High $T_c$ superconductivity arises as a consequence of hole(or electron) doping in the parent cuprate oxides which are Mott insulators with antiferromagnetic long-range order. The observed phase diagram in the plane of temperature $T$ vs. hole doping rate $\delta$ shows the bose condensation(supercconducting temperature) curve of an ‘arch’ shape rather than the often predicted linear decrease, by manifesting the presence of the optimal doping rate of $0.16$ to $0.2$. On the other hand, the observed pseudogap temperature displays nearly a linear decrease with $\delta$. Various U(1) slave-boson approaches to the t-J Hamiltonian were able to predict such a linear decrease in the pseudogap temperature as a function of $\delta$. On the other hand, these theories failed to predict the experimentally observed bose condensation curve(temperature $T_c$) vs. hole doping rate as a function of $\delta$. Instead a linear increase of $T_c$ with $\delta$ was predicted. Further the pseudogap phase was shown to disappear when the gauge fluctuations are introduced into the U(1) slave-boson mean field theory. Most recently Wen and Lee proposed an SU(2) theory to readily estimate the low energy phase fluctuations of order parameters and made a brief discussion on the possibility of holon(boson) pair condensation. In view of failure in the correct prediction of the bose condensation temperature $T_c$ in the phase diagram with earlier theories, in the present study we examine the variation of the holon-pair condensation temperature with the hole doping rate, by treating the phase fluctuations of the order parameters in the SU(2) slave boson theory. The present work differs from our previous U(1) slave-boson study of the phase diagram involving the holon-pair boson condensation and other earlier studies involving the single holon condensation in that coupling between the holon and spinon degrees of freedom in the slave-boson representation of the Heisenberg term of the t-J Hamiltonian is no longer neglected. We find from the treatment of the coupling that the predicted phase diagram displays the arch-shaped bose condensation curve(temperature $T_c$) as a function of hole doping rate in both treatments of the U(1) and SU(2) slave-boson approaches. In addition, comparison between the two approaches will be made to reveal the importance of the low energy phase fluctuations of the order parameters. It is noted that such phase fluctuations are not taken into account in the usual treatment of the U(1) slave-boson mean field theory.

We write the t-J Hamiltonian,

$$H = -t \sum_{<i,j>} (c_i^\dagger c_{j\sigma} + c.c.) + J \sum_{<i,j>} (S_i \cdot S_j - \frac{1}{4} n_i n_j). \quad (1)$$

Here $S_i$ is the electron spin operator at site $i$, $S_i = \frac{1}{2} c_i^\dagger \sigma_{\alpha\beta} c_i^{\dagger \alpha \beta}$ with $\sigma_{\alpha\beta}$ the Pauli spin matrix element and $n_i$, the electron number operator at site $i$, $n_i = c_i^\dagger c_i$. We note that $S_i \cdot S_j = \frac{1}{2} (c_i^\dagger \sigma_{\alpha\beta} c_j^{\dagger \alpha \beta})(c_j c_i - c_j c_i - c_j c_i)$. The t-J Hamiltonian above can be written,

$$H = -t \sum_{<i,j>\sigma} \left[ (f_{\sigma \alpha}^\dagger f_{\sigma \beta}) (b_{i\sigma}^\dagger b_{j\beta} - b_{j\beta}^\dagger b_{i\sigma}) 
+ (f_{\sigma \alpha}^\dagger f_{\sigma \beta})(b_{i\sigma}^\dagger b_{j\beta} - b_{j\beta}^\dagger b_{i\sigma}) $$
$$+ (f_{\sigma \alpha}^\dagger f_{\sigma \beta})(b_{j\beta}^\dagger b_{i\sigma} + b_{i\sigma}^\dagger b_{j\beta}) $$
$$+ (f_{\sigma \alpha}^\dagger f_{\sigma \beta})(b_{j\beta}^\dagger f_{i\sigma} - f_{i\sigma}^\dagger b_{j\beta}) \right].$$

