Quantized control of non-Lipschitz nonlinear systems: a novel control framework with prescribed transient performance and lower design complexity

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Abstract—A novel control design framework is proposed for a class of non-Lipschitz nonlinear systems with quantized states, meanwhile prescribed transient performance and lower control design complexity could be guaranteed. Firstly, different from all existing control methods for systems with state quantization, global stability of strict-feedback nonlinear systems is achieved without requiring the condition that the nonlinearities of the system model satisfy global Lipschitz continuity. Secondly, a novel barrier function-free prescribed performance control (BFPPC) method is proposed, which can guarantee prescribed transient performance under quantized states. Thirdly, a new \$W \$-function-based control scheme is designed such that virtual control signals are not required to be differentiated repeatedly and the controller could be designed in a simple way, which guarantees global stability and lower design complexity compared with traditional dynamic surface control (DSC). Simulation results demonstrate the effectiveness of our method.

Index Terms—Uncertain nonlinear systems, dynamic surface control, prespecified/prescribed performance control (PPC), state quantization.

I. INTRODUCTION

In the past decades, controlling uncertain nonlinear systems has always been a hot research topic in the control community [1]-[6]. Among all the control problems of uncertain nonlinear systems, two of them are particularly interested, i.e., how to achieve better control performance and how to reduce the complexity of control design. For the former problem, a significant progress has been made in [7], in which prescribed performance has been achieved by combining barrier functions such that the output error converges to a predefined arbitrarily small residual set, with convergence rate no less than a given prescribed value, and maximum overshoot less than a preassigned level. Inspired by this barrier functions-based control design method, many remarkable results have been obtained [8]-[12], [20]. Though many theoretical problems have been successfully solved using PPC, there are still some problems that should not be neglected, yet remain unsolved. The mechanism of PPC method is actually based on the fact that the barrier functions in controller will approach infinite when the designed error approaches to the barrier value, and thus the controller will always have large enough actions responding to the increasing error. However, the barrier functions will bring some problems for a control system. Firstly, just as pointed out in [13], the system signals must be continuous, since the discontinuity will make the barrier functions-based controller singular. Secondly, high precision measurement of system states is necessary for PPC method, since low precision measurement may include some noises which results in discontinuity. Unfortunately, it is always hard to ensure high precision measurement to satisfy the requirement of barrier functions in practical systems.

On the other aspect, it is well known that backstepping control method faces the problem of “explosion of complexity” resulting from the repeated differentiation of virtual controls. Dynamic surface control (DSC) method [14] was therefore designed to solve this problem, where a compact set is introduced and it is finally proved to be an invariant set. DSC method has been widely used for many control problems [14]-[19], since it significantly reduces the complexity of control design. However, only semi-global results can be achieved since some constraints on initial conditions should be strictly satisfied in DSC. The parameter tuning of the DSC method is also time-consuming and there is no guideline in general.

With the digital implementation of control algorithms and development of networked control systems, signal quantization widenly exists in practical systems [21]-[23], which is owing to the widespread use of digital processors that employ a finite-precision arithmetic, meanwhile it requires less communication resources. Progress has been made on the quantized control of uncertain systems with input or state quantization. For a control system with state quantization, the state measurements are processed by quantizer, which are discontinuous maps from continuous spaces to finite sets. The discontinuous property will make the control design and stability analysis difficulty, and it will lead to the failure of barrier functions-based control method. A novel adaptive backstepping control is proposed for quantized systems with matched uncertainties in [24]. Employing neural networks [25]-[26] or command filters [27], semi-global control schemes have been designed for systems with quantized states. However, it is worth mentioning that, all existing results of state quantization require that system nonlinearities always satisfy global Lipschitz continuity condition so as to achieve global stability. Therefore, how to achieve or ensure global control with quantized states for nonlinear systems without global Lipschitz continuity condition is still an unsolved problem.

Inspired by the aforementioned problems, this paper aims to propose a low-complexity prescribed performance control method without using barrier functions, and apply this method to solve the outstanding problem for uncertain nonlinear systems under state quantization, with prescribed control performance being guaranteed. It is worthy emphasizing that no existing control method is available for this problem. To achieve the control objective, two main challenges encountered must be tackled. 1) Since discontinuous state quantization is applied, the traditional barrier functions are not applicable. Therefore new control scheme to guarantee transient performance must be proposed; 2) Without global Lipschitz continuity for the system nonlinearities, how to solve the global control with state quantization remains unknown and thus is tricky. To address these issues, we propose a novel control method for uncertain
nonlinear systems with guaranteed transient performance. The main contributions of this paper are summarized as follows:

1) Without using barrier functions, a novel barrier function-free prescribed performance control (BFPPC) method is proposed by constructing an invariant set and adequately using the upper bounds of states and errors on the invariant set. Comparing with PPC method, the BFPPC method allows controller to be designed by using discontinuous quantized system states. This enables the BFPPC to have broader application potential than PPC in practical systems as discontinuous states can be considered.

2) Comparing with all the existing results on nonlinear systems with quantized states, two obvious achievements are made. Firstly, global boundedness of strict-feedback nonlinear systems with quantized states is achieved, without requiring system nonlinearities to satisfy the global Lipschitz continuity conditions. Secondly, prescribed performance for system output and states is guaranteed.

3) Combining with the properties of W-functions, the BFPPC controller is low-complex in the sense that virtual control signal is only the function of state error, and it does not need to be differentiated repeatedly in the controller design. Compared with the DSC-based method, global stability is achieved and an explicit guideline for tuning the control parameters is provided.

II. BFPPC WITH QUANTIZED STATES

In this section, we will show the BFPPC design for uncertain nonlinear systems with quantized states in order to show the advantage of BFPPC comparing to PPC methods.

