GENERALIZATION OF RESULTS ABOUT THE BOHR RADIUS FOR POWER SERIES.

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Abstract. The Bohr radius for power series of holomorphic functions mapping Reinhardt domains $\mathcal{D} \subset \mathbb{C}^n$ into the convex domain $G \subset \mathbb{C}$ is independent of the domain $G$.

1. Preliminaries

Let us recall the theorem of H. Bohr [13] in 1914.

Theorem 1.1. If a power series

$$f(z_1) = \sum_{k=0}^{\infty} c_k z_1^k \quad (1.1)$$

converges in the unit disk $U_1$ and its sum has modulus less than 1, then

$$\sum_{k=0}^{\infty} |c_k z_1^k| < 1, \quad (1.2)$$

if $|z_1| < \frac{1}{3}$. Moreover, the constant $\frac{1}{3}$ cannot be improved.

For convenience we write the inequality (1.2) in the following equivalent form

$$\sum_{k=1}^{\infty} |c_k z_1^k| < 1 - |c_0|. \quad (1.3)$$

Later, certain generalizations of this result were obtained.

1°. ([24]) If the sum of the series (1.1) is such that $|\Re f(z_1)| < 1$ in $U_1$ and $c_0 > 0$, then for $|z_1| < \frac{1}{3}$ the inequality (1.3) holds.

2°. (23, 22) If $\Re f(z_1) < 1$ in $U_1$ and $c_0 > 0$, then for $|z_1| < \frac{1}{3}$ the inequality (1.3) holds.

3°. (24) If $\Re \{[\exp(-i\arg f(0))] f(z_1)\} < 1$ in $U_1$ (here we assume that $\arg f(0) > 0$, if $f(0) = 0$), then for $|z_1| < \frac{1}{3}$ the inequality (1.3) is valid.

AMS classification number: 32A05.

Keywords: Bohr radius, Reinhardt domains, power series.
Formulations of Bohr’s theorem in several complex variables appeared very recently. We recall some of them.

Given a complete Reinhardt domain $D$, we denote by $R_1(D)$ (or by $R_2(D)$) the largest nonnegative number $r$ with the property that if the power series

$$f(z) = \sum_{|\alpha|\geq 0} c_\alpha z^\alpha, \ z \in D,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and all $\alpha_i$ are nonnegative integers, converges in $D$ and the modulus of its sum is less than 1, then

$$\sum_{|\alpha|\geq 1} |c_\alpha z^\alpha| < 1 - |c_0|$$

in the homothetic domain $D_r = rD$. Here $c_0 = c_{0,0,\ldots,0}$. Correspondingly, if we consider a bounded domain $D$ for $R_2(D)$ we have

$$\sum_{|\alpha|\geq 1} \sup_{D_r} |c_\alpha z^\alpha| < 1 - |c_0|.$$

Let

$$D^n_p = \{ z \in \mathbb{C}^n : |z_1|^p + \cdots + |z_n|^p < 1 \},$$

where $0 < p \leq \infty$. The domain $D^n_\infty$ is the poly-disk $\{ z \in \mathbb{C}^n : |z_j| < 1, j = 1, \ldots, n \}$.

**Theorem 1.2.** ([12], see also [18]) For $n > 1$ one has

$$\frac{1}{3\sqrt{n}} < R_1(D^n_\infty) < \frac{2\sqrt{\log n}}{\sqrt{n}}. \tag{1.5}$$

**Theorem 1.3.** ([1]) For $n > 1$ one has

$$\frac{1}{3\sqrt{e}} < R_1(D^n_1) \leq \frac{1}{3}. \tag{1.6}$$

The estimates (1.5) and (1.6) were generalized for $R_1(D^n_p)$, for $1 \leq p < \infty$ in [11] and for $0 < p \leq 1$ in [3]. We point out the next new remarkable result which improves the lower estimate in (1.5).

**Theorem 1.4.** ([15]) For $n > 1$ one has

$$C \sqrt{\frac{\log n}{n \log \log n}} < R_1(D^n_\infty),$$

where the constant $C$ is independent of $n$. 
Both Bohr radii coincide in the case the domain is a polydisk, and in the case $n = 1$ they do coincide with the classical Bohr radius $\frac{1}{3}$. If the domain $\mathcal{D}$ is not a polydisk, then naturally $R_2(\mathcal{D})$ is smaller than $R_1(\mathcal{D})$.

