Realizing an $n$-target-qubit controlled-phase gate in cavity QED: An approach without classical pulses

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We propose a way to realize a multiqubit controlled-phase gate with one qubit simultaneously controlling $n$ target qubits using atoms in cavity QED. In this proposal, there is no need to use classical pulses during the entire gate operation. The gate operation time scales as $\sqrt{n}$ only and thus the gate can be performed faster when compared with sending atoms through the cavity one at a time. In addition, only three operational steps are required to realize this $n$-target-qubit controlled-phase gate. This proposal is quite general and can be applied to other physical systems, such as various superconducting qubits coupled to a resonator, NV centers coupled to a microsphere cavity, or quantum dots in cavity QED.

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1. Introduction

Quantum computing attracts a great deal of attention, since quantum computers can, in principle, solve computational problems much more efficiently than classical computers, or process some computational tasks that are intractable with their classical counterparts [1–3]. In the past decade, various physical systems have been considered for building up quantum information processors. Among them, cavity QED with neutral atoms is a very promising approach for quantum information processing, because a cavity can act as a quantum bus to couple atoms and information can be stored in certain atomic energy levels with long coherence time.

It is known that a quantum computation network can be constructed using one-qubit and two-qubit logic gates [4–6]. Based on the cavity QED technique, many theoretical methods have been proposed for implementing a two-qubit controlled-phase or controlled-NOT gate with atoms [7–16]. Moreover, a two-qubit quantum-controlled-phase gate between a cavity mode and an atom has been experimentally demonstrated [17].

Research on quantum computing has recently moved toward the physical realization of multiqubit controlled quantum gates, which are useful in quantum information processing. In principle, any multiqubit gate can be decomposed into two-qubit gates and one-qubit gates. However, when using the conventional gate-decomposition protocols to build up a multiqubit controlled gate [6], the procedure usually becomes complicated as the number $n$ of qubits increases and the number of single-qubit and two-qubit gates required for the gate implementation heavily depends on the number $n$ of qubits.
During the past few years, many schemes have been proposed for implementing multiqubit controlled gates in different physical systems, e.g., atoms in cavity QED [18–20], trapped ions [21], atomic ensembles [22], superconducting qubits coupled to a cavity or resonator [23,24], and nitrogen-vacancy (NV) centers [25]. The proposals [18–25] are mainly for implementing a multiqubit controlled-phase or controlled-NOT gate with multiple-control qubits acting on one target qubit (Fig. 1). This type of multiqubit controlled gate is of significance in quantum information processing, such as quantum algorithms (e.g., Refs. [2,26]) and quantum error-correction protocols [26–28].

In this work, we focus on another type of multiqubit controlled gate, i.e., a multiqubit phase or CNOT gate with one qubit simultaneously controlling n target qubits (Fig. 2). This type of controlled gate with n target qubits is useful in quantum information processing. For instance, it has applications in error correction [29], quantum algorithms (e.g., the discrete cosine transform [30]), and quantum cloning [31]. In addition, it can be used to prepare Greenberger–Horne–Zeilinger (GHZ) states [32]. It should be mentioned here that the gate in Fig. 1 can be used to prepare states that are locally equivalent to W-class states [33], but cannot be applied to create GHZ states. As is well known, GHZ states and W-class states cannot be interchanged with each other, and both of them play an important role in quantum information processing and communication.

