Spin critical opalescence in zero-temperature Bose-Einstein condensates

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received 23 August 2011; accepted in final form 5 January 2012
published online 6 February 2012

PACS 67.85.Fg - Multicomponent condensates; spinor condensates
PACS 67.85.-d - Ultracold gases, trapped gases
PACS 64.70.Tg - Quantum phase transitions

Abstract – Cold-atom developments suggest the prospect of measuring scaling properties and long-range fluctuations of continuous phase transitions at zero temperature. We discuss the conditions for characterizing the phase separation of Bose-Einstein condensates of boson atoms in two distinct hyperfine spin states. The mean-field description breaks down as the system approaches the transition from the miscible side. An effective spin description clarifies the ferromagnetic nature of the transition. We show that a difference in the scattering lengths for the bosons in the same spin state leads to an effective internal magnetic field. The point at which the internal magnetic field vanishes (i.e., equal values of the like-bosons scattering lengths) is a special point. We show that the long-range density fluctuations are suppressed near that point, while the effective spin exhibits the long-range fluctuations that characterize critical points. The zero-temperature system exhibits critical opalescence with respect to long-wavelength waves of impurity atoms that interact with the bosons in a spin-dependent manner.

Introduction. – Now that cold-atom technology has realized uniform trapping potentials bounded by sharp edges [1,2], the scaling of near zero-temperature phase transitions can be explored in the laboratory. We consider spin domain formation in a Bose-Einstein condensate (BEC) of atoms in two distinct hyperfine states, which we call the “spin-up” (|\(\uparrow\rangle\)) and “spin-down” (|\(\downarrow\rangle\)) states. The Feshbach tuning of one of the scattering lengths can trigger this transition in the quantum (zero-temperature) regime. In situ images of the atoms in one of the spin states can reveal spin density fluctuations at a characteristic length scale (the correlation length) that diverges as the scattering length is tuned near a critical value. We show that the mean-field description breaks down near the transition so that the critical exponents may differ from their mean-field values. To provide a reference to future experiments and to reveal trends with respect to polarization, scattering lengths, and density, we calculate fluctuation properties in mean field. Using a spin description, we find that, for Ising spin-spin interactions with equal |\(\uparrow\uparrow\rangle\) and |\(\downarrow\downarrow\rangle\) scattering lengths, the spins exhibit long-range fluctuations, whereas the long-range density fluctuations are suppressed near the transition. The system then remains transparent to distinguishable low-energy atoms that interact with the bosons in a spin-independent manner, whereas it turns opaque to atoms that experience a spin-dependent interaction. This system can be realized with \(^{87}\text{Rb}\) atoms that support a resonance in the interaction of atoms in different spin states [3,4].

Switching ground states. – The phase separation of boson superfluids was predicted in [5], its cold-atom realization, dynamics and surface tension were predicted and described in [6] and the transition was demonstrated in [7]. Here, we show that the ground-state configuration of a dilute homogeneous BEC of \(N_{\downarrow}\) spin-down bosons and \(N_{\uparrow}\) spin-up bosons of mass \(m\) confined to a macroscopic volume \(\Omega\) alters from a homogeneous “miscible” mixture to an “immiscible” separated state. Bosons at positions \(x\) and \(x'\) interact via effective short-range potentials \(\lambda_{\uparrow\uparrow}(x-x')\) if they occupy the down (up) spin state and via \(\lambda_{\uparrow\downarrow}(x-x')\) if their spins differ. The interaction strengths \(\lambda\) are proportional to the respective scattering lengths, \(\lambda_{\uparrow\downarrow} = (4\pi\hbar^2/m)a_{\uparrow\downarrow},\) one of which can be Feshbach-tuned across the phase boundary.

