Global Fluctuation Spectra in Big Crunch/Big Bang String Vacua

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ABSTRACT

We study Big Crunch/Big Bang cosmologies that correspond to exact world-sheet superconformal field theories of type II strings. The string theory spacetime contains a Big Crunch and a Big Bang cosmology, as well as additional “whisker” asymptotic and intermediate regions. Within the context of free string theory, we compute, unambiguously, the scalar fluctuation spectrum in all regions of spacetime. Generically, the Big Crunch fluctuation spectrum is altered while passing through the bounce singularity. The change in the spectrum is characterized by a function $\Delta$, which is momentum and time-dependent. We compute $\Delta$ explicitly and demonstrate that it arises from the whisker regions. The whiskers are also shown to lead to “entanglement” entropy in the Big Bang region. Finally, in the Milne orbifold limit of our superconformal vacua, we show that $\Delta \to 1$ and, hence, the fluctuation spectrum is unaltered by the Big Crunch/Big Bang singularity. We comment on, but do not attempt to resolve, subtleties related to gravitational backreaction and light winding modes when interactions are taken into account.

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1 Introduction

The theory of Ekpyrotic cosmology \[1, 2, 3\], as well as pre Big Bang scenarios \[4\], has emphasized the idea that the Universe did not begin at the Big Bang but, rather, had a long prior history. Although the details of this theory are far from understood, it is not unreasonable to assume, since the Universe had to pass through the Big Bang where densities and temperatures are set by the Planck scale, that superstrings play a fundamental role. If this is the case, then one expects the natural setting for cosmology to be, not four dimensions, but the higher-dimensional spacetime of string theory. Ekpyrotic cosmology introduced the idea that the Big Bang perhaps resulted from the catastrophic collision of two brane solitons in this higher dimensional space. Colliding branes and the associated bulk space geometry correspond to vacuum solutions of the string theory equations of motion. These vacua may solve equations which are valid to some finite order in a string expansion parameter, such as five-brane/nine-brane collisions in heterotic \(M\)-theory \[5, 6\] which are valid to order \(\kappa_{11}^2\), or they may be exact conformal field theories, which solve the string equations to all orders in the string parameter \(\alpha'\). The second type of vacua were emphasized in the Big Crunch/Big Bang \[7\] realizations of Ekpyrotic theory. Cyclic models \[8\] are based on Big Crunch/Big Bang theories. All of these vacua have in common the property that, prior to the Big Bang, the relevant region of spacetime, our past, is contracting toward a singular brane collision. In Big Crunch/Big Bang scenarios, the Universe then expands outward from this singularity as the Big Bang. The singular point is called the “bounce”.

Of particular importance in all cosmologies, and no less so in Ekpyrotic theories, is the origin and momentum spectra of both scalar field and gravitational quantum fluctuations. These are of the utmost importance since they produce inhomogeneities in the cosmic microwave background (CMB) that have already been observed. These observations are being increasingly refined and offer an experimental window into whatever fundamental physics is responsible for the present Universe. It was shown in \[1, 3\] that nearly scale invariant scalar perturbations are generated in the Big Crunch phase of Ekpyrotic theories prior to the Big Bang. Indeed, a near scale invariant spectrum of perturbations is observed in the CMB, but this is in the post Big Bang phase. To make contact with these observations, one one must conclusively demonstrate that the pre Big Bang fluctuation spectrum in Ekpyrotic theories is propagated, nearly unchanged, through the Big Bang. This, despite
the fact that Big Crunch/Big Bang geometries are singular at the bounce. Although motivated by string theory, almost all previous attempts to address this issue have been carried out within the context of four-dimensional toy model geometries, which do not solve the string equations of motion, even to lowest order \[3, 9, 10\]. Recently, this problem was studied in a five-dimensional Milne background \[11, 12\], and it was emphasized that the five-dimensional structure is important near the Crunch. However, in all cases the problem was studied using the techniques of quantum field theory, with no string theory input. With no further information at their disposal, the authors of these papers had to proceed by matching the relevant pre Big Bang and post Big Bang wavefunctions at the bounce, where the geometry is highly singular. The choice of boundary conditions is, by itself, conjectural, being motivated by differing physical arguments. This is made all the harder by the singular nature of the geometry at the bounce. The result is that some authors claimed that the pre Big Bang scale invariant fluctuations are radically altered when they pass through the singularity \[9\] while others claim they are unaltered \[3, 12\]. The ambiguous nature of these attempts was demonstrated in \[10\], who showed that, although highly restricted, alternative boundary conditions are possible, leading to the contradictory claims in the literature. How, then, can one resolve this ambiguity?

It seems clear from the previous discussion that string theory should lead to a unique solution of this problem. The reason for this is that string theory vacua are, in general, globally defined. First of all, their geometric manifolds include both the Big Crunch and Big Bang regions of cosmology, in addition to other regions. Secondly, the wavefunctions of these vacua are defined everywhere on the geometry. That is, knowing a wavefunction in the Big Crunch region, for example, uniquely specifies the wavefunction in the Big Bang regime. Clearly, this exactly specifies the boundary conditions for the wavefunctions at the singularity, completely resolving the ambiguities present in previous work. In this paper, we will show that this is indeed the case, at least in the limit of zero string coupling. Working within the framework of both supercritical \[13, 14\] and critical type II superstrings\(^1\), we present a class of Big Crunch/Big Bang cosmological vacua, called “generalized” Milne orbifolds \[15\]. The geometry of these vacua includes both the past Big Crunch and the future Big Bang regions. There are, in addition, four other regions, often called “whiskers”: an additional early time region,

\(^1\)Note that “type II” refers to a fermionic string with a chiral GSO projection.
an additional late time region and two intermediate regions with closed time-like curves. The intermediate regions connect to the Big Crunch/Big Bang regions at the bounce singularity. These vacua are exact superconformal field theories and our results are valid to all orders in the string worldsheet parameter $\alpha'$ [15]. In this paper, we will present our discussion within the context of supercritical string theory, since the corresponding spacetimes are manifestly homogeneous and isotropic. However, we can show that the results are identical for certain solutions of critical string theory. These ten-dimensional vacua contain, in addition to the generalized Milne directions, two non-compact spatial directions described by the two-dimensional “cigar” conformal field theory [16], which we take to be two of the three large space directions. These vacua are not homogeneous and isotropic, but approach these properties, as closely as one likes, if a parameter is taken to be small. The Milne orbifold can be obtained as a specific limit of these generalized Milne orbifolds. Working within this context, we find that the wavefunctions are, indeed, globally defined. This follows from the fact that the generalized Milne vacua all descend from a “covering space” by the process of cosetting out a gauge action and orbifolding. The invariant globally defined wavefunctions on the covering space then descend to globally defined wavefunctions on the generalized Milne orbifolds. Therefore, to all orders in $\alpha'$, the boundary condition ambiguities inherent in previous work have been resolved.

What, then, are the results for the fluctuation spectrum? We find that the fluctuation spectrum in the Big Crunch regime is, in general, altered by its passage through the Big Crunch/Big Bang singularity. The change in the spectrum can be calculated unambiguously at any time after the bounce. In the far future, it can be expressed by an explicit momentum and time-dependent function $\Delta(\vec{k}, t)$, which multiplies the early time pre Big Bang fluctuation spectrum. In the Milne orbifold limit, we find that $\Delta \to 1$ and, hence, the Big Crunch fluctuation spectrum is preserved as it passes through the singularity. This proves the conjecture first introduced in [3] and shown within a five-dimensional field theory context in [12]. If this result survives corrections due to gravitational backreaction and stringy effects, which we will comment on at the end of this Introduction, it means that the fluctuation spectrum of Ekpyrotic cosmology may well be consistent with observation. However, for generalized Milne orbifolds $\Delta$ is not unity and the fluctuation spectrum changes as it passes through the bounce singularity. We will show that this change is entirely due to the existence of the whisker regions. Specifically, the quantum mechanics of these stringy regions is inextricably
linked to the quantum mechanics of the Big Crunch/Big Bang geometry. The point of linkage is at the bounce singularity, where the whiskers and the Big Crunch/Big Bang regions touch. The result is that there is, in general, particle production in the future Big Bang region, even though the vacuum in the Big Crunch was empty. This particle creation affects all of the correlation functions in the future. In particular, it affects the two-point correlation function from which \( \Delta \) is calculated. We find that \( \Delta \) is an explicit function of momentum, with both time-independent and time-dependent components. This calculation exposes a subtlety that arises whenever the geometry has whiskers of the type discussed in this paper. That is, that the early time in-vacuum is ambiguous. If we define this vacuum to be such that an observer in the Big Crunch regime sees no particles, then we find the associated in-vacuum in the early time whisker is not completely fixed. This ambiguity in the in-vacuum can be parameterized by two constrained complex functions \( \gamma(\vec{k}) \) and \( \tilde{\gamma}(\vec{k}) \). The parameter \( \gamma(\vec{k}) \) explicitly enters the expression for \( \Delta \).

One natural choice turns out to correspond to \( \gamma(\vec{k}) = 1, \tilde{\gamma}(\vec{k}) = 0 \) and leads to particle creation and a deviation of \( \Delta \) from unity. However, as \( \gamma \to 0 \), the particle production in the Big Bang region decreases to zero and \( \Delta \to 1 \). Therefore, for a large choice of in-vacua, the change in the fluctuation spectrum by the singularity is small. This might open the possibility of measuring string effects, namely the existence of whisker regions connected to the Big Crunch/Big Bang geometry, as small momentum and time-dependent deviations from the scale invariance of the inhomogeneities in the CMB. We hope to discuss this further elsewhere.

The linkage of the quantum mechanics of the Big Crunch/Big Bang and whisker regions has a second, unanticipated effect. As viewed in the Hilbert space of the complete quantum mechanics, the in-vacuum is a pure state. However, we find that it is “entangled”, that is, contains correlations between the late time Big Bang and whisker regions. Therefore, tracing its density matrix over the states of the unobserved whisker, one obtains a non-trivial density matrix in the observable Big Bang region. That is, an observer in the Big Bang region, with no access to information about the whiskers, finds himself in a mixed state. This state has “entanglement” entropy, which manifests itself explicitly in the expression for \( \Delta \). If the entropy were zero, then the expression for \( \Delta \) should be compatible with a Bogolubov transformation linking the “in” and “out” states of the Big Crunch and Big Bang regions. However, non-vanishing entanglement entropy will ruin this compatibility.
We find that our explicit expression for $\Delta$ indicates non-vanishing entanglement entropy, except in the limit $\gamma \to 0$ where the whisker effects decouple. It follows that, not only are the changes in the spectrum calculable, but they explicitly exhibit entropy induced by the existence of the stringy whisker regions. This is not dissimilar to recent discussions of entangled states within the context of BTZ black holes \[17\], see also \[18\].

We have presented the results of this paper mostly in the context of Ekpyrotic Big Crunch/Big Bang transitions. Indeed, Ekpyrotic theory has inspired much of the recent interest in Big Crunch/Big Bang singularities in string theory \[7\;19\] and presents a context where it is particularly important to know what happens to fluctuations at such a singularity. However, Big Crunch/Big Bang transitions are of more general interest and are especially appealing in string theory, where observables are usually defined using simple asymptotic regions. See, for instance, \[4\]. Therefore, we believe that the results of this paper, such as information loss due to the existence of whisker regions, might be of interest in more general cosmological scenarios.

Specifically, in this paper we do the following. Section 2 is devoted to a discussion of a specific four-dimensional quantum field theory involving gravity and three scalar fields. In subsection 2.1, we present a cosmological background solution of this theory and explore various physical properties. It is shown that there are two independent branches to this solution, one representing a Big Crunch universe, evolving from the past and ending in a singularity, and the other a Big Bang universe, beginning in a singularity and evolving into the future. Scalar field quantum fluctuations, in each of the Big Crunch and Big Bang regions, are discussed in subsection 2.2. Both the in-vacuum and the out-vacuum are defined, and we explicitly compute the scalar two-point correlation function with respect to each of these vacua. In subsection 2.3, we relate both the classical and quantum theories of the Big Crunch and Big Bang regions by connecting them at the singularity. Within this context, we compute the two-point correlation function with respect to the in-vacuum in both regions and compare them. We find that the scalar fluctuation spectrum is potentially altered when it passes from the past Big Crunch region through the bounce singularity. In the far future, the change in the spectrum can be expressed by a momentum and time-dependent function. We compute this function explicitly and show that it depends on the Bogolubov coefficients relating the “in” and the “out” states of the Hilbert space. The meaning of these coefficients, and how one should compute them, is discussed. Finally, in subsection 2.4, we show that our
four-dimensional quantum field theory is the low energy effective theory for supercritical type II strings.

Section 3 is devoted to superstring cosmology within the context of type II supercritical string theories. In subsection 3.1, we review the “generalized” Milne orbifold solutions of these theories first presented in [15]. These solutions are exact superconformal field theories. The various regions of the associated spacetimes are discussed in detail, including the additional whisker regions. It is shown that, at low energy, these string vacua give rise to the Big Crunch/Big Bang theories introduced in section 2. We study the scalar fluctuation spectrum in subsection 3.2. The structure of generalized Milne orbifolds as the coset and orbifold of a $PSL(2, \mathbb{R})$ covering space is reviewed and the relationship of these vacua to the Milne orbifold is discussed. In subsection 3.2.1, a basis of wavefunctions, each defined on every region of the spacetime, is presented and studied. We quantize the scalar fluctuations in subsection 3.2.2. This is accomplished by expanding the scalar fluctuations in this basis and canonically quantizing the coefficients. We define two natural vacua, the in-vacuum and the out-vacuum, and compute the Bogolubov coefficients relating them [15]. Using these results, we compute the particle production in the future regions and the scalar two-point correlation function with respect to the in-vacuum. This result is valid in any region, allowing us to compare the spectrum in the future Big Bang region with the early time spectrum during the Big Crunch. We find that, generically, the spectrum is altered. In the far future, we can express the change in the spectrum in terms of a function $\Delta$, which is momentum and time-dependent. We compute this function explicitly. In subsection 3.2.3, we show that there is, in fact, a family of in-vacua, which we specify with two constrained complex functions $\gamma(\vec{k})$ and $\tilde{\gamma}(\vec{k})$. We repeat the calculation of the Bogolubov coefficients, particle production, scalar two-point correlation function and $\Delta$ in this context. Again, in general, the fluctuation spectrum is changed as it passes through the bounce singularity. However, in the limit that $\gamma \rightarrow 0$, $\Delta$ approaches unity and the spectrum is conserved. In this limit, $\Delta$ can be expressed in terms of the pure Big Crunch/Big Bang Bogolubov coefficients. However, for any finite $\gamma$, this is no longer the case. This is explained in subsection 3.3, where it is shown that the generalized in-vacuum is an entangled state in the future. This implies that, from the point of view of an observer in the Big Bang region, this vacuum has entanglement entropy. This is equivalent to quantum mechanical “information loss” into the whisker regions. It is this information loss that obstructs writing $\Delta$ in terms of the pure Big
Crunch/Big Bang Bogolubov coefficients. In subsection 3.4, we discuss the limit of our superconformal vacua to the Milne orbifold. It is shown that, in this limit, the factor $\Delta$ goes to unity for any $\gamma$ in-vacuum. This proves that, in the Milne orbifold cosmology, the fluctuation spectrum is unaltered by the bounce singularity. In subsection 3.5, we include some comments on backreaction. Finally, in Appendices A, B, and C, we outline the theory of global wavefunctions, present the expressions for a basis of these wavefunctions in all regions of the spacetime and discuss the Milne limit of these wavefunctions respectively.

Before proceeding, we would like to make several important comments. First, note that the generalized Milne orbifolds have a smooth circle in the spatial Milne-direction. However, it is not hard to show that all of the results of this paper will be unchanged if we allow a further $\mathbb{Z}_2$ orbifolding in this direction. In this case, this smooth circle becomes a finite interval, with new twisted sector states appearing on each of the two boundaries. This new $\mathbb{Z}_2$ orbifolded vacuum is again an exact superconformal field theory, which is closer in spirit to the notion of colliding branes.

As we have stated previously, our results are computed with respect to exact superconformal field theories, so $\alpha'$ corrections are under control. However, it is well-known that these theories suffer from severe backreaction problems, which could conceivably modify our results. Indeed, string perturbation theory gives rise not only to an expansion in $\alpha'$ (higher derivative corrections to the classical action), but also to an expansion in the string coupling $g_s$ (quantum corrections). The results of [15, 20] on global wavefunctions refer only to the first term in the $g_s$ expansion. It has, in fact, been shown in [21], following [22], that classical string scattering amplitudes exhibit divergences associated with a large backreaction of the spacetime geometry to small perturbations. For a non-perturbative manifestation of gravitational instability, see [23]. Large gravitational backreaction (or the absence thereof) in this and related models [20, 24] was also studied in [26]. The tree-level divergences of [21, 22] indicate a breakdown of string perturbation theory. That is, it is not consistent to ignore $g_s$ corrections if $g_s$ is small but non-zero. For a non-technical discussion of this, see [27]. Therefore, there is at present no fully controlled computational framework, and it is conceivable that our detailed results will receive significant corrections when backreaction is taken into account. One may hope that at least certain important qualitative features, such as the peculiar causal structure of the string solutions and its effect on four-dimensional cosmology, will survive $g_s$ corrections. However, it will take
significant advances in string theory to either establish or refute this.

It has been suggested, see for instance [28], that string winding modes that become light near the bounce might play a role in resolving the singularity in string theory. In the case of the Milne orbifold, the description of these winding modes is somewhat complicated, because the size of the Milne circle grows without bound away from the singularity. See [29] for a very recent discussion and [30] for earlier work. However, in the generalized Milne orbifold the radius of the Milne circle approaches a constant asymptotically, and the vertex operators for the winding modes are explicitly known [15]. In the present paper, we will ignore winding modes, except to note that they become light near the Big Crunch/Big Bang singularity and invalidate a four-dimensional description there. They are heavy away from the bounce and, thus, do not appear in the effective action describing our four-dimensional cosmology. However, we should stress that the computations we do for the generalized Milne orbifold can easily be extended to include winding modes [15]. Of course, the most interesting effects of winding modes should involve string interactions, and these have not yet been computed for the generalized Milne orbifold.

