OPTIMAL REINSURANCE-INVESTMENT AND DIVIDENDS PROBLEM WITH FIXED TRANSACTION COSTS

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ABSTRACT. In this paper, we consider the dividend optimization problem for a financial corporation with fixed transaction costs. Besides the dividend control, the financial corporation takes proportional reinsurance to reduce risk and invests its reserve in a financial market consisting of a risk-free asset (bond) and a risky asset (stock). Because of the presence of the fixed transaction costs, the problem becomes a mixed classical-impulse stochastic control problem. We solve this problem explicitly and construct the value function together with the optimal policy.

1. Introduction. Applying continuous time stochastic processes to model financial problems and in particular to (re)insurance problems has attracted significant interest over recent decades. There exists an extensive literature on applications of stochastic calculus and stochastic control techniques to various insurance problems such as maximizing the expected present value of the dividends paid out up to the time of bankruptcy.

Recently, the dynamic programming approach is becoming an important tool in solving optimal reinsurance, dividends and portfolio selection problems, see [6, 24, 30, 28, 33, 27], etc. An excellent survey of stochastic models for the optimal dividend policy can be found in [26]. For other contributions to this problem,
see [2, 23, 18, 15, 16, 1, 10, 11, 29]. In all of these papers, the reserve process of an insurance company is driven by a Brownian motion with constant drift and diffusion coefficients. Moreover they assume costless trading and the optimal strategy consisting of trading or intervention can act at every time instant.

Taking transaction costs into account, how to solve the optimal dividend problem lies in the theory of impulse control (see [5]). There are two common transaction costs: proportional cost (see for instance [7, Section 4]) and fixed costs (see [7, Section 6]). The impulse control theory was introduced by [4] and extended by [25]. Other important theoretical contributions include [14, 5, 12, 13] and [20, 21, 31, 22]. [19] first applies the impulse control theory to obtain the optimal dividend policy when there is a fixed cost each time for the dividend paid out. [9] applies the classical and impulse control to deal with exchange rate problems. The classical and impulse stochastic control for the optimization of the dividend and risk policy is studied by [8]. [32] investigates the optimal impulsive dividend problem for the classical risk model perturbed by diffusion.

In this paper, we extend the results of [8] by involving the investment choice, that is, besides the proportional reinsurance and dividends payment, the financial corporation has an option of investing its reserve in a financial market which is described by a classical Black-Scholes model. Just as in [8], the distribution of dividends is controlled by a sequence of stopping times \( \{\tau_i; i = 1, 2, \cdots\} \) and a sequence of random variables \( \{\xi_i; i = 1, 2, \cdots\} \). The net amount of dividends during \([0, t]\) equals \( \sum_{i=1}^{\infty} (-K + k \xi_i) I_{\{\tau_i \leq t\}} \). We can interpret \( 1 - k \) as tax rate and \( -K \) as the fixed cost when the dividends are paid out. Our objective is to find the optimal control strategy which maximizes the expected present value of the dividends paid out up to the time of bankruptcy and find the optimal control strategy. In [8], the convexity of the derivative of the value function \( V(x) \) (see (6) for the definition of \( V(x) \)) in the continuation region plays important role for characterizing the optimal control strategy. However, in our case, due to incorporate the investment control, the equation satisfied by the value function in the continuation region becomes to a second order differential equation with the coefficients depend on both the reinsurance and investment control. The solution of this equation for the value function includes an infinite interval integral, which makes it hard to characterize its properties. In particular, we can not prove the convexity of \( V'(x) \) in our case due to the complex expression of \( V'(x) \), which is caused by involving the investment control. Fortunately, we find that if we weak the requirement of the convexity of \( V'(x) \) to require \( V''(x) > 0 \) for \( x \) greater than some constant, all the results in [8] for characterizing the optimal control strategy still hold true. Thus, the monotonicity of \( V'(x) \) for \( x \) greater than some constant is essential for obtaining the optimal control strategy in our model. The difference between the value function \( V(x) \) of our model in continuation region and that of [17] is just a positive constant \( C \), which is part of the characterizations of the optimal control strategy. However, we can not get the expression of the value function in our model directly from [17] and the proof of the existence of the positive constant \( C \) is not easy. Therefore, we develop some new properties (see Proposition 5.2 and 5.3) of the value function \( V(x) \) for proving the existence of the positive constant \( C \) and other characterizations of the optimal control strategy (see Theorem 6.1).

This paper is organized as follows. In Section 2, a rigorous mathematical formulation of the model is given. In Section 3, we present some characterizations of the value function and associate it to the quasi-variational inequalities (QVI).
The structure of the optimal control is conjectured in Section 4. A solution of QVI equations in the continuation region and some characterizations of the conjectured parameters of the optimal control are present in Section 5. In Section 6, we give the explicit expression for the value function and the corresponding optimal policy. Some numerical analysis are present in Section 7.

2. The model. To give a rigorous mathematical formulation of the optimization problem, we start with a probability space \((\Omega, \mathcal{F}, P)\) and a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). The filtration \(\mathcal{F}_t\) represents the information available at time \(t\) and any control made is based upon this information. We assume that the reserve process of an insurance company \(\{R_t\}\) evolves according to

\[
dR_t = \mu dt + \sigma dB_t,
\]

where \(\mu > 0, \sigma > 0, \{B_t\}\) is a standard Brownian motion with respect to \(\{\mathcal{F}_t\}\). The form of proportional reinsurance is taken by the insurance company for reducing risk. Mathematically, if an insurance company applies dynamic proportional reinsurance rule which is governed by a parameter \(a(t) \in [0, 1]\), then the reserve process of an insurance company after proportional reinsurance is

\[
dR_t = a(t)(\mu dt + \sigma dB_t).
\]

The distribution of dividends is controlled by a sequence of increasing stopping times \(\{\tau_i; i = 1, 2, \ldots\}\) and a sequence of nonnegative random variables \(\{\xi_i; i = 1, 2, \ldots\}\) which are associated with the times and amounts of dividends paid out to the shareholders.

Furthermore we have a classical Black-Scholes market, that is, we have a risk-free asset whose price process \(\{S^0_t\}\) satisfies

\[
dS^0_t = r_0 S^0_t dt,
\]

and a risky asset whose price process \(\{S^1_t\}\) satisfies

\[
dS^1_t = S^1_t(r_1 dt + \sigma_p dW_t),
\]

where \(\sigma_p > 0, r_1 > r_0, \{W_t\}\) is another standard Brownian motion with respect to \(\{\mathcal{F}_t\}\) independent of \(\{B_t\}\).

