AROUND THE UNIFORM RATIONALITY I

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Abstract. We prove that there exist rational but not uniformly rational smooth algebraic varieties. The proof is based on computing certain numerical obstruction developed in the case of compactifications of affine spaces. We show that for some particular compactifications this obstruction behaves differently compared to the uniformly rational situation.

1. Introduction

1.1. Let $X$ be a complex projective manifold of dimension $n \geq 2$. Recall that rationality of $X$ (i.e. the existence of a birational map $X \dashrightarrow \mathbb{P}^n$) yields a Zariski open subset $U \subset X$ isomorphic (as an affine scheme) to a domain in $\mathbb{C}^n$. Any rational $X$ obviously carries a family of (very free in the terminology of e.g. [8]) rational curves and one may try to obtain a family of holomorphic maps $\mathbb{C} \rightarrow X$ (called sprays in [4]) such that near each of its points $X$ is (algebraically) $h$-Runge (see [4] for precise definitions and results). The main expectation is that Zariski locally near every point $X$ should actually look like an open subset $U \subseteq \mathbb{C}^n$. One refers to the latter property as uniform rationality (of $X$), the notion introduced recently in [1] (following [4]), where some examples and basic properties of uniformly rational (or u.r. for short) manifolds have been established. The ultimate goal was to approach the following:

Question G (cf. [1], [4]). Is it true that every rational manifold is uniformly rational?

Note that spherical (e.g. toric) varieties and blowups of u.r. varieties at smooth centers are easily seen to be u.r. (all this is contained in [1], together with examples of $X$ being the intersection of two quadrics, small resolution of a singular cubic threefold, and some other instances) [1]. This immediately gives positive answer to Question G in the case when $n = 2$. It is also easy to see that all points on a rational manifold $X$ which may not admit affine neighborhoods $U \subseteq \mathbb{C}^n$ form a locus of codimension $\geq 2$ (compare with Proposition 2.12 below).

The goal of the present paper is to prove the following:

Theorem 1.2. In the previous notation, there exists rational, but not uniformly rational $X$, whenever $n$ is at least 4.

MS 2010 classification: 14E08, 14M20, 14M27.

Key words: rational variety, uniform rationality, compactification.

1) We do not treat here two very interesting questions (both discussed in [4]) on rectifiability of divisorial families and on the local regularization of an arbitrary birational map $X \dashrightarrow \mathbb{P}^n$. Instead we rather concentrate on the “negative side” of Question G (see below). In addition, recall that small resolution of a Lefschetz cubic is (Moishezon and) not u.r., which shows that projectivity assumption on $X$ is crucial for Question G to be of any content.
Thus Theorem 1.2 answers Question 3 negatively. But still it would be interesting to find out whether a sufficiently large power $X^N := X \times X \times \ldots$ of any rational manifold $X$ is u.r. (same question for the “stabilization” $X \times \mathbb{P}^N$ of $X$). More examples and discussion will be provided in the forthcoming paper [7].

1.3. We proceed with a description of the proof of Theorem 1.2. First of all, in view of the above discussion, it is reasonable to treat only rational manifolds which are “minimal” in certain sense (like those that are not blowups of other manifolds for instance). The most common ones are compactifications of affine $\mathbb{C}^n$ with Pic $\simeq \mathbb{Z}$ (see [2,1] below for a setup). Next, one may guess that being u.r. for a rational manifold $X$ results in “homogeneity property” for the underlying set of points (compare with [1] 3.5.E’[1]). The latter means (ideally) that an appropriate test function $f : X \to \mathbb{R}$ (“Gibbs distribution”) must be constant on uniformly rational $X$. More precisely, as soon as just the points on $X$ are concerned, it is natural to look just for such $f$ that are conformally invariant (these constitute a class of the so-called asymptotic invariants of $X$).

Now, if $L$ is an ample (or, more generally, nef) line bundle on $X$, then the very first candidate for $f$ one can think of would be the Seshadri constant $s_L(o)$ of $L$. Namely, for any point $o \in X$ let us consider the blowup $\beta : Y \to X$ of $o$, with exceptional divisor $E := \beta^{-1}(o)$, and then put

$$f(o) := s_L(o) := \max \{ \lambda \in \mathbb{R} \mid \text{the divisor} \beta^*L - \lambda E \text{ is nef} \}.$$  

We will also write simply $s(o)$ instead of $s_L(o)$ when Pic$X = Z \cdot L$ (resp. when $L$ is clear from the context). Further, conformal invariance of $f = s_L(\cdot)$ may be seen via another definition of it as follows (see [2]):

$$s_L(o) := \sup_{k} \frac{\mult_k M}{k},$$

where the supremum is taken over all $k \in \mathbb{Z}$ and the linear subsystems $M \subseteq |kL|$ having isolated base locus near $o$. One of the key ingredients in establishing the expression (1.4) is the Poincaré-Lelong formula

$$\mult_o D = \lim_{r \to 0} \frac{1}{r^{2n-2}} \text{Vol}(D \cap B(r)) = \lim_{r \to 0} \frac{1}{r^{2n-2}} \int_{D \cap B(r)} \omega^{2n-2}$$

for the multiplicity of a hypersurface $D \subseteq X$ at the point $o$, written in terms of volumes of intersections with small balls $B(r)$ centered at $o$.

