Backwards-induction outcome in a quantum game.

A. Iqbal and A.H. Toor
Electronics Department, Quaid-i-Azam University,
Islamabad, Pakistan
email: qubit@isb.paknet.com.pk

March 31, 2022

Abstract
In economics duopoly is a market dominated by two firms large enough to influence the market price. Stackelberg presented a dynamic form of duopoly that is also called ‘leader-follower’ model. We give a quantum perspective on Stackelberg duopoly that gives a backwards-induction outcome same as the Nash equilibrium in static form of duopoly also known as Cournot’s duopoly. We find two qubit quantum pure states required for this purpose.

PACS: 03.67.Lx 02.50.Le 87.23.Kg
Keywords: Quantum game theory, Nash equilibrium, Backwards-induction outcome, Stackelberg and Cournot models of duopoly game.

1 Introduction
Quantum game theory started from a seminal paper by Meyer [1]. Later Eisert et. al. [2] studied the important bimatrix game of Prisoner’s Dilemma (PD) while focusing on the concept of Nash equilibrium (NE) from noncooperative game theory. This concept attracted much attention in other recent works in quantum game theory [2, 3, 4, 5]. In fact Cournot (1838) [6] anticipated Nash’s definition of equilibrium by over a century but only in the context of a particular model of duopoly. In economics, an oligopoly is a form of market in which a number n of producers, say, n ≥ 2, and no others, provide the market with a certain commodity. In the special case where n = 2, it is called a duopoly. Cournot’s work [6] is one of the classics of game theory and also a cornerstone of the theory of industrial organization [7]. In Cournot model of duopoly two-firms simultaneously put certain quantities of a homogeneous product in the market. Cournot obtained an equilibrium value for the quantities both firms will decide to put in the market. This equilibrium value was based on a rule of behavior which says that if all the players except one abide by it, the remaining player
cannot do better than to abide by it too. Nash gave a general concept of an equilibrium point in a noncooperative game but existence of an equilibrium in duopoly game was known much earlier. The “Cournot equilibrium” refers to NE in noncooperative form of duopoly that Cournot considered.

In an interesting later development Stackelberg (1934) proposed a dynamic model of duopoly in which, contrary to Cournot’s assumption of simultaneous moves, a leader (or dominant) firm moves first and a follower (or subordinate) firm moves second. A well known example is the General Motors playing this leadership role in the early history of U.S. automobile industry when more than one firms like Ford and Chrysler acted as followers. In this sequential game a “Stackelberg equilibrium” is obtained using the backwards-induction outcome of the game. In fact Stackelberg equilibrium refers to the sequential-move nature of the game and this is a stronger solution concept than NE because sequential move games sometimes have multiple NE, only one of which is associated with the backwards-induction outcome of the game.

The seminal paper by Meyer has initiated the new field of quantum games and has motivated many people to look at games from quantum perspectives. Eisert et. al. quantized the famous bimatrix game of Prisoner’s Dilemma (PD) and showed that dilemma disappears in quantum world. They allowed players an access to a maximally entangled state that can be generated from a system of two qubits. Each player then unitarily manipulates a qubit in his possession. Players apply a unitary operator from a particular subset of the general set of unitary operators that also forms a group. Benjamin later showed that if the players have access to the set of general unitary operators then there is no NE in pure strategies in the PD game. Benjamin implied that Eisert’s set of unitary operators was carefully chosen to generate a NE that has no classical counterpart.

Eisert et. al. used an unentangling gate in their scheme to be put before the quantum state is forwarded to measuring apparatus that collapses the wave function and gives the payoffs. The unentangling gate in Eisert’s scheme ensured that classical game could be reproducible but it motivated Marinatto to question the necessity of its presence in the scheme to play a quantum game. Marinatto and Weber came up with a solution where an initial quantum state, they called it an “initial strategy”, is made available to the players. This initial strategy is then unitarily manipulated by the players in the “tactics” phase of the game that consisted of applying two unitary and Hermitian operators (the identity and an inversion operator) with classical probabilities on the initial quantum strategy. This inversion operator reverts the quantum state just like the Pauli’s spin flip operator does. Marinatto and Weber showed that for an initial strategy that is a maximally entangled state a unique NE for the game of Battle of Sexes can be obtained. Later in an interesting comment Benjamin considered the players’ access to apply only two unitary and Hermitian operators with classical probabilities on a quantum state a severe restriction on all quantum mechanically possible manipulations. Marinatto and Weber replied that the only restriction on a quantum form of a game is that corresponding classical game must be reproducible as a special case of the quantum form.
Agreeing with this reply we studied the concept of evolutionary stability in asymmetric as well as symmetric quantum games [16, 17]. Our prime motivation was an important and interesting element in Marinatto and Weber’s scheme. It was the fact that a switch-over between a classical and a non-classical form of a game could be achieved by having a control over the parameters of the initial quantum state or initial strategy. In the scheme of Eisert, Wilkens, and Lewenstein [2] such a switch-over takes place when the players assign specific values to the parameters of the unitary operators in their possession; while the initial two-qubit quantum state always remains to be maximally entangled.

