DESIGNING COMMUNICATION NETWORKS VIA
HILBERT MODULAR FORMS

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Abstract. Ramanujan graphs, introduced by Lubotzky, Phillips and Sarnak, allow the design of efficient communication networks. In joint work with B. Jordan we gave a higher-dimensional generalization. Here we explain how one could use this generalization to construct efficient communication networks which allow for a number of verification protocols and for the distribution of information along several channels. The efficiency of our network hinges on the Ramanujan-Petersson conjecture for certain Hilbert modular forms. We obtain this conjecture in sufficient generality to apply to some particularly appealing constructions, which were not accessible before.

The concept of a Ramanujan graph was introduced and studied by Lubotzky, Phillips and Sarnak (LPS) in [16]. These are \( r \)-regular graphs for which the nontrivial eigenvalues \( \lambda \neq \pm k \) of the adjacency matrix satisfy the bounds \(|\lambda| \leq 2\sqrt{r - 1}\). In many aspects these bounds are optimal and natural. For example, the adjacency matrix is the combinatorial analog of the Laplace operator, and the bounds parallel the (conjectured) Selberg bounds for the Laplacian on Riemann surfaces. The main result of LPS was an explicit construction of \( p + 1 \)-regular such graphs, \( p \equiv 1 \mod 4 \) a prime, through the arithmetic of quaternion algebras over the rational numbers. From the point of view of Communication Network Theory, the arithmetic examples are particularly interesting: all Ramanujan graphs are super-expanders; but in addition the examples have many other useful properties, for example very good expansion constants and large girth. Thus they can be used to design efficient communication networks.

The Ramanujan property for the LPS examples hinges on the truth of the Ramanujan-Petersson conjecture for an appropriate space of modular forms of weight 2 over \( \mathbb{Q} \). Let \( f \) be a weight 2 holomorphic cuspidal Hecke eigenform on a congruence subgroup. The conjecture is that for any prime \( p \) not dividing the level the Hecke eigenvalue \( a_p \) satisfies \(|a_p| \leq 2\sqrt{p}\). Eichler and Shimura reduced the conjecture to Weil’s...
results on the absolute value of Frobenius eigenvalues for curves over a finite fields. They achieved this for a fixed form and all but a finite, unspecified set of primes \( p \). The proof requires a deep study of the reduction modulo \( p \) of modular curves and Hecke correspondences. Igusa subsequently showed that the method applied for all primes not dividing the level of the form (see [4] for a more modern exposition and a generalization to forms over \( \mathbb{Q} \) of any weight). Igusa’s part is essential for the applications to communication networks because there one first chooses the prime \( p \) and only then an increasing sequence of levels prime to it. This gives an infinite family of Ramanujan graphs of fixed regularity \( p + 1 \).

In a joint work with B. Jordan ([13]) we gave a higher dimensional generalization of this theory to \((r_1, \ldots, r_g)\)-regular cubical complexes. These are cell complexes locally isomorphic to a product of \( r_i \)-regular trees. Here there are partial Laplacians, one for each tree factor. The parallel bounds for their eigenvalues define the notion of being Ramanujan. We then constructed infinite towers of explicit arithmetic examples whenever each \( r_i \) is of the form \( q_i + 1 \) for a prime power \( q_i \). We used quaternion algebras over totally real fields and hence our generalization required the Ramanujan-Petersson conjecture for Hilbert modular forms of multi-weight \((2, \ldots, 2)\). In place of the work of Eichler, Shimura and Igusa we invoked analogous results of Carayol ([3]) on the reduction modulo \( p \) of Shimura curves over totally real number fields, which as before allow to use Weil’s results.

Carayol’s method imposes on the form a technical assumption which forced us to exclude certain natural and particularly appealing examples. Building on ideas of Langlands, Brylinski and Labesse ([2]) obtained results which are free from such assumptions, by reducing the conjecture to Deligne’s bounds for Frobenius eigenvalues on the intersection homology of Hilbert-Blumenthal varieties. However in their proof they used in an essential way the Satake-Baily-Borel compactification which was available only over \( \mathbb{Q} \). This forced them to exclude for any fixed form an unspecified finite set of primes, which makes their results inapplicable to us as was explained above. One of our aims here is to point out that a modification of these arguments proves the following

**Theorem 0.1.** Let \( F \) be a totally real field of degree \( d \) over \( \mathbb{Q} \). Let \( f \) be a holomorphic Hilbert cuspidal Hecke eigenform over \( F \) of multi-weight \((k_1, \ldots, k_d)\), where the \( k_i \)'s are integers \( \geq 2 \) and all of the same parity. Let \( v \) be a prime of \( F \) of degree \( d_v = [F_v : \mathbb{Q}_p] \) whose residual characteristic \( p \) is prime to the discriminant \( \text{Disc} F \) of \( F \) and to the level
$N = N(f)$ of $f$. Let $\lambda_v(f)$ denote the eigenvalue of the Hecke operator $T_v$ belonging to $f$. Then the Ramanujan-Petersson conjecture holds for $f$ and $v$, namely we have $|\lambda_v(f)| \leq 2p^{(k-1)d_v/2}$, where $k = \max_i k_i$.

For the theorem to be meaningful we must specify the normalizations made in defining $T_v$ and $\lambda_v$. However Hilbert modular forms are best approached through representation theory, and we have in fact opted to define $T_v$ and $\lambda_v$ only representation-theoretically. This spares us the tedious task of making “classical” definitions and then comparing them to the representation-theoretic ones. The version of the theorem we will actually prove is therefore Theorem 2.4 below.

Here is an outline of the article. In Section 1 we discuss the significance of the higher dimensional theory of \cite{13} to communication networks, especially in the two-dimensional case. In Section 2 we prove the above theorem. This enables us to handle in Section 3 a particularly pretty example which was left unsettled in \cite{13}. The result is a mixture of graph theory, automorphic forms, algebraic geometry, and the arithmetic of quaternions over number fields. In our attempt to make this work accessible to a mixed audience and yet not too lengthy we have undoubtedly failed to give the right level of detail to any particular reader. We can only ask for indulgence in this matter.

The debt this work owes to my long term collaboration with B. Jordan, in particular to \cite{13}, should be evident. I started thinking about this problem at the instigation of P. Sarnak. Different approaches to the problem were subsequently suggested by D. Blasius and C.-L. Chai (independently). P. Deligne suggested to use compactly supported cohomology. Conversations with them and with J. Bernstein, G. Faltings, G. Harder, and N. Katz were encouraging and helpful. It is a pleasure to thank them all.

1. Cubical complexes and communication networks

A basic problem in communication networks is to design explicit super-expanders. For a given, arbitrarily large set of nodes one wants to connect each node to a fixed number $r$ of “neighbor” nodes, so that information can spread fast over the resulting network $N$. A commonly used quantitative measure of efficiency is the expansion constant — the largest real constant $c > 0$ so that each subset $A$ of the nodes of $N$ having $|A| \leq |N|/2$ nodes has at least $c|A|$ (new) neighbors. One seeks sequences of networks $N_k$ where $|N_k| \to \infty$ with $k$, while $c$ is independent of $k$ and is as large as possible. Even though one can define the meaning of a random network on $n$ vertices and prove that random networks are good expanders, it is quite hard to get explicit
examples. The problem whether a random network is Ramanujan is open, although it is known to be “almost Ramanujan” as \( r \to \infty \). More precisely, \( \lambda \leq 2\sqrt{r-1} + \log(r-1) + \text{Const} \) with probability \( \to 1 \) with \( N \) (see [3]).

