NON-STATIONARY RUIJSENAARS FUNCTIONS FOR $\kappa = t^{-1/N}$ AND INTERTWINING OPERATORS OF DING-IOHARA-MIKI ALGEBRA

MASAYUKI FUKUDA, YUSUKE OHKUBO AND JUN’ICHI SHIRAISHI

ABSTRACT. We construct the non-stationary Ruijsenaars functions (affine analogue of the Macdonald functions) in the special case $\kappa = t^{-1/N}$, using the intertwining operators of the Ding-Iohara-Miki algebra (DIM algebra) associated with $N$-fold Fock tensor spaces. By the $S$-duality of the intertwiners, another expression is obtained for the non-stationary Ruijsenaars functions with $\kappa = t^{-1/N}$, which can be regarded as a natural elliptic lift of the asymptotic Macdonald functions to the multivariate elliptic hypergeometric series. We also investigate some properties of the vertex operator of the DIM algebra appearing in the present algebraic framework: an integral operator which commutes with the elliptic Ruijsenaars operator, and the degeneration of the vertex operators to the Virasoro primary fields in the conformal limit $q \to 1$.

1. Introduction

The non-stationary Ruijsenaars function $f_{\gamma}^N(x, p, s, \kappa | q, t)$ introduced by one of the authors in [1] is by definition given in the form of the Nekrasov partition function,

$$f_{\gamma}^N(x, p, s, \kappa | q, t) = \sum_{\lambda(1), \ldots, \lambda(N) \in \mathbb{P}} N^{(j-i)/2} \left( t_{ij} / s_{ij} | q \right) \cdot \prod_{\beta=1}^{N} \prod_{\alpha=1}^{N} \left( px_{\alpha+\beta} / tx_{\alpha+\beta-1} \right)^{\lambda_{ij}^{(\beta)}}. \quad (1.1)$$

Here $x = (x_1, \ldots, x_N)$, $s = (s_1, \ldots, s_N)$, and $N \in \mathbb{Z}_{\geq 1}$. As for the detail, see Definition 3.2.1. Our goal in the present paper is to establish the transformation formula stated in Theorem 3.30 below for the non-stationary Ruijsenaars function $f_{\gamma}^N$ in the special case $\kappa = t^{-1/N}$, by using the $S$-duality of the intertwining operators of the Ding-Iohara-Miki (DIM) algebra.

Define the elliptic shifted product $\Theta(a; q, p)$ as the ratio of the Ruijsenaars elliptic gamma function $\Gamma(a; q, p)$ by

$$\Theta(a; q, p) := \frac{\Gamma(q^n a; q, p)}{\Gamma(a; q, p)}, \quad \Gamma(a; q, p) := \frac{(qp/a; q, p)_{\infty}}{(a; q, p)_{\infty}}. \quad (1.2)$$

Definition (Definition 3.2.1). Define $f_{\gamma}^{\text{ell}}(x; s | q, t, p)$ in $\mathbb{Q}(q, t, s)[[p, x_2/x_1, \ldots, x_N/x_{N-1}]]$ by

$$f_{\gamma}^{\text{ell}}(x; s | q, t, p) = \sum_{\theta \in \mathcal{M}_N} c_{\gamma}^{\text{ell}}(\theta; s | q, t, p) \prod_{1 \leq i < j \leq N} (x_j / x_i)^{\theta_{ij}}, \quad (1.3)$$

where $\mathcal{M}_N = \{ (\theta_{ij})_{1 \leq i, j \leq N} \}$ is the set of $N \times N$ strictly upper triangular matrices with nonnegative integer entries ($\theta_{ij} \in \mathbb{Z}_{\geq 0}$, $i \geq j$ implies $\theta_{ij} = 0$), and

$$c_{\gamma}^{\text{ell}}(\theta; s | q, t, p) = \prod_{k=2}^{N} \prod_{1 \leq i < j \leq k} \frac{\Theta(q^{\sum_{a<k}(\theta_{ia} - \theta_{ja})} t_{ij}/s_{ij}; q, p)_{\theta_{ik}}}{\Theta(q^{\sum_{a<k}(\theta_{ja} - \theta_{ia})} q s_{ij}/s_{ij}; q, p)_{\theta_{ik}}} \cdot \frac{\Theta(q^{-\theta_{jk} + \sum_{a<k}(\theta_{ia} - \theta_{ja})} q s_{ij}/t s_{ij}; q, p)_{\theta_{ik}}}{\Theta(q^{-\theta_{jk} + \sum_{a<k}(\theta_{ja} - \theta_{ia})} s_{ij}/s_{ij}; q, p)_{\theta_{ik}}}. \quad (1.4)$$

Theorem (Theorem 3.30). As a formal series in $p$, $s_{i+1}/s_i$, $x_{i+1}/x_i$ ($i = 1, \ldots, N-1$) and $px_1/x_N$, we have

$$f_{\gamma}^N(x', p'x/N | s', t' - \frac{1}{N} | q, t) = \mathcal{C} \times f_{\gamma}^{\text{ell}}(s; x, t, p), \quad (1.5)$$

where $\mathcal{C}$ is a constant.
where
\[
\mathcal{C} := \left( \frac{(pq/t; q, p)_{\infty}}{(p; p)_{\infty}(pt; q)_{\infty}} \right)^{N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(tx_{j}/x_{i}; q, p)}{\Gamma(qx_{j}/x_{i}; q, p)} \prod_{1 \leq i < j \leq N} \frac{(ts_{j}/s_{i}; q)_{\infty}}{(qs_{j}/s_{i}; q)_{\infty}},
\]
and
\[
s' = (s_{1}', \ldots, s_{N}'), \quad s'_{k} = t^{k} s_{k},
\]
\[
\mathbf{x}' = (x'_{1}, \ldots, x'_{N}), \quad x'_{k} = p^{- \frac{k}{t}} x_{k}.
\]  

Contrary to the case \( N \geq 2 \), the case \( N = 1 \) seems somewhat special from the point of view of our vertex operator approach, and the parameter \( \kappa \) can be treated as an arbitrary constant. As a result, we have the following summation formula, which has been already proved by \[2, 3\].

**Theorem (Theorem 3.41).** We have
\[
\exp \left( \sum \frac{1}{n} \frac{(1 - q^{n} \kappa^{n})(1 - \kappa^{n}/t^{n})(\kappa^{-n}p^{n})}{(1 - q^{n})(1 - t^{n})(1 - p^{n})} \right)
= \sum_{\lambda \in \mathcal{P}} \left( \prod \frac{\Gamma(\kappa q^{-\lambda_{i}+\lambda_{j}+t^{-i+j}+1}; q)\lambda_{j}-\lambda_{j+1}}{\Gamma(\kappa q^{\lambda_{i}-\lambda_{j}+t^{-i+j}+1}; q)\lambda_{j}-\lambda_{j+1}} \right).
\]

Remark that setting \( \kappa = t^{-1} \) in (1.9), we recover (1.5) for \( N = 1 \). To prove Theorem 3.30 and Theorem 3.41 we use the technique of the topological vertex operator. This consists of the DIM algebra, the trivalent vertex operators, and the web diagrams encoding the structure of the Fock tensor spaces which the DIM algebra is acting on.

To fix a good starting point, we need to recall some facts about the asymptotically free eigenfunctions \( f^{\theta N}(x, s|q, t) \) for the Macdonald \( q \)-difference operator. See \[4, 5, 6\] and Appendix A as to the basic facts.

**Definition (Definition 3.12).** Define the formal series \( f^{\theta N}(x, s|q, t) \in \mathcal{Q}(q, t, s)[[x_{2}/x_{1}, \ldots, x_{N}/x_{N-1}]] \) by
\[
f^{\theta N}(x, s|q, t) = \sum_{\theta \in \mathcal{M}_{N}} c_{N}(\theta; s|q, t) \prod_{1 \leq i < j \leq N} (x_{j}/x_{i})^{\theta_{ij}},
\]
where the coefficient \( c_{N}(\theta; s|q, t) \) is defined by
\[
c_{N}(\theta; s|q, t) = \prod_{k=2}^{N} \prod_{1 \leq i < j \leq k} \frac{\left( \sum_{a>k}(a_{i}-a_{j})ts_{j}/s_{i}; q)\theta_{ik} \right)}{\left( \sum_{a>k}(a_{i}-a_{j})qs_{j}/s_{i}; q)\theta_{ik} \right)} \\
\times \prod_{k=2}^{N} \prod_{1 \leq i < j < k} \frac{\left( q^{-\theta_{ik}} + \sum_{a>k}(a_{i}-a_{j})qs_{j}/s_{i}; q)\theta_{ik} \right)}{\left( q^{-\theta_{ik}} + \sum_{a>k}(a_{i}-a_{j})s_{j}/s_{i}; q)\theta_{ik} \right)}.
\]

Note that the \( f^{\theta N} \) enjoys the eigenvalue equation (Fact \( \Lambda, 2 \)), the analytic property (Fact \( \Lambda, 6 \)), the bispectral duality (Fact \( \Lambda, 3 \)), and the Poincaré duality (Fact \( \Lambda, 4 \)). We have a “web diagrammatic description” of the combinatorial structure (Definition 3.12), based on the DIM algebra. In other words, the \( f^{\theta N} \) has an interpretation as a certain Nekrasov partition function associated with the web diagram. See \[7, 8\].

The DIM algebra has two kinds of intertwining operators, \( \Phi : \mathcal{F}(0,1) \otimes \mathcal{F}(1,0) \to \mathcal{F}(1,1) \) and \( \Phi^{*} : \mathcal{F}(1,1) \to \mathcal{F}(1,0) \otimes \mathcal{F}(0,1) \) among the triple of Fock spaces, introduced in \[9\]. As for the definition of the modules \( \mathcal{F}(1, M), \mathcal{F}(0,1) \) and the intertwiners, see Fact 2.3, 2.4, and 2.6. By using physics terminology, we refer to the modules \( \mathcal{F}(0,1) \) as the preferred directions. The matrix elements of these intertwiners are identical to the refined topological vertex. We express the composition \( \Phi^{*} \circ \Phi \) by the cross diagram in FIGURE 1 left. Compose these operators reticulately as in FIGURE 2 right. Specialize spectral parameters in a certain manner. Attach empty diagrams to all the external edges. Then we have the Macdonald function \( f^{\theta N} \) as thus constructed matrix element (See Fact 3.13).

Suppose we consider the trace in the preferred or vertical direction, instead of the matrix element as above. See FIGURE 2 below. We will prove that the \( p \)-trace associated with the web on a cylinder

\[1f^{\theta N}(x, s|q, t) \text{ coincides with } p_{N}(x, s|q, t) \text{ in } [7].\]
gives us the non-stationary Ruijsenaars function $\hat{f}_{\kappa}^{gl_N}$ for the special parameter $\kappa = t^{-1/N}$ for $N \geq 2$, and for generic $\kappa$ for $N = 1$.

Thanks to the $S$-duality, we can flip the diagram, obtaining the picture as in FIGURE 3. Finally, the trace in the horizontal direction can be calculated by the standard technique, thereby giving the elliptic lift $f_{N}^{\text{ellip}}$ of the asymptotically free eigenfunction $f_{\kappa}^{gl_N}$ for the Macdonald $q$-difference operator. In Section 3 are given our proofs of Theorems 3.30 and 3.41 which go along this idea.

Some explanations are in order, concerning the physical background and recent related works on the non-stationary systems, including the non-stationary Heun, Lamé, and elliptic Calogero-Sutherland equations. These non-stationary equations have been extensively studied based on the perturbative approach by Atai and Langmann in [10]. Recently in [11], they obtained an integral
formula for the eigenfunctions for the non-stationary elliptic Calogero-Sutherland equation for some special choices of the parameter in the “time derivative” term, using the kernel function technique.

Note that no explicit equations have been obtained for the non-stationary Ruijsenaars function $f^{2/N}$ unfortunately. So the authors have been lead to use the representation theories of the DIM algebra, to bypass the troublesome situation without any eigenvalue equations or associated kernel functions. One may find, nevertheless, the resulting formula in Theorem 3.30 can be regarded as a $q$-analogue of Atai and Langmann’s integral formula in [11], suggesting the existence of a $q$-analogue of the kernel functions.

In this occasion, we address some other related problems in the representation theory of the DIM algebra. First, we derive an integral operator $I(s_1/s_0, \ldots, s_N/s_0)$ introduced in [12, 13, 14] from the intertwining operators of the DIM algebra. As for the definition of $I(s_1/s_0, \ldots, s_N/s_0)$, see Definition 4.5 In [12, 13, 14], it was conjectured that the integral operator $I(s_1/s_0, \ldots, s_N/s_0)$ commutes with Macdonald’s difference operator. A proof has been given in [5]. We provide yet another perspective based on the vertex operator. We also discuss the elliptic analogue of the integral operator. Our elliptic integral operator can be constructed by taking the loop of intertwining operators. It commutes with the Ruijsenaars operator.

Secondly, we study the conformal limit $q \to 1$ of the $2N$-valent intertwining operator $V(z)$, which is a main object in the authors’ previous paper [7], restricting ourselves to the simplest nontrivial case $N = 2$. The operator $V(z)$ is defined by the relations with the $q$-Vir algebra for $N = 2$. We will derive the well-known relation of the Virasoro primary fields from those defining relations in the limit $q \to 1$. In [7], a formula for the matrix elements of $V(z)$ with respect to generalized Macdonald functions (Fact 2.21) are proved. The result Theorem 5.25 and the matrix elements formula for $V(z)$ prove that the generalized Macdonald functions are reduced to Alba, Fateev Litvinov and Tarnopolskii’s basis [15]. Hence, the matrix element formula for $V(z)$ provides us with another proof of the 4-dimensional AGT correspondence [10].

This paper is organized as follows. Basic facts with respect to the intertwining operators of the DIM algebra are summarized in Section 2 In Section 3, we reproduce the non-stationary Ruijsenaars functions from the intertwiners on the cylindrical web diagrams. We also prove Theorem 3.30 by using the $S$-duality. In Section 4, we derive the integral operator $I(s_1/s_0, \ldots, s_N/s_0)$ and prove the commutativity with the Macdonald $q$-difference operator. The elliptic extension of the integral operator is also discussed. We take the conformal limit $q \to 1$ in Section 5. In Appendix A, some facts with respect to Macdonald functions $f^{2/N}$ are explained. In Appendix B, we discuss the construction of the non-stationary Ruijsenaars functions from the affine screening operators in the case of general $\kappa$. In Appendix C are given some straightforward but cumbersome details needed for our proofs.

**Notation.** We use the following standard symbols for the $q$-shifted factorials, the theta functions and the elliptic gamma functions [17]:

\[
(a; q)_\infty := \prod_{n=1}^{\infty} (1 - q^n a), \quad (a; q)_m := \frac{(a; q)_{\infty}}{(aq^m; q)_{\infty}},
\]

(1.12)

\[
\theta_q(a) := (a; q)_\infty (qa^{-1}; q)_\infty,
\]

(1.13)

\[
(a; q, p)_\infty := \prod_{n,m=1}^{\infty} (1 - q^{n-1} p^{m-1} a),
\]

(1.14)

\[
\Gamma(a; q, p) := \frac{(qp/a; q, p)_\infty}{(a; q, p)_\infty}.
\]

(1.15)

For integers $n \leq m$, we use the following symbols for the tensor product and the ordered products

\[
\bigotimes_{n \leq i \leq m} A_i := A_n \otimes A_{n+1} \otimes \cdots \otimes A_m, \quad \bigwedge_{n \leq i \leq m} A_i := A_n \times A_{n+1} \times \cdots \times A_m,
\]

(1.16)

(1.17)
\[
\prod_{n \leq i \leq m} A_i := A_m \times A_{m-1} \times \cdots \times A_n. \tag{1.18}
\]

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a sequence of nonnegative integers such that \( \lambda_1 \geq \lambda_2 \geq \cdots \) with finitely many nonzero parts. The empty diagram is denoted by \( \emptyset = (0,0,\ldots) \). \( \mathcal{P} \) denotes the set of all partitions. For \( \lambda \in \mathcal{P} \), we write \( |\lambda| = \sum_{i \geq 1} \lambda_i \) and \( \ell(\lambda) = \# \{ i | \lambda_i \neq 0 \} \). The conjugate partition of \( \lambda \) is denoted by \( \lambda' \). If \( \lambda_i \leq \mu_i \) for all \( i \), we write \( \lambda \subset \mu \). For an \( N \)-tuple of partitions \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}) \in \mathbb{P}^N \), write \( |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(N)}| \). For a pair of positive integers \( (i,j) \in \mathbb{Z}_{>0}^2 \), the arm length \( a_\lambda(i,j) \) and the leg length \( \ell_\lambda(i,j) \) are defined by
\[
a_\lambda(i,j) = \lambda_i - j, \quad \ell_\lambda(i,j) = \lambda'_j - i. \tag{1.19}
\]

2. DIM algebra, intertwiners, and Mukadé operators

We briefly recall the definition of the Ding-Iohara-Miki (DIM) algebra \([IS][19]\), its intertwining operators and the S-duality formula for the intertwining operators. Let \( q \) and \( t \) be generic complex parameters with \( |q|, |t|^{-1} < 1 \).

**Definition 2.1.** The DIM algebra, which we denote by \( \mathcal{U} = \mathcal{U}_{q,t} \), is a unital associative algebra generated by the currents \( x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n} \), \( \psi^\pm(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \psi_n^\pm z^{-n} \) and the central elements \( c^{\pm 1/2} \). The defining relations are

\[
\begin{align*}
\psi^+(z)x^+(w) &= g(c^{1/2}w/z)x^+\psi^+(z), \\
\psi^-(z)x^+(w) &= g(c^{1/2}w/z)x^+\psi^-(z), \\
\psi^\pm(z)\psi^\pm(w) &= \psi\psi^\pm(z), \\
\psi^+(z)\psi^-(w) &= \frac{g(c+1w/z)}{g(c-1w/z)}\psi^-(w)\psi^+(z), \\
[x^+(z),x^-(w)] &= \frac{(1-q)(1-1/t)}{1-q/t}\left(\delta(c+1/w)\psi^+(c+1/w) - \delta(cz/w)\psi^-(c-1/w)\right), \\
G^+(z/w)x^+(z)x^+(w) &= G^+(z/w)x^+(w)x^+(z),
\end{align*}
\]

where
\[
g(z) = \frac{G^+(z)}{G^-(z)}, \quad G^\pm(z) = (1-q^{\pm 1}z)(1-t^{\pm 1}z)(1-q^{\pm 1}t^{\pm 1}z), \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n. \tag{2.6}
\]

**Fact 2.2 \([IS]\).** The Drinfeld coproduct

\[
\begin{align*}
\Delta(c^{\pm 1/2}) &= c^{\pm 1/2} \otimes c^{\pm 1/2}, \\
\Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(c^{1/2}z) \otimes x^+(c^{1/2}z), \\
\Delta(x^-(z)) &= x^-(c^{1/2}z) \otimes \psi^+(c^{1/2}z) + 1 \otimes x^-(z), \\
\Delta(\psi^\pm(z)) &= \psi^\pm(c^{1/2}z) \otimes \psi^\pm(c^{1/2}z),
\end{align*}
\]

gives rise to a bialgebra structure. Further, \( \mathcal{U} \) has a Hopf algebra structure. We omit the counit and the antipode.

A \( \mathcal{U} \)-module is called of level-\((n,m)\) if the central elements act as \( c = (t/q)^{n/2} \) and \( (\psi^0/\psi^-)^{1/2} = (q/t)^{m/2} \). In this paper, we use two kinds of \( \mathcal{U} \)-modules. The first one is a free field representation with the following boson. Let \( \mathcal{H} \) be the Heisenberg algebra generated by \( \{a_n|n \in \mathbb{Z}\} \) with the commutation relation
\[
[a_n,a_m] = n\frac{1-q^n}{1-t^n} \delta_{n+m,0}. \tag{2.11}
\]

Let \( |0\rangle \) and \( |0\rangle \) be the highest weight vectors defined by \( a_n|0\rangle = 0 \) \((n \geq 0)\) and \( 0|a_n = 0 \) \((n \leq 0)\), respectively. Denote by \( \mathcal{F} \) (resp. \( \mathcal{F}^* \)) the Fock space generated from the highest weight vector \( |0\rangle \) (resp. \( \langle 0| \)). The bilinear form \( \mathcal{F}^* \otimes \mathcal{F} \rightarrow \mathbb{C} \) is defined by setting \( \langle 0|0\rangle = 1 \).
Definition 2.3. Define the vertex operators \( \eta(z), \xi(z) \) and \( \varphi^\pm(z) \in \End(F)[[z^{\pm 1}]] \) by

\[
\eta(z) = \exp\left( \sum_{n=1}^{\infty} \frac{1 - t^n}{n} z^n \right) \exp\left( - \sum_{n=1}^{\infty} \frac{1 - t^n}{n} z^{-n} \right),
\]

(2.12)

\[
\xi(z) = \exp\left( - \sum_{n=1}^{\infty} \frac{1 - t^n}{n} q^{-n/2} t^{n/2} z^n \right) \exp\left( \sum_{n=1}^{\infty} \frac{1 - t^n}{n} q^{-n/2} t^{-n/2} z^{-n} \right),
\]

(2.13)

\[
\varphi^+(z) = \exp\left( - \sum_{n=1}^{\infty} \frac{1 - t^n}{n} (1 - t^n q^{-n/4}) t^{-n/4} z^n \right),
\]

(2.14)

\[
\varphi^-(z) = \exp\left( \sum_{n=1}^{\infty} \frac{1 - t^n}{n} (1 - t^n q^{-n/4}) t^{-n/4} z^{-n} \right).
\]

(2.15)

Fact 2.4 (20). Let \( u \) be an indeterminate and \( M \) be an integer. The algebra homomorphism \( \rho_u : U \rightarrow \End(F) \) defined by

\[
c^{1/2} \mapsto (t/q)^{1/4}, \quad x^+(z) \mapsto uz^{-M} q^{-M/2} t^{M/2} \eta(z), \quad x^-(z) \mapsto u^{-1} z^M q^{M/2} t^{-M/2} \xi(z),
\]

(2.16)

\[
\psi^+(z) \mapsto q^{M/2} t^{-M/2} \varphi^+(z), \quad \psi^-(z) \mapsto q^{-M/2} t^{M/2} \varphi^-(z)
\]

(2.17)

endows \( F \) with the level \((1,M)\)-module structure.

We denote by \( F_u^{(1,M)} \) the Fock space endowed with the level \((1,M)\)-module structure. The dual space \( F^* \) can also be endowed with the right \( U \)-module structure through \( \rho_u \). Then it is denoted by \( F_u^{(1,M)^*} \). The \( \rho_u \) is called the horizontal representation.

Next, we consider the level \((0,1)\)-module. Let \( F^{(0,1)} \) be the vector space spanned by the vectors \( |\lambda\rangle \) with \( \lambda \in P \). Define \( \langle \lambda | \mu \rangle = \delta_{\lambda,\mu} \).

Fact 2.5 (21 22). Let \( u \) be an indeterminate. The following action gives the level \((0,1)\)-module structure to \( F^{(0,1)} \):

\[
c^{1/2} |\lambda\rangle = |\lambda\rangle,
\]

(2.18)

\[
x^+(z) |\lambda\rangle = \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i} \delta(q^{\lambda_i} t^{-i+1} u/z) |\lambda + 1_i\rangle,
\]

(2.19)

\[
x^-(z) |\lambda\rangle = q^{1/2} t^{-1/2} \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i} \delta(q^{\lambda_i} t^{i+1} u/z) |\lambda - 1_i\rangle,
\]

(2.20)

\[
\psi^+(z) |\lambda\rangle = q^{1/2} t^{1/2} B^+_{\lambda} (u/z) |\lambda\rangle,
\]

(2.21)

\[
\psi^-(z) |\lambda\rangle = q^{-1/2} t^{-1/2} B^-_{\lambda} (z/u) |\lambda\rangle.
\]

(2.22)

Here, \( A^\pm_{\lambda,i} \in \mathbb{Q}(q,t) \) and \( B^\pm_{\lambda} (z) \in \mathbb{Q}(q,t)[[z]] \) are defined by

\[
A^+_{\lambda,i} = (1 - t) \prod_{j=1}^{i-1} \frac{1 - q^{\lambda_{i-1+j}} t^{-i+j+1}}{1 - q^{\lambda_{i-1+j}} t^{i+j+1} - q^{\lambda_{i-1+j}} t^{-i+j+1}},
\]

(2.23)

\[
A^-_{\lambda,i} = (1 - t^{-1}) \frac{1 - q^{\lambda_{i+1-\lambda_i}}}{1 - q^{\lambda_{i+1-\lambda_i} + 1} t^{-i-1}} \prod_{j=i+1}^{\infty} \frac{1 - q^{\lambda_j} t^{-j-1}}{1 - q^{\lambda_j} t^{j-1}} (1 - q^{\lambda_j+1} t^{-j-1})(1 - q^{\lambda_j-1} t^{j+1}),
\]

(2.24)

\[
B^+_{\lambda} (z) = \frac{1 - q^{\lambda_1 - 1} t z}{1 - q^{\lambda_1} t z} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_{i+1} t^{-i-1}}}{1 - q^{\lambda_{i+1} t^{-i-1}} + 1} (1 - q^{\lambda_{i+1}} t^{-i-1}),
\]

(2.25)

\[
B^-_{\lambda} (z) = \frac{1 - q^{\lambda_1 - 1} t z}{1 - q^{\lambda_1} t z} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_{i+1} t^i}}{1 - q^{\lambda_{i+1} t^i}} (1 - q^{\lambda_{i+1}} t^{-i-1}) (1 - q^{\lambda_{i+1}} t^{-i-1} + 1).
\]

(2.26)

We denote this module by \( F_u^{(0,1)} \). This is called the vertical representation or the preferred direction. By using the two representations, the trivalent intertwiners \( \Phi, \Phi^* \) of the DIM algebra were introduced in 9.
Fact 2.6. Let \( M \) be an integer. If \( w = -vu \), there exists a unique linear operator

\[
\Phi \left[ \begin{array}{c} (1, M + 1), w \\ (0, 1), v; (1, M), u \end{array} \right] : \mathcal{F}_v^{(1,1)} \otimes \mathcal{F}_u^{(1,M)} \rightarrow \mathcal{F}_w^{(1,M+1)}
\]

such that \( \langle 0 | \Phi(|\emptyset\rangle \otimes |0\rangle) = 1 \) and

\[
a \Phi = \Phi \Delta(a) \quad (\forall a \in \mathcal{U}).
\]

Similarly, there exists a unique linear operator

\[
\Phi^* \left[ \begin{array}{c} (1, M), u; (0, 1), v' \\ (1, M + 1), -vu \end{array} \right] : \mathcal{F}_w^{(1,M+1)} \rightarrow \mathcal{F}_u^{(1,M)} \otimes \mathcal{F}_v^{(0,1)}
\]

such that \( (|0\rangle \otimes \langle \emptyset|) \Phi^* |0\rangle = 1 \) and

\[
\Delta(a) \Phi^* = \Phi^* a \quad (\forall a \in \mathcal{U}).
\]

It is known that these intertwining operators can be realized as follows.

