ON THE CENTRAL ROLE OF SCALE INVARIANT
POISSON PROCESSES ON \((0, \infty)\)

RICHARD ARRATIA

ABSTRACT. The scale invariant Poisson processes on \((0, \infty)\) play a central but mildly disguised role in number theory, combinatorics, and genetics. They give the continuous limits which underly and unify diverse discrete structures, including the prime factorization of a uniformly chosen integer, the factorization of polynomials over finite fields, the decomposition into cycles of random permutations, the decomposition into components of random mappings, and the Ewens sampling formula. They deserve attention as one of the fundamental and central objects of probability theory.

0. Introduction

The scale invariant Poisson processes on \((0, \infty)\) have almost always been overlooked, even though these processes play a fundamental role in combinatorics, number theory, and genetics. Quantities and relations which are most easily explained in terms of these Poisson processes have been studied directly without mentioning the Poisson connection. Examples in number theory (18.1) include Dickman’s function \(\rho\) from [17] in 1930, Buchstab’s function \(\omega\) from [15] in 1937, and the “convolution powers of the Dickman function” from [14, 29]. The examples from combinatorics and genetics involve the Poisson-Dirichlet process and the GEM process from the 1970’s, which are usually discussed in terms of products of independent, Beta-distributed random variables, or constructed by conditioning a Poisson process which is not scale invariant. We want to reveal the presence of the scale invariant Poisson processes, as well as their intrinsic beauty and simplicity, for an audience including probabilists, number theorists, geneticists, and others, without assuming background such as knowledge of point processes.

The sections correspond, more or less, to the transparencies from the lecture, except for the last two sections, which match discussions from the problem sessions, and have been substantially expanded.

1. Scale invariant Poisson processes on \((0, \infty)\)

The only reference to “scale invariant Poisson processes” per se that we have found is Daley and Vere Jones [16], page 325, where the scale invariant Poisson processes on \(\mathbb{R}\) are fully classified. We requested help from the audience in compiling a bibliography on scale invariant Poisson processes; C. Newman supplied a reference to [1]. Pitman and Yor [45] is a general study of spacings in scale invariant random closed subsets of \((0, \infty)\), including examples like the zeroes of Brownian motion, and the scale invariant Poisson processes.

Date: September 15, 1997; revision of January 16, 1998.
1991 Mathematics Subject Classification. primary 60-02, secondary 60F17.
Work supported in part by NSF grant DMS 96-26412.
The first subtle thing to realize about scale invariant Poisson processes is that the plural is essential! As with translation invariant Poisson processes, there is a nonnegative parameter $\theta$ which appears as a scalar factor for the intensity measure, but unlike the translation invariant case, varying $\theta$ leads to major qualitative changes. In particular, for certain natural aspects of qualitative behavior, there are phase transitions at $\theta = 1$ (see (16.2)), and at $\theta = 1/\log 2$ (see section 22.)

Our notation for scaling is $cX := \{cx : x \in X\}$, defined for any $X \subset R^d$ and $c > 0$. For a random set $X$, scale invariance is the property that for all $c > 0$, $cX =_d X$, i.e. all scalings of the set have the same distribution.

In case $X \subset (0, \infty)$, one can take logarithms: let $L = \log(X) := \{\log(x) : x \in X\}$. In this situation, clearly, scale invariance for $X$ is equivalent to translation invariance for $L$.

2. Intensity

For us, a point process is a random set or multiset $X$ such that for sets $I$ in an appropriate class, $X \cap I$ is a finite set or multiset, with finite mean size; the points of $X$ are not labelled. The intensity measure $\mu$ is then defined by $\mu(I) = E[\chi \cap I]$, the expected number of points falling in $I$. For instance, if $X$ is a translation invariant point process on $R$, then its intensity is a translation invariant, nonnegative measure, and hence has the form $\theta$ times Lebesgue measure — we will say that the intensity is $\theta \, dx$ on $R$. The interesting intensity measures for our purposes are, each with parameter $\theta > 0$:

- $\theta \, dx$ on $(−\infty, \infty)$ translation invariant,
- $(\theta/x) \, dx$ on $(0, \infty)$ scale invariant,
- $(\theta e^{-x/x}) \, dx$ on $(0, \infty)$ “Gamma” or “Moran” subordinator.

One can easily verify that $(\theta/u) \, du$ gives the intensity of scale invariant point processes $X$ on $(0, \infty)$, starting from the translation invariance of $L := \log(X)$, as follows. For $a = e^x < b = e^y$, $|X \cap (a, b)| = |L \cap (x, y)|$, with expectation

$$\int_a^b \theta \, dt = \theta(y - x) = \theta \log(b/a) = \int_a^b (\theta/u) \, du.$$  

Another way to compute this is to start with the candidate for a scale invariant intensity, $\theta/x \, dx$, and the map $x \mapsto cx$, and to apply the change of variables formula correctly — an exercise which the author usually gets wrong at first! Note that scaling $(x \mapsto cx)$ the process with intensity $\theta e^{-x/x} \, dx$ yields the same $\theta$ but a different exponential decay, i.e. the new intensity is $\theta e^{-x/c}/x \, dx$, for $0 < c < \infty$.

3. Poisson processes, inversion invariance

The characterizing property of a Poisson process is that for disjoint $I_1, I_2, \ldots, I_k$, the number of points in $I_j$ for $j = 1$ to $k$ are independent, Poisson distributed random variables. This property is preserved by any mapping, not necessarily one-to-one; the relevant examples here are $x \mapsto x/\theta$ for $\theta > 0$, $x \mapsto -x$, $x \mapsto e^x$, $x \mapsto e^{-x}$, $x \mapsto \log x$, and $x \mapsto -\log x$.

Thus in checking that the following four statements about a point process $X = \{X_i\}$ are equivalent, one ingredient is calculating intensities, and the other ingredient, mapping the Poisson properties, is automatic:

- $\{X_i\}$ is scale invariant Poisson $(\theta/x) \, dx$ on $(0, \infty)$,
- $\{\log X_i\}$ is translation invariant Poisson $(\theta \, dx)$ on $(-\infty, \infty)$,
\{−\log X_i\} is translation invariant Poisson \((\theta \, dx)\) on \((−\infty, \infty)\),\n\{1/X_i\} is scale invariant Poisson \((\theta/x) \, dx\) on \((0, \infty)\).

The above shows how the reflection invariance of a translation invariant Poisson process on \(\mathbb{R}\) is directly equivalent to the inversion invariance of a scale invariant Poisson process on \((0, \infty)\).

4. Favorite labellings

For the translation invariant Poisson process, label the points \(L_i\) for \(i \in \mathbb{Z}\) with \(L_i < L_{i+1}\) and \(L_0 \leq 0 < L_1\). This seems like the only decent choice.

\[−\infty < \cdots < L_{-1} < L_0 \leq 0 < L_1 < L_2 < \cdots < \infty.\]

For the scale invariant Poisson processes, there are two natural choices for labelling, and it does not seem reasonable to try to make one notation fit all situations.

If we focus from one down to zero, use \(X_i := e^{-L_i}\), so that \(X_i < X_1\), \(X_1\) is the first point to the left of one, and \(X_n \downarrow 0\) as \(n \to \infty\).

\[0 < \cdots < X_3 < X_2 < X_1 < 1 \leq X_0 < X_{-1} < X_{-2} < \cdots < \infty.\]

If instead we focus from one up to infinity, use \(X_i := e^{L_i}\), so that \(X_i < X_{i+1}\) and \(X_1\) is the first point to the right of one:

\[0 < \cdots < X_{-1} < X_0 \leq 1 < X_1 < X_2 < \cdots < \infty.\]

5. Spacings — for the translation invariant process

With the labelling of (4.1), the spacings of the points of the translation invariant process are \(L_{i+1}−L_i\) for \(i \in \mathbb{Z}\). Because of the exceptional behavior of \(L_1−L_0\), these spacings are not distributed the same as \(\cdots, L_{-2}−L_{-1}, L_{-1}−L_0, 0−L_0, L_1−0, L_2−L_1, \cdots\), which are iid, each distributed like an exponentially distributed random variable \(W\) with mean \(1/\theta\) and \(\mathbb{P}(\theta W > t) = e^{-t}\) for \(t > 0\). Note the familiar “waiting time for a bus” paradox: the interval that covers the origin has length \(L_1−L_0\) which can be expressed as the sum of two independent exponentials, \(L_1−0\) and \(0−L_0\), and which is equal in distribution to the size biased spacing \(W^*\), discussed in a different context in (11.1).

Write \(W_1 := L_1, W_2 := L_2−L_1, W_3 := L_3−L_2\) etc. so that for \(k = 1, 2, \ldots\), \(L_k = W_1 + \cdots + W_k\). Recall that “the (negative) exponential of an exponential is uniform”, at least for \(\theta = 1\). To check this, note that \(t > 0 \Leftrightarrow e^{-t} \in (0, 1)\) and \(\mathbb{P}(\theta W_i < e^{-t}) = \mathbb{P}(\theta W_i > t) = e^{-t}\). Let

\[U_i := e^{-W_i} = (e^{-\theta W_i})^{1/\theta} \Rightarrow (\text{UNIFORM})^{1/\theta}\]

where UNIFORM denotes a random variable uniformly distributed on \([0,1]\).

In lecture, we offered the following closed book QUIZ: the distribution of \(U_i\) is either Beta\((1, \theta)\) or Beta\((\theta, 1)\), but which one?

Use the notation (4.2) which focusses on the scale invariant process near zero, i.e. \(X_i = \exp(-U_i)\). Then for \(k = 1, 2, \ldots\) the expression

\[L_k = W_1 + W_2 + \cdots + W_k\]

for a sum of independent exponential mean \(1/\theta\) random variables is immediately equivalent to

\[X_k = U_1 U_2 \cdots U_k\]

with a product of independent \((\text{UNIFORM})^{1/\theta}\) random variables.
6. SPACINGS — FOR THE SCALE INVARIANT PROCESS

The process of all spacings for the process on \((0, \infty)\) with the notation \((4.2)\) is defined by

\[ Y_k := X_{k-1} - X_k \in (0, \infty), \quad k \in \mathbb{Z}. \]

However, when considering the scale invariant process restricted to \((0,1)\) or \((0,1]\), the natural notation gives a different definition for the first spacing:

\[(6.1) \quad Y_k := X_k - 1 - X_k, \quad k = 2, 3, \ldots, \]

but \(Y_1 := 1 - X_1\).