In this work we shall explain how to design any number \( g \geq 1 \) of efficient communication networks on the same arbitrarily large set \( N_k \) of nodes. Thinking of the connections of each network as having a different color, each node will have a fixed number \( r_i \) of color \( i \) neighbors. Each color will be a Ramanujan network, and hence a super-expander. But in addition, any set of \( h \) edges of different colors starting at the same vertex can be completed to an \( h \)-cube. For example, given a red edge \( e_r \) and a blue edge \( e_b \) emanating from the same vertex, there will be a unique pair of red and blue edges \( e'_r, e'_b \) completing them to a square. In other words, the origins \( o(.) \) and ends \( t(.) \) of the edges will satisfy \( o(e'_r) = t(e_r), o(e'_b) = (e_b), \) and \( t(e'_r) = t(e'_b) \). We say that the network satisfy the cube property (or the square property for 2 colors). Like the LPS examples, the ones we give have good expansion properties and large girth.

In practical applications the cube property permits a variety of potential applications

1. To send information along (say) the red network, and verify, using a hash function, a compatibility condition via the blue channel.
2. To send divide the information to \( g \) parts and send each part individually. One could arrange so that only the combination of the various elements would lead to a meaningful whole.

It seems a difficult problem to construct by random methods systems of \( (k_1, \ldots, k_g) \)-regular networks satisfying the cube condition or to parametrize the space of such systems. The key point is to realize that the cube condition makes natural the introduction of higher dimensional cubical complexes into the situation. Let \( r_i \geq 3 \) be a sequence of length \( g \) of integers, let \( T_{r_i} \) be a regular tree of regularity \( r_i \), and set \( T = \prod_i T_{r_i} \). Then an \( (r_1, \ldots, r_g) \)-regular cubical complex in the sense of [13] is a complex \( X \) in which each connected component is isomorphic to a quotient \( \Gamma \backslash T \) for a discrete, torsion free subgroup of \( \prod_i \text{Aut} T_{r_i} \). For simplicity we also require the parity condition: the \( i \)th component of any element of \( \Gamma \) moves each vertex of \( T_{r_i} \) to an even distance. The 1-skeleton of \( X \) is a collection of \( g \) graphs, or communication networks, on the same set of vertices. The parity assumption makes these graphs bipartite. The vertices have a “multiparity”, an element of \( \{0, 1\}^g \); the \( i \)th graph is \( r_i \)-regular, and its \( 2^{g-1} \) connected components consist of the vertices in which all parities except for the
ith have been fixed. Its edges come from the $i$th tree factor of $T$. Most importantly, the $g$ graphs satisfy the cube property.

Theorem 3.1 of loc.cit. provides us with the following general class of arithmetic examples:

**Theorem 1.1.** Let $B$ be a totally definite quaternion algebra over a totally real field $F$, and let $S = \{ v_1, \ldots, v_g \}$ be a nonempty set of $g$ distinct finite primes of $F$ so that $B$ is split at each $v_i$. Let $q_i$ be the cardinality of the residue field of $F_{v_i}$ and set $r_i = q_i + 1$. Let $\Gamma$ be an $S$-arithmetic torsion-free congruence subgroup of the algebraic group over $\mathbb{Q}$ of the norm 1 elements in $B$. Then $\Gamma$ acts on $T = \prod_i T_{r_i}$ and the resulting quotient $\Gamma \backslash T$ is an irreducible $(r_1, \ldots, r_g)$-regular complex with parities. Moreover if an appropriate space of Hilbert cusp forms (of weight $(2, \ldots, 2)$) satisfies the Ramanujan-Petersson conjecture, then $\Gamma \backslash T$ is Ramanujan.

The assumptions on $B$ mean that $B \otimes F_v$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$ for any infinite prime $v$ of $F$, and that $B \otimes F_{v_i}$ is isomorphic to $\text{GL}(2, F_{v_i})$. Let $\mathcal{O}$ be the order of elements of $F$ integral away from $v_1, \ldots, v_g$ and let $\mathcal{M}$ be a maximal $\mathcal{O}$-order of $B$. Then $\Gamma$ is taken to be a sufficiently small congruence subgroup of the group of norm 1 elements of $\mathcal{M}$.

The description of our complexes as quotients of infinite complexes by infinite groups can be replaced by a finite description, which is more elementary and very convenient for explicit calculation. This finite description takes a particularly simple shape if certain assumptions are made (see [13] for the details): let us assume there is an ideal $N_0 \neq 0$ of $\mathcal{O}_F$, prime to the $v_i$’s (we allow $N_0 = \mathcal{O}_F$), such that the following holds:

**Conditions 1.2.**

1. Every ideal of $F$ has a totally positive generator $\equiv 1 \mod N_0$.
2. The class number of $B$ is 1.
3. The units $\mathcal{M}^\times$ of a maximal order $\mathcal{M}$ of $B$ map onto $(\mathcal{M}/N_0\mathcal{M})^\times$, with the kernel being contained in the center $\mathcal{O}_F^\times$ of $\mathcal{M}^\times$.

Fix a totally positive generator $\pi_j$ which is $\equiv 1 \mod N_0$ for each prime ideal $v_j$. We then have the following:

**Proposition 1.3.**

1. For each $1 \leq j \leq g$ there are exactly $r_j$ (principal) ideals $P_{j,i}$, $1 \leq i \leq r_j$ of $\mathcal{M}$ with norm $v_j$. We can moreover choose generators $\omega_{j,i} \equiv 1 \mod N_0\mathcal{M}$ for $P_{j,i}$ whose norm is $\pi_j$.
2. For every permutation $\sigma$ of $\{ 1, \ldots, g \}$ and any sequence of indices $i_1, \ldots, i_g$, with $1 \leq i_j \leq r_j$, there is a (unique) sequence $i'_1, \ldots, i'_g$, with
1 \leq i'_j \leq r_j$, and a (unique) unit $u \in \mathcal{O}_F^\times$, satisfying $u \equiv 1 \mod N_0$, so that we have the following arithmetic counterpart of the cube condition:

\begin{equation}
\varpi_{\sigma(1),i_1} \cdots \varpi_{\sigma(g),i_g} = u \varpi_{1,i'_1} \cdots \varpi_{g,i'_g}.
\end{equation}

Let $N_1$ be a prime ideal of $\mathcal{O}_F$ prime to the $v_j$'s and to $N_0$, and set $N = N_0N_1$. Let $A$ be the subgroup of $(\mathcal{O}_F/N_1)^\times$ generated by the images modulo $N_1$ of the $\pi_j$'s. Let $B$ be the subgroup of scalars in $(\mathcal{M}/N_1\mathcal{M})^\times$ generated by the images modulo $N_1\mathcal{M}$ of the $\pi_j$'s and by those units of $\mathcal{O}_F$ which are congruent to 1 modulo $N_0$. Set $H = \{ g \in (\mathcal{M}/N_1\mathcal{M})^\times \mid \text{Nm}(g) \in A \}/B$. It is isomorphic to one of the groups $\text{SL}_2(\mathcal{O}_F/N_1)$, $\text{PSL}_2(\mathcal{O}_F/N_1)$, or $\text{PGL}_2(\mathcal{O}_F/N_1)$. (The examples of Section 3 are of the $\text{PSL}_2$ type.) We now have the following

**Proposition 1.4.**

1. The vertices of $X(N)$ are the elements of $H$.
2. The (oriented) edges of direction $j$ of $X(N)$ are the pairs $(v,i_j)$, where $v \in H = \text{Ver} X(N)$ and $1 \leq i_j \leq r_j$.

In other words, each of the graphs is the Cayley graph for the same group $H$ for a different set of generators. This description is particularly convenient for explicit calculation. Let us emphasize that finite description exists even when condition (1.2) does not hold (see [12] for the form it takes in the prototypical one-dimensional case), but it is messier. By [13, Theorem 3.1(4)] the resulting graphs $\text{Gr}_i(X)$ are Ramanujan provided that appropriate spaces of cusp forms satisfy the Ramanujan-Petersson conjecture. The statement that fixing the parities gives connectedness follows from Theorem 3.1(1) there.

