Strategic investments in multi-stage General Lotto games

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Abstract—In adversarial interactions, one is often required to make strategic decisions over multiple periods of time, wherein decisions made earlier impact a player’s competitive standing as well as how choices are made in later stages. In this paper, we study such scenarios in the context of General Lotto games, which models the competitive allocation of resources over multiple battlefields between two players. We propose a two-stage formulation where one of the players has reserved resources that can be strategically pre-allocated across the battlefields in the first stage. The pre-allocation then becomes binding and is revealed to the other player. In the second stage, the players engage by simultaneously allocating their real-time resources against each other. The main contribution in this paper provides complete characterizations of equilibrium payoffs in the two-stage game, revealing the interplay between performance and the amount of resources expended in each stage of the game. We find that real-time resources are at least twice as effective as pre-allocated resources. We then determine the player’s optimal investment when there are linear costs associated with purchasing each type of resource before play begins, and there is a limited monetary budget.

Index Terms—game theory, General Lotto games, resource allocation, pre-commitments

I. INTRODUCTION

In resource allocation problems, system planners must make investment decisions to mitigate the risks posed by disturbances or strategic interference. In many practical settings, these investments are made over several stages leading up to the actual time of allocation. For example, security measures in cyber-physical systems are deployed over long periods of time. As such, attackers can use knowledge of pre-deployed elements to identify vulnerabilities and exploits in the defender’s strategy [1], [2]. As another example, power grid operators must bid on forward-capacity (e.g., day-ahead, hour-ahead and real-time) markets to fulfill future demand. Although grid operators can significantly reduce energy prices and carbon emissions by procuring capacity in day-and-hour-ahead markets, they still rely on real-time markets to account for uncertainty in the demand signal [3], [4]. Further examples include R&D contests, team management in competitive sports, and political lobbying [5].

Indeed, there are numerous real-world examples of systems in which both early and late investments contribute to the system performance. Notably, many of these scenarios consist of strategic interactions between competitors, and exhibit trade-offs when investing in pre-allocated and real-time resources (e.g., resource costs vs. flexibility in deployment, long-term vs. short-term gains). In such scenarios, system planners must choose their dynamic investments while accounting for their competitors’ decision making, and balancing the trade-offs in early and late investment.

In this manuscript, we seek to characterize the interplay between early and late investment in competitive resource allocation settings. We pursue this research agenda in the context of General Lotto games, a game-theoretic framework that explicitly describes the competitive allocation of resources between opponents. The General Lotto game is a popular variant of the classic Colonel Blotto game, wherein two budget-constrained players, A and B, compete over a set of valuable battlefields. The player that deploys more resources to a battlefield wins its associated value, and the objective for each player is to win as much value as possible. Outcomes in the standard formulations are determined by a single, simultaneous allocation of resources. In the novel formulation introduced in this paper, one of the players can strategically decide how to deploy resources before the actual engagement takes place. The placement of the pre-allocated resources thus has an effect on how the allocation decisions are made at the time of competition.

Specifically, we consider the following two-stage scenario. Player A is endowed with $P \geq 0$ resources to be pre-allocated, and both players possess real-time resources $R_A, R_B \geq 0$ to be allocated at the time of competition. In the first stage, player A decides how to deterministically deploy the pre-allocated resources $P$ over the battlefields. Player A’s endowments and pre-allocation decision then become known to player B. In the second stage, both players engage in a General Lotto game where they simultaneously decide how to deploy their real-time resources, and payoffs are subsequently derived. We assume player B does not have any pre-allocated resources at its disposal, and only has real-time resources to compete with. Each player can randomize the deployment of her real-time resources. Here, player B must overcome both the pre-allocated and real-time resources deployed by player A to secure a battlefield. A full summary of our contributions is provided below:

Our Contributions: Our main contribution in this paper is a full characterization of equilibrium payoffs to both players in our two-stage General Lotto game, given player A has $P$ pre-allocated resources, $R_A$ real-time resources, and player B has $R_B$ real-time resources (Theorem 3.1). This result also specifies how player A should optimally deploy its pre-allocated resources to the battlefields, each of which has
an arbitrary associated value. Our characterization explicitly reveals the relative effectiveness of pre-allocated and real-time resources – for any desired performance level $\Pi \geq 0$ against $R_B$, we provide the set of all pairs $(P, R_A)$ that achieve the payoff $\Pi$ for player $A$ (Theorem 3.2). As a consequence, we show that, to achieve the same performance $\Pi > 0$ using only one type of resource, player $A$ needs at least double the amount of pre-allocated resources than the amount of real-time resources (Corollary 3.1).

