PAINELEVÉ-II PROFILE OF THE SHADOW KINK IN THE
THEORY OF LIGHT-MATTER INTERACTION IN NEMATIC
LIQUID CRYSTALS

CHRISTOS SOURDIS

Abstract. We confirm a prediction that the recently introduced shadow kink
defect in the theory of light-matter interaction in nematic liquid crystals is
described to main order by a solution of the Painlevé-II equation which changes
sign once in the whole real line. Our result implies that such a solution of
the latter equation is energy minimizing with respect to compactly supported
perturbations.

1. Introduction

Motivated by experiments in light-matter interaction in nematic liquid crystals,
the authors of [5] considered energy minimizing in $H^1(\mathbb{R})$ solutions of the following
singular perturbation problem:

$$\epsilon^2 v''(x) + \mu(x)v(x) - v^3(x) + \epsilon af(x) = 0, \quad x \in \mathbb{R},$$

where $a > 0$ is a parameter, and the fixed functions $\mu, f$ satisfy

$$\begin{cases}
\mu \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is even}, \quad \mu' < 0 \text{ in } (0, \infty), \\
\mu(\xi) = 0 \text{ for some } \xi > 0;
\end{cases}$$

$$\begin{cases}
f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}) \text{ is odd}, \quad f > 0 \text{ in } (0, \infty).
\end{cases}$$

It was shown that, for any $\epsilon, a > 0$, minimizers $v_\epsilon$ always exist and have a unique
zero $\rho_\epsilon$. The main result of [5] can be summarized as follows. There exist positive
numbers $a_* < a^*$ (independent of $\epsilon$ that can be characterized variationally) such
that:

- If $a \in (0, a_*)$, then minimizers $v_\epsilon$ converge pointwise to $\text{sgn}(x)\sqrt{\mu^+(x)}$
as $\epsilon \to 0$ (the convergence being uniform away from the origin, and so
$\rho_\epsilon \to 0$). There is a transition layer around the origin of width $O(\epsilon)$, where
the behaviour of $v_\epsilon$ is governed by a usual squeezed hyperbolic tangent
profile (see also (2.22) below).
- If $a \in (a^*, \infty)$, then $v_\epsilon$ converges uniformly to $\sqrt{\mu^+(x)}$ and $\rho_\epsilon \to -\xi$ as
$\epsilon \to 0$ (without loss of generality).

Date: November 20, 2018.
In any case, $v_\epsilon$ has steep corner layers of width $O(\epsilon^{\frac{2}{3}})$ around $\pm \xi$, where the behaviour of $v_\epsilon$ is governed by a suitable blow-down of a minimizing (in the sense of Morse) solution of the Painlevé-II equation:

$$y''(s) - sy(s) - 2y^3(s) - \alpha = 0, \quad s \in \mathbb{R},$$

with

$$\alpha = \frac{af(-\xi)}{\sqrt{2\mu'(-\xi)}} < 0,$$

after appropriate reflections and rescalings of constants. More precisely, assuming that $v_\epsilon \to \sqrt{\mu^+}$ (without loss of generality), it holds

$$2^{-\frac{1}{2}} (\mu'(-\xi)\epsilon^{-\frac{1}{4}} v_\epsilon \left(-\xi - \frac{\epsilon^{\frac{2}{3}} s}{(\mu'(-\xi))^{\frac{1}{2}}}\right) \to -Y(s) \text{ in } C^1_{\text{loc}}(\mathbb{R}) \text{ as } \epsilon \to 0,$$

where $Y$ is an energy minimizing solution of (1.3) with $\alpha$ as in (1.4). We clarify that here minimality is understood in the sense of Morse or De Giorgi, that is with respect to compact perturbations. In fact, such solutions automatically satisfy either the plus or the minus set of asymptotic conditions in (1.6) below (see again [5]).

It is known that for any $\alpha \leq 0$ the equation (1.3) admits a unique positive solution $Y_+$ which is defined in the whole real line (see [8]). In fact, it holds $Y'_+ < 0$, which implies that $Y_+$ is linearly nondegenerate, in the sense that the linearization of (1.3) about $Y_+$ does not have bounded elements in its kernel. On the other hand, for $\alpha < 0$ in some neighborhood of zero, the problem (1.3) has also a solution $Y_-$ which changes sign once [4, 15]. We point out that it holds

$$Y_\pm(s) \sim \pm \sqrt{\frac{-s}{2}} + O\left(\frac{1}{s}\right) \text{ as } s \to -\infty; \quad Y_\pm(s) \sim \frac{|\alpha|}{s} \text{ as } s \to +\infty.$$ (1.6)

A natural question that arises is whether the minimizing solution $Y$ of (1.3) that appears in (1.5) is $Y_+$ or a solution that changes sign exactly once (recall that $v_\epsilon$ changes sign exactly once but its zero $\rho$ could escape at infinity in the blow-up (1.5)). In this paper we will show that it is indeed the latter, and more interesting, scenario that occurs. Thus, we describe the microstructure of $v_\epsilon$ near $-\xi$, and verify that the microscopic phase transition that takes place near $-\xi$, is a new type of defect that does not involve the standard hyperbolic tangent.

Our main result is the following.

**Theorem 1.1.** If in addition to (1.2) we assume that $\mu, f \in C^2$ near $-\xi$, then there exists an $a^{**} > a^*$ such that the minimizing solution $Y$ of (1.3) that appears in (1.5) changes sign exactly once, provided that $a > a^{**}$.

Our result when combined with Theorems 1.2 and 1.3 in [5] implies at once the following.
Corollary 1.1. There exists an $\alpha_\ast < 0$ such that, if $\alpha \in (\alpha_\ast, 0)$, the problem (1.3) admits a solution $Y_-$ with exactly one sign change that is energy minimizing with respect to compactly supported perturbations. Moreover, the solution $Y_-$ satisfies the corresponding conditions in (1.6).

In relation to the aforementioned papers [4, 15], our main contribution is that we associated to at least one of such solutions $Y_-$ of the Painlevé-II transcendent the strong property of energy minimality (uniqueness seems to be unknown). In general, in problems of the calculus of variations, the standard method for showing this property is by constructing a calibration. However, it is not clear to us how to apply this considerably more direct approach to the problem at hand. Nevertheless, we note in passing that this is indeed possible for $Y_+$ because it is sign definite.

