ANALOGUE OF THE DUISTERMAAT-VAN DER KALLEN THEOREM FOR GROUP ALGEBRAS

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Abstract. Let $G$ be a group, $R$ an integral domain, and $V_G$ the $R$-subspace of the group algebra $R[G]$ consisting of all the elements of $R[G]$ whose coefficient of the identity element $1_G$ of $G$ is equal to zero. Motivated by the Mathieu conjecture [M], the Duistermaat-van der Kallen theorem [DK], and also by recent studies on the notion of Mathieu subspaces introduced in [Z4] and [Z6], we show that for finite groups $G$, $V_G$ under certain conditions also forms a Mathieu subspace of the group algebra $R[G]$. We also show that for the free abelian groups $G = \mathbb{Z}^n$ ($n \geq 1$) and any integral domain $R$ of positive characteristic, $V_G$ fails to be a Mathieu subspace of $R[G]$, which is equivalent to saying that the Duistermaat-van der Kallen theorem [DK] cannot be generalized to any field or integral domain of positive characteristic.

1. Introduction

Let’s first recall the following notion introduced recently by the first author in [Z4] and [Z6], which can be viewed as a natural generalization of the notion of ideals.

Definition 1.1. Let $R$ be a commutative ring and $A$ an associative $R$-algebra. A $R$-submodule or $R$-subspace $M$ of $A$ is said to be a left (resp., right; two-sided) Mathieu subspace of $A$ if for any $a, b, c \in A$ with $a^m \in M$ for all $m \geq 1$, we have $ba^m \in M$ (resp., $a^mb \in M$; $ba^mc \in M$) when $m \gg 0$, i.e., there exists $N \geq 1$ such that $ba^m \in M$ (resp., $a^mb \in M$; $ba^mc \in M$) for all $m \geq N$.

Two-sided Mathieu subspaces will also simply be called Mathieu subspaces. A $R$-subspace $M$ of $A$ is said to be a pre-two-sided Mathieu...
subspace of $A$ if it is both left and right Mathieu subspace of $A$. Note that the pre-two-sided Mathieu subspaces were previously called two-sided Mathieu subspace or Mathieu subspaces in $\mathbb{Z}_4$.

The introduction of the notion of Mathieu subspaces in $\mathbb{Z}_4$ and $\mathbb{Z}_6$ was mainly motivated by the studies of the Jacobian conjecture $\mathbb{K}$ (see also $\mathbb{BCW}$ and $\mathbb{E}_1$), the Mathieu conjecture $\mathbb{M}$, the vanishing conjecture $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{EW}_{\mathbb{Z}}$, and more recently, the image conjecture $\mathbb{Z}_3$ as well as many other related open problems. For some recent developments on Mathieu subspaces, see $\mathbb{Z}_6$, $\mathbb{FPYZ}$, $\mathbb{E}_{\mathbb{W}:\mathbb{Z}_{\mathbb{1}}}$, $\mathbb{E}_{\mathbb{W}:\mathbb{Z}_{\mathbb{2}}}$, $\mathbb{EZ}$ and $\mathbb{Z}_7$. For a recent survey on the image conjecture and its connections with some other problems, see $\mathbb{E}_2$.

The notion was named after Olivier Mathieu in $\mathbb{Z}_4$ due to his conjecture mentioned above, which now in terms of the new notion can be re-stated as follows.

**Conjecture 1.2.** (The Mathieu Conjecture) Let $G$ be a compact connected real Lie group with the Haar measure $\sigma$. Let $A$ be the algebra of complex-valued $G$-finite functions on $G$, and $M$ the subspace of $A$ consisting of $f \in A$ such that $\int_G f d\sigma = 0$. Then $M$ is a Mathieu subspace of $A$.

J. Duistermaat and W. van der Kallen $\mathbb{DK}$ proved the Mathieu conjecture for the case of tori, which now can be re-stated as follows.

**Theorem 1.3.** (Duistermaat and van der Kallen) Let $z = (z_1, z_2, ..., z_n)$ be $n$ commutative free variables and $V$ the subspace of the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ consisting of the Laurent polynomials with no constant term. Then $V$ is a Mathieu subspace of $\mathbb{C}[z^{-1}, z]$.

Note that despite its innocent looking, the proof of the theorem above is surprisingly difficult. The proof in $\mathbb{DK}$ uses some heavy machineries such as toric varieties, resolutions of singularities, etc.

To discuss the main motivations and results of this paper, we start with the following observation on the Duistermaat-van der Kallen Theorem above.

Let $G$ be the free abelian group $\mathbb{Z}_n$ ($n \geq 1$). Then the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ can be identified in the obvious way with the group algebra $\mathbb{C}[G]$. Under this identification, the subspace $V \subset \mathbb{C}[z^{-1}, z]$ in the theorem corresponds to the subspace $V_G$ of the group algebra $\mathbb{C}[G]$ consisting of the elements of $\mathbb{C}[G]$ whose “constant term” (i.e., the coefficient of the identity element $1_G$ of $G$) is equal to zero. So, we are naturally led to the following (open) problem.

**Problem 1.4.** Let $R$ be a commutative ring and $G$ a group. Let $V_G$ be the $R$-subspace of the elements of the group algebra $R[G]$ with no
"constant term", i.e., the coefficient of the identity element $1_G$ of $G$ is equal to zero. Then under what conditions on $R$ and $G$, $V_G$ forms a Mathieu subspace of the group algebra $R[G]$?

The problem above not only provides a different point of view to get further understanding on the remarkable Duistermaat-van der Kallen Theorem, but also gives a family of candidates for Mathieu subspaces, which may provide some new understandings on the still very mysterious notion of Mathieu subspaces. This makes the problem itself very interesting and worthy to investigate.

