On the Harder-Narasimhan Filtration for Coherent Sheaves on \( \mathbb{P}^2 \): I

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Abstract

Let \( E \) be a torsion-free sheaf on \( \mathbb{P}^2 \). We give an effective method which uses the Hilbert function of \( E \) to construct a weak version of the Harder-Narasimhan filtration of a torsion-free sheaf on \( \mathbb{P}^2 \) subject only to the condition that \( E \) be sufficiently general among sheaves with that Hilbert function. This algorithm uses on a generalization of Davis’ decomposition lemma to higher rank.

Consider the following problem. Let \( E \) be an explicit torsion-free sheaf on \( \mathbb{P}^2 \) given by a presentation

\[
0 \to \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \xrightarrow{\phi} \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to E \to 0. \tag{0.1.1}
\]

How does one go about effectively computing the Harder-Narasimhan filtration of \( E \), i.e. the unique filtration

\[
0 = F_0(E) \subset F_1(E) \subset \cdots \subset F_s(E) = E
\]

such that the graded pieces \( \text{gr}_i(E) := F_i(E)/F_{i-1}(E) \) are semistable in the sense of Gieseker-Maruyama and their reduced Hilbert polynomials \( P_i(n) = \chi(\text{gr}_i(E)(n))/\text{rk}(\text{gr}_i(E)) \) satisfy \( P_1(n) > P_2(n) > \cdots > P_s(n) \) for \( n \gg 0 \)?

In this paper and its planned sequel we consider the problem under the simplifying assumption that the matrix \( \phi \) of homogeneous polynomials is general, i.e. that \( E \) is general among torsion-free sheaves with the same Hilbert function as \( E \). Our solution to the problem then divides into two parts. In this first part we construct a filtration of \( E \) of the type

\[
0 \subset E_{\leq \tau_1} \subset \cdots \subset E_{\leq \tau_s} \subset E \tag{0.1.2}
\]

where \( E_{\leq n} \) denotes the subsheaf of \( E \) which is the image of the natural evaluation map \( H^0(E(n)) \otimes \mathcal{O}_{\mathbb{P}^2}(-n) \to E \). We give an algorithm for picking the \( \tau_i \) so that the filtration approximates the true Harder-Narasimhan filtration but groups together all pieces of the Harder-Narasimhan filtration with slopes between two consecutive integers. The associated graded sheaves \( \text{gr}_i(E) := E_{\leq \tau_i}/E_{\leq \tau_{i-1}} \) are not always semistable, but they do share a number

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We will show in Lemma 1.2 that if a function \( t + i \) with graded pieces \( \text{gr}_i \) then \( \text{gr}_i(\mathcal{E}) \) is permitted to have torsion supported along a curve. For example, the sheaf \( \mathcal{E} = \mathcal{O}_P \oplus \Omega_P(2) \) has Harder-Narasimhan filtration \( 0 \subset \mathcal{O}_P(2) \subset \mathcal{E} \) but is unfilterable by subsheaves of the form \( \mathcal{E} \). The true Harde r-Narasimhan filtration is not always given by subsheaves of the form \( \mathcal{E} \leq n \) and can therefore be much harder to compute. For example, the sheaf \( \mathcal{E} = \mathcal{O}_P \oplus \Omega_P(2) \) has Harder-Narasimhan filtration \( 0 \subset \mathcal{O}_P(2) \subset \mathcal{E} \) but is unfilterable by subsheaves of the form \( \mathcal{E} \leq n \) since \( \mathcal{E} \leq n = 0 \) for \( n \leq -1 \) and \( \mathcal{E} \leq 0 = \mathcal{E} \) for \( n \geq 0 \).

In the planned part II we will show how to refine the WHN filtration of a sufficiently general sheaf to the true Harder-Narasimhan filtration using exceptional objects and mutations.

The precise formulation of the WHN filtration requires a certain number of numerical definitions. We consider a general sheaf \( \mathcal{E} \) with a presentation of the form (0.1.1) for given functions \( a(n) \) and \( b(n) \) of finite support. We define \( r(n) \) and \( h(n) \) as the first and second integrals of \( a(n) - b(n) \), i.e.

\[
\begin{align*}
    r(n) & := \sum_{m \leq n} \{a(m) - b(m)\}, \\
    h(n) & := \sum_{m \leq n} r(m) = \sum_{m \leq n} (n - m + 1) \{a(m) - b(m)\}.
\end{align*}
\]

The function \( h, r, \) and \( a - b \) are respectively the first, second, and third differences of the Hilbert function of \( \mathcal{E} \) defined by \( n \mapsto h^n(\mathcal{E}(n)) \). We will assume that the \( a(n) \) and \( b(n) \) are such that \( r(n) \geq 0 \) for all \( n \). The general \( \phi: \bigoplus \mathcal{O}(-n)^{b(n)} \rightarrow \bigoplus \mathcal{O}(-n)^{a(n)} \) is injective if and only if this is the case (see [3] or Theorem 2.2 below). Depth considerations show that the cokernel \( \mathcal{E} \) of such an injective \( \phi \) will have no subsheaves supported at isolated points, but \( \mathcal{E} \) is permitted to have torsion supported along a curve.

We now define further auxiliary functions by

\[
\begin{align*}
    \tilde{h}(n) & := \max \{h(m) + (n - m)r(m) \mid m \geq n\}, \\
    t(n) & := \max \{m \geq n \mid \tilde{h}(n) = h(m) - (m - n)r(m)\} \in \mathbb{Z} \cup \{+\infty\}.
\end{align*}
\]

We will show in Lemma 1.2 that if \( a(n) \) and \( b(n) \) are such that \( r(n) \geq 0 \) for all \( n \), then the function \( t \) is nondecreasing and takes only finitely many values \( \tau_0 < \tau_1 < \cdots < \tau_s < \tau_{s+1} = +\infty \). These \( \tau_i \) may be effectively computed by an algorithm we will give in paragraph (1.8). We set also \( \tau_{-1} = -\infty \). Then we define the WHN filtration of \( \mathcal{E} \) as the filtration

\[
0 = \mathcal{E}_{\leq \tau_{-1}} \subset \mathcal{E}_{\leq \tau_0} \subset \cdots \subset \mathcal{E}_{\leq \tau_s} \subset \mathcal{E}_{\leq \tau_{s+1}} = \mathcal{E}
\]

with graded pieces \( \text{gr}_i(\mathcal{E}) := \mathcal{E}_{\leq \tau_i} / \mathcal{E}_{\leq \tau_{i-1}} \) for \( 0 \leq i \leq s+1 \). Our main result is:
Theorem 0.2. Let $a, b : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be functions of finite support such that the function $r(n)$ of (1.1.3) is nonnegative. Let $E$ be the cokernel of an injection $\phi : \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)}$. If $\phi$ is sufficiently general, then the WHN filtration of $E$ defined in (1.1.7) has the following properties:

(i) For all $0 \leq i \leq s + 1$ the sheaf $\text{gr}_i(E)$ has resolution

$$ 0 \to \bigoplus_{\tau_{i-1} < n \leq \tau_i} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_{\tau_{i-1} < n \leq \tau_i} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to \text{gr}_i(E) \to 0. $$

(ii) The subsheaf $E_{\leq \tau_0} = \text{gr}_0(E)$ is the torsion subsheaf of $E$.

