Dynamics of quadratic gravitation theory with pseudoscalar torsion and its cosmological perturbations

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An exact analytical solution of the quadratic gravity with torsion is presented. This solution gives a different description of the universe from the ΛCDM. According this solution, the vacuum spacetime possesses a torsion which plays the role of the cosmological constant or the dark energy—the torsion dark energy. The torsion of the spacetime can be produced by the energy and the pressure besides the spin of matter. The density and the pressure play the opposite roles in the expansion of the universe, while the density decelerates the expansion (the attractive effect) the pressure accelerates it (the repulsive effect). The cosmic acceleration depends only on the contribution of matter. Radiation has no effect on the acceleration. The deceleration of the universe expansion can take place only in the earlier matter dominated era. The universe undergoes a phase transformation from a decelerating to an accelerating expansion when the density of matter drops to a critical value.

The linear perturbations is examined, the equation of the structure growth is obtained, a damped solution is given. These results distinguishes our theory from the ΛCDM.

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I. Introduction

In the last few years the realization that the universe is currently undergoing an accelerated expansion phase and the quest for the nature of dark energy has renewed interest in so-called modified gravity theories (for a review see [1]). In these theories one modifies the laws of gravity so that a late-time accelerated expansion is produced without recourse to a dark energy component, a fact which renders these models very attractive. The simplest family of modified gravity theories is obtained by replacing the Ricci scalar $R$ in the usual Hilbert- Einstein Lagrangian with some function $f(R)$ (for reviews, see [2]). Recently, they have been generalized to $f(R, R_{\mu\nu}R^{\mu\nu})$ theories containing higher in curvature corrections to the Einstein-Hilbert Lagrangian [3] and $f(R)$ theories with non-symmetric connections, i.e. theories that allow for torsion [4].

$f(R, R_{\mu\nu}R^{\mu\nu})$ theories have been widely studied over the years. The higher curvature terms naturally arise in the string theory (see e.g. [5] and references therein). Cosmology in the theories with the Lagrangian including $R^2$ and $R_{\mu\nu}R^{\mu\nu}$ terms have been studied in the purely metric formulation (for example see [6]). However, the similar cosmological models in the metric-affine formulation have not been discussed thoroughly in the literature. Especially,
the cosmological effect of torsion in metric-affine theories of gravity has not been explored extensively. We have not
known whether the dynamical torsion could lead to a de Sitter solution and then be used to explain the observed
acceleration of the universe.

It has been shown that even in the most general case of $f(R)$ gravity, the connection can still be algebraically
eliminated in favor of the metric and the matter fields [7]. Clearly, $f(R)$ actions do not carry enough dynamics
to support an independent connection which carries dynamical degrees of freedom. However, this is not a generic
property of generalized gravity. The addition of the $R_{\mu\nu}R^{\mu\nu}$ term to the Lagrangian radically changes the situation
and excites new degrees of freedom in the connection. The new dynamical degree of freedom can be torsion which
plays a fundamental role in the theories. A metric-affine theory includes torsion naturally [8]. Propagating torsion is
the key feature of these theories [9].

Torsion proves to be essential for total angular momentum conservation when intrinsic spin angular momentum is
relevant (for reviews on torsion, see [10]). It has been argued that torsion must be present in a fundamental theory
of gravity [11]. In the teleparallel gravity, for example, torsion plays a central role (for a shorter review see [12]).
Recently, models based on modified teleparallel gravity, namely $f(T)$, were presented. In these models the torsion
proves to be the responsible of the observed acceleration of the universe [13].

In recent years metric-affine theories have been used in cosmology to interpret the accelerating expansion of the universe
[14][15]. In these theories the structure of the gravitational equations and physical consequences of cosmology, in
particular, the situation concerning the accelerating expansion depend essentially on the form of the Lagrangian.
The metric-affine gravity can be divided into different sectors in dependence on the number of nonvanishing components
of the torsion tensor and the order of the differential equations. One sector of the metric-affine gravity is so-called
dynamical scalar torsion sector considered in [14]. Starting from a Lagrangian consisting of $R^2$ and the quadratic
torsion terms a cosmological model has been constructed. This model can contribute an oscillating aspect to the
expansion rate of the universe. A different model of acceleration with torsion but without dark matter and dark
ergy has been presented in [15]. The Lagrangian of it is the most general form including the linear in the scalar
curvature term as well as 9 quadratic terms (6 invariants of the curvature tensor and 3 invariants of the torsion
tensor with indefinite parameters). Its Lagrangian involves too many terms and indefinite parameters, which make
the field equations complicated and difficult to solve and the role of each term obscure. In order to simplify the field
equations some restrictions on indefinite parameters have to be imposed. Under these restrictions, especially, all the
higher derivatives of the scale factor are excluded from the cosmological equations. The question is whether such a
complicated Lagrangian is necessary. Can we use a simpler Lagrangian to construct a model of cosmic acceleration?
In fact all the indefinite parameters in the Lagrangian in [15] have been combined into four new ones, which implies
that some terms are not necessary and the Lagrangian can be simplified. It has been shown [16] that a rather simpler
Lagrangian, $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma T^\mu\nu T^\mu\nu$, is sufficient and necessary to construct a model of cosmic acceleration.
The terms $\beta R_{\mu\nu}R^{\mu\nu}$ and $\gamma T^\mu\nu T^\mu\nu$ play different roles in the theory: the former determines the structure of the field equations while the latter determines the behavior and the stability of the solutions. The $\beta R_{\mu\nu}R^{\mu\nu}$ term leads
to different structure of the cosmological equations from the one in [14]. If $\gamma = 0$, there exists no solution describing
an accelerating universe. In other words, the term $\gamma T^\mu\nu T^\mu\nu$ is necessary to the existence of the solutions describing
an accelerating universe. In addition to the simplicity the main advantage of this Lagrangian is to permit exact or
analytic solutions which have not been found in previous works. In contrast with [15], the field equations are allowed
to contain higher derivatives in [16]. When the macroscopic spacetime average of the spin vanishes, the solutions of the cosmological equations are divided into two classes. Each of them is related with only one torsion function, the scalar or the pseudoscalar torsion function. The first class has been investigated in detail in [16], the second class corresponding to the pseudoscalar torsion function will be studied in a totally different way in this paper. Some meaningful consequences can be inferred from the solutions obtained.

The paper is organized as follows. In section II the gravitational field equations are derived following the approach of [8], [14] and [15]. Using them to the spatially flat Friedmann-Robertson-Walker metric a system of cosmological equations is obtained. Since the spin orientation of particles in ordinary matter is random, the macroscopic spacetime average of the spin vanishes. In this case, the solutions of the cosmological equations are divided into two classes. We will concentrate on the solution corresponding to the pseudoscalar torsion function in section III. An exact analytic solution of the cosmological equations is presented. In terms of this solution the acceleration and the phase transformation from decelerating to accelerating expansion of the universe can be explained. This solution indicates that in vacuum the spacetime possesses an intrinsic torsion which does not originate from the spin of matter. It is the torsion that causes the acceleration of the cosmological expansion. The torsion of the spacetime can be produced by the energy and pressure besides the spin of matter. In section IV the linear perturbations corresponding to the pseudoscalar torsion function are investigated, the growth of structure is studied. The section V is devoted to conclusions.