Starting a quantum game from a general pure state of two qubits that have interacted previously we were able to show the possibility that an unstable symmetric NE of a bimatrix classical game becomes stable in a quantum form of the same game [18]. Our assumption in this approach was that a replacement of the maximally entangled two qubit quantum state, that Marinatto and Weber used to get a unique NE, with a general two qubit pure quantum state also results in another form of the same game. This assumption led us to find the conditions on the constants of a general three qubit pure state that made it possible to counter against coalition formation in a three player symmetric game [19]. The person responsible for preparing quantum states can thus make vanish the motivation to make a coalition in the cooperative game.

Motivated by these recent developments in quantum games as well as the notion of backwards-induction outcome of a dynamic game of complete information [7] we present a quantum perspective on the interesting game of Stackelberg duopoly. In present paper we start with the same assumption that a game is decided only by players’ unitary manipulations, payoff operators, and the measuring apparatus deciding payoffs. When these are same a different input quantum initial state gives only a different form of the same game. This was our assumption when we studied evolutionary stability of a mixed NE in Rock-Scissors-Paper game [18]. Therefore all the games that can be obtained by using a general two qubit pure state are only different forms of the same game if the rest of the procedures in playing the quantum game are same. For example in Marinatto and Weber’s Battle of Sexes [3] the game remains Battle of Sexes for all two qubit pure quantum states.

With this assumption we start an analysis of Stackelberg duopoly by asking a fundamental question: Is it possible to find a two qubit pure quantum state that generates the classical Cournot equilibrium as a backwards-induction outcome of the quantum form of Stackelberg duopoly? Why this question can be of interest? For us it is interesting because in case the answer is yes then the very important resource in quantum game theory i.e. entanglement can potentially be a particularly useful element for ‘follower’ in the leader-follower model of Stackelberg duopoly [7]. This is because in classical settings when static duopoly changes itself into a dynamic form the follower becomes worse-off compared to leader who becomes better-off. We find that under certain restrictions it is possible to find the needed two qubit quantum states. Therefore a quantum form of a dynamic game of complete information gives out an equilibrium that corresponds to classical static form of the same game. In our analysis the
equilibrium can be obtained for a certain range of the constant of the duopoly game. This restriction appears only due to an assumption that we introduce to simplify calculations. This fact, however, does not rule out the possibility of getting a quantum form of Stackelberg duopoly game with no such restriction.

2 Backwards-induction outcome

To make this paper self-contained we give an introduction of the backwards-induction outcome of a sequential game. For this we find very useful the ref. [7]. Consider a simple three step game

1. Player 1 chooses an action $a_1$ from the feasible set $A_1$
2. Player 2 observes $a_1$ and then chooses an action $a_2$ from the feasible set $A_2$
3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$

This game is an example of the dynamic games of complete and perfect information. Key features of such games are

1. the moves occur in sequence
2. all previous moves are known before next move is chosen, and
3. the players’ payoffs are common knowledge. At the second stage of the game when player 2 gets the move he or she faces the following problem, given the action $a_1$ previously chosen by the player

$$Max_{a_2 \in A_2} u_2(a_1, a_2)$$

(1)

Assume that for each $a_1$ in $A_1$, player 2’s optimization problem has a unique solution $R_2(a_1)$ also called the best response of player 2. Now player 1 can also solve player 2’s optimization problem by anticipating player 2’s response to each action $a_1$ that player 1 might take. So that player 1 faces the following problem

$$Max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$$

(2)

Suppose this optimization problem also has a unique solution for player 1 and is denoted by $a_1^*$. The solution $(a_1^*, R_2(a_1^*))$ is the backwards-induction outcome of this game. In a simple version of the Cournot’s model two firms simultaneously decide the quantities $q_1$ and $q_2$ respectively of a homogeneous product they want to put into the market. Suppose $Q$ is the aggregate quantity i.e. $Q = q_1 + q_2$ and $P(Q) = a - Q$ be the market-clearing price, the price at which all products or services available in a market will find buyers. Assume the total cost to a
firm producing quantity \( q_i \) is \( cq_i \). I.e., there are no fixed costs and the marginal cost is a constant \( c \) with \( c < a \). In a two-player game theoretical model of this situation a firm’s payoff or profit can be written as:

\[
P_i(q_i, q_j) = q_i [P(Q) - c] = q_i [a - c - (q_i + q_j)] = q_i [k - (q_i + q_j)]
\]

