Counting vertices in plane and $k$-ary trees with given outdegree

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January 28, 2015

Abstract: We count the number of vertices in plane trees and $k$-ary trees with given outdegree, and prove that the total number of vertices of outdegree $i$ over all plane trees with $n$ edges is $\binom{2n-i-1}{n-1}$, and the total number of vertices of outdegree $i$ over all $k$-ary trees with $n$ edges is $\binom{k}{i} \binom{kn-i}{n-i}$. For both results we give bijective proofs as well as generating function proofs.

Keywords: Plane trees, $k$-ary trees, generating function, bijective proof.

AMS Classification: 05A15, 05C05.

1 Introduction

Let us first review some terminology related to trees. A tree is an acyclic connected graph. In this paper we will assume all the trees are (unlabelled) plane trees, i.e., rooted trees whose vertices are considered to be indistinguishable, but the subtrees at any vertex are linearly ordered. For each vertex $v$ of a tree, we say that $v$ is of outdegree $i$ if it has $i$ subtrees, and call vertices of outdegree 0 leaves. Vertices that are not leaves are called internal vertices.

A complete $k$-ary tree is a plane tree for which each internal vertex has outdegree $k$. If we remove all the leaves and the edges incident to leaves from a completely $k$-ary tree, we get an (ordinary) $k$-ary tree.

The set of plane trees is one of the most well-known and well-studied combinatorial structures. Existing results have focused on studying various statistics on plane trees and finding bijections between plane trees and other structures. In this paper we focus on the outdegree of vertices in plane trees and $k$-ary trees, our main results are the following two theorems, for which we provide both generating function proofs and bijective proofs.

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Theorem 1.1 For integers \( n \geq 1 \) and \( i \geq 0 \), the total number of vertices of outdegree \( i \) over all plane trees with \( n \) edges is
\[
\binom{2n - i - 1}{n - 1}.
\]

Theorem 1.2 For integers \( n, k \geq 1 \) and \( i \geq 0 \), the total number of vertices of outdegree \( i \) over all \( k \)-ary trees with \( n \) edges is
\[
\binom{k}{i} \binom{kn}{n - i}.
\]

The enumeration of plane trees by outdegree sequences has been well studied in the literature, see [2, 3, 7]. Given a sequence of nonnegative integers \((r_0, r_1, \ldots, r_n)\) with \( \sum_{j=0}^{n} r_j = n + 1 \) and \( \sum_{j=0}^{n} jr_j = n \). The number of plane trees with \( n + 1 \) vertices and exactly \( r_j \) vertices have outdegree \( j \) is given by
\[
\frac{1}{n+1} \binom{n + 1}{r_0, r_1, \ldots, r_n}.
\]

A more general version of the above result that enumerating plane forests (graphs such that every connected component is a plane tree) by outdegree sequence is given in [9, Theorem 5.3.10]. From Theorem 1.1 and the above formula we have the following identity
\[
\sum r_i \frac{r_i}{n+1} \binom{n + 1}{r_0, r_1, \ldots, r_n} = \binom{2n - i - 1}{n - 1},
\]
where the sum is over all sequences of nonnegative integers \((r_0, r_1, \ldots, r_n)\) such that \( \sum_{j=0}^{n} r_j = n + 1 \) and \( \sum_{j=0}^{n} jr_j = n \).

Counting vertices in plane trees according to the outdegrees was also studied more recently by other authors. In [4] Deutsch and Shapiro proved that the total number of vertices of odd outdegree over all plane trees with \( n \) edges is
\[
\frac{2}{3} \binom{2n - 1}{n} + \frac{1}{3} F_{n-1},
\]
where \( F_n \) is the \( n \)-th Fine number (Sequence A000957 in [8]). Therefore from Theorem 1.1 we immediately have the following formula for Fine numbers.
\[
F_n = 3 \sum_{k \geq 0} \binom{2n - 2k}{n} - 2 \binom{2n + 1}{n}.
\]

