Neutron transfer reactions in halo effective field theory

M. Schmidt,¹,² L. Platter,²,³ and H.-W. Hammer¹,⁴

¹Institut für Kernphysik, Technische Universität Darmstadt, 64289 Darmstadt, Germany
²Department of Physics and Astronomy, University of Tennessee, Knoxville, TN 37996, USA
³Physics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA
⁴ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum für Schwerionenforschung GmbH, 64291 Darmstadt, Germany

(Dated: December 24, 2018)

Abstract

Direct reaction experiments provide a powerful tool to probe the structure of neutron-rich nuclei like Beryllium-11. We use halo effective field theory to calculate the cross section of the deuteron-induced neutron transfer reaction \(^{10}\)Be(d, p)\(^{11}\)Be. The effective theory contains dynamical fields for the Beryllium-10 core, the neutron, and the proton. In contrast, the deuteron and the Beryllium-11 halo nucleus are generated dynamically from contact interactions using experimental and ab-initio input. The reaction amplitude is constructed up to next-to-leading order in an expansion in the ratio of the length scales characterizing the core and the halo. The Coulomb repulsion between core and proton is treated perturbatively. Finally, we compare our results to cross section data and other calculations.
I. INTRODUCTION

Nuclear processes such as capture and transfer reactions are one focus of ongoing research at existing and forthcoming experimental facilities with radioactive ion beams [1]. However, the consistent theoretical description of such reactions in ab initio calculations poses significant challenges. Tremendous progress has been made for lighter systems in calculating elastic nucleus-nucleon scattering processes by combining the variational approach of the resonating group model and the no-core shell model in the no-core shell model with continuum [2]. However, for larger systems it remains a challenging task to calculate reactions in a controlled way and with reliable uncertainty estimates; see for example Refs. [3–7].

One alternative approach is to reduce the number of dynamical degrees of freedom. A process can then be described as an effective two- or three-body problem using a Lippmann-Schwinger or Faddeev equation. The remaining challenge is to model the interaction between the degrees of freedom appropriately. A reduction to the minimal degrees of freedom required to obtain a certain observable is frequently the starting point of an effective field theory (EFT) treatment of a system. EFTs can be applied if a system displays two disparate scales that can be combined to form a small expansion parameter. The large scale can for example be the excitation energy of a degree of freedom or a heavy state not included in the approach. EFT is the theory in which these high energy modes are integrated out.

Halo nuclei display such a separation of scales [8–11]. They consist of a tightly bound core with large excitation energy $E_x$ and some weakly bound valence nucleons. The EFT that has been developed for these systems is called halo effective field theory (Halo EFT) [12, 13]. It treats the core as a fundamental degree of freedom, which is a valid approximation as long as energies smaller than $E_x$ are considered. Halo EFT has been applied to a variety of processes including electromagnetic transitions and Coulomb dissociation of one-neutron halo nuclei. The formalism has been extended to one-neutron and two-neutron halo nuclei. For a recent review, see Ref. [14].

In this work, we explore the potential of Halo EFT to describe the experimentally important process of a deuteron-induced transfer reaction. Such a calculation has not been carried out yet due to the challenging continuum structure of the reaction. As a test case, we consider $^{10}$Be(d, p)$^{11}$Be. The effective three-body system is given by a $^{10}$Be core, a neutron, and a proton. The one-neutron halo nucleus $^{11}$Be represents a neutron-core state with a binding energy much smaller than the $2^+$ core excitation energy $E_x = 3.37$ MeV; see Fig. 1. This intrinsic scale separation reflects itself also in the small core radius $R_c \sim 2$–3 fm and the large halo radius $R_h \sim 7$ fm [15]. Exploiting these length scales, we calculate the reaction cross section up to first order in $R_c/R_h$. We find that dynamical core excitations, excited $^{11}$Be states and proton-core resonances in Fig. 1 can be neglected at leading order (LO).

We expect that the expansion works best for center-of-mass energies $E$ well below $E_x = 3.37$ MeV; see Fig. 1. However, in the absence of appropriate data, we compare our theory to data at $E \geq 7.78$ MeV, measured by Schmitt et al. at Oak Ridge National Laboratory [17]. In fact, previous works suggest that Halo EFT could still be appropriate for the lower experimental energies. For example, Deltuva et al. calculated the differential cross section in a Faddeev approach, using model interactions that reproduce elastic proton-core scattering data and optical potentials that account for loss channels [18]. Their work suggests that core excitations barely influence the cross section for $E \lesssim 10$ MeV. More recently, Yang and Capel [19] reanalyzed the reaction by combining the adiabatic distorted wave approximation (ADWA) reaction model with a Halo EFT description of $^{11}$Be. They found out
that, for the lower beam energies and forward angles, the reaction is purely peripheral. I.e., it only depends on the asymptotic form of the $^{11}$Be wave function, while being independent of short-range details. Indeed, we will be able to describe data for the lower beam energies.

This manuscript is structured as follows. In section II, we present the EFT Lagrangian. Strong interactions between core, neutron, and proton are described by contact forces and the Coulomb interaction follows from photon couplings. Section III explains how the two-body states $^{11}$Be, $^{11}$Be$^*$ and the deuteron dynamically emerge from the given interactions. We then turn to the three-body system in section IV. A Faddeev equation for the reaction will be constructed up to next-to-leading order (NLO) in the $R_c/R_h$ expansion. Following work carried out for the three-nucleon sector [20, 21], it will include the dominant Coulomb contributions. After presenting results for the reaction cross section, we summarize our work and give an outlook in section V.

II. EFT LAGRANGIAN

The EFT Lagrangian $\mathcal{L}$ can be written as the sum

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_\gamma$$

Figure 1: Thresholds relative to $^{10}$Be + n + p. The center column shows the ground and first excited state of $^{10}$Be. Bound and resonance states of the core-neutron ($^{11}$Be) and core-proton ($^{11}$B) systems are depicted in the left and right columns, respectively. We only show $^{11}$B levels, which have been seen in the $^{10}$Be(p, $\gamma$)$^{11}$B experiment of Ref. [16]. In this work, we explicitly include those states with thick lines.
of one-, two-, and three-body interactions and a photon part. The kinetic part reads

\[ \mathcal{L}_1 = n_0able to renormalize the reaction amplitude. We write

\[ \mathcal{L}_1 = n_0^{\dagger} \left( iD_0 + \frac{D^2}{2M_N} \right) n_0 + p_0^{\dagger} \left( iD_0 + \frac{D^2}{2M_N} \right) p_0 + e^\dagger \left( iD_0 + \frac{D^2}{2mc} \right) c. \] (2)

It introduces fields \( n_0, p_0 (\alpha \in \{-1/2, +1/2\}) \) and \( c \) for neutron, proton, and the \(^{10}\)Be core. They are treated as distinguishable particles. Sums over doubly appearing indices are implicit. Masses are taken to be \( m_N \equiv 938.918 \text{ MeV} \) and \( m_c \equiv 10m_N \).

The photon's kinetic and gauge fixing terms are given by

\[ \mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left( \partial_\mu A^\mu - \eta_\mu \eta_\nu \partial^\nu A^\mu \right)^2 \] (3)

with time-like unit vector \( \eta_\mu \). We only consider Coulomb photons, which induce a static potential. The covariant derivative \( D_\mu \equiv \partial_\mu + ie_{A_i} \hat{Q}_i \) in Eq. (2) with charge operator \( \hat{Q}_i \) induces respective photon couplings \( +ie_{Q_p/c} \) with \( Q_p = 1 \) and \( Q_c = 4 \). As done in Ref. [22], we introduce a screened Coulomb photon propagator

\[ iG_\gamma(p) \equiv i \left[ p^2 + \lambda^2 - ic \right]^{-1}. \] (4)

The artificial photon mass \( \lambda \) has to be taken to zero at the end of each calculation.

The two-body part \( \mathcal{L}_2 \) involves the auxiliary fields \( \sigma_\alpha (\alpha \in \{-1/2, +1/2\}) \) and \( d_i (i \in \{-1, 0, +1\}) \) for the shallow bound states \(^{11}\)Be and deuteron, respectively. It reads

\[ \mathcal{L}_2 = \sigma_0^{\dagger} \left[ \Delta_\sigma^{(0)} - \left( i\partial_0 + \frac{\nabla^2}{2M_{Nc}} \right) \right] \sigma_0 - g_\sigma \left[ \sigma_0^{\dagger} (n_0c) + \text{H.c.} \right] \\
+ d_i^{\dagger} \left[ \Delta_d^{(0)} - \left( i\partial_0 + \frac{\nabla^2}{4m_N} \right) \right] d_i - g_d C_{1/2\alpha,1/2\beta}^{11} \left[ d_i^{\dagger} (p_0n_0) + \text{H.c.} \right] \\
+ \mathcal{L}_{2,11\text{Be}^*} + \cdots \] (5)

with \( M_{Nc} \equiv m_N + m_c \) and a Clebsch-Gordan coefficient \( C_{s1m_1, s2m_2}^{11} \). The expression "H.c." denotes the Hermitian conjugate. The regularization-dependent parameters \( \Delta_a^{(0)}, g_a \in \mathbb{R} \) \( (a \in \{\sigma, d\}) \) will be matched to experiment. Derivatives in Eq. (5) induce range corrections at NLO. Higher-order terms in the ellipses are negligible at NLO. The part \( \mathcal{L}_{2,11\text{Be}^*} \) accounts for NLO contributions from the first excited state \(^{11}\)Be*. It is discussed in appendix D.

The three-body part \( \mathcal{L}_3 \) contains an \( s \)-wave deuteron-core interaction \( C_0 \) which will be used to renormalize the reaction amplitude. We write

\[ \mathcal{L}_3 = -g_d^2 C_0 (d_i c)^{\dagger}(d_i c) + \cdots. \] (6)

III. TWO-BODY STATES

Due to the small neutron separation energies of deuteron and \(^{11}\)Be, three-body processes like deuteron breakup are crucial for the transfer reaction; see for example Refs. [23, 24]. In this section, we summarize how \(^{11}\)Be, the deuteron, and \(^{11}\)Be* emerge dynamically from contact interactions of the EFT Lagrangian. Moreover, we explain the effective treatment of core excitation effects in the \(^{11}\)Be system.
Figure 2: The full $^{11}\text{Be}$ propagator $iG_{\sigma}$ (solid-dashed double line with filled circle) is given by the bare one (with empty circle) and the neutron-core self-energy loop, where the solid (dashed) line represents the neutron (core). A similar expression for the deuteron propagator can be obtained by replacing all dashed core lines by dotted proton lines.