II. Cosmological equations

Following the approach of [8], [14] and [15] we choose the tetrad $\epsilon^I_{\mu}$ and the spin connection $\Gamma^{IJ}_{\mu}$ instead of the metric $g_{\mu\nu}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$ as the dynamical variables. The descriptions in terms of the variables $(\epsilon^I_{\mu}, \Gamma^{JK}_{\nu})$ and $(g_{\mu\nu}, T^\lambda_{\mu\rho\sigma})$ are equivalent in our approach (the argument by Shapiro in detail see [10]). We will concentrate on the role of torsion subject to the metricity. In this case only the torsion part of the connection is independent of the metric (or tetrad).

We start from the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma T^\mu_{\nu\rho} T^{\nu\rho}_{\mu} \right] + L_m ,$$

where $L_m$ is the matter part Lagrangian, $\alpha$, and $\beta$ are two parameters with the dimension of $[L]^2$, $\gamma$ is a parameter of dimensionless, $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\rho\sigma} - \Gamma^\lambda_{\nu\rho\sigma}$ is the torsion tensors. The variational principle yields the field equations for the tetrad $\epsilon^I_{\mu}$ and the spin connection $\Gamma^{IJ}_{\mu}$:

$$\epsilon^I_{\nu} R_{\nu\mu} - \frac{1}{2} \epsilon^J_{\mu} R = E^I_{\mu} - \alpha (4 \epsilon^{l\nu} R_{\nu\mu} - \epsilon^I_{\mu} R) R - \beta (2 \epsilon^{l\sigma} R^\rho_{\sigma\mu} R_{\rho\mu} + 2 \epsilon^J_{\rho} R^\rho_{\sigma\mu} R^I_{\mu\sigma} - \epsilon^I_{\mu} R_{\rho\sigma} R^{\rho\sigma})$$

$$+ \gamma (4 \partial_\nu (e^{\lambda\tau} T^\mu_{\lambda\tau}) - 4 e^K_{\tau} e^{\lambda\nu} T^\mu_{\lambda\nu} \partial_\nu e^K_{\tau} + e^I_{\mu} T^\lambda_{\rho\sigma} T^{\nu\rho\sigma} - 4 e^{I\nu} T^\lambda_{\nu\tau} T^\tau_{\mu\lambda}) ,$$

(2)
\[ e_\mu^I e_\nu^J e^K_\mu \partial_\nu e_\tau^\lambda + e_\nu^I e_\mu^J \Gamma^\mu_{\nu\tau} + e_\mu^I e_\nu^J \Gamma^\lambda_{\nu\mu} = s_{IJ} \mu - 4 \alpha \left( e_\mu^I e_\nu^J e^K_\mu R + e_\nu^I e_\mu^J \left( \Gamma^\lambda_{\nu\mu} R - \partial_\nu R \right) + e_\mu^I e_\nu^J e^K_\mu R \partial_\nu e_\tau^\lambda \right) \\
- 4 \beta e_\mu^I e^K_\mu \partial_\nu R^I_\nu + e_\nu^I e^K_\mu e^K_\tau \partial_\nu e_\tau^\lambda + e_\mu^I \Gamma_\nu^I e^K_\mu R + e_\nu^I e^K_\nu \Gamma^\tau_\nu \lambda \right) \\
- 4 \gamma e_\mu^I e_\nu^J e^K_\tau T^\mu_\tau. \]  

(3)

where \( E^I_\mu \) and \( s_{IJ} \mu \) are energy- momentum and spin tensors of the matter source, respectively. We use the Greek alphabet (\( \mu, \nu, \rho, \ldots = 0, 1, 2, 3 \)) to denote (holonomic) indices related to spacetime, and the Latin alphabet (\( I, J, K, \ldots = 0, 1, 2, 3 \)) to denote algebraic (anholonomic) indices, which are raised and lowered with the Minkowski metric \( \eta_{IJ} = \text{diag} (-1, +1, +1, +1) \). If \( \alpha = \beta = \gamma = 0 \), these equations become the field equations of Einstein-Cartan-Sciama-Kibble theory. Especially, (2) becomes the Einstein equation. To understand these equations, we will do a translation of (2, 3) into a certain effective Riemannian form—transcribing from quantities expressed in terms of the tetrad \( e_\mu^I \) and spin connection \( \Gamma^I_\mu^J \) into the ones expressed in terms of the metric \( g_{\mu\nu} \) and torsion \( T^\lambda_\mu^\nu \) (or contortion \( K^\lambda_\mu^\nu \)), as was done in [4]. It should be noted that the set \( (e^I_\mu, \Gamma^I_\mu^J \nu) \) corresponds to the first order formalism, while the set \( (g_{\mu\nu}, T^\lambda_\mu^\nu) \) to the second order formalism. The origin of this is that in the last case the non-torsional part of the affine connection is a function of the metric, while, within the gauge approach, the variables \( (e^I_\mu, \Gamma^J_\nu) \) are mutually independent completely.

Subject to the metricity, the affine connection \( \Gamma^\lambda_\mu^\nu \) is related to the tetrad \( e^I_\mu \) and the spin connection \( \Gamma^I_\mu^J \) by

\[ \Gamma^\lambda_\mu^\nu = e^I_\mu \partial_\nu e^I_\lambda + e^J_\nu e^I_\mu \Gamma^J_\mu^I_\lambda, \]

(4)

where \( \{ \mu^\lambda_\nu \}, K^\lambda_\mu^\nu \) are the Levi-Civita connection and the contortion, separately, with

\[ K^\lambda_\mu^\nu = - \frac{1}{2} \left( T^\lambda_\mu^\nu + T^\nu^\mu_\lambda - T^\nu^\lambda_\mu \right), \]

\[ T^\lambda_\mu^\nu = e_\mu^I e^K_\nu \gamma^\lambda_\mu^\nu, \]

\[ T^I_\mu^\nu = \partial_\mu e^K_\nu - \partial_\nu e^K_\mu + \Gamma^I_\mu^J_\nu e^K_\nu - \Gamma^I_\nu^J_\mu e^K_\nu. \]

(5)

Accordingly the curvature \( R^\rho_\sigma_\mu^\nu \) can be represented as

\[ R^\rho_\sigma_\mu^\nu = e_\mu^I e^K_\nu \rho R^I_\mu^\nu = \partial_\mu e^K_\nu \rho - \partial_\nu e^K_\mu \rho + \Gamma^\rho_\lambda_\mu e^K_\nu \lambda - \Gamma^\rho_\lambda_\nu e^K_\mu \lambda + \{ \rho^\lambda_\mu \} K^\lambda_\sigma_\nu - \{ \sigma^\rho_\nu \} K^\lambda_\sigma_\mu + \{ \nu^\mu_\rho \} K^\lambda_\sigma_\mu - \{ \lambda^\mu_\rho \} K^\rho_\mu_\nu, \]

(6)

where \( R^\rho_\sigma_\mu^\nu = \partial_\mu \{ \sigma^\rho_\nu \} - \partial_\nu \{ \sigma^\rho_\mu \} + \{ \lambda^\rho_\mu \} \{ \sigma^\lambda_\nu \} - \{ \lambda^\rho_\nu \} \{ \sigma^\lambda_\mu \} \) is the Riemann curvature of the Levi-Civita connection. In view of this, we can identify the actual degrees of freedom of the theory with the (independent) components of the metric \( g_{\mu\nu} \) and the tensor \( K^\lambda_\mu^\nu \).

