The two-phase approximation for black hole collisions: Is it robust?

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Recently Abrahams and Cook devised a method of estimating the total radiated energy resulting from collisions of distant black holes by applying Newtonian evolution to the holes up to the point where a common apparent horizon forms around the two black holes and subsequently applying Schwarzschild perturbation techniques. Despite thecrudeness of their method, their results for the case of head-on collisions were surprisingly accurate. Here we take advantage of the simple radiated energy formula devised in the close-slow approximation for black hole collisions to test how strongly the Abrahams-Cook result depends on the choice of moment when the method of evolution switches over from Newtonian to general relativistic evolution. We find that their result is robust, not depending strongly on this choice.

Black hole collisions are considered to be a likely source of gravitational waves, expected to be detectable by the next generation of gravitational wave antennas which will begin operation in a few years. Hence, there is currently strong motivation within the relativistic astrophysics community to solve Einstein’s equations for the case of two colliding black holes. The full, numerical solution of Einstein’s equations for a black hole collision is a daunting problem, and it has yet to be achieved for any realistic configuration. It is thus useful to apply approximation techniques wherever possible both to build intuition and as alternative calculations against which the numerical results can be checked. If possible it would be desirable to have approximate estimates of how much radiation will be generated by various types of collisions in order to identify the most interesting problems for more detailed numerical study.

One useful approximation technique is the close-limit approximation. For extremely close black holes we can treat the system as a linear perturbation of a single Schwarzschild black hole. This approximation is naturally expected to work well for evolution from initial data in which the two initial black holes are sufficiently close to be well inside a common horizon, but for the cases studied so far the results have shown good agreement with the numerical calculations even to separations where no common horizon exists. At the other extreme, if the black holes are far apart and moving slowly we might apply the Newtonian approximation, whereby the holes are treated simply as point particles moving in a flat background. Clearly, we can never understand the later and most important features of the collapse in only this far-limit. On the other hand, Fig. 1 schematically depicts the picture which emerges upon considering these approximations together. In a “realistic” head-on black hole collision the black holes are initially well-separated as is indicated on the left side of the figure. For this part of the in-fall we can treat the holes as Newtonian point particles. In the subsequent evolution the black holes grow closer and their interaction grows stronger until the two holes merge into a single distorted black hole. This black hole radiates away its distortions, ultimately leaving a Schwarzschild black hole as the outcome. Between the Newtonian region and the region where the close-limit approximation applies there is a complicated transition period which can only be understood by the exact numerical evolution. However we may note that at least for axisymmetric collisions nearly all of the radiation resulting from the collision is released in the final moments of collapse, when the black holes have already merged.

FIG. 1. A cartoon illustration of how the two-phase approximation is constructed.
and the close-limit approximation works well. One is then led to wonder if the essential characteristics of the radiation can be captured by stretching the range of the Newtonian and close-limit approximations until they meet to construct a two-phase approximation. If we restrict consideration to the total energy of the radiation generated by the collapse, might we achieve a good estimate of this radiation by using Newtonian mechanics to evolve the system of black holes until they are on the verge of coalescence, and subsequently evolving from this point using the close-limit approximation?

