AN EFFECTIVE ANALYSIS OF THE DENJOY RANK

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Abstract. We analyze the descriptive complexity of several $\Pi^1_1$ ranks from classical analysis which are associated to Denjoy integration. We show that $VBG, VBG_*, ACG$ and $ACG_*$ are $\Pi^1_1$-complete, answering a question of Walsh in case of $ACG_*$. Furthermore, we identify the precise descriptive complexity of the set of functions obtainable with at most $\alpha$ steps of the transfinite process of Denjoy totalization: if $| \cdot |$ is the $\Pi^1_1$-rank naturally associated to $VBG, VBG_*$ or $ACG_*$, and if $\alpha < \omega^\omega$, then $\{ F \in C(I) : |F| \leq \alpha \}$ is $\Sigma^0_2$-complete. These finer results are an application of the author’s previous work on the limsup rank on well-founded trees. Finally, $\{(f,F) \in M(I) \times C(I) : F \in ACG_* and F' = f$ a.e.\}$ and $\{f \in M(I) : f$ is Denjoy integrable\}$ are $\Pi^1_1$-complete, answering more questions of Walsh.

Real analysis of the early 20th century featured a number of naturally occurring $\Pi^1_1$-complete sets. The most prominent example may be the set of differentiable functions on the unit interval $I$, considered as a subset of $C(I)$, the metric space of continuous functions on $I$ with the supremum norm.

Any $\Pi^1_1$ set $A$ may be decomposed as a transfinite union $\bigcup_{\alpha<\omega_1} A_\alpha$, where each $A_\alpha$ is Borel. When this is done in a sufficiently uniform way, the function which maps each $f \in A$ to the least $\alpha$ such that $f \in A_\alpha$ is called a $\Pi^1_1$-rank. Sometimes a $\Pi^1_1$ set has an obvious and natural $\Pi^1_1$-rank. This is the case for the set of well-founded trees $T \subseteq \omega^{<\omega}$ (the usual well-founded tree rank), the collection of countable closed subsets of $I$ (the Cantor-Bendixson rank), and the collection of continuous functions obtainable by Denjoy totalization, as we shall see shortly. Other times, finding a rank that could be considered natural is more difficult. For example, in [KW86] Kečrich and Woodin defined a suitably natural rank on the set of differentiable functions.

When the rank on $A$ is natural, it becomes meaningful to ask for the precise descriptive complexity of the initial segments $A_\alpha$. This was done implicitly for the well-founded tree rank in [GMS13], for the Cantor-Bendixson rank in [Lem87], and for the Kečrich-Woodin rank on differentiable functions in [Wes14]. The purpose of this paper is to show that the method used in [Wes14] generalizes to give descriptive complexities for three hierarchies from classical analysis related to Denjoy totalization. We also give a new proof of Lempp’s result on the descriptive complexity of the initial segments of the Cantor-Bendixson rank.

Denjoy totalization is a transfinite integration process developed by Denjoy in 1912 to solve the problem of recovering $F$ from $F'$ whenever $F$ is an everywhere differentiable function in $C(I)$. The process does a little more, recovering also some a.e. differentiable $F$, but not all of them. The set of $F \in C(I)$ recoverable from $F'$ by Denjoy totalization is denoted $ACG_*$. The related sets $ACG, VBG_*$ and $VBG$ are described in the next section.

The main result is the following. Let $| \cdot |_{VB}, | \cdot |_{VB_*}, | \cdot |_{AC}$ and $| \cdot |_{AC_*}$ denote the natural $\Pi^1_1$ ranks on $VBG, VBG_*, ACG$ and $ACG_*$ respectively.

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Theorem 1. Let \( X = VB, VB_*, \text{ or } AC_*, \) let \( Y \subseteq 2^{\omega}, \) let \( 1 < \alpha < \omega_1^Y, \) and let
\[
A_\alpha = \{ F \in C(I) : |F|_X \leq \alpha \}.
\]
Then \( A_\alpha \) is \( \Sigma^0_{2\alpha}(Y), \) and for any \( \Sigma^0_{2\alpha}(Y) \) set \( B, \) there is a \( Y \)-computable reduction from \( B \) to \( A_\alpha. \) In particular, \( A_\alpha \) is \( \Sigma^0_{2\alpha} \)-complete, and if \( \alpha < \omega_1^{CK} \), then \( A_\alpha \) is \( \Sigma^0_{2\alpha} \)-complete.

Let \( M(I) \) denote the Polish space of measurable functions on the unit interval, with metric given by \( d(f, g) = \int_1 \min(1, |f - g|) \). Our next theorem answers three questions in [Wal17].

Theorem 2. The following sets are all \( \Pi^1_1 \)-complete:
\[
\begin{align*}
(1) & \quad VB, VB_*, ACG \text{ and } ACG_* \\
(2) & \quad \{ (f, F) \in M(I) \times C(I) : F \in ACG_* \text{ and } F' = f \text{ a.e.} \} \\
(3) & \quad \{ f \in M(I) : F \text{ is Denjoy integrable} \}
\end{align*}
\]

The sets \( \{ F \in C(I) : |F|_X \leq 1 \} \) are better known as the collection of functions of bounded variation when \( X = VB, VB_* \) and the collection of absolutely continuous functions when \( X = AC, AC_* \). The following results filling in the \( \alpha = 1 \) case of Theorem 1 have routine proofs but their statements may be of interest.

Theorem 3. (1) The set of continuous functions of bounded variation is \( \Sigma^0_2 \)-complete. (2) The set of absolutely continuous functions is \( \Pi^0_3 \)-complete.

In Section 1 after some preliminaries we review the main tool in [Wes14], an analysis of the descriptive complexity of the limsup rank on the set of well-founded trees. We also give the background on Denjoy totalization and the hierarchies \( ACG_*, ACG, VB_* \) and \( VB. \) In Section 2 we apply this analysis to the simpler case of the Cantor-Bendixson derivative, recovering the aforementioned result of Lempp. In Section 3 we prove hardness results for the hierarchies on \( ACG_*, ACG, VB_* \) and \( VB, \) obtaining Theorem 2 and one direction of Theorem 1. In Section 4 we obtain matching descriptive results for all of these hierarchies except \( ACG, \) giving the other direction of Theorem 1. Section 5 contains open questions.

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1. Preliminaries

1.1. Notation. We use standard computability-theoretic notation. Trees \( T \subseteq \omega^{<\omega} \) can be encoded by elements of Cantor space. Let \( Tr \subseteq 2^{\omega} \) be the set of codes for such trees. Given \( T \in Tr, \) we usually forget the encoding and treat \( T \) as a subset of \( \omega^{<\omega} \) anyway. The set \( [T] \subseteq \omega^\omega \) is the set of paths through \( T. \) Let \( \sigma^\tau \) denote the concatenation of \( \sigma \) and \( \tau. \) If \( \sigma \in T, \) let \( T_\sigma \) denote \( \{ \tau : \sigma^\tau \in T \}, \) and for any \( \sigma \in \omega^{<\omega}, \) let \( \sigma^\tau \) denote \( \{ \sigma^\tau : \tau \in T \}. \) Let \( [\sigma] \) denote \( \{ X \in \omega^\omega : \sigma < X \} \) or \( \{ X \in 2^{\omega} : \sigma < X \}, \) depending on the context. When a tree is in fact a subset of \( 2^{<\omega}, \) we usually give it the name \( S, \) reserving \( T \) for trees in Baire space.

The complement of a set \( A \subseteq \omega^\omega \) is denoted \( \neg A. \) If \( A, B, C, D \subseteq \omega^\omega \) and \( f \) is a computable function from \( A \cup B \) to \( C \cup D \) with \( f^{-1}(C) = A \) and \( f^{-1}(D) = B, \) we say that \( f \) is a computable reduction from \( (A, B) \) to \( (C, D). \) If \( B = \neg A \) and \( D = \neg C, \) we say \( f \) is a reducible reduction from \( A \) to \( C. \)

A tree \( T \subseteq \omega^{<\omega} \) is well-founded, denoted \( T \in WF, \) if \( [T] = \emptyset. \) A set \( A \subseteq \omega^\omega \) is \( \Pi^1_1 \) if there is a computable reduction from \( A \) to \( WF. \) A \( \Pi^1_1 \) set \( A \) is \( \Pi^1_1 \)-complete if there is a
computable function from $WF$ to $A$. The boldface notions $\Pi_1^1$ and $\Pi_1^1$-complete are obtained by replacing “computable” with “continuous”, or equivalently, relativizing to an oracle.

