Rigid Limit in $\mathcal{N} = 2$ Supergravity
and Weak-gravity Conjecture

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arXiv:0706.2114

June 2007
1. Introduction

2. Rigid Limit and the New Scale

3. Existence of Logarithmic Periods

4. Application: RG equation

5. Summary
1. Introduction

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5. Summary
Some Success of String Theory

- Include quantized graviton.
- Many consistent models:
  - 10d flat IIA / IIB
  - 4d $\mathcal{N} = 8$, type II on $T^6$
  - 4d $\mathcal{N} = 2$, type II on CY = het on $K3 \times T^2$
Some Success of String Theory

- Include quantized graviton.
- Many consistent models:
  - 10d flat IIA / IIB
  - 4d $\mathcal{N} = 8$, type II on $T^6$
  - 4d $\mathcal{N} = 2$, type II on CY = het on $K3 \times T^2$
- Models with Less SUSY:
  - 4d $\mathcal{N} = 1$ from type II on CY with Fluxes
  - $10^{300}$ of them ???
  - Non-perturbative effects which select the Real World ????

Yuji Tachikawa (SNS, IAS)
June 2007, DAMTP
If there’re really $10^{300}$ consistent models, can arbitrary low energy Lagrangian be UV-completed with gravity?

The answer is **NO**.

- Anomaly cancelation.
- More criteria?

Yuji Tachikawa (SNS, IAS)
June 2007, DAMTP
Swampland

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Swampland

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- More criteria?
Weak-gravity Conjecture

Standard Lore

String theory cannot have continuous global symmetry.

- Gauge theory with $g \ll 1$ indistinguishable from global sym.
- there should be lower bound for $g$.
Weak-gravity Conjecture

**Standard Lore**
String theory cannot have continuous global symmetry.

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- There should be lower bound for $g$.

**Claim [Arkani-Hamed, Motl, Nicolis, Vafa]**
Gravity + $U(1)$ gauge field with c.c. $g$ $\rightarrow$ Extra physics at $gM_{\text{planck}}$

- $g_{\text{grav}} = \frac{E}{M_{\text{planck}}} < g$ if $E < gM_{\text{planck}}$
- I.e. gravity is weaker than $U(1)$, before new physics comes in.
Motivation behind the Conjecture

- perturbative string compactification \( M_{\text{string}} = g M_{\text{planck}} \)
- Simplest monopole should not be black:

\[
m_{\text{monopole}} \sim \frac{E_{\text{cutoff}}}{g^2} \quad \text{and} \quad R_{\text{monopole}} \sim \frac{1}{E_{\text{cutoff}}}
\]

then \( R_{\text{Schwarzshild}} < R_{\text{monopole}} \) leads

\[
\frac{m_{\text{monopole}}}{M_{\text{planck}}^2} < \frac{1}{E_{\text{cutoff}}} \quad \rightarrow \quad E_{\text{cutoff}} < g M_{\text{planck}}
\]

- see the original article for more.
To confirm the scale $gM_{\text{planck}}$ in $\mathcal{N} = 2$ Yang-Mills + supergravity.

- Why $\mathcal{N} = 2$?
- because we have a lot more control.
  - Everything comes from Prepotential, which is holomorphic
  - Singularity structure (cuts, poles ...) determines the function
- Holomorphy in SW told us a lot about the dynamics of Yang-Mills
$\mathcal{N} = 2$ SUGRA and Weak-Gravity Conjecture

**Objective**

To confirm the scale $gM_{\text{planck}}$ in $\mathcal{N} = 2$ Yang-Mills + supergravity.

