IDENTITIES OF SYMMETRY FOR GENERALIZED TWISTED BERNOULLI POLYNOMIALS TWISTED BY RAMIFIED ROOTS OF UNITY

BY

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Abstract. We derive eight identities of symmetry in three variables related to generalized twisted Bernoulli polynomials and generalized twisted power sums, both of which are twisted by ramified roots of unity. All of these are new, since there have been results only about identities of symmetry in two variables. The derivations of identities are based on the $p$-adic integral expression of the generating function for the generalized twisted Bernoulli polynomials and the quotient of $p$-adic integrals that can be expressed as the exponential generating function for the generalized twisted power sums.

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1. Introduction and preliminaries

Let $p$ be a fixed prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ will respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Assume that $|\cdot|_p$ is the normalized absolute value of $\mathbb{C}_p$ such that $|p|_p = \frac{1}{p}$. The group $\Gamma$ of all roots of unity of $\mathbb{C}_p$ is the direct product of its subgroups $\Gamma_u$ (the subgroup of unramified roots of unity) and $\Gamma_r$ (the subgroup of ramified roots of unity). Namely, $\Gamma = \Gamma_u \cdot \Gamma_r$, $\Gamma_u \cap \Gamma_r = \{1\}$, where $\Gamma_u = \{\xi \in \mathbb{C}_p | \xi^r = 1 \text{ for some } r \in \mathbb{Z}_{>0} \text{ with } (r, p) = 1\}$, $\Gamma_r = \{\xi \in \mathbb{C}_p | \xi^{p^s} = 1 \text{ for some } s \in \mathbb{Z}_{>0}\}$. Let $d$ be a fixed positive integer. Then we let $X = X_d = \lim_{\leftarrow \mathbb{N}} \mathbb{Z}/dp^N\mathbb{Z}$, and let $\pi : X \to \mathbb{Z}_p$ be the map given by the inverse limit of the natural maps $\mathbb{Z}/dp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}$. 

If \( g \) is a function on \( \mathbb{Z}_p \), we will use the same notation to denote the function \( g \circ \pi \). Let \( \chi : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{Q}^* \) be a (primitive) Dirichlet character of conductor \( d \). Then it will be pulled back to \( X \) via the natural map \( X \to \mathbb{Z}/d\mathbb{Z} \). Here we fix, once and for all, an imbedding \( \mathbb{Q} \to \mathbb{C}_p \), so that \( \chi \) is regarded as a map of \( X \) to \( \mathbb{C}_p \) (cf. [11]). For a uniformly differentiable function \( f : X \to \mathbb{C}_p \), the \( p \)-adic Volkenborn-type integral of \( f \) is defined (cf. [7]) by

\[
\int_X f(z) d\mu(z) = \lim_{N \to \infty} \frac{1}{dP_N} \sum_{j=0}^{dp^{n-1}} f(j).
\]

Then it is easy to see that

\[
(1.1) \quad \int_X f(z + 1) d\mu(z) = \int_X f(z) d\mu(z) + f'(0).
\]

More generally, we deduce from (1.1) that, for any positive integer \( n \),

\[
(1.2) \quad \int_X f(z + n) d\mu(z) = \int_X f(z) d\mu(z) + \sum_{a=0}^{n-1} f'(a).
\]

Throughout this paper, we let \( \xi \in \Gamma_r \) be any fixed root of unity, and let

\[
(1.3) \quad E = \{ t \in \mathbb{C}_p | |t|_p < p^{-1} \}.
\]

Then, for each fixed \( t \in E \), the function \( e^{zt} \) is analytic on \( \mathbb{Z}_p \) and hence considered as a function on \( X \) and, by applying (1.2) to \( f \) with \( f(z) = \chi(z)\xi^z e^{zt} \), we get the \( p \)-adic integral expression of the generating function for the generalized twisted Bernoulli numbers \( B_{n,\chi,\xi} \) attached to \( \chi \) and \( \xi \);

\[
(1.4) \quad \int_X \chi(z) \xi^z e^{zt} d\mu(z) = \frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at} = \sum_{n=0}^{\infty} B_{n,\chi,\xi} \frac{t^n}{n!} (t \in E).
\]

So we have the following \( p \)-adic integral expression of the generating function for the generalized twisted Bernoulli polynomials \( B_{n,\chi,\xi}(x) \) attached to \( \chi \) and \( \xi \):

\[
(1.5) \quad \int_X \chi(z) \xi^z e^{(x+z)t} d\mu(z) = \frac{te^{xt}}{e^{dt} - 1} \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!} (t \in E, x \in \mathbb{Z}_p).
\]
Also, from (1.1) we have:

\[(1.6) \quad \int_X \xi e^{zt} d\mu(z) = \frac{t}{\xi e^{t} - 1}.\]

Let \(S_k(n; \chi, \xi)\) denote the \(k\)th generalized twisted power sum of the first \(n + 1\) nonnegative integers attached to \(\chi\) and \(\xi\), namely

\[(1.7) \quad S_k(n; \chi, \xi) = \sum_{a=0}^{n} \chi(a)\xi^{a}a^{k} = \chi(0)\xi^{0}0^{k} + \chi(1)\xi^{1}1^{k} + \ldots + \chi(n)\xi^{n}n^{k}.\]