Solving for the NE easily gives the Cournot equilibrium

\[
q_i^* = q_j^* = \frac{k}{3}
\]

At this equilibrium the payoffs to both firms from eq. (3) are

\[
P_1(q_1^*, q_2^*)_{\text{Cournot}} = P_2(q_1^*, q_2^*)_{\text{Cournot}} = \frac{k^2}{9}
\]

We now come to consider the classical form of duopoly game when it becomes dynamic. The game becomes dynamic but the payoffs to players are given by the same eq. (3) as for the case of the Cournot’s game. We find backwards-induction outcome in classical and a quantum form of Stackelberg’s duopoly. Taking advantage from bigger picture given to this dynamic game by Hilbert space structure we then find two qubit pure quantum states that give classical Cournot’s equilibrium as the backwards-induction outcome of the quantum game of Stackelberg’s duopoly.

3 Stackelberg duopoly

3.1 Classical form

A leader (or dominant) firm moves first and a follower (or subordinate) firm moves second in Stackelberg model of duopoly. The sequence of events is

1. firm A chooses a quantity \( q_1 \geq 0 \)
2. firm B observes \( q_1 \) and then chooses a quantity \( q_2 \geq 0 \)
3. the payoffs to firms A and B are given by their respective profit functions as

\[
P_A(q_1, q_2) = q_1 [k - (q_1 + q_2)]
\]

\[
P_B(q_1, q_2) = q_2 [k - (q_1 + q_2)]
\]

The backwards-induction outcome is found by first finding firm B’s reaction to an arbitrary quantity by firm A. Denoting this quantity as \( R_2(q_1) \) we find
\[ R_2(q_1) = \max_{q_2 \geq 0} q_2 = \frac{k - q_1}{2} \]  

with \( q_1 < k \). Now the interesting aspect of this game is that firm A can solve the firm B’s problem as well. Therefore firm A can anticipate that a choice of the quantity \( q_1 \) will meet a reaction \( R_2(q_1) \). In the first stage of the game firm A can then compute a solution to his/her optimization problem as

\[ \max_{q_1 \geq 0} P_A[q_1, R_2(q_1)] = \max_{q_1 \geq 0} q_1 \left( k - \frac{q_1}{2} \right) \]

It gives

\[ q_1^* = \frac{k}{2} \quad \text{and} \quad R_2(q_1^*) = \frac{k}{4} \]  

(8)

It is the classical backwards-induction outcome of dynamic form of the duopoly game. At this equilibrium payoffs or profits to players A and B are given by eqs. (6) and (8)

\[ P_A[q_1^*, R_2(q_1^*)]_{Stackelberg} = \frac{k^2}{8}, \quad P_B[q_1^*, R_2(q_1^*)]_{Stackelberg} = \frac{k^2}{16} \]  

(9)

From eq. (9) find the following ratio

\[ \frac{P_A[q_1^*, R_2(q_1^*)]_{Stackelberg}}{P_B[q_1^*, R_2(q_1^*)]_{Stackelberg}} = 2 \]  

(10)

showing that with comparison to Cournot game in Stackelberg game the leader firm becomes better-off and the follower firm becomes worse-off. This aspect also hints an important difference between single and multi-person decision problems. In single-person decision theory having more information can never make the decision maker worse-off. In game theory, however, having more information (or, more precisely, having it made public that one has more information) can make a player worse-off [7].

Now we look at backwards-induction outcome in a quantum perspective. Our motivation is an interesting aspect that quantum form can bring into backwards-induction outcome. It is the possibility that the ‘extra information’ that firm B has does not make firm B worse-off.

### 3.2 Quantum form

Stackelberg duopoly is a two player sequential game. Meyer [1] considered quantum form of sequential game of PQ Penny Flip by unitary operations on single
qubit. Important difference between Meyer’s PQ Penny Flip and Stackelberg duopoly is that at the second stage player in PQ Penny Flip doesn’t know the previous move but in Stackelberg duopoly he knows that.

Marinatto and Weber [3] used two qubits to play the noncooperative game of Battle of Sexes. Players apply unitary operators $I$ and $C$ with classical probabilities on a two-qubit pure quantum state. $I$ is identity and $C$ is inversion or Pauli’s spin-flip operator. We prefer this scheme to play the sequential game of Stackelberg duopoly for two reasons:

1. Interesting feature in this scheme that classical game is reproducible on making initial state unentangled.

2. We assumed that other games obtained from every pure two qubit initial state are only quantum forms of the classical game provided players’ actions and payoff generating measurement are exactly same [18]. This assumption reduces the problem of finding a quantum form of Stackelberg duopoly with property that its equilibrium is same as in Cournot’s duopoly to the problem of finding conditions on parameters of two-qubit pure quantum state. If the conditions are realistic then the corresponding quantum game will give Cournot’s equilibrium as a backwards-induction outcome.

