Pivotal Test Statistic for Nonparametric Cointegrating Regression Functions

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Abstract

This article focuses on cointegrating regression models in which covariate processes exhibit long range or semi-long range memory behaviors, and may involve endogeneity in which covariate and response error terms are not independent. We assume semi-long range memory is produced in the covariate process by tempering of random shock coefficients. The fundamental properties of long memory processes are thus retained in the covariate process. We modify a test statistic proposed for the long memory case by Wang and Phillips (2016) to be suitable in the semi-long range memory setting. The limiting distribution is derived for this modified statistic and shown to depend only on the local memory process of standard Brownian motion. Because, unlike the original statistic of Wang and Phillips (2016), the limit distribution is independent of the differencing parameter of fractional Brownian motion, it is pivotal. Through simulation we investigate properties of nonparametric function estimation for semi-long range memory cointegrating models, and consider behavior of both the modified test statistic under semi-long range memory and the original statistic under long range memory. We also provide a brief empirical example.

Keywords: Fractional differencing parameter; pivotal test statistic; standard Brownian motion; simulation studies; tempered process.

JEL classification codes: C01, C12, C14, C15, C22.

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1 Introduction

Statistical inference for nonlinear cointegrating regression models have been intensively studied in the past decade. These models have been applied to problems in financial markets, stock prices, heavy traffic, and energy markets (e.g., Fasen (2013) and references therein). Let

\[ y_k = f(x_k) + u_k, \quad k = 1, \ldots, N, \]

be a nonlinear cointegrating regression model with regressor process \( x_k \), an error process \( u_k \) and an unknown real function \( f(\cdot) \). Much of the asymptotic theory in the literature has assumed strict exogeneity in which the regressor \( x_k \) is assumed to be uncorrelated with the regression error \( u_k \) (e.g., Karlsen et al. (2007), Cai et al. (2009), Wang and Phillips (2009a), and Wang (2014)). Often, the covariate \( x_k \) is also assumed to be short memory process.

Extending this framework by allowing the regressor \( x_k \) to be driven by long memory innovations and permitting correlation with \( u_k \) (so-called endogeneity) has received less attention and produces some technical problems that have not been completely resolved. Recently, Wang and Phillips (2016) established a limit theory for nonparametric nonstationary regression that allows for both endogeneity and long memory. The test of Wang and Phillips (2016) is a modification of a test statistic given by Hardle and Mammen (1993) for the random sample case. The test was also used in Gao et al. (2012) for a nonlinear cointegrating model with a martingale error structure and no endogeneity.

The limit theory of Wang and Phillips (2016) was then used to develop a model specification test for parametric forms of regression functions of the type \( f(x) = g(x, \theta_0) \) where \( g(\cdot, \theta_0) \) represents a parametric family of functions with unknown parameter \( \theta_0 \in \Theta \). These authors assumed that \( \Theta \) was a compact subspace of \( \mathbb{R}^m \) for some finite \( m \). The specification test of Wang and Phillips (2016) involved local alternative models \( f(x) = g(x, \theta_0) + \rho_N m(x) \), where \( \rho_N \) is a sequence of constants measuring local deviations from the null hypothesis of \( f(x) = g(x, \theta_0) \), and \( m(x) \) is a real function.

A limitation of the test under long memory and endogeneity is that it is not pivotal, and its limit distribution depends on the value of the fractional differencing parameter in the long memory \( x_k \) process. The motivation for the work reported here was that the applicability of the Wang and Phillips (2016) test statistic might be improved if one employs the idea of tempering and assumes that there are semi-long memory input shocks to the regressor process \( x_k \). We describe this idea in detail and list the consequences, which are related to the asymptotic theory for nonparametric cointegrating regression functions.

We now summarize the main contributions of this paper. First, we demonstrate asymptotic properties of kernel estimators of the unknown regression function \( f(x) \) in [1]. Then
we consider limit theory for the test statistic of Wang and Phillips (2016) under the assumption that the regressor process involves semi-long memory input shocks. We show that the limit theory of this test statistic under semi-long memory involves the local time of standard Brownian motion, so that the limit distribution in this case does not involve the fractional differencing parameter $d$. These findings extend the theory developed in Wang and Phillips (2009a,b, 2016), making the methods more applicable because the statistic is pivotal under semi-long memory. In addition, we investigate the properties of regression function estimators and the performance of the test statistic using simulation and present an empirical example related to the Carbon Kuznets curve previously investigated in a number of works (e.g., Müller-Fürstenberger and Wagner (2007), Wang et al. (2018)).

The remainder of the article is organized as follows. The general model and notation is described in Section 2. In Section 3, we provide some limit behaviors, which are essential for developing the limit theory presented in following sections. In Section 4, we introduce nonlinear cointegrating regression models under the assumption that the regressor process involves strongly tempered shocks. We develop a consistent estimator of the regression function and establish its limit distribution. In Section 5, we present a test for parametric forms of the regression function, show that it is based on a pivotal quantity, and consider its power under local alternatives. Section 6 contains the results of simulation studies for the regression function and proposed test statistic. Section 7 contains an empirical example, where the fractional differencing parameter is greater than $1/2$. Section 8 contains concluding remarks and mentions possible directions for future work. The proofs of all technical results are contained in the Appendix.

2 The cointegrating regression context

The models we will consider throughout this article are of the general form (1) in which the regressor process $x_k$ is a sum of input shocks that result in either a long memory or a semi-long memory process. In the long memory case we take, for $k = 1, \ldots, N$,

$$x_k = \sum_{s=1}^{k} X_d(s),$$

$$X_d(s) = \sum_{j=0}^{\infty} b_d(j) \zeta(s-j), \quad (2)$$

where $\zeta(k)$ is an independent and identically distributed (i.i.d.) noise such that $\mathbb{E}\zeta(0) = 0$ and $\mathbb{E}\zeta^2(0) = 1$. The coefficients $b_d(j)$ regularly vary at infinity as $j^{d-1}$, viz.,
b_d(j) \sim \frac{c_d}{\Gamma(d)^{j-1}}, \quad j \to \infty, \quad c_d \neq 0, \quad d \neq 0,

where 0 < d < 1/2 is the fractional differencing parameter.

In contrast to this long memory process, a semi-long memory setting contains strongly tempered shocks as,

\[
x_k = \sum_{s=1}^{k} X_{d,\lambda}(s),
\]

\[
X_{d,\lambda}(s) = \sum_{j=0}^{\infty} e^{-\lambda j} b_d(j) \zeta(s-j),
\]

where \(\zeta(k)\) is the same as for the long memory case. In (3), \(\lambda \equiv \lambda_N > 0\) is a sample size dependent parameter and satisfies the following main assumption:

• **Main Assumption:** The tempering parameter \(\lambda \to 0\) and \(N\lambda \to \infty\) as \(N \to \infty\).

We note that the tempering of the coefficients in expression (3) extends the range of fractional differencing parameter \(d\) from \((0, 1/2)\) to \((0, \infty)\). This means, in contrast with the stationary long memory time series \(X_d(.)\), the semi-long memory time series \(X_{d,\lambda}(.)\) forms a stationary process for \(d\) belonging to any positive value on the real line. The strongly tempered process (3) belongs to the general class of stochastic processes called tempered linear processes, see Sabzikar and Surgailis (2018). These processes have a semi-long memory property in the sense that their autocovariance functions initially resemble that of a long memory process but eventually decay fast at an exponential rate. One special case of such processes is the autoregressive tempered fractionally integrated moving average ARTFIMA\((p, d, \lambda, q)\) process, which has been studied by Meerschaert et al. (2014) and Sabzikar et al. (2019).

Throughout the paper, we denote \(C, C_1, C_2, ...\) as generic constants which may differ at each appearance. We use \(\to_P\) for convergence in probability, \(\Rightarrow\) for weak convergence of the associated probability measures, \(\to_D\) for convergence in distribution, and \(=_{D}\) for equivalence in distribution. For any two functions \(f\) and \(g\), \(f \simeq g\) means \(C_1 \leq f/g \leq C_2\). We use \(||x|| = \max_i |x_i|\) for vector \(x = (x_i)\), \(\wedge\) for the minimum between two numbers, and a.s. for almost surely. Finally, i.i.d. and f.d.d. mean independent and identically distributed and finite dimensional distribution, respectively.
3 Initial results

Throughout this section, set \( d_N := \left[ \mathbb{E}(x_N^2) \right]^{1/2} \). Under the semi-long memory setting, the asymptotic form of \( d_N \) is given by \( d_N \sim \sqrt{N/\lambda} \) as \( N \to \infty \). Then,

\[
\frac{x_N}{d_N} := \frac{\lambda^d}{\sqrt{N}} \sum_{k=1}^{N} X_{d,\lambda}(k) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \zeta(k) + o_P(1),
\]

see expression (A.7) in Sabzikar et al. (2020). For any \( t \in [0,1] \), the following weak convergence applies on \( D[0,1] \):

\[
\frac{x_{\lfloor Nt \rfloor}}{d_N} \Rightarrow B(t),
\]

where \( \lfloor Nt \rfloor \) is the floor function, and \( B(t) \) is the standard Brownian motion.

**Remark 3.1.** Under the long memory case, \( d_N \sim N^{d+1/2} \) as \( N \to \infty \) and \( x_{\lfloor Nt \rfloor}/d_N \Rightarrow B_{d+1/2}(t) \), where \( B_{d+1/2}(t) \) is fractional Brownian motion with parameter \( d + 1/2 \). This makes the process \( B_{d+1/2}(t) \) depend on the unknown fractional differencing parameter \( d \). This is the reason the test statistic of Wang and Phillips (2016) is not pivotal under long memory. On the other hand, the process \( B(t) \) given in expression (5) is free of the unknown parameter \( d \), and this will result in the test statistic being pivotal under semi-long memory.

Consider \( x_{k,N} := x_k/d_N \) for \( 1 \leq k \leq N \) to be a triangular array. A function of \( x_{k,N} \) that will be used in the sequel takes the form of a sample average of functions of \( x_{k,N} \),

\[
S_N := \frac{c_N}{N} \sum_{k=1}^{N} g(c_N x_{k,N}),
\]

where \( g(x) \) is a bounded function such that \( \int_{\mathbb{R}} |g(x)| \, dx < \infty \) and \( c_N := d_N/h \) where \( c_N \to \infty \) and \( c_N/N \to 0 \). The bandwidth parameter is denoted by \( h \) and satisfies \( h \equiv h_N \to 0 \) as \( N \to \infty \). Functions in the form of \( S_N \) commonly arise in nonlinear cointegrating regressions, which could be the kernel function \( K(.) \) or its squared \( K^2(.) \); see Karlsen and Tjøstheim (2001), Karlsen et al. (2007), and Wang and Phillips (2009a).

The limit behavior of \( S_N \) is important in order to establish the limit behavior of the kernel estimate of \( f(x) \) in the nonparametric regression context. The limit distribution of \( S_N \) is as follows

\[
S_N \to_D \int_{\mathbb{R}} g(x) \, dx L_B(1,0),
\]

where \( L_B(t, s) \) is the local time of standard Brownian motion \( B(t) \) at the spatial point \( s \); see Proposition A.9 part (i) to follow. When the function \( g \) is a kernel density, the integral in
expression (6) is unity, and the limit is then the local time of $B$ at the origin; see Jeganathan (2004) and Wang and Phillips (2009a) for some of the related results.

**Definition 3.1.** The local time process of a stochastic process $G(x)$ is defined as

$$L_G(t, s) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t I\{|G(r) - s| \leq \epsilon\}dr,$$

where $(t, s) \in \mathbb{R}_+ \times \mathbb{R}$.

Under long memory case, the local time given in expression (6) is in the form of $L_B d + 1/2$, and therefore depends on the unknown parameter $d$. This complicates the limit theory of the associated test statistic; see Remark 3.1.

### 4 Nonparametric regression function estimation

Assume model (1) where $f(\cdot)$ is an unknown real regression function. To induce endogeneity, let $\eta_k = (\xi_k, \epsilon_k)'$ be a sequence of i.i.d. random vectors with $\mathbb{E}(\eta_0) = 0$ and $\mathbb{E}(\eta_0\eta_0') = \Sigma$ where

$$\Sigma = \begin{pmatrix} 1 & \mathbb{E}(\xi_0\epsilon_0) \\ \mathbb{E}(\epsilon_0\xi_0) & \mathbb{E}(\epsilon_0^2) \end{pmatrix},$$

and take $u_k$ in (1) to be $u_k = \sum_{j=0}^\infty \psi_j \eta_{k-j}$ for $\psi_j = (\psi_{j1}, \psi_{j2})$. Now, assume $\eta_0$ and $\psi_j$ satisfy the following assumption.

