Abstract. This paper establishes non-asymptotic convergence of the cutoffs in Random serial dictatorship mechanism (RSD), in an environment with many students, many schools, and arbitrary student preferences. Convergence is shown to hold when the number of schools, $m$, and the number of students, $n$, satisfy the relation $m \ln m \ll n$, and we provide an example showing that this result is sharp.

We differ significantly from prior work in the mechanism design literature in our use of analytic tools from randomized algorithms and discrete probability, which allow us to show concentration of the RSD lottery probabilities and cutoffs even against adversarial student preferences.

Overview. Random serial dictatorship (RSD) and deferred acceptance (DA) are two classical and widely-employed mechanisms for assignment under ordinal preferences and non-transferrable utility. In particular, they have found use in such diverse and substantive applications as public school choice, assignment of medical residents to hospitals, and allocation of asylum seekers to host countries. This has motivated econometricians to exploit the randomized nature of these mechanisms to estimate important economic parameters, such as school value-added.

Cutoff representations for large matching markets, proposed by Azevedo and Leshno (2016) and Menzel (2015), have become popular as a tractable analytical framework for studying large matching markets. They also have become the basis for empirical studies of assignment markets (Abdulkadri et al., 2017).

Thus far, however, validity of the cutoff representation has only been shown for markets near a limit with infinite students and a finite number of schools. Whether cutoffs provide an adequate description of real-world matching markets, and consequently whether empirical estimates based on the cutoff representation are consistent, remains unresolved. This shortcoming is especially dire in settings where the number of students and the number of schools are of similar size.

To address this, I will conduct a non-asymptotic analysis of cutoff convergence in large matching markets. Namely, I will prove probabilistic bounds on the distance between the observed cutoffs and their large-market counterparts as a function of $n$, the number of students, and $m$, the number of schools. These bounds may be used to prove finite-sample accuracy of statistical estimators as well as asymptotic validity of the cutoff representation in settings where both the number of students and the number of schools grow to infinity.

Model. There are $n$ students and $m \leq n$ schools, and each school $k$ has capacity $\alpha_k$. Often, we will consider an even simpler setting where $\alpha_k = \alpha = m/n$ for all schools. As mentioned in the intro, we will allow the each student’s preferences to be completely arbitrary.

Under the random serial dictatorship mechanism, a permutation $\Pi : [n] \to [n]$ is selected uniformly at random. Then, sequentially for $t = 1, 2, \ldots$, student $\Pi(t)$ selects their most preferred school among those who are not already at capacity. A school is at capacity when $\alpha_k$ students have selected it. We define a school’s cutoff to be the normalized rank of its lowest-ranked admitted student (which is the rank divided by $n$).

Random serial dictatorship

We will now our main result, which quantifies the accuracy of the cutoff representation in a large market.
Interestingly, the behavior of the RSD mechanism exhibits a phase transition in the region where
\[ m \lg m \asymp n. \]
Below this threshold, cutoffs converge. Above this threshold, cutoffs may not converge. In particular, without further assumptions a cutoff representation is not valid in models where schools have asymptotically bounded size.

**Cutoff Convergence.** We will begin with a toy example, which illustrates the behavior of cutoffs in the RSD mechanism surprisingly well. Let us suppose as above that there are \( n \) students and \( m \) schools.

Let each student be assigned a lottery number \( X_i \in [0, 1] \) independently and uniformly at random. Suppose that each school has capacity \( c = n/m \) and moreover that for each school \( k, 1 \leq k \leq m \), exactly \( n/m \) students list school \( k \) as their only acceptable school.

To simplify this discussion, we will define the **cutoff** of school \( k \), \( \gamma_k \), to be the lowest lottery number of a student assigned under the RSD mechanism to school \( k \).

The difference between this notion of cutoff and the normalized rank used elsewhere in the article is immaterial.