## 2. Hilbert modular forms

### 2.0 Hilbert-Blumenthal schemes

For the generalities on Hilbert-Blumenthal schemes which follow see [4, 21]. Let $\mathcal{H}_\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ be the union of the upper and the lower half complex planes. Let $\Sigma$ be the set of the $d$ embeddings of $F$ into a Galois closure $F^{\text{Gal}}$, itself considered as a subfield of $\mathbb{R}$. Denote by $G$ the group scheme $\text{Res}_{\mathcal{O}_F/\mathbb{Z}}\text{GL}_2$ over $\mathbb{Z}$. Then $G(\mathbb{R})$ is isomorphic to $\text{GL}_2(\mathbb{R})^\Sigma$, and $G(\mathbb{R})$ acts on $\mathcal{D} = (\mathcal{H}_\pm)^\Sigma$ componentwise through the resulting $d$ Möbius transformations. Let $U$ be a compact open subgroup of the finite adeles $G(\mathbb{A}_f) = \text{GL}_2(\mathbb{A}_f \otimes F)$. Then the Hilbert-Blumenthal complex space $X^\text{an}_U = G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/U \times \mathcal{D}$ is nonsingular if $U$ is sufficiently small. It has a natural structure of a quasi-projective variety which admits a canonical model over $\mathbb{Q}$ in the sense of Shimura ([3]). In fact, if $U$ contains the principal congruence subgroup of level $N$ and $D = \text{Disc} F$ then there is a moduli-theoretic interpretation of $X^\text{an}_U$ as
a coarse moduli space, parametrizing principally polarized abelian \(d\)-folds with an \(\mathcal{O}_F\)-action and some level \(N\) structure. This yields a model \(\mathcal{X}_U\) over \(\mathbb{Z}[1/ND]\) so that \(\mathcal{X}_U \times \mathbb{Q}\) is Shimura’s canonical model. If \(U\) is sufficiently small then \(\mathcal{X}_U\) is a fine moduli scheme, smooth and quasi-projective over \(\mathbb{Z}[1/ND]\). The Hecke algebra \(\mathbb{T}_U\) of distributions on \(U \backslash G(\mathbb{A}_f)/U\) acts on \(\mathcal{X}_U \times \mathbb{Q}\) through correspondences. We shall assume that \(U\) is a product \(U_p \cup p\) of a part \(U_p \subset G(\mathbb{Q}_p)\) at \(p\) and a part \(U_\infty \subset G(\mathbb{A}_f^p)\) away from \(p\). Then an element of \(\mathbb{T}_U\) coming from \(U_p \backslash G(\mathbb{Q}_p)/U_p\) (which we will call a Hecke operator at \(p\)) acts on \(\mathcal{X}_U \times \mathbb{Z}[1/NDp]\) through its standard moduli interpretation.

Now let \(p\) be a rational prime not dividing \(ND\). Then \(\mathcal{X}_U\) has good reduction modulo \(p\). In \([14]\) Langlands computed the number of fixed points \(T \times \text{Frob}_p\) on \(\mathcal{X}_U \times \overline{\mathbb{F}}_p\) for any Hecke operator \(T\) away from \(p\), i.e. \(T\) coming from \(U_\infty \backslash G(\mathbb{A}_f)/U_\infty\). In fact Langlands allows more general groups than \(G_{\text{GL}_2}\) and he also allows algebraic local system on \(\mathcal{X}_U\). This second generalization enables one to handle cusp forms of weight \((k_1, \ldots, k_d)\) rather than \((2, \ldots, 2)\). Even though the trivial local system in the \(G_{\text{GL}_2}\) case is all that we need here, there is no advantage in restricting to it, and it has a potential use for the Ramanujan local systems of \([13]\). On the other hand we decided to restrict to the \(G_{\text{GL}_2}\) case so as to simplify our exposition by avoiding all mention of \(L\)-packets and endoscopy. The interested reader can see \([2, 14]\) that this restriction is unnecessary. Thus let \(\xi\) be an irreducible algebraic representation of \(G\) (all are defined over the Galois closure \(F^{\text{Gal}}\) of \(F\)). We will always assume that \(U\) is sufficiently small and that \(\xi\) is trivial on the \((\text{Zariski closure})\) of the units \(\mathbb{Z}(\mathcal{O}_F)\) in the center \(\mathbb{Z}\) of \(G\). Then \(\xi\) defines a local system

\[V_{\xi}^{\text{an}} = G(\mathbb{Q}) \backslash (G(\mathbb{A}_f)/U \times \mathcal{D} \times \xi)\]

on \(\mathcal{X}_U^{\text{an}}\). The possible \(\xi\)'s can be described explicitly: identify \(G \times F \simeq \text{GL}_2^d\) and let \(\text{Symm}^k\) be the representation of \(\text{GL}_2\) on the polynomials of degree \(k \geq 0\) in two variables. Then \(\xi\) is a tensor product \(\otimes_{i=1}^d \xi_i\) of irreducible representations \(\xi\) of the \(\text{GL}_2\) factors, and \(\xi_i \simeq \text{Symm}^{k_i-2} \otimes \text{det}^{\alpha_i}\), with \(k_i \geq 2\). The condition of being trivial on the units of \(\mathbb{Z}\) translates into \(k_i - 2 + 2\alpha_i = C\) with \(C\) a constant. The \(k_i\)'s must all have the same parity as \(C\), and \(\alpha_i = (C + 2 - k_i)/2\). Observe that the central character of \(\xi\) maps \(t = (t_1, \ldots, t_\ell)\) to \(\text{Nm}(t)^C\), where \(\text{Nm}(t) = t_1 \cdots t_\ell\).

The \(\xi\)'s above have a moduli theoretic interpretation and hence \(\ell\)-adic analogs \(V_{\xi, \lambda}\), which are lisse sheaves of 2-dimensional \(F^{\text{Gal}}\) vector spaces over \(\mathcal{X}_U[1/\ell]\). Here \(\lambda\) is any prime of \(F^{\text{Gal}}\) lying above \(\ell\). For this let \(f: \mathcal{A} \rightarrow \mathcal{X} = \mathcal{X}_U[1/\ell]\) be the universal abelian variety (which we
may assume exists by adding auxiliary level structure). Then \( R^1 \mathcal{F}_\ell \) is a lisse sheaf of \( F \otimes \mathcal{O}_\ell \)-modules of rank 2. For every \( \sigma \in \Sigma \) we have an \( F \otimes \mathcal{O}_\ell \)-action on \( F^\text{Gal}_\sigma \), and we set \( \mathbf{V}_{\sigma, \lambda} = R^1 f_* \mathcal{F}_\ell \otimes F^\text{Gal}_\sigma \mathcal{F}^\text{Gal}_\lambda \), with the tensor product taken relative to this action and. Then for \( \xi \) as before we set \( \mathbf{V}_\xi, \lambda = \otimes_i (\text{Symm}^i \mathbf{V}_{\sigma_i, \lambda} \otimes \det \mathbf{V}_{\sigma_i, \lambda}) \) with \( \sigma_i \in \Sigma \) the embedding corresponding to the index \( 1 \leq i \leq d \). The moduli description gives that \( \mathbf{V}_\xi, \lambda \) is pure (in the sense of \([6]\)) of weight \( \sum_{i=1}^d (k_i - 2 + 2\alpha_i) = dC \).

