Unitary representations of three dimensional Lie groups revisited: An approach via harmonic functions

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October 30, 2014

Abstract

Harmonic functions of the three dimensional Lie groups defined on certain manifolds related to the Lie groups themselves and carrying all their unitary representations are explicitly constructed. The realisations of these Lie groups are shown to be related with each other by either natural operations as real forms or In"on"u-Wigner contractions.

1 Introduction

Since its first appearance in the frame of Classical Mechanics, three dimensional spherical functions have shown to constitute a valuable and elegant tool at both the mathematical and physical levels, leading to an extensive and profound theory with multiple ramifications in Geometry, Fourier analysis, the theory of special functions and differential equations\,[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In the physical context, spherical functions appear naturally in the construction of bases of eigenstates in the theory of angular momentum, providing therefore a powerful technique to describe accurately the representations of the rotation groups $SO(n)$. This approach has been central to the application of $SO(3)$ to Atomic and Nuclear physics, constituting the basis of the theory of spherical tensor operators and the Racah-Wigner algebraic formalism\,[11, 12]. The generalization of spherical functions to higher dimensions and indefinite metric tensors emerged naturally within the transition of non-relativistic to relativistic physics, the case of the Lorentz group $SO(1, 3)$ and the problem of induced/subduced representations with respect to its various subgroups being exceptionally important\,[2, 13, 14, 15, 16]. In this enlarged context, the theory of spherical harmonics establishes an equivalence between irreducible representations of the rotation groups $SO(N)$ acting on the sphere $S^{N-1}$ and eigenfunctions of the spherical Laplacian, while for pseudo-orthogonal groups $SO(p, q)$, a similar formalism has been developed, taking into account their action on hyperboloids (see e.g.\,[17] and references therein). Beyond the (pseudo-) orthogonal groups, and usually motivated by specific physical situations, generalized harmonic or hyper-spherical functions have been considered in the representation theory of other types of Lie groups, such as the group $SU(3)$ in the study of strong interactions\,[18, 19], the quantum mechanics of three bodies\,[20], or the noncompact Lie group $Sp(6, \mathbb{R})$ in the frame of the translationally invariant shell model\,[21]. Further applications of the method of harmonics in combination with the internal labeling problem can be found\,[e.g. in the nuclear collective model\,[22] or the construction of coherent states on simple Lie groups\,[23, 24, 25]. It may be mentioned that spherical harmonics, in combination with Clifford analysis, has recently been extended to integration problems in superspace\,[26].

In this work, we develop an alternative analysis of the representations of three dimensional semisimple Lie groups from the point of view of harmonic functions. To this extent, we consider the complex Lie group
$SL(2, \mathbb{C})$ and its real compact and split forms $SU(2)$ and $SL(2, \mathbb{R})$, respectively, constructing a realisation of these groups acting on the Lie group themselves. Moreover the latter are identified with what is sometimes called the parameter space of the Lie group. In particular we have the following identification

$$SL(2, \mathbb{C}) \cong S^3 = \left\{ z_1, z_2, z_3, z_4 \in \mathbb{C}, \ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1 \right\},$$

$$SU(2) \cong S^3 = \left\{ x_1, x_2, x_3, x_4 \in \mathbb{R}, \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\},$$

$$SL(2, \mathbb{R}) \cong \mathbb{H}_{2,2} = \left\{ x_1, x_2, x_3, x_4 \in \mathbb{R}, \ x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \right\}.$$  

(1.1)

Considering a complex vector space (respectively a manifold) $E$ (resp. $\mathcal{M}$) and a real vector space (resp. a real manifold) $F$ (resp. $\mathcal{N}$), we say that $F$ (resp. $\mathcal{N}$) is a real form of the vector space $E$ (resp. the manifold $\mathcal{M}$) if we have $E \cong F \otimes \mathbb{R}$ (resp. $\mathcal{M} \cong \mathcal{N} \otimes \mathbb{R}$ locally). In particular when the manifolds $\mathcal{M}$ and $\mathcal{N}$ have a structure of complex or real Lie group this definition coincides with the usual definition of real forms of Lie groups. With this terminology, we have $S^3 \cong S^3 \otimes \mathbb{R} \cong \mathbb{H}_{2,2} \otimes \mathbb{R}$ or at the level of the Lie group obviously, $SL(2, \mathbb{C}) \cong SU(2) \otimes \mathbb{R} \cong SL(2, \mathbb{R}) \otimes \mathbb{R}$. Thus by abuse of language we call $S^3$ the “complex unit three-sphere”. Endowing the manifolds (1.1) with a system of coordinates, the parameterisation turns out to be bijective if appropriate regions are removed (circles $S^1$ or cylinders $C_2$) leading to the following bijective parameterisations

$$S^3 \setminus \left( C_2 \times C_2 \right) \text{ for } SL(2, \mathbb{C}),$$

$$S^3 \setminus \left( S^1 \times S^1 \right) \text{ for } SU(2),$$

$$\mathbb{H}_{2,2} \setminus S^1 \text{ for } SL(2, \mathbb{C}).$$

Next these parameterisations enable us to obtain a differential realisation of the Lie algebras $\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$. These parametrisations have besides the very interesting property in the sense of real forms of vector spaces (manifolds) given above. Indeed, it will be shown that the differential realisations defined respectively for the Lie algebras $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$ are obtained from the differential realisation defined for $\mathfrak{sl}(2, \mathbb{C})$ considering appropriate real forms. Next, with these explicit differential realisation it is shown that all unitary irreducible representations of these Lie groups can be obtained, in unified manner, living on their corresponding manifolds (1.1). The case of $SL(2, \mathbb{R})$ is somewhat peculiar since it is known that the Lie group $SL(2, \mathbb{R})$ admits some ($p$-sheeted, universal) covering groups. Interestingly the differential realisation of $SL(2, \mathbb{R})$ extends to a suitable covering of $\mathbb{H}_{2,2} \setminus S^1$ in such a way that it acts naturally on unitary representations of the corresponding covering of $SL(2, \mathbb{R})$. Note that the unitary representations of $\mathfrak{sl}(2, \mathbb{C})$ can be seen as an extrapolation of the Gel’fand formulæ to $S^3 \setminus \left( C_2 \times C_2 \right)$.  

In addition, it is shown that the formalism can be enlarged to cover the unitary representations of the non-compact non-semisimple Euclidean group $E_2$ by means of appropriate contractions of realisations. It is known that the Lie group $E_2$ can be obtained by some contractions of $SU(1, 1)$ [27]. This relation extends also at the level of the parameter space. The parameter space of $E_2$ is $\mathbb{R}^2 \times [0, 2\pi]$ which is related to $[0, 2\pi] \times [0, 2\pi] \times \mathbb{R}^+$, and allows a parameterisation of the cone $C_{2,2}$ of equation

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0.$$  

The parameterisation becomes bijective on $C_{2,2} \setminus \{0\}$, i.e., on the cone with one point removed. Since $C_{2,2} \setminus \{0\}$ can be seen as a singular limit of $\mathbb{H}_{2,2} \setminus S^1$, all unitary representations of the Euclidean group $E_2$ are obtained by a contraction procedure of $SU(1, 1)$ representations as functions on $C_{2,2} \setminus \{0\}$. Such a process to obtain Lie groups with a semi-direct structure from semisimple Lie groups is known as confluence (see e.g. [28]). Note that in [29], realisations of the Lie groups $SO(3), SO(1, 2)$ and $E_2$ where obtained in unified manner for the description of the Landau quantum systems, albeit these realisations differ from our construction.

Finally, all the irreducible representations of the simple Lie groups obtained so far have the very interesting property of corresponding to harmonic functions on the appropriate spaces.
Considering a differentiable manifold \( \mathcal{M} \) endowed with a non-singular metric tensor \( g_{ij} \), the Laplacian is defined by

\[
\Delta = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \right),
\]

where \( g = \det(g_{ij}) \) and \( g^{ij} \) is the inverse of \( g_{ij} \). A harmonic is simply an eigenfunction of \( \Delta \). The fact that the representations of the three-dimensional (real and complex) Lie groups we will considered in this paper are harmonic on appropriate manifolds naturally means that the Laplacian is proportional to the Casimir operator (the case of \( E_2 \) is somehow particular, as we will see).

All our approach and relationships between the various Lie groups can be summarised in the following diagram

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2 The \( \mathfrak{sl}(2, \mathbb{C}) \) algebra

In this section we briefly review the main facts concerning the representation theory of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) and the theory of harmonic functions, and fix some of the notation that will be used in later paragraphs. For details on the construction the reader is referred to the main references [2, 8, 30] (see [31] for an English translation of [30]). An alternative construction of the representations presented in this paper can be found in [32, 33, 34].

