Invariant Measures and Orbit Closures on Homogeneous Spaces for Actions of Subgroups Generated by Unipotent Elements

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Abstract

The theorems of M. Ratner, describing the finite ergodic invariant measures and the orbit closures for unipotent flows on homogeneous spaces of Lie groups, are extended for actions of subgroups generated by unipotent elements. More precisely: Let $G$ be a Lie group (not necessarily connected) and $\Gamma$ a closed subgroup of $G$. Let $W$ be a subgroup of $G$ such that $\text{Ad}_G(W)$ is contained in the Zariski closure (in $\text{Aut}(\text{Lie} G)$) of the subgroup generated by the unipotent elements of $\text{Ad}_G(W)$. Then any finite $W$-invariant $W$-ergodic measure on $G/\Gamma$ is a homogeneous measure (i.e., it is supported on a closed orbit of a subgroup preserving the measure). Moreover, if $G/\Gamma$ has finite volume (i.e., has a finite $G$-invariant measure), then the closure of any orbit of $W$ on $G/\Gamma$ is a homogeneous set (i.e., a finite volume closed orbit of a subgroup containing $W$). Both the above results hold if $W$ is replaced by any subgroup $\Lambda \subset W$ such that $W/\Lambda$ has finite volume.

1 Introduction

In [Ra2, Ra3] Ratner showed the validity of Raghunathan’s conjecture [4] describing orbit closures for actions of unipotent subgroups on homogeneous spaces of Lie groups, and its analogous conjecture, due to Dani [D2], describing ergodic invariant measures for such actions. Earlier in [M1, M2] Margulis had conjectured that the conclusions of the orbit closure and the ergodic invariant measure conjectures should hold also for the actions of subgroups generated by unipotent elements, as compared to the subgroups themselves being unipotent. In fact for actions of connected subgroups generated by unipotent elements, this conjecture was also verified to be true in Ratner’s
above mentioned papers. Using Ratner’s theorems for actions of unipotent one-parameter subgroups, in this article we show the validity of the generalized conjecture. This also answers a question raised by Ratner in [Ra4, End of Section 4].

**Notation.** Let $G$ be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\text{Ad}_G : G \to \text{GL}(\mathfrak{g})$ denote the Adjoint representation of $G$ on $\mathfrak{g}$. An element $u \in G$ is called $\text{Ad}_G$-unipotent, if $\text{Ad}_G(u)$ is a unipotent linear transformation. A subgroup of $G$ consisting of $\text{Ad}_G$-unipotent elements is called an $\text{Ad}_G$-unipotent subgroup.

Let $\langle \mathcal{U} \rangle$ denote the subgroup generated by a subset $\mathcal{U}$ in $G$. Let $\text{Zcl}(X)$ denote the Zariski closure of a subset $X$ in $\text{GL}(\mathfrak{g})$. For a subgroup $F$ of $G$, let $F^0$ denote the connected component of $F$ containing the identity element.

For a Borel measure $\mu$ on a second countable topological space, we denote by $\text{supp}(\mu)$ the closed subset which is the complement of the union of all open sets with zero $\mu$-measure.

**Theorem 1.1** Let $G$ be a Lie group and $\Gamma$ a closed subgroup of $G$. Let $W$ be a subgroup of $G$ and $\mathcal{U} \subset W$ such that $\mathcal{U}$ consists of $\text{Ad}_G$-unipotent elements and $\text{Ad}_G(W) \subset \text{Zcl}(\text{Ad}_G(\langle \mathcal{U} \rangle))$. Let $\mu$ be a finite $W$-invariant $W$-ergodic Borel measure on $G/\Gamma$. Then there exists a closed subgroup $H$ of $G$ containing $W$ such that $\mu$ is $H$-invariant and $\text{supp}(\mu)$ is a closed $H$-orbit.

A Borel measure on a locally compact second countable topological space is called locally finite, if it is finite on compact sets.

**Theorem 1.2** Let $G$, $\Gamma$, and $W$ be as in Theorem 1.1. Suppose that $G/\Gamma$ has a finite $G$-invariant measure. Let $\mu$ be a locally finite $W$-invariant $W$-ergodic measure on $G/\Gamma$. Then there exists a closed subgroup $H$ of $G$ containing $W$ such that $\mu$ is $H$-invariant and $\text{supp}(\mu)$ is a closed $H$-orbit.

**Theorem 1.3** Let $G$, $\Gamma$, and $W$ be as in Theorem 1.1. Suppose that $G/\Gamma$ has a finite $G$-invariant measure. Then for any $x \in G/\Gamma$, there exists a closed subgroup $F$ of $G$ containing $W$ such that $Wx = Fx$.

Moreover, $F^0x$ has a finite $F^0$-invariant measure (cf. Conjectures 1.1 and 4 below). Also the action of $W$ is ergodic with respect to a locally finite $F$-invariant measure on $Fx$. 

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We may note that Theorem 1.1 and Theorem 1.3 have already been proved in the above mentioned papers of Ratner in the following special case: $G$ is connected, $W$ is of the form $W = \bigcup_{i=1}^{\infty} w_i W^0$, where $w_i$ is $\text{Ad}_G$-unipotent, $i = 1, 2, \ldots$, $W/W^0$ is finitely generated, and $W^0$ is generated by one-parameter $\text{Ad}_G$-unipotent subgroups contained in $W^0$. In the case when $G$ is not connected and $W$ is a nilpotent $\text{Ad}_G$-unipotent subgroup of $G$, Theorem 1.3 was proved by Witte [W, Theorem 1.2]. In the case when $G$ is connected and $W$ is a $\text{Ad}_G$-unipotent subgroup, it was shown by Dani [D4, Theorem 4.3] that if $G/\Gamma$ has a finite invariant measure then any locally finite $W$-invariant $W$-ergodic measure is finite. In [M1, Remarks 3.12], Margulis observed that the same holds for connected $W$. Thus for connected $W$, Theorem 1.2 reduces to Theorem 1.1, which was proved by Ratner (for connected $W$).

The following result is deduced from that above results using the ‘suspension techniques’ (cf. Witte [W, Corollary 5.8]).

**Corollary 1.4** Let $G$ and $W$ be as in any one of the theorems stated above. Assume that $W$ is closed and let $\Lambda$ be a closed subgroup of $W$ such that $W/\Lambda$ has a finite $W$-invariant measure. Then all the theorems stated above are true for $\Lambda$ in place of $W$.

Further, if $G/\Gamma$ admits a finite $G$-invariant measure and $W$ is connected, then we have the following additional information:

1. Any locally finite $\Lambda$-invariant $\Lambda$-ergodic measure on $G/\Gamma$ is finite.
2. For $x \in G/\Gamma$, if $\Lambda x = F x$ for a closed subgroup $F$ of $G$ then $F x$ has a finite $F$-invariant measure.

From this corollary, we deduce the following.

**Corollary 1.5** Let $G$ be a connected semisimple Lie group without nontrivial compact factors. Let $\Gamma$ and $\Lambda$ be lattices in $G$ such that at least one of them is irreducible in $G$; (see [R, Sect. 5.20] for definition). Then either $\Lambda \Gamma$ is dense in $G$ or $\Lambda \cap \Gamma$ is a subgroup of finite index in $\Gamma$, as well as $\Lambda$.

In view of the above results we may ask if the following is true.

**Conjecture 1.1** Let the notation be as in Theorem 1.4. Suppose further that $G/\Gamma$ has a finite $G$-invariant measure. Then the following statements hold:

1. Any locally finite $W$-invariant $W$-ergodic measure on $G/\Gamma$ is finite.
2. For any \( x \in G/\Gamma \), if \( \overline{Wx} = Fx \) for a closed subgroup \( F \) of \( G \), then \( Fx \) has a finite \( F \)-invariant measure.

3. The closure of any \( W \)-orbit has finitely many connected components.

Note that by the above stated theorems and by Hedlund’s lemma \([2.1]\), the three statements in the above conjecture are equivalent.

**Remark 1.6** If the above conjecture is valid for the diagonal action of \( W \) on \( W/\Lambda \times G/\Gamma \), then it holds for the action of \( \Lambda \) on \( G/\Gamma \), where \( W \) and \( \Lambda \) are as in corollary \([1.4]\).

It seems that the generalized Raghunathan conjecture due to Margulis already includes Conjecture \([1]\). Using some standard arguments, as in proof of Theorem \([7.1]\), one can reduce this conjecture to the case of \( G \) being a semisimple group with no nontrivial compact factors and trivial center. Then one can express \( G \) as a product of semisimple subgroups each intersecting \( \Gamma \) in an irreducible lattice. Using the structure of the cusps in the quotient of the \( \mathbb{R} \)-rank one factors, one can take care of those factors. Thus the conjecture remains to be proved for higher rank semisimple groups \( G \). We use the arithmeticity theorem of Margulis, and reduce the conjecture to its following typical case.

**Conjecture 1.2**\(^1\) Let \( G = \text{SL}_n(\mathbb{R}) \), \( \Gamma = \text{SL}_n(\mathbb{Z}) \), and \( W \subset \text{SL}_n(\mathbb{Q}) \) a closed subgroup of \( G \) such that \( W \) is contained in the Zariski closure of a subgroup generated by \( \text{Ad}_G \)-unipotent elements of \( W \). If \( WT \) is discrete, then \( W \cap \Gamma \) is of finite index in \( W \).

Finally, a challenging question is to describe invariant measures and orbit closures for actions of subgroups \( H \) whose Zariski closure is generated by unipotent elements, which are not necessarily contained in \( H \). For example, let \( H \) be a Zariski dense subgroup of \( \text{SL}_2(\mathbb{R}) \) not containing any unipotent elements, and consider the action of \( H \) on \( \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z}) \).

**2 Preliminary Results**

In this section we recall some standard results or their modifications about orbit closures on homogeneous spaces, and Zariski density of certain discrete subgroups as in Borel’s density theorem.

\(^1\)Recently Alex Eskin and G. A. Margulis informed the author that they can prove this conjecture.
Lemma 2.1 (Hedlund’s Lemma) Let $X$ be a second countable topological space and $W$ be a group of homeomorphisms of $X$. Let $\mu$ be a $W$-invariant $W$-ergodic Borel measure on $X$. Then $Wx = \text{supp} \mu$ for $\mu$-almost all $x \in X$.

Lemma 2.2 Let $G$ be a locally compact second countable group, $\Gamma$ a discrete subgroup of $G$, and $\pi : G \to G/\Gamma$ the natural quotient map. Let $F$ be a Borel measurable subgroup of $G$. Suppose there exists a locally finite $F$-invariant Borel measure concentrated on $\pi(F)$. Then $F$ is closed, $\pi(F)$ is closed, and $\text{supp}(\mu) = \pi(F)$.

Proof Since $\mu$ is locally finite, by dominated convergence theorem (see [Ra1, Proposition 1.4]), $\mu$ is invariant under the closure of $F$ in $G$, say $H$. We have a natural inclusion $H/H \cap \Gamma \hookrightarrow G/\Gamma$, which is $H$-equivariant. Since $\mu$ is concentrated on $\pi(F) \subset \pi(H)$, we can treat $\mu$ as a locally finite $H$-invariant Borel measure on $H/H \cap \Gamma$. Since $\mu$ is concentrated on an orbit of $F$, we conclude that a Haar measure on $H$ is strictly positive on $F$. Since $F = FF^{-1}$, we have that $F$ is an open, and hence a closed subgroup of $H$. Thus $F = H$.

Now since $F$ is closed, the result follows from the proof of [R, Theorems 1.12-1.13] (cf. [Ra2, Proposition 1.4]). Although these references assume that $\mu$ is finite, the local nature of the conclusion requires only the assumption that $\mu$ is locally finite.

Lemma 2.3 Let $G$ be a Lie group and $\Gamma$ be a discrete subgroup of $G$. Let $\pi : G \to G/\Gamma$ be the quotient map. Let $F$ be a subgroup of $G$ such that $\text{Ad}_{G}(F) \subset \text{Zel}(\text{Ad}_{G}(F \cap \Gamma))$.

Then $\pi(Z_G(F))$ is closed.

Proof Take any $\gamma \in \Gamma$, then $Z_G(\gamma)\Gamma$ is the inverse image of the discrete set $\{\lambda^{-1}\gamma\lambda : \lambda \in \Gamma\} \subset \Gamma$ under the continuous map $G \ni g \mapsto g^{-1}\gamma g \in G$. Hence $\pi(Z_G(\gamma))$ is dense in $G$. Note that if $F_1$ and $F_2$ are closed subgroups of $G$ such that $\pi(F_1)$ and $\pi(F_2)$ are closed, then $\pi(F_1 \cap F_2)$ is closed. Therefore we conclude that $\pi(Z_G(F \cap \Gamma))$ is closed.

By our Zariski closure hypothesis, $Z_G(F \cap \Gamma)^0 \subset Z_G(F)$. Therefore the result follows from an observation that for any closed subgroup $H$ of $G$, if
\( \pi(H) \) is closed then \( \pi(H_1) \) is closed for any subgroup \( H_1 \) of \( H \) containing \( H^0 \).

**Definition** Let \( F \) be a connected subgroup of a Lie group \( G \) and \( \mathfrak{f} \) the Lie algebra associated to \( F \). Let \( N_G \) denote the normalizer of \( F \) in \( G \). We define

\[
N^1_G(F) = \{ g \in N_G(F) : \det(\text{Ad}(g)|_{\mathfrak{f}}) = 1 \}.
\]

**Remark 2.4** Note that all \( \text{Ad}_G \)-unipotent elements of \( N_G(F) \) are contained in \( N^1_G(F) \). Now suppose \( W \subset G \) and \( U \subset W \) such that \( U \) consists of \( \text{Ad}_G \)-unipotent elements and \( \text{Ad}_G(W) \subset \text{Zcl}(\text{Ad}_G(U)) \). Then

\[
U \subset N_G(F) \implies W \subset N^1_G(F).
\]

**Proposition 2.5** Let \( G \) be a Lie group, \( \Gamma \) a discrete subgroup of \( G \), and \( \pi : G \to G/\Gamma \) the natural quotient map. Let \( U \) be a subgroup of \( G \) generated by one-parameter \( \text{Ad}_G \)-unipotent subgroups of \( U \). Suppose \( F \) is a closed connected subgroup of \( G \) such that \( \pi(F) \) has a finite \( F \)-invariant measure and \( \pi(U) = \pi(F) \). Then \( \pi(N^1(F)) \) is closed in \( G/\Gamma \).

**Proof** Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( \mathfrak{f} \) the Lie algebra associated to \( F \). Let \( d = \dim \mathfrak{f} \). Consider the action of \( G \) on \( \wedge^d \mathfrak{g} \) via the \( d \)-exterior power of \( \text{Ad}_G \). Let \( p \in \wedge^d \mathfrak{g} \setminus \{0\} \). Then by [DM, Theorem 3.4], the orbit \( \Gamma \cdot p \) is closed (in the reference it is assumed that \( G \) is connected, but their proof is valid without this assumption).

Observe that the stabilizer of \( p \) in \( G \) is \( N^1_G(F) \). Therefore \( \Gamma N^1_G(F) \) is a closed subset of \( G \), and hence the same holds for \( N^1_G(F) \Gamma \).