In string theory, the observables are $S$-matrix elements. At zero string coupling, the regime in which we will be working, they are determined by a Bogolubov matrix. Our strategy is to compute this matrix using the globally defined vertex operators [15, 20]. More specifically, we calculate the Bogolubov matrix to leading order in $\alpha'$ from the associated globally defined wavefunctions. However, as was mentioned in [15], the higher $\alpha'$ corrections modify this matrix in a very trivial way, simply multiplying some of the entries by a phase. Therefore, the particle creation rates we compute are exact to all orders in $\alpha'$. In this paper, we will go a step beyond computing $S$-matrix elements and compute correlation functions at a fixed finite time, using the global wavefunctions we obtained from string theory.

For additional recent string theory work related to Big Crunch/Big Bang singularities, see [31].

2 Four-Dimensional Cosmology

In this section, we will explore the cosmological properties of a specific four-dimensional quantum field theory coupled to gravity. This theory consists of three scalar fields, denoted by $\sigma_T$, $\sigma_R$ and $\phi$ respectively, coupled to the
usual Einstein gravity. The action for this theory is given by

\[ S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \mathcal{L}, \tag{1} \]

where

\[ \mathcal{L} = R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_T \partial_\nu \sigma_T - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_R \partial_\nu \sigma_R - 4 Q^2 e^\phi. \tag{2} \]

There are two dimensionful parameters in this action, Newton’s constant \( \kappa_4^2/8\pi \) (\( \kappa_4 \) having the dimension of a length) and a mass scale \( Q \), which we will think of as small compared to the momenta of interest.

### 2.1 The Background Spacetime

We will be interested in cosmological solutions of the equations of motion of this theory that are spatially homogeneous and isotropic, that is, of the Friedman-Robinson-Walker (FRW) type. We find the following general class of such solutions. The scalar fields are independent of all spatial coordinates and evolve as follows. The field \( \sigma_T \) simply vanishes,

\[ \sigma_T = 0, \tag{3} \]

whereas

\[ \sigma_R = \sqrt{2} \log |\tanh(Qt)| \tag{4} \]

and

\[ \phi = 2 \phi_0 - \log |\sinh(2Qt)|. \tag{5} \]

Here \( \phi_0 \) is an arbitrary integration constant. These fields drive a time-varying metric which is independent of all spatial coordinates and given by

\[ g_{\mu\nu} = a(t)^2 \eta_{\mu\nu}, \tag{6} \]

where

\[ a^2 = e^{-2\phi_0} |\sinh(2Qt)| \tag{7} \]

and \( \eta_{\mu\nu} \) is the metric of flat Minkowski space. It follows from the form of the metric that \( t \) is conformal time.

What are the cosmological properties of this class of solutions? Perhaps the most salient feature is that the conformal factor of the metric, \( a(t)^2 \), is
monotonically decreasing in the time interval $-\infty < t < 0$ and monotonically increasing for $0 < t < +\infty$. We will refer to the spacetime in the negative time interval as Region I and the spacetime for the positive time interval as Region II. Let us be more specific about the geometry of these regions. To make contact with the conventional analysis of FRW cosmologies, we will first compute the equation of state

$$w = \frac{P}{\rho}, \quad (8)$$

where $P$ is the pressure and $\rho$ the energy density. These quantities are defined by the Einstein equations

$$\mathcal{H}^2 = \frac{\kappa^2}{3} a^2 \rho \quad (9)$$

and

$$\mathcal{H}' = -\frac{\kappa^2}{6} a^2 (\rho + 3P), \quad (10)$$

where

$$\mathcal{H} = \frac{a'}{a} \quad (11)$$

and the prime denotes differentiation with respect to conformal time $t$. Using (7), we find from (9) and (10) that

$$\rho = \frac{3Q^2}{\kappa^2} e^{2\phi_0} \frac{\cosh^2(2Qt)}{|\sinh^3(2Qt)|} \quad (12)$$

and

$$P = \frac{Q^2}{\kappa^2} e^{2\phi_0} \left( \frac{4 - \cosh^2(2Qt)}{|\sinh^3(2Qt)|} \right) \quad (13)$$

respectively. It follows that

$$w = \frac{1}{3} \left( \frac{4 - \cosh^2(2Qt)}{\cosh^2(2Qt)} \right). \quad (14)$$

The first thing to notice is that $w$ is not constant in time. In the far past and future

$$w \longrightarrow -\frac{1}{3}, \quad t \longrightarrow \pm \infty. \quad (15)$$

As the cosmology evolves, one finds that

$$w = 0, \quad t_0 = \pm \frac{1}{2Q} \log(2 + \sqrt{3}) \Leftrightarrow \cosh(2Qt_0) = 2 \quad (16)$$
and
\[ w = \frac{1}{3}, \quad t_{1/3} = \pm \frac{1}{2Q} \log(1 + \sqrt{2}) \Leftrightarrow |\sinh(2Qt_{1/3})| = 1. \tag{17} \]
Finally, as one approaches the origin, either from negative time or positive time,
\[ w \longrightarrow 1 - \frac{16Q^2}{3}t^2 + \cdots, \quad t \longrightarrow \pm 0. \tag{18} \]
It follows that
\[ w \approx 1, \quad |t| \ll \sqrt{\frac{3}{4Q}}. \tag{19} \]
What is the interpretation of this somewhat unusual FRW cosmology? To elucidate this, note that in a scalar dominated phase the energy density and pressure are given by
\[ \rho = \frac{1}{2\kappa^2} \left( \sum_{i=1}^{3} \psi_i'^2 + V \right), \quad p = \frac{1}{2\kappa^2} \left( \sum_{i=1}^{3} \psi_i'^2 - V \right), \tag{20} \]
where \( \psi_i, i = 1, 2, 3 \) are the fields \( \sigma_T, \sigma_R \) and \( \phi \) respectively and \( V \) is the total potential energy. Using (2), (3), (4) and (5), it is easy to show that \( \rho \) and \( p \) in (20) are identical to expressions (12) and (13). It follows that in both Regions I and II our cosmology is scalar dominated with the equation of state \( w \) changing as the scalar fields evolve. In the conformal time coordinate \( t \), only in the regions described by (15) and (19) does \( w \) behave approximately as a constant, \( w = -1/3 \) and \( w = 1 \) respectively. Note that, in the latter time region, the conformal factor \( a(t) \) in (7) becomes
\[ a \approx \left| \frac{t}{e^{2\phi_0}/2Q} \right|^q, \quad q = \frac{1}{2}, \tag{21} \]
which is consistent with constant \( w = 1 \).
A second important feature of FRW cosmologies is the Hubble parameter
\[ H = \frac{a'}{a^2} \tag{22} \]
and its inverse, \( R_H = |H|^{-1} \), the Hubble radius. Using (7), one finds that
\[ H = Qe^{\phi_0} \frac{\cosh(2Qt)}{\sinh(2Qt)|\sinh(2Qt)|^q}. \tag{23} \]
Again, note that $H$ and, hence, $R_H$ are not constants but, rather, evolve with conformal time $t$. Let us focus on the behaviour of the Hubble radius. In the far past and future

$$R_H \to \infty, \quad t \to \pm \infty.$$  

(24)

The Hubble radius then changes monotonically, taking the value, for example,

$$R_H = \frac{e^{-\phi_0}}{\sqrt{2Q}}, \quad t_{1/3} = \pm \frac{1}{2Q} \log(1 + \sqrt{2})$$  

(25)

before behaving as

$$R_H \to \frac{|2Qt|^{3/2}e^{-\phi_0}}{Q}(1 - Q^2t^2 + \cdots), \quad t \to \pm 0$$  

(26)

as the time approaches the origin. Hence,

$$R_H \approx \frac{|2Qt|^{3/2}e^{-\phi_0}}{Q}, \quad |t| \ll \frac{1}{Q},$$  

(27)

which vanishes at $t = 0$. Note that this behaviour is consistent with the monotonic evolution of the conformal factor $a^2$ and is valid in both Region I and II. From (7) and (27) it is clear that the Hubble radius goes to zero more quickly than the physical wavelength of an excitation, which is proportional to $a$. Therefore, all modes are “frozen” outside of the horizon at the Big Crunch/Big Bang transition. Similarly, we conclude from (7) and (23) that at very early and very late times, modes are inside or outside the horizon depending on whether their comoving momentum $\vec{k}$ satisfies $\vec{k}^2 > Q^2$ or $\vec{k}^2 < Q^2$, respectively. In the following, we will focus on fluctuations with $\vec{k}^2 \gg Q^2$. Thus, these modes start out inside the horizon at very early times and freeze as they approach the Big Crunch. Similarly, in the Big Bang region of spacetime they start out frozen near the Big Bang and enter the horizon at some later time.

A third important quantity to consider is the scalar curvature, $R$. In conformal time, the scalar curvature in FRW spacetimes is given by

$$R = \frac{6a''}{a^3}. $$  

(28)

It then follows from (7) that

$$R = 3Q^2e^{-2\phi_0}\frac{\sinh^2(2Qt) - 1}{|\sinh^3(2Qt)|},$$  

(29)
an expression that is valid in both Regions I and II. Key values of the scalar curvature occur at precisely the same times, namely \( \pm \infty, t_0, t_{1/3} \) and \( \pm 0 \) defined in (16) and (17) respectively, that indicated the behaviour of the equation of state \( w \). In the far past and future

\[
R \rightarrow 0, \quad t \rightarrow \pm \infty.
\]  

(30)

As \( |t| \) decreases from infinity toward zero, \( R \) has the following properties. To begin with, \( R \) is positive and increasing until \( |t_0| = \frac{1}{2Q}\log(2 + \sqrt{3}) \), after which \( R \) monotonically decreases. The scalar curvature vanishes at

\[
R = 0, \quad t_{1/3} = \pm \frac{1}{2Q}\log(1 + \sqrt{2}).
\]  

(31)

For smaller values of \( |t| \), \( R \) is negative. As one approaches the origin,

\[
R \rightarrow -\frac{3}{4} e^{2\phi_0} \frac{1}{|t|^3} (1 - 6Q^2t^2 + \cdots), \quad t \rightarrow 0
\]  

(32)

which diverges as

\[
R \cong -\frac{3}{4} e^{2\phi_0} \frac{1}{|t|^3}, \quad |t| \ll \frac{1}{\sqrt{6Q}}.
\]  

(33)

The above results allow us to give a concise description of our cosmological solution. First consider Region I, corresponding to the negative time interval \( -\infty < t < 0 \). The associated geometry is that of a spatially homogeneous and isotropic FRW spacetime. In the distant past, the manifold has vanishing scalar curvature and a divergent Hubble radius. As time progresses, the scalar curvature first grows positively, reaches a maximum and then begins to decrease, vanishing at a finite time \( t_{1/3} \). Henceforth, the curvature is negative, diverging as \( t^{-3} \) as \( t \) approaches the origin \( t = 0 \). Throughout the entire time interval \( -\infty < t < 0 \), the Hubble radius is monotonically shrinking from infinity to zero. Both the vanishing of the Hubble radius and, particularly, the divergence of the scalar curvature as \( t \rightarrow 0 \), tells us that Region I terminates abruptly at \( t = 0 \). Region I, therefore, is a classic example of what is called a “Big Crunch” cosmology. Region II, on the other hand, corresponding to the positive time interval \( 0 < t < +\infty \), is the exact mirror image of Region I in the time direction. That is, Region II is a spatially homogeneous and isotropic FRW spacetime that starts abruptly at
$t = 0$ with vanishing Hubble radius and negatively infinite scalar curvature and then expands outward. The scalar curvature increases from negative to zero to positive, reaches a maximum and then decreases to zero as $t \to +\infty$. During the entire time interval $0 < t < +\infty$, the Hubble radius monotonically increases from zero to infinity. Region II, therefore, is a classic example of a “Big Bang” cosmology. It is essential to note that in general relativity, because of the curvature singularity at $t = 0$, there is no relationship between Region I and Region II, each representing an independent cosmology.

2.2 Fluctuations

We now turn to a discussion of quantum fluctuations of the scalar fields on the Big Bang/Big Crunch geometries presented above. To do this, we must first expand $\sigma_T$, $\sigma_R$ and $\phi$ around their classical values, which we now denote as $\langle \sigma_T \rangle$, $\langle \sigma_R \rangle$ and $\langle \phi \rangle$, given in (3), (4) and (5) respectively. That is,

$$\sigma_T = \langle \sigma_T \rangle + \delta \sigma_T, \quad \sigma_R = \langle \sigma_R \rangle + \delta \sigma_R, \quad \phi = \langle \phi \rangle + \delta \phi. \quad (34)$$

Inserting each of these fields into its equation of motion, using expressions (6) and (7) for the background metric and assuming the fluctuations are of the form

$$\delta \sigma_T = \delta T(t)e^{i\vec{k} \cdot \vec{x}}, \quad \delta \sigma_R = \delta R(t)e^{i\vec{k} \cdot \vec{x}}, \quad \delta \phi = \delta \Phi(t)e^{i\vec{k} \cdot \vec{x}}, \quad (35)$$

we find that the fluctuation $\delta T$ is a solution of

$$\delta T'' + 2Q \coth(2Qt)\delta T' + \vec{k}^2 \delta T = 0, \quad (36)$$

$\delta R$ solves the same equation

$$\delta R'' + 2Q \coth(2Qt)\delta R' + \vec{k}^2 \delta R = 0 \quad (37)$$

whereas

$$\delta \Phi'' + 2Q \coth(2Qt)\delta \Phi' + (\vec{k}^2 + 4Q^2)\delta \Phi, \quad \delta \Phi \ll 1. \quad (38)$$

Note that the $\delta \Phi$ fluctuations will satisfy the same equation as $\delta T$ and $\delta R$ for momenta

$$\vec{k}^2 \gg 4Q^2. \quad (39)$$
Henceforth, we will restrict our discussion to this momentum regime. Since, in this case, all three fluctuations are specified by the same equation, we will simply focus on one of them, which we choose to be $\delta T$.

Let us search for solutions of the $\delta T$ fluctuation equation (36). To do this, we must first specify the region of spacetime in which we want to work. Begin by considering Region I, with negative conformal time in the interval $-\infty < t < 0$. For very early times, (36) simplifies to

$$
\delta T'' - 2Q\delta T' + \vec{k}^2 \delta T = 0.
$$

(40)

It is easy to see that this has plane wave solutions of the form

$$
\delta T^\pm_{\vec{k}} = C_{\vec{k}} e^{\pm i E_{\vec{k}} t},
$$

(41)

where

$$
E_{\vec{k}} = \sqrt{\vec{k}^2 - Q^2}
$$

(42)

is the energy associated with momentum $\vec{k}$ and $C_{\vec{k}}$ is a normalization constant. Note that $E_{\vec{k}}$ is a positive real number in the momentum regime (39) in which we are working. The normalization constant can be determined using the scalar product

$$
(\phi_1, \phi_2) = -i \int_{\Sigma} \phi_1 \partial_{\mu} \phi_2^* \sqrt{g_{\Sigma}} d\Sigma^\mu
$$

(43)

where $\Sigma$ is a space-like three-surface, $\sqrt{g_{\Sigma}}$ is the volume element on that surface,

$$
d\Sigma^\mu = \delta_{\mu 0} \sqrt{-g_{00}} d\vec{x}
$$

(44)

and $\phi_i$, $i = 1, 2$ are any two scalar functions. Using the metric given in (6) and (7), expression (11) for $\delta T$ and (43), we find that

$$
(\delta T^\pm_{\vec{k}} e^{i \vec{k} \cdot x}, \delta T^\pm_{\vec{k}'} e^{i \vec{k}' \cdot \vec{x}}) = \pm (2\pi)^3 \delta(\vec{k} - \vec{k}') |C_{\vec{k}}|^2 e^{-2\phi_0} E_{\vec{k}}.
$$

(45)

Note that the Klein-Gordon norm obtained from (43) can be either positive or negative depending on the frequency of the scalar function. Setting the right hand side of (45) equal to $\pm (2\pi)^3 \delta(\vec{k} - \vec{k}')$, it follows that the normalization constant, up to a phase, is given by

$$
C_{\vec{k}} = \frac{e^{\phi_0}}{\sqrt{E_{\vec{k}}}}.
$$

(46)
Therefore, the normalized asymptotic plane wave solutions of (40) are given by

$$\delta T^{\pm}_{k} = \frac{e^{i\phi_{0}}}{\sqrt{E_{k}}} e^{Q_{t}} e^{\mp iE_{k}t}, \quad t \rightarrow -\infty. \tag{47}$$

One can, in fact, solve the $\delta T$ fluctuation equation (36) for any value of $t$ in Region I. The result is found to be

$$\delta T^{+}_{k} = \frac{4j e^{i\phi_{0}}}{\sqrt{E_{k}}} (-z)^{j} F(-j, -j; -2j; \frac{1}{z}), \tag{48}$$

where

$$j = -\frac{1}{2} + i\frac{E_{k}}{2Q}, \quad z = -\sinh^{2}(Q_{t}) \tag{49}$$

and $F(a, b; c; x)$ is the hypergeometric function $_{2}F_{1}$ (see Appendix B for its definition and some properties). In addition, the hermitian conjugate

$$\delta T^{-}_{k} = \delta T^{+*}_{k} \tag{50}$$

is an independent solution of (36). Using the facts that

$$z \rightarrow -\frac{e^{-2Q_{t}}}{4}, \quad F(-j, -j; -2j; \frac{1}{z}) \rightarrow 1 \quad \text{as} \quad t \rightarrow -\infty, \tag{51}$$

we see that (48) and (50) approach the plane waves solutions (47) in the far past. One can show that (48) and (50) diverge logarithmically at $t = 0$.

Combining these results with the first expression in (35), we see that any fluctuation $\delta \sigma_{T}$ in Region I can be written as

$$\delta \sigma_{T}^{I} = \int \frac{d^{3}k}{(2\pi)^{3/2}} (a_{k}^{I} \delta T^{+}_{k}(t)e^{i\vec{k} \cdot \vec{x}} + a_{k}^{I*} \delta T^{-}_{k}(t)e^{-i\vec{k} \cdot \vec{x}}), \tag{52}$$

where $a_{k}^{I}$ are arbitrary complex coefficients. Note that the first function in this expansion, $\delta T^{+}_{k}(t)e^{i\vec{k} \cdot \vec{x}}$, corresponds to a pure positive frequency plane wave in the far past. Similarly, in this limit, $\delta T^{-}_{k}(t)e^{-i\vec{k} \cdot \vec{x}}$ becomes a negative frequency plane wave.

