Shuangjian Guo; Xiu-Li Chen
A Maschke type theorem for relative Hom-Hopf modules

_Czechoslovak Mathematical Journal_, Vol. 64 (2014), No. 3, 783–799

Persistent URL: [http://dml.cz/dmlcz/144058](http://dml.cz/dmlcz/144058)

Terms of use:

© Institute of Mathematics AS CR, 2014

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
A MASCHKE TYPE THEOREM FOR RELATIVE HOM-HOPF MODULES

SHUANGJIAN GUO, GUIYANG, XIU-LI CHEN, NANJING

(Received July 4, 2013)

Abstract. Let \( (H, \alpha) \) be a monoidal Hom-Hopf algebra and \( (A, \beta) \) a right \( (H, \alpha) \)-Hom-comodule algebra. We first introduce the notion of a relative Hom-Hopf module and prove that the functor \( F \) from the category of relative Hom-Hopf modules to the category of right \( (A, \beta) \)-Hom-modules has a right adjoint. Furthermore, we prove a Maschke type theorem for the category of relative Hom-Hopf modules. In fact, we give necessary and sufficient conditions for the functor that forgets the \( (H, \alpha) \)-coaction to be separable. This leads to a generalized notion of integrals.

Keywords: monoidal Hom-Hopf algebra; separable functors; Maschke type theorem; total integral; relative Hom-Hopf module

MSC 2010: 16T05

INTRODUCTION

The present paper investigates variations on the theme of Hom-algebras, a topic which has recently received much attention from various researchers. The study of Hom-associative algebras originates with the work by Hartwig, Larsson and Silvestrov in the Lie case [9], where a notion of Hom-Lie algebra was introduced in the context of studying deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlouf and Silvestrov in [10]–[11]. Now the associativity is replaced by Hom-associativity \( \alpha(a)(bc) = (ab)\alpha(c) \). Hom-coassociativity for a Hom-coalgebra can be considered in a similar way, see [11]. Caenepeel and Goywaerts [1]...
studied Hom-structures from the point of view of monoidal categories. This leads to the natural definition of monoidal Hom-algebras, Hom-coalgebras, etc. They constructed a symmetric monoidal category, and then introduced monoidal Hom-algebras, Hom-coalgebras, etc. as algebras, coalgebras, etc. in this monoidal category.

The notion of a relative \((H, B)\)-Hopf module, where \(H\) is a Hopf algebra over a field \(k\) and \(B\) is a right coideal subalgebra of \(H\), was introduced and studied by Takeuchi in [12]. Later, in [5] (see also [4]), Doi noted that the notion of an \((H, B)\)-Hopf module works well if \(B\) is a right \(H\)-comodule algebra. Using this module, he proved that the existence of a total integral \(\phi: H \to B\) is equivalent to \(B\) being a relative injective \(H\)-comodule, and it is also equivalent to any \((H, B)\)-Hopf module \(M\) being a relative injective \(H\)-comodule in [3]. Also, in [3], using a commutative assumption for \(H\), he deduced a version of the Maschke type theorem for \((H, B)\)-Hopf modules which states that every exact sequence of \((H, B)\)-Hopf modules which splits \(B\)-linearly, also splits \((H, B)\)-linearly. Afterwards, Doi proved in [3] that the commutative condition can be removed and replaced by some technical conditions involving the center of \(B\). Caenepeel et al. [2] proved a Maschke type theorem for the category of relative Hopf modules. In fact, they gave necessary and sufficient conditions for the functor that forgets the \(H\)-coaction to be separable. This leads to a generalized notion of integrals of Doi [3].

In this paper we study the generalization of the previous results to the Hom-Hopf algebras. In Section 2, we introduce the notion of a relative Hom-Hopf module and prove that the functor \(F\) from the category of relative Hom-Hopf modules to the category of right \((A, \beta)\)-Hom-modules has a right adjoint (see Proposition 2.3). In Section 3, we introduce the notion of total integrals for Hom-comodule algebras, which is an effective tool for investigating properties of relative Hom-Hopf modules. As an important application, we investigate the injectivity of relative Hom-Hopf modules (see Proposition 3.3), which generalizes the main result in [5]. In Section 4, we obtain the main result of this paper. We give necessary and sufficient conditions for the functor that forgets the \((H, \alpha)\)-coaction to be separable (see Theorem 4.2), and we prove a Maschke type theorem for the category of relative Hom-Hopf modules as an application. In fact, let \((A, \beta)\) be a right \((H, \alpha)\)-Hom-comodule algebra with a total integral \(\phi: (H, \alpha) \to (A, \beta)\). If \(\phi: (H, \alpha) \to (Z(A), \beta)\) (the center of \((A, \beta)\)) is a multiplication map, then every short exact sequence of relative Hom-Hopf modules

\[
0 \longrightarrow (M, \mu) \xrightarrow{f} (N, \nu) \xrightarrow{g} (P, \pi) \longrightarrow 0
\]

which splits as a sequence of \((A, \beta)\)-Hom-modules also splits as a sequence of relative Hom-Hopf modules.
1. Preliminaries

Throughout this paper we work over a commutative ring \( k \) we recall from [1] some information about Hom-structures which are needed in what follows.

Let \( C \) be a category. We introduce a new category \( \tilde{\mathcal{H}}(C) \) as follows: the objects are couples \((M, \mu)\), with \( M \in C \) and \( \mu \in \text{Aut}_C(M) \). A morphism \( f: (M, \mu) \to (N, \nu) \) is a morphism \( f: M \to N \) in \( C \) such that \( \nu \circ f = f \circ \mu \).

Let \( \mathcal{M}_k \) denote the category of \( k \)-modules. \( \mathcal{H}(\mathcal{M}_k) \) will be called the Hom-category associated with \( \mathcal{M}_k \). If \((M, \mu) \in \mathcal{M}_k\), then \( \mu: M \to M \) is obviously a morphism in \( \mathcal{H}(\mathcal{M}_k) \). It is easy to show that \( \tilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (I, I), \tilde{a}, \tilde{l}, \tilde{r}) \) is a monoidal category by Proposition 1.1 in [1]: the tensor product of \((M, \mu)\) and \((N, \nu)\) in \( \tilde{\mathcal{H}}(\mathcal{M}_k) \) is given by the formula
\[
(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu).
\]

Assume that \((M, \mu), (N, \nu), (P, \pi) \in \tilde{\mathcal{H}}(\mathcal{M}_k)\). The associativity and unit constraints are given by the formulas
\[
\tilde{a}_{M,N,P}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \pi^{-1}(p)),
\]
\[
\tilde{l}_M(x \otimes m) = \tilde{r}_M(m \otimes x) = x\mu(m).
\]

An algebra in \( \tilde{\mathcal{H}}(\mathcal{M}_k) \) will be called a monoidal Hom-algebra.