Leveraging the main results, we then derive the optimal investment pair $(P, R_A)$ for player $A$ when there are linear per-unit costs to invest in both types of resources and a limited monetary budget is available. We note that it is optimal to invest in both resources only if the per-unit cost of pre-allocated resources is lower than real-time resources. Indeed, pre-allocated resources are less effective than real-time resources, since their deployment is not randomized and player $B$ knows their placement.

**Related works:** This manuscript takes preliminary steps towards developing analytical insights about competitive resource allocation in multi-stage scenarios. There is widespread interest in this research objective, where the focus ranges from zero-sum games [6], [7], [8], and dynamic games [9], [10], to Colonel Blotto games [11], [12], [13], [14]. The goal of many of these works is to develop computational tools to compute decision-making policies for agents in adversarial and uncertain environments. In comparison, our work provides explicit, analytical characterizations of equilibrium strategies, allowing for insights relating the players’ performance with various elements of adversarial interaction to be drawn. As such, our work is related to a recent research thread studying Colonel Blotto games in which allocation decisions are made over multiple stages [15], [16], [17], [18], [19].

Our work also draws significantly from the primary literature on Colonel Blotto and General Lotto games [20], [21], [22], [23]. In particular, the simultaneous-move subgame played in the second stage of our model was first proposed by Vu and Loiseau [23], and is known as the General Lotto game with favoritism (GL-F). Favoritism refers to the fact that pre-allocated resources provide an inherent advantage to one player’s competitive chances. Their work establishes existence of equilibria and develops computational methods to calculate them to arbitrary precision. However, this prior work considers pre-allocated resources as exogenous parameters of the game. In contrast, we model the deployment of pre-allocated resources as a strategic element of the competitive interaction. Furthermore, we provide the first analytical characterization of equilibria and the corresponding payoffs in GL-F games.

**II. Problem Formulation**

The General Lotto game with pre-allocations (GL-P) is a two-stage game with players $A$ and $B$, who compete over a set of $n$ battlefields, denoted as $B$. Each battlefield $b \in B$ is associated with a known valuation $w_b > 0$, which is common to both players. Player $A$ is endowed with a pre-allocated resource budget $P > 0$ and a real-time resource budget $R_A > 0$. Player $B$ is endowed with a real-time resource budget $R_B > 0$, but no pre-allocated resources. The two stages are played as follows:

- **Stage 1:** Player $A$ decides how to allocate her $P$ pre-allocated resources to the battlefields, i.e., it selects a vector $p = (p_1, \ldots, p_n) \in \Delta_n(P) := \{p' \in \mathbb{R}_+^n : \|p'\|_1 = P\}$. We term the vector $p$ as player $A$’s pre-allocation profile. No payoffs are derived in Stage 1, and $A$’s choice $p$ becomes binding and common knowledge.

- **Stage 2:** Players $A$ and $B$ compete in a simultaneous-move sub-game $\Gamma$ over $B$ with their real-time resource budgets $R_A$, $R_B$. Here, both players can randomly allocate these resources as long as their expenditure does not exceed their budgets in expectation. Specifically, a strategy for player $i \in \{A, B\}$ is an $n$-variate (cumulative) distribution $F_i$ over allocations $x_i \in \mathbb{R}_+^n$ that satisfies

$$\mathbb{E}_{x_i \sim F_i} \left[ \sum_{b \in B} x_{i,b} \right] \leq R_i.$$  
(1)

We use $\mathcal{L}(R_i)$ to denote the set of all strategies $F_i$ that satisfy (1). Given that player $A$ chose $p$ in Stage 1, the expected payoff to player $A$ is given by

$$u_A(p, F_A, F_B) := \mathbb{E}_{x_A \sim F_A, x_B \sim F_B} \left[ \sum_{b \in B} w_b \cdot I(x_{A,b} = p_b, qx_{B,b}) \right]$$  
(2)

where $I(a, b) = 1$ if $a > b$, and $I(a, b) = 0$ otherwise for any two numbers $a, b \in \mathbb{R}_+$. In words, player $B$ must overcome player $A$’s pre-allocated resources $p_b$ as well as player $A$’s allocation of real-time resources $x_{A,b}$ in order to win battlefield $b$. The parameter $q > 0$ is the relative quality of player $B$’s real-time resources against player $A$’s resources. When $q < 1$ (resp. $q > 1$), they are less (resp. more) effective than player $A$’s resources. The payoff to player $B$ is $u_B(p, F_A, F_B) = W - u_A(p, F_A, F_B)$, where we denote $W = \sum_{b \in B} w_b$.