One of the main results of this paper is that for any finite group $G$ and an integral domain $R$ of characteristic $p = 0$ or $p > |G|$ (the order of $G$), the $R$-subspace $V_G$ does form a Mathieu subspace of $R[G]$ (see Theorem 3.5), i.e., Problem 1.4 in this case can be solved completely.

However, for the case that $0 < \text{char. } R = p \leq |G|$, the situation becomes much more subtle. For example, the magic condition $p \nmid |G|$ for the group algebras of finite groups $G$ (e.g., see [P]) does not resolve the difficulty completely, i.e., under this condition $V_G$ still may or may not be a Mathieu subspaces of $R[G]$ (e.g., see Theorem 4.1 and Example 4.2).

In this paper, we first study Problem 1.4 for the group algebras of finite groups $G$ over integral domains $R$ of any characteristics. In particular, besides the main result mention above, for finite abelian groups we also give a complete solution of Problem 1.4 for the case that the base integral domain $R$ satisfies certain primitive root of unity conditions (see Theorems 3.5 and 4.1), e.g., when $R$ is an algebraically closed field.

We then show that for the group algebras of the free abelian groups $G = \mathbb{Z}^n$ ($n \geq 1$) over any integral domain $R$ of positive characteristic, $V_G$ is not a Mathieu subspace of $R[G]$, by showing that an example suggested by Arno van den Essen does provide a desired counter-example. Consequently, it follows that the Duistermaat-van der Kallen theorem, Theorem 1.3, cannot be generalized to the Laurent polynomial algebra $R[z^{-1}, z]$ over any field or integral domain $R$ of positive characteristic.

The arrangement of this paper is as follows.

In Section 2, we recall some general results on Mathieu subspaces obtained in [Z4] and [Z6], which will be needed later in this paper. In Section 3 we prove some results on Problem 1.4 for the group algebras of finite groups $G$ over arbitrary commutative rings or integral domains. In particular, we show in Theorem 3.5 that when the base ring $R$ is an integral domain of characteristic $p = 0$ or $p > |G|$, the subspace $V_G$ is always a Mathieu subspace of $R[G]$. 

In Section 4, we focus on the group algebras of finite abelian groups $G$ over integral domains $R$ of characteristic $p > 0$. The main results of this section is Theorem 4.1, which combining with Theorem 3.5 provides a complete solution of Problem 1.4 for the group algebras of finite abelian groups $G$ over the integral domains $R$ which satisfies a primitive root of unity condition, e.g., when $R$ is an algebraically closed field.

In Section 5, we consider Problem 1.4 for the group algebras of the free abelian groups $\mathbb{Z}^n (n \geq 1)$ over an integral domain $R$ of characteristic $p > 0$. We prove that $V_G$ in this case fails to be a Mathieu subspace of $R[\mathbb{Z}^n]$ by showing that the example in Lemma 5.2, which was suggested by Arno van den Essen to the authors, does provide a desired counter-example.

2. Some Results on Mathieu Subspaces

In this section, we recall some general facts on Mathieu subspaces which will be needed later in this paper. Although all the results below with certain modifications hold for all types of Mathieu subspaces (one-sided, pre-two-sided, etc.) We here only focus on the two-sided case, which by Corollary 3.2 in the next section will be enough for our purpose.

Throughout this paper, unless stated otherwise, $R$ and $K$ always stand respectively for a unital commutative ring and a field of any characteristic, and $\mathcal{A}$ a unital algebra over $R$ or $K$.

Following [Z6], we define for any $R$-subspace $V$ of a $R$-algebra $\mathcal{A}$ the radical, denoted by $\sqrt{V}$, to be the set of $a \in \mathcal{A}$ such that $a^m \in V$ when $m \gg 0$.

We start with the following equivalent formulation of Mathieu subspaces, which was given in Proposition 2.1 in [Z6].

**Proposition 2.1.** Let $\mathcal{A}$ be a $R$-algebra and $V$ a $R$-subspace of $\mathcal{A}$. Then $V$ is a Mathieu subspace of $\mathcal{A}$ iff for any $a \in \sqrt{V}$ and $b, c \in \mathcal{A}$, we have $ba^nc \in V$ when $m \gg 0$.

The following characterization of the Mathieu subspaces with algebraic radicals was also proved in Theorem 4.2 in [Z6].

**Theorem 2.2.** Let $\mathcal{A}$ be a $K$-algebra and $V$ a $K$-subspace of $\mathcal{A}$ such that $\sqrt{V}$ is algebraic over $K$ (i.e., every element of $\sqrt{V}$ is algebraic over $K$). Then $V$ is a Mathieu subspace of $\mathcal{A}$ iff for any idempotent $e \in V$ (i.e., $e^2 = e$), we have $\langle e \rangle \subseteq V$, where $\langle e \rangle$ denotes the ideal of $\mathcal{A}$ generated by $e$. 
The next proposition is easy to check directly (or see Proposition 2.7 in [Z6]).

**Proposition 2.3.** Let \( I \) be an ideal of \( A \) and \( V \) a \( R \)-subspace of \( A \) such that \( I \subseteq V \). Then \( V \) is a Mathieu subspace of \( A \) iff \( V/I \) is a Mathieu subspace of the quotient algebra \( A/I \).

Finally, let’s recall the following family of Mathieu subspaces of the polynomial algebra \( K[z] \) in \( n \) variables \( z := (z_1, z_2, \ldots, z_n) \), which was given in Proposition 4 in [Z4].