In terms of the independent \(U_1, U_2, \ldots = d\) (UNIFORM)\(^{1/\theta}\), \((6.1)\) is \(Y_1 = 1 - U_1,\)

\[ Y_2 = U_1 - U_1 U_2 = U_1 (1 - U_2), \]

and in general for \(n \geq 1,\)

\[(6.2) \quad Y_n = U_1 U_2 \cdots U_{n-1} (1 - U_n). \]

7. RESIDUAL ALLOCATION, GEM

Most geneticists have used the notation

\[ Y_1 = U_1, \quad Y_2 = (1 - U_1) U_2, \quad Y_3 = (1 - U_1) (1 - U_2) U_3, \ldots \]

which is the opposite of \((6.2)\). We wish to argue that the notation \((6.2)\), which respects the sum \((5.2)\), is preferable.

The product form above, with either notation, is referred to as a “residual allocation model”, from Halmos 1944 \([27]\). The distribution of \((Y_1, Y_2, \ldots)\) in \((6.1)\) is called the GEM with parameter \(\theta\), after Griffiths, Engen, and McCloskey; —\([39]\) is the unpublished 1965 thesis by McCloskey; these historical notes are from chapter 41 by Ewens and Tavaré in \([33]\).

The Beta function is defined, for \(a, b > 0,\) by

\[ B(a, b) := \int_0^1 (1 - x)^{a-1} x^{b-1} \, dx \]

and the corresponding distribution, \(\text{Beta}(a, b)\), has density \((1 - x)^{a-1} x^{b-1} / B(a, b)\) on \((0,1)\). Since \(B\) is \(\text{Beta}(a, b)\) if and only if \(1 - B\) is \(\text{Beta}(b, a)\), we see how \(\text{Beta}(1, \theta)\) and \(\text{Beta}(\theta, 1)\) can easily be confused.

8. THE POISSON-DIRICHLET PROCESS

We write \((V_1, V_2, \ldots)\) for the Poisson-Dirichlet process with parameter \(\theta\). Survey references for the Poisson-Dirichlet process include \([46, 7]\), and Chapter 41 in \([33]\).

We do not assume the reader knows this process; we will give several characterizations of it, including one that relates it to the scale invariant Poisson process with parameter \(\theta\), following a review of its history. When the Poisson-Dirichlet process was first studied, none of its connections to the scale invariant Poisson process was explicitly noted.

One might view the Poisson-Dirichlet process with \(\theta = 1\) as implicit in the 1930 paper of Dickman \([17]\) describing the limit distribution of the largest prime factor of a randomly chosen integer. However if one insists on an explicit description of the process, it seems that the first appearance of the Poisson-Dirichlet is in Billingsley 1972 \([13]\), which proves that

\[(8.1) \quad \left( \frac{\log P_i}{\log n}, \frac{\log P_i}{\log n}, \ldots \right) \Rightarrow (V_1, V_2, \ldots), \]

where the limit is the Poisson-Dirichlet process with \(\theta = 1\). Here, \(P_i\) is the \(i^{th}\) largest prime factor of an integer \(N\) chosen uniformly from 1 to \(n\), using \(P_i = 1\) when \(i\) is greater than the number \(\Omega(N)\) of prime factors including multiplicities.
(For example, if the random integer is 12, then \( P_1 = 3, P_2 = P_3 = 2, 1 = P_4 = P_5 = \cdots \) ) Billingsley specified the limit process in terms of the joint density of \((1/V_1, 1/V_2, \ldots, 1/V_k)\). The second published proof of Billingsley’s result, \([18]\), is based on a size-biased permutation, and is especially robust, leading to a proof \([4]\) of the analogous Poisson-Dirichlet convergence for general \( \theta \neq 1 \), in which the random integer is conditioned on the large deviation that the number of distinct prime factors is \( \theta \log \log n \).

Ferguson \([25]\) in 1973 described a class of processes that includes what we now know as the Poisson-Dirichlet; a recent survey of this is Pitman\([43]\).

Kingman \([35]\) in 1975 and Ignatov \([32]\) in 1982 gave the first direct connection between the Poisson-Dirichlet process and the scale invariant Poisson process: they can be coupled, with \( V_i = \text{the } i^{th} \text{ largest of the spacings } Y_1, Y_2, \ldots \) as in (6.1), i.e.

\[
(V_1, V_2, \ldots) =_d \text{RANK}(1 - X_1, X_1 - X_2, X_2 - X_3, \ldots).
\]

See \([45]\) for generalizations.

Vershik and Schmidt \([53, 54]\) in 1977 showed that

\[
(\theta = 1) \left( \frac{L_1}{n}, \frac{L_2}{n}, \ldots \right) \Rightarrow (V_1, V_2, \ldots)
\]

where \( L_i \) is the length of the \( i^{th} \) longest cycle of a random permutation of \( n \) objects, choosing with all \( n! \) possibilities equilikely.

Aldous in 1983 \([2]\) gave the analogous Poisson-Dirichlet limit for the component sizes in a random mapping on \( n \) points, with all \( n^n \) maps equilikely; here \( \theta = \frac{1}{2} \).

Hansen \([28]\) gave a general treatment of decomposable combinatorial structures having a Poisson-Dirichlet limit; see also \([8]\).

The following comes from the 1993 book \([36]\), “Poisson processes,” by Kingman. In 1968 Moran considered the Poisson process with intensity \( \theta \exp(-x)/x \, dx \) on \((0, \infty)\). The points of this process can be labelled \( \sigma_i \) for \( i = 1, 2, \ldots \) and \( 0 < \cdots < \sigma_2 < \sigma_1 \). The sum \( \sigma := \sigma_1 + \sigma_2 + \cdots \) has a Gamma distribution with parameter \( \theta \), and is independent of the rescaled vector \((\sigma_1/\sigma, \sigma_2/\sigma, \ldots)\), which has the Poisson-Dirichlet distribution. There is a somewhat similar model in physics, discussed in \([47]\), 227-228, with a Poisson process on \((0, \infty)\) having intensity \( cx^{-c-1} \, dx \) for a constant \( c \in (0, 1) \), instead of \( \theta \exp(-x)/x \, dx \). It is still the case that a.s. the points can be labelled \( \sigma_i \) for \( i = 1, 2, \ldots \) with \( 0 < \cdots < \sigma_2 < \sigma_1 \), and \( \sigma := \sigma_1 + \sigma_2 + \cdots < \infty \), and one may form the rescaled vector \((\sigma_1/\sigma, \sigma_2/\sigma, \ldots)\). However, in this model, \( \mathbb{E}\sigma = \infty \), in contrast to \( \mathbb{E}\sigma = \theta \) for the Moran model. A unified treatment which includes the two models is \([44]\).

9. Joint density for the Poisson-Dirichlet

The joint density \( f_k \) of the first \( k \) coordinates \((V_1, V_2, \ldots, V_k)\) of the Poisson-Dirichlet distribution has support \( \{(x_1, \ldots, x_k) : 1 > x_1 > \cdots > x_k > 0 \text{ and } x_1 + \cdots + x_k < 1\} \). At such points, for the case \( \theta = 1 \),

\[
f_k(x_1, \cdots, x_k) = \rho \left( \frac{1 - x_1 - \cdots - x_k}{x_k} \right) \frac{1}{x_1 x_2 \cdots x_k}
\]

where \( \rho \) is Dickman’s function \([17, 51]\), characterized by \( \rho = 0 \) on \((-\infty, 0)\), \( \rho = 1 \) on \([0,1]\), \( \rho \) continuous on \((0, \infty)\), and \( \rho'(u) = -\rho(u-1)/u \) for \( u > 1 \).
The area under the graph of $\rho$ is $e^\gamma$, where $\gamma$ is Euler’s constant, and $\rho \geq 0$ everywhere, so $g(u) := e^{-\gamma} \rho(u)$ is a probability density, which might naturally be called the Dickman distribution.

10. What is the “Dickman distribution”?

Dickman in 1930 showed that $\log P_{1/ \log n} \Rightarrow V_1$, where $P(V_1 < 1/u) = \rho(u)$, i.e. the Dickman function $\rho$ gives the tail probabilities for $1/V_1$. So if you hear someone refer to a random variable having “the Dickman distribution,” you may be confident that the speaker refers to one of the following three, but which one?

- the random variable $T$ with density $g(u) := e^{-\gamma} \rho(u)$ on $(0, \infty)$
- $V_1$, with density $\rho((1 - x)/x)/x$ on $(0, 1)$
- $1/V_1$, with density $\rho(u - 1)/u$ on $(1, \infty)$

The Dickman function decays superexponentially fast. Hildebrand 1990 [30] gives the following asymptotic expansion of the Dickman function. Write $L := \log u$ and $M := \log \log u$. As $u \to \infty$,

$$\frac{-\log \rho(u)}{u} = L + M - 1 + \frac{M}{L} - \frac{1}{L} + \frac{M^2}{2L^2} + \frac{M}{L} - \frac{2}{L^2} + O\left(\frac{M^2}{L^3}\right),$$

so that $\rho(u) = \exp(-u \log u - u \log \log u + u - \cdots)$.

11. A Key Random Variable: $T$, the Sum of the Points in $(0, 1)$

Let $T$ be the sum of the locations of all points of the scale invariant Poisson process, with intensity $\theta/x \, dx$, restricted to $(0, 1)$. For the case $\theta = 1$, this is the random variable in the previous section, with density $e^{-\gamma} \rho(u)$. For any $\theta > 0$,

$$T := X_1 + X_2 + X_3 + \cdots$$
$$= U_1 + U_1U_2 + U_1U_2U_3 + \cdots$$
$$= U_1 \left(1 + U_2 + U_2U_3 + \cdots\right)$$
$$= \text{d} U \left(1 + T'\right)$$

with $T', U$ independent, $T' = \text{d} T$, and $U = \text{d} (\text{UNIFORM})^{1/\theta}$. Thus it is elementary to get an integral equation involving the density $g$ of $T$, although this equation is not especially tractable.