**Proposition 2.6 (Dani)** Let \( G \) be a Lie group, \( \Gamma \) a closed subgroup of \( G \), and \( \pi : G \to G/\Gamma \) the natural quotient map. Let \( u \in G \) be an \( \text{Ad}_G \)-unipotent element and \( \mu \) a finite \( u \)-invariant measure on \( G/\Gamma \). Then

\[
\text{Ad}_G(u) \in \text{Zcl}(\text{Ad}_G(g\Gamma g^{-1})), \quad \forall g \in \pi^{-1}(\text{supp}(\mu)).
\]

In particular, if \( H \) is a closed subgroup of \( G \) containing \( u \) such that \( \text{supp}(\mu) \subset \pi(H) \), then

\[
\text{Ad}_G(u) \in \text{Zcl}(\text{Ad}_G(h(H \cap \Gamma)h^{-1})), \quad \forall h \in H \cap \pi^{-1}(\text{supp}(\mu)).
\]

**Proof** This follows from Dani’s version of Borel’s density theorem [DX, Corollary 2.6] (see [W, Proof of Corollary 4.3] for details).
3 Extension of a Discrete Unipotent Flow to a Continuous Unipotent Flow

*Notation.* Let $G$ be a Lie group and $\Gamma$ be a discrete subgroup of $G$ such that $G = G^0\Gamma$. Let $\pi : G \to G/\Gamma$ be the natural quotient map and $x_0 = \pi(e)$. Let $u \in G$ be an $\text{Ad}_G$-unipotent element and $\mathfrak{g}$ be the Lie algebra of $G$.

Let $\rho : G_0 \to G^0$ be the universal covering homomorphism. Let $\{\tilde{u}(t)\}$ be the one-parameter subgroup of $\text{Aut}(G_0)$ such that

$$\{D(\tilde{u}(t)) |_{\pi(G_0) = \mathfrak{g}}\}_{t \in \mathbb{R}} = \text{Zcl}(\langle \text{Ad}_G u \rangle) \subset \text{Aut}(\mathfrak{g})$$

and $D(\tilde{u}(1)) |_{\mathfrak{g}} = \text{Ad}_G u$.

Consider the semidirect product $\tilde{G} = \mathbb{R} \cdot G_0$, where $t \in \mathbb{R}$ acts as $\tilde{u}(t)$ on $G_0$; in other words, $tg(-t) = \tilde{u}(t)(g)$ for all $g \in G_0$. Note that

$$\rho(1g(-1)) = u\rho(g)u^{-1}, \quad \forall g \in G_0.$$  

Therefore we can extend $\rho : \mathbb{Z} \cdot G_0 \to \langle u \rangle G^0$ such that $\rho(1) = u$. Let $
\Gamma_1 = \rho^{-1}(\Gamma \cap \langle u \rangle G^0)$. Since $G = G^0\Gamma$, we have

$$G/\Gamma \cong \langle u \rangle G^0 / (\Gamma \cap \langle u \rangle G^0) \cong \mathbb{Z} \cdot G_0 / \Gamma_1 \subset \tilde{G}/\Gamma_1. \quad (1)$$

Under this identification, the action of $u$ on $G/\Gamma$ and the action of $u(1)$ on $\mathbb{Z} \cdot G_0 / \Gamma_1$ are identical, where $u(t) = t \in \tilde{G}$ for all $t \in \mathbb{R}$ and $\{u(t)\}$ is a one-parameter $\text{Ad}_G$-unipotent subgroup of $\tilde{G}$.

Thus we can treat a discrete unipotent flow as a restriction of a continuous unipotent flow. Now we will deduce the algebraic properties of the invariant measures and orbit closures for the discrete unipotent flows using the analogous properties of the continuous unipotent flows.

Let $\mu$ be a finite $u$-invariant $u$-ergodic Borel measure on $G/\Gamma$. By Hedlund’s lemma [2,3], there exists $g \in G^0$ such that $\text{supp}(\mu) = \langle u \rangle g x_0$. Let $w = g^{-1}ug$ and $\lambda = g^{-1}\mu$; where by definition, $g^{-1}\mu(E) = \mu(\text{Ad}_G g E)$ for any Borel subset $E \subset G/\Gamma$. Then $\lambda$ is $w$-invariant, $w$-ergodic, and $\text{supp}(\lambda) = \pi(\langle w \rangle)$. Let $\tilde{g} \in \rho^{-1}(g)$. Put $w(t) = \tilde{g}^{-1}u(t)\tilde{g}$. We can treat $\lambda$ as a Borel measure on $\tilde{G}/\Gamma_1$. Note that the action of $w$ on $G/\Gamma$ and the action of $w(1)$ on $\mathbb{Z} \cdot G_0 / \Gamma_1 \subset \tilde{G}/\Gamma_1$ are isomorphic. Let $\tilde{\lambda}$ be the measure on $\tilde{G}/\Gamma_1$ such that for any compactly supported continuous function $f$ on $\tilde{G}/\Gamma_1$, we have

$$\int_{\tilde{G}/\Gamma_1} f \, d\tilde{\lambda} = \int_{0}^{1} \left( \int_{\tilde{G}/\Gamma_1} f(w(t)x) \, d\lambda(x) \right) \, dt. \quad (2)$$
Then $\lambda$ is finite, $\{w(t)\}$-invariant, and $\{w(t)\}$-ergodic. Therefore by Ratner’s measure classification theorem [Ra2, Theorem 1], there exists a closed connected subgroup $\tilde{H}$ of $G$ containing $\{w(t)\}$ such that $\lambda$ is $\tilde{H}$-invariant and supp$(\lambda) = \tilde{H}x_0$. Put $H = \rho(Z \cdot G_0 \cap \tilde{H})$. Then $H$ is a closed subgroup of $G$ containing $w$ and $\lambda$ is a finite $H$-invariant measure on $Hx_0$. Therefore by Lemma 2.2, $\pi(H)$ is closed, supp$(\lambda) = \pi(H)$, and $H \cap \Gamma$ is a lattice in $H$.

We shall describe orbit closures under the assumption that $\Gamma$ is a lattice in $G$. Let $g \in G_0$ and $Y = (u)gx_0$. Let $Z = g^{-1}Y$ and $w = g^{-1}ug$. Then $Z = \langle w \rangle x_0$. Let $\tilde{g} \in \rho^{-1}(g)$ and $w(t) = \tilde{g}^{-1}u(t)\tilde{g}$. In view of Equation 1, $Z = \langle w(1) \rangle x_0$. Put $\tilde{Z} = w([0, 1])Z$. Then $\tilde{Z} = \{w(t)\}x_0 \subset G/\Gamma_1$. By Ratner’s description of orbit closures of continuous unipotent flows [Ra3] the following holds: There exists a closed connected subgroup $\tilde{H}$ of $G$ containing $\{w(t)\}$ such that $\tilde{Z} = \tilde{H}x_0$ and $\tilde{Z}$ has a finite $\tilde{H}$-invariant Borel measure, say $\tilde{\lambda}$. Also the trajectory $\{w(t)x_0 : t \geq 0\}$ is uniformly distributed with respect to $\tilde{\lambda}$. Put $H = \rho(Z \cdot G_0 \cap \tilde{H})$. Then $H$ is a closed subgroup of $G$ containing $w$ such that $Z = \pi(H)$, and $Z$ has a finite $H$-invariant Borel measure, say $\lambda$. Also the trajectory $\{w^n x_0 : n > 0\}$ is uniformly distributed with respect to $\lambda$.

**Definition** Let the notation be as in the beginning of this section. Let $\mathcal{H}_u$ be the collection of subgroups $H$ of $G$ such that $H = \langle w \rangle H^0$, $H \cap \Gamma$ is a lattice in $H$, and $\langle w \rangle x_0 = Hx_0$, where $w := g^{-1}ug \in H$ for some $g \in G^0$. Let $\lambda_H$ denote a unique $H$-invariant Borel probability measure on $Hx_0$, for all $H \in \mathcal{H}_u$. Note that $\pi(H)$ has finitely many connected components.

In view of the above discussion and the definitions, we have the following results:

**Theorem 3.1 (Ratner)** Let the notation be as in the beginning of this section. Let $\mu$ be a $u$-invariant $u$-ergodic Borel probability measure on $G/\Gamma$. Then there exists $g \in G^0$ and $H \in \mathcal{H}_u$ such that $ug \in gH$ and $\mu = g\lambda_H$.

**Theorem 3.2 (Ratner)** Let the notation be as in the beginning of this section. Further assume that $\Gamma$ is a lattice in $G$. Let $g \in G^0$. Then there exists $H \in \mathcal{H}_u$ such that $ug \in gH$ and $\langle u \rangle \pi(u) = g\pi(H)$. Moreover, the trajectory $\{u^n \pi(u) : n > 0\}$ is uniformly distributed with respect to $g\lambda_H$.

**Proposition 3.3 (Ratner)** The collection $\mathcal{H}_u$ is countable.
Proof Let $\mathcal{H}$ be the collection of all closed connected subgroups $\bar{H}$ of $\bar{G}$ such that $\bar{H} \cap \Gamma_1$ is a lattice in $\bar{H}$ and for a one-parameter $\text{Ad}_{\bar{G}}$-unipotent subgroup, say $\{w(t)\} \subset \bar{H}$, we have $\{w(t)\}x_0 = \bar{H}x_0$. Then by Proposition 2.6 and the countability theorem of Ratner [Ra2, Theorem 1] (see [DM, Proposition 2.1] for another proof), $\mathcal{H}$ is countable. From the above discussion $\mathcal{H}_u = \{\rho(\mathbb{Z} \cdot G_0 \cap H) : H \in \mathcal{H}\}$. Hence $\mathcal{H}_u$ is countable. \hfill $\square$

4 Singular Subsets of $G$ Associated to the $u$-action

Notation. Let $G$ be a Lie group and $\Gamma$ be a discrete subgroup of $G$ such that $G = G^0 \Gamma$. Let $\pi : G \to G/\Gamma$ be the natural quotient map and $x_0 = \pi(e)$. Let $u \in G$ be an $\text{Ad}_G$-unipotent element.

Definition. For $H \in \mathcal{H}_u$, we say that $F < H$ (or $H > F$) if and only if $F \in \mathcal{H}_u$, $F \subset H$, and $\pi(F) \neq \pi(H)$. If $F < H$, then either $\dim F < \dim H$, or the number of connected components of $\pi(F)$ is less than the number of connected components of $\pi(H)$. Therefore any decreasing sequence $H > F_1 > F_2 > \cdots$ is finite.

For $H \in \mathcal{H}_u$, define

$$N(H, u) = \{g \in G^0 : ug \in gH\},$$
$$S(H, u) = \bigcup_{F \subset H} N(F, u) \quad \text{and} \quad N^*(H, u) = N(H, u) \setminus S(H, u).$$

Note that for any $\gamma \in \Gamma$,

$$N(H, u)\gamma = N(\gamma^{-1}H\gamma, u) \quad \text{and} \quad S(H, u)\gamma = S(\gamma^{-1}H\gamma, u).$$

Proposition 4.1 Let $H \in \mathcal{H}_u$ and $g \in N(H, u)$. Then

$$g \in N^*(H, u) \iff \langle u \rangle \pi(g) = g\pi(H).$$

Proof Replacing $u$ by $g^{-1}ug$, without loss of generality we may assume that $g = e$. Since $H \cap \Gamma$ is a lattice in $H$, by Theorem 3.2 there exists $F \subset H$ such that $F \in \mathcal{H}_u$ and $\pi(\langle u \rangle) = \pi(F)$. Now by definition,

$$e \in N^*(H, u) \iff \pi(F) = \pi(H).$$
Clearly,  
\[ \pi(F) = \pi(H) \iff \overline{\pi(\langle u \rangle)} = \pi(H). \]
This completes the proof of the proposition.

**Proposition 4.2** Let \( \lambda \) be a \( u \)-invariant \( u \)-ergodic Borel probability measure on \( G/\Gamma \). Then there exist \( H \in \mathcal{H}_u \) and \( g \in N^*(H, u) \) such that \( \lambda = g\lambda_H \), where \( \lambda_H \) denotes a unique \( H \)-invariant Borel probability measure on \( \pi(H) \).

**Proof** By Theorem 3.1, there exist \( H \in \mathcal{H}_u \) and \( g_1 \in N(H, u) \) such that \( \lambda = g_1\lambda_H \) and \( \text{supp} \mu = g_1\pi(H) \). By Hedlund’s lemma, there exists \( h \in H \) such that \( \langle u \rangle\pi(g_1h) = \text{supp}(\mu) \). Put \( g = g_1h \). Then \( g \in N(H, u) \), \( \lambda = g\lambda_H \), and \( \langle u \rangle\pi(g) = g\pi(H) \). Now the proposition follows from Proposition 4.1.

**Proposition 4.3** Suppose \( g \in N^*(H, u) \) and \( \gamma \in \Gamma \) such that \( g\gamma \in N(H, u) \). Then:

1. \( \gamma \in N^1_G(H^0) \);
2. \( \pi(H) = \pi(\gamma H\gamma^{-1}) \);
3. \( g\gamma \in N^*(H, u) \); and
4. \( N(H, u) \) contains an open closed subset of \( N(\gamma H\gamma^{-1}, u) \) containing \( g \).

**Proof** Replacing \( u \) by \( g^{-1}ug \), we may assume that \( g = \{e\} \). Since \( e \in N^*(H, u) \) and \( \gamma \in N(H, u) \), by Proposition 4.1,
\[ \pi(H) = \overline{\pi(\langle u \rangle)} = \overline{\pi(\langle u \rangle\gamma)} \subset \gamma\pi(H) = \pi(\gamma H\gamma^{-1}). \]
By the dimension consideration, \( \pi(H^0) = \pi(\gamma H^0\gamma^{-1}) \), and hence \( H^0 = \gamma H^0\gamma^{-1} \). Since the action of \( \gamma \) on \( G/\Gamma \) is a homeomorphism, the number of connected components of \( \pi(H) \) and \( \gamma\pi(H) \) are the same. Therefore \( \pi(H) = \gamma\pi(H) \). Hence \( \gamma \in N^*(H, u) \). Moreover, since with respect to the Haar measures, \( \text{vol}(\pi(\gamma H^0\gamma^{-1})) = \text{det}(\text{Ad}_G(\gamma)|_{\text{Lie}(H^0)}) \text{vol}(\pi(H^0)) \), we obtain that \( \gamma \in N^1_G(H^0) \).

By definition, the sets
\[ \{h \in G^0 : h^{-1}uh \in uH^0\} \text{ and } \{h \in G^0 : h^{-1}uh \in u\gamma H^0\gamma^{-1}\} \]
are open closed in \( N(H, u) \) and \( N(\gamma H\gamma^{-1}, u) \), respectively. Since \( \gamma \in N^1_G(H^0) \), statement (4) follows. \( \square \)
Proposition 4.4 Let $H \in \mathcal{H}_u$ and $\lambda$ be a $u$-invariant $u$-ergodic Borel probability measure on $\pi(N^*(H,u))$. Then $\lambda = g\lambda_H$, for any $\lambda = g\lambda_H$ for any $g \in N^*(H,u) \cap \pi^{-1}(\text{supp} \lambda)$.

Proof By Hedlund’s lemma, there exists $g_0 \in N^*(H,u)$ such that $\langle u \rangle \pi(g_0) = \text{supp}(\lambda)$. Therefore by Proposition 4.3, $\text{supp}(\lambda) = g_0 \pi(H)$. Now by Ratner’s theorem as discussed in the preceding subsection, we have that $\lambda$ is $g_0 H g_0^{-1}$-invariant. Hence $\lambda = g_0 \lambda_H$.

Now $\pi^{-1}(\text{supp} \mu) = g_0 \pi(H) \cap N^*(H,u)$, then $\pi(H) = g_0 \pi(H)$. Hence $\lambda_H = g_0 \lambda_H = \lambda$. This completes the proof of the proposition.

5 Abundance of Unipotent Subgroups

The following main technical result of this section is used in the proofs of Theorems 1.1 and 1.3.