Stackelberg duopoly is a dynamic game of complete information. Its quantum form in Marinatto and Weber’s scheme [3] starts by preparing a pure two-qubit initial quantum state. Both these qubits are then forwarded to two players.
in the game called Alice and Bob. Marinatto and Weber expanded on the ear-
lier scheme proposed by Eisert, Wilkens, and Lewenstein using a two-qubit
system in a maximally entangled quantum state. The fundamental idea remains
the same i.e. to exploit entanglement to get new results for a simultaneous-move
game. Suppose first move is to be played by Alice and she plays her strategy
by applying two operators in her possession on her qubit with classical proba-
bilities. She also announces her move immediately so that Bob knows Alice’s
strategy before playing his move. Bob plays his strategy on his qubit and both
Alice and Bob forward their qubits to a setup where measurement can be done
to decide payoffs to both.

The concept of information about the previous moves is crucial for the un-
derstanding of the present paper. A comparison between the sequential game of
Stackelberg duopoly with the simultaneous moves of the Battle of Sexes makes
clear the different information structure between the two games. For example
in the Battle of Sexes, when played sequentially, Alice does not announce her
first move to Bob before he makes his move. In this way the game becomes
sequential but the information structure of the game is still the same as in its
static form. Consequently, the sequential Battle of Sexes in the above form has
the same NE as in its static form. An unobserved-action form of a game has
the same NE as its simultaneous-move form. This observation led us to play a
quantum form of Stackelberg duopoly while keeping intact the original struc-
ture of a scheme designed for simultaneous moves. A consideration of playing a
sequential game in a quantum way brings to mind the Meyer’s PQ Penny Flip
where only one qubit is used in the game. Contrary to this, in present paper
we use the two-qubit system of a simultaneous moves game to play a sequential
game.

One can ask what is the point of taking extra pains by using two qubits when
a quantum form of this sequential game can also be played by only one qubit
in similar way as Meyer’s PQ Penny Flip. We prefer two qubits for a reason
that appeared to us quite important. In two-qubit case a comparison between
classical and a quantum form of the game translates itself into comparing two
games resulting from using unentangled and entangled initial quantum states.
We do not rule out the possibility that a consideration of the dynamic game
using only single qubit gives equally or even more interesting results.

The quantum form of Marinatto and Weber’s Battle of Sexes game re-
duces to its classical form for the initial state $|\psi_{ini}\rangle = |OO\rangle$ where O represents
the pure classical strategy called ‘Opera’. In the same spirit we let classical
payoffs in Stackelberg duopoly given by eq. reproduced when the initial
state $|\psi_{ini}\rangle = |11\rangle$ is used to play the game. The state $|11\rangle$ means that both
qubits are in lower state. We represent the upper state of a qubit by number 2.
For $|\psi_{ini}\rangle = |11\rangle$ the corresponding density matrix is

$$
\rho_{ini} = |11\rangle \langle 11|
$$

(11)

When players apply the unitary operators $I$ and $C$ such that Alice and Bob
apply $I$ with probabilities $x$ and $y$ respectively the density matrix \( \rho \) changes to

\[
\rho_{\text{fin}} = xyI_A \otimes I_B \rho_{\text{ini}} I^\dagger_A \otimes I^\dagger_B + x(1-y)I_A \otimes C_B \rho_{\text{ini}} I^\dagger_A \otimes C^\dagger_B + y(1-x)C_A \otimes I_B \rho_{\text{ini}} C^\dagger_A \otimes I^\dagger_B + (1-x)(1-y)C_A \otimes C_B \rho_{\text{ini}} C^\dagger_A \otimes C^\dagger_B
\]

(12)

where the inversion operator $C$ interchanges the vectors $|1\rangle$ and $|2\rangle$ i.e. $C |1\rangle = |2\rangle, C |2\rangle = |1\rangle$ and $C^\dagger = C^{-1}$. We now assume that in Stackelberg duopoly also players’ moves are again given by probabilities lying in the range $[0, 1]$. The moves by Alice and Bob in classical duopoly game are given by quantities $q_1$ and $q_2$ where $q_1, q_2 \in [0, \infty)$. We assume that Alice and Bob agree on a function that can uniquely define a real positive number in the range $(0, 1]$ for every quantity $q_1, q_2$ in $[0, \infty)$. A simple such function is \( \frac{1}{1+q_i} \), so that Alice and Bob find $x$ and $y$ respectively as

\[
x = \frac{1}{1+q_1}, \quad y = \frac{1}{1+q_2}
\]

