The distribution functions of $\sigma(n)/n$ and $n/\varphi(n)$, II

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1. Introduction

Let

$$A(t) := \lim_{N \to \infty} \frac{1}{N} \left| \{n \leq N : \sigma(n)/n \geq t \} \right|,$$

where $\sigma(n)$ is the sum of the positive divisors of $n$, and

$$B(t) := \lim_{N \to \infty} \frac{1}{N} \left| \{n \leq N : n/\varphi(n) \geq t \} \right|,$$

where $\varphi$ denotes Euler’s totient function. Both of these limits exist and are continuous functions of $t$ \[11, 3\].

We are interested in the size of $A(t)$ and $B(t)$ as $t$ tends to infinity. From the work of Erdős \[2\] it follows that

$$B(t) = \exp \left\{ -e^t e^{-\gamma} (1 + o(1)) \right\} \quad (t \to \infty),$$

which was sharpened and extended to $A(t)$ by the author \[6\] with the result

$$A(t), B(t) = \exp \left\{ -e^t e^{-\gamma} \left( 1 + O \left( \frac{t^{-2}}{1} \right) \right) \right\} \quad (t \to \infty)$$

where $\gamma = 0.5772...$ is Euler’s constant.

The purpose of this note is to make further improvements to the error term.

**Theorem 1.** We have

$$A(t), B(t) = \exp \left\{ -e^t e^{-\gamma} \left( 1 + \sum_{j=2}^{m} \frac{a_j}{t^j} + O_m \left( \frac{1}{t^{m+1}} \right) \right) \right\},$$

where

$$a_2 = -\frac{\pi^2}{6} e^{2\gamma}, \quad a_3 = \frac{\pi^2}{6} e^{3\gamma}, \quad a_4 = -\left( \frac{\pi^2}{6} + \frac{37\pi^4}{360} \right) e^{4\gamma}.$$

Additional coefficients $a_i$ can be determined without major difficulties by following the proofs of Lemma \[5\], Lemma \[6\] and Section \[5\], starting with the coefficients $b_i$ from Lemma \[5\].

Throughout we will use the notation

$$y = y(t) := e^t e^{-\gamma}.$$
We can further decrease the size of the error term in Theorem 1 in exchange for a more complex main term. Let
\begin{equation}
I(y, s) := \int_y^\infty \log \left(1 + x e^{-s/x}\right) \frac{dx}{\log x} + \int_y^{\log y} \log \left(1 + x^{-1} e^{s/x}\right) \frac{dx}{\log x},
\end{equation}
and
\begin{equation}
L(y) := \exp \left\{ \frac{(\log y)^{3/5}}{(\log \log y)^{1/5}} \right\}.
\end{equation}

**Theorem 2.** There exists a positive constant \(c\) such that
\[ A(t), B(t) = \exp \left\{ -y + \min_{s \in J} I(y, s) + R(y) \right\}, \]
where \(J = [y \log y - y, y \log y + y]\) and
\[ R(y) = O \left( \frac{y}{L(y)^c} \right). \]
Assuming the Riemann hypothesis we have
\[ R(y) = O \left( \sqrt{y} (\log y)^2 \right). \]

The behavior of \(B(t)\) near \(t = 1\) is described by Tenenbaum and Toulmonde [4, Thm. 1.2], who show that
\begin{equation}
1 - B(1 + 1/(\sigma - 1)) = \sum_{j=1}^{m} \frac{g_j}{(\log \sigma)^j} + O \left( \frac{|g_{m+1}|}{(\log \sigma)^{m+1}} + \frac{1}{L(\sigma)^c} \right),
\end{equation}
for some \(c > 0\), where
\[ g_1 = e^{-\gamma}, \quad g_2 = 0, \quad g_3 = -\frac{1}{12} \pi^2 e^{-\gamma}, \]
and
\[ g_j = \{1 + O(j^{-1})\} e^{-\gamma} (-1)^{j+1} (j - 3)! \quad (j \geq 3). \]

A classic result (see e.g. [3]) states that for all \(s \in \mathbb{C}\) we have
\begin{equation}
W(s) := \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \left( \frac{n}{\varphi(n)} \right)^s = \prod_p \left( 1 + \frac{(1 - p^{-1})^{-s} - 1}{p} \right)
\end{equation}
and thus
\begin{equation}
\int_0^\infty B(x) x^{s-1} dx = 0 - \frac{1}{s} \int_0^\infty x^s dB(x) = \frac{W(s)}{s}, \quad (\Re(s) > 0).
\end{equation}

Hence \(\frac{W(s)}{s}\) is the Mellin transform of \(B(t)\). The method used in [4] to establish (4) is essentially that of inversion of the Mellin transform with the abscissa of integration moved to \(-\sigma\). For large \(t\) on the other hand, we find that \(W(s) t^{-s}\) is small when \(\Re(s)\) is close to \(y \log y\). It turns out that the minimum of \(W(s) t^{-s}\) with respect to \(s\) along the positive real axis is already an excellent estimate for \(B(t)\) (see Lemma 2), and it appears that inversion is not a natural choice in this case because of the slower convergence of the product in (5) when \(\Re(s) > 0\). Therefore we will restrict our investigation to \(s \in [0, \infty)\).

