Global solution and long-time behavior
for a problem of phase segregation of the Allen-Cahn type

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Abstract

In this paper we study a model for phase segregation consisting in a sistem of a partial
and an ordinary differential equation. By a careful definition of maximal solution to the
latter equation, this system reduces to an Allen-Cahn equation with a memory term.
Global existence and uniqueness of a smooth solution are proven and a characterization of
the \(\omega\)-limit set is given.

Key words: Allen-Cahn equation, integrodifferential equations, well posedness, long-time
behavior.

AMS (MOS) Subject Classification: 74A15, 35K55, 35A05, 35B40.

1 Problem setting

The Allen-Cahn equation

\[
\kappa \partial_t \rho - \Delta \rho + f'(\rho) = 0
\]  

(1.1)
is meant to describe evolutionary processes in a two-phase material body, including phase segregation: \(\rho\), with \(\rho(x,t) \in [0,1]\), is an order-parameter field interpreted as the scaled volumetric
density of one of the two phases, $\kappa > 0$ is a mobility coefficient, and $f$ denotes a double-well potential confined in $(0, 1)$ and singular at endpoints. The derivation of this equation offered by Gurtin in [3] is based on a balance of contact and distance microforces:

$$\text{div} \xi + \pi + \gamma = 0$$

(1.2)

and on a ‘purely mechanical’ dissipation inequality restricting the free-energy growth:

$$\partial_t \psi \leq w, \quad w := -\pi \partial_t \rho + \xi \cdot \nabla (\partial_t \rho),$$

(1.3)

where the distance microforce is split in an internal part $\pi$ and an external part $\gamma$; $\xi$ denotes the microscopic stress vector, and $w$ specifies the (distance and contact) internal microworking.

The Coleman-Noll compatibility of the constitutive choices

$$\pi = \hat{\pi}(\rho, \nabla \rho, \partial_t \rho), \quad \xi = \hat{\xi}(\rho, \nabla \rho, \partial_t \rho),$$

(1.4)

with the dissipation inequality (1.3) yields

$$\hat{\pi}(\rho, \nabla \rho, \partial_t \rho) = -f'(\rho) - \hat{\kappa}(\rho, \nabla \rho, \partial_t \rho) \partial_t \rho, \quad \hat{\xi}(\rho, \nabla \rho, \partial_t \rho) = \nabla \rho,$$

(1.5)

and hence the Allen-Cahn equation (1.1), for $\hat{\kappa}(\rho, \nabla \rho, \partial_t \rho) = \kappa$ and $\gamma = 0$.

One of us proposed in [7] a modified version of Gurtin’s derivation, where the dissipation inequality (1.3) is dropped and the microforce balance (1.2) is coupled with the microenergy balance

$$\partial_t \varepsilon = e + w, \quad e := -\text{div} \bar{h} + \bar{\sigma},$$

(1.6)

and the microentropy imbalance

$$\partial_t \eta \geq -\text{div} \bar{h} + \sigma, \quad \bar{h} := \mu \bar{\bar{h}}, \quad \bar{\sigma} := \mu \bar{\bar{\sigma}}.$$

(1.7)

The salient new feature of this approach to phase-segregation modeling is that the microentropy inflow $(\bar{h}, \sigma)$ is deemed proportional to the microenergy inflow $(\bar{\bar{h}}, \bar{\bar{\sigma}})$ through the chemical potential $\mu$, a positive field; consistently, the free energy is defined to be

$$\psi := \varepsilon - \mu^{-1} \eta,$$

(1.8)

with the chemical potential playing the same role as coldness in the deduction of the heat equation. Combination of (1.6)-(1.8) gives:

$$\partial_t \psi \leq -\eta (\mu^{-1}) \bar{\bar{h}} \cdot \nabla \mu - \pi \partial_t \rho + \xi \cdot \nabla (\partial_t \rho),$$

(1.9)

an inequality that replaces for (1.3) in restricting constitutive choices that can now be more general than those in (1.4). On taking all of the constitutive mappings delivering $\pi, \xi, \eta$, and $\bar{h}$ depending on the list $\rho, \nabla \rho, \partial_t \rho,$ and the chemical potential $\mu$, and on choosing

$$\psi = \hat{\psi}(\rho, \nabla \rho, \mu) = -\mu \rho + f(\rho) + \frac{1}{2} |\nabla \rho|^2,$$

(1.10)
compatibility with \( (1.9) \) yields

\[
\hat{\pi}(\rho, \nabla \rho, \partial_t \rho, \mu) = \mu - f' (\rho) - \hat{\kappa}(\rho, \nabla \rho, \partial_t \rho) \partial_t \rho, \quad \hat{\xi}(\rho, \nabla \rho, \partial_t \rho, \mu) = \nabla \rho, \quad \hat{\eta}(\rho, \nabla \rho, \partial_t \rho, \mu) = \nabla \rho, \quad \hat{\kappa}(\rho, \nabla \rho, \partial_t \rho, \mu) = \nabla \rho, \quad \hat{\xi}(\rho, \nabla \rho, \partial_t \rho, \mu) = \nabla \rho, \quad \hat{\eta}(\rho, \nabla \rho, \partial_t \rho, \mu) = \nabla \rho,
\]

\((1.11)\)

With the use of \( (1.11) \) and, once again, of the additional constitutive assumptions that the mobility \( \kappa \) is a positive constant and the external distance microforce \( \gamma \) is null, the microforce balance \( (1.2) \) and the energy balance \( (1.6) \) become, respectively,

\[
\kappa \partial_t \rho + \Delta \rho + f' (\rho) = \mu \quad (1.12)
\]

and

\[
\partial_t (-\mu^2 \rho) = \mu \left( \kappa (\partial_t \rho)^2 + \bar{\sigma} \right). \quad (1.13)
\]

This nonlinear system consists of a parabolic PDE and a first-order-in-time ODE and is to be solved for the order-parameter field \( \rho \) and the chemical potential field \( \mu \); formally, setting \( \mu \equiv 0 \) restitutes the standard Allen-Cahn equation \( (1.1) \). We supplement system \( (1.12)-(1.13) \) with the homogeneous Neumann condition

\[
\partial_n \rho = 0 \quad \text{on the body’s boundary} \quad (1.14)
\]

(\( \partial_n \) denotes the outward normal derivative) and with the initial conditions

\[
\rho|_{t=0} = \rho_0 \quad \text{bounded away from 0}, \quad \mu|_{t=0} = \mu_0 \geq 0. \quad (1.15)
\]

Note that, in view of the third of relations \( (1.11) \), the microentropy cannot exceed the level 0 from below anywhere at any time, and that the corresponding prescribed initial field

\[
\eta|_{t=0} = \eta_0 = -\mu_0^2 \rho_0 \quad (1.16)
\]

is nonpositive-valued.

**Remark 1.1.** The last of \( (1.11) \) implies that both the energy influx and the entropy influx are everywhere null for all times. This result, that has a pivotal role in reducing the energy balance to an ODE, is a direct consequence of assuming that the energy-influx mapping \( \hat{h} \), just as all the other constitutive mappings, be independent of the gradient of the chemical potential. This assumption, that precludes energy and entropy diffusion, does not seem appropriate in the case of higher-order phase segregation models of Cahn-Hilliard type, like the one proposed in \([7]\).

## 2 Solution strategy and summary of contents

The aim of our paper is a mathematical investigation of problem \( (1.12)-(1.15) \). The key idea of our strategy is to attack the problem sequentially, the ODE first, then the PDE.

To do so, we introduce a change of variable that is expedient to give \( (1.13) \) plus \( (1.16) \) the form of a parametric initial-value problem. The change of variable in question is:

\[
\xi := -\eta, \quad \xi_0 := -\eta_0, \quad (2.1)
\]
whence
\[ \mu = \sqrt{\xi/\rho}; \]  
(2.2)
it leads to
\[ \partial_t \xi + \frac{\kappa (\partial_t \rho)^2 + \sigma}{\sqrt{\rho}} \sqrt{\xi} = 0, \quad \xi|_{t=0} = \xi_0, \]  
(2.3)
a Cauchy problem for \( \xi(x, \cdot) \) parameterized on the space variable \( x \) and on the field \( \rho(x, \cdot) \). The general form of this problem is discussed in the Appendix. Suffice it to note here that (2.3) exhibits the Peano phenomenon and has infinitely many solutions; among them, we pick a suitably defined maximal solution \( \xi \) (or \( \sqrt{\xi} \), see definition (3.6)-(3.7)), having the desirable property to stay positive as long as is possible. Next, we transform (1.12) into
\[ \kappa \partial_t \rho - \Delta \rho + f'(\rho) - \sqrt{\xi} \frac{1}{\sqrt{\rho}} = 0, \]  
(2.4)
that is, an Allen-Cahn equation for \( \rho(x, \cdot) \) with the additional term \( -\sqrt{\xi/\rho} \); since the factor \( \sqrt{\xi} \) is implicitly defined in terms of \( \rho \) as the maximal solution of (2.3), (2.4) must be regarded as an integrodifferential equation. We prove existence, regularity and uniqueness of the solution to (2.3) subject to the boundary condition (1.14) and the initial condition (1.15) by means of a fixed-point argument, taking advantage of the iterated Contraction Mapping Principle. Crucial to success is that \( \partial_t \rho \) be a priori uniformly bounded in the space-time domain; we show that this is the case by applying standard regularity arguments for parabolic equations.