A. The Beryllium-11 ground state

In Halo EFT, the $^{11}\text{Be}$ ground state ($1/2^+$) is treated as a pure neutron-core $s$-wave state. Already at LO, its propagator $iG_{\sigma}$, depicted as a solid-dashed double line, contains iterations of the neutron-core self-energy loop to all orders; see Fig. 2. As a consequence of the EFT’s Galilean invariance, $iG_{\sigma}$ is a function of the center-of-mass energy $E_{\text{cm}} \equiv p^0 - p^2/(2M_Nc)$ only, where $p^\mu$ denotes the total four-momentum. After resumming the self-energy, $iG_{\sigma}$ takes the well-known effective range expansion form [25]

$$iG_{\sigma}(E_{\text{cm}}) = -ig_{\sigma}^{-2} \frac{2\pi}{\mu_Nc} \left[ \frac{a_{\sigma}^{-1}}{2} + \frac{r_{\sigma}^2}{2} + \cdots - ik \right]^{-1},$$

(7)

where $k \equiv i[-2\mu_Nc(E_{\text{cm}} + ie)]^{1/2}$ is the on-shell relative momentum and the ellipses denote higher order terms. In the Power Divergence Subtraction (PDS) scheme with mass scale $\Lambda_{\text{PDS}}$ [26, 27], the scattering length $a_{\sigma}$ and effective range $r_{\sigma}$ are given by

$$a_{\sigma}^{-1} = \frac{2\pi}{\mu_Nc} \frac{\Delta^{(0)}_{\sigma}(\Lambda_{\text{PDS}})}{g_{\sigma}^2} + \Lambda_{\text{PDS}},$$

(8)

$$r_{\sigma} = \frac{2\pi}{\mu_Nc^2} g_{\sigma}^{-2}.$$  

(9)

The propagator has a pole at $E_{\text{cm}} = -B_{\sigma}$, or equivalently at $k = i\gamma_{\sigma}$, where $B_{\sigma} = 0.50 \text{ MeV}$ [28] and $\gamma_{\sigma} \equiv (2\mu_NcB_{\sigma})^{1/2} \approx 29 \text{ MeV}$ are the small binding energy and binding momentum. Thus, Eq. (7) can be rearranged by writing

$$iG_{\sigma}(E_{\text{cm}}) = ig_{\sigma}^{-2} \frac{2\pi}{\mu_Nc} \left[ \gamma_{\sigma} + ik - \frac{r_{\sigma}^2}{2} \left( k^2 + \gamma_{\sigma}^2 \right) + \cdots \right]^{-1}.$$

(10)

Note that the coupling $g_{\sigma}$ is not an observable. We thus eliminate it using the redefined auxiliary fields $\tilde{\sigma}^{(1)} \equiv g_{\sigma} \sigma^{(1)}$; see for example Ref. [29]. Consequently, we multiply $G_{\sigma}$ by $g_{\sigma}^{-2}$ and each (neutron-core) $^{11}\text{Be}$ vertex by $g_{\sigma}^{-1}$.

1. Halo EFT counting & ANC

In Halo EFT, all parameters in Eqs. (7)–(10) scale with certain powers of the large halo radius $R_h \sim 7 \text{ fm}$ and the small core radius $R_c$, which represents the natural nuclear physics.

---

1 The propagator is diagonal in spin space. Respective factors $\delta^{\alpha\alpha'}$ will be omitted in the following.
length scale [30]. We may estimate $R_c \sim (2\mu NcE_x)^{-1/2} \approx 2.6\text{ fm}$ from the core excitation energy $E_x = 3.37\text{ MeV}$. Thus, the EFT expansion parameter is $R_c/R_h \sim 0.4$.

As one of the first applications of Halo EFT to electromagnetic processes, Hammer and Phillips used data of the low-energy $E1$ strength of $^{11}\text{Be}$ breakup, to determine a value for $r_\sigma$ [30]. Their result $2.7\text{ fm}$ scales like $R_c$. In contrast, the binding momentum $\gamma_\sigma \approx 29\text{ MeV}$ is as small as $R_h^{-1} \approx 28\text{ MeV}$. It follows that for low momenta $k \sim R_h^{-1}$, the effective range term $\sim R_c/R_h^2$ in Eq. (10) is of NLO compared to $\gamma_\sigma + ik \sim R_h^{-1}$. Higher-order terms in the ellipses are of the order $R_c^3/R_h^4$ (N$^3$LO) at most [30].

Once physics in the pole region is reproduced at a desired accuracy, it becomes obsolete to scale the $^{11}\text{Be}$ ground state wave function with a spectroscopic factor. Such experimentally nonobservable quantities are not required in Halo EFT. Instead, Eq. (10) yields an asymptotic normalization coefficient (ANC)

$$A_\sigma = \sqrt{\frac{2\gamma_\sigma}{1 - \gamma_\sigma r_\sigma + \mathcal{O}(R_c^3/R_h^3)}}$$

for the radial wave function $u_\sigma(r) = A_\sigma \exp(-\gamma_\sigma r)$, which is fully determined by low-energy observables [30].

Recently, Calci et al. were able to calculate the ANC using the no-core shell model with continuum (NCSMC) [31]. Their result $A_\sigma = 0.786\text{ fm}^{-1/2}$ was afterwards confirmed by Yang and Capel in Ref. [19], who extracted the value $(0.785 \pm 0.03)\text{ fm}^{-1/2}$ from cross section data [17]. We will use the ANC of Calci et al. as an input parameter at NLO. Equation (11) can then be inverted to give a value for the effective range, which reads

$$r_\sigma \equiv \left(\gamma_\sigma^{-1} - \frac{2}{A_\sigma^2}\right) \left(1 + \mathcal{O}\left(R_c^2/R_h^2\right)\right) \approx 3.5\text{ fm}.$$ (12)

This value is larger than the one obtained by Hammer and Phillips in Ref. [30]. It will still be counted as $R_c$, since $\gamma_\sigma r_\sigma \approx 0.52$ differs by only $0.12 \lesssim R_c^2/R_h^2$ from $R_c/R_h \sim 0.4$.

2. Propagator expansion

From NLO, the propagator in Eq. (10) exhibits spurious deep poles in addition to the physical one representing $^{11}\text{Be}$ [32]. We solve this issue by expanding $iG_\sigma$ around $k = i\gamma_\sigma$ in terms of $R_c/R_h$, yielding the series

$$iG_\sigma(E_{cm}) = i\frac{2\pi}{\mu Nc} \left[\gamma_\sigma - \sqrt{-2\mu Nc(E_{cm} + i\epsilon)}\right]^{-1} \times \left(1 + \frac{r_\sigma}{2} \left(\gamma_\sigma + \sqrt{-2\mu Nc(E_{cm} + i\epsilon)}\right) + \mathcal{O}\left(R_c^2/R_h^2\right)\right).$$ (13)

The residue of $G_\sigma$ has an analogue expansion and reads

$$Z_\sigma = \left[\frac{\partial G_\sigma^{-1}}{\partial E_{cm}}\right]_{E_{cm} = -B_\sigma}^{-1} = \frac{2\pi}{\mu Nc} \gamma_\sigma \left(1 + \gamma_\sigma r_\sigma + \mathcal{O}\left(R_c^2/R_h^2\right)\right).$$ (14)

In section IV, $G_\sigma$ will enter the three-body Faddeev equation and $Z_\sigma$ is needed to normalize the reaction amplitude. At LO, we will truncate Eqs. (13)–(14) after the leading term.
yielding expressions $G_\sigma^{(\text{LO})}$ and $Z_\sigma^{(\text{LO})}$. The NLO forms $G_\sigma^{(\text{NLO})}$ and $Z_\sigma^{(\text{NLO})}$ also include
the terms linear in $r_\sigma$. We will follow Bedaque et al. by replacing $G_\sigma^{(\text{LO})} \to G_\sigma^{(\text{NLO})}$ in the
Faddeev kernel at NLO [33]. This straightforward technique is often referred to as "partial resummation", because it induces specific amplitude terms proportional to $r_\sigma^n$, $n \geq 2$. In principle, such terms only occur at higher orders. However, for natural cutoffs, they are smaller than NLO terms and do not undermine the validity of the NLO calculation [32, 34].

3. Core excitation effects

So far, we have treated $^{11}\text{Be}$ as a pure $1/2^+ \otimes 0^+$ neutron-core state. However, in principle, it also couples to the $1/2^+ \otimes 2^+$ configuration of a neutron and a core excitation $^{10}\text{Be}^*$. Note that this threshold resides far above the pole at an energy separation $E_\pi + B_\sigma \gg B_\sigma$; see Fig. 1. Close to the pole, $G_\sigma$ is insensitive to nonanalyticities of this remote channel.

Instead, it only receives residual modifications, which are automatically taken into account by renormalization onto low-energy observables like $\gamma_\sigma$, $r_\sigma$, etc. Indeed, Deltuva et al. confirmed that dynamical core excitations within the $^{11}\text{Be}$ bound state barely influence the reaction cross section [18]. In other words, our effective single-channel description readily contains all the relevant core excitation information in the pole regime. For illustration, we show in appendix A that our approach is equivalent to a theory with an explicit $^{10}\text{Be}^*$ field.

B. The deuteron

The deuteron is treated as an $s$-wave neutron-proton bound state with binding energy $B_d = 2.22\text{ MeV}$ [35]. The product $\gamma_d r_d \approx 0.40$ of the small binding momentum $\gamma_d \equiv (m_d B_d)^{1/2} \approx 46\text{ MeV}$ and the effective range $r_d = 1.75\text{ fm}$ [35] is as small as $R_c/R_h$.

Up to NLO ($\sim \gamma_d r_d$), the deuteron can be solved in analogy to $^{11}\text{Be}$, including field redefinitions $\delta_i^{(1)} \to d_i^{(1)} \equiv g_d^{(1)}$. Expressions for the propagator $G_d$ around the pole, its residue $Z_d$, and respective truncations, can be obtained from Eqs. (13)–(14) by replacing all subscripts "\sigma" by "d", the total mass $M_{Nc}$ by $2m_N$, and the reduced mass $\mu_{Nc}$ by $m_N/2$. Relativistic effects and $s$-$d$ mixing are negligible up to NLO, as shown by Chen et al. [36].