Deutsch and Shapiro also counted the number of vertices of odd degree in [4]. Here the degree of a vertex \( v \) means the total number of vertices adjacent to \( v \). Therefore if \( v \) is the root, then the degree of \( v \) is the same as the outdegree of \( v \). Otherwise, the degree of \( v \) is one plus the outdegree of \( v \). Deutsch and Shapiro proved that over all plane trees with \( n \) edges, the total number of vertices of odd degree is twice the total number of vertices of odd outdegree. Their proof is based on generating function method. Later in [6], Eu, Liu and Yeh gave a bijective proof of this result by defining a two-to-one correspondence. Although the authors didn’t point this out in [6] explicitly, their correspondence showed that over all plane trees with \( n \) edges, the number of vertices of degree \( i \) is twice the number of vertices of out degree \( i \). Therefore from their result we have the following corollary.
Corollary 1.3 For integers $n \geq 1$ and $i \geq 1$, the total number of vertices of degree (i.e., outdegree +1, except at the root) $i$ over all plane trees with $n$ edges is

$$2\binom{2n-i-1}{n-1}.$$

2 Generating function proofs for the main results

We will need the following form of the Lagrange inversion formula in our generating function proofs for both plane trees and $k$-ary trees. The proof of this Lemma can be found in [9, p42].

Lemma 2.4 Let $f(z) = zG[f(z)], G(0) \neq 0$. Then

$$[z^n]H(f(z)) = \frac{1}{n} [z^{n-1}]H(z)^l G(z)^n. \quad (2.1)$$

2.1 Generating function proof for plane trees

It is a well-known result that the number of plane trees with $n$ edges ($n$-plane trees, for short) is counted by the $n$-th Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$, and the generating function of the Catalan number, $C(z) := \sum_{n \geq 0} c_n z^n$ satisfies

$$C(z) = 1 + zC(z)^2. \quad (2.2)$$

From Lemma 2.4 we get the following property for $C(z)$.

Lemma 2.5 For any positive integer $l$, we have

$$[z^n]C^d(z) = \frac{l}{2n+l} \binom{2n+l}{n}. \quad (2.3)$$

Proof. From (2.2) we know that $C(z) - 1 = z((C(z) - 1) + 1)^2$. By applying (2.1) we have

$$[z^n]C(z)^l = \frac{1}{n} [z^{n-1}] \left( \frac{d}{dz} (z+1)^l \right) (z+1)^{2n}$$

$$= \frac{l}{n} [z^{n-1}] (z+1)^{2n+l-1}$$

$$= \frac{l}{n} \binom{2n+l-1}{n-1} = \frac{l}{2n+l} \binom{2n+l}{n}.$$ 

Remark: Equation (2.3) appears as an exercise in [1, Ex3.59].

Generating function proof of Theorem 1.1:

Let $a_i(m, n)$ denote the number of $n$-plane trees with exactly $m$ vertices of outdegree $i$, and $G_i(t, z)$ its generating function, i.e., $G_i = G_i(t, z) = \sum_{m,n \geq 0} a_i(m, n) t^m z^n$. 

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By considering the outdegree of the root, we can easily derive that \( G_i = G_i(t, z) \) satisfies
\[
G_i = 1 + zG_i + z^2G_i^2 + \cdots + z^{i-1}G_i^{i-1} + tz^iG_i^i + z^{i+1}G_i^{i+1} + \cdots
\]
\[
= \frac{1}{1-zG_i} + (t-1)z^iG_i^i.
\]

Differentiating \( G_i \) with respect to \( t \), setting \( t = 1 \), and making use of the fact that
\[
G_i(1, z) = \sum_{m,n \geq 0} a_i(m, n)z^n = \sum_{n \geq 0} c_n z^n = C(z),
\]
We have
\[
\left. \frac{\partial G_i(t, z)}{\partial t} \right|_{t=1} = \left. \frac{z}{(1-zC(z))^2} \frac{\partial G_i(t, z)}{\partial t} \right|_{t=1} + z^i C(z)^i.
\]

Moreover, from (2.2) we know that \( 1/C(z) = 1 - zC(z) \). Substituting \( 1 - zC(z) \) by \( 1/C(z) \) in the above equation we have
\[
\left. \frac{\partial G_i(t, z)}{\partial t} \right|_{t=1} = \frac{z^i C(z)^i}{1 - zC(z)^2} = \sum_{m \geq 0} z^{m+i} C(z)^{2m+i}.
\]