Let \(b(t)\) be a nonnegative process denoting the amount of money invested in the risky asset at time \(t\). In our model we will not allow \(b(t)\) to exceed the reserve at time \(t\), that is,

\[
0 \leq b(t) \leq R_t.
\]

Thus after proportional reinsurance, dividends control and investment, the dynamics of the controlled state process is given by

\[
R_t = x + \int_0^t (\mu a(t) + r_1 b(t) + r_0(R_t - b(t)))dt + \int_0^t \sigma a(t) dB_t + \int_0^t \sigma_p b(t) dW_t - \sum_{i=1}^{\infty} I_{(\tau_i \leq t)} \xi_i,
\]

where \(x > 0\) is the initial reserve. The time of the bankruptcy is the stopping time defined by

\[
\tau = \tau^x := \inf\{t \geq 0 : R_t = 0\}.
\]
Since we only deal with the optimization problem during the time interval \([0, \tau]\), we shall assume that \(R_t\) vanishes for \(t \geq \tau\). Thus the reserve process is given by

\[
R_t = \begin{cases} 
  x + \int_0^t (\mu a(t) + r_1 b(t) + r_0 (R_t - b(t))) dt + \int_0^t \sigma a(t) dB_t \\
  0, \quad &\text{if } t < \tau, \\
  \sum_{i=1}^\infty I_{\{\tau_i \leq t\}} \xi_i, \quad &\text{if } t \geq \tau.
\end{cases}
\]

**Definition 2.1.** A fourfold mixed classical-impulse stochastic control

\[
\pi := (a, b, T, \xi) = (a(t); b(t); \tau_1, \tau_2, \ldots, \tau_n; \xi_1, \xi_2, \ldots, \xi_n, \ldots)
\]
is an admissible control if it satisfies the following conditions:
1. The processes \(\{a(t)\}_{t \geq 0} \) and \(\{b(t)\}_{t \geq 0} \) are predictable with respect to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) such that \((a(t), b(t)) \in [0, 1] \times [0, R_t]\) for every \(t \geq 0\);
2. \(\{\tau_i\}_{i \geq 1}\) is an increasing sequence of stopping times with respect to \(\{\mathcal{F}_t\}_{t \geq 0}\);
3. For each \(i \geq 1\), \(\xi_i \in \mathcal{F}_{\tau_i}\) and \(0 < \xi_i \leq R_{\tau_i} \);
4. The reserve process \(\{R_t\}_{t \geq 0}\) under the control \(\pi\) satisfies the following inequality

\[
E \left[ \int_0^\infty e^{-ct} R_t^2 dt \right] < \infty.
\]

In what follows, we shall denote by \(\mathcal{A}(x)\) the class of all admissible mixed classical-impulse stochastic controls.

Define the function \(g : [0, +\infty) \rightarrow (-\infty, +\infty)\) by

\[
g(\eta) = k\eta - K.
\]

where \(K \in (0, +\infty)\) and \(k \in (0, 1)\) are constants. The constant \(K\) represents the fixed setup cost which is incurred each time that a dividend is paid out and the constant \(1 - k\) as the tax rate at which the dividends are taxed. Therefore, at each time that a dividend \(\eta\) is paid out, the shareholder finally received only the amount \(g(\eta)\) because of the the transaction cost and the tax. Naturally the shareholder of the insurance company wants to maximize the expected present value of the received dividends and so we can formulate this problem as follows.

**Problem 1.** Given any \(\pi \in \mathcal{A}(x)\), the performance functional for the insurance company is defined by

\[
J(x, \pi) := E \left[ \sum_{i=1}^\infty e^{-c\tau_i} g(\xi_i) I_{\{\tau_i \leq \tau\}} \right],
\]

where \(c > r_1 > 0\) and \(g(\cdot)\) is defined by (5). The objective of the insurance company is to select an optimal control \(\pi \in \mathcal{A}(x)\) so as to maximize the performance functional \(J(x, \pi)\) which represents the expected present value of the dividends received by the shareholder until the time of bankruptcy. That is to determine the value function

\[
V(x) := \sup_{\pi \in \mathcal{A}(x)} J(x, \pi) = \sup_{\pi \in \mathcal{A}(x)} E \left[ \sum_{i=1}^\infty e^{-c\tau_i} g(\xi_i) I_{\{\tau_i \leq \tau\}} \right],
\]

and an optimal control \(\pi^* \in \mathcal{A}(x)\) such that \(V(x) = J(x, \pi^*)\).

**Remark 1.** Note that if \(c < r_1\), we can prove that the optimal value function is infinite and so we restrict our problem under the assumption of \(c > r_1 > 0\). Please see more details in the appendix for the proof of the infinity of the optimal value function.
3. Characterizations of the value function. In this section, we shall present some characterizations of the value function. To this aim, we need to define the maximum utility operator $M$ by

$$M \phi(x) := \sup \{ \phi(x - \eta) + g(\eta) : \eta > 0, x \geq \eta \},$$

where $g(\cdot)$ is given by (5) and $\phi$ is a function mapping from $[0, \infty)$ to $\mathbb{R}$. $MV(x)$ represents the value of the strategy that consists of choosing the best intervention times and the amounts of the future dividend payments.

Let $L^{a,b}$ be the differential operator defined by

$$L^{a,b} \psi(x) := \frac{1}{2} (a^2 \sigma^2 + \sigma^2 \rho^2) \psi''(x) + (\mu a + r_1 b + r_0(x - b)) \psi'(x) - c \psi(x).$$

Now similar to [9] or [8], we intend to derive the inequalities satisfied by the value function $V$ and the corresponding optimal control. Let

$$C(x) = \{(a, b) \in \mathbb{R}^2 : a \in [0, 1], b \in [0, x]\}. \quad (7)$$

**Definition 3.1. (QVI)** We say that a function $W : [0, \infty) \rightarrow [0, \infty)$ satisfies the quasi-variational inequalities for Problem 1 if for every $x \in [0, \infty)$ and $(a, b) \in C(x)$,

$$L^{a,b} W(x) \leq 0, \quad (8)$$

$$MW(x) \leq W(x), \quad (9)$$

$$(W(x) - MW(x)) \left( \max_{(a, b) \in C(x)} L^{a,b} W(x) \right) = 0, \quad (10)$$

$$W(0) = 0. \quad (11)$$

We observe that a solution $W$ of the QVI separates the interval $[0, \infty)$ into two disjoint regions: a continuation region

$$C := \left\{ x \in (0, \infty) : MW(x) < W(x) \text{ and } \max_{(a, b) \in C(x)} L^{a,b} W(x) = 0 \right\}$$

and an intervention region

$$\Sigma := \left\{ x \in (0, \infty) : MW(x) = W(x) \text{ and } \max_{(a, b) \in C(x)} L^{a,b} W(x) \leq 0 \right\}.$$

Given a solution $W$ to the QVI, it is possible to construct the following mixed classical-impulse stochastic control associated with $W$.

**Definition 3.2.** Let $W$ be a solution of the QVI. The control

$$\pi^W = (a^W, b^W, T^W, \xi^W) = (a^W, b^W, \tau_1^W, \tau_2^W, \ldots, \tau_n^W, \ldots ; \xi_1^W, \xi_2^W, \ldots)$$

is called the QVI control associated with $W$ if for any $t > 0$

$$P \left\{ (a^W(t), b^W(t)) \neq \arg \max_{(a, b) \in \mathcal{C}(R^W_t)} L^{a,b} W(R^W_t), R^W_t \in C \right\} = 0,$$

$$\tau_1^W := \inf \{ t \geq 0 : W(R^W_t) = MW(R^W_t) \},$$

$$\xi_1^W := \arg \sup_{\eta > 0, \eta \leq R^W_{\tau_1^W}} \{ W(R^W_{\tau_1^W} - \eta) + g(\eta) \},$$

and for $n \geq 2$

$$\tau_n^W := \inf \{ t > \tau_{n-1}^W : W(R^W_t) = MW(R^W_t) \};$$

$$\xi_n^W := \arg \sup_{\eta > 0, \eta \leq R^W_{\tau_n^W}} \{ W(R^W_{\tau_n^W} - \eta) + g(\eta) \},$$
with \( \tau_0^W := 0, \xi_0^W := 0 \). Here \( R_t^W \) is the trajectory determined by \( \pi^W \).