- Why $\mathcal{N} = 2$?
- because we have a lot more control.
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**Another Objective**

dynamics of $\mathcal{N} = 2$ gravity from holomorphy + monodromy
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5. Summary
$\mathcal{N} = 2$ Yang-Mills from type IIA

- type IIA on Calabi-Yau $\rightarrow$ $\mathcal{N} = 2$ sugra in 4d, a lot of $U(1)^n$

$$ C_3 = A^I_{4d} \wedge \omega^I_{\mathrm{CY}} $$

- D2-brane couples to $C_3$ $\rightarrow$ charged solitons in 4d

$$ \int_S C_3 = A^I \cdot \int_S \omega_I $$

charge of D2 wrapping $S$

- $S^2$ shrinks $\rightarrow$ nearly massless charged particles = W-boson

- $SU(2)$ gauge theory appears.
\( N = 2 \) Yang-Mills from type IIB

- type IIA: worldsheet instantons of order \( e^{-\text{Area of } S^2} \)
  - large corrections for \( S^2 \) small

| \( \alpha' / R^2 \)  | IIA        | IIB        |
|----------------------|------------|------------|
| \( g_{\text{string}} \) | vector    | hyper     |
|                      | hyper     | hyper     |

- Mirror symmetry: IIA on \( M = \text{IIB on } W \)
- No correction to vector kinetic terms in IIB
- Classical geometry of \( W \) should capture the \( SU(2) \) gauge dynamics
$\mathcal{N} = 2$ Yang-Mills from type IIB

- type IIA: worldsheet instantons of order $e^{-\text{Area of } S^2}$
  - large corrections for $S^2$ small

|       | IIA  | IIB   |
|-------|------|-------|
| $\alpha'/R^2$ | vector | hyper |
| $g_{\text{string}}$ | hyper | hyper |

- Mirror symmetry: IIA on $M = $ IIB on $W$
- No correction to vector kinetic terms in IIB
- Classical geometry of $W$ should capture the $SU(2)$ gauge dynamics coupled to $\mathcal{N} = 2$ gravity!
Recap: $\mathcal{N} = 2$ rigid SUSY

- $\mathcal{N} = 2$ pure $SU(n+1)$ gauge theory: complex adjoint scalar $\phi$
- $\langle \phi \rangle \neq 0 \quad \rightarrow \quad$ Higgsed to $U(1)^n$
- special coordinates $a^i, i = 1, \ldots, n$
- dual special coordinates
  
  $$a^D_i = \frac{\partial \mathcal{F}_{\text{gauge}}}{\partial a^i}$$

- $K = \text{Im}(a^D_i)^* a^i$
- mass of BPS particles: $|q_i a^i + m^i a^D_i|$
Recap: Seiberg-Witten theory

- Monodromy determines $a, a^D$
- SW curve
  \[ w + \frac{\Lambda^2}{w} = x^2 - u \]
  and then
  \[ a = \int_A x \frac{dw}{w}, a^D = \int_B x \frac{dw}{w}; \]
  \[ \tau = \frac{\partial a^D}{\partial a} \]
- Curve singular at the monopole/dyon points
Recap: $\mathcal{N} = 2$ Supergravity

- $N \ U(1)$ vector multiplets $\rightarrow N + 1$ gauge fields
  (remember graviphoton !)
- $N$ scalars $\Phi^I$ ($I = 1, \ldots, N$)
- special coordinates, or periods $X^a$ and $F_a$ ($a = 1, \ldots, N + 1$)

$$e^{-K} = \text{Im} \ F^*_a X^a$$

- Kähler transformation: $K \rightarrow K - f - f^*$ and
  $X^a, F_a \rightarrow e^f X^a, e^f F_a$
- mass of BPS particle $m^2 = e^K |q_a X^a + m^a F_a|^2$
Recap: $\mathcal{N} = 2$ supergravity from IIB on CY

- $(2N + 2)$ 3-cycles on CY $\rightarrow$ Canonical basis $A^a, B_a$
  s.t. $A^a \cap A^b = B^a \cap B^b = 0$, $A^a \cap B_b = \delta^a_b$