A. Problem statement and preliminaries

Consider the following system

\[
\begin{aligned}
\dot{x}_i &= f_i(x_i) + x_{i+1}, \quad i = 1, \ldots, n - 1 \\
\dot{x}_n &= f_n(x_n) + u \\
y &= x_1
\end{aligned}
\]  

where \( \dot{x}_i = [x_1, x_2, \ldots, x_i]^T \in R^i \) denotes the state vector of the system; \( u \in R \) is system control input; \( y \in R \) is system output; \( f_i(\cdot) \) are uncertain system nonlinearities, \( i = 1, \ldots, n \).

**Assumption 1:** \( f_i(\cdot), i = 1, \ldots, n \), is a continuous function satisfying \( |f_i(x_1, \ldots, x_i)| \leq f_i^*(x_1, \ldots, x_i) \) where \( f_i^*(\cdot) \) is known function.

**Assumption 2** [24]: Only quantized states \( q(x_1), q(x_2), \ldots, q(x_n) \) are measurable and available for control design, instead of the states \( x_1, x_2, \ldots, x_n \).

The quantizer \( q(x) \) considered in this paper can be either of uniform quantizer, hysteretic-uniform quantizer or logarithmic-uniform quantizer mentioned in [24], where the quantizer \( q(x) \) has the following property

\[
|q(x) - x| \leq \delta_0, \quad \forall x \in R
\]  

where \( \delta_0 > 0 \) is a known constant representing the quantization bound. The map of the uniform quantizer \( q(x) \) for \( x > 0 \) is shown in Fig. 1.

A uniform quantizer can be modeled as

\[
q(x) = \begin{cases} 
  l_i - \frac{l_0}{2} & x < l_i + \frac{l_0}{2} \\
  0 & -\frac{l_0}{2} \leq x < \frac{l_0}{2} \\
  -l_i - \frac{l_0}{2} & x < -l_i - \frac{l_0}{2}
\end{cases}
\]  

where \( l_{i+1} = l_i + \frac{l_0}{2}, i = 0, 1, 2, \ldots \), and \( l_0 \) is the length of the quantization interval. \( q(x) \) is in the set \( U = \{0, \pm l_i\} \). The quantization error is bounded by (2), where \( \delta_0 \geq \frac{\sqrt{2}}{2}. \) From Fig. 1 and (3), it can be seen that the quantized states \( q(x_1), q(x_2), \ldots, q(x_n) \) for control design are not continuous signals, which will make the barrier function-based methods fail since PPC or other barrier function-based methods require the system states to be continuous signals. Now we define the global control in the sense of prescribed performance as follows.

**Global barrier-function-free prescribed performance control (GBPPC) problem:** Consider a class of uncertain nonlinear systems. Without using barrier functions, design a controller \( u \) such that for any initial conditions \( x_i(0) = x_i^0, i = 1, \ldots, n \), all signals in the closed-loop system remain bounded and the system output is confined to prescribed area with prescribed convergence rate and maximum overshoot.

The control objective of this section is to solve GBPPC problem of system (1) with quantized states. To this end, the W-function is proposed as follows, which is the key to achieve the control objective.

**Definition 1** (W-function): Let \( k \) be any positive integer and define a continuous differentiable function \( F(z_1, z_2, \ldots, z_k) \): \( R^+ \times R^+ \times \cdots \times R^+ \rightarrow [0, +\infty) \), where \( R^+ = [0, +\infty) \). Then, \( F(\cdot) \) is called a W-function, if \( \frac{\partial F(z_1, z_2, \ldots, z_k)}{\partial z_j} \geq 0 \) and \( F(z_1, z_2, \ldots, z_k) > 0 \) for \( \forall z_j \in R^+, j = 1, \ldots, k \).

**Lemma 1** [28] (Separation theorem): For any real-valued continuous function \( f(x, y) \), where \( x \in R^m, y \in R^n \), there are smooth scalar functions \( a(x) \geq 0, b(y) \geq 0, c(x) \geq 1 \) and \( d(y) \geq 1 \) such that

\[
\begin{aligned}
|f(x, y)| &\leq a(x) + b(y) \\
|f(x, y)| &\leq c(x) d(y)
\end{aligned}
\]  

**Remark 1:** For an arbitrary continuous function \( f(\cdot) \), it is easy to find a W-function \( F(\cdot) \) such that \( |f(\cdot)| \leq F(\cdot) \), as illustrated in the following examples, 1) if \( f_2(x_1, x_2) = x_1^2 + x_1 \sin x_1 + x_1 x_2 \), then, the W-function \( F_2([x_1], [x_2]) = x_1^2 + |x_1| + |x_1 x_2| \) satisfies \( |f_2(x_1, x_2)| \leq F_2([x_1], [x_2]), \) 2) if \( f_2(x_1, x_2) = x_2^2 + e^{x_2} \cos x_2 \), then, the W-function \( F_2([x_1], [x_2]) = |x_2|^2 x_2^2 + e^{|x_2|} \) satisfies \( |f_2(x_1, x_2)| \leq F_2([x_1], [x_2]) \).

**Remark 2:** It should be noted that, though there are many different methods that achieved prescribed performance for tracking error, such as low-complexity control [12], [20], prescribed performance control [7], [8], [10], and barrier function-based control [9]. It should be noted that, they are all barrier functions-based methods because barrier functions is critical and essential in their methods in confining the tracking error within the prescribed performance functions. However, we want to strengthen that all the control schemes mentioned above work based on the fact that the control signals approach infinity if some states approach the pre-defined “barrier” and all the control schemes mentioned above fail to work in discrete control environment.
such as state quantization. For example, for the PPC-based control scheme proposed in [7] in face of state quantization, the matrix R in (8) of [7] may be singular. For the barrier Lyapunov function-based control, this problem is more obvious. Therefore the PPC-based control scheme in [7] is also called “Barrier Lyapunov function-based control”, and it is not applicable in discrete case such as state quantization. This paper will give a prescribed performance control to solve this problem without using barrier function for the first time.