**Assumption 4.1.** $\sum_{j=0}^\infty \psi_j \neq 0$, $\sum_{j=0}^\infty j^{1/4}(|\psi_{j1}| + |\psi_{j2}|) < \infty$, and $\mathbb{E}|\eta_0|^\alpha < \infty$ for $\alpha > 2$.

Therefore, $\mathbb{E}(u_0^2) = \sum_{j=0}^\infty \psi_j^2 \Sigma \psi_j'$ and $\text{cov}(u_k, x_k) \neq 0$. Additionally, assume the characteristic function $\varphi(t)$ of $\xi_0$ satisfies $\int_{\mathbb{R}} (1 + |t|)|\varphi(t)|dt < \infty$, which ensures the smoothness in the corresponding kernel density (see Wang and Phillips (2016)).

The Nadaraya-Watson regression estimator of $f(x)$ is

$$\hat{f}(x) = \frac{\sum_{k=1}^N y_k K_h(x_k - x)}{\sum_{k=1}^N K_h(x_k - x)},$$

where $K_h(s) = h^{-1}K(s/h)$, and $K(\cdot)$ is a non-negative bounded continuous function. In order to establish the asymptotic behavior for the kernel estimate $f(x)$, $\{x_{k,N}\}_{k \geq 1, N \geq 1}$ needs to be a strong smooth array (see Proposition A.1). Now, we impose the following two assumptions on the kernel $K(\cdot)$:

**Assumption 4.2.** $K(x)$ is a non-negative bounded continuous function satisfying $\int_{\mathbb{R}} K(x)dx = 1$ and $\int_{\mathbb{R}} |\hat{K}(x)|dx < \infty$, where $\hat{K}(x) = \int_{\mathbb{R}} e^{ixt}K(t)dt$. 


Assumption 4.3. For a given \( x \), there exists a real positive function \( f_1(s,x) \) and \( \gamma \in (0,1] \) such that when \( h \) is sufficiently small, \( |f(hy + x) - f(x)| \leq h^\gamma f_1(y,x) \forall y \in \mathbb{R} \), and \( \int_{\mathbb{R}} K(s)[f_1(s,x) + f_1^2(s,x)]ds < \infty \).

The following theorem is the main result on the Nadaraya-Watson kernel estimator (8).

Theorem 4.1. Under Assumptions 4.1–4.3 and for any \( h \) satisfying \( \sqrt{N\lambda} d h \to \infty \) and \( \sqrt{N\lambda} d h^{1+2\gamma} \to 0 \), we have

\[
\left\{ \sqrt{N\lambda d h} \right\}^{1/2} \left( \hat{f}(x) - f(x) \right) \to_D \frac{d_0}{\sqrt{d_1}} N(0,1) L_B^{-1/2}(1,0),
\]

where \( d_0^2 = \mathbb{E}(u_0^2) \int_{\mathbb{R}} K^2(s)ds, \) \( d_1 = \int_{\mathbb{R}} K(s)ds, \) and \( N(0,1) \) denotes a standard normal which is independent from the local time of Brownian motion \( L_B(1,0) \). If we normalize the limit form \( \{9\} \), we have:

\[
\left\{ h \sum_{k=1}^{N} K_h(x_k - x) \right\}^{1/2} \left( \hat{f}(x) - f(x) \right) \to_D N(0,\sigma^2),
\]

where \( \sigma^2 = d_0^2 / d_1 \); see also expression \( \{A.6\} \) given in the Appendix.

Remark 4.1. The local time given in Theorem 4.1 is the local time of standard Brownian motion, which is independent of any unknown parameters. This is a direct consequence of tempering the time series in the regressor \( x_k \) (see Remark 3.1). In this regard, the result of Theorem 4.1 is different from Theorem 2.1 in Wang and Phillips (2016), which involves the local time process \( L_{B_d+1/2}(1,0) \), which is related to a fractional Brownian motion with parameter \( d + 1/2 \).

The condition of Theorem 4.1 that \( \sqrt{N\lambda} d h \to \infty \) is required for the consistency of \( \hat{f}(x) \). The normalized limit \( \{10\} \) is pivotal upon the estimation of \( \mathbb{E}(u_0^2) \) by

\[
\hat{\sigma}_N^2 = \frac{\sum_{k=1}^{N} [y_k - \hat{f}(x_k)]^2 K_h(x_k - x)}{\sum_{k=1}^{N} K_h(x_k - x)}.
\]

Expression \( \{11\} \) is appropriate for use in interval estimation and other inferential procedures. To investigate the bias of \( \hat{f}(x) \), we make use of Assumption 4.1 but replace Assumptions 4.2 and 4.3 for some \( p \geq 2 \) with the following stronger assumptions:

Assumption 4.4. \( \int_{\mathbb{R}} y^i K(y)dy = 0 \) for \( i = 1, ..., p - 1 \) and \( \int_{\mathbb{R}} y^p K(y)dy \neq 0 \). Also, \( K(x) \) has a compact support which is twice continuously differentiable on \( \mathbb{R} \).
Assumption 4.5. For a fixed value of $x$, $f(x)$ has a continuous $p + 1$ derivative in a small neighbourhood of $x$. For $h$ satisfying $\sqrt{N}\lambda^d h \to \infty$ and $\sqrt{N}\lambda^d h^{2(p+1)+1} \to 0$, we have
\[
\left\{ \sqrt{N}\lambda^d h \right\}^{1/2} \left[ \hat{f}(x) - f(x) - \frac{h^p f^{(p)}(x)}{p!} \int_{\mathbb{R}} y^p K(y) dy \right] \to_D N(0, \sigma^2) L_B^{-1/2}(1,0), \quad \text{and}
\]
\[
\left\{ h \sum_{k=1}^N K_h(x_k - x) \right\}^{1/2} \left[ \hat{f}(x) - f(x) - \frac{h^p f^{(p)}(x)}{p!} \int_{\mathbb{R}} y^p K(y) dy \right] \to_D N(0, \sigma^2).
\]
The proof of these follow from a Taylor series expansion, and the details are similar to the ones given in Theorem 2.4 of Wang (2014). Hence, we omit them. For any $h$ satisfying $\sqrt{N}\lambda^d h \to \infty$ as $h \to 0$, we have $\hat{f}(x) \to_P f(x)$. Therefore, $\hat{f}(x)$ is a consistent estimator of $f(x)$.

5 Specification test for the regression function

When there is no reason to believe that $f(x)$ in (1) follows a particular parametric form, the use of a nonparametric estimator is attractive, but nonparametric estimators generally have a slow convergence rate in comparison with the parametric estimators. It is often possible to determine a plausible parametric regression function, and we then might wish to conduct a test of the posited parametric specification, formulated as a hypothesis as,
\[
H_0 : f(x) = g(x, \theta_0),
\]
where $\theta_0$ in (12) is a vector of unknown parameters that belong to a compact and convex space $\Omega_0$. Tests of (12) have been previously considered by Hardle and Mammen (1993), Horowitz and Spokoiny (2001), Gao et al. (2009), Wang and Phillips (2012), and Wang and Phillips (2016) under different assumptions on the data generating mechanism. Indeed, Gao et al. (2009) and Wang and Phillips (2012) considered kernel-smoothed U statistic of the form $\sum_{j,k=1,j \neq k}^N u_k u_j K[(x_k - x_j)/h]$ with $u_k = y_k - g(x_k, \hat{\theta}_N)$.

The asymptotics for U statistics are hard to be extended to the case of endogenous regressors. Thus, Wang and Phillips (2016) modified a test statistic suggested previously by Hardle and Mammen (1993) for the consideration of endogenous regressors with long memory. In that case, the test statistic is not pivotal and its limit theory and convergence rate both depend on the memory parameter $d$ through the local time of fractional Brownian motion $L_{B_{d+1/2}}(t,s)$, which complicates its use in actual problems. In this section, we consider the statistic under an assumption of semi-long memory input shocks to the regressors $x_k$. Both the test statistic of Wang and Phillips (2016) and our modification for use with semi-long
memory regressors are based on the statistic,

\[ T_N := \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K\left[ \frac{x_k - x}{h} \right] [y_k - g(x_k, \hat{\theta}_N)] \right\}^2 \pi(x)dx. \]  \hspace{1cm} (13)

For estimating \( \hat{\theta}_N \), we use a non-linear least squares method by setting \( Q_N(\theta) = \sum_{k=1}^{N} (y_k - g(x_k, \theta))^2 \) and minimizing it over \( \theta \in \Theta \) as follows \( \hat{\theta}_N = \arg\min_{\theta \in \Theta} Q_N(\theta) \). Additionally, \( \pi(x) \) in (13) is a positive integrable weight function with a compact support. To develop the asymptotic theory of \( T_N \) in the context of semi-long memory regressors and endogeneity, we provide three assumptions as follows.

**Assumption 5.1.** \( K(x)\pi(x) \) has a compact support such that \( \int_{\mathbb{R}} K(x)dx = 1 \) and \( |K(x) - K(y)| \leq C|x - y| \) whenever \( |x - y| \) is sufficiently small.

**Assumption 5.2.** There exist \( g_1(x) \) and \( g_2(x) \) such that for each \( \theta, \theta_0 \in \Omega_0 \), \( |g(x, \theta) - g(x, \theta_0)| \leq C||\theta - \theta_0||g_1(x) \), and for some \( 0 < \beta \leq 1 \), \( |g_1(x + y) - g_1(x)| \leq C|y|^\beta g_2(x) \), whenever \( y \) is sufficiently small. Additionally, \( \int_{\mathbb{R}} [1 + g_1^2(x) + g_2^2(x)] \pi(x)dx < \infty \) holds.

**Assumption 5.3.** Under \( H_0 \), \( ||\hat{\theta}_N - \theta_0|| = o_P\left(\sqrt{N}\lambda^d h\right)^{-1/2} \).

As noted in Wang and Phillips (2016), Assumption 5.2 covers a wide range of functions \( g(x, \theta) \) and \( \pi(x) \). Typical examples of \( g(x, \theta) \) include \( \theta e^x/(1+e^x), e^{-|\theta|x}, e^{-\theta x^2}, e^{\theta|x|/(1+e^{|\theta|x})}, \theta \log|x|, (x + \theta)^2 \), etc. Note that assumptions 5.1-5.3 may be compared with those imposed by Wang and Phillips (2016). We now consider the asymptotic behavior of \( T_N \) in (13) and its convergence rate in the semi-long memory setting.

**Theorem 5.1.** Let Assumptions 4.1 and 5.1-5.3 hold. Then, under \( H_0 \), we have

\[ T_{\lambda,d} := \frac{1}{\sqrt{N}\lambda^d h} T_N \rightarrow_D d_{(0)}^2 L_B(1, 0), \]  \hspace{1cm} (14)

where \( d_{(0)}^2 = \mathbb{E}(u_0^2) \int_{\mathbb{R}} K^2(s)ds \int_{\mathbb{R}} \pi(x)dx \) and \( N^{1/2 - \delta_0}\lambda^d h \rightarrow \infty \) for a small enough \( \delta_0 \).

**Remark 5.1.** (i) Under semi-long memory for \( d \in \mathbb{R}_+ \), the limiting null distribution of (14) is free of the unknown fractional differencing parameter \( d \), therefore it is a pivotal quantity. (ii) Under long memory for \( d \in (0, 1/2) \), the test statistic and limit distribution given by Wang and Phillips (2016) are \( T_{N,d} := (d_N/Nh)T_N \rightarrow_D d_{(0)}^2 L_{B_{d+1/2}}(1, 0) \), where \( d_N \sim N^{d+1/2} \) as \( N \rightarrow \infty \), and \( T_N \) is as given in (13). The limiting null distribution of this version of the test statistic relies on \( d \) through the fractional Brownian motion process \( B_{d+1/2} \). (iii) Under short memory for \( d = 0 \) and \( d_N \sim \phi\sqrt{N} \), the test statistic and limit distribution are: \( T_{N,0} = (\phi/\sqrt{N}h)T_N \rightarrow_D d_{(0)}^2 L_B(1, 0) \). (iv) Note that the limiting null distribution of test
statistic (14) formulated under semi-long memory is similar to the short memory case. In neither case does the limiting distribution depend on $d$.

