The forced axisymmetric oscillations of an oblate drop sandwiched between different inhomogeneous surfaces under AC vibrational force

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Abstract. The forced oscillations of an incompressible fluid drop in the uniform AC electric field are considered. The external electric field acts as an external force that causes motion of the contact line. In order to describe this contact line motion the modified Hocking boundary condition is applied: the velocity of the contact line is proportional to the deviation of the contact angle and the rate of the fast relaxation processes, whose frequency is proportional to twice the frequency of the electric field. The equilibrium drop has the form of a cylinder bounded by axially parallel solid inhomogeneous planes. These plates have different surface (wetting etc.) properties. The solution of the problem is represented as a Fourier series in eigenfunctions of the Laplace operator. The resulting system of heterogeneous equations for unknown amplitudes was solved numerically. The amplitude-frequency characteristics and the evolution of the drop shape are plotted for different values of the problem parameters.

1. Introduction
A change of a contact angle under the influence of an external force is one way to change of the wetting of a solid surface by a liquid. An electric field can perform the role of such a force – electrowetting (EW). If conduct plate (substrate) is covered with a dielectric layer, then this is electrowetting-on-dielectric (EWOD). Now EWOD has found wide application in various fields, such as electronic display technology [1,2], variable-focus liquid lenses [3,4], digital (droplet) microfluidic devices for bioanalysis (lab-on-a-chip) [5,6], etc.

For described of a dynamics of a contact line (and contact line) the effective boundary condition is proposed in [7] taking as a premise the Hocking equation [8] for a cylindrical drop:

$$\frac{\partial \zeta}{\partial t} = \pm \Lambda \left( \frac{\partial \zeta}{\partial z} + A \cos(2\omega t) \right),$$

where \(\zeta\) is the deviation of the drop interface from the equilibrium position, \(z\) is the axial coordinate, \(\Lambda\) is a phenomenological constant (the so-called wetting parameter or Hocking parameter), having the dimension of velocity, \(A\) is the effective AC amplitude, \(\omega\) is the AC frequency. The Hocking equation and other aspects of contact line motion are reviewed in greater detail in [9-15].

This paper is intended as an extension of work [7]. In contrast to all other papers [7, 14-17], we consider the case of different plates, i.e. the plates have different surface (wetting etc.) properties. Also these plates have non-uniform surfaces (as opposed to [18]). In order to describe the motion of the contact line, the modified boundary condition (1) is used: the velocity of the contact line is
proportional to the deviation of the contact angle and the rate of fast relaxation processes, whose frequency is proportional to twice the electric field frequency.

2. Problem formulation

The problem formulation largely coincides with that developed in articles [11,14,15]. An drop of incompressible liquid is surrounded by another liquid (see figure 1). The density of the drop is \( \rho_i^* \) and the density of the surrounding liquid is \( \rho_e^* \). In what follows, the quantities with subscript \( i \) refer to the drop, and those with subscript \( e \) – to the surrounding liquid. Both liquids are bounded by two parallel solid surfaces at a distance \( h^* \) from one another. The equilibrium contact angle \( \theta_0 \) between the lateral surface of the drop and the solid surface is equal to \( \pi/2 \). The external uniform alternating electric field acts as an external force that causes the contact line motion.

![Figure 1. Problem geometry (1 – electrode, 2 – dielectric layer).](image)

Let the surface of the droplet be described by the equation \( r^* = R_0^* + \zeta^*(\alpha, z^*, t^*) \) according to the cylindrical coordinates \( r^*, \alpha, z^* \). The azimuthal angle \( \alpha \) is reckoned from the x-axis. Taking the length \( R_0^* \), the height \( h^* \), the density \( \rho_i^* + \rho_e^* \), the time \( \sigma^{-1/2}(\rho_i^* + \rho_e^*)R_0^{3/2} \), the velocity \( A^* \sqrt{\sigma((\rho_i^* + \rho_e^*)R_0^*)^{-1/2}} \), the pressure \( A^* \sigma(R_0^*)^{-2} \), and the deviation of the surface \( A^* \) as characteristic quantities, we pass to dimensionless variables and obtain the following linear problem:

\[
p_j = -\rho_j \varphi_{jj}, \quad \Delta \varphi_j = 0, \quad j = i, e, \tag{2}
\]