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In the immiscible ground-state configuration, \( N \) spin-down bosons reside in a volume \( \Omega \) and \( N \) spin-up bosons occupy \( \Omega \). Assuming the volumes are sufficiently large to neglect surface effects, the immiscible ground-state energy is

\[
E_{\text{sep}} = \frac{\lambda_{\downarrow}}{2\Omega_{\downarrow}} N_{\downarrow}^2 + \frac{\lambda_{\uparrow}}{2(\Omega - \Omega_{\downarrow})} N_{\uparrow}^2. \tag{1}
\]

We determine \( \Omega \) by minimizing \( E_{\text{sep}} \), which gives \( \lambda_{\downarrow} N_{\downarrow}^2/[2\Omega_{\downarrow}] = \lambda_{\uparrow} N_{\uparrow}^2/[2(\Omega - \Omega_{\downarrow})] \), implying equal pressures in \( \Omega \) and \( \Omega_{\downarrow} \). Inserting the corresponding volume fraction, \( \Omega/\Omega = 1 + (N_{\downarrow}/N_{\uparrow})\sqrt{\lambda_{\downarrow}/\lambda_{\uparrow}} \) into \( E_{\text{sep}} \),

\[
E_{\text{sep}} = \Omega_{\downarrow} \lambda_{\downarrow} N_{\downarrow}^2 + \Omega_{\uparrow} \lambda_{\uparrow} N_{\uparrow}^2 = \Omega_{\downarrow} \lambda_{\downarrow} N_{\downarrow}^2 + \Omega_{\uparrow} \lambda_{\uparrow} N_{\uparrow}^2, \tag{2}
\]

we find that the separated ground-state energy

\[
E_{\text{sep}} = \frac{\lambda_{\downarrow}}{2\Omega_{\downarrow}} N_{\downarrow}^2 + \frac{\lambda_{\uparrow}}{2\Omega_{\downarrow}} N_{\uparrow}^2 + \frac{\sqrt{\lambda_{\downarrow}/\lambda_{\uparrow}} N_{\downarrow} N_{\uparrow}}, \tag{3}
\]

drops below the mean-field energy \( E_{\text{mix}} \) of the homogeneous mixture,

\[
E_{\text{mix}} = \frac{\lambda_{\downarrow}}{2\Omega_{\downarrow}} N_{\downarrow}^2 + \frac{\lambda_{\uparrow}}{2\Omega_{\downarrow}} N_{\uparrow}^2 + \frac{\lambda_{\downarrow}}{2\Omega_{\downarrow}} N_{\downarrow} N_{\uparrow}, \tag{4}
\]

if \( \sqrt{\lambda_{\downarrow}/\lambda_{\uparrow}} < \lambda_{\downarrow} \). Characterizing the competition between like and unlike interactions by

\[
g = \frac{\lambda_{\uparrow}}{\lambda_{\downarrow}}, \tag{5}
\]

the BEC ground state switches at \( g = 1 \).

**Divergence of single-component compressibility.** – We determine the spin-down density response \( \delta \rho_{\downarrow} \) to a potential perturbation \( \delta V_{\downarrow} \). The low-frequency, long-wavelength response follows from the Thomas-Fermi description that minimizes the homogeneous free energy \( F_0 = E_{\text{mix}} - \mu_{\downarrow} N_{\downarrow} - \mu_{\uparrow} N_{\uparrow} \) at fixed chemical potentials \( \mu_{\downarrow} \) and \( \mu_{\uparrow} \) and replaces \( \mu_{\downarrow} \rightarrow \mu_{\downarrow} - \delta V_{\downarrow} \). We solve the resulting equations

\[
\lambda_{\downarrow} \rho_{\downarrow} + \lambda_{\uparrow} \rho_{\uparrow} = \mu_{\downarrow} - \delta V_{\downarrow},
\]

\[
\lambda_{\uparrow} \rho_{\downarrow} + \lambda_{\downarrow} \rho_{\uparrow} = \mu_{\uparrow} \tag{6}
\]

by linearizing \( \rho_{\downarrow}(\delta V) = \rho_{\downarrow}(\delta V) + \delta \rho_{\downarrow}(\delta V) \) around the homogeneous (\( \delta V = 0 \)) equilibrium densities \( \rho_{\downarrow}(\delta V) \). The \( \delta \rho_{\downarrow} \) result gives a response function (proportional to the compressibility)

\[
\chi_{\downarrow} = -\frac{\delta \rho_{\downarrow}}{\delta V_{\downarrow}} = \frac{1}{\lambda_{\downarrow}} \frac{1}{1 - g}, \tag{7}
\]