**Definition 1.1.** A monoidal Hom-algebra is an object \((A, \alpha) \in \tilde{\mathcal{H}}(\mathcal{M}_k)\) together with a \( k \)-linear map \( m_A: A \otimes A \to A \) and an element \( 1_A \in A \) such that
\[
\alpha(ab) = \alpha(a)\alpha(b); \quad \alpha(1_A) = 1_A,
\]
\[
\alpha(a)(bc) = (ab)\alpha(c); \quad a1_A = 1_Aa = \alpha(a),
\]
for all \( a, b, c \in A \). Here we use the notation \( m_A(a \otimes b) = ab \).

**Definition 1.2.** A monoidal Hom-coalgebra is an object \((C, \gamma) \in \tilde{\mathcal{H}}(\mathcal{M}_k)\) together with \( k \)-linear maps \( \Delta: C \to C \otimes C, \Delta(c) = c(1) \otimes c(2) \) (summation implicitly understood) and \( \gamma: C \to C \) such that
\[
\Delta(\gamma(c)) = \gamma(c(1)) \otimes \gamma(c(2)); \quad \varepsilon(\gamma(c)) = \varepsilon(c),
\]
and
\[
\gamma^{-1}(c(1)) \otimes c(2)(1) \otimes c(2)(2) = c(1)(1) \otimes c(1)(2) \otimes \gamma^{-1}(c(2)),
\]
\[
\varepsilon(c(1))c(2) = \varepsilon(c(2))c(1) = \gamma^{-1}(c)
\]
for all \( c \in C \).
Definition 1.3. A monoidal Hom-bialgebra \( H = (H, \alpha, m, \eta, \Delta, \varepsilon) \) is a bialgebra in the symmetric monoidal category \( \widetilde{\mathcal{H}}(\mathcal{M}_k) \). This means that \((H, \alpha, m, \eta)\) is a Hom-algebra, \((H, \Delta, \alpha)\) is a Hom-coalgebra and that \(\Delta\) and \(\varepsilon\) are morphisms of Hom-algebras, that is,

\[
\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}; \quad \Delta(1_H) = 1_H \otimes 1_H, \\
\varepsilon(ab) = \varepsilon(a)\varepsilon(b); \quad \varepsilon(1_H) = 1_H.
\]

Definition 1.4. A monoidal Hom-Hopf algebra is a monoidal Hom-bialgebra \((H, \alpha)\) together with a linear map \(S: H \to H\) in \(\widetilde{\mathcal{H}}(\mathcal{M}_k)\) such that

\[
S \ast I = I \ast S = \eta\varepsilon, S\alpha = \alpha S.
\]

Definition 1.5. Let \((A, \alpha)\) be a monoidal Hom-algebra. A right \((A, \alpha)\)-Hom-module is an object \((M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)\) consisting of a \(k\)-module and a linear map \(\mu: M \to M\) together with a morphism \(\psi: M \otimes A \to M, \psi(m \cdot a) = m \cdot a\) in \(\widetilde{\mathcal{H}}(\mathcal{M}_k)\) such that

\[
(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab); \quad m \cdot 1_A = \mu(m)
\]

for all \(a \in A\) and \(m \in M\). The fact that \(\psi \in \widetilde{\mathcal{H}}(\mathcal{M}_k)\) means that

\[
\mu(m \cdot a) = \mu(m) \cdot \alpha(a).
\]

A morphism \(f: (M, \mu) \to (N, \nu)\) in \(\widetilde{\mathcal{H}}(\mathcal{M}_k)\) is called right \(A\)-linear if it preserves the \(A\)-action, that is, \(f(m \cdot a) = f(m) \cdot a\). \(\widetilde{\mathcal{H}}(\mathcal{M}_k)_A\) will denote the category of right \((A, \alpha)\)-Hom-modules and \(A\)-linear morphisms.

Definition 1.6. Let \((C, \gamma)\) be a monoidal Hom-coalgebra. A right \((C, \gamma)\)-Hom-comodule is an object \((M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)\) together with a \(k\)-linear map \(\varrho_M: M \to M \otimes C\) notation \(\varrho_M(m) = m_{[0]} \otimes m_{[1]}\) in \(\widetilde{\mathcal{H}}(\mathcal{M}_k)\) such that

\[
m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); \quad m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m)
\]

for all \(m \in M\). The fact that \(\varrho_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)\) means that

\[
\varrho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).
\]

Morphisms of right \((C, \gamma)\)-Hom-comodules are defined in the obvious way. The category of right \((C, \gamma)\)-Hom-comodules will be denoted by \(\widetilde{\mathcal{H}}(\mathcal{M}_k)^C\).
2. Adjoint functor

**Definition 2.1.** Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra \((A, \beta)\) is called a right \((H, \alpha)\)-Hom-comodule algebra if \((A, \beta)\) is a right \((H, \alpha)\) Hom-comodule with coaction \(\rho_A: A \to A \otimes H, \rho_A(a) = a_{[0]} \otimes a_{[1]}\) such that the conditions
\[
\rho_A(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]},
\]
\[
\rho_A(1_A) = 1_A \otimes 1_H
\]
are satisfied for all \(a, b \in A\).

**Definition 2.2.** Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-module algebra. A relative Hom-Hopf module \((M, \mu)\) is a right \((A, \beta)\)-Hom-module which is also a right \((H, \alpha)\)-Hom-comodule with the coaction structure \(\rho_M: M \to M \otimes H\) defined by \(\rho_M(m) = m_{[0]} \otimes m_{[1]}\) such that the following compatible condition holds: for all \(m \in M\) and \(a \in A\),
\[
\rho_M(ma) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]}a_{[1]}.
\]

A morphism between two right relative Hom-Hopf modules is a \(k\)-linear map which is a morphism in the categories \(\widetilde{\mathcal{H}}(\mathcal{M}_k)_A\) and \(\widetilde{\mathcal{H}}(\mathcal{M}_k)_H^A\) at the same time. \(\mathcal{H}(\mathcal{M}_k)_A^H\) will denote the category of right relative Hom-Hopf modules and morphisms between them.