Definition 2.7. For a partition \( \lambda \), define the \( \lambda \)-component \( \Phi_\lambda \) of \( \Phi \)

\[
\Phi_\lambda \left[ \begin{array}{c} (1, M + 1), -uv \\ (0, 1), v; (1, M), u \end{array} \right] : \mathcal{F}_v^{(1,M)} \rightarrow \mathcal{F}_u^{(1,M+1)}
\]

by

\[
\Phi_\lambda(\alpha) = \Phi(|\lambda\rangle \otimes \alpha) \quad (\forall \alpha \in \mathcal{F}_u^{(1,M)}).
\]

Similarly, define the \( \lambda \)-component \( \Phi_\lambda^* \) of \( \Phi^* \)

\[
\Phi_\lambda \left[ \begin{array}{c} (1, M), u; (0, 1), v' \\ (1, M + 1), -vu \end{array} \right] : \mathcal{F}_w^{(1,M+1)} \rightarrow \mathcal{F}_v^{(1,M)}
\]

by

\[
\Phi_\lambda^*(\alpha) = \sum_\lambda \Phi_\lambda^*(\alpha) \otimes |\lambda\rangle \quad (\forall \alpha \in \mathcal{F}_w^{(1,M+1)}).
\]

Notation 2.8. For \( \lambda \in \mathcal{P} \), set

\[
n(\lambda) = \sum_{i \geq 1} (i - 1) \lambda_i, \quad f_\lambda = (-1)^{|\lambda|} q^{n(\lambda') + |\lambda|/2} t^{-n(\lambda) - |\lambda|/2},
\]

\[
c_\lambda := \prod_{(i,j) \in \lambda} \left( 1 - q^{\alpha(i,j) + \lambda(i,j) + 1} \right), \quad c'_\lambda := \prod_{(i,j) \in \lambda} \left( 1 - q^{\alpha(i,j) + 1 + \lambda(i,j)} \right).
\]

Fact 2.9. \( \Phi_\lambda \) is of the form

\[
\Phi_\lambda \left[ \begin{array}{c} (1, M + 1), -vu \\ (0, 1), v; (1, M), u \end{array} \right] = \hat{\Phi}(\lambda, u, v, M) \Phi_\lambda(v),
\]

where

\[
\hat{\Phi}(\lambda, u, v, M) = (-vu)^{|\lambda|} (-v^{M+1})^{|\lambda|} f_\lambda^{M-1} q^{n(\lambda')} / c_\lambda,
\]

\[
\Phi_\lambda(v) = \Phi_\emptyset(v) \eta_\lambda(v) :,
\]

\[
\Phi_\emptyset(v) = \exp\left( \sum_{n=1}^{\infty} \frac{1}{n} a_n v^n \right) \exp\left( \sum_{n=1}^{\infty} \frac{1}{n} q^n a_n v^{-n} \right),
\]

\[
\eta_\lambda(v) = \prod_{i=1}^{\infty} \prod_{j=1}^{\lambda_j} \eta(q^{\ell-i+1} t - v^{i+j}) :.
\]

The symbol : \( \cdots : \) means the usual normal ordering product. Similarly, \( \Phi_\lambda^* \) is of the form

\[
\Phi_\lambda^* \left[ \begin{array}{c} (1, M), u; (0, 1), v \\ (1, M + 1), -vu \end{array} \right] = \hat{\Phi}^*(\lambda, u, v, M) \Phi_\lambda^*(u),
\]
where

\[ \hat{t}^* \hat{t}^* = (q^{-1}v)^{-|\lambda|}(-u)^{|\lambda|} \frac{f^M_{\lambda} q^{n(\lambda)}/c_\lambda}{\hat{t}^* \hat{t}^*}, \]

\[ \Phi^*_\lambda(u) = \Phi^*_0(u) \xi_\lambda(u) :. \]

\[ \Phi^*_\mu(u) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1 - q^n q^{-n/2} t^{n/2} a_n u - n} \right) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n q^{-n/2} t^{n/2} a_n u - n}{1 - q^n q^{-n/2} t^{n/2} a_n u - n} \right), \]

\[ \xi_\lambda(u) = : \prod_{i=1}^\infty \prod_{j=1}^\infty \xi(q^{j-1} t^{-i+1} u) :. \]

**Figure 4.** The trivalent intertwining operators. Left: \( \Phi \), Right: \( \Phi^* \). Note that the diagrams in this paper are flipped horizontally for the ones in [7].

Throughout the paper, the case \( M = 0 \) is concerned. The intertwining operators \( \Phi \) and \( \Phi^* \) are expressed as the trivalent diagrams in Figure 4. It is known that their matrix elements coincide with the Iqbal, Kozcaz and Vafa’s or Awata and Kanno’s refined topological vertices [9, 23, 24], expressed as the trivalent diagrams in Figure 4. It is known that their matrix elements coincide with the Iqbal, Kozcaz and Vafa’s or Awata and Kanno’s refined topological vertices [9, 23, 24], and the vertical representation corresponds to the preferred direction. We prepare the formula for the normal ordering of the intertwiners, in which the Nekrasov factor appears.

**Definition 2.10.** Define the Nekrasov factor to be

\[ N_{\lambda\mu}(u) := \prod_{(i,j) \in \lambda} \left( 1 - uq^{-a_{\mu(i,j)}-1} t^{-\ell_{\lambda(i,j)}} \right) \prod_{(i,j) \in \mu} \left( 1 - uq^{a_{\lambda(i,j)} t^{\ell_{\mu(i,j)}+1}} \right) \]

\[ = \prod_{(i,j) \in \lambda} \left( 1 - uq^{-a_{\mu(i,j)} t^{\ell_{\mu(i,j)}+1}} \right) \prod_{(i,j) \in \mu} \left( 1 - uq^{a_{\lambda(i,j)}-1} t^{-\ell_{\lambda(i,j)}} \right). \]

**Fact 2.11.** Put \( \gamma = (t/q)^{1/2} \). We have

\[ \Phi_{\lambda(i)}(v_i) \Phi^*_{\mu(j)}(u_j) = G(u_j/\gamma v_i)^{-1} N_{\lambda(i)\mu(j)}(u_j/\gamma v_i) \Phi_{\lambda(i)}(v_i) \Phi^*_{\mu(j)}(u_j) :, \]

\[ \Phi^*_{\mu(j)}(u_j) \Phi_{\lambda(i)}(v_i) = G(v_i/\gamma u_j)^{-1} N_{\lambda(i)\mu(j)}(v_i/\gamma u_j) \Phi^*_{\mu(j)}(u_j) \Phi_{\lambda(i)}(v_i) :, \]

\[ \Phi_{\lambda(i)}(v_i) \Phi_{\lambda(j)}(v_j) = \frac{G(v_j/\gamma 2 v_i)}{N_{\lambda(i)\lambda(i)}(v_j/\gamma 2 v_i)} \Phi_{\lambda(i)}(v_i) \Phi_{\lambda(j)}(v_j) :, \]

\[ \Phi^*_{\mu(j)}(u_j) \Phi^*_{\mu(j)}(u_j) = \frac{G(u_j/u_i)}{N_{\mu(j)\mu(j)}(u_j/u_i)} \Phi^*_{\mu(j)}(u_j) \Phi^*_{\mu(j)}(u_j) :, \]

where \( G(z) = \prod_{i=0}^{\infty} (1 - z q^i t^{-j}) = (z; q, t^{-1})_{\infty}. \)

Furthermore, we introduce the following operators for convenience, which is expressed by the cross diagram in Figure 4.

**Definition 2.12.** Define

\[ \Phi_{cr}^* \left[ \begin{array}{c} u,s/v \\ v,s/u \end{array} \right] = \Phi^* \left[ \begin{array}{c} (1,0), v;(0,1), u/s/v \\ (1,1), -u/s \\ (1,1), -us \end{array} \right] \circ \Phi \left[ \begin{array}{c} (1,0), -us/v; (0,1), s; (1,0), u \\ (1,0), -us \end{array} \right], \]

\[ \langle 0 | \Phi^*_0 \left[ \begin{array}{c} (1,0), v;(0,1), u/s/v \\ (1,1), -u/s \end{array} \right] \Phi_0 \left[ \begin{array}{c} (1,0), -us/v; (0,1), s; (1,0), u \\ (1,0), -us \end{array} \right] |0 \rangle, \]

Note that we modify the normalization of \( \Phi^*_\lambda \) from the previous paper [7] by \( c_\lambda \) and \( c^*_\lambda \).

---

2 Note that we modify the normalization of \( \Phi^*_\lambda \) from the previous paper [7] by \( c_\lambda \) and \( c^*_\lambda \).
Here, we have set \( \langle 0 | \Phi_w^0 \rangle \equiv \langle \Phi^0 | 0 \rangle \) as the vacuum expectation value of the operator \( \Phi_w^0 \). Define the operator

\[
\Phi_{w'} \equiv \sum_{\mu \in \mathbb{Z} \cap \mathbb{Z}_+} \Phi_{w'}^{(0, \mu)}.
\]

Furthermore, we define the normalized operator

\[
T^H(u, v, w) : \bigotimes_{1 \leq i \leq N} F_{v_i}^{(0, 1)} \rightarrow \bigotimes_{1 \leq i \leq N} F_{u_i}^{(0, 1)}.
\]

as the vacuum expectation value \( \langle 0 | \cdots | 0 \rangle \) of the operator \( \bigotimes_{1 \leq i \leq N} \Phi_{w'} \), with respect to the level \((1,0)\) representation. Furthermore, we define the normalized operator

\[
T^H(u, v, w) = \frac{T^H(u, v, w)}{\langle 0 | T^H(u, v, w) | 0 \rangle}.
\]

Here, we have set \( \langle 0 \rangle = \langle 0 | \otimes \cdots \otimes | 0 \rangle \).

**Definition 2.13.** Take the composition

\[
\Phi_{w'}^{(0,1)} \bigotimes_{1 \leq i \leq N} F_{v_i}^{(0,1)} \xrightarrow{id \otimes \cdots \otimes \Phi_{w'}^{(1,0)}} \bigotimes_{1 \leq i \leq N} F_{u_i}^{(0,1)}.
\]

Here \( w' = \frac{w_{v_1 \cdots v_N}}{v_{1 \cdots v_N}} \). Define the operator

\[
T^H(u, v, w) : \bigotimes_{1 \leq i \leq N} F_{v_i}^{(0, 1)} \rightarrow \bigotimes_{1 \leq i \leq N} F_{u_i}^{(0, 1)}.
\]

as the vacuum expectation value \( \langle 0 | \cdots | 0 \rangle \) of the operator \( \bigotimes_{1 \leq i \leq N} \Phi_{w'} \), with respect to the level \((1,0)\) representation. Furthermore, we define the normalized operator

\[
T^H(u, v, w) = \frac{T^H(u, v, w)}{\langle 0 | T^H(u, v, w) | 0 \rangle}.
\]

Here, we have set \( \langle 0 \rangle = \langle 0 | \otimes \cdots \otimes | 0 \rangle \).

**Definition 2.14.** Define the operators

\[
T^V(u, v, w), T^V(u, v, w) : \bigotimes_{1 \leq i \leq N} F_{v_i}^{(1, 0)} \rightarrow \bigotimes_{1 \leq i \leq N} F_{u_i}^{(1, 0)}.
\]

by

\[
T^V(u, v, w) = \sum_{\mu^{(1)}, \cdots, \mu^{(N-1)} \in \mathbb{Z}_+} \Phi^{(1, 0)}_{v_k, \mu^{(k)}}, \mu^{(k-1)} = \mu^{(1)} \in \mathbb{Z}_+.
\]

\[
T^V(u, v, w) = \frac{T^V(u, v, w)}{\langle 0 | T^V(u, v, w) | 0 \rangle}.
\]

Here, \( w_k = \frac{w_{v_{1 \cdots v_{k-1}}}}{v_{1 \cdots v_{k-1}}} w_k \).

We prepare some notations to treat the \( N \)-fold Fock tensor spaces.
Notation 2.15. We write
\[ \mathcal{F}_u^{(N,0)} = \bigotimes_{1 \leq i \leq N} \mathcal{F}_u^{(1,0)}, \quad \mathcal{F}_u^{(N,0)*} = \bigotimes_{1 \leq i \leq N} \mathcal{F}_u^{(1,0)*}, \]
(2.60)
\[ \mathcal{F}_u^{(0,N)} = \bigotimes_{1 \leq i \leq N} \mathcal{F}_u^{(0,1)} \quad (u = (u_1, \ldots, u_N)), \]
(2.61)
\[ a_n^{(i)} = \underbrace{1 \otimes \cdots \otimes 1}_{N-i} \otimes a_n \otimes \underbrace{1 \otimes \cdots \otimes 1}_{i-1}. \]
(2.62)
For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)}) \in \mathbb{R}^N \), set
\[ |\lambda\rangle = \bigotimes_{1 \leq i \leq N} |\lambda^{(i)}\rangle, \quad \langle \lambda | = \bigotimes_{1 \leq i \leq N} \langle \lambda^{(i)} |, \]
(2.63)
\[ |a\lambda\rangle = a_1^{(1)} a_2^{(1)} \cdots a_1^{(2)} a_2^{(2)} \cdots a_1^{(N)} a_2^{(N)} |0\rangle, \]
(2.64)
\[ \langle a\lambda | = \langle 0 | \cdots a_1^{(N)} a_2^{(N)} \cdots a_1^{(2)} a_2^{(2)} \cdots a_1^{(1)} a_2^{(1)} |. \]
(2.65)

In [7], the S-duality formula for the matrix elements of \( T^V \) and \( T^H \) is proved. Recall that the matrix elements of \( T^H \) with respect to the basis \( (|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(N)}\rangle) \) can be easily calculated by operator products. On the other hand, the basis on \( \mathcal{F}_u^{(N,0)} \) which corresponds to \( |\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(N)}\rangle \) is defied as the eigenfunctions of the operator \( X_0^{(1)} \) given as follows.

**Definition 2.16.** Define the operator \( X^{(1)}(z) = \sum_{n \in \mathbb{Z}} X_n^{(1)} z^{-n} \in \text{End}(\mathcal{F}_u^{(N,0)}) \) by
\[ X^{(1)}(z) = (\rho_{a_1} \otimes \rho_{a_2} \otimes \cdots \otimes \rho_{a_N}) \circ \Delta^{(N)}(x^+(z)). \]
(2.66)
Here,
\[ \Delta^{(1)} := \text{id}, \quad \Delta^{(N)} := (\text{id} \otimes \cdots \otimes \text{id} \otimes \Delta) \circ \cdots \circ (\text{id} \otimes \Delta) \circ \Delta \quad (N \geq 2). \]
(2.67)

**Definition 2.17.** For \( k = 1, 2, \ldots, N \), set
\[ \Lambda^{(i)}(z) := \varphi^{-}(\gamma^{1/2}z) \otimes \cdots \otimes \varphi^{-}(\gamma^{i-3/2}z) \otimes \eta(\gamma^{i-1}z) \otimes 1 \otimes \cdots \otimes 1. \]
(2.68)

**Fact 2.18.** On \( \mathcal{F}_u^{(N,0)} \), we get
\[ X^{(1)}(z) = \sum_{i=1}^{N} u_i \Lambda^{(i)}(z). \]
(2.69)

Let \( \Lambda \) be the ring of symmetric functions, and \( p_n \) be the power sum symmetric function of degree \( n \). Then the map
\[ \mathcal{F} \ni |a\lambda\rangle \mapsto \prod_{i \geq 1} p_{\lambda_i} \in \Lambda \]
(2.70)
gives the isomorphism as graded vector spaces between \( \mathcal{F} \) and \( \Lambda \). If \( N = 1 \), the operator \( X_0^{(1)} \) is essentially the same as Macdonald’s difference operator under this isomorphism [25]. Therefore, its eigenfunctions can be identified with the ordinary Macdonald functions. In the case of general \( N \), the eigenfunctions of \( X_0^{(1)} \) can be viewed as a generalization of Macdonald functions. Their existence theorem is given in terms of the following generalized dominance partial ordering.

**Definition 2.19.** We write \( \lambda \geq^L \mu \) (resp. \( \lambda \geq^R \mu \)) if and only if \( |\lambda| = |\mu| \) and
\[ |\lambda^{(N)}| + \cdots + |\lambda^{(j+1)}| + \sum_{k=1}^{i} \lambda^{(j)}_k \geq |\mu^{(N)}| + \cdots + |\mu^{(j+1)}| + \sum_{k=1}^{i} \mu^{(j)}_k \]
(2.71)
Definition 2.22. Similarly, there exists a unique vector for the operator product formulas for the affine screenings given in [1] are reproduced in Appendix B. Stationary Ruijsenaars functions in terms of the Mukadé operators. To help the interested readers, the following is the existence theorem of the generalized Macdonald functions.

Fact 2.21 (Existence and Uniqueness [26, 27]). For an N-tuple of partitions \( \lambda \), there exists a unique vector \( |P_\lambda(u)\rangle \in \mathcal{F}_u^{(N,0)} \) such that

\[
|P_\lambda(u)\rangle = \prod_{i=1}^{N} m_{\lambda^{(i)}}(a_{-n_i}^{(i)})|0\rangle + \sum_{\mu \lessdot \lambda} v_{\lambda,\mu} \prod_{i=1}^{N} m_{\mu^{(i)}}(a_{-n_i}^{(i)})|0\rangle, \quad v_{\lambda,\mu} \in \mathbb{C}(u); \tag{2.73}
\]

\[
X_0^{(1)}|P_\lambda(u)\rangle = \epsilon_\lambda(u)|P_\lambda(u)\rangle, \quad \epsilon_\lambda(u) \in \mathbb{C}(u). \tag{2.74}
\]

Similarly, there exists a unique vector \( \langle P_\lambda| = \langle P_\lambda(u)| \in \mathcal{F}_u^{(N,0)*} \) such that

\[
\langle P_\lambda(u)| = (0) \prod_{i=1}^{N} m_{\lambda^{(i)}}(a_{n_i}^{(i)}) + \sum_{\mu \lessdot \lambda} v_{\lambda,\mu}^* (0) \prod_{i=1}^{N} m_{\mu^{(i)}}(a_{n_i}^{(i)}) \quad v_{\lambda,\mu}^* \in \mathbb{C}(u); \tag{2.75}
\]

\[
\langle P_\lambda(u)| X_0^{(1)} = \epsilon_\lambda^*(u) \langle P_\lambda(u)|, \quad \epsilon_\lambda^*(u) \in \mathbb{C}(u). \tag{2.76}
\]

The eigenvalues \( \epsilon_\lambda \) and \( \epsilon_\lambda^* \) are of the forms

\[
\epsilon_\lambda(u) = \epsilon_\lambda^*(u) = \sum_{k=1}^{N} u_k \epsilon_{\lambda^{(k)}}, \quad \epsilon_\lambda := 1 + (t - 1) \sum_{i \geq 1} (q^{|\lambda|} - 1)t^{-i}. \tag{2.77}
\]

Definition 2.22. Set

\[
|Q_\lambda := \prod_{i=1}^{N} C_{\lambda^{(i)}}^\lambda |P_\lambda\rangle. \tag{2.78}
\]

Fact 2.23 ([26]). It follows that

\[
\langle P_\lambda|Q_\mu \rangle = \delta_{\lambda,\mu}. \tag{2.79}
\]

The following is the S-duality formula for changing the preferred directions. See also Figure 6.

Theorem 2.24 ([7]). We have

\[
\langle \mu | T^H(u,v; w) | \lambda \rangle = \langle P_\lambda | T^V(u,v; w) |Q_\mu \rangle \times (-1)^{|\lambda| + |\mu|}. \tag{2.80}
\]

Theorem 2.24 is essentially proved in [7]. See Appendix C.2 as to the appearance of the factor \((-1)^{|\lambda| + |\mu|}\). For the explicit form of the matrix elements, see Fact 3.33.

3. Proofs of Main Theorems

3.1. Non-stationary Ruijsenaars Functions and Intertwining Operators. In [1], an operator formula is given for the non-stationary Ruijsenaars functions by using the affine screening currents [28, 29, 30]. In this subsection, we show that the affine screening currents can be reproduced from the intertwiners of the DIM algebra in the special case of \( \kappa = t^{-1} \), giving an expression of the non-stationary Ruijsenaars functions in terms of the Mukadé operators. To help the interested readers, the operator product formulas for the affine screenings given in [1] are reproduced in Appendix B.
Definition 3.1. Define
\[
\mathcal{A}(z) = \exp \left( -\sum_{n>0} \frac{1 - t^{-n}}{n(1 - q^{-n})} a_{-n} z^n \right) \exp \left( \sum_{n>0} \frac{1 - t^{n}}{n(1 - q^{-n})} a_n z^{-n} \right), \quad (3.1)
\]
\[
\mathcal{A}^*(z) = \exp \left( \sum_{n>0} \frac{1 - t^{-n}}{n(1 - q^{n})} \gamma^n a_{-n} z^n \right) \exp \left( -\sum_{n>0} \frac{1 - t^{-n}}{n(1 - q^{n})} \gamma^n a_n z^{-n} \right). \quad (3.2)
\]

These operators \(\mathcal{A}(z)\) and \(\mathcal{A}^*(z)\) appear in the following decomposition of the intertwiners \(\tilde{\Phi}_\lambda(z)\) and \(\tilde{\Phi}^*_\lambda(z)\).

Proposition 3.2. For a partition \(\lambda = (m_1, \ldots, m_\ell)\),
\[
\tilde{\Phi}_{(m_1, \ldots, m_\ell)}(z) = \tilde{\Phi}_0(t^{-\ell}z) A(q^{m_1}z) \cdots A(q^{m_\ell}z) : \quad (3.3)
\]
\[
\tilde{\Phi}^*_{(m_1, \ldots, m_\ell)}(z) = \tilde{\Phi}^*_0(t^{-\ell}z) A^*(q^{m_1}z) \cdots A^*(q^{m_\ell}z) : \quad (3.4)
\]

Let \(N \geq 2\) in this subsection. The case \(N = 1\) will be considered in Subsection 3.4. Define the screening currents as follows.

Notation 3.3. For an \(N\)-tuple of the parameters \(u = (u_1, \ldots, u_N)\), we write
\[
t^{\alpha_i} \cdot u := (u_1, \ldots, u_{i-1}, tu_i, t^{-1}u_{i+1}, u_{i+2}, \ldots, u_N), \quad 1 \leq i \leq N - 1, \quad (3.5)
\]
\[
t^{\alpha_0} \cdot u := (t^{-1}u_1, u_2, \ldots, u_{N-1}, tu_N). \quad (3.6)
\]

Here, \(\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\) are regarded as the classical part of the real simple roots of the affine Lie algebra \(\mathfrak{gl}_N\).

Definition 3.4. Define the screening currents \(S^{(i)}(y) : \mathcal{F}_{t^{\alpha_i}u} \to \mathcal{F}_u\) by
\[
\tilde{S}^{(i)}(z) := 1 \otimes \cdots \otimes 1 \otimes A^*(\gamma^{-i}z) \otimes \mathcal{A}(\gamma^{-i}z) \otimes 1 \otimes \cdots \otimes 1, \quad i = 1, \ldots, N - 1, \quad (3.7)
\]
\[
\tilde{S}^{(0)}(z) := \mathcal{A}(z) \otimes 1 \otimes \cdots \otimes 1 \otimes \mathcal{A}^*(\gamma^{-N}t^{-1}z). \quad (3.8)
\]

We cyclically identify \(\tilde{S}^{(i+N)}(z) = \tilde{S}^{(i)}(z)\).