(iii) For $1 \leq i \leq s + 1$ the sheaf $\text{gr}_i(E)$ is torsion-free and of rigid splitting type, i.e. if $L$ is a general line of $\mathbb{P}^2$, then $\text{gr}_i(E)|_L \cong \mathcal{O}_L(-\nu_1)_{\beta_1} \oplus \mathcal{O}_L(-\nu_1 - 1)_{\beta_1}$ for some integers $\nu_1, \beta_1$ and $\beta_i$. Moreover $\nu_1 < \nu_2 < \cdots < \nu_{s+1}$ and $E|_L \cong \bigoplus_{i=0}^{s+1} \text{gr}_i(E)|_L$.

(iv) For $i \leq j$ we have $\text{Hom}(\text{gr}_i(E), \text{gr}_j(E)(-1)) = 0$.

(v) The Harder-Narasimhan filtration of $E/E_{\leq \tau_0}$ for Gieseker-Maruyama stability is a refinement of the filtration (1.1.7) of $E/E_{\leq \tau_0}$. Indeed $\text{gr}_i(E)$ collects all pieces of the Harder-Narasimhan filtration with slopes $\mu$ satisfying $-\nu_1 - 1 < \mu < -\nu_i$ as well as some of those of slopes $-\nu_i - 1$ and $-\nu_i$.

The outline of the paper is as follows. In the first section we prove a number of numerical lemmas leading to a method of filtering the Hilbert function of $E$. The key definition is that the Hilbert function of $E$ (or its differences $h(n)$, $r(n)$ or $a(n) - b(n)$ as defined above) is filterable at $m$ if the function $r(n)$ of (1.1.3) satisfies $r(n) \geq r(m)$ for all $n \geq m$. The $a(n)$ and $b(n)$ then split into

$$ a^\text{sub}_m(n) := \begin{cases} a(n) & \text{if } n \leq m, \\ 0 & \text{if } n \geq m, \end{cases} $$

and analogous functions $b^\text{sub}_m(n)$, $a^\text{quot}_m(n)$, $b^\text{quot}_m(n)$. The $r^\text{sub}_m$, $r^\text{quot}_m$, $h^\text{sub}_m$ and $h^\text{quot}_m$ are defined by integrating. If a Hilbert function is filterable at several integers $m_i$, it may be split into several graded pieces this way. The lemmas of the section show that the Hilbert function of $E$ is filterable at the $\tau_i$ and that its graded pieces satisfy conditions analogous to the conditions of parts (ii)-(v) of Theorem 0.2.

In the second section we show that such filtrations of Hilbert functions correspond to filtrations of $E$ by subsheaves of the form $E_{\leq m}$ if $E$ is sufficiently general among coherent sheaves with the same Hilbert function. The key lemma is the following which may be regarded as a generalization of Davis’ decomposition lemma [1] to higher rank.

Lemma 2.4. Suppose that $E$ is a coherent sheaf on $\mathbb{P}^2$ without zero-dimensional associated points such that the Hilbert function of $E$ is filterable at an integer $m$. Write $E$ as the cokernel of an injection $\phi : \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)}$. If the matrix $\phi$ is sufficiently general, then $E_{\leq m}$ and $E/E_{\leq m}$ have resolutions

$$ 0 \to \bigoplus_{n \leq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_{n \leq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to E_{\leq m} \to 0, $$

$$ 0 \to \bigoplus_{n > m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_{n > m} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to E_{> m} \to 0. $$

The rest of the section is devoted to showing that Theorem 0.2 follows from this lemma and from the numerical lemmas proved in the first section.

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1 Filtering Hilbert Functions

This section contains the purely combinatorial part of the proof of the Theorem 0.2. It consists of a number of numerical lemmas on Hilbert functions of coherent sheaves on $\mathbb{P}^2$. We begin by fixing some terminology. We use the notation $(x)_+ = \max(x, 0)$.

Our fundamental invariant is the difference $a(n) - b(n)$ between the functions of (1.1).

We assume that $a(n) - b(n)$ is an integer for all $n$ and vanishes for all but finitely many $n$ and that the associated function $r(n) = \sum_{m \leq n} \{a(n) - b(n)\} \geq 0$ for all $n$. We call the associated function $h(n) = \sum_{m \leq n} r(n)$ of (1.1.4) the FDH function (or first difference of a Hilbert function). It is the FDH functions which will play the major role in our computations. An intrinsic definition is:

**Definition 1.1.** An FDH function is a function $h: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ such that $r(n) = \Delta h(n) \geq 0$ for all $n$, $h(n) = 0$ for $n \ll 0$, and $h(n)$ is linear of the form $pn + \sigma$ for $n \gg 0$. We call $\sigma$ the rank of $h$, $\sigma - \rho$ its degree, and $\sum_{n \in \mathbb{Z}} \{(pn + \sigma)_+ - h(n)\}$ its deficiency.

An FDH function is *torsion-free* if $r(m) \geq 1$ implies that $r(n) \geq 1$ for all $n \geq m$. An FDH function $h$ is *locally free* if $h$ is torsion-free and additionally $r(m) \geq 2$ implies that $r(n) \geq 2$ for all $n \geq m$. (This terminology will be justified by Theorem 1.2.)

These functions have the following basic properties:

**Lemma 1.2.** Suppose $h$ is an FDH function of rank $\rho$ and degree $\sigma - \rho$. Let $r = \Delta h$, and let $\hat{h}$ and $t$ be as in (0.1.5) and (0.1.6). Then

(i) For all $n$ one has $0 \leq h(n) \leq \hat{h}(n)$ and $t(n) > n$.

(ii) If $m > t(n)$, then $r(m) > r(t(n))$.

(iii) The function $t$ is nondecreasing and takes only finitely many distinct values. If we write these as $\tau_0 < \tau_1 < \cdots < \tau_s < \tau_{s+1} = \infty$, then $0 = r(\tau_0) < r(\tau_1) < \cdots < r(\tau_s) < \rho$.

(iv) Let $\nu_i = \min\{n \mid t(n) = \tau_i\}$. If $\nu_i \leq n < \nu_{i+1}$, then $t(n) = \tau_i$ and $h(n) = nr(\tau_i) + (h(\tau_i) - \tau_i r(\tau_i))$. Moreover $n < \nu_i$ if and only if

$$nr(\tau_{i-1}) + (h(\tau_i) - \tau_i r(\tau_{i-1})) > nr(\tau_i) + (h(\tau_i) - \tau_i r(\tau_i))$$

(v) If $\nu_1 \leq n \leq \tau_1$, then $h(\tau_1) + (n - \tau_1) r(\tau_1) \geq h(m) + (n - m) r(m)$.

(vi) If $n < \nu_1$, then $t(n) = \tau_1 = \max\{n \mid r(n) = 0\}$ and $\hat{h}(n) = h(\tau_0)$. In particular, if $h$ is torsion-free then $\hat{h}(n) = 0$ and $t(n) = \max\{m \mid h(m) = 0\}$ for $n < \nu_1$.

(vii) For $n \geq \nu_{s+1}$ one has $\hat{h}(n) = \rho n + \sigma$ and $t(n) = \tau_{s+1} = +\infty$.

**Proof.**

(i) From the definitions we see that $h(n) = h(n+1) - r(n+1) \leq \hat{h}(n)$ and that this implies that $t(n) \geq n + 1$.

(ii) We go by induction on $m$. Thus we assume that $r(i) > r(t(n))$ for $t(n) < i < m$, and we will show that $r(m) > r(t(n))$ as well. But the definitions of $\hat{h}(n)$ and $t(n)$ yield immediately

$$h(t(n)) + (n - t(n)) r(t(n)) = \hat{h}(n) > h(m) - (m - n) r(m).$$
Finally if
\[ (m - n)r(m) > (t(n) - n)r(t(n)) + h(m) - h(t(n)) \]
\[ = (t(n) - n)r(t(n)) + \sum_{i=t(n)+1}^{m} r(i) \]
\[ \geq (t(n) - n)r(t(n)) + (m - t(n) - 1)r(t(n)) + r(m) \]
whence (ii).

