MULTIPLICATION OPERATORS ON THE BERGMAN SPACE
OF BOUNDED DOMAINS

HANSONG HUANG AND DECHAO ZHENG

Abstract: In this paper we study multiplication operators on Bergman spaces
of high dimensional bounded domains and those von Neumann algebras induced
by them via the geometry of domains and function theory of their symbols. In
particular, using local inverses and $L^2_{a}$-removability, we show that for a holomorphic
proper map $\Phi = (\phi_1, \phi_2, \cdots, \phi_d)$ on a bounded domain $\Omega$ in $\mathbb{C}^d$, the dimension of
the von Neumann algebra $\mathcal{V}^*(\Phi, \Omega)$ consisting of bounded operators on the Bergman
space $L^2_a(\Omega)$, which commute with both $M_{\phi_j}$ and its adjoint $M_{\phi_j}^*$ for each $j$, equals
the number of components of the complex manifold $S_\Phi = \{(z, w) \in \Omega^2 : \Phi(z) = \Phi(w), z \notin \Phi^{-1}(\Phi(Z))\}$, where $Z$ is the zero variety of the Jacobian $J\Phi$ of $\Phi$. This
extends the main result in [16] in high dimensional complex domains. Moreover
we show that the von Neumann algebra $\mathcal{V}^*(\Phi, \Omega)$ may not be abelian in general
although Douglas, Putinar and Wang [15] showed that $\mathcal{V}^*(\Phi, \mathbb{D})$ for the unit disk
$\mathbb{D}$ is abelian.

1. Introduction

Let $\Omega$ be a bounded domain in the $d$-dimensional complex space $\mathbb{C}^d$, and $dV$ the
Lebesgue measure on $\Omega$. The Bergman space $L^2_a(\Omega)$ is the Hilbert space consisting
of all holomorphic functions on $\Omega$ which are square integrable with respect to $dV$.
For a bounded holomorphic function $\phi$ on $\Omega$, let $M_\phi$ be the multiplication operator
with the symbol $\phi$ on $L^2_a(\Omega)$, given by
$$M_\phi f = \phi f, \ f \in L^2_a(\Omega).$$
For a tuple $\Phi = (\phi_1, \cdots, \phi_n)$, let $\mathcal{V}^*(\Phi, \Omega)$ denote the von Neumann algebra consisting of bounded operators on $L^2_a(\Omega)$ which commute with both $M_{\phi_j}$ and its adjoint $M_{\phi_j}^*$ for each $j$. In fact, as the range of an orthogonal projection in $\mathcal{V}^*(\Phi, \Omega)$
must be a joint reducing subspaces of $\{M_{\phi_j} : 1 \leq j \leq n\}$, and vice versa, we
have a natural correspondence between orthogonal projections in $\mathcal{V}^*(\Phi, \Omega)$ and
joint reducing subspaces of $\{M_{\phi_j} : 1 \leq j \leq n\}$ [16, 18, 21]. Many people have
made investigations on commutants, reducing subspaces of multiplication operators
and von Neumann algebras induced by those operators for the single-variable case [11, 15, 18, 19, 20, 21, 23, 24, 33, 34, 35, 38] and some multi-variable cases
have been studied in [3, 14, 22, 36, 37].

In this paper we study the von Neumann algebra $\mathcal{V}^*(\Phi, \Omega)$ via studying the
geometric properties of $\Omega$, analytic properties of $\Phi$ and $L^2_a$-removable sets.

Given two domains $\Omega$ and $\Omega'$ in $\mathbb{C}^d$, recall that a holomorphic map $\Psi : \Omega \to \Omega'$
is said to be a proper map if for each compact subset $K$ of $\Omega'$, $\Psi^{-1}(K)$ is compact.

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A holomorphic map $\Psi$ on $\Omega$ is said to be proper if $\Psi(\Omega)$ is open and $\Psi : \Omega \to \Psi(\Omega)$ is proper. The following theorem is one of our main results.

**Theorem 1.1.** Suppose $\Phi$ is a holomorphic proper map on $\Omega$. The dimension of $\mathcal{V}^*(\Phi, \Omega)$ equals the number of components of $\mathcal{S}_\Phi$. Here $\mathcal{S}_\Phi$ is a complex manifold in $\mathbb{C}^{2d}$:

$$\mathcal{S}_\Phi = \{(z, w) \in \Omega^2 : \Phi(z) = \Phi(w), \ z \not\in \Phi^{-1}(\Phi(Z))\}$$

where $Z$ denotes the zero variety of the Jacobian $J\Phi$ of $\Phi$.

On the unit disk $\mathbb{D}$, a holomorphic proper map $\phi$ is a finite Blaschke product up to composition of a biholomorphic map and thus Theorem 1.1 generalizes the main result on the unit disk in [16]. In [15], Douglas, Putinar and Wang proved that $\mathcal{V}^*(\phi, \mathbb{D})$ is abelian. In Section 6 we will give examples that $\mathcal{V}^*(\Phi, \Omega)$ is not abelian for some $\Phi$, $\Omega$ in high dimensional spaces (for details, see Example 6.2). We will give an example that $\Phi$ is not a holomorphic proper map on $\Omega$ but $\mathcal{V}^*(\Phi, \Omega)$ is of infinite dimension in Example 4.4.

We will show that the dimension of $\mathcal{V}^*(\Phi, \Omega)$ equals the number of equivalent classes of local inverses of $\Phi$. Local inverses and admissible local inverses (for its definition see the end of Section 2) will play an important role in our study. In the proof of Theorem 1.1 in Section 5, we will show that all local inverses of a holomorphic proper map must be admissible.

We say that $\mathcal{V}^*(\Phi, \Omega)$ is trivial if $\mathcal{V}^*(\Phi, \Omega) = CL$. This is equivalent to that $\{M_{\phi_j} : 1 \leq j \leq n\}$ has no nontrivial joint reducing subspace. The following result is a consequence of Theorem 1.1 which was contained in Theorems 2.2 and 2.4 in [17].

**Theorem 1.2.** [17] Suppose $\Phi : \Omega \to \Omega'$ is a holomorphic proper map. Then $\mathcal{V}^*(\Phi, \Omega)$ is nontrivial if and only if $\Phi$ is not biholomorphic.

Since a holomorphic proper map is onto [31 Proposition 15.1.5], it is biholomorphic if and only if it is univalent. In multi-variable cases “nontrivial” holomorphic proper maps may arise from polynomials. For example, both $(z_1 + z_2, z_1 z_2)$ and $(z_1^2 z_2, z_1 + z_2^2)$ are holomorphic proper maps on the bidisk $\mathbb{D}^2$. On the other hand, $(z_1 + z_2, z_1 z_2)$ is also a holomorphic proper map on the unit ball $\mathbb{B}_2$ of $\mathbb{C}^2$, but $(z_1^2 z_2, z_1 + z_2^2)$ is not in $\mathbb{B}_2$. More details about these maps will be discussed in Section 6.

For a local inverse, let $[\rho]$ denote the equivalent class of $\rho$, which consists of all analytic continuations of $\rho$. We define an operator $\mathcal{E}_{[\rho]}$ on the Bergman space $L^2_\alpha(\mathbb{D})$ to be

$$\mathcal{E}_{[\rho]} h = \sum_{\sigma \in [\rho]} h \circ \sigma, J\sigma, h \in L^2_\alpha(\mathbb{D}), \quad (1.1)$$

where the sum involves only finitely many terms and $J\sigma$ denotes the determinant of the Jacobian of $\sigma$ for a holomorphic map $\sigma$. The local inverses can be defined for a holomorphic map $F : \Omega \to \mathbb{C}^d$. To extend the definition of the operator $\mathcal{E}_{[\rho]}$ in (1.1) to high dimensional domains, we need to study $L^2_\alpha$-removable sets. Indeed, Douglas, Sun and the second author [16] have proved that $\mathcal{V}^*(\phi, \mathbb{D})$ equals the linear span of $\mathcal{E}_{[\rho]}$ where $\rho$ runs over all admissible local inverses of each finite Blaschke product $\phi$, the dimension of $\mathcal{V}^*(\phi, \mathbb{D})$ equals the number of components of a Riemann surface $\mathcal{S}_\phi$ defined as in Theorem 1.1.

For general holomorphic maps, we have the following theorem.
Theorem 1.3. Suppose that $\Omega$ is a bounded domain in $\mathbb{C}^d$. If the interior of $\overline{\Omega}$ equals $\Omega$, and $\Phi : \Omega \to \mathbb{C}^d$ is holomorphic on a neighborhood of $\overline{\Omega}$ such that the image of $\Phi$ has an interior point, then $\mathcal{V}^*(\Phi, \Omega)$ is a finite dimensional von Neumann algebra. Moreover $\mathcal{V}^*(\Phi, \Omega)$ is generated by $\mathcal{E}_{[\rho]}$, where $\rho$ run over admissible local inverses of $\Phi$.

Remark 1.4. (1) If the image of $\Phi$ has no interior point, Example 4.4 shows that $\mathcal{V}^*(\Phi, \Omega)$ may be of infinite dimension.

(2) The condition on $\Omega$ in the above theorem is equivalent to that the boundary $\partial \Omega$ of $\Omega$ is not ”huge”, i.e., each point of $\partial \Omega$ is a limit point of the interior points of $\mathbb{C}^d \setminus \Omega$. This observation is contained in the following equivalent statements:

(A) The interior of $\overline{\Omega}$ equals $\Omega$;

(B) For each point $\zeta \in \partial \Omega$ and any open neighborhood $O(\zeta)$ of $\zeta$, $O(\zeta) \setminus \Omega$ contains an open ball.

(B) immediately implies that domains with $C^1$-boundary, star-shaped domains, circled bounded domains, convex domains (bounded symmetric domains), and strictly pseudoconvex domains satisfy the condition on $\Omega$ in Theorem 1.3.

To get the above equivalence, first suppose that (B) holds. If (A) fails, then there is a point $\lambda$ in the interior of $\overline{\Omega}$ and in $\partial \Omega$. So there is a neighborhood $N$ of $\lambda$ such that $N \subseteq \Omega$. Thus the point $\lambda$ is not a limit point of $\mathbb{C}^d \setminus \Omega$. This contradicts (B) that $\lambda$ can be approximated by a sequence of interior points in $\mathbb{C}^d \setminus \Omega$.

Conversely, suppose (A) holds. If (B) fails, then for some point $\zeta \in \partial \Omega$, there is a neighborhood $O(\zeta)$ of $\zeta$ such that $O(\zeta) \setminus \Omega$ contains no open ball. Thus there is no interior point of $O(\zeta) \setminus \Omega$ in $O(\zeta)$, and so we have that $O(\zeta)$ is contained in $\overline{\Omega}$. This gives that $\zeta$ is an interior point of $\overline{\Omega}$, hence by (A), $\zeta$ is an interior point of $\Omega$. But it contradicts that $\zeta$ is in the boundary $\partial \Omega$. Thus (B) holds.

(3) Although a pseudo-convex domain with $C^1$ boundary satisfies the condition in Theorem 1.3, the following example shows that some pseudo-convex domains may not enjoy (A).

Example 1.5. The domain $D \setminus [0,1]$ is a pseudo-convex domain whose interior of its closure does not equal itself as it is conformally equivalent to the unit disk $\mathbb{D}$.

Let $\Omega = (D \setminus [0,1]) \times \mathbb{D}$. Since the biholomorphic image of a pseudo-convex domain is pseudo-convex and $\Omega$ is biholomorphic to $D^2$, $\Omega$ is pseudo-convex. But the interior of $\overline{\Omega}$ is not $\Omega$.

(4) In [36, Theorem 6.1], for bounded smooth pseudoconvex domains with the following two assumptions, Tikaradze showed that $\mathcal{V}^*(\Phi, \Omega)$ is isomorphic to the algebra of locally constant functions on $W_\Phi$ under convolution product where $W_\Phi$ is a subset of the complex manifold:

$$S_\Phi = \{(z, w) \in \Omega^2 : \Phi(z) = \Phi(w), z \notin \Phi^{-1}(\Phi(Z))\},$$

and $W_\Phi$ is defined in [36, Definition 3.2].

Assumption 1. For any $z \in \partial \Omega$, $A^\infty(\Omega) \cap I(z)$ is dense in $L^2_a(\Omega)$ where $A^\infty(\Omega)$ is the set of all holomorphic functions on $\Omega$ which are $C^\infty$.
smooth on \( \overline{\Omega} \), and \( I(z) \) denotes the ideal of holomorphic functions on \( \overline{\Omega} \) which vanish on \( z \).

Assumption 2. There exists a nonzero function \( g \in A^*(\Omega) \) such that \( g \) vanishes on \( \Phi^{-1}(\Phi(Z)) \) where \( Z \) the zero variety of the Jacobian \( J\Phi \) of \( \Phi \).

As a domain \( \Omega \) in \( \mathbb{C}^d \) with \( C^1 \)-boundary has the property that the interior of \( \overline{\Omega} \) equals \( \Omega \) \([23] \ p. 117, \ [27] \ p. 52\), Theorem 1.3 suggests that for connected bounded domains with \( C^1 \)-boundary, Theorem 6.1 in \([36]\) may hold.

Theorem \([13]\) also gives the following criterion when \( \mathcal{V}^*(\Phi, \Omega) \) is nontrivial.

Corollary 1.6. Suppose that \( \Omega \) is a bounded domain. If the interior of \( \overline{\Omega} \) equals \( \Omega \), and \( \Phi : \Omega \rightarrow \mathbb{C}^d \) is holomorphic on a neighborhood of \( \overline{\Omega} \) such that the image of \( \Phi \) has an interior point, then \( \mathcal{V}^*(\Phi, \Omega) \) is nontrivial if and only if there exists an admissible local inverse of \( \Phi \) distinct from the identity map.

On the complex plane \( \mathbb{C} \), the assumption in Corollary 1.6 says that \( \Phi \) is nonconstant and holomorphic over \( \overline{\Omega} \). On the unit disk, Corollary 1.6 is implicitly contained in \([34]\). On a polygon \( \Omega \), \( \mathcal{V}^*(\Phi, \Omega) \) is often trivial and the nontrivialness of \( \mathcal{V}^*(\Phi, \Omega) \) implies some geometric property of \( \Omega \) \([23] \ [24]\).