For $n < \omega$, a set $X \subseteq 2^n$ is $\Sigma_n^0$-complete if $X \equiv_1 \emptyset^{(n)}$. To extend the notion of completeness to the ordinals, we assume some familiarity with hyperarithmetic theory: Kleene’s $\mathcal{O}$, ordinal notations, the sets $H_a$ for $a \in \mathcal{O}$. For $a \in \mathcal{O}$, we use $|a|_\mathcal{O}$ to denote the ordinal coded by $a$. The supremum of ordinals represented in $\mathcal{O}$ is $\omega_1^{CK}$. Following [AK00], if $n \geq \omega$ we say that a subset $X \subseteq \omega$ is $\Sigma_n^0$-complete if $X \equiv_1 H_{2^n}$, where $a \in \mathcal{O}$ is any notation with $|a|_\mathcal{O} = \alpha$. All these notions can relativize to an oracle $Y \in 2^\omega$, using notation $\mathcal{O}^Y$, $|a|^Y_\mathcal{O}$, $H_a^Y$, $\omega_1^Y$, $\Sigma_0^0(Y)$.

These computability notions on subsets of $\omega$ can be extended to effective topological notions on subsets of $\omega^\omega$. A set $A \subseteq \omega^\omega$ is $\Sigma_0^\omega(Y)$ if $\omega \leq \alpha < \omega_1^Y$ and there is an index $e_0$ and a notation $a \in \mathcal{O}^Y\oplus\emptyset$ such that for all $X \subseteq \omega^\omega$, $a \in \mathcal{O}^Y\oplus X$ with $|a|^Y\oplus X = \alpha$ and

$$X \in A \iff e_0 \in H_{2_\alpha}^{Y\oplus X}.$$ 

If $\alpha < \omega$, replace $H_{2_\alpha}^{Y\oplus X}$ with $H_a^{Y\oplus X}$. When showing that a set is $\Sigma_0^\omega(Y)$, we usually use effective transfinite recursion. We assume familiarity with this method and sometimes avoid explicit mention of the sets $H^{Y\oplus X}$ while using it.

It is well-known that $\Sigma_0^\omega = \bigcup_{\alpha < \omega_1^\omega} \Sigma_0^\alpha(Y)$, where $\Sigma_0^\alpha$ refers to the $\alpha$th level of the Borel hierarchy.

We use $I$ to refer to the unit interval, $C(I)$ for the space of continuous real-valued functions on $I$ with the supremum norm, and $M(I)$ for the space of measurable functions on the unit interval, with metric given by $d(f, g) = \int_I \min(1, |f - g|)$. In $C(I)$, the piecewise linear functions with rational endpoints form a countable dense subset; computing an element $F$ of $C(I)$ means providing, uniformly in $\varepsilon$, a piecewise linear function within $\varepsilon$ of $F$ in the supremum norm. In $M(I)$, the step functions with rational values and discontinuities at finitely many rational points form a countable dense subset, and can be used as the ideal points in a representation of $M(I)$ as an effectively presented metric space. Thus computing an element $f$ of $M(I)$ means providing, uniformly in $\varepsilon$, an ideal point of the type described above which is $\varepsilon$-close to $f$.

1.2. Limsup rank. Our main tool is the limsup rank on well-founded trees, which was defined in [Wes14] as follows.

**Definition 1.** Let $T \subseteq \omega^{<\omega}$ be a well-founded tree. If $T = \emptyset$, define $|T|_{ls} = 0$. Otherwise, define

$$|T|_{ls} = \max(\sup_n |T_n|_{ls}, (\limsup_n |T_n|_{ls}) + 1).$$

This rank is designed to line up nicely with Cantor-Bendixson type derivation processes in a way that will be explained below.

**Theorem 4 ([Wes14]).** For all constructive $\alpha > 1$, \{e : $\phi_e$ codes a tree $T$ with $|T|_{ls} \leq \alpha$\} is $\Sigma_{2\alpha}$-complete.

The only reason for not allowing $\alpha = 1$ in the above theorem is that it is $\Pi_2$ to tell whether $\phi_e$ is total, that is, whether it codes anything at all.

Almost for free, we can make topological claims in addition to computational ones. The next theorem follows from Theorem 4 by relativization.

**Theorem 5.** For any nonzero $\alpha < \omega_1$, let

$$A_\alpha = \{T \in Tr : |T|_{ls} \leq \alpha\}.$$
Then if \( \alpha < \omega^Y \), we have \( A_\alpha \in \Sigma_{2\alpha}^0(Y) \), and for any set \( B \in \Sigma_{2\alpha}^0(Y) \), there is a \( Y \)-computable reduction from \( B \) to \( A_\alpha \). In particular, \( A_\alpha \) is \( \Sigma_{2\alpha}^0 \)-complete, and if \( \alpha < \omega_1^{CK} \), then \( A_\alpha \) is \( \Sigma_{2\alpha}^0 \)-complete.

Proof. Let \( a \) be a notation such that for all \( X, a \in \mathcal{O}^{X \oplus Y} \), and \( |a|_{\mathcal{O}^{X \oplus Y}} = 2\alpha \). For all \( X \), by relativization we have

\[
H_{2\alpha}^{X \oplus Y} = \{ e : \phi_e^{X \oplus Y} \text{ codes a tree } T \text{ with } |T|_{\ell} \leq \alpha \},
\]

where the pair of reductions witnessing the 1-equivalence do not depend on \( X \).\footnote{Because both sets consist of machine indices for machines with access to \( X \oplus Y \), the existence of a single pair of computable reductions follows from the existence of a single pair of uniformly \( X \oplus Y \)-computable reductions.}

Letting \( d_0 \) be such that \( \phi_{d_0}^{X \oplus Y} = X \), the reverse reduction in (1) provides \( e_0 \) such that for all \( X \),

\[
X \in A_\alpha \iff \phi_{d_0}^{X \oplus Y} \in A_\alpha \iff e_0 \in H_{2\alpha}^{X \oplus Y},
\]

so \( A_\alpha \in \Sigma_{2\alpha}^0(Y) \).

To show that a given \( B \in \Sigma_{2\alpha}^0(Y) \) can be \( Y \)-computably reduced to \( A_\alpha \), it suffices to consider \( B = \{ X : n_0 \in H_{2\alpha}^{X \oplus Y} \} \) where \( n_0 \) is chosen to make \( B \) \( \Sigma_{2\alpha}^0(Y) \)-universal. The forward reduction in (1) provides \( m_0 \) such that \( X \in B \iff \phi_{m_0}^{X \oplus Y} \in A_\alpha \), and the mapping \( X \mapsto \phi_{m_0}^{X \oplus Y} \) is \( Y \)-computable. \( \square \)

1.3. **Denjoy totalization.** Classical real analysis includes the study of absolutely continuous functions, functions of bounded variation, and countable generalizations of these notions. We consider four classes of real-valued functions on \( I \). They are \( VBG \) (generalized bounded variation), \( VBG_* \) (generalized bounded variation in the restricted sense), \( ACG \) (generalized absolutely continuous) and \( ACG_* \) (generalized absolutely continuous in the restricted sense).

To define these classes it is necessary to generalize the well-known definitions of bounded variation and absolute continuity to take into account also a closed set \( E \) to which the function \( F \) should be in some sense restricted. The non-asterisk definitions are the literal restrictions. The others take into account also the values of \( F \) outside of \( E \). The oscillation of a function \( F \) on an interval \((a, b)\) is defined as \( \omega(F, a, b) = \sup_{x,y \in (a, b)} |F(x) - F(y)| \).

**Definition 2.** Let \( F \in C(I) \) and let \( E \subseteq I \).

1. We say \( F \) is \( VB \) (respectively \( VBG_* \)) on \( E \) if there is an \( N \) such that for all non-decreasing sequences \( a_0, b_0, \ldots, a_k, b_k \in E \), we have \( \sum_i |F(b_i) - F(a_i)| < N \) (respectively \( \sum_i \omega(F, a_i, b_i) < N \)).

2. We say \( F \) is \( AC \) (respectively \( AC_* \)) on \( E \) if for all \( \varepsilon \) there is a \( \delta \) such that for all non-decreasing sequences \( a_0, b_0, \ldots, a_k, b_k \in E \), if \( \sum_i |b_i - a_i| < \delta \), then \( \sum_i |F(b_i) - F(a_i)| < \varepsilon \) (respectively \( \sum_i \omega(F, a_i, b_i) < \varepsilon \)).

Observe that if \( E \) is an interval, then being \( VB \) on \( E \) is the same thing as being \( VBG_* \) on \( E \), and similarly for absolute continuity. We can also understand what it means for a function to satisfy these properties on a closed set \( E \) with reference to simplified functions \( F_E \) and \( F_{E,*} \) defined as follows.