Another approach is via size biasing; the following is taken from [7]. In general, if $X$ is a non-negative random variable with mean $\mu \in (0, \infty)$, then the size biased random variable $X^*$ is characterized by

$$\mathbb{E} h(X^*) = \mathbb{E}(X h(X))/\mu;$$

if further $X$ has density $g$ then $X^*$ has density $xg(x)/\mu$. If $Z$ is a Poisson distributed random variable then $Z^* = \text{d} 1 + Z$ and for any $x > 0$, $(xZ)^* = \text{d} x + (xZ)$. If $Z$ is a sum of independent nonnegative random variables, with $\mathbb{E} Z < \infty$, then $Z^*$ is likewise a sum of independent random variables, using the same summands except that one summand is size biased, and the choice of which summand to bias is made with probability proportional to its contribution to $\mu$. Thus if $T$ is the sum of locations of points in a Poisson process on $(0, \infty)$ with intensity $f(x) \, dx$, such that $\mu := \mathbb{E} T = \int xf(x) \, dx < \infty$, then $T^*$ is formed by choosing a location $x$
with probability $x f(x) \, dx/\mu$ where the summand is size biased by deterministically adding in $x$. Hence the density of $T^*$ at $t$ is

$$
\frac{tg(t)}{\mu} = \int_0^\infty \frac{x f(x) \, dx}{\mu} \, g(t-x)
$$

For the case of the scale invariant Poisson process restricted to $(0,1)$ we have $\mathbb{E} T = \int_0^1 x(\theta/x) \, dx = \theta$, and

$$
\frac{tg(t)}{\theta} = \int_0^1 \frac{x(\theta/x)}{\theta} \, dx \, g(t-x) = \int_{t-1}^{t} g(u)du, \quad t > 0,
$$

Hence the differential-difference equation satisfied by $g$ is

$$
tg'(t) + (1 - \theta) g(t) + \theta g(t-1) = 0, \quad t > 0.
$$

Another approach is that since $T$ is the sum of locations of the Poisson process with intensity function $f(x) = \mathbb{1}(0 < x < 1)\theta/x$, it has Laplace transform

$$
\mathbb{E} \exp(-sT) = \exp \left( -\int_0^\infty (1 - e^{-sx}) f(x) \, dx \right) = \exp \left( -\theta \int_0^1 \frac{1 - e^{sx}}{x} \, dx \right)
$$

This leads to properties of the density $g$; see Vervaat 1972 [55], Watterson [56], and Hensley [29].

12. THE SCALE INVARIANT PROCESS OF SUMS, AND CLASS L

For any fixed $\theta > 0$, our favorite random variable $T$, which is the sum of locations of points in $(0,1)$. It satisfies $T = d T_1 = d (1/t)T_1$, where for $t > 0$, $T_1$ is defined as the sum of locations of all points in $(0,t]$, for the scale invariant Poisson process with intensity $\theta/x \, dx$. Of course we take $T_0 \equiv 0$. As a process $(T_t)_{t \geq 0}$

- increases by jumps
- at time $t$, can only stay constant or jump up by $t$
- has a jump occurring in $(t, t+dt)$ with probability $(\theta/t) \, dt \, (1 + o(1))$
- has independent increments
- is self-similar with index 1: for $t > 0$, $(T_{ts})_{s \geq 0} = d t^1 (T_s)_{s \geq 0}$

In particular, writing $g_t$ for the density of $T_t$ and $g$ for the density of $T$, from $T_0 = d T = d T_T$ we have, for any $t > 0$,

$$
g_t(x) = \frac{1}{t} \, g\left(\frac{x}{t}\right).
$$

From Feller II [24], the Lévy class L of infinitely divisible distributions consists of those which are limit distributions of $\{S_n^*\}$ where $S_n := X_1 + \cdots + X_n$, with independent (not necessarily identically distributed) $X_1, X_2, \ldots$, and $S_n^* = (S_n - b_n) / a_n$ for constants $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that $a_n \to \infty$, $a_{n+1}/a_n \to 1$. To see that for each $\theta > 0$ the random variable $T$ is in class L, take $X_n := T_n - T_{n-1}$, so that $S_n = T_n$, and take $b_n = 0$, $a_n = n$ so that $S_n^* := T_n/n = d T$ for all $n$. We thank Larry Shepp for an enjoyable conversation on this topic.

We cannot understand a technical/aesthetic issue involving the process $(T_t)$: why does the natural random Stieltjes measure $dT(\cdot)$ always lose out to the competing counting measure? That is, to encode the scale invariant Poisson process as a random measure, the usual choice is the counting measure $\sum_{i \in \mathbb{Z}} \delta X_i(\cdot)$. This has to be taken as a random measure on $(0, \infty)$ with the point at zero not in the underlying space; there is infinite mass in every neighborhood of zero. Another
sensible choice would be $dT(\cdot) = \sum_{i \in \mathbb{Z}} X_i \delta_{X_i}(\cdot)$, which gives a sigma-finite random measure on $[0, \infty)$, with finite mass near zero.

13. Conditioning on $T = s$ in general

Consider any Poisson process on $(0,1]$, having intensity $f(x) \, dx$, such that $T_1$, defined to be the sum of the locations of all points in $(0,t)$, has a density $g_t$. Write $T := T_1$ and $g := g_1$. Assume that $g$ is strictly positive on $(0,\infty)$. Label the points in $(0,1)$ so that $1 > X_1 > X_2 > \cdots > 0$.

The joint density of $X_1, \ldots, X_k$ is

$$f(x_1)f(x_2) \cdots f(x_k) \exp \left( - \int_{x_k}^1 f(u) \, du \right),$$

supported by points in $(0,1)^k$ where $x_1 > x_2 \cdots > x_k$. The first $k$ factors correspond to requiring points at $x_1, \ldots, x_k$, and the last factor corresponds to demanding no other points in $(x_k,1)$. Thus for any $s > 0$ the joint density of $X_1, \ldots, X_k$ conditional on $T = s$ is

$$g_{x_k}(s - x_1 - \cdots - x_k) g(s) f(x_1)f(x_2) \cdots f(x_k) \exp \left( - \int_{x_k}^1 f(u) \, du \right),$$

supported at $1 > x_1 > \cdots > x_k > 0$ with $x_1 + \cdots + x_k < s$. For the special case of the scale invariant Poisson process with $f(x) = \theta/x$, this is

$$\frac{1}{x_k} g \left( \frac{(s - x_1 - \cdots - x_k)/x_k}{s} \right) \frac{\theta}{x_1} \frac{\theta}{x_2} \cdots \frac{\theta}{x_k} e^{-\theta \log(1/x_k)}$$

14. Conditioning on $T = 1$ for the scale invariant process

For the special case $s = 1$ the conditional joint density of $X_1, \ldots, X_k$ given $T = 1$ simplifies to

$$g \left( \frac{(1 - x_1 - \cdots - x_k)/x_k}{1} \right) \frac{\theta^k}{x_1 x_2 \cdots x_k} e^{\theta^k \Gamma(\theta)} x_k^{\theta - 1}$$

$$= g \left( \frac{1 - x_1 - \cdots - x_k}{x_k} \right) e^{\theta^k \Gamma(\theta)} x_k^{\theta - 1} x_1 x_2 \cdots x_k,$$

which is the joint density of the Poisson-Dirichlet; the final equality is just giving an explicit formula for the normalizing constant $g(1)$, which for $\theta \neq 1$ comes from [56]. In the special case $\theta = 1$, (14.1) simplifies to (9.1).

In summary, if one starts from the joint density of the Poisson-Dirichlet, with a differential-difference equation or Laplace transform to characterize the difficult factor, then it is very easy to prove that the Poisson-Dirichlet process is the scale-invariant Poisson process, restricted to $(0,1)$, and conditioned on the event $T = 1$: for any $\theta > 0$

$$V_1, V_2, \ldots =_{d} (X_1, X_2, \ldots | T = 1), \quad \text{i.e. PD}(\theta) =_{d} (PP(\theta) | T = 1).$$

The above result first was written in the 1996 version of [7]. One motivation for considering it is the analogy to the result that

$$C_1(n), \ldots, C_n(n) =_{d} (Z_1, \ldots, Z_n | T_n = n),$$

relating the joint distribution of cycle counts for a uniformly distributed random permutation, to the joint distribution of $Z_1, Z_2, \ldots$, a sequence of independent
Poisson random variables with $\mathbb{E} Z_i = 1/i$, conditioned on $T_n = n$, where $T_n := Z_1 + 2Z_2 + \cdots + nZ_n$.

Just as (14.2) is a continuum analog of (14.3), the Moran process representation implies

$$
(14.4) \quad (V_1, V_2, \ldots) = d (\ (\sigma_1, \sigma_2, \ldots) \ | \ \sigma = 1 ),
$$

which is a continuum analog of the relation exploited by Shepp and Lloyd [48] in 1966 to analyze random permutations,

$$
(14.5) \quad (C_1(n), \ldots, C_n(n), 0, 0, \ldots) = d (\ (Z_1, Z_2, \ldots) \ | \ T_\infty = n ).
$$

Here $T_\infty = Z_1 + 2Z_2 + \cdots$, and the $Z_i$ are independent Poisson with $\mathbb{E} Z_i = z^i / i$; this is valid for any $z \in (0, 1)$ but especially useful with $z = 1 - 1/n$. Furthermore, (14.2) is to (14.4) as (14.3) is to (14.5). Loosely speaking, (14.4) and (14.5) both avoid the divergence of $\int_\infty x^{-1}$ by throwing in an exponentially decaying factor which does not affect the conditional distribution, while (14.2) and (14.3) avoid the divergence of $\int_\infty x^{-1}$ by not going out to infinity.

Another elementary proof of (14.2) can be given by comparison to the Moran representation; this is done in [9]. The general conditioning formula (13.1) is also used in [40].

15. More discrete analogs

The role of the scale invariant Poisson process in number theory may be found explicitly in DeKoninck and Galambos [37], with a theorem showing that the process of logarithms of the intermediate prime divisors of a random integer chosen uniformly from 1 to $n$ converges in distribution to the scale invariant Poisson process, with $\theta = 1$. An attempt to metrize this result [3] leads to the following, which shows how the scale invariant Poisson process with $\theta = 1$ is very close to the discrete limit process that arises in number theory, with the number $Z_p$ of occurrences of each prime $p$ being independent, with $\mathbb{P}(Z_p = k) = (1 - 1/p)p^{-k}$. Theorem: it is possible to couple random values $Q_i$ for $i \in \mathbb{Z}$, and the scale invariant Poisson process, so that

$$
\mathbb{E} \sum_{i \in \mathbb{Z}} |X_i - \log Q_i| < \infty
$$

with each $Q_i$ either one or prime, and for each $p$, $Z_p = \sum_{i \in \mathbb{Z}} \mathbb{I}(Q_i = p)$.