**Remark 5.2.** The role of tempering in the normalizing term $\{\sqrt{N}\lambda^d h\}^{-1}$ in (14) is intermediate; i.e., under long memory and in the presence of tempering parameter $\lambda$, the rate of convergence of $\{\sqrt{N}\lambda^d h\}^{-1} \to 0$ is faster compared to the case without a tempering parameter. On the other hand, under short memory and in the presence of tempering parameter $\lambda$, the rate of convergence of $\{\sqrt{N}\lambda^d h\}^{-1} \to 0$ is slower compared to the case without a tempering parameter.

To verify that a test has nontrivial power, one can assess the null hypothesis (12) against a local alternative

$$H_A : f(x) = g(x, \theta) + \rho_N m(x), \quad \text{for } x \in \mathbb{R}. \quad (15)$$

Here, $\rho_N$ is a sequence of numerical constants which measures the local deviation from the null hypothesis. Also in (15), $m(x)$ is a real function free of $\theta$ without lying in the span of $g(x, \theta)$ and its derivative functions. To make $m(x)$ smooth enough under $H_A$ for the sake of asymptotic power development, we give the following two assumptions.

**Assumption 5.4.** (i) There exist $m_1(x)$ and $\gamma \in (0, 1]$ such that for any $y$ sufficiently small, $|m(x+y) - m(x)| \leq C|y|^\gamma m_1(x)$ holds. (ii) Let $\int_{\mathbb{R}}[1 + m^2(x) + m_1^2(x)]\pi(x)dx < \infty$ and $\int_{\mathbb{R}}m^2(x)\pi(x)dx > 0$ hold. (iii) The function $m(x)$ is not an element of the space spanned by $g(x, \theta_0)$ and its derivative functions.

**Assumption 5.5.** Under $H_A$, $||\hat{\theta}_N - \theta_0|| = o_P(\{\sqrt{N}\lambda^d h\}^{-1/2})$.

**Theorem 5.2.** Let Assumptions 4.1, 5.1–5.2, and 5.4–5.5 hold. Then, under $H_A$, we have

$$\lim_{N \to \infty} P\left(T_{\lambda,d} \geq T_0\right) = 1,$$

where $T_0$ is positive, and $h \to 0$ satisfying $N^{1/2-\delta_0}\lambda^d h \to \infty$ for a small enough $\delta_0$ and any $\rho_N$ satisfying $N^{1/2}\lambda^d h\rho_N^2 \to \infty$.

**Remark 5.3.** Based on Theorem 5.2, the test statistic $T_{\lambda,d}$ has nontrivial power against the local alternatives in the form of (15) whenever $\rho_N \to 0$ at a rate that is slower than $\{\sqrt{N}\lambda^d h\}^{-1/2}$ as $\{\sqrt{N}\lambda^d h\}^{-1} \to 0$. Since $m(x)$ moves faster for a nonstationary process $x_k$, it is natural to expect a higher power under the alternative $H_A$ in (15). Proof of Theorem 5.2 ensures the divergence of normalized test statistic $T_{\lambda,d}$ and test consistency under $H_A$; see Remarks 3.2 and 3.3 in Wang and Phillips (2016) for a related discussion.
6 Simulation studies

In this section, we illustrate the theory of the previous sections through the use of several Monte Carlo studies. We first examine properties of nonparametric function estimators and then turn our attention toward behavior of specification tests for hypothesized parametric regression functions.

6.1 Regression function estimation

To examine the behavior of regression function estimators, we generated data sets from a model with the form of (1), where we incorporate $\sigma$ as follows

$$y_k = f(x_k) + \sigma u_k.$$  

(16)

The regressor process $x_k$ is defined for the long memory setting (LM) in (2) and for the semi-long memory setting (SLM) in (3). Let $u_k = \psi u_{k-1} + \epsilon_k$ with $\psi = 0.25$ and $E(\xi_k \epsilon_k) =: \rho_{\xi,\epsilon}$ (c.f., (7)). Also, let $\rho_{\xi,\epsilon} = 0.5$ and $\sigma = 0.2$ in (16). We consider the following regression function, which was also used by Wang and Phillips (2016),

$$f(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \sin(j\pi x)}{j^2}.$$  

(17)

We set the sample size as $N = 1000$, and the number of Monte Carlo replications as $R = 10,000$.

We use the estimator (8) with an Epanechnikov kernel in the form of $K(u) = 0.75(1 - u^2)1_{|u|\leq 1}$. We take the bandwidth $h = \{N^{-1/3}, N^{-1/4}, N^{-1/5}, N^{-1/6}\}$ and the tempering parameter $\lambda$ to be $\{N^{-1/3}(SLM1), N^{-1/4}(SLM2), N^{-1/5}(SLM3), N^{-1/6}(SLM4)\}$ under the assumption of $N\lambda \to \infty$. In Table 1 we give the Monte Carlo approximations of bias, standard deviation (Std) and root of mean squared error (RMSE) of $f(x)$ for the true regression function (17) over the interval $[0, 1]$. The values of Table 1 result from computing bias, Std, and RMSE at 100 points equally spaced between 0 and 1 and then averaging. Errors of the Monte Carlo approximations of Table 1 were generally less than 0.003, and were nearly uniformly smaller for SLM settings than for the LM case. Bias is small and similar for both LM and SLM settings and across values of $d$. Values for Std are uniformly smaller for SLM than for LM processes, increase somewhat as $d$ increases, and are larger for the larger bandwidth used. As a result, values for RMSE follow the same pattern, being smaller for SLM, smaller for smaller $d$, and smaller for shorter bandwidth.

To construct point-wise confidence intervals for the regression function $f(x)$, we use the
limit distribution given in (10). An asymptotic $100(1 - \alpha)\%$ level confidence interval for $f(x)$ is then given by

$$
\hat{f}(x) \pm z_{\alpha/2} \left\{ \frac{\hat{\sigma}_N^2 \int R K^2(s)ds}{\sum_{k=1}^{N} K\left[\frac{x_k - x}{h}\right] \int R K(s)ds} \right\}^{1/2},
$$

where $\hat{\sigma}_N^2$ is given in expression (11). The observed coverage probabilities of confidence intervals are given in Figure 1 as well as Table 2 for $(1 - \alpha) = 0.95$. Observe that, as the bandwidth increases, the coverage probabilities become closer to the nominal level of 95%. Also, as the tempering parameter $\lambda$ increases, the coverage probabilities increases.

Table 1: Comparison of Bias, Std and RMSE for $\hat{f}(x)$ between LM and SLM processes under the assumption of $\rho_{\xi,\epsilon} = 0.5$.

| Criterion | Regressor | Bandwidth | $d = 0$ | $d = 0.1$ | $d = 0.2$ | $d = 0.3$ | $d = 0.4$ |
|-----------|-----------|-----------|---------|-----------|-----------|-----------|-----------|
|            |           | $N^{-1/3}$ | 0.007   | 0.007     | 0.007     | 0.005     | 0.000     |
|            |           | $N^{-1/5}$ | 0.046   | 0.046     | 0.046     | 0.043     | 0.043     |
| Bias       | SLM1      | $N^{-1/3}$ | 0.007   | 0.007     | 0.007     | 0.007     | 0.008     |
|            |           | $N^{-1/5}$ | 0.046   | 0.047     | 0.047     | 0.047     | 0.047     |
|            | SLM3      | $N^{-1/3}$ | 0.007   | 0.007     | 0.007     | 0.007     | 0.006     |
|            |           | $N^{-1/5}$ | 0.046   | 0.047     | 0.047     | 0.047     | 0.046     |
|            |           | $N^{-1/3}$ | 0.114   | 0.127     | 0.139     | 0.150     | 0.157     |
|            |           | $N^{-1/5}$ | 0.125   | 0.147     | 0.165     | 0.184     | 0.200     |
| Std        | SLM1      | $N^{-1/3}$ | 0.114   | 0.121     | 0.128     | 0.135     | 0.140     |
|            |           | $N^{-1/5}$ | 0.125   | 0.134     | 0.144     | 0.154     | 0.164     |
|            | SLM3      | $N^{-1/3}$ | 0.114   | 0.120     | 0.123     | 0.129     | 0.132     |
|            |           | $N^{-1/5}$ | 0.125   | 0.131     | 0.138     | 0.142     | 0.148     |
|            |           | $N^{-1/3}$ | 0.116   | 0.129     | 0.141     | 0.152     | 0.160     |
|            |           | $N^{-1/5}$ | 0.151   | 0.173     | 0.191     | 0.209     | 0.226     |
| RMSE       | SLM1      | $N^{-1/3}$ | 0.116   | 0.123     | 0.130     | 0.137     | 0.143     |
|            |           | $N^{-1/5}$ | 0.151   | 0.161     | 0.170     | 0.179     | 0.189     |
|            | SLM3      | $N^{-1/3}$ | 0.116   | 0.122     | 0.125     | 0.131     | 0.134     |
|            |           | $N^{-1/5}$ | 0.151   | 0.157     | 0.165     | 0.168     | 0.174     |

In Figure 1, we compare the coverage probabilities between LM and SLM settings for different values of $d$. Coverage probabilities based on the SLM process remain close to the nominal level across values of $d$, but coverage probabilities for the LM process drift further away from the nominal level as $d$ increases. Additionally, we demonstrate the width of confidence intervals in Figure 2 as well as Table 2. We observe that, under LM, changes in $d$ affect the width of confidence intervals, with wider intervals resulting from larger $d$. In contrast, under SLM, changes in $d$ have relatively little influence on the width of computed...
intervals. In Table 2 it can be seen that as bandwidth increases, the widths of confidence intervals decrease and this holds for both LM and SLM settings and across all levels of the differencing parameter \( d \). As the tempering parameter \( \lambda \) increases, the confidence interval becomes shorter.

Figure 1: Comparison of coverage probabilities for the regression function \( f(x) \) between LM (left) and SLM (right) cases under the assumption of \( \rho_{\xi,\epsilon} = 0.5 \) for different values of \( d \). We assume the bandwidth is \( h = N^{-1/3} \) and the tempering parameter for the SLM is \( \lambda = N^{-1/6} \).

Figure 2: Comparison of empirical confidence intervals for the regression function \( f(x) \) between LM (left) and SLM (right) cases under the assumption of \( \rho_{\xi,\epsilon} = 0.5 \) for different values of \( d \). We assume the bandwidth is \( h = N^{-1/3} \) and the tempering parameter for the SLM is \( \lambda = N^{-1/6} \).
Table 2: Comparison of coverage probabilities and expected lengths of confidence intervals between LM and SLM processes for $f(x)$ given some selected points of $x$ on the interval $[0,1]$, where we assume $\rho_{\xi,\epsilon} = 0.5$. The values in the parentheses are the expected lengths of confidence intervals.