that diverges when \( g \rightarrow 1 \). The divergence of the compressibility implies large-scale fluctuations that cause critical opalescence [8] near ordinary critical points. The contribution of \( [-\lambda_{\downarrow} g] \) to the denominator of eq. (7) describes the long-wavelength \( \downarrow \rightarrow \uparrow \) attraction mediated by the \( \uparrow \) BEC that competes with the short-range \( \downarrow \rightarrow \downarrow \) repulsion described by \( \lambda_{\downarrow} \). The divergence implies that the smallest of \( V_{\downarrow} \)-potential variations induces a large \( \rho_{\downarrow} \) density response. Equating the flat potential requirement for simulating infinite systems to \( \delta \rho_{\downarrow}/\rho_{\downarrow} = 0 \) (12) suggests that \( V_{\downarrow} \) control should ensure that

\[
\Delta V_{\downarrow} \ll \mu_{\downarrow} [1 - g], \tag{8}
\]

where \( \Delta V_{\downarrow} \) represents the \( \downarrow \) potential variation over \( \Omega \).

**Mean-field breakdown.** – The mean-field description predicts the boundary of its validity regime [9]. The miscible \( g < 1 \) mean-field densities are homogeneous, except for small-amplitude fluctuations. This assumption breaks down when \( \delta \rho \) quantum fluctuations induce an effective \( \delta V_{\downarrow} = \lambda_{\downarrow} \rho_{\downarrow} \rho_{\uparrow} \) variation sufficiently large to give \( \delta \rho_{\downarrow} \sim \rho_{\downarrow,0} \). Estimating the quantum fluctuation as \( \Delta \rho_{\downarrow} \sim \sqrt{\lim_{\lambda \rightarrow 0} \langle \delta \rho_{\downarrow}^{\uparrow} (\delta \rho_{\downarrow}^{\uparrow}) \rangle} \) so that \( \Delta \rho_{\downarrow} \sim (\rho_{\downarrow,0} a_{\downarrow}^{\uparrow})^{1/4} \), we expect large \( \delta \rho_{\downarrow} \) fluctuations when

\[
[1 - g] < \left( \frac{\lambda_{\downarrow}}{\lambda_{\uparrow}} \right) \left( \frac{\rho_{\downarrow,0} a_{\downarrow}^{\uparrow} \rho_{\uparrow,0} a_{\uparrow}}{\rho_{\downarrow} a_{\downarrow}^{\uparrow}} \right)^{1/4}, \tag{9}
\]

where \( a_{\downarrow} \) \( (a_{\uparrow}) \) is the scattering length of spin-up \( (\downarrow) \) state. A similar relationship follows for \( \delta \rho_{\uparrow} \). Assuming similar densities and scattering lengths, \( \rho_{\downarrow} \sim \rho_{\uparrow} \sim \rho/2 \), where \( \rho \) is the total density, \( \rho = \rho_{\downarrow} + \rho_{\uparrow}, a_{\downarrow} \sim a_{\uparrow} \sim a_{\downarrow} \), and introducing \( a = (a_{\downarrow} a_{\uparrow} a_{\downarrow})^{1/3} \) we suggest that

\[
[1 - g] > (\rho a^{3})^{1/4}, \tag{10}
\]

is a necessary condition for mean-field approximation validity.

**Feshbach steering of the system and required magnetic-field control.** – In a magnetically controlled Feshbach resonance it is the strength of a homogeneous magnetic field \( B \) near the resonant value \( B_{0} \) that varies the scattering length. How accurately does the magnetic field have to be controlled to avoid the effects of a fluctuating interaction? The magnetic-field dependence of the scattering length

\[
a = a_0 \left[ 1 - \frac{\Delta}{B - B_0} \right], \tag{11}
\]

where \( a_0 \) denotes the background scattering length and \( \Delta \) represents the width, implies a scattering length variation, \( \delta a \), induced by a magnetic-field variation \( \delta B \) equal to

\[
\delta a = a_0 \left[ \frac{\Delta}{B - B_0} \right] \frac{\delta B}{\Delta}. \tag{12}
\]

For the \(^{87}\)Rb case, the near-equality of the triplet and singlet scattering lengths leads to \( a_0 \approx a_1 \approx a_2 \approx 5 \text{nm} \) within a few percent. Hence \( B \) should be tuned far from resonance, \( |B - B_0| \sim \Delta/\delta \) to encounter the phase boundary. This is important as the resonance of refs. [3,4] has been found to be quite “lossy”. At \( |B - B_0| \sim 50 \Delta - 100 \Delta \), however, particle loss should not play a role. This implies that the transition condition is achieved far from
the resonance with $\frac{\Delta}{B-B_0} = \delta$ a few percent. The relative scattering length variation is, then, of order $\delta^2$ even if $|\delta B| \sim \Delta$.

$$\left| \frac{\delta a}{a} \right| = \delta^2 \left| \frac{\delta B}{\Delta} \right|.$$  

(13)

Hence, it should be feasible to ensure that $|\delta g/g| = 2|\delta a/\mu| \ll [1 - g]$.

