INFINITELY MANY RADIAL SOLUTIONS FOR A \( p \)-LAPLACIAN PROBLEM WITH INDEFINITE WEIGHT

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Abstract. We prove the existence of infinitely many sign changing radial solutions for a \( p \)-Laplacian Dirichlet problem in a ball. Our problem involves a weight function that is positive at the center of the unit ball and negative in its boundary. Standard initial value problems-phase plane analysis arguments do not apply here because solutions to the corresponding initial value problem may blow up near the boundary due to the fact that our weight function is negative at the boundary. We overcome this difficulty by connecting the solutions to a singular initial value problem with those of a regular initial value problem that vanishes at the boundary.

1. Introduction. We study the quasilinear Dirichlet problem

\[
\begin{cases}
\Delta_p u + W(x)g(u) = 0 & \text{in } B_1(0) \subset \mathbb{R}^N, \\
u = 0 & \text{on } \partial B_1(0),
\end{cases}
\]  

(1)

where \( N \geq 2, p > 1, \Delta_p u = \text{div} (|\nabla u|^{p-2}\nabla u) \) denotes the \( p \)-Laplacian operator, and \( B_1(0) \) denotes the unit ball in \( \mathbb{R}^N \) centered at the origin.

We assume that \( g \) is a non-decreasing locally Lipschitzian continuous function and there exists \( C > 0 \) such that

\[
|g(s)| \leq C|s|^{p-1} \quad \text{for all } s \in [-1,1].
\]  

(2)

For the sake of simplicity in the calculations we assume that \( sg(s) > 0 \) for \( s \neq 0 \). We also assume that there exist \( q_1, q_2 \in (p-1, \infty) \) and \( A_1, A_2 \in (0, \infty) \) such that

\[
\lim_{s \to -\infty} \frac{g(s)}{|s|^{q_1-1}s} := A_1, \quad \text{and} \quad \lim_{s \to -\infty} \frac{g(s)}{|s|^{q_2-1}s} := A_2.
\]  

(3)
If $p \in (1, N)$ we assume that either

\[(i) \quad q_1 < \frac{N(p-1)}{N-p} \quad \text{or} \quad (ii) \quad p-1 < q_1, q_2 < p^* - 1,\]  

(4)

where $p^* = Np/(N - p)$. Note that for $p \geq N$ the assumption $q_1, q_2 \in (p - 1, \infty)$ implies

\[N + \frac{q_i(p - N)}{p - 1} \geq p, \quad \text{for } i = 1, 2.\]  

(5)

Also,

if $p < N$ and $q_1 < N(p - 1)/(N - p)$ then

\[N + \frac{q_1(p - N)}{p - 1} > 0.\]  

(6)

Finally, we assume that the weight function $W \in C^1[0, 1]$ and there exists $X \in (0, 1)$ such that

\[W(X) = 0, \quad W'(X) < 0, \quad W > 0 \quad \text{in } [0, X), \quad \text{and} \quad W < 0 \quad \text{in } (X, 1].\]  

(7)

For the sake of simplicity in the presentation we assume $W$ is decreasing in $[0, X)$ (see Remark 2).

Over the last fifty years the study of radial solutions to elliptic boundary value problems has been very active going back to papers such as [2] and [4]. Our approach here is inspired by the methods in [4], where Pohozaev energy and phase plane arguments applied to the solutions to a related singular ordinary differential equations are used to prove the existence of solutions to the boundary value problem by a simple application of the intermediate value theorem (see also [8]). The main difficulty of the problem we study here is that, because the weight function $W$ changes sign, some of the solutions to a related initial value problem blow up preventing the use of continuity properties for such problems. We overcome such a difficulty by following the arguments in [4] in a region where the solutions to the initial value problem do not blow up and connecting them to solutions that satisfy the boundary condition. For examples of applications of problem with indefinite weight the reader is referred to [9]. For recent results on quasilinear problems with weight see [1, 5, 11, 14]. For related results on the existence of infinitely many radial solutions to quasilinear problems see [6, 3, 10].

Our main result is the following theorem.

**Theorem 1.1.** If (3), (4) and (7) hold, then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, the problem (1) has a solution with $k$ nodal sets in the unit ball with $u(0) > 0$. In particular, the problem (1) has infinitely many radial solutions satisfying $u(0) > 0$.

**Remark 1.** Interchanging $q_1$ and $q_2$ en (4) we have $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ the problem (1) has a solution with $k$ nodal sets and $u(0) < 0$. In particular, the problem (1) has infinitely many radial solutions satisfying $u(0) < 0$.

This article is organized as follows. In Section 2 we show that all solutions to (10) below are defined in $[0, X]$ and that for each $a \in \mathbb{R}$ there exists a unique $\zeta$ such that the solution to (17) below satisfies the boundary condition $u(1) = 0$, see Theorem 2.7. In Section 3, we prove that our hypotheses imply if $u$ is a solution to (10) with large $d$ then $u^2(r) + (u'(r))^2$ remains large in an interval $[0, T_1] \subset [0, X]$ with $T_1 > 0$ independent of $d$. In Section 4, we present the phase plane analysis of the solutions to (10) in $[0, X]$, and in Section 5 we prove our main result.
2. The initial value problem. The radial solutions to (1) are the solutions to
\[ \begin{cases} \left( |u'|^{p-2} u' \right)'(r) + \frac{N-1}{r} |u'(r)|^{p-2} u'(r) + W(r)g(u(r)) = 0, & 0 < r < 1, \\ u'(0) = 0, & u(1) = 0, \end{cases} \] \tag{8}

That is, \( v : B_1(0) \to \mathbb{R} \) is a radial solution to (1) if and only if the function \( u : [0, 1] \to \mathbb{R} \) defined by \( u(\sqrt{x_1^2 + \cdots + x_N^2}) := v(x_1, \ldots, x_N) \) satisfies (9). Due to the singularity given by the zeros of \( u' \), the solutions to (8) need not be of class \( C^2 \).

In fact, regularity theory for quasilinear problems indicates that the solutions to (8) may only be expected to be in the Holder space \( C^{1,\mu} \) for some \( \mu \in (0, 1) \), see [7, 12].

It fits our purposes to regard (8) as
\[ \begin{cases} \left( r^{N-1} |u'(r)|^{p-2} u'(r) \right)' + r^{N-1} W(r)g(u(r)) = 0, & 0 < r < 1, \\ u'(0) = 0, & u(1) = 0. \end{cases} \] \tag{9}

Our technique is based on the analysis of the solutions to the initial value problem
\[ \begin{cases} \left( r^{N-1} |u'(r)|^{p-2} u'(r) \right)' + r^{N-1} W(r)g(u(r)) = 0, & 0 < r < 1, \\ u(0) = d, & u'(0) = 0. \end{cases} \] \tag{10}

Throughout this paper we write \( u(r, d) := u(r) \) if the dependence of \( u \) on \( d \) is clear from the context. Letting \( \Gamma(x) = x|x|^{p-2} \) one sees that, for each \( d \in \mathbb{R} \), a continuous function \( u \) satisfies the integral equation
\[ u(r) = d - \int_0^r \Gamma^{-1} \left( s^{1-N} \int_0^{s} t^{N-1} W(t)g(u(t)) dt \right) ds \] \tag{11}
if and only if it is a solution to (10). More generally, for any \( r_0 \in [0, 1) \), \( a \in \mathbb{R} \), \( b \in \mathbb{R} \), a continuous function \( u \) satisfies
\[ u(r) = a + \int_{r_0}^{r} \Gamma^{-1} \left( s^{1-N} \left[ r_0^{N-1} \Gamma(b) - \int_{r_0}^{s} t^{N-1} W(t)g(u(t)) dt \right] \right) ds \] \tag{12}
if and only if it satisfies
\[ \begin{cases} \left( r^{N-1} |u'|^{p-2} u' \right)' + r^{N-1} W(r)g(u(r)) = 0, & r_0 \leq r < 1, \\ u(r_0) = a, & u'(r_0) = b. \end{cases} \] \tag{13}