Thus far, our discussion of the fluctuations $\delta \sigma_{T}$ has been strictly classical. However, the theory can be easily quantized by demanding that $\delta \sigma_{T}$ and, hence, the coefficients $a_{k}^{I}$ be operators in a Hilbert space. For simplicity,
we will continue to denote these operators by $\delta \sigma^I_T$ and $a^I_{\vec{k}}$, suppressing the usual “hat” notation. The quantization will be canonical if we assume that

$$[a^I_{\vec{k}}, a^{\dagger I}_{\vec{k}'}] = \delta^3(\vec{k} - \vec{k}'), \quad [a^I_{\vec{k}}, a^I_{\vec{k}'}] = [a^{\dagger I}_{\vec{k}}, a^{\dagger I}_{\vec{k}'}] = 0.$$

(53)

The vacuum state of the quantum theory is then defined as the normalized state $|0\rangle_{in}$ satisfying

$$a^I_{\vec{k}}|0\rangle_{in} = 0$$

(54)

for all momenta $\vec{k}$. There are many objects that can now be discussed in this context. In this paper, we will focus primarily on the two-point correlation function

$$in\langle 0|\delta \sigma^I_T(t, \vec{x})\delta \sigma^I_T(t, \vec{x} + \vec{r})|0\rangle_{in}.$$

(55)

Using (52), (53) and (54), we find that this function is independent of $\vec{x}$ and given by

$$in\langle 0|\delta \sigma^I_T(t, \vec{x})\delta \sigma^I_T(t, \vec{x} + \vec{r})|0\rangle_{in} = \int \frac{d^3k}{(2\pi)^3} |\delta T^+_k| e^{-i\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{2\pi^2} \int dk |\vec{k}|^2 |\delta T^+_k|^2 \frac{\sin(|\vec{k}|\vec{r})}{|\vec{k}|\vec{r}}.$$

(56)

In the following, we will always set $\vec{r} = 0$ and consider

$$in\langle 0|\delta \sigma^I_T(t, \vec{x})^2|0\rangle_{in} = \int \frac{d^3k}{(2\pi)^3} |\delta T^+_k|^2,$$

(57)

since this is sufficient for determining the fluctuation spectrum. The result for general $\vec{r}$ can be obtained by multiplying the integrand by $\sin(|\vec{k}|\vec{r})/|\vec{k}|\vec{r}$.

Equation (57) can be computed for any time $t$ in Region I using the expression given in (48). However, for our purposes, it is most illuminating to evaluate it in the distant past. Inserting expression (47), we find that the correlation function (57) becomes

$$in\langle 0|\delta \sigma^I_T(t, \vec{x})^2|0\rangle_{in} = f(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k},$$

(58)

where

$$f(t) = 2e^{2\phi_0} e^{2Qt}.$$

(59)
For $k^2 \gg Q^2$, the momentum regime \[(32)\] in which we are working, expression \[(58)\] becomes, to next to leading order,

$$\langle 0| \delta \sigma^I(t, \vec{x})^2 |0 \rangle_{in} = f(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left( \frac{1}{2} + \frac{Q^2/2}{2|k|^2} \right).$$

(60)

The momentum dependence of the first term of the integrand is simply that of zero-point fluctuations in Minkowski space. On the other hand, the second term corresponds precisely to a scale invariant fluctuation spectrum. Note, however, that since $k^2 \gg Q^2$, the second term is subdominant to the Minkowski fluctuations. The time-dependent factor $f(t)$ is not canonical. To understand its origin, we note that the kinetic energy term for $\sigma_T$ in the Lagrangian \[(2)\] is not canonically normalized in the gravitational background given in \[(6)\] and \[(7)\]. This kinetic energy term can be canonically normalized by defining a new scalar field $\Sigma^I_T$ as

$$\Sigma^I_T = e^{-\phi_0} |\sinh(2Qt)|^{1/2} \sigma_T.$$  

(61)

Note that in the far past this expression becomes

$$\Sigma^I_T = f(t)^{-1/2} \sigma_T, \quad t \rightarrow -\infty,$$

(62)

where $f(t)$ is given in \[(59)\]. It follows from this and \[(60)\] that at early times

$$\langle 0| \delta \Sigma^I_T(t, \vec{x})^2 |0 \rangle_{in} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left( \frac{1}{2} + \frac{Q^2/2}{2|k|^2} \right).$$

(63)

We conclude that the factor $f(t)$ in the correlation function \[(60)\] is simply a scale factor that can be absorbed by a field redefinition or not, depending on taste. This concludes our analysis of quantum fluctuations in Region I.

We now want to discuss the quantum fluctuations of $\delta \sigma_T$ in Region II, that is, for conformal time in the positive interval $0 < t < +\infty$. Note that the fluctuation equation \[(36)\] is independent of which region is being considered. It follows that the analysis of quantum fluctuations in Region II is essentially identical to that in Region I. For this reason, we will simply present our results. To begin with, the fluctuations $\delta T^+_k$ and $\delta T^-_k$ given in \[(18)\] and \[(50)\] respectively remain a complete set of solutions of equation \[(36)\]. It follows that any quantum fluctuation in Region II can be written as

$$\delta \sigma_T^{II} = \int \frac{d^3k}{(2\pi)^{3/2}} (a^I_k \delta T^+_k(t)e^{ik\cdot\vec{x}} + a^{II}_k \delta T^-_k(t)e^{-ik\cdot\vec{x}}),$$

(64)
where $a_{\vec{k}}^{II}$ and $a_{\vec{k}}^{II†}$ satisfy the canonical commutation relations

$$\left[a_{\vec{k}}^{II}, a_{\vec{k}'}^{II†}\right] = \delta^3(\vec{k} - \vec{k}'), \quad \left[a_{\vec{k}}^{II}, a_{\vec{k}}^{II} \right] = \left[a_{\vec{k}}^{II†}, a_{\vec{k}}^{II†}\right] = 0. \quad (65)$$

The vacuum state is then defined as the normalized state $|0\rangle_{out}$ satisfying

$$a_{\vec{k}}^{II} |0\rangle_{out} = 0 \quad (66)$$

for all momenta $\vec{k}$. Again, there are many objects that one may wish to compute at this point. For example, in analogy with Region I, we find that in the distant future

$$\langle 0| \delta\sigma_T^{II}(t, \vec{x})^2 |0\rangle_{out} = g(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left( \frac{1}{2} + \frac{Q^2/2}{|k|^2} \right), \quad (67)$$

where

$$g(t) = 2e^{2\phi_0} e^{-2Qt}. \quad (68)$$

As discussed previously, the factor of $g(t)$ arises from the non-canonical normalization of the $\sigma_T$ kinetic energy term in [2] with respect to the geometric background [3] and [7]. Proper normalization of this term can be achieved by defining

$$\Sigma_T^{II} = e^{-\phi_0} |\sinh(2Qt)|^{1/2} \sigma_T^{II} \quad (69)$$

for any positive conformal time $t$. Note that in the far future this relation becomes

$$\Sigma_T^{II} = g(t)^{-1/2} \sigma_T^{II}, \quad t \rightarrow +\infty, \quad (70)$$

where $g(t)$ is given in [68]. In terms of this canonically normalized scalar field, the fluctuation spectrum (67) becomes

$$\langle 0| \delta\Sigma_T^{II}(t, \vec{x})^2 |0\rangle_{out} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left( \frac{1}{2} + \frac{Q^2/2}{|k|^2} \right). \quad (71)$$

Therefore, the $g(t)$ factor in the two-point correlation function (67) is simply a scale factor. It can be absorbed or not, depending on taste. This concludes our analysis of quantum fluctuations in Region II.

As stated earlier, because of the curvature singularity at $t = 0$, there is no classical relationship between Region I and Region II. Each represents an independent cosmology. The same is true for the quantum fluctuations that we have just discussed. The creation operators $a_{\vec{k}}^{II†}$ acting on the vacuum
$|0\rangle_{in}$ create a Hilbert space of states $\mathcal{H}^I$ representing the quantum theory in the Big Crunch geometry of Region I. Similarly, the operators $a_{\vec{k}}^{I\dagger}$ acting on $|0\rangle_{out}$ create a Hilbert space $\mathcal{H}^{II}$ representing the quantum theory of the Big Bang geometry of Region II. A priori, there is absolutely no relation between $\mathcal{H}^I$ and $\mathcal{H}^{II}$.

### 2.3 Relating the Big Bang to the Big Crunch

It is the thesis of Ekpyrotic cosmology that the Universe did not begin at the Big Bang. Rather, it had a prior history, connected to our present geometry via some catastrophic event. In the Big Crunch/Big Bang versions of Ekpyrotic theory, this catastrophic event is a spacetime singularity at $t = 0$. This singularity is of the type found in the curvature scalar in Region I and Region II as $t \to -0$ and $t \to +0$ respectively. We will, therefore, within the context of the theory described by (2), construct a Big Crunch/Big Bang scenario by connecting Region I and Region II classically at the singular point $t = 0$. Having done this, one must also specify a relation between the quantum theories on these two regions. The most naive approach, and the one we will adopt in this section, is to identify the two Hilbert spaces. That is, assume that

$$\mathcal{H}^I = \mathcal{H}^{II} \equiv \mathcal{H}. \quad (72)$$

One consequence of this is that the creation/annihilation operators $a_{\vec{k}}^{I}, a_{\vec{k}}^{I\dagger}$ and $a_{\vec{k}}^{II}, a_{\vec{k}}^{II\dagger}$ all act on the same Hilbert space $\mathcal{H}$. In “normal” quantum field theory, that is, when there is no geometric singularity or event horizon separating the past from the future, the “out” creation/annihilation operators are linearly related to the “in” creation/annihilation operators via a so-called Bogolubov transformation. We will assume that the same is true in our theory, despite the existence of a curvature singularity at $t = 0$. That is, we postulate that

$$\begin{pmatrix} a_{\vec{k}}^{II\dagger} \\ a_{-\vec{k}}^{II\dagger} \end{pmatrix} = \begin{pmatrix} X^{*}(\vec{k}) & Y^{*}(\vec{k}) \\ Y(-\vec{k}) & X(-\vec{k}) \end{pmatrix} \begin{pmatrix} a_{\vec{k}}^{I\dagger} \\ a_{-\vec{k}}^{I\dagger} \end{pmatrix}. \quad (73)$$

The complex Bogolubov coefficients $X$ and $Y$ are not completely independent. They are constrained by the requirement that the Region I and Region II creation/annihilation operators continue to satisfy the canonical commutation relations (53) and (65) respectively. It follows that

$$|X|^2 - |Y|^2 = 1. \quad (74)$$
This is the only constraint on these coefficients. However, for simplicity, we will further assume that

\[ X(\vec{k}) = X(-\vec{k}), \quad Y(\vec{k}) = Y(-\vec{k}). \quad (75) \]

Relaxing these assumptions will not change any of our conclusions.

Having postulated the relations (72) and (73), one can now compute correlation functions that were not defined separately in Region I and Region II. Specifically, we want to calculate the two-point function

\[ \langle 0 | \delta \sigma_{T}^{II}(t, \vec{x})^2 | 0 \rangle_{in}, \quad (76) \]

where \( \delta \sigma_{T}^{II} \) is the Region II field operator given in (64), whereas \( |0\rangle_{in} \) is the Region I vacuum defined in (54). This is easily accomplished using (53), (54), (64), (73) and (74). The result is

\[ \langle 0 | \delta \sigma_{T}^{II}(t, \vec{x})^2 | 0 \rangle_{in} = \int \frac{d^3k}{(2\pi)^3} \left(1 + 2|Y|^2|\delta T_{-}^{k}|^2 + XY\delta T_{+}^{k} + X^*Y^*\delta T_{-}^{k} \right), \quad (77) \]

with the fluctuations \( \delta T_{+}^{k} \) and \( \delta T_{-}^{k} \) given by (48) and (50) respectively. Of particular physical importance is the form of this correlation function in the distant future. As \( t \to +\infty \), expression (77) becomes

\[ \langle 0 | \delta \sigma_{T}^{II}(t, \vec{x})^2 | 0 \rangle_{in} = g(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left(\frac{1}{2} + \frac{Q^2/2}{2|k|^2} \right) \Delta(\vec{k}, t), \quad (78) \]

where the scale factor \( g(t) \) is given in (68) and

\[ \Delta(\vec{k}, t) = 1 + 2|Y|^2 + XYe^{-2iE_{k}t} + X^*Y^*e^{2iE_{k}t}. \quad (79) \]

Written in terms of the scalar field \( \Sigma_{T}^{II} \) defined in (69), and using (70), this simplifies to

\[ \langle 0 | \delta \Sigma_{T}^{II}(t, \vec{x})^2 | 0 \rangle_{in} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left(\frac{1}{2} + \frac{Q^2/2}{2|k|^2} \right) \Delta(\vec{k}, t). \quad (80) \]

We want to compare these results to another correlation function, namely

\[ \langle 0 | \delta \sigma_{T}^{I}(t, \vec{x})^2 | 0 \rangle_{in}. \quad (81) \]
Note that, in this case, both the field operator $\delta \sigma^I_T$ and the vacuum $|0\rangle_{in}$ are the Region I quantities defined in (52) and (54) respectively. For arbitrary time $t$, this two-point function was evaluated in (56) and its functional form in the limit $t \to -\infty$ presented in (60). Finally, in terms of the scalar field $\Sigma^I_T$ defined in (61), and using (62), the $t \to -\infty$ limit of this correlation function becomes

$$i_{in}\langle 0 | \delta \Sigma^I_T(t, \vec{x})^2 | 0 \rangle_{in} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left( \frac{1}{2} + \frac{Q^2/2}{2|k|^2} \right).$$

(82)

Although these expressions were first derived strictly within the context of Region I, they remain valid for the complete Hilbert space $\mathcal{H}$. First note that, for arbitrary conformal time $t$, correlation functions (77) and (56) are identical only if the complex Bogolubov coefficient $Y = 0$. This remains true when comparing the $t \to +\infty$ limit of $i_{in}\langle 0 | \delta \sigma^{II}_T(t, \vec{x})^2 | 0 \rangle_{in}$ given in (78) to the $t \to -\infty$ limit of $i_{in}\langle 0 | \delta \sigma^{I}_T(t, \vec{x})^2 | 0 \rangle_{in}$ presented in (54). The simplest comparison can be made between the future correlation function (80) and the past correlation function (82), since the irrelevant scale factors have been removed. As stated earlier, the form of the argument of the momentum integral in (82) is that of Minkowski fluctuations plus a subdominant scale invariant contribution in the far past. However, the $\Delta(\vec{k}, t)$ factor in (80) potentially modifies the fluctuation spectrum in the far future. If coefficient $Y = 0$, then it follows from (79) that $\Delta = 1$ and one again obtains the spectrum (82). However, if $Y \neq 0$, then $\Delta \neq 1$ and the fluctuation spectrum is modified. What are the physical consequences of this?

A central assertion of Ekpyrotic Big Crunch/Big Bang theories is that a scale invariant fluctuation spectrum is generated in the Big Crunch geometry prior to the singularity which is then transmitted, without modification, to the Big Bang geometry after the singularity. It is these scale invariant fluctuations that are assumed to account for the observed fluctuations in the cosmic microwave background. But is this true? Or is the fluctuation spectrum modified by the presence of the singularity? There has been considerable controversy regarding this, with some authors concluding that the spectrum is transmitted unchanged, some authors claiming it is greatly modified and further literature showing that this question is ambiguous as posed, requiring more physics input to uniquely resolve it. All of this literature has attempted to confront this issue by imposing explicit boundary conditions to match the incoming and outgoing wavefunctions at, or near, the singularity. However, attempting to study the vicinity of a singularity is difficult since
one expects short distance effects to greatly modify the geometry and the physics. This is the source of the ambiguities. Our point of view is different. We see that the question of the persistence of the spectrum through the singularity is precisely expressed by whether or not the Bogolubov coefficient $Y$ vanishes. If it vanishes, the spectrum is transmitted through the singularity into the future. If it does not vanish, the spectrum is modified after the Big Bang. We conclude that to study this problem, one must, unambiguously, compute the Bogolubov coefficient $Y$. But how? One could, of course, attempt to compute $Y$ by matching the wavefunctions at the singularity. But, as we have said, this process would be ambiguous. A far more concrete way would be to compute $Y$ directly from string theory, where at least for certain types of singularities global wavefunctions can be unambiguously defined. Given such globally defined wavefunctions, one can perform calculations in the asymptotic past and future, far away from the singularity at $t = 0$. This is the approach we will follow in the remainder of this paper. However, to compute the Bogolubov coefficient $Y$ in this manner, it is necessary to demonstrate that the field theory defined by (2) is, in fact, the low energy four-dimensional effective theory of some specific string theory. This is indeed the case, as we will now show.

### 2.4 Four-Dimensional Cosmology from String Theory

In $(d+1)$-dimensional spacetime, string theory gives rise to a classical effective action of the form

$$S_{d+1} = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \sqrt{-g} e^{-2\Phi} (R_{d+1} + 4g^{IJ} \partial_I \Phi \partial_J \Phi - 2\Lambda),$$

(83)

where $I, J = 0, 1, \ldots, d$, $g_{IJ}$ and $\Phi$ are the $(d+1)$-dimensional string frame metric and dilaton respectively and all other massless and massive string modes have been set to zero. A positive tree level cosmological constant $\Lambda > 0$ will arise, for example, in supercritical type II string theories in

$$d + 1 = 26, 42, 58, \ldots$$

(84)

dimensions [14].

