Supersymmetry in nuclear physics

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Abstract. I discuss several new developments in nuclear supersymmetry, in particular the identification of a new supersymmetric quartet of nuclei in the $A \sim 190$ region of the nuclear mass table consisting of the $^{192,193}\text{Os}$ and $^{193,194}\text{Ir}$ nuclei, and a study of correlations between different transfer reactions by means of generalized $F$-spin and $SU(3)$ isoscalar factors. The relevant $SU(3)$ isoscalar factors are derived in explicit form.

1. Introduction

Nuclear supersymmetry was proposed more than twenty five years ago [1] in the context of the interacting boson model (IBM) and the interacting boson-fermion model (IBFM) which have proved remarkably successful in providing a unified framework of even-even [2] and odd-even nuclei [3], respectively. One of its most attractive features is that it gives rise to a simple algebraic description, in which dynamical symmetries and supersymmetries play a central role, both as a way to improve our basic understanding of the importance of (super)symmetry in nuclear dynamics, and as a starting point for more precise calculations. Nuclear supersymmetry provides a theoretical framework in which different nuclei are treated as members of the same supermultiplet and whose spectroscopic properties are described by a single Hamiltonian and a single set of transition and transfer operators.

Aside from their esthetic appeal, (super)symmetries provide energy formula, selection rules and closed expressions for electromagnetic transition rates and transfer strengths which can be used as benchmarks to study and interpret the experimental data, even if these symmetries may be valid only approximately. Historically, symmetries have played an important role in nuclear physics. Examples are isospin symmetry, the Wigner supermultiplet theory, special solutions to the Bohr Hamiltonian, the Elliott model, pseudo-spin symmetries and the dynamical symmetries and supersymmetries of the IBM and its extensions.

Nuclear supersymmetry is a composite-particle phenomenon that should not be confused with fundamental supersymmetry, as used in particle physics and quantum field theory where it is postulated as a generalization of the Lorentz-Poincaré invariance as a fundamental symmetry of Nature and predicts the existence of supersymmetric particles, such as the photino and the selectron, for which experimental evidence is yet to be found. If experiments about to start at the LHC at CERN find evidence of supersymmetric particles, supersymmetry would be badly broken, as their masses must be much higher than those of their normal partners. In contrast to particle physics, nuclear supersymmetry has been verified experimentally.

Originally, nuclear supersymmetry was formulated as a symmetry among pairs of nuclei consisting of an even-even and an odd-even nucleus [1, 4, 5]. Subsequently, by including the
neutron-proton degree of freedom it was extended to quartets of nuclei, in which an even-even, an odd-proton, an odd-neutron and an odd-odd nucleus form a supermultiplet [6]. A detailed study using state-of-the-art experimental techniques [7, 8, 9] showed that the odd-odd nucleus $^{196}$Au, together with the odd-neutron nucleus $^{195}$Pt, the odd-proton nucleus $^{195}$Au and the even-even nucleus $^{194}$Pt constitute an example of a supersymmetric quartet of nuclei [6, 10]. Actually, the interpretation of these four nuclei as members of a supersymmetric quartet made it possible to predict [6] the properties of $^{196}$Au almost 15 years before they were measured experimentally [7, 8].

The purpose of this contribution is twofold. First, the concept of nuclear supersymmetry is reviewed briefly before discussing several new developments, in particular the identification of a new supersymmetric quartet in the $A \sim 190$ mass region, consisting of the $^{192, 193}$Os and $^{193, 194}$Ir nuclei, and the study of correlations between different one- and two-nucleon transfer reactions.

2. Nuclear supersymmetry

Supersymmetry was introduced [1] in nuclear physics in 1980 by Iachello in the context of the Interacting Boson Model (IBM) [2] and its extensions. The spectroscopy of atomic nuclei is characterized by the interplay between collective (bosonic) and single-particle (fermionic) degrees of freedom. The IBM describes collective excitations in even-even nuclei in terms of a system of interacting monopole and quadrupole bosons with angular momentum $l = 0, 2$. The bosons are associated with the number of correlated valence proton and neutron pairs, and hence the number of bosons $N$ is half the number of valence nucleons. Since it is convenient to express the Hamiltonian and other operators of interest in second quantized form, one introduces creation, $s^\dagger$ and $d^\dagger_{m}$, and annihilation, $s$ and $d_{m}$, operators, which altogether can be denoted by $b^\dagger_{i}$ and $b_{i}$ with $i = l, m$ ($l = 0, 2$ and $−l \leq m \leq l$). The operators $b^\dagger_{i}$ and $b_{i}$ satisfy the commutation relations

$$ [b_{i}, b^\dagger_{j}] = \delta_{ij} , \quad [b^\dagger_{i}, b^\dagger_{j}] = [b_{i}, b_{j}] = 0 . \quad (1) $$

The bilinear products

$$ B_{ij} = b^\dagger_{i} b_{j} , \quad (2) $$

generate the algebra of $U(6)$ the unitary group in 6 dimensions

$$ [B_{ij}, B_{kl}] = B_{il} \delta_{jk} - B_{kj} \delta_{il} . \quad (3) $$

The IBM Hamiltonian and other operators of interest are expressed in terms of the generators of $U(6)$. In general, the Hamiltonian has to be diagonalized numerically to obtain the energy eigenvalues and wave functions. There exist, however, special situations in which the eigenvalues can be obtained in closed, analytic form. These special solutions provide a framework in which energy spectra and other nuclear properties (such as quadrupole transitions and moments) can be interpreted in a qualitative way. These situations correspond to dynamical symmetries of the Hamiltonian [2]. The concept of dynamical symmetry has been shown to be a very useful tool in different branches of physics. A well-known example in nuclear physics is the Elliott $SU(3)$ model [11] to describe the properties of light nuclei in the $sd$ shell. Another case is the $SU(3)$ flavor symmetry of Gell-Mann and Ne’eman [12] to classify the baryons and mesons into flavor octets, decuplets and singlets and to describe their masses with the Gell-Mann-Okubo mass formula.

A dynamical symmetry arises, when the Hamiltonian is expressed in terms of Casimir invariants of a chain of subgroups of $G = U(6)$, $G \supset G_1 \supset G_2 \supset \ldots$ only. The eigenstates can then be classified uniquely according to the irreducible representations of $G$ and its subgroups $G_1$, $G_2$, $\ldots$. The different representations of $G$, $G_1$, $G_2$, $\ldots$ are split but not admixed by the Hamiltonian. The energy eigenvalues are given by the expectation values of the Casimir
operators. In addition, by using standard group theoretical techniques it is possible to obtain analytic expressions for electromagnetic transition rates and quadrupole moments, etc.

