An Algorithm for the Optimal Consistent Approximation to a Pairwise Comparisons Matrix by Orthogonal Projections

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Abstract

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The algorithm for finding the optimal consistent approximation of an inconsistent pairwise comparisons matrix is based on a logarithmic transformation of a pairwise comparisons matrix into a vector space with the Euclidean metric. Orthogonal basis is introduced in the vector space. The orthogonal projection of the transformed matrix onto the space formed by the images of consistent matrices is the required consistent approximation.

1 Triad Inconsistency in Pairwise Comparisons

Triad inconsistency was introduced in [7] and generalized in [2]. Its convergency analysis was published in [5]. The reader's familiarity with [5] is assumed due to space limitations. Only the essential concepts of the pairwise comparison method are recalled here.

The method of pairwise comparisons was introduced in embryonic form by Fechner (see [3]) and after considerable extension, made popular by Thurstone (see [10]). It can be used as a powerful inference tool and knowledge acquisition technique in knowledge-based systems and data mining.

For the sake of our exposition we define an $N \times N$ pairwise comparison matrix simply as a square matrix $M = [m_{ij}]$ such that $m_{ij} > 0$ for every $i, j = 1, \ldots, n$. A pairwise comparison matrix $M$ is called reciprocal if $m_{ij} = \frac{1}{m_{ji}}$ for every $i, j = 1, \ldots, n$ (then automatically $m_{ii} = 1$ for every $i = 1, \ldots, n$). Let

$$M = \begin{bmatrix}
1 & m_{12} & \cdots & m_{1n} \\
\frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1
\end{bmatrix}$$

where $m_{ij}$ expresses an expert’s relative preference of stimuli $s_i$ over $s_j$.

A pairwise comparison matrix $M$ is called consistent if $m_{ij} \cdot m_{jk} = m_{ik}$ for every $i, j, k = 1, \ldots, n$. While every consistent matrix is reciprocal, the converse is false in general. Consistent matrices correspond to the ideal situation

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in which there are exact values \( s_1, \ldots, s_n \) for the stimuli. The quotients \( m_{ij} = s_i/s_j \) then form a consistent matrix. Conversely, the starting point of the pairwise comparisons inference theory is Saaty’s theorem (see [9]) which states that for every \( N \times N \) consistent matrix \( M = [m_{ij}] \) there exist positive real numbers \( s_1, \ldots, s_n \) such that \( m_{ij} = s_i/s_j \) for every \( i, j = 1, \ldots, n \). The vector \( s = [s_1, \ldots, s_n] \) is unique up to a multiplicative constant. The challenge to the pairwise comparisons method comes from the lack of consistency of the pairwise comparisons matrices which arise in practice (while as a rule, all the pairwise comparisons matrices are reciprocal). Given an \( N \times N \) matrix \( M \) which is not consistent, the theory attempts to provide a consistent \( N \times N \) matrix \( C \) which differs from matrix \( M \) “as little as possible”. Algorithms for reducing the triad inconsistency in pairwise comparisons can be significantly improved by orthogonal projections.

2 The Definition of triad \( L \)-consistency

Let us recall that the matrices in the original space consist of positive elements. The problem of the best approximation of a given matrix \( M = [m_{ij}] \) by a consistent matrix is transformed into a similar problem of approximating a matrix \( M' = [\log m_{ij}] \) by a logarithmic image of a consistent matrix. The benefit of such an approach is that the logarithmically transformed images of consistent matrices form a linear subspace \( L \) in \( R^{N \times N} \). Each matrix in the subspace \( L \) is called a triad \( L \)-consistent matrix \( M' = [m'_{ij}] \) and satisfies the condition: \( m'_{ik} + m'_{kj} = m'_{ij} \) for every \( i, j, k = 1 \ldots n \). It is much easier to work with linear spaces and to use the tools of linear algebra than to work in manifolds (topological or differential). Also the notion of closeness of matrices is translated from one space to the other since the logarithmic transformation is homeomorphic (one-to-one continuous mapping with a continuous inverse; see [1], Vol. II, page 593 for details). In other words two matrices are close to each other in the sense of the Euclidean metric if their logarithmic images are also close in the Euclidean metric.

Let us recall that matrices in the original space have all positive elements. The approximation problem is reduced to the problem of finding the orthogonal projection of the matrix \( M' \) on \( L \) since we opt for the least square approximation in the space of logarithmic images of matrices. The following algorithm is proposed to solve the above problem:

1. Find a basis in \( L \).
2. Orthogonalize it (or orthonormalize it)
3. Compute a projection \( M'' \) of \( M' \) on \( L \) using the orthonormal basis of \( L \) found in step 2.

