ON THE NON-ORTHONORMALITY OF LIPPMANN-SCHWINGER-LOW STATES

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Abstract

It is pointed out that for a general short-ranged potential the Lippmann-Schwinger-Low scattering state $|\psi_k^L\rangle$ does not strictly satisfy the Schrodinger eigen equation, and the pair $|\psi_n^L\rangle, |\psi_k^L\rangle$ is mutually nonorthogonal if $E_n = E_k$. For this purpose, we carefully use an infinitesimal adiabatic parameter $\epsilon$, a nonlinear relation among transition amplitudes, and a separable interaction as illustration.

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Introduction

The Lippmann-Schwinger-Low (LSL) integral equations for state vectors and transition matrices form the backbone of quantum scattering theory [1]. They provide the basis for deriving the Born series in wave mechanics [2], reaction amplitudes in rearrangement collisions [3], Dyson’s perturbation expansion in the Dirac picture [4], and various cross sections in old-fashioned quantum electrodynamics [5]. The aim of the present paper is to examine some features of the LSL equations which have not been treated adequately in the existing literature. To be more precise, Lemmas A, B, C and D below answer the following four questions: (i) Are the LSL representations strictly equivalent to the underlying Schrodinger eigen equations? (ii) What is a general off/on energy-shell unitarity-like relation obeyed by the LSL transition amplitudes? (iii) Do various LSL state vectors accurately satisfy the orthonormality relations mentioned by Goldberger-Watson [6]? (iv) Can we confirm the results explicitly in the case of a separable potential for which the LSL solutions can be obtained in closed form [7]?

Preliminaries

We denote the free and full Hamiltonian operators by $H^0$ and $H \equiv H^0 + V$ respectively with $V$ being a short-range interaction. Their continuum eigenkets obey the Schrodinger (superscript S) equations

\[
(E_k - H^0)|k\rangle = 0 \quad (1)
\]

\[
(E_k - H)|\psi^S_k\rangle = 0 \quad (2)
\]

where the masses are assumed to be renormalized so that energies do not shift. For later convenience we also introduce the free resolvent $G^0_k$, the complex projector $\eta^0_k$ onto free
states of energy $E_k$, $\pi$ times a Dirac delta $D_k^o$, related functions $\mu_{nk}$ and $d_{nk}$ along with a useful identity via

$$G_k^o = \frac{1}{E_k - H^o + i\epsilon} ; \quad \eta_k^o = i\epsilon G_k^o \quad ; \quad \mu_{nk} = \frac{i\epsilon}{E_k - E_n + i\epsilon}$$ (3)

$$D_k^o = \pi\delta(E_k - H^o) = \epsilon G_k^{o\dagger} G_k^o ; \quad d_{nk} = \frac{\epsilon}{(E_k - E_n)^2 + \epsilon^2}$$ (4)

$$\frac{G_n^{o\dagger} - G_k^o}{E_k - E_n + i\epsilon} = E_k - E_n + 2i\epsilon \quad ; \quad G_n^{o\dagger} G_k^o = (1 + \mu_{nk}) G_n^{o\dagger} G_k^o$$ (5)

where $\epsilon \to +0$ is an adiabatic parameter, and $\mu_{nk}$ and $d_{nk}$ vanish if $E_n \neq E_k$. It is customary to replace Eq.(1) & (2) by the LSL representations (labeled by the superscript $L$)

$$|\psi_k^L\rangle = |k\rangle + G_k^o V |\psi_k^L\rangle \quad ; \quad LS$$ (6)

$$= |k\rangle + (E_k - H + i\epsilon)^{-1} V |k\rangle \quad ; \quad Low$$ (7)

obeying plane + outgoing boundary conditions. Our objective is to propose a few Lemmas on some algebraic properties of $|\psi_k^L\rangle$ below by paying careful attention to the $\epsilon$ factors.