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + b^2 \frac{\partial^2}{\partial z^2},
\]

\[
r = 1: \left[ \varphi_y \right] = 0, \quad \zeta_r = \varphi_r, \quad \left[ p \right] = \zeta + \zeta_{ao} + b^2 \zeta_{zo}, \tag{3}
\]

\[
z = \pm 0.5: \quad \varphi_z = 0, \tag{4}
\]

\[
r = 1, \ z = \pm 0.5: \quad \zeta_r = \mp \Lambda_{h, k}(\alpha)(\zeta_z + a \cos(2\omega t)), \tag{5}
\]

where \( p \) is the fluid pressure, \( \varphi \) is the velocity potential, \( f(\alpha) \) is the function of the non-uniform electric field, the square brackets denote the jump in the quantity at the interface between the external liquid and the drop, \( \Lambda_{h, k}(\alpha) \) and \( \Lambda_{h, k}(\alpha) \) are the Hocking parameter of the “top” \( (z = 0.5) \) and “bottom” \( (z = -0.5) \) substrate, respectively. The subscripts \( (t, r, \alpha \) or \( z) \) at the unknown functions denote differentiation with respect to the corresponding variables. The boundary-value problem (2)–(5) involves six parameters: the aspect ratio, the dimensionless density, the wetting parameter, the AC frequency and amplitude.
\[ b = R_0 b^{1}, \quad \rho_l = \rho_l^{\star} \left( \rho_l^{*} + \rho_l^{*} \right)^{-1}, \quad \rho_e = \rho_e^{\star} \left( \rho_e^{*} + \rho_e^{*} \right)^{-1}, \quad \Lambda = \Lambda^{2} b \sqrt{\left( \rho_e^{*} + \rho_l^{*} \right) R_0^{2}}, \]

\[ \omega = \omega^{2} \left( \rho_e^{*} + \rho_l^{*} \right)^{R_0^{2}}, \quad a = 0.5 A C \sigma^{2} \sqrt{\left( \rho_e^{*} + \rho_l^{*} \right) R_0^{2}}, \]

where \( C = \varepsilon_{\varepsilon} \varepsilon_{d} d^{-1} \) is capacitance per unit area, \( \varepsilon_0 \) and \( \varepsilon_d \) are the vacuum and the dielectric layer permittivity, respectively, \( d \) is the thickness of the dielectric film.

3. Forced oscillations

The functions \( \Lambda_n(\alpha) \) and \( \Lambda_b(\alpha) \) are expanded in the Fourier series in terms of the eigenfunctions of the Laplace operator. Let us consider a particular case of the inhomogeneity of the plate surface \( \Lambda_{n,b}(\alpha) = \tilde{\Lambda}_{n,b} \cos(\alpha) \), where \( \tilde{\Lambda}_{n,b} \) are the constant amplitudes. The solutions for the velocity potential and the surface deviation are written as

\[ \phi_l(r, \alpha, z, \omega) = \text{Re} \left( i 2 \omega \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left( a_{m,k} R_{1,0}^{m,k}(r) (2 k + 1) \pi z + a_{2,0} R_{2,0}^{m,k}(r) \cos(2 \pi k z) \right) \right), \]

\[ \phi_e(r, \alpha, z, \omega) = \text{Re} \left( i 2 \omega \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left( b_{m,k} R_{1,0}^{m,k}(r) (2 k + 1) \pi z + b_{2,0} R_{2,0}^{m,k}(r) \cos(2 \pi k z) \right) \right), \]

\[ \zeta(z, \omega) = \text{Re} \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left( c_{m,k} \cos(2 \pi k z) \right) \right), \]

\[ + d_{1,0} \sin \left( \frac{z}{b} \right) + d_{2,0} \cos \left( \frac{z}{b} \right) + \sum_{m=1}^{\infty} \left( d_{m,0} \sin \left( \frac{4 m^2 - 1}{b} \right) \right) + \sum_{m=1}^{\infty} \left( d_{2,m} \cos \left( \frac{4 m^2 - 1}{b} \right) \right) \]

where \( R_{1,0}^{m,k}(r) = I_{m}(2 k + 1) \pi br \), \( R_{2,0}^{m,k}(r) = r^m \), \( I_{m}(2 k \pi br) \), \( K_{m}(2 k \pi br) \), \( I_{m} \) and \( K_{m} \) are the modified Bessel functions of the m-th order. Substituting solutions (6)–(8) into (2)–(5), we obtain the expressions for the unknown amplitudes \( a_{m,k} \), \( b_{m,k} \), \( c_{m,k} \), \( d_{1,0} \) and \( d_{2,0} \). These expressions are equivalent to the similar solutions obtained in [7] for \( \Lambda_n(\alpha) = \Lambda_b(\alpha) = \tilde{\Lambda} \).

