}{\partial p_i} = 0, \quad i = 1, 2, \cdots, n,
\]

or, in a compact form:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}} \right) - \frac{\partial L}{\partial p} = 0, \quad i = 1, 2, \cdots, n,
\]

which are the Lagrange equations of motion [Goldstein 1981].

We can introduce dissipation forces in the Hamilton’ principle by adding a velocity-dependent term in the scalar potential of the system. So, let us consider the general form for the Lagrangian:

\[
L(p, \dot{p}, t) = T(\dot{p}) - \left( U(p) + F(p, \dot{p}) \right),
\]

where, like before, \( T \) is the kinetic energy but now possibly dependent from both \( p \) and \( \dot{p} \), \( U \) is the potential related to the conservative forces and \( F \) is a velocity-dependent potential to account for dissipative effects.
So, substituting this expression in the Euler-Lagrange equations (10) renders:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{p}} - \frac{\partial F}{\partial \dot{p}} \right) - \left( \frac{\partial T}{\partial p} - \frac{\partial U}{\partial p} - \frac{\partial F}{\partial p} \right) = 0 \] (12)

In general, mechanical systems undergoes effects of internal and external forces. Therefore, it is useful to decompose the potential \( U(p) \) into two terms named \( E_{int} \) and \( E_{ext} \), which will account for the internal and external forces, respectively:

\[ U(p) = E_{int}(p) + E_{ext}(p). \] (13)

By substituting expression (13) into the equations (12), we get:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{p}} - \frac{\partial F}{\partial \dot{p}} \right) + \left( \frac{\partial E_{int}}{\partial p} \right) = -\frac{\partial E_{ext}}{\partial p} + \left( \frac{\partial T}{\partial p} - \frac{\partial F}{\partial p} \right), \] (14)

which gives the general form of Euler-Lagrange equations.

### 3.1 Lagrange Equations with Constraints

Now, let us extend the Hamilton’ principle in order to cover constraints. We focus on holonomic constraints; or holonomic system, for which the constraints may be expressed by:

\[ f_1(p_1, p_2, \cdots, p_n, t) = 0, \]
\[ f_2(p_1, p_2, \cdots, p_n, t) = 0, \]
\[ \cdots \]
\[ f_m(p_1, p_2, \cdots, p_n, t) = 0, \] (15)

where \( f_l, l = 1, 2, \cdots, m, \) is a general expression connecting the generalized coordinates. In this case, we can take the differential \( df_l \):

\[ df_l = \sum_{k=1}^{n} \frac{\partial f_l}{\partial p_k} dp_k + \frac{\partial f_l}{\partial t} dt = 0, \ l = 1, 2, \cdots, m. \] (16)

If we consider \( dt = 0 \) and replace \( dp_k \) by the corresponding virtual displacement \( \delta p_k \) we can rewrite expression (16) as:

\[ \sum_{k=1}^{n} a_{lk} \delta p_k = 0, \ l = 1, 2, \cdots, m, \] (17)

where \( a_{lk} = \frac{\partial f_l}{\partial p_k} \). Expression (17) implies a dependence between the virtual displacements \( \delta p_k \). In order to reduce the number of virtual displacements to only independent ones we can use Lagrange multipliers \( \lambda_1, \lambda_2, \cdots, \lambda_m \). So, we can put together the equations (17) using the expression:

\[ \int_{t_1}^{t_2} \sum_{k=1}^{n} \sum_{l=1}^{m} \lambda_l a_{lk} \delta p_k = 0. \] (18)

Therefore, by assuming that the Hamilton’s principle holds for holonomic systems we can incorporate expression (18) in the variational technique used to get Lagrange equations (9) and to obtain:

\[ \int_{t_1}^{t_2} dt \sum_{k=1}^{n} \left( \frac{\partial L}{\partial p_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} \right) \delta p_k = 0. \] (19)
We shall remember that the virtual displacements $\delta q_k$ are connected by the $m$ equations (17). Besides, the Lagrange multipliers $\lambda_1, \lambda_2, \cdots, \lambda_m$ remains at our disposal. So, let us suppose that we can choose these multipliers such that:

$$\frac{\partial L}{\partial p_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial p_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0, \quad k = n - m + 1, \cdots, m. \quad (20)$$

By substituting this expression in the integral (19) we render:

$$\int_{t_1}^{t_2} dt \sum_{k=1}^{n-m} \left( \frac{\partial L}{\partial p_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial p_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} \right) \delta p_k = 0. \quad (21)$$

Once we have $m$ constraint equations in (17) the only virtual displacements $\delta p_k$ involved in expression (21) are the independent ones. Therefore, it follows that:

$$\frac{\partial L}{\partial p_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial p_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0, \quad k = 1, 2, \cdots, n - m. \quad (22)$$

Expressions (20) and (22) give the complete set of Lagrange’s equations for holonomic systems. However, the expressions involves $n + m$ unknowns, namely the $n$ coordinates $p_k$ and the $m$ multipliers $\lambda_l$. So, we must add to the final result the constraints give by expression (16).

Therefore, by putting together expressions (20), (22) and (15) we find that the desired solution must satisfies the equations:

$$\frac{\partial L}{\partial p_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial p_i} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0, \quad k = 1, 2, \cdots, n, \quad f_i(p_1, p_2, \cdots, p_n, t) = 0, \quad l = 1, 2, \cdots, m. \quad (24)$$

4 D-NURBS Formulation

The idea is to submit an initial NURBS curve, given by expression (6) to a Newtonian dynamics generated by an external potential, internal (elastic) and dissipation forces. Therefore, a natural way to parameterize the evolution of the curve along the time is:

$$c(u, t) = \frac{\sum_{i=0}^{n} p_i(t) w_i(t) B_{i,k}(u)}{\sum_{i=0}^{n} w_i(t) B_{i,k}(u)} \quad (25)$$

So, the control points $p_i(t)$ and the weights $w_i(t)$ becomes time-dependent while the rational functions remain $u$-dependent only. Therefore, the control points and the weights become the degrees of freedom of the system evolution and so, they compose the generalized coordinates which are concatenated as follows [Terzopoulos and Qin 1994]:

$$p(t) = \left[ (p_0^T, w_0), (p_1^T, w_1), \cdots, (p_n^T, w_n) \right]^T \in \mathbb{R}^{4(n+1)}, \quad (26)$$

where we have the control points vector and the weights vector specified, respectively, by:

$$p_b(t) = \left[ p_0^T, p_1^T, \cdots, p_n^T \right]^T \in \mathbb{R}^{3(n+1)}, \quad (27)$$

$$p_w(t) = \left[ w_0, w_1, \cdots, w_n \right]^T \in \mathbb{R}^{(n+1)}. \quad (28)$$