It is not obvious how the usual model of number theory, picking an integer uniformly from 1 to $n$, is related to the independent, geometrically distributed $Z_p$ by something like conditioning; this is explained in [6]. The overall theme there is that discrete “logarithmic” combinatorial structures, such as random permutations, random mappings, and random polynomials over finite fields, together with prime factorizations of uniformly chosen random integers, have a dependent component size counting process, that is derived from an independent process by conditioning or something analogous. The discrete dependent processes, rescaled, are close to the Poisson-Dirichlet processes. The discrete independent processes, rescaled, are close to the scale invariant Poisson processes. For both the discrete and continuous situations, the dependent processes are obtained from the independent processes by conditioning, or something like conditioning — this is the point of studying (14.2).
16. Dependent versus independent: total variation distance

Recall, the dependent process is the Poisson-Dirichlet \((V_1, V_2, \ldots)\) with \(V_1 + V_2 + \cdots = 1\), and the related independent process is the scale invariant Poisson, restricted to \((0, 1)\), with \(X_1 + X_2 + \cdots = T\), a random variable having a strictly positive density on \((0, \infty)\). For \(\beta \in [0, 1]\) let

\[ H_\theta(\beta) := d_{TV}(\{V_1, V_2, \ldots\} \cap [0, \beta], \{X_1, X_2, \ldots\} \cap [0, \beta]) . \]

Note that \(H_\theta(0) = d_{TV}(\emptyset, \emptyset) = 0\), and

\[ H_\theta(1) = d_{TV}(\{V_1, V_2, \ldots\}, \{X_1, X_2, \ldots\}) = 1 \]

because \(1 = \Pr(V_1 + V_2 + \cdots = 1)\) and \(0 = \Pr(X_1 + X_2 + \cdots = 1)\). For the case \(\theta = 1\)

\[ H_1(1/1.9) = .4968 \ldots, H_1(1/2) = .4454 \ldots, H_1(1/3) = .1114 \ldots, \]

\[ H_1(1/3.5) = .0471 \ldots, H_1(1/4) = .0184 \ldots. \]

Informally, \(H_1(.25) = .0184 \ldots\) means that if you are shown one sample of a process which is either the Poisson-Dirichlet \((\theta = 1)\) or else the scale invariant Poisson \((\theta = 1)\), but you only get to observe the set of components of size at most \(.25\), then your edge over the house is at most 1.84 percent: if the examiner picks his distribution using a fair coin, then you have at most a 50.92 percent chance of answering correctly.

There is a phase transition in the qualitative behavior of \(H_\theta(0+)\), with linear decay for \(\theta \neq 1\), and superexponential decay for \(\theta = 1\). In detail, from [11], p. 1368, combined with [9, 50, 10], as \(\beta \to 0+\),

\[ \frac{1}{\beta} H_\theta(\beta) \to |1 - \theta| e^{-\gamma \theta} \frac{\theta^\theta}{\Gamma(\theta)} \frac{1}{1 + \theta} > 0 \quad \text{for} \quad \theta \neq 1, \]

\[ -\beta \log H_1(\theta) \sim \log(1/\beta) \to \infty. \]

17. Invariance principle for total variation distance

The function \(H_\theta(\beta)\) gives the total variation distance between the dependent and independent processes, i.e. the Poisson-Dirichlet process with parameter \(\theta\), and the scale invariant Poisson process with parameter \(\theta\), when observing components of size at most \(\beta\). These functions \(H_\beta(\cdot)\), which can be most easily defined in terms of the scale invariant Poisson, showed up first in combinatorics. Consider random permutations of \(n\) objects, observing cycles of length at most \(\beta n\): write \(C_i(n)\) for the number of cycles of length \(i\), and \(Z_i\) for a random variable which is Poisson, mean \(1/i\), with \(Z_1, Z_2, \ldots\) independent. Then [12, 50] for any fixed \(\beta \in [0,1]\), as \(n \to \infty\),

\[ d_{TV}(C_1(n), C_2(n), \ldots, C_{\lfloor \beta n \rfloor}(n)), (Z_1, Z_2, \ldots, Z_{\lfloor \beta n \rfloor}) ) \to H_1(\beta). \]

Similarly [50] with \(\theta = \frac{1}{\beta},\) for the component counts for a random mapping, comparing to the independent limit process, and observing components of size at most \(\beta n\),

\[ d_{TV} \to H_{\frac{1}{\beta}}(\beta), \]

and also similarly with \(\theta = 1\) for the factorization of a random polynomial of degree \(n\) over a finite field, observing factors of degree at most \(\beta n\).

For the factorization of an integer chosen uniformly from 1 to \(n\) the analogous result holds [10, 52], again with \(\theta = 1\). Here, we observe prime factors, with
multiplicity, jointly for all primes \( p \leq n^\beta \). The independent comparison process has \( Z_p \) which are geometrically distributed with \( \mathbb{P}(Z_p \geq k) = p^{-k} \).

18. Buchstab and the Explicit Formula for \( d_{TV} \)

The possibility of numerically evaluating the limiting total variation distance \( H_\theta(\beta) \) arises thanks to relation (14.2), which makes it possible to equate \( H_\theta(\beta) \), defined as the total variation distance between processes, with the total variation distance between two random variables. Recall that the process \((T_s)_{s \geq 0}\) has independent increments, so that the \( T \equiv T_1 \) in conditioning on \( T = 1 \) can be expressed as the sum \( T = T_\beta + (T - T_\beta) \) with two independent summands. This leads to

\[
H_\theta(\beta) := d_{TV} \quad \text{for processes}
\]

\[
= d_{TV} \quad \text{for random variables} \quad \equiv d_{TV}(T_\beta, (T_\beta \mid T = 1))
\]

Manipulation of the densities of the independent summands leads to the explicit expression for \( H_\theta(\beta) \), which for \( \theta = 1 \) can be expressed as:

\[
2H_1(\beta) = e^\gamma \mathbb{E}[\omega(u - T) - e^{-\gamma}] + \rho(u)
\]

\[
= \int_{t \geq 0} |\omega(u - t) - e^{-\gamma}|\rho(t) \, dt + \rho(u).
\]

Here, \( \rho \) is Dickman’s function, and and \( \omega \) is Buchstab’s function, with \( \omega \) continuous on \((1, \infty)\), with \((w\omega(u))' = \omega(u - 1)\) for \( u > 2 \) and \( w\omega(u) = 1 \) for \( 1 \leq u \leq 2 \); see e.g. [51]. Both functions were described some sixty years ago in number theory, and each can also be described in terms of the scale invariant Poisson process for \( \theta = 1 \): \( \rho \) is \( e^\gamma \) times the density function of \( T \), and for \( u > 1 \) with \( \beta := 1/u \), \( \omega(u) \) equals the density of \( T - T_\beta \), evaluated at \( 1^- \), i.e. \( \omega(u) = \lim_{\Delta t \to 0^+} \mathbb{P}(T - T_{1/u} \in (1 - \Delta t, 1))/\Delta t \). To state the connection with number theory, \( \Psi(x, y) \) counts positive integers less than or equal to \( x \) whose largest prime factor is less than or equal to \( y \). \( \Phi(x, y) \) counts positive integers less than or equal to \( x \) with no prime factor less than or equal to \( y \), and for \( u > 1 \) as \( n \to \infty \),

\[
(18.1) \quad \frac{1}{n} \, \Psi \left( n, n^{1/u} \right) \to \rho(u), \quad \frac{\log n}{n} \, \frac{1}{n} \, \Phi \left( n, n^{1/u} \right) \to \omega(u).
\]

For every \( \theta, H_\theta \) is continuous and strictly monotone, mapping \([0,1]\) into itself, with \( H_\theta(0) = 0 \) and \( H_\theta(1) = 1 \), but only for \( \theta = 1 \) is it the case that \( H_\theta(\beta) = \alpha(\beta) \) as \( \beta \downarrow 0 \).

From \( H_\theta(\beta) \to 0 \) as \( \beta \downarrow 0 \) it follows that the Poisson-Dirichlet process, “blown up”, converges in distribution to the scale invariant Poisson. That is, with \( \mathcal{V} := \{V_1, V_2, \ldots\} \subset (0, 1) \), and \( \mathcal{X} := \{X_i : i \in \mathbb{Z}\} \subset (0, \infty) \), as \( v \to \infty \), \( v\mathcal{V} \Rightarrow \mathcal{X} \). For a proof: for any fixed \( x > 0 \), for \( v \geq x \) we have

\[
(18.2) \quad d_{TV}(v\mathcal{V} \cap (0, x), \mathcal{X} \cap (0, x)) = d_{TV}(v\mathcal{V} \cap (0, x/v), (v^{-1}\mathcal{X}) \cap (0, x/v))
\]

\[
= H_\theta(x/v).
\]

where we use the scale invariance of \( \mathcal{X} \) to see that last equality. Thus for fixed \( x \), \( d_{TV}(v\mathcal{V} \cap (0, x), \mathcal{X} \cap (0, x)) \to 0 \) as \( v \to \infty \), which is quite a bit stronger than distributional convergence.
19. Insertion—deletion distance

That $H_0(1) = 1$ says that by looking at a single sample one can a.s. tell the random set $V := \{V_1, V_2, \ldots \}$ from $Y := \{X_1, X_2, \ldots \}$. Could the two random sets nevertheless be close? Consider the Wasserstein metric $d_W$ using counts of insertions and deletions needed to convert one set to the other:

$$ d_W := \min \mathbb{E}|V \triangle Y| $$

$$ = \min( \mathbb{E}|V \setminus Y| + \mathbb{E}|Y \setminus V| ) . $$

The minimum is taken over all couplings of the Poisson-Dirichlet process and the scale invariant process, both with parameter $\theta$. For $\theta = 1$, this can be fully understood: $d_W = 2$, and there is a coupling such that $V \setminus \{V_J\} = Y \setminus \{X_1, X_2, \ldots \}$ where the number $D$ of points of the scale invariant Poisson process restricted to $(0,1)$ that need to be deleted is random, with $\mathbb{E}D = 1$.

That $d_W \geq 2$ for $\theta = 1$ starts with a proof that a.s. at least one deletion from $V$ is needed, because with probability one no subsum of the points of $X$ has the value $V_1 + V_2 + \cdots = 1$. This argument requires only that $\theta \leq 1/\log 2$; see problem 22.1.

For the other half of the lower bound, $\theta = 1$ is needed, so that $X$ and $V$ have the same intensity, and hence the one required deletion from $V$ must be matched by deletions from $X$, averaging one in number.

The coupling showing that $d_W \leq 2$ for $\theta = 1$ can be given explicitly. In this coupling, the deleted component $V_J$ is simply the first pick from $V$ in a size biased permutation, i.e. given the value of $(V_1, V_2, \ldots)$, the conditional probability that $V_J = V_k$ is $V_k$. The entire coupling is the continuum analog of the Feller coupling for random permutations [5], and depends on the “scale invariant spacing lemma”.

