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QUASI-IN Variant AND SUPER-COINVARIANT POLYNOMIALS FOR THE GENERALIZED SYMMETRIC GROUP

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ABSTRACT. The aim of this work is to extend the study of super-coinvariant polynomials, introduced in [2, 3], to the case of the generalized symmetric group $G_{n,m}$, defined as the wreath product $C_m \wr S_n$ of the symmetric group by the cyclic group. We define a quasi-symmetrizing action of $G_{n,m}$ on $\mathbb{Q}[x_1, \ldots, x_n]$, analogous to those defined in [12] in the case of $S_n$. The polynomials invariant under this action are called quasi-invariant, and we define super-coinvariant polynomials as polynomials orthogonal, with respect to a given scalar product, to the quasi-invariant polynomials with no constant term. Our main result is the description of a Gröbner basis for the ideal generated by quasi-invariant polynomials, from which we deduce that the dimension of the space of super-coinvariant polynomials is equal to $m^n C_n$ where $C_n$ is the $n$-th Catalan number.

1. INTRODUCTION

Let $X$ denote the alphabet in $n$ variables $(x_1, \ldots, x_n)$ and $\mathbb{C}[X]$ denote the space of polynomials with complex coefficients in the alphabet $X$. Let $G_{n,m} = C_m \wr S_n$ denote the wreath product of the symmetric group $S_n$ by the cyclic group $C_m$. This group is sometimes known as the generalized symmetric group (cf. [1]). It may be seen as the group of $n \times n$ matrices in which each row and each column has exactly one non-zero entry (pseudo-permutation matrices), and such that the non-zero entries are $m$-th roots of unity. The order of $G_{n,m}$ is $m^n n!$. When $m = 1$, $G_{n,m}$ reduces to the symmetric group $S_n$, and when $m = 2$, $G_{n,m}$ is the hyperoctahedral group $B_n$, i.e. the group of signed permutations, which is the Weyl group of type $B$ (see [14]).
for example for further details). The group $G_{n,m}$ acts classically on $\mathbb{C}[X]$ by the rule

$$\forall g \in G_{n,m}, \forall P \in \mathbb{C}[X], \ g.P(X) = P(X^t g),$$

where $g$ is the transpose of the matrix $g$ and $X$ is considered as a row vector. Let

$$Inv_{n,m} = \{ P \in \mathbb{C}[X] / \forall g \in G_{n,m}, \ g.P = P \}$$

denote the set of $G_{n,m}$-invariant polynomials. Let us denote by $Inv^+_{n,m}$ the set of such polynomials with no constant term. We consider the following scalar product on $\mathbb{C}[X]$: \( (1.2) \)

$$\langle P, Q \rangle = P(\partial X)Q(X) \big|_{x=0}$$

where $\partial X$ stands for $(\partial x_1, \ldots, \partial x_n)$ and $X = 0$ stands for $x_1 = \cdots = x_n = 0$. The space of $G_{n,m}$-coinvariant polynomials is then defined by

$$Cov_{n,m} = \{ P \in \mathbb{C}[X] / \forall Q \in Inv_{n,m}, \ Q(\partial X)P = 0 \}
= \langle Inv^+_{n,m} \rangle \perp \mathbb{C}[X]/\langle Inv^+_{n,m} \rangle$$

where $\langle S \rangle$ denotes the ideal generated by a subset $S$ of $\mathbb{C}[X]$.

A classical result of Chevalley \cite{6} states the following equality:

$$\dim Cov_{n,m} = |G_{n,m}| = m^n n!$$

which reduces when $m = 1$ to the theorem of Artin \cite{1} that the dimension of the harmonic space $H_n = Cov_{n,1}$ (cf. \cite{3}) is $n!$.

Our aim is to give an analogous result in the case of quasi-symmetrizing action. The ring $Qsym$ of quasi-symmetric functions was introduced by Gessel \cite{11} as a source of generating functions for $P$-partitions \cite{18} and appears in more and more combinatorial contexts \cite{3, 18, 19}. Malvenuto and Reutenauer \cite{16} proved a graded Hopf duality between $QSym$ and the Solomon descent algebras and Gelfand et. al. \cite{10} defined the graded Hopf algebra $NC$ of non-commutative symmetric functions and identified it with the Solomon descent algebra.

In \cite{2, 3}, Aval et. al. investigated the space $SH_n$ of super-coinvariant polynomials for the symmetric group, defined as the orthogonal (with respect to (1.2)) of the ideal generated by quasi-symmetric polynomials with no constant term, and proved that its dimension as a vector space equals the $n$-th Catalan number:

$$\dim SH_n = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Our main result is a generalization of the previous equation in the case of super-coinvariant polynomials for the group $G_{n,m}$.