Proposition 5.1 Let $G$ be a Lie group, $H$ a closed connected subgroup of $G$, $u \in N_G(H)$ an $\text{Ad}_G$-unipotent element, and $U$ the subgroup generated by all one-parameter $\text{Ad}_G$-unipotent subgroups of $H$. Then the set

$$\{ g \in N_G(U) : g^{-1}ugu^{-1} \in U \}$$

contains a neighbourhood of $e$ in the set

$$\{ g \in N_G(U) : g^{-1}ugu^{-1} \in H \}$$

Proof Let $\tilde{U} = \text{Ad}_G(U)$. Then $\tilde{U}$ is a connected real algebraic group generated by one-parameter unipotent subgroups of $\text{GL}(g)$ (cf. [Sh, Proof of Lemma 2.9]). Put $\tilde{L} = N_{\text{GL}(g)}(\tilde{U})$. Then $\tilde{L}$ and $\tilde{L}/\tilde{U}$ are real algebraic groups, and the natural quotient homomorphism $\tilde{q} : \tilde{L} \to \tilde{L}/\tilde{U}$ is algebraic. Put $L = N_G(U)$.

Claim 5.1.1 There exists a neighbourhood $\Omega$ of $e$ in $L$ such that for any $h \in H \cap \Omega$, if $\tilde{q}(\text{Ad}(h))$ is an algebraic unipotent element of $\tilde{L}/\tilde{U}$, then $h \in U$. 11
To show this, let \( \tilde{\Omega} \) be a neighbourhood of the identity in \( \tilde{L}/\tilde{U} \) such that the following holds: For any one-parameter subgroup \( \tilde{h}(t) \subset \tilde{L}/\tilde{U} \), if \( \tilde{h}([0,1]) \subset \tilde{\Omega} \) and \( \tilde{h}(1) \) is an algebraic unipotent element, then \( \{ \tilde{h}(t) \} \) is algebraic unipotent subgroup. In this case, there exists a unipotent one-parameter subgroup \( \{ h(t) \} \subset \tilde{L} \) such that \( \tilde{q}(h(t)) = \tilde{h}(t) \). Note that

\[
H \subset L = N_G(U) \subset \operatorname{Ad}_G^{-1}(\tilde{L}).
\]

Let \( \Omega_1 = \operatorname{Ad}_G^{-1} \circ \tilde{q}^{-1}(\tilde{\Omega}) \). Let \( \Omega \subset \Omega_1 \) be a neighbourhood of \( e \) in \( L \) such that the following holds: for any \( h \in H \cap \Omega \), there exists a one-parameter subgroup \( \{ h(t) \} \subset H \) such that \( h = h(1) \) and \( h([0,1]) \subset \Omega_1 \). Now since \( \tilde{U} \subset \operatorname{Ad}_G(H) \), the claim follows from the above construction.

Let \( l \) and \( u \) denote the Lie algebras corresponding to \( L \) and \( U \), respectively. We identify the Lie algebra of \( L/U \) with \( l/u \). Let \( q : L \to L/U \) be the natural quotient homomorphism.

Now suppose that the proposition is not true. Then there exists a sequence \( g_k \to e \) in \( L \) such that \( g_k^{-1}ug_ku^{-1} \in H \setminus U \) for all \( k \in \mathbb{N} \). By passing to a subsequence, there exists \( X_k \in l/u \) such that

\[
q(g_k) = \exp_{L/U}(X_k)
\]

for all \( k \in \mathbb{N} \), and \( X_k \to 0 \) as \( k \to \infty \).

Consider the linear action of \( L \) on \( l/u \) via the representation \( \operatorname{Ad}_{L/U} \circ q \). Since \( ug_ku^{-1} \not\in g_kU \), we have that \( u \cdot X_k \neq X_k \). Now since \( \operatorname{Ad}_{L/U}(q(u)) \) is unipotent, \( u^n \cdot X_k \to \infty \) as \( n \to \infty \). Therefore there exists a sequence \( n_k \to \infty \) such that after passing to a subsequence \( u^{n_k} \cdot X_k \to X \), where \( X \in l/u \setminus 0 \). We can choose \( \{ n_k \} \) such that \( \exp_{L/U}(X) = q(h) \neq e \) for some \( h \in \Omega \). Thus \( q(u^{n_k}g_ku^{-n_k}) \to q(h) \) as \( k \to \infty \). Now \( U \subset H \), \( g_k^{-1}ug_ku^{-1} \in H \) and \( g_k \to e \) as \( k \to \infty \). Therefore \( h \in H \setminus U \).

Now

\[
(\tilde{q}(\operatorname{Ad}_G u))^{n_k} \tilde{q}(\operatorname{Ad}_G g_k)(\tilde{q}(\operatorname{Ad}_G u))^{-n_k} \to \tilde{q}(\operatorname{Ad}_G h) \quad \text{as} \quad k \to \infty.
\]

Since \( \tilde{q}(\operatorname{Ad}_G g_k) \to e \) as \( k \to \infty \), it follows that \( \tilde{q}(\operatorname{Ad}_G h) \) is an algebraic unipotent element of \( \tilde{L}/\tilde{U} \). Hence by Claim 5.1.1, \( h \in U \), which is a contradiction. \( \square \)

The following simple observation is useful.

**Lemma 5.2** Let \( G \) be a Lie group (so that \( G^0 \) is analytic), \( F \) a connected (analytic) Lie group, and \( \rho : F \to G^0 \) an analytic map. Let \( \sigma_F \) denote a Haar
measure on $F$, and $M$ be an analytic submanifold of $G^0$. If $\sigma_F(\rho^{-1}(M)) > 0$, then $\rho(F) \subset M$.

In particular, if $F$ is an analytic subgroup of $G$ such that $N(H,u)$ contains a neighbourhood of $e$ in $F$ then $F \subset N(H,u)$, where $N(H,u)$ is as defined in Section 4.

Corollary 5.3 Let $G$ be a Lie group, $H$ a closed connected subgroup of $G$, $u \in N_G(H)$ an Ad$_G$-unipotent element, and $U$ be a the closed connected subgroup of $H$ generated by all one-parameter Ad$_G$-unipotent subgroups of $H$. Then $h^{-1}uh^{-1} \in U$ for all $h \in H$. In particular, $u \in N_G(F)$ for any subgroup $F$ of $H$ containing $U$.

Proof Let $\rho : H \to G$ be the map defined by

$$\rho(h) = h^{-1}uh^{-1} \text{ for all } h \in H.$$ Since $\rho(H) \subset H$, by Proposition 5.1 $\rho^{-1}(U)$ contains a neighbourhood of $e$ in $H$. Therefore by Lemma 5.2 $h^{-1}uh^{-1} \in U$ for all $h \in H$. \[\Box\]

6 Proofs of Theorem 1.1 and Theorem 1.2

For locally finite $u$-invariant measures, we have the following result due to Dani:

Theorem 6.1 (\cite{D4}, Theorem 4.3) Let $G$ be a Lie group and $\Gamma$ a closed subgroup such that $G/\Gamma$ has a finite $G$-invariant measure. Let $u \in G$ be an Ad$_G$-unipotent element, and $\mu$ be a $u$-invariant locally finite Borel measure on $G/\Gamma$. Then there exist a partition of $G/\Gamma$ into countably many $u$-invariant Borel measurable subsets, say $X_i$ ($i \in \mathbb{N}$), such that $\mu(X_i) < \infty$, $\forall i \in \mathbb{N}$.

Due to this result, the Theorem 1.1 and Theorem 1.2 are special cases of the following.

Theorem 6.2 Let $G$, $\Gamma$, $W$, and $\mathcal{U}$ be as in Theorem 1.1. Let $\mu$ be a locally finite $W$-invariant $W$-ergodic Borel measure on $G/\Gamma$. Suppose that for any Ad-unipotent element $u \in \mathcal{U}$, there exists a partition of $G/\Gamma$ into countably many Borel measurable $u$-invariant subsets $X_i$ ($i \in \mathbb{N}$) such that $\mu(X_i) < \infty$, $\forall i \in \mathbb{N}$. Then there exists a closed subgroup $H$ of $G$ containing $W$ such that $\text{supp}(\mu)$ is closed $H$-orbit.
We intend to prove this theorem by induction on the dimension of $G^0$. Note that the theorem is obvious, if $G$ is a discrete group; that is, if $\text{dim}(G^0) = 0$.

The rest of the proof of this theorem is a series of claims and propositions.

**Claim 6.2.1** We may assume that $\mathcal{U}$ is finite.

**Proof** Since the Zariski closure of a cyclic group generated by a unipotent linear transformation is a connected group, we have that the Zariski closure of $\langle \text{Ad}_G \mathcal{U} \rangle$ is a connected real algebraic group of dimension, say $d$. Hence there is a subset $\mathcal{U}_1 \subset \mathcal{U}$ consisting of at most $d$ elements such that $\text{Zcl}(\langle \text{Ad}_G \mathcal{U}_1 \rangle) = \text{Zcl}(\langle \text{Ad}_G \mathcal{U} \rangle)$. Thus without loss of generality we may replace $\mathcal{U}$ by $\mathcal{U}_1$ and assume that $\mathcal{U}$ is finite. \hfill $\square$

Let $\pi : G \to G/\Gamma$ denote the natural quotient map.

**Claim 6.2.2** We may assume that for each $u \in \mathcal{U}$, there exists a $u$-invariant Borel measurable subset $X_u$ of $G/\Gamma$ such that $\mu(X_u) < \infty$, and $\pi(e)$ belongs to the support of the restriction of $\mu$ to $X_u$, and $\text{supp}(\mu) \subset \pi(W)$.

**Proof** Using Hedlund’s lemma and Claim 6.2.1, it is straightforward to obtain the conclusion of the claim for $\pi(g)$ in place of $\pi(e)$ for some $g$ in $G$. Now if we work with $g\Gamma g^{-1}$, in place of $\Gamma$, without loss of generality we may assume that $\pi(e)$ belongs to the support of the restriction of $\mu$ to $X_u$. \hfill $\square$

**Claim 6.2.3** We may assume that $\Gamma$ is a discrete subgroup of $G$.

**Proof** For each $u \in \mathcal{U}$, by Claim 6.2.2, $\pi(e)$ belongs to the support of a finite $u$-invariant Borel measure on $G/\Gamma$. Therefore by Proposition 2.4, we have that $\text{Ad}_G(\langle u \rangle) \subset \text{Zcl}(\text{Ad}_G(\Gamma))$ for all $u \in \mathcal{U}$. Therefore $W \subset N_G(\Gamma^0)$. Since $\text{supp}(\mu) \subset \pi(W)$, replacing $G$ by $N_G(\Gamma^0)$, without loss of generality we may assume that $\Gamma^0$ is normal in $G$. Now again replacing $G$ by $G/\Gamma^0$ and $\Gamma$ by $\Gamma/\Gamma^0$, the claim holds. \hfill $\square$

**Claim 6.2.4** We may assume that if $\mu(\pi(F)) > 0$ for any closed connected subgroup $F$ of $G$ such that $W \subset N_G(F)$, then $F = G^0$. 

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Proof For any \( w_1, w_2 \in W \), either
\[
w_1 \pi(F) = w_2 \pi(F) \text{ or } w_1 \pi(F) \cap w_2 \pi(F) = \emptyset.
\]
Since \( \mu \) is locally finite, and \( \mu(\pi(F)) > 0 \), the group \( WF/F \) is countable. Put \( H = WF \), and extend the topology of \( F \) to \( H \) such that \( H^0 = F^0 \). Let \( \Lambda = H \cap \Gamma \), and consider the natural continuous inclusion \( \rho : H/\Lambda \to G/\Gamma \).
By the ergodicity of \( W \)-action, \( \mu \) is concentrated on \( \pi(H) \). Hence \( \mu \) can be treated as a locally finite \( W \)-invariant \( W \)-ergodic Borel probability measure on \( H/\Lambda \). Suppose that \( F \neq G^0 \). Then \( \dim(H^0) < \dim(G^0) \), and hence by induction hypothesis there exists a closed subgroup \( H_1 \) of \( H \) containing \( W \) such that \( \mu \) is \( H_1 \)-invariant and \( \text{supp}(\mu) \) is a closed \( H_1 \)-orbit in \( H/\Lambda \). Thus \( H_1 \) is a measurable subgroup of \( G \) containing \( W \) such that \( \mu \) is \( H_1 \)-invariant and concentrated on an orbit of \( H_1 \). Now the theorem follows from Lemma 2.2. Thus without loss of generality we may assume that \( F = G^0 \).
\[
\square
\]
Claim 6.2.5 We may assume that \( G = WG^0 = G^0\Gamma \).

Proof Since \( \pi(WG^0) = W \pi(G^0) \) is a closed \( W \)-invariant subset of \( G/\Gamma \) with strictly positive \( \mu \)-measure, by ergodicity, we have that
\[
\text{supp}(\mu) \subset \pi(WG^0).
\]
Therefore replacing \( G \) by \( WG^0 \), without loss of generality we may assume that \( G = WG^0 \).
For any \( w_1, w_2 \in W \), either
\[
w_1 \pi(G^0) = w_2 \pi(G^0) \text{ or } w_1 \pi(G^0) \cap w_2 \pi(G^0) = \emptyset.
\]
Let \( \mu_0 \) be the restriction of \( \mu \) to \( \pi(G^0) \). Define
\[
W_0 = \{ w \in W : w\mu_0 = \mu_0 \} = \{ w \in W : \pi(w) \in \pi(G^0) \}.
\]
Let \( u \in U \). By Claim 6.2.2 \( \mu_0(X_u) > 0 \). Therefore there exists \( k \in \mathbb{N} \) such that \( u^k \in W_0 \). Note that \( \text{Zcl}(\langle \text{Ad}_G u \rangle) = \text{Zcl}(\langle \text{Ad}_G u^k \rangle) \). Put
\[
U_0 = W_0 \cap \left( \bigcup_{u \in U} \langle u \rangle \right).
\]
Then \( U_0 \) consists of \( \text{Ad}_G \)-unipotent elements, and
\[
\text{Ad}_G(W) \subset \text{Zcl}(\langle \text{Ad}_G U_0 \rangle). \tag{3}
\]
Now suppose we can prove the theorem for the action of $W_0$ on $\mu_0$. Then there exists a closed subgroup, say $F$, such that $W_0 \subset F$, $\mu_0$ is $F$-invariant and $\text{supp}(\mu_0) = \pi(F)$. By Equation 3 we have that $W \subset N_G(F^0)$. Therefore by Claim 6.2.4, $F^0 = G^0$. Now since $w\mu_0$ is $G^0$-invariant for any $w \in W$, we have that $\mu$ is $G^0$-invariant. Thus $\mu$ is $G$-invariant, and the theorem follows. Thus without loss of generality we may assume that $G/\Gamma = \pi(G^0)$.

**Proposition 6.3** There exist $H_u \in \mathcal{H}_u$ for all $u \in \mathcal{U}$ such that

$$\mu(\pi(\cap_{u \in \mathcal{U}} N^*(H_u, u))) > 0.$$  

**Proof** Let $\Omega \subset G$ be a Borel measurable subset such that $\mu(\pi(\Omega)) > 0$. Express $\mathcal{U} = \{u_1, \ldots, u_k\}$ for some $k \in \mathbb{N}$. There exists a $u_1$-invariant Borel measurable subset $X_1$ of $G/\Gamma$ such that $\mu(X_1) < \infty$, and $\mu(\pi(\Omega) \cap X_1) > 0$. Let $\mu_1$ denote the restriction of $\mu$ to $X_1$. Then $\mu_1$ is a finite $u_1$-invariant measure.

Now $\mu_1$ is a direct integral of finite $u_1$-ergodic $u_1$-invariant measures; (see [D1] Section 1.4 for a precise statement). Therefore by Proposition 4.2 and Proposition 3.3, there exists $H_1 \in \mathcal{H}_u$ such that

$$\mu_1(\pi(N^*(H_1, u_1)) \cap \pi(\Omega)) > 0.$$  

Therefore there exists $\gamma_1 \in \Gamma$ such that

$$\mu_1(\pi(N^*(H_1, u_1) \gamma_1 \cap \Omega)) > 0.$$  

Put $H_{u_1} = \gamma_1^{-1} H_1 \gamma_1$ and $\Omega_1 = N^*(H_{u_1}, u_1) \cap \Omega$. Then $\mu(\pi(\Omega_1)) > 0$.