2.1 Linear representations of \( \mathfrak{sl}(2, \mathbb{C}) \)

The real \( \mathfrak{sl}(2, \mathbb{C}) \) algebra is the six-dimensional Lie algebra generated by \( J_\pm, J_0, K_\pm, K_0 \) with non-vanishing commutation relations

\[
\begin{align*}
[J_0, J_+] &= J_+ , & [J_0, K_+] &= K_+ , & [K_0, K_+] &= -J_+ , \\
[J_0, J_-] &= -J_- , & [J_0, K_-] &= -K_- , & [K_0, K_-] &= J_- , \\
[J_+, J_-] &= 2J_0 , & [J_+, K_-] &= 2K_0 , & [K_+, K_-] &= -2J_0 , \\
[J_-, K_+] &= 2K_0 . & 
\end{align*}
\]

(2.1)

The two (quadratic) Casimir operators are given by

\[
\begin{align*}
Q_1 &= J_0^2 + \frac{1}{2} (J_+ J_- + J_- J_+) - K_0^2 - \frac{1}{2} (K_+ K_- + K_- K_+) , \\
Q_2 &= J_0 K_0 + \frac{1}{2} (J_+ K_- + J_- K_+) + K_0 J_0 + \frac{1}{2} (K_+ J_- + K_- J_+) .
\end{align*}
\]

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1The Lie group \( SL(2, \mathbb{C}) \) is the universal covering group of the Lorentz group \( SO(1, 3) \) generated by \( J_{\mu\nu} = -J_{\nu\mu}, 0 \leq \mu, \nu \leq 3 \) and satisfying

\[
[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma}) ,
\]

with \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). The relationship with the generators (2.1) is given by \( J_0 = J_{03}, J_+ = J_{23} + iJ_{31}, J_- = J_{23} - iJ_{31} \) (generators of rotations) and \( K_0 = J_{03}, K_+ = J_{01} + iJ_{02}, K_- = J_{01} - iJ_{02} \) (generators of Lorentz boosts).
Introducing the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ \sigma_+ = \sigma_1 \pm i \sigma_2, \quad z^1, z^2 \in \mathbb{C} \] and

\[ Z = (z^1 \ z^2), \quad \bar{Z} = \left( \bar{z}_1 \ \bar{z}_2 \right), \quad \partial_Z = \left( \frac{\partial}{\partial z_1} \ \frac{\partial}{\partial z_2} \right), \quad \partial_{\bar{Z}} = \left( \frac{\partial}{\partial \bar{z}_1} \ \frac{\partial}{\partial \bar{z}_2} \right), \]

a very convenient and efficient realisation can be given by

\[ \mathcal{J}_+ = Z \sigma_+ \partial_Z - \partial_Z \sigma_+ Z = z^1 \partial_2 - z_2 \partial_1, \]
\[ \mathcal{J}_- = Z \sigma_- \partial_Z - \partial_Z \sigma_- Z = z^2 \partial_1 - z_1 \partial_2, \]
\[ \mathcal{K}_+ = i (Z \sigma_+ \partial_Z + \partial_Z \sigma_+ \bar{Z}) = i (z^1 \partial_2 + z_2 \partial_1), \]
\[ \mathcal{K}_- = i (Z \sigma_- \partial_Z + \partial_Z \sigma_- \bar{Z}) = i (z^2 \partial_1 + z_1 \partial_2), \]
\[ \mathcal{K}_0 = \frac{i}{2} Z \sigma_0 \partial_Z + \frac{i}{2} \partial_Z \sigma_0 \bar{Z} = \frac{i}{2} (z^1 \partial_1 - z^2 \partial_2) + \frac{i}{2} (\bar{z}^1 \partial_1 - \bar{z}^2 \partial_2). \]

With these notations, representations of \( \mathfrak{sl}(2, \mathbb{C}) \) are given by the set of homogeneous functions in two complex variables [8]

\[ \mathcal{D}_{p,q} = \left\{ f \mid f(\lambda z^1, \lambda z^2, \lambda \bar{z}_1, \lambda \bar{z}_2) = \lambda^p \bar{\lambda}^q f(z^1, z^2, \bar{z}_1, \bar{z}_2) \right\}. \]

In order to avoid the monodromy problem, we have to assume that the condition \( p - q \in \mathbb{Z} \) holds [8].

The irreducible representations were originally obtained by Gel’fand (see e.g. [2, 35, 36, 37] and references therein), and are characterised by two numbers \( \ell_0, \ell_1 \), whereas the pair \([\ell_0, \ell_1]\) denotes the representation. Explicitly, they are given by the set of functions [38]

\[ \psi_{\ell_0, \ell_1}^{s,m}(Z, \bar{Z}) = A_{s, \ell_1}^{\ell_0, \ell_1} \sqrt{(2s + 1)(s + m)!(s - m)!(s + \ell_0)!(s - \ell_0)!} \times \]
\[ \times (Z\bar{Z})^{\ell_1 - s - 1} \sum_k \gamma_{s,m+\ell_0+k}(\bar{z}_1)^{s-m-k}(\bar{z}_2)^{s-\ell_0-k} (m+\ell_0+k)!k!(s-m-k)!(s-\ell_0-k)! , \]  

(2.2)

where

\[ A_{s, \ell_1}^{\ell_0, \ell_1} = \frac{\Gamma(s - \ell_1 + 1)\Gamma(\ell_0 + \ell_1 + 1)}{\Gamma(s + \ell_1 + 1)\Gamma((\ell_0 + \ell_1 + 1)}} = \frac{(s - \ell_1)\cdots(\ell_0 + \ell_1 + 1)}{(s + \ell_1)\cdots(\ell_0 + \ell_1 + 1)}. \]

In the sum (2.2), the index \( k \) is such that all powers are positive. This in particular means that

\[ \max(0, -\ell_0 - m) \leq k \leq \min(s - \ell_0, s - m). \]

Observe that the functions \( \psi_{\ell_0, \ell_1}^{s,m} \) belong to the space \( \mathcal{D}_{\ell_0 + \ell_1 - 1, -\ell_0 + \ell_1 - 1} \), thus we must have \( 2\ell_0 \in \mathbb{Z} \). Finally, \( s = |\ell_0|, |\ell_0| + 1, \cdots \) and \(-s \leq m \leq s \). It has been further proved that the following isomorphism of representations holds [8]

\[ [\ell_0, \ell_1] \cong [-\ell_0, -\ell_1]. \]

Since \( \ell_0 \in 2\mathbb{Z} \), we assume now that \( \ell_0 \geq 0 \). A representation of \( \mathfrak{sl}(2, \mathbb{C}) \) is then characterised by a positive half-integer number \( \ell_0 \) and a complex number \( \ell_1 \).
The action of the $\mathfrak{sl}(2, \mathbb{C})$ generators (2.1) gives [38]

$$
\begin{align*}
\mathcal{J}_+\psi_{l_0, \ell_1}^s(Z, \bar{Z}) &= \sqrt{(s-m)(s+m+1)}\psi_{l_0, \ell_1}^{s,m+1}(Z, \bar{Z}), \\
\mathcal{J}_-\psi_{l_0, \ell_1}^s(Z, \bar{Z}) &= \sqrt{(s+m)(s-m+1)}\psi_{l_0, \ell_1}^{s,m-1}(Z, \bar{Z}), \\
\mathcal{J}_0\psi_{l_0, \ell_1}^s(Z, \bar{Z}) &= m\psi_{l_0, \ell_1}^s(Z, \bar{Z}), \\
K_+\psi_{l_0, \ell_1}^s(Z, \bar{Z}) &= C_s\sqrt{(s-m)(s-m-1)}\psi_{l_0, \ell_1}^{s-1,m+1}(Z, \bar{Z}) - \frac{\ell_0\ell_1}{s(s+1)}\sqrt{(s-m)(s+m+1)}\psi_{l_0, \ell_1}^{s,m+1}(Z, \bar{Z}) \\
&\quad + C_{s+1}\sqrt{(s+m+1)(s+m+2)}\psi_{l_0, \ell_1}^{s+1,m+1}(Z, \bar{Z}), \\
K_-\psi_{l_0, \ell_1}^s(Z, \bar{Z}) &= -C_s\sqrt{(s+m)(s-m-1)}\psi_{l_0, \ell_1}^{s-1,m-1}(Z, \bar{Z}) - \frac{\ell_0\ell_1}{s(s+1)}\sqrt{(s+m)(s-m+1)}\psi_{l_0, \ell_1}^{s,m-1}(Z, \bar{Z}) \\
&\quad - C_{s+1}\sqrt{(s+m+1)(s+m+2)}\psi_{l_0, \ell_1}^{s+1,m-1}(Z, \bar{Z}), \\
K_0\psi_{l_0, \ell_1}^s(Z, \bar{Z}) &= C_s\sqrt{(s-m)(s+m)}\psi_{l_0, \ell_1}^{s-1,m}(Z, \bar{Z}) - \frac{\ell_0\ell_1}{s(s+1)}\psi_{l_0, \ell_1}^{s,m}(Z, \bar{Z}) \\
&\quad - C_{s+1}\sqrt{(s+m+1)(s+m+2)}\psi_{l_0, \ell_1}^{s+1,m}(Z, \bar{Z}),
\end{align*}
$$

where

$$
C_s = i\sqrt{(s^2 - \ell_0^2)(s^2 - \ell_1^2)} \sqrt{4s^2 - 1}.
$$

The Casimir operators are given respectively by

$$
\begin{align*}
Q_1 &= \ell_0^2 + \ell_1^2 - 1, \\
Q_2 &= -2\ell_0\ell_1.
\end{align*}
$$

We observe that for a given $s = \ell_0, \ldots, \ell_1$, the functions $\psi_{l_0, \ell_1}^{s,\ell_0,\ldots,\ell_1}$ span the spin-$s$ representation of the subalgebra $\mathfrak{su}(2) \subset \mathfrak{sl}(2, \mathbb{C})$. The representation $[\ell_0, \ell_1]$ is generally infinite dimensional. However, if both $\ell_0$ and $\ell_1$ are simultaneously half-integers and $\ell_1 \geq \ell_0 + 1$, the representation is finite dimensional and its spin content is given by $s = \ell_0, \ell_0 + 1, \ldots, \ell_1 - 1$. In this case we observe that $C_{\ell_0} = C_{\ell_1} = 0$, or $\ell_0 + \ell_1 - 1, q = -\ell_0 + \ell_1 - 1$ are both integers and positive. Moreover, the power of $ZZ$ in (2.2) ranges from $\ell_1 - 1$ to 0 and thus is always positive.

To make contact with more familiar notations, the finite dimensional representations can be rewritten as

$$
D_{p,q} = \left\{(z^1)^{s-m}(z^2)^m(z_2)^{q-n}(z_1)^n, 0 \leq m \leq p, 0 \leq n \leq q\right\},
$$

with $p = \ell_0 + \ell_1 - 1, q = \ell_1 - \ell_0 - 1$. In particular, $D_{1,0} \cong [1/2, 3/2]$ and $D_{0,1} \cong [-1/2, 3/2] \cong [1/2, -3/2]$ correspond respectively to left- or right-handed spinors.