Remark 3.5. Note that these screening currents essentially coincide with those in \([28, 29, 30]\) when \(\kappa = t^{-1}\).

Proposition 3.6. We have
\[
\mathcal{A}(z) A(w) = \frac{(q w/z; q)_\infty}{(w q/z; q)_\infty} : A(z) A(w) :, \quad A^*(z) A^*(w) = \frac{(w/z; q)_\infty}{(tw/z; q)_\infty} : A^*(z) A^*(w) :, \quad (3.9)
\]
\[ A(z)A^*(w) = \frac{(q^\gamma w/z; q)_\infty}{(q^\gamma w/tz; q)_\infty} : A(z)A^*(w) :, \quad A^*(z)A(w) = \frac{(q^\gamma w/z; q)_\infty}{(q^\gamma w/tz; q)_\infty} : A^*(z)A(w) :: \] (3.10)

and for \( N \geq 3 \) we obtain

\[ \tilde{S}^{(i)}(z)\tilde{S}^{(i)}(w) = (1 - w/z)\frac{(qw/tz; q)_\infty}{(tw/z; q)_\infty} : \tilde{S}^{(i)}(z)\tilde{S}^{(i)}(w) : \quad (i = 0, \ldots, N - 1), \] (3.11)
\[ \tilde{S}^{(i)}(z)\tilde{S}^{(i+1)}(w) = \frac{(qw/tz; q)_\infty}{(w/z; q)_\infty} : \tilde{S}^{(i)}(z)\tilde{S}^{(i+1)}(w) : \quad (i = 0, \ldots, N - 2), \] (3.12)
\[ \tilde{S}^{(i+1)}(z)\tilde{S}^{(i)}(w) = \frac{(tw/z; q)_\infty}{(w/z; q)_\infty} : \tilde{S}^{(i+1)}(z)\tilde{S}^{(i)}(w) : \quad (i = 0, \ldots, N - 2), \] (3.13)
\[ \tilde{S}^{(0)}(z)\tilde{S}^{(N-1)}(w) = \frac{(t^2w/z; q)_\infty}{(tw/z; q)_\infty} : \tilde{S}^{(0)}(z)\tilde{S}^{(N-1)}(w) :, \] (3.14)
\[ \tilde{S}^{(N-1)}(z)\tilde{S}^{(0)}(w) = \frac{(qw/tz; q)_\infty}{(qw/t^2z; q)_\infty} : \tilde{S}^{(N-1)}(z)\tilde{S}^{(0)}(w) :, \] (3.15)
\[ \tilde{S}^{(i)}(z)\tilde{S}^{(j)}(w) = \tilde{S}^{(i)}(z)\tilde{S}^{(j)}(w) : \quad \text{for} \ 0 \leq |i - j| \leq 2. \] (3.16)

Let us introduce the following vertex operator 4.

**Notation 3.7.** Write

\[ \gamma^{-1}_t \cdot u := (\gamma^{-1}_1 u_1, \ldots, \gamma^{-1}_N u_N). \] (3.17)

**Definition 3.8.** Define \( \phi_0(z) : \mathcal{F}^{(N,0)}_{\gamma^{-1}_t \cdot k, u} \to \mathcal{F}^{(N,0)}_{\gamma^{-1}_t} \) by

\[ \phi_0(z) = \bigotimes_{1 \leq k \leq N} : \Phi_0^*(t^{-1}\gamma_k z) \Phi_0(t^{-1}\delta_{k,1} \gamma_k^1 - 1) z :. \] (3.18)

**Proposition 3.9.** We have

\[ \phi_0(z)\tilde{S}^{(1)}(w) = \frac{(qw/z; q)_\infty}{(qw/tz; q)_\infty} : \phi_0(z)\tilde{S}^{(1)}(w) :, \] (3.19)
\[ \phi_0(z)\tilde{S}^{(i)}(w) = : \phi_0(z)\tilde{S}^{(i)}(w) :, \quad (2 \leq i \leq N - 1), \] (3.20)
\[ \tilde{S}^{(1)}(z)\phi_0(w) = \frac{(tw/z; q)_\infty}{(w/z; q)_\infty} : \tilde{S}^{(1)}(z)\phi_0(w) :, \] (3.21)
\[ \tilde{S}^{(i)}(z)\phi_0(w) = \tilde{S}^{(i)}(z)\phi_0(w) : \quad (2 \leq i \leq N - 1), \] (3.22)
\[ \phi_0(z)\tilde{S}^{(0)}(w) = \frac{(w/z; q)_\infty}{(tw/z; q)_\infty} : \phi_0(z)\tilde{S}^{(0)}(w) :, \] (3.23)
\[ \tilde{S}^{(0)}(z)\phi_0(w) = \frac{(qw/tz; q)_\infty}{(qw/t^2z; q)_\infty} : \tilde{S}^{(0)}(z)\phi_0(w) :, \] (3.24)
\[ \phi_0(z)\phi_0(w) = \frac{(qw/tz; q)_\infty}{(tw/z; q)_\infty} : \phi_0(z)\phi_0(w) :. \] (3.25)

These screening currents and \( \phi_0(z) \) can be obtained by a specialization of the Mukadé operators. Firstly, we consider the non-affine case and derive the Macdonald functions from specialized Mukadé operators to fix our starting point for making the \( p \)-traces (FIGURE 1).

**Definition 3.10.** For \( 1 \leq i \leq N \), define

\[ \tilde{T}_i^V(z) = \tilde{T}_i^V(u; z) := T^V(v, u; z) \bigg|_{v_k \to \gamma^{-1}_t \delta_{k,1} u_k, (1 \leq k \leq N)} \] (3.26)

\[ ^3 \text{For } N = 2, \text{ we have a different form of the normal ordering between } S^{(0)} \text{ and } S^{(1)}. \text{ However, our results in what follows hold for general } N \geq 2. \]

\[ ^4 \text{Comparing the notation in [7], we have } \phi_0(z) = \Phi^{(0)}(t^{-1}z) \text{ with exception for the spectral parameters of the Fock space.} \]
When we construct $T^V$, we need to compose many $\Phi^\text{tr}$'s producing a big summation running over the set of the partitions in $P^{N-1}$. By giving a certain condition to the spectral parameters attached to the internal edges, we have the “restricted operator” $\tilde{T}^V_i(z)$. Then, one finds that all the internal partitions are allowed to run over the one row diagrams satisfying certain interlacing conditions among them.

**Fact 3.11** (Appendix A in [7]). We have

$$\tilde{T}^V_i(u; z) = \left(\frac{(q/t, q)_{\infty}}{(q, q)_{\infty}}\right)^{i-1} \sum_{0 \leq m_1 \leq m_2 \leq \cdots \leq m_{i-1} < \infty} \phi_0(z) \prod_{1 \leq j \leq i-1} \tilde{S}^{(j)}(q^{m_j}z) \cdot \prod_{k=1}^{i-1} (u_{k+1}/u_k)^{m_k}. \quad (3.28)$$

We call $\tilde{T}^V_i$ the “screened vertex operator”. From these screened vertex operators, we can construct the Macdonald functions.

**Definition 3.12.** Let $s = (s_1, \ldots, s_N)$, $x = (x_1, \ldots, x_N)$ be $N$-tuples of indeterminates. Define the formal series $f_{\text{gl}}(x; s|q,t) \in \mathbb{Q}(q,t,s) [x_1/x_1, \ldots, x_N/x_N - 1]$ by

$$f_{\text{gl}}(x; s|q,t) = \sum_{\theta \in M_N} c_N(\theta; s|q,t) \prod_{1 \leq i < j \leq N} (x_j/x_i)^{\theta_{ij}}, \quad (3.29)$$

where $M_N = \{ (\theta_{ij})_{1 \leq i, j \leq N} \}$ is the set of $N \times N$ strictly upper triangular matrices with nonnegative integer entries ($\theta_{ij} \in \mathbb{Z}_{\geq 0}$, $i \geq j$ implies $\theta_{ij} = 0$), and the coefficient $c_N(\theta; s|q,t)$ is defined by

$$c_N(\theta; s|q,t) = \prod_{k=1}^{N} \prod_{2 \leq i < j \leq k} \left(\frac{q^{\sum_{a>k}(\theta_{ia}-\theta_{ja})}t s_j/s_i; q_{a_{ik}}}{q^{\sum_{a>k}(\theta_{ia}-\theta_{ja})}q s_j/s_i; q_{a_{ik}}}\right)^{\theta_k} \times \prod_{k=2}^{N} \prod_{2 \leq i < j \leq k} \left(\frac{q^{-\theta_{jk}+\sum_{a>k}(\theta_{ia}-\theta_{ja})}q s_j/s_i; q_{a_{ik}}}{q^{-\theta_{jk}+\sum_{a>k}(\theta_{ia}-\theta_{ja})}s_j/s_i; q_{a_{ik}}}\right)^{\theta_k}. \quad (3.30)$$

It is known that $f_{\text{gl}}(x; s|q,t)$ is an eigenfunction of Macdonald’s difference operator [1, 5, 6]. For some basic facts about $f_{\text{gl}}(x; s|q,t)$, see Appendix A. This function can be reproduced as follows.

**Fact 3.13** (Appendix A of [7]). It follows that

$$\langle 0 | \tilde{T}^V_1(x; s_1) \tilde{T}^V_2(s_2) \cdots \tilde{T}^V_N(s_N) | 0 \rangle = \prod_{1 \leq i < j \leq N} \left(\frac{q x_j/x_i; q_{\infty}}{(x_j/x_i; q_{\infty})}\right) \cdot f_{\text{gl}}(s; x|q,t). \quad (3.31)$$

In Appendix A in [7], (3.31) was proved up to proportionality. We can easily calculate the proportional constant by taking the constant term of $s_i$’s and using $q$-binomial theorem.

**Remark 3.14.** By using the bispectral duality proved in [5], the right hand side in (3.31) can be rewritten as

$$\prod_{1 \leq i < j \leq N} \left(\frac{q x_j/x_i; q_{\infty}}{(x_j/x_i; q_{\infty})}\right) \cdot f_{\text{gl}}(s; x|q,t) = \prod_{1 \leq i < j \leq N} \left(\frac{q s_j/s_i; q_{\infty}}{(s_j/s_i; q_{\infty})}\right) \cdot f_{\text{gl}}(x; s|q,t). \quad (3.32)$$

We can also obtain this equation by applying the S-duality formula for the intertwiners (Theorem 2.27) to the left hand side in (3.31) and using Fact 3.13 again.

The formula (3.31) should be understood as the equation as formal power series in $s_{i+1}/s_i$ and $x_{i+1}/x_i$ ($i = 1, \ldots, N-1$). By Fact 3.14, we can also treat the variables $x_i$ and $s_i$ as complex numbers. We will give an affine analogue of the above facts. Since analyticity of the non-stationary Ruijsenaars functions has not been clarified, we treat $x_i$’s and $s_i$’s as indeterminates in the affine case.

Let $p$ be an indeterminate, and consider the following “loop operator” obtained by the loop of the Mukadé operator. (See FIGURE 7)

$^5$ $f_{\text{gl}}(x; s|q,t)$ coincides with $p_N(x; s|q,t)$ in [7].
Definition 3.15. Define $\tilde{T}_i^{\text{loop}}(x, p; s) : \mathcal{F}_{\gamma, t^{-1}}^{(N, 0)} \rightarrow \mathcal{F}_{x}^{(N, 0)}$ by

$$
\tilde{T}_i^{\text{loop}}(x, p; s) = \sum_{\mu^{(0)} \cdots \mu^{(N)} \in \mathcal{P}} p^{\mu^{(N)}} \bigg\langle \prod_{1 \leq k \leq N} \Phi_{k}^{\text{cr}} \left[ x_{k}; \frac{y_{1} \cdots y_{s}}{\lambda_{1} \cdots \lambda_{k}}, \mu^{(k)} ; y_{k} \right] \bigg\rangle_{y_{k} \rightarrow \gamma^{-1} t^{-\delta_{i},k} x_{k}}.
$$

$$
\sum_{\mu^{(0)}, \ldots, \mu^{(N)} \in \mathcal{P}} p^{\mu^{(N)}}
$$

The operator $\tilde{T}_i^{\text{loop}}$ can be expressed as follows. This is an affine analogue of Fact 3.11.

Definition 3.16. Set the shifted screening currents

$$
S_{i}(z) = \tilde{S}^{(i)}(t^{-1/N} z), \quad 0 \leq i \leq N - 1.
$$

The screening currents $S_{i}(z)$ are the realization of the operator in Appendix B in the case of $\kappa = t^{-1}$. In Fact 3.11 we expressed $\tilde{T}_i^{V}$ by composition of screening currents $\tilde{S}^{(i)}(z)$ and $\phi_{0}(z)$. In the affine case, we compose the screening currents as follows.

Definition 3.17. Define the affine screened vertex operators

$$
\phi_{i}(z) = :\phi_{i-1}(t^{-\gamma_{i}} z) S_{i}(z) : \quad (i = 1, \ldots, N - 1),
$$

$$
\phi_{i+N}(z) = \phi_{i}(z),
$$

$$
\phi_{\lambda}^{i}(z) = \phi_{i-\ell(\lambda)}(t^{-\ell(\lambda) + 1/N} z) \prod_{1 \leq j \leq \ell(\lambda)} S_{i-j+1}(t^{-j/N} q^{\lambda} z),
$$

$$
\Phi_{\lambda}^{i}(z) = \frac{(q/t; q)_{\infty}(q; q)_{\infty}}{(q; q)_{\infty}} \phi_{\lambda}^{i}(z).
$$

Proposition 3.18. Let $i = 1, \ldots, N$. Then we have

$$
\tilde{T}_i^{\text{loop}}(x, p; s) = \sum_{\lambda \in \mathcal{P}} p^{\lambda^{(i-1)}} \Phi_{\lambda}^{i-1}(t^{i/N} s) \prod_{1 \leq k \leq N} x_{k}^{\lambda^{(i-k-1)} - |\lambda^{(i-k-1)}|},
$$

where $|\lambda^{(i)}| = \sum_{j \equiv i+1 \text{ (mod $N$)}} \lambda_{j}$.

For the proof, we prepare two lemmas.

Lemma 3.19. Let $i = 1, \ldots, N$. For a partition $\lambda \in \mathcal{P}$, set $\mu^{(k)} = (\lambda_{j}; 1 \leq j \leq \ell(\lambda), i - j \equiv k \text{ mod $N$}).$ Then we have

$$
\prod_{1 \leq k \leq N} \frac{N_{\mu^{(k)}}, \mu^{(k)} (t^{\delta_{k+i}})}{N_{\mu^{(k)}}, \mu^{(k)} (1)} = \frac{N_{\lambda}^{(0)} (t^{1/N})}{N_{\lambda}^{(0)} (1)},
$$

$$
(\mu^{(0)} = \mu^{(N)})
$$
where $N_{\mu}[z|q,\kappa]$ is defined in Definition [3.21].

The proof is given in Section [C.1].

**Lemma 3.20.** Let $i = 1, \ldots, N$. Then we have
\[
\Phi^{i-1}(z) = t^{-1}N_{\lambda}^{[z|q,\kappa]}(1|t^{-1}/N)\cdot \phi^{i-1}(z).
\] (3.41)

The proof is given in [1 Section 2.5].

**Proof of Proposition 3.18.** First, the Nekrasov factors $\prod_{k=1}^{N} N_{\mu(k-1),\mu(k)}(t^{k}:i)$ arise from the normal ordering product of the operator
\[
\phi^{i-1}(z) = t^{-1}N_{\lambda}^{[z|q,\kappa]}(1|t^{-1}/N)\cdot \phi^{i-1}(z).
\]
(See Fact 2.11.) In general, for partitions $\nu$ and $\rho$, we have $N_{\nu,\rho}(1) \neq 0$ (resp. $N_{\rho,\nu}(t) \neq 0$) if and only if $\nu \subset \rho$ (resp. $\rho \subset \nu$). Here, we put $\bar{\nu} = (\nu_2, \nu_3 \ldots)$ for $\nu = (\nu_1, \nu_2, \nu_3 \ldots)$. Thus the partitions $\mu(k)$ in (3.33) are restricted by the cyclic interlacing conditions
\[
\mu(k-1) \subset \mu(k), \quad k = (1, \ldots, N) \quad (k \neq i);
\]
(3.43)
\[
\bar{\mu}^{(i-1)} \subset \bar{\mu}^{(i)}.
\]
Therefore, the $\mu(k)$'s can be expressed by the single partition
\[
\lambda = (\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_1, \mu_1), \quad \lambda = (\mu_1, \mu_2, \mu_2, \ldots, \mu_1, \mu_1, \ldots).
\]
(3.45)
By using this $\lambda$, the partitions $\mu(k)$'s can be written as
\[
\mathcal{M}(k) = (\lambda, 1 \leq j \leq \ell(\lambda), i - j = k \mod N).
\]
(3.46)
Recalling Proposition 3.2 and Definition 3.4, we have
\[
\phi^{i-1}(z) = t^{-1}N_{\lambda}^{[z|q,\kappa]}(1|t^{-1}/N)\cdot \phi^{i-1}(z).
\]
(3.47)
where we put $t' = \left[\frac{\ell(\lambda)-1}{N}\right] + 1$. For $a \in \mathbb{Q}$, $|a|$ is defined to be the integer $n$ satisfying $n \leq a < n + 1$. Lemma 3.19 and the equation
\[
c_{\lambda}c'_{\lambda} = (-1)^{(|\lambda|)}q^{n(\lambda)}N_{\lambda,\lambda}(1)
\]
show that
\[
\prod_{1 \leq k \leq N} N_{\mu(k-1),\mu(k)}(t^{k}:i)q^{2n(\mu(k))'}c_{\mu(k)}c'_{\mu(k)}(\gamma^{-1}t^{-\delta_k;i}x_k)x_k^{-|\mu(k)|}
\]
(3.48)
Furthermore, by using the shifted screening current $S_k(z)$'s, the operator part in (3.47) can be rewritten as
\[
: \phi_0(t^{-t'}s) \prod_{k=1}^{N} \sum_{\alpha=0}^{t'} S^{(k)}(q^{\alpha+1}t^{-\alpha}s) \prod_{k=1}^{N} \sum_{\alpha=0}^{t'} S^{(k)}(q^{\alpha+1}t^{-\alpha}s) : \]
(3.49)
Here, [a] is the integer satisfying that 0 ≤ [a] < N and [a] ≡ a mod N. Therefore, by Lemma 3.20 we can show that (3.47) coincides with the RHS of (3.39). □

This proposition says that the vertex operators $\tilde{T}_1^{\text{loop}}(x, p; s)$ can be identified with the screened vertex operators which are used to construct the non-stationary Ruijsenaars function in [11], though in our case, $\kappa$ should be specialized to $t^{-1}/N$. (See Appendix B.) This motivates us to state the affine analogue of Fact 3.13 that is, to construct the non-stationary Ruijsenaars function as the matrix element of the composition of $\tilde{T}_1^{\text{loop}}(x, p; s)'s$.

In order to state the claim, we introduce the non-stationary Ruijsenaars function.

**Definition 3.21** ([11]). Let $s = (s_1, \ldots, s_N)$, $x = (x_1, \ldots, x_N)$ be $N$-tuples of indeterminates. Define $f^{\hat{\Phi}}(x, p|s, \kappa, q, t) \in \mathbb{Q}(q, t, s)|[px_2/x_1, \ldots, px_N/x_{N-1}, px_1/x_N]|$ by

$$
\left. f^{\hat{\Phi}}(x, p|s, \kappa, q, t) \right| = \sum_{\lambda^{(1)}, \ldots, \lambda^{(N)}} \prod_{i=1}^{N} \frac{\mathcal{N}_{\lambda^{(1)}, \lambda^{(N)}}^{j-i(N)}(ts_j/s_i|q, \kappa)}{\mathcal{N}_{\lambda^{(1)}, \lambda^{(N)}}^{j-i(N)}(s_j/s_i|q, \kappa)} \cdot \prod_{\beta = 1}^{N} \prod_{a \geq 1} \frac{(px_{a+\beta}/tx_{a+\beta-1})^{i(\beta)}}{\kappa}. 
$$

We cyclically identify $x_{i+N} = x_1$ and put

$$
\mathcal{N}_{\lambda, \mu}^{(k)}(u|q, \kappa) = \mathcal{N}_{\lambda, \mu}^{(k)}(u|q, \kappa)
$$

$$
= \prod_{j \geq 1 \beta \geq 2 \beta - \alpha \equiv -k+1 \text{ (mod } N)} \left(uq^{-\mu_j+1}k^{-i+j}|q\right), \lambda_j - \lambda_j+1 \prod_{\beta = 1}^{N} \prod_{a \geq 1} (uq^{\lambda_{\alpha}} - \mu_{\beta} - \mu_{\beta+1})^{i(\beta)}.
$$

Then, we obtain the following theorem. (See also FIGURE 3)

**Theorem 3.22.** Let

$$
x(i) = (\gamma^{-i}t^{-1}x_1, \ldots, \gamma^{-i}t^{-1}x_i, \gamma^{-i}x_{i+1}, \ldots, \gamma^{-i}x_N), \quad 0 \leq i \leq N.
$$

Then we obtain

$$
\langle 0| \tilde{T}_1^{\text{loop}}(x^{(0)}, p; s_1) \tilde{T}_2^{\text{loop}}(x^{(1)}, p; s_2) \cdots \tilde{T}_N^{\text{loop}}(x^{(N-1)}, p; s_N) |0\rangle = \prod_{1 \leq i < j \leq N} \frac{(qs_j/s_i|q)_{\infty}}{(ts_j/s_i|q)_{\infty}} \cdot f^{\hat{\Phi}}(x', p|s', t^{-1/N}|q, t).
$$

Here, we set

$$
s' = (s_1', \ldots, s_N'), \quad s_k' = t^{\frac{k}{N}}s_k,
$$

$$
x' = (x_1', \ldots, x_N'), \quad x_k' = p^{\frac{k}{N}}x_k.
$$

**Proof.** Proposition 3.15 gives

$$
\langle 0| \tilde{T}_1^{\text{loop}}(x^{(0)}, p; s_1) \tilde{T}_2^{\text{loop}}(x^{(1)}, p; s_2) \cdots \tilde{T}_N^{\text{loop}}(x^{(N-1)}, p; s_N) |0\rangle = \sum_{\lambda^{(1)}, \ldots, \lambda^{(N)}} \langle 0| \Phi^{0}_{\lambda^{(1)}}(t^{1/N} s_1) \Phi^{1}_{\lambda^{(2)}}(t^{2/N} s_2) \cdots \Phi^{N-1}_{\lambda^{(N)}}(t^{N/N} s_N) |0\rangle \times \prod_{i=1}^{N} \prod_{k=1}^{N} (\gamma^{-i+1}t^{-\delta_{a,b}}t_{a,b}^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - 1) \times \prod_{i=1}^{N} \prod_{b=1}^{N} (\gamma^{-i+1}t^{-\delta_{a,b}}t_{a,b}^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - 1).
$$

Here, $\delta_{a,b}$ is 1 if $a < b$ or 0 if $a = b$. Then, we have

$$
\prod_{i=1}^{N} \prod_{k=1}^{N} (\gamma^{-i+1}t^{-\delta_{a,b}}t_{a,b}^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - |\lambda(i)(i-k)|^{i(\beta)} - 1) = 1,
$$

(3.58)
Finally, it can be easily shown that the non-stationary Ruijsenaars function in (3.61) coincides with
\[
\prod_{i=1}^{N} \prod_{k=1}^{N} (t^{-\delta_{k,i-1}})|\lambda^{(i)}|^{(i-k)}|\lambda^{(i-1)}|^{(i-k-1)} = \prod_{i=1}^{N} t^{-|\lambda^{(i)}|^{(i-1)}+|\lambda^{(i)}|^{(0)}}
\]  
(3.59)
and
\[
\prod_{i=1}^{N} \prod_{k=1}^{N} x_{k}^{(i-k)}|\lambda^{(i)}|^{(i-k-1)} \cdot \prod_{i=1}^{N} p|\lambda^{(i)}|^{(i-1)} = \prod_{i=1}^{N} \prod_{j=1}^{N} (x_{i-j+1}/x_{i-j})^{\lambda_{j}} \cdot \prod_{i=1}^{N} \prod_{j=1}^{N} (p^{1/N} x_{i-j+1}/x_{i-j})^{\lambda_{j}}.
\]  
(3.60)
Therefore, Fact 3.4 shows that (3.57) is equal to
\[
\prod_{1 \leq i < j \leq N} (qs_{j}/s_{i}; q)_{\infty} \cdot f_{\ell}^{\bar{N}}((1/x_{1}', \ldots, 1/x_{N}'), p\bar{N} |(1/s_{1}', \ldots, 1/s_{N}'), t^{-1/2} | q, t).
\]  
(3.61)
Finally, it can be easily shown that the non-stationary Ruijsenaars function in (3.61) coincides with
\[
f_{\ell}^{\bar{N}}((x', p\bar{N} |s', t^{-1/2} | q, t).
\]

**Remark 3.23.** The LHS in this theorem can be rewritten by the trace of the operators \( \hat{T}^{H} \). For an operator \( A \in \text{End}((\mathcal{H}_{(0,1)})^\otimes N) \), set the formal power series
\[
\text{Tr}_{p}(A) = \sum_{\lambda \in \mathbb{P}^N} p^{\lambda} \langle \lambda | A | \lambda \rangle.
\]  
(3.62)
Then it is clear that
\[
\langle 0 | \hat{T}_{1}^{\text{loop}}(x^{(0)}, p; s_{1}) \cdots \hat{T}_{(N-1)}^{\text{loop}}(x^{(N-1)}, p; s_{N}) | 0 \rangle = \text{Tr}_{p} \left( \hat{T}_{N}^{H} \left( s^{(N-1)}; x_{N} \right) \cdots \hat{T}_{1}^{H} \left( s^{(0)}; x_{1} \right) \right).
\]  
(3.63)
Here,
\[
s^{(i)} = (\gamma^{-i} t^{-1} s_{1}, \ldots, (\gamma^{-i} t^{-1} s_{i}, (\gamma^{-i} s_{i+1}, \ldots, (\gamma^{-i} s_{N}), \quad 0 \leq i \leq N.
\]  
(3.64)

### 3.2. Lift to Elliptic Hypergeometric Series and Non-stationary Ruijsenaars Function

In the previous subsection, we took the loop at vertical direction of the reticulate diagram. Next, we calculate a loop at horizontal direction. This loop can reproduce the lift \( f_{N}^{\text{ell}}(x, s|q, t, p) \) of the Macdonald function \( f_{\ell}^{\bar{N}} \) by the elliptic gamma functions. Let us recall the definition of \( f_{N}^{\text{ell}}(x, s|q, t, p) \).

