Explosive character of instability development for the free surface of a conducting liquid in an electric field

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Abstract. In this work, a perfectly conducting liquid with a free surface, placed in an external uniform electric field, is considered. For a symmetric spatially localized perturbation of the surface, which is directed upwards, it is proved that the part of the potential energy functional which is responsible for nonlinear wave interactions is negatively defined. It is important that this result is obtained without any restrictions on the amplitude of the boundary perturbations, i.e., it takes into account high-order nonlinearities. A general conclusion is that the nonlinearity plays a destabilizing role accelerating the linear instability development of the boundary and defining its explosive character.

1. Introduction
It is known \cite{1-5} that the flat surface of a conducting liquid (e.g., liquid metal for a number of applications \cite{6-11}) is unstable in a strong external electric field. Interaction of the electric field and charges induced by this field on the surface of the liquid leads to an exponential growth of surface perturbations. The system in a finite time reaches a state where the perturbation amplitude becomes comparable with the characteristic wavelength and, consequently, it is not sufficient to use linear theory to describe the evolution of the system. Indeed, it is known from experiments \cite{7, 12} that the instability development results in the formation of cusps at the surface. Local amplification of the electric field strength at the cuspidal points leads, in particular, to the initiation of the explosive electron emission \cite{13, 14}. It becomes necessary to consider nonlinear wave processes for describing such phenomena.

The traditional approach for describing the dynamics of the free surface of a liquid is based on the use of the perturbation theory in a small parameter, viz., the amplitude of the boundary perturbation. For the limit of a strong electric field, where the effects of gravitational and capillary forces are negligibly small as compared to electrostatic ones, it was demonstrated \cite{15-17} that the nonlinearity defines the tendency to the formation of weak root singularities on the surface at which the boundary curvature becomes infinite in a finite time and the surface itself remains smooth. In the near-critical electric field (the plane surface of the conducting liquid is stable with respect to small perturbations in a field below some critical value), the equations have been derived \cite{5, 18, 19} describing the evolution of boundary perturbation amplitudes and taking into account the nonlinearity in the first nonvanishing order. This corresponds to a cubic nonlinearity for the plane symmetry of the problem \cite{20} and to a quadratic nonlinearity for the

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hexagonal symmetry [19]. From the point of view of the Hamiltonian formalism [21–23], which is effective for the analysis of such problems, this corresponds to four- and, respectively, three-wave processes, which are related to the four-order and three-order nonlinearities in the Hamiltonian (it coincides with the total energy of the system, i.e., the sum of the kinetic and potential energies of the liquid in an applied electric field). In the framework of amplitude equations, it was shown that the nonlinearity does not suppress the development of linear instability, but rather accelerates it. This distinguishes the behavior of conducting liquids in an electric field from dielectric ones, for which the nonlinearity suppresses the instability development resulting in the formation of steady-state structures on the surface (this occurs in the case of relatively small density of polarization or free surface charges) [18, 22, 24–26]. At the same time, there remains an open question concerning the influence of the higher-order nonlinearities, which were not taken into account in all mentioned works.

In the present work, we will show that the analysis of the potential energy functional allows making a number of non-trivial conclusions concerning the system behavior. For a symmetric spatially localized perturbation of the surface, which is directed upwards, it is proved that the part of the functional which is responsible for nonlinear interactions is negatively defined, i.e., the nonlinearity plays a destabilizing role accelerating the development of linear instability. This determines the tendency observed in experiments and calculations to the explosive growth of the boundary instability, the formation of singular surface profile in a finite time [4,27–34], and the loss of simple connectivity of the system [35–38].