For the spatially flat Friedmann-Robertson-Walker metric

\[ g_{\mu\nu} = \text{diag} \left( -1, a(t)^2, a(t)^2, a(t)^2 \right), \]

(7)

the non-vanishing components of the Levi-Civita connection are

\[ \{ 0^0_0 \} = 0, \{ 0^0_1 \} = \{ i^0_0 \} = 0, \{ 0^0_2 \} = \{ i^0_2 \} = a \delta_{ij}, \]

\[ \{ 0^i_0 \} = 0, \{ i^0_0 \} = \{ 0^i_0 \} = \frac{\dot{a}}{a} \delta_{ij}, \{ i^0_1 \} = 0, \{ i^0_2 \} = \frac{\dot{a}}{a} \delta_{ij}, \{ i^0_3 \} = 0, i, j, k, \ldots = 1, 2, 3. \]

(8)
The non-vanishing torsion components with holonomic indices are given by two functions, the scalar torsion $h$ and the pseudoscalar torsion $f$:

$$T_{ij0} = a^2 h \delta_{ij}, T_{ijk} = 2a^3 f \epsilon_{ijk},$$

and then the contortion components are

$$K_{ji0} = 0, K_{i j} = h \delta_{ij}, K_{0 ij} = a^2 h \delta_{ij}, K_{1 23} = -a f \epsilon_{ijk}.$$  \(\text{(10)}\)

The non-vanishing components of the curvature $R^{\rho}_{\sigma \mu \nu}$ and the Ricci curvature $R_{\mu \nu}$ are

$$R_{010} = R_{020} = R_{030} = a^2 \left( H + H^2 + H h + \dot{h} \right),$$
$$R_{012} = R_{021} = R_{031} = 2a^3 f \left( H + h \right),$$
$$R_{120} = R_{130} = R_{230} = -a \left( H f + \dot{f} \right),$$
$$R_{121} = R_{131} = R_{232} = a^2 \left( (H + h)^2 - f^2 \right).$$  \(\text{(11)}\)

$$R_{00} = -3 \dot{H} - 3 \dot{h} - 3H^2 - 3H h,$$
$$R_{11} = R_{22} = R_{33} = a^2 \left( H + \dot{h} + 3H^2 + 5H h + 2h^2 - f^2 \right),$$
$$R = 6 \dot{H} + 6 \dot{h} + 12H^2 + 18H h + 6h^2 - 3f^2,$$  \(\text{(12)}\)

where $H = \dot{a} / a$ is the Hubble parameter, $\dot{H} = dH/dt$. Using these results and supposing the matter source is a fluid characterized by the energy density $\rho$, the pressure $p$ and the spin $s_{f, j}^{\mu}$, we obtain four independent equations from (2) and (3):

$$(H + h)^2 - f^2 - \frac{\rho}{3}$$
$$-4 \left( 3\alpha + \beta \right) \left( H + \dot{h} - H h - h^2 + f^2 \right) \left( H + \dot{h} + 2H^2 + 3H h + h^2 - f^2 \right)$$
$$+ 2\gamma \left( h^2 + 4f^2 \right) = 0$$  \(\text{(14)}\)

$$(\beta + 6\alpha) \left( H + \dot{h} \right) + 6 \left( \beta + 4\alpha \right) \left( H + h \right) \dot{H} + (5\beta + 18\alpha) \left( H + h \right) \dot{h} - 4 \beta + 3\alpha \right) f \dot{f}$$
$$+ 3 \left( \beta + 4\alpha \right) H H^2 + (5\beta + 18\alpha) h^2 H + 2 \left( \beta + 3\alpha \right) h^3 - 2 \left( \beta + 3\alpha \right) h f^2 + \frac{1}{4} h + \frac{1}{2} s_{01} = 0,$$  \(\text{(16)}\)
\[ f \left( 2 (\beta + 6\alpha) \left( \dot{H} + \dot{h} \right) \right) + 6 (\beta + 4\alpha) H^2 + 2 (5\beta + 18\alpha) H h + (\beta + 3\alpha) (4h^2 - 4f^2) - 4\gamma + \frac{1}{2} - \frac{1}{2} s_{12} \beta^3 = 0. \]  

(17)

When \( \alpha = \beta = \gamma = 0, h = f = 0, \) (14) and (15) lead to the Friedmann equation and the Raychaudhuri equation. The system of the equations (14)–(17) has the similar structure as the system of gravitational equations for homogeneous isotropic cosmological models in except the coefficients. However, it is the differences in coefficients that make the system of the equations (14)–(17) easy to handle and possible to obtain some exact or analytic solutions in several cases.

Since the spin orientation of particles in ordinary matter is random, the macroscopic spacetime average of the spin vanishes, we suppose \( s_{IJ} = 0, \) henceforth. Then, the equation, (17) has the solutions

\[ f = 0, \]  

and

\[ f^2 = \frac{(\beta + 6\alpha)}{2(\beta + 3\alpha)} \left( \dot{H} + \dot{h} \right) + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)} H^2 + \frac{(5\beta + 18\alpha)}{2(\beta + 3\alpha)} H h + h^2 - \frac{\gamma}{(\beta + 3\alpha)} + \frac{1}{8(\beta + 3\alpha)}. \]  

(19)

The solution (18) corresponds to the scalar torsion function \( h \) which has been investigated in detail in \[16\]. We will concentrate on the solution (19).

### III. Analytic solutions with pseudoscalar torsion function

Differentiating (19) gives

\[ f \ddot{f} = \frac{\beta + 6\alpha}{4(\beta + 3\alpha)} \left( \ddot{H} + \ddot{h} \right) + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)} H \dot{H} + \frac{5\beta + 18\alpha}{4(\beta + 3\alpha)} H h + \frac{5\beta + 18\alpha}{4(\beta + 3\alpha)} H h + h \dot{h}. \]  

(20)

Substituting (19) and (20) into (16) (noting \( s_{01} = 0 \) ) gives

\[ h = 0. \]  

(21)

Then the equations (14), (15) lead to

\[ 2 \dot{H} + 4H^2 - 2f^2 + p - \frac{\rho}{3} = 0, \]  

(22)

\[ 2 \dot{H} + 3H^2 - f^2 + p + 4(3\alpha + \beta) \left( \dot{H} + f^2 \right) \left( \dot{H} + 2H^2 - f^2 \right) - 8\gamma f^2 = 0, \]  

(23)
and (19) become
\[ f^2 = \frac{(6\alpha + \beta)}{2(3\alpha + \beta)} H^2 + \frac{3(4\alpha + \beta)}{2(3\alpha + \beta)} H^2 - \frac{\gamma}{3\alpha + \beta} + \frac{1}{8(3\alpha + \beta)}. \] (24)