- Covariantly constant $(3, 0)$-form $\Omega$

$$X^a = \int_{A^a} \Omega, \quad F_a = \int_{B_b} \Omega$$

- Kähler tr. = overall factor in $\Omega$

- $\delta \Omega$ is $(3, 0) + (2, 1)$ $\rightarrow \int_{CY} \Omega \wedge \partial_{\Phi^I} \Omega = 0$

$$\rightarrow \sum_a X^a \frac{\partial F_a}{\partial \Phi^i} - \sum_a \frac{\partial X^a}{\partial \Phi^i} F_a = 0$$
Embedding $\mathcal{N} = 2$ pure $SU(2)$ in supergravity

- $u$: moduli for $SU(2)$, normalized to have monopole pt. at $u = 1$
- $\epsilon$: hierarchy between gauge theory and gravity
- two moduli $\rightarrow$ three gauge fields $\rightarrow$ six periods
- $SU(2)$ theory has two periods $\rightarrow$ need extra four periods
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- two moduli $\rightarrow$ three gauge fields $\rightarrow$ six periods
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- does new scale $gM_{\text{planck}}$ appear? if so, how?
the Calabi-Yau

\[
\frac{B}{8} x_1^8 + \frac{B}{8} x_2^8 + \frac{1}{4} x_3^4 + \frac{1}{4} x_4^4 + \frac{1}{4} x_5^4 - \psi_0 x_1 x_2 x_3 x_4 x_5 - \frac{1}{4} \psi_2 (x_1 x_2)^4 = 0
\]

in \( \mathbb{CP}^{1,1,2,2,2} \); \([B : \psi_0 : \psi_2]\) gives two parameters; singular when

\[
B^2 (B^2 - \psi_2^2) (B^2 - (\psi_2 + \psi_0^4)^2) = 0
\]

[Candelas, de la Ossa, Font, Katz, Morrison]
[Hosono, Klemm, Theisen, Yau]
[Billó, Denef, Frè, Pesando, Troost, Van Proeyen, Zanon]
Moduli of the Calabi-Yau

- $\epsilon = B$, $\epsilon u = \psi_2 + \psi_0^4$
- $\epsilon = 0$: rigid limit locus,
- CY becomes

$$w + \frac{1}{w} + x^2 + y^2 + z^2 - u + O(\epsilon) = 0$$

- the SW curve!
- $u$-plane embedded!
• \( \epsilon = B, \epsilon u = \psi_2 + \psi_0^4 \)
• \( \epsilon = 0 \): rigid limit locus,
• CY becomes

\[
\frac{1}{w} + x^2 + y^2 + z^2 - u + O(\epsilon) = 0
\]

• the SW curve !
• \( u \)-plane embedded !
• one-year before SW !
Behavior of the periods

- $X^1 \sim \epsilon^{1/2} a$, $F_1 \sim \epsilon^{1/2} a^D$
- $X^{2,3} \sim O(1)$, $F_{2,3} \sim \log \epsilon$
- the Kähler potential is then

$$e^{-K} \sim (\log 1/|\epsilon|) + |\epsilon| \text{Im}(a^D)^* a$$

that is

$$K \sim \log(\log 1/|\epsilon|) + \frac{|\epsilon|}{\log 1/|\epsilon|} \text{Im}(a^D)^* a$$
• $X^1 \sim \epsilon^{1/2} a$,  
  $F_1 \sim \epsilon^{1/2} a^D$ where $a, a^D$ determined from

\[
w + \frac{1}{w} + x^2 + y^2 + z^2 - u = 0
\]

• Usually the curve is

\[
w + \frac{\Lambda^2}{w} + x^2 + y^2 + z^2 - u = 0
\]

$\epsilon^{1/2} \sim \Lambda = M_{\text{cutoff}} \exp \frac{1}{4} \left( \theta i - \frac{8\pi^2}{g^2} \right)$  [Vafa et al.]  

• Let us define $S = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \frac{1}{\pi i} \log \epsilon$
The mass scale of gauge theory I

- $X^1 \sim \epsilon^{1/2} a, \quad F_1 \sim \epsilon^{1/2} a^D$ where $a, a^D$ determined from

$$w + \frac{1}{w} + x^2 + y^2 + z^2 - u = 0$$

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$$w + \frac{\Lambda^2}{w} + x^2 + y^2 + z^2 - u = 0$$

$\epsilon^{1/2} \sim \Lambda = M_{\text{cutoff}} \exp \frac{1}{4} \left( \theta i - \frac{8\pi^2}{g^2} \right)$ [Vafa et al.]

