Kinematic-wave model of viscous fingers with mixing layer

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Abstract. A new one-dimensional kinematic model of viscous fingers growth is proposed. This model is based on the assumption of an intermediate layer formation. The shear instability of the flow is developed in this layer due to intensive mixing of liquids of different viscosities. The thickness of the intermediate layer between outer ones is determined by the velocity difference in the layers. The results of the calculation of the growth rate of the fingers in the framework of this model are in a good agreement with the calculations carried out on the basis of more general two-dimensional equations as well as with the known Koval model.

1. Introduction

It is well known that a displacement process involving two fluids is often unstable if the displacing fluid has lower viscosity than the displaced one. The resulting instability developing at the interface between two fluids is often called the viscous fingering [1, 2]. A number of works devoted to understanding of various aspects of such form of instability is available in the literature [3, 4]. It should be noted that a numerical modelling of the displacement process requires the wasteful expenses of computer time at high Peclet numbers. Moreover, it is difficult to reproduce the detailed fingering pattern. As it was shown in [5, 6] it is simpler to describe the concentration of solute averaged across the fingers. In such cases, the mixing zone is an important feature for determining the extent of mixing. Despite the considerable work in this field was presented, the spreading and growth of the mixing zone is the important opened question. There are the several empirical models for the modelling of the evaluation of mixing zone in unstable, miscible displacements. An empirical model has been suggested by Koval [7] to give a basis for computation of miscible displacement. This model suffers from adoption of empiricism. The involved principal parameters have insignificant or indirect physical value. However, an experimental study of the mixing zone growth in miscible viscous fingering at high Peclet numbers [8] shows that the trend is consistent with Koval’s model [7]. Further development of averaged models of fingers formation is represented in [9, 10]. All these models are based on the assumption that pressure is constant in the transverse direction of the main flow, as well as an empirical information about the displacing and displaced fluids distribution in the region of intensive viscous fingering.

In this paper we propose a 1D kinematic-wave model of viscous fingers taking into account the formation of an intermediate mixing layer. This model includes an empirical parameter
depending on the viscosity ratio and related with the thickness of the mixing zone. The velocity of propagation and the thickness of the viscous finger in the framework of the kinematic-wave model coincide with the corresponding calculations on the basis of the 2D equations [10]. This makes it possible to predict the parameters of viscous fingers avoiding the time-consuming calculations. We also show that the proposed 1D model is in a good agreement with the known Koval model [7].

2. Equations of motion
A Newtonian weakly-compressible flow in a Hele-Shaw cell (the area between two parallel plates separated by a small gap of constant thickness \( b \) in the \( z \) direction) is described by the equations

\[
(\rho v)_t + \text{div}(\rho v \otimes v) + \nabla p = \mu v_{zz}, \quad \rho_t + \text{div}(\rho v) = 0, \quad \left. v \right|_{z=\pm b/2} = 0.
\]

Here \( p \) is the pressure, \( \rho \) is the density, and \( \mu \) is the dynamic viscosity. The second derivatives of the velocity vector \( v = (u, v, w) \) with respect to the variables \( x \) and \( y \) are negligible compared to the derivatives in the vertical direction \( z \). They are assumed to be zero in the suggested model. We put on that the velocity field can be represented as

\[
u = \frac{3}{2} \left( 1 - \left( \frac{2z}{b} \right)^2 \right) u'(t, x, y), \quad v = \frac{3}{2} \left( 1 - \left( \frac{2z}{b} \right)^2 \right) v'(t, x, y), \quad w = 0.
\]

The functions \( p \) and \( \rho \) are supposed to be independent on \( z \). Averaging equations (1) through the gap we obtain (primes are omitted)

\[
(\rho u)_t + (\beta \rho u^2 + p) x + (\beta \rho uv) y = -\mu u, \\
(\rho v)_t + (\beta \rho v^2 + p) x + (\beta \rho uv) y = -\mu v, \\
\rho_t + (u \rho)_x + (v \rho)_y = 0.
\]

(2)

Here and below \( \mu \) denotes the viscosity of the fluid divided by the medium permeability \( b^2/12 \) (further in the text we will call it simply “viscosity”); coefficient \( \beta \) is equal to \( 5/3 \).

