ENumerative Geometry of Plane Curves of Low Genus

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Abstract. We collect various known results (about plane curves and the moduli space of stable maps) to derive new recursive formulas enumerating low genus plane curves of any degree with various behaviors. Recursive formulas are given for the characteristic numbers of rational plane curves, elliptic plane curves, and elliptic plane curves with fixed complex structure. Recursions are also given for the number of elliptic (and rational) plane curves with various “codimension 1” behavior (cuspidal, tacnodal, triple pointed, etc., as well as the geometric and arithmetic sectional genus of the Severi variety). We compute the latter numbers for genus 2 and 3 plane curves as well. We rely on results of Caporaso, Diaz, Getzler, Harris, Ran, and especially Pandharipande.

1. Introduction

Let $\overline{M}_g(\mathbb{P}^2, d)^*$ be the component of the stack $\overline{M}_g(\mathbb{P}^2, d)$ generically parametrizing maps from irreducible curves. (All stacks will be assumed to be Deligne-Mumford stacks.) On the universal curve $U$ over $\overline{M}_g(\mathbb{P}^2, d)^*$ (with structure map $\pi$) there are two natural divisors, the pullback $D$ of $\mathcal{O}_{\mathbb{P}^2}(1)$, and the relative dualizing sheaf $\omega$. Following the notation of [DH1], let $A = \pi_*(D^2)$, $B = \pi_*(D \cdot \omega)$, $C = \pi_*(\omega^2)$, and $TL = A + B$. Let $\Delta_0$ be the divisor generically parametrizing maps from irreducible nodal curves, and let $\Delta_{i,j}$ ($0 < j < d$) be the divisor generically parametrizing maps from a reducible curve, one component of genus $i$ and mapping with degree $j$, and the other of genus $g-i$ mapping with degree $d-j$. Let $\Delta = \Delta_0 + \sum \Delta_{i,j}$. ([DH1] deals with Severi varieties, but all arguments carry over to this situation.) Then $TL$ is the divisor class corresponding to curves tangent to a fixed line. Call irreducible divisors on $\overline{M}_g(\mathbb{P}^2, d)^*$ whose general map contracts a curve enumeratively meaningless; call other divisors enumeratively meaningful. Call enumeratively meaningful irreducible divisors whose general
source curve is singular boundary divisors; these are the components of $\Delta$.

When $g \leq 2$, $C$ can be expressed as a sum of boundary divisors. When $g = 0$ (resp. 1), $TL - (d-1)A$ (resp. $TL - A = B$) can be expressed as a sum of boundary divisors. By restricting this identity to the one-dimensional family in $\overline{\mathcal{M}}_g(\mathbb{P}^2, d)^*$ generically corresponding to curves through a general points and tangent to $3d + g - 2$ general lines, we find recursions for characteristic numbers (when $g \leq 1$). Recursions for the genus 0 characteristic numbers are well-known ([P1], [EK1], [EK2]). Algorithms to determine genus 1 characteristic numbers are known ([GP2] via descendents and topological recursions; [V2] by degenerations), but the formulas given here seem less unwieldy and more suitable for computation.

In [DH1] and [DH2], many divisors on the Severi variety are expressed as linear combinations of $A$, $B$, $C$ and boundary divisors. (Diaz and Harris conjecture that up to torsion, any divisor can be so expressed.) Modulo enumeratively meaningless divisors, their arguments carry over to $\overline{\mathcal{M}}_g(\mathbb{P}^2, d)^*$. Now restrict these divisors to the one-parameter family corresponding to curves through $3d + g - 2$ general points. If $X$ is a divisor on a curve, denote its degree by $|X|$. When $g \leq 1$, there are simple recursions for $|A|$, $|B|$, $|C|$, and any boundary divisor, so we get similar recursions for enumerative “divisor-related behavior” (e.g. the geometric and arithmetic sectional genus of the Severi variety, or the number of cuspidal or triple-pointed curves, or the number of curves through $3d - 1$ general points and with 3 collinear nodes). Some of these formulas were known earlier. When $g = 2$ or 3, $|A|$ and $|B|$ can be found using [R2] or [CH], and $|C|$ is simple to compute using [M] (and, if $g = 3$, Graber’s algorithm [Q] for counting hyperelliptic plane curves). (When $g = 2$, the number $|A|$, and possibly $|B|$, can be quickly computed by the recursions of Belorousski and Pandharipande ([BeP]). Hence these “codimension 1” numbers (e.g. counting cuspidal, tacnodal, or triple-pointed genus 2 or 3 curves, or computing the geometric and arithmetic sectional genus of the Severi variety $V^{d,2}$ or $V^{d,3}$) can be computed.

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2.8. A maple program implementing all algorithms described here is available upon request.

1.1. **Historical notes.** For a more complete historical background and references, see the introduction to [A1].

Characteristic number problems date from the last century, and were studied extensively by Schubert, Chasles, Halphen, Zeuthen, and others. Much of their work is collected in [S].

A modern study of the enumerative geometry of cubics was undertaken successfully in the 1980’s. Among the highlights: Sacchiero and Kleiman-Speiser independently verified Maillard and Zeuthen’s results for cuspidal and nodal plane cubics, and Kleiman and Speiser calculated the characteristic numbers of smooth plane cubics ([KiSp]). Sterz ([St]) and Aluffi ([Al]) independently constructed a smooth variety of “smooth cubics”, and Aluffi used this variety to compute the characteristic numbers of smooth plane cubics and other enumerative information.

The advent of the moduli space of stable maps has had tremendous applications in enumerative algebraic geometry; as an example pertaining to this article, Pandharipande calculated the characteristic numbers of rational curves in $\mathbb{P}^n$ in [P1], and Ernström and Kennedy showed that the characteristic numbers of rational curves in $\mathbb{P}^2$ were encoded in a “contact cohomology ring” of $\mathbb{P}^2$ that is the deformation of the quantum cohomology ring ([EK1], [EK2]).

1.2. **Gromov-Witten theory.** Although it isn’t evident in the presentation, the main idea came from an attempt to understand geometrically why Gromov-Witten invariants determine gravitational descendents in genus 1 (see [KaKi]). This fact should really be seen as related to a more elementary fact of Kodaira’s, that the relative dualizing sheaf of a family of elliptic curves can be expressed as a sum of boundary divisors (cf. [H]). Kodaira’s relation can also be used enumeratively, by restricting it to one-parameter families, as in this article. For the same reason, Belorousski and Pandharipande’s new relation in $\mathcal{M}_{2,3}$ together with Getzler’s genus 2 descendent relations ([G]) yields recursions for all $g = 2$ descendent integrals on $\mathbb{P}^2$. However, full reconstruction in $g = 2$ has not yet been shown for arbitrary spaces – additional universal relations are needed. These results may be interpreted to suggest
the existence of recursive formulas for characteristic numbers of genus 2 curves, although the recursions are likely quite messy. (This is quite speculative, of course.)