For simplicity, we denote this multiqubit phase gate with one qubit simultaneously controlling n target qubits as an n-target-qubit controlled-phase gate. It can be seen from Fig. 2(a) that the n-target-qubit controlled-phase gate consists of n two-qubit controlled-phase (CP) gates. Each two-qubit CP gate involved in this multiqubit gate has a shared control qubit (labeled 1) but a different target qubit (labeled 2, 3, ..., or n + 1). For two qubits, there are a total of four computational basis states |00⟩, |01⟩, |10⟩, and |11⟩. The two-qubit CP gate acting on qubit 1 and qubit j (j = 2, 3, ..., n + 1) is defined as |0⟩1|0⟩j → |0⟩1|0⟩j, |0⟩1|1⟩j → |0⟩1|1⟩j, |1⟩1|0⟩j → |1⟩1|0⟩j, and |1⟩1|1⟩j → − |1⟩1|1⟩j, which implies that if and only if the control qubit 1 is in the state |1⟩, a phase flip happens to the state |1⟩ of the target qubit j, but nothing happens otherwise. According to the definition of a two-qubit CP gate here, it is easy to see that

\[ |0⟩1|0⟩j \rightarrow |0⟩1|0⟩j, \]
\[ |0⟩1|1⟩j \rightarrow |0⟩1|1⟩j, \]
\[ |1⟩1|0⟩j \rightarrow |1⟩1|0⟩j, \]
\[ |1⟩1|1⟩j \rightarrow − |1⟩1|1⟩j. \]
Fig. 2. (a) Circuit of a phase gate with qubit 1 simultaneously controlling \( n \) target qubits (2, 3, \ldots, \( n+1 \)). This \( n \)-target-qubit controlled-phase gate is equivalent to \( n \) two-qubit controlled-phase (CP) gates each having a shared control qubit (qubit 1) but a different target qubit (qubit 2, 3, \ldots, \( n+1 \)). Here, \( Z \) represents a controlled-phase flip on each target qubit. Namely, if the control qubit 1 is in the state \( |1\rangle \), then the state \( |1\rangle \) at each \( Z \) is phase-flipped as \( |1\rangle \rightarrow -|1\rangle \), while the state \( |0\rangle \) remains unchanged. (b) Relationship between an \( n \)-target-qubit controlled-NOT gate and the \( n \)-target-qubit controlled-phase gate. The circuit on the left-hand side of (b) is equivalent to the circuit on the right-hand side of (b). For the circuit on the left-hand side, the symbol \( \oplus \) represents a CNOT gate on each target qubit. If the control qubit 1 is in the state \( |1\rangle \), then the state at \( \oplus \) is bit-flipped as \( |1\rangle \rightarrow |0\rangle \) and \( |0\rangle \rightarrow |1\rangle \). However, when the control qubit 1 is in the state \( |0\rangle \), the state at \( \oplus \) remains unchanged. On the other hand, for the circuit on the right-hand side, the part enclosed in the red dashed-line box represents the \( n \)-target-qubit controlled-phase gate shown in (a). The element containing \( H \) corresponds to a Hadamard transformation described by \( |0\rangle \rightarrow (1/\sqrt{2})(|0\rangle + |1\rangle) \), and \( |1\rangle \rightarrow (1/\sqrt{2})(|0\rangle - |1\rangle) \).

This \( n \)-target-qubit controlled-phase gate with one qubit 1 simultaneously controlling \( n \) target qubits (2, 3, \ldots, \( n+1 \)) is described by the following unitary operator:

\[
U_p = \prod_{j=2}^{n+1} \left( I_j - 2 |1\rangle \langle 1| \right)_j |1\rangle \langle 1|_j ,
\]  

where the subscript 1 represents the control qubit 1, while \( j \) represents the target qubit \( j \); and \( I_j \) is the identity operator for the qubit pair \( (1, j) \), which is given by \( I_j = \sum_{rs} |r\rangle \langle r|_j \langle s|_1 (s|_1 \langle r|_j \), with \( r, s \in \{0, 1\} \). From Eq. (1), it can be seen that the operator \( U_p \) induces a phase flip (from the + sign to the − sign) to the logical state \( |1\rangle \) of each target qubit when the control qubit 1 is initially in the state \( |1\rangle \), and nothing happens otherwise.