Stages 1 and 2 of GL-P are illustrated in Figure 1a. We denote an instance of GL-P as $\text{GL-P}(q, P, R_A, R_B, \mathbb{w})$, and note that the Stage 2 sub-game (i.e., the game with fixed pre-allowing profile) is an instance of the General Lotto game with favoritism [23]. For a given GL-P instance $G$, we define an equilibrium as any joint strategy profile $(p^*, F_A^*, F_B^*) \in \Delta_n(P) \times \mathcal{L}(R_A) \times \mathcal{L}(R_B)$ that satisfies

$$u_A(p^*, F_A^*, F_B^*) \geq u_A(p, F_A, F_B^*) \quad \text{and} \quad u_B(p^*, F_A^*, F_B^*) \geq u_B(p^*, F_A^*, F_B)$$  
(3)

for any $p \in \Delta_n(P)$, $F_A \in \mathcal{L}(R_A)$ and $F_B \in \mathcal{L}(R_B)$.

Notably, player $A$’s strategy consists of her deterministic pre-allocation profile $p$ in Stage 1, as well as her randomized allocation decisions over multiple stages. This property is common in the General Lotto literature, see, e.g., [22], [23].
allocation of real-time resources $F_A$ in Stage 2. It follows from the results in [23] that an equilibrium exists in any GL-P instance $G$, and that the players’ equilibrium payoffs $\pi^*_i(G) = u_i(p^*, F_A^*, F_B^*)$, $i \in \{A, B\}$, are unique. For ease of notation, we will use $\pi^*_i(P, R_A, R_B)$, $i \in \{A, B\}$, to denote the players’ equilibrium payoffs in $G$ when the dependence on the scalar $q$ and the vector $w$ is clear.

### III. MAIN RESULTS

In this section, we present our main result: the characterization of players’ equilibrium payoffs in the GL-P game. We then use this result to derive an expression for the level sets of the function $\pi^*_A(P, R_A, R_B)$ in $(P, R_A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, and to compare the relative effectiveness of pre-allocated and real-time resources.

For any given GL-P game instance, the result below provides an explicit characterization of player $A$’s equilibrium payoff $\pi^*_A(P, R_A, R_B)$. Note that player $B$’s equilibrium payoff is simply $\pi^*_B(P, R_A, R_B) = W - \pi^*_A(P, R_A, R_B)$.

**Theorem 3.1.** Consider a GL-P game instance with $q, P, R_A, R_B > 0$, and $w \in \mathbb{R}^n_{\geq 0}$. The following conditions characterize player $A$’s equilibrium payoff $\pi^*_A(P, R_A, R_B)$:

1. If $qR_B < P$, or $qR_B \geq P$ and $R_A \geq \frac{2(qR_B - P)^2}{qR_B}$, then $\pi^*_A(P, R_A, R_B)$ is
   \[ W \cdot \left(1 - \frac{qR_B}{2R_A} \frac{R_A + \sqrt{R_A(R_A + 2P)}}{P + R_A + \sqrt{R_A(R_A + 2P)}} \right)^2. \]
   \[ (4) \]
2. Otherwise, $\pi^*_A(P, R_A, R_B)$ is
   \[ W \cdot \frac{R_A}{2(qR_B - P)}. \]
   \[ (5) \]

The derivation of the above result is challenging because explicit expressions for the players’ payoffs in the Stage 2 sub-game are generally not attainable for arbitrary vector $p \in \Delta_n(P)$. Moreover, these payoffs are not generally concave. Our approach is to show that for any $p \neq p^*$, the payoff is nondecreasing in the direction pointing to $p^*$. The full proof is provided in Appendix B of the full version [24], and relies on methods developed in [23] (reviewed in Appendix A).

Interestingly, expressions (4) and (5) *do not* depend on the specific vector $w$, and rather depend only on the cumulative value $W = \sum_{b \in B} w_b$. It immediately follows that – keeping $q$, $P$, $R_A$ and $R_B$ fixed – the players’ equilibrium payoffs remain the same under any vector $w'$ such that $\sum_{b} w'_b = W$. Recall that this property also applies to the equilibrium payoffs in the (one-stage) General Lotto game [22], and follows because it is optimal for player $A$ to deploy the pre-allocated resources proportionally to the battlefield values in Stage 1, i.e., the deployment $p^* = (P/W) \cdot w$ is optimal.

