Supmech: the Geometro-statistical Formalism Underlying Quantum Mechanics

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Abstract. As the first step in an approach to the solution of Hilbert’s sixth problem, a general scheme of mechanics, called ‘supmech’, is developed integrating noncommutative symplectic geometry and noncommutative probability theory in an algebraic framework; it has quantum mechanics (QM) and classical mechanics as special subdisciplines and facilitates an autonomous development of QM and satisfactory treatments of quantum-classical correspondence and quantum measurements (including a straightforward derivation of the von Neumann reduction rule). The scheme associates, with every ‘experimentally accessible’ system, a symplectic superalgebra and operates essentially as noncommutative Hamiltonian mechanics incorporating the extra condition that the sets of observables and pure states be mutually separating. The latter condition serves to smoothly connect the algebraically defined quantum systems to Hilbert space-based ones; the rigged Hilbert space - based Dirac bra-ket formalism naturally appears. The formalism has a natural place for commutative superselection rules. Noncommutative analogues of objects like the momentum map and the Poincaré-Cartan form are introduced and some related symplectic geometry developed.

Key Words: supmech; Hilbert’s sixth problem; noncommutative symplectic geometry; measurement problem; autonomous quantum mechanics; noncommutative probability.
Underlying everything you see
There is motion
Governed by
Noncommutative symplectics.

1. INTRODUCTION

The statement of the famous sixth problem of Hilbert (Hilbert 1902; Wightman 1976), henceforth referred to as H6, reads:

“To treat in the same manner, by means of axioms, the physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.”

It appears reasonable to have a somewhat augmented version of H6; the following formal statement is being hereby proposed for this:

“To evolve an axiomatic scheme covering all physics including the probabilistic framework employed for the treatment of statistical aspects of physical phenomena.”

A solution of this problem must include a satisfactory treatment of the dynamics of the universe and its subsystems. Since all physics is essentially mechanics, the formalism underlying such a solution must be an elaborate scheme of mechanics (with elements of probability incorporated). Keeping in view the presently understood place of quantum mechanics (QM) in the description of nature, such a scheme of mechanics must incorporate, at least as a subdiscipline or in some approximation, an ambiguity-free development of QM without resorting to the prevalent practice of quantization of classical systems. One expects that, such a development will, starting with some appealing basics, connect smoothly to the traditional Hilbert space QM and facilitate a satisfactory treatment of measurements.

Since a large class of systems in nature admit a classical description to a very high degree of accuracy, the envisaged mechanics must also facilitate a transparent treatment of quantum-classical correspondence. For this to be feasible, the underlying framework must be such that both quantum and classical mechanics can be described in it (Dass 2002).

The literature on the foundations of QM (Dirac 1958; von Neumann 1955; Jordan, von Neumann and Wigner 1934; Segal 1947; Mackey 1963; Jauch
(1968; Jammer 1974; Holevo 1982; Ludwig 1985; Bell 1988; Bohm and Hiley 1993; Peres 1993; Busch, Grabowski and Lahti 1995) is quite rich and full of valuable insights; it appears, however, fair to say that, at present, there does not exist a formalism satisfying the conditions stated above. In this paper, we shall evolve a formalism of the desired sort which does the needful relating to (non-relativistic) QM and quantum-classical correspondence mentioned above and holds promise to provide a base for a solution of (the augmented) H6.

The desired formalism must have an all-embracing underlying geometry. An appealing choice for the same is noncommutative geometry (NCG) (Connes 1994; Dubois-Violette 1991; Madore 1995; Landi 1997; Gracia-Bondía, Varilly and Figuerra 2001). Noncommutativity is the hallmark of QM. Indeed, the central point made in Heisenberg’s (1925) paper that marked the birth of QM was that the kinematics underlying QM must be based on a non-commutative algebra of observables. This idea was developed into a scheme of mechanics — called matrix mechanics — by Born, Jordan, Dirac and Heisenberg (Born and Jordan 1925; Dirac 1926; Born, Heisenberg and Jordan 1926). The proper geometrical framework for the construction of the quantum Poisson brackets of matrix mechanics is provided by non-commutative symplectic structures (Dubois-Violette 1991,1995,1999; Dubois-Violette, Kerner and Madore 1994; Djemai 1995). The NCG scheme employed in these works is a straightforward generalization of the scheme of commutative differential geometry in which the algebra $C^\infty(M)$ of smooth functions on a manifold $M$ is replaced by a general (not necessarily commutative) complex associative $^*$-algebra $\mathcal{A}$ and the Lie algebra of smooth vector fields on $M$ by that of derivations on $\mathcal{A}$.

While Heisenberg presented the quantum view of observables, Schrödinger (1926) dealt with wave functions which, through the Born (1926) interpretation, brought out the important aspect of QM as an intrinsically probabilistic theory. Noncommutativity of the algebra of observables has important implications relating to the basic character of the operative probability theory — the so-called ‘quantum probability’ of which a variety of versions/formulations have appeared in literature (von Neumann 1955, Jordan, von neumann and Wigner 1934; Segal 1947; Mackey 1963; Jauch 1968; Varadarajan 1985; Accardi 1981; Parthasarathy 1992; Meyer 1995). Among these formulations, the one that suits our needs best is the one provided by the observable-state framework based on complex associative, unital (topological) $^*$-algebras (Meyer 1995; Dubin and Hennings 1990; Lassner 1984;
Inoue 1998). This choice serves the important purpose of allowing us to adopt the strategy of combining elements of noncommutative symplectic geometry and noncommutative probability in an algebraic framework.

The scheme based on normed algebras (Jordan, von Neumann and Wigner 1934; Segal 1947, 1963; Haag and Kastler 1964; Emch 1972, 1984; Bratteli and Robinson 1979, 1981; Haag 1992; Araki 1999; Bogolubov et al. 1990), although it makes use of observables and states, does not serve our needs because it is not suitable for the treatment of noncommutative symplectic geometry. Iguri and Castagnino (1999) have analyzed the prospects of a more general class of algebras (nuclear, barreled b*-algebras) as a mathematical framework for the formulation of quantum principles prospectively better than that of the normed algebras. These algebras accommodate unbounded observables at the abstract level. Following essentially the ‘footsteps’ of Segal (1947), they obtain results parallel to those in the C*-algebra theory — an extremal decomposition theorem for states, a functional representation theorem for commutative subalgebras of observables and an extension of the classical GNS theorem. In a sense, this work is complementary to the present one where the emphasis is on the development of noncommutative Hamiltonian mechanics. We have employed locally convex (super-)algebras restricted by a condition of ‘compatible completeness’ (referred to as the ‘CC condition’) on the collections of observables and pure states (it is satisfied in classical Hamiltonian mechanics and the traditional Hilbert space QM) which plays a crucial role in connecting the basic algebraic scheme of mechanics to the traditional Hilbert space QM.

In section 4 of (Dass 2002), a scheme of mechanics based on noncommutative symplectic geometry was introduced; it was designed to provide a proper geometrical setting for the matrix mechanics mentioned above. States were, however, not introduced in the algebraic setting. This work, therefore, falls short of a proper realization of the strategy mentioned above. In the present work, this deficiency has been removed and a proper integration of noncommutative symplectic geometry and noncommutative probability has been achieved. The improvement in the definition of noncommutative differential forms introduced in (Dubois-Violette 1995,1999) [i.e. demanding \( \omega(\ldots, KX, \ldots) = K\omega(\ldots, X, \ldots) \) where K is in the center of the algebra; for notation, see section 3] is also incorporated. Moreover, to accommodate fermionic objects on an equal footing with the bosonic ones, the scheme developed here is based on superalgebras. The scheme developed along the above lines is given, for easy reference, the name ‘Supmech’.
Supmech has quantum and classical mechanics as special subdisciplines. This fact appears to open the prospects of a consistent treatment of the interaction of a quantum and a classical system. In (Dass 2006), the author applied such a formalism to the treatment of measurements in QM providing what appeared to be the most natural solution to the measurement problem in QM. An important ingredient in this work was the Poisson bracket on the tensor product of two algebras [the non-super version of the formula (154) below]. Shortly after that paper appeared in the arXiv, M.J.W. Hall pointed out, in a private communication to the author, that the ‘Poisson bracket’ mentioned above does not satisfy the Jacobi identity in some cases as shown, for example, in (Caro and Salcedo 1999). A revised calculation by the author [correcting the mistake resulting from not realizing that an equation of the form (136) below need not always represent a derivation of the tensor product algebra] produced results which were partly discouraging [in that a ‘natural’/‘canonical’ symplectic structure on the tensor product of a (super-)commutative and a non- (super-)commutative (super-)algebra (both the (super-)algebras being of the above mentioned type) does not exist] and partly very very interesting: a symplectic structure of the above sort on the tensor product of two non-(super)commutative (super-)algebras exists if and only if each of the (super-)algebras has a ‘quantum symplectic structure’ [i.e. one which gives a Poisson bracket which is a (super-)commutator up to multiplication by a constant $(i\lambda^{-1})$ where $\lambda$ is a real-valued constant of the dimension of action] characterized by a universal parameter $\lambda$. The formalism, therefore, has a natural place for the Planck constant as a universal constant — just as special relativity has a natural place for a universal speed. In fact, the situation in supmech is somewhat better because, whereas, in special relativity, the existence of a universal speed is postulated, in supmech, the existence of a universal Planck-like constant is dictated/predicted by the formalism.

The negative result about the possibility of a consistent quantum-classical interaction in the supmech framework is by no means ‘fatal’ for the treatment of measurement problem in supmech. It turns out that it is adequate to treat the apparatus as a genuine quantum system approximated well by a classical system (in the setting of, for example, phase space descriptions of quantum and classical dynamics); the fact that supmech has both quantum and classical mechanics as special subdisciplines facilitates such a treatment.

The detailed plan of the rest of the paper is as follows. In section 2, we present arguments, based on some fundamentals relating to theory con-
struction, for adopting the kind of formalism that we do. These arguments make it quite plausible that the formalism being evolved is appropriate for doing physics at the deepest level. In section 3, essential developments in the (super-) derivation-based noncommutative differential calculus and symplectic structures are presented. The induced mappings on (super-) derivatives and differential forms (\(\Phi_*\) and \(\Phi^*\)— analogues of the push-forward and pull-back mappings induced by diffeomorphisms in the traditional differential geometry) are described; they play important roles in supmech. Section 4 is devoted to the development of the formalism of supmech; it includes, besides the basic formalism as noncommutative Hamiltonian mechanics (incorporating the CC condition), a treatment of symplectic actions of Lie groups and noncommutative avatars of the momentum map, Poincaré - Cartan form and the symplectic version of Noether’s theorem. A general treatment of localizable systems (more general and simpler than that in the traditional approaches) is also given. In section 5, elementary systems are defined in supmech and the special cases of nonrelativistic and relativistic elementary systems are treated. The role of relativity groups in the identification of fundamental observables of elementary systems is emphasized. Particles are treated as localizable elementary systems. In section 6, coupled systems are treated and the results about the symplectic structure on the tensor products of the superalgebras mentioned above are obtained. [A reasonably self-contained presentation of the non-super version of these results was given in (Dass 2007).]

Section 7 is devoted to the treatment of quantum and classical systems as special categories of systems in supmech. Quantum systems are taken up before classical systems to emphasize the autonomous nature of the treatment of QM. ‘Standard quantum systems’ are defined in the algebraic setting; the CC condition ensures the existence of their Hilbert space-based faithful realizations. [An advantage of adopting the plan outlined above is that when, treating material particles as localizable elementary quantum systems, the Schrödinger wave functions are introduced at an appropriate place, their traditional Born interpretation is obvious and the Schrödinger equation appears as a matter of course — without ever using the classical Hamiltonian or Lagrangian in the process of obtaining it.] The formalism is shown to have a natural place for commutative superselection rules. A transparent treatment of quantum - classical correspondence is given emphasizing some formal aspects. The superclassical extension of classical mechanics (incorporating fermionic objects in the setting of a supercommutative superalgebra)
is treated and is shown (for the case of a finite number of fermion generators) to generally violate the CC condition which disqualifies it from being a bonafide subdiscipline of supmech. In section 8, measurements in quantum systems are treated and a straightforward solution of the measurement problem is given along the lines mentioned above; the von Neumann reduction rule, traditionally postulated, is derived. The treatment of the apparatus (properly as a system) automatically incorporates the desirable decoherence effects to suppress the unwanted macroscopic quantum interference terms. In section 9, a set of axioms underlying the work presented in sections 4-8 is given. The last section contains some concluding remarks.

As the subject matter treated here is of considerably wide interest and some mathematical techniques employed are not sufficiently well known, the author has chosen to keep the presentation below somewhat easy paced.

This paper is a refined version of the one which appeared, with the same title, in arXiv: 0807.3604 v1 (quant-ph).

2. FROM BASICS TO ALGEBRAS

In this section, we shall present arguments based on fundamentals relating to physical theories, for making the choice of the type of formalism for supmech.

We look for the ingredients that should go into the formalism that is intended to cover all physics. To this end, we start by considering the primitive elements which every physical theory — classical, quantum, or more general — is expected to possess (explicitly or implicitly). The author came across the term ‘primitive elements of physical theory’ (PEPT) in a not so well known but an instructive and insightful paper by Houtappel, Van Dam and Wigner (HVW)(1965) which aimed at a treatment of symmetries [especially the ‘geometric invariance principles’ (space-time symmetries)] in a very general setting involving the PEPT which, according to HVW, were ‘measurements and their results’. We, however, would like to have a reasonably ‘complete’ (in the intuitive sense of ‘good enough for doing some concrete physics’) minimal set of such primitive elements. To this end, it serves well to have a close look at the ingredients going into the construction of the mathematical objects employed by HVW (the II-functions — the ‘forefathers’ of the objects presently known as histories (Griffiths 1984, 2002; Omnes 1992; Dass 2005). Doing this, one finds that such a minimal set may be taken as (Dass and Joglekar 2001, last section)
(i) observations/measurements;
(ii) description of evolution of systems (typically in terms of a discrete or continuous parameter called ‘time’);
(iii) conditional predictions about systems: given some information about a system (typically in terms of values of appropriate measurable quantities at one or more instants of time), to make predictions/retrodictions about its behavior.

Item (iii) above generally involves elements of probability. It is not difficult to see why Hilbert, while formulating his VIth problem, chose to put probability along with physics. Any formal axiomatization of probability must include hypotheses about the way uncontrollable random influences affect outcomes of experiments. Since these influences have physical causes, any comprehensive (theory construction)/axiomatization of physics must appropriately treat these causes. In any physical theory, the theoretical apparatus employed to cover item (iii) above is very much a part of the theory and, in case a (partial or total) failure of the theory occurs, may well have to be subjected to scrutiny along with other ingredients of the theory.

The desired formalism must incorporate the above three ingredients in as general a setting as possible. Introduction of appropriate mathematical objects to represent observables (measurable quantities) appears to be a must. When that has been done, a useful concept that serves to introduce elements of probability in a sufficiently general way and integrate items (i)-(iii) above is that of state. A state of a system encodes available information about the system in terms of values of appropriate observables and concretely describes preparation of a system before a measurement. Evolution of systems can be described in terms of change of state with time. Problems of conditional predictions can be formulated in terms of probabilities of transitions of systems prepared in given states to various possible states.

This reasoning leads to the prospects of a reasonably economical and general description of systems in terms of observables and states. In contrast to, for example, Araki’s (1999) book, where these objects were introduced through analysis of experiments, we have introduced them by considering some basics of theory construction. This is obviously more in tune with the general theme of the present work.

The traditional algebraic schemes (generally based on C*-algebras) have employed these objects as basic structures and have achieved some good results. They, however, do not realize the true potential of such an approach.
In these works, one generally puts the Weyl form of commutation relations (for finite as well infinite number of degrees of freedom) ‘by hand’ without connecting them to some underlying geometry. This deficiency can be overcome by dropping the restriction to normed algebras. As pointed out in the previous section, the underlying geometry of QM is noncommutative symplectic geometry whose vehicles are complex associative algebras. A program based on such algebras, integrating noncommutative symplectic geometry with noncommutative probability (arrived at in the previous section as an intuitively appealing step taken after noting the two roles of these algebras), therefore, appears to be the most appropriate for the desired formalism also from considerations based on physics fundamentals.

Taking an appropriate class of (super-)algebras as the basic objects, we shall define noncommutative symplectic structure on them. The promised mechanics (Supmech) will be developed in the form of Hamiltonian mechanics in the setting of this structure. We shall try to exploit the underlying noncommutative symplectic geometry as much as possible. For systems admitting a space-time description, for example, we shall insist on the action of the relativity group on the system algebra to be a hamiltonian action extendable, in favourable situations, to a Poisson action (for the definitions of these actions, see section 4.4) so that their infinitesimal actions are generated, through Poisson brackets, by some observables. We shall use this feature to identify the fundamental observables for appropriately defined elementary systems (material particles will be in the localizable subclass of these systems); their system algebras may then be taken as those generated by the fundamental observables. Employing appropriate tensor products, one then has a canonical procedure for setting up the system algebras for systems of a finite number of particles.

A point worth noting is the generality of the reasoning employed above. We did not restrict ourselves to any distinguished class of physical systems (particles, fields, . . .) nor did we talk about space as the arena for all dynamics. In fact, the formalism evolved will be general enough to permit, in principle, construction of theories in which one starts (for the treatment of the dynamics of the universe as a whole) with matter and its dynamics and space as an arena appears in the description of a later stage in its evolution.

3. SUPERDERIVATION-BASED DIFFERENTIAL CALCULUS; SYMPLECTIC STRUCTURES
In this section the geometrical setting underlying the main work is presented. It is a scheme of non-commutative differential calculus which is a superalgebraic version of Dubois-Violette’s scheme of noncommutative geometry (henceforth referred to as DVNCG) along with some supplementary developments by the author (Dass 1993a,2002). The induced mappings on (super-)derivations and differential forms ($\Phi_*$ and $\Phi^*$), which play an important role in the scheme of mechanics to be developed, are treated in some detail. A generalization of DVNCG, which replaces a superalgebra $\mathcal{A}$ by a pair ($\mathcal{A}$, $\mathcal{X}$) (where $\mathcal{X}$ is a Lie sub-superalgebra of the Lie superalgebra $\text{SDer}(\mathcal{A})$ of superderivations of $\mathcal{A}$) as the basic entity, is also described. This generalization will be employed in the treatment of general quantum systems admitting superselection rules.

Note. In most applications of supmech, the non-super version of the formalism developed below is adequate; this can be obtained by simply putting, in the formulas below, all the epsilons representing parities equal to zero. The reference (Dass 2007) contains a brief account of the non-super version.

3.1. Superalgebras and superderivations

In superalgebra (Manin 1988; Berezin 1987; Leites 1980; Scheunert 1979), all mathematical structures for which addition is defined (vector spaces, algebras, derivations, differential forms etc), have a $\mathbb{Z}_2$-grading where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. This means that each element of such a structure can be uniquely written as a sum of two parts each of which is assigned a definite parity (0 or 1; correspondingly it is called even or odd). Elements with definite parity are called homogeneous. When a multiplicative operation is defined between homogeneous elements of the same or different mathematical types, the product is a homogeneous element (of appropriate mathematical type) and its parity is the sum (mod 2) of the parities of the multiplicands. We shall denote the parity of a homogeneous object $w$ by $\epsilon(w)$ or $\epsilon_w$ according to convenience.

A supervector space is a (complex) vector space $V$ admitting a direct sum decomposition $V = V^{(o)} \oplus V^{(1)}$ into spaces of even and odd vectors; a vector $v \in V$ can be uniquely expressed as a sum $v = v_0 + v_1$ of even and odd vectors. A superalgebra $\mathcal{A}$ is a supervector space which is an associative algebra with identity; it becomes a $^*$-superalgebra if an antilinear $^*$-operation or involution $*: \mathcal{A} \to \mathcal{A}$ is defined which satisfies the relations

$$(AB)^* = \eta_{AB} B^* A^*, \quad (A^*)^* = A, \quad I^* = I$$
where I is the identity/unit element and $\eta_{AB} = (-1)^{c_Ac_B}$. An element $A \in \mathcal{A}$ will be called hermitian if $A^* = A$.

The supercommutator of two elements $A, B$ of a superalgebra is defined as $[A, B] = AB - \eta_{AB}BA$. For ordinary (anti-)commutators, we shall employ the notations $[A, B]|_\pi = AB \mp BA$. A superalgebra $\mathcal{A}$ is said to be supercommutative if the supercommutator of every pair of its elements vanishes.

The graded center of $\mathcal{A}$, denoted as $Z(\mathcal{A})$, consists of those elements of $\mathcal{A}$ which have vanishing supercommutators with all elements of $\mathcal{A}$; it is clearly a supercommutative superalgebra. Writing $Z(\mathcal{A}) = Z_0(\mathcal{A}) \oplus Z_1(\mathcal{A})$, the object $Z_0(\mathcal{A})$ is the traditional center of $\mathcal{A}$.

A (*-)homomorphism of a superalgebra $\mathcal{A}$ into $\mathcal{B}$ is a linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ which preserves products, identity elements, parities (and involutions):

$$\Phi(AB) = \Phi(A)\Phi(B), \quad \Phi(I_A) = I_B, \quad \epsilon(\Phi(A)) = \epsilon(A), \quad \Phi(A^*) = (\Phi(A))^*;$$

if it is, moreover, bijective, it is called a (*-)isomorphism.

A Lie superalgebra is a supervector space $\mathcal{L}$ with a superbracket operation $[ \ , \ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which is (i) bilinear, (ii) graded skew-symmetric which means that, for any two homogeneous elements $a, b \in \mathcal{L}$, $[a, b] = -\eta_{ab}[b, a]$ and (iii) satisfies the super Jacobi identity

$$[a, [b, c]] = [[a, b], c] + \eta_{ab}[b, [a, c]].$$

A (homogeneous) superderivation of a superalgebra $\mathcal{A}$ is a linear map $X : \mathcal{A} \to \mathcal{A}$ such that $X(AB) = X(A)B + \eta_{XA}AX(B)$; this is the superalgebraic generalization of the concept of derivation of an algebra. Introducing the multiplication operator $\mu$ on $\mathcal{A}$ defined as $\mu(A)B = AB$, a necessary and sufficient condition that a linear map $X : \mathcal{A} \to \mathcal{A}$ is a superderivation is

$$X \circ \mu(A) - \eta_{X\mu(A)}(A) \circ X = \mu(X(A)). \quad (1)$$

Proof. In the equation defining the superderivation $X$ above, express every term as a sequence of mappings acting on the element $B$; the resulting equation is precisely the equation obtained by operating each side of Eq.(1) on $B$. □

The set of all superderivations of $\mathcal{A}$ constitutes a Lie superalgebra $SDer(\mathcal{A})$ [$= SDer(\mathcal{A})^{(0)} \oplus SDer(\mathcal{A})^{(1)}$]; this is the superalgebraic generalization of
the Lie algebra $\text{Der}(\mathcal{A})$ of all derivations of the algebra $\mathcal{A}$. The inner superderivations $D_A$ defined by $D_A B = [A, B]$ are easily seen to satisfy the relation

$$[D_A, D_B] = D_{[A,B]}$$

and constitute a Lie sub-superalgebra $\text{ISDer}(\mathcal{A})$ of $\text{SDer}(\mathcal{A})$.

As in DVNCG, it will be implicitly assumed that the superalgebras being employed have a reasonably rich supply of superderivations so that various constructions involving them have a nontrivial content. Some discussion and useful results relating to the precise characterization of the relevant class of algebras may be found in (Dubois-Violette et al. 2001).

The following two facts involving the graded center and the superderivations will be useful [in proving the subcomplex property of $\Omega(\mathcal{A})$ below]:

(i) If $K \in Z(\mathcal{A})$, then $X(K) \in Z(\mathcal{A})$ for all $X \in \text{SDer}(\mathcal{A})$.

(ii) For any $K \in Z(\mathcal{A})$ and $X, Y \in \text{SDer}(\mathcal{A})$, we have

$$[X, KY] = X(K)Y + \eta_{XK} K [X, Y]. \quad (2)$$

Proof. (i) Expand the two sides of the relation $X(AK) = \eta_{AK} X(KA)$ (for any $A \in \mathcal{A}$) and cancel the terms containing $K$ on the two sides.

(ii) Expand $[X, KY](A)$ (for any $A \in \mathcal{A}$). □

Corollary. $\text{SDer}(\mathcal{A})$ is a $Z(\mathcal{A})$-module.

An involution $^*$ on $\text{SDer}(\mathcal{A})$ is defined by the relation $X^*(A) = [X(A^*)]^*$. We have

(i) $[X, Y]^* = [X^*, Y^*]$; (ii) $(D_A)^* = -D_A^*$.

Proof. In each case, apply the left hand side to a general element $B \in \mathcal{A}$ and follow the definitions. [For an illustration of the kind of steps involved, see the proof of the equations (4) below.] □

A superalgebra-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ induces a mapping

$$\Phi_* : \text{SDer}(\mathcal{A}) \rightarrow \text{SDer}(\mathcal{B}) \text{ given by } (\Phi_* X)(B) = \Phi(X[\Phi^{-1}(B)]) \quad (3)$$

for all $X \in \text{SDer}(\mathcal{A})$ and $B \in \mathcal{B}$. It is the analogue (and a generalization) of the push-forward mapping induced by a diffeomorphism between two manifolds on the vector fields and satisfies the expected relations (with $\Psi : \mathcal{B} \rightarrow \mathcal{C}$)

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*; \quad \Phi_*[X, Y] = [\Phi_* X, \Phi_* Y]. \quad (4)$$
Proof: (i) For any $X \in S_{der}(\mathcal{A})$ and $C \in \mathcal{C}$,

\[
[(\Psi \circ \Phi)_*X](C) = (\Psi \circ \Phi)(X[(\Psi \circ \Phi)^{-1}(C)])
= \Psi[\Phi(X[\Phi^{-1}(\Psi^{-1}(C))])]
= \Psi[(\Phi_*X)(\Psi^{-1}(C))]
= [\Psi(\Phi_*X)](C).
\]

(ii) For any $B \in \mathcal{B}$

\[
(\Phi_*[X,Y])(B) = \Phi([X,Y](\Phi^{-1}(B)))
= \Phi[X(Y(\Phi^{-1}(B))) - \eta_{XY}Y(X(\Phi^{-1}(B)))].
\]

Now insert $\Phi^{-1} \circ \Phi$ between $X$ and $Y$ in each of the two terms on the right and follow the obvious steps. □.

Note that $\Phi_*$ is a Lie superalgebra isomorphism (i.e. it is bijective and linear and preserves superbrackets).

3.2. The cochain complex $C(\text{SDer}(\mathcal{A}), \mathcal{A})$

In DVNCG, one starts with a complex associative algebra $\mathcal{A}$ and constructs a differential calculus on it which means a formalism involving differential form like objects on $\mathcal{A}$ with analogues of exterior product, exterior derivative and involution defined on them. For noncommutative $\mathcal{A}$, the choice of differential calculus is not unique; a systematic discussion of the variety of choices may be found in (Dubois-Violette 1995). In applications of NCG, one makes a choice according to convenience. Our needs are best served by a DVNCG type formalism.

For the constructions involving the superalgebraic generalization of DVNCG given in this subsection, some relevant background is provided in (Dubois-Violette 1999; Grosse and Reiter 1999; Scheunert 1979a,1979b,1983,1998). Grosse and Reiter (1999) have given a detailed treatment of the differential geometry of graded matrix algebras [generalizing the treatment of differential geometry of matrix algebras in (Dubois-Violette, Kerner and Madore 1994)]. Some related work on supermatrix geometry has also appeared in (Dubois-Violette, Kerner and Madore 1991; Kerner 1993); however, the approach adopted below is closer to (Grosse and Reiter 1999).

The central object in the developments presented below is a superalgebra $\mathcal{A}$ [complex, associative, unital (i.e. possessing a unit element), not necessarily supercommutative]; it is the counterpart of $C^\infty(M)$, the commutative
algebra of complex smooth functions on the manifold M, in commutative
groupoid. The Lie superalgebra \( SDer(A) \) is the analogue of the Lie algebra
\( \mathcal{X}(M) \) of smooth vector fields on M.

Recalling that, in the commutative differential geometry, the differential p-forms are defined as skew-symmetric multilinear maps of \( \mathcal{X}(M)^p \) into \( C^\infty(M) \), the natural first choice for the space of (noncommutative) differential p-forms is the space

\[
C^p(SDer(A), A) = C^{p,0}(SDer(A), A) \oplus C^{p,1}(SDer(A), A)
\]
of graded skew-symmetric multilinear maps (for \( p \geq 1 \)) of \( [SDer(A)]^p \) into \( A \) (the space of \( A \)-valued p-cochains of \( SDer(A) \)); it is the super-analogue of the Chevalley-Eilenberg p-cochain space (Cartan and Eilenberg 1956; Weibel 1994). We have \( C^0(SDer(A), A) = A \). For \( \omega \in C^{p,s}(SDer(A), A) \), we have

\[
\omega(\ldots, X, Y, \ldots) = -\eta_{XY} \omega(\ldots, Y, X, \ldots).
\] (5)

For a general permutation \( \sigma \) of the arguments of \( \omega \), we have

\[
\omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) = \kappa_\sigma \gamma_p(\sigma; \epsilon_{X_1}, \ldots, \epsilon_{X_p}) \omega(X_1, \ldots, X_p)
\] (6)

where \( \kappa_\sigma \) is the parity of the permutation \( \sigma \) and

\[
\gamma_p(\sigma; s_1, \ldots, s_p) = \prod_{j, k = 1, \ldots, p; j < k, \sigma^{-1}(j) > \sigma^{-1}(k)} (-1)^{s_j s_k}.
\] (7)

An involution * on the cochains is defined by the relation \( \omega^*(X_1, \ldots, X_p) = [\omega(X_1^*, \ldots, X_p^*)]^* \); \( \omega \) is said to be real (imaginary) if \( \omega^* = \omega(-\omega) \).

The exterior product

\[
\wedge : C^{p,r}(SDer(A), A) \times C^{q,s}(SDer(A), A) \to C^{p+q,r+s}(SDer(A), A)
\]
is defined as

\[
(\alpha \wedge \beta)(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \kappa_\sigma \gamma_{p+q}(\sigma; \epsilon_{X_1}, \ldots, \epsilon_{X_{p+q}})(-1)^{\sum_{j=1}^p \epsilon_{X_{\sigma(j)}}} \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).
\] (8)
With this product, the graded vector space
\[ C(SDer(A), A) = \bigoplus_{p \geq 0} C^p(SDer(A), A) \]
becomes an \( N_0 \times \mathbb{Z}_2 \)-bigraded complex algebra. (Here \( N_0 \) is the set of non-negative integers.)

The Lie superalgebra \( SDer(A) \) acts on itself and on \( C(SDer(A), A) \) through Lie derivatives. For each \( Y \in SDer(A)^{(r)} \), one defines linear mappings \( L_Y : SDer(A)^{(s)} \rightarrow SDer(A)^{(r+s)} \) and \( L_Y : C^{p,s}(SDer(A), A) \rightarrow C^{p,r+s}(SDer(A), A) \) such that the following three conditions hold:

\[ L_Y(A) = Y(A) \text{ for all } A \in A \] (9)
\[ L_Y[X(A)] = (L_Y X)(A) + \eta_{XY} X[L_Y(A)] \] (10)
\[ L_Y[\omega(X_1, \ldots, X_p)] = (L_Y \omega)(X_1, \ldots, X_p) + \sum_{i=1}^{p} (-1)^{\epsilon_Y(\epsilon_\omega + \epsilon_{X_1} + \cdots + \epsilon_{X_{i-1}})} \omega(X_1, \ldots, [X_i, X_{i+1}, \ldots, X_p]) \] (11)

The first two conditions give
\[ L_Y X = [Y, X] \] (12)
which, along with the third, gives
\[ (L_Y \omega)(X_1, \ldots, X_p) = Y[\omega(X_1, \ldots, X_p)] - \sum_{i=1}^{p} (-1)^{\epsilon_Y(\epsilon_\omega + \epsilon_{X_1} + \cdots + \epsilon_{X_{i-1}})} \omega(X_1, \ldots, [X_i, X_{i+1}, \ldots, X_p]) \] (13)

Two important properties of the Lie derivative are, in obvious notation,
\[ [L_X, L_Y] = L_{[X,Y]} \] (14)
\[ L_Y(\alpha \wedge \beta) = (L_Y \alpha) \wedge \beta + \eta_{Y\alpha} \alpha \wedge (L_Y \beta). \] (15)

For each \( X \in SDer(A)^{(r)} \), we define the interior product
\[ i_X : C^{p,s}(SDer(A), A) \rightarrow C^{p-1,r+s}(SDer(A), A) \text{ (for } p \geq 1 \text{) by} \]
\[ (i_X \omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}). \] (16)
One defines $i_X(A) = 0$ for all $A \in \mathcal{A}$. Note that there is no $\eta_{X \omega}$ factor on the right in Eq.(16). A more appropriate notation (from the point of view of proper/unambiguous placing of symbols) for $i_X \omega$ is $\omega X$. [See (Hochscild and Serre 1953).] Some important properties of the interior product are:

$$i_X \circ i_Y + \eta_{XY} i_Y \circ i_X = 0$$  \hspace{1cm} (17)

$$i_X(\alpha \wedge \beta) = \eta_{X \beta}(i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X \beta)$$  \hspace{1cm} (18)

$$(L_Y \circ i_X - i_X \circ L_Y) = \eta_X \omega i_{[X,Y]} \omega.$$  \hspace{1cm} (19)

The exterior derivative $d : C^{p,r}(SDer(\mathcal{A}), \mathcal{A}) \to C^{p+1,r}(SDer(\mathcal{A}), \mathcal{A})$ is defined through the relation

$$(i_X \circ d + d \circ i_X) \omega = \eta_{X \omega} L_X \omega.$$  \hspace{1cm} (20)

For $p = 0$, it takes the form $(dA)(X) = \eta_{X A} X(A)$.

