ENGEL CONDITION ON ENVELOPING ALGEBRAS OF LIE SUPERALGEBRAS

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Abstract. Let $L$ be a Lie superalgebra over a field of characteristic $p \neq 2$ with enveloping algebra $U(L)$ or let $L$ be a restricted Lie superalgebra over a field of characteristic $p > 2$ with restricted enveloping algebra $u(L)$. In this note, we establish when $u(L)$ or $U(L)$ is bounded Lie Engel.

1. Introduction

Recall that an associative ring $R$ is said to satisfy the Engel condition if $R$ satisfies the identity

$$[x,y,\ldots, y]_n = 0,$$

for some $n$. It follows from Zel’manov’s celebrated result about the restricted Burnside problem [19] that every finitely generated Lie ring satisfying the Engel condition is nilpotent. Kemer in [5] proved that if $R$ is an associative algebra over a field of characteristic zero that satisfies the Engel condition then $R$ is Lie nilpotent. This result was later proved by Zel’manov in [18] for all Lie algebras. However these results fail in positive characteristic, see [17, 11]. Nevertheless, Shalev in [13] proved that every finitely generated associative algebra over a field of characteristic $p > 0$ satisfying the Engel condition is Lie nilpotent. This result was further strengthened by Riley and Wilson in [10] by proving that if $R$ is a $d$-generated associative $C$-algebra, where $C$ is a commutative ring, satisfying the Engel condition of degree $n$, then $R$ is upper Lie nilpotent of class bounded by a function that depends only on $d$ and $n$. Hence, in the positive characteristic case one would need to assume that $R$ is also finitely generated.

Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field $\mathbb{F}$ of characteristic $p \neq 2$ with bracket $(\cdot,\cdot)$. The adjoint map of $x \in L$ is denoted by $\text{ad} \, x$. We denote the enveloping algebra of $L$ by $U(L)$. In case $p = 3$ we
add the condition \(((y,y),y) = 0\), for every \(y \in L_1\). This identity is necessary to embed \(L\) in \(U(L)\).

The Lie bracket of \(U(L)\) is denoted by \([a,b] = ab - ba\), for every \(a, b \in U(L)\). We are interested to know when \(U(L)\) satisfies the Engel condition. Note that the Engel condition is a non-matrix identity, that is a polynomial identity not satisfied by the algebra \(M_2(\mathbb{F})\) of \(2 \times 2\) matrices over \(\mathbb{F}\). The conditions for which \(U(L)\) satisfies a non-matrix identity are given in [2]. It follows from Zel’manov’s Theorem [18] that over a field of characteristic zero \(U(L)\) satisfies the Engel condition if and only if \(U(L)\) is Lie nilpotent. The characterization of \(L\) when \(U(L)\) is Lie nilpotent over any field of characteristic not 2 is given in [2]. Hence, we have

**Corollary 1.1.** Let \(L = L_0 \oplus L_1\) be a Lie superalgebra over a field of characteristic zero. The following conditions are equivalent:

1. \(U(L)\) is Lie nilpotent;
2. \(U(L)\) is bounded Lie Engel;
3. \(L_0\) is abelian, \(L\) is nilpotent, \((L,L)\) is finite-dimensional, and either \((L_1,L_1) = 0\) or \(\dim L_1 \leq 1\) and \((L_0,L_1) = 0\).

However this result is no longer true in positive characteristic as our following theorem shows (see also Example 2.5).

**Theorem 1.2.** Let \(L = L_0 \oplus L_1\) be a Lie superalgebra over a field of characteristic \(p \geq 3\). The following conditions are equivalent:

1. \(U(L)\) is bounded Lie Engel;
2. \(U(L)\) is PI, \(L_0\) is abelian, \(ad x\) is nilpotent for every \(x \in L_0\), and either \((L_1,L_1) = 0\) or \(\dim L_1 \leq 1\) and \((L_0,L_1) = 0\);
3. \(U(L)\) is PI, \(L_0\) is abelian, \(L\) is nilpotent, and either \((L_1,L_1) = 0\) or \(\dim L_1 \leq 1\) and \((L_0,L_1) = 0\).