**LEMMA A (COMPARISON WITH SCHRODINGER) :**

"In sharp contrast to the underlying Eq.(1) the LSL states satisfy

$$(E_k - H + i\epsilon) |\psi_k^L\rangle = i\epsilon |k\rangle ,$$ (8)

or equivalently

$$(E_k - H) |\psi_k^L\rangle = -\eta_k^o V |\psi_k^L\rangle "$$ (9)"
\textbf{Proof}

Eq. (8) follows from the application of the operator \((E_k - H^o + i\epsilon)\) on Eq. (6) or \((E_k - H + i\epsilon)\) on Eq. (7). It suggests that \(|\psi^L_k\rangle\) is not a strict eigenket of \(H\) for any nonzero infinitesimal \(\epsilon\). Eq. (9) is an outcome of the fact that \(i\epsilon |\psi^L_k\rangle = \eta^o_k V |\psi^L_k\rangle\) is generally a nonzero ket. Indeed, the matrix element \(\langle n|\eta^o_k V |\psi^L_k\rangle = \mu_{nk}\langle n|V |\psi^L_k\rangle\) becomes the on-shell transition amplitude if \(E_n = E_k\).

\textbf{LEMMA B (NONLINEAR RELATION FOR T-MATRIX) :}

\text{"The amplitudes} \(T^L_{nk} \equiv \langle n|V |\psi^L_k\rangle\) \text{ fulfill a nonlinear relation}

\(\frac{(T^L_{nk} - T^L_{kn})}{(E_k - E_n + i\epsilon)} = -(1 + \mu_{nk})C^L_{nk}\) \hspace{1cm} (10)

\(C^L_{nk} = \langle \psi^L_n | V G^o_n G^o_k V |\psi_k\rangle \) \hspace{1cm} (11)

\textbf{Proof}

From Eq. (6) we first obtain \(\langle n|\) and thereby write

\(T^L_{nk} = \langle \psi^L_n | V |\psi^L_k\rangle - \langle \psi^L_n | V G^o_n G^o_k V |\psi^L_k\rangle \) \hspace{1cm} (12)

Subtracting a similar expression for \(T^L_{kn} \equiv \langle \psi^L_n |V |k\rangle\) and employing the identity (5) the desired Lemma follows.

Incidentally, in the special case of \(E_n = E_k\) our Eqs. (10), (11) reduce to the usual on-shell unitarity relation [2-5] viz.

\[\left[T^L_{nk} - T^L_{kn}\right]_{E_n = E_k} = -2i A^L_{nk}\] \hspace{1cm} (13)

\[A^L_{nk} = \left[\epsilon C^L_{nk}\right]_{E_n = E_k} = \langle \psi^L_n | V D^o_k V |\psi^L_k\rangle \] \hspace{1cm} (14)
LEMMA C (NONORTHONORMALITY):

"Consider the overlap $I_{nk}^L \equiv \langle \psi_n^L | \psi_k^L \rangle$ between two arbitrary outgoing LSL states. In sharp contrast to the conventional erroneous value [6] $\langle n | k \rangle$ for the overlap its correct value is

$$I_{nk}^L = \langle n | k \rangle - d_{nk} A_{nk}^L$$

(15)

with $d_{nk}$ given by Eq.(4) and $A_{nk}^L$ by Eq.(14).

Proof

Upon using the Low form for $\langle \psi_n^L |$ and the LS form for $| \psi_k^L \rangle$ (cfs. Eqs. 6,7) one finds

$$\langle \psi_n^L | \psi_k^L \rangle = \langle n | \psi_k^L \rangle + \langle n | V (E_n - H - i\epsilon)^{-1} | \psi_k^L \rangle$$

(16)

