SOME UNSTABLE CRITICAL METRICS FOR THE $L^{\frac{n}{2}}$-NORM OF THE CURVATURE TENSOR

ATREYEE BHATTACHARYA AND SOMA MAITY

Abstract. We consider the Riemannian functional defined on the space of Riemannian metrics with unit volume on a closed smooth manifold $M$ given by $R_p(g) := \int_M |R(g)|^p dv_g$ where $R(g)$, $dv_g$ denote the Riemannian curvature and volume form corresponding to $g$. We show that there are locally symmetric spaces which are unstable critical points for this functional.

1. Introduction

Let $M$ be a closed smooth manifold of dimension $n \geq 3$ and $\mathcal{M}_1$ be the space of Riemannian metrics with unit volume on $M$ endowed with the $C^{2,\alpha}$-topology for any $\alpha \in (0,1)$. In this paper we study the following Riemannian functional

$$R_p(g) = \int_M |R(g)|^p dv_g$$

where $R(g)$ and $dv_g$ denote the corresponding Riemannian curvature tensor and volume form, $p \in [2, \infty)$. Let $S^2(T^*M)$ be the space of symmetric two tensors on $M$ and $W$ be the subspace of $S^2(T^*M)$ orthogonal to the tangent space of the orbit of $g$ under the action of the group of diffeomorphisms of $M$ at $g$. Let $H$ denote the Hessian of $R_p$ at any critical metric. For definitions of critical metric and Hessian we refer to section 2.

Definition 1.1. Let $g$ be a critical point for $R_p|_{\mathcal{M}_1}$. $g$ is stable for $R_p$ if there is an $\epsilon > 0$ such that for every element $h$ in $W$,

$$H(h, h) \geq \epsilon \|h\|^2$$

where $\|\cdot\|$ denote the $L^2$-norm on $S^2(T^*M)$ defined by $g$.

Spherical space forms are stable for $R_p$ for $p \geq 2$ and hyperbolic manifolds are stable for $p \geq \frac{n}{2}$ [2]. The stability of locally symmetric spaces is not known in general. In this paper we prove the following.

Theorem 1. Let $(M, g)$ be an irreducible locally symmetric space of compact type. If the universal cover of $M$ is one of the following then $(M, g)$ is not stable for $R_{\frac{n}{2}}$:

- $SU(q)(q \geq 3)$, $Sp(q)(q \geq 2)$, $Spin(5)$, $Spin(6)$, $SU(2q + 2)/Sp(q + 1)$,
- $Spin(q + l)/Sp(q) \times Sp(l)(l, q \geq 1)$, $E_6/F_4$, $F_4/Spin(9)$

Moreover, $(M, g)$ is a saddle point for $R_{\frac{n}{2}}$.

The theorem follows by restricting $H$ to the space of conformal variations of any irreducible symmetric space and using an estimate for the first positive eigenvalue of the Laplacian of $(M, g)$.

Let $(M, g)$ be a simply connected irreducible symmetric space of compact and $\lambda_1$ and $s$ denote its first positive eigenvalue of the Laplacian and scalar curvature of it. We prove that if $\frac{\lambda_1}{s} \geq \frac{2}{n}$

Key words and phrases. Riemannian functional, stability.
then \( (M, g) \) is stable for \( \mathcal{R}_{\frac{n}{2}} \) restricted to the conformal variations of \( g \). The above condition is also a necessary and sufficient criterion for the stability of the identity map of \( (M, g) \) as a harmonic map. In [2] the stability of the identity map of these spaces has been studied in detail. We observe that if \( (M, g) \) is not a sphere then \( g \) is stable for \( \mathcal{R}_{\frac{n}{2}} \) if and only if it is stable for the identity map.

Let \( (M, g) \) be an irreducible symmetric space of compact type or a compact quotient of an irreducible locally symmetric space of non-compact type. From the proof of the theorem we observe that if \( (M, g) \) is neither one of the type in Theorem 1 then \( (M, g) \) is stable for \( \mathcal{R}_{p}(p \geq \frac{n}{2}) \) restricted to the conformal variations of \( (M, g) \).

