The Self-Dual Critical $N=2$ String$^{**}$

Olaf Lechtenfeld

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
http://www.itp.uni-hannover.de/~lechtenf/

Abstract

I review the covariant quantization of the closed fermionic string with (2,2) extended world-sheet supersymmetry on $\mathbb{R}^{2,2}$. Results on $n$-point scattering amplitudes are presented, for tree- and one-loop world-sheets with arbitrary Maxwell instanton number. I elaborate the connection between Maxwell moduli, spectral flow, and instantons. It is argued that the latter serve to extend the Lorentz symmetry from $U(1,1)$ to $SO(2,2)$ by undoing the choice of spacetime complex structure.

$^{*}$ Talk given at the International Conference ‘Problems of Quantum Field Theory’ in Alushta, Crimea, Ukraine, 16 March 1996

$^{**}$ Supported by the ‘Volkswagen Foundation’ and the ‘Deutsche Forschungsgemeinschaft’
1. Introduction
In this talk I am going to report on recent results [1, 2, 3, 4, 5] obtained in collaboration with Jan Bischoff, a PhD student, and Sergei Ketov, a senior postdoc at Hannover, Germany. Fermionic strings with extended world-sheet supersymmetry are much less understood than the usual bosonic or superstrings, although their discovery now dates back twenty years. For a review, see [6, 7]. Responsible for this gap is the fact that the physical spectrum of extended fermionic strings is not rich enough to be of immediate relevance for particle phenomenology. Although strings with \( N=2 \) world-sheet supersymmetry are critical in four dimensions, the spacetime signature must be (2,2), and the excitation spectrum contains only a single massless scalar field. Thus, such strings are particular point particle theories in disguise and describe self-dual gravitational and gauge fields [7]. On the other hand, these peculiarities attract the interest of mathematical physicists. For example, \( N=2 \) strings should teach us something about the quantization of integrable systems. Furthermore, they can be embedded into a twisted \( N=4 \) string which hints at a topological interpretation [8, 9, 10]. Finally, \( N=2 \) heterotic strings form a building block in the recently proposed F-theory [11] unifying superstring compactification schemes, and they should play some role in the emerging picture of string-string duality.

It is well-known that the low-energy effective actions (LEEAs) of \( N=0 \) (bosonic) open and closed strings is Yang-Mills and gravity (plus other fields), respectively, in 1+25 spacetime dimensions. In the \( N=1 \) case, the five cases of supergravity and/or super-Yang-Mills in 1+9 spacetime dimensions emerge in the low-energy limit. In contrast, open and closed \( N=2 \) strings do not give rise to spacetime supersymmetry. Astoundingly, their LEEAs represent \textit{self-dual} Yang-Mills and \textit{self-dual} gravity in 2+2 spacetime dimensions, respectively. The coupling of both from open-string loops or heterotic versions, however, has not been fully clarified.

2. Covariant Quantization
The Polyakov formulation of the (2, 2) fermionic string involves complexified world-sheet scalar string coordinates \( X^\mu \) and their complexified world-sheet Majorana spinor partners \( \psi^\mu \), with \( \mu = 0, 1 \), coupled to the \( N=2 \) supergravity multiplet consisting of the real world-sheet metric \( h_{\alpha\beta} \), complex gravitino \( \chi_\alpha \), and real Maxwell field \( A_\alpha \) (graviphoton), with \( \alpha, \beta = 0, 1 \). The Brink-Schwarz action [12] possesses \( N=2 \) super-diffeomorphism and \( N=2 \) super-Weyl invariance on the world-sheet as well as global \( U(1, 1) \times \mathbb{Z}_2 \) target-space symmetry. It is important to note that only the world-sheet spinors \( \psi^\mu \) and \( \chi_\alpha \) are charged under the local \( U(1) \) Maxwell symmetry, whereas global \( U(1, 1) \) Lorentz transformations affect only the matter fields \( X^\mu \) and \( \psi^\mu \).