Our proof of Theorem 1.1 goes by contradiction. Assuming that the assertion is false puts us sufficiently far from $-\xi$, in some sense, which allows us to compare $v_\varepsilon$ to a sign definite solution $\eta_\varepsilon$ of (1.1) by examining the quotient

$$w_\varepsilon = \frac{v_\varepsilon}{\eta_\varepsilon}.$$ 

This division trick was introduced by [11] in a different context and has been used extensively in the study of vortices in Bose-Einstein condensates (see [1] and the references therein); the simplest case being (1.1) with $a = 0$ and $-\mu$ being a trapping potential. In this regard, we point out that the connection between (1.1) and the aforementioned studies was already discussed in [5]. However, to the best of our knowledge, this important division trick is applied here for the first time in this context. Thankfully, the required estimates for $\eta$ and its derivatives are readily derivable from [10] or [13]. These rely on the fact that the positive solution $Y_+$ of (1.3) is linearly nondegenerate. The quotient $w$ satisfies a weighted Allen-Cahn equation with a weight that degenerates at $-\xi$ (see (2.15) and the discussion leading to (2.24)). However, as we remarked, we will be working sufficiently away from this degeneracy. It boils down to showing that $w$ cannot have a sharp transition layer from $1$ to $-1$ at $\rho$. For this purpose, there are several approaches in the literature for related problems of Allen-Cahn type. For instance, see [3, Thm. 1.5] for a nonlinear Schrödinger equation with a potential, [2, Thm 4.1] for a phase field model of phase transitions, and [12, Thm. 1] for a spatially inhomogeneous Allen-Cahn equation. On the one hand, the equation for $w$ resembles more the aforementioned phase field model. On the other hand, armed with our estimates for $w$, we found it more convenient to adapt the corresponding proof of [3] with our own twist.

We close this introduction by expressing our hope that some of our arguments can be extended to describe the core of the ’shadow vortex’ defect in the recent paper [7] or the ’shadow domain wall’ in [6].
In the rest of the paper we will prove our main result Theorem 1.1. Following it, in Remark 2.1, we will give an applied mathematicians point of view which motivated our rigorous analysis.

2. Proof of the main result

In order to prove Theorem 1.1, as was already observed at the end of Section 3 of [5], it suffices to establish the following (recall also the preamble to the aforementioned theorem).

**Theorem 2.1.** Let $\mu, f$ satisfy (1.2) with $\mu, f \in C^2$ near $-\xi$. Then, there exists an $a^{**} > a^*$ such that, if $a > a^{**}$, the unique root $\rho_\epsilon$ of $v_\epsilon$ satisfies

$$\rho_\epsilon + \xi = O(\epsilon^{2/3}) \quad \text{as } \epsilon \to 0.$$  

**Proof.** Throughout the proof, we will denote by $C/c > 0$ a large/small generic constant that is independent of small $\epsilon > 0$, and whose value will increase/decrease as we proceed.

Firstly, we will show that

$$\rho_\epsilon \geq -\xi - C\epsilon^{2/3}. \quad (2.1)$$

To this end, let us consider the algebraic equation that comes from (1.1) when we ignore the term $\epsilon^2 v''$:

$$\mu(x)\nu - \nu^3 + \epsilon af(x) = 0. \quad (2.2)$$

Based on the fact that $\mu'(-\xi) > 0$, it is straightforward to verify that there exists a large $R > 0$ such that the following properties hold. The above equation admits a unique solution $\nu_\epsilon$ for $x \leq -\xi - R\epsilon^{2/3}$, provided that $\epsilon > 0$ is sufficiently small. In fact, it holds

$$\nu_\epsilon(x) = -\epsilon a\frac{f(x)}{\mu(x)} + O \left( \frac{\epsilon^3}{|x + \xi|^4} \right) < 0, \quad (2.3)$$

uniformly on $[-1, -\xi - R\epsilon^{2/3}]$, as $\epsilon \to 0$, and

$$|\nu'_\epsilon(x)| \leq C \frac{\epsilon}{|x + \xi|^2}, \quad |\nu''_\epsilon(x)| \leq C \frac{\epsilon}{|x + \xi|^3}, \quad x \in [-1, -\xi - R\epsilon^{2/3}].$$

Let

$$\varphi_\epsilon(x) = \nu_\epsilon(x) - \nu_\epsilon^*(x), \quad x \in [-1, -\xi - R\epsilon^{2/3}]. \quad (2.4)$$

It follows readily that $\varphi_\epsilon$ satisfies the following linear equation:

$$-\epsilon^2 \varphi'' + Q(x)\varphi = O \left( \frac{\epsilon^3}{|x + \xi|^3} \right), \quad (2.5)$$

with

$$Q(x) := \frac{v^3_\epsilon(x) - \mu(x)v_\epsilon(x) - a\epsilon f(x)}{v_\epsilon(x) - \nu_\epsilon(x)} \geq c|x + \xi|, \quad x \in [-1, -\xi - R\epsilon^{2/3}], \quad (2.6)$$
having used that $|v_\epsilon| \leq C\epsilon^{\frac{1}{3}}$ in $(-\infty, -\xi)$ (see Lemma 3.1 in [5]) and possibly increased the value of $R > 0$. In particular, it follows from the latter estimate that

$$\left| v_\epsilon(-\xi - R\epsilon^{\frac{2}{3}}) \right| \leq C\epsilon^{\frac{1}{3}}. \quad (2.7)$$

On the other side, it follows as in Proposition 2.1 in [9] that

$$\varphi_\epsilon(-1) = O(\epsilon^2) \text{ as } \epsilon \to 0. \quad (2.8)$$

In light of (2.5), (2.6), (2.7), (2.8), we deduce by a standard barrier argument that

$$|\varphi_\epsilon(x)| \leq C\epsilon^{\frac{1}{3}} e^{c\frac{x + \xi}{\epsilon^{\frac{2}{3}}}} + C\epsilon^2 + C\frac{\epsilon^3}{|x + \xi|}, \quad x \in [-1, -\xi - R\epsilon^{\frac{2}{3}}].$$

By combining (2.3), (2.4) and the above relation, we arrive at

$$v_\epsilon(x) = -\epsilon a f(x) \mu(x) \left[ 1 + O\left( \frac{\epsilon^2}{|x + \xi|^3} \right) + O\left( \frac{|x + \xi|}{\epsilon^{\frac{2}{3}}} e^{c\frac{x + \xi}{\epsilon^{\frac{2}{3}}}} \right) + O(\epsilon |x + \xi|) \right],$$

uniformly on $[-1, -\xi - R\epsilon^{\frac{2}{3}}]$, as $\epsilon \to 0$. Assuming that (2.1) does not hold. Then, we can evaluate the above relation at $x = \rho_\epsilon$, passing to a contradicting sequence of $\epsilon$’s that tend to zero along which it holds

$$\frac{\rho_\epsilon + \xi}{\epsilon^{\frac{2}{3}}} \to -\infty.$$

This gives us that

$$1 + O\left( \frac{\epsilon^2}{\rho_\epsilon + \xi)^3} \right) + O\left( e^{c\frac{\rho_\epsilon + \xi}{\epsilon^{\frac{2}{3}}}} \right) + o(\epsilon) = 0,$$

which is clearly a contradiction. We have thus established the desired lower bound in (2.1).

