THE SCALAR CURVATURE DEFORMATION EQUATION ON LOCALLY CONFORMALLY FLAT MANIFOLDS

YU YAN

Abstract. We study the equation \( \Delta_g u - \frac{n-2}{4(n-1)} R(g)u + Ku^p = 0 \) (1 + \( \zeta \leq p \leq \frac{n+2}{n-2} \)) on locally conformally flat compact manifolds \( (M^n, g) \). We prove the following: (i) When the scalar curvature \( R(g) > 0 \) and the dimension \( n \geq 4 \), under suitable conditions on \( K \), all positive solutions \( u \) have uniform upper and lower bounds; (ii) When the scalar curvature \( R(g) \equiv 0 \) and \( n \geq 5 \), under suitable conditions on \( K \), all positive solutions \( u \) with bounded energy have uniform upper and lower bounds. We also give an example to show that the energy bound condition for the uniform estimates in [18] is necessary.

1. Introduction

Let \( (M^n, g) \) be an \( n \)-dimensional compact manifold with metric \( g \), and we use \( R(g) \) to denote the scalar curvature of \( g \). Let \( u \) be a positive function defined on \( M \). The scalar curvature of the conformally deformed metric \( u^{\frac{4}{n-2}} g \) is given by

\[
R(u^{\frac{4}{n-2}} g) = -c(n)^{-1} u^{-\frac{n+2}{n-2}} (\Delta_g u - c(n) R(g)u) \quad \text{where} \quad c(n) = \frac{n-2}{4(n-1)}.
\]

The Yamabe Theorem, which was proved by the work of Trudinger [17], Aubin [1] and Schoen [11], says that there exists \( u > 0 \) such that \( R(u^{\frac{4}{n-2}} g) \) is equal to some constant \( K \). The P.D.E. formulation of this theorem is that the equation

\[
\Delta_g u - c(n) R(g)u + c(n) Ku^{\frac{n+2}{n-2}} = 0
\]

has a positive solution for some constant \( K \).

In [4], J. Escobar and R. Schoen extended this result to the case when \( K \) is a function on \( M \). They proved that under certain conditions on \( K \), the above equation has a positive solution \( u \) when \( R(g) > 0 \) or \( R(g) \equiv 0 \).

In fact, in those existence results the solution minimizes the associated constraint variational problem and can be obtained as a limit of a sequence of solutions of the corresponding subcritical equations. Therefore, a natural question is whether non-minimal solutions can also be produced from solutions of the subcritical equations. We would like to know if there are uniform estimates for solutions of the equation

\[
(1) \quad \Delta_g u - c(n) R(g)u + Ku^p = 0 \quad \text{where} \quad 1 + \zeta \leq p \leq \frac{n+2}{n-2}.
\]

This was proved to be true by R. Schoen [12, 16] when \( K \) is a positive constant, \( R(g) > 0 \), and \( (M^n, g) \) is locally conformally flat and not conformally diffeomorphic to \( S^n \). By the work of Y. Li and M. Zhu [9], this is also true when \( K \) is a positive function on a 3-dimensional compact manifold \( (M^3, g) \) which has \( R(g) > 0 \) and is not conformally diffeomorphic to \( S^3 \). In the case when \( K \) is a positive constant, this result by Li and Zhu
was extended to dimensions $n = 4, 5$ by O. Druet in [2, 3]. Then it was extended further to dimensions $n \leq 7$ independently by Y. Li and L. Zhang [7] and F.C. Marques [10]; when the dimension $n \geq 8$, it was proved to be true by Li and Zhang [7] under an additional assumption on the Weyl tensor of the background metric $g$.

In [18] we proved uniform estimates for solutions with bounded energy when $K$ is a function satisfying certain conditions on a 3 or 4 dimensional locally conformally flat manifold with zero scalar curvature. In this paper we study this problem when $K$ is a function on locally conformally flat manifolds $(M^n, g)$. We consider two separate cases: $R(g) > 0$ and $R(g) \equiv 0$.

1.1. Manifolds of Zero Scalar Curvature. When the scalar curvature $R(g) \equiv 0$ on the manifold $M$, equation (1) becomes

\[ \Delta_g u + K^p u = 0 \quad \text{where} \quad 1 + \zeta \leq p \leq \frac{n+2}{n-2}. \]

The necessary conditions for the existence of a solution $u > 0$ are that $K$ changes sign on $M$ and $\int_M K dv < 0$.

The corresponding existence result is the following theorem in [4]:

**Theorem 1.1.** (Escobar–Schoen [4]). Suppose $M$ is locally conformally flat with zero scalar curvature. Suppose $K$ is a nonzero smooth function on $M$ satisfying the condition that there is a maximum point $P_0 \in M$ of $K$ at which all derivatives of $K$ of order less than or equal to $(n-3)$ vanish. Then $K$ is the scalar curvature of a metric $\bar{g} = u^{\frac{4}{n-2}} g$ for some $u > 0$ on $M$ if and only if $K$ satisfies

(i) $K$ changes sign

(ii) $\int_M K dv < 0$.

When the dimension $n = 3, 4$, the flatness condition on $K$ is automatically satisfied and the locally conformally flat assumption on $M$ can be removed.

In [18], we proved a compactness theorem when the dimension of $M$ is equal to 3 or 4.

**Theorem 1.2.** ([18]). Let $(M, g)$ be a three or four dimensional locally conformally flat compact manifold with $R(g) \equiv 0$. Let $K := \{ K \in C^3(M) : K > 0$ somewhere on $M, \int_M K dv < -C_K^{-1} < 0, \| K \|_{C^3(M)} \leq C_K \}$ for some constant $C_K$, and $S_{\Lambda} := \{ u : u > 0 \text{ solves (2)} \text{ with } K \in K, \text{ and } E(u) := \int_M |\nabla u|^2 dv \leq \Lambda \}$. Then there exists $C = C(M, g, C_K, \Lambda, \zeta) > 0$ such that $u \in S_{\Lambda}$ satisfies $\| u \|_{C^3(M)} \leq C$ and $\min_M u \geq C^{-1}$.

In Section 2 we will give an example which shows that these estimates cannot be improved to be independent of the energy $E(u)$.

Next we give a similar theorem on manifolds of dimension $n \geq 5$. We first need to define a flatness condition on $K$ as follows.

**Definition 1.3.** A function $K \in C^{n-2}(M)$ is said to satisfy the flatness condition $(\ast)$ if near each critical point $P$ of $K$ where $K(P) > 0$, there exist a neighborhood and a constant $C_0$ such that in that neighborhood

\[ |\nabla^p K| \leq C_0 |\nabla K|^{\frac{n-2-p}{n-3}} \quad \text{for} \quad 2 \leq p \leq n-3, \]

where $\nabla^p K$ is the $p$-th covariant derivative of $K$. 
Note that this implies in particular all partial derivatives of $K$ up to order $n - 3$ vanish at those critical points, and the order of flatness is the same as that in Theorem 1.1. A simple example of a function satisfying this condition is a function which can be expressed near the critical points as $K(z) = a + b|z|^{n-2}$, where $a, b$ are two constants and $z$ is a local coordinate system centered at the critical point. This type of flatness condition also appeared in [6] and [8], where Y. Li studied the problem of prescribing scalar curvature functions on $S^n$.

We are ready to state the theorem:

**Theorem 1.4.** Let $(M^n, g)$ be a locally conformally flat compact manifold with $R(g) \equiv 0$, and its dimension $n \geq 5$. Let $K \in C^{n-2}(M)$ be a function which satisfies the flatness condition (**)\;; additionally, $K$ is positive somewhere on $M$ and $\int_M K dv_g < 0$. If $u$ is a positive solution of equation (2) with bounded energy $E(u) := \int_M |\nabla u|^2 dv_g \leq \Lambda$, then there exists a positive constant $C$ such that $\|u\|_{C^2(M)} \leq C$ and $\min_M u \geq C^{-1}$, where $C$ depends on $M, g, \|K\|_{C^{n-2}(M)}, \int_M K dv_g, \Lambda$, and $\zeta$.

### 1.2. Manifolds of Positive Scalar Curvature

When the scalar curvature $R(g) > 0$, the necessary condition for equation (1) to have a positive solution is that $K > 0$ somewhere on the manifold. The following existence result was proved in [4].

**Theorem 1.5.** (Escobar–Schoen [4]). Suppose $M$ is a locally conformally flat manifold with positive scalar curvature which is not simply connected, and $K$ is a smooth function on $M$ which is somewhere positive, and there is a maximum point $P_0$ of $K$ at which all partial derivatives of $K$ of order less than or equal to $(n - 2)$ vanish. Then equation (1) has a positive solution.

When the dimension $n = 3$, the flatness condition on $K$ is automatically satisfied and the locally conformally flat assumption on $M$ can be removed.

The compactness result when $n = 3$ was proved in [9].

**Theorem 1.6.** (Li–Zhu [9]). Let $(M^n, g)$ be a three dimensional smooth compact Riemannian manifold with positive scalar curvature which is not conformally equivalent to the standard $S^3$. Then for any $1 < p \leq 5$ and positive function $K \in C^2(M)$, there exists some constant $C$ depending only on $M, g, \|K\|_{C^2(M)}$, and the positive lower bound of $K$ and $p - 1$ such that

$$\frac{1}{C} \leq u \leq C \quad \text{and} \quad \|u\|_{C^3(M)} \leq C$$

for all positive solutions $u$ of $\Delta_g u - \frac{1}{8} R(g) u + K u^p = 0$.

We will give a compactness theorem when the dimension $n \geq 4$. But $K$ needs to satisfy a flatness condition near its critical points.

**Definition 1.7.** A function $K \in C^{n-1}(M)$ is said to satisfy the flatness condition (**) if near each critical point of $K$, there exist a neighborhood and a constant $C_0$ such that in that neighborhood

$$|\nabla^p K| \leq C_0 |\nabla K|^{\frac{n-1-p}{n-2}} \quad \text{for} \quad 2 \leq p \leq n - 2,$$

where $\nabla^p K$ is the $p$-th covariant derivative of $K$. 
Under this condition all partial derivatives of \( K \) up to order \( n - 2 \) vanish at the critical points, which is consistent with the condition given in Theorem 1.5. A simple example of a function satisfying this condition is a function which can be expressed near the critical points as \( K(z) = a + b|z|^n - 1 \), where \( a, b \) are two constants and \( z \) is a local coordinate system centered at the critical point.

Our theorem is:

**Theorem 1.8.** Let \((M^n, g)\) be a locally conformally flat compact manifold with \( R(g) > 0 \). Assume \( M \) is not conformally diffeomorphic to \( S^n \), and its dimension \( n \geq 4 \). Let \( K \in C^{n-1}(M) \) be a positive function which satisfies the flatness condition (**). There exists a positive constant \( C \) such that \( \|u\|_{C^3(M)} \leq C \) and \( \min u \geq C^{-1} \) for any positive solution \( u \) of equation (\( \Box \)), where \( C \) depends on \( M, g, \zeta \) and \( \|K\|_{C^{n-1}(M)} \).

Note that because we assume \( K > 0 \) in this theorem, there is no assumption on the energy of \( u \), which was introduced in the scalar-flat case to overcome the difficulty caused by the sign changing of \( K \).

2. The Example and Some Notations

Let \((M^n, g)\) be a compact manifold with \( R(g) \equiv 0 \) and \( n = 3 \) or \( 4 \). (In fact in this example \( M \) does not need to be locally conformally flat.) We choose \( K \in C^3(M) \) satisfying the following conditions:

- \( K > 0 \) somewhere on \( M \),
- \( \int_M Kdv_g \leq -C_K^{-1} < 0 \) and \( \|K\|_{C^3(M)} \leq C_K \), where \( C_K \) is a positive constant,
- the set \( \{x \in M : K(x) = 0\} = \overline{U} \) for some open set \( U \subset M \).

We define

\[
K_i(x) = \begin{cases} 
\frac{K(x)}{i} & \text{if } K(x) > 0 \\
K(x) & \text{if } K(x) \leq 0 
\end{cases}
\]

Since on \( \partial U \) all derivatives of \( K \) up to order 3 are zero, it follows that \( K_i \in C^3(M) \). Furthermore, by this definition \( K_i \in K \), where \( K \) is as defined in Theorem 1.2. Then by Theorem 1.1 there exists \( u_i > 0 \) which satisfies \( \Delta_g u_i + K_i u_i^{\frac{n+2}{n-2}} = 0 \).

Now suppose there is a constant \( C \) independent of \( i \) such that \( \max_{M} u_i \leq C \). As proved in Section 2 of [18], this implies that \( \{u_i\} \) is uniformly bounded away from 0 and \( \|u_i\|_{C^3(M)} \) is bounded above uniformly. Then passing to a subsequence \( \{u_i\} \) converges in the \( C^2 \)-norm to a function \( u > 0 \), and \( u \) satisfies \( \Delta_g u + \tilde{K} u^{\frac{n+2}{n-2}} = 0 \) where

\[
\tilde{K}(x) = \lim_{i \to \infty} K_i(x) = \begin{cases} 
0 & \text{if } K(x) > 0 \\
K(x) & \text{if } K(x) \leq 0 
\end{cases}
\]

However, because \( \tilde{K} \) is nowhere positive and somewhere negative, the equation \( \Delta_g u + \tilde{K} u^{\frac{n+2}{n-2}} = 0 \) cannot have a positive solution by Theorem 1.1. This contradiction shows that estimates like the ones in Theorem 1.2 can not be true without the energy bound assumption on \( u \).

Next we prove Theorems 1.4 and 1.8. We will prove Theorem 1.4 in Sections 3 to 5 and the proof of Theorem 1.8 will be given in Section 6. We first give some definitions and a lemma which will be used in both proofs.
Definition 2.1. We call a point $\bar{x}$ on a manifold $M$ a blow-up point of a sequence $\{u_i\}$ if $\bar{x} = \lim_{i \to \infty} x_i$ for some $\{x_i\} \subset M$ and $u_i(x_i) \to \infty$.

Definition 2.2. Suppose $u_i$ satisfies $\Delta_g u_i - c(n) R(g_i) u_i + K_i u_i^{p_i} = 0$, where $\{g_i\}$ converges to some metric $g_0$. A point $\bar{x} \in M$ is called an isolated blow-up point of $\{u_i\}$ corresponding to $\{g_i\}$ if there exist local maximum points $x_i$ of $u_i$ and a fixed radius $r_0 > 0$ such that

- $x_i \to \bar{x}$,
- $u_i(x_i) \to \infty$,
- $u_i(x) \leq C (d_{g_i}(x, x_i))^{-\frac{2}{p_i - 1}}$ for any $x \in B_{r_0}(x_i)$, where the constant $C$ is independent of $i$.

Lemma 2.3. If $\bar{x} = \lim_{i \to \infty} x_i$ is an isolated blow-up point of $\{u_i\}$ corresponding to $\{g_i\}$, and $K_i$ is uniformly bounded, then there exists a constant $C$ independent of $i$ and $r$ such that

$$\max_{\partial B_r(x_i)} u_i(x) \leq C \min_{\partial B_r(x_i)} u_i(x)$$

for any $0 < r \leq r_0$.

This can be proved as in [18] in the proof of Lemma 5.2.

Definition 2.4. $\bar{x}$ is called a simple blow-up point of $\{u_i\}$ if it is an isolated blow-up point and there exists $\bar{r} > 0$ independent of $i$ such that $\bar{w}_i(r)$ has only one critical point for $r \in (0, \bar{r})$. Here $\bar{w}_i(r) := r^{\frac{2}{p_i - 1}} \bar{u}_i(r) = \text{Vol}(S_r)^{-1} \int_{S_r} |z|^{\frac{2}{p_i - 1}} u_i(z) d\Sigma_g$ and $z$ is the conformally flat coordinate system centered at each $x_i$.

3. Initial Steps of the Proof of Theorem 1.4.

The proof of Theorem 1.4 follows along the same line of reasoning as the proof of Theorem 1.2 which is done in [18]. As proved in Section 2 of [18], a lower bound on $u$ follows directly if there is a uniform upper bound on $u$. By the standard elliptic theory and Sobolev embedding theorem, a bound on the $C^0$-norm of $u$ easily implies a bound on its $C^3$-norm. Therefore, to prove Theorem 1.4 we only need to show that there is a uniform upper bound on $u$.

By an argument identical to that in Section 3 of [18], we can show that there exists a positive constant $\eta = \eta(M, g, n, \|K\|_{C^{n-2}(M)}, \Lambda)$, such that on the set $K_\eta := \{x \in M : K(x) < \eta\}$, $u$ has a uniform upper bound depending only on $M, g, n, \|K\|_{C^{n-2}(M)}$, and $\Lambda$. Thus it is left to show that $u$ is uniformly bounded on the set where $K \geq \eta$. We have the following proposition.

Proposition 3.1. Given $\epsilon > 0, R \gg 0$, there exists $C = C(\epsilon, R)$ such that if $u$ is a solution of equation (4) and

$$\max_{x \in M} \left( \left( d_g(x, K_2) \right)^{\frac{2}{p_i - 1}} u(x) \right) > C,$$

then there exists $\{x_1, ..., x_N\} \subset M \setminus K_2$ with $N$ depending on $u$, and

- Each $x_i$ is a local maximum point of $u$ and the geodesic balls $\{B_{\frac{R}{u(x_i)}^\frac{2}{p_i - 1}}(x_i)\}$ are disjoint.
\[ \left| \frac{n+2}{n-2} - p \right| < \epsilon \] and in the coordinate system \( y \) so chosen that \( z = \frac{y}{u(x_i)^{\frac{2}{p-1}}} \) is the conformally flat coordinate system centered at \( x_i \), we have
\[
\left\| u(x_i)^{-1} u \left( \frac{y}{u(x_i)^{\frac{2}{p-1}}} \right) - \tilde{v}(y) \right\|_{C^2(B_{2R}(0))} < \epsilon
\]
on the ball \( B_{2R}(0) \subset \mathbb{R}^n(y) \), where
\[
\tilde{v}(y) = \left( 1 + \frac{K(x_i)}{n(n-2)}|y|^2 \right)^{-\frac{n-2}{n}}.
\]

- There exists \( C = C(\epsilon, R) \) such that
\[
u(x) \leq C \left( d_g(x, K\frac{2}{d^2} \cup \{x_1, ..., x_N\}) \right)^{-\frac{2}{p-1}}.
\]

This can be proved as in [18] in the proof of Proposition 4.2, so we omit the details.

