Combinatorics of \( n \)-point functions via Hopf algebra in quantum field theory

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24 May 2005
27 January 2006 (v2)

Abstract
We use a coproduct on the time-ordered algebra of field operators to derive simple relations between complete, connected and 1-particle irreducible \( n \)-point functions. Compared to traditional functional methods our approach is much more intrinsic and leads to efficient algorithms suitable for concrete computations. It may also be used to efficiently perform tree level computations.

1 Introduction

It may be said that time-ordered \( n \)-point functions are at the heart of (perturbative) quantum field theory. These determine the S-matrix that in turn allows to calculate experimentally observable scattering cross sections. Besides the complete \( n \)-point functions which are expectation values of time-ordered products of field operators, a key role is played by connected and by 1-particle irreducible (1PI) \( n \)-point functions. In particular, the latter play a prominent role in the process of renormalization, as it is enough to renormalize 1PI \( n \)-point functions. Furthermore, they are intimately related to the effective action. The different classes of \( n \)-point functions correspond directly to different sums over Feynman graphs, namely all graphs (excluding vacuum graphs), connected graphs and 1PI graphs.

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The relations between these different classes of \( n \)-point functions (and thus sums over Feynman graphs) are traditionally expressed by using functional methods. While having an undeniable elegance, a disadvantage of these methods is the need for auxiliary sources and associated generating functions. These, as well as functional derivatives used in the process are often purely formal and have no rigorous mathematical existence. Consequently, they are only indirectly related to concrete calculations of \( n \)-point functions one may wish to perform.

In the present article we describe the relation between these classes of \( n \)-point functions directly on the level of the algebra of time-ordered field operators. The key structure we will make use of is the natural coproduct on this algebra, making it into a Hopf algebra. Ensembles of time-ordered \( n \)-point functions are simply linear forms on this algebra. We show that the convolution product (induced by the coproduct) provides an extremely concise and elegant way of relating complete and connected \( n \)-point functions. Indeed, this relation is simply given by the convolution exponential (or, conversely, the logarithm).

Our second (and perhaps main) result concerns the relation of connected and \( 1\Pi \) \( n \)-point functions. As is well known, the former are expressible in terms of the latter as the sum over all tree graphs with \( 1\Pi \) vertices. We present a simple recursion formula using the coproduct, which generates exactly all tree diagrams. Moreover, the result takes an algebraic form which can be directly evaluated on \( 1\Pi \) functions so as to yield the connected functions. We proceed to derive an alternative recursion formula which relates directly components of connected \( n \)-point functions ordered by vertex number. Moreover, these same formulas can be alternatively applied to calculate the tree level contribution to the connected \( n \)-point functions, using the interaction terms of the Lagrangian.

A key feature of our results is their close relation to algorithmic descriptions of the computations involved. Indeed, it is easy to read off from our recursion relations not only algorithms to perform the computations, but even data structures relevant for an implementation. Our algorithm for generating trees also seems to be particularly efficient as it allows to impose a lower bound on the number of legs per vertex from the outset.

The present article may be seen as part of a programme, laid out in [1] and rooted in [2, 3, 4], with the aim of formulating and understanding the key structures of quantum field theory and their combinatorics in terms of the Hopf algebra of field operators. The focus of [1] were the different products of field operators (normal product, canonical product and time-ordered product), uncovering their relation through Drinfeld twists via cer-
tain 2-cocycles in Hopf algebra cohomology. At the same time the relation between products and associated \( n \)-point functions was elucidated, again using Hopf algebra cohomology. The present article complements this by investigating with the same means the relation between different classes of \( n \)-point functions that correspond to different classes of Feynman diagrams.

While Hopf algebras and coproducts have a long history in combinatorics, their use in combinatorial problems in quantum field theory is rather recent. The probably first instance was Kreimer’s Hopf algebraic explanation of the Bogoliubov formula of renormalization and of Zimmermann’s solution \[5\], subsequently developed together with Connes \[6\]. We caution the reader, however, that the Hopf algebras used by Kreimer and Connes, while also being related to Feynman graphs, are quite distinct from the Hopf algebra of field operators used here.

Section 2 starts with recalling the different classes of \( n \)-point functions and their relation to Feynman graphs. Then, the basic algebraic formalism used in this article is introduced, in particular the coproduct. Section 3 deals with the relation between complete and connected \( n \)-point function, Section 4 with that between connected and 1PI \( n \)-point functions. An alternative recursion formula for the latter is derived in Section 5. In Section 6 various generalizations (e.g. tree level calculations) and related issues are discussed. Some Conclusions are offered in Section 7. The appendix shows all tree graphs with up to 7 vertices with weight factors computed according to Section 4.

No knowledge of Hopf algebras is required to read this article.

\section{Basic definitions}

We shall be concerned in the following with a generic perturbative quantum field theory. We denote the basic field operators by \( \phi(x) \), where \( x \) represents a label that completely determines the operator. In a position representation \( x \) would specify a point in Minkowski space, possibly together with internal indices. While our notation suggests a field theory with a single scalar field, this is just a convenience. Although our results are general, we limit ourselves in the following exposition to a purely bosonic theory for simplicity. We return in Section 6.4 to a discussion of the general case, including fermionic fields.
2.1 Feynman graphs and $n$-point functions

We review here essentials about (classes of) Feynman graphs and $n$-point functions. For more information on these and the standard functional approach used to manipulate them we refer the reader to standard text books such as [7].

**Definition 1.** A graph is a finite collection of vertices and edges (also called legs), such that any end of an edge may be connected to a vertex. Edges that are connected to vertices at both ends are called internal, while edges with at least one free end are called external. The valence of a vertex is the number of ends of edges connected to the vertex. A tree (graph) is a connected graph that has no cycles.

Feynman graphs are graphs that carry certain labels on vertices and edges. The former correspond to the interaction terms of the Lagrangian while the latter may correspond to momenta and internal indices (usually indicated by different styles of lines, e.g., straight for fermions, wiggly for bosons etc.). We shall only consider labels attached to (open ends) of external legs, assuming internal labels to be summed or integrated over.

**Definition 2.** A labeled graph is a graph whose (ends of) external legs are labeled by field operator labels.

In the following we shall consider only such labeled graphs, i.e., from now on graph really means labeled graph. A Feynman graph has a (usually complex) value as a function of the labels on the external legs.

We denote by $G^{(n)}(x_1, \ldots, x_n)$ the complete $n$-point function. This is the vacuum expectation value of the time-ordered product of $n$ field operators, i.e.,

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle.$$

In terms of Feynman graphs it is the sum of the values of all graphs with external legs labeled by $x_1, \ldots, x_n$. Let us denote by $\Gamma^n$ the set of all Feynman graphs of the given theory\(^1\) with $n$ legs (with vacuum graphs, i.e., graphs containing pieces not connected to any external leg already excluded). For a given graph $\gamma \in \Gamma^n$ we denote by $\gamma(x_1, \ldots, x_n)$ its value for the given labelings. Then,

$$G^{(n)}(x_1, \ldots, x_n) = \sum_{\gamma \in \Gamma^n} \gamma(x_1, \ldots, x_n). \quad (1)$$

\(^1\)Whether one considers the bare or renormalized (including counter terms) theory does not matter to us here. Of course, in the former case quantities might be infinite and manipulations therefore formal.
Figure 1: Decomposition of the complete 4-point function in terms of connected functions.

