1 Introduction

It is our common household experience that when the voltage drop across a fuse exceeds a limit, the fuse burns out. A fuse is nothing but a conductor which conducts uniform current under an applied voltage up to a certain limit beyond which it burns out and becomes nonconducting. This is called fuse failure. Similarly in a dielectric breakdown, a dielectric starts to conduct electricity when the voltage drop across it attains certain threshold value. The above two phenomena are examples of breakdown process which is described broadly as the failure of a physical attribute when the perturbing force driving it goes beyond a limiting value. The most common example of the process is the breaking of a material at a high stress beyond its strength.

Naturally occurring solids are almost always inhomogeneous and have defects like vacancies, microcracks or impurities. These defects are weak points across which stress fields or electric fields or current densities concentrate. Depending on the geometry of the defect, the concentrated field can be very high. If the field or density exceeds the material fatigue limit, failure nucleates locally around the defect and starts to propagate. The propagation can be arrested, or can spread to the entire system leading to the global failure of the system, depending on the field strength and the defect structure in the material. The failure of a material, thus, depends largely on the disorder present in it.

The simplest form of the weak points is substitutional disorder which can be realised by the inclusion of nonconducting material in a conducting material.
or vice versa. Since we are interested in the macroscopic properties of global failure, we consider failures at length scale larger than the regions in which the local failure appears (i.e., atomic distances). At this scale, the disordered fuse system can be modelled by a lattice whose bonds are conducting with a certain probability $p$ and nonconducting otherwise.

For such models, one can apply the principles of percolation theory [45]. Below the percolation threshold $p_c$ (for $p < p_c$) the system is not connected globally by channels of conducting bonds and so nonconducting, whereas for $p > p_c$ the system is conducting (at least one continuous path exists across the system via the conducting bonds).

![Phase diagram of a mixture containing $p$ fraction of conductors and $(1 - p)$ fraction of insulators at random. On the left side, the sample is insulating for voltage $V$ less than the breakdown voltage $V_b(p)$ and conducting otherwise. On the right side, the system is conducting for current $I$ less than the fuse current $I_f(p)$ and insulating otherwise.]

Fig. 1. Phase diagram of a mixture containing $p$ fraction of conductors and $(1 - p)$ fraction of insulators at random. On the left side, the sample is insulating for voltage $V$ less than the breakdown voltage $V_b(p)$ and conducting otherwise. On the right side, the system is conducting for current $I$ less than the fuse current $I_f(p)$ and insulating otherwise.

For $p > p_c$, when conducting channels span across the system and the system is conducting, one can ask what is the fuse current $I_f$ required so that the system becomes nonconducting. At $I > I_f(p)$ not a single spanning path of conducting bonds exists. At $p = 1$ all the conducting bonds are present and $I_f(1)$ normalized by the sample size is simply the fuse threshold current for each of the conducting bonds. For $p < p_c$, the conducting bonds do not form a continuous path across the sample, and the sample is insulating. With increasing voltage $V$ across the sample, one can get a continuous path through original conductors and broken dielectric for $V \geq V_b(p)$, where $V_b(p)$ is the dielectric breakdown voltage. At $p = 0$, all the nonconducting bonds are there in the system and $V_b(0)$ normalized by the system size is simply the breakdown threshold of each of the nonconducting bonds. On the other hand, as $p \to p_c$, both the dielectric breakdown voltage and fuse failure current tend to zero. The corresponding phase diagram is shown in fig. 1.

In this article, we will review the basic ideas in breakdown problems in terms of the fuse failure and dielectric breakdown problems and then discuss how these ideas can be extended to the breakdown in quantum mechanical
systems such as Anderson insulators. In disordered electronic systems above two dimensions, the electronic states below the mobility edge are all localised (the system with Fermi level within this range behaves as an insulator) and the states above the mobility edge are extended (the system turns to a conductor for Fermi level in this range). The Anderson transition from insulating to conducting phase across the mobility edge is well studied [26]. We discuss here the possibility of a breakdown from insulating to conducting phase by applying strong electric field and compare the quantum breakdown with the classical dielectric or fuse breakdown in disordered materials.

2 Analysis of the fuse problem

2.1 Disordered fuse network

In a pure conductor placed between two electrodes with a potential difference, the field lines within the conductor are all parallel to each other and perpendicular to the electrode surfaces. In presence of a disorder in the form of an insulating region, the field lines get deformed around the defect. As a result in the vicinity of the defect current density increases to \( i_e \) from \( i \), the current density value far away from the defect. So one can write

\[
i_e = i(1 + k)
\]

where \( k \) is the enhancement factor which depends on the geometry of the defect. As an example, \( k \) becomes \( l/b \) for an elliptic defect of semi-major axis \( l \) and semi-minor axis \( b \) [12]. For electrode surface area \( S \) total current \( I \) is

\[
I = Si = \frac{Si_e}{1 + k}
\]

Failure occurs for the first time when \( i_e \) becomes equal to \( i_0 \) which is the fatigue limit of the sample material. The failure current is then

\[
I_f = \frac{Si_0}{1 + k}.
\]

Larger the enhancement factor \( k \), smaller is the failure current \( I_f \). For an example, if \( l >> b \) for an elliptic defect, \( I_f \) may get reduced by a large extent. So the presence of defects in the material facilitates failure and the presence of sharper edges of defects make the system more vulnerable to failure. The failure makes the defect bigger and hence \( k \) larger. This means that the current density around the defect enhances further causing failure again. The process causes rapid failure of the whole sample. This means that external voltage for the global failure is the same as the voltage for the first local failure and a local failures once started leads to the failure of the entire sample. This type of failures are called brittle failures.
So far we have discussed the influence of a single defect of a regular size within the sample. Natural and engineering samples usually contain a large number of defects of irregular shapes and sizes at random. To get some quantitative estimate of the failure criteria in terms of defect parameters we idealize the solid to lattice model and use percolation theory for defects [37, 36, 18].

We start with a hypercubic lattice in 2d or 3d with all conducting bonds. The simplest defect can be introduced by removing one bond parallel to the direction of the current flow. In this case, failure current is calculated as

\[ I_f = \frac{\pi}{4} L i_0 \]  

in 2d, where \( L i_0 \) is the fracture current of the lattice in absence of any defects and \( L \) is the linear size of the lattice. The enhancement factor is \( 4/\pi \) here. We can introduce randomness in shape and size of the defect by removing \( (1 - p) \) fraction of the bonds randomly. It is no longer possible to determine the most vulnerable defect and to calculate the enhancement factor. We consider two limits: (1) dilute limit \( (p \to 1) \) when the defect density is small and (2) near the critical point \( p_c \) where the defect density is large and beyond which the lattice loses its connectivity.