Let us now assume the existence of solutions of the string equations of motion with metrics of the form

$$g_{IJ} dx^I dx^J = g'_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma_T} \delta_{ab} dx^a dx^b + e^{2\sigma_R} (dx^d)^2,$$

(85)
where \( \mu, \nu = 0, 1, 2, 3 \), indices \( a, b = 4, \ldots, d - 1 \) and \( g'_{\mu \nu}, \sigma'_T \) and \( \sigma'_R \) are all functions of the four-dimensional coordinates \( x^\mu, \mu = 0, 1, 2, 3 \) only. In addition, we will take the dilaton \( \Phi \) to be a function only of these four-dimensional coordinates. The coordinates \( x^a \) parameterize a \((d - 4)\)-torus with a reference radius \( r_0 \). This radius will typically be chosen to be the string scale, \( \sqrt{\alpha'} \), or perhaps a few orders of magnitude larger. Similarly, the \( d \)-direction is compactified on a circle or an interval, either having a reference radius which, for simplicity, we also choose to be \( r_0 \). For length scales much larger than this radius, all heavy modes will decouple and we will arrive at a four-dimensional effective theory describing the light modes. These light modes will necessarily include \( g'_{\mu \nu}, \sigma'_T, \sigma'_R \) and \( \Phi \). However, one should also make sure that there are no additional “stringy” light modes, such as winding modes around one of the internal circles. For the moment, let us ignore this important subtlety, returning to it at the end of this section. The associated action is most easily computed from (83) by Weyl rescaling the four-dimensional metric as

\[
g'_{\mu \nu} = \Omega^2 g_{\mu \nu}, \tag{86}\]

where

\[
\Omega^2 = e^{2\Phi_4} \tag{87}\]

and

\[
\Phi_4 = \Phi - \frac{1}{2}(\sigma'_R + (d - 4)\sigma'_T). \tag{88}\]

Then, making the field redefinitions

\[
\phi = 2\Phi_4, \quad \sigma_T = \sqrt{2(d - 4)}\sigma'_T, \quad \sigma_R = \sqrt{2}\sigma'_R, \tag{89}\]

defining

\[
\frac{1}{\kappa_4^2} = \frac{V_T V_R}{\kappa_{d+1}^2} \tag{90}\]

where \( V_T \) and \( V_R \) are the volumes of the \((d - 4)\)-torus and the \( d \)-direction circle/interval respectively and dropping all higher derivative terms, we find that the low energy action is given by

\[
S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \mathcal{L}, \tag{91}\]

with

\[
\mathcal{L} = R - \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu \nu} \partial_\mu \sigma_T \partial_\nu \sigma_T - \frac{1}{2} g^{\mu \nu} \partial_\mu \sigma_R \partial_\nu \sigma_R - 2e^\phi \Lambda. \tag{92}\]
Note that this is exactly the four-dimensional theory introduced in (1) and (2) if we set\(^2\)

\[ Q = \sqrt{\frac{\Lambda}{2}}. \]  

(93)

Recall, however, that this result is predicated on the assumption that there are no additional stringy light modes. We must now check to see under what conditions this will be the case.

To analyze this, we must be more careful in our discussion of decoupling. It is clear from (85) that, in the string frame (83), the physical radii of the \((d-4)\)-torus and the \(d\)-direction circle are

\[ e^{\sigma_T r_0}, \quad e^{\sigma_R r_0} \]  

(94)

respectively. From (3) and (89) we see that, for the background geometries of interest in this paper, \(\sigma_T\) vanishes and, hence, the physical \((d-4)\)-torus radius is time-independent and fixed at \(r_0\). On the other hand, it follows from (4) and (89) that the physical radius of the \(d\)-direction circle is time-dependent and given by

\[ r_R(t) = |\tanh(Qt)|r_0. \]  

(95)

For very early and late times, \(r_R(t)\) approaches \(r_0\). Note, however, that

\[ r_R(t) \rightarrow 0, \quad t \rightarrow \pm 0. \]  

(96)

Therefore, we must be concerned that stringy modes winding around the \(d\)-direction circle will become very light as \(t \rightarrow \pm 0\), thus substantially changing the low energy theory given in (91) and (92). Can we estimate the time regime for which this effective theory is no longer valid? To do this, note that the winding modes around the \(d\)-direction circle typically have a mass

\[ m_{\text{winding}} \approx \frac{r_R(t)}{\alpha'}. \]  

(97)

It then follows from (95), and the fact that \(r_0\) is of the order of the string scale or a few orders of magnitude larger, that the effective field theory (91)

\(^2\)In the simple supercritical string models we are considering, the cosmological constant \(\Lambda\) is of order the string scale, which is at odds with our assumption (39) that \(Q\) is small compared to the momenta of interest. However, in the critical “cigar geometry” mentioned in the Introduction, it is possible to choose \(Q\) to be arbitrarily small. For this reason, we will simply ignore this issue in our analysis.
and (92) will be valid for momenta $\vec{k}$ satisfying

$$\vec{k}^2 \ll \frac{r_0^2 \tanh^2(Qt)}{(\alpha')^2}. \tag{98}$$

Since we are interested in momenta satisfying\(^3\)

$$Q^2 < \vec{k}^2, \tag{99}$$

we have to make sure that there is a window of momenta satisfying both (98) and (99). Since we imagine $Q$ to be much below the string scale, $\alpha'Q^2 \ll 1$, and for $r_0^2 \approx \alpha'$, such a window exists as long as

$$t^2 \gg \alpha', \tag{100}$$

that is, as long as we stay more than a few string times away from the singularity. Note that this does not alter the discussion and conclusions associated with the theory given in (1) and (2). To begin with, that theory is singular at the origin, so we could not really discuss the $t \to \pm 0$ regime, nor did we. Indeed, the stringy effects that descend from (83) in the time region (100) could conceivably “regulate” this singularity, making the theory well-defined at the origin. We will not explore this possibility in this paper. Henceforth, we will simply recall that the low energy theory (1), (2) or, equivalently, (91), (92) is not valid too close to the singularity.

Subject to this caveat, we conclude that the four-dimensional cosmology described in this section descends from a class of string theories in $d + 1$ dimensions. We will now examine these string theories and their cosmology in detail, with the aim of eventually using them to compute the Bogolubov coefficients discussed above.

### 3 String Theory Cosmology

As we have just shown, the four-dimensional action (1), (2) is the low energy limit of the supercritical string theory (83) dimensionally reduced to

---

\(^3\)Massless modes with momenta below this scale correspond to growing modes and were discussed in section 5 of [15]. They lead to infrared divergences in string perturbation theory which will be ignored in this paper but would be interesting to understand better. We are not aware of any direct relation to the divergences in string perturbation theory mentioned at the beginning of section 8.
four dimensions on a \((d - 4)\)-torus times a circle. The \((d + 1)\)-dimensional action \(S_3\) is itself an effective theory, all higher \(\alpha'\), string loop and non-perturbative effects being ignored. These effects are important, and we will comment on them in the appropriate places later in this paper. However, we will begin by studying the classical string action \(\mathcal{S}_3\) and, specifically, the \((d + 1)\)-dimensional cosmological solutions of its equations of motion. One such solution, the “periodically identified generalized Milne solution”, or the “generalized Milne orbifold” for short, was first presented in \(\text{(15)}\). This turns out to be an exact solution of classical string theory, that is, it does not receive \(\alpha'\) corrections \(\text{(32)}\). We will see that, upon dimensional reduction to four dimensions, this solution includes the background spacetime studied in subsection \(\text{2.1}\). This makes the generalized Milne orbifold an excellent context to try and answer the questions raised in the previous section.

3.1 The Background Spacetime

Our starting point is the following solution to the equations of motion associated with the classical string action \(\mathcal{S}_3\). The metric and dilaton are found to be

\[
\begin{align*}
\text{ds}_{d+1}^2 &= \frac{1}{Q^2} \frac{dudv}{1 - uv} + \sum_{l=1}^{d-1} (dx^l)^2, \\
\Phi &= -\frac{1}{4} \log(1 - uv)^2 + \Phi_0.
\end{align*}
\]

(101)

Here

\(-\infty < u, v < \infty\)

(102)

are global two-dimensional coordinates with the identification

\((u, v) \sim (ue^{-2\pi Q_\infty}, ve^{2\pi Q_\infty})\).

(103)

The parameter \(Q\) is related to the cosmological constant \(\Lambda\) by expression \(\text{(93)}\). The first term in the metric describes the generalized Milne orbifold \(\text{(15)}\). The remaining terms describe flat/torus-compactified \((d - 1)\)-dimensional space. The two-dimensional generalized Milne directions exhibit an intricate causal structure that is schematically represented in Fig. \(\text{1}\).

In Regions I and II of Fig. \(\text{1}\)

\[uv < 0\]

(104)
and the identification (103) is spacelike. These regions can also be described by the coordinates $t, x$ defined by

$$u = \sinh(Qt)e^{-Qx}, \quad v = -\sinh(Qt)e^{Qx}$$

(105)

where

$$x \sim x + 2\pi r_0.$$  

(106)

In these coordinates, the metric and dilaton solutions become

$$ds_{d+1}^2 = -dt^2 + \tanh^2(Qt)dx^2 + \sum_{I=1}^{d-1}(dx^I)^2,$$

$$\Phi = -\log \cosh(Qt) + \Phi_0.$$  

(107)

For regions III and IV in Fig. 1

$$0 < uv < 1.$$  

(108)

The identification (103) is now timelike. Thus, these regions contain closed timelike curves. In regions V and VI, one has

$$1 < uv$$

(109)
and the identification again becomes spacelike.

Region I describes a circle that starts out at some fixed radius \( r_0 \), collapses to zero size and then, in Region II, expands again to radius \( r_0 \). The Big Crunch/Big Bang singularity occurs when

\[
uv = 0. \tag{110}
\]

At this singularity, regions I and II touch the “whisker” regions IV and III which, in turn, are connected to a second pair of asymptotic early and late time regions, VI and V respectively. In addition, the separation boundary between regions VI and IV and between regions V and III, defined by

\[
uv = 1, \tag{111}
\]

is singular. Fig. 1 is intended to give a rough idea of the structure of the asymptotic regions, but is less precise about the structure near the singularities. For example, the metric in Region VI is actually such that the radius of the circle is increasing as one moves in towards the singularity. The string coupling \( g_s = \exp(\Phi) \) grows as well. In drawing Fig. 1 we have implicitly performed a T-duality locally on that region, to bring it to a form more like that of Region I. Close to the singularity at \( uv = 0 \), where regions I, II, III and IV meet, the \( u, v \) piece of (101) reduces to the Milne orbifold, a two-dimensional Minkowski space with the points related by (103) identified. See, for example, [24, 30, 21, 29] for recent discussions. Deep into regions I and II, that is, as \( uv \to -\infty \), the dilaton becomes linear in time with the string coupling approaching zero. At \( uv = 1 \), the separation between regions VI and IV and between regions V and III, both the curvature and the string coupling diverge.

A priori, one would expect stringy, higher derivative corrections to (83) to become important near the singularities at \( uv = 0 \) and \( uv = 1 \). However, it was shown in [15] that the background (101) defines an exact superconformal field theory with the correct central charge, \( \hat{c} = 10^4 \). That is, background (101) is an exact solution of the string equations of motion to all orders in \( \alpha' \).

The coordinates \( x^1, x^2, x^3 \) range from \( -\infty \) to \( +\infty \). The coordinates \( x^a, i = 4, \ldots, d-1 \) are taken to be periodic, \( x^a \sim x^a + 2\pi r_0 \), so they describe a \((d-4)\)-torus with constant radii \( r_0 \). These radii are chosen to be

\[4\text{The central charge deficit of the } (u, v) \text{ directions is compensated by considering supercritical string theory with a positive cosmological constant.}\]
of the order of the string length or a few orders of magnitude larger. As mentioned previously, the coordinate $x$ defined in (105) is also compact although, in string frame, the corresponding circle has a time-dependent radius $|\tanh(Qt)|r_0$. The coordinate $t$ will be the time coordinate of an observer living in regions I and II.

We close this subsection by showing that in regions I and II of the $(d+1)$-dimensional cosmological solution given in (101), there exists an effective four-dimensional description of this background. This effective solution is valid everywhere except very close to the Big Crunch/Big Bang singularity, where it is expected to break down due to additional light stringy modes. It turns out that the dangerous modes are winding modes around the $x$ direction, which, as we have previously discussed, indeed become light near $t = 0$. With this caveat in mind, we proceed to dimensionally reduce the metric and dilaton solutions in regions I and II from $(d + 1)$-dimensions to four dimensions by compactifying them on the $(d - 4)$-torus times a circle. Recall that in regions I and II the metric and dilaton can be written as in (107). Using the definitions in subsection 2.4, we find that, at low energy, (107) corresponds to

$$
\begin{align*}
\sigma_T &= 0, \\
\sigma_R &= \sqrt{2 \log|\tanh(Qt)|}, \\
\phi &= 2\Phi_0 + \log 2 - \log|\sinh(2Qt)|, \\
g_{\mu\nu} &= e^{-\phi}\eta_{\mu\nu},
\end{align*}
$$

(112)

which is exactly the four-dimensional background given in (8), (4), (5) and (6) of subsection 2.1 if we identify

$$
\phi_0 = \Phi_0 + \frac{1}{2}\log 2.
$$

It is crucial to note that (107) only describes regions I and II of the $(d + 1)$-dimensional spacetime (101). The full $(d + 1)$-dimensional string background contains two additional asymptotic regions V and VI, as well as the intermediate regions III and IV. The four-dimensional spacetime (112) is a low energy description of regions I and II only. However, as we have pointed out before, this four-dimensional effective description breaks down near the Big Crunch/Big Bang singularity, which is exactly where regions I and II are connected to the other regions. To understand what happens near the Big Crunch/Big Bang singularity, as well as in regions III, IV, V and VI, it is
clearly necessary to use the full \((d+1)\)-dimensional string theory background. This is precisely what we will do throughout the remainder of this paper.

### 3.2 Fluctuations

In this subsection, we study fluctuations around the background \((101)\). The key ingredient in our discussion is that string theory allows one to determine globally defined wavefunctions, despite the singularities that prevent doing so in general relativity \([15, 20]\). The underlying reason is that the spacetime \((101)\) corresponds to an orbifold of a coset conformal field theory.\(^5\) There is a well-defined procedure to determine at least the “untwisted” globally defined vertex operators in such theories. The zero mode parts of these vertex operators are the globally defined wavefunctions we are interested in.

This procedure consists of two steps. In the first step, one determines the vertex operators of the coset conformal field theory, which in our case is \(PSL(2, \mathbb{R})/U(1)\) at negative level.\(^6\) One describes the coset conformal field theory as a gauged Wess-Zumino-Witten (WZW) model. The vertex operators of the ungauged \(PSL(2, \mathbb{R})\) WZW model correspond to wavefunctions on the smooth \(PSL(2, \mathbb{R})\) group manifold. These can be found in \([36]\). The vertex operators of the \(SL(2, \mathbb{R})/U(1)\) WZW model can then be viewed as those vertex operators of the ungauged model that are invariant under the \(U(1)\) gauge group. See for example \([15, 20, 35]\).

The second step is the standard string theory orbifold procedure \([37]\). Roughly speaking, an orbifold is obtained from a covering space, in our case the \(PSL(2, \mathbb{R})/U(1)\) coset spacetime at negative level, by identifying points related by the action of a discrete group. Here, the group is \(\mathbb{Z}\) with the action \((103)\), which turns a line into a circle (the circles visible in Fig. 1) and causes the Big Crunch/Big Bang singularity at \(uv = 0\). The untwisted vertex operators\(^7\) are those vertex operators on the covering space that are

---

\(^5\)For early work on applications of coset conformal field theories to cosmology, see for example \([33, 34]\).

\(^6\)\(PSL(2, \mathbb{R})/U(1)\) at positive level corresponds to a two-dimensional black hole geometry \([16, 35]\). \(PSL(2, \mathbb{R})/U(1)\) at negative level can be obtained from the black hole geometry by double Wick rotation \([34]\). It is described by the \(u, v\) piece of \((101)\), without the identification \((103)\). In this spacetime, \(u = v = 0\) is a smooth point (the Big Crunch/Big Bang singularity only arises after the discrete identification \((103)\)), while there are singularities at \(uv = 1\).

\(^7\)There are also twisted vertex operators, corresponding to strings winding around the
invariant under (103). From (106), we see that this amounts to momentum quantization in the $x$ direction. In this paper, we will only be interested in zero momentum in the $x$ direction since the $x$ circle is part of the internal space. For the same reason, we will not be interested in winding modes except to note that they cause the effective four-dimensional description to break down near the Big Crunch/Big Bang.

We would like to comment on the relationship of our theory to another model, which has the same Big Crunch/Big Bang singularity. This is the Milne orbifold $(\mathbb{R}^{(1,1)}/\mathbb{Z}) \times \mathbb{R}^8$, where $\mathbb{Z}$ is generated by the boost transformation $\text{transformation (103)}$ on two-dimensional Minkowski space and $\mathbb{R}^8$ denotes eight additional flat directions, some of which may be compactified. The associated metric is

$$ds^2 = \frac{1}{Q^2} du dv + \sum_{i=1}^{8} (dx^i)^2.$$  

This spacetime is very similar to (101) near the Big Crunch/Big Bang singularity $u = v = 0$, but differs significantly from it away from this point, in particular in the structure of the additional regions. The Milne orbifold consists of four cones touching at $u = v = 0$, that is, regions I, II, III and IV, but the radius of the circle of the cones grows to infinite size as one goes infinitely far away from the singularity. There are no regions V and VI. They can be thought of as having been pushed to infinity by focussing in on the manifold around $u = v = 0$.

Wavefunctions on the Milne orbifold have been discussed from a string theory point of view in [30] and have been used to compute string scattering amplitudes in [21]. There is no coset CFT involved in this spacetime and the untwisted wavefunctions can be easily obtained from the string orbifold procedure. That is, consider wavefunctions on Minkowski space and demand that they be invariant under the action (103) of the orbifold group. Actually, such invariant Minkowski space wavefunctions either grow or decay in regions III and IV. Because these regions are non-compact in the Milne orbifold, one, often implicitly, further restricts to wavefunctions that decay in those regions. There will be no such additional restriction in the spacetime (101), since regions III and IV do not extend to infinity there. As a consequence, more wavefunctions are necessary in our theory than one finds in the Milne orbifold.

\[\text{circles, see [15].}\]
Global wavefunctions for the Milne orbifold with regions III and IV omitted were presented in [11] based on a construction that does not refer to string theory, and applied in [12] to cosmology. The global wavefunctions agree with the restriction to regions I and II of the stringy wavefunctions discussed in the previous paragraph. Also, the vacuum state implicitly defined by the wavefunctions of [30] and more explicitly used in [21] corresponds to the vacuum state defined in [11] upon deleting regions III and IV of the string theory spacetime (or from the point of view of [11], upon adding those regions). It is of interest to see what happens to the wavefunctions of the generalized Milne orbifold upon taking the limit to the Milne orbifold. We analyze this in detail in subsection 2.4 and Appendix C.