After \( N=2 \) superconformal gauge fixing of the Brink-Schwarz action, the local residual gauge transformations decompose chirally. The left- and right-moving parts
each comprise an $N=2$ superconformal algebra whose generators $T, G^\pm$, and $J$ act as constraints in the quantized theory. Proper gauge fixing in the euclidean path integral expression of a closed string amplitude induces Faddeev-Popov determinants, which are rewritten in terms of real conformal ($b_{\alpha \beta}, c^\alpha$), complex (spinorial) superconformal ($\beta_\alpha, \gamma$), and real Maxwell ($\tilde{b}_\alpha, \tilde{c}$) ghost systems. The anti-ghost zero modes are projected out by appropriate anti-ghost insertions into the path integral; eventual ghost zero modes are associated with global residual gauge symmetries and have to be absorbed by vertex operators. The implementation of the first-class constraints in the ghost-extended version is most elegantly performed in the BRST framework.

Cancellation of conformal and chiral anomalies requires the flat target space to be complex two-dimensional. In the following I shall specialize to the simplest case of $C^{1,1}$. Then, world-sheet bosons must be single-valued, whereas non-trivial monodromies are still allowed for world-sheet fermions $f \in \{\psi, \chi, \beta, \gamma\}$. The latter being complex (which I denote by a $\pm$ superscript), one may have

$$f^\pm \rightarrow e^{\pm 2\pi i \rho} f^\pm$$

for the transportation around a non-trivial loop. Thus, the theory sports a continuous set of sectors parametrized by $\rho$ and interpolating between the usual NS ($\rho=\frac{1}{2}$) and R ($\rho=0$) sectors. This phenomenon is well-known as *spectral flow* in $N=2$ superconformal models.

The computation of a closed string $n$-point amplitude requires an integration over all $N=2$ superconformal structures, or supermoduli, of $2d$ $n$-punctured orientable $N=2$ supermanifolds $s\Sigma_n$. The universal supermoduli space decomposes into a sum indexed by the genus $g \in \mathbb{Z}_+$ and the instanton number $c \in \mathbb{Z}$. This reflects the topological classification of the tangent and principal $U(1)$ bundles over ordinary compact Riemann surfaces $\Sigma$ through

$$\frac{1}{2\pi} \int_\Sigma R = 2 - 2g \quad \text{and} \quad \frac{1}{2\pi} \int_\Sigma F = c$$

where $R$ and $F$ are the curvature two-forms of the spin and Maxwell connections of $N=2$ world-sheet supergravity, respectively.

The supermoduli spaces have a natural complex structure and are parametrized by the meromorphic differentials of order 2, $\frac{3}{2}$, and 1, on $\Sigma$, with Maxwell charges of 0, $\pm \frac{1}{2}$, and 0, respectively. Locally, each supermoduli space is a direct sum of metric, fermionic, and Maxwell moduli spaces, with complex dimensions of $3g-3+n$, $2g-2\pm c+n$, and $g-1+n$, respectively. Metric and fermionic moduli may be treated like in the $(1,1)$ fermionic string. Explicitly integrating out the fermionic moduli leads to additional insertions into the path-integral measure, which combine
with certain anti-ghost insertions to picture-changing operators. If $|c| > 2g - 2 + n$, fermionic zero modes can no longer be soaked up in the path integral, which then must vanish. Hence, only a finite number of instanton sectors contribute at a given genus.

In contrast, the Maxwell moduli are specific to the $N=2$ string. They parametrize the space of flat connections or harmonic one-forms $H$ on the $n$-punctured worldsheet $\Sigma_n$ of genus $g$. Since $H = m + \tilde{m}$ where $m$ is a meromorphic one-form with a single pole at each puncture $p_\ell$, a basis is provided by the normalized first abelian differentials and the normalized third abelian differentials. Thus, the Maxwell moduli space is a complex $(g + n - 1)$-dimensional torus, parametrized by $\int \! d\tau A^n_{g,c}(k_1, \ldots, k_n|\tau)$ (3)

where $\kappa$ and $\lambda$ are the gravitational and Maxwell string couplings, respectively, and $\tau$ collectively denotes the metric and Maxwell moduli. The fixed-moduli correlation function reads

$$A^n_{g,c}(k_1, \ldots, k_n|\tau) = \int \! D[X_0] \psi_{\beta\gamma} \tilde{b} \, AZI \cdot PCO \prod_{\ell=1}^{n} V_\ell(k_\ell) e^{i S[X_0|\tau]}$$ (4)

with anti-ghost zero mode and picture-changing insertions $AZI$ and $PCO$, vertex operators $V_\ell$ and the gauge-fixed action $S$.

