Classifying the provably total set functions of $\mathbf{KP}$ and $\mathbf{KP}(\mathcal{P})$

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Abstract

This article is concerned with classifying the provably total set-functions of Kripke-Platek set theory, $\mathbf{KP}$, and Power Kripke-Platek set theory, $\mathbf{KP}(\mathcal{P})$, as well as proving several (partial) conservativity results. The main technical tool used in this paper is a relativisation technique where ordinal analysis is carried out relative to an arbitrary but fixed set $x$.

A classic result from ordinal analysis is the characterisation of the provably recursive functions of Peano Arithmetic, $\mathbf{PA}$, by means of the fast growing hierarchy $[10]$. Whilst it is possible to formulate the natural numbers within $\mathbf{KP}$, the theory speaks primarily about sets. For this reason it is desirable to obtain a characterisation of its provably total set functions. We will show that $\mathbf{KP}$ proves the totality of a set function precisely when it falls within a hierarchy of set functions based upon a relativised constructible hierarchy stretching up in length to any ordinal below the Bachmann-Howard ordinal. As a consequence of this result we obtain that $\mathbf{IKP} + \forall x \forall y (x \in y \lor x \not\in y)$ is conservative over $\mathbf{KP}$ for $\Pi^2_2$-formulae, where $\mathbf{IKP}$ stands for intuitionistic Kripke-Platek set theory.

In a similar vein, utilising $[56]$, it is shown that $\mathbf{KP}(\mathcal{P})$ proves the totality of a set function precisely when it falls within a hierarchy of set functions based upon a relativised von Neumann hierarchy of the same length. The relativisation technique applied to $\mathbf{KP}(\mathcal{P})$ with the global axiom of choice, $\mathbf{AC}_{\text{global}}$, also yields a parameterised extension of a result in $[58]$, showing that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{\text{global}}$ is conservative over $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ and $\mathbf{CZF} + \mathbf{AC}$ for $\Pi^2_2$ statements. Here $\mathbf{AC}$ stands for the ordinary axiom of choice and $\mathbf{CZF}$ refers to constructive Zermelo-Fraenkel set theory.

1 Introduction

A major application of the techniques of ordinal analysis has been the classification of the provably total recursive functions of a theory. Usually the theories to which this methodology has been applied have been arithmetic theories, in that context it makes most sense to speak about arithmetic functions. The concept of a recursive function on natural numbers can be extended to a more general recursion theory on arbitrary sets. For more details see $[38]$, $[39]$ and $[59]$. Since $\mathbf{KP}$ speaks primarily about sets, it is perhaps desirable to obtain a classification of its provably total recursive set functions.

To provide some context we first state a classic result from proof theory, the classification of the provably total recursive functions of $\mathbf{PA}$. A classification can be gleaned from Gentzen's 1938 $[25]$ and 1943 $[20]$ papers. The first explicit characterization of these functions as those definable by
recursions on ordinals less than \( \varepsilon_0 \) was given by Kreisel \([31, 32]\) in the early 1950s. Many people re-proved or provided variants of this classification result (see \([64\), Chap. 4\]) for the history. As to techniques for extracting numerical bounds from infinite proofs, Schwichtenberg’s \([63]\) and the considerably more elegant approach by Buchholz and Wainer in \([10]\) and its generalization and simplification by Weiermann in \([66]\) are worth mentioning. For the following definitions, suppose we have an ordinal representation system for ordinals below \( \varepsilon_0 \), together with an assignment of fundamental sequences to the limit ordinal terms. For an ordinal term \( \alpha \), let \( \alpha_n \) denote the \( n \)-th element of the fundamental sequence for \( \alpha \), i.e. \( \alpha_n + 1 < \alpha \) and \( \sup_{n<\omega}(\alpha_n) = \alpha \). There are certain technical properties that such an assignment must satisfy, these will not be gone into here, for a detailed presentation see \([10]\).

**Definition 1.1.** For each \( \alpha < \varepsilon_0 \) we define the function \( F_\alpha : \omega \to \omega \) by transfinite recursion as follows

\[
F_0(n) := n + 1 \\
F_{\alpha+1}(n) := \underbrace{F_{\alpha} \circ \ldots \circ F_{\alpha}}_{n+1}(n) \\
F_\alpha(n) := F_{\alpha_n}(n) \quad \text{if} \ \alpha \ \text{is a limit.}
\]

This hierarchy is known as the **fast growing hierarchy**. Given unary functions on the natural numbers \( f \) and \( g \), we say that \( f \) majorises \( g \) if there is some \( n \) such that \((\forall m>n)(g(m) < f(m))\). For a recursive function \( f \) let \( A_f(n, m) \) be the \( \Sigma \) formula expressing that on input \( n \) the Turing machine for computing \( f \) outputs \( m \), to avoid frustrating counter examples let us suppose \( A_f \) does this in some ‘natural’ way.

**Theorem 1.2.** Suppose \( f : \omega \to \omega \) is a recursive function. Then

i) If \( \text{PA} \vdash \forall x \exists ! y A_f(x, y) \) then \( f \) is majorised by \( F_\alpha \) for some \( \alpha < \varepsilon_0 \).

ii) \( \text{PA} \vdash \forall x \exists ! y A_{F_\alpha}(x, y) \) for every \( \alpha < \varepsilon_0 \).

**Proof.** This classic result is proved in full in \([10]\). \( \square \)

This chapter will be focused on obtaining a similar result for the provably total set functions of \( \text{KP}^1 \). A similar role to the fast growing hierarchy in Theorem 1.2 will be played by the **relativised constructible hierarchy**.

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\( ^1 \)There are many papers concerned with the provably recursive number-theoretic functions of \( \text{KP} \) and much stronger theories. The basic idea consists in adding another layer of control to the ordinal analysis that allows one to extract bounds for numerical witnesses. These techniques were initially engineered by Buchholz, Wainer \([10]\) and Weiermann \([62]\) and then got extended by Blankertz, Weiermann \([4, 5, 7]\), Michelbrink \([37]\), Pohlers and Stegert \([42]\) to ever stronger theories. Another route for obtaining classifications of provably numerical functions proceeds as follows. The ordinal analysis of a set theory \( T \) shows that the arithmetic part of \( T \) can be reduced to \( \text{PA} \) plus transfinite induction for every ordinal below the proof-theoretic ordinal of \( T \). Thus it suffices to characterize the provably numerical functions of the latter system. This leads to the descent recursive functions in the sense of \([23]\). That this method is perfectly general was first sketched in \([23]\) and then proved rigorously in \([12]\). The latter approach has the advantage that the ordinal analysis of \( T \) needn’t be burdened with the extra task of controlling numerical witnesses.
Definition 1.3. Let $X$ be any set. We may relativise the constructible hierarchy to $X$ as follows:

$$
L_0(X) := \text{TC}(\{X\}) \quad \text{the transitive closure of } \{X\}
$$

$$
L_{\alpha+1}(X) := \{B \subseteq L_\alpha(X) : B \text{ is definable over } \langle L_\alpha(X), \in \rangle\}
$$

$$
L_\theta(X) := \bigcup_{\xi < \theta} L_\xi(X) \quad \text{when } \theta \text{ is a limit.}
$$

In section 2 we build an ordinal notation system relativised to an arbitrary set $X$, this will be used for the rest of the article. In section 3 we define the infinitary system $\text{RS}_\Omega(X)$, based on the relativised constructible hierarchy and show that we can eliminate cuts for derivations of $\Sigma$ formulae. In section 4 we embed $\text{KP}$ into $\text{RS}_\Omega(X)$, allowing us to obtain cut free infinitary derivations of $\text{KP}$ provable $\Sigma$ formulae. Technically we use Buchholz’ operator controlled derivations (see [11]) which are also used in [41]. In section 5 we give a well ordering proof in $\text{KP}$ for the ordinal notation system given in section 2. Finally we combine the results of this chapter to give a classification of the provably total set functions of $\text{KP}$ in section 6. This result, whilst perhaps known to those who have thought hard about these things, has not appeared in the literature to date. Section 7 contains applications to semi-intuitionistic Kripke-Platek set theory. Section 8 carries out a relativised ordinal analysis of Power Kripke-Platek set theory, $\text{KP}(P)$, from which ensues a classification of its provable set functions. This closely follows the treatment in [56]. In section 9, a further ingredient is added to the infinitary system by incorporating a global choice relation. Due to the relativisation one gets partial conservativity results for $\text{KP}(P) + AC_{\text{global}}$ over $\text{KP}(P) + AC$ and $\text{CZF} + AC$ that provide improvements on [58, Theorem 3.3] and [58, Corollary 5.2]. These theories can also be added to the list of theories [57, Theorem 15.1] with the same proof-theoretic strength.

2 A relativised ordinal notation system

The aim of this section is to relativise the construction of the Bachmann-Howard ordinal to contain an arbitrary set $X$ or rather its rank $\theta$. We will construct an ordinal representation system that will be set primitive recursive given access to an oracle for $X$. Here the notion of recursive and primitive recursive is extended to arbitrary sets, see [39] or [59] for more detail. The construction of an ordinal representation system for the Bachmann-Howard ordinal is now fairly standard in proof theory, carried out for example in [9]. Intuitively our system will appear similar, only the ordering $W$ will be inserted as an initial segment before new ordinals start being ‘named’ via the collapsing function.

Before defining the formal terms and the procedure for computing their ordering, it is informative to give definitions for the corresponding ordinals and ordinal functions themselves. To this end we will begin working in $\text{ZFC}$, later it will become clear that the necessary ordinals can be expressed as formal terms and comparisons between these terms can be made primitive recursively relative to $W$.

In what follows $\text{ON}$ will denote the class of all ordinals. First we require some information about the $\varphi$ function on ordinals. These definitions and results are well known, see [62].

Definition 2.1. For each $\alpha \in \text{ON}$ we define a class of ordinals $Cr(\alpha) \subseteq \text{ON}$ and a class function

$$
\varphi_\alpha : \text{ON} \rightarrow \text{ON}
$$
by transfinite recursion.

i) \( Cr(0) := \{ \omega^\beta \mid \beta \in \text{ON} \} \) and \( \varphi_0(\beta) := \omega^\beta \).

ii) For \( \alpha > 0 \) \( Cr(\alpha) := \{ \beta \mid (\forall \gamma < \alpha)(\varphi_\gamma(\beta) = \beta) \} \).

iii) For each \( \alpha \in \text{ON} \) \( \varphi_\alpha(\cdot) \) is the function enumerating \( Cr(\alpha) \).

The convention is to write \( \varphi_\alpha \beta \) instead of \( \varphi_\alpha(\beta) \). An ordinal \( \beta \in Cr(0) \) is often referred to as additive principal, since for all \( \beta_1, \beta_2 < \beta \) we have \( \beta_1 + \beta_2 < \beta \).

**Theorem 2.2.**

i) \( \varphi_1 \beta_1 = \varphi_2 \beta_2 \) if and only if \( \begin{cases} \alpha_1 < \alpha_2 \text{ and } \beta_1 = \varphi_2 \beta_2 \\ \alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2 \\ \alpha_2 < \alpha_1 \text{ and } \varphi_\alpha \beta_1 = \beta_2. \end{cases} \)

ii) \( \varphi_1 \beta_1 < \varphi_2 \beta_2 \) if and only if \( \begin{cases} \alpha_1 < \alpha_2 \text{ and } \beta_1 < \varphi_2 \beta_2 \\ \alpha_1 = \alpha_2 \text{ and } \beta_1 < \beta_2 \\ \alpha_2 < \alpha_1 \text{ and } \varphi_\alpha \beta_1 < \beta_2. \end{cases} \)

iii) For any additive principal \( \beta \) there are unique ordinals \( \beta_1 \leq \beta \) and \( \beta_2 < \beta \) such that \( \beta = \varphi_\beta \beta_2 \).

*Proof.* This result is proved in full in [62]. \( \Box \)

**Definition 2.3.** We define \( \Gamma(\cdot) : \text{ON} \to \text{ON} \) to be the class function enumerating the ordinals \( \beta \) such that for all \( \beta_1, \beta_2 < \beta \) we have \( \varphi_\beta \beta_1 \beta_2 < \beta \). Ordinals of the form \( \Gamma_\beta \) will be referred to as strongly critical.

Now let \( \theta \in \text{ON} \) be the unique ordinal that is the set-theoretic rank of \( X \).

**Definition 2.4.** Let \( \Omega \) be the least uncountable cardinal greater than \( \theta \). The sets \( B_\theta(\alpha) \subseteq \text{ON} \) and ordinals \( \psi_\theta(\alpha) \) are defined by transfinite recursion on \( \alpha \) as follows:

\[
B_\theta(\alpha) := \text{Closure of } \{0, \Omega\} \cup \{\Gamma_\beta : \beta \leq \theta\} \text{ under } +, \varphi \text{ and } \psi_\theta|_\alpha \\
\psi_\theta(\alpha) := \min\{\beta : \beta \notin B_\theta(\alpha)\}
\]

For the remainder of this section, since \( \theta \) remains fixed, the subscripts will be dropped from \( B_\theta \) and \( \psi_\theta \) to improve readability. At first glance it may appear strange having the elements from \( \theta \) inserted into the \( \Gamma \)-numbers. Ultimately we aim to have + and \( \varphi \) as primitive symbols in our notation system, simply having \( \theta \) as an initial segment here would cause problems with unique representation. Some ordinals could get a name directly from \( \theta \) and other names by applying + and \( \varphi \) to smaller elements.

**Lemma 2.5.** For each \( \alpha \in \text{ON} \):

i) The cardinality of \( B(\alpha) \) is \( \max\{\aleph_0, |\theta|\} \), where \( |\theta| \) denotes the cardinality of \( \theta \).

ii) \( \psi \alpha < \Omega \).
Proof. i) Let

\[ B^0(\alpha) := \{0, \Omega\} \cup \{\Gamma_\beta : \beta \leq \theta\} \]
\[ B^{n+1}(\alpha) := B^n(\alpha) \cup \{\xi + \eta : \xi, \eta \in B^n(\alpha)\} \]
\[ \cup \{\varphi_\xi \eta : \xi, \eta \in B^n(\alpha)\} \]
\[ \cup \{\psi_\xi : \xi \in B^n(\alpha) \cap \alpha\}. \]

Observe that \( B(\alpha) = \cup_{n<\omega} B^n(\alpha) \), this can be proved by a straightforward induction on \( n \).

If \( \theta \) is finite then, again by induction on \( n \), we can show that each \( B^n(\alpha) \) is also finite. Since \( B(\alpha) \) is a countable union of finite sets and \( \omega \subseteq B(\alpha) \) it follows that it must have cardinality \( \aleph_0 \).

Now suppose \( \theta \) is infinite, so \( B(\alpha) \) is the countable union of sets of cardinality \( |\theta| \) and thus also has cardinality \( |\theta| \).

ii) If \( \psi \alpha \geq \Omega \) then \( \Omega \subset B(\alpha) \) contradicting i). \( \square \)

Lemma 2.6.

i) If \( \gamma \leq \delta \) then \( B(\gamma) \subseteq B(\delta) \) and \( \psi \gamma \leq \psi \delta \).

ii) If \( \gamma \in B(\delta) \cap \delta \) then \( \psi \gamma \leq \psi \delta \).

iii) If \( \gamma \leq \delta \) and \([\gamma, \delta) \cap B(\gamma) = \emptyset\) then \( B(\gamma) = B(\delta) \).

iv) If \( \xi \) is a limit then \( B(\xi) = \cup_{\eta < \xi} B(\eta) \).

v) \( \psi \gamma \) is a strongly critical and \( \psi \gamma \geq \Gamma_{\theta+1} \).

vi) \( B(\gamma) \cap \Omega = \psi \gamma \).

vii) If \( \xi \) is a limit then \( \psi \xi = \sup_{\eta < \xi} \psi \eta \).

viii) \( \psi(\gamma + 1) \leq (\psi \gamma)^\Gamma \), where \( \delta^\Gamma \) denotes the smallest strongly critical ordinal above \( \delta \).

ix) If \( \alpha \in B(\alpha) \) then \( \psi(\alpha + 1) = (\psi \alpha)^\Gamma \).

x) If \( \alpha \notin B(\alpha) \) then \( \psi(\alpha + 1) = \psi \alpha \) and \( B(\alpha + 1) = B(\alpha) \).

xi) If \( \gamma \in B(\gamma) \) and \( \delta \in B(\delta) \) then \([\gamma < \delta \text{ if and only if } \psi \gamma \leq \psi \delta]\).

Proof. i) Suppose \( \gamma \leq \delta \), now note that \( B(\delta) \) is closed under \( \psi|_\delta \) which includes \( \psi|_\gamma \) so \( B(\gamma) \subseteq B(\delta) \). From this it immediately follows from the definition that \( \psi \gamma \leq \psi \delta \).

ii) From \( \gamma \in B(\delta) \cap \delta \) we get \( \psi \gamma \in B(\delta) \), thus \( \psi \gamma \leq \psi \delta \) by the definition of \( \psi \delta \).

iii) It is enough to show that \( B(\gamma) \) is closed under \( \psi|_{\delta} \). Let \( \beta \in B(\gamma) \) and \( \beta < \delta \), then by assumption \( \beta < \gamma \), thus \( \psi \beta \in B(\gamma) \).
iv) By i) we have \( \cup_{\eta<\xi}B(\eta) \subseteq B(\xi) \). It remains to verify that \( Y := \cup_{\eta<\xi}B(\eta) \) is closed under \( \psi|_\xi \). So let \( \delta \in Y \cap \xi \), since \( \xi \) is a limit there is some \( \xi_0 < \xi \) such that \( \delta \in Y \cap \xi_0 \) and there is some \( \xi_1 < \xi \) such that \( \delta \in B(\xi_1) \). Therefore \( \delta \in B(\xi^*) \cap \xi^* \) where \( \xi^* = \max\{\xi_0, \xi_1\} \), thus \( \psi \delta \in B(\xi^*) \subseteq Y \).

v) We may write the ordinal \( \psi \alpha \) in Cantor normal form, so that \( \psi \alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \) with \( \alpha_1 \geq \ldots \geq \alpha_n \). If \( n > 1 \) then \( \alpha_1, \ldots, \alpha_n < \psi \alpha \) which implies by the definition of \( \psi \alpha \) that \( \alpha_1, \ldots, \alpha_n \in B(\alpha) \).

But by closure of \( B(\alpha) \) under + and \( \varphi \) we get \( \varphi \alpha_1 + \ldots + \varphi \alpha_n = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in B(\alpha) \) contradicting \( \psi \alpha \not\in B(\alpha) \). Thus \( \psi \alpha \) is additive principal and it follows from Theorem 2.2(ii) that we may find ordinals \( \gamma \leq \psi \alpha \) and \( \delta < \psi \alpha \) such that \( \psi \alpha = \varphi \gamma \delta \). If \( \delta > 0 \) then \( \gamma < \psi \alpha \) since \( \gamma \leq \varphi \gamma 0 < \varphi \gamma \delta \), but if \( \delta, \gamma < \psi \alpha \) then we have \( \delta, \gamma \in B(\alpha) \) and hence \( \varphi \gamma \delta \in B(\alpha) \) contradicting \( \psi \alpha \not\in B(\alpha) \). Thus \( \psi \alpha = \varphi \gamma 0 \), but if \( \gamma < \psi \alpha \) then again we get \( \varphi \gamma 0 \in B(\alpha) \); a contradiction. So it must be the case that \( \psi \alpha = \gamma \), ie. \( \psi \alpha \) is strongly critical.

For the second part note that \( \psi \alpha \neq \Gamma_\beta \) for any \( \beta \leq \theta \) since by definition each such \( \Gamma_\beta \in B(\alpha) \).

vi) By 2.5(i) and the definition of \( \psi \) it is clear that \( \psi \alpha \subseteq B(\alpha) \cap \Omega \). Now let \( Y := \psi \alpha \cup \{ \delta \geq \Omega \mid \delta \in B(\alpha) \} \) by v) \( Y \) contains \( 0, \Omega \) and \( \Gamma_\beta \) for \( \beta \leq \theta \), moreover it is closed under + and \( \varphi \). It remains to show that \( Y \) is closed under \( \psi|_\alpha \), this follows immediately from ii).

vii) Let \( \xi \) be a limit ordinal. Using parts vi), iv) and i) we have

\[ \psi \xi = B(\xi) \cap \Omega = (\cup_{\eta<\xi}B(\eta)) \cap \Omega = \cup_{\eta<\xi}(B(\eta) \cap \Omega) = \cup_{\eta<\xi}\psi \eta = \sup_{\eta<\xi} \psi \eta. \]

viii) Let \( Y := (\psi \alpha)^\Gamma \cup \{ \delta \geq \Omega \mid \delta \in B(\alpha) \} \).

\( Y \) is closed under + and \( \varphi \), also it contains \( \Gamma_\beta \) for any \( \beta \leq \theta \) by v). Moreover it contains \( \psi \gamma \) for any \( \gamma \leq \alpha \) by i), so it is closed under \( \psi|_{(\alpha+1)} \). Therefore \( Y \) must contain \( B(\alpha + 1) \), and so \( \psi(\alpha + 1) \leq (\psi \alpha)^\Gamma \).

ix) From \( \alpha \in B(\alpha) \) we get \( \alpha \in B(\alpha + 1) \), it then follows from ii) that \( \psi \alpha < \psi(\alpha + 1) \). Thus \( \psi(\alpha + 1) \leq (\psi \alpha)^\Gamma \) by viii) and \( \psi(\alpha + 1) \geq (\psi \alpha)^\Gamma \) from v), so it must be the case that \( \psi(\alpha + 1) = (\psi \alpha)^\Gamma \).

x) Suppose \( \alpha \not\in B(\alpha) \), then \( (\alpha, \alpha + 1) \cap B(\alpha) = \emptyset \) so we may apply iii) to give \( B(\alpha + 1) = B(\alpha) \) from which \( \psi(\alpha + 1) = \psi \alpha \) follows immediately.

xi) Suppose \( \gamma \in B(\gamma) \) and \( \delta \in B(\delta) \). If \( \gamma < \delta \) then from ix) we get \( \psi(\gamma + 1) = (\psi \gamma)^\Gamma > \psi \gamma \), but by i) \( \psi(\gamma + 1) \leq \psi \delta \).

