Noisy DPC and Application to a Cognitive Channel

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Abstract—In this paper, we first consider a channel that is contaminated by two independent Gaussian noises $S \sim \mathcal{N}(0, Q)$ and $Z_0 \sim \mathcal{N}(0, N_0)$. The capacity of this channel is computed when independent noisy versions of $S$ are known to the transmitter and/or receiver. It is shown that the channel capacity is greater then the capacity when $S$ is completely unknown, but is less then the capacity when $S$ is perfectly known at the transmitter or receiver. For example, if there is one noisy version of $S$ known at the transmitter only, the capacity is $\frac{1}{2} \log(1 + \frac{P}{\|h\|^2 N_0 + Q})$, where $P$ is the input power constraint and $N_1$ is the power of the noise corrupting $S$. We then consider a Gaussian cognitive interference channel (IC) and propose a causal noisy dirty paper coding (DPC) strategy. We compute the achievable region using this noisy DPC strategy and quantify the regions when it achieves the upper bound on the rate.

I. INTRODUCTION

Consider a channel in which the received signal, $Y$, is corrupted by two independent additive white Gaussian noise (AWGN) sequences, $S \sim \mathcal{N}(0, QI_n)$ and $Z_0 \sim \mathcal{N}(0, N_0I_n)$, where $I_n$ is the identity matrix of size $n$. The received signal is of the form,

$$Y = X + S + Z_0$$  \hspace{1cm} (1)

where $X$ is the transmitted sequence for $n$ uses of the channel. Let the transmitter and receiver each have knowledge of independent noisy observations of $S$. We quantify the benefit of this additional knowledge by computing the capacity of the channel in (1) and presenting the coding scheme that achieves capacity. Our result indicates that the capacity is of the form $C(\frac{P}{\mu Q + N_0})$, where $C(x) = 0.5 \log(1 + x)$ and $0 \leq \mu \leq 1$ is the residual fraction (explicitly characterized in Sec. II.C) of the interference power, $Q$, that can not be canceled with the noisy observations at the transmitter and receiver.

We then consider the network in Fig. 1 in which the primary transmitter (node $A$) is sending information to its intended receiver (node $B$). There is also a secondary transmitter (node $C$) who wishes to communicate with its receiver (node $D$) on the same frequency as the primary nodes. We focus on the case when nodes $C$ and $D$ are relatively closer to node $A$ than node $B$. Such a scenario might occur for instance when node $A$ is a cellular base station and nodes $C$ and $D$ are two nearby nodes, while node $B$ is at the cell-edge.

Let node $A$ communicate with its receiver node $B$ at rate $R$ using transmit power $P_A$. Let the transmit power of node $C$ equal $P_C$. Since we assumed that node $B$ is much farther away from the other nodes, we do not explicitly consider the interference that $P_C$ causes at node $B$. A simple lower bound, $R_{CD-ib}$ on the rate that nodes $C$ and $D$ can communicate is

$$R_{CD-ib} = C(|h_{CD}|^2 P_C/(N_D + |h_{AD}|^2 P_A))$$  \hspace{1cm} (2)

which is achieved by treating the signal from node $A$ as noise at node $D$. Similarly, a simple upper bound on this rate is obtained (if either nodes $C$ or $D$ has perfect, noncausal knowledge of node $A$’s signal) as

$$R_{CD-ub} = C(|h_{CD}|^2 P_C/N_D).$$  \hspace{1cm} (3)

We propose a new causal transmission scheme based on the noisy DPC strategy derived in Sec. II. This new scheme achieves the upper bound (3) in some scenarios, which are quantified.

II. NOISY DIRTY PAPER CODING

A. System Model

![Fig. 1. A channel with noise observed at both encoder and decoder.](image_url)

The channel model is depicted in Fig. 1. The transmitter sends an index, $W \in \{1, 2, \ldots, K\}$, to the receiver in $n$ uses of the channel at rate $R = \frac{1}{n} \log_2 K$ bits per transmission. The output of the channel in (1) is contaminated by two independent AWGN sequences, $S \sim \mathcal{N}(0, QI_n)$ and $Z_0 \sim \mathcal{N}(0, N_0I_n)$. Side information $M_1 = S + Z_1$, which is noisy observations of the interference is available at the transmitter. Similarly, noisy side information $M_2 = S + Z_2$, is available at the receiver. The noise vectors are distributed as $Z_1 \sim \mathcal{N}(0, N_1I_n)$ and $Z_2 \sim \mathcal{N}(0, N_2I_n)$.