**Definition 3.24.** Define \( f_{N}^{\text{ell}}(x, s|q, t, p) \in \mathbb{Q}(q, t, s)[[p, x_{2}/x_{1}, \ldots, x_{N}/x_{N-1}]] \) by
\[
f_{N}^{\text{ell}}(x, s|q, t, p) = \sum_{\theta \in \mathcal{M}_{N}} c_{N}^{\text{ell}}(\theta; s|q, t, p) \prod_{1 \leq i < j \leq N} (x_{j}/x_{i})^{\theta_{ij}}.
\]  
(3.65)
where
\[
c^{\text{ellip}}_N(\theta; s|q,t,p) = \prod_{k=2}^{N} \prod_{1 \leq i < j \leq k} \frac{\Theta(q^{n>0} \theta_{ij})}{\Theta(q^{n>0} \theta_{ji})} \quad (3.66)
\]

\[\Theta(-; q, p)_n \text{ is defined in (1.2)}.\]

It is clear that the function \( f^N(x, s|q,t,p) \) is reduced to the ordinary Macdonald function, i.e.,
\[
f^N(x, s|q,t,p) \rightarrow f^N(x, s|q/t). \quad (3.67)
\]

We give a realization of this elliptic lift by taking the trace of the Mukadé operators.

**Definition 3.25.** For \( A \in \text{End}(\mathcal{F}^\otimes N) \), define the \( p \)-trace \( \text{tr}(p^d A) \) to be
\[
\text{tr}(p^d A) = \sum_{\lambda \in P^N} p^{\lambda|Aa_\lambda} a_\lambda \quad (3.68)
\]
Note that the trace \( \text{tr}(p^d \cdot) \) certainly does not depend on bases.

Our main purpose in this subsection is to compute the trace of the following operator. See also FIGURE 9.

**Definition 3.26.** Define the \( N \)-compositions of operators \( \tilde{T}^V_i \) as
\[
\tilde{T}^N(s; x) := \tilde{T}_1^V(s^{(0)}; x_1) \tilde{T}_2^V(s^{(1)}; x_2) \cdots \tilde{T}_N^V(s^{(N-1)}; x_N),
\]
where
\[
s^{(i)} = (\gamma^{-i} t^{-1} s_1, \ldots, \gamma^{-i} t^{-1} s_i, \gamma^{-i} s_{i+1}, \ldots, \gamma^{-i} u_N).
\]

![Figure 9](image)

Figure 9. \( \text{tr}(p^d \tilde{T}^N(s; x)) \).

We state the key property of the \( p \)-trace.

**Notation 3.27.** Write
\[
\hat{R}^N[z] := \left\{ \sum_{n>0, 1 \leq i \leq N} R^{(i)}_n \alpha^{(i)}_{n-z} \in \text{End}(\mathcal{F}^\otimes N) \right\} \quad (3.71)
\]

**Lemma 3.28.** Let \( A_i(z) \in \hat{R}^N[z] \) \((i = 1, \ldots, n)\) be operators which satisfy
\[
A_i(z)A_j(w) = \prod_i \frac{1 - a_{ij}^{w/z}}{1 - b_{ij}^{w/z}} : A_i(z)A_j(w) : \quad (a_{ij}^{w/z}, b_{ij}^{w/z} \in \mathbb{C}, i, j = 1, \ldots, n),
\]
(3.72)
where the product with respect to \( l \) can be either finite or infinite if it converges. Then, it follows that

\[
\text{tr}\left(p^d A_1(z_1) \cdots A_n(z_n)\right) = \frac{1}{(p;p)^N} \prod_{1 \leq i < j \leq n} \prod_l \left(\frac{a_{ij}^l z_j / z_i; p)_{\infty}}{(b^l_{ij} z_j / z_i; p)_{\infty}}\right) \prod_{1 \leq j \leq n} \prod_l \left(\frac{p a_{ij}^l z_j / z_i; p)_{\infty}}{(p b^l_{ij} z_j / z_i; p)_{\infty}}\right). \tag{3.73}
\]

In particular, if the operators satisfy

\[
A_1(z) A_2(w) = \frac{(aw/z; q)_{\infty}}{(bw/z; q)_{\infty}} A_1(z) A_2(w), \quad A_2(w) A_1(z) = \frac{(qz/bw; q)_{\infty}}{(qz/aw; q)_{\infty}} A_2(w) A_1(z), \tag{3.74}
\]

then we can rewrite the result by the elliptic gamma functions:

\[
\text{tr}\left(p^d A_1(z) A_2(w)\right) = \frac{1}{(p;p)^N} \frac{\Gamma(bw/z; q, p)}{(aw/z; q, p)} \prod_{i=1}^2 \prod_l \left(\frac{p a_{ij}^l z_j / z_i; p)_{\infty}}{(p b^l_{ij} z_j / z_i; p)_{\infty}}\right). \tag{3.75}
\]

Proof. Let \( A_i^\pm(z) \) be the operators of the forms

\[
A_i^- (z) = \sum_{n \geq 0} \sum_{1 \leq i \leq n} \frac{R_i^{(i)} a_n^{(i)} z^n}{n!} \in \mathbb{C}, \quad A_i^+ (z) = \sum_{n \geq 0} \sum_{1 \leq i \leq n} \frac{R_i^{(i)} a_n^{(i)} z_n}{n!} \in \mathbb{C}
\]

such that \( A_i(z) = A_i^-(z) A_i^+(z) \). Then we have

\[
\text{tr}\left(p^d A_1(z_1) \cdots A_n(z_n)\right) = \prod_{1 \leq i < j \leq n} \prod_l \frac{1 - a_{ij}^l z_j / z_i}{1 - b^l_{ij} z_j / z_i} \cdot \text{tr}\left(p^d A_1^- (z_1) \cdots A_n^- (z_n) A_1^+ (z_1) \cdots A_n^+ (z_n)\right)
\]

\[
= \prod_{1 \leq i < j \leq n} \prod_l \frac{1 - a_{ij}^l z_j / z_i}{1 - b^l_{ij} z_j / z_i} \cdot \text{tr}\left(p^d A_1^+ (p^{-1} z_1) \cdots A_n^+ (p^{-1} z_n) A_1^- (z_1) \cdots A_n^- (z_n)\right)
\]

\[
= \prod_{1 \leq i < j \leq n} \prod_l \frac{1 - a_{ij}^l z_j / z_i}{1 - b^l_{ij} z_j / z_i} \prod_{i,j=1}^n \frac{1 - p a_{ij}^l z_j / z_i}{1 - p b^l_{ij} z_j / z_i} \times \text{tr}\left(p^d A_1^- (z_1) \cdots A_n^- (z_n) A_1^+ (p^{-1} z_1) \cdots A_n^+ (p^{-1} z_n)\right). \tag{3.77}
\]

Since \( \text{tr}(p^d 1) = \frac{1}{(p;p)_{\infty}} \), by repeating the calculation above, it can shown that for any \( m \in \mathbb{Z}_{>0} \),

\[
\text{tr}\left(p^d A_1(z_1) \cdots A_n(z_n)\right) = \frac{1}{(p;p)^N} \prod_{1 \leq i < j \leq n} \prod_l \frac{(a_{ij}^l z_j / z_i; p)_m}{(b^l_{ij} z_j / z_i; p)_m} \prod_{1 \leq j \leq n} \prod_l \frac{(p a_{ij}^l z_j / z_i; p)_m}{(p b^l_{ij} z_j / z_i; p)_m} + O(p^m). \tag{3.79}
\]

This completes the proof. \( \Box \)

Proposition 3.29. We obtain

\[
\text{tr}\left(p^d \bar{\mathcal{T}}^N(s,x)\right) = \left(\frac{(pq/t; q)_{\infty}}{(pt; q)_{\infty}}\right)^N \prod_{1 \leq i < j \leq N} \frac{\Gamma(t x_j / x_i; q, p)}{\Gamma(q x_j / x_i; q, p)} \cdot f_N^{\text{ellip}}(s;x| q,t,p). \tag{3.80}
\]

Proof. Using Fact 3.11 and Lemma 3.28, we can compute the trace as

\[
\text{tr}\left(p^d \bar{\mathcal{T}}^N(s;x)\right) = \left(\frac{(pq/t; q)_{\infty}}{(pt; q)_{\infty}}\right)^N \prod_{1 \leq i < j \leq N} \frac{\Gamma(t x_j / x_i; q, p)}{\Gamma(q x_j / x_i; q, p)} \sum_{(m_{i,j})_{1 \leq i < j \leq N}} \prod_{(m_{i,j})_{1 \leq i < j \leq N}} \frac{\Gamma(q q^{m_{i,j}-m_{l,j}; t; q, p})}{\Gamma(q q^{m_{i,j}-m_{l,j}; t; q, p})} \prod_{1 \leq i < j \leq N} \frac{\Gamma(q q^{m_{i,j}-m_{l,j}; t; q, p})}{\Gamma(q q^{m_{i,j}-m_{l,j}; t; q, p})}
\]

\[
\times \prod_{1 \leq i < j \leq N} \frac{\Gamma(q q^{m_{i,j}-m_{k,j}; t; q, p})}{\Gamma(q q^{m_{i,j}-m_{k,j}; t; q, p})} \times \prod_{k=2}^{N-1} \frac{\Gamma(q q^{m_{i,j}-m_{k-1,j}; t; q, p})}{\Gamma(q q^{m_{i,j}-m_{k-1,j}; t; q, p})} \times \frac{\Gamma(q q^{m_{i,j}-m_{k-1,j}; x_j / x_i; q, p})}{\Gamma(q q^{m_{i,j}-m_{k-1,j}; x_j / x_i; q, p})} \tag{3.80}
\]

Moreover, we have

\[
\prod_{k=2}^{N-1} \prod_{k \leq i < j \leq N} \frac{\Gamma(q^{m_{k-2,j}-m_{k-1,i}}x_j/x_i; q, p)}{\Gamma(q^{m_{k-1,i}}x_i/x_j; q, p)} \Gamma(q^{m_{k,j}-m_{k-1,i}}x_j/x_i; q, p) \Gamma(q^{m_{k,j}-m_{k-1,i}}x_j/x_i; q, p) \\
\times \prod_{k=2}^{N-1} \prod_{k \leq i < j \leq N} \left( t^{\theta_{i,j}-1}s_{i+1}/s_i \right)^{m_{i,k}}. \tag{3.81}
\]

Here, the summation runs over all integers such that \( m_{j-1,j} \geq m_{j-2,j} \geq \cdots \geq m_{1,j} \geq 0 \) \((j = 2, \ldots, N)\). Then, it can be shown that

\[
\left( \frac{(q/t; q)_{\infty}}{(q; q)_{\infty}} \right)^{N(N-1)/2} \left( \frac{pq/t; q)_{\infty}}{(pq; q)_{\infty}} \right)^{N(N-3)/2} \left( \frac{(pq/t; q)_{\infty}}{(pq; q)_{\infty}} \right)^{N(N+1)/2} \\
= \prod_{1 \leq i < j \leq N} \frac{\Theta(qx_j/x_i; q, p)\theta_{i,j}}{\Theta(qx_i/x_j; q, p)\theta_{i,j}}. \tag{3.82}
\]

Put \( \theta_{1,j} := m_{1,j} \) \((j = 2, \ldots, N)\) and \( \theta_{i,j} := m_{i,j} - m_{i-1,j} \) \((j \geq 3, i = 2, \ldots, j - 1)\). We have

\[
\prod_{1 \leq i < j \leq N} \frac{\Gamma(q^{m_{i,j}}x_j/x_i; q, p)}{\Gamma(q^{m_{i,j}}x_i/x_j; q, p)} = \prod_{1 \leq i < j \leq N} \frac{\Theta(qx_j/x_i; q, p)\theta_{i,j}}{\Theta(qx_i/x_j; q, p)\theta_{i,j}}. \tag{3.83}
\]

and

\[
\prod_{k=2}^{N-1} \prod_{k \leq i < j \leq N} \frac{\Gamma(q^{m_{k-2,j}-m_{k-1,i}}x_j/x_i; q, p)}{\Gamma(q^{m_{k-1,i}}x_i/x_j; q, p)} \Gamma(q^{m_{k,j}-m_{k-1,i}}x_j/x_i; q, p) \\
\times \prod_{k=2}^{N-1} \prod_{k \leq i < j \leq N} \frac{\Gamma(q^{m_{k-2,j}-m_{k-1,i}}x_j/x_i; q, p)}{\Gamma(q^{m_{k-1,i}}x_i/x_j; q, p)} \Gamma(q^{m_{k,j}-m_{k-1,i}}x_j/x_i; q, p) \\
= \prod_{k=2}^{N-1} \prod_{k \leq i < j \leq N} \frac{\Theta(q^{m_{k-2,j}-m_{k-1,i}}x_j/x_i; q, p)\theta_{k-1,j}}{\Theta(q^{m_{k-1,i}}x_i/x_j; q, p)\theta_{k-1,j}} \frac{\Theta(q^{m_{k,j}-m_{k-1,i}}x_j/x_i; q, p)\theta_{k,j}}{\Theta(q^{m_{k-1,i}}x_i/x_j; q, p)\theta_{k,j}}. \tag{3.85}
\]

The exponent of \( q \) in (3.85) can be rewritten as

\[
m_{k-2,j} - m_{k-1,i} = -\theta_{k-1,i} + \sum_{a=1}^{k-2} (\theta_{a,j} - \theta_{a,i}), \tag{3.86}
\]

\[
m_{k-1,j} - m_{k-1,i} = \sum_{a=1}^{k-1} (\theta_{a,j} - \theta_{a,i}). \tag{3.87}
\]

Moreover, we have

\[
\prod_{k=2}^{N} \prod_{k \leq i < j \leq N} \left( t^{\theta_{i,j}-1}s_{i+1}/s_i \right)^{m_{i,k}} = \prod_{1 \leq i < j \leq N} t^{\theta_{i,j}}(s_j/s_i)^{\theta_{i,j}}. \tag{3.88}
\]

By the calculation above, we obtain

\[
\text{tr} \left( p^d \mathcal{T}^N(s; x) \right) = \left( \frac{(pq/t; q)_{\infty}}{(pq; q)_{\infty}} \right)^{N(N-1)/2} \left( \frac{(pq/t; q)_{\infty}}{(pq; q)_{\infty}} \right)^{N(N-3)/2} \left( \frac{(pq/t; q)_{\infty}}{(pq; q)_{\infty}} \right)^{N(N+1)/2} \\
\times \prod_{1 \leq i < j \leq N} \frac{\Theta(qx_j/x_i; q, p)\theta_{i,j}}{\Theta(qx_i/x_j; q, p)\theta_{i,j}} \frac{\Theta(qx_j/x_i; q, p)\theta_{i,j}}{\Theta(qx_i/x_j; q, p)\theta_{i,j}} \\
\times \prod_{k=2}^{N-1} \prod_{k \leq i < j \leq N} \frac{\Theta(q^{m_{k-2,j}-m_{k-1,i}}x_j/x_i; q, p)\theta_{k-1,j}}{\Theta(q^{m_{k-1,i}}x_i/x_j; q, p)\theta_{k-1,j}} \frac{\Theta(q^{m_{k,j}-m_{k-1,i}}x_j/x_i; q, p)\theta_{k,j}}{\Theta(q^{m_{k-1,i}}x_i/x_j; q, p)\theta_{k,j}}.
\]
\begin{equation}
\times \prod_{1 \leq i < j \leq N} t^{\delta_{i,j}} (s_j/s_i)^{\theta_{i,j}}
= \left( \frac{(pq/t; q, p)_\infty}{(p; p)_\infty (pq; p)_\infty} \right)^N \prod_{1 \leq i < j \leq N} \frac{\Gamma(t x_j/x_i; q, p)}{\Gamma(q x_j/x_i; q, p)}
\times \sum_{\theta} \prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq N} \Theta(t q^{\theta_{k,i}+\sum_{n=1}^{k-1}(\theta_{n,j}-\theta_{n,i})} x_j/x_i; q, p)\theta_{k,i}
\times \prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq N} \Theta(q^{\sum_{n=1}^{k-1}(\theta_{n,j}-\theta_{n,i})} x_j/x_i; q, p)\theta_{k,i} .
\end{equation}

(3.89)

Since \( \theta_{i,j}, x_i \) and \( s_i \) correspond to \( \theta_{N-j+1,N-i+1}, 1/s_{N-i+1} \) and \( 1/x_{N-i+1} \) in Definition 3.24, respectively, we have

\[
\text{tr} \left( p^d \ellip^{N}(s; x) \right) = \left( \frac{(pq/t; q, p)_\infty}{(p; p)_\infty (pq; p)_\infty} \right)^N \prod_{1 \leq i < j \leq N} \frac{\Gamma(t x_j/x_i; q, p)}{\Gamma(q x_j/x_i; q, p)}
\times f_N^{\ellip}(1/s_N, \ldots, 1/s_1; 1/x_N, \ldots, 1/x_1|q, q/t, p).
\]

(3.90)

Therefore, Proposition 3.29 follows from the symmetry

\[
f_N^{\ellip}(1/x_N, \ldots, 1/x_1; 1/s_N, \ldots, 1/s_1|q, t, p) = f_N^{\ellip}(x_1, \ldots, x_N; s_1, \ldots, s_N|q, t, p).
\]

(3.91)

\[\square\]

Now, we obtain a relationship between the non-stationary Ruijsenaars functions and the functions \( f_N^{\ellip} \).

**Theorem 3.30.** As formal series in \( p, s_{i+1}/s_i, x_{i+1}/x_i \) \((i = 1, \ldots, N-1)\) and \( px_N/x_N \), we obtain

\[
f^{g_{\infty}}(x', p^{\frac{x}{N}}|s', t^{\frac{x}{N}}; q, t) = C \times f_N^{\ellip}(s; x|q, t, p),
\]

(3.92)

where we put

\[
C := \left( \frac{(pq/t; q, p)_\infty}{(p; p)_\infty (pq; p)_\infty} \right)^N \prod_{1 \leq i < j \leq N} \frac{\Gamma(t x_j/x_i; q, p)}{\Gamma(q x_j/x_i; q, p)}
\times \prod_{1 \leq i < j \leq N} \frac{(ts_j/s_i; q)_\infty}{(qs_j/s_i; q)_\infty}.
\]

(3.93)

\( x' \) and \( s' \) are the same ones in Theorem 3.22.

\[
s' = (s'_1, \ldots, s'_N), \quad s'_k = \frac{x_k}{N} \quad s_k,
\]

(3.94)

\[
x' = (x'_1, \ldots, x'_N), \quad x'_k = p^{\frac{x}{N}} x_k.
\]

(3.95)

For the proof, we prepare the following lemma.

**Lemma 3.31.** Let \( 1 \leq k \leq N \). The vacuum expectation values of the Mukadé operators are

\[
\langle \emptyset | \ellip_k^{H}(u, ; x) | \emptyset \rangle = \prod_{1 \leq i < k} \frac{(qu_k/tu_i; q)_\infty}{(u_k/u_i; q)_\infty},
\]

(3.96)

\[
\langle \emptyset | \ellip_k^{V}(u, ; x) | \emptyset \rangle = \prod_{1 \leq i < k} \frac{(qu_k/tu_i; q)_\infty}{(u_k/u_i; q)_\infty}.
\]

(3.97)

**Proof.** The first vacuum expectation value can be directly calculated as

\[
\langle \emptyset | \ellip_k^{H}(u, ; x) | \emptyset \rangle = \prod_{1 \leq i < k} \frac{G(t^{\delta_{i,k}+\delta_{k,i}} u_j/u_i)}{G(t^{\delta_{i,k}+\delta_{k,i}})} \frac{G(qu_j/tu_i)}{G(q^{t^{\delta_{i,k}+\delta_{k,i}}-1} u_j/u_i)}
= \prod_{1 \leq i < k} \frac{G(u_j/tu_i)}{G(u_i/u)} \frac{G(u_j/tu_i)}{G(u_j/t^2 u_i)}
= \prod_{1 \leq i < k} \frac{(qu_j/tu_i; q)_\infty}{(u_k/u_i; q)_\infty}.
\]

(3.98)
The second one can be calculated by using the $q$-binomial theorem:
\[
\langle 0 | \tilde{T}^V_k(u; x) | 0 \rangle = \left( \frac{q/t; q_{\infty}}{q; q_{\infty}} \right)^{k-1} \sum_{0 \leq m_1 \leq \ldots \leq m_{k-1}} \frac{(qq^{m_1}; q_{\infty})_{k-1}}{(qq^{m_1}/t; q_{\infty})_{k-1}} \prod_{i=2}^{k-1} \frac{(qq^{m_i-m_{i-1}}; q_{\infty})_{k-1}}{(qq^{m_i-m_{i-1}}/t; q_{\infty})_{k-1}} \prod_{i=1}^{k-1} \left( \frac{u_{i+1}}{u_i} \right)^{m_i}
\]
\[
= \sum_{n_1, \ldots, n_{k-1} \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{k-1} \frac{(q/t; q)_{n_i}}{(q; q)_{n_i}} \prod_{i=1}^{k-1} \left( \frac{u_k}{u_i} \right)^{n_i}
\]
\[
= \prod_{i=1}^{k-1} \frac{(qu_k/tu_i; q_{\infty})}{(u_k/u_i; q_{\infty})}.
\]
(3.99)

Proof of Theorem 3.30. In this proof, we use the same notation as in (3.70). For the sketch of the proof, see FIGURE 3 in Introduction. By virtue of Theorem 2.24 we have
\[
\text{tr} \left( p^d \tilde{T}^N(s; x) \right) = \sum_{\mu_0, \ldots, \mu_{N-1} \in \mathbb{P}^N} p^{\mu_0} \prod_{i=1}^{N} \langle P_{\mu_{i-1}} | \tilde{T}^V_i(s^{(i-1)}; x_i) | Q_{\mu_i} \rangle
\]
\[
= \sum_{\mu_0, \ldots, \mu_{N-1} \in \mathbb{P}^N} p^{\mu_0} \prod_{i=1}^{N} \langle \mu_i | \tilde{T}^H_i \left( s^{(i-1)}; x_i \right) | \mu_{i-1} \rangle \prod_{i=1}^{N} (-1)^{\mu_i+|\mu_{i-1}|}
\]
\[
\times \prod_{i=1}^{N} \frac{\langle 0 | \tilde{T}^V_i(s^{(i-1)}; x_i) | 0 \rangle}{\langle 0 | \tilde{T}^H_i \left( s^{(i-1)}; x_i \right) | 0 \rangle}.
\]
(3.100)

where $\mu_i = (\mu^{(i,1)}, \ldots, \mu^{(i,N)}) \in \mathbb{P}^N$ and $\mu_0 = \mu_N$. It is clear that
\[
\prod_{i=1}^{N} (-1)^{\mu_i+|\mu_{i-1}|} = 1.
\]
(3.101)

By Lemma 3.31 we have
\[
\prod_{i=1}^{N} \frac{\langle 0 | \tilde{T}^V_i(s^{(i-1)}; x_i) | 0 \rangle}{\langle 0 | \tilde{T}^H_i \left( s^{(i-1)}; x_i \right) | 0 \rangle} = 1.
\]
(3.102)

Therefore, it follows that
\[
\text{tr} \left( p^d \tilde{T}^N(s; x) \right) = \sum_{\mu_0, \ldots, \mu_{N-1} \in \mathbb{P}^N} p^{\mu_0} \prod_{i=1}^{N} \langle \mu_i | \tilde{T}^H_i \left( s^{(i-1)}; x_i \right) | \mu_{i-1} \rangle
\]
\[
= \text{Tr}_p \left( \tilde{T}^H_N \left( s^{(N-1)}; x_N \right) \cdots \tilde{T}^H_1 \left( s^{(0)}; x_1 \right) \right).
\]
(3.103)

As a result, by Theorem 3.22 (see also Remark 3.23), we obtain
\[
\text{tr} \left( p^d \tilde{T}^N(s; x) \right) = \prod_{1 \leq i < j \leq N} \frac{(qs_j/s_i; q_{\infty})_\infty}{(qs_j/s_i; q_{\infty})_\infty} \cdot \tilde{f}_{\tilde{g}N}(x', p^\perp|s', t^{-\perp}|q, t).
\]
(3.104)

Combining Proposition 3.29 and (3.104) yields Theorem 3.30. □

3.3. Another Expression. In the previous subsection, we have established the relationship between $f_{\tilde{g}N}$ and $f_{\text{ellip}}$ by taking traces of intertwiners. By changing the computation method to take the trace, another expression can be obtained. That is, we use the generalized Macdonald functions as a basis. We first fix the normalization of the generalized Macdonald functions $|P\rangle$, which simplifies the matrix elements of the Mukadé operators.
Definition 3.32. Define
\[
C_\lambda^{(+)}(u) := \xi_\lambda^{(+)}(u) \times \prod_{1 \leq i < j \leq N} N_{\lambda(i), \lambda(j)}(qu_i/tu_j) \cdot \prod_{k=1}^N c_{\lambda(k)},
\]
(3.105)
\[
C_\lambda^{(-)}(u) := \xi_\lambda^{(-)}(u) \times \prod_{1 \leq i < j \leq N} N_{\lambda(i), \lambda(j)}(qu_j/tu_i) \cdot \prod_{k=1}^N c_{\lambda(k)},
\]
(3.106)
\[
\xi_\lambda^{(+)}(u) := \prod_{i=1}^N (-1)^{(N-i+1)|\lambda(i)|} u_i^{-N+1} |\lambda| + \sum_{k=1}^N |\lambda(k)|
\times \prod_{i=1}^N \left( \frac{q}{t} \right)^{(i-1)|\lambda(i)|} \left( \frac{i-1}{q} \right)^{(N-i+1)(n(\lambda(i)) + |\lambda(i)|)} t^{(N-i+1)(n(\lambda(i)) + |\lambda(i)|)},
\]
(3.107)
\[
\xi_\lambda^{(-)}(u) := \prod_{i=1}^N (-1)^{|\lambda(i)|} u_i^{-N+1} |\lambda| + \sum_{k=1}^N |\lambda(k)|
\times \prod_{i=1}^N \left( \frac{q}{t} \right)^{(i-1)|\lambda(i)|} \left( \frac{i-1}{q} \right)^{(N-i+1)(n(\lambda(i)) + |\lambda(i)|)} t^{-(N-i+1)(n(\lambda(i)) + |\lambda(i)|)}.
\]
(3.108)

Definition 3.33. Define \(|K_\lambda| = |K_\lambda(u)| \in F_u\) and \(\langle K_\lambda, |K_\lambda| > C_\lambda(u) \in F^*_u\) by
\[
\langle K_\lambda(u) \rangle := c_\lambda^{(+)}(u) |P_\lambda(u)|, \quad \langle K_\lambda(u) \rangle := c_\lambda^{(-)}(u) |P_\lambda(u)|.
\]
(3.109)
This normalization is based on our yet unfinished study of Conjecture 3.38 in [7]. Note, however, that we do not need the conjecture itself here.