Now if \( \psi \gamma < \psi \delta \) then from the contraposition of i) we get \( \gamma < \delta \).

\[ \square \]

**Definition 2.7.** We write

i) \( \alpha = \alpha_1 + \ldots + \alpha_n \) if \( \alpha = \alpha_1 + \ldots + \alpha_n \), \( n > 1 \), \( \alpha_1, \ldots, \alpha_n \) are additive principal numbers and \( \alpha_1 \geq \ldots \geq \alpha_n \).
Lemma 2.8.

i) If $\alpha =_{NF} 0$ if $\alpha =_N \varphi \gamma \delta$ and $\gamma, \delta < \varphi \gamma \delta$.

ii) $\alpha =_{NF} \psi \gamma$ if $\alpha = \psi \gamma$ and $\gamma \in B(\gamma)$

Proof. i) Suppose $\alpha =_{NF} \alpha_1 + \ldots + \alpha_n$, then for any $\eta \in ON$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad \alpha_1, \ldots, \alpha_n \in B(\eta).$$

ii) If $\alpha =_{NF} \varphi \gamma \delta$ then for any $\eta \in ON$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad \gamma, \delta \in B(\eta).$$

iii) If $\alpha =_{NF} \psi \gamma$ then for any $\eta \in ON$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad \gamma \in B(\eta) \cap \eta.$$

Proof. i) Suppose $\alpha =_{NF} \alpha_1 + \ldots + \alpha_n$, the $\Leftarrow$ direction is clear from the closure of $B(\eta)$ under $\varphi$. For the other direction let

$$AP(\alpha) := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \{ \alpha \} & \text{if } \alpha \text{ is additive principal} \\ \{ \alpha_1, \ldots, \alpha_n \} & \text{if } \alpha =_{NF} \alpha_1 + \ldots + \alpha_n \end{cases}$$

$AP(\alpha)$ stands for the additive predecessors of $\alpha$. Now let

$$Y := \{ \gamma \in B(\eta) \mid AP(\gamma) \subseteq B(\eta) \}.$$ 

Observe that $0, \Omega \in Y$ and $\{ \Gamma_\beta \mid \beta \leq \theta \} \subseteq Y$. Now choose any $\gamma, \delta \in Y$, we have $AP(\gamma + \delta) \subseteq AP(\gamma) \cup AP(\delta) \subseteq B(\eta)$, thus $Y$ is closed under $\varphi$. Now $AP(\varphi \gamma \delta) = \{ \varphi \gamma \delta \}$ since the range of $\varphi$ is the additive principal numbers thus $Y$ is closed under $\varphi$. Finally $AP(\psi \gamma) = \{ \psi \gamma \}$ for any $\gamma \in Y \cap \eta$ so $Y$ is closed under $\psi|_\eta$. It follows that $B(\eta) \subseteq Y$ and thus the other direction is proved.

ii) Again the $\Leftarrow$ direction follows immediately from the closure of $B(\eta)$ under $\varphi$. For the other direction we let

$$PP(\alpha) := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \{ \alpha \} & \text{if } \alpha \text{ is strongly critical} \\ \{ \gamma, \delta \} & \text{if } \alpha =_{NF} \varphi \gamma \delta \\ \{ \alpha_1, \ldots, \alpha_n \} & \text{if } \alpha =_{NF} \alpha_1 + \ldots + \alpha_n. \end{cases}$$

for want of a better phrase $PP(\alpha)$ stands for the predicative predecessors of $\alpha$. Now set

$$Y := \{ \gamma \in B(\eta) \mid PP(\gamma) \subseteq B(\eta) \}.$$ 

It is easily seen that $Y$ contains $0, \Omega$ and $\Gamma_\beta$ for any $\beta \leq \theta$. $PP(\gamma + \delta) \subseteq PP(\gamma) \cup PP(\delta)$ so $Y$ is closed under $\varphi$. Finally $PP(\psi \gamma) = \{ \psi \gamma \}$ for any $\gamma < \eta$ by [2,4)]. It follows that $Y$ must contain $B(\eta)$, which proves the $\Rightarrow$ direction.

iii) Suppose $\alpha =_{NF} \psi \gamma$, the $\Leftarrow$ direction is clear by the closure of $B(\eta)$ under $\psi|_\eta$. For the other direction suppose $\alpha \in B(\eta)$, from this we get $\psi \gamma < \psi \eta$ which gives us $\gamma < \eta$. Now by assumption $\gamma \in B(\gamma)$, and $B(\gamma) \subseteq B(\eta)$ so $\gamma \in B(\eta) \cap \eta$. 

$\square$
In order to create an ordinal notation system from the ordinal functions described above, we single out a set \( R(\theta) \) of ordinals which have a unique canonical description.

**Definition 2.9.** We give an inductive definition of the set \( R(\theta) \), and the complexity \( G_\alpha < \omega \) for every \( \alpha \in R(\theta) \)

- (R1) \( 0, \Omega \in R(\theta) \) and \( G_0 := G_\Omega := 0 \).
- (R2) For each \( \beta \leq \theta \), \( \Gamma_\beta \in R(\theta) \) and \( GT_\beta := 0 \).
- (R3) If \( \alpha = NF \alpha_1 + \ldots + \alpha_n \) and \( \alpha_1, \ldots, \alpha_n \in R(\theta) \) then \( \alpha \in R(\theta) \) and \( G_\alpha := \max\{G_{\alpha_1}, \ldots, G_{\alpha_n}\} + 1 \).
- (R4) If \( \gamma, \delta < \Omega \), \( \alpha = NF \varphi_{\gamma}^{\delta} \) and \( \gamma, \delta \in R(\theta) \) then \( \alpha \in R(\theta) \) and \( G_\alpha := \max\{G_\gamma, G_\delta\} + 1 \).
- (R5) If \( \gamma \geq \Omega \), \( \alpha = NF \varphi_0^{\gamma} \) and \( \gamma \in R(\theta) \) then \( \alpha \in R(\theta) \) and \( G_\alpha := G_\gamma + 1 \).
- (R6) If \( \alpha = NF \psi^{\gamma} \) and \( \gamma \in R(\theta) \) then \( \alpha \in R(\theta) \) and \( G_\alpha := G_\gamma + 1 \).

**Lemma 2.10.** Every element \( \alpha \in R(\theta) \) is included due to precisely one of the rules (R1)-(R6) and thus the complexity \( G_\alpha \) is uniquely defined.

*Proof.* This follows immediately from (R6). \( \Box \)

Our goal is to turn \( R(\theta) \) into a formal representation system, the main obstacle to this is that it is not immediately clear how to deal with the constraint \( \gamma \in B(\gamma) \) in a computable way. This problem leads to the following definition.

**Definition 2.11.** To each \( \alpha \in R(\theta) \) we assign a set \( K_\alpha \) of ordinal terms by induction on the complexity \( G_\alpha \):

- (K1) \( K_0 := K_\Omega := K\Gamma_\beta := \emptyset \) for all \( \beta \leq \theta \).
- (K2) If \( \alpha = NF \alpha_1 + \ldots + \alpha_n \) then \( K_\alpha := K_{\alpha_1} \cup \ldots \cup K_{\alpha_n} \).
- (K3) If \( \alpha = NF \varphi_{\gamma}^{\delta} \) then \( K_\alpha := K_\gamma \cup K_\delta \).
- (K4) If \( \alpha = NF \psi^{\gamma} \) then \( K_\alpha := \{\gamma\} \cup K_\gamma \).

\( K_\alpha \) consists of the ordinals that occur as arguments of the \( \psi \) function in the normal form representation of \( \alpha \). Note that each ordinal in \( K_\alpha \) belongs to \( R(\theta) \) itself and has complexity lower than \( G_\alpha \).

**Lemma 2.12.** For any \( \alpha, \eta \in R(\theta) \)

\[ \alpha \in B(\eta) \quad \text{if and only if} \quad (\forall \xi \in K_\alpha)(\xi < \eta) \]

*Proof.* The proof is by induction on \( G_\alpha \). If \( G_\alpha = 0 \) then \( \alpha \in B(\eta) \) for any \( \eta \), and \( K_\alpha = \emptyset \) by (K1) so the result holds.

Case 1. If \( \alpha = NF \alpha_1 + \ldots + \alpha_n \) then \( \alpha \in B(\eta) \) iff \( \alpha_1, \ldots, \alpha_n \in B(\eta) \) by (K3). Now inductively \( \alpha_1, \ldots, \alpha_n \in B(\eta) \) iff \( (\forall \xi \in K_{\alpha_1} \cup \ldots \cup K_{\alpha_n})(\xi < \eta) \), but by (K2) \( K_\alpha = K_{\alpha_1} \cup \ldots \cup K_{\alpha_n} \).
Case 2. If $\alpha = N_F \varphi \gamma \delta$ we may argue in a similar fashion to Case 1, using (2.8ii) and (K3) instead.

Case 3. If $\alpha = N_F \psi \gamma$ then $\alpha \in B(\eta)$ iff $\gamma \in B(\eta) \cap \eta$ by (2.8ii). Now by induction hypothesis $\gamma \in B(\eta) \cap \eta$ iff $(\forall \xi \in K\gamma)(\xi < \eta)$ and $\gamma < \eta$, and by (K4) this occurs precisely when $(\forall \xi \in K\alpha)(\xi < \eta)$. \hfill \Box

Recall that $\theta$ is the rank of $X$. Let

$L_\theta : = \{0, \Omega, +, \varphi, \psi\} \cup \{\Gamma_\xi : \xi \leq \theta\}$ and

$L_\theta^* : = \{s \mid s \text{ is a finite string of symbols from } L_\theta\}$.

Now let $T(\theta) \subseteq L_\theta^*$ be the set of strings that correspond to ordinals in $R(\theta)$ expressed in normal form. Owing to Lemma 2.10 there is a one to one correspondence between $T(\theta)$ and $R(\theta)$. The ordering on $T(\theta)$ induced from the ordering of the ordinals in $R(\theta)$ will be denoted $\prec$. To differentiate between elements of the two sets, Greek letters $\alpha, \beta, \gamma, \eta, \xi, \ldots$ range over ordinals and Roman letters $a, b, c, d, e, \ldots$ range over finite strings from $L_\theta^*$.

**Theorem 2.13.** The set $T(\theta)$ and the relation $\prec$ on $T(\theta)$ are set primitive recursive in $\theta$.

**Proof.** Below a $\theta$-primitive recursive procedure means a procedure that is primitive recursive in the two parameters $\theta$ and the ordering $\prec_\theta$ on the ordinals $\xi \leq \theta$. We need to provide the following two procedures

A) A $\theta$-primitive recursive procedure which decides for $a \in L_\theta^*$ whether $a \in T(\theta)$.

B) A $\theta$-primitive recursive procedure which decides for non-identical $a, b \in T(\theta)$ whether $a \prec b$ or $b \prec a$.

We define A) and B) simultaneously by induction on the term complexity $Ga$.

For the base stage of A) we have $0, \Omega \in T(\theta)$ and $\Gamma_\xi \in T(\theta)$ for all $\xi \leq \theta$.

For the base stage of B) we have $0 \prec \Gamma_\xi \prec \Omega$ for all $\xi \leq \theta$ and the terms $\Gamma_\xi$ inherit the ordering from $\theta$, for which we have access to an oracle.

For the inductive stage of A) we require the following 3 things:

A1) A $\theta$-primitive recursive procedure that on input $a_1, \ldots, a_n \in T(\theta)$ decides whether $a_1 + \ldots + a_n \in T(\theta)$.

A2) A $\theta$-primitive recursive procedure that on input $a_1, a_2 \in T(\theta)$ decides whether $\varphi a_1 a_2 \in T(\theta)$.

A3) A $\theta$-primitive recursive procedure that on input $a \in T(\theta)$ decides whether $\psi a \in T(\theta)$.

For A1) we need to decide if $n > 1$ and if $a_1 \geq \ldots \geq a_n$, which we can do by the induction hypothesis. We also need to decide if $a_1, \ldots, a_n$ are additive principal; all terms other than those of the form $b_1 + \ldots + b_m$ ($m > 1$) and 0 are additive principal.
For A2), first let $ORD_\theta$ denote the set of $L_\theta$ strings which represent an ordinal (not necessarily in normal form), i.e. each function symbol has the correct arity. Next we define the set of strings which correspond to the strongly critical ordinals, where $\equiv$ signifies identity of strings.

$$SC_\theta := \{\Omega\} \cup \{\Gamma_\xi : \xi \leq \theta\} \cup \{a \in ORD_\theta : a \equiv \psi b\}$$

We may decide membership of $SC_\theta$ in a $\theta$-primitive recursive fashion. For the decision procedure we split into cases based upon the form of $a_2$:

i) If $a_2 \equiv 0$ then $\varphi a_1 a_2 \in T(\theta)$ whenever $a_1 \notin SC_\theta$

ii) If $a_2 \in SC_\theta$ then $\varphi a_1 a_2 \in T(\theta)$ whenever $a_1 \succeq a_2$ and $a_2 \neq \Omega$.

iii) If $a_2 \succ \Omega$ then $\varphi a_1 a_2 \in T(\theta)$ exactly when $a_1 = 0$.

iv) If $a_2 \equiv b_1 + \ldots + b_n < \Omega$, with $n > 1$ then $\varphi a_1 a_2 \in T(\theta)$ regardless of the form of $a_1$.

v) If $a_2 \equiv \varphi b_1 b_2 < \Omega$ then $\varphi a_1 a_2 \in T(\theta)$ whenever $a_1 \succeq b_1$.

For a rigorous treatment of the $\varphi$ function see [62].

The function $K$ from Definition 2.11 lifts to a $\theta$-primitive recursive function on $T(\theta)$. Moreover every $b \in Ka$ is a member of $T(\theta)$ of lower complexity than $a$. Owing to Lemma 2.12 for the decision procedure A3) we may first compute $Ka$, then check whether $(\forall b \in Ka)(b \prec a)$, which we may do by the induction hypothesis.

Finally for the inductive stage of B), given two elements of $T(\theta)$ we may decide their ordering using the following procedure.

B1) $0 \prec a$ for every $a \neq 0$.

B2) $\Gamma_\xi \prec \Omega$ for every $\xi \leq \theta$.

B3) The elements $\Gamma_\xi$ inherit the ordering from $\theta$.

B4) If $a \in SC_\theta$ or $a \equiv \varphi bc$ then $a_1 + \ldots + a_n \prec a$ if $a_1 \prec a$.

B5) If $a \in SC_\theta$ then $\varphi bc \prec a$ if $b, c \prec a$.

B6) $\psi b \prec \Omega$ for all $b$.

B7) $\psi a \succ \Gamma_\xi$ for all $\xi \leq \theta$.

B8) $a_1 + \ldots + a_n \prec b_1 + \ldots + b_m$ if $n < m$ and $(\forall i \leq n)[a_i \equiv b_i]$ or $\exists i \leq \min(n, m)[\forall j < i(a_j = b_j) \text{ and } a_i < b_i]$.

B9) $\varphi a_1 b_1 \prec \varphi a_2 b_2$ if $a_1 \prec a_2$ and $b_1 \prec \varphi a_2 b_2$ or $a_1 = a_2$ and $b_1 \prec b_2$ or $a_2 < a_1$ and $\varphi a_1 b_1 < b_2$.

B10) $\psi a \prec \psi b$ if $a \prec b$. \qed
3 The proof theory of $\text{RS}_\Omega(X)$

3.1 A Tait-style sequent calculus formulation of KP

**Definition 3.1.** The language of KP consists of free variables $a_0, a_1, \ldots$, bound variables $x_0, x_1, \ldots$, the binary predicate symbols $\in, \notin$ and the logical symbols $\lor, \land, \forall, \exists$ as well as parentheses $),($.

The atomic formulas are those of the form

\[(a \in b), (a \notin b)\]

The formulas of KP are defined inductively by:

i) Atomic formulas are formulas.

ii) If $A$ and $B$ are formulas then so are $A \lor B$ and $A \land B$.

iii) If $A(x)$ is a formula in which the bound variable $x$ does not occur, then $\forall x A(x)$, $\exists x A(x)$, $(\forall x \in a) A(x)$ and $(\exists x \in a) A(x)$ are all formulas.

Quantifiers of the form $\exists x$ and $\forall x$ will be called unbounded and those of the form $(\exists x \in a)$ and $(\forall x \in a)$ will be referred to as bounded quantifiers.

A formula is said to be $\Delta_0$ if it contains no unbounded quantifiers. A formula is said to be $\Sigma$ (II) if it contains no unbounded universal (existential) quantifiers.

The negation $\neg A$ of a formula $A$ is obtained from $A$ by undergoing the following operations:

i) Replacing every occurrence of $\in, \notin$ with $\notin, \in$ respectively.

ii) Replacing any occurrence of $\land, \lor, \forall x, (\forall x \in a), (\exists x \in a)$ with $\lor, \land, \exists x, \forall x, (\exists x \in a), (\forall x \in a)$ respectively.

Thus the negation of a formula $A$ is in negation normal form. The expression $A \rightarrow B$ will be considered shorthand for $\neg A \lor B$.

The expression $a = b$ is to be treated as an abbreviation for $(\forall x \in a)(x \in b) \land (\forall x \in b)(x \in a)$.

The derivations of KP take place in a Tait-style sequent calculus, finite sets of formulae denoted by Greek capital letters are derived. Intuitively the sequent $\Gamma$ may be read as the disjunction of formulae occurring in $\Gamma$.

The axioms of KP are:
Logical axioms: $\Gamma, A, \neg A$ for any formula $A$.

Extensionality: $\Gamma, a = b \land B(a) \rightarrow B(b)$ for any formula $B(a)$.

Pair: $\Gamma, \exists z(a \in z \land b \in z)$.

Union: $\Gamma, \exists z(\forall x \in y)(\forall z \in y)(x \in z)$.

$\Delta_0$-Separation: $\Gamma, \exists y[(\forall x \in y)(x \in a \land B(x)) \land (\forall x \in a)(B(x) \rightarrow x \in y)]$ for any $\Delta_0$-formula $B(a)$.

Set Induction: $\Gamma, \forall x[(\forall y \in x)F(y) \rightarrow F(x)] \rightarrow \forall xF(x)$ for any formula $F(a)$.

Infinity: $\Gamma, \exists x[(\exists z \in x)(z \in x) \land (\forall y \in x)(\exists z \in x)(y \in z)]$.

$\Delta_0$-Collection: $\Gamma, (\forall x \in a)\exists yG(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for any $\Delta_0$-formula $G$.

The rules of inference are

\[
\begin{align*}
(\land) & \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \land B} \\
(\lor) & \quad \frac{\Gamma, A}{\Gamma, A \lor B} \quad \frac{\Gamma, B}{\Gamma, A \lor B} \\
(b\exists) & \quad \frac{\Gamma, a \in b \land F(a)}{\Gamma, (\exists x \in b)F(x)} \\
(\exists) & \quad \frac{\Gamma, F(a)}{\Gamma, \exists xF(x)} \\
(b\forall) & \quad \frac{\Gamma, a \in b \rightarrow F(a)}{\Gamma, (\forall x \in b)F(x)} \\
(\forall) & \quad \frac{\Gamma, F(a)}{\Gamma, \forall xF(x)} \\
(Cut) & \quad \frac{\Gamma, A}{\Gamma, \neg A}
\end{align*}
\]

In both $(b\forall)$ and $(\forall)$, the variable $a$ must not be present in the conclusion, such a variable is referred to as the eigenvariable of the inference.

The minor formulae of an inference are those rendered prominently in the premises, the other formulae in the premises will be referred to as side formulae. The principal formula of an inference is the one rendered prominently in the conclusion. Note that the principal formula can also be a side formula of that inference, when this happens we say that there has been a contraction. The rule (Cut) has no principal formula.

Formally, bounded and unbounded quantifiers are treated as logically separate operations. However, it is important to know and ensure that they interact with one another as expected.

**Lemma 3.2.** The following are derivable within KP:

i) $(\forall x \in b)F(x) \leftrightarrow \forall x(x \in b \rightarrow F(x))$.

ii) $(\exists x \in b)F(x) \leftrightarrow \exists x(x \in b \land F(x))$.

**Proof.** We verify only i) as the proof of ii) is very similar. First note that $a \in b \land \neg F(a), a \in b \rightarrow F(a)$ is a logical axiom of KP, we have the following derivation in KP.
For any formula \( \forall \) we define
\[
(b \exists) \quad a \in b \land \neg F(a), a \in b \Rightarrow F(a)
\]
\[
(\forall) \quad (\exists x \in b) \neg F(x), a \in b \Rightarrow F(a)
\]
\[
(\forall) \quad (\forall x \in b) \neg F(x), \forall x \in b \Rightarrow F(x)
\]
\[
(\forall) \quad (\forall x \in b) F(x) \rightarrow \forall x \in b \Rightarrow F(x)
\]
\[
(\forall) \quad \forall x \in b \Rightarrow F(x) \leftrightarrow \forall x \in b \Rightarrow F(x)
\]

### 3.2 The infinitary system RS\(\Omega\)(\(X\))

Let \( X \) be an arbitrary (well founded) set and let \( \theta \) be the set-theoretic rank of \( X \) (hereby referred to as the \( \varepsilon \)-rank). Henceforth all ordinals are assumed to belong to the ordinal notation system \( T(\theta) \) developed in the previous section. The system \( \text{RS}_\Omega(X) \) will be an infinitary proof system based on \( L_\Omega(X) \); the relativised constructible hierarchy up to \( \Omega \).

#### Definition 3.3.

We give an inductive definition of the set \( \mathcal{T} \) of \( \text{RS}_\Omega(X) \) terms, to each term \( t \in \mathcal{T} \) we assign an ordinal level \( | t | \)

1. For every \( u \in \text{TC}(\{X\}) \), \( \bar{u} \in \mathcal{T} \) and \( | \bar{u} | := \Gamma_{\text{rank}(u)} \) [here rank(\( u \)) is the \( \varepsilon \)-rank of \( u \) and TC denotes the transitive closure operator.] Note that rank(\( u \)) \( \leq \theta \).

2. For every \( \alpha < \Omega \), \( \mathbb{L}_\alpha(X) \in \mathcal{T} \) and \( | \mathbb{L}_\alpha(X) | := \Gamma_{\theta+1} + \alpha \).

3. If \( \alpha < \Omega \), \( A(a_1, b_1, \ldots, b_n) \) is a formula of KP with all free variables displayed and \( s_1, \ldots, s_n \) are terms with levels less than \( \Gamma_{\theta+1} + \alpha \) then

\[
[x \in \mathbb{L}_\alpha(X)] A(x, s_1, \ldots, s_n)^{\mathbb{L}_\alpha(X)}
\]

is a term of level \( \Gamma_{\theta+1} + \alpha \). Here the superscript \( \mathbb{L}_\alpha(X) \) indicates that all unbounded quantifiers occurring in \( A \) are replaced by quantifiers bounded by \( \mathbb{L}_\alpha(X) \).

The terms of \( \text{RS}_\Omega(X) \) are to be viewed as purely formal, syntactic objects. However their names are highly suggestive of the intended interpretation in the relativised constructible hierarchy up to \( \Omega \).

#### Definition 3.4.

The formulae of \( \text{RS}_\Omega(X) \) are of the form \( A(s_1, \ldots, s_n) \), where \( A(a_1, \ldots, a_n) \) is a formula of KP with all free variables displayed and \( s_1, \ldots, s_n \) are \( \text{RS}_\Omega(X) \) terms.

Formulae of the form \( \bar{u} \in \bar{v} \) and \( \bar{u} \notin \bar{v} \) will be referred to as basic. The properties \( \Delta_0 \), \( \Sigma \) and \( \Pi \) are inherited from KP formulae.

Note that the system \( \text{RS}_\Omega(X) \) does not contain free variables.

For the remainder of this section we shall refer to \( \text{RS}_\Omega(X) \) terms and formulae simply as terms and formulae.

For any formula \( A \) we define
\[
k(A) := \{| t | \mid t \text{ occurs in } A, \text{ subterms included} \}
\]
\[
\cup \{ \Omega \mid \text{if } A \text{ contains an unbounded quantifier} \}.
\]
If $\Gamma$ is a finite set of the $\text{RS}_\Omega(X)$ formulae $A_1, \ldots, A_n$ then we define

$$k(\Gamma) := k(A_1) \cup \ldots \cup k(A_n).$$

**Abbreviations 3.5.**

i) For $\text{RS}_\Omega(X)$ terms $s$ and $t$, the expression $s = t$ will be considered as shorthand for

$$(\forall x \in s)(x \in t) \land (\forall x \in t)(x \in s).$$

ii) If $|s| < |t|$, $A(s,t)$ is an $\text{RS}_\Omega(X)$ formula and $\Diamond$ is a propositional connective we define:

$$s \Diamond t \iff A(s,t) := \begin{cases} 
  s \in t \land A(s,t) & \text{if } t \equiv \bar{u} \\
  A(s,t) & \text{if } t \equiv L_\alpha(X) \\
  B(s) \land A(s,t) & \text{if } t \equiv [x \in L_\alpha(X) \mid B(x)] 
\end{cases}$$

Our aim will be to remove cuts from certain $\text{RS}_\Omega(X)$ derivations of $\Sigma$ sentences. In order to do this we need to express a certain kind of uniformity in infinite derivations. The right tool for expressing this uniformity was developed by Buchholz in [11] and is termed *operator control*.