Given \( d_0 \in \mathbb{R} - \{0\} \), since \( g \) is a locally Lipschitzian function, there exists \( \tau > 0 \) such that for each \( d \in [d_0 - \tau, d_0 + \tau] \), equation (11) has a unique solution \( u_d \) in the space of continuous functions defined on \([0, \tau]\). This and the continuity of the right hand side in (11) on \((d, u)\), imply that \( u_d \) continuously depends on \( d \). If \( \tau = 1 \) such a solution is a solution to (10). If \( \tau \in (0, 1) \), we obtain a solution on \([0, \tau_1]\) for some \( \tau_1 > \tau \) by applying the same argument to (12) with \( a = u_d(\tau) \) and \( b = u_d'(\tau) \). The function \( u_d \) may be extended to a maximal interval which is either \([0, 1]\) or \([0, \hat{\tau}(d)]\) with \( \lim_{t \to \hat{\tau}(d)^-} [u^2(t) + (u'(t))^2] = +\infty \). We note that, due to hypothesis (2), no solution to (13) satisfies \( \lim_{t \to \hat{\tau}(d)^-} [u^2(t) + (u'(t))^2] = 0 \) if \( (a, b) \neq (0, 0) \). For a comprehensive study of existence, uniqueness and continuous dependence, we refer the reader to [13]. See also [6] for some details in the case \( W = 1 \).

In our next lemma we prove that \( \hat{\tau}(d) > X \). Since \( d_0 \in \mathbb{R} - \{0\} \) is arbitrary, this show the existence of a unique solution to (10) on \([0, X]\) that depends continuously on \( d \).
From now on we define
\[
G(t) = \int_0^t g(s) ds \quad \text{and} \quad p' = p/(p-1).
\]

**Lemma 2.1.** For each \(d \in \mathbb{R} \) the solution to (10) is defined in \([0, X]\).

**Proof.** Let \( u \) be a solution to (10) defined in \([0, t] \) with \( t \leq X \), and
\[
\mathcal{E}(r, d) \equiv \mathcal{E}(r) := \frac{p-1}{p} |u'(r)|^p + W(r)G(u(r)).
\]

Observe \(|u'|^p = |u'|^{p-2}u'|^{p/(p-1)}\) and the function \(|u'|^{p-2}u'\) is differentiable in \((0, t)\) (see (10)). Moreover, function \( h(s) = |s|^{p/(p-1)} \) is differentiable on \(\mathbb{R} \) and \(h'(s) = \frac{p}{p-1}|s|^{p/(p-1)-2} = \frac{p}{p-1}|s|^{(2-p)/(p-1)}s \) for all \( s \neq 0 \) and \( h'(0) = 0 \). Hence, \( \mathcal{E} \) is differentiable on \((0, t)\) and
\[
\mathcal{E}'(r) = \left( \frac{p-1}{p} |u'(r)|^{p-2}u'(r)\right)' + W'(r)G(u(r)) + W(r)g(u(r))u'(r)
\]
\[
= \left( |u'(r)|^{p-2}u'(r)\right)'|^{(2-p)/(p-1)} |u'(r)|^{p-2}u'(r) \left( |u'(r)|^{p-2}u'(r)\right)'
\]
\[
+ W'(r)G(u(r)) + W(r)g(u(r))u'(r)
\]
\[
= |u'(r)|^{2-p}u'(r)|^{p-2}u'(r) \left( |u'(r)|^{p-2}u'(r)\right)'
\]
\[
+ W'(r)G(u(r)) + W(r)g(u(r))u'(r)
\]
\[
= u'(r) \left( - \frac{N-1}{r} |u'(r)|^{p-2}u'(r) - W(r)g(u(r)) \right)
\]
\[
+ W'(r)G(u(r)) + W(r)g(u(r))u'(r) \quad \text{(from (8))}
\]
\[
= - \frac{N-1}{r} |u'(r)|^p + W'(r)G(u(r))
\]
\[
= \frac{p(N-1)}{(p-1)r} \mathcal{E}(r) + G(u(r)) \left[ \frac{p(N-1)}{(p-1)r} W(r) + W'(r) \right] \quad \text{(from (15))}
\]
\[
\leq W'(r)G(u(r)).
\]

Hence \( \mathcal{E} \) decreases on \([0, t)\) which implies \(|u'(r)|^p \leq pW(0)G(d)\) for all \( r \in [0, t) \). Thus \( \lim_{r \to t-} u(r) := u(t) \in \mathbb{R} \) and hence, \( \lim_{r \to t-} u'(r) := u'(t) \in \mathbb{R} \). Therefore \( u \) may be extended to an interval \([0, t + \varepsilon_0)\) for some \( \varepsilon_0 > 0 \). Since this is valid for any \( t \in [0, X] \) we conclude that the solution to (10) may be extended to \([0, X + \varepsilon_0]\) with \( \varepsilon_0 \) depending on \( d \). This proves the lemma. \( \square \)

**Remark 2.** The assumption \( W'(r) \leq 0 \) in \([0, X]\) may be eliminated by observing that \( \frac{p(N-1)}{(p-1)r} W(r) + W'(r) < 0 \) in an interval of the form \([X - \delta, X + \delta]\) and that \( W'(r)G(u(r)) \leq CE(r) \) for \( r \in [0, X - \delta] \) for some constant \( C \) depending only on \( W \).

For \( a, \zeta \in \mathbb{R} \) let us consider
\[
\begin{cases}
  \left( r^{N-1}|u'|^{p-2}u' \right)' + r^{N-1}W(r)g(u(r)) = 0, & X < r < 1, \\
  u(X) = a, & u'(X) = \zeta.
\end{cases}
\]

As mentioned above, due to our assumptions on \( g \), the initial value problem (17) has a unique solution \( u = u(a, \zeta) \) on a maximal interval \( I \), which is denoted by \([X, R_{a, \zeta}] := I \).
Lemma 2.2. For \( a > 0 \) let
\[
\eta_a = \max \left\{ \frac{2^{p/(p-1)} a}{(1-X)X^{(N-1)/(p-1)}}, \left( \frac{2\|W\|\infty(1-X)g(a)}{X^{N-1}} \right)^{1/(p-1)} \right\}. \tag{18}
\]
If \( u \) is the solution to (17) with \( \zeta = -\eta_a \) then there exists \( \hat{r} \in I \cap [0,1) \) such that \( u(\hat{r}) = 0 \) and \( u \) decreases in \([X, \hat{r}]\).

Proof. Let \( r > 0 \) be such that \( 0 \leq u(s) \leq a \) for every \( s \in [X, r] \). The existence of such an \( r \) is guaranteed by the initial conditions in (17) and the fact that \( \zeta = -\eta_a < 0 \). Thus, integrating the differential equation in (17) on \([X, r]\),
\[
r^{-N-1}|u'(r)|^{p-2}u'(r) - X^{-N-1}|u'(X)|^{p-2}u'(X) = - \int_X^r s^{N-1}W(s)g(u(s))ds.
\]
Hence, the definition of \( \eta_a \) implies
\[
|u'(r)|^{p-2}u'(r) = - \left( \frac{X}{r} \right)^{N-1} \eta_a^{p-1} - \int_X^r \left( \frac{s}{r} \right)^{N-1} W(s)g(u(s))ds \\
\leq -X^{-N-1}\eta_a^{p-1} + (1-X)\|W\|\infty g(a) \tag{19}
\]
Thus \( u \) decreases in \([X, r]\). Therefore \( u \) is bounded in \([X, r]\), which implies that \([X, r] \subset I \). Let
\[
\hat{r} = \sup \{ r \in I : 0 \leq u(s) \leq a \text{ for all } s \in [X, r] \} := \sup B.
\]
Due to the continuity of \( u \), if \( r \in B \) then \( u(r) \geq 0 \). Applying again the continuity of \( u \) we have \( u(\hat{r}) \geq 0 \). Since \([X, r] \subset I \) for all \( r \in B \), we have \([X, \hat{r}] \subset I \). Assuming that \( u(\hat{r}) > 0 \), the continuity of \( u \) implies that there exists \( \delta > 0 \) such that \( u(s) > 0 \) for all \( s \in [X, \hat{r} + \delta] \) contradicting the definition of \( \hat{r} \). Hence \( u(\hat{r}) = 0 \).