**Theorem 3.3.** Let \( W \in C^1(0, \infty) \) be a nonnegative increasing solution of the QVI (8)-(11). Suppose that there exists \( L > 0 \) such that \( W \) is twice continuously differentiable on \((0, L)\) and \( W \) is linear on \([L, \infty)\). Then for every \( x \in (0, \infty) \), the QVI control associated with \( W \) is optimal and \( W \) coincides with the value function, i.e.

\[
V(x) = W(x) = J(x, \pi^W).
\]

**Proof.** The idea of the proof is similar to that of Theorem 3.4 of [8]. See also [9].

Let \( \pi \) be an admissible policy and denote \( \{R_t\}_{t \geq 0} \) by the trajectory determined by \( \pi \). Let \( \tau_0 = 0 \) and \( \tau^\pi \) be the first hitting time of \([0, \varepsilon]\) by \( \{R_t\} \). Since \( R \) is a continuous semimartingale on the stochastic interval \([\tau_{i-1}, \tau_i]\) and \( W \) is twice continuous differentiable on \((0, \infty) \setminus \{L\} \), we may apply Itô’s formula to get

\[
e^{-c(t \wedge \tau^\pi \wedge \tau_i)}W(R_{t \wedge \tau^\pi \wedge \tau_i}) - e^{-c(t \wedge \tau^\pi \wedge \tau_{i-1})}W(R_{t \wedge \tau^\pi \wedge \tau_{i-1}})
\]

\[
= \int_{[t \wedge \tau^\pi \wedge \tau_i-1, t \wedge \tau^\pi \wedge \tau_i]} e^{-cs} \left[ \frac{1}{2} (a(s)^2 \sigma^2 + \sigma^2_b(s)^2) W''(R_s) - cW(R_s) \right] ds
\]

\[
+ \int_{[t \wedge \tau^\pi \wedge \tau_i-1, t \wedge \tau^\pi \wedge \tau_i]} e^{-cs} \{ma(s) + r_1b(s) + r_0[R_s - b(s)]\} W'(R_s) ds
\]

\[
+ \int_{[t \wedge \tau^\pi \wedge \tau_i-1, t \wedge \tau^\pi \wedge \tau_i]} e^{-cs} W'(R_s) \sigma a(s) dB_s
\]

\[
+ \int_{[t \wedge \tau^\pi \wedge \tau_i-1, t \wedge \tau^\pi \wedge \tau_i]} e^{-cs} W'(R_s) \sigma_p b(s) dW_s
\]

\[
\leq \int_{[t \wedge \tau^\pi \wedge \tau_i-1, t \wedge \tau^\pi \wedge \tau_i]} e^{-cs} W'(R_s) \sigma a(s) dB_s
\]

\[
+ \int_{[t \wedge \tau^\pi \wedge \tau_i-1, t \wedge \tau^\pi \wedge \tau_i]} e^{-cs} W'(R_s) \sigma_p b(s) dW_s,
\]

where the last inequality follows from (8) and becomes an equality for the QVI control \( \pi^W \) associated with \( W \). According to inequality (9), we have

\[
e^{-ct} \{W(R_{t \wedge \tau^\pi \wedge \tau_i}) - W(R_{t \wedge \tau^\pi \wedge \tau_{i-1}})\} \leq -e^{-ct} g(\xi_i)
\]

on the event \( \{\tau_i \leq t \wedge \tau^\pi\} \) and the inequality also becomes an equality for the QVI control associated with \( W \). Thus for each \( n \geq 1 \),

\[
e^{-c(t \wedge \tau^\pi \wedge \tau_n)}W(R_{t \wedge \tau^\pi \wedge \tau_n}) - W(R_0)
\]

\[
= \sum_{i=1}^{n} \left\{ e^{-c(t \wedge \tau^\pi \wedge \tau_i)}W(R_{t \wedge \tau^\pi \wedge \tau_i}) - e^{-c(t \wedge \tau^\pi \wedge \tau_{i-1})}W(R_{t \wedge \tau^\pi \wedge \tau_{i-1}}) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ e^{-c(t \wedge \tau^\pi \wedge \tau_i)}W(R_{t \wedge \tau^\pi \wedge \tau_i}) - e^{-c(t \wedge \tau^\pi \wedge \tau_{i-1})}W(R_{t \wedge \tau^\pi \wedge \tau_{i-1}}) \right\}
\]
\[ + \sum_{i=1}^{n} I_{\{\tau_i \leq t \wedge \tau^\ast\}} e^{-ct_{\tau_i}} \{W(R_{\tau_i}) - W(R_{\tau_i -})\} \leq \sum_{i=1}^{n} \int_{t \wedge \tau_i \wedge \tau^\ast \wedge \tau_{i-1}} e^{-ct} W'(R_s) \sigma a(s) dB_s \]
\[ + \sum_{i=1}^{n} \int_{t \wedge \tau_i \wedge \tau^\ast \wedge \tau_{i-1}} e^{-ct} W'(R_s) \sigma b(s) dW_s \]
\[ - \sum_{i=1}^{n} I_{\{\tau_i \leq t \wedge \tau^\ast\}} e^{-ct_{\tau_i}} g(\xi_i). \]

Note that the continuous differentiability of \( W \) implies its continuity and therefore its boundedness in the compact interval \([0, L]\). Furthermore, since \( W' \) is a continuous function and it is a constant for \( x \geq L \), \( W' \) is bounded in \([\varepsilon, \infty)\), where \( \varepsilon \) is a small positive constant. The inequality of (4) and the boundedness of \( W' \) on \([\varepsilon, \infty)\) imply that
\[ E \left[ \int_{0}^{\tau^\ast} \{e^{-ct} R_t W'(R_t)\}^2 dt \right] < \infty. \]

Therefore, taking expectations on both side of (12) leads to
\[ W(x) - E \left[ e^{-c(t \wedge \tau^\ast \wedge \tau_n)} W(R_{t \wedge \tau^\ast \wedge \tau_n}) \right] \geq E \left[ \sum_{i=1}^{n} I_{\{\tau_i \leq t \wedge \tau^\ast\}} e^{-ct_{\tau_i}} g(\xi_i) \right], \]
where the inequality becomes equality for the QVI control associated with \( W \).

We want to prove that for every admissible control \( \pi \in \mathcal{A}(x) : W(x) \geq J(x, \pi) \). Due to the nonnegativity of \( W(x) \), we only need to consider those controls \( \pi \) for which \( J(x, \pi) \geq 0 \). For such controls, we can easily verify that \( P(\lim_{n \to \infty} \tau_n \geq \tau^\ast) = 1 \) (otherwise, due to the fixed transaction cost, we can show that \( J(x, \pi) = -\infty \)).