**Definition 3.6.** Let $\mathcal{P}($ON$) := \{ Y : Y$ is a set of ordinals $\}$. A class function

$$\mathcal{H} : \mathcal{P}($ON$) \to \mathcal{P}($ON$)$$

is called an Operator if the following conditions are satisfied for $Y, Y' \in \mathcal{P}($ON$)$

(H1) $0 \in \mathcal{H}(Y)$ and $\Gamma_\beta \in \mathcal{H}(Y)$ for any $\beta \leq \theta + 1$.

(H2) If $\alpha = \text{NF} \alpha_1 + \ldots + \alpha_n$ then $\alpha \in \mathcal{H}(Y)$ iff $\alpha_1, \ldots, \alpha_n \in \mathcal{H}(Y)$.

(H3) If $\alpha = \text{NF} \varphi \alpha_1 \alpha_2$ then $\alpha \in \mathcal{H}(Y)$ iff $\alpha_1, \alpha_2 \in \mathcal{H}(Y)$

(H4) $Y \subseteq \mathcal{H}(Y)$

(H5) $Y' \subseteq \mathcal{H}(Y) \Rightarrow \mathcal{H}(Y') \subseteq \mathcal{H}(Y)$

Note that this definition of operator, as with the infinitary system $\text{RS}_\Omega(X)$ is dependent on the set $X$ and its $\in$-rank $\theta$.

**Abbreviations 3.7.** For an operator $\mathcal{H}$:

i) We write $\alpha \in \mathcal{H}$ instead of $\alpha \in \mathcal{H}(\emptyset)$.

ii) Likewise $Y \subseteq \mathcal{H}$ is shorthand for $Y \subseteq \mathcal{H}(\emptyset)$.

iii) For any $\text{RS}_\Omega(X)$ term $t$, $\mathcal{H}([t](Y)) := \mathcal{H}(Y \cup |t|)$.

iv) If $\exists$ is an $\text{RS}_\Omega(X)$ formula or set of formulae then $\mathcal{H}([\exists](Y)) := \mathcal{H}(Y \cup k(\exists))$.

**Lemma 3.8.** Let $\mathcal{H}$ be an operator $s$ an $\text{RS}_\Omega(X)$ term and $\exists$ an $\text{RS}_\Omega(X)$ formula or set of formulae.
i) If $Y \subseteq Y'$ then $\mathcal{H}(Y) \subseteq \mathcal{H}(Y')$.

ii) $\mathcal{H}[s]$ and $\mathcal{H}[\bar{x}]$ are operators.

iii) If $|s| \in \mathcal{H}$ then $\mathcal{H}[s] = \mathcal{H}$.

iv) If $k(\bar{x}) \subseteq \mathcal{H}$ then $\mathcal{H}[\bar{x}] = \mathcal{H}$.

Proof. These results are easily checked, they are proved in full in [50].

\[\square\]

**Definition 3.9.** If $\mathcal{H}$ is an operator, $\alpha$ an ordinal and $\Gamma$ a finite set of $\text{RS}_\Omega(X)$-formulae, we give an inductive definition of the relation $\mathcal{H} \vdash^\alpha \Gamma$ by recursion on $\alpha$. ($\mathcal{H}$-controlled derivability in $\text{RS}_\Omega(X)$.)

We require always that

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}$$

this condition will not be repeated in the inductive clauses pertaining to the axioms and inference rules below. We have the following axioms:

$$\mathcal{H} \vdash^\alpha \Gamma, \bar{u} \in \bar{v} \quad \text{if} \quad u, v \in TC(X) \text{ and } u \in v$$

$$\mathcal{H} \vdash^\alpha \Gamma, \bar{u} \notin \bar{v} \quad \text{if} \quad u, v \in TC(X) \text{ and } u \notin v.$$

The following are the inference rules of $\text{RS}_\Omega(X)$, the column on the right gives the requirements on the ordinals, terms and formulae for each rule.

\[(\land)\quad \frac{\mathcal{H} \vdash^\alpha_0 \Gamma, A \quad \mathcal{H} \vdash^\alpha_1 \Gamma, B}{\mathcal{H} \vdash^\alpha \Gamma, A \land B} \quad \alpha_0, \alpha_1 < \alpha\]

\[(\lor)\quad \frac{\mathcal{H} \vdash^\alpha \Gamma, C \quad \text{for some } C \in \{A, B\}}{\mathcal{H} \vdash^\alpha \Gamma, A \lor B} \quad \alpha_0 < \alpha\]

\[(\notin)\quad \frac{\mathcal{H}[s] \vdash^\alpha_0 \Gamma, s \in t \to r \neq s \quad \text{for all } |s| < |t|}{\mathcal{H} \vdash^\alpha \Gamma, r \notin t} \quad \alpha_s < \alpha \quad r \in t \text{ is not basic}\]

\[(\in)\quad \frac{\mathcal{H} \vdash^\alpha_0 \Gamma, s \in t \land r = s}{\mathcal{H} \vdash^\alpha \Gamma, r \in t} \quad |s| < |t| \quad |s| < \Gamma_{\theta+1} + \alpha \quad r \in t \text{ is not basic}\]

\[(b\forall)\quad \frac{\mathcal{H}[s] \vdash^\alpha_0 \Gamma, s \in t \to A(s) \quad \text{for all } |s| < |t|}{\mathcal{H} \vdash^\alpha \Gamma, (\forall x \in t)A(x)} \quad \alpha_s < \alpha\]

\[(b\exists)\quad \frac{\mathcal{H} \vdash^\alpha_0 \Gamma, s \in t \land A(s)}{\mathcal{H} \vdash^\alpha \Gamma, (\exists x \in t)A(x)} \quad \alpha_0 < \alpha \quad |s| < |t| \quad |s| < \Gamma_{\theta+1} + \alpha\]
\[ \forall \alpha_{1} < \alpha \]

\[ \exists \alpha \leq \alpha \]

\[ \text{Cut} \]

\[ \Sigma - \text{Ref}_{\Delta}(X) \]

\[ A^z \text{ results from } A \text{ by restricting all unbounded quantifiers in } A \text{ to } z. \]

The reason for the condition preventing the derivation of basic formulas in the rules (\(\varepsilon\)) and (\(\notin\)) is to prevent derivations of sequents which are already axioms, as this would cause a hindrance to cut-elimination. The condition that \(|s| < \Gamma_{\theta+1} + \alpha\) in (\(\varepsilon\)) and (\(\exists\)) inferences will allow us to place bounds on the location of witnesses in derivable \(\Sigma\) formulas.

### 3.3 Cut elimination for RS\(\Omega\)(\(X\))

We need to keep track of the complexity of cuts appearing in a derivation, to this end we define the rank of an RS\(\Omega\)(\(X\)) formula.

**Definition 3.10.** The rank of a term or formula is defined by recursion on the construction as follows:

1. \(rk(\bar{u}) := \Gamma_{\text{rank}(u)}\)
2. \(rk(\exists \alpha(X)) := \Gamma_{\theta+1} + \omega \cdot \alpha\)
3. \(rk([x \in \exists \alpha(X)F(x)]) := \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, rk(F(\bar{0})) + 2)\)
4. \(rk(s \in t) := \max(rk(t) + 1, rk(s) + 6)\)
5. \(rk((\exists x \in \bar{u})F(x)) := \max(rk((\forall x \in \bar{u})F(x)) + 3, rk(F(\bar{0})) + 2))\)
6. \(rk((\exists x \in t)F(x)) := \max(rk((\forall x \in t)F(x)) + 2) \text{ if } t \text{ is not of the form } \bar{u}.\)
7. \(rk(\forall x F(x)) := \max(\Omega, rk(F(\bar{0})) + 1)\)
8. \(rk(A \land B) := \max(rk(A), rk(B)) + 1\)

\(\mathcal{H} \vdash_{\rho} \Gamma\) will be used to denote that \(\mathcal{H} \vdash_{\rho} \Gamma\) and all cut formulas appearing in the derivation have rank \(< \rho.\)

**Observation 3.11. i)** For each term \(t, rk(t) = \omega \cdot |t| + n \text{ for some } n < \omega.\)
ii) For each formula $A$, $rk(A) = \omega \cdot \max(k(A)) + n$ for some $n < \omega$.

iii) $rk(A) < \Omega$ if and only if $A$ is $\Delta_0$.

The next Lemma shows that the rank of a formula $A$ is determined only by $\max(k(A))$ and the logical structure of $A$.

**Lemma 3.12.** For each formula $A(s)$, if $|s| < \max(k(A(s)))$ then $rk(A(s)) = rk(A(\emptyset))$.

**Proof.** The proof is by induction on the complexity of $A$.

Case 1. If $A(s) \equiv s \in t$ then by assumption $|s| < |t|$, so $rk(A(s)) = rk(t) + 1 = rk(A(\emptyset))$.

Case 2. If $A(s) \equiv t \in s$ we may argue in a similar fashion to Case 1.

Case 3. It cannot be the case that $A(s) \equiv s \in s$.

Case 4. If $A(s) \equiv (\exists y \in \bar{u})B(y, s)$ then

$$rk(A(s)) = \max(rk(\bar{u}) + 3, rk(B(\emptyset, s)) + 2)$$

and

$$rk(A(\emptyset)) = \max(rk(\bar{u}) + 3, rk(B(\bar{u}, \emptyset)) + 2).$$

4.1 If $|\bar{u}| > \max(k(B(\emptyset, \emptyset)))$ then $|s| < |\bar{u}|$ by assumption, so using observation 3.11ii) gives us

$$rk(A(s)) = rk(\bar{u}) + 3 = rk(A(\emptyset)).$$

4.2 If $|\bar{u}| \leq \max(k(B(\emptyset, \emptyset)))$ then $|s| < \max(k(B(\emptyset, \emptyset)))$ by assumption, so by induction hypothesis

$$rk(B(\emptyset, s)) = rk(B(\emptyset, \emptyset))$$

and hence using Observation 3.11i) gives us

$$rk(A(s)) = rk(B(\emptyset, \emptyset)) + 2 = rk(A(\emptyset)).$$

Case 5. If $A(s) \equiv (\exists y \in t)B(y, s)$ for some $t$ not of the form $\bar{u}$, we may argue in a similar way to case 4.

Case 6. $A(s) \equiv (\exists y \in s)B(y, s)$, now $|s| < \max(k(A(\emptyset))) = \max(k(B(\emptyset, \emptyset)))$, so by induction hypothesis

$$rk(B(\emptyset, s)) = rk(B(\emptyset, \emptyset))$$

and hence using observation 3.11i) we see that

$$rk(A(s)) = rk(B(\emptyset, s)) + 2$$

$$= rk(B(\emptyset, \emptyset)) + 2$$

$$= rk(A(\emptyset)).$$

Case 7. If $A(s) \equiv \exists x B(x, s)$ then by assumption $|s| < \max(k(A(s))) = \max(k(B(\emptyset, s)))$ so we may apply the induction hypothesis to see that $rk(A(s)) = \max(\Omega, rk(B(\emptyset, s)) + 1) = \max(\Omega, rk(B(\emptyset, \emptyset)) + 1)$. 





1) = \text{rk}(A(\emptyset)).

Case 8. All other cases are either propositional in which case we may just use the induction hypothesis directly or are dual to cases already considered. □

**Definition 3.13.** To each non-basic formula \( A \) we assign an infinitary disjunction \((\lor A_i)_{i \in y}\) or conjunction \((\land A_i)_{i \in y}\) as follows:

1. \( r \in t \) := \( \lor s \mid s \in t \land r = s \mid s < |t| \) provided \( r \in t \) is not a basic formula.
2. \( (\exists x \in t)B(x) \) := \( \lor (s \mid s \in t \land B(s)) \mid s < |t| \)
3. \( \exists xB(x) \) := \( \lor (B(s))_{s \in T} \)
4. \( B_0 \lor B_1 \) := \( \lor (B_i)_{i \in \{0,1\}} \)
5. \( \neg B \) := \( \land (\neg B_i)_{i \in y} \) if \( B \) is of the form considered in 1.-4.

The idea is that the infinitary conjunction or disjunction lists the premises required to derive \( A \) as the principal formula of an \( \text{RS}_\Omega(X) \)-inference different from \( (\Sigma\text{-Ref}_\Omega(X)) \) or \( (\text{Cut}) \).

**Lemma 3.14.** If \( A \simeq (\lor A_i)_{i \in y} \) or \( A \simeq (\land A_i)_{i \in y} \) then

\[ \forall i \in y(\text{rk}(A_i) < \text{rk}(A)) \]

**Proof.** We need only treat the case where \( A \simeq (\lor A_i)_{i \in y} \) since the other case is dual to this one. We proceed by induction on the complexity of \( A \).

Case 1. Suppose \( A \equiv r \in t \) then by assumption either \( r \) or \( t \) is not of the form \( \tilde{u} \), we split cases based on the form of \( t \).

1.1 If \( t \equiv \tilde{u} \) then \( r \) is not of the form \( \tilde{v} \) and \( \text{rk}(A) = \text{rk}(r) + 6 \). In this case \( A_i \equiv \tilde{v} \in \tilde{u} \land \tilde{v} = r \) for some \( |\tilde{v}| < |\tilde{u}| \) and we have

\[
\text{rk}(A_i) = \max(\text{rk}(\tilde{v} \in \tilde{u}), \text{rk}(\tilde{v} = r)) + 1 \\
= \text{rk}(\tilde{v} = r) + 1 \\
= \max(\text{rk}(\forall x \in \tilde{v})(x \in r), \text{rk}(\forall x \in r)(x \in \tilde{v})) + 2 \\
= \text{rk}(r) + 5 < \text{rk}(r) + 6 = \text{rk}(A)
\]

1.2 If \( t \equiv \mathbb{L}_\alpha(X) \) then \( A_i \equiv s \in r \) for some \( |s| < |t| \). So we have

\[
\text{rk}(A_i) = \text{rk}(\forall x \in s)(x \in r) \land (\forall x \in r)(x \in s)) \\
= \max(\text{rk}(s) + 4, \text{rk}(r) + 4) \\
< \max(\text{rk}(r) + 1, \text{rk}(t) + 6) = \text{rk}(A)
\]

1.3 If \( t \equiv [x \in \mathbb{L}_\alpha(X)|B(x)] \) then \( A_i \equiv B(s) \land s = r \) for some \( |s| < |t| \). So we have

\[
\text{rk}(A_i) = \max(\text{rk}(B(s)) + 1, \text{rk}(r = s) + 1).
\]
First note that $rk(r = s) + 1 = \max(rk(s) + 5, rk(r) + 5) < rk(A)$. So it remains to verify that
$rk(B(s)) + 1 < rk(A)$, for this it is enough to show that $rk(B(s)) < rk(t)$.

1.3.1 If $max(k(B(s))) \leq |s|$ then by Observation 3.11(i) we have $rk(B(s)) + 1 < \omega \cdot |s| + \omega \leq rk(t)$.

1.3.2 Otherwise $max(k(B(s))) > |s|$ then by Lemma 3.12 we have

$$rk(B(s)) + 1 = rk(B(\emptyset)) + 1$$

$$< \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, rk(B(\emptyset)) + 2) = rk(t)$$

Case 2. Suppose $A \equiv (\exists x \in t)B(x)$, we split into cases based on the form of $t$.

2.1 If $t \equiv \bar{u}$ then $rk(A) := \max(rk(\bar{u}) + 3, rk(B(\emptyset)) + 2)$. In this case $A_i \equiv \bar{v} \in \bar{u} \land B(\bar{v})$ for some $|\bar{v}| < |\bar{u}|$, so we have

$$rk(A_i) = \max(rk(\bar{u}) + 2, rk(B(\bar{v})) + 1).$$

Clearly $rk(\bar{u}) + 2 < rk(\bar{u}) + 3$ so it remains to verify that $rk(B(\bar{v})) + 1 < rk(A)$

2.1.1 If $|\bar{v}| \geq max(k(B(\bar{v})))$ then by Observation 3.11 $rk(B(\bar{v})) + 1 < rk(\bar{u}) < rk(\bar{u}) + 3$.

2.1.2 If $|\bar{v}| < max(k(B(\bar{v})))$ then by Lemma 3.12 $rk(B(\bar{v})) + 1 = rk(B(\emptyset)) + 1 < rk(B(\emptyset)) + 2$.

2.2 Now suppose $t \equiv L_\alpha(X)$, so $rk(A) = \max(rk(t), rk(B(\emptyset)) + 2)$. In this case $A_i = B(s)$ for some $|s| < |t|$.

2.2.1 If $|s| \geq max(k(B(s)))$ then $rk(B(s)) < rk(t)$ by Observation 3.11.

2.2.2 If $|s| < max(k(B(s)))$ then by Lemma 3.12 $rk(B(s)) = rk(B(\emptyset)) < rk(A)$.

2.3. Now suppose $t \equiv [y \in L_\alpha(X) \mid C(y)]$, so we have

$$rk(A) := \max(rk(t), rk(B(\emptyset)) + 2)$$

$$= \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, rk(C(\emptyset)) + 2, rk(B(\emptyset)) + 2).$$

In this case $A_i \equiv C(s) \land B(s)$ for some $|s| < |t|$.

2.3.1 If $|s| < max(k(B(s)))$ then $rk(B(s)) + 1 = rk(B(\emptyset)) + 1 < rk(B(\emptyset)) + 2$. It remains to show that $rk(C(s)) < rk(A)$.

2.3.1.1 If $max(k(C(s))) < |t|$ then $rk(C(s)) + 1 < rk(t)$ by Observation 3.11.

2.3.1.2 Now if $max(k(C(s))) \geq |t|$ then we may apply Lemma 3.12 to give

$$rk(C(s)) + 1 = rk(C(\emptyset)) + 1 < rk(C(\emptyset)) + 2 \leq rk(A).$$

2.3.2 If $|s| \geq max(k(B(s)))$ then $rk(B(s)) < \Gamma_{\theta+1} + \omega \cdot \alpha$ by Observation 3.11. Now we may apply the same argument as in 2.3.1.1 and 2.3.1.2 to yield $rk(C(s)) + 1 < rk(A).$
Case 3. If \( A \equiv \exists x B(x) \) then \( rk(A) := \max(\Omega, rk(B(\emptyset)) + 1) \). In this case \( A_i \equiv B(s) \) for some term \( s \).

3.1 If \( B \) contains an unbounded quantifier then by Lemma \[3.12\] \( rk(B(s)) = rk(B(\emptyset)) < rk(A) \).

3.2 If \( B \) does not contain an unbounded quantifier then \( rk(B(s)) < \Omega \) by Observation \[3.11\] iii)

Case 4. If \( A \equiv B \lor C \) then the result is clear immediately from the definition of \( rk(A) \).

\[ \square \]

Lemma 3.15. Let \( \mathcal{H} \) be an arbitrary operator.

i) If \( \alpha \leq \alpha' \in \mathcal{H}, \rho \leq \rho', k(\Gamma') \subseteq \mathcal{H} \) and \( \mathcal{H} \models_\rho^\alpha \Gamma \) then \( \mathcal{H} \models_\rho^\alpha' \Gamma, \Gamma' \).

ii) If \( C \) is a basic formula which holds true in the set \( X \) and \( \mathcal{H} \models_\rho^\alpha \Gamma, \neg C \) then \( \mathcal{H} \models_\rho^\alpha \Gamma \).

iii) If \( \mathcal{H} \models^\alpha \Gamma, A \lor B \) then \( \mathcal{H} \models_\rho^\alpha \Gamma, A, B \).

iv) If \( A \simeq \bigwedge(A_i)_{i \in y} \) and \( \mathcal{H} \models^\alpha \Gamma, A \) then \((\forall i \in y) \mathcal{H}[i] \models_\rho^\alpha \Gamma, A_i \).

v) If \( \gamma \in \mathcal{H} \) and \( \mathcal{H} \models_\rho^\alpha \Gamma, \forall x F(x) \) then \( \mathcal{H} \models_\rho^\alpha \Gamma, (\forall x \in L_{\gamma}(X)) F(x) \).

Proof. All proofs are by induction on \( \alpha \).

i) If \( \Gamma \) is an axiom then \( \Gamma, \Gamma' \) is also an axiom, and since \( \{\alpha'\} \cup k(\Gamma') \subseteq \mathcal{H} \) there is nothing to show.

Now suppose \( \Gamma \) is the result of an inference

\[
\begin{array}{l}
\mathcal{H} \models_\rho^\alpha \Gamma_i \ldots \\
\hline
\mathcal{H} \models_\rho^\alpha \Gamma
\end{array}
\]

\((i \in y) \quad \alpha_i < \alpha \)

Using the induction hypothesis we have

\[
\begin{array}{l}
\mathcal{H} \models_{\rho'}^\alpha \Gamma_i, \Gamma_i' \ldots \\
\hline
\mathcal{H} \models_{\rho'}^\alpha \Gamma_i, \Gamma'
\end{array}
\]

\((i \in y) \quad \alpha_i < \alpha \)

It’s worth noting that \( k(\Gamma') \subseteq \mathcal{H}_i \), since \( \mathcal{H}_i(\emptyset) \supseteq \mathcal{H}(\emptyset) \), this can be observed by looking at each inference rule.

Finally we may apply the inference (I) again to obtain

\[ \mathcal{H} \models_\rho^\alpha' \Gamma, \Gamma' \]

as required.

ii) If \( \Gamma, \neg C \) is an axiom then so is \( \Gamma \) so there is nothing to show.

Now suppose \( \Gamma, \neg C \) was derived as the result of an inference rule (I), then \( \neg C \) cannot have been the principal formula since it is basic so we have the premise(s)

\[ \mathcal{H}_i \models_\rho^\alpha_i \Gamma, \neg C \quad \alpha_i < \alpha. \]
Now by induction hypothesis we obtain
\[ H_i \models_{\propto} \Gamma_i \quad \alpha_i < \alpha \]
to which we may apply the inference rule (I) to complete the proof.

iii) If \( \Gamma, A \lor B \) is an axiom then \( \Gamma, A, B \) is also an axiom. If \( A \lor B \) was not the principal formula of the last inference then we can apply the induction hypothesis to its premises and then the same inference again.

Now suppose that \( A \lor B \) was the principal formula of the last inference. So we have
\[ H \models_{\propto} \Gamma, C \quad \text{or} \quad H \models_{\propto} \Gamma, C, A \lor B \quad \text{where} \quad C \in \{ A, B \} \quad \text{and} \quad \alpha_0 < \alpha \]
By i) we may assume that we are in the latter case. By the induction hypothesis, and a contraction, we obtain
\[ H \models_{\propto} \Gamma, A, B \]
Finally using i) yields
\[ H \models_{\propto} \Gamma, A, B . \]

iv) If \( \Gamma, A \) is an axiom, then \( \Gamma \) is also an axiom since \( A \) cannot be the active part of an axiom, so \( \Gamma, A \) is an axiom for any \( i \in y \). If \( A \) was not the principal formula of the last inference then we may apply the induction hypothesis to its premises and then use that inference again.

