Entanglement and local information access for graph states

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We exactly evaluate a number of multipartite entanglement measures for a class of graph states, including d-dimensional cluster states (d = 1, 2, 3), the Greenberger-Horne-Zeilinger states, and some related mixed states. The entanglement measures that we consider are continuous, ‘distance from separable states’ measures, including the relative entropy, the so-called geometric measure, and robustness of entanglement. We also show that for our class of graph states these entanglement values give an operational interpretation as the maximal number of graph states distinguishable by local operations and classical communication (LOCC), as well as supplying a tight bound on the fixed letter classical capacity under LOCC decoding.

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I. INTRODUCTION

The understanding and quantification of entanglement can be said to be one of the most fundamental problems in quantum information [1]. Entanglement measures often have operational meanings. The distillable entanglement, for example, is the asymptotic number of Bell pairs extractable by local operation and classical communication (LOCC). Entanglement measures can also be used to classify quantum resources, such as the necessary condition presented in [2] for universal resources in one-way quantum computation [3]. Theoretical knowledge of entanglement values for interesting states may also enable us to estimate those of experimentally prepared states only via measurements of linear witness operators [4]. However, apart from bipartite scenarios, the calculation of truly multipartite entanglement measures is generally considered to be formidable even for pure states (cf. [5]).

We will primarily be interested in a set of simple multi-qubit entangled states known as “graph states” [6, 7], or stabilizer states (up to local unitaries), which have proven useful in a variety of quantum information tasks. They include the Greenberger-Horne-Zeilinger (GHZ) state, cluster states (a universal resource for one-way quantum computing [3]), and Calderbank-Shor-Steane (CSS) error correction codeword states. Graph states themselves can be seen as algorithmic specific resources in the framework of one-way computing [3], due to their common simple prescription to prepare. Closely related weighted graph states have recently found use in approximating ground states for strongly-interacting spin Hamiltonians [8]. Furthermore, small graph states, in particular cluster states, are a current topic in the laboratory [9] and have been used for one-way quantum computation [10].

One of the main purposes of this paper is to exactly evaluate continuous, distance-like multipartite entanglement measures for graph states, on the way giving them an operational interpretation. Our idea is to utilize the connection between these widely-studied entanglement measures and LOCC state discrimination (see e.g. [11] and refs. therein). This will allow us to give upper and lower bounds to the entanglement of any graph state by using a simple graphical interpretation of these states. In several interesting cases these upper and lower bounds match, hence giving the exact entanglement values. It is remarkable that our argument does not require any difficult calculations despite the generally troublesome optimization that is usually involved in calculating the entanglement measures that we consider. Our method is advantageous as not only does its graphical nature mean that it is often easy to obtain the answer (namely the entanglement values of interest), but it also provides intuition about the nature of multipartite entanglement, for example, how the amount of multipartite entanglement is related to the way Bell pairs are stored in it.

We begin by giving some background definitions and results in section II. We then present the calculation of entanglement and LOCC discrimination protocols for sets of pure graph states in III. In section IV this is extended to several related mixed states. These results are given an operational interpretation in section V where we use previous results to give tight fixed letter channel capacities for LOCC decoding. We finish with conclusions.

II. GRAPH STATES AND ENTANGLEMENT MEASURES

Graph states |G_{k_1 \ldots k_n}\rangle of n qubits can be described pictorially by a graph \( G \) of \( n \) vertices, with \( n \) binary indices \( \{k_1, \ldots, k_n\} \) such that \( k_i = 0, 1 \) [12]. Let us denote the Pauli matrices at the \( i \)-th qubit by \( X_i, Y_i, Z_i \) with the identity \( I_i \). The vertices of \( G \) represent qubits, each of which is initially prepared in the \((-1)^{k_i}\) eigenstate of \( X_i \), i.e. \( \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{k_i}|1\rangle) \). The graph states are then defined by performing 2-qubit Control-Z oper-
ations \((CZ_{ij} = \text{diag}(1,1,1,-1)\) in the \(Z\) basis), e.g. via an Ising interaction, on all pairs of qubits joined by the edges \(E_d\) of \(G\). It can be shown that the \(2^d\) graph states \(\{G_{k_1\ldots k_n}\}\) are the joint eigenstates of the \(n\) independent commuting operators (called stabilizer generators):

\[
K_i := X_i \bigotimes_{(i,j) \in E_d} Z_j \quad i = 1, \ldots, n
\]