Based on index $W$ and $M_1$, the encoder transmits one codeword, $X$, from a $(2^W, n)$ code book, which satisfies average power constraint, $\frac{1}{n} \|X\|^2 \leq P$. Let $W$ be the estimate of $W$ at the receiver; an error occurs if $W \neq W$.

B. Related Work

One special case of (1) is when a noisy version of $S$ is known only to the transmitter; our result in this case is a generalization of Costa’s celebrated result [1]. In [1], it is shown that the achievable rate when the noise $S$ is perfectly known at the transmitter is equivalent to the rate when $S$...
Thus, a capacity bound of the channel can be calculated as both fractional power of the interference that cannot be canceled by the noisy observations at the transmitter and receiver.

The capacity with knowledge of a noisy version of to the case of noisy interference knowledge. We show that the variance of clearly the capacity of this channel equals the transmitter is equal to the capacity with knowledge of a statistically equivalent noisy version of \( \text{S} \) at the receiver. The noise vectors have the following distributions: the interference \( \text{S} \) at the transmitter is known to the encoder and a noisy version \( \text{S} \) at the receiver. The noise vectors have the following distributions:

In contrast, in this paper we study the case when only noisy observations \( \text{S} \) are known at the transmitter and receiver. The only result in [7] on discrete memoryless channels. The only result in [7] for Gaussian channel reveals no additional gain due to the presence of the noisy estimate at the decoder, since perfect knowledge is available at the encoder and DPC can be used. In contrast, in this paper we study the case when only noisy knowledge of \( \text{S} \) is available at both transmitter and receiver.

C. Channel Capacity

**Theorem 1:** Consider a channel of the form \((\text{U}, \text{Y})\) with an average transmit power constraint \( P \). Let independent noisy observations \( \text{M}_1 = \text{S} + \text{Z}_1 \) and \( \text{M}_2 = \text{S} + \text{Z}_2 \) of the interference \( \text{S} \) be available, respectively, at the transmitter and receiver. The noise vectors have the following distributions: \( \text{Z}_i \sim \mathcal{N}(0, \text{N}_i) \), \( i = 0, 1, 2 \) and \( \text{S} \sim \mathcal{N}(0, \text{Q}) \). The capacity of this channel equals \( C \left( \frac{P}{\mu Q + N_0} \right) \), where \( 0 \leq \mu = \frac{1}{1 + \frac{Q}{N_0}} \leq 1 \).

**Remark:** Clearly \( \mu = 0 \) when either \( N_1 = 0 \) or \( N_2 = 0 \) and the capacity is \( C(P/N_0) \), which is consistent with [1]. Further, \( \mu = 1 \) when \( N_1 \to \infty \) and \( N_2 \to \infty \), and the capacity is \( C(P/(Q + N_0)) \), which is the capacity of a Gaussian channel with noise \( Q + N_0 \). Thus, one can interpret \( \mu \) as the residual fractional power of the interference that cannot be canceled by the noisy observations at the transmitter and receiver.

**Proof:** We first compute an outer bound on the capacity of this channel. It is clear that the channel capacity can not exceed \( \max_{P(x|m_1, m_2)} I(X; Y|M_1, M_2) \), which is the capacity when both \( M_1 \) and \( M_2 \) are known at the transmitter and receiver. Thus, a capacity bound of the channel can be calculated as

\[ I(X; Y|M_1, M_2) = I(X; Y|M_1, M_2) - I(X; M_1, M_2) \leq I(X; Y, M_1, M_2) \]

\[ = H(X) + H(Y, M_1, M_2) - H(X, Y, M_1, M_2) \]

\[ = \frac{1}{2} \log(2\pi e)^4 P \left| \begin{array}{cccc} P & Q & N_0 & Q \\ Q & Q + N_1 & Q \\ Q & Q & Q + N_2 \end{array} \right| - \frac{1}{2} \log(2\pi e)^4 P \left| \begin{array}{cccc} P & 0 & 0 \\ P & P & 0 & 0 \\ 0 & Q & Q + N_1 & Q \end{array} \right| - \frac{1}{2} \log(2\pi e)^4 P \left| \begin{array}{cccc} 0 & Q & Q & Q + N_2 \end{array} \right| \]

\[ = C \left( \frac{P}{\mu Q + N_0} \right) . \] (5)

where \( \mu = \frac{P}{1 + \frac{Q}{N_0}} \). Note that the inequality in (4) is actually a strict equality since \( I(X; M_1, M_2) = 0 \).

