Peiffer Elements in Simplicial Groups and Algebras

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Abstract

The main objectives of this paper are to give general proofs of the following two facts:
A. For an operad $\mathcal{O}$ in $\text{Ab}$, let $A$ be a simplicial $\mathcal{O}$-algebra such that $A_m$ is the $\mathcal{O}$-subalgebra generated by $(\sum_{i=0}^m s_i(A_{m-1}))$, for every $n$, and let $N A$ be the Moore complex of $A$. Then
\[ d(N_m A) = \sum_I \gamma(\mathcal{O}_p \otimes \bigcap_{i \in I_1} \ker d_i \otimes \cdots \otimes \bigcap_{i \in I_p} \ker d_i) \]
where the sum runs over those partitions of $[m-1]$, $I = (I_1, \ldots, I_p)$, $p \geq 1$, and $\gamma$ is the action of $\mathcal{O}$ on $A$.

B. Let $G$ be a simplicial group with Moore complex $NG$ in which the normal subgroup of $G_n$ generated by the degenerate elements in dimension $n$ is the proper $G_n$. Then
\[ d(N_n G) = \prod_{I,J} \left[ \bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j \right], \]
for $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$. In both cases, $d_i$ is the $i$-th face of the corresponding simplicial object.

The former result completes and generalizes results from Akça and Arvasi, and Arvasi and Porter; the latter, results from Mutlu and Porter. Our approach to the problem is different from that of the cited works. We have first succeeded with a proof for the case of algebras over an operad by introducing a different description of the adjoint inverse of the normalization functor $N : \text{Ab}^{\Delta^{op}} \to \text{Ch}_{\geq 0}$. For the case of simplicial groups, we have then adapted the construction for the adjoint inverse used for algebras to get a simplicial group $G \boxtimes \Lambda$ from the Moore complex of a simplicial group $G$. This construction could be of interest in itself.

Key words: simplicial algebras, simplicial groups, Dold-Kan functor, operads, near-ring.

1991 MSC: 18G30.
1 Introduction

R. Brown and J. L. Loday have noted [3] that if the second dimension, $G_2$, of a simplicial group $(G_*, d_*, s_*)$ is generated by degenerate elements, then

$$d(N_2G) = [\ker d_0, \ker d_1]$$

Here $N_2G = \ker d_0 \cap \ker d_1$, $d$ is here induced by $d_2$ and the square brackets denote the commutator subgroup. Thus, this subgroup of $N_1G$ is generated by elements of the form $s_0d_1(x)y^{-1}s_0d_1(x^{-1})(xyx^{-1})^{-1}$, and it is just the Peiffer subgroup of $N_1G$, the vanishing of which is equivalent to $d_1 : N_1G \to N_0G$ being a crossed module.

Arvasi and Porter [2] have shown that if $A$ is a simplicial commutative algebra with Moore complex $N_A$, and for $n > 0$ the ideal generated by the degenerate elements in dimension $n$ is $A_n$, then

$$d(N_nA) \supseteq \sum_{I,J} K_I K_J$$

This sum runs over those $\emptyset \neq I, J \subset [n-1] = \{0, ..., n-1\}$ with $I \cup J = [n-1]$, and $K_I := \cap_{i \in I} \ker d_i$. They have also shown the equality for $n = 2, 3$ and 4, and argued for its validity for all $n$. A similar result for simplicial Lie algebras was obtained by Arvasi and Akça in [1].

Mutlu [10] and Mutlu and Porter [11], have adapted Arvasi’s method to the case of simplicial groups. They succeeded to prove that for $n = 2, 3$ and 4,

$$d(N_nG) = \prod_{I,J} [K_I, K_J]$$

and that the inclusion $d(N_nG) \supseteq \prod_{I,J} [K_I, K_J]$ holds for every $n$.

The objective of this paper is to give a general proof for the inclusions partially proved in [1], [2], [10] and [11]. Our approach to the problem is different from that of the cited papers. We have first succeeded with a proof for the case of algebras over an operad $O \in \text{Op}(\text{Ab}^\text{op})$, by introducing a new description of the adjoint inverse of the normalization functor $N : \text{Ab}^\text{op} \to \text{Ch}_{\geq 0}$. We have then adapted this construction to get a simplicial group $G\Lambda A$ from the Moore complex of a simplicial group $G$, which was used in the case of groups. This construction could be of interest in itself.