Now we are going to prove that \( u \) is uniformly bounded on \( M \setminus K_\eta \). Suppose it is not, then there are sequences \( \{u_i\} \) and \( \{p_i\} \) such that
\[
\Delta g u_i + K u_i^{p_i} = 0 \quad \text{and} \quad \max_{M \setminus K_\eta} u_i \to \infty \quad \text{as} \quad i \to \infty.
\]
Therefore \( \max_{M \setminus K_\eta} \left( d_g(x, K\frac{2}{d^2} \cup \{x_1, ..., x_N\}) \right)^{\frac{2}{p_i-1}} u_i(x) \to \infty \) as \( i \to \infty \). Then for fixed \( \epsilon > 0 \) and \( R >> 0 \) we can apply Proposition 3.1 to each \( u_i \) and find \( x_{1,i}, ..., x_{N(i),i} \) such that
\[
(3) \quad \text{each} \quad x_{j,i} \quad (1 \leq j \leq N(i)) \quad \text{is a local maximum point of} \quad u_i;
\]
\[
(4) \quad \text{the balls} \quad B_{\frac{R}{u_i(x_{j,i})^{\frac{2}{p_i-1}}}}(x_{j,i}) \quad \text{are disjoint};
\]

for coordinates \( y \) centered at \( x_{j,i} \) such that \( \frac{y}{u_i(x_{j,i})^{\frac{2}{p_i-1}}} \) is the conformally flat coordinate system,
\[
\left\| u_i(x_{j,i})^{-1} u_i \left( \frac{y}{u_i(x_{j,i})^{\frac{2}{p_i-1}}} \right) - \left( 1 + \frac{K(x_{j,i})}{n(n-2)}|y|^2 \right)^{-\frac{n-2}{2}} \right\|_{C^2(B_{2R}(0))} < \epsilon;
\]

and
\[
(6) \quad u_i(x) \leq C \left( d_g(x, K\frac{2}{d^2} \cup \{x_1, ..., x_{N(i),i}\}) \right)^{-\frac{2}{p_i-1}} \quad \text{for a constant} \quad C = C(\epsilon, R).
\]

Let \( \sigma_i = \min \{ d_g(x_{\alpha,i}, x_{\beta,i}) : \alpha \neq \beta, 1 \leq \alpha, \beta \leq N(i) \} \). Without lost of generality we can assume \( \sigma_i = d_g(x_{1,i}, x_{2,i}) \). There are two possibilities which could happen.

**Case I:** \( \sigma_i \geq \epsilon > 0 \).

Then the points \( x_{j,i} \) have isolated limiting points \( x_1, x_2, ..., \) which are isolated blow-up points of \( \{u_i\} \) as defined above.

**Case II:** \( \sigma_i \to 0 \).

Then we rescale the coordinates to make the minimal distance 1: let \( y = \sigma_i^{-1}z \) where \( z \) is
Let $z$ be a simple blow-up point, without loss of generality we assume it to be $x$ such that $\lim_{y \to \infty} g(\sigma_i y) = 0$. By this definition $\Delta g(\sigma_i y) = g_{\alpha\beta}(\sigma_i y)dy^\alpha dy^\beta$. As proved in Section 4 of [18], 0 is an isolated blow-up point of $\{v_i\}$.

In Sections 4 and 5 we are going to prove that neither Case I nor Case II can happen.

4. Ruling out Case I

If the blow-up points are all isolated, then same argument as that in Section 6 of [18] shows that among the isolated blow-up points $\{x_1, x_2, \ldots\}$, there must be one which is not a simple blow-up point, without loss of generality we assume it to be $x_1$. To simplify the notations we are going to rename it to be $x_0$. Let $x_i$ be the local maximum point of $u_i$ such that $\lim_{i \to \infty} x_i = x_0$.

Let $z = (z_1, \ldots, z_n)$ be the conformally flat coordinates centered at each $x_i$. Since $x_0$ is not a simple blow-up point, as a function of $|z|$, $|z|^2 \sum_i u_i(|z|)$ has a second critical point at $|z| = r_i$ where $r_i \to 0$. Let $y = \frac{z}{r_i}$ and define $v_i(y) = r_i^{\frac{n}{n-1} - 1} u_i(r_i y)$. Then $v_i(y)$ satisfies

$$\Delta g(\sigma_i y) v_i(y) + K_i(y) v_i(y)^{p_i} = 0$$

where $g(\sigma_i y) = g_{\alpha\beta}(r_i y)dy^\alpha dy^\beta$ and $K_i(y) = K(r_i y)$.

By this definition $|y| = 1$ is the second critical point of $|y|^{\frac{2}{p_i - 1}} v_i(|y|)$. As shown in Section 6 of [18], 0 is a simple blow-up point of $\{v_i\}$.

4.1. Estimates for $v_i$. The following estimates are essentially the same as Proposition 5.3 in [18], except for a slightly different choice of parameters, but for completeness we repeat the proof.

**Proposition 4.1.** There exists a constant $C$ independent of $i$ such that

- if $0 \leq |y| \leq 1$, then
  $$v_i(y) \geq Cv_i(0) \left(1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{\frac{1}{n-2}}|y|^2\right)^{-\frac{n+2}{2}}$$

- if $0 \leq |y| \leq \frac{R}{v_i(0)^{\frac{n}{n-2}}}$, then
  $$v_i(y) \leq Cv_i(0) \left(1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1}|y|^2\right)^{-\frac{n+2}{2}}$$

- if $\frac{R}{v_i(0)^{\frac{n}{n-2}}} \leq |y| \leq 1$, then
  $$v_i(y) \leq Cv_i(0)^{t_i} |y|^{-t_i}$$

where $l_i, t_i$ are so chosen that $\frac{2n-5}{2} < \lim_{i \to \infty} l_i < n - 2$, and $t_i = 1 - \frac{(p_i - 1)t_i}{2}$. 

Proof: By Proposition 3.1 when $0 \leq |y| \leq \frac{R}{v_i(0)^{\frac{n-1}{2}}}$,

$$
(1 + \epsilon)v_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2 \right)^{-\frac{n-2}{2}} \\
\geq v_i(y) \\
\geq (1 - \epsilon)v_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2 \right)^{-\frac{n-2}{2}} \\
\geq (1 - \epsilon)v_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{\frac{1}{n-2}} |y|^2 \right)^{-\frac{n-2}{2}}.
$$

So we only need to find the upper and lower bounds on $v_i(y)$ when $\frac{R}{v_i(0)^{\frac{n-1}{2}}} \leq |y| \leq 1$.

First the lower bound.

Let $G_i$ be the Green’s function of $\Delta_{g^{(i)}}$ which is singular at 0 and $G_i = 0$ on $\partial B_1$. Since $\{g^{(i)}\}$ converges uniformly to the Euclidean metric, there exist constants $C_1$ and $C_2$ independent of $i$ such that

$$
C_1 |y|^{2-n} \leq G_i(y) \leq C_2 |y|^{2-n}.
$$

When $|y| = Rv_i(0)^{-\frac{n-1}{2}}$,

$$
v_i(y) \geq (1 - \epsilon) \frac{v_i(0)}{\left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2 \right)^{\frac{n-2}{2}}} \\
= (1 - \epsilon) \frac{v_i(0)}{\left( 1 + \frac{K_i(0)}{n(n-2)} R^2 \right)^{\frac{n-2}{2}}} \\
= (1 - \epsilon) \left( R^{-2} + \frac{K_i(0)}{n(n-2)} \right)^{-\frac{n-2}{2}} R^{2-n} v_i(0) \\
\geq CR^{2-n} v_i(0) \\
\geq C R^{2-n} v_i(0)^{\frac{(n-2)(p_i-1)}{2}} - 1 \leq 1 \\
= C v_i(0)^{-1} |y|^{2-n} \\
\geq C v_i(0)^{-1} G_i(y)
$$

With this constant $C$, when $|y| = 1$, $C v_i(0)^{-1} G_i(y) = 0 < v_i(y)$.

We know that

$$
\Delta_{g^{(i)}} \left( v_i(y) - C v_i(0)^{-1} G_i(y) \right) = \Delta_{g^{(i)}} v_i(y) = -K_i(y) v_i(y)^{p_i} < 0
$$
on $B_1 \setminus B_{\frac{R}{v_i(0)^{\frac{n-1}{2}}}}$. Therefore, by the maximal principle, when $\frac{R}{v_i(0)^{\frac{n-1}{2}}} \leq |y| \leq 1$,

$$
v_i(y) > C v_i(0)^{-1} G_i(y) \\
\geq C v_i(0)^{-1} |y|^{2-n}.
$$

Now we need to compare $|y|^{2-n} v_i(0)^{-1}$ with $v_i(0) \cdot \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2 \right)^{-\frac{n-2}{2}}$ in order to get the desired lower bound.
Thus on \( \{ y \} \) and consequently

\[

v_i(0)^2 |y|^{n-2} \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{\frac{4}{n-2}} |y|^2 \right)^{-\frac{n-2}{2}} \leq v_i(0)^2 \left( \frac{K_i(0)}{n(n-2)} v_i(0)^{\frac{4}{n-2}} \right)^{-\frac{n-2}{2}} \leq C
\]

for a constant \( C \) independent of \( i \). Therefore

\[
v_i(0)^{-1} |y|^{2-n} \geq C v_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{\frac{4}{n-2}} |y|^2 \right)^{-\frac{n-2}{2}},
\]

and consequently

\[
v_i(y) \geq C v_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{\frac{4}{n-2}} |y|^2 \right)^{-\frac{n-2}{2}}
\]

when \( \frac{R}{v_i(0)^{\frac{n-2}{2}}} \leq |y| \leq 1 \).

Next the upper bound.

We are going to apply the same strategy of constructing a comparison function and using the maximal principle.

Define \( \mathcal{L}_i \varphi := \Delta g^{(i)} \varphi + K_i v_i^{p_i-1} \varphi \). By this definition \( \mathcal{L}_i v_i = 0 \). Let \( M_i = \max_{\partial B_1} v_i \) and \( C_i = (1 + \epsilon) \left( \frac{K_i(0)}{n(n-2)} \right)^{-\frac{n-2}{2}} \). Note that \( C_i \) is bounded above and below by constants independent of \( i \). Consider the function

\[
M_i |y|^{-n+2+l_i} + C_i v_i(0)^{t_i} |y|^{-l_i}.
\]

When \( |y| = \frac{R}{v_i(0)^{\frac{n-2}{2}}} \),

\[
v_i(y) \leq (1 + \epsilon) \frac{v_i(0)}{\left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2 \right)^{\frac{n-2}{2}}} \leq (1 + \epsilon) \frac{v_i(0)}{\left( 1 + \frac{K_i(0)}{n(n-2)} R^2 \right)^{\frac{n-2}{2}}} \leq C_i v_i(0) R^{-(n-2)} \leq C_i v_i(0) R^{-l_i} = C_i v_i(0)^{t_i} |y|^{-l_i}.
\]

When \( |y| = 1 \), by the definition of \( M_i \), \( v_i(y) \leq M_i = M_i |y|^{-n+2+l_i} \).

Thus on \( \{|y| = 1\} \cup \{|y| = R v_i(0)^{-\frac{n-2}{2}}\} \),

\[
v_i(y) \leq M_i |y|^{-n+2+l_i} + C_i v_i(0)^{t_i} |y|^{-l_i}.
\]

In the Euclidean coordinates, \( \Delta |y|^{-l_i} = -l_i(n-2-l_i) |y|^{-l_i-2} \) and \( \Delta |y|^{-n+2+l_i} = -l_i(n-2-l_i) |y|^{-n+l_i} \). When \( i \) is sufficiently large, \( g^{(i)} \) is close to the Euclidean metric. Therefore
\[(8) \quad \Delta g^{(i)} |y|^{-l_i} \leq -\frac{1}{2} l_i (n-2-l_i) |y|^{-(l_i+2)} \]

and

\[(9) \quad \Delta g^{(i)} |y|^{-n+2+l_i} \leq -\frac{1}{2} l_i (n-2-l_i) |y|^{-n+l_i}. \]

Thus

\[
\mathcal{L}_i (C_i v_i (0)^{l_i} |y|^{-l_i}) = C_i v_i (0)^{l_i} \Delta g^{(i)} |y|^{-l_i} + C_i v_i (0)^{l_i} K_i v_i (0)^{p_i-1} |y|^{-l_i} \\
\leq -C l_i (n-2-l_i) v_i (0)^{l_i} |y|^{-(l_i+2)} + C' v_i (0)^{l_i} v_i (y)^{p_i-1} |y|^{-l_i}
\]

for some constants \(C, C'\) independent of \(i\).

Lemma 2.3 and the upper bound on \(v_i(y)\) when \(|y| \leq R v_i (0)^{-\frac{p_i-1}{2}}\) imply that

\[
\bar{v}_i \left( R v_i (0)^{-\frac{p_i-1}{2}} \right) \leq \frac{C}{(1 + \epsilon) v_i (0)} \left[ 1 + \frac{K_i (0)}{n(n-2)} v_i (0)^{p_i-1} \left( R v_i (0)^{-\frac{p_i-1}{2}} \right)^2 \right]^{\frac{n-2}{2}} \\
\leq C v_i (0) R^{2-n}.
\]

Then since 0 is a simple blow-up point and \(r^{\frac{2}{p_i-1}} \bar{v}_i (r)\) is decreasing from \(R v_i (0)^{-\frac{p_i-1}{2}}\) to 1,

\[
|y|^{\frac{2}{p_i-1}} \bar{v}_i (|y|) \leq \left( R v_i (0)^{-\frac{p_i-1}{2}} \right)^{\frac{2}{p_i-1}} \bar{v}_i \left( R v_i (0)^{-\frac{p_i-1}{2}} \right) \\
\leq C R^{\frac{2}{p_i-1}+2-n}.
\]

Thus again by Lemma 2.3

\[(10) \quad v_i (y)^{p_i-1} \leq C \bar{v}_i (|y|)^{p_i-1} \leq C |y|^{-(n-2)(p_i-1)} \]

and hence

\[
v_i (y)^{p_i-1} |y|^{-l_i} \leq C |y|^{-(n-2)(p_i-1)} R^{2-(n-2)(p_i-1)}.
\]

Therefore

\[
\mathcal{L}_i \left( C_i v_i (0)^{l_i} |y|^{-l_i} \right) \\
\leq \left( -C l_i (n-2-\tilde{l}_i) + C' R^{2-(n-2)(p_i-1)} \right) v_i (0)^{l_i} |y|^{-(l_i+2)}
\]

By our choice of \(l_i\), \(l_i(n-2-\tilde{l}_i)\) is always bounded below by some positive constant independent of \(i\). When \(i\) is sufficiently large, \(2-(n-2)(p_i-1) < 0\), so we can choose \(R\) big enough such that \(-C l_i (n-2-\tilde{l}_i) + C' R^{2-(n-2)(p_i-1)} < 0\), which implies \(\mathcal{L}_i (C_i v_i (0)^{l_i} |y|^{-l_i}) < 0\).

Similarly,

\[
\mathcal{L}_i \left( M_i |y|^{-n+2+l_i} \right) = M_i \Delta g^{(i)} |y|^{-n+2+l_i} + M_i K_i v_i^{p_i-1} |y|^{-n+2+l_i} \\
\leq -\frac{1}{2} l_i (n-2-\tilde{l}_i) M_i |y|^{-(n+l_i)} + K_i M_i R^{2-(n-2)(p_i-1)} |y|^{-(n+l_i)}
\]
by equations (9) and (10). We can choose $R$ large enough such that $-\frac{1}{2}l_i(n - 2 - l_i) + K_i R^{2-(n-2)(p_i-1)} < 0$ and hence

$$\mathcal{L}_i(M_i |y|^{-n+2+l_i}) < 0.$$  

Therefore when $Rv_i(0)^{-\frac{n-1}{2}} \leq |y| \leq 1$,

$$\mathcal{L}_i \left( M_i |y|^{-n+2+l_i} + C_i v_i(0)^{l_i} |y|^{-l_i} \right) < 0.$$  

Then by the maximal principle

$$v_i(y) \leq M_i |y|^{-n+2+l_i} + C_i v_i(0)^{l_i} |y|^{-l_i}.$$  

By Lemma 2.3 and because 0 is a simple blow-up point, for $\frac{R}{v_i(0)^{\frac{n-1}{2}}} \leq \theta \leq 1$,

$$M_i \leq C \theta^{\frac{n-1}{n-1-l_i}} v_i(\theta) \leq C \theta^{\frac{n-1}{n-1-l_i}} (M_i \theta^{-n+2+l_i} + C_i v_i(0)^{l_i} \theta^{-l_i}) = C \theta^{\frac{n-1}{n-1-l_i}} \theta^{-n+2+l_i} M_i + C \theta^{\frac{n-1}{n-1-l_i}} \cdot C_i v_i(0)^{l_i} \theta^{-l_i}$$

for some constant $C$ independent of $i$.

Note that

$$\lim_{t \to \infty} \left( \frac{2}{p_i - 1} - n + 2 + l_i \right) = -\frac{n - 2}{2} + \lim_{t \to \infty} l_i > -\frac{n - 2}{2} + \frac{2n - 5}{2} > 0$$

because $n \geq 5$.

