Quasi-Exactly Solvable Models with Spin-Orbital Interaction

Alexander Ushveridze
Theoretical Physics Institute,
University of Minnesota
and
Department of Theoretical Physics, University of Lodz,
Pomorska 149/153, 90-236 Lodz, Poland

Abstract
First examples of quasi-exactly solvable models describing spin-orbital interaction are constructed. In contrast with other examples of matrix quasi-exactly solvable models discussed in the literature up to now, our models admit infinite (but still incomplete) sets of exact (algebraic) solutions. The Hamiltonians of these models are Hermitian operators of the form $H = -\Delta^2 + V_1(r) + (s \cdot l) V_2(r) + (s + l) \cdot h V_3(r)$ where $V_1(r)$, $V_2(r)$ and $V_3(r)$ are scalar functions, $l$ is a vector of the angular momentum operator, $s$ is a matrix-valued vector spin-operator and $h$ is an external (constant) vector magnetic field.

1 Introduction
Quasi-exactly solvable (QES) problems are distinguished by the fact that only some of their energy levels and corresponding wavefunctions admit explicit construction. In the last decade a big progress has been achieved in elaborating the concepts of the phenomenon of quasi-exact solvability and formulating methods of constructing and solving QES models of a variety of types (for more detail see e.g. the reviews [4], [7], [3], [5] and book [8]). In this paper we undertake a new step in this direction and construct new classes of QES models which (up to now) have never been discussed in the QES-literature. These are QES models with a spin-orbital interaction. The potentials of such models have the following general form

$$V(x, y, z) = V_1(r) + (s \cdot l)V_2(r) + (j \cdot h)V_3(r)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is a radial coordinate, $l = (l_x, l_y, l_z) = (i(y\partial_z - z\partial_y), i(z\partial_x - x\partial_z), i(x\partial_y - y\partial_x))$ is hermitian (vector) operator of angular momentum, $s = (s_x, s_y, s_z)$ is hermitian (vector) spin-operator whose components realize a certain unitary finite-dimensional representation of algebra $so(3)$, $j = s + l$ is the operator of total momentum, and $h = (h_x, h_y, h_z)$ is an external (constant) magnetic field interacting with the total momentum.

One of the most important distinguishing features of QES models which we intend to present here is that they have an infinite number of exactly (algebraically) constructable

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2E-mail address: alexush@mvii.uni.lodz.pl and alexush@krysia.uni.lodz.pl
energy levels and corresponding eigenvalues. They are however quasi-exactly solvable because the set of their exact solutions is still incomplete and does not fill all the spectrum of a model. First examples of such models were presented in our recent work [1] where they have been called "infinite QES models ".

Another unusual feature of these models is that they (in contrast with models discussed in paper [1]) are matrix models with physically realistic hermitian hamiltonians. Everybody who has some experience with quasi-exact solvability in the matrix (multi-channel) case knows how difficult is to satisfy the condition of hermiticity when constructing such models. It is hardly neccessary to remind the reader that up to now only a couple of hermitian matrix QES models have been constructed (see e.g. [4], [2]).

2 Starting point

To demonstrate how does our construction procedure work we start with the simplest one-dimensional QES model with hamiltonian

\[ H = -\frac{\partial^2}{\partial r^2} + \frac{(c-1/2)(c-3/2)}{r^2} + [b^2 - 2a(2m + c + 1)]r^2 + 2abr^4 + a^2r^6 \]  

acting in Hilbert space of functions defined on the positive half axis \( r \in [0, \infty) \) and vanishing sufficiently fast at its ends \( r = 0 \) and \( r = \infty \). Here \( a, b, c \) are real parameters satisfying the conditions \( a > 0, c > 0 \) and \( m \) is a non-negative integer. As it was demonstrated in [8], for any fixed \( m \) the Schroedinger equation

\[ H\psi(r) = E\psi(r) \]  

for model [1] admits algebraic solutions whose general form is given by the formulas

\[
\psi(r) = r^{c-1/2} \prod_{i=1}^{m} \left( r^2/2 - \xi_i \right) \exp \left( -\frac{ar^4}{4} - \frac{br^2}{2} \right) 
\]

\[
E = 2b(2m + c) + 8a \sum_{i=1}^{m} \xi_i 
\]

The \( m \) complex numbers \( \xi_i \) in expressions [3] and [4] satisfy the system of \( m \) algebraic equations

\[
\sum_{k=1, k \neq i}^{m} \frac{1}{\xi_i - \xi_k} + \frac{c}{2\xi_i} - b - 2a\xi_i = 0, \quad i = 1, \ldots, m. \]  