C. The Beryllium-11 excited state

A second neutron-core state close to threshold is the first excited state $^{11}\text{Be}^* (1/2^-)$. In Halo EFT, it is treated as a $p$-wave bound state [30] with binding energy $B_\pi = 0.18\text{ MeV}$ [28], or binding momentum $\gamma_\pi \equiv (2\mu_{Nc} B_\pi)^{1/2} \approx 18\text{ MeV}$. The Lagrangian part $L_{2,^{11}\text{Be}^*}$ is given in appendix D. As shown in Ref. [12], shallow $p$-wave states require the inclusion of at least two low-energy parameters. Close to the pole, we choose $\gamma_\pi \sim R_h^{-1}$ and the $p$-wave effective range $r_\pi \sim R_c^{-1}$. The propagator expansion then reads

$$iG_\pi(E_{\text{cm}}) = i \frac{6\pi}{\mu_{Nc}} \frac{2}{-r_\pi} \left[ \gamma_\pi^2 + 2\mu_{Nc}(E_{\text{cm}} + i\epsilon) \right]^{-1} (1 + \mathcal{O}(R_c/R_h)) \cdot (15)$$

2 The deuteron propagator is diagonal in spin space, i.e., it has to be multiplied by $\delta_{i,i'}$ in diagrams.
Similar to the ground state, \( r_\pi \) can be obtained from the respective ANC \( A_\pi [30] \). Taking the value \( A_\pi = 0.129 \text{ fm}^{-1/2} \) of Calci et al. [31], we find

\[
    r_\pi = -\frac{2\gamma_\pi^2}{A_\pi^2} (1 + \mathcal{O}(R_c/R_h)) \approx -0.95 \text{ fm}^{-1}.
\]

In the transfer reaction \(^{10}\text{Be}(d, p)^{11}\text{Be} \), intermediate \(^{11}\text{Be}^* \) states represent NLO corrections to the reaction amplitude since \( G_\pi \propto R_c \), and higher orders in Eq. (15) are at most of \( \text{N}^2\text{LO} \). For the moment, we neglect the excited state. It will be subject to the NLO discussion in section IV D.

We note that there are further two-body states, which are neglected in this work. They include the \(^1S_0 \) virtual neutron-proton state and strong proton-core resonances shown in Fig. 1. Corrections from these states to the reaction amplitude are stronger suppressed than such from \(^{11}\text{Be}^* \) in our scheme. Further details will be given at the end of section IV D.

IV. THREE-BODY SYSTEM

In this section, we derive an integral integration for the reaction cross section from interactions of the Lagrangian \( \mathcal{L} \) up to NLO in the \( R_c/R_h \) expansion. Firstly, we show which three-body diagrams are induced by strong and Coulomb interactions of the Lagrangian \( \mathcal{L} \). Secondly, we construct the LO transfer amplitude and present results for the cross section. At the end of the section, we discuss NLO corrections.

A. Power counting & LO diagrams

The transfer amplitude \( T_{\sigma d} \) connects the two states

\[
    \sigma \equiv |p + ^{11}\text{Be}\rangle, \quad d \equiv |^{10}\text{Be} + d\rangle
\]

through strong and Coulomb interactions. In EFT calculations, these interactions have to be classified in a power counting, which exploits the typical momentum scales of the system.

1. Momentum scales

The typical momentum scales of the three-body system are given by the small binding momentum scale \( \gamma \sim \gamma_d \sim \gamma_\sigma \sim R_h^{-1} \) and the core radius \( R_c \). The largest subleading corrections in the strong sector are suppressed by \( \gamma_\sigma r_\sigma \approx 0.52 \sim \gamma_d r_d \approx 0.40 \); see above.

Coulomb interactions additionally introduce the small "Coulomb momentum"

\[
    p_c \equiv Q_c \alpha \mu_{Ne} \approx 25 \text{ MeV} \lesssim \gamma,
\]

where \( \alpha = e^2/(4\pi) \approx 1/137 \) is the fine structure constant. Moreover, Rupak and Kong pointed out that external momenta \( p \) have to be counted separately from \( \gamma \) in the presence of Coulomb photons [20]. In this work, we calculate cross sections for center-of-mass energies \( E \geq 7.78 \text{ MeV} \). Thus, \( p \) is of the order \( p \sim (2m_N E)^{1/2} \geq 120 \text{ MeV} > \gamma \). The two scales \( p_c \) and \( p \) form a second expansion parameter \( p_c/p < 0.2 \), which we will count like \( (R_c/R_h)^2 \).
2. Strong interaction

In Fig. 3, we display the neutron exchange diagrams that form the elementary building blocks of the strong interaction part of the reaction amplitude. We denote them by $-iV_{sd}^{S,m,1\sigma}$ and $-iV_{ds}^{1m,S'm'}$, where $S, S' \in \{0, 1\}$ and $m, m'$ represent total incoming and outgoing spins and their projections, respectively. Let $p (q)$ be the incoming (outgoing) relative momentum and $E$ the center-of-mass energy. We then find

$$V_{sd}^{S,m,1\sigma}(p, q; E) = -\delta^{S1}\delta^{m1} m_N \left[ p \cdot q + p^2 + \frac{1+y}{2} q^2 - m_N(E+i\epsilon) \right]^{-1},$$

$$V_{ds}^{1m,S'm'}(p, q; E) = V_{sd}^{S'm',1m}(q, p; E),$$

where $y \equiv m_N/m_c$ is the mass ratio. Due to the $s$-wave nature of the short-range interactions, only transitions between spin states $S = S' = 1$ with projections $m = m'$ are possible. In the following, we will refer to the functions in Eqs. (19)–(20) as "neutron exchange potentials".

For neutron exchange iterations, we use the standard power counting of pionless EFT, which counts all momenta formally like $\gamma \sim R^{-1}$. Loops, one-body propagators, and $s$-wave two-body propagators then count like $\gamma^4/m_N$, $m_N/\gamma^2$, and $1/(\gamma m_N)$, respectively. It follows that all neutron exchange iterations are of order $m_N R^2$ and have to be resummed at LO.

3. Coulomb contributions

Next, we consider the Coulomb force, whose repulsion is expected to lower the reaction probability. In calculations, it is usually included as a static two-body potential in addition to some nuclear model interaction. In a strict EFT approach, however, Coulomb interactions can be analyzed in a systematic power counting, which exploits the system’s momentum scales. This procedure reveals the relative importance of strong and Coulomb interactions.

Photon couplings in $\mathcal{L}$ induce the diagrams $-i\Gamma_{ab}^{Sm,S'm'} (a, b \in \{d, \sigma\})$ in Fig. 4. Their mathematical expressions are given in appendix C. In the following, we analyze the diagrams using the Coulomb power counting developed by Rupak and Kong [20].

**Bubble diagrams** The one-loop diagrams (a) and (b) in Fig. 4 are proportional to the photon propagator ($\sim p^{-2}$) and to $p_c$; see Eq. (18). All momenta in the loop ("bubble") may be counted like $\gamma$.\(^4\) I.e., we count one-body propagators like $m_N/\gamma^2$ and the

\(^3\)In this work, relative momenta in the three-body center-of-mass system are defined as the momentum of the respective spectator particle. I.e., they equal $p(^{10}\text{Be}) = -p(d)$ in $|d\rangle$, or $p(p) = -p(^{11}\text{Be})$ in $|\sigma\rangle$.

\(^4\)This statement can be verified by applying the so-called "bubble approximation" [21]; see appendix C.
loop integration by $\gamma^5/m_N$. The resulting scaling $m_N p_c/(p^2 \gamma)$ suggests that bubble diagrams are small compared to neutron exchanges ($\sim m_N/\gamma^2$) since $p > \gamma \gtrsim p_c$.

**Box diagrams** In the box diagrams of Fig. 4 (c) and (d), the photon is part of a loop. In this case, it is not straightforward to see if the corresponding integral is governed by powers of $p$ or $\gamma$. Since $p > \gamma$ in our case, the safest option is to count the loop like $m_N/\gamma^3$. This scheme is in line with Ref. [21]. The overall scaling $m_N p_c/\gamma^3$ implies that box diagrams are of the same order as neutron exchanges since $p_c \lesssim \gamma$.

In summary, the Rupak and Kong counting suggests that box diagrams should be iterated at LO, while bubble diagrams are subleading ($\sim \gamma p_c/p^2$). However, one important feature of the bubble diagrams is not captured by this counting. Their photon propagators exhibit infrared divergences at small momentum transfers in the limit of vanishing photon mass; see Eqs. (C1)–(C2). In principle, this enhancement could compensate for the discussed suppression. We account for this possibility by including the bubble diagrams already in the LO calculation, as was also done in Ref. [21]. We will then critically assess this choice by comparing the numerical influence of the box and bubble diagrams on the cross section.

Note that we only consider diagrams with one photon exchange between two strong interactions. Corrections from two or more successive exchanges should be small since they involve further powers of the small Coulomb momentum $p_c$. In principle, they could be included by replacing each photon propagator with the full Coulomb $T$ matrix; see for example [21]. We have checked that, for example, $-i \Gamma_{dd}$ would be modified by around 20% $\sim p_c/p \sim (R_c/R_h)^2$ in the on-shell case. Such effects are neglected in this work.

### B. Transfer amplitude at LO

By iterating neutron exchanges, Coulomb bubble, and box diagrams to all orders we obtain the LO transfer amplitude $T_{\sigma d}^{(\text{LO})}$. The corresponding Faddeev equation (without three-body force) is shown diagrammatically in Fig. 5.

#### 1. Partial wave channels

It is beneficial for our purposes to perform a partial wave projection onto the total angular momentum $J = L + S$ with total spin $S$ and total orbital angular momentum $L$. 
The respective neutron exchange potentials

\[
V_{\sigma d}^{2S+1L_j,3L_j'} (p, q; E) = \delta_{J'L'} \frac{m_N}{pq} \left( -\frac{p^2 + \frac{1}{2} q^2 - m_N (E + i\epsilon)}{pq} \right),
\]

\[
V_{d\sigma}^{3L_j,2S'+1L_j'} (p, q; E) = V_{\sigma d}^{2S+1L_j,3L_j} (q, p; E),
\]

depend on Legendre functions of the 2nd kind

\[
Q_L(x_0) \equiv \frac{1}{2} \int_{-1}^{1} dx \frac{P_L(x)}{x + x_0}.
\]

Unfortunately, partial wave expressions of the Coulomb interactions in Eqs. (C1)–(C4) are impractically lengthy. Instead, we calculate them numerically via

\[
\Gamma_{ab}^{2S+1L_j,2S'+1L_j'} (p, q; E) = \delta_{JJ'} \frac{1}{2} \int_{-1}^{1} dx \frac{P_L(x)}{x + x_0} \Gamma_{ab}^{S0,S'0} (p, q; E) \quad (a \in \{d, \sigma\})
\]

with \(x \equiv p \cdot q/(pq)\).