(13)

and use these real numbers as probabilities with which they apply the identity operator $I$ on the quantum state at their disposal. With a substitution from eqs. (11,13) the final density matrix (12) can be written as

\[
\rho_{\text{fin}} = \frac{1}{(1+q_1)(1+q_2)} [11 \langle 11 | + q_1q_2 |22 \rangle \langle 22 | + q_1 |21 \rangle \langle 21 | + q_2 |12 \rangle \langle 12 |]
\]

(14)

We now also suppose that in the measurement and payoffs finding phase the quantities $q_1$ and $q_2$ are known to the ‘agent’ doing this action. The agent applies the payoff operators $(P_A)_{\text{oper}}, (P_B)_{\text{oper}}$ given as follows

\[
(P_A)_{\text{oper}} = (1+q_1)(1+q_2)q_1 |k \rangle \langle 11 | - |21 \rangle \langle 21 | - |12 \rangle \langle 12 |
\]

\[
(P_B)_{\text{oper}} = (1+q_1)(1+q_2)q_2 |k \rangle \langle 11 | - |21 \rangle \langle 21 | - |12 \rangle \langle 12 |
\]

(15)

Note that the classical payoffs of eq. (6) are reproduced with the initial state $|\psi_{\text{ini}}\rangle = |11\rangle$ with the following trace operations

\[
PA(q_1, q_2) = \text{Trace} [(P_A)_{\text{oper}} \rho_{\text{fin}}]
\]

\[
PB(q_1, q_2) = \text{Trace} [(P_B)_{\text{oper}} \rho_{\text{fin}}]
\]

(16)

A more general form of quantum duopoly can now be played by keeping the payoff operators of eq. (15) in the agent’s possession and preparing a general initial pure state of the following form
The classical payoffs of duopoly game given in eqs. (6) are recovered from the quantity

\[ q \]

chosen by Alice. Denoting this quantity as \( \mid q(t) \rangle \), the classical payoff operators of eqs. (15) can now be written as \( \mid 2 \rangle \). The payoffs to Alice and Bob can now be obtained in this more general quantum game from eqs. (16) that use the same payoff operators of eqs. (15). The payoffs to Alice and Bob can now be written as

\[
\begin{align*}
|q_{tm}\rangle &= |11\rangle + c_{12} |12\rangle + c_{21} |21\rangle + c_{22} |22\rangle \\
\text{where} \quad |c_{ij}|^2 + |c_{jk}|^2 + |c_{ki}|^2 + |c_{kl}|^2 &= 1
\end{align*}
\]

where \( c_{ij} \) for \( i, j = 1 \) or 2, are complex numbers and \( |ij\rangle \) are orthonormal basis vectors in \( 2 \otimes 2 \) dimensional Hilbert space. The payoffs to Alice and Bob can now be obtained in this more general quantum game from eqs. (16) that use the same payoff operators of eqs. (15). The payoffs to Alice and Bob can now be written as

\[
\begin{align*}
|P_A(q_1, q_2)\rangle_{qtm} &= (\omega_{11} + \omega_{12} q_2) + q_1(\omega_{21} + \omega_{22} q_2) \\
(1 + q_1)(1 + q_2) \\
|P_B(q_1, q_2)\rangle_{qtm} &= (\chi_{11} + \chi_{12} q_2) + q_1(\chi_{21} + \chi_{22} q_2) \\
(1 + q_1)(1 + q_2)
\end{align*}
\]

where the subscript \( qtm \) is for ‘quantum’ and

\[
\begin{align*}
\begin{bmatrix}
\omega_{11} \\
\omega_{12} \\
\omega_{21} \\
\omega_{22}
\end{bmatrix} &= 
\begin{bmatrix}
|c_{11}|^2 & |c_{12}|^2 & |c_{21}|^2 & |c_{22}|^2 \\
|c_{12}|^2 & |c_{11}|^2 & |c_{22}|^2 & |c_{21}|^2 \\
|c_{21}|^2 & |c_{22}|^2 & |c_{11}|^2 & |c_{12}|^2 \\
|c_{22}|^2 & |c_{21}|^2 & |c_{12}|^2 & |c_{11}|^2
\end{bmatrix} \\
\begin{bmatrix}
\chi_{11} \\
\chi_{12} \\
\chi_{21} \\
\chi_{22}
\end{bmatrix} &= 
\begin{bmatrix}
|c_{11}|^2 & |c_{12}|^2 & |c_{21}|^2 & |c_{22}|^2 \\
|c_{12}|^2 & |c_{11}|^2 & |c_{22}|^2 & |c_{21}|^2 \\
|c_{21}|^2 & |c_{22}|^2 & |c_{11}|^2 & |c_{12}|^2 \\
|c_{22}|^2 & |c_{21}|^2 & |c_{12}|^2 & |c_{11}|^2
\end{bmatrix}
\end{align*}
\]

The classical payoffs of duopoly game given in eqs. (6) are recovered from the initial quantum state becomes entangled and given by \( |\psi_{ini}\rangle = |11\rangle \). Classical duopoly is, therefore, a subset of its quantum version.