Recall that the basic play ground for our approach are those $X$ containing $U := \mathbb{C}^n$ as a Zariski open subset. We also require the boundary $\Gamma := X \setminus U$ to be (of pure codimension 1 and) irreducible. Assuming such $X$ uniformly rational, we claim that $s(\cdot)$ attains the same value at some points on $U$ and $\Gamma$, respectively (see Corollary 2.15 for a precise statement). One thus gets a relatively simple numerical criterion to test uniform rationality of the manifolds in question. The main issue then is to find a particular $X$ for which this obstruction actually gives something non-trivial.

For $n = 3$, as we show in [7], one does not obtain anything interesting. However, the needed examples, for any $n \geq 4$, are constructed in Section 3 below. The idea behind our construction is to mimic the one for the fourfold $V_5^4$ from [22]. Namely, we start by blowing up $\mathbb{P}^n$ at a smooth cubic of dimension $n - 2$ and contracting the proper transform of a hyperplane, which gives a singular $n$-fold $Y$. The only singularity of $Y$ happens to be of the form $\mathbb{C}^n/\mu_2$ and so an appropriate double covering $X \to Y$ makes $(\Gamma \text{ Cartier and}) Y$ smooth. Here $X$ is an index $n - 3$ Fano manifold having Pic$X = Z \cdot O_X(\Gamma)$ (we keep the same notation for the images of $\Gamma$ on $Y, X$, etc). It remains then to estimate the function $s(\cdot)$ on $X$ and to show that $X$ indeed compactifies $\mathbb{C}^n$.

The first issue is resolved in Corollary 3.6 by an explicit computation, where we show that $s(\cdot) = 1$ on $\Gamma$, while $s(\cdot) \geq 2$ on the complement $X \setminus \Gamma$. In turn, the second issue (that is $X \setminus \Gamma \simeq \mathbb{C}^n$) reduces to finding a cubic
polynomial \( P \) such that the double cover of \( \mathbb{C}^n \) with ramification in \( P \), i.e. Spec \( \mathbb{C}[\mathbb{C}^n][\sqrt{P}] \), is also isomorphic to \( \mathbb{C}^n \). Theorem 1.2 for the given \( X \) now follows from Corollary 3.10 by combining the two mentioned properties of \( X \) with Corollary 2.15.

Remark 1.5. The assumption on \( \Gamma \) to be irreducible is crucial in our approach (cf. Remark 2.8). In fact, the surface \( X := F_1 \) is uniformly rational (as a toric surface) and compactifies \( \mathbb{C}^2 \), with \( \Gamma \) being the union of the \((-1)\) - curve \( Z \) and a ruling \( R \) of the natural projection \( F_1 \to \mathbb{P}^1 \). Then one can easily see that \( s(o) = 3 \) (with respect to \( -K_X = 2Z + 3R \)) for any point \( o \not\in Z \). Otherwise we have \( s(o) = 1 \) — in contradiction with what happens for irreducible \( \Gamma \). Anyhow, \( X \) is an equivariant compactification of \( \mathbb{C}^2 \), and it would be interesting to find out whether all such compactifications of \( \mathbb{C}^n \) are u. r. (cf. 3 and 5).

Conventions. All varieties, unless stated otherwise, are defined over the complex field \( \mathbb{C} \) and are assumed to be normal and projective. We will be using freely standard notation, notions and facts (although we recall some of them for convenience) from [6], [8] and [10].

Acknowledgments. I am grateful to C. Birkar, F. Bogomolov, A.I. Bondal, S. Galkin, M. Romo, and J. Ross for their interest and helpful comments. Some parts of the paper were written during my visits to CIRM, Università degli Studi di Trento (Trento, Italy), Cambridge University (Cambridge, UK) and Courant Institute (New York, US). The work was supported by World Premier International Research Initiative (WPI), MEXT, Japan, Grant - in - Aid for Scientific Research (26887009) from Japan Mathematical Society (Kakenhi), and by the Russian Academic Excellence Project 5 - 100.

2. Beginning of the proof of Theorem 1.2: an obstruction

2.1. Let \( X \) be a Fano manifold with Pic \( X \simeq \mathbb{Z} \) compactifying \( \mathbb{C}^n \). In other words, there exists an affine open subset \( U \subset X, U \simeq \mathbb{C}^n \), such that Pic \( X = \mathbb{Z} \cdot \mathcal{O}_X(\Gamma) \) for the boundary \( \Gamma := X \setminus U \). We will also assume that \( \Gamma \) is an irreducible hypersurface.