Since $\frac{R}{v_i(0)^{\frac{n-1}{2}}} \to 0$, we can choose $\theta$ small enough (fixed and independent of $i$) to absorb the first term on the right hand side of the above inequality into the left hand side to get

$$M_i \leq 2 C \theta^{\frac{n-1}{n-1-l_i}} \cdot C_i v_i(0)^{l_i} \theta^{-l_i} \leq C v_i(0)^{l_i}.$$  

Therefore

$$v_i(y) \leq M_i |y|^{-n+2+l_i} + C_i v_i(0)^{l_i} |y|^{-l_i} \leq M_i |y|^{-l_i} + C_i v_i(0)^{l_i} |y|^{-l_i} \leq C v_i(0)^{l_i} |y|^{-l_i}.$$  

\[\square\]

4.2. A Preliminary Estimate for $\delta_i := \frac{n+2}{n-2} - p_i$. First we prove a technical lemma.

**Lemma 4.2.** When $\sigma < 1$ and $0 \leq \kappa \leq n - 2$,

$$\int_{|y| \leq \sigma} |y|^\kappa v_i(y)^{p_i+1} dy \leq C v_i(0)^{-\frac{2n}{n-2} + \frac{n-2+\kappa}{2} + \delta_i},$$

where $C$ is independent of $i$.

**Proof:** By Proposition 4.1

$$\int_{|y| \leq \frac{R}{v_i(0)^{\frac{n-1}{2}}}} |y|^\kappa v_i(y)^{p_i+1} dy \leq C v_i(0)^{p_i+1} \int_{|y| \leq \frac{R}{v_i(0)^{\frac{n-1}{2}}}} |y|^\kappa dy \leq C v_i(0)^{p_i+1-\frac{(n+1)(p_i-1)}{2}} \leq C v_i(0)^{-\frac{2n}{n-2} + \frac{n-2+\kappa}{2} + \delta_i}.$$
Since \( n \geq 5 \), by our choice of \( l_i \)
\[
\lim_{i \to \infty} \left( n + \kappa - l_i(p_i + 1) \right) = n + \kappa - \frac{2n}{n-2} \lim_{i \to \infty} l_i
\]
\[
< n + \kappa - \frac{2n}{n-2} \cdot \frac{2n-5}{2}
\]
\[
\leq n + (n-2) - \frac{n(2n-5)}{n-2}
\]
\[
< 0.
\]

Therefore
\[
\int_{\frac{\sigma}{\nu_1(0)}}^{\frac{n-1}{2}} |y|^\kappa v_i(y)^{p_i+1} dy \leq C \int_{\frac{\sigma}{\nu_1(0)}}^{\frac{n-1}{2}} |y|^\kappa (v_i(0)^{t_i}|y|^{-l_i})^{p_i+1} dy
\]
\[
\leq C v_i(0)^{t_i(p_i+1) - \frac{2}{n-2}} (n-l_i(p_i+1)+\kappa)
\]
\[
= C v_i(0)^{p_i+1} \frac{(n+\kappa)(p_i-1)}{2} \quad \text{(by the definition of } t_i) \]
\[
= C v_i(0)^{-\frac{2n}{n-2} + \frac{n-2+\kappa}{n-2}}.
\]

Thus
\[
\int_{|y| \leq \sigma} |y|^\kappa v_i(y)^{p_i+1} dy \leq C v_i(0)^{-\frac{2n}{n-2} + \frac{n-2+\kappa}{n-2}}.
\]

The next proposition is a preliminary estimate for \( \delta_i := \frac{n+2}{n-2} - p_i \), we will also derive a refined estimate in a later part of this paper.

**Proposition 4.3.** \( \lim_{i \to \infty} v_i(0)^{\delta_i} = 1. \)

**Proof:** Since the original metric is locally conformally flat, locally it can be written as \( \lambda(z)^{\frac{2}{n-2}} dz^2 \). Let \( \lambda_i(y) = \lambda(r_iz) \), then \( g_i^{(i)}(y) = \lambda_i(y)^{\frac{4}{n-2}} dy^2 \). Let \( \sigma < 1 \), the Pohozaev identity in [14] says that for a conformal Killing field \( X \) on \( B_\sigma \),

\[
(11) \quad \frac{n-2}{2n} \int_{B_\sigma} X(R_i) dv_{g_i} = \int_{\partial B_\sigma} T_i(X, \nu_i) d\Sigma_i
\]

where the notations are
\[
g_i = \nu_i^{\frac{4}{n-2}}g_i^{(i)} = (\lambda_i\nu_i)^{\frac{4}{n-2}} dy^2,
\]
\[
R_i = R(g_i) = c(n)^{-1} K_i \nu_i^{-\delta_i},
\]
\[
dv_{g_i} = (\lambda_i\nu_i)^{\frac{2n}{n-2}} dy,
\]
\[
\nu_i = (\lambda_i\nu_i)^{-\frac{2}{n-2}} \sigma^{-1} \sum_j y^j \frac{\partial}{\partial y_j}
\]

is the unit outer normal vector on \( \partial B_\sigma \) with respect to \( g_i \),
\[
d\Sigma_i = (\lambda_i\nu_i)^{\frac{2(n-1)}{n-2}} d\Sigma_\sigma
\]

where \( d\Sigma_\sigma \) is the surface element of the standard \( S^{n-1}(\sigma) \),
\[
T_i = \operatorname{Ric}(g_i) - n^{-1} R(g_i) g_i \quad \text{is the traceless Ricci tensor with respect to } g_i.
\]
$T_i$ can also be expressed as (see [15])

$$(n - 2)(\lambda_i v_i)^{\frac{2}{n-2}} \left( \text{Hess} \left((\lambda_i v_i)^{-\frac{2}{n-2}}\right) - \frac{1}{n} \Delta \left((\lambda_i v_i)^{-\frac{2}{n-2}}\right) \, dy^2 \right)$$

where Hess and $\Delta$ are taken with respect to the Euclidean metric $dy^2$.

We choose $X = \sum_{j=1}^n y^j \partial / \partial y^j$. The left hand side is

$$\frac{n - 2}{2n} \int_{B_\sigma} X(R_i)dv_{gi} = \frac{2(n-1)}{n} \int_{B_\sigma} X(K_i v_i^{-\delta_i})(\lambda_i v_i)^{\frac{2n}{n-2}}dy$$

$$= \frac{2(n-1)}{n} \int_{B_\sigma} X(K_i v_i^{-\delta_i} v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}}) - \frac{2(n-1)}{n} \delta_i \int_{B_\sigma} K_i v_i^{p_i+1} X(\lambda_i)^{\frac{2n}{n-2}}dy$$

By the divergence theorem it is equal to

$$= \frac{2(n-1)}{n} \int |y| \frac{\partial K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}}}{\partial r} dy + \frac{2(n-1)}{n} \delta_i \left( \int_{B_\sigma} r \frac{\partial K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}}}{\partial r} dy \right)$$

$$+ \int_{B_\sigma} K_i v_i^{p_i+1} r \frac{\partial \lambda_i^{\frac{2n}{n-2}}}{\partial r} dy + \int_{B_\sigma} \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} \text{div} X \, dy$$

$$- \frac{2(n-1)}{n} \delta_i \int_{\partial B_\sigma} K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} X \cdot \left( \sum_{j=1}^n y^j \frac{\partial}{\partial y^j} \right) d\Sigma_\sigma,$$

which can be further written as

$$= \frac{2(n-1)}{n} \left(1 + \frac{\delta_i}{p_i+1}\right) \int_{B_\sigma} |y| \frac{\partial K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}}}{\partial r} dy$$

$$+ \frac{2(n-1)}{n} \delta_i \int_{B_\sigma} |y| K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} \frac{\partial}{\partial r} dy$$

$$+ \frac{2(n-1)}{n} \delta_i \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy - \frac{2(n-1)}{n} \delta_i \int_{\partial B_\sigma} \sigma K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} d\Sigma_\sigma.$$

(12)
The right hand side of (11) is
\[
\int_{\partial B_\sigma} T_i(X, \nu_i) d\Sigma_i = \int_{\partial B_\sigma} (n-2)(\lambda_i \nu_i)^{\frac{n-2}{2}} \left[ \text{Hess} \left( (\lambda_i \nu_i)^{-\frac{2}{n-2}} \right) \left( \frac{\partial}{\partial r} (\lambda_i \nu_i)^{-\frac{2}{n-2}} \sigma^{-1} r \frac{\partial}{\partial r} \right) \right. \\
- \frac{1}{n} \Delta \left( (\lambda_i \nu_i)^{-\frac{2}{n-2}} \right) \left( r \frac{\partial}{\partial r} (\lambda_i \nu_i)^{-\frac{2}{n-2}} \sigma^{-1} r \frac{\partial}{\partial r} \right) \left. \right] (\lambda_i \nu_i)^{\frac{2(n-1)}{n-2}} d\Sigma_i
\]
(where $\langle \cdot, \cdot \rangle$ is the Euclidean metric)
\[
= (n-2) \int_{\partial B_\sigma} \sigma^{-1} \text{Hess} \left( (\lambda_i \nu_i)^{-\frac{2}{n-2}} \right) \left( r \frac{\partial}{\partial r}, r \frac{\partial}{\partial r} \right) \\
- \frac{\sigma}{n} \Delta \left( (\lambda_i \nu_i)^{-\frac{2}{n-2}} \right) \left( \lambda_i \nu_i \right)^{\frac{2(n-1)}{n-2}} d\Sigma_i
\]
\[
= (n-2) \int_{\partial B_\sigma} \sigma^{-1} \left[ - \frac{2}{n-2} (\lambda_i \nu_i) \sum_{j,k} y^j y^k \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k} (\lambda_i \nu_i) + \frac{2n}{(n-2)^2} \sum_j \left( \frac{\partial (\lambda_i \nu_i)}{\partial y^j} \right)^2 \right] d\Sigma_i
\]
(13)
\[
= \frac{2n}{(n-2)^2} \sum_{j,k} y^j y^k \frac{\partial^2 (\lambda_i \nu_i)}{\partial y^j \partial y^k} + \frac{2}{n-2} \sum_j \left( \frac{\partial (\lambda_i \nu_i)}{\partial y^j} \right)^2 \right] d\Sigma_i
\]

Next we are going to study the decay rate of each term in (12) and (13).
On $\partial B_\sigma$, by Proposition 4.1 $v_i \leq C v_i(0)^{t_i}$, then by the elliptic regularity theory [5] $\|v_i\|_{C^2(\partial B_\sigma)} \leq C v_i(0)^{t_i}$. Thus we know (13) decays in the rate of $v_i(0)^{2t_i}$.
The fourth term in (12) decays in the order of $\delta_i v_i(0)^{(p_i+1)}$ by Proposition 4.1. By Lemma 4.2 we know that the second term in (12) is bounded above by
\[
C \delta_i \int_{B_\sigma} |y| v_i^{p_i+1} dy \leq C \delta_i v_i(0)^{-\frac{2}{n-2} + \frac{n+1}{2} \delta_i}.
\]
Therefore the sum of the first and the third terms in (12), which is
\[
\frac{n}{2(n-1)} \left( 1 + \frac{\delta_i}{p_i + 1} \right) \int_{B_\sigma} |y| \frac{\partial K_i}{\partial r} v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy + \frac{n}{2(n-1)} \frac{\delta_i}{p_i + 1} n \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy
\]
is bounded above by $C v_i(0)^{2t_i} + C \delta_i v_i(0)^{(p_i+1)} + C \delta_i v_i(0)^{-\frac{2}{n-2} + \frac{n+1}{2} \delta_i}$.
By our choice of $l_i$ and $t_i$, as $i \to \infty$,
\[
t_i = 1 - \frac{(p_i - 1)l_i}{2} \to 1 - \frac{2}{n-2} \lim_{i \to \infty} l_i < 1 - \frac{2}{n-2} \cdot \frac{2n-5}{2} < 0.
\]
Thus $C v_i(0)^{2t_i} + C \delta_i v_i(0)^{(p_i+1)} \leq C v_i(0)^{2t_i} + C v_i(0)^{(p_i+1)} \leq C v_i(0)^{2t_i}$.
On the other hand
\[
\delta_i \frac{n}{p_i + 1} \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy \geq C \delta_i \int_{B_\sigma} v_i^{p_i+1} dy.
\]
When $|y| \leq \frac{R}{v_i(0)^{\frac{n-2}{2}}}$, by Proposition 4.1

$$v_i(y) \geq (1 - \epsilon) \frac{v_i(0)}{\left(1 + \frac{K_i(0)}{n(n-2)} v_i(0) p_i - 1 \right)^{\frac{n-2}{2}}} \geq (1 - \epsilon) \frac{v_i(0)}{\left(1 + \frac{K_i(0)}{n(n-2)} R^2 \right)^{\frac{n-2}{2}}} \geq C v_i(0),$$

so

$$\int_{B_\sigma} v_i^{p_i+1} dy > \int_{|y| \leq \frac{R}{v_i(0)^{\frac{n-2}{2}}}} v_i^{p_i+1} dy \geq C v_i(0)^{p_i + 1 - \frac{n}{2}(p_i - 1)} = C v_i(0)^{1 - \frac{n}{2} \delta_i} \geq C.$$

(14)

This implies that the third term in (12) is bounded below by $C \delta_i$. Then by comparing the decay rates of the terms in (12) and (13),

$$\delta_i \leq C \left( v_i(0)^{2t_i} + \delta_i v_i(0)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i} + \left| \int_{B_\sigma} \frac{\partial K_i}{\partial r} |y| v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy \right| \right).$$

Since $v_i(0)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i} \to 0$, the second term on the right hand side can be absorbed into the left hand side. Thus we conclude that

$$\delta_i \leq C \left( v_i(0)^{2t_i} + \left| \int_{B_\sigma} \frac{\partial K_i}{\partial r} |y| v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy \right| \right).$$

(15)

By Lemma 4.2

$$\int_{B_\sigma} \frac{\partial K_i}{\partial r} |y| v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy \leq C v_i(0)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i},$$

thus

$$\delta_i \leq C \left( v_i(0)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i} + v_i(0)^{2t_i} \right).$$

This implies that

$$\delta_i \ln v_i(0) \leq C \left( v_i(0)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i} + v_i(0)^{2t_i} \right) \ln v_i(0) \to 0$$

as $i \to \infty$. Therefore $\lim_{i \to \infty} v_i(0)^{\delta_i} = 1$. Consequently, we have

$$\delta_i \leq C \left( v_i(0)^{-\frac{2}{n-2} + v_i(0)^{2t_i}} \right).$$

(16)
4.3. A Preliminary Estimate for $|\nabla K_i|$. We will again study the Pohozaev identity (11), but with a different choice of the conformal Killing field $X = \frac{\partial}{\partial y^i}$.

Direct calculation, as that in the proof of Proposition 4.3, shows that the right hand side of the identity is equal to

$$(n-2) \int_{\partial B_\sigma} \sum_j \frac{y^j}{\sigma} \left( \frac{-2}{n-2} (\lambda_i v_i) \frac{\partial^2 (\lambda_i v_i)}{\partial y^j \partial y^j} + \frac{2n}{(n-2)^2} \frac{\partial (\lambda_i v_i)}{\partial y^j} \frac{\partial (\lambda_i v_i)}{\partial y^j} \right)$$

$$- \frac{y^i}{\sigma} \sum_j \left( \frac{-2}{n(n-2)} (\lambda_i v_i) \frac{\partial^2 (\lambda_i v_i)}{\partial y^j \partial y^j} + \frac{2}{(n-2)^2} \left( \frac{\partial (\lambda_i v_i)}{\partial y^j} \right)^2 \right) d\Sigma_\sigma,$$

and decays in the rate of $v_i(0)^{2t_i}$.

The left hand side of this identity is

$$(n-2) \int_{\partial B_\sigma} \frac{\partial}{\partial y^i} (R_i) dv_{g_i} = \frac{n-2}{2n} c(n)^{-1} \int_{B_\sigma} \frac{\partial}{\partial y^i} (K_i v_i^{-\delta_i}) (\lambda_i v_i)^{\frac{2n}{n-2}} dy$$

$$= \frac{n-2}{2n} c(n)^{-1} \int_{B_\sigma} \left( 1 + \frac{\delta_i}{p_i + 1} \right) \lambda_i^{\frac{2n}{n-2}v_i^{p_i}+1} \frac{\partial K_i}{\partial y^i} dy$$

$$+ \frac{n-2}{2n} c(n)^{-1} \int_{B_\sigma} \frac{\delta_i}{p_i + 1} K_i v_i^{p_i+1} \frac{\partial \lambda_i^{\frac{2n}{n-2}}}{\partial y^i} dy$$

$$- \frac{n-2}{2n} c(n)^{-1} \frac{\delta_i}{p_i + 1} \int_{\partial B_\sigma} \lambda_i^{\frac{2n}{n-2}v_i^{p_i}+1} \frac{1}{\sigma} d\Sigma_\sigma.$$ (17)

By Proposition 4.1, the last term in (17) is bounded above by

$$C \delta_i \cdot v_i(0)^{t_i(p_i+1)} \leq C \delta_i v_i(0)^{2t_i}$$

since $t_i < 0$ and $v_i(0) \to \infty$.