We are interested to describe the process of displacement of a fluid with the viscosity \( \mu_1 \) by a less viscous fluid with viscosity \( \mu_2 \). We utilize for this description equations (2) with variable viscosity depending on the concentration function \( c \). This function is scaled such that it is equal to unity in the displaced fluid \( (\mu = \mu_2) \) and zero in the displacing one \( (\mu = \mu_1) \). We should add the transport equation for the concentration \( c \) to the governing equations. It has the divergent form (the diffusion terms are assumed to be neglected)

\[
(\rho c)_t + (u \rho c)_x + (v \rho c)_y = 0.
\]

(3)

Following [11] we use a monotonic relationship between the viscosity and the concentration in the form \( \mu(c) = \mu_1^{1-c} \mu_2^c \). In order to close model (2), (3) we should specify the equation of state \( p = p(\rho) \). A weak compressibility of the medium given by the equation of state \( p = p(\rho) \) provides hyperbolicity of the model. If the condition \( u^2 + v^2 \ll p'(\rho) \) keeps then the results depend on the choice of \( p = p(\rho) \) insignificantly. Therefore, for the numerical simulation of the 2D flows we assume

\[
p(\rho) = \frac{a^2 \rho^2}{2}, \quad (a^2 = c_0^2/\rho_0),
\]

(4)

where the constants \( \rho_0 \) and \( c_0 \) mean characteristic density and speed of sound in the medium.

System (2)–(4) was recently proposed in [10] for modelling of the viscous fingers taking into account the inertia of fluid. It may be important for the high finger velocities. It was also pointed out that for the process of unidirectional displacement the pressure variation in the transverse direction is small \( (p_y = 0) \).
2.1. Layered flow
We study an incompressible Hele-Shaw flow in a two-dimensional rectangular domain. The size of the domain is $L \times H$. We assume $\rho = 1$ without loss of generality. We replace the second of momentum equation in (2) by $p_y = 0$. It corresponds to transverse flow equilibrium. We consider the class of viscosity-stratified flows

$$u = u_i(t, x), \quad c = c_i = \text{const}, \quad (y \in (y_{i-1}, y_i), \quad i = 1, \ldots, N),$$

wherein $0 = y_0 < y_1(t, x) < \ldots < y_N(t, x) = H$. In this case equations (2) and (3) take the form (see [10, 12] for the details)

$$u_{it} + \beta u_i u_{ix} + p_x = -\mu_i u_i, \quad h_{it} + (u_i h_i)_x = 0, \quad (i = 1, \ldots, N),$$

(5)

Here $h_i(t, x) = y_i(t, x) - y_{i-1}(t, x)$ is the depth of $i$–th liquid layer of viscosity $\mu_i$ and velocity $u_i(t, x)$, $Q = \text{const}$ is the total flow rate through the cell. In deriving equations (5) the kinematic condition at the layers interface is utilized.

For the average velocity $U = Q/H$ of fluid injection, one can use the following simplified version of governing equations (5). The simplification is based on the replacement of the momentum equations by the corresponding Darcy laws:

$$p_x = -\mu_i u_i \quad (i = 1, \ldots, N).$$

(6)

The remaining equations of system (5) do not vary. In what follows, we assume that $H = 1$ and $Q = 1$.

2.2. Two-layer flow
It is interesting to note that in the case of two-layer flow system (5), where the momentum equations are replaced by Darcy laws (6), is reduced to the so-called naive Koval model [9]

$$\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x} \left( \frac{M h_1}{M h_1 + 1 - h_1} \right) = 0.$$  

(7)

Here $M = \mu_2/\mu_1$ is the viscosity ratio. Indeed, taking into account the relations

$$\mu_1 u_1 = \mu_2 u_2, \quad h_1 + h_2 = 1, \quad u_1 h_1 + u_2 h_2 = 1$$

we can represent equation $h_{1t} + (u_1 h_1)_x = 0$ as equality (7).

