Phase Transition for the Mixing Time of the Glauber Dynamics for Coloring Regular Trees

Prasad Tetali *      Juan C. Vera †      Eric Vigoda ‡      Linji Yang ‡

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Abstract

We prove that the mixing time of the Glauber dynamics for random $k$-colorings of the complete tree with branching factor $b$ undergoes a phase transition at $k = b(1 + o_b(1))/\ln b$. Our main result shows nearly sharp bounds on the mixing time of the dynamics on the complete tree with $n$ vertices for $k = Cb/\ln b$ colors with constant $C$. For $C \geq 1$ we prove the mixing time is $O(n^{1+o_b(1)} \ln^2 n)$. On the other side, for $C < 1$ the mixing time experiences a slowing down, in particular, we prove it is $O(n^{1/C+o_b(1)} \ln^2 n)$ and $\Omega(n^{1/C-o_b(1)})$. The critical point $C = 1$ is interesting since it coincides (at least up to first order) to the so-called reconstruction threshold which was recently established by Sly. The reconstruction threshold has been of considerable interest recently since it appears to have close connections to the efficiency of certain local algorithms, and this work was inspired by our attempt to understand these connections in this particular setting.

*School of Mathematics and School of Computer Science, Georgia Institute of Technology, Atlanta GA 30332. Email: tetali@math.gatech.edu. Research supported in part by NSF grant DMS-0701043.

†Department of Management Sciences, University of Waterloo, Waterloo ON. Email: jvera@waterloo.edu.

‡School of Computer Science, Georgia Institute of Technology, Atlanta GA 30332. Email: {vigoda,lijyang}@gatech.edu. Research supported in part by NSF grant CCF-0830298 and CCF-0455666.
1 Introduction

There has been considerable interest in recent years in understanding the mixing time of Markov chains arising from single-site updates (known as Glauber dynamics) for sampling spin systems on finite graphs. The Glauber dynamics is well-studied both for its computational purposes, most immediately its use in Markov chain Monte Carlo (MCMC) algorithms, and for its physical motivation as a model of how physical systems reach equilibrium. Several works in this topic focus on exploring the dynamical and spatial connections between the mixing time and equilibrium properties of the spin system. A notable example of such equilibrium properties is the uniqueness of the infinite volume Gibbs measure, which very roughly speaking corresponds to the influence of a worst-case boundary condition. Recently a related weaker notion known as the reconstruction threshold has been the focus of considerable study. Reconstruction considers the influence of a “typical” boundary condition (we define it more precisely momentarily).

Much of the recent interest in reconstruction stems from its conjectured connections to the efficiency of local algorithms on trees and tree-like graphs, such as sparse random graphs. The Glauber dynamics is one particular example of such a local algorithm, another important example is belief propagation algorithms. The recent work of Acheniewicz and Coja-Oghlan [1] gives strong evidence for the “algorithmic barriers” that arise in the reconstruction phase for several constraint satisfaction problems, including colorings, on sparse random graphs. In this paper we show the mixing time of the Glauber dynamics for random colorings of the complete tree undergoes a phase transition, and the critical point appears to coincide with the reconstruction threshold.

We study the heat-bath version of the Glauber dynamics on the complete tree with branching factor $b$ for the case of (proper vertex) $k$-colorings. Proper colorings correspond in the physics community to the zero-temperature limit of the anti-ferromagnetic Potts model, and the infinite complete tree is known as the Bethe lattice. Let $C = \{1, 2, \ldots, k\}$ denote the set of $k$ colors, and $T_\ell = (V, E)$ denote the complete tree with branching factor $b$, height $\ell$ and $n$ vertices. We are looking at the set $\Omega$ of proper vertex $k$-colorings which are assignments $\sigma : V \to C$ such that for all $(v, w) \in E$ we have $\sigma(v) \neq \sigma(w)$. The Glauber dynamics for colorings is a Markov chain $(X_t)$ whose state space is $\Omega$ and transitions $X_t \to X_{t+1}$ are defined as follows:

- Choose a vertex $v$ uniformly at random.
- For all $w \neq v$ set $X_{t+1}(w) = X_t(w)$.
- Choose $X_{t+1}(v)$ uniformly at random from its set of available colors $C \setminus X_t(N(v))$ where $N(v)$ denotes the neighbors of $v$.

For the complete tree, when $k \geq 3$ the dynamics is ergodic where the unique stationary distribution is the uniform distribution over $\Omega$. The mixing time is the number of steps, from the worst initial state, to reach within variation distance $\leq 1/2e$ of the stationary distribution. We also consider the relaxation time which is the inverse of the spectral gap of the transition matrix. We formally define these notions in Section 3.

For general graphs of maximum degree $b$, the Glauber dynamics is ergodic when $k \geq b + 2$ and the best result for arbitrary graphs proves $O(n^2)$ mixing time when $k > 11b/6$ [20]. There are a variety of improvements for classes of graphs with high degree or girth, see [11] for a survey, and recently, Mossel and Sly [27] proved polynomial mixing time for sparse random graphs $G(n, d/n)$ for constant $d > 1$ for some constant number of colors.

There are two phase transitions of primary interest in the tree $T_\ell$ – uniqueness and reconstruction. These phase transitions are realized by analyzing the influence of the boundary condition,
Theorem 1. For all $C > 0$, there exists $b_0$ such that, for all $b > b_0$, for $k = Cb/\ln b$, the Glauber dynamics on the complete tree $T$ on $n$ vertices with branching factor $b$ and height $H = \lfloor \log_b n \rfloor$ satisfies the following:

1. For $C \geq 1$:

$$\Omega \left( n \ln n / (b \text{ poly}(\log b)) \right) \leq T_{\text{mix}} \leq O(n^{1+o_b(1)} \ln^2 n)$$

$$\Omega (n) \leq T_{\text{relax}} \leq O(n^{1+o_b(1)})$$

2. For $C < 1$:

$$\Omega (n^{1/C-o_b(1)}) \leq T_{\text{mix}} \leq O(n^{1/C+o_b(1)} \ln^2 n)$$

$$\Omega (n^{1/C-o_b(1)}) \leq T_{\text{relax}} \leq O(n^{1/C+o_b(1)})$$

where the $o_b(1)$ functions are $O(\ln \ln b / \ln b)$ for the upper bounds, $b^{1-1/C}/C$ for the lower bounds when $1/2 < C < 1$ and exactly zero for the lower bounds when $0 < C \leq 1/2$. The constants in the $\Omega ()$ and $O ()$ are universal constants.
Remark. When $C \geq 1$, the lower bound of the mixing time is proved by Hayes and Sinclair in a more general setting, and for the particular case of the heat-bath version of the Glauber dynamics on the complete tree, we believe it can be improved to $\Omega(n \ln n/\text{poly}(\log b))$ by the same proof. The lower bound of the relaxation time simply follows from the fact that the probability of selecting a specific vertex to recolor in one step of the dynamics is $1/n$. Note, the results of Berger et al. imply a lower bound of $T_{\text{relax}} \geq \omega(n)$ for the case $C < 1$ since reconstruction holds in this region.

Our result extends to more general $k$ and $b$, thereby refining the general picture provided by [12] and [21].

**Theorem 2.** There exists $b_0$ such that, for all $k, b$ satisfying $b/(k \ln b) > 2$ and $b > b_0$, the Glauber dynamics on the complete tree of $n$ vertices with branching factor $b$ satisfies the following:

$$
\Omega(n^{b/(k \ln b)}) \leq T_{\text{mix}} \leq O(n^{b/(k \ln b) + \gamma \ln^2 n})
$$

$$
\Omega(n^{b/(k \ln b)}) \leq T_{\text{relax}} \leq O(n^{b/(k \ln b) + \gamma}),
$$

where

$$
\gamma = \gamma(b) = 1 - \frac{\ln k}{\ln b} + \frac{\ln \ln b}{\ln b} + O(1)
$$

is at most a small constant.

**Remark.** The constants in the $\Omega()$ and $O()$ of Theorem 2 are universal constants. Also, note that when $k = b^\alpha$ for constant $\alpha < 1$, then $\lim_{b \to \infty} \gamma = 1 - \alpha$, and when $k$ is constant, then $\lim_{b \to \infty} \gamma = 1$.

## 2 Proof Overview

We now give an outline of the proofs of Theorem 2. Readers can refer to Section 3 for the definitions and background materials.

### 2.1 Upper bounds

We first sketch the proof approach for upper bounding the mixing time and relaxation time. Let $G^* = (V, E)$ be the star graph on $b + 1$ vertices, i.e., the complete tree $T_1$ of height 1 with $b$ leaves, and $H$ be the height of the complete tree $T_H$, i.e., $H = \lfloor \log_b n \rfloor$. Let $\tau^*$ be the relaxation time of the Glauber dynamics on the star graph $G^*$ using $k$ colors.

We use the following decomposition result of Lucier and Molloy, which is an application of the block dynamics technique (see, Proposition 3.4 in [23]) to the Glauber dynamics on the complete trees combined with Lemma 2 in Mossel and Sly.

**Theorem 3.** The relaxation time $T_{\text{relax}}$ of the Glauber dynamics on the complete tree of height $H$ with branching factor $b$ satisfies

$$
T_{\text{relax}} \leq (\max\{b, \tau^*\})^H.
$$

Therefore, proving the upper bounds in Theorem 2 reduces to the problem of getting tight upper bounds of the relaxation time $\tau^*$ of the Glauber dynamics on $G^*$. In [21], the authors used a canonical path argument to bound $\tau^* = O(b^{2+1/C}k)$ for any $C > 0$. Instead, here we use two different coupling arguments to show the following two theorems for $\tau^*$.

**Theorem 4.** For any $C < 1$, there exists $b_0 > 0$ such that, for any $b > b_0$, the mixing and relaxation times of the Glauber dynamics on $G^*$ using $k = Cb/\ln b$ colors are $O(b^{1/C} \ln^2 b)$. When $C = 1$, the mixing and relaxation times are $O(b \ln^4 b)$. 


Theorem 5. For any $C > 1$, there exists $b_0 > 0$ such that, for any $b > b_0$, the mixing and relaxation times of the Glauber dynamics on $G^*$ using $k \geq C b / \ln b$ colors are $O(b \ln b)$.

Remark. It can be shown that the relaxation time is actually $O(b)$ when $C > 1$, from our analysis. However, unless we can also eliminate the constant factors and thereby show a very sharp bound of at most $b$, the extra $\ln b$ factor makes little difference to the relaxation time of the dynamics on the whole tree.

The most difficult (and also interesting) case turns out to be when $C \leq 1$. We will prove Theorem 4 in Section 4 and Theorem 5 in Section 5. We sketch the high-level idea of the proof of Theorem 4 in Section 4.1. Having Theorems 4 and 5 in hand, we can then apply Theorem 3 to get the upper bounds on the relaxation time as stated in Theorem 1. We get

$$T_{\text{relax}} = \begin{cases} O(b \ln b)^H = O(\frac{\ln b + O(1)}{b}), & \text{if } C > 1; \\ O(b \ln^4 b)^H = O(\frac{\ln b + O(1)}{b}), & \text{if } C = 1; \\ O(b^{1/C} \ln^2 b)^H = O(\frac{\ln b + O(1)}{b}), & \text{if } C < 1. \end{cases}$$

To then get the desired upper bounds on the mixing time of the whole tree we need a slightly more advanced tool, the log-Sobolev constant of the Markov chain. By adapting Theorem 5.7 in Martinelli, Sinclair and Weitz [24] to our setting of colorings, we establish (in Section 8) the following relationship between the inverse of the log-Sobolev constant $c_{\text{sob}}^{-1}$ and the relaxation time $T_{\text{relax}}$ of the Glauber dynamics on trees.

Theorem 6.

$$c_{\text{sob}}^{-1} \leq T_{\text{relax}}(2 \log_b(n) \ln(k)) \leq T_{\text{relax}}(2 \ln(n)).$$

Since the inverse of the log-Sobolev constant gives a relatively tight upper bound on the mixing time (see Inequality (2) in Section 3), using Theorem 6 we are able to complete the proofs of the upper bounds in Theorem 1.