In the usual Goldberger-Watson treatment (labeled by the superscript G) one erroneously assumes that $H | \psi_k^L \rangle = E_k | \psi_k^L \rangle$ and reduces Eq.(16) into [6]

$$I_{nk}^G = \langle n | k \rangle + \langle n | V \left( \frac{1}{E_k - E_n + i\epsilon} + \frac{1}{E_n - E_k - i\epsilon} \right) | \psi_k^L \rangle = \langle n | k \rangle$$

(17)

In our opinion the use of Eqs. (8), (9) as eigenket statement is quite risky and it is much safer to employ the LS representations (6) for both $\langle \psi_n^L |$ and $| \psi_k^L \rangle$. Then

$$I_{nk}^L = \langle n | k \rangle + \langle n | G_k^o V | \psi_k^L \rangle + \langle \psi_n^L | V G_n^o | k \rangle$$

$$+ \langle \psi_n^L | VG_n^o G_k^o V | \psi_k^L \rangle$$

(18)

which is readily shown to coincide with the Lemma (15) in view of the properties (Eq.(10)) and (Eq.(14)). The fact that $I_{nk}^L$ reduces to $\langle n | k \rangle$ if $E_n \neq E_k$ but fails to
do so if \( E_n = E_k \) is very disturbing because it implies that the set of LSL states \( |\psi^L_n \rangle \) which are degenerate at a given collision energy \( E_k \) are mutually nonorthogonal.

**LEMMA D (ILLUSTRATION):**

"Consider a rank 1 separable potential \([7]\) \( V = \lambda |g \rangle \langle g | \) with \( \lambda \) being a real coupling and \( |g \rangle \) a wave packet. Then, the overlap \( \langle \psi^L_n | \psi^L_k \rangle \) can be independently shown to be

\[
I^L_{nk} = \langle n | k \rangle - g^2 \frac{\langle g | D^o_k | g \rangle}{\Delta_n \Delta_k} \tag{19}
\]

where the form factor \( g_k \) and Fredholm determinant \( \Delta_k \) are defined by

\[
g_k = \langle k | g \rangle ; \quad \Delta_k = 1 - \lambda \langle g | G^o_k | g \rangle \tag{20}
\]

**Proof**

With \( V = \lambda |g \rangle \langle g | \), Eq.(6) is readily solved in closed form as

\[
|\psi^L_k \rangle = |k \rangle + G^o_k | g \rangle \left( \lambda g^* / \Delta_k \right)
\]

\[
\langle \psi^L_n | = \langle n | + \left( \lambda g_n / \Delta_n^* \right) \langle g | G_n^o \right) \tag{21}
\]

Then it is straightforward to compute

\[
I^L_{nk} = \langle n | k \rangle - \lambda g_n g^*_k \left[ \frac{\Delta_k - \Delta_n^*}{\Delta_n \Delta_k} \right] \frac{\langle g | D^o_k | g \rangle}{\lambda (E_k - E_n + i \rho)} - \langle g | G_n^o G^o_k | g \rangle \right] \tag{22}
\]

which coincides with the stated lemma in view of the useful identity \((5)\). Of course, the illustrative Eq.(19) and the general result Eq.(15) are in complete agreement although they were derived by different methods.

**CONCLUSIONS**

The main findings of the present paper are contained in Lemmas A, B, C, and D. The
nonorthogonality of the LSL states (for $E_n = E_k$, $n \neq k$) implies that, even in the absence of bound states, the Moller operator connecting $|k\rangle$ to $|\psi^L_k\rangle$ may be nonunitary and $\sum_k |\psi^L_k\rangle\langle\psi^L_k|$ may lose its interpretation as the unit matrix. Several standard results of scattering perturbation theory [1-7] based on the LSL states may require re-examination. Before ending, it may be added that the present work is not concerned with another peculiarity of the LS equation - the Faddeev ambiguity [8] - arising from the noncompactness of the kernel. We also believe that the time-dependence of the LSL states will be much richer than the standard Schrodinger kets $|\psi^S_k(t)\rangle$ but this aspect will be dealt-with in a future communication.

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