In Section 2, we define and study a “quasi-symmetrizing” action of $G_{n,m}$ on $\mathbb{C}[X]$. We also introduce invariant polynomials under this action, which are called quasi-invariant, and polynomials orthogonal to quasi-invariant polynomials, which are called super-coinvariant. The Section 3 is devoted to the proof of our main result (Theorem 2.4), which gives the dimension of the space $SCov_{n,m}$ of super-coinvariant polynomials for $G_{n,m}$: we construct an explicit basis for $SCov_{n,m}$ from which we deduce its Hilbert series.
2. A quasi-symmetrizing action of $G_{n,m}$

We use vector notation for monomials. More precisely, for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$, we denote $X^\nu$ the monomial
\[(2.1) \quad x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}.
\]
For a polynomial $P \in \mathbb{Q}[X]$, we further denote $[X^\nu] P(X)$ as the coefficient of the monomial $X^\nu$ in $P(X)$.

Our first task is to define a quasi-symmetrizing action of the group $G_{n,m}$ on $\mathbb{C}[X]$, which reduces to the quasi-symmetrizing action of Hivert (cf. [12]) in the case $n = 1$. This is done as follows. Let $A \subset X$ be a subalphabet of $X$ with $l$ variables and $K = (k_1, \ldots, k_l)$ be a vector of positive ($> 0$) integers. If $B$ is a vector whose entries are distinct variables $x_i$ multiplied by roots of unity, the vector $B_<$ is obtained by ordering the elements in $B$ with respect to the variable order. Now the quasi-symmetrizing action of $g \in G_{n,m}$ is given by
\[(2.2) \quad g \cdot A^K = w(g)^{c(K)} (A.|g|)_<^K
\]
where $w(g)$ is the weight of $g$, i.e. the product of its non-zero entries, $|g|$ is the matrix obtained by taking the modules of the entries of $g$, and the coefficient $c(K)$ is defined as follows:
\[c(K) = \begin{cases} 
0 & \text{if } \forall i, \ k_i \equiv 0 \ [m] \\
1 & \text{if not}. 
\end{cases}
\]

**Example 2.1.** If $m = 3$ and $n = 3$, and we denote by $j$ the complex number $j = e^{2\pi i/3}$, then for example
\[
\begin{pmatrix} 
0 & 0 & j \\
1 & 0 & 0 \\
0 & j & 0 
\end{pmatrix} \cdot \begin{pmatrix} x_1^2 \ x_2 
\end{pmatrix}
\]
\[= (j^2)^1 \begin{pmatrix} 
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 
\end{pmatrix} \cdot (x_1, x_2)_{(2,1)}
\]
\[= j^2 (x_3, x_1)_{(2,1)}
\]
\[= j^2 (x_1, x_3)_{(2,1)}
\]
\[= j^2 x_1^2 x_3.
\]

It is clear that this defines an action of the generalized symmetric group $G_{n,m}$ on $\mathbb{C}[X]$, which reduces to Hivert’s quasi-symmetrizing action (cf. [12], Proposition 3.4) in the case $m = 1$.

Let us now study its invariant and coinvariant polynomials. We need to recall some definitions.

A **composition** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of a positive integer $d$ is an ordered list of positive integers ($> 0$) whose sum is $d$. For a vector $\nu \in \mathbb{N}^n$, let $c(\nu)$ represent the composition obtained by erasing zeros (if any) in $\nu$. A polynomial $P \in \mathbb{Q}[X]$ is said
to be quasi-symmetric if and only if, for any \( \nu \) and \( \mu \) in \( \mathbb{N}^n \), we have
\[
[X^\nu]P(X) = [X^\mu]P(X)
\]
whenever \( c(\nu) = c(\mu) \). The space of quasi-symmetric polynomials in \( n \) variables is denoted by \( \text{Qsym}_n \).

The polynomials invariant under the action (2.2) of \( G_{n,m} \) are said to be quasi-invariant and the space of quasi-invariant polynomials is denoted by \( \text{QInv}_{n,m} \), i.e.
\[
P \in \text{QInv}_{n,m} \iff \forall g \in G_{n,m}, \; g \cdot P = P.
\]
Let us recall (cf. [12], Proposition 3.15) that \( \text{QI} \text{nv}_{n,1} = \text{QSym}_n \). The following proposition gives a characterization of \( \text{QInv}_{n,m} \).

**Proposition 2.2.** One has
\[
P \in \text{QInv}_{n,m} \iff \exists Q \in \text{QSym}_n / P(X) = Q(X^m)
\]
where \( Q(X^m) = Q(x_1^m, \ldots, x_n^m) \).

