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AIMD scheduling and resource allocation in distributed computing systems

Eleftherios Vlahakis, Nikolaos Athanasopoulos, Seán McLoone

Abstract—We consider the problem of simultaneous scheduling and resource allocation of an incoming flow of requests to a set of computing units. By representing each computing unit as a node, we model the overall system as a multi-queue scheme. Inspired by congestion control approaches in communication networks, we propose an AIMD-like (additive increase multiplicative decrease) admission control policy that is stable irrespective of the total number of nodes and AIMD parameters. The admission policy allows us to establish an event-driven discrete model, triggered by a locally identifiable enabling condition. Subsequently, we propose a decentralized resource allocation strategy via a simple nonlinear state feedback controller, guaranteeing global convergence to a bounded set in finite time. Last, we reveal the connection of these properties with Quality of Service specifications, by calculating local queuing time via a simple formula consistent with Little’s Law.

I. INTRODUCTION

Distributed computing is a new paradigm emerging to address the growing demand for extensive, real-time computations at the edge as a result of the growing number of end-users (e.g., smart devices, sensors) connected to the edge of the Internet. Although this emerging technology opens new opportunities for more sophisticated applications (see, e.g., [1]–[3]), it presents several research challenges, especially in the context of resource allocation and control of edge-servers due to factors such as the need to take account of latency constraints, limited capacity of edge-servers, and its inherent decentralized structure.

Feedback control has been a powerful mathematical tool for tackling management problems in the context of modern computer systems [4]. Given a representative dynamical model, control theory allows analytical derivation of formal guarantees and certificates. However, modelling computer systems is a formidable task by itself, thus, many works rely on application-specific models obtained via system identification methods. See for example [5]–[7]. Focusing on a more abstract modelling paradigm agnostic to each individual node specificities, we follow a queueing system modelling approach that enhances scalability, naturally, at the expense of accuracy loss. Notable works avoiding application-specific modelling can be found in [8]–[10].

The control problem considered in this paper consists of 1) the scheduling of a stream of requests, and 2) the resource allocation of a set of computing units associated with a specific application. Representing each computing unit as a node and associating each node with a queue, we model the entire scheme as a multi-queue system. We assume that there is no interaction between nodes and computing units are independent from each other. A central node acts as an aggregation point, receiving all requests and dispatching them to individual nodes. Queues in this work are consistent with the First Come First Served (FCFS) selection policy.

Our approach to scheduling and resource allocation is motivated by the Additive Increase Multiplicative Decrease (AIMD) algorithm, a celebrated method in network management. The AIMD algorithm was originally introduced in [11] for tackling congestion phenomena in computer networks in a robust and decentralized manner requiring minimum interaction between nodes. Since then, it has become a fundamental building block of the Transmission Control Protocol (TCP) widely used across the Internet. An excellent and comprehensive study of the AIMD algorithm with several extensions and applications can be found in [12].

In this paper, we study how an AIMD-inspired simple admission control policy can be utilized for general scheduling problems. A typical AIMD model results in an event-driven discrete controller which is triggered by a capacity event associated with constraints, e.g., bandwidth constraints. Berman et al. in [13] and Shorten et al. in [14] show that such a control scheme can be formulated as positive system, thus, stability and convergence properties can be derived from the Perron–Frobenius Theorem. A first challenge we face is that a positive system formulation is not possible in our case due to the absence of a capacity constraint in a scheduling task. Instead, we consider a queue clearance event and manage to show stability via a significant result in Linear Algebra (cf. [15], [16]) involving the eigenproblem of rank-one perturbations of symmetric matrices (Theorems 1 and 2). This formulation leads to a new admission control algorithm with AIMD structure which is stable irrespective of the AIMD tuning and the number of nodes, and inherits attractive features of the standard AIMD algorithm (e.g., fairness among nodes, tunable convergence rate) [17]. To the best of our knowledge, this paper presents a new admission control policy with AIMD dynamics for scheduling tasks.

As a result of the simplicity of the AIMD scheduling policy proposed, we formulate a resource allocation strategy defined as a decentralized globally stabilizing nonlinear feedback controller. Following a set-theoretic approach, we show that under the proposed resource allocation law, individual queues are bounded, and, further, converge in finite time to a
well-defined interval which is invariant [18]. This effectively permits a priori analysis of Quality of Service (QoS) metrics, such as queueing time. Overall, scheduling and resource allocation lie in the same control loop leading to a simple decentralized system which is stable, scalable, and locally configurable.