Fact 3.34 ([7]). We have
\[
\langle K_\lambda(u)|K_\lambda(u) \rangle = \mathcal{M}(u; \lambda) \cdot \prod_{i,j=1}^N N_{\lambda(i), \lambda(j)}(qu_i/tu_j),
\]
(3.110)
where
\[
\mathcal{M}(u; \lambda) = \left( (-1)^N \gamma^2 e_N(u) \right)^{|\lambda|} \prod_{i=1}^N \left( u_i^{\lambda(i)} \gamma^{2\lambda|\lambda(i)|} g_{\lambda(i)} \right)^{(2-N)}, \quad g_{\lambda} = q^{n(\lambda)} t^{-n(\lambda)},
\]
(3.111)
with \(e_N(u) = u_1 \cdots u_N\).

Fact 3.35 ([7]). We have
\[
\langle K_\lambda(v) | T^V(u, v, x) | K_\mu(u) \rangle = \mathcal{M}(u, v; \lambda, \mu; x) \cdot \prod_{i,j=1}^N N_{\lambda(i), \mu(j)}(v_i/g_{u_j}).
\]
(3.112)
Here
\[
\mathcal{M}(u, v; \lambda, \mu; x) = \frac{\left( (-\gamma)^N e_N(u,x) \right)^{|\mu|}}{(\gamma^2 x)^{|\mu|}} \prod_{i=1}^N \frac{u_i^{\mu(i)} g_{\mu(i)}}{u_i^{\lambda(i)} g_{\lambda(i)}} N_{\lambda(i), \mu(i)}(v_i/g_{u_j}).
\]
(3.113)
By this matrix element formula, the trace of \(\mathcal{T}^N(u; x)\) can be calculated as follows.

Proposition 3.36. It follows that
\[
\text{tr} \left( p^\rho \mathcal{T}^N(u; x) \right) = \gamma^{-N^2} t^{-N} \prod_{1 \leq i < j \leq N} \frac{(qs_j/s_i; q)_{\infty}}{(ts_j/s_i; q)_{\infty}}
\end{align}
\]
(3.114)
\[
\times \sum_{\lambda_0, \ldots, \lambda_{N-1} \in P_{N-1}} \frac{N_{\lambda(k), \lambda(k+1)}(u_{k+1}/\gamma u_{k+1})}{N_{\lambda(k), \lambda(k)}(qu_k/tu_{k+1})} \cdot (p_{\gamma} e_N^k) |\lambda_k| \prod_{k=0}^{N-1} \left( x_{k+1}/\gamma^2 t x_k \right)^{|\lambda_k|}.
\]
(3.115)


**Proof.** By Fact 3.35 and Lemma 3.31, we have

\[
\text{tr} \left( p^d \sum \lambda_0 \left( \frac{K_{\lambda_0}(u(0))}{K_{\lambda_0}(u(0))} \right) \right) = \sum \lambda_0 \left( \frac{\xi(\lambda_0)(u(0))}{\xi(\lambda_0)(u(0))} \right) \frac{K_{\lambda_0}(u(0))}{K_{\lambda_0}(u(0))}
\]

Furthermore, it can be shown that

\[
\prod_{k=0}^{N-1} \frac{M(u(k+1); u(k); \lambda_k, \lambda_{k+1}; x_{k+1})}{M(u(k); \lambda_k)} = \gamma^{-N^2} t^{-N} \prod_{k=1}^{N} \left( x_{k+1}/\gamma^{2^N} t x_k \right)^{|\lambda_k|}.
\]

Thus, we have (3.114).

**Corollary 3.37.** We obtain

\[
f^N(x, p^{\frac{1}{2}}|s', t^{-\frac{1}{2}}|q, \gamma) = \prod_{1 \leq i < j \leq N} \frac{(ts_j/s_i; q)_{\infty}}{(qs_j/s_i; q)_{\infty}} \cdot \gamma^{-N^2} t^{-N} \prod_{k=0}^{N-1} \frac{N_{(\lambda_{k-1}); \lambda_{k}+1; \lambda_{k+1}+1; x_{k+1}+1}}{N_{(\lambda_k); \lambda_{k-1}+1; \lambda_{k+2}+1; x_{k+1}}}. \] (3.117)

**Proof.** Combine Proposition 3.35, Theorem 3.30, and Proposition 3.38.

**3.4. Case** $N = 1$. In this subsection, we treat the case $N = 1$. This case is special in the sense that the $\kappa$ parameter is not specialized. This is because in this case, the ratio of spectral parameters $v/u$ is the free parameter, and it becomes the $\kappa$ parameter.

**Definition 3.38.** We put

\[
\Phi^{-}(s) = \exp \left( -\sum_n \frac{1 - (1/\gamma \kappa)^n}{1 - q^n} a_n s^n \right), \quad \Phi^{+}(s) = \exp \left( \sum_n \frac{1 - (t/\gamma \kappa) q^n}{1 - q^n} a_n s^{-n} \right),
\]

with $\kappa = qv/tu$ so that

\[
\Phi^{\text{cr}} \left[ t_{uk}/q; q s/t_k, 0; s, 0; u \right] = \Phi^{-}(s) \Phi^{+}(s),
\]

First, we take the trace in the horizontal direction. We obtain the next lemma.

**Lemma 3.39.** We have

\[
\text{tr} \left( p^d \Phi^{\text{cr}} \left[ t_{uk}/q; q s/t_k, 0; s, 0; u \right] \right) = \exp \left( \sum_n \frac{1 - (1 - q^n)(1 - \gamma \kappa^n/t^n)(\gamma \kappa)^{n-1} p^n}{(1 - q^n)(1 - t^n)(1 - p^n)} \right).
\]

**Proof.** We note the formula for the normal ordering:

\[
\Phi^{+}(p^{-1}s) \Phi^{-}(s) = g(p) : \Phi^{+}(p^{-1}s) \Phi^{-}(s) :,
\]

where $g(p)$ is a function of $p$.
where
\[
g(p) = \exp \left( \sum_n \frac{(1 - (\gamma \kappa)^n)(1 - q^n t^{-n}(\gamma \kappa)^{-n})}{(1 - t^n)(1 - q^n)} p^n \right).
\]  

(3.123)

By Lemma 3.28, we can show that the given trace is
\[
\frac{1}{(p;p)_\infty} \prod_{k=1}^\infty g(p^k) = \exp \left( \sum_n \frac{1}{n} \frac{(1 - q^n(\gamma \kappa)^n)(1 - (\gamma \kappa)^n/t^n)(\gamma \kappa)^{-n}p^n}{(1 - q^n)(1 - t^n)(1 - p^n)} \right).
\]

(3.124)

\[
\square
\]

Next, we make the loop in the vertical direction. We obtain the following lemma.

**Lemma 3.40.**
\[
\text{tr} \left( p^d \Phi_{cr} \left[ tu_k/q; s, \emptyset : u \right] \right) = \sum_{\lambda} p^{\Delta(\gamma \kappa)} N_{\lambda\lambda}(\gamma \kappa) \frac{\Delta(\gamma \kappa)}{\Delta(1)}.
\]

(4.125)

**Proof.** Use the S-duality formula
\[
\text{tr} \left( p^d \Phi_{cr} \left[ tu_k/q; s, \emptyset : u \right] \right) = \sum_{\lambda} p^{\Delta(\gamma \kappa)} \langle P_{\lambda} \Phi_{cr} \left[ tu_k/q; s, \emptyset : u \right] | Q_{\lambda} \rangle
\]  
\[
= \sum_{\lambda} p^{\Delta(\gamma \kappa)} \langle 0 \Phi_{cr} \left[ s; u, \lambda : tu_k/q, \emptyset : qs/tk \right] | 0 \rangle,
\]

(4.126)

and take the normal ordering (Fact 2.11). \(\square\)

Combining these two lemmas results in the following summation formula.

**Theorem 3.41.** We have
\[
\exp \left( \sum_n \frac{1}{n} \frac{(1 - q^n \kappa^n)(1 - \kappa^n/t^n)\kappa^{-n}p^n}{(1 - q^n)(1 - t^n)(1 - p^n)} \right)
\]
\[
= \sum_{\lambda \in \mathcal{P}} (p/\kappa)^{\Delta(\lambda)} \prod_{1 \leq i \leq j} (\kappa q^{-\lambda_i - \lambda_j + j} t^{-j}; q)_{\lambda_i - \lambda_j + 1} (\kappa q^{\lambda_i - \lambda_j - i - j + 1}; q)_{\lambda_j - \lambda_i + 1}.
\]

(3.127)

This gives the proof of the conjecture in [31], which claims the two different forms of the mixed Hodge polynomials of certain twisted GL(\(n, \mathbb{C}\))-character varieties of Riemann surfaces with \(g = 1\). The similar proof is given in [2, 3]. Physically, this relates the partition function of the 5d \(\mathcal{N} = 1^*\) \(U(1)\) gauge theory to that of the 6d theory with one tensor multiplet.

## 4. Integral operators

### 4.1. Integral Operator of Macdonald Functions.

We return to the non-affine case with \(N \geq 2\). In Fact 3.13, the ordinary Macdonald functions were constructed from the screened vertex operators. In this section, an integral operator introduced in [12, 13, 14] will be constructed from them. We treat the spectral parameters \(s = (s_1, \ldots, s_N)\) as generic complex variables in this section. First, we rewrite the screened vertex operators (non-affine case) by the contour integrals.

**Definition 4.1.** For \(k = 0, \ldots, N - 1\), define \(\phi_{k,N}^{\text{cont}}(x) = \phi_{k,N}^{\text{cont}}(s; x) : F^{(N,0)}_{\gamma - 1 - t - s_{k+1}, s} \rightarrow F^{(N,0)}_s\) by
\[
\phi_{k,N}^{\text{cont}}(x) := K(s) \int_C \prod_{i=1}^k \frac{dy_i}{2\pi \sqrt{1-y_i}} \varphi_0(x) \tilde{S}(1)(y_1) \cdots \tilde{S}(k)(y_k) g(s; x, y_1, \ldots, y_k),
\]

(4.1)

\[
K(s) := \prod_{i=1}^k \frac{(q; q)_\infty (q/t; q)_\infty}{(q s_i; q)_\infty (q s_i + 1; q)_\infty},
\]

(4.2)

\[
g(s; x, y_1, \ldots, y_k) := \frac{\theta_q(q s_i y_1/s_{k+1})}{\theta_q(q y_1/x)} \prod_{i=1}^{k-1} \frac{\theta_q(q s_i + 1 y_{i+1}/s_{k+1} + 1 y_i)}{\theta_q(q y_{i+1}/y_i)}.
\]

(4.3)
Here, the contour of the integration $C$ is taken such that $|q/t| < |x/y_1| < |q|$ and $|t^{-1}| < y_i/y_{i+1} < 1$ for $i = 1, \ldots, k$.

This operator can be expanded as follows.

**Proposition 4.2.** We have

$$\phi_{k,N}^\text{cont.}(x) = \oint_C \prod_{i=1}^k \frac{dy_i}{2\pi i - 1y_i} \sum_{r_1, \ldots, r_k \in \mathbb{Z}} \prod_{i=1}^k \frac{(ts_i/s_{k+1}; q)_{r_i} (q y_i/ty_{i-1})^{r_i}}{(qs_i/s_{k+1}; q)_{r_i}} \psi_0(x) S(1)(y_1) \cdots S(k)(y_k).$$  \hfill (4.4)

**Proof.** It follows from the operator product formulas (Proposition 3.9) and Ramanujan’s $1 \psi 1$ summation formula ((5.2.1) in [17]):

$$1 \psi_1(a; b; q; z) := \sum_{n=-\infty}^{\infty} \frac{(a; q)_n z^n}{(b; q)_n} = \frac{(q; q)_\infty (b/a; q)_\infty (az; q)_\infty (aq/az; q)_\infty}{(b; q)_\infty (q/a; q)_\infty (z; q)_\infty (b/a; q)_\infty (|b/a| < z < 1).}$$ \hfill (4.5)

**Proposition 4.3.** $\phi_{k,N}^\text{cont.}(x)$ is given by $\tilde{T}^V$ as

$$\phi_{k,N}^\text{cont.}(s, x) = \prod_{i=1}^k \frac{(s_{k+1}/s_i; q)_{\infty}}{(qs_{k+1}/ts_i; q)_{\infty}} \tilde{T}^V_{k+1}(s; x).$$ \hfill (4.6)

**Proof.** First, we adjust the contour of the integration to the condition $|q| < |y_{i-1}/y_i| < 1$ ($i = 1, \ldots, k$). Here, we put $y_0 = x$. Note that no pole affects this change. Then we have

$$g(s; x, y_1, \ldots, y_k) = \prod_{i=0}^{k-1} \left( \frac{y_{i+1}}{y_i} \right)^{1-q^n} \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^n s_{k+1}} (y_i/y_{i+1})^n.$$ \hfill (4.7)

By the deformation of the formal series

$$\sum_{n \in \mathbb{Z}} \frac{1}{1 - q^n s_{k+1}} (y_i/y_{i+1})^n = \sum_{n \in \mathbb{Z}} \sum_{m \geq 0} \left( q^n \frac{s_{k+1}}{s_{i+1}} \right)^m (y_i/y_{i+1})^n,$$

we have

$$\phi_{k,N}^\text{cont.}(x) = K(s) \prod_{i=0}^{k-1} \left( \frac{s_{k+1}}{s_{i+1}}; q \right)_\infty \left( \frac{s_{k+1}/s_{i+1}}{q} ; q \right)_\infty,$$

$$\times \sum_{m_0, \ldots, m_{k-1} \geq 0} \int_C \prod_{i=1}^k \frac{dy_i}{2\pi i - 1y_i} \psi_0(x) S(i)(q^{m_0+\cdots+m_{i-1}}x) \cdot \prod_{i=0}^{k-1} \delta(q^{m_i} y_i/y_{i+1}) \left( \frac{s_{k+1}}{s_{i+1}} \right)^{m_i},$$

$$= K(s) \prod_{i=1}^k \left( \frac{s_{k+1}}{s_i}; q \right)_\infty \left( \frac{s_{k+1}/s_{i+1}}{q} ; q \right)_\infty \cdot \prod_{0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k \leq k} \phi_0(x) S(i)(q^{i}x) \cdot \prod_{i=1}^{k-1} \left( \frac{s_{k+1}}{s_i} \right)^{\ell_i},$$

$$= \prod_{i=1}^k \left( \frac{s_{k+1}/s_i; q} {qs_{k+1}/ts_i; q} \right)_\infty \cdot \tilde{T}^V_{k+1}(s; x).$$ \hfill (4.9)

The deformation (4.8) itself is not well-defined because $\sum_{m \geq 0} (q^n s_{k+1})^m$ does not converge for arbitrary $n \in \mathbb{Z}$. However, considering the matrix elements of the operators, we can justify the calculation (4.9). For more detail, see Remark 2.2 in [17].

**Fact 3.13** can be rewritten as follows.

---

6 This screened vertex operator corresponds to $\Phi^{(k)}(x)$ in [17] after transformation $x \rightarrow t^{-1}x$ and $y_i \rightarrow (q/t)^i t^{-1}y_i$ and modification of the integration contour. Actually, a more strict condition is imposed for integration contour in [17] in order to show that the screening currents commute with $X(r)(z)$ ($r = 1, \ldots, N$). However, only commutativity with $X(1)(z)$ is required to show Fact 4.4. Hence we adopt this integration contour in this paper.
Proposition 4.7. It follows that
\[ \langle 0 | \phi_{-1,N}^{\text{cont}}(x_1, x_2) \ldots \phi_{-N+1,N}^{\text{cont}}(x_N) | 0 \rangle = f_{N}(x, s| q/t) . \] (4.10)

We introduce the following integral operator, which is essentially the same as the one in [14].

Definition 4.5. Define the integral operator \( I(s_1/s_0, \ldots, s_N/s_0) \) on \( \mathbb{C}[[x_1, x_2, \ldots, x_N/x_{N-1}]] \) by
\[ I(s_1/s_0, \ldots, s_N/s_0)(f(x_1, \ldots, x_N)) = K(s) \prod_{1 \leq i < j \leq N} \frac{(q x_j/t x_i; q)^{\infty}}{(t x_j/x_i; q)^{\infty}} \]
\[ \times \int_{C'} \prod_{i=1}^{N} \frac{d y_i}{2 \pi \sqrt{-1} y_i} \cdot \Pi^{(q,t)}(x|y) \prod_{i=1}^{N} \frac{\theta_{q}(q s_0 y_i/s_i x_i)}{\theta_{q}(y_i/s_i x_i)} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta_{q}(t x_j/y_i) \theta_{q}(y_i/x_i)}{\theta_{q}(t x_j/y_i)} \]
\[ \times \prod_{1 \leq i < j \leq N} (1 - y_j/y_i) \cdot f(y_1, \ldots, y_N) . \] (4.11)

Here, we put
\[ K(s) = K(s_1/s_0, \ldots, s_N/s_0) = \prod_{i=1}^{k} \frac{(q; q)^{\infty}}{(q s_0/s_i q)^{\infty}} \cdot \frac{(q^{s_i}/t s_0 q)^{\infty}}{(q^{s_i}/t s_0 q)^{\infty}} . \] (4.12)

\( \Pi^{(q,t)}(x|y) \) is the kernel function:
\[ \Pi^{(q,t)}(x|y) = \prod_{i,j=1}^{N} \frac{(q y_j/t x_i; q)^{\infty}}{(q y_j/x_i; q)^{\infty}} . \] (4.13)

We chose the integration contour \( C' \) so that \( |y_{k-1}/q y_k| < |t^{-1}| \) (\( k = 1, \ldots, N-1 \)) and \( |t^{-1}| < |x_i/y_i| < |q| \) (\( i = 1, \ldots, N \)), regarding the variables \( x_i \)'s as complex variables satisfying \( |x_{k+1}/x_k| < |t^{-1}| \) (\( k = 1, \ldots, N-1 \)).

Remark 4.6. In what follows, we assume \( |t^{-1}| < |q| \) so that the integration contour is well-defined.

Consider the \( N + 1 \) fold Fock tensor spaces
\[ \mathcal{F}^{(N+1,0)}_{s_0, s_1, \ldots, s_N} = \mathcal{F}^{(1,0)}_{s_0} \otimes \mathcal{F}^{(N,0)}_{s_1, \ldots, s_N} . \] (4.14)
and naturally extend the screened vertex operators \( \phi_{k,N}^{\text{cont}} \)'s to this space. Then we can construct the Macdonald functions from \( \mathcal{F}^{(N,0)}_{s_1, \ldots, s_N} \) and reproduce the integral operator from the additional Fock space \( \mathcal{F}^{(1,0)}_{s_0} \). That is to say, the matrix element in the following proposition can be viewed as the action of \( I \) on Macdonald functions. See also FIGURE [14]

Proposition 4.7. Let \( s^+ = (s_0, s_1, \ldots, s_N) \). Then we have
\[ \langle 0 | \phi_{1,N+1}^{\text{cont}}(s^+; x_1) \ldots \phi_{N,N+1}^{\text{cont}}(x_N) | 0 \rangle = I(s_1/s_0, \ldots, s_N/s_0)(f_{N}(x_1, \ldots, x_N); (s_1, \ldots, s_N)| q/q/t) . \] (4.15)

Proof. In this proof, we put \( s_{i,j} := \gamma^{-i+1} t^{-\delta_{i,j}} s_j \). Here \( \delta_{a,b} \) is 1 if \( a > b \) or 0 if \( a \leq b \). By taking the normal ordering, we have
\[ \langle 0 | \phi_{1,N+1}^{\text{cont}}(s^+; x_1) \ldots \phi_{N,N+1}^{\text{cont}}(x_N) | 0 \rangle = \prod_{0 \leq i < j \leq N} \frac{(q; q)^{\infty}}{(q^{y_i}/y_j q)^{\infty}} \cdot \prod_{1 \leq i < j \leq N} \frac{(q x_j/t x_i; q)^{\infty}}{(t x_j/x_i; q)^{\infty}} \]
\[ \times \int \prod_{i=1}^{N} \frac{d y_{i,1}}{2 \pi \sqrt{-1} y_{i,1}} \prod_{1 \leq i < j \leq N} \frac{(q y_{i,1}/x_i; q)^{\infty}}{(q y_{i,1}/x_i; q)^{\infty}} \frac{(t x_j/y_i; q)^{\infty}}{(x_j/y_i; q)^{\infty}} \prod_{1 \leq i < j \leq N} (1 - y_{j,1}/y_{i,1}) \frac{(q y_{j,1}/y_{i,1}; q)^{\infty}}{(y_{j,1}/y_{i,1}; q)^{\infty}} \]
\[ \times \int \prod_{2 \leq i < j \leq N} \frac{d y_{i,j}}{2 \pi \sqrt{-1} y_{i,j}} \prod_{1 \leq i < j \leq N} \frac{(q y_{j,2}/y_{i,1}; q)^{\infty}}{(q y_{j,2}/y_{i,1}; q)^{\infty}} \cdot \prod_{2 \leq i < j \leq N} \frac{(q y_{i,2}/y_{j,1}; q)^{\infty}}{(y_{i,2}/y_{j,1}; q)^{\infty}} . \]
For the proof, we need to take care of the analyticity of the domain. Hence, let us define the following

\[
\text{Therefore, Proposition 4.7 follows.}
\]

\[\text{Theorem 4.9.}\]

Set

\[
\text{As in Fact 4.4, we can rewrite the inner integrals by Macdonald functions, and we can show that}
\]

\[
\prod_{1 \leq i < j \leq N} \frac{\theta_q(q^{s_{i,j}} y_{i,j}^{-1}; y_{i,j})}{\theta_q(q^{-1}; y_{i,j})} \prod_{i=1}^{N} \frac{\theta_q(q^{N_{i,0}} y_{i,1}; q)}{\theta_q(q^{N_{i,1}} y_{i,1}; q)}.
\]  

(4.16)

As in Fact 4.4, we can rewrite the inner integrals by Macdonald functions, and we can show that

\[
\prod_{1 \leq i < j \leq N} \frac{\theta_q(q^{s_{i,j}} y_{i,j}^{-1}; y_{i,j})}{\theta_q(q^{-1}; y_{i,j})} \prod_{i=1}^{N} \frac{\theta_q(q^{N_{i,0}} y_{i,1}; q)}{\theta_q(q^{N_{i,1}} y_{i,1}; q)}
\]

\[
\times \langle 0 | S_{2}(y_{2,2}) S_{3}(y_{3,3}) \cdots S_{N}(y_{N,2}) \cdots S_{N}(y_{N,N}) | 0 \rangle \cdot \prod_{1 \leq i < j \leq N} \frac{\theta_q(q^{s_{i,j}} y_{i,j}^{-1}; y_{i,j})}{\theta_q(q^{-1}; y_{i,j})} \prod_{i=1}^{N} \frac{\theta_q(q^{N_{i,0}} y_{i,1}; q)}{\theta_q(q^{N_{i,1}} y_{i,1}; q)}.
\]

(4.17)

Therefore, Proposition 4.7 follows.