2.2 Lower bounds

Our proof of the lower bound in Theorem 1 when $C < 1$ builds upon the approach used in [12]. They lower bounded the relaxation time by upper bounding the conductance of the Glauber dynamics on the subset $S \subseteq \Omega$ where the root is frozen (meaning that the configuration at the leaves uniquely determine the color of the root) to some color in $\{1, 2, ..., [k/2]\}$. They showed the conductance of $S$ satisfies $\Phi_S = O(n^{-\frac{1}{2C}})$ when $0 < C < 1/2$, which implies (by 11 and 3 in Section 3) that $T_{\text{mix}} \geq T_{\text{relax}} - 1 = \Omega(n^{1/6C})$.

We improve their bound on the conductance of $S$ by analyzing the probability that for a given leaf $z$, in a random coloring $\sigma$ of the complete tree, the root is frozen and changing the color of $z$ in $\sigma$ to some other color unfreezes the root. We prove that the number of such leaves in most colorings that freeze the root is $O(n^{-1/C+1+o_1(1)})$. Since the probability of recoloring a specific leaf is $1/n$, then intuitively we have $\Phi_S = O(n^{-1/C+o_1(1)})$, and hence $T_{\text{mix}} \geq T_{\text{relax}} - 1 = \Omega(n^{1/C-o_1(1)})$. A complete analysis of the lower bound is in Section 6 and in the analysis we will see that the $o_1(1)$ error term is $b^{1-1/C}/C$ when $1/2 < C < 1$ and zero when $C \leq 1/2$.

Finally, we will show in Section 7 how all of the proofs generalize for $k = o(b/\ln b)$, and thus prove Theorem 2.
3 Technical Preliminaries

Let $P(\cdot, \cdot)$ denote the transition matrix of the Glauber dynamics, and $P^t(\cdot, \cdot)$ denote the $t$-step transition probability. The total variation distance at time $t$ from initial state $\sigma$ is defined as

$$\|P^t(\sigma, \cdot) - \pi\|_{TV} := \frac{1}{2} \sum_\eta |P^t(\sigma, \eta) - \pi(\eta)|.$$

The mixing time $T_{\text{mix}}$ for a Markov chain is then defined as

$$T_{\text{mix}} = \min_t \{ \max_\sigma \{ \|P^t(\sigma, \cdot) - \pi\|_{TV} \} \leq 1/2e \}.$$

Given two copies, $(X_t)$ and $(Y_t)$, of the Markov chain at time $t > 0$, recall that a (one-step) coupling of $(X_t)$ and $(Y_t)$, is a joint distribution whose left and right marginals are identical to the (one-step) evolution of $(X_t)$ and $(Y_t)$, respectively. The Coupling Lemma [2] (c.f., Theorem 5.2 in [20]) guarantees that if there is a coupling and time $t > 0$, so that for every pair $(X_0, Y_0)$ of initial states, $\Pr [X_t \neq Y_t] \leq 1/2e$ under the coupling, then $T_{\text{mix}} \leq t$.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\Omega|}$ be the eigenvalues of the transition matrix $P$. The spectral gap $c_{\text{gap}}$ is defined as $1 - \lambda$ where $\lambda = \max \{ \lambda_2, |\lambda|_{\Omega} \}$ denotes the second largest eigenvalue in absolute value. The relaxation time $T_{\text{relax}}$ of the Markov chain is then defined as $c_{\text{gap}}^{-1}$, the inverse of the spectral gap. It is an elementary fact that the mixing time gives a good upper bound on the relaxation time (see, e.g., [10] for the following bound), which we will use in our analysis:

$$T_{\text{relax}} \leq T_{\text{mix}} + 1. \quad (1)$$

For the upper bounds on the mixing time of the dynamics on the whole tree, we also use the following well-known relationship between the mixing time and the inverse of the log-Sobolev constant (see e.g. [6]):

$$T_{\text{mix}} = O\left( c_{\text{sob}}^{-1} \ln \ln \frac{1}{\min_\sigma \{ \pi(\sigma) \}} \right). \quad (2)$$

Readers can refer to Section 8 or [6] for definitions and more details about the log-Sobolev constant.

To lower bound the mixing and relaxation times we analyze the conductance. The conductance of the Markov chain on $\Omega$ with transition matrix $P$ is given by $\Phi = \min_{S \subseteq \Omega} \{ \Phi_S \}$, where $\Phi_S$ is the conductance of a specific set $S \subseteq \Omega$ defined as

$$\Phi_S = \frac{\sum_{\sigma \in S} \sum_{\eta \in \bar{S}} \pi(\sigma)P(\sigma, \eta)}{\pi(S)\pi(\bar{S})}.$$

Thus, a general way to find a good upper bound on the conductance is to find a set $S$ such that the probability of escaping from $S$ is relatively small. The well-known relationship between the relaxation time and the conductance is established in [18] and [28] and we will use the form

$$T_{\text{relax}} = \Omega(1/\Phi), \quad (3)$$

for proving the lower bounds.

Finally, in much of our analysis below, we use the following version of a Chernoff-type bound; see, e.g., Theorem 4.4 and 4.5 in [26].

**Proposition 7** (Chernoff bound). Let $X_1, ..., X_n$ be independent Bernoulli trials with $\Pr [X_i = 1] = p_i$. Then if $X = \sum X_i$ and $\mu = \E[X]$, for any $\delta < 2e - 1$, we have

$$\Pr [X > (1 + \delta)\mu] \leq \exp(-\delta^2 \mu / 4).$$
4 Upper Bound on Mixing Time for $C \leq 1$: Proof of Theorem 4

In this section, we upper bound the mixing time of the Glauber dynamics on the star graph $G* = (V,E)$ when $k = Cb/\ln b$ for any $C \leq 1$. To be more precise, let $V = \{r, \ell_1, ..., \ell_b\}$, where $r$ refers to the root and $\ell_1, ..., \ell_b$ are the $b$ leaves and $E = \{(r, \ell_1), ..., (r, \ell_b)\}$. For convenience, here we let

$$\epsilon := 1/C - 1,$$

and hence $k = b/((1 + \epsilon) \ln b)$.

We use the maximal one-step coupling, originally studied for colorings by Jerrum [16] to upper bound the mixing time of the Glauber dynamics on general graphs. For a coloring $X \in \Omega$, let $A_X(v)$ denote the set of available colors of $v$ in the coloring $X$, i.e., $A_X(v) = \{c \in \mathcal{C} : \forall u \in N(v), \sigma(u) \neq c\}$. The coupling $(X_t, Y_t)$ of the two chains is done by choosing the same random vertex $v_t$ for recoloring at step $t$ and maximizing the probability of the two chains choosing the same update for the color of $v_t$. Thus, for each color $c \in A_X(v_t) \cap A_Y(v_t)$, with probability $1/\max\{|A_X(v_t)|, |A_Y(v_t)|\}$ we set $X_{t+1}(v_t) = Y_{t+1}(v_t) = c$. With the remaining probability, the color choices for $X_{t+1}(v_t)$ and $Y_{t+1}(v_t)$ are coupled arbitrarily.

We prove the theorem by analyzing the coupling in rounds, where each round consists of $T := 20b \ln b$ steps. Our main result is the following lemma which says that in each round we have a good probability of coalescing (i.e., achieving $X_t = Y_t$).

**Lemma 8.** For all $\epsilon \geq 0$, there exists $b_0$ such that for all $b > b_0$ if $k = b/((1 + \epsilon) \ln b)$ and $T = 20b \ln b$ for all $(x_0, y_0) \in \Omega \times \Omega$, the following holds:

$$\Pr[X_T = Y_T \mid X_0 = x_0, Y_0 = y_0] \geq \begin{cases} (20(1 + \epsilon)b^\epsilon \ln b)^{-1}, & \text{if } \epsilon > 0; \\ (20 \ln^3 b)^{-1}, & \text{if } \epsilon = 0. \end{cases}$$

It is then straightforward to prove Theorem 4.

**Proof of Theorem 4.** For $\epsilon > 0$, let $p_T := (20(1 + \epsilon)b^\epsilon \ln b)^{-1}$; and for $\epsilon = 0$ let $p_T := (20 \ln^3 b)^{-1}$. By repeatedly applying Lemma 8 we have, for all $(x_0, y_0)$,

$$\Pr[X_{2iT} \neq Y_{2iT} \mid X_0 = x_0, Y_0 = y_0] \leq (1 - p_T)^{2i} \leq 1/2e$$

for $i = 1/p_T$. Therefore, by applying the Coupling Lemma, mentioned in Section 3 the mixing time is $O((1 + \epsilon)b^{1+\epsilon} \ln^2 b)$ for $\epsilon > 0$ and $O(b \ln^4 b)$ for $\epsilon = 0$. \hfill \Box

4.1 Overview of the Coupling Argument

Before formally proving Lemma 8 we give a high-level overview of its proof. We will analyze the maximal one-step coupling on the star graph $G*$. We say a vertex $v$ “disagrees” at time $t$ if $X_t(v) \neq Y_t(v)$, otherwise we say the vertex $v$ “agrees”. We denote the set of disagreeing vertices at time $t$ of our coupled chains by

$$D_t = \{v \in V : X_t(v) \neq Y_t(v)\},$$

and we use $D^L_t = D_t \setminus \{r\}$ to represent the set of disagreeing leaves. When we use the term “with high probability” in this section, it means that the probability goes to 1 as $b$ goes to infinity.

In the maximal one-step coupling of the Glauber dynamics for the star graph, at each step we select a random vertex to recolor in both chains, and then we use the best way to couple the colors of that vertex. Therefore, if the coupling selects a leaf $\ell$ to recolor at time $t$, then the probability that
\( \ell \) will be disagree in \( X_t \) and \( Y_t \) is at most \( 1/(k-1) \), and with probability at least \( (k-2)/(k-1) \) the leaf will use the same color which is chosen uniformly random from \( \mathcal{C} \setminus \{X_t(r), Y_t(r)\} \). We also know that if we simply assign a random color from \( \mathcal{C} \) to each leaf, with probability at least \( \Omega(1/(b^e \ln b)) \) there is a color in \( \mathcal{C} \) which is unused in any leaf. This last point hints at the success probability in the statement of Lemma 8.

We analyze the \( T \)-step epoch in four stages. The warm-up round is of length \( T_0 := 4(b + 1) \ln b \) steps. In the warm-up round we just want to make sure we recolor each leaf at least once and we recolor the root at most \( 20 \ln b \) times. This is straightforward to prove via Chernoff bounds. Proposition 10 is the formal statement. We then run for a further \( 4(b + 1) \ln b \) steps, during which we prove (in Lemma 11 for \( \epsilon > 0 \) and Lemma 12 for \( \epsilon = 0 \)) that with high probability for \( \epsilon > 0 \) and with probability at least \( 1/(2 \ln^2 b) \) for \( \epsilon = 0 \), the root does not change colors in either chain, and each leaf is recolored at least once. Consequently at the end of these \( T_w := 8(b + 1) \ln b \) steps, with a good probability, all of the leaf disagreements will be of the same form in the sense that they will have the same pair of colors.

The next stage is of a random length \( T_1 \), which is defined as the first time (after \( T_w \)) where we are recoloring the root and the root has a common available color in \( (X_t) \) and \( (Y_t) \). We prove in Lemma 13, that with probability \( \Omega(1/b^e \ln b) \), \( T_1 < 4(b + 1) \ln b \). We then have probability at least \( 1/2 \) of the root agreeing after the update, and then after at most \( T_2 := 4(b + 1) \ln b \) further steps we are likely to coalesce since we just need to recolor each leaf at least once before the root changes back to a disagreement.

4.2 Coupling Argument: Proof of Lemma 8

We begin with a basic observation about the maximal one-step coupling.

**Proposition 9.** Let \( \mathcal{C}(D^t_L) := \bigcup_{\ell \in D^t_L} \{X_t(\ell), Y_t(\ell)\} \) denote the set of colors that appear in the disagreeing leaves at time \( t \). Then, \( A_{X_t}(r) \oplus A_{Y_t}(r) \subseteq \mathcal{C}(D^t_L) \).

This is simply because those colors that appear on the leaves with agreements are both unavailable in \( X_t \) and \( Y_t \) for the root.

We now analyze the first stage of the \( T \)-step epoch.