2. Characteristic numbers

We work over an algebraically closed field of characteristic 0. Let $R_d(a,b)$ be the number of irreducible degree $d$ rational curves through $a$ fixed general points and tangent to $b$ fixed general lines if $a+b = 3d-1$, and 0 otherwise. Let $R_d := R_d(3d-1,0)$ be the number with no tangency conditions. Let $NL_d(a,b)$ be the number of irreducible degree $d$ rational curves through $a$ fixed general points and tangent to $b$ fixed general lines and with a node on a fixed line if $a+b = 3d-2$, and 0 otherwise. By [DHI] (1.4) and (1.5),

$$NL_d(a,b) = (d-1)R_d(a+1,b) - R_d(a,b+1)/2.$$  

Let $NP(a,b)$ be the number of irreducible degree $d$ rational curves through $a$ fixed general points and tangent to $b$ fixed general lines and with a node at a fixed point if $a+b = 3d-3$, and 0 otherwise. Let $NP_d := NP_d(3d-3,0)$ be the number with no tangency conditions. Let $E_d(a,b)$ be the number of irreducible degree $d$ elliptic curves through $a$ fixed general points and tangent to $b$ fixed general lines if $a+b = 3d$, and 0 otherwise. Let $E_d := E_d(3d,0)$ be the number with no tangency conditions.

The algorithm involves six different recursions, three of them well-known and three quite simple:

1. Calculating $R_d$ using Kontsevich’s recursion [2].
2. Calculating $NP_d$, in essence by using Kontsevich’s recursion on the convex rational ruled surface $F_1$.
3. Calculating $E_d$ using the recursion of Eguchi, Hori, and Xiong.
4. Calculating the characteristic numbers $R_d(a,b)$ using the characteristic numbers of lower degree curves, or curves of the same degree with fewer tangency conditions.
5. The same thing for $NP_d(a,b)$.
6. The same thing for $E_d(a,b)$.

2.1. Bertini-type preliminaries. Assume that $W$ is a variety defined over an algebraically closed field of characteristic 0. Consider a
family of maps from nodal curves to $\mathbb{P}^2$:
\[
\begin{array}{c}
\mathcal{U} \\
\pi \\
W
\end{array}
\xrightarrow{\rho}
\begin{array}{c}
\rho^*\mathcal{O}_{\mathbb{P}^2}(1) \\
\mathbb{P}^2
\end{array}
\]

We say that a map has a tangent line $l \subset \mathbb{P}^2$ if the pullback of $l$ to $\mathcal{U}$ contains a point with multiplicity at least 2; similar definitions apply for flex lines and bitangents.

Let $\omega$ be the relative dualizing sheaf of $\pi$, and $D = \rho^*\mathcal{O}_{\mathbb{P}^2}(1)$. Let $A = \pi^*(D^2)$ and $B = \pi^*(D \cdot \omega)$ for convenience. By the Kleiman-Bertini theorem ([Kl]) applied to $\mathcal{U}$, $D$ is base-point free, and if $V$ is any irreducible substack of $W$, a general representative of $\pi^*D^2$ intersects $V$ properly and transversely. (Strictly speaking, Kleiman-Bertini should be applied to $W \times PGL_2$ with group $PGL_2$ as follows. There is a universal curve $(\pi, 1) : \mathcal{U} \times PGL_2 \to W \times PGL_2$, and the universal map to $\mathbb{P}^2$ is given by $(p, g) \mapsto g \circ p$. For the sake of brevity, we will elide this discussion when we invoke Kleiman-Bertini in the future.)

Next, assume that $W$ is irreducible and $\mathcal{U}_w$ is smooth for general $w \in W$. Let $L$ be the divisor on $\mathcal{U}$ that is the pullback of a general line $l$ in $\mathbb{P}^2$ (so $[L] = D$). Then $L$ has the same dimension as $W$, its ramification divisor is in the divisor class $(D + \omega)|_L$, and the branch divisor is in class $A + B = \pi^*(D \cdot (D + \omega))$.

**Lemma 1.** If the general curve is smooth, and the general map in the family factors as a simply ramified multiple cover followed by an immersion, then:

(a) the branch divisor is reduced, and

(b) if $V$ is any irreducible subvariety of $W$, then (for a general $L$) the branch divisor intersects $V$ properly.

**Proof.** For part (a), we must show that the general point of any component of the branch divisor corresponds to a map simply tangent to the line $l$ (i.e. $l$ is not a bitangent or a flex).

The general map in the family has a finite number of bitangent and flexes. (The image curve has a finite number of bitangents and flexes, as the dual of a reduced curve is a reduced curve in characteristic 0. The only additional bitangents and flexes must involve the simple ramification of the map from the source to the image. This will yield only a finite number of each.) By a similar argument, any particular
map has at most a one-dimensional family of bitangent lines or flex lines; call the locus with a positive-dimensional family of such lines $B$, a proper subvariety of $W$. Then (for dimensional reason), the branch divisor of the pullback of a general line $l$ to the family meets each each component of $B$ properly. Hence (a) follows.

Part (b) is similar, and omitted for the sake of brevity. \qed

Hence in a one-parameter family of maps (satisfying the conditions of the lemma), the number of curves tangent to a general fixed line is $|A + B| = |D \cdot (D + \omega)|$.

**Lemma 2.** Let $W$ be an irreducible reduced substack of $\overline{M}_g(\mathbb{P}^2, d)^*$ whose generic member corresponds to a map from a smooth curve. Then the subset of $W$ corresponding to maps through a fixed general point (resp. tangent to a fixed general line) is of pure codimension 1, each component generically corresponds to a map from a smooth curve, and the corresponding Weil divisor is in class $A|_W$ (resp. $(A + B)|_W$).

