On Tverberg’s conjecture

Siniša T. Vrećica *

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Abstract

In 1989 Helge Tverberg proposed a quite general conjecture in Discrete geometry, which could be considered as the common basis for many results in Combinatorial geometry and at the same time as a discrete analogue of the common transversal theorems. It implies or contains as the special cases many classical ”coincidence” results such as Radon’s theorem, Rado’s theorem, the Ham-sandwich theorem, the ”nonembeddability” results (e.g. nonembeddability of graphs $K_5$ and $K_{3,3}$ in $\mathbb{R}^2$) etc.

The main goal of this short note is to verify this conjecture in one new, non-trivial case. We obtain the continuous version of the conjecture. So, it is not surprising that we use the topological methods, or more precisely the methods of equivariant topology and the theory of characteristic classes.

1 Introduction

Establishing the relation between Rado’s theorem on general measure (see [3]) and the Ham sandwich theorem, the following result is proved in [8] stating that these two results belong to the same family.

**Theorem 1.1** Let $0 \leq k \leq d - 1$, and let $\mu_0, \mu_1, \ldots, \mu_k$ be $\sigma$-additive probability measures on $\mathbb{R}^d$. Then there is a $k$-flat $F$ with the property that every closed halfspace containing $F$, has $\mu_i$-measure at least $\frac{1}{d-k+1}$ for all $i$, $0 \leq i \leq k$.

Namely, this theorem reduces to Rado’s theorem in the case $k = 0$ and to the Ham sandwich theorem in the case $k = d - 1$.

The proof of the above theorem uses the topological result claiming the non-existence of a non-zero section of a certain vector bundle over the Grassmann manifold. Helge Tverberg observed that the special case $k = 0$ (Rado’s theorem) follows easily from his result in combinatorial geometry from [4].

**Theorem 1.2** Let $S$ be a set of $(r - 1)(d + 1) + 1$ points in $\mathbb{R}^d$. Then one can split it into subsets $S_1, S_2, \ldots, S_r$ so that

$$\cap_{i=1}^r \text{conv } S_i \neq \emptyset.$$

This observation motivated him to suppose that a general result should exist which would generalize his result [1.2] and at the same time it would imply theorem 1.1 in the same way as his result implied Rado’s theorem. So, he formulated the following **Tverberg’s conjecture**:

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Conjecture 1.3 Let \(0 \leq k \leq d - 1\) and let \(S_0, S_1, \ldots, S_k\) be finite sets of points in \(\mathbb{R}^d\), with \(\left| S_i \right| = (r_i - 1)(d - k + 1) + 1\) for \(i = 0, 1, \ldots, k\). Then \(S_i\) can be split into \(r_i\) sets, \(S_{i1}, S_{i2}, \ldots, S_{ir_i}\), so that there is a \(k\)-flat \(F\) meeting all the sets \(\text{conv} S_{ij}, 0 \leq i \leq k, 1 \leq j \leq r_i\).

It is easy to see that this conjecture implies the theorem \(\text{[1]}\) (\(\text{[8]}\), see also \(\text{[1]}\)) and its special case \(k = 0\) is the theorem \(\text{[2]}\). This conjecture unifies two important themes of Combinatorial geometry: Helly-type theorems (the special case \(r = 2\) of the theorem \(\text{[2]}\) is the well known Radon’s theorem), and the common transversal theorems. Moreover, it was shown in \(\text{[8]}\) that this result could be considered as an example of the whole family of results of the “combinatorial geometry on vector bundles”, and that these results would also generalize many coincidence results such as the nonembeddability of the graphs having minor \(K_5\) or \(K_{3,3}\) in the plane etc.

In paper \(\text{[8]}\) this conjecture was verified in some special cases. It was proved that, besides the already known case \(k = 0\) the conjecture is true in the case \(k = d - 1\), and in the case \(k = 1\) when \(r_0 = 1\) or \(r_1 = 1\) or \(r_0 = r_1 = 2\). Also, slightly weakened version of the conjecture is proved in the case \(k = d - 2\), obtained when \(3r_i\) points in \(S_i\) were considered instead of \(3r_i - 2\) of them. This version still suffices to imply theorem \(\text{[7]}\) in the case \(k = d - 2\).

