Regularizing transformations of polygons

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Abstract. We start with a generic \( n \)-gon \( Q_0 \) with vertices \( q_{j,0} (j = 0, \ldots, n - 1) \) in the \( d \)-dimensional Euclidean space \( \mathbb{E}^d \). Additionally, \( m + 1 \) real numbers \( u_0, \ldots, u_m \in \mathbb{R} (m < n) \) with \( \sum_{\mu=0}^{m} u_{\mu} = 1 \) are given. From these initial data we iteratively define generations of \( n \)-gons \( Q_k \) in \( \mathbb{E}^d \) for \( k \in \mathbb{N} \) with vertices \( q_{j,k} := \sum_{\mu=0}^{m} u_{\mu} q_{j+\mu,k-1} \). We can show that this affine iteration generally regularizes in an affine sense.

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1. Introduction

Schoenberg [6], Ziv [7], Nicoller [2] and Donisi et al. [1] studied geometric iteration processes starting with a generic \( n \)-gon \( Q_0 \) in \( \mathbb{E}^2 \). They use homotheties to construct vertices of a next generation polygon \( Q_1 \). Reiterating this process creates a series of generations \( Q_k \). This iteration, in general, has a regularizing effect on the polygon. Surprisingly, the result for \( n \)-gons in the plane \( \mathbb{E}^2 \) presented by Roeschel in [5] is also valid for \( n \)-gons in higher dimensions. In [5] the proof for \( \mathbb{E}^2 \) is based on the fact that the space of planar \( n \)-gons is spanned by the planar prototype \( n \)-gons of \( \mathbb{E}^2 \). As this does not hold for higher dimensions the proof for \( \mathbb{E}^d \) with \( d \geq 3 \) demands another approach with different arguments. We prove an affine regularization theorem: these iterations in higher dimensions also deliver generations \( Q_k \) approaching the affine shape of regular planar polygons.

2. The spatial affine iteration

We use vectors in \( \mathbb{R}^d \) to describe points of the \( d \)-dimensional Euclidean space \( \mathbb{E}^d (d > 2) \) with respect to a Cartesian coordinate frame \( \{O;x_1, \ldots, x_d\} \). We start with some spatial \( n \)-gon \( Q_0 \subset \mathbb{E}^d \) with vertices \( \{q_{0,0}, q_{1,0}, \ldots, q_{n-1,0}\} \)
Figure 1 An example for \(n = 8\) and \(m = d = 3\): the polygon \(Q_0\) with vertices of a cube and the first generation polygon \(Q_1\) for \((u_0, u_1, u_2, u_3) = (0.2, -0.35, 0.75, 0.4)\)

((n > 2, q_{j,0} \in \mathbb{R}^d)). Our starting polygon \(Q_0\) shall be called \textit{polygon of generation 0}.

On the other hand in an \(m\)-dimensional affine space \(\mathbb{R}^m\) \((0 < m < n)\) with a simplex \(S := \{a_0, \ldots, a_m\}\) we choose a \textit{reference point} \(z^*\) with respect to \(S\): Let \(z^* := \sum_{\mu=0}^{m} u_{\mu} a_{\mu}\) be given by its barycentric coordinates \((u_0, \ldots, u_m) \in \mathbb{R}^{1 \times (m+1)}\) with \(\sum_{\mu=0}^{m} u_{\mu} = 1\).

Let \(\alpha_{j,1}\) be the affine mappings from the ordered reference simplex vertex set \(S\) to ordered sets of \(m\) consecutive vertices \(q_{j,0}, \ldots, q_{j+m,0}\) of \(Q_0\) \((j \in \mathbb{J} := \{0, \ldots, n-1\}; \text{first index } \text{mod } n\). Each of these \(n\) affine mappings is applied to the reference point \(z^*\); this way we get \(n\) image points \(q_{j,1} := \alpha_{j,1}(z^*) = \sum_{\mu=1}^{m} u_{\mu} q_{j+\mu-1,0}\) which form a new \(n\)-gon \(Q_1\) called the \textit{generation 1 polygon}.