**Proposition 2.3.** The forgetful functor \(F: \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H \to \widetilde{\mathcal{H}}(\mathcal{M}_k)_A\) has a right adjoint \(G: \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \to \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H\). \(G\) is defined by
\[
G(M) = M \otimes H,
\]
with structure maps
\[
(m \otimes h) \cdot a = m \cdot a_{[0]} \otimes ha_{[1]},
\]
\[
\rho_{G(M)}(m \otimes h) = (\mu^{-1}(m) \otimes h_{(1)}) \otimes \alpha(h_{(2)})
\]
for all \(a \in A\) and \(m \in M, h \in H\).

**Proof.** Let us first show that \(G(M)\) is an object of \(\widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H\). It is routine to check that \(G(M)\) is a right \((H, \alpha)\)-Hom-comodule and a right \((A, \beta)\)-Hom-module.
Now we only check the compatibility condition, for all \(a \in A\). Indeed,

\[
\varrho_G((m \otimes h) \cdot a) = \varrho_G(m \cdot a_{[0]} \otimes ha_{[1]}) \\
= \mu^{-1}(m) \cdot (a_{[0]}[0] \otimes h a_{[0]}[1] \otimes \alpha(h_{(2)}) a_{[1]}) \\
= (m \otimes h)[0] \cdot a_{[0]} \otimes (m \otimes h)(1) a_{[1]} \\
= \varrho(m \otimes c) \cdot a.
\]

This is exactly what we have to show.

For an \(A\)-linear map \(\varphi: (M, \mu) \rightarrow (N, \nu)\), we put

\[
G(\varphi) = \varphi \otimes \text{id}_H : M \otimes H \rightarrow N \otimes H.
\]

Standard computations show that \(G(\varphi)\) is a morphism of right \((A, \beta)\)-Hom-modules and right \((H, \alpha)\)-Hom-comodules. Let us describe the unit \(\eta\) and the counit \(\delta\) of the adjunction. The unit is described by the coaction: for \((M, \mu) \in \mathcal{H}(\mathbb{M}_k)_A^H\), we define \(\eta_M : M \rightarrow M \otimes H\) as follows: for all \(m \in M\),

\[
\eta_M(m) = m_{[0]} \otimes m_{[1]}.
\]

We can check that \(\eta_M \in \mathcal{H}(\mathbb{M}_k)_A^H\). For any \((N, \nu) \in \mathcal{H}(\mathbb{M}_k)_A\), we define \(\delta_N : N \otimes H \rightarrow N\) for all \(n \in N\) and \(h \in H\) by

\[
\delta_N(n \otimes h) = \varepsilon(h)n;
\]

we can check that \(\delta_N\) is \((A, \beta)\)-linear. It is easy to check that \(\eta_M \in \mathcal{H}(\mathbb{M}_k)_A^H\). We can check that \(\eta\) and \(\delta\) defined above are all natural transformations and satisfy

\[
G(\delta_N) \circ \eta_{G(N)} = I_{G(N)}, \\
\delta_{F(M)} \circ F(\eta_M) = I_{F(M)}
\]

for all \(M \in \mathcal{H}(\mathbb{M}_k)_A^H\) and \(N \in \mathcal{H}(\mathbb{M}_k)_A\). \(\square\)

3. Structure type theorem and injective type properties for relative Hom-Hopf modules

**Definition 3.1.** Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra. The map \(\phi : (H, \alpha) \rightarrow (A, \beta)\) is called a total integral such that the following conditions are satisfied:

\[
\varrho_A \phi = (\phi \otimes \text{id}_H) \Delta_H, \quad \phi \alpha = \beta \phi, \quad \phi(1_H) = 1_A.
\]
Lemma 3.2. Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra with a total integral \(\phi\) : \((H, \alpha) \to (A, \beta)\) and \((M, \mu) \in \mathcal{H}(\mathcal{M}_k)_H^H\),

\[\lambda_M : M \otimes H \to M, \quad m \otimes h \mapsto \mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]}) \alpha(h)).\]

Then the following assertions hold:

1. \(\lambda_M \varrho_M = \text{id}_M\);
2. \(\lambda_M\) is a morphism of right \((H, \alpha)\)-Hom-comodules, and the right \((H, \alpha)\)-Hom-coaction on \(M \otimes H\) is given by \(\varrho(m \otimes h) = (\mu(m) \otimes h_{(1)}) \otimes \alpha^{-1}(h_{(2)})\) for any \(m \in M\) and \(h \in H\);
3. if \(\phi : (H, \alpha) \to (Z(A), \beta)\) (the center of \(A\)) is a multiplication map, then \(\lambda_M\) is a morphism in \(\mathcal{H}(\mathcal{M}_k)_H^H\).

Proof. (1) For any \(m \in M\), we have

\[\lambda_M \varrho_M(m) = \lambda_M(m_{[0]} \otimes m_{[1]}) = \mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[0]_{[1]}}) \alpha(m_{[1]})) = m_{[0]} \cdot \phi(S(m_{[1]_{[1]}})m_{[1]_{[2]}}) = m_{[0]} \cdot \phi(\varepsilon(m_{[1]})) = \mu^{-1}(m) \cdot 1_A = m.\]

(2) For any \(m \in M\) and \(h \in H\), we have

\[\varrho_M \lambda_M(m \otimes h) = \varrho_M(\mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]}) \alpha(h))) = \mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]_{[2]}}) \alpha(h_{[1]})) \otimes \alpha^{-1}(m_{[0]_{[1]}})(S(m_{[1]_{[1]}}) \alpha(h_{[2]})) = \mu^{-2}(m_{[0]}) \cdot \phi(\alpha(S(m_{[1]_{[2]}}) \alpha(h_{[1]}))) \otimes \alpha^{-1}(m_{[0]_{[1]}})(S(m_{[1]_{[2]}}) \alpha(h_{[2]})) = \mu^{-2}(m_{[0]}) \cdot \phi(\alpha^{-1}(S(m_{[1]}))) \otimes \alpha(h_{[1]})) \otimes \alpha(h_{[2]})) = (\lambda_M \otimes \text{id}_H)((\mu^{-1}(m) \otimes h_{[1]})) \otimes \alpha(h_{[2]})) = (\lambda_M \otimes \text{id}_H)\varrho_M \otimes H(m \otimes h).\]