A fundamental element in the D-NURBS development is the associated Jacobian, defined as follows:

$$J = \left[ \left[ B_0(u, p), \frac{\partial e}{\partial w_0} \right], \left[ B_1(u, p), \frac{\partial e}{\partial w_1} \right], \cdots, \left[ B_n(u, p), \frac{\partial e}{\partial w_n} \right] \right] \in \mathbb{R}^{3 \times 4(n+1)} \quad (29)$$

where:
with (see expression 7):

\[ N_{i,k} (u, p_w) = \frac{w_i(t)B_{i,k} (u)}{\sum_{j=0}^{n} w_j(t)B_{j,k} (u)} \]  

and:

\[ \frac{\partial c}{\partial w_i} = \frac{\sum_{j=0}^{n} (p_i(t) - p_j(t))w_j B_{i,k} (u) B_{j,k} (u)}{\left( \sum_{j=0}^{n} w_j(t)B_{j,k} (u) \right)^2} \]  

We shall observe that \( B_i (u, p) \in \mathbb{R}^{3 \times 3} \) and \( \frac{\partial c}{\partial w_i} \in \mathbb{R}^{3} \), com \( i = 0, \ldots, n \) and consequently \( J \in \mathbb{R}^{3 \times 4(n+1)} \). We can concatenate the \( B_i \)'s and \( \frac{\partial c}{\partial w_i} \) according to the following matrices:

\[ B = [B_0 (u, p), B_1 (u, p), \cdots, B_n (u, p)] \in \mathbb{R}^{3 \times 3(n+1)} \]  

\[ W = \left[ \frac{\partial c}{\partial w_0}, \frac{\partial c}{\partial w_1}, \cdots, \frac{\partial c}{\partial w_n} \right] \in \mathbb{R}^{3 \times (n+1)} \]  

The advantages of defining the matrices \( J, B, W \) and the vectors \( p_b \) and \( p_w \) becomes clear by observing that:

\[
J \mathbf{p} = \begin{bmatrix} B_0 (u, p) \cdot \frac{\partial c}{\partial w_0}, & B_1 (u, p) \cdot \frac{\partial c}{\partial w_1}, & \cdots, & B_n (u, p) \cdot \frac{\partial c}{\partial w_n} \end{bmatrix} = 
\]

\[
= \frac{\sum_{i=0}^{n} w_i(t)B_{i,k}(u)p_i(t)}{\sum_{j=0}^{n} w_j(t)B_{j,k}(u)} + \sum_{i=0}^{n} \left( \frac{\sum_{j=0}^{n} (p_i(t) - p_j(t))w_j(t)B_{i,k}(u)B_{j,k}(u)}{\left( \sum_{j=0}^{n} w_j(t)B_{j,k}(u) \right)^2} \right) w_i(t) = 
\]

\[ B \mathbf{p}_b + W \mathbf{p}_w. \]  

But, with a simple algebra we can show that:

\[ W \mathbf{p}_w = 0. \]  

Therefore,

\[ J \mathbf{p} = B \mathbf{p}_b. \]  

However, by remembering expression (25) it is clear that:
\[ c(u, t) = Bp_b. \] (38)

Henceforth, from expressions (37) and (38) we get that:
\[ c(u, p) = Jp. \] (39)

Other important properties that can be easily proved are:
\[ \frac{dJ}{dt} \cdot p(t) = 0. \] (40)
\[ \frac{dc(u, p)}{dt} = J \cdot \frac{dp}{dt}. \] (41)

The next step is to compute the kinetic and (generalized) potential terms to be inserted in the Lagrangian given by expression (11).

### 4.1 Kinetic Energy \( T \)

In this work we focus on the D-NURBS formulation for a continuous parametric curve subject to a force field. So, we shall consider a (constant) linear mass density distribution \( \mu \). Therefore, the kinetic energy is computed by:
\[ T = \frac{1}{2} \int_\mu \left\| \frac{dc}{dt} \right\|^2 du, \] (42)
where \( \frac{dc}{dt} \) is the curve velocity. By applying expression (41) we observe that:
\[ \left\| \frac{dc}{dt} \right\|^2 = (Jp)T \cdot (Jp). \] (43)

So, if we insert expression (43) into kinetic energy (42) we obtain:
\[ T = \frac{1}{2} \int_\mu (Jp)^T \cdot (Jp) du. \] (44)

which becomes:
\[ T = \frac{1}{2} \int_\mu \mu p^T J^T J p du, \] (45)

Once \( p \) does not depend on the parameter \( u \), we can rewrite expression (45) as:
\[ T = \frac{1}{2} \hat{p}^T M \dot{\hat{p}}, \] (46)

where:
\[ M = M(p) = \int_\mu J^T J du \in \mathbb{R}^{(n+1) \times (n+1)}, \] (47)
is called the mass matrix.

### 4.2 Energy Dissipation \( F \)

Formally, the idea is to consider a velocity-dependent potential \( F \) such that, when introduced in the Euler-Lagrange equations (14) generates a velocity-dependent dissipative force. In order to perform this task let us suppose that \( F \) satisfies:
\[ \frac{dF}{dt} = -\frac{1}{2} \int_\mu \gamma \left\| \frac{dc}{dt} \right\|^2 du \quad \Rightarrow \quad F(t) = -\frac{1}{2} \int_{t=0}^t \int_\mu \gamma \left\| \frac{dc}{dt} \right\|^2 dudt, \] (48)
where the constant $\gamma$ is the damping density. By performing an analogous development of section 4.1 we obtain:

$$\frac{dF}{dt} = -\frac{1}{2}\dot{p}^T D \dot{p},$$  

where $D \in \mathbb{R}^{4(n+1) \times 4(n+1)}$, the damping matrix, is computed by:

$$D = D(p) = \int_u \gamma J^T J du.$$  

4.3 Potential for Conservative Forces

The internal and external conservative forces are introduced in the D-NURBS Lagrangian through the potentials $E_{\text{int}}$ and $E_{\text{ext}}$, respectively. We compute the former by using the thin-plate model [Terzopoulos and Fleischer 1988]:

$$E_{\text{int}}(p) = \frac{1}{2} \int_u \left( \alpha \left\| \frac{dc}{du} \right\|^2 + \beta \left\| \frac{d^2 c}{du^2} \right\|^2 \right) du,$$  

where $\alpha$ is the elasticity and $\beta$ the rigidity parameter of the curve. Using the expression (39) and the fact that the generalize coordinates vector $p$ does not depends on the parameter $u$ (see expression (26)) we can show that:

$$\frac{dc}{du} = \frac{d}{du} (Jp) = J_u p.$$  

Obviously the same is true for the second derivative respect to the parameter $u$. Therefore:

$$E_{\text{int}}(p) = \frac{1}{2} \int_u \left( \alpha p^T J_u^T J_u p + \beta p^T J_{uu}^T J_{uu} p \right) du,$$  

and, consequently:

$$E_{\text{int}}(p) = \frac{1}{2} p^T K p,$$  

where the matrix $K = K(p) \in \mathbb{R}^{4(n+1) \times 4(n+1)}$, named the stiffness matrix, is given by:

$$K(p) = \int_u \left( \alpha J_u^T J_u + \beta J_{uu}^T J_{uu} \right) du.$$  

The external potential $E_{\text{ext}}$ generates the force fields, like gravity, that act on the system. According to expression (14), they are computed by the gradient of the potential $E_{\text{ext}}$ respect to the generalized coordinates:

$$\frac{\partial E_{\text{ext}}}{\partial p} = \frac{1}{2} \left( \frac{\partial E_{\text{ext}}}{\partial p_{0x}}, \frac{\partial E_{\text{ext}}}{\partial p_{0y}}, \frac{\partial E_{\text{ext}}}{\partial p_{0z}}, \frac{\partial E_{\text{ext}}}{\partial w_0}, \frac{\partial E_{\text{ext}}}{\partial p_{1x}}, \frac{\partial E_{\text{ext}}}{\partial p_{1y}}, \frac{\partial E_{\text{ext}}}{\partial p_{1z}}, \frac{\partial E_{\text{ext}}}{\partial w_1}, \cdots, \frac{\partial E_{\text{ext}}}{\partial p_{nx}}, \frac{\partial E_{\text{ext}}}{\partial p_{ny}}, \frac{\partial E_{\text{ext}}}{\partial p_{nz}}, \frac{\partial E_{\text{ext}}}{\partial w_n} \right)^T.$$  

4.4 Euler-Lagrange Equations for D-NURBS

Now, we insert the kinetic energy and potentials just computed in the Euler-Lagrange equations given by expression (14). Besides, we must observe that the matrices $M$, $D$ and $K$ are all symmetric and for a quadratic form $g = p^T A p$ with $A$ symmetric we have $\frac{\partial g}{\partial p} = 2Ap$. Therefore:
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \right) + \left( \frac{\partial E_{\text{int}}}{\partial \mathbf{p}} \right) = -\frac{\partial E_{\text{ext}}}{\partial \mathbf{p}} + \left( \frac{\partial T}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \right),
\]

we get:

\[
\left( \mathbf{M}\ddot{\mathbf{p}} + \dot{\mathbf{M}}\dot{\mathbf{p}} - \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} \right) = -\frac{\partial E_{\text{ext}}}{\partial \mathbf{p}} + \left[ \frac{1}{2} \mathbf{P}^T \frac{\partial \mathbf{K}}{\partial \mathbf{p}} \mathbf{P} \right]^T + \frac{1}{2} \int_{t_0}^t \left( \int_{u} \mathbf{P}^T \frac{\partial \mathbf{D}}{\partial \mathbf{p}} \mathbf{p} \, du \right) dt,
\]

which can be rewritten as follows by just re-arranging the terms:

\[
\mathbf{M}\ddot{\mathbf{p}} + \dot{\mathbf{M}}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = -\frac{\partial E_{\text{ext}}}{\partial \mathbf{p}} + \left[ \frac{1}{2} \mathbf{P}^T \frac{\partial \mathbf{K}}{\partial \mathbf{p}} \mathbf{P} \right]^T - \dot{\mathbf{M}}\dot{\mathbf{p}} - \left[ \frac{1}{2} \mathbf{P}^T \frac{\partial \mathbf{K}}{\partial \mathbf{p}} \mathbf{P} \right]^T + \frac{1}{2} \int_{t_0}^t \left( \int_{u} \mathbf{P}^T \frac{\partial \mathbf{D}}{\partial \mathbf{p}} \mathbf{p} \, du \right) dt.
\]

However, in the Appendices A and B we shown that:

\[
\mathbf{I}\dot{\mathbf{p}} = \mathbf{M}\ddot{\mathbf{p}} - \frac{1}{2} \left( \mathbf{P} \right)^T \frac{\partial \mathbf{M}}{\partial \mathbf{p}} \mathbf{P}.
\]

\[
\left[ \frac{1}{2} \mathbf{P}^T \frac{\partial \mathbf{K}}{\partial \mathbf{p}} \mathbf{P} \right] = 0,
\]

where:

\[
\mathbf{I} = \int_{u} \mu \mathbf{J}^T \mathbf{J} \, du.
\]

Therefore, if we neglect the effects of the last integral term, we can finally write the governing equation for D-NURBS as:

\[
\mathbf{M}\ddot{\mathbf{p}} + \dot{\mathbf{M}}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = -\frac{\partial E_{\text{ext}}}{\partial \mathbf{p}} - \mathbf{I}\dot{\mathbf{p}},
\]

where the matrix \( \mathbf{I} \) is computed by equation (63).

### 4.5 External Forces

In this section we consider an external potential which in cartesian coordinates has the general form:

\[
E_{\text{ext}} = \int_{u} P(x, y, z) \, du
\]

where \( P(x, y, z) \) is a potential density function.

So,

\[
E_{\text{ext}}(\mathbf{p}) = \int_{u} P(x(\mathbf{p}, u), y(\mathbf{p}, u), z(\mathbf{p}, u)) \, du
\]
In cartesian coordinates, the external force is given by:

\[
F_{\text{ext}} = \int_u \left[ \begin{array}{c} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{array} \right] \, du. \tag{67}
\]

However, we must write the external force respect to the generalized coordinates, following the expression (57). For instance, let us consider the term:

\[
\frac{\partial E_{\text{ext}}}{\partial p_{0x}} = \int_u \left[ \frac{\partial}{\partial p_{0x}} P(x(p, u), y(p, u), z(p, u)) \right] \, du \tag{68}
\]

From the Chain-Rule:

\[
\frac{\partial E_{\text{ext}}}{\partial p_{0x}} = \int_u \left[ \frac{\partial P}{\partial x} \frac{\partial x}{\partial p_{0x}} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial p_{0x}} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial p_{0x}} \right] \, du. \tag{69}
\]

But, from the expression (25) we observe that:

\[
\frac{\partial y}{\partial p_{0x}} = \frac{\partial z}{\partial p_{0x}} = 0. \tag{70}
\]

Therefore:

\[
\frac{\partial E_{\text{ext}}}{\partial p_{0x}} = \int_u \left[ \frac{\partial P}{\partial x} \right] \, du \tag{71}
\]