We now find the backwards-induction outcome in this quantum form of Stackelberg duopoly. We proceed in exactly the same way as it is done in the classical game except that players’ payoffs are now given by eqs. (19) and not by eqs. (6). The first step in backwards-induction in quantum game is to find Bob’s reaction to an arbitrary quantity \( q_1 \) chosen by Alice. Denoting this quantity as \( |R_2(q_1)\rangle_{qtm} \) we find

\[
|R_2(q_1)\rangle_{qtm} = \text{Max}_{q_2 \geq 0} |P_B(q_1, q_2)\rangle_{qtm} = \frac{q_1 \Delta_1 + \Delta_2}{-2 \{q_1 \Delta_3 + \Delta_4\}} \quad \text{where}
\]

\[
\begin{align*}
|c_{11}|^2 + |c_{22}|^2 - k |c_{21}|^2 &= \Delta_1, & |c_{12}|^2 + |c_{21}|^2 - k |c_{11}|^2 &= \Delta_2 \\
|c_{12}|^2 + |c_{21}|^2 - k |c_{22}|^2 &= \Delta_3, & |c_{11}|^2 + |c_{22}|^2 - k |c_{12}|^2 &= \Delta_4
\end{align*}
\]
This reaction reduces to its classical value of eq. (6) when $|c_{11}|^2 = 1$. Similar to classical game Alice can now solve Bob’s problem as well. Therefore Alice can anticipate that a choice of the quantity $q_1$ will meet a reaction $[R_2(q_1)]_{qtm}$ in the first stage of the game like its classical version Alice can compute a solution to her optimization problem as

$$\max_{q_1 \geq 0} \left[ P_A \left\{ q_1, \{ R_2(q_1) \} \right\}_{qtm} \right]$$

(22)

To find it Alice calculates the following quantity

$$d \left[ P_A \left\{ q_1, \{ R_2(q_1) \} \right\}_{qtm} \right]_{qtm} dq_1 |_{q_1 = q^*_1} = (c_{11}^2 + c_{22}^2 - c_{12}^2 - c_{21}^2) \left\{ -2q_1^2 + q_1(k - 2) + k \right\}$$

$$+ (1 + 2q_1) \left\{ (k - 1)|c_{21}|^2 - |c_{12}|^2 \right\} + k(|c_{12}|^2 - |c_{22}|^2)$$

$$- q_1 \frac{dq_2}{dq_1} \left\{ \triangle_4 + q_1 \triangle_3 \right\} - q_2 \left\{ 2q_1 \triangle_3 + \triangle_4 \right\}$$

(23)

and replaces $q_2$ in eq. (23) with $[R_2(q_1)]_{qtm}$ given by eq. (21) and then equates eq. (23) to zero to find a $q^*_1$ that maximizes her payoff $[P_A(q_1, q_2)]_{qtm}$. For a maxima she would ensure that the second derivative of $P_A \left\{ q_1, \{ R_2(q_1) \} \right\}_{qtm}$ with respect to $q_1$ at $q_1 = q^*_1$ is a negative quantity. The $q^*_1$ together with $[R_2(q^*_1)]_{qtm}$ will form the backwards-induction outcome of the quantum game.

An interesting situation is when the backwards-induction outcome in quantum version of Stackelberg duopoly becomes same as the classical Cournot equilibrium of duopoly. The classical situation of leader becoming better-off while the follower becoming worse-off is then avoided in the quantum form of Stackelberg duopoly. To look for this possibility we need such an initial state $|\psi_{ini}\rangle = c_{11}|11\rangle + c_{12}|12\rangle + c_{21}|21\rangle + c_{22}|22\rangle$ that at $q^*_1 = q^*_2 = \frac{k}{3}$ we should have following relations holding true also with the normalization condition given in eq. (18)

$$d \left[ P_A \left\{ q_1, \{ R_2(q_1) \} \right\}_{qtm} \right]_{qtm} dq_1 |_{q_1 = q^*_1} = 0$$

(24)

$$\left[ \frac{d^2}{dq_1^2} \left[ P_A \left\{ q_1, \{ R_2(q_1) \} \right\}_{qtm} \right]_{qtm} \right] |_{q_1 = q^*_1} < 0$$

(25)

$$q^*_2 = [R_2(q^*_1)]_{qtm}$$

(26)

