RILEY’S CONJECTURE ON SL(2,R) REPRESENTATIONS OF 2-BRIDGE KNOTS

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1. Introduction. In [R1] and [R2] Riley investigated representations of 2-bridge knot groups in $SL(2,F)$ for various fields $F$. In particular, he considered non-abelian representations in which the meridians go to parabolic elements, calling these parabolic representations. He showed that, for a given 2-bridge knot $K$, such representations correspond to the roots of a certain polynomial $\lambda_K(x) \in \mathbb{Z}[x]$, the Riley polynomial; see [R1, Theorem 2]. Thus the real roots of $\lambda_K(x)$ give parabolic $SL(2,\mathbb{R})$ representations. In [R2], Riley states “Some of our computer calculations made in 1972-73 ... suggested that the number of real roots of $[\lambda_K(x)]$ is not less than $|\sigma|/2.$” Here $\sigma = \sigma(K)$ is the signature of $K$. Following [Tr2], we will refer to this as the

Riley Conjecture. The number of real roots of the Riley polynomial of a 2-bridge knot $K$ is at least $|\sigma(K)|/2$.

Note that since $\lambda_K(x)$ has no multiple roots [R1, Theorem 3], the statement is unambiguous.

Our main result is

Theorem 1.1. The Riley Conjecture is true.

For double twist knots, the Riley Conjecture was proved by Tran [Tr2].

One of our interests in the Riley Conjecture is its connection with the question of when the $n$-fold cyclic branched cover $\Sigma_n(K)$ of a knot $K$ has left-orderable fundamental group. More precisely, as pointed out in [Tr2], Hu’s argument in [H] shows that Theorem 1.1 has the following corollary.

Corollary 1.2. Let $K$ be a 2-bridge knot with $\sigma(K) \neq 0$. Then $\Sigma_n(K)$ has left-orderable fundamental group for $n$ sufficiently large.

By contrast, there are 2-bridge knots $K$ such that $\Sigma_n(K)$ has non-left-orderable fundamental group for all $n$, by [Te, Proof of Theorem 2] and [BGW, Theorem 4].

For any knot $K$, the determinant and signature are related by the following congruence [M, Theorem 5.6]

$$\det(K) \equiv (-1)^{\sigma(K)/2} \pmod{4}$$

If $K$ is the 2-bridge knot corresponding to $p/q \in \mathbb{Q}$, $p > 0$, then $\det(K) = p$. Hence if $p \equiv -1 \pmod{4}$ then $\sigma(K) \equiv 2 \pmod{4}$, and Corollary 1.2 applies. In this case the conclusion of Corollary 1.2 was proved by Hu [H].

For other results on the left-orderability of the fundamental groups of cyclic branched covers of knots see [GL] and [Tr1].

Question 1.3. Does Corollary 1.2 hold without the assumption that $K$ is 2-bridge?
The proof of Theorem 1.1 uses a variant of the classical theorem of Sturm on the number of real roots of a polynomial with real coefficients. This is treated in Section 2. In Section 3 we prove the Riley Conjecture, and in Section 4 we discuss Corollary 1.2.

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2. Sturm’s theorem. Sturm’s theorem gives a way of determining the number of real roots of a polynomial with real coefficients; for a nice discussion of this, including some history, see [GR]. The method depends on constructing a sequence of polynomials $f_0, f_1, ..., f_n = f$ with certain properties (we find it convenient to reverse the usual numbering convention). In Theorem 2.1 we prove a version of Sturm’s theorem where the key properties of $f_0, f_1, ..., f_{n-1}$ are as in the classical setting, but the hypothesis on the relation between $f_n$ and $f_{n-1}$ is weakened. The conclusion is then an inequality rather than an equality.

Let $\alpha = (\alpha_k) = (\alpha_0, \alpha_1, ..., \alpha_n)$ be a sequence of non-zero real numbers. Define the variation $\text{var}(\alpha)$ of $\alpha$ to be the number of changes in the corresponding sequence of signs $\text{sign}(\alpha) = (\text{sign}(\alpha_k))$, i.e.

$$\text{var}(\alpha) = \# \{ k : \alpha_k \alpha_{k+1} < 0, \; 0 \leq k < n \}$$

Let $f = (f_0, f_1, ..., f_n)$ be a sequence of polynomials in $\mathbb{R}[X]$. If $x \in \mathbb{R}$, set $f(x) = (f_0(x), f_1(x), ..., f_n(x)) \in \mathbb{R}^{n+1}$.