The same process can now be applied, in turn, to the polygon \(Q_1\) with the same reference simplex \(S\) and the same reference point \(z^*\), creating a subsequent polygon \(Q_2\). Iteration yields a series of polygons. \(Q_k := \{q_{0,k}, \ldots, q_{n-1,k}\}\) is the \(k\)th \textit{generation polygon} with vertices

\[
q_{j,k} = \sum_{\mu=0}^{m} u_{\mu} q_{j+\mu,k-1} \in \mathbb{R}^d \quad (j \in \mathbb{J}, k \in \mathbb{N}\{0\}).
\] (2.1)

The procedure is a \(d\)-dimensional generalisation of the geometric iteration presented in [5]. Figure 1 shows the first iteration step for an example with \(n = 8\) and \(m = d = 3\).

3. The iteration process

We describe the polygons \(Q_k\) by \(d \times n\)-matrices \(Q_k := (q_{0,k}, \ldots, q_{n-1,k})\) in \(\mathbb{R}^{d \times n}\) with \(q_{j,k}\) (2.1). Formula (2.1) can be rewritten as a product of matrices \(Q_k := Q_{k-1} \cdot M\) with the circulant \(n \times n\)-matrix \(M \in \mathbb{R}^{n \times n}\):
The $n$th complex roots of unity $\in \mathbb{C}$ shall be termed $\zeta_j := \exp(i \frac{2j\pi}{n}) = \cos \frac{2j\pi}{n} + i \sin \frac{2j\pi}{n}$ ($j \in \mathbb{Z}$). We define the vectors

$$P_j := (\zeta_j^0, \ldots, \zeta_j^{n-1}) \in \mathbb{C}^{1 \times n} \quad (j \in \mathbb{Z})$$

(3.2)

and have $P_j \cdot M = P_j \sum_{\mu=0}^m u_{\mu} \zeta_j^\mu$ and $M \cdot P^t_{n-j} = (\sum_{\mu=0}^m u_{\mu} \zeta_j^\mu) P^t_{n-j}$. Thus, the vectors $P_j$ and $P^t_{n-j}$ ($j \in \mathbb{J}$) are left and right eigenvectors of $M$. The corresponding eigenvalue is

$$\lambda_j := \sum_{\mu=0}^m u_{\mu} \zeta_j^\mu \quad (j \in \mathbb{J}).$$

(3.3)

As $(u_0, \ldots, u_m) \in \mathbb{R}^{1 \times (m+1)}$ and $\zeta_j^{m+1} = \zeta_j^{-j}$ we have $\lambda_j = \lambda_{n-j}$ for all $j \in \mathbb{J}\setminus\{0\}$.

We now regard two matrices out of $\mathbb{C}^{n \times n}$

$$L := \frac{1}{\sqrt{n}} \left( \begin{array}{c} P_0 \\ \vdots \\ P_{n-1} \end{array} \right) \quad \text{and} \quad R := \frac{1}{\sqrt{n}} \left( \begin{array}{c} P_n \\ \vdots \\ P_1 \end{array} \right).$$

(3.4)

$L$ and $R$ are symmetric and regular for $n > 1$ (see [3,5,6] and [7]). We have: $L = \overline{R}$ and $L \cdot R = I_{n,n}$ with the $n \times n$- unit matrix $I_{n,n}$; the matrices $L$ and $R$ are unitary $n \times n$-matrices in $\mathbb{C}^{n \times n}$. We have $L \cdot M \cdot R = D(\lambda_0, \ldots, \lambda_{n-1})$ with the diagonal matrix $D(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{C}^{n \times n}$ containing the eigenvalues $\lambda_j$ of $M$ as its elements in the main diagonal. This yields $M = R \cdot D(\lambda_0, \ldots, \lambda_{n-1}) \cdot L$ and

$$Q_k \cdot R = Q_{k-1} \cdot R \cdot D(\lambda_0, \ldots, \lambda_{n-1}) \quad \text{and} \quad Q_k \cdot R = Q_0 \cdot R \cdot D(\lambda_0, \ldots, \lambda_{n-1})^k \quad \text{for } k \in \mathbb{N}\setminus\{0\}.$$ 

(3.5)