- Let us define $S = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \frac{1}{\pi i} \log \epsilon$

- At which scale $g$ is defined ??
Low energy coupling is determined by

\[ \tau = \frac{\partial a^D}{\partial a} = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2(m_W)} \]

in the weak coupling regime \( u \gg \Lambda^2 \),

\[ a \sim \sqrt{u}, \quad a^D \sim \sqrt{u} \log u \quad \rightarrow \quad e^{-2\pi^2/g^2(m_W)} = u^{-1/2} \]

\( g^2(m_W) \) is defined at the scale of W-boson, of mass

\[ m_W^2 = e^{-K} |X^1|^2 = \frac{1}{\log 1/|\epsilon|} |\epsilon u|. \]

Thus, the RG-invariant scale is

\[ \Lambda_{\text{gauge}} = m_W e^{-2\pi^2/g^2(m_W)} = \frac{|\epsilon|^{1/2}}{(\log 1/|\epsilon|)^{1/2}} M_{\text{planck}} \]
Result.

\[ \Lambda_{\text{gauge}} = \frac{|\epsilon|^{1/2}}{(\log \frac{1}{|\epsilon|})^{1/2}} M_{\text{planck}} = e^{-2\pi^2/g^2} g M_{\text{planck}} \]

- recall \( S = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \frac{1}{\pi i} \log \epsilon \) is the natural UV coupling.
- it is determined at \( g M_{\text{planck}} \)!
The mass scale of gauge theory III

Result.

\[ \Lambda_{\text{gauge}} = \frac{|\epsilon|^{1/2}}{(\log 1/|\epsilon|)^{1/2}} M_{\text{planck}} = e^{-2\pi^2/g^2} g M_{\text{planck}} \]

- recall \( S = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \frac{1}{\pi i} \log \epsilon \) is the natural UV coupling.

- it is determined at \( g M_{\text{planck}} \)!

- came essentially from \( m_W^2 = e^K |X^1|^2 = e^K |\epsilon u| \),

- \( e^{-K} = \text{Im} F_\alpha^* X^\alpha \) is dominated by \( \alpha = 3, 4 \) with \( F \sim \log \epsilon \).
kinetic term for $S$

- $S = \frac{1}{\pi i} \log \epsilon$

- $e^{-K} \sim \log |\epsilon| = \frac{1}{\pi i} (S - S^*) \quad \Rightarrow \quad K = \log \text{Im} S$

- $g_{SS^*} \partial_\mu S \partial_\mu S^* = \frac{\partial_\mu S \partial_\mu S^*}{(\text{Im} S)^2}$

Looks like a dilaton! if the CY has a heterotic dual, it is the dilaton.
kinetic term for $S$

- $S = \frac{1}{\pi i} \log \epsilon$

- $e^{-K} \sim \log |\epsilon| = \frac{1}{\pi i} (S - S^*) \quad \rightarrow \quad K = \log \text{Im} S$

\[ g_{\mu \mu'} S \partial_\mu S \partial_{\mu'} S^* = \frac{\partial_\mu S \partial_{\mu'} S^*}{(\text{Im} S)^2} \]

- Looks like a dilaton!
- if the CY has a heterotic dual, it is the dilaton.
\[
\begin{align*}
S &= \frac{1}{\pi i} \log \epsilon \\
\tau &= \frac{\partial^2 \mathcal{F}}{\partial \alpha^2} = -\frac{1}{\pi i} \log u \\
\text{New scale at } gM_{\text{planck}}
\end{align*}
\]
\begin{itemize}
\item $S = \frac{1}{\pi i} \log \epsilon$
\item $\tau = \frac{\partial^2 F}{\partial a^2} = -\frac{1}{\pi i} \log u$
\item New scale at $gM_{\text{planck}}$!
\item Gauge theory periods $X^1, F_1 \sim \epsilon^{1/2}$
\item Extra periods $X^{3,4} \sim O(1), F_{3,4} \sim \log \epsilon$
\item which is the source of the new scale!
\end{itemize}
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Periods of $\mathcal{N} = 2$ pure SYM + sugra