Fix one particular such \( X \not= \mathbb{P}^n \) (see Section 3 for some examples). Let \( H \) be a generator of Pic \( X \) and \( x_1, \ldots, x_n \) be affine coordinates on \( U \). Then, for some (minimal) \( r \), there exist sections \( s_i \in H^0(X, H^r) \) such that \( s_i = x_i \) on \( U \). Indeed, with \( H^r \) very ample, \( s_i \big|_U \) induce an identification \( U \simeq \mathbb{C}^n \). We may also assume without loss of generality that \( X \subset \mathbb{P}^{\dim(H^r)} \) is projectively normal.

Now pick a point \( p \in \Gamma \) and a rational function \( t \in \mathcal{O}_{X,p} \subset \mathbb{C}(U) \) defining \( \Gamma \) in an affine neighborhood \( U' \subset X \) of \( p \).

Lemma 2.2. We have \( t^{-1} \in \mathbb{C}[U] \). More precisely, \( t^{-1} \big|_U \) is an irreducible polynomial in \( x_1, \ldots, x_n \).

Proof. The function \( t \) does not have any zeroes on \( U = X \setminus \Gamma \) by construction. Hence \( t^{-1} \) is a polynomial \( \in \mathbb{C}[U] \). Its irreducibility follows from that of \( \Gamma \). \( \square \)
2.3. Now suppose that $X$ is uniformly rational. Let $U' \ni p$ be as above. Then $U'$ embeds into $\mathbb{C}^n$.

Let $s \in H^0(X, H)$ be the section whose zero locus equals $\Gamma$. By definition of $H$ we have $s|_{U'} = t$ and $s|_U = 1$ (cf. Lemma 2.2), so that the functions $s|_{U'}, s|_U \in \mathbb{C}(x_1, \ldots, x_n)$ are identified on $U' \cap U$ via $s|_{U'} = ts|_U$. This yields

$$y_i := s_i|_{U'} = t' x_i \in \mathcal{O}_{X,p}$$

for all $i$. Indeed, both line bundles $H^|_{U'}$ and $H^r|_U$ are trivial on $U'$ and $U$, respectively, for $s_i|_{U'}$ and $s_i|_U$ regarded as rational functions on $\mathbb{C}^n$, satisfying $s_i|_{U'} \in \mathbb{C}[U']$ and $s_i|_U = x_i$ by construction. Then $H^r|_{U'}$ and $H^r|_U$ are glued over $U' \cap U$ via the multiplication by $t'$ as (2.4) indicates.

Lemma 2.5. In the previous setting, if $y_i \neq \text{const}$ for all $i$, then $y_1, \ldots, y_n$ are local parameters on $U' \subseteq \mathbb{C}^n$ generating the maximal ideal of the $\mathbb{C}$-algebra $\mathcal{O}_{X,p}$.

Proof. Notice that

$$\mathbb{C}(x_1, \ldots, x_n) = \mathbb{C}(U) = \mathbb{C}(U') = \mathbb{C}(y_1, \ldots, y_n)$$

by construction, i.e. $x_i = y_i/t'$ (resp. $y_i$) are (birational) coordinates on $U'$, defined everywhere out of $\Gamma$ (resp. everywhere on $U'$). This implies that the morphism $\xi: \mathbb{C}^n \cap (t^{-1} \neq 0) \rightarrow (U' \subseteq \mathbb{C}^n)$, given by

$$(x_1, \ldots, x_n) \mapsto (y_1 = x_1 t', \ldots, y_n = x_n t'),$$

is birational $^2$

Functions $y_i$ do not have a common codimension 1 zero locus on $U$ (cf. Lemma 2.2). Hence $\xi$ does not contract any divisors. In particular, $\xi^{-1}$ is well-defined near $\Gamma$ by Hartogs, which shows that $y_1, \ldots, y_n$ are the claimed local parameters.

Lemma 2.6. $y_i = \text{const}$ for at most one $i$.

Proof. Indeed, otherwise (2.4) gives $s_i = s_j$ on $X$ for some $i \neq j$, a contradiction.

Lemma 2.7. Let $y_1 = \text{const}$. Then $t', y_2, \ldots, y_n \in \mathcal{O}_{X,p}$ are local parameters on $U' \subseteq \mathbb{C}^n$ generating the maximal ideal of the $\mathbb{C}$-algebra $\mathcal{O}_{X,p}$.

Proof. One may assume that $y_1 = 1$. Then $y_i \neq \text{const}$ for all $i \geq 2$ by Lemma 2.6, and a similar argument as in the proof of Lemma 2.5 shows that birational morphism $\eta: \mathbb{C}^n \cap (t^{-1} \neq 0) \rightarrow (U' \subseteq \mathbb{C}^n)$, given by

$$\eta: (x_1, \ldots, x_n) \mapsto (t' = 1/x_1, y_2 = x_2 t', \ldots, y_n = x_n t'),$$

does not contract any divisors. Hence again $t', y_2, \ldots, y_n$ are the asserted local parameters.