**Proposition 10.** The probability that in \( T_0 = 4(b + 1) \ln b \) steps, the coupling \((X_t, Y_t)\) (or the Glauber dynamics \((X_t)\)) will recolor the root at most \( 20 \ln b \) times and recolor every leaf at least once is at least \( 1 - 2b^{-3} \).

**Proof.** It is a simple fact which follows from the Chernoff bound and the coupon collector problem. Using the union bound the probability that there is a leaf which is not recolored in \( T_0 \) steps is at most
\[
b \left( 1 - \frac{1}{b+1} \right)^{4(b+1) \ln b} \leq b^{-3}.
\]
Now, let \( N \) be the number of times the root is recolored in \( T_0 \) steps. The expectation \( E[N] \) is simply \( 4 \ln b \). Then, by the Chernoff bound
\[
\Pr[N \geq 20 \ln b] \leq \Pr[N \geq (1 + 4)E[N]] \leq b^{-3}.
\]
Therefore the lemma holds by the union bound.

Then we will prove that after \( T_w = T_0 + 4(b + 1) \ln b \) steps, with high probability all of the leaf disagreements are of the same type when \( \epsilon > 0 \).
**Lemma 11.** When $\epsilon > 0$, starting from any pair of initial states $(x_0, y_0)$, after $T_w$ steps, with high probability, for all $i \in D^L_{T_w}$, $X_{T_w}(\ell_i) = Y_{T_w}(r)$ and $Y_{T_w}(\ell_i) = X_{T_w}(r)$.

**Proof.** The idea is that if we just look at one chain, say $(X_i)$, then after $T_0$ steps, with high probability the root is frozen. Moreover, the root is likely to continue to be frozen for the remainder of the $T_w$ steps since we recolor the root at most $O(\ln b)$ times. In the worst case the root is frozen to a disagreement, say $X_t(r) = 2$ and $Y_t(r) = 1$. Then after recoloring a leaf $\ell$ at time $t'$ where $t < t' < T_w$, the only possible disagreement is $X_{t'}(\ell) = 1, Y_{t'}(\ell) = 2$. Hence, it suffices to recolor each leaf at least once.

Let $E$ be the event that in the first $T_0$ steps, every leaf is recolored at least once and in another $4(b+1)\ln b$ steps, every leaf is recolored again at least once and the root is recolored at most $20\ln b$ times. We are first going to bound that for $t > T_0$,

$$\Pr[|A_{X_t}(r)| > 1 \mid E] \leq \frac{1}{(1 + \epsilon)b^t \ln b} := p_0,$$

and the same thing happens for $Y_t$.

Let $G_W$ be the graph with $b$ isolated vertices $\{v_1, ..., v_b\}$, corresponding to the leaves $\{\ell_1, ..., \ell_b\}$. Let $(W_t)$ be a Glauber process on $G_W$ using $k - 1$ colors from another color set $C_W$. We are going to define $W_0$ and couple $(W_t)$ with $(X_i)$ such that $|A_{X_t}(r)| = |A_{W_t}| + 1$ at any time $t$, where $A_{W_t} := \{c \in C_W : \forall v_i, W_t(v_i) \neq c\}$.

To do this, for every $t$ we are going to define a bijection $f_t : C \setminus \{X_t(r)\} \rightarrow C_W$ such that $f_t(X_t(\ell_i)) = W_t(v_i)$ for all $i$. Notice that if such a bijection exists then $|A_{X_t}(r)| = |A_{W_t}| + 1$.

At time $t = 0$, pick any bijection $f_0$ from $C_W$ to $C \setminus \{X_0(r)\}$. Define $W_0$ by $W_0(v_i) = f_0(X_0(\ell_i))$ for all $i$. We will update $f_t$ only when we choose the root to recolor at time $t$ in the coupling of $(W_t)$ and $(X_t)$. To do the coupling at time $t + 1$, we first choose a vertex $v$ in $G^*$ to recolor:

- if $v = \ell_i$, then we choose a random color $c$ different from $X_t(r)$ to recolor $v$. Correspondingly, we choose the vertex $v_i$ in $G_W$ to recolor using color $f_t(c)$.

- if $v = r$, then we choose a random color $c$ from $A_{X_t}(r)$ to recolor the root in $G^*$. Correspondingly, we update the mapping $f_i$ in the following natural way: $f_t(X_{t-1}(r)) = f_{t-1}(c)$, (and $f_t(c)$ is undefined).

Since $(W_t)$ itself is a Glauber process that recolors the vertices of $G_W$ uniformly at random from $C_W$, conditioning on $E$, simple calculations yield that for any $t > T_0$,

$$\Pr[|A_{W_t}| \geq 1 \mid E] \leq \frac{1}{(1 + \epsilon)b^t \ln b}.$$

Then (4) follows by coupling.

Since the same thing happens for $(Y_t)$ and the root is recolored at most $20\ln b$ times, then by the union bound, conditioning on $E$, the probability that at each time we try to recolor the root after $T_0$ steps, the root is always frozen in both copies is at least $1 - (40\ln b)p_0 = 1 - 40/(1 + \epsilon)b^t$. Finally, by Proposition 11, $E$ happens with high probability, and hence the lemma. \qed

For the threshold case $\epsilon = 0$, we prove a slightly weaker lemma for the warm-up stage, in the sense that the successful probability will only be at least $\Omega(1/\ln^2 b)$.

**Lemma 12.** When $\epsilon = 0$, starting from any pair of initial states $(x_0, y_0)$, after $T_w' = T_0 + 2b\ln b$ steps, with probability at least $1/(2\ln^2 b)$, for all $i \in D^L_{T_w'}$, $X_{T_w'}(\ell_i) = Y_{T_w'}(r)$ and $Y_{T_w'}(\ell_i) = X_{T_w'}(r)$. 

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Proof. We use a different approach to prove the lemma, since it is not true that the root will still always be frozen during $T^*_{\omega}$ steps with high probability.

Let $T_0 = 4(b + 1) \ln b$. We first bound

$$\Pr \left[ |D_{T_0}^Z| \geq 4 \ln b \mid X_0 = x_0, Y_0 = y_0 \right] \leq \frac{2}{b^2},$$

which says that after $T_0$ steps, with high probability the number of disagreeing leaves is at most $O(\ln b)$. We also know that with probability at least $1 - 1/b^2$ all leaves will be recolored at least once after $T_0$ steps. This event is denoted as $E$. Conditioning on $E$, we know that each leaf will have at most $1/(k - 1)$ probability to be a disagreement independently. From this, intuitively the expected number of disagreeing leaves will be at most $b/(k - 1) \approx \ln b$, then the lemma holds by using concentration inequalities. To formally argue this, we need to construct a new stochastic process that stochastically upper bounds the number of disagreements in $(X_t, Y_t)$.

Let $(U_i)$ be a Glauber process on the graph $G_U$ with $b + 1$ isolated vertices $\{v_0, v_1, ..., v_b\}$ using two colors $\{0, 1\}$. Each step, the process selects a random vertex $v_i$. If the vertex selected is $v_0$, the process does nothing. Otherwise, with probability $1/(k - 1)$, the vertex $v_i$ will be recolored 1 and with probability $1 - 1/(k - 1)$ it will be recolored 0. It is not hard to couple this process $(U_i)$ with $(X_t, Y_t)$ such that in step $t$, for any leaf $\ell_i$ in $G^*$, if $X_t(\ell_i) \neq Y_t(\ell_i)$ then $Z_t(v_i) = 1$. Let $D_{T}^Z = \{ v \in \{v_1, v_2, ..., v_b\} : Z_t(v) = 1 \}$ be the set of vertices in $G_U$ that are colored 1. By our construction of the coupling, $|D_{T}^Z| \geq |D_{T}^L|$. Thus, to show Inequality (5) is enough to show the following statement for the process $(U_t)$ on the graph $G_U$.

$$\Pr \left[ |D_{T_0}^Z| \geq 4 \ln b \mid U_0 = u_0 \right] \leq \frac{2}{b^2},$$

where $u_0$ is constructed in the following way: For any $i$,

- when $i > 0$, $u_0(v_i) = 1 \iff x_0(\ell_i) \neq y_0(\ell_i)$;
- $u_0(v_0) = 0$.

Therefore conditioned on $E$, the expected size of $|D_{T_0}^Z|$ is $b/(k - 1) \approx \ln b$. Then we have

$$\Pr \left[ |D_{T_0}^Z| \geq 4 \ln b \mid u_0 \right] \leq \Pr \left[ |D_{T_0}^Z| \geq 4 \ln b \mid E, u_0 \right] + \frac{1}{b^2} \leq \exp(-8 \ln b/4) + \frac{1}{b^2} \quad \text{(by the Chernoff bound)} \leq \frac{2}{b^2}.$$

Now with high probability there are $O(\ln b)$ many disagreeing leaves in $G^*$. Then the idea is that we would like to recolor the leaves at least $2b \ln b$ times continuously without recoloring the root. The probability is at least

$$(1 - \frac{1}{b + 1})^{2b \ln b} \geq \frac{1}{\ln^2 b}.$$

Conditioning on this, the leaves being recolored are still uniformly random. Thus, the probability that in these $2b \ln b$ steps, all of those $O(\ln b)$ disagreeing leaves are picked to recolor at least once is at least

$$1 - O(\ln b)(1 - 1/b)^{2b \ln b} \geq 1 - \frac{O(1)}{\ln b}.$$
which will be larger than $2/3$ for big enough $b$.

Hence, if the root was an agreement at the last time $(X_t, Y_t)$ recolored it before $T_0$ steps, then after another $2b \ln \ln b$ steps with probability at least $2/(3 \ln^2 b)$, $X_t = Y_t$. In this case we are done. The worst case is if the root disagrees at the last time $t$ that we recolored the root prior to time $T_0$. In this case, after another $2b \ln \ln b$ steps with probability at least $2/(3 \ln^2 b)$ all of the leaf disagreements will be of the same type, by the definition of the maximal one-step coupling of the Glauber dynamics.

In conclusion, combining with Inequality [5], we proved that with probability at least $1/(2 \ln^2 b)$, all of the uncoupled leaves are of the same type after $T^*_{0}$ steps. \qed

After we succeed in the warm-up stage meaning that all of the leaf disagreements are of the same type, we enter the root-coupling stage, where we try to couple the root. Let $T_1$ be the first time that there is a common available color in the root and the coupling chain select the root to recolor, that is

$$T_1 := T_1^{XY} = \arg \min_{t} \{A_{X_t}(r) \cap A_{Y_t}(r) \neq \emptyset \text{ and the root } r \text{ is selected at step } t \}.$$ 

Clearly $T_1$ is a stopping time.

**Lemma 13.** For $\epsilon \geq 0$, for any pair of initial states $(x_0, y_0)$ where all of the leaf disagreements are of the same type (i.e., there is a pair of colors $c_1, c_2$ such that for all $\ell \in D_0^t$, we have $x_0(\ell) = c_1$ and $y_0(\ell) = c_2$), we have

$$\Pr \left[ T_1^{XY} < 4(b + 1) \ln b \mid (X_0, Y_0) = (x_0, y_0) \right] > \frac{1}{4(1 + \epsilon)b^\epsilon \ln b}.$$ 

*Proof.* First of all, by Proposition [9] $|A_{X_0}(r) \oplus A_{Y_0}(r)| \leq 2$. We are interested in the time $t$ when there is a common color available for the root in $(X_t, Y_t)$.

Let $(Z_t)$ be a Glauber process on the graph $G_Z$ of $b + 1$ isolated vertices $\{v_0, v_1, v_2, ..., v_b\}$ in which $v_0$ corresponds to the root and $v_i$ corresponds to the leaves $\ell_i$ for any $i > 0$. The color set used in the process $(Z_t)$ is $C_Z = [k] \setminus \{c_1, c_2\}$. Each step, $(Z_t)$ chooses a random vertex and recolors it with a random color from the set $C_Z$. Let $T_Z$ be the stopping time on $Z$ satisfying:

$$T_Z^2 = \arg \min_{t \geq 2(b+1) \ln b} \{|A_{Z_t}| \geq 1 \text{ and } v_0 \text{ is selected at the step } t\},$$

where $A_{Z_t} = \{c \in C_Z : \forall i \in [1, ..., b], Z_t(v_i) \neq c\}$ is the set of unused colors in the vertices $\{v_1, v_2, ..., v_b\}$. We want to couple $(Z_t)$ with $(X_t, Y_t)$ in such a way that $T_Z^2 \geq T_1^{XY}$ for all the runs, and then if we show that for any initial state $z_0$, we have

$$\Pr \left[ T_Z^2 < 4(b + 1) \ln b \mid Z_0 = z_0 \right] > \frac{1}{4(1 + \epsilon)b^\epsilon \ln b}.$$ 

Then by the coupling, we know that the lemma is also true.