We get $Q_k \cdot R = \frac{1}{\sqrt{n}} \left( \sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_n^\nu, \sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_{n-1}^\nu, \ldots, \sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_1^\nu \right).$

Then (3.5) yields

$$\sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_{n-j}^\nu = \lambda_j^k \sum_{\nu=0}^{n-1} q_{\nu,0} \zeta_{n-j}^\nu \quad \forall j \in \mathbb{J}.$$ 

(3.6)
Due to $\lambda_0 = \sum_{\mu=0}^{m} u_{\mu} \zeta_0^\mu = 1$ and $\zeta_0 = 1$, the index $j = 0$ in (3.6) delivers $\sum_{\nu=0}^{n-1} q_{\nu,k} = \sum_{\nu=0}^{n-1} q_{\nu,0}$ for all $k \in \mathbb{N}\setminus\{0\}$: All polygons $Q_k$ have the same center of gravity.

From now on let the initial polygon $Q_0$ have its center of gravity in the origin $O := (0, \ldots, 0)^t$. So we can be sure that for all $k \in \mathbb{N}$

$$\frac{1}{n} \sum_{\nu=0}^{n-1} q_{\nu,k} = o_d := (0, \ldots, 0)^t. \quad (3.7)$$

As the matrix $R$ is regular the initial polygon $Q_0$ can explicitly be retrieved from the $d \times n$ matrix

$$Q_0 \cdot R =: B = (b_0, \ldots, b_{n-1}) \in \mathbb{C}^{d \times n} \quad \text{with} \quad b_j = \frac{1}{\sqrt{n}} \sum_{\nu=0}^{n-1} q_{\nu,0} \zeta_{n-j}^\nu \in \mathbb{C}^d. \quad (3.8)$$

From $q_{\nu,0} \in \mathbb{R}^d$ and $\zeta_{n-j}^\nu = \overline{\zeta_j}^\nu$ we get $b_j = b_{n-j}$ for all $j \in \mathbb{J}^* := \{1, \ldots, n-1\}$. Because of (3.7) the first column vector is zero: $b_0 = o_d$. Equation (3.5) yields

$$Q_k \cdot R = B \cdot D(\lambda_0, \ldots, \lambda_{n-1})^k = (o_d, \lambda_1^k b_1, \ldots, \lambda_{n-1}^k b_{n-1}). \quad (3.9)$$

Thus, we do not alter the recursion in any way if we replace the diagonal matrix $D(\lambda_0, \ldots, \lambda_{n-1})$ in (3.5) by the diagonal matrix $D^* := D(0, \lambda_1, \ldots, \lambda_{n-1})$.

With this in mind, the iteration process can be described by

$$Q_k = B \cdot D^* \cdot L = \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} \lambda_\nu^k b_\nu P_\nu \iff q_{j,k} = \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} \lambda_\nu^k b_\nu \zeta_j^\nu \quad (3.10)$$

for $j \in \mathbb{J}$. Note that $b_\nu P_\nu \in \mathbb{C}^{d \times n}$ for $\nu \in \mathbb{J}^*$.

4. Prototype polygons

The Gaussian plane of complex numbers $\mathbb{C}$ can be interpreted as a Euclidean plane $\mathbb{E}^2$ with a Cartesian coordinate frame $\{O, 1, i\}$. We embed $\mathbb{E}^2$ into $\mathbb{E}^d$ by identifying 1 and $i$ with the $d$-dimensional unit vectors $e_1 := (1, 0, 0, \ldots, 0)^t$ and $e_2 := (0, 1, 0, \ldots, 0)^t$, respectively. The elements of $P_j$ (3.2) can be viewed as a collection of $n$ points $\zeta_\nu^j$ ($\nu \in \mathbb{J}$) equally distributed on the unit circle of $\mathbb{E}^2 \subset \mathbb{E}^d$ centered in $O$ with $j \in \mathbb{J}^* := \{1, \ldots, n-1\}$. Its points can be written as

$$T_j = e_1 \frac{P_j + \overline{P_j}}{2} + e_2 \frac{P_j - \overline{P_j}}{2i} = e_1 \frac{P_j + P_{n-j}}{2} + e_2 \frac{P_j - P_{n-j}}{2i}. \quad (4.1)$$