D. Achievability of Capacity

We now prove that (5) is achievable. The codebook generation and encoding method we use follows the principles in [2], [3]. The construction of auxiliary variable is similar to [1].

**Random codebook generation:**

1) Generate \( 2^{nI(U; Y, M_1)} \) i.i.d. length-\( n \) codewords \( \text{U} \), whose elements are drawn i.i.d. according to \( U \sim \mathcal{N}(0, P + \alpha^2(Q + N_1)) \), where \( \alpha \) is a coefficient to be optimized.

2) Randomly place the \( 2^{nI(U; Y, M_2)} \) codewords \( \text{U} \) into \( 2^nR \) cells in such a way that each of the cells has the same number of codewords. The codewords and their assignments to the \( 2^nR \) cells are revealed to both the transmitter and the receiver.

**Encoding:**

1) Given an index \( W \) and an observation, \( M_1 = M_1(i) \), of the Gaussian noise sequence, \( \text{S} \), the encoder searches among all the codewords \( \text{U} \) in the \( W^{th} \) cell to find a codeword that is jointly typical with \( M_1(i) \). It is easy to show using the joint asymptotic equipartition property (AEP) [8] that if the number of codewords in each cell is at least \( 2^nR \), the probability of finding such a codeword \( \text{U} = \text{U}(i) \) exponentially approaches 1 as \( n \to \infty \).

2) Once a jointly typical pair \( (\text{U}(i), M_1(i)) \) is found, the encoder calculates the codeword to be transmitted as \( \text{X}(i) = \text{U}(i) - \alpha M_1(i) \). With high probability, \( \text{X}(i) \) will be a typical sequence which satisfies \( \frac{1}{n} ||\text{X}(i)||^2 \leq P \).

**Decoding:**

1) Given \( \text{X}(i) \) is transmitted, the received signal is \( \text{Y}(i) = \text{X}(i) + \text{S} + \text{Z}_0 \). The decoder searches among all \( 2^{nI(U; Y, M_2)} \) codewords \( \text{U} \) for a sequence that is jointly typical with \( \text{Y}(i) \). By joint AEP, the decoder will find \( \text{U}(i) \) as the only jointly typical codeword with probability approaching 1.

2) Based on the knowledge of the codeword assignment to the cells, the decoder estimates \( W \) as the index of the cell that \( \text{U}(i) \) belongs to.

**Proof of achievability:**

Let \( \text{U} = \text{X} + \alpha M_1 = \text{X} + \alpha (\text{S} + \text{Z}_1) \), \( \text{Y} = \text{X} + \text{S} + \text{Z}_0 \) and \( \text{M}_2 = \text{S} + \text{Z}_2 \), where \( \text{X} \sim \mathcal{N}(0, P) \), \( \text{S} \sim \mathcal{N}(0, Q) \) and \( \text{Z}_i \sim \mathcal{N}(0, N_i) \), \( i = 0, 1, 2 \) are independent Gaussian random variables. To ensure that with high probability, in each of the \( 2^nR \) cells, at least one jointly typical pair of \( \text{U} \) and \( \text{M}_1 \) can be found. The rate, \( R \), which is a function of \( \alpha \), must satisfy

\[ R(\alpha) \leq I(U; Y, M_2) - I(U; M_1). \] (6)
The two mutual informations in (6) can be calculated as

\[
I(U; Y, M_2) = H(U) + H(Y, M_2) - H(U, Y, M_2) = \frac{1}{2} \log \left( \frac{(P + \alpha^2(Q + N_1))}{P + Q + N_0} \right) + \frac{1}{2} \log \left( \frac{(P + \alpha^2(Q + N_1))}{P + Q + N_2} \right)
\] (7)

\[
I(U; M_1) = \frac{1}{2} \log \left( \frac{P + \alpha^2(Q + N_1)}{P} \right)
\] (8)