Now we are going to construct the coupling between $(Z_t)$ and $(X_t, Y_t)$ for $t \leq T_1^{XY}$. Let $z_0$ be the initial state satisfying that for any $i \in [1, ..., b]$, if $x_0(\ell_i) = y_0(\ell_i) \in C_Z$ then $z_0(v_i) = x_0(\ell_i)$, otherwise we give an arbitrary color to the vertex $v_i$. On each step $t$, we first randomly select a vertex in $G^*$ to update in $(X_t, Y_t)$ and accordingly we select the corresponding vertex in $G_Z$ to update in $Z_t$:

- If the vertex is a leaf $\ell_i$:

  $(X_t, Y_t)$ selects a random color $c$ or a disagreement to update. If $c \in C_Z$ then we give the same color to $v_i$ in $Z_t$, otherwise we give a random color to $v_i$. 

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• If the vertex is the root \( r \):

  Recolor the root on \((X_t, Y_t)\) according to the maximal one-step coupling and pick a random color in \( C_Z \) to recolor \( v_0 \) in \( Z \).

  Observe that, \( A_Z \subseteq A_X(r) \cap A_Y(r) \) for any \( 0 \leq t \leq T^{XY}_1 \), which implies that \( T^Z_1 \geq T^{XY}_1 \) holds with probability 1. Now we will show that inequality (6) holds.

  Let \( t_z \) be the first time when we hit the root after \( 2(b+1) \ln b \) steps. Since \( Z \) is a purely random process, we know that with probability at least \( 1/(4(1+\epsilon)b^\epsilon \ln b) \), \( T^Z_1 = t_z \). This is because, similar to Proposition 10, by the coupon collector problem, we can easily prove that:

  **Observation 14.** With high probability, every vertex in the graph \( G_Z \) will be recolored at least once after \( 2(b+1) \ln b \) steps and they are recolored to a random color in \( C_Z \).

  Therefore, by the fact that for each color \( c \), the indicator random variable of whether \( c \) is used by some leaves or not is negatively associated to each other (c.f., Theorem 14 in [8]), it follows by some elementary calculations that for any fixed \( t_z \) and large enough \( b \),

  \[
  \Pr \left[ A_Z \neq \emptyset \mid t_z = \tau \right] \geq 0.99 \left( 1 - (1 - \frac{1}{|C_Z|})^{b^\epsilon |C_Z|} \right) 
  \geq \frac{1}{3(1+\epsilon)b^\epsilon \ln b},
  \]

  where the constant .99 in the penultimate inequality follows from Observation 14 for \( b \) sufficiently large. Moreover, for \( t_z \), by simple calculations, we know that:

  **Observation 15.** \( t_z \) is with high probability less than \( 4(b+1) \ln b \).

  Thus, by applying (17), we have

  \[
  \Pr \left[ T^Z_1 < 4(b+1) \ln b \mid Z_0 = z_0 \right] \geq \sum_{2(b+1) \ln b \leq \tau < 4(b+1) \ln b} \Pr \left[ A_Z \neq \emptyset \mid t_z = \tau \right] \cdot \Pr \left[ t_z = \tau \right]
  \geq \frac{1}{3(1+\epsilon)b^\epsilon \ln b} \cdot \Pr \left[ t_z \in [2(b+1) \ln b, 4(b+1) \ln b] \right]
  \geq \frac{1}{4(1+\epsilon)b^\epsilon \ln b},
  \]

  where the last inequality follows in this case from Observation 15 for \( b \) sufficiently large.

  This completes the proof of Lemma 13.

  We also know that when the root is recolored, if \( |A_X(r) \oplus A_Y(r)| \leq 2 \) and \( |A_X(r) \cap A_Y(r)| \geq 1 \) holds, then the probability that the root will be recolored to the same color in both \( X \) and \( Y \) is at least \( 1/2 \). Hence, at time \( T_1 = T^{XY}_1 \), with probability at least \( 1/2 \) the root will become an agreement. Combining with Lemma 11, we proved that with probability at least \( 1/O((1+\epsilon)b^\epsilon \ln b) \) when \( \epsilon > 0 \), starting from arbitrary initial states \( (x_0, y_0) \), the root will couple in at most \( 12(b+1) \ln b \) steps and by that time all the disagreements (if there is any) in the leaves are of the same type. When \( \epsilon = 0 \), combining with Lemma 12, we get that the probability of the same event happening is at least \( 1/O(\ln^3 b) \).

  The last step is to let all of the disagreements in the leaves go away without changing the root to a disagreement, again with constant probability, after \( T_2 = 4(b+1) \ln b \) more steps. Here is the precise statement of the lemma.
Lemma 16. For \( \epsilon \geq 0 \), consider a pair of initial states \((x_0, y_0)\) where the root \( r \) agrees (i.e., \( x_0(r) = y_0(r) \)) and all of the leaf disagreements are of the same type (i.e., there is a pair of colors \( c_1, c_2 \) such that for all \( \ell \in D_r \), we have \( x_0(\ell) = c_1 \) and \( y_0(\ell) = c_2 \)). Then, with probability at least \( 1/2 \) after \( T_2 = 4(b + 1) \ln b \) steps, we have \( X_{T_2} = Y_{T_2} \).

Proof. First, observe that with high probability after \( T_2 \) steps all of the leaves will be recolored at least once. Assuming all of the leaves are recolored at least once, if the root does not become a disagreement within these \( T_2 \) steps, then all of the leaves will be agreements.

Therefore, we first use a coupling argument to show that the root will not change to a disagreement in \( T_2 \) steps with probability at least \( 3/5 \). Then, since we also know that with high probability each vertex will be recolored at least once before time \( T_2 \), by the union bound, we can conclude that with probability at least \( 1/2 \), \( X_{T_2} = Y_{T_2} \), which proves the lemma.

It remains to show that the root will not change to a disagreement in \( T_2 \) steps with probability at least \( 3/5 \). Let \( t_2 \) be the first time when the root becomes a disagreement, that is, \( X_{t_2}(r) \neq Y_{t_2}(r) \).

Note, since any disagreements on the leaves are colored \( c_1 \) in \( X_0 \) and \( c_2 \) in \( Y_0 \), either \( X_{t_2}(r) = c_2 \) and/or \( Y_{t_2}(r) = c_1 \). Therefore, we define

\[
T^X_2 = \arg \min_t \{ X_t(r) = c_2 \}
\]

be the stopping time when the root is first recolored to \( c_2 \) in \((X_t)\). Similarly, let

\[
T^Y_2 = \arg \min_t \{ Y_t(r) = c_1 \}
\]

be the stopping time when the root is first recolored to \( c_1 \) in \((Y_t)\). We can assume without loss of generality that neither \( X_0(r) \) nor \( Y_0(r) \) are colors \( c_1 \) or \( c_2 \). Since otherwise, by the hypothesis of the lemma, there are no disagreements in the leaves and hence \( X_0 = Y_0 \), in which case the lemma is trivially true. Hence, our goal is to show that

\[
\Pr \left[ T^X_2 \leq T_2 \text{ or } T^Y_2 \leq T_2 \right] < \frac{2}{5}.
\]

And the main step is to show that

\[
\Pr \left[ T^X_2 \leq T_2 \right] < \frac{1}{5}. \tag{8}
\]

Let \((S_t)\) be a random subset process on \( V(G^*) \). Each time it picks a vertex \( v \),

- If \( v \neq r \), with probability \( 1/(k - 1) \), \( S_{t+1} = S_t \cup \{ v \} \) and with probability \( 1 - 1/(k - 1) \), \( S_{t+1} = S_t \setminus \{ v \} \);
- If \( v = r \), if \( S_t = \emptyset \) then \( S_{t+1} = \{ r \} \) otherwise \( S_{t+1} = S_t \).

Let us define

\[
T^S = \arg \min_t \{ r \in S_t \}.
\]

We are going to couple \((S_t)\) with \((X_t)\) such that \( \{ v \in V(G^*) : X_t(v) = c_2 \} \subseteq S_t \). This implies \( T^S \leq T^X_2 \). And if we can show that

\[
\Pr \left[ T^S \leq T_2 \right] \leq \frac{1}{5}
\]

then we have proved inequality (8).

The coupling \((X_t, S_t)\) is defined as follows. We start with \( S_0 = X_0^{-1}(c_2) \), the set of vertices of color \( c_2 \) in the initial coloring. Each time both processes picks the same vertex \( v \) to update.
• If $v = r$, $X_t$ and $S_t$ act independently at this time.

• If $v \neq r$ and $X_t(r) \neq c_2$, then $X_t$ chooses a random color different from the root to recolor $v$ and if that color is not $c_2$, $S_{t+1} = S_t \setminus \{v\}$ otherwise $S_{t+1} = S_t \cup \{v\}$.

• If $v \neq r$ and $X_t(r) = c_2$, then $X_t$ chooses a random color different from $c_2$ to recolor $v$ and if that color is not $c_1$, $S_{t+1} = S_t \setminus \{v\}$ otherwise $S_{t+1} = S_t \cup \{v\}$.

It is easy to see that this is a valid coupling. More importantly, it satisfies $X_{t+1}^{-1}(c_2) \subseteq S_t$.

Now we are going to show that $\Pr[T^S \leq T_2] < 1/5$ holds. Again, by the proof of Proposition 10 we know that with probability at most $b^{-3} \leq 1/5$, the root is updated more than $20 \ln b$ times. And since with probability

$$
(1 - \frac{1}{b})^{0.1b} \approx e^{-0.1} \geq 0.9,
$$

the first time when the root is updated is later than $0.1b$ steps. We now condition on this event. The indicators of whether each leaf is in $S_t$ or not during those $0.1b$ steps are negatively associated (c.f., Theorem 14 in [8]). Then by using the Chernoff bound with negative association among the random variables (c.f., Proposition 7 in [8]), it is not hard to show that with high probability at least $\geq 0.01b$ many different leaves are recolored before the first time we recolor the root. Thus, by the union bound and (9), we can claim that with probability at least 0.85, before the first $t$ such that $r \in S_t$, at least $0.01b$ many leaves have been recolored and root will be recolored at most $20 \ln b$ times before $T_2$. Denote this event as $\mathcal{E}$. We have

$$
\Pr[T^S \leq T_2] \leq \Pr[T^S \leq T_2 \mid \mathcal{E}] + \Pr[\bar{\mathcal{E}}] \leq \Pr[T^S \leq T_2 \mid \mathcal{E}] + 0.15.
$$

Now we just need to prove that $\Pr[T^S \leq T_2 \mid \mathcal{E}]$ can actually be arbitrarily small when $b$ grows. At each time $t$ we update the root in $(S_t)$, we know that the probability of $S_{t-1} = \emptyset$ is at most

$$
(1 - \frac{1}{k-1})^{0.1b} \leq b^{-0.01(1+\epsilon)}.
$$

And we know that we update the root at most $20 \ln b$ times. By the union bound, we can conclude that

$$
\Pr[T^S \leq T_2 \mid \mathcal{E}] < \frac{20 \ln b}{b^{0.01(1+\epsilon)}}.
$$

In conclusion of the first part, we proved inequality (5). Note that we also prove the same thing for $Y_t$. By the union bound

$$
\Pr[T_2^X \leq T_2 \text{ or } T_2^Y < T_2] \leq \frac{2}{5}.
$$

Finally, by combining Lemmas 11, 13 and 16 together, we can conclude that: when $\epsilon > 0$, with probability at least $1/(20(1 + \epsilon) b^2 \ln b)$ after $t = T_w + T_1 + T_2 < T$ steps of the coupling, we have $X_t = Y_t$; when $\epsilon = 0$, from Lemmas 12, 13 and 16, we have that with probability at least $1/(20 \ln^3 b)$ after $t = T_w + T_1 + T_2 < T$ steps of the coupling, we have $X_t = Y_t$, which proves Lemma 8.
5 Upper Bound on Mixing Time for $C > 1$: Proof of Theorem 5

In this section we analyze the upper bound of the mixing time of the Glauber dynamics on the star graph $G^*$ when $k = Cb/\ln b$ for $C > 1$. Here, let

$$\epsilon := C - 1,$$

and hence, $k = (1 + \epsilon)b/\ln b$.