To get Theorem 1.3 we need to study \( L^2_0 \)-removable sets. Let \( A = \Phi^{-1}(\Phi(Z)) \), where \( Z \) is the zero variety of the Jacobian \( J\Phi \) of \( \Phi \). \( A \) is a relative closed subset of \( \Omega \) but “small”. For each function \( h \) in \( L^2_0(\Omega) \), \( \mathcal{E}[\rho|h \) is a well-defined holomorphic function on \( \Omega \setminus A \). The function \( \mathcal{E}[\rho|h \) lies in \( L^2_0(\Omega \setminus A) \). In the case of the unit disk, \( A \) is a discrete subset of \( \mathbb{D} \), and thus by complex analysis one easily gets that every function \( f \in L^2_0(\mathbb{D} \setminus A) \) can extend analytically to \( \mathbb{D} \) and the extension belongs to \( L^2_0(\mathbb{D}) \). However, if \( \Omega \) is a higher dimensional domain, \( A \) is more complicated than a zero variety, and hence \( A \) is far from discrete. We must show that each function in \( L^2_0(\Omega \setminus A) \) extends holomorphically to \( \Omega \). This naturally involves a problem of removability. Precisely, a relatively closed subset \( E \) of \( \Omega \) is called \( L^2_0 \)-removable in \( \Omega \) if each function in \( L^2_0(\Omega \setminus E) \) extends holomorphically to a function in \( L^2_0(\Omega) \) \([5] \ [8]\).

An improved version of the Riemann Removable Singularity Theorem \([2] \ [6]\) states that a zero variety of a domain \( \Omega \) in \( \mathbb{C}^d \) is \( L^2_0 \)-removable in \( \Omega \).

For a subset \( E \) of \( \Omega \), \( \overline{E} \) denotes the closure of \( E \) in \( \overline{\Omega} \). A subset \( E \) of \( \Omega \) is called a zero variety of \( \Omega \) if there is a nonconstant holomorphic function \( f \) on \( \Omega \) such that \( E = \{ z \in \Omega : f(z) = 0 \} \).

A subset \( E \) of \( \Omega \) is called a local zero variety if for each point \( \lambda \in \Omega \), there are an open neighborhood \( U \) of \( \lambda \) and a nonconstant holomorphic function \( h_\lambda \) on \( U \) such that

\[
U \cap \overline{E} \subset \{ z \in U \cap \overline{\Omega} : h_\lambda(z) = 0 \}.
\]

Clearly, every analytic set in \([9]\) of \( \Omega \) is a local zero variety. A local zero variety \( E \) of \( \Omega \) is said to be good if \((1.2) \) holds for each point \( \lambda \in \overline{\Omega} \). For example, if \( h \) is a nonconstant holomorphic function over \( \overline{\Omega} \), then \( \{ z \in \Omega : h(z) = 0 \} \) is a good local zero variety of \( \Omega \). For a bounded planar domain \( \Omega \), a subset of \( \Omega \) is a good local zero variety if and only if it is a finite set. To see this, we need to show that a good local zero variety \( E \) of a bounded planar domain \( \Omega \) is a finite subset. Otherwise, if \( E \) were an infinite subset of \( \Omega \), then \( E \) would contain one accumulation point \( w \) on \( \overline{\Omega} \). By definition, there is a unit disk \( U \) centered at \( w \).
such that \( U \cap \overline{E} \subset \{ z \in U \cap \overline{\Omega} : h(z) = 0 \} \) for some holomorphic function \( h \) over \( U \). Then \( h \) would be identically zero as \( w \) were an accumulation point, which is a contradiction.

We need the following theorem about \( L^2_a \)-removable sets, which generalizes the improved version of the Riemann Removable Singularity Theorem \([2, 6]\). For a map \( F : \Omega \to \mathbb{C}^d \) holomorphic over \( \Omega \), let

\[
F^{-1}(F(E)) = \{ z \in \Omega : F(z) \in F(E) \}.
\]

**Theorem 1.7.** Let \( \Omega \) be a domain in \( \mathbb{C}^d \). Suppose that a local zero variety \( E \) of \( \Omega \) is good and \( F : \Omega \to \mathbb{C}^d \) is holomorphic on \( \Omega \) such that the image of \( F \) contains an interior point. Then \( F^{-1}(F(E)) \) is \( L^2_a \)-removable in \( \Omega \).

Theorem 1.7 immediately gives that each good local zero variety of \( \Omega \) is \( L^2_a \)-removable in \( \Omega \).

This paper is arranged as follows. Section 2 contains some preliminaries such as the notion of admissible local inverse, and some properties of holomorphic proper maps. In Section 3 a new approach is presented to prove the improved version of the Riemann Removable Singularity Theorem \([2, 6]\), and using some ideas of the approach, we will present the proof of Theorem 1.7. In Section 4 we will present the proof of Theorem 1.3 and show how admissible local inverse matches \( L^2_a \)-removable property. Section 5 contains the proofs of Theorems 1.1 and 1.2. In Section 6 we will provide many examples of \( V^*(\Phi, \Omega) \) and show that \( V^*(\Phi, \Omega) \) has fruitful structures.

### 2. Preliminaries

In this section, we will introduce some notations and give definitions of local inverses and the admissible local inverse in multi-variable case. Some properties of holomorphic proper maps in \([31]\) will be stated as they will be used later.

Let \( F : \Omega \to \mathbb{C}^d \) be a holomorphic map. Let \( Z \) be the set \( Z(JF) \) of the zeros of the determinant of the Jacobian \( JF \) of the map \( F \). The image \( F(Z) \) is called the critical set of \( F \), each point in \( F(Z) \) is called a critical value, and each point in \( F(\Omega) \setminus F(Z) \) is called a regular point.

Let \( \Omega \) and \( \Omega' \) be two domains in \( \mathbb{C}^d \). In the introduction, it is mentioned that a holomorphic map \( \Psi : \Omega \to \Omega' \) is called a proper map if for each compact subset \( K \) of \( \Omega' \), \( \Psi^{-1}(K) \) is compact. For example, a holomorphic proper function from the unit disk \( \mathbb{D} \) to \( \mathbb{D} \) is just a finite Blaschke product. More generally, a holomorphic map \( \Phi \) from the polydisk \( \mathbb{D}^d \) to \( \mathbb{D}^d \) is proper if and only if

\[
\Phi(z) = (\phi_1(z_1), \cdots, \phi_d(z_d))
\]

up to a permutation of coordinates where \( \phi_j (1 \leq j \leq d) \) are finite Blaschke products \([30\, \text{Theorem 7.3.3}]\). A holomorphic proper map is a surjective open map \([31\, \text{Proposition 15.1.5}]\). The following result is contained in Theorem 15.1.6 in \([31]\).

**Theorem 2.1.** Suppose \( F : \Omega \to \mathbb{C}^d \) is a holomorphic function and for each point \( w \in \mathbb{C}^d \), \( F^{-1}(w) \) is compact. Then \( F \) is an open map.

Moreover a holomorphic proper map is an \( m \)-folds map and its critical set enjoys the following good properties \([31\, \text{Theorem 15.1.9}]\).
Theorem 2.2. For two domains $\Omega$ and $\Omega'$ in $\mathbb{C}^d$, suppose $F : \Omega \to \Omega'$ is a holomorphic proper function. Let $\sharp(w)$ denote the number of elements in $F^{-1}(w)$ with $w \in \Omega'$. Then the following hold:

1. There is an integer $m$ such that $\sharp(w) = m$ for all regular values $w$ of $F$ and $\sharp(w') < m$ for all critical values $w'$ of $F$;
2. The critical set of $F$ is a zero variety in $\Omega'$.

To study local inverses, we need some notations about analytic continuation [32, Chapter 16]. A function element is an ordered pair $(f, D)$, where $D$ is a simply-connected domain and $f$ is a holomorphic function on $D$. Two function elements $(f_0, D_0)$ and $(f_1, D_1)$ are called direct continuations if $D_0 \cap D_1$ is not empty and $f_0(z)$ equals $f_1(z)$ on $D_0 \cap D_1$. A curve is a continuous map from $[0, 1]$ into $\mathbb{C}^d$.

Given a function element $(f_0, D_0)$ and a curve $\gamma$ with $\gamma(0) \in D_0$, if there are a partition of $[0, 1]$:

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

and function elements $(f_j, D_j)(0 \leq j \leq n)$ such that

1. $(f_j, D_j)$ and $(f_{j+1}, D_{j+1})$ are direct continuation for all $j$ with $0 \leq j \leq n - 1$;
2. $\gamma[s_j, s_{j+1}] \subseteq D_j(0 \leq j \leq n - 1)$ and $\gamma(1) \in D_n$,

then $(f_n, D_n)$ is called an analytic continuation of $(f_0, D_0)$ along $\gamma$; and $(f_0, D_0)$ is called to admit an analytic continuation along $\gamma$. In this case, we write $f_0 \sim f_n$.

Clearly, $\sim$ defines an equivalence and we write $[f]$ for the equivalent class of $f$.

Given a family $\{\phi_j\}$ of holomorphic maps over $\Omega$, and a subdomain $\Delta$ of $\Omega$, a holomorphic function $\rho : \Delta \to \Omega$ is said to be a local inverse of $\{\phi_j\}$ on $\Delta$ if $\phi_j \circ \rho = \phi_j$ for all $j$. In particular, if $\Phi = (\phi_1, \cdots, \phi_d)$, $\rho$ is said to be a local inverse of $\Phi$ if $\Phi \circ \rho = \Phi$.

For a holomorphic map $\Phi : \Omega \to \mathbb{C}^d$, assume the image of $\Phi$ contains an interior point. Let

$$A = \Phi^{-1}(\Phi(Z(J\Phi)))$$

where $Z(J\Phi)$ is the set of zeros of the determinant of the Jacobian $J\Phi$ of $\Phi$. A local inverse $\rho$ of $\Phi : \Omega \to \mathbb{C}^d$ is said to be admissible if for each curve $\gamma$ in $\Omega \setminus A$, $\rho$ admits analytic continuation with values in $\Omega$. The notion of admissible local inverse on the unit disk $\mathbb{D}$ was first introduced by Thomson [34] for a discrete set $A$.

3. $L^2_\sigma$-removable sets

In this section we will present the proof of Theorem 1.7 and show that some sets slightly more general than a zero variety enjoy the $L^2_\sigma$-removable property, which is needed in the proofs of Theorems 1.1 and 1.3.

For completeness using a different approach we present a proof of the following result in [26]. Some ideas in the proof will be used in the proof of Theorem 1.7.

Theorem 3.1. Suppose that $E$ is a zero variety of a domain $\Omega$ in $\mathbb{C}^d$. Then $E$ is $L^2_\sigma$-removable in $\Omega$.

Proof. Without loss of generality, we may assume $d > 1$. First let us consider the special case that $\Omega$ is the polydisk $\mathbb{D}^d$ and $E$ is equal to $\{z \in \Omega : z_d = 0\}$. Then $\Omega \setminus E = \mathbb{D}^{d-1} \times (\mathbb{D} \setminus \{0\})$. 
where $\sum_{\alpha \in \mathbb{Z}_+^d} |c_\alpha|^2 \|z^\alpha\|^2 < \infty$.

Let $g$ be in $L^2_a(\mathbb{D}^{d-1} \times \{0\})$. We can write in Laurent series:

$$g(z) = \sum_{\alpha \in \mathbb{Z}^d_+} c'_\alpha z^\alpha, \ z \in \mathbb{D}^d \times \{0\}.$$

Since

$$\|g\|^2_{L^2_a(\mathbb{D}^{d-1} \times \{0\})} = \int_{\mathbb{D}^{d-1} \times \{0\}} |g(z)|^2 dV(z) = \sum_{\alpha \in \mathbb{Z}^d_+} |c'_\alpha|^2 \|z^\alpha\|^2 + \sum_{\alpha \in \mathbb{Z}^d_+} |c'_\alpha|^2 \int_{\mathbb{D}^{d-1} \times \{0\}} |z^\alpha|^2 dV(z),$$

we have that $c'_\alpha = 0$ for $\alpha$ in $\mathbb{Z}^d_+ \times \mathbb{Z}_-$ and hence

$$\|g\|^2_{L^2_a(\mathbb{D}^{d-1} \times \{0\})} = \sum_{\alpha \in \mathbb{Z}^d_+} |c'_\alpha|^2 \|z^\alpha\|^2.$$

This gives that

$$g(z) = \sum_{\alpha \in \mathbb{Z}^d_+} c'_\alpha z^\alpha,$$

and thus $g$ is in $L^2_a(\mathbb{D}^d)$.