Note that the above theorem does not follow from Zel’manov or Riley and Wilson’s results because \(U(L)\) is not necessarily finitely generated.

Now let \(L = L_0 \oplus L_1\) be a restricted Lie superalgebra over a field of characteristic \(p > 2\) with enveloping algebra \(u(L)\). In our next result we characterize \(L\) for which \(u(L)\) satisfies the Engel condition. Our results complement the results of [15] [16] where it is determined when \(u(L)\) satisfies a non-matrix identity or when \(u(L)\) is Lie solvable, Lie nilpotent, or Lie super-nilpotent. Similar results for group rings and enveloping algebras of restricted Lie algebras were carried out in [3] [6] and [8], respectively.

**Theorem 1.3.** Let \(L = L_0 \oplus L_1\) be a restricted Lie superalgebra over a field of characteristic \(p > 2\). The following conditions are equivalent:
(1) \(u(L)\) is bounded Lie Engel;
(2) \(u(L)\) is PI, \((L_0, L_0)\) is \(p\)-nilpotent, there exists an integer \(n\) such that \((ad x)^n = 0\) for every \(x \in L_0\), and either \((L_1, L_1)\) is \(p\)-nilpotent or \(\dim L_1 \leq 1\) and \((L_1, L_0) = 0\);
(3) \(u(L)\) is PI, \(L\) is nilpotent, \((L_0, L_0)\) is \(p\)-nilpotent, and either \((L_1, L_1)\) is \(p\)-nilpotent or \(\dim L_1 \leq 1\) and \((L_1, L_0) = 0\).

We refer the reader to [1] for basic background about Lie superalgebras and their enveloping algebras and to [2, 15, 16] for notation.

2. Proof of Theorem 1.2

Throughout the paper, all Lie superalgebras are defined over a field of characteristic \(p \geq 3\).

Let \(\mathfrak{D}\) be a subset of an associative algebra \(\mathfrak{U}\) over a field \(\mathbb{F}\). Recall that \(\mathfrak{D}\) is called weakly closed if for every pair of elements \((a, b) \in \mathfrak{D} \times \mathfrak{D}\), there exists an element \(\gamma(a, b) \in \mathbb{F}\) such that \(ab + \gamma(a, b)ba \in \mathfrak{D}\). This notion is applicable to our setting with a Lie superalgebra \(L\). We take \(\mathfrak{U}\) to be the associative subalgebra of \(\text{End}_\mathbb{F}(L)\) generated by all \(ad x\), where \(x \in L\), and \(\mathfrak{D}\) to be the subset of \(\mathfrak{U}\) consisting of all \(ad x\), where \(x\) is a homogeneous element in \(L\). Then \(\mathfrak{D}\) is weakly closed. Now we recall the Jacobson’s Theorem on weakly closed sets.

**Theorem 2.1** (Jacobson [4]). Let \(\mathfrak{D}\) be a weakly closed subset of an associative algebra \(\mathfrak{U}\) of linear transformations of a finite-dimensional vector space \(V\) over \(\mathbb{F}\). Assume that every \(T \in \mathfrak{D}\) is nilpotent. Then the non-unital associative subalgebra generated by \(\mathfrak{D}\) is associative nilpotent.

Next we recall Corollary 2.5 from [7]. Note that the derived subalgebra of \(L\) will be also denoted by \(L'\).

**Theorem 2.2.** Let \(L = L_0 \oplus L_1\) be a Lie superalgebra over a field of characteristic \(p > 2\). Then \(U(L)\) is PI if and only if there exist homogeneous ideals \(B \subseteq A \subseteq L\) such that

1. \(\dim L/A < \infty\), \(\dim B < \infty\);
2. \(A' \subseteq B\);
3. \(B = B_1\);
4. All inner derivations \(ad z, z \in L_0\), defined over \(L\) are algebraic and their degrees are bounded by some constant.

**Lemma 2.3** ([2]). If \(U(L)\) satisfies a non-matrix polynomial identity then \((L_0, L_0) = (L_0, L_1, L_1) = 0\).

**Lemma 2.4.** If \(U(L)\) is bounded Lie Engel then either \((L_1, L_1) = 0\) or \(\dim L_1 \leq 1\) and \((L_0, L_1) = 0\).
Proof. The proof follows exactly as in Lemma 5.1 of [2].

