ON THE FINITE-DIMENSIONAL REPRESENTATIONS OF
THE DOUBLE OF THE JORDAN PLANE

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Abstract. We continue the study of the Drinfeld double of the Jordan plane, denoted by \( \mathcal{D} \) and introduced in [AP]. The simple finite-dimensional modules were computed in [ADP]; it turns out that they factorize through \( U(\mathfrak{sl}_2(k)) \). Here we introduce the Verma modules and the category \( \mathcal{O} \) for \( \mathcal{D} \), which have a resemblance to the similar ones in Lie theory but induced from indecomposable modules of the 0-part of the triangular decomposition. Accordingly, there is the notion of highest weight rank (hw-rk). We classify the indecomposable modules of hw-rk one and find families of hw-rk two. The Gabriel quiver of \( \mathcal{D} \) is computed implying that it has a wild representation type.

Contents

1. Introduction
2. Preliminaries
3. Highest weight modules
4. Modules of highest weight rank one
5. Modules of highest weight rank two
6. Extensions of simple modules
References

1. Introduction

Let \( k \) be an algebraically closed field of characteristic 0. There were substantial advances in the ongoing classification of Nichols algebras over abelian groups with finite Gelfand-Kirillov dimension (abbreviated as GKdim), an important step towards the classification of Hopf algebras with finite GKdim, see [AAH, AAM, AGI]. Recall that Nichols algebras of diagonal type correspond to infinitesimal braidings that are semisimple as Yetter-Drinfeld

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modules (over abelian groups). In the context of finite GKdim, Nichols algebras with non-semisimple infinitesimal braidings appear naturally. The well-known Jordan plane, denoted by $J$, is a paradigmatic example.

Quantized enveloping algebras can be defined via Drinfeld doubles of (bosonizations of) suitable Nichols algebras of diagonal type. A natural question is to investigate the behaviour of the Drinfeld doubles of other Nichols algebras, specifically those with non-semisimple infinitesimal braidings. In this direction, we studied in [AP, ADP] a Hopf algebra $D$ thought of as the Drinfeld double of the bosonization $J\#k\mathbb{Z}$, which differs from quantized enveloping algebras for some distinctive features:

- **[AP]** The Hopf algebra paired with $J\#k\mathbb{Z}$ in order to build the double has the form $J^d\#U(h)$ where $J^d$ is the graded dual of $J$ and $\dim h = 1$.
- **[AP]** The Hopf algebra $D$ is not simple and, in fact, it fits into an abelian exact sequence of Hopf algebras $O(G)\xrightarrow{\pi} D \xrightarrow{\pi} U(sl_2(k))$ where $O(G)$ is the algebra of regular functions on $G = (G_a \times G_a) \rtimes G_m$.
- **[AP]** The analogue of the Cartan subalgebra of $D$ splits as $D^0 \simeq U(h) \otimes k\Gamma$.
- **[ADP]** The finite-dimensional simple modules over $D$ are in bijective correspondence with those of $U(sl_2(k))$ via $\pi$.

The next step is to understand the indecomposable modules. We observe that the action of $D^0$ is not semisimple, what leads us to introduce Verma modules inducing from indecomposable $D^0$-modules; highest weight modules and the category $O$ they belong to. The notions of hw-rk and hw-series arise naturally. See Section 3. In Sections 4 and 5 we study the indecomposable modules with hw-rk one and two respectively. In the case of hw-rk = 1 we get the complete classification by the family of uniserial indecomposable modules $T(n,m)$, see Theorem 4.7. In Section 6 we classify extensions of simple modules using the families described previously. Thus we compute the Gabriel quiver of $D$ and deduce that $D$ has wild representation type, see Proposition 6.5. Section 2 is devoted to preliminaries on the algebra $D$.

There are convincing reasons to believe that the representation theory of the Drinfeld doubles of many of the Nichols algebras discovered in [AAH, AAM] will have the features described above.

**Conventions.** If $\ell < n \in \mathbb{N}_0$, we set $I_{\ell,n} = \{\ell, \ell + 1, \ldots, n\}$, $I_n = I_{1,n}$. If $Y$ is a subobject of an object $X$ in a category $\mathcal{C}$, then we write $X \leq Y$.

Let $A$ be an algebra. Given $a_1, \ldots, a_n \in A$, $n \in \mathbb{N}$, $k\langle a_1, \ldots, a_n \rangle$ denotes the subalgebra generated by $a_1, \ldots, a_n$. Let $A\Mod$ (respectively $A\mod$, irrep $A$) denote the category of left $A$-modules (respectively, the full subcategory of finite-dimensional ones, the set of isomorphism classes of simple objects in $A\mod$). Often we do not distinguish a class in Irrep $A$ and one of its representatives. If $M \in A\Mod$ and $m_1, \ldots, m_n \in M$, $n \in \mathbb{N}$, then $\langle m_1, \ldots, m_n \rangle$ denotes the submodule generated by $m_1, \ldots, m_n$. Given a
Representations of the double of the Jordan plane

Subalgebra $B$, of $A$, $\text{Ind}^A_B : B\text{Mod} \rightarrow A\text{Mod}$ and $\text{Res}^B_A : A\text{Mod} \rightarrow B\text{Mod}$ denote the induction and restriction functors, e.g. $\text{Ind}^A_B(M) = A \otimes_B M$.

Let $L$ be a Hopf algebra. The kernel of the counit $\varepsilon$ is denoted $L^+$, the antipode (always assumed bijective) by $S$, the space of primitive elements by $\mathcal{P}(L)$ and the group of group-likes by $G(L)$. The space of $(g,h)$-primitives is $\mathcal{P}_{g,h}(L) = \{x \in L : \Delta(x) = x \otimes h + g \otimes x\}$ where $g,h \in G(L)$. The category of Yetter-Drinfeld modules over $L$ is denoted by $L\text{YD}$. We refer to [R] for unexplained terminology on Hopf algebras.

2. Preliminaries

2.1. The double of the Jordan plane. The well-known Jordan plane, the quadratic algebra $J = \mathbb{k}\langle x, y | xy - yx - \frac{1}{2}x^2 \rangle$, is a braided Hopf algebra where $x$ and $y$ are primitive. The Hopf algebra $D$ was introduced in [AP, Proposition 2.3], where it is denoted $\tilde{D}$. See [AP, ADP] for properties of $D$, some of which are listed below. The algebra $D$ is presented by generators $u, v, \xi, g^{\pm 1}, x, y$ and relations

\begin{align*}
(1) & \quad g^{\pm 1} g^{\mp 1} = 1, \quad \xi g = g \xi, \\
& \quad gx = xg, \quad gy = yg + xg, \quad \xi y = y \xi - 2y, \quad \xi x = x \xi - 2x, \\
& \quad ug = gu, \quad vg = gv + gu, \quad v \xi = \xi v - 2v, \quad u \xi = \xi u - 2u, \\
& \quad yx = xy - \frac{1}{2} x^2, \quad vu = uv - \frac{1}{2} u^2, \\
(2) & \quad ux = xu, \quad vx = xv + (1 - g) + xu, \\
& \quad uy = yu + (1 - g), \quad vy = yv + \frac{1}{2} g \xi + yu. 
\end{align*}

Notice that we have replaced the generator $\zeta$ by $\xi = -2\zeta$ in the presentation of [AP] and adjusted the relations accordingly. The Hopf algebra structure is determined by $g \in G(D), u, \xi \in \mathcal{P}(D), x, y \in \mathcal{P}_{g,1}(D)$ and

$$\Delta(v) = v \otimes 1 + 1 \otimes v - \frac{1}{2} \xi \otimes u.$$ 

The following set is a PBW-basis of $D$:

$$B = \{x^n y^r \xi^m \xi^k u^i v^j : i, j, k, n, r \in \mathbb{N}_0, \quad m \in \mathbb{Z}\}.$$ 

We shall consider the following subalgebras of $D$:

$$D^{<0} := \mathbb{k}\langle x, y \rangle, \quad D^0 := \mathbb{k}\langle g^{\pm 1}, \xi \rangle, \quad D^{\leq 0} := \mathbb{k}\langle g^{\pm 1}, \xi, x, y \rangle, \quad D^{>0} := \mathbb{k}\langle u, v \rangle$$

and

$$D^\leq := \mathbb{k}\langle g^{\pm 1}, \xi, u, v \rangle.$$ 

The algebra $D$ has a $\mathbb{Z}$-grading $D = \bigoplus_{n \in \mathbb{Z}} D^{[n]}$ given by

\begin{align*}
(4) & \quad \deg x = \deg y = -2, \quad \deg u = \deg v = 2, \quad \deg g = \deg \xi = 0. 
\end{align*}

The algebra $D$ has a triangular decomposition $D \simeq D^{<0} \otimes D^0 \otimes D^{>0}$. 