From (19),
\[
-u'(r) \geq \left( \frac{X^{-N-1}\eta_a^{p-1}}{2} \right)^{1/(p-1)} \text{ for all } r \in [X, \hat{r}].
\]
Integrating on \([X, \hat{r}]\),
\[
0 = u(\hat{r}) = u(X) + \int_X^{\hat{r}} u'(r)dr \leq a - \frac{(\hat{r} - X)X^{(N-1)/(p-1)}\eta_a}{2^{1/(p-1)}}. \tag{20}
\]
This and the definition of \( \eta_a \) yield
\[
\hat{r} \leq X + \frac{2^{1/(p-1)}a}{X^{(N-1)/(p-1)}\eta_a} \leq X + \frac{1 - X}{2} < 1. \tag{21}
\]
Thus, from (21) and (19), \( \hat{r} \in (X, 1) \), \( u(\hat{r}) = 0 \) and \( u \) decreases in \((X, \hat{r})\) proving the lemma.

Lemma 2.3 (Comparison principle). Let \( a, y_1, y_2 \in \mathbb{R} \). Let \( u_1 \) satisfy
\[
\left\{ \begin{array}{l}
(r^{N-1}|u'_1|^{p-2}u'_1)' + r^{N-1}W(r)g(u_1(r)) = 0, \quad X < r < R_1 := R_{a, y_1}, \\
u_1(X) = a, \quad u'_1(X) = y_1,
\end{array} \right.
\]
\[
(22)
\]
and \( u_2 \) satisfy
\[
\begin{cases}
(r^{N-1}|u_2|^{p-2}u_2)' + r^{N-1}W(r)g(u_2(r)) = 0, & X < r < R_2 := R_{a,y_2}, \\
u_2(X) = a, & u_2'(X) = y_2.
\end{cases}
\] (23)

If \( y_1 < y_2 \), then \( u_1(t) < u_2(t) \) for every \( t \in [X, R_1] \cap [X, R_2] \).

**Proof.** Assuming to the contrary there exists \( t \in [X, R_1] \cap [X, R_2] \) such that
\[
u_1(t) = u_2(t) \quad \text{and} \quad u_1(r) < u_2(r) \quad \text{for all} \quad r \in (X, t).
\] (24)

Then, \( u_2'(t) \leq u_1'(t) \). Since \( \Gamma \) is an increasing function,
\[
|u_2'(t)|^{p-2}u_2'(t) \leq |u_1'(t)|^{p-2}u_1'(t).
\]

This, (22) and (23) yield
\[
X^{N-1}|y_2|^{p-2}y_2 - \int_X^t s^{N-1}W(s)g(u_2(s))ds \\
\leq X^{N-1}|y_1|^{p-2}y_1 - \int_X^t s^{N-1}W(s)g(u_1(s))ds.
\] (25)

On the other hand, since \( y_1 < y_2 \) and \( \Gamma \) is strictly increasing,
\[
X^{N-1}|y_1|^{p-2}y_1 < X^{N-1}|y_2|^{p-2}y_2.
\] (26)

Moreover, since \(-W \geq 0\) on \([X, 1]\) and \( g \) is non-decreasing,
\[
-\int_X^t s^{N-1}W(s)g(u_1(s))ds \leq -\int_X^t s^{N-1}W(s)g(u_2(s))ds.
\] (27)

Since (25) together with (26) contradict (27) the lemma is proven. \( \square \)

**Lemma 2.4.** Let \( r_* \in [X, 1] \), \( b > 0 \) and \( y \geq 0 \). If \( u \) satisfies
\[
\begin{cases}
(r^{N-1}|u'|^{p-2}u')' + r^{N-1}W(r)g(u(r)) = 0, & r_* \leq r < R := R_{b,y}, \\
u(r_*) = b, & u'(r_*) = y.
\end{cases}
\] (28)

then \( u'(r) > 0 \) for all \( r \in [r_*, R] \).

**Proof.** Let \( t \in [r_*, R] \). From (28),
\[
t^{N-1}|u'(t)|^{p-2}u'(t) = r_*^{N-1}|u'(r_*)|^{p-2}u'(r_*) - \int_{r_*}^t s^{N-1}W(s)g(u(s))ds \\
= r_*^{N-1}y^{p-1} + \int_{r_*}^t s^{N-1}(-W(s))g(u(s))ds.
\] (29)

Since \( b > 0 \), if \( t \) is close to \( r_* \), \( g(u(r_*)) \approx g(b) > 0 \) for all \( r \in [r_*, t] \). Hence, (29) implies \( u'(t) > 0 \). Now, assume \( t \in [r_*, R] \) satisfies \( u'(t) = 0 \) and \( u'(r) > 0 \) for every \( r \in [r_*, t] \). Then \( u(r) \geq u(r_*) = b \). From (29),
\[
0 = t^{N-1}|u'(t)|^{p-2}u'(t) = r_*^{N-1}y^{p-1} + \int_{r_*}^t s^{N-1}(-W(s))g(u(s))ds > 0.
\]

This contradiction shows \( u'(t) > 0 \) for every \( t \in [r_*, R] \), proving the lemma. \( \square \)
Remark 3. Let \( y \) be the unique solution to the initial value problem (30) with \( u'(X) = y \) and \( u(X) = a \). Then \( y \) is defined on a maximal interval \([X, R_y]\), \((R_y := R_{a,y})\).

Let
\[ A := \{ y \geq -\eta_a : u_y \text{ has a zero} \ r_y \text{ in} \ (X,1) \} \quad \text{and} \quad \hat{\zeta}(a) = \sup A. \tag{31} \]

Applying Lemma 2.4 with \( r_x = X \) and \( b = a \), we observe \( A \subseteq (-\infty, 0) \). Thus, \( \hat{\zeta}(a) \leq 0 \) for all \( a > 0 \).

Remark 3. From the comparison principle (Lemma 2.3), if \( y_1, y_2 \in A \) with \( y_1 < y_2 \) and \( r_1 \equiv r_{y_1}, r_2 \equiv r_{y_2} \) in \((X,1)\) are the corresponding first zeros of \( u_{y_1} \) and \( u_{y_2} \), then \( r_1 < r_2 \).

Theorem 2.5. Let \( a > 0 \). If \( u \) is the solution to (17) with \( u'(X) = \hat{\zeta}(a) \) then \( u(1) = 0 \) and \( u \) is positive in \([X,1]\).

Proof. Let \( \{ y_j \} \subset A \) be an increasing sequence converging to \( \hat{\zeta}(a) \). Let \( u \) be the solution to the initial value problem (30) with \( y = \hat{\zeta}(a) \) and let \( u_j \) be the solution to the initial value problem (30) with \( y = y_j \). Let \( r_j \in (X,1) \) be such that \( u_j(s) > 0 \) for all \( s \in (X,r_j) \) and \( u_j(r_j) = 0 \). By Lemma 2.3, \( \{ r_j \} \) is an increasing sequence bounded above by 1. This and the continuous dependence of solutions on initial conditions imply \( \zeta(a) \leq 1 \). Let \( \tau = \lim_{j \to \infty} r_j \in [X,1] \). By the continuity of \( u \) and Lemma 2.3,
\[ u(\tau) = \lim_{j \to \infty} u(r_j) \geq \lim \inf_{j \to \infty} u_j(r_j) = 0. \tag{32} \]

Let \( \varepsilon > 0 \). By the continuous dependence of solutions to initial value problems on initial conditions, there exists \( j_0 \) such that if \( j \geq j_0 \) then \( r_j \in (\tau - \varepsilon, \tau) \) and \( |u'(t) - u'_j(t)| < \varepsilon \) for all \( j \geq j_0 \) and \( t \in [X, \tau - \varepsilon] \). From
\[ t^{N-1}|u'_n(t)|^{p-2}u'_n(t) = X^{N-1}|y_n|^{p-2}y_n - \int_X^t s^{N-1}W(s)g(u_n(s))ds, \]
and the fact that \( \{ y_j \} \) is a bounded sequence, there exists \( M > 0 \) such that
\[ |u'_j(t)| \leq M, |u'(t)| \leq M \quad \text{for all} \quad j \geq j_0, \ t \in [X, \tau - \varepsilon]. \tag{33} \]

Hence, for \( j \geq j_0 \), \( u_j(\tau - \varepsilon) \leq M(r_j - \tau + \varepsilon) \leq M\varepsilon \). Thus
\[ u(\tau) = a + \int_X^\tau u'(s)ds = a + \int_{\tau-\varepsilon}^\tau u'_j(s)ds + \int_X^{\tau-\varepsilon} (u'(s) - u'_j(s))ds \]
\[ + \int_{\tau-\varepsilon}^\tau u'(s)ds \]
\[ \leq u_j(\tau - \varepsilon) + \varepsilon(\tau - X) + M\varepsilon \]
\[ \leq \varepsilon(2M + \tau - X). \]

Since \( \varepsilon > 0 \) is arbitrary we have \( u(\tau) = 0 \). Since \( u(t) > u_j(t) \) for all \( t \in [X, r_j] \), \( u \) is positive in \([X, \tau]\).