It turns out that system [3] has only \( m + 1 \) permutationally invariant solutions for any given \( m \) which are represented by the sets of real points \( \xi_i \). Each solution is completely characterized by a (quantum) number \( k = 0, 1, \ldots, m \) which indicates the number of positive \( \xi_i \)-points. According to formula [3], the number of positive \( \xi_i \)-points determines the number of (real) wavefunction zeros, which, in turn, determines the ordinal number of an excitation (oscillation theorem). This means that model [1] has \( m + 1 \) exactly constructable solutions describing the ground state and \( m \) first excited states. A more detailed exposition of properties of model [1] and its algebraic solutions can be found in the book [8].
3 The modified equation

It is not difficult to see that the transformation

$$\psi(r) = r \varphi(r)$$  \hspace{1cm} (6)

reduces the equation \[2\] to the form

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2} + V(r,l,m)\right) \varphi(r) = E \varphi(r)$$ \hspace{1cm} (7)

in which we used the notation

$$V(r,l,m) = [b^2 - 2a(2m + 5/2 + l)]r^2 + 2abr^4 + a^2r^6$$ \hspace{1cm} (8)

and

$$l = c - 3/2$$ \hspace{1cm} (9)

Hereafter we shall consider \(l\) as a new independent parameter taking (by agreement) only non-negative integer values. The form of the first three terms in the equation \[3\] coincides with the form of the radial part of a three-dimensional Laplace operator. For this reason it seems quite natural to interpret \(l\) as the 3-dimensional angular momentum and try to relate the equation \[3\] to a certain 3-dimensional quantum problem. In the following three sections we show that there are three such possibilities leading to three different kinds of quasi-exactly solvable problems in the 3-dimensional space.

4 The first possibility

One of the simplest possibilities of interpreting equation \[3\] is based on the assumption that function \(8\) entering into \[3\] is \(l\) independent:

$$V(r,l,m) = V_0(r,N) = [b^2 - 2a(N + 5/2)]r^2 + 2abr^4 + a^2r^6$$ \hspace{1cm} (10)

For this the number

$$N = l + 2m$$ \hspace{1cm} (11)

must be fixed. In this case, equation \[3\] takes the form of a typical radial Schroedinger equation for a spherically symmetric 3-dimensional equation

$$(-\Delta + V_0(r,N)) \Psi(x,y,z) = E \Psi(x,y,z).$$ \hspace{1cm} (12)

Since both \(m\) and \(l\) are assumed to be positive, the condition \[11\] leads to a finite number of possibilities with \(m = 0,1,...,[N/2]\), and \(l = N,N-2,...,N-2[N/2]\), respectively. For this reason, for any given \(N\), the model \[12\] is quasi-exactly solvable and has (as usually) only a finite \([N/2][([N/2] + 1)/2]\) number of explicit solutions. The model of such a form and even its more complicated spherically non-symmetric versions were considered many years ago in papers \[[8],[9]\].
Another possibility of interpreting equation 7 is to consider \( m \) as a fixed number not restricting the value of \( l \). In this case the function \( \Phi \) becomes linearly dependent on \( l \) and can be represented in the form

\[
V(r, l, m) = V_1(r, m) - l \cdot V_2(r) = \{[b^2 - 2a(2m + 5/2)]r^2 + 2abr^4 + a^2r^6\} - l \cdot \{2ar^2\}
\]  

(13)

It is quite obvious that, in order to associate the equation \( 13 \) with a certain 3-dimensional Schroedinger equation, we must find a proper 3-dimensional source for the term which is linear in the momentum \( l \). The first think which comes in ones head is to look for the 3-dimensional scalar operators \( O \) which would commute with both the Laplace operator and \( r \) and would have the eigenvalues linear in \( l \). In this case we could consider \( 8 \) as a reduction of a 3-dimensional problem

\[
(-\Delta + V_1(r, m) - O \cdot V_2(r)) \Psi(x, y, z) = E\Psi(x, y, z)
\]  

(14)

The linearity in \( l \) means that the operator must be proportional to the operator of the angular momentum

\[
l = (l_x, l_y, l_z) = (i(y\partial_z - z\partial_y), i(z\partial_x - x\partial_z), i(x\partial_y - y\partial_x)).
\]  

(15)

But this is a 3-dimensional vector while the operator we are looking for must be a scalar. The only possibility to construct a scalar from \( 8 \) is to take a scalar product of \( l \) with another vector operator. It is quite obvious that there is no such operator if we restrict ourselves to the single-channel problems. However, if we admit the consideration of multi-channel problems, then a good candidate for the second operator can immediately be found. This is obviously the spin operator \( s \). Restricting ourselves (for the sake of simplicity) to the 1/2-spin case (2 by 2 matrices), we can easily check that the spectrum of the operator

\[
O = 2 \cdot s \cdot 1
\]  

(16)