2. Initial equations
Let a perfectly conducting liquid with the free surface be placed in an external uniform electric field with magnitude $E$ as it is shown in figure 1. We assume that the problem has a planar symmetry: all quantities depend only on the pair of coordinates $x$ and $y$. The direction of the $y$ axis of a rectangular coordinate system coincides with the external-field direction. The free surface in the unperturbed state is the $y = 0$ plane (the $x$ axis lies in this plane and the $y$ axis is directed along the normal to this plane).
The perturbed (deformed by the applied electric field) free surface of the conducting liquid is given by the equation $y = \eta(x)$, i.e., the function $\eta$ specifies the deviation of the surface from its initial flat state. Since the electric field does not penetrate into the perfectly conducting medium, it occupies the domain bounded by the free surface,

$$\eta(x) \leq y < \infty, \quad -\infty < x < \infty.$$  

In the absence of space charges, the distribution of the electric field potential $\Phi$ (the field strength is defined by the expression $E = -\nabla \Phi$) is described by the Laplace equation,

$$\Phi_{xx} + \Phi_{yy} = 0,$$  

which should be solved jointly with the condition of equipotentiality of the conductor surface (without loss of generality, the potential value can be chosen equal to zero),

$$\Phi|_{y=\eta} = 0,$$  

and the condition of the uniformity of the field at infinity,

$$\Phi|_{y=\infty} \rightarrow -Ey.$$  

In the unperturbed state of the considered system, where $\eta(x) = 0$, the electric field distribution is specified by the simple formula

$$\Phi = -Ey.$$  

The perturbation of the potential energy of the system under deformation of the boundary is given by the integral

$$P = P_g + P_\sigma + P_E,$$  

where

$$P_g = \int_{-\infty}^{\infty} \frac{\rho \eta^2}{2} dx,$$

$$P_\sigma = \int_{-\infty}^{\infty} \sigma \left( \sqrt{1 + \eta_x^2} - 1 \right) dx,$$

$$P_E = -\int_{-\infty}^{\infty} dx \int_{\eta(x)}^{\infty} \frac{\varepsilon_0}{2} \left[ (\nabla \Phi)^2 - E^2 \right] dy,$$

where the first term corresponds to the energy of the liquid in the gravity field ($\rho$ is the medium density and $g$ is the acceleration of gravity), the second term gives the surface energy of the medium ($\sigma$ is the surface tension coefficient), and the third term is responsible for the energy of the electric field ($\varepsilon_0$ is the electric constant). It is clear from the boundary problem (2)–(4) that the value of the energy $P$ is completely determined by the shape of the free surface $\eta(x)$.

For the further analysis, it is convenient to represent the potential energy functional in the form

$$P = P_l + P_n,$$

where $P_l$ is the part of the functional $P$ quadratic with respect to $\eta$, and $P_n$ is the rest of the functional, whose expansion starts from cubic term. The pressure at the free boundary is determined by varying the surface profile as $\delta P/\delta \eta$, whence it is clear that the functionals $P_l$ and $P_n$ are responsible for the linear and, respectively, nonlinear contributions to the free energy of the system.

If $P_l > 0$, then, under the boundary deformation, the potential energy of a conservative system increases, and the kinetic one decreases, which indicates to linear stability of the liquid boundary. On the contrary, if $P_l < 0$, then the potential energy decreases and, consequently, the kinetic energy grows that corresponds to the development of linear instability of the boundary.
We now analyze the functional $P_n$ which takes into account the contribution of nonlinearities to the potential energy of the system. If $P_n > 0$ and, simultaneously, $P_l < 0$, then the nonlinear interaction of waves suppresses the development of linear instability (linear and nonlinear contributions to the potential energy are oppositely directed). Otherwise, if both parts of the functional $P$ are negative (linear and nonlinear contributions have the same direction), $P_l < 0$ and $P_n < 0$, nonlinearity accelerates the instability development. Below we will demonstrate that $P_n \leq 0$ under reasonable assumptions about the geometry of perturbations of the conducting liquid boundary, i.e., the nonlinearity plays a destabilizing role, determining the tendency to explosive growth of the boundary instability.