Unlike standard stochastic methods, see, e.g., [19], [20]), we follow a deterministic approach to workload modelling. This choice simplifies the simultaneous scheduling and resource allocation problem, and most importantly, leads to deterministic performance certificates. A relaxation of our results towards a non-deterministic workload as well as the incorporation of constraints will be considered in future work.

The remainder of the paper is organized as follows. The notation used in the paper is introduced in Section II while underpinning definitions and assumptions are given in Section III. The main results of the paper, namely, the AIMD scheduling strategy, the resource allocation control, and the calculation of queueing time are then presented in Sections IV, V and VI respectively. In Section VII we highlight our results via an illustrative numerical example. Finally, Section VIII discusses our main results and future research directions.

II. NOTATION

The field of real numbers is denoted by $\mathbb{R}$. $\mathbb{R}^n$ denotes the $n$-dimensional vector space over the field $\mathbb{R}$, and $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices. The transpose of $\xi$ is denoted by $\xi^\top$. Let $x_1, \ldots, x_n$ be vectors not necessarily of the same dimensions. Then, $\hat{\xi} = \text{Col}(x_1, \ldots, x_n) = (x_1' \cdots x_n')'$. Let $a_1, \ldots, a_n \in \mathbb{R}$, then, $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and $A = \text{diag}(a_1, \ldots, a_n)$ is a diagonal matrix, with $a_1, \ldots, a_n$ as its diagonal entries. We denote by $\text{det}(A)$ the determinant of a square matrix $A$. The identity matrix of dimension $m \times m$ is denoted by $I_m \in \mathbb{R}^{m \times m}$ unless the dimensions are obvious in which case the subscript will be omitted. Matrix $\Sigma \in \mathbb{R}^{n \times n}$ is called symmetric if $\Sigma^\top = \Sigma$. Let $\lambda_i(\Phi)$ be the $i$th eigenvalue of matrix $\Phi \in \mathbb{R}^{m \times m}$, with $i = 1, \ldots, m$. Then, the spectrum of $\Phi$ is denoted by $\sigma(\Phi) = \{\lambda_1(\Phi), \ldots, \lambda_m(\Phi)\}$. A matrix $\Phi \in \mathbb{R}^{m \times m}$ is Schur if all its eigenvalues strictly lie inside the unit circle, i.e., $|\lambda_i(\Phi)| < 1, i = 1, \ldots, m$.

III. DEFINITIONS AND BASIC ASSUMPTIONS

A. Single-queue system

We define a request as an individual demand for computing resources provided by a computing node. A computing node is defined as the physical (or virtual) computing environment, consisting of hardware, software, and network resources, whereby a request is executed. A queue is defined as the waiting mechanism whereby a request arriving at a node is temporarily put on hold until it is selected for service from among other requests that are waiting. Here, we consider queues consistent with the First Come First Served (FCFS) selection principle.

A queueing system [21], [22] is defined as the dynamic relationship that is developed between a stream of request arrivals at and a flow of request departures from a computing device, respectively, in the presence of a queue. From a mathematical perspective, a queue acts as an integrator of the difference between arrival and departure rates. A simple queueing system is depicted in Fig. 1 which is consistent with the following notation.

Denoting by $X(t)$ and $Y(t)$ the arrivals at and departures from a queue, respectively, in interval $[0, t]$, the number of queued requests at time $t$, $Q(t)$, is defined as the difference between arrivals and departures in interval $[0, t]$. Note that exact knowledge of $X(t)$ and $Y(t)$ is typically impossible in real applications, with arrivals and departures considered as stochastic processes described by appropriate probability distributions. A comprehensive overview of stochastic queueing systems can be found in [21]. Here, to highlight the admission and resource allocation control strategies proposed in the paper, we simplify our model structure following a deterministic approach. Specifically, we assume the following.

Assumption 1

(A1) The arrival rate, denoted by $\lambda(t) = \frac{d}{dt}X(t)$, is constant.

(A2) Requests arriving at a queueing system are identical in terms of the combination of computing resources (CPU time, memory, disk space) required to serve them.