By the uniqueness of solutions to initial value problems and the assumption \( g(0) = 0 \), we have \( u'(\tau) < 0 \). Assuming that \( \tau < 1 \), there exists \( \varepsilon \in (0, 1 - \tau) \) such that \( u(x) < 0 \) for \( x - \tau \in (0, \varepsilon) \). Let \( \{ \varepsilon_j \} \) be a decreasing sequence converging to
\( \hat{\zeta}(a) \) and let \( v_j \) be the solution to the initial value problem (30) with \( y = z_j \). By continuous dependence of solutions to initial value problems on initial conditions, there exists \( j \) such that \( v_j(\tau + \varepsilon/2) < 0 \). Since \( v_j \) is positive in \([X, \tau]\), there exists \( r_1 \in (\tau, \tau + \varepsilon/2) \subset [X, 1) \) such that \( v_j \) is positive in \([X, r_1]\) and \( v_j(r_1) = 0 \). Hence \( z_j \in A \) and \( z_j > \hat{\zeta}(a) \) contradicting the definition of \( \hat{\zeta}(a) \). This contradiction proves that \( \tau = 1 \) and, therefore, the theorem.

**Theorem 2.6.** The function \( \hat{\zeta} : [0, \infty) \to (-\infty, 0] \) defined on \((0, +\infty)\) by Theorem 2.5 and by \( \hat{\zeta}(0) = 0 \) is a decreasing continuous function.

**Proof.** Let \( a_1 < a_2 \). Let \( u \) be the solution to the second order differential equation in (30) that satisfies the initial condition \( u(X) = a_1, u'(X) = \hat{\zeta}(a_1) \) and similarly \( v \) for \((a_2, \hat{\zeta}(a_2))\). Because \( a_1 < a_2 \), \( v(r) > u(r) \) for \( r \) near \( X \). This and \( u(1) = v(1) \) imply that there exists \( \sigma \in (X, 1] \) such that \( v(s) > u(s) \) for all \( s \in (X, \sigma) \) and \( v(\sigma) = u(\sigma) \). By uniqueness of solutions to initial value problems, \( v'(\sigma) < u'(\sigma) \).

Assuming that \( \hat{\zeta}(a_1) \leq \hat{\zeta}(a_2) \), we have

\[
0 < \sigma^{N-1}(|u'(\sigma)|^{p-2}u'(\sigma) - |v'(\sigma)|^{p-2}v'(\sigma)) = X^{N-1}(|\hat{\zeta}(a_1)|^{p-2}\hat{\zeta}(a_1) - |\hat{\zeta}(a_2)|^{p-2}\hat{\zeta}(a_2)) - \int_X^\sigma s^{N-1}W(s)(g(u(s)) - g(v(s)))ds
\]

\[
< 0.
\]

This contradiction proves that \( \hat{\zeta} \) is a decreasing function.

Let \( \{a_n\} \) be a decreasing sequence of non-negative numbers converging to \( a \geq 0 \). Let \( \hat{\zeta}(a_n) := \xi_n \), and \( u_n \) be the solution to the second order differential equation in (30) that satisfies the initial condition \( u_n(X) = a_n, u_n'(X) = \hat{\zeta}(a_n) \). Let \( u \) be the solution to the second order differential equation in (30) that satisfies the initial condition \( u(X) = a, u'(X) = \hat{\zeta}(a) \). Since \( \{\xi_n\} \) is an increasing sequence bounded by \( \hat{\zeta}(a) \), \( c = \lim_{n \to +\infty} \hat{\zeta}(a_n) \leq \hat{\zeta}(a) \). Let \( w \) denote the solutions to the second order differential equation in (30) with \( w(X) = a \) and \( w'(X) = c \). By continuous dependence on initial conditions we have

\[
0 = \lim_{n \to \infty} u_n(1) = w(1).
\]

Therefore \( c = \hat{\zeta}(a) \). Thus \( \lim_{n \to +\infty} \hat{\zeta}(a_n) = \hat{\zeta}(a) \). Similarly, if \( \{a_n\} \) is an increasing sequence converging to \( a \) then \( \lim_{n \to +\infty} \hat{\zeta}(a_n) = \hat{\zeta}(a) \). This proves that \( \hat{\zeta} \) is continuous on \([0, \infty)\).

Imitating the proofs in Theorem 2.5 and Theorem 2.6 one proves that \( \hat{\zeta} \) may be extended to \((-\infty, \infty)\). That is we have the following result.

**Theorem 2.7.** There exists a continuous function \( \hat{\zeta} : \mathbb{R} \to \mathbb{R} \) with \( \hat{\zeta}(0) = 0 \) such that if \( u \) is the solution to (17) with \( u'(X) = \hat{\zeta}(a) \) then \( u(1) = 0 \) and, \( u \) is positive in \([X, 1) \) if \( a > 0 \) and \( u \) is negative in \([X, 1) \) if \( a < 0 \).

3. Energy analysis. If (4) (ii) is satisfied then there exists \( \delta > 0 \) such that

\[
(\delta + 1)(q_i + 1) < p^*, \quad i = 1, 2.
\]

We choose \( \varepsilon > 0 \) such that

\[
\varepsilon < \min \left\{ \frac{A_1[p^* - (\delta + 1)(q_1 + 1)]}{p^* + (\delta + 1)(q_1 + 1)}, \frac{A_2[p^* - (\delta + 1)(q_2 + 1)]}{p^* + (\delta + 1)(q_2 + 1)} \right\}.
\]
Letting $C_1 := A_1 - \varepsilon, C_2 := A_1 + \varepsilon, C_3 := A_2 - \varepsilon, C_4 := A_2 + \varepsilon,$ by (3) there exists $M > 0$ such that
\begin{align*}
\forall s \geq 0, \quad C_1 s^{q_1 + 1} - M &\leq s g(s) \leq C_2 s^{q_1 + 1} + M, \\
\forall s \leq 0, \quad C_3 s^{q_2 + 1} - M &\leq s g(s) \leq C_4 s^{q_2 + 1} + M,
\end{align*}
(39)
(40)
(41)
(42)
where $G$ is a primitive of $g$ such that $G(0) = 0$. Hence there exist $D > 0, \tilde{C}_1, \tilde{C}_2$ such that
\begin{equation}
\tilde{C}_1 |s|^{q_i + 1} \leq s g(s) \leq \tilde{C}_2 |s|^{q_i + 1} \quad i = 1, 2, \text{ for } |s| \geq D. \tag{43}
\end{equation}

Note that the monotonicity of $g$ implies $G(s) \geq 0$ for all $s \in \mathbb{R}$ and $sg(s) \geq 0$ for all $s \neq 0$.