For general case, let $E$ be a zero variety of a domain $\Omega$ in $\mathbb{C}^d$. We may assume

$$E = \{ z \in \Omega | f(z) = 0 \},$$

for some nonconstant holomorphic function $f$ on $\Omega$. Let $h$ be a function in $L^2_a(\Omega \setminus E)$. We need only to show that $h$ extends holomorphically to $\Omega$ since $E$ has Lebesgue measure zero. Let $H = (z_1, \cdots, z_d-1, f)$. Then

$$JH(z) = \frac{\partial f}{\partial z_d}(z)$$

for $z \in \Omega$. For each $\lambda \in E$, if $\frac{\partial f}{\partial z_\lambda}(\lambda)$ does not equal 0, $JH(\lambda)$ does not equal 0. Thus there is a neighborhood $U_1$ containing $\lambda$ so that $H$ is biholomorphic on a neighborhood of $\overline{U_1}$. So there is a biholomorphic map from $U_1 \setminus E$ onto $\mathbb{D}^{d-1} \times \{0\}$. Using such the biholomorphic map we can establish an isomorphism between $L^2_a(U_1 \setminus E)$ and $L^2_a(\mathbb{D}^{d-1} \times \{0\})$. Since we have shown above that each holomorphic function in $L^2_a(\mathbb{D}^{d-1} \times \{0\})$ extends holomorphically to a function in $L^2_a(\mathbb{D}^d)$, each holomorphic function in $L^2_a(U_1 \setminus E)$ extends holomorphically to a function in $L^2_a(U_1)$. Thus each holomorphic function $h \in L^2_a(\Omega \setminus E)$ extends holomorphically to the subset $\{ z \in E | \frac{\partial f}{\partial z_\lambda} \neq 0 \} \cup (\Omega \setminus E) = \{ z \in \Omega | \frac{\partial f}{\partial z_d} \neq 0 \} \cup (\Omega \setminus E)$ of $\Omega$. Let

$$\Lambda_j = \{ w \in \Omega | \frac{\partial f}{\partial z_j}(w) \neq 0 \}, \ 1 \leq j \leq d,$$
and for $r = 1, 2, \cdots$, 

$$\Lambda_{j_1, \cdots, j_r} = \{ w \in \Omega | \frac{\partial^r f}{\partial z_{j_1} \cdots \partial z_{j_r}}(w) \neq 0 \}, 1 \leq j_k \leq d, 1 \leq k \leq r.$$ 

Repeating the above argument with respect to each variable $z_j$, we have that $h$ extends holomorphically to 

$$\bigcup_{1 \leq j \leq d} \Lambda_j \cup (\Omega \setminus E).$$ 

Replacing $f$ by $\frac{\partial^r f}{\partial z_{j_1} \cdots \partial z_{j_r}}$ inductively and repeating the above argument give that $h$ extends holomorphically to 

$$(\Omega \setminus E) \bigcup \bigcup_{1 \leq j \leq d} \Lambda_j \bigcup \bigcup_{1 \leq j, k \leq d} \Lambda_{j, k} \bigcup \cdots$$ 

To finish the proof, we will show that the union equals $\Omega$. To do this, for a point $w_0$ in the complement of 

$$(\Omega \setminus E) \bigcup \bigcup_{1 \leq j \leq d} \Lambda_j \bigcup \bigcup_{1 \leq j, k \leq d} \Lambda_{j, k} \bigcup \cdots,$$ 

we have 

$$\frac{\partial^r f}{\partial z_{j_1} \cdots \partial z_{j_r}}(w_0) = 0, 1 \leq j_1, \cdots, j_r \leq d.$$ 

This gives that the power expansion of $f$ at $w_0$ equal 0 and hence $f$ is identically zero on a neighborhood of $w_0$. Thus $f \equiv 0$ on $\Omega$. This contradicts that $f$ is nonconstant on $\Omega$. $\square$

A closed set $A$ is said to be strongly $L_a^2$-removable in $\Omega$ if for each subdomain $U$ in $\Omega$, $U \cap A$ is $L_a^2$-removable in $U$. Clearly, each closed subset of a strongly $L_a^2$-removable set in $\Omega$ is $L_a^2$-removable in $\Omega$. Theorem 3.1 implies that a zero variety is strongly $L_a^2$-removable in $\Omega$. The following example gives that the strongly $L_a^2$-removable property is strictly stronger than the $L_a^2$-removable property.

**Example 3.2.** Let $\Omega = \mathbb{D}^2$, and $A = \frac{1}{2} \mathbb{D} \times \frac{1}{2} \mathbb{D}$. We will show that each function $f \in L_a^2(\Omega \setminus A)$ extends to a function $\tilde{f}$ in $L_a^2(\Omega)$. To do this, let $V = (\mathbb{D} \setminus \frac{1}{2}\mathbb{D})^2$. An computation gives that there is a constant $C > 0$ such that 

$$\int_{\Omega} |z^\alpha|^2 dV(z) \leq C \int_{V} |z^\alpha|^2 dV(z), \alpha \in \mathbb{Z}_+^2.$$ 

By Hartogs’ extension theorem, each holomorphic function in $\Omega \setminus A$ extends holomorphically to $\Omega$. Thus each function $f$ in $L_a^2(\Omega \setminus A)$ extends to a holomorphic function $\tilde{f}$ on $\Omega$, and so we have the Taylor expansion of $\tilde{f}$:

$$\tilde{f}(z) = \sum_{\alpha \in \mathbb{Z}_+^2} c_\alpha z^\alpha, z \in \Omega.$$
Also we have
\[
\int_{\Omega} |\tilde{f}(z)|^2 dV(z) = \sum_{\alpha \in \mathbb{Z}^n_+} |c_\alpha|^2 \int_{\Omega} |z^\alpha|^2 dV(z) \\
\leq \sum_{\alpha \in \mathbb{Z}^n_+} C|c_\alpha|^2 \int_{\Omega} |z^\alpha|^2 dV(z) \\
= C \int_{\Omega} |f(z)|^2 dV(z) < \infty.
\]
This gives that \( \tilde{f} \) is in \( L^2_a(\Omega) \), and hence \( A \) is \( L^2_a \)-removable in \( \Omega \).

On the other hand, \( A \) is not strongly \( L^2_a \)-removable in \( \Omega \). To see this, let
\[
U = \frac{1}{2} \mathbb{D} \times \mathbb{D},
\]
and define
\[
h(z) = \frac{1}{z_2}, \ (z_1, z_2) \in U \setminus A.
\]
Since \( h \) is bounded on \( U \setminus A \), we have that \( h \) is in \( L^2_a(U \setminus A) \). But it is clear that \( h \) cannot extend to a holomorphic function on \( U \), and \( h \) does not extend to a function in \( L^2_a(U) \). Therefore, \( A \) is not strongly \( L^2_a \)-removable in \( \Omega \).

We have the following lemma about finite union of strongly \( L^2_a \)-removable sets.

**Lemma 3.3.** Suppose \( A_1, \ldots, A_n \) are strongly \( L^2_a \)-removable sets in \( \Omega \) with Lebesgue measure zero. Then the union of \( A_1, \ldots, A_n \) is also strongly \( L^2_a \)-removable in \( \Omega \).

**Proof.** By induction we need only show that if both \( A_1 \) and \( A_2 \) are strongly \( L^2_a \)-removable in \( \Omega \) and have Lebesgue measure zero, then \( A_1 \cup A_2 \) is strongly \( L^2_a \)-removable in \( \Omega \). To see this, let \( U \) be a subdomain in \( \Omega \). Note that \( A_2 \) is relatively closed in \( \Omega \) and \( U \setminus A_2 = U \cap (\Omega \setminus A_2) \) is open and hence \( U \setminus (A_1 \cup A_2) \) is also open. Suppose \( f \) is a function in \( L^2_a(U \setminus (A_1 \cup A_2)) \). First we prove that \( f \) is in \( L^2_a(U \setminus A_2) \). Since \( U \setminus A_2 \) is open,
\[
U \setminus A_2 = \bigcup_i V_i
\]
where each \( V_i \) is a connected component of \( U \setminus A_2 \). Since the restriction of \( f \) on each \( V_i \setminus A_1 \) is in \( L^2_a(V_i \setminus A_1) \) and \( A_1 \) is strongly \( L^2_a \)-removable sets in \( \Omega \), \( f \) extends analytically to a function in \( L^2_a(V_i) \). Thus \( f \) is holomorphic in \( U \setminus A_2 \). This implies
\[
f \in L^2_a(U \setminus A_2)
\]
as the Lebesgue measure of \( A_2 \) equals zero. Since \( A_2 \) is strongly \( L^2_a \)-removable in \( \Omega \), \( f \) extends holomorphically to a function in \( L^2_a(U) \). We conclude that \( A_1 \cup A_2 \) is strongly \( L^2_a \)-removable in \( \Omega \). \( \square \)

The following lemma will be used in the proof of Theorem 1.7.

**Lemma 3.4.** Let \( E \) be a closed and strongly \( L^2_a \)-removable set in \( \mathbb{C}^d \) with the Lebesgue measure zero. Suppose \( F : \Omega \to \mathbb{C}^d \) is a holomorphic map on \( \Omega \) such that the image of \( F \) has an interior point. Then \( F^{-1}(E) \) is strongly \( L^2_a \)-removable in \( \Omega \).
Proof. Let $E$ be a closed and strongly $L^2_a$-removable set in $\mathbb{C}^d$ with the Lebesgue measure zero. Since $F(\Omega)$ has an interior point, $JF$ does not equal identically zero. We will show that $F^{-1}(E)$ has Lebesgue measure zero. To do this, first we observe

$$F^{-1}(E) \subseteq Z(JF) \cup \{ z \in \Omega : JF(z) \neq 0 \text{ and } F(z) \in E \}.$$ 

Since $\{ z \in \Omega : JF(z) \neq 0 \text{ and } F(z) \in E \}$ is contained in a union of countably many biholomorphic images of subsets in $E$, it must have Lebesgue measure zero. Noting that a zero variety $Z(JF)$ has the Lebesgue measure zero, we have that $F^{-1}(E)$ has Lebesgue measure zero.

Let $V$ be an open subdomain in $\Omega$. We will show that $F^{-1}(E) \cap V$ is $L^2_a$-removable in $V$. To do this, we note that for each $\lambda \in \Omega$, if $JF(\lambda) \neq 0$, then there is an open neighborhood $U(\lambda)$ of $\lambda$ such that $F$ is injective on $\overline{U(\lambda)}$. Since $F^{-1}(E) \cap U(\lambda) = \{ z \in U(\lambda) : F(z) \in E \} = (F|_{U(\lambda)})^{-1}(E)$, and $E$ is strongly $L^2_a$-removable in $\mathbb{C}^d$, $F^{-1}(E) \cap U(\lambda)$ is $L^2_a$-removable in $U(\lambda)$. Thus for each $h \in L^2_a(V \setminus F^{-1}(E))$, $h$ extends holomorphically to

$$\cup_{\lambda \in \{ z : JF(z) \neq 0 \}} U(\lambda).$$

Since

$$V \setminus Z(JF) = V \setminus \{ z : JF(z) \neq 0 \} \subset \cup_{\lambda \in \{ z : JF(z) \neq 0 \}} U(\lambda),$$

$h$ extends holomorphically to $V \setminus Z(JF)$. Since we have shown above that $\{ z \in \Omega : JF(z) \neq 0 \text{ and } F(z) \in E \}$ has the Lebesgue measure zero, $h$ is in $L^2_a(V \setminus Z(JF))$. As $Z(JF)$ is a zero variety, by Theorem 3.1 $h$ extends holomorphically to $V$. Hence we obtain that $F^{-1}(E) \cap V$ is $L^2_a$-removable in $V$. This means that $F^{-1}(E)$ is strongly $L^2_a$-removable in $\Omega$.

To prove Theorem 1.7, we need some lemmas from several complex analysis.

**Lemma 3.5.** If $E$ is a relatively closed subset of a domain $D$ in $\mathbb{C}^d$ and of zero Hausdorff measure $h_{2d-2}(E)$, then each function holomorphic on $D \setminus E$ has a holomorphic continuation to $D$.

**Lemma 3.6.** Let $A$ be an analytic subvariety of a domain in $\mathbb{C}^d$, and $F : A \rightarrow \mathbb{C}^n$ be a holomorphic map. Then $F(A)$ is contained in a union of at most countable analytic subvarieties of domains in $\mathbb{C}^n$, of dimensions not larger than the dimension $\text{dim } F$. Consequently, $\text{dim } F(A) \leq \text{dim } A$.

**Lemma 3.7.** Let $F : \overline{D}^{d-1} \rightarrow \mathbb{C}^d$ be a holomorphic map. Then $F(\overline{D}^{d-1})$ is strongly $L^2_a$-removable in $\mathbb{C}^d$.

**Proof.** Let $r$ denote the maximal rank of Jacobian matrix of $F$ on $\overline{D}^{d-1}$. If $r \leq d-2$, then the dimension dim $F$ of $F$ is no larger than $d-2$ [9] p. 40]. Then by Lemma 3.6 $F(\overline{D}^{d-1})$ is contained in a union of at most countable analytic subvarieties $E_j$ of domains, which are of dimensions $\leq d-2$, and hence their Hausdorff measures $h_{2d-2}(E_j)$ are zero. Therefore $h_{2d-2}F(\overline{D}^{d-1}) = 0$. Since the Lebesgue measure of $F(\overline{D}^{d-1})$ equals zero, by Lemma 3.5 we immediately have that $F(\overline{D}^{d-1})$ is strongly $L^2_a$-removable in $\mathbb{C}^d$. 


Now assume that the maximal rank of Jacobian matrix of $F$ on $\mathbb{D}^{d-1}$ is $d - 1$. Letting $F = (f_1, \cdots, f_d)$, we may assume that the holomorphic function
\[
\varphi(z_1, \cdots, z_{d-1}) = \det \frac{\partial (f_1, \cdots, f_{d-1})}{\partial (z_1, \cdots, z_{d-1})}
\]
is not identically zero on $\mathbb{D}^{d-1}$. Let
\[
\mathbb{Z} = \{ z \in \mathbb{D}^{d-1} : \varphi(z) = 0 \}.
\]
Let $\Omega$ be any domain in $\mathbb{C}^d$. For each $\varepsilon > 0$, let $V(\varepsilon)$ denote the $\varepsilon$-neighborhood of $F(\mathbb{Z})$,
\[
V(\varepsilon) = \{ w \in \mathbb{C}^d : \inf_{\lambda \in F(\mathbb{Z})} |w - \lambda| < \varepsilon \}
\]
and let
\[
\Omega_\varepsilon = \Omega \setminus \overline{V(\varepsilon)}.
\]
As $F$ is a continuous map, there is a positive integer $k$ such that the $\frac{1}{k}$-neighborhood $U(k)$ of $\mathbb{Z}$ satisfies
\[
F(U(k)) \subseteq V(\varepsilon).
\]
Thus we have
\[
F(\mathbb{D}^{d-1}) \setminus V(\varepsilon) = F(\mathbb{D}^{d-1} \setminus U(k)) \setminus V(\varepsilon). \tag{3.1}
\]
Let $F_0 = (f_1, \cdots, f_{d-1})$. For each point $w \in \mathbb{D}^{d-1} \setminus U(k)$, the Jacobian of $F_0$ at $w$ does not vanish, and then there is a ball $U_w$ centered at $w$ such that $F_0$ is univalent on a neighborhood of $U_w$. Since $F_0 : U_w \to F_0(U_w)$ is a biholomorphic map, for each $(\lambda_1, \cdots, \lambda_{d-1}, \lambda_d) \in F(U_w)$ we get
\[
\lambda_1 = f_1(z_1, \cdots, z_{d-1}), \lambda_2 = f_2(z_1, \cdots, z_{d-1}), \cdots, \lambda_{d-1} = f_{d-1}(z_1, \cdots, z_{d-1}),
\]
\[
\lambda_d = f_d(z_1, \cdots, z_{d-1}) = f_d(F_0^{-1}(\lambda_1, \lambda_2, \cdots, \lambda_{d-1})).
\]
Thus $F(U_w)$ is a zero variety. Since $\mathbb{D}^{d-1} \setminus U(k)$ is compact, $\mathbb{D}^{d-1} \setminus U(k)$ is contained in a finite union of open balls $\{ U_w \}$. Then $F(\mathbb{D}^{d-1} \setminus U(k))$ is contained in a union of finitely many zero varieties $F(U_w)$, and by Theorem 3.1 and Lemma 3.3 $F(\mathbb{D}^{d-1} \setminus U(k))$ is strongly $L^2_\alpha$-removable in $\Omega_\varepsilon$. By (3.1), $F(\mathbb{D}^{d-1}) \setminus V(\varepsilon)$ is strongly $L^2_\alpha$-removable in $\Omega_\varepsilon$. Thus $F(\mathbb{D}^{d-1})$ is strongly $L^2_\alpha$-removable in $\Omega_\varepsilon$.