As indicated in Fig. 5, the LO elastic and transfer amplitudes can be summarized into an amplitude vector \(\vec{T}^{(LO)}\). Due to the fact that the total spins \(S_d = S_\sigma = 1\) and orbital angular momenta \(L_d = L_\sigma \equiv L \in \{J - 1, J, J + 1\}\) are conserved at LO, we identify a specific partial wave system by the superscript \(^\prime[L, J]^{\prime}\). For incoming (outgoing) relative momenta \(p (p')\), we finally obtain the scattering equations

\[
\vec{T}^{(LO)}[L,J] (p, p'; E) = -K^{(LO)}[L,J] (p, p'; E) \cdot \vec{e}_1 \\
+ 4\pi \int \frac{dq \, q^2}{(2\pi)^3} K^{(LO)}[L,J] (p, q; E) \cdot G^{(LO)} (q; E) \cdot \vec{T}^{(LO)}[L,J] (q, p'; E)
\]

with LO kernel and propagator matrices

\[
\vec{T}^{(LO)}[L,J] \equiv \begin{pmatrix} T^{(LO)}_{dd} & T^{(LO)}_{d\sigma} \\ T^{(LO)}_{d\sigma} & T^{(LO)}_{\sigma d} \end{pmatrix},
\]

\[
K^{(LO)}[L,J] \equiv \begin{pmatrix} \Gamma_{dd} & V_{d\sigma} + \Gamma_{d\sigma} \\ V_{d\sigma} + \Gamma_{d\sigma} & \Gamma_{\sigma\sigma} \end{pmatrix}^{3L_j,3L_j},
\]

\[
G^{(LO)} \equiv \text{diag} \left[ G_d^{(LO)}, G_\sigma^{(LO)} \right],
\]
and \( \vec{e}_1 \equiv (1, 0)^T \) in channel space. For convenience, we introduced the new functions

\[
G_a^{(\text{N}^\text{N} \text{LO})} (q; E) \equiv G_a^{(\text{N}^\text{N} \text{LO})} \left( E - q^2 / (2 \mu_a) \right) \quad (a \in \{d, \sigma\}, \ n \in \mathbb{N}_0),
\]

where \( \mu_d \equiv 2m_N m_c / (2m_N + m_c) \) and \( \mu_\sigma \equiv (m_N + m_c) m_N / (2m_N + m_c) \).

The full transfer amplitude is given as a sum over the partial wave amplitudes and respective projection operators as shown in appendix B. In all calculations, we truncate the sum at some maximal orbital angular momentum \( L_{\text{max}} \) and increase this value toward convergence. Similarly, whenever including Coulomb diagrams, we decrease the photon mass \( \lambda \to 0 \). We find that the cross section convergences at \( L_{\text{max}} = 12 \) and \( \lambda = 0.1 \text{ MeV} \).

2. Unphysical deep bound states

To see if Eq. (25) requires a three-body force for renormalization, we perform an asymptotic analysis for large incoming and loop momenta \( p, q \gg \gamma_d, \gamma_\sigma, (m_N|E|)^{1/2} \) similar to Ref. [37]. In this limit, the nucleon exchange potential (\( \sim q^{-2} \)) dominates over the Coulomb interactions (\( \sim q^{-3} \)) [21]. Thus, we may neglect the Coulomb force for the moment. It turns out that for \( L \geq 1 \), the potentials in Eq. (25) fall of fast enough to produce unique amplitudes solutions. In the \( L = 0, J = 1 \) system, however, that is not the case. Instead, the amplitudes approach a power law behavior \( \sim p^{-1+s} \) with \( s = 0 \).

It follows that the system exhibits an Efimov effect, i.e., a geometric spectrum of three-body bound states at energies \( E = -B_d - B_3 \) [38–40]. We note that \( \exp(\pi/|s|) \approx 140 \) reproduces the universal scaling factor of three distinguishable particles with mass ratio \( y = 0.1 \) presented in Ref. [39].

In the following, we equip Eq. (25) with a momentum cutoff \( \Lambda \gg \gamma_d, \gamma_\sigma, (m_N|E|)^{1/2} \). The resulting spectrum is shown in Fig. 6 as dashed lines. Coulomb interactions do not influence the large momentum behavior of the system qualitatively. They only push the Efimov states to higher cutoffs (solid lines in Fig. 6). The system will be renormalized using the three-body coupling \( C_0(\Lambda) \) of Eq. (6). It enters the kernel matrix of Eq. (27) as a constant \( s \)-wave potential like

\[
K_{dd}^{(\text{LO})} [0, 1] \to K_{dd}^{(\text{LO})} [0, 1] + C_0(\Lambda).
\]

Note that the choice of this specific three-body force is not unique. One could also introduce it in the transfer or the \(|\sigma\rangle \) elastic channel.

The quantum numbers of the Efimov states correspond to those of a \( J^\pi = 1^+ \) level in \( ^{12}\text{B} \). Experimentally, three such states are known [41]. In a deuteron-\( ^{10}\text{Be} \) cluster picture, their binding energies \( B_3^{(\text{phys})} \geq 5.77 \text{ MeV} \) correspond to spatial separations \( R_3 = (2\mu_d B_3)^{-1/2} \leq 1.5 \text{ fm} \) of the deuteron-\( ^{10}\text{Be} \) pair. Being of the order \( R_c \), they do not reflect a separation of scales in the three-body sector. Thus, the cluster picture is not justified and the Efimov states can be understood as artifacts of the short-range approach. However, although unphysical, they do not impose a problem as long as they lie outside the EFT’s region of applicability. Indeed, after renormalization onto cross section data, all three-body states will occur at binding energies \( B_3 > 19 \text{ MeV} \) and thus far away from the low-energy region; see Fig. 8 (b).
Figure 6: Unrenormalized three-body spectra at LO without (dashed lines) and with (solid lines) Coulomb interactions ($\lambda = 0.1$ MeV, converged) for various cutoffs $\Lambda$.

C. Cross section

The differential cross section of the reaction $^{10}$Be(d, p)$^{11}$Be at a deuteron beam energy

$$E_d = \frac{2m_N}{\mu_d} (E + B_d)$$

(31)

can be obtained by multiplying the transfer amplitude by the residue factor $(Z_a Z_d)^{1/2}$ and evaluating it at on-shell relative momenta

$$\bar{p}_a \equiv \sqrt{2\mu_a(E + B_a + i\epsilon)} , \ (a \in \{d, \sigma\}).$$

(32)

The cross section depends on the center-of-mass angle $\theta_{cm}$ with $\cos\theta_{cm} \equiv \hat{p}(d) \cdot \hat{p}(p)$. In the $|d\rangle$ channel, we set the relative momentum to $\bar{p}_d \equiv -\bar{p}_d \hat{p}(d)$ and in the $|\sigma\rangle$ channel we take $\bar{p}_\sigma \equiv \bar{p}_\sigma \hat{p}'(p)$. The spin-averaged reaction cross section then reads

$$\left( \frac{d\sigma}{d\Omega} \right)(\theta_{cm}; E) = \frac{1}{3} \sum_{m, S', m'} \frac{\mu_d \mu_\sigma}{4\pi^2} \bar{p}_d Z_d Z_\sigma \left| T^{1m, S'm'}_{d\sigma} (\bar{p}_d, \bar{p}_\sigma; E) \right|^2 ,$$

(33)

where $|T^{1m, S'm'}_{d\sigma} (\bar{p}, \bar{p}'; E) |^2 = |T^{S'm',1m}_{\sigma d} (\bar{p}', \bar{p}; E) |^2$.

Table I summarizes the input parameters needed for the calculation of the reaction cross section up to NLO in the $R_c/R_h$ expansion. At LO, only the binding energies $B_d$ and $B_\sigma$ are required. At NLO, also the effective range $r_d$, the ANC $A_\sigma$ of $^{11}$Be, and the binding energy $B_\pi$ and ANC $A_\pi$ of $^{11}$Be$^*$ enter.

1. Coulomb suppression & improved LO system

Our first goal is to critically assess the Coulomb power counting performed above. In particular, we would like to validate the proposed LO nature of the Coulomb force in general and of the bubble diagrams specifically, for the experimental energies used by Schmitt et al. [17]. Given the cutoff-dependence of the $L = 0$ channel, we vary $\Lambda$ in the large range.
Table I: EFT inputs for the calculation of the reaction cross section up to NLO.

| Order            | deuteron | $^{11}$Be | $^{11}$Be$^*$ |
|------------------|----------|-----------|---------------|
| LO $[\mathcal{O}(1)]$ | $B_d = 2.22$ MeV [35] | $B_\sigma = 0.50$ MeV [28] | – |
| NLO $[\mathcal{O}(R_c/R_h)]$ | $r_d = 1.75$ fm [35] | $A_\sigma = 0.786$ fm$^{-1/2}$ [31] | $B_\pi = 0.18$ MeV [28], $A_\pi = 0.129$ fm$^{-1/2}$ [31] |

Figure 7: LO cross section of $^{10}$Be(d, p)$^{11}$Be as function of the center-of mass angle $\theta_{\text{cm}}$. For different deuteron energies $E_d$ (lab frame), the results are compared to data (black points) from Ref. [17]. All bands show cutoff variations $\Lambda \in [300, 1500]$ MeV. Each single curve is converged at $L_{\text{max}} = 12$ and $\lambda = 0.1$ MeV. Hatched bands exclude Coulomb interactions. Light (dark) filled bands enclosed by dotted lines include the Coulomb box (and bubble) diagrams. Dash-dotted curves represent a $\chi^2$-fit of the full equation system in Fig. 5 onto the depicted $E_d = 12$ MeV data using the three-body force $C_0(\Lambda)$; see also Fig. 8. The fit is cutoff-independent for $\Lambda \geq 500$ MeV.
$\Lambda \in [300, 1500]$ MeV in each calculation. This procedure reveals the potential impact of the $s$-wave three-body force $C_0(\Lambda)$ on the LO reaction cross sections.

In a first step, we switch off all Coulomb interactions, which yields the uppermost bands (hatched) in Fig. 7. Each curve is converged at percent level for $L_{\text{max}} = 12$. At all four deuteron beam energies $E_d \in \{12, 15, 18, 21.4\}$ MeV (lab frame), the bands lie high above the experimental data by Schmitt et al. [17]. Apparently, the strong interaction alone does not produce enough repulsion between the scattering partners, even if $C_0(\Lambda)$ is included.

In order to understand the relative importance of the Coulomb box and bubble diagrams, we add them successively to the Faddeev equation. The light bands surrounded by dotted lines in Fig. 7 show that the box diagrams alone lower the cross sections drastically at all beam energies as expected. Indeed, it is important to include them at LO. Further repulsion comes from the bubble diagrams. Their inclusion yields the dark lowermost bands in Fig. 7. Apparently, the influence of the bubble diagrams on the cross section is $\lesssim 40\%$ smaller than the one of the box diagrams. Thus, it seems as if we have overestimated the enhancement due to the bubble diagrams’ infrared divergences by one order in $R_h/R_c$. A posteriori, the bubble diagrams are of NLO and could in principle be neglected at LO. The ”pure LO“ system then only contains neutron transfer and box diagrams.