For a detailed discussion of the problem transformation sketched here above, and for a precise exposition of our main results, we refer the reader to Section 3. Our well-posedness results are proved in Section 4. Section 5 is devoted to an investigation of the long-time behavior of the solution; we prove that \( \sqrt{\xi} \) uniquely converges to some function \( \varphi_\infty \) and that any element \( \rho_\infty \) of the \( \omega \)-limit set solves the stationary problem
\[ \kappa \partial_t \rho - \Delta \rho_\infty + f'(\rho_\infty) - \varphi_\infty \frac{1}{\sqrt{\rho_\infty}} = 0, \]  
(2.5)
supplemented by suitable homogeneous Neumann boundary conditions.

3 Main results

We begin by specifying once and for all the class of data we consider; further assumptions of local importance will be stated when needed. Our problem is formulated over a space-time cylinder
\[ Q_T = \Omega \times [0, T) \quad \text{with} \quad T \in (0, +\infty), \]  
(3.1)
where \( \Omega \) is an open, bounded and connected set of \( \mathbb{R}^N (N \geq 1) \), with a smooth boundary \( \Gamma \) (we use the notation \( Q_t := \Omega \times [0, t) \) for every \( t \in (0, +\infty) \)). As to the coarse-grain free energy \( f \), we split it as follows:
\[ 0 \leq f = f_1 + f_2, \quad \text{where} \quad f_1, f_2 : (0, 1) \to \mathbb{R} \quad \text{are} \ C^2 \text{-functions,} \]  
(3.2)
\( f_1 \) is convex, \( f_2 \) is bounded, \( \lim_{r \searrow 0} f'(r) = -\infty \), and \( \lim_{r \nearrow 1} f'(r) = +\infty \).
We regard \( f_2 \) as a smooth perturbation of the singular convex part \( f_1 \) of \( f \), which is well exemplified by

\[
f_1(r) = r \ln r + (1 - r) \ln(1 - r) \quad \text{for } r \in (0, 1).
\]  

(3.4)

As to the energy source and the initial data, we assume that

\[
\bar{\sigma} \in L^2(\Omega_T), \quad \rho_0, \xi_0 \in L^\infty(\Omega), \quad 0 < \rho_0 < 1 \quad \text{and} \quad \xi_0 \geq 0 \quad \text{a.e. in } \Omega.
\]

(3.5)

Finally, we recall that the mobility \( \kappa \) is a given positive constant.

With this, we take up the forward Cauchy problem \((2.3)\). Clearly, \( \xi \) must be nonnegative. We notice that, if we look for a strictly positive \( \xi \) (for given \( \rho > 0 \) and \( \xi_0 > 0 \)), the Cauchy problem \((2.3)\) has a unique local solution. On the contrary, uniqueness is no longer guaranteed if we allow \( \xi \) to be just nonnegative. On the other hand, every nonnegative local solution can be extended to a global solution. Therefore, we select a (global) solution to problem \((2.3)\) according to the following maximality criterion (for a justification, see the Appendix):

\[
\sqrt{\xi(x,t)} = \sup \{w(x,t) : w \in S^*(\bar{\sigma}, \xi_0, \rho)\} \quad \text{for } (x,t) \in \Omega_T,
\]

(3.6)

where

\[
S^*(\bar{\sigma}, \xi_0, \rho) := \left\{ w \in W^{1,1}(0,T;L^1(\Omega)) : w(0) = \sqrt{\xi_0}, \; w \geq 0 \; \text{a.e. in } \Omega_T, \; \partial_t w = -\left(\kappa (\partial_t \rho)^2 + \bar{\sigma}\right)/(2\rho^{1/2}) \; \text{a.e. where } w > 0 \right\}.
\]

(3.7)

Accordingly, the maximal \( \xi \) satisfies:

\[
\sqrt{\xi(x,t)} = \sqrt{\xi_0(x)} - \int_0^t a^*(x,s) \, ds,
\]

(3.8)

where

\[
a^*(x,s) := \begin{cases} 
\frac{\kappa |\partial_t \rho(x,s)|^2 + \bar{\sigma}(x,s)}{2 \sqrt{\rho(x,s)}} & \text{if } \xi(x,s) > 0, \\
0 & \text{otherwise}. 
\end{cases}
\]

(3.9)

At this point, we replace \( \mu \) by \( \sqrt{\xi/\rho} \) in \((1.2)\) and supplement the equation \((2.3)\) we get with the boundary and initial conditions for \( \rho \) given by, respectively, \((1.14)\) and the first of \((1.15)\). Of the so-obtained initial/boundary value problem we give the following variational formulation: for

\[
V := H^1(\Omega) \quad \text{and} \quad H := L^2(\Omega),
\]

(3.10)

seek a field \( \rho \) such that:

\[
\rho \in H^1(0,T;H) \cap C^0([0,T];V); \\
\rho(0) = \rho_0, \quad 0 < \rho < 1 \quad \text{a.e. in } \Omega_T, \\
\kappa \int_\Omega \partial_t \rho(t) z + \int_\Omega \nabla \rho(t) \cdot \nabla z + \int_\Omega f'(\rho(t)) z - \int_\Omega (\xi(t)/\rho(t))^{1/2} z = 0 \quad \text{for } \text{a.a. } t \in (0,T), \text{ for every } z \in V, \text{ and for } \xi \text{ given by } \text{(3.6).}
\]

(3.13)

Remark 3.1. We regard the initial/boundary value problem \((3.11) - (3.13)\) as an essentially integrodifferential Allen-Cahn equation in the sole unknown \( \rho \). We note, in particular, that \((3.13)\) has a precise meaning, because \( \xi^{1/2} \in L^2(\Omega_T) \) and \( \rho^{-1/2} \in L^\infty(\Omega_T) \) (at least) whenever \( \rho \) satisfies \((3.11)\) and \( \bar{\sigma} \in L^2(\Omega_T) \).
Here is our first result (the symbol $(\cdot)^-$ denotes the negative part).

**Theorem 3.2.** Assume that:

\[ \bar{\sigma} \in L^\infty(Q_\infty) \quad \text{and} \quad \bar{\sigma}^- \in L^1(0, \infty; L^\infty(\Omega)) ; \quad \frac{1}{\rho_0} + \frac{1}{1 - \rho_0} \in L^\infty(\Omega), \]  
\[ \rho_0 \in H^2(\Omega), \quad \partial_{n}\rho_0 = 0 \quad \text{on} \ \Gamma, \quad \text{and} \ \Delta\rho_0 \in L^\infty(\Omega). \]  

Then, for every \( T \in (0, +\infty) \), problem (3.11)–(3.13) has a unique solution. Furthermore,

\[ \rho \in L^p(0, T; W^{2,p}(\Omega)) \quad \text{for every} \ p < +\infty, \quad \partial_t \rho \in L^\infty(Q_T), \quad \text{and} \ \xi \in L^\infty(Q_T). \]  

Finally, there exist constants \( \rho_*, \rho^* \in (0, 1) \) and \( \xi^* \geq 0 \) such that

\[ \rho_* \leq \rho \leq \rho^* \quad \text{and} \quad \xi \leq \xi^* \quad \text{a.e. in} \ Q_T; \]  

these constants can be chosen independently of \( T \).

**Remark 3.3.** Let \( M_0 \) be an upper bound for \(|\bar{\sigma}|\). Assume that \( \xi_* := \inf \xi_0 \) is strictly positive, and take \( M \) such that \(|\partial_t \rho| \leq M \) a.e. in \( Q_T \) (see (3.10)). Then, relation (6.10) in the Appendix implies that \( \xi(t) > 0 \) at least for \( t < 2\sqrt{\xi_* \rho_0} / (\kappa M^2 + M_0) \). In such a case, the pair \((\rho, \xi)\) yields a (local) solution \((\rho, \mu, \eta)\) to the original problem in a strong sense.

Our second result concerns the long-time behavior of the solution \( \rho \) to problem (3.11)–(3.13); it ensures that the elements of the \( \omega \)-limit of every trajectory are steady states. To state this result properly, we have to describe the stationary problem associated to (3.11)–(3.13). We introduce \( \varphi_\infty : \Omega \to [0, +\infty) \), by means of the following formula:

\[ \varphi_\infty(x) := \lim_{t \to +\infty} \sqrt{\xi(x, t)} \quad \text{for a.a.} \ x \in \Omega, \quad \text{where} \ \xi \ \text{is given by (3.6)–(3.9)} \]  

(our next theorem ensures that such a limit actually exists, under form of a bounded function on \( \Omega \)). The stationary problem consists in finding

\[ \rho_\infty \in V \quad \text{and} \quad \rho_* \leq \rho_\infty \leq \rho^* \quad \text{a.e. in} \ \Omega \]  

such that

\[ \int_\Omega \nabla \rho_\infty \cdot \nabla z + \int_\Omega f'(\rho_\infty) z - \int_\Omega \frac{\varphi_\infty}{\sqrt{\rho_\infty}} z = 0 \quad \text{for every} \ z \in V. \]  

**Theorem 3.4.** Under the assumptions of Theorem 3.2, let \( \rho \) be the unique global solution to problem (3.11)–(3.13). Then, the limit \( \varphi_\infty \) exists for a.a. \( x \in \Omega \) and \( \varphi_\infty \in L^\infty(\Omega) \). Moreover, the \( \omega \)-limit defined by

\[ \omega(\rho) := \{ \rho_\infty \in H : \rho_\infty = \lim_{n \to -\infty} \rho(t_n) \ \text{strongly in} \ H \ \text{for some} \ \{t_n\} \ \searrow +\infty \} \]  

is non-empty, compact, and connected in the strong topology of \( H \). Finally, every element \( \rho_\infty \in \omega(\rho) \) coincides with a solution \( \rho_\infty \) to the stationary problem (3.19)–(3.20).