(iii) Since \( t(n - 1) \geq n \) by (i), we see from the definitions that
\[ \tilde{h}(n) \geq h(t(n - 1)) + (n - t(n - 1))r(t(n - 1)) = \tilde{h}(n - 1) + r(t(n - 1)). \]
Thus \( r(t(n - 1)) \leq \Delta \tilde{h}(n) \). Similarly
\[ \tilde{h}(n - 1) \geq h(t(n)) + (n - 1 - t(n))r(t(n)) = \tilde{h}(n) - t(n) \]
and \( \Delta \tilde{h}(n) \leq r(t(n)) \). Hence \( r \circ t \) is nondecreasing. Because of (ii) this implies that \( t \) is nondecreasing. The function \( t \) can only take finitely many values since by (ii) \( r \) takes a different value at each value of \( t \), and the values of \( r \) are bounded since \( r(n) \) is constant for \( n \ll 0 \) and for \( n \gg 0 \). If \( \tau_i = t(n) < +\infty \), then for \( m \gg \tau_i \) (ii) yields \( \rho = r(m) > r(\tau_i) \). Finally for \( n \ll 0 \) one has \( 0 \leq \tilde{h}(n) = h(\tau_0) + (n - \tau_0)r(\tau_0) \) which implies that \( r(\tau_0) \leq 0 \). But since \( h \) is an FDH function, \( r(\tau_0) \geq 0 \). So \( r(\tau_0) = 0 \). This finishes (iii).

(iv) If \( \nu_i \leq n < \nu_{i+1} \), then \( t(n) = \tau_i \). According to the definitions, this implies
\[ \tilde{h}(n) = h(\tau_i) - (\tau_i - n)r(\tau_i) = nr(\tau_i) + (h(\tau_i) - \tau_i r(\tau_i)). \]
As for the inequality (1.2.1), because both its sides are linear and the slope on the left side is less than that on the right side, it is enough to show that the inequality holds for \( n = \nu_i - 1 \) but fails for \( n = \nu_i \). But because of the definition of \( t \), this follows immediately from \( t(\nu_i - 1) = \tau_i - 1 = \nu_i \).

(v) The proof is divided into several cases. First if \( n < \nu_{i+1} \), then by (iv) the inequality becomes \( \tilde{h}(n) \geq h(m') - (m - n)r(m) \) which follows from the definition of \( \tilde{h}(n) \). If \( n \geq \nu_{i+1} \) but \( r(m) \leq r(\tau_i) \), then the inequality follows from the case \( n = \nu_i \) by
\[ h(\tau_i) + (n - \tau_i)r(\tau_i) = \tilde{h}(\nu_i) + (n - \nu_i)r(\tau_i) \]
\[ \geq h(m) + (\nu_i - m)r(m) + (n - \nu_i)r(\tau_i) \]
\[ \geq h(m) + (n - m)r(m). \]
Finally if \( n \geq \nu_{i+1} \) but \( r(m) > r(\tau_i) \), then let \( m' := \min\{M > m \mid r(M) \leq r(\tau_i)\} \leq \tau_i \). Then using the previous case applied with \( m' - 1 \) substituted for \( n \) and \( m' \) substituted for \( m \) we see that
\[ h(m) + (n - m)r(m) = [h(m') - r(m')] - \sum_{i=m+1}^{m'-1} r(i) - (m - n)r(m) \]
\[ \leq [h(\tau_i) + (m' - 1 - \tau_i)r(\tau_i)] - (m' - 1 - n)r(\tau_i) \]
\[ = h(\tau_i) + (n - \tau_i)r(\tau_i). \]
(vi) If \( n < \nu_1 \), then \( t(n) = \tau_0 \) by (iv), \( \tau_0 = \max \{ n \mid r(n) = 0 \} \) by (iii) and (ii), and \( \hat{h}(n) = h(\tau_0) \) by (iv).

(vii) For \( n \geq \nu_{s+1} \) we have \( t(n) = +\infty \). This means that there exists a sequence of integers \( m_i \to +\infty \) such that

\[
\hat{h}(n) = h(m_i) - (m_i - n)r(m_i) = (\rho m_i + \sigma) - (m_i - n)\rho = \rho n + \sigma.
\]

1.3 Filtering FDH Functions. Let \( h \) be an FDH function. We will say that \( h \) is filterable at \( m \) if the associated function \( r \) satisfies \( r(n) \geq r(m) \) for all \( n \geq m \). A filtration of \( h \) is a sequence of integers \( m_0 < m_1 < \cdots < m_s \) at which \( h \) is filterable. Given such a filtration we decompose \( h \) into a sum of \( s + 2 \) function \( h_0, \ldots, h_{s+1} \) defined as follows. We set \( m_{-1} = -\infty \) and \( m_{s+1} = +\infty \). Then the second difference \( \Delta^2 h(n) = a(n) - b(n) \) may be decomposed by

\[
a_i(n) - b_i(n) := \begin{cases} 
  a(n) - b(n) & \text{if } m_{i-1} < n \leq m_i, \\
  0 & \text{otherwise},
\end{cases}
\]

with \( r_i(n) = \sum_{m \leq n} \{ a_i(m) - b_i(m) \} \) and \( h_i(n) = \sum_{m \leq n} r_i(n) \) defined as in (0.1.3) and (0.1.4). If we write \( H_i(n) := h(m_i) + (n - m_i)r(m_i) \), then the \( r_i(n) \) and \( h_i(n) \) satisfy

\[
\begin{align*}
  r_i(n) & = \begin{cases} 
  0 & \text{if } n \leq m_{i-1} \\
  r(n) - r(m_{i-1}) & \text{if } m_{i-1} < n \leq m_i, \\
  r(m_i) - r(m_{i-1}) & \text{if } n > m_i,
\end{cases} \\
  h_i(n) & = \begin{cases} 
  0 & \text{if } n \leq m_{i-1}, \\
  h(n) - H_{i-1}(n) & \text{if } m_{i-1} < n \leq m_i, \\
  H_i(n) - H_{i-1}(n) & \text{if } n > m_i.
\end{cases}
\end{align*}
\]

The filterability of \( h \) at the \( m_i \) implies that \( r_i(n) \geq 0 \) for all \( n \) and \( i \). So the \( h_i(n) \) are all FDH functions.

We call the functions \( h_i(n) \) the graded pieces of the filtration. We will say that a filtration is trivial if all but one of its graded pieces vanish.

Now let us consider the associated function \( t \) of (0.1.6). By Lemma 1.2(iii) the sequence \( \tau_0 < \tau_1 < \cdots < \tau_s \) of all distinct finite values of \( t \) form a filtration of \( h \) which we call the WHN filtration (or weak Harder-Narasimhan filtration). Some of the properties of this filtration are

**Lemma 1.4.** Let \( h \) be an FDH function, let \( \tau_0 < \tau_1 < \cdots < \tau_s \) be the WHN filtration of \( h \), and let \( h_0, h_1, \ldots, h_{s+1} \) be the graded pieces of the filtration. For each \( i \) let \( \nu_i = \min \{ n \mid t(n) = \tau_i \} \). Then

(i) The FDH function \( h_0 \) is torsion. It vanishes if \( h \) is torsion-free.

(ii) The FDH functions \( h_1, h_2, \ldots, h_{s+1} \) are torsion-free.