2.1 The dual group

Let \( p \) be a rational prime not dividing \( \text{Disc} \ F \) and fix some embedding of \( \mathbb{R} \) into \( \overline{\mathbb{Q}}_p \). Then \( G(\mathbb{Q}_p) \simeq \prod_{v \mid p} G_v(\mathbb{Q}_p) \), where \( G_v = \text{res}_{F_v/\mathbb{Q}_p} \text{GL}_2 \). The set \( \Sigma \) is a disjoint union \( \sqcup_v \Sigma_v \), where \( \Sigma_v \) is the set of \( d_v = [F_v : \mathbb{Q}_p] \) embeddings of \( F_v \) in \( \overline{\mathbb{Q}}_p \). The connected component \( L^v G_v(\mathbb{Q}_p)^0 \) of the Langlands dual to \( G_v(\mathbb{Q}_p) \) is then the quotient of \( \text{GL}^\Sigma_2 \) by the \( \Sigma_v \)-tuples \((\ldots, z_\sigma, \ldots)_{\sigma \in \Sigma_v} \) of scalar matrices whose product is the identity. Similarly \( L^v G(\mathbb{Q}_p)^0 \) is the quotient of \( \text{GL}^\Sigma_2 \) by the \( \Sigma \)-tuples \((\ldots, z_\sigma, \ldots)_{\sigma \in \Sigma} \) of scalar matrices whose product is the identity. In particular, \( L^v G(\mathbb{Q}_p)^0 \) is a quotient of \( \prod_{v \mid p} L^v G_v(\mathbb{Q}_p)^0 \).

The Galois group \( \text{Gal}_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) acts naturally on each \( \Sigma_v \) and hence on their union \( \Sigma \), and it is clear that we get an action of \( \text{Gal}_{\mathbb{Q}_p} \) on \( L^v G_v(\mathbb{Q}_p) \) and on \( L^v G(\mathbb{Q}_p) \). Then \( L^v G_v(\mathbb{Q}_p) \) is the semidirect product \( L^v G_v(\mathbb{Q}_p)^0 \rtimes \text{Gal}_{\mathbb{Q}_p} \) and likewise \( L^v G(\mathbb{Q}_p) = L^v G(\mathbb{Q}_p)^0 \rtimes \text{Gal}_{\mathbb{Q}_p} \). It is clear that \( L^v G_v(\mathbb{Q}_p)^0 \) acts naturally on \( \mathbb{C}^{2d_v} \) and that this extends to an embedding \( r_v \) of \( L^v G_v(\mathbb{Q}_p) \) into \( \text{GL}_{2d_v}(\mathbb{C}) \). Likewise we get an injective representation

\[ r : L^v G(\mathbb{Q}_p) \to \text{GL}_{2d}(\mathbb{C}). \]

Now let \( \pi_p \) be an irreducible admissible representation of \( G(\mathbb{Q}_p) \) unramified at \( p \). Then \( \pi_p \) is a product \( \prod_{v \mid p} \pi_v \) of unramified principal series representations \( \pi_v \simeq \pi(\mu_{1,v}, \mu_{2,v}) \) of \( G_v(\mathbb{Q}_p) = \text{GL}_2(F_v) \), with \( \mu_{1,v} \) unramified characters of \( F_v^\times \) (see \([1] \text{ Ch. I.3}\)). We will always assume that each \( \pi(\mu_{1,v}, \mu_{2,v}) \) is infinite-dimensional. Let \( \varpi_v \) be a uniformizer at \( v \) and put \( a_v = \mu_{1,v}(\varpi_v) \) and \( b_v = \mu_{2,v}(\varpi_v) \). Since the central character of \( \pi_v \) maps each \( t \in F_v^\times \) to \( t^C \), it follows that \( |a_v| b_v| = p^{d_v C} \). We shall be interested in the size \( |\lambda_v| \) of the Hecke eigenvalue \( \lambda_v = a_v + b_v \) on \( \pi_v \). Recall that \( \pi_p \) is tempered if and only if all the \( \pi_v, v \mid p \) are. This means that \( |a_v| = |b_v| \) for all \( v \mid p \). If this holds — which is the assertion of the Ramanujan-Petersson conjecture if \( \pi_p \) is a local component of a cuspidal automorphic representation — then \( |\lambda_v| \leq 2p^{d_v C/2} \).
Set \( t_v = \begin{bmatrix} a_v & 0 \\ 0 & b_v \end{bmatrix} \). Then the Satake isomorphism associates to \( \pi_v \) the conjugacy class of the image in \( L G_v(\mathbb{Q}_p) \) of the element

\[ t(\pi_v) = (0^t_v, \text{Frob}_p) \in GL_2(\mathbb{C})^d_v \times \text{Gal}_{\mathbb{Q}_p}, \]

where \( 0^t_v = (0, 1, \ldots, 1, 0) \). (see e.g. [21].) It follows that the Satake isomorphism associates to \( \pi_p \) the element

\[ t(\pi_p) = (0^t_p, \text{Frob}_p) \in L G(\mathbb{Q}_p)^0 \times \text{Gal}_{\mathbb{Q}_p}, \]

where \( 0^t_p \) is the image of \( (\ldots, 0^t_v, \ldots) \) in \( L G(\mathbb{Q}_p)^0 \).

We now recall the formula for the trace of \( r(t(\pi_p))^m \) for any integer \( m \geq 1 \). Put \( l_v = \gcd(d_v, m) \), and write \( m = q_v d_v + r_v \) with integers \( q_v \geq 0 \) and \( 0 \leq r_v < d_v - 1 \). We then have

\[ t'(\pi_v)^m = \left(\prod_{v \leq r_v} \frac{q_v + 1}{q_v}, \ldots, \frac{q_v + 1}{q_v}, 0, \ldots, 0^t_v, \right) \text{Frob}_p^m. \]

A straightforward calculation (\cite{14} or loc. cit.) then gives

(2) \[ \text{Tr} \ r(t(\pi_p))^m = \prod_v \text{Tr} \ r_v(t(\pi_v))^m = \prod_v (a_v^m + b_v^m)^{l_v}. \]

### 2.2 The naive Lefschetz number

Write \( U = U^p U_p \) with \( p \) as before, so that in particular \( U_p \subset G(\mathbb{Q}_p) \) and \( U^p \subset G(\mathbb{A}_f^p) \). Let \( m \geq 0 \) be an integer and take \( a \in G(\mathbb{A}_f^p) \). Following [14], one defines functions \( f_\xi \) on \( G(\mathbb{R}) \), \( \phi_a = \phi_{U^p a U^p} \) on \( G(\mathbb{A}_f^p) \), \( h_p^m \) on \( G(\mathbb{Q}_p) \) and \( f^G(m, a) = f_\xi \phi_a h_p^m \) on \( G(\mathbb{A}) = G(\mathbb{R}) G(\mathbb{A}_f^p) G(\mathbb{Q}_p) \) having the following properties:

1. \( \text{Tr} \pi_\infty(f_\xi) \) is the multiplicity of \( \xi \) in \( \pi_\infty \) for each representation \( \pi_\infty \) of \( G(\mathbb{R}) \);
2. \( \phi_a \) is the characteristic function of \( U^p a U^p \);
3. \( \text{Tr} \pi_p(h_p^m) = p^{md/2} \text{Tr} r(t(\pi_p))^m \) for each infinite-dimensional unramified representation \( \pi_p \).