We will analyze the maximal one-step coupling using a weighted Hamming distance. The root $r$ will have weight $w(r) = b^{1/2} > 1$ and the leaves will have weight $w(v) = 1$. For a set of vertices $S$, let $w(S) = \sum_{v \in S} w(v)$. Recall that the set of disagreeing vertices at time $t$ of our coupled chains is denoted as

$$D_t = \{v \in V : X_t(v) \neq Y_t(v)\},$$

and the set $D^L_t = D_t \setminus \{r\}$ denotes the set of disagreeing leaves. Let $D^r_t$ denote whether there is a disagreement at the root.

We want to show that the coupling decreases the distance in expectation. Hence, we say a pair of colorings $(X_0, Y_0)$ are $\eta$-distance-decreasing if there exists a coupling $(X_0, Y_0) \rightarrow (X_1, Y_1)$ such that:

$$E[w(D_1) \mid X_0, Y_0] < (1 - \eta)w(D_0).$$

To simplify the analysis of the coupling, we will use the following theorem of Hayes and Vigoda [15] to utilize properties of the stationary distribution. The quantity $diam(\Omega)$ is the diameter of $\Omega$ with respect to the Glauber dynamics. In our case, a trivial bound is $diam(\Omega) \leq 2b$.

**Theorem 17.** [15 Theorem 1.2] Let $\eta > 0$. Suppose $S \subseteq \Omega$ such that every $(X_0, Y_0) \in S \times \Omega$ is $\eta$-distance-decreasing, and

$$\pi(S) \geq 1 - \frac{\eta}{16diam(\Omega)},$$

then the mixing time is

$$T_{\text{mix}} \leq \frac{3\lfloor \ln(32diam(\Omega)) \rfloor}{\eta}.$$

We use $S$ as the set of colorings where the root has many available colors. For a coloring $X$, denote the available colors for the root by:

$$A_X(r) := [k] \setminus X(N(r)).$$

Along the lines of the Dyer-Frieze [9] local uniformity results, we will prove the following statement about the available colors for the root $r$ in a random coloring.

**Lemma 18.** Let $X$ be a random coloring of the star graph on $b$ vertices. For every $\epsilon > 0$, there exists $b_0$, such that for all $b > b_0$,

$$\Pr\left[|A_X(r)| > b^{9\epsilon}\right] > 1 - \exp(-b^{9\epsilon}/100).$$

Hence, we let the set $S$ be those colorings $X \in \Omega$ where $|A_X(r)| \geq b^{9\epsilon}$. 

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5.1 Analyzing the coupling

We need to analyze \( E[w(D_1) \mid X_0, Y_0] \). Note, when a leaf \( v \) is recolored, if the root is a disagreement (i.e., \( X_0(r) \neq Y_0(r) \)) then with probability \( 1/(k-1) \) we have \( X_1(v) \neq Y_1(v) \). Hence,
\[
E[w(D^L_1) \mid X_0, Y_0] = \sum_{v \in V \setminus \{r\}} w(v) \left[ \Pr[v \text{ is recolored}] \Pr[X_1(v) \neq Y_1(v) \mid v \text{ is recolored}, X_0, Y_0] + (1 - \Pr[v \text{ is recolored}]) 1[X_0(v) \neq Y_0(v)] \right]
\]
\[
= \frac{b}{b+1} \frac{1}{k-1} + \left( 1 - \frac{1}{b+1} \right) w(D^L_0).
\]
When the root \( r \) is recolored, there is probability at most \( |D^L_0|/\max\{|A_{X_0}(r)|, |A_{Y_0}(r)|\} \) that \( X_1(r) \neq Y_1(r) \). Hence, for \( X_0 \in S \), we have:
\[
E[w(D^L_1) \mid X_0, Y_0] \leq \frac{w(r)}{b+1} \frac{|D^L_0|}{\max\{|A_{X_0}(r)|, |A_{Y_0}(r)|\}} + \left( 1 - \frac{1}{b+1} \right) w(D^L_0)
\]
\[
\leq \frac{b^{\epsilon^2}}{b+1} \frac{|D^L_0|}{|A_{X_0}(r)|} + \left( 1 - \frac{1}{b+1} \right) w(D^L_0)
\]
\[
\leq \frac{|D^L_0| b^{-\epsilon^3}}{b+1} + \left( 1 - \frac{1}{b+1} \right) w(D^L_0).
\]
Therefore, for \( (X_0, Y_0) \in S \times \Omega \), we have:
\[
E[w(D_1) \mid X_0, Y_0] \leq \frac{1}{b+1} \left( 1 \right) + \left( 1 - \frac{1}{b+1} \right) w(D_0)
\]
\[
\leq w(D_0) + \frac{1}{b+1} \left( -w(D_0) + 1 \right) \]
\[
\leq w(D_0) + \frac{1}{b} \left( -1 + b^{-\epsilon^3/4} \right) w(D_0).
\]
Thus, they are \( \eta \)-distance-decreasing for \( \eta = (1 - b^{-\epsilon^3/4})/b \).

Now applying Theorem \([17]\) by Lemma \([18]\) we have the necessary bound on \( \pi(S) \), and thus conclude for \( b \) sufficiently large we have:
\[
T_{\text{mix}} \leq (6b \ln b)/(1 - b^{-\epsilon^3/4}) \leq 12b \ln b.
\]
This completes the proof of Theorem \([5]\) except for the proof of Lemma \([18]\).

Proof of Lemma \([18]\) Fix the color of the root to be \( c \). Let \( \sigma \) be a random coloring conditional on the root receiving color \( c \). We are going to prove that
\[
\Pr[|A_{\sigma}(r)| \leq b^{0.9e} \mid \sigma(r) = c] < \exp(-b^{0.9e}/100).
\]
For each color \( i \in C \setminus \{c\} \), let \( Z_i \) be the indicator function that \( c \in A_{\sigma}(r) \). \( |A_{\sigma}(r)| = \sum_{i \in C} Z_i \).

By Theorem 14 in \([8]\), the \( Z_i \)'s are negatively associated with each other once the root is fixed. Note,
\[
E[|A_{\sigma}(r)|] = (k - 1)E[Z_i]
\]
\[
= k \left( 1 - \frac{1}{k - 1} \right)^b
\]
\[
\geq k \left( 1 - \frac{1}{k - 1} \right) \exp(-b/(k - 1))
\]
\[
\geq b^{0.9e},
\]
for $b$ sufficiently large. Now applying the Chernoff bound, which hold for negatively associated random variables (c.f., Proposition 7 in [8]), we have:

$$\Pr \left[ |A_\sigma(r)| \leq b^{0.9\epsilon} \mid \sigma(r) = c \right] < \exp(-b^{0.9\epsilon}/100).$$

\[ \square \]

6 Proof of the Lower Bounds Below the Threshold in Theorem 1

In this section we prove that when $C < 1$:

$$T_{relax} = \Omega(n^{1/C - o(1)}). \quad (10)$$

In the remainder of this section, $T = T_H$ denotes a complete tree of height $H = \lfloor \log_b n \rfloor$ where the root is denoted by $r$. Let $L(T)$ or simply $L$ denote the leaves of $T$. For a vertex $v$ of $T$, let $T_v$ denote the subtree of $T$ rooted at $v$ and $T_v^*$ denote $T_v \setminus \{v\}$. For the convenience, in this section, let $\epsilon := 1/C - 1,$

and hence $k = b/(1 + \epsilon) \ln b$.

In coloring $\sigma \in \Omega(T)$, we say a vertex $v$ is frozen in $\sigma$ if in the subtree $T_v$ the coloring $\sigma(L(T_v))$ of the leaves of $T_v$ forces the color for $v$. In other words, $v$ is frozen in $\sigma$ if: for all $\eta \in \Omega$ where $\eta(L(T_v)) = \sigma(L(T_v))$, we have $\eta(v) = \sigma(v)$. Note, by definition, the leaves are always frozen. Observe that for a vertex to be frozen its frozen children must “block” all other color choices. This is formalized in the following observation as in [12].

Observation 19. A vertex $v$ where $h(v) > 0$ is frozen in coloring $\sigma$ if and only if, for every color $c \neq \sigma(v)$, there is a child $w$ of $v$ where $\sigma(w) = c$ and $w$ is frozen.

Using this inductive way of defining a vertex being “frozen” in a coloring, we can further show the following lemma, which is a generalization of Lemma 8 in [12] to the case when $0 < \epsilon < 1$, i.e., $1 > C > 1/2$.

Lemma 20. In a random coloring of tree $T$, the probability that a vertex of $T$ is not frozen is at most $b^{-c}$. For the leaves in $T$, by definition, they are always frozen.

6.1 Upper Bound on the Conductance

Let $S_c = S_c(T)$ denote those colorings in $\Omega(T)$ where the root of $T$ is frozen to color $c$. Let $S = \cup_{1 \leq c \leq k/2} S_c$. We will analyze the conductance of $S$ to lower bound the mixing time.

To upper bound the conductance of $S$ we need to bound the number of colorings $\sigma \in S$ which can leave $S$ with one transition, and in that case how many transitions leave $S$. To unfreeze the root, we need to recolor a leaf. Thus, we need to bound the number of colorings frozen at the root which can become unfrozen by one recoloring, and in that case, we need to bound the number of leaves which can be recolored to unfreeze the root. For a coloring $\sigma$, vertex $v$ and color $c$, let $\sigma^{v \rightarrow c}$ denote the coloring obtained by recoloring $v$ to $c$.

We capture the colorings on the “frontier” of $S$ as follows. For tree $T$, coloring $\sigma \in \Omega(T)$, a vertex $v$ and a leaf $z$ of $T_v$, let $\mathcal{E}_{v,z}^\sigma$ denote the event that the coloring $\sigma$ is frozen at the vertex $v$ of $T$ and there exists a color $c$ where the coloring $\sigma^{z \rightarrow c}$ is not frozen at the vertex $v$. By definition, this event only depends on the configurations at the leaves of the subtree $T_v$. In particular, for the root of the tree, let $\mathcal{E}(\sigma, z) := \mathcal{E}_{r,z}^\sigma$ and $1_{\sigma,z}$ be the indicator of it.
We can convert the above intuition into the following upper bound on conductance of $S$ (similar to Lemma 10 in [12]):

**Lemma 21.**

$$
\Phi_S \leq \frac{6}{n} \sum_{z \in L(T)} \sum_{\sigma \in \Omega(T)} \frac{1}{|\Omega(T)|} E_{\sigma,z} = \frac{6}{n} \sum_{z \in L(T)} \sum_{\sigma \in \Omega(T)} \Pr_{\sigma \in \Omega} [E(\sigma, z)]
$$

(11)

Now if we can prove that

$$
\Pr_{\sigma \in \Omega} [E(\sigma, z)] \leq b^{-(1+\epsilon-o(1))H},
$$

(12)

where $o(1)$ is an inverse polynomial of $b$ when $\epsilon < 1$ and equals to zero when $\epsilon \geq 1$. This will be clarified later in Lemma 22. Then by plugging this back into the upper bound (11) we get

$$
\Phi_S \leq \frac{6}{n} \cdot b^H \cdot b^{-(1+\epsilon-o(1))H} \leq 20n^{-1-\epsilon+o(1)}.
$$

Therefore, we can conclude that the conductance of this Glauber dynamics is $O(n^{-1-\epsilon+o(1)})$, and hence by (11) and (3), the mixing time and the relaxation time is $\Omega(n^{1/C-o(1)})$.

**6.2 Proof of (12)**

Fix the color of the root to be color $c^* \in C$. Let $\Omega^* = \{ \sigma \in \Omega : \sigma(r) = c^* \}$ be the set of colors where the root is colored $c^*$. The conditional probability $\Pr_{\sigma \in \Omega^*} [E(\sigma, z)] := \Pr_{\sigma \in \Omega} [E(\sigma, z) | \sigma(w_0) = c^*]$ will be the same for all $c^*$. Hence,

$$
\Pr_{\sigma \in \Omega} [E(\sigma, z)] = \Pr_{\sigma \in \Omega^*} [E(\sigma, z)].
$$

For the remainder of the proof we condition on the root being colored $c^*$.