$T_j$ is represented by a matrix in $\mathbb{R}^{d \times n}$ with columns

$$t_{\nu,j} := (\cos \frac{2\pi \nu j}{n}, \sin \frac{2\pi \nu j}{n}, 0, \ldots, 0)^t \in \mathbb{R}^d (\nu \in \mathbb{J}). \quad (4.2)$$

$T_j$ forms the so-called ‘regular prototype $n$-gon of $j$th kind’. The regular $n$-gon $T_{n-j}$ is symmetric to $T_j$ w.r.t. the axis $e_1$ and thus affinely equivalent to $T_j$. If $j$ and $n$ are relatively prime the polygon $T_j$ is either a regular $n$-gon or an
n-sided regular star. If \( j \) is a divisor of \( n \) with \( n = jp \) the polygon \( T_j \) is either a regular \( p \)-gon or an ordinary regular star with \( p \) vertices, each of the vertices being multiply counted (\( j \) times).

5. The concept of affine regularization

An affine mapping of \( \mathbb{E}^d \) keeping the origin \( O \) in its place is described by

\[
\beta : \mathbb{E}^d \rightarrow \mathbb{E}^d, \ x \mapsto \beta(x) = Cx \quad \text{with} \quad C \in \mathbb{R}^{d \times d}.
\]

The affine image of the polygon \( Q_k = (q_{0,k}, \ldots, q_{n-1,k}) \) is \( \beta(Q_k) = C \cdot Q_k \). Our iteration (2.1) seems to regularize for certain \((u_0, \ldots, u_m) \in \mathbb{R}^{1 \times (m+1)}\) irrespective of the choice of the starting polygon \( Q_0 \). In order to examine this interesting peculiarity we compare the \( n \)-gons \( Q_k \) with a regular prototype \( n \)-gon \( T_j \) (4.1) of \( j \)th kind:

**Definition 5.1.** We call the iteration (2.1) affinely regularizing of kind \( j \) with \( 1 \leq j \leq n/2 \) if, for any generic initial polygon \( Q_0 \), there exist affine mappings \( \beta_k : \mathbb{E}^d \rightarrow \mathbb{E}^d \) transforming \( Q_k = (q_{0,k}, \ldots, q_{n-1,k})^t \) into polygons \( \beta_k(Q_k) \) with the property that the series \( \Delta_k \) of sums of the squared distances

\[
\Delta_k := \sum_{\nu=0}^{n-1} \| \beta(q_{\nu,k}) - t_{\nu,j} \|^2 = \text{tr} \left( (T_j - \beta_k(Q_k))^t \cdot (T_j - \beta_k(Q_k)) \right)
\]

of respective vertices of \( \beta(Q_k) \) and of the regular prototype polygon \( T_j \) of \( j \)th kind is a null series: \( \lim_{k \to \infty} \Delta_k = 0 \).

6. The affine regularization theorem

The shape of the polygons \( Q_k \) depends on the input data set \( Q_0 \) and on the barycentric coordinates \((u_0, \ldots, u_m)\) of the reference point \( z^* \) with \( \sum_{\mu=0}^{m} u_\mu = 1 \). The latter determine the matrix \( M (3.1) \), the eigenvalues \( \lambda_j \) and the diagonal matrix \( D^* = D(0, \lambda_1, \ldots, \lambda_{n-1}) \). The norms \( n_j := |\lambda_j| \) of \( \lambda_j \) for \( j \in \mathbb{J}^* \) are given by

\[
n_j^2 = \lambda_j \sum_{\mu,\nu=0}^{m} u_\mu u_\nu \zeta_j^{\mu-\nu}.
\]

We put \( N := \max \{ n_1, \ldots, n_{n-1} \} \). Let the barycentrics \((u_0, \ldots, u_m)\) be chosen generally such that not all \( \lambda_1, \ldots, \lambda_{n-1} \) vanish. \( N = 0 \) is equivalent with \( \lambda_1 = \cdots = \lambda_{n-1} = 0 \) and can only occur if \( m = n - 1 \) and, additionally, \((u_0, \ldots, u_{n-1}) = (1/n, \ldots, 1/n)\). This case of iterated series of ‘degenerate \( n \)-gons’ \( Q_k \), all collapsing into the center of gravity \( O \) shall be excluded further on. For \( 0 < N < 1 \) the series \( Q_k \) gradually contracts for increasing \( k \) and tends towards the center of gravity \( O \). For \( N = 1 \) the series \( Q_k \) remains finite, \( ^1 \)