Note that unlike the characteristic zero case, \( U(L) \) satisfying the Engel condition does not necessarily imply that \( U(L) \) is Lie nilpotent. In fact, in characteristic \( p \geq 3 \), \( U(L) \) can be \( p \)-Engel and yet \( (L, L) \) be infinite-dimensional.

**Example 2.5.** Let \( L = L_0 \oplus L_1 \), where \( (L_0, L_0) = (L_1, L_1) = 0 \), \( L_0 \) has a basis \( x_1, x_2, \ldots \) and \( L_1 \) has a basis \( y, z_1, z_2, \ldots \) with \( z_i = (x_i, y) \). Let \( m \) be a positive integer and set \( v = x_2 \cdots x_m \). Note that \( [x_i, y, y, y] = 0 \), for all \( x_i \). Thus, by the Leibniz formula we have,

\[
[x_1 \cdots x_m, y] = x_1[v, y] + p[x_1, y][v, y] + \left( \frac{p}{2} \right) [x_1, y, y][v, y] = x_1[v, y] = \cdots = x_1 \cdots x_{m-1}[x_m, y] = 0.
\]

It follows that \( U(L) \) is \( p \)-Engel.

We just recall the following identity that follows from super-Jacobi identity:

\[
(1) \quad (\text{ad} \ z)^2 = \frac{1}{2} \text{ad} \ (z, z), \quad \text{for every } z \in L_1.
\]

**Lemma 2.6.** Suppose \( U(L) \) is PI, \( (L_1, L_1) = (L_0, L_0) = 0 \), and \( \text{ad} \ x \) is a nilpotent transformation on \( L \), for every \( x \in L_0 \). Then \( L \) is nilpotent.

**Proof.** Note that, by Theorem 2.2 \( L \) contains a homogeneous ideal \( A \) of finite codimension such that \( A' \) is finite-dimensional. First we show that \( A \) is nilpotent. Note that the restriction of \( \text{ad} \ x \) to \( A' \) is a nilpotent transformation acting on a finite-dimensional vector space, for every \( x \in A_0 \). It follows from Theorem 2.1 that the (non-unital) associative algebra generated by all \( \text{ad} \ x \) with \( x \in A_0 \) acting on \( A' \) is associative nilpotent. This means that

\[
(A, A, A_0, \ldots, A_0) = 0,
\]

for some \( t \). Since \( (L_1, L_1) = 0 \), we deduce that \( A \) is nilpotent. To prove \( L \) is nilpotent, we argue by induction on \( \dim L / A \). By Lemma 5.4 in [14], it suffices to show that \( L / A' \) is nilpotent. So we can replace \( L \) with \( L / A' \) and assume that \( A \) is an abelian ideal of \( L \) of finite codimension. Let \( z \) be a homogeneous element in \( L \setminus A \) and denote by \( N \) the ideal of \( L \) generated by \( A \) and \( z \). If \( z \in L_1 \) then we deduce from Equation (1) and the hypothesis \( (L_1, L_1) = 0 \) that \( (A, z, z) = 0 \). It is now easy to see that \( N \) is nilpotent. On the other hand, if \( z \in L_0 \) then \( \text{ad} \ z \) is a nilpotent transformation on \( L \). Since \( A \) is abelian, we can observe that

\[
\text{ad} \ z = \sum_{i=0}^{t} \frac{(z, \ldots, z)}{i!} \text{ad} \ (z, \ldots, z)
\]

where \( (z, \ldots, z) \) is the ideal generated by \( z \) in \( L \). Since \( A \) is abelian, we have \( (A, z, \ldots, z) = 0 \). Therefore, \( N \) is nilpotent.
N is nilpotent. Now we note that dim \( L/N < \dim L/A \) and it follows by induction that \( L \) is nilpotent.