| $x$ | Regressor | Bandwidth | $d = 0$ | $d = 0.1$ | $d = 0.2$ | $d = 0.3$ | $d = 0.4$ |
|-----|-----------|-----------|---------|-----------|-----------|-----------|-----------|
| 0.25 | SLM1 | $N^{-1/3}$ | 0.854  | 0.828  | 0.790  | 0.755  | 0.706  |
| | | | (1.053) | (1.117) | (1.194) | (1.273) | (1.336) |
| | | $N^{-1/5}$ | 0.926  | 0.913  | 0.893  | 0.876  | 0.849  |
| | | | (0.829) | (0.907) | (0.973) | (1.048) | (1.132) |
| | SLM3 | $N^{-1/3}$ | 0.854  | 0.837  | 0.818  | 0.797  | 0.768  |
| | | | (1.053) | (1.101) | (1.161) | (1.218) | (1.231) |
| | | $N^{-1/5}$ | 0.926  | 0.918  | 0.906  | 0.896  | 0.882  |
| | | | (0.829) | (0.885) | (0.932) | (0.976) | (1.020) |
| 0.50 | SLM1 | $N^{-1/3}$ | 0.870  | 0.838  | 0.806  | 0.7693 | 0.724  |
| | | | (1.687) | (1.818) | (1.943) | (2.081) | (2.186) |
| | | $N^{-1/5}$ | 0.945  | 0.933  | 0.917  | 0.902  | 0.881  |
| | | | (1.286) | (1.398) | (1.521) | (1.629) | (1.767) |
| | SLM3 | $N^{-1/3}$ | 0.870  | 0.851  | 0.832  | 0.807  | 0.786  |
| | | | (1.687) | (1.768) | (1.860) | (1.944) | (2.010) |
| | | $N^{-1/5}$ | 0.945  | 0.936  | 0.930  | 0.918  | 0.910  |
| | | | (1.286) | (1.359) | (1.472) | (1.506) | (1.596) |
| 0.75 | SLM1 | $N^{-1/3}$ | 0.867  | 0.836  | 0.799  | 0.762  | 0.721  |
| | | | (1.926) | (2.059) | (2.187) | (2.349) | (2.397) |
| | | $N^{-1/5}$ | 0.935  | 0.921  | 0.906  | 0.885  | 0.866  |
| | | | (1.326) | (1.452) | (1.565) | (1.668) | (1.812) |
| | SLM3 | $N^{-1/3}$ | 0.867  | 0.850  | 0.829  | 0.805  | 0.779  |
| | | | (1.926) | (1.996) | (2.098) | (2.200) | (2.301) |
| | | $N^{-1/5}$ | 0.935  | 0.928  | 0.919  | 0.910  | 0.895  |
| | | | (1.326) | (1.411) | (1.488) | (1.559) | (1.633) |
Table 2. (continued)

| x  | Regressor | Bandwidth | $d = 0$ | $d = 0.1$ | $d = 0.2$ | $d = 0.3$ | $d = 0.4$ |
|----|-----------|-----------|--------|----------|----------|----------|----------|
| LM | $N^{-1/3}$ | 0.796     | 0.701  | 0.596    | 0.501    | 0.397    |
|    |           | (0.855)   | (0.974)| (1.032)  | (1.160)  | (1.240)  |
|    | $N^{-1/5}$ | 0.875     | 0.833  | 0.777    | 0.718    | 0.623    |
|    |           | (0.989)   | (1.183)| (1.377)  | (1.568)  | (1.733)  |
| SLM1 | $N^{-1/3}$ | 0.796     | 0.735  | 0.716    | 0.872    | 0.629    |
|     |           | (0.855)   | (0.902)| (0.978)  | (1.027)  | (1.053)  |
|     | $N^{-1/5}$ | 0.875     | 0.840  | 0.842    | 0.822    | 0.804    |
|     |           | (0.989)   | (1.085)| (1.256)  | (1.374)  |           |
| SLM3 | $N^{-1/3}$ | 0.796     | 0.773  | 0.744    | 0.721    | 0.690    |
|     |           | (0.855)   | (0.907)| (0.921)  | (0.957)  | (1.009)  |
|     | $N^{-1/5}$ | 0.875     | 0.870  | 0.858    | 0.847    | 0.831    |
|     |           | (0.989)   | (1.045)| (1.118)  | (1.161)  | (1.234)  |

6.2 Performance of test

The test statistic $T_{\lambda,d}$ has asymptotic reference distribution given in Theorem 5.1, and the statistic $T_{N,d}$ has the distribution provided by Wang and Phillips (2016). Both of these limiting distributions are of complex form, and are not easy to use in practice. In the case of a LM assumption for the regressor variables $x_k$, the limit distribution of $T_{N,d}$ depends on the differencing parameter $d$. The limit distribution of both statistics involve the term $d^2_{(0)}$, which does not have a simple form and depends on $x$ through $\sigma^2_N$.

Under appropriate conditions, we can approximate the sampling distribution of test statistics through the use of subsampling Politis and Romano (1994), which can make the testing procedure more feasible for use with real applications. This allows us to be able to study the properties of test statistic under both long memory and semi-long memory input shocks to the regressor. The crux of the method is to recompute the test statistic on smaller blocks or “subsamples” of the observed data to construct the empirical distribution of the subsample test estimates, and thus find the critical values.

We use blocks of size $b$ of consecutive observations as subsamples. Define the following vectors for each block, $x_{b,t} := (x_t, ..., x_{t+b-1})$ and $y_{b,t} := (y_t, ..., y_{t+b-1})$ for $t = 1, ..., N - b + 1$. Based on the subsample $x_{b,t}$ and $y_{b,t}$, the test statistic is:

$$T_{b,t} := \int_{\mathbb{R}} \left\{ \sum_{a=t}^{t+b-1} K \left[ \frac{x_{b,a} - x}{h} \right] \left[ y_{b,a} - g(x_{b,a}, \hat{\theta}_{b,t}) \right] \right\}^2 \pi(x)dx,$$

where $\hat{\theta}_{b,t} = \hat{\theta}_b(x_{b,t}, y_{b,t})$ is a parametric estimator of $\theta_0$ based on the information in block
Then, the subsampling estimator \( \hat{F}_{\lambda,b}(x) \) of the distribution function \( F_{\lambda}(x) \) with \( F_{\lambda}(x) = \text{Prob}\{(\sqrt{N \lambda d}h)^{-1}T_N \leq x\} \) can be defined as

\[
\hat{F}_{\lambda,b}(x) = \frac{1}{N-b+1} \sum_{t=1}^{N-b+1} 1\left\{ \frac{1}{\sqrt{b \lambda d h}} T_{b,t} \leq x \right\},
\]

where \( \lambda_b \) and \( h_b \) are both dependent on the block size \( b \). Under an assumption of long memory input shocks to the regressors, we remove the tempering parameter.

In this section, we conduct a Monte Carlo to find the empirical size of the test statistic based on subsamples. The design is similar to the previous Monte Carlo exercise of Section 6.1. To keep the computational cost feasible, we assume the sample size is \( N = 500 \) and the number of replications from the Monte Carlo is \( R = 2000 \). We use several block sizes, \( b = \{0.5N^{0.5}, N^{0.5}, 2N^{0.5}, 4N^{0.5}\} \). The null hypothesis to be tested can be stated as,

\[
H_0 : f(x, \theta) = \theta_0 + \theta_1 x. \tag{19}
\]

The true values of \((\theta_0, \theta_1)\) are set to \((0, 1)\). For nonparametric function estimation we used a Gaussian kernel with \( \pi(x) = 1 \) for \(-100 \leq x \leq 100\) and 0 otherwise such that Assumption 5.1 holds.

Monte Carlo distributions of test statistics for both LM and SLM under the null hypothesis are given in Figure 3. We observe that the histograms under SLM are almost overlapped unlike the histograms under LM. In order to study the empirical size of test, we generate data under the null hypothesis \( H_0 \) and find the proportion of times where \( T_{\lambda,d} \) falls above the \((1 - \alpha)\)-th quantiles of its subsampling distribution, where \( \alpha \) is the nominal significance level of Type I error. The results are given in Table 3.

We observe that the size of the test is not maintained. The Type I error rate is large for all the cases and both test statistics, although under SLM regressors, the values are smaller. Additionally, as bandwidth \( h \) and tempering parameter \( \lambda \) increase, the values of the size of the test become smaller, but are still not close to the nominal level. When \( d = 0 \), the values of the size of the test between LM and SLM processes are equal. Since the values of empirical size of the test are significantly large, it is meaningless to study the power of the test with subsampling technique, even though the power is expected to be large based on Remark 5.3. Whether the cause of this poor performance is due to a subtle scaling issue or some other problem is an open question. If some modification of one or both of the statistics would lead to tests with the ability to maintain size, then the power of these procedures would become an issue worth addressing.
Table 3: Empirical test sizes. The given values under \(d\) correspond to the 4 block sizes in the manuscript; i.e. \([0.5N^{0.5}], [N^{0.5}], [2N^{0.5}], [4N^{0.5}]\).

| Regressor | Bandwidth | Levels | \(d = 0\) | \(d = 0.1\) | \(d = 0.2\) | \(d = 0.3\) | \(d = 0.4\) |
|-----------|-----------|--------|------------|------------|------------|------------|------------|
| **LM**    | 0.01      | 0.939, 0.884 | 0.998, 0.994 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.05      | 1.000, 0.998 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.10      | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
| **N^{-1/3}** | 0.01 | 0.997, 0.970 | 0.999, 0.991 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.10      | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
| **N^{-1/5}** | 0.05 | 0.997, 0.970 | 0.999, 0.991 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.10      | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
| **SLM1**  | 0.01      | 0.939, 0.884 | 0.996, 0.997 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.05      | 1.000, 0.998 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.10      | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
| **SLM3**  | 0.01      | 0.939, 0.884 | 0.996, 0.997 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.05      | 1.000, 0.998 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
|           | 0.10      | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 | 1.000, 1.000 |
Figure 3: Comparison of Monte Carlo histograms of test statistic under the null hypothesis $H_0$ between LM (left) and SLM (right) cases under the assumption of $\rho_{\xi,\epsilon} = 0.5$ for different values of $d$. We assume the bandwidth is $h = N^{-1/3}$ and the tempering parameter for the SLM is $\lambda = N^{-1/6}$.

7 Application: Carbon Kuznets Curve

The Carbon Kuznets Curve (CKC) relates the per capita CO$_2$ emissions of countries to their per capita GDP. Basically, the CKC hypothesis states that there is an inverted-U shaped relationship between economic activities and per capita CO$_2$ emissions, where the upward slope of the CKC can be explained by the increase in natural resources depletion as economic activities grow, and the downward slope of the CKC results from a reduction in the emission of air pollutants as the country continues to develop technological advance and stricter regulatory policies; see Chan and Wang (2015) and references therein.

It is believed that the GDP variable has nonstationary behavior over time, and the assumption of exogeneity may not hold because of measurement error and other sources of errors; see Wang et al. (2018) for a related discussion. In the usual CKC formulation, the logarithm of per capita CO$_2$ emissions and GDP are assumed to be integrated processes, and there exists a nonlinear link function, see Müller-Fürstenberger and Wagner (2007). Here, we consider 3 countries (Belgium, Denmark, and France), which are shown in Figure 4 and are given in Chan and Wang (2015) and Piaggio and Padilla (2012). The annual data set is from 1950 to 2008 and contains 59 observations. The CO$_2$ emission data come from the Carbon Dioxide Information Analysis Center (Boden et al. 2009), and the GDP data come from Maddison (2003).
The range of our data do not exactly match with those of Chan and Wang (2015) as they modified the data set by transformations. We fit an ARTFIMA model with $p = q = 0$ to the data set using the artfima package from the statistical software R developed by Sabzikar et al. (2019). The Whittle estimators of fitted parameters $d$ and $\lambda$ for $\log(x_k)$ and $\log(y_k)$ of 3 countries are given in Table 4. Also, we fit an ARFIMA model with $p = q = 0$ to the data. The values of MSE for fitting ARFIMA model and ARTFIMA model are given in Table 4. We observe that the MSEs based on fitting the ARTFIMA model are much smaller than fitting the ARFIMA model. Thus, we use the test statistic under the assumption of semi-long memory input shocks to the regressors given in (14). All the values of $d$ given in Table 4 are substantially greater than 0.5, and our proposed test statistic that assumes a semi-long memory regressor process is applicable for $d > 0.5$. Note that this may be a larger problem for the test statistic given by Wang and Phillips (2016) where $d$ has to belong to the interval $(0, 0.5)$.

Table 4: The Whittle estimators of fitted parameters from an ARTFIMA model for 3 countries as well as the MSE of fitting an ARFIMA model versus an ARTFIMA model.

| Country | Variable | $\lambda$ | $d$ | ARFIMA | ARTFIMA |
|---------|----------|-----------|-----|--------|---------|
| Belgium | $\log(x_k)$ | 0.129 | 1.086 | 0.977 | 0.411 |
|         | $\log(y_k)$ | 0.118 | 1.000 | 0.235 | 0.026 |
| Denmark | $\log(x_k)$ | 0.154 | 1.142 | 0.858 | 0.350 |
|         | $\log(y_k)$ | 0.124 | 1.128 | 1.415 | 0.286 |
| France  | $\log(x_k)$ | 0.138 | 1.093 | 0.976 | 0.395 |
|         | $\log(y_k)$ | 0.071 | 1.133 | 1.013 | 0.188 |

The plots in Figure 4 (top) exhibit some of the quadratic behavior anticipated by the theory of CKC curves. We formulate the following hypotheses:

\[
H_1 : z_k = \theta_0 + \theta_1 e_k + u_k,
\]

\[
H_2 : z_k = \theta_0 + \theta_1 e_k + \theta_2 e_k^2 + u_k,
\]

where $y_k$ and $x_k$ are per capita emission of CO$_2$ and GDP in year $k = 1, \ldots, 59$, $z_k = \log(y_k)$, and $e_k = \log(x_k)$. We set the bandwidth at $h = \{1/\sqrt{N}, 1/N\}$ and give the related plots in Figure 4 (bottom). We consider the Gaussian kernel with $\pi(x) = 1$ for $-100 \leq x \leq 100$ and 0 otherwise. We use the subsampling technique described in Section 6.2 to find the p-values for the given hypotheses. We consider the following block sizes: $b = \{2N^{0.5}, 4N^{0.5}, 6N^{0.5}\}$.