Note that $\lambda_i(y) = \lambda(r_i y)$, the second term in (17) is bounded above by

$$C \delta_i r_i \int_{|y| \leq \sigma} v_i(y)^{p_i+1} dy,$$

which is further bounded by $C \delta_i r_i v_i(0)^{n-2\delta_i} \leq C \delta_i$ by Lemma 4.2 and Proposition 4.3. Therefore the first term in (17) which is

$$\frac{n-2}{2n} c(n)^{-1} \int_{B_\sigma} \left( 1 + \frac{\delta_i}{p_i + 1} \right) \lambda_i^{\frac{2n}{n-2}v_i^{p_i}+1} \frac{\partial K_i}{\partial y^i} dy$$

is bounded above by $C (v_i(0)^{2t_i} + \delta_i v_i(0)^{2t_i} + \delta_i r_i) \leq C (\delta_i r_i + v_i(0)^{2t_i})$. This shows that

$$\left| \int_{B_\sigma} \lambda_i^{\frac{2n}{n-2}v_i^{p_i}+1} \frac{\partial K_i}{\partial y^i} dy \right| \leq C (\delta_i r_i + v_i(0)^{2t_i}).$$ (18)

By the Taylor expansion

$$\frac{\partial K_i}{\partial y^i}(y) = \frac{\partial K_i}{\partial y^i}(0) + \nabla \left( \frac{\partial K_i}{\partial y^i} \right) (\varsigma) \cdot y$$

for some $|\varsigma| \leq |y|$.
Note that $K_i(y) = K(r, y)$. By Lemma 4.2 and Proposition 4.3

$$
\int_{B_r} \lambda_i^{2n} v_i^{p_i+1} |\nabla \left( \frac{\partial K_i}{\partial y^i} \right) (\cdot) \cdot y | \, dy \leq C r_i \int_{B_r} v_i^{p_i+1} |y| \, dy \\
\leq C r_i v_i(0)^{-\frac{n-2}{n-2} + \frac{n-1}{2} \delta_i} \\
\leq C r_i v_i(0)^{-\frac{2}{n-2}}.
$$

Thus we know

$$
\left| \frac{\partial K_i}{\partial y^i}(0) \right| \int_{B_r} v_i^{p_i+1} \, dy \leq C \left| \int_{B_r} \lambda_i^{2n} v_i^{p_i+1} \frac{\partial K_i}{\partial y^i}(0) \, dy \right| \\
\leq C \left( r_i v_i(0)^{-\frac{2}{n-2}} + (\delta_i r_i + v_i(0)^{2i}) \right) \\
\leq C \left( r_i v_i(0)^{-\frac{2}{n-2}} + r_i v_i(0)^{2i} + v_i(0)^{2i} \right) \quad \text{(by inequality (10))} \\
\leq C \left( r_i v_i(0)^{-\frac{2}{n-2}} + v_i(0)^{2i} \right).
$$

Then by (14)

$$
(19) \quad \left| \frac{\partial K_i}{\partial y^i}(0) \right| \leq C \left( r_i v_i(0)^{-\frac{2}{n-2}} + v_i(0)^{2i} \right).
$$

The same estimate holds for $|\frac{\partial K_i}{\partial y^j}(0)|$, $j = 2, \ldots, n$ as well, since we can also choose $X = \frac{\partial}{\partial y^j}$ in the above calculation.

4.4. **Location of the Blow-up.** Choose a point $\bar{y}$ with $|\bar{y}| = 1$. It is proved in Section 6 of [18] that $\frac{v_i}{v_i(\bar{y})}$ converges in $C^2$-norm to a function $h$ on any compact subset of $\mathbb{R}^n \setminus \{0\}$, and $h = \frac{1}{2} + \frac{1}{2} |y|^{2-n}$.

Recall that we chose the coordinate systems $z = (z^1, \ldots, z^n)$ and $y = \frac{x_i}{r_i}$ to be centered at each $x_i \in M$, thus $\nabla K_i(0) = r_i \nabla K(x_i)$. Here we write $\nabla K(x_i)$ instead of $\nabla K(0)$ to emphasize the fact that $\nabla K$ is evaluated at different point $x_i$ as $i \to \infty$. We claim that this blow-up must occur at a critical point of $K$, i.e.,

**Proposition 4.4.** $\nabla K(x_0) = \lim_{i \to \infty} \nabla K(x_i) = 0$.

**Proof:** Suppose this is not true, then there exists some $j \in \{1, \ldots, n\}$, such that $|\frac{\partial K}{\partial y^j}(x_i)| \geq \varepsilon$ for a constant $\varepsilon$ independent of $i$. Without loss of generality we assume $j = 1$. Then from inequality (19) we know that $\varepsilon r_i \leq C \left( r_i v_i(0)^{-\frac{2}{n-2}} + v_i(0)^{2i} \right)$. Therefore

$$
(20) \quad r_i \leq C v_i(0)^{2i}
$$

when $v_i(0)^{-\frac{2}{n-2}}$ is sufficiently small.

Once more we look at the Pohozaev identity (11) with $X = \sum_j y^j \frac{\partial}{\partial y^j}$. We divide both sides of it by $v_i^2(\bar{y})$ so it becomes

$$
(21) \quad \frac{n-2}{2n} \frac{1}{v_i^2(\bar{y})} \int_{B_r} X(R_i) \, dv_i = \frac{1}{v_i^2(\bar{y})} \int_{\partial B_r} T_i(X, v_i) \, d\Sigma_i
$$
Its right hand side is

\[
\frac{1}{v_i^2(y)} \int_{\partial B_\sigma} T_i(X, \nu_i) d\Sigma_i \\
= \frac{1}{v_i^2(y)} \int_{\partial B_\sigma} (\text{Ric}(g_i) - n^{-1} R(g_i) g_i) (X, \nu_i) d\Sigma_i \\
= \frac{1}{v_i^2(y)} \int_{\partial B_\sigma} \left[ \text{Ric} \left( \left( \lambda_i v_i \right)^{\frac{4}{n-2}} dy \otimes dy \right) - n^{-1} R \left( \left( \lambda_i v_i \right)^{\frac{4}{n-2}} dy \otimes dy \right) \right] (X, \nu_0)(\lambda_i v_i)^2 d\Sigma_\sigma
\]

(22) \quad = \int_{\partial B_\sigma} \left( \lambda_i v_i \right)^{\frac{4}{n-2}} \left[ \text{Ric} \left( \left( \frac{\lambda_i v_i}{v_i(y)} \right)^{\frac{4}{n-2}} dy \otimes dy \right) - n^{-1} R \left( \left( \frac{\lambda_i v_i}{v_i(y)} \right)^{\frac{4}{n-2}} dy \otimes dy \right) \right] (X, \nu_0) d\Sigma_\sigma

where \( \nu_0 = \sigma^{-1} \sum_j y^j \frac{\partial}{\partial y^j} \) is the unit outer normal on \( \partial B_\sigma \) with respect to the Euclidean metric \( dy \otimes dy \).

When \( i \to \infty \), for \( |y| = \sigma \), \( \lambda_i(y) = \lambda(x_0) \). Thus when \( i \) goes to \( \infty \), up to a constant (22) converges to

\[
\int_{\partial B_\sigma} h^2 \left( \text{Ric} \left( h^{\frac{4}{n-2}} dy \otimes dy \right) - n^{-1} R \left( h^{\frac{4}{n-2}} dy \otimes dy \right) h^{\frac{4}{n-2}} dy \otimes dy \right) (X, \nu_0) d\Sigma_\sigma \\
= \int_{\partial B_\sigma} h^2 \cdot (n-2) h^{\frac{2}{n-2}} \left[ \text{Hess} \left( h^{-\frac{2}{n-2}} \right) (X, \nu_0) - \frac{1}{n} \Delta \left( h^{-\frac{2}{n-2}} \right) < X, \nu_0 > \right] d\Sigma_\sigma \\
(23) = (n-2)\sigma^{-1} \int_{\partial B_\sigma} h^{\frac{2(n-1)}{n-2}} \left[ \text{Hess} \left( h^{-\frac{2}{n-2}} \right) (X, X) - \frac{1}{n} \Delta \left( h^{-\frac{2}{n-2}} \right) \sigma^2 \right] d\Sigma_\sigma
\]

We know that

\[
h^{-\frac{2}{n-2}} = \left( \frac{1}{2} \left( 1 + |y|^{2-n} \right) ^{-\frac{2}{n-2}} = 2^{\frac{2}{n-2}} |y|^2 - \frac{2^{\frac{n}{n-2}}}{n-2} |y|^n + O \left( |y|^{2(n-1)} \right),
\]

and by direct computation

\[
\text{Hess} \left( 2^{\frac{2}{n-2}} |y|^2 - \frac{2^{\frac{n}{n-2}}}{n-2} |y|^n \right) (X, X) - \frac{1}{n} \Delta \left( 2^{\frac{2}{n-2}} |y|^2 - \frac{2^{\frac{n}{n-2}}}{n-2} |y|^n \right) \sigma^2 = -2^{\frac{n}{n-2}} (n-1)\sigma^n.
\]

Therefore

\[
\text{Hess} \left( h^{-\frac{2}{n-2}} \right) (X, X) - \frac{1}{n} \Delta \left( h^{-\frac{2}{n-2}} \right) \sigma^2 = -2^{\frac{n}{n-2}} (n-1)\sigma^n + O \left( \sigma^{2(n-1)} \right).
\]

Also we know

\[
h^{-\frac{2(n-1)}{n-2}} = \left( \frac{1}{2} \right) \left( \frac{2^{(n-1)}}{n-2} \right)^{\frac{n}{n-2}} |y|^{-2(n-1)} \left( 1 + O(|y|^{n-2}) \right).
\]
Thus we can conclude that (23) is equal to
\[ -\frac{1}{2} (n-1)(n-2) \sigma^{-1} \int_{\partial B_\sigma} \left( |y|^{-2(n-1)} + O(|y|^{-n}) \right) \left( |y|^n + O(|y|^{2(n-1)}) \right) \sigma^{-1} d\Sigma_1 \]
\[ = -\frac{1}{2} (n-1)(n-2) + O(\sigma^{n-2}) \]

Therefore the limit of the right hand side of (21) is strictly less than 0 when we choose \( \sigma \) to be sufficiently small.

On the other hand, the left hand side of (21) is
\[ \frac{n-2}{2n} \varphi(n)^{-1} \frac{1}{v_i^2(y)} \int_{B_\sigma} X(K_i v_i^{-\delta_i})(\lambda_i v_i)^{\frac{2n}{n-2}} dy. \]

We write
\[ \frac{1}{v_i^2(y)} \int_{B_\sigma} X(K_i v_i^{-\delta_i})(\lambda_i v_i)^{\frac{2n}{n-2}} dy = \frac{1}{v_i^2(y)} \int_{B_\sigma} X(K_i) v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy - \frac{\delta_i}{v_i^2(y)} \int_{B_\sigma} K_i \lambda_i^{\frac{2n}{n-2}} v_i^{p_i} X(v_i) dy. \]

The second term of (24)
\[ = -\frac{\delta_i}{p_i+1} \frac{1}{v_i^2(y)} \int_{B_\sigma} K_i \lambda_i^{\frac{2n}{n-2}} X(v_i^{p_i+1}) dy \]
\[ = -\frac{\delta_i}{p_i+1} \frac{1}{v_i^2(y)} \int_{B_\sigma} \left[ \text{div} \left( K_i \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} X \right) - K_i \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \text{div} X \right. \]
\[ - \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} X(K_i) - K_i v_i^{p_i+1} (\lambda_i^{\frac{2n}{n-2}}) \right] dy \]
\[ = -\frac{\delta_i}{p_i+1} \frac{\sigma}{v_i^2(y)} \int_{\partial B_\sigma} K_i \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} d\Sigma \]
\[ + \frac{\delta_i}{p_i+1} \frac{1}{v_i^2(y)} \int_{B_\sigma} K_i \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \left( n + X(\ln K_i) + \frac{2n}{n-2} X(\ln \lambda_i) \right) dy \]

On \( \partial B_\sigma \), \( \frac{v_i(y)}{v_i(y)} \to h(\sigma) \) and \( v_i \to 0 \) uniformly, so
\[ \frac{1}{v_i^2(y)} \int_{\partial B_\sigma} K_i \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} d\Sigma \to 0. \]

Since \( X = r \frac{\partial}{\partial r} \) and \( |\frac{\partial}{\partial r}(\ln K_i)|, |\frac{\partial}{\partial r}(\ln \lambda_i)| \) are uniformly bounded, we can choose \( \sigma \) to be small (independent of \( i \)) to make \( n + X(\ln K_i) + \frac{2n}{n-2} X(\ln \lambda_i) > 0 \). Thus when \( i \to \infty \), the limit of the second term of (24) is greater than or equal to 0.

Next we will show that the limit of the first term of (24) is 0, or equivalently,
\[ \lim_{i \to \infty} v_i^2(0) \int_{B_\sigma} X(K_i) v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy = 0, \]
since \( v_i(y) \geq C v_i(0)^{-1} \) by Proposition 4.1. This then will end the proof because it implies that the limit of the left hand side of (21) is greater than or equal to 0, contradicting the sign of the right hand side.
Note that
\[ X(K_i)(y) = \left( \sum_j y^j \frac{\partial K_i}{\partial y^j} \right)(y) \]
\[ = \left( \sum_j y^j \frac{\partial K_i}{\partial y^j} \right)(0) + \sum_k \frac{\partial}{\partial y^k} \left( \sum_j y^j \frac{\partial K_i}{\partial y^j} \right)(\zeta) y^k \quad \text{for some } |\zeta| \leq |y| \]
\[ = \sum_j \frac{\partial K_i}{\partial y^j}(\zeta) y^j + \sum_{j,k} \frac{\partial^2 K_i}{\partial y^j \partial y^k}(\zeta) \zeta_j y^k \]

Therefore
\[ v^2_i(0) \left| \int_{B_\sigma} X(K_i)v^{p_i+1}_i \lambda^{\frac{2n}{n-2}}_i dy \right| \]
\[ \leq v^2_i(0) \int_{B_\sigma} \left| \frac{\partial K_i}{\partial y^j}(\zeta) \right| |y| v^{p_i+1}_i \lambda^{\frac{2n}{n-2}}_i dy + v^2_i(0) \int_{B_\sigma} \sum_{j,k} \frac{\partial^2 K_i}{\partial y^j \partial y^k}(\zeta) |y|^2 v^{p_i+1}_i \lambda^{\frac{2n}{n-2}}_i dy \]
\[ \leq C v^2_i(0) r_i \int_{B_\sigma} |y| v^{p_i+1}_i dy + C v^2_i(0) v^2_i \int_{B_\sigma} |y|^2 v^{p_i+1}_i dy \]
\[ \leq C v^2_i(0) r_i v_i(0)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i} + C v^2_i(0) r_i^{-\frac{4}{n-2} + \frac{n}{4} \delta_i} \quad \text{by Lemma 4.2} \]
\[ \leq C v_i(0)^{2+2t_i - \frac{2}{n-2} + C v_i(0)^{2+4t_i - \frac{4}{n-2}} \quad \text{by Proposition 4.3 and Inequality (20)} \]

By the definition of \( t_i \),
\[ \lim_{i \to \infty} t_i = \lim_{i \to \infty} \left( 1 - \frac{(p_i - 1)l_i}{2} \right) = 1 - \frac{2}{n-2} \lim_{i \to \infty} l_i < 1 - \frac{2}{n-2} \cdot \frac{2n - 5}{2} = \frac{3 - n}{n-2} \]

Thus
\[ \lim_{i \to \infty} \left( 2 + 2t_i - \frac{2}{n-2} \right) < 2 + 2 \cdot \frac{3 - n}{n-2} - \frac{2}{n-2} = 0 \]

and
\[ \lim_{i \to \infty} \left( 2 + 4t_i - \frac{4}{n-2} \right) < 2 + 4 \cdot \frac{3 - n}{n-2} - \frac{4}{n-2} = \frac{4 - 2n}{n-2} < 0. \]

Since these are all strict inequalities, we know that
\[ \lim_{i \to \infty} \left( C v_i(0)^{2+2t_i - \frac{2}{n-2} + C v_i(0)^{2+4t_i - \frac{4}{n-2}} \right) = 0 \]
and consequently
\[ \lim_{i \to \infty} v^2_i(0) \left| \int_{B_\sigma} X(K_i)v^{p_i+1}_i \lambda^{\frac{2n}{n-2}}_i dy \right| = 0. \]
\[ \square \]
4.5. **Refined Estimates for** $\delta_i$ **and** $|\nabla K_i|$. Now because $x_0 = \lim_{i \to \infty} x_i$ is a critical point of the function $K$, which satisfies the flatness condition $(\ast)$, we have $|\nabla^p K(x_i)| \leq C_0|\nabla K(x_i)|^{\frac{n-2-p}{n-3}}$ when $2 \leq p \leq n - 3$. When $p = 2$, since $g = \lambda \frac{1}{n-2} dz^2$, this implies

\[
|\nabla^2 K \left( \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_2} \right) (x_i)| = \left| \frac{\partial^2 K}{\partial z_l \partial z_2} (x_i) - \Gamma_{l12}^i (x_i) \frac{\partial K}{\partial z_l} (x_i) \right| \leq C |\nabla K(x_i)|^{\frac{n-4}{n-3}},
\]

where $l, l_1, l_2 = 1, 2, \ldots, n$. Therefore

\[
\left| \frac{\partial^2 K}{\partial z_l \partial z_2} (x_i) \right| \leq C |\nabla K(x_i)| + C |\nabla K(x_i)|^{\frac{n-4}{n-3}} \leq C |\nabla K(x_i)|^{\frac{n-4}{n-3}},
\]

since $|\nabla K(x_i)| < 1$ for sufficiently large $i$. That is, $|\frac{\partial^p K}{\partial z_\alpha} (x_i)| \leq C |\nabla K(x_i)|^{\frac{n-2-|\alpha|}{n-4}}$ for $|\alpha| = 2$. Here we have used the notations that

\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ with each } \alpha_i \geq 0, \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n,
\]

and

\[
\frac{\partial^\alpha K}{\partial z_\alpha} = \frac{\partial^{\alpha_1} \partial^{\alpha_2} \ldots \partial^{\alpha_n} K}{(\partial z_1)^{\alpha_1} (\partial z_2)^{\alpha_2} \ldots (\partial z_n)^{\alpha_n}}.
\]

Generally, when $2 \leq p < q \leq n - 3$, we have $|\nabla K(x_i)|^{\frac{n-2-p}{n-3}} < |\nabla K(x_i)|^{\frac{n-2-q}{n-3}}$, so by similar computations we have

\[
\left| \frac{\partial^\alpha K}{\partial z_\alpha} (x_i) \right| \leq C |\nabla K(x_i)|^{\frac{n-2-|\alpha|}{n-3}} \quad \text{for} \quad 2 \leq |\alpha| \leq n - 3.
\]

Then since $K_i(y) = K(r_i y)$, $|\frac{\partial^p K_i}{\partial y^\alpha} (0)| = r_i^{|\alpha|} |\frac{\partial^p K}{\partial z_\alpha} (x_i)|$ and $|\nabla K_i(0)| = r_i |\nabla K(x_i)|$. Thus

\[
\left| \frac{\partial^\alpha K_i}{\partial y^\alpha} (0) \right| \leq r_i^{|\alpha|} C |\nabla K(x_i)|^{\frac{n-2-|\alpha|}{n-3}} = C r_i^{\frac{(|\alpha|-1)(n-2)}{n-3}} |\nabla K_i(0)|^{\frac{n-2-|\alpha|}{n-3}} < C r_i |\nabla K_i(0)|^{\frac{n-2-|\alpha|}{n-3}}
\]

(27)

where the last step comes from the fact that $\frac{(|\alpha|-1)(n-2)}{n-3} > 1$ and $r_i < 1$. With this flatness condition on $K_i$, we can refine the estimates for $\delta_i$ and $|\nabla K_i|$ as follows.