3. Quadratic part of the potential energy functional

In order to find the expression for $P_l$, we pass to the limit of small surface-slope angles in the functional of the potential energy $P$, i.e., we consider the situation where $|\eta_x| \ll 1$. This limit corresponds to considering linear waves on the free liquid boundary.

First, let us consider the surface energy of the system. The integrand in $P_\sigma$ can be expanded in a series in $\eta_x$. We obtain in the leading order

$$\sqrt{1 + \eta_x^2} - 1 \approx \eta_x^2/2.$$

As for the energy of the liquid in the gravity field, the integrand in $P_g$ is originally quadratic nonlinear and it remains unchanged in the small-angle approximation.

For considering the electrostatic energy of the system, it is convenient to introduce the perturbation of the electric field potential as

$$\varphi(x, y) = \Phi(x, y) + E\eta,$$

($\varphi = 0$ for $\eta = 0$). It is clear that the perturbed potential $\varphi$, as well as the original one $\Phi$, satisfies the Laplace equation. The boundary conditions for $\varphi$, as follows from (3) and (4), have the form

$$\varphi|_{y=\eta} = E\eta(x),$$

(6)

$$\varphi|_{y\to\infty} \to 0.$$

(7)

Using the Green’s formulas, the electrostatic energy can be represented as an integral over the free surface,

$$P_E = \frac{\varepsilon_0}{2} \int_s \varphi \frac{\partial \varphi}{\partial n} \, ds,$$

(8)

where $ds$ is the surface differential and $\partial/\partial n$ denotes the normal derivative.

For the function satisfying the Laplace equation at $y > 0$ and decaying at infinity $y \to \infty$, the following solution is valid:

$$\varphi(x, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y\varphi(x', 0)}{(x' - x)^2 + y^2} \, dx'.$$

Also we can take $\partial\varphi/\partial n \approx \partial\varphi/\partial y$ in the small-angle approximation. All this allows us to calculate the normal derivative in (8). Applying the boundary condition (6), we obtain in the leading (quadratic) order of the expansion in $\eta$:

$$\int_s \varphi \frac{\partial \varphi}{\partial n} \, ds \approx -E^2 \int_{-\infty}^{\infty} \eta k_x \eta \, dx,$$
where we put $d_s \approx d_x$. Here $\hat{k}_x$ is the integral operator, whose Fourier transform is equal to the modulus of the wave vector. It can be expressed in terms of the Hilbert transform $\hat{H}_x$ as

$$\hat{k}_x = -\frac{\partial}{\partial x} \hat{H}_x, \quad \hat{H}_x \eta = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{+\infty} \frac{\eta(x')}{x' - x} \, dx', \quad \text{where P.V. denotes the principal value of the integral.}$$

Collecting the resulting expressions together, we find that, in the linear approximation, the potential energy of the system is given by the following quadratic functional:

$$P_l = \int_{-\infty}^{\infty} \left[ \frac{\rho g \eta^2}{2} + \frac{\sigma \eta_x^2}{2} - \frac{\varepsilon_0 E^2 \eta k_x \eta}{2} \right] \, dx. \quad (9)$$

After passing to $k$-representation, the functional $P_l$ takes the form

$$P_l = \left( \frac{\rho g}{2} - \frac{\varepsilon_0 E^2}{2} k + \frac{\sigma}{2} k^2 \right) \int_{-\infty}^{\infty} |\eta_k|^2 \, dk.$$ 

One can see that $P_l < 0$ and, hence, linear instability of the boundary will develop under the condition

$$\rho g - \varepsilon_0 E^2 k + \sigma k^2 < 0,$$

which is known from the linear theory [2, 39]. It follows from this condition that, for the instability onset, the electric field strength must exceed the critical value $E_c$ given by the expression

$$\varepsilon_0 E_c^2 = \sqrt{4 \rho g \sigma}.$$ 

Then the harmonics with the wavenumbers in the range $k_1 < k < k_2$, where

$$k_1 = \frac{\varepsilon_0 E^2 - \sqrt{\varepsilon_0^2 E^4 - 4 \rho g \sigma}}{2\sigma}, \quad k_2 = \frac{\varepsilon_0 E^2 + \sqrt{\varepsilon_0^2 E^4 - 4 \rho g \sigma}}{2\sigma}, \quad (10)$$

become unstable. In the following section, we will consider the effect of nonlinearity on the character of the instability development without restrictions on the boundary perturbation amplitude (that is, in fact, taking into account arbitrary orders of the expansion in $\eta$).

