Exact solutions of the (2 + 1)-dimensional Dirac oscillator under a magnetic field in the presence of a minimal length in the noncommutative phase-space

Abdelmalek Boumali

Laboratoire de Physique Appliquée et Théorique,
Université de Tébessa, 12000, W. Tébessa, Algeria.

Hassan Hassanabadi

Department of Physics, Shahrood University of Technology,
Shahrood, Iran P.O. Box 3619995161-316 Shahrood Iran.

Abstract

We consider a two-dimensional Dirac oscillator in the presence of magnetic field in noncommutative phase space in the framework of relativistic quantum mechanics with minimal length. The problem in question is identified with a Poschl-Teller potential. The eigenvalues are found and the corresponding wave functions are calculated in terms of hypergeometric functions.

PACS numbers: 03.65.Ge;

Keywords: Dirac oscillator; minimal length; noncommutative space
I. INTRODUCTION

The Dirac relativistic oscillator is an important potential both for theory and application. It was for the first time studied by Ito et al.\cite{1}. They considered a Dirac equation in which the momentum \( \vec{p} \) is replaced by \( \vec{p} - im\beta \omega \vec{r} \), with \( \vec{r} \) being the position vector, \( m \) the mass of particle, and \( \omega \) the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak \cite{2}, who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Physically, it can be shown that the (DO) interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field \cite{3, 4}. The electromagnetic potential associated with the DO has been found by Benitez et al.\cite{5}. The Dirac oscillator has attracted a lot of interest both because it provides one of the examples of the Dirac’s equation exact solvability and because of its numerous physical applications (see \cite{6} and references therein). Recently, Franco-Villafane et al.\cite{7} exposed the proposal of the first experimental microwave realization of the one-dimensional DO. The experiment relies on a relation of the DO to a corresponding tight-binding system. The experimental results obtained, concerning the spectrum of the one-dimensional DO with and without the mass term, are in good agreement with those obtained in the theory. In addition, Quimbay et al.\cite{8, 9} show that the Dirac oscillator can describe a naturally occurring physical system. Specifically, the case of a two-dimensional Dirac oscillator can be used to describe the dynamics of the charge carriers in graphene, and hence its electronic properties. Also, the exact mapping of the DO in the presence of a magnetic field with a quantum optics leads to regarding the DO as a theory of an open quantum systems coupled to a thermal bath (see Ref.\cite{6} and references therein).

The unification between the general theory of relativity and the quantum mechanics is one of the most important problems in theoretical physics. This unification predicts the existence of a minimal measurable length on the order of the Planck length. All approaches of quantum gravity show the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be reformulated. The minimal length uncertainty relation has appeared in the context of the string theory, where it is a consequence of the fact that the string cannot probe distances smaller than the string scale \( h\sqrt{\beta} \), where \( \beta \) is a small positive parameter called the deformation parameter. This minimal length can be introduced as an additional
uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operators becomes $[\hat{x}, \hat{p}] = i\hbar (1 + \beta p^2)$. This commutation relation leads to the standard Heisenberg uncertainty relation $\Delta \hat{x} \Delta \hat{p} \geq \frac{i\hbar}{2} (1 + \beta (\Delta p)^2)$, which clearly implies the existence of a non-zero minimal length $\Delta x_{\text{min}} = \hbar \sqrt{\beta}$. This modification of the uncertainty relation is usually termed the generalized uncertainty principle (GUP) or the minimal length uncertainty principle. Investigating the influence of the minimal length assumption on the energy spectrum of quantum systems has become an interesting issue primarily for two reasons. First, this may help to set some upper bounds on the value of the minimal length. In this context, we can cite some studies of the hydrogen atom and a two dimensional Dirac equation in an external magnetic field. Moreover, the classical limit has also provided some interesting insights into some cosmological problems. Second, it has been argued that quantum mechanics with a minimal length may also be useful to describe non-point-like particles, such as quasi-particles and various collective excitations in solids, or composite particles (see Ref 14 and references therein).

Nowadays, the reconsideration of the relativistic quantum mechanics in the presence of a minimal measurable length have been studied extensively. In this context, many papers were published where a different quantum system in space with Heisenberg algebra was studied. They are: the Abelian Higgs model, the thermostatics with minimal length, the one-dimensional Hydrogen atom, the casimir effect in minimal length theories, the effect of minimal lengths on electron magnetism, the Dirac oscillator in one and three dimensions, the solutions of a two-dimensional Dirac equation in presence of an external magnetic field, the noncommutative phase space Schrödinger equation, the Schrödinger equation with Harmonic potential in the presence of a Magnetic Field.

The study of non-commutative spaces and their implications in physics is an extremely active area of research. It has been argued in various instances that non-commutativity should be considered as a fundamental feature of space-time at the Planck scale. On the other side, the study of quantum systems in a non-commutative (NC) space has been the subject of much interest in last years, assuming that non-commutativity may be, in fact, a result of quantum gravity effects. In these studies, some attention has been given to the models of non-commutative quantum mechanics (NCQM). The interest in this approach lies on the fact that NCQM is a fruitful theoretical laboratory where we can get some insight on the consequences of non-commutativity in field theory by using standard calculation techniques.
of quantum mechanics. Various non-commutative field theory models have been discussed as well as many extensions of quantum mechanics. Of particular interest is the so-called phase space non-commutativity which has been investigated in the context of quantum cosmology, black holes physics and the singularity problem. This specific formulation is necessary to implement the Bose-Einstein statistics in the context of NCQM (see Refs. [28, 29] and references therein).