(2.4)

On the other hand, we know that
\[
\left. \frac{\partial G_i(t, z)}{\partial t} \right|_{t=1} = \left. \left( \sum_{m,n \geq 0} ma_i(m, n)t^{m-1}z^n \right) \right|_{t=1} = \sum_{m,n \geq 0} ma_i(m, n)z^n.
\]

(2.5)

Comparing (2.4) and (2.5) we have that the total number of vertices of outdegree \( i \) over all \( n \)-plane trees is
\[
\sum_{m \geq 0} ma_i(m, n) = [z^n] \left. \frac{\partial G_i(t, z)}{\partial t} \right|_{t=1} = \sum_{m \geq 0} [z^{n-m-i}] C(z)^{2m+i}.
\]

Applying Lemma 2.5 and the Chu-Vandermonde identity \( \binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k} \) we get
\[
\sum_{m \geq 0} ma_i(m, n) = \sum_{m \geq 0} \frac{2m+i}{2(n-m-i) + (2m+i)} \binom{2(n-m-i) + (2m+i)}{n-m-i}
\]
\[
= \sum_{m \geq 0} \frac{2m+i}{2n-i} \binom{2n-i}{n-m-i}
\]
\[
= \frac{1}{2n-i} \sum_{m \geq 0} \left[ \binom{n+m}{2n-i} - \binom{n-m-i}{n-i} \right]
\]
\[
= \frac{1}{2n-i} \sum_{m \geq 0} \left[ \binom{2n-i}{n+m} - \binom{2n-i}{n-m-1} \right]
\]
\[
= \sum_{m \geq 0} \binom{2n-i-1}{n+m-1} - \sum_{m \geq 0} \binom{2n-i-1}{n+m}
\]
\[
= \binom{2n-i-1}{n-1}.
\]
2.2 Generating function proof for $k$-ary trees

Let $b_k(n)$ denote the number of $k$-ary trees with $n$ edges (with the convention $b_k(0) = 1$), and $B_k(z)$ its generating function, i.e., $B_k(z) = \sum_{n \geq 0} b_k(n) z^n$. By considering the outdegree of the root of $k$-ary trees, we have

$$B_k(z) = 1 + \binom{k}{1} zB_k(z) + \frac{k}{2} z^2 B_k(z)^2 + \cdots \binom{k}{k} z^k B_k(z)^k = (1 + zB_k(z))^k.$$  \hfill (2.6)

Similar to what we did for plane trees, we can get the following property for $B_k(z)$.

Lemma 2.6 For any positive integer $l$, we have

$$[z^n]B_k(z)^l = \frac{l}{n} \binom{kn}{n}.$$  \hfill (2.7)

Proof. From (2.6) we know that $zB_k(z) = z(1 + zB_k(z))^k$. Applying Lemma 2.4 we have

$$[z^n](z^l B_k(z)^l) = \frac{1}{n}[z^{n-1}] \left( \frac{d}{dz} z^l \right) (1 + z)^{kn}.$$

Therefore we have

$$[z^n]B_k(z)^l = \frac{l}{n}[z^n](1 + z)^{kn} = \frac{l}{n} \binom{kn}{n}.$$  \hfill \qed

Generating function proof of Theorem 1.2:

Let $a_{k,i}(m,n)$ denote the number of $k$-ary trees with $n$ edges and contains exactly $m$ vertices of out outdegree $i$, and $A_{k,i}(t,z)$ its generating function, i.e.,

$$A_{k,i}(t,z) = \sum_{m,n \geq 0} a_{k,i}(m,n) t^m z^n.$$

Considering the outdegree of the root of a $k$-ary tree, we have the following recurrence relation:

$$A_{k,i}(t,z) = \left[ 1 + \binom{k}{1} zA_{k,i}(t,z) + \frac{k}{2} z^2 A_{k,i}(t,z)^2 + \cdots \binom{k}{k} z^k A_{k,i}(t,z)^k \right].$$

Therefore we have

$$A_{k,i}(t,z) = (1 + zA_{k,i}(t,z))^k + (t - 1) \binom{k}{i} z^i A_{k,i}(t,z)^i.$$  \hfill (2.8)