**Dilute limit \( (p \to 1) \)**

In this limit there are only a few isolated defects (insulating bonds) in the sample. The current density around a defect is effectively independent of the presence of other defects. Under such considerations the most vulnerable of the defects will lead to the failure of the system. So our primary task is to identify the most probable dangerous defect (the weakest point which causes the largest concentration of the current density).

The ensemble of \( n \) successively removed bonds, far from boundaries [18, 19], in a plane perpendicular to the current flow, act as dangerous defect. In 2d it is a linear defect and in 3d the defect has the shape of a disc. The current through the conducting bonds at the immediate neighbourhood of the defect is

\[ i_e = i(1 + k_{2n}) \quad (\text{in 2d}), \quad i_e = i(1 + k_{3n}^{1/2}) \quad (\text{in 3d}). \]  

Here \( i \) is current density through the bonds far away from the defect. Enhancement factor \( k \) contains \( \sqrt{n} \) term because current is diverted by \( n \) defects to spread uniformly around the perimeter of the disc, which is proportional to \( \sqrt{n} \). The probability of appearance of a defect of \( n \) successive insulating bonds is

\[ P(n) \sim (1 - p)^n L^d, \]  

where \( L \) is the lattice size and \( d \) is the dimension of the lattice. \( L^d \) provides the number of places that the defects can occupy. The approach of \( P(n) \approx 1 \) gives the size of most probable dangerous defect:
\[ n_c = - \frac{2}{\ln(1-p)} \ln L \quad \text{(in 2d)}, \quad n_c = - \frac{3}{\ln(1-p)} \ln L \quad \text{(in 3d)}. \quad (7) \]

Combining (5) and (7) \( i_e \) becomes

\[
\begin{align*}
  i_e &= i \left[ 1 + k_2 \left( -\frac{2 \ln L}{\ln(1-p)} \right) \right] \quad \text{(in 2d)}, \\
  i_e &= i \left[ 1 + k_3 \left( -\frac{3 \ln L}{\ln(1-p)} \right) \right] \quad \text{(in 3d)}. \quad (8)
\end{align*}
\]

Here, the total current is \( i_L(d-1) \). Equating \( i_e \) with the threshold value \( i_0 \), the expression for the failure current becomes

\[
\begin{align*}
  I_f &= \frac{i_0 L}{1 + 2 k_2 \left[ \frac{\ln L}{\ln(1-p)} \right]} \quad \text{(in 2d)}, \\
  I_f &= \frac{i_0 L^2}{1 + \sqrt{3} k_3 \left[ \frac{\ln L}{\ln(1-p)} \right]^2} \quad \text{(in 3d)}. \quad (9)
\end{align*}
\]

From the equations it is clear that as \( p \to 1 \), \( I_f \) reduces to \( i_0 L^{d-1} \), the value of the current in the lattice with all the bonds conducting. The slope of the \( I_f(p) \) vs. \( p \) curve at \( p = 1 \) is infinite (see fig. 1). It is expected because even the presence of a single defect enhances the process of breakdown through cascade. The most important thing is the size dependence of \( I_f \). For large enough \( L \) and \( p \) not very close to 1 (such that the absolute value of \( \ln(1-p) \) is not too large) the failure current per bond \( i_f = I_f / L^{d-1} \) decreases as \( 1 / \ln L \) and \( 1 / (\ln L)^{1/2} \) in 2d and 3d respectively.

**Critical behavior \( (p \to p_c) \)**

Near criticality \( (p \to p_c) \), the material is strongly disordered and can be described by node-link-blob picture [43, 41, 17] of percolation theory (see fig. 2). Close to \( p_c \) the conducting part of the material extends over the sample to form percolating cluster which is self-similar up to the length scale \( \xi_p \), the percolation correlation length. The geometry of the cluster at length scale \( \xi_p \) appears same as that of the original cluster at smaller scale. This means that the infinite cluster may be divided into cells of size \( \xi_p \). Each cell consists of backbone bonds (which takes part in current conduction) and numerous dangling bonds (bonds which do not take part in current conduction). Only backbone is important because it takes part in conductivity. The backbone is made of two kinds of bonds: multiply connected bonds appear as blobs; and singly connected bonds (called red bonds), appear as links. Distribution of current in the sample is solely determined by links; being singly connected, each link has to carry the full current inside a cell. Since at threshold \( (p_c) \) correlation length \( (\xi_p) \) spans the sample (large enough), it is reasonable to assume that the failure current approaches zero as \( p \to p_c \).
Fig. 2. A portion of the node-link-blob superlattice model near $p_c$. The distance between two nodes of the lattice is $\xi_p$, while chemical length of the tortuous link of the super lattice is $L_c$.

To determine quantitatively the critical behavior of the failure current $I_f$, we consider the voltage $V$ applied across the node-link-blob network. Since $V$ is distributed among $L/\xi_p$ number of links (or cells) in series, the average voltage across each link is $V_L \sim \xi_p V$. The resistance of the sample $R$ is related to average link resistance $R_L$ through $R \sim \xi_p^{d-2} R_L$, since there are $L \xi_p^{-1}$ number of links in series in the length $L$ of the sample and there are $\xi_p^{-1}$ number of parallel such links. So the mean current in a link appears to be $i_L \sim \xi_p^{d-2} V/R$ through the relation $i_L = V_L/R_L$. $V$ equals to failure or fuse voltage $V_f$ when $i_L$ reaches the maximum current which a link can stand. One can assume fuse voltage behaves as

$$V_f \sim (p - p_c)^{-\tau_f},$$

(10)

Using the relation $\xi_p \sim (p - p_c)^{-\nu}$ and $R \sim (p - p_c)^{-\tau_c}$, $\tau_f$ comes out as $\tau_f = \tau_c - (d-1)\nu$. From the relation $I_f = V_f/R$, we get

$$I_f \sim (p - p_c)^{(d-1)\nu}.$$  

(11)

The values of the correlation length exponent $\nu$ are 1.33 in $d = 2$ and 0.88 in $d = 3$ and the values of the conductivity exponent $\tau_c$ are 1.33 in $d = 2$ and 2 in $d = 3$ [45]. So it is clear that fuse voltage $V_f$ attains finite value in $2d$ and diverges with the exponent 0.2 in $3d$, in contrary to the failure current which always approaches zero as $p \to p_c$.