B. Controller design

In view of the quantized states, introduce the following change of coordinates

\[ e_1^{(q)} = q(x_1(t)) - \rho(t)x_1^{(q)}(0) \]

\[ e_i^{(q)} = q(x_i(t)) - c_i^{(q)} - \rho(t)x_i^{(q)}(0), i = 2, ..., n \]

(6)

(7)

where \( x_i^{(q)}(0) = q(x_i(0)), i = 1, ..., n \), and \( \rho(t) \) is a performance function. For the sake of brevity, in this section, choose

\[ \rho(t) = \begin{cases} \frac{1}{t} & t < \pi \tau s \\ 0 & t \geq \pi \tau s \end{cases} \]

where \( \tau s \) is an arbitrary positive constant. It can be easily seen that \( 0 \leq \rho(t) \leq 1 \) and \( |\rho(t)| \leq \rho_M > 0 \) being a known constant. More details on the design of \( \rho(t) \) will be shown in Section III. Then, the virtual and actual controller are designed as follows

\[ \alpha_i^{(q)} = -\gamma_i H_i(\rho(t))e_i^{(q)} - c_i e_i^{N_i}, i = 1, ..., n \]

\[ u = \alpha_i^{(q)} \]

(9)

(10)

where \( \gamma_i \) and \( c_i \) are positive constants, \( N_i > 1 \) is an arbitrary odd integer, and \( H_i(\rho(t)) \) is a design function chosen as (25), (31) and (37) in the following part.

C. Main results

For the designed control method, we have the following results.

**Theorem 1:** Consider the closed loop system consisting of system (1) and the controller proposed as in (9) and (10). Then, under Assumption 1 and 2, there exist \( \gamma_i, c_i, N_i \) and \( H_i(\rho(t)) \) such that:

1. The GBPPC problem of system (1) is solved.
2. The output system satisfies the prescribed performance. Specifically, \( \rho(t)x_i(0) - \delta_M - e_i \leq x_i(t) \leq \rho(t)x_i(0) + \delta_M + e_i \), where \( \delta_M \) is a constant satisfying \( \delta_M \geq (1 + \rho(t))\delta_0 \) and \( e_i \) is an arbitrary small positive constant.

**Proof:** Since it is difficult to analyze system stability with quantized errors \( e_i^{(q)} \), to facilitate the system stability analysis, define \( e_i = x_i(t) - \rho(t)x_i(0) \),

\[ e_i = x_i(t) - \alpha_{i-1} - \rho(t)x_i(0), i = 2, ..., n, \]

\[ \alpha_i(t) = -\gamma_i H_i(\rho(t))e_i - c_i e_i^{N_i}, i = 1, ..., n \]

(11)

(12)

(13)

and a compact set

\[ \Omega = \{(e_1, ..., e_n) ||e_i|| \leq p_i(t), i = 1, ..., n\} \]

(14)

where \( p_1(t) = \delta_M + e_1, p_i(t) = \delta_{i-1}(t) + \delta_M + e_i, \delta_1(t) = \gamma_i H_i(\rho(t))\delta_M + c_0, \delta_i(t) = \gamma_i H_i(\rho(t))(\delta_M + \delta_{i-1} + c_0, c_i = 2, ..., n, with c_0 and c_i being arbitrary positive constants. \( H_i(\rho(t)) \) will be designed as continuous functions, and \( H_i(\rho(t)) \) are bounded by noting \( 0 \leq \rho(t) \leq 1 \), and thus \( \delta_i(t) \) and \( p_i(t) \) are bounded.

Denote \( e = [e_1, ..., e_n]^T \). Then, we will prove that \( \{e(t) \in \Omega \) for all \( t \geq 0, i.e., \Omega \) is an invariant set for \( e(t) \). It is easily verified that

\[ e(0) \in \Omega \] since \( e_i(0) = e_i^{(q)}(0) = 0 \) for \( i = 1, ..., n \). Then we prove the results through the following recursive steps.

Step 1: Consider the following Lyapunov function

\[ V_1 = \frac{1}{2}e_1^2 \]

(15)

It follows from (1) and (11) that the time derivative of \( V_1 \) is

\[ \dot{V}_1 = e_1 e_1 = e_1 (f_1(x_1) + x_2 - \rho(t)x_1(0)) = e_1 f_1(x_1) - e_1 \rho(t)x_1(0) + e_1(\alpha_1 + \rho(t)x_2(0) + e_2) \]

(16)

Noting \( x_1 = e_1 + \rho(t)x_1(0) \), it follows from Assumption 1 that

\[ |f_1(x_1)| \leq f_1'(x_1) = \phi_1(\rho(t), e_1, x_1(0)) \]

(17)

where \( \phi_1(\cdot) \) is a known function. Obviously, there exists a W-function \( \phi_1(\cdot) \) such that

\[ \phi_1(\rho(t), e_1, x_1(0)) \leq \phi_1(1, \rho(t), e_1, x_1(0)) \]

(18)

which means

\[ |f_1(x_1)| \leq \phi_1(1, \rho(t), e_1, x_1(0)) \]

(19)

Then, there exists a W-function \( h_1(\cdot) \) such that

\[ h_1(\rho(t), |\rho(t)|, |e_1|, |x_1(0)|, |x_2(0)|, \delta_0) \geq f_1(x_1) - \rho(t)x_1(0) + \rho(t)x_2(0) \]

(20)