**Proof.** Let \( P \) be an element of \( \text{QInv}_{n,m} \). Let us denote by \( \zeta \) the \( m \)-th root of unity \( \zeta = e^{2\pi i/m} \) and by \( g_j \) the element of \( G_{n,m} \) whose matrix is
\[
\begin{pmatrix}
\zeta & 0 \\
0 & 1 \\
0 & \ddots \\
0 & 1
\end{pmatrix}
\]
with the \( \zeta \) in place \( j \). Then we observe that the identities
\[
\forall j = 1, \ldots, n, \; \frac{1}{m} (P + g_j \cdot P + g_j^2 \cdot P + \cdots + g_j^{m-1} \cdot P) = P
\]
imply that every exponents appearing in \( P \) are multiples of \( m \). Thus there exists a polynomial \( Q \in \mathbb{C}[X] \) such that \( P(X) = Q(X^m) \). To conclude, we note that \( \mathcal{S}_n \subset G_{n,m} \) implies that \( P \) is quasi-symmetric, whence \( Q \) is also quasi-symmetric.

The reverse implication is obvious. \( \square \)

Let us now define super-coinvariant polynomials:
\[
\text{SCov}_{n,m} = \{ P \in \mathbb{C}[X] / \forall Q \in \text{QInv}_{n,m}, \; Q(\partial X)P = 0 \}
\]
with the scalar product defined in (1.2). This is the natural analogous to \( \text{Cov}_n \) in the case of quasi-symmetrizing actions and \( \text{SCov}_{n,m} \) reduces to the space of superharmonic polynomials \( \text{SH}_n \) (cf. [3]) when \( m = 1 \).

**Remark 2.3.** It is clear that any polynomial invariant under (2.2) is also invariant under (1.1), i.e. \( \text{Inv}_{n,1} \subset \text{QInv}_{n,m} \). By taking the orthogonal, this implies that \( \text{SCov}_{n,m} \subset \text{Cov}_{n,m} \). These observations somewhat justify the terminology.

Our main result is the following theorem which is a generalization of equality (1.4).
Theorem 2.4. The dimension of the space $\text{Scov}_{n,m}$ is given by
\begin{equation}
\dim \text{Scov}_{n,m} = m^n C_n = m^n \frac{1}{n+1} \binom{2n}{n}.
\end{equation}

Remark 2.5. In the case of the hyperoctahedral group $B_n = G_{n,2}$, C.-O. Chow defined a class $BQSym(x_0, X)$ of quasi-symmetric functions of type $B$ in the alphabet $(x_0, X)$. His approach is quite different from ours. In particular, one has the equality:
\begin{equation}
BQSY_m(x_0, X) = QSym(X) + QSym(x_0, X).
\end{equation}

In the study of the coinvariant polynomials, it is not difficult to prove that the quotient $\mathbb{C}[x_0, X]/\langle BQSym^+ \rangle$ is isomorphic to the quotient $\mathbb{C}[X]/\langle QSym^+ \rangle$ studied in [3]. To see this, we observe that if $\mathcal{G}$ is the Gröbner basis of $\langle QSym^+ \rangle$ constructed in [3] (see also the next section), then the set $\{x_0, \mathcal{G}\}$ is a Gröbner basis (any syzygy is reducible thanks to Buchberger's first criterion, cf. [8]).

The next section is devoted to give a proof of Theorem 2.4 by constructing an explicit basis for the quotient $\mathbb{C}[X]/\langle QInv^+_{n,m} \rangle$.

3. Proof of the main theorem

Our task is here to construct an explicit monomial basis for the quotient space $\mathbb{C}[X]/\langle QInv^+_{n,m} \rangle$. Let us first recall (cf. [3]) the following bijection which associates to any vector $\nu \in \mathbb{N}^n$ a path $\pi(\nu)$ in the $\mathbb{N} \times \mathbb{N}$ plane with steps going north or east as follows. If $\nu = (\nu_1, \ldots, \nu_n)$, the path $\pi(\nu)$ is
\begin{align*}
(0, 0) &\rightarrow (\nu_1, 0) \rightarrow (\nu_1, 1) \rightarrow (\nu_1 + \nu_2, 1) \rightarrow (\nu_1 + \nu_2, 2) \rightarrow \cdots \\
&\quad \rightarrow (\nu_1 + \cdots + \nu_n, n - 1) \rightarrow (\nu_1 + \cdots + \nu_n, n).
\end{align*}

For example the path associated to $\nu = (2, 1, 0, 3, 0, 1)$ is
\begin{equation*}
\pi(\nu) = \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & 8 \\
\hline
9 & 10 & 11 & 12 \\
\hline
\end{array}
\end{equation*}

We distinguish two kinds of paths, thus two kinds of vectors, with respect to their “behavior” regarding the diagonal $y = x$. If the path remains above the diagonal, we call it a Dyck path, and say that the corresponding vector is Dyck. If not, we say that the path (or equivalently the associated vector) is transdiagonal. For example $\eta = (0, 0, 1, 2, 0, 1)$ is Dyck and $\varepsilon = (0, 3, 1, 1, 0, 2)$ is transdiagonal.
We then have the following result which generalizes Theorem 4.1 of [3] and which clearly implies the Theorem 2.4.