If $M > 1$ we construct the solution of (7) in the form of a simple wave

$$h_1(\xi) = \frac{1}{M-1} \left( \sqrt{\frac{M}{\xi}} - 1 \right), \quad \xi = \frac{x - x_0}{t} \quad (M^{-1} \leq \xi \leq M).$$

The previous formulae give the solution of equation (7) with discontinuous initial data

$$h_1|_{t=0} = \begin{cases} 1, & x < x_0 \\ 0, & x > x_0. \end{cases}$$

Let us note, that for $M < 1$ solution of the same Cauchy problem is given by the shock wave moving with the average flow velocity $D = U = 1$. In this case the Saffman–Taylor instability does not develop.
It is known that the growth rate of the viscous fingers calculating with the help of the naïve Koval model is significantly higher than it is observed experimentally. Koval [7] postulates empirically that equation (7) is valid if the viscosity ratio $M$ is replaced by an effective viscosity ratio $M_e$, where

$$M_e = \left(\frac{M^{1/4}}{c_e} + 1 - c_e\right)^4, \quad c_e = 0.22.$$  

(8)

It should be noted that formula (8) is used in practice [8] but not justified theoretically. Below we propose a way to correct model (7) by using an intermediate mixing layer.

3. Three-layer flow with mixing

We use the notation $(\mu_1, u_1, h_1)$, $(\bar{\mu}, v, \eta)$ and $(\mu_2, u_2, h_2)$ for the fluid viscosities, velocities and depths in the displacing, intermediate and the displaced layers respectively ($\mu_1 < \bar{\mu} < \mu_2$). The cell height $H$ and the flow rate $Q$ are assumed to be unity as before. Then within the kinematic model $(p_x = -\mu_i u_i)$ we have

$$\mu_1 u_1 = \bar{\mu} v = \mu_2 u_2, \quad h_1 + \eta + h_2 = 1, \quad u_1 h_1 + v \eta + u_2 h_2 = 1.$$  

(9)

Following [13, 14] we consider the equilibrium between generation and dissipation of energy of the small-scale motions in a shear flow. In this case the thickness of the intermediate layer can be expressed in the form

$$\eta = \frac{\delta}{\bar{\mu} v}((u_1 - v)^2 + (u_2 - v)^2),$$  

(10)

where $\delta$ is an empirical parameter. Here we do not present the derivation of relation (10). Therefore, this formula can be regarded as an additional hypothesis reducing the equations of fingers evolution in the development of the Saffman–Taylor instability to the Hopf equation for the averaged thickness of a finger.

Let us define the middle-line $z = h_1 + \eta/2$ and formulate the law of mass conservation for the liquid layer of depth $z$

$$z_t + (\psi(z))_x = 0, \quad \text{where} \quad \psi = u_1 h_1 + v \eta/2.$$  

(11)

Using the notation

$$a_1 = 1 - \frac{\bar{\mu}}{2\mu_1} - \frac{\bar{\mu}}{2\mu_2}, \quad a_2(z) = \frac{\bar{\mu}}{\mu_2} + \left(\frac{\bar{\mu}}{\mu_1} - \frac{\bar{\mu}}{\mu_2}\right)z,$$

$$a_3 = \left(\left(\frac{\bar{\mu}}{\mu_1} - 1\right)^2 + \left(\frac{\bar{\mu}}{\mu_2} - 1\right)^2\right)\frac{\delta}{\bar{\mu}}$$

and formulas (9) we express the fluid velocity $v$ in the mixing layer by means of the variables $\eta$ and $z$:

$$v = (a_1 \eta + a_2(z))^{-1}.$$  

In this case formula (10) takes the form $\eta = a_3 v$ and allows one to obtain the dependence $\eta(z)$

$$\eta(z) = -a_2(z) + \sqrt{a_2^2(z) + 4a_1 a_3}.$$  

Substitution of $\eta(z)$ into the previous formula expresses velocities $v$ and $u = \bar{\mu} v/\mu_1$ as the functions of the middle-depth $z$. As the result we derive the relation for the flux $\psi$ in the layer of depth $z$:  

$$\psi(z) = \left(\frac{\bar{\mu}}{\mu_1} z + \left(1 - \frac{\bar{\mu}}{\mu_1}\right)\frac{\eta(z)}{2}\right)\frac{1}{a_1 \eta(z) + a_2(z)}.$$  