Inequality (15) gives

\[
\delta_i \leq C \left( v_i(0)^{2t_i} + \int_{B_{r_i}} \frac{\partial K_i}{\partial r} |y|^p v_i^{n+1} \lambda_i^{\frac{2n}{n-3}} dy \right) = C \left( v_i(0)^{2t_i} + \int_{B_{r_i}} r_i^p v_i^{n+1} \lambda_i^{\frac{2n}{n-3}} dy \right).
\]

We write $r \frac{\partial K_i}{\partial r} = \sum_j y_j \frac{\partial K_i}{\partial y_j}$. For each $j = 1, \ldots, n$,

\[
\frac{\partial K_i}{\partial y_j} (y) = \frac{\partial K_i}{\partial y_j} (0) + \sum_{|\beta| = 1} \frac{\partial^\beta K_i}{\partial y^\beta} (0) y^\beta + \frac{1}{2!} \sum_{|\beta| = 2} \frac{\partial^\beta K_i}{\partial y^\beta} (0) y^\beta + \cdots + \frac{1}{(n-4)!} \sum_{|\beta| = n-4} \frac{\partial^\beta K_i}{\partial y^\beta} (0) y^\beta + \frac{1}{(n-3)!} \sum_{|\beta| = n-3} \frac{\partial^\beta K_i}{\partial y^\beta} (\zeta) y^\beta
\]

and
where $|\varsigma| \leq |y|$, and $y^\beta = y_1^{\beta_1} y_2^{\beta_2} \cdots y_n^{\beta_n}$ for $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. Therefore
\[
\int_{B_r} \left| \frac{r}{\partial r} \right| v_i^{p_i+1} \chi_i^{2n-2} dy
\leq C \left( \int_{B_r} \left| \frac{\partial K_i}{\partial y^j}(0) \right| |y| v_i^{p_i+1} dy + \sum_{|\beta|=1}^{n-4} \int_{B_r} \left| \frac{\partial^{\beta} \partial K_i}{\partial y^\beta \partial y^j}(0) \right| |y|^{|\beta|+1} v_i^{p_i+1} dy \right.
\left. + \sum_{|\beta|=n-3} \int_{B_r} \left| \frac{\partial^{\beta} \partial K_i}{\partial y^\beta \partial y^j}(\varsigma) \right| |y|^{n-2} v_i^{p_i+1} dy \right).
\]
By Lemma 4.2 and Proposition 4.3, the first term
\[
\int_{B_r} \left| \frac{\partial K_i}{\partial y^j}(0) \right| |y| v_i^{p_i+1} dy \leq C |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}},
\]
and the last term
\[
\sum_{|\beta|=n-3} \int_{B_r} \left| \frac{\partial^{\beta} \partial K_i}{\partial y^\beta \partial y^j}(\varsigma) \right| |y|^{n-2} v_i^{p_i+1} dy \leq C r_i^{n-2} v_i(0)^{-2}.
\]
In addition, by (27), for any $1 \leq |\beta| \leq n-4$,
\[
\int_{B_r} \left| \frac{\partial^{\beta} \partial K_i}{\partial y^\beta \partial y^j}(0) \right| |y|^{|\beta|+1} v_i^{p_i+1} dy
\leq C r_i \int_{B_r} |\nabla K_i(0)| \left| \frac{n-3-|\beta|}{n-3} |y|^{|\beta|} \cdot |y| v_i^{p_i+1} dy \right.
\left. = C r_i \int_{B_r} |\nabla K_i(0)| \left| \frac{n-3-|\beta|}{n-3} |y|^{|\beta|} \cdot |y| v_i^{p_i+1} dy \right.
\right.
\left. \leq C r_i \left( \int_{B_r} |\nabla K_i(0)| \cdot |y| v_i^{p_i+1} dy + \int_{B_r} |y|^{n-2} v_i^{p_i+1} dy \right)
\leq C r_i |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}} + C r_i v_i(0)^{-2}.
\]
Thus
\[
\int_{B_r} \left| \frac{\partial K_i}{\partial r} \right| v_i^{p_i+1} \chi_i^{2n-2} dy
\leq C |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}} + \left( C r_i |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}} + C r_i v_i(0)^{-2} \right) + C r_i^{n-2} v_i(0)^{-2}
\]
(28) $\leq C |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}} + C r_i v_i(0)^{-2}$.

Plugging this back into (15) we now have a refined estimate
\[
\delta_i \leq C \left( v_i(0)^{2\delta_i} + |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}} + r_i v_i(0)^{-2} \right).
\]
This will enable us to also refine the estimate for $|\nabla K_i(0)|$. 

Inequality (18) gives \[ \left| \int_{B_r} \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \frac{\partial K_i}{\partial y^j} dy \right| \leq C (\delta_i r_i + v_i(0)^{2t_i}). \]

Again we write
\[
\frac{\partial K_i}{\partial y^j}(y) = \frac{\partial K_i}{\partial y^j}(0) + \sum_{|\beta| = 1} \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) y^\beta + \frac{1}{2!} \sum_{|\beta| = 2} \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) y^\beta + \ldots
\]
\[+ \frac{1}{(n - 4)!} \sum_{|\beta| = n - 4} \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) y^\beta + \frac{1}{(n - 3)!} \sum_{|\beta| = n - 3} \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) y^\beta.\]

Therefore we have
\[
\int_{B_r} \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \left| \frac{\partial K_i}{\partial y^j}(0) \right| dy
\]
\[\leq \int_{B_r} \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} dy + C \sum_{|\beta| = 1} \int_{B_r} \left| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \right| \left| y \right|^{|\beta|} v_i^{p_i+1} dy
\]
\[+ C \sum_{|\beta| = n - 3} \int_{B_r} \left| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \right| \left| y \right|^{n - 3} v_i^{p_i+1} dy.
\]

By (14) this implies
\[
\left| \frac{\partial K_i}{\partial y^j}(0) \right| \leq C (\delta_i r_i + v_i(0)^{2t_i}) + C \sum_{|\beta| = 1} \int_{B_r} \left| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \right| \left| y \right|^{|\beta|} v_i^{p_i+1} dy
\]
\[+ C \sum_{|\beta| = n - 3} \int_{B_r} \left| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \right| \left| y \right|^{n - 3} v_i^{p_i+1} dy.
\]

By Lemma 4.2 Proposition 4.3 (27), and Young’s Inequality, when \(1 \leq |\beta| \leq n - 4\),
\[
\int_{B_r} \left| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \right| \left| y \right|^{|\beta|} v_i^{p_i+1} dy
\]
\[\leq C r_i \int_{B_r} \left| \nabla K_i(0) \right|^{\frac{n - 2 - (|\beta| + 1)}{n - 3}} \left| y \right|^{n - 3} v_i^{p_i+1} dy
\]
\[= C r_i \int_{B_r} \left| \nabla K_i(0) \right|^{\frac{n - 3 - |\beta|}{n - 3}} \left| y \right|^{|\beta|} v_i^{p_i+1} dy
\]
\[\leq C r_i \int_{B_r} \left( \left| \nabla K_i(0) \right|^{\frac{n - 3 - |\beta|}{n - 3}} \left| y \right|^{n - 3} v_i^{p_i+1} dy + \left| y \right|^{n - 3} v_i^{p_i+1} dy \right)
\]
\[= C r_i \left( \int_{B_r} \left| \nabla K_i(0) \right| v_i^{p_i+1} dy + \int_{B_r} \left| y \right|^{n - 3} v_i^{p_i+1} dy \right)
\]
\[\leq C r_i \left| \nabla K_i(0) \right| + C r_i v_i(0)^{-\frac{2(n - 3)}{n - 2}}.
\]
Furthermore,
\[
\sum_{|\beta|=n-3} \int_{B_r} \left| \frac{\partial^\beta}{\partial y^\beta} K_i(s) \right| |y|^{n-3} v_i^{p_i+1} dy \leq C r_i^{n-2} \int_{B_r} |y|^{n-3} v_i^{p_i+1} dy \leq C r_i^{n-2} v_i(0) \frac{2(n-3)}{n-2}.
\]

Therefore
\[
\left| \frac{\partial K_i}{\partial y^j}(0) \right| \leq C (\delta_i r_i + v_i(0)^{2t_i}) + \left( C r_i |\nabla K_i(0)| + C r_i v_i(0) \frac{2(n-3)}{n-2} \right) + C r_i^{n-2} v_i(0) \frac{2(n-3)}{n-2}.
\]

The same estimate also holds for \( \left| \frac{\partial K_i}{\partial y^j}(0) \right| \), where \( j = 2, \ldots, n \), so we know
\[
|\nabla K_i(0)| \leq C \delta_i r_i + C v_i(0)^{2t_i} + C r_i |\nabla K_i(0)| + C r_i v_i(0) \frac{2(n-3)}{n-2}.
\]

When \( i \) is large enough, all the terms involving \( |\nabla K_i(0)| \) can be absorbed into the left hand side of this inequality, therefore we get a refined estimate
\[
|\nabla K_i(0)| \leq C r_i v_i(0)^{2t_i} + C r_i^2 v_i(0)^{-2} + C v_i(0)^{2t_i} + C r_i v_i(0) \frac{2(n-3)}{n-2}.
\]

Finally, we are going to prove that (25) holds. As in the proof of Proposition 4.4, this will give the desired contradiction by comparing the signs of both sides of (21), which rules out case I.

We know
\[
v_i^2(0) \int_{B_r} |X(K_i)| v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy = v_i^2(0) \int_{B_r} \left| r \frac{\partial K_i}{\partial r} \right| v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy \leq C v_i^2(0) \left( |\nabla K_i(0)| v_i(0)^{-\frac{2}{n-2}} + r_i v_i(0)^{-2} \right) \quad \text{(by (28))}
\]
\[
\leq C v_i^2(0) \left( \left( r_i^2 v_i(0)^{-2} + v_i(0)^{2t_i} + r_i v_i(0) \frac{2(n-3)}{n-2} \right) v_i(0)^{-\frac{2}{n-2}} + r_i v_i(0)^{-2} \right) \quad \text{(by (30))}
\]
\[
= C \left( r_i^2 v_i(0)^{-\frac{2}{n-2}} + v_i(0)^{2+2t_i-\frac{2}{n-2}} + 2r_i \right).
\]

By (26) we know \( \lim_{i \to \infty} \left( 2 + 2t_i - \frac{2}{n-2} \right) < 0 \), therefore \( \lim v_i(0)^{2+2t_i-\frac{2}{n-2}} = 0 \). It follows from this and \( \lim r_i^2 v_i(0)^{-\frac{2}{n-2}} = \lim r_i = 0 \) that
\[
\lim_{i \to \infty} v_i^2(0) \int_{B_r} |X(K_i)| v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy = 0.
\]

This completes the proof in case I.
5. Ruling Out Case II

Now we consider Case II, which has been reduced to the following: there is a sequence of functions \( \{ v_i \} \), each satisfies
\[
\Delta_{g_i} v_i + K(\sigma_i y) v_i^{p_i} = 0
\]
where \( \sigma_i \to 0 \) and \( g^{(i)}(y) = g_{\alpha\beta}(\sigma_i y) dy^\alpha dy^\beta \). The sequence \( \{ v_i \} \) has isolated blow-up point(s) \( \{ 0, \ldots \} \).

If 0 is not a simple blow-up point, then we can do another rescaling and repeat the argument in the previous section, with \( r_i \) replaced by \( r_i \sigma_i \), to get a contradiction. Therefore 0 must be a simple blow-up point for \( \{ v_i \} \). Then we can still repeat the argument in the previous section, with \( r_i \) replaced by \( \sigma_i \). The only difference is in the expression of
\[
h = \lim_{i \to \infty} v_i(y)
\]
As shown in Section 7 of [18], because here \( |y|^{p_i - 1} v_i(|y|) \) doesn’t have a second critical point at \( |y| = 1 \), we have a different expression of \( h \): near 0,
\[
h(y) = c_1 |y|^{2-n} + A + O(|y|)
\]
where \( A \) is a positive constant. This positive “mass” term \( A > 0 \) guarantees that the limit of the boundary term of the Pohozaev identity (21) is negative, i.e.,
\[
\lim_{i \to \infty} \frac{1}{v_i^2(\bar{y})} \int_{\partial B_{\epsilon}} T_i(X, \nu_i) d\Sigma_i < 0.
\]
The other parts of the proof remain the same. Therefore Case II can also be ruled out.

Thus we have finished the proof of Theorem 1.4.

6. Proof of Theorem 1.8

In this section we will prove Theorem 1.8. There are many parallels between the proofs of Theorem 1.4 and Theorem 1.8. Therefore we are going to emphasize the differences between the two proofs and omit the details of some of the steps if they can be obtained using essentially the same argument as in Theorem 1.4.

By the standard elliptic theory, a bound on \( \| u \|_{C^3(M)} \) can be easily obtained provided there is a uniform bound on \( \| u \|_{C^0(M)} \). Following from the Sobolev inequality and strong maximal principle, a uniform upper bound on \( u \) would also imply a uniform lower bound away from 0. Therefore the main issue is to establish a uniform upper bound on all positive solutions \( u \); again we prove this by contradiction.

Suppose this is not true, then there are sequences \( \{ u_i \} \) and \( \{ p_i \} \) such that
\[
\Delta_g u_i - c(n) R(g) u_i + K u_i^{p_i} = 0 \quad \text{and} \quad \max_M u_i \to \infty \quad \text{as} \quad i \to \infty.
\]
By similar arguments as in the scalar-flat case, we can show that for fixed \( \epsilon > 0 \) and \( R >> 0 \) we can find \( x_{1,i}, \ldots, x_{N(i),i} \) on \( M \) for each function \( u_i \) such that
\[
(31) \quad \text{each } x_{j,i} \ (1 \leq j \leq N(i)) \text{ is a local maximum point of } u_i;
\]
\[
(32) \quad \text{the balls } B_{\frac{R}{u_i(x_{j,i})^{\frac{1}{p_i - 1}}}}(x_{j,i}) \text{ are disjoint};
\]
for coordinates \( y = (y^1, ..., y^n) \) such that \( \frac{y}{u_i(x_{j,i})^{\frac{n-1}{2}}} \) are the conformally flat coordinates centered at \( x_{j,i} \),

\[
    u_i(x_{j,i})^{-1} u_i \left( \frac{y}{u_i(x_{j,i})^{\frac{n-1}{2}}} \right) - \left( 1 + \frac{K(x_{j,i})}{n(n-2)} |y|^2 \right)^{-\frac{n-2}{2}} \in C^2(B_{2R}(0))
\]

and

\[
    u_i(x) \leq C \left( d_g(x, \{x_{1,i}, ..., x_{N(i),i}\}) \right)^{-\frac{2}{n-2}} \quad \text{for a constant } C = C(\epsilon, R).
\]

Let \( \sigma_i = \min\{d_g(x_{\alpha,i}, x_{\beta,i}) : \alpha \neq \beta, 1 \leq \alpha, \beta \leq N(i)\} \). Without lost of generality we can assume \( \sigma_i = d_g(x_{1,i}, x_{2,i}) \). As before there are two possibilities.

**Case I:** \( \sigma_i \geq \epsilon > 0 \).
Then the points \( x_{j,i} \) have isolated limiting points \( P_1, P_2, ..., \) which are isolated blow-up points of \( \{u_i\} \).

**Case II:** \( \sigma_i \to 0 \).
Then we rescale the coordinates to make the minimal distance 1: let \( y = \sigma_i^{-1} z \) where \( z = (z^1, ..., z^n) \) are the conformally flat coordinates centered at \( x_{1,i} \). We also rescale the function by defining

\[
    v_i(y) = \sigma_i^{\frac{2}{n-2}} u_i(\sigma_i y).
\]

\( v_i \) satisfies

\[
    \Delta_{g^{(i)}} v_i - c(n) R \left( g^{(i)} \right) v_i + K_i v_i^n = 0
\]

where \( g^{(i)}(y) = g_{\alpha\beta}(\sigma_i y) dy^\alpha dy^\beta \), \( R(g^{(i)})(y) = \sigma_i^2 R(g(\sigma_i y)) \) and \( K_i(y) = K(\sigma_i y) \).

We can prove as in Section 4 of [18] that 0 is an isolated blow-up point of \( \{v_i\} \).

### 6.1. Ruling Out Case I

Now assume we are in Case I, i.e., all the blow-up points \( \{P_1, P_2, ...\} \) are isolated blow-up points.

6.1.1. **Simple Blow-up.** Next we need to study the behavior of the functions around simple blow-up points. If any of the points, say \( P_1 \), is a simple blow-up point, then let \( x_i \) be the local maximal point of \( u_i \) such that \( \lim_{i \to \infty} x_i = P_1 \). Let \( z \) be the conformally flat coordinates centered at each \( x_i \). The next proposition is analogous to Proposition 4.1.