Let \( T(a) \) be the Hecke operator \( T(a) \) associated to \( U^p a U^p \). Then \( T(a) \times \text{Frob}_p^m \) acts as a correspondence on \( X_U \times \mathbb{F}_p \). If \( m \) is sufficiently large then the set of fixed points of this correspondence is finite, and its graph on \( (X_U \times \mathbb{F}_p)^2 \) is transversal to the diagonal. This enables to define its Lefschetz number by

\[ \text{Lef}(T(a) \times \text{Frob}_p^m, X_U \times \mathbb{F}_p, \mathbb{V}_{\xi, \lambda}) = \sum_{t} \text{Tr} (T(a) \times \text{Frob}_p^m, (\mathbb{V}_{\xi, \lambda})_t), \]

the sum taken over the (finite) fixed point set of \( T(a) \times \text{Frob}_p^m \) on \( S(X_U \times \mathbb{F}_p) \).
Let $\omega_\xi$ be the central character $\omega_\xi(t_1, \ldots, t_d) = (t_1 \ldots t_d)^C$ of $\xi$, and let $L_2^\text{dis}(G(\mathbb{Q})(Z(A_f) \cap U) \backslash G(\mathbb{A}), \omega_\xi)$ be the space of functions $f$ on $G(\mathbb{A})$ satisfying

1. $f(g_Q z_U z_\infty g) = \omega_\xi(z_\infty)^{-1} f(g)$ for all $g_Q \in G(\mathbb{Q})$, $z_U \in Z(A_f) \cap U$, $z_\infty \in Z(\mathbb{R})$ and $g \in G(\mathbb{A})$.

2. $f| \det |^{C/2}$ is square integrable on $Z(\mathbb{R})(Z(A_f) \cap U) G(\mathbb{Q}) \backslash G(\mathbb{A})$.

As the notation indicates, $L_2^\text{dis}(G(\mathbb{Q})(Z(A_f) \cap U) \backslash G(\mathbb{A}), \omega_\xi)$ is a discrete sum of representations of $G(\mathbb{A})$.

Using the moduli interpretation (see [17] and [21, Exposé V]), the results of Langlands (loc.cit.) together with [2, Section 3.3] give the following key

**Theorem 2.1.** We have

$$\text{Lef}(T(a) \times \text{Frob}_p^a, \mathcal{X}_U \times \overline{\mathbb{F}}_p, V_{\xi, \lambda}) = \sum \text{Tr} \Pi(f^G(m, a)),$$

where the sum is over all irreducible representations $\Pi$ of $G(\mathbb{A})$ which occur in the discrete spectrum $L_2^\text{dis}(G(\mathbb{Q})(Z(A_f) \cap U) \backslash G(\mathbb{A}), \omega_\xi)$.

(In this formula we are implicitly using the strong multiplicity one theorem for $GL_2$.)

**2.3 The true Lefschetz number** The problem now is to identify Langlands’s “naive” Lefschetz number as the sum of the local terms in an actual Lefschetz trace formula. An important technical issue is the contribution of the boundary. A conjecture of Deligne asserts it is 0 for $T(a) \times \text{Frob}_p^a$, with a given $a$, if $p$ is sufficiently large. In his thesis Rapoport constructed smooth (toroidal) compactifications of $\mathcal{X}_U$ over $\mathbb{Z}[1/ND]$ in which the complement of $\mathcal{X}_U$ is a relative normal crossing divisor [21, Corollaire 5.3]. For all but finitely many primes $p$ there also exists the Baily-Borel compactification, whose boundary is the finite set of cusps, and the toroidal compactification is a blow-up of it. At such primes $p$, Brylinski and Labesse could prove Deligne’s conjecture ([2, Theorem 2.3.3]). According to C.-L. Chai (private communication) this holds for all $p$ prime to $ND$. Alternatively we can use the results of [19], since Condition 7.2.1 there, which suffices for Deligne’s conjecture, holds for these compactifications, provided we know also that the monodromy of our sheaves around the toroidal resolutions of the cusps are tame. This is manifest from the higher dimensional Mumford-Raynaud-Tate parametrization near the cusps [20, Théorème 5.1 and 4.11]. The idea is that the $\ell$-adic Tate module of a universal abelian variety in a punctured $p$-adic analytic neighborhood of a cusp admits a canonical exact sequence having an étale quotient and a multiplicative sub-object (which are moreover Cartier dual to one...
COMMUNICATION NETWORKS AND HILBERT MODULAR FORMS 11

another). The ℓ-adic monodromy is therefore manifestly tame at p. Finally, we could appeal to [9], where Deligne’s conjecture is proved in general. Either way we get the following

**Corollary 2.2.** For every a as above there exists an integer m_0(a) such that for each integer m ≥ m_0(a) we have

\[
\operatorname{Lef}(T(a) \times \text{Frob}_p, X_U \times \mathbb{F}_p, V_{\xi, \lambda}) = \text{Tr}(T(a) \times \text{Frob}_p^m|H^*_e(X_U \times \mathbb{F}_{\ell}, V_{\xi, \lambda}),
\]

where as usual \(\text{Tr}(\cdot|H^*_e)\) means the alternating sum \(\sum_{i=0}^{2d} (-1)^i \text{Tr}(\cdot|H^*_i)\).

### 2.4 Cuspidality and compact support

Assume that \(\pi = \otimes_p \pi_p\) is an irreducible cuspidal representation of \(G(\mathbb{A})\). In particular each \(\pi_p\) (and \(\pi_v\)) is infinite dimensional, and \(\pi \otimes \det^{-C/2}\) is unitary. Then \(\pi\) contributes to \(H^*(X_U, V_{\xi})\) if and only if \(\pi\) has a \(U\)-invariant vector and if \(\pi_\infty\) is isomorphic to

\[
\prod_{i=1}^{d} \pi(\cdot, (1-k_i)/2, |\cdot|^{(k_i-1)/2} \text{sign}^{k_i}) \otimes \det C.
\]

We will then say that \(\pi\) is of type \(\xi\). In this case \(\pi\) contributes to \(H^i(X_U, V_{\xi})\) if and only if \(i = d\). In fact taking the inverse limit over \(U\) we set

\[
U^d(\pi_f, \xi) = \operatorname{Hom}_{G(\mathbb{A}_f)}(\pi_f, \lim_{\leftarrow} H^d(X_U, V_{\xi})).
\]

Then \(U^d(\pi_f, \xi)\) is a \(2^d\)-dimensional space, (see [2, Section 3.4], and notice that by the strong multiplicity 1 theorem for \(\text{GL}_2\) we do not have multiplicities in our case). The ℓ-adic analog likewise gives a \(2^d\)-dimensional \(\text{Gal}_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) representation \(U^d_\lambda(\pi_f, \xi)\). We now need the following

**Proposition 2.3.** If \(\pi\) is a cusp form as above, then \(U^d(\pi_f, \xi)\) comes from the compactly supported cohomology; in other words, the natural map of forgetting supports

\[
\operatorname{Hom}_{G(\mathbb{A}_f)}(\pi_f, \lim_{\leftarrow} H^d_e(X_U, V_{\xi})) \to U^d(\pi_f, \xi)
\]

is an isomorphism.

**Proof:** See [1, Cor. 5.5] (and also the last comment in [10]). Here is the general idea of the proof. For any cusp \(c\) of \(X_U\) let \(X_{U,c}\) be the boundary component corresponding to \(c\) in the Borel-Serre compactification of \(X_U\). The cuspidality of \(\pi\) implies that the (de-Rham) cohomology classes of the vector-valued differential forms \(\{ \omega \} \) associated to it restrict to 0 on each \(X_{U,c}\). This can be seen also by verifying
that the periods $\int_\beta \omega$ of these differential forms around each (vector-valued Borel-Moore) $d$-dimensional cycle $\beta$ is 0. In fact, we can move $\beta$ towards the cusp; then on the one hand the period stays constant, but on the other hand the cuspidality of $\pi$ makes the $\omega$ and with it the period decrease exponentially. This forces the period to be 0, and we conclude that our $\{\omega\}'s$ are cohomologous to compactly supported forms.