The main effort will be to establish the 'opposite direction' of (2.1). To this end, we will argue by contradiction. Assuming that the assertion of the theorem is false would give us a sequence $\epsilon_n \to 0$ such that

$$\frac{\rho_{\epsilon_n} + \xi}{\epsilon_n^{\frac{2}{3}}} \to +\infty. \quad (2.9)$$

Abusing notation, we will drop from now on the subscript $n$ and assume that all the following $\epsilon$’s are along this sequence or a subsequence of it.

Let $\eta_\epsilon$ be the unique negative solution of

$$\begin{cases}
\epsilon^2 \eta''(x) + \mu(x)\eta(x) - \eta^3(x) + \epsilon af = 0, & x < 0; \\
\eta(-\infty) = 0, & \eta(0) = 0.
\end{cases}$$
In passing, we note that uniqueness follows as in Remark 9 in [14]. For future purposes, we point out that the method of the aforementioned reference yields that

\[
\frac{v_\epsilon(x)}{\eta_\epsilon(x)} < 1, \quad x < \rho_\epsilon.
\]

(2.10)

Fine estimates for the convergence \( \eta_\epsilon \to -\sqrt{\mu^+} \) uniformly as \( \epsilon \to 0 \) are available from [10] (because the positive solution of (1.3) is nondegenerate, as we have remarked). In particular, \( \eta_\epsilon \) satisfies (1.5) with \( Y = Y_+ \). Essentially we will only need that there exist constants \( C, \delta > 0 \) such that

\[
-C\sqrt{x + \overline{\xi}} \leq \eta_\epsilon(x) \leq -\frac{1}{C}\sqrt{x + \overline{\xi}}, \quad \overline{\epsilon^2} \leq x + \overline{\xi} \leq \delta,
\]

\[
\eta_\epsilon(x) \sim -\sqrt{\mu'(0)(x + \overline{\xi})}, \quad \epsilon^\frac{2}{3} \ll x + \overline{\xi} \leq \delta,
\]

(2.11)

and that the above relations can be differentiated with respect to \( x \) in the obvious way (see also Appendix A in [1]). For a bit more refined estimates and a sketch of their derivation we refer to Remark 2.1 below.

Using a well known trick from [11], suppressing the dependence on \( \epsilon \), we write

\[
v = \eta w,
\]

(2.12)

for some \( w \) such that

\[
w > 0 \text{ for } x < \rho; \quad w < 0 \text{ for } x > \rho.
\]

(2.13)

For future purposes, let us note that the upper bound

\[
|v(x)| \leq C \left( \sqrt{x + \overline{\xi} + \epsilon^\frac{2}{3}} \right), \quad x \in (-\overline{\xi}, 0),
\]

which follows at once from Lemma 3.1 in [5], together with the identical lower bound which follows from (2.11) imply that

\[
|w(x)| \leq C, \quad x \in (-\overline{\xi}, 0).
\]

(2.14)

We find that \( w \) satisfies

\[
\epsilon^2 w'' + 2\epsilon^2 \frac{\eta'}{\eta} w' + \eta^2 (w - w^3) + \epsilon a \frac{f(x)}{\eta} (1 - w) = 0, \quad x < 0.
\]

(2.15)

Next, we stretch variables around \( x = \rho \), setting

\[
W(y) = w(x) \text{ where } y = \frac{x - \rho}{\tilde{\epsilon}}
\]

with \( \tilde{\epsilon} = \tilde{\epsilon}(\epsilon) > 0 \) to be determined such that

\[
\tilde{\epsilon} \ll \epsilon^\frac{2}{3}.
\]

(2.16)
We will denote with \( \dot{\cdot} \) the derivative with respect to \( y \). We obtain from (2.15) that
\[
\ddot{W} + 2\epsilon \eta' (\rho + \tilde{\epsilon} y) \dot{W} + \frac{\epsilon^2}{\epsilon^2 + \eta^2} (\rho + \tilde{\epsilon} y) (W - W^3) \\
+ a \frac{\epsilon^3}{\epsilon + \eta} (\rho + \tilde{\epsilon} y) (1 - W) = 0
\]
for \( y < -\rho/\tilde{\epsilon} \). In light of (2.9), (2.11) and the working assumption (2.16), it holds
\[
\frac{\epsilon^2}{\epsilon^2 + \eta^2} (\rho + \tilde{\epsilon} y) \sim \mu'(-\xi) \frac{\epsilon^2}{\epsilon^2} (\rho + \xi) \quad \text{as} \ \epsilon \to 0
\]
(for fixed \( y \)). Therefore, we choose
\[
\tilde{\epsilon} = \frac{\epsilon}{\sqrt{\rho + \xi}}.
\]
Recalling (2.9), this choice clearly satisfies (2.16). The point is that we want the third term in (2.17) to be of the same order as the first one; the other terms will turn out to be of smaller order. Then, equation (2.17) becomes
\[
\ddot{W} + 2\epsilon \sqrt{\rho + \xi} \frac{\eta'}{\eta} (\rho + \tilde{\epsilon} y) \dot{W} + \frac{\eta^2 (\rho + \tilde{\epsilon} y)}{\rho + \xi} (W - W^3) \\
+ a \frac{\epsilon^3}{\rho + \xi} \frac{f}{\eta} (\rho + \tilde{\epsilon} y) (1 - W) = 0
\]
(2.20)

By virtue of (2.14), we have that \( W \) is bounded locally with respect to \( \epsilon \). Hence, using standard elliptic estimates and the usual diagonal argument, keeping in mind (2.13), passing to a further subsequence if necessary, we find that
\[
W(y) \to W_0(y) \text{ in } C^1_{\text{loc}}(\mathbb{R}) \text{ as } \epsilon \to 0,
\]
(2.21)
where \( W_0 \) satisfies
\[
\frac{1}{\mu'(-\xi)} \ddot{W}_0 + W_0 - W_0^3 = 0, \quad y \in \mathbb{R}; \quad yW_0(y) \leq 0, \quad y \in \mathbb{R}.
\]