Due to the continuity of $W$ at zero, there exists $T \in (0, X)$ (see (7)) so that
\begin{equation}
\forall r \in [0, T], \quad W(r) \geq \frac{W(0)}{2} := \frac{m}{2}. \tag{44}
\end{equation}

Given $d > 0$, let $u$ be the solution to (10) defined on $[0, X]$ (see Lemma 2.1 above). It follows that
\begin{equation}
-r^{N-1} |u'(r)|^{p-2} u'(r) = \int_0^r s^{N-1} W(s) g(u(s)) \, ds. \tag{45}
\end{equation}

Due to $d > 0$ and the continuity of $u$, we have $u > 0$ near $r = 0$. Since $g$ is an increasing function, $g(0) = 0$, (44) and (45) then $u'(r) < 0$ for $r > 0$ small. Let
\begin{equation*}
 r_0 = r_0(d) := \sup \{ r > 0 : \forall s \in [0, r], \, u(s) \geq d/2 \}.
\end{equation*}

Note that, from (45), $u'(r) < 0$ for all $r \in (0, r_0)$.

**Lemma 3.1.** There exist positive constants $K_0$ and $K_1$ independent of $d$ such that
\begin{equation}
K_0 d^{ \frac{p-1-q_1}{p} } \leq r_0 \leq K_1 d^{ \frac{p-1-q_1}{p} } \quad \text{for } d \gg 1. \tag{46}
\end{equation}

**Proof.** For $d > 2D$, let us define $\tau = \tau(d) := \min \{ r_0(d), T \}$. By (43), (44) and the fact that $u'(r) < 0$ for $r \in (0, \tau]$,
\begin{align*}
 r^{N-1} |u'(r)|^{p-1} &= \int_0^r s^{N-1} W(s) g(u(s)) \, ds \geq \frac{m\tilde{C}_1}{2} \int_0^r s^{N-1} (u(s))^{q_1} \, ds \\
&\geq \frac{m\tilde{C}_1}{2} \left( \frac{d}{2} \right)^{q_1} r^{N} = K_1 d^{q_1} r^{N}.
\end{align*}

Hence, $-u'(r) \geq K_1^{1/(p-1)} d^{ \frac{q_1}{p} } r^{ \frac{p}{p-1} }$. Integrating on $[0, \tau]$, we have
\begin{align*}
d - d/2 &\geq d - u(\tau) = - \int_0^\tau u'(r) \, dr \geq \frac{p-1}{p} K_1^{1/(p-1)} d^{ \frac{q_1}{p} } r^{ \frac{p}{p-1} } \bigg|_0^\tau \\
&= \frac{p-1}{p} K_1^{1/(p-1)} d^{ \frac{q_1}{p} } \tau^{ \frac{p}{p-1} }.
\end{align*}

Thus,
\begin{equation*}
\tau \leq K_1 d^{ \frac{p-1-q_1}{p-1} } \quad \text{for } d > 2D,
\end{equation*}
where \( K_1 = \frac{p-1}{p} K_1^{1/(p-1)} \). Therefore \( \tau(d) \to 0 \) as \( d \to +\infty \). Then, for \( d \gg 1 \),
\[
\tau \leq K_1 d^{\frac{p-1-q_1}{p}} \leq T/2.
\]
Hence, \( \tau < T \) and thus \( \tau = r_0 \). Consequently, for \( d \gg 1 \),
\[
r_0 \leq K_1 d^{\frac{p-1-q_1}{p}}.
\]
On the other hand, for \( d \gg 1 \) and \( r \in [0, r_0] \), using again (43),
\[
r^{N-1} |u'(r)|^{p-1} = \int_0^r s^{N-1} W(s) g(u(s)) \, ds \leq \tilde{C}_2 \|W\|_\infty \int_0^r s^{N-1} (u(s))^{q_1} \, ds
\]
\[
\leq \tilde{C}_2 \|W\|_\infty d^{q_1} r^{N-1}.
\]
As above, integrating on \([0, r_0]\) we have \( d/2 \leq \tilde{K}_0 d^{1/(p-1)} \frac{d^{q_1} \tilde{C}_2^{q_1} r}{N} \) which proves the first inequality in (46). Hence the lemma has been proved.

**Lemma 3.2.** There exists \( C > 0 \), independent of \( d \), such that for \( r \in [0, r_0] \),
\[
\mathcal{E}(r, d) \geq C d^{q_1+1}, \quad \text{for } d \gg 1.
\]

**Proof.** Without loss of generality we can assume that \( r_0 < T \). For every \( r \in [0, r_0] \) we have
\[
G(u(r)) \geq C |u(r)|^{q_1+1} - M \quad (\text{see } (41))
\]
\[
\geq C d^{q_1+1} - M \geq \frac{C}{2} d^{q_1+1} \quad (d \gg 1).
\]
Since, \( \mathcal{E}(r, d) \geq W(r) G(u(r)) \geq mC d^{q_1+1}/4 \), the lemma follows.

Since \( W \) is of class \( C^1 \), there exists \( T_1 \leq T \) such that for \( r \in (0, T_1] \),
\[
\frac{p'(N-1)}{r} W(r) + W'(r) > 0, \quad (47)
\]
where \( p' = \frac{p}{p-1} \). Note that by Lemma 3.1 we may assume \( r_0(d) < T_1 \) for \( d \gg 1 \).

**Lemma 3.3.** If either \( p \geq N \) or \( p < N \) and (i) in (4) hold, then \( \lim_{d \to +\infty} \mathcal{E}(r, d) = +\infty \) uniformly for \( r \in [0, T_1] \).

**Proof.** From Lemma 3.2, it follows that \( \lim_{d \to +\infty} \mathcal{E}(r, d) = +\infty \), uniformly for \( r \in [0, r_0] \). Due to (16), (47) and \( G(t) \geq 0 \) for all \( t \in \mathbb{R} \), we have
\[
\mathcal{E}'(r) + \frac{p'(N-1)}{r} \mathcal{E}(r) = G(u) \left[ \frac{p'(N-1)}{r} W(r) + W'(r) \right] \geq 0, \quad (48)
\]
for every \( r \in (0, T_1] \). Therefore, \( \left( r^{p'(N-1)} \mathcal{E}(r) \right)' \geq 0 \). From Lemmas 3.1 and 3.2, since \( N + q_1(p-N)/(p-1) > 0 \) (see (5) and (6)), we get
\[
\mathcal{E}(r) \geq r_0^{p'(N-1)} \mathcal{E}(r_0) \geq C d^{N+q_1(p-N)/(p-1)} \to +\infty, \quad (49)
\]
as \( d \to +\infty \) uniformly for \( r \in [r_0, T_1] \), which proves the lemma.
For \( u(r, d) := u(r) \) the solution to (10) we define:

\[
H(r, d) := r \mathcal{E}(r, d) + \frac{N - p}{p} |u'(r, d)|^{p-2} u'(r, d) u(r, d),
\]

\[
P(r, d) := \int_0^r s^{N-1} \left[ \left( NW(s) + sW'(s) \right) G(u(s)) - \frac{N - p}{p} W(s) g(u(s)) u(s) \right] ds.
\]

(50)

The quantities in (50) are related by the Pohozaev-type identity (see [3, 4, 11]):

\[
r^{N-1} H(r, d) - t^{N-1} H(t, d)
= \int_t^r s^{N-1} \left[ \left( NW(s) + sW'(s) \right) G(u) - \frac{N - p}{p} W(s) g(u) u \right] ds.
\]

(51)

Taking \( t = 0 \) in equation (51), we have the following Pohozaev identity

\[
r^{N-1} H(r, d) = P(r, d),
\]

equivalently

\[
r^N \left[ \frac{p-1}{p} |u'(r)|^p + W(r) G(u(r)) \right] + \frac{N - p}{p} r^{N-1} |u'(r)|^{p-2} u'(r) u(r)
= \int_0^r s^{N-1} \left[ \left( NW(s) + sW'(s) \right) G(u) - \frac{N - p}{p} W(s) g(u) u \right] ds = P(r, d).
\]

(52)

We recall that \( r_0(d) \to 0 \) as \( d \to \infty \). Let \( \delta > 0 \) be as in (61). By further restricting \( T_1 \) we may assume that

\[
N + s \frac{W'(s)}{W(s)} > \frac{N}{1 + \delta}, \quad \text{for } s \in [0, T_1].
\]

(53)