The purpose of this work is to investigate the formulation of a two-dimensional Dirac oscillator in the presence of a magnetic field by solving fundamental equations in the framework of relativistic quantum mechanics with minimal length in the non-commutative phase space. To do this we first mapped the problem in question into a commutative space by using an appropriate transformations. Then, we solved it in the presence of a minimal length. We would like mentioned here that the origin of relativistic Landau problem and the Dirac oscillator is entirely different in the former case the magnetic field is introduced via minimal coupling while in the latter case the interaction is introduced via non-minimal coupling and can be viewed as anomalous magnetic interaction [1, 30].

The paper is organized as follows. In section II, we solve the Dirac oscillator in the presence of magnetic field in noncommutative phase space. Then, in section III, we study this problem in the framework of relativistic quantum mechanics with minimal length. Finally, in section IV, we present the conclusion.

II. THE SOLUTIONS IN NONCOMMUTATIVE PHASE SPACE

To begin with we note that the non-commutative phase space is characterized by the fact that their coordinate operators satisfy the equation [28, 29]

\[
\left[ x_i^{(NC)} , x_j^{(NC)} \right] = i \tilde{\theta}_{ij} , \quad \left[ p_i^{(NC)} , p_j^{(NC)} \right] = i \tilde{\theta}_{ij} , \quad \left[ x_i^{(NC)} , p_j^{(NC)} \right] = i \hbar \delta_{ij} ,
\]

where \( \tilde{\theta}_{ij} \) and \( i \tilde{\theta}_{ij} \) are an antisymmetric tensor of space dimension. In order to obtain a theory which preserve the unitary and causal, we choose \( \tilde{\theta}_{0j} = 0 \) which implies that the time remains as a parameter and the non-commutativity affects only the physical space. By replacing the normal product with star product, the Dirac equation in commuting space will change into the Dirac equation in NC space.

\[
\hat{H}_D (p, x) \star \psi_D (x) = E \psi_D (x) ,
\]
where the $\star$-product Moyal between two functions is defined by
\[
(f \star g)(x) = \exp \left[ \frac{i}{2} \tilde{\theta}_{ab} \partial_{x_a} \partial_{x_b} \right] f(x) g(y) \big|_{x=y}.
\] (3)

Instead of solving the NC Dirac equation by using the star product procedure, we use Bopp’s shift method, that is, we replace the star product by the usual product by making a Bopp’s shift
\[
x^{(NC)}_i = x_i - \frac{1}{2\hbar} \tilde{\theta}_{ij} p_j, \quad p^{(NC)}_i = p_i + \frac{1}{2\hbar} \tilde{\theta}_{ij} x_j.
\] (4)

So, in the two dimensional non-commutative phase-space, Eq. (4) becomes
\[
x^{(NC)} = x - \frac{\tilde{\theta}}{2\hbar} p_y, \quad y^{(NC)} = y + \frac{\tilde{\theta}}{2\hbar} p_x, \quad p^{(NC)}_x = p_x + \frac{\tilde{\theta}}{2\hbar} y, \quad p^{(NC)}_y = p_y - \frac{\tilde{\theta}}{2\hbar} x.
\] (5)

In this case, the two-dimensional Dirac oscillator equation, in commutative space, which is written by
\[
\left\{ c_{\alpha_x} \left( p_x - im_0 \omega \tilde{\beta} x \right) + c_{\alpha_y} \left( p_y - im_0 \omega \tilde{\beta} y \right) + \tilde{\beta} m_0 c^2 \right\} \psi_D = E \psi_D,
\] is modified and transformed into
\[
\left\{ c_{\alpha_x} \left( p_x^{(NC)} - im_0 \omega \tilde{\beta} x^{(NC)} \right) + c_{\alpha_y} \left( p_y^{(NC)} - im_0 \omega \tilde{\beta} y^{(NC)} \right) + \tilde{\beta} m_0 c^2 \right\} \psi_D = E_{NC} \psi_D.
\] (6)

Using the following representation of Dirac matrices:
\[
\alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\] (7)
and with $\psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^T$, equation (6) becomes
\[
\begin{pmatrix} m_0 c^2 & c_+ \\ c_- & -m_0 c^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E_{NC} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\] (8)
or
\[
m_0 c^2 \psi_1 + c_- \psi_2 = E \psi_1, \\
c_+ \psi_1 - m_0 c^2 \psi_2 = E \psi_2,
\] (9)
(10)
where
\[
p_- = p_x^{(NC)} - ip_y^{(NC)} + im_0 \omega \left( x^{(NC)} - iy^{(NC)} \right) = \varrho_1 (p_x - ip_y) + im_0 \varrho_2 (x - iy),
\] (11)
\[ p_+ = p_x^{(NC)} + ip_y^{(NC)} - im_0\omega \left( x^{(NC)} + iy^{(NC)} \right) = q_1 (p_x + ip_y) - im_0\omega q_2 (x + iy), \] (12)

and where
\[ q_1 = 1 + \frac{m_0\omega}{2\hbar} \hat{\theta}, \quad q_2 = 1 + \frac{\hat{\theta}}{2m_0\omega\hbar}. \] (13)

From equations (9) and (10), we have
\[ \{ c^2 p_- p_+ - (E^2 - m_0^2 c^4) \} \psi_1 = 0. \] (14)

Now, in order to solve the last equation, and for the sake of simplicity, we bring the problem into the momentum space.