Abrahams and Cook first performed such a calculation in their study of black holes with initial inward momentum. They applied the close-limit approximation to nearby black holes with just enough inward momentum to guarantee that they share a common apparent horizon in the initial data reasoning that the horizon formation point was an identifiable marker which roughly indicated the greatest separation at which the close-limit calculation would be successful. Then, they compared their results with those of the “exact” numerical calculation for the case of initially stationary black holes released from such a separation that Newtonian in-fall would provide the specified momentum at the moment of apparent horizon formation. The result was excellent agreement with the numerical calculation. The Abrahams-Cook calculation seems to demonstrate the effectiveness of two-phase approximate evolution for treating initially distant black hole collisions. However, let us suppose the numerical result didn’t exist. Could we then claim that the two phase approximation was generating a plausible estimate of the radiation? This is an important question because the close-limit approximation is currently being applied to problems which have not yet been solved numerically. The critical question in determining the usefulness of the two-phase approximation is whether the result is sensitive to how the Newtonian first phase is joined to the close limit second phase. For collapse from large separations, we know that the close-limit approximation will wildly overestimate the radiation. Similarly, we do not expect a reasonable result if we allow the Newtonian phase of the evolution to proceed too far. Unfortunately, we are left to choose the point between these extremes where we will switch from one type of evolution to the next. Thus it may turn out that our result will be strongly dependent on when we choose to make the transition. Our method is not useful if our uncertainty about the transition prevents us from making a sufficiently precise estimate of the radiation. Alternatively, if our prediction for the radiation were very insensitive to our choice of transition, then for a wide range of choices of transition, we would find only a narrow range of energy predictions. Then it would be compelling, though not rigorously justified, to infer that the correct result lies in that range. The goal of the work reported here is to determine whether there is such a narrow range of energies resulting from a wide class of transition choices, or whether the success of the Abrahams-Cook calculation was the result of a fortuitous choice of a Newtonian to close-limit transition at the point of apparent horizon formation.

Our method is straightforward. We repeat the Abrahams-Cook calculation varying the point along the in-fall trajectory at which Newtonian evolution ends and the close-limit evolution begins in order to see how strongly the result depends on this choice. A quick Newtonian-physics calculation shows that for equal mass particles colliding head-on,

$$\left( \frac{l}{m} \right) = \frac{l/m}{1 - 8(P/m)^2(l/m)}.$$  

where \((l/m)_0\) is the initial separation from which the particles (black holes) are released at rest, \(l/m\) is the separation at some later point, and \(P/m\) is the magnitude of each hole’s momentum at that point. All quantities are scaled by \(m\), the combined bare mass of the black holes. The relation above defines a curve in the \(l-P\) parameter space, the Newtonian trajectory of a system of in-falling particles. Thus, we have described the Newtonian phase of the evolution. At some point when the black holes are close, we will halt the Newtonian evolution and begin close-limit evolution.

In principle, at this transition we will switch from a description of the two black holes as a system of particles to a field-theoretic description, where the fields obey an approximate form of Einstein’s equations. This requires us to specify values for the metric and its time-derivative on the initial time-slice in terms of the parameters \(l/m\) and \(p/m\) which have resulted from the Newtonian particle calculation. In principle there is no unique way make this specification: we must interpret the meaning of the particle parameters in the context of initial field configuration. In our case, to find the radiation, we will be using the results of the “close-slow” approximation which has been shown to agree well with the numerical results for a broad range of initial momenta when the black holes are nearly close enough to have coalesced. Implicitly, then, we will use the same description of the initial data as has been used in the close-slow calculation wherein the data are specified according to an approximation of the procedure developed by Bowen and York and numerically solved for a wide variety of initial configurations by Cook. In this formalism, the initial slice is assumed to be conformally flat with traceless extrinsic curvature. A particular family of solutions for the extrinsic curvature, labelled by \(P\), is chosen, and an elliptic boundary value problem determines the initial metric. After this, we must still interpret the quantities \(l\) and \(m\) as they will be realized for strongly interacting black holes. While it is natural to take \(l\) to be the proper separation between the minimum area throats associated with
FIG. 2. Here we see that the amount of radiated energy is not sensitive to the choice of transition. The horizontal axis indicates the separation at which we switch from Newtonian to close-limit evolution. The curves show results for the radiation according to the two-phase approximation for collisions of black holes released from several different initial separations, \( \mu_0 \). It is evident that for holes released from significant initial distances the radiation result is not sensitive to the choice of transition as long as the transition is made somewhere in the vicinity of the separation at which a horizon first forms.