In this case, \( W(s) > m/2 > 0 \) and hence (52) is equivalent to

\[
P(r, d) = \int_0^r s^{N-1} W(s) \left[ \left( N + s \frac{W'(s)}{W(s)} \right) G(u) - \frac{N - p}{p} g(u) u \right] ds.
\]

(54)

**Lemma 3.4.** If \( p < N \) and (ii) in (4) holds then \( P(r_0, d) \to \infty \) as \( d \to \infty \).

**Proof.** Due to (53), \( G(\cdot) \geq 0, (41), (43) \) and \( u(s) \geq d/2 \), we have

\[
\left( N + s \frac{W'(s)}{W(s)} \right) G(u) - \frac{N - p}{p} g(u) u
\geq \left( \frac{d}{2} \right)^{q_1 + 1} \left[ \frac{NC_1}{(1 + \delta)(q_1 + 1)} - \frac{N - p}{p} C_2 \right] - \frac{C N}{1 + \delta}.
\]

By (38), the expression inside brackets is positive. Hence, for \( d \gg 1 \),

\[
\left( N + s \frac{W'(s)}{W(s)} \right) G(u) - \frac{N - p}{p} g(u) u > 0, \quad \text{for every } s \in [0, r_0].
\]
Hence, by (46), for $d \gg 1$, 

\[ P(r_0, d) \geq \frac{m}{2N} \left( \frac{d}{2} \right)^{q_1+1} \left[ \frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p} C_2 \right] r_0^{N} - \frac{m}{2} \frac{C}{1+\delta} r_0^{N} \]

\[ \geq \frac{mC}{2} \left( \frac{d}{2} \right)^{q_1+1} \left[ \frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p} C_2 \right] \delta r_0^{N} \]

\[ - \frac{mC_1}{2} \left( \frac{d}{2} \right)^{q_1+1} \left[ \frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p} C_2 \right] \delta r_0^{N} \]

\[ = d^{N[1-(q_1+1)/p]} \left( C - C_0 d^{-(q_1+1)} \right) \geq C d^{N[1-(q_1+1)/p]}. \]

This proves the lemma. \( \square \)

**Lemma 3.5.** If $p < N$ and (ii) in (4) holds then $P(r, d) \to \infty$ as $d \to \infty$ uniformly for $r \in [r_0, T_1]$.

**Proof.** Note that

\[ P(r, d) = P(r_0, d) + \int_{r_0}^{r} s^{N-1} W(s) \left[ \left( N + s \frac{W'(s)}{W(s)} \right) G(u) - \frac{N-p}{p} g(u) u \right] ds \quad (55) \]

where

\[ I^+ = \int_{\{u(s) \geq 0\}} s^{N-1} W(s) \left[ \left( N + s \frac{W'(s)}{W(s)} \right) G(u) - \frac{N-p}{p} g(u) u \right] ds, \]

and

\[ I^- = \int_{\{u(s) \leq 0\}} s^{N-1} W(s) \left[ \left( N + s \frac{W'(s)}{W(s)} \right) G(u) - \frac{N-p}{p} g(u) u \right] ds. \]

Using (39), (41) and arguing as above, we get

\[ I^+ \geq \int_{\{u(s) \geq 0\}} s^{N-1} W(s) \left[ \frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p} C_2 \right] s^{q_1+1} ds \quad (56) \]

\[ - \int_{\{u(s) \geq 0\}} s^{N-1} W(s) \left[ \frac{NM}{(1+\delta)(q_1+1)} - \frac{N-p}{p} M_1 \right] ds \]

\[ \geq - \int_{\{u(s) \geq 0\}} s^{N-1} W(s) \left[ \frac{NM_3}{(1+\delta)(q_1+1)} - \frac{N-p}{p} M_1 \right] ds \]

\[ \geq - \frac{NM_3}{(1+\delta)(q_1+1)} - \frac{N-p}{p} M_1 \left\| \|W\| \right\| \int_{0}^{1} s^{N-1} ds = -C. \]

In a similar way, using (40) and (42), we have $I^- \geq -C$. This, (55) and (56) imply $P(r, d) \to \infty$ as $d \to \infty$ for every $r \in [r_0, T_1]$. \( \square \)

**4. Phase plane analysis.** Recall that, given $d > 0$, the problem (10) has a unique solution $u(r, d)$ defined for all $r \in [0, X]$.

Since $g(0) = 0, (u(r, d), u'(r, d)) \neq (0, 0)$ for all $r \in [0, X]$. Hence there exists a continuous function $\phi(r, d)$, for $r \in [0, X]$, such that $\phi(0, d) = 0$,

\[ u(r, d) = \rho(r, d) \cos \phi(r, d), \]

\[ u'(r, d) = -\rho(r, d) \sin \phi(r, d), \]

where $\rho(r, d) = \sqrt{(u(r, d))^2 + (u'(r, d))^2}$. Moreover, $\phi(\cdot, d)$ is differentiable at every $r \in [0, X]$ such that $u'(r) \neq 0$. 

Differentiating the first equation in (57) with respect to \( r \), for \( u'(r) \neq 0 \),
\[
u'(r) = \rho'(r, d) \cos(\phi(r, d)) - \rho(r, d) \sin(\phi(r, d)) \cdot \phi'(r, d).
\]
(58)
Let \( T > 0 \) and \( m \) be as in (44). We recall that our problem has a singularity at \( r = 0 \) (see (8)) and, if \( u'(r) = 0, u''(r) \) may not exist since
\[
u''(r) = -\frac{N - 1}{(p - 1)r} u'(r) - \frac{W(r)g(u(r))}{(p - 1)|u'(r)|^{p-2}}.
\]
(59)
However, if \( u'(r) \neq 0 \) then \( u''(r) \) is defined by (59). Combining (57) and the first equation in (10), we have
\[
\phi'(r, d) = \frac{(u'(r, d))^2}{\rho^2(r, d)} + \frac{W(r)u(r)g(u(r))}{(p - 1)\rho^2(r, d)|u'(r)|^{p-2}} + \frac{(N - 1)u(r)u'(r)}{r(1)\rho^2(r, d)},
\]
(60)
for \( r \in (0, X] \) with \( u'(r) \neq 0 \).

**Remark 4.**

(i) By Lemmas 3.3, 3.4 and 3.5, \( E(r, d) \to +\infty \) as \( d \to +\infty \) uniformly for \( r \in [0, T_1] \), and therefore \( \rho(r, d) \to +\infty \) as \( d \to +\infty \) uniformly for \( r \in [0, T_1] \).

(ii) From (58), if \( j \) is a non-negative integer and \( \phi(r_1, d) = j\pi + \pi/2 \) for some \( r_1 \in (0, X] \) then \( \phi'(r_1, d) = 1 \). Hence \( \phi(r, d) > j\pi + \pi/2 \) for every \( r \in [r_1, X] \) (see also [3, p. 750] and Corollary 1 below).

(iii) If \( u \) has no zero in \( (0, T_1/2) \) then \( u'(t) < 0 \) for all \( t \in (0, T_1/2) \), which implies \( \sin(\phi(t)) \in (0, 1) \), see (57). Hence \( \phi(t) > 0 \) for all \( t \in (0, T_1/2) \). On the other hand, if \( u \) vanishes in \( r_1 \in (0, T_1/2) \) then taking \( r_1 \) as the smallest zero of \( u \) we have \( \phi(r_1, d) = \pi/2 \). This and (ii) imply \( \phi(T_1/2, d) > \pi/2 \). Thus in any case \( \phi(T_1/2, d) > 0 \).