Observing that \( \tau^\ast \leq \tau \) and so we have
\[ \lim_{n \to \infty} \left\{ W(x) - E[\{e^{-c(t \wedge \tau^\ast \wedge \tau_n)} W(R_{t \wedge \tau^\ast \wedge \tau_n})\}] \right\} = W(x) - E[\{e^{-c(t \wedge \tau^\ast)} W(R_{t \wedge \tau^\ast})\}]. \]

Thus,
\[ W(x) - E[\{e^{-c(t \wedge \tau^\ast)} W(R_{t \wedge \tau^\ast})\}] \geq E \left[ \sum_{i=1}^{n} I_{\{\tau_i \leq t \wedge \tau^\ast\}} e^{-ct_{\tau_i}} g(\xi_i) \right], \]
with equality again for the QVI control \( \pi^W \).

Next we shall prove that
\[ \lim_{t \to \infty} \{W(x) - E[\{e^{-c(t \wedge \tau^\ast)} W(R_{t \wedge \tau^\ast})\}]\} = W(x) - E[\{e^{-c(t \wedge \tau^\ast)} W(R_{t \wedge \tau^\ast})\}]. \]
(14)

Note that \( W \) is twice continuously differentiable on \((0, L)\) and linear on \([L, \infty)\). Therefore there exists a constant \( M_0 \) such that \( 0 \leq W(R_t) \leq M_0(|R_t| + 1) \). From (4), we can deduce that \( E[\int_{0}^{\infty} e^{-ct} |R_t| dt] < \infty \) and so \( \lim_{t \to \infty} E[|e^{-ct} |R_t|] = 0 \). Thus, we have
\[ 0 \leq \lim_{t \to \infty} E[\{e^{-c(t \wedge \tau^\ast)} W(R_{t \wedge \tau^\ast})\}] \leq M_0 \lim_{t \to \infty} E[\{e^{-c(|R_t| + 1)}\}] = 0. \]

Therefore
\[ \lim_{t \to \infty} E[\{e^{-c(t \wedge \tau^\ast)} W(R_{t \wedge \tau^\ast})\}] \]
Zapatero, we conjecture that the optimal control \( \hat{\pi} \) structure of optimal control by conjecture.

4. Structure of optimal control by conjecture. Inspired by Cadenillas and Zapatero [9], we conjecture that the optimal control \( \hat{\pi} = (\hat{a}, \hat{T}, \hat{\xi}) \) is characterized by two parameters \( \beta, L \) with \( 0 < \beta < L < \infty \) and in a form that can be described as follows. If the reserve process \( R \) reaches boundary \( L \), then the control should be exercised to push it instantaneously to level \( \beta \). That is, 

\[
\hat{\tau}_i = \inf\{t > \hat{\tau}_{i-1} : \hat{R}_t \geq L\}
\]

and 

\[
\hat{R}_{\hat{\tau}_i} = \hat{R}_{\hat{\tau}_i} - \hat{\xi}_i = \beta I_{\{\hat{R}_{\hat{\tau}_i} = L\}}.
\]

Thus, under the above conjecture, the continuation region should be the interval \((0, L)\), that is, as long as the reserve process \( R \) stays in the interval \((0, L)\), the controller should make no intervention. Therefore, in the continuation region \((0, L)\), we have, for any \( x \in (0, L) \), \( V(x) \) should satisfy the following equation

\[
\max_{(a,b) \in C(x)} \left\{ \frac{1}{2} \left( a^2 \sigma^2 + b^2 \right) V''(x) + (\mu a + r_1 b + r_b(x-b))V'(x) - cV(x) \right\} = 0.
\]

Moreover, if \( x \geq L \), then the reserve process for the optimal control would jump to \( \beta \) by making a dividend payment. Therefore, the value function would satisfy

\[
\forall x \in [L, \infty) : \quad V(x) = V(\beta) + g(x - \beta) = V(\beta) + k(x - \beta) - K.
\]

If \( V \) were differentiable at \( L \) and \( \beta \), then from equation (19), we would get \( V'(L) = k \), \( V' (\beta) = k \).

In summary, we conjecture that the solution \( V(x) \) of the QVI equation is determined by (18) for \( x \in [0, L] \) and given by (19) for \( x \geq L \), where \( L, \beta \) are two parameters of the optimal control and can be determined by

\[
V'(\beta) = k, \quad V'(L) = k.
\]
In what follows, we try to find a solution satisfied by (18)-(19) with constraint (20)-(21) and verify that all the above conjectures are correct.

5. Solution of the QVI equation in the continuation region. [17] proved that there exist \( u_1 > 0 \) and a twice continuously differentiable function \( W(x) \) satisfies (18) with \( W'(u_1) = 1, W''(u_1) = 0 \). Based on this result, we shall construct a solution of (18)-(19) with constraint (20)-(21). For the further analysis of our problem, we need first to summarize the results of [17] by a theorem. Before we present the summary theorem, we need some notations defined below.

Set
\[
\begin{align*}
m_p & = \frac{r_1 - r_0}{\sigma_p^2}, \quad \gamma = \frac{\mu^2}{2\sigma^2}, \quad \gamma_1 = \frac{(r_1 - r_0)m_p}{2}, \\
\rho & = \frac{1}{2} \left( \sqrt{\left(\frac{2r_1}{\sigma_p^2} - 1\right)^2 + \frac{8c}{\sigma_p^2}} - \left(1 + \frac{2r_1}{\sigma_p^2}\right) \right), \quad (22) \\
K(t) & = \left(\sigma_p^2 + \sigma^2\right)^{-(\rho+1+\gamma_1/\rho)} \exp \left\{-\frac{2\mu}{\sigma_p} \arctan \left(\frac{\sigma_p t}{\sigma}\right)\right\}, \\
D(x, \nu) & = \int_x^\infty (t-x)^\nu K(t) dt, \quad -1 < \nu < 1 + 2\rho + \frac{2r_1}{\sigma_p^2}, \quad (23) \\
E(x, \nu) & = \int_{-\infty}^x (x-t)^\nu K(t) dt, \quad -1 < \nu < 1 + 2\rho + \frac{2r_1}{\sigma_p^2}. \quad (24)
\end{align*}
\]

and let \( \alpha \in (0,1) \) be the unique root of \( h(s) \) defined by
\[
\begin{align*}
h(s) &= r_0 s^2 - (\gamma + \gamma_1 + r_0 + c)s + c.
\end{align*}
\]

Moreover, we let \( x_0 = \frac{\sigma^2(1-\alpha)}{\mu} \) and \( b(x) \) be the unique solution of the ordinary differential equation
\[
b'(x) = \left(\gamma_1 + c - r_0\right)b^2(x) + m_p(\mu + r_0)x b(x) - \sigma^2 m_p^2/2
\]
\[
\frac{(r-r_0)\sigma^2}{2\sigma_p^2} + \frac{r-\mu}{2} b^2(x)
\]
with \( b(x_0-) = b(x_0+) = \frac{m_p \sigma^2}{\mu} \).