\( ^1 \)As the prototypes \( T_j \) and \( T_{n-j} \) are affinely equivalent, an iteration regularizing of \( j \)th kind will also be regularizing of kind \( n-j \) and we can confine ourselves to \( 1 \leq j \leq n/2 \).
but in general still may change its shape and its position from generation to generation. For $N > 1$ the series $Q_k$ gradually expands for increasing $k$.

We will prove that for any $N > 0$, the algorithm is—in general—affinely regularizing. We divide the set of indices into two distinct subsets:

$$\mathbb{J}_1 := \{ j \in \mathbb{J}^* / |\lambda_j| = N \} \neq \emptyset \quad \text{and} \quad \mathbb{J}_2 := \mathbb{J}^* \setminus \mathbb{J}_1. \quad (6.2)$$

According to (3.3), for any $j^* \in \mathbb{J}_1$ the index $n - j^*$ is also contained in $\mathbb{J}_1$; for even $n$ and $j^* = n/2$ these two indices coincide. We have

$$\frac{|\lambda_j|}{N} = 1 \forall j \in \mathbb{J}_1 \quad \text{and} \quad 0 \leq \frac{|\lambda_j|}{N} < 1 \forall j \in \mathbb{J}_2. \quad (6.3)$$

Equations (3.10) yield

$$Q_k = \frac{N^k}{\sqrt{n}} \left( \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} \right) \Leftrightarrow q_{j,k} = \frac{N^k}{\sqrt{n}} \left( \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{\nu}^j + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{\nu}^j \right). \quad (6.4)$$

Regardless of the input data $b_j$ (3.8) the coefficients $(\lambda_{\nu}/N)^k$ form null series for all $\nu \in \mathbb{J}_2$ and $k \to \infty$; the coefficients $(\lambda_{\nu}/N)^k$ for all $\nu \in \mathbb{J}_1$ are complex numbers of norm 1 for all $k \in \mathbb{N}$.

$Q_k$ and any homothetic image $\rho_k(Q_k)$ have the same affine shape. Following Definition 5.1 we can apply homotheties $\rho_k : \mathbb{E}^d \to \mathbb{E}^d$ with $x \mapsto x \sqrt[n]{n}$. These homotheties $\rho_k$ turn (6.4) into

$$\rho_k(Q_k) = \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} \Leftrightarrow \rho_k(q_{j,k}) = \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{\nu}^j + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{\nu}^j. \quad (6.5)$$

With reference to the cardinal number of the index set $\mathbb{J}_1$ we have three cases:

**Case A:** The index set $\mathbb{J}_1$ contains just one element. This can only happen if $n$ is an even integer and the barycentrics $(u_0, \ldots, u_n)$ lead to $\mathbb{J}_1 = \{ n/2 \}$. We have $\zeta_{n/2} = -1$, and $\lambda_{n/2} = \sum_{\mu=0}^{m} u_\mu (-1)^\mu \in \mathbb{R}$. As $N = |\lambda_{n/2}| > 0$ and therefore $\lambda_{n/2} = \pm N \neq 0$ formula (6.5) reads as

$$\rho_k(Q_k) = (\pm 1)^k b_{n/2} P_{n/2} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu}. \quad (6.6)$$

For every $k$ we apply a further homothety $\sigma_k : \mathbb{E}^d \to \mathbb{E}^d$ with

$$\sigma_k(x) = (\pm 1)^k x \Rightarrow \sigma_k(\rho_k(Q_k)) = b_{n/2} P_{n/2} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\pm \lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu}. \quad (6.7)$$