The fact that the wavefunctions that descend from a covering conformal field theory solution of string theory are globally defined on the associated orbifold is of fundamental importance for the results of this paper. For that reason, we outline in Appendix A, in more detail than discussed here, the procedure for constructing these global wavefunctions. In the remainder of this subsection, we will review and extend the results of [15] on global wavefunctions and quantum vacuum states in the spacetime (101). The global structure of the spacetime, in particular the presence of the additional asymptotic regions, will be shown to have important implications for the physics of the Big Crunch/Big Bang transition.

### 3.2.1 Global Wavefunctions on the Generalized Milne Orbifold

We restrict our discussion to those fluctuations that are relevant for the four-dimensional cosmology introduced in section 2. That is, the only non-zero momentum components are in the three non-compact space dimensions, and we ignore all winding and excited string modes. In particular, momentum and winding in the $x$ direction are set to zero,

$$p = w = 0$$

in the notation of [15]. The justification for this is that all the modes we ignore have masses on the order of the string scale, except very close to the Big Bang/Big Crunch singularity where winding modes around the $x$ direction become light, as we have previously mentioned.

In subsection 2.2, we solved the wave equation (36) for the fluctuations $\delta T$ defined in (35). Recall that from the point of view of the four-dimensional
quantum field theory (11) and (2), there were two classically independent regions, Region I and Region II. In each of these regions, we found the same two independent solutions of (36) for a given momentum $\vec{k}$. These solutions are $\delta T^+_k$ in (18) and $\delta T^-_k (= \delta T^+_k)$. Far from $t = 0$, one solution, $\delta T^+_k$, reduces to a positive frequency wave, while its conjugate has negative frequency. However, as we have shown in subsection 3.1 this four-dimensional theory arises as the low energy limit of the generalized Milne orbifold solution of the $(d + 1)$-dimensional classical string action (83). The full string theory background (101) has, in addition to regions I and II, the regions III, IV, V and VI discussed in detail above. Therefore, in string theory, one must solve the $\delta T$ fluctuation equation in each of these six regions. Recall that the generalized Milne orbifold arises from a covering $PSL(2, \mathbb{R})$ manifold by constructing the coset $PSL(2, \mathbb{R})/U(1)$ and then identifying points related by a $\mathbb{Z}$ group action. It follows from this structure that one can find solutions of the $\delta T$ fluctuation equation on the generalized Milne orbifold by first finding solutions of the fluctuation equation on $PSL(2, \mathbb{R})$ and then restricting to those solutions that are invariant under the action of both $U(1)$ and $\mathbb{Z}$. Such solutions, since they are globally defined on $PSL(2, \mathbb{R})$, remain globally defined on the generalized Milne orbifold. That is, each such solution is defined in each of the six regions I, II, III, IV, V and VI. As explained in [15, 20], there are four independent wavefunctions of this type for each momentum $\vec{k}$, which we denote by

$$K^+_{\pm, \vec{k}}(uv), K^-_{\pm, \vec{k}}(uv), K^0_{\pm, \vec{k}}(uv) \text{ and } K^0_{-\vec{k}}(uv).$$

Note that, in addition to their $\vec{k}$ dependence, the argument of these functions is the coordinate $uv$. More precisely, they depend not only on $uv$ but, also, on the region I, II, III, IV, V or VI. This latter dependence is suppressed in (116). In regions I and II, it follows from (105) that

$$uv = -\sinh^2(Qt).$$

Similarly, in regions V and VI we can write

$$1 - uv = -\sinh^2(Qt).$$

Hence, in these regions the wavefunctions are dependent on $t$, as we would expect them to be. The four independent wavefunctions in (116) can be written as

$$K_{\pm, \vec{k}} = \mathcal{N} K_{\pm, \vec{k}},$$

34
where $N$ is a $\vec{k}$-dependent normalization constant given in (234) and the expressions for each of the four functions $K_{\pm \pm, \vec{k}}$ in the six regions are given in Appendix B. That is, the independent wavefunctions in (116) are explicitly known. The $K_{\pm \pm, \vec{k}}$ are defined such that they are orthonormal, with norm squared $\pm 1$, with respect to the appropriate generalization of the Klein-Gordon inner product (43). The generalization is that $\Sigma$ should be a global “Cauchy” surface, intersecting regions I and VI or II and V, which turn out to be equivalent. The $K_{\pm \pm, \vec{k}}$ given in Appendix A have the same norm up to a sign, so, by dividing out a common factor, we can obtain the normalized $K_{\pm \pm, \vec{k}}$. For example, $K_{++, \vec{k}}$ coincides with $\delta T_{\vec{k}}^+$ in Region I and vanishes in Region VI, so it indeed has unit Klein-Gordon norm.

A key point of this paper is that the wavefunctions $K_{\pm \pm, \vec{k}}$ are globally defined on the generalized Milne orbifold. This follows from the fact that they descend from globally defined functions on the $PSL(2, \mathbb{R})$ covering space. The expressions for each of the four independent wavefunctions, in each of the six regions, are presented in Appendix B. It is, however, useful at this point to discuss several of these wavefunctions in more detail. First consider $K_{++, \vec{k}}$. In Region I, it is found to be

$$K_{++, \vec{k}} = \delta T_{\vec{k}}^+, \tag{120}$$

where $\delta T_{\vec{k}}^+$ is given in (48). This function is purely positive frequency in the asymptotic region $t \ll -1/Q$, while it diverges logarithmically near the Big Crunch singularity, $t \to -0$. In Region II, one has

$$K_{++, \vec{k}} \propto (-z)^{-j-1} F(-j, -j; -2j; \frac{1}{z}), \tag{121}$$

where $j$ and $z$ are defined in (49). In this region, $K_{++, \vec{k}}$ is purely positive frequency asymptotically and diverges logarithmically near the Big Bang as $t \to +0$. In the intermediate Region III,

$$K_{++, \vec{k}} \propto (uv)^{j} F(-j, -j; 1; 1 - \frac{1}{uv}), \tag{122}$$

which diverges logarithmically near the Big Crunch/Big Bang singularity at $uv = 0$, while it approaches a constant near the $uv = 1$ singularity that separates regions III and V. Similarly, in Region IV

$$K_{++, \vec{k}} \propto (uv)^{-j-1} F(j + 1, j + 1; 1; 1 - \frac{1}{uv}). \tag{123}$$
This diverges logarithmically near $uv = 0$ and approaches a constant near $uv = 1$. In Region V, $\mathcal{K}_{++,\vec{k}}$ is a mixture of positive and negative frequency components with equal amplitude in the asymptotic region $uv \gg 1/Q^2$, while it approaches a constant near $uv = 1$. The exact expression in Region V is

$$\mathcal{K}_{++,\vec{k}} \propto F(-j, j + 1; 1; 1 - uv). \quad (124)$$

Finally, and importantly, in Region VI

$$\mathcal{K}_{++,\vec{k}} = 0. \quad (125)$$

The global behavior of $\mathcal{K}_{+-,\vec{k}}$ can be obtained by replacing $I \leftrightarrow VI$, $II \leftrightarrow V$ and $uv \leftrightarrow 1 - uv$ in the above expressions for $\mathcal{K}_{++,\vec{k}}$. Of particular importance for us is the form of $\mathcal{K}_{+-,\vec{k}}$ in regions I and VI. We find that

$$\mathcal{K}_{+-,\vec{k}} = 0 \quad (126)$$

and

$$\mathcal{K}_{+-,\vec{k}} = \delta T_{\vec{k}}^{++}, \quad (127)$$

in regions I and VI respectively. Similarly, the global structure of $\mathcal{K}_{--,\vec{k}}$ is obtained from that of $\mathcal{K}_{+-,\vec{k}}$ by replacing $I \leftrightarrow II$, $V \leftrightarrow VI$ and “positive frequency” $\leftrightarrow$ “negative frequency”. Specifically, we will use the fact that

$$\mathcal{K}_{--,\vec{k}} = \delta T_{\vec{k}}^{++*} \quad (128)$$

and

$$\mathcal{K}_{--,\vec{k}} = 0 \quad (129)$$

in regions II and V respectively. Finally, the behavior of $\mathcal{K}_{++,\vec{k}}$ is obtained from that of $\mathcal{K}_{--,\vec{k}}$ by replacing $I \leftrightarrow VI$, $II \leftrightarrow V$ and $uv \leftrightarrow 1 - uv$. In particular, we find that

$$\mathcal{K}_{--,\vec{k}} = 0 \quad (130)$$

in Region II, whereas

$$\mathcal{K}_{--,\vec{k}} = \delta T_{\vec{k}}^{++*} \quad (131)$$

in Region V.
3.2.2 Quantization

We would now like to quantize the scalar fluctuations $\delta \sigma_T$ on the generalized Milne orbifold. Generically, this can be done by expanding

$$
\delta \sigma_T = \int \frac{d^3 \vec{k}}{(2\pi)^{3/2}} \left( a_{1, \vec{k}} K_{1, \vec{k}}(uv) e^{i\vec{k} \cdot \vec{x}} + a_{2, \vec{k}} K_{2, \vec{k}}(uv) e^{i\vec{k} \cdot \vec{x}} + a_{1, \vec{k}}^\dagger K^*_{1, \vec{k}}(uv) e^{-i\vec{k} \cdot \vec{x}} + a_{2, \vec{k}}^\dagger K^*_{2, \vec{k}}(uv) e^{-i\vec{k} \cdot \vec{x}} \right),
$$

(132)

where $\{ K_{1, \vec{k}}, K_{2, \vec{k}} \}$ is some appropriate pair of orthonormal wavefunctions. As compared to (52), we have twice as many functions in the expansion (132). The reason is that there are twice as many independent global wavefunctions on the generalized Milne orbifold as there are independent wavefunctions in the effective four-dimensional theory. We now impose the canonical commutation relations

$$
[a_i, \vec{k}, a_j, \vec{k}'] = \delta \delta^3(\vec{k} - \vec{k}'), \quad [a_i, \vec{k}, a_j, \vec{k}'] = [a_i, \vec{k}, a_j, \vec{k}'] = 0.
$$

(133)

and define the vacuum state $|0\rangle$ by

$$a_1|0\rangle = a_2|0\rangle = 0.$$

(134)

To proceed, we now must ask the question: what is a natural vacuum state to choose, or equivalently, what is a natural set of wavefunctions $\{ K_{1, \vec{k}}, K_{2, \vec{k}} \}$? In [15], two natural vacua were defined and shown to be inequivalent. The first vacuum state corresponds to the choice $K_{1, \vec{k}} = K^*_{++}, K_{2, \vec{k}} = K^*_{+-}$. The associated fluctuation expansion is

$$
\delta \sigma_T = \int \frac{d^3 \vec{k}}{(2\pi)^{3/2}} \left( a^L_{\vec{k}} K^*_{++, \vec{k}}(uv) e^{i\vec{k} \cdot \vec{x}} + a^{VI}_{\vec{k}} K^*_{+-, \vec{k}}(uv) e^{i\vec{k} \cdot \vec{x}} + a^L_{\vec{k}}^\dagger K^*_+ K_{++, \vec{k}}(uv) e^{-i\vec{k} \cdot \vec{x}} + a^{VI}_{\vec{k}}^\dagger K^*_+ K_{+-, \vec{k}}(uv) e^{-i\vec{k} \cdot \vec{x}} \right). \quad (135)
$$

Recall that at early times, $K_{++, \vec{k}}$ is purely positive frequency in Region I and, from (125), vanishes in Region VI, and vice versa for the wavefunction $K_{+-, \vec{k}}$. It follows that the vacuum state constructed from $a^L_{\vec{k}}$ and $a^{VI}_{\vec{k}}$ would indeed be called empty by early time observers in regions I and VI. Therefore, we denote this state by $|0\rangle_{in}$. The second vacuum state, specified by $|0\rangle_{out}$, corresponds to the choice $K_{1, \vec{k}} = K^*_{-+, \vec{k}}$ and $K_{2, \vec{k}} = K^*_{++, \vec{k}}$ and the associated expansion

$$
\delta \sigma_T = \int \frac{d^3 \vec{k}}{(2\pi)^{3/2}} \left( a^L_{\vec{k}} K^*_{-+, \vec{k}}(uv) e^{i\vec{k} \cdot \vec{x}} + a^{VI}_{\vec{k}} K^*_+ K_{-+, \vec{k}}(uv) e^{i\vec{k} \cdot \vec{x}} + a^L_{\vec{k}}^\dagger K^*_{-+} K_{-+, \vec{k}}(uv) e^{-i\vec{k} \cdot \vec{x}} + a^{VI}_{\vec{k}}^\dagger K^*_{-+} K_{-+, \vec{k}}(uv) e^{-i\vec{k} \cdot \vec{x}} \right).
$$
\[ +a_{k\bar{k}}^{H\dagger}\mathcal{K}_{-,k}(uv)e^{-i\bar{k}\cdot\bar{x}} + a_{k\bar{k}}^{V\dagger}\mathcal{K}_{+,k}(uv)e^{-i\bar{k}\cdot\bar{x}} \].  
\(136\)

Since, at late times, \(\mathcal{K}_{-,\bar{k}}^*\) is purely positive frequency in Region II and, from (129), vanishes in Region V, and vice versa for \(\mathcal{K}_{+,\bar{k}}^*\), this state would similarly be called empty by late time observers in regions II and V.

The relation between \(|0\rangle_{in}\) and \(|0\rangle_{out}\) can be determined from
\[
\begin{pmatrix}
\mathcal{K}_{-,\bar{k}} \\
\mathcal{K}_{+,\bar{k}} \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{-,\bar{k}}^*
\end{pmatrix} = \begin{pmatrix}
A & C & 0 & B \\
C & A & B & 0 \\
0 & B^* & A^* & C^* \\
B^* & 0 & C^* & A^*
\end{pmatrix} \begin{pmatrix}
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{-,\bar{k}}^*
\end{pmatrix},
\]
\(137\)

where the Klein-Gordon orthonormality of the wavefunctions implies that \(A, B\) and \(C\) satisfy the conditions
\[|A|^2 + |C|^2 - |B|^2 = 1, \quad AC^* + A^*C = 0.\]  \(138\)

For the special case (115) we are considering, one finds
\[A = -1, \quad B = C = \frac{1}{i\sinh\left(\frac{\pi E \bar{k}}{2\hbar}\right)}\]  \(139\)

Note that \(B\) is a function of \(\bar{k}\) and that \(B(\bar{k}) = B(-\bar{k})\). Then, (137) simplifies to
\[
\begin{pmatrix}
\mathcal{K}_{-,\bar{k}} \\
\mathcal{K}_{+,\bar{k}} \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{-,\bar{k}}^*
\end{pmatrix} = \begin{pmatrix}
-1 & B & 0 & B \\
B & -1 & B & 0 \\
0 & -B & -1 & -B \\
-B & 0 & -B & -1
\end{pmatrix} \begin{pmatrix}
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{-,\bar{k}}^*
\end{pmatrix},
\]
\(140\)

which can be inverted to
\[
\begin{pmatrix}
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{-,\bar{k}}^*
\end{pmatrix} = \begin{pmatrix}
-1 & B & 0 & B \\
B & -1 & B & 0 \\
0 & B & -1 & B \\
B & 0 & B & -1
\end{pmatrix} \begin{pmatrix}
\mathcal{K}_{-,\bar{k}} \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{+,\bar{k}}^* \\
\mathcal{K}_{-,\bar{k}}^*
\end{pmatrix}.
\]
\(141\)

It is straightforward to check these relations using the formulas in Appendix B. Inserting (141) into (136), we find the Bogolubov transformation
\[
\begin{pmatrix}
a_{k\bar{k}}^{H\dagger} \\
a_{k\bar{k}}^{V\dagger} \\
a_{k\bar{k}}^{L\dagger} \\
a_{k\bar{k}}^{R\dagger}
\end{pmatrix} = \begin{pmatrix}
-1 & B & 0 & B \\
-B & -1 & B & 0 \\
0 & -B & -1 & B \\
-B & 0 & B & -1
\end{pmatrix} \begin{pmatrix}
a_{k\bar{k}}^{H} \\
a_{k\bar{k}}^{V} \\
a_{k\bar{k}}^{L} \\
a_{k\bar{k}}^{R}
\end{pmatrix}.
\]
\(142\)

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Note that the Bogolubov transformation \ref{eq:bogolubov} mixes creation with annihilation operators. This implies particle creation at late times \cite{15}. To see this, note that for each momentum $\vec{k}$

\begin{equation}
    \langle 0 | a_{\vec{k}}^{I\dagger} a_{\vec{k}}^{I} | 0 \rangle_{in} = |B|^2 = \frac{1}{\sinh^2 \left( \frac{\pi E_{\vec{k}}}{2Q} \right)}.
\end{equation}

and

\begin{equation}
    \langle 0 | a_{\vec{k}}^{V\dagger} a_{\vec{k}}^{V} | 0 \rangle_{in} = |B|^2 = \frac{1}{\sinh^2 \left( \frac{\pi E_{\vec{k}}}{2Q} \right)}.
\end{equation}

The physical interpretation is that, at late times, the vacuum $|0\rangle_{in}$ has $1/\sinh^2(\pi E_{\vec{k}}/2Q)$ particles with momentum $\vec{k}$ in Region I and the same number of particles with momentum $\vec{k}$ in Region V for each value of $\vec{k}$. Note that the number of particles created goes to zero\textsuperscript{8} in the limit that $E_{\vec{k}}/Q \rightarrow \infty$.

\begin{equation}
    E_{\vec{k}}/Q \rightarrow \infty.
\end{equation}

We will use this result in subsection \ref{subsection:3.4}

Let us now, within the context of the generalized Milne orbifold, compute the analogue of expression \ref{eq:two_point_function}, that is, the two-point correlation function

\begin{equation}
    \langle 0 | \delta \sigma_T(uv, \vec{x}) \delta \sigma_T(uv, \vec{x} + \vec{r}) | 0 \rangle_{in}.
\end{equation}

Since this function is correlated with respect to the in-vacuum, it is simplest to expand the fluctuations as in \ref{eq:fluctuations_expansion}. Using this and \ref{eq:fluctuations_expansion}, we find that

\begin{equation}
    \langle 0 | \delta \sigma_T(uv, \vec{x}) \delta \sigma_T(uv, \vec{x} + \vec{r}) | 0 \rangle_{in} = \frac{1}{2\pi^2} \int d\vec{k} |\vec{k}|^2 \left( |\mathcal{K}_{+,\vec{k}}|^2 + |\mathcal{K}_{-\vec{k}}|^2 \right) \frac{\sin(|\vec{k}| |\vec{r}|)}{|\vec{k}| |\vec{r}|}.
\end{equation}

\textsuperscript{8}At least for the modes \ref{eq:modes} we are considering. The generalization of \ref{eq:generalized modes} is \cite{15}

\begin{equation}
    |B|^2 = \frac{\cosh^2(\pi m)}{\sinh^2 \left( \frac{\pi E_{\vec{k}}}{2Q} \right)}
\end{equation}

with

\begin{equation}
    m \equiv \frac{1}{2} \left( \frac{n}{Qr_0} - \frac{wr_0}{Q\alpha'} \right),
\end{equation}

$n$ and $w$ being integers labelling momentum and winding in the $x$ direction and $E$ being the energy.
As in section 1, we will always set $\vec{r} = 0$ and consider
\[ \langle 0 | \delta \sigma_T(uv, \vec{x})^2 | 0 \rangle_{\text{in}} = \int \frac{d^3k}{(2\pi)^3} \left( |\mathcal{K}_{++,\vec{k}}|^2 + |\mathcal{K}_{+-,\vec{k}}|^2 \right). \tag{150} \]

Since our wavefunctions are globally defined, (150) is valid for $(u, v)$ in any of the six regions of the orbifold. Of course, the explicit form of $\mathcal{K}_{++,\vec{k}}$ and $\mathcal{K}_{+-,\vec{k}}$ change from region to region and, hence, so will the expression for the correlation function. Let us begin by calculating (150) in Region I. It follows from (120) and (126) that
\[ \langle 0 | \delta \sigma_T(t, \vec{x})^2 | 0 \rangle_{\text{in}} = \int \frac{d^3k}{(2\pi)^3} |\delta T^+_{\vec{k}}|^2. \tag{151} \]
The agreement with (56) illustrates that the in-vacuum defined by (135) is indeed the correct vacuum in Region I. As previously, for momenta $\vec{k}^2 \gg Q^2$ in the distant past, the two-point function in Region I becomes
\[ \langle 0 | \delta \sigma_T(t, \vec{x})^2 | 0 \rangle_{\text{in}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\vec{k}|} \left( \frac{1}{2} + \frac{Q^2/2}{2|\vec{k}|^2} \right), \tag{152} \]
where we have ignored a momentum-independent factor that can be removed by a field redefinition. The same expression for the correlation function and conclusions hold in Region VI.