![Fig. 1: A simple queueing system.](image)

B. Queueing time

Queueing time (also termed waiting time or latency [21]) is the main performance metric of a queueing system (e.g., in edge computing applications), expressing the time that a request is expected to be queued before processed. Given the knowledge of arrivals at and departures from a queue in interval $[0, t]$, we define queueing time as

$$T_q(t) = \frac{\int_0^t Q(s)ds}{X(t)},$$

(1)

where $Q(t) = X(t) - Y(t)$. Note that the integral in the numerator on the right side of (1) expresses the aggregate queueing time of all requests arriving in $[0, t]$ measured in requests $\times$ seconds. By averaging the aggregate queueing time and the request arrivals, respectively, by the length of interval $[0, t]$ as $T_q(t) = \frac{1}{X(t)} \int_0^t Q(s)ds$, it is easy to see that definition (1) is in agreement with Little’s Law, which states that the average number of queued requests $\frac{1}{t} \int_0^t Q(s)ds/t$ is equal to the product of average arrival rate $X(t)/t$ and queueing time $T_q(t)$. A detailed description of Little’s Law can be found in [21, Chapter 2].
C. Event-driven discretization and event generator

An event generator is introduced as the mechanism indicating time instants at which a well-defined (triggering) condition \( C_k \) is satisfied. Condition satisfaction can be written as

\[
C_k(t_k) = \text{true},
\]

where \( t_k \) denotes the time instant at which the \( k \)th event occurs (is generated). Note that time events can be modelled by casting the continuous time as an autonomous state variable, namely,

\[
t(k + 1) = t(k) + T(k),
\]

where \( T(k) \) is the time-varying sampling period.

We emphasize that the facilitation of an aperiodic model (with respect to time) derivation will be the result of two main design strategies, namely,

1) the introduction of a batch queue into the system,
2) the adoption of an AIMD admission control policy.

This strategic choice is now exemplified via a simple tandem queueing system.

D. Tandem queueing system with AIMD dynamics

We consider the two-queue system (also termed tandem queueing system) shown in Fig. 2, where \( \lambda(t) \) is a piecewise differentiable function representing workload, \( \delta(t) \) is the number of queued requests waiting at queue \( Q_1 \) to be dispatched to queue \( Q_2 \) at an admission rate \( u(t) \), while \( w(t) \) and \( \gamma(t) \) represent queued requests and service rate, respectively. Using this notation, the continuous-time dynamics of the two-queue system can be written in a compact form as

\[
\begin{bmatrix}
\dot{\delta}(t) \\
\dot{w}(t)
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
\lambda(t) \\
u(t) \\
\gamma(t)
\end{bmatrix}.
\]

Fig. 2: A tandem queueing system.

Before proceeding with the discretization of the model, we define a triggering condition that enables the generation of an event indicating the commencement of a new cycle. Let \( u(t) \) be an admission control policy such that

\[
\delta(t_k) = 0,
\]

i.e., all the requests that have arrived at queue \( Q_1 \) by time \( t_k \) have been admitted to queue \( Q_2 \). Hence, in this regard, \( Q_1 \) instantaneously becomes empty at \( t_k \). To ensure that condition (5) can always be satisfied at a finite time for a constant \( \lambda(t) \), we design \( u(t) \) as an AIMD controller as follows.

\[
u(t_k^+) = \lim_{t \to t_k^-} u(t), \quad u(t_k^-) = \lim_{t \to t_k^+} u(t),
\]

where \( t_k^- \) is the ending time of the \((k-1)\)th cycle, and \( t_k^+ \) the starting time of the \(k\)th cycle. Since \( \delta(t_k^+) = 0 \), we let

\[
u(t_k^+) = \beta u(t_k^-), \quad \delta(t_k^+) = 0,
\]

where \( 0 < \beta < 1 \) is called the backoff parameter. While queue \( Q_1 \) remains empty, admission control \( u(t^+) \) shrinks to a fraction of \( u(t^-) \) according to (6). This is called the Multiplicative Decrease (MD) phase of the cycle. By the time queue \( Q_1 \) starts growing, admission rate \( u(t) \) increases in a ramp fashion as

\[
u(t) = \beta u(t^-) + \alpha(t - t_k), \quad t \geq t_k,
\]

where the slope of the ramp \( \alpha > 0 \) is called the growth rate. Let now \( t_{k+1} \) be the ending time of the \(k\)th cycle, i.e., \( \delta(t_{k+1}) = 0 \). Then, the duration of the \(k\)th cycle is called the cycle period and is denoted by \( T(k) = t_{k+1} - t_k \). Similarly, the interval \([t_k^-, t_{k+1}^-]\) is called the Additive Increase phase. Denoting time instants by \( k = t_k \), with \( k \geq 0 \), an event-driven discrete model is derived as

\[
\begin{bmatrix}
\delta(k+1) \\
u(k+1) \\
\gamma(k+1)
\end{bmatrix} =
\begin{bmatrix}
\delta(k) \\
u(k) \\
\gamma(k)
\end{bmatrix} +
\begin{bmatrix}
\lambda(k) \\
u(k) \\
\gamma(k)
\end{bmatrix} +
\begin{bmatrix}
\lambda(k) \\
u(k) \\
\gamma(k)
\end{bmatrix},
\]

where \( u(k) \) is an AIMD controller with triggering condition

\[
\delta(k) = 0.
\]

Next, we generalize the AIMD admission control approach to a system with multiple queues, and examine the properties of the AIMD algorithm and its effect on the entire system dynamics.