such that the graph states satisfy the eigenequations

\[
K_i |G_{k_1\ldots k_n}\rangle = (-1)^{k_i} |G_{k_1\ldots k_n}\rangle,
\]

i.e. the index \(k_i\) gives the stabilizer eigenvalue \((-1)^{k_i}\) for \(K_i\). Hence if we can measure \(K_i\), we can determine \(k_i\). We will later utilize this feature to design protocols for discriminating sets of these states using LOCC. Note that for a given \(G\), the graph states \(\{G_{k_1\ldots k_n}\}\) construct a complete orthonormal basis of dimension \(D_H = 2^n\). They are local unitarily equivalent as \(|G_{k_1\ldots k_n}\rangle = \prod_{i=1}^n Z_i^{k_i} |G_{0\ldots 0}\rangle\), hence they all have equal entanglement.

Our methods can be used to derive additive bounds on entanglement values for any graph state. However most of the graphs that we will consider explicitly in this paper are examples of “two-colourable” graph states. These are defined as graph states where it is possible to assign one of two colours (say Amber and Blue) to each qubit, such that no two qubits connected by an edge have the same colour. Many of the graph states that have found significance in quantum information are of the two-colourable kind - e.g. GHZ states, CSS codeword states, and cluster states are all two-colourable.

Among the graph states that we consider, we are able to derive exact entanglement values for \(dD\) cluster states (i.e. the \(d\)-Dimensional cubic grid graphs of Figs. 1 and 3), the GHZ states (i.e. the tree graph of Fig. 2 with one centre vertex and \(n - 1\) vertex ‘leaves’), and the Steane [7,1,3] codeword state. To our knowledge, the entanglement values for such graph states have been calculated only for the discrete Schmidt measure (i.e. the minimum number of terms required in a product state expansion of the state) in Ref. [7] (cf. [14]). A merit of continuous entanglement measures is that their values are stable under small deviation in the state space.

The entanglement measures that we consider in this paper are defined as follows. For a state \(\rho\), the relative entropy of entanglement [15] is defined as

\[
E_R(\rho) = \min_{\omega \in \text{SEP}} \text{tr} \rho (\log_2 \rho - \log_2 \omega),
\]

where the minimum is taken over all fully separable mixed states \(\omega\). The global robustness of entanglement [16] is defined as

\[
R(\rho) = \min_{\omega} t
\]

such that there exists a state \(\omega\) such that \((\rho + t\omega)/(1 + t)\) is separable. For convenience we also define an extension of the geometric measure [17] as

\[
E_g(\rho) = \min_{\omega \in \text{SEP}} - \log_2 (\text{tr} \omega)
\]

(note that \(E_g\) is an entanglement monotone only for pure states \(\rho\)).

In Ref. [11] it has been shown that the maximum number \(N\) of pure states in the set \(\{|\psi_i\rangle| i = 1, \ldots, N\}\), that can be discriminated perfectly by LOCC (in fact by separable [18] measurements), is bounded hierarchically (cf. [19]) by the amount of entanglement they contain:

\[
N \leq \frac{D_H}{1 + R(|\psi_i\rangle)} \leq \frac{D_H}{2E_R(|\psi_i\rangle)} \leq \frac{D_H}{2E_g(|\psi_i\rangle)}
\]

where \(D_H = 2^n\) is the total dimension of the Hilbert space, and \(\mathbf{t_i} = \frac{1}{n} \sum_{i=1}^n x_i\) denotes the “average”. In fact it can be shown, using a local symmetry argument (“twirling”), that for all pure stabilizer states the two rightmost inequalities of Eq. (6) collapse to equalities [20]. However, although these inequalities give powerful insight into the relationship between LOCC state discrimination and the quantification of entanglement, computing any of the measures in Eq. (6) is usually extremely difficult. Hence it is not clear to what extent these relationships will enable quantitative progress. The aim of our paper is to show that in fact, for large classes of graph states, these inequalities may be exploited to derive strong explicit bounds that in many cases are exact.

As many of the graph states that we consider have played a diverse and important role in the literature, it is likely that our computations will prove useful in understanding the role of these entanglement measures in multiparty quantum information scenarios.