Finally, it is known that the representation is unitary [2, 35, 36, 37] whenever one of the following conditions holds:

1. $\ell_0 \in \frac{1}{2}\mathbb{N}$ and $\ell_1 = i\sigma, \sigma \in \mathbb{R}$ (principal series);
2. $\ell_0 = 0$ and $0 < \ell_1 \leq 1$ (complementary series).

For the principal series, the Hilbert space is defined as follows: Replacing $z_1 \to z$ and $z_2 \to 1$ in (2.2), the Hilbert space turns out to coincide with the space of square integrable functions $L^2(\mathbb{C})$, where

$$
(f, g) = \frac{1}{\pi} \int_{\mathbb{R}^2} \overline{f(z)}g(z)dx\,dy,
$$

with $z = x + iy$. The functions $\Psi_{l_0, \ell_1}^{s,m}$ are orthogonal with respect to this scalar product [38]. The case of the complementary series is more involved, and explicit constructions can be found in [8, 30, 31].

### 2.2 Realisation of the representations of $\mathfrak{sl}(2, \mathbb{C})$ on $S^3_\mathbb{C} \setminus (C_2 \times C_2)$

The $SL(2, \mathbb{C})$–Lie group is the set of two-by-two unimodular complex matrices of determinant one

$$
U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad \alpha\delta - \beta\gamma = 1.
$$

If we set $\alpha = z_1 - iz_2, \beta = z_1 + iz_2, \delta = z_4 - iz_3, \gamma = z_4 + iz_3$, we observe that $SL(2, \mathbb{C})$ can be identified with the complex unit sphere $S^3_\mathbb{C}$. 

5
Recall that in the previous section we have briefly reviewed the Gel’fand construction of unitary representations of $\mathfrak{sl}(2, \mathbb{C})$ in terms of appropriate homogeneous functions on \( \mathbb{C}^4 \). It is therefore natural to ask whether the Gel’fand formulæ can be extended to the space \( S_\mathbb{C}^3 \). In other words, it is worthy to be analysed whether the $\mathfrak{sl}(2, \mathbb{C})$ representations obtained so far can be described by means of functions on \( S_\mathbb{C}^3 \). This is equivalent to finding a differential realisation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $SL(2, \mathbb{C})$ by a left action and determining left-invariant vector fields.

The complex unit three-sphere itself can obtained by complexification (in the sense given in the introduction) of the ordinary three-sphere, which can be parameterised by \( \vartheta \) leading to the identity

$$z_\pm = \cos \Theta, \quad z_0 = \sin \Theta, \quad \zeta_\pm = e^{i\Phi}.$$

the complex unit three-sphere \( S_\mathbb{C}^3 \subset \mathbb{C}^4 \) can be parameterised by

$$z_+ = z_0 z_+ = \cos(\Theta)e^{i\Phi_+},$$
$$z_- = z_0 z_- = \sin(\Theta)e^{i\Phi_-},$$

leading to the identity

$$z_+ \zeta_+ + z_- \zeta_- = 1.$$

Equation (2.6) implies in particular that \( z_+ \zeta_+ = \cos^2 \Theta, z_- \zeta_- = \sin^2 \Theta \) with \( \Theta \in \mathbb{C} \). Starting from \( z_+ = \zeta_+ = \cos \Theta \) and \( z_- = \zeta_- = \sin \Theta \), generic points can be obtained as follows:

$$z_\pm \rightarrow w_\pm z_\pm, \quad \zeta_\pm \rightarrow \frac{\zeta_\pm}{w_\pm},$$

with \( w_\pm \neq 0 \). Thus we can choose \( w_\pm = \zeta_\pm \) leading to the parameterisation (2.5). Finally, using the properties of the elementary trigonometric functions it is sufficient to assume

$$\left( \vartheta_0, \varphi_{\pm 0}, \varphi_{\pm 1}, \vartheta_1, \varphi_{\pm 1} \right) \in [0, \pi/2] \times [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi].$$

Thus, the parameterisation above covers \( S_\mathbb{C}^3 \). If we now remove the two cylinders \( C_2 \) from \( S_\mathbb{C}^3 \) defined by

$$\Theta = 0, \quad (\varphi_{\pm 0}, \varphi_{\pm 1}) \in [0, 2\pi],$$
$$\Theta = \pi/2, \quad (\varphi_{-0}, \varphi_{-1}) \in [0, 2\pi] \times [0, 2\pi].$$

and we denote \( I_\mathbb{R}^3 = \left\{ (\vartheta_0, \varphi_{\pm 0}, \varphi_{\pm 1}, \vartheta_1, \varphi_{\pm 1}) \in [0, \pi/2] \times [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] : s.t. (\vartheta_0, \vartheta_1) \neq (0, 0), (\pi/2, 0) \right\} \) we obtain a bijective map from \( I_\mathbb{R}^3 \) onto \( S_\mathbb{C}^3 \setminus \left( C_2 \times C_2 \right) \). We observe that the application from \( I_\mathbb{R}^3 \) to \( S_\mathbb{C}^3 \setminus \left( C_2 \times C_2 \right) \) is continuous, although the reciprocal application is obviously not continuous, for which reason the considered map does not constitute a homeomorphism.

A direct computation shows that the differential operators defined on \( S_\mathbb{C}^3 \setminus \left( C_2 \times C_2 \right) \)

$$L_+ = \frac{1}{4} e^{i(\varphi_{-0} - i\varphi_{-1} + \varphi_{-0} + i\varphi_{-1})} \left( \tan(\vartheta_0 + i\vartheta_1) \left( -i \varphi_{-0} - \varphi_{-1} \right) + \right)$$
$$\partial_\vartheta_0 - i \partial_\varphi_1 + \cot(\vartheta_0 + i\vartheta_1) \left( -i \varphi_{-0} - \varphi_{-1} \right) \right)$$

$$= \frac{1}{2} e^{i(\Phi_+ + \Phi_-)} \left( -i \tan \Theta \Phi_+ + \partial_\vartheta_0 - i \cot \Theta \partial_\varphi_- \right),$$

$$L_- = \frac{1}{4} e^{i(\varphi_{-0} + i\varphi_{-1} - \varphi_{-0} - i\varphi_{-1})} \left( \tan(\vartheta_0 + i\vartheta_1) \left( -i \varphi_{-0} - \varphi_{-1} \right) + \right)$$
$$\partial_\vartheta_0 + i \partial_\varphi_1 + \cot(\vartheta_0 + i\vartheta_1) \left( -i \varphi_{-0} - \varphi_{-1} \right) \right)$$

$$= \frac{1}{2} e^{i(\Phi_- + \Phi_+)} \left( -i \tan \Theta \Phi_+ + \partial_\vartheta_0 - i \cot \Theta \partial_\varphi_- \right),$$

$$L_0 = -i \left( \partial_{\varphi_{-0}} - i \partial_{\varphi_{-1}} - \partial_{\varphi_{-0}} - i \partial_{\varphi_{-1}} \right) = -\frac{i}{2} \left( \partial_{\Phi_+} - \partial_{\Phi_-} \right),$$
Within this realisation, it turns out that the spinor representations are given by
\[ \psi \]
and
\[ \bar{\psi} \]
for the left-handed spinors, whereas for the right-handed spinors we obtain
\[ S \]
We now define a scalar product on \( C_\ell \) s, m
\[ \sum (\) commutation relations
\[ \partial \bar{\Phi} \]
\[ L_0 = -i (\partial_{\psi^+} + i \partial_{\psi^-} - \partial_{\psi^-} - i \partial_{\psi^-}) = -i \left( \partial_{\psi^+} - \partial_{\psi^-} \right), \]

satisfy the \( \mathfrak{sl}(2, \mathbb{C}) \) commutation relations
\[ [L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_0, \quad [\bar{L}_0, L_\pm] = \pm \bar{L}_\pm, \quad [\bar{L}_+, \bar{L}_-] = 2\bar{L}_0, \quad [L_0, \bar{L}_0] = 0. \]

In order to make contact with the notations considered previously, we observe that
\[ J_\pm = L_\pm + \bar{L}_\pm, \quad J_0 = L_0 + \bar{L}_0, \quad K_\pm = -i(L_\pm - \bar{L}_\pm), \quad K_0 = -i(L_0 - \bar{L}_0). \]

Within this realisation, it turns out that the spinor representations are given by
\[ D_{\pm,0} = \{ \{ z_-, \bar{z}_+ \} \} \cong \{ \bar{z}_+, z_- \}, \]
\[ D_{0,1} = \{ \{ \bar{z}_+, \bar{z}_1 \} \} \cong \{ z'_+, \bar{z}'_- \}. \]

In particular, the action is given by
\[ J_+ z_+ = 0, \quad J_+ z_- = z_+, \quad J_+ z'_+ = -z'_-, \quad J_+ z'_- = 0, \]
\[ J_- z_+ = \bar{z}_-, \quad J_- z_- = 0, \quad J_- z'_+ = -z'_-, \quad J_- z'_- = z'_-, \]
\[ J_0 z_+ = \frac{1}{2} z_+, \quad J_0 z_- = -\frac{1}{2} z_-, \quad J_0 z'_+ = -\frac{1}{2} z'_+, \quad J_0 z'_- = \frac{1}{2} z'_-. \]

for the left-handed spinors, whereas for the right-handed spinors we obtain
\[ \bar{J}_+ z'_+ = 0, \quad \bar{J}_+ z'_- = z'_+, \quad \bar{J}_+ z'_+ = -\bar{z}'-, \quad \bar{J}_+ z'_- = 0, \]
\[ \bar{J}_- z'_+ = \bar{z}'-, \quad \bar{J}_- z'_- = 0, \quad \bar{J}_- z'_+ = -\bar{z}'-, \quad \bar{J}_- z'_- = \bar{z}'+, \]
\[ \bar{J}_0 z'_+ = \frac{1}{2} z'_+, \quad \bar{J}_0 z'_- = -\frac{1}{2} z'_-, \quad \bar{J}_0 z'_+ = -\frac{1}{2} z'_+, \quad \bar{J}_0 z'_- = \frac{1}{2} z'_-. \]

This explicitly shows that the complex unit three-sphere (2.6) is invariant under the action of the generators of \( \mathfrak{sl}(2, \mathbb{C}) \), hence we are able to extend safely the unitary representations to \( \mathbb{S}^3 \) \( \mathbb{C} \times \mathbb{C} \).