**Proof.** The Kleiman-Bertini argument for incidence conditions is well-known (see [FP] Section 9). We show the result for the locus $T$ in $W$ corresponding to maps tangent to a fixed general line. By purity of branch locus, $T$ is pure codimension 1 in $W$. By Lemma 1 (a), $T$ (as a Weil divisor) is in class $(A + B)|_W$. The irreducible components of the (proper) substack corresponding to maps from singular curves all meet $T$ properly by Lemma 1 (b), so the general point of each component of $T$ corresponds to a map from a smooth curve. \qed

**Corollary 3.** For $W$ as in Lemma 2, such that the generic map in $W$ has trivial automorphism group, $A^a(A + B)^{\dim W - a}$ is the solution to the enumerative problem: how many maps in $W$ pass through a general points and are tangent to $(\dim W - a)$ general lines?

We will need to understand the divisor $TL = A + B$ on maps from nodal curves as well.

**Lemma 4.** If $W$ is an irreducible family of maps and $U_w$ is a curve with one node for a general $w \in W$, then the divisor $\pi_*(D \cdot (D + \omega))$ is the divisor corresponding to where the map from the normalization is tangent to a fixed general line $l$, plus twice the divisor corresponding to where the node maps to $l$. If $V$ is any irreducible subvariety of $W$, then this divisor meets $V$ properly (for general $l$).
Proof. Compare the relative dualizing sheaf of the nodal curve with
the relative dualizing sheaf of the normalization. □

Next, we recall relevant facts about the moduli stack of stable maps.
The stack $\mathcal{M}_0(\mathbb{P}^2, d)$ is smooth of dimension $3d-1$. The stack $\mathcal{M}_1(\mathbb{P}^2, d)^*$ is the closure (in $\mathcal{M}_1(\mathbb{P}^2, d)$) of maps that collapse no elliptic component. It has dimension $3d$, and it is smooth away from the divisor where an elliptic component is collapsed ([V1] Lemma 3.13). In particular, if $\Delta$ is the union of divisors corresponding to maps from nodal curves with no collapsed elliptic component, then $\mathcal{M}_1(\mathbb{P}^2, d)^*$ is smooth at the generic point of each component of $\Delta$.

Lemma 5. Suppose $\Delta$ is the locus in $\mathcal{M}_1(\mathbb{P}^2, d)^*$ described above, or the locus in $\mathcal{M}_0(\mathbb{P}^2, d)$ generically corresponding to maps from curves with one node. Fix a general points and $b$ general lines, where $a + b = \dim \Delta$. Then the intersection $\Delta \cdot A^a T^b L$ is equal to the number of maps where the map from the normalization passes through the $a$ points and is tangent to the $b$ lines; plus twice the number where the node maps to one of the $b$ lines, and the curve passes through the $a$ points and is tangent to the remaining $b-1$ lines; plus four times the number where the node maps to the intersection of two of the $b$ lines, and the curve passes through the $a$ points and is tangent to the remaining $b-2$ lines.

Proof. This follows from the fact that the condition of requiring the node to map to a fixed general line is transverse to any subvariety (by Kleiman-Bertini), and Lemma [4]. □

2.2. Incidences only. We begin by considering cases with no tangencies.

Clearly $R_1 = 1$. There is a well-known formula ([KoM] Claim 5.2.1 or [RuT]) for computing $R_d$ inductively:

\begin{equation}
R_d = \sum_{i+j=d} i^2 j \left( j \left( \frac{3d-4}{3i-2} \right) - i \left( \frac{3d-4}{3i-1} \right) \right) R_i R_j.
\end{equation}

One proof involves studying rational curves through $3d-2$ fixed points, two of which are marked $p$ and $q$, and two marked points $r$ and $s$ on fixed general lines, and pulling back an equivalence on $\text{Pic} \mathcal{M}_{0,4}$. The
same "cross-ratio" trick gives a recursion for $NP_d$:

$$NP_d = \sum_{i+j=d} (ij-1)i \left( j \left( \frac{3d-6}{3i-3} \right) - i \left( \frac{3d-6}{3i-2} \right) \right) R_i R_j$$

$$+ \sum_{i+j=d} ij \left( 2ij \left( \frac{3d-6}{3i-4} \right) - i^2 \left( \frac{3d-6}{3i-3} \right) - j^2 \left( \frac{3d-6}{3i-5} \right) \right) NP_i R_j. \tag{3}$$

(Pandharipande gives another recursion for $NP_d$ in [P3] Section 3.4.)

The Eguchi-Hori-Xiong formula (proved by Pandharipande in [P3] using Getzler’s relation) gives $E_d$:

$$E_d = \frac{1}{12} \binom{d}{3} R_d + \sum_{i+j=d} \frac{ij(3i-2)}{9} \binom{3d-1}{3j} R_i E_j. \tag{4}$$

(Remarkably, there is still no geometric proof known of this result.)

2.3. Swapping incidences for tangencies: genus 0. From [P1] Lemma 2.3.1, in $\text{Pic}(\mathcal{M}_0(\mathbb{P}^2, d)) \otimes \mathbb{Q}$,

$$TL = \frac{d-1}{d} A + \sum_{j=0}^{\lfloor d/2 \rfloor} \frac{j(d-j)}{d} \Delta_{0,j}. \tag{5}$$

Apply this rational equivalence to the one-parameter family corresponding to degree $d$ rational curves through $a$ general points and tangent to $b$ general lines (where $a + b = 3d - 2$) to get:

$$R_d(a, b + 1) = \frac{d-1}{d} R_d(a + 1, b)$$

$$+ \sum_{i+j=d} \frac{ij}{2d} \left[ \sum_{a_i + a_j = a \atop b_i + b_j = b} \binom{a}{a_i} \binom{b}{b_i} \binom{b-1}{b_j} (ij) R_i(a_i, b_i) R_j(a_j, b_j) \right. \left. + 4b \sum_{a_i + a_j = a+1 \atop b_i + b_j = b-1} \binom{a}{a_i} \binom{b-1}{b_j} iR_i(a_i, b_i) R_j(a_j, b_j) \right. \left. + 4 \binom{b}{2} \sum_{a_i + a_j = a+2 \atop b_i + b_j = b-2} \binom{a}{a_i-1} \binom{b-2}{b_j} R_i(a_i, b_i) R_j(a_j, b_j) \right].$$

In each sum, it is assumed that $i, j > 0$; $a_i, a_j, b_i, b_j \geq 0$; $a_i + b_i = 3i - 1$; $a_j + b_j = 3j - 1$; and that all of these are integers. The large bracket corresponds to maps from reducible curves. The first sum in the large bracket corresponds to the case where no tangent lines pass
through the image of the node; the second sum corresponds to when one tangent line passes through the image of the node; and the third to when two tangent lines pass through the image of the node (see Lemma [3]). Note that in the second sum, $3i - 1$ of the $a + b$ conditions fix the component corresponding to $R_i$ (up to a finite number of possibilities). The component corresponding to $R_j$ is specified by the remaining $3j - 2$ conditions, plus the condition that it intersect the other component on a fixed line.