For odd-mass nuclei the IBM has been extended to include single-particle degrees of freedom of the odd nucleon [3]. The Interacting Boson-Fermion Model (IBFM) has as its building blocks a set of \( N \) bosons with \( l = 0, 2 \) and \( M = 1 \) fermion (either a proton or a neutron) occupying single-particle orbits with angular momenta \( j = j_1, j_2, \ldots \). The components of the fermion angular momenta span the \( \Omega \)-dimensional space of the group \( U(\Omega) \) with \( \Omega = \sum_j (2j + 1) \). We introduce, in addition to the boson operators for the collective degrees of freedom, fermion creation \( a_i^\dagger \) and annihilation \( a_i \) operators for the extra nucleon. The fermion operators satisfy anti-commutation relations

\[
\{ a_i, a_j^\dagger \} = \delta_{ij} , \quad \{ a_i^\dagger, a_j^\dagger \} = \{ a_i, a_j \} = 0 . \tag{4}
\]

The bilinear products

\[
A_{ij} = a_i^\dagger a_j , \tag{5}
\]

generate the algebra of \( U(m) \), the unitary group in \( m \) dimensions

\[
[A_{ij}, A_{kl}] = A_{il} \delta_{jk} - A_{kj} \delta_{il} . \tag{6}
\]

By construction, the fermion operators commute with the boson operators.

\[
[B_{ij}, A_{kl}] = 0 . \tag{7}
\]

The operators \( B_{ij} \) and \( A_{ij} \) generate the Lie algebra of the symmetry group \( G = U^B(6) \otimes U^F(\Omega) \) of the IBFM. The dynamical symmetries that can arise in the IBFM are known under the name of dynamical boson-fermion symmetries for odd-mass nuclei.

Boson-fermion symmetries can further be extended by introducing the concept of supersymmetries [4], in which states in both even-even and odd-even nuclei are treated in a single framework. So far, I have discussed the symmetry properties of a mixed system of boson and fermion degrees of freedom for a fixed number of bosons \( N \) and one fermion \( M = 1 \). The operators \( B_{ij} \) and \( A_{ij} \) can only change bosons into bosons and fermions into fermions. In addition to \( B_{ij} \) and \( A_{ij} \), one can introduce operators that change a boson into a fermion and vice versa, but conserve the total number of bosons and fermions

\[
F_{ij} = b_i^\dagger a_j , \quad G_{ij} = a_i^\dagger b_j . \tag{8}
\]

The enlarged set of operators \( B_{ij}, A_{ij}, F_{ij} \) and \( G_{ij} \) forms a closed algebra which consists of both commutation and anticommutation relations

\[
[B_{ij}, B_{kl}] = B_{il} \delta_{jk} - B_{kj} \delta_{il} , \quad [B_{ij}, A_{kl}] = 0 , \quad [B_{ij}, F_{kl}] = F_{il} \delta_{jk} , \quad [B_{ij}, G_{kl}] = -G_{kj} \delta_{il} ,
\]

\[
[A_{ij}, A_{kl}] = A_{il} \delta_{jk} - A_{kj} \delta_{il} , \quad [A_{ij}, F_{kl}] = -F_{kj} \delta_{il} , \quad [A_{ij}, G_{kl}] = G_{il} \delta_{jk} ,
\]

\[
\{ F_{ij}, F_{kl} \} = 0 , \quad \{ F_{ij}, G_{kl} \} = B_{il} \delta_{jk} + A_{kj} \delta_{il} , \quad \{ G_{ij}, G_{kl} \} = 0 . \tag{9}
\]
This algebra can be identified with that of the graded Lie group $G = U(6/\Omega)$. It provides an elegant scheme in which the IBM and IBFM can be unified into a single framework \[4\]

$$G = U(6/\Omega) \supset U^B(6) \otimes U^F(\Omega) \ .$$ \hspace{1cm} (10)

In this supersymmetric framework, even-even and odd-mass nuclei form the members of a supermultiplet which is characterized by $[N]$ where $N = N + M$, i.e. the total number of bosons and fermions. Thus, supersymmetry distinguishes itself from other symmetries in that it includes, in addition to transformations among fermions and among bosons, also transformations that change a boson into a fermion and vice versa (see Table 1).

The Hamiltonian of nuclear supersymmetry is written in terms of the generators of the graded Lie algebra of $U(6/\Omega)$ of Eq. (9). A dynamical supersymmetry arises when the Hamiltonian can be expressed in terms of Casimir operators of a chain of subgroups of $U(6/\Omega)$. Dynamical nuclear supersymmetries correspond to very special cases of the Hamiltonian which may not be applicable to all regions of the nuclear chart, but nevertheless several nuclei in the Os-Ir-Pt-Au region have been found to provide experimental evidence for the approximate occurrence of supersymmetries in nuclei.

### 2.1. U(6/4) supersymmetry

The Os-Ir-Pt-Au mass region provides ample experimental evidence for the occurrence of dynamical (super)symmetries in nuclei. The even-even nuclei $^{194,196}\text{Pt}$ are the standard examples of the $SO(6)$ limit of the IBM [13] and the odd proton, in first approximation, occupies the single-particle level $2d_{3/2}$. In this special case, the boson and fermion groups, $SO^B(6)$ and $SU^F(4)$, are locally isomorphic and can be combined into the spinor group $Spin(6)$. The odd-proton nuclei $^{191,193}\text{Ir}$ and $^{193,195}\text{Au}$ were suggested as examples of the $Spin(6)$ limit of the IBFM [1, 14]. The appropriate extension to a supersymmetry is by means of the graded Lie group $U(6/4)$

$$U(6/4) \supset U^B(6) \otimes U^F(4) \supset SO^B(6) \otimes SU^F(4) \supset Spin(6) \supset Spin(5) \supset Spin(3) \ .$$ \hspace{1cm} (11)

A dynamical supersymmetry arises when the Hamiltonian is expressed in terms of the Casimir invariants of the subgroups of $U(6/4)$

$$H = a C_{2 Spin(6)} + b C_{2 Spin(5)} + c C_{2 Spin(3)} \ .$$ \hspace{1cm} (12)

The energy spectrum is given by the expectation value of the Casimir invariants of the spinor groups

$$E = a \left[ \sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2 \right] + b \left[ \tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1) \right] + c J(J + 1) \ ,$$ \hspace{1cm} (13)

which simultaneously describes the spectra of both the even-even and the odd-proton nucleus with a single set of parameters $a$, $b$ and $c$. The pairs of nuclei $^{190}\text{Os} - ^{191}\text{Ir}$, $^{192}\text{Os} - ^{193}\text{Ir}$, $^{192}\text{Pt} - ^{193}\text{Au}$ and $^{194}\text{Pt} - ^{195}\text{Au}$ were analyzed as examples of a $U(6/4)$ supersymmetry with $N = 9$, 8, 8 and 7, respectively \[4\].