We now define a scalar product on \( \mathbb{S}^3 \) \( \mathbb{C} \times \mathbb{C} \) according to the following prescription:
\[ (f, g) = \frac{1}{(2\pi)^2} \int_{\mathbb{S}^3} \cos \Theta \sin \Theta \cos \Theta \sin \Theta d\theta_0 d\varphi_0 d\varphi_1 d\varphi_0 d\varphi_{+1} d\varphi_{-0} d\varphi_{-1} \]
\[ \times f(\theta_0, \varphi_0, \varphi_1, \varphi_{+1}, \varphi_{-0}, \varphi_{-1}) g(\theta_0, \varphi_0, \varphi_1, \varphi_{+1}, \varphi_{-0}, \varphi_{-1}). \]

It is important to observe that this scalar product is not suitable to be adapted to the \( \psi \)–functions, since in this case the integrals will generally be divergent (see e.g. [38] for a suitable scalar product for this case). This will however not constitute a serious constraint, as we will show that the corresponding scalar product adapted to the suitable real form of \( \mathbb{S}^3 \) \( \mathbb{C} \times \mathbb{C} \) will circumvent this difficulty.
2.3 Harmonics of $S^3 \setminus \left( C_2 \times C_2 \right)$

The differential realisation of sl(2, C) is of interest for several reasons. Firstly, the representations are obtained by functions living naturally on $S^3 \setminus \left( C_2 \times C_2 \right)$. In addition, the functions (2.11) have a further interesting property, namely that they are harmonic.

From the relation $z_+z'_+ + z_-z'_- = 1$ we get

$$d^2s = dz_+dz'_+ + dz_-dz'_- = d^2\Theta + \cos^2\Theta d^2\Phi_+ + \sin^2\Theta d^2\Phi_-, \quad d^2\bar{s} = d^2\bar{\Theta} + \cos^2\bar{\Theta} d^2\bar{\Phi}_+ + \sin^2\bar{\Theta} d^2\bar{\Phi}_-.$$

Introducing the metric tensor $g_{ij}$ together with its inverse $g^{ij}$ and $g = \det(g_{ij})$, we can define the Laplacian

$$\Delta = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}g^{ij} \partial_j) + \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}g^{ij} \partial_j) = \frac{1}{\cos \Theta \sin \Theta} \partial_\Theta (\cos \Theta \sin \Theta \partial_\Theta) + \frac{1}{\cos^2 \Theta} \partial_{\Phi_+}^2 + \frac{1}{\sin^2 \Theta} \partial_{\Phi_-}^2 + \frac{1}{\cos \Theta \sin \Theta} \partial_\bar{\Theta} (\cos \bar{\Theta} \sin \bar{\Theta} \partial_\bar{\Theta}) + \frac{1}{\cos^2 \bar{\Theta}} \partial_{\bar{\Phi}_+}^2 + \frac{1}{\sin^2 \bar{\Theta}} \partial_{\bar{\Phi}_-}^2.$$

A direct computation shows that the Laplacian reduces to a Casimir operator, i.e.

$$-\frac{1}{2} \Delta = L_0^2 + \frac{1}{2}(L_+ L_- + L_- L_+) + \bar{L}_0^2 + \frac{1}{2}(\bar{L}_+ \bar{L}_- + \bar{L}_- \bar{L}_+) = \frac{1}{2}[j_0^2 + \frac{1}{2}(j_+ j_- + j_- j_+)] - \frac{1}{2}[\bar{j}_0^2 + \frac{1}{2}(\bar{j}_+ \bar{j}_- + \bar{j}_- \bar{j}_+)].$$

Therefore, for a given monomial we have

$$\Lambda(z_+^{a} z'_+^{b} z_-^{c} z'_-^{d}) + (a + b)(a + b + 2) + (c + d)(c + d + 2) z_+^{a} z'_+^{b} z_-^{c} z'_-^{d} = 0 \ .$$

This implies that the functions $\psi_{l_0, l_1}^{s, m}$ are harmonic and satisfy the constraint

$$\Delta \psi_{l_0, l_1}^{s, m} + 2(l_0^2 + l_1^2 - 1) \psi_{l_0, l_1}^{s, m} = 0 \ .$$

3 The su(2) algebra

The su(2) algebra is the so-called compact real form of the sl(2, C) algebra, where the latter is considered now as a complex three-dimensional algebra. The results of the previous section can be directly applied to obtain a differential realisation of su(2) on the Lie group SU(2) itself, that is, the three-sphere $S^3$, corresponding to a real form of the complex unit three-sphere $S^3_{\mathbb{C}}$ (corresponding to a real form of the Lie group SL(2, C)).

3.1 Unitary representations of su(2) on $S^3 \setminus S^1$

The SU(2)–Lie group is the set of two-by-to unitary complex matrices of determinant one

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \ \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1 \ .$$

If we consider the real form of $S^3_{\mathbb{C}}$ parametrised by

$$\Theta = \theta, \ \Phi_+ = \varphi_+ = \varphi_+ , \ \Phi_- = \varphi_- = \varphi_- ,$$

in the formulæ of Section 2.2 we obtain

$$z_+, z'_+ \rightarrow w_+ = \cos \theta e^{i\varphi_+}, \ z_-, z'_- \rightarrow w_- = \sin \theta e^{i\varphi_-}, \ 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi_+ < 2\pi ,$$

from which we derive the identity

$$|w_+|^2 + |w_-|^2 = 1 .$$
Note that the angle $\theta$ above must belong to the interval $[0, \pi/2]$ if the moduli of $w_+$ and $w_-$ are positive. Thus $w_+$ and $w_-$ naturally parameterise the three-sphere. Now, removing the two circles from $S^3$ defined by

$$\theta = 0, \quad \phi_+ \in [0, 2\pi],$$
$$\theta = \frac{\pi}{2}, \quad \phi_- \in [0, 2\pi],$$

we have a bijection between $C_3 = [0, \pi/2] \times [0, 2\pi] \times [0, 2\pi]$ and $S^3 \setminus \{S^1 \times S^1\}$ such that the direct application is continuous whereas the reciprocal application is not. Note also that the manifold $S^3 \setminus \{S^1 \times S^1\}$ is a real form of the manifold $S^3 \setminus \{C_2 \times C_2\}$ in the sense given in the introduction.

With this parameterisation the matrix elements of $U$ in (3.1) become $\alpha = \cos \theta e^{i\varphi_+}$ and $\beta = \sin \theta e^{i\varphi_-}$. It is clear from this construction that our parameterisation of $SU(2)$ differs from the usual one in terms of the Euler angles [39, 40].

In the same real form, the generators of the $su(2)$–algebra are simply obtained, substituting (3.2) into (2.7) and (2.8) (i.e. $L_\pm = L_\pm \rightarrow R_\pm, L_0 = L_0 \rightarrow R_0$)

$$R_+ = \frac{1}{2} e^{i(\phi_- - \phi_+)} \left( -i \tan \theta \partial_{\phi_+} + \partial_\theta - i \cot \theta \partial_{\phi_-} \right),$$
$$R_- = \frac{1}{2} e^{i(\phi_+ - \phi_-)} \left( -i \tan \theta \partial_{\phi_+} - \partial_\theta - i \cot \theta \partial_{\phi_-} \right),$$
$$R_0 = \frac{-i}{2} \left( \partial_{\phi_+} - \partial_{\phi_-} \right),$$

and satisfying

$$[R_0, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = 2R_0 \quad (3.3)$$

The spinor representation is given by

$$D_\varphi = \{w_+, w_-\} \cong \{\bar{w}_-, w_+\},$$

whereas the finite dimensional representations are determined by

$$D_{\ell} = \left\{ \Phi_{\ell,m} = \sqrt{\frac{(2\ell + 1)!}{(\ell + m)!(\ell - m)!}} w_+^{\ell+m} w_-^{\ell-m} \cos^{\ell+m} \theta \sin^{\ell-m} \theta, \ell - m \le m \le \ell \right\},$$

with $\ell \in \mathbb{N}$. We have the action

$$R_+ \Phi_{\ell,m} = \sqrt{(\ell - m)(\ell + m + 1)} \Phi_{\ell,m+1},$$
$$R_- \Phi_{\ell,m} = \sqrt{(\ell + m)(\ell - m + 1)} \Phi_{\ell,m-1},$$
$$R_0 \Phi_{\ell,m} = m \Phi_{\ell,m}.$$

Introducing the $SU(2)$–invariant scalar product on the three-sphere (or more precisely on $S^3 \setminus \{S^1 \times S^1\}$), which can be naturally obtained from the scalar product (2.11) using the corresponding real form [10]).

$$\langle f, g \rangle = \frac{1}{2\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} \sin \theta d\phi_+ \int_0^{2\pi} \sin \theta d\phi_- g(\theta, \phi_+, \phi_-) f(\theta, \phi_+, \phi_-),$$

the orthogonality relation

$$\langle \Phi_{\ell,m}, \Phi_{\ell',m'} \rangle = \delta_{\ell\ell'} \delta_{mm'},$$

can be easily shown, as well as the fact that the operators satisfy $R^*_{\pm} = R_\pm, R_0 = R_0$ with respect to the scalar product. As a consequence, the representation $D_{\ell}$ is unitary. We emphasise again that the Haar measure of $SU(2)$ considered here is different from the usual Haar measure, as follows from the choice of parameterisation taken above.
3.2 Harmonics on $S^3 \setminus S^1$