This completes the computation of the characteristic numbers for rational plane curves.

Pandharipande earlier obtained (by topological recursion methods and descendants) what can be seen to be the same recursion in the form of a differential equation ([P4]): if

$$R(x, y, z) = \sum_{a, b, d} R_d(a, b) \frac{x^a y^b}{a! b!} e^{dz},$$

then

$$R_{yz} = -R_x + R_{xz} - \frac{1}{2} R_{zz}^2 + (R_{zz} + yR_{xx})^2.$$

A similar argument applied to the one-parameter family corresponding to degree $d$ rational curves with a node at a fixed point, through $a$ general points and tangent to $b$ general lines (where $a + b = 3d - 4$) gives the formula shown in Appendix A. The corresponding differential equation is:

$$NP_{yz} = -NP_x + NP_{xx} - \frac{1}{2} NP_{zz}^2 + (NP_{xx} + yNP_{xx})^2$$

$$+ 2(NP_{zz} + yNP_{xx})(NP_{zz} + yNP_{xx}) - NP_{zz}NP_{zz}.$$

2.4. Swapping incidences for tangencies: genus 1. On the universal curve over $M_1(\mathbb{P}^2, d)^*$, let $Q$ be the divisor corresponding to nodal irreducible fibers. Following [R3], let $R$ be the divisor corresponding to rational components of reducible fibers. Then

$$(6) \quad \omega \simeq \frac{Q}{12} + R$$

(Kodaira’s formula for the canonical bundle of an elliptic surface; see [BPV] Theorem 12.1 for a proof over $\mathbb{C}$). Hence $B = \pi_*(D \cdot \omega) = \ldots$
\[ \frac{d}{12} \Delta + \sum_i i \Delta_{0,i}, \text{ so} \]

\[ TL = A + \frac{d}{12} \Delta + \sum_i i \Delta_{0,i}. \]

(7)

Restricting this identity to the one-parameter family corresponding to degree \( d \) elliptic curves through \( a \) general points and tangent to \( b \) general lines (where \( a + b = 3d - 1 \)) gives:

\[ E_d(a, b + 1) = E_d(a + 1, b) \]

\[ + \frac{d}{12} \left( \binom{d-1}{2} R_d(a, b) + 2bNL_d(a, b - 1) + 4 \left( \frac{b}{2} \right) NP_d(a, b - 2) \right) \]

\[ + \sum_{i+j=d} \sum_{a_i+a_j=a \atop b_i+b_j=b} \left( \begin{array}{c} a \\ a_i \\ \end{array} \right) \left( \begin{array}{c} b \\ b_i \\ \end{array} \right) (ij) R_i(a_i, b_i) E_j(a_j, b_j) \]

\[ + 2b \left( \sum_{a_i+a_j=a+1 \atop b_i+b_j=b-1} \left( \begin{array}{c} a \\ a_i \\ \end{array} \right) \left( \begin{array}{c} b-1 \\ b_i \\ \end{array} \right) jR_i(a_i, b_i) E_j(a_j, b_j) \right) \]

\[ + \sum_{a_i+a_j=a+1 \atop b_i+b_j=b-1} \left( \begin{array}{c} a \\ a_i \\ \end{array} \right) \left( \begin{array}{c} b-1 \\ b_i \\ \end{array} \right) iR_i(a_i, b_i) E_j(a_j, b_j) \right) \]

\[ + 4 \left( \frac{b}{2} \right) \sum_{a_i+a_j=a+2 \atop b_i+b_j=b-2} \left( \begin{array}{c} a \\ a_i-1 \\ \end{array} \right) \left( \begin{array}{c} b-2 \\ b_i \\ \end{array} \right) R_i(a_i, b_i) E_j(a_j, b_j) \right]. \]

\( NL_d(a, b - 1) \) can be found using (1). The large square bracket corresponds to maps of reducible curves. The first sum corresponds to the case when no tangent line passes through the image of the node, the next two sums correspond to when one tangent line passes through the image of the node, and the last sum corresponds to when two tangent lines pass through the image of the node.

The corresponding differential equation is:

\[ E_y = E_x + \Delta + 2(R_{zz} + R_{xx})(E_z + E_x) - R_{zz} E_z \]

where

\[ \Delta = \frac{1}{12} \left( \frac{1}{2} (R_{zzz} - 3R_{zz} + 2R_z) + 2yNL_z + 2y^2NP_z \right). \]
This completes the computation of the characteristic numbers of elliptic plane curves.

2.5. Characteristic numbers of elliptic curves with fixed $j$-invariant ($j \neq \infty$). Let $M_j$ be the Weil divisor on $\overline{M}_1(\mathbb{P}^2, d)^*$ corresponding to curves whose stable model has fixed $j$-invariant $j$. Then $M_j \cong M_{\infty}$ if $j \neq 0, 1728, 0$, and $M_{1728} \cong M_{\infty}/2$ (Lemma 4). If $a+b = 3d-1$, define $J_d(a, b) := M_{\infty,aTL^b}$. By Corollary [3] if $d \geq 3$, the characteristic numbers of curves with fixed $j$-invariant $j \neq 0, 1728, \infty$ are given by $J_d(a, b)$, and if $j = 0$ or $j = 1728$, the characteristic numbers are one third and one half $J_d(a, b)$ respectively. But $M_{\infty}$ parametrizes maps from nodal rational curves, so we can calculate $M_{\infty,aTL^b}$ using Lemma [3]

$$J_d(a, b) = \left(\frac{d-1}{2}\right) R_d(a, b) + 2b NL_d(a, b - 1) + 4 \left(\frac{b}{2}\right) NP_d(a, b - 2).$$

2.6. Numbers. Using the recursions given above, we find the following characteristic numbers for elliptic curves. (The first number in each sequence is the number with only incidence conditions; the last is the number with only tangency conditions.)

**Conics:** 0, 0, 0, 2, 10, 45/2.

**Cubics:** 1, 4, 16, 64, 256, 976, 3424, 9766, 21004, 33616.

**Quartics:** 225, 1010, 4396, 18432, 73920, 280560, 994320, 3230956, 9409052, 23771160, 50569520, 129996216.

**Quintics:** 87192, 412306, 1873388, 8197344, 3424992, 136396752, 512271756, 1802742368, 5889847264, 17668868832, 48034140112, 116575540736, 248984451648, 463227482784, 747546215472, 1048687299072.