4. Nonlinear contribution to the potential energy

Let us consider the role of nonlinearity in the instability development of the boundary of a conducting liquid in an applied electric field. The nonlinearity contribution to the potential energy of the system can be found as the difference between the initial exact expression (5) for the system free energy and the quadratic approximate expression (9) derived in the previous section,

$$P_n = P - P_l = \frac{\varepsilon_0}{2} F + \frac{\sigma}{2} G, \quad (11)$$

$$F = \int_{s} \varphi \frac{\partial \varphi}{\partial n} \, ds + E^2 \int_{-\infty}^{\infty} \eta k_x \eta \, dx, \quad G = \int_{-\infty}^{\infty} \left[ 2 \left( \sqrt{1 + \eta_x^2} - 1 \right) - \eta_x^2 \right] \, dx.$$ 

Here the functionals $F$ and $G$ take into account the contributions of nonlinearity to the electric field energy and, respectively, to the surface energy.

Now we perform the conformal mapping of the domain (1) onto the half-plane

$$0 \leq v < \infty, \quad -\infty < u < \infty.$$
The boundary shape \( y = \eta(x) \) in terms of new variables is given parametrically as
\[
y = \eta(x(u)) = \eta(u), \quad x = x(u) = u + H_u y(u).
\]
The function \( \varphi \) remains harmonic after the transformation, so that it satisfies the Laplace’s equation in the variables \( u \) and \( v \),
\[
\varphi_{uu} + \varphi_{vv} = 0.
\]
The corresponding boundary conditions follow from (6) and (7),
\[
\varphi|_{v=0} = E \eta(x(u)), \quad \varphi|_{v \to \infty} \to 0.
\]
The first integral in the expression for \( F \) rewrites as
\[
\int_{s} \eta \frac{\partial \varphi}{\partial n} ds = \int_{\infty}^{-\infty} y(u) \frac{\partial \varphi}{\partial v} du,
\]
where we have used the formula
\[
\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}
\]
that follows from the Cauchy–Riemann relations for the conjugate harmonic quantities \( u \) and \( v \).

As a result, we find that the functional \( F \) can be written in the form
\[
F = E^2 \int_{-\infty}^{\infty} \eta(x) \hat{k}_x \eta(x) dx - E^2 \int_{-\infty}^{\infty} \eta(x(u)) \hat{k}_u \eta(x(u)) du,
\]
or, which is the same, in the form
\[
F = \frac{E^2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\eta(x) \eta(x')}{(x - x')^2} dx dx' - \frac{E^2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\eta(x(u)) \eta(x(u'))}{(u - u')^2} du du'.
\]

Redefining the variables in the first term of this expression as \( x \to u \) and \( x' \to u' \), we finally get
\[
F = \frac{E^2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L(u, u')}{(u - u')^2} du du',
\]
where
\[
L(u, u') = \eta(u) \eta(u') - \eta(x(u)) \eta(x(u')).
\]
It is clear that if, for example, \( L(u, u') \leq 0 \) for any \( u \) and \( u' \), then the inequality \( F \leq 0 \) holds.

Let us consider how the function \( L \) value depends on the perturbation shape of the boundary \( \eta \).