Taking, in Eq.(20), $\omega$ a homogeneous $p$-form and contracting both sides with homogeneous derivations $X_1, \ldots, X_p$ gives the quantity $(d\omega)(X, X_1, \ldots, X_p)$ in terms of evaluations of exterior derivatives of lower degree forms. This determines $d \omega$ recursively giving

$$(d\omega)(X_0, X_1, \ldots, X_p) = \sum_{i=0}^{p} (-1)^{i+a_i} X_i[\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)]$$

$$+ \sum_{0 \leq i < j \leq p} (-1)^{i+b_{ij}} \omega(X_0, \ldots, X_{i-1}, [X_i, X_j], X_{i+1}, \ldots, \hat{X}_j, \ldots, X_p)$$  \hspace{1cm} (21)

where the hat indicates omission and

$$a_i = \epsilon_{X_i} (\epsilon_{\omega} + \sum_{k=0}^{i-1} \epsilon_{X_k}); \quad b_{ij} = \epsilon_{X_j} \sum_{k=i+1}^{j-1} \epsilon_{X_k}.$$  \hspace{1cm}

Some important properties of the exterior derivative are

$$d^2(= d \circ d) = 0; \quad d \circ L_Y = L_Y \circ d$$  \hspace{1cm} (22)

and

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$$  \hspace{1cm} (23)
where \( \alpha \) is a p-cochain. The first of equations (22) shows that the pair, 
\((C(SDer(A), A), d)\) constitutes a cochain complex. We shall call a cochain \( \alpha \) closed if \( d\alpha = 0 \) and exact if \( \alpha = d\beta \) for some cochain \( \beta \).

The cochain complex obtained above is a special case of the Chevalley-
Eilenberg cochain complex. For later use, we collect here some results relating
the Chevalley-Eilenberg cohomology (employing ordinary Lie algebras and
vector spaces; no gradings are involved).

Let \( G \) be a Lie algebra over the field \( K \) (which may be \( \mathbb{R} \) or \( \mathbb{C} \)) and \( V \) a \( G \)-module which means it is a vector space over \( K \) having defined on it a \( G \)-action associating a linear mapping \( \Psi(\xi) \) on \( V \) with every element \( \xi \) of \( G \) such that

\[
\Psi(0) = id_V \quad \text{and} \quad \Psi([\xi, \eta]) = \Psi(\xi) \circ \Psi(\eta) - \Psi(\eta) \circ \Psi(\xi).
\]

The analogue of Eq.(21) for a \( V \)-valued p-cochain \( \lambda^{(p)} \) of \( G \) is

\[
(d\lambda^{(p)})(\xi_0, \xi_1, ..., \xi_p) = \sum_{i=0}^{p}(-1)^i\Psi(\xi)[\lambda^{(p)}(\xi_0, ..., \hat{\xi}_i, ..., \xi_p)] + \sum_{0 \leq i < j \leq p}(-1)^j\lambda^{(p)}(\xi_0, ..., \xi_{i-1}, [\xi_i, \xi_j], \xi_{i+1}, ..., \hat{\xi}_j, ..., \xi_p) \quad (24)
\]

for \( \xi_0, ..., \xi_p \in G \). The subspaces of the vector space \( C^p(G, V) \) consisting of
closed cochains (cocycles) and exact cochains (coboundaries) are denoted as
\( Z^p(G, V) \) and \( B^p(G, V) \) respectively; the quotient space \( H^p(G, V) \equiv Z^p(G, V)/B^p(G, V) \)
is called the p-th cohomology group of \( G \) with coefficients in \( V \).

For the special case of the trivial action of \( G \) on \( V \) [i.e. \( \Psi(\xi) = 0 \ \forall \xi \in G \)],
a subscript zero is attached to these spaces \([C^p_0(G, V) \text{ etc}]\). In this case, for
\( p = 1 \) and \( p = 2 \), Eq.(24) takes the form

\[
d\lambda^{(1)}(\xi_0, \xi_1) = -\lambda^{(1)}([\xi_0, \xi_1])
\]

\[
d\lambda^{(2)}(\xi_0, \xi_1, \xi_2) = -[\lambda^{(2)}([\xi_0, \xi_1], \xi_2) + \text{cyclic terms in } \xi_0, \xi_1, \xi_2]. \quad (25)
\]

### 3.3. Differential forms

Taking clue from (Dubois-Violette 1995, 1999) [where the subcomplex of
\( Z(A) \)-linear cochains (\( Z(A) \) being, in his notation, the center of the algebra
\( A \) was adopted as the space of differential forms], we consider the subset
\( \Omega(A) \) of \( C(SDer(A), A) \) consisting of \( Z_0(A) \)-linear cochains. Eq.(2) ensures
that this subset is closed under the action of \( d \) and, therefore, a subcomplex.
We shall take this space to be the space of differential forms in subsequent
geometrical work. We have, of course,
\[ \Omega(A) = \oplus_{p \geq 0} \Omega^p(A) \]
with \( \Omega^0(A) = A \) and \( \Omega^p(A) = \Omega^{p,0}(A) \oplus \Omega^{p,1}(A) \) for all \( p \geq 0 \).

### 3.4. Induced mappings on differential forms

A superalgebra *-isomorphism \( \Phi : A \to B \) induces, besides the Lie
superalgebra-isomorphism \( \Phi_* : SDer(A) \to SDer(B) \), a mapping
\[ \Phi^* : C^{p,s}(SDer(B), B) \to C^{p,s}(SDer(A), A) \]
given, in obvious notation, by
\[ (\Phi^* \omega)(X_1, \ldots, X_p) = \Phi^{-1}[\omega(\Phi_* X_1, \ldots, \Phi_* X_p)]. \]  

The mapping \( \Phi \) preserves (bijectively) all the algebraic relations. Eq.(3)
shows that \( \Phi_* \) preserves \( Z_0(A) \)-linear combinations of the superderivations.
It follows that \( \Phi^* \) maps differential forms onto differential forms.

These mappings are analogues (and generalizations) of the pull-back mappings on differential forms (on manifolds) induced by diffeomorphisms. They satisfy the expected relations [with \( \Psi : B \to C \)]
\[ (\Psi \circ \Phi)^* = \Phi^* \circ \Psi^* \]  
\[ \Phi^* (\alpha \wedge \beta) = (\Phi^* \alpha) \wedge (\Phi^* \beta) \]  
\[ \Phi^*(d\alpha) = d(\Phi^* \alpha). \]

**Outlines of proofs of Eqs.(27-29):**

Eq.(27) : For \( \omega \in C^{p,s}(SDer(C), C) \) and \( X_1, \ldots, X_p \in SDer(A) \),
\[ [(\Psi \circ \Phi)^* \omega](X_1, \ldots, X_p) = (\Phi^{-1} \circ \Psi^{-1})[\omega(\Phi_* (\Phi_* X_1), \ldots, \Phi_* (\Phi_* X_p))]
= \Phi^{-1}[(\Psi^* \omega)(\Phi_* X_1, \ldots, \Phi_* X_p)]
= [\Phi^*(\Psi^* \omega)](X_1, \ldots, X_p). \] \( \square \)
Eq.(28): For \( \alpha \in C^{p,r}(SDer(\mathcal{B}), \mathcal{B}) \), \( \beta \in C^{q,s}(SDer(\mathcal{B}), \mathcal{B}) \) and \( X_1, \ldots, X_{p+q} \in SDer(\mathcal{A}) \),

\[
\Phi^*(\alpha \wedge \beta)(X_1, \ldots, X_{p+q}) = \Phi^{-1}[(\alpha \wedge \beta)(\Phi_*X_1, \ldots, \Phi_*X_{p+q})].
\]

Expanding the wedge product and noting that

\[
\Phi^{-1}[\alpha(\ldots)\beta(\ldots)] = \Phi^{-1}[\alpha(\ldots)]\Phi^{-1}[\beta(\ldots)],
\]

the right hand side is easily seen to be equal to \( [(\Phi^*\alpha)\wedge(\Phi^*\beta)](X_1, \ldots, X_{p+q}) \). \( \square \)

Eq.(29): We have

\[
\Phi^*((d\alpha))(X_0, \ldots, X_p) = \Phi^{-1}[(d\alpha)(\Phi_*X_0, \ldots, \Phi_*X_p)].
\]

Using Eq.(21) for \( d\alpha \) and noting that

\[
\Phi^{-1}[(\Phi_*X_i)(\alpha(\Phi_*X_0, \ldots))] = \Phi^{-1}[\Phi(X_i[\Phi^{-1}(\alpha(\Phi_*X_0, \ldots))])]
\]

and making similar (in fact, simpler) manipulations with the second term in the expression for \( d\alpha \), it is easily seen that the left hand side of Eq.(29), evaluated at \( (X_0, \ldots, X_p) \), equals \( [(d(\Phi^*\alpha))(X_0, \ldots, X_p)]. \) \( \square \)

Now, let \( \Phi_t : \mathcal{A} \to \mathcal{A} \) be a one-parameter family of transformations (i.e. superalgebra isomorphisms) given, for small \( t \), by

\[
\Phi_t(A) \simeq A + tg(A) \tag{30}
\]

where \( g \) is some (linear, even) mapping of \( \mathcal{A} \) into itself. The condition \( \Phi_t(AB) = \Phi_t(A)\Phi_t(B) \) gives \( g(AB) = g(A)B + Ag(B) \) implying that \( g(A) = Y(A) \) for some even superderivation \( Y \) of \( \mathcal{A} \) (to be called the infinitesimal generator of \( \Phi_t \)). From Eq.(3), we have, for small \( t \),

\[
(\Phi_t)_*X \simeq X + t[Y, X] = X + tL_YX. \tag{31}
\]

Similarly, for any \( p \)-form \( \omega \), we have

\[
\Phi_t^*\omega \simeq \omega - tL_Y\omega. \tag{32}
\]

**Proof**: We have

\[
(\Phi_t^*\omega)(X_1, \ldots, X_p) = \Phi_t^{-1}[\omega((\Phi_t)_*X_1, \ldots(\Phi_t)_*X_p)]
\]

\[
\simeq \omega(X_1, \ldots, X_p) - tY\omega(X_1, \ldots, X_p)
\]

\[
+ t \sum_{i=1}^{p} \omega(X_1, \ldots, [Y, X_i], \ldots, X_p)
\]

\[
= [\omega - tL_Y\omega](X_1, \ldots, X_p). \quad \square
\]
Note that (as in the commutative geometry), the Lie derivative term appears with a plus sign in Eq.(31) and a minus sign in Eq.(32). This is because, in the pull-back action, the effective mapping is $\Phi_t^{-1}$.

3.5. A generalization of the DVNCG scheme (Dass 1993, 2002)

In the formula (21) for $d\omega$, the superderivations $X_j$ appear on the right either singly or as supercommutators. It should, therefore, be possible to restrict them to a Lie sub-superalgebra $X$ of $SDer(A)$ and develop the whole formalism with the pair $(A, X)$ obtaining thereby a useful generalization of the formalism developed in the previous three subsections. Working with such a pair is the analogue of restricting oneself to a leaf of a foliated manifold as the first example below indicates.

Examples: (i) $A = C^\infty(R^3); X = \text{the Lie subalgebra of the Lie algebra } \mathcal{X}(R^3)$ of vector fields on $R^3$ generated by the Lie differential operators $L_j = \epsilon_{jkl}x_k\partial_l$ for the $SO(3)$-action on $R^3$. These differential operators, when expressed in terms of the polar coordinates $r, \theta, \phi$, contain only the partial derivatives with respect to $\theta$ and $\phi$; they, therefore, act on the 2-dimensional spheres that constitute the leaves of the foliation $R^3 - \{(0,0,0)\} \simeq S^2 \times R$. The restriction [of the pair $(A, \mathcal{X}(R^3))$] to $(A, X)$ amounts to restricting oneself to a leaf $(S^2)$ in the present case.

(ii) $A = M_4(C)$, the algebra of complex $4 \times 4$ matrices. The vector space $C^4$ on which these matrices act serves as the carrier space of the spin $s = 3/2$ projective irreducible representation of the rotation group $SO(3)$. Denoting by $S_j$ (j=1,2,3) the $4 \times 4$ matrices representing the generators of the Lie algebra so(3), let $X$ be the real Lie algebra generated by the inner derivations $D_{S_j}$. The pair $(A, X)$ is relevant for the treatment of spin dynamics for $s = 3/2$.

In the generalized formalism, one obtains the cochains $C^{p,s}(\mathcal{X}, A)$ for which the formulas of sections 3.2 and 3.3 are valid (with the $X_j, s$ restricted to $\mathcal{X}$). The differential forms $\Omega^{p,s}(A)$ will now be replaced by the objects $\Omega^{p,s}(\mathcal{X}, A)$ obtained by restricting the cochains to the $Z_0(A)$-linear ones. [In the new notation, the objects $\Omega^{p,s}(A)$ will be called $\Omega^{p,s}(SDer(A), A)$].

Note. In (Dass 2002), the notation $\Omega^p(A, \mathcal{X})$ was used for the space of differential p-forms [which appeared natural in view of the notation $(A, \mathcal{X})$ for the pairs called algebraic differential systems there]. In the present work, we have changed it to $\Omega^p(\mathcal{X}, A)$ to bring it in tune with the notation for Lie algebra cochains in the mathematics literature.
To define the induced mappings $\Phi_*$ and $\Phi^*$ in the present context, one must employ a *pair-isomorphism* $\Phi : (A, X) \to (B, Y)$ which consists of a superalgebra *- isomorphism $\Phi : A \to B$ such that the induced mapping $\Phi_* : SDer(A) \to SDer(B)$ restricts to an isomorphism of $X$ onto $Y$. Various properties of the induced mappings hold as before.

Given a one-parameter family of transformations $\Phi_t : (A, X) \to (A, X)$, the condition $(\Phi_t)_*X \subset X$ implies that the infinitesimal generator $Y$ of $\Phi_t$ must satisfy the condition $[Y, X] \in X$ for all $X \in X$. In practical applications one will generally have $Y \in X$ which trivially satisfies this condition.

This generalization will be used in sections 4.5 and 7.5 below.

### 3.6. Symplectic structures

*Note.* The sign conventions about various quantities adopted below are parallel to those of Woodhouse (1980). This results in a (super-) Poisson bracket which, when applied to classical Hamiltonian mechanics, gives one differing from the Poisson bracket in most current books on mechanics by a minus sign. [See Eq.(46).] The main virtue of the adopted conventions is that Eq.(38) below has no unpleasant minus sign.

A *symplectic structure* on a superalgebra $A$ is a 2-form $\omega$ (the *symplectic form*) which is even, closed and non-degenerate in the sense that, for every $A \in A$, there exists a unique superderivation $Y_A$ in $SDer(A)$ [the (globally) Hamiltonian superderivation corresponding to A] such that

$$i_{Y_A}\omega = -dA.$$ (33)

The pair $(A, \omega)$ will be called a *symplectic superalgebra*. A symplectic structure is said to be *exact* if the symplectic form is exact ($\omega = d\theta$ for some 1-form $\theta$ called the *symplectic potential*).

A *symplectic mapping* from a symplectic superalgebra $(A, \alpha)$ to another one $(B, \beta)$ is a superalgebra isomorphism $\Phi : A \to B$ such that $\Phi^*\beta = \alpha$. (If the symplectic structures involved are exact, one requires a symplectic mapping to preserve the symplectic potential under the pull-back action; Eq.(29) then guarantees the preservation of the symplectic form.) A symplectic mapping from a symplectic superalgebra onto itself will be called a *canonical/symplectic transformation*. The symplectic form and its exterior powers are invariant under canonical transformations.

If $\Phi_t$ is a one-parameter family of canonical transformations generated by
$X \in SDer(A)$, the condition $\Phi^*_t \omega = \omega$ implies, with Eq.(32),

$$L_X \omega = 0.$$  \tag{34}

The argument just presented gives Eq.(34) with $X$ an even superderivation. More generally, a superderivation $X$ (even or odd or inhomogeneous) satisfying Eq.(34) will be called a \textit{locally Hamiltonian} superderivation. Eq.(20) and the condition $d\omega = 0$ imply that Eq.(34) is equivalent to the condition

$$d(i_X \omega) = 0.$$  \tag{35}

The (globally) Hamiltonian superderivations defined by Eq(33) constitute a subclass of locally Hamiltonian superderivations for which $i_X \omega$ is exact. Note from Eq(33) that $\epsilon(Y_A) = \epsilon(A)$. In analogy with the commutative case, the supercommutator of two locally Hamiltonian superderivations is a globally Hamiltonian superderivation. Indeed, given two locally Hamiltonian superderivations $X$ and $Y$, we have, recalling Equations (19) and (20),

$$\eta_{XY}i_{[X,Y]} \omega = (L_Y \circ i_X - i_X \circ L_Y) \omega$$
$$= \eta_{XY}(i_Y \circ d + d \circ i_Y)(i_X \omega)$$
$$= \eta_{XY}d(i_Y i_X \omega)$$

which is exact. It follows that the locally Hamiltonian superderivations constitute a Lie superalgebra in which the globally Hamiltonian superderivations constitute an ideal.

The \textit{Poisson bracket} (PB) of two elements $A$ and $B$ of $\mathcal{A}$ is defined as

$$\{A, B\} = \omega(Y_A, Y_B) = Y_A(B) = -\eta_{AB}Y_B(A).$$  \tag{36}

It obeys the superanalogue of the Leibnitz rule:

$$\{A, BC\} = Y_A(BC) = Y_A(B)C + \eta_{AB}BY_A(C)$$
$$= \{A, B\}C + \eta_{AB}B\{A, C\}.$$  \tag{37}

As in the classical case, we have the relation

$$[Y_A, Y_B] = Y_{\{A,B\}}.$$  \tag{38}
Eqn.(38) follows by using the equation for $i_{[X,Y]}\omega$ above with $X = Y_A$ and $Y = Y_B$ and equations (16), (36) and (33), remembering that Eq.(33) determines $Y_A$ uniquely. The super-Jacobi identity
\begin{equation}
0 = \frac{1}{2}(d\omega)(Y_A, Y_B, Y_C) = \{A, \{B, C\}\} + (-1)^{\epsilon_A(\epsilon_B+\epsilon_C)}\{B, \{C, A\}\} \\
+(-1)^{\epsilon_C(\epsilon_A+\epsilon_B)}\{C, \{A, B\}\}
\end{equation}
(39)
is obtained by using Eq.(21) and noting that
\begin{align*}
Y_A[\omega(Y_B, Y_C)] &= \{A, \{B, C\}\} \\
\omega([Y_A, Y_B], Y_C) &= \omega(Y_{(A,B)}, Y_C) = \{\{A, B\}, C\}.
\end{align*}
Clearly, the pair $(A, \{, \})$ is a Lie superalgebra. Eq.(38) shows that the linear mapping $A \mapsto Y_A$ is a Lie-superalgebra homomorphism.

An element $A$ of $\mathcal{A}$ can act, via $Y_A$, as the infinitesimal generator of a one-parameter family of canonical transformations. The change in $B \in \mathcal{A}$ due to such an infinitesimal transformation is
\begin{equation}
\delta B = \epsilon Y_A(B) = \epsilon\{A, B\}.
\end{equation}
(40)
In particular, if $\delta B = \epsilon I$ (infinitesimal ‘translation’ in B), we have
\begin{equation}
\{A, B\} = I
\end{equation}
(41)
which is the noncommutative analogue of the classical PB relation $\{p, q\} = 1$.

A pair $(A,B)$ of elements of $\mathcal{A}$ satisfying the condition (41) will be called a canonical pair.

3.7. Reality properties of the symplectic form and the Poisson bracket

For classical superdynamical systems, conventions about reality properties of the symplectic form are based on the fact that the Lagrangian is a real, even object (Berezin and Marinov 1977; Dass 1993b). The matrix of the symplectic form is then real-antisymmetric in the ‘bosonic sector’ and imaginary-symmetric in the ‘fermionic sector’ (which means anti-Hermitian in both sectors). Keeping this in view, it appears appropriate to impose, in supmech, the following reality condition on the symplectic form $\omega$:
\begin{equation}
\omega^*(X,Y) = -\eta_{XY}\omega(Y,X) \text{ for all } X, Y \in SDer(\mathcal{A});
\end{equation}
(42)
but this means, by Eq.(5), that $\omega^* = \omega$ (i.e. $\omega$ is real) which is the most natural assumption to make about $\omega$. Eq.(42) is equivalent to the condition

$$\omega(X^*, Y^*) = -\eta_{XY} [\omega(Y, X)]^*.$$  \hfill (43)

Now, for arbitrary $A, B \in \mathcal{A}$, we have

$$\{A, B\}^* = Y_A(B)^* = Y_A^*(B^*) = \eta_{AB} dB^*(Y_A^*)$$

$$= -\eta_{AB} \omega(Y_{B^*}, Y_A^*) = \omega(Y_A, Y_{B^*})^*$$

$$= -[dA(Y_{B^*})]^* = -\eta_{AB} [Y_{B^*}^*(A)]^*$$

$$= -\eta_{AB} Y_{B^*}(A^*)$$

giving finally

$$\{A, B\}^* = -\eta_{AB} \{B^*, A^*\}.$$ \hfill (44)

Eq.(44) is consistent with the reality properties of the classical and quantum Poisson brackets. [See equations (46) and (54) below.]

### 3.8. The algebra $\mathcal{A}_\mathrm{cl}$ and the classical Poisson bracket

Classical symplectic structure, traditionally treated in the framework of a symplectic manifold $(M, \omega_{cl})$ [where $M$ is a differentiable manifold of even dimension (say, 2n) and $\omega_{cl}$, the classical symplectic form, a nondegenerate differential 2-form on $M$], can be realized as a special case of the symplectic structure treated above by taking $\mathcal{A} = C^\infty(M, C) \equiv \mathcal{A}_{cl}$, the commutative algebra of smooth complex-valued functions on $M$ [with pointwise product: $(fg)(x) = f(x)g(x)$]. The star operation in this case is the complex conjugation $f^*(x) = f(x)$. The derivations of $\mathcal{A}_{cl}$ are the smooth complex vector fields. The differential forms of subsection C above are easily seen, for $\mathcal{A} = \mathcal{A}_{cl}$, to be the traditional differential forms on $M$. The symplectic structure is defined in terms of the classical differential form (in canonical coordinates)

$$\omega_{cl} = \sum_{j=1}^{n} dp_j \wedge dq_j;$$ \hfill (45)

in terms of the general local coordinates $\xi^a (a = 1, .., 2n)$, we have $\omega_{cl} = (\omega_{cl})_{ab} d\xi^a \wedge d\xi^b$. The Hamiltonian derivation corresponding to $f \in \mathcal{A}_{cl}$ is the Hamiltonian vector field

$$X_f = \omega_{cl}^{ab} \frac{\partial f}{\partial \xi^a} \frac{\partial}{\partial \xi^b}$$
where \((\omega_{ab}^{cl})\) is the inverse of the matrix \((\omega_{cl})_{ab}\). The classical PB is given by

\[
\{f, g\}_{cl} = X_f(g) = \omega_{cb}^{cl} \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b};
\]

in canonical coordinates, it takes the traditional form

\[
\{f, g\}_{cl} = \sum_{j=1}^{n} \left( \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right). \tag{46}
\]

Noting that the form \(\omega_{cl}\) and the coordinates \(\xi^a\) are real, the reality properties embodied in the equations (42) and (44) are obvious in the present case.

### 3.9. Special algebras; the canonical symplectic form

In this subsection, we shall consider a distinguished class of superalgebras (Dass 1993a, 2002; Dubois-Violette 1995, 1999) which have a canonical symplectic structure associated with them. As we shall see in the next subsection and in section 7, these superalgebras play an important role in Quantum mechanics.

A complex, associative, non-supercommutative *-superalgebra will be called \textit{special} if all its superderivations are inner. The differential 2-form \(\omega_c\) defined on such a superalgebra \(\mathcal{A}\) by

\[
\omega_c(D_A, D_B) = [A, B] \tag{47}
\]

is said to be the \textit{canonical form} on \(\mathcal{A}\). It is easily seen to be closed [the equation \((d\omega_c)(D_A, D_B, D_C) = 0\) is nothing but the Jacobi identity for the supercommutator], imaginary (i.e. \(\omega_c^* = -\omega_c\)) and dimensionless. For any \(A \in \mathcal{A}\), the equation

\[
\omega_c(Y_A, D_B) = -(dA)(D_B) = [A, B]
\]

has the unique solution \(Y_A = D_A\). (To see this, note that, since all derivations are inner, \(Y_A = D_C\) for some \(C \in \mathcal{A}\); the condition \(\omega_c(D_C, D_B) = [C, B] = [A, B]\) for all \(B \in \mathcal{A}\) implies that \((C - A) \in Z(\mathcal{A})\). But then \(D_C = D_A\).) This gives

\[
i_{D_A}\omega_c = -dA. \tag{48}
\]
The closed and non-degenerate form \( \omega_c \) defines, on \( \mathcal{A} \), the canonical symplectic structure. It gives, as Poisson bracket, the supercommutator:

\[
\{ A, B \} = Y_A(B) = D_A(B) = [A, B].
\] (49)

Using equations (48), (20) and (22), it is easily seen that the form \( \omega_c \) is invariant in the sense that \( L_X \omega_c = 0 \) for all \( X \in S\text{Der}(\mathcal{A}) \). The invariant symplectic structure on the algebra \( M_n(C) \) of complex \( n \times n \) matrices obtained by Dubois-Violette and coworkers (1994) is a special case of the canonical symplectic structure on special algebras described above.

If, on a special superalgebra \( \mathcal{A} \), instead of \( \omega_c \), we take \( \omega = b\omega_c \) as the symplectic form (where \( b \) is a nonzero complex number), we have

\[
Y_A = b^{-1}D_A, \quad \{ A, B \} = b^{-1}[A, B].
\] (50)

We shall make use of such a symplectic structure (with \( b = -i\hbar \)) in the following subsection and in the treatment of quantum mechanics in section 7. (Note that \( b \) must be imaginary to make \( \omega \) real.) Such a symplectic structure with general nonzero \( b \) will be referred to as the quantum symplectic structure with parameter \( b \).

### 3.10. The quantum symplectic form

In the traditional QM of a non-relativistic spinless particle, one generally operates in the Schrödinger representation which employs the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^3, dx) \) of complex square-integrable functions on \( \mathbb{R}^3 \) and a suitable dense domain \( \mathcal{D} \) in \( \mathcal{H} \) which is generally taken to be the space \( \mathcal{S}(\mathbb{R}^3) \) of Schwartz functions. The fundamental observables of such a particle are the Cartesian components \( X_j, P_j (j = 1, 2, 3) \) of position and momentum vectors which are self-adjoint linear operators represented as

\[
(X_j \phi)(x) = x_j \phi(x); \quad (P_j \phi)(x) = -i\hbar \frac{\partial \phi}{\partial x_j} \text{ for all } \phi \in \mathcal{D}.
\] (51)

These operators satisfy the canonical commutation relations (CCR)

\[
[X_j, X_k] = 0 = [P_j, P_k]; \quad [X_j, P_k] = i\hbar I \quad (j, k = 1, 2, 3)
\] (52)

where \( I \) is the unit operator. Other operators appearing in QM of the particle belong to the algebra \( \mathcal{A} \) generated by the operators \( X_j, P_j (j= 1,2,3) \) and \( I \).
subject to the CCR (52)]. The space \( \mathcal{D} = \mathcal{S}(R^3) \) is clearly an invariant domain for all elements of \( \mathcal{A} \). Defining a *-operation on \( \mathcal{A} \) by \( A^* = A^\dagger|\mathcal{D} \) (where \( A^\dagger \) is the Hilbert space adjoint of \( A \)), the Hermitian elements of \( \mathcal{A} \) represent the general observables of the particle.

The algebra \( \mathcal{A} \) obtained above is ‘special’ in the sense defined in the previous subsection (Dubois-Violette 1995,1999); one has, therefore, a canonical form \( \omega_c \) defined on it. The quantum symplectic structure is defined on \( \mathcal{A} \) by employing the quantum symplectic form

\[
\omega_Q = -i\hbar \omega_c. \tag{53}
\]

Note that the factor \( i \) serves to make \( \omega \) real and \( \hbar \) to give it the dimension of action (which is the correct dimension of a symplectic form in mechanics). The minus sign is a matter of convention. Eq.(50) now gives the quantum Poisson bracket

\[
\{A, B\}_Q = (-i\hbar)^{-1}[A, B]. \tag{54}
\]

The CCR (52) can now be expressed as the quantum Poisson brackets involving the canonical pairs \( (X_j, P_j) \):

\[
\{X_j, X_k\}_Q = 0 = \{P_j, P_k\}_Q; \quad \{P_j, X_k\}_Q = \delta_{jk} I \quad (j, k = 1, 2, 3). \tag{55}
\]

4. THE FORMALISM OF SUPMECH

We shall now develop an algebraic scheme of mechanics employing noncommutative differential forms and symplectic structures of the previous section. Most developments are parallel to those in classical Hamiltonian mechanics. States are defined as in traditional algebraic approaches; a condition of ‘compatible completeness’ between observables and pure states is an important extra input. The treatment of symmetries and conservation laws is along the lines dictated by the formalism. Lie group actions on a symplectic superalgebra are treated in some detail. Augmented symplectics including time are treated which includes noncommutative generalizations of the Poincaré-Cartan form and the symplectic version of Noether’s theorem. Systems with configuration space are treated and observables corresponding to localization are introduced.

4.1. The system algebra and states
Supmech associates, with every physical system, a symplectic superalgebra \((\mathcal{A}, \omega)\) of the type considered in the previous section. Here we shall treat the term ‘physical system’ informally as is traditionally done; some formalities in this connection will be taken care of in section 9 where the axioms are stated. The even Hermitian elements of \(\mathcal{A}\) represent observables of the system. The collection of all observables in \(\mathcal{A}\) will be denoted as \(\mathcal{O}(\mathcal{A})\).

To take care of limit processes and continuity of mappings, we must employ topological algebras. The choice of the admissible class of topological algebras must meet the following reasonable requirements:

(i) It should be closed under the formation of (a) topological completions and (b) tensor products. [Both are nontrivial requirements (Dubin and Hennings 1990).]

(ii) It should include (a) the Op\(^*\)-algebras (Horuzhy 1990) based on the pairs \((\mathcal{H}, \mathcal{D})\) where \(\mathcal{H}\) is a separable Hilbert space and \(\mathcal{D}\) a dense linear subset of \(\mathcal{H}\). [Recall that such an algebra is an algebra of operators which, along with their adjoints, map \(\mathcal{D}\) into itself. The \(*\)-operation on the algebra is defined as the restriction of the Hilbert space adjoint on \(\mathcal{D}\) as in section 3.10. These are the algebras of operators (not necessarily bounded) appearing in the traditional Hilbert space QM; for example, the algebra \(\mathcal{A}\) in section 3.10 belongs to this class.]; (b) Algebras of smooth functions on manifolds (to accommodate classical dynamics).

(iii) The GNS representations of the system algebra (in the non-supercommutative case) induced by various states must have separable Hilbert spaces as the representation spaces.

The right choice appears to be the \(\hat{\otimes}\)-(star-)algebras of Helemskii (1989) (i.e. locally convex \(*\)-algebras which are complete and Hausdorff with a jointly continuous product) satisfying the additional condition of being separable. [Note. The condition of separability may have to be dropped in applications to quantum field theory.] Henceforth all (super-)algebras employed will be assumed to belong to this class. For easy reference, unital \(*\)-algebras of this class will be called supmech-admissible. Mappings between topological spaces should henceforth be understood as continuous.

A state on a (unital) \(*\)-algebra \(\mathcal{A}\) is a linear functional \(\phi\) on \(\mathcal{A}\) which is (i) positive [which means \(\phi(A^*A) \geq 0\) for all \(A \in \mathcal{A}\)] and (ii) normalized [i.e. \(\phi(I) = 1\)]. Given a state \(\phi\), the quantity \(\phi(A)\) for any observable \(A\) is real (this can be seen by considering, for example, the quantity \(\phi[(I + A)^*(I + A)]\)) and
is to be interpreted as the expectation value of $A$ in the state $\phi$. Following
general usage in literature, we shall call observables of the form $A^*A$ or
sum of such terms positive (strictly speaking, the term ‘non-negative’ would
be more appropriate); states assign non-negative expectation values to such
observables. The family of all states on $\mathcal{A}$ will be denoted as $\mathcal{S}(\mathcal{A})$. It is easily
seen to be closed under convex combinations: given $\phi_i \in \mathcal{S}(\mathcal{A})$, $i = 1,..,n$
and $p_i \geq 0$ with $p_1 + .. + p_n = 1$, we have $\phi = \sum_{i=1}^{n} p_i \phi_i$ also in $\mathcal{S}(\mathcal{A})$. [More
generally, the integral of an $\mathcal{S}(\mathcal{A})$-valued function on a probability space
integrated with respect to the probability measure is an element of $\mathcal{S}(\mathcal{A})$.]
States which cannot be expressed as nontrivial convex combinations of other
states will be called pure states and others mixed states or mixtures. The
family of pure states of $\mathcal{A}$ will be denoted as $\mathcal{S}_1(\mathcal{A})$. The triple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega)$
will be referred to as a symplectic triple.

Note. In physics literature, it is sometimes found convenient to include,
among states, those for which the magnitudes of expectation values of some
observables are infinite. For example, in practical work in QM, one employs
wave functions which give infinite values for the expectation values of some
unbounded observables like position, momentum or energy. It needs to be
made clear, however, that physical states (or physically realizable states) must
be restricted to those for which the expectation values of all observables are
finite. In practical quantum mechanical work this would mean that only
wave functions lying in a common invariant dense domain of all observables
must be treated as representing (pure) physical states. Our formal definition
of state, in fact, allows only physical states. (This is because the domain of
definition of the functionals defining states is the whole algebra $\mathcal{A}$.)