The cubic numbers agree with those found by Aluffi in [A1]. The quartic numbers agree with the predictions of Zeuthen (see [S] p. 187).

Using the recursion of Subsection 2.5, we find the following characteristic numbers for elliptic curves with fixed $j$-invariant ($j \neq 0, 1728, \infty$).

**Conics:** 0, 0, 0, 12, 48, 75.

**Cubics:** 12, 48, 192, 768, 2784, 8832, 21828, 39072, 50448.
Quartics: 1860, 8088, 33792, 134208, 5193768, 13954512, 31849968, 60019872, 92165280, 115892448.

The cubic numbers agree with those found by Aluffi in [A2] Theorem III(2). The incidence-only numbers necessarily agree with the numbers found by Pandharipande in [P2], as the formula is the same.

2.7. **Characteristic numbers in** $\mathbb{P}^n$. The same method gives a program to recursively compute characteristic numbers of elliptic curves in $\mathbb{P}^n$ that should be simpler than the algorithm of [V2]. Use Kontsevich’s cross-ratio method to count irreducible nodal rational curves through various linear spaces and where the node is required to lie on a given linear space (analogous to the derivation of (3)). Use (6) to compute all the characteristic numbers of each of these families of rational curves. Use [V1] to compute the number of elliptic curves through various linear spaces. Finally, use (7) to compute all characteristic numbers of curves in $\mathbb{P}^n$. The same calculations also allow one to compute characteristic numbers of elliptic curves in $\mathbb{P}^n$ with fixed $j$-invariant.

2.8. **Covers of** $\mathbb{P}^1$. By restricting Pandharipande’s relation (3) and relation (7) to degree $d$ covers of a line by a genus 0 and 1 curve respectively (so $A$ restricts to 0), where all but 1 ramification are fixed, we obtain recursions for $M^g_d$ ($g = 0, 1$), the number of distinct covers of $\mathbb{P}^1$ by irreducible genus $g$ curves with $2d + 2g - 2$ fixed ramification points:

$$M^0_d = \frac{(2d - 3)}{d} \sum_{j=1}^{d-1} \binom{2d - 4}{2j - 2} M^0_j M^0_{d-j} j^2 (d - j)^2$$

$$M^1_d = \frac{d}{6} \binom{d}{2} (2d - 1) M^0_d + \sum_{j=1}^{d-2} 2j(2d - 1) \binom{2d - 2}{2j - 2} M^0_j M^1_{d-j} (d - j) j.$$

The first equation was found earlier by Pandharipande and the second by Pandharipande and Graber ([GP2]); their proofs used an: analysis of the divisors on $M_{g,n}(\mathbb{P}^1, d)$. The closed-form expression $M^0_d = d^{d-3}(2d - 2)!/d!$ follows by an easy combinatorial argument from the first equation using Cayley’s formula for the number of trees on $n$ vertices. (This formula was first proved in [CrTa]. A more general formula was stated by Hurwitz and was first proved in [GoJ]. For more on this problem, including history, see [Gor].)
Graber and Pandharipande have conjectured a similar formula for $g = 2$:

$$M_d^2 = d^2 \left( \frac{97}{136}d - \frac{20}{17} \right) M_d^1 + \sum_{j=1}^{d-1} M_j^0 M_{d-j}^2 \left( \frac{2d}{2j-2} \right) j(d-j) \left( -\frac{115}{17}j + 8d \right) + \sum_{j=1}^{d-1} M_j^1 M_{d-j}^1 \left( \frac{2d}{2j} \right) j(d-j) \left( \frac{11697}{34}j(d-j) - \frac{3899}{68}d^2 \right).$$

It is still unclear why a genus 2 relation should exist (either combinatorially or algebro-geometrically). The relation looks as though it is induced by a relation in the Picard group of the moduli space, but no such relation exists.

2.9. **Divisor theory on $\overline{M}_1(\mathbb{P}^2, d)^*$.** In [P1], Pandharipande determined the divisor theory on $\overline{M}_0(\mathbb{P}^n, d)$ (including the top intersection products of divisors). The divisor theory of $\overline{M}_1(\mathbb{P}^2, d)^*$ is more complicated. In addition to the divisor $A$ and the enumeratively meaningful boundary divisors, there are potentially three other enumeratively meaningless divisors (see [V1] Lemma 3.14):

1. points corresponding to cuspidal rational curves with a contracted elliptic tail,
2. points corresponding to a contracted elliptic component attached to two rational components, where the images of the rational components meet at a tacnode, and
3. points corresponding to contracted elliptic components attached to three rational components.

The stack $\overline{M}_1(\mathbb{P}^2, d)^*$ is smooth away from these divisors. $\overline{M}_1(\mathbb{P}^2, d)$ is unibranch at the third type of divisor; Thaddeus has shown that $\overline{M}_1(\mathbb{P}^2, d)^*$ is singular there ([T]). There are several natural questions to ask about the geometry and topology of $\overline{M}_1(\mathbb{P}^2, d)^*$. Is it smooth at the other two divisors? Is the normalization of $\overline{M}_1(\mathbb{P}^2, d)^*$ smooth? If $d = 3$, how does it compare to Aluffi’s space of complete cubics? What are the top intersection products of these divisors? (The arguments here allow us to calculate $A^a B^{3d-a}$ and $A^a B^{3d-1-a} D$ where $D$ is any boundary divisor.) What about $\overline{M}_1(\mathbb{P}^n, d)^*$?
3. “Codimension 1” Numbers

Fix a degree \(d\) and geometric genus \(g\). In [DH1], Diaz and Harris express over twenty divisors on the normalization of the Severi variety as linear combinations of \(A, B, C\), and boundaries \(\Delta_0\) and \(\Delta_{i,j}\) (and conjecture that all divisors are linear combinations). For example, if \(CU\) is the divisor of cuspidal curves, then \(CU = 3A + 3B + C - \Delta\) ([DH1] (1.1)). If \(K_W\) is the canonical bundle of the (normalization of the) Severi variety, then \(K_W = -3A/2 + 3B/2 + 11C/12 - 13\Delta/12\) ([DH1] (1.17)).