(22), (23) and (24) give the solutions
\[ H^2 = \frac{(8\gamma - 1)^2}{32\gamma\beta} + \frac{\rho}{24\gamma} - (8\gamma - 1) \left( \frac{4\alpha + \beta}{16\gamma\beta} (\rho - 3p) + (3\alpha + \beta) \frac{4\alpha + \beta}{24\gamma\beta} (\rho - 3p)^2 \right) \] (25)

\[ \dot{H} = -\frac{(16\gamma - 1)(8\gamma - 1)}{32\beta\gamma} - \frac{\rho}{24\gamma} \]
\[ - \frac{3(4\alpha + \beta) - 8\gamma (18\alpha + 5\beta)}{48\beta\gamma} (\rho - 3p) - (3\alpha + \beta) \frac{4\alpha + \beta}{24\gamma\beta} (\rho - 3p)^2 \] (26)

\[ f^2 = \frac{1 - 8\gamma}{32\beta\gamma} + \frac{\rho}{24\gamma} + \frac{3(4\alpha + \beta) - 16\gamma (3\alpha + \beta)}{48\beta\gamma} (\rho - 3p) + (3\alpha + \beta) \frac{4\alpha + \beta}{24\gamma\beta} (\rho - 3p)^2 \] (27)

One can find that although (14) and (15) lead to the Friedmann equation and the Raychaudhuri equation in General Relativity when \( \alpha = \beta = \gamma = 0, h = f = 0 \), \( H \) and \( \dot{H} \) satisfy the equation (25) and (26) instead of the Friedmann equation and the Raychaudhuri equation in the case \( f \neq 0 \). The equation (27) indicates that the torsion of the spacetime can be produced by the energy and pressure besides the spin of matter. Even in vacuum the spacetime possesses the torsion \( f = \sqrt{\frac{1 - 8\gamma}{32\beta\gamma}} \), which has been found in [18]. Hence the conception of the vacuum as physical notion is changed essentially. Instead of it as passive receptacle of physical objects and processes, the vacuum assumes a dynamical properties as a gravitating object. We find this picture very appealing and physical since (27) seems to indicate that in metric-affine gravity as matter tells spacetime how to curve, matter will also tell spacetime how to twirl.

The combination of (25) and (26) yields the acceleration equation
\[ \frac{\ddot{a}}{a} = \frac{1 - 8\gamma}{4\beta} + \frac{3\alpha + \beta}{3\beta} (\rho - 3p). \] (28)

This means that \( \rho \) and \( p \) play the opposite roles in the expansion of the universe, while \( \rho \) decelerates the expansion (the attractive effect) \( p \) accelerates it (the repulsive effect). If \( \rho \) and \( p \) include the radiation parts \( \rho_r, p_r \) and the matter parts \( \rho_m, p_m \):

\[ \rho = \rho_r + \rho_m, p = p_r + p_m, \]
\[ p_r = \frac{1}{3} \rho_r, p_m = 0, \] (29)

(28) becomes
\[ \frac{\ddot{a}}{a} = \frac{1 - 8\gamma}{4\beta} + \frac{3\alpha + \beta}{3\beta} \rho_m. \] (30)

We obtain an important conclusion: the acceleration of the universe, \( \ddot{a} /a \) is independent of the contribution of radiation and depends only on the contribution of matter.

Some important consequences can be obtained from (30):
i) The term \( \frac{1-8\gamma}{32\beta} \) (the torsion of the vacuum) plays the role of the cosmological constant or the dark energy, which agrees with the result in [31]. If \( \frac{1-8\gamma}{32\beta} > 0 \), \( \rho_m = 0 \), then \( \ddot{a} > 0 \), the acceleration of cosmological expansion, the inflation in the early universe acquires the vacuum origin. We can call \( f = \sqrt{\frac{1-8\gamma}{32\beta}} \) the torsion dark energy.

ii) If

\[-3\alpha < \beta < 0, \tag{31}\]

\( \rho_m \) decelerates the expansion of the universe. Especially, when \( \beta = -2\alpha, \gamma = 1/8 \), (30) becomes the acceleration equation of the \( \Lambda \)CDM in dust matter. In other words, the latter is only a special case of the former.

iii) If

\[-3\alpha < \beta < 0, \gamma > \frac{1}{8}, \tag{32}\]

the universe can undergo a phase transformation from a decelerating to an accelerating expansion when \( \rho_m = \frac{3(8\gamma - 1)}{4(\beta + 3\alpha)} \).

Now (25) becomes

\[
H^2 = \frac{(8\gamma - 1)^2}{32\gamma\beta} + \frac{\rho_r}{24\gamma} + \frac{12\alpha + 5\beta - 24\gamma (4\alpha - \beta)}{48\beta\gamma} \rho_m + \frac{(3\alpha + \beta)(4\alpha + \beta)}{24\gamma\beta} \rho_m^2. \tag{33}\]

In the early radiation-dominated epoch, \( \rho_r \gg \rho_m \), we have

\[H^2 \approx \frac{\rho_r}{24\gamma}.\]

Since

\[\rho_r = \rho_0 a^{-4},\]

we obtain

\[a = \left( \frac{\rho_0}{6\gamma} \right)^{1/4} t. \tag{34}\]

In the early matter-dominated epoch, \( \rho_r \sim \rho_m \ll \rho_m^2 \), we have

\[H^2 \approx \frac{(3\alpha + \beta)(4\alpha + \beta)}{24\gamma\beta} \rho_m^2.\]

Since

\[\rho_m = \rho_m a^{-3},\]

we obtain

\[a = \sqrt{\frac{3(3\alpha + \beta)(4\alpha + \beta)}{8\gamma\beta}} \rho_m t^{1/3}. \tag{35}\]

In the present epoch,

\[
\frac{(8\gamma - 1)^2}{32\gamma\beta} \gg \frac{12\alpha + 5\beta - 24\gamma (4\alpha - \beta)}{48\beta\gamma} \rho_m,
\]

we have

\[H^2 \approx \frac{(8\gamma - 1)^2}{32\gamma\beta}.\]
and then
\[ a = a_0 e^{\frac{(\gamma - 1)}{2} (t - t_0)}. \]  
(36)

Recalling
\[ 1 + z = \frac{1}{a}, \]
we have
\[ \dot{z} = \frac{dz}{dt} = -(1 + z) H, \]
and then
\[ \dot{H} = \frac{dH}{dt} = \frac{dH}{dz} \dot{z} = -H (1 + z) \frac{dH}{dz}. \]  
(37)

(30) can be written as
\[ \dot{H} + H^2 = -\frac{8\gamma - 1}{4\beta} + \frac{\beta + 3\alpha}{3\beta} \rho_m. \]  
(38)

(37) and (38) lead to
\[ \frac{1}{2} (1 + z) \frac{d}{dz} \left( \frac{H^2}{2} \right) = H^2 - \frac{1 - 8\gamma}{4\beta} - \frac{\beta + 3\alpha}{3\beta} \rho_m. \]
and then
\[ \int \frac{d(H^2)}{H^2 - \frac{1 - 8\gamma}{4\beta} - \frac{\beta + 3\alpha}{3\beta} \rho_m} = \ln (1 + z)^2. \]  
(39)