E. Multi-queue system

We consider a set of \( n \) computing nodes represented by a multi-queue system, where each node is modelled by a queue combined with a (physical or virtual) computing environment. We assume that a constant workload \( \lambda \) enters the system via a batch queue, which is independent of the computing nodes. The workload is manifested as a flow of requests that are dispatched to \( n \) nodes according to an admission control policy \( u_i(t) \), \( i = 1, \ldots, n \), with each node representing a computing unit. We denote the number of queued requests that have not yet been admitted at time \( t \) by \( \delta(t) \), and the number of admitted requests waiting to be selected for service by the \( i \)th node at time \( t \) by \( w_i(t) \). Service rate of the \( i \)th node is denoted by \( \gamma_i \), \( i = 1, \ldots, n \). The entire system is depicted in Fig. 3. Next, we examine the aggregate dynamics of a large-scale AIMD admission controller.

IV. AIMD Admissions Control

We consider the system of \( n \) computing nodes depicted in Fig. 3. The number of queued (unadmitted) requests at the beginning of the \((k+1)\)th event is given by

\[
\delta(k+1) = \delta(k) + \lambda T(k) - \int_{t_k}^{t_{k+1}} \sum u_i(t) \, dt.
\]

We recall that at each event \( k, k+1, \ldots \), we have

\[
\delta(k) = \delta(k+1) = \cdots = 0.
\]
The AIMD formulation of the admission controller yields an exact formula for the cycle period $T(k)$ permitting a closed form of the aggregate admission control system. To this purpose, during the AI phase, the $i$th admission rate ramps up as follows,

$$u_i(t) = \beta_i u_i(t_k) + \alpha_i(t - t_k), \quad i = 1, \ldots, n,$$

which is a continuous-time controller for $t \in [t_k, t_{k+1})$. Based on condition (12), the event-based dynamics of the $i$th admission controller is written as:

$$u_i(t) = \beta_i u_i(t) + \alpha_i T(k), \quad i = 1, \ldots, n.$$

In view of the triggering condition (12) and using (14), we get

$$\lambda T(k) = \sum_{i=1}^{n} (2\beta_i u_i(t) + \alpha_i T(k)) \frac{T(k)}{2}$$

or

$$\lambda = \sum_{i=1}^{n} (\beta_i u_i(t) + \frac{\alpha_i}{2} T(k)),$$

from which, the cycle period is defined as

$$T(k) = \frac{\lambda - \sum_{i=1}^{n} \beta_i u_i(t)}{\sum_{i=1}^{n} \frac{\alpha_i}{2}}.$$

From (17), we may write that

$$u_i(t+1) = \beta_i u_i(t) + \alpha_i - \frac{\sum_{i=1}^{n} \beta_i u_i(t)}{\sum_{i=1}^{n} \frac{\alpha_i}{2}}.$$

Now, defining

$$U(k) = \text{Col}(u_1(k), \ldots, u_n(k)),$$

$$\bar{\alpha} = \frac{1}{\sum_{j=1}^{n} \alpha_j} \text{Col}(\alpha_1, \ldots, \alpha_n),$$

$$B = \text{diag}(\beta_1, \ldots, \beta_n),$$

the aggregate admission control system can be expressed as

$$U(k+1) = \Phi U(k) + 2\bar{\alpha} \lambda,$$

where $\Phi = B - 2\bar{\alpha} \beta'$. With $\bar{\alpha} = 1$. Next, we show that system (23) is stable, thus, the $i$th AIMD admission controller $u_i(k)$ converges to a unique equilibrium point $u_i^*$. We first present the following result, which appears in several works in the context of Linear Algebra, see, e.g., [16, Theorem 1], [15, Section 5].

**Theorem 1** Let $C = D + \rho z'z$, where $D \in \mathbb{R}^{n \times n}$ is diagonal, $\rho \in \mathbb{R}$, and $z \in \mathbb{R}^n$. Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the eigenvalues of $D$, and $c_1 \leq c_2 \leq \ldots \leq c_n$ be the eigenvalues of $C$. Then,

i.) $d_1 \leq c_1 \leq d_2 \leq c_2 \leq \ldots \leq d_n \leq c_n$ if $\rho > 0$.

ii.) $c_1 \leq d_1 \leq c_2 \leq d_2 \leq \ldots \leq c_n \leq d_n$ if $\rho < 0$.