**Theorem 5.1.** Depend on \( m_p \) less or greater than \( 1 - \alpha \), we have the following two results:

**Case I:** \( m_p \leq 1 - \alpha \). Let \( x_1 \) be the unique root of \( b(x) = x \) and
\[
\begin{align*}
\bar{K} & = \frac{E(x_1, \rho) + \frac{m_p}{m_p} E(x_1, \rho - 1)}{D(x_1, \rho) - \frac{m_p}{m_p} D(x_1, \rho - 1)}, \\
\bar{C} & = \frac{1}{(1 + \rho)(E(u_1, \rho) - \bar{K} D(u_1, \rho))}, \\
\hat{k} & = \frac{E(x_1, \rho) - \bar{K} D(x_1, \rho)}{E(u_1, \rho) - \bar{K} D(u_1, \rho)} \exp \left\{ \int_{x_0}^{x_1} \frac{m_p}{b(y)} dy \right\},
\end{align*}
\]
with \( u_1 \) determined by
\[
\begin{align*}
\bar{K} = -\frac{E(u_1, \rho - 1)}{D(u_1, \rho - 1)}. \quad (25)
\end{align*}
\]
Thus, we guess that a solution of (18) for 

\[ W(x) = \left\{ \begin{array}{ll}
\hat{k} \alpha x \left(\frac{x}{x_0}\right)^\alpha, & 0 \leq x < x_0, \\
\hat{k} \left( \int_{x_0}^x e^{-\frac{\alpha x}{\alpha}} dy + \frac{\alpha x}{\alpha} \right), & x_0 \leq x < x_1, \\
C(\hat{K}D(x, \rho + 1) + E(x, \rho + 1)), & x \geq x_1,
\end{array} \right. \] (26)

is a solution of (18) with conditions \( W'(u_1) = 1, W'''(u_1) = 0 \). Moreover, the corresponding optimal control pair \((\hat{a}, \hat{b})\) is given by

\[ \hat{a}(t) = \left\{ \begin{array}{ll}
\frac{\mu \hat{R}_t}{\sigma^2(1-\alpha)}, & \hat{R}_t < x_0, \\
1, & x_0 \leq \hat{R}_t \leq L,
\end{array} \right. \]  \( \hat{b}(t) = \left\{ \begin{array}{ll}
\frac{m \hat{R}_t}{\sigma^2}, & \hat{R}_t < x_0, \\
\hat{b}(\hat{R}_t), & x_0 \leq \hat{R}_t < x_1, \\
\hat{R}_t, & x_1 \leq \hat{R}_t \leq L.
\end{array} \right. \] (27)

**Case II:** \( m_p > 1 - \alpha \). Let \( \eta \in (0, 1) \) be the unique root of the function \( h(y) \) defined by

\[ h(y) = \frac{\alpha^2}{2} y^3 + (r_1 - \sigma_p^2)y^2 + \left(\frac{\alpha^2}{2} - \gamma - r_1 - c\right)y + c, \]

and

\[ \hat{K} = \frac{E(x_0, \rho) + \frac{\rho x_0}{1-\alpha} E(x_0, \rho - 1)}{D(x_0, \rho) - \frac{\rho x_0}{1-\alpha} D(x_0, \rho - 1)}, \]
\[ \hat{C} = \frac{1}{(1+\rho)(E(u_1, \rho) - \hat{K}D(u_1, \rho))}, \]
\[ \hat{k} = \frac{E(x_0, \rho) - \hat{K}D(x_0, \rho)}{E(u_1, \rho) - \hat{K}D(u_1, \rho)}, \]

with \( u_1 \) determined by

\[ \hat{K} = - \frac{E(u_1, \rho - 1)}{D(u_1, \rho - 1)}. \] (28)

Then the function \( W(x) \) defined by

\[ W(x) = \left\{ \begin{array}{ll}
\hat{k} \alpha x \left(\frac{x}{x_0}\right)^\alpha, & 0 \leq x < x_0, \\
\hat{k} \left( \int_{x_0}^x e^{-\frac{\alpha x}{\alpha}} dy + \frac{\alpha x}{\alpha} \right), & x_0 \leq x < x_1, \\
\hat{C} \left[ \hat{K}D(x, \rho + 1) + E(x, \rho + 1) \right], & x \geq x_1,
\end{array} \right. \] (29)

is a solution of (18) with conditions \( W'(u_1) = 1, W'''(u_1) = 0 \). Moreover, the corresponding optimal control pair \((\hat{a}, \hat{b})\) is given by

\[ \hat{a}(t) = \left\{ \begin{array}{ll}
\frac{\mu \hat{R}_t}{\sigma^2(1-\eta)}, & \hat{R}_t < x_0, \\
1, & x_0 \leq \hat{R}_t \leq L,
\end{array} \right. \]  \( \hat{b}(t) = \hat{R}_t. \) (30)

The above theorem presents a solution of (18) with conditions

\[ W'(u_1) = 1, \quad W'''(u_1) = 0. \]

However in our problem, we need to find a solution of (18) with conditions

\[ V'(\beta) = V'(L) = k. \]

Thus, we guess that a solution of (18) for \( x \in (0, L) \) in our problem may be in the form of \( V(x) = CW(x) \), where \( C > 0 \) and \( W(x) \) is given in the Theorem 5.1.

In the following, we shall present some characterizations of \( \beta, L, C \) under the assumption that there exist \( \beta, L, C > 0 \) such that \( V(x) = CW(x) \) is a solution of (18) with conditions \( V'(\beta) = V'(L) = k \). The existence of \( \beta, L, C \) will be proved
in the next section. We first present a property of the solution to (18) for \( x > u_1 \) with \( u_1 \) defined in Theorem 5.1.

**Proposition 5.2.** Let \( W(x) \) and \( u_1 \) be defined in the Theorem 5.1 with different cases respectively. Then, we have \( W''(x) < 0 \) for \( 0 \leq x < u_1 \) and \( W''(x) > 0 \) for \( x > u_1 \).

**Proof.** Since the proof for the case of \( m_p > 1 - \alpha \) is similar to the case of \( m_p \leq 1 - \alpha \), we only give a proof for the case of \( m_p \leq 1 - \alpha \).

The first part of \( W''(x) < 0 \) for \( 0 \leq x < u_1 \) follows from Lemma 3.4 of [17]. From Lemma 3.5 of [17], we know that \( W(x) \) for \( x \geq x_1 \) defined by (26) is a solution of the following differential equation

\[
\frac{1}{2}(\sigma^2 + x^2\sigma_p^2)W''(x) + (\mu + r_1x)W'(x) - cW(x) = 0.
\]  

(31)

Differentiating the above equation yields

\[
\frac{1}{2}(\sigma^2 + x^2\sigma_p^2)W'''(x) + (\mu + (r_1 + \sigma_p^2)x)W''(x) - (c - r_1)W'(x) = 0.
\]  

(32)

Setting \( x = u_1 \) in the above equation leads to

\[
\frac{1}{2}(\sigma^2 + u_1^2\sigma_p^2)W'''(u_1) - (c - r_1)W'(u_1) = 0.
\]

It is easy to see that \( W'''(u_1) > 0 \).