Now suppose \( A \) was the principal formula of the last inference. With the possible use of part i), we may assume we are in the following situation:
\[ H[i] \models_{\propto} \Gamma, A, A_i \quad (\forall i \in y) \quad \alpha_i < \alpha . \]
Inductively and via a contraction we obtain
\[ H[i] \models_{\propto} \Gamma, A_i . \]
Here it is important to note that \( H[i][i] \equiv H[i] \). To which we may apply part i) to obtain
\[ H[i] \models_{\propto} \Gamma, A_i \]
as required.

v) The interesting case is where \( \forall x F(x) \) was the principal formula of the last inference. In this case we may assume we are in the following situation:
(1) \[ H[s] \models_{\propto} \Gamma, \forall x F(x), F(s) \quad \text{for all terms} \quad s, \text{with} \quad \alpha_s < \alpha . \]
Using the induction hypothesis yields
(2) \[ H[s] \models_{\propto} \Gamma, (\forall x \in L_\gamma(X)) F(x), F(s) \]
Note that for \( |s| < \Gamma_{\gamma+1} + \gamma \) we have \( s \in L_\gamma(X) \rightarrow F(s) \equiv F(s) \). So as a subset of (2) we have
\[ H[s] \models_{\propto} \Gamma, (\forall x \in L_\gamma(X)) F(x), s \in L_\gamma(X) \rightarrow F(s) \quad \text{for all} \quad |s| < \Gamma_{\gamma+1} + \gamma, \text{with} \quad \alpha_s < \alpha . \]
From which one application of (b\forall) gives us the desired result. \( \square \)
Lemma 3.16 (Reduction for RS_Ω(X)). Suppose \( C \equiv \bar{u} \in \bar{v} \) or \( C \simeq \bigvee_{i \in y}(C_i) \) and \( rk(C) := \rho \neq \Omega \).

If \( [\mathcal{H} \vdash^\alpha \Lambda, \neg C \quad \& \quad \mathcal{H} \vdash^\beta \Gamma, C] \) then \( \mathcal{H} \vdash^\alpha+\beta \Lambda, \Gamma \).

Proof. If \( C \equiv \bar{u} \in \bar{v} \) then by Lemma 3.15(ii) we have either \( \mathcal{H} \vdash^\alpha \Lambda \) or \( \mathcal{H} \vdash^\beta \Gamma \). Hence using Lemma 3.15(i) we obtain \( \mathcal{H} \vdash^\alpha+\beta \Lambda, \Gamma \) as required.

Now suppose \( C \simeq \bigvee_{i \in y}(C_i) \), we proceed by induction on \( \beta \). We have

(1) \( \mathcal{H} \vdash^\alpha \Lambda, \neg C \)

(2) \( \mathcal{H} \vdash^\beta \Gamma, C \).

If \( C \) was not the principal formula of the last inference in (2), then we may apply the induction hypothesis to the premises of that inference and then the same inference again. Now suppose \( C \) was the principal formula of the last inference in (2). If \( B \) was the principal formula of the inference (\( \Sigma\text{-Ref}_\Omega(X) \)), then \( B \) is of the form \( \exists z F(s_1, \ldots, s_n) \), which implies \( rk(B) = \Omega \), therefore the last inference in (2) was not (\( \Sigma\text{-Ref}_\Omega(X) \)). So we have

(3) \( \mathcal{H} \vdash^{\beta_0} \Gamma, C, C_{i_0} \) for some \( i_0 \in y \), \( \beta_0 < \beta \) with \( |i_0| < \Gamma_{\theta+1} + \beta \).

The induction hypothesis applied to (2) and (3) yields

(4) \( \mathcal{H} \vdash^{\alpha+\beta_0} \Lambda, \Gamma, C_{i_0} \).

Now applying Lemma 3.15(v) to (1) provides

(5) \( \mathcal{H}[i_0] \vdash^\alpha \Lambda, \neg C_{i_0} \).

But \( |i_0| \in \mathcal{H} \) by (4), which means \( \mathcal{H}[i_0] = \mathcal{H} \) by Lemma 3.8(v), so in fact we have

(6) \( \mathcal{H} \vdash^\alpha \Lambda, \neg C_{i_0} \).

Thus we may apply (Cut) to (4) and (6) (noting that \( rk(C_{i_0}) < rk(C) := \rho \) by Lemma 3.14) to obtain

\( \mathcal{H} \vdash^{\alpha+\beta} \Lambda, \Gamma \)

as required. \( \square \)

Theorem 3.17 (Predicative cut elimination for RS_Ω(X)).

If \( \mathcal{H} \vdash^\beta \Gamma \) and \( \Omega \notin [\rho, \rho + \omega^\alpha) \) then \( \mathcal{H} \vdash^{\rho+\omega^\alpha} \Gamma \).

Proof. The proof is by main induction on \( \alpha \) and subsidiary induction on \( \beta \). If \( \Gamma \) is an axiom then the result is immediate. If the last inference was anything other than (Cut) we may apply the subsidiary induction hypothesis to its premises and then the same inference again. The crucial case is where the last inference was (Cut), so suppose there is a formula \( C \) with \( rk(C) < \rho + \omega^\alpha \) such that

(1) \( \mathcal{H} \vdash^{\beta_0} \Gamma, C \) with \( \beta_0 < \beta \).

(2) \( \mathcal{H} \vdash^{\beta_0} \Gamma, \neg C \) with \( \beta_0 < \beta \).
Applying the subsidiary induction hypothesis to (1) and (2) yields

\[ \mathcal{H} \vdash \varphi_{\alpha \beta} \rho \Gamma, C. \]
\[ \mathcal{H} \vdash \varphi_{\alpha \beta} \rho \Gamma, -C. \]

Case 1. If \( rk(C) < \rho \) then we may apply (Cut) to (3) and (4), noting that \( \varphi_{\alpha \beta} + 1 < \varphi_{\alpha \beta} \in \mathcal{H} \), to give the desired result.

Case 2. Now suppose \( rk(C) \in [\rho, \rho + \omega] \), so we may write \( rk(C) \) in the following form:

\[ rk(C) = \rho + \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \quad \text{with} \quad \alpha > \alpha_1 \geq \ldots \geq \alpha_n. \]

If \( n = 0 \), this means that \( rk(C) = \rho \). From (3) we know that \( k(C) \subseteq \mathcal{H} \) and thus \( rk(C) \in \mathcal{H} \). Now (5) and (H2) and (H3) from Definition 3.6 give us \( \alpha_1, \ldots, \alpha_n \in \mathcal{H} \). Since \( rk(C) \neq \Omega \) we may apply the Reduction Lemma 3.16 to (3) and (4) to obtain

\[ \mathcal{H} \vdash \varphi_{\alpha \beta} \rho + \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \Gamma. \]

Now \( \varphi_{\alpha \beta} + \varphi_{\alpha \beta} < \varphi_{\alpha \beta} \), so by Lemma 3.15(i) we have

\[ \mathcal{H} \vdash \varphi_{\alpha \beta} \rho + \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \Gamma. \]

Applying the main induction hypothesis (since \( \alpha_n < \alpha \)) to (7) gives

\[ \mathcal{H} \vdash \varphi_{\alpha_n}(\varphi_{\alpha \beta}) \rho + \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \Gamma. \]

But since \( \varphi_{\alpha \beta} \) is a fixed point of the function \( \varphi_{\alpha_n}(\cdot) \) we have

\[ \mathcal{H} \vdash \varphi_{\alpha \beta} \rho + \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \Gamma. \]

Now since \( \alpha_1, \ldots, \alpha_{n-1} < \alpha \) we may repeat this application of the main induction hypothesis a further \( n - 1 \) times to obtain

\[ \mathcal{H} \vdash \varphi_{\alpha \beta} \rho \Gamma \]

as required. \( \square \)

Lemma 3.18 (Boundedness for RS_\Omega(X)). If \( C \) is a \( \Sigma \) formula, \( \alpha \leq \beta < \Omega \), \( \beta \in \mathcal{H} \) and \( \mathcal{H} \vdash \omega \rho \Gamma, C \) then \( \mathcal{H} \vdash \omega \rho \Gamma, C^{L_\beta(X)} \).

Proof. The proof is by induction on \( \alpha \). If \( C \) is basic then \( C \equiv C^{L_\beta(X)} \) so there is nothing to show. If \( C \) was not the principal formula of the last inference then we may apply the induction hypothesis to its premises and then the same inference again. Now suppose \( C \) was the principal formula of the last inference. The last inference cannot have been (\( \Sigma \)-Ref_\Omega(X)) since \( \alpha < \Omega \).
Case 1. Suppose $C \simeq \bigwedge(C_i)_i \in y$ and $H[\|i\| \Gamma, C, C_i]$ with $\alpha_i < \alpha$. Since $C$ is a $\Sigma$ formula, there must be some $\eta \in H(\emptyset) \cap \Omega$ such that $(\forall s \in y)(|s| < \eta)$. Therefore $C^{\beta}(X) \simeq \bigwedge(C_i^{\beta}(X))_i \in y$. Now two applications of the induction hypothesis gives

$$H[\|i\| \Gamma, C, C_i]$$

to which we may apply the appropriate inference to gain the desired result.

Case 2. Now suppose $C \simeq \bigvee(C_i)_i \in y$ and $H(\alpha) \Gamma, C, C_i$, with $i_0 \in y$, $|i_0| < \Gamma_{\theta+1} + \alpha < \Gamma_{\theta+1} + \beta$. In this case $C^{\beta}(X) \simeq \bigvee(C_i)_i \in y'$ where either $y' = y$ or $y' = \{i \in y \mid |i| < \Gamma_{\theta+1} + \beta\}$. Now by assumption $|i_0| < \Gamma_{\theta+1} + \alpha < \Gamma_{\theta+1} + \beta$, so $i_0 \in y'$. Thus using the same inference again, or (b∃) in the case that the last inference was (∃), we obtain

$$H \Gamma, C^{\beta}(X)$$

as required. 

**Definition 3.19.** For each $\eta \in T(\theta)$ we define

$$H[\eta] : P(\text{ON}) \mapsto P(\text{ON})$$

$$H[\eta](Y) := \bigcap\{B(\alpha) \mid Y \subseteq B(\alpha) \text{ and } \eta < \alpha\}$$

**Lemma 3.20.** For any $\eta$, $H[\eta]$ is an operator.

**Proof.** We must verify the conditions (H1) - (H5) from Definition 3.6.

(H1) Clearly $0 \in H[\eta](Y)$ and $\{\Gamma_\beta \mid \beta \leq \theta\} \subseteq H[\eta](Y)$ since these belong in any of the sets $B(\alpha)$. It remains to note that $H[\eta](Y) \supseteq B(1)$ and since $\Gamma_{\theta+1} = \psi(1) \in B(1)$ we have $\Gamma_{\theta+1} \in H[\eta](Y)$.

(H2) and (H3) follow immediately from Lemma 2.8(i) and ii) respectively.

(H4) is clear from the definition. Now for (H5) suppose $Y' \subseteq H[\eta](Y)$, then $Y' \subseteq B(\alpha)$ for every $\alpha$ such that $\eta < \alpha$ and $Y \subseteq B(\alpha)$. It follows that $H[\eta](Y') \subseteq H[\eta](Y)$.

**Lemma 3.21.** i) $H[\eta](Y)$ is closed under $\varphi$ and $\psi|_{\eta+1}$.

ii) If $\delta < \eta$ then $H[\delta](Y) \subseteq H[\eta](Y)$

iii) If $\delta < \eta$ and $H[\delta] \Gamma, C_i \Gamma$ then $H[\eta] \Gamma, C_i \Gamma$

**Proof.** i) Note that for any $X$, $H[\eta](X) = B(\alpha)$ for some $\alpha \geq \eta + 1$.

ii) follows immediately from the definition of $H[\eta]$ and iii) follows easily from ii).

**Lemma 3.22.** Suppose $\eta \in B(\eta)$ and for any ordinal $\beta$ let $\bar{\beta} := \eta + \omega^{\Omega+\beta}$.

i) If $\alpha \in H[\eta]$ then $\bar{\alpha}, \psi \bar{\alpha} \in H[\bar{\eta}]$
If \( \alpha_0 \in H_\eta \) and \( \alpha_0 < \alpha \) then \( \psi \hat{\alpha}_0 < \psi \hat{\alpha} \).

**Proof.**

i) First note that \( H_\eta(\emptyset) = B(\eta + 1) \). Now from \( \alpha, \eta \in B(\eta + 1) \) we get \( \hat{\alpha} \in B(\eta + 1) \) and thus \( \hat{\alpha} \in B(\hat{\alpha}) \). It follows that \( \psi \hat{\alpha} \in B(\psi \hat{\alpha}) \).

ii) Suppose that \( \alpha_0 \in H_\eta \) and \( \alpha_0 < \alpha \), using the preceding argument we get that \( \psi \hat{\alpha}_0 \in B(\hat{\alpha}_0 + 1) \subseteq B(\hat{\alpha}) \), thus \( \psi \hat{\alpha}_0 < \psi \hat{\alpha} \).

**Theorem 3.23** (Collapsing for \( RS_\Omega(X) \)). Suppose \( \Gamma \) is a set of \( \Sigma \) formulae and \( \eta \in B(\eta) \).

\[
\text{If } H_\eta \vdash_\Omega \Gamma \text{ then } H_\hat{\alpha} \vdash_\psi \hat{\alpha} \Gamma.
\]

**Proof.** We proceed by induction on \( \alpha \). First note that from \( \alpha \in H_\eta \) we get \( \hat{\alpha}, \psi \hat{\alpha} \in H_\hat{\alpha} \) from Lemma 3.22i).

If \( \Gamma \) is an axiom then the result follows by Lemma 3.15i). So suppose \( \Gamma \) arose as the result of an inference, we shall distinguish cases according to the last inference of \( H_\eta \vdash_\Omega \Gamma \).

**Case 1.** Suppose \( A \simeq \bigwedge (A_i)_{i \in y} \in \Gamma \) and \( H_\eta \vdash_\Omega \Gamma, A_i \) with \( \alpha_i < \alpha \) for each \( i \in y \). Since \( A \) is a \( \Sigma \) formula, we must have \( \sup \{|i| \mid i \in y\} < \Omega \), therefore as \( k(A) \subseteq H_\eta = B(\eta + 1) \) we must have \( \sup \{|i| \mid i \in y\} < \psi(\eta + 1) \). It follows that for any \( i \in y \mid i \in H_\eta \) and thus \( H_\eta \vdash_\Omega i \) = \( H_\eta \). This means that we may use the induction hypothesis to give

\[
H_\hat{\alpha}_i \vdash_{\psi \hat{\alpha}} \Gamma, A_i \quad \text{for all } i \in y.
\]

Now applying Lemma 3.21i) we get

\[
H_\hat{\alpha} \vdash_{\psi \hat{\alpha}} \Gamma, A_i \quad \text{for all } i \in y.
\]

Upon noting that \( \psi \hat{\alpha}_i < \psi \hat{\alpha} \) by 3.22i) we may apply the appropriate inference to obtain

\[
H_\hat{\alpha} \vdash_{\psi \hat{\alpha}} \Gamma.
\]

**Case 2.** Now suppose that \( A \simeq \bigvee (A_i)_{i \in y} \in \Gamma \) and \( H_\eta \vdash_\Omega \Gamma, A_{i_0} \) with \( i_0 \in y \), \( |i_0| \in H_\eta \) and \( \alpha_0 < \alpha \).

We may immediately apply the induction hypothesis to obtain

\[
H_\hat{\alpha} \vdash_{\psi \hat{\alpha}} \Gamma, A_{i_0}
\]

Now we want to be able to apply the appropriate inference to derive \( \Gamma \) but first we must check that \( |i_0| < \Gamma_{\theta + 1} + \psi \hat{\alpha} \). Since \( |i_0| \in H_\eta = B(\eta + 1) \) we have

\[
|i_0| < \psi(\eta + 1) < \psi \hat{\alpha} \leq \Gamma_{\theta + 1} + \psi \hat{\alpha}.
\]

Therefore we may apply the appropriate inference to yield

\[
H_\hat{\alpha} \vdash_{\psi \hat{\alpha}} \Gamma.
\]
Case 3. Now suppose the last inference was \(\Sigma\)-Ref \(\Omega\)(\(X\)) so we have \(\exists z F^z \in \Gamma\) and \(\mathcal{H}_\eta \frac{\alpha_0}{\Omega+1} \Gamma, F\) with \(\alpha_0 < \alpha\) and \(F\) a \(\Sigma\) formula. Applying the induction hypothesis we have
\[
\mathcal{H}_{\hat{\alpha}} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, F.
\]
Applying Boundedness \(3.18\) we obtain
\[
\mathcal{H}_{\hat{\alpha}} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, F^{L_{\psi_{\alpha_0}}}(X).
\]
Now by Lemma \(3.22\) \(|L_{\psi_{\alpha_0}}(X)| = \Gamma_{\theta+1} + \psi_{\alpha_0} < \Gamma_{\theta+1} + \psi_{\hat{\alpha}}\), so we may apply \((\exists)\) to obtain
\[
\mathcal{H}_{\hat{\alpha}} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, \exists z F^z
\]
as required.

Case 4. Finally suppose the last inference was (Cut), so for some \(A\) with \(rk(A) \leq \Omega\) we have
\[
\begin{align*}
(1) & \quad \mathcal{H}_\eta \frac{\alpha_0}{\Omega+1} \Gamma, A \quad \text{with } \alpha_0 < \alpha. \\
(2) & \quad \mathcal{H}_\eta \frac{\alpha_0}{\Omega+1} \Gamma, \neg A \quad \text{with } \alpha_0 < \alpha.
\end{align*}
\]
4.1 If \(rk(A) < \Omega\) then \(A\) is \(\Delta_0\). In this case both \(A\) and \(\neg A\) are \(\Sigma\) formulae so we may immediately apply the induction hypothesis to both \((1)\) and \((2)\) giving
\[
\begin{align*}
(3) & \quad \mathcal{H}_{\hat{\alpha}_0} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, A \\
(4) & \quad \mathcal{H}_{\hat{\alpha}_0} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, \neg A.
\end{align*}
\]
Since \(k(A) \subseteq \mathcal{H}_\eta(\emptyset) = B(\eta + 1)\) and \(A\) is \(\Delta_0\) it follows from Observation \(3.11\) that \(rk(A) \in B(\eta + 1) \cap \Omega\). Thus \(rk(A) < \psi(\eta + 1) < \psi_{\hat{\alpha}}\), so we may apply (Cut) to complete this case.

4.2 Finally suppose \(rk(A) = \Omega\). Without loss of generality we may assume that \(A \equiv \exists z F(z)\) with \(F\) a \(\Delta_0\) formula. We may immediately apply the induction hypothesis to \((1)\) giving
\[
\mathcal{H}_{\hat{\alpha}_0} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, A.
\]
Applying Boundedness \(3.18\) to \((5)\) yields
\[
\mathcal{H}_{\hat{\alpha}_0} \frac{\psi_{\alpha_0}}{\psi_{\alpha_0}} \Gamma, A^{L_{\psi_{\alpha_0}}}(X).
\]
Now using Lemma \(3.15v)\) on \((2)\) yields
\[
\mathcal{H}_{\hat{\alpha}_0} \frac{\alpha}{\Omega+1} \Gamma, \neg A^{L_{\psi_{\alpha_0}}}(X).
\]
Observe that since \(\eta, \alpha_0 \in \mathcal{H}_\eta\) we have \(\hat{\alpha}_0 \in B(\eta + 1) \subseteq B(\hat{\alpha}_0)\). So since \(\Gamma, \neg A^{L_{\psi_{\alpha_0}}}(X)\) is a set of \(\Sigma\)-formulae we may apply the induction hypothesis to \((7)\) giving
\[
\mathcal{H}_{\alpha_1} \frac{\psi_{\alpha_1}}{\psi_{\alpha_1}} \Gamma, \neg A^{L_{\psi_{\alpha_0}}}
\]
where \(\alpha_1 := \hat{\alpha}_0 + \omega^{\Omega+\alpha_0}\).
Now \[
\alpha_1 = \alpha_0 + \omega^{\Omega+\alpha_0} = \eta + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \eta + \omega^{\Omega+\alpha} := \alpha.
\]
Owing to Lemma 3.22 ii) we have \(\psi\alpha_0, \psi\alpha_1 < \psi\alpha\), thus we may apply (Cut) to (6) and (8) giving
\[
H_{\psi\alpha} \vdash_{\psi\alpha} \Gamma
\]
as required.

4 Embedding KP into RS\(_{\Omega}(X)\)

**Definition 4.1.** i) Given ordinals \(\alpha_1, \ldots, \alpha_n\). The expression \(\omega^{\alpha_1} \# \ldots \# \omega^{\alpha_n}\) denotes the ordinal \(\omega^{\alpha_p(1)} + \ldots + \omega^{\alpha_p(n)}\), where \(p : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}\) such that \(\alpha_p(1) \geq \ldots \geq \alpha_p(n)\). More generally \(\alpha \# 0 := 0 \# \alpha := 0\) and \(\alpha \# \beta := \omega^{\alpha_1} \# \ldots \# \omega^{\alpha_n} \# \omega^{\beta_1} \# \ldots \# \omega^{\beta_m}\) for \(\alpha = \text{NF} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}\) and \(\beta = \text{NF} \omega^{\beta_1} + \ldots + \omega^{\beta_m}\).

ii) If \(A\) is any \(\text{RS}\(_{\Omega}(X)\)-formula then \(\text{no}(A) := \omega^{rk(A)}\).

iii) If \(\Gamma = \{A_1, \ldots, A_n\}\) is a set of \(\text{RS}\(_{\Omega}(X)\)-formulae then \(\text{no}(\Gamma) := \text{no}(A_1) \# \ldots \# \text{no}(A_n)\).

iv) \(\models \Gamma\) will be used to abbreviate that
\[
\mathcal{H}(\Gamma) \vdash_{\text{no}(\Gamma)} 0 \Gamma\]
holds for any operator \(\mathcal{H}\).

v) \(\models^\alpha \Gamma\) will be used to abbreviate that
\[
\mathcal{H}(\Gamma) \vdash_{\text{no}(\Gamma) \# \alpha} 0 \Gamma\]
holds for any operator \(\mathcal{H}\).

As might be expected \(\models^\alpha \Gamma\) and \(\models \rho \Gamma\) stand for \(\models^0 \Gamma\) and \(\models^0 \rho \Gamma\) respectively.

The following lemma shows that under certain conditions we may use \(\models\) as a calculus.

**Lemma 4.2.** i) If \(\Gamma\) follows from premises \(\Gamma_i\) by an \(\text{RS}\(_{\Omega}(X)\)-inference other than (Cut) or (\(\Sigma\)-Ref\(_{\Omega}(X)\)) and without contractions then
\[
\text{if } \models^\alpha \Gamma_i \text{ then } \models^\alpha \Gamma
\]
ii) If \(\models^\rho \Gamma, A, B\) then \(\models^\rho \Gamma, A \lor B\).

*Proof. Part i*) follows from Lemma 3.14. It also needs to be noted that if the last inference was universal with premises \(\{\Gamma_i\}_{i \in Y}\), then \(\mathcal{H}[\Gamma_i] \subseteq \mathcal{H}[i]\).