**Influence of the sample size**

Size dependence of $I_f$ is related to the notion of the most dangerous defect present in the sample. In the present context the most dangerous defect is a 'cell' of the infinite cluster with length $\xi_p$ in the direction parallel to the
applied voltage and \( l_{\text{max}} \) in the perpendicular direction. The total probability of having a defect of size \( l \) is, \( P = g(l)(L/\xi_p)^d \), where \( g(l) \) is the probability density of defect cluster of linear size \( l \). Percolation theory predicts [45, 38] that

\[
P \sim \exp \left( -\frac{l}{\xi_p} \right) \left( \frac{L}{\xi_p} \right)^d.
\] (12)

\( l_{\text{max}} \) is obtained when \( P \approx 1 \) and

\[
l_{\text{max}} \sim \xi_p \ln L.
\] (13)

Now the current that flows through the side link of the defect is proportional to \((l_{\text{max}})^{d-1}I\) and one obtains

\[
I_f \sim \frac{(p - p_c)^{(d-1)\nu}}{(\ln L/\xi_p)^{(d-1)}}
\] (14)

So this is the correction over equation (11) due to finite size of the sample. Bergman and Stroud [7] gave an idea about the competition between extreme statistics and percolation statistics. The extreme statistics (size dependence of the most probable defect and failure current) is expected to dominate for \( \ln L > \xi_p \) (or, when Lifshitz scale is greater than the connectedness correlation length). The dominance of extreme statistics is expected for \( p \) far away from percolation threshold when the correlation length \( \xi_p \) is small.

Li and Duxbury [27] proposed the dependence of \( I_f \) on \( L \) through \((\ln L)^{-\psi_f}\). The approximate range of \( \psi_f \) is

\[
\frac{1}{2(d-1)} < \psi_f < 1.
\] (15)

Considering the results of the dilute limit and in the critical region, the combined form of (9) and (11) is

\[
I_f = I_0 \left[ \frac{(p - p_c)}{(1 - p_c)} \right]^{\phi_f} \left[ \ln(\ln L/\xi_p) \right]^{\psi_f}
\] (16)

The value of the different exponents (see table 1) and the constant \( K \) depend on the dimension and on the type of percolation. From (9) and (14) one can see that the combined result is valid only for \( 2d \). The expression has three obvious features:

1. For \( p = 1 \), \( I_f = I_0 \), as expected.
2. Near \( p = 1 \), \((p - p_c)\) is almost constant and we get back the expression (9).
3. Near \( p_c \), the denominator of (16) is of the order of unity and we recover the expression (11) with \( \phi_f = (d - 1)\nu \).
2.2 Distribution of the failure current

dilute limit \((p \to 1)\)

In random fuse networks the failure current \(I_f\) shows large sample to sample fluctuations. Since the failure current is determined by the weakest defect in the sample, the fluctuations in the failure currents do not come down with the system size. The average failure current is not a self-averaging quantity.

Distribution of the failure currents in a system of size \(L\) follows as (see ref. [12])

\[
F_L(I) = 1 - \exp \left[ -A_d L^d \exp \left\{ -d A \ln L \left( \frac{I_f I_0}{I_f - I} \right)^{d-1} \right\} \right].
\] (17)

The derivative of this cumulative failure probability distribution \(F_L(I)\) with respect to current \(I\) provides the current \(I\) at which the system fails. Certainly, at most probable failure current the derivative becomes maximum. Here \(I_0\) is the failure current for pure sample. It may become obvious with a simple calculation that \(I_f\) appears to be the most probable failure current (as was assumed in calculation) only in the limit of large enough system size.

Though the current \(I\) can vary from zero to infinity, the form of \(F_L(I)\) is meaningful only for \(I\) up to \(I_0\). One should expect the value unity for \(F_L(I)\) at \(I = I_0\) and \(I = \infty\), but it is true only for large system size. \(F_L(I)\) suffers from size and defect concentration dependence through \(I_f\) (see equation 9).

The above expression is referred to as the Gumbel distribution [23]. Another well known one which is very often used in engineering is Weibull distribution

\[
F_L(I) = 1 - \exp \left[ -r L^d \left( \frac{I}{I_f} \right)^m \right].
\] (18)

Here \(m\) is a constant and for large \(m\) (say more than 5) \(I_f\) refers to most probable failure current.

At critical region \((p \to p_c)\)

Near criticality the cumulative failure distribution function is

\[
F_L(I) = 1 - \exp \left[ -A'_d L^d \exp \left( \frac{k'(p - p_c)\nu}{I_f} \right) \right],
\] (19)

where \(A'_d\) and \(k'\) are two constants.

2.3 Continuum model

One can extend the ideas of lattice percolation to continuum conducting medium. A material at the scale of the size of the defects can be looked
upon as a continuous field with some defects as inclusions in the field. In continuum model of lattice percolation, insulating spherical holes (circular holes in 2d) are punched at random as defects in a uniform conducting sample. The holes can overlap (Swiss-cheese model) and two non-overlapping neighbouring holes have a conducting region between them. These regions constitute conducting channels of cross-section $\delta$. Somewhat similar to the breakdown problem discussed above, the transport properties of any such channel depend on the transport capacities (cross-section and length) of the narrowest part (the weakest bond) of the channel. With some reasonable assumptions, one \cite{24} can express the transport capacities of the weakest region in terms of percolation cluster statistics on the lattice (particularly in terms of the percolation correlation length $\xi_p$).

Both the discrete and continuum type are almost same in the dilute limit ($p \to 1$). So all the results derived earlier for discrete model are valid here. But near criticality an infinite percolation cluster, with the links of mean length $\xi_p$ and of different cross-sectional width $\delta$, is formed. The backbone \cite{17, 41} of this cluster is represented by a superlattice (see sec. 2.1) of tortuous singly connected links and blobs crossing at nodes at a separation

$$\xi_p \sim |p - p_c|^{-\nu}. \quad (20)$$

The chemical length between any two nodes is

$$L_c \sim |p - p_c|^{-\zeta}. \quad (21)$$

For singly connected bonds (or sites) on the percolating backbone, $\zeta = 1$ in all dimensions \cite{24} except $d = 1$ \cite{45, 15}. There are $(L/\xi_p)^{d-1}$ number of parallel links exist between two electrodes, where $L$ is the size of the sample. The current in a link is given by $i_L = (\xi_p/L)^{d-1} I$. The current density $i$ in a channel of cross-section $\delta$ is given by

$$i \sim \frac{\xi_p^{d-1} I}{\delta^{d-1}}. \quad (22)$$

The maximum current density is obtained for minimum cross-section $\delta_{min}$ which is inversely proportional to the shortest chemical length $L_c$ \cite{24, 4, 6}. If $i_0$ is the threshold current density at which the material fails, then

$$i_0 \sim \xi_p^{d-1} L_c^{d-1} I_f \quad (23)$$

and we get,

$$I_f \sim (p - p_c)^{\nu+1} \quad \text{(in 2d)}, \quad I_f \sim (p - p_c)^{2(\nu+1)} \quad \text{(in 3d)}. \quad (24)$$