**Definition 3.** Let \( F \in C(I) \), and \( E \subseteq I \) a closed set. Then let \( F_E \) and \( F_{E,*} \) denote the functions satisfying

1. \( F_E(x) = F_{E,*}(x) = F(x) \) for \( x \in E \), and
2. If \((c, d)\) is a connected component of \( I \setminus E \),
Proposition 6. Let $F_E$ be linear on $[c, d]$, and

(a) let $F_E$ be linear on $[c, d]$, and
(b) let

$$F_{E,*} \left( \frac{2c + d}{3} \right) = \sup F([c, d])$$

$$F_{E,*} \left( \frac{c + 2d}{3} \right) = \inf F([c, d]),$$

and let $F_{E,*}$ be linear on $[c, \frac{2c+d}{3}, \frac{2c+d}{3}, \frac{c+2d}{3}]$ and $[\frac{c+2d}{3}, d]$.

Note that $\omega(F_{E,*}, [c, d]) = \omega(F, [c, d])$ for $(c, d)$ a connected component of $I \setminus E$.

Then the following proposition holds:

Proposition 6. Let $F \in C(I)$ and $E \subseteq I$ be closed. Then

1. $F$ is $VB$ (resp. $AC$) on $E$ if and only if $F_E$ is $VB$ (resp. $AC$) on $I$.
2. $F$ is $VB_*$ (resp. $AC_*$) on $E$ if and only if $F_{E,*}$ is $VB$ (resp. $AC$) on $I$.

Now we can define the main notions.

Definition 4. A function $F \in C(I)$ is $VBG$ (respectively $ACG$, $VBG_*$, $ACG_*$) if there is a countable sequence of closed sets $E_n$ such that $\bigcup_n E_n = I$ and $F$ is $VB$ (respectively $AC$, $VB_*$, $AC_*$) on each $E_n$.

It is immediate that $VBG_* \subseteq VBG$ and $ACG_* \subseteq ACG$. Recall the relationship between absolute continuity and bounded variation: a continuous function of bounded variation is absolutely continuous if and only if it satisfies Lusin’s condition $(N)$.

Definition 5. A function $F : I \to \mathbb{R}$ satisfies $(N)$ if for every Lebesgue null set $A \subseteq I$, its image $F(A)$ is also null.

If $\bigcup_n E_n = I$, then $F$ satisfies $(N)$ if and only if it satisfies $(N)$ on each $E_n$. Therefore, $F \in ACG$ if and only if $F \in VBG$ and $F$ satisfies $(N)$. We also have (see [Sak64, Thm VII.8.8, pg. 233]) that $ACG_* = VBG_* \cap ACG$. It follows that $F \in ACG_*$ if and only if $F \in VBG_* \cap ACG_*$.

Based on the definitions above, it would seem that these sets are $\Sigma^1_1$. However, the following equivalent characterization shows each of these classes is in fact $\Pi^1_1$.

Theorem 7 (see [Sak64, Thm VII.9.1, pg 233]). For a function $F$ to be $VBG$ (respectively $VBG_*$, $ACG$, $ACG_*$) it is necessary and sufficient that for every closed $E \subseteq I$, there is an interval $[a, b] \subseteq I$ such that $(a, b) \cap E \neq \emptyset$ and $F$ is $VB$ (respectively $VB_*$, $AC$, $AC_*$) on $[a, b] \cap E$.

Corollary 8. The sets $VBG, VBG_*, ACG, ACG_* \subseteq C(I)$ are all $\Pi^1_1$.

The previous theorem also suggests a derivation process.

Definition 6. Let $X$ stand for $VB, VBG_*, AC$ or $AC_*$. Given $F \in C(I)$, define $P_{F,X}^0 = I$.

Define

$$P_{F,X}^{\alpha+1} = P_{F,X}^\alpha \setminus \{ (a, b) : F \text{ is } X \text{ on } [a, b] \cap P_{F,X}^\alpha \}$$

and for a limit ordinal $\lambda$, define $P_{F,X}^\lambda = \cap_{\alpha < \lambda} P_{F,X}^\alpha$. Define a rank $| \cdot |_X$ by letting $|F|_X$ be the least $\alpha$ such that $P_{F,X}^\alpha = \emptyset$, if such $\alpha$ exists.
If $F \in VBG$, then the only way for $P_{F,VB}^{\alpha+1} = P_{F,VB}^\alpha$ is if $P = \emptyset$; and furthermore, the countable sequence of sets $[a,b] \cap P_{F,VB}^\alpha$ for which $(a,b)$ were removed over the course of the derivations would serve as the sequence $E_n$ required in the definition of $VBG$. Reasoning similarly about all four hierarchies, it is immediate that a given $F \in C(I)$ belongs to one of the classes $VBG, VBG_*, ACG, ACG_*$ if and only if the associated derivation process eventually produces the empty set.

**Definition 7.** Let $VBG_\alpha$ (respectively $VBG_{*\alpha}, ACG_\alpha, ACG_{*\alpha}$) denote the sets $\{F \in C(I) : |F|_{VB} \leq \alpha\}$ (respectively $|F|_{VB_*}, |F|_{AC_*}, |F|_{AC_*}$).

Recall from the introduction that $ACG_*$ is exactly the set of functions $F \in C(I)$ which can be recovered from $F'$ by Denjoy totalization. We will not give the definition of the Denjoy totalization process; the interested reader may consult [Sak64, Section VIII.5]. What matters for us is that Denjoy totalization is a transfinite procedure which terminates at some countable ordinal stage, and $ACG_{*\alpha}$ consists of precisely the functions $F \in ACG_*$ which are recovered from $F'$ in at most $\alpha$ steps of Denjoy totalization. Therefore, the sets $ACG_{*\alpha}$ have a meaningful interpretation in terms of Denjoy totalization.

Note that there are actually two transfinite procedures which are sometimes called “Denjoy totalization”: the narrow Denjoy integral, which coincides with the integrals of Perron, Kurzweil and Henstock, and the wide Denjoy integral, sometimes known as the Denjoy-Khintchine integral. In this paper, “Denjoy totalization” always refers to the narrow Denjoy integral. The narrow Denjoy integral has the same relationship to the class $ACG_*$ as the wide Denjoy integral has to the class $ACG$.

### 2. Cantor-Bendixson rank

In this section we analyze the initial segments of the Cantor-Bendixson hierarchy. The theorems of this section are not used in later sections. Let $S \subseteq 2^{<\omega}$ be a tree with no dead ends. Let $[S]$ denote the set of paths in $S$. The Cantor-Bendixson derivative $D(S)$ is defined as the tree without dead ends such that $[D(S)]$ consists of exactly the paths not isolated in $[S]$. Define $D^0(S) = S$, $D^{\alpha+1}(S) = D(D^\alpha(S))$, and $D^\lambda(S) = \bigcap_{\alpha<\lambda} D^\alpha(S)$ for $\lambda$ a limit.

**Definition 8.** The Cantor-Bendixson rank of a tree $S \subseteq 2^\omega$, denoted $\|S\|_{CB}$, is the least $\alpha$ such that $D^\alpha(S) = \emptyset$, if such exists. Otherwise we say $\|S\|_{CB} = \infty$.

Other authors define $\|T\|_{CB}$ to be the least $\alpha$ such that $D^\alpha(S) = D^{\alpha+1}(S)$, so that a set with a perfect subset also has a Cantor-Bendixson rank. Others define their rank to be always one less than ours, so that every ordinal is used.

**Proposition 9.** Let $Y \in 2^\omega$ and $\alpha < \omega^Y$. Then $\{S : \|S\|_{CB} \leq \alpha\}$ is $\Sigma^0_2(Y)$ in $\{S : S$ is a tree with no dead ends\}.

**Proof.** The proof is by effective transfinite recursion. Because checking whether a no-dead-ends tree is empty can be accomplished by checking the root, $\{S : \|S\|_{CB} = 0\}$ is $\Sigma^0_0$.

A tree $S$ has $D^{\alpha+1}(S) = \emptyset$ if and only if $D^\alpha(S)$ has only finitely many branches. If $D^\alpha(T)$ has at least $k$ branches, then by going up to a height $n$ at which the branches have separated, we may find at least $k$-many $\sigma$ of length $n$ such that $D^\alpha(T_\sigma) \neq \emptyset$. And if there are $k$ incomparable $\sigma$ such that $D^\alpha(T_\sigma) \neq \emptyset$, then $D^\alpha(T)$ has at least $k$ branches. Thus

$$D^{\alpha+1}(S) = \emptyset \iff \exists k \forall n (\text{there are at most } k \text{ many } \sigma \text{ of length } n \text{ for which } D^\alpha(S_\sigma) \neq \emptyset).$$
Assuming \( \{ S : D^\alpha(S) = \emptyset \} \) is \( \Sigma^0_{2\omega}(Y) \) uniformly in \( \alpha \), this shows that \( D^{\alpha+1}(S) = \emptyset \) is \( \Sigma^0_{2\omega+2}(Y) \) uniformly in \( \alpha \).