For each holomorphic function $h$ in $L^2_\alpha(\mathbb{D}^{d-1})$, $h$ is in $L^2_\alpha(\Omega \setminus F(\mathbb{D}^{d-1}))$. Since $F(\mathbb{D}^{d-1})$ is strongly $L^2_\alpha$-removable in $\Omega_\varepsilon$, $h$ is in $L^2_\alpha(\Omega_\varepsilon)$, and by arbitrariness of $\varepsilon$, $h$ is holomorphic on $\Omega \setminus F(\mathbb{D}^{d-1})$. Noting that $\dim \mathbb{Z} \leq d - 2$, by Lemma 3.6 we have
\[
\dim F(\mathbb{Z}) \leq d - 2.
\]
Then the Hausdorff measure $h_{2d-2}(F(\mathbb{Z})) = 0$, and by Lemma 3.5 $h$ extends holomorphically to $\Omega$. Noting that $h \in L^2_\alpha(\mathbb{D}^{d-1})$ and the Lebesgue measure of $F(\mathbb{D}^{d-1})$ equals zero, we have $h \in L^2_\alpha(\Omega)$. Thus $F(\mathbb{D}^{d-1})$ is strongly $L^2_\alpha$-removable in $\Omega$, and so it is strongly $L^2_\alpha$-removable in $\mathbb{C}^d$ to complete the proof. \qed
Now we are ready to present the proof of Theorem 1.7.

**Proof of Theorem 1.7.** Let $E$ be a good local zero variety of a domain $\Omega$. We will investigate the structure of $\overline{E}$. To do this, let $\mathcal{R}_E$ be the set of these points $\lambda$ in $\overline{E}$ with the following property:

There is an open neighborhood $W$ of $\lambda$ and a holomorphic function $h$ on $\overline{W}$ such that

$$\overline{E} \cap W = \overline{\Omega} \cap \{z \in W : h(z) = 0\},$$

and $\{z \in W : h(z) = 0\}$ is biholomorphic to an open set in $\mathbb{C}^{d-1}$.

Then $\mathcal{R}_E$ is a relatively open set in $\overline{E}$. Let $\mathcal{S}_E$ denote the complement of $\mathcal{R}_E$ in $\overline{E}$. Hence $\mathcal{S}_E$ is a closed set.

We will show that the Hausdorff dimension of $\mathcal{S}_E$ is at most $2d - 4$. To do so, let $\lambda$ be a point in $\overline{E}$. Since $E$ is a good local zero variety, there are a holomorphic function $h$ and an open neighborhood $W$ of $\lambda$ such that

$$W \cap \overline{E} = \{z \in W \cap \overline{\Omega} : h(z) = 0\}.$$

Let $E_W = W \cap \overline{E}$, $(\mathcal{R}_E)_W = \mathcal{R}_E \cap W$, and

$$(\mathcal{S}_E)_W = \mathcal{S}_E \cap W.$$

The proposition in [9, p. 22] gives that $\mathcal{R}_E$ is a complex manifold with dimension $d - 1$ and the proposition in [9, p. 20] implies that $(\mathcal{S}_E)_W$ is the singular locus of $E_W$ which is made of the singular points of $E_W$ and finitely many complex submanifolds with dimensions less than $d - 1$. By Corollary 3 [9, p. 24] we have that the Hausdorff dimension of $(\mathcal{S}_E)_W$ is at most $2d - 4$.

Since $\mathcal{S}_E$ is compact, $\mathcal{S}_E$ is covered by finitely many open sets $\{W_i\}_{i=1}^n$ and the Hausdorff dimension of each of $\{(\mathcal{S}_E)_{W_i}\}_{i=1}^n$ is at most $2d - 4$. Noting

$$\mathcal{S}_E = \bigcup_{i=1}^n (\mathcal{S}_E)_{W_i},$$

we get that the Hausdorff dimension of $\mathcal{S}_E$ is at most $2d - 4$.

The rest of the proof is similar to the proof of Lemma 3.7. For each $\varepsilon > 0$, let $V(\varepsilon)$ be the $\varepsilon$-neighborhood of $F(\mathcal{S}_E)$. Since $F$ is holomorphic on $\overline{\Omega}$, there is a positive integer $k$ such that for each point $w$ in the $\frac{1}{k}$-neighborhood $\mathcal{N}(k)$ of $\mathcal{S}_E$,

$$F(w) \in V(\varepsilon). \quad (3.2)$$

This gives that for each point $\lambda$ in $\overline{E} \setminus \mathcal{N}(k)$, $\lambda$ is in $\mathcal{R}_E$. Thus there is a neighborhood $U_\lambda$ of $\lambda$ such that $U_\lambda \cap E$ is contained in the image of $\overline{D}^{d-1}$ under a biholomorphic map, and so $F(U_\lambda \cap E)$ is also contained in the image of $\overline{D}^{d-1}$ under a holomorphic map. By Lemma 3.7, $F(U_\lambda \cap E)$ is strongly $L_2$-removable in $\mathbb{C}^d$. Since $\overline{E} \setminus \mathcal{N}(k)$ is compact, we have that $\overline{E} \setminus \mathcal{N}(k)$ is contained in a union of finitely many sets $U_\lambda \cap E$. Thus Lemma 3.3 gives that $F(\overline{E} \setminus \mathcal{N}(k))$ is strongly $L_2$-removable in $\mathbb{C}^d$. By (3.2),

$$F(\mathcal{N}(k)) \subseteq V(\varepsilon),$$

we have

$$F(\overline{E}) \setminus V(\varepsilon) \subseteq F(\overline{E} \setminus \mathcal{N}(k)),$$

to obtain that $F(\overline{E}) \setminus V(\varepsilon)$ is strongly $L_2$-removable in $\mathbb{C}^d$.

For any domain $U$ in $\mathbb{C}^d$ and a function $h$ in $L_2^2(U \setminus F(\overline{E}))$, $h$ belongs to $L_2^2(U \setminus V(\varepsilon) \setminus F(\overline{E}))$. Since $F(\overline{E}) \setminus V(\varepsilon)$ is strongly $L_2$-removable, we have that $h$ is in $L_2^2(U \setminus V(\varepsilon))$ to get that $h$ is holomorphic in $U \setminus F(\mathcal{S}_E)$. Since $F$ is holomorphic
on a neighborhood of $E$, $F$ satisfies the Lipschitz condition on some neighborhood of the subset $S_E$ of $E$. Thus [31, p. 347, Property 4] gives that the Hausdorff dimension of $F(S_E)$ is not larger than that of $S_E$, which is at most $2d - 4$. This gives that the Hausdorff measure

$$h_{2d-2}(F(S_E)) = 0.$$  

Thus by Lemma 3.3, we have that $h$ extends holomorphically to $U$ and hence $h$ is in $L^2_a(U)$, to get that $F(E)$ is strongly $L^2_a$-removable in $C^d$.

Noting that $F(E)$ has the Lebesgue measure 0, by Lemma 3.3 we conclude that $F^{-1}(F(E))$ is strongly $L^2_a$-removable in $\Omega$ and hence $L^2_a$-removable in $\Omega$ to complete the proof of Theorem 3.1. $\square$

**Remark 3.8.** The last two paragraphs above indeed show that if $E$ is a good local zero variety of a domain $\Omega$ in $\mathbb{C}^d$, and if $F : \Omega \to \mathbb{C}^d$ is holomorphic on $\Omega$ such that the image of $F$ contains an interior point, then $F(E)$ is strongly $L^2_a$-removable in $\mathbb{C}^d$, and $F^{-1}(F(E))$ is strongly $L^2_a$-removable in $\Omega$.

A simple version of Remmert’s Proper Mapping theorem ([9, p. 65] or [28, 29]) says that if $f : \Omega_1 \to \Omega_2$ is a holomorphic proper map and $Z$ is a subvariety of $\Omega_1$, then $f(Z)$ is a subvariety of $\Omega_2$. Since a subvariety can be locally represented as the intersection of finitely many (locally-defined) zero varieties, $f(Z)$ is $L^2_a$-removable in $\Omega_2$.

We also have a corollary of Theorem 3.1.

**Corollary 3.9.** Suppose that $E$ is a zero variety of a domain $\Omega$ in $\mathbb{C}^d$, and $F$ is a holomorphic proper map on $\Omega$. Then $F^{-1}(F(E))$ is relatively closed and $L^2_a$-removable in $\Omega$.

**Proof.** Suppose that $E$ is a zero variety of a domain $\Omega$, and $F$ is a holomorphic proper map on $\Omega$. Note that by Remmert’s Proper Mapping theorem $F(E)$ is a subvariety of $F(\Omega)$. Thus $F(E)$ is relatively closed and is a local zero variety, and so is $F^{-1}(F(E))$. By Theorem 3.1, $F^{-1}(F(E))$ is strongly $L^2_a$-removable in $\Omega$. Thus each function in $L^2_a(\Omega \setminus F^{-1}(F(E)))$ extends holomorphically to $\Omega$. Noting that the Lebesgue measure of $F^{-1}(F(E))$ equals zero, we conclude that $F^{-1}(F(E))$ is $L^2_a$-removable to complete the proof. $\square$

It is known that if $E$ is a zero variety of a domain $\Omega$, then $\Omega \setminus E$ is connected [31, Chapter 14]. We need the following result.

**Proposition 3.10.** Suppose that $E$ is a local zero variety of a domain $\Omega$ in $\mathbb{C}^d$. Then we have the following:

(i) if $E$ is good and $F : \Omega \to \mathbb{C}^d$ is holomorphic on $\overline{\Omega}$ such that the image of $F$ has an interior point, then $\Omega \setminus F^{-1}(F(E))$ is connected.

(ii) if $F$ is a holomorphic proper map on $\Omega$, then $\Omega \setminus F^{-1}(F(E))$ is connected.

**Proof.** To prove (i) we suppose that $E$ is a good local zero variety of a domain $\Omega$, and $F : \Omega \to \mathbb{C}^d$ is a holomorphic map such that the image of $F$ has an interior point. First we note that

$$F^{-1}(F(E)) \subseteq Z(JF) \cup \{z \in \Omega : JF(z) \neq 0, F(z) \in F(\overline{E})\}.$$  

If $JF(\lambda) \neq 0$, there is an open neighborhood $U(\lambda)$ of $\lambda$ such that $F$ is biholomorphic on $U(\lambda)$. This gives that $\{z \in \Omega : JF(z) \neq 0, F(z) \in F(\overline{E})\}$ is contained in a
union of countably many sets \((F|_{U(\lambda)})^{-1}(F(\overline{E}))\). The proof of Theorem 1.7 gives that \(F(\overline{E})\) is contained in a union of finitely many sets whose Hausdorff dimensions are at most \(2d - 2\). Hence the Hausdorff measure \(h_{2d-2}(F(\overline{E}))\) is finite. Thus \(\{z \in \Omega : JF(z) \neq 0, F(z) \in F(\overline{E})\}\) is contained in a union of countably many sets whose Hausdorff dimensions are at most \(2d - 2\). As [31] Theorem 14.4.9 states that a zero variety of a domain in \(\mathbb{C}^d\) can be represented as a union of countably compact sets \(L_n\) whose Hausdorff measures \(h_2(L_n)\) is finite, we obtain that the Hausdorff dimension of \(F^{-1}(F(\overline{E}))\) is at most \(2d - 2\). [31] Theorem 14.4.5 states that for a connected domain \(U\) in \(\mathbb{R}^{2d}\), if a relatively closed set \(G\) can be written as the union of countably many compact sets \(K_n\) with the Hausdorff measure \(h_t(K_n)\) is finite for some \(t \in (0, 2d - 1)\), then \(U \setminus G\) is connected. Then we conclude that \(\Omega \setminus F^{-1}(F(\overline{E}))\) is connected.

To prove (ii), by Corollary 3.10 and its proof, we have that if \(F\) is a holomorphic proper map on \(\Omega\), then \(F^{-1}(F(\overline{E}))\) is relatively closed, and is a local zero variety. Then by the same argument above, we conclude that \(\Omega \setminus F^{-1}(F(\overline{E}))\) is connected.

4. Proof of Theorem 1.3

In this section we will present the proof of Theorem 1.3. We also need the notion of representing local inverse, which is of great importance in the proof of Theorem 1.3. If for some operator \(T\) in \(\mathcal{V}^*(\Phi, \Omega)\), \(T\) has the following representation:

\[
Th(w) = \sum_{j=1}^{N} c_j h(\rho_j(w))(J\rho_j)(w), \quad h \in L^2_a(\Omega), \quad w \in \Delta,
\]

on an open domain \(\Delta\) where \(c_j\) are constants and all \(\rho_j\) are local inverses of \(\Phi\) on \(\Delta\), then \(\rho_k\) is called a representing local inverse for \(\mathcal{V}^*(\Phi, \Omega)\) on \(\Delta\) provided that \(c_k\) does not equal 0. The representing local inverse is first introduced in [34] on the unit disk \(\mathbb{D}\).

From now on, let \(\Phi\) denote \((\phi_1, \cdots, \phi_d)\), where each \(\phi_j(1 \leq j \leq d)\) is a holomorphic function on \(\Omega\). Let \(Z\) denote the set of zeros of the determinant of the Jacobian \(J\Phi\) of \(\Phi\).