### III. LOCC Discrimination and Entanglement of Graph States

We are now ready to apply Eq. (6) to graph states with a given graph \(G\). In order for the states to perfectly distinguishable, they must be orthogonal. Hence we consider the perfect LOCC discrimination of a subset of the complete orthonormal basis \(\{|G_{k_1k_2\ldots k_n}\rangle\}\). Since all states in the set have equal entanglement, i.e. \(E(|G_{k_1k_2\ldots k_n}\rangle) = E(|G_{0\ldots 0}\rangle) \forall k_i\), the average in Eq. (6) can be replaced by \(E(|G_{0\ldots 0}\rangle)\).

We will evaluate the hierarchy of inequalities in Eq. (6) from above and below in terms of graph problems. Adopting the notation of equation (6), for a given choice of graph \(G\) let \(N\) denote the largest number of graph states associated with \(G\) that may be perfectly discriminated using LOCC. In subsection (A) below we will derive a simple lower bound on \(N\) by finding sets of graph states that can be constructed by explicitly distinguished by LOCC. In subsection (B) we will provide an upper bound on the rightmost term of equation (6) based upon a very elementary analysis of the bipartite entanglement present across certain splittings of the graph states. We
will then show that these two bounds meet for many sets of two-colourable graph states mentioned above, hence giving both their exact entanglement values as well as the maximal possible $N$. Our methods are also applicable to certain mixed states, which we will discuss in a later section.

A. Lower “colouring” bound

A lower bound for $N$ can be given by maximizing the number $m_c$ of stabilizer generators $\{K_i\}$ that can be determined simultaneously in a single setting of LOCC measurements. If we can evaluate $m_c$ eigenvalues of stabilizer generators by LOCC, we know that we can discriminate deterministically at least $2^{m_c}$ states, by picking only one state from each subspace determined by the $m_c$ eigenvalues. For example, by finding the eigenvalues of $\{K_i\}_{i=1}^{m_c}$, we can discriminate the set of $2^{m_c}$ states $\{|G_{k_1,k_2...k_{m_c}}0...0\rangle\}$. Therefore, we have a lower bound as

$$2^{m_c} \leq 2^{\max m_c} \leq N.$$  \hspace{1cm} (7)

One approach is to identify a set of $|A|$ qubits that are not connected to each other and are thus coloured by a single colour (we call these the Amber qubits). By locally measuring $X$ on all the Amber qubits, and locally measuring $Z$ on all the others, one can use Eq. (1) and classical communication to determine the eigenvalues $\{k_i|i \in A\}$ corresponding to the subset of generators $\{K_i|i \in A\}$. We can see this as follows. Without loss of generality we label the amber qubits $i = 1,\ldots,|A|$. By Eq. (1) we see that the Amber generators $\{K_i\}_{i=1}^{|A|}$ have only $\mathds{1}$ or $X$ on the first $|A|$ qubits, and then either $Z$ or $\mathds{1}$ on the remaining ones. Hence the measurement of the generators from the set $\{K_i\}_{i=1}^{|A|}$ can be simulated by LOCC measurement by locally measuring $X$ on all the Amber qubits, and locally measuring $Z$ on all the others, and then communicating the outcomes. In order to get as tight a bound as possible we would like to have $m_c = |A|$ as large as possible. Hence Eq. (7) translates into a bound

$$2^{|A|} \leq N,$$  \hspace{1cm} (8)

where $|A|$ is the (maximum) number of mutually disconnected qubits, i.e., the (maximum) number of vertices which can be coloured by the same single colour.

In graph theoretic terminology the set of vertices that achieves such a maximum is known as the maximum independent set.

In the case of two colourable graphs this analysis can be particularly simple. For a given 2-colouring of the graph one can set the colour with the larger number of vertices to be Amber and the other to be Blue, $B$, i.e., set $m_c = |A| \geq |B|$. We illustrate this colouring strategy in Fig. 1-3. In those figures we readily find that the optimal lower bound that can be obtained using this approach is given for GHZ states by setting all the leaf vertices as Amber, i.e. max $m_c = n - 1$, whereas for dD cluster states we find that max $m_c = \lceil \frac{n}{2} \rceil$. The best lower bounds for ring states (closed 1D chain graph) and Steane codeword states can be shown in a similar way - the results are summarized in Table I.