If \( \lambda \) is a limit, a tree has \( D^\lambda(S) = \emptyset \) if and only if there is an \( \alpha < \lambda \) such that \( D^\alpha(S) = \emptyset \), by compactness. Assuming \( D^\alpha(S) = \emptyset \) is \( \Sigma^0_{2\omega}(Y) \) uniformly in \( \alpha < \lambda \), and supposing a sequence \( \alpha_n \to \lambda \) is \( Y \)-effectively given, we have \( D^\lambda(S) = \emptyset \) if and only if \( \exists \alpha < \lambda : [D^\alpha(S) = \emptyset] \), a \( \Sigma^0_\lambda(Y) \) statement. Note \( \Sigma^0_\lambda(Y) = \Sigma^0_{2\omega}(Y) \) for \( \lambda \) a limit.

\[ \square \]

**Proposition 10.** There is a computable reduction \( T \mapsto S_T \) from trees \( T \subseteq \omega^{<\omega} \) to no-dead-end trees \( S \subseteq 2^{<\omega} \), satisfying

1. If \( T \) is not well-founded, \( [S_T] \) contains a perfect set.
2. If \( T \) is well-founded with \( |T|_I = \alpha \), then \( [S_T] \) is countable, and \( |S_T|_{CB} = \alpha \).

**Proof.** The idea is that each node of a tree \( T \subseteq \omega^{<\omega} \) should correspond to a branching of paths in \( S_T \subseteq 2^{<\omega} \), with the topological clustering of the paths provided by the hierarchical structure of \( T \).

Define \( S_T \) by

\[ S_T = \{ 0^{2n_0+i_0} \cdots 0^{2n_k+i_k} 10^m : (n_0, \ldots, n_k) \in T, i_0, \ldots, i_k \in \{0,1\} \} \]

Note that \( 0^m \in S_T \) if and only if the empty node is in \( T \). If \( T \not\in WF \), then if \( P \in [T] \) the following subset of \( [S_T] \) is perfect:

\[ \{ 0^{2P(0)+X(0)} \cdots X(1) \} \subseteq \omega \].

Supposing now that \( T \) is well-founded, we claim that \( |T|_I = |S_T|_{CB} \). The proof is by induction on the usual rank of \( T \). If \( T = \emptyset \) then also \( S_T = \emptyset \), so \( |S_T|_{CB} = |T|_I = 0 \).

Suppose \( |T|_I = \alpha+1 \). (This is the only case because the limsup rank is always a successor.) Then there is an \( N \) such that for \( n \geq N, |T_n|_I \leq \alpha \). Observe that if \( C \subseteq 2^{<\omega} \) is any finite prefix-free collection of strings such that \( \cup_{\sigma \in C} [\sigma] \) covers \( [S] \), then

\[ D^\alpha(S) = \bigcup_{\sigma \in C} \sigma \cdot D^\alpha(S_\sigma). \]

Letting \( C = \{ 0^{2N} \} \cup \{ 0^{2n+1} : n < N, i \in \{0,1\} \} \), we have

\[ D^{\alpha+1}(S_T) = \bigcup_{\sigma \in C} \sigma \cdot D^{\alpha+1}((S_T)_\sigma). \]

By induction, for \( n < N \) and \( i \in \{0,1\}, |0^{2n+1} \cdot S_T|_{CB} \leq \alpha + 1 \), so

\[ D^{\alpha+1}(S_T) = 0^{2N} \cdot D^{\alpha+1}((S_T)_{0^{2N}}). \]

Also, for \( n \geq N \), we have \( |0^{2n+1} \cdot S_T|_{CB} \leq \alpha \), so for each \( k, D^\alpha((S_T)_{0^{2N+k}}) = \emptyset \), so \( [D^\alpha((S_T)_{0^{2N+k}})] \subseteq \{0^\alpha \} \). Therefore \( D^{\alpha+1}((S_T)_{0^{2N}}) = \emptyset \), so \( |S_T|_{CB} \leq \alpha + 1 \).

Now we need \( |S_T|_{CB} \geq \alpha + 1 \). If \( |T_n|_I = \alpha + 1 \) for some \( n \), then by induction \( |0^\alpha \cdot S_T|_{CB} = \alpha + 1 \), which suffices. If \( \limsup_n |T_n|_I = \alpha \), then for every \( \beta < \alpha \), there are infinitely many \( n \) such that \( |T_n| > \beta \), so there are infinitely many \( n \) for which \( |0^\alpha \cdot S_T|_{CB} > \beta \). Therefore, for all \( \beta < \alpha \), \( 0^\alpha \) is not an isolated path of \( D^\beta(S_T) \), so \( 0^\alpha \in [D^\beta(S_T)] \), and \( |S_T|_{CB} > \alpha \). \( \square \)

\[ 2 \text{The more familiar option, } S'_T = \{ 0^{n_0} \cdots 0^{n_k} : (n_0, \ldots, n_k) \in T \}, \text{ satisfies part (2) but not part (1) of the theorem; consider the case when } [T] \text{ consists of a single path.} \]
Theorem 11. For any nonzero $\alpha < \omega_1$, let

$$A_\alpha = \{ S \subseteq 2^\omega : S \text{ is a no dead end tree and } |S|_{CB} \leq \alpha \}.$$ 

Then all the conclusions of Theorem 5 hold. In particular, if $\alpha < \omega_1^{CK}$, $A_\alpha$ is $\Sigma^0_{2\alpha}$-complete.

A special case is the following.

Corollary 12. [Lem87] For each constructive $\alpha > 1$, the sets

$$\{ e : \phi_e \text{ codes a tree } S \text{ which has no dead ends and } |S|_{CB} \leq \alpha \}$$

are $\Sigma^0_{2\alpha}$-complete.

This result differs only cosmetically from the result as it was stated in [Lem87] because he considered trees which might have dead ends and because he used a definition of $\Sigma^0_{2\alpha}$-complete which is off by one from our definition for $\alpha \geq \omega$.

3. Completeness and hardness on the Denjoy hierarchies

In this section we present a construction which provides a computable reduction from $(WF, \neg WF)$ to $(ACG_*, \neg VBG_*)$. Because $ACG_* \subseteq ACG \cap VBG_*$ and $ACG \cup VBG_* \subseteq VBG$, this shows that all four classes are $\Pi^1_1$-complete. Additionally, this reduction serves as a simultaneous uniform reduction from $(A_\alpha, A_{\alpha+1})$ to $(ACG_{\alpha*}, \neg VBG_\alpha)$, where $A_\alpha = \{ T \in WF : |T|_{\ell_\delta} \leq \alpha \}$. We have $ACG_{\alpha*} \subseteq ACG_{\alpha} \cap VBG_{\alpha}$ and $ACG_{\alpha} \cup VBG_{\alpha*} \subseteq VBG_\alpha$, so all these sets are at least as complex as $A_\alpha$, namely at least $\Sigma^0_{2\alpha}(Y)$-hard.

The idea is that each node of the given tree should contribute a finite length to the variation of the constructed function. In most cases the total variation will be infinite as a result, but the way in which that infinite length is distributed will determine the rank of the function.

The following notation will be useful: if $J, K \subseteq I$ are two intervals, then $J \langle K \rangle$ denotes the interval that has the same relation to $K$ as $J$ has to $I$; that is, if $J = [a, b]$ and $K = [c, d]$ then $J \langle K \rangle = [c + a(d - c), c + b(d - c)]$.

To define this reduction, it will be useful to have a way to tell $F_T$ to increase its variation in a given interval $J$ in response to seeing more of $T$. Given an interval $J \subseteq [0, 1]$, we define the wiggle function $W(J)$ as follows. The goal is to have a function whose variation is at least 1 (regardless of how small $J$ is), but which only takes values in $[0, |J|]$, where $|J|$ is the length of the interval $J$. Therefore, for small $J$, the function should oscillate intensely. We also leave some space at the top and bottom of each oscillation to give room for adding some more oscillations; this sets us up for a typical method of producing a function of high Denjoy rank.