(iii) For \( i = 1, \ldots, s \) the function \( t_i \) associated to \( h_i \) by (0.1.6) satisfies \( t_i(n) = \tau_{i-1} \) for \( n < \nu_i \), and \( t(n) = +\infty \) for \( n \geq \nu_i \). Thus the \( h_i \) are FDH functions with trivial WHN filtrations.
Proof.  (i) This is a direct translation of Lemma 1.2(vi).

(ii) This follows directly from Lemma 1.2(ii) and the formula for \( r_i(n) \).

(iii) We first suppose \( n < \nu_i \). We will compute \( \tilde{h}_i(n) \) and \( t_i(n) \) according to the definitions (0.1.3) and (0.1.6). This means first computing \( h_i(m) + (n-m)r_i(m) \) for all \( m \geq n \).

If \( m \leq \tau_{i-1} \), then \( h_i(m) = r_i(m) = 0 \) and so \( h_i(m) + (n-m)r_i(m) = 0 \).

If \( \tau_{i-1} < m \leq \tau_i \), then

\[
h_i(m) + (n-m)r_i(m) = \{h(m) + (n-m)r(m)\} - \{h(\tau_{i-1}) + (n-\tau_{i-1})r(\tau_{i-1})\}.
\]

But since \( m > \tau_{i-1} \), the definitions of \( \tilde{h}(n) \) and \( t(n) \) imply that the right side of this equation is negative for all \( n \) such that \( t(n) = \tau_{i-1} \), including \( n = \nu_i - 1 \). The right hand side is also linear in \( n \) with slope \( r(m) - r(\tau_{i-1}) \) which is positive by Lemma 1.2(ii), so it must be negative for all \( n < \nu_i \). Thus \( h_i(m) + (n-m)r_i(m) < 0 \) if \( \tau_{i-1} < m \leq \tau_i \).

If \( m \geq \tau_i \), then \( h_i(m) + (n-m)r_i(m) = h_i(\tau_i) + (n-\tau_i)r_i(\tau_i) < 0 \) because \( h_i \) is linear in this range.

So by the definitions we have \( \tilde{h}_i(n) = 0 \) and \( \nu_i(n) = \tau_{i-1} \) for \( n < \nu_i \).

Now we suppose that \( \nu_i \leq n \leq \tau_i \). Then after subtracting \( h(\tau_{i-1}) + (n-\tau_{i-1})r(\tau_{i-1}) \) from both sides of the inequality of Lemma 1.2(v), we see that for all \( n \leq m \leq \tau_i \) we have

\[
h_i(\tau_i) + (n-\tau_i)r_i(\tau_i) \geq h_i(m) + (n-m)r_i(m).
\]

And for all \( m \geq \tau_i \) we have equality in (1.4.1) because \( h_i \) is linear in this range. So by the definitions, \( \tilde{h}_i(n) = h_i(m) + (n-m)r_i(m) \) for all \( m \geq \tau_i \), and \( t_i(n) = +\infty \).

Finally if \( n \geq \tau_i \), then for all \( m \geq n \) we have equality in (1.4.1), so again we have \( t_i(n) = +\infty \). \( \square \)

1.5 Torsion-free FDH functions \( h \) with trivial WHN-filtrations. We wish to decompose an \( h \) of this type in a certain way. For \( n \gg 0 \) the function \( h(n) \) is linear, so we may write it in the form \( \rho(n-\nu) + \beta \) with \( \rho, \nu \), and \( \beta \) integers such that \( 0 \leq \beta < \rho \). But then if \( n < \nu_1 \) we have by Lemma 1.2(vi) that \( h(n) = \tilde{h}(n) = 0 \) and \( t(n) = \max\{n \mid h(n) = 0\} \). If \( n \geq \nu_1 \) then by Lemma 1.2(vii) we have \( t(n) = +\infty \) and \( 0 \leq h(n) \leq \tilde{h}(n) = \rho(n-\nu) + \beta \). Moreover, \( \nu_1 = \nu \) by Lemma 1.2(iv).

We now define

\[
\gamma^i(n) = \begin{cases} 
(n-\nu+1)_+ & \text{for } i = 1, \ldots, \beta, \\
(n-\nu)_+ & \text{for } i = \beta + 1, \ldots, \rho,
\end{cases}
\]

and

\[
h^i(n) = \min \left\{ \gamma^i(n), \left[ h(n) - \sum_{k=1}^{i-1} \gamma^k(n) \right]_+ \right\}.
\]

These functions have the following properties:

**Lemma 1.6.** Suppose \( h \) is a torsion-free FDH function such that the associated function \( t \) of (0.1.0) takes only two distinct values. For \( i = 1, \ldots, \rho \) let \( h^i : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0} \) be the function defined in (1.5.2). Then

(i) The \( h^i \) satisfy \( \sum_{i=1}^{\rho} h^i = h \),

(ii) The \( h^i \) are torsion-free FDH functions of rank 1. The degree of \( h^i \) is \(-\nu\) for \( i = 1, \ldots, \beta \), and \(-\nu - 1\) for \( i = \beta + 1, \ldots, \rho \).

(iii) The deficiency of \( h^i \) is positive for \( i = 1, \ldots, \beta \).
Lemma 1.7. Let $h$ be an FDH function, let $h_0, h_1, \ldots, h_{s+1}$ be the graded pieces of the WHN filtration of $h$ of Lemma [1.4]. For $i = 1, \ldots, s + 1$, let $\rho_i$ denote the rank of $h_i$, and let $h_i = \sum_{j=1}^{\rho_i} h_i^j$ be the decomposition of $h_i$ of Lemma [1.4]. Then

(i) $h_1^i \geq h_2^i \geq \cdots \geq h_{s+1}^i$ for $i = 1, \ldots, s + 1$, and

(ii) $h_1^i > h_{i+1}^i$ for $i = 1, \ldots, s$.

Proof. (i) First note that if $n < \nu$, then $\gamma^i(n) = h^i(n) = 0$ for all $i$. But $h(n) = 0$ as well. So this case is fine. If $n \geq \nu$, then we have $0 \leq h(n) \leq \tilde{h}(n) = \sum_{i=1}^{\rho} \gamma^i(n)$. So there exists a $k$ such that $\sum_{i=1}^{k-1} \gamma^i(n) \leq h(n) \leq \sum_{i=1}^{k} \gamma^i(n)$. Then

$$h^i(n) = \begin{cases} \gamma^i(n) & \text{for } i = 1, \ldots, k-1 \\ h(n) - \sum_{i=1}^{k-1} \gamma^i(n) & \text{for } i = k \\ 0 & \text{for } i = k+1, \ldots, \rho \end{cases}$$  \hspace{1cm} (1.6.1)

and the sum is $h(n)$. This completes the proof of (i).

For (ii) we first show that the $h^i$ are torsion-free FDH functions, i.e. that $\Delta h^i(n) > 0$ for all $n$ such that $h^i(n) > 0$. To verify this, we may clearly assume that $n \geq \nu$ since otherwise $h^i(n) = 0$. Now note that if one has a function of the form $h^i = \min(f,g)$, then in order to show that $h^i(n) > 0$ implies $\Delta h^i(n) > 0$ it is enough to show that $f(n) > 0$ implies $\Delta f(n) > 0$ and that $g(n) > 0$ implies $\Delta g(n) > 0$. So now consider the case $i = 1, \ldots, \beta$. The function $f(n) := \gamma^i(n)$ satisfies $\Delta \gamma^i(n) = 1 > 0$. And if $g(n) := \left(h(n) - \sum_{i=1}^{k-1} \gamma^i(n)\right)_+$, then $h(n) > (n - \nu + 1)$. But from the definition of $\tilde{h}(n)$ we have

$$0 = \tilde{h}(\nu - 1) \geq h(n) - (n - \nu + 1)\Delta h(n).$$

So $(n - \nu + 1)\Delta h(n) \geq h(n)$. Thus $\Delta h(n) > i - 1$, and $\Delta g(n) > 0$. This proves that $h^i$ is a torsion-free FDH function for $i = 1, \ldots, \beta$.