If $d_1, \ldots, d_n$ are distinct and all the elements of $z$ are nonzero, then $c_1, \ldots, c_n$, namely the eigenvalues of $C$, strictly separate the eigenvalues of $D$.

We are in a position to state the first main result, namely, the stability of the AIMD scheduling policy.

**Theorem 2** Let vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, where $0 \leq \alpha_i \leq 1$, $0 < \beta_i < 1$, $\forall i = 1, \ldots, n$, and $1' \alpha = 1$, with $1 = (1, \ldots, 1)$. Let also $B = \text{diag}(\beta_1, \ldots, \beta_n)$. Then,

$$\Phi = B - 2\alpha \beta',$$

is a Schur matrix.

**Proof:** Matrix $\Phi$ can also be written as

$$\Phi = (I - 2A)B,$$

where $A = \alpha 1'$ is a rank-one matrix with $\sigma(A) = \{1', 0, \ldots, 0\}$, and $1' \alpha = 1$ by definition. In the sequel, we denote by $\sigma(\Phi) = \{\phi_1, \ldots, \phi_n\}$ the spectrum of $\Phi$. Clearly,

$$\sigma(B) = \{\beta_1, \ldots, \beta_n\}.$$

Also, it is easy to show that $\sigma(I - 2A) = \{-1, 1, 1, \ldots, 1\}$. We can also write that

$$\det(\Phi) = \phi_1 \phi_2 \cdots \phi_n,$$

and $\det(\Phi) = \det(B) = \det(I - 2A)$. Thus,

$$\phi_1 \phi_2 \cdots \phi_n = -\beta_1 \beta_2 \cdots \beta_n.$$

Let now $\hat{A} = \text{diag}(\alpha_1, \ldots, \alpha_n)$, and

$$\hat{\Phi} = B - 2\alpha \hat{A} - \hat{A} - \frac{1}{2} \hat{A}^2 = B - 2\alpha \hat{A}^2 - \frac{1}{2} \hat{A}^2.$$

Clearly, matrices $\Phi$ and $\hat{\Phi}$ are similar, and therefore have identical eigenvalues. Note also that $\hat{\Phi}$ can be written as

$$\hat{\Phi} = B - 2zz'$$

which is clearly a symmetric matrix, where

$$z = \left[\sqrt{\alpha_1} \beta_1, \sqrt{\alpha_2} \beta_2, \ldots, \sqrt{\alpha_n} \beta_n\right].$$

Without loss of generality, let $b_1 \leq \ldots \leq b_n$, and $\phi_1 \leq \ldots \leq \phi_n$. Then, from Theorem 1 and since all elements of $z$ are nonzero, we may write that

$$\phi_1 \leq \beta_1 \leq \phi_2 \leq \beta_2 \leq \ldots \leq \phi_n \leq \beta_n.$$

From (20) and (50), we can conclude that $0 < \beta_2, \phi_3, \ldots, \phi_n < 1$, and $\phi_1$ is a negative real number. From (50), we have that

$$\phi_2 \phi_3 \cdots \phi_n \geq \beta_1 \beta_2 \cdots \beta_{n-1}.$$
However, due to (26), (31) implies that

\[ |\phi_i| \leq \beta_i, \]

i.e., \(-\beta_i \leq \phi_i < 0\). Thus, all the eigenvalues of \(\Phi\) (hence \(\Phi\)) strictly lie inside the unit circle (specifically on the real axis between \(-1\) and \(1\)). This proves the theorem.

The main deductions that follow from the analysis presented in this section are as follows:

1) AIMD parameters \(\alpha_i, \beta_i, i = 1, \ldots, n\), can be locally selected at each individual node. Thus, system (14) represents a decentralized admission control policy.

2) In view of Theorem 2, the aggregate system (23) is stable regardless of the choice of AIMD parameters.

3) Since \(\Phi\) in (23) is a Schur matrix, the \(i\)th AIMD admission rate converges to

\[ u_i^* = \frac{\alpha_i}{1 - \beta_i} \tau^*, \]

where \(\tau^* = \sum_{i=1}^n (\alpha_i \frac{1 + \beta_i}{1 - \beta_i})^{-1} \lambda\).