Differentiating $A_{k,i}(t,z)$ with respect to $t$ on both sides, setting $t = 1$, and using the fact that $A_{k,i}(1,z) = B_k(z)$, we get

$$\left. \frac{\partial A_{k,i}(t,z)}{\partial t} \right|_{t=1} = k(1 + zB_k(z))^{k-1} z \left. \frac{\partial A_{k,i}(t,z)}{\partial t} \right|_{t=1} \left. \frac{\partial A_{k,i}(t,z)}{\partial t} \right|_{t=1} + \binom{k}{i} z^i B_k(z)^i.$$

\hfill 5
Applying (2.6) to the above identity, we get
\[
\left. \frac{\partial A_{k,i}(t, z)}{\partial t} \right|_{t=1} = \frac{kzB_k(z)}{1 + zB_k(z)} \left. \frac{\partial A_{k,i}(t, z)}{\partial t} \right|_{t=1} + \binom{k}{i} z^i B_k(z)^i.
\]
Solving the above equation, we get
\[
\left. \frac{\partial A_{k,i}(t, z)}{\partial t} \right|_{t=1} = \frac{(1 + zB_k(z)) \binom{k}{i} (zB_k(z))^i}{1 - (k - 1)zB_k(z)}
= \binom{k}{i} ((1 + zB_k(z))z^i B_k(z)^i) \sum_{r \geq 0} (k - 1)^r z^r B_k(z)^r
= \binom{k}{i} \sum_{r \geq 0} (k - 1)^r (z^{i+r} B_k(z)^{i+r} + z^{i+r+1} B_k(z)^{i+r+1}).
\]
Applying Lemma 2.6 we have that the total number of vertices of outdegree $i$ over all $k$-ary trees with $n$ edges is
\[
\sum_{m \geq 0} a_{k,i}(m, n) = \left[ z^n \right] \left. \frac{\partial A_{k,i}(t, z)}{\partial t} \right|_{t=1}
= \binom{k}{i} \sum_{r \geq 0} (k - 1)^r \left( [z^{n-r-i}] B_k(z)^{i+r} + [z^{n-r-i-1}] B_k(z)^{i+r+1} \right)
= \binom{k}{i} \sum_{r \geq 0} (k - 1)^r \left[ \frac{k(i + r)}{kn} \left( \frac{kn}{n - r - i} \right) + \frac{k(i + r + 1)}{kn} \left( \frac{kn}{n - r - i - 1} \right) \right]
= \binom{k}{i} \sum_{r \geq 0} (k - 1)^r \left[ \frac{r + i}{n} \left( \frac{kn}{n - r - i} \right) + \frac{i + r + 1}{kn} \left( \frac{n}{n - r - i - 1} \right) \right].
\]
Here
\[
\sum_{r \geq 0} (k - 1)^r \frac{r + i}{n} \left( \frac{kn}{n - r - i} \right) = \sum_{r \geq 0} (k - 1)^{r-i} \frac{r}{n} \left( \frac{kn}{n - r} \right)
= (k - 1)^{-i} \sum_{r \geq 0} (k - 1)^r \frac{n - (n - r)}{n} \left( \frac{kn}{n - r} \right)
= (k - 1)^{-i} \sum_{r \geq 0} (k - 1)^r \left[ \left( \frac{kn}{n - r} \right) - \frac{1}{n} \left( \frac{n - r}{n} \right) \left( \frac{kn}{n - r} \right) \right].
\]
Applying the Chu-Vandermonde identity
\[
\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}
\]
and the identity
\[
\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}.
\]
we have
\[ \sum_{r \geq 0} (k - 1)^r \frac{r + i}{n} \binom{kn}{n - r - i} \]
\[ = (k - 1)^{-i} \sum_{r \geq i} (k - 1)^r \left[ \binom{kn - 1}{n - r} + \binom{kn - 1}{n - r - 1} - k \binom{kn - 1}{n - r - 1} \right] \]
\[ = (k - 1)^{-i} \sum_{r \geq i} (k - 1)^r \left[ \binom{kn - 1}{n - r} - (k - 1)^{r+1} \binom{kn - 1}{n - r - 1} \right] \]
\[ = \binom{kn - 1}{n - i}. \]