**Proposition 2.4.** Let \( n, d \geq 1 \) and \( R \) an arbitrary integral domain. Let \( S = \{v_1, v_2, \ldots, v_d\} \subseteq R^n \) (with \( d \) distinct elements) and \( 0 \neq c_i \in R \) (\( 1 \leq i \leq d \)). Denote by \( V \) the subspace of \( f(z) \in R[z] \) such that

\[
\sum_{i=1}^{d} c_i f(v_i) = 0. 
\]

Then \( V \) is a Mathieu subspace of \( R[z] \) iff for any non-empty subset \( J \subset \{1, 2, \ldots, d\} \), we have

\[
\sum_{i \in J} c_i \neq 0. 
\]

Note that the proposition above was only proved in [Z4] under the condition that \( R \) is a field. But, it is easy to see that the same proof actually goes through equally well for all integral domains.

### 3. Some General Results for the Case of Finite Groups

Throughout the rest of this paper, unless stated otherwise, \( G \) stands for a finite group, \( R \) a commutative ring, and \( K \) a field of any characteristic. We denote by \( R[G] \) and \( K[G] \) the group algebra of \( G \) over \( R \) and \( K \), respectively. Furthermore, we also fix the following terminologies and notations.

\( i \) ) We denote by 1 or \( 1_G \) the identity element of the group \( G \) and also the identity element of the group algebra \( R[G] \).

\( ii \) ) For any \( u \in R[G] \), we denote by \( \text{Const}(u) \) the coefficient of \( 1_G \) of \( u \), and call it the constant term of \( u \).

\( iii \) ) The set of all the elements of \( R[G] \) with no constant term will be denoted by \( V_{G,R} \), or simply by \( V_G \) if the base ring \( R \) is clear in the context.

\( iv \) ) When \( R \) is an integral domain, by the characteristic of \( R \) (denoted by \( \text{char} \ R \)) we mean the characteristic of the field of fractions of \( R \).

\(^1\)Note that Eq. (2.2) in [Z4] had been misprinted.
Next, we start with the following equivalent formulation of Problem 1.4 for the group algebras of finite groups.

**Proposition 3.1.** Let $R$ be any commutative ring and $G$ a finite group. Then $V_G$ is a Mathieu subspace of any fixed type of $R[G]$ iff all elements of $\sqrt{V_G}$ are nilpotent.

**Proof:** First, it is easy to see that the $(\Leftarrow)$ part follows directly from the assumption and Definition 1.1.

For the $(\Rightarrow)$ part, here we only give a proof for the left Mathieu subspace case. The proofs of the other three cases are similar.

Assume that $V_G$ is a left Mathieu subspace and let $u \in \sqrt{V_G}$. Replacing $u$ by a positive power of $u$, if necessary, we may assume that $u^m \in V_G$ for all $m \geq 1$.

Now, since $G$ is finite, by Definition 1.1 there exists $N \geq 1$ such that $g^{-1}u^m \in V_G$ for all $g \in G$ and $m \geq N$. In particular, for each $g \in G$, the constant term of $g^{-1}u^N$, which is the same as the coefficient of $g$ in $u^N$, is equal to 0, whence $u^N = 0$, i.e., $u$ is nilpotent.

Another way to show the $(\Rightarrow)$ part is as follows.

Assume otherwise and let $u \in \sqrt{V_G}$ such that $u^m \neq 0$ for all $m \geq 1$. Since $G$ is finite, there exists $g \in G$ such that the coefficient of $g$ in $u^m$ is nonzero for infinitely many $m \geq 1$. Then the constant term of $g^{-1}u^m$ is nonzero for infinitely many $m \geq 1$. Then by Definition 1.1 $V_G$ is not a Mathieu subspace of $R[G]$, which is a contradiction.

Two immediate consequences of Proposition 3.1 are the following two corollaries.

**Corollary 3.2.** Let $R$ and $G$ be as in Proposition 3.1. Then $V_G$ is a Mathieu subspace of any fixed type of $R[G]$ iff $V_G$ is a (two-sided) Mathieu subspace of $R[G]$.

Therefore, throughout the rest of this paper we may and will focus only on the two-sided case.

**Corollary 3.3.** Let $R$ and $G$ be as in Proposition 3.1. Assume that $V_G$ is a Mathieu subspace of $R[G]$. Then $V_G$ contains no nonzero idempotent of $R[G]$.

**Proof:** Assume otherwise. Let $e \in V_G$ be a nonzero idempotent, i.e., $e^2 = e \neq 0$. Then for any $m \geq 1$, we have $e^m = e \in V_G$, whence $e \in \sqrt{V_G}$. But, since $e$ is clearly not nilpotent, by Proposition 3.1 $V_G$ is not a Mathieu subspace of $R[G]$, which is a contradiction.

When the base ring $R$ is a field, we show next that the converse of Corollary 3.3 actually also holds.
Proposition 3.4. Let $K$ be a field and $G$ a finite group. Then $V_G$ is a Mathieu subspace of $K[G]$ iff $V_G$ contains no nonzero idempotent of $K[G]$.

Proof: The ($\Rightarrow$) part is a special case of Corollary 3.3. To show the ($\Leftarrow$) part, note that $K[G]$ is algebraic over $K$, since it is of finite dimension over $K$. In particular, the radical $\sqrt{V_G}$ of $V_G$ is algebraic over $K$. Then by Theorem 2.2, $V_G$ is a Mathieu subspace of $K[G]$. $\square$

Next, we show that Problem 1.4 can be solved for the group algebras of all finite groups $G$ over integral domains $R$ such that char. $R = 0$ or char. $R = p > |G|$.