**Lemma 1.** For a given school, \( k \), the cutoff \( \gamma_k \) has cdf \( F(x) = x^c \) and expectation \( 1 - 1/c \)

**Proof.** The cutoff in this model is distributed as the maximum of \( c \) independent copies of \( X \sim \text{Unif}([0, 1]) \), corresponding to those \( c \) students who find school \( k \) acceptable. We have that
\[
\mathbb{P}\left( \max_{i \leq c} X_i \leq x \right) = \prod_{i=1}^{c} \mathbb{P}(X_i \leq x) = x^c.
\]
Since the cutoff \( \gamma_k \) is a positive random variable, its expectation can be computed as
\[
E[\gamma_k] = \int_0^\infty \mathbb{P}(\gamma_k > x) \, dx = \int_0^1 1 - x^c \, dx = 1 - \frac{1}{c}.
\]
Taken together, these facts imply that \( \gamma_k \uparrow 1 \) in probability as \( n \uparrow \infty \). Thus, a given school’s cutoff can be shown to lie close to its expectation with high probability, irrespective of the number of schools. A more interesting question is whether this phenomenon can be expected to hold *simultaneously for all schools*. To this end, we can show the following.

**Proposition 1.** The distribution of \( \min_k \gamma_k \) undergoes a phase transition in the region \( n \asymp m \ln m \). In particular, when \( n \leq \alpha m \ln m \) for any finite \( \alpha \) the cutoffs do not converge uniformly to 1. On the other hand, whenever \( n \gg m \ln m \) the cutoffs do converge uniformly to 1.

**Proof.** Suppose that \( n = m \ln m / \alpha \). We have
\[
\mathbb{P}\left( \min_{k \leq m} \gamma_k > t \right) = \prod_{k=1}^{m} \mathbb{P}(\gamma_k > t) = (1 - t^c)^m.
\]
Considering that we have \( c = n/m = \ln m / \alpha \), we can write
\[
(1 - t^c)^m = \left( 1 - \frac{m^{n/mL}}{m} \right)^m = \left( 1 - \frac{m^{1+\ln t/\alpha}}{m} \right)^m.
\]
Choosing $t = \exp(-\alpha)$ and taking the limit as $n \uparrow \infty$ gives us the equality
\[
\lim_{n \uparrow \infty} P \left( \min_{k \leq m} \gamma_k > \exp(-\alpha) \right) = \lim_{m \uparrow \infty} \left( 1 - \frac{1}{m} \right)^m = \frac{1}{e}
\]
Thus, for any constant value of $\alpha$, the smallest cutoff will not converge in probability to 1, implying that the cutoffs do not converge uniformly.

On the other hand, choosing $t = \exp(- (1 + \epsilon) \alpha)$ and noting that $(1 + x/m)^m \to e^x$ uniformly for $x \in (-1, 1)$, we get
\[
\lim_{n \uparrow \infty} P \left( \min_{k \leq m} \gamma_k > \exp(-(1 + \epsilon) \alpha) \right) = \lim_{m \uparrow \infty} e^{m - \epsilon} = 1.
\]
This implies that when $n \gg m \ln m$ the cutoffs do converge uniformly, almost surely.

**General Result.** The prior example is extremely simple. In particular, the list of student preferences is prescribed to have a very special form, which notably implies that the resulting assignment is completely deterministic. It is therefore not an adequate model of real-world assignment.

Nevertheless, we will subsequently show that the behavior of exhibited by the cutoffs generalizes to the setting when school capacities are heterogeneous and student preferences are arbitrary. For uniform school capacity ($\alpha = n/m$), our results imply that the cutoffs of a given school $k$ converge whenever $n \gg m$, and that the cutoffs converge uniformly in the regime $n \gg m \ln m$.

**Theorem 1.** Let $(\pi_1, \ldots, \pi_n)$ be fixed preferences, and consider the random serial dictatorship allocation $\mu$ for these preferences with uniform random ordering $\Pi$. Let the cutoff at school $k$ be given by
\[
\gamma_k = \sup \left\{ \frac{\Pi(i)}{n} \left| \mu(i) = k \right. \right\}.
\]
Then there exist some deterministic cutoffs $\bar{\gamma}_k$ such that
\[
P \left( |\gamma_k - \bar{\gamma}_k| \geq \varepsilon \right) \leq 17 \exp \left\{ - \frac{\varepsilon \alpha_k}{32 \bar{\gamma}_k} \right\}
\]
In particular, since $\bar{\gamma}_k \leq 1$, we can expect cutoffs to be stable for a single school whenever its capacity satisfies $\alpha_k \gg 32$.