Interestingly, however, the inclusion of the bubble diagrams as one specific NLO correction leads to a surprisingly good agreement with the cross section data at lower beam energies and forward angles. Thus, choosing the ”improved LO“ system of Fig. 5 significantly accelerates the EFT convergence. This statement will be verified later by including the remaining NLO corrections. Moreover, the improved LO system, unlike the pure one, can be renormalized onto data at $E_d = 12$ MeV since the respective band comprises all data points.

2. Peripherality regions

Although subleading in a strict sense, the bubble diagrams do not introduce any new parameters like, for example, effective range coefficients. Thus, the improved LO system stays independent of short-range details. Cross sections are then only affected by the tail of the $^{11}$Be wave function, i.e., the reaction is purely ”peripheral“. Yang and Capel argued that such a description is sufficient to describe the reaction at lower beam energies and forward angles [19]. Our results provide clear evidence for this claim since the improved LO band for $E_d = 12$ MeV perfectly describes the whole data region ($4.7^\circ \leq \theta_{\text{cm}} \leq 10.4^\circ$).

Moreover, according to Yang and Capel, the peripherality region increases (decreases) in size for lower (higher) energies. Indeed, at $E_d = 15$ MeV, only forward scattering ($\theta_{\text{cm}} \leq 4.6^\circ$) is captured by the improved LO band. Deviations at larger angles are of NLO size. At even higher energies $E_d \geq 18$ MeV, however, the bands deviate from data by 40-80%. We conclude that the reaction is indeed only peripheral at forward angles and low energies. That implies that our short-range EFT may fail at energies above $E_d \approx 15$ MeV.

3. Cutoff-dependence & renormalizability

Out of all components $L \leq L_{\text{max}} = 12$, only the $L = 0$ part is cutoff-dependent. Due to this circumstance, the band widths in Fig. 7 are only 20% the size of the box diagram shift (LO). Such contributions are negligible up to NLO. Thus, in principle, each curve within the filled bands represents an LO result itself and renormalization is not required. Let us
emphasize that the only inputs to our LO system are then given by the binding energies $B_d$ and $B_\sigma$; see Tab. I. At astrophysical energies, however, the $L = 0$ component is of much greater importance, leading to a much stronger cutoff dependence.

We demonstrate the renormalizability of the improved LO system using the three-body force $C_0(\Lambda)$. For various cutoffs $\Lambda \geq 300$ MeV, we adjust it in a $\chi^2$-fit to the depicted $E_d = 12$ MeV data set. This procedure yields the two solutions for $C_0(\Lambda)$ shown in Fig. 8 (a). Their fit values $\chi^2 \approx 2.29$ (solid curve) and $\chi^2 \approx 2.23$ (dot-dashed curve) are, within numerical uncertainties, equal in size and respectively constant for $\Lambda \geq 500$ MeV. For illustration, we show fit results for $\Lambda = 500$ MeV in Fig. 7 as dot-dashed curves. The first three-body state occurs at $\Lambda \approx 300$ MeV (or $\Lambda \approx 7$ GeV); see Fig. 8 (b). It lies above $B_3 \approx 19$ MeV (or $B_3 \approx 28$ GeV) and converges to even higher values as $\Lambda \to \infty$.

D. Corrections at NLO and beyond

We now discuss NLO contributions to the reaction cross section in the $R_c/R_h$ expansion, stemming from range corrections in the two-body sectors and from the excited state $^{11}$Be$^*$. 

1. Effective range corrections

A straightforward way to include effective range corrections in the deuteron and $^{11}$Be is to replace the LO propagators $G_a^{(LO)}$ by $G_a^{(NLO)}$ ($a \in \{d, \sigma\}$) in Eq. (25) [33].\footnote{Correspondingly, one has to use the residues $Z_a^{(NLO)}$ in the calculation of the cross section in Eq. (33).} Note that this approach reintroduces a cutoff-dependence in the $L = 0$ channel. In principle, it could be cured by readjusting the three-body force $C_0(\Lambda)$ [42]. In order to see the impact of the additional cutoff-dependence, we include effective range corrections in the renormalized
improved LO system for various $\Lambda \in [500, 1500]$ MeV. Figure 9 shows that the resulting red hatched bands lie well within the ±40% LO uncertainty band (blue, enclosed by thin solid lines) of the improved LO estimates (blue dot-dashed curves). The band widths are comparably small, giving rise to a mild cutoff dependence.

It has to be mentioned that a small fraction of the band widths stems from an unexpected

\[ \frac{d\sigma}{d\Omega} (\theta_{\text{cm}}) \] / mb

\[ \theta_{\text{cm}}/\text{deg} \]

\[ \frac{d\sigma}{d\Omega} (\theta_{\text{cm}}) \] / mb

\[ \theta_{\text{cm}}/\text{deg} \]

---

6 Below $\Lambda = 500$ MeV, the renormalized improved LO result is not yet converged. Note that the cutoff variation up to 1500 MeV is only used to estimate higher-order corrections. It does, however, not reveal the necessity of additional counter terms.
The one-neutron exchange diagrams (a) $-iV_{\pi d}$ and (b) $-iV_{d\pi}$ induce contributions to the transfer amplitude $T^{(NLO)}_{\sigma d}$. The exemplary diagram in (c) contains a propagator $G^{(LO)}_{\pi d}$ depicted as a thickened solid-dashed double line.

cutoff dependence in the $L = 1$ sector. It can be understood as an artifact of the choice, not to perturb the amplitude itself to first order in $R_c/R_h$, but the integration kernel. That modifies the UV behavior of the partial wave amplitudes, leading to a divergence in the $L = 1$ sector. This divergence would not be present in a strictly perturbative approach [37]. Even though desirable, such a more involved NLO treatment lies beyond the scope of this work. In fact, we have checked that the influence of the cutoff on the $L = 1$ amplitude is less than 2% over the range $\Lambda \in [500, 1500]$ MeV. Thus, this issue can be neglected at NLO.

2. The Beryllium-11 excited state

The excited state $^{11}\text{Be}^*$ introduces a third channel $|\pi\rangle \equiv |p + ^{11}\text{Be}\rangle$ to the three-body system. It couples to $|d\rangle$ via the diagrams $-iV_{\pi d}, -iV_{d\pi}$ shown in Fig. 10 (a,b). Their mathematical forms and partial wave projections are given in appendix D. We note that $|\pi\rangle$ only occurs as an intermediate state in the reaction. Thus, the NLO nature of $^{11}\text{Be}^*$ follows from the propagator scaling $G^{(LO)}_{\pi d} \sim R_c/(\gamma^2 m_N)$; see section III. A typical contribution to the reaction amplitude is given by Fig. 10 (c). Again, we count all loop momenta like $\gamma \sim R_h^{-1}$. Note that the two (neutron-core)-$^{11}\text{Be}^*$ vertices contribute a factor $\gamma^2$. The overall scaling $m_N R_c R_h$ is then one order smaller than the LO scaling $m_N R_h^2$.

We complete the NLO system by inserting both effective range corrections in $G_d$ and $G_{\sigma}$, and the potentials $V_{\pi d}, V_{d\pi}$ into the integration kernel. The resulting Faddeev equations are given in appendix E. Similar to the previous calculation, we vary $\Lambda \in [500, 1500]$ MeV and include the LO three-body force $C_0(\Lambda)$. Figure 9 shows that the results of the previous calculation (hatched bands) get shifted back toward the improved LO results, ending up as red bands enclosed by dotted lines. Thus, the influence of $^{11}\text{Be}^*$ is indeed of NLO, in agreement with our power counting. The remaining cutoff dependences of the $L = 0$ and $L = 1$ sectors are negligible compared to $N^2$LO corrections ($\pm 16\%$, red uncertainty bands enclosed by thick solid lines). Thus, no further renormalization is needed at NLO.

Recall that the NLO parameters $r_\sigma = 3.5\text{ fm}$ and $r_\pi = -0.95\text{ fm}^{-1}$ were calculated in Eqs. (12) and (16) from the ANCs of Calci et al. [31]. Instead, one could directly use the Halo EFT values $r_\sigma = 2.7\text{ fm}$ and $r_\pi = -0.66\text{ fm}^{-1}$ of Hammer and Phillips [30]. The relative differences 30% and 40% are of size $R_{c}/R_h$ and should thus be negligible at NLO. We have checked that the final NLO bands would only indeed only change by ca. 5%. Thus, both choices for $r_\sigma, r_\pi$ are consistent with the proposed power counting.
Figure 11: The excited state $^{11}\text{Be}^*$ allows transitions from total spin $S = 1$ to $S = 1$ ($|d\rangle \rightarrow |\pi\rangle \rightarrow |d\rangle$) or to $S = 0$ ($|d\rangle \rightarrow |\pi\rangle \rightarrow |\text{np}(1S_0) + ^{10}\text{Be}\rangle$). The thickened solid-dotted double line represents the neutron-proton $1S_0$ virtual state. Multiple transitions via $|\pi\rangle$ are negligible at NLO.

3. Higher-order channels

At higher orders in Halo EFT, additional scattering channels enter the calculation. For example, the proton-neutron sector exhibits a shallow $1S_0$ virtual state [26, 27]. It does not occur at LO, because the total neutron-proton spin $S = 0$ is conserved if all interactions are of $s$-wave type. In the presence of the $p$-wave state $^{11}\text{Be}^*$, however, $S$ may change, and transitions $|d\rangle \rightarrow |\pi\rangle \rightarrow |\text{np}(1S_0) + ^{10}\text{Be}\rangle$ become possible; see Fig. 11. However, the virtual state is not only suppressed due to the intermediate $|\pi\rangle$ channel. Since multiple spin changes ($\sim (R_c/R_h)^2$ or smaller) are negligible at NLO, a virtual state leads to $S = 0$ in the final state of $^{10}\text{Be}(d, p)^{11}\text{Be}$. The corresponding phase space is $1/3$ the size of $S = 1$, yielding a suppression of $R_c/(3R_h) \lesssim (R_c/R_h)^2$. We thus neglect virtual states at NLO.