Invoking

\[ h_1(\rho(t), |\rho(t)|, |e_1|, |x_1(0)|, |x_2(0)|, \delta_0) \leq \delta_0, \]

(21)

\[ x_1(0) = x_1^{(q)}(0) + \theta_1 \delta_0 \]

(22)

\[ x_2(0) = x_2^{(q)}(0) + \theta_2 \delta_0 \]

(23)

with constants \(-1 \leq \theta_1 \leq 1, -1 \leq \theta_2 \leq 1.\)

Using Lemma 1 and (21)-(22), there exists W-functions \( h_2(\cdot) \) and \( h_3(\cdot) \) such that

\[ h_2(\rho(t), \rho(t)|e_1|, x_1(0), x_2(0), \delta_0) \geq h_3(\rho(t)|e_1|, x_1(0), x_2(0), \delta_0) \]

(24)

Then, choose \( H_1(\cdot) \) and \( H_3(\cdot) \) as

\[ H_1(\rho(t), |\rho(t)|, |e_1|, x_1^{(q)}(0), |x_2^{(q)}(0)|, \delta_0) = h_3(\delta_0, \delta_0) + h_2(\rho(t), \rho(t)|e_1|, x_1^{(q)}(0), |x_2^{(q)}(0)|, \delta_0) \]

(25)

Using (20) and (23)-(25), we have

\[ H_1(\rho(t)) > |f_1(x_1) - \rho(t)x_1(0) + \rho(t)x_2(0)| \]

(26)

on \( \Omega. \)

Noting \( \|e_2\| \leq p_2 \) on \( \Omega, \) substituting (13) and (26) into (16) yields

\[ \dot{V}_1 = -\gamma_1 H_1(\rho(t))e_1^2 - c_1 e_1^{N_1+1} + e_1 e_2 + e_1 f_2(x_1) - \rho(t)x_1(0) + \rho(t)x_2(0) \]

(27)

\[ \leq e_1 \left( H_1(\rho(t)) - H_1(\rho(t)) |e_1| - c_1 e_1^{N_1 + p_2} \right) \]

Step i (\( i = 2, ..., n-1 \)): Consider the following Lyapunov function

\[ V_i = \frac{1}{2}e_i^2 \]

(28)

It follows from (1) and (12) that the time derivative of \( V_i \) is

\[ \dot{V}_i = e_i e_i = e_i (f_i(x_i) + x_{i+1} - \rho(t)x_i(0) - \delta_i) \]

(29)
Similar to Step 1, there exists a $W$-function $H_0^0(t)$ such that

$$H_0^0(\rho(t), \rho(t)|, |e_1|, \ldots, |e_n|, [x^{(q)}_{i+1}(0)], \ldots, [x^{(q)}_{i+1}(0)], \delta_0) \geq |f_i(x_i) - \rho(t)x_i(0) - \delta_i - 1 + \rho(t)x_{i+1}(0)|. \tag{30}$$

Therefore, choose

$$H_1(\rho(t)) = H_0^0(\rho(t), \rho(t), \mu, \ldots, p_i, x^{(q)}_1(0), \ldots, x^{(q)}_{i+1}(0), \delta_0) \tag{31}$$

Using (30), (31) and the property of $W$-function, we have

$$H_i(\rho(t)) \geq |f_i(x_i) - \rho(t)x_i(0) - \delta_i - 1 + \rho(t)x_{i+1}(0)| \tag{32}$$

on $\Omega$.

Noting $|e_{i+1}| \leq p_i + 1$ on $\Omega$, substituting (13) and (32) into (29) yields

$$\dot{V}_i = -\gamma_i H_i(\rho(t)) \frac{e_i^2}{e_i} + c_i e_i^{N_i} + e_i e_{i+1} + e_i f_i(x_i) - \rho(t)x_i(0) + e_i \alpha_i + \gamma_i H_i(\rho(t)) |e_i| - c_i |e_i|^{N_i} + p_i + 1 \tag{33}$$

Step $n$: Consider the following Lyapunov function

$$V_n = \frac{1}{2} e_n^2 \tag{34}$$

It follows from (1), (10) and (13) that the time derivative of $V_i$ is

$$\dot{V}_n = e_n \dot{e}_n = e_n f_n(x_n) + u - \alpha_n - 1 - \rho(t)x_n(0) \tag{36}$$

Therefore, choose

$$H_n(\rho(t)) = H_0^0(\rho(t), \rho(t), \mu, \ldots, p_n, [x^{(q)}_1(0)], \ldots, x^{(q)}_n(0), \delta_0) \tag{37}$$

Then, we have

$$\dot{V}_n \leq |e_n| \left( H_n(\rho(t)) - \gamma_n H_n(\rho(t)) |e_n| - c_n |e_n|^{N_n} + \frac{\alpha_n^{(q)} - \alpha_n}{\alpha_n^{(q)} - \alpha_n} \right) \tag{38}$$

on $\Omega$.

Choose parameters $c_i > 0$ and $\gamma_i > 0$, $i = 1, \ldots, n$, to satisfy

$$0 < c_i \leq c_0 \left( N_i \delta_i + p_i + (\mu_0 + \delta_M)N_i^{N_i} \right)^{-1}$$

Therefore, choose

$$0 < c_i \leq c_0 \left( N_i (\delta_i + \mu_0 + \delta_M)N_i^{N_i} \right)^{-1} \tag{39}$$

where $H_i^*$ is a positive constant, since $H_i^*$ is designed as the upper bound functions for some uncertain functions and thus we can always increase $H_i^*$ to avoid $H_i^* = 0$.