If
\[ \frac{1 - 8\gamma}{4\beta} + \frac{\beta + 3\alpha}{3\beta} \rho_m = b \]
vary slowly, (39) gives
\[ H^2 = (H_0^2 - b) (1 + z)^2 + b. \]

Taking
\[ H_0 = 74.2 \text{ km sec}^{-1} \text{Mpc}^{-1} \]
we can compute \( H(z) \) as a function of \( z \) for \( b = 0, 1000, 2000, 3000 \text{ km}^2 \text{ sec}^{-2} \text{Mpc}^{-2} \), respectively:

\begin{align*}
\begin{array}{cccccccccccc}
z & b & 0 & 0.1 & 0.17 & 0.27 & 0.4 & 0.48 & 0.88 & 0.9 & 1.3 & 1.43 & 1.53 & 1.75 \\
H(z) & 0 & 74.2 & 81.6 & 86.8 & 94.2 & 103.9 & 109.6 & 139.5 & 141.0 & 170.7 & 180.3 & 187.7 & 204.1 \\
H(z) & 1000 & 74.2 & 80.3 & 84.7 & 90.9 & 99.2 & 103.0 & 117.8 & 131.4 & 157.6 & 166.2 & 172.7 & 187.3 \\
H(z) & 2000 & 74.2 & 79.0 & 82.5 & 87.5 & 94.2 & 97.3 & 120.0 & 121.1 & 143.3 & 150.7 & 156.3 & 168.9 \\
H(z) & 3000 & 74.2 & 77.7 & 80.2 & 83.9 & 88.9 & 91.3 & 108.9 & 109.8 & 127.5 & 133.4 & 138.0 & 148.2 \\
H(z) \text{ observed} & 74.2 & 69 \pm 12 & 83 \pm 8 & 77 \pm 14 & 95 \pm 17 & 97 \pm 60 & 90 \pm 40 & 117 \pm 23 & 168 \pm 17 & 177 \pm 23 & 140 \pm 14 & 202 \pm 40
\end{array}
\end{align*}
The last line lists the observational data \[10\]. We can find that

i). \( H(z) \) is not very sensible of the change of \( b \);

ii). when \( z < 1 \) the values of \( H(z) \) corresponding to \( b = 2000 \) agree with the observational data, \( \rho_m = -\frac{3\beta}{\beta + 3\alpha} \left( \frac{1 - 8\gamma}{4\beta} - 2000 \right) \);

iii). when \( z > 1 \) the values of \( H(z) \) corresponding to \( b = 0 \) agree with the observational data, \( \rho_m = -\frac{3\beta}{\beta + 3\alpha} \left( \frac{1 - 8\gamma}{4\beta} - 0 \right) = -\frac{3(1 - 8\gamma)}{4(\beta + 3\alpha)} \).

In the case \(-3\alpha < \beta < 0, \gamma > \frac{1}{8}\),

\[
\rho_m|_{z<1} < \rho_m|_{z>1}.
\]

This means that (39) agrees with the observational data in both cases \( z < 1 \) and \( z > 1 \).

### IV. Perturbation theory

The perturbed equations can be obtained as a result of straightforward but rather tedious calculations, following the approach of \(20\). In the case \( h = 0 \), the unperturbed field equations (14) and (15) become

\[
H^2 = \frac{\rho + \rho_g}{3},
\]

\[
2 \dot{H} + 3H^2 = -(p + p_g).
\]

where

\[
\rho_g = 12(3\alpha + \beta) \left( \dot{H} + f^2 \right) \left( \dot{H} + 2H^2 - f^2 \right) - 3(8\gamma - 1) f^2,
\]

\[
p_g = 4(3\alpha + \beta) \left( \dot{H} + f^2 \right) \left( \dot{H} + 2H^2 - f^2 \right) - (8\gamma + 1) f^2.
\]

Using (25), (26), (27) and (29) we compute

\[
\rho_g = \frac{3(8\gamma - 1)^2}{32\beta \gamma} - \left( 1 - \frac{1}{8\gamma} \right) \rho_r - \frac{(8\gamma - 1)(12\alpha + 5\beta)}{16\beta \gamma} \rho_m + \frac{(3\alpha + \beta)(4\alpha + \beta)}{8\beta \gamma} \rho_m^2,
\]

\[
p_g = \frac{(8\gamma + 1)^2}{32\beta \gamma} - \frac{1}{3} \left( 1 + \frac{1}{8\gamma} \right) \rho_r - \frac{12\alpha + 5\beta + 8\beta \gamma}{48\beta \gamma} \rho_m - \frac{(3\alpha + \beta)(4\alpha + \beta)}{24\beta \gamma} \rho_m^2.
\]

The unperturbed field equation (2) can be written as the form

\[
G_{\mu \nu} = T_{\mu \nu} + T_g \rho_{\mu \nu}.
\]

where \( G_{\mu \nu} \) is the Einstein tensor, \( T_{\mu \nu} \) is the energy -momentum of the ordinary matter and radiation, \( T_g \rho_{\mu \nu} \) is the energy -momentum given by (44) and (45). Let us consider the scalar perturbations of a flat FRW metric in the longitudinal gauge and in conformal time \( \eta \):

\[
ds^2 = a^2(\eta)[(1 + 2\phi)d\eta^2 - (1 - 2\psi)\gamma_{ij}dx^idx^j].
\]
The equations of motion for small perturbations linearized about the background metric are

$$\delta G_{\mu \nu} = \delta T_{\mu \nu} + \delta T_g^{\mu \nu}. \quad (48)$$

For scalar type metric perturbations with a line element given in (47) (in conformal time), the perturbed field equations can be obtained following the approach of [20]. In the case

$$\rho_m \gg \rho_r, \quad (49)$$

the equation

$$\delta G^0_{0} = -\delta \rho_m - \delta \rho_g \quad (50)$$

takes the form

$$2a^{-2} \left(3 \mathcal{H} \left(\mathcal{H} \phi + \psi'\right) - \nabla^2 \psi\right)$$

$$+ \left(\frac{3(1 - 8\gamma)}{32\beta \gamma} + \frac{12\alpha + 5\beta - 16\gamma}{48\beta \gamma} \rho_m + \frac{(3\alpha + \beta)4\alpha + \beta}{24\gamma \beta \rho_m^2} \right)\psi$$

$$= - \frac{12\alpha + 5\beta - 16\gamma}{8\beta \gamma} \rho_m \delta - \frac{(3\alpha + \beta)4\alpha + \beta}{2\beta \gamma \rho_m^2} \rho_m \delta, \quad (51)$$

where \( \mathcal{H} \equiv \alpha' / a = aH \) with prime denoting derivative with respect to conformal time \( \eta \). The equation

$$\delta G_{ij} = \delta p_g \delta_{ij} \quad (52)$$

gives

$$-2a^{-2} \left(2\mathcal{H'} + \mathcal{H}^2\right) \phi + \mathcal{H} \phi' + \psi'' + 2\mathcal{H} \psi' + \frac{1}{2} \nabla^2 \left(\phi - \psi\right) \right) \delta^i_j - \frac{1}{2} \left(\phi - \psi\right)_{,ij} \right)$$

$$+ 2 \left(\frac{1 - 8\gamma}{32\beta \gamma} + \frac{12\alpha + 5\beta - 16\gamma}{48\beta \gamma} \rho_m + \frac{(3\alpha + \beta)4\alpha + \beta}{24\gamma \beta \rho_m^2} \right) \psi \delta^i_j$$

$$= \left(- \frac{12\alpha + 5\beta - 24\alpha\gamma - 4\beta\gamma}{24\beta \gamma} \rho_m \delta - \frac{(3\alpha + \beta)4\alpha + \beta}{6\beta \gamma \rho_m^2} \rho_m \delta \right) \delta^i_j. \quad (53)$$