To enumerate the spectrum of physical excitations of $N=2$ strings, one needs first to determine their BRST cohomology $H(Q) = \ker Q/\im Q = \{\text{phys}\}$ and then to list additional identifications among physical states, e.g. by picture-changing or spectral flow. This investigation has not been completed, but partial results are very suggestive [1, 13, 14]. Namely, for a given lightlike momentum $k^{\pm\mu}$ one finds a single bosonic scalar state $|k^{\pm\mu}\rangle$, but no states whatsoever at any massive level! This agrees with the naive expectation that 2+2 dimensions leave no room for transverse vibrations. In essence, the field theory of $N=2$ strings is really just the field theory of a scalar particle, described by the spacetime field $\Phi$! Therefore, one should expect the interaction of $N=2$ strings to be fully given by the exact LEEA. Accordingly, the task is to compute all scattering amplitudes for $\Phi$ at tree-, loop-, and multi-instanton-level in $N=2$ strings.

3. Amplitudes

Let me present the results [3, 4] of direct computations of closed $N=2$ on-shell string
amplitudes (at fixed metric and Maxwell moduli). The lightlike momenta will be denoted by \( k_\ell^{\pm\mu} \), with \( \ell = 1, \ldots, n \), \( \mu = 0,1 \), and the \( \pm \) superscript indicating the charge \( q = \pm \frac{1}{2} \) under the \( U(1) \) factor of the \( U(1,1) \) Lorentz group. With the 2d invariant tensors \( \eta_{\mu\nu} \) and \( \epsilon_{\mu\nu} \) the on-shell condition reads \( \eta_{\mu\nu} k_\ell^{\pm\mu} k_\ell^{-\nu} = 0 \). Momentum conservation, \( \sum_\ell k_\ell^{\pm\mu} = 0 \), will be assumed. We introduce the following three antisymmetric bilinears,

\[
c_0^{\ell m} := \eta_{\mu\nu} (k_\ell^{+\mu} k_m^{-\nu} - k_\ell^{-\mu} k_m^{+\nu}) \quad \text{and} \quad c_\pm^{\ell m} := \epsilon_{\mu\nu} k_\ell^{\pm\mu} k_m^{\pm\nu} ,
\]

of which the first one is fully \( U(1,1) \) invariant while the other two are only inert under \( SU(1,1) \) as their charge index indicates.

For less than three external legs,

\[
A^0 = 0 \quad , \quad A^1 = 0 \quad , \quad A^2 = 1 \quad ,
\]

including loop and instanton corrections. The tree-level three-point function is

\[
A_{g=0}^3 = \lambda^{+1} (c_{23}^{-})^2 + \lambda^{0} (c_{23}^{0})^2 + \lambda^{-1} (c_{23}^{+})^2
\]

with the observation that Lorentz invariance requires the coupling \( \lambda \) to acquire a \( U(1) \) charge of two units! Using the standard Mandelstam variables \( s, t, \) and \( u \), the first non-trivial amplitude is

\[
A_{g=0}^4 \propto \lambda^{+2} [c_{12}^{-} c_{34}^{0} t + c_{23} c_{41}^{-} s - stu]^2 + \lambda^{+1} [c_{12}^{-} c_{34}^{0} t + c_{23} c_{41}^{0} s - stu]^2 + \lambda^{0} [c_{12}^{0} c_{34}^{0} t + c_{23} c_{41}^{0} s - stu]^2 + \lambda^{-1} [c_{12}^{+} c_{34}^{0} t + c_{23} c_{41}^{+} s - stu]^2
\]

\[
= 0
\]

due to the very special kinematics in 2+2 dimensions! It is believed that \( A_{n \geq 4} = 0 \) to all orders, so that string duality is consistent with the conjectured complete absence of massive poles in intermediate channels.