Therefore, \( W''(x) > 0 \) in the right neighborhood of \( u_1 \). If there exists a constant \( \tilde{x} > u_1 \) such that \( W''(\tilde{x}) \leq 0 \), then \( \tilde{x} = \inf\{x > u_1 : W''(x) \leq 0\} < \infty \). Taking \( x = \tilde{x} \) in equation (32), we can easily have \( W'''(\tilde{x}) > 0 \). Therefore \( W''(x) < 0 \) in the left neighborhood of \( \tilde{x} \) which is contradictory to the definition of \( \tilde{x} \). Thus we must have \( \tilde{x} = \infty \) and then we obtain the desired result, i.e., for \( x > u_1, W''(x) > 0 \). \( \Box \)

In what follows, we set

\[
v(x) = \begin{cases} 
C W(x), & 0 \leq x < L, \\
v(\beta) + k(x - \beta) - K, & x \geq L,
\end{cases}
\]

(33)

where \( W(x) \) is defined in the Theorem 5.1 with different cases respectively. Thus from the above proposition, we have

\[
v''(x) \begin{cases} < 0, & x < u_1, \\
> 0, & u_1 < x < L.
\end{cases}
\]

(34)

Thus \( v'(x) \) strictly decreases on the interval \((0, u_1)\) and strictly increases on interval \((u_1, L)\). So if there exist \( \beta, L, C \) satisfying (20)-(21), we must have

\[
\beta < u_1 < L.
\]

(35)

**Proposition 5.3.** Suppose there exist \( \beta, L, C \) satisfying (20)-(21) and let \( \hat{x} \) be defined by

\[
\hat{x} := \inf\{x \geq 0 : v(x) = Mv(x)\}.
\]

(36)

Then \( \hat{x} = L \).

**Proof.** Let \( H : [0, x] \to \mathbb{R} \) be defined by

\[
H(y) = v(y) + k(x - y) - K,
\]

and so it is easy to obtain

\[
H'(y) = v'(y) - k.
\]
Thus from (20)-(21), (34)-(35), we can easily to get
\[ v'(x) = \begin{cases} > k, & 0 \leq x < \beta, \\ < k, & \beta < x < \beta + K. \end{cases} \] (37)

Therefore, \( H(y) \) is increasing in \([0, x]\) if \( x \in (0, \beta) \). Moreover, if \( x \in (\beta, L) \), \( H(y) \) is increasing in \([0, \beta]\) and decreasing in \([\beta, x]\).

By the definition of the maximum utility operator \( M \) and above analysis, we obtain
\[ Mv(x) = \begin{cases} v(x) - K, & 0 \leq x < \beta, \\ v(\beta) + k(x - \beta) - K, & \beta \leq x \leq L. \end{cases} \] (38)

Thus for \( 0 \leq x < \beta \),
\[ v(x) - Mv(x) = K > 0. \] (39)

For \( \beta \leq x \leq L \),
\[ v(x) - Mv(x) = v(x) - v(\beta) - k(x - \beta) + K. \]

Let \( Z(x) := v(x) - Mv(x) \) and so \( Z'(x) = v'(x) - k \). From (37), we obtain that \( Z(x) \) is decreasing in \([\beta, L]\). Noting that \( Mv(L) = v(L) = v(\beta) + k(L - \beta) - K \) and so
\[ Z(L) = v(L) - Mv(L) = 0. \] (40)

Therefore,
\[ Z(x) = v(x) - Mv(x) > 0 \text{ for } \beta \leq x < L. \] (41)

From (39), (40) and (41), we obtain that \( L \) is the first time for \( v(x) = Mv(x) \), i.e. \( L = \inf \{ x \geq 0 : v(x) = Mv(x) \} \). So \( \hat{x} = L \).

**Remark 2.** Note that the proof of Proposition 5.3 does not use any information of \( m_p \), therefore no matter \( m_p < 1 - \alpha \) or \( m_p \geq 1 - \alpha \) we still have \( \hat{x} = L \). The only thing we need to care is that \( \hat{x}, L, \beta, v(x) \) must be corresponded to the same case.

6. **Solution to the QVI and the optimal policy.** The results of the previous section provide the conjectured structure of the solution \( v(x) \) to the QVI equations (8)-(11), that is,
\[ v(x) = \begin{cases} CW(x), & 0 \leq x \leq L, \\ CW(\beta) + k(x - \beta) - K, & x \geq L, \end{cases} \] (42)

where \( W(x) \) is defined in Theorem 5.1 with different cases respectively and constants \( \beta, L, C \) are determined by
\[ \begin{align*}
\frac{v'(\beta)}{v'(L)} &= k, \\
\frac{v'(L)}{v(L)} &= k, \\
v(L) &= v(\beta) + k(L - \beta) - K.
\end{align*} \] (43)

6.1. **Existence of \( \beta, L, C \).** In this subsection, we will show that there exist \( \beta, L, C \) which satisfy (43). Since the proof of the existence of \( \beta, L, C \) for the case of \( m_p > 1 - \alpha \) is similar to the case of \( m_p \leq 1 - \alpha \), we only give a proof for the case of \( m_p \leq 1 - \alpha \). We start with a function
\[ F(x) := W'(x) = \begin{cases} kx_0 \left( \frac{x}{x_0} \right)^{\alpha-1}, & 0 \leq x < x_0, \\ kE - \int_{x_0}^{x} \frac{m_p}{x} dy, & x_0 \leq x < x_1, \\
(\rho + 1)C(KD(x, \rho) + E(x, \rho)), & x \geq x_1, \end{cases} \]
where $W(x)$ is defined by (26) in Theorem 5.1. From the definition of $D(x, \nu), E(x, \nu)$ in (23)-(24), it is easy to check that
\[
D(x, \nu) > 0, \quad E(x, \nu) > 0
\]
\[
\lim_{x \to +\infty} D(x, \nu) = 0, \quad \lim_{x \to +\infty} E(x, \nu) = +\infty.
\]
Therefore, from the relation between $\tilde{F}$ and the fact that $\beta$ is strictly increasing on $[0, 1]$, we have
\[
\tilde{K} = -\frac{E(u_1, \rho - 1)}{D(u_1, \rho - 1)} < 0,
\]
and so
\[
\tilde{C} = \frac{1}{(1 + \rho)(E(u_1, \rho) - K D(u_1, \rho))} > 0,
\]
\[
k = \frac{E(x_1, \rho) - K D(x_1, \rho)}{E(u_1, \rho) - K D(u_1, \rho)} \exp\{\int_{x_0}^{x_1} m_p \frac{1}{b(y)} dy\} > 0.
\]
Thus combining the above analysis, we can obtain that,
\[
\lim_{x \to 0} F(x) = \lim_{x \to 0} k x_0 \left(\frac{x}{x_0}\right)^{\alpha - 1} = +\infty
\]
\[
\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} (\rho + 1) C (K D(x, \rho) + E(x, \rho)) = +\infty.
\]
Let $u_1$ be defined in Theorem 5.1 and so $F(u_1) = W'(u_1) = 1$. By Proposition 5.2 and the fact that $F(x) = W'(x)$, we have $F(x)$ is strictly decreasing on $[0, u_1]$ and strictly increasing on $[u_1, +\infty]$. Thus if $0 < C < k$, there must exist two points $\beta^C < u_1 < L^C$ such that $CF(\beta^C) = CF(L^C) = k$. Obviously, if $C = k$, then $\beta^C = L^C = u_1$. It is easy to see that $\beta^C$ is an increasing function of $C$, while $L^C$ is a decreasing function of $C$, for $C \in (0, k]$.