(which, obviously commutes with both \( \Delta \) and \( r \)) is linear in \( l \). Indeed, representing operator \( 16 \) in the form

\[
O = j^2 - l^2 - s^2
\]  

(17)

(where \( j = l + s \) is a total momentum) and taking for concreteness a particular case with \( j = l + 1/2 \) we easily find the corresponding branch of the spectrum

\[
o = j(j + 1) - l(l + 1) - s(s + 1) = (l + 1/2)(l + 3/2) - l(l + 1) - 3/4 = l.
\]  

(18)

This finally leads us to a 3-dimensional matrix QES models

\[
\left( -\Delta + \{[b^2 - 2a(2m + 5/2)]r^2 + 2abr^4 + a^2r^6\} - 2(s \cdot 1) \cdot \{2ar^2\} \right) \Psi(x, y, z) = E\Psi(x, y, z)
\]  

(19)

describing spin-orbital interaction.

It is a time to ask ourselves of what kind of models did we obtain? First of all, one should stress again that these models are actually quasi-exactly solvable. This follows from the fact that for any given \( m \) and \( l \) they have an infinite number of normalizable solutions, but only \( m + 1 \) of them are exactly (algebraically) constructable. Second, and this is may be the most important thing, despite the fact that the set of exactly constructable solutions is incomplete,
this set is infinitely large. This is so because the number \( l \) is not fixed by the 3-dimensional model \( \mathbb{R}^3 \). It appears as a solution of the eigenvalue problem for operators \( O \) and may take arbitrary non-negative integer values.

In conclusion of this section note that the hamiltonians of models we obtained are hermitian by construction.

\section{The third possibility}

The last interesting possibility of reducing the equation \( \mathbb{R}^3 \) to a 3-dimensional form appears when the function \( \mathbb{R}^3 \) depends on both parameters \( l \) and \( m \). In this case the function \( \mathbb{R}^3 \) becomes linearly dependent on both \( l \) and \( m \) and can be represented in the form

\[
V(r, l, m) = V_1(r) - l \cdot V_2(r) - m \cdot V_3(r) = \{(b^2 - 5a)r^2 + 2abr^4 + a^2r^6\} - l \cdot \{2ar^2\} - m \cdot \{4ar^2\} \quad (20)
\]

By analogy with the previous section we can consider the numbers \( l \) as the eigenvalues of the operator of spin-orbital interaction, and the only thing which remains to do is to interpret \( m \) as an independent quantum number appearing in equation \( \mathbb{R}^3 \) as an eigenvalue of a certain operator \( M \) commuting with the variable \( r \), Laplasian \( \Delta \) and the spin-orbital operator \( \mathbf{s} \cdot \mathbf{l} \).

A good candidate for such an operator is the \( z \)-projection of the total momentum \( \mathbf{s} + \mathbf{l} \). In fact, it should not necessarily be a \( z \)-projection. Because of the spherical symmetry, it could be equally well a \( x \)- or \( y \)-projection, or any other projection. We can therefore represent this operator in a covariant form

\[
M = 2(\mathbf{s} + \mathbf{l}) \cdot \mathbf{h} \quad (21)
\]

where \( \mathbf{h} \) is a unit magnetic field. We introduced an additional factor 2 to make the eigenvalues of operator \( M \) integer rather than half integer. Of course, the negative integers are not interesting for us, because, as we remember, only for non-negative integer values of \( m \) the system admits algebraic solution. Summarizing, we can consider \( \mathbb{R}^3 \) as a reduction of a 3-dimensional problem

\[
(-\Delta + V_1(r) - (\mathbf{s} \cdot \mathbf{l}) \cdot V_2(r) - 2(\mathbf{s} + \mathbf{l}) \cdot \mathbf{h} \cdot V_3(r)) \Psi(x, y, z) = E\Psi(x, y, z) \quad (22)
\]

which can be treated as a spectral problem for a matrix quantum model describing spin-orbital interaction together with the interaction of a total momentum with an external magnetic field. It is remarkable, that the model \( \mathbb{R}^3 \) does not contain any integer parameters anylonger. All these parameters appear dynamically as solutions of the eigenvalue problems for additionally introduced symmetry operators. At the same time, the model \( \mathbb{R}^3 \) remains quasi-exactly solvable, because for any particular values of these eigenvalues the equation \( \mathbb{R}^3 \) has only a certain incomplete set of solutions.

\section{Conclusion}

The method of construction infinite (matrix) QES models exposed in this paper is, obviously, quite general and can easily be used for building other spin-orbital models with more complicated potentials and higher matrix dimensions. For this it is sufficient to start with other known one-dimensional QES models first rewriting them in the form of a radial Schroedinger equation and then interpreting the \( l \)-dependent terms appearing in their potential as the eigenvalues of a spin-orbital operators.
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