2.2. **An exact sequence of Hopf algebras.** Let \( \mathcal{O} := k\langle x, u, g\rangle \); this is a commutative Hopf subalgebra of \( \mathcal{D} \), hence \( \mathcal{O} \cong \mathcal{O}(G) \), where \( G \) is the algebraic group as in the Introduction. Let \( e, f, h \) be the Chevalley generators of \( \mathfrak{sl}_2(k) \), i.e. \( [e, f] = h \), \( [h, e] = 2e \), \( [h, f] = -2f \). The Hopf algebra map \( \pi: \mathcal{D} \to U(\mathfrak{sl}_2(k)) \) determined by
\[
(5) \quad \pi(v) = \frac{1}{2} e, \quad \pi(y) = f, \quad \pi(\xi) = h, \quad \pi(u) = \pi(x) = \pi(g-1) = 0,
\]
induces an isomorphism of Hopf algebras \( \mathcal{D}/\mathcal{D}^+ \cong U(\mathfrak{sl}_2(k)) \). Thus we have an exact sequence as mentioned in the Introduction.

2.3. **Simple modules.** The simple objects in \( \mathcal{D} \text{-mod} \) are classified in \[ADP\]; the starting point is the following fact that we will use later.

**Proposition 2.1.** \[ADP\] Proposition 3.9] If \( M \in \mathcal{D} \text{-mod} \), then \( g-1, x \) and \( u \) act nilpotently on \( M \).

Let \( L(n) \) be the simple \( \mathfrak{sl}_2(k) \)-module with highest weight \( n \). Then \( L(n) \) becomes a simple \( \mathcal{D} \)-module via the projection \( \pi \) in \[5\]. Precisely, \( L(n) \) has a basis \( \{z_{n,0}, \ldots, z_{n,n}\} \) where the action is given by
\[
(6) \quad \xi \cdot z_{n,i} = (n-2i)z_{n,i}, \quad v \cdot z_{n,i} = \frac{i(n-i+1)}{2}z_{n,i-1}, \quad y \cdot z_{n,i} = z_{n,i+1},
\]
\[
x \cdot z_{n,i} = 0, \quad u \cdot z_{n,i} = 0, \quad g \cdot z_{n,i} = z_{n,i},
\]
i.e.\( \in \mathbb{N}_0 \), where \( z_{n,-1} = z_{n,n+1} = 0 \) by convention.

**Theorem 2.2.** \[ADP\] 3.11] The family \( \{L(n)\}_{n \in \mathbb{N}_0} \) parametrizes \( \text{irrep} \mathcal{D} \). \( \square \)

**Remark 2.3.** If \( M \in \mathcal{D} \text{-mod} \), then \( y \) and \( v \) act nilpotently on \( M \) and the eigenvalues of the action of \( \xi \) are integers, but the action of \( \xi \) is not necessarily semisimple (consider a Jordan-Hölder series of \( M \) and apply Theorem 2.2).

3. **Highest weight modules**

3.1. **Weight decompositions.** Proposition \[2.1\] and Remark \[2.3\] lead us to the following considerations. Given \( P \in \mathcal{D}_0 \text{-Mod} \) and \( n \in \mathbb{Z} \), we set
\[
P^{(n)} = \{m \in P: (\xi - n)^a \cdot m = 0, (g-1)^b \cdot m = 0 \text{ for some } a, b \in \mathbb{N}_0\}.
\]
By a standard argument, see e.g. \[D\] 1.2.13], the sum \( \sum_{n \in \mathbb{Z}} P^{(n)} \) is direct. Let \( \mathcal{D}_0 \mathcal{S} \) be the full subcategory of \( \mathcal{D}_0 \text{-Mod} \) consisting of those \( P \in \mathcal{D}_0 \text{-Mod} \) such that \( P = \bigoplus_{n \in \mathbb{Z}} P^{(n)} \). Notice that \( \mathcal{D}_0 \text{-mod} \) is a subcategory of \( \mathcal{D}_0 \mathcal{S} \).

Given \( M \in \mathcal{D} \text{-Mod} \), we set by abuse of notation
\[
M^{(n)} = (\text{Res}_{\mathcal{D}}^{\mathcal{D}_0}(M))^{(n)},
\]
The weights of \( M \in \mathcal{D} \text{-Mod} \) are the elements of \( \Pi(M) := \{n \in \mathbb{Z}: M^{(n)} \neq 0\} \); the \( M^{(n)} \)'s are called weight subspaces even when they are 0.

**Definition 3.1.** A module \( M \in \mathcal{D} \text{-Mod} \) is **suitably graded** if
\[
M = \bigoplus_{n \in \mathbb{Z}} M^{(n)} \quad \text{and} \quad \dim M^{(n)} < \infty \quad \text{for all } n \in \mathbb{Z}.
\]
Let \( \mathcal{D} \mathcal{S} \) be the full subcategory of \( \mathcal{D} \text{-Mod} \) of suitably graded modules.
Remark 3.2. (i) From the defining relations we get that

\[ x \cdot M^{(n)} \subset M^{(n-2)} \supset y \cdot M^{(n)}, \quad g \cdot M^{(n)} = M^{(n)} \supset \xi \cdot M^{(n)}, \]

\[ u \cdot M^{(n)} \subset M^{(n+2)} \supset v \cdot M^{(n)}, \quad n \in \mathbb{Z}. \]

(ii) Morphisms of \( \mathcal{D} \)-modules preserve the weight subspaces. Thus submodules and quotients of suitably graded modules are suitably graded.

Given an exact sequence of \( \mathcal{D} \)-modules \( N \rightarrowtail M \twoheadrightarrow S \), the sequence

\[ N^{(n)} \rightarrowtail M^{(n)} \twoheadrightarrow S^{(n)} \]

of \( \mathcal{D}^0 \)-modules is exact for any \( n \in \mathbb{Z} \).

(iii) Any \( M \in \mathcal{D} \text{mod} \) is suitable graded by Remark 2.3. If \( n \in \Pi(M) \), then \( \dim M^{(n)} = \dim M^{(-n)} \) and \( \{n, n-2, n-4, \ldots, 2-n, -n\} \subseteq \Pi(M) \).

Proof. Let \( \lambda \in k \). Then \( (\xi - \lambda + 2)y = y(\xi - \lambda) \) by (2); hence by induction \( (\xi - \lambda + 2)^ay = y(\xi - \lambda)^a \) for any \( a \in \mathbb{N} \). Also, \( (g - 1)y = y(g - 1) + gx \), hence by induction \( (g - 1)^ay = y(g - 1)^a + axy(g - 1)^{a-1} \) for any \( a \in \mathbb{N} \). Therefore \( y \cdot M^{(n)} \subset M^{(n-2)} \); the rest is similar. The proof of (iii) follows from the representation theory of \( \mathfrak{sl}_2 \).

Let \( M \in \mathcal{D} \mathcal{S} \mathcal{G} \). Given \( n \in \mathbb{Z} \), one identifies the dual \( (M^{(n)})^* \) with the subspace \( \left( \bigoplus_{\ell \neq a \in \mathbb{Z}} M^{(a)} \right)^{-1} = \{f \in M^*: f_{M(a)} = 0, a \neq n\} \) of \( M^* \). Clearly the sum of the various duals \( (M^{(n)})^* \) is direct. The graded dual of \( M \) is

\[ M^\vee = \bigoplus_{n \in \mathbb{Z}} (M^{(n)})^* \hookrightarrow M^*. \]

We need the formula for the antipode:

\[ S(g) = g^{-1}, \quad S(x) = -g^{-1}x, \quad S(y) = -g^{-1}y, \]

\[ S(\xi) = -\xi, \quad S(u) = -u, \quad S(v) = -v - \frac{1}{2} \xi u. \]

Lemma 3.3. If \( M \in \mathcal{D} \mathcal{S} \mathcal{G} \), then \( M^\vee \in \mathcal{D} \mathcal{S} \mathcal{G} \) and

\[ (M^\vee)^{(n)} = (M^{(-n)})^*, \quad n \in \mathbb{Z}. \]

Proof. Fix \( q \in \mathbb{Z} \). Notice that \( (M^{(q)})^* \) is stable under the action of \( g \) and \( \xi \). We claim that

\[ x \cdot (M^{(q)})^* \subset (M^{(q+2)})^* \supset y \cdot (M^{(q)})^*, \]

\[ u \cdot (M^{(q)})^* \subset (M^{(q+2)})^* \supset v \cdot (M^{(q)})^*. \]