**Remark 3.5.** One can wonder whether \( f_1 \) can be a more general potential from Convex Analysis. Actually this is the case, since we may replace the monotone part \( f_1' \) of \( f' \) by a graph \( \alpha \). Precisely, we may assume that

\[ \alpha \text{ is a maximal monotone graph in} \ \mathbb{R} \times \mathbb{R}, \]  

with \( D(\alpha) = (0, 1), \ \lim_{r \to 0} \alpha^0(r) = -\infty, \quad \text{and} \ \lim_{r \to 1} \alpha^0(r) = +\infty, \]
where $D(\alpha)$ stands for the domain of $\alpha$ and $\alpha^0(r)$ denotes the element of $\alpha(r)$ having minimum modulus for $r \in (0, 1)$ (see, e.g., [11, p. 28]). Accordingly, $f_1$ is replaced by a convex l.s.c. function $\hat{\alpha} : \mathbb{R} \to (-\infty, +\infty]$ such that $\partial \hat{\alpha} = \alpha$, so that $f := \hat{\alpha} + f_2 \geq 0$.

When thinking of such a generalization, we have to introduce a selection $\zeta$ of $\alpha(\rho)$ with some regularity and we have to come up with a convenient replacement for the variational equation (3.13). Here is a suitably general formulation:

\[
\text{find } \zeta \in L^2(Q_T) \text{ and } \zeta \in \alpha(\rho) \text{ a.e. in } Q_T, \text{ such that } \quad (3.24)
\]

\[
k \int_\Omega \partial_t \rho(t) v + \int_\Omega \nabla \rho(t) \cdot \nabla v + \int_\Omega (\zeta(t) + f_2(\rho(t))) v - \int_\Omega (\xi(t)/\rho(t))^{1/2} v = 0 \quad \text{for a.a. } t \in (0, T), \text{ for every } v \in V, \text{ and for } \xi \text{ given by } (3.6). \quad (3.25)
\]

An analogous modification is due for the stationary problem (3.19)–(3.20), that may be replaced by the following problem:

\[
\text{find } \zeta_\infty \in L^2(\Omega) \text{ and } \zeta_\infty \in \alpha(\rho_\infty) \text{ a.e. in } \Omega, \text{ such that } \quad (3.26)
\]

\[
\int_\Omega \nabla \rho_\infty \cdot \nabla z + \int_\Omega (\zeta_\infty + f_2(\rho_\infty)) z - \int_\Omega \varphi_\infty \sqrt{\rho_\infty} z = 0 \quad \text{for every } z \in V. \quad (3.27)
\]

With such measures, Theorem 3.2 can be extended as far as existence and regularity are concerned. Precisely, we can prove that there is a global solution $(\rho, \zeta)$, with $\zeta \in L^\infty(Q_\infty)$ and $\rho$ satisfying the same regularity requirements and bounds as in Theorem 3.2. However, we cannot prove uniqueness. Furthermore, Theorem 3.4 holds for every (possibly nonunique) global solution to the generalized problem satisfying the same bounds as above. We sketch how to achieve such generalizations in the forthcoming Remarks 4.9 and 5.3.

4 Existence and uniqueness

In this section, we prove Theorem 3.2. This is a rather complicated task, that we now delineate. First of all, we show that every solution satisfies the last part of the statement, i.e., that some kind of maximum principle holds. Then, we show that we can count a priori on more regularity than that specified in (3.16). Finally, by looking for solutions satisfying such stronger properties, only, we prove existence and uniqueness using a fixed point argument. In the preliminary steps, we find convenient to deal with an auxiliary problem. In the whole section, it is understood that the assumptions of Theorem 3.2 are satisfied.

Construction of the crucial constants. We find the constants $\rho_\ast$, $\rho^*$, and $\xi^*$, noting that our procedure actually yields values that do not depend on $T$, as stated in the last part of Theorem 3.2. For convenience, we set

\[
M_2 := \sup_{r \in (0, 1)} |f_2'(r)|, \quad (4.1)
\]

and choose $\rho_\ast \in (0, 1)$ and $\xi^0 \geq 0$ so as to have

\[
f_2'(\rho_\ast) \leq -M_2, \quad \rho_0 \geq \rho_\ast \quad \text{and} \quad \xi_0 \leq \xi^* \text{ a.e. in } \Omega, \quad (4.2)
\]

due to (3.8), (3.9), and (3.14). Moreover, on accounting for the second of (3.14), we define $\xi^* \geq 0$ as follows:

\[
\sqrt{\xi^*} \triangleq \sqrt{\xi^0} + \frac{1}{2\sqrt{\rho_\ast}} \|\sigma^-\|_{L^1(0, \infty; L^\infty(\Omega))}. \quad (4.3)
\]
Finally, using the last of (3.3) and (3.5) once more, we choose \( \rho^* \in (0, 1) \) such that
\[
f_1'(\rho^*) - \frac{\sqrt{\xi^*}}{\sqrt{\rho^*}} \geq M_2 \quad \text{and} \quad \rho_0 \leq \rho^* \quad \text{a.e. in } \Omega.
\] (4.4)

At this point, we are ready to prove the last part of Theorem 3.2. At the same time, with a view towards the fixed point argument we are going to use later on, we prepare some auxiliary material. So, we show that
\[
\rho_* \leq \rho \leq \rho^* \quad \text{and} \quad \xi \leq \xi^* \quad \text{a.e. in } Q_T
\] (4.5)
for any solution \( \rho \) to the variational equation (3.13), but with (3.6)–(3.9) replaced by something else.

**The auxiliary problem.** From the previous section, we see that \( \sqrt{\xi} \), rather than \( \xi \), plays the main role. Hence, we define \( \Phi : D(\Phi) \to W^{1,1}(0, T; L^1(\Omega)) \cap L^\infty(Q_T) \) as follows. We set:
\[
D(\Phi) := \{ v \in H^1(0, T; H), \quad v > 0 \quad \text{a.e. in } Q_T, \quad 1/v \in L^\infty(Q_T) \}
\] (4.6)
and, for \( v \in D(\Phi) \), we denote by \( \Phi(v) \) the function \( \varphi \) given by
\[
\varphi(x, t) = \sup \{ w(x, t) : \quad w \in \mathcal{F}(v) \} \quad \text{for } (x, t) \in Q_T,
\] (4.7)
where we have set
\[
\mathcal{F}(v) := \{ w \in W^{1,1}(0, T; L^1(\Omega)) : \quad w(0) = \sqrt{\xi_0}, \quad w \geq 0 \quad \text{a.e. in } Q_T, \quad \partial_t w = -(\kappa (\partial_t v)^2 - \bar{\sigma})/(2v^{1/2}) \quad \text{a.e. where } w > 0 \}.
\] (4.8)
By arguing as in the Appendix, we see that \( \varphi := \Phi(v) \) is the square root of the maximal solution to the Cauchy problem:
\[
\partial_t \xi = -\left( (\kappa (\partial_t v)^2 + \sigma) / (2v^{1/2}) \right) \sqrt{\xi} \quad \text{and} \quad \xi(0) = \xi_0,
\]
and it is characterized by
\[
\varphi(x, t) = \sqrt{\xi_0(x)} - \int_0^t a^*(x, s) \, ds,
\] (4.9)
where
\[
a^*(x, s) := \frac{\kappa |\partial_t v(x, s)|^2 + \bar{\sigma}(x, s)}{2 \sqrt{v(x, s)}} \quad \text{if} \quad \varphi(x, t) > 0, \quad a^*(x, s) := 0 \quad \text{otherwise.} \] (4.10)
Note that \( \varphi \) actually belongs to \( W^{1,1}(0, T; L^1(\Omega)) \cap L^\infty(Q_T) \). Then, the auxiliary problem is obtained as follows. For a given \( v \in D(\Phi) \), we require that \( \rho \) satisfies:
\[
\rho \in H^1(0, T; H) \cap C^0([0, T]; V), \quad \rho(0) = \rho_0, \quad 0 < \rho < 1 \quad \text{a.e. in } Q_T, \quad f'(u) \in L^2(Q_T), \quad \text{and} \quad \rho^{-1/2} \in L^2(Q_T); \]
(4.11)
\[
\kappa \int_0^t \partial_s \rho(t) z + \int_\Omega \nabla \rho(t) \cdot \nabla z + \int_\Omega f'(\rho(t)) z - \int_\Omega \frac{\varphi(t)}{\sqrt{\rho(t)}} z = 0
\] (4.13)
for a.a. \( t \in (0, T) \) and every \( z \in V \), where \( \varphi = \Phi(v) \).

Therefore, problem (3.11)–(3.13) is equivalent to the auxiliary problem, provided \( v = \rho \) and \( \varphi = \sqrt{\xi} \), and provided that some stronger regularity requirements are granted.
Lemma 4.1. Let $v \in D(\Phi)$ and assume that $\rho$ satisfies (4.11)–(4.14). Then, we have that

$$\rho \geq \rho^* \quad \text{a.e. in } Q_T.$$ (4.14)

In particular, this is true if $v$ is any solution $\rho$ to problem (3.11)–(3.13).