In Ref. [16], several $^{11}\text{B}$ resonances have been observed in $^{10}\text{Be}(p, \gamma)^{11}\text{B}$; see Fig. 1. The lowest one ($1/2^+$) occurs at a proton-core center-of-mass energy $E_c = (1.33 \pm 0.04)$ MeV. It has a total width $\Gamma = (230 \pm 65)$ keV and the branching ratio for decay into $^{10}\text{Be} + p$ is close to 1 [16]. The resonance represents a pole at $E_{cm} = E_r - i\Gamma/2$ in the Coulomb-modified resonance propagator; see for example Refs. [43, 44]. This pole position implies effective range terms $a_c^{-1} = ((-2.7 \pm 0.8) \text{ fm})^{-1}$ and $r_c/2 (2\mu_{NC}E_r) = ((-3.5 \pm 1.4) \text{ fm})^{-1}$, which scale like $R_c^{-1}$. Moreover, in three-body diagrams, the resonance propagator comes along with a Gamow-Sommerfeld factor $0 < C_\eta ^2 < 1$ [45]. It gives the probability of two charged particles to meet in one point. At resonance, it takes the small value $0.13 \lesssim (R_c/R_h)^2$. It follows that the influence of the resonance propagator on the reaction is suppressed by three orders in $R_c/R_h$ compared to $G_\sigma$. Note that there are more $^{11}\text{B}$ states around $\bar{E} = 0$, which could possibly couple strongly to the proton-core system. However, transitions to those states would involve even smaller Gamow factors $C_\eta ^2 < (R_c/R_h)^2$. Thus, we neglect strong proton-core interactions at NLO.

During the reaction process, the $^{11}\text{Be}$ state may break up into an excited core $^{10}\text{Be}^*$ and a neutron. Thus, $|\sigma\rangle$ in principle couples to the additional intermediate channel $|^{10}\text{Be}^* + d\rangle$ via neutron exchanges. However, each such channel comes along with two couplings of order $R_c^2$; see appendix A for details. Thus, dynamical core excitations can be neglected at NLO.

V. SUMMARY & OUTLOOK

In this work, we carried out the first Halo EFT calculation of deuteron-induced transfer reactions. As a working example, we considered $^{10}\text{Be}(d, p)^{11}\text{Be}$, involving the one-neutron halo nucleus $^{11}\text{Be}$. The degrees of freedom in this approach are the $^{10}\text{Be}$ core, the neutron, and the proton. Strong interactions are described by contact forces alone. To obtain the
differential cross section, the reaction amplitude was constructed diagrammatically in an expansion in the ratio $R_c/R_h \sim 0.4$ of core and halo radius. The corresponding Faddeev equation contains all dynamical features of a transfer reaction. A three-body force ensures internal consistency. We included the Coulomb force by considering the dominant photon exchange diagrams, which were iterated to all orders in the Faddeev equation.

The differential cross section was compared to experimental data by Schmitt et al. [17]. In agreement with Yang and Capel [19], who calculated the cross section in the adiabatic distorted wave approximation, we found that Halo EFT is able to describe scattering at low beam energies $E_d \lesssim 15$ MeV (center-of-mass energies $E \lesssim 10$ MeV). In this regime, the reaction can be considered peripheral, i.e., it predominantly depends on the long-range tail of the $^{11}$Be wave function. This part is systematically reproduced by the $R_c/R_h$ expansion.

Our theory contains only few information on the spectra of the involved particles. We included, in particular, only two-body states with a binding momentum $\gamma$ clearly smaller than the respective momentum scale of short-range physics; see Fig. 1. The influence of such states should be enhanced by powers of $\gamma^{-1}$ compared to those far away from the two-body threshold. As a consequence of this reduction, we were able to describe data using only a minimal amount of experimental input. At leading order (LO) in the $R_c/R_h$ expansion, only the binding energies of deuteron and $^{11}$Be are needed; see Table I. Next-to-leading-order (NLO) corrections arise from respective effective ranges and the first exited state $^{11}$Be*. The effective ranges of the $^{11}$Be states were extracted from the ANCs of the ab-initio calculation by Calci et al. [31]. Both NLO corrections modify the cross section at a $40\%$ level, as predicted by the power counting.

While our results describe data at $E_d \lesssim 15$ MeV fairly well, they strongly overestimate the cross section at higher beam energies. Apparently, the low-energy expansion of Halo EFT converges, if at all, slowly at these energies. In order to improve the expansion, it might be necessary to modify the three-body power counting, which, at the moment, counts loop momenta like small binding momenta. For example, dynamical core excitations might then already occur at lower orders. Further hints on missing ingredients in our approach can be inferred from previous theoretical analyses, e.g. by Schmitt et al. in Ref. [17], Deltuva et al. in Ref. [18] or Yang and Capel in Ref. [19], which were successfull in describing also scattering for $E_d \geq 15$ MeV. We note that the model used in Ref. [19] contains the same amount of information on the $^{11}$Be spectrum as our work. Thus, we do not expect the inclusion of $^{11}$Be levels beyond the first excited state to be of prime importance.

Instead, we might need to consider not explicitly measured loss channels, in particular due to deep $^{11}$B states indicated in Fig. 1, at these energies. Usually, such effects are included using optical model potentials, adjusted to, for example, proton-core scattering data. In the future, we will instead introduce imaginary contact terms to the strong Lagrangian, a method called "Open EFT" [46]. It was applied successfully to a broad range of inelastic processes including quarkonium decays in nonrelativistic QCD [47] and three-body recombinations of ultracold atoms [48]. Let us emphasize again, that Halo EFT is ideally suited for the description of strong interactions at low energies. In this sense, our long-term goal is to apply the developed framework to the astrophysical regime. Although Coulomb interactions become nonperturbative then, short-range effects like higher-order states should be even less important. Moreover, it will be interesting to calculate cross sections for other deuteron-induced reactions like $^{14}$C(d, p)$^{15}$C.
Acknowledgments

We thank D. R. Phillips for giving valuable feedback on the manuscript and S. König for providing information on the calculation of the Coulomb box diagrams. M. S. appreciates stimulating discussions with I. Thompson, D. Baye, and other participants of the INT Program INT-17-1a "Toward Predictive Theories of Nuclear Reactions Across the Isotopic Chart". Moreover, M. S. sincerely thanks the Nuclear Theory groups of UT Knoxville and Oak Ridge National Laboratory for their kind hospitality and support during his research stay. This work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 279384907 – SFB 1245, by the National Science Foundation under Grant No. PHY-1555030, and by the Office of Nuclear Physics, U.S. Department of Energy under Contract No. DE-AC05-00OR22725.

Appendix A: Core excitation effects

In this section, we show that core excitation effects in the pole region are taken care of in this work due to renormalization onto low-energy observables. For that, we consider a theory with an explicit $^{10}\text{Be}^\ast$ field $C_m$ ($m \in \{-2, \ldots, 2\}$) by adding a piece

$$\mathcal{L}_{1,^{10}\text{Be}^\ast} = C_m^\dagger \left( \partial_0 + \frac{\nabla^2}{2m_c} - E_x \right) C_m$$

(A1)

to the Lagrangian. A similar approach has been chosen by Zhang et al. to analyze effects of the core excitation $^{7}\text{Li}^\ast$ on the $^{7}\text{Li}(n, \gamma)^8\text{Li}$ reaction [49]. Moreover, Zhang et al. and Ryberg et al. used a $^{7}\text{Be}^\ast$ core excitation field in their calculation of the $S$-factor of $^{7}\text{Be}(p, \gamma)^8\text{B}$ [50, 51]. In both systems, the core excitation occurs at low energies. That, however, is not true in our case since $(2\mu Nc E_x)^{1/2} \sim R_c^{-1}$.

Together with a neutron, $^{10}\text{Be}^\ast$ couples to the $^{11}\text{Be}$ ground state in a $d$-wave. In terms of the redefined field $\tilde{\sigma}_\alpha$, we thus write

$$\mathcal{L}_{2,^{10}\text{Be}^\ast} = -\sum_{s \in \{3/2, 5/2\}} \frac{g_{s,x}^{(s)}}{g_\sigma} C_{1/2m_1 m_2}^{2m_1, s} \left[ \tilde{\sigma}_\alpha^{\dagger} \left( n_\alpha \left[ -i \nabla \right]_{2m_1} C_m \right) + \text{H.c.} \right].$$

(A2)

The vertex term contains a Galilei-invariant derivative $\nabla \equiv \mu_{Nc}(m_N^{-1} \nabla - m_c^{-1} \nabla)$. It is embedded in the tensor structure

$$[\mathcal{O}]_{lm_l} \equiv \sqrt{\frac{4\pi}{2l+1}} |\mathcal{O}|^l Y_{lm_l}^\ast(\hat{\mathcal{O}})$$

(A3)

with $l = 2$, where $Y_{lm}^\ast(\hat{\mathcal{O}})$ denotes a spherical harmonic, evaluated at $\hat{\mathcal{O}} \equiv \mathcal{O} / |\mathcal{O}|$.

The mass difference $E_x + B_\sigma \gg B_\sigma$ in the transition is of natural size. Thus, we assume no fine-tuning in this scattering channel and count $g_{s,x}^{(s)} \sim R_c^{3/2}$. It follows that the overall coupling $g_{s,x}^{(s)}/g_\sigma \sim R_c^2$ is natural as well, since $g_\sigma \sim r_\sigma^{1/2} \sim R_c^{-1/2}$; see Eq. (9).

The core excitation modifies the $^{11}\text{Be}$ propagator through the $^{10}\text{Be}^\ast$-neutron self-energy loop $-i\Sigma_{s,x}^{\sigma, \sigma'}$. It resembles the neutron-core self-energy loop in Fig. 2, but all core lines
have to be replaced by core excitation lines. Using the PDS scheme, we find

\[ \Sigma_{\sigma,x}(E_{cm}) = - \sum_s \left( \frac{g_{s,x}^{(s)}}{g_{\sigma}} \right)^2 \frac{\mu_{Nc}}{10\pi} \left[ 2\mu_{Nc}(E_{cm} - E_x + i\epsilon) \right]^2 \]

\[ \times \left( \Lambda_{\text{PDS}} - [-2\mu_{Nc}(E_{cm} - E_x + i\epsilon)]^{1/2} \right) \]

\[ \equiv - g_{\sigma}^{-2} \sum_n \Delta_{\sigma,x}^{(n)}(E_{cm} + i\epsilon)^n. \] (A4)

Note that \( \Sigma_{\sigma,x} \) is analytic for \( E_{cm} < E_x \), i.e., it can be expanded at \( E_{cm} = 0 \). The resulting coefficients \( \Delta_{\sigma,x}^{(n)} \) then contribute to the unrenormalized parameters \( \Delta_{\sigma}^{(1)}(\Delta_{\sigma}^{(1)} \equiv -1) \) of the bare \( ^{11}\text{Be} \) propagator; see Eq. (5). Thus, renormalization onto observables \( \gamma_{\sigma} \) (or \( a_{\sigma} \), \( r_{\sigma} \), etc.) automatically takes care of core excitation effects at small \( E_{cm} \), where the pole is located.

In other words, \( C_m \) does not introduce any new information to the two-body sector and can be integrated out.