Moreover, we will prove a slight generalization of this result. Namely, we could consider the family of \((d - k + 1)\)-dimensional simplicies \(\Delta_0, \Delta_1, \ldots, \Delta_k\), where \(\Delta_i\) is the simplex spanned by the vertices \(e_0^i, e_1^i, \ldots, e_{d-k+1}^i\) for \(i = 0, 1, \ldots, k\). We consider then the linear mapping sending the vertices of \(\Delta_i\) to the points in \(S_i\) and note that the sets \(\text{conv} S_{i1}\) and \(\text{conv} S_{i2}\) are actually the images of the pair of disjoint faces of the simplex \(\Delta_i\) under the considered linear mapping. We observe that our argument also works for any continuous mapping and not only for linear ones. So, we prove the following theorem.
Theorem 2.2 Let $0 \leq k \leq d - 1$ and let $\varphi_i : \Delta_i \to \mathbb{R}^d$, for $i = 0, 1, ..., k$ be continuous mappings. Then there is an affine $k$-flat $F$ which intersects the images of two disjoint faces of every simplex $\Delta_i$.

Proof: Let us denote with $\text{Gr}_{d,d-k}$ the Grassmann manifold of $(d-k)$-dimensional linear subspaces of $\mathbb{R}^d$, and for any $L \in \text{Gr}_{d,d-k}$, with $\pi_L : \mathbb{R}^d \to L$ the orthogonal projection.

Let us consider $(d-k+1)$-dimensional simplicies $\Delta_i = \text{conv} \{ e_i^0, e_i^1, ..., e_i^{d-k+1} \}$, $i = 0, 1, ..., k$. With $(\Delta_i)^2_*$ we denote the deleted square of the simplex $\Delta_i$, i.e. the set of ordered pairs of points in $\Delta_i$ having disjoint supports. It is easy to verify that the deleted square $(\Delta_i)^2_*$ is a $(d-k)$-dimensional manifold which we denote by $M$. It could be proved that $M$ is actually homeomorphic to the sphere $S^{d-k}$, but we don’t need this fact here. (It is shown in [3] that $M$ is $(d-k-1)$-connected.)

The mapping $\varphi_i : \Delta_i \to \mathbb{R}^d$ induces the mapping

$$\tilde{\varphi}_i : (\Delta_i)^2_* \to \mathbb{R}^d \times \mathbb{R}^d, \quad \tilde{\varphi}_i(x, y) = (\varphi_i(x), \varphi_i(y)).$$

Now we consider the product of these induced mappings $\tilde{\varphi}_i$ and get the mapping

$$\tilde{\varphi} : (\Delta_0)^2_* \times \cdots \times (\Delta_k)^2_* \to (\mathbb{R}^d)^2 \times \cdots \times (\mathbb{R}^d)^2,$$

$$\tilde{\varphi}((x_0, y_0), ..., (x_k, y_k)) = (\varphi_0(x_0, y_0), ..., \varphi_k(x_k, y_k)).$$

The statement of the theorem reduces to the claim that there exists $L \in \text{Gr}_{d,d-k}$ such that

$$\pi_L(\varphi_0(x_0)) = \pi_L(\varphi_0(y_0)) = \cdots = \pi_L(\varphi_k(x_k)) = \pi_L(\varphi_k(y_k)),$$

for some $((x_0, y_0), ..., (x_k, y_k)) \in (\Delta_0)^2_* \times \cdots \times (\Delta_k)^2_*$. Here $L = F^1$, i.e. $L$ is the orthogonal complement to the affine $k$-flat $F$ claimed to exist in the statement of the theorem.

Let us denote with $\xi$ the canonical vector bundle over $\text{Gr}_{d,d-k}$. For every $L \in \text{Gr}_{d,d-k}$, we have the mapping

$$\psi_L = \pi_L \circ \tilde{\varphi} : (\Delta_0)^2_* \times \cdots \times (\Delta_k)^2_* \to L^{2k+2},$$

$$\psi_L((x_0, y_0), ..., (x_k, y_k)) = (\pi_L(\varphi_0(x_0)), \pi_L(\varphi_0(y_0)), ..., \pi_L(\varphi_k(x_k)), \pi_L(\varphi_k(y_k))).$$

The group $G = \mathbb{Z}/2 \oplus \cdots \mathbb{Z}/2$ acts on these spaces, freely on $(\Delta_0)^2_* \times \cdots \times (\Delta_k)^2_* \approx M^{k+1}$ and fiberwise on $\xi^{2k+2}$ (i.e. trivially on $\text{Gr}_{d,d-k}$). The above mapping is equivariant and it induces a section $s$ of the vector bundle

$$\xi^{2k+2} \times_G M^{k+1} \to \left( \text{Gr}_{d,d-k} \times M^{k+1} \right) / G = \text{Gr}_{d,d-k} \times M^{k+1} / G.$$

The fiber over $[L, (x_0, y_0, ..., x_k, y_k)]$ could be identified with $L^{2k+2}$. The statement of the theorem reduces now to the claim that the section $s$ intersects the diagonal $\Delta$ in some fiber. Let us suppose, to the contrary, that $s$ does not intersect the diagonal in any fiber. Projecting to the orthogonal complement of the diagonal and then radially to its sphere (in each fiber), we obtain the non-zero section of the vector bundle with the fiber $L^{2k+1}$ and the section of the associated sphere bundle whose fiber is homeomorphic with $S^{(d-k)(2k+1)-1}$. 