**Corollary 1.** Letting $\eta = \min_k \alpha_k / \bar{\gamma}_k \geq \min_k \alpha_k$, we have
\[
P \left( \max_{k \leq m} |\gamma_k - \bar{\gamma}_k| \geq \varepsilon \right) \leq 17 \exp \left\{ \ln m - \frac{\varepsilon \eta}{32} \right\}
\]
Thus, in an asymptotic sequence where capacities are $n/m$ and $m \ln m \ll n$, the full vector of cutoffs converges uniformly with probability 1.

**Remark.** By virtue of our earlier example, this result is best-possible in its dependence on $n$ and $m$.

Let $\tau_k(t, \Pi)$ be the number of the first $t$ students, ranked according to $\Pi$, who weakly prefer $k$ to their selected school. By this construction $\tau_k(t, -)$ is increasing in $t$ and satisfies
\[
\gamma_k = \inf_{t \geq 0} \left\{ \tau_k(t, \Pi) = \alpha \right\} / n.
\]
First, we will show that for a given $t$, the quantity $\tau_k(t, \Pi)$ is close to its expectation with very high probability.
Rearranging, we conclude that

Let us define $\bar{\tau}$

Proof of Theorem 1.

Using the increasing differences property again, we can deduce that

The increasing first differences shown in Proposition 3 implies

$s$

curve

Thus, by Proposition 2,

$\alpha_k \leq {\mathbb{E}}[\tau_k(n\bar{\gamma}_k, \Pi)] = \sum_{s=1}^{n\bar{\gamma}_k} {\mathbb{E}}[\tau_k(s, \Pi)] - {\mathbb{E}}[\tau_k(s-1, \Pi)]$

$\leq (n\bar{\gamma}_k) ( {\mathbb{E}}[\tau_k(n\bar{\gamma}_k, \Pi)] - {\mathbb{E}}[\tau_k(n\bar{\gamma}_k - 1, \Pi)] ).$

Rearranging, we conclude that

$\frac{\alpha_k}{n\bar{\gamma}_k} \leq {\mathbb{E}}[\tau_k(n\bar{\gamma}_k, \Pi)] - {\mathbb{E}}[\tau_k(n\bar{\gamma}_k - 1, \Pi)].$

Using the increasing differences property again, we can deduce that

$\mathbb{E}[\tau_k(n(\bar{\gamma}_k + \varepsilon), \Pi)] \geq \alpha_k + \frac{\varepsilon \alpha_k}{\bar{\gamma}_k}.$

Thus, by Proposition 2

$1 \quad \mathbb{P}(\gamma_k \geq \bar{\gamma}_k + \varepsilon) \leq \mathbb{P}
\left(X_{n(\bar{\gamma}_k + \varepsilon)} \geq \mathbb{E}X_{n(\bar{\gamma}_k + \varepsilon)} + \frac{\varepsilon \alpha_k}{n\bar{\gamma}_k}\right) \leq 8 \exp\left(-\frac{\varepsilon \alpha_k}{32\bar{\gamma}_k}\right).$

Similarly, increasing differences upper bounded by 1 implies that

$\mathbb{E}[\tau_k(n(\bar{\gamma}_k - \varepsilon), \Pi)] \leq \alpha_k + 1 - \frac{\alpha_k \varepsilon}{\bar{\gamma}_k},$

since the secant connecting $\tau_k(0, \Pi) = 0$ to $\tau_k(n\bar{\gamma}_k, \Pi) \leq \alpha_k + 1$ must lie above the curve $s \mapsto \tau_k(s, \Pi)$ at $s = n(\bar{\gamma}_k - \varepsilon)$. By Proposition 2