Restricting these divisors to the one-dimensional family of geometric genus \(g\) degree \(d\) plane curves through \(3d + g - 2\) general points (which misses the enumeratively meaningless divisors), we obtain recursive equations for the number of such curves with various geometric behaviors (e.g. with a tacnode, three collinear nodes, etc.). We will give examples from the literature that turn out to be immediate consequences of [DH1].

3.1. Geometric and arithmetic sectional genera of the Severi variety. We also obtain recursions for versions of the geometric and arithmetic sectional genera. Following [P3] Section 3, consider the curves \(C_d\) (the intersection of the Severi variety with \(3d + g - 2\) hyperplanes corresponding to requiring the curve to pass through \(3d + g - 2\) general points \(p_1, \ldots, p_{3d+g-2}\), \(\hat{C}_d\) (the one-parameter family of \(\overline{M}_g(\mathbb{P}^2, d)\) corresponding to requiring the image curve to pass through \(3d + g - 2\) general points), and \(\tilde{C}_d\) (the normalization of \(\hat{C}_d\)). Let the arithmetic genera of these curves be \(g_d, \hat{g}_d,\) and \(\tilde{g}_d\) respectively. There are natural maps \(\tilde{C}_d \rightarrow \hat{C}_d \rightarrow C_d\). The singularities of \(\hat{C}_d\) are simple nodes, which occur when the image curve has a simple node at one of the general points \(p_i\) ([P3] Section 3; the argument holds for any \(g\)). The singularities of \(C_d\) are the above, plus simple cusps corresponding to cuspidal curves, plus singularities of the type of the coordinate axes at the origin in \(\mathbb{C}^{i+j}\) corresponding to curves with two components (of degrees \(i, j\)) whose geometric genera add to \(g\), plus the singularities of the type of the coordinate axes in \(\mathbb{C}^{(d-1)-(g-1)}\), corresponding to irreducible curves of geometric genus \(g - 1\) ([DH1] Section 1).
Thus
\[ g_d - \hat{g}_d = CU_{d,g} + \frac{1}{2} \sum_{i+j=d \atop g_i+g_j=g} (ij - 1)(3d + g - 2)N^{i,g_i}N^{j,g_j} \]
\[ + \left( \left( \frac{d-1}{2} \right) - g \right) N^{d,g-1}, \]
(8)
where \( CU_{d,g} \) is the number of irreducible degree \( d \) geometric genus \( g \) cuspidal curves through \( 3d + g - 2 \) fixed general points, and \( N^{d,g} \) is the number of irreducible degree \( d \) geometric genus \( g \) curves through \( 3d + g - 1 \) points.

Also, Pandharipande’s genus 0 argument of \([P3]\) 3.4 works for any genus, and shows that
\[ \hat{g}_d - \tilde{g}_d = (3d + g - 2)NP_{d,g} \]
(9)
where \( NP_{d,g} \) is the number of irreducible degree \( d \) geometric genus \( g \) plane curves through \( 3d + g - 3 \) fixed general points with a node at another fixed point.

The arithmetic (resp. geometric) sectional genus of a variety \( V \subset \mathbb{P}^e \) of dimension \( e \) is defined to be the arithmetic (resp. geometric) genus of the curve obtained by intersecting \( V \) with \( e - 1 \) general hyperplanes.

**Proposition 6.** The geometric sectional genus is \( \hat{g}_d \).

**Proof.** From \([DH1]\) Section 1, the only codimension 1 singularities of the Severi variety \( V^{d,g} \) are those corresponding (generically) to cuspidal curves and curves with \( \delta + 1 \) nodes (\( \delta := \binom{d-1}{2} - g \)), and the singularities are as described above. If \( V^{d,g} \) is intersected with (special) hyperplanes corresponding to requiring the curve to pass through various generally chosen fixed points, the intersection picks up new singularities, corresponding to curves with a node at one of the fixed points. Hence the geometric sectional genus is the genus of the partial normalization of \( C_d \) corresponding to normalizing the singularities corresponding to cuspidal and \( \ delta + 1 \)-nodal curves, which is the arithmetic genus of \( \hat{C}_d \).

**Notational caution:** In \([P3]\), \( \hat{g}_d \) is called the “arithmetic genus”.

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3.2. **Genus 0.** Clearly, \(|A| = R_d\). By (3),
\[
|B| = -\frac{R_d}{d} + \frac{1}{2d} \sum_{i+j=d} \left(\frac{3d-2}{3i-1}\right)i^2j^2R_iR_j.
\]
It is simple to show (e.g. [P1] Lemma 2.1.2) that \(C = -\Delta\), so
\[
|C| = -\frac{1}{2} \sum_{i+j=d} \left(\frac{3d-2}{3i-1}\right)ijR_iR_j.
\]
Note that Kontsevich’s recursion (2) can be rewritten as
\[
9(d-2)A = 3(d+2)B + 2dC
\]
(or \(\pi_*(3D + \omega) \cdot (3(d-2)D - 2d\omega)\) restricted to the one-parameter family is 0).

The formula of Katz-Qin-Ruan for the number of degree \(d\) triple-pointed rational curves ([KQRu], Lemma 3.2) can be rewritten as
\[
(d^2 - 6d + 10)|A|/2 - (d - 6)|B|/2 + |C|
\]
which is the \(g = 0\) case of [DH1] (1.3). Pandharipande’s formula for the number of degree \(d\) rational cuspidal curves ([P1] Prop. 5) can be rewritten as \(3|A| + 3|B| + |C| - |\Delta|\), which is the \(g = 0\) case of [DH1] (1.1). Ran’s formula for the cuspidal number ([R3] Theorem (ii) (2)) yields the same numbers for small \(d\), and presumably is the same formula after a substitution.

By adjunction, the geometric sectional genus \(\hat{g}_d\) of the Severi variety is given by \(2\hat{g}_d - 2 = |K_W + (3d-2)A|\). The formula of Pandharipande for \(\hat{g}_d\) ([P3] Section 3.2) can be rewritten as
\[
2\hat{g}_d - 2 = (-3|A|/2 + 3|B|/2 + 11|C|/12 - 13|\Delta|/12) + (3d - 2)|A|,
\]
which is the \(g = 0\) case of [DH1] (1.17). Pandharipande then computes the arithmetic sectional genus \(g_d\) using (8). His computation of \(\hat{g}_d\) by other means gives his recursive formula for \(NP_d\) (mentioned in Subsection 2.2) via (9).