Analogously we can find:

\[
\frac{\partial E_{\text{ext}}}{\partial p_{0y}} = \int_u \left[ \frac{\partial P}{\partial y} \right] \, du, \quad \frac{\partial E_{\text{ext}}}{\partial p_{0z}} = \int_u \left[ \frac{\partial P}{\partial z} \right] \, du. \tag{72}
\]

On the other hand:

\[
\frac{\partial E_{\text{ext}}}{\partial w_0} = \int_u \left[ \frac{\partial}{\partial w_0} P(x(p, u), y(p, u), z(p, u)) \right] \, du, \tag{73}
\]

and so:

\[
\frac{\partial E_{\text{ext}}}{\partial w_0} = \int_u \left[ \frac{\partial P}{\partial x} \frac{\partial x}{\partial w_0} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial w_0} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial w_0} \right] \, du = \int_u \left[ \frac{\partial x}{\partial w_0} \frac{\partial y}{\partial w_0} \frac{\partial z}{\partial w_0} \right] \left[ \frac{\partial P}{\partial x} \frac{\partial P}{\partial y} \frac{\partial P}{\partial z} \right]^T \, du \tag{74}
\]

So, by using expression (30), we find that the above results can be grouped in the following matricial expression:

\[
\left[ \begin{array}{c} \frac{\partial E_{\text{ext}}}{\partial p_{0x}} \\ \frac{\partial E_{\text{ext}}}{\partial p_{0y}} \\ \frac{\partial E_{\text{ext}}}{\partial p_{0z}} \\ \frac{\partial E_{\text{ext}}}{\partial w_0} \end{array} \right] = \int_u \left[ \begin{array}{cccc} N_{0,k} & 0 & 0 & 0 \\ 0 & N_{0,k} & 0 & 0 \\ 0 & 0 & N_{0,k} & 0 \\ \frac{\partial x}{\partial w_0} & \frac{\partial y}{\partial w_0} & \frac{\partial z}{\partial w_0} & \frac{\partial x}{\partial w_0} \end{array} \right] \left[ \begin{array}{c} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{array} \right] \, du \tag{75}
\]

Generalizing for \( i = 0, n \) we have:

\[
\frac{\partial E_{\text{ext}}}{\partial p} = \int_u \left[ \begin{array}{cccc} N_{0,k} & N_{0,k} & \frac{\partial x}{\partial w_i} & \frac{\partial y}{\partial w_i} & \frac{\partial z}{\partial w_i} \\ N_{0,k} & N_{0,k} & \frac{\partial x}{\partial w_i} & \frac{\partial y}{\partial w_i} & \frac{\partial z}{\partial w_i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ N_{n,k} & N_{n,k} & \frac{\partial x}{\partial w_n} & \frac{\partial y}{\partial w_n} & \frac{\partial z}{\partial w_n} \end{array} \right] \left[ \begin{array}{c} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{array} \right] \, du = \int_u J^T \, f(x, y, z) \, du, \tag{76}
\]
where \( f(x, y, z) \) is the external force field density defined by the gradient of the potential density \( P \) in cartesian coordinates \((x, y, z)\). Therefore, the external force field in the generalized coordinates is given by:

\[
f_{p}(\mathbf{p}) = \int_{u} J^{T} f(x(\mathbf{p}, u), y(\mathbf{p}, u), z(\mathbf{p}, u)) \, du.
\]  

(77)

Particularly, in the case of the gravitational potential for a particle we have:

\[
E = -mg y,
\]

(78)

where \( m \) is the mass particle, \( g \) is the gravitational field intensity and \( y \) gives the particle position (its height) in the vertical axes. For a 1D continuous system, a curve in the two-dimensional Euclidean space, the gravitational potential can be computed by a generalization of expression (78) given by:

\[
E_{ext} = -\int_{u} \mu g y \, du
\]

(79)

where \( \mu \) is the linear mass density (constant), like before. Therefore, the potential density is:

\[
P(x, y) = -\mu g y.
\]

(80)

and the force field density is given by:

\[
\nabla P(x, y) = -\mu g \begin{pmatrix} \frac{\partial y}{\partial x} \\ \frac{\partial y}{\partial y} \end{pmatrix} T = -\mu g (0, 1) T
\]

(81)

So, according to expression (77), the external force field \( f_{p} \) is:

\[
f_{p}(\mathbf{p}) = -\int_{u} \mu J^{T} \begin{pmatrix} 0 \\ g \end{pmatrix} \, du = -\int_{u} \mu J^{T} \begin{pmatrix} 0 \\ g \end{pmatrix} \, du.
\]

(82)

4.6 D-NURBS with Constraints

In the case of linear constraints, equations (15) become:

\[
A \mathbf{p} + \mathbf{d} = 0,
\]

(83)

where \( A \in \mathbb{R}^{m \times n} \), with \( m < n \), is a constant matrix and \( \mathbf{d} \in \mathbb{R}^{m} \) is a constant vector. In this case, we can choose a set of \( n - m \), say \( \mathbf{q} = (q_{1}, q_{2}, \cdots, q_{n-m}) \) independent variables and explicitly write the \( m \) remaining ones as a function of the \( \mathbf{q} \) vector, which will be the new generalized coordinates. In fact, if we write equation (83) in the form:

\[
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} p_{1} \\ p_{2} \\ \vdots \\ p_{m} \end{pmatrix} + \begin{pmatrix} a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,n} \\ a_{2,m+1} & a_{2,m+2} & \cdots & a_{2,n} \\ \vdots \\ a_{m,m+1} & a_{m,m+2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} p_{m+1} \\ p_{m+2} \\ \vdots \\ p_{n} \end{pmatrix} = -\mathbf{d},
\]

(84)

or simply:

\[
G_{1} \overline{\mathbf{q}} + G_{2} \mathbf{q} = -\mathbf{d}
\]

where \( G_{1} \in \mathbb{R}^{m \times m} \) and \( G_{2} \in \mathbb{R}^{m \times (n-m)} \) are the first and second matrices of expression (84) and \( \overline{\mathbf{q}} = (p_{1}, p_{2}, \cdots, p_{m})^{T} \), \( \mathbf{q} = (p_{m+1}, p_{m+2}, \cdots, p_{n})^{T} \). Then, by supposing that \( p_{1}, \cdots, p_{m} \) can be choosen such that \( G_{1} \) is non-singular, we have:

\[
\overline{\mathbf{q}} = -G_{1}^{-1} G_{2} \mathbf{q} - G_{1}^{-1} \mathbf{d}.
\]

(85)
Let $I \in \mathbb{R}^{(n-m)\times(n-m)}$ and from the observation that $G_1^{-1}G_2 \in \mathbb{R}^{m\times(n-m)}$ and using expression (85) it is clear that the matrix:

$$G = \begin{bmatrix} -G_1^{-1}G_2 \\ I \end{bmatrix} \in \mathbb{R}^{n \times (n-m)},$$

(86)

allows to write:

$$p = G q + d_0$$

(87)

with $d_0 = [G_1^{-1} d, 0]^T \in \mathbb{R}^{n \times 1}$.