Steps (1) and (2) produce a basis of the space \( L \). This is done once only for a matrix of a given size \( N \). A Gram-Schmidt orthogonalization procedure is used for constructing an orthogonal basis in \( L \). The actual algorithm for finding a triad \( L \)-consistent approximation is based on step (3); therefore the approximation problem is reduced to: given a matrix \( A \) (a logarithmic image of the matrix to be approximated by a consistent matrix), find the orthogonal projection \( A' \) of \( A \) onto \( L \).

The most natural way of solving this problem is to project the matrix \( A \) on the one dimensional subspaces of \( L \) generated by each vector in the orthogonal basis of \( L \) and then sum these projections. While most of the computation is routine, the problem of finding an orthogonal basis in the space \( L \) is somewhat challenging. For every \( N \times N \) consistent matrix \( A \) there exists a vector of stimuli \( (s_1, s_2, \ldots, s_N) \), unique up to a multiplicative constant such that \( a_{ij} = s_i/s_j \). One may thus infer that the dimension of the space \( L \) is \( N - 1 \). As a consequence, this observation stipulates that the space \( L \) has to have a basis comprised of \( N - 1 \) elements.

Analysis of numerous examples has led to the discovery of the following basis matrices \( B_k = [b^k_{ij}] \)

\[
b^k_{ij} = \begin{cases} 
1, & \text{for } 1 \leq i \leq k < j \leq N \\
-1, & \text{for } 1 \leq j \leq k < i \leq N \\
0, & \text{otherwise}
\end{cases}
\]

Fig. 1 illustrates the basis matrices for \( N = 7 \). In essence each basis matrix \( B_k \) contains two square blocks of 0s (situated symmetrically about the main diagonal) of size \( k \) and \( N - k \), a block of 1s of size \( k \) by \( N - k \) above the main diagonal, and a block of \(-1\)s of size \( N - k \) below the main diagonal, where \( k = 1, \ldots, N - 1 \).
Proposition 1. The matrices $B_k$ are linearly independent.

Proof. The rank of the following matrix containing as its rows the enlisted (by rows) matrices $B_k$ is $N-1$ because the determinant of the submatrix formed by column 12, 13, \ldots, 1N is equal to 1:

$$
\begin{array}{cccccccccccccccc}
11 & 11 & 13 & 14 & \ldots & 1N & 21 & 22 & 23 & \ldots & N1 & N2 & N3 & \ldots & NN \\
B_1 & 0 & 1 & 1 & 1 & \ldots & 1 & -1 & 0 & 0 & \ldots & -1 & 0 & 0 & \ldots & 0 \\
B_2 & 0 & 0 & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & \ldots & -1 & -1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
B_{N-1} & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & -1 & -1 & -1 & \ldots & 0 \\
\end{array}
$$

Proposition 2. In an antisymmetric matrix, the set of conditions:

1. $x_{pq} + x_{qs} = x_{ps}$ where $p, q, s$ are pairwise different is equivalent to:

2. $x_{ij} + x_{jk} = x_{ik}$ where $i < j < k$.

Proof. Let us assume that $s$ is between $p$ and $q$. If $p < q$ then (1) can be written as:

$$x_{pq} = x_{ps} - x_{qs}$$

and by symmetry:

$$x_{pq} = x_{ps} + x_{qs}$$

which is exactly (2) if we set $(p, s, q) = (i, j, k)$.

The reasoning in other cases (for $p$ or $q$ in the middle) is the same because of the symmetry of condition (1) with respect to the coefficients $(p, q, s)$.

As a consequence of Proposition 2 we do not need to check all matrix elements. It is enough to check the elements above the main diagonal. Proposition 2 is used in the proof of Proposition 3.

Let us now check if the proposed basis matrices are triad $L$-consistent. It is sufficient (in light of the above proposition and because of their symmetry) to check that they are triad $L$-consistent with respect to the entries above the diagonal.
**Proposition 3.** All matrices $B_k$ satisfy the following condition:

$$x_{ij} + x_{jk} = x_{ik} \quad \text{where } i < j < k$$

**Proof.** The above condition stipulates that the value in the right upper corner of the rectangle (see Fig. 2) is the sum of values from the left-upper and bottom right corner:

![Partitioning the matrix](image)

Fig 2. Partitioning the matrix

Each of the basis matrices satisfies this condition. There are two cases to be considered:

- **case 1** - the “starred” corners are outside the rectangle of 1s in the matrix $B_k$.
- **case 2** - a “starred” corner is in the rectangle of 1s in the matrix $B_k$.