Having established this, we now calculate the object of real interest, namely, the two-point function (150) in Region II. Using (141) and the expressions for the wavefunctions given in the previous subsection, we find that
\[ \langle 0 | \delta \sigma_T(uv, \vec{x})^2 | 0 \rangle_{\text{in}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\vec{k}|} \left( \frac{1}{2} + \frac{Q^2/2}{2|\vec{k}|^2} \right) \Delta(\vec{k}, t), \tag{153} \]
As $t \to +\infty$, this expression becomes
\[ \langle 0 | \delta \sigma_T(uv, \vec{x})^2 | 0 \rangle_{\text{in}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\vec{k}|} \left( \frac{1}{2} + \frac{Q^2/2}{2|\vec{k}|^2} \right) \Delta(\vec{k}, t), \tag{154} \]
where we have ignored a momentum-independent factor and
\[ \Delta(\vec{k}, t) = 1 + 2|B|^2 + |B|^2 e^{-2iE_{\vec{k}}t} + |B|^2 e^{2iE_{\vec{k}}t}. \tag{155} \]
This should be compared with the field theory result given in (79). Clearly, it is impossible to find coefficients $X$ and $Y$ such that (79) reproduces (155). This discrepancy has an interesting origin and interpretation, which will be the subject of subsection 3.3.
3.2.3 A Family of Vacuum States

So far, we have considered the two vacuum states of \([15]\). One of them is empty in each of the early time regions I and VI, while the other is empty in the late time regions II and V. Using these vacua, we found that there is non-trivial particle creation in Region II, described by \((143)\). These particles influence the two-point function of the corresponding field, leading to the non-trivial factor of \(\Delta(\vec{k}, t)\) in \((155)\). Upon reflection, however, one is led to wonder whether the choice of vacua of \([15]\) is really unique and, in particular, whether there exists alternative vacuum states for which the early time two-point function \((151)\) is less drastically altered upon going through the Big Crunch/Big Bang transition.

Since we are interested in an observer who starts out in Region I and ends up in Region II after the Big Crunch/Big Bang transition, we will continue to impose the condition that any alternative in-vacuum state should be empty in Region I. That is, the two-point function in Region I should equal \((151)\). However, we will no longer impose a similar condition in the other in-region, Region VI, since our observer does not live there. We now construct a family of such generalized in-vacua and compute, in each case, the analogues of \((143)\) and \((154)\). These correspond to quantities our observer could, in principle, measure in Region II.

We continue to use the expansion \((132)\) with

\[
\mathcal{K}_{1,\vec{k}} = \mathcal{K}_{++,\vec{k}},
\]

which has positive frequency in Region I and vanishes in Region VI. However, we now allow a more general wavefunction for \(\mathcal{K}_{2,\vec{k}}\), which vanishes in Region I but does not necessarily have positive frequency in Region VI. The most general such wavefunction, for a given momentum, is an arbitrary normalized linear combination of \(\mathcal{K}_{+-,\vec{k}}\) and \(\mathcal{K}^*_{+-,\vec{k}}\) specified by

\[
\mathcal{K}_{\gamma\tilde{\gamma},\vec{k}} = \gamma(\vec{k})\mathcal{K}_{+-,\vec{k}} + \tilde{\gamma}(\vec{k})(\mathcal{K}_{+-,\vec{k}} + \mathcal{K}^*_{+-,\vec{k}}),
\]

where \(\gamma(\vec{k})\) and \(\tilde{\gamma}(\vec{k})\) are complex numbers satisfying

\[
|\gamma(\vec{k}) + \tilde{\gamma}(\vec{k})|^2 - |\gamma(\vec{k})|^2 = 1.
\]

For each

\[
\gamma(\vec{k}) \neq 0,
\]
there exists at least one $\tilde{\gamma}$ such that (158) is satisfied. Note that $\gamma(\tilde{k}) = 1$, $\tilde{\gamma}(\tilde{k}) = 0$ corresponds to the natural in-vacuum $|0\rangle_{in}$ defined using (135). For simplicity, we will assume that

$$
\gamma(\tilde{k}) = \gamma(-\tilde{k}), \quad \tilde{\gamma}(\tilde{k}) = \tilde{\gamma}(-\tilde{k}),
$$

(160)

although more general choices would not change our conclusions. Henceforth, we will take

$$
K_{2,\tilde{k}} = K_{\gamma,\tilde{k}},
$$

(161)

and use the expansion

$$
\delta \sigma_T = \int \frac{d^3 k}{(2\pi)^{3/2}} \left( a_k^I K_{++,\tilde{k}}(uv) e^{i\tilde{k} \cdot \tilde{x}} + a_k^V I K_{\gamma,\tilde{k}}(uv) e^{i\tilde{k} \cdot \tilde{x}} + a_k^I K_{\gamma,\tilde{k}}^*(uv) e^{-i\tilde{k} \cdot \tilde{x}} + a_k^V I K_{\gamma,\tilde{k}}^*(uv) e^{-i\tilde{k} \cdot \tilde{x}} \right). \tag{162}
$$

We can now repeat the computations of subsection 32.2 for the new in-vacuum states $|0\rangle_{\gamma,\tilde{k}}$ defined by (132), (134), (156), (157) and (158). We continue to use the natural out-vacuum defined using (150). It follows from the expressions for the wavefunctions given in Appendix B and (157) that

$$
\begin{pmatrix}
K_{++,\tilde{k}}^* \\
K_{\gamma,\tilde{k}}^* \\
K_{++,\tilde{k}} \\
K_{\gamma,\tilde{k}}
\end{pmatrix} = \begin{pmatrix}
-1 & -B & 0 & -B \\
-\gamma^* B & -(\gamma^* + \tilde{\gamma}) & -\gamma^* B & -\tilde{\gamma} \\
0 & B & -1 & B \\
\gamma B & -\tilde{\gamma} & \gamma B & -(\gamma + \tilde{\gamma})
\end{pmatrix} \begin{pmatrix}
K_{--,\tilde{k}} \\
K_{-+,\tilde{k}}^* \\
K_{*-+,\tilde{k}} \\
K_{*-+\tilde{k}}
\end{pmatrix}, \tag{163}
$$

where $B$ is given in (139). Inserting this into (162) and comparing to (156), we find the Bogolubov transformation

$$
\begin{pmatrix}
\ a_k^I \\
\ a_k^V I \\
\ a_{\tilde{k}}^I \\
\ a_{\tilde{k}}^V I
\end{pmatrix} = \begin{pmatrix}
-1 & -\gamma^* B & 0 & \gamma B \\
-\gamma^* B & -(\gamma^* + \tilde{\gamma}) & B & -\tilde{\gamma} \\
0 & -\gamma^* B & -1 & \gamma B \\
-\gamma B & -\tilde{\gamma} & B & -(\gamma + \tilde{\gamma})
\end{pmatrix} \begin{pmatrix}
\ a_k^I \\
\ a_k^V I \\
\ a_{\tilde{k}}^I \\
\ a_{\tilde{k}}^V I
\end{pmatrix}. \tag{164}
$$

This can be inverted to give

$$
\begin{pmatrix}
\ a_k^I \\
\ a_k^V I \\
\ a_{\tilde{k}}^I \\
\ a_{\tilde{k}}^V I
\end{pmatrix} = \begin{pmatrix}
-1 & B & 0 & -B \\
\gamma B & -(\gamma + \tilde{\gamma}) & -\gamma B & \tilde{\gamma} \\
0 & B & -1 & -B \\
\gamma^* B & \tilde{\gamma} & -\gamma^* B & -(\gamma^* + \tilde{\gamma})
\end{pmatrix} \begin{pmatrix}
\ a_k^I \\
\ a_k^V I \\
\ a_{\tilde{k}}^I \\
\ a_{\tilde{k}}^V I
\end{pmatrix}. \tag{165}
$$
Using (164), we can now compute the occupation numbers
\[ \langle \gamma \tilde{\gamma} | a^{II \dagger}_{\vec{k}} a^{II}_{\vec{k}} | 0 \rangle = |\gamma B|^2 \frac{|\gamma(\vec{k})|^2}{\sinh^2 \left( \frac{\pi E_{\vec{k}}}{2Q} \right)} \] (166)
in Region II and
\[ \langle \gamma \tilde{\gamma} | a^{V \dagger}_{\vec{k}} a^{V}_{\vec{k}} | 0 \rangle = |B|^2 + |\tilde{\gamma}|^2 \] (167)
in Region V. For the reasons given at the beginning of this subsection, we are principally interested in the number of particles in Region II. We see from (166) that this number can be made arbitrarily small by choosing \( \gamma(\vec{k}) \) to be close to zero. Note, however, that we cannot set \( \gamma(\vec{k}) = 0 \), since this would be inconsistent with the constraint (158).

Next we compute
\[ \langle \gamma \tilde{\gamma} | \delta \sigma_T(uv, \vec{x})^2 | 0 \rangle_{\gamma \tilde{\gamma}} = \int \frac{d^3k}{(2\pi)^3} \left( |\mathcal{K}_{++,\vec{k}}|^2 + |\mathcal{K}_{++,\vec{k}}|^2 \right) \] (168)
In Region I, \( \mathcal{K}_{++-\vec{k}} \) vanishes. It follows that \( \mathcal{K}_{++-\vec{k}} \) also vanishes and, hence, (168) reduces to the familiar result (151). From (223) in Appendix B, we see that \( \mathcal{K}_{++-\vec{k}} \) is purely imaginary in Region II. In fact, this is also the case in the other regions that touch \( uv = 0 \). Therefore, \( \mathcal{K}_{++-\vec{k}} \) reduces to \( \gamma \mathcal{K}_{++-\vec{k}} \) in those regions. Using this, (168) takes the following form in Region II,
\[ \langle \gamma \tilde{\gamma} | \delta \sigma_T(uv, \vec{x})^2 | 0 \rangle_{\gamma \tilde{\gamma}} = \int \frac{d^3k}{(2\pi)^3} \left( (1 + 2|\gamma B|^2) |\delta T^+_{\vec{k}}|^2 + |\gamma B(\vec{k})|^2 |\mathcal{K}_{\gamma,\tilde{\gamma},\vec{k}}|^2 \right) \] (169)
In the far future where \( t \to +\infty \), and for \( \vec{k}^2 \gg Q^2 \), this expression becomes
\[ \langle \gamma \tilde{\gamma} | \delta \sigma_T(uv, \vec{x})^2 | 0 \rangle_{\gamma \tilde{\gamma}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\vec{k}|} \left( \frac{1}{2} + \frac{Q^2/2}{2|\vec{k}|^2} \right) \Delta(\vec{k}, t), \] (170)
where we have dropped a momentum-independent scale factor and
\[ \Delta(\vec{k}, t) = 1 + 2|\gamma B|^2 + |\gamma B|^2 e^{-2iE_{\vec{k}}t} + |\gamma B|^2 e^{2iE_{\vec{k}}t}. \] (171)
In the limit \( \gamma(\vec{k}) \to 0 \), this approaches the early time result (152). Therefore, for this limiting choice of in-vacuum, an observer in Region II would
say that the fluctuations in Region I went through the Big Crunch/Big Bang unchanged. For this special limiting case, the result (171) can be reformulated in the framework of subsection 2.3. It corresponds to the Bogolubov coefficients $X$ and $Y$ with the properties

\[ |X| = 1, \quad Y = 0 \quad \text{in the limit } \gamma(\vec{k}) \to 0. \]  

However, this limiting case is the only one for which the framework of subsection 2.3 can be made to reproduce (171). This apparent discrepancy is the subject of the next subsection.

### 3.3 Information Loss

It is useful to use the Bogolubov transformation (165) to explicitly write the relation between the in-vacua $|0\rangle_{\gamma\tilde{\gamma}}$ and the natural out-vacuum $|0\rangle_{\text{out}}$. We find that (remember (160))

\[ |0\rangle_{\gamma\tilde{\gamma}} = N_{\gamma\tilde{\gamma}} \exp \left\{ \int d^3k \left( \theta a^{\dagger}_{\vec{k}} a^{\dagger}_{-\vec{k}} + \lambda a^{\dagger}_{\vec{k}} a^{\dagger}_{-\vec{k}} + \mu a^{\dagger}_{\vec{k}} a^{\dagger}_{-\vec{k}} \right) \right\} |0\rangle_{\text{out}}, \]  

(173)

where

\[ \theta = -\frac{\gamma B^2}{2(\gamma^*(1 - B^2) + \gamma^*)}, \]

\[ \lambda = \frac{\tilde{\gamma}^* - \gamma B^2}{2(\gamma^*(1 - B^2) + \gamma^*)}, \]

\[ \mu = \frac{\gamma B}{\gamma^*(1 - B^2) + \gamma^*}. \]  

(174)

To check this, use (165) to verify that $|0\rangle_{\gamma\tilde{\gamma}}$ defined by (173) is indeed annihilated by $a^{\dagger}_{\vec{k}}$ and $a^{\dagger}_{-\vec{k}}$ given that $|0\rangle_{\text{out}}$ is annihilated by $a^{\dagger}_{\vec{k}}$ and $a^{\dagger}_{-\vec{k}}$. $N_{\gamma\tilde{\gamma}}$ is a normalization constant, which we will determine for a special case. It is useful to keep in mind that $B$ given in (139) is purely imaginary.

For the natural in-vacuum $|0\rangle_{\text{in}}$ defined in (135), which corresponds to choosing $\gamma(\vec{k}) = 1, \tilde{\gamma}(\vec{k}) = 0$, the coefficients in (174) become

\[ \theta = -\frac{B^2}{2(1 - B^2)}, \]

\[ \lambda = -\frac{B^2}{2(1 - B^2)}, \]

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\[ \mu = \frac{B}{1 - B^2}. \]  

(175)

In this case, we find

\[ N_{10} = \prod_{\vec{k}} \left(1 - B^2(\vec{k})\right)^{-1/2} \equiv \exp \left\{ V \int \frac{d^3k}{(2\pi)^3} \log \left(1 - B^2(\vec{k})\right)^{-1/2} \right\}, \]

(176)

with \( V \) the volume of space. For strictly infinity \( V \), \( N_{10} \) vanishes, so we will assume \( V \) to be large but finite instead. Then (176) can be derived as follows. Define the unitary matrix

\[ S = \exp \left\{ \int d^3k \left(B(a^H_k a^V_{-\vec{k}} + a^H_{-\vec{k}} a^V_k - a^H_k a^V_{-\vec{k}} - a^H_{-\vec{k}} a^V_k)\right) \right\}. \]

(177)

Using the formula

\[ e^{-A}Be^A = B + [B, A] + \frac{1}{2!}[B, [B, A]] + \cdots, \]

(178)

it is straightforward to verify that

\[ a^I_k = -Sa^H_k S^{-1}, \quad a^V_k = -Sa^V_k S^{-1}. \]

(179)

This implies that, up to a phase

\[ |0\rangle_{in} = S|0\rangle_{out}. \]

(180)

The normalization factor (176) can then be obtained by determining the coefficient of \( |0\rangle_{out} \) upon expanding the right hand side of (180).\(^9\)

As another special case, consider the limiting values \( \gamma(\vec{k}) \to 0 \), \( \tilde{\gamma}(\vec{k}) \to \infty \). This was the limit that turned off particle creation in Region II, as we discussed at the end of subsection 3.2.3. This corresponds to

\[ \theta \to 0, \]

\[ \lambda \to \frac{1}{2}, \]

\[ \mu \to 0, \]

(181)

\(^9\)In doing this, it is convenient to work with rescaled oscillators \( \alpha_{\vec{k}} \equiv ((2\pi)^3/V)^{1/2} a_{\vec{k}}, \)

which satisfy the canonical commutation relation \([\alpha_{\vec{k}}, \alpha_{\vec{k}}^\dagger] = 1\).
However, it turns out that in the limit (181) the normalization factor $N_{0,\infty}$ vanishes, even for finite $V$.