From (1.4), (1.6), and (1.7), one easily derives the following identities: for \(w \in \mathbb{Z}_{>0}\),

\[(1.8) \quad \frac{dw \int_X \chi(x)\xi e^{zt} d\mu(x)}{\int_X \xi^{dwy}e^{dvy} d\mu(y)} = \frac{\xi^{dw}e^{dt} - 1}{\xi^{d}e^{t} - 1} \sum_{a=0}^{d-1} \chi(a)\xi^{a}e^{at}
\]

\[(1.9) \quad = \sum_{a=0}^{d-1} \chi(a)\xi^{a}e^{at}
\]

\[(1.10) \quad = \sum_{k=0}^{\infty} S_k(dw - 1; \chi, \xi) \frac{t^{k}}{k!} \quad (t \in E).\]

In what follows, we will always assume that the \(p\)-adic integrals of the various (twisted) exponential functions on \(X\) are defined for \(t \in E\) (cf. (1.3)), and therefore it will not be mentioned.

[1], [2], [8], [10], [12] and [13] are some of the previous works on identities of symmetry in two variables involving Bernoulli polynomials and power sums. For the brief history, one is referred to those papers. For the first time, the idea of [8] was extended in [6] to the case of three variables so as to yield many new identities with abundant symmetry. This added some new identities of symmetry even to the existing ones in two variables as well. Also, see [4] and [5], for some other extensions of the idea of [8] to the case of three variables.

In this paper, we will produce 8 basic identities of symmetry in three variables \(w_1, w_2, w_3\) related to generalized twisted Bernoulli polynomials and generalized twisted power sums, both of which are twisted by ramified roots of unity (i.e., \(p\)-power roots of unity) (cf. (4.8)-(4.11), (4.14)-(4.17)). All of these seem to be new, since there have been results only about identities
of symmetry in two variables in the literature ([9]). On the other hand, in
[3] the measure introduced by Koblitz (cf. [11]) was exploited in order to
treat the unramified roots of unity case (i.e., the orders of the roots of unity
are prime to \(p\) and the conductors of Dirichlet characters).

The following is stated as Theorem 4.5 and an example of the full six
symmetries in the positive integers \(w_1, w_2, w_3, \ldots\).

\[
\begin{align*}
&w_1^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{d_{w_1-1}} \chi(a) \xi_a \zeta_{w_2} u_3 (w_2 y_1 + \frac{w_2}{w_1} a) \\
&\quad \cdot S_{n-k}(d_{w_3} - 1; \chi, \xi_{w_1} w_2) w_2^{n-k} w_3^{k-1} \\
&= w_1^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{d_{w_2-1}} \chi(a) \xi_a \zeta_{w_1} \zeta_{w_2} u_3 (w_3 y_1 + \frac{w_3}{w_2} a) \\
&\quad \cdot S_{n-k}(d_{w_2} - 1; \chi, \xi_{w_1} w_3) w_3^{n-k} w_2^{k-1} \\
&= w_2^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{d_{w_3-1}} \chi(a) \xi_a \zeta_{w_1} \zeta_{w_2} u_2 (w_1 y_1 + \frac{w_1}{w_2} a) \\
&\quad \cdot S_{n-k}(d_{w_3} - 1; \chi, \xi_{w_1} w_2) w_1^{n-k} w_3^{k-1} \\
&= w_3^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{d_{w_1-1}} \chi(a) \xi_a \zeta_{w_1} \zeta_{w_3} u_2 (w_1 y_1 + \frac{w_1}{w_3} a) \\
&\quad \cdot S_{n-k}(d_{w_1} - 1; \chi, \xi_{w_1} w_3) w_3^{n-k} w_1^{k-1} \\
&= w_3^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{a=0}^{d_{w_2-1}} \chi(a) \xi_a \zeta_{w_1} \zeta_{w_3} u_1 (w_2 y_1 + \frac{w_2}{w_3} a) \\
&\quad \cdot S_{n-k}(d_{w_2} - 1; \chi, \xi_{w_1} w_3) w_2^{n-k} w_1^{k-1}.
\end{align*}
\]

The derivations of identities are based on the \(p\)-adic integral expression
of the generalized twisted Bernoulli polynomials in (1.5) and the quotient
of integrals in (1.8)-(1.10) that can be expressed as the exponential generating function for the generalized twisted power sums. These abundance of symmetries would not be unearthed if such \(p\)-adic integral representations
had not been available. We indebted this idea to the paper [9].
2. Several types of quotients of $p$-adic integrals

Here we will introduce several types of quotients of $p$-adic integrals on $X$ or $X^3$ from which some interesting identities follow owing to the built-in symmetries in $w_1, w_2, w_3$. In the following, $w_1, w_2, w_3$ are all positive integers and all of the explicit expressions of integrals in (2.2), (2.4), (2.6), and (2.8) are obtained from the identity in (1.4) and (1.6). To ease notations, from now on we suppress $\mu$ and denote, for example, $d\mu(x)$ and $d\mu(x_1)d\mu(x_2)d\mu(x_3)$ respectively simply by $dx$ and $dX$.