In case 1, the entries in “starred” corners are all 0s and the condition in question is satisfied since:

$$x_{ij} + x_{jk} = 0 + 0 = 0 = x_{ik}$$

In Case 2, always two (but never three) “starred” corners are in the rectangle of 1s of $B_k$ and one of them is $x_{ik}$. Therefore the LHS of the expression is 1 and so is the RHS.

$$x_{ij} + x_{jk} = 0 + 1 = 1 = x_{ik}$$

or by symmetry

$$x_{ij} + x_{jk} = 1 + 0 = 1 = x_{ik}$$

**Proposition 4.** A linear combination of triad $L$–consistent matrices is triad $L$–consistent.

**Proof.** This follows from elementary algebra. A linear combination of objects satisfying a linear condition in Proposition 3 satisfy the same condition, i.e.:

if $x_{ij} + x_{jk} = x_{ik}$ and $y_{ij} + y_{jk} = y_{ik}$ then $(ax_{ij} + by_{ij}) + (ax_{jk} + by_{jk}) = ax_{ik} + by_{ik}$

The above considerations lead to formulation of the following Theorem.

**Theorem.** Every triad $L$–consistent $N \times N$ matrix is a linear combination of the basis matrices $B_k = [b_{ij}^k]$ for $k = 1, 2, \ldots, N - 1$. 
3 The Orthogonalization Algorithm

The Gram-Schmidt orthogonalization process (see, for example, [1]) can be used to construct the basis. The fairly "regular" form of the basis matrices suggests that the orthogonal basis should also be quite regular. Indeed, solving a system of \( N - 2 \) linear equations produces the following orthogonal basis matrices \( T_k = [t^k_{ij}] \):

\[
t^k_{ij} = \begin{cases} 
-\frac{N-k}{N-k+1} & \text{for } i < k = j \\
\frac{1}{N-k+1} & \text{if } i < k < j \leq N \\
-\frac{1}{N-k+1} & \text{if } i = k < j \leq N \\
-t^k_{ji} & \text{if } t_{ij} \neq 0 \text{ and } j < i \\
0 & \text{otherwise}
\end{cases}
\]

This is equivalent to the following simpler non-recursive definition:

\[
T_k = B_k - \frac{N-k}{N-k+1}B_{k-1}
\]

where \( B_0 \) is a matrix with all zero elements.

Space limitations force the authors to rely on the reader’s knowledge of basic linear algebra to limit this presentation to the final formula and an example. However, the detailed computation leading to the above formula is available by internet in each author’s WEB page. The orthogonal basis for the case \( N = 7 \) is presented in Fig. 3.

\[
T_1 = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
T_2 = \begin{bmatrix}
0 & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
-\frac{5}{6} & 0 & 1 & 1 & 1 & 1 \\
-\frac{5}{6} & -1 & 0 & 0 & 0 & 0 \\
-\frac{5}{6} & -1 & 0 & 0 & 0 & 0 \\
-\frac{5}{6} & -1 & 0 & 0 & 0 & 0 \\
-\frac{5}{6} & -1 & 0 & 0 & 0 & 0 \\
-\frac{5}{6} & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
T_3 = \begin{bmatrix}
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & -\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}
\end{bmatrix}
\]

\[
T_4 = \begin{bmatrix}
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & -\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7}
\end{bmatrix}
\]

\[
T_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7}
\end{bmatrix}
\]

\[
T_6 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{7}
\end{bmatrix}
\]

Fig. 3. An example of orthogonal basis for \( N = 7 \)

The Euclidean norms of the basis matrices can be computed by the following formula:

\[
|T_k|^2 = 2((k-1) \cdot \left[ \frac{N-k}{(N-k+1)^2} + \frac{(N-k)^2}{(N-k+1)^2} \right] + N-k) = \frac{2(N-k)}{(N-k+1)}
\]

The orthogonal basis for the space \( L \) is given above (see the formulas for \( t_{ij} \) and \( T_k \)) and for a given \( N \) one can produce the \( N-1 \) matrices \( T_k \). Once the matrices \( T_k \) are determined we may compute the following values for a given matrix \( A \) (note that operation \( \cdot \) is a dot product; not a regular matrix product):

\[
\forall(k = 1, \ldots, N-1): t_k = \frac{T_k \cdot A}{|T_k|^2}
\]
The next step is to compute the linear combination

\[ A' = \sum_{k=1}^{N-1} t_k \times T_k \]

where the operation \( \times \) is a scalar multiplication.

The result is the required projection of \( A \) into \( L \). It is easy to see that the complexity of computing the coefficients \( t_k \) and hence the matrix \( A' \) is \( O(n^2) \).

4 Conclusions

The triad inconsistency definition provides an opportunity for reducing the inconsistency of the experts’ judgements. It can also be used as a technique for data validation in the knowledge acquisition process. The inconsistency measure of a comparison matrix can serve as a measure of the validity of the knowledge.

The technique presented here for calculating a consistent approximation to a pairwise comparisons matrix is an important step forward. The use of an orthogonal basis simplifies the computation of the mapping of a given matrix since it is just a linear combination of the basis matrices which need be computed only once for a problem of a given size.

A convincing argument for using an orthogonal basis is a consideration of the complication that arises in ordinary geometry when oblique axes are used instead of orthogonal axes.

The use of an orthogonal basis leads to an algorithm that is simple to implement (especially in a language supporting matrix operations).

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