Proof. The proof we give is quite standard. Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, nondecreasing, and such that $g(r) < 0$ for $r < \rho^*$ and $g(r) = 0$ for $r \geq \rho^*$; furthermore, let $G$ be the primitive of $g$ that vanishes at $\rho^*$. Now, we write (4.14) at $t = s$ and test it by $z := g(\rho(s))$. Then, we integrate over $(0,t)$ with respect to $s$, where $t \in (0,T)$ is arbitrary. We have:

$$\kappa \int_{\Omega} G(\rho(t)) + \int_{Q_t} \nabla \rho \cdot \nabla g(\rho) + \int_{Q_t} (f'(\rho) - \varphi / \sqrt{\rho}) g(\rho) = \kappa \int_{\Omega} G(\rho_0).$$ (4.15)

The right-hand side vanishes by (4.2) and the first two terms on the left-hand side are non-negative. As to the third, we can replace $Q_t$ by its subset where $\rho < \rho^*$. Due to (3.2)–(3.3) and (4.1)–(4.2), a.e. in such a subset we have that:

$$f'(\rho) - \frac{\varphi}{\sqrt{\rho}} \leq f'_*(\rho^*) + M_2 \leq 0 \quad \text{and} \quad g(\rho) \leq 0,$$ (4.16)

whence the nonnegativity of the third integral in (4.15). We conclude that $G(\rho(t)) = 0$ a.e. in $\Omega$ for every $t \in [0,T]$, i.e., that $\rho \geq \rho^*$ a.e. in $Q_T$.

Lemma 4.2. Assume $v \in D(\Phi)$ and $v \geq \rho^*$. Then, we have that

$$\Phi(v) \leq \sqrt{\xi^*} \quad \text{a.e. in } Q_T.$$ (4.17)

In particular, $\xi \leq \xi^*$ a.e. in $Q_T$ for every solution $(\rho,\xi)$ to problem (3.11)–(3.13).

Proof. We set $\varphi := \Phi(v)$ for brevity and stipulate that

$$\chi$$ is the characteristic function of the subset of $Q_T$ where $\varphi > 0.$ (4.18)

Then, (4.9)–(4.10) yield

$$\varphi(t) = \sqrt{\xi_0} - \int_0^t \chi(s) \kappa |\partial_t v(s)|^2 + \bar{\sigma}^+(s) - \bar{\sigma}^-(s) \frac{ds}{2v(s)}$$

$$\leq \sqrt{\xi_0} + \frac{1}{2} \int_0^t \chi(s) \frac{\bar{\sigma}^-(s)}{v(s)} ds \leq \sqrt{\xi_0} + \frac{1}{2\sqrt{\rho^*}} \|\bar{\sigma}^-\|_{L^1(0,\infty;L^\infty(\Omega))} = \sqrt{\xi^*}. \quad \text{In particular, } \xi \leq \xi^* \text{ if } v = \rho \text{ and } (\rho,\xi) \text{ solves problem (3.11)–(3.13)}, \text{ because the inequality } \rho \geq \rho^* \text{ has already been proved for every solution}$.

Lemma 4.3. Assume $v \in D(\Phi)$ and $v \geq \rho^*$ a.e. in $Q_T$ and let $\rho$ satisfy (4.11)–(4.14). Then, we have that

$$\rho \leq \rho^* \quad \text{a.e. in } Q_T.$$ (4.19)

In particular, this is true if $v$ is any solution $\rho$ to problem (3.11)–(3.13).
Let \( g : \mathbb{R} \to \mathbb{R} \) be Lipschitz continuous, nondecreasing, such that \( g(r) > 0 \) for \( r > \rho^* \) and \( g(r) = 0 \) for \( r \leq \rho^* \); and let \( G \) be the primitive of \( g \) that vanishes at \( \rho^* \). Then, (4.14) holds with the new \( g \) and \( G \); once again, the only trouble comes from the third term in the left-hand side. However, (a.e.) in the subset of \( Q_t \) where \( \rho > \rho^* \), we have that

\[
g(\rho) \geq 0 \quad \text{and} \quad f'(\rho) - \frac{\varphi}{\sqrt{\rho}} \geq f_1'(\rho) - \frac{\sqrt{\xi}}{\sqrt{\rho}} + f_2'(\rho) \geq f_1'(\rho^*) - \frac{\sqrt{\xi}}{\sqrt{\rho^*}} - M_2 \geq 0, \tag{4.20}
\]

thanks to the estimate (4.17) and the definition of \( \rho^* \) given by (4.4). We conclude that \( G(\rho(t)) = 0 \) a.e. in \( \Omega \) for every \( t \in [0,T] \), i.e., that \( \rho \leq \rho^* \) a.e. in \( Q_T \).

**Remark 4.4.** We note that the last sentence of Theorem 3.2 is completely proved.

**Lemma 4.5.** Let \( v \in D(\Phi) \) be such that \( \rho_* \leq v \leq \rho^* \) a.e. in \( Q_T \), and let \( \rho \) satisfy (4.11)–(4.14). Then, we have that

\[
\|\partial_t \rho\|_{L^p(Q_T)} \leq R_p \quad \text{for every} \quad p \in (1, +\infty),
\]

\[
\|\rho\|_{L^p(0,T; W^{2,p}(\Omega))} \leq R_p' \quad \text{for every} \quad p \in (1, +\infty),
\]

where the constants \( R_p \) and \( R_p' \) depend only on the structure of our problem, the initial data \( \rho_0 \) and \( \xi_0 \), and \( p \). In particular, all this is true if \( v \) is any solution \( \rho \) to problem (3.11)–(3.13).

**Proof.** Observe that

\[
\int_\Omega \partial_t \rho(t)z + \int_\Omega \nabla \rho(t) \cdot \nabla z = \int_\Omega F(t)z \quad \text{and} \quad \rho(0) = \rho_0
\]

for a.a. \( t \in (0,T) \), for every \( z \in V \), and for \( F := \varphi/\sqrt{\rho} - f'(\rho) \). Owing to (4.14), (4.17), (4.19), and to our assumptions on \( f \) and \( \rho_0 \), we have that

\[
|F| \leq \frac{\sqrt{\xi}}{\sqrt{\rho^*}} + \sup_{\rho_* \leq \rho \leq \rho^*} |f'(\rho)| \quad \text{a.e. in} \quad Q_T \quad \text{and} \quad \|\rho_0\|_{L^\infty(\Omega)} \leq \rho^*.
\]

(4.23)

Therefore, we can apply the general \( L^p \)-regularity theory (see, e.g., [3, Thm. 9.1, p. 341]) and deduce that (4.21)–(4.22) hold. Moreover, thanks to the above lemmas, the first of (3.13) is proved.

**Lemma 4.6.** Let \( v \in D(\Phi) \) be such that \( \rho_* \leq v \leq \rho^* \) a.e. in \( Q_T \), and let \( \rho \) satisfy (4.11)–(4.14). Moreover, assume that

\[
\|\partial_t v\|_{L^p(Q_T)} \leq R_p \quad \text{for every} \quad p \in (1, +\infty),
\]

with the same \( R_p \) as in (4.21). Then,

\[
\|\partial_t \rho\|_{L^\infty(Q_T)} \leq R_\infty,
\]

(4.25)

for some constant \( R_\infty \) depending only on the structure of our problem and the initial data \( \rho_0 \) and \( \xi_0 \). In particular, this is true if \( v \) is any solution \( \rho \) to problem (3.11)–(3.13).
Proof. Here we use the stronger regularity assumption (3.15). We proceed formally, because the argument that would make the calculation rigorous is quite standard. We differentiate (4.14) with respect to time and obtain:

\[ \kappa \int_{\Omega} \partial_t u(t) z + \int_{\Omega} \nabla u(t) \cdot \nabla z + \int_{\Omega} u(t) z = \int_{\Omega} F(t) z \quad \text{for a.a. } t \in (0,T) \text{ and every } z \in V, \]

where we have set \( u := \partial_t \rho \) and

\[ F := \partial_t \rho - f''(\rho) \partial_t \rho - \frac{1}{2} \varphi \rho^{-3/2} \partial_t \rho - \frac{1}{2} \chi (|\partial_t v|^2 + \bar{\sigma}) (v \rho)^{-1/2}, \]

with \( \chi \) as in (4.18). Now, the initial value of \( u \) is known explicitly through the differential equation for \( \rho \), and is given by the formula:

\[ \kappa u(0) = \Delta \rho_0 - f'(\rho_0) + \sqrt{\xi_0}/\sqrt{\rho_0}. \]

Hence, \( u(0) \) belongs to \( L^\infty(\Omega) \) and its \( L^\infty \)-norm is estimated by a known constant, due to (3.14). Now, we observe that the following estimate holds (see [5, Thm. 7.1, p. 181]):

\[ \|u\|_{L^\infty(Q_T)} \leq C_q \max \{ \|u(0)\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(Q_T)} \}, \]

where \( q \) and \( C_q \) are as follows: \( q \) is any real number satisfying \( q > r_N^2/(r_N - 1) \), where \( r_N = \max\{2,1+(N/2)\} \); \( C_q \) is a constant depending only on \( \Omega \) and \( q \). Hence, we are led to find an \( L^q \)-estimate of \( F \) for some \( q \) satisfying the above inequality. Actually, we can choose any of such values of \( q \) and get an estimate using the previous lemma and assumptions (3.14) and (4.24).