The proof that $h^i$ is a torsion-free FDH function for $i = \beta + 1, \ldots, \rho$ is similar except that one uses $\tilde{h}(\nu) = \beta$ to obtain $(n - \nu + 1)\Delta h(n) \geq h(n) - \beta$.

For the rank and degree of $h^i$ note that for $n \gg 0$ the formula (1.5.2) becomes

$$h^i(n) = \min \left\{ \gamma^i(n), \sum_{k=1}^{\rho} \gamma^k(n) \right\} = \gamma^i(n) = \begin{cases} n - \nu + 1 & \text{for } i = 1, \ldots, \beta, \\ n - \nu & \text{for } i = \beta + 1, \ldots, \rho. \end{cases}$$

For (iii) note that for $i = 1, \ldots, \beta$ the deficiency of $h^i$ is $\sum_{n \geq \nu} (n - \nu + 1 - h^i(n))$. All the terms in this sum are nonnegative, so it is enough to show that $h^i(\nu) = 0$. But recall that $t(\nu - 1) = \max\{n \mid h(n) = 0\}$. And by Lemma [1.2](i) $t(\nu - 1) \geq \nu$. Thus $h(\nu) = 0$, from which $h^i(\nu) = 0$ for all $i$ by (i). Part (iii) now follows. \hfill \Box

As a final numerical result we wish to compare the decompositions of Lemmas [1.4] and [1.6]. To do this we introduce an order on torsion-free FDH functions. Namely if $h$ and $h'$ are torsion-free FDH functions of ranks $\rho$ and $\rho'$, degrees $d$ and $d'$, and deficiencies $\delta$ and $\delta'$, then $h \succeq h'$ (resp. $h \succ h'$) if $d/\rho > d'/\rho'$ or if $d/\rho = d'/\rho'$ and $\delta/\rho \leq \delta'/\rho'$ (resp. $\delta/\rho < \delta'/\rho'$).
Proof. First we introduce some notation. As in Lemma 1.6 if \( n \gg 0 \), then we may write \( h_i(n) = \rho_i(n - \nu_i) + \beta_i \) with \( 0 \leq \beta_i < \rho_i \). For each \( i \) and \( j = 1, \ldots, \rho_i \) we define

\[
\gamma_i^j := \min\{n \mid h_i^j(n) > 0\} = \min\{n \mid h_i(n) > \sum_{k=1}^{j-1} \gamma_k^i(n)\},
\]

\[
\zeta_i^j := \min\{n \mid h_i^j(n) = \gamma_i^j(n) > 0\} = \min\{n \mid h_i(n) \geq \sum_{k=1}^{j-1} \gamma_k^i(n) > 0\}.
\]

where \( \gamma_i^k \) is as in Lemma 1.6. Then if \( 1 \leq j \leq \beta_i \) (resp. if \( \beta_i + 1 \leq j \leq \rho_i \)), the FDH function \( h_i \) has rank 1, degree \( d_i^j = -\nu_i \) (resp. \( d_i^j = -\nu_i - 1 \)), and deficiency \( \delta_i^j \) satisfying \((n_i^j + d_i^j + 1) \leq \delta_i^j \leq (\zeta_i^j + d_i^j + 1)\).

(i) For all \( i \) and \( j = 1, \ldots, \rho_i \) we have \( \gamma_i^j \leq \eta_i^{j+1} \). Hence \( h_i^1 \geq h_i^2 \geq \cdots \geq h_i^{\beta_i} \) for all \( i \) because all these functions have the same rank, the same degree \( \alpha_i \), and nondecreasing deficiencies. Similarly \( h_i^\beta_i + 1 \geq h_i^{\beta_i + 2} \geq \cdots \geq h_i^{\beta_i} \). And \( h_i^\beta_i > h_i^{\beta_i + 1} \) by reason of degree.

(ii) Note that by Lemmas 1.2(v) and 1.4 the FDH function \( h_i^{\rho_i} \) has degree \( -\nu_i - 1 \) and has

\[
\zeta_i^{\rho_i} = \min\{n > \nu_i \mid h(n) = h(\tau_i) + (n - \tau_i)\nu_i\} \leq \tau_i - 1,
\]

while \( h_i^1 \) has degree \( -\nu_i + 1 \) or \( -\nu_i + 1 \) and has

\[
\eta_i^{\nu_i+1} = \min\{n > \nu_i \mid h(n) > h(\tau_i) + (n - \tau_i)\nu_i\} = \tau_i + 1.
\]

Since \( \nu_i < \nu_i + 1 \), the degree of \( h_i^{\rho_i} \) is at least that of \( h_i^{\nu_i+1} \), and in case of equality the former function has a smaller deficiency than the latter. \( \square \)

1.8 Effective computation of the \( \tau_i \) and \( \nu_i \). The \( \tau_i \) and \( \nu_i \) defined in Lemma 1.2 and referred to in the statement of Theorem 0.2 may be effectively computed from \( h(n) \) or \( r(n) = \Delta h(n) \). Note that \( \Delta r(n) \) is 0 for all but finitely many \( n \). For \( t = +\infty \) we write \( r(t) = \rho \) and \( h(t) - tr(t) = \sigma \).

According to Lemma 1.2(ii),

\[ \{\tau_i\}^{s+1}_{s=0} \subset T := \{n \mid r(m) > r(n) \text{ for all } m > n\} \cup \{+\infty\}. \]

The set \( T \) may be computed by passing through the finite set

\[ T' := \{n \mid \Delta r(n + 1) > 0\} \cup \{+\infty\} \]

in descending order and purging those \( n \in T' \) such that \( r(n) \geq r(m) \) where \( m \) is the smallest unpurged element of \( T' \) larger than \( n \). The minimal element of \( T' \) is \( \tau_0 \) since it is the unique \( t \in T \) such that \( r(t) = 0 \). The other \( \tau_i \) and \( \nu_i \) may be computed recursively as follows.

Suppose we have computed \( \tau_0, \ldots, \tau_{i-1} \) and \( \nu_1, \ldots, \nu_{i-1} \). We now need to find for which \( x > \nu_{i-1} \) there is a \( t \in T \) with \( t > \tau_{i-1} \) such that

\[ h(t) + (x - t)r(t) \geq h(\tau_{i-1}) + (x - \tau_{i-1})r(\tau_{i-1}). \]

For \( t = +\infty \), the left side should be read as \( \rho x + \sigma \) according to our above conventions. Each of the inequalities is equivalent to

\[ x \geq x_t := \frac{(h(t) - tr(t)) - (h(\tau_{i-1}) - \tau_{i-1}r(\tau_{i-1}))}{r(t) - r(\tau_{i-1})}. \]
So $\nu_i = \min \{[x_t] \mid t \in T \text{ and } t > \tau_{i-1}\}$ where the notation $[x_t]$ means the smallest integer greater than or equal to $x_t$. We then look at those $t$ such that $[x_t] = \nu_i$, and pick out those among them for which $h(t) + (\nu_i - t)r(t)$ is maximal. The largest of these $t$ is $\tau_i$.