We claim that \( W_0 \) is nontrivial, which would imply that
\[
W_0(y) \equiv -\tanh \left( \sqrt{\frac{\mu'(-\xi)}{2}} y \right).
\]
(2.22)

In passing, we note that the explicit formula for \( W_0 \) is not important for our purposes, we will essentially use that
\[
W_0(y) \to \mp 1 \quad \text{as} \quad y \to \pm \infty.
\]
(2.23)

To this end, let \( Z_\epsilon \) be the unique positive solution of the following boundary value problem:
\[
\ddot{Z} + 2\epsilon \frac{\eta'}{\sqrt{\rho + \xi} \eta} (\rho + \tilde{\epsilon} y) \dot{Z} + \frac{\eta^2 (\rho + \tilde{\epsilon} y)}{\rho + \xi} (Z - Z^3) = 0, \quad y \in \left( \frac{-\xi + \epsilon^2}{\tilde{\epsilon} \rho}, 0 \right);
\]
As will become apparent shortly, the specific value \(1/4\) is not of importance. The existence of such a \(Z_\epsilon\) follows by directly minimizing the associated energy functional

\[
E_\epsilon(Z) = \int_{(\xi+\epsilon^2\rho)\epsilon^{-1}}^0 \left\{ \frac{\eta^2(\rho+\epsilon y)}{2} (\dot{Z})^2 + \frac{\eta^4(\rho+\epsilon y)(1-Z^2)^2}{\rho+\xi} \right\} dy \quad (2.24)
\]

subject to the above boundary conditions. Moreover, simple energy considerations give that \(0 < Z_\epsilon < 1\). The uniqueness comes from the observation that \(\{tZ_\epsilon : t \in (0,1)\}\) is a family of lower solutions to the above boundary value problem and Serrin’s sweeping principle (see [14] and the references therein). In fact, by virtue of (1.2), we note that these are also lower solutions to the equation (2.20) in the same interval. As a consequence of the assumption (2.9), the blow-up analysis in [5] yields that \(\eta_\epsilon\) and \(v_\epsilon\) share the same first order term in their corresponding inner expansions around \(-\xi\). Hence, both \(\eta_\epsilon\) and \(v_\epsilon\) satisfy (1.5) with \(Y = -Y_+\). In particular, given any \(L > 0\), it holds

\[
W_\epsilon \left( \frac{-\xi - \rho + L\epsilon^2}{\epsilon} \right) \to 1 \quad \text{as} \quad \epsilon \to 0. \quad (2.25)
\]

Thus, since \(W_\epsilon \left((\xi+\epsilon^2\rho)\epsilon^{-1}\right) > 1/2\), \(W_\epsilon(0) = 0\) and \(W_\epsilon > 0\) in between, we deduce by Serrin’s sweeping principle that

\[
W_\epsilon(y) > Z_\epsilon(y), \quad y \in \left((-\xi+\epsilon^2\rho)\epsilon^{-1}, 0\right).
\]

On the other hand, as in Proposition 2.3 in [12], we have

\[
Z_\epsilon \to -\tanh \left( \sqrt{\frac{\mu'(-\xi)}{2}} y \right) \quad \text{in} \quad C^1_{loc} ((-\infty, 0]) \quad \text{as} \quad \epsilon \to 0.
\]

The above two relations imply that \(W_0\) is nontrivial, and thus is given by (2.22) as claimed.

Let us consider now (2.20) without derivatives, which recalling (2.19) and after dividing by \(1-W\) becomes

\[
\Sigma^2 + \Sigma + a\epsilon \frac{f}{\eta^3}(\rho + \epsilon y) = 0.
\]

The above algebraic equation can be solved explicitly and has the following two solutions:

\[
\Sigma_{\pm}(y) = \frac{-1 \pm \sqrt{1 - 4a\epsilon \frac{f}{\eta^3}(\rho + \epsilon y)}}{2}.
\]
It follows from (2.9) and (2.11) that there is some fixed large $D > 0$ such that these are real valued for

$$-\xi + D\epsilon^{\frac{2}{3}} \leq \rho + \tilde{\epsilon}y \leq \delta,$$

provided that $\epsilon > 0$ is sufficiently small. From now on let us fix a $D$ such that

$$0 < 1 + \Sigma_-(y) < \frac{1}{100} \quad \text{and} \quad -\frac{1}{100} < \Sigma_+(y) < 0$$  \hspace{1cm} (2.26)

in the aforementioned interval, for sufficiently small $\varepsilon > 0$ (the number 100 has no significance here).

Let

$$\Phi(y) = 1 - W(y), \quad y \in \left[ D\sqrt{\rho + \xi\epsilon^{-\frac{1}{3}} - \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon}}, 0 \right].$$  \hspace{1cm} (2.27)

We know that

$$0 < \Phi < 1$$  \hspace{1cm} (2.28)

(recall (2.10)). We can write (2.20) as

$$-\ddot{\Phi} - \frac{2\epsilon}{\sqrt{\rho + \xi\eta}} \frac{\eta'}{\rho + \xi} (\rho + \tilde{\epsilon}y) \Phi + \frac{\eta^2(\rho + \tilde{\epsilon}y)}{\rho + \xi} (W - \Sigma_-(y)) (W - \Sigma_+(y)) \Phi = 0.$$  \hspace{1cm} (2.29)

By virtue of (2.21), (2.23), (2.25) with $L = D$, and (2.26), we deduce from (2.20) via the maximum principle that there exists an $M > 0$ such that

$$0 < 1 - W(y) < \frac{1}{100}, \quad y \in \left[ D\sqrt{\rho + \xi\epsilon^{-\frac{1}{3}} - \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon}}, -M \right],$$  \hspace{1cm} (2.30)

provided that $\epsilon > 0$ is sufficiently small. More precisely, we observe that $W_t = 1 - t$, $\frac{1}{100} < t \leq 1$, is a family of lower solutions to (2.20) which allows to use Serrin’s sweeping principle since $W > 0$, see [14] and the references therein. Concerning the second term in (2.29), it follows from (2.11) that

$$0 < \frac{\epsilon}{\sqrt{\rho + \xi\eta}} \frac{\eta'}{\rho + \xi} (\rho + \tilde{\epsilon}y) \leq C \frac{\epsilon^{\frac{3}{2}}}{\sqrt{\rho + \xi}}, \quad y \in \left[ D\sqrt{\rho + \xi\epsilon^{-\frac{1}{3}} - \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon}}, 0 \right];$$  \hspace{1cm} (2.31)

concerning the last term, we obtain from (2.11) and (2.30) that

$$C \geq \frac{\eta^2(\rho + \tilde{\epsilon}y)}{\rho + \xi} (W - \Sigma_-) (W - \Sigma_+) \geq \frac{1}{C}, \quad y \in \left[ -\frac{2(\rho + \xi)^{\frac{3}{2}}}{3\epsilon}, -M \right].$$  \hspace{1cm} (2.32)