$\mathbb{P}(\gamma_k \leq \bar{\gamma}_k - \varepsilon) \leq \mathbb{P}
\left(X_{n(\bar{\gamma}_k - \varepsilon)} \geq \mathbb{E}X_{n(\bar{\gamma}_k - \varepsilon)} + \frac{\varepsilon \alpha_k}{n\bar{\gamma}_k}\right)$

$\leq 8 \exp\left(\frac{1 - \varepsilon \alpha_k}{32\bar{\gamma}_k}\right)$

Since $8e^{1/32} \leq 9$, we have by a union bound that

$\mathbb{P}(\mid \gamma_k - \bar{\gamma}_k \mid \geq \varepsilon) \leq 17 \exp\left(-\frac{\varepsilon \alpha_k}{32\bar{\gamma}_k}\right).$

This is what we aimed to show. $\square$
Proof of Corollary 1. By a union bound,
\[
P(\max_{k \leq m} |\gamma_k - \bar{\gamma}_k| \geq \varepsilon) \leq \sum_{k=1}^{m} 17 \exp \left\{ -\frac{\varepsilon \alpha_k / \bar{\gamma}_k}{32} \right\}
\]
\[
\leq 17 m \exp \left\{ -\frac{\varepsilon \eta}{32} \right\} = 17 \exp \left\{ \ln m - \frac{\varepsilon \eta}{32} \right\}
\]
In the case that each \(\alpha_k = n/m\) and \(m \ln m \ll n\) we have \(\eta = n/m \gg \ln m\), so the result follows. \(\square\)

Proof of Proposition 2. Let \(\nu\) be a median of the random variable \(X_t\) and \(S\) (resp. \(T\)) be the set of \(\pi\) such that \(\tau_k(\pi, t) \leq n\nu\) (resp. \(\tau_k(\pi, t) \geq n\nu\)). The proof rests on the following crucial lemmas.

**Lemma 2.** The function \(\pi \rightarrow \tau_k(t, \pi)\) is 2-Lipschitz with respect to the Hamming distance,
\[
(2) \quad \delta(\pi, \sigma) = \# \{ k \mid \pi(k) \neq \sigma(k) \}.
\]

**Lemma 3** (Talagrand (1995), Theorem 5.1). Let \(\Pi : [n] \rightarrow [n]\) be a permutation chosen uniformly at random. Then
\[
(3) \quad \mathbb{E} \left[ \exp \left\{ \frac{1}{16} \delta(\Pi, S) \right\} \right] \leq 2.
\]

By Markov’s inequality and (3), we then have that
\[
P(2\delta(\Pi, S) \geq s) = \mathbb{P}\left( \exp \left\{ \frac{1}{16} \delta(\Pi, S) \right\} \geq \exp \left\{ \frac{s}{32} \right\} \right)
\]
\[
\leq \mathbb{E} \left[ \exp \left\{ \frac{1}{16} \delta(\Pi, S) \right\} \right] \exp \left\{ -\frac{s}{32} \right\}
\]
\[
\leq 2 \exp \left\{ -\frac{s}{32} \right\}.
\]

Since \(\tau_k(t, -)\) is 2-Lipschitz for \(\delta\), it follows that
\[
n(X_t - \nu) = \tau_k(\Pi, t) - n\nu \leq 2\delta(\Pi, S).
\]

So
\[
P(X_t - \nu \geq s) = \mathbb{P}(n(X_t - \nu) \geq ns) \leq \mathbb{P}(2\delta(\Pi, S) \geq ns) \leq 2 \exp \left\{ -\frac{ns}{32} \right\}.
\]

Finally, by Lemma 2.6.2 of Vershynin (2018), we can replace \(\nu\) by \(\mathbb{E}X_t\) for the price of a factor of 2 on the right-hand side. We can repeat the argument with \(S\) replaced by \(T\) and \(X_t - \nu\) by \(\nu - X_t\). Taking a union bound yields Proposition 2. \(\square\)

Proof of Lemma 4. The proof hinges on the following lemmas.