We continue until some $\tau_i = +\infty$.

The other invariants in the statement of Theorem 1.2 and the proof of Lemma 1.7 may be computed as
\[
\rho_i = r(\tau_i) - r(\tau_{i-1}), \quad \beta_i = (h(\tau_i) + (\nu_i - \tau_i)r(\tau_i)) - (h(\tau_{i-1}) + (\nu_i - \tau_{i-1})r(\tau_{i-1})).
\]

2 A Special Filtration on $\mathbb{P}^2$

In this section we proceed to give sheaf-theoretic significance to the numerical computations of the previous section. We do this by introducing the WHN filtration on the general torsion-free sheaf $\mathcal{E}$ with a given Hilbert function. The Hilbert function of the graded pieces $\text{gr}_i(\mathcal{E})$ of the filtration are those given by Lemma 1.4. Lemma 1.6 is then used to show that the graded pieces satisfy $\text{Hom}(\text{gr}_j(\mathcal{E}), \text{gr}_j(\mathcal{E})(-1)) = 0$ for all $i \leq j$. Lemma 1.7 is used to show that the WHN filtration is compatible with the Harder-Narasimhan filtration for Gieseker-Maruyama stability.

2.1 Hilbert Functions. Recall that any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^2$ without zero-dimensional torsion has a free resolution of the form
\[
0 \to \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \overset{\phi}{\to} \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to \mathcal{E} \to 0.
\]

The $a(n)$ and $b(n)$ are related to the Hilbert function $n \mapsto h^0(\mathcal{E}(n))$ of $\mathcal{E}$ via
\[
\sum_n h^0(\mathcal{E}(n)) t^n = (1 - t)^{-3} \sum_n \{a(n) - b(n)\} t^n \quad (2.1.2)
\]

or $a(n) - b(n) = \Delta^3 h^0(\mathcal{E}(n))$. So the $a(n)$ and $b(n)$ determine the Hilbert function of $\mathcal{E}$ which conversely determines the differences $a(n) - b(n)$. If $\mathcal{E}$ is sufficiently general, then for all $n$ either $a(n) = 0$ or $b(n) = 0$ according to the sign of $a(n) - b(n)$, so the Hilbert function then actually determines the $a(n)$ and $b(n)$.

The next theorem is the filtered Bertini theorem as applied to the special case of $\mathbb{P}^2$:

**Theorem 2.2.** (Chang [C]) A general map $\phi: \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)}$ is injective if and only if the function $h$ whose Poincaré series is
\[
\sum_n h(n) t^n = (1 - t)^{-2} \sum_n \{a(n) - b(n)\} t^n
\]
is an FDH function. Moreover, the cokernel $\mathcal{E}$ of a general $\phi$ is

- torsion-free with at worst singular points of multiplicity 1 if $h$ is torsion-free,
- locally free if $h$ is locally free,
- a line bundle on a curve with normal crossings if $h$ is torsion,
- a line bundle on a smooth curve if $h$ is torsion and unfilterable.
Here the \textit{multiplicity} of a singular point \( P \) of a torsion-free sheaf \( \mathcal{E} \) on a smooth surface is the length of \( \mathcal{E}_{P}^{\vee}/\mathcal{E}_{P} \).

Theorem 2.2 and formula (2.1.2) allow us to define the \textit{FDH function of a coherent sheaf} \( \mathcal{E} \) without zero-dimensional torsion as \( h := \Delta h^0(\mathcal{E}(n)) \). The rank of the FDH function \( h \) is then \( \operatorname{rk}(\mathcal{E}) \), and the degree of \( h \) is \( c_1(\mathcal{E}) \). If this rank is \( \rho \), and the degree is written as \( \rho \alpha + \beta \) with \( \alpha \) and \( \beta \) integers such that \( 0 \leq \beta < \rho \), then the deficiency of \( h \) is \( c_2(\mathcal{E}) - c_2(\mathcal{F}) \) where \( \mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(\alpha + 1)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(\alpha)^{\rho - \beta} \).

We may speak of a generic sheaf with FDH function \( h \) because of the following fact, which is well known and which we therefore state without proof:

\textbf{Lemma 2.3.} The coherent sheaves without zero-dimensional torsion on \( \mathbb{P}^2 \) with a fixed Hilbert function or FDH function form an irreducible and smooth locally closed substack of the stack of coherent sheaves on \( \mathbb{P}^2 \).

Our next lemma relates the filterability of the FDH function \( h \) to sheaf theory. It is essentially a generalization of Davis’ decomposition lemma [D] to higher rank.

\textbf{Lemma 2.4.} Suppose \( h \) is an FDH function which is filterable at an integer \( m \). Suppose \( \mathcal{E} \) is a general coherent sheaf without zero-dimensional torsion with FDH function \( h \). Let \( \mathcal{E}_{\leq m} \) be the subsheaf of \( \mathcal{E} \) generated by \( H^0(\mathcal{E}(m)) \). Then \( \mathcal{E}_{\leq m} \) has FDH function \( h_{\text{sub}}^m \), and \( \mathcal{E}/\mathcal{E}_{\leq m} \) has FDH function \( h_{\text{quot}}^m \). Moreover, if \( \mathcal{E} \) is generic, then so are \( \mathcal{E}_{\leq m} \) and \( \mathcal{E}/\mathcal{E}_{\leq m} \). If \( \mathcal{E} \) is the cokernel of an injection \( \phi: \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \rightarrow \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \), then \( \mathcal{E}_{\leq m} \) and \( \mathcal{E}/\mathcal{E}_{\leq m} \) have resolutions

\[ 0 \rightarrow \bigoplus_{n\leq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \rightarrow \bigoplus_{n\leq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \rightarrow \mathcal{E}_{\leq m} \rightarrow 0 \]
\[ 0 \rightarrow \bigoplus_{n\geq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \rightarrow \bigoplus_{n\geq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \rightarrow \mathcal{E}_{\geq m} \rightarrow 0 \]

\textbf{Proof.} Consider the morphism of exact sequences

\[ 0 \rightarrow \bigoplus_{n\leq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \rightarrow \bigoplus_{n\leq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \rightarrow \mathcal{E}_{\leq m} \rightarrow 0 \]
\[ 0 \rightarrow \bigoplus_{n\geq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \rightarrow \bigoplus_{n\geq m} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \rightarrow \mathcal{E}_{\geq m} \rightarrow 0. \]

The Poincaré series associated to \( \phi' \) and \( \phi'' \) are, respectively,

\[ (1 - t)^{-2} \sum_{n>m} (a(n) - b(n)) t^n = \sum_n h_{\text{quot}}^m(n) t^n, \]
\[ (1 - t)^{-2} \sum_{n\leq m} (a(n) - b(n)) t^n = \sum_n h_{\text{sub}}^m(n) t^n. \]