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**Table 1.** Overview of algebraic models

| Model | Generators | Invariant | Symmetry |
|-------|------------|-----------|-----------|
| IBM   | $b_i^a b_j$ | $N$       | $U(6)$    |
| IBFM  | $b_i^a b_j$, $a_\alpha^a a_\nu$ | $N$, $M$ | $U(6) \otimes U(\Omega)$ |
| SUSY  | $b_i^a b_j$, $a_\alpha^a a_\nu$, $b_i^a a_\mu$, $a_\mu^a b_i$ | $N$       | $U(6/\Omega)$ |

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2.2. $U(6/12)$ supersymmetry

Another example of a dynamical supersymmetry in this mass region is that of the Pt isotopes. The even-even nuclei are well described by the $SO(6)$ limit of the IBM and the odd neutron mainly occupies the negative parity orbits $3p_{1/2}$, $3p_{3/2}$ and $3f_{5/2}$ which can be decomposed into a pseudo-orbital part $k = 0, 2$ and a pseudo-spin part $s = 1/2$. Since the pseudo-orbital part of the odd neutron has the same values of angular momentum as the bosons, the boson and fermion chains can be combined into a single one. The corresponding graded Lie group is $U(6/12)$ which can be reduced into

$$
U(6/12) \supset U^B(6) \otimes U^F(12) \supset U^B(6) \otimes U^F(6) \otimes U^F(2)
$$

$$
\supset U^B(6) \otimes U^F(2) \supset SO^{BF}(6) \otimes U^F(2)
$$

$$
\supset SO^{BF}(5) \otimes U^F(2) \supset SO^{BF}(3) \otimes SU^F(2)
$$

$$
\supset Spin(3) .
$$

In this case, the Hamiltonian

$$
H = \alpha C_{2U^{BF}(6)} + \beta C_{2SO^{BF}(6)} + \gamma C_{2SO^{BF}(5)} + \delta C_{2SO^{BF}(3)} + \epsilon C_{2Spin(3)} ,
$$

simultaneously describes the excitation spectra of both the even-even and the odd-neutron nucleus with a single set of parameters $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$. The energy spectrum is given by the eigenvalues of the Casimir operators

$$
E = \alpha [N_1(N_1 + 5) + N_2(N_2 + 3)] + \beta \left[ \sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2 \right]
$$

$$
+ \gamma \left[ \tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1) \right] + \delta L(L + 1) + \epsilon J(J + 1) .
$$

The pair of nuclei $^{194}$Pt - $^{195}$Pt was studied as an example of a $U(6/12)$ supersymmetry with $\mathcal{N} = 7$ [5, 15, 16, 17].

3. Supersymmetric quartets of nuclei

The concept of nuclear SUSY was extended in 1985 to include the neutron-proton degree of freedom [6]. In this case, a supermultiplet consists of an even-even, an odd-proton, an odd-neutron and an odd-odd nucleus. The first experimental evidence of a supersymmetric quartet was found in the $A \sim 190$ mass region in the $^{194,195}$Pt and $^{195,196}$Au nuclei as an example of the $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry [7, 8, 9, 18, 19], in which the odd neutron is allowed to occupy the $3p_{1/2}$, $3p_{3/2}$ and $2f_{5/2}$ orbits of the 82-126 shell, and the odd proton the $2d_{3/2}$ orbit of the 50-82 shell. The interpretation of these four nuclei as members of a supersymmetric quartet made it possible to predict [6] the properties of $^{196}$Au almost 15 years before they were measured experimentally [7, 8].

The relevant subgroup chain of the $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry is a combination of the group reductions in Eqs. (11) and (14) (see Table 2). If the Hamiltonian is expressed in terms of Casimir invariants of the subgroups appearing in the reduction shown in Table 2, a dynamical supersymmetry arises. For example, the Hamiltonian

$$
H = A C_{2U^{BF}(6)} + B C_{2SO^{BF}(6)} + B' C_{2Spin(6)} + C C_{2Spin(5)} + E C_{2Spin(3)} + F C_{2SU(2)} ,
$$

describes the excitation spectra of a quartet of nuclei characterized by $N_\nu$ and $N_\pi$ [6]. The quartet consists of an even-even nucleus with $N_\nu = N_\pi$, $M_\nu = 0$ and $M_\pi = 0$, an odd-proton nucleus with $N_\nu = N_\pi$, $M_\nu = 0$ and $N_\pi = N_\pi - 1$, $M_\pi = 1$, an odd-neutron nucleus with $N_\nu = N_\nu - 1$, $M_\nu = 1$ and $N_\pi = N_\pi$, $M_\pi = 0$, and an odd-odd nucleus with $N_\nu = N_\nu - 1$, $M_\nu = 1$ and $N_\pi = N_\pi - 1$, $M_\pi = 1$. Here, all terms that only contribute to binding energies were...
Table 2. Group lattice for the $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry

| $U(6/12)_\nu$ | $\otimes$ | $U(6/4)_\pi$ |
|----------------|-------------|----------------|
| $[N_\nu]$     |             | $[N_\pi]$     |
| $\downarrow$  |             | $\downarrow$  |
| $U_{F\nu}(12)$| $\otimes$   | $U_{B\nu}(6)$ |
| $[1^M_{\nu}]$ |             | $[N_\nu]$     |
| $\downarrow$  |             | $\downarrow$  |
| $U_{F\nu}(2)$ | $\otimes$   | $U_{F\nu}(6)$ |
| $[n_1, n_2]$  |             | $[N - i, i]$  |
| $\downarrow$  |             | $\downarrow$  |
| $U_{B\nu}(6)$ |             |               |
| $[N_1, \ldots, N_6]$ | | $\downarrow$ |
| $\downarrow$  |             |               |
| $SO_{B\nu}(6)$ | $\otimes$  | $SU_{F\nu}(4)$ |
| $(\Sigma_1, \Sigma_2, \Sigma_3)$ | |             |
| $\downarrow$  |             |               |
| $Spin(6)$     | $\otimes$   | $Spin(3)$     |
| $(\sigma_1, \sigma_2, \sigma_3)$ | | $J$           |
| $\downarrow$  |             | $\downarrow$  |
| $Spin(5)$     |             |               |
| $(\tau_1, \tau_2)$ | | $J$           |
| $\downarrow$  |             | $\downarrow$  |
| $SU_{F\nu}(2)$ |             |               |
| $S$            | $\otimes$   |               |
| $\downarrow$  |             |               |
| $SU(2)$       |             |               |
| $L$            |             |               |

neglected. The energy spectra of the four nuclei belonging to the supersymmetric quartet are described simultaneously by a single energy formula in terms of the eigenvalues of the Casimir operators

$$E = A [N_1(N_1 + 5) + N_2(N_2 + 3) + N_1(N_1 + 1)]$$
$$+ B [\Sigma_1(\Sigma_1 + 4) + \Sigma_2(\Sigma_2 + 2) + \Sigma_3^2]$$
$$+ B' [\sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2]$$
$$+ C [\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1)]$$
$$+ D L(L + 1) + E J(J + 1).$$