For part ii) suppose \(\models^\rho \Gamma, A, B\), so we have
\[
\mathcal{H}[\Gamma] \vdash_{\rho}^{\text{no}(\Gamma, A, B) \# \alpha} 0 \Gamma, A, B.
\]
Two applications of \((\lor)\) and a contraction yields
\[
\mathcal{H}[\Gamma] \vdash_{\rho}^{\text{no}(\Gamma, A, B) \# \alpha+2} 0 \Gamma, A \lor B.
\]
It remains to note that since $\omega^{rk(A \lor B)}$ is additive principal, Lemma 3.14 gives us
\[ no(\Gamma, A, B) \# \alpha + 2 = no(\Gamma) \# \alpha \# \omega^{rk(A)} \# \omega^{rk(B)} + 2 < no(\Gamma) \# \alpha \# \omega^{rk(A \lor B)} = no(\Gamma, A \lor B) \# \alpha. \]
So we may complete the proof with an application of Lemma 3.15. \hfill \Box

**Lemma 4.3.** Let $A$ be an $\mathbf{RS}_\Omega(X)$ formula and $s, t$ be $\mathbf{RS}_\Omega(X)$ terms.

i) $\vdash A, \neg A$

ii) $\vdash s \notin s$

iii) $\vdash s \subseteq s$ where $s \subseteq s :=(\forall x \in s)(x \in s)$

iv) If $|s| < |t|$ then $\vdash \exists t \rightarrow s \in t$ and $\vdash \neg(s \in t), s \in t$

v) $\vdash s \neq t, t = s$

vi) If $|s| < |t|$ and $\vdash \Gamma, A, B$ then $\vdash \exists r, \exists s \in t \rightarrow A, s \in t \land B$

vii) If $|s| < \Gamma_{\theta +1} + \alpha$ then $\vdash s \in L_\alpha(X)$

**Proof.** i) We use induction of $rk(A)$, and split into cases based upon the form of $A$:

Case 1. Suppose $A \equiv \bar{u} \in \bar{v}$. In this case either $A$ or $\neg A$ is an axiom so there is nothing to show.

Case 2. Suppose $A \equiv r \in t$ where $\max(|r|, |t|) \geq \Gamma_{\theta +1}$. By Lemma 3.14 and the induction hypothesis we have $\vdash \exists t \land r = s, s \in t \rightarrow r \neq s$ for all $|s| < |t|$. Thus we have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

\[
\begin{array}{c}
\vdash s \in t \land r = s, s \in t \rightarrow r \neq s \\
\vdash \exists t \land r = s, s \in t \rightarrow r \neq s \\
\vdash r \in t, s \in t \rightarrow r \neq s \\
\vdash r \in t, r \notin t
\end{array}
\]

Case 3. Suppose $A \equiv (\exists x \in t)F(x)$. By Lemma 3.14 and the induction hypothesis we have $\vdash \exists s \in t \land F(s), s \in t \rightarrow \neg F(s)$ for all $|s| < |t|$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

\[
\begin{array}{c}
\vdash s \in t \land F(s), s \in t \rightarrow \neg F(s) \\
\vdash (\exists x \in t)F(x), s \in t \rightarrow \neg F(s) \\
\vdash (\exists x \in t)F(x), (\forall x \in t)\neg F(x)
\end{array}
\]

Case 4. $A \equiv A_0 \lor A_1$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

\[
\begin{array}{c}
\vdash A_0, \neg A_0 \\
\vdash A_0 \lor A_1, \neg A_0 \\
\vdash A_1, \neg A_1 \\
\vdash A_0 \lor A_1, \neg A_1 \\
\vdash A_0 \lor A_1, \neg A_0 \land \neg A_1
\end{array}
\]
All other cases may be seen as variations of those above.

ii) We proceed by induction on \( rk(s) \). If \( s \) is of the form \( \bar{u} \) then \( s \not \in s \) is already an axiom. Inductively we have \( \vdash r \not \in r \) for all \( |r| < |s| \). Now suppose \( s \) is of the form \( \mathbb{L}_\alpha(X) \), in this case \( r \not \in r \equiv r \notin s \wedge r \not \in r \) so we have the following template for derivations in \( \mathbf{RS}_\Omega(X) \):

\[
\begin{align*}
\text{(bΞ) } & \quad \vdash r \in s \wedge r \not \in r \quad \vdash (\exists x \in s)(x \not \in r) \\
\text{(v) } & \quad \vdash s \not = r \\
\text{(3.5ii) } & \quad \vdash r \in s \rightarrow s \not = r \\
\text{(ς) } & \quad \vdash s \not \in s
\end{align*}
\]

Now suppose \( s \) is of the form \( \{x \in \mathbb{L}_\alpha(X) \mid B(x)\} \), by i) we have \( \vdash B(r), \neg B(r) \) for any \( |r| < |s| \). We have the following template for derivations in \( \mathbf{RS}_\Omega(X) \):

\[
\begin{align*}
\text{(∧) } & \quad \vdash r \not \in r \quad \vdash B(r), \neg B(r) \quad \text{for any } |r| < |s| \\
(\text{bΞ}) & \quad \vdash B(r) \wedge r \not \in r, \neg B(r) \\
\text{(v) } & \quad \vdash (\exists x \in s)(x \not \in r), \neg B(r) \\
\text{(3.5ii) } & \quad \vdash s \not = r, \neg B(r) \\
\text{(ς) } & \quad \vdash B(r) \rightarrow s \not = r \\
\text{(ς) } & \quad \vdash s \not \in s
\end{align*}
\]

Lemma 4.2i)

\[
\begin{align*}
(\text{bΓ}) & \quad \vdash \bar{v} \not \in \bar{u}, \bar{v} \in \bar{u} \\
\text{(v) } & \quad \vdash \bar{v} \in \bar{u} \rightarrow \bar{v} \in \bar{u} \\
\text{(3.5ii) } & \quad \vdash (\forall x \in s)(x \in s)
\end{align*}
\]

Suppose \( s \equiv \mathbb{L}_\alpha(X) \), by the induction hypothesis we have \( \vdash r \subseteq r \) for all \( |r| < |s| \). We have the following template for derivations in \( \mathbf{RS}_\Omega(X) \):

\[
\begin{align*}
\text{(∧) } & \quad \vdash r \subseteq r \\
(\text{bΞ}) & \quad \vdash r = r \\
\text{(3.5ii) } & \quad \vdash r \in s \\
\text{(bΨ) } & \quad \vdash r \in s \rightarrow r \in s \\
\text{(3.5ii) } & \quad \vdash (\forall x \in s)(x \in s)
\end{align*}
\]

Finally suppose \( s \equiv [x \in \mathbb{L}_\alpha(X) \mid B(x)] \), again by the induction hypothesis we have \( \vdash r \subseteq r \) for all \( |r| < |s| \). Also by part i) we have \( \vdash \neg B(r), B(r) \) for all such \( r \). We have the following template for derivations in \( \mathbf{RS}_\Omega(X) \):

\[
\begin{align*}
\text{(∧) } & \quad \vdash \neg B(r), r \subseteq r \\
\text{(3.5ii) } & \quad \vdash \neg B(r), r = r \\
\text{(bΨ) } & \quad \vdash \neg B(r), B(r) \wedge r = r \\
\text{Lemma 4.2i) } & \quad \vdash \neg B(r), r \in s \\
\text{(bΨ) } & \quad \vdash B(r) \rightarrow r \in s \\
\text{(3.5ii) } & \quad \vdash (\forall x \in s)(x \in s)
\end{align*}
\]

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iv) Was shown whilst proving iii).

v) By part i) we have $\vdash \neg(s \subseteq t), s \subseteq t$ and $\vdash \neg(t \subseteq s), t \subseteq s$ for all $|s| < |t|$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$.

$$
\begin{align*}
\begin{array}{c}
\vdash \neg(s \subseteq t), s \subseteq t \\
\vdash \neg(s \subseteq t) \lor \neg(t \subseteq s), s \subseteq t \\
\vdash \neg(t \subseteq s) \lor \neg(s \subseteq t), t \subseteq s \\
\end{array}
\end{align*}
$$

vi) If $t \equiv L_\alpha(X)$ then this result is trivial since $s \in t \rightarrow A := A$ and $s \in t \wedge B := B$.

Now if $t \equiv \bar{u}$ then $s \in t := s \in t$ and if $t \equiv [x \in L_\alpha(X) | C(x)]$ then $s \in t := C(s)$. In either case we have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$
\begin{align*}
\begin{array}{c}
\vdash \Gamma, A, B \\
\vdash \Gamma, s \in t \rightarrow A, B \\
\vdash \Gamma, s \in t \rightarrow A, s \in t \\
\end{array}
\end{align*}
$$

vii) By part iii) we have $\vdash s = s$ for all $|s| < \Gamma_{\theta+1} + \alpha$ which means we have $\vdash s \in L_\alpha(X) \wedge s = s$ for all such $s$. From which one application of ($\varepsilon$) gives the desired result. \qed

**Lemma 4.4** (Extensionality). For any $\mathbf{RS}_\Omega(X)$ formula $A(s_1, \ldots, s_n)$,

$$
\vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).
$$

Where $[s_i \neq t_i] := \neg(s_i \subseteq t_i), \neg(t_i \subseteq s_i)$.

**Proof.** The proof is by induction on $rk(A(s_1, \ldots, s_n)) \# rk(A(t_1, \ldots, t_n))$.

Case 1. Suppose $A(s_1, s_2) \equiv s_1 \subseteq s_2$. By the induction hypothesis we have $\vdash [s_1 \neq t_1], [s \neq t], s_1 \neq s, t_1 = t$ for all $|s| < |s_2|$ and all $|t| < |t_2|$. What follows is a template for derivations in $\mathbf{RS}_\Omega(X)$, for ease of reading the principal formula of each inference is underlined (some lines do not necessarily represent single inferences, but in these cases it is clear how to extend the concept of "principal formula" in a sensible way).

**Lemma 4.3**

$$
\begin{align*}
\begin{array}{c}
\vdash [s_1 \neq t_1], [s \neq t], s_1 \neq s, t_1 = t \\
\vdash [s_1 \neq t_1], s \neq t, s_1 \neq s, t_1 = t \\
\vdash [s_1 \neq t_1], t \in t_2 \rightarrow s \neq t, s_1 \neq s, t \in t_2 \wedge t_1 = t \\
\vdash [s_1 \neq t_1], t \in t_2 \rightarrow s \neq t, s_1 \neq s, t_1 \in t_2 \\
\vdash [s_1 \neq t_1], s \neq s_2, s_1 \neq s, t_1 \in s_2 \\
\vdash [s_1 \neq t_1], s \neq s_2, s_1 \neq s, t_1 \in t_2 \\
\vdash [s_1 \neq t_1], s \neq s_2 \wedge s \notin s_2, s \in s_2 \rightarrow s_1 \neq s, t_1 \in t_2 \\
\vdash [s_1 \neq t_1], s \neq s_2 \wedge s \notin s_2, s \in s_2 \rightarrow s_1 \neq s, t_1 \in t_2 \\
\vdash [s_1 \neq t_1], s_2 \neq s_2, s_1 \neq s_2, t_1 \in t_2 \\
\vdash [s_1 \neq t_1], s_2 \neq t_2, s_1 \neq s_2, t_1 \in t_2
\end{array}
\end{align*}
$$

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Case 2. Suppose \( A(s_1) \equiv s_1 \in s_1 \). In this case \( \neg A(s_1) \equiv s_1 \notin s_1 \) so the result follows from Lemma 4.3(i).

Case 3. Suppose \( A(s_1, \ldots, s_n) \equiv (\exists y \in s_i)(B(y, s_1, \ldots, s_n)) \) for some \( 1 \leq i \leq n \). Inductively we have

\[
\vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], \neg B(r, s_1, \ldots, s_n), B(r, t_1, \ldots, t_n)
\]

for all \( |r| < |s| \). Now by applying (b3v) we obtain

\[
\vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], r \notin s_i \rightarrow \neg B(r, s_1, \ldots, s_n), r \in s_i \land B(r, t_1, \ldots, t_n)
\]

To which we may apply (b∃) followed by (b∀) to arrive at the desired conclusion.

Case 4. Suppose \( A(s_1, \ldots, s_n) \equiv (\exists x \in r)B(x, s_1, \ldots, s_n) \) for some \( r \) not present in \( s_1, \ldots, s_n \). From the induction hypothesis we have

\[
\vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], p \vDash r \rightarrow \neg B(p, s_1, \ldots, s_n), p \vDash r \land B(p, t_1, \ldots, t_n)
\]

for all \( |p| < |r| \).

Applying (b∃) followed by (b∀) gives us the desired result.

The cases where \( A(s_1, \ldots, s_n) \equiv \exists x B(x, s_1, \ldots, s_n) \) or \( A(s_1, \ldots, s_n) \equiv B \lor C \) may be treated in a similar manner to case 4. All other cases are dual to one of the ones considered above. \( \square \)

**Lemma 4.5** (Set Induction). For any \( \mathbf{RS}_\Omega(X) \)-formula \( F \):

\[
\vdash_r^{\text{rk}(A)} \forall x[(\forall y \in x)F(y) \rightarrow F(x)] \rightarrow \forall x F(x)
\]

where \( A := \forall x[(\forall y \in x)F(y) \rightarrow F(x)] \).

*Proof.* Claim:

\[ (*) \quad \mathcal{H}[A, s] \vdash_0^{\text{rk}(A) + \# s + 1} \neg A, F(s) \quad \text{for any term } s. \]

We begin by verifying (*) using induction on \( |s| \). From the induction hypothesis we know that

\[ \mathcal{H}[A, t] \vdash_0^{\text{rk}(A) + \# t + 1} \neg A, F(t) \quad \text{for all } |t| < |s|. \]  

By applying (v) if necessary to (1) we obtain

\[ \mathcal{H}[A, t, s] \vdash_0^{\text{rk}(A) + \# s + 1} \neg A, t \in s \rightarrow F(t) \quad \text{for all } |t| < |s|. \]

To which we may apply (b∀) yielding

\[ \mathcal{H}[A, s] \vdash_0^{\# s + 2} \neg A, (\forall y \in s)F(y) \quad \text{where } \eta := \omega^{\text{rk}(A) + \# s}. \]

Observe that \( \eta \omega(-F(s), F(s)) < \omega^{\text{rk}(A)} \), so by Lemma 4.3 we have

\[ \mathcal{H}[A, s] \vdash_0^{\# s + 2} \neg F(s), F(s). \]
Applying \((\land)\) to (3) and (4) yields
\[
H[A, s] \models_0^{n+3} \neg A, (\forall y \in s)F(y) \land \neg F(s), F(s).
\]
To which we may apply (3) to obtain
\[
H[A, s] \models_0^{n+4} \neg A, \exists x[(\forall y \in x)F(y) \land \neg F(x)], F(s).
\]
It remains to observe that \(\neg A \equiv \exists x[(\forall y \in x)F(y) \land \neg F(x)]\) and that \(\eta + 4 < \omega^{\kappa(A)}\), and hence we may apply Lemma 3.15ii to provide
\[
H[A, s] \models_0^{\omega^{\kappa(A)}|s|+1} \neg A, F(s)
\]
so the claim is verified.

Applying \((\forall)\) to (*) gives
\[
H[A] \models_0^{\omega^{\kappa(A)}|\Omega|} \neg A, \forall x F(x).
\]
Now by two applications of \((\forall)\) we may conclude
\[
H[A] \models_0^{\omega^{\kappa(A)}|\Omega|+2} A \to \forall x F(x).
\]
It remains to note that \(\text{no}(A \to \forall x F(x)) \geq \omega^{\Omega+1} > \Omega + 2\), so we have
\[
\models_0^{\omega^{\kappa(A)}} A \to (\forall x \in L\alpha(X))F(x)
\]
as required.

\[\square\]

**Lemma 4.6** (Infinity). Suppose \(\omega < \mu < \Omega\), then
\[
\models (\exists x \in L\mu(X))[(\exists z \in x)(z \in x) \land (\forall y \in x)(\exists z \in x)(y \in z)].
\]

**Proof.** The following gives a template for derivations in \(\text{RS}_{\Omega}(X)\), the idea is that \(L\omega(X)\) serves as a witness inside \(L\mu(X)\).

\[
\text{Lemma 4.3(ii)}
\]

| Rule | Description |
|------|-------------|
| (3.5i) | \(\models_0^{s \in L_k(X)}\) for any \(|s| < |L_k(X)|\) and \(k < \omega\). |
| (b\(\equiv\)) | \(\models_0^{L_k(X) \subseteq L_\omega(X) \land s \in L_k(X)}\) |
| (3.5ii) | \(\models_0^{(\exists z \in L_\omega(X))(s \in L_k(X))}\) |
| (bv) | \(\models_0^{(\forall y \in L_\omega(X))(\exists z \in L_\omega(X))(y \in z)}\) |
| (\(\land\)) | \(\models_0^{(\forall y \in L_\omega(X))(\exists z \in L_\omega(X))(y \in z)}\) |
| (3.5i) | \(\models_0^{\text{no}(X) \in L_\omega(X)}\) |
| (b\(\equiv\)) | \(\models_0^{\text{no}(X) \subseteq L_\omega(X) \land \text{no}(X) \in L_\omega(X)}\) |
| (3.5ii) | \(\models_0^{(\exists z \in L_\omega(X))(z \in L_\omega(X))}\) |
| (bv) | \(\models_0^{(\forall y \in L_\omega(X))(\exists z \in L_\omega(X))(y \in z)}\) |
| (\(\land\)) | \(\models_0^{(\forall y \in L_\omega(X))(\exists z \in L_\omega(X))(y \in z)}\) |

\[\square\]
Lemma 4.7 (Δ₀-Separation). Suppose $A(a,b_1,\ldots,b_n)$ be a Δ₀-formula of $\text{KP}$ with all free variables indicated, $\mu$ a limit ordinal and $|s|, |t_0|,\ldots, |t_n| < \Gamma_{\theta+1} + \mu$.

$$\vdash (\exists y \in \mathbb{L}_\mu(X))[\forall x \in y](x \in s \land A(x,t_1,\ldots,t_n)) \land (\forall x \in s)(A(x,t_1,\ldots,t_n) \rightarrow x \in y)$$

Proof. Let $\alpha := \max\{|s|, |t_0|,\ldots, |t_n|\} + 1$ and note that $\alpha < \Gamma_{\theta+1} + \mu$ since $\mu$ ia a limit. Now let $\beta$ be the unique ordinal such that $\alpha = \Gamma_{\theta+1} + \beta$ if such an ordinal exists, if not set $\beta := 0$. Now define

$$t := [z \in \mathbb{L}_\beta(X) \mid z \in s \land B(z)]$$

where $B(z) := A(z,t_1,\ldots,t_n)$. We have the following templates for derivations in $\text{RS}_\Omega(X)$:

\begin{align*}
\text{Lemma 4.7 i} & \quad \vdash -r \in s \land B(r), r \in s \land B(r) \quad \text{for all } |r| < |s| \\
\text{Lemma 4.7 ii} & \quad \vdash (r \in s \land B(r)) \rightarrow r \in s \land B(r) \\
\text{Lemma 4.7 iii} & \quad \vdash \forall x \in t)(x \in s \land B(r))
\end{align*}

In the following derivation $r$ ranges over terms $|r| < |s|.$

\begin{align*}
\text{Lemma 4.7 iv} & \quad \vdash -(r \in s), r \in s \\
\text{Lemma 4.7 iii} & \quad \vdash -B(r), B(r) \\
\text{Lemma 4.7 i} & \quad \vdash r = r
\end{align*}

\begin{align*}
\text{Lemma 4.7 ii} & \quad \vdash -(r \in s), -B(r), r \in s \land B(r) \\
\text{Lemma 4.2ii} & \quad \vdash (B(r) \rightarrow r \in t) \\
\text{Lemma 4.2ii} & \quad \vdash r \in s \land B(r), r \in t
\end{align*}

Now applying $(\land)$ to the two preceding derivations and noting that $|t| < \Gamma_{\theta+1} + \mu$ gives us

$$\vdash t \in \mathbb{L}_\mu(X) \land [(\forall x \in t)(x \in s \land B(r)) \land (\forall x \in s)(B(x) \rightarrow x \in t)]$$

to which we may apply $(b\exists)$ to obtain

$$\vdash (\exists y \in \mathbb{L}_\mu(X))(\forall x \in y)(x \in s \land B(x)) \land (\forall x \in s)(B(x) \rightarrow x \in y)].$$

It should also be checked that

$$t \in \mathcal{H}[(\exists y \in \mathbb{L}_\mu(X))(\forall x \in y)(x \in s \land B(x)) \land (\forall x \in s)(B(x) \rightarrow x \in y)]$$

but this is the case since

$$|s|, |t_0|,\ldots, |t_n| \in k((\exists y \in \mathbb{L}_\mu(X))(\forall x \in y)(x \in s \land B(x)) \land (\forall x \in s)(B(x) \rightarrow x \in y))]$$

and $|t| = \max\{|s|, |t_0|,\ldots, |t_n|\} + 1, \Gamma_{\theta+1}$. □
Lemma 4.8 (Pair and Union). Let $\mu$ be a limit ordinal and let $s, t$ be $RS_\Omega(X)$-terms such that $|s|, |t| < \Gamma_{\beta+1} + \mu$, then

i) $\vdash (\exists z \in L_\mu(X))(s \in z \land t \in z)$

ii) $(\exists z \in L_\mu(X))(\forall y \in s)(\forall x \in y)(x \in z)$

Proof. Let $\alpha := \max\{|s|, |t|\} + 1$, now let $\beta$ be the unique ordinal such that $\alpha = \Gamma_{\beta+1} + \beta$ if such an ordinal exists, otherwise set $\beta := 0$. Now by Lemma 4.3(ii) we have

$\vdash s \in L_\beta(X)$ and $\vdash t \in L_\beta(X)$.

Now by ($\land$) and noticing that $\beta < \mu$ since $\mu$ is a limit, we have

$\vdash L_\beta(X) \in L_\mu(X) \land (s \in L_\beta(X) \land t \in L_\beta(X))$.

To which we may apply ($b\exists$) to obtain the desired result.

ii) Let $\beta$ be the unique ordinal such that $|s| = \Gamma_{\beta+1} + \beta$ if such an ordinal exists, otherwise let $\beta = 0$. By Lemma 4.3(ii) we have $\vdash r \in L_\beta(X)$ for any $|r| < |s|$. In the following template for derivations in $RS_\Omega(X)$, $r$ and $t$ range over terms such that $|r| < |t| < |s|$:

\[\begin{align*}
\text{(v) if necessary} & \quad \vdash r \in L_\beta(X) \\
(b\forall) & \quad \vdash r \in t \rightarrow r \in L_\beta(X) \\
\text{(v) if necessary} & \quad \vdash (\forall x \in t)(x \in L_\beta(X)) \\
(b\forall) & \quad \vdash t \in s \rightarrow (\forall x \in t)(x \in L_\beta(X)) \\
\text{(3.3)ii) } & \quad \vdash (\forall x \in s)(\forall y \in y)(x \in L_\beta(X)) \\
(b\exists) & \quad \vdash L_\beta(X) \in L_\mu(X) \land (\forall y \in s)(\forall x \in y)(x \in L_\beta(X)) \quad \text{since $\beta < \mu$} \\
\end{align*}\]

\[\vdash (\exists z \in L_\mu(X))(\forall y \in s)(\forall x \in y)(x \in z)\]

\[\square\]

Lemma 4.9 ($\Delta_0$-Collection). Suppose $F(a, b)$ is any $\Delta_0$ formula of $KP$.

$\vdash (\forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y)$

Proof. By Lemma 4.3, we have

$\vdash \neg(\forall x \in s)\exists y F(x, y), (\forall x \in s)\exists y F(x, y)$.

Applying ($\Sigma$-Ref$_\Omega(X)$) yields

$\mathcal{H}[(\forall x \in s)\exists y F(x, y)] \vdash^{\alpha+1}_\Omega \neg(\forall x \in s)\exists y F(x, y), \exists z(\forall x \in s)(\exists y \in z)F(x, y)$

where $\alpha := \omega^r((\forall x \in s)\exists y F(x, y)) + \#(\forall x \in s)\exists y F(x, y))$. Now two applications of ($\lor$) provides

$\mathcal{H}[(\forall x \in s)\exists y F(x, y)] \vdash^{\alpha+3}_6 (\forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y)$.