20. Scale invariant spacing lemma

Label the scale invariant Poisson process as in (4.3), with $X_i < X_{i+1}$ for $i \in \mathbb{Z}$, and let $Y_i := X_{i+1} - X_i$.

LEMMA. For any $\theta > 0$, \{ $X_i : i \in \mathbb{Z}$ \} =_d \{ $Y_i : i \in \mathbb{Z}$ \}

and a.s. all the $Y_i$ are distinct.

This is the continuous analog of a property of the Feller coupling: if $\xi_1, \xi_2, \ldots$ are independent Bernoulli with $\mathbb{P}(\xi_i = 1) = \theta/(\theta + i - 1)$, and $Z_k$ is defined as the number of $k$-spacings between consecutive ones in $\xi_1, \xi_2, \ldots$, then the $Z_1, Z_2, \ldots$ are independent, Poisson, with $\mathbb{E}Z_k = \theta/k$. In both the discrete and continuous cases, one starts with an independent process, and the surprise is that the spacings also form an independent process. In the discrete case, while each of the processes $\xi_1, \xi_2, \ldots$ and $Z_1, Z_2, \ldots$ has independent coordinates, the two processes are not the same.

The proof [3] of the scale invariant spacing lemma is based on the Poisson process on $(0, \infty)^2$ with intensity $\theta e^{-wy} dw \, dy$, whose projections on each coordinate axis give realizations of the scale invariant Poisson process. Labelling the points in decreasing order of their $w$ coordinates gives the $y$ coordinates in a permutation such that $X_n := \sum_{i \leq n} Y_i$ forms a simple point process $\mathcal{X} = \{X_n : n \in \mathbb{Z} \}$ on $(0, \infty)$ whose spacings, by construction, are the points $Y_i$ of the scale invariant Poisson process. It is then a calculation that the distribution of $\mathcal{X}$ is also scale invariant.
Problem 21. What other intensity functions \( f(x) \) on \((0, \infty)\) beside those of the form \( f(x) = \theta/x \) lead to Poisson processes whose spacings are again Poisson processes?

Tom Kurtz, upon being asked the above question, which is still open, immediately countered with a very different question:

Problem 21.2. What point processes on \((0, \infty)\) are equal in distribution to their own spacings? Formally, what point processes with points \( X_i \) for \( i \in \mathbb{Z} \) with \( X_i < X_{i+1} \) for all \( i \) have

\[
Y_i := X_{i+1} - X_i \quad \text{are all distinct, a.s., and} \quad \{X_i : i \in \mathbb{Z}\} =_d \{Y_i : i \in \mathbb{Z}\}.
\]

The above question is reminiscent of an intriguing, still open 1978 question from Liggett \cite{38}, about invariant random measures for independent particle systems, a conjecture that \( MP = M \) in distribution implies the existence of a nontrivial solution of \( MP = M \) almost surely.

The answer to problem 21.2 involves mixtures of distributional solutions, such as the scale invariant Poisson processes and perhaps some others, and deterministic solutions. During a problem session at the DIMACS workshop, I asked what are all simple deterministic solutions, i.e. doubly infinite sequences, such as \( x_i := 2^i \), with \( 0 < \cdots < x_{-1} < x_0 < x_1 < x_2 < \cdots < \infty \) such that

\[
y_i := x_{i+1} - x_i \quad \text{are all distinct, and} \quad \{y_i : i \in \mathbb{Z}\} = \{x_i : i \in \mathbb{Z}\}.
\]

Gábor Tardos and László Lovász from the audience quickly contributed a partial solution: for any fixed \( k \geq 0 \) take

\[
x_i := b^i, \quad \text{where} \; b > 1 \; \text{solves} \; b^{k+1} - b^k = 1,
\]

so that \( b \) is 2 when \( k = 0 \), and \( b \) is the golden ratio when \( k = 1 \). They observed that \( y_i := x_{i+1} - x_i < x_{i+1} \) implies \( y_i = x_{i+1} - x_i \leq x_i \), i.e. \( x_{i+1} \leq 2x_i \), so that the example with \( x_i = 2^i \) is extreme. They also showed how “entrance solutions” may be extended: working from left to right, the next spacing is chosen from the previously defined \( x_k \), excluding those already used as spacings, i.e. \( y_i \in \{\ldots, x_{i-2}, x_{i-1}, x_i\} \setminus \{\ldots, y_{i-2}, y_{i-1}\} \). For example, \( b = (1 + \sqrt{5})/2 \) and \( x_i = b^i \) for \( i \leq 0 \) is an entrance solution with \( y_i = x_{i-1} \) for \( i < 0 \), which may be extended with an infinite series of two-way choices: first \( x_1 = x_0 + y_0 \) where \( y_0 \in \{x_0, x_{-1}\} \), then \( x_2 = x_1 + y_1 \) where \( y_1 \in \{x_1, x_0, x_{-1}\} \setminus \{y_0\} \), and so on. To describe their idea in more detail, focus on the implied permutation, as follows.

Any solution of (21.2) determines a permutation \( \pi \) of \( \mathbb{Z} \), such that \( \forall i \in \mathbb{Z}, y_i = x_{\pi(i)} \), i.e. \( \forall i \in \mathbb{Z} \),

\[
x_{i+1} = x_i + x_{\pi(i)} \in (0, \infty).
\]

Note that always \( \pi(i) \leq i \) since \( y_i \leq x_i \). The geometric solutions in (21.3) correspond to permutations \( \pi \) with \( \pi(i) = i - k \). In constructing deterministic solutions,
for any given $\pi$ with

\[(21.5) \quad \pi \text{ permutes } \mathbb{Z}, \text{ and } \pi(i) \leq i \text{ } \forall i \in \mathbb{Z},\]

if there is an “entrance solution” from zero, then the solution can be uniquely extended out to infinity. That is, if for some $k$ there are $0 < \cdots < x_{k-2} < x_{k-1} < x_k$ satisfying (21.4) for all $i < k$, then recursively defining $x_{k+1}, x_{k+2}, \ldots$ by (21.4) for $i = k, k+1, \ldots$ involves no problem — the solution cannot explode to plus infinity in finite time. Running this argument in the opposite direction is not so easy: extending backwards from $x_k, x_{k+1}, \ldots$ involves taking differences, which can produce values $\leq 0$. We can show that entrance solutions exist by using a compactness argument on the sequence of ratios between successive points:

**Theorem 21.3.** If $\pi$ satisfies (21.5), then there exists at least one solution of (21.4).

**Proof.** Given $x = (x_i)_{i \in \mathbb{Z}} \in (0, \infty)^\mathbb{Z}$, let $V_i := (x_{i-1}/x_i, x_{i-2}/x_{i-1}, x_{i-3}/x_{i-2}, \ldots) \in (0, \infty)^\mathbb{N}$. Note that if the sequence $x$ satisfies (21.4), then for all $i \in \mathbb{Z}$, $V_i \in [\frac{1}{2}, 1]^\mathbb{N}$.

If $V_i = (r_1, r_2, \ldots)$ then, factoring out $c := x_i$, we have $(x_1, x_{i-1}, x_{i-2}, \ldots) = c(1, r_1, r_1 r_2, \ldots)$. If $i+1 = x_i + x_{i-d}$ then $x_{i+1} = c(1 + \prod_{1 \leq j \leq d} r_j) = c/r_0$; this defines $r_0$, such that $V_{i+1} = (r_0, r_1, r_2, \ldots)$. [In case $d = 0$ the product is empty and has value 1, and $x_{i+1} = 2x_i, r_0 = \frac{1}{2}$.]

This motivates us to define, for each $d \geq 0$, a function $f^{(d)}$, $(0, \infty)^\mathbb{N} \rightarrow (0, \infty)^\mathbb{N}$ by $(r_1, r_2, r_3, \ldots) \mapsto (1 + \prod_{1 \leq j \leq d} r_j)^{-1}, r_1, r_2, \ldots)$. Note that these functions are continuous and map $K$ into itself. Given $\pi$ satisfying (21.5), define $d(i) := i - \pi(i)$, and for $n = 1, 2, \ldots$ define a map $T_{-n, 0} := f^{(d(1))} \circ f^{(d(2))} \circ \cdots \circ f^{(d(n))} \circ f^{(d)}$. Note that if $x$ satisfies (21.4) then $T_{-n, 0}(V_{-n}) = V_0$. Conversely, given any $(x_{-n-2}, x_{-n-1}, x_{-n}) \in (0, \infty)$, (not necessarily satisfying (21.4),) the map $T_{-n, 0}$ gives a recipe for extending the sequence with values $x_{-(n-1)}, \ldots, x_{-1}, x_0$ such that $x_{i+1} = x_i + x_{\pi(i)} \in (0, \infty)$ for $i = -n, \ldots, -2, -1$. For $n = 1, 2, \ldots$ let $S_n$ be the image of the compact set $K$ under $T_{-n, 0}, \ldots, S_2 \subset S_1 \subset K$. By the finite intersection property, $\cap_{n \geq 1} S_n \neq \emptyset$. Any point $(r_1, r_2, \ldots) \in \cap_{n \geq 1} S_n$ yields a sequence $x_0 = 1, x_{-1} = r_1, x_{-2} = r_1 r_2, \ldots$ satisfying $x_{i+1} = x_i + x_{\pi(i)}$ for $i = -1, -2, \ldots$, and this entrance solution can be extended to a full solution of (21.4).

Clearly if a sequence $(x_i)$ satisfies (21.4) then so does any scalar multiple $(c x_i)$ for any $c > 0$. Are solutions uniquely determined by $\pi$, up to such scalar multiples? At least from the point of view taken in the proof of the following theorem, the situation resembles time inhomogeneous renewal chains; and we can only handle the bounded case. Do transient chains somehow correspond to counterexamples to uniqueness?

**Theorem 21.4.** Assume $\pi$ satisfies (21.5) and $k := \sup\{i - \pi(i) : i \leq 0\} < \infty$. Then solutions of (21.4) are unique up to a scalar multiple.

**Proof.** We consider the effect of using $x_{i+1} = x_i + x_{\pi(i)}$ for $i = 0, -1, \ldots, -n$ to express $x_1$ and $x_0$ as linear combinations, with nonnegative integer weights, of $x_{-n}, x_{-n-1}, \ldots, x_{-n-k}$, viewed as indeterminates. The idea is to show that for large $n$, the weights in the combination for $x_1$ are close to being a multiple of the weights for $x_0$, so that the ratio $x_1/x_0$ is close to being determined by $\pi$.