As a corollary we get that $U^d(\pi_f, \xi)$, which initially was derived from $H^d(\mathcal{X}_U, \mathbb{V}_\xi)$, is in the image of the “forget support” map from $H^d_c(\mathcal{X}_U, \mathbb{V}_\xi)$. This image is the parabolic cohomology $\tilde{H}^d(\mathcal{X}_U, \mathbb{V}_\xi)$, and it exists in $\ell$-adic cohomology. As the sheaf $\mathbb{V}_{\xi, \lambda}$ is pure of weight $d_C$, a weight argument (see [6]) gives that $\tilde{H}^d(\mathcal{X}_U, \mathbb{V}_\xi)$ is pure of weight $d(C + 1)$. The same is therefore true for $U^d_{\lambda}(\pi_f, \xi)$.

2.5 The Ramanujan-Petersson Conjecture

We partially recall our previous notation. With $\xi$ as before (in particular, of central character $\omega_\xi = Nm_C$), let $\pi = \prod_p \pi_p$ be an irreducible cuspidal representation of $G(\mathbb{A})$ of infinity type $\xi$, so that $\pi_\infty$ is the discrete series representation corresponding to $\xi$ as above. Let $p$ be a rational prime which is prime to the level of $\pi$ and to $Disc F$. For a prime $v$ of $F$ above $p$ let $d_v$ denote the degree of $F_v/Q_p$, and let $T_v$ be the Hecke operator at $v$, defined as the action of the double class $U_v \begin{bmatrix} \varpi_v & 0 \\ 0 & 1 \end{bmatrix} U_v$, with $U_v \simeq \text{GL}_2(O_{F,v})$ having measure 1. Here $\varpi_v$ is a uniformizer at $v$. The Satake parameters of $\pi_v$ are given as before in terms of $a_v$ and $b_v$ (up to order). The eigenvalue of $T_v$ on $\pi$ (or on $\pi_v$) is $\lambda_v = a_v + b_v$. We now obtain the following version of the Ramanujan-Petersson Conjecture:

**Theorem 2.4.** The representations $\pi_p$ and $\pi_v$ for $v|p$ are tempered. More precisely, each of $a_v$, $b_v$ is a Weil number of weight $d_v + C$, i.e. $a_v$ and $b_v$ are algebraic numbers such that under any embedding into $\mathbb{C}$ we have

$$|a_v| = |b_v| = p^{d_v C/2}.$$ 

Consequently, $|\lambda_v| \leq 2p^{d_v C/2}$.

**Proof:** Using our previous notation, there exists a finite linear combination $T = \sum c_i T(g_i)$ of Hecke operators away from $p$ which acts as projection onto the $U^p$ invariants of the $\pi_f^p$-isotypical part. By the strong multiplicity one theorem this is a projection onto the $U^p$ invariants of the $\pi_f$ isotypical part. Let $m_0$ be an integer such that Deligne’s conjecture holds for $T(g_i) \times \text{Frob}_p^m$ on $\mathcal{X}_U \times \mathbb{F}_p$ for all $i$ and for all $m \geq m_0$. From Theorem 2.1 and Corollary 2.2 applied to $T \times \text{Frob}_p^m$,
with $T$ viewed as the above linear combination, and from the definition of $U^d_{\lambda}(\pi_f, \xi)$ we get
\[ \text{Tr}(\text{Frob}_p^m|U^d_{\lambda}(\pi_f, \xi)) = \text{Tr} \pi_p(h^m_p). \]

The purity of the parabolic cohomology and the property of $h^m_p$ from Section 2.3 give the existence of $2^d = \dim U^d_{\lambda}(\pi_f, \xi)$ Weil numbers $\{w_i\}_i$ of weight $d(C + 1)$ so that for every large enough $m$ we have
\[ \sum_i w_i^m = p^{md} \text{Tr} r(t(\pi_p)^m) = p^{md} \prod_{v|p} (a_v^m/l_v + b_v^m/l_v)^{l_v}, \]
by equation (2), where as before $l_v = \gcd(m, d_v)$. As in [2, Theorem 3.4.6] this implies that each $\pi_v$ is tempered, namely that $|a_v| = |b_v|$ for every embedding of $a_v, b_v$ into $\mathbb{C}$. That the weight is as it should be then follows from our knowledge of the product $|a_v b_v|$, as was explained in Section 2.2.

Theorem 0.1 follows when one makes the necessary normalizations. In particular one takes $C = \max_i k_i$.

3. Special cases

We now give two special cases when the conditions 1.2 are satisfied. The first one was accessible in [13], while the second one was not.

A. Take $F = \mathbb{Q}$ and let $B$ be a quaternion algebra over $\mathbb{Q}$ of discriminant 2 (it is unique up to an isomorphism). A model for $B$ is given by the usual rational quaternion algebra generated over $\mathbb{Q}$ by $\tilde{i}, \tilde{j}$ satisfying $\tilde{i}^2 = \tilde{j}^2 = -1$ and $\tilde{i} \tilde{j} = -\tilde{j} \tilde{i}$. It is very easy to see that Conditions 1.2 are satisfied with $N_0 = 2\mathbb{Z}$. Taking $g = 1$ prime $p$ satisfying $p \equiv 1 \mod 4$, one gets the Lubotzky-Phillips-Sarnak graphs [16]. These are Cayley graphs on $\text{PSL}(2, F_N)$, for any prime $N \equiv 1 \mod 4$ which is different from $p$ and $a$ a square modulo $p$. The set of generators is the reductions modulo $N$ of the $p + 1$ norm $p$ integral quaternions $\equiv 1 \mod 2$.

The case of $g = 2$ distinct primes $p, q$ satisfying $p \equiv q \equiv 1 \mod 4$ appeared in a different context in [18]. The resulting two (blue and red) communication networks are realized as the same Cayley graphs as before on $\text{PSL}(2, F_N)$, for any prime $N \equiv 1 \mod 4$ which is different from $p$ and $q$ and is a a square modulo both. The sets of generators are the reductions modulo $N$ of the $p + 1$ (respectively $q + 1$) norm $p$ (respectively norm $q$) integral quaternions $\equiv 1 \mod 2$. They have the square property. Thus each node has $p + 1$ blue neighbors and $q + 1$ red ones.
B. We shall now construct a blue and a red Ramanujan communication networks having the square property over the same set of vertices PSL(2, \(\mathbb{F}_N\)) in which each node has \(p + 1\) blue and \(p + 1\) red neighbors. Here \(p\) is any prime \(\equiv 1, 9 \mod 20\) and \(N \neq p\) is a prime satisfying \(N \equiv 1, 9 \mod 20\) and which is a square modulo \(p\). There are other conditions which \(N\) must satisfy. These cannot be stated as simply, but we will see that they are satisfied by a set of primes \(N\) of positive density which can be explicitly described.

Take \(F = \mathbb{Q}(\sqrt{5}) \subset \mathbb{R}\). Let \(\infty_1 = \text{Id}, \infty_2\) be the real primes of \(F\). The fundamental unit of \(F\) is \(\tau = (1 + \sqrt{5})/2\). We will need the following lemma from Class Field Theory:

**Lemma 3.1.** 1. Let \(N\) be a rational prime such that \((\frac{4}{N}) = (\frac{5}{N}) = 1\) (in other words \(N \equiv 1, 9 \mod 20\)). It is then possible to write \(N = a^2 - 20b^2\) with \(a, b\) integers and \(a > 0\) odd. Then \(\tau\), considered modulo \(N\) via either choice of \(\sqrt{5}\) mod \(N\), is a square modulo \(N\) if and only if \(a + 2b \equiv 1\) mod 4, and this holds for a set of primes \(N\) of Dirichlet density 1/8.