We have $b_{n/2} = \sum_{\nu=0}^{n-1} q_{\nu,0} \zeta_{n/2} = \sum_{\nu=0}^{n-1} (-1)^\nu q_{\nu,0} \in \mathbb{R}^d$. For a generic input polygon $Q_0$ we can assume $b_{n/2} \neq 0_d$. In this case we choose an affine mapping $\tau$ with fixed point $O$ and $b_{n/2} \mapsto c_1 \in \mathbb{R}^d$. The mapping $\tau$ induces an affine mapping $\mathbb{C}^d \to \mathbb{C}^d$ transforming $b_{\nu}$ ($\nu \in \mathbb{J}_2$) into $b_{\nu}^* := \tau(b_{\nu}) \in \mathbb{C}^d$; the
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vectors $b_ν^∗$ do not depend on $k$. The affine mapping $β_k := τ ∘ σ_k ∘ ρ_k$ places the $k$th generation polygon $Q_k$ into

$$β_k(Q_k) = e_1 P_{n/2} + \sum_{ν ∈ ℐ_2} (±\frac{λ_ν}{N})^k b_ν^* P_ν. \quad (6.8)$$

The distance vectors $d_{j,k}$ of the vertices of $β_k(Q_k)$ to the respective vertices of the prototype polygon $T_{n/2} = e_1 P_{n/2}$ (4.1) are the columns of $D_k = (d_{0,k}, \ldots, d_{n−1,k})$ with

$$D_k = \sum_{ν ∈ ℐ_2} \left(±\frac{λ_ν}{N}\right)^k b_ν^* P_ν ⇔ d_{j,k} = \sum_{ν ∈ ℐ_2} \left(±\frac{λ_ν}{N}\right)^k b_ν^* \zeta_j^ν \quad (j ∈ ℤ). \quad (6.9)$$

The vectors $b_ν^∗, ζ_j^ν$ are independent from $k$. As the norms of $(\frac{λ_ν}{N})^k$ form null series for all $ν ∈ ℐ_2$ we can be sure that $\lim_{k→∞} d_{j,k} = 0$ for all $j ∈ ℤ$. The sum of the squared distances $Δ_k := \sum_{j=0}^{n−1} \|d_{j,k}\|^2$ is a null series: $\lim_{k→∞} Δ_k = 0$. Thus, according to our Definition 5.1 the iteration process in case A is affinely regularizing of kind $n/2$. For generic input $Q_0$ the polygons $Q_k$ approach the shape of the $n$-gon $T_{n/2}$. The straight lines approximating the polygons $Q_k$ tend towards the straight line through $O$ with direction vector $b_{n/2}$.

Case B: The index set $ℐ_1$ contains exactly two different elements: $ℐ_1 = \{j^∗, n−j^∗\}$ with $1 ≤ j^∗ < n/2$. In a way, this could be considered the general case. We put $λ_{j^∗} = Ne^{iϕ}$ and $λ_{n−j^∗} = Ne^{−iϕ}$ with some real angle $ϕ ∈ [0, 2π)$ and define $W := \sum_{ν ∈ ℐ_2} \left(±\frac{λ_ν}{N}\right)^k b_ν^* P_ν$. Then (6.5) yields

$$ρ_k(Q_k) = e^{iϕ} b_{j^∗} P_{j^∗} + e^{−iϕ} b_{n−j^∗} \overline{P_{j^∗}} + W. \quad (6.10)$$

Let $b_{j^∗} := x + iy$ with $x, y ∈ ℜ^d$. We then have $b_{n−j^∗} = \overline{b_{j^∗}} = x − iy$ and

$$ρ_k(Q_k) = x(e^{ikϕ} P_{j^∗} + e^{−ikϕ} \overline{P_{j^∗}}) + iy(e^{ikϕ} P_{j^∗} − e^{−ikϕ} \overline{P_{j^∗}}) + W. \quad (6.11)$$

For a generic input $n$-gon $Q_0$ the two vectors $x, y ∈ ℜ^d$ are linearly independent. Let $σ : ℜ^d → ℜ^d$ be any affine mapping that maps the two vectors $x, y$ into $σ(x) := e_{1/2}$ and $σ(y) := −e_{2/2}$. $σ$ induces an affine mapping $C^d → C^d$ transforming $b_ν(ν ∈ ℐ_2)$ into $σ(b_ν)$. We have

$$σ(ρ_k(Q_k)) = (e_1 \cos kϕ + e_2 \sin kϕ) \frac{P_{j^∗} + \overline{P_{j^∗}}}{2} + (−e_1 \sin kϕ + e_2 \cos kϕ) \frac{P_{j^∗} − \overline{P_{j^∗}}}{2i} + \sum_{ν ∈ ℐ_2} \left(\frac{λ_ν}{N}\right)^k σ(b_ν) P_ν. \quad (6.12)$$