The following result shows that \(A(t)\) and \(B(t)\) are close enough so that it suffices to show that Theorems 1 and 2 hold for \(B(t)\), which is the simpler object since \(\varphi(n)\) does not depend on the multiplicities of the prime factors of \(n\).
Theorem 3. For \( t \geq t_0 \) we have
\[
A(t) \leq B(t) < e^{3\sqrt{y}} A\left(t - \frac{5e^\gamma}{\sqrt{y}}\right)
\]

Another arithmetic function closely related to \( \varphi \) and \( \sigma \) is Dedekind’s \( \psi \) function, defined by
\[
\psi(n) = n \prod_{p|n} (1 + p^{-1}).
\]

With
\[
D(t) := \lim_{N \to \infty} \frac{1}{N} \left| \{n \leq N : \psi(n)/n \geq t \} \right|,
\]
one can show that \( D(t/\zeta(2)) \) also satisfies Theorems 1 and 2. It is easy to see that \( D(t/\zeta(2)) \geq B(t) \) using the definition of \( \psi \) and \( \varphi \). For the upper bound of \( D(t/\zeta(2)) \) one can consider the analog of Lemma 2 (i) below.

2. Proof of Theorem 3

The inequality \( A(t) \leq B(t) \), valid for all \( t \), follows from
\[
\frac{\sigma(n)}{n} = \prod_{p^r|n} \frac{1 + p + \ldots + p^r}{p^r} = \prod_{p^r|n} \frac{1 - p^{-\nu - 1}}{1 - p^{-1}} < \prod_{p|n} \frac{1}{1 - p^{-1}} = \frac{n}{\varphi(n)}.
\]

To establish the second inequality of Theorem 3, we let
\[
m = m(t) = \prod_{p \leq \sqrt{y}} p^{h_p}, \quad \text{where } h_p = \left\lfloor \frac{\log y}{\log p} \right\rfloor.
\]

For every \( n \) that satisfies
\[
\frac{n}{\varphi(n)} = \prod_{p|n} \frac{1}{1 - p^{-1}} \geq t,
\]
nm will satisfy
\[
\frac{\sigma(nm)}{nm} = \prod_{p^k|nm} \frac{1 - p^{-k-1}}{1 - p^{-1}} = \prod_{p|nm} \frac{1}{1 - p^{-1}} \prod_{p^k|nm} (1 - p^{-k-1}) \geq tP,
\]
where
\[
P = \prod_{p^k|nm} (1 - p^{-k-1}) \geq \prod_{p \leq \sqrt{y}} \left(1 - \frac{1}{y} \right) \prod_{p > \sqrt{y}} \left(1 - \frac{1}{p^2} \right) \geq 1 - \frac{5}{\sqrt{y} \log y},
\]
for \( t \geq t_0 \), by a standard application of the prime number theorem. Thus
\[
\frac{\sigma(nm)}{nm} \geq t \left(1 - \frac{5}{\sqrt{y} \log y} \right) = t - \frac{5e^\gamma}{\sqrt{y}},
\]
which implies
\[
A \left(t - \frac{5e^\gamma}{\sqrt{y}}\right) \geq \frac{1}{m} B(t).
\]
The result now follows since, for \( t \geq t_0 \),
\[
\log m = \sum_{p \leq \sqrt{y}} h_p \log p \leq \sum_{p \leq \sqrt{y}} \log y < 3\sqrt{y}.
\]
3. The relation between \( B(t) \) and \( W(s) \).