We prove the commutativity between the integral operator \(I\) and Macdonald’s difference operator. For the proof, we need to take care of the analyticity of the domain. Hence, let us define the following region.

\textbf{Notation 4.8.} Define the projection \( \pi : (\mathbb{C}^*)^N \to \mathbb{C}^{N-1} \) by

\[
\pi(a_1, \ldots, a_N) = (a_2/a_1, \ldots, a_N/a_{N-1}).
\]  

(4.18)

Set

\[
\begin{align*}
U_r^N &:= \{(z_1, \ldots, z_{N-1}) \in (\mathbb{C})^{N-1} | z_i < r, i = 1, \ldots, N - 1\}, \\
B_r^N &:= \{(z_1, \ldots, z_{N-1}) \in (\mathbb{C})^{N-1} | z_i \leq r, i = 1, \ldots, N - 1\}.
\end{align*}
\]  

(4.19)

(4.20)

so that

\[
\pi^{-1}(U_r^N) = \{(x_1, \ldots, x_N) \in (\mathbb{C}^*)^N | x_j/x_i < r^{j-i}, 1 \leq i < j \leq N\}.
\]  

(4.21)

Define the subset \( \mathcal{O}(U_r^N) \subset \mathbb{C}[x_2/x_1, \ldots, x_N/x_{N-1}] \) to be the set of power series which absolutely convergent on the \( B_r^N \) for any \( r' < r \), i.e., which can be regarded as a holomorphic function on \( U_r^N \).

\textbf{Theorem 4.9.} On \( \mathcal{O}(U_r^N) \), we obtain

\[
\left[ D_x(s_1, \ldots, s_N; q, q/t), I(s_1/s_0, \ldots, s_N/s_0) \right] = 0.
\]  

(4.22)
Here, $D_x(s; q, t)$ is the Macdonald $q$-difference operator:

$$ D_x(s; q, t) := \sum_{k=1}^{N} s_k \prod_{1 \leq \ell < k} \frac{1 - tx_k/x_\ell}{1 - x_k/x_\ell} \prod_{k < \ell \leq n} \frac{1 - xt_k/x_\ell}{1 - x_k/x_\ell} T_{q,x_k}. \quad (4.23) $$

In the proof, we use the following fact.

**Fact 4.10** (32). It follows that

$$ D_x(1, t^{-1}, \ldots, t^{-N+1}; q, t) \Pi(q,t)(x|y) = D_y^{-1}(1, t^{-1}, \ldots, t^{-N+1}; q, t) \Pi(q,t)(x|y). \quad (4.24) $$

Here, we put

$$ D_y^{-1}(s; q, t) := \sum_{k=1}^{n} s_k \prod_{1 \leq \ell < k} \frac{1 - ty_k/y_\ell}{1 - y_k/y_\ell} \prod_{k < \ell \leq n} \frac{1 - y_k/y_\ell}{1 - y_k/y_\ell} T_{q^{-1},y_k}. \quad (4.25) $$

**Proof of Proposition 4.9.** A direct calculation gives

$$ D_x(s; q, q/t) \prod_{1 \leq i < j \leq N} \frac{(qx_j/tx_i; q)_\infty}{(tx_j/x_i; q)_\infty} = \prod_{1 \leq i < j \leq N} \frac{(qx_j/tx_i; q)_\infty}{(tx_j/x_i; q)_\infty} D_x(s; q, t), \quad (4.26) $$

and

$$ D_x(s; q, t) \prod_{i=1}^{N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_i/x_i)} \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_j/y_i)} = \prod_{i=1}^{N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_i/x_i)} \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_j/y_i)} s_0 D_x((1, t^{-1}, \ldots, t^{-N+1}; q, t)). \quad (4.27) $$

By these equations and Fact 4.10, we can show that for a function $f(x_1, \ldots, x_N) \in \mathcal{O}(U_{[t^{-1}]})$,

$$ D_x(s; q, q/t) I(s_1/s_0, \ldots, s_N/s_0) f(x_1, \ldots, x_N) $$

$$ = \mathcal{K}(s) \prod_{1 \leq i < j \leq N} \frac{(qx_j/tx_i; q)_\infty}{(tx_j/x_i; q)_\infty} \cdot \int \prod_{i=1}^{N} \frac{dy_i}{2\pi \sqrt{-1} y_i} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_i/x_i)} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_j/y_i)} $$

$$ \times \prod_{1 \leq i < j \leq N} (1 - y_j/y_i) \cdot f(y_1, \ldots, y_N) s_0 D_y^{-1}(1, t^{-1}, \ldots, t^{-N+1}; q, t) \Pi(q,t)(x|y) $$

$$ = \mathcal{K}(s) \prod_{1 \leq i < j \leq N} \frac{(qx_j/tx_i; q)_\infty}{(tx_j/x_i; q)_\infty} \cdot \int \prod_{i=1}^{N} \frac{dy_i}{2\pi \sqrt{-1} y_i} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_i/x_i)} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_j/y_i)} $$

$$ \times \prod_{1 \leq i < j \leq N} (1 - y_j/y_i) \cdot f(y_1, \ldots, y_N) s_0 \sum_{k=1}^{n} \prod_{1 \leq i < j \leq n} \frac{1 - y_k/y_\ell}{1 - y_k/y_\ell} T_{q^{-1},y_k} \Pi(q,t)(x|y). \quad (4.28) $$

We have poles in $y_k$ of the each term containing the difference operator $T_{q^{-1},y_k}$ at $y_k = q^a x_k$, $y_k = q^{-a} x_k$, $y_k = q^a x_k$, $y_k = q^{-a} x_k$, $y_k = q^{-a} x_k$ ($a = 0, 1, 2, \ldots, i < k < j$). Therefore, by the change of variable $y_k \rightarrow q y_k$ in the each term (note that there is no pole between $y_k$ and $q y_k$), we can show that (4.28) is equal to

$$ \mathcal{K}(s) \prod_{1 \leq i < j \leq N} \frac{(qx_j/tx_i; q)_\infty}{(tx_j/x_i; q)_\infty} \cdot \int \prod_{i=1}^{N} \frac{dy_i}{2\pi \sqrt{-1} y_i} \cdot \Pi(q,t)(x|y) $$

$$ \times \sum_{k=1}^{n} t^{-k+1} T_{q,y_k} \prod_{1 \leq \ell < k} \frac{1 - ty_\ell/y_k}{1 - y_\ell/y_k} \prod_{k < \ell \leq n} \frac{1 - y_k/y_\ell}{1 - y_k/y_\ell} \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_i/x_i)} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta(qs_0 y_i/s_i x_i)}{\theta(q y_j/y_i)} $$

$$ \times \prod_{1 \leq i < j \leq N} (1 - y_j/y_i) \cdot f(y_1, \ldots, y_N) s_0 \]
Since by fixing $3.29$, we can show that

$$
\Pi(q,t)(x|y) \prod_{i=1}^{N} \frac{\theta_q(y_{0i}/x_i)}{\theta_q(qy_i/x_i)} \cdot \prod_{1 \leq i < j \leq N} \frac{\theta_q(tx_j/y_i)}{\theta_q(x_j/y_i)}
\times \prod_{1 \leq i < j \leq N} (1 - y_j/y_i) \cdot D_y(s_1, \ldots, s_N|q, q/t)f(y_1, \ldots, y_N).
$$

(4.29)

This completes the proof.

**Corollary 4.11.** We have

$$
I(s_1/s_0, \ldots, s_N/s_0) \left( f^{\ellip N}(x_1, \ldots, x_N; s_1, \ldots, s_N|q, q/t) \right) = f^{\ellip N}(x_1, \ldots, x_N; s_1, \ldots, s_N|q, q/t).
$$

(4.30)

**Proof.** Since by fixing $s$, the Macdonald function $f^{\ellip N}(x; s|q, q/t)$ is in $O(U_{t^{-1}}^{N-1})$ (Fact A.6), we can use Theorem A.9. Moreover, by the uniqueness of Fact A.2, we can show that $f^{\ellip N}(x; s|q, q/t)$ is an eigenfunction of $I(s_1/s_0, \ldots, s_N/s_0)$:

$$
I(s_1/s_0, \ldots, s_N/s_0) \left( f^{\ellip N}(x; s|q, q/t) \right) \propto f^{\ellip N}(x; s|q, q/t).
$$

(4.31)

The expansion (Proposition 4.12) makes it clear that the constant term of the LHS is 1. Hence, the eigenvalue is 1. \hfill \square

4.2. Integral Operator in Elliptic Case. In this subsection, we give a brief discussion on the elliptic lift of the integral operator. Namely, consider the trace at the horizontal representation of the operator in Proposition 4.11. Then we can derive the following integral operator.

**Definition 4.12.** Define the operator $I^{\ellip}$ on $\mathbb{C}[[x_2/x_1, \ldots, x_N/x_{N-1}]]$ by

$$
I^{\ellip}(s_1/s_0, \ldots, s_N/s_0)(f(x_1, \ldots, x_N))
= \left( \frac{(pq/t; q, p)_{\infty}}{(pt; q, p)_{\infty}} \right)^N \prod_{1 \leq i < j \leq N} \frac{\Gamma(tx_j/x_i; q, p)}{\Gamma(qx_j/x_i; q, p)}
\times \prod_{1 \leq i < j \leq N} \frac{\theta_p(y_j/y_i)}{\theta_p(x_j/y_i)} \cdot f(y_1, \ldots, y_N).
$$

(4.32)

Here, $K(s)$ is defined in Definition 4.5. Further we set

$$
\Pi^{(q,t,p)}(x|y) = \prod_{i,j=1}^{N} \frac{\Gamma(qy_j/x_i; q, p)}{\Gamma(qy_j/x_i; q, p)}.
$$

(4.33)

The following trace can be viewed as the action of $I^{\ellip}$ on the non-stationary Ruijsenaars functions.

**Proposition 4.13.** Let $s^+ = (s_0, s_1, \ldots, s_N)$. Then we have

$$
\text{tr} \left( p^d \phi_{1,N+1}^{\text{cont}}(s^+; x_1) \ldots \phi_{N,N+1}^{\text{cont}}(x_N) \right)
= I^{\ellip}(s_1/s_0, \ldots, s_N/s_0) \left( f^{\ellip N}(x^t, p^{1/N}|s^t, t^{-1/N}|q, t) \right),
$$

(4.34)

where we use the same notation in Theorem 3.22.

**Proof.** This can be proved similarly to the one of Proposition 4.11. By Lemma 3.22 and Proposition 3.29, we can show that

$$
\text{tr} \left( p^d \phi_{1,N+1}^{\text{cont}}(s^+; x_1) \ldots \phi_{N,N+1}^{\text{cont}}(x_N) \right) = \left( \frac{(pq/t; q, p)_{\infty}}{(pt; q, p)_{\infty}} \right)^N
\times I^{\ellip}(s_1/s_0, \ldots, s_N/s_0) \left( \prod_{1 \leq i < j \leq N} \frac{\Gamma(tx_j/x_i; q, p)}{\Gamma(qx_j/x_i; q, p)} \cdot \prod_{1 \leq i < j \leq N} \frac{\Gamma(ts_j/s_i; q)_{\infty}}{\Gamma(qs_j/s_i; q)_{\infty}} \cdot f^{\ellip N}(s; x|q, t, p) \right).
$$

(4.35)
Applying Theorem 5.30 we obtain the claim.

We can obtain the commutativity between the integral operator \( I_{\text{ellip}} \) and the Ruijsenaars operator.

**Proposition 4.14.** We obtain

\[
\left[ D_x(s_1, \ldots, s_N; q, q/t, p), I_{\text{ellip}} (s_1/s_0, \ldots, s_N/s_0) \right] = 0.
\]  

Here,

\[
D_x(s; q, t, p) := \sum_{k=1}^{n} s_k \prod_{1 \leq \ell < k} \theta_p(t_{k}x_{k}/x_{\ell}) \prod_{k < \ell \leq n} \theta_p(x_{k}/x_{\ell}) T_{q, x_{k}}.
\]  

Since the proof is quite similar to that of Theorem 4.9, we omit it. Concerning the equation with respect to the kernel function, we can use the following fact.

**Fact 4.15** ([33, 34]). It follows that

\[
D_x(1, t^{-1}, \ldots, t^{-N+1}; q, t, p) \Pi^{(q, t, p)}(x|y) = D_{y^{-1}}(1, t^{-1}, \ldots, t^{-N+1}; q, t, p) \Pi^{(q, t, p)}(x|y).
\]  

Here, we put

\[
D_{y^{-1}}(s; q, t, p) := \sum_{k=1}^{n} s_k \prod_{1 \leq \ell < k} \theta_p(t_{y_k}/y_{\ell}) \prod_{k < \ell \leq n} \theta_p(y_k/y_{\ell}) T_{q, y_{k}}.
\]

In this subsection, we have derived the integral operator \( I_{\text{ellip}} \) and given commutativity with the Ruijsenaars operator. Unfortunately, the non-stationary Ruijsenaars functions \( f_{\theta} \) are not the eigenfunctions of the Ruijsenaars operator. So, neither for \( I_{\text{ellip}} \). It is left to a future study to find an operator whose eigenfunctions are \( f_{\theta} \).

5. Conformal Limit \( q \to 1 \)

5.1. Preparation. In this section, we will derive the relation of the Virasoro algebra and the primary field from the relation of the \( q \)-Virasoro algebra and the Mukadé operator \( T^V \). Firstly, we define the following algebra.

**Definition 5.1.** For \( k = 2, \ldots, N \), define

\[
X^{(k)}(z) = \sum_{n \in \mathbb{Z}} X_n^{(k)} z^{-n} = X^{(1)}(\gamma^{2(1-k)}z) X^{(1)}(\gamma^{2(2-k)}z) \cdots X^{(1)}(z) \in \text{End} (\mathcal{F}_u)[[z^{\pm 1}]].
\]  

**Fact 5.2** ([20]). The operator \( X^{(k)}(z) \) is of the form

\[
X^{(k)}(z) = \sum_{1 \leq j_1 < \cdots < j_k \leq N} : \Lambda^{(j_1)}(z) \cdots \Lambda^{(j_k)}((q/t)^{-k-1}z) : u_{j_1} \cdots u_{j_k}.
\]

Here, \( \Lambda^{(j)}(z) \) is defined in Definition 2.17.

The algebra generated by \( X^{(k)}(z) \) can be regarded as the tensor product of the \( q \)-deformed \( W_N \) algebra and some Heisenberg algebra [35] (See Proposition 5.20 in the next subsection). Their PBW-type basis is well-understood.

**Definition 5.3.** For an \( N \)-tuple of partitions \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}) \), we define the vectors \( |X_\lambda\rangle = |X_\lambda(u)\rangle \in \mathcal{F}_u \) and \( \langle X_\lambda| = \langle X_\lambda(u)| \in \mathcal{F}_u^\ast \) by

\[
|X_\lambda(u)\rangle := X^{(1)}_{-\lambda^{(1)}} X^{(2)}_{-\lambda^{(2)}} \cdots X^{(N)}_{-\lambda^{(N)}} |0\rangle,
\]  

\[
\langle X_\lambda(u)| := \langle 0 | X^{(N)}_{\lambda^{(N)}} X^{(N)}_{\lambda^{(N)}} \cdots X^{(2)}_{\lambda^{(2)}} X^{(2)}_{\lambda^{(2)}} \cdots X^{(1)}_{\lambda^{(1)}} X^{(1)}_{\lambda^{(1)}}.
\]

**Fact 5.4** ([36, 71]). The set \( \{|X_\lambda\rangle\} \) (resp. \( \{\langle X_\lambda|\} \)) form a PBW-type basis of \( \mathcal{F}_u \) (resp. \( \mathcal{F}_u^\ast \)), if \( u_i \neq q^{t-r} u_j \) and \( u_i \neq 0 \) for all \( i, j \) and \( r, s \in \mathbb{Z} \).
Here, we set $P$ and we parametrize $v$.

Remark 5.8. There is an ambiguity of the choice of parametrization satisfying (5.10). No matter how we choose it, final results (Theorem 5.25) are not affected if (5.10) is satisfied. For example, even if we add an arbitrary parameter to (5.12) and (5.13) to keep the degree of freedom of $v$.

Fact 5.6 (7). It follows that
\[
\langle K_\lambda(v) \mid V(x) \mid K_\mu(u) \rangle = \left( \frac{-\gamma^2}{(2\pi)^N} e^{N(u,x)} \right)^{|x|} \prod_{i=1}^{N} \frac{u_{i}^{(i)}(\alpha) v_{i}^{(i)}(\alpha)}{v_{i}^{(i)}(\alpha) u_{i}^{(i)}(\alpha)} \prod_{j=1}^{N} N_{\lambda(\alpha), \mu(\alpha)}(qu_{j}/tu_{j}). \tag{5.6}
\]

Remark 5.7. The operator $T^V(u)$ satisfies (7)
\[
\left(1 - \frac{w}{z}\right) X^{(i)}(z) T^V(u, v; w) = \gamma^{-i} \left(1 - \gamma^{-2i} \frac{w}{z}\right) T^V(u, v; w) X^{(i)}(z). \tag{5.7}
\]
Thus, the $V(x)$ can be realized by $T^V$ with some modifications to spectral parameters.

5.2. Conformal Limit $q \to 1$. Consider the limit $q \to 1$ of $V(x) = V(u; x)$ in the case $N = 2$. Put
\[
q = e^b, \quad t = e^{-b}, \tag{5.8}
\]
\[
u_i = e^{u_i b}, \quad v_i = e^{v_i b} \quad (i = 1, 2), \tag{5.9}
\]
and we parametrize $v'_i$ and $u'_i$ by the parameters $P', P$ and $\alpha$ such that
\[
v'_i - u'_j = P'_i - P_j - \alpha, \quad i, j = 1, 2. \tag{5.10}
\]
Here, we set $P_1 = -P_2 = P$, $P'_1 = -P'_2 = P'$. Write
\[
Q := b + b^{-1}. \tag{5.11}
\]

Remark 5.8. There is an ambiguity of the choice of parametrization satisfying (5.10). No matter how we choose it, final results (Theorem 5.25) are not affected if (5.10) is satisfied. For example, the simplest parametrization is
\[
u'_1 = P + \alpha/2, \quad u'_2 = -P + \alpha/2, \tag{5.12}
\]
\[
\nu'_1 = P' - \alpha/2, \quad v'_2 = -P' - \alpha/2. \tag{5.13}
\]
Even if we add an arbitrary parameter to (5.12) and (5.13) to keep the degree of freedom of $u'_i$ and $v'_i$, the results stay the same.

It is easy to show the following lemma.

Lemma 5.9. If (5.10) is satisfied, it follows that
\[
u'_i - u'_j = 2P, \quad v'_i - v'_j = 2P', \tag{5.14}
\]
\[
u'_1 + u'_2 - v'_1 - v'_2 = 2\alpha. \tag{5.15}
\]

The parametrization above is designed so that the following factor appears in the limit of the Nekrasov factor.

Definition 5.10. Set
\[
Z_{\text{lif}}(\alpha; P, \lambda; P, \mu) = \prod_{i,j=1}^{2} \prod_{(k,l) \in \lambda^{(o)}} \left( P'_i - P_j - \alpha + b^{-1}(a_{\lambda(0)}(k,l) + 1) - b \ell_{\mu(0)}(k,l) \right) \prod_{(k,l) \in \mu^{(o)}} \left( P'_i - P_j - \alpha - b^{-1} a_{\mu(0)}(k,l) + b(\ell_{\lambda(0)}(k,l) + 1) \right). \tag{5.16}
\]

It is known that the operator $V(x)$ exists uniquely [7]. Moreover, their matrix element formula is proved.
Proposition 5.11. We have
\[
\sum_{i,j=1}^{2} N_{\lambda^{(i)}, \mu^{(i)}} (q v_i / t u_j) = Z_{bf}(a | q', \lambda | q', \mu) \cdot h^{2(|\lambda| + |\mu|)} + \mathcal{O} \left( h^{2(|\lambda| + |\mu|) + 1} \right).
\] (5.18)

Proof. Follows by direct calculation.

Let us define bosons independent of \( h \).

Definition 5.12. Define the Heisenberg algebra \( a_n \) \((n \in \mathbb{Z}_{\neq 0})\) by the commutation relation
\[
[a_n, a_m] = -b^{-2} n \delta_{n+m, 0},
\] (5.19)
and we assume that
\[
a^{(i)}_{-n} = a^{(i)}_n, \quad a^{(i)}_{-n} = (-h^2)^{1 - \frac{q^n}{1 - p^n}} a^{(i)}_n \quad (n > 0),
\] (5.20)
\[
\lim_{h \to 0} a^{(i)}_{-n} = a_{-n}, \quad \lim_{h \to 0} a^{(i)}_n = a_n.
\] (5.21)

It is known that the generalized Macdonald functions are reduced to the generalized Jack functions \({\mathfrak{J}}\) in the limit \( h \to 0 \). They are defined as eigenfunctions of the following Hamiltonian. The existence theorem also follows similarly to the generalized Macdonald functions.

Definition 5.13. Define the operator \( H_b \) by
\[
H_b = H^{(1)}_b + H^{(2)}_b - (1 + b^{-2}) \sum_{n=1}^{\infty} n a^{(1)}_n a^{(2)}_n,
\] (5.22)
where \( H^{(i)}_b \) \((i = 1, 2)\) is a modified Hamiltonian of the Calogero-Sutherland model:
\[
H^{(i)}_b := \frac{1}{2} \sum_{n,m} \left( a^{(i)}_{n} a^{(i)}_{-n} a^{(i)}_{n+m} + a^{(i)}_{-n} a^{(i)}_{n} a^{(i)}_{n+m} \right) - \sum_{n=1}^{\infty} \left( b^{-1} u'_i + \frac{1 + b^{-2}}{2 n} \right) a^{(i)}_{-n} a^{(i)}_n.
\] (5.23)

Fact 5.14 \([\mathbb{3}]\). There exists a unique function \(|J_\lambda\rangle = |J_\lambda(u'_1, u'_2)\rangle\) satisfying the following two conditions:
\[
|J_\lambda\rangle = \prod_{i=1}^{2} m_{\lambda^{(i)}}(a_{-n}) |0\rangle + \sum_{\mu < \lambda} d_{\lambda\mu} \prod_{i=1}^{2} m_{\mu^{(i)}}(a_{-n}) |0\rangle, \quad d_{\lambda\mu} \in \mathbb{C};
\] (5.24)

\[
H_b |J_\lambda\rangle = e'_\lambda |J_\lambda\rangle, \quad e'_\lambda \in \mathbb{C}.
\] (5.25)

Similarly, there exists a unique function \( \langle J_\lambda | = \langle J_\lambda | u'_1, u'_2 |\rangle \) such that
\[
\langle J_\lambda | = \prod_{i=1}^{2} m_{\lambda^{(i)}}(a_n) + \sum_{\mu < \lambda} d'_{\lambda\mu} \prod_{i=1}^{2} m_{\mu^{(i)}}(a_n), \quad d'_{\lambda\mu} \in \mathbb{C};
\] (5.26)

\[
\langle J_\lambda | H_b = e'_\lambda \langle J_\lambda |, \quad e'_\lambda \in \mathbb{C}.
\] (5.27)

We call the eigenfunctions \(|J_\lambda\rangle\) and \( \langle J_\lambda | \) the generalized Jack functions.

Fact 5.15 \([\mathbb{3}]\). We have
\[
|P_\lambda(\mathbf{u})\rangle \xrightarrow{h \to 0, u_i := e^{t_1 h}, q = e^{-1}, t = e^{-b h}} |J_\lambda(u'_1, u'_2)\rangle,
\] (5.28)
\[
\langle P_\lambda(\mathbf{u}) | \xrightarrow{h \to 0, u_i := e^{t_1 h}, q = e^{-1}, t = e^{-b h}} \langle J_\lambda(u'_1, u'_2) |.
\] (5.29)

In fact, the generalized Jack functions are defined for general \( N \). Also for general \( N \), they correspond to the limit of the generalized Macdonald function.