Thus the exponents for the failure current are higher than those of the discrete model.
Using \( I_f = V_f / R \) and \( R \sim |p - p_c|^{-\hat{t}_c} \) with equations (20) and (21) the expression for fuse voltage becomes \( V_f \sim |p - p_c|^{\hat{t}_f} \) with \( \hat{t}_f \) equal to \((d - 1)(\nu + 1) - \hat{t}_c \) [12]. \( \hat{t}_c \) is the conductivity exponent: \( \hat{t}_c \simeq 1.3 \) and 2.5 in 2d and 3d respectively. Consequently, \( \hat{t}_f \) becomes approximately 1 and 1.3 in 2d and 3d respectively. Unlike discrete percolation where failure current at \( p_c \) attains a finite value in 2d and vanishes only in 3d, in continuum percolation failure voltage always vanishes at \( p_c \).

| Dimension | Lattice percolation | Continuum percolation |
|-----------|---------------------|-----------------------|
| 2         | \( \nu (= 4/3) \)  | \( \nu + 1 (= 7/3) \) |
| 3         | \( 2\nu (= 1.76) \) | \( 2(\nu + 1) (= 3.76) \) |

2.4 Electromigration

Electromigration is an example where we find the practical application of the concepts we have hitherto discussed. Miniaturization of the circuits and gadgets have become the norm of the day. Very thin metallic films are used as interconnects among the active parts of devices, resulting in large current densities through the metallic films. Having large enough momentum, the free electrons become able to displace metallic ions from their equilibrium positions. Thus depending upon the material a net material transport occurs [48] through grain boundary diffusion, surface diffusion or lattice diffusion. This ionic displacement and the accumulated effect of material transport due to large current densities (more than \( 10^4 \text{ A/cm}^2 \)) are called electromigration. Over time such electromigration leads to void formation at the cathode and extrusion at the anode in thin film interconnects. Such void formation and hillock formation cause an open circuit and a short circuit to neighbouring connecting wires respectively. The problem of material accumulation can be suppressed by the layers of other material around and above the interconnects. The problem of open circuit due to void formation has received much attention as the random failures of several interconnects finally lead to failure of the entire device.

Let a fraction \((1 - p)\) of the resistors are removed from random resistor network (say in 2d). A random walker starts its walk from one of the lateral sides (those without electrodes) of the network, and jumps from one cell to a nearest neighbour cell by crossing the bonds irrespective of occupied or unoccupied bonds, intending to reach the opposite side, with the constraint that it never visits a cell more than once. Let the whole path consists of \( n_0 \) occupied and \( n_1 \) unoccupied bonds. For a given configuration of missing resistors the minimum value of \( n_0 \) is the shortest path. The mean shortest path \( < n_0 > \) (for
theoretical study see ref. [14, 46]) can be realised by considering the average over a large number of configurations. An electromigration induced failure of the network can be realised if the walker fuses the resistors as it crosses them. The criterion for a resistor to fuse is not a particular value of current (as in the fuse model), rather a particular value of threshold charge $Q_0$ that attains the resistor from the time of application of the constant current $I_0$ to the network (for details of the model see ref. [10, 52]). If $t_1$ is the failure time of an arbitrary resistor of the network which is subjected to a constant current $I_0$, then the resistor must satisfy
\[
\int_0^{t_1} I(t)dt = Q_0
\]
to fail.

So shortest path for a given impurity concentration is the path which corresponds to the smallest number of resistors to fuse. The problem is to study the variation of failure time of the whole network $\tau$ with $p$. For a network having an isolated defect of length $n$, $\tau$ is given by
\[
\tau = (L - n)Q_0/I_0,
\]
where $LQ_0/I_0$ is the $\tau$ value for the pure network. In pure limit the mean failure time $<\tau>$ is related to the longest probable defect with size $n_c$. This means that the bonds which fuse are also the bonds with largest current.

$\tau$ decreases as $p$ decreases from pure limit, and approaches zero at $p_c$, since the number of bonds to be broken goes also to zero. The shortest path $n_0$, the failure current $I_f$, the number of broken bonds $N_f$ and the failure time $<\tau>$, all these quantities tend to go to zero at $p_c$ with the correlation length exponent $\nu$.

### 2.5 Numerical simulations of random fuse network

The numerical simulation of failure of random fuse network in $2d$ was carried out by de Arcangelis et al. [16]. Here, one starts with a lattice, the bonds of which are conductors and present with a probability $p$. An increasing external voltage is applied across the lattice and the voltage $V^f_1$ when there is first local failure is recorded. The fused bond is removed and the voltage is raised till there is a second failure at $V^f_2$. The process is continued till the global failure at a voltage $V^{fin}_f$ occurs. The variation of $V^f_1$ and $V^{fin}_f$ with $p$ are studied. $V^f_1$ decreases with decreasing $p$ till $p \sim 0.7$ whereupon it attains minimum value and then it starts increasing again. $V^{fin}_f$, on the contrary, increases monotonically and behaves almost identically as $V^f_1$ for $|p - p_c| < 0.08$. Both approaches $p_c$ with the diverging exponent 0.48. It seems that $V^f_1$ exhibits pseudodivergence in $2d$. 
Duxbury et al. [19] performed similar type of simulations covering the whole range of \( p \) from 1 to \( p_c = 0.5 \). Their simulation results concerning \( I_f \) vs. \( p \) graph fit well with the interpolation formula (16) with the exponent value \( \phi_f = 1 \), whereas the theoretically predicted value is 1.33. This may be due to the finite size effect.

![Graph showing \( L/I_f \) vs. \( \ln L \) expressing their linear dependence (after Duxbury et al. [19]). The curves from top to bottom correspond to initial impurity probability \( p = 0.6, 0.7, 0.8 \) and 0.9 respectively.]

They also demonstrated the finite value of \( V^1_f \) at \( p_c \) by looking into the variation of \( I_f \) and the conductance near \( p_c \). Following equation (9) they checked the linear dependence of \( L/I_f \) on \( \ln L \) successfully for several values of \( p \) varying \( L \) from 10 to 200 and determined \( \psi_f = 1 \) from their slopes. The slope increases as \( p \) approaches \( p_c \). Instead of (17) they preferred

\[
F_L(V) = 1 - \exp \left[ -AL^2 \exp \left( \frac{KL}{V} \right) \right]
\]

(27)
as cumulative distribution function of failure voltage.

Arcangelis et al. [16] and Duxbury et al. [19] determined the number of fused bonds \( N_f \) upto complete failure and they found that \( N_f \) goes to zero algebraically as \( p \to p_c \) with an exponent seemingly equal to the correlation length exponent \( \nu \).

3 Dielectric breakdown problem

In dielectric breakdown problem the sample is insulating and conducting material acts as defects in the composite. The volume fraction \( (p) \) of the conducting material is less than the critical volume fraction \( (p_c) \) so that the system overall is not conducting. The dielectric portions can withstand electric field
upto $e_c$, at and beyond which they become conducting (local breakdown). Global breakdown occurs when the conducting portions span the sample with the succession of local breakdown under the influence of an increasing external field.