Theorem 3.5. Let $G$ be a finite group and $R$ an integral domain such that char. $R = 0$ or char. $R = p > |G|$. Then $V_G$ is a Mathieu subspace of $R[G]$.

Proof: Let $u \in \sqrt{V_G}$. Then by Proposition 3.1 it suffices to show that $u$ is nilpotent. Note that by replacing $u$ by a positive power of $u$, if necessary, we may assume $u^m \in V_G$, i.e., Const($u^m$) = 0, for all $m \geq 1$.

Let $\mu : R[G] \rightarrow \text{End}_R(R[G])$ be the $R$-algebra homomorphism which maps each $v \in R[G]$ to the $R$-endomorphism $m_v \in \text{End}_R(R[G])$ defined by the left multiplication by $v$ on $R[G]$. Then it is easy to check that for any $v \in R[G]$, the trace of the linear map $\mu(v) = m_v$ is equal to $|G|\text{Const}(v)$. Consequently, for the $u \in \sqrt{V_G}$ fixed at the beginning and any $m \geq 1$, the trace of the $m$-th power $(\mu(u))^m = \mu(u^m)$ of the linear transformation $\mu(u)$ is equal to zero.

On the other hand, since char. $R = 0$ or char. $R = p > |G|$, it is well-known in linear algebra that in this case the linear transformation $\mu(u)$ must be nilpotent, i.e., $(\mu(u))^m = \mu(u^m) = 0$ for $m \gg 0$. Since $\mu$ is clearly injective (e.g., by applying $\mu(v)$ to 1 $\in R[G]$ for all $v \in R[G]$), we also have $u^m = 0$ when $m \gg 0$, i.e., $u$ is nilpotent, as desired. $\square$

One remark on Theorem 3.5 is that when the conditions char. $R = 0$ and char. $R = p > |G|$ fail, i.e., when $0 < \text{char. } R = p \leq |G|$, the situation for Problem 1.4 becomes much more complicated.

For instance, as shown by the next lemma and also by Theorem 4.1 in Section 4, the magic condition $p \nmid |G|$ for the theory of group algebras $R[G]$ of finite groups $G$ (e.g., see [P]) does not resolve the difficulty completely for Problem 1.4.

Lemma 3.6. Let $G$ be any finite group with $|G| \geq 2$, and $R$ an integral domain of char. $R = p > 0$. Assume $p \mid (|G| - 1)$ (hence, $p \nmid |G|$). Then $V_G$ is not a Mathieu subspace of $R[G]$. 
Proof: Let \( u = - \sum_{g \in G \setminus \{1_G\}} g \in V_G \) and \( v = 1_G - u = 1 - u \). Note that \( v \) is the sum of all the distinct elements of \( G \) in \( R[G] \). Hence, for any \( g \in G \), we have \( vg = gv = v \). Consequently, we have \( v^2 = |G|v \), which in terms of \( u \) is the same as
\[
(1 - u)^2 = 1 - 2u + u^2 = |G|(1 - u).
\]
Solving \( u^2 \) from the equation above, we get
\[
(3.1) \quad u^2 = (|G| - 1) - (|G| - 2)u.
\]
Since \( p \mid (|G| - 1) \), we have \((|G| - 1) = 0 \) and \((|G| - 2) = -1 \). Then by Eq. (3.1), we have \( u^2 = u \). Since \( u \neq 0 \), by Corollary 3.3 \( V_G \) is not a Mathieu subspace of \( R[G] \). □

Next, we show the following lemma that will be needed later.

Lemma 3.7. Let \( R \) be any commutative ring and \( G \) any group (not necessarily finite). Assume that \( V_G \) is a Mathieu subspace of \( R[G] \). Then for each subgroup \( H \) of \( G \), \( V_H \) is a Mathieu subspace of \( R[H] \).

Proof: Assume otherwise. Let \( H \) be a subgroup of \( G \) such that \( V_H \) is not a Mathieu subspace of \( R[H] \). Then by Definition 1.1 and the definition of \( V_H \), there exist \( u, v \in R[H] \) such that \( \text{Const}(u^m) = 0 \) for all \( m \geq 1 \), but \( \text{Const}(u^m v) \neq 0 \) for infinitely many \( m \geq 1 \).

Since \( R[H] \subseteq R[G] \), we have \( u, v \in R[G] \), and \( u^m \in V_G \) for all \( m \geq 1 \), but \( u^m v \notin V_G \) for infinitely many \( m \geq 1 \). Hence, \( V_G \) is not a Mathieu subspace of \( R[G] \), which is a contradiction. □

Corollary 3.8. Let \( R \) and \( G \) be as in Lemma 3.7 and \( H \) a subgroup of \( G \). Assume that \( V_H \) is not a Mathieu subspace of \( R[H] \). Then \( V_G \) is not a Mathieu subspace of \( R[G] \).

As an application of Lemma 3.7 or Corollary 3.8, we derive the following necessary condition for \( V_G \) to be a Mathieu subspace of \( R[G] \) over integral domains \( R \) of positive characteristic.

Proposition 3.9. Let \( R \) be an integral domain of characteristic \( p > 0 \) and \( G \) an arbitrary finite group. Write \( |G| = p^r d \) for some \( r \geq 0 \) and \( d \geq 1 \) with \( p \nmid d \). Assume that \( R \) contains a primitive \( d \)-th root of unity and \( V_G \) is a Mathieu subspace of \( R[G] \). Then for each prime divisor \( q \) of \( |G| \), we have \( p \geq q \).