**Lemma 1.** Let \( s \geq 1 \). If
\[
B(t)t^{s-1} = \max_{x \geq 0} B(x)x^{s-1}
\]
then
\[
s = y \log y + O(y).
\]

**Proof.** Assume \( B(t)t^{s-1} \geq B(t+h)(t+h)^{s-1} \) for \( |h| \leq 1 \). After taking logarithms we use (1) to obtain
\[
y(e^{he^{-\gamma}} - 1) + O(yt^{-2}) \geq (s-1)\log(1 + ht^{-1}),
\]
and hence
\[
yhe^{-\gamma} \geq (s-1)ht^{-1} + O(sh^2t^{-2} + yh^2 + yt^{-2}).
\]
The result now follows if we first let \( h = t^{-1} \), and then \( h = -t^{-1} \), and multiply the last inequality by \( h^{-1}t \) in each case. \( \Box \)

**Lemma 2.**

(i) For all \( s \geq 0 \), \( t > 0 \) we have
\[
B(t) \leq \frac{W(s)}{t^s}.
\]

(ii) Let \( s \geq 1 \) and \( t \geq t_0 \). If \( B(t)t^{s-1} = \max_{x \geq 0} B(x)x^{s-1} \), then
\[
\frac{W(s)}{3st^s} \leq B(t)
\]
and
\[
\log B(t) = O(t) + \min_{u \geq 0} \log \frac{W(u)}{tu}.
\]

**Proof.** (i) For all \( s \geq 0 \),
\[
B(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N, n \geq t \varphi(n)} 1 \leq \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \left( \frac{n}{t \varphi(n)} \right)^s = \frac{W(s)}{t^s}.
\]

(ii) From (6) we have
\[
\frac{W(s)}{s} = \int_0^t B(x)x^{s-1}dx + \int_t^{2t} B(x)x^{s-1}dx + \int_{2t}^\infty B(x)x^{s-1}dx =: I_1 + I_2 + I_3
\]
Since \( \max_{x \geq 0} B(x)x^{s-1} = B(t)t^{s-1} \), we have \( I_1, I_2 \leq tB(t)t^{s-1} = B(t)t^s \). If \( c \) is the implied constant in the error term of (1), then for \( x \geq t \)
\[
B(2x) \leq \exp \left\{ -e^{2xe^{-\gamma}}(1 - cx^{-2}) \right\} \leq \exp \left\{ -e^{xe^{-\gamma}}(y/2)(1 + cx^{-2}) \right\}
\]
\[
\leq B(x)^{y/2} \leq B(x)^{1+\log y} \leq B(x)B(t)^{\log y}
\]
\[
\leq B(x) \exp \left\{ -y \log y + O(y/\log y) \right\} = B(x) \exp \left\{ -s + O(y) \right\}
\]
\[
\leq \frac{B(x)}{2^{s+1}},
\]
since \( s = y \log y + O(y) \) by Lemma 1. We conclude that for \( k \geq 1 \)
\[
\int_{t^{2k+1}}^{t^{2k+1}} B(x)x^{s-1}dx = 2^s \int_{t^{2k-1}}^{t^{2k}} B(2x)x^{s-1}dx \leq \frac{1}{2} \int_{t^{2k-1}}^{t^{2k}} B(x)x^{s-1}dx,
\]
and thus \( I_3 \leq I_2 \leq B(t)t^s \).

The second assertion in (ii) follows from the first and (i), since \( s = y \log y + O(y) \). \( \Box \)
4. The study of the product $W(s)$.

Let

$$t_u := \prod_{p \leq u} \frac{1}{1-p^{-1}}, \quad P_u := \prod_{p \leq u} p.$$

**Lemma 3.** Let $2 \leq u \leq v$. For $s \ll v$ we have

$$\frac{W(s)}{t_u^s} = \frac{t_u}{t_v P_u} \left( 1 + O\left( \frac{s}{v \log v} \right) \right) \prod_{p \leq u} \left( 1 + p \left( 1 - p^{-1} \right)^{s+1} \right) \prod_{u < p \leq v} \left( 1 + p^{-1} \left( 1 - p^{-1} \right)^{-s-1} \right).$$

**Proof.** The contribution from primes $p > v$ to the product (5) is

$$\prod_{p > v} \left( 1 + \frac{(1-p^{-1})^{-s-1}}{p} \right) = \prod_{p > v} \left( 1 + \frac{1}{p} \left( e^{O\left( \frac{s}{p^2} \right)} - 1 \right) \right) = \prod_{p > v} \left( 1 + O\left( \frac{s}{p^2} \right) \right) = 1 + O\left( \frac{s}{v \log v} \right).$$

For primes $p$ in the range $u < p \leq v$ we write

$$\prod_{u < p \leq v} \left( 1 + \frac{(1-p^{-1})^{-s-1}}{p} \right) = \prod_{u < p \leq v} \left( 1 - p^{-1} \right) \prod_{u < p \leq v} \left( 1 + p^{-1} \left( 1 - p^{-1} \right)^{-s-1} \right).$$