Remark 2.8. An upshot of the previous considerations is that the whole “analysis” on $X$, encoded in the line bundle $H$, can be captured just by two charts, like $U$ and $U'$, with a transparent gluwing (given by $t$) on the overlap $U \cap U'$. Let us stress one more time that this holds under the assumption that $X$ is u. r. In addition, as will also be seen in 2.9 a similar property does not extend directly to the case of $X$ with reducible boundary $\Gamma$ (compare Proposition 2.12 and Remark 1.5).

$^2$ More specifically, dividing all the $x_i$ by $x_1$, say, one may assume $x_i = y_i$ on $U' \subseteq \mathbb{C}^n$ for all $i \geq 2$. Then $\xi$ is simply the multiplication of $x_1$ by $t'$.
2.9. Let \( h \in H^0(X, H^r) \) be any section. One may write (cf. 2.3)

\[
\frac{h}{s'}|_{U'} = \sum_{0 \leq i_1 + \ldots + i_n \leq m} a_{i_1, \ldots, i_n} x_1^{i_1} \ldots x_n^{i_n},
\]

where \( a_{i_1, \ldots, i_n} \in \mathbb{C} \), \( m = m(h) \geq 0 \) and \( i_j \) are non-negative integers. Now, it follows from (2.4) and Lemmas 2.5 and 2.7 that

\[
h|_{U'} = \sum_{0 \leq i_1 + \ldots + i_n \leq m} a_{i_1, \ldots, i_n} y_1^{i_1} \ldots y_n^{i_n} t^{2r-i_1-\ldots-i_n}.
\]

Conversely, starting with any function on \( U' \) as in (2.11), with \( m \leq 2 \), we can find \( h \in H^0(X, H^r) \) such that \( h|_{U'} = \text{RHS of (2.11)} \) (cf. Remark 2.8). Indeed, in this way we get a global section of \( H^r \), regular away the codimension \( \geq 2 \) locus \( X \setminus U \cup U' \) (recall that \( \Gamma \) is irreducible), hence regular on the entire \( X \).

This discussion condensates to the next

**Proposition 2.12.** There exist a point \( o \in U \) and a point \( p = p(o) \in \Gamma \cap U' \) such that for any hypersurface

\[ \Sigma \sim r\Gamma \]

having prescribed multiplicity \( \text{mult}_o \Sigma > 0 \) at \( o \), there is a hypersurface \( \hat{\Sigma} \sim r\Gamma \) such that \( \text{mult}_p \hat{\Sigma} \geq \text{mult}_o \Sigma \).

**Proof.** Set \( o \in U = \mathbb{C}^n \) to be the origin with respect to \( x_i \).

**Lemma 2.13.** The loci \( H_i := (s_i = 0) \), \( 1 \leq i \leq n \), have a common intersection point, denoted \( p \), on \( \Gamma \).

**Proof.** Assume the contrary. Then all \( y_i \neq \text{const} \) (cf. Lemma 2.6), for otherwise \( \Gamma = (s_1 = 0) \), say, and so \( \cap H_i = \deg \Gamma \neq 0 \). Further, by construction \( \cap H_i \) is a (reduced) point, which immediately gives \( X = \mathbb{P}^n \) (recall that \( H^r \) is very ample according to the setting in (2.11), a contradiction.

Let the section \( h \in H^0(X, H^r) \) correspond to \( \Sigma \). We may assume without loss of generality all but one \( a_{i_1, \ldots, i_n} \) in (2.10) and (2.11) to be zero. Let also \( p \) be as in Lemma 2.13. Then, since \( t(p) = 0 \) by definition, one may take \( \hat{\Sigma} := \Sigma \) whenever all \( y_i \neq \text{const} \). Finally, if \( y_1 = 1 \) (and \( i_1 \neq 0 \)), say, then from (2.4) and Lemmas 2.6 and 2.7 we obtain

\[ \text{mult}_o \Sigma = i_1 \text{mult}_o t^{-r} + i_2 + \ldots + i_n \leq 2. \]

It is thus sufficient to take any \( \hat{\Sigma} \ni p \) with \( i_1 = 0 \) and \( i_2i_3 \neq 0 \).

**Remark 2.14.** The proof of Proposition 2.12 shows that both \( \Sigma \) and \( \hat{\Sigma} \) can actually be taken to vary in some linear systems, having isolated base loci near \( o \) and \( p \), respectively. Furthermore, the value \( s(o) \) is attained on a linear system \( \mathcal{M} \) (cf. 1.3), with isolated base point at \( o \), iff the value \( s(p) \) is attained on a similar linear system for \( p \) and \( \hat{\Sigma} \).