We are ready to present the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Suppose that the interior of \(\overline{\Omega}\) equals \(\Omega\) and \(\Phi : \Omega \to \mathbb{C}^d\) is holomorphic on a neighborhood of \(\overline{\Omega}\) such that the image of \(\Phi\) has an interior point. First we show that \(\mathcal{V}^*(\Phi, \Omega)\) is of finite dimension. Since the image of \(\Phi\) has an interior point, the complex dimension of \(\Phi(\Omega)\) equals \(d\). This gives that \(J\Phi\) does not identically vanish. Then \(\Phi(\overline{Z})\) is a closed set with the Lebesgue measure zero. As \(\Phi\) may map some points in \(\Omega\) to some points in \(\partial\Omega\), for each \(\lambda \in \Omega \setminus \overline{\Phi^{-1}(\Phi(\overline{Z}))}\), \(\Phi^{-1}(\Phi(\lambda)) \cap \partial\Omega\) may not be empty. Write \(\Phi^{-1}(\Phi(\lambda)) \cap \overline{\Omega} = \{\lambda_1, \cdots, \lambda_N\}\). Some of \(\lambda_j\) may be in \(\partial\Omega\). We will use the property of \(\Omega\) that the interior of \(\overline{\Omega}\) equals
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Ω to show that for some neighborhood of λ, the images of an open subset of the neighborhood under local inverses are contained either in Ω or in \( C^d \setminus \overline{\Omega} \).

For each point \( \lambda_j \), since the Jacobian \( J\Phi \) does not vanish at \( \lambda_j \), there exists a neighborhood of \( \lambda_j \) which is mapped biholomorphically to a neighborhood of \( \Phi(\lambda_j) = \Phi(\lambda) \). Therefore there exist a disk \( \Delta \) containing \( \lambda, N \) domains \( \Delta_1, \cdots, \Delta_N \) and biholomorphic maps \( \rho_1, \cdots, \rho_N \) such that

\[
\bigcup_{j=1}^{N} \Delta_j = \Phi^{-1}(\Phi(\Delta)),
\]

(4.1)

where \( \rho_1(z) = z, \Delta_1 = \Delta \) and \( \Phi \circ \rho_j = \Phi, 1 \leq j \leq N \).

For each point \( \lambda_j \), since the Jacobian \( J\Phi \) does not vanish at \( \lambda_j \), there exists a neighborhood of \( \lambda_j \) which is mapped biholomorphically to a neighborhood of \( \Phi(\lambda_j) = \Phi(\lambda) \). Therefore there exist a disk \( \Delta \) containing \( \lambda, N \) domains \( \Delta_1, \cdots, \Delta_N \) and biholomorphic maps \( \rho_1, \cdots, \rho_N \) such that

\[
\bigcup_{j=1}^{N} \Delta_j = \Phi^{-1}(\Phi(\Delta)),
\]

(4.1)

where \( \rho_1(z) = z, \Delta_1 = \Delta \) and \( \Phi \circ \rho_j = \Phi, 1 \leq j \leq N \).

Note some \( \Delta_i \) may have nonempty intersection with \( \partial \Omega \).

In fact, \( \Phi^{-1}(\Phi(\lambda)) \cap \partial \Omega \) may not be empty. If for some \( i \), \( \rho_i(\lambda) \) is in \( \partial \Omega \), then \( \rho_i(\Delta_1) \cap \partial \Omega \neq \emptyset \). We will obtain some formula like

\[
Sg(w) = \sum_j c_j g(\rho_j(w))(J\rho_j)(w), \quad w \in \Delta,
\]

which \( c_i = 0 \) if \( \rho_i(\lambda) \in \partial \Omega \) and \( g \in L^2_a(\Omega) \). Here the disk \( \Delta \) may shrink.

We will derive the above formula and show \( c_2 = 0 \) if \( \rho_2(\lambda) \in \partial \Omega \). Since the interior of \( \Omega \) equals \( \Omega \), by Remark 1.4(2) there is a domain \( U \) contained in \( \Delta_2(\Delta_2 = \rho_2(\Delta)) \) such that \( U \) has no intersection with \( \Omega \). Set \( \tilde{\Delta} = \rho_2^{-1}(U) \subseteq \Delta \), and we have \( \lambda \not\in \tilde{\Delta} \). By shrinking \( \tilde{\Delta} \) finitely many times (we still keep the notation \( \tilde{\Delta} \)), one has that for each \( j \) either \( \rho_j(\tilde{\Delta}) \subseteq \Omega \) or \( \rho_j(\tilde{\Delta}) \cap \overline{\Omega} = \emptyset \). By (4.1), we immediately get

\[
\bigcup_{j=1}^{N} \tilde{\Delta}_j = \Phi^{-1}(\Phi(\tilde{\Delta}))
\]

(4.2)

Let \( \Lambda = \{j : 1 \leq j \leq N, \rho_j(\tilde{\Delta}) \subseteq \Omega\} \). Thus

\[
\bigcup_{j \in \Lambda} \tilde{\Delta}_j = \Phi^{-1}(\Phi(\tilde{\Delta})) \cap \Omega.
\]

We will give the representation of those operators in the von Neumann algebra \( \mathcal{V}^*(\Phi, \Omega) \) and the idea comes from [18]. To do this, let \( S \) be a unitary operator in \( \mathcal{V}^*(\Phi, \Omega) \). Then \( S \) commutes with every operators in \( \{M_{\phi_j} : 1 \leq j \leq d\} \). For any functions \( g \) and \( h \) in \( L^2_a(\Omega) \), let

\[
\Phi(g) = S g \quad \text{and} \quad \Phi(h) = Sh.
\]

Thus for any polynomial \( P \) of \( z_1, \cdots, z_d \), letting

\[
P(\Phi(z)) = P(\phi_1(z), \cdots, \phi_d(z))
\]

we have

\[
M_{P(\Phi)}S = SM_{P(\Phi)},
\]

so for any two polynomials \( P \) and \( Q \) of \( z_1, \cdots, z_d, \)

\[
\langle P(\Phi)g, Q(\Phi)h \rangle = \langle M_{P(\Phi)}S g, M_{Q(\Phi)}Sh \rangle = \langle SM_{P(\Phi)}g, SM_{Q(\Phi)}h \rangle = \langle P(\Phi)g, Q(\Phi)h \rangle.
\]
This implies
\[\int_\Omega \left( (P\Omega) \circ \Phi(w)g(w)\overline{h}(w) - (P\Omega) \circ \Phi(w)\overline{g(w)h(w)} \right) dV(w) = 0. \quad (4.3)\]
Now let
\[X = \text{span} \{ p\Omega : p, q \text{ are polynomials in } d \text{ variables} \}. \]
By the Stone-Weierstrass Theorem, each continuous function on \(\Phi(\Omega)\) can be uniformly approximated by members in \(X\). Thus \((4.3)\) gives
\[\int_\Omega \left( u(\Phi(w))g(w)\overline{h}(w) - u(\Phi(w))\overline{g(w)h(w)} \right) dV(w) = 0, \quad u \in C(\Phi(\Omega)). \quad (4.4)\]
By Lebesgue’s Dominated Convergence Theorem, \((4.4)\) holds for all \(u \in L^\infty(\Phi(\Omega))\). Thus for each \(u \in L^\infty(\Phi(\Delta))\), \((4.4)\) gives that
\[\int_\Omega \chi_{\Phi(\Delta)}(\Phi(w))u(\Phi(w))g(w)\overline{h}(w) dV(w) = \int_\Omega \chi_{\Phi(\Delta)}(\Phi(w))u(\Phi(w))\overline{g(w)h(w)} dV(w), \quad \text{and hence} \]
\[\int_{\Phi^{-1}(\Phi(\Delta)) \cap \Omega} u(\Phi(w))g(w)\overline{h}(w) dV(w) = \int_{\Phi^{-1}(\Phi(\Delta)) \cap \Omega} u(\Phi(w))\overline{g(w)h(w)} dV(w). \]
By \((4.2)\) we get
\[\int_{\bigcup_{j \in A} \overline{J}_j} u(\Phi(w))g(w)\overline{h}(w) dV(w) = \int_{\bigcup_{j \in A} \overline{J}_j} u(\Phi(w))\overline{g(w)h(w)} dV(w). \]
and then making changes of variables gives
\[\int_\Delta u(\Phi(z)) \sum_{j \in \Lambda} (\overline{g(h) \circ \rho_j(z)})(J \rho_j(z))|^2 dV(z) = \int_\Delta u(\Phi(z)) \sum_{j \in \Lambda} (\overline{g(h) \circ \rho_j(z)})(J \rho_j(z))|^2 dV(z). \]
Noting that \(\Phi\) is injective on \(\Phi(\Omega)\) and \(u\) can be an arbitrary function in \(L^\infty(\Phi(\Delta))\), we immediately have
\[\sum_{j \in \Lambda} (\overline{g(h) \circ \rho_j(z)})(J \rho_j(z))|^2 = \sum_{j \in \Lambda} (\overline{g(h) \circ \rho_j(z)})(J \rho_j(z))|^2, \quad z \in \Delta. \quad (4.5)\]
In fact, first \((4.5)\) holds almost everywhere on \(\Delta\), and then by continuity \((4.5)\) holds on \(\Delta\). Let \(\mathcal{H}\) be the Bergman space over \(\Delta\). Let
\[c_g^j = g(\rho_j(z))(J \rho_j(z)) \quad \text{and} \quad f^j_g = \overline{g(\rho_j(z))(J \rho_j(z))}, 1 \leq j \leq N, \quad g \in L^2_\Delta(\Omega). \]
These functions \(c_g^j\) and \(f^j_g\) are in \(\mathcal{H}\). By \((4.5)\), \(\sum_{k \in \Lambda} e_k^g \otimes e_k^h\) and \(\sum_{k \in \Lambda} f_k^g \otimes f_k^h\) have the same Berezin transform. Since the Berezin transform is injective, we have
\[\sum_{k \in \Lambda} e_k^g \otimes e_k^h = \sum_{k \in \Lambda} f_k^g \otimes f_k^h, \quad g, h \in L^2_\Delta(\Omega). \]
Let \(K = \mathbb{Z}A\). By Lemma 3.5 in [11] there is an \(K \times K\) unitary numerical matrix \(W\) such that
\[\left( g(\rho_k(w))(J \rho_k(w)) \right)_{k \in \Lambda} = \left( \overline{g(\rho_k(w))(J \rho_k(w))} \right)_{k \in \Lambda}, \quad w \in \Delta, \]
Note that \(1 \in \Lambda\). By expanding the first row of \(W\), we have that there exist \(N\) constants \(c_1, \cdots, c_N\) such that
\[\overline{g(\rho_1(w))(J \rho_1(w))} = \sum_{j \in \Lambda} c_j g(\rho_j(w))(J \rho_j)(w) = \sum_{j=1}^N c_j g(\rho_j(w))(J \rho_j)(w), \]
where \( c_j = 0 \) if \( j \not\in \Lambda \), thus to get

\[
Sg(w) = \sum_{j=1}^{N} c_j g(\rho_j(w))(J\rho_j)(w), \ w \in \tilde{\Delta}, g \in L^2_a(\Omega)
\]

as \( \rho_1(w) \equiv w \). In particular, \( c_2 = 0 \). Thus we get

\[
Sg(w) = \sum_{j \in \Lambda} c_j g(\rho_j(w))(J\rho_j)(w), \ w \in \Delta,
\]

where \( g \) ranges over analytic polynomials. Since each \( \rho_j \) is holomorphic on \( \Delta (\Delta \subseteq \Omega) \), both sides of the above equality are holomorphic on \( \Delta \). Thus the above equality also holds for each \( w \in \Delta \). As \( \rho_2(\lambda) \in \partial \Omega \), we have that \( c_2 = 0 \). Similarly we conclude that \( c_i = 0 \) if \( \rho_i(\lambda) \in \partial \Omega \). That is, for each \( j \) such that \( c_j \neq 0 \), \( \rho_j(\lambda) \in \Omega \).

Then

\[
Sg(w) = \sum_{j \in \Lambda} c_j g(\rho_j(w))(J\rho_j)(w), \ w \in \Delta, g \in L^2_a(\Omega), \quad (4.6)
\]

holds for each \( g \in L^2_a(\Omega) \).

In summary, we write

\[
Sg(w) = \sum_{j=1}^{N} c_j g(\rho_j(w))(J\rho_j)(w), \ w \in \Delta, g \in L^2_a(\Omega)
\]

where it is assumed the symbols \( j \) are rearranged such that for such \( j \) satisfying \( c_j \neq 0 \), one has \( \rho_j(w) \in \Omega \) for each \( w \in \Delta \).