Define
\[
I(C) := \int_{\beta^C}^{L^C} (k - CF(y)) dy.
\]
Since the limits in the integral and the integrand are continuous function of $C$, the function $I(C)$ is also a continuous function of $C$. The fact that both the integrand and the interval $[\beta^C, L^C]$ are decreasing with respect to $C$, implies that $I(C)$ is a decreasing function of $C$. Due to the fact that $CF(x) \to 0$ uniformly on any compact set of $(0, +\infty)$, we can see that $L^C \to +\infty$ and $(k - CF(y)) \to k$ as $C \to 0$. Therefore, $I(C) \to +\infty$ as $C \to 0$. Because $I(k) = 0$ there exists $\hat{C} < k$, such that
\[
I(\hat{C}) = \int_{\beta^C}^{L^C} (k - \hat{C} F(y)) dy = K.
\]
Note that $\hat{C} F(x) = v'(x)$, therefore the above equality is equal to
\[
v(L^C) = v(\beta^C) + k(L^C - \beta^C) - K.
\]
Thus the equation (43) is satisfied with the choice of $C = \hat{C}, \beta = \beta^C$ and $L = L^C$.

**Theorem 6.1.** Let $v(x)$ be defined by (42) with $W(x)$ defined in Theorem 5.1 for the case of $m_p \leq 1 - \alpha$ and $m_p > 1 - \alpha$ respectively. Then there exist constants $\beta, L, C$ such that the equation (43) holds.
6.2. Solution to the QVI equations and the optimal policy. In this subsection, we will prove that the function defined by (42) with \( \beta, L, C \) determined by (43) is indeed the solution to the QVI equations (8)-(11). Moreover, we also provide the explicit expression of the optimal control policy.

**Theorem 6.2.** The function \( v(x) \) defined by (42) with \( \beta, L, C \) determined by (43) and \( W(x) \) given by (26) and (29) for the case of \( m_p \leq 1 - \alpha \) and \( m_p > 1 - \alpha \) respectively is a solution to QVI equations (8)-(11).

**Proof.** Since the proof of this theorem for the case of \( m_p > 1 - \alpha \) is similar to the case of \( m_p \leq 1 - \alpha \), we only give a proof for the case of \( m_p \leq 1 - \alpha \).

We first show that for \( x \in [0, L] \), \( v(x) \) satisfies (18) and \( Mv(x) < v(x) \). From Theorem 3.1 of [17], we know that \( W(x) \) satisfies (18) for \( 0 \leq x \leq u_1 \). If \( u_1 < x < L \), by Proposition 5.2, we have \( W''(x) > 0 \) and \( W'(x) \) is an increasing function. Therefore, \( (a, b) = (1, x) \) is a maximizer of the left-hand side of (18). Substituting the maximizer \( (a, b) = (1, x) \) into (18) leads to the equation (31). Noting that \( W(x) \) defined by (26) is a solution of the equation (31). Therefore, \( W(x) \) satisfies (18) on \([u_1, L]\). Observed that for \( x \in [0, L] \), \( v(x) = CW(x) \) and so \( v(x) \) satisfies the equation (18) on \([0, L]\).

From Proposition 5.3 and the definition of \( v(x) \), we can easily obtain that
\[
Mv(x) < v(x), \text{ for } x \in [0, L).
\]

In the next, we will show that for \( x \in [L, \infty) \),
\[
Mv(x) = v(x),
\]
and
\[
\max_{(a,b) \in \mathcal{C}(x)} \mathcal{L}^{a,b}v(x) \leq 0.
\]

Let \( Z(\eta) = v(x - \eta) + g(\eta) \) for \( x \geq L \). Then
\[
Z(\eta) = \begin{cases} 
CW(\beta) + k(x - \beta) - 2K, & x - \eta \geq L, \\
CW(x - \eta) + kn - K, & x - \eta \in [0, L]. 
\end{cases}
\]

From (37), it is easy to see that the maximum value of \( Z(\eta) \) in interval \([0, x]\) is
\[
CW(\beta) + k(x - \beta) - K.
\]

Therefore, for \( x \geq L \), \( Mv(x) = \sup_{\eta > 0, \eta < x} Z(\eta) = W(\beta) + k(x - \beta) - K = v(x) \).

From the definition of \( v(x) \) on the interval \([L, \infty)\), it is easy to see that \( v(x) \) is an increasing function for \( x \in [0, \infty) \) and
\[
\mathcal{L}^{a,b}v(x) = \frac{1}{2}(a^2 \sigma^2 + \sigma_p r^2 b^2)v''(x) + (\mu a + r b + r_0(x - b))v'(x) - cv(x) \\
= (a^2 \sigma^2 + \sigma_p r^2 b^2)v''(L) + (\mu a + r b + r_0(x - b))k - cv(L) \\
< \frac{1}{2}(a^2 \sigma^2 + \sigma_p r^2 b^2)v''(L) + (\mu a + r b + r_0(x - b))k - cv(L) \\
\leq \max_{(a,b) \in \mathcal{C}(x)} \mathcal{L}^{a,b}v(L) = 0,
\]

where we used the facts of \( v'(L) = k \) and \( v''(L) > 0 \) to obtain the last two inequalities. Thus, taking the maximum over \( (a, b) \in \mathcal{C}(x) \) leads to
\[
\max_{(a,b) \in \mathcal{C}(x)} \mathcal{L}^{a,b}v(x) \leq 0.
\]

Thus the function \( v(x) \) given by (42) with \( W(x) \) given by (26) satisfies QVI equations (8)-(11). \( \square \)
Next we will present a verification theorem to show that the solution \( v(x) \) of the QVI equation is indeed the value function \( V(x) \).

**Theorem 6.3.** Depend on \( m_p \) less or greater than \( 1 - \alpha \), we have the following two results:

**Case I:** \( m_p \leq 1 - \alpha \). The control

\[
\hat{\pi} = (\hat{a}, \hat{b}, \hat{T}, \hat{\xi}) = (\hat{a}, \hat{b}, \hat{\tau}_1, \ldots, \hat{\tau}_n, \hat{\xi}_1, \ldots, \hat{\xi}_n, \ldots)
\]

with \((\hat{T}, \hat{\xi})\) defined by (16)-(17) and \((\hat{a}, \hat{b})\) defined by (27) is the QVI control associated with the function \( v(x) \) defined by (42) with \( W(x) \) given by (26). This control is the optimal control and the function \( v(x) \) coincides with the value function \( V(x) \), that is,

\[
V(x) = v(x) = J(x, \hat{\pi}) = J(x, \hat{a}, \hat{b}, \hat{T}, \hat{\xi}).
\]

**Case II:** \( m_p > 1 - \alpha \). The control

\[
\hat{\pi} = (\hat{a}, \hat{b}, \hat{T}, \hat{\xi}) = (\hat{a}, \hat{b}, \hat{\tau}_1, \ldots, \hat{\tau}_n, \hat{\xi}_1, \ldots, \hat{\xi}_n, \ldots)
\]

with \((\hat{T}, \hat{\xi})\) defined by (16)-(17) and \((\hat{a}, \hat{b})\) defined by (30) is the QVI control associated with the function \( v(x) \) defined by (42) with \( W(x) \) given by (29). This control is the optimal control and the function \( v(x) \) coincides with the value function \( V(x) \), that is,

\[
V(x) = v(x) = J(x, \hat{\pi}) = J(x, \hat{a}, \hat{b}, \hat{T}, \hat{\xi}).
\]

**Proof.** We only give a proof for the case of \( m_p \leq 1 - \alpha \). By Theorem 6.2, the function \( v(x) \) defined by (42) with \( W(x) \) given by (26) satisfies all the conditions of Theorem 3.3. From the definition of control associated with the solution to the QVI and the discussion in previous section, we know that the control \( \hat{\pi} \) with \((\hat{T}, \hat{\xi})\) defined by (16)-(17) and \((\hat{a}, \hat{b})\) defined by (27) is the QVI control associated with \( v(x) \). Furthermore, it is easy to verify that \( \hat{\pi} \) is an admissible control. Therefore by applying Theorem 3.3, we obtain that \( v \) is the value function and \( \hat{\pi} \) is the optimal strategy.

\[\square\]

7. **Numerical simulation.** In this section, we present a numerical example to illustrate our results. To focus on the main point of our paper, we first draw the figure of \( V'(x) \) to get some better understanding about the fixed cost \( K \) and value function. Without loss of generality, we show the simulation of the case I.