Definition 9. Given an interval $J \subseteq I$, let $M$ be the least integer large enough that $2M|J| > 1$, and define

1. $W(J)(x) = 0$ if $x \not\in J$.
2. $W(J)(x) = 0$ if $x \in \left[ \frac{4k}{4M+1}, \frac{4k+1}{4M+1} \right]\langle J \rangle$ for $k \leq M$.
3. $W(J)(x) = |J|$ if $x \in \left[ \frac{4k+2}{4M+1}, \frac{4k+3}{4M+1} \right]\langle J \rangle$ for $k < M$.
4. If $x \in \left[ \frac{4k+i}{4M+1}, \frac{4k+i+1}{4M+1} \right]\langle J \rangle$ for $k < M$ and $i = 1, 3$, determine the value of $W(J)(x)$ by linear interpolation from the values already defined.
Define
\[ \text{Flat}(J) = \{ K \subseteq J : K \text{ is a maximal interval on which } W(J) \text{ is constant} \} \]
\[ = \left\{ \left[ \frac{4k + i}{4M + 1}, \frac{4k + i + 1}{4M + 1} \right] : k \leq M, i = 0, 2 \right\} \]

Fix an infinite sequence of disjoint closed interval subsets of \( I \), decreasing in size as they approach 0, with gaps between them. For specificity, say \( J_n = [\frac{1}{3n+2}, \frac{1}{3n+1}] \). For any interval \( L \), we define the function \( \tilde{F}(T, L) \) by recursion on the usual rank of \( T \) as follows.

**Definition 10.** Given an interval \( L \subseteq I \), let \( \tilde{F}(\emptyset, L) \) be the constant function 0, and for nonempty \( T \), let
\[ \tilde{F}(T, L) = W(L) + \sum_{K \in \text{Flat}(L), n \in \omega} \tilde{F}(T_n, J_n(K)). \]

Observe that the \( \tilde{F}(T_n, J) \) are being copied onto the plateaus of the wiggle functions.

It is possible to extend this recursive definition of \( \tilde{F} \) so that any tree \( T \) (even ill-founded) can be used. Each \( \sigma \in T \) contributes some wiggle to \( \tilde{F}(T, L) \), in locations which can be described as follows. Let \( C^L_\emptyset = \{ L \} \), and given \( C^L_\sigma \) for \( |\sigma| \geq 1 \), define
\[ C^L_{\sigma^{-n}} = \bigcup_{H \in C^L_\sigma} \{ J_n(K) : K \in \text{Flat}(H) \}. \]

**Definition 11.** Given an interval \( L \subseteq I \), define
\[ F(T, L) = \sum_{H \in C^L_\sigma} W(H). \]

**Proposition 13.** For all trees \( T \subseteq \omega^{<\omega} \), \( F(T, L) \) is well-defined and uniformly \( T \)-computable. For \( T \in WF \), \( F(T, L) = \tilde{F}(T, L) \).

*Proof.* We may decompose \( F(T, L) = \bigcup_{\ell \in \omega} F_\ell(T, L) \), where
\[ F_\ell(T, L) = \sum_{H \in C^L_\sigma} W(H), \]
\[ \ell \in \omega \]

Observe that since each \( J_n \) satisfies \(|J_n| < 1/2\), each \( H \in C^L_\sigma \) satisfies \(|H| < 2^{-|\sigma|} \). Furthermore, the intervals of \( \bigcup_{|\sigma| = \ell} C^L_\sigma \) are disjoint. Therefore, since \( W(H) \) has its range in \([0, |H|]\), each \( F_\ell(T, L) \) has its range in \([0, 2^{-\ell}]\). Also, the \( F_\ell(T, L) \) are uniformly computable. Therefore, \( F(T, L) \) is the effective uniform limit of computable functions, and is therefore computable.

Now suppose \( T \in WF \). If \( T \) is just a root, it is clear that \( F(T, L) = \tilde{F}(T, L) \). Assuming that each \( F(T_n, K) = \sum_{\sigma \in T_n, H \in C^L_\sigma} W(H) \), the agreement of the two definitions follows in general because for each \( n \),
\[ C^L_{n^{-\sigma}} = \bigcup_{K \in \text{Flat}(L)} C^J_{n^{-\sigma}}(K). \]

**Proposition 14.** If \( T \) is not well-founded, then \( F(T, I) \notin VBG \).
Proof. Let $Z$ be a path in $T$, and consider

$$P_Z = \bigcap_{\ell \in \omega} \left( \bigcup_{\ell' \in \omega} C_{Z|\ell'}^\ell \right)$$

This is an intersection of a decreasing sequence of closed sets in a compact space, so $P_Z$ is closed and non-empty. We claim that there is no open interval $U$ such that $P_Z \cap U$ is nonempty and $F(T, I)$ has bounded variation on $P_Z \cap U$. By Theorem 7 this suffices.

Fix an open interval $U$ and a number $N$. Since $P_Z \cap U \neq \emptyset$, there is an $M$ and $H_0 \in C_{Z|M}$ such that $P_Z \cap H_0 \subseteq U$. We will show that the variation of $F(T, I)$ on $P_Z \cap H_0$ is at least $2^N$. Let $C = \{ H \in C_{Z|(M+N)} : H \subseteq H_0 \}$. Since each interval of $C_{Z|\ell}^\ell$ is broken into multiple intervals in $C_{Z|(\ell+1)}^\ell$, there are at least $2^N$ intervals in $C$.

Let the functions $F_{\ell}$ be as in the previous proposition. For each $H \in C$, $\sum_{\ell < M+N} F_{\ell}(T, I)$ is constant on $H$, the function $W(H)$ has variation at least 1, and $F_{M+N}(T, I)(x) = W(H)(x)$ for all $x \in H$. Therefore, the function $G$ defined by $G = \sum_{\ell \leq M+N} F_{\ell}(T, I)$ has variation at least 1 on $H$. In fact, $G$ has variation at least 1 on any subset of $H$ which contains at least one point of $K$ for each $K \in \text{Flat}(H)$. Therefore, it suffices to show that each such $K$ contains a point $x \in P_Z$ such that $G(x) = F(T, I)(x)$.

We claim that $x = \min(P_Z \cap K)$ is such a point. For each $\ell > M + N$, there is a unique interval $J \in \bigcup_{\sigma \in T : |\sigma| = \ell \in C_{\sigma}^\ell}$ such that $x \in J$. All those intervals are disjoint, so $F_{\ell}(T, I)(x) = W(J)(x)$. We claim that $W(J)(x) = 0$. Since $x \in P_Z$, $J \in C_{Z|\ell}^\ell$. Because $x$ is the minimum of $P_Z \cap K$, it is also the minimum of $P_Z \cap J$. But $P_Z$ contains points from every $L \in \text{Flat}(J)$. Therefore, $x$ is an element of the first such interval $L \subseteq J$. Since $W(J) \equiv 0$ on its first interval, $W(J)(x) = 0$, as required. Since this is true for arbitrary $\ell$, we have $G(x) = F(T, I)(x)$ for all such $x$. Therefore, the variation of $F(T, I)$ on $H_0$ is at least $2^N$, so $F(T, I) \notin VBG$. \qed

Proposition 15. If $T$ is well-founded and $|T|_{ls} \geq 1$, then $F(T, I) \in ACG_*$, and $|F(T, I)|_X = |T|_{ls}$, where $X$ is any of $VB,VB_*,AC,AC_*$.

Proof. We proceed by induction on the usual rank of the tree, starting with the root-only tree $T = \{ \emptyset \}$. The statement we prove inductively is slightly stronger: for all intervals $L$, if $|T|_{ls} = \alpha + 1$ then $|F(T, L)|_X = \alpha + 1$, and furthermore there is a point $x \in P_\alpha^{F(T, L), X}$ such that $F(T, L)(x) = 0$. From here forward, we drop the subscripts on $P_\alpha^{F(T, L), X}$, whenever possible, writing instead $P_\alpha$. When subscripts are included, it is because we reference the derivation applied to a different function, typically $F(T_n, K)$ for some $n$ and some $K$.

In the base case, the tree $T = \{ \emptyset \}$ has $|T|_{ls} = 1$, and $F(T, L) = W(L)$, so it also has rank 1. Let $T$ be given, with $|T|_{ls} = \alpha + 1$. By induction, assume that for each $n$ and $L'$, $|T_n|_{ls} = |F(T_n, L')|_X$. We know that

$$P_\alpha \subseteq \bigcup_{K \in \text{Flat}(L)} \{ \min K \} \cup \bigcup_n J_n(K)$$

because $F(T, L)$ vanishes outside this set. Because $|T_n|_{ls} \leq \alpha + 1$ for all $n$, we have $P_{\alpha+1} \cap J_n(K) = \emptyset$ for all $n$. Similarly, because $|T_n|_{ls} \leq \alpha$ for sufficiently large $n$, we also have $P_\alpha \cap J_n(K) = \emptyset$ for sufficiently large $n$. Therefore, if min $K \in P_\alpha$, it is isolated, so min $K \notin P_{\alpha+1}$. This shows that $|F(T, L)|_X \leq \alpha + 1$.