One may enquire whether these lower bounds may be improved upon by taking into account the fact that different graphs may correspond to physical states that are equivalent under local unitary transformations (i.e. with the same entanglement properties). Indeed, as we show in Fig. 2, for the GHZ state there are two different graphs - the fully connected one, and the usual ‘tree’ graph. The fully connected graph has a maximum independent set of 1 vertex, whereas the ‘tree’ gives a maximum independent set of $n - 1$. Hence by considering different graphs corresponding to the same graph state (up to local unitary transformations) one may obtain vastly improved lower bounds. However, it turns out that for the examples in this paper this freedom does not lead to better lower bounds than the ones that we present in Table I. This includes in particular the ring graph state for which

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & $2^{\max m_c}$ & $2^{n-\max m_c}$ & $E_R$, $E_R$, $\log_2(1+R)$ \\
\hline
\hline
2D cluster & $2^{\frac{n}{2}}$ & $2^{n-\frac{n}{2}} = 2^{\frac{n}{2}}$ & $[\frac{n}{2}]$ \\
GHZ & $2^{n-1}$ & $2^{n-1}$ & 1 \\
ring & $2^{\frac{n}{2}}$ & $2^{n-\frac{n}{2}} = 2^{\frac{n}{2}}$ & $[\frac{n}{2}] \leq E \leq [\frac{n}{2}]$ \\
Steane code & $2^4$ & $2^{7-3}$ & 3 \\
\hline
\end{tabular}
\caption{Summary of the lower and upper bounds, and entanglement values [21]. Information encoded on these states, can be decoded by LOCC with capacity $C = n - E$.}
\end{table}
there exists a gap between the lower and upper bounds. We will however use this freedom in the next section (via the method of Local Complementation) where it will enable us to improve the upper bounds that we will obtain for certain types of graph.

Note that the LOCC identification of stabilizer elements is also an important primitive in entanglement distillation [13, 22]. The only real differences being the fact that in distillation protocols stabilizer eigenvalues are determined indirectly (i.e., their parities) in order not to destroy entanglement, and furthermore all stabilizer eigenvalues are evaluated to get a specific target pure graph state from an ensemble of identical noisy copies.

### B. Upper “matching” bound

We will obtain an upper bound to the rightmost term of Eq. (6), by weakening the constraint of full-separability. If we define $E_{gs}$ as the geometric measure with respect to some bipartition, we have that $E_g \geq E_{gs}$, since the set of fully separable states is a subset of the bipartite separable states. Hence,\[ \frac{D_H}{2^{E_g(|G_{oo...o}))}} = 2^{n - E_g(|G_{oo...o})} \leq 2^{n - \max E_{gs_i}(|G_{oo...o})}. \] (9)

Our strategy is to try to find a suitable bipartition across which the entanglement is as large as possible. Once we have specified a bipartition we can readily calculate the entanglement across it by several methods (cf. Refs. [7, 23]). However, we will pose this question as a graph “matching” problem in order to gain intuition into when the bounds can be tight.

In particular, we will consider transforming a graph by bipartite LOCC into another graph made only of $m_p$ disjoint “matched” Bell pairs (we may extend these ideas also to other tree-type units). Since local unitaries leave bipartite entanglement unchanged, the entanglement is then simply $E_{gs_i} = m_p$.

The simplest cases arise when local applications of Control-Z can be used to erase edges within each partition, as we illustrate in figures (1-3). This simple approach is often sufficient to match the upper bounds we obtained in section (A). For example, this approach works for even cluster states (cf. Figs. 1-3), even ring states, GHZ states, and the Steane code.

For more complicated graphs such an elementary approach does not always work, and so we must utilize the so-called local complementation (LC) of graph states [6], which corresponds to a multi-local unitary operation $V_i$ on the $i$-th qubit and its neighbors, defined as $V_i = \sqrt{\prod_{i,j} \exp}(\frac{i\pi}{4} X_i) \prod_{(i,j) \in E_G} \exp(\frac{i\pi}{4} Z_j)$. LC centres on a qubit $i$ is visualized readily as a transformation of the subgraph of $i$-th qubit’s neighbours, such that an edge between two neighbours of $i$ is deleted if the two neighbours are themselves connected, or an edge is added otherwise. The use of LC to transform the odd 2D cluster state into a bunch of Bell pairs is illustrated in Fig. 4.

The technique of LC often enables us to match the lower bounds derived using the approach of section (A). For example, although we do not present further details, it turns out that LC can easily be used to derive the optimum number of Bell pairs for all 1D, 2D or 3D cluster states, including those with an odd total number of qubits. In the case of ring graphs with an odd number of vertices, examples of non-two-colourable graphs, LC does not enable us to exactly match the bounds of section (A), but nevertheless enables us to achieve quite close bounds on the exact value (see Table 1). In all of the cases discussed above, except for the odd ring states, this bound matches the lower colouring bound and we have equivalence for all of the measures in equation (6) (cf. Table 1).