Of particular importance is the Feynman propagator $G_F(x, y)$ which is the value of the graph that consists of an edge only, its two ends labeled by $x$ and $y$ respectively. Note that the Feynman propagator is symmetric in its arguments $G_F(x, y) = G_F(y, x)$ as suggested by the corresponding symmetry of the graph. We also define its inverse $G_F^{-1}$ which is determined by the equation\footnote{Here as in the following we use a notation that suggests merely an integration over space-time. However, appropriate summations over internal indices are also implied, but not written explicitly.}

$$\int dy G_F(x, y) G_F^{-1}(y, z) = \delta(x, z).$$ \hfill (2)

Consider the restricted class $\Gamma_c$ of Feynman graphs that are connected. The $n$-point functions $G^{(n)}_c$ defined by the corresponding restriction of (1) are called the connected $n$-point functions. The relation between complete and connected $n$-point functions can be described in a simple way. Partition the set of external legs of the complete $n$-point function in all possible ways. The sum over the product of connected functions for each partition yields the complete $n$-point function,

$$G^{(n)}(x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{I_1 \cup \cdots \cup I_k = \{x_1, \ldots, x_n\}} \prod_{j=1}^{k} G_c(I_j).$$ \hfill (3)

Here $I_1, \ldots, I_k$ denote non-empty subsets of $\{x_1, \ldots, x_n\}$ forming a partition. Note also that the partitions are unordered, i.e., the subsets $I_1, \ldots, I_k$ are not distinguished. As an example, Figure 1 shows the decomposition of the complete 4-point function in terms of connected functions. The former is indicated by an empty circle, while the latter are indicated by circles carrying the letter “c”.

We turn to consider the restriction of the class of connected Feynman graphs to that of 1-particle irreducible (1PI) Feynman graphs, denoted by...
\[ \Gamma_{\text{1PI}}. \] These are Feynman graphs which are connected and remain connected when any one of their internal edges is cut. We define \( \Gamma_{\text{1PI}} \) in analogy to (1) for this restricted class. The relation between 1PI-functions and connected ones may now be described as follows. The connected \( n \)-point function is the sum over all tree graphs with the given external legs, where the value of each vertex is given by \( G_{\text{1PI}} \). Since the latter carry Feynman propagators on all their legs, internal edges connecting two vertices need to carry the inverse Feynman propagator \( \Gamma_{\text{F}}^{-1} \) to cancel one superfluous Feynman propagator. As an example, Figure 2 shows the infinite decomposition of the connected 2-point function (propagator) in terms of 1PI ones. The 1PI vertices are drawn as shaded discs.

One variant of the 1PI-functions is of interest. Namely, replace the Feynman propagator \( \Gamma_{\text{F}} \) on the external legs of a 1PI-function by the connected propagator \( \Gamma_{\text{c}}^{(2)} \). We denote these modified 1PI-functions by \( \hat{\Gamma}_{\text{1PI}} \). The relation between \( \Gamma_{\text{c}} \) and \( \hat{\Gamma}_{\text{1PI}} \) is the same as that between \( \Gamma_{\text{c}} \) and \( \Gamma_{\text{1PI}} \) described above, except that internal edges now carry the inverse \( (\Gamma_{\text{c}}^{(2)})^{-1} \) of the connected propagator (defined analogous to (2)). A crucial property of the modified 1PI-functions is that the 2-point function \( \hat{\Gamma}_{\text{1PI}}^{(2)} \) vanishes by definition. This means that only trees with vertices that have valence at least three can occur in the sum. In particular, this makes the sum over trees finite for any given set of external legs. Figure 3 shows the decomposition of the connected 4-point function in terms of modified 1PI functions. The fact that the legs now carry the connected propagator is indicated with little circles.

Note that we assume all 1-point functions to vanish.
2.2 Field operator algebra and coproduct

We turn to introduce the basic algebraic definitions and elementary formalism employed in this article. While our basic setup is largely the same as that in [1] we give an adapted and self-contained description here.

Let $V$ be the vector space of linear combinations of elementary field operators $\phi(x)$. That is, elements of $V$ take the form $\lambda_1 \phi(x_1) + \lambda_2 \phi(x_2) + \cdots + \lambda_n \phi(x_n)$, where $\lambda_i$ are complex numbers and $x_i$ denote field operator labels. Consider now the commutative algebra $S(V)$ generated by those field operators with the time-ordered product. A general element in $S(V)$ takes the form

$$\lambda_1 \phi(x_{1,1})\phi(x_{1,2})\cdots \phi(x_{1,k_1}) + \lambda_2 \phi(x_{2,1})\phi(x_{2,2})\cdots \phi(x_{2,k_2}) + \cdots .$$

Note that we do not explicitly indicate the time-ordering prescription, but it is always understood. Let us denote by $V^k$ the vector space of $k$-fold products of field operator. Then, $S(V)$ is the direct sum of the spaces $V^k$, i.e., $S(V) = \bigoplus_{k=0}^{\infty} V^k$. We denote the identity operator spanning $V^0$ by $1$.

In mathematical terms, $S(V)$ is called the symmetric algebra over $V$. In our current notation a time-ordered $n$-point function is a linear map $V^n \to \mathbb{C}$. Thus, the ensemble of time-ordered $n$-point functions (of a given type) determines a linear map $S(V) \to \mathbb{C}$. Recalling the various types of $n$-point functions introduced in the previous section we denote the corresponding linear maps $S(V) \to \mathbb{C}$ as follows,

$$\rho(\phi(x_1)\cdots \phi(x_n)) := G^{(n)}(x_1,\ldots,x_n)$$
$$\sigma(\phi(x_1)\cdots \phi(x_n)) := G^{(n)}_c(x_1,\ldots,x_n)$$
$$\tau(\phi(x_1)\cdots \phi(x_n)) := G^{(n)}_{1PI}(x_1,\ldots,x_n)$$
$$\hat{\tau}(\phi(x_1)\cdots \phi(x_n)) := \hat{G}^{(n)}_{1PI}(x_1,\ldots,x_n).$$

The assumption that all 1-point functions vanish means that $\rho(\phi(x)) = \sigma(\phi(x)) = \tau(\phi(x)) = \hat{\tau}(\phi(x)) = 0$. For completeness, we also need to define 0-point functions. Since $\rho(1) = \langle 0|0 \rangle$ we need to set $\rho(1) = 1$. We shall see that the consistent choice for the other 0-point functions is $\sigma(1) = \tau(1) = \hat{\tau}(1) = 0$. Also, the 2-point function $\hat{\tau}(\phi(x)\phi(y))$ vanishes by construction.