(18)

The coefficients $A$, $B$, $B'$, $C$, $D$, and $E$ can be determined in a fit of the experimental excitation energies.
Table 3. The number of bosons and fermions in the supersymmetric quartets of Pt-Au and Os-Ir nuclei which are characterized by \( N_\pi = 2, N_\nu = 5 \) and \( N_\pi = 3, N_\nu = 5 \), respectively.

| Nucleus | \( N_\pi \) | \( M_\pi \) | \( N_\nu \) | \( M_\nu \) | Nucleus | \( N_\pi \) | \( M_\pi \) | \( N_\nu \) | \( M_\nu \) |
|---------|-------------|-------------|-------------|-------------|---------|-------------|-------------|-------------|-------------|
| \(^{194}\)Pt\(_{116}\) | 2 | 0 | 5 | 0 | \(^{192}\)Os\(_{116}\) | 3 | 0 | 5 | 0 |
| \(^{195}\)Pt\(_{117}\) | 2 | 0 | 4 | 1 | \(^{193}\)Os\(_{117}\) | 3 | 0 | 4 | 1 |
| \(^{195}\)Au\(_{116}\) | 1 | 1 | 5 | 0 | \(^{193}\)Ir\(_{116}\) | 2 | 1 | 5 | 0 |
| \(^{196}\)Au\(_{117}\) | 1 | 1 | 4 | 1 | \(^{194}\)Ir\(_{117}\) | 2 | 1 | 4 | 1 |

The number of bosons and fermions are related to the number of valence nucleons, i.e. the number of protons and neutrons outside the closed shells. For nuclei in the \( A \sim 190 \) mass region, the relevant closed shells are \( Z = 82 \) for protons and \( N = 126 \) for neutrons. The number of bosons and fermions for the \(^{194,195}\)Pt and \(^{195,196}\)Au quartet of nuclei are obtained as follows. For the even-even nucleus \(^{194}\)Pt\(_{116}\) the number of bosons are \( N_\pi = (82 - 78)/2 = 2 \) and \( N_\nu = (126 - 116)/2 = 5 \). There are no unpaired nucleons \( M_\pi = M_\nu = 0 \). For the odd-neutron nucleus \(^{195}\)Pt\(_{117}\) there are 9 valence neutrons which leads to \( N_\nu = 4 \) neutron bosons and \( M_\nu = 1 \) unpaired neutron. The \(^{79}\)Au isotopes have 3 valence protons which are divided over \( N_\pi = 1 \) proton boson and \( M_\pi = 1 \) unpaired proton. This supersymmetric quartet of nuclei is characterized by \( N_\pi = N_\pi + M_\pi = 2 \) and \( N_\nu = N_\nu + M_\nu = 5 \). The number of bosons and fermions are summarized in Table 3.

The odd-odd nucleus \(^{196}\)Au, together with the odd-neutron nucleus \(^{195}\)Pt, the odd-proton nucleus \(^{195}\)Au and the even-even nucleus \(^{194}\)Pt, have been verified experimentally using state-of-the-art techniques [7, 8, 9] to closely fulfill the rules that define a supersymmetric quartet \([6,10]\). Figure 1 shows a comparison of the experimental spectrum of the odd-odd nucleus \(^{196}\)Au with the theoretical spectrum for the \( U(6/12) \otimes U(6/4) \) supersymmetry calculated with the energy formula of Eq. (18). The values of the parameters are given in Table 4.

The \( A \sim 190 \) mass region is a particularly complex one, displaying transitional behavior such as prolate-oblate deformed shapes, \( \gamma \)-unstability, triaxial deformation and/or coexistence.
of different configurations which present a daunting challenge to nuclear structure models. Nevertheless, despite its complexity, this mass region has been a rich source of empirical evidence for the existence of dynamical symmetries in nuclei both for even-even, odd-proton, odd-neutron and odd-odd nuclei, as well as supersymmetric pairs [1, 5] and quartets of nuclei [6, 7].

Recently, the structure of the odd-odd nucleus $^{194}$Ir was investigated by a series of transfer and neutron capture reactions [21]. The odd-odd nucleus $^{194}$Ir differs from $^{196}$Au by two protons, the number of neutrons being the same. The latter is crucial, since the dominant interaction between the odd neutron and the core nucleus is of quadrupole type, which arises from a more general interaction in the IBFM for very special values of the occupation probabilities of the $3p_{1/2}$, $3p_{3/2}$ and $2f_{5/2}$ orbits, *i.e.* to the location of the Fermi surface for the neutron orbits [22]. This situation is satisfied to a good approximation by the $^{195}$Pt and $^{196}$Au nuclei which both have the 117 neutrons. The same is expected to hold for the isotones $^{193}$Os and $^{194}$Ir. For this reason, it is reasonable to expect the odd-odd nucleus $^{194}$Ir to provide another example of

![Comparison between the theoretical and experimental spectrum of $^{194}$Ir.](image)

**Table 4.** Values of the parameters in keV

|       | A     | B   | B'   | C     | D   | E   | Ref. |
|-------|-------|-----|------|-------|-----|-----|------|
| Pt-Au | 52.5  | 8.7 | -53.9| 48.8  | 8.8 | 4.5 | [8]  |
| Os-Ir | 41.0  | -6.0| -29.0| 38.0  | 6.3 | 4.5 | [20] |
Figure 3. Prediction of the spectrum of $^{193}$Os for the $U_\nu(6/12) \otimes U_\pi(6/4)$ supersymmetry.

...a dynamical symmetry in odd-odd nuclei.