In this paper, we will present a way to implement the \( n \)-target-qubit controlled-phase gate with \( (n+1) \) atoms in cavity QED. Here, the \( (n+1) \) atoms are one control atom acting as a control qubit and \( n \) target atoms each playing the role of a target qubit. This proposal has the following features: (i) there is no need to use classical pulses during the entire operation; (ii) the gate operation time scales as \( \sqrt{n} \) only and thus the gate can be performed faster when compared with sending atoms...
through the cavity one by one; (iii) the \( n \) two-qubit CP gates involved can be simultaneously performed; and (v) the gate implementation requires only three operational steps. This proposal is quite general and can be applied to other physical systems, such as various superconducting qubits coupled to a resonator, nitrogen-vacancy (NV) centers coupled to a microsphere cavity, or quantum dots in cavity QED.

Note that an \( n \)-target-qubit CNOT gate, shown in Fig. 2(b), can also be achieved using the present proposal. This is because the \( n \)-target-qubit CNOT gate is equivalent to the \( n \)-target-qubit controlled-phase gate discussed above, plus two Hadamard gates on each target qubit (Fig. 2(b)).

This paper is organized as follows. In Sect. 2, we briefly review the basic theory of atom–cavity resonant interaction and atom–cavity off-resonant interaction. In Sect. 3, we show how to realize an \( n \)-target-qubit controlled-phase gate using atoms in cavity QED. In Sect. 4, we study the fidelity of the gate operation. A brief discussion and a summary are given in Sect. 5.

2. Basic theory

The gate implementation requires two types of atom–cavity interaction, which are described in the following subsections.

2.1. Atom–cavity resonant interaction

Consider a two-level atom, say, atom 1, with a ground level \( |0\rangle \) and an excited level \( |1\rangle \). Assume that the cavity mode is resonant with the \( |0\rangle \leftrightarrow |1\rangle \) transition of the atom. The interaction Hamiltonian in the interaction picture can be written as

\[
H = \hbar g_r (a^+ \sigma^-_{01} + \text{H.c.}),
\]

where \( a^+ \) and \( a \) are the photon creation and annihilation operators of the cavity mode, \( g_r \) is the resonant coupling constant between the cavity mode and the \( |0\rangle \leftrightarrow |1\rangle \) transition of the atom, and \( \sigma^-_{01} = |0\rangle \langle 1| \). It is easy to find that the time evolution of the states \( |1\rangle_c |0\rangle_c \) and \( |0\rangle_c |1\rangle_c \) of the atom and the cavity mode, governed by the Hamiltonian (2), is described by

\[
\begin{align*}
|1\rangle_c |0\rangle_c &\rightarrow -i \sin(g_r t) |0\rangle_c |1\rangle_c + \cos(g_r t) |1\rangle_c |0\rangle_c, \\
|0\rangle_c |1\rangle_c &\rightarrow \cos(g_r t) |0\rangle_c |1\rangle_c - i \sin(g_r t) |1\rangle_c |0\rangle_c,
\end{align*}
\]

while the state \( |0\rangle_c |0\rangle_c \) remains unchanged. Here and below, \( |0\rangle_c \) and \( |1\rangle_c \) are the vacuum state and the single-photon state of the cavity mode, respectively.

2.2. Atom–cavity off-resonant interaction

Consider \( n \) atoms \((2, 3, \ldots, n + 1)\) each having three levels \( |0\rangle, |1\rangle, \) and \( |2\rangle \) (Fig. 3). Suppose that the cavity mode is coupled to the \( |1\rangle \leftrightarrow |2\rangle \) transition of each atom but highly detuned (decoupled) from the transition between any other two levels (Fig. 3). In the interaction picture, the interaction Hamiltonian of the whole system is given by

\[
H = \hbar \sum_{k=2}^{n+1} g (e^{-i \Delta_c t} a^+ \sigma^-_{12,k} + \text{H.c.}),
\]