Let $Z_k = \{ \text{real roots of } f_k \} \subset \mathbb{R}$, $0 \leq k \leq n$, and let $Z = \bigcup_{k=0}^{n} Z_k$. Choose $x_+$ and $x_- \in \mathbb{R}$ such that $Z \subset (x_-, x_+)$. Then $\text{sign}(f(x_+))$ is independent of the choice of $x_+$, so we denote it by $\text{sign}(f(\infty))$, and write $\text{var}(f(\infty)) = \text{var}(f(x_+))$. Similarly, we write $\text{sign}(f(-\infty)) = \text{sign}(f(x_-))$ and $\text{var}(f(-\infty)) = \text{var}(f(x_-))$.

**Theorem 2.1.** Let $f = (f_0, f_1, ..., f_n)$ be a sequence of polynomials in $\mathbb{R}[X]$ such that

1. $f_0$ is constant and non-zero, and
2. if $f_k(x_0) = 0$ for some $0 < k < n$ and $x_0 \in \mathbb{R}$, then $f_{k-1}(x_0)f_{k+1}(x_0) < 0$.

Then $f_n$ has at least $|\text{var}(f(-\infty)) - \text{var}(f(\infty))|$ distinct real roots.

**Proof.** The theorem is trivially true when $n = 0$ so we assume $n \geq 1$.

Define $v : \mathbb{R} \setminus Z \rightarrow \{0, 1, ..., n\}$ by $v(x) = \text{var}(f(x))$. Note that $v$ is constant on each component of $\mathbb{R} \setminus Z$.

Suppose $x_0 \in Z$, so $x_0 \in Z_k$ for some $k$ (not necessarily unique) with $1 \leq k \leq n$. If $k < n$ then by condition (2) there exists $\delta > 0$ such that $f_{k-1}(x)f_{k+1}(x) < 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Hence, as $x$ passes through $x_0$ the signs of $(f_{k-1}(x), f_k(x), f_{k+1}(x))$ change as $(\pm, \epsilon, \mp) \rightarrow (\pm, \epsilon', \mp)$, where $\epsilon, \epsilon' \in \{+, -\}$. This contributes 0 to the change in $v(x)$.

Suppose $k = n$. Note that $f_{n-1}(x_0) \neq 0$, by (1) if $n = 1$ and by (2) if $n > 1$. Hence, as $x$ passes through $x_0$ the signs of $(f_{n-1}(x), f_n(x))$ change as $(\pm, \epsilon) \rightarrow (\pm, \epsilon')$. Thus the corresponding change in $v(x)$ is 0 or $\pm 1$.

Therefore $|\text{var}(f(-\infty)) - \text{var}(f(\infty))|$ is at most the number of distinct real roots of $f_n$. \qed
3. The Riley Conjecture. Let $K$ be the 2-bridge knot corresponding to $p/q \in \mathbb{Q}$, where $p$ and $q$ are coprime, and $p$ is odd and $> 1$. Let $n = (p - 1)/2$. Then (see [R1, Proposition 1]) there exist $\epsilon_i, \eta_i \in \{\pm 1\}$, with $\epsilon_i = \eta_{n+1-i}$, $1 \leq i \leq n$, such that $\pi(K) = \pi_1(S^3 \setminus K)$ has presentation

$$<a, b : wa = bw >,$$

where $a$ and $b$ are meridians and $w = \prod_{i=1}^{n} a^{\epsilon_i} b^{\eta_i}$.

Also, $\sigma(K) = \sum_{i=1}^{n}(\epsilon_i + \eta_i)$ [S]. Hence $\sum_{i=1}^{n} \epsilon_i = \sigma(K)/2$.

Riley considers parabolic representations of $\pi(K)$ into $SL(2, \mathbb{C})$, where

$a \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A$, and

$b \rightarrow \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = X$.

Let $W_k = \prod_{i=1}^{k} A^{\epsilon_i} X^{\eta_i}$, $1 \leq k \leq n$, and set $W_0 = I$.

Write $W_k = \begin{pmatrix} a_k & b_k \\ * & * \end{pmatrix}$, $a_k, b_k \in \mathbb{Z}[x]$, $0 \leq k \leq n$.