Proof: Assume otherwise and let \( q \) be a prime divisor of \( |G| \) such that \( p < q \). Then we have \( q \mid d \), whence \( R \) also contains a primitive \( q \)-th root of unity.
Write $|G| = q^n s$ with $s, n \geq 1$ such that $q \nmid n$. Then by the well-known Sylow's theorem in the theory of finite groups (e.g., see p.105, Theorem 2.11.7 in [He]), $G$ has at least one $q$-Sylow subgroup $H$, i.e., a subgroup $H$ of $G$ with $|H| = q^n$.

Now, pick up any non-identity element $h \in H$. Then $h$ has order $q^k$ for some $1 \leq k \leq r$. Let $g = h$ if $k = 1$; and $g = h^{k-1}$ if $k \geq 2$. Then $g$ has order $q$ and hence, generates a cyclic subgroup $C_q$ of $G$ of order $|C_q| = q$. Then by Theorem 4.1 to be proved in Section 4, $V_{C_q}$ is not a Mathieu subspace of $R[C_q]$. Hence, by Corollary 3.8, $V_G$ is not a Mathieu subspace of $R[G]$ either, which is a contradiction. \(\Box\)

Finally, we point out that when the finite group $G$ in Proposition 3.9 is abelian, a much stronger condition will be given in Theorem 4.1 of the next section.

4. The Case for Finite Abelian Groups

In this section, we study Problem 1.4 for finite abelian groups over certain integral domains. The main result of this section is the following theorem.

**Theorem 4.1.** Let $R$ be an integral domain of characteristic $p > 0$, and $G$ a finite abelian group with $|G| = p^r d$ for some $r \geq 0$ and $d \geq 1$ with $p \nmid d$. Assume that $R$ contains a primitive $d$-th root of unity. Then $V_G$ is a Mathieu subspace of $R[G]$ if and only if $p > d$.

Two remarks on Theorem 4.1 are as follows.

First, when the integral domain $R$ has char. $R = 0$ (or char. $R = p > |G|$), Problem 1.4 has been solved by Theorem 3.5 together with which Theorem 4.1 provides a complete solution of Problem 1.4 for the group algebras of all finite abelian groups when the base integral domain $R$ satisfies the primitive root of unity condition in Theorem 4.1, e.g., when $R$ is an algebraically closed field.

Second, from the example below we see that the $d$-th primitive root of unity condition on the integral domain $R$ in Theorem 4.1 is necessary.

**Example 4.2.** Let $\mathbb{F}_3$ be the field with three elements. Note that $\mathbb{F}_3$ obviously does not contain any primitive 5th root of unity. But, $V_{\mathbb{Z}_5}$ is a Mathieu subspace of $\mathbb{F}_3[\mathbb{Z}_5]$, although char. $\mathbb{F}_3 = 3 < d = 5$.

**Proof:** Assume otherwise. Then by Proposition 3.4, there exists a nonzero idempotent $f \in V_{\mathbb{Z}_5}$. By identifying the group algebra $\mathbb{F}_3[\mathbb{Z}_5]$ with the quotient algebra $\mathbb{F}_3[t]/(t^5 - 1)$ of the polynomial algebra $\mathbb{F}_3[t]$ in one variable $t$, we may write $f = c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$. Then it is
easy to check that the following equations hold:

\[ \text{Const}(f^2) = 2(c_1c_4 + c_2c_3), \]
\[ f^3 = c_1^3 t^3 + c_2^3 t^3 + c_3^4 t^4 + c_4^3 t^2. \]

Since \( f^2 = f^3 = f \in V_{\mathbb{Z}_5} \), hence we also have

\[ c_1c_4 = -c_2c_3, \] (4.1)
\[ c_1 = c_2^3; \quad c_2 = c_4^3; \quad c_3 = c_1^3; \quad c_4 = c_3^2. \] (4.2)

From the four equations in Eq. (4.2), it is easy to see that if one of the \( c_i \)'s is equal to zero, then so are all the \( c_i \)'s. Since \( f \neq 0 \), we see that all the \( c_i \)'s are nonzero.

By combining equations in Eqs. (4.1)-(4.2), it is also easy to see that \((c_2c_3)^3 = -(c_2c_3)\), whence \((c_2c_3)^2 = -1\). However, the base field \( \mathbb{F}_3 \) contains no square root of \(-1\). Hence, we get a contradiction. ✷

Next, we will devote the rest of this section to give a proof for Theorem 4.1. First, we need to show the following reduction lemma.

**Lemma 4.3.** Let \( R \) be an integral domain of characteristic \( p > 0 \) and \( H \) a finite abelian group. Let \( q = p^r \) for some \( r \geq 1 \) and \( G = H \times \mathbb{Z}_q \). Then \( V_H \) is a Mathieu subspace of \( R[H] \) iff \( V_G \) is a Mathieu subspace of \( R[G] \).

**Proof:** For convenience, we identify \( \mathbb{Z}_q \) with the multiplicative cyclic group \( C_q \) with \( q \)-element. We also identify \( H \) and \( C_q \) with the subgroups \( H \times \{ 1_{C_q} \} \) and \( \{ 1_H \} \times C_q \) of \( G \), respectively.

Under these identifications, \( G \) is also the inner product of its subgroups \( H \) and \( C_q \), and the group algebras \( R[H] \) and \( R[C_q] \) become subalgebras of \( R[G] \). Then the \((\Rightarrow)\) part of the lemma follows immediately from Lemma 3.1.

To show the \((\Rightarrow)\) part, pick up any \( u \in \sqrt{V_G} \). Then by Proposition 3.1, it suffices to show that \( u \) is nilpotent. To do so, replacing \( u \) by a positive power of \( u \), if necessary, we assume that \( u^m \in V_G \) for all \( m \geq 1 \).