Since each operator in a von Neumann algebra is a linear combination of four unitary operators in the von Neumann algebra (\[10\), Proposition 13.3], \[39\), Theorem 10.6]) each operator \( S \) in \( V^*(\Phi, \Omega) \) has the same form as (4.10). Thus \( S \) is completely determined by \( (c_1, \cdots, c_N) \) in the formula (4.6) on \( \Delta \). Noting that vectors \( (c_1, \cdots, c_N) \) belong to a subspace of \( \mathbb{C}^N \), we have

\[
\dim V^*(\Phi, \Omega) \leq N.
\]

Next we will find finitely many generators of \( V^*(\Phi, \Omega) \). To do so, recall that a local inverse \( \rho \) of \( \Phi : \Omega \to \mathbb{C}^d \) is called admissible if for each curve \( \gamma \) in \( \Omega \setminus \Phi^{-1}(\Phi(Z)) \), \( \rho \) admits analytic continuation with values in \( \Omega \). First we will show that if \( \rho \) is a representing local inverse for \( V^*(\Phi, \Omega) \), then \( \rho \) is admissible. By Proposition 3.10 \( \Omega \setminus \Phi^{-1}(\Phi(Z)) \) is connected. Since \( \Phi \) is holomorphic on \( \Omega \), \( \Phi^{-1}(\Phi(Z)) \) is relatively closed in \( \Omega \). Thus

\[
\Omega \setminus \Phi^{-1}(\Phi(Z)) = \Omega \setminus \Phi^{-1}(\Phi(Z)).
\]

Let \( \rho \) be a representing local inverse for \( V^*(\Phi, \Omega) \). Letting

\[
A_0 = \Phi^{-1}(\Phi(Z)),
\]

we will see that \( \rho \) is admissible with respect to \( A_0 \). Proposition 3.10 gives that \( \Omega \setminus A_0 \) is connected. Since \( \rho \) is a representing local inverse for \( V^*(\Phi, \Omega) \), there are an operator \( S \) in \( V^*(\Phi, \Omega) \) and an open ball \( \Delta \) such that

\[
Sg(w) = \sum_{j=1}^{N} c_j g(\rho_j(w))(J\rho_j)(w), \ w \in \Delta, g \in L^2_a(\Omega), \quad (4.7)
\]
where \( \rho_{j_0} = \rho \) for some integer \( j_0 \) and \( c_j \neq 0 \). This can be reformulated as

\[
S^* K_w = \sum_{j=1}^{N} c_j (J\rho_j)(w) K_{\rho_j(w)}, \; w \in \Delta,
\]

where \( K_w \) denotes the reproducing kernel of \( L^2_\rho(\Omega) \) at \( w \in \Omega \). For each point \( \lambda \in \Omega \setminus A_0 \), there is an open ball \( \Delta_\lambda \) containing \( \lambda \) where a similar representation as (4.7) holds for \( S \) and hence we get

\[
S^* K_w = \sum_{j=1}^{N_\lambda} c_j (J\rho_j)(w) K_{\rho_j(w)}, \; w \in \Delta_\lambda. \tag{4.8}
\]

Let \( \gamma \) be an arbitrary curve in \( \Omega \setminus A_0 \) and \( \gamma(0) \in \Delta \). Since the union of all these open balls \( \Delta_\lambda \) covers \( \gamma \), applying the Henie-Borel theorem shows that there exist finitely many such balls whose union covers \( \gamma \). If the intersection of two such balls is not empty, then by the uniqueness of the representation (4.8) we get direct continuations of the tuple \( \{\rho_j^\lambda \} : 1 \leq j \leq N \} \) and all \( N_\lambda \) are equal. Then it follows that \( \rho \) admits analytic continuation along \( \gamma \). Furthermore,

\[
c_j^\lambda \neq 0, \; 1 \leq j \leq N_\lambda,
\]

as we showed before, for these \( j \) the images \( \rho_j^\lambda(\Delta_\lambda) \) lie in \( \Omega \). Thus the images of \( \rho \) and its continuations lie in \( \Omega \). By arbitrariness of \( \gamma \), \( \rho \) is admissible with respect to \( A_0 \). So a representing local inverse for \( \mathcal{V}_*^*(\Phi, \Omega) \) is admissible.

For each point in \( \Omega \setminus A_0 \), there is a neighborhood where all members in \( [\rho] \) are holomorphic. As \( \Omega \setminus A_0 \) is connected, if \( \rho \) is a representing local inverse for \( \mathcal{V}_*^*(\Phi, \Omega) \), then the number \( k([\rho]) \) of different members in \( [\rho] \) (defined on a same domain \( \Delta \)) does not depend on the choice of the domain \( \Delta \). In this sense, we denote this integer \( k([\rho]) \) by \( \sharp [\rho] \), called the multiplicity of \( [\rho] \) [20]. Now fix a representing local inverse \( \rho \) of \( \Phi \). As done in [16] or [20], define

\[
\mathcal{E}_{[\rho]} h(w) = \sum_{\sigma \in [\rho]} h \circ \sigma(w) J\sigma(w), \; w \in \Omega \setminus A_0, \tag{4.9}
\]

where \( h \) is an arbitrary function over \( \Omega \setminus A_0 \) or \( \Omega \). In the case of \( \Phi \) being holomorphic over \( \Omega \), the right hand side of (4.9) is a finite sum. Also by the above paragraph \( \sigma(z) \in \Omega \setminus A_0 \) if \( z \in \Omega \setminus A_0 \) and \( \sigma \in [\rho] \). Then the formula (4.9) makes sense. For a local inverse \( \rho \) of \( \Phi \), let \( \rho^- \) denote the inverse of \( \rho \).

Next, we will see that if \( \rho \) is admissible, then both \( \mathcal{E}_{[\rho]} \) and \( \mathcal{E}_{[\rho^-]} \) are in \( \mathcal{V}_*^*(\Phi, \Omega) \), and \( \mathcal{E}_{[\rho]} = \mathcal{E}_{[\rho^-]} \). Also, one will see that if \( \rho \) is admissible, \( \rho \) is representing for \( \mathcal{V}_*^*(\Phi, \Omega) \). In fact, the proof of the theorem in [34, p. 526] shows that the class of all admissible local inverses of \( \Phi \) is closed under composition; if \( \rho \) is an admissible local inverse, then its inverse \( \rho^- \) is also admissible. Suppose that \( \rho \) is an admissible local inverse of \( \Phi \) with respect to \( A_0 \), defined as above. By the proof of [20, Lemma 6.3], \( \mathcal{E}_{[\rho]} \) maps each function in \( L^2_\rho(\Omega) \) to a function in \( L^2_\rho(\Omega \setminus A_0) \); and furthermore, there exists a constant \( C \) such that

\[
\|\mathcal{E}_{[\rho]} g\| \leq C \|g\|, \; g \in L^2_\rho(\Omega),
\]

Theorem [17] says that \( A_0 \) is \( L^2_\rho \)-removable in \( \Omega \), and thus

\[
L^2_\rho(\Omega \setminus A_0) = L^2_\rho(\Omega).
\]
So $E_{[\rho]}$ defines a bounded operator on $L^2_a(\Omega)$. Again by the proof of [20, Lemma 6.3], we get

$$E_{[\rho]}^* = E_{[\rho^-]}.$$  

Since both $E_{[\rho]}$ and $E_{[\rho^-]}$ commute with $M_\Phi = \{M_{\phi_j} : 1 \leq j \leq d\}$, they are in $V^*(\Phi, \Omega)$. This shows that $\rho$ is a representing local inverse for $V^*(\Phi, \Omega)$. Thus, all admissible local inverses are representing for $V^*(\Phi, \Omega)$.

Finally, we will derive a delicate form of (4.6). If $S$ is in $V^*(\Phi, \Omega)$, it has the form as (4.10)

$$Sg(w) = \sum_{j=1}^N c_j g(\rho_j(w))(J\rho_j)(w), \ w \in \Delta, \ g \in L^2_a(\Omega),$$

where $\Delta$ is a subdomain of $\Omega$. By applying techniques of analytic continuation, if $\rho_k$ and $\rho_l$ lie in the same equivalent class, then their coefficients are equal [10]; that is, $c_k = c_l$. To do this, let $\rho_l$ be the analytic continuation of $\rho_k$ along a loop $\gamma$, and for each $\rho_j$, let $\tilde{\rho}_j$ denote the analytic continuation of $\rho_j$ along $\gamma$. Then we get

$$\sum_{j=1}^N c_j g(\rho_j(w))(J\rho_j)(w) = \sum_{j=1}^N c_j g(\tilde{\rho}_j(w))(J\tilde{\rho}_j)(w), \ w \in \Delta, \ g \in L^2_a(\Omega).$$

By the uniqueness of coefficients $c_j$ and noting $\tilde{\rho}_k = \rho_l$ we have $c_k = c_l$, as desired.

Hence each operator $S$ in $V^*(\Phi, \Omega)$ can be represented as a linear span of $E_{[\rho]}$, where $\rho$ are representing local inverses for $V^*(\Phi, \Omega)$. Also, we have shown that those $\rho$ are exactly admissible local inverses of $\Phi$, and each $E_{[\rho]}$ is a well-defined bounded operator in $V^*(\Phi, \Omega)$ to complete the proof of Theorem 1.3.

Recall that for a local inverse $\rho$ of $\Phi$, $\rho^-$ denotes the inverse of $\rho$. The proof of Proposition 4.1 gives the following result.

**Proposition 4.1.** Suppose both $\Omega$ and $\Phi$ satisfy the assumptions in Theorem 1.3. Then $\rho$ is a representing local inverse for $V^*(\Phi, \Omega)$ if and only if $\rho$ is admissible. In this case, both $E_{[\rho]}$ and $E_{[\rho^-]}$ are in $V^*(\Phi, \Omega)$, and $E_{[\rho]}^* = E_{[\rho^-]}$.

In particular, $\rho$ is admissible if and only if $\rho^-$ is admissible.

We immediately have the following corollaries.

**Corollary 4.2.** Suppose both $\Omega$ and $\Phi$ satisfy the assumptions in Theorem 1.3. Then the dimension of $V^*(\Phi, \Omega)$ equals the number of equivalent classes of admissible local inverses of $\Phi$ on $\Omega$.

**Corollary 4.3.** Let $\Omega$ be a domain in Theorem 1.3. For $n$ ($n \geq d$) holomorphic functions $\phi_1, \ldots, \phi_n$ on $\overline{\Omega}$, if there are $d$ members $\phi_{i_1}, \ldots, \phi_{i_d}$ among them such that

$$J(\phi_{i_1}, \ldots, \phi_{i_d}) \neq 0$$

on $\Omega$, then $V^*(\phi_1, \ldots, \phi_n, \Omega)$ is of finite dimension, and $V^*(\phi_1, \ldots, \phi_n, \Omega)$ is generated by $E_{[\rho]}$, where $\rho$ are admissible local inverse of $\phi_1, \ldots, \phi_n$.

On the other hand, Theorem 1.3 may fail if the assumption on $\Phi$ is not satisfied.
Example 4.4. Let \( p(z_1, z_2) = z_1 z_2 \) and \( \Omega = \mathbb{B}_{2} \). Then \( \dim V^*(p, \mathbb{B}_{2}) = \infty \) as \( M_p \) has infinitely many pairwise orthogonal subspaces:

\[
\text{span} \{ p^k z_1^n : k = 0, 1, \cdots, n \in \mathbb{Z}_+ \}.
\]

Let \( \Phi = (p^2, p^3) \). Thus the image \( \Phi(\Omega) \) of \( \Phi \) is contained in

\[
\{(z^2, z^3) : z \in \mathbb{D}\},
\]

and thus \( \Phi(\Omega) \) has no interior point. Since

\[
V^*(p, \mathbb{B}_{2}) \subseteq V^*(\Phi, \mathbb{B}_{2}),
\]

\[
\dim V^*(\Phi, \mathbb{B}_{2}) = \infty.
\]

However, for a single polynomial \( q \) it may happen that \( \dim V^*(q, \Omega) < \infty \) and even \( V^*(q, \Omega) = CI \). For instance, there exist abundant polynomials \( q \) of degree one such that \( \mathcal{V}^*(q, \mathbb{D}^d) = CI \) for \( d \geq 1 \) [27, Theorem 5.1]. Also, it is shown that for positive integers \( k \) and \( l, 2 \leq \dim V^*(z^k + w^l, \mathbb{D}^d) < \infty \) [14, Theorem 1.1].

5. Proofs of Theorems 1.1 and 1.2

In this section we will present the proofs of Theorems 1.1 and 1.2 and assume that \( \Omega \) is a bounded domain in \( \mathbb{C}^d \).

Proof of Theorem 1.1 First we show that each local inverse \( \sigma \) of \( \Phi \) is admissible in \( \Omega \). To do this, let

\[
\mathcal{E} = \Phi^{-1}(\Phi(\mathcal{Z})).
\]

By Corollary 3.9 \( \mathcal{E} \) is relatively closed and \( L^2_\rho \)-removable in \( \Omega \). We will show that for each curve \( \gamma \subseteq \Omega \setminus \mathcal{E}, \sigma \) admits analytic continuation with values in \( \Omega \). In fact, by Proposition 3.10 \( \Omega \setminus \mathcal{E} \) is connected. Given a curve \( \gamma \) in \( \Omega \setminus \mathcal{E} \), Theorem 2.2 shows that for each point \( \lambda \) on \( \gamma \) there exists an enough small ball \( B_\lambda \) centered at \( \lambda \) such that

\[
\Phi^{-1}(\Phi(B_\lambda)) = \bigcup_{j=1}^{n} U_j(\lambda),
\]

where \( U_1(\lambda) = B_\lambda \), and \( \{U_j(\lambda)\}_{j=1}^{n} \) are disjoint domains on which \( \Phi \) is biholomorphic. This integer \( n \) only depends on \( \Phi \). Then it is easy to define \( n \) local inverses of \( \Phi \):

\[
\rho_1^\lambda, \rho_2^\lambda, \cdots, \rho_n^\lambda,
\]

which map \( B_\lambda \) bijectively to \( U_1(\lambda), \cdots, U_n(\lambda) \), respectively. Since \( \gamma \) is compact, there are finitely may balls \( \{B_{\lambda_k}\}_{k=1}^{N} \) whose union covers \( \gamma \). After reordering them, we may require that

\[
B_{\lambda_k} \cap B_{\lambda_{k+1}} \neq \emptyset, k = 1, \cdots, N - 1.
\]

Clearly, those local inverses \( \{\rho_{j+1}^\lambda\}_{j=1}^{n} \) on \( B_{\lambda_k} \) are direct continuations of \( \{\rho_j^\lambda\}_{j=1}^{n} \) defined on \( B_{\lambda_{k+1}} \), up to a permutation. Thus each local inverse admits analytic continuation along \( \gamma \) with values in \( \Omega \), as desired. So by arbitrariness of \( \gamma \) all local inverses of \( \Phi \) are admissible.

We claim that each operator \( S \) in \( V^*(\Phi, \Omega) \) can be represented as a linear span of \( \mathcal{E}_{[\rho]} \), where \( \rho \) are local inverses of \( \Phi \). For this, recall that

\[
\mathcal{E}_{[\rho]}h(w) = \sum_{\sigma \in [\rho]} h \circ \sigma(w)J\sigma(w), w \in \Omega \setminus \mathcal{E}, h \in L^2_\rho(\Omega),
\]
and we have just shown that for each local inverse $\rho$, $\rho$ is admissible. Using Corollary 3.9 and the discussions of the paragraph below (4.9), we have that both $E[\rho]$ and $E[\rho^{-1}]$ are bounded operators in $V^* (\Phi, \Omega)$. Besides, for a fixed point $\lambda \in \Omega \setminus \mathcal{E}$ with $\mathcal{E} = \Phi^{-1}(\Phi(Z))$, we have (5.1)

$$\Phi^{-1}(\Phi(B_\lambda)) = \bigoplus_{j=1}^n U_j(\lambda).$$

This, along with the discussions below (4.2), enable us to obtain a similar formula as (4.6) for each operator $S$ in $V^* (\Phi, \Omega)$,

$$Sg(w) = \sum_{j=1}^N c_j g(\rho_j(w)) (J\rho_j)(w), \quad w \in \Delta, \ g \in L^2_a(\Omega),$$

where $\Delta$ is a neighborhood of $\lambda$. By the technique of analytic continuation, we get that if $\rho_i$ and $\rho_k$ lie in the same equivalence $[\rho_i]$, then $c_i = c_k$. This means that $S$ can be represented as a linear span of $E[\rho]$, where $\rho$ are local inverses of $\Phi$ as desired.