**Theorem 1.5.** (\[1\]) The inequality

$$1 - \sqrt[3]{\frac{2}{3}} < R_2(\mathcal{D})$$

is true for every complete bounded Reinhardt domain $\mathcal{D}$.

**Theorem 1.6.** (\[1\]) There holds the inequality

$$R_2(\mathcal{D}_1^n) < \frac{0.44663}{n}.$$ 

The radius $R_2(\mathcal{D})$ was a subject of investigation in \[11, 17\]. Other results about the Bohr radius for holomorphic functions can be found in \[2, 5, 6, 8, 9, 10, 16\].

2. **Generalized Bohr radii**

One of the proofs of Bohr’s theorem (Theorem 1.1) is based on the Landau inequality [21]: if the function $f(z_1)$ satisfies in $U_1$ the inequality $|f(z_1)| < 1$, then $|c_k| \leq 2(1 - |c_0|)$ holds for every $k \geq 1$. This inequality can be obtained as a simple consequence of the Caratheodory inequality [14]: if the function $f(z_1)$ satisfies in $U_1$ the inequality $\Re f(z_1) > 0$, then $|c_k| \leq 2\Re c_0$ is true for every $k \geq 1$. Both inequalities are particular cases of a more general assertion.

Let $\bar{G}$ be the convex hull of $G$.

**Proposition 2.1.** (\[3\]) If $f(U_1) \subset G$, then

$$|c_k| \leq 2\text{dist}(c_0, \partial \bar{G}),$$

for all $k \geq 1$.

Now it is not difficult to prove a generalization of Theorem 1.1. Let $G \subset \mathbb{C}$ be any domain. A point $p \in \partial G$ is called a point of convexity if $p \in \partial \bar{G}$. A point of convexity $p$ is called regular if there exists a disk $U \subset G$ so that $p \in \partial U$.

**Theorem 2.1.** If the function $f(z_1)$ is such that $f(U_1) \subset G$, with $\bar{G} \neq \mathbb{C}$, then for $|z_1| < \frac{1}{3}$ the inequality

$$\sum_{k=1}^{\infty} |c_k z_1^k| < \text{dist}(c_0, \partial \bar{G})$$

(2.2)
is valid. The constant $\frac{1}{3}$ cannot be improved if $\partial G$ contains at least one regular point of convexity.

**Proof:** 1) If $|z_1| < \frac{1}{3}$ then (2.1) yields
\[
\sum_{k=1}^{\infty} |c_kz_1^k| < 2\text{dist}(c_0, \partial G) \sum_{k=1}^{\infty} \frac{1}{3^k} = \text{dist}(c_0, \partial \tilde{G}).
\]

2) We will prove the exactness of the constant $\frac{1}{3}$ in the case the boundary contains at least one regular point of convexity. In the classical case of Bohr’s Theorem 1.1 this is obtained by considering the family of functions (21)
\[
f(z_1) = \frac{\alpha - z_1}{1 - \alpha z_1}, \ 0 < \alpha < 1.
\]

Here
\[
\sum_{k=1}^{\infty} |c_kz_1^k| = 1
\]
if and only if $|z_1| = \frac{1}{1+2\alpha}$. Furthermore, taking $\alpha \rightarrow 1$, we obtain the desired result. Note that instead of the family (2.3) one can use the family $e^{i\phi}f(z_1)$, where $f(z_1)$ is taken from (2.3). In this case it follows that $c_0 = e^{i\phi} \alpha$, and when $\alpha \rightarrow 1$ we get that $c_0$ tends to $\partial U_1$ along the radius of argument $\phi$. If $G$ is an arbitrary disk $U$, then, remarking that (2.2) does not change under homotheties and translations, we deduce the exactness of $\frac{1}{3}$ in the case of any disk. Let $\zeta$ be a regular point of convexity, then there exists a disk $U \subset G$ such that $\zeta \in (\partial U) \cap (\partial G)$. Consider the functions $f$ in (1.1) such that $f(U_1) \subset U$. For suitable $c_0$ (see above) we will have $\text{dist}(c_0, \partial U) = \text{dist}(c_0, \partial G) = \text{dist}(c_0, \partial \tilde{G})$. Therefore, in the inequality (2.2) one cannot take $|z_1| < r$, where $r > \frac{1}{3}$. ♦

We remark that Theorem 1.1, the assertion $3^0$, as well as the generalized assertions $1^0$ and $2^0$ are contained in Theorem 2.1. For example, in $1^0$ no need in assuming $c_0 > 0$, and instead of (1.3) one gets
\[
\sum_{k=1}^{\infty} |c_kz_1^k| < 1 - |\Re c_0|.
\]