**Spin analogy.** — Scaling exponents have been determined for finite-temperature transitions of spin lattices. We introduce the effective spin operator $\hat{\sigma}$ so that $\hat{\sigma}_z \uparrow = +|\uparrow \rangle$, $\hat{\sigma}_z \downarrow = -|\downarrow \rangle$, and the $\hat{\sigma}_z$ and $\hat{\sigma}_y$ operators are represented by the Pauli matrices in the $|\uparrow \rangle$, $|\downarrow \rangle$ basis. The interaction of bosons with coordinates $r$ and $r'$, where $r$ and $r'$ represent both location $x$ and $x'$ and spin $\sigma$ and $\sigma'$, is [10]

$$V_{\text{eff}}(r,r') = \frac{\delta(x-x')}{4} \left[ (\lambda_\uparrow + \lambda_\downarrow + 2\lambda_u) \right.$$  

$$+ (\lambda_\uparrow - \lambda_\downarrow) (\hat{\sigma}_z + \hat{\sigma}'_z) + (\lambda_\uparrow + \lambda_\downarrow - 2\lambda_u) \hat{\sigma}_z \hat{\sigma}'_z \left].

(14)

The term linear in $\hat{\sigma}_z$ describes an effective short-range magnetic field carried by the particles. This effective field interacts with the other spins. The corresponding interaction term contributes a mean-field energy that is indistinguishable from that of an effective magnetic field $\mathbf{h}_{\text{eff}}$, $\mathbf{h}_{\text{eff}} \cdot \hat{\sigma}$, where $\mathbf{h}_{\text{eff}} = [(\lambda_\uparrow - \lambda_\downarrow)/4] \hat{\mathbf{\rho}} \hat{\mathbf{\rho}}$. Characterizing the interactions by

$$\lambda_L = \frac{\lambda_\uparrow + \lambda_\downarrow}{2}, \quad d = \frac{\lambda_\uparrow - \lambda_\downarrow}{\lambda_\uparrow + \lambda_\downarrow}, \quad \text{and} \quad r = \frac{\lambda_u}{\lambda_L},$$

(15)

the spin-spin interaction potential reads

$$V_{\text{eff}}(r,r') = \frac{\delta(x-x')}{2} \left[ (1 + r) \right.$$  

$$+ d(\hat{\sigma}_z + \hat{\sigma}'_z) + (1 - r)(\hat{\sigma}_z \hat{\sigma}'_z) \left].

(16)$

g takes the form $g = r^2/[1 - d^2]$ and $\mathbf{h}_{\text{eff}} = (d/2)\lambda_L \hat{\mathbf{\rho}} \hat{\mathbf{\rho}}$. The zero internal magnetic-field condition, $d = 0$, provides a special point: only for $\mathbf{h}_{\text{eff}} = 0$ do the ground-state spins locally align when the spin-spin coupling term turns ferromagnetic as illustrated in fig. 1. The spin analogy suggests that $\langle \hat{\sigma}_z \rangle$ plays the role of the order parameter$^1$, possibly offset by its value in the homogeneous mixture.

$^1$Measurements of $\langle \hat{\sigma}_z \rangle$ in the $\Omega_f$ and $\Omega_i$ volumes would record the different branches of the magnetization curve. At finite temperature, $\langle \hat{\sigma}_z \rangle$ could vary discontinuously across the transition as a function of $T$, except for $P = 0$ at $d = 0$ — the critical point. In analogy with finite-temperature phase separation transitions, we suggest that there may be a line (plotted as a function of $P$) of spinodal and a coexistence line that touch at the critical point. The second-order nature of the zero-temperature transition (independent of $P$) may be a consequence of these lines approaching each other as $T \rightarrow 0$. At $P = 0$, $d = 0$, and fixed temperature value, the $\langle \hat{\sigma}_z \rangle$ variation as a function of $r$ tends to its asymptotic value $\pm 1$ over an $r$ interval of magnitude $k_BT/|\lambda_L\rho|$, so that for $T = 0$, $\langle \hat{\sigma}_z \rangle$ "jumps" from 0 to $\pm 1$, but this quantity still provides a legitimate order parameter.