3.3. **Genus 1.** Clearly \(|A| = E_d\) and
\[
|\Delta| = \left(\frac{d-1}{2}\right)R_d + \sum_{i+j=d} \frac{ij(3d-1)}{3i-1}R_iE_j.
\]
From Subsection 2.4, $B = \frac{d}{12} \Delta_0 + \sum_i i \Delta_{0,i}$, so

$$|B| = \frac{d}{12} \binom{d-1}{2} R_d + \sum_{i+j=d} i^2 j \binom{3d-1}{3i-1} R_i E_j.$$

From the description of $\omega$ in Subsection 2.4,

$$|C| = - \sum_{i+j=d} ij \binom{3d-1}{3i-1} R_i E_j.$$

Note that the Eguchi-Hori-Xiong recursion can be rewritten as $9A - 3B - 2C = 0$ (or $\pi_* (3D + \omega) \cdot (3D - 2\omega)$ restricted to the one-parameter family is numerically 0, cf. (10)).

Ran’s formula for the number of degree $d$ cuspidal elliptic curves ([R3] Theorem (ii) (3)) can be rewritten as $|3A + 3B + C - \Delta|$, which is the $g = 1$ case of [DH1] (1.1). Call this number $CU_{d,1}$.

Using [DH1] as in the genus 0 case, we find the geometric sectional genus of the Severi variety $\hat{g}_d$:

$$2\hat{g}_d - 2 = (-3|A|/2 + 3|B|/2 + 11|C|/12 - 13|\Delta|/12) + (3d - 1)|A|,$$

$$= \left(3d - \frac{5}{2}\right) E_d + \left(\frac{3d - 26}{24}\right) \binom{d-1}{2} R_d$$

$$+ \sum_{i+j=d} ij \binom{3d-1}{3i-1} R_i E_j \left(\frac{3}{2i-2}\right).$$

This formula is identical to that of Ran’s Theorem (ii) of [R3]. Via (8), this yields a recursion for the arithmetic sectional genus of the Severi variety $g_d$:

$$g_d = \hat{g}_d + CU_{d,1} + \sum_{i+j=d} (ij - 1) \binom{3d-1}{3i-1} R_i E_j + \left(\binom{d-1}{2} - 1\right) R_d.$$

The values of $\hat{g}_d$ for $3 \leq d \leq 7$ are: 0, 486, 410439, 395296561, 534578574561. The values of $g_d$ for $3 \leq d \leq 7$ are: 0, 2676, 1440874, 1117718773, 1317320595961.

3.4. Genus 2. Let $T_d$ be the number of irreducible degree $d$ geometric genus 2 plane curves through $3d+1$ fixed general points ($d > 2$). From [R2] or [CH], the numbers $|A|$ and $|B|$ can be found (the latter by computing $|TL| = |A| + |B|$, the number of irreducible geometric genus 2 plane curves through $3d$ points tangent to a fixed line). The number $|A|$ can be computed more easily by the recursion of Belorousski and
Pandharipande \[\text{[BeP]}\]. (Their ideas should also lead to a recursive calculation for \(|B|\).) Also,

\[|\Delta| = \left(\left(\frac{d-1}{2}\right) - 1\right) E_d + \sum_{i+j=d} \frac{ij}{3i-1} \left(3d E_i T_j + \frac{1}{2} \left(\frac{3d}{3i-1}\right) E_i E_j\right).\]

To compute \(|C|\), consider the family of genus 2 curves to be pulled back from the universal curve over the moduli stack \(\mathcal{M}_2\), blown up at a finite number of points (corresponding to the points in the family where the curve is a genus 2 curve and a genus 0 curve intersecting at a node). If \(\rho: \mathcal{U} \to \mathcal{M}_2\) is the universal curve over \(\mathcal{M}_2\), and \(\omega_\rho\) is the relative dualizing sheaf, then by \[\text{[M]}\] (8.5),

\[\rho_*(\omega_\rho^2) = (\delta_0 + 7\delta_1)/5\]

where \(\delta_0\) is the divisor corresponding irreducible nodal curves and \(\delta_1\) is the divisor corresponding to reducible nodal curves (with each component of genus 1). Hence \(|C|\) can be expressed in terms of previously-known quantities:

\[|C| = \frac{1}{5} \left(\left(\frac{d-1}{2}\right) - 1\right) E_d + \frac{7}{10} \sum_{i+j=d} \frac{ij}{3i-1} \left(3d E_i T_j - \sum_{i+j=d} \frac{3d}{3i-1} R_i U_j\right).\]

Examples are given at the end of the section.

3.5. Genus 3. Once again, \(|A|\) and \(|B|\) can be calculated by the algorithm of \[\text{[R2]}\] or \[\text{[CH]}\], and \(|\Delta|\) can be inductively calculated. Graber has found a recursive method of counting the number of genus \(g\) hyperelliptic plane curves through \(3d+1\) general points (\[\text{[R3]}\]) by relating these numbers to the Gromov-Witten invariants of the Hilbert scheme of two points in the plane. (The algorithm is effective, and maple code is available.) Call the genus 3 hyperelliptic numbers \(H_d\); the smallest non-zero values are \(H_5 = 135\), \(H_6 = 3929499\), \(H_7 = 23875461099\) (\[\text{[R3]}\]). If \(h\) is the reduced divisor of the hyperelliptic locus on the stack \(\mathcal{M}_3\), then \(h = 9\lambda - \delta_0 - 3\delta_1\) (see \[\text{[H]}\] appendix for explanation and proof). As in the genus 2 case, if \(\rho\) is the structure map of the universal curve over \(\mathcal{M}_3\), \(\rho_*(\omega_\rho^2) = 12\lambda - \delta_0 - \delta_1\) (see \[\text{[M]}\] p. 306), so \(\rho_*(\omega_\rho^2) = (4h + \delta_0 + 9\delta_1)/3\). Hence

\[|C| = \frac{4}{3} H_d + \frac{1}{3} \left(\left(\frac{d-1}{2}\right) - 2\right) T_d + \sum_{i+j=d} \frac{ij}{3i-1} \left(3 \left(\frac{3d+1}{3i}\right) E_i T_j - \left(\frac{3d+1}{3i-1}\right) R_i U_j\right).\]
In this way, all codimension 1 numbers for genus 2 and 3 curves can be computed. As examples, for $4 \leq d \leq 6$, $|A|$, $|B|$, $|C|$, $|\Delta|$, and $|TL|$ are given as well as $|CU|$, the number of cuspidal curves, and $\hat{g}$ and $g$, the geometric and arithmetic sectional genera of the Severi variety.