Indeed, if \( f \in (M^{(q)})^* \), \( p \in \mathbb{Z} \) and \( m \in M^{(p)} \), then

\[ \langle x \cdot f, m \rangle = -\langle f, x \cdot m \rangle = 0 \text{ unless } x \cdot m \in M^{(q)} \text{ that is } p - 2 = q. \]

The other inclusions are similar. Thus \( M^\vee \) is a submodule of \( M^* \). We next claim that \( (M^{(q)})^* \subset (M^*)^{(-q)} \). Since \( \dim M^{(q)} < \infty \), there exist \( a, b \in \mathbb{N} \) such that \( (\xi - q)^a \) and \( (g - 1)^b \) act by 0 on \( M^{(q)} \). Pick \( f \in (M^{(q)})^* \). By (8) we have, for any \( p \in \mathbb{Z} \) and \( m \in M^{(p)} \), that

\[ \langle (\xi + q) \cdot f, m \rangle = \langle f, (-\xi + q) \cdot m \rangle \]
\[
\langle (\xi + q)^a \cdot f, m \rangle = (-1)^a \langle f, (\xi - q)^a \cdot m \rangle = 0; \\
\langle (g - 1) \cdot f, m \rangle = \langle f, (g^{-1} - 1) \cdot m \rangle \\
\implies \langle (g - 1)^b \cdot f, m \rangle = (-1)^b \langle f, (g - 1)^b \cdot (g^{-b} \cdot m) \rangle = 0
\]

The claim follows. Together with the first claim, this implies \(\square\).

**Remark 3.4.** If \(M, N \in \mathcal{DS}\mathcal{G}\), then \(M \otimes N = \oplus_{n \in \mathbb{Z}} (M \otimes N)^{(n)}\) and
\[
(M \otimes N)^{(n)} = \oplus_{p+q=n}(M^{(p)} \otimes N^{(q)}).
\]
Notice however that \(\dim (M \otimes N)^{(p+q)}\) is not necessarily finite.

**Proof.** It suffices to show that \(M^{(p)} \otimes N^{(q)} \subset (M \otimes N)^{(p+q)}\) for any \(p, q \in \mathbb{Z}\); this follows from the equalities
\[
\Delta(\xi - \lambda)^a = \sum_{0 \leq c \leq a} \binom{a}{c} (\xi - \mu)^c \otimes (\xi - \nu)^{a-c}, \quad \mu + \nu = \lambda; \\
\Delta(g - 1)^b = \sum_{0 \leq d \leq b} \binom{b}{d} (g - 1)^d \otimes g^d(g - 1)^{b-d}, \quad a, b \in \mathbb{N}. \quad \square
\]

### 3.2. Highest weight modules.

Let us now consider \(P \in \mathcal{D}^{\text{mod}}\) such that \(P = P^{(n)}\) for some \(n \in \mathbb{Z}\). Then \(P\) becomes a \(\mathcal{D}^{\geq 0}\)-module by \(u \cdot m = 0, v \cdot m = 0, m \in P\). We define the Verma module
\[
\mathbb{M}(P) := \text{Ind}_{\mathcal{D}^{\geq 0}}^{\mathcal{D}}(P) \simeq \mathcal{D} \otimes_{\mathcal{D}^{\geq 0}} P.
\]
Thus \(\mathbb{M}(P) \simeq \mathcal{D}^{<0} \otimes_k P\) as vector space.\(^1\)

**Remark 3.5.** The Verma module \(\mathbb{M}(P)\) is suitable graded. Indeed we have \(P = P^{(n)} \subset \mathbb{M}(P)^{(n)}\) by definition, hence \(y^i x^j P \subset \mathbb{M}(P)^{(n-2(i+j))}\) for any \(i, j \in \mathbb{N}_0\) by \(\square\). That is, \(\mathbb{M}(P)^{(n-2k)} = \oplus_{i+j=k} y^i x^j P\) and
\[
\mathbb{M}(P) = \bigoplus_{k \in \mathbb{N}_0} \mathbb{M}(P)^{(n-2k)}.
\]

**Definition 3.6.** Let \(\mathcal{O}\) be the full subcategory of \(\mathcal{DS}\mathcal{G}\) consisting of those \(M\) such that \(\Pi(M)\) is bounded above. For \(M \in \mathcal{O}\), its highest weight is \(\text{hw} M := \text{supp} \Pi(M)\); we introduce \(\text{hw-rk} M = \dim M^{(\text{hw} M)}\).

**Definition 3.7.** If \(M \in \mathcal{O}\) is generated by \(M^{(\text{hw} M)}\), then we say that \(M\) is a **highest weight module**.

**Remark 3.8.**

(i) If \(M \in \mathcal{O}\) has highest weight \(n\) and \(P := M^{(n)}\), then there is a morphism of \(\mathcal{D}\)-modules \(\Phi : \mathbb{M}(P) \to M\) which is the identity on \(P\); indeed, \(u \cdot M^{(n)} = v \cdot M^{(n)} = 0\). Furthermore, \(\text{hw}(M/\text{Im} \Phi) < \text{hw} M\).

\(^1\)Notice that the Verma modules in [ADP] are induced from one-dimensional modules where \(g\) and \(\xi\) act by arbitrary eigenvalues, thus more and less general than the previous definition.
(ii) The Verma modules $M(P)$ with $P = P^{(n)} \in \mathcal{D}_{\mathfrak{o}}\text{mod}$ are highest weight modules. Any highest weight module $M$ is the quotient of a Verma module, namely of $M(P)$ with $P = M^{(\text{hw} M)}$.

(iii) The category $\mathcal{D}_{\mathfrak{o}}\text{mod}$ is a subcategory of $\mathcal{D}$. Thus any $M \in \mathcal{D}_{\mathfrak{o}}\text{mod}$ has a unique series of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \cdots \subseteq M_T = M$ (that we call the $\text{lw-series}$) such that $M_i/M_{i-1}$ is a highest weight module, $i \in I_T$, and $\text{hw-rank} M_i/M_{i-1} > \text{hw-rank} M_{i+1}/M_i$, $i \in I_{T-1}$.

(iv) If $M, N \in \mathcal{D}$, then $M \otimes N \in \mathcal{D}$. This follows from Remark 3.4: an elementary argument shows that the set of pairs $(p, q) \in \Pi(M) \times \Pi(N)$ such that $p + q = n$ for a given $n$, is finite when both $\Pi(M)$ and $\Pi(N)$ are bounded above (or below).

3.3. Lowest weight modules. We also have the full subcategory $\mathfrak{o}$ of $\mathcal{D}_{\mathfrak{SG}}$ consisting of those $M$ such that $\Pi(M)$ is bounded below; for $M \in \mathfrak{o}$, we set $\text{lw } M := \inf \Pi(M)$ and $\text{lw-rank } M := \dim M^{(\text{lw} M)}$. If $M \in \mathcal{D}$, then $M' = \mathfrak{o}$. This gives a contravariant equivalence of categories between $\mathcal{D}$ and $\mathfrak{o}$.

Given $P \in \mathcal{D}_{\mathfrak{o}}\text{mod}$ such that $P = P^{(n)}$ for some $n \in \mathbb{Z}$, $P$ becomes a $\mathcal{D}^{\leq 0}$-module by $x \cdot m = 0, y \cdot m = 0, m \in P$. The opposite Verma module is $\overline{M}(P) := \text{Ind}_{\mathcal{D}^{\leq 0}}^{\mathcal{D}}(P) \simeq \mathcal{D} \otimes_{\mathcal{D}^{\leq 0}} P$.

Clearly $\overline{M}(P) \simeq \mathcal{D}^{>0} \otimes_k P$ as vector spaces. Then $M \in \mathfrak{o}$ is a lowest weight module if it is generated by $M^{(\text{lw} M)}$. We have the following properties.

(i) If $M \in \mathfrak{o}$ and $P := M^{(\text{lw} M)}$, then there is a morphism of $\mathcal{D}$-modules $\varphi : \overline{M}(P) \to M$, $\varphi_P = \text{id}_P$. Moreover, $\text{lw}(M/\text{Im } \varphi) > \text{lw } M$.

(ii) The opposite Verma modules are lowest weight modules and any lowest weight module is the quotient of one of them.

(iii) The category $\mathcal{D}_{\mathfrak{o}}\text{mod}$ is a subcategory of $\mathfrak{o}$ and any object belonging to $\mathfrak{o}$ and $\mathcal{D}$ is in $\mathcal{D}_{\mathfrak{o}}\text{mod}$. Thus any $M \in \mathcal{D}_{\mathfrak{o}}\text{mod}$ has a unique series of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \cdots \subseteq M_T = M$ (that we call the $\text{lw-series}$) such that $M_i/M_{i-1}$ is a lowest weight module, $i \in I_T$, and $\text{hw-rank} M_i/M_{i-1} < \text{hw-rank} M_{i+1}/M_i$, $i \in I_{T-1}$.