Fig. 1: (Colour online) In spin language, the difference of $\uparrow \uparrow$ and $\downarrow \downarrow$ scattering lengths translates into a short-range internal magnetic field carried by the particles interacting with the other spins. This internal magnetic field can locally align the effective ground-state spins, even if the spin-spin coupling is antiferromagnetic ($\lambda_L(1-r) < 0$). In this figure, the $a_\uparrow$, $a_\downarrow$ phase diagram for fixed $a_\uparrow$ value is shown. We illustrate the $\lambda_\uparrow - \lambda_\downarrow$ effect on the zero-temperature transition by showing that the immiscible regime (shaded by horizontal lines) and the ferromagnetic coupling regime (shaded by diagonal lines) only intersect along the $a_\uparrow = a_\downarrow$ line. Only when $a_U$ is varied while $a_\uparrow = a_\downarrow$, does phase separation at zero temperature take place when the spin coupling switches from ferro- to antiferromagnetic.

**Zero-temperature itinerant ferromagnet-like transition while avoiding “lower-branch” physics.** — The effective spin description reveals that the transition combines ingredients of phase separation and ferromagnetic transitions [11]. The fermion analogue, discussed in the pioneering work on quantum phase transitions [12], was reported in a cold-atom trap [13]. That transition, however, requires strong repulsion as the components separate when the inter-particle interaction energy outweighs the kinetic energy. In this strong-coupling regime, two-particle bound states (dimers) form involving “lower-branch” physics [14–19]. In contrast, BEC phase separation is triggered by the competition of different short-range interactions, all of which can be weak, allowing the system to remain in its metastable BEC state.

**Mean-field fluctuations.** — We describe the correlations induced by the quantum fluctuations of the normal modes that diagonalize the free-energy operator

$$\hat{F} = \hat{H} - \mu_\uparrow N_\uparrow - \mu_\downarrow N_\downarrow,$$

(17)

where $\hat{H}$ denotes the Hamiltonian. The zero-momentum replacement of the creation and annihilation operators $\hat{c}_j \downarrow k_\uparrow \rightarrow \sqrt{N} \downarrow$, where $j = \uparrow \downarrow$ leads to the above mean-field free energy

$$F_0 = E_{\text{mix}} - \mu_\uparrow N_\uparrow - \mu_\downarrow N_\downarrow.$$

(18)
The next-order contribution in the Bogoliubov expanded free energy $\delta F = \delta F + F_0$ keeps the quadratic terms

$$\delta F = \sum_{k,j=\uparrow,\downarrow} \left[ e_k \left( \hat{c}_{j,k}^\dagger \hat{c}_{j,k} \right) + \lambda_j \rho_{0,j} \left( \frac{\hat{c}_{j,k}^\dagger + \hat{c}_{j,-k}}{\sqrt{2}} \right) \left( \frac{\hat{c}_{j,-k}^\dagger + \hat{c}_{j,k}}{\sqrt{2}} \right) \right] + \sum_k 2 \lambda_U [\rho_{0,j} \rho_j^*] \left( \frac{\hat{c}_{j,k}^\dagger + \hat{c}_{j,-k}}{\sqrt{2}} \right) \left( \frac{\hat{c}_{j,-k}^\dagger + \hat{c}_{j,k}}{\sqrt{2}} \right),$$

(19)

where $e_k = \hbar^2 k^2/2m$, and the chemical potentials, $\mu_\uparrow$, $\mu_\downarrow$, were replaced by eq. (6) with $\delta V_\uparrow = 0$. Two-component BEC Bogoliubov transformations were constructed long ago, but these treatments obscure the underlying oscillator structure [20,21]. As in ref. [10,22] we introduce the phase-space position/momentum-like operators

$$\phi_{j,k} = \frac{b_{j,k}^\dagger + b_{j,-k}}{\sqrt{2}}; \quad \pi_{j,k} = \frac{b_{j,k}^\dagger - b_{j,-k}}{i\sqrt{2}}, \quad j = \uparrow, \downarrow,$$