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2 The authors were supported by MCYT, Grant BSM2003-04686-C02 (European FEDER support included). The first author would also like to thank to the Xunta de Galicia for the financial support provided during his stay in Santiago de Compostela (Sept.- Dec. 2004).
In section 2 we give an alternative description of the Dold-Kan functor, which we shall use later. In section 3 we take an operad \( \mathcal{O} \in \text{Op}(Ab^\Delta^{op}) \), an \( \mathcal{O} \)-simplicial algebra, and we study what happens when we apply the normalization functor to this algebra. We finally give a description of the kind of algebras one gets in \( Ch_{\geq 0} \).

Section 4 is devoted to state and prove the first important result, Theorem 8. In section 5 we introduce a simplicial group build up from a chain of groups (Def. 11) and prove some properties of this construction when applied to the Moore complex of a simplicial group (Prop. 18 and Rem. 19).

Finally, in section 6, we prove the other main result of this paper, Lemma 21, which completes the proof of Theorem 25.

2 The inverse of the normalization functor

In this section we give a description of the Dold-Kan functor \( Ch_{\geq 0} \to Ab^\Delta^{op} \), suitable for the use we shall make of it. This description is in the spirit of that given in [7] for the inverse of the conormalization functor.

We write \( \Delta \) for the simplicial category, and \( Fin \) for the category with the same objects as \( \Delta \), but where the homomorphisms \([m] \to [n]\) are just the set maps. We associate to each \( n \) the free abelian group \( \mathbb{Z}[n] := \mathbb{Z}e_0 + \cdots + \mathbb{Z}e_n \cong \mathbb{Z}^{n+1} \), and to each \( \alpha : [m] \to [n] \in Fin, \alpha : \mathbb{Z}[m] \to \mathbb{Z}[n] \) given by

\[
\alpha(e_i) := e_{\alpha(i)}
\]

In this way we have a cosimplicial abelian group.

Put \( v_i := e_i - e_n \) in \( \mathbb{Z}[n] \) and write \( \mathbb{Z}[[n]] \) for the \( \mathbb{Z} \)-module freely generated by \( \{v_0, \ldots, v_{n-1}\} \). Since \( v_n = e_n - e_n = 0 \) and for \( \alpha : \mathbb{Z}[m] \to \mathbb{Z}[n] \) we have that

\[
\alpha(v_i) = \alpha(e_i - e_n) = e_{\alpha(i)} - e_{\alpha(m)} = e_{\alpha(i)} - e_{\alpha(n)} + e_{\alpha(n)} - e_{\alpha(m)} = v_{\alpha(i)} - v_{\alpha(m)}
\]

we conclude that \( \mathbb{Z}[[n]] \) is a \( Fin \)-subgroup of \( \mathbb{Z}[n] \).

Write \( \mathbb{Z}(n) \) for the abelian group \( \text{hom}_\mathbb{Z}(\mathbb{Z}[[n]], \mathbb{Z}) \). For \( \varphi \in \mathbb{Z}(n) \) and \( \alpha : [m] \to [n] \in Fin \) we take \( \alpha(\varphi) := \varphi \alpha \in \mathbb{Z}(m) \).

In this way, we equip the sequence \( \mathbb{Z}(*) \) with a \( Fin^{op} \) structure. Observe that \( \text{hom}_\mathbb{Z}(\mathbb{Z}[[n]], \mathbb{Z}) \) is freely generated by the morphisms \( \varphi_j, 0 \leq j \leq n-1 \), of the form

\[
\varphi_j(v_i) := \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}
\]
Thus, we shall identify \( \mathbb{Z}(n) \) with \( \mathbb{Z}\varphi_0 \oplus ... \oplus \mathbb{Z}\varphi_{n-1} \).

In particular, if we restrict to those arrows in \( \Delta \), then \( n \mapsto \mathbb{Z}(n) \) is a simplicial abelian group. Faces and degeneracies with source \( \mathbb{Z}(n) \) are explicitly given by

\[
\begin{align*}
    s_j(\varphi_i) &= \begin{cases} 
        \varphi_i & \text{if } i < j \\
        \varphi_i + \varphi_{i+1} & \text{if } i = j \\
        \varphi_{i+1} & \text{if } i > j
    \end{cases} \\
    d_j(\varphi_i) &= \begin{cases} 
        0 & \text{if } i = j \\
        \varphi_{i-1} & \text{if } i > j
    \end{cases}
\end{align*}
\]

for \( j \neq n \), and

\[
\begin{align*}
    s_n(\varphi_i) &= \begin{cases} 
        \varphi_i & \text{if } i < n \\
        0 & \text{if } i = n
    \end{cases} \\
    d_n(\varphi_i) &= \begin{cases} 
        \varphi_i & \text{if } i < n - 1 \\
        0 & \text{if } i = n - 1
    \end{cases}
\end{align*}
\]