Finally, the product over small primes is

$$\prod_{p \leq u} \left( \frac{(1-p^{-1})^{-s}}{p} \right) \prod_{p \leq u} \left( 1 + p \left( 1 - p^{-1} \right)^{s+1} \right) = \frac{t_u}{P_u} \prod_{p \leq u} \left( 1 + p^{-1} \left( 1 - p^{-1} \right)^{s+1} \right).$$

**Lemma 4.** Let $2 \leq u \leq v$. For $v \gg s = u \log u + O(u)$ we have

$$\frac{W(s)}{t_u^s} = \frac{t_u}{t_v P_u} \left( 1 + O\left( \frac{1}{\log u} \right) \right) \prod_{p \leq u} \left( 1 + p e^{-s/p} \right) \prod_{u < p \leq v} \left( 1 + p^{-1} e^{s/p} \right).$$

**Proof.** We write

$$\prod_{u < p \leq v} \left( 1 + p^{-1} \left( 1 - p^{-1} \right)^{-s-1} \right) = \prod_{u < p \leq v} \left( 1 + p^{-1} \exp\left( \frac{s}{p} + O\left( \frac{s}{p^2} \right) \right) \right) = \prod_{u < p \leq v} \left( 1 + p^{-1} e^{s/p} \right) \left( 1 + O\left( \frac{s}{p^3} e^{s/p} \right) \right).$$

After taking the logarithm of the last expression, the contribution from the error term is

$$\ll \sum_{p > u} \frac{s}{p^3} e^{s/p} \int_u^\infty \frac{s}{x^3} e^{s/x} \frac{dx}{\log x} \times \frac{1}{u \log u} \int_u^\infty \frac{s}{x^3} e^{s/x} \frac{dx}{\log x} = \frac{1}{u \log u} e^{s/u} \times \frac{1}{\log u}.$$

Thus

$$\prod_{u < p \leq v} \left( 1 + p^{-1} \left( 1 - p^{-1} \right)^{-s-1} \right) = \left( 1 + O\left( \frac{1}{\log u} \right) \right) \prod_{u < p \leq v} \left( 1 + p^{-1} e^{s/p} \right).$$
Similarly,
\[
\prod_{p \leq u} (1 + p (1 - p^{-1})^{s+1}) = \prod_{p \leq u} \left( 1 + p \exp \left( -\frac{s}{p} + O \left( \frac{s}{p^2} \right) \right) \right) \\
= \prod_{p \leq u} \left( 1 + p e^{-s/p} \right) \left( 1 + O \left( \frac{s}{p} e^{-s/p} \right) \right).
\]

The contribution from the error term to the logarithm of the last expression is
\[
\times \sum_{p \leq u} \frac{s}{p} e^{-s/p} \times \int_{\sqrt{2}}^u \frac{s}{x} e^{-s/x} \frac{dx}{\log x} \tag{8}
\]
\[
\times \frac{u}{\log u} \int_2^u \frac{s}{x^2} e^{-s/x} dx \times \frac{1}{\log u}.
\]

Thus
\[
\prod_{p \leq u} (1 + p (1 - p^{-1})^{s+1}) = \left( 1 + O \left( \frac{1}{\log u} \right) \right) \prod_{p \leq u} \left( 1 + p e^{-s/p} \right).
\]

The result now follows from Lemma 3. \(\square\)

**Lemma 5.** Let \(s \geq e\) and define \(z\) by \(s = z \log z\). For \(m \geq 2\) we have
\[
W(s) = \exp \left( z \log z \log (e^\gamma \log z) - z + z \sum_{j=2}^{m} \frac{b_j}{(\log z)^j} + O_m \left( \frac{z}{(\log z)^{m+1}} \right) \right),
\]
where
\[
b_2 = \frac{\pi^2}{6}, \quad b_3 = -\frac{\pi^2}{6}, \quad b_4 = \frac{\pi^2}{6} + \frac{7\pi^4}{60}.
\]