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\[ -\frac{J}{2} \sum_{<i,j>} (1 - \chi_{h_i})(1 - \chi_{h_j}) \times \]
\[ (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) (f_{i,j} f_{i,j} - f_{j,i} f_{i,j}) \]
\[ + \frac{t}{2} \sum_{<i,j>} \left[ \sum_{\alpha, \beta} \lambda_{i}^{(\alpha, 2)} (f_{i,j}^{\dagger} f_{i,j} + b_{i,j}^{\dagger} b_{i,j}) \right] + \frac{t^2}{2} \sum_{<i,j>} \left[ \sum_{\alpha, \beta} \lambda_{i}^{(\alpha, 2)} (f_{i,j}^{\dagger} f_{i,j} + b_{i,j}^{\dagger} b_{i,j}) \right]. \]

Here \( f_{i,j} \) (\( f_{i,j}^{\dagger} \)) is the spinon annihilation(creation) operator and \( h_i \equiv \left( \begin{smallmatrix} b_{i,1} & b_{i,2} \end{smallmatrix} \right) \left( \begin{smallmatrix} b_{i,1}^{\dagger} & b_{i,2}^{\dagger} \end{smallmatrix} \right) \), the doublet of holon annihilation(creation) operators. \( \lambda_i^{(1, 2, 3)} \) are the real Lagrangian multipliers to enforce the local single occupancy constraint in the SU(2) slave-boson representation [3].

The Heisenberg interaction term (the second term in Eq.[8]) above can be decomposed into terms involving mean fields and fluctuations respectively,

\[ -\frac{J}{2} (1 - \chi_{h_i})(1 - \chi_{h_j}) \times \]
\[ (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) (f_{i,j} f_{i,j} - f_{j,i} f_{i,j}) \]
\[ \times \left( (1 - \chi_{h_i})(1 - \chi_{h_j}) \right) \]
\[ + \frac{J}{2} \left( (1 - \chi_{h_i})(1 - \chi_{h_j}) \right) \times \]
\[ \left( (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) (f_{i,j} f_{i,j} - f_{j,i} f_{i,j}) \right) \]
\[ - \frac{J}{2} \left( (1 - \chi_{h_i})(1 - \chi_{h_j}) \right) \times \]
\[ \left( (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) (f_{i,j} f_{i,j} - f_{j,i} f_{i,j}) \right) \]
\[ - \left( (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) (f_{i,j} f_{i,j} - f_{j,i} f_{i,j}) \right). \]

By introducing the Hubbard-Stratonovich fields, \( \chi_{ij} \) and \( \Delta_{ij} \) in association with the direct, exchange and pairing channels of the spinon, we obtain the effective Hamiltonian from Eq.[8],

\[ H_{eff} = \]
\[ \frac{J(1 - \delta)^2}{2} \sum_{<i,j>} \left[ |\Delta_{ij}|^2 + \frac{1}{2} |\chi_{ij}|^2 + \frac{1}{4} \right] \]
\[ + \frac{J(1 - \delta)^2}{4} \sum_{<i,j>} \left[ |\chi_{ij}|^2 - |f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}| \right] \]
\[ + \frac{2t}{J(1 - \delta)^2} \left[ (b_{i,j}^{\dagger} b_{i,j} - b_{j,i}^{\dagger} b_{j,i}) \right] \chi_{ij} - \text{c.c.} \]

\[ + \frac{J(1 - \delta)^2}{2} \sum_{<i,j>} \left[ \left( f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j} \right) \right] \]
\[ - \frac{t}{J(1 - \delta)^2} \left( b_{i,j}^{\dagger} b_{i,j} + b_{j,i}^{\dagger} b_{j,i} \right) \Delta_{ij} - \text{c.c.} \]

\[ - \frac{J}{2} \sum_{<i,j>} |\Delta_{ij}|^2 \left[ \sum_{\alpha, \beta} \lambda_{i}^{(\alpha, 2)} (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) \right] \Delta_{ij} - \text{c.c.} \]

\[ - \frac{J(1 - \delta)^2}{2} \sum_{<i,j>} \left[ \sum_{\alpha, \beta} \lambda_{i}^{(\alpha, 2)} (f_{i,j}^{\dagger} f_{i,j}^{\dagger} f_{i,j} f_{i,j}) \right] \Delta_{ij} - \text{c.c.} \]