(12)
Note that it is necessary to require the fulfilment of the inequalities
\[ a_2^2(z) + 4a_1a_3 \geq 0, \quad z - \eta(z)/2 \geq 0, \quad z + \eta(z)/2 \leq 1. \]  
(13)
The last two inequalities are the consequence of \( h_1 \geq 0 \) and \( h_1 + \eta \leq 1 \). To determine the depth \( z = z(t, x) \) we use conservation law (11) with function \( \tilde{\psi}(z) \) given by formula (12).

Let us construct a solution of equation (11) with piecewise constant initial data \( z(0, x) = 1 \) for \( x < x_0 \) and \( z(0, x) = 0 \) for \( x > x_0 \). This formulation means that the domain \( x < x_0 \) is filled by the liquid of viscosity \( \mu_1 \) \((h_1 = 1, \eta = h_2 = 0)\) whereas the liquid of viscosity \( \mu_2 \) \((h_2 = 1, \eta = 0)\) fills the domain \( x > x_0 \). Since the fluid of viscosity \( \mu_1 \) moves in the positive direction of the \( x \)-axis with the average velocity \( U = 1 \) then if \( \mu_1 < \mu_2 \) the Saffman–Taylor instability develops at the interface.

Due to constraints (13) the function \( \psi(z) \) does not define on the whole interval \([0, 1]\). Typical behavior of the function \( \psi(z) \) is given in Figure 1 (curve 1) for the following parameters \( \mu_1 = 2, \mu_2 = 8, \mu = \sqrt{\mu_1\mu_2} = 4 \) and \( \delta = 1 \). Therefore, in order to solve the problem we should define the function \( \psi(z) \) on the entire interval \([0, 1]\). Taking into account that \( \psi(z) \) is the flow rate in the layer of the depth \( z = h_1 + \eta/2 \) the conditions \( \psi(0) = 0 \) and \( \psi(1) = 1 \) should be satisfied.

We assume that a function \( \tilde{\psi}(z) \) is the extension of \( \psi(z) \) on the entire interval \([0, 1]\). Note that the function \( \psi(z) \) can be extended in various ways. If the plot of constructed extension \( \tilde{\psi}(z) \) lies between the convex hull of the plot \( \psi(z) \) (Figure 1, line 1) and the diagonal of the unit square connecting points \((0, 0)\) and \((1, 1)\) (Figure 1, line 4) then the solution of the problem does not depend on the choice of extension [15]. Let us construct the tangents to the plot of \( \psi(z) \) from the origin and from the point \((1, 1)\) \((z_1 \) and \( z_2 \) denote the abscissae of the points of tangency). Thus, the function \( \tilde{\psi}(z) \) is given by formula (12) on the interval \([z_1, z_2]\). On the intervals \([0, z_1]\) and \((z_2, 1]\) it is represented by the tangent lines. Using the function \( \tilde{\psi}(z) \) we construct a self-similar solution of equation (11) with discontinuous initial data. This solution is represented by the simple wave

\[ \xi = \psi(z), \quad \psi'(z_2) < \xi < \psi'(z_1), \quad \xi = (x - x_0)/t. \]

This wave is joined to the strong discontinuities \( \xi_1 = \psi'(z_1) \) and \( \xi_2 = \psi'(z_2) \) (see [15] for details).
3.1. A comparison with the Koval model

Let us verify kinematic-wave model (11) by means of comparing with the well-approved Koval model (7). The effective viscosity ratio $M_e$ given by formula (8) is used instead of $M$. We choose the following parameters: $\mu_1 = 2$, $\mu_2 = 8$. It means that the viscosity ratio $M = 4$ and $M_e = 1.417$ according to (8). Self-similar solution of the Koval model for these parameters is shown in Figure 2 (solid curve). Corresponding solution of the scalar conservation law (11) (with $\delta = 1.8$) is presented in Figure 2 by the dashed curve. As we can see in the framework of these models the growth rates of viscous fingers are similar. Although the parameter $\delta$ is the function of $M$, the values of $\delta(M)$ vary weakly for $M \in (1, 8)$. Moreover, proposed model (11) describes the behavior of the finger near its tip better than the Koval model.