Noting (6), (11), the first inequality of Lemma 3 in [29] and employing $|e_1| \leq p_1$ on $\Omega$, we have

$$\alpha_i^{(q)} - \gamma_1 \leq \delta_i(t) = \gamma_1 H_1(\rho(t)) \delta_M + c_0, \tag{41}$$

Similarly, noting (9), (13), (39) and (41), we establish that

$$\alpha_i^{(q)} - \gamma_i \leq \delta_i(t) = \gamma_i H_i(\rho(t)) (\delta_M + \delta_1 - \delta_i(t)) + c_0, \tag{42}$$

Substituting (42) into (38) yields

$$\dot{V}_n \leq |e_n| \left( H_n(\rho(t)) - \gamma_n H_n(\rho(t)) |e_n| - c_n |e_n|^{N_n} + \delta_n(t) \right) \tag{43}$$

on $\Omega$.

Noting $H_i(\rho(t)) \geq H_i^*$ and invoking the definition of $p_i$, it follows from the third inequality of (39) that

$$H_i(\rho(t)) + \gamma_i H_i(\rho(t)) (\delta_M + \delta_1 - \delta_i(t)) + e_i e_{i+1} + \delta_M + c_0 - c_i p_i^{N_i} \leq \gamma_i H_i(\rho(t)) (\delta_M + \delta_i(t) - \delta_i(t)) \leq H_i(\rho(t)) p_i \tag{44}$$

Using (44) and noting $p_{i+1} = \gamma_i H_i(\rho(t)) (\delta_M + \delta_1 - \delta_i(t)) + c_0 + e_i + \delta_M$ and $\delta_n = \gamma_n H_n(\rho(t)) (\delta_M + \delta_1 - \delta_n(t)) + c_0$, one obtain

$$H_i(\rho(t)) - \gamma_i H_i(\rho(t)) (\delta_M + \delta_1 - \delta_i(t)) + c_0 + e_i + \delta_M \leq \gamma_n H_n(\rho(t)) (\delta_M + \delta_1 - \delta_n(t)) + c_0 \tag{45}$$

It follows from (27) and (45) that, if $|e_1| \geq p_1$, then $V_1 \leq 0$. Similarly, it follows from (33), (43) and (45) that, if $|e_i| \geq p_i$, then $V_i \leq 0$ for $i = 2, \ldots, n$. Thus, we have

$$|e_i| \leq p_i, \quad i = 1, \ldots, n \tag{46}$$

which implies $e(t) \in \Omega$ for all $t \geq 0$, i.e., $\Omega$ is an invariant set for $e(t)$. Then, it is not difficult to conclude the boundedness of all the closed-loop signals.

From (46), it can be observed that

$$|x_1(t) - \rho(t)x_1(0)| \leq p_1 \tag{47}$$

which further implies

$$\rho(t)x_1(0) - \delta_M - \delta_1 \leq x_1(t) \leq \rho(t)x_1(0) + \delta_M + \delta_1 \tag{48}$$

This completes the proof.

### III. BFPPC OF GENERAL NONLINEAR SYSTEMS

In this section, we will generalize the above method to design a tracking controller for the following nonlinear systems

$$\begin{cases}
\dot{x}_i = f_i(x_i) + g_i(x_i)x_{i+1}, & i = 1, \ldots, n - 1 \\
\dot{x}_n = f_n(x_n) + g_n(x_n)u \\
y = x_1
\end{cases} \tag{49}$$

where $f_i(\cdot)$ and $g_i(\cdot)$ are uncertain system nonlinearities, $i = 1, \ldots, n$. The control objective is to design a controller $u$ such that system output $y$ can track a given trajectory $y_{gt}$, and the tracking error is confined to a prescribed area.

**Assumption 3:** $f_i(\cdot)$ and $g_i(\cdot)$ are continuous functions satisfying $|f_i(x_1, \ldots, x_i)| \leq f_i^*(x_1, \ldots, x_i)$ and $0 < u \leq g_i(x_1, \ldots, x_i) \leq g_i^*(x_1, \ldots, x_i)$ where $f_i^*(\cdot)$ and $g_i^*(\cdot)$ are known functions for $i = 1, \ldots, n$. $u$ is a known constant.

**Assumption 4:** The desired trajectory $y_{gt}$ is continuously differentiable, and $y_{gt}$ and $y_{gt}$ are bounded, i.e., $|y_{gt}| \leq Y_0, |y_{gt}| \leq Y_1$ with $Y_0$ and $Y_1$ being known positive constants.
A. BFPPC controller

Introduce the following error variables and change of coordinates
\[ e_i = x_i - y_d - \rho_i(t)(x_i(0) - y_d(0)) \]  
where \( \rho_i(t) \) is a continuously differentiable function satisfying \( \rho_i(0) = 1 \) and \( \rho_i(t) \geq 0 \) for all \( t \geq 0 \). Then, the error variables can be designed as follows
\[ \rho_i(t) = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{t}{\pi s} \right) \right), & t < \pi s \\ 0, & t \geq \pi s \end{cases} \]  
where \( s > 0 \) is a virtual controller parameter. To achieve a small error variable, \( \rho_i(t) \) can be designed as follows
\[ \rho_i(t) = 1 - \rho_{i-1} e^{-\rho_i 0 t} + \rho_i 1 \]  
where \( t_s \), \( \rho_{i-1} \) and \( \rho_i 1 \) are arbitrary positive constants.

The BFPPC controller \( u \) and virtual controllers \( \alpha_i \) are designed as follows
\[ \alpha_i = -k_i e_i - M_i \tanh \left( \frac{M_i e_i}{c_i} \right) - c_i e_i N_i, \quad i = 2, \ldots, n - 1 \]  
\[ u = -k_n e_n - M_n \tanh \left( \frac{M_n e_n}{c_n} \right) - c_n e_n N_n \]  
where \( k_i, c_i, M_i \) and \( e_i \) are positive design parameters, with \( N_i > 1 \) being arbitrary odd integers.