**Corollary 2.15.** For \( o \in U \) and \( p \in \Gamma \) as above we have \( s(p) \geq s(o) \).

**Proof.** Fix some \( h \in \mathcal{M} \) and \( k := k_i \) as in (1.3). We may assume w.l.o.g. that \( r = 1 \) because \( s_{H^r}(\cdot) = rs(\cdot) \). We may also take \( h = h(h_1, \ldots, h_n) \) to be a homogeneous polynomial in some \( h_i \in H^0(X, H) \) for \( X \subset \mathbb{P}^{\dim |H|} \) being projectively normal (cf. 2.1). Let \( m_j(k) := \text{mult}_o h_j \) be such that \( \text{mult}_o h = \sum_j m_j(k) \). One may assume that \( \text{sup lim of } \sum_j m_j(k)/k \) exists and equals \( s(o) \).

3) "\( \sim \)" denotes the linear equivalence of divisors on \( X \).
Now, Proposition 2.12 provides some sections $\tilde{h}_1, \ldots, \tilde{h}_n \in H^0(X, H)$, having $\text{mult}_p \tilde{h}_j \geq m_i(k)$ for all $j$. Then we obtain $h := h(\tilde{h}_1, \ldots, \tilde{h}_n) \in H^0(X, H^k)$ and $\text{mult}_p h \geq \sum_j m_j(k)$. Thus by (1.3) and Remark 2.14 we get $s(p) \geq s(o)$ as wanted.

With Corollary 2.15 we conclude our construction of a necessary condition for the manifold $X \subset \mathbb{C}^n$ in 2.1 to be u.r. Let us now construct those $X$ that do not pass through this simple obstruction.

3. End of the proof of Theorem 1.2

3.1. Take the projective space $\mathbb{P} := \mathbb{P}^n$, $n \geq 4$, with a hyperplane $H \subset \mathbb{P}$ and a cubic hypersurface $S \subset H$. Let $\sigma : V \rightarrow \mathbb{P}$ be a blowup of (the ideal defining) $S$. More specifically, for the reasons that will become clear in 3.3 below, we assume $V \subset \mathbb{P}^1 \times \mathbb{P}$ to be given by (local) equation

$$wt_0 = (wx_1^2 + F)t_1,$$

where $t_i$ are projective coordinates on the first factor and $H, S$ are given by $w = 0, w = F(x_1, \ldots, x_n) = 0$, respectively, in projective coordinates $w, x_1$ on $\mathbb{P}$. Furthermore, we take $F$ in the form

$$x_1x_3^2 + x_2^2x_4 + F_3,$$

with a general homogeneous cubic $F_3 \in \mathbb{C}[x_3, \ldots, x_n]$. This easily shows (Bertini) that $V$ is smooth.\footnote{Note once again that $V$ is glued out of local charts of the form $wt_0 = \tilde{F}t_1$ for various (smooth) cubics ($\tilde{F} = 0) \subset \mathbb{P}$ containing $S$. These charts form a smooth cover of $V$.}

Put $E := \sigma^{-1}S$ and $H^* := \sigma^*H$. Notice that $\sigma$ resolves the indeterminacies of the linear system $|3H - S|$. Let $\varphi : V \rightarrow Y$ be the corresponding morphism onto some variety $Y$ with very ample divisor $O_Y(1)$ pulling back to $3H^* - E$.

**Lemma 3.2.** $\varphi$ is birational and contracts the divisor $H_V := \sigma^{-1}H \simeq \mathbb{P}^{n-1}$ to a point.

**Proof.** By construction of $|3H - S|$ the map $\varphi$ coincides with the Veronese embedding (with respect to $2H$) on an affine open subset in $\mathbb{P}$. Hence $\varphi$ is birational.

Now let $Z \subset \mathbb{P}$ be the image of a curve contracted by $\varphi$. Suppose that $Z \not\subset H$. Then, since $\sigma^{-1}Z$ is contracted by $\varphi$, we have $3H \cdot Z = \deg S \cdot Z$. On the other hand, we obviously have $\deg S \cdot Z \leq H \cdot Z$, a contradiction. Thus every curve contracted by $\varphi$ belongs to $H_V \simeq \mathbb{P}^{n-1}$.

Note that $Y$ has exactly one singular point (cf. Lemma 3.3 below). More precisely, $\varphi \circ \sigma^{-1}$ induces an isomorphism between $\mathbb{P} \setminus H \simeq \mathbb{C}^n$ and $Y \setminus \varphi(E)$, so that $Y$ can be singular only at the point $o := \varphi(H_V)$ on the boundary $\varphi(E)$ (cf. Lemma 3.2).