\[ - \sum_{\alpha, \beta} \frac{J}{2} |\Delta_{ij}|^2 \left[ \Delta_{ij}^{\alpha} (b_{i,j}^{\dagger} b_{i,j} + b_{i,j}^{\dagger} b_{i,j}) \right] + \text{c.c.} \]

\[ - \sum_{\alpha, \beta} \frac{J}{2} |\Delta_{ij}|^2 \left[ \Delta_{ij}^{\alpha} (b_{i,j}^{\dagger} b_{i,j} + b_{i,j}^{\dagger} b_{i,j}) \right] - \text{c.c.} \]

\[ + i \lambda_{i}^{(3)} (f_{i,j}^{\dagger} f_{i,j} - f_{i,j}^{\dagger} f_{i,j}) + \text{c.c.} \]

\[ + i \lambda_{i}^{(3)} (f_{i,j}^{\dagger} f_{i,j} - f_{i,j}^{\dagger} f_{i,j}) + \text{c.c.} \]
\[ -\frac{t}{2} \sum_{<i,j>} \left( \Delta_{ij} - (f_{i1}f_{2i} - f_{2j}f_{i1}) \right) \chi_{ij}^{b_{i}b_{j}} - \text{c.c.} \]
\[ + \frac{t^2}{2J(1 - \delta)^2} \sum_{<i,j>} \left| \chi_{ij}^{b_{i}b_{j}} - (b_{i1}b_{j1} + b_{i2}b_{j2}) \right|^2 \]
\[ + \frac{t^2}{2J(1 - \delta)^2} \sum_{<i,j>} (b_{i1}b_{j1} - (b_{i2}b_{j2})) (b_{i1}b_{j1} - b_{i2}b_{j2}) , \]
where \( \chi_{ij} \) is \( \sim \alpha \), the d-wave spinon pairing order parameter, \( \Delta_{ij} = \pm \Delta \), with the sign \( + \) for the nearest neighbor link parallel to \( \hat{x} \) \((\hat{y})\) and the s-wave holon pairing order parameter, \( \Delta_{ij} = \Delta_{\alpha \beta} \) with the boson indices \( \alpha \) and \( \beta \). For the case of \( \Delta_{\alpha \alpha} = 0 \), \( \lambda(k) = 0 \) and \( \Delta' \leq \chi \), the \( b_{1}\)-bosons are populated at and near \( k = (0,0) \) in the momentum space and the \( b_{2}\)-bosons, at and near \( k = (\pi, \pi) \). Pairing of two different \( (\alpha \neq \beta) \) bosons(holons) gives rise to the non-zero center of mass momentum. On the other hand, the center of mass momentum is zero only for pairing between identical \( (\alpha = \beta) \) bosons. Thus writing \( \Delta_{\alpha \beta} = \Delta_{\beta \alpha} \) for pairing between the identical holons and allowing the uniform chemical potential, \( \mu = \mu \), the mean field Hamiltonian from Eq.(3) is derived to be,
\[ H^{MF} = NJ(1 - \delta)^2 \left( \frac{1}{2} \chi^2 + \Delta^2 \right) + NJ \Delta_f^2 (2\Delta^2 + \delta^2) \]
\[ + \sum_k E_k^f \left( \alpha_k^\dagger \alpha_{k1} - \alpha_k \alpha_{k1}^\dagger \right) \]
\[ + \sum_{k,s=1,2} \left[ E_{ks}^h \beta_k^\dagger \beta_{ks} + \frac{1}{2} (E_{ks}^h + \mu) \right] + \mu N \delta . \]

Here \( E_k^f \) and \( E_k^h \) are the quasiparticle energies of spinon and holon respectively. \( \alpha_k(\alpha_k^\dagger) \) and \( \beta_{ks}(\beta_{ks}^\dagger) \) are the annihilation(creation) operators of the spin quasiparticles and the holon quasiparticles respectively.