Similarly we have
\[ \sum_{r \geq 0} (k - 1)^r \frac{r + i + 1}{n} \binom{kn}{n - r - i - 1} = \sum_{r \geq i+1} (k - 1)^{r-i-1} \frac{r}{n} \binom{kn}{n - r} = \binom{kn - 1}{n - i - 1}. \]

Therefore we proved that
\[ \sum_{m,n \geq 0} ma_{k,i}(m,n) = [z^n] \frac{\partial A_{k,i}(t,z)}{\partial t} \bigg|_{t=1} = \binom{k}{i} \left[ \binom{kn - 1}{n - i} + \binom{kn - 1}{n - i - 1} \right] = \binom{k}{i} \binom{kn}{n - i}. \]

\section{Bijective proofs for the main results}

\subsection{Bijective proof for plane trees}

Let \( T_{n,i} \) be the set of ordered pairs \((T,v)\) such that \( T \) is an \( n \)-plane tree, and \( v \) a vertex in \( T \) of outdegree \( i \). We will associate with each pair \((T,v) \in T_{n,i} \) a composition of nonnegative integer \( n \). Here a composition of \( n \) can be thought of as an expression of \( n \) as an ordered sum of integers. More precisely, a sequence \( \alpha = (a_1, \ldots, a_k) \) of nonnegative integers satisfying \( \sum_{j=1}^k a_j = n \) is called a \( k \)-composition of \( n \). We use \( A_{n,k} \) to denote the set of all \( k \)-composition of \( n \).

For each \( \alpha \in A_{n,k} \) we define \( f(\alpha) = \sum_{j=1}^k (a_j - 1) \). If a \( k \)-composition \( \alpha = (a_1, \ldots, a_k) \) satisfy \( f(a_1, \ldots, a_j) \geq 0 \) for each \( j, 1 \leq j \leq k - 1 \) and \( f(\alpha) = -1 \), then we call \( \alpha \) a unit composition. If \( \alpha = (a_1, \ldots, a_k) \) satisfy \( f(a_1, \ldots, a_j) \geq 0 \) for each \( j, 1 \leq j \leq k \), then we call \( \alpha \) a positive composition. We use \( B \) to denote the set of all unit compositions and \( B_{n,k} \) to denote the set of all unit \( k \)-compositions of integer \( n \).

For each \( n \)-plane tree \( T \), we label the \( n + 1 \) vertices of \( T \) with numbers 1 to \( n + 1 \) in the depth-first order, or preorder (The definition of the depth-first order can be found in [5, p.336] or [9, p.33]). Now we associate with \( T \) a composition \( \delta(T) \) by \( \delta(T) = (d_1, d_2, \ldots, d_{n+1}) \), here \( d_j \) is the outdegree of the vertex in \( T \) labeled \( j \).

\begin{lemma}
The map \( T \mapsto \delta(T) \) is a bijection from the set of plane trees to \( B_{n,n+1} \).
\end{lemma}
Example 3.8 Let \( n = 14 \). Figure 1 shows a plane tree \( T \) with 14 edges, whose vertices are labeled in depth-first order from 1 to 15, and we have \( \delta(T) = (3, 2, 0, 2, 0, 0, 0, 3, 0, 0, 2, 0, 0) \in \mathcal{B}_{14, 15} \).

![Figure 1: A plane tree with 14 edges, with one of its vertices of outdegree 2 circled.](image-url)

Lemma 3.7 has a fairly straightforward proof by induction and will be omitted here. And there is a more general result that plane forests are in bijection with a sequence of unit compositions. More details can be found in [9, p.34], in which the author use the terminology of Lukasiewicz words instead of compositions.

In the following of this section we concentrate on plane trees with specified vertex of given outdegree. Now we are ready to establish our bijection.