For $2 \leq i \leq k$, we inductively carry out the same procedure with $u_i$ in place of $u_1$, and $\Omega_{i-1}$ in place of $\Omega$. We obtain $H_{u_i} \in \mathcal{H}_u$ such that if we put $\Omega_i = N^*(H_{u_i}, u_i) \cap \Omega_{i-1}$, then $\mu(\pi(\Omega_i)) > 0$.

Now $\Omega_k \subset \cap_{u \in \mathcal{U}} N^*(H_u, u)$ and $\mu(\pi(\Omega_k)) > 0$. This completes the proof.

**Proposition 6.4** Let $F$ be a connected Lie subgroup of $G$ such that $\pi(F)$ has an $F$-invariant probability measure, say $\lambda_F$. Suppose for some $g \in G$, $(g\lambda_F)(\pi(\cap_{u \in \mathcal{U}} N(H_u, u))) > 0$. Then there exists $\gamma \in \Gamma$ such that

$$gF \gamma \subset \cap_{u \in \mathcal{U}} N(H_u, u).$$
Proof Let \( \tilde{\lambda}_F \) denote a Haar measure on \( F \). Then
\[
\tilde{\lambda}_F(\{h \in F : g\pi(h) \in \pi(\cap_{u \in \mathcal{U}} N(H_u, u))\}) > 0.
\]
Therefore there exists \( \gamma \in \Gamma \) such that
\[
\tilde{\lambda}_F(\{h \in F : gh\gamma \in \cap_{u \in \mathcal{U}} N(H_u, u)\}) > 0.
\]
Take any \( u \in \mathcal{U} \). Then the map \( \rho : F \to G \), given by
\[
\rho(h) = (gh\gamma)^{-1}u(gh\gamma)u^{-1}
\]
for all \( h \in F \), is analytic. Since \( \tilde{\lambda}(\rho^{-1}(H_u)) > 0 \), by Lemma 5.2 \( \rho(F) \subset H_u \). Hence \( gF\gamma \subset N(H_u, u) \). This completes the proof. \( \Box \)

**Proposition 6.5** For any \( g \in \cap_{u \in \mathcal{U}} N^*(H_u, u) \) and \( v \in \mathcal{U} \), if
\[
g\lambda_{H_v}(\pi(\cap_{u \in \mathcal{U}} N(H_u, u))) > 0
\]
then
\[
gH_v^0 \subset \cap_{u \in \mathcal{U}} N(H_u, u) \tag{4}
\]
and
\[
g\lambda_{H_v}(\pi(\cup_{u \in \mathcal{U}} S(H_u, u))) = 0. \tag{5}
\]

Proof By Proposition 6.3 there exists \( \gamma \in \Gamma \) such that
\[
gH_v^0 \gamma \subset \cap_{u \in \mathcal{U}} N(H_u, u).
\]
Hence
\[
gH_v^0 \subset \cap_{u \in \mathcal{U}} N(\gamma H_u \gamma^{-1}, u).
\]
By Proposition 4.3, the set \( N(H_u, u) \) contains the connected component of \( g \) in \( N(\gamma H_u \gamma^{-1}, u) \). Therefore \( gH_v^0 \subset N(H_u, u) \) for all \( u \in \mathcal{U} \). Hence (4) holds.

Now suppose \( g\lambda_{H_v}(\cup_{u \in \mathcal{U}} \pi(S(H_u, u))) > 0 \). Then there exist \( u \in \mathcal{U} \) and \( F_u < H_u \) such that \( g\lambda_{H_v}(\pi(N(F_u, u))) > 0 \). By Proposition 6.3, there exists \( \gamma \in \Gamma \) such that \( gH_u^0 \gamma \subset N(F_u, u) \). Therefore \( g\gamma \in N(F_u, u) \). Thus \( g\gamma \notin N^*(H_u, u) \). Since \( g \in N^*(H_u, u) \), this contradicts Proposition 4.3. \( \Box \)
Proposition 6.6 There exists a Borel set $T^* \subset \cap_{u \in U} N^*(H_u, u)$ such that $\mu(\pi(\cap_{u \in U} N^*(H_u, u) \setminus T^*)) = 0$ and $g \lambda_{H^0_u}(\pi(T^*)) = 1$, $\forall u \in U$, $\forall g \in T^*$.

**Proof** Put $T^*_0 = \cap_{u \in U} N^*(H_u, u)$. Let $\mu^*$ be the restriction of $\mu$ to $\pi(T^*_0)$. Let $T^*_1 = \{g \in T^*_0 : g \lambda_{H^0_u}(\pi(T^*_0)) > 0, \forall u \in U\}$.

Considering the ergodic decomposition of $\mu^*$ restricted to $\pi(N(H_u, u))$ for all $u \in U$, and applying Proposition 4.4, we obtain that $\mu(\pi(T^*_1)) = 0$.

By Proposition 6.5, $T^*_1 = \{g \in T^*_0 : g \lambda_{H^0_u}(\pi(T^*_0)) = 1, \forall u \in U\}$.

Therefore $\mu$ restricted to $\pi(T^*_1)$ is $\mu^*$. Now for any $u \in U$, by Proposition 4.4, the measure $\mu^*$ is a direct integral of measures of the form $g \lambda_{H^0_u}$, where $g \in T^*_1$.

By the above procedure we can obtain a decreasing sequence of Borel subsets $\{T^*_i\}$ such that

$T^*_i = \{g \in T^*_1 : g \lambda_{H^0_u}(\pi(T^*_0)) = 1, \forall u \in U\}$,

and $\mu(\pi(T^*_i \setminus T^*_{i+1})) = 0$ for all $i \geq 0$. Thus $T^* = \cap_{i \geq 0} T^*_i$ has the desired properties.

Claim 6.6.1 We may assume that $e \in \cap_{u \in U} N^*(H_u, u)$, $\pi(W) = \text{supp}(\mu)$, and $\pi(e)$ is in the support of the restriction of $\mu$ to $\pi(\cap_{u \in U} N^*(H_u, u))$.

**Proof** Let $T^*$ be as in Proposition 6.6. By Hedlund’s lemma, there exists $g \in T^*$ such that $W = \pi(g)$ and $\pi(g)$ is a density point of $\mu^*$. Replacing $\Gamma$ by $g\Gamma g^{-1}$, and $H_u$ by $gH_u g^{-1}$ for all $u \in U$, without loss of generality we may assume that $e \in T^*$, $\pi(W) = \text{supp}(\mu)$, and $\pi(e)$ is a density point of $\mu^*$.

Proposition 6.7 $H^0_v \subset \cap_{u \in U} N(H_u, u)$ for all $v \in U$.

**Proof** Since $e \in T^*$, by Proposition 6.6, we have that $\lambda_{H^0_u}(\pi(\cap_{u \in U} N(H_u, u))) = 1$ for all $v \in U$. Now the proposition follows from Proposition 5.5. \qed
Definition 6.8 For each $u \in U$, let $U_u$ be the subgroup generated by all one-parameter $\text{Ad}_G$-unipotent subgroups of $H_u$. Then $U_u$ is normal in $H_u$. Let $F_u$ be the connected component of the identity in the closure of the subgroup $U_u(H_u \cap \Gamma)$. Note that $\pi(U_u) = \pi(F_u)$.

Proposition 6.9

$$F_v \subset N_G(H_u^0) \cap N_G(F_u) \subset N_G(U_u).$$

Proof Since $e \in \cap_{u \in U} N^*(H_u, u)$, by Proposition 4.3,

$$\Gamma \cap N(H_u, u) \subset N_G(H_u^0).$$

Note that $\Gamma \cap N_G(U_u) \subset N_G(F_u)$. Now since $N_G(H_u^0) \subset N_G(U_u)$, we have

$$\Gamma \cap N(H_u, u) \subset N_G(F_u).$$

Let $v \in U$. By Proposition 6.5, $F_v \subset H_u^0 \subset \cap_{u \in U} N(H_u, u)$. Hence $F_v \cap \Gamma \subset N_G(H_u^0) \cap N_G(F_u)$. By Proposition 2.6, $Z\text{cl}(\text{Ad}_G(F_v \cap \Gamma)) \supset \text{Ad}_G(F_v)$. Now the proposition follows.

Proposition 6.10 Express $U = \{u_1, \ldots, u_k\}$ and put

$$F = \{f_1 \cdots f_k \in G : f_i \in F_{u_i}, 1 \leq i \leq k\}.$$

Then $F$ is a closed subgroup of $G$, $\pi(F)$ is closed, and $F \cap \Gamma$ is a lattice in $F$.

Proof Put $F_i = F_{u_i}$. Due to Proposition 6.3, we can define the semidirect product $\tilde{F} = F_k \times \cdots \times F_1$, where $F_i$ acts on $F_{i-1} \times \cdots \times F_1$ by conjugation on each factor ($i = 2, \ldots, k$). Clearly $\tilde{F}$ is a connected Lie group. Let $\Lambda = (F_k \cap \Gamma) \times \cdots \times (F_1 \cap \Gamma)$. Since $F_i \cap \Gamma$ is a lattice in $F_i$, we have that $\Lambda$ is a lattice in $\tilde{F}$. Let $\tilde{\sigma}$ be a finite $\tilde{F}$-invariant measure on $\tilde{F}/\Lambda$. By the definition of semidirect product, the map $\rho : \tilde{F} \to G$, given by $\rho(f_k, \ldots, f_1) = f_k \cdots f_1$ for all $f_i \in F_i$ ($i = 1, \ldots, k$), is a continuous homomorphism. Note that $F = \rho(\tilde{F})$. Since $\rho(\Lambda) \subset \Gamma$, the map $\rho$ determines a continuous $\rho$-equivariant map $\hat{\rho} : \tilde{F}/\Lambda \to G/\Gamma$. Then the push-forward of $\tilde{\sigma}$ under $\hat{\rho}$ is a finite $F$-invariant measure concentrated on $\pi(F) = \hat{\rho}(\tilde{F}/\Lambda)$. Now by Lemma 2.2, $F$ is closed and $\pi(F)$ is closed. Since $\pi(F)$ has a finite $F$-invariant measure, $F \cap \Gamma$ is a lattice in $F$. \( \square \)
Proposition 6.11 $W \subset N_G^1(F)$.

**Proof** Take any $u, v \in \mathcal{U}$. Define the map $\rho : F_v \to G$ by

$$\rho(f) = f^{-1}ufu^{-1} \text{ for all } f \in F_v.$$ 

Since $F_v \subset N(H_u, u)$ and $F_v$ is connected, we have $\rho(f) \in H_u^0$ for all $f \in F_v$. Now since $F_v \subset N_G(U_u)$, by Proposition 6.11, $\rho^{-1}(U_u)$ contains a neighbourhood of $e$ in $F_v$. Therefore by Lemma 5.2, $f^{-1}ufu^{-1} \in U_u \subset F$, $\forall f \in F_v$. Hence $u \in N_G(F)$. Now the proposition follows. $\square$

Claim 6.11.1 We may assume that $F$ is a normal subgroup of $G$.

**Proof** By Proposition 2.7, $\pi(N_G^1(F))$ is closed. Now by Proposition 6.11, we have $\text{supp}(\mu) = \pi(W) \subset \pi(N_G^1(F))$. Therefore without loss of generality we can replace $G$ by $N_G^1(F)$. Now the claim follows. $\square$

Proposition 6.12 The measure $\mu$ is $F$-invariant.

**Proof** For any $u \in \mathcal{U}$, since $F_u \subset H_u^0$ and $\pi(F_u)$ has a finite $F_u$-invariant measure, by Proposition 4.4 and Proposition 6.6, $\mu^*$ is a direct integral of measures of the form $g \cdot \lambda_{F_u}$, where $g \in T^*$. Therefore by arguing as in Proposition 6.6 for $F_u$ in place of $H_u^0$, we obtain a Borel set $T' \subset T^*$ such that the following holds: $\mu^*(g \cdot \lambda_{F_u}(T')) = 1$, $\forall g \in T'$, $\forall u \in \mathcal{U}$. Hence $g \cdot \lambda_F(\pi(T')) = 1$ ($\forall g \in T'$), and the measure $\mu^*$ is a direct integral of measures of the form $g \cdot \lambda_F$ ($g \in T'$).

Since $F$ is a normal subgroup of $G$, the measure $g\lambda_F$ is $F$-invariant for all $g \in G$. Therefore $\mu^*$ is $F$-invariant. Hence $w\mu^*$ is $F$-invariant for all $w \in W$. Since $\mu$ is $W$-ergodic, and $\mu^* \neq 0$, we conclude that $\mu$ is $F$-invariant. $\square$

Claim 6.12.1 We may assume that $\mu(\pi(Z_G(W)^0)) > 0$.

**Proof** Due to Claim 5.11.1 and Proposition 3.12, without loss of generality we can pass to the quotient of $G$ by $F$ and assume that $F = \{e\}$.

Take any $u \in \mathcal{U}$. Since $U_u = \{e\}$, by Proposition 5.1, there exists a neighbourhood $\Omega_u$ of $e$ in $N(H_u, u)$ such that $\omega^{-1}u\omega u^{-1} \in \{e\}$ for all $\omega \in \Omega_u$. Thus $Z_G(u)^0$ contains a neighbourhood of $e$ in $N(H_u, u)$. Therefore $Z_G(\mathcal{U})^0$ contains a neighbourhood of $e$ in $\cap_{u \in \mathcal{U}}N(H_u, u)$. Now since $Z_G(W)^0 = Z_G(\mathcal{U})^0$, by Claim 5.6.3, we have that $\mu(\pi(Z_G(W)^0)) > 0$. $\square$

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Claim 6.12.2 $G^0 \subset Z_G(W)$.

**Proof** This follows from Claims 6.2.4 and 6.12.1. □

Claim 6.12.3 We may assume that $G^0 \cap \Gamma$ is contained in the center of $G$.

**Proof** By Lemma 2.3, the orbit $\pi(Z_G(\gamma))$ is closed for any $\gamma \in \Gamma$. By Claim 6.12.2, for any $\gamma \in G^0 \cap \Gamma$, we have $W \subset Z_G(\gamma)$. Therefore $\text{supp}(\mu) = \pi(W) \subset \pi(Z_G(\gamma))$.

By Claim 6.2.4, $G^0 \subset Z_G(\gamma)$, and by Claim 6.2.5 $Z_G(\gamma) = G$. □

**Completion of the proof of the theorem** Put $Z = G^0/G^0 \cap \Gamma$. Then $Z$ is a locally compact group. Consider the natural inclusion $\psi : Z \to G/\Gamma$. Since $G = G^0\Gamma$, the map $\psi$ is a homeomorphism. Let the map $\rho : W \to Z$ be defined by $\rho(w) = \psi^{-1}(\pi(w))$ ($\forall w \in W$). By Claim 6.12.2, we have $w\psi(z) = \psi(z\rho(w))$ for all $z \in Z$. Therefore the action of $w \in W$ on $G/\Gamma$ corresponds to the right action of $\rho(w)$ on $Z$. Let $\tilde{\mu}$ be the projection of $\mu$ under $\psi^{-1}$. Then $\tilde{\mu}$ is ergodic under the right action of $\rho(W)$ on the locally compact group $Z$. By Claim 6.6.1, $\text{supp}(\tilde{\mu})$ contains the identity element of $Z$. Hence $\tilde{\mu}$ is a Haar measure on the closed subgroup $\rho(W)$. Let $H$ be the inverse image of $\rho(W)$ in $G^0$. Since $\mu = \psi_*(\tilde{\mu})$, we have that $\mu$ is $H$-invariant and $\text{supp}(\mu) = \pi(H)$. This completes the proof of Theorem 5.2. □

### 7 Orbit Closures in Finite Volume Homogeneous Spaces

One of the main purposes of this section is to prove the following result, which will be used in the proof of Theorem 1.3.