Recalling that
\[ x = i\hbar \frac{\partial}{\partial p_x}, \quad y = i\hbar \frac{\partial}{\partial p_y}, \quad \hat{p}_x = p_x, \quad \hat{p}_y = p_y, \] (15)

and passing onto polar coordinates with the following definition [25]
\[ p_x = p \cos \theta, \quad p_y = p \sin \theta, \quad p^2 = p_x^2 + p_y^2, \] (16)

\[ \hat{x} = i\hbar \frac{\partial}{\partial p_x} = i\hbar \left( \cos \theta \frac{\partial}{\partial p} - \frac{i \sin \theta}{p} \frac{\partial}{\partial \theta} \right), \] (17)

\[ \hat{y} = i\hbar \frac{\partial}{\partial p_y} = i\hbar \left( \sin \theta \frac{\partial}{\partial p} + \frac{\cos \theta}{p} \frac{\partial}{\partial \theta} \right), \] (18)

Eqs. (11) and (12) transform into
\[ p_- = e^{-i\theta} \left\{ q_1 p - \lambda \left( \frac{\partial}{\partial p} - \frac{i}{p} \frac{\partial}{\partial \theta} \right) \right\}, \] (19)

\[ p_+ = e^{i\theta} \left\{ q_1 p + \lambda \left( \frac{\partial}{\partial p} + \frac{i}{p} \frac{\partial}{\partial \theta} \right) \right\}, \] (20)

where
\[ \lambda = \left( 1 + \frac{\hat{\theta}}{2m_0\omega\hbar} \right) m_0\hbar\omega. \] (21)

With the aid of these expressions, the \( p_- p_+ \) term, appears in the Eq. (14), can be written by
\[ p_- p_+ = q_1^2 p^2 - 2q_1 \lambda - \lambda^2 \frac{\partial^2}{\partial p^2} - \lambda^2 \frac{\partial^2}{\partial \theta^2} - \lambda^2 \frac{\partial}{\partial p} + 2i\lambda q_1 \frac{\partial}{\partial \theta}. \] (22)

So, Eq. (14) becomes
\[ \left\{ q_1^2 p^2 - \lambda^2 \left( \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \theta^2} \right) + 2i\lambda q_1 \frac{\partial}{\partial \theta} - 2\lambda q_1 - \zeta \right\} \psi_1 = 0, \] (23)

with
\[ \zeta = \frac{E^2 - m_0^2 c^4}{c^2}. \] (24)
With the help of the following relation\(^{35}\)

\[
\psi_1 (p, \theta) = f (p) e^{im\theta},
\]

Eq. (25) is modified and transforms into

\[
\left( \frac{d^2 f (p)}{dp^2} + \frac{1}{p} \frac{df (p)}{dp} - \frac{m^2}{p^2} f (p) \right) + \left( \kappa^2 - k^2 p^2 \right) f (p) = 0,
\]

with

\[
\kappa^2 = \frac{2 \lambda \varrho_1 (m + 1) + \zeta}{\lambda^2}, \quad k^2 = \frac{\varrho_1^2}{\lambda^2}.
\]

Putting that

\[
f (p) = p^m e^{-k^2 p^2} F (p),
\]

then, the differential equation

\[
F'' + \left( \frac{2m + 1}{p} - 2kp \right) F' - \left[ 2k (m + 1) - \kappa^2 \right] F = 0,
\]

is obtained for \( F (p) \) which by using, instead of \( p \), the variable \( xt = kp^2 \), is transformed into the Kummer equation

\[
t \frac{d^2 F}{dt^2} + \left( m + 1 - t \right) \frac{dF}{dt} - \frac{1}{2} \left( m + 1 - \frac{\kappa^2}{4k} \right) F = 0,
\]

whose solution is the confluent series \( {}_1F_1 (a; m + 1; t) \), with

\[
a = \frac{1}{2} (m + 1) - \frac{\kappa^2}{4k}.
\]

The confluent series becomes a polynomial if and only if \( a = -n \), \( (n = 0, 1, 2, \ldots) \).

According\(^{35}\) we have

\[
\psi_1 (p, \theta) = C_{n,m} p^m e^{-\frac{k^2}{4} p^2} {}_1F_1 (-n; |m| + 1; kp^2) e^{im\theta},
\]

\[
E_n = \pm m_0 c^2 \sqrt{1 + 4 \left( 1 + \frac{m_0 \omega}{2 \hbar} \right) \left( 1 + \frac{\bar{\theta}}{2m_0 \omega \hbar} \right) n}.
\]

The total associated wave function is

\[
\psi_{n,m} (p, \theta) = \left( \frac{1}{c \rho_+ \sqrt{E + m_0 c^2}} \right) \psi_1.
\]