If we denote by \( C \) a suitable constant that could be computed in terms of our assumptions on the problem structure and the constants \( \rho_*, \rho^*, \) and \( \xi^* \), we have indeed that

\[ \|F\|_{L^q(Q_T)} \leq C (\|\partial_t \rho\|_{L^q(Q_T)} + \|\nabla v\|^2_{L^q(Q_T)} + \|\bar{\sigma}\|_{L^q(Q_T)}) \]

\[ \leq C (R_q + R^2_q + \|\bar{\sigma}\|_{L^q(Q_T)}), \]

for any \( q \in (1,\infty) \). Hence, (4.24) provides a value of \( R_\infty \) satisfying (4.26).

\[ \square \]

Remark 4.7. We note, in particular, that now all the conditions listed in (3.16) are proved.

Towards the fixed point argument. Due to the above lemmas, we can confine ourselves to look for solutions satisfying all the bounds we have proved, i.e., belonging to the set \( \mathcal{X} \) defined here below:

\[ \mathcal{X} := \{ v \in \mathcal{V} : \rho_* \leq v \leq \rho^* \text{ a.e. in } Q_T, \|\partial_t v\|_{L^p(Q_T)} \leq R_p \quad \forall \, p \in (1,\infty) \}, \]

where the constants involved are of course the same as before, and where

\[ \mathcal{V} := H^1(0,T; H) \cap C^0([0,T]; V). \]

We regard \( \mathcal{X} \) as a metric subspace of the Banach space \( \mathcal{V} \), i.e., we consider the metric \( d \) on \( \mathcal{X} \) defined by

\[ d^2(u,v) := \int_{Q_T} |\partial_t u - \partial_t v|^2 + \sup_{t \in [0,T]} \|u(t) - v(t)\|^2 \quad \text{for } u,v \in \mathcal{X}. \]
Thus, $\mathcal{X}$ is complete: each one of the conditions specified in the definition (1.25) of $\mathcal{X}$ defines a closed subset of $\mathcal{V}$, and such is $\mathcal{X}$, being the intersection of a family of closed sets. Next, for $v \in \mathcal{X}$, we observe that $v \in D(\Phi)$, and we define the mapping $\Psi : \mathcal{X} \to \mathcal{X}$ as follows:

$$
\Psi(v) \text{ is the unique solution } \rho \text{ to problem (1.11)–(1.14).}
$$

(4.31)

It is clear that, for every $v \in \mathcal{X}$, the auxiliary problem (1.11)–(1.14) has a unique solution $\rho$. Indeed, if we set $\varphi := \Phi(v)$, we see that, for a.a. $(x,t) \in Q_T$, the nonlinear function $r \mapsto f'(r) - \varphi(x,t)/\sqrt{r}$, defined for $\rho_* \leq r \leq \rho^*$, is Lipschitz continuous; moreover, $\varphi$ is bounded. Hence, the standard theory for regular parabolic equations yields existence and uniqueness and we conclude that $\Psi(v)$ is well defined. Furthermore, the precise choice of all constants in (1.25) exactly ensures that $\Psi(v)$ belongs to $\mathcal{X}$, due to the previous lemmas, whence $\Psi$ actually maps $\mathcal{X}$ into itself. Finally, it is clear that an element of $\mathcal{X}$ is a solution to (1.11)–(1.14) if and only if it is a fixed point for $\Psi$. Hence, it suffices to prove that $\Psi^k := \Psi \circ \cdots \circ \Psi$ ($k$ times) is a contraction for $k$ large enough. The rest of the proof is devoted to that. A key point of our argument is the following lemma.

**Lemma 4.8.** For $i = 1,2$, pick $a_i \in L^1_{\text{loc}}[0,+\infty)$, and let $y_i$ be the maximal solution to the Cauchy problem (6.1) with $a = a_i$. Then, we have

$$
\sup_{s \in [0,t]} |\sqrt{y_1(s)} - \sqrt{y_2(s)}| \leq \int_0^t |a_1(s) - a_2(s)| \, d\tau \quad \text{for every } t \geq 0.
$$

(4.32)

**Proof.** We first prove that

$$
\frac{d}{dt} |\sqrt{y_1} - \sqrt{y_2}| \leq |a_1 - a_2| \quad \text{a.e. in } (0, +\infty).
$$

(4.33)

To this end, we choose everywhere defined representatives of $a_i$, set $\varphi_i := \sqrt{y_i}$ for brevity, and notice that the functions $\varphi_i$ and $|\varphi_1 - \varphi_2|$ are locally absolutely continuous on $[0, +\infty)$. Therefore, there exists an exceptional set $E \subset [0, +\infty)$ having zero Lebesgue measure, such that the derivatives at $t$ of the above functions exist for every $t \in (0, +\infty) \setminus E$. Moreover, we have that $\varphi'_i(t) = -a_i(t)$ if $\varphi_i(t) > 0$. Let us fix a point $t$ outside of $E$ and prove (4.33) at $t$. We distinguish three cases. In the first one, we have $\varphi_i(t) > 0$ for $i = 1, 2$. Then,

$$
|\varphi_1 - \varphi_2| = \pm (\varphi_1 - \varphi_2) \text{ in a neighborhood of } t \text{ and } \varphi'_i(t) = -a_i(t) \text{ for } i = 1, 2, \text{ whence }
$$

$$
|\varphi_1 - \varphi_2|'(t) = |\varphi_1 - \varphi_2|'(t) = \mp (a_1 - a_2)(t) \leq |a_1(t) - a_2(t)|.
$$

Assume now $\varphi_i(t) = 0$ for $i = 1, 2$. Then, $\varphi'_i(t) = 0$ (see (6.7)) and the desired inequality trivially follows. In the last case, we have, e.g., $\varphi_1(t) > 0 = \varphi_2(t)$. We derive that $\varphi'_1(t) = -a_1(t)$. On the other hand, as $\varphi_2(t) = 0$, we see that (6.8) implies $a_2(t) \geq 0$. Therefore, noting that $\varphi_1 - \varphi_2 > 0$ in a neighborhood of $t$, we deduce that

$$
|\varphi_1 - \varphi_2|'(t) = (\varphi_1 - \varphi_2)'(t) = \varphi'_1(t) - \varphi'_2(t) \leq -a_1(t) + a_2(t) \leq |a_1(t) - a_2(t)|,
$$

and (4.33) is completely proved. Now, we derive (4.32). We fix $t \geq 0$ and $s \in [0, t]$. Then, (4.33) yields

$$
|\sqrt{y_1(s)} - \sqrt{y_2(s)}| = \int_s^t |\sqrt{y_1} - \sqrt{y_2}|'(\tau) \, d\tau \leq \int_0^t |a_1(\tau) - a_2(\tau)| \, d\tau \leq \int_0^t |a_1(\tau) - a_2(\tau)| \, d\tau,
$$

and (4.32) immediately follows. $\square$
Conclusion. We take $v_1, v_2 \in X$ and set for convenience $\varphi_i := \Phi(v_i)$ and $\rho_i := \Psi(v_i)$ for $i = 1, 2$. Then, we write the equality in (4.11) for $v = v_i$ and test the difference by $\partial_t (\rho_1 - \rho_2)$. Then, we integrate over $(0, t)$ for an arbitrary $t \in (0, T)$ and add the same integral to both sides for convenience. If we set $\rho := \rho_1 - \rho_2$ and use a similar notation for $\varphi$ and $v$, we obtain:

$$
\kappa \int_{Q_t} |\partial_t \rho|^2 + \frac{1}{2} \|\rho(t)\|^2_{V} = \int_{Q_t} (\rho - f'(\rho_1) + f'(\rho_2)) \partial_t \rho + \int_{Q_t} (\varphi_1 \rho_1^{-1/2} - \varphi_2 \rho_2^{-1/2}) \partial_t \rho \quad (4.34)
$$

We now pass to estimate the right-hand side of the last relation. In order to simplify the notation, we use the same symbol $c$ for different constants (even in the same formula) that only depend on the problem structure, the data, and $T$; $c_3$ denotes those constants that depend, in addition, on the parameter $\delta \in (0, 1)$. As the functions $f$ and $(\cdot)^{\pm 1/2}$ are Lipschitz continuous on the interval $[\rho_*, \rho^*]$, and as estimate (4.17) holds for both $\varphi_i$, we have that

$$
\int_{Q_t} (\rho - f'(\rho_1) + f'(\rho_2)) \partial_t \rho + \int_{Q_t} (\varphi_1 \rho_1^{-1/2} - \varphi_2 \rho_2^{-1/2}) \partial_t \rho
$$

$$
\leq c \int_{Q_t} |\rho| |\partial_t \rho| + \int_{Q_t} \varphi_1 |\rho| |\partial_t \rho| + \int_{Q_t} \rho_2^{-1/2} |\varphi| |\partial_t \rho|
$$

$$
\leq \delta \int_{Q_t} |\partial_t \rho|^2 + c_3 \int_{Q_t} (|\rho|^2 + c_4 \int_{Q_t} |\varphi|^2),
$$

for every $\delta \in (0, 1)$. By combining with (4.34) and choosing $\delta$ small enough, we easily deduce that

$$
\int_{Q_t} |\partial_t \rho|^2 + \|\rho(t)\|^2_{V} \leq c \int_0^t \|\rho(s)\|^2_{H} \, ds + c \int_0^t \|\varphi(s)\|^2_{H} \, ds.
$$