Let the following relations be true:
\[
\eta(x) \geq 0, \quad \eta(x) = \eta(-x), \quad \eta(x) \geq \eta(x') \quad \text{for} \quad |x'| > |x|.
\]
decreases with distance from the point \( x = 0 \) (i.e., the surface perturbation is directed upwards). It is easy to notice that the inequality \( L \leq 0 \) is satisfied if
\[
|x(u)| \leq |u|.
\] (16)

We need to consider whether this requirement is met in our case, i.e., under the conditions (13)–(15).

Differentiating \( x(u) \) with respect to \( u \), we get
\[
x_u = 1 - \hat{k}_x \eta(x(u)) = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta(x(u'))}{(u - u')^2} \, du'.
\]
Since \( \eta \) is the non-negative function, the following inequality holds: \( x_u \leq 1 \). The variable \( u \) can be always chosen so that \( x(0) = 0 \) (i.e., the plane \( u = 0 \) in the new coordinates coincides with the plane \( x = 0 \) in the original ones). Then, integrating this inequality over \( u \), we arrive at the required condition (16).

Thus, we have proved that, under the conditions (13)–(15) for the boundary shape, the inequality \( L(u, u') \leq 0 \) holds for any \( u \) and \( u' \). Then, as it was already indicated (it can be seen immediately from (12)), the following inequality is fulfilled: \( F \leq 0 \).

Now let us consider the functional \( G \). After a simple algebra, it takes the form
\[
G = - \int_{-\infty}^{\infty} \eta_x^4 \left( \sqrt{1 + \eta_x^2} + 1 \right)^{-2} \, dx.
\]
It is obvious that \( G \leq 0 \) for any \( \eta \).

It follows from (11) that \( P_n \leq 0 \) if the conditions \( F \leq 0 \) and \( G \leq 0 \) are fulfilled. Then we can conclude that \( P_n \leq 0 \) under the conditions (13)–(15), i.e., the part \( P_n \) of the potential energy functional \( P \) responsible for the influence of nonlinearity is negatively defined. As a consequence, the nonlinearity plays a destabilizing role, accelerating the development of linear instability of the initially plane boundary of the liquid.

5. Conclusion

In the present work, it has been demonstrated that the part of the potential energy functional of the system (conducting liquid with the free surface in an external electric field) responsible for nonlinear wave interactions is negatively defined. This result is obtained for a symmetric spatially localized perturbation of the surface (it is directed upwards) without restrictions on its amplitude. Previously, a similar analysis was performed only within a weakly-nonlinear approximation, where the influence of nonlinearities was taken into account in the first nonvanishing order (see, for example, [5, 19]).

A decrease in the potential energy of a conservative system leads, obviously, to an increase in the kinetic energy and, as can be expected, to an acceleration of the electrohydrodynamic instability development. The boundary perturbations grow exponentially in the linear stage of the instability. Then, due to destabilizing effect of nonlinearity, the explosive growth of the perturbations occurs in the advanced instability stages. This determines the specific features of the system behavior, namely, the tendency to developing singularities on the free surface in a finite time. Particular self-similar solutions of the motion equations found in papers [4, 30, 31] (they describe the dynamics of the formation of the so-called Taylor cones [40] on the charged boundary of a liquid) also indicate to the presence of this tendency.

Note that the main results of the present work are applicable to the liquid helium with the free surface charged by electrons in the presence of an external electric field. The basic difference is that we should consider the boundary perturbation directed downwards. This is due to the fact that the situation, where the electric field is present in the bulk of the liquid and it is...
fully shielded over the surface by the surface charge, can be realized for liquid helium [25, 41]. There arises the tendency to the formation of cuspidal dimples [42–44] instead of the cusps for a perfectly conducting liquid.

Acknowledgments

The authors are grateful to E A Kuznetsov for stimulating discussions. This work was supported in part by the Russian Academy of Sciences (programs of the Presidium RAS No. 2 and 11 and program of the Presidium UB RAS “Physical models, theories, devices”, project 18-2-2-15) and by the Russian Foundation for Basic Research (projects No. 17-08-00430 and 19-08-00098).

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