$S(V)$ is not only an algebra, but also a coalgebra in a natural way. This means that there exists a linear coproduct map $\Delta : S(V) \to S(V) \otimes S(V)$

\[\begin{array}{l}
\rho(\phi(x_1)\cdots \phi(x_n)) := G^{(n)}(x_1,\ldots,x_n) \\
\sigma(\phi(x_1)\cdots \phi(x_n)) := G^{(n)}_c(x_1,\ldots,x_n) \\
\tau(\phi(x_1)\cdots \phi(x_n)) := G^{(n)}_{1PI}(x_1,\ldots,x_n) \\
\hat{\tau}(\phi(x_1)\cdots \phi(x_n)) := \hat{G}^{(n)}_{1PI}(x_1,\ldots,x_n). \\
\end{array}\]

\[\begin{array}{l}
\rho(\phi(x)) = \langle 0|0 \rangle = 1 \\
\sigma(\phi(x)) = \tau(\phi(x)) = \hat{\tau}(\phi(x)) = 0. \\
\end{array}\]

In [1] the same algebra was considered, but with the normal ordered product. This makes no difference to its structure. It is just more convenient in the present context to start immediately with time-ordered product.

\[\Delta(x_1\cdots x_n) = x_1\cdots x_n \otimes 1 + 1 \otimes x_1\cdots x_n.
\]
with certain properties. One way to think about this coproduct map is as a way to split a product of field operators into two parts in all possible ways. For example,

\[ \Delta(1) = 1 \otimes 1, \]
\[ \Delta(\phi(x)) = \phi(x) \otimes 1 + 1 \otimes \phi(x), \]
\[ \Delta(\phi(x)\phi(y)) = \phi(x)\phi(y) \otimes 1 + \phi(x) \otimes \phi(y) + \phi(y) \otimes \phi(x) + 1 \otimes \phi(x)\phi(y). \]

The general formula for the coproduct is

\[ \Delta(\phi(x_1) \cdots \phi(x_n)) = \sum_{I_1 \cup \cdots \cup I_{k+1} = \{\phi(x_1), \ldots, \phi(x_n)\}} T(I_1) \otimes \cdots \otimes T(I_{k+1}). \]

Here the sum runs over partitions of the set of field operators \( \{\phi(x_1), \ldots, \phi(x_n)\} \) into two sets \( I_1 \) and \( I_2 \). \( T \) denotes the time-ordered product of the field operators in the corresponding partition.

The coproduct has the property that it is an algebra map. This means that \( \Delta(\phi(x_1) \cdots \phi(x_n)) = \Delta(\phi(x_1) \cdots \phi(x_k)) \cdot \Delta(\phi(x_{k+1}) \cdots \phi(x_n)) \). Here, the product in \( S(V) \) is extended to a product in \( S(V) \otimes S(V) \) in the obvious way. The property of the coproduct to be an algebra map together with (4) and (5) completely determines it. Formula (7) can be derived from these properties.

Another important structure is the counit \( \epsilon : S(V) \rightarrow \mathbb{C} \). It is defined by \( \epsilon(1) = 1 \) and \( \epsilon(\phi(x_1) \cdots \phi(x_n)) = 0 \) for \( n > 0 \). The characterizing property of the counit is the equality \( (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta \). The algebra \( S(V) \) together with unit, counit, coproduct and antipode (which is another map that we do not need here) forms a Hopf algebra, see [1].

We shall also need iterated coproducts. First note that the coproduct satisfies the equality \( (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \). That is, after applying the coproduct once, a second application either on the first or on the second component yield the same result. This is called coassociativity. We define the map \( \Delta^k : S(V) \rightarrow S(V)^{\otimes k+1} \) as the \( k \)-fold application of the coproduct. Here, \( S(V)^{\otimes k+1} \) denotes the \((k + 1)\)-fold tensor product of \( S(V) \). Thus, \( \Delta^0 = \text{id} \) and \( \Delta^{k+1} = (\Delta \otimes \text{id}^{\otimes k}) \circ \Delta^k \). Here, the latter equation could be written in \( k + 1 \) different ways (corresponding to different positions of the application of the coproduct) which are all equivalent due to coassociativity.

The map \( \Delta^k \) generalizes (7) as follows:

\[ \Delta^k(\phi(x_1) \cdots \phi(x_n)) = \sum_{I_1 \cup \cdots \cup I_{k+1} = \{\phi(x_1), \ldots, \phi(x_n)\}} T(I_1) \otimes \cdots \otimes T(I_{k+1}). \]
The difference to the single coproduct is that the set of field operators is now split into \( k + 1 \) partitions. Note also that the partitions are ordered, i.e., the sets \( I_1 \ldots I_{k+1} \) are distinguishable.

A further definition we will require is the following. Given linear maps \( \alpha : S(V) \to \mathbb{C} \) and \( \beta : S(V) \to \mathbb{C} \) their convolution product is the map \( \alpha \ast \beta : S(V) \to \mathbb{C} \) defined by \( (\alpha \otimes \beta) \circ \Delta \). That is, we apply the coproduct followed by the application of \( \alpha \) to its first and \( \beta \) to its second component. Note that the coassociativity of the coproduct implies associativity of the convolution product. Thus, we can write multiple convolution products without needing to specify brackets. In particular, we may write an iterated convolution product using the iterated coproduct,

\[
\alpha_1 \ast \cdots \ast \alpha_k = (\alpha_1 \otimes \cdots \otimes \alpha_k) \circ \Delta^{k-1}.
\]

The convolution product makes \((S(V))^*\), the space of complex linear functions on \( S(V) \), into an algebra. This algebra has a unit which is given by the counit \( \epsilon \) of \( S(V) \). Furthermore, any element \( \alpha \) which satisfies \( \alpha(1) \neq 0 \) is invertible in this algebra, i.e., has an inverse with respect to the convolution product.

### 3 Complete and connected \( n \)-point functions

The first instance where we shall apply the Hopf algebraic approach to capture the combinatorics of quantum field theory is in the relation between the complete and the connected \( n \)-point functions.

As the (iterated) coproduct \( \Delta \) is intimately related to partitions it seems predestined to express the relation between complete and connected \( n \)-point functions \( \mathcal{Y} \). Indeed, the relation between the two is very compactly and elegantly expressed using the convolution product (and thus implicitly the coproduct).

**Proposition 3.** The complete \( n \)-point functions may be expressed in terms of the connected ones through the convolution exponential, \( \rho = \exp \ast \sigma \). The convolution exponential is defined in terms of its power series expansion.

**Proof.** We explicitly perform the power series expansion on a given argument
in the subspace \( V^n \subset S(V) \) \((n > 0)\),

\[
(\exp_*(\sigma))(\phi(x_1) \cdots \phi(x_n)) = \sum_{k=0}^{\infty} \frac{1}{k!} \sigma^{*k}(\phi(x_1) \cdots \phi(x_n))
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{k!} (\sigma \otimes \cdots \otimes \sigma) \circ \Delta^{k-1}(\phi(x_1) \cdots \phi(x_n))
\]

\[
= \sum_{k=1}^{n} \frac{1}{k!} \sum_{I_1 \cup \cdots \cup I_k = \{\phi(x_1), \ldots, \phi(x_n)\}} \prod_{j=1}^{k} \sigma(T(I_j)).
\]

Here, \( \sigma^{*k} \) denotes the \( k \)-fold convolution product of \( \sigma \) with itself. In particular, \( \sigma^{*0} = \epsilon \) by definition. Since \( \epsilon(\phi(x_1) \cdots \phi(x_n)) = 0 \) the first summand is zero in the first line and we may omit it. In going from the second to the third line we insert the definition of the iterated coproduct \([3]\). Note that by definition \( \sigma(1) = 0 \). This implies that all partitions where at least one partitioning set \( I_j \) is empty do not contribute. In particular, all summands of the outer sum with \( k > n \) must vanish. The only difference to the partitioning in \([3]\) is that the latter are unordered. However, the number of occurrences of each unordered partition in the set of ordered ones is exactly \( k! \). Thus, the factor \( 1/k! \) establishes equality with \( \rho(\phi(x_1) \cdots \phi(x_n)) \). To complete the proof, note that \( \rho(1) = \sigma^{*0}(1) = \epsilon(1) = 1 \) since \( \sigma(1) = 0 \) and thus \( \sigma^{*k}(1) = 0 \) for \( k > 0 \).