- special coordinates of pure SYM $a^i, a^D_i$ ($i = 1, \ldots, r$)
- embedding into sugra: $S = \frac{\theta}{2\pi} + \frac{4\pi i g^2}{g^2}$ is also a field
- $r + 1$ parameters
- $2r + 4$ periods, i.e. FOUR extra periods
- gauge theory periods $X^i \sim \epsilon^{1/h} a^i$ and $F_i \sim \epsilon^{1/h} a^D_i$
- FOUR extra periods
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- FOUR extra periods

- we’ll show two behave as $O(1)$, the other two $\log \epsilon$
- New Scale $g M_{\text{planck}}$
CY which is K3 fibration

\[ w + \frac{\mu^2}{w} + W_{K3}(x, y, z; u_i, v_\alpha) = 0 \]

- K3 changes as \( w \) changes
- Suppose a two-cycle \( S \) shrinks at \( w_i + \mu^2/w_i = k_i \)
- \( \rightarrow \) two three-cycles in CY!
- \( X, F = \int_C \frac{dw}{w} \int_S \Omega_{K3} \)
\[ w + \frac{\mu^2}{w} = x^2 + y^2 + z^n \]

\[ \rightarrow SU(n) \text{ gauge group; } r = (n - 1) \text{ two-cycles} \]

\[ w + \frac{\mu^2}{w} = x^2 + y^2 + z^n + u_2 z^{n-2} + u_3 z^{n-3} + \cdots + u_n + O(\mu^{1+1/n}) \]

- Rescale according to the mass dimension:

\[ \mu^2 = \epsilon^2, \quad x = \epsilon^{1/2} \tilde{x}, \quad y = \epsilon^{1/2} \tilde{y}, \quad z = \epsilon^{1/n} \tilde{z}, \]

\[ w = \epsilon \tilde{w}, \quad u_i = \epsilon^{i/n} \tilde{u}_i \]

\[ \tilde{w} + \frac{1}{\tilde{w}} = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^n + u_2 \tilde{z}^{n-2} + \cdots + u_n + O(\epsilon^{1/n}) \]

- \((n - 1)\) 2-cycles are at finite values of \(\tilde{x}, \tilde{y}, \tilde{z}\) \(\rightarrow x, y, z \ll 1\)
\( A_r \) singularity

- \( w + \mu^2/w = x^2 + y^2 + z^n \)
  \( \rightarrow SU(n) \) gauge group; \( r = (n - 1) \) two-cycles

\[
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  \( w = \epsilon \tilde{w}, \quad u_i = \epsilon^{i/n} \tilde{u}_i \)

\[
\tilde{w} + \frac{1}{\tilde{w}} = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^n + u_2 \tilde{z}^{n-2} + \cdots + u_n + O(\epsilon^{1/n})
\]

- \( (n - 1) \) 2-cycles are at finite values of \( \tilde{x}, \tilde{y}, \tilde{z} \)

\( \rightarrow \) \( x, y, z \ll 1 \)

- do they exhaust all the K3 cycles? \( \text{NO!} \)
Recap: K3

- CY manifold of real dimension 4, i.e. $SU(2)$ holonomy
- 22 2-cycles
- two 2-cycles $S, S'$ intersection number $(S, S') = \#(S \cap S')$
- Signature is $(3, 19)$
- some 2-cycles $C$ are holomorphically embedded $\int_C \Omega_{2,0} = 0$
- other 2-cycles $S$ are called transcendental; has signature $(2, x)$
- 2-cycles collapsible at a point (e.g. ADE singl.) have negative self-intersection rank $r$ singularity: $x \geq r$. 
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- at least two 2-cycles $T_a$ ($a = 1, 2$) with positive self-intersection.
  FOUR extra CY periods.
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- 2-cycles collapsible at a point (e.g. ADE singl.) have negative self-intersection rank $r$ singularity: $x \geq r$.
- at least two 2-cycles $T_a$ ($a = 1, 2$) with positive self-intersection.
  → FOUR extra CY periods.
- Another nice matching of pure math and supergravity.
Logarithmic periods