**Proof of Theorem 1.2.** The implication (1) \( \Rightarrow \) (2) follows from Lemmas 2.3 and 2.4 while (2) \( \Rightarrow \) (3) follows from Lemma 2.6. It remains to prove (3) \( \Rightarrow \) (1). Note that, by Theorem 1.1 of [2], \( U(L) \) satisfies a non-matrix PI and thus \( R = [U(L), U(L)]U(L) \) is nil of bounded index, say \( p^m \). Since \( L \) is nilpotent, there exists an integer \( n \) such that \( L_{p^n} \) is contained in the centre \( Z(L) \) of \( L \). Let \( X \) be an ordered basis for \( L_0 \) and \( Y \) an ordered basis for \( L_1 \). Let \( w \) be an element in the augmentation ideal \( \omega(L) \) of \( U(L) \). Then by the PBW Theorem, \( w \) is a linear combination of PBW monomials of the form \( x_1^{a_1} \cdots x_i^{a_i} y_1 \cdots y_j \).

First suppose that \( (L_1, L_1) = 0 \). Then, modulo \( R \), \( w^{p^n} \) is a linear combination of monomials of the form \( x_1^{a_1 p^n} \cdots x_i^{a_i p^n} \).

Since \( L_{p^n} \subseteq Z(L) \), we deduce that \( w^{p^n} = u + v \), where \( u \) is a central element in \( U(L) \) and \( v \in R \). Hence, \( w^{p^m+n} = w^{p^m} \) is a central element in \( U(L) \). Clearly, \( m \) and \( n \) are independent of \( w \) and so \( U(L) \) is \( p^{m+n} \)-Engel in this case.

The case when \( \dim L_1 \leq 1 \) and \( (L_0, L_1) = 0 \) can be handled similarly.

\( \square \)

### 3. Proof of Theorem 1.3

We recall that Engel’s Theorem holds for Lie superalgebras (see [12], for example).

**Theorem 3.1** (Engel’s Theorem). Let \( L \) be a finite-dimensional Lie superalgebra such that \( \text{ad} \ x \) is nilpotent, for every homogeneous element \( x \in L \). Then \( L \) is nilpotent.

**Lemma 3.2.** If \( u(L) \) is bounded Lie Engel then either \( (L_1, L_1) \) is \( p \)-nilpotent or \( \dim L_1 \leq 1 \) and \( (L_0, L_1) = 0 \).

**Proof.** The proof follows exactly as in Lemma 4.1 of [16]. \( \square \)

**Theorem 3.3** ([7]). Let \( L = L_0 \oplus L_1 \) be a restricted Lie superalgebra. Then \( u(L) \) satisfies a PI if and only if there exist homogeneous restricted ideals \( B \subseteq A \subseteq L \) such that

1. \( L/A \) and \( B \) are both finite-dimensional.
2. \( A' \subseteq B, B' = 0 \).
3. The restricted Lie subalgebra \( B_0 \) is \( p \)-nilpotent.
Theorem 3.4 ([15]). Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra over a perfect field and denote by $M$ the subspace spanned by all $y \in L_1$ such that $(y, y)$ is $p$-nilpotent. The following statements are equivalent:

1. $u(L)$ satisfies a non-matrix PI.
2. $u(L)$ satisfies a PI, $(L_0, L_0)$ is $p$-nilpotent, $\dim L_1/M \leq 1$, $(M, L_1)$ is $p$-nilpotent, and $(L_1, L_0) \subseteq M$.
3. The commutator ideal of $u(L)$ is nil of bounded index.

Lemma 3.5. Suppose that $u(L)$ is PI, $(L_1, L_1)$ is $p$-nilpotent or $(L_0, L_1) = 0$, and ad $x$ is a nilpotent transformation on $L$, for every $x \in L_0$. Then $L$ is nilpotent.

Proof. Note that, by Theorem 3.3, $L$ contains a homogeneous ideal $A$ of finite codimension such that $A'$ is finite-dimensional and $(A', A') = 0$. First we show that $A$ is nilpotent. Let $a \in A'$ and $y \in A_1$. By Equation (1), we have $(a, y, y) = 0$.