For the considered real form, the previous metric becomes
\[ d^2 s = d^2 \theta + \cos^2 \theta \, d^2 \varphi_+ + \sin^2 \theta \, d^2 \varphi_-, \]
while the Laplacian on $S^3 \setminus S^1$ reduces to
\[ \Delta = \frac{1}{\cos \theta \sin \theta} \partial_\theta (\cos \theta \sin \theta \partial_\theta) + \frac{1}{\cos^2 \theta} \partial^2 \varphi_+ + \frac{1}{\sin^2 \theta} \partial^2 \varphi_- . \]

A routine computation shows that the Laplacian is related to the Casimir operator by
\[ \Delta = -4(J_0^2 + \frac{1}{2}(J_+ J_- + J_- J_+)) = 4Q . \]

It follows at once that the functions $\Phi_{\ell,m}$ are harmonic:
\[ \Delta \Phi_{\ell,m} + 4\ell(\ell+1)\Phi_{\ell,m} = 0. \]

4 The $sl(2, \mathbb{R})$ algebra

The group $SL(2, \mathbb{R})$ is the group of two-by-two real unimodular matrices. By means of the unitary matrix
\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \tag{4.1} \]
it can be easily shown that this group is isomorphic to the group $SU(1, 1)$ defined by
\[ U = \begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1 . \]

The isomorphism is given by the relation (see e.g. [23])
\[ SU(1, 1) = A \, SL(2, \mathbb{R}) \, A^\dagger . \tag{4.2} \]

It turns out that the Lie group $SU(1, 1)$ can be identified as topological space with the hyperboloid $H_{2,2}$ of signature $(2, 2)$.

We denote by $J_{\pm}, J_0$ the $sl(2, \mathbb{R}) \ni su(1, 1)$-generators that satisfy the commutators
\[ [J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_0 . \]

Over this basis, the Casimir operator is given by
\[ Q = J_0^2 - \frac{1}{2}(J_+ J_- + J_- J_+) . \]

4.1 Unitary representations

Since $SL(2, \mathbb{R})$ is a non-compact group, its unitary representations are infinite dimensional. They have largely been studied by different authors, and classifications can be found in [6, 8, 9, 41, 42]. Representations of the first type are either bounded from below or bounded from above, corresponding to the so-called discrete series of Bargmann [6].
where \( s \in \mathbb{R} \). It can be shown that these representations are finite-dimensional whenever \( s \) is a negative half-integer number. Since \( n \in \mathbb{N} \), the representation \( D^+_{\lambda} \) is bounded from below and the representation \( D^-_{\lambda} \) bounded from above.

If we perform the substitution \( J_0 \to -J_0, J_\pm \to -J_\pm \), it can be shown that these two representations are isomorphic. In addition, they are unitary and can be exponentiated when \( s > 0 \).

Representations of the second type are neither bounded from below nor bounded from above but unbounded. They correspond to the continuous series

\[
D_{\lambda, \mu} \left\{ \begin{array}{ll}
J_+ \lambda, \mu, n \rangle = \sqrt{(2\mu + n + 1)(n - 2\lambda)} \lambda, \mu, n + 1 \rangle, \\
J_- \lambda, \mu, n \rangle = \sqrt{(2\mu + n)(n - 1 - 2\lambda)} \lambda, \mu, n - 1 \rangle, \\
J_0 \lambda, \mu, n \rangle = (n - \lambda + \mu) \lambda, \mu, n - 1 \rangle, \\
Q \lambda, \mu, n \rangle = (\lambda + \mu)(\lambda + \mu + 1) \lambda, \mu, n - 1 \rangle,
\end{array} \right.
\]

with \( \mu, \nu \in \mathbb{C} \) and \( n \in \mathbb{Z} \). The representations are unitary whenever all operators are Hermitian. As a consequence, the eigenvalues of \( J_0 \) are real, and \( \mu - \lambda \in \mathbb{R} \). If we set \( \lambda + \mu = \Phi_0 + i\Phi_1 \), the Casimir operator reduces to

\[
Q = \Phi_1^2 + \Phi_1 - \Phi_2^2 + i\Phi_2(2\Phi_1 + 1).
\]

We observe that the representations \( D_{\mu, \nu} \) and \( D_{(\mu + 1/2, (\nu + 1/2)} \) are isomorphic, thus we can restrict ourselves to the case \(-1/2 < \mu - \lambda \leq 1/2 \). Two continuous representations must be distinguished:

- the continuous principal series

\[
-1/2 < \mu - \lambda \leq 1/2, \quad \Phi_1 = -\frac{1}{2} \text{ and } \Phi_2 = \sigma > 0,
\]

with

\[
Q = -\frac{1}{4} - \sigma^2 < -\frac{1}{4}.
\]

- the continuous supplementary series

\[
-1/2 < \mu - \lambda \leq 1/2, \quad \Phi_2 = 0, \quad \Phi_1 = \lambda + \mu \in \mathbb{R}, \quad \text{with } |\lambda + \mu + 1/2| < 1/2 - |\lambda - \mu|.
\]

Observe that, in contrast to the \( \mathfrak{sl}(2, \mathbb{C}) \) or \( \mathfrak{su}(2) \) Lie algebras, the spin can adopt an arbitrary real value. This is due to the fact that the first homotopy group of \( SU(1, 1) \) is isomorphic to \( \mathbb{Z} \), as well as that the representations considered are defined on some appropriate covering group of \( SU(1, 1) \) (see e.g. [43] for details).

### 4.2 Realisation of Unitary representations of \( \mathfrak{sl}(2, \mathbb{R}) \) on \( \mathbb{H}_{2,2} \setminus S^1 \)

The Lie algebra \( \mathfrak{su}(1, 1) \) is a real form of the three-dimensional complex Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). Correspondingly, we now show that all its unitary representations can be obtained on a appropriate covering of a real form of \( S^3 \setminus (C_2 \times C_2) \).

As observed before, we can identify the Lie group \( SU(1, 1) \) with the hyperboloid \( \mathbb{H}_{2,2} \) of signature \((2, 2)\), which is a real form of the Lie group \( SL(2, \mathbb{C}) \), i.e., the unit complex three-sphere \( S^3 \) that we parameterise using

\[
\Theta = i\theta_1 = i\rho, \quad \Phi_+ = \varphi_{+0} = \varphi_+, \quad \Phi_- = \varphi_{-0} = \varphi_-, 
\]

in the formul\'e of Section 2.2, the complex numbers \( z_\pm \) and \( z'_\pm \) reducing to (we have multiplied \( \zeta_- \) by \(-i\) for convenience)

\[
\zeta_+ = \cosh \rho e^{i\varphi_+}, \quad \zeta_- = \sinh \rho e^{i\varphi_-}, \quad \rho \in \mathbb{R}_+, \quad 0 \leq \varphi_+ < 2\pi,
\]

which obviously satisfy

\[
|\zeta_+|^2 - |\zeta_-|^2 = 1.
\]

The variables \( \zeta_\pm \) clearly parameterise the hyperboloid \( \mathbb{H}_{2,2} \). If we now remove one circle from \( \mathbb{H}_{2,2} \)

\[
\rho = 0, \quad \varphi_+ \in [0, 2\pi],
\]

we obtain a bijective map between \( \mathcal{T}_{2,2} = [0, 2\pi] \times [0, 2\pi] \times \mathbb{R}_+ \) and \( \mathbb{H}_{2,2} \setminus S^1 \) such that the direct application is continuous, the reciprocal application being not continuous. Note also that the manifold \( \mathbb{H}_{2,2} \setminus S^1 \) is a real form of
the manifold $S^2 \setminus \{C_2 \times C_2\}$ in the sense of the introduction. Now it can be shown that the manifold $]1, +\infty[ \times S^1 \times S^1$ is homeomorphic to $\mathbb{H}_{2,2} \setminus S^1$. Indeed,

$$f : ]1, +\infty[ \times S^1 \times S^1 \to \mathbb{H}_{2,2} \setminus S^1$$

$$(r, u_+, u_-) \mapsto \left\{ \begin{array}{l}
\zeta_+ = ru_+ \\
\zeta_- = \sqrt{r^2 - 1}u_-
\end{array} \right.$$

is clearly bijective. Recalling that $\mathbb{H}_{2,2} \setminus S^1$ is given by $|\zeta_+|^2 - |\zeta_-|^2 = 1$ with $|\zeta_+| > 1$, the reciprocal application is defined by

$$f^{-1} : \mathbb{H}_{2,2} \setminus S^1 \to ]1, +\infty[ \times S^1 \times S^1$$

$$(\zeta_+, \zeta_-) \mapsto \left\{ \begin{array}{l}
t = |z_+| \\
u_+ = \frac{\zeta_+}{\sqrt{|\zeta_+|^2 - 1}} \\
u_- = \frac{\zeta_-}{\sqrt{|\zeta_-|^2 - 1}}
\end{array} \right.$$

It is obvious to show that $f$ and $f^{-1}$ are continuous. Since $]1, +\infty[$ is contractible, $\pi_1([1, +\infty[ \times S^1 \times S^1) = \pi_1(S^1 \times S^1) = Z \times Z$ and we have $\pi_1(\mathbb{H}_{2,2} \setminus S^1) = Z \times Z$. We are thus able to consider appropriate covering of $\mathbb{H}_{2,2} \setminus S^1$ which are defined by their parameterisation:

- the $(p_+, p_-)$-sheeted covering $\mathbb{H}_{2,2} \setminus S^1$ parameterised by $\rho \in \mathbb{R}_+^*$, $0 \leq \varphi_+ < 2p_+ \pi$
- the $(\infty, p_-)$-sheeted covering $\mathbb{H}_{2,2} \setminus S^1$ parameterised by $\rho \in \mathbb{R}_+^*$, $\varphi_+ \in \mathbb{R}$, $0 \leq \varphi_- < 2p_- \pi$
- the $(p_+, \infty)$-sheeted covering $\mathbb{H}_{2,2} \setminus S^1$ parameterised by $\rho \in \mathbb{R}_+^*$, $\varphi_- \in \mathbb{R}$, $0 \leq \varphi_+ < 2p_+ \pi$
- the $(\infty, \infty)$-sheeted covering $\mathbb{H}_{2,2} \setminus S^1$ parameterised by $\rho \in \mathbb{R}_+^*$, $\varphi_+, \varphi_- \in \mathbb{R}$.