(20)

that map $\delta F$ onto a sum of oscillator Hamiltonians. Below, the $k$-subscript and the sum over repeated indices will be tacitly understood. Unlike ref. [19], we make use of the simplectic approach to phase-space transformations [23]. We denote the phase-space vector $\zeta \equiv (\phi_\uparrow, \phi_\downarrow, \pi_\uparrow, \pi_\downarrow)$ and write $\delta F = (1/2)\zeta K_j \zeta$, where

$$K \equiv \begin{pmatrix} K_\phi & 0 \\ 0 & K_\pi \end{pmatrix},$$

(21)

$$K_\phi \equiv \begin{pmatrix} e + \lambda_\pi \pi_\uparrow/\pi_\downarrow & \lambda_\nu \sqrt{\pi_\uparrow \pi_\downarrow} \\ \lambda_\nu \sqrt{\pi_\uparrow \pi_\downarrow} & e - \lambda_\pi \pi_\uparrow/\pi_\downarrow \end{pmatrix}, \quad K_\pi \equiv \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}.$$

(22)

The Bogoliubov transformation is a linear, canonical point transformation $\zeta \rightarrow \eta, \eta \equiv (\phi_\uparrow, \phi_\downarrow, \pi_\uparrow, \pi_\downarrow)$ and $\zeta_\pi = M_j \eta_j$, where

$$M \equiv \begin{pmatrix} M_\phi & 0 \\ 0 & M_\pi \end{pmatrix},$$

(23)

resulting in a transformed $K$-matrix of the form

$$K' = \tilde{M}KM = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},$$

(24)

where $\tilde{M}$ represents the transpose of $M$ and $E$ is the diagonal matrix of collective mode eigenvalues,

$$E \equiv \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}. $$

(25)

The eigenvalue equations (23) and (24) take the form

$$E = \tilde{M} \phi K_\phi M_\phi,$$

(26)

$$E = \tilde{M} \pi K_\pi M_\pi.$$  

(27)

The $K'$ is then equivalent (up to a constant) to the diagonalized Bogoliubov Hamiltonian $\delta F' = \sum_{\sigma=\pm} E_{\sigma} b_{\sigma}^\dagger b_{\sigma}$, where $b_{\sigma}^\dagger = (\phi_\sigma + i\pi_\sigma)/\sqrt{2}, b_{\sigma} = (\phi_\sigma - i\pi_\sigma)/\sqrt{2}$. To ensure that $b_{\sigma}, b_{\sigma}^\dagger$ satisfy the boson commutator relations, the $M$ transformation has to be canonical [23] implying

$$M_\phi \tilde{M}_\pi = M_\pi \tilde{M}_\phi = I.$$

(28)

Writing $M_\phi$ as the product of a rotation matrix $R$ ($R = R^{-1}$) and a diagonal scaling matrix $S$, we have $M_\phi = RS$ with

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad S = \begin{pmatrix} \Gamma_+ & 0 \\ 0 & \Gamma_- \end{pmatrix}. $$

(29)

We satisfy eq. (28) by choosing $M_\pi = RS^{-1}$. As $K_\pi = eI$, eq. (27) leads to $S^{-2} = E/e$ or $\Gamma_\pm = \sqrt{E}/E_{\pm}$. Inserting the corresponding $S$ into eq. (26) results in

$$\tilde{R}K_\phi R = S^{-1}ES^{-1} = \begin{pmatrix} E_+^2/e & 0 \\ 0 & E_-^2/e \end{pmatrix}.$$

(30)

so that $E$ follows from the diagonalization of $K_\phi$ and $E_\pm$ and $E_\pm^2/e$ are the eigenvalues of $K_\phi$.