Let us emphasize that although the power series defining the exponential is formally infinite, on any given element of \( S(V) \) it is truncated to a finite and thus well defined sum as shown in the proof above. Thus, the relation \( \rho = \exp_* \sigma \) is completely well defined algebraically. Indeed, we may even invert it.

**Corollary 4.** The connected \( n \)-point functions may be expressed in terms of the complete ones through the convolution logarithm, \( \sigma = \log_\ast \rho \). The convolution logarithm is defined in terms of its power series expansion in \( \rho - \epsilon \).

**Proof.** By definition

\[
\log_\ast \rho = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k} (\rho - \epsilon)^* k.
\]

Note that \( (\rho - \epsilon)(1) = 0 \). This implies as in the proof of Proposition \([3]\) that the sum truncates to a finite sum on any given element in \( S(V) \). One may
show that the operations $\exp_x$ and $\log_x$ are mutually inverse by inserting one power series into the other. Since the procedure and result is exactly the same as in the usual arithmetic of complex numbers (say) we do not perform it explicitly here. However, it is crucial that the truncation of the power series for any given argument in $S(V)$ makes it algebraically well defined in the present context.

The attentive reader will notice that the present method of relating the complete and connected $n$-point functions shows certain similarities to the conventional one. Namely, in the conventional method one also finds that the relation is given by the exponential and the logarithm respectively, see e.g. [7]. The difference is of course that the conventional method uses sources while the present one uses the coproduct to define exponential and logarithm.

4 Connected and 1PI $n$-point functions

We now turn to the relation between connected and 1PI $n$-point functions. To state our Hopf algebraic formulation of this relation we need to define a few auxiliary structures first.

Define the formal element $R \in S(V) \otimes S(V)$ using the inverse Feynman propagator (2) as follows,\footnote{$R$ is formal insofar as it really lives in a completion of the tensor product $S(V) \otimes S(V)$. However, this fact is largely irrelevant for our purposes.}

$$R := \int dx \, dy \, G_F^{-1}(x, y) \, (\phi(x) \otimes \phi(y)).$$

(10)

Using the product in $S(V) \otimes S(V)$ we may view $R$ as an operator acting on this space by multiplication. Define now the map $Q : S(V) \rightarrow S(V) \otimes S(V)$ as the composition of $R$ with the coproduct together with a factor of 1/2:

$$Q := \frac{1}{2} R \circ \Delta.$$

(11)

We generalize $Q$ to maps $Q_i : S(V)^{\otimes k} \rightarrow S(V)^{\otimes k+1}$. Namely, let $Q_i$ be the application of $Q$ on the $i^{th}$ component only, i.e.,

$$Q_i := \text{id}^{\otimes i-1} \otimes Q \otimes \text{id}^{\otimes k-i}.$$  

Finally, we define maps $\Lambda^k : S(V) \rightarrow S(V)^{\otimes k+1}$ for $k \in \mathbb{N}_0$ recursively as follows,

$$\Lambda^0 := \text{id},$$

$$\Lambda^k := \Lambda^{k-1} \circ Q.$$  

11
\[ \Lambda^k := \frac{1}{k} \sum_{i=1}^{k} Q_i \circ \Lambda^{k-1}. \quad (12) \]

We are now ready to state our main result.

**Theorem 5.** The connected \( n \)-point functions may be expressed in terms of the 1PI ones through the formula

\[ \sigma = \sum_{k=1}^{\infty} \sigma^k, \quad \text{with} \quad \sigma^k := \tau^\otimes k \circ \Lambda^{k-1}. \quad (13) \]

The remainder of this section will be devoted to the proof of this result.

Recall from the description in Section 2.1 that the connected \( n \)-point functions are expressible as the sum over all tree graphs with 1PI \( n \)-point functions as vertices. Since the latter are represented by \( \tau \) we see that the sum in formula (13) must correspond to the sum over the number of such vertices \( k \). In turn, the map \( \Lambda^k \) must contain the information about all tree graphs with \( k \) vertices. We proceed to explain this.

We first generalize the definition of \( R \) given in (10) to an element (or operator) \( R_{i,j} \in S(V)^\otimes k \) with \( 1 \leq i < j \leq k \) via

\[ R_{i,j} := \int dx \, dy \, G_{F^{-1}}(x, y) \left( 1^{\otimes i-1} \otimes \phi(x) \otimes 1^{\otimes j-i-1} \otimes \phi(y) \otimes 1^{\otimes k-j} \right). \quad (14) \]

In other words, the field operators \( \phi(x) \) and \( \phi(y) \) are inserted at the \( i \)th and \( j \)th position respectively.

We proceed to establish a correspondence between graphs with \( k \) vertices and certain elements of \( S(V)^\otimes k \). Each tensor factor of \( S(V)^\otimes k \) corresponds to one vertex. A product \( \phi(x_1) \cdots \phi(x_n) \) in a given tensor factor corresponds to external legs labeled by \( x_1, \ldots, x_n \). The element \( R_{i,j} \in S(V)^\otimes k \) corresponds to an internal edge connecting the \( i \)th vertex with the \( j \)th vertex. Combining several internal edges and external legs by multiplying the respective expressions in \( S(V)^\otimes k \) allows to build arbitrary graphs with \( k \) vertices. Figure 4 shows some examples. Applying \( \tau^\otimes k \) to the resulting expression obviously yields the value of the respective graph with the vertices being 1PI functions. Thus, the graphs we just discussed are exactly those that are to enter into expressing the connected functions in terms of the 1PI ones.

The ordering of the tensor factors of \( S(V)^\otimes k \) induces an ordering of the vertices of the graph, i.e., we may think of them as labeled with numbers \( 1, \ldots, k \). However, when applying \( \tau^\otimes k \) the ordering is “forgotten”. Indeed, it is not relevant for the interpretation of graphs, but only plays a role at
the level of their algebraic representation here. For short, we call a graph ordered if its vertices are ordered. In the following, we will encounter elements of $S(V)^\otimes k$ that are linear combinations of expressions corresponding to ordered graphs. Alternatively, we might think of such elements as linear combinations of unordered graphs by considering different ordered graphs that correspond to the same unordered graph as the same. We call weight of the graph the scalar multiplying the expression for a given graph. Clearly, the weight of an unordered graph is the sum of the weights of all corresponding ordered graphs.