B. Main results

For system (49) with our designed BFPPC controller, we have the following results.

Theorem 2: Consider the closed-loop system consisting of uncertain nonlinear system (49) satisfying Assumption 3-4, the virtual control signals (54) and the actual controller (55). Then, there exist \( k_i, c_i, N_i, M_i \) and \( e_i \) such that the following properties hold:
1. The GBPPC problem of system (49) is solved.
2. The tracking error satisfies the prescribed performance, specifically, \( \rho_i(t) - p_i \leq x_i - y_d \leq \rho_i(t) + p_i \) where \( \rho_i(t) = \rho_1(t)(x_1(0) - y_d(0)) \).

Proof: Define a compact set
\[ \Omega = \{(e_1, \ldots, e_n) | |e_i| \leq p_i, \quad i = 1, \ldots, n\} \]  
where \( p_i \) are arbitrary small constants.

Denote \( e_2 = [e_1, \ldots, e_n]^T \). It can be easily verified that \( e(0) \in \Omega \) by noting \( e_2(0) = e_2(0) \). In the following, we will prove that \( e(t) \in \Omega \) for all \( t \geq 0 \), i.e., \( \Omega \) is an invariant set for \( e(t) \).

Step 1: Consider the following Lyapunov function
\[ V_1 = e_2^T e_2 \]  
where \( e_2 = [e_1, \ldots, e_n]^T \) is the time derivative of \( e_i \) is
\[ \dot{e}_i = f_i(\bar{x}_i) (e_{i+1} + \rho_{i+1}(t)x_{i+1}(0) + \alpha_i \bar{x}_i - \rho_i(t)x_i(0) \]  
where \( \alpha_0 = |x_0(t) - y_d(0)| \) and \( \epsilon_n = x_n(0) = 0 \). And denote \( \alpha_0 = |x_0(t) - y_d(0)| \).

Similarly as Theorem 1, there exist W-functions \( \varphi_{i,1}(\cdot) \) and \( \varphi_{i,2}(\cdot) \) satisfying
\[ |f_i(\bar{x}_i)| \leq \varphi_{i,1}(Z_i^T) \]  
where \( Z_i = [\rho_i(t), \ldots, \rho_i(t), |e_i|, \ldots, |e_i|, |y_d|, a_0, |x_2(0)|, \ldots, |x_i(0)|]^T \).

Thus, there exists a W-function \( F_i^*(\cdot) \) such that
\[ F_i^*(\rho_i(t), \ldots, \rho_i(t), |e_i|, \ldots, |e_i+1|, |y_d|, a_0, |x_2(0)|, \ldots, |x_i(0)| + |\dot{\rho}_i(t)x_i(0)|) \]  
Define constant \( F_i^* \) as
\[ F_i^* = F_i^0 \left( \rho_1(0), \ldots, \rho_i(0), |e_i|, \ldots, |e_i|, |y_d|, a_0, |x_2(0)|, \ldots, |x_i(0)| \right) \]  
Therefore, it follows from (59)-(62) and the property of W-function that
\[ F_i^* \geq \frac{1}{g_m}(\{f_i(\bar{x}_i)\} + |g_i(\bar{x}_i)| + |\rho_i(t)x_i(0)| + |\dot{\rho}_i(t)x_i(0)|) \]  

on \( \Omega \).

Similarly, using (57), (58) and (63) and employing \( -M_i e_i \tanh(\frac{M_i e_i}{c_i}) \leq -M_i |e_i| + 0.3 \varepsilon_i \) according to Lemma 2 in [15], the time derivative of \( V_i \) satisfies
\[ \dot{V}_i \leq -k_i e_i e_i + g_m |e_i| F_i^* + 0.3 \varepsilon_i \]  
Choose \( k_i, M_i \) and \( \varepsilon_i \) to satisfy
\[ k_i p_i + M_i c_i \geq F_i^* + 0.3 \frac{1}{p_i} \varepsilon_i \]  
Choose \( k_i, M_i \) and \( \varepsilon_i \) to satisfy
\[ k_i p_n + M_n c_n \geq F_n^* + 0.3 \frac{1}{p_n} \varepsilon_n \]  
It is easily known that (71) can be satisfied by increasing either of \( k_n, M_n \) or \( c_n \).
It follows from (70) and (71) that \( \dot{V}_n \leq 0 \) when \( |e_n| \geq p_n \). Thus, we have

\[
|e_n| \leq p_n \tag{72}
\]

From (66) and (72), it can be seen that \( e(t) \in \Omega \) for all \( t \geq 0 \), namely, \( \Omega \) is an invariant set for \( e(t) \). Using (50) and (66), we can further have

\[
|x_1 - y_d - \rho_1(t)(x_1(0) - y_d(0))| \leq p_1 \tag{73}
\]

which implies

\[
\rho_1^*(t) - p_1 \leq x_1 - y_d \leq \rho_1^*(t) + p_1 \tag{74}
\]

where \( \rho_1^*(t) = \rho_1(t)(x_1(0) - y_d(0)) \). This completes the proof.

Remark 3: The invariant set \( \Omega \) plays a key role in reducing the complexity of virtual and actual controller. Noting \( F_0^*(\cdot, \cdot) \) is an increasing function with respect to \( \rho_1(t), \rho_2(t), |\rho_1(t)|, |\rho_2(t)|, |e_1|, |e_2|, |y_d|, |y_d| \), we use its maximum, \( F_0^* \), since all the variables have maximum on the introduced invariant set \( \Omega \), i.e., \( |e_i| \leq p_i \). It can be seen that the time-varying terms \( \rho_1(t), \rho_2(t), |\rho_1(t)|, |\rho_2(t)|, |e_1|, |e_2|, |y_d|, |y_d| \) do not appear in \( \alpha_i \) by using their maxima on \( \Omega \), and thus \( \alpha_i \) is only a function of \( e_i \) and therefore it is a low-complexity controller.