In a sensible physical theory, the collection of pure states must be rich
enough to distinguish between two different observables. (Mixtures represent
averaging over ignorances over and above those implied by the irreducible
probabilistic aspect of the theory; they, therefore, are not the proper objects
for a statement of the above sort.) Similarly, there should be enough ob-
servables to distinguish between different pure states. These requirements
are taken care of in supmech by stipulating that the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$ be
compatibly complete in the sense that
(i) given $A, B \in \mathcal{O}(\mathcal{A}), A \neq B$, there should be a state $\phi \in \mathcal{S}_1(\mathcal{A})$ such that
$\phi(A) \neq \phi(B)$;
(ii) given two different states $\phi_1$ and $\phi_2$ in $\mathcal{S}_1(\mathcal{A})$, there should be an $A \in \mathcal{O}(\mathcal{A})$ such that
$\phi_1(A) \neq \phi_2(A)$. 

29
We shall refer to this condition as the ‘CC condition’ for the pair $\left( \mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}) \right)$.

Expectation values of all even elements of $\mathcal{A}$ can be expressed in terms of those of the observables (by considering the breakup of such an element into its Hermitian and anti-Hermitian part). This leaves out the odd elements of $\mathcal{A}$. It appears reasonable to demand that the expectation values $\phi(A)$ of all odd elements $A \in \mathcal{A}$ must vanish for all pure states (and, therefore, for all states).

Next, we consider the relation between states and traditional probability measures. We shall introduce classical probabilities in the formalism through a straightforward formalization of a measurement situation. To this end, we consider a measurable space $(\Omega, \mathcal{F})$ and associate, with every measurable set $E \in \mathcal{F}$, a positive observable $\nu(E)$ such that

(i) $\nu(\emptyset) = 0$,  
(ii) $\nu(\Omega) = I$,  
(iii) $\nu(\cup_i E_i) = \sum_i \nu(E_i)$ (for disjoint unions).

(The last equation means that, in the relevant topological algebra, the possibly infinite sum on the right hand side is well defined and equals the left hand side.) Then, given a state $\phi$, we have a probability measure $p_\phi$ on $(\Omega, \mathcal{F})$ given by

$$p_\phi(E) = \phi(\nu(E)) \quad \forall E \in \mathcal{F}. \quad (56)$$

The family $\{\nu(E), E \in \mathcal{F}\}$ will be called a positive observable-valued measure (POObVM) on $(\Omega, \mathcal{F})$. It is the abstract counterpart of the ‘positive operator-valued measure’ (POVM) employed in Hilbert space QM (Holevo 1982; Busch et al. 1995). The objects $\nu(E)$ may be called supmech events (representing possible outcomes in a measurement situation); a state assigns probabilities to these events. Eq.(56) represents the desired relationship between the supmech expectation values and classical probabilities.

Denoting the algebraic dual of the superalgebra $\mathcal{A}$ by $\mathcal{A}^*$, an automorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ induces the transpose mapping $\tilde{\Phi} : \mathcal{A}^* \rightarrow \mathcal{A}^*$ such that

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \quad \text{or} \quad <\tilde{\Phi}(\phi), A> = <\phi, \Phi(A)>. \quad (57)$$

where the second alternative has employed the dual space pairing $<,>$. The mapping $\tilde{\Phi}$ (which is easily seen to be linear and bijective) maps states (which form a subset of $\mathcal{A}^*$) onto states. To see this, note that

(i) $\tilde{\Phi}(\phi)(A^*A) = \phi(\Phi(A^*A)) = \phi(\Phi(A)\Phi(A)) \geq 0; \quad (i)$

30
(ii) $[\Phi(\phi)](I) = \phi(\Phi(I)) = \phi(I) = 1$.

The linearity of $\Phi$ (as a mapping on $A^*$) ensures that it preserves convex combinations of states. In particular, it maps pure states onto pure states. We have, therefore, a bijective mapping $\Phi : S_1(A) \to S_1(A)$.

When $\Phi$ is a canonical transformation, the condition $\Phi^*\omega = \omega$ gives, for $X, Y \in SDer(A)$,

$$\omega(X, Y) = (\Phi^*\omega)(X, Y) = \Phi^{-1}[\omega(\Phi_*X, \Phi_*Y)]$$

which gives

$$\Phi[\omega(X, Y)] = \omega(\Phi_*X, \Phi_*Y). \quad (58)$$

Taking expectation value of both sides of this equation in a state $\phi$, we get

$$(\Phi\phi)[\omega(X, Y)] = \phi[\omega(\Phi_*X, \Phi_*Y)]. \quad (59)$$

The dependence on $X,Y$ in this equation can be gotten rid of by defining $\omega_\Phi$ by

$$\omega_\Phi(X, Y) = \omega(\Phi_*X, \Phi_*Y). \quad (60)$$

Eq(59) can now be written as

$$(\Phi\phi)[\omega(., .)] = \phi[\omega_\Phi(., .)]. \quad (61)$$

It is generally simpler to use Eq.(59). When $\Phi$ is an infinitesimal canonical transformation generated by $G \in A$, we have

$$\Phi(\phi)(A) = \phi(\Phi(A)) \simeq \phi(A + \epsilon\{G, A\}). \quad (62)$$

Putting $\Phi(\phi) = \phi + \delta\phi$, we have

$$(\delta\phi)(A) = \epsilon\phi(\{G, A\}). \quad (63)$$

4.2. Dynamics

Dynamics is described by specifying an observable $H$, called the Hamiltonian; the evolution of the system is given in terms of the one-parameter family $\Phi_t$ of canonical transformations generated by $H$. (The parameter $t$ is
supposed to be an evolution parameter which need not always be the conventional time.) Writing $\Phi_t(A) = A(t)$ and recalling Eq.(40), we have the *Hamilton’s equation* of supmech:

$$\frac{dA(t)}{dt} = Y_H[A(t)] = \{H, A(t)\}.$$  \hfill (64)

The triple $(A, \omega, H)$ [or, more appropriately, the quadruple $(A, S_1(A), \omega, H)$] will be called a *supmech Hamiltonian system*; it is the analogue of a classical Hamiltonian system $(M, \omega_{cl}, H_{cl})$ [where $(M, \omega_{cl})$ is a symplectic manifold and $H_{cl}$, the classical Hamiltonian (a smooth real-valued function on $M$); note that the specification of the symplectic manifold $M$ serves to define both observables and pure states in classical mechanics]. As far as the evolution is concerned, the Hamiltonian is, as in the classical case, arbitrary up to the addition of a constant multiple of the identity element. We shall assume that $H$ is bounded below in the sense that its expectation values in all pure states (hence in all states) must be bounded below.

This is the analogue of the Heisenberg picture in traditional QM. An equivalent description is obtained by transferring the time dependence to states through the relation [see Eq.(57)]

$$<\phi(t), A> = <\phi, A(t)>$$

where $\phi(t) = \tilde{\Phi}_t(\phi)$. The mapping $\tilde{\Phi}_t$ satisfies the condition (61) which [with $\Phi = \Phi_t$] may be said to represent the canonicality of the evolution of states. With $\Phi = \Phi_t$ and $G = H$, Eq.(63) gives the *Liouville equation* of supmech:

$$\frac{d\phi(t)}{dt}(A) = \phi(t)(\{H, A\}) \quad \text{or} \quad \frac{d\phi(t)}{dt}(.) = \phi(t)(\{H, .\}).$$  \hfill (65)

This is the analogue of the Schrödinger picture in traditional QM.

### 4.3. Equivalent descriptions; Symmetries and conservation laws

By a ‘description’ of a system, we shall mean specification of its triple $(A, S(A), \omega)$. Two descriptions are said to be *equivalent* if they are related through a pair of isomorphisms $\Phi_1 : A \to A$ and $\Phi_2 : S(A) \to S(A)$ such that the symplectic form and the expectation values are preserved:

$$\Phi_1^*\omega = \omega; \quad \Phi_2(\phi)[\Phi_1(A)] = \phi(A)$$  \hfill (66)
for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$. The second equation above and Eq(57) imply that we must have $\Phi_2 = (\bar{\Phi}_1)^{-1}$. Two equivalent descriptions are, therefore, related through a canonical transformation on $\mathcal{A}$ and the corresponding inverse transpose transformation on the states. An infinitesimal transformation of this type generated by $G \in \mathcal{A}$ takes the form

$$
\delta A = \epsilon\{G, A\}, \quad (\delta \phi)(A) = -\epsilon\phi(\{G, A\}) \quad (67)
$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$.

These transformations may be called symmetries of the formalism; they are the analogues of simultaneous unitary transformations on operators and state vectors in a Hilbert space preserving expectation values of operators. Symmetries of dynamics are the subclass of these which leave the Hamiltonian invariant:

$$
\Phi_1(H) = H. \quad (68)
$$

For an infinitesimal transformation generated by $G \in \mathcal{A}$, this equation gives

$$
\{G, H\} = 0. \quad (69)
$$

It now follows from the Hamilton equation (64) that (in the ‘Heisenberg picture’ evolution) $G$ is a constant of motion. This is the situation familiar from classical and quantum mechanics: generators of symmetries of the Hamiltonian are conserved quantities and vice-versa.

**Note.** Noting that a symmetry operation is uniquely defined by any one of the two mappings $\Phi_1$ and $\Phi_2$, we can be flexible in the implementation of symmetry operations. It is often useful to implement them such that the symmetry operations act, in a single implementation, *either* on states *or* on observables, and the two actions are related as the Heisenberg and Schrödinger picture evolutions above [see Eq.(57)]; we shall refer to this type of implementation as *unimodal*. In such an implementation, the second equation in (67) will not have a minus sign on the right.

For future reference, we define equivalence of supmech Hamiltonian systems. Two supmech Hamiltonian systems

$$(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H) \quad \text{and} \quad (\mathcal{A}', \mathcal{S}_1(\mathcal{A}'), \omega', H')$$

are said to be equivalent if they are related through a pair $\Phi = (\Phi_1, \Phi_2)$ of bijective mappings such that $\Phi_1 : (\mathcal{A}, \omega) \to (\mathcal{A}', \omega')$ is a symplectic mapping
connecting the Hamiltonians [i.e. \[\Phi_1^* \omega = \omega \text{ and } \Phi_1(H) = H'] and \(\Phi_2 : S_1(A) \to S_1(A')\) such that \(<\Phi_2(\phi), \Phi_1(A)>> = <\phi, A>.

### 4.4. Symplectic actions of Lie groups

In this subsection and the next section, we shall generally employ bosonic objects. The square brackets will, therefore, be commutators; the subscript \(-\) (minus) for the latter will be omitted.

The study of symplectic actions of Lie groups in supmech proceeds generally parallel to the classical case (Sudarshan and Mukunda 1974; Arnold 1978; Guillemin and Sternberg 1984; Woodhouse 1980) and promises to be quite rich and rewarding. Here we shall present the essential developments mainly to provide background material for the next section.

Let \(G\) be a connected Lie group with Lie algebra \(G\). Elements of \(G, G\) and \(G^*\) (the dual space of \(G\)) will be denoted, respectively, as \(g, h, \ldots, \xi, \eta, \ldots\) and \(\lambda, \mu, \ldots\). The pairing between \(G^*\) and \(G\) will be denoted as \(<.,.>\). Choosing a basis \(\{\xi^a; a = 1, \ldots, r\}\) in \(G\), we have the commutation relations \([\xi_a, \xi_b] = C^c_{ab} \xi_c\).

The dual basis in \(G^*\) is denoted as \(\{\lambda^a\}\) (so that \(<\lambda^a, \xi^b> = \delta^a_b\)). The action of \(G\) on \(G\) (adjoint representation) will be denoted as \(Ad_g : G \to G\) and that on \(G^*\) (the coadjoint representation) by \(Cad_g : G^* \to G^*\); the two are related as \(<Cad_g \lambda, \xi> = <\lambda, Ad_g^{-1} \xi>\). With the bases chosen as above, the matrices in the two representations are related as \((Cad_g)_{ab} = (Ad_g^{-1})_{ba}\).

Recalling the mappings \(\Phi_1\) and \(\Phi_2\) of the previous subsection, a symplectic action of \(G\) on a symplectic superalgebra \((A, \omega)\) is given by the assignment, to each \(g \in G\), a symplectic mapping (canonical transformation) \(\Phi_1(g) : A \to A\) which is a group action [which means that \(\Phi_1(g) \Phi_1(h) = \Phi_1(gh)\) and \(\Phi_1(e) = id_A\) in obvious notation]. The action on the states is given by the mappings \(\Phi_2(g) = [\Phi_1(g)]^{-1}\).

A one-parameter subgroup \(g(t)\) of \(G\) generated by \(\xi \in G\) induces a locally Hamiltonian derivation \(Z_\xi \in SDer(A)\) as the generator of the one-parameter family \(\Phi_1(g(t)^{-1})\) of canonical transformations of \(A\): For small \(t\)

\[\Phi_1(g(t)^{-1})(A) \simeq A + tZ_\xi(A)\]

**Note.** We employed \(\Phi_1(g(t)^{-1})\) (and not \(\Phi_1(g(t))\)) for defining \(Z_\xi\) above because the former correspond to the right action of \(G\) on \(A\).

The correspondence \(\xi \to Z_\xi\) is a Lie algebra homomorphism:

\[Z_{[\xi, \eta]} = [Z_\xi, Z_\eta]\]

34
Proof. The objects $Z_\xi$ are linear in $\xi$. To see this, note that, for infinitesimal group parameters $\epsilon^a$, we have

$$\Phi_1(\exp(\epsilon^a \xi_a)^{-1})(A) \simeq A + \epsilon^a Z_{\xi_a}(A).$$

With $\xi = \alpha^a \xi_a$ and $g(t) = \exp(t \xi)$, we have $\epsilon^a = t \alpha^a$; this gives $tZ_\xi(A) = t\alpha^a Z_{\xi_a}(A)$ for all $A \in \mathcal{A}$, hence $Z_\xi = \alpha^a Z_{\xi_a}$.

Defining the group actions $L_g, R_g$ and $A_g$ by $L_g(h) = gh, R_g(h) = hg, A_g(h) = g^{-1}hg$ and recalling that $\xi \in \mathcal{G}$ is a left-invariant vector field on $G$ (i.e. $(L_g)_* \xi = \xi \forall \xi \in \mathcal{G}$), we note that, for any $h \in G, \xi' \equiv (R_h)_* \xi = (A_h)_* \xi$ generates the one-parameter subgroup $g'(t) = h^{-1}g(t)h$. Now, recalling Eq.(3), we have, for small $t$,

$$\Phi_1(g'(t)^{-1})(A) = \Phi_1(h^{-1}g(t)^{-1}h)(A) = \Phi_1(h^{-1})\Phi_1(g(t)^{-1})\Phi_1(h)(A) \simeq A + t\Phi_1(h^{-1})Z_\xi[\Phi_1(h)(A)] = A + t\Phi_1(h^{-1})Z_\xi(A) = A + tZ_{\xi'}(A)$$

which gives

$$Z_{\xi'} = \Phi_1(h^{-1})_* Z_\xi.$$

Replacing $h$ above by the general element $h(s)$ of the one-parameter subgroup generated by $\eta \in \mathcal{G}$, we have

$$\xi'(s) \equiv (R_{h(s)})_* \xi \simeq \xi + sL_\eta(\xi) = \xi + s[\eta, \xi]. \quad (*)$$

Moreover

$$Z_{\xi'(s)} = \Phi_1(h(s)^{-1})_* Z_\xi \simeq Z_\xi + sL_{\eta}(Z_\xi) = Z_\xi + s[Z_\eta, Z_\xi]. \quad (**$$

Recalling the linearity of $Z_{\xi'(s)}$ in $\xi'(s)$, equations $(*)$ and $(**)$ give the desired result. □

The G-action is said to be Hamiltonian if the derivations $Z_\xi$ are Hamiltonian, i.e. for each $\xi \in \mathcal{G}, Z_\xi = Y_{h_\xi}$ for some $h_\xi \in \mathcal{A}$ (called the Hamiltonian
corresponding to $\xi$). These hamiltonians are arbitrary up to addition of multiples of the unit element. This arbitrariness can be somewhat reduced by insisting that $h_\xi$ be linear in $\xi$. (This can be achieved by first defining the hamiltonians for the members of a basis in $G$ and then defining them for general elements as appropriate linear combinations of these.) We shall always assume this linearity.

A hamiltonian $G$-action satisfying the additional condition

$$\{h_\xi, h_\eta\} = h_{[\xi, \eta]}$$

for all $\xi, \eta \in G$ (70)

is called a Poisson action. The hamiltonians of a Poisson action have the following equivariance property:

$$\Phi_1(g)(h_\xi) = h_{Ad_g(\xi)}. \quad (71)$$

Since $G$ is connected, it is adequate to verify this relation for infinitesimal group actions. Denoting by $g(t)$ the one-parameter group generated by $\eta \in G$, we have, for small $t$,

$$\Phi_1(g(t))(h_\xi) \simeq h_\xi + t\{h_\eta, h_\xi\} = h_\xi + th_{[\eta, \xi]} = h_{\xi + t[\eta, \xi]} \simeq h_{Ad_{g(t)}(\xi)}$$

completing the verification.

A Poisson action is not always admissible. The obstruction to such an action is determined by the objects

$$\alpha(\xi, \eta) = \{h_\xi, h_\eta\} - h_{[\xi, \eta]}$$

which are easily seen to have vanishing Hamiltonian derivations:

$$Y_{\alpha(\xi, \eta)} = [Y_{h_\xi}, Y_{h_\eta}] - Y_{h_{[\xi, \eta]}} = [Z_\xi, Z_\eta] - Z_{[\xi, \eta]} = 0$$

and hence vanishing Poisson brackets with all elements of $A$. (This last condition defines the so-called neutral elements (Sudarshan and Mukunda 1974) of the Lie algebra($A, \{,\}$). They clearly form a complex vector space which will be denoted as $N$. We also have

$$\alpha([\xi, \eta], \zeta) + \alpha([\eta, \zeta], \xi) + \alpha([\zeta, \xi], \eta) = 0.$$ 

The derivation (Woodhouse 1980) of this result in classical mechanics employs only the standard properties of PBs and remains valid in supmech.
Comparing this equation with Eq. (25), we see that \( \alpha(.,.) \in Z^2_0(\mathcal{G}, \mathcal{N}) \). A redefinition of the hamiltonians \( h_\xi \to h'_\xi = h_\xi + k_\xi I \) (where the scalars \( k_\xi \) are linear in \( \xi \)) changes \( \alpha \) by a coboundary term:

\[
\alpha'(\xi, \eta) = \alpha(\xi, \eta) - k[\xi, \eta]I
\]

showing that the obstruction is characterized by a cohomology class of \( \mathcal{G} \) [i.e. an element of \( H^2_0(\mathcal{G}, \mathcal{N}) \)]. A necessary and sufficient condition for the admissibility of Poisson action of \( \mathcal{G} \) on \( \mathcal{A} \) is that it should be possible to transform away all the obstruction 2-cocycles by redefining the hamiltonians, or, equivalently, \( H^2_0(\mathcal{G}, \mathcal{N}) = 0 \).

We restrict ourselves to the special case, relevant for application in section 5, in which \( \mathcal{G} \) is a finite dimensional real Lie algebra and the cocycles \( \alpha \) are multiples of the unit element:

\[
\alpha(\xi, \eta) = \alpha(\xi, \eta)I;
\]

here the quantities \( \alpha(\xi, \eta) \) must be real numbers because the set of observables is closed under Poisson brackets. This implies \( \mathcal{N} = \mathbb{R} \), the set of real numbers. In this case, the relevant cohomology group \( H^2_0(\mathcal{G}, \mathbb{R}) \) is a real finite dimensional vector space; we shall take it to be \( \mathbb{R}^m \). In this case, as in classical symplectic mechanics (Sudarshan and Mukunda 1974; Cariñena and Santander 1975; Alonso 1979), Hamiltonian group actions (more generally, Lie algebra actions) with nontrivial neutral elements can be treated as Poisson actions of a (Lie group with a) larger Lie algebra \( \hat{\mathcal{G}} \) obtained as follows: Let \( \eta_r(.,.) (r = 1, .., m) \) be a set of representatives in \( Z^2_0(\mathcal{G}, \mathbb{R}) \) of a basis in \( H^2_0(\mathcal{G}, \mathbb{R}) \). We add extra generators \( M_r \) to the basis \( \{\xi_a\} \) of \( \mathcal{G} \) and take the commutation relations of the larger Lie algebra \( \hat{\mathcal{G}} \) as

\[
[\xi_a, \xi_b] = C^c_{ab} \xi_c + \sum_{r=1}^{m} \eta_r(\xi_a, \xi_b) M_r; \quad [\xi_a, M_r] = 0 = [M_r, M_s]. \tag{73}
\]

The simply connected Lie group \( \hat{\mathcal{G}} \) with the Lie algebra \( \hat{\mathcal{G}} \) is called the projective group (Alonso 1979) of \( \mathcal{G} \); it is generally a central extension of the universal covering group \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \).

The hamiltonian action of \( \mathcal{G} \) with the cocycle \( \alpha \) now becomes a Poisson action of \( \hat{\mathcal{G}} \) with the Poisson bracket relations (writing \( h_{M_r} = h_r \))

\[
\{h_a, h_b\} = C^c_{ab} h_c + \sum_{r=1}^{m} \eta_r(\xi_a, \xi_b) h_r
\]

\[
\{h_a, h_r\} = 0 = \{h_r, h_s\}. \tag{74}
\]

37
Momentum map. In classical mechanics, given a Poisson action of a connected Lie group $G$ on a symplectic manifold $(M, \omega_{cl})$ [with hamiltonians/comoments $h^{(cl)}_\xi \in C^\infty(M)$], a useful construction is the so-called momentum map (Souriau 1997; Arnold 1978; Guillemin and Sternberg 1984)

$$P : M \to G^*$$

given by

$$< P(x), \xi > = h^{(cl)}_\xi(x) \forall x \in M \text{ and } \xi \in G. \quad (75)$$

This map relates the symplectic action $\Phi_g$ of $G$ on $M$ ($\Phi_g : M \to M, \Phi_g^*\omega_{cl} = \omega_{cl} \forall g \in G$) and the transposed adjoint action on $G^*$ through the equivariance property

$$P(\Phi_g(x)) = Ad_g^*(P(x)) \forall x \in M \text{ and } g \in G. \quad (76)$$

Noting that points of $M$ are pure states of the algebra $A_{cl} = C^\infty(M)$, the map $P$ may be considered as the restriction to $M$ of the dual/transpose $\tilde{h}^{(cl)} : A^*_{cl} \to G^*$ of the linear map $h^{(cl)} : G \to A_{cl}$ [given by $h^{(cl)}(\xi) = h^{cl}_\xi$]:

$$< \tilde{h}^{(cl)}(u), \xi > = < u, h^{(cl)}(\xi) > \forall u \in A^*_{cl} \text{ and } \xi \in G.$$

The analogue of $M$ in supmech is $S_1 = S_1(A)$. Defining $h : G \to A$ by $h(\xi) = h_\xi$, the analogue of the momentum map in supmech is the mapping $\tilde{h} : S_1 \to G^*$ (considered as the restriction to $S_1$ of the mapping $\tilde{h} : A^* \to G^*$) given by

$$< \tilde{h}(\phi), \xi > = < \phi, h(\xi) > = < \phi, h_\xi >. \quad (77)$$

Recalling the symplectic mappings $\Phi_1$ and $\Phi_2$ and Eq.(71), we have

$$< \tilde{h}(\Phi_2(g)\phi), \xi > = < \Phi_2(g)\phi, h_\xi > = < \phi, \Phi_1(g^{-1})(h_\xi) > = < \phi, h_{Ad_{g^{-1}}}(\xi) > = < \phi, h(Ad_{g^{-1}}(\xi)) > = < Cad_g(\tilde{h}(\phi)), \xi >$$

giving finally

$$\tilde{h}(\Phi_2(g)\phi) = Cad_g(\tilde{h}(\phi)) \quad (78)$$

which is the supmech analogue of Eq.(75). [Note. In Eq.(78), the co-adjoint (and not the transposed adjoint) action appears on the right because $\Phi_2(g)$ is inverse transpose (and not transpose) of $\Phi_1(g).$]
4.5. Generalized symplectic structures and Hamiltonian systems

The generalization of the DVNCG scheme introduced in section 3.5 can be employed to obtain the corresponding generalization of the supmech formalism. One picks up a distinguished Lie sub-superalgebra $\mathcal{X}$ of $SDer(A)$ and restricts the superderivations of $A$ in all definitions and constructions to those in $\mathcal{X}$. Thus, a symplectic superalgebra $(A,\omega)$ is now replaced by a generalized symplectic superalgebra $(A,\mathcal{X},\omega)$ and a symplectic mappings $\Phi : (A,\mathcal{X},\alpha) \rightarrow (B,\mathcal{Y},\beta)$ is restricted to a superalgebra-isomorphism $\Phi : A \rightarrow B$ such that $\Phi^*\beta = \alpha$. A supmech Hamiltonian system $(A,S_1(A),\omega,H)$ is now replaced by a generalized supmech Hamiltonian system $(A,S_1(A),\mathcal{X},\omega,H)$. In section 7.5, we shall employ the pairs $(A,\mathcal{X})$ with $\mathcal{X} = ISDer(A)$ to define quantum symplectic structure on superalgebras admitting outer as well as inner superderivations.

Note. In (Dass 2002), the formalism in section IV (of that paper) was developed right from the beginning in terms of the pairs $(A,\mathcal{X})$ (called ‘algebraic differential systems’ there; we have chosen to dispense with this nomenclature to avoid confusion of the term ‘differential system’ with its use elsewhere in mathematics literature). This has the advantage of extra generality; however, noting that this generality is needed only at very few places and its use everywhere would make the formalism unnecessarily more complicated, the author has not opted for it in the present work.

4.6. Augmented symplectics including time; the generalized Poincaré-Cartan form

We shall now augment the kinematic framework of supmech by including time and obtain the non-commutative analogues of the Poincaré-Cartan form and the symplectic version of Noether’s theorem (Souriau 1997).

For a system $S$ with associated symplectic superalgebra $(A,\omega)$ we construct the extended algebra $A^e = C^\infty(R) \otimes A$ (where the real line $R$ is the carrier space of the ‘time’ $t$) whose elements are finite sums $\sum_i f_i \otimes A_i$ (with $f_i \in C^\infty(R) \equiv A_0$) which may be written as $\sum_i f_i A_i$. This algebra is the analogue of the algebra of functions on the evolution space of Souriau (the Cartesian product of the time axis and the phase space — often referred to as the phase bundle). The superscript $e$ in $A^e$, may, therefore, also be taken to refer to ‘evolution’.

Derivations on $A_0$ are of the form $g(t)\frac{d}{dt}$ and one-forms of the form $h(t)dt$.
where $g$ and $h$ are smooth functions; there are no nonzero higher order forms. We have, of course, $dt \left( \frac{d}{dt} \right) = 1$.

A (super-)derivation $D_1$ on $A_0$ and $D_2$ on $A$ extend trivially to (super-)derivations on $A^e$ as $D_1 \otimes id_A + id_{A_0} \otimes D_2$ respectively (where $id_A$ is the identity mapping on $A$); these trivial extensions may be informally denoted as $D_1$ and $D_2$. With $f \otimes A$ written as $fA$, we can write $D_1(fA) = (D_1f)A$ and $D_2(fA) = f(D_2A)$.

The mapping $\Xi : A \rightarrow A^e$ given by $\Xi(A) = 1 \otimes A (= A)$ is an isomorphism of the algebra $A$ onto the subalgebra $\tilde{A} \equiv 1 \otimes A$ of $A^e$ and can be employed to pull back the differential forms on $A$ to those on $\tilde{A}$. We write, for a $p$-form $\alpha$ on $A$, $(\Xi^{-1})^* (\alpha) = \tilde{\alpha}$ and extend this form on $\tilde{A}$ to one on $A^e$ by defining $\tilde{\alpha}(\frac{d}{dt}, \ldots) = 0$. We shall generally skip the tilde. Similarly, we may extend the one-form $dt$ on $A_0$ to one on $A^e$ by defining $(dt)(X) = 0$ for all $X \in SDer(A)$.

The symplectic structure $\omega$ on $A$ induces, on $A^e$, a generalized symplectic structure (of the type introduced in section 4.5) with the distinguished Lie sub-superalgebra $X$ of $SDer(A^e)$ taken to be the one consisting of the objects $\{ id_{A_0} \otimes D ; D \in SDer(A) \}$ which constitute a Lie sub-superalgebra of $SDer(A^e)$ isomorphic to $SDer(A)$, thus giving a generalized symplectic superalgebra $(A^e, X, \tilde{\omega})$. The corresponding PBs on $A^e$ are trivial extensions of those on $A$ obtained by treating the ‘time’ $t$ as an external parameter; this amounts to extending the C-linearity of PBs on $A$ to what is essentially $A_0$-linearity:

$$\{ fA + gB, hC \}_{A^e} = fh\{ A, C \}_A + gh\{ B, C \}_A$$

where, for clarity, we have put subscripts on the PBs to indicate the underlying superalgebras. We shall often drop these subscripts; the underlying (super-)algebra will be clear from the context.

To describe dynamics in $A^e$, we extend the one-parameter family $\Phi_t$ of canonical transformations on $A$ generated by a Hamiltonian $H \in A$ to a one-parameter family $\Phi^e_t$ of transformations on $A^e$ (which are ‘canonical’ in a certain sense, as we shall see below) given by

$$\Phi^e_t(fA) \equiv (fA)(t) = f(t)A(t) \equiv (\Phi_t^{(0)} f)\Phi_t(A)$$

where $\Phi_t^{(0)}$ is the one-parameter group of translations on $A_0$ generated by the derivation $\frac{d}{dt}$. An infinitesimal transformation under the evolution $\Phi^e_t$ is given by

$$\delta(fA)(t) \equiv (fA)(t + \delta t) - (fA)(t)$$

40
\[ \frac{df}{dt} A + f\{H,A\}_A \delta t = \dot{Y}_H(fA) \delta t \]

where

\[ \dot{Y}_H = \frac{\partial}{\partial t} + \tilde{Y}_H. \]  

(79)

Here \( \frac{d}{dt} \) is the derivation on \( \mathcal{A}^e \) corresponding to the derivation \( \frac{d}{dt} \) on \( \mathcal{A}_0 \) and

\[ \tilde{Y}_H = \{H,\cdot\}_{\mathcal{A}^e}. \]

(*)

Note that

(i) \( dt(\dot{Y}_H) = 1; \)

(ii) the right hand side of the equation (*) remains well defined if \( H \in \mathcal{A}^e \) (‘time dependent’ Hamiltonian). Henceforth, in various formulas in this subsection, \( H \) will be understood to be an element of \( \mathcal{A}^e \).

The obvious generalization of the supmech Hamilton equation (64) to \( \mathcal{A}^e \) is the equation

\[ \frac{dF(t)}{dt} = \dot{Y}_H F(t) = \frac{\partial F(t)}{\partial t} + \{H(t), F(t)\}. \]

(80)

We next consider an object in \( \mathcal{A}^e \) which contains complete information about the symplectic structure and dynamics [i.e. about \( \tilde{\omega} \) and \( H \) (up to an additive constant multiple of \( I \))] and is canonically determined by these objects. It is the 2-form

\[ \Omega = \tilde{\omega} + dt \wedge dH \]

(81)

which is ‘obviously’ closed. [To have a formal proof, apply Eq.(133) below with \( \Omega = 1 \otimes \omega + d_1 t \otimes d_2 H \).] If the symplectic structure on \( \mathcal{A} \) is exact (with \( \omega = d\theta \), we have (‘obviously’) \( \Omega = d\Theta \) where

\[ \Theta = \tilde{\theta} - Hdt \]

(82)

is the supmech avatar of the Poincaré-Cartan form in classical mechanics. [Again, for a formal derivation, use Eq.(133) with \( \Theta = 1 \otimes \theta - dt \otimes H \).]

The closed form \( \Omega \) is generally not non-degenerate. It defines what may be called a presymplectic structure (Souriau 1997) on \( \mathcal{A}^e \). In fact, we have here the noncommutative analogue of a special type of presymplectic structure called contact structure (Abraham and Marsden 1978; Berndt 2001); it may
be called the Poincaré-Cartan contact structure. We shall, however, not attempt a formal development of noncommutative contact structures here.

A symplectic action of a Lie group $G$ on the presymplectic space $(\mathcal{A}^e, \Omega)$ is the assignment, to every $g \in G$, an automorphism $\Phi(g)$ of the superalgebra $\mathcal{A}^e$ having the usual group action properties and such that $\Phi(g)^*\Omega = \Omega$. This implies, as in section 4.4, that, to every element $\xi$ of the Lie algebra $\mathcal{G}$ of $G$, corresponds a derivation $Z_\xi$ such that $L_{Z_\xi}\Omega = 0$ which, in view of the condition $d\Omega = 0$, is equivalent to the condition

$$d(i_{Z_\xi}\Omega) = 0. \quad (83)$$

We shall now verify that the one-parameter family $\Phi^{(e)}_t$ of transformations on $\mathcal{A}^e$ is symplectic/canonical. For this, it is adequate to verify that Eq.(83) holds with $Z_\xi = \hat{Y}_H$. We have, in fact, the stronger relation

$$i_{\hat{Y}_H}\Omega = 0. \quad (84)$$

Indeed

$$i_{\hat{Y}_H}\Omega = i_{\partial/\partial t}\Omega + i_{\hat{Y}_H}\Omega = i_{\partial/\partial t}(dt \wedge dH) + i_{\hat{Y}_H}\hat{\omega} + i_{\hat{Y}_H}(dt \wedge dH) = dH - dH - i_{\hat{Y}_H}(dH)dt = [i_{\hat{Y}_H}(i_{\hat{Y}_H}\hat{\omega})]dt = 0. $$

The equation in note (i) above and Eq.(84) are analogous to the properties of the ‘characteristic vector field’ of a contact structure. The derivation $\hat{Y}_H$ may, therefore, be called the characteristic derivation of the Poincaré-Cartan contact structure.