We see now what is required to prove Theorem 5. Namely, we need to show that $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n)) \in S(V)^\otimes k$ corresponds exactly to the sum over all tree graphs with $k$ (unordered!) vertices and external legs labeled by $x_1, \ldots, x_n$, each with weight one. Actually, we need to show something slightly weaker, namely, the statement needs to apply only to graphs where all of the vertices have valence at least two. This is because the 0-point and 1-point 1PI functions are zero. Indeed, we will see in the process that the more general statement is false.

Consider the coproduct applied to the $i$th component of $S(V)^\otimes k$, i.e., $\Delta_i := \text{id}^\otimes i-1 \otimes \Delta \otimes \text{id}^\otimes k-i$ as a map $S(V)^\otimes k \to S(V)^\otimes k+1$. Recalling the formula (7) we see that $\Delta_i$ converts a graph with $k$ vertices into a sum over graphs with $k+1$ vertices by splitting the $i$th vertex into two in all possible ways. That is, the $i$th vertex is replaced by two vertices (numbered $i$ and $i+1$) and its legs (considered as distinguishable) are distributed between the two new vertices in all possible ways. Note that the two new vertices are distinguished due to the ordering of the tensor factors. Thus, to obtain

$\phi(x_1)\phi(x_2)\phi(x_3) \quad R \cdot (\phi(x_1)\phi(x_2) \otimes \phi(x_3)\phi(x_4))$

Figure 4: Examples for the correspondence of graphs with elements of $S(V)^\otimes k$. 

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the corresponding operation for unordered graphs we need to divide by a factor of 2. This factor corresponds to the two different relative orderings of the new vertices with which each unordered configuration occurs. The only exception to this is the case when the split vertex has no legs at all. No overcounting happens in this case. The meaning of the map $Q_i$ given by (11) becomes clear now in terms of graphs. Namely, it splits the $i^{th}$ vertex into two and subsequently reconnects the two new vertices with an edge. Dividing by 2 compensates for the double counting as described above if we are interested in unordered graphs (assuming the set of legs of the split vertex is not empty).

**Lemma 6.** Fix integers $k \geq 1$ and $n \geq 0$ as well as field operator labels $x_1, \ldots, x_n$. (a) $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$ corresponds in the manner described above to a sum of weighted graph with $k$ vertices and $n$ external legs labeled by $x_1, \ldots, x_n$. (b) Each of these graphs is connected. (c) Each of these graphs is a tree graph. (d) Any tree graph with $k$ unlabeled vertices and the given external legs occurs in $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$ with some positive weight.

**Proof.** Firstly, it is clear that $\Lambda^0(\phi(x_1) \cdots \phi(x_n))$ corresponds to the graph with one vertex and the external legs labeled by $x_1, \ldots, x_n$. Secondly, $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$ is generated from this by sums of multiple applications of maps $Q_i$ and scalar factors. But $Q_i$ converts a term corresponding to a graph to a sum over terms corresponding to graphs. Thus, $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$ is a sum of terms each of which corresponds to a graph (with some weight). This completes the proof of (a).

Given a connected graph, splitting a vertex produces at most two disconnected pieces. Reconnecting the new vertices with an edge yields again a connected graph. Thus, $Q_i$ produces connected graphs from connected ones. Evidently, $\Lambda^0(\phi(x_1) \cdots \phi(x_n))$ corresponds to a connected graph. This proves (b).

Splitting a vertex of a tree graph necessarily yields two disconnected graphs. Reconnecting the new vertices with an edge thus cannot introduce a cycle. Therefore, $Q_i$ produces tree graphs from tree graphs. Evidently, $\Lambda^0(\phi(x_1) \cdots \phi(x_n))$ corresponds to a tree graph. This proves (c).

To prove (d) we use a recursive argument. Evidently, for $k = 1$ the statement is true. Now assume that any tree graph with $k - 1$ unlabeled vertices and the given external legs occurs in $\Lambda^{k-2}(\phi(x_1) \cdots \phi(x_n))$ with positive weight. Consider a tree graph with $k$ unlabeled vertices and the given external legs. Choose an arbitrary internal edge. Shrinking this edge and fusing the vertices it connects yields a tree graph that corresponds by assumption to a term in $\Lambda^{k-2}(\phi(x_1) \cdots \phi(x_n))$. Say, the fused vertex has
position $i$. Applying $Q_i$ to this term will yield a sum over terms one of which will correspond to the original tree graph. By the recursive definition of $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$ it thus contains this term with positive weight. This completes the proof.

What remains in order to prove Theorem 5 is to show that the term corresponding to each tree graph (with all vertices of valence at least two) has weight exactly 1. We start with a more restricted result.

**Lemma 7.** Fix integers $k \geq 1$ and $n \geq k$ and field operator labels $x_1, \ldots, x_n$. Consider a tree with $k$ vertices, external legs labeled by $x_1, \ldots, x_n$ and the property that each vertex has at least one external leg. Then, the term in $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$ corresponding to that tree has weight 1.

**Proof.** We proceed by induction on the number of vertices. Clearly, the statement is true for $k = 1$. Now assume that $\Lambda^{k-2}(\phi(x_1) \cdots \phi(x_n))$ contains each tree graph with $k - 1$ vertices and external legs labeled by $x_1, \ldots, x_n$ and the property that each vertex carries at least one external leg with weight exactly 1. (Of course it may in addition contain terms corresponding to other graphs.) Consider a tree graph $\gamma$ with $k$ vertices, external legs labeled by $x_1, \ldots, x_n$ and the property that each vertex carries at least one external leg. We proceed to show that it occurs with weight exactly 1 in $\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n))$. To this end we check from which graphs with $k - 1$ vertices $\gamma$ is generated by the recursion formula (12) and in how many ways. Translated into the language of graphs, the formula prescribes that a given graph is split and reconnected at every of its vertices. The resulting terms are summed over and multiplied by $1/k$. Conversely, this implies that $\gamma$ is generated from all the graphs that are obtained by shrinking one of its internal edges. Since there are $k$ edges these are a priori $k$ graphs. These graphs are indeed all distinct, since the external legs attached to each vertex force them to be distinguishable. Furthermore, each of these graphs generates $\gamma$ in only one way, i.e., as one resulting graph in the splitting and reconnecting of only one of its vertices (again due to the forced distinguishability of the vertices). Since by assumption each generating graph has weight one, the multiplicity $k$ cancels exactly with the factor $1/k$ in (12) to produce weight 1 for $\gamma$. This completes the proof.

To describe the weight of arbitrary tree graphs we will need to consider symmetries of graphs.

**Definition 8.** Consider a tree graph $\gamma$ with ordered vertices. A symmetry of $\gamma$ is a permutation of the ordering of its vertices that yields the (topologically)
same ordered graph. The number of symmetries, i.e., order of the group of
permutations leaving the graph invariant, is called the symmetry factor
of the graph.

Since the symmetry factor is the same for any ordering of the vertices of
a graph the concept makes sense for unordered graphs as well.