So, by a standard barrier argument, we deduce from (2.28), (2.29), (2.31) and the above relation that

$$0 < \Phi(y) \leq e^{cy} e^{-\epsilon \left(y + \frac{2(\rho + \xi)^{\frac{3}{2}}}{3\epsilon}\right)}, \quad y \in \left[ -\frac{2(\rho + \xi)^{\frac{3}{2}}}{3\epsilon}, -M \right].$$
Consequently, by recalling the definition of $\Phi$ from (2.27), we infer that

$$0 < 1 - W(y) \leq e^{cy} + e^{-c\left(\frac{\rho + \xi}{2\epsilon}\right) \frac{\xi}{y}}, \quad y \in \left[-\frac{(\rho + \xi)^{\frac{3}{2}}}{2\epsilon}, 0\right]. \quad (2.33)$$

In turn, by standard elliptic estimates, via (2.29), (2.31) and the upper bound in (2.32), we get that

$$\left|\dot{W}(y)\right| \leq Ce^{cy} + Ce^{-c\left(\frac{\rho + \xi}{2\epsilon}\right) \frac{\xi}{y}}, \quad y \in \left[-\frac{(\rho + \xi)^{\frac{3}{2}}}{2\epsilon}, 0\right]. \quad (2.34)$$

Let us write (2.20) as

$$\ddot{W} + 2\frac{\epsilon}{\sqrt{\rho + \xi}} \eta' (\rho + \tilde{\epsilon}y) \dot{W} + \frac{\partial}{\partial W} G(W, y) = 0, \quad (2.35)$$

with

$$G(W, y) = -\frac{\eta^2 (\rho + \tilde{\epsilon}y)}{\rho + \xi} \frac{(1 - W^2)^2}{4} - a \frac{\epsilon f (\rho + \tilde{\epsilon}y)}{(\rho + \xi) \eta} \frac{(1 - W^2)^2}{2}. \quad (2.36)$$

When multiplied by $\dot{W}$, equation (2.35) reads as

$$\frac{d}{dy} \left(\frac{(\dot{W})^2}{2}\right) + 2\frac{\epsilon}{\sqrt{\rho + \xi}} \eta' (\rho + \tilde{\epsilon}y) (\dot{W})^2 + \frac{d}{dy} (G(W, y))$$

$$+ \frac{\epsilon}{(\rho + \xi)^{\frac{3}{2}}} \eta \eta' (\rho + \tilde{\epsilon}y) \frac{(1 - W^2)^2}{2}$$

$$- a \frac{\epsilon^2 f \eta' (\rho + \tilde{\epsilon}y)}{(\rho + \xi)^{\frac{3}{2}} \eta^2} \frac{(1 - W^2)^2}{2} + a \frac{\epsilon^2 f' (\rho + \tilde{\epsilon}y)}{(\rho + \xi)^{\frac{3}{2}} \eta} \frac{(1 - W^2)^2}{2} = 0. \quad (2.37)$$

We shall next integrate the above relation over

$$I = \left(-\frac{(\rho + \xi)^{\frac{3}{2}}}{2\epsilon}, (\delta - \rho)\frac{\sqrt{\rho + \xi}}{\epsilon}\right) = (\alpha, \beta).$$

The following fact will be useful in the sequel. By the definition (2.12) of $w$, and standard estimates for $v$ and $\eta$ that hold uniformly away from their corner layer at $-\xi$ (see Proposition 2.1 in [9]), we infer that

$$\|w(x) - \Sigma_- (\frac{x - \rho}{\epsilon})\|_{C^1[-\xi + \frac{\xi}{2}, -\xi + 2\delta]} \leq C\epsilon^2. \quad (2.38)$$
The integral of the first term in (2.37) is plainly the half of

\[
(W')^2 (\beta) - (\bar{W})^2 (\alpha) = \frac{c^2}{\rho + \xi} (w')^2 (-\xi + \delta) - (\bar{W})^2 \left( -\frac{(\rho + \xi)^3}{2\epsilon} \right)
\]

\[
= O \left( \frac{c^4}{\rho + \xi} \right) + O \left( \left( \frac{\epsilon}{\rho + \xi} \right)^{10} \right)
\]

\[
= O \left( \left( \frac{\epsilon}{\rho + \xi} \right)^6 \right),
\]

where we used (2.34), and (2.38) which implies that \( w' (-\xi + \delta) = O(\epsilon) \) (recall also the formula for \( \Sigma_- \)). We point out in passing that the rough estimate \( w' (-\xi + \delta) = O(1) \) would have actually sufficed for our purposes. Regarding the second term in (2.37), the fact that it is nonnegative will suffice for the time being. Using (2.33) and (2.38), we find from the definition of \( G \) in (2.36) that the integral of the third term in (2.37) can be estimated as follows:

\[
\int_I \frac{d}{dy} (G(W, y)) dy = G(W(\beta), \beta) - G(W(\alpha), \alpha) = O \left( \frac{\epsilon}{\rho + \xi} \right).
\]

Concerning the fourth term in (2.37), thanks to (2.11), we find that

\[
\int_I \eta \eta' (\rho + \bar{\epsilon}y) \frac{(1 - W^2)^2}{2} dy \geq c \int_I (1 - W^2)^2 dy.
\]

Concerning the fifth term in (2.37), recalling (2.14) and using once more (2.11), we have

\[
\left| \int_I \frac{f'}{\eta^2} (\rho + \bar{\epsilon}y) \frac{(1 - W^2)^2}{2} dy \right| \leq C \int_I (\rho + \xi + \bar{\epsilon}y)^{-\frac{3}{2}} dy
\]

\[
\leq C \bar{\epsilon}^{-1} (\rho + \xi)^{-\frac{3}{2}}
\]

\[
\leq \frac{C}{\epsilon}.
\]