where \( \Delta_c = \omega_{21} - \omega_c \) is the detuning between the cavity mode frequency \( \omega_c \) and the \( |1\rangle \leftrightarrow |2\rangle \) transition frequency \( \omega_{21} \) of the atoms, \( g \) is the coupling constant between the cavity mode and the \( |1\rangle \leftrightarrow |2\rangle \) transition, and \( \sigma^-_{12,k} = |1\rangle_k \langle 2| \).
Fig. 3. (Atom–cavity off-resonant interaction. The cavity mode is off-resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition with a detuning $\Delta_c = \omega_{21} - \omega_c$. Here, $\omega_c$ is the cavity mode frequency while $\omega_{21}$ is the $|1\rangle \leftrightarrow |2\rangle$ transition frequency of the atom. In addition, $g$ is the off-resonant coupling constant between the cavity mode with the $|1\rangle \leftrightarrow |2\rangle$ transition. To allow the quantum information of a qubit to be stored in the two lowest levels for a long time, atoms can be chosen for which the transition between the two lowest levels is forbidden due to the selection rule, or the two lowest levels are chosen to be hyperfine levels of an atom.

For the case of $\Delta_c \gg \sqrt{n}g$ (i.e., the cavity mode is off-resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of each atom), there is no energy exchange between the atoms and the cavity mode. However, the coupling between the atoms and the cavity mode may induce a phase to the atomic states. This kind of coupling, which can induce a phase but does not cause energy exchange, is often called “dispersive coupling” in quantum optics, or it is said that the cavity mode is dispersively coupled to the atoms. In this case, based on the Hamiltonian (4), one can obtain the following effective Hamiltonian \[ H_{\text{eff}} = -\hbar \sum_{k=2}^{n+1} \frac{g^2}{\Delta_c} (a^+a\sigma_{11,k} - a^+a\sigma_{22,k}) + \hbar \sum_{k\neq k'}^{n+1} \frac{g^2}{\Delta_c} (\sigma_{12,k}\sigma_{12,k'} + \sigma_{12,k}\sigma_{12,k'}) \] (5),

where the two terms in the first line represent the photon-number-dependent Stark shifts, while the two terms in the second line describe the “dipole” coupling between the two atoms \((k, k')\) mediated by the cavity mode. When the level $|2\rangle$ of each atom is not occupied, the Hamiltonian (5) reduces to

\[ H_{\text{eff}} = -\hbar \sum_{k=2}^{n+1} \frac{g^2}{\Delta_c} a^+a\sigma_{11,k} \] (6).

The time-evolution operator for the Hamiltonian (6) is

\[ U(t) = \otimes_{k=2}^{n+1} U_{kc}(t) \] (7).

Here, $U_{kc}(t)$ is the time-evolution operator acting on the cavity mode and the atom $k$ \((k = 2, 3, \ldots, n+1)\), which is given by

\[ U_{kc}(t) = \exp[i \left( \frac{g^2}{\Delta_c} a^+a\sigma_{11,k}t \right) ] \] (8).

One can easily find that the operator $U_{kc}(t)$ results in the following state transformation:

\[ |0\rangle_k |0\rangle_c \rightarrow |0\rangle_k |0\rangle_c , \]
\[ |1\rangle_k |0\rangle_c \rightarrow |1\rangle_k |0\rangle_c , \]
\[ |0\rangle_k |1\rangle_c \rightarrow |0\rangle_k |1\rangle_c , \]
\[ |1\rangle_k |1\rangle_c \rightarrow e^{i\phi_k(t)} |1\rangle_k |1\rangle_c , \] (9)

where $\phi_k(t) = \frac{g^2t}{\Delta_c}$. This result (9) demonstrates that a phase shift $\phi_k(t)$ is induced to the state $|1\rangle$ of the atom $k$ in the case when the cavity mode is in the single-photon state $|1\rangle_c$. For $t = \pi \Delta_c / g^2$, $\phi_k(t) \approx \frac{\pi}{2}$. 