It should be noted that another kinematic-wave model has been recently obtained in [10] by taking into account the friction between the fluid layers of different viscosities. This effect is important when the finger become thin at the vicinity of the tip.

3.2. A comparison with 2D model (2)–(4)

We compare the results of numerical simulations of the viscous fingers formation based on two-dimensional hyperbolic system of equations (2)–(4) with the results obtained by using kinematic-wave model (11). The calculations are performed in the Hele-Shaw cell with sizes $20 \times 2$ in the coordinate system moving with the average flow velocity $U = 1$ with respect to the $x$-axis. The viscosities of the fluids are equal to $\mu_1 = 2$ and $\mu_2 = 8$. At the initial moment of time the interface $x = x_0 = 10$ is perturbed with respect to the formula $x = x_0 + 4\Delta_1(\exp(-10(y-1)^2) - 1/2)$ where $\Delta_1$ is the resolution of the uniform grid in $x$. For discretization with respect to $x$ and $y$ we use 400 and 50 nodes respectively. On the left and right boundaries of the computational domain the reflection conditions are imposed. We assume that in the framework of kinematic model (11) in the mixing layer the variable $c$ is equal to $1/2$. So, the viscosity in this layer is given by formula $\bar{\mu} = \sqrt{\mu_1 \mu_2}$ by virtue of the fact in two-dimensional model (2)–(4) the viscosity is $\mu(c) = \mu_1^{1-c} \mu_2^c$. The calculations are carried out at $\beta = 1$, $\rho_0 = 1$ and $c_0 = 75$. In this case the change in density is not more than 0.15%. At the same time the condition $p_y = 0$ is fulfilled with high accuracy. It means that the above proposed one-dimensional model is suitable for describing a such flow. The given perturbation of the interface leads to the formation of a single finger. It is symmetric with respect to the line $y = 1$. The results of the concentration $c$ calculations using model (2)–(4) at time $t = 10$ and $t = 20$ are shown in Figure 3.

![Figure 3](image)

Figure 3. The growth of viscous finger: comparison of the results for two-dimensional equations (2)–(4) and one-dimensional kinematic-wave model (11) with $\delta = 1.8$. 
Self-similar solutions of kinematic-wave model (11) with $\delta = 1.8$ is shown in Figure 3 at $t = 10$ and $t = 20$ by solid (the middle-line $1 \pm z$) and dashed (the outer $1 \pm h_1$ and inner $1 \pm (h_1 + \eta)$ boundaries of the finger) curves. The self-similar variable $\xi$ is replaced by $\xi + U$, $U = 1$. It corresponds to a transition into the moving coordinate system. As we can see, the velocity of viscous finger propagation and its transverse diffusion are in a good agreement with calculations provided by equations (2)--(4).

4. Conclusion
The main result of this paper is the construction of a new kinematic-wave model of viscous finger growth (11). The model is based on the assumption about an intermediate mixing layer formation between two layers of liquid with different viscosities and velocities. Relation (10) gives the thickness of the mixing layer by means of the fluid velocity in the outer layers. This formula is not derived in the paper and can be considered as an additional hypothesis. It allows describing the evolution of the viscous fingers by using Hopf-like equation. The proposed model (11) is verified with the help of the comparison with the well-tested Koval model (7) and (8). The results are also confirmed by the numerical simulation on the basis of 2D equations (2)--(4). Figure 3 shows that the velocity of the viscous finger propagation are the same for the both models.

Thus, the suggested 1D model predicts the propagation velocity of the viscous fingers to a sufficient degree of accuracy and do not require large processing power. The model also provides an additional information concerning to the transverse diffusion of viscous finger.

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