**Appendix B: Partial wave expansion**

Let us consider a general interaction \( I \), which could be an amplitude \( T \), a neutron exchange potential \( V \) or a Coulomb interaction \( \Gamma \). We expand \( I \) in tensor spherical harmonics

\[ (Y_{(L,S)Jm_j}(\hat{p}))^m \equiv \sqrt{4\pi} \sum_{m_L} C_{LmL,Sm}^{Jm_j} Y_{L}^{m_L}(\hat{p}) \] (B1)

by writing

\[ \mathcal{I}^{S_m,S_{m'}^{m'}}(p, q; E) = \sum_{J,L,L'} \sum_{J^*} \mathcal{I}^{2S+1L_J,2S'+1L_{J'}^*}(p, q; E) P_{2S+1L_J,2S'+1L_{J'}^*}^{m,m'}(\hat{p}, \hat{q}), \] (B2)

\[ P_{2S+1L_J,2S'+1L_{J'}^*}^{m,m'}(\hat{p}, \hat{q}) \equiv \sum_{m_J} (Y_{(L,S)Jm_j}(\hat{p}))^m (Y_{(L',S')Jm_j}(\hat{q}))^{m*}. \] (B3)

Specific partial waves can be extracted via

\[ \mathcal{I}^{2S+1L_J,2S'+1L_{J'}^*}(p, q; E) = \frac{(4\pi)^{-2}}{2J+1} \sum_{m_{m'}^{m',m}} \int_{\Omega_{\hat{p}},\Omega_{\hat{q}}} P_{2S+1L_J,2S'+1L_{J'}^*}^{m,m'}(\hat{q}, \hat{p}) \mathcal{I}^{S_m,S_{m'}^{m'}}(p, q; E). \] (B4)

**Appendix C: Coulomb diagrams**

The Coulomb interactions in Fig. 4 resemble such considered by König et al. for the three-nucleon system [22]. However, they exhibit nontrivial dependencies on the mass ratio
$y \equiv m_N/m_c$. The bubble interactions read

$$\Gamma_{dd}^{1m,1m'}(p, q; E) = \delta^{mm'} \frac{Q_c \alpha m_N^2}{(p - q)^2 + \lambda^2 - i\epsilon} \times f\left(p - q, A_d(p; E), A_d(q; E)\right), \quad (C1)$$

$$\Gamma_{\sigma\sigma}^{Sm, S'm'}(p, q; E) = \delta^{SS'} \delta^{mm'} \frac{Q_c \alpha (2\mu_N)^2}{(p - q)^2 + \lambda^2 - i\epsilon} \times f\left(\frac{y}{\xi} (p - q), A_\sigma(p; E), A_\sigma(q; E)\right), \quad (C2)$$

and the box interactions are given by

$$\Gamma_{\sigma d}^{Sm,1m'}(p, q; E) = \Gamma_{\sigma d}^{Sm,1m'}(q, p; E), \quad (C4)$$

where we defined $\xi \equiv (1 + y)/2$. Moreover, $\alpha \equiv e^2/(4\pi) \approx 1/137$ is the fine structure constant and $Q_c = 4$ is the core charge. All interactions involve the function

$$f(\Delta, A_1, A_2) \equiv \frac{1}{|\Delta|} \tan^{-1}\left(\frac{A_1 - A_2 + \Delta^2/4}{|\Delta| \sqrt{A_2}}\right) + [A_1 \leftrightarrow A_2], \quad (C5)$$

whose arguments involve the expressions

$$A_d(p; E) \equiv \frac{1 + 2y}{4} p^2 - m_N(E + i\epsilon) \xrightarrow{\text{on-shell}} \gamma_d^2, \quad (C6)$$

$$A_\sigma(p; E) \equiv \xi^{-2} \frac{1 + 2y}{4} p^2 - \xi^{-1} m_N(E + i\epsilon) \xrightarrow{\text{on-shell}} \gamma_\sigma^2. \quad (C7)$$

The form of $\Gamma_{\sigma\sigma}$ can be simplified significantly by neglecting terms of order $O\left(y^2\right)$; see Eq. (C2). This approximation is justified since $y^2 = 0.01$ is a tiny number. The only angular dependence then comes from the photon propagator, which can be projected onto certain partial waves analytically.

The bubble diagrams $-i\Gamma_{dd}^{1m,1m'}$ and $-i\Gamma_{\sigma\sigma}^{Sm, S'm'}$ are linear in the Coulomb propagator. Thus, their largest contributions to the transfer reaction comes from the region of small momentum transfers $p - q$. For $p = q$, the values of the function $f$ in Eqs. (C1)–(C2) collapse to $[A_a(p; E)]^{-1/2}/2 \xrightarrow{\text{on-shell}} 1/(2\gamma_a) \ (a \in \{d, \sigma\})$. Thus, the deuteron and halo loops of the LO bubble diagrams in Fig. 4 may be counted like $m_N/\gamma$. 

23
The Coulomb interactions can be connected to the $s$-wave projected functions $K_{\text{bubble}}$ and $K_{\text{box}}$ of Ref. [22] by taking the limits $y, Q_c \to 1$. We find

$$\int_{-1}^{1} \frac{dx}{x} \Gamma_{aa}^{10,10} (p, q; E) \bigg|_{y, Q_c \to 1} = - \frac{m_N}{4\pi} K_{\text{bubble}} (E; p, q) \ (a \in \{d, \sigma\}), \quad (C8)$$

$$\int_{-1}^{1} \frac{dx}{x} \Gamma_{sd}^{10,10} (p, q; E) \bigg|_{y, Q_c \to 1} = - \frac{m_N}{2\pi} K_{\text{box}} (E; p, q), \quad (C9)$$

where $x \equiv p \cdot q / (pq)$.

**Appendix D: Excited state of Beryllium-11**

In this section, we discuss the inclusion of the excited state $^{11}\text{Be}^*$ at NLO in the reaction calculation. The Lagrangian part

$$L^{^{11}\text{Be}^*} = \pi^\dagger_{\alpha} \left[ \Delta_\pi^{(0)} + \left( i\partial_0 + \frac{\nabla^2}{2M_{Nc}} \right) \right] \pi_{\alpha}$$

$$- g_\pi C^{1/2\alpha_\prime}_{1/2\alpha,1m_l} \left[ \pi^\dagger_{\alpha_\prime} \left( n_\alpha \left[ -i \nabla \right]_{1m_l} c \right) \right] + \text{H.c.} \quad (D1)$$

of Eq. (5) contains an auxiliary field $\pi_{\alpha}$ ($\alpha \in \{-1/2, 1/2\}$) for $^{11}\text{Be}^*$ with renormalization-dependent parameters $\Delta_\pi^{(0)}$, $g_\pi \in \mathbb{R}$. The Galilei-invariant derivative $\nabla$ and the $p$-wave tensor structure $[O]_{1m_l}$ are defined in appendix A. Unlike in the $s$-wave case, both the constant and derivative part of the bare propagator term in Eq. (D1) are needed to describe the shallow $p$-wave state [12, 30]. The full $^{11}\text{Be}^*$ propagator can be obtained by resumming all two-body loops, similarly to Fig. 2. For more details, we refer to Ref. [30]. After proper renormalization and field redefinitions $\pi_{\alpha} \to \pi_{\alpha} \equiv g_\pi \pi_{\alpha}$, the propagator $G_\pi$ around the pole at $E_{\text{cm}} = -B_\pi$ is given by Eq. (15).

In the NLO three-body system, the intermediate state $|\pi\rangle \equiv |p + ^{11}\text{Be}^*\rangle$ couples to $|d\rangle$ via neutron exchange potentials shown in Fig. 10. They read

$$V_{\pi d,1m'}^{S m,1m'} (p, q; E) = m_N \sqrt{6} \begin{pmatrix} S & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\times \sum_{m_l} C_{1m_l,1m'}^{S m,1m'} \left[ \frac{1}{1+y} p + q \right]^{*}_{1m_l}$$

$$V_{d \pi}^{1m,Sm'} (p, q; E) = \left[ V_{\pi d,1m'}^{S m',1m} (q, p; E) \right]^* \quad (D2)$$

with $S \in \{0, 1\}$ in the $|\pi\rangle$ channel and involve a $6j$-symbol. Partial wave projections are
Table II: Subsystems of fixed $J$ after including the excited state channel $|\pi\rangle$. Subsystems (1) and (2a) require $J \geq 1$. The quantum numbers $\bar{3}$ and $\bar{1}$ in subsystems (2a) and (2b) refer to rotated spin states of $|\pi\rangle$; see Eq. (D6).

| Subsystem | $L_d = L_\sigma$ | $S_d = S_\sigma$ | $L_\pi$ | $S_\pi$ |
|-----------|----------------|----------------|--------|--------|
| (1)       | $J$            | 1              | $J \pm 1$ | 1      |
| (2a)      | $J - 1$        | 1              | $J$     | $3 - \sqrt{J + 1} \times 3 + \sqrt{J} \times 1$ |
| (2b)      | $J + 1$        | 1              | $J$     | $1 - \sqrt{J} \times 3 + \sqrt{J + 1} \times 1$ |

A direct transition potential between $|\sigma\rangle$ and $|\pi\rangle$ is not induced by the Lagrangian, i.e., these states can only be connected via an intermediate state $|d\rangle$.

The Clebsch-Gordan coefficient and the $6j$-symbols in Eq. (D4) imply some selection rules. Firstly, only transitions with $|\Delta L| = 1$ are allowed. It follows that for $J = 0$, we have $L_d = L_\sigma = 1$, $L_\pi = 0$, and for fixed $J \geq 1$, the system decouples into the two subsystems (1) $L_d = L_\sigma = J$, $L_\pi = J \pm 1$ and (2) $L_d = L_\sigma = J \pm 1$, $L_\pi = J$. Secondly, $S_\pi = 1$ is fixed in subsystem (1), while both options $S_\pi \in \{0, 1\}$ are allowed in subsystem (2). Lastly, in subsystem (2), the two channels $L_d = L_\sigma = J \pm 1$ further decouple after defining rotated spin states

$$\left(\begin{array}{c} \pi, \bar{3}J_j \\ \pi, \bar{1}J_j \end{array}\right) \propto \frac{1}{\sqrt{2J + 1}} \left(\begin{array}{cc} \sqrt{J + 1} & \sqrt{J} \\ -\sqrt{J} & \sqrt{J + 1} \end{array}\right) \left(\begin{array}{c} \pi, \bar{3}J_j \\ \pi, \bar{1}J_j \end{array}\right).$$