**Theorem 3.1.** The set of monomials

\[
\mathcal{B}_{n,m} = \{ (X_n)^{\eta + \alpha} / \pi(\eta) \text{ is a Dyck path, } 0 \leq \alpha_i < m \}
\]

is a basis for the quotient \( \mathbb{C}[X_n]/\langle \text{Inv}^+_n \rangle \).

To prove this result, the goal is here to construct a Gröbner basis for the ideal \( \mathcal{J}_{n,m} = \langle \text{Inv}^+_n \rangle \). We shall use results of [2, 3].

Recall that the lexicographic order on monomials is

\[(3.1) \quad X^\nu >_{\text{lex}} X^\mu \quad \text{iff} \quad \nu >_{\text{lex}} \mu,
\]

if and only if the first non-zero part of the vector \( \nu - \mu \) is positive.

For any subset \( S \) of \( \mathbb{Q}[X] \) and for any positive integer \( m \), let us introduce \( S^m = \{ P(X^m) , P \in S \} \). If we denote by \( G(I) \) the unique reduced monic Gröbner basis (cf. [3]) of an ideal \( I \), then the simple but crucial fact in our context is the following.

**Proposition 3.2.** With the previous notations,

\[(3.2) \quad G(\langle S^m \rangle) = G(\langle S \rangle)^m.
\]

**Proof.** This is a direct consequence of Buchberger’s criterion. Indeed, if for every pair \( g, g' \) in \( G(\langle S \rangle) \), the syzygy \( S(g, g') \) reduces to zero, then the syzygy \( S(g(X^m), g'(X^m)) \) also reduces to zero in \( G(\langle S^m \rangle) \) by exactly the same computation. \( \square \)

Let us recall that in [2] is constructed a family \( \mathcal{G} \) of polynomials \( G_\varepsilon \) indexed by transdiagonal vectors \( \varepsilon \). This family is constructed by using recursive relations of the fundamental quasi-symmetric functions and one of its property (cf. [2]) says that the leading monomial of \( G_\varepsilon \) is: \( \text{LM}(G_\varepsilon) = X^\varepsilon \). Since \( \mathcal{G} \) is a Gröbner basis of \( \mathcal{J}_{n,1} \), the following result is a consequence of Propositions 2.2 and 3.2.

**Proposition 3.3.** The set \( \mathcal{G}^m \) is a Gröbner basis of the ideal \( \mathcal{J}_{n,m} \).

To conclude the proof of Theorem 3.1, it is sufficient to observe that the monomials not divisible by a leading monomial of an element of \( \mathcal{G}^m \), i.e. by \( X^{m\varepsilon} \) for \( \varepsilon \) transdiagonal, are precisely the monomials appearing in the set \( \mathcal{B}_{n,m} \).
As a corollary of Theorem 3.1, one gets an explicit formula for the Hilbert series of $SCov_{n,m}$. For $k \in \mathbb{N}$, let $SCov_{n,m}^{(k)}$ denote the projection

$$SCov_{n,m}^{(k)} = SCov_{n,m} \cap \mathbb{Q}^{(k)}[X]$$

where $\mathbb{Q}^{(k)}[X]$ is the vector space of homogeneous polynomials of degree $k$ together with zero.

Let us denote by $F_{n,m}(t)$ the Hilbert series of $SCov_{n,m}$, i.e.

$$F_{n,m}(t) = \sum_{k \geq 0} \dim SCov_{n,m}^{(k)} t^k.$$

Let us recall that in [3] is given an explicit formula for $F_{n,1}$:

$$F_{n,1}(t) = F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k$$

using the number of Dyck paths with a given number of factors (cf. [13]).

The Theorem 3.1 then implies the

**Corollary 3.4.** With the notations of (3.3), the Hilbert series of $SCov_{n,m}$ is given by

$$F_{n,m}(t) = \frac{1 - t^m}{1 - t} F_n(t^m)$$

from which one deduces the close formula

$$\sum_n F_{n,m}(t) x^n = \frac{(1-t) - \sqrt{(1-t)(1-t - 4t^m x (1-t^m))} - 2x(1-t^m)}{(1-t)(2t^m - 1) - x(1-t^m)}.$$

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