The corresponding p-values for both the bandwidths are displayed in Table 5. For most block sizes and both hypothesized models the p-values are very small, rejecting both straight
Figure 4: Plots of log(CO$_2$) versus log(GDP) for Belgium, Denmark, and France based on the observed values (top) and selected bandwidths (bottom).

line and quadratic models. When the block size is increased to the largest size considered, the quadratic model fares better under the assumption of $h = 1/N$, as does the straight line model for France only, which is not clear given the scatterplots of Figure 4 (top). Overall, we conclude that the test has poor performance, this time in an actual application.
Table 5: Approximate p-values based on subsampling distribution.

| $h$   | Block Size | Belgium | Denmark | France |
|-------|------------|---------|---------|--------|
|       |            | $H_1$   | $H_2$   |        |
| $1/\sqrt{N}$ | $2N^{0.5}$ | < 0.0001 | < 0.0001 | < 0.0001 | < 0.0001 | < 0.0001 |
|       | $4N^{0.5}$ | < 0.0001 | < 0.0001 | < 0.0001 | < 0.0001 | < 0.0001 |
|       | $6N^{0.5}$ | < 0.0001 | 0.300    | < 0.0001 | < 0.0001 | < 0.0001 |
| $1/N$ | $2N^{0.5}$ | < 0.0001 | < 0.0001 | < 0.0001 | < 0.0001 | < 0.0001 |
|       | $4N^{0.5}$ | < 0.0001 | 0.225    | < 0.0001 | < 0.0001 | < 0.0001 |
|       | $6N^{0.5}$ | < 0.0001 | 0.800    | < 0.0001 | 0.600    | 0.500    | 0.300 |

8 Conclusion

In this paper, we have given a pivotal test statistic for the non-parametric cointegrating regression function, which extends the earlier work of Wang and Phillips (2016). The limit theory for the test involves local time of standard Brownian motion and is free of any unknown parameters. This development has relied on the idea of tempering (in the strong sense) such that there are semi-long memory input shocks to the regressor. In addition, the tempering of the coefficients in the regressors can also extend the range of fractional differencing parameter $d$ from $(0, 1/2)$ to any positive value on the real line.

We adopted a strategy of subsampling to approximate the reference distribution of our test statistic, as well as that of Wang and Phillips (2016), which increases the applicability of tests based on these statistics. The existence of limiting distributions for the statistics is an important part of ensuring that a subsampling approach is justified. Our work has also indicated that, for the most part, tests based on these statistics are unable to maintain size, which calls their use in applications into question. Additional investigation is required for a modified test statistic such that it can have the correct size. This is beyond the scope of this article, and hence we leave it for future work.

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Appendix A: Technical Details and Proofs

A.1 Preliminaries and Lemmas

We can decompose \( x_k \) as follows:

\[
x_k \equiv \sum_{j=1}^{k} X_{d \lambda}(j) = \sum_{j=1}^{k} \sum_{i=-\infty}^{j} \xi(i) \phi_N(j-i), \quad \phi_N(k) := e^{-\lambda k} b_d(k)
\]

\[
\begin{align*}
&= x_s + \sum_{j=s+1}^{k} \sum_{i=-\infty}^{s} \xi(i) \phi_N(j-i) + \sum_{j=s+1}^{k} \sum_{i=s+1}^{j} \xi(i) \phi_N(j-i) \\
&=: x_{s,k} + x'_{s,k}, \quad s < k,
\end{align*}
\]

such that \( x_{s,k} = x_s + \sum_{j=s+1}^{k} \sum_{i=-\infty}^{s} \xi(i) \phi_N(j-i) \) and it depends on \( \{\ldots, \xi(s-1), \xi(s)\} \), and \( x'_{s,k} = \sum_{j=1}^{k-s} \sum_{i=1}^{j} \xi(i+s) \phi_N(j-i) \). If we define \( a_N := \sum_{j=0}^{N} \phi_N(j) \), then we can show \( a_N \asymp O(\lambda^{-d}) \) as \( N \lambda \to \infty \) and recall that \( \lambda \) is a sample size dependent parameter; see Lemma A.5 part (b) in [Sabzikar et al. (2020)].

**Definition A.1.** Let \( \Omega_N(\eta) \equiv \{(l,k) : \eta N \leq k \leq (1-\eta)N, k + \eta N \leq l \leq N \} \) for all \( 0 \leq k < l \leq N \) and \( N \geq 1 \), where \( 0 < \eta < 1 \). A random array \( \{x_{k,N}\}_{k \geq 1, N \geq 1} \) is called **strong smooth** if there exists a sequence of constants \( d_{l,k,N} > 0 \) and a sequence of \( \sigma \)-fields \( \mathcal{F}_{k,N} \) (define \( \mathcal{F}_{0,N} = \sigma\{\phi, \Omega \} \), the trivial \( \sigma \)-field) such that

(a) For some \( m_0 > 0 \) and \( C > 0 \), \( \inf_{(l,k) \in \Omega_N(\eta)} d_{l,k,N} \geq \eta^{m_0}/C \) as \( N \to \infty \),

(i) \( \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_{l=(1-\eta)N}^{N} (d_{l,0,N})^{-1} = 0 \),

(ii) \( \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \max_{0 \leq k \leq (1-\eta)N} \sum_{l=k+1}^{k+\eta N} (d_{l,k,N})^{-1} = 0 \), and

(iii) \( \limsup_{N \to \infty} \frac{1}{N} \max_{0 \leq k \leq N-1} \sum_{l=k+1}^{N} (d_{l,k,N})^{-1} < \infty \).

(b) \( x_{k,N} \) is adapted to \( \mathcal{F}_{k,N} \), and conditional on \( \mathcal{F}_{k,N} \), \( (x_{l,N} - x_{k,N})/d_{l,k,N} \) has a density \( h_{l,k,N}(x) \) which is uniformly bounded by a constant \( K \) and

\[
\lim_{\kappa \to 0} \lim_{N \to \infty} \sup_{(l,k) \in \Omega_N[1/(2m_0)]} \sup_{|u| \leq \kappa} |h_{l,k,N}(u) - h_{l,k,N}(0)| = 0.
\]

**Lemma A.1.** Let’s define \( \eta_i = (\xi_i, \epsilon_i)' \) where \( \mathbb{E}(\eta_0) = 0 \) and \( \mathbb{E}|\eta_0|^2 < \infty \). Let \( \mathcal{F}_s = \sigma(\eta_s, \eta_{s-1}, \ldots) \), \( \Lambda(.) \) is a real function of its components where \( \mathbb{E}(\Lambda(\eta_1)) = 0 \), and \( g(x) \) is a bounded function such that \( \int_{\mathbb{R}} |g(x)|dx < \infty \).
(i) For a constant $H_0$, we have

$$\sup_x \mathbb{E} \left| \sum_{k=1}^N g \left[ \frac{x_k - x}{h} \right] \right|^m \leq H_0^m (m+1)! \left( \sqrt{N} \lambda^d h \right)^m, \quad m \geq 1. \tag{A.1}$$

(ii) As $\{\sqrt{N} \lambda^d h\}^{-1} \to 0$ and $N \lambda \to \infty$, we have

$$\sup_x \mathbb{E} \left| \sum_{k=1}^N \Lambda(\eta_{k-j}) g \left[ \frac{x_k - x}{h} \right] \right|^2 \leq C \mathbb{E} \Lambda^2(\eta_1) \sqrt{N} \lambda^d h \left\{ 1 + h \sqrt{j} \right\}. \tag{A.2}$$

(iii) If we define $u_k := \sum_{j=0}^\infty \psi_j \eta_{k-j}$, then

$$\mathbb{E} \left| \sum_{k=1}^N u_k g \left( \frac{x_k}{h} \right) \right|^2 \leq C \mathbb{E} \|\eta_1\|^2 \sqrt{N} \lambda^d h. \tag{A.3}$$

(iv) If we define $u_{k,m_0} := \sum_{j=m_0}^\infty \psi_j \eta_{k-j}$ where $\psi_j = (\psi_{1j}, \psi_{2j})$ and $\Lambda_i = (\Lambda_1(\eta_i), \Lambda_2(\eta_i))$, then we have

$$\sup_x \mathbb{E} \left| \sum_{k=1}^N u_{k,m_0} g \left[ \frac{x_k - x}{h} \right] \right|^2 \leq C \mathbb{E} \|\Lambda_1\|^2 \sqrt{N} \lambda^d h \left\{ \sum_{j=m_0}^\infty j^{1/4}(|\psi_{1j}| + |\psi_{2j}|) \right\}^2. \tag{A.4}$$

Proof. For the proof of expression (A.1), see Lemma 5.1. of Chan and Wang (2014). To prove (A.2), note that some of the details are very similar to the ones given for the proof of expression (8.11) in Lemma 8.2. of Wang and Phillips (2016), and therefore we only give an outline for the different parts as follows. Let

$$\Delta_N \equiv \left| \sum_{k=1}^N \Lambda(\eta_{k-j}) g \left[ \frac{x_k - x}{h} \right] \right|^2 \leq 2 \sum_{k=A_0}^N \Lambda(\eta_{k-j}) g \left[ \frac{x_k - x}{h} \right]^2 + C \left( \sum_{k=1}^{A_0} |\Lambda(\eta_{k-j})| \right)^2$$

$$= 2 \left( \sum_{k=A_0}^N \sum_{l=1,k-l<A_0}^N + 2 \sum_{k=A_0}^{N-1} \sum_{l=k+A_0}^N \right) \Lambda(\eta_{k-j}) \Lambda(\eta_{l-j})$$

$$\times g \left[ \frac{x_k - x}{h} \right] g \left[ \frac{x_l - x}{h} \right] + C \left( \sum_{k=1}^{A_0} |\Lambda(\eta_{k-j})| \right)^2$$

$$= \Delta_{1N} + \Delta_{2N} + \Delta_{3N},$$

where $A_0$ is a positive constant. Then, we have
\[
\sup_x \mathbb{E} \Delta_{1N} \leq C \mathbb{E} \Lambda^2(\eta_1) \sqrt{N} \lambda^d h,
\]
\[
\sup_x \mathbb{E} \Delta_{2N} \leq C \mathbb{E} \Lambda^2(\eta_1) h^2 \sum_{k=A_0}^{N-1} \sum_{l=k+1}^{N \wedge (k+j)} d_k^{-1} \left( \sum_{l=k+1}^{j} \phi_N(k) \sum_{l=k+j}^{N} d_{l-k}^{-2} \right)
\leq C \mathbb{E} \Lambda^2(\eta_1) \sqrt{N} \lambda^d h^2 \left( \frac{j}{d_j} + \sum_{k=0}^{j} |\phi_j(k)| \sum_{l=k+j}^{N} \frac{\lambda_{l-k}^{2d}}{l-k} \right)
\leq C \mathbb{E} \Lambda^2(\eta_1) \sqrt{N} \lambda^d h^2 \left( \frac{j}{d_j} + \sum_{k=0}^{j} |\phi_j(k)| \right)
\leq C \mathbb{E} \Lambda^2(\eta_1) \sqrt{N} \lambda^d h^2 \left( \frac{\lambda_j^d}{\sqrt{j}} j + \lambda_j^{-d} \right)
\leq C \mathbb{E} \Lambda^2(\eta_1) \sqrt{N} \lambda^d h^2 \sqrt{j} \quad \text{for} \quad j \geq 1, \quad \text{and}
\]
\[
\sup_x \mathbb{E} \Delta_{3N} \leq C \mathbb{E} \Lambda^2(\eta_1).
\]

Eventually, we can conclude expression (A.2). Expression (A.3) easily follows from expression (A.2), and expression (A.4) can be followed by a similar path from the proof of expression (8.13) in Lemma 8.2. of Wang and Phillips (2016).