Similarly in $2^0$ no need in assuming $c_0 > 0$, and instead of (1.3) one gets
\[
\sum_{k=1}^{\infty} |c_kz_1^k| < 1 - |\Re c_0|.
\]
Let us recall another fact, known earlier:  

$4^0$. ([4]) If $\Re f(z_1) > 0$ in $U_1$ and $c_0 > 0$, then for $|z_1| < \frac{1}{3}$ the inequality  

$$\sum_{k=1}^{\infty} |c_k z_1^k| < c_0$$  

holds, and the constant $\frac{1}{3}$ cannot be improved. I thought before that Theorem 1.1. and $4^0$ are two different facts, having the same Bohr radius. In the light of Theorem 2.1, I know now that both results are particular cases of this theorem. Now, in the case of $4^0$ without the assumption $c_0 > 0$, we get  

$$\sum_{k=1}^{\infty} |c_k z_1^k| < \Re c_0$$

instead of (2.4).

Theorem 2.1 motivates the following generalization of the first and second Bohr radii. Denote by $R_1(\mathcal{D}, G)$ (or by $R_2(\mathcal{D}, G)$), where $G \subset \mathbb{C}$, $\tilde{G} \neq \mathbb{C}$, and $\mathcal{D}$ is a complete Reinhardt domain (bounded complete Reinhardt domain) in $\mathbb{C}^n$ the largest $r \geq 0$ such that if the function (1.4) is holomorphic in $\mathcal{D}$ and $f(\mathcal{D}) \subset G$ then  

$$\sum_{|\alpha| \geq 1} \sup_{\mathcal{D}_r} |c_\alpha z^\alpha| < \text{dist}(c_0, \partial \tilde{G})$$

in a homothety $\mathcal{D}_r$ (or correspondingly  

$$\sum_{|\alpha| \geq 1} \sup_{\mathcal{D}_r} |c_\alpha z^\alpha| < \text{dist}(c_0, \partial \tilde{G})$$

Theorem 2.1 and the result from [7] about the Rogosinski radius allow one to hope that the two Bohr radii $R_1(\mathcal{D}, G)$ and $R_2(\mathcal{D}, G)$ are independent of the convex domain $G$. The main result of the present paper is the proof of the validity of this more general assertion.

3. THE MAIN RESULT

Let $M$ be a complex manifold, $\mathcal{H}(M)$ be the space of holomorphic on $M$ functions equipped with the natural topology of uniform convergence over compact subsets of $M$. Let $\| \cdot \|_r$, $r \in (0, 1)$, be a one-parameter family of semi-norms in $\mathcal{H}(M)$ that are continuous with respect to the topology of $\mathcal{H}(M)$. In what
follows we always assume that
\[ a) \|f\|_r \leq \|f\|_{r_2} \text{ if } r_1 \leq r_2. \]
\[ b) \|f \cdot g\|_r \leq \|f\|_r \cdot \|g\|_r \ \forall r \in (0,1). \]

There exists a point \( z_0 \in M \) such that
\[ c) \|f\|_r \rightarrow |f(z_0)| \text{ as } r \rightarrow 0, \ \forall f \in \mathcal{H}(M). \]
\[ d) \|f\|_r = |f(z_0)| + \|f - f(z_0)\|_r, \ \forall f \in \mathcal{H}(M). \]

Denote by \( B(\| \cdot \|_r, G) \) the largest \( r \geq 0 \) such that for \( f \in \mathcal{H}(M) \) and \( f(M) \subset G \) one has
\[ \|f - f(z_0)\|_r < \text{dist}(f(z_0), \partial \tilde{G}), \tag{3.1} \]
where \( \tilde{G} \) is the convex hull of the domain \( G \subset \mathbb{C} \).