V. RESOURCE ALLOCATION CONTROL

Resource allocation in a queueing system pertains to a strategy ensuring that computing nodes provide incoming requests with adequate resources so that the number of queued requests is not increasing indefinitely as more requests are added to the system. Stabilizing the overall system, minimizing queueing and idle times, providing a trade-off between server utilization and application performance, and maximizing system throughput and output are essential objectives of resource allocation strategies in queueing systems. Here, we focus on stability as a fundamental qualitative property, which if not present, may make it impossible for a queueing scheme to achieve any other desirable objective.

We follow a bottom-up approach for designing a decentralized resource allocation control strategy as follows. Let \((\alpha_i, \beta_i)\), \(\gamma\) denote the AIMD parameters, and the service rate, respectively, associated with the \(i\)th node. Recall that, for \(\tau \in [0, T(k)]\),

\[ u_i(\tau) = \beta_i u_i(k) + \alpha_i \tau, \]

is the rate at which requests are admitted to the \(i\)th node, while

\[ w_i(\tau) = w_i(k) + \beta_i u_i(k) \tau + \frac{\alpha_i}{2} \tau^2 - \gamma(k) \tau, \]

is the number of queued requests waiting in the \(i\)th node, during the \(k\)th cycle, respectively. We define by

\[ y_i^k(\tau) = w_i(\tau) + \gamma(k) \tau, \]

\[ z_i^k(\tau) = \gamma(k) \tau, \]

the total number of requests that have been admitted by time \(\tau\), and the number of requests that can be served at most by time \(\tau\), respectively. Let also \(\hat{\gamma}(k)\) be the slope of a line segment starting from the origin tangent to parabola \(y_i^k(\tau)\) (see OA in Fig. 4). By letting \(\gamma(k) = \hat{\gamma}(k)\), thus selecting \(z_i^k\) as the line tangent to \(y_i^k(\tau)\) at point \(t_i^k \in [0, T(k)]\), as shown in Fig. 4 we effectively guarantee that the maximum number of requests that can be served, during the \(k\)th cycle, never exceeds the actual number of admitted requests, avoiding, thus, node under-utilization. In Theorem 3 below, we also show that this resource allocation choice is stabilizing. Note also that if \(\gamma(k) > \hat{\gamma}(k)\) (see red dashed line in Fig. 4) there is always a nonzero interval that the queue of the \(i\)th node remains empty, i.e., resources are over-provisioned. Similarly, by letting \(0 < \gamma(k) < \hat{\gamma}(k)\) (see blue dashed line in Fig. 4) there is no stability guarantee that the \(i\)th queue remains bounded.

We now show how to obtain a closed formula for \(\hat{\gamma}(k)\). We first find the intersection point \(B\), as shown in Fig. 4, where

\[ y_i^k(t_i^k) = z_i^k(t_i^k), \]

\[ \frac{dy_i^k}{dt_i^k} = \frac{dz_i^k}{dt_i^k}, \]

From (39), we get

\[ t_i^k = \frac{\hat{\gamma}(k) - \beta_i u_i(k)}{\alpha_i}, \]

while, after a few calculations, using (40) in (38), we have

\[ \gamma(k) = \beta_i u_i(k) + \sqrt{2\alpha_i w_i(k)}, \]

which is a nonlinear, discrete-time state-feedback controller. Finally, using (41), (40) becomes

\[ t_i^k = \sqrt{\frac{2w_i^2(k)}{\alpha_i}}. \]

We are now in a position to state the second main result of our work, namely, the proposed resource allocation strategy along with its stability properties.

Theorem 3 Let

\[ w_i(k+1) = w_i(k) + (\beta_i u_i(k) + \frac{\alpha_i}{2} T(k) - \gamma(k)) T(k), \]

with \(w_i(0) \geq 0\), be the queue dynamics of the \(i\)th node, where \(\alpha_i, \beta_i\) are the AIMD parameters, \(T(k)\) is the cycle period, and

\[ \gamma(k) = \beta_i u_i(k) + \sqrt{2\alpha_i w_i(k)}, \]
is a feedback resource allocation policy. Then, the following are true.

i.) System (43)-(44) is nonnegative for all $w_i(k) \geq 0$.

ii.) The set $\mathcal{W}_i(k) = [0, \frac{\alpha}{2} T(k)^2]$ is invariant with respect to system (43)-(44).

iii.) For $w_{i,0} \notin \mathcal{W}_i(k)$, there is an integer $k^* > 0$ such that $w_i(k^*) \in \mathcal{W}_i(k)$.