**Proof.** We apply Lemma 4 with \(u = z\) and \(v = s\) to obtain
\[
\log W(s) = -s \sum_{p \leq z} \log (1 - p^{-1}) - \sum_{p \leq z} \log p + O \left( \frac{\log_2 z}{\log z} \right) \\
+ \sum_{p \leq z} \log \left( 1 + p e^{-s/p} \right) + \sum_{z < p \leq s} \log \left( 1 + p^{-1} e^{s/p} \right) \\
= z \log z \log (e^\gamma \log z) - z + O \left( \frac{z}{\exp(\sqrt{\log z})} \right) \\
+ \int_\epsilon^z \log \left( 1 + x e^{-s/x} \right) \frac{dx}{\log x} + \int_z^s \log \left( 1 + x^{-1} e^{s/x} \right) \frac{dx}{\log x},
\]
by a strong form of Mertens’ Theorem \([5]\) and a standard application of the prime number theorem. We need to estimate the two integrals in \((9)\). The first integral is
\[
\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \int_\epsilon^z x^k e^{-s/x} \frac{dx}{\log x} = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} I_k(k, 1),
\]
where
\[
I_k(a, b) := \int_e^z x^a e^{-skx} \frac{dx}{(\log x)^b} = \frac{1}{sk} \int_e^z \left( \frac{sk}{x^2} e^{-skx} \right) \frac{x^{a+2}}{(\log x)^b} dx
\]
(11)
\[
\leq \frac{z^{a+2}}{sk(k \log z)^{b+1}} \int_e^z \frac{sk}{x^2} e^{-skx} dx \leq \frac{z^{1+a-k}}{k(k \log z)^{b+1}},
\]
for \(a \geq b\), since \(x \log x\) is increasing for \(x \geq e\). Integration by parts applied to the second integral in (11) shows that
\[
I_k(a, b) = \frac{z^{1+a-k}}{k(k \log z)^{b+1}} - \frac{a+2}{sk} I_k(a+1, b) + \frac{b}{sk} I_k(a+1, b+1) + O_m(1/(sk)),
\]
for \(a \leq k + m\). After \(m - 1\) iterations of (12), starting with \(I_k(k, 1)\), we find that
\[
I_k(k, 1) = \sum_{j=2}^{m} \frac{z}{(\log z)^j} q_j(k) + O_m \left( \frac{z}{k(k \log z)^{m+1}} \right),
\]
where \(q_j(k)\) is a rational function of \(k\) with \(q_j(k) = O(1/k)\). In particular,
\[
q_2(k) = \frac{1}{k}, \quad q_3(k) = -\frac{k+2}{k^2}, \quad q_4(k) = \frac{1}{k^2} + \frac{(k+2)(k+3)}{k^3}.
\]
Inserting (13) into (10) gives
\[
\int_e^z \log \left( 1 + x e^{-s/x} \right) \frac{dx}{\log x} = z \sum_{j=2}^{m} \frac{\theta_j}{(\log z)^j} + O_m \left( \frac{z}{(\log z)^{m+1}} \right),
\]
where
\[
\theta_j = \sum_{k \geq 1} (-1)^{k+1} \frac{q_j(k)}{k}.
\]
Similarly, the second integral in (11) is
\[
\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \int_e^z x^{-k} e^{skx} \frac{dx}{\log x} = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} J_k(k, 1),
\]
where
\[
J_k(a, b) := \int_e^z x^a e^{skx} \frac{dx}{(\log x)^b} = \frac{1}{sk} \int_e^z \left( \frac{sk}{x^2} e^{skx} \right) \frac{x^{-a+2}}{(\log x)^b} dx
\]
(16)
\[
\times \frac{1}{sk(k \log z)^{b+1}} \int_k^k e^w \left( \frac{sk}{w} \right)^{2-a} dw = O_m \left( \frac{z^{1+k-a}}{(k \log z)^{b+1}} \right),
\]
for \(a \geq -m\). Integration by parts applied to the second integral in (16) shows that
\[
J_k(a, b) = \frac{z^{1+k-a}}{k(k \log z)^{b+1}} + \frac{2-a}{sk} J_k(a-1, b) - \frac{b}{sk} J_k(a-1, b+1) + O_m \left( k^{-1} e/s)^{a-1} \right),
\]
for \(k - a \leq m\). After \(m - 1\) iterations of (17), starting with \(J_k(k, 1)\), we find that
\[
J_k(k, 1) = \sum_{j=2}^{m} \frac{z}{(\log z)^j} r_j(k) + O_m \left( \frac{z}{k(k \log z)^{m+1}} \right),
\]
where \(r_j(k)\) is a rational function of \(k\) with \(r_j(k) = O(1/k)\). In particular,
\[
r_2(k) = \frac{1}{k}, \quad r_3(k) = \frac{2-k}{k^2}, \quad r_4(k) = \frac{(2-k)(3-k)}{k^3} - \frac{1}{k^2}.
\]
Inserting (18) into (15) gives

\[
\int_{z}^{s} \log \left( 1 + x^{-1} e^{s/x} \right) \frac{dx}{\log x} = z \sum_{j=2}^{m} \frac{\rho_j}{(\log z)^j} + O_m \left( \frac{z}{(\log z)^{m+1}} \right),
\]
where

\[
\rho_j = \sum_{k \geq 1} \frac{(-1)^{k+1} r_j(k)}{k}.
\]