The new data from the polarized $(\vec{d}, \alpha)$ transfer reaction provided crucial new information about and insight into the structure of the spectrum of $^{194}$Ir which led to significant changes in the assignment of levels [21, 20] as compared to previous work [23]. Figure 2 shows the negative parity levels of $^{194}$Ir in comparison with the theoretical spectrum in which it is assumed that these levels originate from the $\nu 3p_{1/2}$, $\nu 3p_{3/2}$, $\nu 2f_{5/2} \otimes \pi 2d_{3/2}$ configuration. The theoretical energy spectrum is calculated using the energy formula of Eq. (18) with the parameter values of Table 4. This parameter set is a lot closer to the parameter values used for $^{196}$Au [8] than the ones in [23], indicating systematics in this zone of the nuclear chart. Given the complex nature of the spectrum of heavy odd-odd nuclei, the agreement is remarkable. There is an almost one-to-one correlation between the experimental and theoretical level schemes [21].

The successful description of the odd-odd nucleus $^{194}$Ir opens the possibility of identifying a second quartet of nuclei in the $A \sim 190$ mass region with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry. The new quartet consists of the nuclei $^{192,193}$Os and $^{193,194}$Ir and is characterized by $N_\pi = 3$ and $N_\nu = 5$ (see Table 3). Whereas the $^{192}$Os and $^{193,194}$Ir nuclei are well-known experimentally, the available data for $^{193}$Os is rather scarce. Figure 3 shows the predicted spectrum for $^{193}$Os obtained from Eq. (18) using the same parameter set as for $^{194}$Ir [21]. The ground state of $^{193}$Os has spin and parity $J^P = \frac{3}{2}^-$, which seems to imply that the second band with labels $[7,1]$, $(7,1,0)$ is the ground state band, rather than $[8,0]$, $(8,0,0)$. This ordering of bands is supported by preliminary results from the one-neutron transfer reaction $^{192}$Os($\vec{d},p$)$^{193}$Os [24].

An analysis of the energy spectra of the four nuclei that make up the quartet shows that the parameter set obtained in 1981 for the pair $^{192}$Os-$^{193}$Ir [4] is very close to that of $^{194}$Ir [21], which indicates that the nuclei $^{192,193}$Os and $^{193,194}$Ir may be interpreted in terms of a quartet of nuclei with $U(6/12)_\nu \otimes U(6/4)_\pi$ supersymmetry.

4. Correlations

The nuclei belonging to a supersymmetric quartet are described by a single Hamiltonian, and hence the wave functions, transition and transfer rates are strongly correlated. As an example of these correlations, I consider transfer reactions between the $^{194,195}$Pt and $^{192,193}$Os nuclei. The Pt...
and Os nuclei are connected by one-neutron transfer reactions within the same supersymmetric quartet $^{194}$Pt $\leftrightarrow$ $^{195}$Pt and $^{192}$Os $\leftrightarrow$ $^{193}$Os, whereas the transitions between the Pt and Os nuclei involve the transfer of a proton pair between different quartets $^{194}$Pt $\leftrightarrow$ $^{192}$Os and $^{195}$Pt $\leftrightarrow$ $^{193}$Os.

4.1. Generalized $F$-spin

The correlations between different transfer reactions can be derived in an elegant and explicit way by a generalization of the concept of $F$-spin which was introduced in the neutron-proton IBM [25] in order to distinguish between proton and neutron bosons.

The eigenstates of the $U(6/12)_\rho \otimes U(6/4)_\pi$ supersymmetry are characterized by the irreducible representations $[N_1, N_2, N_3]$ of $U^{BF_2}(6)$ which arise from the coupling of three different $U(6)$ representations, $[N_\nu]$ for the neutron bosons, $[N_\rho]$ for the pseudo-orbital angular momentum of the odd neutron ($N_\rho = 0$ for the even-even and odd-proton nucleus of the quartet, and $N_\rho = 1$ for the odd-neutron and the odd-odd nucleus). In analogy with the three quark flavors in the quark model ($u, d$ and $s$), also in this case there are three different types of identical objects ($\pi, \nu$ and $\rho$), which can be distinguished by $F$-spin and hypercharge $Y$.

The two kinds of bosons form an $F$-spin doublet, $F = \frac{1}{2}$, with charge states $F_z = \frac{1}{2}$ for protons ($\pi$) and $F_z = -\frac{1}{2}$ for neutrons ($\nu$) [25]. In addition, the bosons carry hypercharge $Y = \frac{1}{2}$. The pseudo-orbital part ($\rho$) of the angular momentum of the odd neutron has $F = F_z = 0$ and $Y = -\frac{3}{2}$ [26].

Group theoretically, the generalized $F$-spin is defined by the reduction

$$U(18) \supset U(6) \otimes U(3)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$[N] \quad [N_1, N_2, N_3] \quad [N_1, N_2, N_3]$$

Here $U(6)$ is to be identified with the $U^{BF_2}(6)$ of the group reduction of Table 2 which is the result of first coupling the bosons at the level of $U(6)$, followed by coupling the orbital part

$$[[N_\nu], [N_\pi]; [N_\nu + N_\pi - i, i], [N_\rho]; [N_1, N_2, N_3]] .$$

This sequence of $U(6)$ couplings can be described in a completely equivalent way by the three-dimensional index group $U(3)$ of Eq. (19) which can be reduced to

$$U(3) \supset SU(3) \supset [SU(2) \supset SO(2)] \otimes U(1)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[N_1, N_2, N_3] \quad (\lambda, \mu) \quad F \quad F_z \quad Y$$

The relation between the two sets of quantum numbers is given by

$$(\lambda, \mu) = (N_1 - N_2, N_2 - N_3) ,$$

$$F = \frac{1}{2} (N_\pi + N_\nu - 2i) ,$$

$$F_z = \frac{1}{2} (N_\pi - N_\nu) ,$$

$$Y = \frac{1}{3} (N_\pi + N_\nu - 2N_\rho) .$$

As a result, matrix elements between states with the same quantum numbers but different $U(6)$ couplings are then related by $SU(3)$ isoscalar factors, and hence correlations between different transfer reactions can be derived in terms of these isoscalar factors by means of the concept of generalized $F$-spin. Some $SU(3)$ isoscalar factors relevant for nuclear supersymmetry are derived in Appendix A.
4.2. One-neutron transfer
In a study of the $^{194}\text{Pt} \rightarrow ^{195}\text{Pt}$ stripping reaction it was found [15] that one-neutron $j = 3/2, 5/2$ transfer reactions can be described by the operator

$$P^{(j)\dagger} = \frac{\alpha_j}{\sqrt{2}} \left[ (\hat{s}_\nu \times \hat{a}^{\dagger}_{\nu,j})^{(j)} - (\hat{a}_\nu \times \hat{a}^{\dagger}_{\nu,\frac{3}{2}})^{(j)} \right].$$