Set \( r_1 = 0.043; r_0 = 0.0422; \sigma_p = \sqrt{0.05}; c = 0.05; \mu = 1; \sigma = \sqrt{2}; k = 0.97; K = 0.0575 \). By calculation, we have that when \( C = 0.96 \), the area composed by \( V'(x) \) and \( K \) is the fixed cost \( K \).

Next, we draw a figure to show the performance of the value function. The strategy is a triple which contains reinsurance and investment as well as dividend strategy. Since the optimal reinsurance and investment strategy is the same as the reference, we refer the readers to [17]. We only focus on the optimal impulsive dividend strategy. In the above setting at the beginning of this section, if the initial value is less than 10.3000, whenever the reserve process hits 10.3000, the amount of 5.898 is payed as dividend and the reserve is draw back to 4.4020, and the process continues. If the initial value \( x \) is larger than 10.3000, the amount of \( x - 4.4020 \) is payed as dividend and the reserve is draw back to 4.4020, and then the process continues.
8. **Conclusion.** In this paper, we consider the dividend optimization problem with fixed transaction costs and both the proportional reinsurance and investment can be adopted by the financial corporation. Due to the existence of the fixed transaction costs, the problem becomes a mixed classical-impulse stochastic control problem. We make a conjecture on the optimal dividend strategy and give some characterizations of the conjectured parameters of the optimal control. Through the dynamic
programming theory and by solving the quasi-variational inequalities (QVI) equation, we finally obtain explicit expression of the value function and the corresponding optimal policy.

**Appendix.** In this appendix, we prove that if $c < r_1$, then the optimal value function will be infinite. To see this, we divide the company into two departments, one of them deals with the investments and the other with the reinsurance business. Suppose the investment department starts with capital $x_0 \in (0, x)$, invests all the surplus on risky assets and diverts to the reinsurance department a constant flow $0 < p < r_1 x_0$. Let $\{Y_t^{(1)}\}$ denote the surplus process of the investment department and then we have

$$Y_t^{(1)} = x_0 - \int_0^t p \, ds + \int_0^t r_1 Y_s^{(1)} \, ds + \int_0^t \sigma_p Y_s^{(1)} \, dW_s.$$  

We assume that the reinsurance department keeps always receiving the constant flow $p$ no matter what happened of $\{Y_t^{(1)}\}$. Furthermore, we assume that the reinsurance department starts with capital $x - x_0$ and chooses no reinsurance ($a(t) \equiv 1$). Thus the surplus process $\{Y_t^{(2)}\}$ of the reinsurance department is

$$Y_t^{(2)} = x - x_0 + \int_0^t p \, ds + \int_0^t \mu \, ds + \int_0^t \sigma dB_s = x - x_0 + (\mu + p)t + \sigma B_t.$$  

Let $\hat{\tau}_i := \inf\{t > 0 : Y_t^{(i)} = 0\}$, $i = 1, 2$ and it is easy to see that $\hat{\tau}_2$ is independent of both $\hat{\tau}_1$ and the process $\{Y_t^{(1)}\}$. Set $X_t := Y_t^{(1)} + Y_t^{(2)}$ and for any given $t_0 > 0$, we consider control strategy $\pi_{t_0}$ described as follows. Let $\tau_1 = t_0 \wedge \hat{\tau}_1 \wedge \hat{\tau}_2, \xi_1 = X_{\tau_1} = Y_{\tau_1}^{(1)} + Y_{\tau_1}^{(2)}$, and $\tau_i, i \geq 2$ be any increasing stopping time sequence and $\xi_i = 0, i \geq 2$.

Further more, we choose $a(t) \equiv 1$ and $b(t) = Y_t^{(1)}$ for any $t \leq \tau_1$. Observing that $X_{\tau_1 \wedge \hat{\tau}_2} = Y_{\tau_1 \wedge \hat{\tau}_2}^{(1)} + Y_{\tau_1 \wedge \hat{\tau}_2}^{(2)} \geq 0$ and so

$$X_{\tau_1} = X_{t_0} 1_{t_0 \leq \hat{\tau}_1 \wedge \hat{\tau}_2} + X_{\hat{\tau}_1 \wedge \hat{\tau}_2} 1_{t_0 > \hat{\tau}_1 \wedge \hat{\tau}_2} \geq X_{t_0} 1_{t_0 \leq \hat{\tau}_1 \wedge \hat{\tau}_2} \geq Y_{t_0}^{(1)} 1_{t_0 \leq \hat{\tau}_1 \wedge \hat{\tau}_2}.$$  

Thus under the above control strategy $\pi_{t_0}$, we have

$$\max J(x, \pi) \geq J(x, \pi_{t_0}) = E[e^{-c \tau_1}(kX_{\tau_1} - K)] \geq kE[e^{-c \tau_1} X_{\tau_1}] - K \geq kE[e^{-c t_0} Y_{t_0}^{(1)} 1_{t_0 \leq \hat{\tau}_1 \wedge \hat{\tau}_2}] - K = kE[e^{-c t_0} Y_{t_0}^{(1)} 1_{\hat{\tau}_1 \geq t_0, \hat{\tau}_2 \geq t_0}] - K \geq kE[e^{-c t_0} Y_{t_0}^{(1)}] P(\hat{\tau}_2 \geq t_0) - K \geq kE[e^{-c t_0} Y_{t_0}^{(1)}] P(\hat{\tau}_2 \geq t_0) - K, \quad (44)$$  

where in the last inequality we used the fact that $Y_{t_0}^{(1)} \leq 0$ for $\hat{\tau}_1 < t_0$. From the standard theory of diffusion process and after some calculations, we have

$$E[e^{-c t_0} Y_{t_0}^{(1)}] = e^{-(c-r_1)t_0}(x_0 - \frac{p(1 - e^{-r_1 t_0})}{r_1}).$$  

Noting that $0 < p < r_1 x_0, c < r_1$ and letting $t_0 \to +\infty$ in the above equality yield

$$\lim_{t_0 \to +\infty} E[e^{-c t_0} Y_{t_0}^{(1)}] = +\infty.$$
On the other hand, from Example 4.3 of [3, Chapter II], we obtain

\[
\lim_{t_0 \to +\infty} P(\tilde{\tau}_2 \geq t_0) = 1 - e^{-2(\mu + \rho}(x - x_0) > 0.
\]

Thus combining above analysis and letting \( t_0 \to +\infty \) in the equation (44) lead to the result that the optimal value function is infinite in the case of \( c < r_1 \).

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