2. Proof

Let \( \{e_i\} \) be an orthonormal basis at a point of \( M \). \( \hat{R} \) is a symmetric 2-tensor defined by

\[
\hat{R}(x, y) = \sum R(x, e_i, e_j)R(y, e_i, e_j).
\]

Let \( D \) and \( D^* \) be the Riemannian connection, its formal adjoint and \( s \) denote the scalar curvature. \( d^D : S^2(T^*M) \to \Gamma(T^*M \otimes \Lambda^2 M) \) and its formal adjoint \( \delta^D \) are defined by

\[
d^D \alpha(x, y, z) := (D_y \alpha)(x, z) - (D_z \alpha)(x, y)
\]

\[
\delta^D(A)(x, y) = \sum \{D_{e_i}A(x, y, e_i) + D_{e_i}A(y, x, e_i)\}
\]

where \( \Lambda^2 M \) and \( \Gamma(T^*M \otimes \Lambda^2 M) \) denote alternating two forms and sections of \( T^*M \otimes \Lambda^2 M \). Let \( g_t \) be a one-parameter family of metrics with \( \frac{dt}{dt}(g_t)_{t=0} = h \) and \( T(t) \) be a tensor depending on \( g_t \). Then \( \frac{dt}{dt}T(t)_{t=0} \) is denoted by \( T'(h) \). Define \( \Pi_h(x, y) = \frac{dt}{dt}D_x y_{t=0} \) where \( x, y \) are two fixed vector fields. The suffix \( h \) will be omitted when there will not be any ambiguity. Consider any \( f \in C^\infty(M) \). Note that

\[
g(\Pi_{fg}(x, y), z) = \frac{1}{2}[D_x fg(y, z) + D_y fg(x, z) - D_z fg(x, y)]
\]

\[
= \frac{1}{2}[df(x)g(y, z) + df(y)g(x, z) - df(z)g(x, y)]
\]

Let \( \Delta \) denote the Laplace operator which acts on \( C^\infty(M) \). We use the following definition.

\[
\Delta f = -\text{tr}(Ddf)
\]

\( (, ), |\cdot|, \langle \cdot, \cdot \rangle, ||\cdot|| \) denote point-wise inner product, point-wise norm, global inner product and global norms induced by \( g \).

\( \nabla \mathcal{R}_p(g) \) in \( S^2(T^*M) \) is called the gradient of \( \mathcal{R}_p \) at \( g \) if for every \( h \in S^2(T^*M) \),

\[
\frac{d}{dt}|_{t=0} \mathcal{R}_p(g + th) = \mathcal{R}_p'(g)h = \langle \nabla \mathcal{R}_p(g), h \rangle
\]

g is called a critical point for \( \mathcal{R}_{p|\mathcal{M}_1} \) if the component of \( \nabla \mathcal{R}_p(g) \) along the tangent space of \( \mathcal{M}_1 \) at \( g \) is zero. The Hessian at a critical point of \( \mathcal{R}_p \) is given by

\[
H(h_1, h_2) = \langle (\nabla \mathcal{R}_p)'_g(h_1), h_2 \rangle \quad \forall h_1, h_2 \in S^2(T^*M)
\]

**Proposition 2.1.** Let \( (M, g) \) be a compact irreducible symmetric space and \( f \in C^\infty(M) \). Then

\[
H(fg, fg) = p|R|^{p-2}[a\|\Delta f\|^2 - b\|df\|^2 + c\|f\|^2]
\]
where, $a$, $b$, $c$ is given by,

$$a = n - 1 + (p - 2) \frac{4s^2}{n^2|R|^2},$$
$$b = 4(p - 1) \frac{s}{n},$$
$$c = (p - \frac{n}{2}) |R|^2.$$

Let $(M, g)$ be a closed irreducible symmetric space and $h_1, h_2 \in S^2(T^*M)$. From [2] (4.1) we have,

$$H(h_1, h_2) = -p|R|^{p-2} \langle \delta^D(D^*)'_g(h_1)R, h_2 \rangle - p|\langle |R|^{p-2}((D^*)'_g(h_1)h_2) \rangle - p|R|^{p-2} \langle (D^*)'_g(h_1), h_2 \rangle (2.2)$$

$$+ \frac{1}{2} \langle (|R|^{p-2})'_g(h_1)g, h_2 \rangle + \frac{p}{n} \|R\|^p \langle h_1, h_2 \rangle$$

Next we compute each term of the above equation for conformal variations to obtain the Proposition.