Write \( u = \sum_{s \in C_q} \alpha_s s \) with \( \alpha_s \in R[H] \) for each \( s \in C_q \). Note that for any \( k \geq 1 \) and \( s \in C_q \), we have \( s^{q^k} = 1_{C_q} \), since \( |C_q| = q \). Then by the conditions that \( \text{char.} \ R = p > 0 \) and \( q \) is a positive power of \( p \), for any \( k \geq 1 \) we also have

\[ u^{q^k} = \sum_{s \in C_q} \alpha_s^{q^k} s^{q^k} = \sum_{s \in C_q} \alpha_s^{q^k} \in R[H]. \]
Moreover, since \( u^m \in V_G \) for all \( m \geq 1 \), we have \((u^q)^k = u^{qk} \in R[H] \cap V_G = V_H \) for all \( k \geq 1 \), whence \( u^q \in \sqrt{V_H} \). Since by assumption \( V_H \) is a Mathieu subspace of \( R[H] \), applying Proposition \ref{prop:mathieu-subspace} to the group algebra \( R[H] \) we see that \( u^q \) is nilpotent, whence so is \( u \). \( \square \)

Next, let’s recall the following well-known fundamental theorem of finite abelian groups.

**Theorem 4.4.** Any finite abelian group can be written as a direct product of cyclic groups whose orders are powers of primes.

For the proof of the theorem above, see any abstract algebra textbook (e.g., see Th.2.2, Ch.II, [Hu]).

Note that by applying Theorem \ref{thm:abelian-group-decomposition} and Lemma \ref{lem:mathieu-subspace} (inductively), it is easy to see that we may actually assume that the exponent \( r \) in Theorem \ref{thm:mathieu-characteristic} is equal to zero, i.e., it suffices to show the following lemma.

**Lemma 4.5.** Let \( G \) be a finite abelian group and \( R \) an integral domain of characteristic \( p > 0 \) such that \( p \nmid d := |G| \). Assume that \( R \) contains a primitive \( d \)-th root of unity. Then \( V_G \) is a Mathieu subspace of \( R[G] \) iff \( p > d = |G| \).

From now on and throughout the rest of this section, we let \( G \) and \( R \) be as in the lemma above.

Note first that when \( d = |G| = 1 \), we have \( V_G = \{0\} \), which is obviously a Mathieu subspace of \( R[G] \). Hence, Lemma \ref{lem:mathieu-subspace} holds in this trivial case. So we will assume \( d = |G| \geq 2 \).

Note also that by Theorem \ref{thm:abelian-group-decomposition}, we may (and will) further assume that the abelian group \( G \) is given by

\[
G = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_n}
\]

for some \( n \geq 1 \) and \( d_i \geq 2 \) \((1 \leq i \leq n)\).

But, here we do not need to assume that the integers \( d_i \geq 2 \) \((1 \leq i \leq n)\) are powers of primes.

In order to study the group algebra \( R[G] \) of \( G \) in Eq. \ref{eq:abelian-group-decomposition}, we need to write the factor groups \( \mathbb{Z}_{d_i} \) \((1 \leq i \leq n)\) in Eq. \ref{eq:abelian-group-decomposition} as multiplicative groups \( H_i \) with a fixed generator \( e_i \in H_i \), i.e., for each \( 1 \leq i \leq n \), we let

\[
H_i = \{ e_i^k \mid 0 \leq k \leq d_i - 1 \} \simeq \mathbb{Z}_{d_i}.
\]

For convenience, for each \( 1 \leq i \leq n \), we also identify \( H_i \) (implicitly) with the subgroup of \( G \) in Eq. \ref{eq:abelian-group-decomposition} consisting of all the \( n \)-tuples whose \( j \)-th \((j \neq i)\) component being the identity element of \( H_j \simeq \mathbb{Z}_{d_j} \). Note
that under this identification, we have $H_i \subset G$, whence $G$ is also the inner product of the subgroups $H_i$ ($1 \leq i \leq n$), i.e., with the abusive notations fixed above, we have

$$G = H_1 \cdot H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n \quad (4.5)$$

Furthermore, we also need to introduce the following two sets:

$$D := \{ \beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}^n \mid 0 \leq \beta_i \leq d_i - 1 \} \quad (4.6)$$

$$S := \{ a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n \mid a_i^{d_i} = 1 \} \quad (4.7)$$

Note that since $\mathbb{R}$ contains a primitive $d$-th root of unity, $\mathbb{R}$ also contains a primitive $d_i$-th ($1 \leq i \leq n$) root of unity, since $d_i | d$. Then from Eqs. (4.6) and (4.7), we have $|S| = |D| = |G|$.

Next, with the notations fixed above we give an equivalent formulation of Lemma 4.5 in terms of the polynomial algebra $\mathbb{R}[z]$ over $\mathbb{R}$ in $n$ variables $z := (z_1, z_2, ..., z_n)$.

First, we define and consider the following $\mathbb{R}$-linear functional:

$$\mathcal{L}: \mathbb{R}[z] \to \mathbb{R} \quad f \to \sum_{a \in S} f(a). \quad (4.8)$$

**Lemma 4.6.** Let $G$ and $R$ be fixed as above. Then for any $\alpha \in D$, we have

$$\mathcal{L}(z^\alpha) = \begin{cases} d & \text{if } \alpha = 0; \\ 0 & \text{if } \alpha \neq 0. \end{cases} \quad (4.9)$$

**Proof:** If $\alpha = 0$, then $\mathcal{L}(z^\alpha) = \sum_{a \in S} 1 = |S| = d$. So we let $\alpha \neq 0$. Without losing any generality, we assume that the first component of $\alpha$ is nonzero, and denote it by $k$ (for short).