As a consequence of our main result, we are able to characterize expressions for the level curves of the function $\pi^*_A(P, R_A, R_B)$. That is, for a desired performance level $\Pi \geq 0$ and fixed $R_B$, we provide the set of all pairs $(P, R_A)$ such that $\pi^*_A(P, R_A, R_B) = \Pi$.

**Theorem 3.2.** Given any $R_B > 0$ and $w \in \mathbb{R}^n_{\geq 0}$, fix a desired performance level $\Pi \in [0, W]$. The set of all pairs $(P, R_A) \in \mathbb{R}^+$ that satisfy $\pi^*_A(P, R_A, R_B) = \Pi$ is given by the following conditions:

If $0 \leq \Pi < \frac{W}{2}$, then

\[
R_A = \begin{cases} 
\frac{2W(qR_B - P)}{qR_B W - (W - \Pi)^2} & \text{if } P \in \left[0, \frac{W - 2W(qR_B)}{W - \Pi}\right] \\
\frac{W^2(qR_B W - (W - \Pi)^2)}{2qR_B W - (W - \Pi)^2} & \text{if } P \in \left[0, \frac{W - 2W(qR_B)}{W - \Pi}\right]
\end{cases}
\]
\[ (6) \]

If $\frac{W}{2} \leq \Pi \leq W$, then

\[
R_A = \frac{(qR_B W - (W - \Pi)P)^2}{2qR_B W - (W - \Pi)W}, \text{ if } P \in \left[0, \frac{WqR_B}{W - \Pi}\right]
\]
\[ (7) \]
If $P > \frac{WqR}{W - \Pi}B$, then $\pi^*_A(P, R_A, R_B) > \Pi$ for any $R_A \geq 0$.

We plot the surface $\pi^*_A(P, R_A, R_B)$ for $(P, R_A) \in \mathbb{R}^2_+$ as well as the level curves corresponding to $\Pi \in \{0.250, 0.500, 0.625, 0.750, 0.875\}$ in Figure 1b. Notably, for any $\Pi \in (0, W)$, the level curve $R_A^\Pi(P)$ is strictly decreasing and convex in $P \in [0, \frac{qR_AW}{W - \Pi}]$, where we use $R_A^\Pi(P)$ to explicitly note the dependence on $\Pi$. Hence, the function $\pi^*_A(P, R_A, R_B)$ is quasi-concave in $(P, R_A)$.

We can use the result in Theorem 3.2 to obtain an expression for the relative effectiveness of pre-allocated and real-time resources when these are deployed in isolation. In the following corollary, we provide this expression, and observe that real-time resources are at least twice as valuable as pre-allocated resources, and can be arbitrarily more valuable in specific settings:

**Corollary 3.1.** For given $R_A, R_B > 0$, the unique value $P^w > 0$ such that $\pi^*_A(\frac{P^w}{2}, 0, R_B) = \pi^*_A(0, R_A, R_B)$ is characterized by the following expression:

$$P^w = \frac{2R_A}{2qR_B^2} \text{ if } R_A > qR_B,$$

$$= \frac{2(qR_B)^2}{2qR_B^2 - R_A} \text{ if } R_A \leq qR_B.$$  

(8)

Notably, the ratio $P^w/R_A$ is lower-bounded by 2, and $P^w/R_A \to \infty$ as $R_A \to 0^+$. The table in Figure 1c compares the relative effectiveness of pre-allocated and real-time resources corresponding with the performance levels considered in Figure 1b.

**IV. OPTIMAL INVESTMENT DECISIONS**

In this section, we consider a scenario where player $A$ has an opportunity to make an investment decision regarding its resource endowments. That is, the pair $(P, R_A) \in \mathbb{R}^2_+$ is a strategic choice made by $A$ before the game GL-P$(q, P, R_A, R_B, w)$ is played. Given a monetary budget $X_A > 0$ for player $A$, any pair $(P, R_A)$ must belong to the following set of feasible investments:

$$\mathcal{I}(X_A) := \{(P, R_A) : R_A + cP \leq X_A\}$$  

(9)

where $c \geq 0$ is the per-unit cost for purchasing pre-allocated resources, and we assume the per-unit cost for purchasing real-time resources is 1 without loss of generality. We are interested in characterizing player $A$’s optimal investment subject to the above cost constraint, and given player $B$’s resource endowment $R_B > 0$. This is formulated as the following optimization problem:

$$\pi^*_A := \max_{(P, R_A) \in \mathcal{I}(X_A)} \pi^*_A(P, R_A, R_B).$$  

(10)

In the result below, we derive the complete solution to the optimal investment problem (10).