**Theorem 3.9** There is a bijection between \( \mathcal{T}_{n, i} \) and \( \mathcal{A}_{n-i,n} \);

**Proof.** Given an ordered pair \( (T, v) \in \mathcal{T}_{n,i} \), we label the vertices of \( T \) in depth-first order. Suppose \( v \) is labeled \( j, 1 \leq j \leq n+1 \). We define

\[
\bar{\delta}(T, v) = \alpha = (d_{j+1}, \ldots, d_{n+1}, d_1, \ldots, d_{j-1}).
\]

For example, let \( T \) be the plane tree shown in Figure 1, and the circled vertex with label 4 is the specified vertex \( v \), which has outdegree 2. Then we have \( \bar{\delta}(T, v) = (0, 2, 0, 0, 0, 3, 0, 0, 2, 0, 0, 3, 2, 0) \). Since there are \( n \) edges of \( T \), and \( v \) has outdegree \( i \), it is obvious that \( \sum_{i=1, i \neq j}^{n+1} d_i = n - i \), thus \( \bar{\delta}(T, v) \in \mathcal{A}_{n-i,n} \).

On the other hand, given \( \alpha \in \mathcal{A}_{n-i,n} \), we can uniquely decompose \( \alpha \) into the form

\[
\alpha = \alpha_1 \alpha_2 \ldots \alpha_s \alpha_0
\]

for some nonnegative integer \( s \) such that \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are unit compositions, and \( \alpha_0 \) is a positive composition with \( f(\alpha_0) = s - i \). We call such a decomposition the fundamental decomposition of \( \alpha \). For example, the fundamental decomposition of \( \alpha = (0, 2, 0, 0, 0, 3, 0, 0, 2, 0, 0, 3, 2, 0) \) is the following:

\[
(0) (2, 0, 0) (0) (3, 0, 0, 2, 0, 0) (3, 2, 0).
\]
Now we set $\alpha' = a_0 \ i \ a_1a_2\ldots a_s$. One can easily verify that $\alpha'$ is a unit composition. We set $T = \delta^{-1}(\alpha')$. Suppose $a_0$ is of length $l$. Let $v$ be the $(l+1)$-th vertex of $T$ in the depth-first order. Then we have $(T, v) = \delta^{-1}(\alpha)$. Hence we proved that the map $\delta$ is a bijection.

**Bijective proof of Theorem 1.1:** It is a basic result in enumerative combinatorics that the number of $k$ compositions of $n$, or equivalently, the number of the nonnegative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = n$ is \binom{n+k-1}{n}, our result then follows.

### 3.2 Bijective proof for $k$-ary trees

Let $T_n^k$ denote the set of $k$-ary trees with $n$ edges, and $T_{n,i}^k$ denote the set of ordered pairs $(T, v)$ such that $T \in T_n^k$, and $v$ is a vertex of $T$ of outdegree $i$. Given $(T, v) \in T_{n,i}^k$, let $T'$ denote the complete $k$-ary tree that corresponds to $T$. Note that $v$ becomes an internal vertex in $T'$ (and has outdegree $k$), and we denote it as $v'$. Let $\alpha = \delta(T', v')$. Since there are $n + 1$ internal vertices and $k(n+1)$ edges in $T'$, and all the internal vertices of $T'$ has outdegree $k$, from Theorem 3.9 we have the following corollary.

**Corollary 3.10** The map $\delta$ is a bijection between $T_{n,i}^k$ and compositions $\alpha$ that satisfy the following:

1. $\alpha \in A_{kn,k(n+1)}$, and $\alpha$ consists of exactly $n$ $k$’s and $(kn + k - n)$ 0’s;

2. In the fundamental decomposition $\alpha = \alpha_1\alpha_2\ldots\alpha_s\alpha_0$, we have $s \geq k$, and among the first $k$ unit compositions $\alpha_1, \alpha_2, \ldots, \alpha_k$, exactly $i$ of them begin with $k$.

**Example 3.11** Let $k = 3$, $n = 8$ and $i = 2$. Figure 2 shows a ternary tree $T$ with 8 edges and the corresponding complete ternary tree $T'$ with $k(n+1) + 1 = 28$ vertices. A specified vertex $v$ of outdegree 2 is circled in $T$, and which corresponds to an internal vertex $v'$ in $T'$. We have

$$\alpha = \delta(T', v') = (3, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 0, 0, 3, 3, 0, 0),$$

The fundamental decomposition of $\alpha$ is

$$(3, 0, 0, 0) (0) (3, 0, 0, 0) (0) (3, 0, 0, 3, 3, 0, 0, 0, 3, 0, 0, 0) (3, 3, 0, 0).$$

It is easy to check that $\alpha$ satisfies conditions (i) and (ii).