On the other hand, suppose that $|T|_{ls} = \alpha + 1$ because $|T_n|_{ls} = \alpha + 1$ for some $n$. Then, letting $K \in \text{Flat}(L)$ be leftmost, and letting $x \in P_\alpha^{F(T_n, J_n(K)), X}$ with $F(T_n, J_n(K))(x) = \alpha + 1$.
0, observe that \( x \in P^\alpha \) as well, and \( F(T, L)(x) = W(L)(x) + F(T_n, K)(x) = 0 \), where \( W(L)(x) = 0 \) because \( K \) was chosen leftmost.

Finally, suppose that \( |T|_{ls} = \alpha + 1 \) because for every \( \beta < \alpha \), there are infinitely many \( n \) such that \( |T_n|_{ls} > \beta \). Again let \( K \in \text{Flat}(L) \) be leftmost. For each such \( \beta \), we have \( \min K \in P^\beta \), because for infinitely many \( n \), \( P^\beta \cap J_n(K) \neq \emptyset \). If \( \alpha \) is a limit, it is now immediate that \( \min K \in P^\alpha \). Suppose that \( \alpha = \beta + 1 \). We claim that \( F(T, L) \) is not \( VB \) on \( P^\beta \) in any neighborhood of \( \min K \), which also shows that \( \min K \in P^\alpha \). In either case, observe that \( F(T, L)(\min K) = 0 \).

To prove the claim, given \( N \) and a neighborhood \( U \ni \min K \), let \( n_1 < n_2 \cdots < n_N \) be chosen so that \( J_{n_i}(K) \subseteq U \) and \( |T_{n_i}|_{ls} = \alpha \). Then \( \sum_{i=1}^{N} W(J_{n_i}(K)) \) has variation at least \( N \). If \( P \subseteq K \) is any set containing at least one point from each \( H \in \text{Flat}(J_{n_i}(K)) \) for each \( i \leq N \), then \( \sum_{i=1}^{N} W(J_{n_i}(K)) \) has variation at least \( N \) on \( P \). By the choice of \( n_i \), \( P^\beta \) is such a set, and for each \( H \in \text{Flat}(J_{n_i}(K)) \), there is \( x \in P^\beta \cap H \) such that \( F(T_{n_i}, J_{n_i}(K))(x) = 0 \). Therefore, these points suffice to witness that \( F(T, L) \) has variation at least \( N \) on \( P^\beta \), because \( F(T, L)(x) = \sum_{i=1}^{N} W(J_{n_i}(K))(x) \) for all such \( x \). Therefore, \( |F(T, L)|_{X} = \alpha + 1 \) and \( \min K \in P^\alpha \) satisfies \( F(T, L)(\min K) = 0 \).

**Corollary 16.** Let \( Y \in 2^\omega \) and \( \alpha < \omega^Y \). For any \( \Sigma^0_{2\alpha}(Y) \) set \( B \), there is a \( Y \)-computable reduction from \((B, \neg B)\) to \((ACG_{*\alpha}, \neg \text{VBG}_\alpha)\).

**Theorem 17.** The sets \( \text{VBG}, \text{VBG}_*, \text{ACG}, \text{ACG}_* \) are all \( \Pi^1_1 \)-complete.

This answers a question of Walsh [Wal17], who asked whether \( \text{ACG}_* \) was \( \Pi^1_1 \)-complete. Walsh also showed that the graph of Denjoy integration is a \( \Pi^1_1 \), non-Borel subset of \( M(I) \times C(I) \). He asked whether that set is \( \Pi^1_1 \)-complete, a question which we answer in the affirmative. From here on, let \( F_T = F(T, I) \).

**Lemma 18.** For all trees \( T \subseteq \omega^{<\omega} \), \( F_T \) is a.e differentiable, and the map \( T \mapsto (F'_T, F_T) \in M(I) \times C(I) \) is computable.

**Proof.** For each \( \ell \), let

\[
G_\ell = \sum_{\sigma \in T: |\sigma| < \ell \text{ and } \max(\sigma) < \ell} W(H).
\]

Then \( \lim_{\ell \to \infty} G_\ell = F_T \). Observe that \( G'_\ell \) is a.e. equivalent to an ideal point of \( M(I) \). We claim that in \( M(I) \), \( \lim_{\ell \to \infty} G'_\ell = F'_T \). For this it suffices to observe that \( G_\ell = F_T \) on any interval where \( G'_\ell \neq 0 \), and also on any interval that is disjoint from all intervals of \( C_\sigma \) for \( \sigma \in T \) with \( |\sigma| \geq \ell \) or \( \max(\sigma) \geq \ell \). Regardless of whether \( T \) is well-founded, the measure of these intervals of agreement approaches 1 in the limit, and the convergence is effective. □

**Theorem 19.** The set \( \{(f, F) \in M(I) \times C(I): F \in \text{ACG}_* \text{ and } F' = f \text{ a.e.}\} \) is \( \Pi^1_1 \)-complete.

**Proof.** The map \( T \mapsto (F'_T, F_T) \) provides a computable reduction from \( WF \) to \( \{(f, F) \in M(I) \times C(I): F \in \text{ACG}_* \text{ and } F' = f \text{ a.e.}\} \). □

The next and final result of this section concerns \( \{f \in M(I): f \text{ is Denjoy integrable}\} \). Walsh showed that this set is \( \Sigma^1_2 \) and not \( \Sigma^1_1 \), and asked for better bounds, which we give in Theorem 4. We need a lemma.

**Lemma 20.** If \( T \notin WF \), then \( F'_T \) is not Denjoy integrable.
Theorem 21. The set \( \{ f \in M(I) : f \text{ is Denjoy integrable} \} \) is \( \Pi^1_1 \)-complete.

Proof. Dougherty and Kechris \cite{DK91}, pg. 162] present a proof, originally due to Ajtai, that for sequences \( \mathcal{T} \in C(I)^\omega \), if \( f \) converges to \( f \) and \( f \) is the derivative of an everywhere differentiable function \( F \), then \( F \in \Delta^4_1(f) \). An inspection of the proof shows that the same argument works if the limiting function \( f \) is given as \( f \in M(I) \) (i.e. just its a.e. equivalence class is given) and all that is needed is for \( f \) to be Denjoy integrable (so that \( F \in ACG_\ast \) with \( F' = f \), rather than \( F \) being everywhere differentiable). Therefore, for \( f \in M(I) \),

\[
f \text{is Denjoy integrable} \iff \exists F \in \Delta^4_1(f)[F' = f \text{ a.e.}]
\]

which is \( \Pi^1_1 \).

Now consider the completeness direction. By Lemma \cite{18} the map \( T \mapsto F_T^\prime \) is computable, and \( T \in WF \) if and only if \( F_T^\prime \) is Denjoy integrable by Lemma \cite{20}.

4. Descriptive results on the Denjoy hierarchies

4.1. Descriptive complexity of \( VBG \) and \( VBG_\ast \) hierarchies. Our goal now is to give the precise descriptive complexity of the sets \( VBG_{\alpha}, VBG_\alpha \) and \( ACG_{\alpha} \). For \( ACG_\alpha \), we will only give an upper bound on the descriptive complexity.

Proposition 22. If \( \alpha < \omega^Y_1 \), then \( VBG_{\alpha} \) and \( VBG_\alpha \) are \( \Sigma^\alpha_2(Y) \).

Proof. By effective transfinite recursion. Let \( X \) be one of \( VB, VBG_\ast \), and let \( p, q \in \mathbb{Q} \). When \( \alpha = 1 \), we have \( P^0_{F,X} \cap (p, q) = \emptyset \) if and only if \( F \) has bounded variation on the interval \([p, q]\).

This property is \( \Sigma^0_2 \) uniformly in \( F, p \) and \( q \).

Supposing that \( \{(F, p, q) : [p, q] \cap P^0_{F,X} = \emptyset \} \in \Sigma^\alpha_2(Y) \) uniformly in \( \alpha \), let us show \( \{(F, p, q) : [p, q] \cap P^{\alpha+1}_{F,X} = \emptyset \} \in \Sigma^\alpha_{2\alpha+2}(Y) \) uniformly in \( \alpha \). We have by Proposition \cite{6}

\[
P^{\alpha+1}_{F,VB} \cap [p, q] = \emptyset \iff \exists [p', q'] \supseteq [p, q] \text{ such that } F_{p^\alpha} \text{ is } VB \text{ on } [p', q']
\]

\[
P^{\alpha+1}_{F,VB_\ast} \cap [p, q] = \emptyset \iff \exists [p', q'] \supseteq [p, q] \text{ such that } F_{E^\ast} \text{ is } VB \text{ on } [p', q'], \text{ where } E = P^\alpha.
\]
So it is sufficient to show that there is a uniform procedure for computing, given \( \varepsilon \), approximations to \( F_{p_0} \) and \( F_{p_0, s} \) that are correct to within \( \varepsilon \), using an oracle which can answer \( \Sigma^0_2(F, Y) \) questions, for example questions of the form “\( P_{F,X}^0 \cap [a, b] = \emptyset? \)”.