Now, in the presence of an external magnetic field, Eq. (6) is transformed into

\[
\left\{ c_1 \alpha \left[ \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} \right) - im_0 \omega \tilde{\beta} x^{(NC)} \right] + c_2 \alpha \left[ \left( p_y^{(NC)} - \frac{eB_x^{(NC)}}{2c} \right) - im_0 \omega \tilde{\beta} y^{(NC)} \right] + \tilde{\beta} m_0 c^2 \right\} \psi_D = e \psi_D.
\]
Here the potential vectors is chosen as

$$\vec{A} = \left( -\frac{B_y^{(NC)}}{2}, \frac{B_x^{(NC)}}{2}, 0 \right),$$

and the Eq. (34) can be cast into a detail form as follows:

$$\begin{pmatrix} m_0c^2 & c\tilde{p}_- \\ c\tilde{p}_+ & -m_0c^2 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = \epsilon \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}. \tag{36}$$

with

$$\tilde{p}_- = \left( p_x^{(NC)} + \frac{eBy^{(NC)}}{2c} \right) + im_0\tilde{\omega}x^{(NC)} - i(p_y^{(NC)} - \frac{eBx^{(NC)}}{2c}) + m_0\omega y^{(NC)}, \tag{37}$$

$$\tilde{p}_+ = \left( p_x^{(NC)} + \frac{eBy^{(NC)}}{2c} \right) - im_0\tilde{\omega}x^{(NC)} + i(p_y^{(NC)} - \frac{eBx^{(NC)}}{2c}) + m_0\omega y^{(NC)} \tag{38}$$

or

$$\tilde{p}_- = p_x^{(NC)} - ip_y^{(NC)} + im_0\tilde{\omega} \left( x^{(NC)} - iy^{(NC)} \right), \tag{39}$$

$$\tilde{p}_+ = p_x^{(NC)} + ip_y^{(NC)} - im_0\tilde{\omega} \left( x^{(NC)} + iy^{(NC)} \right), \tag{40}$$

where

$$\tilde{\omega} = \omega - \frac{\omega_c}{2}, \quad \omega_c = \frac{|e|B}{m_0c}. \tag{41}$$

is a cyclotron frequency.

Thus, the (2 + 1)-dimensional Dirac oscillator in a magnetic field is mapped onto the same with reduced angular frequency $\tilde{\omega}$ in absence of magnetic field. Hence, the only role of a magnetic field consists in reducing the angular frequency, and the entire dynamics remains unchanged.

Using the mapping defined by (5), the systems of equations become

$$\tilde{p}_- = \varrho_1 \left( p_x - ip_y \right) + im_0\tilde{\omega}\varrho_2 \left( x - iy \right) \tag{42}$$

$$\tilde{p}_+ = \varrho_1 \left( p_x + ip_y \right) - im_0\tilde{\omega}\varrho_2 \left( x + iy \right). \tag{43}$$

By the same way used above, we obtain

$$\tilde{\psi}_1 (p, \theta) = \tilde{C}_{n,m}p^m e^{-\frac{1}{2}p^2} F_1 \left( -n; |m| + 1; kp^2 \right) e^{im\theta}, \tag{44}$$

$$\epsilon_n = \pm m_0c^2 \sqrt{1 + 4 \left( 1 + \frac{m_0\tilde{\omega}}{2\hbar} \right) \left( 1 + \frac{\theta}{2m_0\tilde{\omega}\hbar} \right) n}. \tag{45}$$
The last equation concerning the eigenvalue is in a good agreement with that obtained in the literature (see Ref. [6] and references therein).

The corresponding total eigenfunction is given by

\[
\psi_{n,m}(p, \theta) = \left( \frac{1}{\sqrt{c^2 + m^2 c^2 + \epsilon}} \right) \tilde{\psi}_1.
\]  

(46)

In this section, we have studied the solutions of the two dimensional Dirac oscillator with or without an external magnetic field by using the same way described in [25]: the authors work within a momentum space representation of the Heisenberg algebra, and by an appropriate transformation, the problem is identified as a Kummer differential equation where the solutions are well-known. The solutions that we have found are in well-agreement with those obtained in the literature. This agreement allow us to extend this method by introducing the concept of the minimal length.

III. THE PROBLEM WITH A MINIMAL LENGTH

In the minimal length formalism, the Heisenberg algebra is given by

\[
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \left( 1 + \beta p^2 \right),
\]

(47)

where $\beta > 0$ is the minimal length parameter. A representation of $\hat{x}_i$ and $\hat{p}_i$ which satisfies Eq. (47), may be taken as

\[
\hat{x}_i = i\hbar \left( 1 + \beta p^2 \right) \frac{d}{dp_i}, \quad \hat{p}_i = p_i,
\]

(48)

or

\[
\hat{x} = i\hbar \left( 1 + \beta p^2 \right) \frac{d}{dp_x}, \quad \hat{y} = i\hbar \left( 1 + \beta p^2 \right) \frac{d}{dp_y},
\]

(49)

\[
\hat{p}_x = p_x, \quad \hat{p}_y = p_y.
\]

(50)

In this case, Eq. (14) is modified and becomes

\[
\left\{ c^2 P_+ P_+ - (\epsilon^2 - m_0^2 c^4) \right\} \psi_1 = 0,
\]

(51)

with

\[
P_+ = \frac{1}{2} \left( p_x - i p_y \right) - \lambda \left( 1 + \beta p^2 \right) \left( \frac{\partial}{\partial p_x} - i \frac{\partial}{\partial p_y} \right),
\]

(52)
\[ P_+ = \varrho_1 (p_x - ip_y) + \lambda (1 + \beta p^2) \left( \frac{\partial}{\partial p_x} + i \frac{\partial}{\partial p_y} \right). \]  
(53)