It remains to note that

$\alpha + 3 < \omega^r((\forall x \in s)\exists y F(x, y)) + 1 = \text{no}((\forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y))$

so the proof is complete.

\[\square\]
Theorem 4.10. If $\text{KP} \vdash \Gamma(a_1, \ldots, a_n)$ where $\Gamma(a_1, \ldots, a_n)$ is a finite set of formulae whose free variables are amongst $a_1, \ldots, a_n$, then there is some $m < \omega$ (which we may compute from the derivation) such that

$$H[s_1, \ldots, s_n] \frac{\Omega^{\omega^m}}{\Omega + m} \Gamma(s_1, \ldots, s_n)$$

for any operator $H$ and any $\text{RS}_{\Omega}(X)$ terms $s_1, \ldots, s_n$.

Proof. Suppose $\Gamma(a_1, \ldots, a_n) \equiv \{A_1(a_1, \ldots, a_n), \ldots, A_k(a_1, \ldots, a_n)\}$. Note that for any choice of terms $s_1, \ldots, s_n$ and each $1 \leq i \leq k$

$$rk(A_i(s_1, \ldots, s_n)) = \omega \cdot \max(k(A_i(s_1, \ldots, s_n))) + m_i \text{ for some } m_i < \omega$$

$$\leq \omega \cdot \Omega + m_i = \Omega + m_i.$$ 

Therefore

$$\text{no}(A_i(s_1, \ldots, s_n)) = \omega^{rk(A_i(s_1, \ldots, s_n))} \leq \omega^{\Omega + m_i} = \omega^{\Omega} \cdot m_i.$$ 

So letting $m = \max(m_1, \ldots, m_k) + 1$ we have

$$\text{no}(\Gamma(s_1, \ldots, s_n)) \leq \Omega \cdot m_1 \# \ldots \# \Omega \cdot \omega^m$$

$$= \Omega \cdot (m_1 \# \ldots \# \omega^m)$$

$$\leq \Omega \cdot \omega^m.$$ 

The proof now proceeds by induction on the $\text{KP}$ derivation. If $\Gamma(a_1, \ldots, a_n)$ is an axiom of $\text{KP}$ then the result follows from $[4.3], [4.4], [4.5], [4.6], [4.8]$ or $[4.9]$. Now suppose that $\Gamma(a_1, \ldots, a_n)$ arises as the result of an inference rule.

Case 1. Suppose the last inference was $(b\forall)$, so $(\forall x \in a_i)F(x, \bar{a}) \in \Gamma(\bar{a})$ and we are in the following situation in $\text{KP}$

$$(b\forall) \quad \Gamma(\bar{a}), c \in a_i \rightarrow F(c, \bar{a})$$

$$\Rightarrow \quad \Gamma(\bar{a})$$

where $c$ is different from $a_1, \ldots, a_n$. Inductively we have some $m < \omega$ such that

$$H[s, r] \frac{\Omega^{\omega^m}}{\Omega + m} \Gamma(s), r \in s_i \rightarrow F(r, \bar{s}) \quad \text{for all } |r| < |s_i|.$$ 

1.1 If $s_i$ is of the form $\bar{u}$ we may immediately apply $(b\forall)$ to complete this case.

Suppose $s_i \equiv \text{L}_\alpha(X)$ for some $\alpha$. Applying Lemma 3.15(ii) to (1) gives

$$H[\bar{s}, r] \frac{\Omega^{\omega^m}}{\Omega + m} \Gamma(\bar{s}), \neg(r \in s_i), F(r, \bar{s}).$$

Since $|r| < |s|$, by Lemma [4.3, ii) we have

$$\vdash r \in s.$$ 

Applying (Cut) to (1) and (2) yields

$$H[\bar{s}, r] \frac{\Omega^{\omega^{m+1}}}{\Omega + m} \Gamma(\bar{s}), F(r, \bar{s}).$$

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To which we may apply \((b\forall)\) to complete this case.

Suppose \(s_i \equiv [x \in L_\alpha(X) \mid B(x)]\), again we may apply Lemma 3.15ii) to (1) to obtain

\[
(5) \quad \mathcal{H}[\bar{s}, r] \models \frac{\omega}{\Omega} \cdot \omega^m \cdot \Gamma(\bar{s}), \neg (r \in s_i), F(r, \bar{s}).
\]

Since \(|r| < |s|\) by Lemma 4.3v) we have

\[
(6) \quad \models \neg (r \in s_i), r \in s.
\]

Applying (Cut) to (5) and (6) yields

\[
(7) \quad \mathcal{H}[\bar{s}, r] \models \frac{\omega}{\Omega} \cdot \omega^m \cdot \Gamma(\bar{s}), r \in s_i \rightarrow F(r, \bar{s}).
\]

To which we may apply \((b\forall)\) to complete this case.

\[\text{Case 2. Suppose the last inference was } (\forall) \text{ so } \forall x A(x, \bar{a}) \in \Gamma(\bar{a}) \text{ and we are in the following situation in } \mathbf{KP}\]

\[
(\forall) \quad \frac{\Gamma(\bar{a}), F(c, \bar{a})}{\Gamma(\bar{a})}
\]

where \(c\) is different from \(a_1, \ldots, a_n\). Inductively we have some \(m < \omega\) such that

\[
\mathcal{H}[\bar{s}, r] \models \frac{\omega^m}{\Omega} \cdot \omega^{m+1} \cdot \Gamma(\bar{s}), F(r, \bar{s}) \quad \text{for all terms } r.
\]

We may immediately apply \((\forall)\) to complete this case.

\[\text{Case 3. Suppose the last inference was } (b\exists) \text{ so } (\exists x \in s_i) F(x, \bar{s}) \in \Gamma(\bar{s}) \text{ and we are in the following situation in } \mathbf{KP}\]

\[
(b\exists) \quad \frac{\Gamma(\bar{a}), c \in a_i \wedge F(c, \bar{a})}{\Gamma(\bar{a})}
\]

3.1 Suppose \(c\) is different from \(a_1, \ldots, a_n\). Using the induction hypothesis we find some \(m < \omega\) such that

\[
(9) \quad \mathcal{H}[\bar{s}] \models \frac{\omega^m}{\Omega} \cdot \omega^m \cdot \Gamma(\bar{s}), \emptyset_e \in s_i \wedge F(\emptyset, \bar{s}).
\]

3.1.1 If \(s_i\) is of the form \(\bar{u}\) we may immediately apply \((b\exists)\) to complete the case.

3.1.2 Suppose \(s_i\) is of the form \(L_\alpha(X)\). Applying Lemma 3.15v) to (1) yields

\[
(10) \quad \mathcal{H}[\bar{s}] \models \frac{\omega^m}{\Omega} \cdot \omega^m \cdot \Gamma(\bar{s}), F(\emptyset, \bar{s}).
\]
Noting that in this case $\bar{\theta} \in s \land F(\bar{\theta}, \bar{s}) \equiv F(\bar{\theta}, \bar{s})$, we may apply ($b\exists$) to complete this case.

3.1.3 Suppose $s_i$ is of the form $[x \in \ll_{\alpha}(X) \mid B(x)]$. First we must verify the following claim

\[ (*) \quad \models \neg(\bar{\theta} \in s_i \land F(\bar{\theta}, \bar{s})), \bar{\theta} \in s_i \land F(\bar{\theta}, \bar{s}). \]

Note that owing to Lemma 4.4 we have $\models [r \neq \bar{\theta}], \neg B(r), B(\bar{\theta})$ for all $|r| < |s_i|$. In the following template for derivations in $RS_\Omega(X)$ $r$ ranges over terms $|r| < |s_i|$.

\[
\begin{array}{ccc}
\text{Lemma 4.2ii) } & \models [r \neq \bar{\theta}], \neg B(r), B(\bar{\theta}) \\
\text{Lemma 4.2ii) } & \models r \neq \bar{\theta}, \neg B(r), B(\bar{\theta}) \\
(\neg) & \models \neg(\bar{\theta} \in s_i), B(\bar{\theta}) \\
(\wedge) & \models \neg(\bar{\theta} \in s_i), \neg F(\bar{\theta}, \bar{s}), B(\bar{\theta}) \land F(\bar{\theta}, \bar{s}) \\
\text{Lemma 4.3i) } & \models \neg F(\bar{\theta}, \bar{s}), F(\bar{\theta}, \bar{s}) \\
\end{array}
\]

Now applying (Cut) to (9) and (*) we get

\[ (11) \quad H[\bar{s}]_{\Omega^{\omega\omega_{\Omega+m}+1}} \Gamma(\bar{s}), \bar{\theta} \in s_i \land F(\bar{\theta}, \bar{s}). \]

Note the possible increase in cut rank. We may apply ($b\exists R$) to (11) to complete this case.

3.2 Suppose $c$ is one of $a_1, \ldots, a_n$, without loss of generality let us assume $c = a_1$. Applying the induction hypothesis we can compute some $m < \omega$ such that

\[ (12) \quad H[\bar{s}]_{\Omega^{\omega\omega_{\Omega+m}}} \Gamma(\bar{s}), s_1 \in s_i \land F(s_1, \bar{s}). \]

Note that in fact 3.2 subsumes 3.1 since we can conclude (12) from the induction hypothesis regardless of whether or not $c$ is a member of $\bar{a}$. To help with clarity 3.1 is left in the proof above, but in later embeddings we shall dispense with such cases.

If $s_1$ and $s_i$ are of the form $\bar{u}$ and $\bar{v}$ with $|s_1| < |s_i|$ then we may immediately apply ($b\exists$) to complete this case. If this is not the case then we verify the following claim

\[ (**) \quad \models \neg(s_1 \in s_i \land F(s_1, \bar{s})), (\exists x \in s_i) F(x, \bar{s}). \]

To prove (**) we split into cases based on the form of $s_i$.

3.2.1 Suppose $s_i$ is of the form $\bar{u}$.

3.2.1.1 If $s_1$ is also of the form $\bar{v}$ [remember that by assumption $|s_1| \geq |s_i|$] then $\neg(s_1 \in s_i), F(s_1, \bar{s}), (\exists x \in s_i) F(x, \bar{s})$ is an axiom so we may apply ($\lor$) twice to complete this case.

3.2.1.2 Now suppose $s_1$ is not of the form $\bar{v}$. We have following template for derivations in $RS_\Omega(X)$, here $r$ ranges over terms with $|r| < |s_i|$.
3.2.2 Now suppose $s_i$ is of the form $\mathbb{L}_\alpha(X)$. In the following template for derivations in $RS^\Omega(X)$ $r$ ranges over terms with $|r| < |s_i|$.

\[ \frac{\text{Lemma 4.3\it{i}}}{\Gamma(\bar{\bar{s}})} \frac{\text{Lemma 4.4}}{\Gamma(\bar{\bar{s}}, s_i)} \]

\[ (\land) \quad \frac{\Gamma(\bar{\bar{s}}), r \in s_i}{\Gamma(\bar{\bar{s}}, r)} \quad (\land) \quad \frac{\Gamma(\bar{\bar{s}}), s_i \neq s_1}{\Gamma(\bar{\bar{s}}, s_i\land F(r, \bar{s}))} \]

\[ \frac{\text{Lemma 4.3\it{i}}}{\Gamma(\bar{\bar{s}})} \frac{\text{Lemma 4.4}}{\Gamma(\bar{\bar{s}}, s_i)} \]

\[ (\forall) \quad \frac{\Gamma(\bar{\bar{s}}), r \in s_i \land F(x, \bar{s})}{\Gamma(\bar{\bar{s}}, F(x, \bar{s}))} \quad (\forall) \quad \frac{\Gamma(\bar{\bar{s}}), \exists x \in s_i F(x, \bar{s})}{\Gamma(\bar{\bar{s}}, \exists x \in s_i F(x, \bar{s}))} \]

3.2.3 Finally suppose $s_i$ is of the form $[x \in \mathbb{L}_\alpha B(x)]$. In the following template for derivations in $RS^\Omega(X)$ $r$ ranges over terms with $|r| < |s_i|$.

\[ \frac{\text{Lemma 4.3\it{i}}}{\Gamma(\bar{\bar{s}})} \frac{\text{Lemma 4.4}}{\Gamma(\bar{\bar{s}}, s_i)} \]

\[ (\land) \quad \frac{\Gamma(\bar{\bar{s}}), s_i \neq s_1}{\Gamma(\bar{\bar{s}}, s_i\land F(r, \bar{s}))} \quad (\land) \quad \frac{\Gamma(\bar{\bar{s}}), B(r)}{\Gamma(\bar{\bar{s}}, F(r, \bar{s}))} \]

\[ \frac{\text{Lemma 4.3\it{i}}}{\Gamma(\bar{\bar{s}})} \frac{\text{Lemma 4.4}}{\Gamma(\bar{\bar{s}}, s_i)} \]

\[ (\forall) \quad \frac{\Gamma(\bar{\bar{s}}), \exists x \in s_i F(x, \bar{s})}{\Gamma(\bar{\bar{s}}, \exists x \in s_i F(x, \bar{s}))} \quad (\forall) \quad \frac{\Gamma(\bar{\bar{s}}), (\exists x \in s_i F(x, \bar{s}))}{\Gamma(\bar{\bar{s}}, (\exists x \in s_i F(x, \bar{s})) )} \]

This completes the proof of the claim (**). It remains to note that we may apply (Cut) to (**), and (12) to complete Case 3.

Case 4. Suppose the last inference was (\exists) so $\exists x F(x, \bar{s}) \in \Gamma(\bar{s})$ and we are in the following situation in $\text{KP}$:

\[ (\exists) \quad \frac{\Gamma(\bar{\bar{s}})}{\Gamma(\bar{\bar{s}})} \]

Let $p = s_j$ if $c = a_j$ otherwise let $p = \emptyset$, from the induction hypothesis we can compute some $m < \omega$ such that

$H(\bar{s}) \frac{\Omega_{m} \omega_{m} \Gamma(\bar{s}), F(p, \bar{s})}{\Omega_{x+m} \Gamma(\bar{s}), F(p, \bar{s})}$.

Applying (\exists) completes this case.

Case 5. If the last inference was (\lor) or (\land) the result follows immediately by applying the corresponding $RS^\Omega(X)$ inference to the induction hypotheses.
Case 6. Finally suppose the last inference was (Cut). So we are in the following situation in \( \mathbf{KP} \)

\[
\begin{array}{c}
\text{(Cut)} \\
\Gamma(\bar{a}), B(\bar{a}, \bar{b}) & \Gamma(\bar{a}), \neg B(\bar{a}, \bar{b}) \\
\hline
\Gamma(\bar{a})
\end{array}
\]

Here \( \bar{b} := b_1, \ldots, b_l \) denotes the free variables occurring in \( B \) that are different from \( a_1, \ldots, a_n \). Let \( \emptyset \) denote the sequence of \( l \) occurrences of \( \emptyset \). From the induction hypothesis we find \( m_1 \) and \( m_2 \) such that

\[
\begin{align*}
\mathcal{H}[\bar{s}] & \Omega^{m_1} \cdot \omega \cdot \mathcal{H}[\bar{s}] \Gamma(\bar{s}), B(\bar{s}, \emptyset) \\
\mathcal{H}[\bar{s}] & \Omega^{m_2} \cdot \omega \cdot \mathcal{H}[\bar{s}], \neg B(\bar{s}, \emptyset)
\end{align*}
\]

To which we may apply (Cut) to complete the proof. \( \square \)

5 A well ordering proof in \( \mathbf{KP} \)

The aim of this section is to give a well ordering proof in \( \mathbf{KP} \) for initial segments of formal ordinal terms from \( T(\theta) \). First let

\[
\begin{align*}
e_0 & := \Omega + 1 \\
e_{n+1} & := \omega^{e_n}.
\end{align*}
\]

Each \( e_n \) is a formal term belonging to every representation system \( T(\theta) \) from \( \mathbf{2.13} \). Although the term is the same, the order type of terms in \( T(\theta) \) below \( e_n \) will be dependent upon \( \theta \). We aim to verify that for every \( n < \omega \)

\[
\mathbf{KP} \vdash A_n(\theta) := \exists \alpha \exists f[\text{dom}(f) = \alpha \land \text{range}(f) = \{ a \in T(\theta) \mid a < \psi_\theta(e_n) \}] \\
\land \forall \gamma, \delta \in \text{dom}(f)(\gamma < \delta \rightarrow f(\gamma) < f(\delta))
\]

where in the above formula \( \prec \) denotes the ordering on \( T(\theta) \). Formally \( A_n(\theta) \) is a \( \Sigma \)-formula of \( \mathbf{KP} \) in which \( \theta \) is a parameter (free variable) ranging over ordinals. For the remainder of this section we argue informally in \( \mathbf{KP} \). The symbols \( \alpha, \beta, \gamma, \ldots \) are to be \( \mathbf{KP} \)-variables ranging over ordinals and are ordered by \( < \), the symbols \( a, b, c, \ldots \) are seen as \( \mathbf{KP} \)-variables ranging over codes of formal terms from \( T(\theta) \), these are ordered by \( \prec \). For the remainder of this section the variable \( \theta \) will remain free as we argue in \( \mathbf{KP} \), for ease of reading we shall simply \( \Omega \) and \( \psi \) instead of \( \Omega_\theta \) and \( \psi_\theta \). This proof is an adaptation to the relativised case of a well ordering proof in \( \mathbf{50} \) or \( \mathbf{51} \).

**Definition 5.1.** The set \( \text{Acc}_\theta \) is defined by

\[
\text{Acc}_\theta := \{ a \in T(\theta) \mid a < \Omega \land \exists \alpha \exists f[\text{dom}(f) = \alpha \land \text{range}(f) = \{ b : b \leq a \}] \\
\land \forall \gamma, \delta \in \text{dom}(f)(\gamma < \delta \rightarrow f(\gamma) < f(\delta)) \}.
\]

**Lemma 5.2** (\( \text{Acc}_\theta \)-induction). For any \( \mathbf{KP} \)-formula \( F(a) \) we have

\[
(\forall a \in \text{Acc}_\theta)[(\forall b \prec a) F(b) \rightarrow F(a)] \rightarrow (\forall a \in \text{Acc}_\theta) F(a).
\]
Proof. For \(a \in \text{Acc}_\theta\) let \(o(a)\) and \(f_a\) be the unique ordinal and function such that \(o(a) = \text{dom}(f_a)\), \(\{b : b \leq a\} = \text{range}(f_a)\) and \(\forall \gamma, \delta \in o(a)(\gamma < \delta \rightarrow f_a(\gamma) < f_a(\delta))\). Now for a contradiction let us assume that
\[
(\forall a \in \text{Acc}_\theta)[(\forall b \prec a) F(b) \rightarrow F(a)] \quad \text{but} \quad \lnot F(a_0) \text{ for some } a_0 \in \text{Acc}_\theta
\]
Using set induction/foundation we may pick \(a_0\) such that \(o(a_0)\) is minimal. (Note that here we must make use of the full set induction schema of KP since the formula \(F\) is of unbounded complexity)
Now for any \(b \prec a_0\) we have \(o(b) < o(a_0)\), thus by our choice of \(a_0\) we get \(F(b)\), thus we have
\[
(\forall b \prec a_0) F(b).
\]
So by assumption we have \(F(a_0)\), contradiction. \(\square\)

Lemma 5.3. \(\text{Acc}_\theta\) has the following closure properties:

i) \(b \in \text{Acc}_\theta \land a \prec b \rightarrow a \in \text{Acc}_\theta\)

ii) \((\forall a \prec b)(a \in \text{Acc}_\theta) \rightarrow b \in \text{Acc}_\theta\)

iii) \(a, b \in \text{Acc}_\theta \rightarrow a + b \in \text{Acc}_\theta\)

iv) \(a, b \in \text{Acc}_\theta \rightarrow \varphi_{ab} \in \text{Acc}_\theta\)

v) \((\forall \beta \leq \theta) \Gamma_\beta \in \text{Acc}_\theta\)

Proof. i) Using the notation defined at the start of the proof of Lemma 5.2 we may define
\[
o(a) := \{\delta \in o(b) \mid f_b(\delta) \leq a\} \quad \text{and} \quad f_a := f_b|_{o(a)+1}
\]
thus witnessing that \(a \in \text{Acc}_\theta\).

ii) Let us assume that \((\forall a \prec b)(a \in \text{Acc}_\theta)\), we must verify that \(b \in \text{Acc}_\theta\). Using \(\Delta_0\)-Separation and Infinity we may form the set \(\{a \mid a \prec b\}\), therefore \(f := \cup_{a \prec b} f_a\) is a set by \(\Delta_0\)-Collection and Union. Let \(\beta := \text{dom}(f)\). Setting \(o(b) := \beta + 1\) and \(f_b := f \cup \{\beta, b\}\) furnishes us with the correct witnesses to confirm that \(b \in \text{Acc}_\theta\).

iii) Firstly we must specify what \(a + b\) means, since it may not be the case that the string \(a + b\) is a term in \(T(\theta)\). However, we may define a \(\theta\)-primitive recursive function \(+ : T(\theta) \times T(\theta) \rightarrow T(\theta)\) which corresponds to ordinal addition.

Let us assume that \((\forall c \prec b)(a + c \in \text{Acc}_\theta)\), now if we can show that \(a + b \in \text{Acc}_\theta\) then the desired result will follow from \(\text{Acc}_\theta\)-induction (5.2). Now let \(d < a + b\), either \(d \leq a\) in which case \(d \in \text{Acc}_\theta\) by i) or \(d > a\) and thus \(d = a + c\) for some unique \(c \prec b\). Such a \(c\) may be determined in a \(\theta\)-primitive recursive fashion, hence \(d \in \text{Acc}_\theta\) by assumption. Thus we have
\[
(\forall d < a + b)(d \in \text{Acc}_\theta).
\]
From which we may use ii) to obtain \(a + b \in \text{Acc}_\theta\), completing the proof.
iv) Again a function \( \varphi : T(\theta) \times T(\theta) \to T(\theta) \) may be defined in a \( \theta \)-primitve recursive fashion. It is our aim to show \( (\forall x, y \in \text{Acc}_\theta)(\varphi xy \in \text{Acc}_\theta) \), to this end let

\[
F(a) := (\forall b \in \text{Acc}_\theta)(\varphi ab \in \text{Acc}_\theta)
\]

and assume

\[\star\] \( (\forall z \prec a)F(z) \]

by 5.2 it suffices to verify \( F(a) \). So let us assume

\[\star\star\] \( a, b \in \text{Acc}_\theta \) and \( (\forall y \prec b)(\varphi ay \in \text{Acc}_\theta) \)

now we must verify \( \varphi ab \in \text{Acc}_\theta \). To do this we prove that

\[
d \prec \varphi ab \Rightarrow d \in \text{Acc}_\theta
\]

by induction on \( Gd \); the term complexity of \( d \).

1) If \( d \) is strongly critical then \( d \preceq a \) or \( d \preceq b \) in which case \( d \in \text{Acc}_\theta \) by \[\star\] or \[\star\star\].

2) If \( d \equiv \varphi d_0 d_1 \) then we have the following subcases:

2.1) If \( d_0 \prec a \) and \( d_1 \prec \varphi ab \) then since \( Gd_1 < Gd \) we get \( d_1 \in \text{Acc}_\theta \) from the induction hypothesis. So by \[\star\] we get \( d \equiv \varphi d_0 d_1 \in \text{Acc}_\theta \)

2.2) If \( d \equiv \varphi ad_1 \) and \( d_1 \prec b \) then \( d \in \text{Acc}_\theta \) by \[\star\star\].