Finally, concerning the last term in (2.37), we obtain similarly that

\[
\left| \int_I \frac{f'}{\eta} (\rho + \bar{\epsilon}y) \frac{(1 - W^2)^2}{2} dy \right| \leq \frac{C}{\epsilon}.
\]

Collecting all of the above, we infer from the integration of (2.37) over \( I \) that

\[
\int_I (1 - W^2)^2 dy \leq C.
\]

(2.39)

We claim that (2.39) implies that there exist \( c, N > 0 \) such that

\[
W(y) \leq -c, \quad y \in \left[ N, 5 \left( \frac{\rho + \xi}{\epsilon} \right)^{\frac{3}{2}} \right],
\]

(2.40)
if $\epsilon > 0$ is sufficiently small. Indeed, if not, then there would exist points

$$p_\epsilon \in \left[0, 5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon}\right] \quad \text{with} \quad p_\epsilon \to +\infty$$

such that

$$W(p_\epsilon) \to 0.$$ 

Let

$$\tilde{W}(y) = W(y + p).$$

Then, we have

$$\dddot{\tilde{W}} + 2 \frac{\epsilon}{\sqrt{\rho + \xi}} \frac{\eta'}{\eta} (\rho + \tilde{\epsilon}p + \tilde{\epsilon}y) \dot{\tilde{W}} + \frac{\eta^2 (\rho + \tilde{\epsilon}p + \tilde{\epsilon}y)}{\rho + \xi} (\tilde{W} - \tilde{W}^3) + a \frac{\epsilon}{\rho + \xi} f (\rho + \tilde{\epsilon}p + \tilde{\epsilon}y) (1 - \tilde{W}) = 0$$

(at least) for $y \in \left[-1, 100 \frac{\epsilon}{(\rho + \xi)^{\frac{3}{2}}}\right], \tilde{W} < 0$ therein, and $\tilde{W}(0) \to 0$. Then, since $\tilde{\epsilon}p \to 0$, we can argue as we did for (2.21) to find that $\tilde{W} \to \tilde{W}_0$ in $C^1_{loc}([-1, \infty))$, where

$$\frac{1}{\mu'(-\xi)} \tilde{W}_0 + \tilde{W}_0 - \tilde{W}_0^3 = 0, \quad y > -1; \quad \tilde{W}_0(y) \leq 0, \quad y > -1, \quad \tilde{W}_0(0) = 0.$$ 

It is clear that $\tilde{W}_0 \equiv 0$. So, given any $K > 1$, it holds

$$\int_I (1 - W^2)^2 dy \geq \int_p^{p+K} (1 - W^2)^2 dy = \int_0^K (1 - \tilde{W}^2)^2 dy \geq \frac{K}{2},$$

provided that $\epsilon > 0$ is sufficiently small, which contradicts (2.39) and establishes the validity of (2.40).

The estimate (2.39) yields that

$$\int_{\frac{5(\rho + \xi)^{\frac{3}{2}}}{\epsilon}}^{10(\rho + \xi)^{\frac{3}{2}}} (1 - W^2)^2 dy \leq C,$$

which implies that

$$|-1 - W(q)| < \frac{1}{100} \quad \text{for some} \quad q \in \left(5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon}, 10 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon}\right).$$

We may further assume that the $N$ in (2.40) is such that

$$-1 < W_0(N) < -1 + \frac{1}{1000}.$$
(recall (2.23)). Then, thanks to (2.21), (2.26) and (2.43), as we did for (2.30) we have that

\[ W(y) > -1 - \frac{1}{100}, \quad y \in \left[ N, 5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon} \right]. \]

On the other side, the relations (2.40) and (2.43) allow us to apply the same argument in the opposite direction, and thus arrive at

\[ |W(y) + 1| < \frac{1}{100}, \quad y \in \left[ N, 5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon} \right]. \] \quad (2.44)

Let now

\[ \Psi(y) = W(y) - \Sigma(y), \quad y \in \left[ 0, 5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon} \right]. \]

We can write (2.20) as

\[
\ddot{\Psi} + 2 \frac{\epsilon}{\sqrt{\rho + \xi}} \frac{\eta'}{\eta} (\rho + \bar{\epsilon}y) \dot{\Psi} + \frac{\eta^2(\rho + \bar{\epsilon}y)}{\rho + \xi} (1 - W) (W - \Sigma(y)) \Psi = \\
-\ddot{\Sigma} - 2 \frac{\epsilon}{\sqrt{\rho + \xi}} \frac{\eta'}{\eta} (\rho + \bar{\epsilon}y) \dot{\Sigma}.
\] \quad (2.45)

We note that

\[ |-1 - \Sigma(y)| \leq C \epsilon (\rho + \xi + \bar{\epsilon}y)^{-\frac{3}{2}}, \] \quad (2.46)

\[ \left| \dot{\Sigma}(y) \right| \leq C \frac{\epsilon^2}{\sqrt{\rho + \xi}} (\rho + \xi + \bar{\epsilon}y)^{-\frac{3}{2}}, \quad \left| \ddot{\Sigma}(y) \right| \leq C \frac{\epsilon^3}{\rho + \xi} (\rho + \xi + \bar{\epsilon}y)^{-\frac{7}{2}}, \] \quad (2.47)

for \( y \in \left[ 0, 5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon} \right] \). Thus, since

\[ 0 < \frac{\eta'}{\eta} (\rho + \bar{\epsilon}y) \leq \frac{C}{\rho + \xi}, \quad y \in \left[ -\frac{(\rho + \xi)^{\frac{3}{2}}}{2 \epsilon}, 5 \frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon} \right], \] \quad (2.48)

equation (2.45) becomes

\[
\ddot{\Psi} + 2 \frac{\epsilon}{\sqrt{\rho + \xi}} \frac{\eta'}{\eta} (\rho + \bar{\epsilon}y) \dot{\Psi} + \frac{\eta^2(\rho + \bar{\epsilon}y)}{\rho + \xi} (1 - W) (W - \Sigma(y)) \Psi = \\
O \left( \frac{\epsilon^3 (\rho + \xi + \bar{\epsilon}y)^{-\frac{7}{2}}} {\rho + \xi} \right),
\] \quad (2.49)
for \( y \in \left[ 0, \frac{5(\rho + \xi)^{3/2}}{\epsilon} \right] \). By virtue of (2.44), as we did previously for \( \Phi \) (also keep in mind (2.14)), we get that

\[
|W - \Sigma_-| \leq C \frac{e^3}{(\rho + \xi)^{3/2}} + Ce^{-cy} + Ce^{\left(y - 5\frac{(\rho + \xi)^{3/2}}{\epsilon}\right)}, \quad y \in \left[ 0, \frac{5(\rho + \xi)^{3/2}}{\epsilon} \right].
\]