Let \( b_j = \theta_j + \rho_j \), then

\[
b_2 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (q_2(k) + r_2(k)) = 2 \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^2} = 2 \sum_{k \geq 1} \frac{1}{k^2} - 4 \sum_{k \geq 1} \frac{1}{(2k)^2} = \frac{\pi^2}{6},
\]
\[
b_3 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (q_3(k) + r_3(k)) = -2 \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^2} = -\frac{\pi^2}{6},
\]
and

\[
b_4 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (q_4(k) + r_4(k)) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \left( \frac{2}{k} + \frac{12}{k^3} \right) = \frac{\pi^2}{6} + \frac{7\pi^4}{60}.
\]

The result now follows from combining (9), (14) and (19).

**Lemma 6.** For \( t \geq 1 \) and \( y = e^t e^{-\gamma} \) we have

\[
\min_{s \geq \frac{y}{t}} \frac{W(s)}{ts} = \exp \left( -y + y \sum_{k=2}^{m} \frac{c_k}{(\log y)^k} + O_m \left( \frac{y}{(\log y)^{m+1}} \right) \right),
\]
where

\[
c_2 = \frac{\pi^2}{6}, \quad c_3 = -\frac{\pi^2}{6}, \quad c_4 = \frac{\pi^2}{6} + \frac{37\pi^4}{360}.
\]

**Proof.** Let \( t \geq 1 \) be given. From Lemma 5, we have

\[
\log \frac{W(s)}{ts} = z \left( \log z \log(\log z/\log y) - 1 + \sum_{k=2}^{m} \frac{b_k}{(\log z)^k} + O_m \left( \frac{1}{(\log z)^{m+1}} \right) \right) =: h(z),
\]
where \( s = z \log z \). We see that \( h(y) \sim -y \) and \( h(z) > 0 \) for \( z \geq ey \), so that the minimum of \( h(z) \) occurs at some \( z \in [e, ey] \), where the error term of (20) is uniformly \( O_m \left( (y/(\log y)^{m+1}) \right) \). Therefore we only need to minimize

\[
f(z) := z \left( \log z \log(\log z/\log y) - 1 + \sum_{k=2}^{m} \frac{b_k}{(\log z)^k} \right).
\]

To that end we set \( f'(z) = 0 \), which is equivalent to

\[
\log y = \log z \exp \left( \sum_{k=2}^{m+1} \frac{\alpha_k}{(1 + \log z) \log^k z} \right),
\]
where \( \alpha_2 = b_2, \alpha_k = b_k - (k-1)b_{k-1} \) for \( k = 3, \ldots, m - 1 \), and \( \alpha_{m+1} = -mb_m \). Thus

\[
\alpha_2 = b_2 = \frac{\pi^2}{6}, \quad \alpha_3 = b_3 - 2b_2 = -\frac{\pi^2}{2}, \quad \alpha_4 = b_4 - 3b_3 = \frac{2\pi^2}{3} + \frac{7\pi^4}{60}.
\]
Since \( f(e) \sim -e \log \log y, \ f(y) \sim -y, \) and \( f(ey) > 0, \) the unique solution to (22) is the minimizer of \( f(z). \) We rewrite (22) as

\[
\log z \log \left( \frac{\log z}{\log y} \right) = -\sum_{k=2}^{m} \frac{\beta_k}{(\log z)^k} + O\left( \frac{1}{(\log z)^{m+1}} \right),
\]

where \( \beta_2 = \alpha_2 \) and \( \beta_k = \alpha_k - \beta_{k-1} \) for \( k = 3, \ldots, m. \) Thus

\[
\beta_2 = \frac{\pi^2}{6}, \quad \beta_3 = \alpha_3 - \beta_2 = -\frac{2\pi^2}{3}, \quad \beta_4 = \alpha_4 - \beta_3 = \frac{4\pi^2}{3} + \frac{7\pi^4}{60}.
\]

To express \( z \) in terms of \( y \) we first write (23) as

\[
\log y = \log z \exp \left( \sum_{k=2}^{m} \frac{\beta_k}{(\log z)^{k+1}} + O\left( \frac{1}{(\log z)^{m+2}} \right) \right)
\]