The equations (23) can be written in compact form as:

$$A^T \lambda = - \left( M \ddot{q} + D \dot{q} + K \dot{q} + \frac{\partial E_{\text{ext}}}{\partial \dot{q}} + I \dot{q} \right).$$

(88)

So, using expressions (87) we can observe that:

$$\dot{p} = G \dot{q}, \quad \ddot{p} = G \ddot{q}.$$ 

(89)

Therefore, by substituting expressions (87) and (89) in equation (88) and using the fact that $A = [G_1 \quad G_2]$ we obtain:

$$\begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} \lambda = - \left( MG \ddot{q} + DG \dot{q} + K (G q + d_0) + \frac{\partial E_{\text{ext}}}{\partial \dot{q}} + IG \dot{q} \right),$$

If we multiply both sides by $G^T$, where the matrix $G$ is defined in expression (86) we obtain:

$$\begin{bmatrix} (G_1^{-1} G_2)^T \\ I \end{bmatrix} \times \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} \lambda = - G^T \left( MG \ddot{q} + DG \dot{q} + K (G q + d_0) + \frac{\partial E_{\text{ext}}}{\partial \dot{q}} + IG \dot{q} \right),$$

$$0 = -G^T M G \ddot{q} - G^T D G \dot{q} - G^T K (G q + d_0) - G^T \frac{\partial E_{\text{ext}}}{\partial \dot{q}} - G^T I G \dot{q}$$

$$G^T M G q + G^T D G q + G^T K G q = -G^T \frac{\partial E_{\text{ext}}}{\partial \dot{q}} - G^T I G \dot{q} - G^T K d_0.$$ 

If we name:

$$M_q = G^T M G; \quad D_q = G^T D G; \quad K_q = G^T K G; \quad f_q = -G^T \frac{\partial E_{\text{ext}}}{\partial \dot{q}}; \quad I_q = G^T I G,$$

then, we get the D-NURBS evolution equation subject to the linear constraints given by:

$$M_q \ddot{q} + D_q \dot{q} + K_q q = f_q - I_q \dot{q} - G^T K d_0.$$ 

(90)

4.7 Numerical Implementation

The equation (64), as well as its constrained counterpart in expression (90), does not have in general analytical solution and so we have to use a numerical approach to solve it with the desired precision. The equation (64) is a second order ordinary differential equation. Besides, it is important to observe that the matrices $M$, $D$, $K$ depends on the integration of products of the rational B-spline functions (7) and their derivatives of first and second order respect to the variable $u$.

Therefore, the numerical solution of expression (64) can be performed by finite difference methods (FDM) in time. Besides, we need a numerical scheme for computing the integrals, as described next.
4.7.1 Matrices Computation

The matrices $M, D, K$ that appears in D-NURBS evolution equation are given by expressions (47), (50), and (56), respectively. They involve derivatives of zero, first and second order of $J$ respect to the variable $u$. For instance, for matrix $K = (k_{ij}) \in \mathbb{R}^{4(n+1) \times 4(n+1)}$ we have:

$$k_{ij} = \hat{u} f_{ij}(u) du = \sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} f_{ij}(u) du,$$

where:

$$f_{ij} = \alpha \left( \frac{\partial j_i}{\partial u} \right)^T \left( \frac{\partial j_i}{\partial u} \right) + \beta \left( \frac{\partial^2 j_i}{\partial u^2} \right)^T \left( \frac{\partial^2 j_i}{\partial u^2} \right),$$

with $j_i$ means the collum $i$ of the Jacobian $J$.

The computation of each term in the summation in expression (91) can be performed by Gauss quadrature [Chapra and Canale 2009]. An analogous scheme can be used to compute the other matrices.

In our implementation we have developed a numerical approach based on isogeometric analysis following the recipe of [Cottrell et al. 2009]. Our implementation avoids the cost of assembling the global matrices $M$, $D$ and $K$. For this, we calculate the matrices of each element individually, where the elements are constructed by partitioning the knots vector. Figure 5 shows building elements ($e_1, e_2, e_3, e_4$) from open knots vector $v = (0, 0, 0, 0.25, 0.50, 0.75, 1, 1, 1)$. Here we will have six control points $(n - k + 1 = 8 - 3 + 1 = 6)$ which according [Cottrell et al. 2009] will be distributed over the elements by following expression

$$\mathbf{E} = \{ e_1 = (p_1, p_2, p_3), e_2 = (p_2, p_3, p_4), e_3 = (p_3, p_4, p_5), e_4 = (p_4, p_5, p_6) \}$$

Generally a open knots vector can be partitioned into $n_e$ elements expressed by

$$n_e = |\mathbf{E}| = n - 2(k - 1)$$

4.7.2 Numerical Scheme for Time Integration

Let the D-NURBS evolution equation:

$$M \ddot{p} + D \dot{p} + K p = f_p(p) - I \dot{p}$$

where $f_p = \int J^T f(x, y, z) du$ and $I(p) = \int \mu J^T \dot{j} du$, according to Appendix A and sections 4.4-4.5.

Let us consider the following numerical scheme:

$$\ddot{p} = \frac{p^{(t+\Delta t)} - 2p^{(t)} + p^{(t-\Delta t)}}{(\Delta t)^2}$$

$$\dot{p} = \frac{p^{(t+\Delta t)} - p^{(t-\Delta t)}}{2\Delta t}$$
If we substitute these expressions in equation (95) we obtain:

\[
M \left( \frac{p^{(t+\Delta t)} - 2p^{(t)} + p^{(t-\Delta t)}}{\Delta t^2} \right) + D \left( \frac{p^{(t+\Delta t)} - p^{(t-\Delta t)}}{2\Delta t} \right) + Kp^{(t+\Delta t)} = f_p - I \left( \frac{p^{(t+\Delta t)} - p^{(t-\Delta t)}}{2\Delta t} \right)
\]