(iv) If $M, N \in \mathfrak{o}$, then $M \otimes N \in \mathfrak{o}$.

(v) Let $P \in \mathcal{D}_{\mathfrak{o}}\text{mod}$ such that $P = P^{(n)}$ for some $n \in \mathbb{Z}$. Then $P^*$ does not generate $\mathcal{M}(P)^\vee$.

(vi) Let $P \in \mathcal{D}_{\mathfrak{o}}\text{mod}$ such that $P = P^{(n)}$ for some $n \in \mathbb{Z}$. Then the natural map of $\mathcal{D}$-modules $\varphi : \overline{M}(P) \to \mathcal{M}(P^*)^\vee$ is not an isomorphism.

Proof of (v) Let $p \in P$, $f \in P^*$ and $a, b \in k$. Because of (iii), we have

$$
\langle (au + bv) \cdot f, x \cdot p \rangle = -\langle f, (au + bv)x \cdot p \rangle = -\langle f, (x(au + b(u + v)) + b(1 - g)) \cdot p \rangle = -\langle f, b(1 - g) \cdot p \rangle;
$$

thus, $\langle (au + bv) \cdot f, x \cdot p \rangle = 0$ if $p \in \ker(1 - g)$. Since $\ker(1 - g) \neq 0$, we conclude that $(x \cdot P)^*$ is not contained in the submodule generated by $P^*$. $\square$
Definition 4.1. The natural isomorphism $P \simeq P^{**}$ is of $D_0$-modules because $S^2_{D_0} = \text{id}$; thus $\varphi$ exists. Now $\text{Im} \varphi$ is generated by $P^{**}$, thus [vi] applies. □

4. Modules of highest weight rank one

We aim to describe indecomposable finite-dimensional highest weight modules; we start in this section with those having highest weight 1. We shall define an indecomposable module $T(n, m) \in \mathcal{P}_{n, m}$ for any $(n, m) \in \mathbb{N}_0^2$.

The only Verma modules of hw-rk = 1, with the conventions of this article, are $\mathcal{M}(n) := \mathcal{M}(\lambda_n)$, $n \in \mathbb{Z}$, where $\lambda_n \in \mathcal{P}_{230}$ has dimension 1, with basis $\{z_n\}$ and action

$$g \cdot z_n = z_n, \quad \xi \cdot z_n = nz_n, \quad u \cdot z_n = 0, \quad v \cdot z_n = 0.$$

The elements $z_{n(i,j)} := y^ix^jz_n$, $i, j \in \mathbb{N}_0$, form a basis of $\mathcal{M}(n)$. Recall that $[t]^{[k]}$ denotes the raising factorial $[t]^{[k]} := \prod_{i=1}^k (t+i-1)$ for $t \in \mathbb{k}$ and $k \in \mathbb{N}_0$. By [ADP] Lemma 2.5] the action of $D$ is explicitly given by

$$x \cdot z_{n(i,j)} = \sum_{k=0}^{i} \binom{i}{k} \frac{[i]^{[k]}}{2^k} z_{n(i-k,j+k+1)},$$

$$y \cdot z_{n(i,j)} = z_{n(i+1,j)},$$

$$g \cdot z_{n(i,j)} = \sum_{k=0}^{i} \binom{i}{k} [2]^{[k]} \frac{[i]^{[k]}}{2^k} z_{n(i-k,j+k)},$$

$$\xi \cdot z_{n(i,j)} = (n-2(i+j))z_{n(i,j)},$$

$$u \cdot z_{n(i,j)} = (-1)^{i-1} \sum_{k=1}^{i} \binom{i}{k+1} \frac{[k+1]!}{2^k} z_{n(i-1-k,j+k)},$$

$$v \cdot z_{n(i,j)} = \frac{i(n-2j-i+1)}{2} z_{n(i-1,j)} + \sum_{k=1}^{i-1} \frac{i(i-j-i+1)}{(n-k-1)2^{k+1}} z_{n(i-1-k,j+k)}.$$

It was shown in [ADP] that the Verma module $\mathcal{M}(n)$ has a unique simple quotient $\mathcal{M}(n)/\langle z_{n(0,1)}, z_{n(n+1,0)} \rangle$ which is isomorphic to $L(n)$, cf. Theorem 2.2. Now $\mathcal{M}(n)$ is presented by the generator $z_n$ with defining relations

$$g \cdot z_n = z_n, \quad \xi \cdot z_n = nz_n, \quad u \cdot z_n = 0, \quad v \cdot z_n = 0.$$

Similarly $L(n)$ is presented by $z_n$ with defining relations (11) and

$$x \cdot z_n = 0, \quad y^{n+1} \cdot z_n = 0.$$

Definition 4.1. Let $(n, m) \in \mathbb{N}_0^2$. We define $T(n, m) := \mathcal{M}(n+2m)/\mathcal{N}(n, m)$ where $\mathcal{N}(n, m)$ is the submodule of $\mathcal{M}(n+2m)$ generated by

$$z_{n+2m(0,m+1)} = x^{m+1} \cdot z_{n+2m},$$

$$z_{n+2m(n+2(m-j)+1,j)} = y^{m+2(m-j)+1} x^j \cdot z_n, \quad j \in \mathbb{N}_0.$$


Thus $T(n, m)$ is presented by $z = z_{n+2m}$ with defining relations
\begin{align*}
\tag{14} g \cdot z_{n+2m} &= z_{n+2m}, & \xi \cdot z_{n+2m} &= (n + 2m)z_{n+2m}, \\
\tag{15} u \cdot z_{n+2m} &= 0, & v \cdot z_{n+2m} &= 0, \\
\tag{16} x^{n+1} \cdot z_{n+2m} &= 0, & y^{n+2(m-j)+1} \cdot z_{n+2m} &= 0, & j \in \mathbb{I}_0, m.
\end{align*}

Let $z_{n,m,(i,j)}$ be the image of $z_{n+2m,(i,j)}$ in $T(n, m)$. For simplicity, we denote $z_{n+2m,(i,j)}$ by $z_{(i,j)}$ and $z_{n,m,(i,j)}$ by $z_{(i,j)}$.

\begin{lemma}
The set $B = \{z_{(i,j)} : j \in \mathbb{I}_0, i \in \mathbb{I}_{0, n+2(m-j)}\}$ is a basis of $T(n, m)$, thus
\[\dim T(n, m) = (m + 1)(n + m + 1).\]
\end{lemma}

\textbf{Proof.} Let $N$ be the vector subspace of $M(n + 2m)$ generated by the elements
\begin{equation}
\tag{17} z_{(i,j)} \quad \text{with either} \quad \begin{cases} 
  j \geq m + 1 \quad \text{and} \quad i \in \mathbb{N}, \ \text{or else} \\
  0 \leq j \leq m \quad \text{and} \quad i \geq n + 2(m-j) + 1.
\end{cases}
\end{equation}

We claim that $N(n, m) = N$; clearly this equality implies the Lemma. For this claim, since the generators (13) belong to $N$, we are reduced to prove

1. $N(n, m) \supset N$.
2. $N$ is a submodule of $M(n + 2m)$.

\textbf{[i]} is clear: if $j \geq m + 1$, then $z_{(i,j)} = y^j x^{j-m-1} \cdot z_{(0, m+1)}$; if $j \in \mathbb{I}_0$ and $i \geq n + 2(m-j) + 1$, then $z_{(i,j)} = y^{n+2(m-j)-1} \cdot z_{(n+2(m-j)+1, j)}$.

\textbf{[ii]} We use the formulas (10) to show that the generators leave $N$ invariant. Fix $z_{(i_0, j_0)}$ as in (17). If $j_0 \geq m+1$, then by (10) the actions of the generators involve linear combinations of $z_{(i,j)}$ with $j \geq j_0$ hence they are in $N$. So we assume that $0 \leq j_0 \leq m$ and $i_0 \geq n + 2(m-j_0) + 1$.