In 2d the solution for dielectric problem can be obtained from the solution of fuse problem with the aid of duality relation [33, 9]. We follow Bowman and Stroud [9] and consider the case where $p$ is smaller than $p_c$ and the sample is macroscopically insulating. The equations for the induction vector $D$ and the field $E$ are:

$$\nabla \cdot D = 0$$
$$\nabla \times E = 0$$
$$D(r) = \epsilon(r)E(r),$$

where $\epsilon(r)$ is local dielectric constant of the insulating portion and $E$ is irrotational. With $E = -\nabla \phi$ equations (28) yields

$$\frac{\partial}{\partial x} \left[ \epsilon(r) \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \epsilon(r) \frac{\partial \phi}{\partial y} \right] = 0,$$

where $\phi$ is the scalar potential. Now consider the dual composite of the original, where insulator phase of the original is replaced by conducting phase and vice versa. The sample is macroscopically conducting and the relevant quantities are current density $i$ and electric field $\bar{E}$. These satisfy the equations

$$\nabla \cdot i = 0$$
$$\nabla \times \bar{E} = 0$$
$$i(r) = \sigma(r)\bar{E}(r)$$

Since $i$ is divergenceless, $i$ can be expressed as a curl of a vector potential $V$. $V (V = V_z = \psi(x, y))$ is chosen in such a way, only $z$-component of $i$ vanishes. Assuming local conductivity as $\sigma(r) = 1/\epsilon(r)$ equations (30) yields

$$\frac{\partial}{\partial x} \left[ \epsilon(r) \frac{\partial \psi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \epsilon(r) \frac{\partial \psi}{\partial y} \right] = 0.$$ 

Comparing (29) and (31), one has

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial y}.$$ 

With all these, the components of current density $i$ in dual composite become

$$i_x = \frac{\partial \psi}{\partial y} = E_y, \quad i_y = -\frac{\partial \psi}{\partial x} = -E_x.$$ 

Thus we see that the magnitude of the current density in the dual composite is equal to the magnitude of the electric field in the original one and the
direction is rotated by $90^\circ$. It can also be shown that the field $\mathbf{E}$ of the dual problem is equal to vector $\mathbf{D}$ of the dielectric problem and is rotated by $90^\circ$. Now the physical correspondence of the two pictures can easily be established [12, 39].

All the results of fuse problem can be utilized for obtaining the solution for dielectric problem. For example, the equations (16) and (19) can be used in a straightforward way on replacing $I_0$ and $I_f$ by $V_0$ and $V_b$ respectively and also $(1 - p)$ by $p$.

### 3.1 Dielectric breakdown problem

As in the fuse problem, dielectric breakdown problem can be viewed on a discrete lattice with disorder in the framework of percolation theory. Basically, a lattice of insulating bonds is considered out of which $p$ ($p < p_c$) fraction of bonds are conducting at random. For this purpose resistor (capacitor) of different resistivity (capacitance) [8, 9] ([5]) can be used. Resistors (capacitors) of smaller (higher) resistance (capacitance) are used for conducting bonds.

#### Dilute limit ($p \to 0$)

The problem is very similar to the fuse problem. The enhanced local field due to the presence of long defect (made of $n$ number of larger valued capacitors (conducting bonds) perpendicular to the electrode, is (following Beale and Duxbury [5])

$$E_e = E(1 + kn),$$  \hspace{1cm} (34)

where $E$ is the externally applied field and $k$ is the enhancement factor. It is valid for all dimension. The probability to find a long defect made of $n$ conducting bonds is

$$P(n) \sim p^n L^d.$$  \hspace{1cm} (35)

The most probable defect size is

$$n_c = -\frac{d}{\ln p} \ln L.$$  \hspace{1cm} (36)

The enhanced field near the insulating bond adjacent to the most probable defect is now

$$E_e = E \left[ 1 + K_d \left( -\frac{\ln L}{\ln p} \right) \right].$$  \hspace{1cm} (37)

When enhanced field $E_e$ of any bond attains threshold value of dielectric breakdown, local breakdown takes place and the bond becomes conducting. So breakdown voltage becomes

$$V_b = \frac{E_0 L}{1 + K_d \left( \frac{\ln L}{\ln p} \right)},$$  \hspace{1cm} (38)

where $E_0$ is the breakdown field without defect.
Close to critical point \((p \to p_c)\)

In this limit, consider the conducting defects which are on the average the percolation correlation length \(\xi_p\) distance apart. Hence, average field is \(V_1/\xi_p = V/L\), where \(V\) and \(V_1\) are externally applied voltage and potential difference between any two conducting defects respectively. The maximum attainable field is \(V_1/a\), where \(a\) is the minimum available separation (bond length) between the defects in the network. When the local electric field \(V_1/a\) reaches the bond-threshold value \(e_c\), breakdown occurs. Now the required average electric field for local breakdown is \(E_b = (a/\xi_p)e_c\). Near \(p_c, \xi_p\) diverges as \(\xi_p(p) \propto a(p-p_c)^{-\nu}\). So the critical behavior of the average breakdown field as derived by Lobb et al. [28] appears as

\[
E_b \sim (p_c - p)^\nu. \tag{39}
\]

in all dimensions. This can also be derived [46] from the concept of minimum-gap \(g(p)\), as \(E_b\) can be considered as proportional (see sec. 2.4) to \(g(p)\) and \(g(p)\) is proportional to \(\xi_p^{-1}\) (see relation 20), the result above follows. Since \(E_b\) is inversely proportional to the linear dimension of the most vulnerable defect and the typical size of the defect or conducting clusters diverges at \(p_c, E_b\) approaches zero in this limit.

Influence of sample size

We consider a critical defect which is a pair of longitudinal very closely spaced clusters of conducting material (almost linear in shape) of the order of size \(l\) situated one after one, parallal to the direction of the applied field \(E\) in a \(L^d\) lattice. The field between these two clusters enhanced by a factor \(l\) time \(E\). Far from \(p_c, \) these clusters appear with the probability \((1/\xi_p)\exp(-l/\xi_p)\). Thus the most dangerous defect (the largest cluster) is of the order of \(l_{\text{max}} \sim \xi_p \ln(L^d)\). The first breakdown field \(E_b^1\) (field needed to break the first bond) scales as \(1/l_{\text{max}}\) in a finite but large system [5]. Since, in this particular case breakdown field for the sample is same as for the first breakdown, \(E_b\) becomes

\[
E_b \sim \frac{(p_c - p)^\nu}{\ln L}. \tag{40}
\]

This approximate expression for the average breakdown field was derived by Beale and Duxbury [5].