Substituting (7) and (8) into (6), we find

\[
R(\alpha) \leq \frac{1}{2} \log P[(Q + P + N_0)(Q + N_2) - Q^2] - \frac{1}{2} \log \left( \frac{2T}{\alpha^2} \right)
\] (9)

After simple algebraic manipulations, the optimal coefficient, \(\alpha^*\), that maximizes the right hand side of (9) is found to be

\[
\alpha^* = \frac{QN_2}{Q(P + N_0)(N_1 + N_2) + (Q + P + N_0)N_1N_2}
\] (10)

Substituting for \(\alpha^*\) in (9), the maximal rate equals

\[
R(\alpha^*) = C\left(\frac{P}{\mu Q + N_0}\right)
\] (11)

with \(\frac{1}{\mu} = 1 + \frac{Q}{N_1} + \frac{Q}{N_2}\), which equals the upper bound (5).

E. Special cases

Noisy estimate at transmitter/receiver only: When the observation of \(S\) is only available at the transmitter or receiver, the channel is equivalent to our original model when \(N_2 \to \infty\) and \(N_1 \to \infty\), respectively. Their capacity are, respectively

\[
I(X; Y | M_1) = C\left(\frac{P}{\mu Q + N_0}\right)
\] (12)

\[
I(X; Y, M_2) = C\left(\frac{P}{\mu Q + N_2}\right)
\] (13)

Note that when \(N_1 = 0\), the channel model further reduces to Costa's DPC channel model [1]. This paper extends that result to the case of noisy interference. Indeed, by setting \(N_1 = N_2 = 0\) in (12) and (13), we can see that the capacity with noisy interference known to transmitter only equals the capacity with a statistically similar noisy interference known to receiver only.

From (12), one may intuitively interpret the effect of knowledge of \(M_1\) at the transmitter. Indeed, a fraction \(\frac{Q}{Q + N_2}\) of the interfering power can be canceled using the proposed coding scheme. The remaining \(\frac{N_2}{Q + N_2}\) fraction of the interfering power, \(Q\), is treated as 'residual' noise. Thus, unlike Costa's result [1], the capacity in this case depends on the power \(Q\) of the interfering source: For a fixed \(N_1\), as \(Q \to \infty\), the capacity decreases and approaches \(C\left(\frac{P}{N_1 + N_0}\right)\).

Multiple Independent Observations: Let there be \(n_1\) independent observations \(M_1, M_2, \ldots, M_{n_1}\) of \(S\) at the transmitter and \(n_2\) independent observations \(M_{n_1+1}, M_{n_1+2}, \ldots, M_{n_1+n_2}\) at the receiver. It can be easily shown that the capacity in this case is given by

\[
C\left(\frac{P/(\mu Q + N_0)}{1 + \frac{Q}{N_1} + \frac{Q}{N_2} + \ldots + \frac{Q}{N_{n_1+n_2}}}ight)
\] and \(N_1, N_2, \ldots, N_{n_1+n_2}\) are the variances of the Gaussian noise variables, corresponding to the \(n_1 + n_2\) observations. The proof involves calculating maximum likelihood estimates (MLE) of the interference at both the transmit and receive nodes and using these estimates in Theorem 1. To avoid repetitive derivations, the proof is omitted.

It is easy to see that the capacity expression is symmetric in the noise variances at the transmitter and receiver. In other words, having all the \(n_1 + n_2\) observations at the transmitter would result in the same capacity. Thus, the observations of \(S\) made at the transmitter and the receiver are equivalent in achievable rate, as long as the corrupting Gaussian noises have the same statistics.

In this section, we assumed non-causal knowledge of the interference at the transmitter and receiver nodes. In the next section, we propose a simple and practical transmission scheme that uses causal knowledge of the interference to increase the achievable rate.