Thus, it holds

\[
|W - \Sigma_-| \leq C \frac{e^3}{(\rho + \xi)^{3/2}} + Ce^{-cy}, \quad y \in \left[ 0, \frac{4(\rho + \xi)^{3/2}}{\epsilon} \right]. \quad (2.50)
\]

In turn, by using the upper bound

\[
\frac{y^2(\rho + \tilde{c}y)}{\rho + \xi} \leq C, \quad y \leq \frac{5(\rho + \xi)^{3/2}}{\epsilon},
\]

we deduce from (2.48), (2.49), (2.50), and standard elliptic estimates that

\[
|\dot{W} - \dot{\Sigma}_-| \leq C \frac{e^3}{(\rho + \xi)^{3/2}} + Ce^{-cy}, \quad y \in \left[ 0, \frac{(\rho + \xi)^{3/2}}{\epsilon} \right]. \quad (2.51)
\]

By combining (2.46) and (2.50), we find that

\[
|W + 1| \leq C \frac{e^3}{(\rho + \xi)^{3/2}} + Ce^{-cy}, \quad y \in \left[ 0, \frac{(\rho + \xi)^{3/2}}{\epsilon} \right]. \quad (2.52)
\]

By combining (2.47) and (2.51), we get that

\[
|\dot{W}| \leq C \frac{e^2}{(\rho + \xi)^3} + Ce^{-cy}, \quad y \in \left[ 0, \frac{(\rho + \xi)^{3/2}}{\epsilon} \right]. \quad (2.53)
\]

We shall integrate the above relation over

\[
J = \left( -\frac{(\rho + \xi)^{3/2}}{2\epsilon}, \frac{(\rho + \xi)^{3/2}}{2\epsilon} \right) = (\bar{\alpha}, \bar{\beta}).
\]

Below we will carefully estimate each term in the resulting relation. Thanks to (2.34) and (2.53), in regards to the first term in (2.37), we find that

\[
(\dot{W})^2 \left( \frac{(\rho + \xi)^{3/2}}{2\epsilon} \right) - (\dot{W})^2 \left( -\frac{(\rho + \xi)^{3/2}}{2\epsilon} \right) = O \left( \frac{e^4}{(\rho + \xi)^6} \right),
\]

Regarding the second term in (2.37), thanks to (2.11) and (2.21), we have

\[
0 < \frac{e^3}{\epsilon} \frac{\eta'}{\sqrt{\rho + \xi}} \eta' (\rho + \tilde{c}y) (\dot{W})^2 \to \frac{1}{2}(\dot{W}_0)^2 \text{ in } C_{loc}(\mathbb{R}) \text{ as } \epsilon \to 0.
\]
Therefore, we obtain from (2.34), (2.48), (2.53), and Lebesgue’s dominated convergence theorem that

$$
\frac{(\rho + \xi)^{\frac{3}{2}}}{\epsilon} \frac{\epsilon}{\sqrt{\rho + \xi}} \int_J \frac{\eta'}{\eta} (\rho + \tilde{\epsilon}y)(\tilde{W})^2 dy \to \frac{1}{2} \int_{-\infty}^{\infty} (\tilde{W}_0)^2 dy.
$$

Recalling (1.2), (2.33) and (2.36), the integral of the third term in (2.37) can be estimated from above as follows:

$$
\int_J \frac{d}{dy} \left( G(W, y) \right) dy = G \left( W(\tilde{\beta}), \tilde{\beta} \right) - G \left( W(\tilde{\alpha}), \tilde{\alpha} \right) \leq -G \left( W(\tilde{\alpha}), \tilde{\alpha} \right) = O \left( e^{-c\frac{\rho + \xi}{\epsilon^2}} \right).
$$

Concerning the fourth term in (2.37), by working as we did for (2.54), using (2.33) and (2.52), we find that

$$
\int_J \eta' (\rho + \tilde{\epsilon}y) \frac{(1 - W^2)}{2} dy \to \frac{\mu'(-\xi)}{4} \int_{-\infty}^{\infty} (1 - W_0^2)^2 dy = \frac{1}{2} \int_{-\infty}^{\infty} (\tilde{W}_0)^2 dy.
$$

Concerning the fifth term in (2.37), recalling that \( f(x) < 0 \) for \( x < 0 \) while \( W(y) < 0 \) for \( y > 0 \), and keeping in mind (2.11), we have

$$
\int_J \frac{f'}{\eta^2} (\rho + \tilde{\epsilon}y) \frac{(1 - W^2)}{2} dy \geq \int_0^{\rho + \xi} \frac{f'}{\eta^2} (\rho + \tilde{\epsilon}y) \frac{(1 - W^2)}{2} dy \geq \frac{c}{(\rho + \xi)^{\frac{3}{2}}} \int_0^{\rho + \xi} \frac{1 dy}{2} \geq \frac{c}{\epsilon}.
$$

Concerning the last term in (2.37), by using (2.14) and that \( -\eta \geq c\sqrt{\rho + \xi} \) in \( J \), we get that

$$
\left| \int_J \frac{f'}{\eta} (\rho + \tilde{\epsilon}y) \frac{(1 - W^2)}{2} dy \right| \leq C \frac{\rho + \xi}{\epsilon}.
$$

Putting all the above in the integral of (2.37) over \( J \) gives us that

$$
ca \frac{\epsilon}{(\rho + \xi)^{\frac{3}{2}}} \leq aC \frac{\epsilon}{\sqrt{\rho + \xi}} + \frac{\epsilon}{(\rho + \xi)^{\frac{3}{2}}} \left( \frac{3}{2} \int_{-\infty}^{\infty} (\tilde{W}_0)^2 dy + o(1) \right),
$$

with constants \( c, C \) independent of both \( \epsilon \) and \( \alpha \). We see that the above relation cannot hold if \( a^{**} \) in the assertion of the theorem is chosen sufficiently large, provided that \( \epsilon > 0 \) is sufficiently small. We have thus reached a contradiction which completes the proof.
Remark 2.1. Let us give the formal argument that led us to the rigorous analysis following (2.21). According to folklore, we write

$$W(y) = W_0(y) + \hat{\epsilon} W_1(y) + \text{higher order terms},$$

and try to find $\hat{\epsilon}$ and $W_1$ by plugging this into (2.20).