Let $\xi_1$ be a primitive $d_1$-th root of unity in $\mathbb{R}$. Then we have $\xi_1^k \neq 1$, since $1 \leq k \leq d_1 - 1$. Note that for each root $1 \neq r \in \mathbb{R}$ of the polynomial $z_1^{d_1} - 1 \in \mathbb{R}[z_1]$, $r$ is also a root of the polynomial $\sum_{\ell=0}^{d_1-1} z_1^\ell$, for $z_1^{d_1} - 1 = (z_1 - 1) \sum_{\ell=0}^{d_1-1} z_1^\ell$. Therefore, for the fixed primitive $d_1$-th root of unity $\xi_1 \in \mathbb{R}$, we have

$$\sum_{\ell=0}^{d_1-1} (\xi_1^\ell)^k = \sum_{\ell=0}^{d_1-1} (\xi_1^{d_1})^\ell = 0. \quad (4.10)$$

Now, for each $1 \leq i \leq n$, set $C_i := \{ \xi_i^\ell \mid 0 \leq \ell \leq d_i - 1 \}$, where $\xi_i$ is any fixed primitive $d_i$-th root of unity in $\mathbb{R}$. Then from the definition of the set $S$ in Eq. (4.7), we have $S = C_1 \times C_2 \times \cdots \times C_n$. By taking the sum $\mathcal{L}(z^\alpha) = \sum_{a \in S} a^\alpha$ first over the set $C_1$, it follows immediately from Eq. (4.10) that $\mathcal{L}(z^\alpha) = 0$. \qed
Next, we define the following $R$-algebra homomorphism:

$$
\varphi : R[z] \to R[G]
$$

$$
z_i \mapsto e_i.
$$

Note that the kernel of the $R$-algebra homomorphism $\varphi$ above is the ideal of $R[z]$ generated by the polynomials $z_i^{d_i} - 1 \ (1 \leq i \leq n)$. We will denote this ideal by $I_{\vec{d}}$, where $\vec{d}$ stands for the $n$-tuple $(d_1, d_2, ..., d_n)$.

The pre-image of $V_G \subset R[G]$ under the linear map $\varphi$ is given by the following lemma.

**Lemma 4.7.** With the setting above, we have

$$
\varphi^{-1}(V_G) = \ker L.
$$

**Proof:** First, let $V_0$ be the $R$-subspace of $R[z]$ spanned by $z^\alpha \ (0 \neq \alpha \in D)$ and $V := R \cdot 1 \oplus V_0$. Then by the definition of $\varphi$ in Eq. (4.11), it is easy to see that we have

$$
\varphi^{-1}(V_G) = \{ f \in R[z] \mid f \equiv r \pmod{I_{\vec{d}}} \ \text{for some } r \in V_0 \}.
$$

Therefore, it suffices to show that $\ker L$ coincides with the set on the right-hand side of the equation above.

Now, let $f \in R[z]$. Then there exists a unique $r \in V$ such that $f \equiv r \pmod{I_{\vec{d}}}$. By Eq. (4.13) we have

$$
f \in \varphi^{-1}(V_G) \iff r \in V_0.
$$

Furthermore, since $S$ is the zero-set of the ideal $I_{\vec{d}}$ in $R^n$, we have $f(a) = r(a)$ for all $a \in S$. In particular, we have $L(f) = L(r)$ and hence,

$$
f \in \ker L \iff r \in \ker L.
$$

Write $r(z) = \sum_{\alpha \in D} c_{\alpha} z^\alpha$. Then by Eq. (4.9) we have

$$
L(r) = L(c_0) + \sum_{0 \neq \alpha \in D} c_\alpha L(z^\alpha) = dc_0.
$$

Since $p \nmid d$, we see that $r \in \ker L$ iff $c_0 = 0$ iff $r \in V_0$. Then by the equivalences in Eqs. (4.14) and (4.15), we have that $f \in \varphi^{-1}(V_G)$ iff $f \in \ker L$, whence the lemma follows. \[\square\]

Finally, we can give a proof for Lemma 4.5 as follows, from which the proof of the main result Theorem 4.1 will be completed.

**Proof of Lemma 4.5:** Note that the ($\Leftarrow$) part of the lemma follows directly from Theorem 4.1, which actually does not need the primitive
root of unity condition on $R$ in the lemma. But, with the primitive root of unity condition on $R$ it also follows from the arguments below.

First, we consider the $R$-homomorphism $\varphi: R[z] \to R[G]$ defined in Eq. (4.11). Note that $\varphi$ is surjective with the kernel $I_{\vec{d}}$. Hence, from Eq. (4.12) we have $I_{\vec{d}} \subseteq \ker L$ and $\varphi(\ker L) = V_G$. Therefore, we may identify $R[G]$ with the quotient algebra $R[z]/I_{\vec{d}}$, and $V_G$ with $\ker L/I_{\vec{d}}$. Via these identifications and by Proposition 2.3, we have that $V_G$ is a Mathieu subspace of $R[G]$, iff $\ker L$ is a Mathieu subspace of the polynomial algebra $R[z]$.

Second, by applying Proposition 2.4 to the set $S$ in Eq. (4.7) with $c_i = 1$ (1 $\leq i \leq d$), we have that $\ker L$ is a Mathieu subspace of $R[z]$, iff for any non-empty subset $J \subseteq \{1, 2, ... , d\}$, the cardinal number $|J| \neq 0$ in $R$, i.e., $|J| \neq 0 \mod p$. Furthermore, it is easy to see that the latter property holds iff $p > d = |G|$.