Recall

$$S \Phi = \{ (z, w) \in \Omega^2 : \Phi(z) = \Phi(w), \ z \notin \Phi^{-1}(\Phi(Z)) \}.$$

To complete the proof of Theorem 1.1, we need show that the number of equivalent classes of local inverses of $\Phi$ equals the number of components of $S \Phi$. To do this, first we have two observations. On one hand, each point $(z, w)$ in $S \Phi$ has the form $(z, \rho(z))$ for some local inverse $\rho$ of $\Phi$, and there is a neighborhood $O$ of $(z, \rho(z))$ such that each point in $O \cap S \Phi$ has the form $(\lambda, \rho(\lambda))$. On the other hand, for two local inverses $\rho$ and $\sigma$ of $\Phi$, they are equivalent if and only if there is a curve $\gamma$ along which $\rho$ admits analytic continuation $\sigma$. The curve $\gamma$ gives a curve in $S \Phi$ to joint $(z, \rho(z))$ and $(w, \sigma(w))$. Thus if $\rho$ is equivalent to $\sigma$, then their images lie in the same component of $S \Phi$. By a similar discussion the converse is also true. So the number of equivalent classes of local inverses of $\Phi$ equals the number of components of $S \Phi$. This completes the proof.

In general, if $\Phi : \Omega \to \mathbb{C}^d$ is holomorphic on a neighborhood of $\overline{\Omega}$, it is likely that not all local inverses of $\Phi$ are admissible. Most of them appear to be “local” rather than “global” (the precise term is “admissible”). However, all local inverses of a holomorphic proper map are global, as shown in the proof of Theorem 1.1. Some words are in order.

**Remark 5.1.**

1. Suppose $\Phi$ is a holomorphic proper map from $\Omega$ to $\Phi(\Omega)$. Then for each curve $\gamma$ in $\Omega$, $\Phi^{-1}(\Phi(\gamma))$ is contained in $\Omega$ and hence for each local inverse $\rho$ of $\Phi$, its analytic continuation along $\gamma$ must lie in $\Omega$ (see Theorem 1.3). This enables us to drop the condition on the domain $\Omega$ that the interior points of the closure of $\Omega$ equals $\Omega$.

2. In the proof of Theorem 1.3, we assume that $\Phi$ is holomorphic on $\overline{\Omega}$ and in that case $\Phi^{-1}(\Phi(Z))$ is relatively closed in $\Omega$. Thus

$$\Omega \setminus \Phi^{-1}(\Phi(Z)) = \Omega \setminus \Phi^{-1}(\Phi(Z)).$$

Later, we do analytic continuation of local inverses of $\Phi$ in $\Omega \setminus \Phi^{-1}(\Phi(Z))$. In the situation where $\Phi$ is a holomorphic proper map on $\Omega$, we need verify
a similar statement as (5.2),
\[ \Omega \setminus \Phi^{-1}(\Phi(Z)) = \Omega \setminus \Phi^{-1}(\Phi(Z)). \]

For this, just note that by Corollary 3.9 \( \Phi^{-1}(\Phi(Z)) \) is relatively closed in \( \Omega \), which forces the above equality to hold.

(3) The proof of Corollary 3.9 gives that for a holomorphic proper map \( \Phi \) over \( \Omega \) and a zero variety \( E \) of \( \Omega \), \( \Phi^{-1}(\Phi(E)) \) is locally contained in a zero variety. Thus, for a holomorphic proper map \( \Phi \) on \( \Omega \) and a local inverse \( \rho \) of \( \Phi \), there is a relatively closed subset \( A \) of \( \Omega \) such that \( A \) is locally contained in a zero variety and for each curve \( \gamma \) in \( \Omega \setminus A \), \( \rho \) admits analytic continuation with values in \( \Omega \).

We proceed to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** If \( \Phi \) is biholomorphic, the only local inverse of \( \Phi \) is the identity map and hence by the proof of Theorem 1.3, (4.1) will become \( \Delta = \Phi^{-1}(\Phi(\Delta)) \).

The resulting formula (4.6)
\[ Sg(w) = \sum_{j=1}^{N} c_j g(\rho_j(w))(J\rho_j)(w), \quad w \in \Delta, g \in L^2_a(\Omega) \]
contains exactly one term; that is, each operator \( S \) in \( \mathcal{V}^*(\Phi, \Omega) \) is a constant tuple of the identity. Thus \( \mathcal{V}^*(\Phi, \Omega) \) is trivial.

Conversely we will show that if \( \Phi : \Omega \to \Omega' \) is a non-biholomorphic proper map then \( \mathcal{V}^*(\Phi, \Omega) \) is nontrivial. Since \( \Phi \) is not biholomorphic, Theorem 2.2 gives that there exists a nontrivial local inverse \( \rho \) of \( \Phi \). Recall that \( E[\rho]h(w) = \sum_{\sigma \in [\rho]} h \circ \sigma(w)J\sigma(w), \quad w \in \Omega \setminus \mathcal{E}, \ h \in L^2_a(\Omega) \).

Using Corollary 3.9 and the discussions of the paragraph below (4.9), we have that both \( E[\rho] \) and \( E[\rho]^{-1} \) are bounded operators in \( \mathcal{V}^*(\Phi, \Omega) \). Since \( \rho \) is not the identity map, \( E[\rho] \) is not a scalar multiple of the identity operator. This implies that \( \mathcal{V}^*(\Phi, \Omega) \) is nontrivial to complete the proof of Theorem 1.2.

\[ \Box \]

6. Some examples of \( \mathcal{V}^*(\Phi, \Omega) \)

In this section, we will show some examples of \( \mathcal{V}^*(\Phi, \Omega) \). In [15], Douglas, Puchin, and Wang showed that \( \mathcal{V}^*(\Phi, \Omega) \) is abelian if \( \Phi \) is a finite Blaschke product. Hence by Thomson’s commutant theorem on multiplication operators [34], if \( \phi \) is holomorphic on the closed unit disk \( \overline{\mathbb{D}} \) and \( \phi \) is not constant, then there is a finite Blaschke product \( B \) such that \( \mathcal{V}^*(\phi, \mathbb{D}) = \mathcal{V}^*(B, \mathbb{D}) \) is always abelian. But Example 6.2 shows that \( \mathcal{V}^*(\Phi, \Omega) \) may be not abelian in multi-variable case and Examples 6.4 and 6.5 show that \( \mathcal{V}^*(\Phi, \Omega) \) may be abelian and nontrivial in multi-variable case also.

For each map \( \Phi : \Omega \to \mathbb{C}^d \), the deck transformation group \( G(\Phi) \) of \( \Phi \) consists of all holomorphic automorphism \( \rho \) of \( \Omega \) satisfying \( \Phi \circ \rho = \Phi \). If all local inverses of a holomorphic map \( \Phi : \Omega \to \mathbb{C}^d \) lie in \( G(\Phi) \), then \( \Phi \) is called a regular map [26]. One also say that the proper holomorphic map \( \Phi \) is factored by automorphisms [11, 13] if \( \Phi \) is regular and the deck transformation group \( G(\Phi) \) is finite. For more information of such groups, we call the reader’s attention to [11, 13]. An extensive study was
made in [3], and especially in Sections 5 and 6 there, for the joint reducing subspaces of multiplication operators defined by a class of proper holomorphic maps, whose deck transformation groups are generated by so called pseudoreflections on $\mathbb{C}^n$.

It is known that a finite Blaschke product is regular if and only if it is of the form

$$m_1 \circ \varphi \circ m_2,$$

where both $m_1$ and $m_2$ are in Aut$(D)$, and $\varphi(z) = z^n$ for some positive integer $n$ [20]. However, in multi-variable case we will see more examples induced by polynomials or those arising from finite reflection groups [4]. Given a discrete group $\Gamma$, recall that the group von Neumann algebra $\mathcal{L}(\Gamma)$ is the weak closure of the linear span of all left regular representations $\{L_\rho : \rho \in \Gamma\}$ of $\Gamma$ on $l^2(\Gamma)$, defined by

$$L_\rho g(\sigma) = g(\rho^{-1}\sigma), \sigma \in \Gamma, g \in l^2(\Gamma).$$

Theorem 6.1 has an immediate corollary.

**Corollary 6.1.** Suppose $\Phi$ is a holomorphic regular proper map on $\Omega$. Then $V^*(\Phi, \Omega)$ is *-isomorphic to $\mathcal{L}(G(\Phi))$, where $G(\Phi)$ is the deck transformation group of $\Phi$.

**Proof.** Suppose $\Phi$ is a holomorphic regular proper map on $\Omega$. Then each local inverse of $\Phi$ corresponds to a member in $G(\Phi)$. Let

$$G(\Phi) = \{\rho_j : 1 \leq j \leq n\}.$$

For each local inverse of $\Phi$, $\rho$ is in $G(\Phi)$, and thus $E_\rho$ defines a unitary operator in $V^*(\Phi, \Omega)$. Letting $E_\rho$ denote $E_{\rho_j}$, Theorem 6.1 gives that $V^*(\Phi, \Omega)$ is generated by $E_\rho$ for $\rho$ in $G(\Phi)$. Noting

$$E_\rho^* E_\sigma = E_{\rho \circ \sigma}, \rho, \sigma \in G(\Phi),$$

we have that the map given by

$$\sum_{j=1}^n c_j E_{\rho_j}^* \mapsto \sum_{j=1}^n c_j L_{\rho_j},$$

is a *-isomorphism from $V^*(\Phi, \Omega)$ to $\mathcal{L}(G(\Phi))$. \qed

Note that the assumption of Corollary 6.1 can be reformulated as: $\Phi$ is a holomorphic proper map and each local inverse of $\Phi$ is holomorphic in $\Omega$. To see this, suppose $\rho$ is a local inverse of $\Phi$, then so is its inverse $\rho^-$. Since both $\rho \circ \rho^-$ and $\rho^- \circ \rho$ locally are identity, and all local inverses of $\Phi$ are holomorphic in $\Omega$, it follows that both $\rho$ and $\rho^-$ are holomorphic automorphisms of $\Omega$, forcing $\rho \in G(\Phi)$. Thus $\Phi$ is regular, as desired.

**Example 6.2.** Let $\Omega$ be $\mathbb{D}^2$ or $\mathbb{B}_2$. Let

$$p(z_1, z_2) = z_1^2 + z_2^2 \quad \text{and} \quad q(z_1, z_2) = z_1^2 z_2^2,$$

and

$$\Phi = (p, q).$$

There are exactly eight admissible local inverses of $\Phi$: $\rho_1, \cdots, \rho_8$, which are defined by

$$\rho_1(z_1, z_2) = (z_1, z_2), \rho_2(z_1, z_2) = (-z_1, z_2), \rho_3(z_1, z_2) = (z_1, -z_2),$$

$$\rho_4(z_1, z_2) = (-z_1, -z_2); \rho_5(z_1, z_2) = (z_2, z_1), \rho_6(z_1, z_2) = (-z_2, z_1),$$

$$\rho_7(z_1, z_2) = (z_1, z_2), \rho_8(z_1, z_2) = (-z_1, -z_2).$$
and
\[ \rho_7(z_1, z_2) = (z_2, -z_1), \quad \rho_8(z_1, z_2) = (-z_2, -z_1). \]
Each \( \rho_j \) induces a unitary operator \( U_j \) on \( L^2_{\Omega}(\Omega) \):
\[ U_j f = f \circ \rho_j, \quad f \in L^2_{\Omega}(\Omega). \]

By Theorem 1.3, the von Neumann algebra \( \mathcal{V}^*(\Phi, \Omega) \) is generated by \{\( U_j : 1 \leq j \leq 8 \}\). Noting that the deck transformation group \( G(\Phi) \) equals \{\( \rho_j : 1 \leq j \leq 8 \}\), we have that \( \mathcal{V}^*(\Phi, \Omega) \) is not abelian since \( G(\Phi) \) is not abelian.

The following proposition tells us that the map \( \Phi \) in Example 6.2 is a holomorphic proper map.

**Proposition 6.3.** Let \( \Omega \) be \( \mathbb{D}^2 \) or \( \mathbb{B}^2 \), and let \( \Phi(z_1, z_2) = \left( z_1^2 + z_2^2, z_1^2 z_2^2 \right) \). Then \( \Phi(\Omega) \) is open and \( \Phi \) is a holomorphic proper map from \( \Omega \) to \( \Phi(\Omega) \).

**Proof.** Let \( \Psi(z_1, z_2) = (z_1 + z_2, z_1 z_2) \). First we will show that \( \Psi \) is an open map in \( \mathbb{C}^2 \); that is, \( \Psi \) maps each open ball to an open set. If so, noting that \( \Phi(z_1, z_2) = \Psi(z_1^2, z_2^2) \), \( \Phi \) is also an open map.

To show that \( \Psi \) is open, we just need show that for a given point \( w = (w_1, w_2) \) and for any neighborhood \( U \) of \( w \), \( \Psi(U) \) contains an open ball centered at \( \Psi(w) \). Let \( \Psi(w) = (s_1, s_2) \). Noting that \( w_1 \) and \( w_2 \) are zeros of the polynomial \( v \) defined by
\[ v(x) = x^2 - s_1 x + s_2, \]
we can find two disks \( D_1 \) and \( D_2 \) centered at \( w_1 \) and \( w_2 \), respectively, such that
\[ D_1 \times D_2 \subseteq U. \]

If \( w_1 \neq w_2 \), we can require that \( \overline{D_1 \cap D_2} = \emptyset \). By applications of Rouche’s theorem on \( \partial D_1 \) and \( \partial D_2 \), respectively, if \( s' = (s'_1, s'_2) \) is close to \( \Psi(w) \), then the polynomial \( u(x) = x^2 - s'_1 x + s'_2 \) has exactly two zeros: one in \( D_1 \) and the other in \( D_2 \). If \( w_1 = w_2 \), then let \( D_2 = D_1 \). By similar discussion, the polynomial \( u(x) = x^2 - s'_1 x + s'_2 \) has exactly two zeros in \( D_1 \). In either case, \( \Psi(D_1 \times D_2) \) contains an open ball centered at \( \Psi(w) \), and so does \( \Psi(U) \). Thus both \( \Psi \) and \( \Phi \) are open maps.