**Action of $x$ on $z_{(i_0, j_0)}$.** Here $x \cdot z_{(i_0, j_0)}$ is a linear combination of elements of the form $z_{(i_0-k, j_0+k+1)}$ with $k \in \mathbb{I}_{0, j_0}$. If $j_0 + k + 1 \geq m + 1$, then we are done. Otherwise,
\[i_0 - k \geq n + 2m - 2j_0 + 1 - k \geq n + 2m - 2(j_0 + k + 1) + 1.\]

**Action of $y$ or $\xi$ on $z_{(i_0, j_0)}$.** This is clear.

**Action of $g$ on $z_{(i_0, j_0)}$.** Here $g \cdot z_{(i_0, j_0)}$ is a linear combination of elements of the form $z_{(i_0-k, j_0+k)}$ with $k \in \mathbb{I}_{0, i_0}$. If $j_0 + k \geq m + 1$, then we are done. Otherwise,
\[i_0 - k \geq n + 2m - 2j_0 + 1 - k \geq n + 2m - 2(j_0 + k) + 1.\]

**Action of $u$ on $z_{(i_0, j_0)}$.** Here $u \cdot z_{(i_0, j_0)}$ is a linear combination of elements of the form $z_{(i_0-k-1, j_0+k)}$ with $k \in \mathbb{I}_{i_0-1}$. If $j_0 + k \geq m + 1$, then we are done. Otherwise,
\[i_0 - k - 1 \geq n + 2m - 2j_0 + 1 - k - 1 \geq n + 2m - 2(j_0 + k) + 1.\]
Action of $v$ on $z(i_0,j_0)$. Here $v \cdot z(i_0,j_0)$ is a linear combination of $z(i_0-1,j_0)$ and elements of the form $z(i_0-k-1,j_0+k)$ with $k \in \mathbb{N}_{i_0-1}$. We begin with the latter case. If $j_0 + k \geq m + 1$, then we are done. Otherwise,

$$i_0 - 1 - k \geq n + 2m - 2j_0 + 1 - 1 - k \geq n + 2m - 2(j_0 + k) + 1.$$ 

Now for $z(i_0 - 1, j_0)$, if $i_0 = n + 2(m - j_0) + 1$ then the coefficient that goes with $z(i_0 - 1, j_0)$ is zero. So we can assume $i_0 > n + 2(m - j_0) + 1$. Then $i_0 - 1 \geq n + 2(m - j_0) + 1$ and we are done. \qed

Clearly, $T(n,0) \cong L(n)$.

**Lemma 4.3.** The linear map $\psi : L(n) \hookrightarrow T(n,m)$ given by $z_{n,i} \mapsto z(i,m)$ for $i \in \mathbb{N}_{0,n}$, is a monomorphism of $D$-modules. Let $L := \text{Im } \psi$. If $m \geq 1$, then $T(n,m)/L \cong T(n+2,m-1)$.

**Proof.** Using the formulas (10) we see first that $z_{(0,m)}$ satisfies the defining relations (11) and (12) of $L(n)$, implying the existence of $\psi$ with the desired properties. Second, we see that the class of $z_{(0,0)}$ in $T(n,m)/L$ satisfies the defining relations (14), (15) and (16) of $T(n+2,m-1)$. Thus we have an epimorphism $T(n+2,m-1) \twoheadrightarrow T(n,m)/L$ which is an isomorphism because $\dim T(n+2,m-1) = m(m + m + 2) = \dim T(n,m)/L$. \qed

**Lemma 4.4.** Let $N$ be a non-zero submodule of $T(n,m)$. Then $L \subseteq N$.

**Proof.** We first show that for every $j \in \mathbb{N}_{m}$ and $i \in \mathbb{N}_{0,n+2(m-j)}$ we have

$$v^i \cdot z(i,j) = \left(\frac{(i!)^2}{2^i} \binom{n + 2(m - j)}{i}\right) z_{(0,j)}.$$ 

We argue recursively on $i$. For $i = 0$ the equality is clear. Suppose that $v^\ell \cdot z(\ell,j) = \left(\frac{(\ell!)^2}{2^\ell} \binom{n + 2(m - j)}{\ell}\right) z_{(0,j)}$, for $\ell \in \mathbb{N}_{0,i}$.

Using (10) we see that $v^{i+1} \cdot z(i+1,j)$ is equal to

$$\frac{(i+1)(n+2(m-j)-i)!}{2^{i+1}} v^i z(i,j) + \sum_{k=1}^{i} \frac{(i+1)(n+2m-j-i)(k+1)!}{2^{k+1}} v^k v^{i-k} \cdot z(i-k,j+k)$$

$$= \frac{(i+1)(n+2(m-j)-i)!}{2^{i+1}} \binom{n+2(m-j)}{i} z_{(0,j)}$$

$$+ \sum_{k=1}^{i} \frac{(i+1)(n+2m-j-i)(k+1)!}{2^{k+1}} \binom{n+2(m-j-k)}{i-k} v^k z_{(0,j+k)}$$

$$= \frac{(i+1)!}{2^{i+1}} \binom{n+2(m-j)}{i+1} z_{(0,j)}.$$ 

Clearly (18) implies that $v^k \cdot z(i,j) = 0$ for $k > i$. Now let $z \in N - 0$. Then $z = \sum_{i,j} c_{i,j} z_{(i,j)}$ for some $c_{i,j} \in \mathbb{k}$. Let $i_0 = \max\{i : c_{i,j} \neq 0\}$. Then

$$v^{i_0} z = \sum_{j} c_{i_0,j} \binom{i_0!}{2^{i_0}} \binom{n+2(m-j)}{i_0} z_{(0,j)} \in N.$$
Taking \( j_0 = \max \{ j : c_{i_0,j} \neq 0 \} \) we get that
\[
x^{m-j_0}y^{i_0} \cdot z = c_{i_0,j_0} \frac{(i_0)!^2}{2i_0^i} \left( n + 2(m - j_0) \right) z_{(0,m)} \in N.
\]
Hence \( z_{(0,m)} \in N \) and since \( L \) is simple, this shows that \( L \subseteq N \).

**Proposition 4.5.** The module \( T(n,m) \) is uniserial and indecomposable.

**Proof.** If \( N \) is a simple submodule of \( T(n,m) \), then \( N = L \) by Lemma 4.4 thus \( L \) is the socle of \( T(n,m) \). We conclude from Lemma 4.3 that the socle series is a composition series, hence \( T(n,m) \) is uniserial and indecomposable.

**Remark 4.6.** The dual module \( T(n,m)^* \) is also uniserial and indecomposable but it is not a highest weight module; the subfactors of its hw-series are the simple modules \( L(n) \) etc.

**Theorem 4.7.** Let \( T \) be an indecomposable finite-dimensional highest weight module with hw-rk \( T = 1 \). Then \( T \cong T(n,m) \) for some \( (n,m) \in \mathbb{N}_0 \).

**Proof.** Assume that hw \( T = p \in \mathbb{Z} \). Since \( T \) is generated by \( T(p) \), \( T \) is a quotient of \( M(p) \). Fix \( z \in T(p) - 0 \). Since hw-rk \( T = 1 \), \( z \) generates \( T \), \( \xi \cdot z = pz \) and \( g \cdot z = z \). Given a simple quotient \( L \) of \( T \), \( L \) is then a (finite-dimensional) simple quotient of \( M(p) \). Then \( p \in \mathbb{N}_0 \) and \( L \cong L(p) \) as in [ADP 3.11].

By Proposition 2.1 \( x^m \cdot z \neq 0 \) and \( x^{m+1} \cdot z = 0 \) for some \( m \in \mathbb{N}_0 \). Let \( n := p - 2m \). Applying the relations (10), we see that the submodule generated by \( x^m \cdot z \) is a quotient of \( M(n) \); hence \( n \in \mathbb{N}_0 \) arguing as above.

Let \( j \in \mathbb{N}_0 \). By Remark 2.3 there exists \( a_j \in \mathbb{N}_0 \) such that \( y^{a_j}x^j \cdot z \neq 0 \), \( y^{a_j+1}x^j \cdot z = 0 \). Then the last commutation relation in (10) says that
\[
0 = vy^{a_j+1}x^j \cdot z = \frac{(a_j+1)(p-2j-a_j)}{2} y^{a_j}x^j \cdot z
\]
\[
+ \sum_{k=1}^{a_j} \frac{(a_j+1)(p-j-a_j)}{(p-k-1)!} y^{a_j-k}x^j \cdot z.
\]

Let \( I = \{ k \in \mathbb{N}_0 : y^{a_j-k}x^j \cdot z \neq 0 \} \). Then \( \{ y^{a_j-k}x^j \cdot z : k \in I \} \) is linearly independent. Indeed if \( \sum_{k \in I} c_k y^{a_j-k}x^j \cdot z = 0 \), then applying enough times one can show that each \( c_k \) should be zero. Then equation (19) tells us that \( p - 2j - a_j = 0 \) since \( 0 \in I \). Then \( a_j = p - 2j = n - 2(m - j) \) by definition of \( n \). Hence \( T \) is a quotient of \( T(n,m) \). Let \( z_{i,j} := y^i x^j \cdot z \).

By an argument similar to the one given in Lemma 4.2 one shows that the elements \( \{ z_{i,j} \} \) form a basis of \( T \), hence \( T \cong T(n,m) \).