Further, we want to modify $Y$ into a smooth $n$-fold (our pertinent $X$), yet preserving the properties $\mathbb{C}^n \subset Y$ and $\text{Pic} Y = \mathbb{Z}$. Let us start with the following technical observation:

**Lemma 3.3.** Singularity $o \in Y$ is locally analytically of the form $\mathbb{C}^n/\mu_2$ for the 2-cyclic group $\mu_2$ acting diagonally on $\mathbb{C}^n$.\footnote{Note once again that $V$ is glued out of local charts of the form $wt_0 = \tilde{F}t_1$ for various (smooth) cubics ($\tilde{F} = 0) \subset \mathbb{P}$ containing $S$. These charts form a smooth cover of $V$.}
Proof. Recall that $K_V = -(n + 1)H^* + E$ and $\varphi$ contracts $H_V = \sigma_1^* H \sim H^* - E$ to the point $o$.

One can choose such divisors $D_1, \ldots, D_n$ on $Y$ that $\varphi_1^{-1}D_i \sim H^*$ for all $i$ and the pair

$$(V, \sum_{i=1}^n \varphi_1^{-1}D_i + H_V)$$

is log canonical. Note also that

$$K_V + \sum_{i=1}^n \varphi_1^{-1}D_i + H_V = 0.$$  

Then we apply [10, Lemma 3.38] to deduce that the pair $(Y, \sum_{i=1}^n D_i)$ is log canonical.

It now follows from [9, 18.22] that $o \in Y$ is a toric singularity. In particular, it is of the form $\mathbb{C}^n/\mu_m$ for a cyclic group $\mu_m$ acting diagonally on $\mathbb{C}^n$, and it remains to show that $m = 2$.

For the latter, notice that $\sigma(\varphi_1^{-1}D_i)$ are hyperplanes on $\mathbb{P}$, with the plane $\sigma(\varphi_1^{-1}D_1) \cap \ldots \cap \sigma(\varphi_1^{-1}D_{n-2})$, say, intersecting the cubic $S$ at exactly 3 distinct points. This implies that $H_V \cap \varphi_1^{-1}D_1 \cap \ldots \cap \varphi_1^{-1}D_{n-2}$ is a $(−2)$-curve on the smooth surface $\varphi_1^{-1}D_1 \cap \ldots \cap \varphi_1^{-1}D_{n-2}$ and the equality $m = 2$ follows by varying $D_i$. □

Choose some smooth hypersurface $R \in |3(3H^* - E)|$, with $R \cap H_V = \emptyset$ (cf. Lemma 3.2), and let $\pi : \tilde{V} \rightarrow V$ be the double covering ramified in $R + H_V \sim 10H^* - 4E$. Variety $\tilde{V}$ is smooth and we have

$$-K_{\tilde{V}} = -\pi^*(K_V + \frac{1}{2}(R + H_V)) = \pi^*((n - 4)H^* + E) := (n - 4)\tilde{H} + \tilde{E}$$

by Hurwitz formula, where $\tilde{H}$ and $\tilde{E}$ are the pullbacks to $\tilde{V}$ of $H^*$ and $E$, respectively.

It is immediate from the construction that the group Pic $\tilde{V}$ is generated by $\mathcal{O}_{\tilde{V}}(\pi^{-1}H_V)$ and $\mathcal{O}_{\tilde{V}}(\tilde{E})$ (note that $\pi^*H_V = 2\pi^{-1}H_V$ because $\pi$ ramifies in $H_V$). Indeed, since $\mathcal{O}_{V}(H^*)$ and $\mathcal{O}_{V}(E)$ generate Pic $V$, with intersections $H^* \cap R$ and $E \cap R$ being irreducible (same for $H^* \cap H_V$ and $E \cap H_V$), the line bundles $\mathcal{O}_{\tilde{V}}(\pi^{-1}H_V)$ and $\mathcal{O}_{\tilde{V}}(\tilde{E})$ are the claimed generators of Pic $\tilde{V}$.

**Lemma 3.34.** There exists a birational contraction $f : \tilde{V} \rightarrow X$ of $\pi^{-1}H_V$, given by a multiple of the linear system $|\pi^*(3H^* - E)|$, onto some smooth variety $X$.

**Proof.** Let $Z \subset \pi^{-1}H_V \simeq \mathbb{P}^{n-1}$ be a line. We have

$$K_{\tilde{V}} \cdot Z = -((n - 4)\tilde{H} + \tilde{E}) \cdot Z = 3 - n < 0.$$  

Then [10, Theorem 3.25] delivers the contraction $f$ as stated. Finally, Lemma 3.3 yields

$$\pi^{-1}H_V \cdot Z = \frac{1}{2}\pi^*H_V \cdot Z = \frac{1}{2}H_V \cdot \pi(Z) = -1,$$

which implies that $f$ is just the blowup of the smooth point $f(\pi^{-1}H_V) \in X$. □

It follows from Lemma 3.3 that $X$ is a smooth Fano $n$-fold of index $n - 3$. Namely, we have

$$-K_X = (n - 3)f_\ast \tilde{H} = (n - 3)f_\ast \tilde{E},$$

for Pic $X = \mathbb{Z} \cdot \mathcal{O}_X(f_\ast \tilde{E})$.