(a) Type $\Lambda_{23}^i$ (for $i = 0, 1, 2, 3$)

\begin{equation}
\begin{aligned}
I(\Lambda_{23}^i) &= d^i \int_X \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 w_2 w_3 x_1 + w_1 w_2 x_2 + w_1 w_3 x_3} \\
&\quad \cdot e^{(w_1 w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 (\Sigma_{j=1}^3 y_j))t} dX \\
&= (\xi^{d w_1 w_2 w_3 t} - 1) \cdot \left( \sum_{a=0}^{d-1} \chi(a) \xi^{a w_1 w_3 e^{a w_2 w_3 t}} \right) \left( \sum_{a=0}^{d-1} \chi(a) \xi^{a w_1 w_2 e^{a w_1 w_2 t}} \right). 
\end{aligned}
\end{equation}

(b) Type $\Lambda_{13}^i$ (for $i = 0, 1, 2, 3$)

\begin{equation}
\begin{aligned}
I(\Lambda_{13}^i) &= d^i \int_X \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} \\
&\quad \cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 (\Sigma_{j=1}^3 y_j))t} dX \\
&= (\xi^{d w_1 e^{d w_2} - 1} (\xi^{d w_2 e^{d w_3} - 1} (\xi^{d w_3 e^{d w_3} - 1} \\
&\quad \cdot \left( \sum_{a=0}^{d-1} \chi(a) \xi^{a w_1 e^{a w_3 t}} \right) \left( \sum_{a=0}^{d-1} \chi(a) \xi^{a w_2 e^{a w_2 t}} \right) \left( \sum_{a=0}^{d-1} \chi(a) \xi^{a w_3 e^{a w_3 t}} \right). 
\end{aligned}
\end{equation}

(c-0) Type $\Lambda_{12}^i$
\( I(\Lambda_{12}^0) = \int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dX \)

\( I(\Lambda_{12}^1) = \int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dX \)

\( I(\Lambda_{12}^2) = \int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dX \)

\( I(\Lambda_{12}^3) = \int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dX \)

(c-1) Type \( \Lambda_{12}^1 \)

\[ d^3 \int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dX \]

All of the above \( p \)-adic integrals of various types are invariant under all permutations of \( w_1, w_2, w_3 \), as one can see either from \( p \)-adic integral representations in (2.1), (2.3), (2.5), and (2.7) or from their explicit evaluations in (2.2), (2.4), (2.6), and (2.8).

3. Identities for generalized twisted Bernoulli polynomials

All of the following results can be easily obtained from (1.5) and (1.8)-(1.10). First, let’s consider Type \( \Lambda_{23}^0 \), for each \( i = 0, 1, 2, 3 \).

\( a - 0 \) \( I(\Lambda_{23}^0) = \int_{X^3} \chi(x_1)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dx_1 \)

\( I(\Lambda_{23}^1) = \int_{X^3} \chi(x_1)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dx_2 \)

\( I(\Lambda_{23}^2) = \int_{X^3} \chi(x_1)\xi^{w_1 x_1 + w_2 x_2 + w_3 x_3} 
\cdot e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + \sum_{i=1}^{n} w_i x_i + w_1 w_2 y + w_1 w_3 y + w_2 w_3 y + w_1 w_2 w_3 y) t} dx_3 \)

\[ = \left( \sum_{k=0}^{\infty} \frac{B_{k,x} \xi^{w_1 x_1 + w_2 x_2 + w_3 x_3}}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{B_{l,x} \xi^{w_1 x_1 + w_2 x_2 + w_3 x_3}}{l!} \right) \]
\[
\left(\sum_{m=0}^{\infty} \frac{B_{m,\xi^{w_1w_2}}(w_3y_1)(w_1w_2t)^m}{m!}\right)
= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\xi^{w_2w_3}}(w_1y_1)B_{l,\xi^{w_1w_3}}(w_2y_2) \cdot B_{m,\xi^{w_1w_2}}(w_3y_3) \cdot w_1^{l+m}w_2^lw_3^{k+l} \right) \frac{t^n}{n!},
\]

where the inner sum is over all nonnegative integers \(k, l, m\), with \(k+l+m = n\), and

\[
\binom{n}{k, l, m} = \frac{n!}{k!l!m!}.
\]

(a-1) Here we write \(I(\Lambda_{23}^1)\) in two different ways:

\[
I(\Lambda_{23}^1) = \frac{1}{w_3} \int_X \chi(x_1)\xi^{w_2w_3x_1}e^{w_2w_3(x_1+w_1y_1)t}dx_1
\]
\[
\cdot \int_X \chi(x_2)\xi^{w_1w_3x_2}e^{w_1w_3(x_2+w_2y_2)t}dx_2
\]
\[
dw_3 \int_X \chi(x_3)\xi^{w_1w_3x_3}e^{w_1w_3x_3t}dx_3
\]
\[
\cdot \int_X \xi^{dw_1w_2w_3x_4}e^{dw_1w_2w_3x_4}dx_4
\]
\[
= \frac{1}{w_3} \left( \sum_{k=0}^{\infty} B_{k,\xi^{w_2w_3}}(w_1y_1) \frac{(w_2w_3t)^k}{k!} \right)
\]
\[
\cdot \left( \sum_{l=0}^{\infty} B_{l,\xi^{w_1w_3}}(w_2y_2) \frac{(w_1w_2t)^l}{l!} \right)
\]
\[
\cdot \left( \sum_{m=0}^{\infty} S_m(dw_3 - 1; \chi, \xi^{w_1w_2}) \frac{(w_1w_2t)^m}{m!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} \right.
\]
\[
\cdot B_{k,\xi^{w_2w_3}}(w_1y_1)B_{l,\xi^{w_1w_3}}(w_2y_2)
\]
\[
\cdot S_m(dw_3 - 1; \chi, \xi^{w_1w_2})w_1^{l+m}w_2^lw_3^{k+l-1} \right) \frac{t^n}{n!},
\]
(2) Invoking (1.9), (3.2) can also be written as