Next, in the same manner as for $\mathfrak{su}(2)$, the $\mathfrak{sl}(2, \mathbb{C})$-generators reduce to

$$J_+ = \frac{1}{2} e^{(\varphi_+ - \varphi_-)} \left( - i \tanh(\rho) \partial_{\varphi_+} - \partial_{\rho} + i \coth(\rho) \partial_{\varphi_-} \right),$$
$$J_- = \frac{1}{2} e^{(\varphi_- - \varphi_+)} \left( - i \tanh(\rho) \partial_{\varphi_-} + \partial_{\rho} + i \coth(\rho) \partial_{\varphi_+} \right),$$
$$J_0 = -\frac{i}{2} \left( \partial_{\varphi_+} - \partial_{\varphi_-} \right),$$

leading to

$$\left[ J_0, J_\pm \right] = \pm J_\pm, \quad \left[ J_+, J_- \right] = -2J_0.$$  

Note that in order to reproduce the usual commutation relation, such that for unitary representations we have $J_\pm^\dagger = J_\mp$, in the substitution above we have multiplied $J_\pm$ by a factor $-i$ (see below).

The Casimir operator is given by

$$Q = J_0^2 - \frac{1}{2} \left( J_+ J_- + J_- J_+ \right).$$

Interestingly, the differential realisation (4.8) which defines a left action of $\mathfrak{sl}(2, \mathbb{R})$ on $\mathbb{H}_{2,2} \setminus S^1 \subset SL(2, \mathbb{R})$ extends on an appropriate covering of $\mathbb{H}_{2,2} \setminus S^1$. Furthermore, these definitions enable us to obtain explicit realisations of unitary representations of either the $p$-sheeted covering or the universal covering of $SL(2, \mathbb{R})$.

Since the spinor representation given by

$$\mathcal{D}_\pm = \{ \zeta_+, \zeta_- \} \cong \{ \zeta_+, \zeta_- \},$$

lives on $\mathbb{H}_{2,2} \setminus S^1$, all unitary representations can be defined on $\mathbb{H}_{2,2} \setminus S^1$ (or one of its coverings). We can in principle obtain all representations of $\mathfrak{su}(1, 1)$ with either $\{ \zeta_+, \zeta_- \}$ or $\{ \zeta_-, \zeta_+ \}$ but, depending on the representation considered, only one choice would be consistent with the scalar product on $\mathbb{H}_{2,2} \setminus S^1$ defined below. The unitary representation of Section 4.1 can be naturally defined on $\mathbb{H}_{2,2} \setminus S^1$ (eventually on some $p$-sheeted covering or even on its universal covering).
For the discrete series bounded from below we define

\[ D^+_s = \left\{ \Psi^+_{s,n} = \sqrt{\frac{2\Gamma(n+2s)}{(1+2s)\Gamma(n+1)}} \zeta_+^{-2s-n} \xi_+^{-n} \right\}, \]

while for the discrete series bounded from above

\[ D^-_s = \left\{ \Psi^-_{s,n} = \sqrt{\frac{2\Gamma(n+2s)}{(1+2s)\Gamma(n+1)}} \zeta_-^{-2s-n} \xi_-^{-n} \right\}, \]

with \( s > 0 \). Unbounded representations and the continuous series are defined by

\[ D_{\lambda,\mu} = \left\{ \Psi_{\lambda,\mu,n} = \sqrt{\frac{2\Gamma(-2\lambda+n)}{(2\mu+n+1)\Gamma(-2\mu-2\lambda-1)}} \zeta_+^{2\lambda-n} \xi_-^{2\mu+n} \right\}. \]

Note that we could have defined the continuous series with \( \zeta_\pm \) instead of \( \zeta_\pm \). Both possibilities however lead to identical conclusions. Unitarity is ensured if the parameters \( \lambda, \mu \) are given by (4.5) or (4.6). If \( s, \lambda, \mu \) are not integers, we have to define the \( \Psi \)-functions on a suitable covering space of \( \mathbb{H}_{2,2} \setminus S^1 \). In particular, in order that the formulae above make sense, the representations must be defined on some covering of \( \mathbb{H}_{2,2} \setminus S^1 \). For the discrete series, if \( 2s = p/q \in \mathbb{Q} \), the representations are defined on the \( (q,1) \)-sheeted covering of \( \mathbb{H}_{2,2} \setminus S^1 \), whereas for \( s \in \mathbb{R} \setminus \mathbb{Q} \), the representations are defined on the \((\infty,1)\)-sheeted covering of \( \mathbb{H}_{2,2} \setminus S^1 \). For the continuous series, the representations are defined respectively on the \((p,p'), (\infty, p'), (p, \infty), (\infty, \infty)\)-sheeted covering of \( \mathbb{H}_{2,2} \setminus S^1 \), where \( \text{Re}(\mu), \text{Re}(\lambda) \) are respectively (rational, rational), (irrational, rational), (irrational, rational), (irrational, irrational), with \( \text{Re}(z) \) the real part of \( z \).

We can now introduce an \( SU(1,1) \)-invariant scalar product appropriate to the discrete series respectively on the \((q,1)\)-covering or the \((\infty,1)\)-covering of \( \mathbb{H}_{2,2} \setminus S^1 \) (which can be naturally obtained from the scalar product (2.11) using the corresponding real form)

\[
(f,g)_{(q,1)} = \frac{1}{r^{q+1}} \int_0^{+\infty} \cosh \rho \sinh \rho \, d\rho \int_0^{2\pi} d\varphi_+ \int_0^{2\pi} d\varphi_- \tilde{f}(\rho, \varphi_+, \varphi_-) g(\rho, \varphi_+, \varphi_-),
\]

\[
(f,g)_{(\infty,1)} = \frac{2}{r^{q+1}} \int_0^{+\infty} \cosh \rho \sinh \rho \, d\rho \int_0^{+\infty} d\varphi_+ \int_0^{2\pi} d\varphi_- \tilde{f}(\rho, \varphi_+, \varphi_-) g(\rho, \varphi_+, \varphi_-).
\]

For the discrete series, the integral involving the \( \Psi \)-functions can be computed easily when \( s \) is a rational number, first performing the change of variables \( \cosh \rho = r \) leading to integrals of the form

\[ I_{a,b} = \int_1^{+\infty} r^{2a+1} (r^2 - 1)^b \, dr. \]

The latter integrals are related to hypergeometric functions \([44]\), and one can show that \( I_{a,b} \) is convergent for \( a, b \in \mathbb{R} \) satisfying the inequalities \( a + b < -1, b > -1 \) (which automatically implies that \( -a > 0 \)):

\[ I_{a,b} = \frac{1}{2} \frac{\Gamma(1+b)\Gamma(-a-b-1)}{\Gamma(-a)}.
\]
Applying these results to the unitary representation of \( \mathfrak{sl}(2, \mathbb{R}) \) constructed on \( \mathbb{H}_{2,2} \setminus \mathbb{S}^1 \), we have the following results. For the discrete series bounded from above and below, the integrals converges if \( s > 1/2 \), leading to (for \( s, s' > 1/2 \) and \( s = p/(2q), s' = p'/\langle 2q' \rangle \in \mathbb{Q} \))

\[
(\Psi_{s,m}, \Psi_{s',m'})_{(q',1)} = \delta_{s,s'} \delta_{m,m'},
\]

with \( q' \) the least common multiple of \( q, q' \). Since \( J_{\pm}^1 = J_{\mp}, J_0^1 = J_0 \) for the scalar product, the representations are unitary if \( s > 1/2 \).

Whenever \( s \) or \( s' \) is an irrational number, still in the case of the discrete series, using the integral representation of the Dirac \( \delta \)–distribution implies the identity

\[
(\Psi_{s,m}, \Psi_{s',m'})_{(\infty,1)} = \delta_{s,s'} \delta(s - s') \delta_{m,m'}.
\]

If we proceed along the same lines as for the unbounded representation, i.e. defining a scalar product on an appropriate covering of \( \mathbb{H}_{2,2} \setminus \mathbb{S}^1 \), it turns out that the corresponding integrals diverge. This obstruction can be surmounted by considering an adapted scalar product, like that defined in [8, 35].

### 4.3 Harmonics of \( \mathbb{H}_{2,2} \setminus \mathbb{S}^1 \)

For the real form \( \mathbb{H}_{2,2} \setminus \mathbb{S}^1 \) of \( \mathcal{S}^3 \setminus (C_2 \times C_2) \) the metric becomes

\[
d^2 s = -d^2 \rho + \cosh^2 \rho \, d^2 \varphi_+ - \sinh^2 \rho \, d^2 \varphi_-.\]

A simple computation shows that the Laplacian is related to the Casimir operator

\[
\Delta = -\frac{1}{\cosh \rho \sinh \rho} \partial_\rho (\cosh \rho \sinh \rho \partial_\rho) + \frac{1}{\cosh^2 \rho} \partial^2_{\varphi_+} - \frac{1}{\sinh^2 \rho} \partial^2_{\varphi_-} = -4Q.
\]

In particular we have

\[
\Delta \Psi_{s,m}^\pm + 4s(s+1) \Psi_{s,m}^\pm = 0; \quad \Delta \Psi_{\lambda,m} = 4(\lambda + \mu)(\lambda + \mu + 1) \Psi_{\lambda,m} = 0,
\]

hence the \( \Psi \)–functions are harmonic.