The equation

$$G^l_{\ i} = R^l_{\ i}$$

$$= E^l_{\ i} - 4\alpha R^l_{\ i} R - \beta \left(2R^\mu R_{\ mu} + 2R^\sigma R_{\ \mu\nu\sigma}\right)$$

$$+ \gamma \left(4e^l_{\ i} T^\mu_{\ \nu} \psi^\mu_{\ i} - 4e^l_{\ i} T_{\ mu} \psi^\mu_{\ i} - 4T^\lambda_{\ \mu} T^\mu_{\ i} \psi^\lambda_{\ i}\right), \quad (54)$$

becomes

$$2a^{-2} \left[\mathcal{H} \phi + \psi'\right]_{,i} - a^{-1} \left(8\gamma - 1 - 4\alpha \rho_m\right) \psi'_{,i}$$

$$- 4\beta a^{-1} \left(\frac{(8\gamma - 1)(4\gamma + 1)}{16\beta \gamma} - \frac{12\alpha + 5\beta - 8\gamma(3\alpha + 2\beta)}{24\beta \gamma} \rho_m - 2(3\alpha + \beta) \frac{4\alpha + \beta}{24\gamma \beta \rho_m^2} \right) \mathcal{H} \phi_{,i}$$

$$+ 8\beta a^{-1} \mathcal{H} f_{,i} + 8a^{-1} \mathcal{H} f \xi_{,i} = 0, \quad (55)$$

where \( \xi \) is the perturbation of \( f \).
In the Fourier space $k$, we have

$$
2a^{-2} \left( 3\mathcal{H}(\mathcal{H}\phi + \psi') - k^2 \psi \right)
+ \left( \frac{3(1 - 8\gamma)}{16\beta \gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{8\beta \gamma} \right) \rho_m + \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{4\gamma \beta} \rho_m^2 \psi
= \left(- \frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta \gamma} \rho_m \delta - \frac{3\alpha + \beta}{2\beta \gamma} \left( 4\alpha + \beta \right) \rho_m^2 \delta, \right.
\tag{56}
$$

$$
-2a^{-2} \left[ \left( 2\mathcal{H}' + \mathcal{H}^2 \right) \phi + \mathcal{H}\phi' + \psi'' + 2\mathcal{H}\psi' + \frac{1}{2} k^2 (\phi - \psi) \right] \delta'_{ij} - \frac{1}{2} (\phi - \psi)_{,ij} \right] 
+ 2 \left( \frac{1 - 8\gamma}{32\beta \gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{48\beta \gamma} \rho_m + \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{24\gamma \beta} \rho_m^2 \right) \psi \delta'_{ij}
= \left(- \frac{12\alpha + 5\beta - 24\alpha\gamma - 4\beta\gamma}{24\beta \gamma} \rho_m \delta - \frac{3\alpha + \beta}{6\beta \gamma} \left( 4\alpha + \beta \right) \rho_m^2 \delta \right) \delta_{ij}. \tag{57}
$$

$$
2a^{-2} \left( 3\mathcal{H}(\mathcal{H}\psi + \psi') - k^2 \psi \right)
- 4\beta a^{-1} \left( \frac{8\gamma - 1}{16\gamma} + \frac{12\alpha + 5\beta - 8\gamma (3\alpha + 2\beta)}{24\beta \gamma} \rho_m - \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{12\gamma \beta} \rho_m^2 \right) \mathcal{H}\phi
+ 8\beta a^{-1} \mathcal{H} f^2 \psi + 8a^{-1} \mathcal{H} f \xi = 0. \tag{58}
$$

When $i \neq j$, (57) leads to

$$
\phi = \psi. \tag{59}
$$

So we have the equations of $\psi$ and $\xi$:

$$
2a^{-2} \left( 3\mathcal{H}(\mathcal{H}\psi + \psi') - k^2 \psi \right)
+ \left( \frac{3(1 - 8\gamma)}{16\beta \gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{8\beta \gamma} \right) \rho_m + \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{4\gamma \beta} \rho_m^2 \psi
= \left(- \frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta \gamma} \rho_m \delta - \frac{3\alpha + \beta}{2\beta \gamma} \left( 4\alpha + \beta \right) \rho_m^2 \delta, \right.
\tag{60}
$$

$$
-2a^{-2} \left[ \left( 2\mathcal{H}' + \mathcal{H}^2 \right) \psi + 3\mathcal{H}\psi' + \psi'' \right] 
+ 2 \left( \frac{1 - 8\gamma}{32\beta \gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{48\beta \gamma} \rho_m + \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{24\gamma \beta} \rho_m^2 \right) \psi
= \left(- \frac{12\alpha + 5\beta - 24\alpha\gamma - 4\beta\gamma}{24\beta \gamma} \rho_m \delta - \frac{3\alpha + \beta}{6\beta \gamma} \left( 4\alpha + \beta \right) \rho_m^2 \delta \right). \tag{61}
$$

$$
2a^{-2} \left[ \mathcal{H}\psi + \psi' \right] - a^{-1} \left( \frac{8\gamma - 1}{16\gamma} + \frac{12\alpha + 5\beta - 8\gamma (3\alpha + 2\beta)}{24\beta \gamma} \rho_m - \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{12\gamma \beta} \rho_m^2 \right) \mathcal{H}\psi
+ 8\beta a^{-1} \mathcal{H} f^2 \psi + 8a^{-1} \mathcal{H} f \xi
= 0. \tag{62}
$$