**Remark 4.8.** The previous result shows that the requirement in Definition 3.7 that highest weight modules are generated by their highest weight subspaces is necessary. Indeed, otherwise there would be many more indecomposables than in Proposition 4.7. For instance let \( \widetilde{T} = T(n,m) \oplus S_r(n) \), where \( m \geq 1 \) and \( S_r(n) \) is the indecomposable module defined in Proposition 5.2, let \( N \) be the submodule generated by the elements \( s_i - z_{i,m} \), and let \( T = \widetilde{T}/N \). Then
T is easily seen to be indecomposable since it satisfies a property similar to the one proven in Lemma 4.4. It satisfy $hw-rk T = 1$, but it is not isomorphic to any $T(n', m')$ since it has two copies of $L(n)$ as composition factors.

5. Modules of highest weight rank two

In this Section we introduce a family $S_\gamma(n)$ of highest weight modules of $hw-rk 2$ and use it to classify self extensions of simple modules.

5.1. Highest weight rank 2. Let $n \in \mathbb{N}_0$ and $(\lambda, \mu) \in \mathbb{K}^2$. We consider the $D^0$-module $P_{\lambda, \mu}(n)$ with basis $s, w$ and action

$$(20) \quad g \cdot s = s, \quad g \cdot w = w + \lambda s, \quad \xi \cdot s = ns, \quad \xi \cdot w = nw + \mu s.$$ 

The module $P_{\lambda, \mu}(n)$ is isomorphic to $P_{t\lambda, t\mu}(n)$ for any $t \in \mathbb{K}^\times$ and is indecomposable whenever $(\lambda, \mu) \neq 0$; any indecomposable in $D^0_{\geq 0}$ of dimension 2 has this shape. Let $M_{\lambda, \mu}(n) = \text{Ind}_{D^0_{\geq 0}}(P_{\lambda, \mu}(n))$; it is presented by generators $s$ and $s$ with defining relations (20) and

$$(21) \quad u \cdot s = 0, \quad v \cdot s = 0, \quad u \cdot w = 0, \quad v \cdot w = 0.$$ 

Here is a basis of $M_{\lambda, \mu}(n)$:

$$s_{ij} := y^ix^j \cdot s, \quad w_{ij} := y^ix^j \cdot w, \quad i, j \in \mathbb{N}_0.$$ 

Let $M_1$ be the span of $(s_{i,j})_{i,j \in \mathbb{N}_0}$ and $M_2 = M_{\lambda, \mu}(n)/M_1$. Thus we have a short exact sequence $M(n) \simeq M_1 \longrightarrow M_{\lambda, \mu}(n) \longrightarrow M_2 \simeq M(n)$ in $D\text{Mod}$.

In the rest of this Subsection, $N$ is a submodule of $M_{\lambda, \mu}(n)$,

$$S := M_{\lambda, \mu}(n)/N, \quad S_1 := M_1/M_1 \cap N \hookrightarrow S, \quad S_2 := S/S_1 \simeq M_2/(N/M_1 \cap N).$$

Let $s, w, s_{ij}, w_{ij}$ be the images of $s, w, s_{ij}, w_{ij}$ in $S$; let $\overline{r} \in S_2$ be the image of $r \in S$ under the canonical projection.

5.1.1. $S_1$ and $S_2$ are simple.

Lemma 5.1. Keep the notation above.

(i) Assume that the following relations hold in $S$ and $S_2$:

$$(22) \quad x \cdot s = 0,$$

$$(23) \quad x \cdot \overline{w} = 0.$$ 

Then $S_1$ is spanned by $s_i := s_{i,0} = y^i \cdot s, i \in \mathbb{N}_0$; and $S_2$ is spanned by $w_i := w_{i,0} = y^i \cdot w, i \in \mathbb{N}_0$. Hence

$$(24) \quad S^{(n-2i)} = ks_i + kw_i, \quad i \in \mathbb{N}_0, \quad \text{and} \quad S = \oplus_{i \in \mathbb{N}_0} S^{(n-2i)}.$$ 

In addition, there exists $\gamma \in \mathbb{K}$ such that $x \cdot w = \gamma s_1$. 
(ii) The following relations hold for all $i \in \mathbb{N}_0$:

\begin{align}
(25) & \quad x \cdot s_i = 0, \quad u \cdot s_i = 0, \quad v \cdot s_i = \frac{i(n-i+1)}{2} s_{i-1}, \\
(26) & \quad x \cdot w_i = \gamma s_{i+1}, \\
(27) & \quad \xi \cdot w_i = (n - 2i)w_i + \mu s_i, \\
(28) & \quad g \cdot w_i = w_i + (\lambda + i\gamma)s_i, \\
(29) & \quad u \cdot w_i = -\left(i\lambda + \frac{i(i-1)}{2}\gamma\right) s_{i-1}, \\
(30) & \quad v \cdot w_i = \frac{i(n-i+1)}{2} w_{i-1} + \left(i(n-2i+2)\lambda + i(i-1)(n-2i+2)\gamma - i\mu\right) s_{i-1}.
\end{align}

**Proof.** (i) follows directly from \((22)\) and \((23)\), looking at the basis of $M_{\lambda,\mu}(n)$. Then \((7)\) implies \((24)\). Now $x \cdot w \in S^{(n-2)}_1(n-2)$ hence there exists $\gamma \in \mathbb{k}$ such that $x \cdot w = \gamma s_1$ by \((23)\). The relations in (ii) are proved arguing recursively. For \((25)\) the defining relations of $\mathcal{D}$ are used. For \((26)\) we have

\[x \cdot w_{i+1} = xy \cdot w_i = (y + \frac{1}{2}x) x \cdot w_i = \gamma(y + \frac{1}{2}x) \cdot s_{i+1} = \gamma s_{i+2}.
\]

The proof of \((27)\) is direct starting from \((20)\):

\[
\xi \cdot w_{i+1} = \xi y \cdot w_i = (y \xi - 2y) \cdot w_i = (n - 2(i + 1))w_{i+1} + \mu s_{i+1}.
\]

The proof of \((28)\) also starts from \((20)\):

\[
g \cdot w_{i+1} = gy \cdot w_i = (y + x) g \cdot w_i = (y + x) \cdot (w_i + (\lambda + i\gamma)s_i) = w_{i+1} + (\lambda + i\gamma)s_{i+1}.
\]

To prove \((29)\) we start from \((21)\) and argue:

\[
u \cdot w_{i+1} = uy \cdot w_i = (yu + 1 - g) \cdot w_i = -(i\lambda + \frac{i(i-1)}{2}\gamma - (\lambda + i\gamma)) s_i
\]

as needed. Finally we prove \((30)\). First we observe that

\[-g\zeta \cdot w_i = g \cdot \left(\frac{(n-2i)}{2} w_i - \mu s_i\right) = \frac{(n-2i)}{2} w_i + \frac{(n-2i)}{2} \lambda + \frac{(n-2i)}{2} \gamma - \mu s_i.
\]

Then we proceed recursively:

\[
v \cdot w_{i+1} = vy \cdot w_i = (vy - g\zeta + yu) \cdot w_i
\]

\[
= \frac{i(n-i+1)}{2} w_i + \left(\frac{i(n-2i+2)}{2} \lambda + \frac{i(i-1)(n-2i+2)}{4} \gamma - i\mu\right) s_i
\]

\[
+ \frac{(n-2i)}{2} w_i + \frac{(n-2i)}{2} \lambda + \frac{(n-2i)}{2} \gamma - \mu s_i - \left(i\lambda + \frac{i(i-1)}{2}\gamma\right) s_i;
\]

it remains to perform the routine verification of the following equalities:

\[
\frac{i(n-i+1)}{2} + \frac{n-2i}{2} = \frac{(i+1)(n-(i+1)+1)}{2},
\]

\[
\frac{i(n-2i+2)}{2} + \frac{n-2i}{2} = \frac{(i+1)(n-2(i+1)+2)}{2},
\]

\[
\frac{i(i-1)(n-2i+2)}{4} + \frac{(n-2i)}{2} = \frac{i(i+1)(n-2(i+1)+2)}{4}.
\]
Proof. We keep the notation above and apply Lemma 5.1; thus in (29) and (30), we see that
\begin{align}
\lambda + \frac{n}{2} \gamma &= 0 \\
\mu &= 0.
\end{align}

(iii) For any $\gamma$, $S_\gamma(n)$ is an extension of $L_n$ by $L_n$. Any extension is like this and dim Ext$_D(L(n), L(n)) = 1$, $n \in \mathbb{N}_0$.