We have not computed the normalization factor for generic values of $\gamma$ and $\tilde{\gamma}$, although it could, in principle, be done in the same way. For instance, if $\gamma$ and $\tilde{\gamma}$ are both real and positive, the generalization of (177) is

$$S = \exp \left\{ \int d^3k \left( -\rho \left( a_{\tilde{k}} V a_{\tilde{k}} V + a_{\tilde{k}} V a_{\tilde{k}} V - a_{\tilde{k}} V a_{\tilde{k}} V - a_{\tilde{k}} V a_{\tilde{k}} V \right) + \mu \left( a_{\tilde{k}} V a_{\tilde{k}} V - a_{\tilde{k}} V a_{\tilde{k}} V \right) \right) \right\},$$

with

$$\cosh(2\mu) = \gamma + \tilde{\gamma},$$
$$\sinh(2\mu) = -\gamma,$$
$$\rho \frac{e^{2\mu} - 1}{2\mu} = -\gamma B.$$ (184)

If $\gamma$ and $\tilde{\gamma}$ do not have the same phase, the exponent of $S$ also involves $a_{\tilde{k}}^{II} a_{\tilde{k}}^{II}$ and $a_{\tilde{k}}^{II} a_{\tilde{k}}^{II}$.

The Hilbert space of states $\mathcal{H}$ we have been using can be viewed as the tensor product

$$\mathcal{H} = \mathcal{H}^{II} \otimes \mathcal{H}^{V}$$

of a Hilbert space $\mathcal{H}^{II}$ associated with $a_{\tilde{k}}^{II}$ and $a_{\tilde{k}}^{II}$ and a Hilbert space $\mathcal{H}^{V}$ associated with $a_{\tilde{k}}^{V}$ and $a_{\tilde{k}}^{V}$. These two Hilbert spaces can be thought of as associated with the out-regions II and V respectively. Note that $\mathcal{H}$ could equally well be written as a tensor product of Hilbert spaces associated with in-regions I and VI. The out-vacuum can, accordingly, be expressed as

$$|0\rangle_{out} = |0\rangle_{II} \otimes |0\rangle_{V}. $$ (186)

It is clear that each in-vacuum defined by (173) is a pure state in this tensor product Hilbert space. With a pure state, one can associate a trivial density matrix

$$\rho_{\gamma\tilde{\gamma}} = |0\rangle_{\gamma\tilde{\gamma}} \langle \gamma\tilde{\gamma} |.$$ (187)

When one is only interested in computing correlation functions in Region II, which would be the case for a physicist living in that region and trying to
predict the result of experiments done there, it is convenient (and equivalent) to use the density matrix obtained by tracing (187) over $\mathcal{H}^V$. This is given by

$$\rho_{\gamma\gamma}^{II} = \sum_i V \langle i | \rho_{\gamma\gamma} | i \rangle V,$$  \hfill (188)

where $\{|i\rangle_V\}$ is an orthonormal basis of $\mathcal{H}^V$. Note that $|0\rangle_{\gamma\gamma} \in \mathcal{H} = \mathcal{H}^{II} \otimes \mathcal{H}^V$, whereas $|i\rangle_V \in \mathcal{H}^V$. Therefore, (188) is indeed a density matrix in Region II, that is, an element of $\mathcal{H}^{II*} \otimes \mathcal{H}^{II}$. Now, if $\mu$ in (178) is nonzero (as it is, except in the limit $\gamma \to 0$), $|0\rangle_{\gamma\gamma}$ is an entangled state in $\mathcal{H} = \mathcal{H}^{II} \otimes \mathcal{H}^V$. When one traces the density matrix of an entangled pure state over $\mathcal{H}^V$, one obtains a non-trivial density matrix (188) in Region II. Therefore, an observer in Region II with no access to information about Region V finds himself in a mixed state, that is, a state with entropy. This entropy is called “entanglement entropy” and reflects the ignorance about the correlated Region V. In the limit $\gamma \to 0$, one again obtains a trivial density matrix

$$\rho_{00}^{II} = |0\rangle_{II} \langle 0|,$$  \hfill (189)

which is consistent with (172).

This explains the discrepancy noted at the end of subsection 3.2.3. In subsection 2.3, we assumed unitary evolution from Region I to Region II, that is, a pure state in Region I evolving into a pure (squeezed) state in Region II. This assumption generically turns out to be inconsistent with what we find in the full string theory background, which includes additional regions. We find that a pure in-vacuum state evolves to a pure state in the tensor product Hilbert space $\mathcal{H}$, but that this state generically contains correlations between regions II and V. As a consequence, in a description where Region V is ignored, such as the one appropriate for an observer in Region II, this state is a mixed state. Only in the limit $\gamma \to 0$ does the in-vacuum evolve to a pure state in Region II. As explained in subsection 3.2.3 the limit $\gamma \to 0$ can, strictly speaking, not be reached. The more accurate statement is that the entropy of the state as described in Region II can be made arbitrarily small by taking $\gamma$ to be arbitrarily close to zero.

The above discussion is a generalization of thermofield dynamics as recently applied to the eternal BTZ black hole, where the thermal state of a scalar field outside the black hole is obtained from an entangled state in a tensor product of two Hilbert spaces, the second Hilbert space being associated with an additional asymptotic region behind the horizon of the black hole [17]. Also see [18] and references therein.
3.4 The Milne Orbifold Limit

In the limit that

$$uv \to 0,$$

(190)

the metric in (101) reduces to the Milne orbifold metric (114)

$$ds^2 = \frac{1}{Q^2} dudv + \sum_{I=1}^{8} (dx^I)^2$$

(191)

subject to the identification (103),

$$(u, v) \sim (ue^{-2\pi QR}, ve^{2\pi QR}).$$

(192)

In addition, it follows from (101) that, in this limit,

$$\Phi \to \Phi_0.$$

(193)

Note that the theory we are explicitly discussing in this paper, (101), differs in the number of compactified dimensions from the Milne orbifold (114). Specifically, the generalized Milne orbifold has $(d + 1)$-dimensions, whereas the Milne space has ten dimensions. How, then, can we claim that metric (101) reduces near the singularity to (114)? The naive answer is that this limit does not involve the internal spatial dimensions which, therefore, are irrelevant. A more precise way of understanding the Milne limit is to recognize, as we mention in the Introduction, that two flat spatial directions of the generalized Milne orbifold can be replaced by a “cigar” geometry [16] without changing any of the conclusion of this paper. In this case, we have a critical, ten-dimensional string theory where $Q$ is a free parameter. This cigar geometry has a well-defined smooth limit to the Milne orbifold. Keeping this in mind, we will ignore the number of compact dimensions in this subsection.

If in the limit (190) we keep the ratio

$$\frac{\vec{k}^2 uv}{Q^2} \text{ fixed},$$

(194)

then the fluctuation equation (36) on the generalized Milne orbifold reduces to

$$\delta T'' + \frac{1}{t} \delta T' + \vec{k}^2 \delta T = 0.$$  

(195)
Were we to consider fields with non-vanishing momentum along the $x$ circle as well, we would also be required to keep the product

$$Qr_0 \text{ fixed.} \quad (196)$$

Expression $(195)$ is precisely the fluctuation equation for a scalar mode in the Milne orbifold. The limit defined by $(190)$, $(194)$ and $(196)$ takes the generalized Milne orbifold to the Milne orbifold.

It is instructive to see what our global wavefunctions $K_{\pm \pm \vec{k}}$ reduce to in this limit. In Appendix C, we show that $K_{++ \vec{k}}$ reduces to the standard wavefunctions defining the adiabatic vacuum of the Milne orbifold, that is, superpositions of positive frequency plane waves in Minkowski space. They are Hankel functions in regions I and II and modified Bessel functions that decay asymptotically in regions III and IV, the expressions in the four regions being related by analytic continuation. Near the Big Crunch/Big Bang singularity, these wavefunctions diverge logarithmically. On the other hand, we find that

$$K_{+- \vec{k}} \to 0 \quad (197)$$

everywhere in the Milne limit. More precisely, it is always zero in Region I, is proportional to a finite Bessel function in Region II and corresponds to an asymptotically growing modified Bessel function in regions III and IV. Near the Big Crunch/Big Bang singularity, $K_{+- \vec{k}}$ approaches constants in all regions. However, its overall normalization factor vanishes, suppressing the wavefunction completely. It then follows from $(197)$ that, in this limit,

$$K_{\gamma \tilde{\gamma} \vec{k}} \to 0. \quad (198)$$

Inserting these results into expression $(168)$ for the scalar two-point function, we find that, in the Milne limit, for any choice of in-vacuum

$$\gamma \tilde{\gamma} \langle 0 | \delta \sigma_T (uv, \vec{x})^2 | 0 \rangle_{\gamma \tilde{\gamma}} = \int \frac{d^3 \hat{k}}{(2\pi)^3} |K_{++, \vec{k}}|^2, \quad (199)$$

which is valid in all regions. This is the standard result for the two-point function in the adiabatic vacuum of the Milne orbifold. It is useful to evaluate this correlation function in regions I and II. It is straightforward to do this in Region I using $(120)$. To evaluate $(199)$ in Region II, first note from $(190)$ and $(194)$ that, in the Milne limit,

$$E_{\vec{k}}/Q = \sqrt{\hat{k}^2 - Q^2}/Q \to \infty \quad (200)$$
and, hence, that the Bogolubov coefficient given in (139) satisfies

\[ B \rightarrow 0. \]  

(201)

One can then compute the correlation function in Region II using (128) and the Bogolubov transformation (163). The result is that in both regions I and II of the Milne orbifold,

\[ \langle 0 | \delta \sigma_T(t, \vec{x})^2 | 0 \rangle_{in} = \int \frac{d^3k}{(2\pi)^3} |\delta T_k^+|^2. \]  

(202)

It follows from our previous discussions that, in the distant past and distant future, the spectrum is of the form of Minkowski fluctuations plus a subdominant scale invariant contribution and, therefore,

\[ \Delta(\vec{k}, t) = 1. \]  

(203)

That is, in the Milne orbifold, the spectrum is unchanged by passing through the Big Bang/Big Crunch singularity. Note that this quantum process can be described by the pure Region I and Region II four-dimensional effective theory with the Bogolubov coefficients

\[ |X| = 1, \quad Y = 0. \]  

(204)

Finally, we see that from (143) and (201) that, in the Milne limit,

\[ \langle 0 | a_k^{II\dagger} a_k^{II} | 0 \rangle_{in} \rightarrow 0. \]  

(205)

That is, particle creation is turned off. We should mention that we have assumed that \( \gamma \) is kept fixed during the limiting procedure. It is clear that there would be particle creation if, instead, one kept \( |\gamma B| \) fixed during this limit.

### 3.5 A Note on Backreaction

As we have emphasized at the end of the Introduction, the string theory background (101) has been argued to be unstable to gravitational backreaction. Hence, reliable computations should take this backreaction into account. This seems to be out of reach at present and we will not attempt to resolve the associated deep puzzles in string theory here. However, we would like to offer a few remarks from the point of view taken in this paper.
First, it is worth translating the issue into the four-dimensional language of section 2. Looking at the energy density of the four-dimensional matter fields, one might be tempted to conclude that quantum mechanical particles should not lead to large backreaction. The reason is that the energy density of the classical matter fields, since it is dominated by scalar kinetic energy, scales like $|t|^{-3}$ near the Big Bang/Big Crunch singularity at $t = 0$. The wavefunction of a particle diverges at most logarithmically near $t = 0$, so, according to (20), its energy density cannot diverge more quickly than that of the background matter fields. It would seem, therefore, that a small fluctuation remains small compared to the background. So where is the large backreaction?

The point is that the framework of section 2 is hard to use for computations near the bounce, since the geometry is singular there. The right framework to use is the one of section 3. In these variables, there is not much energy in the classical matter fields near the bounce, where spacetime is locally flat with a slowly varying dilaton. The reflection of this in section 2 is that the sum of the energy densities in the matter and gravitational fields diverges only like $1/|t|$, as one can see from (12) and (29) (divided by $\kappa^2$). This sum is dominated by the potential energy of the four-dimensional dilaton $\phi$ which, in the string theory framework, arises from the cosmological constant. Therefore, the criterion for our computations to be unstable to backreaction is that the energy density of small fluctuations grows faster than $1/|t|$ near the singularity. Unfortunately, this is the case for generic fluctuations, whose wavefunctions diverge logarithmically at the bounce. Hence, we recover the familiar large backreaction problem.

It is interesting to note that there is an important class of fluctuations that do not give rise to large backreaction [21]. Of the modes considered in this paper, those described by $K_{+-\vec{k}}$ are of this type, as can be seen using Appendix B. Their energy density actually vanishes at the Big Bang/Big Crunch singularity.\footnote{This is not true for the $K_{+-\vec{k}}$ of the more generic string modes considered in [15]. For example, $K_{+-\vec{k}}$ does not correspond to a chiral wavefunction near the singularity if it has non-zero momentum along the $x$ circle.}

The issue of backreaction is crucial from various points of view, string theoretic as well as cosmological. A source of criticism of the simple orbifold-like cosmological singularities studied in the string theory literature is that they are unstable against even a single particle being added to the system before
the singularity \cite{23}. Such a particle would change the conical singularity into a genuine curvature singularity. In our model, this is indeed the case for particles with generic wavefunctions, as we have just discussed. Now suppose that, however remote the possibility, no such small fluctuation appeared and the Universe made it through the Crunch singularity. Then, we have shown that the Big Crunch/Big Bang creates a collection of particles with occupation numbers \((166)\) in Region II. One can wonder how these particles backreact on the geometry. First, will the energy density of these particles overwhelm the classical energy density very close to the Big Crunch/Big Bang singularity? The answer is no, as can be seen from \((168)\). The wavefunctions of the created particles are precisely proportional to \(K_{+,-k}\) near the singularity, so their energy density vanishes there.\(^{11}\) Second, it is tempting to speculate that, in some more realistic version of our model, this particle creation might play a role as a reheating mechanism. In fact, reheating by gravitational particle creation has been discussed before. See, for instance, \cite{38}.

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Appendix A: Global Wavefunctions from String Theory

In this Appendix, we will outline the procedure for constructing the globally defined wavefunctions on both the Milne orbifold and the generalized Milne orbifold discussed in this paper.

\(^{11}\)Again, this would be different for the modes we ignored in this paper. In order to turn off their backreaction in Region II, one would have to work in a \(\gamma \to 0\) limit.
Milne Orbifold: We first review how globally defined wavefunctions are constructed on the Milne orbifold $\mathbb{R}^{1,1}/\mathbb{Z} \times \mathbb{R}^8$, an orbifold of Minkowski space.

1. Solve the wave equation

\[(\nabla^2 + m^2)\phi = 0 \quad (206)\]

on the covering space $\mathbb{R}^{1,9}$. A smooth basis of solutions is given by the plane waves

\[\{ e^{i\vec{p} \cdot \vec{X}} e^{i(p^+ X^- + p^- X^+)} | 2p^+ p^- - \vec{p}^2 = m^2 \}. \quad (207)\]

2. Choose an alternative basis of solutions to (206), consisting of continuous superpositions of the smooth solutions (207), such that the orbifold generator $X^\pm \mapsto e^{\pm 2\pi} X^\pm$ is diagonal. Such a basis is given by [30] (see also [21])

\[\phi_{l,\vec{p}} = e^{i\vec{p} \cdot \vec{X}} \int_{-\infty}^{\infty} dw e^{i(p^+ X^- - e^w + p^- X^+ e^w)} e^{ilw} \quad (208)\]

with $l \in \mathbb{R}$. The orbifold generator acts by multiplication by $e^{-2\pi i l}$. The wavefunctions (208) are generically not smooth near the light-cone $X^+ X^- = 0$, which divides the covering space in four regions. When studying string theory on Minkowski space, one could in principle use these singular wavefunctions but, since there exist smooth wavefunctions (207) which can be written as continuous superpositions of (208), it is preferable to work with the latter.

3. Restrict the wavefunctions (208) to the orbifold invariant ones, that is, to those with $l \in \mathbb{Z}$. Using coordinates $t, x$ defined by $X^\pm = t e^{\pm x}/\sqrt{2}$, in terms of which the orbifold generator acts as $x \mapsto x + 2\pi$, it is clear that $l$ is the momentum in the $x$ direction. Therefore, the condition $l \in \mathbb{Z}$ is the usual momentum quantization. Hence, we end up with wavefunctions (208) on the orbifold space. They are not smooth at the singularity $X^+ X^- = 0$, yet they are globally defined. Note that, because of the quantization of $l$, it is not possible to use a basis of smooth wavefunctions on the orbifold space.

4. Introduce twisted sectors (winding modes). We will not discuss those in this paper; see [29] for a very recent discussion.
Generalized Milne Orbifold: We now review the prescription of [20, 15] for constructing global wavefunctions on the generalized Milne orbifold, which in string theory corresponds to a orbifold of a coset conformal field theory at negative level [15],

$$\frac{PSL(2, \mathbb{R})_{k<0}}{U(1)/\mathbb{Z} \times \mathbb{R}^{d-1}}.$$  \hfill (209)

1. Start with wavefunctions on $AdS_3$, that is, the well-known eigenfunctions of the Laplacian on $AdS_3$ which coincides with the $SL(2, \mathbb{R})$ group manifold. $SL(2, \mathbb{R})$ is the group of $2 \times 2$ matrices $g$ with unit determinant,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \hfill (210)$$

$PSL(2, \mathbb{R})$ is obtained from this by identifying $g$ with $-g$. Wavefunctions on $PSL(2, \mathbb{R})$ should be consistent with this identification. The wavefunctions on a group manifold coincide with the matrix elements in the different representations in the group. Therefore, we can write

$$\phi_{j, \alpha, \beta}(g) = \langle j, \alpha | g | j, \beta \rangle, \hfill (211)$$

where $j$ labels a $PSL(2, \mathbb{R})$ representation and $\alpha, \beta$ label states in the representation. In this paper, we will only consider the principal continuous representations, with $j = -1/2 + is$ and $s$ real. They correspond to taking $k^2 > Q^2$. A choice of basis of wavefunctions now corresponds to a choice of states $\alpha, \beta$ in the representation $j$. By choosing an appropriate basis, one can choose a smooth basis of wavefunctions on $PSL(2, \mathbb{R})$. Note that the group manifold $PSL(2, \mathbb{R})$ admits a $PSL(2, \mathbb{R})_L \times PSL(2, \mathbb{R})_R$ symmetry, the two factors acting on $g$ by left and right multiplication respectively. It is clear from (211) that the action of $PSL(2, \mathbb{R})_L$ is determined by the state $\langle j, \alpha \rangle$, while the action of $PSL(2, \mathbb{R})_R$ is determined by $| j, \beta \rangle$. The Laplacian on the $PSL(2, \mathbb{R})$ group manifold is the Casimir operator of $PSL(2, \mathbb{R})$ (acting either from the left or from the right). Its eigenvalue is $j(j+1)$.