Let \( k \) be a positive integer. For \( x_0 > 0 \), let us define
\[
\tilde{m}(x_0) = \min \left\{ \frac{g(x)}{|x|^{p-2}x} : |x| \geq x_0 \right\}.
\]

Due to the \( p \)-superlinearity of \( g \) we have \( \tilde{m}(x_0) \to +\infty \) as \( x_0 \to +\infty \). For \( r > 0 \) and \( \eta > 0 \) we define \( \omega(\rho, \eta) := \tilde{m}(\rho \sin(\eta)) \sin^2(\eta)/(p - 1) \). Now we choose \( \rho_0 > 0 \) and \( \delta \in (0, \pi/4) \) such that
\[
\text{(i)} \quad 0 < \delta < \frac{(p - 1)T_1}{32(N - 1)}, \quad \text{(ii)} \quad \omega(\rho_0, \delta) > \frac{4(N - 1)}{m(p - 1)T_1},
\]
\[
\text{(iii)} \quad \tilde{m}(\rho_0/2) \geq \frac{2(p/2 + k(p - 1))}{m}, \quad \text{(iv)} \quad 16\delta + \frac{8\pi}{m\omega(\rho_0, \delta)} \leq \frac{T_1}{2k}.
\]
(61)
Since \( \lim_{d \to +\infty} \rho(r, d) = +\infty \) uniformly for \( r \in [0, T_1] \), there exists \( d_0 > 0 \) such that if \( d > d_0 \) then \( \rho(r, d) \geq \rho_0 \) for every \( r \in [0, T_1] \).

**Lemma 4.1.** If \( T_1/2 \leq r \leq T_1 \) and \( \phi(r, d) \in [\frac{j\pi}{2} - \delta, \frac{j\pi}{2} + \delta] \) with \( j > 0 \) an odd integer, then \( \phi'(r, d) > 1/4 \).

**Proof.** From (60),
\[
\phi'(r, d) \geq \sin^2 \phi + \frac{W(r)u(r)g(u)}{(p - 1)\rho^2(r, d)|u'(r)|^{p-2}} - \frac{(N - 1)|\cos \phi|}{r(1)}.
\]
Taking into account that \( |\sin(\phi(r, d))| \geq \cos \delta \) and \( |\cos(\phi(r, d))| \leq \sin \delta \),
\[
\phi'(r, d) \geq \cos^2 \delta + \frac{W(r)u(r)g(u)}{(p - 1)\rho^2(r, d)|u'(r)|^{p-2}} - \frac{2(N - 1)\delta}{r(1)}.
\]
Since \( \delta < \min\{\pi/4, (p - 1)T_1/(32(N - 1))\} \), see (61)-(i),

\[
\phi'(r, d) \geq \cos^2(\pi/4) + \frac{W(r)u(r)g(u)}{(p - 1)\rho^2(r, d)|u'|^p} - \frac{1}{16} \geq \frac{7}{16} > \frac{1}{4}. 
\]

(62)

Thus, the lemma is proved.

\[\square\]

**Lemma 4.2.** If \( T_1/2 \leq r \leq T_1 \) and \( \phi(r, d) \in [\frac{j\pi}{2} + \delta, \frac{(j+1)\pi}{2} - \delta] \) with \( j > 0 \) an integer, then \( \phi'(r, d) > m\omega(\rho_0, \delta)/4 \).

**Proof.** We carry out the details of the proof for \( p \geq 2 \). The case \( 1 < p < 2 \) follows similarly. From (60),

\[
\phi'(r, d) \geq \frac{W(r)u(r)g(u(r))}{(p - 1)\rho^2(r, d)|u'|^p} - \frac{(N - 1)}{2r(p - 1)} \geq \frac{W(r)}{p - 1} \frac{g(u(r))}{|u(r, d)|^p} \geq \frac{W(r)}{p - 1} \frac{g(u(r))}{|u(r, d)|^p} \geq \frac{m\omega(\rho_0, \delta)}{4}. 
\]

Due to \( \cos(\phi(r, d)) \geq \sin \delta \) and \( \omega(\rho_0, \delta) > \frac{4(N - 1)}{m(p - 1)T_1} \), see (61)-(ii), it follows that

\[
\phi'(r, d) > \frac{W(r)}{p - 1} \frac{g(u(r))}{|u(r, d)|^p} \geq \frac{W(r)}{p - 1} \frac{g(u(r))}{|u(r, d)|^p} \geq \frac{m\omega(\rho_0, \delta)}{4}. 
\]

Since \( |u| = \rho \cos \phi \geq \rho_0 \sin \delta, g(u)/|u|^{p-2}u \geq \tilde{m}(\rho_0 \sin \delta) \). This and the definition of \( \omega(\rho_0, \delta) \) yield

\[
\phi'(r, d) > W(r)\omega(\rho_0, \delta) - \frac{m\omega(\rho_0, \delta)}{4} \geq \frac{m\omega(\rho_0, \delta)}{4}. 
\]

In the latter inequality we have used \( W(r) \geq m/2 \) for any \( r \in [0, T_1] \). Thus, (63) proves the lemma.

\[\square\]

**Lemma 4.3.** If \( T_1/2 \leq r \leq T_1 \) and \( \phi(r, d) \in [j\pi - \delta, j\pi] \cup (j\pi, j\pi + \delta] \) for some positive integer \( j \), then

\[
\phi'(r, d) \geq 8k|\sin(\phi(r, d))|^{2-p}. 
\]

**Proof.** From \( \delta < \pi/4, (57) \), and \( |\cos(\phi(r, d))| \geq \cos \delta \), it follows

\[
u^2(r) = \rho^2(r, d)(1 - \sin^2(\delta)) \geq \rho^2(r, d)/2. 
\]

This, (61)-(iii), and (60) imply

\[
\phi'(r, d) \geq \frac{W(r)u(r)g(u(r))}{(p - 1)\rho^2(r, d)|\sin(\phi(r, d))|^{p-2}} - \frac{(N - 1)|\sin(\phi(r, d))|}{r(p - 1)} \geq \frac{W(r)u(r)g(u(r))}{2p^2(\rho^2(r, d))^{p-2}} \geq \frac{1}{16} |\sin(\phi(r, d))|^{p-1} \geq \frac{m\tilde{m}(\rho_0/2)}{2p^{(p-2)}/(p - 1)} |\sin(\phi(r, d))|^{2-p} \geq 8k|\sin(\phi(r, d))|^{2-p}, 
\]

which completes the proof of the lemma.

\[\square\]
Corollary 1. Let \( j \) be non-negative integer. If \( \hat{r} \in [T_1/2, T_1] \) then \( \phi(\hat{r}, d) = j\pi/2 \) for all \( r \in (\hat{r}, T_1] \).

Proof. For \( j \) odd, see Remark 4. The case \( j \) even follows from Lemma 4.3.

Proposition 1. For any \( p > 1 \), \( \lim_{d \to +\infty} \phi(T_1, d) = +\infty \).

Proof. Let \( d > d_0 \) and \( k \) as in Lemmas 4.1, 4.2 and 4.3. Hence \( \phi(\cdot, d) \) increases in \([T_1/2, T_1]\). Let \( r_0 \in [T_1/2, T_1] \). Since \( \phi(r_0, d) > 0 \), there exists a non-negative integer \( j \) such that either

\[
\phi(r_0, d) \in [j\pi/2, (j+1)\pi/2], \quad \text{or} \quad \phi(r_0, d) \in [(j+1)\pi/2, (j+1)\pi/2 - \delta].
\]

Suppose \( j \) is odd. If \( \phi(r_0, d) \in [j\pi/2, j\pi/2 + \delta] \) then by Lemma 4.1 and (61) there exists \( r_1 \in (r_0, r_0 + 4\delta) \subset (r_0, r_0 + T_1/(8k)] \) such that \( \phi(r_1, d) = j\pi/2 + \delta \).

By Lemma 4.2 and (61) there is \( r_2 \in (r_1, r_1 + 2\pi/(m\omega(\rho_0, \delta))] \subset [r_1, r_1 + T_1/(8k)] \) such that \( \phi(r_2, d) = (j+1)\pi/2 - \delta \).