For our conventions on extensions, see Subsection 5.1.

\begin{proof}
We keep the notation above and apply Lemma 5.1, thus dim $S \leq n^2$. By (29) and (30), we see that
\begin{align}
\lambda + \frac{n}{2} \gamma &= 0 \\
\mu &= 0.
\end{align}

Then $N \cap \mathbb{M}_{n,n}(n)(n)$, in particular $s \notin N$. Now there are morphisms of $D$-modules $\iota : L(n) \rightarrow S_1$ and $\iota : L(n) \rightarrow S_2$ given by $\iota(z_{n,i}) = s_i$ and $\iota(z_{n,i}) = \overline{w}_i$, $i \in \mathbb{I}_0$. Now $s \neq 0$ in $S$ thus $\iota$ is injective (because $L(n)$ is simple) and $\{s_0, \ldots, s_n\}$ are linearly independent. Similarly $\overline{w} \neq 0$ in $S_2$, $\iota$ is injective and $\{w_0, \ldots, w_n\}$ are linearly independent. Hence dim $S = n^2$ and (3) is proved.

(ii) The $D^0$-module $S_0(n)(n) \simeq \mathbb{P}_{0,0}(n)$ is decomposable while $S_\gamma(n)(n) \simeq \mathbb{P}_{\gamma,0}(n)$ is indecomposable when $\gamma \neq 0$.

Indeed let $\iota : L_n \rightarrow S_\gamma(n)$ given by $s_i \mapsto s_i$ and $\pi : S_\gamma(n) \rightarrow L_n$ given by $s_i \mapsto 0$, $w_i \mapsto s_i$. Then the following sequence is exact
\begin{align*}
L_n \xrightarrow{\iota} S_\gamma(n) \xrightarrow{\pi} L_n,
\end{align*}

$\square$
Remark 5.3. In the context of Lemma 5.1, if dim $S < \infty$ and $x \cdot s = 0$, then $S_1$ actually belongs to $U(\mathfrak{sl}_2)\text{mod}$, thus $y^{n+1} \cdot s = 0$; similarly $\overline{w} = 0$ implies that $y^{n+1} \cdot \overline{w} = 0$, hence $y^{n+1} \cdot w = 0$ by looking at the weights of $S_1$.

Remark 5.4. The set $\{s_0, \ldots, s_n, w_0, \ldots, w_n\}$ is a basis of $S_n(n)$. Set $w_{n+1} = s_{n+1} = w_{-1} = s_{-1} = 0$ by convention. Then the action is given by

$$
\begin{align*}
x \cdot s_i &= 0, & x \cdot w_i &= \gamma s_{i+1}, \\
y \cdot s_i &= s_{i+1}, & y \cdot w_i &= w_{i+1}, \\
g \cdot s_i &= s_i, & g \cdot w_i &= w_i - \frac{n-2i}{2} \gamma s_i, \\
\xi \cdot s_i &= (n-2i)s_i, & \xi \cdot w_i &= (n-2i)w_i, \\
u \cdot s_i &= 0, & u \cdot w_i &= \frac{i(n+i-1)}{2} \gamma s_{i-1}, \\
v \cdot s_i &= \frac{i(n+i+1)}{2} s_{i-1}, & v \cdot w_i &= \frac{i(n+i+1)}{2} w_{i-1} - \frac{i(n-2i+2)(n+1-i)}{4} \gamma s_{i-1}.
\end{align*}
$$

(36)

6. Extensions of simple modules

6.1. Generalities. Let $n, m \in \mathbb{N}_0$. The goal of this Section is to compute the vector spaces $\text{Ext}_D^1(L(n), L(m))$. To fix the notation, an extension of $L(n)$ by $L(m)$ is a short exact sequence

$$
L(n) \xrightarrow{\iota} T \xrightarrow{\pi} L(m).
$$

By abuse of notation we say also that $T$ is an extension of $L(n)$ by $L(m)$.

Remark 6.1. Let $H$ be a Hopf algebra and $M, N, P \in H\text{mod}$. Given an extension $M \xrightarrow{\iota} P \xrightarrow{\pi} N$, the sequence $N^* \xrightarrow{\pi^*} P^* \xrightarrow{\iota^*} M^*$ is exact. Then $\text{Ext}_H^1(M, N) \simeq \text{Ext}_H^1(N^*, M^*)$. Since $L(p)^* \simeq L(p)$, $p \in \mathbb{N}_0$, we have

$$
\text{Ext}_D^1(L(n), L(m)) \simeq \text{Ext}_D^1(L(m), L(n)), \quad m, n \in \mathbb{N}_0.
$$

Fix an extension (37) and identify $L(n)$ as a submodule of $T$ via $\iota$. Pick $w_m(0) \in T$ such that $\pi(w_m(0)) = z_m(0) \in L(m)$ and set

$$
w_m(i) := y^i \cdot w_m(0), \quad i \in \mathbb{Z}_{0,m}.
$$

Then $\{z_{n,0}, \ldots, z_{n,n}, w_m(0), \ldots, w_m(m)\}$ is a linear basis of $T$. Also set

$$
r_d := d \cdot w_m(0), \quad d \in \mathcal{O} = \mathbb{k}\langle x, u, g^{\pm 1} \rangle.
$$

Now $\mathcal{O}$ acts by 0 on $L(m)$, hence $r_d \in L(n)$ for $d \in \mathcal{O}$. Also $v \cdot w_m(0) \in L(n)$.

We start giving restrictions on the existence of non-trivial extensions.

Lemma 6.2. If $\text{Ext}_D^1(L(n), L(m)) \neq 0$, then $m - n \in \{2, 0, -2\}$.

Proof. Let $T$ be an extension of $L(n)$ by $L(m)$ and keep the notation above.

Claim. If $r_x = r_{g-1} = r_u = 0$, then the extension $T$ is trivial.
We claim that $x$, $g-1$ and $u$ act by 0 on $T$. Since they act trivially on $L(n)$, it is enough to consider the action on the $w_m(i)$’s. We argue recursively, the case $i = 0$ being the hypothesis. If $x$, $g-1$ and $u$ act by 0 in $w_m(i)$, then

$$
x \cdot w_m(i + 1) = xy \cdot w_m(i) = (yx + \frac{1}{2} x^2)w_m(i) = 0,
$$

$$
u \cdot w_m(i + 1) = uy \cdot w_m(i) = (yu + (1 - g)) \cdot w_m(i) = 0,
$$

$$(1 - g) \cdot w_m(i + 1) = (1 - g) y \cdot w_m(i) = (y(1 - g) - gx) \cdot w_m(i) = 0.$$

Hence $T \in \mathcal{D}/\mathcal{DO} \mod$; since $\mathcal{D}/\mathcal{DO}^+ \simeq U(\mathfrak{sl}_2(k))$, the extension is trivial.

Thus, if $T$ is a non-trivial extension, then at least one of $r_x$, $r_{g-1}$, $r_u$ is not zero. Let $s \in L(n)$ be such that $\zeta \cdot w_m(0) = mw_m(0) + s$.

**Case 1.** $r_u \neq 0$. Since

$$v \cdot r_u = vu \cdot w_m(0) = (uv - \frac{1}{2} u^2) \cdot w_m(0) = 0,$$

there exists $a \in k^\times$ such that $r_u = az_{n,0}$. We compute in two ways:

$$\xi \cdot r_u = \xi u \cdot w_m(0) = (u \xi + 2u) \cdot w_m(0) = mr_u + u \cdot s + 2r_u = (m + 2)r_u$$

and also $\xi \cdot r_u = a \xi \cdot z_{n,0} = nr_u$. We conclude that $n = m + 2$.

**Case 2.** $r_u = 0$ and $r_{g-1} \neq 0$. Since

$$v \cdot r_{g-1} = v(g-1) \cdot w_m(0) = ((g-1)v + gu) \cdot w_m(0) = g \cdot r_u = 0,$$

there exists $a \in k^\times$ such that $r_{g-1} = az_{n,0}$. We compute in two ways:

$$\xi \cdot r_{g-1} = (g - 1)\xi \cdot w_m(0) = mr_{g-1} + (g-1) \cdot s = nr_{g-1}$$

and also $\xi \cdot r_{g-1} = a \xi \cdot z_{n,0} = nr_{g-1}$. We conclude that $n = m$.