For a mixture of overall polarization, $P = (N_+ - N_-)$, we cast the resulting transformation in terms of the average sound velocity $c = \sqrt{\lambda_L \rho/m}$, the length $\xi = h/(mc)$ and a transition parameter

$$t = (1 - g) \left( \frac{1 - d^2}{1 + dP} \right),$$

(31)

Diagonalizing the $K_\phi$, we find

$$E_{\pm} = h^2 k^2 c_\pm^2 (1 + \xi^2 k^2),$$

(32)

where

$$c_\pm = \sqrt{dP} \left( \frac{1 \pm \sqrt{1 - t}}{2} \right),$$

(33)

$$\xi_\pm = \frac{c}{\sqrt{1 + dP}} \left[ \frac{1}{\sqrt{1 - t}} \right].$$

(34)

The $S$-matrix elements, $\Gamma_\pm$, take the form $\Gamma_\pm = S(\xi_\pm k)$ with $S(x) = \sqrt{\frac{2}{1 + x^2}}$ and the rotation angle $\theta$ in $R$ is determined by

$$\cos(\theta) = \sqrt{\frac{1}{2} \left[ 1 + \frac{d + P}{1 + dP} \right]}.$$ 

(35)

The corresponding transformation $\Phi_{\pm}(\zeta) \rightarrow \Phi_{\pm}(\eta)$, $\pi_{\pm}(\zeta) \rightarrow \Pi_{\pm}(\zeta)$ with $\langle \Pi_{\sigma,k}^\dagger \Pi_{\sigma',k'}\rangle = \langle \Phi_{\sigma,k}^\dagger \Phi_{\sigma',k'} \rangle = (1/2)\delta_{\sigma,\sigma'}\delta_{k,k'}$ determines all correlation and response functions. For instance, the mean-field $\uparrow\downarrow$ density correlation function is

$$\langle \rho_{\uparrow}(\mathbf{x})\rho_{\downarrow}(0) \rangle = \rho_{0,\uparrow} + \rho_{0,\downarrow} \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot \mathbf{x}} \left[ 2\langle \Phi_{\sigma,\uparrow}^\dagger \Phi_{\sigma,\downarrow} \rangle \right].$$

(36)

where

$$[2\langle \Phi_{\sigma,\uparrow}^\dagger \Phi_{\sigma,\downarrow} \rangle] = s(\xi_\pm k) \cos^2(\theta) + s(\xi_\pm k) \sin^2(\theta).$$

(37)
Note that the correlation functions harbor two length scales: $\xi_+$ and $\xi_-$. As $\xi_-$ diverges, $\xi_- \approx t^{-1/2} \xi_0 \sqrt{2/(1 + dP)^2}$, this length scale, the correlation length, exceeds the imaging resolution before the transition is reached.

For the special case of Ising spin-spin interactions, $d = 0$, we determine the long-range part of the density-density and spin-spin correlation functions,

$$\langle \hat{\rho}(x)\hat{\rho}(0) \rangle \approx \rho_0^2 + \rho_0 \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \langle \xi_+ \rangle A_{pp}^-,$$

(38)

$$\langle \hat{\sigma}_z(x)\hat{\sigma}_z(0) \rangle \approx \langle \hat{\sigma}_z \rangle^2 + \rho_0 \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \langle \xi_+ \rangle A_{ss}^-,$$

(39)

where $\rho_0 = \rho_0^+ + \rho_0^-$ and where the $A^-$ amplitudes depend on the Bogoliubov angle, giving

$$A_{ss}^- = \frac{1}{4} \left( \sqrt{1 - P} \left[ 1 + \frac{P}{\sqrt{1 - t}} \right] \right.\notag
$$

$$+ \left. \sqrt{1 + P} \left[ 1 - \frac{P}{\sqrt{1 - t}} \right]^2 \right),$$

(40)

$$A_{pp}^- = \frac{1}{4} \left( \sqrt{1 - P} \left[ 1 + \frac{P}{\sqrt{1 - t}} \right] \right.\notag
$$

$$- \left. \sqrt{1 + P} \left[ 1 - \frac{P}{\sqrt{1 - t}} \right]^2 \right).$$

(41)

Hence, the near-transition spin-spin correlation function exhibits the long-range order parameter correlations typical of a second-order phase transition. In contrast, the long-range part of the density-density correlations are suppressed, $A_{pp}^- \sim t^2$ if $P \neq 0$ near the transition. Even though the mean-field approximation breaks down, we suggest that the near-transition suppression of long-range density fluctuations is an actual feature of $d = 0$ fluctuations.