IV. OPTIMIZING BFPPC CONTROLLER PARAMETERS

Since we enlarge some terms in \( W \)-function in choosing the controller parameters, therefore, the controller parameters are always conservative as they are usually larger or smaller than they should to be. In this part, we set a switching mechanism for the selection of controller parameters so as to find out more suitable controller parameters.

Select a series of constants \( \{p_{i,m}\} \) such that \( 0 < p_{i,1} < p_{i,2} < \cdots < p_{i,K} \) for \( i = 1, \ldots, n \) and \( m = 1, \ldots, K \), where \( K \) is a positive integer specified by designer and it represents the maximum of switching times.

Denote the switching time instant as \( t_{\sigma(t)} \) with \( \sigma(t) \) defined as follows

\[
\sigma(t) = \begin{cases} 
m + 1, & \text{otherwise} \\
m, & \text{if } |e_i| \leq p_{i,m} \text{ for all } i \in \{1, \ldots, n\} 
\end{cases} \tag{75}
\]

which implies \( |e_i| \leq p_{i,m} \) for \( 0 \leq t \leq t_m \), where \( t_0 = 0 \).

Theorem 3: Consider the nonlinear system (49) with Assumption 3-4, the virtual controllers (54), the actual controller (55). The controller parameters are chosen as \( k_i = k_{i,\sigma(t)}, M_i = M_{i,\sigma(t)}, c_i = c_{i,\sigma(t)}, N_i = N_{i,\sigma(t)} \), for \( t \in [t_{\sigma(t)} - 1, t_{\sigma(t)}) \), where \( k_{i,\sigma(t)}, M_{i,\sigma(t)}, c_{i,\sigma(t)} \) and \( N_{i,\sigma(t)} \) are chosen arbitrarily, while \( k_{i,K}, M_{i,K}, c_{i,K} \) and \( N_{i,K} \) are chosen to satisfy (65) and (71), with

\[
F_i^* = F_0^* \left( \rho M, 1, \ldots, \rho M, i, \rho M, 0, \ldots, \rho M, 0, P, K, \ldots, P, i + 1, K; Y_0, Y_1, Y_0, x_2(0), \ldots, x_2(i + 1)(0) \right) \tag{76}
\]

\[
F_n^* = F_0^* \left( \rho M, 1, \ldots, \rho M, i, \rho M, 0, \ldots, \rho M, 0, P, K, \ldots, P, N, K; Y_0, Y_1, Y_0, x_2(0), \ldots, x_2(i + 1)(0) \right) \tag{77}
\]

Then global boundedness of all the system signals is guaranteed.

Proof: By noting \( |e_i| \leq p_{i,K} \) for \( 0 \leq t \leq t_K \), it is not difficult to prove Theorem 3 by following the similar way as the proof of Theorem 2. Thus the proof is omitted.

Remark 4: Theorem 3 is introduced not only to guarantee the global stability of system but also to seek more preferable controller parameters. The controller parameters \( k_{i,\sigma(t)}, M_{i,\sigma(t)}, c_{i,\sigma(t)} \) and \( N_{i,\sigma(t)} \), \( i = 1, \ldots, n \) for \( \sigma(t) \leq K - 1 \) can be chosen arbitrarily, which suggests the control objective may be achieved by the controller parameters determined freely by experienced designer while the system stability is also guaranteed in this method.

Remark 5: Theorem 3 suggests that the prescribed performance may be satisfied at \( \sigma(t) \leq K - 1 \), before \( W \)-functions are used to select the controller parameters, which may result in better control action while prescribed performance are also guaranteed, since the selection of controller parameters may combine the experience of designer.

Remark 6: The essential difference from all the existing prescribed performance control is, that, the controller in this paper does not contain barrier functions, as shown as follows

\[
\alpha_i = -k_i e_i - M_i \tan \left( \frac{M_i e_i}{c_i} \right) - c_i e_i \frac{N_i}{e_i}, i = 1, \ldots, n - 1 \tag{78}
\]

\[
u = -k_n e_n - M_n \tan \left( \frac{M_n e_n}{c_n} \right) - c_n e_n \frac{N_n}{e_n} \tag{79}
\]

All the other prescribed performance control methods are mostly derived from [21], which controller are, for example, given as follows

\[
\alpha_i = -k_i \ln \left( \frac{1 + e_i}{1 - e_i} \right), i = 1, \ldots, n - 1 \tag{80}
\]

\[
u = -k_n \ln \left( \frac{1 + e_n}{1 - e_n} \right) \tag{81}
\]

Comparing with these controllers, the reasons why we design the controller construction as (78) and (79) are that: (1) Firstly, barrier functions, such as \( \ln(\cdot) \), may easily run to an error or singularity in practice. (2) Secondly, expanding (80) and (81) by using Taylor series as follows

\[
\alpha_i = k_i e_i - \frac{e_i}{3} \rho_i - \frac{1}{5} \frac{e_i}{\rho_i} - \frac{5}{3} \frac{e_i}{\rho_i} - \ldots, i = 1, \ldots, n - 1 \tag{82}
\]

\[
u = 2k_n e_n - \frac{1}{3} \frac{e_n}{\rho_n} - \frac{1}{5} \frac{e_n}{\rho_n} - \frac{5}{3} \frac{e_n}{\rho_n} - \ldots \tag{83}
\]

It can be seen that our controller construction preserves some high-order terms of (82) and (83), such as \( c_i e_i \), which makes our controller preserve some quality of traditional PPC, and discards residual high-order terms, which makes our controller avoid singularity facing the case of \( e_i \) being discontinuous. (3) Thirdly, the term, \( M_i \tan \left( \frac{M_i e_i}{c_i} \right), \) is used to avoid the controller action to decreasing dramatically when \( e_i \) is small.