Note that $\bar{3} = 3$ and $\bar{1} = 1$ for $J = 0$. The corresponding partial wave potentials read

$$V_{\pi d}^{2S + 1L_j, \bar{3}L_j'} (p, q; E) = (-1)^{J+1} \sqrt{2(2S + 1)(2L + 1)(2L' + 1)}$$

$$\times C_{L_\sigma, L_\pi}^{10, 10} \left\{ \begin{array}{c} S \ 1 \\ 1/2 \\ 1/2 \end{array} \right\} \left\{ \begin{array}{c} S \ 1 \\ 1 \end{array} \right\} \left\{ \begin{array}{c} L' \ L \ J \\ 1/2 \ 1/2 \end{array} \right\}$$

$$\times \frac{m_N}{pq} \left[ \frac{1}{1 + y} p Q_{J \pm 1} + q Q_J \right] \left( -\frac{p^2 + \frac{1+y}{2} q^2 - m_N(E + i\epsilon)}{pq} \right),$$

$$V_{d\pi}^{2S + 1L_j, 2S' + 1L_j'} (p, q; E) = V_{\pi d}^{2S' + 1L_j', 3L_j} (q, p; E).$$

In summary, for fixed $J \geq 1$, we find the three decoupled subsystems (1), (2a), and (2b) presented in Tab. II. Just as in the LO case, they can be identified by the conserved quantum number (1) $L_d = J$, (2a) $L_d = J - 1$, and (2b) $L_d = J + 1$. In the case $J = 0$, only system (2b) is allowed.
Appendix E: NLO equations

As explained in appendix D, the introduction of the excited state $^{11}\text{Be}^*$ produces three decoupled scattering systems for fixed $J \geq 1$, corresponding to $L_d = L_{\sigma} \in \{J-1, J, J+1\}$, and a single system for $J = 0$ with $L_d = L_{\sigma} = 1$. The respective NLO amplitude vectors $\overline{T}^{(\text{NLO})}[L_d, J]$ read

$$\overline{T}^{(\text{NLO})}[J, J] = \begin{pmatrix} T^{(\text{NLO})}_{\hspace{0.5em}d\hspace{0.5em}d}^{3J, 3J} \\ T^{(\text{NLO})}_{\sigma\sigma}^{3J, 3J} \\ T_{\pi\sigma}^{(J-1), 3J} \\ T_{\pi d}^{(J+1), 3J} \end{pmatrix} (J \geq 1),$$

(E1)

$$\overline{T}^{(\text{NLO})}[J \pm 1, J] = \begin{pmatrix} T^{(\text{NLO})}_{\hspace{0.5em}d\hspace{0.5em}d}^{3J, 3J} \\ T^{(\text{NLO})}_{\sigma\sigma}^{3J, 3J} \\ T_{\pi\sigma}^{(J \pm 1), 3J} \\ T_{\pi d}^{(J \pm 1), 3J} \end{pmatrix} (J \geq 0 \text{ and } J \geq 1).$$

(E2)

They are determined by the kernel and propagator matrices

$$K^{(\text{NLO})}[J, J] = \begin{pmatrix} \Gamma_{dd}^{3J, 3J} \\ (V_{d\sigma} + \Gamma_{\sigma\sigma})^{3J, 3J} \\ V_{\pi\sigma}^{(J-1), 3J} \\ V_{\pi d}^{(J+1), 3J} \end{pmatrix} \begin{pmatrix} (V_{d\sigma} + \Gamma_{\sigma\sigma})^{3J, 3J} \\ \Gamma_{\sigma\sigma}^{3J, 3J} \\ 0 \\ 0 \end{pmatrix},$$

(E3)

$$G^{(\text{NLO})}[J, J] = \text{diag} \left[ G_{d}^{(\text{NLO})}, G_{\sigma}^{(\text{NLO})}, G_{\pi}^{(\text{LO})}, G_{\pi}^{(\text{LO})} \right],$$

(E4)

and

$$K^{(\text{NLO})}[J \pm 1, J] = \begin{pmatrix} \Gamma_{dd}^{3(J \pm 1), 3(J \pm 1)} \\ (V_{d\sigma} + \Gamma_{\sigma\sigma})^{3(J \pm 1), 3(J \pm 1)} \\ V_{\pi\sigma}^{3(J \pm 1), 3(J \pm 1)} \\ V_{\pi d}^{3(J \pm 1), 3(J \pm 1)} \end{pmatrix} \begin{pmatrix} (V_{d\sigma} + \Gamma_{\sigma\sigma})^{3(J \pm 1), 3(J \pm 1)} \\ \Gamma_{\sigma\sigma}^{3(J \pm 1), 3(J \pm 1)} \\ 0 \\ 0 \end{pmatrix},$$

(E5)

$$G^{(\text{NLO})}[J \pm 1, J] = \text{diag} \left[ G_{d}^{(\text{NLO})}, G_{\sigma}^{(\text{NLO})}, G_{\pi}^{(\text{LO})}, G_{\pi}^{(\text{LO})} \right],$$

(E6)

respectively, similar to Eq. (25). The propagator function $G_a^{(\text{LO})}$ is defined via Eq. (29) with $a = \pi$ and reduced mass $\mu_{\pi} = \mu_{\sigma} = m_N(m_N + m_c)/(2m_N + m_c)$.

[1] C. Fahlander and B. Jonson, Phys. Scripta T152 (2013).
[2] P. Navrátil et al., Phys. Scripta 91, 053002 (2016), 1601.03765.
[3] K. Yoshida, M. Gómez-Ramos, K. Ogata, and A. M. Moro, Phys. Rev. C97, 024608 (2018), 1711.04458.
[4] P. Capel, D. R. Phillips, and H.-W. Hammer, Phys. Rev. C98, 034610 (2018), 1806.02712.
[5] G. B. King, A. E. Lovell, and F. M. Nunes, Phys. Rev. C98, 044623 (2018), 1810.06129.
[6] F. M. Nunes et al., EPJ Web Conf. 178, 03001 (2018).
A. E. Lovell and F. M. Nunes, Phys. Rev. C97, 064612 (2018), 1801.06096.

M. V. Zhukov et al., Phys. Rept. 231, 151 (1993).

P. G. Hansen, A. S. Jensen, and B. Jonson, Ann. Rev. Nucl. Part. Sci. 45, 591 (1995).

B. Jonson, Phys. Rep. 389, 1 (2004).

A. S. Jensen, K. Riisager, D. V. Fedorov, and E. Garrido, Rev. Mod. Phys. 76, 215 (2004).

C. A. Bertulani, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A712, 37 (2002), nucl-th/0205063.

P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Lett. B 569, 159 (2003), nucl-th/0304007.

H.-W. Hammer, C. Ji, and D. R. Phillips, J. Phys. G44, 103002 (2017), 1702.08605.

W. Nörtershäuser et al., Phys. Rev. Lett. 102, 062503 (2009), 0809.2607.

D. R. Goosman, E. G. Adelberger, and K. A. Snover, Phys. Rev. C1, 123 (1970).

K. T. Schmitt et al., Phys. Rev. Lett. 108, 192701 (2012), 1203.3081.

A. Deltuva, A. Ross, E. Norvaišas, and F. M. Nunes, Phys. Rev. C94, 044613 (2016), 1610.04448.

J. Yang and P. Capel, Phys. Rev. C98, 054602 (2018), 1805.12074.

G. Rupak and X. Kong, Nucl. Phys. A717, 73 (2003), nucl-th/0108059.

S. König, H. W. Grießhammer, and H.-W. Hammer, J. Phys. G42, 045101 (2015), 1405.7961.

S. König, H. W. Grießhammer, and U. van Kolck, J. Phys. G43, 055106 (2016), 1508.05085.

M. Yilmaz and B. Gonul, Few Body Syst. 29, 223 (2000), nucl-th/0105012.

M. Gomez-Ramos and A. M. Moro, Phys. Rev. C95, 044612 (2017), 1702.04954.

H. A. Bethe, Phys. Rev. 76, 38 (1949).

D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424, 390 (1998), nucl-th/9801034.

D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. B534, 329 (1998), nucl-th/9802075.

TUNL Nuclear Data Evaluation Project, Energy Level Diagram, 11Be (2012), URL http://www.tunl.duke.edu/nucldata/figures/11figs/11_02_2012.pdf.

H. W. Grießhammer, Nucl. Phys. A744, 192 (2004), nucl-th/0404073.

H.-W. Hammer and D. R. Phillips, Nucl. Phys. A865, 17 (2011), 1103.1087.

A. Calci et al., Phys. Rev. Lett. 117, 242501 (2016), 1608.03318.

C. Ji, D. R. Phillips, and L. Platter, Annals Phys. 327, 1803 (2012), 1106.3837.

P. F. Bedaque, G. Rupak, H. W. Grießhammer, and H.-W. Hammer, Nucl. Phys. A714, 589 (2003), nucl-th/0207034.

C. Ji and D. R. Phillips, Few Body Syst. 54, 2317 (2013), 1212.1845.

J. de Swart, C. Terheggen, and V. Stoks (1995), arXiv:nucl-th/9509032.

J.-W. Chen, G. Rupak, and M. J. Savage, Nucl. Phys. A653, 386 (1999), nucl-th/9902056.

H. W. Grießhammer, Nucl. Phys. A760, 110 (2005), nucl-th/0502039.

V. Efimov, Phys. Lett. 33B, 563 (1970).

E. Braaten and H.-W. Hammer, Phys. Rept. 428, 259 (2006), cond-mat/0410417.

P. Naidon and S. Endo, Rept. Prog. Phys. 80, 056001 (2017), 1610.09805.

TUNL Nuclear Data Evaluation Project, Energy Level Diagram, 12B (2017), URL http://www.tunl.duke.edu/nucldata/figures/12figs/12_02_2017.pdf.

H.-W. Hammer and T. Mehen, Phys. Lett. B516, 353 (2001), nucl-th/0105072.

L. P. Kok, J. W. de Maag, H. H. Brouwer, and H. van Haeringen, Phys. Rev. C26, 2381 (1982).

X. Kong and F. Ravndal, Phys. Lett. B450, 320 (1999), [Erratum: Phys.
[45] X. Kong and F. Ravndal, Phys. Lett. B470, 1 (1999), nucl-th/9904066.

[46] E. Braaten, H.-W. Hammer, and G. P. Lepage, Phys. Rev. D 94, 056006 (2016), 1607.02939.

[47] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D51, 1125 (1995), [Erratum: Phys. Rev. D55, 5853 (1997)], hep-ph/9407339.

[48] E. Braaten and H.-W. Hammer, Phys. Rev. Lett. 87, 160407 (2001), cond-mat/0103331.

[49] X. Zhang, K. M. Nollett, and D. R. Phillips, Phys. Rev. C89, 024613 (2014), 1311.6822.

[50] X. Zhang, K. M. Nollett, and D. R. Phillips, Phys. Rev. C89, 051602 (2014), 1401.4482.

[51] E. Ryberg, C. Forssén, H.-W. Hammer, and L. Platter, Eur. Phys. J. A50, 170 (2014), 1406.6908.