On the other hand, we can compare (3.40), (3.43), (3.45), and (3.46)–(3.50) and apply Lemma 1.8. Thus, for every $s \in (0, t)$ and a.e. in $\Omega$, we have that

$$
|\varphi(s)| = |\varphi_1(s) - \varphi_2(s)|
$$

$$
\leq \frac{1}{2} \int_0^s \left| v_1^{-1/2}(\tau) (\kappa |\partial_t v_1(\tau)|^2 + \sigma(\tau)) - v_2^{-1/2}(\tau) (\kappa |\partial_t v_2(\tau)|^2 + \sigma(\tau)) \right| \, d\tau
$$

$$
\leq \frac{1}{2} \int_0^s \left| v_1^{-1/2}(\tau) - v_2^{-1/2}(\tau) \right| (\kappa |\partial_t v_1(\tau)|^2 + \sigma(\tau)) \, d\tau
$$

$$
+ \frac{\kappa}{2} \int_0^s |\partial_t v_1(\tau)|^2 - |\partial_t v_2(\tau)|^2 |
$$

$$
\leq c \int_0^s (1 + \sigma(\tau)) |v(\tau)| \, d\tau + c \int_0^s |\partial_t v(\tau)| \, d\tau \leq c \int_0^s |v(\tau)| \, d\tau + c \int_0^s |\partial_t v(\tau)| \, d\tau,
$$

since $\rho_* \leq v_i \leq \rho^*$. Hence,

$$
\|\varphi(s)\|^2_{H} \leq c \|v\|^2_{Z(Q_s)} + c \|\partial_t v\|^2_{Z(Q_s)} \leq c \int_{Q_s} |\partial_t v|^2 + c \|v\|^2_{C^0(0, s; V)}.
$$

Therefore, we deduce that

$$
\int_{Q_t} |\partial_t \rho|^2 + \|\rho(t)\|^2_{V} \leq c \int_0^t \|\rho(s)\|^2_{V} \, ds + c \left( \int_{Q_s} |\partial_t v|^2 + \|v\|^2_{C^0(0, s; V)} \right) \, ds,
$$

as desired.
and the Gronwall lemma easily yields:

\[
\int_{Q_s} |\partial_t \rho|^2 + \|\rho\|^2_{C^0([0,t];V)} \leq C_* \int_0^t \left( \int_{Q_s} |\partial_t v|^2 + \|v\|^2_{C^0([0,s];V)} \right) \, ds \quad \text{for every } t \in [0,T],
\]

with a precise constant \( C_* \). This means that

\[
\int_{Q_s} |\partial_t (\Psi^k(v_1) - \Psi^k(v_2))|^2 + \|\Psi^k(v_1) - \Psi^k(v_2)\|^2_{C^0([0,t];V)} \leq \frac{C_k}{(k-1)!} \int_0^t s^{k-1} \left( \int_{Q_s} |\partial_t (v_1 - v_2)|^2 + \|v_1 - v_2\|^2_{C^0([0,s];V)} \right) \, ds,
\]

for every \( t \in [0,T] \) with \( k = 1 \). Arguing by induction on \( k \), it is straightforward to prove that the above inequality holds for every \( k \geq 1 \). We conclude that

\[
d^2(\Psi^k(v_1) - \Psi^k(v_2)) \leq \frac{C_k T^k}{k!} \, d^2(v_1, v_2) \quad \text{for every integer } k \geq 1,
\]

whence, \( \Psi^k \) is a contraction on \( X \) for \( k \) large enough. This completes the proof.

**Remark 4.9.** Here, we briefly sketch how to modify the above proof in order to achieve the generalization mentioned in Remark 3.3, where (3.15) was replaced by (3.24)–(3.26). Accordingly, we consider the auxiliary problem obtained by assuming (3.24) for \( \zeta \) and modifying (4.14) as follows:

\[
k \int_{\Omega} \partial_t \rho(t) \, z + \int_{\Omega} \nabla \rho(t) \cdot \nabla z + \int_{\Omega} (\zeta + f'_\epsilon(\rho(t))) \, z = -\int_{\Omega} (\xi(t)/\rho(t))^{1/2} \rho \, z = 0 \quad (4.36)
\]

for a.a. \( t \in (0,T) \) and every \( z \in V \), where \( \xi \) is given by (4.7).

However, it is convenient to consider approximating problems as well. Precisely, we write \( \alpha_\epsilon(\rho) \) instead of \( \zeta \) in equations (8.26) and (11.37), where \( \alpha_\epsilon \) denotes the Yosida regularization of \( \alpha \) at level \( \epsilon \in (0,1) \) (see, e.g., [11, p. 28]). We will speak of the \( \epsilon \)-generalized problem and of the \( \epsilon \)-auxiliary problem, respectively. Now, we briefly show how to obtain first uniform bounds as in the previous lemmas, and then existence, for the generalized problem. As far as the choice of the crucial constants is concerned, in conditions (4.2) and (4.4) we replace \( f'_1 \) and \( M_2 \) by \( \alpha^0 \) and \( M_2 + 1 \), respectively.

As \( |\alpha_\epsilon(r)| \leq |\alpha^0(r)| \), and as \( \alpha_\epsilon(r) \) converges to \( \alpha^0(r) \) when \( \epsilon \) tends to zero for every \( r \in (0,1) \), we see that the inequalities

\[
\alpha_\epsilon(\rho_s) \leq -M_2 \quad \text{and} \quad \alpha_\epsilon(\rho^*) - \frac{\sqrt{\epsilon}}{\sqrt{\rho^*}} \geq M_2
\]

hold true for \( \epsilon \) small enough. Therefore, it is easy to see that Lemmas 4.1–4.3 still hold for each \( \epsilon \)-problem, i.e., that the a priori bounds (4.5) are fulfilled in the new situation.

As to Lemma 4.4, we observe that

\[
\alpha^0(r') \leq \alpha_\epsilon(r) \leq \alpha^0(r'') \quad \text{whenever } 0 < r' < \rho_s \leq r \leq \rho^* < r'' < 1.
\]

Therefore, bounds (4.36) imply uniform bounds for \( \zeta = \alpha_\epsilon(\rho) \) that can play the role of (4.14).

Thus, we see that (4.21) holds for the solution to each of the \( \epsilon \)-problems uniformly with respect to \( \epsilon \).
As far as the generalization of Lemma 4.6 is concerned, time differentiation is allowed for ε-problems. However, we replace (4.26) by the following equality
\[ \kappa \int_{\Omega} \partial_t u(t) z + \int_{\Omega} \nabla u(t) \cdot \nabla z + \int_{\Omega} u(t) z + \int_{\Omega} \alpha'_\varepsilon(\rho) u(t) z = \int_{\Omega} F(t) z, \]
and we modify the previous definition of \( F \) by writing \( f''_2 \) in place of \( f'' \). In other words, we replace the term \( -f''_1(\rho) \partial_t \rho \) on the right-hand side by the term \( \alpha'_\varepsilon(\rho) u \) on the left-hand side.

As a possible technique for (4.27) consists in testing the above equation by \( z := Z(u) \), where \( Z \) is monotone and vanishes at 0 in order to get recursive \( L^p \)-estimates by a Moser’s argument, and as the integral involving \( \alpha'_\varepsilon g \) gives a nonnegative contribution in such a procedure, the type-(4.25) bound one finds for the solutions to our ε-problems is uniform with respect to ε. All this ensures that the definition of \( X \) can be done in the present case, and it is independent of ε. However, the constant \( C_* \) we find in applying the fixed point argument does depend on ε. Nevertheless, this is enough to conclude for the existence of a unique solution to the approximating problem among the functions \( \rho \) belonging to \( X \). Since uniqueness among all solutions can be proved for every fixed \( \varepsilon > 0 \), we see that our argument constructs a global solution \( \rho_\varepsilon \) defined in the whole of \([0, +\infty)\). Moreover, for every fixed \( T \), \( \rho_\varepsilon \) satisfies a number of a priori estimates uniformly with respect to \( \varepsilon \), and (4.17) and (4.37) imply that a uniform \( L^\infty \)-estimate holds for \( \varepsilon \). Therefore, modulo standard arguments, we see that \( (\rho_\varepsilon, \varepsilon) \) sequentially converges in the proper topology to a pair \( (\rho, \varepsilon) \) and that \( (\rho, \varepsilon) \) is a global solution of the generalized problem. Unfortunately, in the new situation the previous uniqueness proof does not work, because (3.26) cannot be differentiated with respect to time.

### 5 Long-time behavior

In this section, we prove Theorem 3.4. Hence, we choose data satisfying the prescribed conditions and pick the corresponding unique global solution \( (\rho, \xi) \). Our proof is organized as follows.

The first lemma establishes the first assertion of the theorem, i.e., that the function \( \varphi_\infty \) given by (3.18) is well-defined and bounded. In the next one, we derive some new a priori estimates, which ensure that the \( \omega \)-limit \( \omega(\rho) \) given by (3.21) has the desired properties. Finally, we conclude by the announced characterization of \( \omega(\rho) \). We find it convenient to set:

\[ \varphi := \sqrt{\xi} \quad \text{and} \quad \chi := \text{the characteristic function of the subset of } Q_\infty \text{ where } \varphi > 0, \quad (5.1) \]

and we notice that the function \( \varphi_\infty \) to be studied is the pointwise limit of \( \varphi \) as time goes to infinity. Furthermore, we recall that estimates (4.14) and (4.19) hold for \( \rho_\varepsilon \).