(24)

\[
= \log z \left( 1 + \sum_{k=2}^{m} \frac{\delta_k}{(\log z)^{k+1}} + O\left( \frac{1}{(\log z)^{m+2}} \right) \right),
\]

where

\[
\delta_2 = \beta_2 = \frac{\pi^2}{6}, \quad \delta_3 = \beta_3 = -\frac{2\pi^2}{3}, \quad \delta_4 = \beta_4 = \frac{4\pi^2}{3} + \frac{7\pi^4}{60}.
\]

Using series inversion on (24) we obtain

\[
\log z = \log y \left( 1 + \sum_{k=2}^{m} \frac{\eta_k}{(\log y)^{k+1}} + O\left( \frac{1}{(\log y)^{m+2}} \right) \right),
\]

(25)

where

\[
\eta_2 = -\delta_2 = -\frac{\pi^2}{6}, \quad \eta_3 = -\delta_3 = -\frac{2\pi^2}{3}, \quad \eta_4 = -\delta_4 = -\frac{4\pi^2}{3} - \frac{7\pi^4}{60}.
\]

We exponentiate (25) to get

\[
z = y \exp \left( \sum_{k=2}^{m} \frac{\eta_k}{(\log y)^k} + O\left( \frac{1}{(\log y)^{m+1}} \right) \right)
\]

(26)

\[
y \left( 1 + \sum_{k=2}^{m} \frac{\lambda_k}{(\log y)^k} + O\left( \frac{1}{(\log y)^{m+1}} \right) \right)
\]

where

\[
\lambda_2 = \eta_2 = -\frac{\pi^2}{6}, \quad \lambda_3 = \eta_3 = -\frac{2\pi^2}{3}, \quad \lambda_4 = \eta_4 = -\frac{4\pi^2}{3} - \frac{7\pi^4}{360}.
\]

Combining (21), (23) and (26) we see that \( \min_z f(z) \) is

\[
y \left( 1 + \sum_{k=2}^{m} \frac{\lambda_k}{(\log y)^k} + O\left( \frac{1}{(\log y)^{m+1}} \right) \right) \left( 1 - 1 + \sum_{k=2}^{m} \frac{b_k - \beta_k}{(\log z)^k} + O\left( \frac{1}{(\log z)^{m+1}} \right) \right)
\]

\[
= y \left( 1 + \sum_{k=2}^{m} \frac{\lambda_k}{(\log y)^k} + O\left( \frac{1}{(\log y)^{m+1}} \right) \right) \left( 1 - 1 + \sum_{k=2}^{m} \frac{\mu_k}{(\log y)^k} + O\left( \frac{1}{(\log y)^{m+1}} \right) \right),
\]

where (24) implies

\[
\mu_2 = b_2 - \beta_2 = 0, \quad \mu_3 = b_3 - \beta_3 = -\frac{\pi^2}{2}, \quad \mu_4 = b_4 - \beta_4 = -\frac{7\pi^2}{6}.
\]
Thus
\[
\min_z f(z) = -y + y \sum_{k=2}^{m} \frac{c_k}{(\log y)^k} + O_m \left( \frac{y}{(\log y)^{m+1}} \right),
\]
where
\[
c_2 = \mu_2 - \lambda_2 = \frac{\pi^2}{6}, \quad c_3 = \mu_3 - \lambda_3 = -\frac{\pi^2}{6}, \quad c_4 = \mu_4 + \mu_2 \lambda_2 - \lambda_4 = \frac{\pi^2}{6} + \frac{37}{360} \pi^4.
\]

\textbf{Lemma 7.} Let \( g(s) = \log W(s) \). We have
\[
g''(s) \asymp \frac{1}{s \log s} \quad (s \geq 2).
\]

**Proof.** From (5) we find that
\[
g''(s) = \sum_p p^{-1} (1 - p^{-1}) \log^2(1 - p^{-1}) \frac{(1 - p^{-1})^{-s}}{(1 + p^{-1}((1 - p^{-1})^{-s} - 1))^2}
\]
\[
\asymp \sum_p p^{-3} \frac{(1 - p^{-1})^{-s}}{(1 + p^{-1}(1 - p^{-1})^{-s})^2}
\]
Let \( u \) be given by \( u^{-1}(1 - u^{-1})^{-s} = 1 \), so that \( s = u \log u + O(\log u) \). Then
\[
g''(s) \asymp \sum_{p \leq u} p^{-1} (1 - p^{-1}) \log s + \sum_{p > u} p^{-3} (1 - p^{-1})^{-s}
\]
\[
\asymp \sum_{p \leq u} p^{-1} e^{-s/p} + \sum_{p > u} p^{-3} e^{s/p} \asymp \frac{1}{s \log s},
\]
where the last two sums are estimated just like in (7) and (8).