Let us now find those curves on $X$ having the smallest intersection number with $f_\ast \tilde{E}$:

**Proposition 3.5.** For every curve $Z \subset X$ we have $f_\ast \tilde{E} \cdot Z \geq 1$ and equality is achieved when $\sigma(\pi(f_\ast^{-1}Z))$ is a point on $\mathbb{P}$. In other words, $f_\ast^{-1}Z \subset \tilde{E}$ is an elliptic curve, which is the preimage of a ruling on $E$.  

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Proof. Notice first that

$$-K_{\tilde{V}} = \frac{n-3}{2}(3H - \tilde{E}) - \frac{n-1}{2}(H - \tilde{E}).$$

In particular, we get $f^*(-K_X) = \frac{n-3}{2}(3H - \tilde{E})$, and hence $f_*\tilde{E} \cdot Z = a$ iff $-K_X \cdot Z = (n-3)a$ iff

$$\frac{n-3}{2}(3H - \tilde{E}) \cdot f_*^{-1}Z = (n-3)a$$

for any $a \in \mathbb{Z}$.

Further, if $\pi(f_*^{-1}Z)$ is a ruling on $E$, then $\tilde{H} \cdot f_*^{-1}Z = 0$ by definition and

$$\tilde{E} \cdot f_*^{-1}Z = \pi^*E \cdot f_*^{-1}Z = E \cdot \pi(f_*^{-1}Z) = 2E \cdot \pi(f_*^{-1}Z) = -2$$

by the projection formula, where $\pi(f_*^{-1}Z)$ has intersection index 4 with ramification divisor $R + H_V$, i.e. $f_*^{-1}Z$ is an elliptic curve. This implies that $a = 1$ for such $Z$ and Proposition 3.5 follows.

\[\square\]

Corollary 3.6. For every point $p \in X$ we have $s(p) = 1$ when $p \in f_*(\tilde{E})$ and $s(p) \geq 2$ otherwise.

Proof. Consider the case when $p = f(\pi^{-1}H_V) \in f_*(\tilde{E})$. Note that the Mori cone $N^2(\tilde{V}) \subset N^1(\tilde{V}) \otimes \mathbb{R} = \mathbb{R}^2$ is generated by the classes of a line in $\pi^{-1}H_V \simeq \mathbb{P}^{n-1}$ and an elliptic curve $Z \subset \tilde{E}$ as in Proposition 3.5. Now, by construction of $\tilde{V}$ via the blowup $f$ of $X$ at $p$ (see Lemma 3.4) we obtain that $s(p) = 1$, since divisor $\tilde{H}$ is nef and

$$f^*f_*\tilde{E} - \lambda \pi^{-1}H_V = \left(\frac{3}{2} - \frac{\lambda}{2}\right)\tilde{H} + \frac{1}{2}(\lambda - 1)\tilde{E}$$

is nef only when $\lambda \leq 1$. Then the estimate $s(p) \geq 1$ holds for any other $p \in f_*(\tilde{E})$ due to the lower semi-continuity of the function $s(\cdot)$ on $X$ (see [11, Example 5.1.11]). But $s(p) > 1$ cannot occur for these $p$ because otherwise the divisor $\beta^*f_*\tilde{E} - \lambda \mathcal{E}$ (we are using the notation of 1.3), with $\lambda > 1$, intersects the curve $\sigma_*^{-1}Z$ as $1 - \lambda < 0$. Thus $s(\cdot) = 1$ identically on $f_*(\tilde{E})$.

Recall further that $\pi$ when considered on $\tilde{V} \setminus \pi^{-1}H_V \cup \tilde{E} = X \setminus f_*(\tilde{E})$ is the double cover of $V \setminus H_V \cup E \simeq \mathbb{C}^n$ ramified in $R$. Also, the proper transform on $\tilde{V}$ of any element $\Sigma \in |mf_*\tilde{E}|, m \in \mathbb{Z}$, is an element from $|\frac{m}{2}(3\tilde{H} - \tilde{E})|$ which maps (via $\sigma$) onto some $\Sigma' \in |m(3H - S)|$ on $\mathbb{P}$. In particular, we get

$$\text{mult}_p \Sigma = \text{mult}_{\pi \circ \sigma(p)} \Sigma' \quad \text{or} \quad \geq \text{mult}_{\pi \circ \sigma(p)} \Sigma'$$

as long as $p \notin f_*(\tilde{E})$ (for $p \in X$ identified with $f^{-1}(p) \in \tilde{V}$), depending on whether $p \notin R$ or $p \in R$, respectively.