Finally, by combining the three equivalences above, we see that the lemma follows. $\square$

5. The Case for the Group Algebra $R[Z^n]$ with $\text{char. } R = p > 0$

In this section, we show that Problem 1.4 has a negative answer for the group algebras of the free abelian groups $Z^n$ ($n \geq 1$) over all integral domains $R$ of positive characteristics. More precisely, we have the following proposition.

Proposition 5.1. For any integral domain $R$ of $\text{char. } R = p > 0$, $V_{Z^n}$ is not a Mathieu subspace of the group algebra $R[Z^n]$.

Note that under the natural identification $R[Z^n] \simeq R[z^{-1}, z]$ (the Laurent polynomial algebra in $n$ variables $z = (z_1, z_2, ..., z_n)$ over $R$), the proposition above is equivalent to saying that for any integral domain $R$ of $\text{char. } R = p > 0$, the subspace $V$ of all the Laurent polynomials in $R[z^{-1}, z]$ with no constant term does not form a Mathieu subspace of the Laurent polynomial algebra $R[z^{-1}, z]$. In particular, it follows that the Duistermaat-van der Kallen Theorem, Theorem 1.3, cannot be generalized to any field of characteristic $p > 0$.

To show Proposition 5.1, note first that we may identify $Z$ as the subgroup of $Z^n$ consisting of all the elements $(a, a, ..., a) \in Z^n$ with $a \in Z$. Then by Corollary 3.8 we may actually assume $n = 1$. Furthermore, via the identification $R[Z] \simeq R[z, z^{-1}]$ mentioned above, it will be enough to show the following lemma. The example in the lemma was suggested to the authors by Arno van den Essen.

Lemma 5.2. Let $p$ be a prime and $z$ a free variable. Set $f := z^{-1} + z^{p-1} \in Z_p[z^{-1}, z]$. Then the following two statements hold:
i) Const\( (f^m) = 0 \) for all \( m \geq 1 \);

ii) Const\( (z^{-1}f^{p^k-1}) = (-1)^{p^k-1} \) for all \( k \geq 1 \).

In order to prove the lemma above, we need first to show the following lemma.

**Lemma 5.3.** For any prime number \( p > 0 \), the following statements hold.

i) For any \( k, a \in \mathbb{N} \) such that \( k \geq 1 \) and \( a \leq p^k - 1 \), we have

\[
(p^k - 1) \equiv (-1)^a \mod p. \tag{5.1}
\]

ii) For any integer \( b \geq 1 \), we have

\[
\binom{bp}{b} \equiv 0 \mod p. \tag{5.2}
\]

**Proof:**

i) Let \( x \) be a free variable. We consider the polynomial \((x - 1)^{p^k-1}\) in the rational function field \( \mathbb{Z}_p(x) \), for which we have the following two equations:

\[
(1 - x)^{p^k-1} = \sum_{a=0}^{p^k-1} (-1)^a \binom{p^k - 1}{a} x^a, \tag{5.3}
\]

\[
(1 - x)^{p^k-1} = \frac{(1 - x)^{p^k}}{1 - x} = \frac{1 - x^{p^k}}{1 - x} = \sum_{a=0}^{p^k-1} x^a. \tag{5.4}
\]

Note that Eq. (5.4) above also holds for the case \( p = 2 \), since \( 1 = -1 \) in \( \mathbb{Z}_2 \). Now, by comparing the coefficients of \( x^a \) in the polynomials on the right-hand sides of Eqs. (5.3) and (5.4), we see that i) follows.

ii) Write \( b = p^rn \) for some \( r \geq 0 \) and \( n \geq 1 \) such that \( p \nmid n \). In particular, we have \( p^{r+1} \nmid b \).

We consider the polynomial \((x + 1)^{bp} \in \mathbb{Z}_p[x]\). Note that the coefficient of \( x^b \) in \((x + 1)^{bp}\) is equal to \( \binom{bp}{b} \). On the other hand, we also have

\[
(x + 1)^{bp} = (x + 1)^{np^{r+1}} = (x^{p^{r+1}} + 1)^n.
\]

Now, assume that \( \binom{bp}{b} \not\equiv 0 \mod p \). Then by the equation above, \( x^b \) appears in the polynomial \((x^{p^{r+1}} + 1)^n\) with a nonzero coefficient, whence \( b = p^{r+1}k \) for some \( 1 \leq k \leq n \). But this implies \( p^{r+1} \mid b \), which is a contradiction. □

**Proof of Lemma 5.2:**

i) Since \( f = z^{-1} + z^{p-1} \), the constant term of \( f^m \) \((m \geq 1)\) is given by the sum of \( \binom{m}{b} \) for all the integers \( 0 \leq b \leq m \).
such that \(- (m - b) + b(p - 1) = 0\), which is the same as \(m = bp\). Therefore, there is at most one such an integer \(b\), which is \(m/p\) if (and only if) \(p \mid m\). Hence we have

\[
\text{Const}(f^m) = \begin{cases} 
(bp/m) & \text{if } p \mid m \text{ and } b = m/p; \\
0 & \text{if } p \nmid m.
\end{cases}
\]

(5.5)

Then from the equation above and Eq. (5.2), we see that \(i)\) follows.

\(ii)\) By a similar argument as in \(i)\), it is easy to check that for any \(k \geq 1\), the coefficient of \(z\) in \(f^p^{k-1}\) is given by \((p^k-1)\), which by Eq. (5.1) is equal to \((-1)^{p^{k-1}}\). Hence, we have \(\text{Const}(z^{-1}f^p^{k-1}) = (-1)^{p^{k-1}}\) for all \(k \geq 1\), i.e., \(ii)\) holds. \(\square\)

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