2.3 If \( a \prec d_0 \) and \( d \prec b \) then \( d \in \text{Acc}_\theta \) since \( b \in \text{Acc}_\theta \).

3. If \( d \equiv d_1 + \ldots + d_n \) and \( n > 1 \) we get \( d_1, \ldots, d_n \in \text{Acc}_\theta \) from the induction hypothesis and thus \( d \in \text{Acc}_\theta \) follows from iii).

Thus we have verified that

\[
(\forall b \in \text{Acc}_\theta)[(\forall y \prec b)(\varphi ay \in \text{Acc}_\theta) \to \varphi ab \in \text{Acc}_\theta]
\]

So, from \( \text{Acc}_\theta \)-induction we get \( (\forall b \in \text{Acc}_\theta)(\varphi ab \in \text{Acc}_\theta) \), ie. \( F(a) \) completing the proof.

v) We aim to show that

\[
(\forall \beta \leq \theta)[(\forall \gamma < \beta)(\Gamma_\gamma \in \text{Acc}_\theta) \to \Gamma_\beta \in \text{Acc}_\theta]
\]

from which we may use transfinite induction along \( \theta \) (since \( \theta \) is an ordinal) to obtain the desired result.

So suppose \( \beta \leq \theta \) and \( (\forall \delta < \beta)(\Gamma_\delta \in \text{Acc}_\theta) \). Now suppose \( b \prec \Gamma_\beta \), by induction on the term complexity of \( b \) we verify that \( b \in \text{Acc}_\theta \).
If \( b \equiv 0 \) we are trivially done by ii) or if \( b \equiv \Gamma_\delta \) for some \( \delta < \beta \) then we know \( b \in \text{Acc}_\theta \) by assumption.

If \( b \equiv b_0 + \ldots + b_n \) or \( b \equiv \varphi b_0 b_1 \) then we may use parts iii) and iv) and the induction hypothesis since the components \( b_i \) have smaller term complexity.

It cannot be the case that \( b \equiv \psi b_0 \) since \( \psi a > \Gamma_\theta \) for every \( a \).

Thus using ii) we get that \( \Gamma_\beta \in \text{Acc}_\theta \) and the proof is complete. \( \square \)

**Definition 5.4.** By recursion through the construction of ordinal terms in \( T(\theta) \) we define the set \( SC_{<\Omega}(a) \) which lists the most recent strongly critical ordinal below \( \Omega \) used in the build up of the ordinal term \( a \):

1) \( SC_{<\Omega}(0) := SC_{<\Omega}(\Omega) := \emptyset \)

2) \( SC_{<\Omega}(a) := \{a\} \) if \( a \equiv \Gamma_\beta \) for some \( \beta \leq \theta \) or \( a \equiv \psi a_0 \).

3) \( SC_{<\Omega}(a_1 + \ldots + a_n) := \cup_{1 \leq i \leq n} SC_{<\Omega}(a_i) \)

4) \( SC_{<\Omega}(\varphi a_0 a_1) := SC_{<\Omega}(a_0) \cup SC_{<\Omega}(a_1) \)

5) \( SC_{<\Omega}(\psi a) := \{\psi a\} \).

Now let

\[ M_\theta := \{a \in T(\theta) \mid SC_{<\Omega}(a) \subseteq \text{Acc}_\theta\} \]

and

\[ a \prec_{M_\theta} b := a, b \in M_\theta \land a < b. \]

Finally for a definable class \( U \) we define the following formula

\[ \text{Prog}_{M_\theta}(U) := (\forall y \in M_\theta)(\forall z \prec_{M_\theta} y)(z \in U \rightarrow (y \in U)) \]

**Lemma 5.5.**

\[ \text{Acc}_\theta = M_\theta \cap \Omega := \{a \in M_\theta \mid a \prec \Omega\} \]

*Proof.* Suppose that \( a \in \text{Acc}_\theta \) and observe that \( (\forall x \in SC_{<\Omega}(a))(x \preceq a) \), thus \( SC_{<\Omega}(a) \subseteq \text{Acc}_\theta \) by [5.3] thus we have verified that \( a \in M_\theta \cap \Omega \).

Now let us suppose that \( a \in M_\theta \cap \Omega \), so we know that \( SC_{<\Omega}(a) \subseteq \text{Acc}_\theta \). By induction on the term complexity \( Ga \) we verify that \( a \in \text{Acc}_\theta \).

Clearly \( 0 \in \text{Acc}_\theta \) and if \( a \equiv \Gamma_\beta \) for some \( \beta \leq \theta \) then \( a \in \text{Acc}_\theta \) by Lemma [5.3i).

If \( a \equiv a_1 + \ldots + a_n \) then we get \( a_1, \ldots, a_n \in M_\theta \cap \Omega \) since \( SC_{<\Omega}(a_i) \subseteq SC_{<\Omega}(a) \) for each \( i \). Now using the induction hypothesis we get \( a_1, \ldots, a_n \in \text{Acc}_\theta \) and so by Lemma [5.3ii) we have \( a \in \text{Acc}_\theta \).

If \( a \equiv \varphi bc \) then we get \( b, c \in M_\theta \cap \Omega \), so using the induction hypothesis we get \( b, c \in \text{Acc}_\theta \) and so by Lemma [5.3iii) we have \( a \in \text{Acc}_\theta \).

If \( a \equiv \psi a_0 \) then \( SC_{<\Omega}(a) = \{a\} \) so we have \( a \in \text{Acc}_\theta \) by assumption. \( \square \)
**Definition 5.6.** For a definable class $U$ let

$$U^\delta := \{ b \in M_\theta \mid (\forall a \in M_\theta)[M_\theta \cap a \subseteq U \rightarrow M_\theta \cap a + \omega^b \subseteq U] \}$$

where $M_\theta \cap a := \{ b \in M_\theta \mid b \prec a \}$.

**Lemma 5.7.** $KP \vdash \text{Prog}_{M_\theta}(U) \rightarrow \text{Prog}_{M_\theta}(U^\delta)$

**Proof.** Assume

1. $\text{Prog}_{M_\theta}(U)$
2. $b \in M_\theta$
3. $(\forall x \prec_{M_\theta} b)(z \in U^\delta)$

Under these assumptions we need to verify that $b \in U^\delta$. Since we already have that $b \in M_\theta$ by (2), it suffices to verify

$$(\forall a \in M_\theta)[M_\theta \cap a \subseteq U \rightarrow M_\theta \cap a + \omega^b \subseteq U]$$

to this end we assume that

4. $a \in M_\theta$ and $M_\theta \cap a \subseteq U$

Now choose some $d \in M_\theta \cap a + \omega^b$, we must show that $d \in U$ under the assumptions (1)-(4).

If $d \prec a$ then we have $d \in U$ by (4).

If $d = a$ then using (1) and (4) we have $a \in U$.

If $d > a$ then since $d \prec a + \omega^b$, we may find $d_1, \ldots, d_k$ such that

$$d = a + \omega^{d_1} + \ldots + \omega^{d_k} \quad \text{and} \quad d_k \leq \ldots \leq d_1 \prec b$$

Since $M_\theta \cap a \subseteq U$ we get $M_\theta \cap a + \omega^{d_i} \subseteq U$ from (3).

In a similar fashion using (3) a further $k - 1$ times we obtain

$$M_\theta \cap a + \omega^{d_1} + \ldots + \omega^{d_k} \subseteq U$$

Finally using one application of $\text{Prog}_{M_\theta}(U)$ (assumption (1)) we have $d \in U$ and thus the proof is complete. \qed

**Definition 5.8.** We define the class $X_\theta$ in $KP$ as

$$X_\theta := \{ a \in M_\theta \mid (\exists x \in Ka)(x \geq a) \vee \psi a \in \text{Acc}_\theta \}$$

Recall that the function $k$ was defined in Definition 2.11 and can be computed in a $\theta$-primitive recursion fashion. The class $X_\theta$ may be thought of as those $a \in M_\theta$ for which either $\psi a$ is undefined or $\psi a \in \text{Acc}_\theta$.

**Lemma 5.9.** $KP \vdash \text{Prog}_{M_\theta}(X_\theta)$. 43
Proof. Assume

(1) \[ a \in M_\theta \]

(2) \[ (\forall z \prec_{M_\theta} a)(z \in X_\theta) \]

We need to verify that \( a \in X_\theta \). If \((\exists x \in Ka)(x \succeq a)\) then we are done, so assume \((\forall x \in Ka)(x < a)\) and thus \(\psi a \in T(\theta)\) and we must verify that \(\psi a \in \text{Acc}_\theta\). To achieve this we verify that

\((*)\) \[ b < \psi a \implies b \in \text{Acc}_\theta \]

from which we would be done by [5.3i]. To verify \((*)\) we proceed by induction on \(Gb\), the term complexity of \(b\).

If \(b \equiv 0\) or \(b \equiv \Gamma_\beta\) for some \(\beta \leq \theta\) we are done by [5.3i).

If \(b \equiv b_0 + \ldots + b_n\) or \(b \equiv \varphi b_0 b_1\) then the result follows by the induction hypothesis and [5.3i) or [5.3iii).

So suppose that \(b \equiv \psi b_0\). It must be the case that \((\forall x \in Kb_0)(x < b_0)\) and \(b_0 \prec a\). We must now show that \(b_0 \in M_\theta\) in order to use (2) to conclude that \(b_0 \in X_\theta\). The claim is that

\((**)\) \[ SC_{<\Omega}(b_0) \subseteq \text{Acc}_\theta \text{ and thus } b_0 \in M_\theta \]

Suppose \(d \in SC_{<\Omega}(b_0)\) then either \(d \equiv \Gamma_\beta\) for some \(\beta \leq \theta\) in which case \(d \in \text{Acc}_\theta\) by [5.3i) or \(d \equiv \psi d_0 \prec \psi a\) for some \(d_0\). But

\[ Gd \leq Gb_0 < Gb \]

and thus \(d \in \text{Acc}_\theta\) by induction hypothesis. Thus the claim \((**)\) is verified. Now using (2) we obtain \(b_0 \in X_\theta\) which implies \(b \equiv \psi b_0 \in \text{Acc}_\theta\). \(\square\)

Lemma 5.10. For any \(n < \omega\) and any definable class \(U\)

\[ \text{KP} \vdash \text{Prog}_{M_\theta}(U) \rightarrow M_\theta \cap e_n \subseteq U \land e_n \in U. \]

Proof. We proceed by induction on \(n\) [outside of \(\text{KP}\)].

If \(n = 0\) then \(\text{Prog}_{M_\theta}(U)\) says that

\[ (\forall a \in \text{Acc}_\theta)[(\forall b < a)(b \in U) \rightarrow a \in U]. \]

So using \(\text{Acc}_\theta\)-induction (Lemma 5.2) we obtain \(\text{Acc}_\theta \subseteq U\). Hence from [5.3] we get \(M_\theta \cap \Omega \subseteq U\). Now \(\Omega, \Omega + 1 \in M_\theta\) so using \(\text{Prog}_{M_\theta}(U)\) a further two times we have \(\Omega + 1 := e_0 \in U\) as required.

Now suppose the result holds up to \(n\); since the induction hypothesis holds for all definable classes we have that that

\[ \text{KP} \vdash \text{Prog}_{M_\theta}(U^\delta) \rightarrow M_\theta \cap e_n \subseteq U^\delta \land e_n \in U^\delta \]

and by Lemma 5.7 we have

(1) \[ \text{KP} \vdash \text{Prog}_{M_\theta}(U) \rightarrow M_\theta \cap e_n \subseteq U^\delta \land e_n \in U^\delta. \]
Now we argue informally in $KP$. Suppose $\text{Prog}_{M^\theta}(U)$, then from (1) we obtain

$$M^\theta \cap e_n \subseteq U^\delta \quad \land \quad e_n \in U^\delta.$$  

This says that

$$(\forall b \in M^\theta \cap (e_n + 1))(\forall a \in M^\theta)[M^\theta \cap a \subseteq U \rightarrow M^\theta \cap a + \omega^b \subseteq U].$$

Now if we put $a = 0$ and $b = e_n$ (noting that $e_n \in M^\theta$) we obtain

$$M^\theta \cap \omega^{e_n} \subseteq U$$

from which $\text{Prog}_{M^\theta}(U)$ implies $\omega^{e_n} \in U$ as required. \hfill \Box

**Theorem 5.11.** For every $n < \omega$

$$KP \vdash \forall \theta \psi(e_n) \in \text{Acc}_\theta$$

and hence $KP \vdash \forall \theta A_n(\theta)$.

**Proof.** By 5.9 we have $\text{Prog}_{M^\theta}(X^\theta)$ recalling that

$$X^\theta := \{a \in M^\theta \mid (\exists x \in Ka)(x \geq a) \lor \psi(a) \in \text{Acc}_\theta\}.$$

So from 5.10 we get $e_n \in X^\theta$ for any $n < \omega$ and thus $\psi(e_n) \in \text{Acc}_\theta$. \hfill \Box

### 6 The provably total set functions of $KP$

At this point we should perhaps remind ourselves that the ordinal $\psi_\alpha$ depends on a parameter $\theta$ which is the rank of $TC(\{X\})$ as $\psi$ is defined simultaneously with the sets $B^\theta(\alpha)$. After Definition 2.4 we adopted the convention to drop the subscript $\theta$ from $\psi^\theta$. For the next application we have to be aware of this dependence. For each $n < \omega$ we define the following recursive set function

$$G_n(X) := L_{\psi(e_n)}(X)$$

where $\theta$ is the rank of $TC(\{X\})$. For a formula $A(a, b)$ of $KP$ let

$$\forall x \exists ! y A(x, y) := \forall x \forall y_1 \forall y_2[A(x, y_1) \land A(x, y_2) \rightarrow y_1 = y_2] \land \forall x \exists y A(x, y).$$

**Definition 6.1.** If $T$ is a theory formulated in the language of set theory, $f$ a set function and $\mathcal{X}$ a class of formulae. We say that $f$ is $\mathcal{X}$ definable in $T$ if there is some $\mathcal{X}$-formula $A_f(a, b)$ with exactly the free variables $a, b$ such that

i) $V \models A_f(x, y) \iff f(x) = y$.

ii) $T \models \forall x \exists ! y A_f(x, y)$.

**Theorem 6.2.** Suppose $f$ is a set function that is $\Sigma$ definable in $KP$, then there is some $n$ (which we may compute from the finite derivation) such that

$$V \models \forall x(f(x) \in G_n(x)).$$

Moreover $G_m$ is $\Sigma$ definable in $KP$ for each $m < \omega$.  

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Proof. Let $A_f(a,b)$ be the $\Sigma$ formula expressing $f$ such that $\text{KP} \vdash \forall x \exists ! y A_f(x,y)$ and fix an arbitrary set $X$. Let $\theta$ be the rank of $X$. Applying Theorem 4.10 we can compute some $k < \omega$ such that

$$\mathcal{H}_0 \frac{\Omega \cdot \omega^k}{\Omega + k} \forall x \exists ! y A_f(x,y).$$

Applying Lemma 3.15 iv) twice we get

$$\mathcal{H}_0 \frac{\Omega \cdot \omega^k}{\Omega + k} \exists y A_f(X,y).$$

Applying Theorem 3.17 (predicative cut elimination) we get

$$\mathcal{H}_0 \frac{\varepsilon_{k+1}}{\varepsilon_{k+1}} \exists y A_f(X,y).$$

Now by Theorem 3.23 (collapsing) we have

$$\mathcal{H}_{\varepsilon_{k+2}} \frac{\psi_{\varphi(e_{k+2})}}{\psi_{\varphi(e_{k+2})}} \exists y A_f(X,y).$$

Applying Theorem 3.17 (predicative cut elimination) again yields

$$\mathcal{H}_\varphi \frac{\psi_{\varphi(e_{k+2})}}{\psi_{\varphi(e_{k+2})}} \exists y A_f(X,y) \quad \text{where } \gamma := e_{k+2}.$$

Now by Lemma 3.18 (boundedness) we obtain

$$(1) \quad \mathcal{H}_\varphi \frac{\psi_{\psi(\varphi)})}{\psi_{\psi(\varphi)}} \exists y A_f(X,y) \quad \text{where } \alpha := \psi_{\varphi}.\psi(\varphi).$$

Since (1) contains no instances of (Cut) or ($\Sigma$-Ref$_\Omega(X)$), it follows by induction on $\alpha$ that

$$L_\alpha(X) \models \exists y A_f(X,y)$$

It remains to note that $L_\alpha(X) \subseteq G_{k+3}(X)$ to complete this direction of the proof.

For the other direction we argue informally in $\text{KP}$. Let $X$ be an arbitrary set, we may specify the rank of $X$ in a $\Delta_0$ manner ([3] p. 29). By Theorem 5.11 we can find an ordinal of the same order type as $\psi_{\varphi}(e_m)$ with $\theta$ being the rank of $TC\{\{X\}\}$. We can now generate $L_{\psi_{\varphi}(e_m)}(X)$ by $\Sigma$-recursion ([3] p. 26 Theorem 6.4).

The comparison of Theorem 1.2 with Theorem 6.2 provides a pleasing relation between the arithmetic and set theoretic worlds.

Remark 6.3. In fact the first part of [6.2] can be carried out inside $\text{KP}$, i.e. If $f$ is $\Sigma$ definable in $\text{KP}$ then we can compute some $n$ such that $\text{KP} \vdash \forall x (\exists ! y \in G_n(x)) A_f(x,y)$. This is not immediately obvious since it appears we need induction up to $\psi_{\varphi}(\varepsilon_{\psi+1})$, which we do not have access to in $\text{KP}$. The way to get around this is to note that we could, in fact, have managed with an infinitary system based on an ordinal representation built out of $B_{\theta}(e_m)$, provided $m$ is high enough, and we may compute how high $m$ needs to be from the finite derivation. We do have access to induction up to $\psi(e_m)$ for any ordinal $\theta$ in $\text{KP}$ by Theorem 5.11.
7 Applications to semi-intuitionistic KP

PA is conservative over its intuitionistic cousin (called Heyting arithmetic, HA) for $\Pi^0_2$-statements. One might wonder whether a corresponding result holds in set theory for $\Pi_2$-statements. As it turns out, such a result does not obtain for KP and its intuitionistic version IKP however, adding the law of excluded middle for atomic formulas to IKP yields conservativity for $\Pi_2$ theorems.

A semi-intuitionistic version of IKP is obtained by assuming the law of excluded middle for atomic formulas, i.e.,

$$\forall x \forall y (x \in y \lor \neg x \in y).$$

Semi-intuitionistic versions of KP have become important in Feferman’s work in connection with discussions of definiteness of concepts and the continuum hypothesis (cf. [17, 18, 19, 20, 51]).

Theorem 7.1. KP is $\Pi_2$ conservative over the semi-intuitionistic theory IKP plus (3).

Proof. Let $T$ be the theory IKP augmented by (3). Assume that $\text{KP} \vdash \forall x \exists y A(x, y)$, where $A(a, b)$ is $\Delta_0$. We now argue in $T$. Let $X$ be an arbitrary set. As in the proof of Theorem 4.10 we can determine an $\alpha$ (uniformly depending on the rank of $\text{TC}(\{X\})$ such that

$$\forall x \forall y (x \in y \lor \neg x \in y).$$

To see that we can do this inside $T$ note that the $m$ in Remark 6.3 does not depend on $\theta$. Since $\text{L}_\alpha(X)$ contains no instances of (Cut) or (Σ-Ref$\Omega_1(X)$), it follows by induction on $\alpha$ that

$$\text{L}_\alpha(X) \models \exists y A(X, y).$$

Excluded middle for atomic formulas is required at several points. For instance it is needed in Lemma 4.3, Case 1. Also when showing that all sequents $\Lambda$ occurring in the derivation (4) are true in $\text{L}_\alpha(X)$ one needs to invoke the law of excluded middle for $\Delta_0$-formulas. The latter follows from (3) with the help of $\Delta_0$-Separation.

8 A relativised ordinal analysis of KP($\mathcal{P}$)

With the help of [56] and the foregoing machinery one can also characterize the provable power recursive set functions of Power Kripke-Platek set theory, KP($\mathcal{P}$). For background on KP($\mathcal{P}$) see [56]. To introduce its axioms we need the notion of subset bounded formula.

Definition 8.1. We use subset bounded quantifiers $\exists x \subseteq y \ldots$ and $\forall x \subseteq y \ldots$ as abbreviations for $\exists x(x \subseteq y \land \ldots)$ and $\forall x(x \subseteq y \rightarrow \ldots)$, respectively.

The $\Delta^P_0$-formulae are the smallest class of formulae containing the atomic formulae closed under $\land, \lor, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a, \exists x \in a, \forall x \subseteq a, \exists x \subseteq a.$$ 

A formula is in $\Sigma^P$ if belongs to the smallest collection of formulae which contains the $\Delta^P_0$-formulae and is closed under $\land, \lor$ and the quantifiers $\forall x \in a, \exists x \in a, \forall x \subseteq a$ and $\exists x$. A formula is $\Pi^P$ if belongs to the smallest collection of formulae which contains the $\Delta^P_0$-formulae and is closed under $\land, \lor, \neg$ and the quantifiers $\forall x \in a, \exists x \in a, \forall x \subseteq a$ and $\forall x$. A formula is $\Pi^P$ if belongs to the smallest collection of formulae which contains the $\Delta^P_0$-formulae and is closed under $\land, \lor, \neg$ and the quantifiers $\forall x \in a, \exists x \in a, \forall x \subseteq a$ and $\forall x$.

2See [12] for a definition of IKP.

3This means that the disjunction over all formulas in $\Lambda$ is true in $L_\alpha(X)$.
Definition 8.2. \(\text{KP}(\mathcal{P})\) has the same language as \(\text{ZF}\). Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, \(\Delta^0\)-Separation, \(\Delta^0\)-Collection and Set Induction (or Class Foundation).

The transitive models of \(\text{KP}(\mathcal{P})\) have been termed \textbf{power admissible} sets in [22].

Remark 8.3. Alternatively, \(\text{KP}(\mathcal{P})\) can be obtained from \(\text{KP}\) by adding a function symbol \(\mathcal{P}\) for the powerset function as a primitive symbol to the language and the axiom
\[
\forall y [y \in \mathcal{P}(x) \iff y \subseteq x]
\]
and extending the schemes of \(\Delta^0\) Separation and Collection to the \(\Delta^0\)-formulae of this new language.

Lemma 8.4. \(\text{KP}(\mathcal{P})\) is not the same theory as \(\text{KP} + \text{Pow}\), where \(\text{Pow}\) denotes the Powerset Axiom. Indeed, \(\text{KP} + \text{Pow}\) is a much weaker theory than \(\text{KP}(\mathcal{P})\) in which one cannot prove the existence of \(V_{\omega+\omega}\).

Proof. [56, Lemma 2.4]. \(\Box\)

8.1 The infinitary proof system \(\text{RS}^\mathcal{P}_\Omega(X)\)

The infinitary proof system \(\text{RS}^\mathcal{P}_\Omega\) of [56] is based on a formal analogue of the von Neumann hierarchy along the Bachmann-Howard ordinal. For our purposes both have to be relativised to a given set \(X\).

Definition 8.5. Let \(X\) be any set. We may relativise the von Neumann hierarchy to \(X\) as follows:
\[
\begin{align*}
V_0(X) & := TC(\{X\}) \quad \text{the transitive closure of } \{X\} \\
V_{\alpha+1}(X) & := \{B : B \subseteq V_\alpha(X)\} \\
V_\theta(X) & := \bigcup_{\xi<\theta} V_\xi(X) \quad \text{when } \theta \text{ is a limit.}
\end{align*}
\]

Let \(X\) be an arbitrary (well founded) set and let \(\theta\) be the set-theoretic rank of \(X\) (hereby referred to as the \(\in\)-rank). Henceforth all ordinals are assumed to belong to the ordinal notation system \(T(\theta)\) developed in section 3. The system \(\text{RS}^\mathcal{P}_\Omega(X)\) will be the relativised version of the infinitary proof system \(\text{RS}^\mathcal{P}_\Omega\) from [56].