**Case 3.** $r_u = r_{g-1} = 0$ and $r_x \neq 0$. Since

$$v \cdot r_x = vx \cdot w_m(0) = (vx + (1 - g) + xu) \cdot w_m(0) = 0,$$

there exists $a \in k^\times$ such that $r_x = az_{n,0}$. We compute in two ways:

$$\zeta \cdot r_x = \zeta x \cdot w_m(0) = (x \xi - x) \cdot w_m(0) = mr_x + x \cdot s - 2r_x = (m - 2)r_x$$

and also $\zeta \cdot r_x = a \xi \cdot z_{n,0} = nr_x$. We conclude that $n = m - 2$. □

6.2. **Extensions of $L(n)$ by $L(n \pm 2)$**. Let $n \in \mathbb{N}_0$. We next introduce the $D$-module $T(n,1)$ generated by $z_n$ with defining relations

\begin{align}
(11) & \quad g \cdot z_n = z_n, \quad \zeta \cdot z_n = -\left(\frac{n+2}{2}\right) z_n, \quad u \cdot z_n = 0, \quad v \cdot z_n = 0, \\
(38) & \quad x^2 \cdot z_n = 0, \quad y^{n+3} \cdot z_n = 0, \quad y^{n+1} x \cdot z_n = 0.
\end{align}

This belongs to the family of $D$-modules $T(n,m)$ studied in Section 4. The set \( \{z_n(i,j) := y^i x^j \cdot z_n; j \in \mathbb{N}_0, i \in \mathbb{Z}_{0,n+2-2j}\} \) is a basis of $T(n,1)$, see Section 4 or prove it directly. Given $b \in k^\times$, let $E(b)$ be the exact sequence

$$L(n) \xrightarrow{\iota_b} T(n,1) \xrightarrow{\pi} L(n + 2),$$

where $\iota_b$ and $\pi$ are determined by $\iota_b(z_n,i) = b z_n(i,1)$ and $\pi(z_n(i,1)) = 0$ for $i \in \mathbb{N}_0$; and $\pi(z_n(k,0)) = z_{n+2}(k)$ for $k \in \mathbb{N}_{0,n+2}$.
Proposition 6.3. Any extension $L(n) \xrightarrow{T} L(n+2)$ is either trivial or else isomorphic to $E(b)$ for a unique $b \in \mathbb{k}^\times$. Hence $T$ is isomorphic either to $L(n) \oplus L(n+2)$ or to $T(n,1)$ and

$$\dim \text{Ext}^1_D(L(n),L(n+2)) = 1.$$  \hfill (39)

Proof. Identify $L(n)$ as a submodule of $T$ via $\iota$ and pick $w \in T$ satisfying

$$\pi(w) = z_{n+2}(0) \in L(n+2).$$

Hence there exist $c_0, \ldots, c_n \in \mathbb{k}$ such that

$$\zeta \cdot w = (n+2)w + \sum_{i=0}^n c_i z_{n,i}.$$ 

Let $w_{n+2}(0) := w + \sum_{i=0}^n \frac{c_i}{2(i+1)} z_{n,i}$. Clearly $\pi(w_0) = z_{n+2}(0)$, but also

$$\zeta \cdot w_{n+2}(0) = (n+2)w + \sum_{i=0}^n c_i z_{n,i} + \sum_{i=0}^n \frac{c_i(n-2i)}{2(i+1)} z_{n,i}$$

$$= (n+2)w + \sum_{i=0}^n \frac{2i+2+n-2i}{2(i+1)} c_i z_{n,i}$$

$$= (n+2) \left( w + \sum_{i=0}^n \frac{c_i}{2(i+1)} z_{n,i} \right) = (n+2)w_{n+2}(0).$$

Hence $w_{n+2}(0) \in T^{n+2}$,

$$w_{n+2}(j) := y^j \cdot w_{n+2}(0) \in T^{n+2-2j}, \quad \pi(w_{n+2}(j)) = z_{n+2}(j), \quad j \in \mathbb{I}_{n+2},$$

and \{ $z_{n,0}, \ldots, z_{n,n}, w_{n+2}(0), \ldots, w_{n+2}(n+2)$ \} is a basis of $T$. Thus

$$T^k = \begin{cases} \mathbb{k} z_{n,\frac{n-k}{2}} \oplus \mathbb{k} w_{n+2}(\frac{n+2-k}{2}), & k \in \mathbb{I}_{-n,n}; \\ \mathbb{k} w_{n+2}(0) & k = n+2; \\ \mathbb{k} w_{n+2}(n+2) & k = -n-2. \end{cases}$$

Now $y^{n+3} \cdot w_{n+2}(0) \in T^{-n-4}$ and $u \cdot w_{n+2}(0), v \cdot w_{n+2}(0) \in T^{n+4}$ by (7), i.e. \hfill (40)

$$y^{n+3} \cdot w_{n+2}(0) = 0, \quad u \cdot w_{n+2}(0) = 0, \quad v \cdot w_{n+2}(0) = 0.$$ 

Also $g \cdot w_{n+2}(0) \in T^{n+2}$, i.e. $g \cdot w_{n+2}(0) = aw_{n+2}(0)$ for some $a \in \mathbb{k}$; applying $\pi$ to both sides of the last equality we get that $a = 1$, that is

$$g \cdot w_{n+2}(0) = w_{n+2}(0).$$ \hfill (41)

Finally $x \cdot w_{n+2}(0) \in L(n) \cap T^n = k z_{n,0}$, i.e. $x \cdot w_{n+2}(0) = cz_{n,0}$ for some $c \in \mathbb{k}$. If $c = 0$, then the subspace with basis $w_{n+2}(0), \ldots, w_{n+2}(n+2)$ is a submodule isomorphic to $L(n+2)$ and the extension is trivial. If $c \neq 0$, then the extension is isomorphic to $E(c^{-1})$. \hfill \Box

Proposition 6.4. If $S$ is an extension of $L(n)$ by $L(n-2)$, then $S$ is isomorphic either to $L(n) \oplus L(n-2)$ or to $T(n-2,1)$. Also, \hfill (42)

$$\dim \text{Ext}^1_D(L(n),L(n-2)) = 1.$$
6.3. The quiver and representation type. Propositions 5.2, 6.3, and 6.4 give us the Gabriel quiver of $\mathcal{D}$, i.e., the quiver $\text{Ext}^1(\mathcal{D})$ with vertices $\mathbb{N}_0$ and $\dim \text{Ext}^1_D(S_i, S_j)$ arrows from the vertex $i$ to the vertex $j$. That is,
\[ \begin{array}{cccccc}
0 & 2 & 4 & \ldots & 2n & 2n+2 \\
1 & 3 & 5 & \ldots & 2n-1 & 2n+1 \\
\end{array} \]

From the analysis of this quiver one concludes:

**Proposition 6.5.** The algebra $\mathcal{D}$ has wild representation type.

**Proof.** This is evident for experts in representation theory of artin algebras but we include a proof for completeness.

Claim 1. Let $A \twoheadrightarrow B$ be a surjective map of algebras and $M, N \in B\text{mod}$. Then the canonical map $\text{Ext}^1_B(M, N) \rightarrow \text{Ext}^1_A(M, N)$ is injective.

Claim 2. Let now $A$ be a (possibly infinite-dimensional) algebra over a field $k$ with $\text{Ext}$-quiver $Q$ such that $\dim_k \text{Ext}^1_A(L, L') < \infty$ for any $L, L' \in \text{irrep} A$. Let $F$ be a finite subset of $\text{irrep} A$ and let $Q_F$ be the (full) subquiver of $Q$ spanned by $F$. Then there exists a finite-dimensional quotient algebra $A \twoheadrightarrow B$ such that the $\text{Ext}$-quiver of $B$ is isomorphic to $Q_F$.

Given $L, L' \in F$, pick a basis $(v_i)$ of $\text{Ext}^1_A(L, L')$ and for each $v_i$ an extension $M_i$ of $L$ by $L'$ representing $v_i$. Let $M$ be the direct sum of all $L, L'$ in $F$ and all the corresponding $M_i$. Clearly $\dim M < \infty$ hence so is the image $B$ of the representation $A \rightarrow \text{End} M$. By construction and Claim 1, the canonical map $\text{Ext}^1_B(L, L') \rightarrow \text{Ext}^1_A(L, L')$ is bijective, hence the $\text{Ext}$-quiver of $B$ is isomorphic to $Q_F$.

By Claim 2 applied to $F = \{0, 2, 4\}$ there exists a surjective algebra map $\mathcal{D} \twoheadrightarrow B$ where $\dim B < \infty$ and the $\text{Ext}$-quiver of $B$ is isomorphic to
\[ \begin{array}{cccc}
0 & 2 & 4 \\
1 & 3 & 5 \\
\end{array} \]

Let $C$ be the basic algebra which is Morita equivalent to $B$ and $\tau = \text{rad} C$. Then $C/\tau^2$ has finite or tame representation type if and only if the separated quiver $\Gamma_s$ of (44) is a disjoint union of Dynkin and affine Dynkin diagrams, see [ARS, Theorem X.2.6]. But $\Gamma_s$ has the form
Thus $C/\mathfrak{r}^2$ has wild representation type, and \textit{a fortiori} $C$, $B$ and $D$ also. □

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