**Proposition 6.1.** There exist a constant \( C \) independent of \( i \) and a radius \( r_1 \leq \bar{r} \) (where \( \bar{r} \) is defined as in Definition 2.4) such that

- if \( 0 \leq |z| \leq r_1 \), then
  \[
  u_i(z) \geq C u_i(x_i) \left( 1 + \frac{K(x_i)}{n(n-2)} u_i(x_i)^{n-2} |z|^2 \right)^{-\frac{n-2}{2}}
  \]

- if \( 0 \leq |z| \leq \frac{R}{u_i(x_i)^{\frac{n-1}{2}}} \), then
  \[
  u_i(z) \leq C u_i(x_i) \left( 1 + \frac{K(x_i)}{n(n-2)} u_i(x_i)^{n-1} |z|^2 \right)^{-\frac{n-2}{2}}
  \]

- if \( \frac{R}{u_i(x_i)^{\frac{n-1}{2}}} \leq |z| \leq r_1 \), then
  \[
  u_i(z) \leq C u_i(x_i) t_i |z|^{-l_i}
  \]

where \( l_i, t_i \) are so chosen that \( \left( \frac{2(n-1)(n-2)}{2n} \right) \lim_{i \to \infty} l_i < n-2 \), and \( t_i = 1 - \frac{(n-1)l_i}{2} \).
Note that here \( \lim_{i \to \infty} l_i \) is slightly different from that in Proposition 4.1. This modification is made to accommodate some adjustments (in a later part of the proof) that are related to \( R(g) > 0 \). However, the proof of this proposition is essentially the same as that of Proposition 4.1. Therefore, in the proof below we will only point out the major steps and the few differences between the two proofs. We refer the readers to the proof of Proposition 4.1 for the details.

Proof: By (33) when \( 0 \leq |z| \leq \frac{R}{u_i(x_i)} \),

\[
(1 - \epsilon)u_i(x_i) \left( 1 + \frac{K(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n+2}{2}} \\
\leq u_i(z) \\
\leq (1 + \epsilon)u_i(x_i) \left( 1 + \frac{K(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n+2}{2}}.
\]

So we only need to find the upper and lower bounds on \( u_i(z) \) when \( \frac{R}{u_i(x_i)} \leq |z| \leq r_1 \).

The lower bound:

Let \( G_i \) be the Green’s function of \( \Delta g - c(n)R(g) \) which is singular at 0 and \( G_i = 0 \) on \( \partial B_{r_1}(x_i) \). (Here the operator is different from the Laplacian operator which is used in the proof of Proposition 4.1.) By Lemma 9.2 in [9], there exist constants \( C_1 \) and \( C_2 \) independent of \( i \) such that

\[
C_1 |z|^{2-n} \leq G_i(z) \leq C_2 |z|^{2-n}.
\]

There exists a constant \( C \) independent of \( i \), such that when \( |z| = Ru_i(x_i)^{\frac{4}{n-2}} \) and \( |z| = r_1 \), \( C u_i(x_i)^{-1} G_i(z) \leq u_i(z) \).

Since

\[
\Delta g u_i - c(n)R(g)u_i = -K u_i^{p_i+1} < 0
\]

and

\[
\Delta g G_i - c(n)R(g)G_i = 0,
\]

we conclude by the maximal principle that

\[
u_i(z) \geq C u_i(x_i)^{-1} G_i(z) \quad \text{when } Ru_i(x_i)^{-\frac{4}{n-2}} \leq |z| \leq r_1.
\]

Finally because \( G_i(z) \geq C_1 |z|^{2-n} \) and

\[
u_i(x_i)^{-1} |z|^{2-n} \geq C u_i(x_i) \left( 1 + \frac{K(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n+2}{2}}
\]

for some constant \( C \), we know

\[
u_i(z) \geq C u_i(x_i) \left( 1 + \frac{K(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n+2}{2}}
\]

when \( \frac{R}{u_i(x_i)} \leq |z| \leq r_1 \).

The upper bound:

Define \( L_i \varphi := \Delta g \varphi - c(n)R(g)\varphi + K u_i^{p_i-1} \varphi \). (The linear term is not in the \( L_i \) in the proof of Proposition 4.1.) By this definition \( L_i u_i = 0 \). Let \( M_i = \max_{\partial B_{r_1}} u_i \) and \( C_i = \)
\[(1 + \epsilon) \left( K(x_i) \right)^{-\frac{n-2}{2}} \]. Note that \(C_i\) is bounded above and below by constants independent of \(i\). Consider the function
\[ M_i |z|^{-n+2+l_i} + C_i u_i(x_i)^{l_i} |z|^{-l_i}.\]

On \(\{|z| = r_1\} \cup \{|z| = Ru_i(x_i)^{-\frac{n-1}{2}}\}\),
\[ u_i(z) \leq M_i |z|^{-n+2+l_i} + C_i u_i(x_i)^{l_i} |z|^{-l_i}.\]

In the Euclidean metric, \(\Delta |z|^{-l_i} = -l_i(n-2-l_i)|z|^{-l_i-2}\) and \(\Delta |z|^{-n+2+l_i} = -l_i(n-2-l_i)|z|^{-n+l_i}\). Although here the metric \(g\) may not be Euclidean, from the local coordinates expression of \(\Delta_g\) it is easy to see that when \(r_1\) is small enough, we can find a constant \(C\) such that when \(|z| \leq r_1\),
\[ \Delta_g |z|^{-l_i} \leq -Cl_i(n-2-l_i)|z|^{-l_i-2} \]
and
\[ \Delta_g |z|^{-n+2+l_i} \leq -Cl_i(n-2-l_i)|z|^{-n+l_i}. \]

This implies
\[ \mathcal{L}_i(C_i u_i(x_i)^{l_i} |z|^{-l_i}) \]
\[ = C_i u_i(x_i)^{l_i} (\Delta_g |z|^{-l_i} - c(n)R(g)|z|^{-l_i} + Ku_i(z)^{p_i-1} |z|^{-l_i}) \]
\[ < C_i u_i(x_i)^{l_i} (\Delta_g |z|^{-l_i} + Ku_i(z)^{p_i-1} |z|^{-l_i}) \quad \text{(since } R(g) > 0) \]
\[ \leq -Cl_i(n-2-l_i)u_i(x_i)^{l_i} |z|^{-l_i-2} + C^0 u_i(x_i)^{l_i} u_i(z)^{p_i-1} |z|^{-l_i} \]
\[ < 0 \]
when \(R\) is large enough, where the last inequality uses Lemma 2.3, the simple blow-up property of \(\{u_i\}\), and the fact that \(l_i(n-2-l_i)\) is always bounded below by some positive constant independent of \(i\).

Similarly, we can prove
\[ \mathcal{L}_i(M_i |z|^{-n+2+l_i}) < 0. \]

Therefore when \(Ru_i(x_i)^{-\frac{n-1}{2}} \leq |z| \leq r_1\),
\[ \mathcal{L}_i \left( M_i |z|^{-n+2+l_i} + C_i u_i(x_i)^{l_i} |z|^{-l_i} \right) < 0, \]
and thus by the maximal principle
\[ u_i(z) \leq M_i |z|^{-n+2+l_i} + C_i u_i(x_i)^{l_i} |z|^{-l_i}. \]

By Lemma 2.3 and the simple blow-up property of \(\{u_i\}\), for \(\frac{R}{u_i(x_i)^{-\frac{n-1}{2}}} \leq \theta \leq r_1\),
\[ M_i \leq C \theta^{\frac{2}{n-1}} u_i(\theta) \]
\[ \leq C \theta^{\frac{2}{n-1}} \left( M_i \theta^{-n+2+l_i} + C_i u_i(x_i)^{l_i} \theta^{-l_i} \right) \]
\[ = C \theta^{\frac{2}{n-1}} \cdot \theta^{-n+2+l_i} M_i + C \theta^{\frac{2}{n-1}} \cdot C_i u_i(x_i)^{l_i} \theta^{-l_i} \]
for some constant \(C\) independent of \(i\).

Because
\[ \lim_{i \to \infty} \left( \frac{2}{p_i - 1} - n + 2 + l_i \right) = \frac{n-2}{2} + \lim_{i \to \infty} l_i > \frac{n-2}{2} + \frac{(2n-1)(n-2)}{2n} > 0 \]
and \( \frac{R}{u_i(x_i) \partial_i} \to 0 \), we can choose \( \theta \) small enough (fixed and independent of \( i \)) to absorb the first term on the right hand side of the above inequality into the left hand side to get \( M_i \leq Cu_i(x_i)^{t_i} \). Therefore

\[
\begin{align*}
  u_i(z) & \leq M_i|z|^{-n+2+l_i} + C_i u_i(x_i)^{t_i} |z|^{-l_i} \\
  & \leq M_i|z|^{-l_i} + C_i u_i(x_i)^{t_i} |z|^{-l_i} \\
  & \leq Cu_i(x_i)^{t_i} |z|^{-l_i}
\end{align*}
\]

The following technical lemma is parallel to Lemma 4.2. Note that because of the modification of \( l_i \) we are able to have the estimate up to \( \kappa = n - 1 \).

**Lemma 6.2.** When \( \sigma < r_1 \) and \( 0 \leq \kappa \leq n - 1 \),

\[
\int_{|z| \leq \sigma} |z|^\kappa u_i(z)^{p_i+1}dz \leq Cu_i(x_i)^{-\frac{2n}{n-2} + \frac{n-2 + \kappa}{2}}
\]

where \( C \) is independent of \( i \) and \( r_1 \) is defined as in Proposition 6.1.

**Proof:** By Proposition 6.1,

\[
\begin{align*}
  \int_{|z| \leq \sigma} |z|^\kappa u_i(z)^{p_i+1}dz & \leq Cu_i(x_i)^{p_i+1} \int_{|z| \leq \sigma} |z|^\kappa dz \\
  & \leq Cu_i(x_i)^{p_i+1 - \frac{(n+\kappa)(n-1)}{2}} \\
  & = Cu_i(x_i)^{-\frac{2n}{n-2} + \frac{n-2 + \kappa}{2}}.
\end{align*}
\]

By our choice of \( l_i \)

\[
\lim_{i \to \infty} \left( n + \kappa - l_i(p_i + 1) \right) = n + \kappa - \frac{2n}{n-2} \lim_{i \to \infty} l_i \\
< n + \kappa - \frac{2n}{n-2} \cdot \frac{(2n-1)(n-2)}{2n} \\
\leq n + (n-1) - (2n-1) \\
= 0.
\]

Therefore

\[
\begin{align*}
  \int_{\frac{R}{u_i(x_i) \partial_i} \leq |z| \leq \sigma} |z|^\kappa u_i(z)^{p_i+1}dz & \leq C \int_{\frac{R}{u_i(x_i) \partial_i} \leq |z| \leq \sigma} |z|^\kappa \left( u_i(x_i)^{t_i} |z|^{-l_i} \right)^{p_i+1}dz \\
  & \leq Cu_i(x_i)^{l_i(p_i+1) - \frac{p_i-1}{2}(n+\kappa-l_i(p_i+1))} \\
  & = Cu_i(x_i)^{p_i+1 - \frac{(n+\kappa)(p_i-1)}{2}} \quad \text{(by the definition of } t_i) \\
  & = Cu_i(x_i)^{-\frac{2n}{n-2} + \frac{n-2 + \kappa}{2} \delta_i}.
\end{align*}
\]

Thus

\[
\int_{|z| \leq \sigma} |z|^\kappa u_i(z)^{p_i+1}dz \leq Cu_i(x_i)^{-\frac{2n}{n-2} + \frac{n-2 + \kappa}{2} \delta_i}.
\]
Let $\delta_i := \frac{n+2}{n-2} - p_i$. Since the background metric $g$ is locally conformally flat, we can write it locally as $\lambda(z)^{-\frac{2}{n-2}}dz^2$. Let $\sigma < r_1$. As in the scalar-flat case, we need to use the Pohozaev identity: for a conformal Killing field $X$ on $B_\sigma(x_i)$,

$$\frac{n-2}{2n} \int_{B_\sigma} X(R_i) dv_{g_i} = \int_{\partial B_\sigma} T_i(X, \nu_i) d\Sigma_i$$

where

$$g_i = u_i^{-\frac{4}{n-2}} = (\lambda u_i)^{-\frac{4}{n-2}} dz^2,$$

$$R_i = R(g_i) = c(n)^{-1} Ku_i^{-\delta_i},$$

$$dv_{g_i} = (\lambda u_i)^{\frac{2n}{n-2}} dz,$$

$$\nu_i = (\lambda u_i)^{-\frac{2}{n-2} + \frac{2}{n-2} \sum_j \partial j}$$

is the unit outer normal vector on $\partial B_\sigma$ with respect to $g_i$,

$$d\Sigma_i = (\lambda u_i)^{\frac{2(n-1)}{n-2}} d\Sigma_\sigma$$

where $d\Sigma_\sigma$ is the surface element of the standard $S^{n-1}(\sigma)$,

$$T_i = \text{Ric}(g_i) - n^{-1} R(g_i) g_i$$

is the traceless Ricci tensor with respect to $g_i$. $T_i$ can also be expressed as

$$(n-2)(\lambda u_i)^{-\frac{2}{n-2}} \left( \text{Hess} \left( (\lambda u_i)^{-\frac{2}{n-2}} \right) - \frac{1}{n} \Delta \left( (\lambda u_i)^{-\frac{2}{n-2}} \right) dz^2 \right)$$

where Hess and $\Delta$ are taken with respect to the Euclidean metric $dz^2$.

Now we choose $X = \sum_{j=1}^n z^j \frac{\partial}{\partial z^j}$. By an argument which is almost identical to that in the proof of Proposition 4.3, we know

$$\frac{n}{2(n-1)} \left( 1 + \frac{\delta_i}{p_i + 1} \right) \int_{B_\sigma} |z| \frac{\partial K}{\partial r} u_i^{p_i + 1} \lambda^{\frac{2m}{n-2}} dz + \frac{n}{2(n-1)} \frac{\delta_i}{p_i + 1} \int_{B_\sigma} Ku_i^{p_i + 1} \lambda^{\frac{2m}{n-2}} dz$$

$$\leq C u_i(x_i)^{2r_i} + C \delta_i u_i(x_i)^{(p_i) - 1} + C \delta_i u_i(x_i)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i}.$$ 

Since

$$\int_{B_\sigma} u_i^{p_i + 1} dz > \int_{|z| \leq \frac{R}{u_i(x_i)^{p_i}} + 1} u_i^{p_i + 1} dz$$

$$\geq C u_i(x_i)^{p_i + 1 - \frac{4}{n-2}(p_i - 1)}$$

(by Proposition 6.1)

$$= C u_i(x_i)^{\frac{n-2}{2} \delta_i}$$

$$\geq C,$$

as before we can argue that

$$\delta_i \leq C \left| \int_{B_\sigma} \frac{\partial K}{\partial r} |z| u_i^{p_i + 1} \lambda^{\frac{2m}{n-2}} dz \right| + C u_i(x_i)^{2r_i} + C \delta_i u_i(x_i)^{(p_i) - 1} + C \delta_i u_i(x_i)^{-\frac{2}{n-2} + \frac{n-1}{2} \delta_i}.$$
Then because
\[
\lim_{i \to \infty} t_i = 1 - \frac{2}{n-2} \lim_{i \to \infty} l_i < 1 - \frac{2(n-1)(n-2)}{2n} = \frac{1-n}{n} < 0,
\]
the last two terms on the right hand side can be absorbed into the left hand side, so we have
\[
\delta_i \leq C \left( \int_{B_r \sigma} \frac{\partial K}{\partial r} |z|u_i^{p_i+1} \lambda(z)^{-\frac{2n}{n-2}} dz + u_i(x_i)^{2t_i} \right).
\]
By Lemma 6.2 this implies
\[
\lim_{i \to \infty} u_i(x_i)^{\delta_i} = 1,
\]
which is parallel to Proposition 4.3, and we also have a preliminary estimate for \(\delta_i\):
\[
\delta_i \leq C \left( u_i(x_i)^{2t_i} + u_i(x_i)^{-\frac{2}{n-2}} \right).
\]

Now suppose the blow-up points \(\{P_1, P_2, \ldots\}\) are all simple blow-up points. Choose a point \(P \in \partial B_{r_1}(P_1)\), by Proposition 6.1 we know \(u_i(P) \to 0\) as \(i \to \infty\). Let \(\Omega\) be any compact subset of \(M \setminus \{P_1, P_2, \ldots\}\) containing \(P\). By Definition 2.2, \(u_i\) is bounded above on \(\Omega\) by some constant \(C\) independent of \(i\) (although it may depend on \(\Omega\)), thus on \(\Omega\) we have the standard Harnack inequality. Therefore
\[
\max_{\Omega} \frac{u_i}{u_i(P)} \leq C \min_{\Omega} \frac{u_i}{u_i(P)} \leq C \frac{u_i(P)}{u_i(P)} = C.
\]
Since \(u_i\) satisfies (1),
\[
\Delta_g \left( \frac{u_i}{u_i(P)} \right) - c(n)R(g) \frac{u_i}{u_i(P)} + u_i(P)^{p_i-1} K \left( \frac{u_i}{u_i(P)} \right)^{p_i} = 0.
\]
Then by the standard elliptic theory, \(\frac{u_i}{u_i(P)}\) converges in \(C^2\)-norm on \(\Omega\) to some function \(G \geq 0\) which satisfies \(\Delta_g G - c(n)R(g)G = 0\) on \(\Omega\). Because \(\Omega\) is arbitrary, \(G\) satisfies \(\Delta_g G - c(n)R(g)G = 0\) on \(M \setminus \{P_1, P_2, \ldots\}\). Since \(R(g) > 0\), \(G\) must be singular at one or more of the points \(\{P_1, P_2, \ldots\}\). Suppose it is singular at \(P_1, \ldots, P_k\), it follows that \(G\) is a linear combination of the positive fundamental solutions \(G_\gamma\) with poles at \(P_\gamma\) for \(\gamma = 1, \ldots, k\), i.e., there exist positive constants \(a_1, \ldots, a_k\) such that \(G = \sum_{\gamma=1}^k a_\gamma G_\gamma\).