(23)

It is convenient to take ratios of intensities, since they do not depend on the value of the coefficient $\alpha_j$ and hence provide a direct test of the wave functions. For the stripping reaction $^{194}\text{Pt} \rightarrow ^{195}\text{Pt}$ (ee $\rightarrow$ on) the ratio of intensities for the excitation of the $(\tau_1, \tau_2) = (1,0)$, $L = 2$ doublet with $J = 3/2, 5/2$ belonging to the first excited band with $[N', -1, 1]$, $[N', -1, 1, 0]$ relative to that of the ground state band $[N]$, $(N', 0, 0)$ is given by [15, 20]

$$R(\text{ee} \rightarrow \text{on}) = \frac{(N - 1)(N + 1)(N + 4)}{2(N + 2)},$$

(24)

which gives $R = 29.3$ for $^{194}\text{Pt} \rightarrow ^{195}\text{Pt}$ ($N = N_\nu + N_\pi = 7$), to be compared to the experimental value of 19.0 for $j = 5/2$, and $R = 37.8$ for $^{192}\text{Os} \rightarrow ^{193}\text{Os}$ ($N = N_\nu + N_\pi = 8$). The equivalent ratio for the inverse pick-up reaction is given by

$$R(\text{on} \rightarrow \text{ee}) = R(\text{ee} \rightarrow \text{on}) \frac{N_\pi}{(N_\nu + N_\pi - 1)N_\nu},$$

(25)

which gives $R = 1.96$ for $^{195}\text{Pt} \rightarrow ^{194}\text{Pt}$ ($N_\pi = 2$ and $N_\nu = 5$) and $R = 3.24$ for $^{193}\text{Os} \rightarrow ^{192}\text{Os}$ ($N_\pi = 3$ and $N_\nu = 5$). This means that the mixed symmetry $L = 2$ state is predicted to be excited more strongly than the first excited $L = 2$ state.

This correlation between pick-up and stripping reactions has been derived in a general way only using the symmetry relations that exist between the wave functions of the even-even and odd-neutron nuclei of the supersymmetric quartet. The factor in the right-hand side of Eq. (25) can be expressed in terms of $SU(3)$ isoscalar factors. It is important to emphasize, that Eqs. (24) and (25) are parameter-independent predictions which are a direct consequence of nuclear SUSY and which can be tested experimentally.

4.3. Two-proton transfer
The two supersymmetric quartets in the mass $A \sim 190$ region differ by two protons. In principle, the connection between the two quartets can be studied by two-proton transfer reactions. In the IBM, two-proton transfer operator is, in first order, given by

$$P^\dagger_\pi = \alpha \hat{s}^\dagger_\pi, \quad P_\pi = \alpha \hat{s}_\pi.$$

(26)

Whereas the operator $s_\pi$ only excites the ground state of the final nucleus, $s^\dagger_\pi$ can also populate excited states.

Table 5 shows the results for ratios of spectroscopic strengths between even-even nuclei. The selection rules of the operator $s^\dagger_\pi$ allow the excitation of states with with $(\tau_1, \tau_2) = (0,0)$ and $L = 0$ belonging to the ground band ($\Sigma_1, \Sigma_2, \Sigma_3$) = $([N' + 1, 0, 0]$ and excited bands with $[N - 1, 0, 0])$. The corresponding ratios for the odd-neutron nuclei are strongly correlated to those of the even-even nuclei (see Tables 5 and 6)

$$S_1 = R_1,$n
$$S_2 = R_2,$n
$$S_{3a} = R_3 \frac{N_\pi + 1}{N_\nu N'},$$n
$$S_{3b} = R_3 \frac{(N_\nu - 1)(N + 1)}{N_\nu N'}.$$n

(27)
Moreover, the summed strength for even-even nuclei \( R_{\nu}(N_\pi, N_\nu \rightarrow N_\pi + 1, N_\nu) \) to final states with \((\tau_1, \tau_2) = (0, 0)\). \( N = N_\nu + N_\pi \) refers to the initial nucleus \((N = 7 \text{ for } ^{194}\text{Pt} \rightarrow ^{192}\text{Os})\).

| \( n \) | \([N_1, N_2]\) | \((\Sigma_1, \Sigma_2, \Sigma_3)\) | \( R_n \) |
|-------|----------------|----------------|--------|
| 1     | \([N + 1, 0]\) | \((N + 1, 0, 0)\) | 1      |
| 2     | \([N + 1, 0]\) | \((N - 1, 0, 0)\) | \( \frac{N(N+1)}{(N+1)^2(N+4)} \) |
| 3     | \([N, 1]\)    | \((N - 1, 0, 0)\) | \( \frac{N_\nu (N-1)(N+2)}{(N_\nu+1)(N+1)^2(N+4)} \) |

Table 6. Ratios of spectroscopic strengths for two-proton transfer reactions between odd-neutron nuclei \( S_{\nu
\nu}(N_\pi, N_\nu \rightarrow N_\pi + 1, N_\nu) \) to final states with \((\tau_1, \tau_2) = (0, 0)\). \( N = N_\nu + N_\pi \) refers to the initial nucleus \((N = 7 \text{ for } ^{195}\text{Pt} \rightarrow ^{193}\text{Os})\).

| \( n \) | \([N - i, i]\) | \([N_1, N_2]\) | \((\Sigma_1, \Sigma_2, \Sigma_3)\) | \( S_n \) |
|-------|----------------|----------------|----------------|--------|
| 1     | \([N, 0]\)    | \([N + 1, 0]\) | \((N + 1, 0, 0)\) | 1      |
| 2     | \([N, 0]\)    | \([N + 1, 0]\) | \((N - 1, 0, 0)\) | \( R_2 \) |
| 3a    | \([N, 0]\)    | \([N - 1, 0]\) | \((N - 1, 0, 0)\) | \( R_3 \frac{N_\nu + 1}{N_\nu N} \) |
| 3b    | \([N - 1, 1]\) | \([N, 1]\)    | \((N - 1, 0, 0)\) | \( R_3 \frac{N_\nu (N-1)(N+1)}{N_\nu(N+1)^2(N+4)} \) |

The coefficients in the right-hand side correspond to the ratio of two \( SU(3) \) isoscalar factors. Moreover, the summed strength for even-even nuclei is the same as that for odd-neutron nuclei \( \sum_n R_n = \sum_n S_n \).