Let $A_{kn,k(n+1)}^*$ denote the set of compositions in $A_{kn,k(n+1)}$ that satisfy (i) and (ii). From Corollary 3.10 we know that to prove Theorem 1.2, we need to show the following:

**Theorem 3.12** For integers $n, k \geq 1$ and $i \geq 0$, there is a bijection between $A_{kn,k(n+1)}^*$ and the set of ordered pairs $(X, Y)$, where $X$ is an $i$-element subset of $[k] := \{1, 2, \ldots, k\}$, and $Y$ is an $(n-i)$-element subset of $[kn]$.

**Proof.** Given $\alpha \in A_{kn,k(n+1)}^*$. Suppose the fundamental decomposition of $\alpha$ is $\alpha = \alpha_1\alpha_2\cdots\alpha_s\alpha_0$. Let $\beta = (b_1, b_2, \ldots, b_{kn})$ be the composition obtained from $\alpha$ by deleting the first number in
each of the first $k$ unit compositions $\alpha_i$, $1 \leq i \leq k$. Then exactly $n - i$ members among $b_1, b_2, \ldots, b_{kn}$ equals $k$, suppose they are $b_1, b_2, \ldots, b_{n-i}$, with $l_1, l_2, \ldots, l_{n-i} \in [kn]$. Let $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_i}$ be the $i$ unit compositions among $\alpha_1, \alpha_2, \ldots, \alpha_k$ that begin with $k$. Now we set $X = \{j_1, j_2, \ldots, j_i\}$, $Y = \{l_1, l_2, \ldots, l_{n-i}\}$ and $\phi(\alpha) = (X, Y)$.

For example, for the composition $\alpha$ in Example 3.11, we have $\phi(\alpha) = (X, Y)$ where $X = \{1, 3\}$, and $Y = \{8, 11, 12, 16, 21, 22\}$.

Now we prove that $\phi$ is a bijection by defining its inverse. Given $(X, Y)$ with $X = \{j_1, j_2, \ldots, j_i\} \subseteq [k]$ and $Y = \{l_1, l_2, \ldots, l_{n-i}\} \subseteq [kn]$, we define a sequence $\beta = (b_1, b_2, \ldots, b_{kn})$ such that $b_j = k$ if $j \in Y$, and $b_j = 0$ otherwise. Similarly we define $\gamma = (c_1, c_2, \ldots, c_k)$ to be a sequence of integers such that $c_j = k$ if $j \in X$, and $c_j = 0$. Now we insert $c_1, c_2, \ldots, c_k$ into $\beta$ one-by-one to get a sequence $\alpha = (a_1, a_2, \ldots, a_{k(n+1)})$ such that in the fundamental decomposition of $\alpha$, the first $k$ unit compositions begin with $c_1, c_2, \ldots, c_k$, respectively. Note that when inserting a 0, it forms a unit composition by itself; when inserting a $k$, it will “use” $k$ 0’s from $\beta$, since there are $j$ k’s among $c_1, c_2, \ldots, c_k$ and $kn - k(n-j) = kj$ “remaining” 0’s (each of the $n - j$ k’s in $\beta$ will “use” k 0’s too), such an insertion is always possible. Thus we proved that $\phi$ is a bijection.

For more example, let $n = 2$, $k = 2$ and $i = 1$. There are eight pairs $(X, Y)$, where $X$ is a one-element subset of $\{1, 2\}$, and $Y$ is a one-element subset of $\{1, 2, 3, 4\}$. Figure 1 shows the corresponding $\alpha$, $(T', v')$ and $(T, v)$ for each pair $(X, Y)$.

Theorem 1.2 follows immediately from Theorem 3.12.
Table 1: The bijection for 8 pairs \((T, v)\) when \(n = 2, k = 2\) and \(i = 1\).

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