We define the complex numbers $θ_{μ, ν} := σ(b_μ)^t \overline{σ(b_ν)}$ for $μ, ν ∈ ℐ_2$. The matrices

$$R_k := \begin{pmatrix} \cos kϕ & \sin kϕ & 0 & \ldots & 0 \\ −\sin kϕ & \cos kϕ & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \quad (6.13)$$
describe rotations $\tau_k$ in $\mathbb{E}^d$. The induced mappings $\tau_k$ in $\mathbb{C}^d$ transform the vectors $\sigma(b_\nu)$ into vectors $\tau_k(\sigma(b_\nu)) \in \mathbb{C}^d (\nu \in \mathbb{J}_2)$. The mappings $\beta_k := \tau_k \circ \sigma \circ \rho_k$ are affine mappings from $\mathbb{E}^d$ into $\mathbb{R}^d$ and deliver

$$
\beta_k(Q_k) = e_1 \frac{P_{j*} + \overline{P}_{j*}}{2} + e_2 \frac{P_{j*} - \overline{P}_{j*}}{2} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_\nu}{N} \right)^k \tau_k(\sigma(b_\nu)) P_\nu.
$$

(6.14)

As every $\tau_k$ preserves scalar products we have $\tau_k(\sigma(b_\mu))^t \tau_k(\sigma(b_\nu)) = \theta_{\mu,\nu}$ for all $\mu, \nu \in \mathbb{J}_2$. According to (5.2) we compute the sum of squared distances of the vertices of $\beta_k(Q_k)$ to the respective vertices of the prototype polygon $T_{j*}$ and arrive at

$$
\Delta_k = \text{tr} \left( (T_{j*} - \beta_k(Q_k))^t (T_{j*} - \beta_k(Q_k)) \right) = n \sum_{\mu \in \mathbb{J}_2} \left( \frac{\lambda_\mu \lambda_\mu}{N^2} \right)^k \theta_{\mu,n-\mu}.
$$

(6.15)

As the values $\theta_{\mu,n-\mu}$ are independent from $k$ and $0 \leq \frac{\lambda_\mu \lambda_\mu}{N^2} < 1$ for all $\mu \in \mathbb{J}_2$ the values $\Delta_k$ ($k \in \mathbb{N}$) form a null series. Accordingly, the corresponding iteration process in case B is regularizing of kind $j*$ with $1 \leq j* < n/2$. For generic input $n$-gons $Q_0$ the two vectors $x$ and $y$ determine a plane $\varepsilon^*$ through $O$. The planes $\varepsilon_k$ approximating the polygon $Q_k$ tend towards $\varepsilon^*$.

**Case C:** The index set $\mathbb{J}_1$ contains more than two different elements. We have $j*, j^{**}, n - j* \in \mathbb{J}_1$ with $1 \leq j* < j^{**} \leq n/2$. According to (6.1) this is characterized by

$$
\sum_{\mu,\nu=0}^m u_\mu u_\nu \zeta_{j*}^{\mu-\nu} = \sum_{\mu,\nu=0}^m u_\mu u_\nu \zeta_{j*}^{\mu-\nu}.
$$

(6.16)

The coefficients of $u_\mu u_\nu$ in (6.16) are $\zeta_{j*}^{\mu-\nu} + \zeta_{j^{**}*}^{\nu-\mu} - \zeta_{j*}^{\nu-\mu} - \zeta_{j^{**}*}^{\mu-\nu} \in \mathbb{R}$. The corresponding barycentrics $(u_0, \ldots, u_m)$ denote points $z^* \in \mathbb{R}^m$ which, in general, are positioned on an $(m - 1)$-dimensional quadric of $\mathbb{R}^m$ containing the vertices of the simplex $S$. In this case we cannot prove any regularizing effect of the affine iteration.