**Proposition 3.1.** If \( U \) is any disk and \( \Pi \) is any half-plane, then
\[ B(\| \cdot \|_r, \Pi) = B(\| \cdot \|_r, U). \tag{3.2} \]

**Proof:** Let \( \Pi_1 = \{z_1 : \Re z_1 > 0\} \), then (\[\Pi\], Theorem 7)
\[ B(\| \cdot \|_r, U_1) = B'(\| \cdot \|_r, \Pi_1), \]
where \( B' \) is defined in the same way as \( B \) but with the additional assumption \( f(z_0) > 0 \). This assumption can be removed as follows. If \( \Re f(z_0) > 0 \) in \( M \) then \( \Re f_1(z_0) > 0 \), where \( f_1(z) = f(z) - \Re f(z_0) \). But \( f_1(z_0) > 0 \), hence
\[ B(\| \cdot \|_r, U_1) = B(\| \cdot \|_r, \Pi_1). \]

We remark that (3.1) does not change under homotheties, translations and rotations of the domain \( G \). Therefore (3.2) holds. \( \diamond \)

**Theorem 3.1.** If \( \tilde{G} \neq \mathbb{C} \), then \( B(\| \cdot \|_r, G) \) is not smaller than (3.2). If \( \partial G \) contains at least one regular point of convexity, then \( B(\| \cdot \|_r, G) \) is equal to (3.2).

**Proof:** Let \( \tilde{G} \neq \mathbb{C} \) and \( f(M) \subset G \). Fix any \( f(z_0) \in G \). On the boundary \( \partial \tilde{G} \) there exists a point \( \zeta \) so that \( \text{dist}(f(z_0), \partial \tilde{G}) = \text{dist}(f(z_0), \zeta) \). Through the point \( \zeta \) the line of support of \( \tilde{G} \) passes which defines the half-plane \( \Pi_0 \supseteq G \). Then
\[ \text{dist}(f(z_0), \partial \tilde{G}) = \text{dist}(f(z_0), \partial \Pi_0). \]
Therefore \( B(\| \cdot \|_r, G) \geq B(\| \cdot \|_r, \Pi_0) \), since \( \{f : f \in \mathcal{H}(M), f(M) \subset G\} \subset \{f : f \in \mathcal{H}(M), f(M) \subset \Pi_0\} \).
Assume now that there is a regular point of convexity in \( \partial G \). Then the proof repeats the proof of part 2) of the Theorem 2.1. Note that there
we did not use the concrete form of the family $(2.3)$, but rather the fact that $c_0$ can lie on any radius emanating from the center of the disk $U$ to its boundary. So, let us assume that $U \subset G$, $\zeta \in (\partial U) \cap (\partial G) \cap (\partial \tilde{G})$. Then consider $f(z_0)$ lying on the radius from the center of the disk $U$ to the point $\zeta$. Now $\text{dist}(f(z_0), \zeta) = \text{dist}(f(z_0), \partial U) = \text{dist}(f(z_0), \partial \tilde{G})$, hence $B(\| \cdot \|_r, G) \leq B(\| \cdot \|_r, U)$, since \{ $f : f \in \mathcal{H}(M), f(M) \subset U$ \} $\subset$ \{ $f : f \in \mathcal{H}(M), f(M) \subset G$ \}. $\diamondsuit$.

**Corollary 3.1.** If the domain $G$ is convex and $G \neq C$, then $B(\| \cdot \|_r, G)$ is independent of the choice of the domain $G$.

**Proof:** There exists disk $U \subset G$ such that $\partial U \cap \partial G \neq \emptyset$. Therefore there exist regular points of convexity on $\partial G$. $\diamondsuit$.

**Corollary 3.2.** The first Bohr radius $R_1(D, G)$ and the second Bohr radius $R_2(D, G)$ are independent of the choice of the convex domain $G$, $G \neq C$.

In particular, the assertions of Theorems 1.2, 1.3 and 1.4 are valid for $R_1(D, G)$ while those of Theorem 1.5 and 1.6 are valid for $R_2(D, G)$ for every convex domain $G \neq C$.

**Some concluding remarks.** If the family of semi-norms $\| \cdot \|_r$ does not satisfy some of the conditions $a) - d)$, then the assertion of Theorem 3.1 is not valid anymore. Examples can be found in [4]. If $\tilde{G} = C$, then the right-hand side of (3.1) is equal to $\infty$, therefore in this case

$$B(\| \cdot \|_r, G) = 1.$$ 

One can also consider different realizations of $B(\| \cdot \|_r, G)$ than the first and second Bohr radii $R_1(D, G)$ and $R_2(D, G)$.

We conclude the present article with formulating an open problem: if $\tilde{G} \neq C$, is it always true that $B(\| \cdot \|_r, G)$ is equal to (3.2)? The same question makes sense for the first and second Bohr radii $R_1(D, G)$ and $R_2(D, G)$.

4. **Acknowledgements**

The author is deeply grateful to E. Liflyand and A. Vidras for their help in preparing the paper and improving the presentation.

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