This is precisely the key difference between the scalar-flat and the scalar-positive cases. Recall that when \(R(g) \equiv 0\), we used a removable singularity theorem for harmonic functions to prove that the isolated blow-up points cannot all be simple (Section 6 of [12]). Here because \(R(g) > 0\), we will need to do more work to show that.

Next we apply the Pohozaev identity (35) to \(X = \frac{\partial}{\partial z}\). As in the scalar-flat case, direct computation shows that the boundary term is equal to
\[(n - 2) \int_{\partial B} \frac{\sigma}{\sigma} \left( -\frac{2}{n - 2} (\lambda u_i) \frac{\partial^2 (\lambda u_i)}{\partial z^1 \partial z^j} + \frac{2}{(n - 2)^2} \frac{\partial (\lambda u_i)}{\partial z^1} \frac{\partial (\lambda u_i)}{\partial z^j} \right) \]

\[-\frac{2^1}{\sigma} \sum_j \left( -\frac{2}{n(n - 2)} (\lambda u_i) \frac{\partial^2 (\lambda u_i)}{(\partial z_j)^2} + \frac{2}{(n - 2)^2} \left( \frac{\partial (\lambda u_i)}{\partial z_j} \right)^2 \right) \, d\Sigma,\]

and it decays in the rate of \(u_i(x_i)^{2t_i}\).

The interior term

\[\frac{n - 2}{2n} \int_{B} \frac{\partial}{\partial z^1} (R_i) \, dv_{\sigma, i} = \frac{n - 2}{2n} c(n)^{-1} \int_{B} \left( 1 + \frac{\delta}{p_i + 1} \right) \lambda^\frac{2n}{n-2} u_i^{p_i+1} \frac{\partial K}{\partial z^1} \, dz \]

\[+ \frac{n - 2}{2n} c(n)^{-1} \int_{B} \frac{\delta}{p_i + 1} K u_i^{p_i+1} \frac{\partial \lambda^\frac{2n}{n-2}}{\partial z^1} \, dz \]

\[- \frac{n - 2}{2n} c(n)^{-1} \frac{\delta}{p_i + 1} \int_{\partial B} \lambda^\frac{2n}{n-2} K u_i^{p_i+1} \frac{\partial \lambda^\frac{2n}{n-2}}{\sigma} \, d\Sigma.\]

By Proposition 6.1, Lemma 6.2 and (39), the second term is bounded by

\[C \delta_i u_i(x_i)^{\frac{n-2}{2} \delta_i} \leq C \delta_i,\]

and the last term is bounded by

\[C \delta_i u_i(x_i)^{t_i(p_i+1)} \leq C \delta_i u_i(x_i)^{2t_i}.\]

Thus we have a bound on the first term:

\[\frac{n - 2}{2n} c(n)^{-1} \int_{B} \left( 1 + \frac{\delta}{p_i + 1} \right) \lambda^\frac{2n}{n-2} u_i^{p_i+1} \frac{\partial K}{\partial z^1} \, dz \leq C \left( u_i(x_i)^{2t_i} + \delta_i u_i(x_i)^{2t_i} + \delta_i \right) \]

\[\leq C \left( u_i(x_i)^{2t_i} + \delta_i \right).\]

This shows that

\[\int_{B} \lambda^\frac{2n}{n-2} u_i^{p_i+1} \frac{\partial K}{\partial z^1} \, dz \leq C \left( u_i(x_i)^{2t_i} + \delta_i \right).\]

By the Taylor expansion,

\[\frac{\partial K}{\partial z^1}(z) = \frac{\partial K}{\partial z^1}(0) + \nabla \left( \frac{\partial K}{\partial z^1} \right)(\varsigma) \cdot z \quad \text{for some } |\varsigma| \leq |z|.\]

By Lemma 6.2 and (39),

\[\int_{B} \lambda^\frac{2n}{n-2} u_i^{p_i+1} \nabla \left( \frac{\partial K}{\partial z^1} \right)(\varsigma) \cdot z \, dz \leq C \int_{B} u_i^{p_i+1} |z| \, dz \]

\[\leq C u_i(x_i)^\frac{n-2}{2}.\]
Together with (36) and (41), this shows that
\[
\frac{\partial K}{\partial z} (x_i) = \frac{\partial K}{\partial z} (0) \leq C \left( u_i (x_i) \frac{2}{\lambda G} + u_i (x_i)^2 \right) + \delta_i
\]
\[
\leq C \left( u_i (x_i) \frac{2}{\lambda G} + u_i (x_i)^2 \right)
\]
by (40).

The same estimate holds for \( \frac{\partial K}{\partial z_i} \), \( j = 2, \ldots, n \) as well, so we know \( |\nabla K (P_i)| = \lim_{i \to \infty} |\nabla K (x_i)| = 0 \). That is, the blow-up point \( P_1 \) is a critical point of \( K \).

In the next step we once again study the Pohozaev identity with \( X = \sum_j z^j \frac{\partial}{\partial z^j} \). We divide both sides of it by \( u_i^2 (P) \), so it becomes

\[
\frac{n - 2}{2n} \frac{1}{u_i^2 (P)} \int_{B_{\sigma} (x_i)} X (R_i) dv_{g_i} = \frac{1}{u_i^2 (P)} \int_{\partial B_{\sigma} (x_i)} T_i (X, \nu_i) d\Sigma_i.
\]

The right hand side (boundary term) is

\[
\frac{1}{u_i^2 (P)} \int_{\partial B_{\sigma} (x_i)} T_i (X, \nu_i) d\Sigma_i
\]
\[
= \frac{1}{u_i^2 (P)} \int_{\partial B_{\sigma} (x_i)} \left[ \text{Ric} \left( (\lambda u_i)^{\frac{4}{n-2}} dz \otimes dz \right) 
- n^{-1} R \left( (\lambda u_i)^{\frac{4}{n-2}} dz \otimes dz \right) (\lambda u_i)^{\frac{4}{n-2}} dz \otimes dz \right] (X, \nu_0) (\lambda u_i)^2 d\Sigma_{\sigma}
\]
\[
= \int_{\partial B_{\sigma} (x_i)} \left( \frac{\lambda u_i}{u_i (P)} \right)^2 \left[ \text{Ric} \left( \left( \frac{\lambda u_i}{u_i (P)} \right)^{\frac{4}{n-2}} dz \otimes dz \right) 
- n^{-1} R \left( \left( \frac{\lambda u_i}{u_i (P)} \right)^{\frac{4}{n-2}} dz \otimes dz \right) \left( \frac{\lambda u_i}{u_i (P)} \right)^{\frac{4}{n-2}} dz \otimes dz \right] (X, \nu_0) d\Sigma_{\sigma}
\]

where \( \nu_0 = \sigma^{-1} \sum_j z^j \frac{\partial}{\partial z^j} \) is the unit outer normal on \( \partial B_{\sigma} (x_i) \) with respect to the Euclidean metric \( dz \otimes dz \).

Recall that on \( B_{\sigma} (P_1) \setminus \{ P_1 \} \), \( \frac{u_i}{u_i (P)} \to G \) as \( i \to \infty \), so the boundary term converges to

\[
\int_{\partial B_{\sigma} (P)} (\lambda G)^2 \left( \text{Ric} \left( (\lambda G)^{\frac{1}{n-2}} dz^2 \right) - n^{-1} R \left( (\lambda G)^{\frac{1}{n-2}} dz^2 \right) (\lambda G)^{\frac{1}{n-2}} dz^2 \right) (X, \nu_0) d\Sigma_{\sigma},
\]

which can be expressed as

\[
(n - 2) \sigma^{-1} \int_{\partial B_{\sigma} (P)} (\lambda G)^{\frac{2(n-1)}{n-2}} \left[ \text{Hess} \left( (\lambda G)^{-\frac{2}{n-2}} \right) (X, X) - \frac{1}{n} \Delta \left( (\lambda G)^{-\frac{2}{n-2}} \right) \sigma^2 \right] d\Sigma_{\sigma}.
\]

Since \( \Delta G - c(n) R (g) G = 0 \) on \( B_{\sigma} (P_1) \setminus \{ P_1 \} \), we know \( G^{\frac{1}{n-2}} g = (\lambda G)^{\frac{1}{n-2}} dz^2 \) has zero scalar curvature. This implies that \( \lambda G \) is a positive Euclidean harmonic function on
Next we calculate (43).

\[ (\lambda G)(z) = a_1|z|^{2-n} + A + h(z) \]

where \( h(z) \) is a harmonic function with \( h(0) = 0 \). Furthermore, the fundamental solution \( G_1 \) satisfies

\[ (\lambda G_1)(z) = |z|^{2-n} + E(P_1) + O(|z|). \]

Here \( E(P_1) \) is the energy at \( P_1 \), and by the Positive Mass Theorem [13] \( E(P_1) > 0 \) since \((M, g)\) is not conformally equivalent to \( S^n \). Then because \( G \geq a_1 G_1 \), we know that \( A \geq a_1 E(P_1) > 0 \).

Next we calculate (43).

\[
(\lambda G)^{-\frac{n-2}{2}} = (a_1|z|^{2-n} + A + O(|z|))^{-\frac{n-2}{2}}
\]

\[
= a_1^{-\frac{n-2}{2}}|z|^2 - \frac{2}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^n + O(|z|^{2n-2}).
\]

Since

\[
\text{Hess} \left( a_1^{-\frac{n-2}{2}}|z|^2 - \frac{2}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^n \right) (X, X) = 2a_1^{-\frac{n-2}{2}}|z|^2 - \frac{2(n^2 - n)}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^n,
\]

we have

\[
\text{Hess} \left( (\lambda G)^{-\frac{n-2}{2}} \right) (X, X) = 2a_1^{-\frac{n-2}{2}}|z|^2 - \frac{2(n^2 - n)}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^n + O(|z|^{2n-2});
\]

and because

\[
\Delta \left( a_1^{-\frac{n-2}{2}}|z|^2 - \frac{2}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^n \right) = 2na_1^{-\frac{n-2}{2}} - \frac{2(2n^2 - 2n)}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^{n-2},
\]

we have

\[
\frac{1}{n}\Delta \left( (\lambda G)^{-\frac{n-2}{2}} \right) \sigma^2 = 2a_1^{-\frac{n-2}{2}} \sigma^2 - \frac{2(2n^2 - 2n)}{n-2}Aa_1^{-\frac{n-2}{2}}|z|^{n-2} \sigma^2 + O(|z|^{2n-4}) \sigma^2.
\]

Therefore on \( \partial B_\sigma(P_1) \),

\[
\text{Hess} \left( (\lambda G)^{-\frac{n-2}{2}} \right) (X, X) - \frac{1}{n}\Delta \left( (\lambda G)^{-\frac{n-2}{2}} \right) \sigma^2 = -2(n - 1)Aa_1^{-\frac{n-2}{2}} \sigma^n + O(\sigma^{2n-2}).
\]

We also know

\[
(\lambda G)^{\frac{2(n-1)}{n-2}} = (a_1|z|^{2-n} + A + O(|z|))^{\frac{2(n-1)}{n-2}}
\]

\[
= a_1^{\frac{2(n-1)}{n-2}}|z|^{-2(n-1)} \left( 1 + \frac{2(n-1)}{n-2} \frac{A}{a_1} |z|^{n-2} + O(|z|^{2n-2}) \right).
\]
Thus (43) is equal to
\[
(n - 2)\sigma^{-1} \cdot a_1^{2(n-1)} \sigma^{-2(n-1)} \left( 1 + \frac{2(n-1)}{n-2} A \sigma^{n-2} \right) \\
+ O(\sigma^{2n-2}) \cdot \left( -2(n-1) A a_1^{-\frac{n}{n-2}} \sigma^n + O(\sigma^{2n-2}) \right) \sigma^{n-1}
\]
\[
= -2(n-1)(n-2) A a_1 + O(\sigma^{n-2})
\]
\[
< 0
\]
when \( \sigma \) is sufficiently small, since \( A > 0 \) and \( a_1 > 0 \).

On the other hand, the left hand side (interior term) of (42) is
\[
\frac{n - 2}{2n} c(n)^{-1} \frac{1}{u_i^2(P)} \int_{B_{\sigma}(x_i)} X(Ku_i^{-\delta_i})(\lambda u_i)^{2n} d\sigma.
\]
Using the divergence theorem we can write
\[
\frac{1}{u_i^2(P)} \int_{B_{\sigma}(x_i)} X(Ku_i^{-\delta_i})(\lambda u_i)^{2n} d\sigma \\
= \frac{1}{u_i^2(P)} \int_{B_{\sigma}(x_i)} X(K)u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\sigma - \frac{\delta_i}{p_i + 1} \int_{\partial B_{\sigma}(x_i)} K u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\Sigma\sigma \\
+ \frac{\delta_i}{p_i + 1} \int_{B_{\sigma}(x_i)} K u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} \left( n + X(\ln K) + \frac{2n}{n-2} X(\ln \lambda) \right) d\sigma.
\]
The second term
\[
- \frac{\delta_i \sigma}{p_i + 1} \int_{\partial B_{\sigma}(x_i)} K u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\Sigma\sigma = - \frac{\delta_i \sigma}{p_i + 1} \int_{\partial B_{\sigma}(x_i)} K \left( \frac{u_i}{u_i(P)} \right)^2 u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\Sigma\sigma
\]
\[
\rightarrow 0,
\]
since \( \frac{u}{u_i(P)} \rightarrow G \) and \( u_i \rightarrow 0 \) uniformly on \( B_{2\sigma}(P_1) \setminus B_\frac{1}{2}(P_1) \).

Since \( X = r \frac{\partial}{\partial r} \ln K \), \( \frac{\partial}{\partial r} \ln \lambda \) are uniformly bounded, we can choose \( \sigma \) small (independent of \( i \)) to make \( n + X(\ln K) + \frac{2n}{n-2} X(\ln \lambda) > 0 \). Thus the limit of the last term is greater than or equal to 0.

We claim that the first term \( \frac{1}{u_i^2(P)} \int_{B_{\sigma}(x_i)} X(K)u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\sigma \rightarrow 0 \). It follows from Proposition 6.4.1 that \( u_i(P) \geq C u_i(x_i)^{-1} \), thus to prove this limit it suffices to show that
\[
\lim_{i \rightarrow \infty} \int_{B_{\sigma}(x_i)} X(K)u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\sigma = 0.
\]

We write \( X(K) = r \frac{\partial K}{\partial r} = \sum_{j=1}^n z_j \frac{\partial K}{\partial z_j} \). Since the coordinates are centered at \( x_i \), for each \( j = 1, \ldots, n \),
\[
\frac{\partial K}{\partial z_j} = \frac{\partial K}{\partial z_j}(x_i) + \sum_{|\beta|=1} \frac{\partial K}{\partial z_j}(x_i) z_\beta \left( \frac{1}{2} \sum_{|\beta|=2} \frac{\partial K}{\partial z_j}(x_i) z_\beta \right) + \cdots \\
+ \frac{1}{(n-3)!} \sum_{|\beta|=n-3} \frac{\partial K}{\partial z_j}(x_i) z_\beta + \frac{1}{(n-2)!} \sum_{|\beta|=n-2} \frac{\partial K}{\partial z_j}(x_i) z_\beta
\]
where $|\varsigma| \leq |z|$. Therefore

$$
\int_{B_r} \left| r \frac{\partial K}{\partial r} \right| u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz
\leq C \left( \int_{B_r} \left| \frac{\partial K}{\partial z^\beta}(x_i) \right| |z| u_i^{p_i+1} dz + \sum_{|\beta|=1}^{n-3} \int_{B_r} \left| \frac{\partial^3 \partial K}{\partial z^\beta \partial z^\gamma}(x_i) \right| |z|^{|\beta|+1} u_i^{p_i+1} dz \right.
\left. + \sum_{|\beta|=n-2} \int_{B_r} \left| \frac{\partial^3 \partial K}{\partial z^\beta \partial z^\gamma}(\varsigma) \right| |z|^{n-1} u_i^{p_i+1} dz \right).
$$

By Lemma 6.2 and (39),

$$
\int_{B_r} \left| \frac{\partial K}{\partial z^\beta}(x_i) \right| |z| u_i^{p_i+1} dz \leq C|\nabla K(x_i)| u_i(x_i)^{-\frac{2}{n-2}}
$$

and

$$
\sum_{|\beta|=n-2} \int_{B_r} \left| \frac{\partial^3 \partial K}{\partial z^\beta \partial z^\gamma}(\varsigma) \right| |z|^{n-1} u_i^{p_i+1} dz \leq Cu_i(x_i)^{-\frac{2(n-1)}{n-2}}.
$$

In addition, because $K$ satisfies the flatness condition (**), as in the scalar-flat case we can show that

$$
\left| \frac{\partial^n K}{\partial z^\alpha}(x_i) \right| \leq C|\nabla K(x_i)|^{\frac{n-1-|\alpha|}{n-2}}
$$

when $2 \leq |\alpha| \leq n-2$. Thus for any $1 \leq |\beta| \leq n-3$,

$$
\begin{align*}
&\int_{B_r} \left| \frac{\partial^3 \partial K}{\partial z^\beta \partial z^\gamma}(x_i) \right| |z|^{|\beta|+1} u_i^{p_i+1} dz \\
&\leq C \int_{B_r} |\nabla K(x_i)|^{\frac{n-1-|\beta|+1}{n-2}} |z|^{|\beta|+1} u_i^{p_i+1} dz \\
&= C \int_{B_r} |\nabla K(x_i)|^{\frac{n-2-|\beta|}{n-2}} |z|^{|\beta|} \cdot |z| u_i^{p_i+1} dz \\
&\leq C \int_{B_r} \left( |\nabla K(x_i)|^{\frac{n-2-|\beta|}{n-2}} |z|^{\frac{n-2}{n-2-|\beta|}} + |z|^{\frac{n-2}{n-2-|\beta|}} \right) \cdot |z| u_i^{p_i+1} dz \\
&\quad \text{(by Young’s Inequality)} \\
&= C \left( \int_{B_r} |\nabla K(x_i)| \cdot |z| u_i^{p_i+1} dz + \int_{B_r} |z|^{n-1} u_i^{p_i+1} dz \right) \\
&\leq C|\nabla K(x_i)| u_i(x_i)^{-\frac{2}{n-2}} + Cu_i(x_i)^{-\frac{2(n-1)}{n-2}}.
\end{align*}
$$