Since \( h \) is filterable at \( m \), \( h_{\text{quot}}^m \) and \( h_{\text{sub}}^m \) are both FDH functions by (1.3). If \( \phi \) is general, then \( \phi' \) is general and so injective by the filtered Bertini Theorem 2.2. In any case the snake lemma yields an exact sequence

\[ 0 \rightarrow \ker(\phi') \rightarrow \cok(\phi'') \rightarrow \mathcal{E} \rightarrow \cok(\phi') \rightarrow 0 \]

such that \( \operatorname{im}(\psi) = \mathcal{E}_{\leq m} \). So if \( \phi \) is general, then \( \cok(\phi'') = \mathcal{E}_{\leq m} \) and \( \cok(\phi') = \mathcal{E}/\mathcal{E}_{\leq m} \), and they have FDH functions \( h_{\text{sub}}^m \) and \( h_{\text{quot}}^m \), respectively. Finally, if \( \mathcal{E} \) is generic, then \( \phi \) is generic, which implies the genericity of \( \phi' \) and \( \phi'' \) and thus of \( \mathcal{E}_{\leq m} \) and \( \mathcal{E}/\mathcal{E}_{\leq m} \). \( \Box \)
2.5 Davis’ Decomposition Lemma. If $\mathcal{E}$ is torsion-free and $r(m) = 1 = \min_{n \geq m} r(n)$, then the lemma holds without any condition that $\mathcal{E}$ be general. This is because the vanishing of $\ker(\phi')$ may be shown without invoking the filtered Bertini theorem. For the condition $r(m) = 1$ implies that $\operatorname{cok}(\phi'')$ is of rank 1. Since it has nonzero image in $\mathcal{E}$ which is torsion-free, it follows that $\ker(\phi')$ is of rank 0. But since $\ker(\phi') \subset \bigoplus_{n > m} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)}$, it is also torsion-free. So it vanishes. In the case of an $\mathcal{E}$ of rank one, this is more or less Davis’ Decomposition Lemma.

2.6 The WHN Filtration of a General Sheaf. Because of the last lemma, if $h$ is an FDH filtration with a filtration $m_0 < m_1 < \cdots < m_s$, then the general sheaf $\mathcal{E}$ with FDH function $h$ will have a filtration

$$0 \subset \mathcal{E}_{\leq m_0} \subset \mathcal{E}_{\leq m_1} \subset \cdots \subset \mathcal{E}_{\leq m_s} \subset \mathcal{E}. \tag{2.6.1}$$

If we write $\operatorname{gr}_i(\mathcal{E}) = \mathcal{E}_{\leq m_i}/\mathcal{E}_{\leq m_{i-1}}$ for $i = 1, \ldots, s$, and $\operatorname{gr}_0(\mathcal{E}) = \mathcal{E}_{\leq m_0}$ and $\operatorname{gr}_{s+1}(\mathcal{E}) = \mathcal{E}/\mathcal{E}_{\leq m_s}$, then the FDH function of $\operatorname{gr}_i(\mathcal{E})$ is the function $h_i$ of (1.3). If $\mathcal{E}$ has resolution

$$0 \to \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_n \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to \mathcal{E} \to 0,$$

then each $\operatorname{gr}_i(\mathcal{E})$ has resolution

$$0 \to \bigoplus_{m_{i-1} < n \leq m_i} \mathcal{O}_{\mathbb{P}^2}(-n)^{b(n)} \to \bigoplus_{m_{i-1} < n \leq m_i} \mathcal{O}_{\mathbb{P}^2}(-n)^{a(n)} \to \operatorname{gr}_i(\mathcal{E}) \to 0.$$  

If we apply this with the WHN filtration of $h$ of Lemma 1.4, then we call the resulting filtration of $\mathcal{E}$ the WHN filtration of the sheaf $\mathcal{E}$. This filtration only exists for a general sheaf with FDH function $h$ because the construction of the filtration of the sheaf depended ultimately on the filtered Bertini theorem.

We now recall some terminology. If $\mathcal{E}$ is a coherent sheaf of rank $\rho > 0$ on $\mathbb{P}^2$, then its reduced Hilbert polynomial is $P_\mathcal{E}(n) := \chi(\mathcal{E}(n))/\rho$. Such polynomials may be ordered by $P_\mathcal{E} \succ P_\mathcal{F}$ (resp. $P_\mathcal{E} \succeq P_\mathcal{F}$) if $P_\mathcal{E}(n) > P_\mathcal{F}(n)$ (resp. $P_\mathcal{E}(n) \geq P_\mathcal{F}(n)$) for $n \gg 0$. This order is compatible with the order on FDH functions of Lemma 1.7 in the sense that if $\mathcal{E}$ has FDH function $h_\mathcal{E}$ and $\mathcal{F}$ has FDH function $h_\mathcal{F}$, then $P_\mathcal{E} \succ P_\mathcal{F}$ if and only if $h_\mathcal{E} > h_\mathcal{F}$, and $P_\mathcal{E} \succeq P_\mathcal{F}$ if and only if $h_\mathcal{E} \succeq h_\mathcal{F}$.

Lemma 2.7. Let $h$ be an FDH function, let the $h_i$, $\rho_i$, $\nu_i$, $\beta_i$, and $h_i^j$ be as in Lemma 1.4. Then there exists a coherent sheaf $\mathcal{F}$ with FDH function $h$ of the form $\mathcal{F} = \bigoplus_{i=0}^{s+1} \mathcal{F}_i$ such that

(i) $\mathcal{F}$ admits the WHN filtration with graded pieces $\operatorname{gr}_i(\mathcal{F}) \cong \mathcal{F}_i$.

(ii) $\mathcal{F}_0$ is the torsion subsheaf of $\mathcal{F}$.

(iii) For $i = 1, \ldots, s + 1$ we have $\mathcal{F}_i = \bigoplus_{j=1}^{\beta_i} \mathcal{I}_{Z_i^j}(-\nu_i) \oplus \bigoplus_{j=\beta_i+1}^{\rho_i} \mathcal{I}_{Z_i^j}(-\nu_i - 1)$ where the $Z_i^j$ are disjoint sets of distinct points. Moreover, $Z_i^j \neq \emptyset$ for $1 \leq j \leq \beta_i$. 


Proof. Let $\mathcal{F}_0$ be a general sheaf with FDH function $h_0$, and let $\mathcal{F}_i$ be a general sheaf with FDH function $h_i$. For $i = 1, \ldots, s + 1$, let $\mathcal{F}_i = \bigoplus_{j=1}^{\rho_i} \mathcal{F}_j^i$ and $\mathcal{F} = \bigoplus_{i=1}^{s+1} \mathcal{F}_i$. Then $\mathcal{F}$ has FDH function $h = \sum_{i=1}^{s+1} \sum_{j=1}^{\rho_i} h_j^i = h_0$. We now verify the three asserted conditions in reverse order.

(iii) By Lemma 1.4 (ii), the $h_j^i$ are torsion-free sheaves of rank 1. So by Theorem 2.6, $\mathcal{F}_i$ is a twist of an ideal sheaf of a set $Z_i^j$ of distinct points. The twist is given by the degree of $h_j^i$, which is $-\nu_i$ if $1 \leq j \leq \beta_i$ (resp. $-\nu_i - 1$ if $\beta_i + 1 \leq j \leq \rho_i$). Replacing the $Z_i^j$ by projectively equivalent sets of points if necessary, we may assume that the $Z_i^j$ are disjoint. Finally, the cardinality of $Z_i^j$ is the deficiency of $h_j^i$, which is positive if $1 \leq j \leq \beta_i$ by Lemma 1.4 (iii).

(ii) The sheaf $\mathcal{F}_0$ is torsion because the function $h_0$ is torsion by Lemma 1.4 (i). The other factors $\mathcal{F}_i$ in the direct sum $\mathcal{F}$ are torsion-free by part (iii) which we just proved. So $\mathcal{F}_0$ is exactly the torsion subsheaf of $\mathcal{F}$.

(i) We need to show that for $i = 0, \ldots, s$, the subsheaf $\mathcal{F}_{i+1, \leq \tau} \subset \mathcal{F}$ is $\bigoplus_{k=0}^i \mathcal{F}_k$. So we show that $\mathcal{F}_{i, \leq \tau} = \mathcal{F}_i$ and $\mathcal{F}_{i+1, \leq \tau} = 0$.