III. APPLYING DPC TO A COGNITIVE CHANNEL

\[
\text{Fig. 2. Cognitive interference channel model.}
\]

**Theorem 2:** Consider the network as shown in Fig. 2, nodes can communicate with node \(D\) at rate given by (14)

\[
\mu = 1 + 0.5 + 0.5 + 0.5 + 0.5 + 0.5 = 0.5 + 0.5 = 0.5
\]

\[
\mu = 1 + 0.5 + 0.5 + 0.5 + 0.5 = 0.5 + 0.5 + 0.5 = 0.5
\]

**Proof:** Consider the various cases as follows:

1. \(|h_{AD}|^2 \geq P(h_{BC})^2 + N_0(e^{2R} - 1)\). Now, consider the multiple access channel from nodes \(A, C\) to node \(D\). Clearly, node \(D\) can decode the signal transmitted by node \(A\) by treating the signal from node \(C\) as noise. Hence, it can easily subtract this signal from the received signal and node \(C\) can achieve its rate upper bound \(C(P_C|h_{CD}|^2/N_C)\).

2. Consider the case \(|h_{AD}|^2 \leq \frac{N_0}{P_e}e^{2R}e^{2R} - 1\) and \(|h_{AC}|^2 \leq \frac{P}{P_e}e^{2R} - 1\). Now, neither node \(C\) nor node \(D\) can perfectly decode the signal from node \(A\). Thus, an achievable rate of \(C\left(\frac{|h_{CD}|^2}{(N_D+I_A)|h_{AD}|^2}\right)\) for node \(C\) is obtained simply by treating the signal from node \(A\) as noise at node \(D\).

3. Now, consider the case \(|h_{AC}|^2 \geq \frac{N_0}{P_e}(e^{2R} - 1)\) and \(|h_{AD}|^2 \geq \frac{N_0}{P_e}(e^{2R} - 1)\). In the following we construct a simple practical scheme in which nodes \(C\) and \(D\) obtain causal, noisy estimates of the signal being sent from node \(A\). Using these estimates and Theorem 1, the nodes cancel out a part of the interference to achieve a higher transmission rate as follows.
\[ R_{CD} = \begin{cases} C\left(\frac{|h_{CD}|^2 P_C}{|h_{CD}|^2 P_C + (N_D + P_A |h_{AD}|^2)}\right) & \text{if } |h_{AD}|^2 \geq \frac{P_C |h_{CD}|^2 + N_D}{P_A} (e^{2R} - 1) \\
C\left(\frac{|h_{CD}|^2 P_C}{|h_{CD}|^2 P_C + (N_D + P_A |h_{AD}|^2)}\right) & \text{if } |h_{AD}|^2 \leq \frac{N_D}{P_A} (e^{2R} - 1) \text{ and } |h_{AC}|^2 \leq \frac{N_C}{P_A} (e^{2R} - 1) \\
C\left(\frac{|h_{CD}|^2 P_C}{|h_{CD}|^2 P_C + (N_D + P_A |h_{AD}|^2)}\right) & \text{if } |h_{AC}|^2 \geq \frac{N_C}{P_A} (e^{2R} - 1) \text{ and } P_C |h_{CD}|^2 + N_D (e^{2R} - 1) \geq |h_{AD}|^2 \geq \frac{N_D}{P_A} (e^{2R} - 1) \\
\left(1 - \frac{m}{n}\right)C\left(\frac{|h_{CD}|^2 P_C (n/n-m)}{\mu_{tr} |h_{AD}|^2 P_A + N_D}\right) & \text{if } |h_{AC}|^2 \geq \frac{N_C}{P_A} (e^{2R} - 1) \text{ and } P_C |h_{CD}|^2 + N_D (e^{2R} - 1) \geq |h_{AD}|^2 \leq \frac{N_D}{P_A} (e^{2R} - 1) \\
\left(1 - \frac{m}{n}\right)C\left(\frac{|h_{CD}|^2 P_C (n/n-m)}{\mu_{tr} |h_{AD}|^2 P_A + N_D}\right) & \text{if } |h_{AC}|^2 \geq \frac{N_C}{P_A} (e^{2R} - 1) \text{ and } P_C |h_{CD}|^2 + N_D (e^{2R} - 1) \geq |h_{AD}|^2 \leq \frac{N_D}{P_A} (e^{2R} - 1) \end{cases} \]

Let us assume that node A uses a code book of size \((2^n R, n)\) where each element is i.i.d. Gaussian distributed. The transmit signal is denoted as \(X_A(i), i = 1, 2 \ldots n\). Nodes C and D listen to the signal transmitted by node A for \(m\) symbols in each block of \(n\) symbols. Based on the received signal, nodes C and D decodes the code word transmitted by node A. Let \(P_{e,C}\) and \(P_{e,D}\) denote, respectively, the probability of decoding error at nodes C and D: These error probabilities depend on the channel gains as well as \(m\). In the remaining \(n - m\) symbols, nodes C and D use their estimate of \(X_A(i), i = m + 1 \ldots n\) to increase their transmission rate.