A symplectic $G$-action (in the present context) is said to be hamiltonian if the 1-forms $i_{Z_\xi}\Omega$ are exact, i.e. to each $\xi \in \mathcal{G}$, corresponds a ‘hamiltonian’ $\hat{h}_\xi \in \mathcal{A}^e$ (unique up to an additive constant multiple of the unit element) such that

$$i_{Z_\xi}\Omega = -d\hat{h}_\xi. \quad (85)$$

These ‘hamiltonians’ (Noether invariants) are constants of motion:

$$\frac{d\hat{h}_\xi(t)}{dt} = \dot{\hat{Y}}_H(\hat{h}_\xi(t)) = (d\hat{h}_\xi)(\dot{\hat{Y}}_H)(t) = -(i_{Z_\xi}\Omega)(\dot{\hat{Y}}_H)(t) = 0 \quad (86)$$
where, in the last step, Eq. (84) has been used. This is the supmech analogue of the symplectic version of Noether’s theorem. For some concrete examples of Noether invariants, see section 5.2.

4.7. Systems with configuration space; localizability

We shall now consider the class of systems each of which has a configuration space (say, M) associated with it and it is meaningful to ask questions about the localization of the system in subsets of M. To start with, we shall take M to be a topological space and take the permitted domains of localization to belong to B(M), the family of Borel subsets of M.

Some good references containing detailed treatment of localization in conventional approaches are (Newton and Wigner 1949; Wightman 1962; Varadarajan 1985; Bacry 1988). We shall follow a relatively more economical path exploiting some of the constructions described above.

We shall say that a system S [with associated symplectic superalgebra (\(A, \omega\))] is localizable in M if we have a positive observable-valued measure (as defined in section 4.1) on the measurable space (M, B(M)), which means that, corresponding to every subset \(D \in B(M)\), there is a positive observable \(P(D)\) in \(A\) satisfying the three conditions

(i) \(P(\emptyset) = 0\);
(ii) \(P(M) = 1\);
(iii) for any countable family of mutually disjoint sets \(D_i \in B(M)\),

\[
P(\bigcup_i D_i) = \sum_i P(D_i). \quad (87)
\]

For such a system, we can associate, with any state \(\phi\), a probability measure \(\mu_\phi\) on the measurable space \((M, B(M))\) defined by [see Eq.(56)]

\[
\mu_\phi(D) = \phi(P(D)), \quad (88)
\]

making the triple \((M, B(M), \mu_\phi)\) a probability space. The quantity \(\mu_\phi(D)\) is to be interpreted as the probability of the system, given in the state \(\phi\), being found (on observation/measurement) in the domain D.

Generally it is of interest to consider localizations having suitable invariance properties under a transformation group G. Typically G is a topological group with continuous action on M assigning, to each \(g \in G\), a bijection \(T_g : M \to M\) such that, in obvious notation, \(T_g T_{g'} = T_{gg'}\) and \(T_e = id_M\); it also has a symplectic action on \(A\) and \(S(A)\) given by the mappings \(\Phi_1(g)\)
and $\Phi_2(g)$ introduced in section 4.4. The localization on $M$ described above will be called $G$-covariant (or, loosely, $G$-invariant) if

$$\Phi_1(g)(P(D)) = P(T_g(D)) \quad \forall g \in G \text{ and } D \in B(M).$$

Equation (89) translates into the following useful relation for the probabilities (88):

$$\mu_{\Phi_2(\phi)}(P(D)) = \mu_{\phi}(T_g(D)).$$

Proof

$$\mu_{\Phi_2(\phi)}(D) = <\Phi_2(g)(\phi), P(D) > = <\phi, \Phi_1(g)(P(D)) > = <\phi, P(T_g(D)) > = \mu_{\phi}(T_g(D)). \quad \square$$

In most practical applications, $M$ is a manifold and $G$ is a Lie group with smooth action on $M$ and a Poisson action on the symplectic superalgebra $(A, \omega)$.

In Hilbert space $QM$, the problem of $G$-covariant localization is traditionally formulated in terms of the so-called ‘systems of imprimitivity’ (Mackey 1949; Varadarajan 1985; Wightman 1962). We are operating in the more general algebraic setting trying to exploit the machinery of noncommutative symplectics developed above. Clearly, there is considerable scope for mathematical developments in this context parallel to those relating to systems of imprimitivity. We shall, however, restrict ourselves to some essential developments relevant to the treatment of localizable elementary systems (massive particles) later.

We shall be mostly concerned with $M = \mathbb{R}^n$ (equipped with the Euclidean metric). In this case, one can consider averages of the form (denoting the natural coordinates on $\mathbb{R}^n$ by $x_j$)

$$\int_{\mathbb{R}^n} x_j d\mu_\phi(x), \quad j = 1, ..., n. \quad (***)$$

It is natural to introduce position/configuration observables $X_j$ such that the quantity (*** is $\phi(X_j)$. Let $E_n$ denote the (identity component of) Euclidean group in n dimensions and let $p_j, m_{jk}(= -m_{kj})$ be its generators satisfying the commutation relations

$$[p_j, p_k] = 0, \quad [m_{jk}, p_l] = \delta_{jl}p_k - \delta_{kl}p_j$$
\[ [m_{jk}, m_{pq}] = \delta_{jp}m_{kq} - \delta_{kp}m_{jq} - \delta_{jq}m_{kp} + \delta_{kq}m_{jp}. \]

We shall say that a system S with configuration space \( R^n \) has \textit{concrete Euclidean-covariant localization} if it is localizable as above in \( R^n \) and

(i) it has position observables \( X_j \in \mathcal{A} \) such that the above mentioned relation holds:

\[ \phi(X_j) = \int_{R^n} x_j d\mu_\phi(x); \quad (91) \]

(The term ‘concrete’ is understood to imply this condition.)

(ii) the group \( E_n \) has a Poisson action on \( \mathcal{A} \) so that we have the hamiltonians \( P_j, M_{jk} \) associated with \( p_j, m_{jk} \) such that

\[ \{P_j, P_k\} = 0, \quad \{M_{jk}, P_l\} = \delta_{jl}P_k - \delta_{kl}P_j \]

\[ \{M_{jk}, M_{pq}\} = \delta_{jp}M_{kq} - \delta_{kp}M_{jq} - \delta_{jq}M_{kp} + \delta_{kq}M_{jp}; \quad (92) \]

(iii) the covariance condition (89) holds with the Euclidean group action on \( R^n \) given by

\[ T_{(R,a)}x = Rx + a, \quad R \in SO(n), \ a \in R^n. \quad (93) \]

Equations (91) and (89) for infinitesimal Euclidean transformations and Eq.(67) then give the analogues of the classical canonical PBs of \( X_j \)s with the Euclidean generators:

\[ \{P_j, X_k\} = \delta_{jk}I, \quad \{M_{jk}, X_l\} = \delta_{jl}X_k - \delta_{kl}X_j. \quad (94) \]

\textit{Proof.} Using Eq.(91) with \( \phi \) replaced by \( \phi' = \Phi_2(g)(\phi) \), we have

\[ \phi'(X_j) = \int x_j d\mu_{\phi'}(x) = \int x_j d\mu_\phi(x') = \int (x'_j - \delta x_j) d\mu_\phi(x') \quad (95) \]

where \( x' \equiv T_g(x) = x + \delta x \) and we have used Eq.(90) to write \( d\mu_{\phi'}(x) = d\mu_\phi(x') \). Writing \( \phi' = \phi + \delta\phi \) and taking \( T_g \) to be a general infinitesimal transformation generated by \( \epsilon \xi = \epsilon^a \xi_a \), we have

\[ \epsilon\phi(\{h_\xi, X_j\}) = \int_{R^n} \delta x_j d\mu_\phi(x). \quad (96) \]
For translations, with $\xi = p_k, h_{pk} = P_k, \delta x_j = \epsilon \delta x_k$, Eq.(96) gives
\[
\phi(\{P_k, X_j\}) = \delta_{jk} = \delta_{jk} \phi(I).
\]
Since this holds for all $\phi \in \mathcal{S}(A)$, we have the first of the equations (94). The second equation is similarly obtained by taking, in obvious notation, $\epsilon \xi = \frac{1}{2} \epsilon_{jk} m_{jk}$ and
\[
\delta x_l = \epsilon_{lk} x_k = \epsilon_{jk} \delta_{ji} x_k = \frac{1}{2} \epsilon_{jk} (\delta_{ji} x_k - \delta_{ki} x_j). \quad \square
\]

The hamiltonians $P_j$ and $M_{jk}$ will be referred to as the momentum and angular momentum observables of the system $S$. It should be noted that the PBs obtained above do not include the expected relations $\{X_j, X_k\} = 0$; these relations, as we shall see in the following section, come from the relativity group. [Recall that, in the treatments of localization based on systems of imprimitivity, the commutators $[X_j, X_k] = 0$ appear because there the analogues of the objects $P(D)$ are assumed to be projection operators satisfying the relation $P(D)P(D') = P(D \cap D')(= P(D')P(D))$. In our more general approach, we have no basis for making such assumptions.]

5. RELATIVITY GROUPS, ELEMENTARY SYSTEMS AND FUNDAMENTAL OBSERVABLES

Having presented the general formalism of supmech, we now proceed to treat concrete systems. We start with the simplest ones: particles. In this section, we take up the question of the definition of a particle and the fundamental observables relating to the characterization/labelling and kinematics of a particle. Relativity group will be seen to play a very important role in this context.

5.1. General considerations about relativity groups and elementary systems

A particle is basically an irreducible entity (in the sense that it cannot be represented as the composite of more than one identifiable entities) localized in what we traditionally call ‘space’ and the description of its dynamics involves ‘time’. We must, therefore, introduce the concepts of space and time or, more generally, space-time before we talk about particles.

In the following developments, space-time will be understood to be a (3+1)-dimensional differentiable manifold equipped with a suitable metric
to define spatial distances and time-intervals. [The breakup (3+1) means that one of the four dimensions is in some way distinguished from the other three. Details about the metric will be given only when needed.] A reference frame is an atlas providing a coordinatization of the space-time points. Observers are supposedly intelligent beings employing reference frames for doing concrete physics; they will be understood to be in one-to-one correspondence with reference frames.

Observables of systems localized in space are generally observer-dependent. This observer dependence is systematically taken into consideration by adopting a relativity scheme which incorporates (i) specification of the geometry of space-time, (ii) selection of a class of reference frames to to be treated as distinguished (all members of the chosen class to be treated as physically equivalent) and (iii) transformation laws between coordinatizations of different members of the chosen class (these transformations constitute a group called the relativity group of the scheme).

Assuming a fixed background space-time $M$, we shall assume the relativity group to be a connected Lie group $G_0$ (with Lie algebra $\mathfrak{g}_0$) acting as a transformation group on $M$. For concrete applications, we shall take $G_0$ to be the Galilean group and the Poincaré group (the inhomogeneous Lorentz group) in the schemes of Galilean relativity and special relativity respectively. Both these groups have the one-parameter group $T$ of time translations as a subgroup.

Treatment of kinematics and dynamics of a system in accordance with a relativity scheme involves the action of $G_0$ on the symplectic algebra $(\mathcal{A}, \omega)$. To exploit the availability of a symplectic framework, we would like this to be a hamiltonian action so that we can associate observables with the infinitesimal generators of $G_0$. We may formally state, in the sub-domain of supmech covering theories admitting a background space-time, the principle of relativity as follows:

(i) There is a preferred class of reference frames whose space-time coordinatisations are related through the action of a connected Lie group $G_0$.

(ii) For a system with the system algebra $\mathcal{A}$, the relativity group $G_0$ has a hamiltonian action on the symplectic algebra $(\mathcal{A}, \omega)$ [or the generalized symplectic superalgebra $(\mathcal{A}, \mathcal{X}, \omega)$ in appropriate situations].

(iii) All reference frames in the chosen class are physically equivalent in the sense that the fundamental equations of the theory are covariant with respect to the $G_0$-transformations of the relevant variables.
We shall call such a scheme $G_0$-relativity and systems covered by it $G_0$-relativistic.

Heisenberg and Schrödinger pictures of dynamics corresponding to two observers $O$ and $O'$ may be related through the symplectic action of $G_0$ by following the strategy of Sudarshan and Mukunda (1974) (referred to as SM below) exploiting the fact that $G_0$ has $\mathcal{T}$ as a subgroup. Showing the observer dependence of the algebra elements explicitly, the two Heisenberg picture descriptions $A(O,t)$ and $A(O',t')$ of an element $A$ of $\mathcal{A}$ can be related through the sequence (assuming a common zero of time for the two observers)

$$A(O,t) \rightarrow A(O,0) \rightarrow A(O',0) \rightarrow A(O',t')$$

where the first and the last steps involve the operations of time translations in the two frames. We shall be concerned only with the symplectic action of $G_0$ involved in the middle step. A similar strategy can be adopted for the Schrödinger picture. Detailed treatments of the relativistic Heisenberg and Schrödinger pictures in the classical hamiltonian formalism may be found in SM.

Construction of Noether invariants, on the other hand, involves explicit consideration of the transformation of the time variable. The formalism of section 4.6 has obvious limitations in this regard because time was treated as an external parameter in the Poisson brackets employed there. We shall, therefore, construct the Noether invariants only for the Galilean group where the only admitted transformations of the time variable are translations.

To formalize the notion of a (relativistic, quantum) particle as an irreducible entity, Wigner (1939) introduced the concept of an ‘elementary system’ as a quantum system whose Hilbert space carries a projective unitary irreducible representation of the Poincaré group. The basic idea is that the state space of an elementary system should not admit a decomposition into more than one invariant (under the action of the relevant relativity group) subspaces. Following this idea, elementary systems in classical mechanics (SM; Alonso 1979) have been defined in terms of a transitive action of the relativity group on the phase space of the system. Alonso (1979) gave a unified treatment of classical and quantum elementary systems by treating them as special cases of (irreducible/transitive) kinematical action of the relativity groups (called ‘invariance groups’ in that work) on the state space of a dynamical system.

In this section, we shall treat elementary systems in the framework of supmech. Traditional classical and quantum elementary systems will be seen
as special cases of these. This treatment goes a step further than that of Alonso in that the unification is achieved in a single *symplectic* framework.

A system $S$ having associated with it the symplectic triple $(\mathcal{A}, S_1, \omega)$ will be called an *elementary system* in $G_0$-relativity if it is a $G_0$-relativistic system such that the action of $G_0$ on the space $S_1$ of its pure states is transitive. Formally, an elementary system may be represented as a collection $\mathcal{E} = (G_0, \mathcal{A}, S_1, \omega, \Phi)$ where $\Phi = (\Phi_1, \Phi_2)$ are mappings as in section 4.4 implementing the $G_0$-actions — $\Phi_1$ describing a hamiltonian action on $(\mathcal{A}, \omega)$ and $\Phi_2$ a transitive action on $S_1$.

The transitive action of $G_0$ on $S_1$ implies that the expectation value of a $G_0$-invariant observable is the same in every pure state (hence in every state). To see this, let $Q$ be such an observable and $\phi_1, \phi_2$ two pure states such that $\Phi_2(g)(\phi_1) = \phi_2$ for some $g \in G_0$. We have

$$<\phi_2, Q> = <\Phi_2(g)(\phi_1), Q> = <\phi_1, \Phi_1(g^{-1})(Q)> = <\phi_1, Q>$$

as desired. Denoting this common expectation value of $Q$ by $q$ (we shall call it the value of $Q$ for the system), we have, by the CC condition, $Q = qI$.

This has the important implication that, for an elementary system, a Poisson action (of $G_0$ or of its projective group $\hat{G}_0$) is always available; this is because, if $G_0$ does not admit Poisson action, the cocycle $\alpha$ of section 4.4 is a multiple of the unit element and the hamiltonian action of $G_0$ can be extended to a Poisson action of $\hat{G}_0$.

Let $\xi_a (a = 1, \ldots, r)$ be a basis in the Lie algebra $\hat{G}_0$ of $\hat{G}_0$ satisfying the commutation relations as in section 4.4. The admissibility of Poisson action of $\hat{G}_0$ on $\mathcal{A}$ implies that, corresponding to the generators $\xi_a$, we have the hamiltonians $h_a \equiv h_{\xi_a}$ in $\mathcal{A}$ satisfying the PB relations

$$\{h_a, h_b\} = C_{ac}^{\ b} h_c. \quad (97)$$

Recalling Eq.(78), the condition of transitive action on $S_1$ implies that the $\hbar$-images of pure states of an elementary system are coadjoint orbits in $\hat{G}_0^*$. In classical mechanics, one has an isomorphism between the symplectic structures on the symplectic manifolds of elementary systems and those on the coadjoint orbits. In our case, the state spaces of elementary systems and coadjoint orbits of relativity groups are generally spaces of different types and the question of an isomorphism does not arise. We can, however, use Eq.(78) to obtain useful information about the transformation properties of
the quantities $h_a$ under the $\mathcal{G}_0$-action. Recalling the notations in section 4.4, writing $\text{Cad}_g[\hat{h}(\phi)] = u_a(g)\lambda^a$, we have

$$u_a(g) = \langle \text{Cad}_g[\hat{h}(\phi)], \xi_a \rangle = \langle \hat{h}[\Phi_2(g)(\phi)], \xi_a \rangle = \langle \phi, \Phi_1(g^{-1})h_a \rangle$$

showing that the transformation properties of Hamiltonians $h_a$ are directly related to those of the corresponding coordinates (with respect to the dual basis) of points on the relevant co-adjoint orbit. This is adequate to enable us to use the descriptions of the relevant co-adjoint actions in (Alonso 1979) and draw parallel conclusions.

We shall adopt the following strategy:

(i) Given a relativity scheme, use the Poisson action of $G_0$ or $\mathcal{G}_0$ on the symplectic superalgebra of an elementary system to obtain the corresponding Hamiltonians and their PBs [Eq.(97)]. These PBs are clearly the same for all elementary systems of the group $G_0$.

(ii) Use these PBs to identify some fundamental observables [i.e. those which cannot be obtained from other observables (through algebraic relations or PBs)]. These include observables (like mass) that Poisson-commute with all $h_a$s and the momentum observables (the group of space translations being a subgroup of both the relativity groups we consider).

(iii) Determine the transformation laws of $h_a$s under finite transformations of $G_0$ following the relevant developments in (SM; Alonso 1979). Use these transformation laws to identify the $G_0$-invariants and some other fundamental observables (the latter will be configuration and spin observables). The values of the invariant observables characterize (or serve to label) an elementary system.

(iv) The system algebra $\mathcal{A}$ for an elementary system is to be taken as the (topological completion of) the one generated by the fundamental observables and the identity element.

(v) Obtain (to the extent possible) the general form of the Hamiltonian as a function of the fundamental observables as dictated by the PB relations (97).

(vi) (For the Galilean group) use the formalism of section 4.6 to consider the action of $G_0$ on the presymplectic space ($\mathcal{A}^e, \Omega$) and, noting that this action satisfies Eq.(85), identify the appropriate Noether invariants.
We shall now obtain an equation that will be useful for this last job. Let \( \xi \in G_0 \) generate an infinitesimal transformation giving \( \delta t = \epsilon f(t) \) (and possibly some changes in other quantities). [In view of the limitations of the formalism of section 4.6 mentioned above, arguments other than \( t \) for the function \( f \) have been excluded.] The relation between the induced derivations \( Z_\xi \) on \( \mathcal{A} \) and \( \hat{Z}_\xi \) on \( \mathcal{A}' \) is given by

\[
\hat{Z}_\xi = Z_\xi + f(t) \frac{\partial}{\partial t}.
\] (99)

We have \( Z_\xi = Y_{\hat{h}} \xi \) (see section 4.4). We look for the quantity \( \hat{h}_\xi \) (the prospective Noether invariant) such that Eq. (85) holds. (Finding such a quantity will establish invariance of \( \Omega \) under the relevant group action and also determine the corresponding Noether invariant.) Equations (99) and (81) now give the desired relation

\[
i \hat{Z}_\xi \Omega = iZ_\xi \omega - iZ_\xi (dH)dt + f(t)dH
= -dh_\xi - Y_{\hat{h}} (H) dt + f(t)dH.
\] (100)

Most of the equations in the following two subsections have the same mathematical form as some of the equations in (Alonso 1977, 1979) and/or SM. The following couple of remarks should serve to clarify the situation.

(a) Classical elementary systems are defined in terms of a transitive canonical action of the relevant relativity group on symplectic manifolds. These are obviously special cases of supmech elementary systems corresponding to commutative system algebras of the type treated in section 3.8. Those results in the treatment of classical elementary systems whose derivation does not use the commutativity of the algebra \( \mathcal{A}_{cl} \) are expected to be valid for general supmech elementary systems.

(b) Quantum elementary systems are defined in terms of projective irreducible unitary representations of the relevant relativity group on separable Hilbert spaces. Keeping the developments in section 3.10 in view, these are seen as special cases of supmech elementary systems when the system algebra \( \mathcal{A} \) is a member of a triple \( (\mathcal{H}, \mathcal{D}, \mathcal{A}) \) [a ‘quantum triple’; see section 7.2] where \( \mathcal{H} \) is a separable Hilbert space, \( \mathcal{D} \) a dense linear subset of \( \mathcal{H} \) and \( \mathcal{A} \) an \( \text{Op}^* \)-algebra based on \( (\mathcal{H}, \mathcal{D}) \). According to theorem (3.2) in (Cariñena and Santander 1975), every projective unitary representation of a relativity group \( G_0 \) can be lifted to a unitary representation of the corresponding projective group \( \hat{G}_0 \) (called the ‘projective covering group’ of \( G_0 \) in that work).
The infinitesimal generators of $\hat{G}_0$ arising in such a (continuous) unitary representation serve as hamiltonians of the corresponding supmech elementary system. Once the infinitesimal generators have been obtained, the Hilbert space goes into the background; the rest of the work is algebraic. All the results obtained, in the traditional treatments of quantum elementary systems, by algebraic manipulations (involving the above mentioned infinitesimal generators) and use of the quantum Poisson brackets (54) are expected to be valid in the treatments of the corresponding elementary systems in supmech.

5.2. Nonrelativistic elementary systems

In this and the following subsection, we shall keep close to the notational conventions of Alonso (1979). Our PBs, however, follow the conventions of Woodhouse 1980) and differ from those of (SM; Alonso 1979) by a sign; moreover, our Galilean generator $K$ differs from that of Alonso by a sign.

The relativity group $G_0$ of the nonrelativistic domain of supmech is (the identity component of) the Galilean group of transformations of the Newtonian space-time $R^3 \times R$ given by

$$x' = Rx + tv + a, \quad t' = t + b$$

(101)

where $R \in SO(3), \ v \in R^3, \ a \in R^3$ and $b \in R$. This group does not admit Poisson action. After a careful consideration of the freedom to modify the hamiltonians by additive terms, the hamiltonians $J_i, K_i, P_i, H$ corresponding to the ten generators $J_i, K_i, P_i (i = 1, 2, 3), \mathcal{H}$ of $G_0$ [so that $h_{P_i} = P_i$, etc] can be shown to satisfy the Poisson bracket relations (SM)

$$\{J_i, J_j\} = -\delta_{ij}J_k, \quad \{J_i, K_j\} = -\delta_{ij}K_k, \quad \{J_i, P_j\} = -\delta_{ij}P_k$$

$$\{K_i, H\} = -P_i, \quad \{K_i, P_j\} = -\delta_{ij}M,$$

(102)

where $M$ is a neutral element; all other PBs vanish. By the argument presented above, we must have $M= mI, m \in R$. We shall identify $m$ as the mass of the elementary system. The condition $m \geq 0$ will follow later from an appropriate physical requirement. The objects $P_i$ and $J_i$, being generators of the Euclidean subgroup $E_3$ of $G_0$, are the momentum and angular momentum observables of section 4.7.

Following the procedure outlined in section 4.4, we augment the Lie algebra $\mathcal{G}_0$ of $G_0$ to a larger Lie algebra $\hat{\mathcal{G}}_0$ by including an additional generator $M$ corresponding to $M$ (which now becomes the Hamiltonian corresponding
to $M \in \hat{\mathcal{G}}_0$; it commutes with all other generators and appears only in the commutator

$$[\mathcal{K}_j, \mathcal{P}_k] = -\delta_{jk}M.$$  

(103)

The remaining commutation relations of $\hat{\mathcal{G}}_0$ are those of $\mathcal{G}_0$ ($[J_j, J_k] = -\epsilon_{jkl}J_l$ etc.). The projective group $\hat{\mathcal{G}}_0$ of the Galilean group $\mathcal{G}_0$ is the connected and simply connected Lie group with the Lie algebra $\hat{\mathcal{G}}_0$.

Representing a general group element of $\hat{\mathcal{G}}_0$ in the form

$$g = (A, v, b, a, \tau) = \exp(-\tau M)\exp(-a.P)\exp(-b.H)\exp(-v.K)A$$  

(104)

where $A \in SU(2)$ and $\tau \in R$, the group law of $\hat{\mathcal{G}}_0$ is obtained, after a straightforward calculation, as

$$g'g = (A' A, v' + R(A')v, b' + b, a' + bv' + R(A')a, \tau' + \tau + R(A')_{jk}v'_{jk}a_k).$$  

(105)

The transformation laws of the hamiltonians of $\hat{\mathcal{G}}_0$ under its adjoint action may be found following the procedure of either SM or Alonso (1979). These transformation laws give the following three independent invariants:

$$M, \quad C_1 \equiv 2MH - P^2, \quad C_2 \equiv (MJ - K \times P)^2.$$  

(106)

Of these, the first one is obvious; the vanishing of PBs of $C_1$ with all the hamiltonians is also easily checked. Writing $C_2 = B_jB_j$ where

$$B_j = MJ_j - \epsilon_{jkl}K_kP_l,$$

it is easily verified that

$$\{J_j, B_k\} = -\epsilon_{jkl}B_l, \quad \{K_j, B_k\} = \{P_j, B_k\} = \{H, B_k\} = 0$$

which finally leads to the vanishing of PBs of $C_2$ with all the hamiltonians. By the argument given above for $M$, the last two invariants also should be scalar multiples of the unit element in $\mathcal{A}$. The values of these three invariants characterize a Galilean elementary system in supmech.

We henceforth restrict ourselves to elementary systems with $m \neq 0$. Defining $X_i = m^{-1}K_i$, we have

$$\{X_j, X_k\} = 0, \quad \{P_j, X_k\} = \delta_{jk}I, \quad \{J_j, X_k\} = -\epsilon_{jkl}X_l.$$  

(107)
Comparing the last two equations above with the equations (94) (for n=3), we identify $X_j$ with the position observables of section 4.7. Note that the fact that the $X_j$s mutually Poisson-commute comes from the relativity group.

Writing $S = J - X \times P$, we have $C_2 = m^2 S^2$. We have the PB relations

$$\{S_i, S_j\} = -\epsilon_{ijk} S_k, \quad \{S_i, X_j\} = 0 = \{S_i, P_j\}. \tag{108}$$

We identify $S$ with the internal angular momentum or spin of the elementary system.

The invariant quantity

$$U \equiv \frac{C_1}{2m} = H - \frac{P^2}{2m} \tag{109}$$

is interpreted as the internal energy of the elementary system. It is the appearance of this quantity (which plays no role in Newtonian mechanics) which is responsible for energy being defined in Newtonian mechanics only up to an additive constant.

Writing $S^2 = \sigma I$ and $U = u I$, we see that Galilean elementary systems with $m \neq 0$ can be taken to be characterized/labeled by the parameters $m$, $\sigma$ and $u$. The fundamental kinematical observables are $X_j, P_j$ and $S_j$ ($j=1,2,3$). Other observables are assumed in supmech to be functions of the fundamental observables.

Henceforth we shall take $u = 0$ (a natural assumption to make if the elementary systems to be treated are particles). Eq.(109) now gives

$$H = \frac{P^2}{2m} \tag{110}$$

which is the Hamiltonian for a free Galilean particle in supmech.

Note. (i) Full Galilean invariance (more generally, full invariance under a relativity group) applies only to an isolated system. Interactions/(external influences) are usually described with (explicit or implicit) reference to a fixed reference frame or a restricted class of frames. For example, the interaction described by a central potential implicitly assumes that the center of force is at the origin of axes of the chosen reference frame.

(ii) In the presence of external influences, translational invariance is lost and the PB $\{H, P_j\} = 0$ must be dropped. For a spinless particle, the Hamiltonian
(assumed to be a function of the fundamental observables $X$ and $P$) then has the general form

$$H = \frac{P^2}{2m} + V(X, P).$$

In most practical situations, $V$ is a function of $X$ only.

We can now rule out the case $m < 0$ on physical grounds because, by Eq.(110), this will allow arbitrarily large negative values for energy. (Expectation values of the observable $P^2$ are expected to have no upper bound.)

Lastly, we consider the action of $G_0$ on the augmented algebra $\mathcal{A}^e$ for a free massive spinless particle. As noted above, it is adequate, for each $\xi$ in the chosen basis of $G_0$, to find a ‘hamiltonian’ $\hat{h}_\xi$ such that Eq.(85) holds; for this we must show the exactness of the form on the right hand side of Eq.(100). We have

(i) for rotations ($\xi = J_i, h_\xi = J_i$) $f(t) = 0, Y_\xi(H) = 0$, giving $\hat{h}_\xi = h_\xi = J_i$;

(ii) for Galilean boosts ($\xi = K_i, h_\xi = K_i = mX_i$) $f(t) = 0, vY_\xi(H) = v\{K_i, H\} = -vP_i$ giving $\hat{h}_\xi = mX_i - P_i t$;

(iii) for space translations ($\xi = P_i, h_\xi = P_i$) $f(t) = 0, Y_\xi(H) = 0$, giving $\hat{h}_\xi = h_\xi = P_i$;

(iv) for time translations ($\xi = H, h_\xi = H$) $f(t) = 1, Y_\xi = 0$, giving $\hat{h}_\xi = H$.

Finally, the Noether invariants of the Galilean group are

$$J, \quad mX - Pt, \quad P, \quad H$$

which are (up-to signs) the supmech avatars of those in (Souriau 1997).

5.3. Relativistic elementary systems

In the scheme of special relativity, the relativity group $G_0$ is the (identity component of) Poincaré group of transformations on the Minkowski space-time $[(R^4, \eta_{\mu\nu})$ where $\mu, \nu = 0,1,2,3$ and $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)]$:

$$x' = \Lambda x + a, \quad \Lambda \in SO(3,1) \quad \text{(with} \Lambda^0_0 \geq 1), \quad a \in R^4.$$

Sudars This group admits Poisson actions (SM; Alonso 1979; Guillemin and Sternberg 1984). Since the general method (of treating elementary systems
in supmech) has been illustrated in the previous subsection, we shall be more brief here. Some more details may be found in (Alonso 1977,1979; SM). We shall generally keep close to the developments in (Alonso 1977,1979).

The generators of \( G_0 \) are \((M_{\mu\nu} = -M_{\nu\mu}, P_{\mu})\) or, equivalently, \((J_i, K_i, P_i, H)\) where \( P_0 = H, M_{0i} = K_i \) and \( M_{ij} = \epsilon_{ijk}J_k \). The hamiltonians \((J_i, K_i, P_i, H)\) arising from the Poisson action of \( G_0 \) on the symplectic superalgebra \((A, \omega)\) of an elementary system satisfy the PB relations

\[
\{J_i, J_j\} = -\epsilon_{ijk}J_k, \quad \{J_i, K_j\} = -\epsilon_{ijk}K_k, \quad \{J_i, P_j\} = -\epsilon_{ijk}P_k, \quad \{K_i, K_j\} = \epsilon_{ijk}J_k, \quad \{K_i, P_j\} = -\delta_{ij}H, \quad \{K_i, H\} = -P_i; \quad (114)
\]

all other PBs vanish. The PBs for the manifestly Lorentz-covariant hamiltonians \( M_{\mu\nu}, P_{\mu} \) are those of Eq.(92) with \( j,k,l,p,q \) replaced by \( \mu, \nu, \lambda, \rho, \sigma \) and \( \delta_{jl} \) by \( \eta_{\mu\lambda} \) etc.

Defining \( W^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}M_{\nu\lambda}P_{\rho} \) (the Pauli-Lubanski vector), the two independent invariants of the \( G_0 \)-action are \( P^2 \equiv P_{\mu}P^\mu \) and \( W^2 \); this can be directly checked from the covariant PBs mentioned above. For elementary systems, they take values \( p^2I \) and \( w^2I \); the real-valued quantities \( p^2 \) and \( w^2 \) characterize the elementary systems. It is useful to note that

\[
W^0 = -J\cdot P, \quad W = -HJ + P \times K. \quad (115)
\]

We shall restrict ourselves to the cases with \( p^2 \leq 0 \) (i.e. \( p^\mu \) non-spacelike) and write \( p^2 = -m^2 \) (with \( m \geq 0 \)). For situations with \( m > 0 \) and \( H \) and \( (H + mI) \) invertible, one can define the position and spin observables as follows:

\[
X = -\frac{1}{2}[K, H^{-1}]_+ + [mH(H + mI)]^{-1}P \times W \quad (116)
\]

\[
S = -m^{-1}W + [mH(H + mI)]^{-1}W \cdot PP. \quad (117)
\]

The expected PB relations hold:

\[
\{X_i, X_j\} = 0, \quad \{P_i, X_j\} = \delta_{ij}I, \quad \{X_i, S_j\} = 0 = \{P_i, S_j\}
\]

\[
\{J_i, X_j\} = -\epsilon_{ijk}X_k, \quad \{J_i, S_j\} = \{S_i, S_j\} = -\epsilon_{ijk}S_k. \quad (118)
\]

We have \( W^2 = m^2S^2 \) and the relations

\[
H^2 = P^2 + m^2I, \quad J = X \times P + S \quad (119)
\]

\[
K = -\frac{1}{2}[X, H]_+ + (H + mI)^{-1}S \times P. \quad (120)
\]
Writing \( S^2 = \sigma I \), the invariant quantities \( m \) (mass) and \( \sigma \) (spin) characterize an elementary system and the fundamental kinematical observables are again \( X_j, P_j, S_j \) (\( j = 1, 2, 3 \)).