It is interesting that for important states such as cluster states, all of these multipartite entanglement values coincide with bipartite entanglement ones (as well as the Schmidt measure values in Ref. [7]). Note however that it is never possible to transform these graph states into
The central equalities follow from a local symmetry in the graph and the proofs illustrated in Figs. 1-4. For two colourable graphs, we can easily extend without much difficulty to prove the matching for 3D cluster states and the Steane code state. Good bounds are given for the ring state. These results are summarised in Table I and the proofs illustrated in Figs. 1-4 (the ring state, 3D cluster and the Steane code state proofs are simple adaptations of the figures and so not shown).

Furthermore, for these states the entanglement quantities are additive, since bipartite pure state entanglement is additive for these measures. This will prove useful in section V where we consider discrimination in an asymptotic setting for calculating classical channel capacities. This additivity also has implications for investigations of asymptotic distillation of multipartite quantum states, where regularized (asymptotic) measures such as the relative entropy of entanglement can play an important role, see e.g. Ref. [24].

IV. CLASSES OF MIXED STATES

The above methods also allow us to compute the entanglement of certain mixtures of graph states. The reason for this is that in obtaining the exact entanglement values for the graph states discussed above, we are actually also able to derive an explicit form for the ‘closest’ separable state in the various entanglement measures. If two or more pure graph states share the same ‘closest’ separable state, then one can exploit the fact that mixtures of these states will also share the same ‘closest’ separable state, and this enables us to compute the entanglement of such mixtures. In more detail, the argument proceeds by the following steps:

1) Each joint eigenspace of the Amber generators \( \{K_i|i \in A\} \) can be spanned by product states. Suppose the set of Amber generators \( \{K_i|i \in A\} \) is simultaneously determined as discussed previously. The outcomes of these measurements determine one of the joint eigenspaces of the operators \( \{K_i|i \in A\} \). Each such eigenspace has dimension \( 2^{|A|} \) and, more crucially, it can be shown that each of these eigenspaces can be spanned by product states. This can be seen as follows - if we pick a new set of stabilizer generators \( \{K_i|i \in A\} \cup \{Z_i|i \notin A\} \), then these operators are all mutually commuting, and it is not too difficult to verify that they are the stabilizers of product states. As this new set of stabilizers contains the Amber generators, \( \{K_i|i \in A\} \), this means that it must be possible to span the joint eigenspaces of the \( \{K_i|i \in A\} \) by product states.

2) Consider any graph for which \( E_R(|G\rangle) = E_g(|G\rangle) = |B\rangle \). In such cases, for each graph state \( |G\rangle \) selected from a given eigenspace \( S_A \) of the operators \( \{K_i|i \in A\} \), a closest separable state for the relative entropy and geometric measures can be taken to be the equal mixture of all product states spanning \( S_A \). The relative entropy between a given pure graph state \( |G\rangle \) and the equal mixture of all pure states spanning \( S_A \) can easily be calculated as:

\[
- \log_2 \left( \frac{1}{2^{n-|A|}} \right) = |B| \tag{11}
\]

As the conditions of the statement assert that this achieves \( E_R(|G\rangle) \), we know that in these cases the equal mixture of states spanning \( S_A \) must provide an optimal separable state. Similar arguments apply for the geometric measure.

3) Consider any graph for which \( E_R(|G\rangle) = E_g(|G\rangle) = |B\rangle \). In such cases, for any mixture of graph states selected from a given eigenspace \( S_A \) of the operators \( \{K_i|i \in A\} \), a closest separable state for the relative...
entropy and geometric measures can be taken to be the equal mixture of all product states spanning $S_A$. This follows straightforwardly from the previous statement, and the fact that if the same separable state $\omega$ is optimal for two states $\rho_1, \rho_2$, then it is also optimal for mixtures $p_1\rho_1 + (1 - p_1)\rho_2$ (this in turn follows from the convexity of both the geometric measure functional and the relative entropy).