- In the $\epsilon \to 0$ limit,

\[
 w + \frac{\epsilon^2}{w} + W_{K3}(x, y, z; \epsilon^{i/h} u_i, v_a) = 0
\]

becomes

\[
 w + \frac{\epsilon^2}{w} + W_{K3}(x, y, z; 0, v_a) = 0
\]

- Suppose $T_a$ shrinks at $w + \epsilon^2/w = k_a$

\[
 w_a^+ \sim k_a, \quad w_a^- \sim \epsilon^2/k_a
\]

\[
 \int \Omega = \int_{w_a^-}^{w_a^+} \frac{dw}{w} \int_{T_a} \Omega_{K3} \sim 2c \log \epsilon
\]
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RG equation

\[
\frac{\partial F_{\text{gauge}}}{\partial \log \Lambda} = u_2 \quad \text{where} \quad u_2 = \langle \text{tr} \phi^2 \rangle
\]

- First noticed by [Matone] for $SU(2)$ using SW curve
- later generalized to classical gauge groups by [Sonnenshein, Theisen, Yankielowicz] and [Eguchi, Yang], using SW curve
- it is built in the instanton calculation [Dorey, Khoze, Mattis] :

\[
\langle \partial_\lambda L_0 \rangle = \partial_\lambda L_{\text{eff}}, \quad \langle \partial_\lambda W_0 \rangle = \partial_\lambda W_{\text{eff}}, \quad \langle \partial_\lambda F_0 \rangle = \partial_\lambda F_{\text{eff}}.
\]

while $F_0 = \tau_0 \text{ tr} \phi^2$, $\log \Lambda \propto \tau_0$

- for $E_{6,7,8}$, curves are known; RG relation not proven
Embedding into compact CY

- $a^i, a^D_i$: periods of gauge theory
- $X^i = \epsilon^{1/h} a^i, F_i = \epsilon^{1/h} a^D_i$
- extra periods $X^a, F_a$: Recall

$$w + \frac{\mu^2}{w} = W_{K3}(x, y, z; \epsilon^{i/h} u_i, v_a)$$

for small $\epsilon$, so that extra cycles are at finite values of $x, y, z$

- one can calculate $X^a, F_a$ perturbatively in $\epsilon^{i/h} u_i$. 
Embedding into compact CY

- $a^i, a^D_i$: periods of gauge theory
- $X^i = \epsilon^{1/h} a^i, F_i = \epsilon^{1/h} a^D_i$
- extra periods $X^a, F_a$: Recall

\[ w + \frac{\mu^2}{w} = W_{K3}(x, y, z; \epsilon^{i/h} u_i, v_a) \]

for small $\epsilon$, so that extra cycles are at finite values of $x, y, z$
- one can calculate $X^a, F_a$ perturbatively in $\epsilon^{i/h} u_i$.

- they have $\log \epsilon$,
- but all $u_i$ dependence is analytic in $\epsilon^{i/h} u_i$. 
Proof.