5. Summary and conclusions

In this contribution, I discussed the concept of supersymmetry as used in nuclear physics. In this application, a supermultiplet consists of a pair or a quartet of nuclei whose wave functions are related by supersymmetry. This has important consequences for energies, transition and transfer strengths. 

It was suggested that there exists a second quartet of nuclei in the \( A \sim 190 \) mass region with \( U(6/12)_c \otimes U(6/4)_\pi \) supersymmetry, consisting of the \( ^{192,193}\text{Os} \) and \( ^{193,194}\text{Ir} \) nuclei. The analysis is based largely on new experimental information on \( ^{194}\text{Ir} \). Given the complexity of the \( A \sim 190 \) mass region, the simple yet detailed description of \( ^{194}\text{Ir} \) in a supersymmetry scheme is remarkable.

Nuclear supersymmetry establishes precise links among the spectroscopic properties of different nuclei. This relation has been used to predict the energies of \( ^{193}\text{Os} \). In order to establish the existence of a second supersymmetric quartet of nuclei in the \( A \sim 190 \) mass region, it is crucial that the nucleus \( ^{193}\text{Os} \) be studied in more detail experimentally. The predictions for correlations between one-neutron transfer reactions in Pt and Os can be tested experimentally by combining for example \( (d,^3p) \) stripping and \( (p,d) \) pick-up reactions.

Since the wave functions of the members of a supermultiplet are connected by symmetry, there exists a high degree of correlations between different one- and two-nucleon transfer reactions not only between nuclei belonging to the same quartet, but also for nuclei from different multiplets. In order to discuss these correlations induced by nuclear supersymmetry, the concept of generalized F-spin was developed. As a consequence, the correlations can be expressed in an elegant and explicit way in terms of \( SU(3) \) isoscalar factors. It was shown that a special class of
these isoscalar factors can be expressed in terms of the usual $SU(2)$ Clebsch-Gordan coefficients. As an example, the correlations between one-neutron transfer reactions and two-proton transfer reactions were studied. In the former case, nuclear supersymmetry predicts that the $L = 2$ mixed symmetry states in the even-even nuclei $^{194}$Pt and $^{192}$Os are excited much stronger (two to three times as strong) than the first excited $L = 2$ state.

It is important to note, that the technique of the generalized $F$-spin is valid for any system whose wave functions are described in terms of couplings of three different symmetric representations of $U(6)$, as for example in nuclear supersymmetry (this contribution), in an algebraic description of the neutron skin in very neutron-rich nuclei [27] and in an isospin-invariant extension of the IBM for light nuclei called IBM-3 [28].

**Appendix A. SU(3) isoscalar factors**

The isoscalar factors can be derived in the standard way from the recursion relations for the $SU(3)$ step operators combined with the phase convention of De Swart [29] and Racah’s factorization lemma

$$
\langle \begin{array}{c}
(\lambda_1, \mu_1) \\
F_1, F_{1z}, Y_1
\end{array} | \begin{array}{c}
(\lambda_2, \mu_2) \\
F_2, F_{2z}, Y_2
\end{array} \rangle 
= \langle \begin{array}{c}
(\lambda_1, \mu_2) \\
F_1, Y_1
\end{array} | \begin{array}{c}
(\lambda_2, \mu_2) \\
F_2, Y_2
\end{array} \rangle \langle \begin{array}{c}
(\lambda, \mu) \\
F, F_z, Y
\end{array} | \begin{array}{c}
(\lambda_1, \mu_2) \\
F_1, F_{1z}, F_{2z}
\end{array} \rangle .
$$

(A.1)

Inspection of the wave functions in nuclear supersymmetry shows that the ground state wave functions of the nuclei that belong to a supersymmetric quartet of nuclei are characterized by the symmetric representation $[N_1, N_2, N_3] = [N, 0, 0]$ of $U(6)$ or, equivalently, by the $SU(3)$ labels $(\lambda, \mu) = (N, 0)$. For the examples discussed in this contribution, the relevant $SU(3)$ couplings are

$$
(N, 0) \otimes (1, 0) = (N + 1, 0) \oplus (N - 1, 1) ,
$$

$$
(N, 0) \otimes (0, 1) = (N, 1) \oplus (N - 1, 0) .
$$

(A.2)

In the special case of the first coupling, the $SU(3)$ isoscalar factors can be derived in terms of $SU(2)$ Clebsch-Gordan coefficients. This can be seen by employing two different realizations of the reduction $SU(3) \supset SU(2) \otimes U(1)$. The one used in Eq. (21) gives rise to the basis $|(\lambda, \mu), F, F_z, Y_F\rangle$ (here I use $Y_F = Y$ to indicate that this is the hypercharge operator associated

![Figure A1](image-url)

**Figure A1.** Weight diagram of the $SU(3)$ multiplet $(\lambda, 0)$
with $F$-spin). Another basis is provided by $V$-spin and the corresponding hypercharge $Y_V$ which leads to $| (\lambda, \mu), V, Z, Y_V \rangle$. The generators of the two realizations of $SU(2) \otimes U(1)$ are given by

\[
\begin{align*}
F_+ &= \pi^+ \nu \\
F_- &= \nu^+ \pi \\
F_z &= (\pi^+ \pi - \nu^+ \nu)/2 \\
Y_F &= (\pi^+ \pi + \nu^+ \nu - 2\rho^+ \rho)/3
\end{align*}
\]

(A.3)

The weights in the two realizations are related to one another by

\[
\begin{align*}
V_z &= \frac{3}{4} Y_F + \frac{1}{2} F_z \\
Y_V &= -\frac{1}{2} V_F + F_z
\end{align*}
\]

(A.4)

For the symmetric representation $(\lambda, \mu) = (N, 0)$, the relation between the two sets of basis states is one-to-one

\[
\begin{align*}
|N, 0\rangle_F &= \left| \frac{N - k}{2}, \frac{N - k}{2} - l, \frac{N}{3} - k \right\rangle_F \\
|N, 0\rangle_V &= \left| \frac{N - l}{2}, \frac{N - l}{2} - k, \frac{N}{3} - l \right\rangle_V
\end{align*}
\]

(A.5)

with $k = 0, 1, \ldots, N$ and $l = 0, 1, \ldots, N - k$.