**Corollary 4.1.** Fix a monetary budget $X_A > 0$, relative per-unit cost $c > 0$, and $R_B > 0$ real-time resources for player $B$. Then, player $A$’s optimal investment in pre-allocated resources for the optimization problem in (10) under the linear cost constraint in (9) is

$$P^* = \begin{cases} \frac{1 - c}{2c} X_A, & \text{if } c < t \\ X_A, & \text{if } c = t \\ 0, & \text{if } c > t \end{cases}$$  

(11)

where $t := \min\{1, \frac{X_A}{qR_B}\}$. The optimal investment in real-time resources is $R_A^P = X_A - cP^*$. The resulting payoff $\pi^*_A$ to player $A$ is given by

$$W \cdot \begin{cases} 1 - \frac{qR_B}{2X_A} (2 - c), & \text{if } c < t \\ \frac{X_A}{qR_B}, & \text{if } c \geq t \text{ and } \frac{X_A}{qR_B} \geq 1 \end{cases}$$  

(12)

The above solution is obtained by leveraging the level set characterization from Theorem 3.2, and the fact that the level sets are strictly decreasing and convex for $\Pi \in (0, W)$. We omit details of the proof for space considerations. A visual illustration of how the optimal investments are determined for specific settings: $c = 0.423$ and $c = 1.333$ is shown in Figure 2. The budget constraint $\mathcal{I}(X_A)$ is a line segment in $\mathbb{R}^2_+$, and we thus seek the level curve that lies tangent to the segment. In cases where the cost $c$ is sufficiently high, no level curve lies tangent to $\mathcal{I}(X_A)$, and, thus, player $A$ invests all of her budget in real-time resources.

**V. CONCLUSION**

In this manuscript, we studied the strategic role of pre-allocations in competitive interactions under a two-stage General Lotto game model. We identified an explicit expression for the set of pre-allocated and real-time budget pairs that correspond with a given desired performance. We then...
used this explicit expression to derive the optimal dynamic investment strategy under a given linear cost constraint, and to compare the relative effectiveness of pre-allocated and real-time resources when deployed in isolation. Exciting directions for future work include studying the strategic outcomes (i.e., equilibria) when both players can make pre-allocations, and introducing heterogeneities in players' battlefield valuations and resource effectiveness to the model.

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Appendix

A. Method to derive equilibria of second-stage subgame

The recent work of Vu and Lisonek [23] provides a general method to derive an equilibrium of the second stage subgame from the GL-P game, which is termed a General Lotto game with favoritism (GL-F). In a GL-F game, the pre-allocation vector $p$ is an exogenous parameter. We denote an instance of this game as GL-F($q, p, R_A, R_B$). The method to calculate an equilibrium involves solving the following system \(^3\) of two equations for two unknowns $(\kappa_A, \kappa_B) \in \mathbb{R}_{++}^2$:

$$R_A = \sum_{b=1}^{n} \frac{\left|h_b(\kappa_A, \kappa_B) - p_b\right|^2}{2q_wk_B}, \quad R_B = \sum_{b=1}^{n} \frac{\left|h_b(\kappa_A, \kappa_B) - p_b\right|^2}{2q_wk_A}$$  \tag{13}

where $h_b(\kappa_A, \kappa_B) := \min\{qw_Bk_B, w_b\kappa_A + p_b\}$ for $b \in B$.

The above equations correspond to the budget constraint (1) for both players. There always exists a solution $(\kappa_A, \kappa_B) \in \mathbb{R}_{++}^2$ to this system [23], and corresponds to the following equilibrium payoffs.

Lemma A.1 (Adapted from [23]). Suppose $(\kappa_A^*, \kappa_B^*) \in \mathbb{R}_{++}^2$ solves (13). Let $B_1 := \{b \in B : h_b(\kappa_A^*, \kappa_B^*) = qw_Bk_B\}$ and $B_2 = B - B_1$. Then there is a corresponding equilibrium $(F_A^*, F_B^*)$ of the game GL-F($q, p, R_A, R_B$) where player A’s equilibrium payoff is given by

$$\pi_A(q, p, R_A, R_B) := \sum_{b \in B_1} w_b \left[ \frac{\kappa_B^*}{2q_B^*} \left( 1 - \frac{w_B^2}{(qw_Bk_B)^2} \right) \right] + \sum_{b \in B_2} w_b \frac{\kappa_A^*}{2q_A^*}$$  \tag{14}

and the equilibrium payoff to player B is

$$\pi_B(q, p, R_A, R_B) = W - \pi_A(p, R_A, R_B).$$

The equilibrium strategies are characterized by marginal distributions detailed in [23].