We call the barycentrics $(u_0, \ldots, u_m)$ ‘generic’ if they do not lead to Case C or, for $m = n - 1$, they are different from $(\frac{1}{n}, \ldots, \frac{1}{n})$. Overall, we have

**Theorem 6.1.** Affine Regularization Theorem. For generic barycentrics the iteration process (2.1) is affinely regularizing according to Definition 5.1. The barycentrics $(u_0, \ldots, u_m)$ ($m < n$) determine the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ given by (3.3) and their maximal norm $N > 0$. If there is exactly one index $j*$ (with $1 \leq j* \leq n/2$) with eigenvalue $\lambda_{j*}$ of norm $N$ the iteration is regularizing of kind $j*$.

If the iteration is affinely regularizing of kind $j*$ then, for a generic input $n$-gon $Q_0$, the shape of $Q_k$ gradually approaches the shape of an affinely transformed prototype $n$-gon $T_{j*}$.
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Figure 2 An example with $n = 8$, $m = d = 3$ with $\mathbb{J}_1 = \{1, 7\}$

Figure 3 An exceptional example for $n = 8$, $m = d = 3$ with $b_4 = \mathbf{0}_d$ and $\mathbb{J}_1 = \{4\}$. For this specific $n$-gon $Q_0$ the algorithm works as if it was affinely regularizing of kind 3

Figure 2 shows an example for the same initial octogon $Q_0$ as in Fig. 1 ($n = 8$, $m = d = 3$). Here the barycentrics of $z^*$ are $(u_0, u_1, u_2, u_3) = (0.4, 0.5, 0.3, -0.2)$. We get $n_1 \approx 1.03, n_2 \approx 0.71, n_3 \approx 0.13, n_4 \approx 0.4$ and therefore we have $\mathbb{J}_1 = \{1, 7\}$. The algorithm is regularizing of kind 1 (case B). The figure shows $Q_0$ and the following generations up to $Q_{16}$.

7. Remarkable exceptions

For specific initial polygons $Q_0$ the algorithm may deliver unexpected results. If the coefficient vectors $b_\nu$ of the regarded eigenvalues $\lambda_\nu$ for $\nu \in \mathbb{J}_1$ in (6.4)
vanish the respective eigenvalues have no influence on the regularizing process. So, for such a specific \( n \)-gon \( Q_0 \), the algorithm works in the same way as if in (3.10) these eigenvalues \( \lambda_\nu \) had been replaced by \( \lambda_\nu = 0 \). The remaining eigenvalues deliver another maximum norm \( N^* < N \) and a different set \( J_1 \). Now our classification (Sect. 6) reveals the affine shape of the series \( Q_k \).

Figure 3 shows such an example for \( n = 8, m = d = 3 \) where we have \((u_0, u_1, u_2, u_3) = (0.5, -0.25, 0.5, 0.25)\). We get \( n_1 \approx 0.52, n_2 = 0.5, n_3 \approx 0.99 < 1, n_4 = 1 \). Hence \( N = 1 \) and we conclude that the algorithm is affinely regularizing of kind 4; \( Q_k \) is expected to approach the shape of the prototype is \( T_4 \) which is a line segment. The special initial octogon \( Q_0 \), however, yields \( b_4 = 0 \); we put \( \lambda_4 := 0 \) and perform a new case study. The affine shape of \( Q_k \) tends towards the prototype \( T_3 \). Figure 3 displays \( Q_0 \) and the following generations up to \( Q_6 \).

8. Conclusion

We studied affine iterations transforming an initial \( n \)-gon \( Q_0 \) in \( \mathbb{E}^d (d > 1) \) into successive generations of \( n \)-gons \( Q_k \). The Affine Regularization Theorem in this paper does not only extend the results in [5] to dimensions \( d > 2 \); surprisingly, even for dimensions \( d > 2 \) the regularization leads to planar, regular prototypes no matter which generic input \( n \)-gon \( Q_0 \) we start with. For very specific input \( n \)-gons \( Q_0 \), though, the same algorithm seems to regularize in a different way. The understanding of this phenomenon completes the results.

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