(98)

where \( M, D, K \) and \( I \) are supposed to be computed at time \( t + \Delta t \). We shall be careful about the term \( I \left( \frac{p^{(t+\Delta t)} - p^{(t-\Delta t)}}{2\Delta t} \right) \). Following its definition in expression (115) and expression (97) we can write:

\[
I \dot{p} = \int \mu \left( J^T \right)^{(t+\Delta t)} \left( \dot{J} \right)^{(t+\Delta t)} du \left( \frac{p^{(t+\Delta t)} - p^{(t-\Delta t)}}{2\Delta t} \right)
\]

(99)

Therefore, we can rewrite equation (99) as:

\[
I \dot{p} = \frac{1}{2\Delta t} \int \mu \left( J^T \right)^{(t+\Delta t)} \left( \left( \dot{J} \right)^{(t+\Delta t)} p^{(t+\Delta t)} - \left( \dot{J} \right)^{(t+\Delta t)} p^{(t-\Delta t)} \right) du.
\]

(100)

By using the fact that \( \dot{J}^{(t+\Delta t)} p^{(t+\Delta t)} = 0 \) we simplify expression (100) to:

\[
I \dot{p} = -\frac{1}{2\Delta t} \int \mu \left( J^T \right)^{(t+\Delta t)} \left( \dot{J} \right)^{(t+\Delta t)} p^{(t-\Delta t)} du.
\]

Using the approximation:

\[
\left( \dot{J} \right)^{(t+\Delta t)} = \left( \frac{J^{(t+\Delta t)} - J^{(t-\Delta t)}}{2\Delta t} \right),
\]

we get:

\[
I \dot{p} = -\frac{1}{2\Delta t} \int \mu \left( J^T \right)^{(t+\Delta t)} \left( \left( \frac{J^{(t+\Delta t)} - J^{(t-\Delta t)}}{2\Delta t} \right) p^{(t-\Delta t)} \right) du
\]

\[
= -\frac{1}{4(\Delta t)^2} \int \mu \left( J^T \right)^{(t+\Delta t)} \left( J^{(t+\Delta t)} - J^{(t-\Delta t)} \right) p^{(t-\Delta t)} du
\]

\[
= -\frac{1}{4(\Delta t)^2} \left( \int \mu \left( J^T \right)^{(t+\Delta t)} J^{(t+\Delta t)} p^{(t-\Delta t)} - \int \mu \left( J^T \right)^{(t+\Delta t)} J^{(t-\Delta t)} p^{(t-\Delta t)} \right) du
\]

\[
= -\frac{1}{4(\Delta t)^2} \left( M^{(t+\Delta t)} p^{(t-\Delta t)} - \int \mu \left( J^T \right)^{(t+\Delta t)} c^{(t-\Delta t)} du \right)
\]

(101)

So, by substituting expression (101) in (98) it renders:

\[
M \left( \frac{p^{(t+\Delta t)} - 2p^{(t)} + p^{(t-\Delta t)}}{\Delta t^2} \right) + D \left( \frac{p^{(t+\Delta t)} - p^{(t-\Delta t)}}{2\Delta t} \right) + Kp^{(t+\Delta t)} =
\]

\[
f_p + \frac{1}{4(\Delta t)^2} \left( M^{(t+\Delta t)} p^{(t-\Delta t)} - \int \mu \left( J^T \right)^{(t+\Delta t)} c^{(t-\Delta t)} du \right),
\]

(102)

If we multiply both sides of expression (98) to \( \times (\Delta t)^2 \) and rearrange the terms we get:
\[
(4M + 2\Delta t D + 4(\Delta t)^2 K) p^{(t+\Delta t)} = 4(\Delta t)^2 f_p + 8M p^{(t)} - (3M - 2\Delta t D) p^{(t-\Delta t)} - \int \mu J^T e^{(t-\Delta t)} du
\] (103)

This expression can be written as:
\[
A_0^{(t+\Delta t)} p^{(t+\Delta t)} = A_1^{(t,t-\Delta t)},
\] (104)

where:
\[
A_0^{(t+\Delta t)} = 4M + 2\Delta t D + 4(\Delta t)^2 K,
\] (105)

and,
\[
A_1^{(t,t-\Delta t)} = 4(\Delta t)^2 f_p + 8M p^{(t)} - (3M - 2\Delta t D) p^{(t-\Delta t)} - \int \mu J^T e^{(t-\Delta t)} du.
\] (106)

Therefore, once initial conditions \(p(0) = p_0\) and \(\dot{p}(0) = v_0\) are given, we can use the approximation:
\[
\frac{p(0) - p(0 - \Delta t)}{\Delta t} = v_0.
\] (107)

to write:
\[
p(-\Delta t) = p(0) - \Delta t v_0,
\] (108)

and, consequently, we can start the iterative scheme given by expression (104).

The complexity for computing the expression (104) depends on the algorithm for calculating the matrices \(M, D, K\) and the method used to solve the linear system. Considering that \(n\) is the number of control points, \(n_e\) is number of elements, \(k\) is the polynomial order of NURBS basis and \(n_g\) is the number of quadrature points, the algorithm implemented to compute the matrices \(M, D, K\) performs the following steps:

1. For \(e = 1 \ldots n_e\) do
   
   (a) Compute the Jacobian matrix block for element “e” (complexity \(O(n_g * n * k)\)).
   (b) Compute mass matrix block for element “e” (complexity \(O(n_g * k^2)\)).
   (c) Compute damping matrix block for element “e” (complexity \(O(n_g * k^2)\)).
   (d) Compute stiffness matrix block for element “e” (complexity \(O(n_g * k^2)\)).

Therefore, the asymptotic complexity of the whole algorithm is given by:
\[
O\left(n_e n_g \left(O(nk) + O(k^2) + O(k^2) + O(k^2)\right)\right) = O(n_e n_g nk).
\] (109)

We highlight that the computational cost of the D-NURBS evolution must also consider the numerical method for solving the linear system (104). To compute (104), we have used conjugate gradient method whose complexity is \(O(n)\). Hence, we can conclude that the expression (104) has final computational complexity equal to \(O(n_e n_g nk)\).

5 Experimental Results

We have developed an experimental environment based on the D-NURBS approach with constraints. In our setting we consider the case of an elastic wire with 10\(m\) length with negligible transverse section fixed at the ends.