}\]
we have $\phi_k(t) = \pi$, i.e., $|1_k\rangle_c \rightarrow -|1_k\rangle_c$, which implies that a phase flip is induced to the state $|1\rangle$ of the atom $k$ by the cavity photon. In addition, the first two lines of Eq. (9) show that the states $|0\rangle$ and $|1\rangle$ of atom $k$ remain unchanged when the cavity mode is in the vacuum state $|0\rangle_c$, and the third line of Eq. (9) shows that the state $|0\rangle$ of atom $k$ remains unchanged when the cavity mode is in the single-photon state $|1\rangle_c$.

The operator $U(t)$ is a product of the operators $U_{2c}(t)$, $U_{3c}(t)$, ..., and $U_{(n+1)c}(t)$, which can be seen from Eq. (7). Based on Eq. (7), Eq. (8), and the result (9), one can easily find that:

(i) The states $|0\rangle$ and $|1\rangle$ of each of atoms (2, 3, ..., $n + 1$) remain unchanged when the cavity mode is in the vacuum state $|0\rangle_c$.

(ii) The state $|0\rangle$ of each atom (2, 3, ..., $n + 1$) remains unchanged when the cavity mode is in the single-photon state $|1\rangle_c$.

(iii) For $t = \pi \Delta_c / g^2$, a phase flip happens to the state $|1\rangle$ of each atom (2, 3, ..., $n + 1$) simultaneously, in the case when the cavity mode is in the single-photon state $|1\rangle_c$. To see this, consider the state $|\psi\rangle = |1_2\rangle |1_3\rangle |1_4\rangle |1_e\rangle$ for three atoms (2, 3, 4) and the cavity mode. One can easily verify that, by applying the operator $U(t) = U_{2c}(t) U_{3c}(t) U_{4c}(t)$ to the state $|\psi\rangle$, the state $|\psi\rangle$ becomes $(−|1_2\rangle)(−|1_3\rangle)(−|1_4\rangle)|1_e\rangle$ for $t = \pi \Delta_c / g^2$, which shows that a phase flip is induced to the state $|1\rangle$ of each of the three atoms (2, 3, 4) at the same time, by the cavity photon.

The results (i–iii) given here will be applied to the gate implementation discussed in the next section.

3. Implementation of an $n$-target-qubit controlled-phase gate

To realize the gate, we will employ a two-level atom 1 and $n$ identical three-level atoms (2, 3, ..., $n + 1$). The three levels of each atom (2, 3, ..., $n + 1$) are shown in Fig. 3. For each atom, the two lowest levels $|0\rangle$ and $|1\rangle$ represent the two logical states of a qubit. In the following, atom 1 acts as a control while each one of the atoms (2, 3, ..., $n + 1$) plays a target role.

We suppose that, during the following gate operation, (i) the cavity mode is resonant with the $|0\rangle \rightarrow |1\rangle$ transition of atom 1; and (ii) the cavity mode is off-resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of atoms (2, 3, ..., $n + 1$) but highly detuned (decoupled) from the transition between any other two levels of atoms (2, 3, ..., $n + 1$). These conditions can in principle be met by prior adjustment of the cavity mode frequency or by appropriately choosing atoms to have the desired level structure. Note that the cavity mode frequency for both optical cavities and microwave cavities can be changed in various experiments (e.g., see Refs. [36–39]).

The cavity mode is initially in the vacuum state $|0\rangle_c$. The procedure for implementing the $n$-target-qubit controlled-phase gate described by Eq. (1) is given below:

Step (i): Send atom 1 through the cavity for an interaction time $t_1 = \pi/(2g_r)$ by choosing the atomic velocity appropriately. After atom 1 exits the cavity, the state $|0\rangle_c$ for atom 1 and the cavity mode remains unchanged, while their state $|1\rangle |0\rangle_c$ changes to $-i |0\rangle |1\rangle_c$, as described by Eq. (3).