\[
I(Λ_{23}^1) = \frac{1}{w_3} \sum_{a=0}^{dw_3-1} \chi(a) ξ^{aw_1w_2} \int_X \chi(x_1) ξ^{w_2w_3x_1} e^{w_2w_3(x_1+w_1y_1)t} dx_1 \\
- \int_X \chi(x_2) ξ^{w_1w_3x_2} e^{w_1w_3(x_2+w_2y_2+w_3a)t} dx_2
\]

\[
= \frac{1}{w_3} \sum_{a=0}^{dw_3-1} \chi(a) ξ^{aw_1w_2} \left( \sum_{k=0}^{∞} B_{k,χ} ξ^{w_2w_3} (w_1y_1) \left(\frac{w_2w_3t}{k!}\right)^{k} \right) \\
- \left( \sum_{l=0}^{∞} B_{l,χ} ξ^{w_1w_3} (w_2y_2) \left(\frac{w_1w_3t}{l!}\right)^{l} \right)
\]

\[(3.4)\]

\[
= \sum_{n=0}^{∞} \left( w_3^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k,χ} ξ^{w_2w_3} (w_1y_1) \\
- \sum_{a=0}^{dw_3-1} \chi(a) ξ^{aw_1w_2} B_{n-k,χ} ξ^{w_1w_3} (w_2y_2 + \frac{w_2}{w_3}a)^{n-k} w_3^k \frac{1}{n!} \right)
\]

(a-2) Here we write I(Λ_{23}^2) in three different ways:

\[
I(Λ_{23}^2) = \frac{1}{w_2w_3} \int_X \chi(x_1) ξ^{w_2w_3x_1} e^{w_2w_3(x_1+w_1y_1)t} dx_1 \\
- \int_X \chi(x_2) ξ^{w_1w_3x_2} e^{w_1w_3(x_2+w_2y_2+w_3a)t} dx_2
\]

\[
= \frac{1}{w_2w_3} \left( \sum_{k=0}^{∞} B_{k,χ} ξ^{w_2w_3} (w_1y_1) \left(\frac{w_2w_3t}{k!}\right)^{k} \right) \\
- \left( \sum_{l=0}^{∞} S_l (dw_2 - 1; χ, ξ^{w_1w_3}) \left(\frac{w_1w_3t}{l!}\right)^{l} \right)
\]

\[(3.5)\]

\[
= \sum_{n=0}^{∞} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,χ} ξ^{w_2w_3} (w_1y_1) \\
- S_l (dw_2 - 1; χ, ξ^{w_1w_3}) \right)
\]

\[(3.6)\]
(2) Invoking (1.9), (3.5) can also be written as

\[ I(\Lambda_{23}^2) = \frac{1}{w_2 w_3} \sum_{a=0}^{dw_3-1} \chi(a) \xi_{cw_1 w_3} \]
\[ \cdot \int_X \chi(x_1) \xi_{cw_2 w_1 x_1} \int e^{cw_1 w_3 (x_1 + w_1 y_1 + \frac{w_3}{w_2} a) t} dx_1 \]
\[ \cdot \frac{dw_3}{x} \chi(x_3) \xi_{cw_1 w_3 x_1} e^{cw_1 w_3 x_3 t} dx_3 \]
\[ \cdot \frac{dx_1}{x} \xi_{cw_1 w_3 x_4} e^{cw_1 w_3 x_4 t} dx_4 \]
\[ = \frac{1}{w_2 w_3} \sum_{a=0}^{dw_3-1} \chi(a) \xi_{cw_1 w_3} \sum_{k=0}^{\infty} B_{k, \chi, \xi_{cw_1 w_3}} (w_1 y_1 + \frac{w_3}{w_2} a) \frac{(w_2 w_3 t)^k}{k!} \]
\[ \cdot \left( \sum_{l=0}^{\infty} S_l (dw_3 - 1; \chi, \xi_{cw_1 w_3}) \left( \frac{w_1 w_2 t^l}{l!} \right) \right) \]
\[ = \sum_{n=0}^{\infty} \left( \frac{w_2}{n-1} \sum_{k=0}^{\infty} \binom{n}{k} \sum_{a=0}^{dw_3-1} \chi(a) \xi_{cw_1 w_3} B_{k, \chi, \xi_{cw_1 w_3}} (w_1 y_1 + \frac{w_3}{w_2} a) \right) \]
\[ \cdot \frac{S_{n-k} (dw_3 - 1; \chi, \xi_{cw_1 w_3}) (w_1^{n-k} w_3^{-k-1})}{n!} \]
(a - 3) \[ I(\Lambda_{23}^0) = \frac{1}{w_1 w_2 w_3} \int_X \chi(x_1) \xi^{u_2 w_3 x_1} e^{w_2 w_3 x_1 t} dx_1 \]
\[ \cdot \frac{w_2 \int_X \chi(x_2) \xi^{u_1 w_3 x_2} e^{w_1 w_3 x_2 t} dx_2}{\int_X \xi^{u_1 w_2 x_1} e^{w_1 w_2 x_1 t} dx_1} \]
\[ \cdot \frac{w_3 \int_X \chi(x_3) \xi^{u_1 w_2 x_3} e^{w_1 w_2 x_3 t} dx_3}{\int_X \xi^{u_1 w_2 x_4} e^{w_1 w_2 x_4 t} dx_4} \]
\[ = \frac{1}{w_1 w_2 w_3} \sum_{k=0}^{\infty} S_k (w_1 - 1; \chi, \xi^{u_1 w_3}) \frac{(w_2 w_3 t)^k}{k!} \]
\[ \cdot \sum_{l=0}^{\infty} S_l (w_2 - 1; \chi, \xi^{u_1 w_3}) \frac{(w_1 w_3 t)^l}{l!} \]
\[ \cdot \sum_{m=0}^{\infty} S_m (w_3 - 1; \chi, \xi^{u_1 w_2}) \frac{(w_1 w_2 t)^m}{m!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} \right) S_k (w_1 - 1; \chi, \xi^{u_1 w_3}) S_l (w_2 - 1; \chi, \xi^{u_1 w_3}) \]
\[ \cdot S_m (w_3 - 1; \chi, \xi^{u_1 w_2}) w_1^{l+m} w_2^{m+1} w_3^{k+l-1} \frac{t^n}{n!} \]