For the event $E(\sigma, z)$ to occur we need that along the path from the leaf $z$ to the root $r$, unfreezing each of these vertices will “free” a color for their parent. More precisely, let $w_0, \ldots, w_H$ where $w_0 = r$ and $w_H = z$ denote the path in $T$ from the root $r$ down to the leaf $z$. For $\sigma$ to be in $E(\sigma, z)$, $w_1$ has to be frozen because the color of $z$ only affects the root through $w_1$, and if $w_1$ is not frozen then it cannot affect the root becoming unfrozen. Moreover, in order for the root to become unfrozen by changing the color of the leaf $z$, it must also occur that $w_1$ becomes unfrozen at the same time, hence $\sigma \in E_{w_1,z}^\sigma$. Applying this argument in a similar manner down to the leaf $z$, we can observe that

$$
\Pr_{\sigma \in \Omega^*} [E_{r,z}^\sigma] \leq \Pr_{\sigma \in \Omega^*} [E_{w_1,z}^\sigma] \leq \cdots \leq \Pr_{\sigma \in \Omega^*} [E_{w_H-1,z}^\sigma].
$$

This suggests an inductive proof to bound $\Pr_{\sigma \in \Omega^*} [E_{r,z}^\sigma]$. Actually, we need a much stronger result:

$$
\Pr_{\sigma \in \Omega^*} [E_{r,z}^\sigma] \leq b^{-(1+\epsilon-o(1))\Pr_{\sigma \in \Omega^*} [E_{w_1,z}^\sigma]} \leq b^{-2(1+\epsilon-o(1))\Pr_{\sigma \in \Omega^*} [E_{w_2,z}^\sigma]} \leq \cdots \leq b^{-H(1+\epsilon-o(1))}.
$$

Intuitively, the event $E_{w_i,z}^\sigma$ implies the fact that $w_{i+1}$ is the only child that causes $w_i$ simultaneously being frozen and being blocked from using color $\sigma(w_{i+1})$, which further implies that both $E_{w_{i+1},z}^\sigma$ and all the siblings of $w_{i+1}$ should be either using colors other than $\sigma(w_{i+1})$ or not frozen. Here, the
sibling’s of \( w_i \) mean the children of \( w_{i-1} \) except \( w_i \). And \( b^{-(1+\epsilon-o(1))} \) comes from the probability of the last event above concerning the siblings of \( w_{i+1} \).

Now, we are going to analyze \( \Pr_{\sigma \in \Omega^*} [ E_{r,z}^\sigma ] \) more carefully and formally. For each \( 1 \leq i \leq H \), let \( A_{i,z}^\sigma \) denote the event that no sibling \( y \) of \( w_i \) satisfies: both \( \sigma(y) = \sigma(w_i) \) and \( \sigma \) is frozen at \( y \). Notice that, if \( \sigma \notin A_{i,z}^\sigma \), meaning that there is a sibling of \( w_1 \) being frozen to the color of \( w_1 \) in \( \sigma \), then changing the colors in the leaves of \( T_{w_1} \) to make \( w_1 \) unfrozen will not be sufficient to make all the children of the root colored with \( \sigma(w_1) \) become unfrozen. Hence, \( \sigma \notin E_{r,z}^\sigma \), and we have \( E_{r,z}^\sigma \) implies \( A_{1,z}^\sigma \). As noted earlier, we also know that \( E_{r,z}^\sigma \) implies \( E_{w_1,z}^\sigma \). Therefore,

\[
\Pr_{\sigma \in \Omega^*} [ E_{r,z}^\sigma ] \leq \Pr_{\sigma \in \Omega^*} [ E_{w_1,z}^\sigma \cap A_{1,z}^\sigma ] \]

(13)

where \( C^* = C - c^* \).

We will bound the terms in (14) separately for each \( c_1 \). We have that for each \( c_1 \in C^* \),

\[
\Pr_{\sigma \in \Omega^*} [(\sigma(w_1) = c_1) \cap E_{w_1,z}^\sigma \cap A_{1,z}^\sigma] = \Pr_{\sigma \in \Omega^*} [E_{w_1,z}^\sigma | \sigma(w_1) = c_1] \Pr_{\sigma \in \Omega^*} [\sigma(w_1) = c_1].
\]

(15)

Consider the following method to generate a random coloring: First we choose a random color \( c_0 \) for the root, then we choose a random color for each child \( v_i \) of the root from \( C^* \), and then we do the same things for each subtree \( T_{w_i} \) recursively, where \( 1 \leq i \leq b \). Hence, we can first generate the configurations \( \eta \in \Omega(T \setminus T_{w_1}^*) \) for those vertices not inside the subtree rooted at \( w_1 \) and then we generate the configurations \( \tau \in \Omega(T_{w_1}^*) \) inside the subtree \( T_{w_1} \). From this perspective, it is clear that the events \( A_{i,z}^\sigma \) and \( E_{w_1,z}^\sigma \) are independent, conditioned on the fixed colors of the root and \( w_1 \). Therefore, we have,

\[
\Pr_{\sigma \in \Omega^*} [E_{w_1,z}^\sigma | \sigma(w_1) = c_1] = \Pr_{\sigma \in \Omega^*} [E_{w_1,z}^\sigma | \sigma(w_1) = c_1] \Pr_{\sigma \in \Omega^*} [A_{1,z}^\sigma | \sigma(w_1) = c_1].
\]

(16)

Observe that, the first term on the right hand side is actually the same as the probability of the event \( E(\sigma,z) \) for a tree of height \( H - 1 \) with the color of the root being fixed to \( c_1 \), i.e. \( \Pr_{\eta \in \Omega(T_{w_1})} [E_{\eta,1,z}^\eta | \eta(r) = c_1] \). As we discussed before, this probability is the same for all \( c_1 \in C^* \) because of symmetry, and hence we denote it as \( \Pr_{\eta \in \Omega^*(T_{w_1})} [E_{\eta,1,z}^\eta] \). Putting equations (14), (15) and (16) together, we have

\[
\Pr_{\sigma \in \Omega^*} [E_{w_1,z}^\sigma | \sigma(w_1) = c_1] \leq \Pr_{\eta \in \Omega^*(T_{w_1})} [E_{\eta,1,z}^\eta] \sum_{c_1 \in C^*} \Pr_{\sigma \in \Omega^*} [A_{1,z}^\sigma | \sigma(w_1) = c_1] \Pr_{\sigma \in \Omega^*} [\sigma(w_1) = c_1].
\]

(17)

Finally, we can calculate a good upper bound on \( \Pr_{\sigma \in \Omega^*} [A_{1,z}^\sigma | \sigma(w_1) = c_1] \) as stated in the following lemma. And by the symmetry, the probabilities are the same for different colors \( c_1 \in C^* \) we are fixing for the vertex \( w_1 \).

**Lemma 22.**

\[
\Pr_{\sigma \in \Omega^*} [A_{1,z}^\sigma | \sigma(w_1) = c_1] \leq b^{-(1+\epsilon-o(1))},
\]

where \( o(1) \) is the function \( (1+\epsilon)/b^\epsilon \) when \( \epsilon < 1 \) and equals to zero when \( \epsilon \geq 1 \).
Plugging Lemma 22 into Inequality (17), we get:

\[
\Pr_{\sigma \in \Omega^*} \left[ E_{r,z}^\sigma \right] \leq \Pr_{\sigma \in \Omega^*} \left[ E_{w_1,z}^\sigma \cap A_{1,z}^\sigma \right] \\
\leq \Pr_{\eta \in \Omega^* (T_{w_1})} \left[ E_{w_1,z}^\eta \right] \cdot b^{-(1+\epsilon-o(1))}.
\]

By induction, applied on \( \Pr_{\eta \in \Omega^* (T_{w_1})} \left[ E_{w_1,z}^\eta \right] \), we have that:

\[
\Pr_{\sigma \in \Omega} \left[ E_{r,z}^\sigma \right] \leq b^{-(1+\epsilon-o(1))H}.
\]

This completes the proof of (12).

To complete the proof of the lower bounds when \( C < 1 \) in Theorem 1 we need to prove Lemmas 20, 21, and 22.

6.3 Proof of Lemmas

Proof of Lemma 20. The proof is very similar to the proof of Lemma 8 in [12]. We include it here for completeness.

Let \( U_\ell \) be the probability that a vertex at the height \( \ell \) is not frozen. We are going to prove that \( U_\ell < \frac{1}{b} \) by induction.

First of all, by definition, \( U_0 = 0 \) since they are leaves. Let \( v \) be a vertex at height \( \ell > 0 \). Since the probability that the color of \( v \) is \( c \) for some \( c \in C \) is independent from the probability that \( v \) is frozen, therefore we can just fixed the color of \( v \) to some \( c^* \in C \), and hence

\[
U_\ell = \Pr [ v \text{ is not frozen in } \sigma \mid \sigma(v) = c^*].
\]

Let \( w \) be a child of \( v \). Again by the same argument using the independency, the probability that \( w \) is frozen to color \( c \) equals \( \frac{1-U_{\ell-1}}{k-1} \). Thus, the probability that all the children of \( v \) are either not frozen or not colored by using \( c \) is

\[
(1 - \frac{1 - U_{\ell-1}}{k-1})^b.
\]

By the union bound, \( U_\ell \), the probability that there exists a color \( c \neq c^* \) such that all the children of \( v \) are not frozen to color \( c \), is bounded by:

\[
(k-1) \left( 1 - \frac{1 - U_{\ell-1}}{k-1} \right)^b \leq (k-1) \exp \left( - \frac{b(1 - U_{\ell-1})}{k-1} \right) \\
\leq (k-1) \exp \left( - \frac{b(1 - b^{-\epsilon})}{k-1} \right) \quad \text{(by induction)} \\
\leq (k-1) \cdot \exp \left( - (1 + \epsilon)(1 - b^{-\epsilon}) \ln b \right) \quad \text{(since } k = b/(1 + \epsilon) \ln b) \\
\leq b^{-\epsilon},
\]

where (18) is true for large \( b \).

Putting everything together, we showed that for every \( \ell \leq H \),

\[
U_\ell \leq b^{-\epsilon}.
\]

Remark. Using the same argument, we can prove a slightly stronger result: for any \( \epsilon > 0 \), \( U_\ell \leq \frac{1}{b \ln b} \), and when \( \epsilon \geq 1 \), \( U_\ell \leq \frac{1}{(1+\epsilon)\epsilon} \).

\( \square \)
Proof of Lemma 21. Let $F := \bigcup_{c \in C} S_c$ be the set of colorings that freeze the root. As we discussed before, by symmetry, $\pi(S_{c_1}) = \pi(S_{c_2})$ for $c_1, c_2 \in C$. Then $\pi(S) \geq 1/2$. Also, by Lemma 20, we know that

$$\sum_{c \in C} \pi(S_c) = \pi(F) \geq 1 - b^{-\epsilon}.$$ 

Therefore for any $\epsilon > 0$, there exists a $b_0$ such that for all $b > b_0$, $\pi(S) \pi(\bar{S}) \geq 1/6$. From the definition of $\Phi_S$, we know that

$$\Phi_S \leq 6 \left( \sum_{\sigma \in S} \sum_{\eta \in \bar{S}} \pi(\sigma)P(\sigma, \eta) \right).$$

Notice that, for any $\sigma \in S_{c_1}$, $\eta \in S_{c_2}$ and $c_1 \neq c_2$, we have $P(\sigma, \eta) = 0$. Since it is impossible to change the color of the frozen root by just one move. Further, in order to unfreeze the root in one step, the Glauber dynamics has to first recolor a leaf and change the color of the leaf so as to unfreeze the root. That is $\eta$ can only be $\sigma^{z \rightarrow c}$ for some $z \in L(T)$ and $c \in C^*$, where $C^* = C - \{\text{the color of the parent of } z \text{ in } \sigma\}$. Therefore,

$$\Phi_S \leq 6 \sum_{\sigma \in F} \sum_{\eta \in F} \pi(\sigma)P(\sigma, \eta) \leq 6 \sum_{\sigma \in F} \sum_{\eta \in F} \pi(\sigma)P(\sigma, \eta) \leq 6 \sum_{\sigma \in F} \sum_{z \in L(T)} \sum_{c \in C^*} \left(1_{\sigma, z, c} \pi(\sigma)P(\sigma, \eta)\right),$$

where in inequality (19), $\eta$ ranges over the set $Q(\sigma) := \{\eta \in F : \eta = \sigma^{z \rightarrow c}, \text{ for some } z \in L(T), c \in C^*\}$, and in equation (20) we just rephrase the equation (19) by using the indicator $1_{\sigma, z, c}$ of the set $Q(\sigma)$.