**Lemma 2.** The function \(\pi \rightarrow \tau_k(t, \pi)\) is 2-Lipschitz with respect to the Hamming distance,
\[
(4) \quad \max_{k \leq m} \left\{ \tau_k(t, \pi) - \tau_k(t, t_{ij}\pi) \right\} \leq 2.
\]

where \(t_{ij}\) denotes the transposition of \(i\) and \(j\).

**Lemma 5.** If \(\delta(\sigma, \pi) = r\) then there exist \(r\) transpositions \(t_1, \ldots, t_r\) such that
\[
\pi = t_r \cdots t_1 \sigma.
\]
Having proved these two statements, we have
\[
\max_{k \leq m} \left\{ \tau_k(t, \pi) - \tau_k(t, \sigma) \right\}
\]
\[
= \max_{k \leq m} \left\{ \sum_{j=1}^{r} \tau_k(t, t_j \cdots t_1\sigma) - \tau_k(t, t_{j-1} \cdots t_1\sigma) \right\}
\]
\[
\leq \sum_{j=1}^{r} \max_{k \leq m} \left\{ \tau_k(t, t_j \cdots t_1\sigma) - \tau_k(t, t_{j-1} \cdots t_1\sigma) \right\}
\]\[
\leq \sum_{j=1}^{r} 2 = 2r
\]
This is what we aimed to show. \(\square\)

**Proof of Lemma 4** We begin by defining the “insertion operator" \((j \to s)\) which is defined for a permutation \(\pi\) by
\[
(j \to s)\pi : i \mapsto \begin{cases} 
\pi(i) - \mathbb{1}\{\pi(i) > \pi(j)\} + \mathbb{1}\{\pi(i) > s\} & i \neq j \\
\mathbb{1}\{s\} & i = j
\end{cases}
\]
In other words \((j \to s)\pi\) preserves the ordering of all indices except \(j\), which is removed and re-inserted at position \(s\). We will first prove that if \(\pi' = (j \to s)\pi\) then for all individuals \(h \neq j\) we have
\[
(5) \quad \max_{k \leq m} \left| \tau_k(\pi(h), \pi) - \tau_k(\pi'(h), \pi') \right| \leq \mathbb{1}\{\pi(h) \leq s\}
\]
Noting that \(t_{ij}\pi = (j \to \pi(i))(i \to \pi(j))\pi\) we obtain that for \(r \neq \pi(i), \pi(j)\),
\[
\max_{k \leq m} \left| \tau_k(r, \pi) - \tau_k(r, t_{ij}\pi) \right| \leq \mathbb{1}\{r \leq \pi(i)\} + \mathbb{1}\{r \leq \pi(j)\} \leq 2
\]
Since \(\tau_k(-, \pi)\), viewed as a function \(\mathbb{N} \to \mathbb{N}\), is non-decreasing and 1-Lipschitz, it follows that this inequality must also hold at \(\pi(i)\) and \(\pi(j)\): the difference cannot grow from 0 to something greater than 1 or from 1 to something greater than 2 over a distance of 1 without the larger function growing at a rate exceeding 1.

We are therefore finished if we can prove \((5)\). Since the statement we are proving is symmetric in \(\pi\) and \((j \to s)\pi\), and since \((j \to \pi(j))(j \to s)\pi = \pi\), it is without loss of generality to assume that \(s < \pi(j)\) so that \(j\) is moved ahead in the ranking by \((j \to s)\). Put \(r' = \pi' \circ \pi^{-1}(r)\). Inequality \((5)\) can then be restated as, for all \(r' \neq s\),
\[
\max_{k \leq m} \left| \tau_k(r, \pi) - \tau_k(r', (j \to s)\pi) \right| \leq \mathbb{1}\{r \leq s\},
\]
This must hold for \(0 \leq r' < s\) since the two orderings and hence the choices made are identical in this region. For \(r' > s\) we will proceed by induction on \(r'\), with the following inductive hypotheses:

1. for all schools \(k\),
\[
0 \leq \tau_k(r', \pi') - \tau_k(r, \pi) \leq \mathbb{1}\{r \leq s\};
\]
2. there is at most one available school \(k'\) under \(\pi\) such that
\[
\tau_{k'}((r', (j \to s)\pi) - \tau_{k'}(r, \pi) = 1.
\]
For \( r' = s + 1 \), note that \( \tau_k((s + 1), (j \rightarrow s)\pi) \) will be exactly one greater than \( \tau_k(s, \pi) \) for all schools weakly preferred to the school chosen by \( j \) under \((j \rightarrow s)\pi\), and the same as \( \tau_k(s, \pi) \) for all other schools. Of the former group, only the school chosen by \( j \) can still be available.