**Lemma 2.1.** $\langle \delta^D(D^*)'_g(fg)R, fg \rangle = 4\frac{s^2}{n} \|df\|^2$

**Proof.** Let $\tilde{g}(t)$ be an one-parameter family of metrics with $\tilde{g}(0) = g$ and $T$ be a $(0, 4)$ tensor independent of $t$. Expressing $D^*$ in a local coordinate chart and differentiating it, we obtain,

$$(D^*)'(x, y, z) = -(\hat{g}^{ij})(D_kT)_{jxyz} + \hat{g}^{ij}[\Pi_{xyz} + T_{jxyz} + T_{jxyz}]$$

Note that, $\Pi$ acting on two vector fields gives a vector field. Now evaluating $(D^*)_g(h)(R)$ on an orthonormal basis we have,

$$(D^*)_g(h)(R)_{jkl} = R_{\Pi_{i,j}kl} + R_{\Pi_{i,j}kl} + R_{ij\Pi_{k}l} + R_{ij\Pi_{k}l}.$$

From the definition of $D^*$ we have,

$$d^D fg(x, y, z) = D_g f g(x, y, z) - D_z f g(x, y, z) = df(y)g(x, z) - df(z)g(x, y)$$

Combining these two,

$$\sum (D^*)_g(h)(R)_{jkl} = 2 \sum [R_{\Pi_{i,j}kl} + R_{\Pi_{i,j}kl} + R_{ij\Pi_{k}l} + R_{ij\Pi_{k}l}]df_k$$

Let $\mu$ be the Einstein constant of $(M, g)$. Now using (2.2) we have,

$$2 \sum R_{\Pi_{i,j}kl}df_k = 2 \sum g(\Pi(e_i, e_i, e_m)R(e_m, e_j, e_j)df(e_k)$$

$$= - (n - 2) \mu \|df\|^2$$

Similarly,

$$2 \sum R_{ij\Pi_{k}l}df_k = n \mu \|df\|^2$$

$$2 \sum R_{i\Pi_{j}kl}df_k = 2 \mu \|df\|^2$$

Combining the equations (2.3) and (2.4) the proof of the lemma follows. \qed

**Lemma 2.2.** $\langle (D^*)'_g(fg), d^D fg \rangle = -(n - 1)\|\Delta f\|^2 + (n - 4)\frac{s^2}{n} \|df\|^2$

**Proof.** By a simple calculation we have,

$$D^2 f g(u, v) = Ddf(x, y)g(u, v)$$

and

$$Dd^D fg(x, y, z, w) = Ddf(x, z)g(y, w) - Ddf(x, w)g(y, z).$$
From [1] 1.174(c) and using (2.5) we have,
\[ R'_g(fg)(x, y, z, u) = -\frac{1}{2} [Ddf(y, z)g(x, u) + Ddf(x, u)g(y, z) - Ddf(x, z)g(y, u) - Ddf(y, u)g(x, z)] + fR(x, y, z, u) \] (2.7)

Therefore,
\[ (R'_g(fg), D^D fg) = Ddf_{ik}R'_g(fg)_{ijkl} - Ddf_{il}R'_g(fg)_{ijkl} = -(n - 2)|Ddf|^2 - |\Delta f|^2 - 2\mu|df|^2 \]

Using Bochner-Weitzenböck formula on the space of one forms we have,
\[ \Delta df = D^*Ddf + (n - 1)cdf \]

Hence the lemma follows. \(\square\)

**Lemma 2.3.** \((\bar{R}'_g(fg), fg) = 4\frac{s}{n}\|df\|^2 - |R^2||f||^2\)

**Proof.**
\[ \bar{R}_{pq} = g^{ij}g^{kl}R_{pij}R_{qkl} \]

Differentiating it with respect to \(t\) and evaluating on an orthonormal basis we have,
\[ (\bar{R}_g; h)'_{pq} = -h_{mn}(R_{pm}R_{nq} + R_{pn}R_{mq} + R_{pq}R_{mn}) + (R'_g, h)_{pijk}R_{qijk} + R_{pijk}(R'_g, h)_{qijk} \]

Therefore,
\[ (\bar{R}'_g(fg), fg) = -3|R|^2\|f\|^2 + 2\langle(R'_g, fg), fR \rangle \]

Using (2.7) we have,
\[ (\bar{R}'_g(fg), R) = \frac{1}{2} \sum [Ddf_{jk}R_{ijkl} + Ddf_{il}R_{ijkl} - Ddf_{ik}R_{ijkl} - Ddf_{jl}R_{ijkl}] + f|R|^2 \] (2.8)
\[ = 2\mu\Delta f + f|R|^2 \]