By the definition of the Glauber dynamics, we know that $\pi(\sigma) = 1/|\Omega|$ and $P(\sigma, \eta) = 1/(n(q-1))$ for the case that the change of color happens at a leaf. Also, observe that the indicator $1_{\sigma, z, c}$ does not depend on the third parameter $c$. Therefore we can simplify equation (20) as:

$$\frac{6}{n} \sum_{\sigma \in \Omega} \sum_{z \in L(T)} \frac{1_{\sigma, z}}{|\Omega|},$$

where $1_{\sigma, z}$ is the indicator for the event that the root in coloring $\sigma$ is frozen and there exists a color $c$ where the root in the coloring $\sigma^{z \rightarrow c}$ is not frozen.

Hence we proved that

$$\Phi_S \leq \frac{6}{n} \sum_{z \in L(T)} \sum_{\sigma \in \Omega(T)} \frac{1_{\sigma, z}}{|\Omega(T)|}.$$ 

\[\square\]

Proof of Lemma 22. When $\epsilon < 1$, the probability that all the siblings of $w_1$ are either not frozen or not colored with $c_1$ is upper bounded by

$$\left(1 - \frac{1 - U_{H-1}}{k-1}\right)^{b-1} \leq \exp\left(-\frac{(b-1)(1 - b^{-\epsilon})}{k-1}\right) \leq b^{-(1+\epsilon)} \cdot b^{\frac{1+\epsilon}{5\epsilon}}.$$
Now we can see that $o(1)$ is actually $(1 + \epsilon)/b^\epsilon$ when $\epsilon < 1$.

Note that, when $\epsilon \geq 1$, by the same way it is easy to see that

$$
(1 - \frac{1 - U_{H-1}}{k-1})^{-1} \leq b^{-(1+\epsilon)}.
$$

7 A Simple Generalization to $k = o(b/\ln b)$: Proof of Theorem 2

In all of the previous sections, we assumed $k = Cb/\ln b$ where $C$ is constant. But we are also interested in the case when $k$ is constant, say a hundred colors, and what the mixing time of the Glauber dynamics will be in this case. Let $\alpha = \alpha(k, b) := b/(k \ln b)$. We would like also to see how to generalize the upper bound and lower bound analysis assuming $\alpha$ is any function growing with $b$, that is when $k = o(b/\ln b)$. Intuitively, it is not hard to see the analysis will go through in the same way since the hardest case is when $\alpha$ is around the non-reconstruction threshold. Actually, all of our proofs will be the same and we just need to modify slightly the statements.

For the upper bound, we change Lemma 8 and Lemma 13 into the following ones.

Lemma 23. Let $T = 20b\ln b$. There exists $b_0$, for all $(x_0, y_0) \in \Omega \times \Omega$, all $\alpha(k, b) \geq 2$, and all $b > b_0$ the following holds:

$$
\Pr \left[ X_T = Y_T \mid X_0 = x_0, Y_0 = y_0 \right] \geq \frac{1}{(20\alpha(k, b)b^{\alpha(k, b)\ln b})}.
$$

Lemma 24. For any pair of initial states $(x_0, y_0)$ where all of the leaf disagreements are of the same type, then

$$
\Pr \left[ T_1^{XY} < 4b\ln b \mid (X_0, Y_0) = (x_0, y_0) \right] \geq \frac{1}{(4\alpha(k, b)b^{\alpha(k, b)-1}\ln b)}.
$$

Then by the same argument as in Section 4, we are able to show that the relaxation time of the Glauber dynamics on $G^*$ is upper bounded by $O(\alpha b^\alpha \ln b)$. Thus, the mixing time of the Glauber dynamics on the complete tree is bounded by

$$
T_{\text{mix}} = O(n^{\alpha + \frac{\ln \alpha + 2\ln \ln b + 20}{\ln b} \ln^2 n}),
$$

and the relaxation time is bounded by

$$
T_{\text{relax}} = O(n^{\alpha + \frac{\ln \alpha + 2\ln \ln b + 20}{\ln b} \ln^2 n}).
$$

For the lower bound, we change Lemma 20 and Lemma 22 into the following lemmas.

Lemma 25. In a random coloring of the tree $T$, the probability that a vertex of $T$ is not frozen is at most $b^{-1}$.

Lemma 26.

$$
\Pr_{\sigma \in \Omega^*} \left[ A_{1,2}^\sigma \mid \sigma(w_1) = c_1 \right] \leq b^{-\alpha(k, b)}.
$$

Then, by exactly the same way as in Section 5, we can show that the mixing time and the relaxation time of the Glauber dynamics on the complete tree $T$ when $\alpha \geq 2$ is lower bounded by $\Omega(n^{\alpha}) = \Omega(n^{b/(k \ln b)})$. 21
8 Bounding the Log-Sobolev Constant: Proof of Theorem 6

In this section we will analyze the log-Sobolev constant \( c_{sob} \) of the heat-bath Glauber dynamics on the complete tree by comparing it with the spectral gap \( c_{\text{gap}} \). For completeness, we prove Theorem 6, which is an adaption of the proof of Theorem 5.7 in Martinelli, Sinclair and Weitz \[24\]. In their paper, they proved it for the case of the Ising model on the complete tree with a fixed boundary condition, although they observed that it holds more generally. For convenience, we will use the same notation for the complete tree and its vertices, that is, \( T_\ell \) stands for both the complete tree of height \( \ell \) and its vertices \( V(T_\ell) \).

Let \( f \) be a function(vector) from \( \Omega \) to \( \mathbb{R} \), \( \pi \) be the uniform distribution over \( \Omega \) and \( \mu \) be any probability distribution over \( \Omega \). Let \( D(f) \) be the standard Dirichlet form of the heat-bath Glauber dynamics defined as:

\[
D(f) = \frac{1}{2} \sum_{\sigma} \sum_{\sigma'} (f(\sigma) - f(\sigma'))^2 \pi(\sigma) P(\sigma, \sigma').
\]

Let \( E_\mu(f) \) be the average of \( f \) under the distribution \( \mu \), and let \( \text{Var}_\mu(f) := E_\mu(f^2) - E_\mu^2(f) \) be the corresponding variance, which can also be written as:

\[
\text{Var}_\mu(f) = \frac{1}{2} \sum_{\sigma} \mu(\sigma) \sum_{\sigma'} (f(\sigma) - f(\sigma'))^2 \mu(\sigma) \mu(\sigma').
\]

Let \( \text{Ent}_\mu(f) := E_\mu(f \log f) - E_\mu(f) \log(E_\mu(f)) \). When it is clear what the underlying distribution is we will drop the subscript \( \mu \) in the notation \( \text{Ent}(f) \).

The spectral gap \( c_{\text{gap}} \) is equivalently defined as (see, e.g., Chapter 13 in \[20\])

\[
c_{\text{gap}} = \inf_f \frac{D(f)}{\text{Var}(f)}
\]

and the log-Sobolev constant \( c_{sob} \) is defined as (see, e.g., \[3\]),

\[
c_{sob} = \inf_{f \geq 0} \frac{D(\sqrt{f})}{\text{Ent}(f)}
\]

where the infimum in both equations is over non-constant functions \( f \).

Let \( B \subseteq A \subseteq T \) be two subsets of the vertices on tree \( T \). Let \( \eta \in \Omega \) be a configuration. Let \( E^\eta_A(f) \) be the expectation of \( f \) under a prefixed distribution \( \mu \) in the region \( A \) with boundary condition \( \eta \). That is

\[
E^\eta_A(f) = \sum_{\sigma} \frac{\mu(\sigma)}{Z} f(\sigma)
\]

where \( \sigma \) ranges over the configurations that are the same as \( \eta \) outside \( A \) (denoted as \( \sigma \sim_A \eta \)) and \( Z \) is the normalizing factor. The quantities \( \text{Var}^\eta_A \) and \( \text{Ent}^\eta_A \) are defined similarly. If we drop \( \eta \) then \( E_A(f), \text{Var}_A(f), \text{Ent}_A(f) \) become functions from \( \Omega \) to \( \mathbb{R} \). The following are standard facts concerning variance and entropy; the first being the chain rule, and the second follows from the so-called tensoring property over a product distribution – see e.g., Proposition 5.6 of \[19\]. In the following, we will use the fact that the distribution on configurations over the tree with the root removed, has a product form over the subtrees rooted at the children of the root, to satisfy the hypothesis for the tensoring property.

**Proposition 27.**

\[
\begin{align*}
\text{Var}^\eta_A(f) &= E^\eta_A(\text{Var}_B(f)) + \text{Var}^\eta_A(E_B(f)). \\
\text{Ent}^\eta_A(f) &= E^\eta_A(\text{Ent}_B(f)) + \text{Ent}^\eta_A(E_B(f)).
\end{align*}
\]
Proposition 28. Let $A = \bigcup A_i$ where $A_i$ are disjoint, and suppose that conditioning on the boundary being $\eta$, the probability of $A_i$’s being in any configuration for different $i$’s is completely independent. Then

$$\text{Var}_A(f) \leq \sum_i E_A^\eta(\text{Var}_{A_i}(f)),$$

and

$$\text{Ent}_A^\eta(f) \leq \sum_i E_A^\eta(\text{Ent}_{A_i}(f)).$$

Lemma 29. Let $c_{sob}(\ell)$ be the log-Sobolev constant of the heat-bath Glauber dynamics on the complete tree of height $\ell > 0$ with the root being attached to an external vertex with a fixed color, then

$$c_{sob}(\ell)^{-1} \leq c_{sob}(\ell - 1)^{-1} + \alpha \cdot c_{gap}(\ell)^{-1},$$

where $\alpha = \frac{\log(k-2)}{1-2\gamma(k-1)} = c_{sob}(0)^{-1}$.

Proof. Let $f$ be any non-negative function. Let $I$ be the set of vertices in the complete tree $T_\ell$ without the root, i.e., $I = T_\ell \setminus \{\text{root}\}$. Let us first use Proposition 27 to analyze the $\text{Ent}(f)$:

$$\text{Ent}(f) = E(\text{Ent}_I(f)) + E(\text{Ent}_T(f))$$

We will bound $E(\text{Ent}_I(f))$ and $E(\text{Ent}_T(f))$ separately. For $E(\text{Ent}_I(f))$, by Proposition 28, it can be upper bounded as

$$E(\text{Ent}_I(f)) \leq \sum_v E(\text{Ent}_{T_v}(f)),$$

where $v$ ranges over all the children of the root of $T_\ell$ and $T_v$ denotes the subtree of $T_\ell$ rooted at the vertex $v$. Let $\eta \in \Omega(T_\ell)$, then for a specific $\text{Ent}_{T_v}^\eta(f)$, we then have

$$\text{Ent}_{T_v}^\eta(f) \leq c_{sob}(\ell - 1)^{-1}D_{T_v}(\sqrt{f}),$$

where $D_{T_v}(\sqrt{f})$ is the corresponding Dirichlet form for the dynamics on the subtree $T_v$. For the heat-bath Glauber dynamics, we can further derive that

$$D_{T_v}(f) = \frac{1}{2} \sum_{\sigma,\tau} (f(\sigma) - f(\tau))^2 \mu(\sigma)P(\sigma,\tau)$$

$$= \frac{1}{2} \sum_{x \in T_v} E_{T_v}^\eta(\text{Var}_{\{x\}}(f)) \quad \text{(since } P(\sigma,\tau) \neq 0 \text{ only if they differ at a single vertex)}$$

where $\mu(\sigma)$ is the marginal distribution with respect to $\eta$.