In the polar coordinates, the equations (52) and (53) can be written as

\[ P_- = e^{-i\theta} \left\{ \varrho_1 p - \lambda (1 + \beta p^2) \left( \frac{\partial}{\partial p} - i \frac{\partial}{p \partial \theta} \right) \right\}, \]  
(54)

\[ P_+ = e^{i\theta} \left\{ \varrho_2 p + \lambda (1 + \beta p^2) \left( \frac{\partial}{\partial p} + i \frac{\partial}{p \partial \theta} \right) \right\}. \]  
(55)

When we evaluate the \( P_- P_+ \) term, we get

\[ P_- P_+ = \varrho_1^2 p^2 + 2 (1 + \beta p^2) \left\{ \lambda \varrho_1 \left( i \frac{\partial}{\partial \theta} - 1 \right) - \beta \lambda^2 \left( p \frac{\partial}{\partial p} + i \frac{\partial}{p \partial \theta} \right) \right\} - \lambda^2 (1 + \beta p^2)^2 \left( \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \theta^2} \right), \]  
(56)

and then we have

\[ \left[ \varrho_1^2 p^2 + 2 (1 + \beta p^2) \left\{ \lambda \varrho_1 \left( i \frac{\partial}{\partial \theta} - 1 \right) - \beta \lambda^2 \left( p \frac{\partial}{\partial p} + i \frac{\partial}{p \partial \theta} \right) \right\} - \lambda^2 (1 + \beta p^2)^2 \left( \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \theta^2} \right) - \xi^2 \right] \psi_1 = 0, \]  
(57)

with

\[ \xi^2 = \frac{\varepsilon^2 - m_0^2 c^4}{c^2}. \]  
(58)

Putting that

\[ \psi_1 = h(p) e^{im\theta}, \]  
(59)

Eq. (58) reads

\[ \left[ \varrho_1^2 p^2 - 2 (1 + \beta p^2) \left\{ \lambda \varrho_1 (m + 1) + \beta \lambda^2 \left( \frac{d}{dp} - m \right) \right\} - \lambda^2 (1 + \beta p^2)^2 \left( \frac{d^2}{dp^2} + \frac{1}{p} \frac{d}{dp} - \frac{m^2}{p^2} \right) - \xi^2 \right] h(p) = 0. \]  
(60)

This equation can be written in another form as follows

\[ \left\{ -a(p) \frac{d^2}{dp^2} + b(p) \frac{d}{dp} + c(p) - \xi^2 \right\} h(p) = 0, \]  
(61)

where

\[ a(p) = \lambda^2 (1 + \beta p^2)^2, \]  
(62)

\[ b(p) = -2 \beta \lambda^2 (1 + \beta p^2) p - \frac{\lambda^2 (1 + \beta p^2)^2}{p}, \]  
(63)
\[ c(p) = \rho_1^2 p^2 - 2\lambda \rho_1 (m + 1) (1 + \beta p^2) + 2\beta \lambda^2 m (1 + \beta p^2) + \frac{\lambda^2 (1 + \beta p^2)^2 m^2}{p^2}. \] (64)

The solutions of Eq. (61) can be found by using the following transformations \[^{34}\]

\[ h(p) = \rho(p) \varphi(p), \quad q = \int \frac{1}{\sqrt{a(p)}} dp, \] (65)

with

\[ \rho(p) = e^{\int \chi(p) dp}. \] (66)

Using these transformations, we obtain a form similar to the Schrödinger differential equation:

\[ \left( -\frac{d^2}{dq^2} + V(q) \right) \varphi(p) = \xi \varphi(p), \] (67)

where

\[ \chi(p) = \frac{2b + a'}{4a} = -\frac{1}{2p}, \] (68)

and

\[ V(p) = \rho_1^2 p^2 - 2\lambda \rho_1 (m + 1) (1 + \beta p^2) + 2\beta \lambda^2 m (1 + \beta p^2) + \beta \lambda^2 (1 + \beta p^2) + \frac{\lambda^2 (1 + \beta p^2)^2 m^2}{p^2} \left( m^2 - \frac{1}{4} \right). \] (69)

Using that

\[ p = \frac{1}{\sqrt{\beta}} \tan \left( q \lambda \sqrt{\beta} \right), \] (70)

we get

\[ V(p) = -\frac{1}{\beta} + \beta \lambda^2 \left\{ \frac{\zeta_1 (\zeta_1 - 1)}{\sin^2 (q \lambda \sqrt{\beta})} + \frac{\zeta_2 (\zeta_2 - 1)}{\cos^2 (q \lambda \sqrt{\beta})} \right\}; \] (71)

with

\[ \zeta_1 (\zeta_1 - 1) = m^2 - \frac{1}{4}, \] (72)

\[ \zeta_2 (\zeta_2 - 1) = \left( m - \frac{\rho_1}{\beta \lambda} + \frac{1}{2} \right) \left( m - \frac{\rho_1}{\beta \lambda} + \frac{3}{2} \right) \] (73)

Thus, we have

\[ \left( -\frac{d^2}{dq^2} + \frac{1}{2} U_0 \left\{ \frac{\zeta_1 (\zeta_1 - 1)}{\sin^2 (\alpha q)} + \frac{\zeta_2 (\zeta_2 - 1)}{\cos^2 (\alpha q)} \right\} \right) \varphi(p) = \tilde{\xi}^2 \varphi(p) \] (74)

with \( \tilde{\xi}^2 = \xi^2 + \frac{\alpha^2}{\beta} \) and \( U_0 = u^2 \) with \( u = \lambda \sqrt{\beta} \).