The Riley polynomial of $K$ is defined to be $\lambda_K = a_n$. Riley showed [R1, Theorem 2] that the above assignment of $a$ and $b$ defines a homomorphism from $\pi(K)$ to $SL(2, \mathbb{C})$ if and only if $\lambda_K(x) = 0$. Thus the real roots of $\lambda_K$ give parabolic representations of $\pi(K)$ into $SL(2, \mathbb{R})$.

Let $\delta_i = \epsilon_i \eta_i$, $1 \leq i \leq n$. Then

$$A^{\epsilon_i} X^{\eta_i} = \begin{pmatrix} 1 + \delta_i x & \epsilon_i \\ \eta_i & 1 \end{pmatrix},$$

giving the recurrence equations, for $1 \leq k \leq n$,

$$(3.1) \quad a_k = (1 + \delta_k x)a_{k-1} + (\eta_k x)b_{k-1}$$

$$(3.2) \quad b_k = \epsilon_k a_{k-1} + b_{k-1}$$

It follows from (3.1) and (3.2) by induction on $k$ that $a_k$ has degree $k$, with leading coefficient $\prod_{i=1}^{k} \delta_i$, and

$$(3.3) \quad a_k(0) = 1$$

Also, since $\det W_k = 1$, we have that for all $x \in \mathbb{C}$,

$$(3.4) \quad a_k(x) \text{ and } b_k(x) \text{ are not both zero, } 0 \leq k \leq n$$

Lemma 3.1. If $0 < k < n$ and $a_k(x_0) = 0$, $x_0 \in \mathbb{R}$, then $a_{k-1}(x_0)$ and $a_{k+1}(x_0)$ are non-zero and $\text{sign}(a_{k-1}(x_0))\text{sign}(a_{k+1}(x_0)) = -\eta_k \eta_{k+1}$. 

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Proof. Suppose $a_k(x_0) = 0$. Then (3.1) gives
\[(1 + \delta_k x_0)a_{k-1}(x_0) + (\eta_k x_0)b_{k-1}(x_0) = 0\]
while from (3.2) we get
\[b_k(x_0) = \epsilon_k a_{k-1}(x_0) + b_{k-1}(x_0)\]
Multiplying both sides of (3.6) by $\eta_k x_0$ and using (3.5) gives
\[a_{k-1}(x_0) = - (\eta_k x_0)b_k(x_0)\]
Replacing $k$ by $k+1$ in (3.1) we obtain
\[a_{k+1}(x_0) = (\eta_{k+1} x_0)b_k(x_0)\]
By (3.3) $x_0 \neq 0$, and by (3.4) $b_k(x_0) \neq 0$. The result now follows from (3.7) and (3.8). □

Proof of Theorem 1.1. Define $f_k = \left( \prod_{i=1}^{k} \eta_i \right) a_k$, $0 \leq k \leq n$. Then $f_0$ is the constant polynomial 1, and Lemma 3.1 implies that if $f_k(x_0) = 0$ for some $0 < k < n$ then $f_{k-1}(x_0)f_{k+1}(x_0) < 0$. Thus $f = (f_k)$ satisfies the hypotheses of Theorem 2.1.

The coefficient of $x^k$ in $f_k$ is $\left( \prod_{i=1}^{k} \eta_i \right)\left( \prod_{i=1}^{k} \delta_i \right) = \prod_{i=1}^{k} \epsilon_i = \mu_k$, say.

Then $\text{sign}(f(\infty)) = (\mu_k)$, and $\text{sign}(f(-\infty)) = ((-1)^k \mu_k)$. Since $\mu_k = \epsilon_k \mu_{k-1}$, we have
\[\text{var}(f(\infty)) = \# \{ k : 1 \leq k \leq n, \epsilon_k = -1 \}\]
and
\[\text{var}(f(-\infty)) = \# \{ k : 1 \leq k \leq n, \epsilon_k = +1 \}\]
Therefore by Theorem 2.1 the number of real roots of $\lambda_K = a_n = \pm f_n$ is at least
\[|\text{var}(f(-\infty)) - \text{var}(f(\infty))| = \left| \sum_{k=1}^{n} \epsilon_k \right| = |\sigma(K)|/2.\]

Remark. The inequality in the Riley Conjecture can be strict. For example, the knot $10_{32}$, which is the 2-bridge knot corresponding to the rational number $69/29$, has $\sigma(10_{32}) = 0$. On the other hand, by [KT1] and [KT2] (see also [ORS]), there is a meridian-preserving epimorphism from $\pi(10_{32})$ to $\pi(3_1)$, the group of the trefoil. Since $\pi(3_1)$ has a real parabolic representation [R1], so does $10_{32}$.