From the diagonalized Hamiltonian Eq.(4), we readily obtain the total free energy,
\[ F = NJ(1 - \delta)^2 \left( \frac{1}{4} + \frac{1}{2} \chi^2 \right) \]
\[ -2k_BT \sum_k \ln[\cosh(\beta E_k^f / 2)] \]
\[ + NJ \Delta_f^2 \left( \Delta^2 + \delta^2 \right) + k_BT \sum_{k,s} \ln[1 - e^{-\beta E_{ks}^h}] \]
\[ + \sum_{k,s} \frac{E_{ks}^h + \mu}{2} + \mu N \delta . \]

The chemical potential is determined from the number constraint of doped holes,
\[ -\frac{\partial F}{\partial \mu} = \sum_k \left[ \frac{1}{e^{\beta E_{k1}^f} - 1} e^{\beta E_{k1}^f} + \frac{1}{2} \frac{-\epsilon_k - \mu}{E_{k1}^f} - 1 \right] \]
\[ + \frac{1}{e^{\beta E_{k2}^f} - 1} e^{\beta E_{k2}^f} + \frac{1}{2} \frac{\epsilon_k - \mu}{E_{k2}^f} - 1 \right] - N \delta = 0, \]
and the Lagrangian multipliers are determined by the following three constraints imposed by the SU(2) slave-boson theory,
\[ \frac{\partial F}{\partial \lambda(k)} = -\sum_k \tanh \frac{\beta E_{k1}^f}{2} \frac{\partial E_{k1}^f}{\partial \lambda(k)} \]
\[ + \sum_{k,s} \frac{e^{\beta E_{ks}^h} + 1}{2(e^{\beta E_{ks}^h} - 1)} \frac{\partial E_{ks}^h}{\partial \lambda(k)} = 0, \quad k = 1, 2, 3. \]

It can be readily proven from Eq.(3) above that \( \lambda(k) = 0 \) satisfies the three constraints above.

By minimizing the free energy, the order parameters, \( \chi \), \( \Delta_f \) and \( \lambda \) are numerically determined as a function of temperature and doping rate. In Fig.1 the mean field results of the U(1) slave-boson theory(dotted lines) are displayed for \( J = 0.2 \), \( J = 0.3 \) and \( J = 0.4 \) for comparison with the predicted phase diagrams(solid lines). The predicted pseudogap(pseudogap) temperature, \( T_{SU(2)}^f \) is consistently higher than \( T_{U(1)}^f \), the U(1) value. We note from the fourth term in Eq.(5) that the holon pairing channel depends on the spinon pairing order parameter \( \Delta_f \). Accordingly the predicted holon pair condensation temperature(superconducting transition temperature) \( T_{SU(2)}^h \) depends on the spin gap(pseudogap) temperature \( T^{*} ; T_{SU(2)}^h \) in the overdoped region decreases with \( T^{*} \). \( T_{SU(2)}^h \) at optimal doping is predicted to be lower than the value of \( T_{U(1)}^h \) predicted by the U(1) theory. The predicted optimal doping rate is shifted to a larger value, showing closer agreement with observation than the U(1) phase diagram. Such discrepancies are attributed to the phase fluctuations of order parameters, which were not treated in the U(1) mean field theory.

In summary, based on the SU(2) slave-boson symmetry conserving t-J Hamiltonian we derived a phase diagram of high \( T_c \) cuprates which displays the bose condensation temperature of an arch shape as a function of hole doping rate. Unlike other previous studies which predicted a linear increase with the hole doping rate, this result is consistent with observation. We showed that the low energy fluctuations cause a shift of the optimal doping rate to a larger value and a suppression of the holon pair bose condensation temperature, allowing closer agreement with observation compared to the U(1) case.
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FIG. 1. Computed phase diagrams with $J = 0.2t$, $J = 0.3t$ and $J = 0.4t$. $T^{\Phi}_{SU(2)}(T^{\Phi}_{U(1)})$ denotes the pseudogap temperature and $T^{\Phi}_{SU(2)}(T^{\Phi}_{U(1)})$, the holon pair bose condensation temperature predicted from the SU(2)(solid lines) and (U(1))(dotted lines) slave-boson theories respectively. The scale of temperature in the figure is based on $t = 0.44eV$[5].