### 5 The algebra of rotations-translations in two dimensions

In the previous sections we have realised all simple real or complex three-dimensional Lie algebras using the topological space underlying the corresponding Lie group. There is one more three-dimensional Lie algebra (with a semi-direct sum structure) which has interesting properties, namely the algebra of translations-rotations in two dimensions. We denote by \( E_2 \) the corresponding group and \( \mathfrak{e}_2 \) its Lie algebra. This group can be obtained, among other possibilities, by an Inönü-Wigner contraction of \( SO(3) \) [27, 45]. Indeed, if we define

\[
J = R_0, \quad P_\pm = \varepsilon R_\pm,
\]

and we take the limit when \( \varepsilon \) goes to zero, then (3.3) reduces to

\[
[J, P_\pm] = \pm P_\pm, \quad [P_+, P_-] = 0.
\]

This means that \( J \) is the generator of rotations and \( P_\pm \) of translations.

Since \( \pi_1(E_2) = \mathbb{Z} \) and \( E_2 \) is non-compact, it shares some properties with the group \( SL(2, \mathbb{R}) \): its unitary representations are infinite dimensional and the eigenvalues of \( J \) can be equal to any real number. More precisely, unitary representations are parameterised by two numbers \( p \in \mathbb{R}, -1/2 < s \leq 1/2 \) and are given by

\[
J[p, s, n] = (s + n)[p, s, n],
\]
\[
P_+[p, s, n] = ip[p, s, n + 1],
\]
\[
P_-[p, s, n] = -ip[p, s, n - 1].
\]

The representations are unbounded from below and above since \( n \in \mathbb{Z} \).
5.1 Representations of \( c_2 \) by contraction

Contractions of Lie algebra representations have been studied from a variety of points of view. In contrast to the contraction of structure tensors, limiting processes for representations are not straightforward, as they implicitly involve topological properties of the corresponding groups \([46]\). In this paragraph, we construct all unitary representations of \( c_2 \) as a contraction of \( \mathfrak{sl}(2, \mathbb{R}) \)-representations, via the realisations considered before.

Considering the \( \mathfrak{sl}(2, \mathbb{R}) \) algebra given by (4.8), and introducing \( r = \cosh \rho \) we get the differential operators

\[
i J_+ = \frac{i}{2} e^{(\varphi_+ - \varphi_-)} \left( -i \frac{\sqrt{r^2 - 1}}{r} \partial_{\varphi_+} - \sqrt{r^2 - 1} \partial_{\varphi_-} + i \frac{r}{\sqrt{r^2 - 1}} \partial_{\varphi_-} \right),
\]

\[
i J_- = \frac{i}{2} e^{(\varphi_- - \varphi_+)} \left( -i \frac{\sqrt{r^2 - 1}}{r} \partial_{\varphi_+} + \sqrt{r^2 - 1} \partial_{\varphi_-} + i \frac{r}{\sqrt{r^2 - 1}} \partial_{\varphi_-} \right),
\]

\[
J_0 = -i \left( \partial_{\varphi_+} - \partial_{\varphi_-} \right).
\]

Taking the limit for \( r \to +\infty \) we obtain the realisation

\[
i J_+^\infty = \frac{i}{2} e^{(\varphi_+ - \varphi_-)} \left( -i \partial_{\varphi_+} - r \partial_{\varphi_-} + i \partial_{\varphi_-} \right),
\]

\[
i J_-^\infty = \frac{i}{2} e^{(\varphi_- - \varphi_+)} \left( -i \partial_{\varphi_+} + r \partial_{\varphi_-} + i \partial_{\varphi_-} \right),
\]

\[
J_0^\infty = -i \left( \partial_{\varphi_+} - \partial_{\varphi_-} \right).
\]

It satisfies the \( \mathfrak{sl}(2, \mathbb{R}) \)-commutation relations. In the same limit \( \zeta_\pm \) reduces to

\[
r_+ = \zeta_+^\infty = re^{i\varphi_+}, \quad r_- = \zeta_-^\infty = re^{i\varphi_-}, \quad r \in \mathbb{R}_+, \quad \varphi_\pm \in [0, 2\pi[,
\]

which belongs to the spinor representation (pay attention to the fact that, due to the \( i \) factor in \( i J_\pm^\infty \), there is some overall \( i \) factor with respect to Section 4.2).

Now we forget that we have taken the limit \( r \to \infty \) and we define the contraction to be

\[
P_+ = -i \frac{1}{2} e^{(\varphi_+ - \varphi_-)} r \partial_{\varphi_+},
\]

\[
P_- = i \frac{1}{2} e^{(\varphi_- - \varphi_+)} r \partial_{\varphi_+},
\]

\[
J = -i \left( \partial_{\varphi_+} - \partial_{\varphi_-} \right).
\]

One can see easily that these generators generate the \( c_2 \) algebra and satisfy (5.1). The parameter space of \( E_2 \) allows a parameterisation of the cone \( C_2 \) since \([0, 2\pi[ \times \mathbb{R}^2 \sim [0, 2\pi[ \times \mathbb{R}_+ \times \mathbb{R}_-\]. Here \([0, 2\pi[ \times \mathbb{R}^2 \) corresponds respectively to the angle of rotation and the space translation, whereas \([0, 2\pi[ \times \mathbb{R}_+ \) allows to obtain a parameterisation of the cone. Indeed (5.3) leads to the equality

\[
|\rho_+|^2 - |\rho_-|^2 = 0.
\]

Denote \( \mathbb{I} \equiv [0, 2\pi[ \times [0, 2\pi[ \times \mathbb{R}_+ \). Now, if we remove the point \( r = 0 \) from the cone, we have a bijection from \( \mathbb{I} \) onto \( C_2 \setminus \{0\} \) such that the direct application is continuous and the reciprocal is not. Furthermore, since the manifold \( C_2 \setminus \{0\} \) is clearly homeomorphic to \( \mathbb{R}^*_+ \times S^1 \times S^1 \), the first homotopy group reduces to \( \mathbb{Z} \times \mathbb{Z} \). This means that one may consider covering spaces for \( C_2 \setminus \{0\} \) is a similar manner as we have considered covering spaces for \( \mathbb{H} \).

Finally, one can check that the cone is stable under the action of the generators of \( c_2 \) (5.4), and that \( r_\pm \) belongs to the parameter space of \( E_2 \) and thus parameterises \( C_2 \setminus \{0\} \). Of course \( (r_+, r_-) \) is not a representation of \( c_2 \), but interestingly all unitary representations of the Euclidean Lie algebra in two dimensions can be obtained in a simple way with \( r_+ \) and \( r_- \). If we set for \( -\frac{1}{2} < s \leq \frac{1}{2}, p \in \mathbb{R} \)

\[
D_{s,p} = \left\{ \Lambda_{p,s,n}(r, \varphi_+, \varphi_-) = \frac{1}{\sqrt{2\pi}} |r_+ r_-|^{-p-s-n} r_+^{2s+n} r_-^{-n} e^{i(2s+n) \varphi_+-i n \varphi_-}, \quad n \in \mathbb{Z} \right\},
\]

we have the action

\[
P_+ \Lambda_{p,s,n}(r, \varphi_+, \varphi_-) = -i p \Lambda_{p,s,n+1}(r, \varphi_+, \varphi_-),
\]

\[
P_- \Lambda_{p,s,n}(r, \varphi_+, \varphi_-) = i p \Lambda_{p,s,n-1}(r, \varphi_+, \varphi_-),
\]

\[
J \Lambda_{p,s,n}(r, \varphi_+, \varphi_-) = (s+n) \Lambda_{p,s,n}(r, \varphi_+, \varphi_-).
\]
The functions $\Lambda_{p,s,n}$ are defined on $C_{2,2} \setminus \{0\}$ when $s = 0,1/2$ or eventually on one of its coverings if $2s$ is not an integer number. More precisely, if $2s = t/q$, the functions are defined on the $(q,1)$–sheeted covering $C_{2,2} \setminus \{0\}$ (with the notations of the covering of $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$) parameterised by $r \in \mathbb{R}_+, 0 \leq \varphi_+ < 2q\pi, 0 \leq \varphi_- < 2\pi$, while for $s$ a real non-rational number, the functions are defined on the $(\infty,1)$–sheeted covering $(\mathbb{C}_{2,2} \setminus \{0\})$ parameterised by $r \in \mathbb{R}_+, \varphi_+ \in \mathbb{R}, 0 \leq \varphi_- < 2\pi$. We observe that isomorphic representations could be obtained with $|r,\varphi_-,\varphi_+|^2p^s-2s-2n\pi^2p^s+n$, where the role of $\varphi_+$ and $\varphi_-$ is permuted. We shall however not consider this possibility further in detail.

Now, with the change of variables $r = \cosh \rho$, the Laplacian (4.9) becomes

$$\Delta = -\frac{1}{r} \partial_r (r^2 - 1) \partial_r + \frac{1}{r^2} \partial_{\varphi_+} - \frac{1}{r^2 - 1} \partial_{\varphi_-},$$

and it reduces in the limit $r \to \infty$ to

$$\Delta_c = -\frac{1}{r} \partial_r (r^2 \partial_r).$$

It is straightforward to observe that the functions $\Lambda_{p,s,n}$ are eigenfunctions of $\Delta_c$

$$\Delta_c \Lambda_{p,s,n}(r, \varphi_+, \varphi_-) = -(2 + 2p) 2p \Lambda_{p,s,n}(r, \varphi_+, \varphi_-).$$

Note that for $\varepsilon_2$, the Casimir operator is given by

$$Q = \frac{1}{2} (P_+ P_- + P_- P_+) = \frac{1}{4} r \partial (r \partial_r),$$

and thus

$$Q \Lambda_{p,s,n}(r, \varphi_+, \varphi_-) = p^2 \Lambda_{p,s,n}(r, \varphi_+, \varphi_-).$$

As it can somehow be expected basing on the non-simplicity of the Euclidean Lie algebra, here $\Delta_c$ is not related to the Casimir operator of $\varepsilon_2$. This result can also be understood a posteriori because the cone $C_{2,2}$ is a singular limit of $\mathbb{H}_{2,2}$.