We also need the following property of \( \Lambda^k \):

**Lemma 9.** Fix integers \( k, n \geq 0 \) and operator labels \( x_1, \ldots, x_n \). Then, \( \Lambda^k \) satisfies the factorization property

\[
\Lambda^k(\phi(x_1) \cdots \phi(x_n)) = \Lambda^k(1) \cdot \Delta^k(\phi(x_1) \cdots \phi(x_n)).
\]

**Proof.** This follows immediately from the multiplicativity of the coproduct
and the recursive definition \([12]\).

We can now state the generalization of Lemma 4.

**Lemma 10.** Fix integers \( k \geq 1 \) and \( n \geq 0 \) and field operator labels \( x_1, \ldots, x_n \).
Consider a tree \( \gamma \) with \( k \) vertices and external legs labeled by \( x_1, \ldots, x_n \). Let \( s \) be the symmetry factor of \( \gamma \). Then, the term in \( \Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n)) \) corresponding to that tree has weight \( 1/s \).

**Proof.** If \( \gamma \) has external legs attached to every of its vertices we simply recall Lemma 7 and note that a graph with this property has no non-trivial symmetries. Thus, we may now assume that \( \gamma \) has \( m \) vertices to which no external leg is attached. Consider a graph \( \gamma' \) which is constructed from \( \gamma \) by attaching an external leg to every vertex without external legs, choosing arbitrary but fixed labels \( y_1, \ldots, y_m \) for the legs in the process. By Lemma 9 and the multiplicativity of \( \Delta \) and thus \( \Delta^j \) we have

\[
\Delta^{k-1}(\phi(x_1) \cdots \phi(x_n)\phi(y_1) \cdots \phi(y_m)) = \\
\Lambda^{k-1}(\phi(x_1) \cdots \phi(x_n)) \cdot \Delta^{k-1}(\phi(y_1) \cdots \phi(y_m)).
\]

By Lemma 7, the graph \( \gamma' \) occurs in the term on left hand side with weight 1. By Lemma 6 the graph \( \gamma \) occurs in the first factor on the right hand side with some non-zero weight, say \( \alpha \). Every summand of \( \Delta^{k-1}(\phi(y_1) \cdots \phi(y_m)) \) (recall the formula \([8]\)) which places the external legs at the designated vertices of \( \gamma \) to produce \( \gamma' \) contributes to the weight of \( \gamma' \) in terms of that of \( \gamma \). Any different ways this can happen define a symmetry of \( \gamma \). Furthermore, \( \gamma \) can have no more than these symmetries, since its vertices that already carry external legs are distinguishable and thus held fixed under any symmetry. Therefore, we obtain the formula \( 1 = \alpha \cdot s \) for the weights, or \( \alpha = 1/s \). This completes the proof. 

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The appendix shows the result of computing all tree graphs without external legs as weighted contributions to $\Lambda^{k-1}(1)$, for vertex number $k \leq 7$.

The proof of Theorem 5 is completed with the following lemma.

**Lemma 11.** Consider a tree graph $\gamma$, all of whose vertices have valence at least two. Then, $\gamma$ has no non-trivial symmetries.

**Proof.** Consider a vertex $v$ of $\gamma$. We show that any symmetry must leave $v$ invariant. If $v$ carries an external leg it must be invariant since it is distinguishable. Thus, assume $v$ carries no external leg. Choose one internal edge $e$ connected to $v$. Cut $\gamma$ into two by removing $e$. This yields two tree graphs $\gamma_1$ and $\gamma_2$. Each of these must have at least one external leg to satisfy the valence requirement. Say $e_1$ is an external leg of $\gamma_1$ and $e_2$ an external leg of $\gamma_2$. Since $\gamma$ is a tree there is exactly one path to connect $e_1$ with $e_2$. Since the vertices connected with $e_1$ and $e_2$ are held fixed under any symmetry so is the whole chain of vertices formed by the path. However, $v$ is part of this chain by construction and thus held fixed by any symmetry. \hfill \qed

5 Further recursion relations

We may extend the results of the previous section to obtain further interesting recursion relations. Recall from (13) the decomposition of $\sigma$ into components $\sigma^k$ according to vertex number $k$.

**Proposition 12.** $\sigma^k$ may be determined recursively via $\sigma^1 = \tau$ and with the recursion equation for $k > 1$,

$$\sigma^k = \frac{1}{k - 1} \sum_{i=1}^{k-1} (\sigma^i \otimes \sigma^{k-i}) \circ Q.$$

Given the definition of $\sigma^k$ in (13) Proposition 12 is implied by the following lemma.

**Lemma 13.**

$$\Lambda^{k-1} = \frac{1}{k - 1} \sum_{i=1}^{k-1} (\Lambda^{i-1} \otimes \Lambda^{k-i-1}) \circ Q \quad \forall k > 1.$$
multiplicativity of the (iterated coproduct) together with the factorization property of \( \Lambda \) (Lemma 9). We obtain the equivalent expression

\[
\Lambda^{k-1} = \frac{1}{2(k-1)} \sum_{i=1}^{k-1} \left( (\Delta^{i-1} \otimes \Delta^{k-i-1}) R \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i-1}) \circ \Delta
\]

\[
= \frac{1}{2(k-1)} \sum_{i=1}^{k-1} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i-1}) \circ \Delta. \tag{15}
\]

Before proceeding with the proof we note that this formula has a straightforward interpretation in terms of sums over weighted tree diagrams following the correspondence of Section 4. Namely, the formula states that the weighted sum over trees with \( k \) vertices is given by summing over all ordered pairs of weighted trees with total number of vertices equal to \( k \), connecting them in all possible ways with an edge and dividing by \( 2(k-1) \).

The equation (15) is proved by induction. We verify it for \( k = 2 \),

\[
\Lambda^1 = \frac{1}{2} R \cdot (\Lambda^0 \otimes \Lambda^0) \circ \Delta = \frac{1}{2} R \cdot \Delta,
\]

and assume it holds for general order \( k \). Then, (12) yields

\[
\Lambda^k = \frac{1}{k} \left( \sum_{j=1}^{k} Q_j \right) \circ \Lambda^{k-1}
\]

\[
= \frac{1}{k} \left( \sum_{j=1}^{k} Q_j \right) \cdot \frac{1}{2(k-1)} \left( \sum_{i=1}^{k-1} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i-1}) \right) \circ \Delta
\]

\[
= \frac{1}{2k(k-1)} \sum_{i=1}^{k-1} \left( \sum_{j=1}^{i} Q_j \right) \cdot \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i-1}) \circ \Delta
\]

\[
+ \frac{1}{2k(k-1)} \sum_{j=i+1}^{k} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i-1}) \circ \Delta
\]

\[
= \frac{1}{2k(k-1)} \sum_{j=i+1}^{k} \left( \sum_{j=1}^{k} \sum_{a=1}^{i} \sum_{b=i+1}^{k} R_{a,b} \right) \cdot Q_j \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i-1})
\]