Step (ii): Send atoms (2, 3, ..., $n + 1$) through the cavity for an interaction time $t_2 = \pi \Delta_c / g^2$ by choosing the atomic velocity appropriately. After atoms (2, 3, ..., $n + 1$) leave the cavity, (i) the states $|0\rangle$ and $|1\rangle$ of each atom remain unchanged in the case when the cavity mode is in the vacuum state $|0\rangle_c$; (ii) the state $|0\rangle$ of each atom remains unchanged when the cavity mode is in the single-photon state $|1\rangle_c$; but (iii) the state $|1\rangle$ of each atom changes to $−|1\rangle$ (i.e., a phase flip happens to the...
state $|1\rangle$ of each of the atoms 2, 3, ... , and $n + 1$) when the cavity mode is in the single-photon state, as discussed in the previous subsection 2.2.

**Step (iii):** Send atom 1 back through the cavity for an interaction time $t_3 = 3\pi/(2g_r)$. After atom 1 leaves the cavity, the state $|0\rangle |0\rangle_c$ of atom 1 and the cavity mode remains unchanged, while their state $|0\rangle |1\rangle_c$ changes to $i |1\rangle |0\rangle_c$.

Note that the level $|0\rangle$ of each of the atoms (2, 3, ... , $n + 1$) is not affected by the cavity mode during the operation of step (ii). This is because the cavity mode was assumed to be highly detuned (decoupled) from the $|0\rangle \leftrightarrow |1\rangle$ transition and the $|0\rangle \leftrightarrow |2\rangle$ transition of each of the atoms (2, 3, ... , $n + 1$), as mentioned before.

One can check that the $n$-target-qubit controlled-phase gate, described by Eq. (1), was obtained with ($n + 1$) atoms (i.e., the control atom 1 and the target atoms 2, 3, ... , and $n + 1$) after the above manipulation.

Let us consider a three-qubit example in order to see more clearly how the multitarget-qubit controlled-phase gate described by Eq. (1) is realized after the operations above. For three qubits, there are a total of eight computational basis states from $|000\rangle$ to $|111\rangle$. Based on the results given above for each operational step, it can be easily found that, for three-qubit computational basis states $|100\rangle, |101\rangle, |110\rangle$, and $|111\rangle$, the time evolution of the states of the whole system after each operational step is given by

$$
\begin{align*}
&\left|\begin{array}{c}
1 \rangle |0\rangle |0\rangle_c \\
1 \rangle |0\rangle |1\rangle_c \\
1 \rangle |1\rangle |0\rangle_c \\
1 \rangle |1\rangle |1\rangle_c \\
\end{array} \right| \\
\xrightarrow{\text{Step (i)}}
&\left|\begin{array}{c}
-i |0\rangle |0\rangle |0\rangle_c \\
-i |0\rangle |0\rangle |1\rangle_c \\
-i |0\rangle |1\rangle |0\rangle_c \\
-i |0\rangle |1\rangle |1\rangle_c \\
\end{array} \right| \\
\xrightarrow{\text{Step (ii)}}
&\left|\begin{array}{c}
-i |0\rangle |0\rangle (-|1\rangle) |1\rangle_c \\
-i |0\rangle (-|1\rangle) |0\rangle |1\rangle_c \\
-i |0\rangle (-|1\rangle)(-|1\rangle) |1\rangle_c \\
\end{array} \right| \\
\xrightarrow{\text{Step (iii)}}
&\left|\begin{array}{c}
1 \rangle |0\rangle |0\rangle_c \\
1 \rangle |0\rangle (-|1\rangle) |0\rangle_c \\
1 \rangle (-|1\rangle) |0\rangle |0\rangle_c \\
1 \rangle (-|1\rangle)(-|1\rangle) |0\rangle_c \\
\end{array} \right|
\end{align*}
$$

(10)

Here and above, $|ijk\rangle$ is an abbreviation of the state $|i\rangle_1 |j\rangle_2 |k\rangle_3$ of atoms (1, 2, 3) with $i, j, k \in \{0, 1\}$. The result (10) shows that when the control atom 1 is initially in the state $|1\rangle$, a phase flip happens to the state $|1\rangle$ of atoms (2, 3) while the cavity mode returns to its original vacuum state, after the above three-step operation.