6. COUPLED SYSTEMS IN SUPMECH

We shall now consider the interaction of two systems \( S_1 \) and \( S_2 \) described individually as the supmech Hamiltonian systems \( (\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)}) \) (\( i = 1, 2 \)) and treat the coupled system \( S_1 + S_2 \) also as a supmech Hamiltonian system. To facilitate this, we must obtain the relevant mathematical objects for the coupled system.

6.1. The symplectic form and Poisson bracket on the tensor product of two superalgebras

The superalgebra corresponding to the coupled system \( (S_1 + S_2) \) will be taken as the (skew) tensor product \( \mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)} \); its elements are finite sums of tensored pairs:

\[
\sum_{j=1}^{m} A_j \otimes B_j, \quad A_j \in \mathcal{A}^{(1)}, \quad B_j \in \mathcal{A}^{(2)}
\]

with the multiplication rule

\[
(\sum_{j=1}^{m} A_j \otimes B_j)(\sum_{k=1}^{n} A_k \otimes B_k) = \sum_{j,k} \eta_{B_j A_k} (A_j A_k) \otimes (B_j B_k).
\] (121)

The superalgebra \( \mathcal{A}^{(1)} \) (resp. \( \mathcal{A}^{(2)} \)) has, in \( \mathcal{A} \), an isomorphic copy consisting of the elements \( (A \otimes I_2, A \in \mathcal{A}^{(1)}) \) (resp. \( I_1 \otimes B, B \in \mathcal{A}^{(2)} \)) to be denoted as \( \tilde{\mathcal{A}}^{(1)} \) (resp. \( \tilde{\mathcal{A}}^{(2)} \)). Here \( I_1 \) and \( I_2 \) are the unit elements of \( \mathcal{A}^{(1)} \) and \( \mathcal{A}^{(2)} \) respectively. We shall also use the notations \( \tilde{A}^{(1)} = A \otimes I_2 \) and \( \tilde{B}^{(2)} = I_1 \otimes B \).

Derivations and differential forms on \( \mathcal{A}^{(i)} \) and \( \tilde{\mathcal{A}}^{(i)} \) (\( i = 1, 2 \)) are formally related through the induced mappings corresponding to the isomorphisms \( \Xi^{(i)} : \mathcal{A}^{(i)} \rightarrow \tilde{\mathcal{A}}^{(i)} \) given by \( \Xi^{(1)}(A) = A \otimes I_2 \) and \( \Xi^{(2)}(B) = I_1 \otimes B \). For example, corresponding to \( X \in S\text{Der}(\mathcal{A}^{(1)}) \), we have \( \tilde{X}^{(1)} = \Xi^{(1)}(X) \) in \( S\text{Der}(\tilde{\mathcal{A}}^{(1)}) \) given by [see Eq.(3)]

\[
\tilde{X}^{(1)}(\tilde{A}^{(1)}) = \Xi^{(1)}(X)(\tilde{A}^{(1)}) = \Xi^{(1)}[X(A)] = X(A) \otimes I_2.
\] (122)

Similarly, corresponding to \( Y \in S\text{Der}(\mathcal{A}^{(2)}) \), we have \( \tilde{Y}^{(2)} \in S\text{Der}(\tilde{\mathcal{A}}^{(2)}) \) given by \( \tilde{Y}^{(2)}(\tilde{B}^{2}) = I_1 \otimes Y(B) \). For the 1-forms \( \alpha \in \Omega^1(\mathcal{A}^{(1)}) \) and \( \beta \in \Omega^1(\mathcal{A}^{(2)}) \),
we have $\tilde{\alpha}^{(1)} \in \Omega^1(\tilde{A}^{(1)})$ and $\tilde{\beta}^{(2)} \in \Omega^1(\tilde{A}^{(2)})$ given by [see Eq.(26)]

$$\tilde{\alpha}^{(1)}(\tilde{X}^{(1)}) = \Xi^{(1)}[\alpha((\Xi^{(1)})^{-1}) \tilde{X}^{(1)}] = \Xi^{(1)}[\alpha(X)] = \alpha(X) \otimes I_2 \quad (123)$$

and $\tilde{\beta}^{(2)}(\tilde{Y}^{(2)}) = I_1 \otimes \beta(Y)$. Analogous formulas hold for the higher forms.

We can extend the action of the superderivations $\tilde{X}^{(1)} \in SDer(\tilde{A}^{(1)})$ and $\tilde{Y}^{(2)} \in SDer(\tilde{A}^{(2)})$ to $\tilde{A}^{(2)}$ and $\tilde{A}^{(1)}$ respectively by defining

$$\tilde{X}^{(1)}(\tilde{B}^{(2)}) = 0, \quad \tilde{Y}^{(2)}(\tilde{A}^{(1)}) = 0 \quad \text{for all} \quad A \in A^{(1)} \quad \text{and} \quad B \in A^{(2)}. \quad (124)$$

Note that an $X \in SDer(A)$ is determined completely by its action on the subalgebras $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$:

$$X(A \otimes B) = X(\tilde{A}^{(1)} \tilde{B}^{(2)}) = (X \tilde{A}^{(1)}) \tilde{B}^{(2)} + \eta_{XA}(X \tilde{B}^{(2)}).$$

With the extensions described above, we have available to us superderivations belonging to the span of terms of the form [see Eq.(122)]

$$X = X^{(1)} \otimes I_2 + I_1 \otimes X^{(2)}. \quad (125)$$

Replacing $I_2$ and $I_1$ in Eq.(125) by elements of $Z(A^{(2)})$ and $Z(A^{(1)})$ respectively, we again obtain superderivations of $A$. We, therefore, have the space of superderivations

$$[SDer(A^{(1)}) \otimes Z(A^{(2)})] \oplus [Z(A^{(1)}) \otimes SDer(A^{(2)})]. \quad (126)$$

This space, however, is generally only a Lie sub-superalgebra of $SDer(A)$. For example, for $A^{(1)} = M_m(C)$ and $A^{(2)} = M_n(C)$, recalling that all the derivations of these matrix algebras are inner and that their centers consist of scalar multiples of the respective unit matrices, we have the (complex) dimensions of $SDer(A^{(1)})$, and $SDer(A^{(2)})$ respectively, $(m^2 - 1)$ and $(n^2 - 1)$ [so that the dimension of the space (126) is $m^2 + n^2 - 2$] whereas that of $SDer(A)$ is $(m^2n^2 - 1)$.

We shall need to employ a class of superderivations more general than (126). To this end, it is instructive to obtain explicit representation(s) for a general derivation of the matrix algebra $A = M_m(C) \otimes M_n(C)$. We have

$$[A \otimes B, C \otimes D]_{ir,js} = A_{ik}B_{rt}C_{kj}D_{ts} - C_{ik}D_{rt}A_{kj}B_{ts}$$

which gives

$$[A \otimes B, C \otimes D]_- = AC \otimes BD - CA \otimes DB \quad (127)$$

$$= [A, C]_- \otimes \frac{1}{2}[B, D]_+ + \frac{1}{2}[A, C]_+ \otimes [B, D]_- \quad (128)$$
This gives, in obvious notation,

\[
D_{A \otimes B} \equiv [A \otimes B, .] = A(\cdot) \otimes B(\cdot) - (\cdot) A \otimes (\cdot) B = D_A \otimes J_B + J_A \otimes D_B
\]  

(129) (130)

where \( J_B \) is the linear mapping on \( A^{(2)} \) given by \( J_B(D) = \frac{1}{2}[B, D]_+ \) and a similar expression for \( J_A \) as a linear mapping on \( A^{(1)} \). Eq.(129) shows that a derivation of the algebra \( A = A^{(1)} \otimes A^{(2)} \) need not explicitly contain those of \( A^{(i)} \). We shall, however, not get involved in the search for the most general expression for a derivation of the tensor product algebra \( A \) (although such an expression would be very useful). The expression (130) is more useful for us; it is a special case of the more general form

\[
X = X_1 \otimes \Psi_2 + \Psi_1 \otimes X_2
\]

(131)

where \( X_i \in SDer(A^{(i)}) \) (i=1,2) and \( \Psi_i : A^{(i)} \rightarrow A^{(i)} \) (i =1,2) are linear mappings. Our constructions below will lead us to the form (131). It is important to note, however, that an expression of the form (131) (which represents a linear mapping of \( A \) into itself) need not always be a derivation as can be easily checked. We shall impose the condition (1) on such an expression to obtain a derivation.

To obtain the differential forms and the exterior product on \( A \), the most straightforward procedure is to obtain the graded differential space \( (\Omega(A), d) \) as the tensor product (Greub 1978) of the graded differential spaces \( (\Omega(A^{(1)}), d_1) \) and \( (\Omega(A^{(2)}), d_2) \). A (homogeneous) differential k-form on \( A \) is of the form (in obvious notation)

\[
\alpha_{k_t} = \sum_{i + j = k; \ r + s = t \ mod(2)} \alpha_{i_r}^{(1)} \otimes \alpha_{j_s}^{(2)}.
\]

(132)

The \( d \) operation on \( \Omega(A) \) is given by \( [\alpha \in \Omega^p(A^{(1)}) \ and \ \beta \in \Omega(A^{(2)})] \)

\[
d(\alpha \otimes \beta) = (d_1 \alpha) \otimes \beta + (-1)^p \alpha \otimes d_2 \beta.
\]

(133)

Given the symplectic forms \( \omega^{(i)} \) on \( A^{(i)} \) (i=1,2) we shall construct the induced symplectic form \( \omega \) on \( A \) satisfying the following conditions :

(a) It should not depend on anything other than the objects \( \omega^{(i)} \) and \( I^{(i)} \) (i=1,2) [the ‘naturality’/’canonicality’ assumption for \( \omega \). (Note that the unit
elements are the only distinguished elements of the algebras being considered].

(b) The restrictions of $\omega$ to $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ be, respectively, $\omega^{(1)} \otimes I_2$ and $I_1 \otimes \omega^{(2)}$.

This determines $\omega$ uniquely:

$$\omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}. \quad (134)$$

To verify that it is a symplectic form, we must show that it is (i) closed and (ii) nondegenerate. Eq.(133) gives

$$d\omega = (d_{1}\omega^{(1)}) \otimes I_2 + \omega^{(1)} \otimes d_{2}(I_2) + d_{1}(I_1) \otimes \omega^{(2)} + I_1 \otimes d_{2}\omega^{(2)} = 0$$

showing that $\omega$ is closed. To show the nondegeneracy of $\omega$, we must show that, given $A \otimes B \in \mathcal{A}$, there exists a unique superderivation $Y = Y_{A \otimes B}$ in $SDer(\mathcal{A})$ such that

$$i_{Y}\omega = -d(A \otimes B) = -(d_{1}A) \otimes B - A \otimes d_{2}B = i_{Y^{(1)}} \omega^{(1)} \otimes B + A \otimes i_{Y^{(2)}} \omega^{(2)}. \quad (135)$$

where $Y^{(1)}_{A}$ and $Y^{(2)}_{B}$ are the Hamiltonian superderivations associated with $A \in \mathcal{A}^{(1)}$ and $B \in \mathcal{A}^{(2)}$. The structure of Eq.(135) suggests that $Y$ must be of the form [see Eq.(131)]

$$Y = Y^{(1)}_{A} \otimes \Psi^{(2)}_{B} + \Psi^{(1)}_{A} \otimes Y^{(2)}_{B} \quad (136)$$

where the linear mappings $\Psi^{(1)}_{A}$ and $\Psi^{(2)}_{B}$ satisfy the conditions $\Psi^{(1)}_{A}(I_1) = A$ and $\Psi^{(2)}_{B}(I_2) = B$. Recalling the discussion after Eq.(131) and Eq.(1) [and denoting the multiplication operators in $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ and $\mathcal{A}$ by $\mu_{1}$, $\mu_{2}$ and $\mu$ respectively], the condition for $Y$ to be a superderivation may be written as

$$Y \circ \mu(C \otimes D) - \eta_{Y,C\otimes D} \mu(C \otimes D) \circ Y = \mu(Y(C \otimes D)). \quad (137)$$

Noting that $\mu(C \otimes D) = \mu_{1}(C) \otimes \mu_{2}(D)$ (the skew tensor product causes no problems here), Eq.(137) with $Y$ of Eq.(136) gives

$$\eta_{BC}([Y^{(1)}_{A} \circ \mu_{1}(C)] \otimes [\Psi^{(2)}_{B} \circ \mu_{2}(D)] + [\Psi^{(1)}_{A} \circ \mu_{1}(C)] \otimes Y^{(2)}_{B} \circ \mu_{2}(D)])$$

$$-(-1)^{c}[[\mu_{1}(C) \circ Y^{(1)}_{A}] \otimes [\mu_{2}(D) \circ \Psi^{(2)}_{B}] + [\mu_{1}(C) \circ \Psi^{(1)}_{A}] \otimes [\mu_{2}(D) \circ Y^{(2)}_{B}]]$$

$$= \eta_{BC} \mu_{1}([A, C]_{1}) \otimes \mu_{2}(\Psi^{(2)}_{B}(D)) + \mu_{1}(\Psi^{(1)}_{A}(C)) \otimes \mu_{2}([B, D]_{2}) \quad (138)$$
where $\epsilon \equiv \epsilon_A \epsilon_C + \epsilon_B \epsilon_D + \epsilon_B \epsilon_C$ and we have used the relations $Y_A^{(1)}(C) = \{A, C\}_1$ and $Y_B^{(2)}(D) = \{B, D\}_2$.

The objects $Y_A^{(1)}$ and $Y_B^{(2)}$, being superderivations, satisfy relations of the form (1):

$$Y_A^{(1)} \circ \mu_1(C) - \eta_{AC} \mu_1(C) \circ Y_A^{(1)} = \mu_1(Y_A^{(1)}(C)) = \mu_1(\{A, C\}_1)$$
$$Y_B^{(2)} \circ \mu_2(D) - \eta_{BD} \mu_2(D) \circ Y_B^{(2)} = \mu_2(\{B, D\}_2).$$  \hspace{1cm} (139)

Putting $D = I_2$ in Eq.(138), we have [noting that $\mu_2(D) = \mu_2(I_2) = id_2$, the identity mapping on $A^{(2)}$ and that $\{B, I_2\}_2 = Y_B^{(2)}(I_2) = 0$]

$$[Y_A^{(1)} \circ \mu_1(C)] \otimes \Psi_B^{(2)} + [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)}$$
$$- \eta_{AC} \{\mu_1(C) \circ Y_A^{(1)} \otimes \Psi_B^{(2)} + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}\}$$
$$= \mu_1(\{A, C\}_1) \otimes \mu_2(B)$$  \hspace{1cm} (140)

which, along with equations (139), gives

$$\mu_1(\{A, C\}_1) \otimes [\Psi_B^{(2)} - \mu_2(B)] =$$
$$- [\Psi_A^{(1)} \circ \mu_1(C) - \eta_{AC} \mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}.  \hspace{1cm} (141)$$

Similarly, putting $C = I_1$ in Eq.(138), we get

$$[\Psi_A^{(1)} - \mu_1(A)] \otimes \mu_2(\{B, D\}_2) =$$
$$- Y_A^{(1)} \otimes [\Psi_B^{(2)} \circ \mu_2(D) - \eta_{BD} \mu_2(D) \circ \Psi_B^{(2)}].  \hspace{1cm} (142)$$

Now, equations (142) and (141) give the relations

$$\Psi_A^{(1)} - \mu_1(A) = \lambda_1 Y_A^{(1)}$$  \hspace{1cm} (143)

$$\Psi_B^{(2)} \circ \mu_2(D) - \eta_{BD} \mu_2(D) \circ \Psi_B^{(2)} = - \lambda_1 \mu_2(\{B, D\}_2)$$  \hspace{1cm} (144)

$$\Psi_B^{(2)} - \mu_2(B) = \lambda_2 Y_B^{(2)}$$  \hspace{1cm} (145)

$$\Psi_A^{(1)} \circ \mu_1(C) - \eta_{AC} \mu_1(C) \circ \Psi_A^{(1)} = - \lambda_2 \mu_1(\{A, C\}_1)$$  \hspace{1cm} (146)

where $\lambda_1$ and $\lambda_2$ are complex numbers.
Equations (136), (143) and (145) now give

\[ Y = Y_A^{(1)} \otimes [\mu_2(B) + \lambda_2 Y_B^{(2)}] + [\mu_1(A) + \lambda_1 Y_A^{(1)}] \otimes Y_B^{(2)} \]
\[ = Y_A^{(1)} \otimes \mu_2(B) + \mu_1(A) \otimes Y_B^{(2)} + (\lambda_1 + \lambda_2) Y_A^{(1)} \otimes Y_B^{(2)}. \]  

(147)

Note that only the combination \((\lambda_1 + \lambda_2) \equiv \lambda\) appears in Eq.(147). To have a unique \(Y\), we must obtain an equation fixing \(\lambda\) in terms of given quantities.

Substituting for \(\Psi_A^{(1)}\) and \(\Psi_B^{(2)}\) from equations (143) and (145) into equations (144) and (146) and using equations (139), we obtain the equations

\[ \lambda \mu_1([A,C]_1) = -\mu_1([A,C]) \text{ for all } A, C \in A^{(1)} \]  
\[ \lambda \mu_2([B,D]_2) = -\mu_2([B,D]) \text{ for all } B, D \in A^{(2)}. \]  

(148)

(149)

We have not one but two equations of the type we have been looking for. This is a signal for the emergence of nontrivial conditions (for the desired symplectic structure on the tensor product superalgebra to exist).

Let us consider the equations (148,149) for the various possible situations (corresponding to whether or not one or both the superalgebras are super-commutative):

(i) Let \(A^{(1)}\) be supercommutative. Assuming that the PB \(\{,\}_1\) is nontrivial, Eq.(148) implies that \(\lambda = 0\). Eq.(149) then implies that \(A^{(2)}\) must also be super-commutative. It follows that

(a) when both the superalgebras \(A^{(1)}\) and \(A^{(2)}\) are super-commutative, the unique \(Y\) is given by Eq.(147) with \(\lambda = 0\);

(b) a ‘natural’/’canonical’ symplectic structure does not exist on the tensor product of a super-commutative and a non-supercommutative superalgebra.

(ii) Let the superalgebra \(A^{(1)}\) be non-supercommutative. Eq.(148) then implies that \(\lambda \neq 0\), which, along with Eq.(149) implies that the superalgebra \(A^{(2)}\) is also non-supercommutative [which is also expected from (b) above]. Equations (148,149) now give

\[ \{A,C\}_1 = -\lambda^{-1}[A,C], \quad \{B,D\}_2 = -\lambda^{-1}[B,D] \]  

(150)

which shows that, when both the superalgebras are non-supercommutative, a ‘natural’/’canonical’ symplectic structure on their (skew) tensor product exists if and only if each superalgebra has a quantum symplectic structure with the same parameter \((-\lambda)\), i.e.

\[ \omega^{(1)} = -\lambda \omega_c^{(1)}, \quad \omega^{(2)} = -\lambda \omega_c^{(2)} \]  

(151)
where $\omega_c^{(i)}$ (i=1,2) are the canonical symplectic forms on the two superalgebras. The traditional quantum symplectic structure is obtained with $\lambda = i\hbar$.

**Note.** The two forms $\omega^{(i)}(i=1,2)$ of Eq.(151) represent bonafide symplectic structures only if the superalgebras $A^{(i)}$ (i=1,2) have only inner superderivations (see section 3.9). More generally, we can have generalized symplectic superalgebras $(A^{(i)}, X^{(i)}, \omega^{(i)})(i=1,2)$ where $X^{(i)} = ISDer(A^{(i)})$. In all the permitted cases, the PB on the superalgebra $A = A^{(1)} \otimes A^{(2)}$ is given by

$$\{A \otimes B, C \otimes D\} = Y_{A\otimes B}(C \otimes D) = \eta_{BC}[\{A, C\}_1 \otimes BD + AC \otimes \{B, D\}_2 + \lambda\{A, C\}_1 \otimes \{B, D\}_2]$$

(152)

where the parameter $\lambda$ vanishes in the super-commutative case; in the non-supercommutative case, it is the universal parameter appearing in the symplectic forms (151).

Noting that, in the non-supercommutative case,

$$\lambda\{A, C\}_1 \otimes \{B, D\}_2 = -[A, C] \otimes \{B, D\}_2 = -\{A, C\}_1 \otimes [B, D]$$

$$= -\frac{1}{2}[A, C] \otimes \{B, D\}_2 - \frac{1}{2}\{A, C\}_1 \otimes [B, D],$$

(153)

the PB of Eq.(152) can be written in the more symmetric form

$$\{A \otimes B, C \otimes D\} = \eta_{BC}[\{A, C\}_1 \otimes \frac{BD + BDD}{2} + \frac{AC + ACA}{2} \times \{B, D\}_2].$$

(154)

Recalling that, for the matrix algebra $M_n(C)(n \geq 2)$, the Poisson bracket (with the canonical symplectic form) is a commutator, Eq.(128) is a special case of Eq.(154). In fact, had we employed supermatrices, we would have got exactly Eq.(154) as can be easily verified using the equation preceding Eq.(127). As shown below, a direct calculation for the tensor product of two classical algebras of observables also gives results consistent with Eq.(154).

**Example [Both algebras commutative]**

$A^{(1)} = C^\infty(R^m)$, $A^{(2)} = C^\infty(R^n)$; (m,n even).

Let $x^i$ and $y^r$ be the coordinates on $R^m$ and $R^n$ respectively and let the Poisson brackets on them be

$$\{f, g\}_1 = \omega_1^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}; \quad \{u, v\}_2 = \omega_2^{rs} \frac{\partial u}{\partial y^r} \frac{\partial v}{\partial y^s}.$$
Let $z^a = (x^i, y^r)$ be the coordinates on $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. The PB on $\mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ is [putting $F(z) \equiv (f \otimes u)(x, y) = f(x)u(y)$ and $G(z) = g(x)v(y)$ and choosing the symplectic form on $\mathbb{R}^{m+n}$ in accordance with Eq.(134)]

$$\{F, G\} = \omega^{ab} \frac{\partial F}{\partial z^a} \frac{\partial G}{\partial z^b}$$

$$= \omega^i_j \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} uv + \omega^r_s \frac{\partial u}{\partial y^r} \frac{\partial v}{\partial y^s} fg$$

$$= \{f, g\}_1 uv + \{u, v\}_2 fg,$$

(155)

which is consistent with Eq.(154).

### 6.2. Dynamics of coupled systems

Given the individual systems $S_1$ and $S_2$ as the supmech Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ ($i = 1, 2$), the coupled system $(S_1 + S_2)$ is a supmech Hamiltonian system with the system algebra and symplectic form as discussed above and the Hamiltonian $H$ given by

$$H = H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + H_{int}$$

(156)

where the interaction Hamiltonian is generally of the form

$$H_{int} = \sum_{i=1}^{n} F_i \otimes G_i.$$  

(157)

The evolution (in the Heisenberg type picture) of a typical observable $A(t) \otimes B(t)$ is governed by the supmech Hamilton’s equation

$$\frac{d}{dt} [A(t) \otimes B(t)] = \{H, A(t) \otimes B(t)\}$$

$$= \{H^{(1)}, A(t)\}_1 \otimes B(t) + A(t) \otimes \{H^{(2)}, B(t)\}_2$$

$$+ \{H_{int}, A(t) \otimes B(t)\}.$$  

(158)

The last Poisson bracket in this equation can be evaluated using Eq.(152) or (154).

In the Schrödinger type picture, the time evolution of states of the coupled system is given by the supmech Liouville equation (65) with the Hamiltonian of Eq.(156).

In favorable situations, the supmech Heisenberg or Liouville equations may be written for finite time intervals by using appropriate exponentiations.
of operators. We shall do this in section 8 below in which a concrete application of the formalism of this section to measurements in quantum mechanics will be described.

The main lesson from this section is that all systems in nature whose interaction with other systems can be talked about must belong to one of the two ‘worlds’: the ‘commutative world’ in which all system superalgebras are super-commutative and the ‘noncommutative world’ in which all system superalgebras are non-supercommutative with a universal quantum symplectic structure. (No restriction is implied by the above requirement on the type of symplectic structure on system superalgebras in the commutative world.) In view of the familiar inadequacy of the commutative world, the ‘real’ world must clearly be the noncommutative (hence quantum) world; its systems will be called quantum systems. (This is formalized as axiom A7 in section 9.) The classical systems with commutative system algebras and traditional symplectic structures will appear only in the appropriately defined classical limit (or, more generally, in the classical approximation) of quantum systems.

This brings us on the threshold of an autonomous development of QM.

7. QUANTUM SYSTEMS, (SUPER-)CLASSICAL SYSTEMS AND QUANTUM-CLASSICAL CORRESPONDENCE

We start by describing what we call ‘standard quantum systems’ (eventually to be seen as quantum systems without superselection rules) in purely algebraic terms.

7.1. Standard quantum systems

By a standard quantum system (SQS) we shall mean a supmech Hamiltonian system \((\mathcal{A}, S_1, \omega, H)\) in which the system algebra \(\mathcal{A}\) is special (in the sense of section 3.9) and has a trivial graded center and \(\omega\) is the quantum symplectic form \(\omega_Q\) given by [see Eq.(53)]

\[
\omega_Q = -i\hbar \omega_c. \tag{159}
\]

(We have, in the terminology of section 3.9, the quantum symplectic structure with parameter \(b = -i\hbar\).) This is the only place where we put the Planck constant ‘by hand’ (the most natural place to do it — such a parameter is needed here); its appearance at all conventional places (canonical
commutation relations, Heisenberg and Schrödinger equations, etc) will be automatic.

The *quantum Poisson bracket* implied by the quantum symplectic form is [see Eq.(54)]

\[ \{A, B\} = (-i\hbar)^{-1}[A, B]. \]  

Recall that the bracket [\cdot,\cdot] represents a supercommutator; it follows that the bracket on the right in Eq.(160) is an anticommutator when both A and B are odd/fermionic and a commutator in all other situations with homogeneous A,B.

A *quantum canonical transformation* is an automorphism \( \Phi \) of the system algebra \( \mathcal{A} \) such that \( \Phi^*\omega_Q = \omega_Q \). Now

\[ (\Phi^*\omega_Q)(X_1, X_2) = \Phi^{-1}[\omega_Q(\Phi^*X_1, \Phi^*X_2)] \]  

where \( X_1, X_2 \) are inner superderivations, say, \( D_A \) and \( D_B \). We have

\[ (\Phi^*D_A)(B) = \Phi[D_A(\Phi^{-1}(B))] = \Phi([A, \Phi^{-1}(B)]) = [\Phi(A), B] \]

which gives

\[ \Phi^*D_A = D_{\Phi(A)}. \]  

Equations (161) and (47) now give

\[ \Phi(i[A, B]) = i[\Phi(A), \Phi(B)] \]

which shows, quite plausibly, that quantum canonical transformations are (in the present algebraic setting — we have not yet come to the Hilbert space) the automorphisms of the system algebra preserving the quantum PBs.

The evolution of an SQS in time is governed, in the Heisenberg picture, by the supmech Hamilton’s equation (64) which now becomes the familiar *Heisenberg equation of motion*

\[ \frac{dA(t)}{dt} = (-i\hbar)^{-1}[H, A(t)]. \]  

In the Schrödinger picture, the time dependence is carried by the states and the evolution equation (65) takes the form

\[ \frac{d\phi(t)}{dt}(A) = (-i\hbar)^{-1}\phi(t)([H, A]) \]
which may be called the generalized von Neumann equation.

We shall call two SQSs \( \Sigma = (A, S_1, \omega, H) \) and \( \Sigma' = (A', S'_1, \omega', H') \) equivalent if they are equivalent as supmech Hamiltonian systems. (See section 4.3.)

Note. In the abstract algebraic framework, the CC condition is to be kept track of. An advantage, as we shall see below, of the Hilbert space based realizations of quantum systems is that the CC condition is automatically satisfied in them.

7.2. Hilbert space based realizations of standard quantum systems

Recalling the brief treatment of the Schrödinger representation in section 3.10, it is useful to introduce the concept of a quantum triple \((\mathcal{H}, \mathcal{D}, \mathcal{A})\) where \(\mathcal{H}\) is a complex Hilbert space, \(\mathcal{D}\) a dense linear subset of \(\mathcal{H}\) and \(\mathcal{A}\) an \(\text{Op}^*\)-algebra of operators based on \((\mathcal{H}, \mathcal{D})\). We shall assume that, for a given \(\mathcal{A}, \mathcal{D}\) is maximal, i.e. largest such domain. When \(\mathcal{A}\) is generated by a finite set of fundamental observables \(F_1, \ldots, F_n\), then in the notation of (Dubin and Hennings 1990), \(\mathcal{D} = C^\infty(F_1, \ldots, F_n)\) (i.e. intersection of the domains of all polynomials in \(F_1, \ldots, F_n\)).

If, in the quantum triple above, we take \(\mathcal{A}\) as our system algebra, then its states are given by the subclass of density operators \(\rho\) on \(\mathcal{H}\) for which \(|\text{Tr}(\rho A)| < \infty\) for all observables \(A\) in \(\mathcal{A}\); the quantity \(\text{Tr}(\rho A) \equiv \phi_\rho(A)\) (where \(\phi_\rho\) is the state represented by the density operator \(\rho\)) is the expectation value of the observable \(A\) in the state \(\phi_\rho\). Pure states are the subclass of these states consisting of one-dimensional projection operators. In view of the maximality of \(\mathcal{D}\), the latter are precisely the one-dimensional projectors \(|\psi><\psi|\) where \(\psi\) is any normalized element of \(\mathcal{D}\) (which means that pure states are the unit rays corresponding to the elements of \(\mathcal{D}\)). [Note. Here \(|\psi><\psi|\) is only a formal notation for the projector \(P_\psi\) defined by \(P_\psi\chi = (\psi, \chi)\psi\) for all \(\chi \in \mathcal{H}\); Dirac bra and ket vectors will be introduced later.]

When the algebra \(\mathcal{A}\) of the quantum triple above is special, we obtain a Hilbert space based SQS by choosing the quantum symplectic form as above and an even Hermitian element \(H\) of \(\mathcal{A}\) as the Hamiltonian. It is clear that, when the choice of Hamiltonian is not under consideration, a Hilbert space-based SQS is adequately described as a quantum triple with the algebra \(\mathcal{A}\) qualified as above.

The CC condition for the pair \((\mathcal{O}(\mathcal{A}), S_1)\) can be explicitly verified for a
Hilbert space based SQS:

(i) Given $A, B \in \mathcal{O}(A)$, and $(\psi, A\psi) = (\psi, B\psi)$ for all normalized $\psi$ in $\mathcal{D}$ (hence for all $\psi$ in $\mathcal{D}$), we have $(\phi, A\psi) = (\phi, B\psi)$ for all $\phi, \psi \in \mathcal{D}$, implying $A = B$. [Hint: Consider the given equality with the state vectors $(\phi + \psi)/\sqrt{2}$ and $(\phi + i\psi)/\sqrt{2}$.]

(ii) Given normalized vectors $\psi_1, \psi_2$ in $\mathcal{D}$ and $(\psi_1, A\psi_1) = (\psi_2, A\psi_2)$ for all $A \in \mathcal{O}(A)$, the equality $\psi_1 = \psi_2$ (up to a phase) can be seen by taking, for $A$, the projection operators corresponding to members of an orthonormal basis in $\mathcal{H}$ containing $\psi_1$ as a member.

Note. We have implicitly assumed above that all elements of $\mathcal{D}$ represent pure states. This excludes the situations when $\mathcal{H}$ is a direct sum of more than one coherent subspaces in the presence of superselection rules.

An interesting feature of the Hilbert space-based SQSs is that we have density operators representing states which, being Hermitian operators, are also observables. A density operator $\rho$ is the observable corresponding to the property of the system being in the state $\phi_\rho$. Given two states represented by density operators $\rho_1$ and $\rho_2$, we have the quantity $w_{12} = Tr(\rho_1\rho_2)$ defined (representing the expectation value of the observable $\rho_1$ in the state $\rho_2$ and vice versa) which has the natural interpretation of transition probability from one of the states to the other (the two are equal because $w_{12} = w_{21}$). When $\rho_i = |\psi_i><\psi_i|$ ($i = 1, 2$) are pure states, we have $Tr(\rho_1\rho_2) = |(\psi_1, \psi_2)|^2$ — the familiar textbook expression for the transition probability between two pure quantum states.