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We reach a contradiction (proving in this way the theorem) by showing that the top-
dimensional Stiefel-Whitney class of this sphere bundle does not vanish. The Poincare dual
of the top-dimensional Stiefel-Whitney class of the sphere bundle coincides with the homol-
ogy class of the zero-set of a section of the associated vector bundle (with the fiber \( L^{2k+1} \)),
which is transversal to the zero section. So, it suffices to find a section of the corresponding
vector bundle which intersects the zero section transversally in an odd number of points.

In order to construct such a section we consider \( k+1 \) parallel \((d-k)\)-dimensional
affine planes \( A_0, A_1, \ldots, A_k \) in \( \mathbb{R}^d \), and \( d-k+2 \) points in each of them being the vertices \( v_0, v_1, \ldots, v_d \) and a barycenter \( \hat{\sigma}_i \) of \((d-k)\)-dimensional simplex \( \sigma_i \), \( 0 \leq i \leq k \). Let us suppose that they are in a generic position meaning that their \( k+1 \) barycenters span an affine \( k \)-dimensional flat \( F \). We also consider linear mappings \( f_i \) which map vertices of \( \Delta_i \) to the vertices and the barycenter of \( \sigma_i \), namely \( f_i(e_j^i) = v_j, 0 \leq j \leq d-k \) and \( f_i(e_{d-k+1}^i) = \hat{\sigma}_i \) for \( i = 0, 1, \ldots, k \).

The images under the linear mapping \( f_i \) of two disjoint faces of the simplex \( \Delta_i \) are the convex hulls of the corresponding vertices of the simplex \( \sigma_i \) and its barycenter \( \hat{\sigma}_i \). The only nonempty intersection of the convex hulls of two disjoint subsets of \( \{v_0^i, v_1^i, \ldots, v_{d-k}^i, \hat{\sigma}_i\} \) is

\[
\text{conv}\{v_0^i, v_1^i, \ldots, v_{d-k}^i\} \cap \text{conv}\{\hat{\sigma}_i\} = \{\hat{\sigma}_i\}.
\]

If some affine \( k \)-dimensional plane in \( \mathbb{R}^d \) intersects the images of two disjoint faces of some simplex \( \Delta_i \), then this plane contains the barycenter \( \hat{\sigma}_i \) or it intersects \( A_i \) in at least 1-dimensional affine plane. If this plane should intersect the images of two disjoint faces of each simplex \( \Delta_0, \Delta_1, \ldots, \Delta_k \), then (because of the generic position of the planes \( A_0, A_1, \ldots, A_k \) it has to contain the barycenters \( \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k \) and it is uniquely determined by them. So, the mappings \( f_0, f_1, \ldots, f_k \) induce the section of the considered vector bundle which intersects the diagonal at a single orbit of the action of the group \( G \), i.e. at the orbit

\[
\left[ L, \left( \frac{e_0^0 + \cdots + e_0^{d-k}}{d-k+1}, e_0^{d-k+1}, \ldots, \frac{e_k^0 + \cdots + e_k^{d-k}}{d-k+1}, e_k^{d-k+1} \right) \right],
\]

where \( L = F^\perp \) is the orthogonal complement to the \( k \)-dimensional flat \( F \) spanned by the points \( \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k \).

So, we found a section which intersects the zero section of the considered vector bundle
in a single point. Obviously, for small perturbation of the mappings \( f_i \), the intersection
with the zero section will remain a single point, and so the obtained section intersects the zero section transversally. This completes our proof. \( \square \)

**Remark 2.3** We note that we proved continuous version of the conjecture, and so our theorem 2.2 also generalizes the result of Bajmóczy and Bárány from [1] in the same way as the result from [7] generalizes the result of Bárány, Shlosman and Szücs from [2].

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Siniša T. Vrečica
Faculty of Mathematics
University of Belgrade
Studentski trg 16, P.O.B. 550
11000 Belgrade, Serbia
vrecica@matf.bg.ac.yu