(3) For any \(m \in M\), \(h \in H\) and \(b \in A\), we have

\[\lambda_M((m \otimes h) \cdot b) = \lambda_M(m \cdot b_{[0]} \otimes hb_{[1]}) = \mu^{-1}(m_{[0]} \cdot b_{[0]_{[0]}}) \cdot \phi(S(m_{[1]}b_{[0]_{[1]}}) \alpha(hb_{[1]})) = \mu^{-1}(m_{[0]} \cdot b_{[0]_{[0]}}) \cdot \phi(S(m_{[1]})S(b_{[0]_{[1]}}) \alpha(hb_{[1]})) = \mu^{-1}(m_{[0]} \cdot b_{[0]_{[0]}}) \cdot \phi(\alpha(S(m_{[1]})[S(b_{[0]_{[1]}})hb_{[1]}])) = \mu^{-1}(m_{[0]} \cdot b_{[0]_{[0]}}) \cdot \phi(\alpha(S(m_{[1]})[S(b_{[0]_{[1]}})(b_{[1]}h)]))\]

789
\[
\begin{align*}
\mu^{-1}(m_0 \cdot b_{0|0}) & \cdot \phi(\alpha(S(m_{1|1}))[\alpha^{-1}(S(b_{0|1}))b_{1|1}h]) \\
= \mu^{-1}(m_0 \cdot b_{0|0}) & \cdot \phi(\alpha(S(m_{1|1}))[\alpha^{-1}(S(b_{1|1}))\alpha^{-1}(b_{1|2})h]) \\
= \mu^{-1}(m_0 \cdot b_{0|0}) & \cdot \phi(\alpha(S(m_{1|1})\alpha^2(h))) \\
= m_0 & \cdot (\beta^{-1}(b)\phi(S(m_{1|1})\alpha(h))) \\
= m_0 & \cdot (\phi(\alpha^{-4}(S(m_{1|1}))\alpha^{-3}(h))\beta^{-1}(b)) \\
= \mu^{-1}(m_0) & \cdot \phi(S(m_{1|1})\alpha(h)) \cdot b \\
= \lambda_M (m \otimes h) \cdot b.
\end{align*}
\]

\[\square\]

**Proposition 3.3.** Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom comodule algebra with a total integral \(\phi: (H, \alpha) \to (A, \beta)\). Then \((M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)^H\) is injective as a right \((H, \alpha)\)-Hom-comodule.

If \(H\) is a Hopf algebra, then we obtain the main result of [5], Theorem 1.

**Corollary 3.4.** Let \(H\) be a Hopf algebra and \(A\) a right \(H\)-comodule algebra. If there is a right \(H\)-comodule map \(\phi: (H, \alpha) \to (A, \beta)\) such that \(\phi(1_H) = 1_A\), then every relative \((H, A)\)-Hopf-module is injective as a right \(H\)-comodule.

Let \(M\) be a relative Hom-Hopf module, and let

\[M_0 = \{m \in M; \varrho_M(m) = \mu^{-1}(m) \otimes 1_H\}\]

be an invariant subspace of \(M\) and a right \((C, \beta)\)-Hom-module, where

\[C = \{b \in A; \varrho_A(b) = \beta^{-1}(b) \otimes 1_H\}\]

is a subalgebra of \(A\).

**Proposition 3.5.** Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra with a total integral \(\phi: (H, \alpha) \to (A, \beta)\) and \((M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)^H\). Assume that \(\phi\) is a multiplication map and let

\[\tau_M: (M, \mu) \to (M, \mu)\]

be the trace map defined by

\[m \mapsto m_{0|0} \cdot \phi(S(m_{1|1})).\]

Then the following assertions hold:
(1) \( \tau_M(m) \in M_0 \) and \( \tau|_{M_0} = \text{id} \);

(2) \( \tau_A : (A, \beta) \to (C, \beta) \) defined by \( b \mapsto b_{[0]} \phi(S(b_{[1]})) \) is a morphism of left \((C, \beta)\)-Hom-modules, so that \((C, \beta)\) is a direct summand of \((A, \beta)\) as a sum of left \((C, \beta)\)-Hom-modules;

(3) if \( \text{Im} \phi \subseteq Z(A) \), then \( \tau_M : (M, \mu) \to (M, \mu) \) is a morphism of right \((C, \beta)\)-Hom-modules.

The exact sequence

\[
(M, \mu) \xrightarrow{\tau_M} (M_0, \mu) \rightarrow 0
\]

thus obtained splits as a sequence of right \((C, \beta)\)-Hom-modules.

Proof. (1) For any \( m \in M \), we have

\[
\varrho(\tau_M(m)) = \varrho(m_{[0]} \phi(S(m_{[1]})))
= m_{[0]} m_{[0]} \phi(S(m_{[1]})) \otimes m_{[0]} m_{[1]} \phi(S(m_{[1]}))
= \mu^{-1}(m_{[0]} \phi(\alpha(S(m_{[1]}))) \otimes m_{[1]} m_{[1]} \phi(\alpha(S(m_{[1]}))))
= \mu^{-1}(m_{[0]} \phi(S(m_{[1]}))) \otimes (m_{[1]} m_{[1]} \phi(\alpha(S(m_{[1]}))))
= \mu^{-1}(m_{[0]} \phi(S(m_{[1]}))) \otimes 1_H
= \mu^{-1}(\tau_M(m)) \otimes 1_H.
\]

For any \( n \in M_0 \),

\[
\tau_M(n) = m_{[0]} \phi(S(m_{[1]}))) = \mu^{-1}(n) 1_A = n.
\]

(2) For any \( c \in C \) and \( a \in A \),

\[
\tau_A(ca) = (c_{[0]} a_{[0]} \phi(S(c_{[1]} a_{[1]}))) = (\beta^{-1}(c) a_{[0]} \phi(\alpha(S(a_{[1]}))))
= c(a_{[0]} \phi(S(a_{[1]}))) = c \tau_A(a),
\]

thus, \( \tau_A : (A, \beta) \to (C, \beta) \) is a morphism of left \((C, \beta)\)-Hom-modules, and by (1), \((C, \beta)\) is a direct summand of \((A, \beta)\) as a sum of left \((C, \beta)\)-Hom-modules.