By Lemma 4.3, if \( p \geq 2 \), there exists \( r_3 \in [r_2, r_2 + \delta/(8k)] \) such that \( \phi(r_3, d) = (j+1)\pi/2 \). On the other hand, if \( p \leq 2 \), from Lemma 4.3 for \( r \geq r_2 \) and \( \phi(r, d) \leq (j+1)\pi/2 \), we have \( \phi'(r, d)\phi^{p-2}(r, d) \geq 8k \). Integration on \([r_2, r] \) and (61) give

\[
8k(p-1)(r-r_2) \leq \left( \frac{(j+1)\pi}{2} \right)^{p-1} - \left( \frac{(j+1)\pi}{2} - \delta \right)^{p-1} \leq 2 \left( \frac{2}{(j+1)\pi} \right)^{2-p} \delta.
\]

Therefore

\[
r - r_2 \leq 2 \left( \frac{2}{(j+1)\pi} \right)^{2-p} \delta \frac{8k(p-1)}{8k} \leq T_1/8k. \tag{68}
\]

Hence there exists \( r_3 \in [r_2, r_2 + T_1/(8k)] \) such that

\[
r_3 \in [r_0, r_0 + 3T_1/(8k)] \subset [r_0, r_0 + T_1/(2k)] \quad \text{and} \quad \phi(r_3, d) = (j+1)\pi/2. \tag{69}
\]

If \( \phi(r_0, d) \in [j\pi/2 + \delta, (j+1)\pi/2 - \delta] \), then placing \( r_0 \) in the role of \( r_1 \) we see that there exists \( r_3 \in [(j+1)\pi/2 - \delta, (j+1)\pi/2] \) that satisfies (69). Similarly if \( \phi(r_0, d) \in [(j+1)\pi/2 - \delta, (j+1)\pi/2] \), placing \( r_0 \) in the role of \( r_2 \) above we find \( r_3 \) satisfying (69).

If \( j \) in (66) is an even positive integer and \( \phi(r_0, d) \in [j\pi/2, j\pi/2 + \delta] \) applying Lemma 4.3 we see that there is \( r_1 \in [r_0, r_0 + T_1/(8k)] \) such that \( \phi(r_1, d) = j\pi/2 + \delta \). Then applying Lemma 4.2 it follows that there exists \( r_2 \in [r_1, r_1 + 2\pi/(m\omega(\rho_0, \delta))] \) such that \( \phi(r_2, d) = (j+1)\pi/2 - \delta \). Finally, applying Lemma 4.1 there exists \( r_3 \in [r_2, r_2 + T_1/(8k)] \) that satisfies (69). That is (69) is satisfied for both \( j \) even and \( j \) odd. Thus \( \phi(r, d) - \phi(t, d) \geq \pi/2 \) if \( r - t \geq T_1/(2k), \) which implies

\[
\phi(T_1, d) \geq \phi(T_1/2) + \frac{k\pi}{2} > \frac{k\pi}{2}. \tag{70}
\]

This proves the proposition.

By Proposition 1 given any positive integer \( k \) there exists \( d_k \) such that if \( d \geq d_k \) then \( \phi(T_1, d) > k\pi/2. \) Since \( \phi(r, d) \) is a continuous function, by the intermediate
value theorem there exists \( \hat{r} \in (0, T_1) \) so that \( \phi(\hat{r}, d) = k\pi / 2 \). By part (ii) of Remark 4,

\[
\phi(X, d) \geq \phi(\hat{r}) \geq k\pi / 2.
\]

Thus, we have proved:

**Proposition 2.** \( \lim_{d \to +\infty} \phi(X, d) = +\infty \).

Now we are ready to prove Theorem 1.1.

5. **Proof of Theorem 1.1.** Let \( u(r, d) \) be the solution to problem

\[
\begin{cases}
(r^{N-1}|u'(r)|^{p-2}u'(r))' + r^{N-1}W(r)g(u(r)) = 0, & 0 < r \leq X, \\
u'(0) = 0, & u(0) = d.
\end{cases}
\]

Let us define \( a := u(X, d) \) and let \( v(r, d) \) be the solution to problem

\[
\begin{cases}
(r^{N-1}|v'|^{p-2}v')' + r^{N-1}W(r)g(v(r)) = 0, & X < r < 1, \\
v(X) = a, & v'(X) = \hat{\zeta},
\end{cases}
\]

where \( \hat{\zeta} := \hat{\zeta}(a) \) is given by Theorem 2.7. Note that \( v(1, d) = 0 \). By Proposition 2 and the continuous dependence of \( \phi(X, d) \) on \( d \), there exists \( K \) such that if \( k \geq K \) then there exist positive real numbers \( d_k \) and \( \hat{d}_k \) such that

\[
d_k < \hat{d}_k, \quad \phi(X, d_k) = k\pi, \quad \text{and} \quad \phi(X, \hat{d}_k) = k\pi + \pi / 2.
\]

Without loss of generality we may assume \( k \) to be even. The case \( k \) odd follows similarly. Since \( k \) is even, \( u(X, d_k) > 0 \) and \( u'(X, d_k) = 0 \). Therefore \( \hat{\zeta}(u(X, d_k)) < 0 = u'(X, d_k) \). Also, \( u'(X, \hat{d}_k) < 0 \) and \( u(X, \hat{d}_k) = 0 \). Hence,

\[
\hat{\zeta}(u(X, \hat{d}_k)) = 0 > u'(X, \hat{d}_k).
\]

Thus, by the intermediate value theorem there exists \( \bar{d}_k \in (d_k, \hat{d}_k) \) such that \( u'(X, \bar{d}_k) = \hat{\zeta}(u(X, \hat{d}_k)) \). Let \( U_k(r) \) be the function defined by

\[
U_k(r) = \begin{cases}
u(r, \bar{d}_k) & r \in [0, X], \\
u(r, \hat{d}_k) & r \in [X, 1],
\end{cases}
\]

where \( v \) is given by (72). Since \( u(X, d) = v(X, d), \ u'(X, d) = v'(X, d), \) and \( v(1, \bar{d}_k) = 0, \ U_k \) is a radial solution to (1). Thus, the sequence \( \{U_k(r)\}_k \) gives us infinitely many radially symmetric solutions to problem (1), which concludes the proof of Theorem 1.1.

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REFERENCES

[1] K. Bal and P. Garain, Nonexistence results for weighted $p$-Laplace equations with singular nonlinearities, *Electron. J. Differ. Equ.*, 95 (2019), 1–12.

[2] H. Berestycki, P.-L. Lions and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$, *Indiana Univ. Math. J.*, 30 (1981), 141–157.

[3] A. Castro, J. Cossio, S. Herrón, R. Pardo and C. Vélez, Infinitely many radial solutions for a sub-super critical $p$-Laplacian problem in a ball, *Annali di Matematica Pura ed Applicata*, 199 (2020), 737–766.

[4] A. Castro and A. Kurepa, Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball, *Proc. Am. Math. Soc.*, 101 (1987), 57–64.

[5] T. Chen and R. Ma, Three positive solutions of $N$-dimensional $p$-Laplacian with indefinite weight, *Electron. J. Qual. Theory Differ. Equ.*, (2019), Paper No. 19, 14 pp.

[6] J. Cossio, S. Herrón and C. Vélez, Infinitely many radial solutions for a $p$-Laplacian problem $p$-superlinear at the origin, *J. Math. Anal. Appl.*, 376 (2011), 741–749.

[7] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, 7 (1983), 827–850.

[8] A. El Hachimi and F. De Thelin, Infinitely many radially symmetric solutions for a quasilinear elliptic problem in a ball, *J. Differ. Equ.*, 128 (1996), 78–102.

[9] W. H. Fleming, A selection-migration model in population genetics, *J. Math. Biol.*, 2 (1975), 219–233.

[10] M. García-Huidobro, R. Manásevich and F. Zanolin, Infinitely many solutions for a Dirichlet problem with a nonhomogeneous $p$-Laplacian-like operator in a ball, *Adv. Differential Equations*, 2 (1997), 203–230.

[11] S. Herrón and E. Lopera, Signed radial solutions for a weighted $p$-superlinear problem, *Electron. J. Differ. Equ.*, 24 (2014), 1–13.

[12] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.*, 12 (1988), 1203–1219.

[13] W. Reichel and W. Walter, Radial solutions of equations and inequalities involving the $p$-Laplacian, *J. Inequal. Appl.*, (1997), 47–71.

[14] A. S. Tersenov, On the existence of radially symmetric solutions of the inhomogeneous $p$-Laplace equation, *Siberian Mathematical Journal*, 57 (2016), 918–928.

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