Larger sample has higher chance of having a larger vulnerable defect. So larger sample requires smaller electric field for breakdown and \(E_b\) becomes zero in the limit \(L \to \infty\). Breakdown path is always on the average perpendicular to the electrode. So the above result is independent of dimension.

Due to the dominating behavior of \((p_c - p)^\nu\) over \(\ln L\), breakdown field tends to zero as \(p\) approaches \(p_c\). Very near to \(p_c\) Chakrabarti and Benguigui ([12] p. 66) proposed a scaling relation as
\[ E_b \sim (p_c - p)^\nu \ln(L/\xi_p) \]  
(41)

According to Bergman and Stroud [7], \( E_b \) may become size independent very close to \( p_c \), and a cross-over from extreme statistics to percolation dominated statistics may be there.

**Summary**

A general formula may be inferred for dielectric breakdown field as:

\[ E_b = E_0 \left[ \frac{(p_c - p)/p_c}{p_c} \right]^{\phi_b} \left[ 1 + K \frac{\ln(L/\xi_p)}{-\ln p} \right], \]

(42)

with the exponent \( \phi_b \) (=\( \nu \) for lattice percolation) dependent on the dimension and on the type of percolation. For the theoretically estimated value for \( \phi_b \) see table 2.

| Dimension | Lattice percolation | Continuum percolation |
|-----------|---------------------|-----------------------|
| 2         | \( \nu \approx 4/3 \) | \( \nu + 1 \approx 7/3 \) |
| 3         | \( \nu \approx 0.88 \) | \( \nu + 1 \approx 1.88 \) |

3.2 Distribution of breakdown field

Distribution function \( F_L(E) \) in dielectric breakdown is defined as the probability of breakdown of a dielectric system of size \( L \) in an external electric field \( E \). Usually, Weibull distribution [50, 49]

\[ F_L(E) = 1 - \exp(-rL^dE^m) \]

(43)

is used to fit the distribution function of breakdown and failure problems. Here \( r \) and \( m \) are constants. Duxbury and his co-workers [18, 19, 20, 5] argued that the distribution function for dielectric (and electrical) breakdown is given by Gumbell distribution:

\[ F_L(E) = 1 - \exp \left[ -AL^d \exp \left( -\frac{K}{E} \right) \right]. \]

(44)

It can be derived [5] from a simple scaling argument based on percolation cluster statistics [44, 21, 25]. It is valid for \( L >> \xi_p \) or in dilute limit. The two expression (43) and (44) are qualitatively same if the Weibull exponent \( m \) is large. Though second one provides better fit to numerical simulation [5]. Sornette [42] argued that the expression (44) is not appropriate in continuum system having percolation type of disorder, rather, Weibull-like distribution is valid there.
3.3 Continuum model

The model consists of dielectric material with the spherical (circular in 2d) conducting inclusions as defects with randomly positioned centres having the possibility to overlap. We assume breakdown field $E_b$ is proportional to the width $\delta$. For breakdown $\xi_p^{-1}$ numbers of link elements to be broken. With the same logic as in sec. 2.3, $E_b \sim \delta_{\text{min}} \xi_p^{-1} \sim L_c^{-1} \xi_p^{-1}$ or

$$E_b \sim (p_c - p)^{\nu + 1}$$

very near to $p_c$ [13, 28]. So continuum system is weaker than the discrete one. This is because the conductivities of the conducting channels increases as $p \to p_c$, whereas, conductivity of the bonds in discrete model is independent of $p$.

3.4 Shortest path

The concept of shortest path is already mentioned in section 2.4 in the context of electromigration. Just like before, we consider a walker starting its walk from one electrode with jump from one site to another and reaches the opposite electrode after executing a self-avoiding walk. The aim of the walker is to create a percolating conducting path by transforming a insulating bond to conducting bond when it jumps between two conducting sites separated by the insulating bond. $n_0$ denotes the total number of insulating bonds which are to be transformed into conducting bonds during walk across the sample. After completion of the walk, the sample experiences a continuous conducting path between the electrodes and in this way dielectric breakdown comes into existence.

The shortest path for a given configuration is defined as the path with the smallest $n_0$. The normalised smallest path is given by $g(p) = < n_0 > / L$, where $< n_0 >$ is obtained by considering the average of $n_0$ over a large number of configurations. Some authors call $g(p)$ by minimum gap.

The behavior of $g(p)$ has been studied [46, 20] extensively in 2d and 3d for regular and directed percolation scenario. As $p$ increases from zero to $p_c$, the minimum gap $g(p)$ decreases from unity to zero with the correlation length exponent $\nu$.

3.5 Numerical simulations in dielectric breakdown

Stochastic models

In these models, stochastic growth processes are considered which mimic the dielectric breakdown processes. For example, Sawada et al. [40] considered a random growth process where growth takes place in two ways: tips of the
pattern grow with a probability $p_0$ and new tips (branching) appear with probability $p_n$ (here, $p_n < p_0$). In their simulation the pattern appears as fractal and fractal dimension can be tuned by the parameter $R \equiv p_0/p_n$. However, results of such a simplification do not satisfy experimental results of dielectric breakdown [35].

To mimic the dielectric discharge pattern in gases Niemeyer et al. [34] suggested a stochastic model where the breakdown pattern generated in turn determines the local electric field and the growth probability. The model could reproduce the fractal properties of dielectric breakdown process by numerical simulations. In this model the breakdown pattern starts growing from the centre of a lattice with insulating bonds. One electrode is placed at the centre and other one is placed at a long distance on the circumference of a circle. In one step only one interface bond (and a point) among all the nearest neighbors of the pattern breaks down depending upon the growth probability $p$ and becomes the member of the pattern. The newly added bond becomes a conductor and the newly added site is shorted to the voltage of the central electrode. The growth starts from the centre and grows radially outwards. The growth probability $p$ depends on the local field (potential) which in turn is controlled by the breakdown pattern via the relation

$$p(i, j \to i', j') = \left(\frac{V_{i', j'}}{\sum V_{i', j'}}\right)^\eta,$$

(46)

where the indices $i, j$ and $i', j'$ represent the discrete lattice coordinates. The electric potential $V$ is defined for all sites of the lattice by the discrete Laplace equation

$$V_{i,j} = \frac{1}{4}(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1})$$

(47)

with the boundary condition $V = 0$ for each point of the discharge pattern and $V = 1$ outside the external circle. $\eta$ describes the relation between local field and growth probability. The fractal structure of the pattern has been seen to obey

$$N(r) \sim r^{d_f},$$

(48)

where $N(r)$ is the total number of discharge points inside a circle of radius $r$ and $d_f$ is the Hausdorff dimensions. In 2d one has $d_f \simeq 2, 1.89 \pm 0.01, 1.75 \pm 0.02, 1.6$ for $\eta = 0, 0.05, 1, 2$ respectively. The structure tends to be more linear with larger $\eta$. The observed value of $d$ for $\eta = 1$ is in good agreement with the experimental result ($\simeq 1.7$) (Niemeyer and Pinnekamp [35]) and the resulting figure is very similar to Lichtenberg figure.