First suppose that $g$ is an FDH function of rank $\rho$ and degree $d$, and if $\tau$ is an integer such that $g(n) = \rho(n + 1) + d$ for all $n \geq \tau - 1$, then $g$ is filterable at $\tau$, and $g_\tau^{\text{quot}} = g$ and $g_\tau^{\text{num}} = 0$. So if $\mathcal{G}$ is a general sheaf with FDH function $g$, then $\mathcal{G}_{\leq \tau} = \mathcal{G}$. If we apply this with $g = h_0$ and $\tau = \tau_0$, we see that $\mathcal{F}_{0, \leq \tau_0} = \mathcal{F}_0$. We may also apply this with $g = h_1^i$ and $\tau = \tau_i$ because $\tau_i \geq \zeta_i^j + 1$ where $\zeta_i^j$ is as in the proof of Lemma 1.7. Thus for $i = 1, \ldots, s$, we have $\mathcal{F}_{i, \leq \tau_i} = \bigoplus_{j=1}^{\rho_i} \mathcal{F}_j^i = \bigoplus_{j=1}^{\rho_i} \mathcal{F}_j^i = \mathcal{F}_i$.

To show that $\mathcal{F}_{i+1, \leq \tau_i} = 0$ we need to show that all $h_{i+1, \leq \tau_i}(\eta_i^j) = 0$. But if $\eta_i^j$ is as in the proof of Lemma 1.7, then $h_{i+1, \leq \tau_i}(\eta_i^j) = 0$. But as we showed there $\eta_i^j \geq \tau_i + 1$. This completes the proof of the lemma. \quad \Box

Proof of Theorem 2.3. Part (i) was shown in (2.6). Parts (ii), (iv), and (v) then describe open properties, so it is enough to verify them for the sheaf $\mathcal{F}$ of Lemma 2.7. Part (ii) then follows from Lemma 2.7 (ii). To derive (iv) from (2.7) (iii), note that the fact that the $Z_i^j$ are all disjoint implies that $\text{Hom}(I_{Z_i^j}, I_{Z_i^j}) \cong I_{Z_i^j}$. So $\text{Hom}(\text{gr}_j(\mathcal{E}), \text{gr}_j(\mathcal{E})(-1))$ is a sum of terms of the forms $H^0(I_{Z_j^i}(\nu_i - \nu_j))$, $H^0(I_{Z_j^i}(\nu_i - \nu_j - 1))$, and $H^0(I_{Z_j^i}(\nu_i + 1 - \nu_j))$. In the first two forms the cohomology vanishes because the twists $\nu_i - \nu_j$ or $\nu_i - \nu_j - 1$ are negative. For the third form the twist $\nu_i + 1 - \nu_j$ is nonpositive, but even if it is zero $H^0(I_{Z_j^i})$ vanishes because this form only occurs with $1 \leq l \leq \beta_j$ and in that case $Z_j^i \neq \emptyset$.

Before beginning on (v) note that since the $\text{gr}_i(\mathcal{F}) = \bigoplus_{j=1}^{\rho_i} \mathcal{F}_j^i$ is a direct sum of semistable sheaves, the graded pieces of the Harder-Narasimhan of $\text{gr}_i(\mathcal{F})$ are direct sums of $\mathcal{F}_j^i$'s with proportional Hilbert polynomials. Hence any non-torsion quotient sheaf $\mathcal{G}$ of $\text{gr}_i(\mathcal{F})$ has $P_{\mathcal{G}} \geq \min_j \{ P_{\mathcal{F}_j^i} \}$, which is $P_{\mathcal{F}_j^i}$ by Lemma 2.7 (i), and any nonzero subsheaf $\mathcal{H}$ has $P_{\mathcal{H}} \leq \max_j \{ P_{\mathcal{F}_j^i} \} = P_{\mathcal{F}_j^i}$.

Now to show that the Harder-Narasimhan filtration of $\mathcal{F}$ is a refinement of the WHN filtration, we need to show that for $1 \leq i \leq s$, if $\mathcal{G}$ is a nonzero torsion-free quotient of $\text{gr}_i(\mathcal{F})$ and $\mathcal{H}$ a nonzero subsheaf of $\text{gr}_{i+1}(\mathcal{F})$, then $P_{\mathcal{G}} > P_{\mathcal{H}}$. But by the previous paragraph and Lemma 2.7 (ii) we have $P_{\mathcal{G}} \geq P_{\mathcal{F}_{i+1}^j} > P_{\mathcal{F}_{i+1}^j} \geq P_{\mathcal{H}}$.

To show the second assertion of (v) we now need to show that every nonzero subsheaf of
\[ \text{gr}_i(\mathcal{F}) \text{ has slope at most } -\nu_i \text{ and every non-torsion quotient sheaf has slope at least } -\nu_i - 1. \]

But this is now clear.

In part (iii) the isomorphisms \( \text{gr}_i(\mathcal{F})|_L \cong \mathcal{O}_L(-\nu_i)^{\beta_i} \oplus \mathcal{O}_L(-\nu_i - 1)^{\rho_i - \beta_i} \) follow from Lemma 2.7(iii). Because these latter sheaves are rigid (i.e. generic in the stack of coherent sheaves on \( L \)), a general \( \mathcal{E} \) must have \( \text{gr}_i(\mathcal{E})|_L \) isomorphic to \( \mathcal{O}_L(-\nu_i)^{\beta_i} \oplus \mathcal{O}_L(-\nu_i - 1)^{\rho_i - \beta_i} \).

We now claim that any filtered sheaf \( \mathcal{H} \) such that \( \text{Ext}^1(\text{gr}_i(\mathcal{H}), \text{gr}_j(\mathcal{H})) = 0 \) for all \( i > j \) has \( \mathcal{H} \cong \bigoplus_i \text{gr}_i(\mathcal{H}) \). This claim can easily be verified by induction on the length of the filtration. To apply this to \( \mathcal{E}|_L \), we need to verify that if \( i > j \), then \( \text{Ext}^1(\text{gr}_i(\mathcal{E})|_L, \text{gr}_j(\mathcal{E})|_L) = 0 \). But \( \text{Ext}^1(\text{gr}_i(\mathcal{E})|_L, \text{gr}_j(\mathcal{E})|_L) \) is a direct sum of terms of the form \( H^1(\mathcal{O}_L(\nu_i - \nu_j + \epsilon)) \) with \( \epsilon \in \{-1, 0, 1\} \). Since the \( \nu_i \) form a strictly increasing sequence of integers, the twists \( \nu_i - \nu_j + \epsilon \) are all nonnegative. So the \( H^1 \) vanish. Therefore \( \mathcal{E}|_L \cong \bigoplus_i \text{gr}_i(\mathcal{E})|_L \), completing the proof of (iii).  

\[ \blacksquare \]

**References**

[C] Chang, M.-C.: A filtered Bertini-type theorem, *J. reine angew. Math.* 397 (1989), 214-219.

[D] Davis, E.: 0-Dimensional Subschemes of \( \mathbb{P}^2 \): New Applications of Castelnuovo’s Function, *Ann. Univ. Ferrara* 32 (1986), 93-107.

[DL] Drezet, J.-M., and Le Potier, J.: Fibrés stables et fibrés exceptionnels sur \( \mathbb{P}^2 \), *Ann. scient. Ec. Norm. Sup.* 18 (1985), 193-244.

[HL] Hirschowitz, A., and Laszlo, Y.: Fibrés génériques sur le plan projectif, *Math. Ann.* 297 (1993), 85-102.