Firstly, we will need the following estimates for $-\eta$ near $-\xi$ that follow essentially from (2.14) and by working as we did to establish (2.51), see also [10, Thm. 1.1]. It holds

$$\eta(x) = \nu(x) + O(\epsilon^2) x^{-\frac{3}{2}}, \quad x \in (-\xi + \epsilon^2, -\xi + \delta),$$

uniformly as $\epsilon \to 0$, for some fixed small $\delta > 0$, where $\nu < 0$ solves the algebraic equation (2.2) near $-\xi$. It follows readily that

$$\nu(x) = -\sqrt{\mu'(-\xi)(x + \xi)} + \frac{af(-\xi)}{2\mu'(-\xi)} \frac{\epsilon}{x + \xi} + O(\epsilon).$$

We note that the relations (2.9), (2.11), yield the following:

$$\frac{1}{\rho + \xi} \eta^2 \sqrt{\rho + \epsilon y} = \frac{\rho(-\xi) + \mu'(-\xi)}{\rho + \xi} y - \frac{af(-\xi)}{\sqrt{\mu'(-\xi)}} \frac{\epsilon}{x + \xi} + O(\epsilon) \quad \text{and} \quad \frac{\epsilon}{\sqrt{\rho + \xi}} \eta' \frac{\epsilon}{\sqrt{\rho + \xi}} = \frac{\epsilon}{2(\rho + \xi)^{\frac{3}{2}}} + O\left(\frac{\epsilon}{(\rho + \xi)^{\frac{3}{2}}}(|y| + 1)\right),$$

$$\frac{\epsilon}{\rho + \xi \eta} \frac{f}{\sqrt{\rho + \xi}} = \frac{\epsilon}{\sqrt{\mu'(-\xi)}} \frac{\epsilon}{\rho + \xi} + O\left(\frac{\epsilon}{(\rho + \xi)^{\frac{3}{2}}}(|y| + 1)\right),$$

uniformly for $|\hat{\epsilon} y| \leq \delta$, as $\epsilon \to 0$.

Taking the above into account, we find that

$$\hat{\epsilon} = \frac{\epsilon}{\rho + \xi}$$

and

$$\hat{W}_1 + (1 - 3W_0^2)W_1 = -\frac{\mu'(-\xi)}{2} y(W_0 - W_0^3) - (\hat{W}_0)^2 + \frac{af(-\xi)}{\sqrt{\mu'(-\xi)}} (W_0 - W_0^3) + \frac{af(-\xi)}{\sqrt{\mu'(-\xi)}} (1 - W_0)$$

for $y \in \mathbb{R}$. However, the last term does not decay to zero as $y \to +\infty$, which is necessary in order to get a solution $W_1$ that decays to zero as $|y| \to \infty$. We point out that in the rigorous proof we took advantage of this non-decay property in (2.55).
References

[1] Amandine Aftalion, Benedetta Noris, and Christos Sourdis. Thomas-Fermi approximation for coexisting two component Bose–Einstein condensates and nonexistence of vortices for small rotation. *Communications in Mathematical Physics*, 336(2):509–579, 2015.

[2] Nicholas D Alikakos and Peter W Bates. On the singular limit in a phase field model of phase transitions. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 5, pages 141–178. Elsevier, 1988.

[3] Antonio Ambrosetti, Andrea Malchiodi, and Wei-Ming Ni. Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, part I. *Communications in Mathematical Physics*, 235(3):427–466, 2003.

[4] Tom Claeys, Arno BJ Kuijlaars, and Maarten Vanlessen. Multi-critical unitary random matrix ensembles and the general Painlevé II equation. *Annals of Mathematics*, pages 601–641, 2008.

[5] Marcel G Clerc, Juan Diego Dávila, Michal Kowalczyk, Panayotis Smyrnelis, and Estefania Vidal-Henriquez. Theory of light-matter interaction in nematic liquid crystals and the second Painlevé equation. *Calculus of Variations and Partial Differential Equations*, 56(4):93, 2017.

[6] Marcel G Clerc, Michal Kowalczyk, and Panayotis Smyrnelis. Gradient theory of domain walls in thin, nematic liquid crystals films. *arXiv preprint arXiv:1809.01034*, 2018.

[7] Marcel G Clerc, Michal Kowalczyk, and Panayotis Smyrnelis. Symmetry breaking and restoration in the Ginzburg–Landau model of nematic liquid crystals. *Journal of Nonlinear Science*, 28(3):1079–1107, 2018.

[8] Stuart P Hastings and John Bryce Mcleod. A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Archive for Rational Mechanics and Analysis*, 73(1):31–51, 1980.

[9] Radu Ignat and Vincent Millot. The critical velocity for vortex existence in a two-dimensional rotating Bose–Einstein condensate. *Journal of Functional Analysis*, 233(1):260–306, 2006.

[10] Georgia Karali and Christos Sourdis. The ground state of a Gross–Pitaevskii energy with general potential in the Thomas–Fermi limit. *Archive for Rational Mechanics and Analysis*, 217(2):439–523, 2015.

[11] Lotfi Lassoued and Petru Mironescu. Ginzburg–landau type energy with discontinuous constraint. *Journal d'Analyse Mathématique*, 77(1):1–26, 1999.

[12] Kinme Nakashima. Multi-layered stationary solutions for a spatially inhomogeneous Allen–Cahn equation. *Journal of Differential Equations*, 191(1):234–276, 2003.

[13] Stephen Schechter and Christos Sourdis. Heteroclinic orbits in slow–fast hamiltonian systems with slow manifold bifurcations. *Journal of Dynamics and Differential Equations*, 22(4):629–655, 2010.

[14] C Sourdis. On the uniqueness of solutions to a class of semilinear elliptic equations by Serrin’s sweeping principle. *Rend. Sem. Mat. Univ. Politec. Torino Bruxelles-Torino Talks in PDEs Turin, May 25, 2016, 74(2).

[15] William C Troy. The role of Painlevé II in predicting new liquid crystal self-assembly mechanisms. *Archive for Rational Mechanics and Analysis*, 227(1):367–385, 2018.

Institute of Applied and Computational Mathematics, FORTH, GR711 10 Heraklion, Greece.

E-mail address: sourdis@uoc.gr