For convenience, as a maximum deviation of the drop surface from the equilibrium position, we prescribe on the “upper” plate \( z = 0.5 - \zeta_u = \max(\zeta(0,0,0),0) \), on the “bottom” plate \( z = -0.5 - \zeta_u = \max(\zeta(0,0,0),0) \) and a “quarter” position \( z = 0.25 - \zeta_v = \max(\zeta(0,0,0),0) \). The values of the internal contact angle \( \gamma \) on the “upper” plate are \( \gamma_u \), at the “bottom” plate \( \gamma_b \), and the deviation from the equilibrium contact angle on the “upper” plate is \( \delta_u = \max(\gamma_u - 0.5 \pi) \) and on the “bottom” plate \( \delta_b = \max(\gamma_b - 0.5 \pi) \).

The dynamics of the drop significantly depends on the amplitude \( \lambda \) of the function \( \Lambda(\alpha) \). First consider the behaviour of the drop in the case of homogeneous plates, i.e. \( \Lambda_{n,b}(\alpha) = \tilde{\Lambda}_{n,b} \) [18]. Figure 2 shows the oscillation amplitude of the drop surface and the deviation of the contact angle as a function of the frequency of the uniform electric field for several values of the Hocking parameters \( \tilde{\Lambda}_n \) and \( \tilde{\Lambda}_b \). The amplitudes of the surface oscillations and the contact angle reach maximum values in a linear resonance. It is also seen from the graphs that the values of the resonant frequencies decrease with an increase of \( \tilde{\Lambda}_n \) or \( \tilde{\Lambda}_b \). Despite weak dissipation at small values of the parameter \( \lambda \), the amplitude of surface oscillations is finite (figure 2a) and the amplitude of oscillations of the contact line is small (figure 2d). The contact angle varies in a wide range (figure 2c, f). It is important to note that if at least one of the parameters \( \tilde{\Lambda}_n \) or \( \tilde{\Lambda}_b \) is finite, the amplitude of the surface oscillations is always finite. Consequently, dissipation is determined by the largest damping parameter.
For clarity, figure 2 shows the case of equality of the Hocking parameters \( \lambda = 1 \). In this situation, the external force excites only odd spatial modes, so that there is no deviation of the drop surface in the center of the layer (figure 2c). Recall that for finite values of the parameter \( \lambda \), a dissipation is maximum during the movement of the contact line, and therefore, the plots do not display pronounced resonance peaks (figure 2b,c). The motion of a drop does not depend on the parameter \( \lambda \) at certain frequencies \( \omega \): the contact line is like a fixed contact line for any values of \( \lambda \) (figure 2b,c). The values of such "anti-resonant" frequencies are determined from the solution (6)–(8).

Figure 3 shows the profile of the lateral surface (figure 3a) and the contact line (figure 3b,c) and changes in the internal contact angle (figure 3d) at different moments of the oscillation period. The shape of the drop surface depends on the frequency of the electric field. For example, in figure 3, most
of the vibration energy at a given frequency \( \omega = 1 \) (see figure 2a) is transmitted to the lowest spatial mode.

The deviation of the contact angle as a function of the square root of the amplitude (\( b = 1 \), \( \rho_i = 0.7 \), \( \Lambda_{\alpha,b} (\alpha) = \lambda_{\alpha,b}, \lambda_{\alpha} = 1 \)), (a) \( \omega = 1 \), (b) \( \omega = 2.5 \), (c) \( \omega = 5 \), (d) \( \omega = 10 \), \( \gamma_{\alpha} : \lambda_{b} = 0.1 \) – solid line, \( \lambda_{b} = 1 \) – dotted, \( \lambda_{b} = 10 \) – dash-2-dotted; \( \gamma_{\beta} : \lambda_{b} = 0.1 \) – dashed line, \( \lambda_{b} = 1 \) – dash-dotted, \( \lambda_{b} = 10 \) – 2-dash-2-dotted

![Figure 4](image)