Thus

$$
\int_{B_r} \left| r \frac{\partial K}{\partial r} \right| u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz \leq C|\nabla K(x_i)| u_i(x_i)^{-\frac{2}{n-2}} + Cu_i(x_i)^{-\frac{2(n-1)}{n-2}}.
$$

(45)

Plugging this back into (44) we now have a refined estimate for $\delta_i$:

$$
\delta_i \leq C \left( u_i(x_i)^{2\delta_i} + |\nabla K(x_i)| u_i(x_i)^{-\frac{2}{n-2}} + u_i(x_i)^{-\frac{2(n-1)}{n-2}} \right).
$$

(46)
To prove (44) we still need to refine the estimate for $|\nabla K(x_i)|$. In (41) we have

$$\int_{B_\epsilon} \lambda_\epsilon^{2n-2} u_i^{p_i+1} \frac{\partial K}{\partial z^1} dz \leq C (u_i(x_i)^{2t_i} + \delta_i).$$

Again we write out the Taylor expansion

$$\frac{\partial K}{\partial z^1}(z) = \frac{\partial K}{\partial z^1}(x_i) + \sum_{|\beta|=1} \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(x_i) z^\beta + \frac{1}{2!} \sum_{|\beta|=2} \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(x_i) z^\beta + \cdots$$

$$+ \frac{1}{(n-3)!} \sum_{|\beta|=n-3} \frac{\partial^3 K}{\partial z^\beta \partial z^1}(x_i) z^\beta + \frac{1}{(n-2)!} \sum_{|\beta|=n-2} \frac{\partial^3 K}{\partial z^\beta \partial z^1}(\varsigma) z^\beta.$$

Therefore we have

$$\int_{B_\epsilon} \lambda_\epsilon^{2n-2} u_i^{p_i+1} \left| \frac{\partial K}{\partial z^1}(x_i) \right| dz$$

$$\leq \int_{B_\epsilon} \lambda_\epsilon^{2n-2} u_i^{p_i+1} \left| \frac{\partial K}{\partial z^1}(x_i) \right| dz + C \sum_{|\beta|=1}^{n-3} \int_{B_\epsilon} \left| \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(x_i) \right| |z|^{|\beta|} u_i^{p_i+1} dz$$

$$+ C \sum_{|\beta|=n-2}^{n-3} \int_{B_\epsilon} \left| \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(\varsigma) \right| |z|^{n-2} u_i^{p_i+1} dz$$

$$\leq C (u_i(x_i)^{2t_i} + \delta_i) + C \sum_{|\beta|=1}^{n-3} \int_{B_\epsilon} \left| \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(x_i) \right| |z|^{|\beta|} u_i^{p_i+1} dz$$

$$+ C \sum_{|\beta|=n-2}^{n-3} \int_{B_\epsilon} \left| \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(\varsigma) \right| |z|^{n-2} u_i^{p_i+1} dz.$$

By (36) this implies

$$\left| \frac{\partial K}{\partial z^1}(x_i) \right| \leq C (u_i(x_i)^{2t_i} + \delta_i) + C \sum_{|\beta|=1}^{n-3} \int_{B_\epsilon} \left| \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(x_i) \right| |z|^{|\beta|} u_i^{p_i+1} dz$$

$$+ C \sum_{|\beta|=n-2}^{n-3} \int_{B_\epsilon} \left| \frac{\partial^\beta K}{\partial z^\beta \partial z^1}(\varsigma) \right| |z|^{n-2} u_i^{p_i+1} dz.$$
By Lemma 6.2 and Young’s Inequality, when \(1 \leq |\beta| \leq n - 3\),
\[
\int_{B_\sigma} \left| \frac{\partial^\beta}{\partial z^\beta} \frac{\partial K}{\partial z^1}(x_i) \right| |z| |\beta| u_i^{n+1} dz \\
\leq C \int_{B_\sigma} |\nabla K(x_i)| \left( \frac{n-1-|\beta|+1}{n-3} \right) |z| \|\nabla u_i^{n+1} dz \\
= C \int_{B_\sigma} |\nabla K(x_i)| \left( \frac{n-2-|\beta|}{n-3} \right) |\nabla u_i^{n+1} dz \\
\leq C \int_{B_\sigma} \left( |\nabla K(x_i)| \left( \frac{n-2-|\beta|}{n-3} \right) + |z| \left( \frac{1}{n-2} \right) \right) |z| \|\nabla u_i^{n+1} dz \\
= C \int_{B_\sigma} |\nabla K(x_i)| |z| \|\nabla u_i^{n+1} dz + \int_{B_\sigma} |\nabla u_i^{n+1} dz \\
\leq C |\nabla K(x_i)| u_i^{n+1} + C \|\nabla u_i^{n+1} dz \\
\leq C |\nabla K(x_i)| u_i^{n+1} + C |\nabla u_i^{n+1} dz \\
\leq C |\nabla u_i^{n+1} dz \\
\leq C |\nabla u_i^{n+1} dz \\
\leq C u_i^{n+1} - 2.
\]

Therefore
\[
\left| \frac{\partial K}{\partial z^1}(x_i) \right| \\
\leq C (u_i^{2n+1} + \delta) + C |\nabla K(x_i)| u_i^{n+1} + C \|\nabla u_i^{n+1} dz + C u_i^{n+1} - 2
\]

The same estimate also holds for \( \left| \frac{\partial K}{\partial z^j}(x_i) \right| \), where \(j = 2, ..., n\), so we know
\[
|\nabla K(x_i)| \
\leq C \delta + C u_i^{2n+1} + C |\nabla K(x_i)| u_i^{n+1} + C |\nabla u_i^{n+1} dz + C u_i^{n+1} - 2 + \frac{2}{n-2} \frac{n-3}{n-2}
\]
\[
+ C u_i^{2n+1} + C |\nabla K(x_i)| u_i^{n+1} + C |\nabla u_i^{n+1} dz + C u_i^{n+1} - 2 + \frac{2}{n-2} \frac{n-3}{n-2} \quad (\text{by (36)}).
\]

When \(i\) is large enough, all the terms involving \(|\nabla K(x_i)|\) can be absorbed into the left hand side of this inequality, therefore we get a refined estimate
\[
|\nabla K(x_i)| \leq C u_i^{2n+1} + C u_i^{2n+1} + C u_i^{n+1} - 2 + \frac{2}{n-2} \frac{n-3}{n-2}
\]

Finally, we are going to prove (44).
By this definition

\[ u_i^2(x_i) \int_{B_\sigma} |X(K)| u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz \]

\[ = u_i^2(x_i) \int_{B_\sigma} \left| \frac{\partial K}{\partial r} \right| u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz \]

\[ \leq Cu_i^2(x_i) \left( |\nabla K(x_i)| u_i(x_i) - \frac{2}{n-2} + u_i(x_i)^{\frac{2(n-1)}{n-2}} \right) \quad (\text{by (45)}) \]

\[ \leq Cu_i^2(x_i) \left( \left( u_i(x_i)^{2+2t_i} + u_i(x_i)^{\frac{2(n-1)}{n-2}} + u_i(x_i)^{2+\frac{2}{n-2}} + u_i(x_i)^{-\frac{2}{n-2}} \right) u_i(x_i)^{-\frac{2}{n-2}} + u_i(x_i)^{-\frac{2(n-1)}{n-2}} \right) \]

( by (47) )

\[ = C \left( u_i(x_i)^{2+2t_i} + u_i(x_i)^{-\frac{2}{n-2}} + u_i(x_i)^{\frac{2(n-1)}{n-2}} + u_i(x_i)^{-\frac{2}{n-2}} \right). \]

By (37) we know

\[ \lim_{i \to \infty} \left( 2 + 2t_i - \frac{2}{n-2} \right) = 2 + \frac{2(1-n)}{n} = \frac{2}{n} \]

\[ \lim_{i \to \infty} u_i(x_i)^{2+2t_i} - \frac{2}{n-2} = 0. \]

Therefore

\[ \lim_{i \to \infty} u_i(x_i)^{2+2t_i} - \frac{2}{n-2} = 0. \]

Then since

\[ \lim_{i \to \infty} u_i(x_i)^{-\frac{4}{n-2}} = \lim_{i \to \infty} u_i(x_i)^{-\frac{2}{n-2}} = \lim_{i \to \infty} u_i(x_i)^{-\frac{2}{n-2}} = 0, \]

we have

\[ \lim_{i \to \infty} u_i^2(x_i) \int_{B_\sigma} |X(K)| u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz = 0. \]

This proves (44). It follows that the limit of the interior term of (12) as \( i \) goes to infinity is greater than or equal to 0. But this is a contradiction because we have shown that the limit of the boundary term is strictly negative. Therefore, at least one of the isolated blow-up points must be non-simple.

6.1.2. Isolated but Non-simple Blow-up. Without loss of generality we assume \( P_1 \) is not a simple blow-up point. Then as a function of \( |z|, \ |z|^{\frac{2}{p_i-1}} \bar{u}_i(|z|) \) has a second critical point at \( |z| = r_i \) where \( r_i \to 0 \). Let \( y = \frac{z}{r_i} \) and define \( v_i(y) = r_i^{p_i-1} u_i(r_i y) \). Then \( v_i(y) \) satisfies

\[ \Delta g(y) v_i - c(n) R(g) v_i + K_i v_i^{p_i} = 0 \]

where \( g(y) = g_{\alpha\beta}(r_i y) dy^\alpha dy^\beta, R(g) = r_i^2 R(g)(r_i y) \) and \( K_i(y) = K(r_i y) \).

By this definition \( |y| = 1 \) is the second critical point of \( |y|^{\frac{2}{p_i-1}} \bar{u}_i(|y|) \). Just as in the scalar-flat case, it can be shown that 0 is a simple blow-up point for \( \{ v_i \} \).

By some calculations which are very similar to the proof of Proposition 6.1, we can prove the following estimates: there exist a constant \( C \) independent of \( i \) and a radius \( \tilde{r} \leq 1 \) such that

- if \( 0 \leq |y| \leq \tilde{r} \), then

\[ v_i(y) \geq Cv_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^\frac{4}{n-4} |y|^2 \right)^{-\frac{n-2}{2}}. \]
\begin{itemize}
  \item if $0 \leq |y| = \frac{R}{v_i(0)}$, then
    \[ v_i(y) \leq C v_i(0) \left( 1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2 \right)^{-\frac{n-2}{2}} \]
  \item if $\frac{R}{v_i(0)} \leq |y| \leq \tilde{r}$, then
    \[ v_i(y) \leq C v_i(0)^{t_i} |y|^{-l_i} \]
\end{itemize}

where $l_i$, $t_i$ are so chosen that $\frac{2(n-1)(n-2)}{2n} < \lim_{i \to \infty} l_i < n-2$, and $t_i = 1 - \frac{(p_i-1)\kappa}{2}$.

It follows that when $\sigma < \tilde{r}$ and $0 \leq \kappa \leq n-1$, there exists a constant $C$ such that
\begin{equation}
\int_{|y| \leq \sigma} |y|^\kappa v_i(0)^{p_i+1} dy \leq C v_i(0)^{-\frac{2\kappa}{n-2} + \frac{n-2+\kappa}{2}}.
\end{equation}

This can be proved by the same calculation as in the proof of Lemma 6.2. Next by an argument that is almost identical to the proof of Proposition 4.3, we can show that
\[ \delta_i \leq C \left( v_i(0)^{2t_i} + \int_{B_r} \frac{\partial K_i}{\partial r} |y| v_i(v_i^p + v_i(0)^{2t_i}) \right). \]

This gives a preliminary estimate
\[ \delta_i \leq C \left( v_i(0)^{-\frac{2}{n-2} + v_i(0)^{2t_i}} \right), \]
and additionally $\lim_{i \to \infty} v_i(0)^{\delta_i} = 1$. Then by the same calculations as those in Section 4.3, we know that for $j = 1, 2, \ldots, n$,
\[ \left| \int_{B_r} \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \frac{\partial K_i}{\partial y^j} dy \right| \leq C \left( \delta_i r_i + v_i(0)^{2t_i} \right), \]
and we have a preliminary estimate
\[ \left| \frac{\partial K_i}{\partial y^j}(0) \right| \leq C \left( r_i v_i(0)^{-\frac{2}{n-2} + v_i(0)^{2t_i}} \right). \]

Choose a point $\bar{y}$ with $|\bar{y}| = \tilde{r}$. We have
\[ \Delta g^{(i)} \frac{v_i}{v_i(\bar{y})} - c(n) R(g^{(i)}) \frac{v_i}{v_i(\bar{y})} + v_i(\bar{y})^{p_i-1} K_i \left( \frac{v_i}{v_i(\bar{y})} \right)^{p_i} = 0. \]

On any compact subset $\Omega$ of $\mathbb{R}^n \setminus \{0\}$ which contains $\bar{y}$, since we have a Harnack inequality for $v_i$, $\frac{v_i}{v_i(\bar{y})}$ is uniformly bounded. Thus because $v_i(\bar{y}) \to 0$ and $g^{(i)}$ converges to the Euclidean metric, $\frac{v_i}{v_i(\bar{y})}$ converges on $\Omega$ in $C^2$-norm to a function $h$ with $\Delta h = 0$, where $\Delta$ is the Euclidean Laplacian. Since $\Omega$ is arbitrary, $\Delta h = 0$ on $\mathbb{R}^n \setminus \{0\}$. Then because 0 is a simple blow-up point of $\{v_i\}$ and $|y|^{\frac{2n}{n-2} \frac{v_i(\bar{y})}{v_i(\bar{y})}}$ has a second critical point at $|y| = 1$, we know $h(y) = \frac{1}{2} + \frac{1}{2} |y|^{2-n}$.

Now as in Section 4.4 we can prove that $\nabla K(P_i) = \lim_{i \to \infty} \nabla K(x_i) = 0$, i.e., $P_i$ is a critical point of $K$. Recall that the proof is by contradiction: suppose $\nabla K(P_i) \neq 0$, we study the Pohozaev identity (divided by $v_i^2(\bar{y})$) with $X = r \frac{\partial}{\partial r}$ and compare the signs of the limits of both sides. The key point is to establish the limit
\[ \lim_{i \to \infty} v_i^2(0) \int_{B_r} r \frac{\partial K_i}{\partial r} v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} dy = 0. \]
In fact, if we have this limit, then by the same argument as in Section 4.5, it will give a contradiction and rule out Case I completely.

Since $P_i$ is a critical point and $K$ satisfies condition (**), we know

$$\left| \frac{\partial^\alpha K}{\partial x_i^\alpha}(x_i) \right| \leq C |\nabla K(x_i)|^{n-1-|\alpha|}$$

when $2 \leq |\alpha| \leq n-2$. Then because $K_i(y) = K(r_i y)$,

$$\left| \frac{\partial^\alpha K_i}{\partial y^\alpha}(0) \right| \leq C r_i \left( \frac{n-1-|\alpha|}{n-2} \right) |\nabla K_i(0)|^{\frac{n-1-|\alpha|}{n-2}}$$

$$< C r_i |\nabla K_i(0)|^{\frac{n-1-|\alpha|}{n-2}}$$

$$< C r_i |\nabla K_i(0)|^{\frac{n-2-|\alpha|}{n-2}}$$,

where the last step uses the fact that $|\nabla K_i(0)| \to 0$ and $\frac{n-1-|\alpha|}{n-2} > \frac{n-2-|\alpha|}{n-3}$. Then we can use exactly the same argument as in Section 4.5 to refine the estimates for $\delta_i$ and $|\nabla K_i(0)|$ and thus prove the key limit. This finishes the proof in Case I.

6.2. Ruling out Case II. Recall that by defining $v_i(y) = \sigma_i^{\frac{2}{n-2}} u_i(\sigma_i y)$ and $y = \frac{\tilde{y}}{r_i}$, we have reduced Case II to the situation that $v_i$ satisfies

$$\Delta_{y(i)} v_i - c(n) R(g^{(i)}) v_i + K_i v_i^{p_i} = 0$$

where $g^{(i)}(y) = g_{\alpha\beta}(\sigma_i y) dy^\alpha dy^\beta$, $R(g^{(i)})(y) = \sigma_i^2 R(g)(\sigma_i y)$ and $K_i(y) = K(\sigma_i y)$, and 0 is an isolated blow-up point of $\{v_i\}$.

If 0 is not a simple blow-up point, then we can do another rescaling and repeat the previous argument in Section 6.1.2 with $r_i$ replaced by $r_i \sigma_i$, to get a contradiction. Therefore 0 must be a simple blow-up point for $\{v_i\}$. Then we can still repeat the argument in Section 6.1.2 with $r_i$ replaced by $\sigma_i$. The only difference is in the expression of $h = \lim_{i \to \infty} \frac{v_i(y)}{v_i(\tilde{y})}$.

As in the scalar-flat case, because here $|y|^{\frac{2}{n-2}} v_i(\|y\|)$ doesn’t have a second critical point at $|y| = 1$, we have a different expression of $h$: near 0,

$$h(y) = c_1 |y|^{2-n} + A + O(|y|)$$

where $A$ is a positive constant. This positive “mass” term $A > 0$ guarantees that the limit of the boundary term of the Pohozaev identity is still negative, i.e.,

$$\lim_{i \to \infty} \frac{1}{v_i(\tilde{y})} \int_{\partial B_{\epsilon}} T_i(X, v_i) d\Sigma_i < 0.$$

The other parts of the proof remain the same. Therefore Case II can also be ruled out. This completes the proof of Theorem 1.8.

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Department of Mathematics, The University of British Columbia, Vancouver, B.C., V6T 1Z2, Canada

E-mail address: yyan@math.ubc.ca