We consider two cases:

i) In sub-horizon approximation, $\partial/\partial \eta \sim \mathcal{H} \ll k$, we have

$$
\left( \frac{2a^{-2}k^2 - 3(1 - 8\gamma)}{16\beta \gamma} - \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{8\beta \gamma} \right) \rho_m - \left( 3\alpha + \beta \right) \frac{4\alpha + \beta}{4\gamma \beta} \rho_m^2 \psi
= \left( \frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta \gamma} \rho_m \delta + \frac{3\alpha + \beta}{2\beta \gamma} \left( 4\alpha + \beta \right) \rho_m^2 \delta \right). \tag{63}
$$
\[ -2a^{-2}\left[ (2H' + H^2) \psi + 3H\psi' + \psi'' \right] \\
= +2\left( \frac{1 - 8\gamma}{32\beta\gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{48\beta\gamma} \rho_m + (3\alpha + \beta) \frac{4\alpha + \beta}{24\beta\gamma} \rho^2_m \right) \psi \\
= \left( -\frac{12\alpha + 5\beta - 24\alpha\gamma - 4\beta\gamma}{24\beta\gamma} \rho_m \delta - \frac{(3\alpha + \beta) (4\alpha + \beta)}{6\beta\gamma} \rho^2_m \delta \right). \tag{64} \]

\[ 2a^{-2} [H\psi + \psi'] - a^{-1} (8\gamma - 1 - 4\alpha\rho_m) \psi' \\
-4\beta a^{-1} \left( \frac{(8\gamma - 1) (4\gamma + 1)}{16\beta\gamma} - \frac{12\alpha + 5\beta - 8\gamma (3\alpha + 2\beta)}{24\beta\gamma} \rho_m - (3\alpha + \beta) \frac{4\alpha + \beta}{12\beta\gamma} \rho^2_m \right) \psi \\
+ 8\beta a^{-1} Hf^2 \psi + 8a^{-1} Hf\xi = 0. \tag{65} \]

ii) The approximation of ultra-relativistic variation, on scales which are much smaller than the Hubble radius \( H \ll \partial/\partial\eta, \partial/\partial\eta \sim k \), we have

\[ 2a^{-2} (3H\psi' - k^2 \psi) \\
+ \left( \frac{3 (1 - 8\gamma)}{16\beta\gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{8\beta\gamma} \rho_m + (3\alpha + \beta) \frac{4\alpha + \beta}{4\beta} \rho^2_m \right) \psi \\
= \left( -\frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta\gamma} \rho_m \delta - \frac{(3\alpha + \beta) (4\alpha + \beta)}{2\beta\gamma} \rho^2_m \delta, \right. \tag{66} \]

\[ -2a^{-2} [2H'\psi + \psi''] \\
+2\left( \frac{1 - 8\gamma}{32\beta\gamma} + \frac{12\alpha + 5\beta - 16\gamma (3\alpha + \beta)}{48\beta\gamma} \rho_m + (3\alpha + \beta) \frac{4\alpha + \beta}{24\beta\gamma} \rho^2_m \right) \psi \\
= \left( -\frac{12\alpha + 5\beta - 24\alpha\gamma - 4\beta\gamma}{24\beta\gamma} \rho_m \delta - \frac{(3\alpha + \beta) (4\alpha + \beta)}{6\beta\gamma} \rho^2_m \delta \right). \tag{67} \]

\[ (2a^{-1} - 8\gamma + 1 + 4\alpha\rho_m) \psi' \\
-4\beta \left( \frac{(8\gamma - 1) (4\gamma + 1)}{16\beta\gamma} - \frac{12\alpha + 5\beta - 8\gamma (3\alpha + 2\beta)}{24\beta\gamma} \rho_m - (3\alpha + \beta) \frac{4\alpha + \beta}{12\beta\gamma} \rho^2_m \right) \psi \\
+ 8\beta Hf^2 \psi + 8Hf\xi = 0. \tag{68} \]

When 

\[ \rho_m \gg 1, \tag{69} \]

(25), (26) and (27) become

\[ \frac{H^2}{a^2} \approx \frac{(3\alpha + \beta) (4\alpha + \beta)}{24\beta\gamma} \rho_m^2, \tag{70} \]

\[ \frac{H'}{a^2} \approx \frac{3\alpha + \beta}{3\beta} \rho_m, \tag{71} \]

\[ f^2 \approx \frac{(3\alpha + \beta) (4\alpha + \beta)}{24\beta\gamma} \rho_m^2 = \frac{H^2}{a^2}. \tag{72} \]

When 

\[ \rho_m \gg 1, \tag{69} \]
\[ \dot{\psi} + \left( \frac{1}{2} \frac{(3\alpha + \beta)(4\alpha + \beta)}{6\gamma\beta} \rho_{m0}a^{-3} \right) \frac{2}{3} \sqrt{\frac{6\beta}{(3\alpha + \beta)(4\alpha + \beta) \rho_{m0}}} \psi + \sqrt{\frac{(3\alpha + \beta)(4\alpha + \beta)}{6\gamma\beta}} \rho_{m0}a^{-3} \delta = 0, \]  
(73)
we compute

\[
\rho_m \ddot{\delta} - \left[ 2 \dot{\rho}_r + 2 \dot{\rho}_g + 6H \left( \frac{4}{3} \rho_r + \rho_g + p_g \right) - 2H \rho_m + \frac{4 \dot{\rho}_r + \dot{p}_g}{\rho_m + \frac{4}{3} \rho_r + \rho_g + p_g} \right] \dot{\delta}
\]

\[
= - \left[ \dot{\rho}_r + \dot{\rho}_g + \left( 9H - \frac{1}{3} \frac{\dot{\rho}_r + \dot{p}_g}{\rho_m + \frac{4}{3} \rho_r + \rho_g + p_g} \right) \left( \dot{\rho}_r + \dot{\rho}_g \right) + 3H \dot{p}_g \right] \delta
\]

\[
- \left[ \frac{1}{2} \left( \rho_m + \frac{4}{3} \rho_r + \rho_g + p_g \right) \rho + \left( \frac{7}{2} \rho_m + 3 \rho_r + \frac{7}{2} \rho_g - \frac{3}{2} p_g \right) \left( \frac{4}{3} \delta \rho_r + \delta \rho_g \right) \right] \delta
\]

\[
+ \frac{\dot{\rho}_r + 3 \dot{p}_g}{\rho_m + \frac{4}{3} \rho_r + \rho_g + p_g} H \left( \frac{4}{3} \rho_r + \rho_g + p_g \right) \delta + \frac{\delta}{\rho_m + \frac{4}{3} \rho_r + \rho_g + p_g}
\]

\[
- \left( \frac{3}{2} \frac{\dot{\rho}_r + \dot{p}_g}{\rho_m + \frac{4}{3} \rho_r + \rho_g + p_g} \left( \dot{\rho}_r + \dot{p}_g \right) + 9H \dot{\rho}_r + 8H \dot{\rho}_g + 3H \dot{p}_g \right) \delta
\]

\[
- \left( \frac{1}{2} \frac{\dot{\rho}_r + 3 \dot{p}_g}{\rho_m + \frac{4}{3} \rho_r + \rho_g + p_g} H - \frac{7}{2} \rho_m + 3 \rho_r - \frac{7}{2} \rho_g + \frac{3}{2} p_g \right) \left( \frac{4}{3} \delta \rho_r + \delta \rho_g \right) \right) \delta
\]

\[
= 0. \quad (81)
\]