2. Let \(p \neq N\) be a rational prime satisfying with \((\frac{5}{p}) = (\frac{1}{p}) = 1\) and use part 1. to write \(p = a^2 - 20b^2\) for integers \(a, b\). Choose a \(\sqrt{5}\) modulo \(N\). Then the set of primes \(N\) satisfying the criterion in part 1. and for which both \(a \pm 2b\sqrt{5}\) are squares modulo \(N\) has Dirichlet density 1/32. Such primes \(N\) satisfy in particular \((\frac{5}{N}) = 1\).

**Proof:** If 5 is a square modulo \(N\) then \(N\) splits in \(F\), and since the class number of \(F\) is 1 we get \(\pm N = \nu\overline{\nu}\), with \(\nu\) and \(\overline{\nu}\) conjugate integers of \(F\). Since \(-1\) is the norm of \(\tau\) and since \(\tau\) reduces modulo 2 (which is inert in \(F\)) to a generator of \(\mathbb{F}_4^\times\), with \(\mathbb{F}_4 = \mathcal{O}_F/2\mathcal{O}_F\), we can assume that the sign is + and that \(\nu \equiv 1\) mod 2\(\mathcal{O}_F\). Writing \(\nu = a + b\sqrt{5}\) we get that \(a\) and \(b\) are integers with \(a\) odd and \(b' = 2b\) even. Multiplying by \(-1\) we may also assume that \(a > 0\). A choice of a square root of 5 modulo \(N\) then determines the sign of \(b\), or equivalently the prime \(\nu = a + 2b\sqrt{5}\) above \(N\). We then ask when is the ideal \(\nu\mathcal{O}_F\) split in \(L = F(\sqrt{\tau})\). Since \(N \equiv 1\) mod 4, it splits in \(\mathbb{Q}(\sqrt{-1})\), so that \(\nu\) (and \(\overline{\nu}\)) split in \(F(\sqrt{-1})\). Therefore \(\nu\) splits in \(F(\sqrt{\tau})\) if and only if \(N\) is completely split in \(\mathbb{Q}(\sqrt{5}, \sqrt{-1}, \sqrt{\tau})\), which is Galois of degree 8 over \(\mathbb{Q}\). By the Čebotarev density theorem, this happens for a set of primes \(N\) of Dirichlet density 1/8 (namely, for half the primes \(\equiv 1, 9 \mod 20\)). It remains to determine the condition for \(\nu\) to split in \(F(\sqrt{\tau})\).

The Idèle class group character \(\chi\) of \(F\) which cuts \(L\) is of order 2 and unramified outside 2. Since \(\infty_1(\tau) > 0\) and \(\infty_2(\tau) < 0\), we see that \(\chi\) is trivial on \(F_{\infty_1}^\times\) but not on \(F_{\infty_2}^\times\). For a prime \(v\) of \(F\) set \(U_v = \mathcal{O}_{F,v}^\times\).
and let $U_{v,n}$ be its subgroup of elements congruent to 1 modulo $v^n$ for any $n \geq 1$. Then $\chi$ must be trivial on the product $U_\infty$ of the connected components of $F^\times_i$ and on $U = \prod_{v \neq 2} U_v$. The units $O_F^\times = \pm \tau^\mathbb{Z}$ surject onto $U_2/U_{2,1} \times \prod_i \pi_0(F^\times_i)$, and the kernel of this surjection is $\tau^{6\mathbb{Z}}$. Since the class number of $F$ is 1, we get that $\chi$ is determined by its values on

$$C = F^\times \backslash \mathbb{A}_F^\times / UU_\infty \simeq \tau^{6\mathbb{Z}} \backslash U_{2,1}/(U_{2,1})^2.$$ 

First we calculate $C$. Through the 2-adic logarithm map, $U_{2,1} \simeq \mu(F) \oplus L$, where $L = 2\mathbb{Z}_2 + 4O_{F,2}$ and $\mu(F) = \{ \pm 1 \}$ is the group of the roots of unity of $F$. Since $\tau^6 = 9 + 4\sqrt{5} \equiv 5 \mod 8O_F$ we get that

$$C \simeq \mu(F) \oplus 2O_F/(2\mathbb{Z} + 4O_F) \simeq (\mathbb{Z}/2\mathbb{Z})^2,$$

with generators $-1$ and $2\tau$ for the respective two factors.

To determine $\chi$ we use the product formula $\prod_v \chi_v(x) = 1$ for any $x \in F^\times$, the product taken over all places of $F$. For $x = -1$ we get

$$\chi(-1) = \chi_2(-1) = \chi_{\infty_2}(-1) = -1,$$ 

and since $\tau^3 = 2 + \sqrt{5} = 1 + 2\tau$, we also get

$$\chi(2\tau) = \chi_2(1 + 2\tau) = \chi_2(\tau^3) = -1.$$ 

In particular, for $x, y \in \mathbb{Z}_2$ we have $\chi_2(1 + 2(x + y\tau)) = (-1)^{x+y}$. It follows the prime $\nu O_F$ splits in $F(\sqrt{\tau})$ if and only if $\chi_\nu(\nu) = -1$. But

$$\chi_\nu(\nu) = \chi_2(\nu)\chi_{\infty_2}(\nu) = (-1)^{(a-1)/2+b} \cdot 1 = (-1)^{(a-1)/2+b}.$$ 

As claimed. (We write $\chi_\nu$ etc. for the local component of $\chi$ at the place defined by the prime $\nu O_F$.)

2. Write $p = \pi \overline{\pi}$, with (say) $\pi = a + 2\sqrt{b}$. Observe that the field $F(\sqrt{\tau}, \sqrt{\overline{\tau}})$ is Galois over $\mathbb{Q}$ and contains $\sqrt{p}$. As an extension of $F$ it is ramified only over $\pi$ and $\overline{\pi}$: this is because $(1 + \sqrt{\tau})/2$ is an algebraic integer, as its trace and norm to $F$ are 1 and $(1 - a)/2 - b\sqrt{5}$. Hence it is linearly disjoint from $F(\sqrt{-1}, \sqrt{\tau})$ over $F$. Applying the Čebotarev density theorem to the degree 32 extension

$$\mathbb{Q}(\sqrt{-1}, \sqrt{5}, \sqrt{p}, \sqrt{\tau}, \sqrt{\overline{\tau}})$$

of $\mathbb{Q}$ gives the required density statement, concluding the proof of the lemma.

Now set $B_F = B \otimes F$, with $B$ the rational quaternions as before. For the facts we need about $B_F$ see [22, Chapter 5]. In particular, $B_F$ is ramified precisely at the two infinite primes of $F$.