Next we show that \( \Phi \) is a proper map. Noting \( \Phi \) is holomorphic on \( \overline{\Omega} \), we have that \( \Phi \) is a proper map if and only if \( \Phi(\partial \Omega) \subseteq \partial \Phi(\Omega) \). Since \( \Phi(\Omega) \) is open, it suffices to show that \( \Phi(\partial \Omega) \cap \Phi(\Omega) = \emptyset \). To do this, fix \( \lambda = (\lambda_1, \lambda_2) \) and \( \mu = (\mu_1, \mu_2) \). Let
\[ x_1 = \lambda_1^2 + \lambda_2^2 \quad \text{and} \quad x_2 = \lambda_1^2 \lambda_2^2. \]
Thus \( \lambda_1^2 \) and \( \lambda_2^2 \) are the solutions of the equation
\[ x^2 - x_1 x + x_2 = 0 \quad (x \in \mathbb{C}). \]
So, \( \Phi(\lambda) = \Phi(\mu) \) if and only if \( (\lambda_1^2, \lambda_2^2) \) is a permutation of \( (\mu_1^2, \mu_2^2) \). This is equivalent to that there exists a member \( \rho \in G(\Phi) \) (see Example 6.2) satisfying
\[ \mu = \rho(\lambda). \]
Since both \( \Omega \) and \( \partial \Omega \) are invariant under the action of \( G(\Phi) \), if \( \Phi(\lambda) = \Phi(\mu) \), we have that \( \lambda \in \partial \Omega \) if and only if \( \mu \in \partial \Omega \). This gives \( \Phi(\partial \Omega) \cap \Phi(\Omega) = \emptyset \), to complete the proof. \( \Box \)
By the proof of Proposition 6.3 we deduce that if Ω is replaced by any domain invariant under the deck transformation group G(Φ), then the same results hold. Similarly, one can prove that \((z_1 + z_2, z_1^2 + z_2^2)\) is a holomorphic proper map on either \(\mathbb{D}^2\) or \(\mathbb{B}_2\). By the same idea, one can also show that for positive integers \(\alpha_1, \ldots, \alpha_d\), \((z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots, z_d^{\alpha_d})\) defines a holomorphic proper map on \(\mathbb{B}_d\). In these cases, a direct application of Corollary 6.1 shows that \(V^*(\Phi, \Omega)\) is *-isomorphic to \(\mathcal{L}(G(\Phi))\). In fact, Proposition 6.3 induces more holomorphic proper maps as below.

**Example 6.4.** Suppose \(\Phi = (z_1^2, z_1 + z_2^2)\). Since \(\Phi\) is the composition of \((z_1, z_2, z_1 + z_2)\) and \((z_1, z_2^2)\), \(\Phi\) is a holomorphic proper map on \(\mathbb{D}^2\). By Theorem 6.4 studying the structure of \(V^*(\Phi, \mathbb{D}^2)\) reduces to determining all admissible local inverses of \(\Phi\) on \(\mathbb{D}^2\). To do so, we will solve the following equation in \(\mathbb{D}^2\):

\[
\Phi(w) = \Phi(z).
\]

Following the proof of Proposition 6.3 we see that \(\Phi(z) = \Phi(w)\) if and only if one of the following holds:

\[
\begin{align*}
&w_1 = z_1, \\
&w_2 = z_2,
\end{align*}
\]

or

\[
\begin{align*}
&w_1 = z_2^2, \\
&w_2 = z_1.
\end{align*}
\]

By solving these equations, we get three equivalent classes of admissible local inverses of \(\Phi\): \(\rho_1(z_1, z_2) = (z_1, z_2)\), \(\rho_2(z_1, z_2) = (z_1, -z_2)\) and \(\{\rho_3, \rho_4\}\) with

\[
\rho_3(z_1, z_2) = (z_2^2, \sigma(z_1)), \quad \text{and} \quad \rho_4(z_1, z_2) = (z_2^2, -\sigma(z_1)),
\]

where \(\sigma\) denotes the branch of \(\lambda \mapsto \sqrt{\lambda}\) defined on a neighborhood of \(\frac{1}{2}\) such that \(\sigma(\frac{1}{2}) = \sqrt{\frac{1}{2}}\). Note that for each curve \(\gamma \subseteq \mathbb{D}\setminus\{0\}\) with \(\gamma(0) = \frac{1}{2}\), \(\sigma\) always admits an analytic continuation along \(\gamma\) and \(-\sigma\) lies in the same equivalence with \(\sigma\); that is, \(-\sigma \sim \sigma\). These local inverses naturally derive three operators in \(V^*(\Phi, \mathbb{D}^2)\): \(I\), \(S_1\) and \(S_2\), defined by

\[
S_1 f(z_1, z_2) = f(z_1, -z_2), \quad (z_1, z_2) \in \mathbb{D}^2,
\]

and

\[
S_2 f = J \rho_3 \cdot f \circ \rho_3 + J \rho_4 \cdot f \circ \rho_4,
\]

for \(f \in L^2_\mathbb{D}(\mathbb{D})\). Formally,

\[
S_2 f(z_1, z_2) = \frac{z_2}{\sigma(z_1)} [ -f(z_2^2, \sigma(z_1)) + f(z_2^2, -\sigma(z_1))]\), \((z_1, z_2) \in (\mathbb{D}\setminus\{0\}) \times \mathbb{D},
\]

for \(f \in L^2_\mathbb{D}(\mathbb{D})\). We emphasize that \(S_2 f\) is locally defined first and then it extends analytically to the whole bidisk \(\mathbb{D}^2\). Thus

\[
\dim V^*(\Phi, \mathbb{D}^2) = 3.
\]

For a finite dimensional von Neumann algebra \(A\) on a Hilbert space \(H\), \(A\) is *-isomorphic to \(\bigoplus_{k=1}^n M_n(C)\) [12, Theorem III.1.2]. Thus \(V^*(\Phi, \mathbb{D}^2)\) is *-isomorphic to \(C \oplus C \oplus C\), and then \(V^*(\Phi, \mathbb{D}^2)\) is abelian.

We have to point out that \(\Phi\) is not a proper map on \(\mathbb{B}_d\). On the other hand, \(\Phi\) has exactly two admissible local inverses on \(\mathbb{B}_d\): \((z_1, z_2), (z_1, -z_2)\). Then by Theorem 6.5 \(\dim V^*(\Phi, \mathbb{B}_2) = 2\), and then \(V^*(\Phi, \mathbb{B}_2)\) is *-isomorphic to \(C \oplus C\).

To contrast with Example 6.4 we have the following interesting example.
Example 6.5. Suppose \( \Phi = (z_1^2 z_2^4, z_1^2 z_2^3) \). Then \( \Phi \) is a holomorphic proper map on \( \mathbb{D}^2 \) but not on \( \mathbb{B}_2 \). By the argument in Example 6.4 we get exactly twelve equivalent classes of admissible local inverses of \( \Phi \) on \( \mathbb{D}^2 \):

\[
(z_1, i^k z_2), (-z_1, -i^k z_2), (1 \leq k \leq 4);
\]

\[
(z_2^2, \pm \sigma(z_1)), (-z_2^2, \pm \sigma(z_1)); (z_2^2, \pm i \sigma(z_1)), (-z_2^2, \pm i \sigma(z_1)),
\]

where \( \sigma \) is the local inverse defined in Example 6.4. By a simple computation, not all these equivalent classes commute with each other under composition. Theorem 6.7 gives that \( \mathcal{V}^\ast(\Phi, \mathbb{D}^2) \) is not abelian.

But there are only eight admissible local inverses of \( \Phi \) on \( \mathbb{B}_2 \): \((z_1, i^k z_2)\) and \((-z_1, i^k z_2)(1 \leq k \leq 4)\), each corresponding to a unitary operator in \( \mathcal{V}^\ast(\Phi, \mathbb{B}_2) \). Applying Theorem 6.5 shows that \( \mathcal{V}^\ast(\Phi, \mathbb{B}_2) \) is \(*\)-isomorphic to \( \mathcal{L}(G(\Phi, \mathbb{B}_2)) \). Note that \( G(\Phi, \mathbb{B}_2) \) is abelian but not cyclic.

In the single-variable case, all known abelian deck transformation groups \( G(\Phi) \) are \(*\)-isomorphic to \( \mathbb{Z}_n \) or \( \mathbb{Z} \), a cyclic group. For example, the deck transformation group \( G(z^n) \) of \( z^n \) over \( \mathbb{D} \) is isomorphic to \( \mathbb{Z}_n \). If \( \Phi(z) = \exp\left(-\frac{1}{z^4}\right) \), then \( \Phi \) is a covering map from \( \mathbb{D} \) onto \( \mathbb{D} \setminus \{0\} \), and \( G(\Phi) \) is isomorphic to \( \pi_1(\mathbb{D} \setminus \{0\}) \cong \mathbb{Z} \). Example 6.3 provides a different example.

The following example comes from Examples 2 and 3 in [1].

Example 6.6. Let \( \Omega_1 \) be the domain

\[
\Omega_1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 6\}.
\]

and

\[
F(z) = (z_1 z_2 + z_1^{-1} z_2 + z_1 z_2^{-1} + z_1^{-1} z_2^{-1}, z_1 + z_1^{-1} + z_2 + z_2^{-1}).
\]

Then \( F : \Omega_1 \to F(\Omega_1) \) is a holomorphic regular proper map. Furthermore, the deck transformation group of \( F \) is the dihedral group \( G_1 \) of order 8 generated by two members

\[
(z_1, z_2) \mapsto (\frac{1}{z_1}, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto (z_2, z_1).
\]

Then by Corollary 6.4 \( \mathcal{V}^\ast(F, \Omega_1) \) is \(*\)-isomorphic to \( \mathcal{L}(G_1) \). It is clear that \( \mathcal{V}^\ast(F, \Omega_1) \) is not abelian.

Let \( \Omega_2 \) be the domain

\[
\Omega_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2|^{-1} + |z_1^2 z_2^{-1}| + |z_1^{-2} z_2| + |z_1 z_2^{-1}| + |z_1^{-1} z_2| < 8\}.
\]

Then the automorphisms of \( \Omega_2 \)

\[
(z_1, z_2) \mapsto (z_2 / z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto (z_1, z_1^2 / z_2)
\]

generate a dihedral group \( D_6 \) of order 12, which is the deck transformation group of the holomorphic regular proper map \( H = (h_1, h_2) : \Omega_2 \to H(\Omega_2) \) where

\[
h_1(z) = z_1 + z_1^{-1} + z_1^{-1} z_2 + z_1 z_2^{-1} + z_1^2 z_2^{-1} + z_1^{-2} z_2,
\]

and

\[
h_2(z) = z_2 + z_2^{-1} + z_2^3 z_2^{-1} + z_1^{-3} z_2 + z_1^{-3} z_2^2 + z_1^3 z_2^{-2}.
\]

Then \( \mathcal{V}^\ast(H, \Omega_2) \) is non-ableian since Corollary 6.4 implies that \( \mathcal{V}^\ast(H, \Omega_2) \) is \(*\)-isomorphic to \( \mathcal{L}(D_6) \).
Inspired by Proposition 6.3, we consider the permutation groups $S_d$ $(d \geq 2)$. Let

$$
\phi_1 = \sum_{1 \leq j \leq d} z_j,
$$

$$
\phi_2 = \sum_{1 \leq i < j \leq d} z_i z_j,
$$

and $\phi_d = z_1 z_2 \cdots z_d$ [21, Chapter 8]. Also put $\Phi = (\phi_1, \cdots, \phi_d)$.

**Proposition 6.7.** As defined above, $\Phi$ is an open map on $\mathbb{C}^d$. Furthermore, $\Phi$ is a holomorphic proper map on $\Omega$ if $\Omega$ is invariant under the action of $S_d$.

**Proof.** First we will prove that $\Phi$ is an open map on $\mathbb{C}^d$. To see this, we first show that for each $\lambda$ in $\mathbb{C}^d$, $\Phi^{-1}(\Phi(\lambda))$ is a finite set. This is equivalent that $\Phi(\lambda) = \Phi(\mu)$ if and only if there is a permutation $\rho$ on $\mathbb{C}^d$ such that $\mu = \rho(\lambda)$. In fact, let $\Phi(\lambda) = w$ and $w = (w_1, \cdots, w_d)$. Consider the following algebraic equation in $x$:

$$
x^d - w_1 x^{d-1} + \cdots + (-1)^{d-1} w_{d-1} x + (-1)^d w_d = 0.
$$

Let $Q$ denote the polynomial of $x$ defined by the left hand side of this equation. Since $\Phi(\lambda) = w$, $\lambda$ are $d$ zeros of $Q$, counting multiplicity. Since $\Phi(\lambda) = \Phi(\mu)$, $\mu$ are also $d$ zeros of $Q$. Thus up to a permutation $\mu$ equals $\lambda$, as desired. So applying Theorem 2.1 shows that $\Phi$ is an open map on $\mathbb{C}^d$.

Using the same argument as in the proof of Proposition 6.3, one can show that $\Phi$ is a holomorphic proper map on $\Omega$ if $\Omega$ is invariant under the action of $S_d$. □

Let $\Phi$ be defined as above Proposition 6.7. The deck transformation group $G(\Phi)$ of $\Phi$ is isomorphic to $S_d$. Since $\Omega$ is invariant under $S_d$, by Corollary 6.4.1 $V^*(\Phi, \Omega)$ is $*$-isomorphic to $L(S_d)$. Therefore, $V^*(\Phi, \Omega)$ is abelian if and only if $S_d$ is abelian, if and only if $d = 1, 2$. The case of $\Omega = \mathbb{B}_d$ or $\mathbb{D}_d$, this was obtained by [21, Proposition 8.4.6].

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School of Mathematics, East China University of Science and Technology, Shanghai, 200237, China

Email address: hs Huang@ecust.edu.cn

Department of Mathematics, Vanderbilt University, Nashville, TN, 37240, United States

Email address: dechao.zhang@vanderbilt.edu