**Lemma 5.1.** The limit (3.18) is well defined and \( \varphi_\infty \) is bounded.

**Proof.** Equations (4.9)–(4.10) and definitions (5.1) yield:

\[ \varphi(t) = \sqrt{\xi_0} - \int_0^t \chi(s) \frac{\kappa |\partial_t \rho(s)|^2 + \bar{\sigma}^+(s) - \bar{\sigma}^-(s)}{2\rho(s)} ds = \lambda_-(t) - \lambda_+(t) \quad (5.2) \]

where we have set:

\[ \lambda_-(x, t) := \sqrt{\xi_0(x)} + \int_0^t \chi(x, s) \frac{\bar{\sigma}^-(x, s)}{2\sqrt{\rho(x, s)}} ds \]
and
\[ \lambda_+(x, t) := \int_0^t \chi(x, s) \frac{\kappa |\partial_t \rho(x, s)|^2 + \bar{\sigma}^+(x, s)}{2\sqrt{\rho(x, s)}} \, ds \]
for a.a. \((x, t) \in Q_\infty\). To prove the assertion, it suffices to show that \(\lambda_{\pm}\) are bounded and that \(\lambda_{\pm}(x, t)\) are convergent as \(t\) tends to infinity for a.a. \(x \in \Omega\). We recall that \(\varphi\) is nonnegative. Hence, in view also of the last of (3.14), we have that
\[ 0 \leq \lambda_+(x, t) \leq \lambda_-(x, t) \leq \frac{\|\xi_0\|_{L^\infty(\Omega)}^{1/2}}{2\sqrt{\rho_{\ast}}} + \| \bar{\sigma}^- \|_{L^1(0, T; L^\infty(\Omega))}^2 \sqrt{\rho_{\ast}}, \quad \text{for a.a. } (x, t) \in Q_\infty, \]
so that \(\lambda_{\pm}\) are bounded. On the other hand, it is clear that both \(\lambda_-\) and \(\lambda_+\) are non-decreasing with respect to time, so that their convergence is ensured.

**Lemma 5.2.** We have that:
\[ \varphi \leq \sqrt{\xi_{\ast}} \text{ a.e. in } Q_\infty \text{ and } \int_{Q_\infty} |\partial_t \varphi| < +\infty; \]
\[ \rho \in L^\infty(0, \infty; V) \text{ and } \int_{Q_\infty} |\partial_t \rho|^2 < +\infty. \]

**Proof.** The first of (5.4) is a consequence of Lemma 4.2. To prove the second one, we observe that the calculation in the previous proof yields:
\[ \int_\Omega \lambda_+(t) \leq \int_\Omega \lambda_-(t) \leq c |\Omega| \quad \text{for every } t > 0, \]
where \(c\) is the right-hand side of (5.3) and |\(\Omega|\) is the measure of \(\Omega\). This clearly implies that
\[ \int_{Q_\infty} \chi |\partial_t \rho|^2 < +\infty \quad \text{and} \quad \int_{Q_\infty} \chi \bar{\sigma}^\pm < +\infty, \]
because \(\rho\) is bounded from below. On the other hand, we have that
\[ \partial_t \varphi = -\chi \frac{\kappa |\partial_t \rho|^2 + \bar{\sigma}}{2\sqrt{\rho}}, \]
whence immediately
\[ \int_{Q_t} |\partial_t \varphi| \leq \int_{Q_t} \chi \frac{\kappa |\partial_t \rho|^2 + \bar{\sigma}^+ + \bar{\sigma}^-}{2\sqrt{\rho}} = \int_\Omega \left( \lambda_+(t) + \lambda_-(t) - \sqrt{\xi_0} \right) \leq 2c |\Omega| \]
for every \(t > 0\) (with the same meaning of \(c\) as before); the second of (5.3) follows. To prove (5.5), we formally test (3.13) by \(\partial_t \rho\) and integrate over \((0, t)\). Then, we perform an integration by parts with respect to time, and obtain:
\[ \kappa \int_{Q_t} |\partial_t \rho|^2 + \int_{Q_t} 2\sqrt{\rho} \partial_t \varphi + \int_{Q_t} \left( \frac{1}{2} |\nabla \rho(t)|^2 + f(\rho(t)) \right) + 2 \int_\Omega \sqrt{\rho_0} \sqrt{\xi_0} \]
\[ = \int_\Omega \left( \frac{1}{2} |\nabla \rho_0|^2 + f(\rho_0) \right) + 2 \int_\Omega \sqrt{\rho_0} \varphi(t) \]
for every \( t > 0 \). By accounting for (5.11), we see that the above equality can be written as follows:

\[
\kappa \int_{Q_t} (1 - \chi) |\partial_t \rho|^2 + \int_{\Omega} \left( \frac{1}{2} |\nabla \rho(t)|^2 + f(\rho(t)) \right) + 2 \int_{\Omega} \sqrt{\rho_0} \sqrt{\xi_0} \\
= \int_{\Omega} \left( \frac{1}{2} |\nabla \rho_0|^2 + f(\rho_0) \right) + 2 \int_{\Omega} \sqrt{\rho(t)} \varphi(t) + \int_{Q_t} \chi \sigma.
\]

(5.9)

Thus, all the terms on the left-hand side of (5.9) are nonnegative. Moreover, the right-hand side is bounded, since \( \rho \) and \( \varphi \) are bounded and \( \chi \sigma \in L^1(Q_\infty) \) by the second of (5.7). We immediately deduce that

\[
\int_{Q_\infty} (1 - \chi) |\partial_t \rho|^2 < +\infty,
\]

i.e., that the first of (5.10) holds. Moreover, on recalling the first of (5.7), we see that the second of (5.5) holds as well.

**Conclusion of the proof.** Properties (5.5) imply, in particular, that \( \rho \) is a bounded weakly continuous \( V \)-valued function on \([0, +\infty)\), due to the compact embedding \( V \subset H \). Therefore, the set \( \omega(\rho) \) is a non-empty compact subset of \( H \). Actually, \( \omega(\rho) \) is also connected, due to the continuity of \( \rho \) from \([0, +\infty)\) to \( H \) and to a standard argument from the theory of dynamical systems (see, for instance, [4, p. 12]). Then, the first properties of the \( \omega \)-limit stated in the theorem follow. It remains for us to characterize \( \omega(\rho) \).

Let \( \rho^\infty \in \omega(\rho) \) and let \( \{t_n\} \) be a diverging sequence such that \( \{\rho(t_n)\} \) converges to \( \rho^\infty \) strongly in \( H \). Then, we define \( \rho_n \) and \( \varphi_n \) by the following formulas:

\[
\rho_n(t) := \rho(t + t_n) \quad \text{and} \quad \varphi_n(t) := \varphi(t + t_n) \quad \text{for} \ t \geq 0,
\]

and we consider their weak limits on a fixed bounded interval \((0, T)\) (e.g., for \( T = 1 \)). Our aim is to prove that such limits do not depend on time and furnish a steady state. First of all, we notice that

\[
\kappa \int_{Q_T} \partial_t \rho_n z + \int_{Q_T} \nabla \rho_n \cdot \nabla z + \int_{Q_T} f'(\rho_n) z - \int_{Q_T} \varphi_n \rho_n^{-1/2} z = 0 \quad \text{for every} \ z \in V,
\]

(5.10)

as one immediately sees by integrating (3.13) over \((t_n, t_n + T)\). Next, we observe that

\[
\|\rho_n\|_{L^\infty(0,T;V)} \leq \|\rho_0\|_{L^\infty(0,T;V)} \quad \text{and} \quad \|\varphi_n\|_{L^\infty(Q_T)} \leq \|\varphi\|_{L^\infty(Q_\infty)}
\]

(5.11)

for every \( n \). Moreover, as \( \varphi(t) \) converges pointwise to \( \varphi_\infty \) as \( t \) tends to infinity, we easily see that the whole sequence \( \{\varphi_n\} \) converges to \( \varphi_\infty \) weakly-star in \( L^\infty(0,T;H) \). Furthermore, \( \partial_t \varphi \) belongs to \( L^1(Q_\infty) \) by (1.17). By accounting for the second of (5.11) as well, we deduce that

\[
\varphi_n \to \varphi_\infty \quad \text{strongly in} \ C^0([0,T];L^p(\Omega)) \quad \text{for any} \ p < +\infty.
\]

(5.12)

Considering now (5.11) once more, we infer that, to within subsequences,

\[
\rho_n \to \rho_\infty \quad \text{weakly in} \ L^\infty(0,T;V) \quad \text{and} \quad \partial_t \rho_n \to 0 \quad \text{strongly in} \ L^2(0,T;H)
\]

(5.13)

for some \( \rho_\infty \). We deduce that (see [9] Sect. 8, Cor. 4)

\[
\rho_n \to \rho_\infty \quad \text{strongly in} \ C^0([0,T];H),
\]

(5.14)
whence, in particular, \( \rho_n(0) \to \rho(0) \) strongly in \( H \). Moreover, (5.13) also imply that \( \rho_\infty \) must be a constant, namely, \( \rho_\infty(t) = \rho_\infty(0) \) for every \( t \in [0,T] \). On the other hand, \( \rho_n(0) \to \rho_\infty \) by assumption. We conclude that \( \rho_\infty(t) = \rho_\infty \) for every \( t \in [0,T] \). Furthermore, as \( f \) and any power are Lipschitz continuous on \([\rho_*, \rho^*]\), we see that (5.14) implies that \( f(\rho_n) \) and \( \rho_n^{-1/2} \) converge to \( f(\rho_\infty) \) and \( \rho_\infty^{-1/2} \), respectively, in the same topology. Therefore, we can pass to the limit in (5.10) and easily conclude that \( \rho_\infty \) satisfies (3.19)–(3.20).