Using Theorem 1 the achievable rate is given by

\[ r = \frac{1}{2} \left(1 - \frac{m}{n}\right) \log \left(1 + \frac{|h_{CD}|^2 P_C (n/n-m)}{\mu_{tr} |h_{AD}|^2 P_A + N_D}\right), \] (15)

where

\[ \frac{1}{\mu_{tr}} = 1 + \frac{|h_{AD}|^2 P_A}{N_1} + \frac{|h_{AD}|^2 P_A}{N_2} \] (16)

The transmit power at node C is increased over the \(n - m\) symbols that it transmits to meet average power constraint \(P_C\). The variance of error in the estimate of \(X_A\) at nodes C and D is given respectively by \(N_1\) and \(N_2\). Because of the i.i.d Gaussian code book being used, \(N_1 = 2P_{e,C}P_A |h_{AD}|^2\) and \(N_2 = 2P_{e,D}P_A |h_{AD}|^2\). The value of \(P_{e,C}\) and \(P_{e,D}\) can be obtained using the theory of error exponent. Specifically, using the random coding bound, we obtain,

\[ P_{e,C} \leq \exp(-mE_C(R)) \] and \[ P_{e,D} \leq \exp(-nE_D(R)) \] (17)

where \(E_C(R)\) and \(E_D(R)\) represent the random coding exponent. \(E_C(R)\) is derived in [9] and shown in [13] for easy reference \(E_D(R)\) is similarly defined. In [18], \(A_1 = \frac{|h_{AC}|^2 P_A}{N_C}, \beta = \exp(2R), \gamma = 0.5(1 + A_1^2 + \sqrt{1 + A_1^2}), \delta = 0.5 \log(0.5 + \frac{A_1}{4} + 0.5 \sqrt{1 + \frac{A_1}{4}}). \) Substituting for \(N_1\) and \(N_2\) into [19], one can obtain the rate given in (14).

Note that there is no constraint that node C must use codes of length \(m-n\) since node A uses codes of length \(n\). Node C can code over multiple codewords of A to achieve its desired probability of error.

The selection of \(m\) critically affects the achievable rates. On the one hand, increasing \(m\) results in lesser fraction of time available for actual data communications between nodes C and D and thus decreasing rate. On the other hand, increasing \(m\) results in improved decoding of node A’s signal at nodes C and D consequently reducing \(P_{e,C}\) and \(P_{e,D}\) and increasing the achievable rate. The optimal value of \(m\) can be obtained by equating the derivative of (15) to 0. Due to the analytical intractability, we resort to simple numerical optimization to find the optimal value of \(m\). For a given \(n\), we evaluate the rate \(r_{CD}\) for all values of \(m = 1, 2 \ldots n\) and then simply pick the largest value. We are currently trying to derive analytical expressions for the optimum value of \(m\).

4. Let \(|h_{AC}|^2 \leq \frac{N_C}{P_A} (e^{2R} - 1)\) and \(P_C |h_{CD}|^2 + N_D (e^{2R} - 1) \geq |h_{AD}|^2 \geq \frac{N_D}{P_A} (e^{2R} - 1)\). In this case, the transmitter node C cannot decode node A’s signal. However, node D uses all received symbols to first decode node A’s signal (with certain error probability) and then cancel its effect from the received signal. Subsequently, node D will decode node C’s signal and the achievable rate is obtained from Theorem 1.

5. Finally, let \(|h_{AC}|^2 \geq \frac{N_C}{P_A} (e^{2R} - 1)\) and \(|h_{AD}|^2 \leq \frac{N_D}{P_A} (e^{2R} - 1)\). In this case, node D cannot decode node A’s signal. However, node C uses the first \(m\) received symbols to first decode node A’s signal (with certain error probability) and then employ a noisy DPC transmission strategy. Subsequently, the achievable rate is obtained from Theorem 1.