On the other hand, it is obvious that the following states of the system

$$
|000\rangle |0\rangle_c, |001\rangle |0\rangle_c, |010\rangle |0\rangle_c, |011\rangle |0\rangle_c
$$

remain unchanged during the entire operation. This is because no photon was emitted to the cavity during the operation of step (i) due to energy conservation, when atom 1 is initially in the state $|0\rangle$. Hence, it can be concluded from Eq. (10) that a two-target-qubit controlled-phase gate, i.e., a phase gate with one qubit simultaneously controlling two target qubits, was achieved with three atoms (i.e., the control atom 1 and the target atoms 2 and 3) after the above process.

One can see that our gate implementation described above requires atoms to be sent through a cavity. We should point out that the technique of sending atoms through a cavity to have the atoms
4. Fidelity

Let us now study the fidelity of the gate operations. We note that, since the resonant interaction between the atom 1 and the cavity is used in steps (i) and (iii), these two steps can be completed within a very short time (e.g., by increasing the resonant atom–cavity coupling constant $g_r$), such that the dissipation of the atom 1 and the cavity is negligibly small. In this case, the dissipation of the system would appear in the operation of step (ii) due to the use of the atom–cavity dispersive interaction. During the operation of step (ii), the dynamics of the lossy system, composed of the cavity mode and the atoms ($2, 3, \ldots, n + 1$), is determined by

$$\frac{d\rho}{dt} = -i[H, \rho] + \kappa L[a] + \sum_{k=2}^{n+1} \gamma_{21}\mathcal{L}\left[\sigma_{12,k}^{-}\right] + \sum_{k=2}^{n+1} \gamma_{20}\mathcal{L}\left[\sigma_{02,k}^{-}\right] + \sum_{k=2}^{n+1} \gamma_{10}\mathcal{L}\left[\sigma_{01,k}^{-}\right],$$

where $H$ is the Hamiltonian (4), $L[a] = (2a\rho a^+ - a^+ a\rho - \rho a^+ a)$, $\mathcal{L}\left[\sigma_{ij,k}^{-}\right] = 2\sigma_{ij,k}\rho \sigma_{ij,k}^+ - \sigma_{ij,k}^+\sigma_{ij,k}\rho - \rho \sigma_{ij,k}\sigma_{ij,k}^+$ (with $\sigma_{ij,k} = |i\rangle_k\langle j|$, $\sigma_{ij,k}^+ = |j\rangle_k\langle i|$, and $i,j \in \{12, 02, 01\}$), $\kappa$ is the decay rate of the cavity mode, $\gamma_{jj}$ is the decay rate of the level $|j\rangle$ of the atoms ($2, 3, \ldots, n + 1$) via the decay path $|j\rangle \rightarrow |i\rangle$ (here, $ji \in \{21, 20, 10\}$). The fidelity of the gate operations is given by

$$\mathcal{F} = |\langle \psi_{id}|\bar{\rho}|\psi_{id}\rangle|,$$
Fidelity of the gate operations versus the ratio $\Delta_c/g$. The parameters used in the numerical calculation are $\gamma^{-1}_{21} = \gamma^{-1}_{20} = \gamma^{-1}_{10} = 3 \times 10^{-2}$ s, $\kappa^{-1} = 3 \times 10^{-2}$ s, and $g = 2\pi \times 50$ kHz.

where $|\psi_{id}\rangle$ is the state of the whole system after the above three-step gate operations, in the ideal case without considering the dissipation of the system during the entire gate operation; and $\tilde{\rho}$ is the final density operator of the whole system after the gate operations are performed in a real situation.