Indeed, it would be possible to base the proof of Lemma 7 on the recursion formula for \( \Lambda \) given here instead of (12). The argument would then roughly proceed by considering all \( k - 1 \) ways to cut a tree with \( k \) vertices into two by removing an internal edge. The factor 2 accounts for the relative ordering of the two subtrees.
\[ + \sum_{j=i+1}^{k} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k} R_{a,b} \right) \cdot Q_j \cdot (\Lambda^{i-1} \otimes \Lambda^{k-1-i}) \circ \Delta \]

\[ = \frac{1}{2k(k-1)} \sum_{i=1}^{k-1} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k+1} R_{a,b} \right) \cdot (\Lambda^i \otimes \Lambda^{k-1-i}) \circ \Delta \]

\[ + (k-i) \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k+1} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i}) \circ \Delta \]

\[ = \frac{1}{2k} \left( \sum_{a=1}^{k} R_{a,k+1} \right) \cdot (\Lambda^{k-1} \otimes \Lambda^0) + \sum_{i=2}^{k-1} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k+1} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i}) \]

\[ + \left( \sum_{b=2}^{k+1} R_{1,b} \right) \cdot (\Lambda^0 \otimes \Lambda^{k-1}) \circ \Delta \]

\[ = \frac{1}{2k} \sum_{i=1}^{k} \left( \sum_{a=1}^{i} \sum_{b=i+1}^{k+1} R_{a,b} \right) \cdot (\Lambda^{i-1} \otimes \Lambda^{k-i}) \circ \Delta. \]

6 Extensions and applications

6.1 Modified 1PI functions

In Section 2.1 we have discussed two versions of 1PI functions, the standard one, denoted \( G_{1PI} \), and a modified one, denoted \( \hat{G}_{1PI} \). Recall that the connected \( n \)-point functions \( G_c \) are expressible in terms of \( G_{1PI} \) and of \( \hat{G}_{1PI} \) essentially in the same way, as a sum over all tree graphs, the difference being that for \( \hat{G}_{1PI} \) the (inverse) Feynman propagator \( G_F \) is replaced by the (inverse) connected propagator \( G_c^{(2)} \) for all edges. Thus, the results of Sections 4 and 5 immediately carry over to the relation between connected functions and modified 1PI functions. We only need to modify the definition of \( R \) by replacing \( G_F^{-1} \) with \( G_c^{(2)} \) in 10; call the new version \( \hat{R} \). We denote the induced modifications of \( Q \) and \( \Lambda \) by \( \hat{Q} \) and \( \hat{\Lambda} \) respectively.

Corollary 14. The connected \( n \)-point functions may be expressed in terms of the modified 1PI ones through the formula

\[ \sigma = \sum_{k=1}^{\infty} \hat{\sigma}^k, \quad \text{with} \quad \hat{\sigma}^k := \hat{\tau}^k \circ \hat{\Lambda}^{k-1}. \quad (16) \]
Corollary 15. \( \hat{\sigma}^k \) may be determined recursively via \( \hat{\sigma}^1 = \hat{\tau} \) and with the recursion equation for \( k > 1 \),

\[
\hat{\sigma}^k = \frac{1}{k-1} \sum_{i=1}^{k-1} (\hat{\sigma}^i \otimes \hat{\sigma}^{k-i}) \circ \hat{Q}.
\]

Recall that for the modified 1PI functions not only 0- and 1-point functions vanish, but also 2-point functions. This implies that only trees contribute which have the property that all their vertices have valence at least three. For a given number of external legs, there are only finitely many such trees. Thus, in contrast to \( \text{(13)} \) the sum in \( \text{(16)} \) is finite for each given set of external legs, i.e., for each element of \( S(V) \) to which it is applied.

6.2 Tree level contributions

Another context in quantum field theory, where we sum over all tree diagrams is of course when we wish to evaluate merely the tree level contribution to an \( n \)-point function. Let us indicate the tree level contribution by a lower index “\( T \)”. Thus, the tree level contributions to the algebraic \( n \)-point functions are denoted \( \rho_T \), \( \sigma_T \) and \( \tau_T \) for the complete, connected and 1PI \( n \)-point functions respectively. Of course, the latter are now nothing but the interaction terms of the Lagrangian. The results of Section 3 carry over immediately.

Corollary 16. \( \rho_T = \exp_\ast \sigma_T \), where the convolution exponential is defined in terms of its power series expansion.

Corollary 17. \( \sigma_T = \log_\ast \rho_T \), where the convolution logarithm is defined in terms of its power series expansion in \( \rho_T - \epsilon \).

The results of Sections 4 and 5 take the following form.

Corollary 18. The tree level connected \( n \)-point functions may be expressed in terms of the interaction vertices through the formula

\[
\sigma_T = \sum_{k=1}^{\infty} \sigma_T^k, \quad \text{with} \quad \sigma_T^k := \tau_T^k \circ \Lambda^{k-1}.
\]

Corollary 19. \( \sigma_T^k \) may be determined recursively via \( \sigma_T^1 = \tau_T \) and with the recursion equation for \( k > 1 \),

\[
\sigma_T^k = \frac{1}{k-1} \sum_{i=1}^{k-1} (\sigma_T^i \otimes \sigma_T^{k-i}) \circ Q.
\]
Not only the 0- and 1-point, but also the 2-point contribution to \( \tau_T \) vanishes by definition. Thus, as in the case of the modified 1PI functions, only finitely many tree graphs with given external legs contribute and the sum \( 17 \) is finite on any given element of \( S(V) \).

### 6.3 Algorithmic Considerations

A key feature of the Hopf algebraic approach to \( n \)-point functions presented here is the close relation of the obtained algebraic relations to concrete algorithms. In Section 4 the recursive definition \( 12 \) mirrors an algorithm to construct all tree graphs. Tree graphs with \( k + 1 \) vertices are created from those with \( k \) vertices by taking each graph in turn and applying the following procedure: Take every vertex of the graph in turn and split it into two vertices, distributing the legs in all possible ways and reconnecting the two new vertices.

Of course, the actual algorithm is slightly more complicated as symmetries have to be taken into account and the correct weights must be obtained. However, the recursion relation \( 12 \) even suggests an implementation of data structures. For example, we may represent \( S(V)^{\otimes n} \) by an array, each element of which corresponds to a graph in the sense of Section 4. Such an element would contain a rational scalar (the weight) and an array, each element of which would correspond to one tensor factor in \( S(V)^{\otimes n} \) or equivalently to one vertex. In turn each element would be a set of symbolic elements representing external or internal legs. The coproduct and the \( R \)-operator are then very simple operations distributing or adding pairs of symbolic elements.

In this context a certain property of the recursion algorithm represented by \( 12 \) is rather interesting. Namely, it is easy to see that a graph containing at least one vertex with valence 1 will in a recursion step only generate graphs that contain at least one vertex with valence 1. Conversely, this means that if we are not interested in such graphs we may exclude them at each recursion step without losing any relevant graphs. What is more, we may implement this removal of “irrelevant” graphs through a truncated coproduct. Recall from \( 7 \) that the coproduct restricted to the subspace \( V^n \subset S(V) \) is a map \( V^n \rightarrow \bigoplus_{i=0}^n V^i \otimes V^{n-i} \). Removing those components of the direct sum where at least one of the target tensor factors is \( V^0 \) we obtain a map \( V^n \rightarrow \bigoplus_{i=1}^{n-1} V^i \otimes V^{n-i} \). We call this the truncated coproduct \( \Delta_{\geq 1} \). For example,

\[
\begin{align*}
\Delta_{\geq 1}(1) &= 0, \\
\Delta_{\geq 1}(\phi(x)) &= 0, \\
\Delta_{\geq 1}(\phi(x)\phi(y)) &= \phi(x) \otimes \phi(y) + \phi(y) \otimes \phi(x)
\end{align*}
\]
In the recursion process (12) the number of legs of a vertex changes when the $Q$-operator is applied to it. The only terms corresponding to 1-point vertices arise from those terms in the coproduct where one of the new vertices receives no legs at all and has thus, after reconnecting with $R$, only one leg. Thus, replacing the coproduct $\Delta$ by the truncated coproduct $\Delta_{\geq 1}$ exactly eliminates the irrelevant trees with 1-point vertices.