(3) For any \( c \in C \) and \( m \in M \),

\[
\tau_M(m \cdot c) = (m_{[0]} \cdot c_{[0]} \phi(S(m_{[1]} c_{[1]}))) = (m_{[0]} \cdot \beta^{-1}(c) \phi(\alpha(S(m_{[1]}))))
= \mu(m_{[0]} \cdot (\beta^{-1}(c) \phi(S(m_{[1]})))) = \mu(m_{[0]} \cdot \phi(S(m_{[1]})) \beta^{-1}(c)
= \mu(m_{[0]} \cdot \phi(S(m_{[1]}))) \cdot c = \tau_M(m) \cdot c.
\]

Thus, \( \tau_M \) is a morphism of right \((C, \beta)\)-Hom-modules, and by (1), the exact sequence

\[
(M, \mu) \xrightarrow{\tau_M} (M_0, \mu) \rightarrow 0
\]

thus obtained splits as a sequence of right \((C, \beta)\)-Hom-modules. \( \square \)
4. A Maschke-type theorem for relative Hom-Hopf modules

In this section, we give necessary and sufficient conditions for the functor $F$ which forgets the $(H, \alpha)$-coaction to be separable, and we prove a Maschke type theorem for relative Hom-Hopf modules as an application.

**Definition 4.1.** Let $(H, \alpha)$ be a monoidal Hom-Hopf algebra and $(A, \beta)$ a right $(H, \alpha)$-Hom-comodule algebra. A $k$-linear map

$$\theta: (H, \alpha) \otimes (H, \alpha) \rightarrow (A, \beta)$$

such that $\theta \circ (\alpha \otimes \alpha) = \beta \circ \theta$ is called a normalized $(A, \beta)$-integral, if $\theta$ satisfies the following conditions:

(1) For all $h, g \in H$,

$$(4.1) \quad \theta (\alpha^{-1}(g) \otimes h_{(1)}) \otimes \alpha(h_{(2)}) = \beta (\theta(g_{(2)} \otimes \alpha^{-1}(h))\{0\}) \otimes g_{(1)} \theta(g_{(2)} \otimes \alpha^{-1}(h))\{1\}.$$

(2) For all $h \in H$,

$$(4.2) \quad \theta(h_{(1)} \otimes h_{(2)}) = 1_A \varepsilon(h).$$

(3) For all $a \in A, h, g \in H$,

$$(4.3) \quad \beta^2(a_{[0]} \otimes \alpha^{-1}(g) a_{[0]} \otimes \alpha^{-1}(h) \alpha^{-1}(a_{[1]})) = \theta(g \otimes h)a.$$

**Theorem 4.2.** Let $(H, \alpha)$ be a monoidal Hom-Hopf algebra and $(A, \beta)$ a right $(H, \alpha)$-Hom-comodule algebra. Then the following assertions are equivalent:

(1) The left adjoint $F$ in Proposition 2.3 is separable.

(2) There exists a normalized $(A, \beta)$-integral $\theta: (H, \alpha) \otimes (H, \alpha) \rightarrow (A, \beta)$.

**Proof.** $(2) \implies (1)$. For any relative Hom-Hopf module $(M, \mu)$, we define

$$\nu_M: M \otimes H \rightarrow M,$$

$$\nu_M(m \otimes h) = \mu(m_{[0]} \otimes \alpha^{-1}(h)),$$

for all $m \in M$ and $h \in H$. Now, we shall check that $\nu_M \in \widehat{\mathcal{H}}(\mathcal{M}_k)^H$. In fact, for all $m \in M, h \in H$ and $a \in A$, it is easy to get that

$$\nu_M(\mu(m) \otimes \alpha(h)) = \mu(\nu_M(m \otimes h)).$$
We also have
\[
\nu_M((m \otimes h) \cdot a) = \nu_M(m a_{[0]} \otimes ha_{[1]})
\]
\[
= (\mu(m_{[0]}) \cdot \beta(a_{[0][0]})) \theta(m_{[1]} a_{[0][1]} \otimes \alpha^{-1}(h)\alpha^{-1}(a_{[1]}))
\]
\[
= \mu^2(m_{[0]}) \cdot (\beta(a_{[0][0]})) \beta^{-1}(\theta(m_{[1]} a_{[0][1]} \otimes \alpha^{-1}(h)\alpha^{-1}(a_{[1]})))
\]
\[
= \mu^2(m_{[0]}) \cdot (\beta(a_{[0][0]})) \theta(\alpha^{-1}(m_{[1]}))\alpha^{-1}(a_{[0][1]} \otimes \alpha^{-2}(h)\alpha^{-2}(a_{[1]}))
\]
\[
(4.3) \Rightarrow \mu^2(m_{[0]}) \cdot (\theta(m_{[1]} \otimes \alpha^{-1}(h))\beta^{-1}(a))
\]
\[
= (\mu(m_{[0]}) \cdot \theta(m_{[1]} \otimes \alpha^{-1}(h))) \cdot a
\]
\[
= (\nu_M(m \otimes h)) \cdot a.
\]
Hence it is a morphism of \((A, \beta)\)-Hom-modules. Next, we shall check that \(\nu_M\) is a morphism of Hom-comodules over \((H, \alpha)\). It is sufficient to check that
\[
\varrho_M \circ \nu_M = (\nu_M \otimes \text{id}_H) \circ \varrho_M
\]
holds. For all \(m \in M\) and \(h \in H\), we have
\[
\varrho_M \circ \nu_M(m \otimes h) = \varrho_M(\mu(m_{[0]}) \theta(m_{[1]} \otimes \alpha^{-1}(h)))
\]
\[
= \mu(m_{[0]}) \theta(m_{[1]} \otimes \alpha^{-1}(h))_{[0]} \otimes (\mu(m_{[0]}) \theta(m_{[1]} \otimes \alpha^{-1}(h)))_{[1]}
\]
\[
= \mu(m_{[0]}) \theta(m_{[1]} \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(m_{[0][1]}) \theta(m_{[1]} \otimes \alpha^{-1}(h))_{[1]}
\]
\[
= m_{[0]} \theta(\alpha(m_{[1][2]}) \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(m_{[1][1]}) \theta(\alpha(m_{[1][2]}) \otimes \alpha^{-1}(h))_{[1]}
\]
\[
(4.1) \Rightarrow m_{[0]} \beta^{-1}(\theta(m_{[1]} \otimes h_{(1)})) \otimes \alpha(h_{(2)})
\]
\[
= m_{[0]} \theta(\alpha^{-1}(m_{[1][1]} \otimes \alpha^{-1}(h_{(1)}))) \otimes \alpha(h_{(2)})
\]
\[
= (\nu_M \otimes \text{id}_H) \circ \varrho_M(m \otimes h).
\]

For all \(m \in M\), we have
\[
\nu_M \circ \eta_M(m) = \nu_M(m_{[0]} \otimes m_{[1]}) = \mu(m_{[0][0]}) \theta(m_{[0][1]} \otimes \alpha^{-1}(m_{[1]}))
\]
\[
= m_{[0]} \theta(m_{[1][1]} \otimes m_{[1][2]}) \quad (4.2) \Rightarrow m.
\]

So the left adjoint \(F\) in Proposition 2.3 is separable by virtue of Rafael theorem.