Proof: i.) In the proof, we denote $\frac{d\phi}{dx}$ by $\phi'(x)$. Substituting (44) in (43), we write

$$w_i(k+1) = w_i(k) + \frac{\alpha_i}{2} T(k)^2 - \sqrt{2\alpha_i w_i(k)} T(k),$$

and we show that $f(w_i(k)) = w_i(k+1)$ is convex in $w_i(k)$. Indeed, $f(w_i(k))$ is convex with respect to $w_i(k)$ since it is a sum of the affine function $w_i(k) + \frac{\alpha_i}{2} T(k)^2$ and the convex function $-\sqrt{2\alpha_i w_i(k)} T(k)$. Note also that $f(w_i(k))$ is convex for all $T(k) \geq 0$. Since, $f(w_i(k))$ is continuously differentiable and convex for $w_i(k) \geq 0$, the unique minimizer is attained by setting $f'(w_i(k)) = 0$, which results in $1 - \frac{\sqrt{2\alpha_i T(k)^2}}{2\sqrt{w_i(k)}},$ or

$$w_i^*(k) = \frac{\alpha_i T(k)^2}{2}.$$ 

Taking into account that $f(w_i^*(k)) = 0$, it holds that $f(w_i(k)) \geq \alpha_i T(k)^2 \geq 0$. Hence, $w_i^*(k)$ is the unique minimizer of $f(w_i(k))$, and $w_i^*(k) \geq w_i(k)$ for all $k \geq 0$.

ii.) The condition $w_i(k+1) \leq w_i(k)$ holds when $w_i(k) + \frac{\alpha_i}{2} T(k)^2 = \sqrt{2\alpha_i w_i(k)} T(k)$, which is obtained when $w_i(k) \geq \frac{\alpha_i}{8} T(k)^2$. Let $\mathcal{W}_i(k) = \{w \in \mathbb{R} : w \geq \frac{\alpha_i}{8} T(k)^2\}$. Since $\mathcal{W}_i(k) \cap \mathcal{W}_i(k) = [\frac{\alpha_i}{8} T(k)^2, \frac{\alpha_i}{2} T(k)^2]$, we need only to verify $w_i(k+1) \leq w_i(k)$ for all $k \geq 0$.

iii.) Consider function $g(w_i(k)) = w_i(k) - f(w_i(k))$. Then, $g'(w_i(k)) = 1 - f'(w_i(k)) = \frac{\alpha_i}{2\sqrt{w_i(k)}} T(k)^2 > 0$ since $T(k) > 0$ for all $k \geq 0$. Moreover, $g(\frac{\alpha_i}{2} T(k)^2) = \frac{\alpha_i}{2} T(k)^2$. Thus, $g(w_i(k)) \geq \frac{\alpha_i}{2} T(k)^2$ for all $k \geq 0$, and $g(w_i(k)) \geq \frac{\alpha_i}{2} T(k)^2$. We now claim that for any $w_i(0) \geq \frac{\alpha_i}{2} T(k)^2$, we have $g\left(w_i(0) - \frac{\alpha_i}{2} T(k)^2\right) \leq g\left(w_i(0) - \frac{\alpha_i}{2} T(k)^2\right)$, or $g(w_i(0)) \leq g(w_i(0))$. We now claim that for any $w_i(0) \geq \frac{\alpha_i}{2} T(k)^2$, we have $g\left(w_i(0) - \frac{\alpha_i}{2} T(k)^2\right) \leq g\left(w_i(0) - \frac{\alpha_i}{2} T(k)^2\right)$, or $g(w_i(0)) \leq g(w_i(0))$. This proves part iii.

In view of (16) and (48), we may write that

$$\sum_{i=1}^{n} u_i^\text{av}(k) = \lambda, \ \forall k \geq 0.$$ 

In view of (49), we can write that $u_i^\text{av}(k) T(k)$ corresponds to the fraction of the total arrivals at the batch queue (namely, $\sum_{i=1}^{n} u_i^\text{av}(k) T(k) = \lambda T(k)$) associated with the $i$th node. Using definition (1), and letting $\delta_i(t) = u_i^\text{av}(k) T(k) - \beta_i u_i(k) T(k)$, with $0 \leq \tau \leq T(k)$, be the number of queued requests waiting in the buffer before being dispatched to the $i$th node, we may write that $T_i(k) = \int_{0}^{T(k)} \delta_i(t) \, dt$. Interestingly enough, integral $\int_{0}^{T(k)} \delta_i(t) \, dt$ is identical to the shaded area $\text{ABTA}$ in Fig. 4. Similarly, using definition (1), we may write that $T_i(k) = \int_{0}^{T(k)} \delta_i(t) \, dt$. Adding the two aforementioned areas and dividing by $u_i^\text{av}(k) T(k)$ clearly yields the queuing time associated with the $i$th node. In other words, $T_i(k)$ is equal to the area of the trapezium $\text{ABTA}$ divided by $u_i^\text{av}(k) T(k)$, i.e.,