Using \( F \) and the compactness of \( I \), compute a number \( N \) large enough that
\[
\omega(F, [k/2^N, (k + 1)/2^N]) < \varepsilon
\]
for all \( k < 2^N \). For each \( \ell < k \leq 2^N \), ask whether \( P_{F,X}^0 \cap [\ell/2^N, k/2^N] = \emptyset \). This identifies (up to an error of \( 1/2^N \) on each side) the connected components of \( I \setminus P_{F,X}^0 \). Let \( E \) be the corresponding approximation to \( P_{F,X}^0 \); that is,
\[
E = I \setminus \bigcup \{ ([\ell/2^N, k/2^N) : P_{F,X}^0 \cap [\ell/2^N, k/2^N] = \emptyset \}.
\]
Let \( F^\varepsilon \) be an approximation of \( F \) correct to within \( \varepsilon \). Then return \( F^\varepsilon_E \) or \( F^\varepsilon_{E,*} \) as appropriate.

The returned function will be correct to within \( \varepsilon \) on \( E \); on components of \( I \setminus E \) that were too small to find, \( F^\varepsilon_{E,*} \) could be off by at most \( 4\varepsilon \); on components of \( I \setminus E \) whose endpoints were only approximated, \( F^\varepsilon_{E,*} \) could be off by at most \( 5\varepsilon \). The errors for \( F^\varepsilon_E \) are less.

Therefore, by induction, \( \{ (F, p, q) : P_{F,X}^{\alpha +1} \cap [p, q] = \emptyset \} \) is \( \Sigma^0_{2\alpha+2}(Y) \), uniformly in \( \alpha \).

Finally, if a limit ordinal \( \lambda \) is given as a \( Y \)-effective sequence \( \alpha_n \) with \( \lim_{n \to \infty} \alpha_n = \lambda \), then \( [p, q] \cap P_{F,X}^\lambda = \emptyset \iff \exists n [p, q] \cap P_{F,X}^{\alpha_n} = \emptyset \). Since the statements \( [p, q] \cap P_{F,X}^{\alpha_n} = \emptyset \) uniformly \( \Sigma^0_{2\alpha_n}(Y) \), we have \( \{ (F, p, q) : [p, q] \cap P_{F,X}^\lambda = \emptyset \} \) is \( \Sigma^0_{\lambda}(Y) = \Sigma^0_{2\lambda}(Y) \), uniformly in \( \lambda \).

Observe that in the proof of Proposition 22, the only place it is used is that \( X = VB \) or \( VB_\epsilon \), rather than \( AC \) or \( AC_\epsilon \), is in counting the quantifiers of bounded variation in the successor step, to conclude that two jumps on an oracle for “\( P_{F,X}^0 \cap [a, b] = \emptyset? \)” questions would suffice to answer \( P_{F,X}^{\alpha +1} \cap [a, b] = \emptyset \) questions. If \( X \) were \( AC \) or \( AC_\epsilon \), this argument gives a bound of 4 jumps. In the case of \( AC_\epsilon \), in the next section we will improve the bound to 2 jumps.

Corollary 23. If \( \alpha < \omega_1^Y \), \( ACG_{\alpha \omega} \) and \( ACG_\alpha \) are \( \Sigma^0_{4\alpha}(Y) \).

4.2. Descriptive complexity of the \( ACG_\alpha \) hierarchy. Next we consider the descriptive complexity of \( ACG_\alpha \) hierarchy. Already at the first level there is a difference, because \( F_{F,X} \) is computable if and only if \( F \) is absolutely continuous, a \( \Pi^0_3 \) property.

Theorem 24. The set \( \{ F \in C(I) : F \text{ is absolutely continuous} \} \) is \( \Pi^0_3 \)-complete.

Proof. Our strategy is to define a computable reduction from \( 2^\omega \) to \( C(I, I) \) whose output approximates a version of the Cantor function (Devil’s Staircase function) which will converge to a Cantor-like function only if the input is an element of a given \( \Pi^0_3 \)-complete subset of \( 2^\omega \).

Let \( A \subseteq 2^\omega \) be such a set and let \( g(n) \) be a computable function such that for all \( X \),
\[
X \in A \iff \forall n [W_{g(n)}^X \text{ is finite}]
\]
We now define a function \( F : [0, 1] \to [0, 1] \), uniformly in \( X \), such that \( F \) is absolutely continuous if and only if \( \forall n [W_{g(n)}^X \text{ is finite}] \) holds.

Effective in \( X \), we define a computable sequence of functions \( F_n \) which converge effectively and uniformly to the desired computable function \( F \). Let \( F_0(x) = x \). Each \( F_n \) will be piecewise linear, containing some pieces of slope zero separated by pieces of positive slope. Wherever \( F_n \) is piecewise constant, it is equal to the limiting function \( F \).

For each \( n \) let \( I_n = \left[ \frac{1}{n+2}, \frac{1}{n+1} \right] \). This is the interval in which \( W_{g(n)}^X \)'s finiteness or lack thereof will be expressed. At stage \( s + 1 \), let \( F_{s+1} \upharpoonright \left[ \frac{1}{n+2}, \frac{1}{n+1} \right] = F_s \upharpoonright \left[ \frac{1}{n+2}, \frac{1}{n+1} \right] \) for all \( n \geq s \) and for all \( n < s \) such that no new element of \( W_{g(n)}^X \) has been enumerated at stage \( s \).
those \( n \) for which a new element is enumerated into \( W^X_{g(n)} \), define \( F_{s+1} \upharpoonright I_n \) as follows. For each maximal interval \( I \subseteq I_n \) on which \( F_s \) is constant, let \( F_{s+1} \equiv F_s \) on \( I \). For each maximal interval \( I \) on which \( F_s \) is linear with positive slope, define \( F_{s+1} \) on \( I \) to satisfy:

1. \( F_{s+1} = F_s \) at the endpoints of \( I \).
2. \( F_{s+1} \) is piecewise linear, continuous, and increasing.
3. \( F_{s+1} \) has slope zero on \( \frac{1}{3} \) of the measure of \( I \), and positive slope everywhere else on \( I \).
4. \( F_s \) and \( F_{s+1} \) differ by no more than \( 2^{-s} \) at any point.

This can be accomplished by letting \( F_{s+1} \upharpoonright I \) resemble a sufficiently fine staircase. The effect is that \( \frac{1}{3} \) of the measure of \( I \) is given to points at which \( F'(x) = 0 \). Thus if \( F' \) was nonzero on a measure \( r \) subset of \( I_n \), then \( F'_{s+1} \) is nonzero on a measure \( \frac{2}{3}r \) subset of \( I_n \).

This completes the construction. One may check that \( F \) is continuous and of bounded variation.

Now suppose that it holds that \( \forall n[W^X_{g(n)} \text{ is finite}] \). Then for each \( n \), there will come a stage \( s \) for which \( F_s \upharpoonright I_n = F \upharpoonright I_n \), and so the final \( F \) is piecewise linear on \( I_n \) for all \( n \). Then \( F \) satisfies the Lusin \((N)\) property because each \( I_n \) satisfies it, and there are only countably many \( I_n \). Thus \( F \) is absolutely continuous.

On the other hand, suppose that \( W^X_{g(n)} \) is infinite for some fixed \( n \). Then letting \( Z = \cup_s \{ x \in I_n : F'_s(x) = 0 \} \), we have \( \mu(Z) = \mu(I_n) \), but \( F(Z) \) is countable, since for each \( s \), \( \{ F_s(x) : F'_s(x) = 0 \} \) is finite. But \( F \) is continuous, so \( F(I_n) = I_n \), so \( F(I_n \setminus Z) \) has measure \( \mu(I_n) \), and \( F \) fails to satisfy Lusin’s \((N)\).

Recall that \( F \in ACG \) (respectively \( ACG_\ast \)) if and only if \( F \) is in \( VBG \) (respectively \( VBG_\ast \)) and \( F \) satisfies \((N)\). We do not know the precise descriptive complexity of the condition \((N)\), so for the purposes of this analysis, it is easier to use a related and strictly stronger condition, Banach’s condition \((S)\).

**Definition 12.** A function \( F : I \to \mathbb{R} \) satisfies \((S)\) if for every \( \varepsilon \) there is a \( \delta \) such that for every set \( A \subseteq I \) of Lebesgue measure less than \( \delta \), its image \( F(A) \) has measure less than \( \varepsilon \).

In case \( F \) is continuous, condition \((S)\) has a \( \Pi^0_3 \) equivalent definition similar to the definition of absolute continuity. For the rest of this section we adopt the convention that if \( b < a \), \([a, b] \) will denote the interval \([b, a]\).