[Note. It is desirable to represent the quantities $w_{12}$ as bonafide probabilities in the standard form (56) employing an appropriate PObVM. It is clearly adequate to have such a representation for the case of pure states with $\rho_j = |\psi_j><\psi_j|$ ($j = 1, 2$, say. To achieve this, let $\phi = \phi_{\rho_1}$ and $\{\chi_r; r = 1, 2, \ldots\}$ an orthonormal basis in $\mathcal{H}$ having $\chi_1 = \psi_2$. The desired PObVM is obtained by taking, in the notation of section 4.1,

$$\Omega = \{\chi_r; r = 1, 2, \ldots\}, \quad \mathcal{F} = \{\text{All subsets of } \Omega\}$$

and, for $E = \{\chi_r; r \in J\} \in \mathcal{F}$ where $J$ is a subset of the positive integers,

$$\nu(E) = \sum_{r \in J} |\chi_r><\chi_r|.$$}

We now have $w_{12} = |(\psi_1, \psi_2)|^2 = p_\phi(E)$ of Eq.(56) with $\phi = \phi_{\rho_1}$ and $E = |\chi_1><\chi_1| = |\psi_2><\psi_2|$.]
We next consider the implementation, in a Hilbert space based SQS, of the mappings \((\Phi_1, \Phi_2)\) representing a quantum canonical transformation. These mappings are subject to the invariance condition

\[
< \Phi_2(\phi_\rho), \Phi_1(A) > = < \phi_\rho, A >
\]

for any \(A \in \mathcal{A}\) and state \(\phi_\rho\). The mapping \(\Phi_2\) maps pure states onto pure states; hence, for \(\rho = |\psi><\psi|\), we have \(\Phi_2(\phi_\rho) = \phi_{\rho'}\) with \(\rho' = |\psi'><\psi'|\) for some \(\psi'\). In this case, writing \(\Phi_1(A) = A'\), the above mentioned condition takes the form

\[
(\psi', A'\psi') = (\psi, A\psi).
\]

These mappings will now be shown to be unitarily implemented, i.e. for any such pair \((\Phi_1, \Phi_2)\), there is a unitary operator \(U\) on \(\mathcal{H}\) such that

\[
A' = UAU^{-1} \quad \text{and} \quad \psi' = U\psi.
\]

**Proof.** In view of the invariance condition (which fixes the mapping \(\Phi_2\) in terms of \(\Phi_1\)), it is adequate to prove the first relation. Let \(\chi_r\) \((r=1,2,\ldots)\) be an orthonormal basis in \(\mathcal{H}\) (with all the \(\chi_r\) in \(\mathcal{D}\)) and \(\chi'_r\) ‘the’ image of \(\chi_r\) under the \(\Phi_2\) action (i.e. \(\chi'_r\) is a representative of the unit ray which is the \(\Phi_2\)-image of the unit ray corresponding to \(\chi_r\)). Expanding \(\chi'_s\) in the original basis, we have

\[
\chi'_s = \sum_r U_{rs} \chi_r \quad \text{where} \quad U_{rs} = (\chi_r, \chi'_s).
\]

Defining an operator \(U\) by \((\chi_r, U\chi_s) = U_{rs}\), we have \(\chi'_s = U\chi_s\) \((s=1,2,\ldots)\). Putting \(\psi = \chi_s\) in the invariance condition above, we have

\[
(\chi_s, U^\dagger A'U\chi_s) = (\chi_s, A\chi_s).
\]

Writing similar equations with \(\chi_s\) replaced by \((\chi_r+\chi_s)/\sqrt{2}\) and \((\chi_r+i\chi_s)/\sqrt{2}\), it is easily seen that

\[
(\chi_r, U^\dagger A'U\chi_s) = (\chi_r, A\chi_s)
\]

which gives \(U^\dagger A'U = A\).

Now, for \(A = I\), we must have \(A' = I\) (the mapping \(\Phi_1\) being an automorphism of the unital algebra \(\mathcal{A}\)); this gives \(U^\dagger U = I\) or, remembering the invertibility of the mapping \(\Phi_2\), \(U^\dagger = U^{-1}\). We have, therefore, \(A' = UAU^{-1}\). The condition (163) implies

\[
U(i[A, B])U^{-1} = i[UAU^{-1}, UB^{-1}]
\]

69
which shows that $U$ must be a unitary operator. □

Note that, while unitary transformations (mapping $\mathcal{D}$ onto itself – only these are permitted to represent quantum symmetries in our formalism) are genuine symmetry operations, the antiunitary transformations are not.

The unitarily implemented $\Phi_2$ actions on states leave the transition probabilities invariant [in fact, they leave transition amplitudes invariant: $(\psi', \chi') = (\psi, \chi)$.] Note that, in contrast with the traditional formalism of QM, invariance of transition probabilities is not postulated but proved in the present setting.

A symmetry implemented (in the unimodal sense, as defined in section 4.3) by a unitary operator $U$ acts on a state vector $\psi \in \mathcal{D}$ according to $\psi \rightarrow \psi' = U\psi$ and (when its action is transferred to operators) on an operator $A \in \mathcal{A}$ according to $A \rightarrow A'$ such that, for all $\phi \in \mathcal{D}$,

$$(\phi', A\phi') = (\phi, A'\phi) \Rightarrow A' = U^{-1}AU.$$ (166)

For an infinitesimal unitary transformation $U \simeq I + i\epsilon G$ where $G$ is an even, Hermitian element of $\mathcal{A}$ [this follows from the condition $(U\phi, U\psi) = (\phi, \psi)$ for all $\phi, \psi \in \mathcal{D}$]. Considering the transformation $A \rightarrow A'$ in Eq.(166) as a quantum canonical transformation, generated (through PBs) by an element $T \in \mathcal{A}$, we have

$$\delta A = -i\epsilon [G, A] = \epsilon \{T, A\}$$ (167)

giving

$$T = -i(-i\hbar)G = -\hbar G$$ (168)

and

$$U \simeq I - i\frac{\epsilon}{\hbar}T.$$ (169)

It is the appearance of $\hbar$ in Eq(169) which is responsible for its appearance at almost all conventional places in QM.

The quantum canonical transformation representing evolution in time of an SQS is implemented on the state vectors by a one-parameter family of unitary operators [in the form $\psi(t) = U(t - s)\psi(s)$] generated by the Hamiltonian operator $H : U(\epsilon) \simeq I - i\frac{\epsilon}{\hbar}H$. This gives, in the Schrödinger
picture, the Schrödinger equation for the evolution of pure states of a Hilbert space based SQS:

$$i\hbar \frac{d\psi(t)}{dt} = H\psi(t).$$

(170)

In the Heisenberg picture, we have, of course, the Heisenberg equation of motion (164), which is now an operator equation.

Quantum triples provide a natural setting for a mathematically rigorous development of the Dirac bra-ket formalism. The essence of this formalism lies in generalizing the orthogonal expansions in a Hilbert space to include integrals over ‘generalized eigenvectors’ (Fourier transforms, for example). This becomes necessary when some observables of interest have (partly or wholly) continuous spectrum. The appropriate formalism for this is provided by a ‘rigged Hilbert space’ [or Gel’fand triple (Gelfand and Vilenkin 1964)]; the latter appears as a natural development once a pair \((\mathcal{H}, \mathcal{D})\) consisting of a Hilbert space \(\mathcal{H}\) and a dense linear subset \(\mathcal{D}\) in it is given.

Given a dense domain \(\mathcal{D}\) in \(\mathcal{H}\), one can define the *-algebra \(L^+(\mathcal{D})\) [in the notation of Lassner (1984)] – the largest Op*-algebra based on \((\mathcal{H}, \mathcal{D})\). [If, instead of choosing \(\mathcal{A}\) first and then constructing \(\mathcal{D}\), we had chosen \(\mathcal{D}\) first and then proceeded to choose \(\mathcal{A}\), then the natural/simplest choice of \(\mathcal{A}\) would be \(L^+(\mathcal{D})\).] On \(\mathcal{D}\), a locally convex topology \(t\) is defined by the seminorms \(\|\cdot\|_A\), given by

$$\|\psi\|_A = \|A\psi\|, \quad A \in L^+(\mathcal{D});$$

we denote the resulting locally convex topological vector space by \(\mathcal{D}[t]\). Let \(\mathcal{D}'[t']\) be the dual space of \(\mathcal{D}[t]\) with the strong topology (Kristensen, Mejlbo and Thue Poulsen 1965) \(t'\) (it is defined by the seminorms

$$p_B(\chi) = \sup_{\psi \in B} |\chi(\psi)|$$

for all bounded subsets \(B\) of \(\mathcal{D}\)). Then the Gelfand triple

\(\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}'[t']\)

constitutes the canonical rigged Hilbert space (Lassner 1984) based on the pair \((\mathcal{H}, \mathcal{D})\). The space \(\mathcal{D}'[t']\) (the space of continuous linear functionals or distributions on the test function space \(\mathcal{D}[t]\)) is the space of bra vectors of Dirac. The space of kets is the space \(\mathcal{D}^\times\) of continuous antilinear functionals.
on $D[t]$. [An element $\chi \in H$ defines a continuous linear functional $F_\chi$ and an antilinear functional $K_\chi$ on $H$ (hence on $D$) given by $F_\chi(\psi) = (\chi, \psi)$ and $K_\chi(\psi) = (\psi, \chi)$; both the bra and ket spaces, therefore, have $H$ as a subset.]

A Hermitian operator $A (= A^\ast)$ in $L^+(D)$ which admits a unique self adjoint extension in $H$ (often called ‘essentially self adjoint’) and is cyclic [i.e. there exists a vector $\psi$ in $D$ such that the vectors $P(A)\psi$, where $P(A)$ is a polynomial in $A$, are dense in $H$] has complete sets of generalized eigenvectors [eigenkets $\{|\lambda >; \lambda \in \sigma(A), \text{the spectrum of } A \}$ and eigenbras $\{<\lambda|; \lambda \in \sigma(A)\}]$:

$$A|\lambda > = \lambda |\lambda >; \quad <\lambda|A = \lambda <\lambda|;$$

$$\int_{\sigma(A)} d\mu(\lambda)|\lambda > <\lambda| = I \quad (172)$$

where $I$ is the unit operator in $H$ and $\mu$ is a unique measure on $\sigma(A)$. These equations are to be understood in the sense that, for all $\phi, \psi \in D$,

$$<\phi|A|\lambda > = \lambda <\phi|\lambda >; \quad <\lambda|A|\phi > = \lambda <\lambda|\phi >;$$

$$\int_{\sigma(A)} d\mu(\lambda) <\phi|\lambda > <\lambda|\psi > = <\phi|\psi > .$$

The last equation implies the expansion (in eigenkets of $A$)

$$|\psi > = \int_{\sigma(A)} d\mu(\lambda)|\lambda > <\lambda|\psi > .$$

More generally, one has complete sets of generalized eigenvectors associated with complete sets of commuting observables. For more details on the mathematically rigorous development of the bra-ket formalism, we refer to the literature (Roberts 1966; Antoine 1969; A. Böhm 1978; de la Madrid 2005).

7.3. Inevitability of the Hilbert space

Having shown the advantages of a Hilbert space-based realization of a standard quantum system, we now proceed to consider the existence and inevitability of such a realization.

Given an (abstract) SQS $\Sigma = (A, S_1, \omega, H)$, by a Hilbert space realization of it we mean an SQS $\hat{\Sigma} = (\hat{A}, \hat{S}_1, \hat{\omega}, \hat{H})$ of the type treated in the previous subsection which is equivalent to $\Sigma$ as a supmech Hamiltonian system. This
amounts to constructing a quantum triple \((\mathcal{H}, \mathcal{D}, \hat{A})\) in which the algebra \(\hat{A}\) is isomorphic, as a topological algebra, to the system algebra \(\mathcal{A}\) and choosing \(\mathcal{D}\) to be maximal (in the sense of section 7.2). Elements of \(\mathcal{D}\) then provide pure states such that the observables (i.e. the Hermitian elements of \(\hat{A}\)) and pure states satisfy the CC condition; moreover, once \(\hat{A}\) has been obtained (as a special algebra isomorphic to \(\mathcal{A}\)), construction of \(\hat{\omega}\) and \(\hat{H}\) is automatic.

From the above definition it is clear that, such a realization, if it exists, is unique up to equivalence.

Mathematically we have the problem of obtaining a faithful irreducible \(*\)-representation of the \(*\)-algebra \(\mathcal{A}\). Good references for the treatment of relevant mathematical concepts are Powers (1971) and Dubin and Hennings (1990). By a \(*\)-representation of a \(*\)-algebra \(\mathcal{A}\) we mean a triple \((\mathcal{H}, \mathcal{D}, \pi)\) where \(\mathcal{H}\) is a (separable) Hilbert space, \(\mathcal{D}\) a dense linear subset of \(\mathcal{H}\) and \(\pi\) a \(*\)-homomorphism of \(\mathcal{A}\) into the operator algebra \(L^+(\mathcal{D})\) (defined in section 7.2) satisfying the relation

\[
(\chi, \pi(A)\psi) = (\pi(A^*)\chi, \psi) \quad \text{for all} \quad A \in \mathcal{A} \quad \text{and} \quad \chi, \psi \in \mathcal{D}.
\]

We shall build up our arguments such that no new assumptions will be involved in going from the abstract algebraic setting to the Hilbert space setting; emergence of the Hilbert space formalism will be automatic.

To this end, we shall exploit the fact that the CC condition guarantees the existence of plenty of (pure) states of the algebra \(\mathcal{A}\). Given a state \(\phi\) on \(\mathcal{A}\), a standard way to obtain a representation of \(\mathcal{A}\) is to employ the so-called GNS construction. Some essential points related to this construction are given below:

(i) Noting that the given algebra \(\mathcal{A}\) is itself a complex vector space, one tries to define a scalar product on it using the state \(\phi\), the obvious choice being \((A, B) = \phi(A^*B)\). This, however, is prevented from being positive definite if the set \(L_\phi = \{A \in \mathcal{A}; \phi(A^*A) = 0\}\) (which can be shown to be a left ideal of \(\mathcal{A}\)) has nonzero elements in it. On the quotient space \(\mathcal{D}_\phi^{(0)} = \mathcal{A}/L_\phi\), the object

\[
([A], [B]) = \phi(A^*B)
\]

is a well defined scalar product. Here \([A] = A + L_\phi\) denotes the equivalence class of \(A\) in \(\mathcal{D}_\phi^{(0)}\).
(ii) One then completes the inner product space \((D_\phi(0), \langle , \rangle)\) to obtain the Hilbert space \(\mathcal{H}_\phi\); it is guaranteed to be separable by the separability of the topological algebra \(\mathcal{A}\).

(iii) One defines a representation \(\pi_\phi^{(0)}\) of \(\mathcal{A}\) on the pair \((\mathcal{H}_\phi, D_\phi(0))\) by

\[
\pi_\phi^{(0)}(A)[B] = [AB];
\]  

(174)

it can be easily checked to be a well defined *-representation.

(iv) The operators \(\pi_\phi^{(0)}(A)\) induce a topology on \(D_\phi(0)\) [see Eq.(171)]; the completion \(D_\phi\) of \(D_\phi(0)\) in this topology acts as the common invariant domain for the operators \(\pi_\phi(A)\) (where \(\pi_\phi\) is the closure of the representation \(\pi_\phi^{(0)}\)).

(v) The original state \(\phi\) is represented as a vector state in the representations \(\pi_\phi^{(0)}\) and \(\pi_\phi\) by the vector \(\chi_\phi = [I]\) (the equivalence class of the unit element of \(\mathcal{A}\)); indeed, we have, from Eq.(173),

\[
\phi(A) = ([I], [A]) = ([I], \pi_\phi^{(0)}(A)[I])
\]

\[
= (\chi_\phi, \pi_\phi^{(0)}(A)\chi_\phi) = (\chi_\phi, \pi_\phi(A)\chi_\phi).
\]  

(175)

(vi) The triple \((\mathcal{H}_\phi, D_\phi, \pi_\phi)\) satisfying Eq.(175), referred to as the GNS representation of \(\mathcal{A}\) induced by the state \(\phi\) [some authors refer to the triple \((\mathcal{H}_\phi, D_\phi(0), \pi_\phi^{(0)})\) as the GNS representation of \(\mathcal{A}\)], is determined uniquely, up to unitary equivalence, by the state \(\phi\).

(vii) The representation \(\pi_\phi\) of \(\mathcal{A}\) is irreducible if and only if the state \(\phi\) is pure.

This construction, however, does not completely solve our problem because a GNS representation is generally not faithful; for all \(A \in L_\phi\), we have obviously \(\pi_\phi(A) = 0\). [For example, a state with zero expectation value for the kinetic energy of a particle will yield a GNS representation which will represent the momentum observable of the particle by the zero operator.]

Note. The GNS representation is faithful if the state \(\phi\) is faithful (i.e. if \(L_\phi = \{0\}\)). Such a state, however, is not guaranteed to exist by our postulates.

A faithful but generally reducible representation of \(\mathcal{A}\) can be obtained by taking the direct sum of the representations of the above sort corresponding to all the pure states \(\phi\). [For the construction of the direct sum of a possibly uncountable set of Hilbert spaces, see Rudin (1974).] Let \(\mathcal{K}\) be the Cartesian product of the Hilbert spaces \(\{\mathcal{H}_\phi : \phi \in \mathcal{S}_1(\mathcal{A})\}\). A general element \(\psi\) of \(\mathcal{K}\)
is a collection \( \{ \psi_{\phi} : \phi \in S_1(A) \} \); here \( \psi_{\phi} \) is called the component of \( \psi \) in \( \mathcal{H}_\phi \).

The desired Hilbert space \( \mathcal{H} \) consists of those elements \( \psi \) in \( \mathcal{K} \) which have an at most countable set of nonzero components \( \psi_{\phi} \) which, moreover, satisfy the condition
\[
\sum_{\phi} \| \psi_{\phi} \|_{\mathcal{H}_\phi}^2 < \infty.
\]

The scalar product in \( \mathcal{H} \) is given by
\[
(\psi, \psi') = \sum_{\phi} (\psi_{\phi}, \psi'_{\phi})_{\mathcal{H}_\phi}.
\]

The direct sum of the representations \( \{ (\mathcal{H}_\phi, \mathcal{D}_\phi, \pi_{\phi}); \phi \in S_1(A) \} \) is the representation \( (\mathcal{H}, \mathcal{D}, \pi) \) where \( \mathcal{H} \) is as above, \( \mathcal{D} \) is the subset of \( \mathcal{H} \) consisting of vectors \( \psi \in \mathcal{D}_\phi \) for all \( \phi \in S_1(A) \) and, for any \( A \in \mathcal{A} \),
\[
\pi(A)\psi = \{ \pi_{\phi}(A)\psi_{\phi}; \phi \in S_1(A) \}.
\]

Now, given any two different elements \( A_1, A_2 \) in \( \mathcal{A} \), let \( \phi_0 \) be a pure state (guaranteed to exist by the CC condition) such that \( \phi_0(A_1) \neq \phi_0(A_2) \). Let \( \psi_0 \in \mathcal{H} \) be the vector with the single nonzero component \( \chi_{\phi_0} \). For any \( A \in \mathcal{A} \), we have
\[
(\psi_0, \pi(A)\psi_0) = (\chi_{\phi_0}, \pi_{\phi_0}(A)\chi_{\phi_0}) = \phi_0(A).
\]
This implies
\[
(\psi_0, \pi(A_1)\psi_0) \neq (\psi_0, \pi(A_2)\psi_0), \quad \text{hence } \pi(A_1) \neq \pi(A_2)
\]
showing that the representation \( (\mathcal{H}, \mathcal{D}, \pi) \) is faithful.

The Hilbert space \( \mathcal{H} \) obtained above may be non-separable (even if the spaces \( \mathcal{H}_\phi \) are separable); this is because the set \( S_1(A) \) is generally uncountable. To obtain a faithful representation of \( \mathcal{A} \) on a separable Hilbert space, we shall use again the separability of \( \mathcal{A} \) as a topological algebra. Let \( \mathcal{A}_0 = \{ A_1, A_2, A_3, \ldots \} \) be a countable dense subset of \( \mathcal{A} \) consisting of nonzero elements. The CC condition guarantees the existence of pure states \( \phi_j \) \( (j=1,2,\ldots) \) such that
\[
\phi_j(A_j^*A_j) \neq 0, \quad j = 1, 2, \ldots \tag{176}
\]
Now consider the GNS representations \( (\mathcal{H}_{\phi_j}, \mathcal{D}_{\phi_j}, \pi_{\phi_j}) \) \( (j=1,2,\ldots) \). Eq.(176) guarantees that
\[
\pi_{\phi_j}(A_j) \neq 0, \quad j = 1, 2, \ldots \tag{177}
\]
Indeed
\[0 \neq \phi_j(A_j^*A_j) = (\chi_{\phi_j}, \pi_{\phi_j}(A_j^*A_j)\chi_{\phi_j}) \]
\[= (\pi_{\phi_j}(A_j)\chi_{\phi_j}, \pi_{\phi_j}(A_j)\chi_{\phi_j}).\]

Now consider the direct sum \((H', D', \pi')\) of these representations. To show that \(\pi'\) is faithful, we must show that, for any nonzero element \(A\) of \(\mathcal{A}\), \(\pi'(A) \neq 0\). This is guaranteed by Eq.(177) because, \(\mathcal{A}_0\) being dense in \(\mathcal{A}\), \(A\) can be arranged to be as close as we like to some \(A_j\) in \(\mathcal{A}_0\).

The representation \(\pi'\), however, is, in general, reducible. To obtain a faithful irreducible representation, we should try to obtain the relations \(\pi(A_j) \neq 0\) for any \(j = 1, 2, \ldots\) in a single representation \(\pi\). To this end, let \(B^{(k)} = A_1A_2\ldots A_k\) and choose \(\phi^{(k)} \in \mathcal{S}_1(\mathcal{A})\) such that
\[\phi^{(k)}(B^{(k)*}B^{(k)}) \neq 0.\]

In the GNS representation \((H_{\phi^{(k)}}, D_{\phi^{(k)}}, \pi_{\phi^{(k)}})\), we have
\[0 \neq \pi_{\phi^{(k)}}(B^{(k)}) = \pi_{\phi^{(k)}}(A_1)\ldots \pi_{\phi^{(k)}}(A_k)\]
which implies
\[\pi_{\phi^{(k)}}(A_j) \neq 0, \quad j = 1, \ldots, k. \quad (178)\]

This argument works for arbitrarily large but finite \(k\). If the \(k \to \infty\) limit of the above construction leading to a limiting GNS representation \((\mathcal{H}, \mathcal{D}, \pi)\) exists, giving
\[\pi(A_j) \neq 0, \quad j = 1, 2, \ldots, \quad (179)\]
then, by an argument similar to that for \(\pi'\) above, one must have \(\pi(A) \neq 0\) for all non-zero \(A\) in \(\mathcal{A}\) showing faithfulness of \(\pi\).

Note. (i) For finitely generated system algebras (this covers all applications of QM in atomic physics), a limiting construction is not needed; the validity of Eq.(178) for sufficiently large \(k\) is adequate. [Hint : Take the generators of the algebra \(\mathcal{A}\) as some of the elements of \(\mathcal{A}_0\).]
(ii) For general algebras, it appears that some extra condition is needed to arrive at a faithful irreducible representation.
(iii) The developments in this subsection did not require the algebra \(\mathcal{A}\) to belong to the restricted class employed for SQSs in section 7.1; the results
obtained are, therefore, valid for more general quantum systems. We shall use this fact in section 7.5.

(iv) In fact, even non-commutativity of the algebra $\mathcal{A}$ was not used above. This, however, is not surprising; commutative algebras, under fairly general conditions, can be realized as algebras of operators in Hilbert spaces.

A more complete treatment of these matters is intended to be presented when the treatment of quantum field theory in an appropriately augmented supmech framework is taken up.

Having shown the existence and desirability of the Hilbert space-based realizations for finitely generated system algebras, we now have a formal justification for the direct route to the Hilbert space taken in the traditional treatment of QM of localizable elementary systems (massive particles), namely, employment of projective unitary irreducible representations of the relativity group $G_0$. This is the simplest way to satisfy the condition of transitive action of $G_0$ on the space of pure states and simultaneously satisfy the CC condition.

We take up the QM of these objects in the next subsection.

7.4. Quantum mechanics of localizable elementary systems (massive particles)

A quantum elementary system is an SQS which is also an elementary system. The concept of a quantum elementary system, therefore, combines the concept of quantum symplectic structure with that of a relativity scheme. The basic entities relating to an elementary system are its fundamental observables which generate the system algebra $\mathcal{A}$. For quantum elementary systems, this algebra $\mathcal{A}$ has the quantum symplectic structure as described in section 7.1. All the developments in section 5 can now proceed with the PBs understood as quantum PBs. We shall employ the Hilbert space-based realizations of these systems (which are guaranteed to exist because the algebra $\mathcal{A}$ is finitely generated).

The relativity group $G_0$ (or its projective group $\hat{G}_0$) has a Poisson action on $\mathcal{A}$ and a transitive action on the set $\mathcal{S}_1(\mathcal{A})$ of pure states of $\mathcal{A}$. We have seen in section 7.2 that, in a Hilbert space based realization of an SQS in terms of a quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$, a symmetry operation can be represented as a unitary operator on $\mathcal{H}$ mapping $\mathcal{D}$ onto itself. A symmetry group is then realized as a unitary representation on $\mathcal{H}$ such that the representative operators map $\mathcal{D}$ onto itself. For an elementary system the
condition of transitive action on $S_1$ implies that this representation must be irreducible. (There is no contradiction between this requirement and that of invariance of $D$ because $D$ is not a closed subspace of $H$ when $H$ is infinite dimensional.)

By a (quantum) particle we shall mean a localizable (quantum) elementary system. We shall first consider nonrelativistic particles. The configuration space of a nonrelativistic particle is the 3-dimensional Euclidean space $R^3$. The fundamental observables for such a system were identified, in section 5.2, as the mass ($m$) and Cartesian components of position ($X_j$), momentum ($P_j$) and spin($S_j$) ($j = 1,2,3$) satisfying the PB relations in equations (107,108,102). The mass $m$ will be treated, as before, as a positive parameter. The system algebra $\mathcal{A}$ of the particle is the *-algebra generated by the fundamental observables (taken as hermitian) and the unit element. Since it is an ordinary *-algebra (i.e. one not having any fermionic objects), the supercommutators reduce to ordinary commutators. The PBs mentioned above now take the form of the commutation relations

$$[X_j, X_k] = 0 = [P_j, P_k], \quad [X_j, P_k] = i\hbar \delta_{jk} I \quad (A)$$

$$[S_j, S_k] = i\hbar \varepsilon_{jkl} S_l, \quad [S_j, X_k] = 0 = [S_j, P_k]. \quad (B) \quad (180)$$

We shall first consider the spinless particles ($S = 0$). We, therefore, need to consider only the Heisenberg commutation relations (180A) [often referred to as the canonical commutation relations (CCR)]. Assuming the existence of a quantum triple $(H, D, \hat{A})$ corresponding to this SQS, we shall employ some results obtained in section 7.2 to obtain the explicit construction. Here $\hat{A}$ is the algebra generated by (representatives of) the fundamental observables $X_j, P_j$ ($j = 1,2,3$) and the unit element $I$ subject to the commutation relations (180A) and the pair $(H, D)$ carries a faithful irreducible representation of the system algebra as explained above. We introduce the bra and ket spaces as in section 7.2. Let $x = (x_1, x_2, x_3), dx = dx_1 dx_2 dx_3$ and $|x>, <x|$ the simultaneous eigenkets and eigenbras of the operators $X_j$ ($j = 1,2,3$):

$$X_j |x> = x_j |x>, \quad <x|X_j = <x|x_j, \quad x_j \in R, \quad j = 1,2,3; \quad (181)$$

they are assumed (with a promise of justification later) to form a complete set providing a resolution of identity in the form

$$I = \int_{R^3} |x> dx <x|. \quad (182)$$
Given any vector $|\psi \rangle \in \mathcal{D}$, the corresponding wave function $\psi(x) = \langle x | \psi \rangle$ must satisfy the relation

$$\langle X_j \psi \rangle(x) = \langle x | X_j | \psi \rangle = x_j \psi(x). \quad (183)$$

Recalling the discussion of localization in section 4.7, the localization observable $P(D)$ corresponding to a Borel set $D$ in $R^3$ is represented as the operator

$$P(D) = \int_D |x > dx < x|.$$ \hspace{1cm} (184)

[The required properties of $P(D)$ are easily verified.] Given the particle in the state corresponding to $|\psi \rangle \in \mathcal{D}$, the probability that it will be found in the domain $D$ is given by

$$\langle \psi | P(D) | \psi \rangle = \int_D \langle \psi | x > dx < x | \psi \rangle = \int_D |\psi(x)|^2 dx \quad (185)$$

giving the traditional Born interpretation of the wave function $\psi$. The integral above is meaningful for all Borel sets $D$ only if $\psi$ is square integrable over $R^3$ which implies $\mathcal{H} = L^2(R^3, dx)$.

To determine the operators $P_j$, we must choose the unitary operators $U(a)$ representing space translations such that the infinitesimal generators satisfy the last two equations in (180A). The simplest choice for $U(a)$, namely,

$$[U(a) \psi](x) = \psi(x - a)$$

[which is a special case of of the relation $[U(g) \psi](x) = \psi(T_g^{-1}x)$; these operators are unitary when the transformation $T_g$ of $R^3$ preserves the Lebesgue measure] happens to be adequate. Recalling Eq.(169), we have, for an infinitesimal translation,

$$\delta \psi = -\frac{i}{\hbar} a.P \psi = -a.\nabla \psi$$

giving the operators $P_j$ representing momentum components as

$$(P_j \psi)(x) = -i\hbar \frac{\partial \psi}{\partial x_j} \quad (186)$$

which satisfy the desired commutation relations.
We now identify the space $\mathcal{D}$ as (Dubin and Hennings 1990)

$$\mathcal{D} = C^\infty(X_j, P_j; j = 1, 2, 3) = \mathcal{S}(R^3).$$

The operators $U(a)$ clearly map this domain onto itself. With this choice of $\mathcal{D}$, the operators $X_j$ and $P_j$ given by equations (183) and (186) are essentially self adjoint. The two triples of operators $\{X_j\}$ and $\{P_j\}$ separately constitute complete sets of commuting operators. The completeness of the $X_j$s can be easily seen by operating in the $X$-representation and taking the harmonic oscillator ground state wave function as the cyclic vector. Similar argument works for the $P_j$s in the momentum representation. This confirms the legitimacy of the Dirac constructions employed above.

The pair $(\mathcal{H}, \mathcal{D}) = (L^2(R^3), \mathcal{S}(R^3))$ with operators $X_j$ and $P_j$ as constructed above is known as the Schrödinger representation of the CCR (180A).

Closures of the operators $P_j, X_j$ (which are self adjoint and are denoted by the same symbols) generate the unitary groups of operators $U(a) = \exp(-ia.P)$ and $V(b) = \exp(-ib.X)$ (where $a.P = \sum_j a_j P_j$ etc. and we have put $\hbar = 1$.) which satisfy the Weyl commutation relations

\[
\begin{align*}
U(a)U(b) &= U(b)U(a) = U(a + b), & V(a)V(b) &= V(b)V(a) = V(a + b) \\
U(a)V(b) &= e^{ia.b}V(b)U(a). & (187)
\end{align*}
\]

For all $\psi \in \mathcal{D}$, we have

\[
(U(a)\psi)(x) = \psi(x - a), \quad (V(b)\psi)(x) = e^{-ib.x}\psi(x); \quad (188)
\]

this is referred to as the Schrödinger representation of the Weyl commutation relations. According to the uniqueness theorem of von Neumann (Varadarajan 1985), the irreducible representation of the Weyl commutation relations is, up to unitary equivalence, uniquely given by the Schrödinger representation (188).

**Note.** (i) Not every representation of the CCR (180A) with essentially self adjoint $X_j$ and $P_j$ gives a representation of the Weyl commutation relation. [For a counterexample, see Inoue (1998), example (4.3.3).] A necessary and sufficient condition for the latter to materialize is that the harmonic oscillator Hamiltonian operator $H = P_j^2/(2m) + kX_j^2/2$ be essentially self adjoint. In the Schrödinger representation of the CCR obtained above, this condition is satisfied (Glimm and Jaffe 1981; Dubin and Hennings 1990).
The von Neumann uniqueness theorem serves to confirm/verify, in the present case, the uniqueness (up to equivalence) of the Hilbert space realization of an SQS mentioned in section 7.2.

Quantum dynamics of a free nonrelativistic spinless particle is governed, in the Schrödinger picture, by the Schrödinger equation (170) with $\psi \in \mathcal{D} = \mathcal{S}(\mathbb{R}^3)$ and with the Hamiltonian (110)[where $P$ is now the operator in Eq.(186)]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi.$$  \hspace{1cm} (189)

Explicit construction of the projective unitary representation of the Galilean group $G_0$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx)$ and Galilean covariance of the free particle Schrödinger equation (189) have been treated in the literature (Bargmann 1954; Varadarajan 1985; Dass and Sharma 1998).

When external forces are acting, the Hamiltonian operator has the more general form (111). Restricting $V$ in this equation to a function of $X$ only (as is the case in common applications), and proceeding as above, we obtain the traditional Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(X) \right] \psi.$$  \hspace{1cm} (190)

Note. In (Dass 2002), the free particle Hamiltonian operator of Eq.(127)(there) was arrived at by direct reasoning from the commutation relations of the projective Galilean group. (This has some instruction value.) Here we have used essentially similar reasoning to arrive at the free particle Hamiltonian of Eq.(110) in the more general context of supmech. In (Dass 2002), the full Hamiltonian of Eq.(128)(there) has $V(X)$ [instead of the $V(X,P)$ of Eq.(111) above]. This is due to a mistake in the last stage of the argument there (which occurred while using the relation $[H', X_k] = 0$ which need not be valid when interactions are present.)

It should be noted that, in the process of obtaining the Schrödinger equation (190) for a nonrelativistic spinless particle with the traditional Hamiltonian operator, we did not use the classical Hamiltonian or Lagrangian for the particle. No quantization algorithm has been employed; the development of the quantum mechanical formalism has been autonomous, as promised.

From this point on, the development of QM along the traditional lines can proceed.
For nonrelativistic particles with \( m > 0 \) and spin \( s \geq 0 \), we have \( \mathcal{H} = L^2(R^3, C^{2s+1}) \) and \( \mathcal{D} = \mathcal{S}(R^3, C^{2s+1}) \). The treatment of spin being standard, we skip the details.

Relativistic elementary quantum systems have been treated extensively in literature (Wigner 1939; Bargmann and Wigner 1948; Varadarajan 1985; Alonsa 1977, 1979). These treatments employ projective irreducible representations of the Poincaré group which can be obtained from the irreducible unitary representations of its covering group. The justification for this already has been given in the supmech formalism in the last para of section 7.3. We shall skip the details.