Consider now the coupling of $(N, 0)$ and $(1, 0)$ to $(N + 1, 0)$. The state $|A\rangle$ in Figure A1 is characterized by

\[
\begin{align*}
|\psi_A\rangle &= \left| \frac{N + 1}{2}, \frac{N + 1}{2}, \frac{N + 1}{3} \right\rangle_F \\
&= \left| \frac{N}{2}, \frac{N + 1}{3} \right\rangle_V |(1, 0), \frac{1}{2}, \frac{1}{2}, \frac{3}{3} \rangle_F
\end{align*}
\]

(A.6)

i.e. it has the same labels in either of the bases. Next, the state $|\psi_B\rangle$ in Figure A1 can be obtained by applying the operator $V_-$ $k$ times. In the $V$-basis, the expansion coefficients are the usual $SU(2)$ Clebsch-Gordan coefficients

\[
\begin{align*}
|\psi_B\rangle &= \left| \frac{N + 1}{2}, \frac{N + 1}{2}, \frac{N + 1}{3} - k, \frac{N + 1}{3} \right\rangle_V \\
&= \left| \frac{N}{2}, \frac{N - k}{2}, \frac{1}{2}, \frac{1}{2}, \frac{N}{2}, \frac{N + 1}{2} - k \right\rangle \\
&+ \left| \frac{N}{2}, \frac{N}{2} + 1 - k, \frac{1}{2}, \frac{1}{2}, \frac{N + 1}{2}, \frac{N + 1}{2} - k \right\rangle \\
&= \left| \frac{N}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{3} \right\rangle_V |(1, 0), \frac{1}{2}, \frac{1}{2}, \frac{3}{3} \rangle_V
\end{align*}
\]

(A.7)

Finally, if we rewrite this equation in the $F$-basis, it is easy to see that its expansion coefficients (or $SU(3)$ isoscalar factors) in the $F$-basis can be expressed in terms of $SU(2)$ Clebsch-Gordan coefficients as

\[
\begin{align*}
\left< \frac{N - k}{2}, \frac{N}{3} - k \right| (1, 0) &\frac{1}{2}, \frac{1}{2}, \frac{3}{3} \left| \frac{N + 1 - k}{2}, \frac{N + 1}{3} - k \right>
\end{align*}
\]
The isoscalar factors for the states with $(\lambda, \mu) = (N-1,1)$ are obtained by the orthogonality condition

\[
\left\langle \begin{array}{c} \frac{N-k}{2}, \frac{N}{3} - k \\ \frac{1}{2}, \frac{1}{3} \end{array} \right| \left( \begin{array}{cc} (N,0) & (1,0) \\ \frac{N+1-k}{3} + 1 - k & 0, -\frac{2}{3} \end{array} \right) \left( \begin{array}{c} \frac{(N-1,1)}{2}, \frac{N+1-k}{3} - k \\ \frac{1}{2} \end{array} \right) \rightangle = -\sqrt{\frac{k}{N+1}},
\]

\[
\left\langle \begin{array}{c} \frac{N-k}{2}, \frac{N}{3} - k \\ \frac{1}{2}, \frac{1}{3} \end{array} \right| \left( \begin{array}{cc} (N,0) & (1,0) \\ \frac{N+1-k}{3} + 1 - k & 0, -\frac{2}{3} \end{array} \right) \left( \begin{array}{c} \frac{(N-1,1)}{2}, \frac{N+1-k}{3} - k \\ \frac{1}{2} \end{array} \right) \rightangle = \sqrt{\frac{N+1-k}{N+1}}. \quad (A.9)
\]

The $SU(3)$ isoscalar factors for the couplings of Eq. (A.2) are presented in Table A1.

| $F_1$ | $Y_1$ | $F_2$ | $Y_2$ | $(\lambda, \mu)$ | $F$ | $Y$ | $\langle (\lambda,0) \rangle$ | $(0,1)$ | $(\lambda, \mu)$ |
|------|------|------|------|-----------------|-----|-----|----------------------------|----------|----------------|
| $\frac{N-k}{2}$ | $\frac{N}{3} - k$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $(N+1,0)$ | $\frac{N+1-k}{2}$ | $\frac{N+1}{3} - k$ | $\sqrt{\frac{N+1-k}{N+1}}$ |
| $\frac{N+1-k}{2}$ | $\frac{N}{3} + 1 - k$ | $0$ | $-\frac{2}{3}$ | $(N+1,0)$ | $\frac{N+1-k}{2}$ | $\frac{N+1}{3} - k$ | $\sqrt{\frac{k}{N+1}}$ |
| $\frac{N-k}{2}$ | $\frac{N}{3} - k$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $(N-1,1)$ | $\frac{N+1-k}{2}$ | $\frac{N+1}{3} - k$ | $-\sqrt{\frac{k}{N+1}}$ |
| $\frac{N+1-k}{2}$ | $\frac{N}{3} + 1 - k$ | $0$ | $-\frac{2}{3}$ | $(N-1,1)$ | $\frac{N+1-k}{2}$ | $\frac{N+1}{3} - k$ | $\sqrt{\frac{N+1-k}{N+1}}$ |
| $\frac{N-k}{2}$ | $\frac{N}{3} - k$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $(N-1,1)$ | $\frac{N+1-k}{2}$ | $\frac{N+1}{3} - k$ | $-1$ |

| $F_1$ | $Y_2$ | $F_2$ | $Y_2$ | $(\lambda, \mu)$ | $F$ | $Y$ | $\langle (\lambda,0) \rangle$ | $(0,1)$ | $(\lambda, \mu)$ |
|------|------|------|------|-----------------|-----|-----|----------------------------|----------|----------------|
| $\frac{N+1-k}{2}$ | $\frac{N}{3} + 1 - k$ | $\frac{1}{2}$ | $-\frac{1}{3}$ | $(N+1,0)$ | $\frac{N+2-k}{2}$ | $\frac{N+2}{3} - k$ | $\sqrt{\frac{N+2-k}{N+2}}$ |
| $\frac{N-k}{2}$ | $\frac{N}{3} - k$ | $0$ | $\frac{2}{3}$ | $(N+1,0)$ | $\frac{N+2-k}{2}$ | $\frac{N+2}{3} - k$ | $\sqrt{\frac{k}{N+2}}$ |
| $\frac{N+1-k}{2}$ | $\frac{N}{3} + 1 - k$ | $\frac{1}{2}$ | $-\frac{1}{3}$ | $(N+1,0)$ | $\frac{N+2-k}{2}$ | $\frac{N+2}{3} - k$ | $\sqrt{\frac{k}{N+2}}$ |
| $\frac{N-k}{2}$ | $\frac{N}{3} - k$ | $0$ | $\frac{2}{3}$ | $(N+1,0)$ | $\frac{N+2-k}{2}$ | $\frac{N+2}{3} - k$ | $\sqrt{\frac{k}{N+2}}$ |
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