**Theorem 7.1** Let $G$, $\Gamma$, and $W$ be as in Theorem 1.3. Let $\pi : G \to G/\Gamma$ be the natural quotient map and $x_0 = \pi(e)$. Let $H$ be the minimal among the closed subgroups $F$ of $G$ such that $W \subset F$ and $F \gamma_0$ is closed. Then $H^0 \gamma_0$ has a finite $H^0$-invariant measure.

To prove this theorem we will reduce this question to homogeneous spaces of semisimple groups. In order to quotient out by solvable factors, first we study the effects on the orbit closures when we pass to the quotients of finite volume homogeneous spaces.
Lemma 7.2 Let $G$ be a locally compact group, $\Lambda$ a closed subgroup $G$, and put $x_0 = e\Lambda$. Let $Z$ be a closed subgroup of $\{g \in G : g\Lambda g^{-1} \subset \Lambda\}$ such that $Zx_0$ is compact. Let $\rho : G/\Lambda \to G/Z\Lambda$ be the natural quotient map. Suppose there exists a closed subgroup $H$ of $G$ such that the orbit $Hx_0$ is open in its closure $\overline{Hx_0}$. Then
\[ \rho(\overline{Hx_0} \setminus Hx_0) = \overline{H\rho(x_0)} \setminus H\rho(x_0). \]
In particular, if $H\rho(x_0)$ is closed, then $Hx_0$ is closed.

Proof Since $Z\Lambda/\Lambda$ is compact, $\rho$ is a proper map. Therefore
\[ \rho(\overline{Hx_0} \setminus Hx_0) \supset \overline{H\rho(x_0)} \setminus H\rho(x_0). \]
To show the inclusion, suppose that there exists $y_0 \in Y = \overline{Hx_0} \setminus Hx_0$ such that $\rho(y_0) = h\rho(x_0)$ for some $h \in H$. Then there exists $z \in G$ such that $y_0 = hz\Lambda$. Hence $z\Lambda x_0 \in Y$.

We claim that $z^kx_0 \in Y$ for all $k \in \mathbb{N}$. To prove this claim by induction, suppose that $z^kx_0 \in Y$. Now sequences $\{h_i\} \subset H$ and $\lambda_i \in \Lambda$ be such that $h_i\lambda_i \to z$ as $i \to \infty$. Then
\[ h_i z^k x_0 = (h_i \lambda_i) \lambda_i^{-1} z^k x_0 = (h_i \lambda_i) z^k x_0 \to z z^k x_0, \]
as $i \to \infty$. Since $Y$ is a closed $H$-invariant set, we have that $z^{k+1}x_0 \in Y$. This proves the claim.

Since $Zx_0$ is compact, and $Z \cap \Lambda$ is a normal subgroup of $Z$, we have that $Z/(Z \cap \Lambda)$ is a compact group. Hence $x_0 \in \{z^kx_0 : k \in \mathbb{N}\}$. Therefore $x_0 \in Y = \overline{Hx_0} \setminus Hx_0$, which is a contradiction. This completes the proof. \qed

Lemma 7.3 Let $G$ be a Lie group and $\Gamma$ a closed subgroup of $G$ such that $G/\Gamma$ has a finite $G$-invariant measure. Let $\Gamma_1 \subset \Gamma$ be a subgroup of finite index in $\Gamma$. Let $\rho : G/\Gamma_1 \to G/\Gamma$ be the natural quotient map, $x_1 = e\Gamma_1$, and $x_0 = \rho(x_1)$. Let $H$ be a closed subgroup of $G$. Then the following statements hold:

1. $\overline{Hx_0} \setminus Hx_0$ closed in $G/\Gamma \iff \overline{Hx_1} \setminus Hx_1$ closed in $G/\Gamma_1$.
2. $Hx_0$ closed in $G/\Gamma \iff Hx_1$ closed in $G/\Gamma_1$. 

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Proof  Let $\Gamma^0$ be the connected component of $e$ in $\Gamma$. Then $\Gamma^0 \subset \Gamma_1$. Let $\Omega$ be a relatively compact open neighbourhood of $e$ in $H$ such that $\Omega^{-1}\Omega \cap \Gamma \subset \Gamma^0$. Then for any $\gamma, \gamma' \in \Gamma$, 
\[ \Omega\gamma x_1 \cap \Omega\gamma' x_1 \neq \emptyset \implies \gamma x_1 = \gamma' x_1. \] (6)

[1:⇒] Since $\Omega x_0$ is open in $H x_0$, we have that $\rho^{-1}(\Omega x_0) \cap \overline{H x_1}$ is open in $\overline{H x_1}$. Now $\rho^{-1}(\Omega x_0) = \bigcup_{\gamma \in \Gamma} \Omega \gamma x_1$. Therefore by Equation (6), $\Omega x_1$ is open in $\overline{H x_1}$. Hence $H x_1$ is open in $\overline{H x_1}$. This proves (1:⇒).

[1:⇔] There exists $\Gamma_2 \subset \Gamma_1$ which is a normal subgroup of finite index in $\Gamma$. Put $x_2 = e\Gamma_2$. By (1:⇒), $\overline{H x_2} \setminus H x_2$ is closed in $G/\Gamma_2$. Now by Lemma 7.2 applied to $\Gamma_2$ in place of $\Lambda$ and $\Gamma$ in place of $Z$, we have that $\overline{H x_0} \setminus H x_0$ is closed in $G/\Gamma$. This proves (1:⇔).

[2:⇒] This holds because $\rho$ is a proper map.

[2:⇒] Since $\overline{H x_0} \setminus H x_0 = \emptyset$ is closed in $G/\Gamma$, by (1:⇒) $\overline{H x_2} \setminus H x_2$ is closed. By Lemma 7.2 applied to $\Gamma_2$ in place of $\Lambda$ and $\Gamma$ in place of $Z$, we get that $H x_2$ is closed. Now (2:⇒) follows from (2:⇔). \( \square \)

Next we recall some of the properties of actions of unipotent subgroup on homogeneous spaces which will be used in the proof of Theorem 7.1. The properties considered here do not involve the description of invariant measures and orbit closures for such actions.

**Nondivergence of unipotent trajectories on finite volume homogeneous spaces and consequences**

**Theorem 7.4 (Dani)** Let $G$ be a connected Lie group and $\Gamma$ a lattice in $G$. Let a compact set $C \subset G/\Gamma$ and an $\epsilon > 0$ be given. Then there exists a compact set $K \subset G/\Gamma$ such that for any $\text{Ad}_G$-unipotent element $u \in G$ and any $x \in C$ the following holds:

\[ \frac{1}{N} \sum_{n=1}^{N} \chi_K(u^n x) > 1 - \epsilon, \quad \forall N \in \mathbb{N}, \] (7)

where $\chi_K$ denotes the characteristic function of $K$ on $G/\Gamma$.

**Proof** For one-parameter unipotent subgroups, the analogous result is essentially proved in Dani [D4]; see [DM, Theorem 6.1] for details. For the
discrete flows, we extend the action of a cyclic unipotent subgroup to the action of a one-parameter unipotent subgroup as in the beginning of Section 3. Now the analogous result for the one-parameter unipotent subgroup action implies the result for action of a cyclic unipotent subgroup.

Certain observations due to Margulis in [M1], relating to Theorem 6.1 and Moore’s version of Mautner phenomenon lead to the following result.

**Theorem 7.5** ([Sh, Theorem 2.3]) Let $G$ be a Lie group, $\Gamma$ a lattice in $G$, and $\pi : G \to G/\Gamma$ be the natural quotient map. Let $U$ be a subgroup of $G$ generated by $\text{Ad}_G$-unipotent one-parameter subgroups. Then there exists the smallest closed connected subgroup $L$ of $G$ containing $U$ such that $\pi(L)$ is closed. Further $\pi(L)$ admits a finite $L$-invariant measure, which is $U$-ergodic.

**Corollary 7.6** Let the notation be as in Theorem 7.5. Then the following statements hold.

1. There exists $g \in L$ such that $\pi(gUg^{-1}) = \pi(L)$.

2. $\text{Ad}_G(L) \subset \text{Zcl}(\text{Ad}_G(L \cap \Gamma))$.

3. $\pi(N^1_G(L))$ is closed.

**Proof** By Theorem 7.5, we have that $U$ acts ergodically on $\pi(L)$ with respect to a finite $L$-invariant measure. Therefore statement 1 follows from Hedlund’s lemma, statement 2 follows from statement 1 and Proposition 2.4, and statement 3 follows from statement 1 and Proposition 2.5.

Next we consider certain properties of actions of subgroups generated by unipotent elements on finite volume homogeneous spaces of semisimple groups.

**Proposition 7.7** Let $G$ be a Lie group and $\Gamma$ a lattice in $G$. Suppose that $G^0$ is a semisimple group with trivial center, $G = \Gamma G^0$, and $Z_G(G^0) \subset \Gamma$. Let $W \subset G$ be a subgroup such that $\text{Zcl}(\text{Ad}_G(W)) = \text{Zcl}(\text{Ad}_G(\langle U \rangle))$, where $U$ consists of $\text{Ad}_G$-unipotent elements of $W$. Then there exists a homomorphism $\rho : W \to G^0$ such that $\text{Ad}_G(w) = \text{Ad}_G(\rho(w))$ and $wx = \rho(w)x$ for all $x \in G/\Gamma$ and $w \in W$. 

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Proof Since $G^0$ is semisimple, $\text{Ad}_G(u) \in \text{Ad}_G(G^0)$ for all $\text{Ad}_G$-unipotent elements of $G$. Since $\text{Ad}_G(G^0)$ is a connected adjoint semisimple group, it is Zariski closed. Therefore $\text{Ad}_G(W) \subset \text{Ad}_G(G^0)$. Now since the center of $G^0$ is trivial, there exists a homomorphism $\rho : W \to G^0$ such that $\text{Ad}_G(w) = \text{Ad}_G(\rho(w))$ for all $w \in W$.

Let $w \in W$. Put $\delta = w^{-1}\rho(w)$. Then $\delta \in Z_G(G^0) \subset \Gamma$. Now for any $g \in G^0$,

$$\rho(w)\pi(g) = w\delta\pi(g) = w\pi(\delta g) = w\pi(g\delta) = w\pi(g).$$

Since $G/\Gamma = \pi(G^0)$, the above equation holds for all $g \in G$. \hfill \Box

Proposition 7.8 Let $G$ be a connected semisimple (real algebraic) group of real rank $\geq 2$ with trivial center and no nontrivial compact factors. Let $\Gamma$ be an irreducible lattice in $G$ and $x_0 = e\Gamma \in G/\Gamma$. Let $\{U_j\}$ be a collection of unipotent one-parameter subgroups of $G$. For each $j$, let $F_j$ be the smallest closed connected subgroup of $G$ containing $U_j$ such that $F_jx_0$ is closed. Let $F$ be the smallest algebraic subgroup of $G$ containing $F_j$ for all $j$. Then $Fx_0$ is closed, $Fx_0$ has a finite $F$-invariant measure, and the solvable radical of $F$ is unipotent.

In particular, $F$ is the smallest closed connected subgroup of $G$ containing $U_j$ for all $j$ such that $Fx_0$ is closed.

Proof By the arithmeticity theorem of Margulis, there exists a semisimple $\mathbb{Q}$-group $\tilde{G}$ and a surjective homomorphism $\phi : \tilde{G} \to G$ of real algebraic groups such that $\ker\phi$ is compact, and $\rho(\tilde{G}(\mathbb{Z}))$ and $\Gamma$ are commensurable (see [Z]). By Lemma 7.3, without loss of generality we may replace $\Gamma$ by $\rho(\tilde{G}(\mathbb{Z}))$ and assume that $\Gamma = \rho(\tilde{G}(\mathbb{Z}))$. Let $\bar{\phi} : \tilde{G}/\tilde{G}(\mathbb{Z}) \to G/\Gamma$ be the quotient map associated to $\rho$. Then $\bar{\phi}$ is a proper map.

For each $j$ there exists a unipotent one-parameter subgroup $\tilde{U}_j$ in $\tilde{G}$ such that $U_j = \phi(\tilde{U}_j)$. Let $\tilde{x}_0 = \bar{\phi}(x_0)$. Let $\tilde{F}_j$ be the smallest closed connected subgroup of $\tilde{G}$ containing $\tilde{U}_j$ such that $F_j\tilde{x}_0$ is closed. Then by Corollary 7.6 (1) and by [Sa, Prop. 3.2], we have that $\tilde{F}_j$ is a $\mathbb{Q}$-subgroup of $\tilde{G}$. If $J$ is the smallest $\mathbb{Q}$-subgroup of $\tilde{G}$ containing $\tilde{U}_j$ then $J\tilde{x}_0$ is closed. Hence $\tilde{F}_j$ is the smallest $\mathbb{Q}$-subgroup of $\tilde{G}$ containing $\tilde{U}_j$.

Let $\bar{F}$ be the smallest algebraic subgroup of $G$ containing $\bar{F}_j$ for all $j$. Then $\bar{F}$ is the smallest algebraic $\mathbb{Q}$-subgroup of $\tilde{G}$ containing all $\tilde{U}_j$. Therefore the radical of $\bar{F}$ is unipotent, and $\bar{F}\tilde{x}_0$ is closed, and has a finite $\bar{F}$-invariant measure.
Since \( \hat{\phi} \) is a proper map, \( \phi(\hat{F})x_0 \) and \( \phi^{-1}(F)\hat{x}_0 \) are closed. Now since \( U_j \subset \phi(F_j) \) and \( \phi(U_j) \subset F_j \), by the minimality, we have that \( F_j = \phi(\hat{F}_j) \).

Since \( \phi(\hat{F}) \) and \( \phi^{-1}(F) \) are algebraic groups, we have that \( \phi(\hat{F}) = F \).

Since \( \phi \) is proper, \( Fx_0 = \phi(\hat{F}x_0) \) is closed, and has a finite \( F \)-invariant measure. Moreover since \( \ker \phi \) is a compact semisimple Lie group,

\[ \text{Rad}(F) = \phi(\text{Rad}(\hat{F})). \]

Hence \( \text{Rad}(F) \) is unipotent. This completes the proof. \( \square \)

**Proof of Theorem 7.1**

The proof is given through a series of claims.

**Claim 7.8.1** We may assume that \( \Gamma \) is a discrete subgroup of \( G \), and

\[ \text{Ad}_G(G) \subset Z^{\text{cl}}(\text{Ad}_G(\Gamma)). \]

**Proof** Since \( G/\Gamma \) admits a finite \( G \)-invariant measure, by Proposition 2.4, we have that \( \text{Ad}_G(U) \subset Z^{\text{cl}}(\text{Ad}_G(\Gamma)). \) Let

\[ G_1 = \text{Ad}_G^{-1}(Z^{\text{cl}}(\text{Ad}_G(\Gamma))). \]

Then \( W \subset G_1, G_1 \subset N_G(\Gamma^0), \) and \( \Gamma \subset G_1. \) Therefore \( G_1/\Gamma \) admits a finite \( G_1 \)-invariant measure, and hence \( \pi(G_1) \) is closed. Hence replacing \( G \) by \( G_1 \), we may assume that \( \Gamma^0 \) is normal in \( G \). Now without loss of generality we can replace \( G \) by \( G/\Gamma^0 \) and \( \Gamma \) by \( \Gamma/\Gamma^0 \), and assume that \( \Gamma \) is a discrete subgroup of \( G \). Moreover, \( \text{Ad}_G(G) \subset \text{Ad}_G(Z^{\text{cl}}(\Gamma)). \)

Now note that if \( F \) is a closed subgroup of \( G \) containing \( W \) such that \( Fx_0 \) is closed, then \( WF^0x_0 \) is closed, and hence replacing \( F \) by \( WF^0 \) we may assume that \( F = WF^0 \).