In practice, the second part of the evolution has already been solved. The results of the close-slow calculation for the energy radiated during the collision can be summarized in the following simple form. \[ E/M = 2.51 \times 10^{-2} \kappa^2_2(\mu_0) - 2.06 \times 10^{-2} \coth \mu_0 \kappa_2(\mu_0) \left( \frac{P}{M} \right) + 5.37 \times 10^{-3} \left( \frac{\coth \mu_0}{\Sigma_1} \right)^2 \left( \frac{P}{M} \right)^2, \] (2)

where \( \mu_0 \) is a parameter specifying the initial separation of the holes and both \( \kappa_2 \) and \( \Sigma_1 \) are known functions of \( \mu_0 \) described in Ref. [8]. Here all quantities are scaled by \( M \), the ADM mass of the spatial slice. This equation gives us the total radiation directly in terms of \( \mu_0 \) and \( P/M \).

However, to use the above result we need know \( \mu_0 \) and \( P/M \) in terms of the parameters \( l/m \) and \( P/m \). Unfortunately, it is not natural in the conformal formalism to calculate the initial data directly from \( l/m \) and \( P/m \) as we interpret them here. Only after calculating the initial data sets in detail in terms of \( \mu_0 \) and \( P \), and then calculating the values of \( l, m \) and \( M \) can we ascertain their relationship among these parameters. Fortunately Cook [7] has performed a large number of such calculations generating tables of \( l, m \) and \( M \) in terms of \( \mu_0 \) and \( P \) for many cases covering the range of the parameter space with which we are interested. Instead of repeating these calculations, we find that we are able to get a sufficient estimate of \( \mu_0 \) and \( P/M \) in terms of \( l/m \) and \( P/m \) by interpolation from Cook’s results. Thus, we can apply the two-phase approximation to the evolution of distant in-falling black holes without really doing any new calculations. Newtonian methods give us a concise equation for the first phase of evolution, the close-slow approximation gives us an equation for the second phase, and the translation between the two methods is accomplished by interpolation from existing studies of axisymmetric initial data.

We seek to determine the sensitivity of the two-phase approximation result for the total radiation on the choice of transition. In Fig. 2 we plot this dependence for several different values of initial separation. The horizontal axis indicates the separation at which we make the transition from Newtonian to close-limit evolution. For example, the curve labelled \( \mu_0 = 2.5 \) shows that for holes released from rest at a separation of \( l \sim 5.5M \) the two-phase technique will find a minimum amount of radiation when the transition is made at \( l = 3.5m \). The excellent agreement with the numerical results initially discovered by Abrahams and Cook was for the choice that the transition occurs at the point where the two black holes are just close enough to share a common apparent horizon. The energies at the horizon...
formation line indicated in the figure thus correspond to the Abrahams-Cook calculation. Note that for large initial separations, the total radiated energy does not depend significantly on the choice of transition point in the vicinity of the horizon formation line. If the initial separation is greater than \( \mu_0 \sim 3.5 \) is then even choosing to start the close-limit phase when the black holes are more than twice as far apart only changes the resulting radiation by less than a factor of two. As the initial separation becomes much closer the sensitivity begins to increase. This is not surprising since for initially close black holes there is essentially no stage of the evolution where the Newtonian particle method is appropriate. Fig. 3 shows a comparison of the two-phase approximation with the Pullin-Price close-limit result \(^1\) and the result of the numerical calculation. \(^2\) The error bars on the two-phase approximation result indicate the range of energies obtained if we allow the evolution transition to occur at any proper separation within \( \frac{1}{2}m \) of the apparent horizon formation point. As Abrahams and Cook have shown previously \(^5\), there is excellent agreement with the numerical results. We conclude that the Abrahams-Cook approximate result for the radiation generated by in-fall from large initial separation is insensitive to the choice of transition from Newtonian to close-limit evolution. Together with the fact that the results for the radiation agree well with the numerical calculation, this suggests that the the two-phase approximation method may possibly be useful as a method for estimating the radiation generated in other types of black hole collisions.

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