The conditions (24,25) simply say that the backwards-induction outcome of the quantum game is the same as Cournot equilibrium in classical game. The
condition (26) says that Bob’s reaction to Alice’s choice of $q^*_1 = \frac{k}{3}$ is $q^*_2 = \frac{k}{3}$. To show that such quantum states can exist for which the conditions (24, 25, 26) with the normalization (18) hold true we give an example where $|c_{11}|^2, |c_{12}|^2$ and $|c_{21}|^2$ are written as functions of $k$ with our assumption that $|c_{22}|^2 = 0$. Though this assumption puts its own restriction on the possible range of $k$ for which the above conditions hold for these functions but still it shows clearly the possibility of finding the required initial quantum states. The functions are found as

$$|c_{12}(k)|^2 = \frac{-f(k) + \sqrt{f(k)^2 - 4g(k)h(k)}}{2g(k)}$$

where

$$f(k) = j(k) \left\{ \frac{-7}{18}k^2 + \frac{1}{3}k + \frac{1}{2} \right\} + \left\{ \frac{k^2}{9} + \frac{k}{3} + \frac{1}{2} \right\} + \left\{ \frac{-1}{9}k^2 - \frac{1}{2}k - \frac{1}{2} \right\}$$

$$g(k) = j(k)^2 \left\{ \frac{-1}{9}k^2 + \frac{7}{18}k^2 - \frac{1}{2} \right\} + j(k) \left\{ \frac{2}{9}k^3 + \frac{5}{18}k^2 - \frac{1}{2}k - 1 \right\} + \left\{ \frac{-1}{9}k^2 - \frac{1}{2}k - \frac{1}{2} \right\}$$

$$h(k) = \frac{-1}{6}k, \quad j(k) = \frac{9 - 4k^2}{k^2 - 9}$$

also

$$|c_{21}(k)|^2 = j(k) |c_{12}(k)|^2$$

(27)

$$|c_{11}(k)|^2 = 1 - |c_{12}(k)|^2 - |c_{21}(k)|^2$$

(28)

Now, interestingly, given that allowed range of $k$ is $1.5 \leq k \leq 1.73205$, all of the conditions (18, 24, 25, 26) hold at $q^*_1 = q^*_2 = \frac{k}{3}$. So that in this range of $k$ a quantum form of Stackelberg duopoly exists that gives the classical Cournot equilibrium as backwards-induction outcome. The restriction on allowed range of $k$ is result of our assumption that $|c_{22}(k)|^2 = 0$. In fig. 2 below $|c_{11}(k)|^2, |c_{12}(k)|^2$ and $|c_{21}(k)|^2$ are plotted against $k$ in the above range.
Fig. 2. $|c_{11}(k)|^2$, $|c_{12}(k)|^2$ & $|c_{21}(k)|^2$ against $k$ when $|c_{22}|^2 = 0$

4 Discussion and Conclusion

What can possibly be a relevance of considering a quantum form of a game that models a competition between two firms in macroscopic world of economics? Quantum mechanics was developed to understand phenomena in the regime of atomic and subatomic interactions and is still mostly used in that domain. What is of interest in extending a game theoretical model of interaction between firms towards quantum domain? These questions naturally arise not only with reference to Stackelberg duopoly considered in this paper but also other related works in quantum games. Apart from exciting new directions that quantum mechanics brings to game theory there is also a fundamental interest in quantum games from the view of quantum information theory. The fact that quantum algorithms may be thought of as games between classical and quantum agents was pointed out by Meyer [1]. Meyer indicated a strong motivation for the study of quantum games by considering zero-sum quantum games in order to have a new starting point at hand to find quantum algorithms that outperform its classical analogue. Many quantum information exchange protocols have been modelled like games. Eavesdropping and optimal cloning are two such often cited examples where objective before a player is to gain as much information as possible. We believe that like other notions of game theory finding some relevance in quantum information a consideration of backwards-induction can be of interest for exactly the same reasons. It does not seem hard to imagine situations in quantum information where moves occur in sequence, all previous moves are observed before the next move is chosen, players’ payoffs from each feasible combination of moves are common knowledge. Interesting questions
then arise about how a quantum version of dynamic game of complete information can influence the outcome. Our primary motivation, however, to study backwards-induction in quantum games is from the view of dynamic stability, especially of symmetric NE, that has important relevance in evolutionary games \[15\] that we found interesting in quantum settings \[16, 17, 18, 20\].