At the one-loop level, we have computed \[4, 15\]

\[
A_{g=1}^3 \propto (A_{g=0}^3)^3
\]

with the help of \( c_{\ell m}^{-} c_{\ell m}^{+} + c_{\ell m}^{0} c_{\ell m}^{0} = 0 \), leading me to the interesting conjecture that

\[
A_{g}^3 \propto (A_{g=0}^3)^{2g+1} .
\]
4. Spectral Flow and Instantons
Since moduli are integrated over, shifting their values cannot affect the final amplitude. It does, however, have an effect on the integrand, $A^a_{g,c}$. The Maxwell moduli couple to the gauge-fixed action as

$$S[X \ldots |\tau] \sim \int \Sigma H \wedge *J$$

so that $H \rightarrow H + h$ changes the action by

$$\int \Sigma h \wedge *J = g \sum_{i=1}^g \left( \oint_{a_i} h \oint_{b_i} *J - \oint_{b_i} h \oint_{a_i} *J \right) + n \sum_{\ell=1}^n \oint_{p_\ell} h \int_{p_0}^* *J$$

where I used the harmonicity of $h$ and $d*J=0$ and introduced a reference point $p_0$. Clearly, my shift has induced twists $\exp\left\{ 2\pi i \theta \int_{z_0}^z *J \right\}$ around the standard homology cycles as well as around the punctures, with $\theta \in [0,1]$. Dependence on $p_0$ drops out because $\sum_{\ell} \text{res}_{p_\ell} h = 0$.

When integrating over the Maxwell moduli, I could fix some $H$ and let $h$ sweep over the complex torus. During this course, the twists extend the already existing sum over spin structures to an integral over spin structures. Integration over matter and ghost fields finally yields Jacobi theta functions, with the prescription to integrate over their characteristics [4]. This is in line with the phenomenon of spectral flow. Indeed, a twist around a puncture at $z$ may be absorbed into a redefined (twisted) vertex operator,

$$V(z) \rightarrow V^{(\theta)}(z) := e^{\frac{2\pi i \theta}{2} \int_{z_0}^z *J} V(z)$$

which can be seen to create a spectrally flowed state from the vacuum. Alternatively, conjugating the generators of the $N=2$ superconformal algebra by the spectral-flow operator

$$\text{SFO}(\theta) := \exp\{2\pi i \theta \int_{z_0}^z *J\}$$

explicitly realizes the spectral flow endomorphism of the algebra.

Furthermore, SFO is BRST-closed but only its zero mode is not BRST-exact, so that its position in an amplitude does not matter. Hence, amplitudes are twist-invariant, $\langle V^{(\theta_1)}_1 \ldots V^{(\theta_n)}_n \rangle = \langle V_1 \ldots V_n \rangle$, as long as the total twist vanishes, $\sum_{\ell} \theta_\ell=0$. Bosonizing the local $U(1)$ current $*J = d\phi$ with a linear combination $\phi = \phi_m + \phi_{gh}$ of compact (matter and ghost) bosons,

$$\text{SFO}(\theta) = \exp\left\{ 2\pi i \theta \left[ \phi(z) - \phi(z_0) \right] \right\} = e^{-2\pi i \theta \phi(z_0)} \exp\left\{ 2\pi i \theta \phi(z) \right\}$$

provides me with a local operator $\exp\{2\pi i \theta \phi\}$, modulo zero-mode ambiguities. The two factors in eq. (15) are separately neutral under the local $U(1)$, but carry opposite global $U(1)$ charges.
Interestingly, spectral flow is related to Maxwell instantons on the world-sheet. Flowing around full circle, i.e. choosing $\theta=1$, one may define an instanton-creation operator

$$ICO := \lambda SFO(\theta=1) = \lambda e^{-2\pi i \phi(z_0)} \exp\{2\pi i \phi(z)\}$$ (16)

which implements a singular gauge transformation $A \rightarrow A + d \ln z$ and changes the world-sheet instanton number via $ICO |c\rangle = |c+1\rangle$. I can now hide the zero-mode or reference-point ambiguity by simply declaring