\section{A.2 Proofs of Theorems}

**Proof of Theorem 4.1.** First, in order to establish the asymptotic distribution of \([\hat{f}(x) - f(x)]\), we decompose it as follows:

\[
\hat{f}(x) - f(x) = \sum_{k=1}^{N} u_k K[\{x_k - x\}/h] + \sum_{k=1}^{N} [f(x_k) - f(x)] K[\{x_k - x\}/h] + \sum_{k=1}^{N} K[\{x_k - x\}/h].
\]

(A.5)

Then, only the first term in expression (A.5) determines the asymptotic distribution of \([\hat{f}(x) - f(x)]\) as the second term is negligible under mild conditions. To prove the self-normalized expression (10), we let \(\Theta_{1N} := \sum_{k=1}^{N} u_k K[\{x_k - x\}/h] \), \(\Theta_{2N} := \sum_{k=1}^{N} (f(x_k) - f(x)) K[\{x_k - x\}/h] \), and \(\Theta_{3N} := 1/\sum_{k=1}^{N} K[\{x_k - x\}/h] \). Now, we have

\[
\left\{ h \sum_{k=1}^{N} K_h(x_k - x) \right\}^{1/2} \left( \hat{f}(x) - f(x) \right)
\]
\[
\frac{\sum_{k=1}^{N} u_k K[(x_k - x)/h]}{\left\{\sum_{k=1}^{N} K[(x_k - x)/h]\right\}^{1/2}} + \frac{\sum_{k=1}^{N} [f(x_k) - f(x)] K[(x_k - x)/h]}{\left\{\sum_{k=1}^{N} K[(x_k - x)/h]\right\}^{1/2}} = \Theta_1N + \Theta_2 \sqrt{\Theta_3N}
\]

where \( \Theta_2 \sqrt{\Theta_3N} \rightarrow P 0 \) from Proposition A.8. Also, the last line in (A.6) comes from (a) Slutsky’s theorem, (b) \( \left\{\sqrt{N\lambda^d h}\right\}^{-1/2} \sum_{k=1}^{N} K[h(x_k - x)/h] \rightarrow D_1(L_B(t, 0))^2 \), and (c) the following result:

\[
\left\{\sqrt{N\lambda^d h}\right\}^{-1/2} \sum_{k=1}^{N} \left[\frac{x_k - x}{h}\right] u_k \rightarrow D_0 N(0, 1)
\]

The convergence in distribution in (A.7) comes from Proposition A.9 part (iv). Expression (10) easily implies expression (9) and the fact that

\[
\left\{\sqrt{N\lambda^d h}\right\}^{-1/2} \left\{\frac{h}{h} \sum_{k=1}^{N} K_h(x_k - x)\right\}^{1/2} = \frac{d_0}{d_1} N(0, 1),
\]

thus the proof is complete. \( \square \)

**Proof of Theorem 5.1.** First, \( T_N \) can be decomposed into three terms under \( H_0 \), i.e. \( y_k = g(x_k, \theta_0) + u_k \), as follows

\[
T_N = \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K_h\left[\frac{x_k - x}{h}\right] \left[g(x_k, \theta_0) + u_k - g(x_k, \hat{\theta}_N)\right]\right\}^2 \pi(x) dx
\]

\[
= T_N^I + T_N^II + T_N^III
\]

such that

\[
T_N^I := \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K_h\left[\frac{x_k - x}{h}\right] u_k\right\}^2 \pi(x) dx
\]
\[ T_{N}^{\text{II}} := \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \left( g(x_k, \theta_0) - g(x_k, \hat{\theta}_N) \right) \right\}^2 \pi(x) \, dx, \quad \text{and} \]

\[ T_{N}^{\text{III}} := 2 \int_{\mathbb{R}} \left\{ \sum_{1 \leq k < j \leq N} K \left[ \frac{x_k - x}{h} \right] K \left[ \frac{x_j - x}{h} \right] u_k \left( g(x_j, \theta_0) - g(x_j, \hat{\theta}_N) \right) \right\} \pi(x) \, dx. \]

Based on Assumption 5.2, there exists \( g_1(x) \) such that for each \( \theta, \theta_0 \in \Omega_0 \), we have \( |g(x, \theta) - g(x, \theta_0)| \leq C||\theta - \theta_0||g_1(x) \). Therefore,

\[ T_{N}^{\text{II}} \leq C||\hat{\theta}_N - \theta_0||^2 \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] g_1(x_k) \right\}^2 \pi(x) \, dx = o_P\left( \sqrt{N}\lambda^d h \right). \quad (A.8) \]

Expression (A.8) comes from (a) assumption 5.3, \( ||\hat{\theta}_N - \theta_0|| = o_P(\{\sqrt{N}\lambda^d h\}^{-1/2}) \) and (b) the following result:

\[ \left\{ \sqrt{N}\lambda^d h \right\}^{-2} \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] m(x_k) \right\}^2 \pi(x) \, dx \rightarrow_D \frac{d^2_{(1)}}{d_1} L^2_B(1, 0), \quad (A.9) \]

where \( d^2_{(1)} = \int_{\mathbb{R}} m^2(x) \pi(x) \, dx \int_{\mathbb{R}} K(s) \, ds \) and \( d_1 = \int_{\mathbb{R}} K(s) \, ds \). Expression (A.9) can be obtained by a similar path from the expression (7.12) in Proposition 7.3 of Wang and Phillips (2016). Also, note that

\[ \sum_{1 \leq k < j \leq N} \tilde{u}_{1k} \tilde{u}_{1j} \left\{ \int_{\mathbb{R}} K \left[ \frac{x_k - x}{h} \right] K \left[ \frac{x_j - x}{h} \right] \pi(x) \, dx \right\} = o_P\left( \sqrt{N}\lambda^d h \right), \quad (A.10) \]

which can be followed by a similar path from the proof of expression (8.18) in Lemma 8.3 of Wang and Phillips (2016). Therefore, we can conclude \( T_{N}^{\text{III}} = o_P(\sqrt{N}\lambda^d h) \). Now, it suffices to show \( \{\sqrt{N}\lambda^d h\}^{-1} T_{N}^{\text{I}} \rightarrow_D d^2_{(0)} L_B(1, 0) \). For a fixed \( A > 0 \), we define

\[ \tilde{\eta}_i = \eta_i I(||\eta_i|| \leq A), \quad \tilde{\eta}_i = \hat{\eta}_i - \mathbb{E}(\hat{\eta}_i), \quad \hat{\eta}_i = \eta_i - \tilde{\eta}_i, \quad \text{and} \]

\[ \tilde{\eta}_i = \sum_{j=0}^{\infty} \psi_j \tilde{\eta}_k-j, \quad \hat{\eta}_i = \sum_{j=0}^{\infty} \psi_j \hat{\eta}_k-j. \]

From expression (A.4) of Lemma A.1, we have
Now based on expression (A.4) in Lemma A.1, we have
\[ \mathbb{E}\left[ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \tilde{u}_k \right]^2 \pi(x) dx \leq C \mathbb{E} ||\eta||^2 \sqrt{N} \lambda^d h \leq C \mathbb{E} ||\eta||^2 I(||\eta|| > A) \sqrt{N} \lambda^d h, \]

where \( \mathbb{E}(\tilde{\eta}) = 0. \) Note that \( \mathbb{E} ||\eta||^2 I(||\eta|| > A) \to 0 \) as \( A \to \infty \) and \( u_k = \tilde{u}_k + \hat{u}_k, \) therefore it suffices to show
\[
\left\{ \sqrt{N} \lambda^d h \right\}^{-1} T_N^I := \left\{ \sqrt{N} \lambda^d h \right\}^{-1} \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \tilde{u}_k \right\}^2 \pi(x) dx \to D d_2^2 L_B(1, 0).
\]

Let \( \bar{u}_{1k} = \sum_{j=0}^{m_0} \psi_j \tilde{\eta}_{k-j} \) and \( \bar{u}_{2k} = \bar{u}_k - \bar{u}_{1k} \). Then, we have \( T_N^I = T_N^{I(a)} + T_N^{I(b)} + T_N^{I(c)} \) where
\[
T_N^{I(a)} = \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \bar{u}_{1k} \right\}^2 \pi(x) dx,
\]
\[
T_N^{I(b)} = \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} \left[ \frac{x_k - x}{h} \right] \bar{u}_{2k} \right\}^2 \pi(x) dx, \quad \text{and}
\]
\[
T_N^{I(c)} = 2 \int_{\mathbb{R}} \left\{ \sum_{1 \leq k < j \leq N} K \left[ \frac{x_k - x}{h} \right] K \left[ \frac{x_j - x}{h} \right] \bar{u}_{1k} \bar{u}_{2k} \right\} \pi(x) dx.
\]

Now based on expression (A.4) in Lemma A.1 we have
\[
\mathbb{E}(T_N^{I(b)}) \leq C \sup_x \mathbb{E} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \bar{u}_{2k} \right\}^2 \]
\[
\leq C \mathbb{E} ||\eta||^2 \sqrt{N} \lambda^d h \left\{ \sum_{k=m_0}^{\infty} k^{1/4} (|\psi_{1k}| + |\psi_{2k}|) \right\}^2, \quad \text{as} \quad N \to \infty \quad \text{and} \quad m_0 \to 0.
\]

Under Assumption 4.1 \( \sum_{k=m_0}^{\infty} k^{1/4} (|\psi_{1k}| + |\psi_{2k}|) < \infty. \) Therefore, we conclude \( T_N^{I(b)} = o_P(\sqrt{N} \lambda^d h). \) Based on expression (A.10), \( T_N^{I(c)} = o_P(\sqrt{N} \lambda^d h). \) We can furtherly write \( T_N^{I(a)} \) as
\[
T_N^{I(a)} = \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \bar{u}_{1k} \right\}^2 \pi(x) dx = \mathbb{E}(\bar{u}_{10}^2) \int_{\mathbb{R}} K^2 \left[ \frac{x_k - x}{h} \right] \pi(x) dx
\]
\[
+ \sum_{k=1}^{N} \left( \bar{u}_{1k}^2 - \mathbb{E}(\bar{u}_{1k}^2) \right) \int_{\mathbb{R}} K^2 \left[ \frac{x_k - x}{h} \right] \pi(x) dx.
\]
\[
+ 2 \sum_{1 \leq k < j \leq N} \tilde{u}_{1k} \tilde{u}_{1j} \int_{\mathbb{R}} K \left[ \frac{x_k - x}{h} \right] K \left[ \frac{x_j - x}{h} \right] \pi(x) dx \\
= R_N^{I(a)} + R_N^{II(a)} + R_N^{III(a)}.
\]

From
\[
\sum_{k=1}^{N} \left( \tilde{u}_{1k}^2 - \mathbb{E}(\tilde{u}_{1k}^2) \right) \int_{\mathbb{R}} K^2 \left[ \frac{x_k - x}{h} \right] \pi(x) dx = O_P \left( \{\sqrt{N} \lambda^d h\}^{1/2} \right),
\]
which can be followed by a similar path from the expression (8.17) in Lemma 8.3 of Wang and Phillips (2016) and expression (A.10), we can conclude
\[
|R_N^{II(a)}| + |R_N^{III(a)}| = o_P \left( \sqrt{N} \lambda^d h \right).
\]
Eventually, by virtue of \( \mathbb{E}(\tilde{u}_{10}^2) \rightarrow \mathbb{E}(\tilde{u}_{0}^2) \), we only need to prove
\[
\{\sqrt{N} \lambda^d h\}^{-1} R_N^{I(a)} \rightarrow_D \mathbb{E}(\tilde{u}_{0}^2) \int_{\mathbb{R}} K^2(x) dx \int_{\mathbb{R}} \pi(s) ds L_B(1, 0).
\]
If we define \( \int_{\mathbb{R}} g_h(y) dy = \int_{\mathbb{R}} K^2(z) dz \int_{\mathbb{R}} \pi(x) dx \) such that \( |g_h(y)| \leq C \int_{\mathbb{R}} |\pi(x)| dx < \infty \), then (A.11) could be followed from the proof of Theorem 2.1 in Wang and Phillips (2009a), and we omit it. Therefore, the proof is complete. \( \square \)

**Proof of Theorem 5.2.** The proof is similar to the proof of Theorem 3.2 in Wang and Phillips (2016). Therefore, we give an outline. We first decompose \( T_N \) under \( H_A \), i.e. \( y_k = g(x_k, \theta_0) + \rho_N m(x_k) + u_k \) as follows:
\[
T_N = \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] \left[ g(x_k, \theta_0) + \rho_N m(x_k) + u_k - g(x_k, \hat{\theta}_N) \right] \right\}^2 \pi(x) dx
\]
\[
= T_N^I + T_N^{II} + T_N^{III} + 2 \rho_N T_N^{IV} + \rho_N^2 T_N^V,
\]
where
\[
T_N^{IV} := \int_{\mathbb{R}} \left\{ \sum_{1 \leq k < j \leq N} K \left[ \frac{x_k - x}{h} \right] K \left[ \frac{x_j - x}{h} \right] m(x_k) \left[ u_j - \left( g(x_j, \hat{\theta}_N) - g(x_j, \theta_0) \right) \right] \right\} \pi(x) dx,
\]
\[
T_N^{V} := \int_{\mathbb{R}} \left\{ \sum_{k=1}^{N} K \left[ \frac{x_k - x}{h} \right] m(x_k) \right\}^2 \pi(x) dx,
\]
such that \( |T_N^{IV}| \leq [T_N^I + T_N^{II} + T_N^{III}]^{1/2} \times [T_N^V]^{1/2} \) by Hölder's inequality. The remained terms have been already defined in the proof of Theorem 5.1. From the proof of Theorem 5.1,
Theorem 1. Let \( x_{k,N} = x_k / d_N \). A random array \( \{x_{k,N}\}_{k \geq 1, N \geq 1} \) is called strong smooth.