(1) \implies (2). We consider the relative Hom-Hopf module \(A \otimes H\), and the \((A, \beta)\)-actions and \((H, \alpha)\)-coaction are defined as follows:

\[
\begin{align*}
(a \otimes h) \cdot b &= ab_{[0]} \otimes hb_{[1]};
\varrho_{A \otimes H}(a \otimes h) &= (\beta^{-1}(a) \otimes h_{(1)}) \otimes \alpha(h_{(2)}),
\end{align*}
\]

for any \(a, b \in A\) and \(h \in H\).
The retraction $\nu$ of the unit of the adjunction in Proposition 2.3 yields a morphism

$$\nu_{A \otimes H} : (A \otimes H) \otimes H \rightarrow A \otimes H$$

such that, for all $a \in A, h \in H$,

$$\nu_{A \otimes H}((a \otimes h_{(1)}) \otimes h_{(2)}) = \beta(a) \otimes h.$$

It can be used to construct $\theta$ as follows:

$$\theta : H \otimes H \rightarrow A,$$

$$\theta(h \otimes g) = r_A(id_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes h) \otimes g),$$

where $r$ means the right unit constraint. For all $h \in H$ we have

$$\theta(h_{(1)} \otimes h_{(2)}) = r_A(id_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes h_{(1)}) \otimes h_{(2)})$$

$$= r_A(id_A \otimes \varepsilon)(1_A \otimes h) = 1_A \varepsilon(h).$$

Hence condition (4.2) follows. It can be seen to obey (4.3) by naturality and the $(A, \beta)$-modules map of $\nu$.

The verification of (4.1) is more involved. For any right $(H, \alpha)$-Hom-comodule $M$, we consider the relative Hom-Hopf module $M \otimes A$, the $(A, \beta)$-action and $(H, \alpha)$-coaction are defined as follows: for all $m \in M$ and $a, b \in A$,

$$\left\{ \begin{array}{l}
(m \otimes a) \cdot b = \mu^{-1}(m) \otimes a \beta(b), \\
\varrho_{M \otimes A}(m \otimes a) = (m_{[0]} \otimes a_{[0]}) \otimes m_{[1]}a_{[1]}.
\end{array} \right.$$ 

In particular, there is a relative Hom-Hopf module $H \otimes A$ and a map

$$\xi : H \otimes A \rightarrow A \otimes H$$

$$\xi(h \otimes a) = \beta(a_{[0]}) \otimes \alpha^{-1}(h)a_{[1]}.$$

Since $\xi$ is both right $(A, \beta)$-linear and right $(H, \alpha)$-colinear, we have

$$\xi(\nu_{H \otimes A}((h \otimes a) \otimes g)) = \nu_{A \otimes H}((\xi \otimes id_H)((h \otimes a) \otimes g))$$

$$= \nu_{A \otimes H}((\beta(a_{[0]}) \otimes \alpha^{-1}(h)a_{[1]}) \otimes g).$$

It is not hard to check that $GF(H \otimes A) = (H \otimes A) \otimes H \in H \mathcal{H}(\mathcal{M}_k)_A^H$, and its left $(H, \alpha)$-Hom comodule structure is given by

$$(h \otimes a) \otimes g \mapsto \alpha(h_{(1)}) \otimes ((h_{(2)} \otimes \beta^{-1}(a)) \otimes \alpha^{-1}(g)).$$
Also, $H \otimes A \in \mathcal{H}(\mathcal{M}_k)^H_A$, and the left $(H, \alpha)$-coaction of $H \otimes A$ is given by

$$h \otimes a \mapsto \alpha(h(1)) \otimes (h(2) \otimes \beta^{-1}(a)).$$

We also get that $\nu_{H \otimes A} : (H \otimes A) \otimes H \to H \otimes A$ is a Hom morphism in $\mathcal{H}(\mathcal{M}_k)^H_A$, which means

\begin{equation}
\nu_{H \otimes A}((h \otimes a) \otimes g)[0] \otimes \nu_{H \otimes A}((h \otimes a) \otimes g)[1] = \alpha(h(1)) \otimes \nu_{H \otimes A}((h(2) \otimes \beta^{-1}(a)) \otimes \alpha^{-1}(g)).
\end{equation}

(4.5)

Thus we conclude that $\nu_{H \otimes A}$ is both left and right $(H, \alpha)$-colinear. Taking $h, g \in H$, and putting

$$\nu_{A \otimes H}((1_A \otimes h) \otimes g) = \sum_i a_i \otimes q_i \in A \otimes H,$$

$$\nu_{H \otimes A}((h \otimes 1_A) \otimes g) = \sum_i p_i \otimes b_i \in H \otimes A,$$

we obtain

\begin{align*}
\beta(\theta(h(2) \otimes \alpha^{-1}(g))[0]) \otimes h(1) \theta(h(2) \otimes \alpha^{-1}(g))[1] & = \beta(r_A(\text{id}_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes h(2)) \otimes \alpha^{-1}(g))[0]) \otimes h(1) \\
& \times (r_A(\text{id}_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes h(2)) \otimes \alpha^{-1}(g))[1]) \\
& = \beta(r_A(\text{id}_A \otimes \varepsilon)\xi_{H \otimes A}((h(2) \otimes 1_A) \otimes \alpha^{-1}(g))[0]) \otimes h(1) \\
& \times (r_A(\text{id}_A \otimes \varepsilon)\xi_{H \otimes A}((h(2) \otimes 1_A) \otimes \alpha^{-1}(g))[1]) \\
& = \sum_i \beta(r_A(\text{id}_A \otimes \varepsilon)(\xi_{H \otimes A}(p_i(2) \otimes \beta^{-1}(b_i))[0]) \\
& \otimes p_i(1)(r_A(\text{id}_A \otimes \varepsilon)(\xi_{H \otimes A}(p_i(2) \otimes \beta^{-1}(b_i))[1]) \\
& \sum_i \beta(b_i[0]) \otimes p_i(1)\varepsilon(p_i(2))(b_i[1]) \\
& = \sum_i \xi(p_i \otimes b_i) = \xi(\nu_{H \otimes A}((h \otimes 1_A) \otimes g)).
\end{align*}

Using the fact that $\nu_{A \otimes H}$ is a morphism of right $(H, \alpha)$-Hom comodules, we also have

$$\theta(\alpha^{-1}(h) \otimes g(1)) \otimes \alpha(g(2))$$

$$= r_A(\text{id}_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes \alpha^{-1}(h)) \otimes g(1)) \otimes \alpha(g(2))$$

$$= \sum_i r_A(\text{id}_A \otimes \varepsilon)(\beta^{-1}(a_i) \otimes q_i(1)) \otimes \alpha(q_i(2))$$
\[
\sum_i a_i \otimes q_i = \nu_{A \otimes H}((1_A \otimes h) \otimes g) \\
\overset{(4.4)}{=} \xi(\nu_{H \otimes A}((h \otimes 1_A) \otimes g)).
\]

Hence, we can get condition (4.1).