The same pattern for $\eta = 1$ has been produced surprisingly by a different growth model: diffusion-limited aggregation model (DLA) of Witten and Sander [51] (For review see Meakin [31]). Niemeyer et al. neither justifies the appearance of $\eta$ nor provides any theoretical explanation regarding explicit
rule for breakdown. Many models (for details see ref. [39]) have been proposed but a model which fully describes dielectric breakdown in solids is still lacking.

**Deterministic models**

Contrary to Niemeyer et al. [34], Takayasu [47] introduced deterministic approach to produce dendritic fractal pattern (as is found in lightning) by considering a priori spatial fluctuation on bond resistances \( r_i = \theta r; \theta \in [0, 1] \) and the nonlinear irreversible characteristics of the resistance: once the potential difference across a resistance \( r_i \) attain a pre-assigned threshold voltage \( v_c \), \( r_i \) reduces to \( \delta r_i \) (\( \delta \) is a small positive quantity) and its resistivity then never changes. This is the breakdown of a resistor in this model. Breakdown of a resistor induces successive breakdown of other resistors leading to the formation of percolation cluster of broken resistors. The pattern appears to be anisotropically fractal with the dimension \( d_f = 1.58 \pm 0.12 \) in 2d. Due to anisotropy this \( d_f \) is less than the fractal dimension 1.89 of a percolating network in 2d.

Family et al. [22] made the stochastic model of Niemeyer et al. [34] deterministic by attaching randomly a breakdown coefficient \( \theta (\theta \in [0, 1]) \) to each insulating bond. There are two versions of this model: In one model, at each time step an interface bond \( ij \) with the largest \( \theta V_{ij}^\eta \) breaks down, whereas in the other model an interface bond breaks down with a probability \( \theta V_{ij}^\eta / p_{max} \), where \( \eta \) is an adjustable constant and \( p_{max} \) is the largest value of \( \theta V_{ij}^\eta \) among all the interface bonds. The patterns appear as tunable (varying \( \eta \)) fractal. In the first model stringy highly anisotropic (with branches) pattern with fractal dimension of about 1.2 appears in 2d, whereas in the second one pattern is strikingly lacking in anisotropy. The shape of the pattern is similar to those found in Niemeyer et al. [34] except the fractal dimension of 1.70 \( \pm 0.05 \) (for \( \eta = 1 \)) in 2d. The results for \( d_f \) appear same to that for DLA on a square lattice [30] and to that for the same model in a homogeneous medium (here, cross and square-shaped patterns appear) [22].

Direct simulations of dielectric breakdown from various groups confirmed the theoretical consideration for size and impurity dependence of breakdown voltages quite successfully. In these simulations, a discrete lattice of insulating bonds is considered with \( p \) fraction of bonds as conductors. Each insulator breaks down to conductor at a threshold value of voltage drop \( v_c \). A macroscopic voltage is applied and the voltage distribution throughout the lattice is computed by solving Laplace equation (47). The insulator with largest voltage drop (at or above \( v_c \)) is converted to a conductor (incidence of first local breakdown). The voltage distribution is then recalculated, the second local breakdown is identified and the process continues. If at any step the applied voltage is not large enough to cause breakdown of any insulator, it is increased gradually. The simulation continues till a sample spanning cluster of conductors appears. The applied voltage \( V_b \) at which global breakdown occurs divided by the sample size \( L \) is identified as breakdown field \( E_b \). It has been
experienced that the breakdown field $E_b$ is same as the field $E_b^1$ required for the occurrence of first local breakdown.

Manna and Chakrabarti [29] determined the $p$ dependence of both $E_b^1$ and $g(p)$ for the entire range of $p$ below $p_c$. They found (see figure 4) that both $E_b^1$ and $g(p)$ go to zero at $p_c$ with the exponent of value almost equal to 1. They argued that the exponent is actually $\nu$ and the smallness of the value is due to the smallness of the lattice size ($L = 25$, where exact $p_c$ is not reachable). Bowman and Stroud [9] found that $E_b^1$ vanishes at $p_c$ with an exponent equal to $1.1 \pm 0.2$ in $2d$ and $0.7 \pm 0.2$ in $3d$ for both site and bond problems. These are consistent with the correlation length exponent ($\nu$) values of $4/3$ and $0.88$ in $2d$ and $3d$ respectively.

![Figure 4](image)

**Fig. 4.** The variation of first breakdown field $E_b$ and the minimum gap $g(p)$ with initial impurity probability $p$, (after Manna and Chakrabarti [29]). The two quantities behave same very near to the percolation threshold $p_c$, whereas far from $p_c$ they behave differently.

Beale and Duxbury [5] proposed a relation for first (local) breakdown field as

$$E_b^1 \sim \frac{1}{A(p) + B(p) \ln L}.$$  (49)
Near $p_c$, they found $A(p)$ and $B(p)$ vary like $(p_c - p)^{-\nu}$ (in the lattice of size $L = 50, 70, 100$) as is expected from eqn. (40). From the plot of $\ln\{B(p) \ln(p)\}$ versus $-\ln(p_c - p)$ their data shows $\nu = 1.46 \pm 0.22$ which is in good agreement with exact value $4/3$ in 2$d$. Their data fits well to the Gumbel double exponential form (44) for cumulative failure distribution. Manna and Chakrabarti [29] and Beale and Duxbury [5] found the exponent $\phi_b$ of eqn. (42) to be about 1.0 and 1.2 respectively.

Acharyya and Chakrabarti [2] found that starting with a concentration $p$ ($p < p_c$) of the conductors, the rate at which the insulating bonds breaks down to conductors as the electric field is raised in a dielectric breakdown diverges at breakdown voltage $V_b$.

This indicates that global breakdown process is highly correlated very near to $V_b$. They defined a new quantity named as breakdown susceptibility $\chi (=dn/dV)$. Here $n(V)$ is the average number of broken bonds at an external voltage $V$ ($< V_b$) for a fixed value of $p$. It is obvious that for a sufficient voltage $V$, $n(V)$ saturates to $L^d$. $\chi$ exhibits a maxima at $V_{b}^{eff}$ which is different from $V_{b}^{fin}$, where $V_{b}^{fin}$ is the voltage required to creat the last member of the sample spanning conducting cluster. $V_{b}^{eff}$ approaches $V_{b}^{fin}$ with increasing sample size $L$. It seems that $\chi$ is a divergent quantity for infinite sample. So there exists a possibility to predict $V_{b}^{fin}$ without going to the complete break-

![Figure 5](image-url)
down point of the sample. Statistics of the growing clusters up to \( V_b \) (minimum external voltage required to have the global connection across the sample via the conducting sites) in 2d site percolation was studied by Acharyya et al. [3]. \( V_b \) is identified not only as the point of global breakdown but also as the point of divergence of the rate of the various statistical quantities like total number of conducting sites, the average size of the conducting cluster and the number of such clusters.