A. Numerical Results

In our numerical results we fix the values for the parameters as: \(P_A = 10, P_C = 2, N_C = N_D = 1\). For simplicity we fix \(|h_{CD}| = 1\) and vary \(h_{AC}\) and \(h_{AD}\).

Fig. 3 shows the variation of the achievable rate with \(m\) for different values of \(n\). As \(n\) increases the fractional penalty on the rate for larger \(m\) is offset by the gains due to better decoding. Thus, the optimum value of \(m\) increases. However,
it turns out that the optimum ratio $m/n$ decreases as $n$ increases. We are currently trying to analytically compute the limit to which the optimum $m$ converges as $n \to \infty$.

\[
E_C(R) = \begin{cases} 
\frac{A_1}{A_2} \left[ (\beta + 1) - (\beta - 1) \sqrt{1 + \frac{4\beta}{A_1(\beta - 1)}} \right] + \frac{1}{2} \log \left( \beta - \frac{A_1(\beta - 1)}{2} \left[ 1 + \frac{4\beta}{A_1(\beta - 1)} - 1 \right] \right) & \text{if } R > C \left( \frac{|h_{AC}|^2 P_A}{N_C} \right) \\
1 - \gamma + \frac{4\beta}{2} + \frac{1}{2} \log (\gamma - \frac{4\beta}{2}) + \frac{1}{2} \log (\gamma) - R & \text{if } \delta \leq R \leq C \left( \frac{|h_{AC}|^2 P_A}{N_C} \right) \\
0 & \text{if } R < \delta
\end{cases}
\]

(18)

Fig. 4 shows the variation of the achievable rate $r_{CD}$ with $h_{AD}$ for different values of $h_{AC}$. Notice the nonmonotonic variation of $r_{CD}$ with $h_{AD}$ which can be explained as follows. First consider $h_{AC} = 0$. In this case, the transmitter cannot reliably decode node A’s signal. If in addition, $h_{AD}$ is also small, then node $D$ cannot decode node $A$’s signal either. Thus, as $h_{AD}$ increases, the interference of node $A$ at node $D$ increases and the achievable rate $r_{CD}$ decreases. Now, as $h_{AD}$ increases beyond a certain value, node $D$ can begin to decode node $A$’s signal and the probability of error is captured by Gallager’s error exponents. In this scenario, as $h_{AD}$ increases, the error probability decreases and thus node $D$ can cancel out more and more of interference from node $A$. Consequently, $r_{CD}$ increases. Similar qualitative behavior occurs for other values of $h_{AC}$. However, for large $h_{AC}$, node $C$ can decode (with some errors) the signal from node $A$ and then use a noisy DPC scheme to achieve higher rates $r_{CD}$. Notice also that as explained before for large $h_{AD}$, the outer bound on the rate is achieved for all values of $h_{AC}$.

The variation of $r_{CD}$ with $h_{AC}$ is given in Fig. 5. First consider the case $|h_{AD}| = 0.2$. In this case, node $D$ cannot decode the signal of node $A$ reliably. Now, for small values of $|h_{AC}|$ node $C$ also cannot decode node $A$’s signal. Hence, the achievable rate equals the lower bound, $R_{CD-LB}$. As $|h_{AC}|$ increases, node $C$ can begin to decode node $A$’s signal and cancel out a part of the interference using the noisy DPC scheme; hence $r_{CD}$ begins to increase. Similar behavior is observed for $|h_{AD}| = 0.6$. However, when $|h_{AD}| = 0.9$, node $D$ can decode node $A$’s signal with some errors and cancel out part of the interference. Hence, in this case, even for small values of $|h_{AC}|$ the achievable rate $r_{CD}$ is greater than the lower bound. As before $r_{CD}$ increases with $|h_{AC}|$ since node $A$ can cancel out an increasing portion of the interference using the noisy DPC technique. Note however, that a larger $h_{AD}$ causes more interference at node $D$, which is reflected in the decrease of the lower bound. Thus, for a given $|h_{AC}|$ the achievable rate can be lower or higher depending on the value of $|h_{AD}|$.

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