Next, we take the limit of the generator \( X^{(i)}(z) \). In advance, we decompose the generator \( X^{(i)}(z) \) into the \( q \)-Virasoro algebra and some Heisenberg algebra in order to obtain the relation of the Virasoro Primary fields. This decomposition can be obtained by the following linear transformation of the bosons.
Definition 5.16. For \( n > 0 \), define
\[
\begin{aligned}
&b_n^\prime := \frac{(1 - t^{-n})}{(1 + (q/t)^n)(a_n^{(1)} - \gamma^{-n}a_n^{(2)})}, \quad b_n^\prime := -\frac{(1 - t^n)}{(1 + (q/t)^n)(a_n^{(1)} - \gamma^{-n}a_n^{(2)})}, \\
&b_n^\prime := (1 - t^{-n})(\gamma^{-2n}a_n^{(1)} + \gamma^{-n}a_n^{(2)}), \quad b_n^" := -(1 - t^n)(a_n^{(1)} + \gamma^n a_n^{(2)}).
\end{aligned}
\] (5.30)

Furthermore, for \( n \in \mathbb{Z}_{\neq 0} \), define
\[
\beta_n^{(1)} := -\frac{\sqrt{-1}}{2}b(a_n^{(1)} - a_n^{(2)}), \quad \beta_n^{(2)} := -\frac{\sqrt{-1}}{2}b(a_n^{(1)} + a_n^{(2)}).
\] (5.32)

Definition 5.17. Define the zero mode by
\[
\begin{aligned}
&\beta_0^{(1)} := \sqrt{-1} \cdot \frac{v_1' - v_2'}{2} = \sqrt{-1}P, \quad \beta_0^{(2)} := \sqrt{-1} \cdot \frac{v_1' + v_2'}{2} \quad \text{on } F_u, \\
&\beta_0^{(1)} := -\sqrt{-1} \cdot \frac{v_1' - v_2'}{2} = \sqrt{-1}P, \quad \beta_0^{(2)} := -\sqrt{-1} \cdot \frac{v_1' + v_2'}{2} \quad \text{on } F_v.
\end{aligned}
\] (5.33)

Proposition 5.18. For \( n, m \in \mathbb{Z} \), it follows that
\[
\begin{aligned}
[b_n^\prime, b_m^\prime] &= -n \frac{(1 - q^n)(1 - t^{-n})}{(1 + (q/t)^n)} \delta_{n+m,0}, \\
[b_n^\prime, b_m^\prime] &= \beta_0^{(1)}, \beta_0^{(2)} = 0, \\
[\beta_0^{(2)}, \beta_m^{(2)}] &= \frac{n}{2} \delta_{n+m,0}.
\end{aligned}
\] (5.35)

Proof. It follows from direct computation. \(\square\)

Definition 5.19. Set
\[
\begin{aligned}
\Lambda_1'(z) := \exp \left( \sum_{n \neq 0} \frac{b_n^\prime}{|n|} z^{-n} \right), \quad \Lambda_2'(z) := \exp \left( -\sum_{n \neq 0} \frac{b_n^\prime}{|n|} (t/q)^n z^{-n} \right), \\
\Lambda'(z) := \exp \left( \sum_{n > 0} \frac{b_n^\prime}{n(1 + (q/t)^n)} z^n \right) \exp \left( \sum_{n > 0} \frac{(q/t)^n b_n^\prime}{n} z^{-n} \right).
\end{aligned}
\] (5.38)

Moreover, define
\[
T(z) = \Lambda_1'(z)e^{-\sqrt{-1}\beta_0^{(1)}} + \Lambda_2'(z)e^{\sqrt{-1}\beta_0^{(1)}}, \quad Y(z) = \Lambda'(z)e^{-\sqrt{-1}\beta_0^{(2)}}.
\] (5.40)

Proposition 5.20. \(X^{(1)}(z)\) can be decomposed as
\[
X^{(1)}(z) = T(z)Y(z).
\] (5.41)

Moreover we have
\[
X^{(2)}(z) := \exp \left( \sum_{n \neq 0} \frac{b_n^\prime}{|n|} z^{-n} \right) : e^{-2\sqrt{-1}\beta_0^{(2)}} :.
\] (5.42)

Proposition 5.21. The operator \(T(z)\) satisfies the defining relation of the \(q\)-Virasoro algebra:
\[
f(w/z)T(z)T(w) - f(z/w)T(w)T(z) = -\frac{(1 - q)(1 - t^{-1})}{1 - (q/t)}(\delta(qw/tz) - \delta(tw/qz)),
\] (5.43)

where
\[
f(z) = \exp \left( \sum_{n > 0} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})}{1 + (q/t)^n} z^n \right).
\] (5.44)

These proposition can be shown by direct calculation. (See also \([35]\)). For representation theory of the \(q\)-Virasoro algebra, we refer the reader to \([39]\). Now, we consider the limit of these generators.
Theorem 5.25. \( \Phi \)

**Define**

there is no pole at \( \hbar \)

irreducible.

For simplicity, we hereafter assume that

Furthermore, we introduce

We have

Proposition 5.26. \( \Phi \)

Moreover, we obtain

Fact 5.22 \([39]\). The \( h \)-expansion of \( T(z) \) is of the form

\[
T(z) = 2 - \left( L(z) - \frac{Q^2}{4} \right) h^2 + \mathcal{O}(h^4). \tag{5.45}
\]

Here \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n} \) is some operator written by the boson \( \beta_n^{(1)} \). Moreover, \( L_n \)'s generate the Virasoro algebra with the central element \( c = 1 + 6Q^2 \):

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{n(n - 1)(n + 2)}{12} \delta_{n+m,0}. \tag{5.46}
\]

Remark 5.23. On \( \mathcal{F}^{(2,0)}_u \), the Virasoro algebra acts as

\[
L_0 |0\rangle = \left( \frac{Q^2}{4} - p^2 \right) |0\rangle, \quad L_n |0\rangle = 0 \quad (n > 0). \tag{5.47}
\]

Similarly, on \( \mathcal{F}^{(2,0)*}_v \)

\[
\langle 0 | L_0 = \left( \frac{Q^2}{4} - p^2 \right) |0\rangle, \quad \langle 0 | L_{-n} = 0 \quad (n > 0). \tag{5.48}
\]

If \( p, p' \) and \( b \) are generic, the following vectors form a basis on \( \mathcal{F}^{(2,0)}_u \) and \( \mathcal{F}^{(2,0)*}_v \), respectively:

\[
|L_{\lambda,\mu}\rangle := L_{-\lambda_1} L_{-\lambda_2} \cdots \beta_{-\mu_1}^{(2)} \beta_{-\mu_2}^{(2)} \cdots |0\rangle, \quad \lambda, \mu \in \mathcal{P}, \tag{5.49}
\]

\[
|L_{\lambda,\mu}\rangle := (0) \cdots \beta_{\mu_1}^{(2)} \beta_{\mu_2}^{(2)} \cdots L_{\lambda_1} L_{\lambda_2}, \quad \lambda, \mu \in \mathcal{P}. \tag{5.50}
\]

Hence, we can identify \( \mathcal{F}^{(2,0)}_u \) with the tensor product of the Verma module of the Virasoro algebra \( \langle L_n \rangle \) and the Fock space of the Heisenberg algebra \( \langle \beta_n^{(2)} \rangle \). Further, \( |0\rangle \) can be regarded as the tensor product of the highest weight vector of the highest weight \( \frac{Q^2}{4} - p^2 \) and the vacuum state of the Fock space. For simplicity, we hereafter assume that \( p, p' \) and \( b \) are generic so that the modules are irreducible.

By Fact 5.6, Proposition 5.11 and Fact 5.16 we can assume the following expansion. Namely, there is no pole at \( h = 0 \).

**Definition 5.24.** Define \( V_i(z) (i = 0, 1, \ldots) \) to be each coefficient in the \( h \)-expansion of \( V(z) = V(\frac{v}{u}; z) \), i.e.,

\[
V(z) = V_0(z) + V_1(z) h + V_2(z) h^2 + \cdots. \tag{5.51}
\]

Furthermore, we introduce

\[
\Phi^{HV}(z) = z^{-p^2+p^2-\alpha(Q-\alpha)}V_0(z). \tag{5.52}
\]

We obtain the result that the operator \( \Phi^{HV}(z) \) corresponds to the Virasoro primary fields.

**Theorem 5.25.** \( \Phi^{HV}(z) \) satisfies the relation

\[
[L_n, \Phi^{HV}(z)] = z^n \left( z \frac{\partial}{\partial z} + \alpha(Q - \alpha) (n + 1) \right) \Phi^{HV}(z). \tag{5.53}
\]

Moreover, we obtain

\[
[\beta_n^{(2)}, \Phi^{HV}(w)] = \sqrt{-1} w^n (Q - \alpha) V_0(w), \quad n > 0, \tag{5.54}
\]

\[
[\beta_{-n}^{(2)}, \Phi^{HV}(w)] = -\sqrt{-1} w^{-n} \alpha V_0(w), \quad n \geq 0. \tag{5.55}
\]

The proof is given in Section 5.5. Let us also state the following proposition.

**Proposition 5.26.** We have

\[
\frac{\langle \tilde{J}_{\lambda}(u'_1, u'_2) | \Phi^{HV}(z) | \tilde{J}_{\mu}(u'_1, u'_2) \rangle}{\langle 0 | \Phi^{HV}(z) | 0 \rangle} = Z_{bf} \langle \alpha | p', \lambda | p, \mu \rangle, \tag{5.56}
\]
where we put
\[ |\tilde{J}_\lambda(u_1, u_2)| = |J_\lambda(u_1, u_2)| \times \lim_{\hbar \to 0} h^{-N|\lambda|} C_\lambda^+(u), \quad (\tilde{J}_\lambda(v_1, v_2)| = (J_\lambda(v_1, v_2)| \times \lim_{\hbar \to 0} h^{-N|\lambda|} C_\lambda^-(v). \] 
\[ (5.57) \]

**Proof.** This is clear by Fact 5.6, Fact 5.15 and Proposition 5.11. \qed

The relations in Theorem 5.25 are the exactly same as the ones in [15] up to notation. We can also check that \( |\tilde{J}_\lambda| \) corresponds to the Alba, Fateev, Litvinov and Tarnopolski’s (AFLT) basis under the identification between the Fock space \( F \) and the Verma module of the algebra (Virasoro) \( \otimes \) (Heisenberg). However, \( |\tilde{J}_\lambda| \) and the AFLT basis are defined in the different ways. While \( |J_\lambda| \) is defined as eigenfunctions of \( H_\lambda \), the AFLT basis is defined by the condition that the matrix elements of the primary fields reproduce the \( Z_{ijf} \). Our matrix elements formula (Fact 5.6) and Theorem 5.25 prove that \( |\tilde{J}_\lambda| \) and the AFLT basis actually coincide. Furthermore, these results also prove the 4D AGT correspondence [16] which states the duality between the (non-deformed) Virasoro algebra and the 4D \( \mathcal{N} = 2 \) gauge theory. We can also expect the similar results for general \( \mathcal{N} \).

5.3. **Proof of Theorem 5.25.**

**Definition 5.27.** Define the currents
\[ J^{(1)}(z) = -2\sqrt{-1} \sum_{n \in \mathbb{Z}} \beta_n^{(2)} z^{-n}, \] 
\[ J^{(2)}(z) = \frac{1}{8} \cdot J^{(1)}(z)^2 - \frac{1}{4} \sum_{n > 0} n \left( (1 + 2b^2) a_n^{(1)} + b^2 a_n^{(2)} \right) z^n \] 
\[ - \frac{1}{4} \sum_{n > 0} n \left( (2 + b^2) a_n^{(1)} + a_n^{(2)} \right) z^{-n}, \] 
\[ J^{(2)}(z) = \frac{1}{2} \cdot J^{(1)}(z)^2 - \frac{1}{2} \sum_{n > 0} n \left( (2 + 3b^2) a_n^{(1)} + (1 + 2b^2) a_n^{(2)} \right) z^n \] 
\[ + \frac{1}{2} \sum_{n > 0} n (-a_n^{(1)} + b^2 a_n^{(2)}) z^{-n}. \] 
\[ (5.58) \]
\[ (5.59) \]
\[ (5.60) \]

These currents appear in the following expansion.

**Lemma 5.28.** We have
\[ Y(z) = 1 + \frac{1}{2} J^{(1)}(z) h + J^{(2)}(z) h^2 + \mathcal{O}(h^3), \] 
\[ (t/q) Y(z) = 1 + \left( -Q + \frac{1}{2} J^{(1)}(z) \right) h + \left( J^{(2)}(z) - \frac{1}{2} Q J^{(1)}(z) + \frac{Q^2}{2} \right) h^2 + \mathcal{O}(h^3), \] 
\[ X^{(2)}(z) = 1 + J^{(1)}(z) h + J^{(2)}(z) h^2 + \mathcal{O}(h^3), \] 
\[ (t/q)^2 X^{(2)}(z) = 1 + \left( -2Q + J^{(1)}(z) \right) h + \left( J^{(2)}(z) - 2Q J^{(1)}(z) + 2Q^2 \right) h^2 + \mathcal{O}(h^3). \] 
\[ (5.61) \]
\[ (5.62) \]
\[ (5.63) \]
\[ (5.64) \]

**Proof.** By direct calculation. \qed

At first, we prove the relation with the Heisenberg algebra \( \beta_n^{(2)} \).

**Proposition 5.29.** We have
\[ [\beta_n^{(2)}, \psi_0(w)] = \sqrt{-1} w^n (Q - \alpha) \psi_0(w), \quad n > 0, \] 
\[ [\beta_n^{(2)}, \psi_0(w)] = -\sqrt{-1} w^{-n} \alpha \psi_0(w), \quad n \geq 0. \] 
\[ (5.65) \]
\[ (5.66) \]

**Proof.** We calculate the \( h \)-expansion of the defining relation
\[ \left( X^{(2)}(z) - \frac{w}{z} X^{(2)}(z) \right) \psi(w) = \psi(w) \left( X^{(2)}(z) - (t/q)^2 \frac{w}{z} X^{(2)}(z) \right). \] 
\[ (5.67) \]
Lemma 5.28 shows that the coefficient in front of $\hbar^1$ in the expansion of (5.67) is
\[
(J^{(1)}(z) - \frac{w}{z} J^{(1)}(z)) V_0(w) = V_0(w) \left( J^{(1)}(z) - \frac{w}{z} \left( -2Q + J^{(1)}(z) \right) \right).
\] (5.68)
The coefficients of (5.68) in front of $z^n$ gives the following relations.

- Coefficient of $z^n$ ($n > 0$):
  \[
  \beta_n^{(2), n}, V_0(w) = w \beta_{n-1}^{(2), n-1}, V_0(w),
  \] (5.69)

- Coefficient of $z^0$:
  \[
  \beta_0^{(2), 0}, V_0(w) = -\sqrt{-1} \alpha \frac{w}{Q} V_0(w),
  \] (5.70)

- Coefficient of $z^{-1}$:
  \[
  \beta_1^{(2), 1}, V_0(w) = \sqrt{-1} w(Q - \alpha) V_0(w),
  \] (5.71)

- Coefficient of $z^{-n}$ ($n > 1$):
  \[
  \beta_n^{(2), n}, V_0(w) = w \beta_{n-1}^{(2), n-1}, V_0(w).
  \] (5.72)

By solving inductively these relations, we can get Proposition 5.29.

In the proof of Proposition 5.29, we computed the coefficient in front of $\hbar^1$ with respect to the relation between $V(w)$ and $X^{(2)}(z)$. Actually, the same relation can be obtained from the relation between $V(w)$ and $X^{(1)}(z)$. (The coefficients of $\hbar^2$ give trivial relations.) Next, we calculate the coefficient of $\hbar^2$, from which the relation with the Virasoro algebra arises. Let us introduce the following operator.

**Definition 5.30.** Define $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n}$ by
\[
L(z) := 2J^{(2)}(z) - \tilde{J}^{(2)}(z)
= \sum_{n,m \in \mathbb{Z}} \beta_n^{(2)} \beta_m^{(2)} : z^{-n-m} - \sqrt{-1} Q \sum_{n \in \mathbb{Z}} n \beta_n^{(2)} z^{-n}.
\] (5.73)

**Lemma 5.31.** We have
\[
\left( -L(z) + L(z) - \frac{w}{z} (-L(z) + L(z)) \right) V_0(w)
= V_0(w) \left( -L(z) + L(z) - \frac{w}{z} \left( -L(z) + L(z) + Q J^{(1)}(z) - Q^2 \right) \right).
\] (5.74)

**Proof.** We calculate the $h$-expansion of the defining relation
\[
\left( 1 - \frac{w}{z} \right) X^{(1)}(z)V(w) = \left( 1 - \frac{t}{q} \frac{w}{z} \right) V(w) X^{(1)}(z).
\] (5.75)

By Fact 5.22 and Lemma 5.28, the coefficient of $\hbar^2$ in the expansion of (5.75) gives the relation
\[
\left( 1 - \frac{w}{z} \right) \left( -L(z) + \frac{Q^2}{4} + 2J^{(2)}(z) \right) V_0(w) + \left( 1 - \frac{w}{z} \right) J^{(1)}(z) V_1(w)
= V_0(w) \left( -L(z) + \frac{Q^2}{4} + 2J^{(2)}(z) - \frac{w}{z} \left( -L(z) + \frac{Q^2}{4} + 2J^{(2)}(z) - QJ^{(1)}(z) + Q^2 \right) \right)
+ V_1(w) \left( J^{(1)}(z) - \frac{w}{z} \left( J^{(1)}(z) - 2Q \right) \right).
\] (5.76)

The coefficient of $\hbar^2$ in the $h$-expansion of the defining relation (5.67) is
\[
\left( 1 - \frac{w}{z} \right) \tilde{J}^{(2)}(z) V_0(z) + \left( 1 - \frac{w}{z} \right) J^{(1)}(z) V_1(w)
= V_0(w) \left( \tilde{J}^{(2)}(z) - \frac{w}{z} \left( \tilde{J}^{(2)}(z) - 2Q J^{(1)}(z) + 2Q^2 \right) \right)
+ V_1(w) \left( J^{(1)}(z) - \frac{w}{z} \left( J^{(1)}(z) - 2Q \right) \right).
\] (5.77)

By subtracting (5.76) from (5.75), we can cancel the terms of $V_1(w)$ and obtain (5.74).
Lemma 5.32. It follows that
\[
\left( \mathcal{L}(z) - \frac{w}{z} \mathcal{L}(z) \right) \mathcal{V}_0(w) - \mathcal{V}_0(w) \left( \mathcal{L}(z) - \frac{w}{z} \left( \mathcal{L}(z) + Q \mathcal{J}^{(1)}(z) - Q^2 \right) \right) = \alpha(Q - \alpha) \mathcal{V}_0(w) \sum_{n \in \mathbb{Z}} (w/z)^n. \tag{5.78}
\]

Proof. By Proposition 5.29 we can show that for \( k > 0 \),
\[
\left[ \sum_{m \in \mathbb{Z}} : \beta_m^{(2)} \beta_m^{(2)} : , \mathcal{V}_0(w) \right] = \mathcal{V}_0(w) \left( -2\sqrt{-1} \alpha \sum_{m \geq k} \beta_m^{(2)} w^{k-m} + 2\sqrt{-1} (Q - \alpha) \sum_{0 < m < k} \beta_m^{(2)} w^{k-m} \right) \nonumber
\]
\[
- (Q - \alpha)^2 (k - 1) w^k \mathcal{V}_0(w) + 2\sqrt{-1} (Q - \alpha) \sum_{m \leq 0} \beta_m^{(2)} w^{k-m} \mathcal{V}_0(w). \tag{5.79}
\]
For \( k = 0 \),
\[
\left[ \sum_{m \in \mathbb{Z}} : \beta_m^{(2)} \beta_m^{(2)} : , \mathcal{V}_0(w) \right] = -2\sqrt{-1} \alpha \mathcal{V}_0(w) \sum_{m > 0} \beta_m^{(2)} w^{-m} + \alpha \left( \frac{u'+u''+v'+v''}{2} \right) \mathcal{V}_0(w) \nonumber
\]
\[
+ 2\sqrt{-1} (Q - \alpha) \sum_{m > 0} \beta_m^{(2)} w^m \mathcal{V}_0(w). \tag{5.80}
\]
For \( k < 0 \),
\[
\left[ \sum_{m \in \mathbb{Z}} : \beta_m^{(2)} \beta_m^{(2)} : , \mathcal{V}_0(w) \right] = -2\sqrt{-1} \alpha \mathcal{V}_0(w) \sum_{0 \leq m} w^{k-m} \beta_m^{(2)} - (k + 1) \alpha^2 w^k \mathcal{V}_0(w) \nonumber
\]
\[
+ \left( -2\sqrt{-1} \alpha \sum_{k \leq m < 0} w^{k-m} \beta_m^{(2)} + 2\sqrt{-1} (Q - \alpha) \sum_{m < k} w^{k-m} \beta_m^{(2)} \right) \mathcal{V}_0(w). \tag{5.81}
\]
By using these commutation relations, we can prove (5.78) at each coefficient of \( z^{-k} \).

By the above lemmas, we can obtain the following relation between the Virasoro algebra \( L_n \) and \( \mathcal{V}_0(w) \).

Proposition 5.33. For \( n \in \mathbb{Z} \), we have
\[
[L_n, \mathcal{V}_0(w)] = w[L_{n-1}, \mathcal{V}_0(w)] + \alpha(Q - \alpha) w^n \mathcal{V}_0(w). \tag{5.82}
\]

Proof. By Lemma 5.31 and Lemma 5.32 we have
\[
\left( 1 - \frac{w}{z} \right) L(z) \mathcal{V}_0(w) - \left( 1 - \frac{w}{z} \right) \mathcal{V}_0(w) L(z) = \alpha(Q - \alpha) \mathcal{V}_0(w) \sum_{n \in \mathbb{Z}} (w/z)^n. \tag{5.83}
\]
By taking the coefficient of \( z^{-n} \), we get (5.82).

We have proved the relations among \( \mathcal{V}_0(w) \), the Heisenberg algebra \( \beta_n^{(2)} \) and the Virasoro algebra \( L_n \). Conversely, we can show that an operator satisfying these relations is unique up to the vacuum expectation value.

Lemma 5.34. Let \( \tilde{\mathcal{V}}_0(w) : \mathcal{F}_u^{(2,0)} \to \mathcal{F}_v^{(2,0)} \) be an operator satisfying the relations obtained by replacing \( \mathcal{V}_0(w) \) with \( \tilde{\mathcal{V}}_0(w) \) in (5.65), (5.66) and (5.82). Then, we have
\[
\tilde{\mathcal{V}}_0(w) = \mathcal{V}_0(w) \times (0| \tilde{\mathcal{V}}_0(w) | 0). \tag{5.84}
\]

Proof. As we explained in Remark 5.28 we can regard \( \mathcal{F}_u^{(2,0)} \) and \( \mathcal{F}_v^{(2,0)*} \) as the modules of the Virasoro algebra \( (L_n) \) and the Heisenberg algebra \( \langle \beta_n^{(2)} \rangle \). Since \( |L_{\lambda,\mu} \rangle \) and \( |L_{\lambda,\mu} \rangle \) form bases on \( \mathcal{F}_u^{(2,0)} \) and \( \mathcal{F}_v^{(2,0)*} \), respectively, the linear operators from \( \mathcal{F}_u^{(2,0)} \) to \( \mathcal{F}_v^{(2,0)} \) can be characterized by the matrix elements with respect to them. By (5.65), (5.66) and (5.82), the matrix elements \( \langle L_{\lambda}| \mathcal{V}_0(w)| L_{\mu} \rangle \) can
be attributed to the matrix elements with respect to partitions of smaller size. Indeed, if \( \lambda^{(1)} \neq 0 \), then (5.85) gives

\[
\langle \lambda^{(1)}, \lambda^{(2)} \rangle \mathcal{V}_0 (w) | L^{(1), \mu^{(2)}} \rangle = \sum_\nu (\text{const.}) \langle \mathcal{L}^{(1) - \lambda^{(1)}} \lambda^{(1)}, \ldots, \lambda^{(2)} \rangle \mathcal{V}_0 (w) | L^{\nu, \mu^{(2)}} \rangle.
\]

(5.85)

Here, \( \nu \) runs partitions of size \( |\nu| = |\mu^{(2)}| - |\lambda^{(1)}| \) or \( |\mu^{(2)}| - |\lambda^{(1)}| + 1 \). Eventually, there remains only the vacuum expectation value \( \langle \mathcal{V}_0 (w) \rangle \) in the calculation of the matrix elements. Therefore, the operator is unique up to the vacuum expectation value. \( \square \)

By the uniqueness of Lemma 5.34, we can prove Theorem 5.25.