**Figure 4.** Deviation of the contact angle as a function of the square root of the amplitude (\( b = 1 \), \( \rho_i = 0.7 \), \( \Lambda_{\alpha,b} (\alpha) = \lambda_{\alpha,b}, \lambda_{\alpha} = 1 \)), (a) \( \omega = 1 \), (b) \( \omega = 2.5 \), (c) \( \omega = 5 \), (d) \( \omega = 10 \), \( \gamma_{\alpha} : \lambda_{b} = 0.1 \) – solid line, \( \lambda_{b} = 1 \) – dotted, \( \lambda_{b} = 10 \) – dash-2-dotted; \( \gamma_{\beta} : \lambda_{b} = 0.1 \) – dashed line, \( \lambda_{b} = 1 \) – dash-dotted, \( \lambda_{b} = 10 \) – 2-dash-2-dotted

![Figure 5](image)

**Figure 5.** The accuracy of representation of the drop surface (a-d) and the contact angle (d-f) vs frequency \( \omega \) (\( \alpha = 5 \), \( b = 1 \), \( \rho_i = 0.7 \), \( \lambda_{\alpha} = 1 \)), \( \lambda_{b} = 0.1 \) – solid line, \( \lambda_{b} = 1 \) – dashed, \( \lambda_{b} = 10 \) – dotted.

The deviation of the contact angle as a function of the square root of the amplitude \( \alpha \) (i.e. proportional to AC potential \( V \)) is given in figure 4 for different values of the Hocking parameter \( \lambda \) and AC frequency \( \omega \). The responses obtained are in qualitative agreement with the experimental data. The maximum deviation of the contact angle tends to \( \pi / 2 \), i.e. \( \theta \to 0 \) or \( \theta \to \pi \) whereas in experiments the contact angle is finite.

The inhomogeneous surface of plate \( \Lambda(\alpha) = \lambda |\cos(\alpha)| \) excites both axisymmetric and azimuthal modes. Consequently the dynamics of the drop differs significantly from its behavior in the uniform field. Similar dependences are shown in figure 5-7. The amplitude of the lateral surface oscillations has identical local maximum unlike the situation with the uniform electric field. Additional resonance
amplitude peaks are associated with the excitation of azimuth modes. A quadrupole mode has a significant effect (figure 6b): the drop is compressed along field inhomogeneity.

![Figure 6](image)

**Figure 6.** Evolution of the drop surface shape (a), the shape of the “upper” contact line (b) and the contact angles (c,d). \( T = 2\pi \omega^{-1} \) is the oscillation period. (\( b = 1, \rho_1 = 0.7, \alpha = 10, \lambda_u = 1, \lambda_b = 0.1, \omega = 1, \varepsilon = 0.1, f(\alpha) = 1 \), (a-c) \( t = 0 \) – solid line, \( t = 0.125T \) – dashed, \( t = 0.25T \) – dotted, \( t = 0.375T \) – dash-dotted.

![Figure 7](image)

**Figure 7.** Deviation of the contact angle as a function of the square root of the amplitude (\( b = 1, \rho_1 = 0.7, \alpha = 10, \lambda_u = 1, \), (a) \( \omega = 1 \), (b) \( \omega = 2.5 \), (c) \( \omega = 5 \), (d) \( \omega = 10 \), \( \gamma_u : \lambda_b = 0.1 \) – solid line, \( \lambda_b = 1 \) – dotted, \( \lambda_b = 10 \) – dash-2-dotted; \( \gamma_b : \lambda_b = 0.1 \) – dashed line, \( \lambda_b = 1 \) – dash-dotted, \( \lambda_b = 10 \) – 2-dash-2-dotted.

### 4. Conclusions

In this paper we studied the behavior of the cylindrical drop between solid plates under the action of the uniform AC electric field taking into account the dynamics of the contact angle. The solid plates had different inhomogeneous surfaces.

The study made allowed us to estimate the deviation of frequency and surface characteristics as a function of the Hocking parameter, frequency and amplitude of the external electric field and geometric parameters of the system. It was found that inhomogeneous surfaces can excite azimuthal modes as was observed, for example, in experiments [19]. The maximum deviation of the contact angle tends to \( \pi/2 \), i.e. \( \vartheta \to 0 \) or \( \vartheta \to \pi \) whereas in experiments the contact angle is always finite. However, the estimates of dimensionless values indicate that the dimensionless amplitude \( \sqrt{\alpha} \) is achieved in experiments \( \sqrt{\alpha} \leq 10 \). In any case, we need to add the mechanism of the contact angle saturation. At this amplitude the contact angle is finite in a large range of the problem parameters. This allows us to expect good agreement between the theoretical predictions and experimental data in the case of a more detailed comparison. The main problem is to measure the value of the Hocking parameter \( \lambda \) at this stage.

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