2. Choose a basis of wavefunctions on $PSL(2, \mathbb{R})$ such that the $U(1)_L \times U(1)_R$ subgroup, with both factors generated by $\sigma_3$, is diagonalized. This amounts to choosing the basis vectors of the representation $j$ such that $\sigma_3$ is diagonal. It turns out [36] that, for the representations we
are considering, there are two states for each $\sigma_3$ eigenvalue $m$. Thus, we find four independent wavefunctions for each $j, m, \bar{m}$

$$K_{\pm\pm}(j, m, \bar{m}; g) = \langle j, m, \pm | g | j, \bar{m}, \pm \rangle. \quad (212)$$

One finds that the $PSL(2, \mathbb{R})$ group manifold splits into six regions where these wavefunctions are smooth. Generically, the wavefunctions are not smooth at the boundaries separating these regions. However, they are still globally defined. When studying $PSL(2, \mathbb{R})$ by itself, one would usually prefer to work with a basis of smooth wavefunctions.

3. The coset conformal field theory $PSL(2, \mathbb{R})_{k<0}/U(1)$, where $k$ is the level of the $PSL(2, \mathbb{R})$ affine Lie algebra and $U(1)$ acts as $g \mapsto e^{\alpha \sigma} g e^{\alpha \sigma}$, describes the generalized Milne spacetime \cite{101} without the identification \cite{113, 34, 15}. Its globally defined wavefunctions are obtained by restricting to those wavefunctions (212) that are invariant under $U(1)$, that is, to those satisfying $m = -\bar{m}$ \cite{35, 15}.

4. The $\mathbb{Z}$ orbifold group is another (discrete) subgroup of $U(1)_L \times U(1)_R$. The globally defined orbifold wavefunctions are obtained by imposing a quantization condition on $m$ (momentum quantization) and by introducing twisted sectors (winding modes). We refer to \cite{15} for details. In the present paper, we are interested in modes with no momentum or winding along the Milne circle.

**Appendix B: More on Global Wavefunctions**

In this Appendix, we provide some technical details on the global wavefunctions $K_{\pm\pm, \bar{k}}$ used in Section 3. For simplicity of notation, we will suppress the subscript $\bar{k}$. As compared to \cite{15, 20}, we restrict the discussion to the case of interest in this paper, as explained at the beginning of subsection 3.2.1. In the notation of \cite{15}, this means that

$$\lambda = \mu = -j, \quad (213)$$

with $j$ defined in (42) and (49) as

$$j = -\frac{1}{2} + \frac{E_{\bar{k}}}{2Q}, \quad E_{\bar{k}} = \sqrt{k^2 - Q^2}. \quad (214)$$

To be precise, $e^{\alpha \sigma}|j, m, \pm\rangle = e^{2ima}|j, m, \pm\rangle$. 

55
We start by recalling some properties of the hypergeometric function
\[ F(a, b; c; z) \equiv _2 F_1(a, b; c; z) \]  \[89\]. Using the notation
\[ (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}, \]  \[215\]
this function has the power series expansion
\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \]  \[216\]
which converges at least for \( |z| < 1 \). It is useful to note that as \( z \to 0 \),
\[ F(a, b; c; z) \to 1 \]  \[217\]
for all values of the parameters \( a \), \( b \) and \( c \). The behavior for large \( |z| \) can be obtained, for \( b - a \) not integer, using the transformation formula
\[ F(a, b; c; z) = \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}(-z)^{-a}F(a, 1 - c + a; 1 - b + a; \frac{1}{z}) \]
\[+ \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}(-z)^{-b}F(b, 1 - c + b; 1 - a + b; \frac{1}{z}), \]  \[218\]
which is valid for \( |\arg(-z)| < \pi \). Equation \[218\] is singular for integer \( b - a \). For \( a = b \), it is replaced by
\[ F(a, a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)}(-z)^{-a}\sum_{n=0}^{\infty} \frac{(a)_n(1 - c + a)_n}{(n!)^2} z^{-n} \]
\[\times (\log(-z) + 2\Psi(n + 1) - \Psi(a + n) - \Psi(c - a - n)),\]  \[219\]
where
\[ \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \]  \[220\]
Expression \[219\] is valid for \( |\arg(-z)| < \pi, |z| > 1 \) and \( c - a \) not integer.
Other transformation formulas we will use are
\[ F(a, b; c; z) = (1 - z)^{-a}F\left(a, c - b; c; \frac{z}{z - 1}\right) \]  \[221\]
and
\[ F(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2}(1 - z)^n \]
\( \times (- \log(1 - z) + 2\Psi(n + 1) - \Psi(a + n) - \Psi(b + n)), (222) \)

the latter of which is valid for \( |\arg(1 - z)| < \pi \) and \( |1 - z| < 1 \). Also, recall that the Euler Beta function \( B(a, b) \) is defined by

\[
B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}.
\] (223)

In terms of these functions, the global wavefunctions \( K_{\pm \pm} \) are defined as follows.

**Region I:**

\[
\begin{align*}
K_{++}(uv) &= \frac{1}{2\pi i} B(-j, -j)(-uv)^{j}F\left(-j, -j; -2j; \frac{1}{uv}\right) \\
K_{--}(uv) &= \frac{1}{2\pi i} B(j + 1, j + 1)(-uv)^{-j - 1}F\left(j + 1, j + 1; 2j + 2; \frac{1}{uv}\right) \\
K_{+-}(uv) &= 0 \\
K_{-+}(uv) &= \frac{1}{\pi i} B(-j, j + 1)F(-j, j + 1; 1; uv) 
\end{align*}
\] (224)

**Region II:**

\[
\begin{align*}
K_{++}(uv) &= \frac{1}{2\pi i} B(j + 1, j + 1)(-uv)^{-j - 1}F\left(j + 1, j + 1; 2j + 2; \frac{1}{uv}\right) \\
K_{--}(uv) &= \frac{1}{2\pi i} B(-j, -j)(-uv)^jF\left(-j, -j; -2j; \frac{1}{uv}\right) \\
K_{+-}(uv) &= \frac{1}{\pi i} B(-j, j + 1)F(-j, j + 1; 1; uv) \\
K_{-+}(uv) &= 0 
\end{align*}
\] (225)

**Region III:**

\[
\begin{align*}
K_{++}(uv) &= \frac{1}{2\pi i} B(-j, j + 1)(uv)^jF\left(-j, -j; 1 - \frac{1}{uv}\right) \\
K_{--}(uv) &= \frac{1}{2\pi i} B(-j, j + 1)(uv)^{-j - 1}F\left(j + 1, j + 1; 1 - \frac{1}{uv}\right) \\
K_{+-}(uv) &= \frac{1}{2\pi i} B(-j, j + 1)(1 - uv)^jF\left(-j, -j; 1; \frac{uv}{uv - 1}\right) \\
K_{-+}(uv) &= \frac{1}{2\pi i} B(-j, j + 1)(1 - uv)^{-j - 1}F\left(j + 1, j + 1; \frac{uv}{uv - 1}\right) 
\end{align*}
\] (226)

**Region IV:**

\[
\begin{align*}
K_{++}(uv) &= \frac{1}{2\pi i} B(-j, j + 1)(uv)^{-j - 1}F\left(j + 1, j + 1; 1 - \frac{1}{uv}\right) 
\end{align*}
\]
\[ K_-(uv) = \frac{1}{2\pi i} B(-j, j+1) (uv)^j F\left(-j, -j; 1; 1 - \frac{1}{uv}\right) \]
\[ K_+(uv) = \frac{1}{2\pi i} B(-j, j+1) (1 - uv)^{-j-1} F\left(j+1, j+1; 1; \frac{uv}{uv-1}\right) \]
\[ K_+(uv) = \frac{1}{2\pi i} B(-j, j+1) (1 - uv)^j F\left(-j, -j; 1; \frac{uv}{uv-1}\right) \]  
(227)

Region V:
\[ K_{++}(uv) = \frac{1}{\pi i} B(-j, j+1) F(-j, j+1; 1 - uv) \]
\[ K_{--}(uv) = 0 \]
\[ K_{+-}(uv) = \frac{1}{2\pi i} B(j + 1, j + 1) (uv - 1)^{-j-1} F\left(j+1, j+1; 2j+2; \frac{1}{1-uv}\right) \]
\[ K_{-+}(uv) = \frac{1}{2\pi i} B(-j, -j) (uv - 1)^j F\left(-j, -j; -2j; \frac{1}{1-uv}\right) \]  
(228)

Region VI:
\[ K_{++}(uv) = 0 \]
\[ K_{--}(uv) = \frac{1}{\pi i} B(-j, j+1) F(-j, j+1; 1 - uv) \]
\[ K_{+-}(uv) = \frac{1}{2\pi i} B(-j, -j) (uv - 1)^j F\left(-j, -j; -2j; \frac{1}{1-uv}\right) \]
\[ K_{-+}(uv) = \frac{1}{2\pi i} B(j + 1, j + 1) (uv - 1)^{-j-1} F\left(j+1, j+1; 2j+2; \frac{1}{1-uv}\right) \]  
(229)

First, we use these explicit formulas to determine the behavior of some of these wavefunctions in the asymptotic early and late time regimes. In regions I and II, we can use (105) to write
\[ uv = -\sinh^2(Qt), \]  
(230)
while in regions V and VI
\[ 1 - uv = -\sinh^2(Qt). \]  
(231)

Let us concentrate on the early time part of Region I (\(t \ll -1/Q\)). It follows from (224), (230) and (216) that as \(t \to -\infty\)
\[ K_{++} \to \frac{1}{2\pi i} B(-j, -j) 2^{-2j} e^{-2jQt} = \frac{1}{2\pi i} B(-j, -j) 4^{-j} e^{Qt} e^{-iE_\ell t}. \]  
(232)
This shows that $K_{++}$ reduces to a positive frequency wave for early times in Region I. Moreover, expression (232) allows us to determine the factor $N$ such that

$$K_{++} = N K_{++}$$  \hspace{1cm} (233)

is normalized (see subsection 3.2.1). The factor $N$ is actually the same for all $K_{\pm \pm}$. Comparing (232) with (47) or (48), we find

$$N = \frac{4 i e^{\phi_0} 2 \pi i}{\sqrt{E_k B(-j, -j)}}.$$  \hspace{1cm} (234)

Using (218), we can rewrite $K_{--}$ in Region I as

$$K_{--} = \frac{1}{\pi i} B(-j, 2j + 1)(-uv)^i F \left( -j, -j; -2j; \frac{1}{uv} \right)$$

$$+ \frac{1}{\pi i} B(j + 1, -2j - 1)(-uv)^{-j+1} F \left( j + 1, j + 1; 2j + 2; \frac{1}{uv} \right).$$  \hspace{1cm} (235)

This shows that asymptotically in Region I, $K_{--}$ is a superposition of positive and negative frequency waves with equal amplitudes.

Equivalently to (235), we can rewrite $K_{+-}$ in Region II as

$$K_{+-} = \frac{1}{\pi i} B(-j, 2j + 1)(-uv)^i F \left( -j, -j; -2j; \frac{1}{uv} \right)$$

$$+ \frac{1}{\pi i} B(j + 1, -2j - 1)(-uv)^{-j+1} F \left( j + 1, j + 1; 2j + 2; \frac{1}{uv} \right),$$  \hspace{1cm} (236)

which shows that asymptotically in Region II, $K_{+-}$ is a superposition of positive and negative frequency waves with equal amplitudes.

Next, we discuss the behavior near the Big Crunch/Big Bang singularity $uv = 0$. First consider $K_{++}$. In Region I, we can use (219) and (224) to write

$$K_{++}(uv) = \frac{1}{2 \pi i} \sum_{n=0}^{\infty} \frac{(-j)^n (j+1)^n}{(n!)^2} (uv)^n (-\log(-uv) + 2\Psi(n+1) - 2\Psi(-j-n)).$$  \hspace{1cm} (237)

We see that in Region I, $K_{++}$ diverges logarithmically near $uv = 0$. Similarly, we find in Region II

$$K_{++}(uv) = \frac{1}{2 \pi i} \sum_{n=0}^{\infty} \frac{(-j)^n (j+1)^n}{(n!)^2} (uv)^n (-\log(-uv) + 2\Psi(n+1) - 2\Psi(j+1+n)).$$  \hspace{1cm} (238)
Applying (221) and (222) to (226), we find that in Region III

\[ K_{++}(uv) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(-j)_n(j+1)_n}{(n!)^2} (uv)^n \times (- \log(uv) + 2\Psi(n+1) - \Psi(-j+n) - \Psi(j+1+n)). \] (239)

The same equation, (239), holds in Region IV as well. Now consider \( K_{+-} \) near \( uv = 0 \). In Region I, it vanishes. The small \( uv \) behavior in Region II is manifest from (216) and (225). In Region III, we can use (221) to rewrite (226) as

\[ K_{+-}(uv) = \frac{1}{2\pi i} B(-j, j+1) F(-j, j+1; 1; uv). \] (240)

The same formula, (240), also holds in Region IV. Therefore, as \( uv \) approaches zero, \( K_{+-} \) approaches a constant in regions III and IV, and twice that constant in Region II.

**Appendix C: Comparison with the Milne Orbifold**

In subsection 3.2 we gave a qualitative discussion of some differences and similarities between the pure and generalized Milne orbifolds. This comparison can be made very concrete at the level of global wavefunctions, which we will do in this Appendix. Some of these observations were first made by the authors of [21]. As in the previous Appendix, we will suppress the subscript \( \vec{k} \) for notational simplicity.

In the limit that

\[ uv \to 0 \, \text{with} \, \frac{\vec{k}^2 uv}{Q^2} \, \text{and} \, Qr_0 \, \text{fixed}, \] (241)

where \( r_0 \) is defined in (103), the equation of motion (106) reduces to that of a massless scalar in the Milne orbifold (195). We now take this limit of the solutions \( \mathcal{K}_{++} \) and \( \mathcal{K}_{+-} \) and see what they correspond to in the pure Milne case. First, note that in the limit (241),

\[ \frac{E_{\vec{k}}}{Q} \to +\infty. \] (242)
It then follows from (241) that
\[ j \to -\frac{1}{2} + i\infty. \]  

(243)

In this limit, the normalization factor (234) becomes
\[ |\mathcal{N}| \to \frac{e^{\phi_0} \sqrt{\pi}}{\sqrt{8Q}}. \]  

(244)

Using the fact that
\[ \Psi(z) \simeq \log(z) \quad \text{for} \quad z \to \infty \quad \text{with} \quad |\arg(z)| < \pi, \]  

(245)

we find from (237) that in regions I and II
\[
K_{++} \to \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( \frac{E_k^2 uv}{4Q^2} \right)^n \left( -\log \left( -\frac{E_k^2 uv}{4Q^2} \right) + \pi i + 2\Psi(n + 1) \right)
\]

and
\[
K_{++} \to \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( \frac{E_k^2 uv}{4Q^2} \right)^n \left( -\log \left( -\frac{E_k^2 uv}{4Q^2} \right) - \pi i + 2\Psi(n + 1) \right)
\]

(246)

(247)

respectively. Since in Region II \( uv = -Q^2 t^2 \) in the limit (241), the right hand side of (247) is precisely proportional to the Hankel function \( H_0^{(2)}(Et) \), that is,
\[ K_{++} \to -\frac{1}{2} H_0^{(2)}(E_k t), \]  

(248)

while (246) is its analytic continuation to Region I with negative \( t \) via the lower half \( t \) plane [11]. Therefore, the limit (241) of \( K_{++} \) in regions I and II is the wavefunction used in [21, 11] and [30] to define the adiabatic vacuum inherited from Minkowski space. In Region III, using (239) and (243), we find that
\[
K_{++} \to \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( \frac{E_k^2 uv}{4Q^2} \right)^n \left( -\log \left( \frac{E_k^2 uv}{4Q^2} \right) + 2\Psi(n + 1) \right),
\]

(249)

which is the expansion of the modified Bessel function \( K_0 \):
\[ K_{++} \to \frac{1}{\pi i} K_0 \left( \sqrt{\frac{uv E_k^2}{Q^2}} \right). \]  

(250)
The function $K_0$ decays asymptotically (that is, for large values of its argument). This is exactly the analytic continuation of (248) to Region III and, additionally, to Region IV, via the lower half $t$ plane. We conclude that in all four regions of pure Milne space I, II, III and IV, the limit (241) of $K_{++}$ exactly coincides with the wavefunctions used in [21] to define the vacuum inherited from Minkowski space. These wavefunctions coincide with those of [11] when restricted to regions I and II.

To determine the behavior of $K_{+-}$, note that in the limit (241)

$$F(-j, j + 1; uv) \to \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( \frac{uvE_k^2}{4Q^2} \right)^n.$$  \hspace{1cm} (251)

It follows that in Region II

$$F(-j, j + 1; uv) \to J_0(E_k t),$$  \hspace{1cm} (252)

with $J_0$ a Bessel function, whereas in regions III and IV

$$F(-j, j + 1; uv) \to I_0 \left( \sqrt{uvE_k^2/Q^2} \right),$$  \hspace{1cm} (253)

with $I_0$ a modified Bessel function which has the property that it grows asymptotically. For this reason, $I_0$ it is usually not considered in discussions of the Milne orbifold, as we mentioned in subsection 3.2. In the generalized Milne orbifold, there is nothing wrong with growing behavior near $uv = 0$ since regions III and IV do not extend to infinity. In the limit (241), where regions III and IV are blown up to infinite size, the prefactor $B(-j, j + 1)$ in (225) and (240) actually goes to zero. Therefore, although in regions III and IV $K_{+-}$ is proportional to (253) in the limit (241), the proportionality factor is zero and $K_{+-}$ vanishes, as it does in regions I and II.

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