4 Zener breakdown in Anderson insulators

From the microscopic point of view, current conduction requires the mobility of the electrons across the material and for composites of metals and dielectrics or insulators this means that there must be percolating paths of conducting material in the system (as long as electron is considered a classical particle). If \( p \) denotes the concentration of the conducting material in the composite and \( p_c \) the percolation threshold, then from our previous discussion we know that the conductivity will go to zero at \( p_c \) as \( |p - p_c|^{\nu} \) [45]. For \( p \) below \( p_c \), the composite is ceases to be conducting and the classical breakdown voltage needed to make the system conducting goes as \( |p - p_c|^\phi_b \) (see sec. 3.1).

Studies on Anderson transition (see ref. [26] for example) show that electron as a quantum particle cannot diffuse even through the geometrically percolating path due to the coherent back scattering (of the wave function) from the random geometry of the cluster in a dimension less than three. Since all the states on any percolating lattice gets localized (exponentially) electrons do not diffuse through the disordered (percolating) lattice.

In a three dimensional disordered systems there exist a mobility edge (a sharp energy level in conduction band) \( \epsilon_c \) below which all the states are localized and above which all the states are extended. Insulating or conducting phase appears depending upon the position of Fermi level \( \epsilon_f \) below or above the mobility edge \( \epsilon_c \). The (Anderson) transition from the insulating phase \( (\epsilon_f < \epsilon_c) \) to metallic phase \( (\epsilon_f > \epsilon_c) \) (where the electrons are quantum mechanically percolating that is their wave functions extend all over) across the mobility edge in a percolating disordered solid, has already been well studied [26]. In metallic phase the conductivity increases as \( |\epsilon_f - \epsilon_c|^{\nu_q} ; \nu_q \neq \nu_c \).

In the case of a classical nonpercolating \( (p < p_c) \) system, an application of an external electric field \( E_b \) forces the system to undergo breakdown following the relation \( E_b \sim |p_c - p|^{\nu_c} \) (see sec. 3.1). In analogy with the above classical dielectric breakdown problem one can think of a problem of quantum dielectric breakdown (possibility of the appearance of a conducting path) in Anderson insulating \( (\epsilon_f < \epsilon_c) \) or quantum mechanically non-percolating phase with an application of strong electric field. This may be thought as a kind of complementary problem of the Zener breakdown in insulator.

In an insulator, states are localised exponentially within the atomic distance \( a \) and effective band width \( w \) is in general larger than \( a \). There exists a
possibility of tunneling of the states across the band gap if the energy gained by an electron in travelling an atomic distance $a$ in an electric field $E$ is larger or equal to the band gap $\Delta \epsilon_b$ (or when $eEA \geq \Delta \epsilon_b$). Here $e$ denotes the electronic charge and the band gap is the separation between conduction and valance band. In Zener breakdown, bands get effectively tilted in strong electric field $E$ in the direction of the field, reducing the band gap effectively from $\Delta \epsilon_b$ to $\Delta \epsilon_{br}$. If the effective width of the reduced band gap ($w = \Delta \epsilon_{br}/eE$) becomes of the order of atomic distance $a$ interband tunnelling takes place for $w \leq a$ and the insulation breaks down. So the Zener breakdown field $E_b$ scales linearly with the band gap: $E_b = \Delta \epsilon_b/ea$.

Similar kind of breakdown of insulation can occur in the case of Anderson insulators with an application of strong electric field $E$ in more than two dimension. States are localized (exponentially) with in the localization length $\xi_q$, where $\xi_q$ varies as $\xi_q \sim |\Delta \epsilon_m|^{-\nu_q}$. Here $\Delta \epsilon_m$ is the mobility gap; $\Delta \epsilon_m = \epsilon_c - \epsilon_f$. One can now easily think of a possibility of tunneling across the mobility gap, if the energy gained by an electron in travelling a distance $\xi_q$ in field $E$ is of the order of the mobility gap $\Delta \epsilon_m$ (or when $eE \xi_q \geq \Delta \epsilon_m$). Thus in contrary to the standered Zener breakdown in semiconductors, here breakdown field $E_b$ scales almost quadratically with mobility gap as [11]

$$E_b = \frac{\Delta \epsilon_m}{e\xi_q} \simeq |\Delta \epsilon_m|^{T_q}; \quad T_q = 1 + \nu_q \quad (50)$$

($\nu_q \sim 0.9$ [26, 32], giving $T_q \sim 1.9$ in $3d$).

Fig. 6. Schematic density of states for Anderson insulators ($d > 2$) shown in (a), where the Fermi label $\epsilon_f$ is below the mobility edge $\epsilon_c$ (metallic phase for $\epsilon_f > \epsilon_c$). For strong electric field $E$, the band of (localized) states get tilted and tunnelling occurs when the effective width $w = |\Delta \epsilon_{mr}|/eE$ of the mobility gap $\Delta \epsilon_{mr}$, is less than or equal to the localization length $\xi_q$. 
The tunnelling probability $g(E)$ decreases exponentially with the width of the effective barrier as $g(E) \sim \exp(-w/\xi_q) \sim \exp(-\Delta \epsilon_m/eE\xi_q) \sim \exp(-|\Delta \epsilon_m|T_q/E)$. The cumulative failure probability of Gumbel form (same as for fracture [20, 18, 19]) for a sample of size $L$ under field $E$ is given by [11]

$$F_L(E) \sim 1 - \exp[-L^d g(E)]$$

$$\sim 1 - \exp \left[-L^d e \exp \left(-\frac{|\Delta \epsilon_m|T_q}{E}\right)\right]. \quad (51)$$

This gives

$$E_b \sim \frac{|\Delta \epsilon_m|T_q}{\ln L}, \quad (52)$$

as the size dependence of the typical breakdown field for and above which $F_L(E)$ is significant.

5 Conclusions

We have discussed the classical failure of the fuse systems, the dielectric breakdown and the quantum breakdown in the Anderson insulators. We have discussed how the extreme value statistics and the resulting Gumbel distribution arises in breakdown and failure processes, especially when the disorder concentration is low. At high concentration of disorder near the percolation threshold, we have discussed how the cross-over might take place from extreme value to percolation statistics. We have discussed the system size dependence that arises at the distribution of the failure current at low disorder regime. Finally, the extension of Zener breakdown phenomenon for band insulators to the disorder-induced Anderson insulators has been discussed in sec. 4.

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