Now suppose \( F_i \) (\( i = 1, 2 \)) are closed subgroups of \( G \) such that \( F_i = WF^0_i \) and \( F_i x_0 \) is closed. Then \( Z = F^0_1 x_0 \cap F_2 x_0 \) is an open an closed subset of the closed set \( F_1 x_0 \cap F_2 x_0 \). Again if we put \( F = F_1 \cap F_2 \) then \( F^0z \) is open in \( Z \) for all \( z \in Z \). Therefore \( F^0z \) is closed in \( Z \) for all \( z \in Z \). Hence \( Fx_0 \) is closed. Moreover \( \dim(F^0) < \min(\dim F^0_1, \dim F^0_2) \), unless \( F_1 \subset F_2 \) or \( F_2 \subset F_1 \). This shows the existence of the minimal \( H \) as assumed in the statement of the theorem.

**Claim 7.8.2** We may assume that \( H = H^0(H \cap \Gamma) \).
Proof Take any \( u \in U \). By Theorem 7.4, there exists a compact set \( K \subset G/\Gamma \) such that the set \( \{ k \in \mathbb{N} : u^k x_0 \in K \} \) is infinite. Therefore there exists \( k_1, \ldots, k_n \in \mathbb{N} \) such that \( K \cap (\langle u \rangle H^0 x_0) \subset \bigcup_{i=1}^n u^{k_i} H^0 x_0 \). Therefore there exists \( k \in \mathbb{N} \) such that \( u^k x_0 \in H^0 x_0 \).

Put \( W_0 = \{ w \in W : wx_0 \in H^0 x_0 \} \). Then \( W_0 H^0 x_0 = H^0 x_0 / H^0 x_0 \). Put \( U_0 = W_0 \cap (\bigcup_{u \in U} \langle u \rangle) \).

Then \( \text{Ad}_G(W) \subset \text{Zcl}(\text{Ad}_G(U_0)) \).

Suppose there exists a closed subgroup \( L \) of \( W_0 H^0 \) containing \( W_0 \) such that \( L x_0 \) is closed. Then by Equation 8, we have that \( F = W L \) is a subgroup of \( G \). Hence \( F x_0 \) is closed in \( H x_0 \), and hence in \( G/\Gamma \). Thus \( F = H \), and \( L = H^0 \). This shows that replacing \( W \) by \( W^0 \), we may assume that \( H^0 x_0 = H^0 x_0 / H^0 x_0 \). This completes the proof of the claim.

In particular, we may assume that \( G = W G^0 \), and \( W x_0 \subset G^0 x_0 \). Thus we may assume that \( G = G^0 \Gamma \).

Claim 7.8.3 We may assume that is no proper subgroup \( L \) of \( G \) containing \( W \) such that \( L x_0 \) has finite \( L \)-invariant measure.

Proof Let \( L \) be such a subgroup. Then the claim follows from replacement of \( G \) by \( L \).

Projecting to semisimple factors

Let \( R \) be the connected solvable radical of \( G \). Put \( R' = R R^0 \). By Auslander’s theorem [8, Theorem 8.24] \( R' \) is solvable. By Zariski density of \( \Gamma \), \( R' \) is normal in \( G \). Therefore \( R = R' \). Hence \( R \Gamma \) is closed. Now since \( R \) is normalized by \( \Gamma \), \( R \cap \Gamma \) is a lattice in \( R \) (see [8, Theorem 1.13]). Therefore by Mostow’s theorem [8, Theorem 3.1], \( R / (R \cap \Gamma) \) is compact.

Let \( \tilde{C} \) be the product of all maximal connected compact normal subgroups of \( G/R \), and let \( \tilde{Z} \) be the center of \( (G^0/R)/\tilde{C} \). Since \( (G^0/R)/\tilde{C} \) is semisimple and \( G = W G^0 \), we have that \( \tilde{Z} \) is central in \( (G/R)/\tilde{C} \). Put \( \tilde{G} = ((G/R)/\tilde{C})/\tilde{Z} \). Then \( G^0 \) is a semisimple group with trivial center and no nontrivial compact normal subgroups. Let \( \sigma : G \to \tilde{G} \) be the natural quotient homomorphism.

Put \( \bar{\Gamma} = \sigma(\Gamma) \). Then \( \bar{\Gamma} \) is a lattice in \( \tilde{G} \), and \( \sigma^{-1}(\bar{\Gamma})/\Gamma \) is compact. Let \( \Lambda = \{ g \in \tilde{G} : g(\tilde{G} \cap \bar{\Gamma}) g^{-1} \subset \tilde{G} \cap \bar{\Gamma} \} \). Then \( \Lambda / \bar{\Gamma} \) is compact. Let
\[ \tilde{\sigma} : G/\Gamma \to \tilde{G}/\Lambda \] be the natural quotient map. Then \( \tilde{\sigma} \) is a proper map. Put \( \bar{x}_0 = \tilde{\sigma}(x_0) \).

**Claim 7.8.4** It is enough to show that \( \sigma(H)^0 \bar{x}_0 \) admits a finite \( \sigma(H) \) invariant measure.

**Proof** Let \( \mu_H \) denote a locally finite \( H^0 \)-invariant Borel measure on \( H^0 x_0 \). Since \( \tilde{\sigma} \) is a proper map, the projected measure \( \bar{\mu}_H = \tilde{\sigma}_*(\mu_H) \) on \( \sigma(H^0) \bar{x}_0 \) is a locally finite and \( \sigma(H^0) \)-invariant. Now the claim follows from the uniqueness, up to constant multiple, of the locally finite \( \sigma(H^0) \)-invariant measures on \( \sigma(H^0) \bar{x}_0 \).

Since \( Z_{\tilde{G}}(\Lambda \cap \tilde{G}^0) \subset Z_{\tilde{G}}(\Gamma \cap \tilde{G}^0) \subset \Lambda \), by Proposition 7.7, there exists a homomorphism \( \rho : W \to \tilde{G}^0 \) such that \( \text{Ad}_{\tilde{G}}(\sigma(w)) = \text{Ad}_{\tilde{G}}(\rho(w)) \) and \( \sigma(w)x = \rho(w)x \) for all \( w \in W \) and \( x \in \tilde{G}/\Lambda \).

There exist closed normal subgroups \( G_1, \ldots, G_k \) of \( \tilde{G}^0 \) such that \( \tilde{G}^0 = G_1 \times \cdots \times G_k \), and if \( p_i : \tilde{G}^0 \to G_i \) is the projection on the \( i \)-th factor, then \( \Lambda_i = p_i(\Lambda) \) is an irreducible lattice in \( G_i \). Let \( \tilde{\rho}_i : \tilde{G}^0/(\tilde{G}^0 \cap \Lambda) \to G_i/\Lambda_i \) be the natural quotient map. Put \( \sigma_i = p_i \circ \sigma_{|\tilde{G}^0} \) and \( x_i = \tilde{\rho}_i(\tilde{\sigma}(x_0)) \).

Fix any \( i \in \{1, \ldots, k\} \). Let \( U \) be the subgroup of \( G_i \) generated by the one-parameter unipotent subgroups \( \{u(t)\} \) associated to all \( u \in U \) such that \( p_i(\rho(u)) = u(1) \). Let \( L_i \) be the smallest closed connected subgroup of \( G_i \) containing \( U \) such that \( L_i x_i \) is closed. Then by Theorem 7.3, \( L_i x_i \) has finite \( L_i \)-invariant measure.

**Claim 7.8.5** \( L_i = G_i \).

**Proof** Since the fibers of \( \tilde{\rho}_i \) have finite invariant measures, we have that \( p_i^{-1}(L_i) \bar{x}_0 = \tilde{\rho}_i(L_i x_i) \) has a finite \( p_i^{-1}(L_i) \)-invariant measure. Therefore \( \sigma(W)p_i^{-1}(L_i) \bar{x}_0 = p_i^{-1}(L_i) \bar{x}_0 \) has a finite \( \sigma(W)p_i^{-1}(L_i) \)-invariant measure. Again \( W(p_i \circ \sigma)^{-1}(L_i) x_0 = \sigma^{-1}(\sigma(W)p_i^{-1}(L_i)) x_0 \) has a finite \( W(p_i \circ \sigma)^{-1}(L_i) \)-invariant measure. Now the claim follows from Claim 7.8.3.

**Claim 7.8.6** The radical of \( H_i = \sigma_i(H^0) \) is compact for all \( 1 \leq i \leq k \).
Proof. For any \( u \in \mathcal{U} \), there exists a closed connected subgroup \( F_u \) of \( G \) such that \( F_u x_0 \) has a finite \( F_u \)-invariant measure, \( u \in N_G(F_u) \), and \( \langle u \rangle x_0 = \langle u \rangle F_u x_0 \). Therefore \( F_u \subset H \).

Now \( \sigma_i(F_u) x_i = \tilde{p}_i(\sigma(F_u) x_0) \). Hence \( \sigma_i(F_u) x_i \) has a finite \( \sigma_i(F_u) \)-invariant measure. Therefore \( \sigma_i(F_u) x_i \) is closed. Let \( \{u(t)\} \) be the one-parameter unipotent subgroup of \( G \) such that \( p_i(\rho(u)) = u(1) \). Since \( \text{Zcl}(\langle u(1) \rangle) = \{u(t)\} \), we have that \( \{u(t)\} \subset N_G(\sigma_i(F_u)) \) and

\[
\{u(t)\} \subset N_G(H_i). \tag{9}
\]

Case of \( \mathbb{R} \)-rank\( (G_i) \geq 2 \): In this case by Proposition 7.8, the group \( L_i \) is the Zariski closure of the subgroup generated by \( \{u(t)\} \sigma_i(F_u) \) for all \( u \in \mathcal{U} \). Therefore by Equation 9 and Claim 7.8.5, \( H_i \) is a normal subgroup of \( G_i \). Hence the claim holds in this case.

Case of \( \mathbb{R} \)-rank\( (G_i) = 1 \): Put \( F = N_{G_i}(H_i) \). Then \( U \subset F \). Now suppose that the unipotent radical of \( F \) is nontrivial. Since \( \text{rank}(G_i) = 1 \), \( F \) is contained in a unique minimal parabolic subgroup, say \( P \) of \( G \). Let \( N \) denote the unipotent radical of \( P \). Then \( U \subset N \) and \( \sigma_i(H) \subset P \). Hence \( \{u(t)\} \sigma_i(F_u) \subset P \) for all \( u \in \mathcal{U} \). Let \( M = Z_{G_i}(A)^0 \). Since

\[
\{u(t)\} x_i = \{u(t)\} \sigma_i(F_u) x_i,
\]

we have that \( \{u(t)\} \sigma_i(F_u) \subset MN \). Therefore \( MN \cap \Lambda_i \neq \{e\} \). Hence \( N x_i \) is compact. Therefore \( \overline{U x_i} \subset N x_i \), and hence

\[
N = L_i = G_i,
\]

which is a contradiction. Thus \( F \) is a reductive subgroup of \( G_i \). Since \( \{e\} \neq U \subset F \), we obtain that the solvable radical of \( H_i \) is compact. This completes the proof of the claim.

Completion of the proof of the theorem Since \( \tilde{\sigma} \) is a proper map, \( \sigma(H^0)\tilde{x}_0 \) is closed. By the Claim 7.8.6, the radical of \( \sigma(H^0) \) is compact. Now by [M3, Theorem 15], \( \sigma(H^0)\tilde{x}_0 \) has a finite \( \sigma(H^0) \)-invariant measure. In view of Claim 7.8.4, this completes the proof of the theorem. \( \square \)

Remark 7.9 We may note that the above results in this section are independent of the classification of ergodic invariant measures and orbit closures for unipotent flows.
Limits of ergodic invariant measures for a discrete unipotent flow

Our proof of Theorem 1.3 is based on the following result.

**Theorem 7.10 (Mozes and Shah)** Let \( G \) be a Lie group and \( \Gamma \) a discrete subgroup of \( G \) such that \( G = G^0 \Gamma \). Let \( u \in G \) be an \( \text{Ad}_G \)-unipotent element. Let \( \mu_i \) be a sequence of \( u \)-invariant \( u \)-ergodic probability measures on \( G/\Gamma \) and such that \( \mu_i \) converges to a probability measure \( \mu \) on \( G/\Gamma \) in the weak*-topology. Suppose that \( x_0 = e\Gamma/\Gamma \in \text{supp}(\mu) \). Then there exists a closed subgroup \( H \) of \( G \) such that the following holds:

1. \( \mu \) is \( L \)-invariant and \( \text{supp}(\mu) = Lx_0 \).
2. For any sequence \( g_i \to e \) in \( G \) such that \( \langle u \rangle g_i x_0 = \text{supp}(\mu_i) \) (such sequences exist due to Hedlund’s lemma), we have
   \[
   g_i^{-1}u g_i \in L, \quad \forall i \gg 0.
   \]
   (Here \( \forall i \gg 0 \) stands for the expression ‘for all \( i \geq i_0 \), for some \( i_0 > 0 \).’)
   In other words, \( u \in L \) and \( g_i \text{supp}(\mu) \subset \text{supp}(\mu), \forall i \gg 0 \).
3. \( \text{Ad}_G(L) \subset \text{Zcl}(\text{Ad}_G(L \cap \Gamma)) \).
4. \( L = \langle u \rangle L^0 \).

**Proof** Using the method of Section 3, we extend the action of \( \langle u \rangle \) to the action of a one-parameter unipotent subgroup. For the action of a one-parameter unipotent subgroup, the analogous result holds [MS, Theorem 1.1]. From that we deduce statements 1 and 2 of the theorem using the intersection with \( \mathbb{Z} \cdot G_0 \) as in Section 3.

By Proposition 2.4, \( \text{Ad}_G(g_i^{-1}ug_i) \subset \text{Zcl}(\text{Ad}_G(L \cap \Gamma)) \) for all \( i \gg 0 \). Put \( L_1 = L \cap \text{Ad}_G^{-1}(\text{Zcl}(\text{Ad}_G(L \cap \Gamma))) \). Then \( g_i^{-1}ug_i \in L_1 \) for all \( i \gg 0 \), and \( L_1x_0 \) is closed. Therefore \( g_i^{-1}\text{supp}(\mu_i) = g_i^{-1}\langle u \rangle g_i x_0 \subset L_1x_0 \). Hence \( \text{supp}(\mu) \subset L_1x_0 \). This shows that \( L \subset L_1 \), which proves statement 3.

To obtain the last statement, note that \( u^{-1}g_i^{-1}ug_i \to e \). Therefore \( u^{-1}g_i^{-1}ug_i \in L^0 \) for all \( i \gg 0 \). Put \( L_2 = \langle u \rangle L^0 \). Since \( L^0x_0 \) is closed and \( u \in L \), we have that \( L_2x_0 \) is closed, and \( g_i^{-1}ug_i \in L_2 \) for all \( i \gg 0 \). Hence \( \text{supp}(\mu) \subset L_2x_0 \). This shows that \( L_2 = L \). This proves statement 4. \( \square \)
Corollary 7.11 Let $G$, $\Gamma$, $x_0$, and $u$ be as in be as in Theorem 7.10. Suppose further that $\Gamma$ is a lattice in $G$. Let $g_i \to e$ be any sequence in $G$. Then after passing to a subsequence, there exists a closed subgroup $L$ of $G$ such that the following holds:

1. $g_i^{-1}ug_i \subset L$ for all $i > 0$.
2. $Lx_0 \subset \bigcup_{i \geq i_0} \langle u \rangle g_i x_0$ for any $i_0 > 0$.
3. $\text{Ad}_G(L) \subset \text{Zcl}(\text{Ad}_G(L \cap \Gamma))$ and $L \cap \Gamma$ is a lattice in $L$.
4. $L = \langle u \rangle L^0$.