Fig. 5. We now numerically calculate the fidelity of the gate operations. As an example, we consider realizing a four-target-qubit controlled-phase gate, using a two-level control atom 1 and four identical three-level target atoms (2, 3, 4, 5). The four identical target atoms (2, 3, 4, 5) are chosen as Rydberg atoms with the principal quantum numbers 49, 50, and 51, which correspond to the three levels $|0\rangle$, $|1\rangle$, and $|2\rangle$, as depicted in Fig. 3, respectively. For the Rydberg atoms chosen here, the energy relaxation time $\gamma^{-1}_1$ of the level $|1\rangle$ and the energy relaxation time $\gamma^{-1}_2$ of the level $|2\rangle$ are both $\sim 3 \times 10^{-2}$ s (e.g., see Refs. [36,44,45]). Without loss of generality, we assume that each of the five atoms is initially in the state $\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$ and the cavity mode is in the vacuum state before the gate. The expression for the ideal state $|\psi_{id}\rangle$ of the system after the entire gate operation is straightforward (not shown here to simplify our presentation). As a conservative estimation, we consider $\gamma^{-1}_{21} = \gamma^{-1}_{20} = \gamma^{-1}_1$ and $\gamma^{-1}_{10} = \gamma^{-1}$. In addition, we choose $g = 2\pi \times 50$ kHz [44] and $\kappa^{-1} = 3.0 \times 10^{-2}$ s. Our numerical calculation shows that a high fidelity $\sim 98\%$ can be achieved when the ratio $\Delta_c/g$ is about 10 (Fig. 5).

For the Rydberg atoms chosen here, the $|1\rangle \leftrightarrow |2\rangle$ transition frequency is $\sim 51.1$ GHz [36]. The cavity mode frequency is then $\sim 51.09$ GHz for $\Delta_c/g = 10$. For this value of the cavity mode frequency and the $\kappa^{-1} = 3.1 \times 10^{-2}$ s chosen in our calculation, the required quality factor $Q$ of the cavity is $\sim 10^{10}$. Note that cavities with a high $Q \sim 4.2 \times 10^{10}$ have been reported [46]. Our analysis given here shows that implementing a phase gate with one qubit simultaneously controlling four target qubits (i.e., a four-target-qubit controlled-phase gate) with atoms is possible within the present cavity QED technique.

We should mention that the motivation for using the circular Rydberg states is that they have long energy relaxation times and have been widely used in quantum information processing [7,36,44,45,47–49].

5. Discussion and conclusion

Before concluding, we should point out that, in 2010, Yang, Liu, and Nori proposed a first scheme for implementing a phase gate of one qubit simultaneously controlling $n$ target qubits based on cavity QED [50]. In the same year, Yang, Zheng, and Nori proposed another scheme for realizing a
multiqubit tunable phase gate of one qubit simultaneously controlling $n$ target qubits within cavity QED [51]. As discussed in Refs. [50,51], these two schemes require a pulse to be applied to each of the qubit systems inside a cavity. The purpose of this work is to present an alternative approach to implementing the proposed gate. As shown above, application of a pulse is not required and thus our present scheme differs from the previous ones in Refs. [50,51]. Because no pulses are needed, the present scheme is much improved when compared with the previous proposals in Refs. [50,51].

We have proposed a way to realize a multiqubit controlled-phase gate with one qubit simultaneously controlling $n$ target qubits using atoms in cavity QED. As shown above, the gate can be implemented: (i) by using one cavity through three-step operations only, (ii) without the need to use classical pulses during the gate operations, (iii) faster when compared with sending atoms through a cavity or loading atoms into a cavity one by one, and (iv) in an operation time that scales as $\sqrt{n}$ only. We believe that this work is of interest because it provides a way to implement the proposed multiqubit gate usefully in quantum information processing. Finally, we note that this proposal is quite general and can be applied to other physical systems such as various superconducting qubits coupled to a resonator, NV centers coupled to a microsphere cavity, or quantum dots in cavity QED.

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