Proof. We verify this proposition from the Definition A.1. Assume \( \{x_{k,N}\}_{k \geq 1, N \geq 1} \) is a random triangular array such that \( x_{0,N} \equiv 0 \). Let \( d_{l,k} \equiv \lambda_{l-k}^{-d} (l-k)^{1/2} \), therefore

\[
d_{l,k} \equiv \frac{d_{l,k}}{d_N} \propto \left( \frac{\lambda_{l-k}}{\lambda_N} \right)^{-d} \sqrt{\frac{l-k}{N}}, \quad N\lambda_N \to \infty.
\]

Recall that \( \lambda_N \equiv \lambda > 0 \) is a sample size dependent parameter.

(a) \( \inf_{(l,k) \in \Omega_{N}(\eta)} d_{l,k} \equiv \inf_{(l,k) \in \Omega_{N}(\eta)} \lambda_{l-k}^{-d} (l-k)^{1/2} \sqrt{(l-k)/N} \geq \sqrt{\frac{\eta N}{N}} \lambda_{\eta N/N}^{-d} > 0 \).

(i) \( \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_{l=(1-\eta)N}^{N} \left\{ \left( \frac{\lambda_l}{\lambda_N} \right)^{-d} \sqrt{l/N} \right\}^{-1} \)

\[
\leq \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \int_{(1-\eta)N}^{N} \left( \frac{\lambda_l}{\lambda_N} \right)^{-d} \sqrt{l} \frac{dl}{l} \]

\[
\leq \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \int_{(1-\eta)N}^{N} \left( \frac{\sqrt{N} \lambda_{N}^{-d} - \sqrt{(1-\eta)N} \lambda_{(1-\eta)N}^{-d}} \right) = 0
\]

(ii) \( \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \max_{0 \leq k \leq (1-\eta)N} \sum_{l=k+1}^{k+\eta N} \left\{ \left( \frac{\lambda_{l-k}}{\lambda_N} \right)^{-d} \sqrt{(l-k)/N} \right\}^{-1} \)

\[
\leq \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{N} \max_{0 \leq k \leq (1-\eta)N} \int_{k+1}^{k+\eta N} \frac{\lambda_{l-k}}{\sqrt{l-k}} \frac{dl}{l-k} \]
\begin{align*}
\lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{\sqrt{N\lambda_N \delta}} \left\{ \sqrt{\eta N \lambda_N \delta} - 1 \right\} = 0
\end{align*}

(iii) \begin{align*}
\limsup_{N \to \infty} \frac{1}{N} \max_{0 \leq k \leq N-1} \sum_{l=k+1}^{N} \left\{ \left( \frac{\lambda_{l-k}}{\lambda_N} \right)^{-d} \sqrt{(l-k)/N} \right\}^{-1}
\leq \limsup_{N \to \infty} \frac{1}{\sqrt{N\lambda_N \delta}} \max_{0 \leq k \leq N-1} \int_{k+1}^{N} \frac{\lambda_{l-k}^d}{\sqrt{l-k}} \, dl
\leq \limsup_{N \to \infty} \frac{1}{\sqrt{N\lambda_N \delta}} \max_{0 \leq k \leq N-1} \left\{ \sqrt{N-k\lambda_N^d} - 1 \right\}
= \limsup_{N \to \infty} \left\{ 1 - \frac{1}{\sqrt{N\lambda_N \delta}} \right\} = 1 < \infty.
\end{align*}

(b) If we define \( f_{l,k,N}(t) = \mathbb{E}\left( e^{it[x_{l,N} - x_{k,N}]/d_{l,k,N}} \right) \), then
\begin{align*}
\sup_{x \in \mathbb{R}} \left| h_{l,k,N}(x) - n(x) \right| \leq C \int_{\mathbb{R}} \left| f_{t,k,N}(t) - e^{-t^2/2} \right| dt \to 0,
\end{align*}
where \( n(x) = e^{-x^2/2}/\sqrt{2\pi} \) is the density of standard normal (see proof of Corollary 2.2 in Wang and Phillips (2009a) for a similar argument). All the conditions given in Definition A.1 hold. Thus, \( x_{k,N} \) is a strong smooth array. The alternative way to prove this proposition is the following. Firstly, note that
\begin{align}
\frac{\lambda_N^d}{\sqrt{N}} \sum_{k=1}^{N} X_{d,\lambda_N}(k) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \zeta(k) + O_P(1), \tag{A.12}
\end{align}
where \( \{\zeta(j)\}_{j \in \mathbb{Z}} \) is a sequence of i.i.d. random variables with \( \mathbb{E}(\zeta_0) = 0 \) and \( \mathbb{E}(\zeta_0^2) = 1 \); see expression (A.7) in Sabzikar et al. (2020). Secondly, the triangular array \( N^{-1/2} \sum_{k=1}^{N} \xi(k) \) satisfies the strong smooth conditions; see Wang and Phillips (2009a). This together with (A.12) completes the proof of Proposition A.1. \( \square \)

**Proposition A.2.** Let
\begin{align*}
S_N(t) = \frac{1}{[\sqrt{N}h]^{3/2}} \sum_{k=1}^{\lfloor Nt \rfloor} u_k K\left[ \frac{x_k - x}{h} \right], \quad \text{and} \quad \psi_N(t) = \frac{1}{\sqrt{Nh}} \sum_{k=1}^{\lfloor Nt \rfloor} u_k^2 K^2\left[ \frac{x_k - x}{h} \right].
\end{align*}
For any fixed \( 0 \leq t \leq 1 \), \( S_N(t) \), \( S_N^2(t) \) and \( \psi_N(t) \) are uniformly integrable.
Proof. The proof is similar to the Proposition 7.3 in [Wang and Phillips (2009b)]; hence, we only give an outline. If we let

$$\psi''_N(t) = \frac{1}{\sqrt{Nh}} \sum_{k=1}^{[Nt]} K^2 \left[ \frac{x_k - x}{h} \right] \mathbb{E}(u_k^2),$$

for $0 \leq t \leq 1$ and define $r_y(t) := K^2 \left( \frac{y}{h} + t \right)$, then $\sup_N \mathbb{E}[\psi''_N(t)]^2 < \infty$ as follows

$$\mathbb{E}[\psi''_N(t)]^2 = \mathbb{E} \left\{ \frac{1}{\sqrt{Nh}} \sum_{k=1}^{[Nt]} K^2 \left[ \frac{x_k - x}{h} \right] \mathbb{E}(u_k^2) \right\}^2 \leq \frac{C}{Nh^2} \left\{ \sum_{k=1}^{N} \mathbb{E} K^4 \left[ \frac{x_k - x}{h} \right] + 2 \mathbb{E} \left\{ K^2 \left[ \frac{x_k - x}{h} \right] K^2 \left[ \frac{x_l - x}{h} \right] \right\} \right\}$$

$$\leq \frac{C}{Nh^2} \left\{ \sum_{k=1}^{N} \sup_y \mathbb{E} r_y^2 \left( \frac{x_{0,k}}{h} \right) + 2 \mathbb{E} \left\{ \sup_y \mathbb{E} r_y \left( \frac{x_{0,k}}{h} \right) \sup_y \mathbb{E} r_y \left( \frac{x'_{k,l}}{h} \right) \right\} \right\} \leq \frac{C}{Nh^2} \left\{ \sum_{k=1}^{N} \frac{h}{\sqrt{k}} + 2 \sum_{1 \leq k < l \leq N} \frac{h^2}{\sqrt{k} \sqrt{l - k}} \right\} < \infty,$$

uniformly on $N$. The last line could be followed by a similar path from the proof of Lemma 7.1 part (a) in [Wang and Phillips (2009b)]. Note that $\sup_{0 \leq t \leq 1} \mathbb{E}|\psi_N(t) - \psi''_N(t)|^2 = o(1)$, which can be verified by a similar argument given in Proposition A.6. These implies $\sup_N \mathbb{E}[\psi_N(t)]^2 < \infty$, and therefore $\psi_N(t)$ is uniformly integrable. Furtherly, we can write

$$|\psi_N(t) - S_N^2(t)| = \frac{2}{\sqrt{Nh}} \sum_{1 \leq k < l \leq [Nt]} u_k u_l K \left[ \frac{x_k - x}{h} \right] K \left[ \frac{x_l - x}{h} \right],$$

which implies $\sup_{0 \leq t \leq 1} \mathbb{E}|\psi_N(t) - S_N^2(t)| = o(1)$ and can be verified by a similar argument given in Proposition A.6. Therefore, $S_N^2(t)$ is uniformly integrable, which implies $S_N(t)$ is uniformly integrable.

Proposition A.3. Let $\Theta_{1N} := \sum_{k=1}^{N} u_k K \left[ \frac{x_k - x}{h} \right]$. Then, $\Theta_{1N} = O_P(\sqrt{Nh})^{1/2}$.

Proof. Based on the Proposition A.2 uniformly in $N$, we have $\mathbb{E}S_N^2(1) \leq C$, where $C$ is a generic constant. Therefore, based on the Markov’s inequality

$$P(\Theta_{1N} \geq \sqrt{Nh})^{1/2}) \leq \frac{\mathbb{E}(\Theta_{1N}^2)}{\sqrt{Nh}} = \mathbb{E}S_N^2(1) \leq C.$$

As a result, $\Theta_{1N} = O_P(\sqrt{Nh})^{1/2}$.

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\textbf{Proposition A.4.} Let $\Theta_{2N} := \sum_{k=1}^{N} [f(x_k) - f(x)] K[(x_k - x)/h]$. Then, $\Theta_{2N} = O_P(N h^{\gamma+1}/d_N)$, where $d_N = \sqrt{N}/\lambda^d$.

\textit{Proof.}

\[
\mathbb{E}[\Theta_{2N}] \leq \sum_{k=1}^{N} \mathbb{E}\left\{ |f(x_{k,N}d_N) - f(x)| K\left[ \frac{x_{k,N}d_N - x}{h} \right] \right\} 
= \sum_{k=1}^{N} \int_{\mathbb{R}} \left\{ |f(d_N d_{k,0,N} y) - f(x)| \times K\left( \frac{d_N d_{k,0,N} y}{h} - \frac{x}{h} \right) \right\} h_{k,0,N}(y)dy
\leq \frac{h}{d_N} \sum_{k=1}^{N} \frac{1}{d_{k,0,N}} \int_{\mathbb{R}} \left\{ |f(hy + x) - f(x)| K(y) \right\} dy
\leq \frac{Nh^{\gamma+1}}{d_N} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{d_{k,0,N}} \int_{\mathbb{R}} K(s)f_1(s,x)ds \leq A \frac{Nh^{\gamma+1}}{d_N},
\]

where $A$ is a constant and $\gamma \in (0,1]$. The last line of proof follows from Assumption 4.3 and note that we have considered $x_{k,N} = d_{k,0,N} y$ and $d_N d_{k,0,N} y = hy + x$. Therefore, $\Theta_{2N} = O_P(N h^{\gamma+1}/d_N)$. \hfill \qed

\textbf{Proposition A.5.} Let $\Theta_{3N} := \{\sum_{k=1}^{N} K[(x_k - x)/h]\}^{-1}$. Then, $\Theta_{3N} = o_P(\{\sqrt{N} \lambda^d h\}^{-1})$.