**Remark 5.3.** We briefly show how to extend Theorem 3.4 to the case mentioned in Remark 3.5. Minor changes in the above proof are necessary. First of all, we just have to write \( f = \widehat{\alpha} + f_2 \) in (5.9) and the same equality holds true. As the sequel does not involve any special property of \( f \), relations (5.4) and (5.5) are proved in the same way. Hence, by going through the above proof, we see that the only point to check is the following: if we define \( \zeta_n \) by the formula \( \zeta_n(t) = \zeta(t + t_n) \), does the uniformly bounded sequence \( \{\zeta_n\} \) converge to a constant function \( \zeta_\infty \) weakly-star in \( L^\infty(Q_T) \), at least for a subsequence, and is the pair \((\rho_\infty, \zeta_\infty)\) a solution to the stationary problem (3.19)–(3.20)? To answer these questions in the positive, we notice that a weak-star limit \( \zeta_\infty \) actually exists. Moreover, (5.14) holds even for any \( z \in L^2(0,T;V) \) (while so far we have written it just for test functions that are constant in time). On the other hand, the convergence (5.14) ensures that \( \zeta_\infty \in \alpha(\rho_\infty) \) a.e. in \( Q_T \) and that we can pass to the limit in (5.10). We obtain that

\[
\kappa \int_{Q_T} \partial_t \rho_\infty z + \int_{Q_T} \nabla \rho_\infty \cdot \nabla z + \int_{Q_T} (\zeta_\infty + f_2(\rho_\infty)) z - \int_{Q_T} \varphi_\infty \rho_\infty^{-1/2} z = 0 \quad (5.15)
\]

for every \( z \in L^2(0,T;V) \). In particular, \(-\Delta \rho_\infty + \zeta_\infty + f_2(\rho_\infty) = -(\varphi_\infty/\rho_\infty)^{1/2} \) at least in the sense of distribution on \( Q_T \), whence we deduce by comparison that \( \zeta_\infty \) does not depend on time. Then, we immediately see that (5.15) becomes (3.27), i.e., that \((\rho_\infty, \zeta_\infty)\) solves (3.19)–(3.20).

### 6 Appendix

We here spend some words on the general forward Cauchy problem

\[
g'(t) + 2a(t)\sqrt{y(t)} = 0, \quad y(0) = y_0, \quad (6.1)
\]

for given \( a \in L^1_{\text{loc}}[0,+) \) and \( y_0 \in [0,+) \).

If \( y_0 > 0 \), there is a **unique, strictly positive, local solution**, that has the form:

\[
\sqrt{y(t)} = \sqrt{y_0} - \int_0^t a(s) \, ds, \quad (6.2)
\]

as long as the right-hand side remains positive; needless to say, \( y \) may happen to tend to zero in a finite time. As to nonnegative solutions, we note that every local solution can be extended to a global solution, because the nonlinearity is sublinear. A sufficient condition for uniqueness is that \( a \) is nonnegative, because the function \(|0,+) \ni y \mapsto a(t)\sqrt{y} \) is nondecreasing for any fixed \( t \). The **unique, nonnegative, global solution** is given by

\[
\sqrt{y(t)} = \left( \sqrt{y_0} - \int_0^t a(s) \, ds \right)^+ \quad \text{for} \, t \in [0,+) , \quad (6.3)
\]
where the symbol \((\cdot)^+\) denotes the positive part (for instance, if \(a = 1\) and \(y_0 = 1\), we have that \(y(t) = ((1 - t)^+)^2\) for every \(t \geq 0\), and that \(y(t)\) vanishes for \(t > 1\)). Actually, to verify that (6.3) always provides a solution is the matter of a simple computation. For a general \(a\), the situation is more complicated, as we briefly explain.

If \(a\) is negative, uniqueness is no longer guaranteed, because a (forward) Peano phenomenon might occur (for instance, if \(a = -1\) and \(y_0 = 0\), the formula \(y(t) = ((t - \lambda)^+)^2\) yields a solution for every \(\lambda > 0\)). More generally, for \(y_0 \geq 0\), if a solution \(y^*\) vanishes at some point \(t_0\) and if \(a\) is negative in a right neighborhood of \(t_0\), then there are infinitely many solutions beside \(y^*\). Therefore, whenever an a priori assumption on the sign of \(a\) is inappropriate (as is the case for the problem we study in our present paper), one would like to select the solution \(y\) that is maximal, i.e., that satisfies \(y(t) \geq z(t)\) for every \(t \geq 0\) and for every solution \(z\) to the same Cauchy problem.

Now, it is known that in general any Cauchy problem for a differential equation of the form \(y' = g(t, y)\) has a unique maximal solution whenever \(g\) is a rather general Carathéodory function (see, e.g., the extension of Peano’s theorem [8, Thm. 4.1, p. 28] mentioned on p. 95 of the same book); the maximal solution can be constructed by taking the pointwise supremum of the solutions. For \(g\) as in (6.1), the square roots of the solutions describe the set

\[
S(a, y_0) := \left\{ w \in W^{1,1}_{\text{loc}}[0, +\infty) : w(0) = \sqrt{y_0}, \ w(t) \geq 0 \text{ for every } t \geq 0, \ w' = -a \text{ a.e. where } w > 0 \right\},
\]

whence the maximal solution \(y\) is characterized by

\[
\sqrt{y(t)} = \sup\{w(t) : w \in S(a, y_0)\} \text{ for every } t \geq 0.
\]  

We note that every solution to the Cauchy problem (6.1) is such that

\[
\sqrt{y(t)} = \sqrt{y_0} - \int_0^t a^*(s) \, ds,
\]

where

\[
a^*(s) := a(s) \quad \text{if} \quad y(s) > 0 \quad \text{and} \quad a^*(s) := 0 \quad \text{otherwise}.
\]  

Generally, this is nothing more than an a posteriori reconstruction of \(y\), because the definition of \(a^*\) depends on \(y\) itself. But, if \(y\) is the maximal solution, an additional property of \(a^*\) holds true, because \(a \geq 0\) a.e. where \(y = 0\). Indeed, by maximality, \(y\) tries to become positive whenever it is possible. Thus, in view of (6.7), we have that

\[
a^*(t) \leq a(t) \quad \text{for a.a. } t \geq 0.
\]  

At this point, we can come back to our problem. A comparison of (2.3) with (6.1) suggests that we take \(x\) as a parameter and set:

\[
a(\cdot) := \frac{\kappa (\partial_1 \rho(x, \cdot))^2 + \bar{\sigma}(x, \cdot)}{2 \sqrt{\rho(x, \cdot)}} \text{ for a.a. } x \in \Omega;
\]

with this, formulas (6.5)–(6.7) become (3.6)–(3.9). Hence, (3.6)–(3.9) provide the maximal global solution to (2.3) for a given \(\rho\); moreover, a sufficient condition for uniqueness is that \(\bar{\sigma}\)
is nonnegative, a case when we have from (6.3) that

\[ \sqrt{\xi(t)} = \left( \sqrt{\xi_0} - \int_0^t \frac{\kappa (\partial_t \rho(s))^2 + \bar{\sigma}(s)}{2\sqrt{\rho(s)}} \, ds \right)^+. \]  

(6.9)

If the initial datum \( \xi_0 \) is strictly positive, the last formula holds for small \( t \) even for a negative \( \bar{\sigma} \); in fact, as long as the right-hand side remains positive (see Remark 3.3), we simply have that

\[ \sqrt{\xi(t)} = \sqrt{\xi_0} - \int_0^t \frac{\kappa (\partial_t \rho(s))^2 + \bar{\sigma}(s)}{2\sqrt{\rho(s)}} \, ds. \]  

(6.10)

Acknowledgements. Special thanks are due to Aldo Pratelli, who pointed out the proof of Lemma 4.8: the authors appreciated a lot his contribution. Some financial support from the MIUR-COFIN 2006 research program on “Free boundary problems, phase transitions and models of hysteresis” and from the IMATI of CNR in Pavia, Italy, is gratefully acknowledged. The work of Podio-Guidugli was supported by the Italian Ministry of University and Research (under PRIN 2005 “Modelli Matematici per la Scienza dei Materiali”) and by the EU Marie Curie Research Training Network MULTIMAT “Multi-scale Modeling and Characterization for Phase Transformations in Advanced Materials”.

References

[1] H. Brezis, “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”, North-Holland Math. Stud. 5, North-Holland, Amsterdam, 1973.

[2] M. Frémond, “Non-smooth Thermomechanics”, Springer-Verlag, Berlin, 2002.

[3] M.E. Gurtin, Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a micro-force balance, Phys. D 92 (1996) 178–192.

[4] A. Haraux, “Systèmes Dynamiques Dissipatifs et Applications”, RMA Res. Notes Appl. Math. 17, Masson, Paris, 1991.

[5] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural’ceva: “Linear and quasilinear equations of parabolic type”, Trans. Amer. Math. Soc. 23, Amer. Math. Soc., Providence, RI, 1968.

[6] A. Miranville, Consistent models of Cahn-Hilliard-Gurtin equations with Neumann boundary conditions, Phys. D 158 (2001) 233–257.

[7] P. Podio-Guidugli, Models of phase segregation and diffusion of atomic species on a lattice, Ric. Mat. 55 (2006) 105–118.

[8] W.T. Reid, “Ordinary differential equations”, Wiley, New York, 1971.

[9] J. Simon, Compact sets in the space \( L^p(0,T;B) \), Ann. Mat. Pura Appl. 146 (1987) 65–96.