(b) For Type \( \Lambda_{13}^i \) \((i=0, 1, 2, 3)\), we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, each of those can be obtained from the corresponding ones in (a-0), (a-1), (a-2), and (a-3). Indeed, if we substitute \( w_2 w_3, w_1 w_3, w_1 w_2 \) respectively for \( w_1, w_2, w_3 \) in (2.1), this amounts to replacing \( t \) by \( w_1 w_2 w_3 t \) and \( \xi \) by \( \xi^{u_1 w_2 w_3} \) in (2.3). So, upon replacing \( w_1, w_2, w_3 \) respectively by \( w_2 w_3, w_1 w_3, w_1 w_2 \), dividing by \( (w_1 w_2 w_3)^n \), and replacing \( \xi^{u_1 w_2 w_3} \) by \( \xi \), in each of the expressions of (3.1), (3.3), (3.4), (3.6), (3.8)-(3.10), we will get the corresponding symmetric identities for Type \( \Lambda_{13}^i \) \((i=0, 1, 2, 3)\).

(c - 0) \[ I(\Lambda_{12}^0) = \int_X \chi(x_1) \xi^{u_1 x_1} e^{u_1 (x_1 + w_2) y} t dx_1 \int_X \chi(x_2) \xi^{u_2 x_2} e^{u_2 (x_2 + w_3) y} dx_2 \]
\[ \cdot \int_X \chi(x_3) \xi^{u_3 x_3} e^{u_3 (x_3 + w_1) y} t dx_3 \]
\[ = \sum_{n=0}^{\infty} \frac{B_{k, \chi, \xi^{u_1} (w_2 y)}}{k!} (w_1 t)^k \left( \sum_{l=0}^{\infty} \frac{B_{l, \chi, \xi^{u_2} (w_3 y)}}{l!} (w_2 t)^l \right) \]
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\[ \sum_{m=0}^{\infty} \frac{B_{m,\xi;w_3}(w_1y)(w_3t)^m}{m!} \]

(3.11)

\[ \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\xi;w_1}(w_2y)B_{l,\xi;w_2}(w_3y) \right) \cdot B_{m,\xi;w_3}(w_1y)w_1^{k}w_2^{l}w_3^{m} \frac{t^n}{n!}. \]

\[ \int_{X} \chi(x_1) \xi^{w_1x_1} e^{w_1x_1t} dx_1 \]

\[ i \int_{X} \xi^{dw_1w_2w_3} e^{dw_1w_2w_3t} dz_3 \]

\[ \int_{X} \chi(x_2) \xi^{w_2x_2} e^{w_2x_2t} dx_2 \]

\[ \int_{X} \chi(x_3) \xi^{w_3x_3} e^{w_3x_3t} dx_3 \]

\[ \int_{X} \xi^{dw_3w_1w_2} e^{dw_3w_1w_2t} dz_2 \]

\[ \frac{1}{w_1w_2w_3} \sum_{k=0}^{\infty} S_k(dw_2 - 1; \chi, \xi^{w_1}) \left( \frac{w_1t^k}{k!} \right) \cdot \left( \sum_{l=0}^{\infty} S_l(dw_3 - 1; \chi, \xi^{w_2}) \left( \frac{w_3t^l}{l!} \right) \right) \]

\[ \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} S_k(dw_2 - 1; \chi, \xi^{w_1})S_l(dw_3 - 1; \chi, \xi^{w_2}) \right) \cdot S_m(dw_1 - 1; \chi, \xi^{w_3})w_1^{k-1}w_2^{l-1}w_3^{m-1} \frac{t^n}{n!}, \]

4. Main theorems

As we noted earlier in the last paragraph of Section 2, the various types of quotients of \( p \)-adic integrals are invariant under any permutation of \( w_1, w_2, w_3 \). So the corresponding expressions in Section 3 are also invariant under any permutation of \( w_1, w_2, w_3 \). Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting \( w_1, w_2, w_3 \) in a single one labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of \( S_3 \). In particular, the number of possible distinct expressions are 1, 2, 3 or 6. (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two
identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4.4 and 4.8, leaving the others as easy exercises for the reader. As for the case of Theorem 4.4, in addition to (4.11)-(4.13), we get the following three ones:

\[(4.1) \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_3}}(w_1 y_1) S_l(dw_3 - 1; \chi, \xi^{w_1})
\cdot S_m(dw_2 - 1; \chi, \xi^{w_3}) w_1^{k+m-1} w_2^{k+l-1},
\]

\[(4.2) \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_3}}(w_2 y_1) S_l(dw_1 - 1; \chi, \xi^{w_3})
\cdot S_m(dw_3 - 1; \chi, \xi^{w_2}) w_2^{l+m-1} w_3^{k+l-1},
\]

\[(4.3) \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi,\xi^{w_3}}(w_3 y_1) S_l(dw_2 - 1; \chi, \xi^{w_3})
\cdot S_m(dw_1 - 1; \chi, \xi^{w_3}) w_3^{l+m-1} w_1^{k+l-1}.
\]