In the case

\[
\rho_m \gg \rho_r, \quad (82)
\]

we have

\[
\rho_m \ddot{\delta} - \left[ 2 \dot{\rho}_g + 6H (\rho_g + p_g) - 2H \rho_m + \frac{\dot{p}_g}{\rho_m + \rho_g + p_g} \right] \dot{\delta}
\]

\[
= - \left[ \ddot{\rho}_g + \left( 9H - \frac{\dot{\rho}_g}{\rho_m + \rho_g + p_g} \right) \dot{\rho}_g + 3H \dot{p}_g \right] \delta
\]

\[
- \left[ \frac{1}{2} (\rho + \rho_g + p_g) \rho_m + \left( \frac{7}{2} \rho_m + 7 \rho_g - \frac{3}{2} p_g \right) (\rho_g + p_g) \right] \delta
\]

\[
+ \frac{3 \dot{p}_g}{\rho_m + \rho_g + p_g} H (\rho_g + p_g) \delta + \frac{\delta}{\rho_m + \rho_g + p_g}
\]

\[
- \left( \frac{3 \dot{p}_g}{\rho_m + \rho_g + p_g} H - \frac{7}{2} \rho_m - \frac{7}{2} \rho_g + \frac{3}{2} p_g \right) (\delta \rho_g + \delta p_g) \right) \delta
\]

\[
= 0. \quad (83)
\]

Using (44) and (45) we obtain

\[
\delta \rho_g = - \frac{(8 \gamma - 1) \left( 12 \alpha + 5 \beta \right)}{16 \beta \gamma} \rho_m \delta + \frac{(3 \alpha + \beta) (4 \alpha + \beta)}{4 \beta \gamma} \rho_m^2 \delta, \quad (84)
\]

\[
\delta p_g = - \frac{12 \alpha + 5 \beta + 8 \beta \gamma}{48 \beta \gamma} \rho_m \delta - \frac{(3 \alpha + \beta) (4 \alpha + \beta)}{12 \beta \gamma} \rho_m^2 \delta, \quad (85)
\]
Then (83) can be written as

\[
\begin{aligned}
\left(1 - \frac{(8\gamma - 1)(12\alpha + 5\beta)}{16\beta\gamma} + \frac{(3\alpha + \beta)(4\alpha + \beta)}{4\beta\gamma}\rho_m \right) \rho_m \frac{\dot{\rho}}{\rho} + 2H \left(1 + \frac{3(12\alpha + 5\beta - 64\alpha\gamma - 24\beta\gamma)}{32\beta\gamma} + \frac{5(3\alpha + \beta)(4\alpha + \beta)}{8\beta\gamma}\rho_m \right) \rho_m \frac{\dot{\rho}}{\rho} \\
- 3H \left(\frac{(8\gamma - 1)^2}{4\beta\gamma} + \frac{(3\alpha + \beta)(4\alpha + \beta)}{2\beta\gamma}\rho_m \right) \rho_m \frac{\dot{\rho}}{\rho} \\
- \frac{p_g}{\rho_m + \rho_g + p_g} \left(\frac{12\alpha + 5\beta - 96\alpha\gamma - 24\beta\gamma}{16\beta\gamma} + \frac{(3\alpha + \beta)(4\alpha + \beta)}{4\beta\gamma}\rho_m \right) \rho_m \frac{\dot{\rho}}{\rho} \\
+ \left[\frac{(3\alpha + \beta)(4\alpha + \beta)}{4\beta\gamma}\rho_m \right] \rho_m \frac{\dot{\rho}}{\rho} \\
+ \left[\frac{(8\gamma - 1)(12\alpha + 5\beta)}{16\beta\gamma} + 3(3\alpha + \beta)(4\alpha + \beta)\rho_m \right] H \rho_m \frac{\dot{\rho}}{\rho} \\
+ \frac{(3\alpha + \beta)}{\beta} \left(\frac{(8\gamma - 1)^2}{8\beta\gamma} + \frac{12\alpha + 5\beta - 8\gamma(18\alpha + 5\beta)}{24\beta\gamma}\rho_m + \frac{(3\alpha + \beta)(4\alpha + \beta)}{12\beta\gamma}\rho_m^2 \right) \rho_m \frac{\dot{\rho}}{\rho} \\
= 0. 
\end{aligned}
\]

where

\[
\frac{\dot{p}_g}{\rho_m + \rho_g + p_g} = \frac{-(12\alpha + 5\beta + 8\beta\gamma) \dot{\rho}_m - 4(3\alpha + \beta)(4\alpha + \beta) \rho_m \frac{\dot{\rho}_m}{\rho} - 4(3\alpha + \beta)(4\alpha + \beta) \rho_m^2}{6(8\gamma - 1)^2 + 2(12\alpha + 5\beta - 8\gamma(18\alpha + 5\beta)) \rho_m + 4(3\alpha + \beta)(4\alpha + \beta) \rho_m^2}. \tag{87}
\]

In the case

\[\rho_m \gg 1\]

(86) becomes

\[\ddot{\delta} + 5H \dot{\delta} + 4H^2 \delta = 0. \tag{88}\]

Introducing the logarithmic time variable

\[N = \ln \alpha\]
we get
\[ \frac{d^2 \delta}{dN^2} + 4 \frac{d\delta}{dN} + 4\delta = 0. \] (89)

This equation has the general solution
\[ \delta = Ae^{-2N} + BN e^{-2N}. \]

and then
\[ \delta = Aa^{-2} + Ba^{-2} \ln a. \] (90)

We find that matter perturbations are always damped. This phenomenon is called phantom damping \[21\].

V. Conclusions

Quadratic theories of gravity described by the Lagrangian \( R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \) have been studied in many works in supergravity, quantum gravity, string theory and M-theory. However, the cosmology in these theories has not been explored extensively, especially, when the torsion of the spacetime is considered. In this paper we show that by only allowing the connection to be asymmetrical and adding a term \( \gamma T^\nu_{\mu\rho} T_\mu^\nu T_\rho^\rho \) to the Lagrangian \( R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \) a meaningful cosmological solution can be obtained. This solution provides a possible explanation to the acceleration of the cosmological expansion and the phase transformation from a decelerating to an accelerating expansion without a cosmological constant or dark energy. One can find that although the field equation (2) becomes Einstein’s equation, (22) and (23) lead to the Friedmann equation and the Raychaudhuri equation when \( \alpha = \beta = \gamma = 0 \), the cosmological equations (25) and (26) are essentially different and then give different description to the evolution of the universe. The acceleration equation (28) is totally different. A series new consequences can be obtained from these equations. The vacuum spacetime possesses the torsion \( f = \sqrt{\frac{1-8\gamma}{32\gamma}} \), which plays the role of the cosmological constant or the dark energy— the torsion dark energy. The torsion of the spacetime can be produced by the energy and pressure besides the spin of matter. The density \( \rho \) and the pressure \( p \) play the opposite roles in the expansion of the universe, while \( \rho \) decelerates the expansion (the attractive effect) \( p \) accelerates it (the repulsive effect). The acceleration \( \ddot{a}/a \) depends only on the contribution of matter. Radiation has no effect on the acceleration. The deceleration of the universe expansion can take place only in the case \(-3\alpha < \beta < 0, \gamma > \frac{1}{2}, \rho - 3p > \frac{3(8\gamma - 1)}{4(\beta + 3\alpha)} \). The universe undergoes a phase transformation from a decelerating to an accelerating expansion when \( \rho - 3p = \frac{3(8\gamma - 1)}{4(\beta + 3\alpha)} \).

The investigation of the linear perturbations corresponding to the pseudoscalar torsion function yields a different equation of the structure growth from the one in the ΛCDM. Its solution describes a damped matter perturbation.

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