$$T_i^\text{tot}(k) = \frac{(w_i(k) + w_i(k+1)) T(k)}{u_i^\text{av}(k) T(k)}.$$ 

Remark 1 The total queuing time of node $i$ can be defined by means of local information without any information pertinent to the batch queue. Also, by defining $w_i^\text{av}(k) =...
(w_i(k) + w_i(k + 1))/2 as the average number of queued requests over the kth cycle, (50) becomes

\[ T_i(k) = \frac{w_{av}^i(k)}{u_{av}^i(k)}, \]

which is clearly consistent with Little’s Law.

VII. NUMERICAL EXAMPLE

We consider a flow of requests with constant flow rate, \( \lambda \) [req/sec], entering a system of four computing nodes. The \( i \)th node is associated with a FCFS queue the length of which is denoted by \( w_i \), \( i = 1, \ldots, 4 \). By tuning parameters \( \alpha_i, \beta_i \) independently, the \( i \)th node admits requests according to AIMD control policy \( u_i \) given in (14), and alters its service rate via discrete nonlinear feedback controller \( \gamma_i \) defined in (44). Simulation results are presented in Fig. 5-10 for the setup parameters shown in Table I.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\lambda & \alpha_i & \beta_i & u_i(0) & w_i(0) \\
\hline
100 & 5i & 0.5 & (i - 1)5 & (2i - 1)7.5 & \{1, 2, 3, 4\} \\
\hline
\end{array}
\]

TABLE I: Simulation parameters

\[
\begin{array}{|c|c|c|c|}
\hline
\mathcal{W}_1(k) & \mathcal{W}_2(k) & \mathcal{W}_3(k) & \mathcal{W}_4(k) \\
\hline
[0, 4.44] & [0, 8.88] & [0, 13.33] & [0, 17.77] \\
\hline
\end{array}
\]

TABLE II: Invariant sets for \( k \geq 15 \)

As can be seen from Fig. 5, the cycle period converges as expected to \( T^* = 1.33 \) [sec]. Viewing Fig. 6, admission rates \( u_i(k), i = 1, \ldots, 4 \), also converge to \( u^*_i = \alpha_i T^*, i = 1, \ldots, 4 \), verifying the validity of Theorem 2. Fig. 7 illustrates typical AIMD behaviour with convergence occurring after approximately 10 events. This convergence rate is related to the particular choice of AIMD parameters. For example, faster convergence is expected if growth rates \( \alpha_i, i = 1, \ldots, 4 \), are selected more aggressively. Service rates depicted in Fig. 8 are calculated according to resource allocation law (44). From the figure, it is evident that the service rate mean value of each node is heavily related to the corresponding average AIMD admission rate. Queue profiles are shown in Fig. 9.
where it is evident that queues are bounded highlighting the stability properties of Theorem 3. Invariant sets $\mathcal{W}_i(k)$ for $T(k) = T^*$ are given in Table II. Overall, we note that under the proposed scheduling and resource allocation strategy, stable operation is guaranteed for all computing nodes regardless of the tuning of individual AIMD parameters. We refer interested readers to [23] for further simulation scenarios with arbitrary number of nodes. Therein, a script for a random arrival process with exponentially distributed inter-arrival times is also available.

VIII. CONCLUSION

We study the problem of simultaneous scheduling and resource allocation of a deterministic flow of requests entering a system of computing nodes which is represented as a multi-queue scheme. Inspired by the well-established AIMD algorithm, we present a new admission control policy for general scheduling problems. Using an interesting property of rank-one perturbations of symmetric matrices, we provide stability guarantees, independent of the overall system dimension and the AIMD tuning. Following a bottom-up approach, we then propose a resource allocation strategy defined as a decentralized nonlinear feedback controller which is globally stabilizing. This effectively guarantees that individual queues are bounded converging in finite time to a well-defined interval. Finally, we associated these properties with Quality of Service specifications, by calculating the local queueing time via a simple formula consistent with Little’s Law. Our method is simple, scalable, and locally configurable. It is worth noting however that further effort is required to formally address two additional challenges, namely, non-deterministic workload and the presence of resource constraints. This is the subject of our immediate future research efforts.

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