These characteristics ensure our controller to have better control performance than traditional PPC and other normal controllers.

V. SIMULATION RESULTS

In this section, two simulation examples are presented to demonstrate the advantages of BFPPC method. Consider the following system with quantized states

\[
\begin{aligned}
\dot{x}_1 &= x_2^2 - \sin x_1 + x_2 \\
\dot{x}_2 &= x_1 x_2 + u
\end{aligned} \tag{84}
\]

For the purpose of simulation, let \( x_1(0) = 1, x_2(0) = 0 \), and use uniform quantizer (3) with \( \ell_0 = 0.1 \). According to our BFPPC method, the \( W \)-function should be chosen to satisfy

\[
H_1(\rho(t)) \geq \left( p_1 + \rho(t) \left( \left| x_1^q(0) \right| + 0.1 \right) \right)^2 + 1 + \frac{1}{2} \left( \left| x_1^q(0) \right| + 0.1 \right) \left( \left| x_2^q(0) \right| + 0.1 \right) \tag{85}
\]
The simulation results are depicted as Figs. 2-5. It can be seen that the barrier functions-based controller unavailable. Then, noting $|\rho(t)| \leq 1$, we can choose the design parameters and functions as $p_1 = \delta_M + 0.05$, $p_2 = 2 + \delta_M + 1$, $H_1(\rho(t)) = 5$, $\gamma_1 = 4$, $c_1 = 0.1$, $N_1 = 3$, $H_2(\rho(t)) = 10$, $\gamma_2 = 0.4$, $c_2 = 1.5$, $N_2 = 3$. Set performance function $\rho(t)$ as (8) with $t_s = 1$. According to Theorem 1, the virtual and actual controller are designed as follows:

$$\alpha_1 = -20c_1^{(q)} - 0.1 c_1^{(q)} \cosh \left( c_1^{(q)} \right),$$

$$u = -4c_2^{(q)} - 1.5 c_2^{(q)} \cosh \left( c_2^{(q)} \right).$$

The simulation results are depicted as Figs. 2-5. It can be seen that from these results that, though system nonlinearities do not satisfy global Lipschitz continuity condition, the prescribed performance for system output is achieved with quantized signals which makes the barrier functions-based controller unavailable.

To further show the advantage of BFPPC, consider the following uncertain nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_1 e^{-0.5x_1} + \left( 1 + \sin x_1^2 \right) x_2 \\
\dot{x}_2 &= x_1 \sin x_2 + x_1 x_2^2 + (3 + \cos x_1) u
\end{aligned}$$

where $y_d = \sin t$, $f_1(x_1) = x_1 + x_1 e^{-0.5x_1}$, $g_1(x_1) = 1 + \sin x_1^2$, $f_2(x_2) = x_1 \sin x_2 + x_1 x_2^2$, $g_2(x_2) = 3 + \cos x_1$, and $f_1^*(x_1) = |x_1| + |x_1| e^{0.5|x_1|}$, $g_1^*(x_1) = 2$, $g_m = 1$, $f_2^*(x_2) = |x_1| + |x_1| x_2^2$, $g_2^*(x_2) = 4$. Let $x_1(0) = 0.5$, $x_2(0) = 0$. Choose $p_1(t)$ and $p_2(t)$ as (52) with $t_s = 1$. According to Theorem 3, the BFPPC controller are constructed as follows:

$$\alpha_1 = -k_{1,\sigma(t)} c_1 - M_{1,\sigma(t)} \tanh \left( \frac{M_{1,\sigma(t)} c_1}{0.5} \right) - c_1 \sigma(t) e_1^{N_{1,\sigma(t)}};$$

$$u = -k_{2,\sigma(t)} c_2 - M_{2,\sigma(t)} \tanh \left( \frac{M_{2,\sigma(t)} c_2}{0.5} \right) - c_2 \sigma(t) e_2^{N_{2,\sigma(t)}};$$

Choose $K = 2$, $p_{1,1} = 0.04$, $p_{2,1} = 1$, $p_{1,2} = 0.05$, $p_{2,2} = 2$, $k_{1,1} = 2$, $c_{1,1} = 0.1$, $M_{1,1} = 6$, $k_{2,1} = 1$, $c_{2,1} = 2$, $M_{2,1} = 0.1$, $N_{1,1} = N_{2,1} = 3$, while $k_{1,2} = k_{2,2} = 2$, $N_{1,2} = 3$, $N_{2,2} = 5$, $M_{1,2}$ and $c_{1,2}, i = 1, 2$, are any constants that satisfying (65) and (71) with $F_1^* = F_1^!(1, 1, 0.05, 2, 1)$ and $F_2^* = F_2^!(1, 1, 1, 0.05, 2, 1)$. The simulation results are depicted as Figs. 6-8. From Fig. 6, it can be seen that the prescribed performance for tracking error is achieved by the proposed controller.

VI. CONCLUSION

This paper proposes BFPPC method based on the an invariant set and $W$-functions. By using the maximums of some variables or functions on the invariant set, the designed controller is simplified such that repeated differentiation of virtual controls is avoided and prescribed performance is also guaranteed. This novel method is extended to solve the control problem of nonlinear system with state quantization. By virtue of BFPPC method, the global Lipschitz continuity condition is cancelled and global bounded of all the closed-loop signals is proved. Simulation results demonstrate the effectiveness of our method.

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Fig. 6. Tracking error and its prespecified bounds

Fig. 7. System output $y$ and desired signal $y_d$

Fig. 8. Control input $u$

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