Proof By theorem 3.2, for each $i > 0$, the trajectory $\{u^n g_i x_0 : n > 0\}$ is uniformly distributed with respect to a probability measure $\mu_i$ such that $\langle u \rangle g_i x_0 = \text{supp}(\mu_i)$. By theorem 7.4, given any $\epsilon > 0$ there exists a compact set $K \subset G/\Gamma$ such that $\mu_i(K) > 1 - \epsilon$ for all $i > 0$. Therefore after passing to a subsequence, there exists a probability measure $\mu$ on $G/\Gamma$ such that $\mu_i \to \mu$. Now the conclusion of the corollary follows immediately from Theorem 7.10. $\square$

8 Proof of Theorem 1.3

Let $\pi : G \to G/\Gamma$ be the natural quotient map. Put $Y = \overline{Wx}$. Let $g \in \pi^{-1}(x)$. Then replacing $Y$ by $g^{-1}Y$ and $W$ by $g^{-1}Wg$, without loss of generality we may assume that $Y = \overline{W}$.

We argue as in the proof of Claim 7.8.1 and assume that $\Gamma$ is discrete.

By Claim 6.2.1 we may assume that $\mathcal{U}$ is finite. Using the arguments of the proof of Claim 7.8.2, we may assume that $Y \subset \pi(G^0)$ and $G = W G^0 = \Gamma G^0$.

Now we divide Theorem 1.3 into the following two complementary theorems.

Theorem 8.1 Let the notation and conditions be as above. Then there exists a closed subgroup $M$ of $G$ containing $W$ such that $\pi(W)$ is $M$-invariant and $\pi(M)$ is open in $\pi(W)$.

Theorem 8.2 Let the notation be as above. Suppose there exists a closed subgroup $M$ of $G$ containing $W$ such that $\pi(W)$ is $M$-invariant, and $\pi(M)$ is open in $\pi(W)$. Then $\pi(W) = \pi(M)$. Also $W$ acts ergodically with respect to a locally finite $M$-invariant measure on $\pi(M)$. 

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We will prove Theorem 8.1 by closely following the arguments of the proof of Theorem 6.2. While the proof of Theorem 8.2 involves Theorem 7.10 on limits of ergodic invariant measures for unipotent flows as a main new ingredient.

**Proof of Theorem 8.1**

The proof is given via a series of claims, which are reductions to special cases, and propositions.

**Claim 8.2.1** We may assume that $G$ contains no proper closed subgroup $F$ containing $W$ such that $\pi(F)$ is closed.

**Proof** By Theorem 7.1 there exists a smallest closed subgroup $F$ of $G$ containing $W$ such that $\pi(F)$ is closed. Moreover, $\pi(F^0)$ has a finite $F^0$-invariant measure. As in Claim 7.8.2 we may assume that $F = WF^0 = F^0(F \cap \Gamma)$. Therefore, without loss of generality, we may replace $G$ by $F$. Now the claim follows. ✷

**Proposition 8.3** There exist $H_u \in \mathcal{H}_u$ for each $u \in \mathcal{U}$ such that $\pi(\cap_{u \in \mathcal{U}} N(H_u, u))$ contains an open subset of $Y$, say $\Psi$, and $\cup_{u \in \mathcal{U}} \pi(S(H_u, u))$ does not contain any open subset of $\Psi$.

**Proof** Let $\Omega$ be any nonempty open subset of $\pi^{-1}(Y) \cap G^0$. Express $\mathcal{U} = \{u_1, \ldots, u_k\}$ for some $k \in \mathbb{N}$. For any $H \in \mathcal{H}_{u_1}$, there exist compact sets $\{C_i(H)\}_{i \in \mathbb{N}}$ such that $C_i(H) \subset C_{i+1}(H)$ for all $i \in \mathbb{N}$, and

$$N(H, u_1) = \cup_{i \in \mathbb{N}} C_i(H).$$

By Theorem 6.2 for every $g \in \Omega$, there exists $H \in \mathcal{H}_{u_1}$ such that $g \in N(H, u_1)$. Hence

$$\pi(\Omega) \subset \bigcup_{H \in \mathcal{H}_{u_1}} \bigcup_{i \in \mathbb{N}} \pi(C_i(H)).$$

By Proposition 8.3, $\mathcal{H}_{u_1}$ is countable. Since $\pi(\Omega)$ is an open subset of $Y$, by Baire’s category theorem, $\pi(C_i(H))$ contains a nonempty open subset of $\pi(\Omega)$ for some $H \in \mathcal{H}_{u_1}$ and some $i \in \mathbb{N}$. Furthermore we can choose the $H$
with an additional property that for any \( F < H \), the set \( \pi(C_j(F)) \) does not contain any nonempty open subset of \( \pi(\Omega) \) for any \( j \in \mathbb{N} \). Then there exists a nonempty open subset \( \Omega' \) of \( \Omega \) such that \( \pi(\Omega') \subset \pi(N(H, u_1)) \) and, due to Baire’s category theorem, \( \pi(S(H, u_1)) \) does not contain any nonempty open subset of \( \pi(\Omega) \).

Now by Baire’s category theorem, there exists \( \gamma_1 \in \Gamma \) such that \( N(H, u_1) \) contains a nonempty open subset \( \Omega_1 \) of \( \Omega' \). Put \( H_{u_1} = \gamma_1^{-1} H \gamma_1 \). Then \( \Omega_1 \subset N(H_{u_1}, u_1) \). Since \( \pi(S(H_{u_1}, u_1)) = \pi(S(H, u_1)) \), we have that

\[
\pi(S(H_{u_1}, u_1))
\]
does not contain a nonempty open subset of \( \pi(\Omega_1) \).

For each \( 2 \leq i \leq k \), repeating this procedure for \( u_i \) in place of \( u_1 \) and \( \Omega_1 \) in place of \( \Omega \), we obtain \( H_i \subset N(H_{u_i}, u_i) \) contains a nonempty open subset \( \Omega_i \) of \( \Omega' \), and \( \pi(S(H_{u_i}, u_i)) \) does not contain a nonempty open subset of \( \pi(\Omega_i) \).

Thus \( \cap_{1 \leq i \leq k} N(H_{u_i}, u_i) \) contains a nonempty open subset \( \Omega_0 = \Omega_k \) of \( \Omega \) and, due to Baire’s category theorem, \( \cup_{i=1}^k \pi(S(H_{u_i}, u_i)) \) does not contain a nonempty open subset of \( \Psi = \pi(\Omega_0) \).

\[\blacksquare\]

Claim 8.3.1 We may assume that \( e \in \cap_{u \in \mathcal{U}} N^*(H_u, u) \), \( \overline{\pi(W)} = Y \), and \( \cap_{u \in \mathcal{U}} N(H_u, u) \) contains a neighbourhood of \( e \) in \( \pi^{-1}(Y) \).

**Proof** By Proposition 8.3, there exists \( g \in \cap_{u \in \mathcal{U}} N^*(H_u, u) \) such that the following holds: \( W \pi(g) = Y \) and \( \cap_{u \in \mathcal{U}} N(H_u, u) \) contains a neighbourhood of \( g \) in \( \pi^{-1}(Y) \). If we replace \( \Gamma \) by \( g \Gamma g^{-1} \) and \( H_u \) by \( g H_u g^{-1} \), the claim follows. \[\blacksquare\]

**Proposition 8.4** Let \( L \) be a closed connected subgroup of \( G \). If \( \pi(L) \subset Y \), then the following holds: \( L \subset \cap_{u \in \mathcal{U}} N(H_u, u) \) and \( L \cap \cap_{u \in \mathcal{U}} N^*(H_u, u) \) is dense in \( L \).

**Proof** By Claim 8.3.1, we have that \( \cap_{u \in \mathcal{U}} N(H_u, u) \) contains a neighbourhood of \( e \) in \( L \). Therefore by Lemma 5.3, we have that \( L \subset \cap_{u \in \mathcal{U}} N(H_u, u) \).

Now suppose that \( \cup_{u \in \mathcal{U}} S(H_u, u) \) contains a nonempty open subset of \( L \). Then by Baire’s category theorem, there exists \( u \in \mathcal{U} \) and \( F < H_u \) such that \( N(F, u) \) contains a nonempty open subset of \( L \). Therefore by Lemma 5.3, \( L \subset N(F, u) \). In particular \( e \in S(H_u, u) \), which contradicts Claim 8.3.1. Therefore \( \cap_{u \in \mathcal{U}} N^*(H_u, u) \) contains a dense subset of \( L \). \[\blacksquare\]
Proposition 8.5 \( H_v^0 \subset \cap_{u \in \mathcal{U}} N(H_u, u) \).

**Proof** By Claim 8.3.1 and Proposition 4.1, for any \( v \in \mathcal{U} \),
\[
\pi(H_v^0) \subset \pi(\langle v \rangle) \subset Y.
\]
Therefore the present proposition follows from Proposition 8.4. \(\square\)

**Definition** Let \( U_u \) and \( F_u \) be as defined before the statement of Proposition 6.9. Using Proposition 8.5, in place of Proposition 6.7, we conclude that Proposition 6.9 is valid. Also Propositions 6.10 and 6.11 are valid; that is, \( F \) is a closed subgroup generated by all \( F_u \) (\( u \in \mathcal{U} \)), \( \pi(F) \) is closed and has a finite invariant measure, and
\[
W \subset N_G^1(F).
\]

Proposition 8.6 \( Y \) is \( F \)-invariant.

**Proof** Express \( \mathcal{U} = \{ u_1, \ldots, u_k \} \). For \( 1 \leq i \leq k \), put \( L_i = F_{u_1} \cdots F_{u_i} \).
Since \( e \in N^*(H_{u_1}, u_1) \), by Proposition 4.1, we have
\[
\pi(L_1) \subset \pi(H_{u_1}) = \pi(\langle u_1 \rangle) \subset Y.
\]
Since \( F = L_k \), to prove that \( \pi(L_k) \subset Y \) by induction, we assume that \( \pi(L_i) \subset Y \) for some \( 1 \leq i \leq k - 1 \). Put
\[
L_i^* = L_i \cap (\cap_{u \in \mathcal{U}} N^*(H_u, u)).
\]
Then for \( f \in L_i^* \), by Proposition 4.1,
\[
f \pi(F_{u_{i+1}}) \subset f \pi(H_{u_{i+1}}) = \langle u_{i+1} \rangle \pi(f) \subset Y.
\]
By Proposition 8.4, \( L_i^* \) is dense in \( L_i \). Hence \( \pi(L_{i+1}) = L_i^* \pi(F_{u_{i+1}}) \subset Y \).
Therefore \( \pi(F) \subset Y \).

Since \( W \in N_G(F) \), we have that
\[
FY = F \pi(W) \subset \pi(FW) = W \pi(F) \subset WW = Y.
\]
By Proposition 2.5, \( \pi(N_G^1(F)) \) is closed. Also \( W \subset N_G^1(F) \). Therefore by Claim 8.2.1, without loss of generality we may assume that \( F \) is a normal subgroup of \( G \).
Claim 8.6.1  We may assume that $Z_G(W)$ contains a neighbourhood of $e$ in $\pi^{-1}(Y)$.

Proof  Due to Proposition 8.4, without loss of generality we can pass to the quotient $G/F\Gamma$ and assume that $F = \{e\}$. Now arguing as in the proof of Claim 6.12.1, we conclude that $Z_G(W)$ contains a neighbourhood of $e$ in $\cap_{u \in U} N(H_u, u)$. Now since $\cap_{u \in U} N(H_u, u)$ contains a neighbourhood of $e$ in $\pi^{-1}(Y)$, the claim follows.

Completion of the proof of the theorem Define $M_1 = \{z \in Z_G(W) : \pi(z) \in \pi(W)\}$. Then by Claim 8.6.1, we have that the closure $\overline{M_1}$ contains a neighbourhood of $e$ in $\pi^{-1}(Y)$. Now for any $z \in M_1$,

$$Y = \overline{W\pi(z)} = \overline{z\pi(W)} = zY.$$  \hspace{1cm} (10)

Therefore if $M$ is the closure of the subgroup generated by $M_1$ and $W$. Then $MY = Y$ and $\pi(M)$ is open in $Y$. This completes the proof of the theorem.

Proof of Theorem 8.2
We intend to prove this theorem by induction on $\dim(G)$. For $\dim(G) = 0$ the theorem is trivial.

Put $Y = \overline{\pi(W)}$. Without loss of generality we may assume that

$$M = \{g \in G : gY = Y\}. \hspace{1cm} (11)$$

Put $Y_1 = Y \setminus \pi(M)$. Note that $Y_1$ is $M$-invariant. We want to show that $Y_1 = \emptyset$. Suppose that

$$Y_1 \neq \emptyset. \hspace{1cm} (12)$$

Then arguing as in Proposition 8.3, for each $u \in U$ there exists $H_u \in \mathcal{H}$ such that $\cap_{u \in U} N(H_u, u)$ contains a nonempty open subset of $\pi^{-1}(Y_1)$, say $\Psi$, such that $\pi(\cup_{u \in U} S(H_u, u))$ does not contain any open subset of $\pi(\Psi)$. Let $g \in \Psi \setminus \cup_{u \in U} S(H_u, u)$. Replacing $\Gamma$ by $g\Gamma g^{-1}$ and putting $x = \pi(g^{-1})$, without loss of generality we may assume that $Y = \overline{Wx}$, $Y_1 = Y \setminus Mx$, and

$$e \in \cap_{u \in U} N^*(H_u, u).$$

Thus $\pi((u)) = \pi(H_u)$. 

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Just as in Claim 8.2.1, we may assume that there is no proper closed subgroup $F$ of $G$ containing $W$ such that $Fx$ is closed.

In view of (10), (11) and (12), there exists a sequence $g_i \to e$ in $G^0 \setminus Z_G(W)$ such that $\pi(g_i) \in Wx$ for all $i \in \mathbb{N}$. By Corollary 7.11, after passing to a subsequence, we have the following: For any $u \in \mathcal{U}$, there exists a closed subgroup $L_u$ of $G$ containing $u$ such that $\pi(L_u) \subset Y$,

$$g_i^{-1}ug_iu^{-1} \in L_u^0, \forall i \gg 0,$$

(13)

$L_u \cap \Gamma$ is a lattice in $L$, and

$$\text{Ad}_G(L_u) \subset \text{Zcl}(\text{Ad}_G(L_u \cap \Gamma)).$$

Since $L_u = \langle u \rangle L_u^0$ and $\pi(\langle u \rangle) = H_u$, we deduce that

$$L_u \cap \Gamma = (L_u^0 \cap \Gamma)(H_u \cap \Gamma).$$

Hence

$$\text{Ad}_G(L_u) \subset \text{Zcl}(\text{Ad}_G(L_u^0 \cap \Gamma)(H_u \cap \Gamma)).$$

(14)

We claim that $\pi(L_u) \subset Y_1$. Because, if $Mx$ contains a nonempty open subset of $\pi(L_u) \subset Y$, then $L_u^0 \subset M$, and hence $L_u \subset M$. But

$$\pi(M) \cap Mx = \emptyset,$$

which proves the claim.