The duopoly game models economic competition between firms and applied economics is the area where it is studied in detail. We considered this game in a scheme that tells how to play a quantum game and gives a Hilbert structure to the strategy space to which players have access to. The fact that quantum game theory can give entirely new views on games important in economics is apparent in recent interesting papers by Piotrowski and Sladkowski \[21, 22\] proposing a quantum-like description of markets and economies where players’ strategies belong to Hilbert space. It shows that quantum games certainly have features of interest to applied economists. Reciprocating with it we showed that games played by firms in economic competition can give counter-intuitive solutions when played in a quantum world.

We conclude our results as follows. A comparison between the NE in Cournot game with backwards-induction outcome in classical Stackelberg duopoly shows that having Alice (or firm A who acts first) know that Bob (or firm B who acts second) knows \(q_1\) (Alice’s move) hurts Bob. In fact in classical Stackelberg game Bob should not believe that Alice has chosen its Stackelberg quantity i.e. \(q_1^* = \frac{k}{2}\). We have shown that there can be a quantum version of Stackelberg duopoly where Bob is not hurt even if he knows the quantity \(q_1\) chosen by Alice. The backwards-induction outcome of this quantum game is the same as the NE in classical Cournot game where decisions are made simultaneously and there is no such information that hurts a player. Though this outcome in quantum game is obtained for a restricted range of the variable \(k\) but it is only because of a simplification in calculations.

5 Acknowledgment

This work is supported by Pakistan Institute of Lasers and Optics, Islamabad.

References

[1] D. A. Meyer. Quantum Strategies. Phy. Rev. Lett. 82, 1052-1055 (1999). quant-ph/9804010. Also D. A. Meyer. Quantum games and quantum algorithms. quant-ph/0004092

[2] J. Eisert, M. Wilkens, M. Lewenstein. Quantum Games and Quantum Strategies. Phys. Rev. Lett. 83, 3077 (1999). quant-ph/9806088. Also J. Eisert, M. Wilkens. Quantum Games. J. Mod. Opt. 47 (2000) 2543. quant-ph/0004076
[3] L. Marinatto, T. Weber. A quantum approach to static games of complete information. Phy. Lett. A 272, 291 (2000). [quant-ph/0004081]

[4] Jiangfeng Du, Xiaodong Xu, Hui Li, Xianyi Zhou, Rongdian Han. Nash Equilibrium in the Quantum Battle of Sexes Game. [quant-ph/0010050]

[5] Jiangfeng Du, Hui Li, Xiaodong Xu, Mingjun Shi, Xianyi Zhou, Rongdian Han. Remark On Quantum Battle of The Sexes Game. [quant-ph/0103004]

[6] A. Cournot. Researches into the Mathematical Principles of the Theory of Wealth. Edited by N. Bacon. New York: Macmillan, 1897

[7] R. Gibbons. Game Theory for Applied Economists. Princeton University Press. 1992

[8] J. Tirole. The theory of industrial organization. Cambridge: MIT Press. 1988

[9] H. von Stackelberg. Marktform und Gleichgewicht. Vienna: Julius Springer. 1934

[10] S. C. Benjamin, P. M. Hayden. Comment on ‘Quantum Games and Quantum Strategies’ [quant-ph/0003036]

[11] S. C. Benjamin. Comment on: "A quantum approach to static games of complete information”. [quant-ph/0008127]

[12] S. C. Benjamin, P. M. Hayden. Multi-Player Quantum Games. [quant-ph/0007038]

[13] L. Marinatto. Private communication.

[14] L. Marinatto, T. Weber. Reply to "Comment on: A Quantum Approach to Static Games of Complete Information". Physics Letters A 277, 183-184 (2000). [quant-ph/0009103]

[15] R. Cressman. The Dynamic (In)Stability of Backwards Induction. Journal of Economic Theory. 83, 260-285 (1998)

[16] A. Iqbal and A. H. Toor. Evolutionarily stable strategies in quantum games. Phys. Lett. A 280 (2001) 249-256. [quant-ph/0007100]

[17] A. Iqbal and A. H. Toor. Entanglement and Dynamic Stability of Nash Equilibria in a Symmetric Quantum Game. Phys. Lett. A 286 (2001) 245-250. [quant-ph/0101106]

[18] A. Iqbal and A. H. Toor. Quantum Mechanics gives Stability to a Nash Equilibrium. Phys. Rev. A 65, 022306 (2002). [quant-ph/0104091]

[19] A. Iqbal and A. H. Toor. Quantum Cooperative Games. Phys. Lett. A 293/3-4 (2002) 103-108. [quant-ph/0108091]
[20] A. Iqbal and A. H. Toor. Evolutionary stability of mixed Nash equilibrium in quantized symmetric bimatrix games. quant-ph/0106056

[21] E. W. Piotrowski, J. Sladkowski. Quantum Market Games. quant-ph/0104006

[22] E. W. Piotrowski, J. Sladkowski. Quantum Bargaining Games. quant-ph/0106140