The last equation is the well-known Schrödinger equation in a Poschl-Teller potential where \[^{33}\]

\[ U = \frac{1}{2} U_0 \left\{ \frac{\zeta_1 (\zeta_1 - 1)}{\sin^2 (uq)} + \frac{\zeta_2 (\zeta_2 - 1)}{\cos^2 (uq)} \right\}. \] (75)
with $\zeta_1 > 1$ and $\zeta_2 > 1$. Thus, following Eqs. (72) and (73), we have

\[
\zeta_1 = m \pm \frac{1}{2}, \quad (76)
\]

\[
\zeta_2 = \frac{1}{2} \pm \left( m + 1 - \frac{\zeta_1}{\beta \lambda} \right), \quad (77)
\]

Introducing the new variable

\[
z = \sin^2(uq), \quad (78)
\]

the Schrödinger equation is transformed into

\[
z (1 - z) \varphi'' + \left( \frac{1}{2} - z \right) \varphi' + \frac{1}{4} \left\{ \frac{\bar{\xi}^2}{u^2} - \frac{\zeta_1 (\zeta_1 - 1)}{z} - \frac{\zeta_2 (\zeta_2 - 1)}{1 - z} \right\} \varphi = 0. \quad (79)
\]

Now, putting that

\[
\varphi = z^{\frac{\zeta_1}{2}} (1 - z)^{\frac{\zeta_2}{2}} \Psi(z), \quad (80)
\]

we arrive at

\[
z (1 - z) \Psi'' + \left[ \left( \zeta_1 + \frac{1}{2} \right) - z (\zeta_1 + \zeta_2 + 1) \right] \Psi' + \frac{1}{4} \left\{ \frac{\bar{\xi}^2}{u^2} - (\zeta_1 + \zeta_2)^2 \right\} \Psi = 0. \quad (81)
\]

The general solutions of this equation are [35, 36]

\[
\Psi = C_{1,2} F_1(a; b; c; z) + C_{2} z^{1-c} F_1(a+1-c; b+1-c; 2-c; z), \quad (82)
\]

where

\[
a = \frac{1}{2} \left( \zeta_1 + \zeta_2 + \frac{\bar{\xi}}{u^2} \right), \quad b = \frac{1}{2} \left( \zeta_1 + \zeta_2 - \frac{\bar{\xi}}{u^2} \right), \quad c = \zeta_1 + \frac{1}{2}. \quad (83)
\]

With the condition $a = -n$, we obtain

\[
\bar{\xi}^2 = u^2 (\zeta_1 + \zeta_2 + 2n)^2. \quad (84)
\]

In order to obtain the energy of spectrum, it should be to note that in the limit $\beta \rightarrow 0$, the energy of spectrum should be covert to no-GUP result. Thus, we choose

\[
\zeta_1 = m + \frac{1}{2}, \quad (85)
\]

\[
\zeta_2 = \frac{1}{2} - \left( m + 1 - \frac{\zeta_1}{\beta \lambda} \right). \quad (86)
\]

Following this, we obtain

\[
\epsilon_n = \pm \sqrt{m_0^2 c^4 + 4c^2 \left( 1 + \frac{m_0 \omega}{2\hbar} \right) \left( 1 + \frac{\bar{\theta}}{2 m_0 \omega \hbar} \right) n + 4c^2 \beta \left( 1 + \frac{\bar{\theta}}{2 m_0 \omega \hbar} \right)^2 n^2}, \quad (87)
\]
where
\[ \beta < \beta_0, \beta_0 = \frac{1}{m + \frac{3}{2}} \left(1 + \frac{m \omega}{2h} \theta \right), \text{ with } m > 0. \]

So, the non-zero minimal length is
\[ \Delta x_{\text{min}} = \hbar \sqrt{\beta} < (\Delta x_{\text{min}})_0 = \sqrt{\frac{1}{m + \frac{3}{2}} \left(1 + \frac{m \omega}{2h} \theta \right)} l_{\text{min}}, \]

with \( l_{\text{min}} = \sqrt{\frac{\hbar}{m \omega}} \) is the characteristic length of the Dirac oscillator, and \( (\Delta x_{\text{min}})_0 \) is the admissible length above it the physics becomes experimentally inaccessible. We can see that the influence of the non-commutative parameters on \( (\Delta x_{\text{min}})_0 \) is very clear. Now, expanding to first order in \( \beta \) we have
\[ \epsilon_n \simeq \pm m_0 c^2 \sqrt{1 + \frac{4}{m_0^2 c^2} \left(1 + \frac{m_0 \omega}{2h} \theta \right) \left(1 + \frac{\theta}{2m_0 \omega \hbar} \right) n} \times \left(1 + \frac{2\beta \left(1 + \frac{\theta}{2m_0 \omega \hbar} \right)}{1 + \frac{4}{m_0^2 c^2} \left(1 + \frac{m_0 \omega}{2h} \theta \right) \left(1 + \frac{\theta}{2m_0 \omega \hbar} \right) n} \right) \]

The first term is the energy spectrum of the usual two-dimensional Dirac oscillator and the second term represents the correction due the presence of the minimal length. As mentioned by \[19\], we note the dependence on \( n^2 \) which is feature of hard confinement. For a large values of \( n \) we have
\[ \epsilon_n = \hbar \bar{\omega} n, \]

which means, according that the energy continuum for large \( n \) for the Dirac oscillator without the minimal length disappears in the presence of the minimal length, and consequently the behavior of the DO can be described by a non-relativistic harmonic oscillator with frequency \( \bar{\omega} = \frac{2\sqrt{\pi}}{\hbar} \left(1 + \frac{\theta}{2m_0 \omega \hbar} \right). \)