The Hilbert projective metric $\rho$ on $(0, \infty)^{k+1}$ (modulo scalar multiples) is defined by $\rho(u, v) := \max_i \log(u_i/v_i) - \min_i \log(u_i/v_i)$. If $u = (x_{-n}, x_{-n-1}, \ldots, x_{-n-k})$
comes from a solution of (21.4) and \( v = (1, 1, \ldots, 1) \), then the property \( x_i < x_{i+1} \leq 2x_i \) implies that \( \rho(u, v) \leq k \log 2 \).

Consider \( k + 1 \) by \( k + 1 \) matrices with nonnegative integer coefficients, indexed by \( 0 \leq i, j \leq k \), as follows. Write \( E^{i,j} \) for the matrix having all zeroes, except for a single one in row \( i \), column \( j \). Write \( B = \sum_{0 \leq i \leq k} E^{i,i-1} \) for the matrix with ones below the diagonal, and let \( C = B + E^{0,0} \). For \( 0 \leq d \leq k \) let \( A^{(d)} = C + E^{(0,d)} \). For \( n \in \mathbb{Z} \) let \( d(n) = n - \pi(n) \), \( F^{(n)} = A^{(d(n))} \), and \( M^{(n)} = F^{(-1)}F^{(-2)} \cdots F^{(-n)} \). Row \( i \) of \( M^{(n)} \) gives the coefficients of \( x_i \), as a linear combination of \( x_{-n}, x_{-n-1}, \ldots, x_{-n-k} \), and row 0 plus row \( d(0) \) gives the coefficients of \( x_1 \).

Note that \( C^k \) is all ones in column zero, and zeroes elsewhere. Any product \( M \) of \( k \) or more factors of the form \( A^{(d(i))} \) has all entries in column zero strictly positive, and any product with \( 2k \) or more factors has the property that every column is either all zeroes, or else has all entries strictly positive. The maximum entry in a product \( M \) with exactly \( 2k \) factors is at most \( s = 2^{2k} \), and the least entry in a non-zero column is at least one; hence if \( r = \rho(u, v) \), then

\[
\rho(Mu, Mv) \leq \log \left( \frac{1 + se^r}{1 + s} \right).
\]

It follows that using \( r = k \log 2 \) and given \( \epsilon > 0 \), we can pick \( n \) so that for all \( u, v \) with \( \rho(u, v) \leq r \) we have \( \rho(M^{(n)}u, M^{(n)}v) < \epsilon \). Using \( v = (1, 1, \ldots, 1) \) and the remarks at the ends of the previous two paragraphs, the ratio \( a := (M^{(n)}v)_d(0)/(M^{(n)}v)_0 \) satisfies \( |\log a - \log((x_1 - x_0)/x_0)| < \epsilon \), for any solution of (21.4). This show that any two solutions of (21.4) have exactly the same ratio \( x_0/x_1 \). The same argument shows that the permutation \( \pi \) determines the ratio \( x_i/x_{i+1} \) for each \( i \in \mathbb{Z} \).

Problem 21.2 also includes solutions placing probability \( 1/m \) on each point in an orbit of period \( m \geq 1 \) of the spacing transformation. The simple deterministic solutions discussed above are the case \( m = 1 \), fixed points. For the general case, there are \( m \) distinct deterministic sequences; the random process picks each of these with probability \( 1/m \) each, and the spacings of the \( k \)th sequence are all distinct and as a set give the points of the \( (k + 1)^{st} \) sequence mod \( m \).

For example, for \( m > 1 \) there are solutions of the form \( \mathbb{P}(X_i = x_i) = 1/m \), where the deterministic sequence is given by \( x_i := b^i \), such that \( \Delta x_i := x_{i+1} - x_i = (b - 1)b^i, \ldots, \Delta^m x_i := (b - 1)^m x_i \) give \( m \) distinct sets of points, and the last of these sets is the same as \( \{x^i : i \in \mathbb{Z} \} \). This is possible iff \( (b - 1)^m = b^k \) for some \( k \in \mathbb{Z} \), \( b \in (1, \infty) \). For example, with \( m = 2 \) and \( k = 1, b = (3 + \sqrt{5})/2 \approx 2.618 \) and with \( m = 2 \) and \( k = -1, b = 1.75487 \). These examples with geometric sequences are misleading in that in general, the \( y_i := x_{i+1} - x_i \) are not in increasing order, and the iterated transformation on sequences does not correspond to \( \Delta^m \), but rather to \( (\text{RANK} \circ \Delta)^m \).

To compare simple deterministic solutions with the scale invariant Poisson processes, write \( \sigma \) for the inverse of \( \pi \). Note that for all \( i, i \leq \sigma(i) \) and \( y_{\sigma(i)} = x_i < x_{i+1} = y_{\sigma(i+1)} \). For the random solutions (21.1) there is a random permutation \( \sigma \) of \( \mathbb{Z} \) such that for all \( i \)

\[
Y_{\sigma(i)} < Y_{\sigma(i+1)},
\]

with \( \sigma \) determined only up to translation. If we write \( \sigma(i) = i + C(i) \), then it is easy to show, for the scale invariant Poisson process, that a.s. \( \limsup C_i = \infty \).
and \( \lim \inf C_t = -\infty \), a qualitative property possessed by no mixture of simple deterministic solutions. Large deviations for the permutation \( \sigma \) are studied in [57].

To summarize the above discussion: the solution to problem 21.2 includes a) the scale invariant Poisson processes, b) simple deterministic solutions, i.e. fixed points of the spacing transformation, c) deterministic orbits of length \( m > 1 \), and d) other extreme points of the set of distributional solutions. For b), it remains to resolve the question of uniqueness relative to permutations satisfying (21.5), but with unbounded displacements; and for c) and d), everything is open.

Peter Baxendale recently asked, for simple deterministic solutions, what are the possible values of the ratio \( r := x_0/x_1 \) of two adjacent points? Is \( \frac{2}{3} \) achievable?

More generally, writing \( r_i := x_{i-1}/x_i \), what are the possible configurations of \( k+1 \) consecutive points \( (x_0, x_1, \ldots, x_k) \), i.e. what points in \((\frac{1}{2}, 1) \) are realizable as the value of \( (r_1, r_2, \ldots, r_k) \), for \( k = 1, 2, \ldots \)? Two other ways to generalize the question about \( r \) are to ask which finite sets \( A \subset (0, \infty) \) can satisfy \( A \subset \{ x_i : i \in \mathbb{Z} \} \) for some solution of (21.2), and similarly which finite sets \( B \subset [\frac{1}{2}, 1) \) can satisfy \( B \subset \{ r_i : i \in \mathbb{Z} \} \)?

### 22. Problem session: A phase transition at \( \theta = 1/\log 2 \)

For any \( \theta > 0 \), starting from a realization \( \{ X_i : i \in \mathbb{Z} \} \) of the scale invariant Poisson process with intensity \( \theta/x \) \( dx \), let \( A \equiv A(\theta) \) be the random closed set which is the closure of the countable set whose points are \( \sum_{i \in I} X_i \) for finite \( I \subset \mathbb{Z} \). From the scale invariance of the underlying \( \{ X_i \} \) it follows easily that \( A \) is also scale invariant: for any \( c > 0 \) and for each \( \theta > 0 \)

\[
cA(\theta) =_d A(\theta).
\]

The process \( (A(\theta))_{\theta > 0} \) has stationary, independent increments in the following sense. The Minkowski sum of two sets is \( A \oplus B := \{ a + b : a \in A, b \in B \} \). With \( \mathcal{X}(\theta) \) to denote the set of points in the scale invariant Poisson process with intensity \( \theta/x \) \( dx \), the process \( \mathcal{X}(\theta)_{\theta > 0} \) and hence also \( (A(\theta))_{\theta > 0} \) can be constructed with increasing sample paths: \( \theta_1 < \theta_2 \) implies \( \mathcal{X}(\theta_1) \subset \mathcal{X}(\theta_2) \), and hence \( A(\theta_1) \subset A(\theta_2) \).

The process \( \mathcal{X} \) has stationary, independent increments in the strong sense that \( \mathcal{X}(\theta_2) \setminus \mathcal{X}(\theta_1) \) is independent of \( \mathcal{X}(\theta_1) \) and equal in distribution to \( \mathcal{X}(\theta_2 - \theta_1) \), but the process \( A(\theta)_{\theta > 0} \) has stationary independent increments in a weaker sense. If \( \theta_1 < \theta_2 \) and if \( A'(\theta_2 - \theta_1) \) is independent of \( A(\theta_1) \) and equal in distribution to \( A(\theta_2 - \theta_1) \), then \( A(\theta_1) \oplus A'(\theta_2 - \theta_1) =_d A(\theta_2) \).

Some of the structure of the set of divisors of a random integer, as in [26], is captured by the scale invariant closed set \( A \subset [0, \infty) \). First, for any \( \theta > 0 \) define a random closed set \( B(\theta) \subset [0, 1] \) by \( B := \{ \sum_{i \in I} V_i : I \subset \mathbb{N} \} \). From the Poisson-Dirichlet convergence in (8.1) and its extension to the large deviation case for \( \theta \neq 1 \), it follows easily that the random finite set

\[
(22.1) \quad \mathcal{D}_n := \{ \log d/\log n \} \subset [0, 1],
\]

where \( d \) runs over the divisors of our random integer, has \( \mathcal{D}_n \Rightarrow B(\theta) \). (See Theorem 22.2 for an extension.) In fact, with the Hausdorff metric for closed subsets of \([0, 1] \), and the \( l_1 \) metric on \( \mathbb{R}^\infty \), the map which produces \( \mathcal{D}_n \) from \( (\log P_1, \log P_2, \ldots) \) and \( B(\theta) \) from \( (V_1, V_2, \ldots) \) is a contraction. Second, just like (18.2) it is easy to see that as \( v \to \infty \), \( vB(\theta) \Rightarrow A(\theta) \). In fact, for any \( 0 < x \leq v \),

\[
(22.2) \quad d_{TV}(vB \cap (0, x), A \cap (0, x)) = d_{TV}(B \cap (0, x/v), (v^{-1}A) \cap (0, x/v))
\]
Thus for fixed $x$, $d_{TV}(vB \cap (0, x), A \cap (0, x)) \to 0$ as $v \to \infty$, which is even stronger than the distributional convergence $v B \Rightarrow A$.