|       | $g = 2$              |       | $g = 3$              |
|-------|----------------------|-------|----------------------|
|       | $d = 4$              | $d = 5$ | $d = 6$              | $d = 4$ | $d = 5$ | $d = 6$ |
| $|A|$  | 27                   | 36855  | 58444767             | 1       | 7915    | 34435125 |
| $|B|$  | 117                  | 166761 | 268149471            | 5       | 41665   | 182133909 |
| $|C|$  | 90                   | 75852  | 73644975             | 9       | 48840   | 154231695 |
| $|\Delta|$ | 450                 | 447300 | 547180713            | 27      | 147900  | 474418485 |
| $|TL|$ | 144                  | 203616 | 326594238            | 6       | 49580   | 216569034 |
| $|CU|$ | 72                   | 239400 | 506246976            | 0       | 49680   | 329520312 |
| $\hat{g}$ | 28                | 166321 | 420645826            | 0       | 30906   | 251620624 |
| $g$   | 325                  | 762994 | 1410743814           | 0       | 191511  | 995749561 |
Appendix A. A recursive formula for \( NP(a, b) \)

\[
NP(a, b + 1) = \frac{d - 1}{d} NP(a + 1, b) + \sum_{i+j=d} \frac{ij}{2d} \left[ \sum_{a_i+a_j=a+2, \quad b_i+b_j=b} \binom{a}{a_i-1} \binom{b}{b_i} \sum_{a_i+a_j=a, \quad b_i+b_j=b} \binom{a}{a_i} \binom{b}{b_i} \right.
\]

\[
\times (ij - 1) R_i(a_i, b_i) R_j(a_j, b_j) + 2 \sum_{a_i+a_j=a+3, \quad b_i+b_j=b-1} \binom{a}{a_i} \binom{b}{b_i} (ij - 1) R_i(a_i, b_i) R_j(a_j, b_j)
\]

\[
+ 4b \sum_{a_i+a_j=a+1, \quad b_i+b_j=b-1} \binom{a}{a_i} \binom{a-1}{a_i} \binom{b}{b_i} R_i(a_i, b_i) R_j(a_j, b_j)
\]

\[
+ 4b \sum_{a_i+a_j=a+1, \quad b_i+b_j=b-1} \binom{a-1}{a_i} \binom{b}{b_i} iNP_i(a_i, b_i) R_j(a_j, b_j)
\]

\[
+ 4b \sum_{a_i+a_j=a+1, \quad b_i+b_j=b-1} \binom{a}{a_i} \binom{b-1}{b_i} iNP_i(a_i, b_i) R_j(a_j, b_j)
\]

\[
+ 4b \sum_{a_i+a_j=a+1, \quad b_i+b_j=b-1} \binom{a}{a_i} \binom{b-2}{b_i} R_i(a_i, b_i) R_j(a_j, b_j)
\]

\[
+ 8\binom{b}{2} \sum_{a_i+a_j=a+2, \quad b_i+b_j=b-2} \binom{a}{a_i-1} \binom{b-2}{b_i} R_i(a_i, b_i) NP_j(a_j, b_j)
\]

In each sum in the large bracket, it is assumed that \( a_i + b_i = 3i - 1 \) if \( R_i(a_i, b_i) \) appears in the sum, and \( a_i + b_i = 3i - 3 \) if \( NP_i(a_i, b_i) \) appears. The same assumption is made when \( i \) is replaced by \( j \).

The large square bracket corresponds to maps from reducible curves. (To avoid confusion: the “image of the node” refers to the image of the node of the source curve. The “fixed node” refers to the node of the image that is required to be at a fixed point.) Zero, one, or two tangent lines can pass through the image of the node of the source curve. The two branches through the fixed node can belong to the
same component, or one can belong to each. The table below identifies which possibilities correspond to which sum in the large bracket.

| sum       | number of tangent lines through image of node of source | number of irreducible components through fixed node |
|-----------|--------------------------------------------------------|-----------------------------------------------------|
| first     | 0                                                      | 2                                                   |
| second    | 0                                                      | 1                                                   |
| third     | 1                                                      | 2                                                   |
| fourth and fifth | 1                                          | 1                                                   |
| sixth     | 2                                                      | 2                                                   |
| seventh   | 2                                                      | 1                                                   |

REFERENCES

[A1] P. Aluffi, *The characteristic numbers of smooth plane cubics*, in *Algebraic geometry Sundance 1986*, A. Holme and R. Speiser eds., LNM 1311, Springer-Verlag: New York, 1988.

[A2] P. Aluffi, *How many smooth plane cubics with given j-invariant are tangent to 8 lines in general position?*, in *Enumerative algebraic geometry*, S. Kleiman and A. Thorup eds., Contemp. Math. 123, AMS: Providence, 1991.

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, New York, 1984.

[BeP] P. Belorousski and R. Pandharipande, personal communication.

[CH] Caporaso-Harris, *Counting plane curves of any genus*, preprint 1996. [alg-geom/9608025]

[CrTa] M. Crescimanno and W. Taylor, *Large N phases of chiral QCD*2, Nuclear Phys. B 437 (1995), no. 1, 3–24.

[DH1] S. Diaz and J. Harris, *Geometry of the Severi variety*, Trans. AMS 309 (1988), no. 1, 1–34.

[DH2] S. Diaz and J. Harris, *Geometry of Severi varieties II: Independence of divisor classes and examples*, in *Algebraic Geometry Sundance 1986*, A. Holme and R. Speiser eds., LNM 1311, Springer-Verlag: New York 1988.

[EK1] L. Ernström and G. Kennedy, *Recursive formulas for the characteristic numbers of rational plane curves*, J. of Alg. Geo. 7 (1998), 141–181.

[EK2] L. Ernström and G. Kennedy, *Contact cohomology of the projective plane*, preprint 1997. [alg-geom/9703013]

[FP] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, preprint 1996. [alg-geom/9608011]

[Ge] E. Getzler, *Topological recursion relations in genus 2*, preprint 1998. [alg-geom/9801003]

[GorL] V. V. Goryunov and S. K. Lando, *On enumeration of meromorphic functions of the line*, preprint.
[V1] R. Vakil, *The enumerative geometry of rational and elliptic plane curves in projective space*, preprint 1997, [alg-geom/9709007](http://arxiv.org/abs/alg-geom/9709007).

[V2] R. Vakil, *Characteristic numbers of rational and elliptic curves in projective space*, manuscript in preparation.