According to the Eqs. (83) and (85), we can see that the parameter \( c = m + 1 \) is an integer: thus either the two solutions of Eq. (82) coincide or one of the solutions will blow up. Now, when \( c \) is an integer greater than 1, which is our case, the second solution diverges. Thus, the component \( \psi_1 \) will has the following form
\[ (\psi_1)_{n,m} (p, \theta, z) = (C_1)_{n,m} p^{-\frac{1}{2}} e^{i m \theta} z^{\frac{\theta}{2}} (1 - z) \frac{\theta}{2} {}_2F_1 (-n; b, |m| + 1; z). \]
Finally, the total associated eigenfunction is done by
\[ \psi_{n,m}(p, \theta, z) = \left( \frac{1}{cP_+ - m_0 c^2} \right) \psi_1. \]  

(93)

Now, in the presence of an uniform magnetic field, Eq. (6) is transformed into
\[ \{ c_\alpha \left[ \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} \right) - im_0 \omega \bar{\beta}_x^{(NC)} \right] + c_\alpha y \left[ \left( p_y^{(NC)} - \frac{eB_x^{(NC)}}{2c} \right) - im_0 \omega \bar{\beta}_y^{(NC)} \right] + \bar{\beta} m_0 c^2 \} \psi_D = \bar{\epsilon} \psi_D. \]

(94)

In this case, Eq. (8) takes the following form
\[ \left( \begin{array}{c} m_0 c^2 \\ c_\bar{\beta}_- \\ c_\bar{\beta}_+ \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \bar{\epsilon} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \]

(95)

with
\[ \bar{\beta}_- = \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} \right) + im_0 \omega \bar{\beta}_x^{(NC)} - iP \left( y^{(NC)} - \frac{eB_x^{(NC)}}{2c} \right) + m_0 \omega y^{(NC)}, \]

(96)

\[ \bar{\beta}_+ = \left( p_x^{(NC)} + \frac{eB_y^{(NC)}}{2c} \right) - im_0 \omega \bar{\beta}_x^{(NC)} + iP \left( y^{(NC)} - \frac{eB_x^{(NC)}}{2c} \right) + m_0 \omega y^{(NC)}, \]

(97)

or
\[ \bar{\beta}_- = \theta \left( p_x - iP \right) + im_0 \tilde{\omega} \theta \left( x - iy \right), \]

(98)

\[ \bar{\beta}_+ = \theta \left( p_x + iP \right) + im_0 \tilde{\omega} \theta \left( x + iy \right), \]

(99)

where
\[ \tilde{\omega} = \omega - \frac{\omega_c}{2} \quad \omega_c = \frac{|e| B}{m_0 c}, \]

(100)

and where \( \omega_c \) is a cyclotron frequency. According to the above case, the eigensolutions are given by
\[ (\psi_1)_{n,m}(p, \theta, z) = \left( \tilde{C}_1 \right)_{n,m} p^{-\frac{1}{2}} e^{im_0 \tilde{\omega} \theta \left( 1 - z \right)^3} 2 F_1 \left( -n; b, |m| + 1; z \right), \]

(101)

\[ \bar{\epsilon}_n = \sqrt{m_0^2 c^4 + 4c^2 \left( 1 + \frac{m_0 \tilde{\omega}}{\hbar} \right) \left( 1 + \frac{\tilde{\theta}}{2m_0 \tilde{\omega} \hbar} \right) n + 4c^2 \beta \left( 1 + \frac{\tilde{\theta}}{2m_0 \tilde{\omega} \hbar} \right)^2 n^2}, \]

where the total wave-function is done by
\[ \psi_{n,m}(p, \theta, z) = \left( \frac{1}{cP_+ - m_0 c^2} \right) \psi_1. \]

(102)
IV. RESULTS AND DISCUSSIONS

Here we have obtained exact solutions of the two-dimensional Dirac oscillator in non-commutative space with the presence of minimal length. Firstly, by adopting the same procedure that used by Menculini et al. [25], we have solved the problem only in the case of noncommutative space. The results found are in well agreement with those obtained in the literature. After that, we have introduced the minimal length in the the problem in question. This introduction has been make as follows: (i) we write the coordinates of the noncommutative space with those in commutative space by using the Bipp shift approximation, and (ii) then we introduce the minimal length in our equation. By these, the problem in question is identified with a Poschl-Teller potential. Also, when $\theta$, and $\bar{\theta}$ tend to zero, we recover exactly the same results of [37].

Finally, let us note that the non-relativistic harmonic oscillator is used as a model for describing the quark’s confinement in mesons and baryons, while the Dirac oscillator is expected to give a good description of the confinement in heavy quark systems. Quimby and Strange suggested that the two-dimensional Dirac oscillator model can be describe some properties of electrons in graphene. This model explains the origin of the left-handed chirality observed for charge carriers in monolayer and bilayer graphene. They have shown that the change of the strength of a magnetic field leads to the existence of a quantum phase transition in the chirality of the systems. In addition, in a recent paper, it has been shown that we can modulate the system of graphene under a magnetic field with a model based on a Dirac oscillator. With this, the author has determine all thermodynamic properties of this system by using the thermal zeta function [38].