The NURBS curve geometry is instantiated using an open knot vector \(v = (0, 0, 0, 0, 0.25, 0.50, 0.75, 1, 1, 1, 1)\) with basis functions of order \(k = 4\) (degree \(k - 1 = 3\)). Therefore, following section 2, the spline space has dimension \(n - k + 1 = 10 - 4 + 1 = 7\), which means that we have seven controls points. Each point of a NURBS curve is influenced by \(k\) control points. Therefore, to set geometric constraints that keep the wire fixed at the ends, we must let \(k - 1\) fixed control points at the ends of the curve. This can be cast in the linear constraint framework for D-NURBS developed in section 4.6.
Table 1: Initial configuration of the control points and weights (generalized coordinates) for wire simulation: \( p(0) = (x_0, y_0, z_0, w_0; x_1, y_1, z_1, w_1; \cdots; x_6, y_6, z_6, w_6) \).

| \( i \) | \( x_i \) | \( y_i \) | \( z_i \) | \( w_i \) |
|-------|-------|-------|-------|
| 0     | -5.00| 5     | 0     | 1     |
| 1     | -4.17| 5     | 0     | 1     |
| 2     | -2.50| 5     | 0     | 1     |
| 3     | 0.00 | 5     | 0     | 1     |
| 4     | 2.50 | 5     | 0     | 1     |
| 5     | 4.17 | 5     | 0     | 1     |
| 6     | 5.00 | 5     | 0     | 1     |

Besides, we consider that the wire is subject to a gravitational field with value \( g = 9.8 \text{m/s}^2 \) and define control points position and weights at \( t = 0 \) according to table 1. Besides, we set \( \dot{p}(0) = 0 \) to complete the initial conditions for time integration.

![iteration: 0 [0.00%]](image)

Figure 6: Initial D-NURBS setup for simulation of the elastic wire fixed at the ends.

Figure 6 demonstrates the environment at time \( t = 0s \). Here physics parameters were defined as: \( \alpha = 35 \), \( \beta = 10 \), \( \mu = 30 \), \( \gamma = 0 \). To perform spatial and time integration we define 10 points in Gauss quadrature and \( \Delta t = 0.008s \), respectively.

In our experiments we observed that the evolution of the weights \( w_i \) may cause unrealistic behaviors and instability, as observed in Figure 7(a). As mentioned in [Terzopoulos and Qin 1994], the weights \( w_i \) may not have arbitrary finite real values. Negative values may vanish the denominator of the rational functions in expression 7. Besides, small weights values may lower the deformation energy [Terzopoulos and Qin 1994]. Therefore, some constraint must be included in order to enforce some control in the weight vector evolution.

In this work we implement this task by a very simple strategy: the generalized coordinate vector is updated by solving the expression 90 but the weight vector is always returned to its initial value; that means, \( w_i = 1 \) for \( i = 0, 1, \cdots, 6 \), following Table 1. As observed in Figure 7(b), the wire evolution becomes (visually) acceptable in this case.
(a) Wire configuration at iteration $t = 1, 10, 22$ without constrain the weights evolution.

(b) System configuration at iteration $t = 1, 10, 22$ when enforcing weights $w_i = 1$ after each iteration.

Figure 7: D-NURBS behavior for unconstrained and constrained weight vector evolution.

To study the dynamic evolution of the D-NURBS curve we choose a point in the center of the wire and followed its amplitude evolution in time. Figure 8a shows its dynamic evolution without the presence of damping, while Figure 8b illustrates the dynamic evolution with damping, where $\gamma = 5$. As expected, the former reports a periodic evolution once there are not dissipative forces and the latter pictures an attenuation of the amplitude along the time due to the damping.

To analyze the effects of the elasticity and stiffness of D-NURBS curve we increasing each parameter separately. First, we leave $\beta$ (see equation 51) with the same value of the initial configuration and modify $\alpha$. The Figures 9a and 9b show the results.

Similarly, we modify $\beta$ while $\alpha$ remains unchanged. The results are shown in figures 9c and 9d. We observe that the system is more sensitive respect to the parameter $\beta$ than the parameter $\alpha$. In fact, when increasing the parameter $\beta$ from 11 to 15 we observe a drastic change in the amplitude evolution as highlighted when comparing Figures 9c and 9d. On the other hand, when changing $\alpha$ from 70 to 105 (Figures 9a and 9b, respectively) we did not observe a similar behavior.

The expression (109) shows that the number of control points has a fundamental role in the computational cost of the D-NURBS algorithm. Therefore, we perform a runtime analysis of D-NURBS evolution for different number of controls points. We set parameters to: $k = 3, \alpha = 35, \beta = 50, \mu = 30, \gamma = 1$ and five points in Gauss quadrature. The host is a Intel Core i5-3210M at 2.5 Ghz, with 6 GB RAM running a Windows 7 (64bit).

We take 360 iterations for each configuration and measure the corresponding CPU time. In order to compare the complexity given by expression (109) and the CPU time for each simulation, we compute the following rates:

$$ P^j = \frac{T^{j+1} - T^j}{T^j}, $$

$$ \hat{P}^j = \frac{C^{j+1} - C^j}{C^j}, $$

(110) (111)
(a) Amplitude evolution for D-NURBS without damping.

(b) Amplitude evolution of D-NURBS with damping.

Figure 8: Amplitude evolution of elastic wire represented by D-NURBS.
Figure 9: Sensitivity of D-NURBS amplitude respect to elasticity $\alpha$ and stiffness $\beta$.
where $T^j$ is the CPU time for configuration $j$ and $C^j$ is the asymptotic complexity for the same configuration; that means:

$$C^j = 360 \times n_e^j n_g^j n_k^j.$$ (112)

The configuration ($j=1$) has 20 control points and $k=4$ and 360 D-NURBS iterations are performed. Next, for $j=2$, we increase the number of control points by 5, keep $k=4$ and perform 360 iterations of the algorithm again, and so on. The result is pictured on Figure 10 where the dot blue curve shows the evolution of expression (111) and the red line shows the evolution of expression (110), both for $j=1, 2, \ldots, 28$ (number of control points 20, 25, ..., 140).

By observing Figure 10 we note that as we increase the control points number we get $P^j \to \hat{P}^j$. This is the expected behavior for asymptotic function, i.e., for large $n$ the similarity between real and predicted time becomes more evident.

### 6 Conclusions and Future Works

We present a review of D-NURBS approach. We emphasize the formulation based on the Lagrangian mechanics followed by detailed development of the governing equations. We used a numerical method based on isogeometric analysis for the spatial integration used to compute the Jacobian, mass, damping and stiffness matrix. For validation we performed experiments with D-NURBS curve and discuss the influence of parameters, effects of NURBS weights and computational cost. For further works we plan to evaluate the D-NURBS for 2D and 3D systems.