The correlation functions can be extracted from the pixel counts of a single in situ $\uparrow$ density image, averaging $\rho_\uparrow(R + x/2)\rho_\uparrow(R - x/2)$ over $R$ (making the self-averaging assumption that the $R$-average is equivalent to averaging over many samples). In accordance with the suppression of long-range density fluctuations, we suggest that $\rho_\uparrow$ for a single image can be extracted from the $\uparrow$ density. A $\rho_\uparrow$ image that resolves features on the $\xi_-$ length scale though not on the $\xi_+$ scale can be converted into a $\rho_\downarrow$ image by $\rho_\downarrow \approx \rho_0 - \rho_\uparrow$. In the case of $^{87}$Rb, $d$ nearly vanishes but is not exactly zero ($d \sim 0.01$). Therefore, we investigate the effect of a small but finite $d$ value on the ratio of the long-range density-density and spin-spin correlation amplitudes in figure 2. In this figure, we plot the $A_{pp}^-/A_{ss}^-$ ratio as a function of $d$ for different $P$-polarizations when $\xi_+ = 10\xi_0$. Note that for $d$ ranging from $-0.03$ to $0.03$, the $A_{pp}^-/A_{ss}^-$ ratio remains smaller than one part in one thousand with $P$ ranging up to $0.7$, suggesting that, near transition, suppression of the long-range density fluctuations remains valid even for small but finite $d$-values.

Spin opalescence. – In a classical system in equilibrium at temperature $T$, the long-wavelength structure factor is proportional to $k_B T\kappa$, where $\kappa$ denotes the isothermal compressibility. Near the critical point of a gas-liquid transition, the paradigm of critical opalescence, $\kappa$ diverges. The compressibility is also the long-wavelength limit of the static density response function. In the ground state of the spin-1/2 BEC system with Ising spin interactions ($d = 0$), it is the spin response that diverges, not the density response. To show that, we consider a weak magnetic-field perturbation of good momentum $k$ described by a contribution $\delta H e^{ik \cdot x} \Sigma_\uparrow(x) / \Omega$ to the energy density, where $\Sigma_\uparrow(x) = [\hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\uparrow(x) - \hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\downarrow(x)] / \Omega$ to the energy density. The long-range part of the mean-field static response $\delta \Sigma_\uparrow(x) = \delta \Sigma_\downarrow(x)$ is given by $\delta \Sigma_\downarrow = \chi_{\sigma,\sigma}^\downarrow \delta H$, where

$$\chi_{\sigma,\sigma}^- \sigma,\sigma = A_{\sigma,\sigma}^- A_{\sigma,\sigma}^+ \frac{\rho}{m c_\sigma^2} 1 + k^2 \xi^2 ,$$

(42)

which diverges $\propto t^{-1}$ at the transition, whereas the analogous long-range density response function $\chi_{\rho,\rho}^- \propto A_{pp}^- / c_\rho^2 \propto t$ vanishes near the transition. As a consequence, the spin fluctuations can mediate interactions in a very pronounced manner and the Hamiltonian terms neglected in the Bogoliubov approximation become important.

Feasibility and summary. – Critical slowing prevents an actual zero-temperature crossing of a second-order phase transition: the temperature $T$ should be lower than an energy scale, $mc^2$, that vanishes at the transition ($mc^2 \sim t$ in mean field). Experimentally, it would be very interesting to reach the mean-field
breakdown regime where the fluctuations acquire large amplitudes, under the conditions of experimental access: \(k_B T < mc^2\), the measuring time is \(\tau_M \gg \hbar / [mc^2]\), the potential variation \(\Delta V\) remains sufficiently small, and the system’s linear size \(L\) significantly exceeds the coherence length. For a \(^{87}\text{Rb}\) two-component BEC of density \(\rho \sim 5 \times 10^{13} \text{cm}^{-3}\), \(P \sim 0.1\), \(a \sim 5\) nm, we estimate that mean field breaks down at \(g \sim 0.95\), where \(\xi \sim 5\). These conditions, while challenging, can be met in experiments.

We have discussed the prospect of characterizing the dilute gas BEC phase separation of bosons in distinct hyperfine states as a zero-temperature second-order phase transition. For equal like-boson scattering lengths, we expect the system to exhibit long-range spin-spin correlations whereas the long-range density fluctuations are suppressed.

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ET’s work was funded by the LDRD-program of Los Alamos National Laboratory. We thank M. Boshier for helpful comments.

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