We will now investigate the relation between the total integrals and the normalized \((A, \beta)\)-integrals. This will explain our terminology, and we will also prove that the forgetful functor is separable if and only if there exists a total integral \(\phi: (H, \alpha) \to (A, \beta)\) such that the image of \(\varrho_A \circ \phi\) is contained in the center of \(H \otimes A\).

**Proposition 4.3.** Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra. If \(\theta: (H, \alpha) \otimes (H, \alpha) \to (A, \beta)\) is a normalized \((A, \beta)\)-integral for \((H, A, H)\), then the \(k\)-linear map
\[
\phi: (H, \alpha) \to (A, \beta), \quad \phi(h) = \theta(1_H \otimes h)
\]
for all \(h \in H\) is a total integral.

**Proof.** Notice first that \(\phi(1_H) = \theta(1_H \otimes 1_H) = \varepsilon_H(1_H)1_A = 1_A\). Hence
\[
\theta(\alpha^{-1}(g) \otimes \alpha^{-1}(h(1))) \otimes \alpha(h(2)) = (\theta(\alpha(g(2)) \otimes \alpha^{-1}(h)))_{(0)} \otimes \alpha(g(1))(\theta(\alpha(g(2)) \otimes \alpha^{-1}(h)))_{(1)}.
\]
It follows by taking \(g = 1_H\) that
\[
\theta(1_H \otimes \alpha^{-1}(h)) \otimes \alpha(h_2) = \theta(1_H \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(\theta(1_H \otimes \alpha^{-1}(h))_{[1]}),
\]
and applying \(\alpha \otimes \alpha^{-1}\) to the above identity, we have
\[
\theta(1_H \otimes \alpha^{-1}(h)) \otimes h_2 = \theta(1_H \otimes \alpha^{-1}(h))_{[0]} \otimes \theta(1_H \otimes \alpha^{-1}(h))_{[1]}.
\]
So we obtain
\[
\phi(h_1) \otimes h_2 = \phi(h)_{[0]} \otimes \phi(h)_{[1]}.
\]
It is easy to check that \(\phi \alpha = \beta \phi\). So \(\phi\) is a total integral.

Let \(\phi: (H, \alpha) \to (A, \beta)\) be a total integral for the right \((H, \alpha)\)-Hom-comodule algebra \((A, \beta)\), and define
\[
\theta: (H, \alpha) \otimes (H, \alpha) \to (A, \beta), \quad \theta(g \otimes h) = \phi(h S^{-1}(g))
\]
for all \(g, h \in H\). \qed
Theorem 4.4. Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra, and \(\phi: (H, \alpha) \rightarrow (A, \beta)\) a total integral. If 
\[ g\phi(h)[1] \otimes \phi(h)[0] = \phi(h)[1]g \otimes \phi(h)[0], \quad \phi(h) \in Z(A), \]
then \(\theta\) is a normalized \((A, \beta)\)-integral.

Proof. For any \(h, g \in H\) and \(a \in A\), we have 
\[
\begin{align*}
\beta^2(a[0][0])\theta(\alpha^{-1}(g)a[0][1] \otimes \alpha^{-1}(h)\alpha^{-1}(a[1])) \\
= \beta(a[0])\theta(\alpha^{-1}(g)a[1(1)] \otimes \alpha^{-1}(h)a[1(2)]) \\
= \beta(a[0])\phi(\alpha^{-1}(h)a[1(2)]S^{-1}(\alpha^{-1}(g)a[1(1)])) \\
= \beta(a[0])\phi(h[(\alpha^{-1}(a[1(2)]S^{-1}(\alpha^{-1}(a[1(1)]))))S^{-1}(\alpha^{-1}(g))]) \\
= a\phi(hS^{-1}(g)) = \theta(g \otimes h)a,
\end{align*}
\]
and
\[
\begin{align*}
\beta(\theta(g[2] \otimes \alpha^{-1}(h))[0]) \otimes g[1]\theta(g[2] \otimes \alpha^{-1}(h))[1] \\
= \beta(\phi(\alpha^{-1}(h)S^{-1}(g[2]))[0]) \otimes \phi(\alpha^{-1}(h)S^{-1}(g[2]))[1]g[1] \\
= \phi(h[1]S^{-1}(\alpha(g[2])) \otimes (\alpha^{-1}(h)S^{-1}(g[2]))g[1] \\
= \phi(h[1]S^{-1}(g[2])) \otimes (\alpha^{-1}(h)S^{-1}(g[2]))g[1]g[1(1)] \\
= \phi(h[1]S^{-1}(g[2])) \otimes h[2](S^{-1}(g[2]))g[1(1)] \\
= \phi(h[1])S^{-1}(\alpha^{-1}(g)) \otimes \alpha(h[2]) \\
= \theta(\alpha^{-1}(g) \otimes h[1]) \otimes \alpha(h[2]),
\end{align*}
\]
\[
\theta(h_1 \otimes h_2) = \phi(h_2S^{-1}(h)) = \varepsilon_H(h)1_A.
\]
It is easy to check that \(\phi\alpha = \beta\phi\). So \(\theta\) is a normalized \((A, \beta)\)-integral. \(\square\)

Since separable functors reflect well the semisimplicity of the objects of a category, by Theorem 4.2, we will prove a Maschke type theorem for relative Hom-Hopf modules as an application.

Lemma 4.5. Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra with a total integral \(\phi: (H, \alpha) \rightarrow (A, \beta)\) and \((M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)_A^H\) and a Hom-morphism \(f: (N, \nu) \rightarrow (M, \mu)\). Let 
\[ f_\phi: N \otimes N, N \otimes H \xrightarrow{f \otimes \text{id}_H} M \otimes H \xrightarrow{\tau} M, \]
that is, 
\[ f_\phi(n) = \mu^{-1}(f(n[0])[0]) \cdot \phi(S(f(n[0])[1])\alpha(n[1])), \]
for any \(n \in N\). Then the following assertions hold:
(1) \( f_\phi \) is a morphism of right \((H, \alpha)\)-Hom-comodules,

(2) if \( f : (N, \nu) \to (M, \mu) \) is a morphism of right \((A, \beta)\)-Hom-modules and \( \phi : (H, \alpha) \to (Z(A), \beta) \) is a multiplication map, then \( f_\phi \) is a morphism of right \((A, \beta)\)-Hom-module.