In our case, a possible application is the determination of the upper limit of the length in comparison with the data found experimentally for the case of graphene: this idea has been used by Menculini et al. [25] in order to obtain an upper bound on the minimal length appearing in the framework of generalized uncertainly principle.

V. CONCLUSION

In this paper, we have exactly solved the Dirac oscillator in two dimensions in the presence of an external magnetic field in the framework of relativistic quantum mechanics with
minimal length and in the noncommutative phase-space. Firstly, the eigensolutions of the
problem in question are obtained in noncommutative space. Then, we extend our study in
the presence of a minimal length. The energy levels, for both cases, show a dependence on
n² in the presence of the minimal length which described a hard confinement. For the large
values of n, our DO become like a non-relativistic harmonic oscillator. the dependence of
the non-zero minimum length on the noncommutativite parameters is very clear. In the
limit where β → 0, and where θ, and ¯θ tend to zero, we recover the results obtained in the
literature

[1] D. Itô, K. Mori and E. Carriere, Nuovo Cimento A, 51, 1119 (1967).
[2] M. Moshinsky and A. Szczepaniak, J. Phys. A: Math. Gen, 22, L817 (1989).
[3] R. P. Martinez-y-Romero and A. L. Salas-Brito, J. Math. Phys, 33, 1831 (1992).
[4] M. Moreno and A. Zentella, J. Phys. A : Math. Gen, 22, L821 (1989).
[5] J. Benitez, P. R. Martinez y Romero, H. N. Nunez-Yepez and A. L. Salas-Brito,Phys. Rev.
Lett, 64, 1643–5 (1990).
[6] A. Boumali and H. Hassanabadi, Eur. Phys. J. Plus. 128 : 124 (2013).4
[7] J. A. Franco-Villafane, E. Sadurni, S. Barkhofen, U. Kuhl, F. Mortessagne, and T. H. Selig-
man, Phys. Rev. Lett. 111, 170405 (2013).
[8] C. Quimbay and P. Strange, arXiv:1311.2021, (2013).
[9] C. Quimbay and P. Strange, arXiv:1312.5251, (2013).
[10] A. Kempf, J. Math. Phys. 35 4483, (1994).
[11] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 , 1108, (1995).
[12] H. Hinrichsen and A. Kempf, J. Math. Phys. 37, 2121, (1996).
[13] A. Kempf, J. Phys. A: Math. Gen. 30, 2093, (1997).
[14] C. Quesne and VM Tkachuk, Sigma. 3, 016, (2007).
[15] P. Gaete and E. Spallucci, EPL 105 , 21002 (2014).
[16] B Vakili and M A Gorji, J. Stat. Mech, P10013, (2012)
[17] P. Pedram, EPL 101 , 30005, (2013).
[18] Kh Nouicer 2005 J. Phys. A: Math. Gen. 38 10027
[19] Khireddine Nouicer 2007 J. Phys. A: Math. Theor. 40 2125
[20] Khireddine Nouicer 2006 J. Phys. A: Math. Gen. 39 5125
[21] C. Quesne and V. M. Tkachuck, J. Phys. A: Math. Gen. 38, 1747 (2005).
[22] TV Fityo, IO Vakarchuk, VM Tkachuk, J. Phys. A: Math. Gen. 39, 2143 (2006).
[23] C Quesne, VM Tkachuk, J. Phys. A: Math. Gen. 39, 10909 (2006).
[24] M. Betrouche, M.Maamache and J. R. Choi, Advances in High Energy Physics, (2013).

[25] L. Menculini, O. Panella and P. Roy, Phys. Rev. D, 87, 065017, (2013).
[26] H. Hassanabadi, Z. Molaee, and S. Zarrinkamar, Advances in High Energy Physics, (2014).
[27] Hassanabadi, E. Maghsoodi, Akpan N. Ikot, and S. Zarrinkamar, Advances in High Energy
Physics, (2013).
[28] O. Bertolami and R. Queiroz, Phys. Lett. A, 375, 4116–4119 (2011).
[29] C. Bastos, O. Bertolami, N. C. Dias and J. N. Prata, Int. J. Mod. Phys. A, 28, 1350064, (2013);
  C. Bastos, O. Bertolami, N. C. Dias and J. N. Prata, J. Math. Phys, 49, 072101 (2008);
  C. Bastos, N. C. Dias and J. N. Prata, Comm. Math. Phys. 299, 709 (2010).
[30] D. Nath and P. Roy, Ann. Phys 351, 13–21 (2014).
[1] L. M. Abreu, E. S. Santos and J. D.M. Vianna, J. Phys. A: Math. Theor.43, 495402 (2010).
[32] C. Bastos, O. Bertolami, N. C. Dias and J. N. Prata, Int. J. Mod. Phys. A, 28, 1350064, (2013).
[33] S. Flugge, Practical Quantum mechanics, Springer-Verlag, Berlin, (1974).
[34] T. K. Jana and P. Roy, Phys. Lett. A, 373, 1239–1241 (2009).
[35] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas,
Graphs, and Mathematical Tables, New York: (1965).
[36] G. E. Andrews, R. Askey and R. Roy (1999). Special functions, Cambridge University Press,
(1999).
[37] A. Boumali and H. Hassanabadi, Can. J. Phys, (2015) (will be published soon)
[38] A. Boumali, arxiv: 1411.1353 (2014);
  A. Boumali, EJTP, 12, 121–130 (2015); arxiv: 1409.6205 (2014).