**Definition 13.** A function \( F : I \to \mathbb{R} \) satisfies interval-\((S)\) if for every \( \varepsilon \) there is a \( \delta \) such that for all non-decreasing sequences \( a_0, b_0, \ldots, a_k, b_k \), if \( \sum_i (b_i - a_i) < \delta \) and the intervals \([F(a_i), F(b_i)]\) are disjoint, then \( \sum_i |F(b_i) - F(a_i)| \leq \varepsilon \).

**Proposition 25.** A function \( F \in C(I) \) satisfies \((S)\) if and only if it satisfies interval-\((S)\).

**Proof.** If \( F \) satisfies \((S)\), then by the continuity of \( F \), we have that \([F(a_i), F(b_i)] \subseteq F([a_i, b_i])\). Therefore, \( F \) satisfies interval-\((S)\) with the same witnesses that it uses to satisfy \((S)\). On the other hand, suppose \( F \) satisfies interval-\((S)\), and let \( \varepsilon \) be given. Let \( \varepsilon_0 < \varepsilon \). Let \( \delta \) be the witness that \( F \) satisfies \((S)\) for \( \varepsilon_0 \), and let \( A \subseteq I \) with \( \mu(A) < \delta \). Then \( A \) can be covered by \( \cup_{i<\omega} (a_i, b_i) \) where \( \mu(\cup_{i<\omega} (a_i, b_i)) < \delta \) as well. It suffices to show that for each \( k \) and for each \( \varepsilon' > 0 \), that \( \mu(B_k) < \varepsilon_0 + \varepsilon' \), where \( B_k = \cup_{i<k} F([a_i, b_i]) \). By continuity of \( F \), \( B_k \) is a finite union of intervals. Let \((c_j, d_j)_{j<k}\) be chosen so that

- For each \( j \), there is an \( i \) such that \([c_j, d_j] \subseteq F([a_i, b_i])\),
- The \([c_j, d_j]\) are disjoint, and
• \( \mu(B_k \setminus \bigcup_{j<\ell}[c_j, d_j]) < \varepsilon' \).

For each \( j \), choose \( a_j' \) and \( b_j' \) so that for some \( i \), \( a_i', b_i' \in [a_i, b_i] \), \( F([a_i', b_i']) = [c_j, d_j] \), and \( F([a_j', b_j']) = [c_j, d_j] \). Then the \( (a_j', b_j')_{i<\ell} \) are set up as in the definition of interval-(S), so \( \mu(\bigcup_{j<\ell}[c_j, d_j]) \leq \varepsilon \), so \( \mu(B_k) < \varepsilon + \varepsilon' \).

**Proposition 26.** If \( F \) is \( ACG_* \), then \( F \) fulfills condition (S).

**Proof.** We make use of Banach’s property \( (T_i) \). By definition, a function \( F \) satisfies Banach’s property \( (T_i) \) if \( \{ x \in \mathbb{R} : F^{-1}(\{ x \}) \text{ is infinite} \} \) is null. It is known (cf. [Sak64, Thm IX.8.4, pg 284; Thm IX.6.3, pg 279]) that a continuous function satisfies (S) if and only if it satisfies \( (N) \) and \( (T_i) \), and that any \( F \in VBG_* \) satisfies \( (T_i) \). Since any \( F \in ACG \) satisfies \( (N) \), it follows that any \( F \in ACG_* \) satisfies \( (N) \) and \( (T_i) \), and therefore (S).

**Proposition 27.** Let \( \alpha < \omega_1^Y \) and \( \alpha > 1 \). Then \( ACG_{\alpha} \) is \( \Sigma_{2\alpha}(Y) \).

**Proof.** Recall that a continuous function \( F \) is absolutely continuous if and only if it is of bounded variation and satisfies property \( (N) \). Furthermore, for any closed set \( E \), the functions \( F_E \) and \( F_{E,*} \) have property \( (N) \) if \( F \) does, because their linear portions cannot contribute to a failure of \( (N) \). Therefore, if \( F \) has the property \( (N) \), then we have not only that \( F \in ACG \) (resp. \( ACG_* \)) if and only if \( F \in VBG \) (resp. \( VBG_* \)), but also that \( |F|_{AC} = |F|_{VB} \) (resp. \( |F|_{AC,*} = |F|_{VB,*} \)) in this case. This is because in the derivation process, the functions \( F_E \) (respectively \( F_{E,*} \)) are absolutely continuous on an interval if and only if they are of bounded variation on that interval. Since \( (N) \) is also necessary for members of \( ACG \) and \( ACG_* \), we have

\[
ACG_{\alpha} = VBG_{\alpha} \cap \{ F \in C(I) : F \text{ satisfies } (N) \}
\]

and similarly for \( ACG_{\alpha} \). However, since the descriptive complexity of \( (N) \) is not known (the naive complexity is not Borel), we cannot use this. What we can use in the case of \( ACG_* \) is the \( \Pi_3^0 \) property interval-(S), which is implied by \( ACG_* \) and implies \( (N) \). Therefore,

\[
ACG_{\alpha} = VBG_{\alpha} \cap \{ F \in C(I) : F \text{ satisfies interval-(S)} \}
\]

and a similar statement does NOT hold for \( ACG \), as there are functions in \( ACG \) that do not satisfy \( (S) \). Since satisfying interval-(S) is only \( \Pi_3^0 \), this equality and Proposition 22 together complete the proof for all \( \alpha > 1 \).

We conclude this section by summarizing the results into the main theorem mentioned in the introduction.

**Theorem 28.** Let \( Y \in 2^\omega \), let \( 1 < \alpha < \omega_1^Y \), and let \( A_\alpha = VBG_{\alpha} \cap ACG_{\alpha} \). Then \( A_\alpha \) is \( \Sigma_{2\alpha}(Y) \), and for any \( \Sigma_{2\alpha}(Y) \) set \( B \), there is a \( Y \)-computable reduction from \( B \) to \( A_\alpha \). In particular, \( A_\alpha \) is \( \Sigma_{2\alpha} \)-complete, and if \( \alpha < \omega_1^{CK} \), then \( A_\alpha \) is \( \Sigma_{2\alpha} \)-complete.

5. Questions

Letting \( |T| \) denote the usual well-founded rank of a tree \( T \in WF \), a proof of the following in slightly different language can be found in [GMS13, Propositions 2.12 & 2.15].

**Theorem 29 (GMS13).** If \( \alpha < \omega_1^{CK} \), then \( \{ T : |T| \leq \omega \alpha \} \) is \( \Sigma_{2\alpha} \)-complete.

Therefore, the complexity of initial segments of the usual well-founded rank increases by two jumps for every increase of \( \omega \) in the rank. By contrast, all of the natural ranks considered in this paper have an increased complexity of two jumps for every increase of 1 in the rank. We wonder if there are examples of natural \( \Pi_1^1 \) ranks that don’t fit one of these two patterns.
Question 30. Is there an example of a $\Pi^1_1$ set $A$, a natural $\Pi^1_1$ rank which decomposes it as $A = \bigcup_{\alpha < \omega_1} A_\alpha$, and ordinals $\beta_\alpha$ such that each $A_\alpha$ is $\Gamma_{\beta_\alpha}$-complete (here $\Gamma = \Sigma$ or $\Pi$) such that one of the following holds:

1. $\beta_\alpha + i = \beta_{\alpha + \delta}$ for some $\delta > \omega$ and $i \leq 2$, or
2. $\beta_\alpha + i = \beta_{\alpha + 1}$ for some $i > 2$.

It is open whether ACG provides an example satisfying part (2) of the above. Naively, the derivation process producing the classical rank on ACG could require $i = 4$.

Question 31. What are the exact descriptive complexities of the sets $ACG_\alpha$?

The difficulty associated with applying the analysis of this paper to ACG is related to the unknown complexity of Lusin’s $(N)$.

Question 32. What is the descriptive complexity of $\{F \in C(I) : F \text{ satisfies Lusin's } (N)\}$?

Finally, although $\{T \in WF : |T|_{ls} \leq \alpha\}$ and $\{T \in WF : |T| \leq \omega_\alpha\}$ are both $\Sigma^0_2$-complete, and thus there is a reduction from one to the other that passes through the universal set $\{X : n_0 \in H^X_{2^a}\}$ for appropriate $a$, we are not aware of a natural reduction between them.

Question 33. Give natural computable functions $f$ and $g$ such that $f^{-1}(\{T : |T|_{ls} \leq \alpha\}) = \{T : |T| \leq \omega_\alpha\}$ and $g^{-1}(\{T : |T| \leq \omega_\alpha\}) = \{T : |T|_{ls} \leq \alpha\}$.

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