7.5. Quantum systems with more general system algebras; Superselection rules

Now we consider general quantum systems which, as already defined in section 6.2, are those with (not necessarily special) non-supercommutative system algebras equipped with quantum symplectic structure. Standard quantum systems are the subclass of these in which the system algebra \( \mathcal{A} \) has a trivial graded center and only inner superderivations. We shall now relax these two conditions. On a noncommutative (super-)algebra \( \mathcal{A} \) having a trivial graded center but having both inner and outer (super-)derivations, a quantum symplectic structure can be defined by employing the generalization of the supmech formalism treated in section 4.5 and operate with the generalized symplectic superalgebra \( (\mathcal{A}, \mathcal{X}, \omega_Q) \) where \( \mathcal{X} = \text{ISDer}(\mathcal{A}) \) (this gives, again, the quantum PBs of section 7.1). For convenience of reference, we shall call this class of systems quasi-standard quantum systems.

We next consider the generalization involving a nontrivial graded center. We shall restrict ourselves to considering a nontrivial center only. The center \( \mathcal{C} \equiv Z_0(\mathcal{A}) \) of \( \mathcal{A} \) is a commutative locally convex algebra. Keeping in view that a faithful (not necessarily irreducible) Hilbert space-based realization of the superalgebra \( \mathcal{A} \) is always possible, we shall operate, in this subsection, in the framework of a quantum triple \( (\mathcal{H}, \mathcal{D}, \hat{\mathcal{A}}) \) where \( \hat{\mathcal{A}} \) is a faithful realization of \( \mathcal{A} \).

A nontrivial center \( \mathcal{C} \) implies the presence of superselection rules and/or external fields. The two are, in fact, related: values of the external fields define superselection rules. Before taking up the general case, we consider a couple of illustrative situations:

(i) Consider first the situation when \( \mathcal{C} \) is generated by a finite number of
self-adjoint operators $Q_s$ (s=1,...,n) each of which has a discrete spectrum (we shall call such observables charge type observables). In this case, we have $\mathcal{H} = \oplus_i \mathcal{H}_i$ where the spaces $\mathcal{H}_i$ are simultaneous eigenspaces of the observables $Q_s$. Defining $\mathcal{D}_i = \mathcal{D} \mid \mathcal{H}_i$ and $\mathcal{A}_i = \mathcal{A} \mid \mathcal{H}_i$, the algebras $\mathcal{A}_i$ have trivial center and the quantum triples $(\mathcal{H}_i, \mathcal{D}_i, \mathcal{A}_i)$ correspond to quasi-standard quantum systems; the operators $Q_s$ act as superselection operators and the spaces $\mathcal{H}_i$ as coherent subspaces labeled by the set of eigenvalues of the superselection operators.

(ii) Functions representing (components of) external fields belong to the commutative algebra of functions on a connected domain of the space-time manifold; in the relevant situations, this algebra obviously belongs to the center of the system algebra.

Regarding the general situation (of which, the treatment below is, admittedly, somewhat heuristic) we note that generalizations of the famous Gel’fand-Naimark theorem on commutative $C^*$-algebras (Bratteli and Robinson 1979) to some classes of commutative locally convex *-algebras have appeared in literature (Iguri and Castagnino 1999; Álvarez 2004). These generalizations relate (through isomorphisms or, more generally, homomorphisms) the latter algebras to those of continuous functions on reasonably ‘nice’ classes of topological spaces [typically Tychonoff spaces (Álvarez 2004)]; these topological spaces will be referred to as the spectral spaces of the respective algebras. In the above-mentioned Hilbert space-based realization, the center $\mathcal{C} \equiv Z_0(\hat{\mathcal{A}})$ will be represented as a commutative algebra of operators with $\mathcal{D}$ as a common invariant domain. Generalization of the spectral theorem to commutative algebras of operators is expected to lead to a representation of $\mathcal{H}$ as a direct integral

$$\mathcal{H} = \int_{\Sigma} \mathcal{H}(\lambda) d\sigma(\lambda)$$

(191)

where $\Sigma$ is the spectral space of $\mathcal{C}$ and $\sigma$ is a measure uniquely determined by the algebra $\mathcal{C}$. Defining $\mathcal{D}(\lambda)$ and $\hat{\mathcal{A}}(\lambda)$ as the restrictions of $\mathcal{D}$ and $\hat{\mathcal{A}}$ to $\mathcal{H}(\lambda)$, we have the quantum triples $(\mathcal{H}(\lambda), \mathcal{D}(\lambda), \hat{\mathcal{A}}(\lambda))$ representing quasi-standard quantum systems. The Hilbert spaces $\mathcal{H}(\lambda)$ are traditionally referred to as coherent subspaces.

Generally the space $\Sigma$ will be disconnected; the integral (191) will then reduce to a sum of integrals of the same type over the connected pieces of $\Sigma$. The examples (i) and (ii) above correspond to the two extreme situations when the space $\Sigma$ is, respectively, discrete and connected.

83
An important class of examples corresponds to the situation when there are one or more mutually commuting operators with continuous spectra defining superselection rules. The mass operator of a Galilean particle is a good example. Had we not invoked, in section 5.2, the CC condition to have $M = mI$ and had treated the mass observable $M$ simply as an element of the center $\mathcal{C}$ (thereby making the latter non-trivial), we could not have treated the system in question as a standard quantum system. Instead, we would have the situation in Eq.(191) with $\lambda$ replaced by the mass parameter $m$ and and $\Sigma$ an interval of the form $[a, \infty)$ with $a > 0$. Galilean relativity provides no clue to the value of $a$; it may tentatively be taken to be the mass of the least massive among the positive mass particles in nature.

A more systematic treatment of the general case is expected to be presented in a future work relating to a supmech-based treatment of quantum field theory.

Before closing this subsection, it is worth emphasizing that

(i) in the present formalism, there is a natural place for superselection rules;
(ii) the superselection rules arising as described above are commutative — a highly desirable feature.

For a somewhat complementary treatment of matters related to this subsection, we refer to the insightful paper of Jauch and Misra (1961).

7.6. Classical Systems

Continuing the treatment, in section 3.8, of classical symplectic structures as special cases of the symplectic structures of section 3.6, we note here that a classical Hamiltonian system $(M, \omega_{cl}, H_{cl})$ is realized in supmech as the Hamiltonian system $(A_{cl}, \omega_{cl}, H_{cl})$ where $A_{cl} = C^\infty(M)$ and $H_{cl}$ a smooth real-valued function which is bounded below. The supmech Hamilton equation (64) is, in the present context, nothing but the traditional Hamilton equation:

$$\frac{df}{dt} = \{H_{cl}, f\}_{cl}. \tag{192}$$

States in the present context are probability measures on $M$; in obvious notation, they are of the form

$$\phi_\mu(f) = \int_M f d\mu. \tag{193}$$
Pure states are Dirac measures (or, equivalently, points of $M$) $\mu_{\xi_0}(\xi_0 \in M)$ for which $\phi_{\xi_0}(f) = f(\xi_0)$. The pair $(\mathcal{O}(A_{cl}), S_1(A_{cl}))$ of classical observables and pure states is easily seen to satisfy the CC condition: Given two different real-valued smooth functions on $M$, there is a point of $M$ at which they take different values; conversely, given two different points of $M$, there is a real-valued smooth function on $M$ which takes different values at those points.

In ordinary mechanics, only pure states are used. Expectation values of all observables in these states are their precise values at the relevant points and the theory is deterministic (obtained here as a special case of a probabilistic theory). More general states are employed in classical statistical mechanics where, in most applications, they are taken to be represented by densities on $M$ $\rho(\xi) d\xi$ where $d\xi = dq dp$ is the Liouville volume element on $M$.

The state evolution equation (65) of supmech gives, in the present context

$$\int_M \left( \frac{\partial \rho(\xi,t)}{\partial t} \right)(\xi) f(\xi) d\xi = \int_M \rho(\xi,t)\{H, f\}_{cl}(\xi) d\xi.$$  (194)

For the remainder of this subsection, we take $M = R^{2n}$. To satisfy the normalization condition, the density $\rho$ must vanish at infinity. Now, using the Poisson bracket of section 3.8 in Eq.(194), performing a partial integration and discarding the (vanishing) surface term, the right hand side of Eq.(194) becomes

$$- \int_M \omega_{ab}^{cl}(\xi) \frac{\partial H}{\partial \xi^a} \frac{\partial \rho}{\partial \xi^b} f d\xi = \int_M \{\rho, H\}_{cl} f d\xi.$$  

Since $f$ is arbitrary, Eq.(194) now gives the traditional Liouville equation

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}_{cl}.$$  (195)

A classical Galilean elementary system is a system characterized/labelled by the three Galilean invariants $m, \sigma, u$. Its fundamental observables other than the invariants are the position, momentum and spin vectors $X, P, S$ satisfying the PB relations of section 5.2 (where the symbols now represent phase space variables). The observable $X$ has the interpretation of the position vector of the center of mass of the system. A particle is an elementary system with the internal energy $u = 0$ and negligible size so that $X$ now refers to the particle position. The free particle Hamiltonian for a spinless particle is given by Eq.(110) and the one with interaction in Eq.(111). For a detailed
treatment of classical Galilean systems we refer to the literature (SM; Alonso
1979).

7.7. Superclassical systems

Superclassical mechanics is an extension of classical mechanics which employs, besides the traditional phase space variables, Grassmann variables \( \theta^\alpha (\alpha = 1,..,n, \text{say}) \) satisfying the relations

\[
\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0 \quad \text{for all } \alpha, \beta;
\]

[in particular \((\theta^\alpha)^2 = 0 \) for all \( \alpha \)]. These objects generate the so-called Grassmann algebra (with \( n \) generators) \( G_n \) whose elements are functions of the form

\[
f(\theta) = a_0 + a_\alpha \theta^\alpha + a_{\alpha\beta} \theta^\alpha \theta^\beta + ...
\]

where the coefficients \( a_\alpha \) are complex numbers. If the coefficients in Eq. (196) are taken to be smooth functions on, say, \( R^m \), the resulting functions \( f(x,\theta) \) are referred to as smooth functions on the superspace \( R^{m|n} \); the algebra of these functions is denoted as \( C^\infty(R^{m|n}) \). With parity zero assigned to the variables \( x^a \) (\( a = 1,..,m \)) and one to the \( \theta^\alpha \), \( C^\infty(R^{m|n}) \) is a supercommutative superalgebra. Restricting the variables \( x^a \) to an open subset \( U \) of \( R^m \), one obtains the superdomain \( U^{m|n} \) and the superalgebra \( C^\infty(U^{m|n}) \) in the above-mentioned sense. Gluing such superdomains appropriately, one obtains the objects called supermanifolds (Leites 1980; Berezin 1987; Vorontov 1992; DeWitt 1984). These are the objects serving as phase spaces in superclassical mechanics. We shall, for simplicity, restrict ourselves to the simplest supermanifolds \( R^{m|n} \) and take, for the development of supermechanics in the present context, \( \mathcal{A} = C^\infty(R^{m|n}) \). The ‘coordinate variables’ \( x^a, \theta^\alpha \) will be jointly referred to as \( \xi^A \). We shall write \( \epsilon(\xi^A) = \epsilon_A \). A \(*\)-operation is assumed to be defined on \( \mathcal{A} \) for which \((\xi^A)^* = \xi^A \).

Left and right differentiations with respect to the odd variables are defined as follows:

\[
\frac{\partial l}{\partial \theta^\alpha} (\theta^{\alpha_1}..\theta^{\alpha_s}) = \delta^{\alpha_1}_{\alpha} \theta^{\alpha_2}..\theta^{\alpha_s} - \delta^{\alpha_2}_{\alpha} \theta^{\alpha_1} \theta^{\alpha_3}..\theta^{\alpha_s} + ..
\]

\[
(-1)^{s-1} \delta^{\alpha_s}_{\alpha} \theta^{\alpha_1}..\theta^{\alpha_s-1}
\]

(197)

\[
\frac{\partial r}{\partial \theta^\alpha} (\theta^{\alpha_1}..\theta^{\alpha_s}) = \delta^{\alpha_s}_{\alpha} \theta^{\alpha_1}..\theta^{\alpha_{s-1}} - \delta^{\alpha_{s-1}}_{\alpha} \theta^{\alpha_1} \theta^{\alpha_{s-2}}..\theta^{\alpha_s} + ..
\]

\[
(-1)^{s-1} \delta^{\alpha_1}_{\alpha} \theta^{\alpha_2}..\theta^{\alpha_s}
\]

(198)
and extended by linearity to general elements of $\mathcal{A}$. Taking

$$\frac{\partial f}{\partial x^a} = \frac{\partial_r f}{\partial x^a} = \frac{\partial f}{\partial x^a},$$

we now have left and right derivatives with respect to $\xi^A$ defined on $\mathcal{A}$. Defining

$$Ae = \frac{\partial_l}{\partial \xi^A}, \quad e_A = \frac{\partial_r}{\partial \xi^A},$$

we have, for any two homogeneous elements $f, g$ of $\mathcal{A}$,

$$Ae(fg) = (Ae f)g + (-1)^{\epsilon f \epsilon A} f(Ae g);$$
$$e_A(fg) = f(e_A g) + (-1)^{\epsilon_A \epsilon A}(e_A f)g.$$  \hspace{1cm} (199)

The objects $Ae$ (but not $e_A$) are superderivations of the superalgebra $\mathcal{A}$. A general element of $SDer(\mathcal{A})$ (called a supervectorfield) is of the form

$$X = X^A(\xi) \cdot Ae = e_A X(\xi).$$  \hspace{1cm} (200)

The differential of a function $f \in \mathcal{A}$ can be written as

$$df = d\xi^A \frac{\partial f}{\partial \xi^A} = \frac{\partial_r f}{\partial \xi^A} d\xi^A$$  \hspace{1cm} (201)

where $d\xi^A$ are symbols serving as basis vectors in the space of 1-forms. We have

$$\frac{\partial_r f}{\partial \xi^A} = (-1)^{\epsilon_A(\epsilon_f + \epsilon_A)} \frac{\partial_l f}{\partial \xi^A}. $$  \hspace{1cm} (202)

A general 1-form $\omega^{(1)}$ and a 2-form $\omega^{(2)}$ can be written as

$$\omega^{(1)} = \omega^{(1)}_A d\xi^A = d\xi^A A \omega^{(1)} $$  \hspace{1cm} (203)

$$\omega^{(2)} = \omega^{(2)}_{AB} d\xi^B d\xi^A = d\xi^B d\xi^A AB \omega = d\xi^A A AB d\xi^B.$$  \hspace{1cm} (204)

Note that, when $A, B$ are odd, $\omega^{(2)}_{AB} = \omega^{(2)}_{BA}$. It follows that, the odd dimension $n$ (in $\mathbb{R}^{m|n}$) need not be an even number for a symplectic form to exist; the number $m$ must, of course, be even.
Given a symplectic form $\omega$ on $\mathcal{A}$, we have, for any $f \in \mathcal{A}$,

$$\omega(X_f, Y) = -(df)(Y)$$  \hfill (205)

where $X_f$ is the Hamiltonian supervector field corresponding to $f$ and $Y \in SDer(\mathcal{A})$. Since $\omega$ is even, Equations (204,205) give (writing $f,A$ for $\frac{\partial f}{\partial \xi^A}$)

$$X_f^A A \omega_B = -f_B.$$  \hfill (206)

On $\mathbb{R}^{m|n}$, the symplectic form can be chosen so that the coefficients $A \omega_B$ are independent of $\xi$. Assuming this and introducing the inverse $(A \omega_B)$ of the matrix $(A \omega_B)$, we have

$$X_f^A = -f_B B \omega^A.$$  \hfill (207)

The Poisson bracket of $f,g \in \mathcal{A}$ is

$$\{f,g\} = X_f(g) = X_f^A \frac{\partial g}{\partial \xi^A} = -\frac{\partial f}{\partial \xi^B} B \omega^A \frac{\partial g}{\partial \xi^A}.$$  \hfill (208)

The dynamics is governed by the supermech Hamilton equation (64) with $H$ an even Hermitian element of $\mathcal{A}$ and the Poisson bracket of Eq.(208).

States in superclassical mechanics are normalized positive linear functionals on $\mathcal{A} = C^\infty(\mathbb{R}^{m|n})$; they are generalizations of the states (193) given by

$$\phi(f) = \int_{\mathbb{R}^{m|n}} f(x,\theta)d\mu(x,\theta)$$  \hfill (209)

where the measure $\mu$ satisfies the normalization and positivity conditions

$$1 = \phi(1) = \int d\mu(x,\theta)$$  \hfill (210)

$$0 \leq \int f f^*d\mu \quad \text{for all} \ f \in \mathcal{A}.$$  \hfill (211)

In the rest of this subsection, we shall consider only states represented by a density function:

$$d\mu(x,\theta) = \rho(x,\theta)d\theta^1...d\theta^n d^m x.$$  \hfill (212)
To ensure real expectation values for observables, \( \rho(.,.) \) must be even(odd) for \( n \) even(odd). The condition (210) implies that

\[
\rho(x, \theta) = \rho_0(x) \theta^n \ldots \theta^1 + \text{terms of lower order in } \theta
\]  

(213)

where \( \rho_0 \) is a probability density on \( \mathbb{R}^m \).

The inequality (211) implies inequalities involving the coefficient functions on the right in Eq.(213). They eventually determine a convex domain \( D \) in a real vector space. Pure states correspond to points on the boundary of \( D \) (which is generally not a manifold).

The CC condition is, unfortunately, not generally satisfied by the pair \((\mathcal{O}(A), \mathcal{S}_1(A))\) in super-classical mechanics. To show this, it is adequate to give an example (Berezin 1987). Taking \( A = C^\infty(\mathbb{R}^3) \equiv G_3 \), we have

\[
\rho(\theta) = \theta^3 \theta^2 \theta^1 + c_\alpha \theta^\alpha.
\]  

(214)

The inequality (211) with \( f = a \theta^1 + b \theta^2 \) (with \( a \) and \( b \) arbitrary complex numbers) implies \( c_3 = 0 \); similarly, \( c_1 = c_2 = 0 \), giving, finally

\[
\rho(\theta) = \theta^3 \theta^2 \theta^1.
\]  

(215)

There is only one possible state which must be pure. This state does not distinguish, for example, observables \( f = a \theta^1 + b \theta^2 \) with the same ‘a’ but different ‘b’, thus verifying the assertion made above.

The fermionic extension of classical mechanics, therefore, appears to have a fundamental inadequacy; no wonder, therefore, that it is not realized by systems in nature.

The argument presented above, however, does not apply to the \( n = \infty \) case.

7.8. Quantum-Classical Correspondence

It will now be shown that supmech permits a transparent treatment of quantum-classical correspondence. In contrast to the general practice in this domain, we shall be careful about the domains of operators and avoid some usual pitfalls in the treatment of the \( \hbar \to 0 \) limit.

Our strategy will be to start with a quantum Hamiltonian system, transform it to an isomorphic supmech Hamiltonian system involving phase space functions and \( \star \)-products [Weyl-Wigner-Moyal formalism (Weyl 1949; Wigner 1932; Moyal 1949)] and show that, in this latter Hamiltonian system, the subclass of phase space functions in the system algebra which go over to smooth
functions in the $\hbar \to 0$ limit yield the corresponding classical Hamiltonian system. For simplicity, we restrict ourselves to the case of a spinless nonrelativistic particle though the results obtained admit trivial generalization to systems with phase space $R^{2n}$.

In the existing literature, the works on quantum-classical correspondence closest to the present treatment are those of Liu (1975,1976), Gracia-Bondía and Várilly (1988) and Hörmander (1979); some results from these works, especially Liu (1975,1976), are used below. The reference (Bellissard and Vitot 1990) is a comprehensive work reporting on some detailed features of quantum-classical correspondence employing some techniques of noncommutative geometry; its theme, however, is very different from ours.

In the case at hand, we have the quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H} = L^2(R^3)$, $\mathcal{D} = S(R^3)$ and $\mathcal{A}$ is the algebra of the spinless Galilean particle treated in section 7.4 as a standard quantum system. As in Eq. (190), we shall take the potential function $V$ to be a function of $X$ only. For $A \in \mathcal{A}$ and $\phi, \psi$ normalized elements in $\mathcal{D}$, we have the well defined quantity

$$ (\phi, A\psi) = \int \int \phi^*(y) K_A(y, y') \psi(y') dy dy' $$

(216)

where the kernel $K_A$ is a (tempered) distribution. Recalling the definition of Wigner function (Wigner 1932; Wong 1998) corresponding to the wave function $\psi$:

$$ W_\psi(x, p) = \int_{R^3} \exp[-ip.y/\hbar] \psi(x + \frac{y}{2}) \psi^*(x - \frac{y}{2}) dy $$

(217)

and defining the quantity $A_W(x, p)$ by

$$ A_W(x, p) = \int \exp[-ip.y/\hbar] K_A(x + \frac{y}{2}, x - \frac{y}{2}) dy $$

(218)

(note that $W_\psi$ is nothing but the quantity $P_W$ where P is the projection operator $|\psi><\psi|$ corresponding to $\psi$) we have

$$ (\psi, A\psi) = \int \int A_W(x, p) W_\psi(x, p) dx dp. $$

(219)

Whereas the kernels $K_A$ are distributions, the objects $A_W$ are well defined functions. For example,

$$ A = I : \quad K_A(y, y') = \delta(y - y') \quad A_W(x, p) = 1 $$
\[ A = X_j : \quad K_A(y, y') = y_j \delta(y - y') \quad A_W(x, p) = x_j \]
\[ A = P_j : \quad K_A(y, y') = -i\hbar \frac{\partial}{\partial y_j} \delta(y - y') \quad A_W(x, p) = p_j. \]

The Wigner functions \( W_\psi \) are generally well-behaved functions. We shall use Eq. (219) to characterize the class of functions \( A_W \) and call them Wigner-Schwartz integrable (WSI) functions [i.e. functions integrable with respect to the Wigner functions corresponding to the Schwartz functions in the sense of Eq. (217)]. For the relation of this class to an appropriate class of symbols in the theory of pseudodifferential operators, we refer to Wong (1998) and references therein.

The operator \( A \) can be reconstructed (as an element of \( \mathcal{A} \)) from the function \( A_W \):

\[
(\phi, A\psi) = (2\pi\hbar)^{-3} \int \int \int \exp\left[ip.(x - y)/\hbar\right] \phi^*(x) A_W(\frac{x + y}{2}, p) \psi(y) d\eta dp d\xi. \tag{220}
\]

Replacing, on the right hand side of Eq. (217), the quantity \( \psi(x + \frac{y}{2})\psi^*(x - \frac{y}{2}) \) by \( K_\rho(x + \frac{y}{2}, x - \frac{y}{2}) \) where \( K_\rho(., .) \) is the kernel of the density operator \( \rho \), we obtain the Wigner function \( \rho_W(x, p) \) corresponding to \( \rho \). Eq. (219) then goes over to the more general equation

\[
\text{Tr}(A\rho) = \int \int A_W(x, p) \rho_W(x, p) dx dp. \tag{221}
\]

The Wigner function \( \rho_W \) is real but generally not non-negative.

Introducing, in \( R^6 \), the notations \( \xi = (x, p) \), \( d\xi = dp dx \) and \( \sigma(\xi, \xi') = p.x' - x.p' \) (the symplectic form in \( R^6 \)), we have, for \( A, B \in \mathcal{A} \)

\[
(AB)_W(\xi) = (2\pi)^{-6} \int \int \exp[-i\sigma(\xi - \eta, \tau)] A_W(\frac{\eta + \frac{\hbar \tau}{4}}{2}) B_W(\eta - \frac{\hbar \tau}{4}) d\eta d\tau \equiv (A_W \ast B_W)(\xi). \tag{222}
\]

The product \( \ast \) of Eq. (222) is the twisted product of Liu (1975, 1976) and the \( \ast \)-product of Bayen et al (1978). The associativity condition \( A(BC) = (AB)C \) implies the corresponding condition \( A_W \ast (B_W \ast C_W) = (A_W \ast B_W) \ast C_W \).
in the space \( \mathcal{A}_W \) of WSI functions which is a complex associative non-commutative, unital \(*\)-algebra (with the star-product as product and complex conjugation as involution). There is an isomorphism between the two star-algebras \( \mathcal{A} \) and \( \mathcal{A}_W \) as can be verified from equations (220) and (218).

Recalling that, in the quantum Hamiltonian system \( (\mathcal{A}, \omega_Q, H) \) the form \( \omega_Q \) is fixed by the algebraic structure of \( \mathcal{A} \) and noting that, for the Hamiltonian \( H \) of Eq.(111) [with \( V = V(X) \)],

\[
H_W(x,p) = \frac{p^2}{2m} + V(x),
\]

we have an isomorphism between the supmech Hamiltonian systems \( (\mathcal{A}, \omega_Q, H) \) and \( (\mathcal{A}_W, \omega_W, H_W) \) where \( \omega_W = -i\hbar \omega_c \). Under this isomorphism, the quantum mechanical PB (160) is mapped to the Moyal bracket

\[
\{A_W, B_W\}_M \equiv (-i\hbar)^{-1}(A_W \star B_W - B_W \star A_W).
\]

For functions \( f, g \) in \( \mathcal{A}_W \) which are smooth and such that \( f(\xi) \) and \( g(\xi) \) have no \( \hbar \)-dependence, we have, from Eq.(222),

\[
f \star g = fg - (i\hbar/2)\{f, g\}_c + O(\hbar^2).
\]

The functions \( A_W(\xi) \) will have, in general, some \( \hbar \) dependence and the \( \hbar \rightarrow 0 \) limit may be singular for some of them (Berry 1991). We denote by \( (\mathcal{A}_W)_{reg} \) the subclass of functions in \( \mathcal{A}_W \) whose \( \hbar \rightarrow 0 \) limits exist and are smooth (i.e. \( C^\infty \) ) functions; moreover, we demand that the Moyal bracket of every pair of functions in this subclass also have smooth limits. This class is easily seen to be a subalgebra of \( \mathcal{A}_W \) closed under Moyal brackets. Now, given two functions \( A_W \) and \( B_W \) in this class, if \( A_W \rightarrow A_cl \) and \( B_W \rightarrow B_cl \) as \( \hbar \rightarrow 0 \) then \( A_W \star B_W \rightarrow A_cl B_cl \); the subalgebra \( (\mathcal{A}_W)_{reg} \), therefore, goes over, in the \( \hbar \rightarrow 0 \) limit, to a subalgebra \( \mathcal{A}_{cl} \) of the commutative algebra \( C^\infty(R^6) \) (with pointwise product as multiplication). The Moyal bracket of Eq.(224) goes over to the classical PB \( \{A_{cl}, B_{cl}\}_c \); the subalgebra \( \mathcal{A}_{cl} \) therefore, is closed under the classical Poisson brackets. The classical PB \( \{,\}_c \) determines the nondegenerate classical symplectic form \( \omega_{cl} \). When \( H_W \in (\mathcal{A}_W)_{reg} \) [which is the case for the \( H_W \) of Eq.(223)], the subsystem \( (\mathcal{A}_W, \omega_W, H_W)_{reg} \) goes over to the supmech Hamiltonian system \( (\mathcal{A}_{cl}, \omega_{cl}, H_{cl}) \).

When the \( \hbar \rightarrow 0 \) limits of \( A_W \) and \( \rho_W \) on the right hand side of Eq.(221) exist (call them \( A_{cl} \) and \( \rho_{cl} \)), we have

\[
Tr(A\rho) \rightarrow \int \int A_{cl}(x,p)\rho_{cl}(x,p)dxdp.
\]
The quantity $\rho_{\text{cl}}$ must be non-negative (and, therefore, a genuine density function). To see this, note that, for any operator $A \in \mathcal{A}$ such that $A_{W} \in (\mathcal{A}_{W})_{\text{reg}}$, the object $A^*A$ goes over to $A_{W} \ast A_{W}$ in the Weyl-Wigner-Moyal formalism which, in turn, goes to $A_{cl}A_{cl}$ in the $\hbar \rightarrow 0$ limit; this limit, therefore, maps non-negative operators to non-negative functions. Now if, in Eq.(226), $A$ is a non-negative operator, the left hand side is non-negative for an arbitrarily small value of $\hbar$ and, therefore, the limiting value on the right hand side must also be non-negative. This will prove the non-negativity of $\rho_{cl}$ if the objects $A_{cl}$ in Eq.(226) realizable as classical limits constitute a dense set of non-negative functions in $C^{\infty}(M)$. This class is easily seen to include non-negative polynomials; good enough.

In situations where the $\hbar \rightarrow 0$ limit of the time derivative equals the time derivative of the classical limit [i.e. we have $A(t) \rightarrow A_{cl}(t)$ and $\frac{dA(t)}{dt} \rightarrow \frac{dA_{cl}(t)}{dt}$], the Heisenberg equation of motion for $A(t)$ goes over to the classical Hamilton’s equation for $A_{cl}(t)$. With a similar proviso, one obtains the classical Liouville equation for $\rho_{cl}$ as the classical limit of the von Neumann equation.

Before closing this section, we briefly discuss an interesting point:

For commutative algebras, the inner derivations vanish and one can have only outer derivations. Classical mechanics employs a subclass of such algebras (those of smooth functions on manifolds). It is an interesting contrast to note that, while the standard quantum systems have system algebras with only inner derivations, classical system algebras have only outer derivations. The deeper significance of this is related to the fact that the noncommutativity of standard quantum algebras is tied to the nonvanishing of the Planck constant $\hbar$. [This is seen most transparently in the star product of Eq.(222) above.] In the limit $\hbar \rightarrow 0$, the algebra becomes commutative (the star product of functions reduces to ordinary product) and the inner derivations become outer derivations (commutators go over to classical Poisson brackets implying that an inner derivation $D_A$ goes over to the Hamiltonian vector field $X_{A_{cl}}$).

8. MEASUREMENTS IN QUANTUM MECHANICS

In this section we shall employ the formalism of section 6.2 to the treatment of measurements in QM taking both the measured system and the apparatus to be quantum systems. We shall, however, not adopt the von Neumann procedure (von Neumann 1955; Wheeler and Zurek 1983) of in-
Introducing vector states for the pointer positions; instead, we shall assign density operators to the pointer states and exploit the fact that the apparatus admits a classical description to a very good approximation. We shall do this by using the phase space description of the QM of the apparatus (the Weyl-Wigner-Moyal formalism) and then go to the classical approximation (exploiting the fact that supmech accommodates both classical and quantum mechanics as special subdisciplines). The undesirable macroscopic superpositions (of system + apparatus pure states) are shown to be suppressed when observations on the apparatus are restricted to macroscopically distinguishable pointer readings.

We shall start by putting the measurement problem in proper perspective.

8.1. The measurement problem in quantum mechanics

A measurement is an activity in which a system [about which some information is desired — to be called the ‘measured system’ (denoted here as S); it may be microscopic or macroscopic], prepared in a specified state, is made to interact with a (generally, but not necessarily, macroscopic) system called the ‘apparatus’ (denoted here as A) so as to eventually produce a phenomenon accessible through sensory perception (typically a pointer reading) or a permanent record (which may be noted at convenience). The pointer reading or record (the ‘measurement outcome’) is a numerical value which is interpreted (by employing an underlying theory and some common sense logic formalizable in terms of classical physics) as the value of some physical quantity (an observable). Thus one can talk about measurement of an observable of a system prepared in a given state.

In quantum theoretic treatments, a value of an observable (a self-adjoint operator) is understood to be a real number in its spectrum (Omnes 1994, p.115). Supmech events of the type $\nu(E)$ of section 4.1 can be associated with domains in the spectrum of an observable $A$ by employing the resolution of identity (172) corresponding to $A$:

$$\nu(\Delta) = \int_{\Delta} d\mu(\lambda) |\lambda><\lambda|$$

(227)

where $\Delta$ is a measurable subset of $\Omega = \sigma(A)$. These $\nu(\Delta)$s should, more appropriately, be called quantum events. We have already used objects of the form (227) [see Eq.(184)]. Given a state $\phi$ in which the system S is
prepared, we have, recalling Eq.(56),

\[ p_{\phi}(\Delta) = \phi(\nu(\Delta)) \] (228)

as the probability that, on measurement of the observable A, its value will be found in the domain \(\Delta\). These probabilities are the predictions of the underlying theory. Verification of the theory consists in comparing these probabilities with the appropriate relative frequencies in repeated measurements with the system prepared in the same state.

We consider, for simplicity, the measurement of an observable (of a quantum system S) represented by a self-adjoint operator \(F\) (acting in an appropriate domain in the Hilbert space \(\mathcal{H}_S\) of S) having a nondegenerate spectrum with the eigenvalue equations \(F|\psi_j\rangle = \lambda_j|\psi_j\rangle\) \((j = 1,2,...)\). The apparatus A is chosen such that, to each of the eigenvalues \(\lambda_j\) corresponds a pointer position \(M_j\). If the system is initially in an eigenstate \(|\psi_j\rangle\), the apparatus is supposedly designed to give, after the measurement interaction, the pointer reading \(M_j\); the outcome of the measurement is then understood as \(\lambda_j\). A question immediately arises: ‘What is the measurement outcome when the initial state of the system S is a superposition state \(|\psi\rangle = \sum_j c_j|\psi_j\rangle\)?’ To find the answer, we must consider the dynamics of the coupled system \((S + A)\) with an appropriate measurement interaction.

The standard treatment of measurements in QM (von Neumann 1955; Wheeler and Zurek 1983; Jauch 1968; Omnes 1994; Dass 2005) is due to von Neumann who emphasized that quantum mechanics being, supposedly, a universally applicable theory, every system is basically quantum mechanical; to have a consistent theory of measurement, we must, therefore, treat the apparatus A also quantum mechanically. Accordingly, one introduces a Hilbert space \(\mathcal{H}_A\) for the apparatus A; the pointer positions \(M_j\) are assumed to be represented by the state vectors \(|\mu_j\rangle\) in this space. The Hilbert space for the coupled system \((S + A)\) is taken to be \(\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A\).