We can now apply to \( \mathbb{Z}(\ast) \) the exterior power algebra functor, \( \Lambda : \text{Ab} \to \text{Ass} \) and then the forgetful functor \( \text{Ass} \to \text{Ab} \) by considering the exterior algebra just as an abelian group. We write \( \Lambda \mathbb{Z}(\ast) \) for the simplicial abelian group so obtained.

**Definition 1** Let \((A_\ast, d)\) be a connected chain complex of abelian groups, and \(B_\ast\) a sequence of \(\mathbb{Z}^+\)-graded abelian groups. We write \(A \boxtimes B\) for the sequence of groups \(n \mapsto \bigoplus_{i \geq 0}(A_i \otimes B^i_n)\).

Write \(K_\ast A := A \boxtimes \Lambda \mathbb{Z}(\ast) = \bigoplus_{i=0}(A_i \otimes \Lambda^i \mathbb{Z}(\ast))\). We can endow \(K_\ast A\) with a \(\text{Fin}^{op}\)-group structure by associating to \(\alpha \in \text{Fin}(m,n)\), the morphism \(K(\alpha) : K_n A \to K_m A\) by the formula

\[
K(\alpha)(a \otimes \varphi) := a \otimes \alpha(\varphi) + dg \otimes \delta(\varphi)
\]

Here \(\delta\) is the \(\alpha\)-derivation \(\Lambda \mathbb{Z}(n) \to \Lambda \mathbb{Z}(m)\) completely characterized by

\[
\delta(\varphi)_i := \begin{cases} 
        0 & \text{if } i \neq \alpha(m) \\
        1 & \text{if } i = \alpha(m)
    \end{cases}
\]

**Proposition 2** Let \([m] \xrightarrow{\alpha} [n] \xrightarrow{\beta} [p] \in \text{Fin}\), then \(K(\beta \alpha) = K(\alpha)K(\beta)\). In consequence, \(K_\ast A\) is a \(\text{Fin}^{op}\)-group. In particular, it is a simplicial abelian group.

**PROOF.** Take \(a \otimes \varphi \in K_p A\). By evaluating both \(K(\beta \alpha)\) and \(K(\alpha)K(\beta)\) at \(a \otimes \varphi\) and comparing, we get that in order the identity to hold, it suffices to verify if \(\delta_{\beta \alpha} = \alpha \delta_{\beta} + \delta_{\alpha} \beta\).
Let us observe that both $\delta_{\beta\alpha}$ and $\alpha\delta_{\beta} + \delta_{\alpha}\beta$ are $\alpha\beta$-derivations. Hence they will agree if they agree on the generators of $\Lambda Z(m)$. Take $\varphi_i \in Z(m)$ as above,

$$(\alpha\delta_{\beta} + \delta_{\alpha}\beta)(\varphi_i) = \alpha\delta_{\beta}(\varphi_i) + \delta_{\alpha}(\varphi_i\beta) \quad \text{(3)}$$

$$\delta_{\alpha}(\varphi_i\beta) = \begin{cases} \sum_{\beta(j) = i} \delta_{\alpha}\varphi_j & \text{if } i \neq \beta(n) \\ -\sum_{\beta(j) \neq \beta(n)} \delta_{\alpha}\varphi_j & \text{if } i = \beta(n) \end{cases} \quad \text{(4)}$$

We have to analyze the following possible cases:

i. If $i = \beta(n)$, then $i \neq \beta\alpha(m)$, $\beta_\alpha(\varphi_i) = 1$. Hence (3) is 0.

ii. If $i \neq \beta(n)$, we have two possibilities, $i = \beta\alpha(m)$ and $i \neq \beta\alpha(m)$. If $i = \beta\alpha(m)$, then (4) is 0, $\beta_\alpha(\varphi_i) = 0$, and in consequence (3) is 0. If $i \neq \beta\alpha(m)$, then (4) is 1, $\beta_\alpha(\varphi_i) = 0$ and (3) is 0.

Thus (3) coincides with $\delta_{\beta\alpha}(\varphi_i)$.