\section{5. Proof of Theorems 1 and 2}

Define the set of maximizers
\[
M := \{ t \geq 1 : \exists s > 1 \text{ with } \max_{x \geq 0} B(x)x^{s-1} = B(t)x^{s-1} \}.
\]

**Lemma 8.** There is a constant \( c > 0 \) such that for every \( t \geq 1 \) there is a \( t_1 \in M \) with
\[
|t - t_1| \leq c \sqrt{t/y}.
\]

**Proof.** Let \( t \geq 1 \) and let \( s \) be given by \( \min_u \frac{W(u)}{t^3} = \frac{W(s)}{t^3} \). Let \( t_1 \in M \) satisfy \( \max_{x \geq 1} B(x)x^{s-1} = B(t_1)x^{t_1^{s-1}} \). Finally, define \( s_1 \) by \( \min_u \frac{W(u)}{t_1^3} = \frac{W(s_1)}{t_1^3} \). From Lemma 2 we find
\[
\frac{W(s)}{3st_1^s} \leq B(t_1) \leq \frac{W(s_1)}{t_1^{s_1}} \leq \frac{W(s)}{t_1^s},
\]
so
\[
\log \frac{W(s)}{t_1^s} = \log \frac{W(s_1)}{t_1^{s_1}} + O(\log s).
\]
By Taylor’s theorem there is an \( s_0 \) between \( s \) and \( s_1 \) with
\[
\log \frac{W(s)}{t_1^s} = \log \frac{W(s_1)}{t_1^{s_1}} + g''(s_0) (s - s_1)^2,
\]

where \( g(u) = \log W(u) \). Combining the last two equations with Lemma 7 we obtain
\[
|s - s_1| = O(\sqrt{s} \log s).
\]

Let \( f(u) = \exp(g'(u)) \). From the definition of \( s \) and \( s_1 \) we have \( t = f(s) \) and \( t_1 = f(s_1) \). Thus \( |t - t_1| \leq |s - s_1| \max_I f'(u) \), where \( I \) is the interval with endpoints \( s, s_1 \). Now \( f'(u) = f(u) g''(u) \), so Lemma 7 yields
\[
|t - t_1| = O\left( \frac{t \sqrt{s} \log s}{s \log s} \right) = O\left( \sqrt{t/y} \right),
\]
by Lemma 1. \( \square \)

Proof of Theorem 1. If \( t \in M \) then the result follows from Lemma 2 (ii) and Lemma 6 with \( a_j = -c je^{\gamma} \). If \( t \notin M \), the result follows from Lemma 8 and the monotonicity of \( B(t) \). \( \square \)

Proof of Theorem 2. We apply Lemma 4 with \( u = y \) and \( v = y \log y \). For \( s = y \log y + O(y) \),
\[
\log \frac{W(s)}{ts} = -s \log t - s \sum_{p \leq y} \log(1 - p^{-1}) - \sum_{p \leq y} \log p + O\left( \frac{\log_2 y}{\log y} \right)
+ \sum_{p \leq y} \log \left( 1 + pe^{-s/p} \right) + \sum_{y < p \leq y \log y} \log \left( 1 + p^{-1} e^{s/p} \right)
= -y + I(y, s) + O\left( \frac{y}{L(y)^c} \right),
\]
by a strong form of Mertens’ Theorem 5 and a standard application of the prime number theorem. Under the assumption of the Riemann hypothesis, the error term can be replaced by \( O(\sqrt{y} (\log y)^2) \). If \( t \in M \) and \( \max_{x \geq 0} B(x) x^{s-1} = B(t) t^{s-1} \), then \( s = y \log y + o(y) \), by an argument like in the proof of Lemma 1 but this time using Theorem 4 with \( m = 2 \) instead of 1. Therefore Lemma 2 implies
\[
\log B(t) = O(t) + \min_{u \in J} \frac{W(u)}{t^u} = O(t) - y + \min_{u \in J} I(y, u) + O\left( \frac{y}{L(y)^c} \right),
\]
where \( J = [y \log y - y, y \log y + y] \). If \( t \notin M \), there is a \( t_1 \in M \) with \( |t - t_1| = O(\sqrt{t/y}) \) by Lemma 8. For \( s = y \log y + O(y) \) and \( y_1 := e^{\gamma} e^{-\gamma} = y + O(\sqrt{t/y}) \) we have \( I(y_1, s) = I(y, s) + O(\sqrt{t/y}) \). Thus the result follows again from the monotonicity of \( B(t) \). \( \square \)

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