Now, suppose the hypotheses hold at \( r' - 1 \) for some \( r' > s + 1 \). If the individual \( i = \pi^{-1}(r) = \pi'^{-1}(r') \) makes the same choice under both orderings, we are done. Otherwise, the school chosen by \( i \) under \( \pi \) is not available under \( \pi' \)—any school available at \( r' \) under \( \pi' \) must be available at \( r \) under \( \pi \), by hypothesis (1). It follows from (2) that this choice must be the single available school \( k' \) at which capacities differ, and that \( i \) must have claimed the last place at this school under \( \pi \).

As such, \( \tau_k \) will increment by one at all schools \( k \) weakly preferred to \( k' \) by \( i \), so \( \tau_k(r, \pi) - \tau_k(r', \pi) \) will remain constant. The same must be true for all schools less preferred than the school \( k'' \) chosen by \( i \) under \( \pi' \), which are neither chosen nor preferred to something chosen. Finally, it follows from (2) that \( \tau_k(r' \pi') - \tau_k(r', \pi) = 0 \), since \( k'' \neq k' \), so \( \tau_k \) is now full under \( \pi' \). Moreover, \( k'' \) is now the only school that is available under \( \pi \) at which capacities differ, so \( \tau_k \) will remain constant.

The proof by induction.

**Proof of Lemma**\(^2\) The permutation \( \pi \sigma^{-1} \) acts as the identity on all elements which do not differ under \( \pi \) and \( \sigma \), hence it only acts non-trivially on \( S \) for some transpositions \( t_1, \ldots, t_r \) that
\[
t_1 \cdots t_r = \pi \sigma^{-1} \iff t_1 \cdots t_r \sigma = \pi.
\]
\(\Box\)

**Proof of Proposition**\(^3\) Let us consider the quantity
\[
E[\tau_k(t, \Pi) - \tau_k(t - 1, \Pi)].
\]
This is equal to the probability that individual \( \Pi^{-1}(t) \) weakly prefers school \( k \) to the set of available alternatives at time \( t \), which is clearly non-negative. Moreover, this can be decomposed as
\[
E \left[ \sum_{i=1}^{n} 1\{\Pi(i) = t\} \{k \geq_i S_t(\Pi)\} \right] = \frac{1}{n} \sum_{i=1}^{n} P(k \geq_i S_t(\Pi) | \Pi(i) = t)
\]
Note that \( S_t(\Pi) \), which is the set of schools available at time \( t \), depends only on \( \Pi_{t-1} = (\Pi^{-1}(1), \ldots, \Pi^{-1}(t - 1)) \), namely the sequence of students ranked in positions 1 through \( t - 1 \).

Moreover, conditional on the event that \( \Pi^{-1}(t) = i \), this sequence has the same distribution as the *unconditional* distribution of \((\Pi)_{t-1}^{-1}\), where \(^*\Pi\) is a uniform random ranking of all individuals excluding \( i \). We can therefore write
\[
E[\tau_k(t, \Pi) - \tau_k(t - 1, \Pi)] = \frac{1}{n} \sum_{i=1}^{n} P(k \geq_i S_t(\Pi) | \Pi(i) = t)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} P(k \geq_i S_t(\Pi))
\]
Since the set of available schools \( S_t(\pi) \) is non-increasing in \( t \) for each given permutation \( \pi \), the indicator \( 1\{k \geq_i S_t(\pi)\} \) is non-decreasing in \( t \), so we conclude that
each summand $P(k \geq i, S_t(\Pi))$ is increasing as a function of $t$. This completes the proof. □

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