**Proof.** (1) For any \( n \in N \), we have

\[
\varrho_M(f_\phi(n)) = \varrho_M(\mu^{-1}(f(n_0)[0]) \cdot \phi(S(f(n_0)[1]) \alpha(n_1)))
\]

\[
= \mu^{-1}(f(n_0)[0][0]) \cdot \phi(S(f(n_0)[1][2]) \alpha(n_1[1]))
\]

\[
\otimes \alpha^{-1}(f(n_0)[0][1])(S(f(n_0)[1][2]) \alpha(n_1[2]))
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot \phi(S(f(n_0)[1][2]) \alpha(n_1[1]))
\]

\[
\otimes \alpha^{-1}(f(n_0)[1][1])(S(f(n_0)[1][2]) \alpha(n_1[2]))
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot \phi(S(f(n_0)[1][2]) \alpha(n_1[1]))
\]

\[
\otimes \alpha^{-1}(f(n_0)[1][1])(S(f(n_0)[1][2]) \alpha(n_1[2]))
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot \alpha(S(f(n_0)[1][2]) \alpha(n_1[1]))
\]

\[
\otimes \alpha^{-1}(f(n_0)[1][1])(S(f(n_0)[1][2]) \alpha(n_1[2]))
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot \alpha^{-1}(S(f(n_0)[1][2]) \alpha(n_1[1])) \otimes \alpha(n_1[2])
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot \phi(S(f(n_0)[1][2]) \alpha(n_1[1])) \otimes \mu^{-1}(f(n_0)[0]) \cdot \phi(S(n_0)[1][1]) \alpha(n_0[1]) \otimes n_1
\]

\[
= (f_\phi \otimes \mathrm{id}_H) \varrho_N(n).
\]

(2) For any \( n \in N \) and \( b \in A \), we have

\[
f_\phi(n \cdot b) = \mu^{-1}(f(n_0)[0]) \cdot b_0[0] \cdot \phi(S(f(n_0)[1]) \alpha(n_1) \beta_n[1])
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot b_0[0] \cdot \phi(S(f(n_0)[1]) \alpha(n_1))
\]

\[
= \mu^{-1}(f(n_0)[0]) \cdot \phi(S(f(n_0)[1]) \alpha(n_1[1])) \otimes \alpha^{-1}(S(b_0[1][2]) \alpha(n_1[1]))
\]

\[
= (\mu^{-1}(f(n_0)[0]) \cdot b_0[0]) \cdot \phi(S(f(n_0)[1]) \alpha(n_1[1])) \otimes \alpha^{-1}(S(b_0[1][2]) \alpha(n_1[1]))
\]

\[
= (f(n_0)[0]) \cdot \beta^{-1}(b) \cdot \phi(S(f(n_0)[1]) \alpha(n_1[1]))
\]

\[
= f(n_0)[0] \cdot (\beta^{-1}(b) \phi(S(f(n_0)[1]) \alpha(n_1[1])))
\]

\[
= f(n_0)[0] \cdot (\phi(S(f(n_0)[1]) \alpha(n_1[1])) \beta^{-1}(b))
\]

\[
= (\mu^{-1}(f(n_0)[0]) \cdot \phi(S(f(n_0)[1]) \alpha(n_1[1]))) \cdot b = f_\phi(n) \cdot b.
\]

\[\square\]
Theorem 4.6. Let \((H, \alpha)\) be a monoidal Hom-Hopf algebra and \((A, \beta)\) a right \((H, \alpha)\)-Hom-comodule algebra with a total integral \(\phi: (H, \alpha) \to (A, \beta)\). If \(\phi: (H, \alpha) \to (Z(A), \beta)\) is a multiplication map, then every short exact sequence of relative Hom-Hopf modules

\[
0 \longrightarrow (M, \mu) \xrightarrow{f} (N, \nu) \xrightarrow{g} (P, \pi) \longrightarrow 0
\]

which splits as a sequence of \((A, \beta)\)-Hom-modules also splits as a sequence of relative Hom-Hopf modules.

Acknowledgement. The authors are grateful to the referee for carefully reading the manuscript and for many valuable comments which largely improved the article.

References

[1] S. Caenepeel, I. Goyvaerts: Monoidal Hom-Hopf algebras. Commun. Algebra 39 (2011), 2216–2240.
[2] S. Caenepeel, G. Militaru, B. Ion, S. Zhu: Separable functors for the category of Doi-Hopf modules, applications. Adv. Math. 145 (1999), 239–290.
[3] Y. Doi: Hopf extensions of algebras and Maschke type theorems. Isr. J. Math. 72 (1990), 99–108.
[4] Y. Doi: Algebras with total integrals. Commun. Algebra 13 (1985), 2137–2159.
[5] Y. Doi: On the structure of relative Hopf modules. Commun. Algebra 11 (1983), 243–255.
[6] Y. Frégié, A. Gohr: On Hom-type algebras. J. Gen. Lie Theory Appl. 4 (2010), Article ID G101001, pages 16.
[7] J. T. Hartwig, D. Larsson, S. D. Silvestrov: Deformations of Lie algebras using \(\sigma\)-derivations. J. Algebra 295 (2006), 314–361.
[8] A. Makhlouf, S. Silvestrov: Hom-algebras and Hom-coalgebras. J. Algebra Appl. 9 (2010), 553–589.
[9] A. Makhlouf, S. D. Silvestrov: Hom-algebra structures. J. Gen. Lie Theory Appl. 2 (2008), 51–64.
[10] M. Takeuchi: Relative Hopf modules-equivalences and freeness criteria. J. Algebra 60 (1979), 452–471.

Authors’ addresses: Shuangjian Guo, School of Mathematics and Statistics, Guizhou University of Finance and Economics in Huaxi University Town, Guiyang, Guizhou Province, 550025, P. R. China; Xiu-Li Chen (corresponding author), Department of Mathematics, Southeast University, Jiujihu Campus, No. 2 Southeast University Road, Nanjing, 210096, P. R. China, e-mail: xiulichen10210126.com.