Let $f(\theta) := \mathbb{P}(1 \in A(\theta))$. Using scale invariance, $\forall x > 0, \mathbb{P}(x \in A(\theta)) = f(\theta)$. Hence with $m(A)$ to denote Lebesgue measure, for $0 \leq a < b < \infty$, $\mathbb{E}m(A \cap (a, b)) = (b - a) f(\theta)$. It is easy to show that for $\theta \leq 1/\log 2$, $f(\theta) = 0$. Note that for any $\theta$, $f(\theta) \leq \mathbb{P}(T < 1) < 1$. In lecture, we asked a semi-open question:

**Problem 22.1.** Prove the conjecture that if $\theta > 1/\log 2$, then $f(\theta) > 0$.

At the time of lecture, the conjecture was semi-open in the following sense: Tenenbaum (private communication) had proved a related property about the set of divisors of a random integer from 1 to $n$ with $\theta \log \log n$ distinct prime divisors. From the early version of this number theory result, it was not yet possible to deduce the continuum result as a corollary, so the conjecture was indeed open. However, it was highly plausible to guess that the underlying principles from Tenenbaum’s proof would also work directly on the continuum problem. Subsequent to the workshop, the conjecture has indeed been proved this way. Further, Tenenbaum gave a sharper version of the number theoretic result, from which the continuum result follows as a corollary. This amounts to using a more complicated process, prime divisors of a random integer, to approximate a simpler process, the scale invariant Poisson!

The event that $A(\theta)$ has positive length is a tail event with respect to the independent exponentials $W_i$ in (5.1), and hence by the Kolmogorov zero-one law and the above, for every $x > 0$ and any $\theta > 1/\log 2$, $1 = \mathbb{P}(m(A \cap (0, x)) > 0)$.

The random sets above are closely related to the theory of Bernoulli convolutions; we learned of the connection thanks to Jim Pitman. For Bernoulli convolutions, the usual setup is to start with a deterministic sequence $r_1, r_2, \ldots > 0$ with $\sum r_n < \infty$ and to consider the random variable $Y = \sum_{i \geq 1} S_i r_i$, where the $S_i$ have values 1, -1, independently with probability 1/2 each. The closed support of the distribution of $Y$ is the set $K$ of all points of the form $\sum s_ir_i$, where the $s_i$ are arbitrarily chosen from $\{1, -1\}$. There is a straightforward translation to the situation where $s_i \in \{0, 1\}$, which is natural for the application to number theory.

Erdős focussed on the case $r_n := \lambda^n$ for a fixed $0 < \lambda < 1$. Here it is obvious that $K$ has zero length for $0 < \lambda < \frac{1}{2}$, and that $K$ is an interval for $\frac{1}{2} \leq \lambda < 1$. Erdős asked, when is the distribution $\mu_\lambda$ of $Y$ absolutely continuous with respect to Lebesgue measure? For $\lambda < \frac{1}{2}$, since the support $K$ has zero length, $\mu_\lambda$ is singular, but for $\lambda > \frac{1}{2}$, even though the support $K$ is an interval, the absolute continuity question is subtle. In 1939 Erdős [21] showed that there are values in $(\frac{1}{2}, 1)$, such as $\lambda = 2/(1 + \sqrt{5})$, for which $\mu_\lambda$ is singular, and in 1940 [22] he showed that there exists some $t < 1$ such that for almost every $\lambda \in (t, 1)$, $\mu_\lambda$ is absolutely continuous. In 1995, Solomyak [49] showed that $t$ in the previous sentence can be taken to be 1/2. Peres and Solomyak [42] give a simple proof of this. In 1958 Kahane and Salem [34] discussed a variant of the problem where the summands $r_n$ are random, of the form $r_n = U_1 U_2 \cdots U_n$ where the $U_i$ are independent, and uniformly chosen from intervals $[a_i, b_i] \subset [\frac{1}{2}, 1]$.

Consider the random closed set $C := \{ \sum_{i \geq 1} c_i X_i : c_i \in \{0, 1\} \}$, with the scale invariant Poisson process labelled as in (4.2), so that $C \subset [0, T]$. This is close to $A$ in that always $A \cap [0, 1) = C \cap [0, 1)$, and related to $B$: as a corollary of (14.2), for each $\theta > 0$, $B =_d (C \mid T = 1)$. Let $J_1, J_2, \ldots$ be fair coins with
values in \(\{0,1\}\), independent of each other and everything else. The random series \(Y^* := \sum_{i \geq 1} J_i X_i\) has values in the random set \(C\), and the random series \(Y := \sum_{i \geq 1} J_i V_i\) has values in the random set \(B\). Consider the conditional distribution \(\mu_{Y^* \mid X}\) of \(Y^*\) given \(X_1, X_2, \ldots\), and the conditional distribution \(\mu_{Y \mid V}\) of \(Y\) given the Poisson-Dirichlet process. These are random probability measures on \([0, \infty)\) and \([0,1]\), respectively, and correspond to the fixed probability distribution \(\mu_\lambda\) from the situation where only the signs are random. The method of [42] can be adapted to show that for \(\theta > 1/\log 2\), a.s. the distributions \(\mu_{Y^* \mid X}\) and \(\mu_{Y \mid V}\) are absolutely continuous with respect to Lebesgue measure, with density in \(L_2\) (Yuval Peres, private communication).

The closed support in \([0, \infty)\) of \(\mu_{Y^* \mid X}\) is \(C\), and that of \(\mu_{Y \mid V}\) is \(B\). The distribution of \(Y\) itself, without conditioning on the value of the Poisson-Dirichlet process, is the probability measure \(\mathbb{E} \mu_{Y \mid V}\), averaging over the values of \((V_1, V_2, \ldots)\). From Hirth [31] and Donnelly-Tavaré [19], the distribution of \(Y\) is simply Beta\((\frac{\theta}{2}, \frac{\theta}{2})\), which for \(\theta = 1\) is the arc-sine distribution. It gives the limit of the distribution of \(\log d/\log n\), and the limit distribution of \(\log d/\log N\), where a random integer \(N\) is chosen uniformly from 1 to \(n\), and then \(d\) is chosen uniformly from the \(\tau(N)\) divisors of \(N\); see [51], II.6.2. All this background suggests looking at the distribution of \((\text{the rescaled log of })\) a randomly chosen divisor of uniformly chosen random integer, as a random probability distribution. We need some notation.

Let \(P\) denote the space of probability measures on \(\mathbb{R}\), with the topology of weak convergence, as given by the Lévy metric \(\rho\); see e.g. [20]. For \(m \geq 1\) let
\[
\mu_m := \left(\frac{\sum_{d|m} \delta_{\log d/\log m}}{\sum_{d|m} 1}\right) \in P
\]
be the probability measure on \([0,1]\) that puts mass \(1/\tau(m)\) on each point of the form \(\log d/\log m\), where \(d\) runs over the divisors of \(m\). (For \(m = 1\), interpret 0/0 as 1, so that \(\mu_1 := \delta_1\).)

For \(n \geq 1\) let \(N \equiv N(n)\) be chosen uniformly from 1 to \(n\), i.e. \(\mathbb{P}_n(N = m) = 1/n\) for \(m = 1, 2, \ldots, n\). Let \(\mu_N\) denote the random measure whose value is \(\mu_m\) on the event \(\{N = m\}\), so that the law of \(\mu_N\) as a random element of \(P\) is \(\sum_{1 \leq m \leq n} \delta_{\mu_m}\).

**Theorem 22.2.** As \(n \to \infty\),
\[\mu_N \Rightarrow \mu_{Y \mid V},\]
\[\text{i.e. as random a element of } P, \mu_{N(n)} \text{ converges in distribution to } \mu_{Y \mid V}, \text{ using the Poisson-Dirichlet process with } \theta = 1.\]

**Proof.** As noted in [31], if \(m = p_1 p_2 \cdots p_k\) is squarefree, then \(\mu_m\), supported at \(2^k\) distinct points, is a Bernoulli convolution, and the essential difficulty is to deal with \(m\) such that \(p^2|m\) for some prime. Write \(P_i(m)\) for the \(i^{th}\) largest prime factor of \(m\), with \(P_i = 1\) if \(i\) is greater than the number \(\Omega(m)\) of prime factors of \(m\), including multiplicities. Let \(J_i\) be \(\{0,1\}\)-valued fair coins, independent of everything else. For \(k = 1, 2, \ldots\) define \(\mu_{(k)}_m \in P\) to be the distribution of \(\sum_1^k J_i (\log P_i(m) / \log m)\), so that \(\mu_{(k)} = \mu_m\) iff \(m\) is squarefree and has at most \(k\) prime factors. Similarly, define \(\mu_{Y \mid V} \in P\) to be the conditional distribution of \(\sum_1^k J_i V_i\) given \((V_1, V_2, \ldots)\). Write \(\mathcal{L}(S)\) for the law of \(S\). The Lévy metric \(\rho\) on \(P\) has the property that for random variables \(S, T, 1 = \mathbb{P}(|S - T| \leq \epsilon)\) implies that \(\rho(\mathcal{L}(S), \mathcal{L}(T)) \leq \epsilon\). Hence with \(S = \sum_1^k J_i v_i\) and \(T = \sum_1^\infty J_i v_i\), it follows that \(\rho(\mu_{(k)}_{Y \mid V}, \mu_{Y \mid V}) \leq V_{k+1} + V_{k+2} + \cdots = 1 - (V_1 + \cdots + V_k)\).
Given $\epsilon > 0$, pick $k$ so that $\mathbb{P}(V_1 + \cdots + V_k < 1 - \epsilon) < \epsilon$. This implies $\mathbb{P}\left( \rho(\mu^{(k)}_{Y|V}, \mu^{(k)}_{Y}) > \epsilon \right) < \epsilon$. For any $m$ write $r = P_1(m) \cdots P_k(m)$ and consider the map $g_m$ with $g_m(d) = (d, r)$, the greatest common divisor. This map, applied to the set of divisors of $m$, is exactly $\tau(m)2^{-k}$ to one, provided that $m$ has at least $k$ prime divisors and the $k$ largest prime divisors are distinct. For $m$ satisfying this condition, $g_k$ applied to a divisor chosen randomly from the $\tau(m)$ divisors of $m$ yields a uniform pick from the $2^k$ divisors of $r$, and hence $\rho(\mu^{(k)}_m, \mu_m) \leq \log(m/r)/\log m$.

We note that Billingsley [13] actually proved both (8.1) and the slightly different form we need here, that $(\log P_i(N)/\log N)_{i \geq 1} \Rightarrow (V_i)_{i \geq 1}$. Thus we can pick $n_1$ so that for $n \geq n_1$, $\mathbb{P}_n(P_1(N) \cdots P_k(N) \geq N^{1-\epsilon}$ and $P_1(N) > \cdots > P_k(N) > 1) > 1 - \epsilon$. This yields, for $n \geq n_1$, that $\mathbb{P}_n(\rho(\mu^{(k)}_N, \mu_N) > \epsilon) < \epsilon$.