The measurement interaction is elegantly described (Omnes 1994; Dass 2005) by a unitary operator \(U\) on \(\mathcal{H}\) which, acting on the initial state of \((S + A)\) (with the system S in the initial state in which it is prepared for the experiment and the apparatus in the ‘ready’ state which we denote as \(|\mu_0\rangle\)) gives an appropriate final state. We shall assume the measurement to be ideal which is supposedly such that (Omnes 1999) ‘when the measured system is initially in an eigenstate of the measured observable, the measurement leaves it in the same state.’ In this case, the measurement outcome must be
the corresponding eigenvalue which must be indicated by the final pointer position. This implies

\[ U(\psi_j \otimes \mu_0) = \psi_j \otimes \mu_j. \]  \hspace{1cm} (229)

For \( S \) in the initial state \( |\psi\rangle = \sum c_j \psi_j > \), the final \((S + A)\)-state must be, by linearity of \( U \),

\[ |\psi_f\rangle \equiv U[(\sum_j c_j \psi_j >) \otimes |\mu_0\rangle] = \sum_j c_j [\psi_j > \otimes |\mu_j\rangle]. \]  \hspace{1cm} (230)

Note that the right hand side of Eq.(230) is a superposition of the quantum states of the (generally macroscopic) system \((S + A)\).

Experimentally, however, one does not observe such superpositions. Instead, one obtains, in each measurement, a definite outcome \( \lambda_j \) corresponding to the final \((S + A)\)-state \( |\psi_j \otimes |\mu_j\rangle > \). Repetitions of the experiment, with system in the same initial state, yield various outcomes randomly such that, when the number of repetitions becomes large, the relative frequencies of various outcomes tend to have fixed values. To account for this, von Neumann postulated that, after the operation of the measurement interaction as above, a discontinuous, noncausal and instantaneous process takes place which changes the state \( |\psi_f\rangle \) to the state represented by the density operator

\[ \rho_f = \sum_i \tilde{P}_i |\psi_f\rangle <\langle \psi_f| \tilde{P}_i = \sum_j |c_j|^2 (|\psi_j\rangle <\langle \psi_j| \otimes |\mu_j\rangle <\langle \mu_j|; \]  \hspace{1cm} (232)

here \( \tilde{P}_i = |\psi_i\rangle <\langle \psi_i| \otimes I_A \) where \( I_A \) is the identity operator on \( \mathcal{H}_A \). This is referred to as von Neumann’s projection postulate and the phenomenon with the above process as the underlying process the state vector reduction or wave function collapse. Eq.(232) represents, in the von Neumann scheme, the \((S + A)\)-state on the completion of the measurement. It represents an ensemble of \((S + A)\)-systems in which a fraction \( p_j = |c_j|^2 \) appears in the \( j \)th product state in the summand. With the projection postulate incorporated, the von Neumann formalism, therefore, predicts that, in a measurement with the system \( S \) initially in the superposition state as above,

(i) the measured values of the observable \( F \) are the random numbers \( \lambda_j \) with respective probabilities \( |c_j|^2 \);
(ii) when the measurement outcome is $\lambda_j$, the final state of the system is $|\psi_j\rangle$.

Both the predictions are in excellent agreement with experiment.

The main problem with the treatment of a quantum measurement given above is the ad-hoc nature of the projection postulate. Moreover, having to invoke a discontinuous, acausal and instantaneous process is a very unpleasant feature of the formalism. The so-called measurement problem in QM is essentially the problem of explaining the final state (232) without introducing anything ad-hoc and/or physically unappealing in the theoretical treatment. This means that one should either give a convincing dynamical explanation of the reduction process or else circumvent it.

A critical account of various attempts to solve the measurement problem and related detailed references may be found in the author’s article (Dass 2005); none of them can be claimed to have provided a satisfactory solution. In the following subsection, a straightforward treatment of a quantum measurement in the supmech framework is presented, making up for some deficiencies in the von Neumann treatment.

8.2. Supmech treatment of a quantum measurement

We shall now treat the (S +A) system in the supmech framework of section 6.2 treating both, the system S and the apparatus A, as quantum Hamiltonian systems. Given the two quantum triples $(H_S, D_S, A_S)$ and $(H_A, D_A, A_A)$ corresponding to S and A, the quantum triple corresponding to (S+A) is $(H_S \otimes H_A, D_S \otimes D_A, A_S \otimes A_A)$.

Von Neumann’s treatment does not do adequate justice to the physics of the apparatus and needs some improvements. We propose to take into consideration the following points:

(i) The apparatus A is a quantum mechanical system admitting, to a very good approximation, a classical description. Even when the number of the effective apparatus degrees of freedom is not large (for example, the Stern-Gerlach experiment, treated in the next section, where the center of mass position vector of a silver atom acts as the effective apparatus variable), a classical description of the relevant variables is adequate. This feature is of more than academic interest and must be incorporated in the theoretical treatment.

(ii) Introduction of vector states for the pointer positions is neither necessary
nor desirable. A better procedure is to introduce density operators for them and take into consideration the fact that the Wigner functions corresponding to them are approximated well by classical density functions.

(iii) The pointer states have a stability property: After the measurement interaction is over, the apparatus, left to itself, settles quickly into one of the pointer positions. It is this process which should replace von Neumann’s ‘instantaneous, non-causal and discontinuous’ process.

Note. A detailed mathematical treatment of this process, as we shall see below, is not necessary; it is adequate to take its effect correctly into account. To get a feel for this, note that, in, for example, the Stern-Gerlach experiment, treated in the next subsection, the measurement interaction is over (ignoring fringe effects) after the atom is out of the region between the magnetic pole pieces. In this case, by ‘the apparatus settling to a pointer position’ one means the movement of the atom from just outside the pole pieces to a detector. In this case, the choice of the detector is decided by the location of the atom just after the measurement interaction. Details of motion of the atom from the magnets to the detector is of no practical interest in the present context. In the case of a macroscopic apparatus, the ‘settling ...’ refers to the process of the apparatus reattaining thermal equilibrium (disturbed slightly during the measurement interaction) after the measurement interaction; again, details of this process are not important in the present context (the eventual pointer position is decided by the region of the apparatus phase space in which the apparatus happens to be immediately after the measurement interaction).

(iv) Observations relating to the apparatus are restricted to the pointer positions $M_j$. A properly formulated Hamiltonian dynamics (classical or quantum) which takes this into consideration (treating the apparatus ‘respectfully’ as a system) would involve, at appropriate stage, averaging over the inoperative part of the phase space of the apparatus [the ‘internal environment’ (Dass 2005) of the apparatus]. It is this averaging, as we shall see below, which produces the needful decoherence effects (Zurek 2003) to wipe out undesirable quantum interferences.

(v) Different pointer positions are macroscopically distinguishable. We shall take this into consideration by employing an appropriate energy-time uncertainty inequality. This condition will be seen to play a crucial role in the demonstration of the demolition of the unwanted quantum interference.
terms.

A general pointer observable for $A$ is of the form

$$J = \sum_j b_j P_j$$

(233)

where $P_j$ is the projection operator onto the space of states in $\mathcal{H}_A$ corresponding to the pointer position $M_j$ [considered as an apparatus property; for a detailed treatment of the relationship between classical properties and quantum mechanical projectors, see Omnes(1994,1999) and references therein] and $b_j$s are real numbers such that $b_j \neq b_k$ for $j \neq k$. In purely quantum mechanical terms, the projector $P_j$ represents the question (von Neumann 1955; Jauch 1968): ‘Is the pointer at position $M_j$?’ The observable $J$ has different ‘values’ at different pointer positions. Since one needs only to distinguish between different pointer positions, any observable $J$ of the above mentioned specifications can serve as a pointer observable.

The phase space function $P_W^j$ corresponding to the projector $P_j$ is supposedly approximated well by a function $P_{cl}^j$ on the phase space $\Gamma$ of the apparatus $A$ (the $\hbar \to 0$ limit of $P_W^j$). Now, in $\Gamma$, there must be non-overlapping domains $D_j$ corresponding to the pointer positions $M_j$. In view of the point (iv) above, different points in a single domain $D_j$ are not distinguished by the experiment. We can, therefore, take $P_{cl}^j$ to be proportional to the characteristic function $\chi_{D_j}$ of the domain $D_j$; it follows that the phase space function $J_W^j$ corresponding to the operator $J$ above is approximated well by the classical pointer observable

$$J_{cl}^j = \sum_j b_j' \chi_{D_j}$$

(234)

where $b_j'$s have properties similar to the $b_j$s above.

The pointer states $\phi_j^{(A)}$ corresponding to the pointer positions $M_j$ are density operators of the form (constant)$P_j$; the phase space functions corresponding to these states are approximated well by the classical phase space density functions $\rho_{cl}^j = V(D_j)^{-1} \chi_{D_j}$, where $V(D)$ is the phase space volume of the domain $D$. [Note. If $V(D_j)$ is infinite, one can treat $\rho_{cl}^j$ as a function on $\Gamma$ which vanishes outside $D_j$ and varies very slowly in $D_j$.]

We shall take $H_{int} = F \otimes K$ (absorbing the coupling constant in $K$) where $F$ is the measured quantum observable and $K$ is a suitably chosen apparatus variable. We shall make the usual assumption that, during the measurement
interaction, $H_{int}$ is the dominant part of the total Hamiltonian ($H \simeq H_{int}$). The unitary operator $U$ of subsection 8.1 describing the measurement interaction in the von Neumann scheme must now be replaced by the measurement operator in supmech [which implements the appropriate canonical transformation on the states of the (S +A) system] given by $M \equiv \exp[\tau \hat{\partial}_H]$ where $\tau = t_f - t_i$ is the time interval of measurement interaction and $\hat{\partial}_H$ is the evolution generator in the supmech Liouville equation (65) [it is the transpose of the operator $\partial_H \equiv Y_H$, the evolution generator in Eq.(64)].

Assuming, again, that the measurement is ideal and denoting the ‘ready state’ of the apparatus by $\phi_0^{(A)}$, we have the following analogue of Eq.(229):

$$M(|\psi_j><\psi_j| \times \phi_0^{(A)}) = |\psi_j><\psi_j| \times \phi_j^{(A)}. \quad (235)$$

Here and in the following paras, we have identified the quantum states of the system S with the corresponding density operators. When the system is initially in the superposition state $|\psi>$ as above, the initial and final (S+A)-states are

$$\Phi_{in} = |\psi><\psi| \times \phi_0^{(A)}; \quad \Phi_f = M(\Phi_{in}). \quad (236)$$

Note that the ‘ready’ state may or may not correspond to one of the pointer readings. (In a voltage type measurement, it does; in the Stern-Gerlach experiment with spin half particles, it does not.) For the assignment of the $\Gamma$-domain to the ‘ready’ state, the proper interpretation (which covers both the situations above) of the ready state is ‘not being in any of the (other) pointer states’. Accordingly, we assign, to this state, the domain

$$\tilde{D}_0 \equiv \Gamma - \bigcup_{j \neq 0} D_j \quad (237)$$

where the condition $j \neq 0$ on the right is to be ignored when the ‘ready’ state is not a pointer state.

We must now take care of the point (iii) above. When the measurement interaction is over, the apparatus, left to itself, will quickly occupy, in any single experiment, a pointer position $M_j$ (depending on the region of the phase space $\Gamma$ it happens to be in after the measurement interaction). For the ensemble of (S +A) systems described by the initial state $\Phi_{in}$, the final state (after ‘settling down’) must be of the form

$$\hat{\Phi}_f = \sum_j p_j \rho_j^{(S)} \times \phi_j^{(A)} \quad (238)$$
where $\rho_j^{(S)}$ are some states of $S$. Since, during the transition from the state $\Phi_f$ to $\hat{\Phi}_f$, the system and the apparatus are left to themselves and only some internal changes take place in the apparatus (which, in view of the stability property (iii) above, are not expected to change the expectation value of a pointer observable $J$), we must have, for an arbitrary system observable $A$,

$$\hat{\Phi}_f(A \otimes J) = \Phi_f(A \otimes J).$$

(239)

Now, $\Phi_f = \Phi'_f + \Phi''_f$ where

$$\Phi'_f = M \left( \sum_j |c_j|^2 [\langle \psi_j | \otimes \phi_0^{(A)}] \right)$$

$$= \sum_j |c_j|^2 [\langle \psi_j | \otimes \phi_j^{(A)}]$$

(240)

(where we have used the fact that, in supmech, a canonical transformation on states preserves convex combinations) and

$$\Phi''_f = M \left( \sum_{j \neq k} c_k^* c_j [\langle \psi_j | \otimes \phi_k^{(A)}] \right) \equiv M(R).$$

(241)

[Note that $R$, the operand of $M$, is not an $(S + A)$-state; here $M$ has been implicitly extended by linearity to the dual space of the algebra $A_S \otimes A_A$.]

We shall now prove that

$$W \equiv \Phi''_f(A \otimes J) \simeq 0.$$  

(242)

Transposing the $M$ operation to the observables, we have

$$W = R[exp(\tau \partial H)(A \otimes J)]$$

$$= \int_{d\Gamma} d\Gamma \rho_0^{(A)W} \sum_{j \neq k} c_k^* c_j < \psi_k | exp(\tau \partial H')(A \otimes J^W) | \psi_j >$$

(243)

where we have adopted the phase space description of the QM of the apparatus, $d\Gamma$ is the phase space volume element, $\rho_0^{(A)W}$ is the Wigner function corresponding to the state $\phi_0^{(A)}$ and $H' = F \otimes K^W$. Using equations (154)
and (224), we have
\[
\partial_H'(A \otimes J^W) = \{ F \otimes K^W, A \otimes J^W \} = (-i\hbar)^{-1} \left( [F, A] \otimes \frac{K^W * J^W + J^W * K^W}{2} \right. \\
\left. + \frac{FA + AF}{2} \otimes (K^W * J^W - J^W * K^W) \right). \tag{244}
\]

Given the fact that the apparatus is well described classically, we have \( K^W \simeq K^{cl} \) and \( J^W \simeq J^{cl} \) to a very good approximation. This gives
\[
\partial_H'(A \otimes J^W) \simeq (-i\hbar)^{-1}K^{cl}J^{cl}[F, A]
\]
which, in turn, implies
\[
<\psi_k|exp(\tau \partial_H')(A \otimes J^W)|\psi_j> \simeq exp[i(\lambda_k - \lambda_j)K^{cl} \tau]J^{cl} <\psi_k|A|\psi_j>.
\]

We now have, replacing \( \rho_0^{(A)W} \) by its classical approximation \( \rho_0^{(A)cl} \),
\[
W \simeq \int_{\tilde{D}_0} d\Gamma \rho_0^{(A)cl} \sum_{j \neq k} c_k^* c_j exp[i(\lambda_k - \lambda_j)K^{cl} \tau]J^{cl} <\psi_k|A|\psi_j>. \tag{245}
\]

Let
\[
< K^{cl} >_0 \equiv \int_{\tilde{D}_0} K^{cl} \rho_0^{(A)cl} d\Gamma \tag{246}
\]
(the mean value of \( K^{cl} \) in the domain \( \tilde{D}_0 \)). Putting \( K^{cl} = < K^{cl} >_0 s \), taking \( s \) to be one of the integration variables and writing \( d\Gamma = dsd\Gamma' \), we have
\[
W \simeq \int_{\tilde{D}_0} dsd\Gamma' \rho_0^{(A)cl} \sum_{j \neq k} c_k^* c_j exp[i\eta_{jk} s]J^{cl} <\psi_k|A|\psi_j> \tag{247}
\]
where
\[
\eta_{jk} = (\lambda_k - \lambda_j) < K^{cl} >_0 \tau. \tag{248}
\]

Note that \( s \) is a real dimensionless variable with domain of integration of order unit length.
We shall now argue that, for \( j \neq k \),

\[ |\eta_{jk}| \gg \hbar. \tag{249} \]

(This is not obvious; when \( F \) is a component of spin, for example, the \( \lambda \)s are scalar multiples of \( \hbar \).) To this end, we invoke the apparatus feature (v) above. A fairly straightforward way of formulating a criterion for macroscopic distinguishability of different pointer positions would be to identify a quantity of the dimension of action which could be taken as characterizing the physical separation between two different pointer positions and show that its magnitude is much larger than \( \hbar \). The objects \( \eta_{jk} \) (for \( j \neq k \)) are quantities of this type. The inequality (249) then follows from the assumed macroscopic distinguishability of different pointer positions. Another, essentially equivalent, way of seeing this is to treat Eq.(249) as the time-energy uncertainty inequality \( |\Delta E \Delta t| \gg \hbar \) where \( \Delta t = \tau \) and \( \Delta E \) is the difference between the energy values corresponding to the apparatus locations in two different domains \( D_j \) and \( D_k \) in \( \Gamma \). Recalling that \( H \simeq H_{int} \) during the relevant time interval, we have \( \Delta E \simeq (\lambda_k - \lambda_j) < K^{cl}_b > 0 \).

The large fluctuations implied by Eq.(249) wipe out the integral above giving \( W \simeq 0 \) as desired.

Equations (239), (238) and (242) now give

\[ 0 = (\hat{\Phi}_f - \Phi'_f)(A \otimes J) = \sum_j \phi_j^{(A)}(J) Tr([p_j \rho_j^{(S)} - |c_j|^2 \psi_j] < \psi_j | A) \]

which must be true for all \( J \) [with arbitrary \( b_j \) in Eq.(233) satisfying the stated condition]. This gives

\[ Tr([p_j \rho_j^{(S)} - |c_j|^2 \psi_j] < \psi_j | A) = 0 \]

for all system observables \( A \) and, therefore,

\[ p_j \rho_j^{(S)} = |c_j|^2 |\psi_j > < \psi_j|. \]

Finally, therefore, we have \( \hat{\Phi}_f = \Phi'_f \) which is precisely the state obtained from \( \Phi_f \) by applying the von Neumann projection.

This completes the derivation of the von Neumann projection rule. This has been obtained through straightforward physics; there is no need to give
any ad hoc prescriptions. The derivation makes it clear as to the sense in which this reduction rule should be understood: it is a prescription to correctly take into consideration the effect of the ‘settling down’ of the apparatus after the measurement interaction for obtaining the final state of the observationally constrained \((S + A)\) system.

Eq.\((247)\), followed by the reasoning above, represents, in a \textit{live} form, the operation of environment-induced decoherence. To see this, note that, the domain \(\tilde{D}_0\) may be taken to represent the internal environment of the apparatus. With this understanding, the mechanism wiping out the unwanted quantum interference terms is, indeed, the environment-induced decoherence. In the treatment presented here this mechanism becomes automatically operative. (Even the external environment can be trivially included by merely saying that the system \(A\) above represents ‘the apparatus and its environment’.)

\section*{8.3. Example: the Stern-Gerlach experiment}

As an illustration, we consider the Stern-Gerlach experiment (Busch, Grabowski and Lahti 1995; Omnes 1994; Cohen-Tannoudji, Diu and Laloë 2005) with, say, silver atoms (which means spin \(s = \frac{1}{2}\)). A collimated beam of (unpolarized) silver atoms is made to pass through inhomogeneous magnetic field after which the beam splits into two beams corresponding to atoms with \(S_z = \pm \frac{\hbar}{2}\). The spin and magnetic moment operators of an atom are \(S = \frac{\hbar}{2} \sigma\) and \(\mu = gS\) (where \(g\) is the magnetogyric ratio). Let the magnetic field be \(B(r) = B(z)\text{e}_3\) (in obvious notation). [Refinements (Potel et al. 2004) introduced to ensure the condition \(\nabla \cdot B = 0\) do not affect the essential results obtained below.] We have

\[H_{\text{int}} = -\mu \cdot B = -gB(z)S_3.\]  

(250)

The force on an atom, according to Ehrenfest’s theorem, is

\[F = -\nabla < -\mu \cdot B > = g \frac{dB(z)}{dz} < S_3 > \text{e}_3\]

(251)

where the average is taken in the quantum state of the atom. During the experiment, the internal state of the atom remains unchanged (to a very good approximation); only its center of mass \(r\) and spin \(S\) have significant dynamics. In this experiment, \(S_3\) is the measured quantum observable and \(r\) acts as the operative apparatus variable.
Let us assume that the beam initially moves in the positive x-direction, the pole pieces are located in the region \( x_1 \leq x \leq x_2 \) and the detectors located in the plane \( x = x_3 > x_2 \) (one each in the regions \( z > 0 \) and \( z < 0 \); these regions contain the emergent beams of silver atoms corresponding, respectively, to \( S_3 = +\hbar/2 \) and \( S_3 = -\hbar/2 \)). We have, in the notation used above, \( F = S_3 \) and \( K = -gB(z) \). Assuming the experiment to start when the beam reaches at \( x = x_1 \), the phase space of the apparatus is

\[
\Gamma = \{(x, y, z, p_x, p_y, p_z) \in \mathbb{R}^6; x \geq x_1, *\}
\]

where * indicates the restriction that, for \( x_1 \leq x \leq x_2 \), the space available for the movement of atoms is the one between the two pole pieces. For the order of magnitude calculation below, we shall ignore the shape of the pole pieces and take * to imply \( z_1 \leq z \leq z_2 \).

The domains \( D_1 \) and \( D_2 \) corresponding to the two pointer positions are

\[
D_1 = \{(x, y, z, p_x, p_y, p_z) \in \Gamma; x > x_2, p_z > 0\}
\]

\[
D_2 = \{(x, y, z, p_x, p_y, p_z) \in \Gamma; x > x_2, p_z < 0\};
\]

the domain \( \tilde{D}_0 = \Gamma - (D_1 \cup D_2) \). For simplicity, let us take \( B(z) = b_0 + b_1z \) where \( b_0 \) and \( b_1 \) are constants. For \( j \neq k \), we have \( \lambda_j - \lambda_k = \pm \hbar \). The relevant integral is [see Eq.(247) above]

\[
I = \int_{z_1}^{z_2} dz (\ldots) \exp[\pm \frac{i}{\hbar} \mu b_1 z \tau]
\]

where \( \mu = gh \). Putting \( z = (z_2 - z_1)u \), the new integration variable \( u \) is a dimensionless variable taking values in a domain of length of order one. The quantity of interest is

\[
|\eta| = \mu |b_1|(z_2 - z_1)\tau.
\]

According to the data in (Cohen-Tannoudji, Diu and Laloë 2005) and (Goswami 1992, problem 4.6), we have (\( v_x \) is the x-component of velocity of the silver atom)

\[
|b_1| \sim \left| \frac{dB}{dz} \right| \sim 10^5 \text{ gauss/cm} \\
z_2 - z_1 \simeq 1 \text{ mm}, \quad v_x \sim 500 \text{ m/sec} \\
x_2 - x_1 = 3 \text{ cm}, \quad x_3 - x_2 = 20 \text{ cm}.
\]
This gives
\[ \tau \sim \frac{x_3 - x_1}{v_x} \sim 5 \times 10^{-4}\text{ sec.} \]

Denoting the Bohr magneton by \( \mu_b \) and putting \( \mu \sim \mu_b \simeq 0.9 \times 10^{-20} \text{ erg/gauss} \), we have \( |\eta| \sim 10^{-19} \text{erg-sec} \). With \( \hbar \sim 1.1 \times 10^{-27} \text{ erg-sec} \), we have, finally \( (|\eta|/\hbar) \sim 10^8 \), confirming the strong suppression of the undesirable quantum interferences.

9. AXIOMS

We shall now write down a set of axioms covering the work presented in the preceding sections. Before the statement of axioms, a few points are in order:

(i) These axioms are meant to be provisional; the ‘final’ axioms will, hopefully, be formulated (not necessarily by the present author) after a reasonably satisfactory treatment of quantum theory of fields and space-time geometry in an appropriately augmented supmech type framework has been given.

(ii) The terms ‘system’, ‘observation’, ‘experiment’ and a few other ‘commonly used’ terms will be assumed to be understood. The term ‘relativity scheme’ employed below will be understood to have its meaning as explained in section 5.1.

(iii) The ‘universe’ will be understood as the largest possible observable system containing every other observable system as a subsystem.

(iv) By an experimentally accessible system we shall mean one whose ‘identical’ (for all practical purposes) copies are reasonably freely available for repeated trials of an experiment. Note that the universe and its ‘large’ subsystems are not included in this class.

(v) The term ‘system’ will, henceforth will normally mean an experimentally accessible one. Whenever it is intended to cover the universe and/or its large subsystems (this will be the case in the first two axioms only), the term system* will be used.

The axioms will be labeled as \( A_1, \ldots, A_8 \).

\( A_1. \) (Probabilistic framework; System algebra and states)
(a) System algebra; Observables. A system* \( S \) has associated with it a supmech-admissible (as defined in section 4.1) superalgebra \( \mathcal{A} = \mathcal{A}^{(5)} \). (Its
elements will be denoted as A, B, ...). Observables of S are elements of the subset \( \mathcal{O}(\mathcal{A}) \) of even Hermitian elements of \( \mathcal{A} \).

(b) States. States of the system* \( \mathcal{A} \), also referred to as the states of the system algebra \( \mathcal{A} \) (denoted by the letters \( \phi, \psi, .. \)), are defined as (continuous) positive linear functionals on \( \mathcal{A} \) which are normalized [i.e. \( \phi(I) = 1 \) where I is the unit element of \( \mathcal{A} \)]. The set of states of \( \mathcal{A} \) will be denoted as \( S(\mathcal{A}) \) and the subset of pure states by \( S_1(\mathcal{A}) \). For any \( A \in \mathcal{O}(\mathcal{A}) \) and \( \phi \in S(\mathcal{A}) \), the quantity \( \phi(A) \) is to be interpreted as the expectation value of \( A \) when the system is in the state \( \phi \).

(c) Expectation values of odd elements of \( \mathcal{A} \) vanish in every pure state (hence in every state).

(d) Compatible completeness of observables and pure states. The pair \( (\mathcal{O}(\mathcal{A}), S_1(\mathcal{A})) \) satisfies the CC condition described in section 4.1.

(e) Experimental situations and probabilities. An experimental situation (relating to observations on the system* \( \mathcal{A} \)) has associated with it a positive observable-valued measure (PObVM) as defined in section 4.1; it associates, with measurable subset of a measurable space, objects called supmech events which have measure-like properties. Given the system prepared in a state \( \phi \), the probability of realization of a supmech event \( \nu(E) \) is \( \phi(\nu(E)) \).

A2. Differential calculus; Symplectic structure. The system algebra \( \mathcal{A} \) of a system* \( \mathcal{A} \) is such as to permit the development of derivation-based differential calculus based on it; moreover, it is equipped with a real symplectic form \( \omega \) thus constituting a symplectic superalgebra \( (\mathcal{A}, \omega) \) [more generally, a generalized symplectic superalgebra \( (\mathcal{A}, \mathcal{X}, \omega) \) when the derivations are restricted to a distinguished Lie sub-superalgera \( \mathcal{X} \) of \( SDer(\mathcal{A}) \)].

A3. Dynamics. The dynamics of a system \( \mathcal{S} \) is described by a one-parameter family of canonical transformations generated by an even Hermitian element \( H \) (the Hamiltonian) of \( \mathcal{A} \) which is bounded below in the sense of section 4.2.

The mechanics described by the above-stated axioms will be referred to as supmech. The triple \( (\mathcal{A}, \omega, H) \) [or the quadruple \( (\mathcal{A}, S_1(\mathcal{A}), \omega, H) \)] will be said to constitute a supmech Hamiltonian system.

A4. Relativity scheme. (a) For systems admitting space-time description, the ‘principle of relativity’ as described in section 5.1, will be operative.

(b) The admissible relativity schemes will be restricted to (i) Galilean relativity, (ii) special relativity.
A5. Elementary systems; Material particles. (a) In each of the admitted relativity schemes, material particles will be understood to be localizable elementary systems (as defined in sections 4.7 and 5.1) with positive mass.
(b) The system algebra for a material particle will be (the topological completion of) the one generated by its fundamental observables (as defined in section 5.1) and the identity element.

A6. Coupled systems. Given two systems $S_1$ and $S_2$ described as supmech Hamiltonian systems $(A^{(i)}, S_1^{(i)}, \omega^{(i)}, H^{(i)})$ ($i=1,2$), the coupled system $(S_1 + S_2)$ will be described as a supmech Hamiltonian system $(A, S_1, \omega, H)$ with

$$A = A^{(1)} \otimes A^{(2)}, \quad S_1 = S_1(A), \quad \omega = \tilde{\omega}^{(1)} + \tilde{\omega}^{(2)}$$

[see Eq.(134)] and $H$ as in Eq.(156).

A7. Quantum systems. All (experimentally accessible) systems in nature have noncommutative system algebras (and hence are quantum systems); they have a quantum symplectic structure (as defined in section 3.9) with the universal parameter $b = -i\hbar$.

Note. (i) The subclass of quantum systems having finitely generated system algebras were shown (in section 7.3) to have Hilbert space based realizations (without introducing additional postulates — explicitly or implicitly).

(ii) Axioms A7 and A5(a) imply that all material particles are localizable elementary quantum systems. Since they have finitely generated system algebras, the corresponding supmech Hamiltonian systems are guaranteed to have Hilbert space based realizations. They can be treated as in section 7.4 without introducing any extra postulates; in particular, introduction of the Schrödinger wave functions with the traditional Born interpretation and the Schrödinger dynamics are automatic.

(iii) General quantum systems were shown in section 7.5 to admit superselection rules.

A8. Measurements. In a measurement involving a ‘measured system’ $S$ and apparatus $A$
(a) both $S$ and $A$ are standard quantum systems (as defined in section 7.1);
(b) the Hamiltonian system $(A^{(A)}, S_1^{(A)}, \omega^{(A)}, H^{(A)})$ corresponding to the apparatus admits an equivalent (in the sense of section 4.3) phase space realization (in the Weyl-Wigner-Moyal scheme) $(A^{(W_A)}, S_1^{(W_A)}, \omega^{(W_A)}, H^{(W_A)})$.
(c) elements of $A_{W}^{(A)}$ and $S_{W}^{(A)}$ appearing in the description of dynamics of the coupled system (S+A) admit $\hbar \rightarrow 0$ limits and are approximated well by these limits;
(d) the various pointer positions of the apparatus (i) have the stability property as stated in section 8.2 and (ii) are macroscopically distinguishable [the macroscopic distinguishability can be interpreted, for example, in terms of an energy-time uncertainty product inequality ($\Delta E \Delta t \gg \hbar$) relevant to the experimental situation];
(e) observations on the apparatus are restricted to readings of the output devices (pointers).

*Note.* Part (b) in this axiom is expected to be redundant and is included ‘to be on the safe side’. (A redundancy is excusable if it serves to bring some extra clarity without introducing any inconsistency.)

10. CONCLUDING REMARKS

1. Sometimes the question is raised: ‘Why algebras?’ The answer emerging from the present work is: ‘Because they provide the right framework for noncommutative symplectic geometry as well as for noncommutative probability and, therefore, are natural objects to employ in the construction of a formalism integrating the two [in the spirit of unification of physics and probability theory envisaged in the formulation of (augmented) Hilbert’s sixth problem].’

   Indeed, as we have seen, for an autonomous development of quantum mechanics, the fundamental objects are algebras and not Hilbert spaces.

2. A contribution of the present work expected to be of some significance to the algebraic schemes in theoretical physics and probability theory is the introduction of the condition of compatible completeness for observables and pure states [axiom A1(d)] which plays an important role in ensuring that the so-called ‘standard quantum systems’ defined algebraically in section 7.1, have faithful Hilbert space-based realizations. It is desirable to formulate necessary and/or sufficient conditions on the superalgebra $\mathcal{A}$ alone (i.e. without reference to states) so that the CC condition is automatically satisfied.

   An interesting result, obtained in section 7.7, is that the superclassical systems with a finite number of fermionic generators generally do not satisfy the CC condition. This probably explains their non-occurrence in nature. It is worth investigating whether the CC condition is related to some stability property of dynamics.
3. Some features of the development of QM in the present work (apart from the fact that it is autonomous) should please theoreticians: there is a fairly broad-based algebraic formalism connected smoothly to the Hilbert space QM; there is a natural place for commutative superselection rules and for the Dirac's bra-ket formalism; the Planck constant is introduced 'by hand' at only one place (at just the right place: the quantum symplectic form) and it appears at all conventional places automatically. Moreover, once the concepts of localization, elementary system and standard quantum system are introduced at appropriate places, it is adequate to define a material particle as a localizable elementary quantum system: 'everything else' — including the interpretation of the Schrödinger wave function and the Schrödinger equation — is automatic.

4. The treatment of quantum-classical correspondence in section 7.8, illustrated with the example of a nonrelativistic spinless particle, makes clear as to how the subject should be treated in the general case: go from the traditional Hilbert space-based description of the quantum system to an equivalent (in the sense of a supmech Hamiltonian system) phase space description in the Weyl-Wigner-Moyal formalism, pick up the appropriate subsets in the observables and states having smooth $\hbar \to 0$ limits and verify that the limit gives a commutative supmech Hamiltonian system (which is generally a traditional classical Hamiltonian system).

5. It is of some interest to note that, in the formalization of a measurement situation in supmech in section 4.1, the concept of PObVM naturally appears; it is the abstract counterpart (in fact, a generalization) of the concept of POVM which has been dealt with by quantum theorists in various contexts during the past three decades.

6. The treatment of measurements in QM in section 8 provides a justification for the collapse postulate through straightforward physics: it is the prescription for obtaining the effective final state of an observationally constrained coupled system. The sight of Eq.(247) where one can see the operation of the decohering effect of 'internal environment' of the apparatus in live action, should please theoreticians. The incorporation of the external environment (for the restricted purpose of realizing von Neumann reduction) has been reduced to a matter of less than two lines: just saying that the symbol 'A' now stands for 'the apparatus and its environment'.
7. The title of the present paper was originally intended to be, as announced in Ref.[47,49], ‘Supmech: a unified symplectic view of physics’; while preparing the final version of it, however, the author felt that the present title would represent the contents better.

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