But, by interchanging $l$ and $m$, we see that (4.1), (4.2), and (4.3) are respectively equal to (4.11), (4.12), and (4.13).

As to Theorem 4.8, in addition to (4.17) and (4.18), we have:

\[(4.4) \sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_2 - 1; \chi, \xi^{w_1}) S_l(dw_3 - 1; \chi, \xi^{w_2})
\cdot S_m(dw_1 - 1; \chi, \xi^{w_3}) w_1^{k-1} w_2^{l-1} w_3^{m-1},
\]

\[(4.5) \sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_3 - 1; \chi, \xi^{w_2}) S_l(dw_1 - 1; \chi, \xi^{w_3})
\cdot S_m(dw_2 - 1; \chi, \xi^{w_1}) w_2^{k-1} w_3^{l-1} w_1^{m-1},
\]

\[(4.6) \sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_3 - 1; \chi, \xi^{w_1}) S_l(dw_2 - 1; \chi, \xi^{w_3})
\cdot S_m(dw_1 - 1; \chi, \xi^{w_2}) w_1^{k-1} w_3^{l-1} w_2^{m-1},
\]

\[(4.7) \sum_{k+l+m=n} \binom{n}{k,l,m} S_k(dw_2 - 1; \chi, \xi^{w_3}) S_l(dw_1 - 1; \chi, \xi^{w_2})
\cdot S_m(dw_3 - 1; \chi, \xi^{w_1}) w_3^{k-1} w_2^{l-1} w_1^{m-1}.
\]
However, (4.4) and (4.5) are equal to (4.17), as we can see by applying the permutations \( k \to l, l \to m, m \to k \) for (4.4) and \( k \to m, l \to k, m \to l \) for (4.5). Similarly, we see that (4.6) and (4.7) are equal to (4.18), by applying permutations \( k \to l, l \to m, m \to k \) for (4.6) and \( k \to m, l \to k, m \to l \) for (4.7).

**Theorem 4.1.** Let \( w_1, w_2, w_3 \) be any positive integers. Then we have:

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, l, m} \cdot B_{l, m} \cdot B_{m, l, w_1 w_2 w_3} (w_1 w_2 w_3 y_1) \cdot B_{l, w_1 w_2 w_3} (w_2 y_2)
\]

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, l, m} \cdot B_{l, m} \cdot B_{m, l, w_1 w_2 w_3} (w_2 y_2) \cdot B_{l, w_1 w_2 w_3} (w_1 y_2)
\]

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, l, m} \cdot B_{l, m} \cdot B_{m, l, w_1 w_2 w_3} (w_1 y_2) \cdot B_{l, w_1 w_2 w_3} (w_3 y_2)
\]

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, l, m} \cdot B_{l, m} \cdot B_{m, l, w_1 w_2 w_3} (w_3 y_2) \cdot B_{l, w_1 w_2 w_3} (w_2 y_2)
\]

**Theorem 4.2.** Let \( w_1, w_2, w_3 \) be any positive integers. Then we have:

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, l, m} \cdot B_{l, m} \cdot B_{m, l, w_1 w_2 w_3} (w_1 y_1) \cdot B_{l, w_1 w_2 w_3} (w_2 y_2)
\]

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, l, m} \cdot B_{l, m} \cdot B_{m, l, w_1 w_2 w_3} (w_3 y_3) \cdot B_{l, w_1 w_2 w_3} (w_2 y_2)
\]
\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, X, \xi} w_3 (w_1 y_1) B_{l, X, \xi} w_2 (w_3 y_2) \\
\cdot S_m (dw_2 - 1; \chi, \xi) w_1^{l+m} w_3^{k+l-1}
\]

(4.9)

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, X, \xi} w_3 (w_2 y_1) B_{l, X, \xi} w_2 (w_1 y_2) \\
\cdot S_m (dw_1 - 1; \chi, \xi) w_2^{l+m} w_1^{k+l-1}
\]

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, X, \xi} w_2 (w_3 y_1) B_{l, X, \xi} w_1 (w_2 y_2) \\
\cdot S_m (dw_3 - 1; \chi, \xi) w_3^{l+m} w_2^{k+l-1}
\]

\[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, X, \xi} w_2 (w_3 y_1) B_{l, X, \xi} w_3 (w_1 y_2) \\
\cdot S_m (dw_2 - 1; \chi, \xi) w_3^{l+m} w_1^{k+l-1}
\]

\[
= \frac{1}{2} \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, X, \xi} w_1 (w_2 y_1) \\
\cdot S_m (dw_3 - 1; \chi, \xi) w_3^{l+m} w_1^{k+l-1}
\]