Let us prove

\[ \frac{\partial \mathcal{F}_{\text{gauge}}}{\partial \log \Lambda} = u_2 \quad \rightarrow \quad \frac{\partial^2 \mathcal{F}_{\text{gauge}}}{\partial u_j \partial \log \Lambda} = \delta_j^i. \]

First, rewrite LHS:

\[
= \frac{\partial}{\partial u_j} \left( 2\mathcal{F}_{\text{gauge}} - \sum_i a^i \frac{\partial \mathcal{F}_{\text{gauge}}}{\partial a^i} \right) = \sum_i \frac{\partial a^i}{\partial u_j} a^D_i - \sum_i a^i \frac{\partial a^D_i}{\partial u_j}.
\]

i.e.

\[
\sum_i F_i \frac{\partial X^i}{\partial u_j} - \sum_i X^i \frac{\partial F_i}{\partial u_j} = \epsilon^{2/h} \frac{\partial^2 \mathcal{F}_{\text{gauge}}}{\partial u_j \partial \log \Lambda} + O(\epsilon^{3/h})
\]
Proof.

Consider the transversality condition

\[ \sum_i F_i \frac{\partial X^i}{\partial u_j} - \sum_i X^i \frac{\partial F_i}{\partial u_j} = -\sum_a F_a \frac{\partial X^a}{\partial u_j} + \sum_a X^a \frac{\partial F_a}{\partial u_j}. \]

which came from \( \int_{CY} \Omega \wedge \partial u_j \Omega = 0 \).

- \( X^a, F_a \) analytic in \( \epsilon^{i/h} u_i \) \( \rightarrow \) RHS = \( \text{const} \cdot \delta_j^2 \cdot \epsilon^{2/h} + O(\epsilon^{3/h}) \)
- \( F_a \) contain \( \log \epsilon \), but they should cancel in RHS
- DONE !
- \( \text{const} \) fixes the proportionality factor between \( u_2 \) in CY and \( \langle \text{tr} \phi^2 \rangle \)
• First proof for $E$-type gauge groups.

• Known proofs for classical gauge groups similar: there, SW differential has a pole at infinity of the SW curve. → Riemann bilinear id. leads to

\[
\sum_i \frac{\partial a_i}{\partial u_j} a_i^D - \sum_i a_i \frac{\partial a_i^D}{\partial u_j} = \text{contribution from the pole}
\]

• cycles at finite $\tilde{x}$, $\tilde{y}$, $\tilde{z}$; poles at infinite $\tilde{x}$, $\tilde{y}$, $\tilde{z}$

↔ gauge-theory cycles at infinitesimal $x$, $y$, $z$; extra cycles at finite $x$, $y$, $z$

• for exceptional gauge groups, the curve is not hyperelliptic → horrible poles! which prevented the proof.
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4. Application: RG equation

5. Summary
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5. Summary
• Weak-gravity conjecture in the framework of $\mathcal{N} = 2$ supergravity.
• Existence of scale $g M_{\text{planck}}$ equivalent to the kinetic term for $S$ being

$$\frac{\partial_{\mu} S \partial_{\mu} S}{(\text{Im } S)^2}.$$ 

• Byproduct: the RG equation

$$\frac{\partial F_{\text{gauge}}}{\partial \log \Lambda} = u_2$$

understood from the embedding into supergravity,
• Weak-gravity conjecture in the framework of $\mathcal{N} = 2$ supergravity.

• Existence of scale $g M_{\text{planck}}$ equivalent to the kinetic term for $S$ being

\[
\frac{\partial_\mu S \partial_\mu S}{(\text{Im } S)^2}.
\]

• Byproduct: the RG equation

\[
\frac{\partial F_{\text{gauge}}}{\partial \log \Lambda} = u_2
\]

understood from the embedding into supergravity,

• Holomorphy was the key.
Summary

\begin{itemize}
  \item Weak-gravity conjecture in the framework of $\mathcal{N} = 2$ supergravity.
  \item Existence of scale $gM_{\text{planck}}$ equivalent to the kinetic term for $S$ being
    \[ \frac{\partial_{\mu} S \partial_{\mu} S}{(\text{Im } S)^2}. \]
  \item Byproduct: the RG equation
    \[ \frac{\partial F_{\text{gauge}}}{\partial \log \Lambda} = u_2 \]
    understood from the embedding into supergravity,
  \item Holomorphy was the key.
  \item Holomorphy will tell us more about quantum gravity!
\end{itemize}