### References

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Let us consider the expression:

$$Y (\mathbf{p}, \dot{\mathbf{p}}) = \dot{\mathbf{M}} \dot{\mathbf{p}} - \frac{1}{2} \left( \dot{\mathbf{p}} \right)^T \frac{\partial \mathbf{M}}{\partial \mathbf{p}} \mathbf{p}. \quad (113)$$

By the product rule we have:

$$\dot{\mathbf{M}} = \frac{d}{dt} \left[ \int_u \mu \mathbf{J}^T \mathbf{J} du \right] = \int_u \mu \mathbf{J}^T \dot{\mathbf{J}} du + \int_u \mu \left( \dot{\mathbf{J}} \right)^T \mathbf{J} du. \quad (114)$$

Now, let us define the expressions $I$ and $\hat{I}$ as:

$$I \equiv \int_u \mu \mathbf{J}^T \dot{\mathbf{J}} du, \quad (115)$$

$$\hat{I} \equiv \int_u \mu \left( \dot{\mathbf{J}} \right)^T \mathbf{J} du. \quad (116)$$

Therefore, we can rewrite expression (113):

$$Y (\mathbf{p}, \dot{\mathbf{p}}) = I \dot{\mathbf{p}} + \hat{I} \dot{\mathbf{p}} - \frac{1}{2} \left( \dot{\mathbf{p}} \right)^T \frac{\partial \mathbf{M}}{\partial \mathbf{p}} \mathbf{p}. \quad (117)$$

Now, we prove that:

$$\hat{I} \dot{\mathbf{p}} = \left[ \int_u \mu \left( \dot{\mathbf{J}} \right)^T \mathbf{J} du \right] \dot{\mathbf{p}} = \frac{1}{2} \left( \dot{\mathbf{p}} \right)^T \frac{\partial \mathbf{M}}{\partial \mathbf{p}} \mathbf{p}. \quad (118)$$

If we name:

$$R = \frac{1}{2} \left( \dot{\mathbf{p}} \right)^T \frac{\partial \mathbf{M}}{\partial \mathbf{p}} (\mathbf{J}^T \dot{\mathbf{J}}) \dot{\mathbf{p}}, \quad (119)$$
then, by applying the product rule we get:

$$R = \frac{1}{2} (\dot{p})^T (J \dot{p}) T + \frac{1}{2} (\dot{p})^T J^T \frac{\partial J}{\partial p_i} \dot{p}. $$ \hspace{1cm} (120)

Let $j_i$ be the column $i$ of the Jacobian $J$. Once $J = J(p)$, where $p = p(t)$, then the Chain-Rule allows to write:

$$\dot{j}_i = \frac{d}{dt} (j_i) = \frac{\partial J}{\partial p_i} \frac{d}{dt} (p) = \frac{\partial J}{\partial p_i} \dot{p}. $$ \hspace{1cm} (121)

Expression (120) can be rewritten as:

$$R = \frac{1}{2} (\dot{p})^T (J \dot{p}) T + \frac{1}{2} (\dot{p})^T J^T \frac{\partial J}{\partial p_i} \dot{p}. $$ \hspace{1cm} (122)

So, by substituting equation (121) in expression in (122) we obtain:

$$R = \frac{1}{2} (\dot{j}_i)^T J \dot{p} + \frac{1}{2} (J \dot{p})^T (\dot{j}_i), $$ \hspace{1cm} (123)

$$R = \frac{1}{2} (\dot{j}_i^T J \dot{p}) + \frac{1}{2} (\dot{j}_i^T J \dot{p})^T = (\dot{j}_i)^T J \dot{p}. $$ \hspace{1cm} (124)

Therefore, from the expressions (119) and (124) we get that:

$$(\dot{j}_i)^T J \dot{p} = \frac{1}{2} (\dot{p})^T \frac{\partial}{\partial p_i} (J^T J) p, \text{ for } i = 0, \ldots, 4(n + 1), $$ \hspace{1cm} (125)

which is equivalent to expression (118).

Therefore, by substitution this result in equation (117) we obtain:

$$Y(p, \dot{p}) = I \dot{p}, $$ \hspace{1cm} (126)

where $I$ is computed by expression (115).

**B Appendix**

In order to prove that:

$$\left[ \frac{1}{2} \dot{p}^T \frac{\partial K}{\partial \dot{p}} \dot{p} \right] = 0, $$ \hspace{1cm} (127)

we must observe that:

$$\frac{\partial c}{\partial p_i} = \frac{\partial J}{\partial p_i} p + \dot{j}_i. $$ \hspace{1cm} (128)

However, due to the definition of Jacobian matrix $J$ we must have $\frac{\partial c}{\partial p_i} = j_i$. Therefore:

$$\frac{\partial J}{\partial p_i} p = 0. $$ \hspace{1cm} (129)

On the other hand:

$$\left[ \frac{1}{2} \dot{p}^T \frac{\partial K}{\partial \dot{p}} \dot{p} \right] = \left[ \frac{1}{2} \dot{p}^T \frac{\partial K}{\partial p_0} \dot{p} \frac{1}{2} \dot{p}^T \frac{\partial K}{\partial p_1} \dot{p} \ldots \frac{1}{2} \dot{p}^T \frac{\partial K}{\partial p_i} \dot{p} \ldots \frac{1}{2} \dot{p}^T \frac{\partial K}{\partial p_{4(n+1)}} \dot{p} \right]. $$ \hspace{1cm} (130)

But, from the definition of $K$ matrix in expression (56):

$$\dot{p}^T \frac{\partial K}{\partial p_i} \dot{p} = \dot{p}^T \left( \frac{\partial}{\partial p_i} \int u \left( \alpha \dot{J}_u^T J_u + \beta \dot{J}_u^T J_{uu} \right) du \right) \dot{p}. $$ \hspace{1cm} (131)
Once the vector \( p \) does not depend on the parameter \( u \) we can write the first term inside the integral as:

\[
\alpha p^T \left( \int \left( \frac{\partial J_u^T}{\partial p_i} J_u + J_u^T \frac{\partial J_u}{\partial p_i} \right) \, du \right) p = \tag{132}
\]

\[
\alpha p^T \left( \int \frac{\partial J_u^T}{\partial p_i} J_u \, du \right) p + \alpha p^T \left( \int J_u^T \frac{\partial J_u}{\partial p_i} \, du \right) p = \tag{133}
\]

\[
\alpha p^T \left( \int \frac{\partial}{\partial u} \left( \frac{\partial J}{\partial p_i} \right) J_u \, du \right) p + \alpha p^T \left( \int J_u^T \frac{\partial}{\partial u} \left( \frac{\partial J}{\partial p_i} \right) \, du \right) p = 0, \tag{134}
\]

\[
\alpha \left( \int \frac{\partial}{\partial u} \left( \frac{\partial J}{\partial p_i} \right)^T J_u \, du \right) p + \alpha \left( \int J_u^T \frac{\partial}{\partial u} \left( \frac{\partial J}{\partial p_i} \right) \, du \right) = 0, \tag{135}
\]

due to equation (129). Therefore, expression (127) has been proved.