Consider now the set $\mathfrak{D}$ of all linear transformations ad $z$ acting on $A'$, where $z$ is a homogeneous element in $A$. Since $\mathfrak{D}$ is weakly closed and consisting of nilpotent linear transformations, we deduce, by Theorem 2.1, that the non-unital associative algebra generated by $\mathfrak{D}$ is associative nilpotent. This means that $(A', A, \ldots, A_{t+1}) = 0$, for some $t$. Hence, $A$ is nilpotent. To prove that $L$ is nilpotent, we use induction on $\dim L/A$. By Lemma 5.4 in [14], it suffices to show that $L/\langle A' \rangle_p$ is nilpotent. So we replace $L$ with $L/\langle A' \rangle_p$ and assume that $A$ is an abelian ideal of $L$ of finite codimension. We claim that $H = L/A$ is nilpotent. Note that either $(H_0, H_1) = 0$ or $(H_1, H_1)$ is $p$-nilpotent. If $(H_0, H_1) = 0$ then clearly every ad $x$ with $x \in H_1$ is nilpotent. On the other hand, if $(H_1, H_1)$ is $p$-nilpotent then, by Equation (1), every ad $x$ with $x \in H_1$ is nilpotent. Hence, Theorem 3.1 applies and we deduce that $H$ is nilpotent. This means that $\gamma_{c+1}(L) \subseteq A$, where $c$ is the nilpotency class of $H$. Let $z$ be a homogeneous element in $\gamma_c(L)$ and denote by $N$ the ideal of $L$ generated by $A$ and $z$. If $z \in L_1$ then, by Equation (1), $(A, z, z) = 0$. On the other hand, ad $z$ is nilpotent if $z \in L_0$. We deduce that $N$ is nilpotent. Now, $\dim L/N < \dim L/A$ and it follows by induction that $L$ is nilpotent. □

Proof of Theorem 1.3. The implication (1) $\Rightarrow$ (2) follows from Lemma 3.2 and Theorem 3.4 while (2) $\Rightarrow$ (3) follows from Lemma 3.5. It remains to prove (3) $\Rightarrow$ (1). Note that, by Theorem 3.4, $u(L)$ satisfies...
a non-matrix PI and thus \( R = [u(L), u(L)]u(L) \) is nil of bounded index. Since \( L \) is nilpotent, there exists an integer \( s \) such that \( L_0^p \subseteq Z(L) \). Let \( X \) be an ordered basis for \( L_0 \) and \( Y \) an ordered basis for \( L_1 \). Let \( w \in \omega(L) \). Then, by the PBW Theorem, \( w \) is a linear combination of PBW monomials of the form \( x_1^{a_1} \cdots x_i^{a_i} y_1 \cdots y_j \). Let \( z_j = \frac{1}{2}(y_j, y_j) \).

Note that \( y_j^p = \frac{p-1}{2} y_j \).

Now suppose that \((L_1, L_1)\) is \( p\)-nilpotent. Then there exists an integer \( n \geq s \) such that \( y_j^p = 0 \), for all \( y_j \in Y \). Now note that, module \( R \), \( w^{p^n} \) is a linear combination of monomials of the form \( x_1^{a_1} \cdots x_i^{a_i} \).

Since, \( L_0^p \subseteq Z(L) \), we deduce that \( w^{p^n} \in Z(u(L)) + R \). Furthermore, \( R \) is nil of bounded index, say \( p^m \). Hence, \( w^{p^{m+n}} \in Z(u(L)) \). Clearly, \( m \) and \( n \) are independent of \( w \) and so \( u(L) \) is \( p^{m+n}\)-Engel in this case.

On the other hand, if \((L_1, L_1)\) is not \( p\)-nilpotent then, by the hypothesis, we must have \( \dim L_1 = 1 \) and \((L_1, L_0) = 0 \). Suppose that \( L_1 \) is spanned by \( y \in L_1 \) and let \( u \) be a PBW monomial of the form \( x_1^{a_1} \cdots x_i^{a_i} y \). We have \( u^p = x_1^{a_1+p} \cdots x_i^{a_i+p} y^p \in Z(u(L)) + R \). Similarly, if \( v \) is a PBW monomial of the form \( x_1^{a_1} \cdots x_i^{a_i} \) then \( v^p \in Z(u(L)) + R \). We deduce that \( w^p \in Z(u(L)) + R \). Clearly, \( m \) and \( s \) are independent of \( w \) and so \( u(L) \) is \( p^{m+s}\)-Engel, completing the proof.

\[\square\]

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