Now by arguments as those involved in the proof of Proposition 8.3, we have that

$$L_u^0 \subset \cap_{v \in \mathcal{U}} N(H_v, v).$$

(15)

For any $v \in \mathcal{U}$, let $U_v$ denote the subgroup generated by all one-parameter $\text{Ad}_G$-unipotent subgroups of $H_v$, and $F_v$ denote the closed connected subgroup of $H_v$ such that $\pi(U_v) = \pi(F_v)$. Then $F = \prod_{v \in \mathcal{U}} F_v$ is a closed subgroup of $G$. Also $Fx$ admits a finite $F$-invariant measure, and $\pi(N_G(F))$ is closed.

Claim 8.6.2 We may assume that $F$ is a normal subgroup of $G$.

Proof Since $e \in \cap_{v \in \mathcal{U}} N^*(H_v, v)$, by (13) and by the arguments as in the proof of Proposition 8.3, we have that

$$L_u^0 \cap \Gamma \subset N_G(H_v^0) \cap N_G(F_v), \forall v \in \mathcal{U}.$$ 

(16)
On the other hand, by the arguments as those involved in Proposition 6.11, we have \( v \in N_G(F) \) for all \( v \in U \). Since \( \pi(N_G(F)) \) is closed, we have \( H_v \subset N_G(F) \) for all \( v \in U \). Therefore by (14),

\[
L_u \subset N_G(H_v^0) \cap N_G(F).
\] (17)

Since \( g_i^{-1}ug_i \in L_u \) for all \( i \gg 0 \), we have that \( u \in g_iN_G(F)g_i^{-1} \) for all \( i \gg 0 \). Therefore \( W \subset g_iN_G(F)g_i^{-1} \) for all \( i \gg 0 \). For each \( i > 0 \) there exists \( w_i \in W \) such that \( w_ix = \pi(g_i) \). Therefore \( W \subset (w_i^{-1}g_iN_G(F)g_i^{-1}w_i) \), and \( (w_i^{-1}g_iN_G(F)g_i^{-1}w_i)x = w_i^{-1}g_i\pi(N_G(F)) \) is closed. Therefore by an assumption made earlier in the proof, \( w_i^{-1}g_iN(F)g_i^{-1}w_i = G \); or in other words, \( F \) is a normal subgroup of \( G \).  

Let \( \bar{G} = G/F \) and \( \rho : G \to \bar{G} \) be the quotient homomorphism. Note that \( \pi(F) \) is closed. Therefore \( \bar{\Gamma} = \rho(\Gamma) \) is a discrete subgroup of \( \bar{G} \). Let \( \bar{\pi} : G \to G/\bar{\Gamma} \) be the natural quotient. Note that if \( \Gamma \) is a lattice in \( G \), then \( \bar{\Gamma} \) is also a lattice in \( \bar{G} \).

**Claim 8.6.3** We may assume that \( \rho(H_v^0) \) contains no nontrivial \( \text{Ad}_{\bar{G}} \)-unipotent one-parameter subgroup for all \( v \in U \).

**Proof** If it does, then by the same arguments as before on \( \bar{G}/\bar{\Gamma} \), we can go modulo another subgroup like \( F \) containing all the one-parameter \( \text{Ad}_{\bar{G}} \)-unipotent subgroups of all \( H_v \).  

By (15) and the arguments as in the proof of Claim 6.12.1, we have

\[
\rho(L_u^0) \subset Z_G(\rho(W)), \quad \forall u \in U.
\] (18)

Therefore by (13), we have that \( \rho(g_i^{-1}ug_i) \) and \( \rho(u) \) commute with each other. Therefore \( \rho(g_i^{-1}ug_iu^{-1}) \) is an \( \text{Ad}_{\bar{G}} \)-unipotent element for all \( i \gg 0 \). Since \( g_i \to e \), for each \( u \in U \) and each \( i \gg 0 \), there exists a unique one-parameter \( \text{Ad}_{\bar{G}} \)-unipotent subgroup \( \{u_i(t)\} \) such that

\[
u_i(1) = \rho(g_i^{-1}ug_iu^{-1}) \].

Therefore \( \{u_i(t)\} \subset \rho(L_u^0) \) for all \( i \gg 0 \). Now since \( \bar{\pi}(\rho(L_u^0)) \) has a finite \( \rho(L_u^0) \)-invariant measure, by Proposition 2.6 and by (16), we have

\[
\{u_i(t)\} \subset N_{\bar{G}}(\rho(H_v^0)), \quad \forall u, v \in U, \forall i \gg 0.
\]

Now by Claims 8.6.3, Proposition 2.6 and (18), we have

\[
\{u_i(t)\} \subset Z_{\bar{G}}(\rho(H_v)), \quad \forall u, v \in U, \forall i \gg 0.
\] (19)
Put
\[ S_1 = \left[ \bigcap_{v \in U} Z_G(\rho(H_v)) \right]^{0} \subset Z_G(\rho(W)). \] (20)

By Lemma 2.3, $\bar{\pi}(S_1)$ is closed. Also $\{u_i(t)\} \subset S_1$ for all $u \in U$ and for all $i \gg 0$. Let $S$ be the smallest closed connected subgroup of $S_1$ such that $\bar{\pi}(S)$ is closed, and $\{u_i(t)\} \subset S$ for all $u \in U$ and all $i \gg 0$.

**Proposition 8.7** \( \text{Ad}_G(S) \subset \text{Zcl}(\text{Ad}_G(S \cap \bar{\Gamma})). \)

**Proof** Since $\bar{\pi}(\rho(L_u^0))$ has a finite $\rho(L_u^0)$-invariant measure, by Theorem 7.5, for each $i \gg 0$, there exists a smallest closed connected subgroup $S_{u, i}$ of $L_u^0$ containing $\{u_i(t)\}$ such that $\bar{\pi}(S_{u, i})$ is closed. And by Corollary 7.6(2),
\[ \text{Ad}_G(S_{u, i}) \subset \text{Zcl}(\text{Ad}_G(S_{u, i} \cap \bar{\Gamma})), \quad \forall u \in U, \forall i \gg 0. \] (21)
Since $\bar{\pi}(S)$ is closed, $\bar{\pi}(S \cap \rho(L_u^0))$ is closed. Therefore by minimality,
\[ S_{u, i} \subset S \quad \text{for all } u \in U \text{ and all } i \gg 0. \]

Put
\[ S' = S \cap \text{Ad}_G^{-1} \text{Zcl}(\text{Ad}_G(S \cap \bar{\Gamma})). \]

Since $\bar{\pi}(S)$ is closed, we have that $\bar{\pi}(S')$ is closed. Since $\{u_i(t)\} \subset S'$ for all $u \in U$ and all $i \gg 0$, by minimality, $S' = S$. \( \square \)

**Claim 8.7.1** We may assume that $S$ is central in $\bar{G}$.

**Proof** By definition of $S$ and Corollary 7.6(3), $\bar{\pi}(N^1_G(S))$ is closed. By (20), we have $\rho(g_i^{-1}ug_i) \in N^1_G(S)$ for all $u \in U$ and $i \gg 0$. Therefore by Zariski density, $\rho(g_i^{-1}Wg_i) \subset N^1_G(S)$ for all $i \gg 0$. Now by the arguments at the end of the proof of Claim 8.6.2, without loss of generality we may assume that $S$ is a normal subgroup of $\bar{G}$. Therefore by Proposition 8.7 and Lemma 2.3, we have $Z_G(S)\bar{\pi}(\rho(g^{-1}))$ is closed. Note that $\rho(W) \subset Z_G(S)$ and $x = \pi(g^{-1})$. Therefore by an earlier assumption, $\bar{G} = Z_G(S)$. \( \square \)

**Proposition 8.8** The orbit $\bar{\pi}(S)$ is compact, and there exists a closed subgroup $T$ of $G$ containing $\rho(W)$ such that $\bar{\pi}(\rho(W)) = \bar{\pi}(T)$. In particular, $\bar{\pi}(ST)$ is closed.
Proof By Claim 8.7.1, we have that $S \cong \mathbb{R}^k$ for some $k$ and $S \cap \bar{\Gamma} \cong \mathbb{Z}^r$ for some $0 \leq r \leq k$. Then for all $u \in U$ and all $i \gg 0$, we have that $S_{u,i}$ is contained in the subgroup of $S$ corresponding to the $\mathbb{R}^r$ with respect to the above isomorphisms. Therefore by the definition of $S$, $S \cong \mathbb{R}^r$; that is $\bar{\pi}(S)$ is compact.

By Theorem 8.1, there exists a closed subgroup $T$ of $\bar{G}$ containing $\rho(W)$ such that $\bar{\pi}(\rho(W))$ is $T$-invariant and $\bar{\pi}(T)$ is open in $\bar{\pi}(\rho(W))$.

Since $g_i \notin Z_G(W)$ for all $i > 0$, we have that $\dim \bar{G}/S < \dim G$. Therefore by the induction hypothesis applied to $\bar{G}/S$, we conclude that $\bar{\pi}(T)$ is closed; that is, $\bar{\pi}(\rho(W)) = \bar{\pi}(T)$. This completes the proof of the proposition.

Completion of the proof of the theorem Since $\rho(g_i^{-1}ug_iu^{-1}) \in S$ for all $u \in U$ and all $i \gg 0$ and $S$ is normal in $\bar{G}$, we deduce that $\rho(g_i^{-1}ug_iw^{-1}) \in S$ for all $w \in W$. This shows that $\rho(g_i^{-1}Wg_i) \subset ST$ for all $i \gg 0$. Hence by the arguments as in the last part of the proof of Claims 8.6.2, we conclude that $\bar{G} = ST$.

Since $g_i \in ST$, $S \subset Z(\bar{G})$ and $\rho(W) \subset T$, we have that $\rho(g_i^{-1}wg_iw^{-1}) \in T$ for all $w \in W$ and all $i \gg 0$.

Now $\bar{\pi}(\rho(W)) = \pi(T)$ and $\bar{\pi}(W)$ is invariant under $F = \ker \rho$. Hence $\bar{\pi}(W) = \pi(G)$. Since $Y_1 \supset \bar{\pi}(W)$ and $Y_1 \cap Mx = \emptyset$, we have a contradiction. Hence $Y_1 = \emptyset$. Thus $\bar{Wx} = Mx$.

Now using the arguments as in the proof of Theorem 6.2, it is straightforward to verify that $W$ acts ergodically with respect to a locally finite $M$-invariant measure on $Mx$. This completes the proof of the theorem. \qed

9 Some Consequences

In this section we derive the corollaries, which are stated in the introduction, of the descriptions of invariant measures and orbit closures for the actions of subgroups generated by unipotent elements.

Proof of Corollary 1.4

The proof given below is due to Dave Witte.

The proof for the description of the ergodic invariant measures follows from Theorem 1.3 and the arguments as in [W, Proof of Corollary 5.8].
Proof for the description of orbit closures

To describe the orbit closures without loss of generality we may assume that
\[ x = e\Gamma \text{ and } G = \overline{W\Gamma} = \Gamma G^0. \]

Now as in Claim 7.8.1, without loss of generality we may assume that \( \Gamma \) is discrete.

Let \( G' = W \times G, \Gamma' = \Lambda \times \Gamma, \) and \( \Delta : W \to G' \) be the diagonal embedding. Note that \( G'/\Gamma' = W/\Lambda \times G/\Gamma. \)

Let \( x_1 = e \Lambda \in W/\Lambda, \) \( x_2 = e\Gamma \in G/\Gamma, \) and \( x' = (x_1, x_2). \)

Note that \( G', \Gamma', \) and \( \Delta(W) \) satisfy the conditions of Theorem 1.3; and hence there exists a closed subgroup \( F' \) of \( G' \) containing \( \Delta(W) \) such that \( \Delta(W)x' = F'x'. \)

We claim that
\[ (x_1, \Lambda x_2) = \Delta(W)x' \cap (x_1, G/\Gamma). \tag{22} \]

Since \( \Delta(W)x' = (x_1, \Lambda x_2), \) we have that \( (x_1, \Lambda x_2) \subset \Delta(W)x' \cap (x_1, G/\Gamma). \)

To prove the opposite inclusion, suppose that \( (x_1, x) \in \Delta(W)x'. \) Let \( \{w_n\} \subset W \) with \( w_n x_1 \to x_1 \) and \( w_n x_2 \to x. \) Because \( w_n x_1 \to x_1, \) there exist sequences \( \{\lambda_n\} \subset \Lambda \) and \( \delta_n \to e \) in \( W \) such that \( w_n = \delta_n \lambda_n \) for all \( n \in \mathbb{N}. \)

Therefore \( \lambda_n x_2 = \delta_n^{-1} w_n x_2 \to x. \) Thus \( x \in \overline{\Lambda x_2}. \) This proves the claim.

Now by Equation 22,
\[ (x_1, \Lambda x_2) = F'x' \cap (x_1, Gx_2) \]
\[ = (F' \cap (\Lambda \times G))x' \]
\[ = (x_1, Lx_2), \]

where \( L \) is the projection of \( F' \cap (\Lambda \times G) \) into \( G. \) Thus \( \overline{\Lambda x_2} = Lx_2. \) Since \( Lx_2 \) is closed, replacing \( L \) by \( \overline{L} \) we have that \( L \) is a closed subgroup of \( G. \)

By Theorem 1.3, \( (F')^0 x' \) has a finite invariant measure. Let \( F_0' \) be any subgroup of \( F' \) containing \( (F')^0 \) such that \( F_0'x' \) has a finite \( F_0'\text{-invariant measure.} \)

Because the stabilizer of \( x' = (x_1, x_2) \) in \( F_0' \) is contained in \( F_0' \cap (\Lambda \times G), \) this implies that \( (F_0' \cap (\Lambda \times G))x' \) also has a finite invariant measure (see [R, Lemma 1.6, p. 20]). Thus, letting \( L' \) be the projection of \( F_0' \cap (\Lambda \times G) \) into \( G, \) we see that \( L'x_2 \) has a finite invariant measure. Because \( L' \) is open in \( L, \) we have that \( L^0 \) is the identity component of \( L'. \) Hence we conclude that \( L^0 x_2 \) has a finite invariant measure. This completes the main part of the proof of the corollary.
Note that if $F'x'$ has a finite $F'$-invariant measure, then $L'=L$. Therefore $Lx_2$ has a finite $L$-invariant measure. Note that if $W$ is connected, then $F'x'$ is connected, and hence $Lx_2$ has a finite $L$-invariant measure. □

Proof of Corollary 1.3

First without loss of generality we may assume that $\Lambda$ is irreducible. Since $G$ is a connected semisimple group with no nontrivial compact factors, $G$ is generated by $\text{Ad}_G$-unipotent one-parameter subgroups. By Corollary 1.4, applied to $W = G$, there exists a closed subgroup $F$ of $G$ containing $\Lambda$ such that $\overline{\Lambda \Gamma} = F \Gamma$. Therefore $F^0$ is normalized by $\Lambda$. By Borel’s density theorem, $\text{Ad}_G(\Lambda)$ is Zariski dense in $\text{Ad}_G(G)$. Therefore $F^0$ is a normal subgroup of $G$.

Note that $\Lambda F^0$ is an open, and hence a closed, subgroup of $F$. Therefore the projection of $\Lambda$ on $G/F^0$ is discrete. Now since $\Lambda$ is an irreducible lattice in $G$, either $F^0 = \{e\}$ or $F = G$. Thus either $\Lambda \Gamma$ is discrete, or $\Lambda \Gamma$ is dense in $G$. Since $W = G$ is connected, by Corollary 1.4, $F \Gamma/\Gamma$ has a finite $F$-invariant measure. Therefore if $F^0 = e$ then $F \Gamma/\Gamma = \Lambda \Gamma/\Gamma$ is finite. This shows that $\Lambda \cap \Gamma$ is of finite index in $\Lambda$. Therefore $\Lambda \cap \Gamma$ is a lattice in $G$, and hence it is a subgroup of finite index in $\Gamma$. □

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