**Proof of Theorem 5.25.** First, we prepare notations of the Verma modules and the Fock space explained in Remark 5.24. Let \( M_{\lambda_1} \) (resp. \( M_{\lambda_2} \)) be the Verma module of the Virasoro algebra with the highest weight vector \( |\lambda_1\rangle \) (resp. \( |\lambda_2\rangle \)) of highest weight \( h_1 = \frac{Q^2}{4} - P^2 \) (resp. \( h_2 = \frac{Q^2}{4} - P^2 \)). Let \( F_u \) be the Fock space of the Heisenberg algebra \( \langle \beta^{(2)}_n \rangle \) in which the zero mode acts as \( \beta^{(2)}_0 = u \). Then we can show that

\[
F_u \cong M_{\lambda_1} \otimes F_{u'}, \quad u' = \sqrt{-1} (u_1' + u_2')/2,
\]

(5.86)

\[
F_v \cong M_{\lambda_2} \otimes F_{v'}, \quad v' = \sqrt{-1} (v_1' + v_2')/2
\]

(5.87)

as representation spaces of the algebra \( \langle \mathcal{L}_n \rangle \otimes \langle \beta^{(2)}_n \rangle \). Further, let \( \langle h_2 \rangle \) be the dual vector such that \( \langle h_2 \rangle | \mathcal{L}_0 = h_2 \langle h_2 \rangle, \langle h_2 \rangle | \mathcal{L}_{-n} = 0 \) \( (n > 0) \), and \( \langle h_2 \rangle | h_2 \rangle = 1 \).

Define \( V^H (w) : F_{u'} \rightarrow F_{v'} \) by

\[
V^H (w) = \exp \left(2\sqrt{-1} (\alpha - Q) \sum_{n > 0} \frac{\beta^{(2)}_n}{n} w^n \right) \exp \left(-2\sqrt{-1} \alpha \sum_{n > 0} \frac{\beta^{(2)}_{-n}}{n} w^n \right).
\]

(5.88)

Then we have

\[
[\beta^{(2)}_n, V^H (w)] = \sqrt{-1} w^n (Q - \alpha) V^H (w), \quad n > 0,
\]

(5.89)

\[
[\beta^{(2)}_{-n}, V^H (w)] = -\sqrt{-1} w^{-n} \alpha V^H (w), \quad n > 0.
\]

(5.90)

Define \( V^{vir} (w) : M_{\lambda_1} \rightarrow M_{\lambda_2} \) to be the Virasoro primary field of conformal dimension \( \alpha (Q - \alpha) \), i.e., it satisfies

\[
[L_n, V^{vir} (w)] = w^n \left( w \frac{\partial}{\partial w} + \alpha (Q - \alpha) (n + 1) \right) V^{vir} (w)
\]

(5.91)

and

\[
\langle h_2 \rangle | V^{vir} (w) | h_1 \rangle = w^{h_2 - h_1 - \alpha (Q - \alpha)} \times (\text{scalar}).
\]

(5.92)

Then it follows that

\[
[L_n, V^{vir} (w)] = w |L_{n-1}, V^{vir} (w)| + \alpha (Q - \alpha) w^n \mathcal{V}_0 (w).
\]

(5.93)

Therefore, Lemma 5.34 shows that

\[
\Phi^{H,V} (z) = V^{vir} (w) \otimes V^H (w) \times (\text{scalar})^{-1}.
\]

(5.94)

This completes the proof. \( \square \)

**Acknowledgment**

The authors would like to thank H. Awata, B. Feigin, A. Hoshino, H. Kanno, Y. Matsuo, M. Noumi, and S. Yanagida for valuable comments. The research of J.S. is supported by JSPS KAKENHI (Grant Numbers 19K03512). Y.O. and M.F. are partially supported by Grant-in-Aid for JSPS Research Fellow (Y.O.: 18J00754, M.F.: 17J02745).
Appendix A. Asymptotically Free Eigenfunction of Macdonald’s Difference Operator

In this appendix, we briefly review basic facts of the asymptotically free eigenfunction of Macdonald’s difference operator. Consider the following modification of Macdonald’s difference operator.

**Definition A.1.** Define the operator \( D_N(s; q, t) \) on \( \mathbb{Q}(q, t) [x_2/x_1, \ldots, x_N/x_{N-1}] \) by

\[
D_N(s; q, t) := \sum_{k=1}^{N} s_k \prod_{1 \leq \ell < k} \frac{1 - t x_k/x_\ell}{1 - x_k/x_\ell} \prod_{k < \ell \leq n} \frac{1 - x_\ell/t x_k}{1 - x_\ell/x_k} T_{q,x_k},
\]

where \( T_{q,x_k} \) is the difference operator defined by

\[
T_{q,x_k} F(x_1, \ldots, x_N) = F(x_1, \ldots, q x_k, \ldots, x_N).
\]

An combinatorial formula for the eigenfunction of \( D_N(s; q, t) \) was given in [5], [6], [4]. That is the function \( f^{\psi_N} \) given in Section 5.1.

**Fact A.2** ([5], [6], [4]). The function \( f^{\psi_N}(x; s|q,t) \) is an unique formal solution to the eigenfunction equation

\[
D_N(s; q, t) f^{\psi_N}(x; s|q,t) = (s_1 + \cdots + s_N) f^{\psi_N}(x; s|q,t)
\]

up to scalar multiples.

The parameters \( s = (s_1, \ldots, s_n) \) and \( x = (x_1, \ldots, x_N) \) are symmetric to each other in the meaning of the following bispectral duality.

**Fact A.3** ([4]). It follows that

\[
\prod_{1 \leq i < j \leq N} \frac{(q s_j/s_i) q}{(q s_i/t s_j) q} \cdot f^{\psi_N}(x; s|q,t) = \prod_{1 \leq i < j \leq N} \frac{(q x_j/x_i) \infty}{(q x_i/t x_j) \infty} \cdot f^{\psi_N}(s; x|q,t).
\]

The following Poincaré duality is also important.

**Fact A.4** ([5]). It follows that

\[
f^{\psi_N}(x; s|q,t) = \prod_{1 \leq i < j \leq N} \frac{(t x_j/x_i) \infty}{(q x_j/t x_i) \infty} \cdot f^{\psi_N}(x; s|q,t).
\]

We defined \( f^{\psi_N}(x; s|q,t) \) as a formal power series. However, its analyticity is well-understood, and we can treat \( f^{\psi_N}(x; s|q,t) \) as a function of several complex variables.

**Notation A.5.** Define the subset \( D_N \subset \mathbb{C}^{N-1} \) to be

\[
D_N = \{(w_1, \ldots, w_{N-1}) \in \mathbb{C}^{N-1} \mid w_i \cdot \cdots \cdot w_{j-1} \neq q^{-Z} \cup \{0\} \quad (1 \leq i < j \leq N)\}
\]

so that

\[
\pi^{-1}(D_N) = \{(s_1, \ldots, s_N) \in (\mathbb{C}^*)^N \mid s_j/s_i \notin q^{-Z} \quad (1 \leq i < j \leq N)\}.
\]

**Fact A.6** ([5]). Let \( \tau \) be a generic complex parameter. We regard \( f^{\psi_N}(x; s|q,\tau) \) as formal power series in

\[
(z_1, \ldots, z_N) = (x_2/x_1, \ldots, x_N/x_{N-1}).
\]

Set \( r_0 = |q/\tau|^{n-2} \) if \( |q/\tau| \leq 1 \), and \( r_0 = |\tau/q| \) if \( |q/\tau| \geq 1 \). Then for any \( r < r_0 \) and any compact subset \( K \subset D_N \), the series \( f^{\psi_N}(x; s|q,\tau) \) is absolutely convergent, uniformly on \( B_r^N \times K \). Hence \( f^{\psi_N}(x; s|q,\tau) \) defines a holomorphic function on \( U_{r_0}^N \times D_N \). (For the notation \( B_r^N \) and \( U_{r_0}^N \), see Notation [2.8])

We have the correspondence between the non-stationary Ruijsenaars functions and the ordinary Macdonald functions.

**Fact A.7** ([4]). Let \( p^\delta x = (p^{N/N} x_1, \ldots, p^{1/N} x_N) \) and \( \kappa^\delta s = (\kappa^{N/N} x_1, \ldots, \kappa^{1/N} x_N) \). Then, it follows that

\[
\lim_{p \to 0} f^{\psi_N}(p^\delta x, p|\kappa^\delta s, \kappa|q,t) = f^{\psi_N}(x; s|q,t).
\]

By this limit, we can check that Theorem 3.22 is certainly consistent with Fact 3.13.
APPENDIX B. NON-STATIONARY RUIJSENAAARS FUNCTIONS AND AFFINE SCREENING CURRENT

We recall the screening currents and vertex operators depending on the parameter $\kappa$. The operators in the main text correspond to the specialized case $\kappa \rightarrow t^{-1}$.

**Definition B.1.** Let $S_i(z), \phi_i(z) \in \mathcal{H}^N[z]$ be operators satisfying

$$ S_i(z)S_i(w) = \frac{(w/z;q)_{\infty}}{(tw/z;q)_{\infty}} \frac{(qw/tz;q)_{\infty}}{(qw/z;q)_{\infty}} : S_i(z)S_i(w) :, $$

$$ S_i(z)S_{i+1}(w) = \frac{(\kappa_{i+1}^{1/N}qw/z;q)_{\infty}}{(k_{i+1}^{1/N}qw/tz;q)_{\infty}} : S_i(z)S_{i+1}(w) :, $$

$$ S_{i+1}(z)S_i(w) = \frac{(k_{i}^{-1/N}tw/z;q)_{\infty}}{(k_{i-1/N}^{1/N}w/z;q)_{\infty}} : S_{i+1}(z)S_i(w) :, $$

$$ \phi_i(z)\phi_j(w) = \frac{(w/z;q,\kappa)_{\infty}}{(tw/z;q,\kappa)_{\infty}} \phi_i(z)\phi_j(w) : \quad (0 \leq i \leq N-1), $$

$$ \phi_i(z)\phi_j(w) = \frac{(\kappa_{i+j}^{-1/N}w/z;q,\kappa)_{\infty}}{(\kappa_{i+j}^{1/N}tw/z;q,\kappa)_{\infty}} \phi_i(z)\phi_j(w) : \quad (0 \leq i < j \leq N-1), $$

$$ \phi_i(z)\phi_{j+1}(w) = \frac{(\kappa_{i+j+1}^{-1/N}w/z;q,\kappa)_{\infty}}{(\kappa_{i+j+1}^{1/N}tw/z;q,\kappa)_{\infty}} \phi_i(z)\phi_{j+1}(w) : \quad (0 \leq j < i \leq N-1), $$

and

$$ \phi_i(z)S_i+1(w) = \frac{(k_{i+1}^{1/N}qw/z;q)_\infty}{(k_{i+1}^{1/N}qw/tz;q)_\infty} : \phi_i(z)S_{i+1}(w) :, $$

$$ S_{i+1}(w)\phi_i(z) = \frac{(k_{i}^{-1/N}tw/z;q)_\infty}{(k_{i-1/N}^{1/N}w/z;q)_\infty} : \phi_i(z)S_{i+1}(w) :, $$

$$ \phi_i(z)S_i(w) = \frac{(w/z;q)_\infty}{(tw/z;q)_\infty} : \phi_i(z)S_i(w) :, \quad S_i(w)\phi_i(z) = \frac{(qz/tw;q)_\infty}{(qz/w;q)_\infty} : \phi_i(z)S_i(w) :, $$

$$ \phi_i(z)\phi_i(w) = : \phi_i(z)\phi_i(w) :, \quad S_i(w)\phi_i(z) = : \phi_i(z)S_i(w) : \quad (j \neq i, i+1). $$

The screened vertex operator having the parameter $\kappa$ is defined as follows.

**Definition B.2.** For $1 \leq i \leq N-1$ and $\lambda \in P$, define

$$ \phi^i_\lambda(z) = \phi_{i-\ell}(\kappa^{(\ell+1)/N}z) \prod_{1 \leq j \leq \ell} S_{i-j+1}(\kappa^{j/N}q^\lambda z), \quad (B.1) $$

$$ \Phi^i_\lambda(z) = \left( \frac{(q/t;q)_\infty}{(q;q)_\infty} \right)^{\ell(\lambda)} \phi^i_\lambda(z), \quad (B.2) $$

$$ \Phi^i(z|x, p) = \sum_{\lambda \in P} \Phi^i_\lambda(z) \prod_{k \geq 1} (p^{1/N}x_{N-i+k}/x_{N-i+k-1})^\lambda. \quad (B.3) $$

**Notation B.3.** For $1 \leq i \leq N$, set

$$ \omega_i = (\omega_1^i, \ldots, \omega_N^i) = (0, \ldots, 0, 1, \ldots) \in \mathbb{Z}^N. \quad (B.4) $$

We write

$$ t^{\omega_i}x = (t^{\omega_i x_1}, \ldots, t^{\omega_i x_N}). \quad (B.5) $$

The non-stationary Ruijsenaars functions can be constructed by the screened vertex operators.

**Fact B.4 (Π).** Let $N \in \mathbb{Z}_{>2}$. Then, it follows that

$$ \langle 0|\Phi^0(1/s N|t^{\omega_N^N}x, p)\Phi^1(1/s N^{-1}|t^{\omega_N^N-1}x, p) \cdots \Phi^{N-1}(1/s_1|t^{\omega_{N-1}^N}x, p)|0 \rangle \quad (B.6) $$
Decomposing factors which will be canceled afterward, we have
\[
\sum_{\lambda(1),\ldots,\lambda(N)\in P} \langle 0| \Phi_{\lambda(N)}^0 (1/s_N) \Phi_{\lambda(N-1)}^1 (1/s_{N-1}) \cdots \Phi_{\lambda(1)}^{N-1} (1/s_1) | 0 \rangle (\lambda) \prod_{j=1}^{N} (p^{1/N} x_{j+i}/x_{j+i-1})^{\lambda(j)}
\]
\[
= \prod_{1 \leq i<j \leq N} \frac{(\kappa(j-i)/N s_j/s_i; q, \kappa)_\infty}{(\kappa(j-i)/N s_j/s_i; q, \kappa)_\infty} \cdot \tilde{f}_0^N (x, p^{1/N} | s, \kappa^{1/N} | q, t).
\]
Here, we put
\[
t(\lambda) := \prod_{i=1}^{N} t^{-|\lambda(i)|} \prod_{0<i<j \leq N-1} t^{\lambda(N-i)(N+i-j)+\lambda(N-j)(i+j-1)}
\]
\[
= \prod_{i=1}^{N} t^{-|\lambda(i)|} \prod_{0<i<j \leq N-1} t^{\lambda(i)(i-1)+|\lambda(i)|}.
\]
(B.7)

Note that there is a typo in (i), and the corresponding formula Theorem 2.13 in (i) should be read as in Fact (ii).

APPENDIX C. SOME COMBINATORIAL FORMULAS

C.1. Proof of Lemma 3.19 By using factorials, we can rewrite the Nekrasov factors as follows. If \( k \neq i \),
\[
N_{\mu(k-1),\mu(k)} (Q) = \prod_{\substack{h \equiv i-k \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{-\lambda_h+\lambda_j+N+1} t^{h-j/N}; q)_{\lambda_j+1-\lambda_j+N+1}
\times \prod_{\substack{h \equiv i-k+1 \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{\lambda_h-\lambda_j-1} t^{i+h-N+1}; q)_{\lambda_j-1-\lambda_j+N+1}.
\]
(C.1)

Decomposing factors which will be canceled afterward, we have
\[
N_{\mu(k-1),\mu(k)} (Q) = A^{(k-1),k} (Q) \cdot \tilde{A}^{(k-1),k} (Q) \quad (k \neq i),
\]
(C.2)
where we put
\[
A^{(k-1),k} (Q) := \prod_{\substack{h \equiv i-k \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{-\lambda_h+\lambda_j+N} t^{h-j/N}; q)_{\lambda_j+1-\lambda_j+N} \prod_{\substack{h \equiv i-k+1 \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{\lambda_h-\lambda_j-1} t^{i+h-N+1}; q)_{\lambda_j-1-\lambda_j+N},
\]
(C.3)
\[
\tilde{A}^{(k-1),k} (Q) := \prod_{\substack{h \equiv i-k \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{-\lambda_j+N+1} t^{h-j/N}; q)_{\lambda_j+1-\lambda_j+N+1} \prod_{\substack{h \equiv i-k+1 \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{\lambda_j-\lambda_j-1} t^{i+h-N+1}; q)_{\lambda_j-1-\lambda_j+N}.
\]
(C.4)

Similarly, we have
\[
N_{\mu(i-1),\mu(i)} (Q) = \prod_{\substack{h \equiv N \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{-\lambda_h+\lambda_i+N} t^{h-j/N}; q)_{\lambda_j-N+1-\lambda_j+1} \prod_{\substack{\alpha \equiv 1 \\ \beta \equiv 0 \\ \beta \geq \alpha \geq 1}} (Q q^{\lambda_a-\lambda_j} t^{\beta+\alpha-1}; q)_{\lambda_j-\lambda_j+N}
\]
\[
= A^{(i-1),i} (Q) \cdot \tilde{A}^{(i-1),i} (Q),
\]
(C.5)
where we put
\[
A^{(i-1),i} (Q) := \prod_{\substack{h \equiv N \\ j-h \equiv 0 \\ j \geq h \geq 1}} (Q q^{-\lambda_j+N} t^{h-j/N}; q)_{\lambda_j-N+1-\lambda_j} \prod_{\substack{\alpha \equiv 1 \\ \beta \equiv 0 \\ \beta \geq \alpha \geq 1}} (Q q^{-\lambda_j} t^{\beta-\alpha+1}; q)_{\lambda_j-1-\lambda_j+N},
\]
(C.6)
\[
\tilde{A}^{(i-1,i)}(Q) := \prod_{\substack{h \equiv N \pm 1 \equiv 0 \quad \text{mod } N \\ j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{-\lambda_h + \lambda_{j+1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+1}} \cdot \prod_{\substack{h \equiv N \pm 1 \equiv 0 \quad \text{mod } N \quad \beta - \alpha \equiv 1 \quad \beta \geq \alpha \geq 1}} (Q q^{\lambda_{\beta} - \lambda_{\alpha}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_{\beta} - \lambda_{\alpha+1}}.
\]

For \( k = 1, \ldots, N \),
\[
N_{\mu(k), \mu(k)}(Q) = \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{-\lambda_h + \lambda_{j+N}, i, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N}} \cdot \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{\lambda_h - \lambda_{j+N+1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N+1}}
= B^{(k)}(Q) \cdot \tilde{B}^{(k)}(Q),
\]

where we put
\[
B^{(k)}(Q) := \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{-\lambda_h + \lambda_{j+N+1}, i, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N+1}},
\]
\[
\tilde{B}^{(k)}(Q) := \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{-\lambda_h + \lambda_{j+1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+1}} \cdot \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{\lambda_h - \lambda_{j+N-1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N-1}}.
\]

Then, it is clear that
\[
\prod_{1 \leq k \leq N} \tilde{B}^{(k)}(1) = N_{\lambda, \lambda}^{(0)}(1|q, t^{-1/N}).
\]

Since it follows that
\[
A^{(i-1,i)}(Q) = \prod_{\substack{h \equiv N \pm 1 \equiv 0 \quad \text{mod } N \\ j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{-\lambda_h + \lambda_{j+N}, i, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N}} \cdot \prod_{h \equiv N \pm 1 \equiv 0 \quad \text{mod } N \quad h \geq 1} (Q q^{\lambda_h - \lambda_{j+N+1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N+1}}
\times \prod_{\alpha \equiv 1 \quad \beta - \alpha \equiv 1 \quad \beta \geq \alpha \geq 1} (Q q^{\lambda_{\beta} - \lambda_{\alpha}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_{\beta} - \lambda_{\alpha+1}},
\]
\[
= \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{-\lambda_h + \lambda_{j+N}, i, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N}} \cdot \prod_{\substack{h \equiv i-k+1 \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (Q q^{\lambda_h - \lambda_{j+N-1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N-1}},
\]
we have
\[
\prod_{1 \leq k \leq N} \frac{A^{(k-1,k)}(t; \delta_{k,i})}{B^{(k)}(1)} = 1.
\]

By the above computation, it follows that
\[
\prod_{1 \leq k \leq N} \frac{N_{\mu(k-1), \mu(k)}(t; \delta_{k,i})}{N_{\mu(k), \mu(k)}(1)} = \frac{1}{N_{\lambda, \lambda}^{(0)}(1|q, t^{-1/N})} \cdot \tilde{A}^{(i-1,i)}(t) \cdot \prod_{k \neq i} \tilde{A}^{(k-1,k)}(1).
\]

Furthermore, it can be shown that for \( k \neq i \),
\[
\tilde{A}^{(k-1,k)}(1)
\]
\[
= \prod_{\substack{h \equiv i-k \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (q^{-\lambda_h + \lambda_{j+N+1}, i, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N+1}} \cdot \prod_{\substack{h \equiv i-k+1 \quad \text{mod } N \quad j-h \equiv 0 \quad \text{mod } N \quad j \geq h \geq 1}} (q^{\lambda_h - \lambda_{j+N-1}, i+1, N \frac{h-j}{N}, N}, q)_{\lambda_j - \lambda_{j+N-1}}
\times \prod_{\substack{h \equiv i-k+1 \quad \text{mod } N \quad h \geq 1}} (q^{\lambda_h - \lambda_{j-N}, i}, q)_{\lambda_{h-1} - \lambda_{h}}.
\]
On the other hand, by using the relation
\[ (P_{\lambda}(v)| T^{V}(u, v, w)| Q_{\mu}(u)) \]
\[ = \zeta^{\mu} \times \prod_{k=1}^{N} N_{\lambda, \mu}^{(i)}(v_{i}/u_{j}) \prod_{1 \leq i < j \leq N} N_{\mu, \lambda}^{(i)}(q_{ij}/t_{ij}) \prod_{1 \leq i < j \leq N} N_{\lambda, \lambda}^{(i)}(q_{ij}/t_{ij}), \tag{C.17} \]
where
\[ \zeta^{\mu} := M(u, v; \lambda, w; \xi^{+}_{\mu}(u) \cdot \xi_{\lambda}^{(-)}(v)^{-1}. \tag{C.18} \]
On the other hand, by using the relation
\[ N_{\lambda, \lambda}(\gamma^{-1}x) = N_{\mu, \lambda}(\gamma^{-1}x^{-1}) |x^{\lambda}+\mu| f_{\lambda}^{(\mu)} \tag{C.19} \]
we have
\[ (\mu | T^{H}(u, v; w)| \lambda) \]
\[ = \zeta^{\lambda} \times \prod_{k=1}^{N} N_{\lambda, \mu}^{(i)}(v_{i}/u_{j}) \prod_{1 \leq i < j \leq N} N_{\mu, \lambda}^{(i)}(q_{ij}/t_{ij}) \prod_{1 \leq i < j \leq N} N_{\lambda, \lambda}^{(i)}(q_{ij}/t_{ij}), \tag{C.20} \]
where
\[ \zeta^{\lambda} := \prod_{i=1}^{N} \hat{t}(\lambda^{(i)}, u_{1}, \ldots, u_{i}, v_{1}, \ldots, v_{i-1}, w, v_{i}, 0, c_{\lambda}^{(i)}, u_{1}, \ldots, u_{i-1}) (u_{i}/v_{i}) \cdot \prod_{1 \leq i < j \leq N} (u_{j}/u_{i})^{\lambda-\mu} \prod_{1 \leq i < j \leq N} (u_{j}/u_{i})^{\lambda-\mu} f_{\lambda}^{(\mu)}. \tag{C.21} \]
A direct calculation gives
\[ \zeta^{\lambda} = \prod_{i=1}^{N} q^{n(\mu^{(i)})+n(\lambda^{(i)})+|\lambda^{(i)}|}(-1)^{N-i} \prod_{i=1}^{N} (u_{i}^{(N-i)+|\mu^{(i)}|} \sum_{k=1}^{N} \lambda^{(k)} + N-i)_{\lambda-\mu}. \tag{C.22} \]
\[ \times \prod_{i=1}^{N} q^{n(\mu^{(i)})+n(\lambda^{(i)})+|\lambda^{(i)}|}(-1)^{N-i} \prod_{i=1}^{N} (u_{i}^{(N-i)+|\mu^{(i)}|} \sum_{k=1}^{N} \lambda^{(k)}). \tag{C.23} \]
This completes the proof of Theorem 2.24.
[37] A. Morozov and A. Smirnov, “Towards the Proof of AGT Relations with the Help of the Generalized Jack Polynomials,” Lett. Math. Phys. 104 no. 5, (2014) 585–612 [arXiv:1307.2576] [hep-th].

[38] Y. Ohkubo, “Generalized Jack and Macdonald polynomials arising from AGT conjecture,” J. Phys. Conf. Ser. 804 no. 1, (2017) 012036 [arXiv:1404.5401] [math-ph].

[39] J. Shiraishi, H. Kubo, H. Awata, and S. Odake, “A Quantum deformation of the Virasoro algebra and the Macdonald symmetric functions,” Lett. Math. Phys. 38 (1996) 33–51 [arXiv:q-alg/9507034] [q-alg].

MF: DEPARTMENT OF PHYSICS, FACULTY OF SCIENCE, THE UNIVERSITY OF TOKYO, HONGO 7-3-1, BUNKYO-KU, TOKYO 113-0033 JAPAN

YO, JS: GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA 3-8-1, MEGURO-KU, TOKYO 153-8914 JAPAN

E-mail address: fukuda@hep-th.phys.s.u-tokyo.ac.jp
E-mail address: yusuke.ohkubo.math@gmail.com
E-mail address: shiraish@ms.u-tokyo.ac.jp