$$\lambda \equiv e^{2\pi i \phi(z_0)}$$ (17)

which gives $ICO$ and $\lambda$ (!) a global $U(1)$ charge of $q=2$. Then, amplitudes with different instanton backgrounds are related through

$$\langle V_1 \ldots V_n \rangle_c = \langle V_1 \ldots V_n (ICO)^c \rangle_{c=0} = \lambda^c \langle V^{(\theta_1)}_1 \ldots V^{(\theta_n)}_n \rangle_{c=0}$$ (18)

with a total twist of $\sum_{\ell} \theta_\ell = c$. This matches the $\lambda$ dependence in eq. (3).

5. Instantons as Lorentz Generators

Finally, I propose a common solution of two related problems. First, I have shown that $A^{3}_{c\neq0}$ is only invariant under the $SU(1,1)$ subgroup of the spacetime Lorentz group. Second, I should even like to extend the Lorentz group from $U(1,1) \simeq U(1) \times SU(1,1)$ to $SO(2,2) \simeq SU(1,1) \times SU(1,1)$, as is proper for a real $(2+2)$-dimensional spacetime and necessary for the existence of spacetime spinors. Surprisingly, my instanton creation and annihilation operators point to a solution. Factorizing into a matter and ghost part, $ICO = ICO_m ICO_{gh}$, the first factor is a genuine current and provides me with new charges, $M_+ = \oint ICO_m$ and $M_- = \oint ICO_{m}^{-1}$. Together with the $U(1)$ Lorentz generator $M_3 = \oint d\phi_m$, they create the second $SU(1,1)$ factor of the extended $SO(2,2)$ Lorentz group!

It is not difficult to check that $M_+$ and $M_-$ act on amplitudes by simply raising and lowering the $U(1)$ charge $q$ of two $c_d^\ell m$ kinematical factors by one unit each. For $n=3$, the range of instanton backgrounds in eq. (3), $|c| \leq 2g+1$, is nicely matched by the allowed range of $U(1)$ charges, $|q| \leq 2(2g+1)$, from eqs. (10) and (7). These charge assignments are also compatible with amplitude factorization.

To conclude, the extended $SO(2,2)$ Lorentz invariance of $N=2$ string amplitudes is not manifest but realized in a hidden way. The inevitable choice of a complex structure in $\mathbb{R}^{2,2}$ selects some $U(1,1)$ subgroup. In essence, world-sheet Maxwell instantons rotate around this subgroup and, with it, the spacetime complex structure. It is therefore natural to view the Maxwell coupling $\lambda$ as a coordinate parametrizing the moduli space of complex structures, which requires it to appropriately transform under part of the Lorentz group. This seems to connect well with the results of [8, 10] and asks for a twistor-like formulation.
References

[1] J. Bischoff, S.V. Ketov and O. Lechtenfeld, Nucl. Phys. B438 (1995) 373.
[2] S.V. Ketov and O. Lechtenfeld, Phys. Lett. 353B (1995) 463.
[3] O. Lechtenfeld, hep-th/9512189, Nucl. Phys. B Suppl., to appear.
[4] J. Bischoff, PhD thesis, www.itp.uni-hannover.de/~lechtenf/theses.html
[5] J. Bischoff and O. Lechtenfeld, in preparation.
[6] N. Marcus, hep-th/9211059.
[7] H. Ooguri and C. Vafa, Nucl. Phys. B361 (1991) 469.
[8] N. Berkovits and C. Vafa, Nucl. Phys. B433 (1995) 123.
[9] N. Berkovits, Phys. Lett. 350B (1995) 28.
[10] H. Ooguri and C. Vafa, Nucl. Phys. B451 (1995) 121.
[11] C. Vafa, hep-th/9602022.
[12] L. Brink and J. Schwarz, Nucl. Phys. B121 (1977) 285.
[13] J. Bienkowska, Phys. Lett. 281B (1992) 59.
[14] H. Lu and C.N. Pope, Nucl. Phys. B447 (1995) 297.
[15] M. Bonini, E. Gava and R. Iengo, Mod. Phys. Lett. A6 (1991) 795.