Now take $m = 1$ and $\Sigma' \in |3H - S|$ satisfying $\text{mult}_{\pi \circ \sigma(p)} \Sigma' = 2$. Such $\Sigma'$ vary in a linear system on $\mathbb{P}$ with isolated base locus near $p^5$. This and (3.13) (cf. (1.13)) imply that $s(p) \geq 2$ for $f_*(\tilde{E}) \sim \Sigma$.

\[\square\]

3.8. It remains to show that $X \setminus f_*(\tilde{E}) \simeq \mathbb{C}^n$ for one particular $R$.

Identifying $\mathbb{P} \setminus H = V \setminus H_V \cup E = Y \setminus \varphi(E)$ with $\mathbb{C}^n = \mathbb{P}^n \cap (w = 1)$ via $\sigma, \varphi$ we observe that there are elements $y_1, \ldots, y_n$ in $|3H^* - E|$, depending on the affine coordinates $x_i$, for which the assignment $x_i \mapsto y_i$, $1 \leq i \leq n$, induces an automorphism on $\mathbb{C}^n = \varphi \circ \sigma^{-1}(\mathbb{C}^n)$. Namely,

$$y_1 := x_1 + F, \quad y_2 := x_2, \quad y_3 := x_2x_3, \quad \ldots, \quad y_n := x_2x_n$$

satisfy this property, since one has induced isomorphism $\mathbb{C}(y_1, \ldots, y_n) \simeq \mathbb{C}(x_1, \ldots, x_n)$ by the choice of $F$ in 3.1. This also shows (as $\sigma^*F|_E \neq 0$ identically) that one may assume $y_i|_E \neq 0$ identically for all $i$.

\[5\] Indeed, if $x_1, \ldots, x_n, w$ are projective coordinates on $\mathbb{P}$, with $H = (w = 0)$ and $S = (w = F(x_1, \ldots, x_n) = 0)$ as in 3.1 then we consider $\Sigma' := (F + wB = 0)$ for an arbitrary quadratic form $B = B(x_1, \ldots, x_n)$ and $p := [0 : \ldots : 0 : 1]$. The case of arbitrary $p \in \mathbb{C}^n$ is easily reduced to this one.
Further, the equation of $R$ on $V \setminus H_V \cup E$ is a cubic polynomial in $y_i$, and we may take

$$R \cap (V \setminus H_V \cup E) := \langle P^1(y_i, \ldots, y_{n-1}, y_n + 1 = 0)$$

for some generic $P$. Notice that this defines a smooth hypersurface in $\mathbb{C}^n$.

Expressing $y_i$ in terms of $x_i$, we identify $R \cap (V \setminus H_V \cup E)$ with a hypersurface in $\mathbb{P} \setminus H$. Then compactifying via $w$, we obtain that $R \subset V$ can only be singular at the locus $y_1 = \ldots = y_{n-1} = w = 0$, i.e. precisely at $S$.

**Lemma 3.9.** $R$ is smooth and $R \cap H_V = \emptyset$.

**Proof.** After the blowup $\sigma$ the only singularities on $E$ that $R = (P + w^2 y_n + w^3 = 0)$ can have belong to the locus $E \cap \bigcap_{i=1}^{n-1} (y_i = 0) \cap H_V$. But the latter is empty by the choice of $y_i$. Hence $R$ is smooth. The claim about $R \cap H_V$ follows from Lemma 3.2 and the fact that $y_i|_E \neq 0$ identically for $y_i \in |3H^* - E|$. $\square$

Lemma 3.9 implies that $\tilde{V}$ is smooth. Then so is $X$ (see Lemma 3.4) and on the open chart $\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E} = X \setminus f_* \tilde{E}$ morphism $\pi : \tilde{V} \to V$ coincides with the projection of

$$\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E} = (T^2 = P(y_1, \ldots, y_{n-1}) + y_n + 1) \subset \mathbb{C}^{n+1}$$

onto $\mathbb{C}^n = V \setminus H_V \cup E$ with affine coordinates $y_i$. This yields $\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E} \simeq \mathbb{C}^n$, if one takes $T, y_1, \ldots, y_{n-1}$ as generators of the affine algebra $\mathbb{C}[\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E}]$.

Finally, since the defining equation of $R|_E$ is $P(y_1, \ldots, y_{n-1}) = 0$, with generic $P$, both cycles $R|_E$ and $\tilde{E}$ are irreducible. Thus $f_* \tilde{E}$ is also irreducible and so $X$ is the required compactification of $\mathbb{C}^n$ (with the boundary divisor $\Gamma = f_* \tilde{E}$). This completes the construction of $X$.

Theorem 1.2 now follows from the next

**Corollary 3.10.** $X$ is not uniformly rational.

**Proof.** Notice first that $X$ satisfies all the assumptions of Section 2. Then Corollary 2.15 applies to $X$ once we assume the latter to be u.r. We obtain $s(p) \geq s(o)$ for some $p \in f_* \tilde{E}$ and $o \in X \setminus f_* \tilde{E}$. At the same time, Corollary 3.6 gives $s(p) = 1$ and $s(o) \geq 2$, a contradiction. Hence $X$ can not be u.r. $\square$

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