To define an invariant scalar product in this case is more involved, as the eigenvalues of the Casimir operator $Q$ are continuous hence the eigenfunctions $\Lambda_{p,s,n}$ cannot be normalised. However, as we now show using an appropriate change of parameterisation, we can still define an invariant scalar product. The first step in the construction is to define the Haar measure for $E_2$ corresponding to our parameterisation. Using that of the cone by the variables $z_\pm$ together with (5.4), we can show that under a translation of vector $(a \cos \theta, a \sin \theta)$ we have

$$\left(r, \varphi_+, \varphi_-\right) \rightarrow \left(ar \cos(-\theta + \varphi_+ - \varphi_- - \frac{\pi}{2}), \varphi_+ - \varphi_-, \varphi_-\right),$$

and therefore

$$d\mu(E_2) = \frac{1}{r} dr d\varphi_+ d\varphi_-,$$

is an $E_2$–invariant measure. Now performing a change of variables $r = e^\rho$ with $\Phi \in \mathbb{R}$, we obtain

$$P_+ = -\frac{i}{2} e^{i(\varphi_+ - \varphi_-)} \partial_\Phi, \quad P_- = \frac{i}{2} e^{i(\varphi_+ - \varphi_-)} \partial_\Phi, \quad J = -\frac{i}{2} (\partial_{\varphi_+} - \partial_{\varphi_-}),$$

$$\Lambda_{p,s,n}(\Phi, \varphi_+, \varphi_-) = e^{2i\Phi} e^{i(n+2s)\varphi_+ - in\varphi_-} \text{ and } d\mu(E_2) = d\Phi d\varphi_+ d\varphi_-.$$ This is however still not sufficient to define an appropriate scalar product. To proceed further we observe that the Lie algebra $\varepsilon_2$ can also be obtained by an Inönü-Wigner contraction of either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$. Note also that both algebras are real forms of the complex $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra. Formally, to go from one real form to the other we just have to perform the following substitution

$$J_{0}^{\mathfrak{sl}(2,\mathbb{R})} \rightarrow J_{0}^{\mathfrak{su}(2)} = \frac{1}{2} J_{0}^{\mathfrak{sl}(2,\mathbb{R})},$$

$$J_{\pm}^{\mathfrak{sl}(2,\mathbb{R})} \rightarrow J_{\pm}^{\mathfrak{su}(2)} = -i J_{\pm}^{\mathfrak{sl}(2,\mathbb{R})},$$

(5.6)
or some other equivalent transformation as those used in the theory of Angular Momentum [40]. In our procedure to construct $\varepsilon_2$ and its unitary representations we have considered the contraction of $\mathfrak{sl}(2, \mathbb{R})$. Going from $\mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{su}(2)$ through the substitution (5.6) can just be realised by $\Phi \rightarrow \Psi = -i\Phi$ with $\Psi \in \mathbb{R}$. With this substitution, the variables $r_\pm$ are now living on the manifold $S^1 \times S^1 \times S^1$. Thus, in the transition from $\Phi \in \mathbb{R}$ to $\Psi \in S^1$ we have substituted $\mathbb{R} \times S^1 \times S^1$ by $S^1 \times S^1 \times S^1$. Meaning that on a formal ground $S^1 \times S^1 \times S^1$ can be seen as the one point compactification of $\mathbb{R} \times S^1 \times S^1$ since the first circle is the one-point compactification of $\mathbb{R}$. Now since the latter manifold is compact a scalar product can be defined. The parameterisation $(\Psi, \varphi_+, \varphi_-) \in [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$ uniquely defines a point on $S^1 \times S^1 \times S^1$, the first homotopy group of which is isomorphic to $Z \times Z \times Z$. This leads to the new realisation of $\varepsilon_2$

$$P_+ = -\frac{1}{2}e^{i(\varphi_+ - \varphi_-)} \partial_\varphi,$$

$$P_- = \frac{1}{2}e^{i(\varphi_- - \varphi_+)} \partial_\varphi,$$

$$J = -i \left( \partial_{\varphi_+} - \partial_{\varphi_-} \right).$$

and of the representations $\mathcal{D}_{p,s}$

$$\Lambda_{p,s,n}(\Psi, \varphi_+, \varphi_-) = e^{i2p\Psi} e^{i(n+s)\varphi_+ - in\varphi_-}.$$

The representations of $E_2$ are defined on some covering of $S^1 \times S^1 \times S^1$ like before, namely $\tilde{S}^1 \times \tilde{S}^1 \times S^1$ or $S^1 \times S^1 \times S^1$, where $\tilde{S}^1$ (resp. $S^1$) is the $p$–sheeted (universal) covering of the circle $S^1$. These spaces are defined by their parameterisation. If $s = 0, 1/2$ the functions $\Lambda_{p,r/q,n}$ are parametrised by $\Psi \in \mathbb{R}, \varphi_+ \in [0, 2\pi], \varphi_- \in [0, 2\pi]$, if $2s = r/q$ by $\Psi \in \mathbb{R}, \varphi_+ \in [0, 2q\pi], \varphi_- \in [0, 2\pi]$ and if $s \in \mathbb{Q} \setminus \mathbb{Q}$ by $\Psi \in \mathbb{R}, \varphi_+ \in \mathbb{R}, \varphi_- \in [0, 2\pi]$.

Now, considering two functions $f$ and $g$ defined on some covering of $S^1 \times S^1 \times S^1$, we can endow the latter space with the scalar product

$$(f, g)_q = \frac{1}{q (2\pi)^2} \int_{-\infty}^{+\infty} d\Psi \int_0^{2\pi} d\varphi_+ \int_0^{2\pi} d\varphi_- f(\Psi, \varphi_+, \varphi_-) g(\Psi, \varphi_+, \varphi_-),$$

and

$$\langle f, g \rangle = \frac{2}{(2\pi)^2} \int_{-\infty}^{+\infty} d\Psi \int_{-\infty}^{+\infty} d\varphi_+ \int_0^{2\pi} d\varphi_- f(\Psi, \varphi_+, \varphi_-) g(\Psi, \varphi_+, \varphi_-).$$

respectively. Using the integral representation of the Dirac $\delta$–distribution we have the following situation: for $s, s' \in \mathbb{Q}$, i.e., $2s = r/q, 2s' = r'/q'$, and considering $q''$ the least common multiple of $q, q'$

$$(\Lambda_{p, s'', n}, \Lambda_{p, s, n})_{q''} = \delta_{n,n'} \delta_{s,s'} \delta(p - p').$$

If $s$ or $s'$ is a real number (but not a rational number)

$$(\Lambda_{p, s', n'}, \Lambda_{p, s, n})_{q''} = \delta_{n,n'} \delta(s' - s) \delta(p - p').$$

The functions $\Lambda_{p,s,n}$ are hence orthonormal, proving that the representations $\mathcal{D}_{p,s}$ are unitary.

6 Conclusions

In this paper the Gel'fand formulæ for the unitary representations of the $SL(2, \mathbb{C})$ Lie group have been extended in such a way that all unitary representations can be seen as harmonic functions on $S^1 \setminus \{C_2 \times C_2 \} \subset SL(2, \mathbb{C})$.

Considering appropriate real forms of $S^1 \setminus \{C_2 \times C_2 \}$ has further enabled us to extend the previous formulæ in such a way that they reduce to unitary representations of the real forms of $SL(2, \mathbb{C})$, say $SU(2)$ and $SU(1, 1)$. At the same time, the corresponding representations become harmonic functions on $S^1 \setminus \{S^1 \times S^1 \} \subset SU(2)$ or $\mathbb{H}_{2, 2} \subset SL(2, \mathbb{R})$ (or one of its covering space) respectively. In addition, considering a contraction of $SU(2)$ allows to derive all unitary representations of the Euclidean group $E_2$ on $C_2 \setminus \{0\}$ or its one-point compactification $S^1 \times S^1 \times S^1$. It is also important to emphasise that the Lie groups $SL(2, \mathbb{R})$ and $E_2$, being both infinitely connected, representations on
their $p$–sheeted covering can be defined (corresponding to a real spin). Interestingly, defining appropriate coverings of either $\mathbb{H}_{2,2} \setminus S^1$ or $\mathbb{C}_{2,2} \setminus \{0\}$, the representations of $p$–sheeted coverings can safely be defined.

The approach could be potentially useful for the analysis of the real forms of other compact Lie groups like $SU(N)$, as well as an alternative procedure to construct basis of eigenstates for physically relevant chains like $SU(N) \supset \cdots \supset SO(3)$ and the study of subduced representations in terms of harmonics, in analogy to the case studied in [22].

Let us finally mention that three-dimensional simple Lie groups are somehow exceptional, in the sense that all its representations can be constructed from the spinor representation(s). Within our construction, spinors are living on a suitable manifold related to Lie groups themselves, and since the spinors are harmonic, all unitary representations will be harmonic on the corresponding manifold. This property is no longer valid for higher dimensional Lie group. For instance, for $SU(3)$, all representations are obtained has harmonic functions on the five-sphere [18], which no longer coincides with a manifold related to the Lie group $SU(3)$, by considering the three-dimensional representation and its complex conjugate. Basing on the latter observation, a similar Ansatz could certainly be considered for Lie groups of rank higher than two and different from $SO(n)$, like $SU(n)$, $n > 3$ or $Sp(N)$, $N > 2$, but in this case, the computational difficulties will certainly prevent us from obtaining all the representations in closed form as harmonic functions on some appropriate manifold.

Acknowledgments

The authors express their gratitude to Marcus Slupinski for fruitful discussions and valuable suggestions that improved the manuscript. This work was partially supported by the research project MTM2010-18556 of the MICINN (Spain).

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