\textit{Proof.} From Proposition \textbf{A.4} we can easily conclude this under the assumption of $\sqrt{N} \lambda^d h \to \infty$. \hfill \qed

\textbf{Proposition A.6.} Assume $h \to 0$ and $\sqrt{N} \lambda^d h \to \infty$. Let

\[
T_{i,N}(t) := \{\sqrt{N} \lambda^d h\}^{-1} \sum_{k=1}^{Nt} K^i\left[ \frac{x_{k,N} - x}{h} \right] \quad \text{and} \quad T'_{i,N}(t) := \{\sqrt{N} \lambda^d h\}^{-1} \sum_{k=1}^{Nt} K^2\left[ \frac{x_{k,N} - x}{h} \right] u_k^2.
\]

Then, $T_{i,N}(t) \to_d d_i L_B(t,0)$ and $T'_{i,N}(t) \to_d d_i^2 L_B(t,0)$ for $i = 1,2$, where $d_i = \int_{\mathbb{R}} K^i(s)ds$ and $d_0^2 = \mathbb{E}(u_0^2) \int_{\mathbb{R}} K^2(s)ds$.

\textit{Proof.} Let’s define $x_{k,N} := \xi_N(k/N)$ for $1 \leq k \leq N$ such that $\xi_N(t) \Rightarrow B(t)$ on $D[0,1]$ following Theorem 4.3 in \textbf{Sabzikar and Surgailis (2018)}. There exists an equivalent process $\hat{\xi}_N(k/N)$ of $\xi_N(k/N)$; i.e. $\hat{\xi}_N(k/N) = D \xi_N(k/N)$ such that $\sup_{0 \leq t \leq 1} |\hat{\xi}_N(t) - B(t)| = o_P(1)$. Now, apply part (i) of Proposition \textbf{A.9} by setting $c_N = \frac{d_N^2}{h}$ and $g(t) = K^i[t - \frac{t}{2}]$ for $i = 1,2$; then we have

\[
\sup_{0 \leq t \leq 1} \left| \{\sqrt{N} \lambda^d h\}^{-1} \sum_{k=1}^{Nt} K^i\left[ \frac{d_N \hat{\xi}_N(k/N) - x}{h} \right] - L_B(t,0) \int_{\mathbb{R}} K^i(s)ds \right| \to_P 0, \quad (A.13)
\]
as \( c_N \to \infty, \frac{c_N}{N} \to 0 \), and \( N \to \infty \). This together with the fact that \( \hat{\xi}_N(\frac{k}{N}) = D \xi_N(\frac{k}{N}) = \frac{a^*}{d_N} \) for \( 1 \leq k \leq N \) implies for \( N \geq 1 \), \( T_{IN}(t) \) is f.d.d. for \( d_iL_B(t, 0) \). Additionally, \( T_{IN}(t) \) is tight for \( N \geq 1 \) following Proposition \(\text{A.7} \) i.e.

\[
\max_{1 \leq k \leq N} \left| K^*[\frac{x_k - x}{h}] \right| = o_P(\sqrt{N} \lambda^d h),
\]

under the assumption of \( \sup_{0 \leq t \leq 1} |x_{[Nt]}, N - B(t)| = o_P(1) \). Therefore,

\[
T_{IN}(t) \to_D d_i L_B(t, 0) \quad \text{for} \quad i = 1, 2.
\]

Now, in order to prove \( T'_{IN}(t) := \{\sqrt{N} \lambda^d h\}^{-1} \sum_{k=1}^{[Nt]} K^2(\frac{x_k - x}{h}) u_k^2 \to_D d_0^2 L_B(t, 0) \), we generally define

\[
T''_{IN}(t) := \{\sqrt{N} \lambda^d h\}^{-1} \sum_{k=1}^{[Nt]} K^2(\frac{x_k - x}{h}) E u_k^2,
\]

and need to show \( \sup_{0 \leq t \leq 1} \mathbb{E}|T'_{IN}(t) - T''_{IN}(t)|^2 = o(1) \). Based on the Preliminary given in Section A.1, \( x_k = x^*_{0,k} + x'_{0,k} \) for \( s = 0 \), we have:

\[
\mathbb{E}\left[ |T'_{IN}(t) - T''_{IN}(t)|^2 \right| \xi_0, \xi_-1, \ldots] \leq \left\{ \frac{1}{\sqrt{N} \lambda^d h} \right\}^2 \sup_{y, 1 \leq M \leq N} \mathbb{E}\left[ \sum_{k=1}^{M} K^2(\frac{y + x'_{0,k}}{h}) (u_k^2 - E(u_k^2))^2 \right] \quad \text{a.s.}
\]

\[
\leq \left\{ \frac{1}{\sqrt{N} \lambda^d h} \right\}^2 \sup_y \left[ \sum_{k=1}^{N} \mathbb{E}r^2(\frac{x'_{0,k}}{h}) g^2(u_k) + 2 \sum_{1 \leq k < l \leq N} \left| \mathbb{E}r(\frac{x'_{0,k}}{h}) r(\frac{x'_{0,l}}{h}) g(u_k) g(u_l) \right| \right]
\]

as \( g(t) = t^2 - E u_k^2 \), \( g_1(t) = t^2 - E u_k^2 \), and \( r(t) = K^2(\frac{y}{h} + t) \). Now based on the last part of proof of proposition 7.2 in \cite{Wang and Phillips 2009b} and as \( \{\sqrt{N} \lambda^d h\}^{-1} \to 0 \), we have

\[
\mathbb{E}\left[ |T'_{IN}(t) - T''_{IN}(t)|^2 \right| \xi_0, \xi_-1, \ldots] \to 0 \quad \text{a.s.}
\]

Therefore, \( T'_{IN}(t) \) and \( T''_{IN}(t) \) have the same f.d.d.. Since \( E u_k^2 = E u_{m_0}^2 \) for \( k \geq m_0 \) and \( T''_{IN}(t) \to_D d_0^2 L_B(t, 0) \) from the modification of expression \(\text{A.11} \), we can easily show that \( T''_{IN}(t) \) is tight from Proposition \(\text{A.7} \). Therefore, \( T'_{IN}(t) \to_D d_0^2 L_B(t, 0) \) on \( D[0, 1] \).

\[\square\]

**Proposition A.7.** \( S_N(t) = \{\sqrt{N}h\}^{-1/2} \sum_{k=1}^{[Nt]} u_k K[\frac{x_k - x}{h}] \) is tight on \( D[0, 1] \).
Proof. According to Theorem 4 of Billingsley (1974), we need to show
\[ \max_{1 \leq k \leq N} |u_k K \left( \frac{x_k - x}{h} \right)| = o_P \left( \sqrt{Nh} \right)^{1/2}. \]  
(A.14)

The proof follows from Proposition 7.4 in Wang and Phillips (2009b), and therefore we give an outline. There exists a sequence of \( \alpha_N(\nu, \kappa) \) such that
\[ \lim_{\kappa \to 0} \limsup_{N \to \infty} \alpha_N(\nu, \kappa) = 0 \]
for each \( \nu > 0 \) and \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq t \leq 1 \) where \( t - t_m \leq \kappa \). We have
\[ P \left[ \left| S_N(t) - S_N(t_m) \right| \geq \nu \left| S_N(t_1) \cdots S_N(t_m) \right| \right] \leq \alpha_N(\nu, \kappa) \text{ a.s.} \]

Therefore, we need to show
\[ \sup_{|t-t_m| \leq \kappa} P \left( \left| \sum_{k=\left[Nt_m\right]+1}^{[Nt]} u_k K \left( \frac{x_k - x}{h} \right) \right| \geq \nu N^{1/2} \lambda^{-d} \left[ \xi_{\left[Nt_m\right]}, \xi_{\left[Nt_m\right]-1}, \cdots, \xi_1 \right] \right) \leq \alpha_N(\nu, \kappa). \]

We may choose \( \alpha_N(\nu, \kappa) \) such that
\begin{align*}
\alpha_N(\nu, \kappa) &:= \frac{1}{\nu^2} \frac{1}{\sqrt{Nh^2}} \sup_{y, 0 \leq t \leq \kappa} \mathbb{E} \left\{ \sum_{k=1}^{[Nt]} u_k K \left[ \frac{y + x_{0,k}}{h} \right] \right\}^2 \\
&
\leq \frac{1}{\nu^2} \frac{1}{\sqrt{Nh^2}} \sup_y \sum_{k=1}^{[N\kappa]} \mathbb{E} \left\{ u_k^2 K^2 \left[ \frac{y + x_{0,k}}{h} \right] \right\} \\
&
+ \frac{1}{\nu^2} \frac{1}{\sqrt{Nh^2}} \sup_{y} \sum_{1 \leq k < l \leq [N\kappa]} \left| \mathbb{E} \left\{ u_k u_l K \left[ \frac{y + x_{0,k}}{h} \right] K \left[ \frac{y + x_{0,l}}{h} \right] \right\} \right|.
\end{align*}

Following expression (A.2) in Lemma A.1 we can show
\[ \alpha_N(\nu, \kappa) \leq \frac{C}{\nu^2 \sqrt{Nh^2}} \times \left[ \left\lfloor N\kappa \right\rfloor h^d \left( \left\lfloor N\kappa \right\rfloor \right)^{1/2} \right] \left[ 1 + h \sqrt{j} \right] = \frac{C}{\nu^2 \sqrt{N}} \times (\left\lfloor N\kappa \right\rfloor)^{1/2} \lambda^d \left[ 1 + h \sqrt{j} \right], \]

which yields \( \lim_{\kappa \to 0} \limsup_{N \to \infty} \alpha_N(\nu, \kappa) = 0 \). Note that
\[ \max_{1 \leq k \leq N} \left| u_k K \left( \frac{x_k - x}{h} \right) \right| \leq \left\{ \sum_{j=1}^{N} u_j^4 K^4 \left[ \frac{x_j - x}{h} \right] \right\}^{1/4}. \]

Following Lemma A.1 one can show that \( \mathbb{E} u_j^4 K^4 \left[ \frac{x_j - x}{h} \right] \leq C h / \sqrt{j} \), which concludes (A.14). As a result, \( S_N(t) \) is tight.

\[ \square \]

**Proposition A.8.** Let \( \Theta_{2N} := \sum_{k=1}^{N} (f(x_k) - f(x)) K \left[ \frac{2x_k - x}{h} \right] \) and \( \Theta_{3N} := 1 / \sum_{k=1}^{N} K \left[ \frac{2x_k - x}{h} \right] \).
Then, $\Theta_{2N} \sqrt{\Theta_{3N}} \rightarrow_P 0$.

Proof. First, note that
\[
\Theta_{2N} \sqrt{\Theta_{3N}} = O_P \left( \frac{Nh^{\gamma + 1}}{d_N} \right) o_P \left( \sqrt{N} \lambda^d h \right)^{-1/2} = o_P \left( (N/d_N)^{1/2} h^{\gamma + 1/2} \right),
\]
which is equal to $o_P((N/d_N)^{1/2} h^{\gamma + 1/2})$. Under assumptions $Nh^{\gamma + 1}/d_N \rightarrow 0$ and $h \rightarrow 0$ as $N \rightarrow \infty$, we conclude $\Theta_{2N} \sqrt{\Theta_{3N}} \rightarrow_P 0$.

Proposition A.9. Assume $g(x)$ is a bounded function such that $\int_R |g(x)|dx < \infty$. Let $c_N := \frac{dN}{h}$ such that $c_N \rightarrow \infty$ and $\frac{c_N}{N} \rightarrow 0$. Then, we have

(i) $\frac{c_N}{N} \sum_{k=1}^N g(c_N x_k, N) \rightarrow_D \int_R g(x) dx L_B(1, 0)$,

(ii) $\frac{dN}{N} \sum_{k=1}^N |g(x_k)| (1 + |u_k|) = O_P(1),$

(iii) $(\frac{dN}{N})^{1/2} \sum_{k=1}^N g(x_k) u_k = O_P(1), \quad \text{and}$

(iv) For $\sqrt{N} \lambda^d h \rightarrow \infty$, we have

\[
\left\{ (\sqrt{N} \lambda^d h)^{-1/2} \sum_{k=1}^N K[(x_k - x)/h] u_k, (\sqrt{N} \lambda^d h)^{-1} \sum_{k=1}^N g \left[ \frac{x_k - x}{h} \right] \right\} \rightarrow_D \left\{ d_0 N(0, 1)L_1^{1/2}(1, 0), \int_R g(s)ds L_B(1, 0) \right\} \quad \text{on} \quad D^2[0, 1].
\]

Proof. Proof of (i) is similar to Theorem 2.1 in Wang and Phillips (2009a) and Lemma 7 of Jeganathan (2004). Proof of (ii) follows from expression (7.2) in Proposition 7.1 of Wang and Phillips (2016), and proof of (iii) follows from expression (A.3) in Lemma A.1. The proof of (iv) is similar to the proof of expression (3.8) in Wang and Phillips (2009b) with some minor modifications. \qed