The map $f_k: (x_1, \ldots, x_k) \mapsto \mathcal{L}(\sum_{i=1}^k J_i x_i)$, from $(\mathbb{R}^k, l_1)$ to $(P, \rho)$ is continuous; in fact it is a contraction. Now $\mu^{(k)}_{Y|V} = f_k(V_1, \ldots, V_k)$, and $\mu^{(k)}_N = f_k(\log N P_1(N), \ldots, \log N P_k(N))$, so using Billingsley’s result again, as $n \to \infty$, $\mu^{(k)}_N \Rightarrow \mu^{(k)}_{Y|V}$. By Skorohod embedding [20] for the complete separable metric space $P$, there exists a coupling in which $\mu^{(k)}_{N(n)} \Rightarrow \mu^{(k)}_{Y|V}$ almost surely. This coupling can be extended to a coupling of $N(1), N(2), \ldots$ and the Poisson-Dirichlet process, so that for one fixed $k$, $\mu^{(k)}_{N(n)}, \mu^{(k)}_N, \mu^{(k)}_{Y|V}$, and $\mu^{(k)}_{Y}$ are all defined on the same probability space, and a.s. $\mu^{(k)}_{N(n)} \Rightarrow \mu^{(k)}_{Y|V}$. Pick $n_2$ such that for $n \geq n_2$, $\mathbb{P}(\rho(\mu^{(k)}_{N(n)}, \mu^{(k)}_{Y|V}) > \epsilon) < \epsilon$. The triangle inequality yields, for $n \geq \max(n_1, n_2)$, that $\mathbb{P}(\rho(\mu_N(n), \mu_{Y|V}) > 3\epsilon) < 3\epsilon$.

\section*{REFERENCES}

[1] Aizenman, M., and Newman, C. (1986) Discontinuity of the percolation density in one-dimensional $1/|x - y|^2$ percolation models. Comm. Math. Phys. 107, 611-647.

[2] Aldous, D. J. (1983) Exchangeability and related topics. Springer, Lecture Notes in Mathematics, vol. 1117.

[3] Arratia, R. (1996) Independence of small prime factors: total variation and Wasserstein metrics, insertions and deletions, and the Poisson-Dirichlet process. Draft, 73 pages, available from rarratia@math.usc.edu.

[4] Arratia, R. (1996) Large and small prime factors conditional on a large deviation for the number of prime factors of a random integer. Draft, 17 pages, available from rarratia@math.usc.edu.

[5] Arratia, R., Barbour, A. D., and Tavaré, S. (1992) Poisson process approximation for the Ewens Sampling Formula. Ann. Appl. Probab. 2, 519-535.

[6] Arratia, R., Barbour, A. D., and Tavaré, S. (1997) Random combinatorial structures and prime factorizations. AMS Notices 44, 903-910.

[7] Arratia, R., Barbour, A. D., and Tavaré, S. (1997) Logarithmic combinatorial structures. Monograph, 187 pages, in preparation.

[8] Arratia, R., Barbour, A. D., and Tavaré, S. (1997) On Poisson-Dirichlet limits for random decomposable combinatorial structures. To appear in Comb., Prob., and Comp.

[9] Arratia, R., Barbour, A.D., and Tavaré, S. (1997) The Poisson-Dirichlet distribution and the scale invariant Poisson process. To appear in Comb., Prob., and Comp.

[10] Arratia, R., and Stark, D. (1997) A total variation distance invariance principle for primes, permutations and Poisson-Dirichlet. Preprint.

[11] Arratia, R., Stark, D., and Tavaré, S. (1995). Total variation asymptotics for Poisson process approximations of logarithmic combinatorial assemblies. Ann. Probab. 23, 1347-1388.

[12] Arratia, R., and Tavaré, S. (1992) The cycle structure of random permutations. Ann. Probab. 20, 1567-1591.
[13] Billingsley, P. (1972) On the distribution of large prime factors. Period. Math. Hungar. 2, 283-289.
[14] de Bruijn, N. G., and van Lint, J. H. (1964) Incomplete sums of multiplicative functions, I. Nederl. Akad. Wetensch. Proc. Ser. A 67, 339-347, 348-359.
[15] Buchstab, A. A. (1937) An asymptotic estimation of a general number-theoretic function. Mat. Sbornik (2) 44, 1239-1246.
[16] Daley, D.J., and Vere-Jones, D. (1988) An introduction to the theory of point processes. Springer-Verlag, Springer series in statistics.
[17] Dickman, K. (1930) On the frequency of numbers containing prime factors of a certain relative magnitude. Ark. Math. Astr. Fys. 22, 1-14.
[18] Donnelly, P., and Grimmett, G. (1993) On the asymptotic distribution of large prime factors. J. London Math. Soc. (2) 47, 395-404.
[19] Donnelly, P., and Tavaré, S. (1987) The population genealogy of the infinitely-many neutral alleles model. J. Math. Biol. 25, 381-391.
[20] Dudley, R. M. (1989) Real Analysis and Probability. Wadsworth and Brooks/Cole.
[21] Erdős, P. (1939). On a family of symmetric Bernoulli convolutions. Am. J. Math. 61, 974-975.
[22] Erdős, P. (1940). On the smoothness properties of Bernoulli convolutions. Am. J. Math. 62, 180-186.
[23] Ewens, W. J. (1972) The sampling theory of selectively neutral alleles. Theoret. Population Biol. 3, 87-112.
[24] Feller, W. (1966) An introduction to probability and its applications, volume II. Wiley.
[25] Ferguson, T. S. (1973) A Bayesian analysis of some nonparametric problems. Ann. Stat. 1, 209-230.
[26] Hall, R.R., and Tenenbaum, G. (1988) Divisors. Cambridge Tracts in Mathematics 90, Cambridge.
[27] Halmos, P. R. (1944) Random Alms. Ann. Math. Stat. 15, 182-189.
[28] Hansen, J. C. (1994). Order statistics for decomposable combinatorial structures. Random Structures and Algorithms 5, 517-533.
[29] Hensley, D. (1986). The convolution powers of the Dickman function. J. London Math. Soc. (2) 33, 395-406.
[30] Hildebrand, A. (1990). The asymptotic behavior of the solutions of a class of differential-difference equations. J. London Math. Soc. (2) 42, 11-31.
[31] Hirth, U. M. (1997) Probabilistic number theory, the GEM/Poisson-Dirichlet distribution and the Arc-sine law. Comb., Prob., and Comp. 6, 57-77.
[32] Ignatov, T. (1982) On a constant arising in the asymptotic theory of symmetric groups, and on Poisson-Dirichlet measures. Theor. Prob. Appl. 27, 136-147.
[33] Johnson, N. S. and Kotz, S., and Balakrishnan, N. (1997). Discrete Multivariate Distributions. Wiley, New York.
[34] Kahane, J. P., and Salem, R. (1958). Sur la convolution d’une infinité de distributions de Bernoulli. Colloquium Mathematicum 6, 193-202.
[35] Kingman, J.F.C. (1975). Random discrete distributions. J. Royal Stat. Soc. 37, 1-22.
[36] Kingman, J.F.C. (1993) Poisson Processes. Oxford Science Publications.
[37] De Koninck, J.-M., and Galambos, J. (1987) The intermediate prime divisors of integers. Proc. Amer. Math. Soc. 101, 213-216.
[38] Liggett, T. (1978) Random invariant measures for Markov chains, and independent particle systems. Z. Wahrsch. verw. Gebiete 45, 297-313.
[39] McCloskey, J. W. (1965) A model for the distribution of individuals by species in an environment, Michigan State University, unpublished Ph.D. thesis.
[40] Peman, M. (1993). Order statistics for jumps of normalized subordinators. Stoch. Pr. Appl. 46, 267-281.
[41] Peman, M., Pitman, J., and Yor, M. (1992). Size-biased sampling of Poisson point processes and excursions. Prob. Th. Rel. Fields 92, 21-39.
[42] Peres, Y., and Solomyak, B. (1996). Absolute continuity of Bernoulli convolutions, a simple proof. Math. Res. Lett. 3, 231-239.
[43] Pitman, J. (1996). Some developments of the Blackwell-MacQueen urn scheme. In T.S. Ferguson et al., editor, Statistics, Probability and Game Theory; Papers in honor of David
Blackwell, Lecture Notes-Monograph Series 30, 245–267. Institute of Mathematical Statistics, Hayward, California, 1996.

Pitman, J. (1997). Coalescents with multiple collisions. Technical report No. 495, Department of Statistics, Berkeley.

Pitman, J., and Yor, M. (1996) Random discrete distributions derived from self-similar random sets. Electron. J. Probab. 1, no. 4.

Pitman, J., and Yor, M. (1997) The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. Ann. Probab. 25, 855-900.

Ruelle, D. (1987) A mathematical reformulation of Derrida’s REM and GREM. Commun. Math. Phys. 108, 225-239.

Shepp, L. A., and Lloyd, S. P. (1966) Ordered cycle lengths in a random permutation. Trans. Amer. Math. Soc. 121, 340-357.

Solomyak, B. (1995). On the random series $\sum \pm \lambda^n$ (An Erdős problem). Ann. Math 142, 611-625.

Stark, D. (1997) Explicit non-zero limits of total variation distance in independent Poisson approximations of logarithmic combinatorial assemblies. Comb., Prob., and Comp. 6, 87-106.

Tenenbaum, G. (1995) Introduction to analytic and probabilistic number theory. Cambridge studies in advanced mathematics, 46. Cambridge University Press.

Tenenbaum, G. (1997). Crible d’Ératosthène et modèle de Kubilius. Preprint.

Vershik, A.M., and Shmidt, A.A. (1977) Limit measures arising in the theory of groups, I, Theory Probab. Appl. 22, 79–85.

Vershik, A.M., and Shmidt, A.A. (1978) Limit measures arising in the theory of groups, II, Theory Probab. Appl. 23, 30–49.

Vervaat, W. (1972) Success Epochs in Bernoulli Trials with Applications in Number Theory, Mathematical Center Tracts, vol. 42, Mathematisch Centrum, Amsterdam.

Watterson, G. A. (1976). The stationary distribution of the infinitely-many alleles diffusion model. J. Appl. Probab. 13, 639-651.

Zeitouni, O. (1997) Superexponential decay for the GEM process. To appear, J. Appl. Prob.

Current address: University of Southern California, Department of Mathematics, DRB 155, Los Angeles CA 90089-1113

E-mail address: rarratia@math.usc.edu