The class number of $B_F$ is 1, and all maximal orders in $B_F$ are conjugate. To describe the sets of generators of our red and blue networks
we need to explicitly compute in one maximal order \( \mathcal{M} \). We take for \( \mathcal{M} \) the \( \mathcal{O}_F \)-submodule of \( B_F \) generated by

\[
\begin{align*}
e_1 &= (1 + \tau^{-1} \hat{\imath} + \tau \hat{j})/2 \\
e_2 &= (\tau^{-1} \hat{\imath} + \hat{j} + \tau \hat{k})/2 \\
e_3 &= (\tau \hat{i} + \tau^{-1} \hat{j} + \hat{k})/2 \\
e_4 &= (\hat{i} + \tau \hat{j} + \tau^{-1} \hat{k})/2
\end{align*}
\]

(3)

We have \( \mathcal{M}/2\mathcal{M} \cong \text{Mat}_{2 \times 2}(\mathbb{F}_4) \), with \( \mathbb{F}_4 = \mathcal{O}_F/2\mathcal{O}_F \) a field with 4 elements. The reduction map induces an exact sequence

\[
1 \rightarrow (\mathcal{O}_F + 2\mathcal{M})^\times \rightarrow \mathcal{M}^\times \rightarrow \text{PGL}(2, \mathbb{F}_4) \rightarrow 1.
\]

From our choice of \( \mathcal{M} \) it follows that \( \mathcal{O}_F + 2\mathcal{M} \) is the \( \mathcal{O}_F \)-sub-order given by

\[
\{ a + b \hat{\imath} + c \hat{j} + d \hat{k} \mid |a, b, c, d \in \mathcal{O}_F, b + \tau c + \tau^{-1} d \in 2\mathcal{O}_F \}.
\]

Since \( \tau \) reduces modulo 2 to a generator (of order 3) of \( \mathbb{F}_4^\times \), we obtain the following

**Lemma 3.2.** Every element in \( \mathcal{M} \) whose norm to \( \mathcal{O}_F \) is odd, namely not in \( 2\mathcal{O}_F \), can be uniquely written as \( u(a + b \hat{\imath} + c \hat{j} + d \hat{k}) \), with \( u \in \mathcal{M}^\times \) and \( a, b, c, d \in \mathcal{O}_F \) satisfying

1. \( b + \tau c + \tau^{-1} d \in 2\mathcal{O}_F \);
2. \( a \equiv 1 \mod 2\mathcal{O}_F \);
3. \( 1 \leq a < \tau^3 \).

When \( u = 1 \) we shall say that our element is in normal form.

As a corollary we see that Conditions [1,2] hold with \( N_0 = 2\mathcal{O}_F \).

Now write \( 4p = a^2 - 20b^2 \), with \( a, b \) integers (say, minimal positive), necessarily of the same parity. The elements \( \pi = (a + 2b\sqrt{5})/2 \) and \( \overline{\pi} = (a - 2b\sqrt{5})/2 \) are the primes above \( p \) in \( \mathcal{O}_F \). Let \( \mathcal{M}_{0,F} = \mathcal{O}_F[\hat{i}, \hat{j}] \) be the order of \( \mathcal{O}_F \)-integral quaternions. We shall say that an element \( x \in \mathcal{M}_{0,F} \) is in normal form if \( x \equiv 1 \mod 2 \) and \( \text{Tr} x = \text{Tr}_{B_F/F} x \) is totally positive (namely positive for both real embeddings) and satisfies \( \tau^{-3} < \text{Tr} x \leq \tau^3 \). We then have the following

**Lemma 3.3.** 1. There are precisely \( p + 1 \) elements in normal form \( \gamma_i \) (respectively \( \overline{\gamma}_i \)), \( 1 \leq i \leq p + 1 \), in \( \mathcal{M}_{0,F} \) whose norm is \( \pi \) (respectively \( \overline{\pi} \)).

2. For any indices \( 1 \leq i, j \leq p + 1 \) there exist unique indices \( 1 \leq i', j' \leq p + 1 \) and a unique unit \( u \in \mathcal{M}_{0,F}^\times = \langle \pm \pi^{3/2} p^{1/2} \rangle \) such that

\[
\gamma_i \overline{\gamma}_j = \overline{\gamma}_{j'} \gamma_{i'} u.
\]
Proof: This is merely an explicit restatement of Proposition 1.3.

Now let \( N \) be a rational prime congruent to 1 or 9 modulo 20. Choosing square roots of \(-1\) and of 5 in the prime field \( \mathbb{F}_N \) allows us to map \( \mathcal{M} \) homomorphically onto \( \text{Mat}_{2 \times 2}(\mathbb{F}_N) \) by

\[
\begin{align*}
\sqrt{5} & \mapsto \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \\
i & \mapsto \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \text{ and } j & \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]

Proposition 1.4 now gives us that we have constructed two Ramanujan communication networks as the two Cayley graphs on \( \text{PSL}(2, \mathbb{F}_N) \) having the reductions modulo \( N \) of the \( \gamma_i \)'s (respectively the \( \overline{\gamma}_j \)'s) for generators.

And this is my Appendix.

References

[1] A. Borel, Stable real cohomology of arithmetic groups II. In Prog. in Math. 14, Birkhauser 1990, Boston, pp 21–55. No. 118 In Oeuvres, vol. III, Springer-Verlag 1983, pp 650–684.

[2] J.-L. Brylinski and J.-P. Labesse, Cohomologie d’intersection et fonctions \( L \) de certaines variétés de Shimura, Ann. Scient. Éc. Norm. Sup. 17 (1984), 361–412.

[3] H. Carayol, Sur les représentations \( \ell \)-adiques associées aux formes modulaires de Hilbert, Ann. scient. Éc. Norm. Sup. 19 (1986), 409–468.

[4] P. Deligne, Formes modulaires et représentations \( \ell \)-adiques, Sém. Bourbaki 355 (1968/9). Lecture Notes in Math., vol. 179, Springer-Verlag and New York, 1971, pp. 139–172.

[5] P. Deligne, Travaux de Shimura, Sém. Bourbaki 1970/71, exp. 389 (1971), 139–172.

[6] P. Deligne, La conjecture de Weil II, Publ. Math. Inst. Hautes Études Sci. 52 (1981), 313–428.

[7] P. Deligne and G. Pappas, Singularités des espaces de modules de Hilbert en les caractéristiques divisant le discriminant, Compositio Math. 90 (1994), no. 1, 59–79.

[8] J. Friedman, On the second eigenvalue and random walks in random \( d \)-regular graphs, Combinatorica 11 (1991), no. 4, 331–362.

[9] K. Fujiwara, Rigid Geometry, Lefschetz-Verdier trace formula and Deligne’s conjecture, Invent. math. 127 (1997), no. 3, 489–533.

[10] G. Harder, On the cohomology of discrete arithmetically defined groups. in: Discrete Subgroups of Lie Groups and Applications to Moduli, Colloq., Bombay, 1973, Oxford Univ. Press, Bombay, 1975, pp. 129–160.

[11] H. Jacquet and R. P. Langlands, Automorphic Forms on \( \text{GL}(2) \), Lecture Notes in Math., vol. 114, Springer-Verlag, Berlin and New York, 1970.

[12] B. W. Jordan and R. Livné, Ramanujan local systems on graphs, Topology 36 (1997), 1007–1024.

[13] B. W. Jordan and R. Livné, The Ramanujan property for regular cubical complexes, Duke Math. J. 105 (2000), 85–103.
[14] R. Langlands, *On the zeta function of some simple Shimura varieties*, Canadian J. math., 31 (1979), 1121–1216.

[15] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Birkhäuser Verlag, Basel-Boston-Berlin, 1994.

[16] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica 8 (1988), 261–278.

[17] J. S. Milne, *Points on Shimura varieties mod p*, in Automorphic Forms, Representations and L-functions, Proc. Symp. in Pure Math., vol 33, part 2, AMS, R.I. 1979, pp. 165–184.

[18] S. Mozes, *Actions of Cartan subgroups*, Israel J. of Math. 90 (1995), 253–294.

[19] Pink, R. *On the calculation of local terms in the Lefschetz-Verdier trace formula and its application to a conjecture of Deligne*, Ann. of Math. (2) 135 (1992), no. 3, 483–525.

[20] M. Rapoport, *Compactifications de l’espace des modules de Hilbert-Blumenthal*, Comp. Math. 36 (1978), 255–335.

[21] L. Breen and J. P. Labesse (editors), *Variétés de Shimura et Fonctions L* (2nd edition), Publications Mathématiques de l’Université Paris VII, 1979.

[22] M.-F. Vignéras, *Arithmétique des Algèbres de Quaternions*, Lecture Notes in Math., vol. 800, Springer-Verlag, Berlin and New York, 1980.

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