Then, from (21), (22) and above we have

$$E(\text{Ent}_I(f)) \leq \sum_v E(\text{Ent}_{T_v}(f)) \quad \text{(by 21)}$$

$$\leq \sum_v c_{sob}(\ell - 1)^{-1}E(D_{T_v}(\sqrt{f})) \quad \text{(by 22)}$$

$$= \sum_v c_{sob}(\ell - 1)^{-1}E(\sum_{x \in T_v} E_{T_v}^\eta(\text{Var}_{\{x\}}(f))) \quad \text{(by 23)}$$

$$= \sum_v c_{sob}(\ell - 1)^{-1} \sum_{x \in T_v} E(\text{Var}_{\{x\}}(f)) \quad \text{(by the linearity of the expectation)}$$

$$= c_{sob}(\ell - 1)^{-1} \sum_{x \in I} E(\text{Var}_{\{x\}}(f))$$

$$\leq c_{sob}(\ell - 1)^{-1}D(\sqrt{f}) \quad \text{(by applying 23 again on the entire tree)}$$
For $\text{Ent}(E_I(f))$, $E_I(f)$ can be viewed as a function from $\{1, 2, \ldots, k-1\}$ to $R$ since those $k-1$ values can represent the colors of the root (boundary). Therefore $\text{Ent}(E_I(f))$ is the entropy of the random variable $E_I(f)$ taking $k-1$ values uniformly at random. It is well-known (see e.g. Appendix of [6]) that \( \log(\frac{k}{1-\log(k-2)/(k-1)}) \) is the inverse of the log-Sobolev constant of the random walk $\mathcal{R}$ on the complete graph $K_{k-1}$, which jumps to stationarity in one step. Thus, letting $\alpha = \frac{\log(k-2)}{1-2/(k-1)}$, we may upper bound $\text{Ent}(E_I(f))$ as follows.

\[
\text{Ent}(E_I(f)) \leq \alpha \mathcal{D}_\mathcal{R}(\sqrt{E_I(f)}) \quad (\text{by the log-Sobolev Inequality})
\]
\[
= \alpha \text{Var}_\mathcal{R}(\sqrt{E_I(f)}) \quad (\text{since the complete graph has transition } P(x,y) = \pi_\mathcal{R}(y))
\]
\[
= \alpha \text{Var}_T(\sqrt{E_I(f)})
\]
\[
\leq \alpha \left( E[E_I(f)] - E^2(\sqrt{E_I(f)}) \right) \quad (\text{by the definition of the variance})
\]
\[
\leq \alpha E(\sqrt{f})^2 - E^2(\sqrt{f}) \quad (\text{by the concavity of } \sqrt{x})
\]
\[
= \alpha \text{Var}(\sqrt{f})
\]
\[
\leq \alpha c_{\text{gap}}(\ell)^{-1} \mathcal{D}(\sqrt{f}) \quad (\text{by the definition of the spectral gap})
\]

Putting everything together, we proved

\[
\text{Ent}(f) = E(\text{Ent}_I(f)) + \text{Ent}(E_I(f)) \leq c_{\text{sob}}(\ell - 1)^{-1} \mathcal{D}(\sqrt{f}) + \alpha c_{\text{gap}}(\ell)^{-1} \mathcal{D}(\sqrt{f}),
\]

then by the definition of $c_{\text{sob}}$, we get

\[
c_{\text{sob}}(\ell)^{-1} \leq c_{\text{sob}}(\ell - 1)^{-1} + \alpha c_{\text{gap}}(\ell)^{-1}.
\]

**Lemma 30.** Let $c_{\text{sob}}(\ell)$ be the spectral gap of the heat-bath Glauber dynamics on the complete tree of height $\ell > 0$ with the root being attached to an external vertex with a fixed color; then for $\ell > 0$ and $k \leq b + 2$, we have $c_{\text{gap}}(\ell) \leq c_{\text{gap}}(\ell - 1)$.

**Proof.** Let $\mathcal{D}_\ell(f)$ and $\text{Var}_\ell(f)$ be the Dirichlet form and the variance of function $f : \Omega(\mathcal{T}_\ell) \to \mathcal{R}$ for the Glauber dynamics on the complete tree of height $\ell$ with the root attached to an external vertex with a fixed color. Let $P_\ell$ denote the probability transition of the dynamics, and let $\pi_\ell$ denote its unique stationary distribution.

Let $g$ be the eigenfunction such that $c_{\text{gap}}(\ell - 1) = \mathcal{D}_{\ell-1}(g)/\text{Var}_{\ell-1}(g)$. Now we are going to construct a function $f : \Omega(\mathcal{T}_\ell) \to \mathcal{R}$, such that $\mathcal{D}_\ell(f) \leq \mathcal{D}_{\ell-1}(g)$ and $\text{Var}_\ell(f) = \text{Var}_{\ell-1}(g)$. Then, since

\[
c_{\text{gap}}(\ell) \leq \frac{\mathcal{D}_\ell(f)}{\text{Var}_\ell(f)} \leq \frac{\mathcal{D}_{\ell-1}(g)}{\text{Var}_{\ell-1}(g)} = c_{\text{gap}}(\ell - 1),
\]

we prove the lemma.

Let $A \subseteq \mathcal{T}_\ell$ be the set of non-leaf vertices of $\mathcal{T}_\ell$, i.e., $A = T_\ell \setminus L(\mathcal{T}_\ell)$, where $L(\mathcal{T}_\ell)$ is the set of leaves in the tree $T_\ell$. There is a natural correspondence between vertices in $A$ and in $T_{\ell-1}$. The function $f$ is then defined as: for $\sigma \in \Omega(\mathcal{T}_\ell)$ and $\sigma' \in \Omega(\mathcal{T}_{\ell-1})$, $f(\sigma) = g(\sigma')$ if the configuration $\sigma$ agrees with $\sigma'$ on the subset $A$.

It is straightforward to show that $\text{Var}_\ell(f) = \text{Var}_{\ell-1}(g)$. We will show $\mathcal{D}_\ell(f) \leq \mathcal{D}_{\ell-1}(g)$ when $k \leq b + 2$. By definition,

\[
\mathcal{D}_\ell(f) = \sum_{\sigma, \eta \in \Omega(\mathcal{T}_\ell)} \pi_\ell(\sigma) P_\ell(\sigma, \eta)(f(\sigma) - f(\eta))^2.
\]
For a subset of vertices $S \subset T_\ell$, let

$$\Omega(S) = \{ \sigma' \in [k]^S : \text{there exists } \sigma \in \Omega(T_\ell) \text{ where } \sigma(A) = \sigma' \}.$$ 

Let $\sigma', \eta' \in \Omega(A)$ be colorings of the internal vertices. Let $\phi, \psi \in \Omega(L(T_\ell))$ be colorings of the leaves. Finally, let $\circ$ be the concatenation operator, thus $\sigma' \circ \psi = \sigma \in \Omega(T_\ell)$ where $\sigma(A) = \sigma'$ and $\sigma(L(T_\ell)) = \phi$. Then we can rewrite the Dirichlet form as:

$$\mathcal{D}_\ell(f) = \sum_{\sigma', \eta' \in \Omega(A)} \sum_{\phi, \psi \in \Omega(L(T_\ell))} \pi_\ell(\sigma' \circ \phi) P_\ell(\sigma' \circ \phi, \eta' \circ \psi)(f(\sigma' \circ \phi) - f(\eta' \circ \psi))^2.$$

According to the definition of the Glauber dynamics, for configurations $\sigma, \eta \in \Omega(T_\ell)$ which differ at more than one vertex, we have $P_\ell(\sigma, \eta) = 0$. Let $\oplus$ denote the symmetric difference. Now we can rewrite the Dirichlet form as:

$$\mathcal{D}_\ell(f) = \sum_{v \in A} \sum_{\sigma', \sigma'' \in \Omega(A)} \sum_{\phi \in \Omega(L(T_\ell))} \sum_{\psi \in \{v\}} \pi_\ell(\sigma' \circ \phi) P_\ell(\sigma' \circ \phi, \eta' \circ \phi)(f(\sigma' \circ \phi) - f(\eta' \circ \phi))^2.$$

Since $g(\sigma' \circ \phi) = g(\sigma' \circ \psi) = f(\sigma')$.

Thus we only need to consider when the sole disagreement is at an internal vertex. We can further decompose based on whether the disagreement is an internal vertex of the tree $T_{\ell-1}$, which we denote as $A_{\ell-1}$, or a leaf of $T_{\ell-1}$.

For $v \in L(T_{\ell-1})$ we have:

$$\sum_{\sigma', \eta' \in \Omega(A)} (g(\sigma') - g(\eta'))^2 \sum_{\phi} \pi_\ell(\sigma' \circ \phi) P_\ell(\sigma' \circ \phi, \eta' \circ \phi)$$

$$\leq \sum_{\sigma', \sigma'' \in \Omega(T_{\ell-1})} (g(\sigma') - g(\eta'))^2 \pi_{\ell-1}(\sigma') \sum_{\phi} \frac{1}{(k-1)|L(T_{\ell-1})||T_\ell|}$$

$$\leq \sum_{\sigma', \xi \in \Omega(T_{\ell-1})} (g(\sigma') - g(\eta'))^2 \pi_{\ell-1}(\sigma') P_{\ell-1}(\sigma', \eta'),$$

where the last inequality is by the fact that $P_{\ell-1}(\sigma', \eta') = 1/|T_{\ell-1}|(k-1) \geq 1/|T_\ell|$ when $k \leq b + 2$. Note that, the condition $k \leq b + 2$ can be weakened to $k \leq \zeta b$ where $\zeta \approx 1.763 \ldots$ is the solution of the equation $e^{-1/\zeta} = 1/\zeta$. This is because, the configurations $\phi$ on the leaves of $T_\ell$ in Inequality (24) range over all of the valid configurations on the leaves for both $\sigma'$ and $\eta'$, the number of which is at most $(k-1)^{|L(T_\ell)|-b}(k-2)^b$. But the probability of getting such a configuration from $\sigma'$ is $\pi_\ell(\sigma' \circ \phi)/\pi_{\ell-1}(\sigma') = (k-1)^{-|L(T_\ell)|}$. Therefore, a simple calculation shows that Inequality (24) holds when $k \leq \zeta b$. 

25
Similarly, for $v \in A_{\ell-1}$ we have:

\[
\sum_{\sigma', \eta' \in \Omega(A')} \left( (g(\sigma') - g(\eta'))^2 \sum_{\phi \in \Omega(L(T_\ell))} \pi_\ell(\sigma' \circ \phi) P_\ell(\sigma' \circ \phi, \eta' \circ \phi) \right) \\
\leq \sum_{\sigma', \eta' \in \Omega(T_{\ell-1})} (g(\sigma') - g(\eta'))^2 \pi_{\ell-1}(\sigma') P_{\ell-1}(\sigma', \eta'). \tag{25}
\]

Combining (24) and (25), and summing over $v \in T_{\ell-1}$ we have shown that $D_\ell(f) \leq D_{\ell-1}(g)$, which implies the lemma.

\[\square\]

Proof of Theorem 6. Now we apply Lemma 29 inductively, and we get

\[c_{\text{sob}}^{-1}(H) \leq \alpha \left( 1 + c_{\text{sob}}^{-1}(1) + \cdots + c_{\text{sob}}^{-1}(|\log_b n|) \right).
\]

Then by applying Lemma 30 on the spectral gaps, we can conclude that

\[c_{\text{sob}}^{-1} \leq (\log_b n + 1) \alpha c_{\text{gap}}^{-1}(H) \leq 2c_{\text{gap}}^{-1} \log_b n \log k.
\]

\[\square\]

9 Conclusions

In the context of spin systems on sparse graphs and more generally for random instances of constraint satisfaction problems (CSPs), an informal conjecture of Andrea Montanari [5] asserts that in the nonreconstruction regime for such models, the Glauber dynamics is always fast (as in $O(n \log n)$) in converging to stationarity on almost all (as in $(1 - o(1))$) of the state space. While the community is far from establishing such a precise connection between the reconstruction threshold and a (dynamical) transition in mixing time, the present contribution provides further evidence towards such a conjecture, by establishing tight estimates on the mixing time of Glauber dynamics on colorings at and near the reconstruction threshold. Results of similar flavor for other instances of CSPs are natural open problems of interest.

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