Thus, consider any mixture $\rho$ of graph states $|G_\vec{k}\rangle$ from $S_A$, i.e.,

$$\rho = \sum_{\vec{k}} \lambda_{\vec{k}} |G_\vec{k}\rangle \langle G_\vec{k}|,$$

where the $\lambda_{\vec{k}}$ are the eigenvalues of $\rho$, and are nonzero only for indices $\vec{k}$ such that the $\{|k_i| i \in A\}$ take constant (but otherwise arbitrary) values. The relative entropy and geometric measures can be computed for such mixed states as

$$E_R(\rho) = |B| - S(\rho),$$
$$E_g(\rho) = |B|,$$

where $S(\rho) = -\text{tr} \rho \log_2 \rho$. Note that these results include any binary mixture of Bell basis states for the 2-qubit case, and these expressions are additive for tensor products of these states.

(4) Computing the robustness $R$ for such mixed states. For such mixed states one can also derive the robustness of entanglement as

$$R(\rho) = 2^{[|B|]} \max_{\vec{k}} \lambda_{\vec{k}} - 1.$$  

This is based on the following lower bound for the robustness of a state given the robustness and weight of one state in its convex decomposition, the derivation of which is shown in footnote [25]:

$$R(\rho) \geq p_0(1 + R(\rho_0)) - 1.$$  

It is easy to show that this lower bound is achieved when $\rho$ is a mixture of graph states from the same joint eigenspace $S_A$, by admixing in the minimal $\omega$ that turns $\rho$ into an equal mixture of pure states spanning $S_A$, thereby giving equation (14).

We can also extend the above analysis to consider some mixtures of pure graph states corresponding to different graphs. Provided that (a) the states being mixed are defined for the same number $n$ of qubits, (b) they all have entanglement $E_R(|G\rangle) = E_g(|G\rangle) = |B|$, (c) the pure graph states being mixed are taken from the eigenspace $S_A$, and (d) the generators for the Amber qubits do not change, then Eqs. (13) still stand. For example, take any two colourable graph $G$ used in the previous sections, and apply the local complementation operation to any of the Amber qubits to generate graph $G'$. Clearly two (potentially non-orthogonal) states $|G_\vec{k}\rangle$ and $|G'_\vec{k}\rangle$ have the same entanglement since the transformation is local. Also, the generators $\{K_i| i \in A\}$ remain unchanged since the edges to neighbors of Amber qubits are unaffected. Thus, following the same logic as above, our entanglement measures $E_R$ and $E_g$ for the mixed state of the form

$$\rho = u|G_\vec{k}\rangle \langle G_\vec{k}| + (1 - u)|G'_\vec{k}\rangle \langle G'_\vec{k}|,$$

are given by the same formula in Eqs. (13). The analysis used to derive the formula for the robustness in Eq. (14) does not seem to be extended to this case straightforwardly.

V. CLASSICAL CAPACITY OF QUANTUM MULTIPARTY CHANNELS

Imagine we have encoded classical information onto multipartite quantum states, we can ask how well we can access information in these states locally, as an application of the preceding results. We begin by extending Eq. (6) to the probabilistic case. Suppose that we have been given a state from an ensemble $\{\rho_i\}$ which we will measure with an LOCC POVM $\{M_j\}$. The conditional probabilities of getting each measurement outcome are given by $p(j|i) := \text{tr}(M_j \rho_i)$. In the manner of [11], we bound these conditional probabilities in terms of the entanglement of each $\rho_i$. Any POVM element in an LOCC measurement can be written as $M_i = s_j \omega_j$, where $s_j = \text{tr} M_j$, and $\omega_j$ is a separable normalized quantum state. The conditional probability of successful discrimination is hence bounded as $p(j|i) = s_j \text{tr}(\omega_j \rho_i) \leq s_i 2^{-E_g(\rho_i)}$, where the last inequality follows as $\omega_i$ must be separable. Due to the completeness of the POVM elements $\sum_i s_i = D_H$, this condition can be rearranged and bounded as

$$\sum_i p(i|i) 2^{E_g(\rho_i)} \leq D_H.$$  

This equation can prove useful in the analysis of fixed-letter channel capacities using graph state codewords with LOCC readout. Following logic similar to [20], suppose that Alice transmits codewords of length $L$ formed from strings of states $\{\rho_i\}$. If we require that the receivers must decode using LOCC measurements with worst case conditional error bounded as $1 - p(i|i) < \epsilon$, then Eq. (17) will give a bound on the rate of the code. If we assume that the geometric measure is additive under tensor products of our signal states, as is the case for all the two-colourable examples discussed in previous sections, then the maximum number $N(L)$ of possible codewords of length $L$ must be bounded, according to Eq. (17) with $D_H = 2^{L_n}$, as:

$$\log_2 N(L)/L \leq n - E_g(\rho_i) - \log_2 \{(1 - \epsilon)/L\}.$$  

In the large blocklength limit $L \to \infty$ the third term vanishes, and this gives a bound on the capacity of:

$$C \leq n - E_g(\rho_i).$$
This general bound holds whenever the states \( \{ \rho_i \} \) in the ensemble have a geometric entanglement that is additive (note that additivity does not always hold \cite{27}, though it does for all states considered here). This capacity bound is achievable (tight) for our examples in Table I by selecting \( \{ \rho_i \} \) from the appropriate subspaces of graph states and discriminating them perfectly using the colouring protocols in section III.

However, for pure graph states the bound of Eq. (19) in all these examples is no better than a similar bound by bipartite entanglement measures derived recently in Ref. \cite{26}, since the geometric entanglement \( E_g \) unfortunately reduces to bipartite entanglement. Nevertheless, our discussion also applies to the mixed states discussed in the previous section, and also shows that no tighter bound than Eq. (19) can be derived using only the entanglement properties (i.e. local unitarily invariant functions) of these states.

VI. CONCLUSION

We have introduced a simple graphical strategy, via Eq. (6), to evaluate several distance-type multipartite entanglement measures for graph states and have shown exact values, seen in Table I, for interesting graph states such as 1, 2, and 3D cluster states with an arbitrary number of qubits. The lower and upper bounds in our evaluation can be formulated as widely-studied graph problems (up to local unitary equivalence of graph states). The lower bound is rephrased as the maximum independent set problem \cite{28} (by associating such a set with Amber) which is known to be NP-complete in general. On the other hand, the upper bound is formulated as the maximum matching problem for which polynomial time algorithms exist in general graphs (cf. Ref. \cite{29} for quantum algorithms), but we have to check additionally whether all erased edges are attributed to local operations in a given matching. Taking advantage of existing approximate algorithms in graph theory, it may be possible to obtain a good estimate for entanglement values for wider graph states. It is also conceivable that Local Complementation may be used to improve both the lower and upper bounds in the cases where they are not tight. Although searching over the complete orbit of a given graph under local complementation appears to be exponentially complicated in the number of qubits \cite{30}, the results that we present in Table I show that for many interesting classes of graph state the canonical choice of graph can often already give tight bounds.

Recently in Ref. \cite{31}, our results have been found to have a direct application to the research of one-way quantum computation, too. Since entanglement is simply consumed in the course of one-way computation, roughly speaking, the amount of multipartite entanglement in the initial resource states must be high enough to be capable to carry on universal computation. It is shown that a suitably chosen entanglement measure (like the geometric measure we addressed) gives a necessary criterion for universal resources. Since the geometric measure for a known universal resource state for one-way computation, namely the 2D cluster state, grows unboundedly with the system size \( n \), any universal resource state must have an unbounded amount of entanglement for the geometric measure, as well. This criterion immediately implies, for example, that the GHZ state, for which \( E_g = 1 \) regardless of \( n \), cannot be a universal resource.

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composition of a state $\rho = \sum_{k=0}^{d} p_k p_k$, a real number $t \in \mathbb{R}^+$, and another state $\omega$ such that:

$$\rho + t\omega \in SEP.$$  

Then we see that:

$$p_0 \rho_0 + t\omega + \sum_{k>0} p_k \rho_k \in SEP \Rightarrow \rho_0 + \left( (t + \sum_{k>0} p_k) / p_0 \right) \tilde{\omega} \in SEP,$$

where $\tilde{\omega}$ is the normalised state:

$$\left( t\omega + \sum_{k>0} p_k \rho_k \right) / \left( t + \sum_{k>0} p_k \right).$$

From these equations and the definition of the robustness we see that:

$$\left( t + \sum_{k>0} p_k \right) / p_0 \geq R(\rho_0)$$

$$\Rightarrow t \geq p_0 R(\rho_0) - \sum_{k>0} p_k$$

$$\Rightarrow t \geq p_0 R(\rho_0) - (1 - p_0)$$

$$\Rightarrow R(\rho) \geq p_0 (1 + R(\rho_0)) - 1.$$

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