If we limit the allowed valence of vertices even more, we can push the truncation prescription even further. Assume we are interested only in trees with vertices of valence at least three (as in the case of modified 1PI functions or tree level calculations). By extension of the above discussion it is clear that the removal of all irrelevant trees is achieved by a further truncation of the coproduct. Namely, remove from the coproduct map $V^n \to \bigoplus_{i=0}^n V^i \otimes V^{n-i}$ the components with tensor factors $V^0$ and $V^1$. We denote the truncated coproduct defined in this way by $\Delta_{\geq 2}$. It is then obvious that using this truncated coproduct to define $Q$ and in turn $\Lambda$ produces only trees all of whose vertices have valence at least three. In particular, this means that for given external legs the algorithm sketched above for calculating all trees terminates after finitely many steps. As in the above case, it never creates a tree that would need to be discarded later. Note that this procedure may be extended to any lower bound $n$ on the valence of vertices by using the corresponding $(n-1)$-truncated coproduct $\Delta_{\geq n-1}$ defined in the obvious way.

### 6.4 Fermions

Recall that we have limited ourselves above to a purely bosonic theory. However, as already mentioned, this limitation is purely one of convenience and simplicity. Indeed, all of our arguments and results apply equally to fermionic fields. However, the underlying formalism becomes slightly more complicated. The vector space $V$ will in general be a $\mathbb{Z}_2$-graded vector space, a direct sum of a bosonic and a fermionic part. In turn, the algebra $\mathcal{S}(V)$ is the $\mathbb{Z}_2$-graded symmetric algebra over $V$. As special cases, if $V$ is completely bosonic we recover the usual commutative symmetric algebra (as above); if $V$ is completely fermionic we recover the usual anti-commutative exterior algebra. As a Hopf algebra $\mathcal{S}(V)$ is in general a $\mathbb{Z}_2$-graded or super-Hopf algebra. In particular, the coproduct becomes graded. This simply means that a minus sign appears as soon as odd elements are commuted, e.g.,

$$\Delta(ab) = ab \otimes 1 + 1 \otimes ab + a \otimes b + (-1)^{|a||b|} b \otimes a.$$
Here, $|a|$ is defined to be 0 or 1 depending on whether $a$ is bosonic or fermionic. We refer to [1] for more details on the structure of $S(V)$ in general.

However, all formulas appearing in Theorems, Propositions and Corollaries generalize completely unchanged. The graded structure is completely implicit there. The only explicitly changing formulas are indeed those that involve explicit evaluations of the coproduct such as (6), (7) and (8). The underlying reason is that our constructions are completely “functorial” and could indeed be generalized to arbitrary (reasonable) symmetric categories. (For a generalization of certain $n$-point functions to non-symmetric categories see [4].)

7 Conclusions

Functional methods used to handle the combinatorics of quantum field theory, while having a certain elegance, have some serious drawbacks. While appearing to be analytic, they are really formal as the mathematical objects involved usually do not actually exist. In particular, the source terms appearing in functional expressions are usually merely a book keeping device, rather than actual mathematical (or physical) entities. We hope to have convinced the reader that at least certain combinatorial aspects of quantum field theory can be handled in a much more intrinsic and (we think) at least as elegant language. Indeed, our main object is nothing but the rather concrete time-ordered algebra of field operators. Its coproduct, while perhaps an unusual structure for quantum field theorists, is well known to mathematicians. It is thus natural to use it instead of more indirect and formal functional methods.

Another advantage of our algebraic approach over a functional one is its closeness to algorithmic descriptions of the processes involved. Recall from Section 6.3 how easy it is to translate the recursion relation underlying the correspondence between connected and 1PI $n$-point functions into an algorithm, which moreover appears to be rather efficient.

As mentioned in the introduction, the present paper shares a common programme with [1], namely to employ the full Hopf algebraic structure of the algebra of field operators in describing and understanding quantum field theory. Thus, it is natural to combine the results of the present paper with those of [1]. Indeed, the functorial nature of the Drinfeld twist employed in [1] to relate different products and their (complete) $n$-point functions should make it possible to induce the corresponding transformation on the
corresponding connected or 1PI $n$-point functions using the results presented here. This, of course, goes beyond the scope of the present paper.

Acknowledgements

Â. M. was supported through fellowships provided by Fundação para a Ciência e a Tecnologia SFRH/BD/1282/2000 and subsequently by Fundação Calouste Gulbenkian 65709.

A Appendix

This appendix shows all tree graphs without external legs and with up to 7 vertices, computed as contributions to $\Lambda^k(1)$ via (12). The factors in front are the inverses of the symmetry factors of Definition 8, see Lemma 10.

$\Lambda^0(1) = \bullet$

$\Lambda^1(1) = \frac{1}{2} \bullet \bullet$

$\Lambda^2(1) = \frac{1}{2} \bullet \bullet \bullet$

$\Lambda^3(1) = \frac{1}{2} \bullet \bullet \bullet + \frac{1}{3!} \bullet \bullet \bullet \bullet$ 

$\Lambda^4(1) = \frac{1}{2} \bullet \bullet \bullet \bullet + \frac{1}{2} \bullet \bullet \bullet \bullet \bullet + \frac{1}{3!} \bullet \bullet \bullet \bullet \bullet$
\[ \Lambda^5(1) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3!} + \frac{1}{2!} + \frac{1}{5!} \]

\[ \Lambda^6(1) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3!} + \frac{1}{3!} + \frac{1}{2} + \frac{1}{2!} + \frac{1}{2 \cdot 3!} + \frac{1}{4!} + \frac{1}{6!} \]
References

[1] C. Brouder, B. Fauser, A. Frabetti, and R. Oeckl, *Quantum field theory and Hopf algebra cohomology*, J. Phys. A 37 (2004), 5895–5927, hep-th/0311253.

[2] B. Fauser, *On the Hopf algebraic origin of Wick normal ordering*, J. Phys. A 34 (2001), 105–115, hep-th/0007032.

[3] C. Brouder, *A quantum field algebra*, Preprint math-ph/0201033, 2002.

[4] R. Oeckl, *Braided Quantum Field Theory*, Commun. Math. Phys. 217 (2001), 451–473, hep-th/9906225.

[5] D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, Adv. Theor. Math. Phys. 2 (1998), 303–334, q-alg/9707029.

[6] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Commun. Math. Phys. 210 (2000), 249–273.

[7] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980.