Hence the lemma follows. \(\square\)

**Lemma 2.4.** \((|R|^p)'(fg) = 2\frac{s}{n}p|R|^{p-2}\Delta f - pf|R|^p\)

**Proof.**
\[ (|R|^p)'(fg) = p|R|^{p-2}(R, R'_g fg) - 2p|R|^{p-2}(\bar{R}, fg) = p|R|^{p-2}(R, R'_g fg) - 2pf|R|^{p-2}tr(\bar{R}) = p|R|^{p-2}(2\mu\Delta f + f|R|^2) - 2pf|R|^p = 2\mu p|R|^{p-2}\Delta f - pf|R|^p \]

Using (2.6) we have,
\[ (D^D fg, R) = 2(Ddf, r) = -2\mu\Delta f \]

Now the Proposition follows from the above lemma and equation.

**Proof of Theorem 1.1:** Let \((M, g)\) is a simply connected irreducible symmetric space of compact type which is not a sphere. Then
\[ R = \frac{s}{n(n - 1)} I + W \]
where $I$ is the curvature of the standard sphere with sectional curvature 1 and $W$ is the Weyl curvature of $(M,g)$. From the above expression we have

$$\frac{s^2}{|R|^2} < \frac{2}{n(n-1)}$$

Let $\lambda_1$ be the first positive eigenvalue of the Laplacian of $(M,g)$ and $f$ be an eigenfunction corresponding to $\lambda_1$. Then from Proposition 2.1 we have,

$$H(fg, fg) = s\lambda_1 |R|^{p-2} [a(\frac{\lambda_1}{s}) - \frac{b}{s}] \|f\|^2$$

$$\leq s\lambda_1 |R|^{p-2} \left[ (n-1 + 4\frac{n-4}{n^3(n-1)}) \frac{\lambda_1}{s} - 2\frac{n-2}{n} \right] \|f\|^2$$

From the Table A.1 and A.2 in [2] we have,

$$H(fg, fg) < 0.$$ 

Next, we choose a sufficiently large eigenvalue $\lambda_i$ such that $a(\frac{\lambda_i}{s}) - \frac{b}{s} < 0$. Let $\tilde{f}$ be an eigenfunction corresponding to $\lambda_i$. Then we have,

$$H(\tilde{f}g, \tilde{f}g) < 0$$

This completes the proof. \qed

**Theorem 2.** Let $(M,g)$ be either a compact quotient of an irreducible symmetric of non-compact type or a compact symmetric space which is not one of the types in Theorem 1. $(M,g)$ is stable for $R_p$ for $p \geq \frac{n}{2}$ when it is restricted to the space of conformal variations of $g$.

**Proof.** If $(M,g)$ is a a compact quotient of an irreducible symmetric of non-compact type then the theorem is an immediate consequence of Proposition 2.1. Otherwise from the Table A.1 and A.2 in [2] we have $\frac{\lambda_1}{s} \geq \frac{2}{n}$. Therefore,

$$H(fg, fg) \geq sp\lambda_1 |R|^{p-2} [a(\frac{\lambda_1}{s}) - \frac{b}{s}] \|f\|^2$$

$$\geq sp\lambda_1 |R|^{p-2} \left[ (n-1 + 4\frac{n-4}{n^3(n-1)}) \frac{\lambda_1}{s} - 2\frac{n-2}{n} \right] \|f\|^2$$

$$\geq 2sp\lambda_1 |R|^{p-2} \left[ \frac{n-1}{n} - 2\frac{n-2}{n} \right] \|f\|^2$$

$$\geq 0$$ 

\qed

**Remark:** Let $(M,g)$ be one of the critical metrics of $R_p$ mentioned in Theorem 2. This is an immediate consequence of the above theorem that $(M,g)$ is a local minimizer for $R_p(p \geq \frac{n}{2})$ restricted to the space of metrics conformal to $g$.

**References**

[1] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3)[Results in Mathematics and Related Areas (3)], Volume 10, Springer-Verlag, Berlin, (1987).

[2] Soma Maity, *On the stability of the $L^p$-norm of curvature tensor*, arXive:1201.1691[math.DG] 15 March 2012.

[2] Hajime Urakawa, *The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold*, Compositio Mathematica, Volume 59, p. 57-71(1986).