Definition 8.6. We give an inductive definition of the set \(\mathcal{T}^\mathcal{P}\) of \(\text{RS}^\mathcal{P}_\Omega(X)\) terms. To each term \(t \in \mathcal{T}^\mathcal{P}\) we assign an ordinal level \(|t|\).

(i) For every \(u \in TC(\{X\})\), \(\bar{u} \in \mathcal{T}^\mathcal{P}\) and \(|\bar{u}| := \Gamma_{\text{rank}(u)}\).

(ii) For every \(\alpha < \Omega\), \(V_\alpha(X) \in \mathcal{T}^\mathcal{P}\) and \(|V_\alpha(X)| := \Gamma_{\theta+1} + \alpha\).

(iii) For each \(\alpha < \Omega\), we have infinitely many free variables \(a^1_\alpha, a^2_\alpha, a^3_\alpha, \ldots\) which are terms of level \(\Gamma_{\theta+1} + \alpha\).

(iv) If \(\alpha < \Omega\), \(A(a,b_1,\ldots,b_n)\) is a \(\Delta^0\)-formula of \(\text{KP}(\mathcal{P})\) with all free variables displayed and \(s_1,\ldots,s_n\) are terms in \(\mathcal{T}^\mathcal{P}\) then
\[
[x \in V_\alpha(X) | A(x,s_1,\ldots,s_n)]
\]
is a term of level \(\Gamma_{\theta+1} + \alpha\).
The $\text{RS}_\Omega^P(X)$-formulae are the expressions of the form $F(s_1,\ldots,s_n)$, where $F(a_1,\ldots,a_n)$ is a formula of $\text{KP}(P)$ with all free variables exhibited and $s_1,\ldots,s_n$ are $\text{RS}_\Omega^P(X)$-terms. We set

$$\left| F(s_1,\ldots,s_n) \right| = \{|s_1|,\ldots,|s_n|\}.$$ 

For a sequent $\Gamma = \{A_1,\ldots,A_n\}$ we define

$$|\Gamma| := |A_1| \cup \ldots \cup |A_n|.$$ 

A formula is a $\Delta_0^P$-formula of $\text{RS}_\Omega^P(X)$ if it is of the form $F(s_1,\ldots,s_n)$ with $F(a_1,\ldots,a_n)$ being a $\Delta_0^P$-formula of $\text{KP}(P)$ and $s_1,\ldots,s_n$ $\text{RS}_\Omega^P(X)$-terms.

As in the case of the Tait-style version of $\text{KP}(P)$ in [56, Sec. 3], we let $\neg A$ be the formula which arises from $A$ by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\wedge,\vee,(\forall x \in s), (\exists x \in s), (\forall x \subseteq s),(\exists x \subseteq s),\forall x,\exists x$ by $\vee,\wedge,(\exists x \in s), (\forall x \in s),(\exists x \subseteq s),\exists x,\forall x$, respectively, and (iii) dropping double negations. $A \to B$ stands for $\neg A \vee B$.

**Remark 8.7.** There is a crucial difference between Definition 3.3 and Definition 8.6 when it comes to measuring the level of a comprehension term. The level of $[x \in \forall_\alpha(X)|A(x,s_1,\ldots,s_n)]$ does not take the terms $s_1,\ldots,s_n$ into account. They may be of arbitrary (especially higher) level.

Since we also want to keep track of the complexity of cuts appearing in derivations, we endow each formula with an ordinal rank.

**Definition 8.8.** The *rank* of a term or formula is determined as follows.

1. $rk(\overline{u}) := \Gamma_{\text{rank}(u)}$ for $u$ in the transitive closure of $X$.
2. $rk(\forall_\alpha(X)) := \Gamma_{\theta+1} + \alpha$.
3. $rk([x \in \forall_\alpha(X) | F(x)]) := \max\{\Gamma_{\theta+1} + \omega \cdot \alpha + 1,rk(F(\overline{0}) + 2}\}$.
4. $rk(s \in t) := rk(s \notin t) := \max\{|s| + 6,|t| + 1\}$.
5. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := \max\{rk(t) + 3,rk(F(\overline{0})) + 2\}$.
6. $rk((\forall x \subseteq t)F(x)) := rk((\exists x \subseteq t)F(x)) := \max\{rk(t) + 3,rk(F(\overline{0})) + 2\}$.
7. $rk(\exists x F(x)) := rk(\forall x F(x)) := \max\{|\Omega|,rk(F(\overline{0}) + 2\}$.
8. $rk(A \wedge B) := rk(A \vee B) := \max\{rk(A),rk(B)\} + 1$.

**Definition 8.9.** The axioms of $\text{RS}_\Omega^P(X)$ are:

(X1) $\Gamma, \overline{u} \in \overline{v}$ if $u,v \in TC(X)$ and $u \in v$.

(X2) $\Gamma, \overline{u} \notin \overline{v}$ if $u,v \in TC(X)$ and $u \notin v$.

(A1) $\Gamma, A, \neg A$ for $A$ in $\Delta_0^P$.

(A2) $\Gamma, t = t$. 

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(A3) θ, s₁ ≠ t₁, ..., sₙ ≠ tₙ, ¬A(s₁, ..., sₙ), A(t₁, ..., tₙ)
    for A(s₁, ..., sₙ) in Δ₀.

(A4) θ, s ∈ Vₐ(X) if |s| < |Vₐ(X)|.

(A5) θ, s ⊆ Vₐ(X) if |s| ≤ |Vₐ(X)|.

(A6) θ, t ∉ [x ∈ Vₐ(X) | F(x, s)], F(t, s)
    whenever F(t, s) is Δ₀ and |t| < |Vₐ(X)|.

(A7) θ, ¬F(t, s), t ∈ [x ∈ Vₐ(X) | F(x, s)]
    whenever F(t, s) is Δ₀ and |t| < |Vₐ(X)|.

We adopt the notion of operator from Definition 3.6. If s is an RSₐ(X)-term, the operator H[s] is defined by

H[s](X) = H(X ∪ {s}).

Likewise, if X is a formula or a sequent we define

H[X](X) = H(X ∪ |X|).

Definition 8.10. Let H be an operator and let Λ be a finite set of RSₐ(X)-formulae. H₁₀ Λ is defined by recursion on α.

If Λ is an axiom and |Λ| ∪ {α} ⊆ H(∅), then H₁₀ Λ.

Moreover, we have inductive clauses pertaining to the inference rules of RSₐ(X), which all come with the additional requirement that

|Λ| ∪ {α} ⊆ H(∅)

where Λ is the sequent of the conclusion. We shall not repeat this requirement below.

Below the third column gives the requirements that the ordinals have to satisfy for each of the inferences. For instance in the case of (∀)ₐ, to be able to conclude that H₁₀ θ, ∀xF(x), it is required that for all terms s there exists αₛ such that H[s]₁₀ θ, F(s) and |s| < αₛ + 1 < α. The side conditions for the rules (b∀)ₐ, (p∀)ₐ, (ζ)ₐ, (Z)ₐ below have to be read in the same vein.

Below we shall write |s| < |t| and |s| ≤ |t| for |s| < max(θ₊₁, |t|) and |s| ≤ max(θ₊₁, |t|), respectively.

The clauses are the following:
Remark 8.11. Suppose $\mathcal{H} \vdash^{\alpha} \Gamma(s_1, \ldots, s_n)$ where $\Gamma(a_1, \ldots, a_n)$ is a sequent of $\text{KP}(P)$ such that all variables $a_1, \ldots, a_n$ do occur in $\Gamma(a_1, \ldots, a_n)$ and $s_1, \ldots, s_n$ are $\text{RS}_\Omega^P(X)$-terms. Then we have that $|s_1|, \ldots, |s_n| \in \mathcal{H}(\emptyset)$. Standing in sharp contrast to the ordinal analysis of $\text{KP}$, however, the terms $s_i$ may and often will contain subterms that the operator $\mathcal{H}$ does not control, that is, subterms $t$ with $|t| \not\in \mathcal{H}(\emptyset)$.

The embedding of $\text{KP}(P)$ into $\text{RS}_\Omega^P(X)$ and the ordinal analysis of $\text{RS}_\Omega^P(X)$ can be carried in much the same way as for $\text{RS}_\Omega^P$ in [56] with only minor amendments necessary to deal with terms and axioms pertaining to the given set $X$. Below we list the main steps.

**Theorem 8.12.** If $\text{KP}(P) \vdash \Gamma(a_1, \ldots, a_n)$ where $\Gamma(a_1, \ldots, a_n)$ is a finite set of formulae whose free variables are amongst $a_1, \ldots, a_n$, then there is some $m < \omega$ (which we may compute from the derivation) such that

$$\mathcal{H}(s_1, \ldots, s_n) \vdash^{\Omega \omega^m} \Gamma(s_1, \ldots, s_n)$$

for any operator $\mathcal{H}$ and any $\text{RS}_\Omega^P(X)$ terms $s_1, \ldots, s_n$.

**Proof.** This can be proved in the same way as [56, Theorem 6.9].

**Theorem 8.13** (Cut elimination I).

$$\mathcal{H} \vdash^{\alpha} \Gamma \implies \mathcal{H} \vdash^{\omega_\alpha} \Gamma$$

where $\omega_0(\beta) := \beta$ and $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

**Proof:** The proof is the special case of Theorem 3.17 when $\rho = \Omega + n$ and $\alpha = 0$. See also [56, Theorem 7.1].

For a formula $C$ of $\text{RS}_\Omega^P(X)$, $C^{\forall \delta}(X)$ is obtained from $C$ by replacing all unbounded quantifiers $Qz$ in $C$ by $(Qz \in \forall \delta(X))$. 

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Lemma 8.14 (Boundedness for $RS^P_\Omega(X)$). If $C$ is a $\Sigma^P$ formula, $\alpha \leq \beta < \Omega$, $\beta \in \mathcal{H}$ and $\mathcal{H} \models^p_\Omega \Gamma, C$ then $\mathcal{H} \models^\alpha_\beta \Gamma, C^{\Omega,\beta}(X)$.

Proof. Similar to Lemma 8.14.

Theorem 8.15 (Collapsing for $RS^P_\Omega(X)$). Suppose $\Gamma$ is a set of $\Sigma^P$ formulae such that $|\Gamma| \subseteq B(\eta)$ and $\eta \in B(\eta)$.

If $\mathcal{H}_\eta \models^\alpha_\Omega \Gamma$ then $\mathcal{H}_{\hat{\alpha}} \models^\psi_{\alpha^+} \Gamma$

where $\hat{\alpha} = \eta + \omega^{\Omega+\alpha}$.

Proof. The proof is essentially the same as that of [54, Theorem 7.4].

For the characterisation theorem for $\mathbf{KP}(P)$, we need to show that derivability in $RS^P_\Omega(X)$ entails truth for $\Sigma^P$-formulae. Since $RS^P_\Omega(X)$-formulae contain variables we need the notion of assignment. Let $\text{VAR}$ be the set of free variables of $RS^P_\Omega(X)$. A variable assignment $\ell$ is a function

\[ \ell : \text{VAR} \rightarrow V_{\psi(\varepsilon^\Omega+1)} \]

satisfying $\ell(a^\alpha) \in V_{\alpha+1}(X)$. $\ell$ can be canonically lifted to all $RS^P_\Omega(X)$-terms as follows:

\[ \ell(\bar{u}) = u \text{ for } u \in TC\{X\} \]
\[ \ell(\forall_a(X)) = V_a(X) \]
\[ \ell([x \in V_a(X) \mid F(x, s_1, \ldots, s_n)]) = \{ x \in V_a(X) : F(x, \ell(s_1), \ldots, \ell(s_n)) \} \]

Note that $\ell(s) \in V_{\psi(\varepsilon^\Omega+1)}(X)$ holds for all $RS^P_\Omega(X)$-terms $s$. Moreover, we have $\ell(s) \in V_{|s|+1}(X)$.

Theorem 8.16 (Soundness). Let $\mathcal{H}$ be an operator with $\mathcal{H}(\emptyset) \subseteq B(\varepsilon^\Omega+1)$ and $\alpha, \rho < \psi(\varepsilon^\Omega+1)$. Let $\Gamma(s_1, \ldots, s_n)$ be a sequent consisting only of $\Sigma^P$-formulae with constants from $TC\{X\}$. Suppose

\[ \mathcal{H} \models^\alpha_\rho \Gamma(s_1, \ldots, s_n) \]

Then, for all variable assignments $\ell$,

\[ V_{\psi(\varepsilon^\Omega+1)}(X) \models \Gamma(\ell(s_1), \ldots, \ell(s_n)) \]

where the latter, of course, means that $V_{\psi(\varepsilon^\Omega+1)}$ is a model of the disjunction of the formulae in $\Gamma(\ell(s_1), \ldots, \ell(s_n))$.

Proof: The proof is basically the same as for [54, Theorem 8.1]. It proceeds by induction on $\alpha$. Note that, owing to $\alpha, \rho < \Omega$, the proof tree pertaining to $\mathcal{H} \models^\alpha_\rho \Gamma(s_1, \ldots, s_n)$ neither contains any instances of ($\Sigma^P$-$\text{Ref}$) nor of ($\forall$)$_\infty$, and that all cuts are performed with $\Delta^P_\Omega$-formulae. The proof is straightforward as all the axioms of $RS^P_\Omega$ are true under the interpretation and all other rules are truth preserving with respect to this interpretation. Observe that we make essential use of the free variables when showing the soundness of ($\forall\bar{v}$)$_\infty$, ($p\forall\bar{v}$)$_\infty$, ($\exists$)$_\infty$ and ($\exists$)$_\infty$. We treat ($p\forall\bar{v}$)$_\infty$ as an example. So assume ($\forall x \subseteq s_1 F(x, \bar{s}) \in \Gamma(\bar{s})$ and

\[ \mathcal{H}[r] \models^\alpha_\rho \Gamma(s_1, \ldots, s_n), r \subseteq s_i \rightarrow F(r, \bar{s}) \]

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holds for all terms \( r \) with \( | r | \leq | s_i | \) for some \( \alpha_r < \alpha \). In particular we have

\[
\mathcal{H}[a^\beta] \models^\alpha \Gamma(s_1, \ldots, s_n), a^\beta \subseteq s_i \to F(a^\beta, \vec{s})
\]

where \( \beta = | s_i | \) and \( a^\beta \) is a free variable not occurring in \( \Gamma(s_1, \ldots, s_n) \) and \( \alpha' = \alpha_{a^\beta} \). By the induction hypothesis we have

\[
V_{\psi \Omega(\varepsilon \Omega+1)}(s_1, \ldots, s_n) \models \Gamma(\ell(s_1), \ldots, \ell(s_n), \ell'(a^\beta) \subseteq \ell(s_i) \to F(\ell'(a^\beta), \ell(s_1), \ldots, \ell(s_n))
\]

where \( \ell' \) is an arbitrary variable assignment. This entails that either

\[
V_{\psi \Omega(\varepsilon \Omega+1)}(s_1, \ldots, s_n) \models (\forall x \subseteq \ell(s_i)) F(x, \ell(s_1), \ldots, \ell(s_n))
\]

or

\[
V_{\psi \Omega(\varepsilon \Omega+1)}(s_1, \ldots, s_n) \models (\exists y A_f(x, y))
\]

for all assignments \( \ell' \). In the former case we have found what we want and in the latter case we arrive at

\[
V_{\psi \Omega(\varepsilon \Omega+1)}(s_1, \ldots, s_n) \models \Gamma(\ell(s_1), \ldots, \ell(s_n)).
\]

\section{8.2 The provably total set functions of \( \text{KP}(\mathcal{P}) \)}

For each \( n < \omega \) we define the following recursive set function

\[
G_n^P(X) := V_{\psi \theta(e_n)}(X)
\]

where \( e_n \) was defined in \( \text{[2]} \) and \( \theta \) stands for the rank of the transitive closure of \( X \).

\textbf{Theorem 8.17.} Suppose \( f \) is a set function that is \( \Sigma^P \) definable in \( \text{KP}(\mathcal{P}) \), then there is some \( n \) (which we may compute from the finite derivation) such that

\[
V \models (\forall x (f(x) \in G_n^P(x))).
\]

Moreover \( G_n^P \) is \( \Sigma^P \) definable in \( \text{KP}(\mathcal{P}) \) for each \( m < \omega \).

\textbf{Proof.} Let \( A_f(a, b) \) be the \( \Sigma^P \) formula expressing \( f \) such that \( \text{KP}(\mathcal{P}) \vdash (\forall x \exists y A_f(x, y)) \) and fix an arbitrary set \( X \). Let \( \theta \) be the rank of \( X \). Applying Theorem \( \text{[8.12]} \) we can compute some \( k < \omega \) such that

\[
\mathcal{H}_0 \models^\Omega \omega_k \exists x \exists y A_f(x, y).
\]

Applying inversion as in Lemma \( \text{[3.15 iv]} \) twice we get

\[
\mathcal{H}_0 \models^\Omega \omega_k \exists y A_f(X, y).
\]

Applying Theorem \( \text{[8.13]} \) we get

\[
\mathcal{H}_0 \models^\Omega \omega_{k+1} \exists y A_f(X, y).
\]

Now by Theorem \( \text{[8.15]} \) (collapsing) we have

\[
\mathcal{H}_{k+2} \models^\psi_{\theta(e_{k+2})} \exists y A_f(X, y).
\]
Now by Lemma 8.14 (boundedness) we obtain
\[ H_{\bar{\psi}(\gamma)} (\exists y \in V_{\bar{\psi}(\gamma)}(X)) A_f(X, y) V_{\bar{\psi}(\gamma)}(\gamma) \] where \( \gamma := e_{k+2} \).

The Soundness Theorem 8.16 applied to (5) now yields that
\[ V_{\bar{\psi}(\gamma)}(\gamma) \models \exists y A_f(X, y). \]

It remains to note that \( V_\alpha(X) \subseteq G^P_{k+3}(X) \) to complete this direction of the proof.

For the other direction we argue informally in \( KP(P) \). Let \( X \) be an arbitrary set. By Theorem 5.11 we can find an ordinal of the same order type as \( \psi(\epsilon^{n+1}) \). We can now generate \( V_{\psi(\epsilon^{n+1})}(X) \) by \( \Sigma^P \)-recursion (similar to [3] p. 26 Theorem 6.4).

Remark 8.18. As was the case for \( KP \), the first part of 6.2 can be carried out inside \( KP(P) \), i.e.

\[ \text{If } f \text{ is } \Sigma^P \text{ definable in } KP(P) \text{ then we can compute some } n \text{ such that} \]
\[ KP(P) \vdash \forall x (\exists! y \in G_n^P(x)) A_f(x, y). \]

This is not immediately obvious since it appears we need induction up to \( \psi(\epsilon^{\Omega+1}) \), which we do not have access to in \( KP(P) \). The way to get around this is to note that we could, in fact, have managed with an infinitary system based on an ordinal representation built out of \( B_\theta(e_m) \), provided \( m \) is high enough, and we may compute how high \( m \) needs to be from the finite derivation. We do have access to induction up to \( \psi(e_m) \) in \( KP(P) \) by Theorem 5.11.

9 Adding global choice: \( KP(P) + AC_{global} \)

Here we extend the relativised ordinal analysis to \( KP(P) \) with global choice. Since the global axiom of choice, \( AC_{global} \), is less familiar, let us spell out the details. By \( KP(P) + AC_{global} \) we mean an extension of \( KP(P) \) where the language contains a new binary relation symbol \( R \) and the axiom schemes of \( KP(P) \) are extended to this richer language and the following axioms pertaining to \( R \) are added:

\[ \begin{align*}
\forall x \forall y \forall z [R(x, y) \land R(x, z) \rightarrow y = z] \\
\forall x [x \neq \emptyset \rightarrow \exists y \in x R(x, y)].
\end{align*} \]

Section 3 of [58] describes an extension of \( RS^P_{\Omega}(R, X) \) that incorporates the new symbol \( R \). We can now relativise this system to a given set \( X \) as we did with \( RS^P_{\Omega} \) in the previous section. Let us call the relativized version \( RS^P_{\Omega}(R, X) \). The ordinal analysis of \( RS^P_{\Omega}(R, X) \) can be performed with almost no changes as for \( RS^P_{\Omega}(X) \) in the foregoing section. On account of the relativization we arrive at stronger versions of [58] Corollary 3.1 and [58] Theorem which incorporate the parameter \( X \). A \( \Pi^P_2 \)-formula is a formula of the form \( \forall y A(y) \) with \( A(y) \) in \( \Sigma^P \).

**Theorem 9.1.** Let \( B \) be \( \Pi^P_2 \)-sentence of the language without the predicate \( R \). If \( KP(P) + AC_{global} \vdash B \), then \( KP(P) + AC \vdash B \).

**Proof.** Basically as in [58] Theorem 3.2. \( \square \)
The acronym CZF stands for Constructive Zermelo-Fraenkel set theory. For details see [1, 2].

**Corollary 9.2.** (i) $\text{KP}(P) + \text{AC}_{\text{global}}, \text{KP}(P) + \text{AC}$, and $\text{CZF} + \text{AC}$ prove the same $\Pi^2_2$-sentences.

(ii) The three theories are of the same proof-theoretic strength as $\text{KP}(P)$. More precisely, they prove the same $\Pi^1_4$-sentences of the language of second order arithmetic when identified with their canonical translation into the language of set theory.

**Proof.** (i) For $\text{KP}(P) + \text{AC}_{\text{global}}$ and $\text{KP}(P) + \text{AC}$ this follows from the foregoing Theorem. A question left open in [55] was that of the strength of constructive Zermelo-Fraenkel set theory with the axiom of choice. There $\text{CZF} + \text{AC}$ was interpreted in $\text{KP}(P) + V = L$ ([55, Theorem 3.5]). However, the realizability interpretation works with $\text{AC}_{\text{global}}$ as well. Moreover, for this notion of realizability, realizability of a $\Pi^2_2$-sentence $B$ entails its truth. Therefore if $\text{CZF} + \text{AC} \vdash B$, then $\text{KP}(P) + \text{AC}_{\text{global}} \vdash B$.

Conversely note that $\text{CZF} + \text{AC}$ proves the law of excluded middle for $\Delta^P_1$-formulae. This amount of classical logic suffices to prove the power set axiom from the subset collection axiom. The proof-theoretic ordinal of $\text{CZF}$ is also the Bachmann-Howard ordinal. Moreover, in Theorem [5, 11], $\text{KP}$ can be replaced by $\text{CZF} + \text{AC}$. As a result, the ordinal analysis for $B$ utilizing $\text{RS}^P(R, X)$, can be carried out in $\text{CZF} + \text{AC}$ itself and the proof of the pertaining soundness is also formalisable in $\text{CZF} + \text{AC}$, whence the latter theory proves $B$.

(ii) follows from (i) viewed in conjunction with [54, Corollary 3.5].

Finally, we remark that the three theories of Corollary 9.2 can be added to the list of proof-theoretically equivalent theories presented in [57, Theorem 15.1].

**10 The provably total set functions of other theories**

Part of the machinery developed here could also be used to give a characterization of the total set functions of extensions of $\text{KP}$ such as the theories $\text{KPi}$ and $\text{KPM}$ that are describing a recursively inaccessible and a recursively Mahlo universe of sets, respectively (see [30, 43, 49]). This however would also require an interpretation of collapsing functions as acting on set-theoretic ordinals along the lines of [40, 44].

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