**Theorem 4.3.** Let \( w_1, w_2, w_3 \) be any positive integers. Then we have:

\[
\frac{1}{w_1} \sum_{k=0}^{n} \binom{n}{k} B_{k, X, \xi} w_2 (w_3 y_1) \\
\cdot \sum_{a=0}^{dw_1 - 1} \chi(a) \xi^{aw_2 w_3} B_{n-k, X, \xi} w_1 (w_2 y_2 + \frac{w_2}{w_1} a) w_3^{n-k} w_2^{k}
\]

\[
= \frac{1}{w_1} \sum_{k=0}^{n} \binom{n}{k} B_{k, X, \xi} w_1 (w_2 y_1) \\
\cdot \sum_{a=0}^{dw_2 - 1} \chi(a) \xi^{aw_2 w_3} B_{n-k, X, \xi} w_1 (w_3 y_2 + \frac{w_3}{w_1} a) w_1^{n-k} w_3^{k}
\]

\[
= \frac{1}{w_2} \sum_{k=0}^{n} \binom{n}{k} B_{k, X, \xi} w_1 (w_3 y_1)
\]
\begin{align*}
&\sum_{a=0}^{w_2-1} \chi(a)\xi^{aw_1w_3} B_{n-k,\chi,\xi^{w_2w_3}} (w_1y_2 + \frac{w_1}{w_2}a)w_3^{n-k}w_1^k \\
&= w_2^{n-1} \sum_{k=0}^{w_2-1} \binom{n}{k} B_{k,\chi,\xi^{w_2w_3}} (w_1y_1) \\
&= w_3^{n-1} \sum_{k=0}^{w_3-1} \binom{n}{k} B_{k,\chi,\xi^{w_1w_3}} (w_2y_1) \\
&= w_3^{n-1} \sum_{k=0}^{w_3-1} \binom{n}{k} B_{k,\chi,\xi^{w_1w_2}} (w_1y_2 + \frac{w_1}{w_2}a)w_2^{n-k}w_1^k \\
&= w_3^{n-1} \sum_{k=0}^{w_3-1} \binom{n}{k} B_{k,\chi,\xi^{w_1w_3}} (w_1y_1) \\
&= w_3^{n-1} \sum_{k=0}^{w_3-1} \binom{n}{k} B_{k,\chi,\xi^{w_1w_3}} (w_2y_2 + \frac{w_3}{w_2}a)w_1^{n-k}w_1^k.
\end{align*}

**Theorem 4.4.** Let $w_1, w_2, w_3$ be any positive integers. Then we have the following three symmetries in $w_1, w_2, w_3$:

\begin{align*}
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\chi,\xi^{w_2w_3}} (w_1y_1) S_l(dw_2 - 1; \chi, \xi^{w_1w_3}) \\
&\cdot S_m(dw_3 - 1; \chi, \xi^{w_1w_2}) w_1^{l+m} w_2^{k-1} w_3^{k+l-1} \\
&= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\chi,\xi^{w_1w_3}} (w_2y_1) S_l(dw_3 - 1; \chi, \xi^{w_1w_2}) \\
&\cdot S_m(dw_1 - 1; \chi, \xi^{w_1w_3}) w_1^{l+m} w_2^{k-1} w_3^{k+l-1} \\
&= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\chi,\xi^{w_1w_2}} (w_3y_1) S_l(dw_1 - 1; \chi, \xi^{w_1w_3}) \\
&\cdot S_m(dw_2 - 1; \chi, \xi^{w_1w_3}) w_1^{l+m} w_2^{k-1} w_3^{k+l-1}.
\end{align*}
Theorem 4.5. Let $w_1, w_2, w_3$ be any positive integers. Then we have:

\[
(w_1 w_2)_{n-1} \sum_{k=0}^{n-1} \binom{n}{k} \sum_{a=0}^{d_{w_1}-1} \sum_{b=0}^{d_{w_2}-1} \chi(ab) \xi^{aw_2+bw_1} B_{n, \chi, \xi, w_1} \left( w_1 y_1 + \frac{w_2}{w_1} a \right) \cdot S_{n-k}(dw_2 - 1; \chi, \xi^{w_1} w_2) w_2^{n-k} w_2^{k-1} \\
= w_1^{n-1} \sum_{k=0}^{n-1} \binom{n}{k} \sum_{a=0}^{d_{w_1}-1} \chi(a) \xi^{aw_2 w_3} B_{n, \chi, \xi^{w_1} w_2} \left( w_3 y_1 + \frac{w_3}{w_1} a \right) \\
\cdot S_{n-k}(dw_2 - 1; \chi, \xi^{w_1} w_2) w_2^{n-k} w_2^{k-1} \\
= w_2^{n-1} \sum_{k=0}^{n-1} \binom{n}{k} \sum_{a=0}^{d_{w_2}-1} \chi(a) \xi^{aw_1 w_3} B_{n, \chi, \xi^{w_1} w_2} \left( w_1 y_1 + \frac{w_1}{w_2} a \right) \\
\cdot S_{n-k}(dw_1 - 1; \chi, \xi^{w_1} w_3) w_3^{n-k} w_3^{k-1} \\
= w_3^{n-1} \sum_{k=0}^{n-1} \binom{n}{k} \sum_{a=0}^{d_{w_3}-1} \chi(a) \xi^{aw_1 w_2} B_{n, \chi, \xi^{w_1} w_2} \left( w_1 y_1 + \frac{w_1}{w_3} a \right) \\
\cdot S_{n-k}(dw_1 - 1; \chi, \xi^{w_1} w_3) w_3^{n-k} w_3^{k-1}.
\]