This also shows that the converse of Corollary 1.2 is not true. In fact, by [GL, Theorem 1.2 and Lemma 9.1], $\Sigma_n(10_{32})$ has left-orderable fundamental group for $n \geq 6$. 

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4. Cyclic branched covers. In this section we indicate how the argument in [H] gives Corollary 1.2.

In [R2] Riley considers arbitrary non-abelian $SL(2,\mathbb{C})$ representations of $\pi(K)$, $K$ a 2-bridge knot. Up to conjugation, we may assume that

$$a \to \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}, \text{ and}$$

$$b \to \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix}.$$  

Riley shows that this defines a representation if and only if $\phi(t, x) = 0$ for a certain polynomial $\phi \in \mathbb{Z}[t^{\pm 1}, x]$. He notes that $\phi(t, x) = \phi(t^{-1}, x)$ [R2, Proposition 1], and therefore $\phi(t, x) = \psi(s, x)$, where $s = t + t^{-1}$, for some $\psi \in \mathbb{Z}[s, x]$. Then $\psi(2, x)$ is the Riley polynomial $\lambda_K(x)$.

Suppose $\lambda_K$ has a real root $x_0$. Since $\lambda_K$ has no repeated factors [R1, Theorem 3], $\frac{\partial \psi}{\partial x}|_{(s=2, x=x_0)}$ is non-zero. It follows that there exists $\delta > 0$ and a continuous function $\gamma : (2-\delta, 2+\delta) \to \mathbb{R}$, with $\gamma(2) = x_0$, such that $\psi(s, \gamma(s)) = 0$ for all $s \in (2-\delta, 2+\delta)$. In particular, for all $s \in (2-\delta, 2)$ there is a non-abelian representation $\rho_s : \pi(K) \to SL(2, \mathbb{R})$ such that $\rho_s(a)$ has trace $s$. Conjugating $\rho_s$ we may assume that

$$\rho_s(a) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

where $s = 2 \cos \theta$.

For $n$ sufficiently large, $s_n = 2 \cos(2\pi/n) \in (2-\delta, 2)$. Then $\rho_{s_n}(a)$ has order $n$. It follows from [H, Theorem 3.1] (see also [BGW, Theorem 6]) that $\pi_1(\Sigma_n(K))$ is left-orderable.

References

[BGW] S. Boyer, C. Gordon and L. Watson, On L-spaces and left-orderable fundamental groups, Math. Ann. 365 (2013), 1213–1245.

[GR] É. Ghys and A. Ranicki, Signatures in algebra, topology and dynamics, arXiv:1512.09258.

[GL] C. Gordon and T. Lidman, Taut foliations, left-orderability, and cyclic branched covers, Acta Math. Vietnam 39 (2014), no.4, 599–635.

[H] Y. Hu, Left-orderability and cyclic branched coverings, Algebr. Geom. Topol. 15 (2015), 399–413.

[KT1] T. Kitano and M. Suzuki, A partial order in the knot table, Exp. Math. 14 (2005), no.4, 385–390.

[KT2] T. Kitano and M. Suzuki, Corrigendum to “A partial order in the knot table”, Exp. Math 20 (2011), no.3, 371.

[M] K. Murasugi, On a certain numerical invariant of link types, Trans. Amer. Math. Soc. 117 (1965), 387–422.

[ORS] T. Ohtsuki, R. Riley and M. Sakuma, Epimorphisms between 2-bridge link groups, Geometry and Topology Monographs 14 (2008), 417–450.

[R1] R. Riley, Parabolic representations of knot groups I, Proc. London Math. Soc. (3) 24 (1972), 217–242.

[R2] R. Riley, Nonabelian representations of 2-bridge knot groups, Quart. J. Math. Oxford Ser. (2) 35 (1984), 191–208.

[S] Y. Shinohara, On the signature of a link with two bridges, Kwansei Gakuin Univ. Annual Studies 25 (1976), 111–119.
[Te] M. Teragaito, *Fourfold cyclic branched covers of genus one two-bridge knots are L-spaces*, Bol. Soc. Mat. Mex. (3) **20** (2014), no.2, 391–403.

[Tr1] A. Tran, *On left-orderability and cyclic branched coverings*, J. Math. Soc. Japan **67** (2015), 1169–1178.

[Tr2] A. Tran, *Nonabelian representations and signatures of double twist knots*, arXiv:1507.05241.

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