Theorem 4.6. Let $w_1, w_2, w_3$ be any positive integers. Then we have the following three symmetries in $w_1, w_2, w_3$:

\[
(w_1 w_2)^{n-1} \sum_{a=0}^{d_{w_1}-1} \sum_{b=0}^{d_{w_2}-1} \chi(ab) \xi^{aw_3+bw_1} B_{n, \chi, \xi, w_1} \left( w_3 y_1 + \frac{w_3}{w_1} a + \frac{w_2}{w_2} b \right) \\
= (w_2 w_3)^{n-1} \sum_{a=0}^{d_{w_2}-1} \sum_{b=0}^{d_{w_3}-1} \chi(ab) \xi^{aw_1+bw_3} B_{n, \chi, \xi, w_1} \left( w_2 y_1 + \frac{w_2}{w_2} a + \frac{w_3}{w_3} b \right)
\]
\[ B_{n, \chi, \xi; w_1 w_2 w_3} \left( \frac{w_1 y_1}{w_2} a + \frac{w_1}{w_3} b \right) \]
\[ = (w_3 w_1)^{n-1} \sum_{a=0}^{d_{w_3-1}} \sum_{b=0}^{d_{w_1-1}} \chi(ab) \xi^{w_2(a w_1 + b w_3)} \cdot B_{n, \chi, \xi; w_1 w_2 w_3} \left( \frac{w_2 y_1}{w_3} a + \frac{w_2}{w_1} b \right). \]

**Theorem 4.7.** Let \( w_1, w_2, w_3 \) be any positive integers. Then we have the following two symmetries in \( w_1, w_2, w_3 \):

\[
\sum_{k,l,m=3}^{n} \binom{n}{k,l,m} B_{k, \chi, \xi; w_1 w_3}(w_1 y_1) \cdot B_{l, \chi, \xi; w_2 w_1}(w_3 y_1) B_{m, \chi, \xi; w_3 w_2}(w_2 y_1)
\]
\[
= \sum_{k,l,m=3}^{n} \binom{n}{k,l,m} B_{k, \chi, \xi; w_1 w_3}(w_1 y_1) \cdot B_{l, \chi, \xi; w_2 w_1}(w_3 y_1) B_{m, \chi, \xi; w_3 w_2}(w_2 y_1).
\]

(4.16)

**Theorem 4.8.** Let \( w_1, w_2, w_3 \) be any positive integers. Then we have the following two symmetries in \( w_1, w_2, w_3 \):

\[
\sum_{k,l,m=3}^{n} \binom{n}{k,l,m} S_k(dw_1 - 1; \chi, \xi^{w_3}) S_l(dw_2 - 1; \chi, \xi^{w_1}) \cdot S_m(dw_3 - 1; \chi, \xi^{w_2}) w_3^{k-1} w_1^{l-1} w_2^{m-1}
\]
\[
= \sum_{k,l,m=3}^{n} \binom{n}{k,l,m} S_k(dw_1 - 1; \chi, \xi^{w_3}) S_l(dw_2 - 1; \chi, \xi^{w_1}) \cdot S_m(dw_3 - 1; \chi, \xi^{w_2}) w_2^{k-1} w_1^{l-1} w_3^{m-1}.
\]

(4.17)

(4.18)

**REFERENCES**

1. **Deeba, E.Y.; Rodriguez, D.M.** – Stirling’s series and Bernoulli numbers, Amer. Math. Monthly, 98 (1991), 423–426.

2. **Howard, F.T.** – Applications of a recurrence for the Bernoulli numbers, J. Number Theory, 52 (1995), 157–172.
3. Kim, D.S. – Identities of symmetry for generalized twisted Bernoulli polynomials twisted by unramified roots of unity, submitted.

4. Kim, D.S. – Identities of symmetry for q-Bernoulli polynomials, Comput. Math. Appl., 60 (2010), 2350–2359.

5. Kim, D.S.; Park, K.H. – Identities of symmetry for Euler polynomials arising from quotients of fermionic integrals invariant under $S_3$, J. Inequal. Appl., 2010, Art. ID 851521, 16 pp.

6. Kim, D.S.; Park, K.H. – Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under $S_3$, Appl. Math. Comput., 219 (2013), 5096–5104.

7. Kim, T. – On a $q$-analogue of the $p$-adic log gamma functions and related integrals, J. Number Theory, 76 (1999), 320–329.

8. Kim, T. – Symmetry $p$-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials, J. Difference Equ. Appl., 14 (2008), 1267–1277.

9. Kim, T.; Kim, Y.-H. – On the symmetric properties for the generalized twisted Bernoulli polynomials, J. Inequal. Appl., 2009, Art. ID 164743, 8 pp.

10. Kim, T. – Symmetry of power sum polynomials and multivariate fermionic $p$-adic invariant integral on $\mathbb{Z}_p$, Russ. J. Math. Phys., 16 (2009), 93–96.

11. Koblitz, N. – A new proof of certain formulas for $p$-adic $L$-functions, Duke Math. J., 46 (1979), 455–468.

12. Tuenter, H.J.H. – A symmetry of power sum polynomials and Bernoulli numbers, Amer. Math. Monthly, 108 (2001), 258–261.

13. Yang, S. – An identity of symmetry for the Bernoulli polynomials, Discrete Math., 308 (2008), 550–554

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