Let us take a closer view on faces and degeneracies in $K_A$. Take $a \otimes \varphi \in K_A$, and write simply $s_i$ and $d_i$ for either $K(s_i)$ and $K(d_i)$. For $0 \leq i \leq n$, we have that $s_i(a \otimes \varphi) = a \otimes s_i(\varphi)$, for $0 \leq i \leq n - 1$, $d_i(a \otimes \varphi) = a \otimes d_i(\varphi)$, and $d_n(a \otimes \varphi) = a \otimes d_n(\varphi) + da \otimes \delta_{d_n}(\varphi)$. So, for $i \neq n$, we can immediately say that $a \otimes \varphi \in \ker d_i$ if $i \in \mathbb{Z}$.

We write for a monomial $\varphi$, $\varphi := \{i_1, ..., i_r\}$ if and only if $\varphi \in \mathbb{Z}$ $\varphi_{i_1} \land ... \land \varphi_{i_r}$.

Proposition 3: Write $N : Ab^{\Delta^0} \rightarrow Ch_{\geq 0}$ the normalization complex and let $K : Ch_{\geq 0} \rightarrow Ab^{\Delta^0}$ be as before. Then $KN \simeq 1_{Ab^{\Delta^0}}$ and $NK \simeq 1_{Ch_{\geq 0}}$. Thus $K$ is (isomorphic to) the classical adjoint inverse of the normalization functor.

Proof. Recall that $N_m A = \bigcap_{i=0}^{m-1} \ker d_i$ for $A \in Ab^{\Delta^0}$. Observe that when $A = K_C$ for some $C \in Ch_{\geq 0}$, then $a \otimes \varphi \in \ker d_i$ iff $i \in \mathbb{Z} \varphi$. On the other hand, we have that $\sum_{\varphi \in \Lambda_m} a_{\varphi} \otimes \varphi$, with $\Lambda_m$ the set of monic monomials in $\Lambda Z(n)$, is in $\ker d_i$ iff each $a \otimes \varphi \in \ker d_i$. Then $x \in N_m K_C$ if and only if $x = a \otimes \varphi_0 \land ... \land \varphi_{m-1}$ for some $a \in C_m$. In this case,

$$d_m(a \otimes \varphi_0 \land ... \land \varphi_{m-1}) = da \otimes \varphi_0 \land ... \land \varphi_{m-2} \in N_{m-1} K_C$$

Since $N_m K_C \simeq C_m$, as $\mathbb{Z}$-modules, and $d_m$ induces $d$, we get that $NK C \simeq C$.

Write $\Gamma$ for the classical adjoint inverse of $N$ (see [8]). For a simplicial module $A$, we have that

$$A_m \simeq \Gamma_m N A = \bigoplus_{\eta : m \rightarrow -k} N_k A[\eta] \simeq \bigoplus_{i=0}^{m} N_i A \otimes \Lambda^i Z(m) = K_m N A$$
Furthermore, this isomorphism of \( \mathbb{Z} \)-modules is compatible with faces and degeneracies of \( \Gamma N A \) and \( K N A \). Hence, it is an isomorphism of simplicial modules.

### 3 Algebras in \( Ab^{\Delta^{op}} \) and \( Ch_{\geq 0} \)

Let \( O \in \text{Op}(Ab) \), the category of operads on \( Ab \). \( O \) induce a unique \( O \in \text{Op}(Ab^{\Delta^{op}}) \). Let \( F \) be the monad associated to \( O \). For any \( A \in Ab \) we have

\[
F(A) := \bigoplus_{n \geq 0} O(n) \otimes_{\Sigma_n} A^\otimes n
\]

and, for any \( A \in Ab^{\Delta^{op}} \),

\[
F_m(A) := \bigoplus_{n \geq 0} O(n) \otimes_{\Sigma_n} A_m^{\otimes n}
\]

We associate to \( \alpha \in \Delta(m, n) \), \( F(\alpha) : F_n A \rightarrow F_m A \) by taking \( \alpha \) degree wise.

Take \( N_m A = \tilde{A}_m \) and \( \Lambda_m^j = \Lambda^j \mathbb{Z}(m) \). Using that \( A_* \simeq K_* N A \), we write

\[
A_m \simeq \bigoplus_{j=0}^m \tilde{A}_j \otimes \Lambda_m^j \tag{5}
\]

Then

\[
F_m A \simeq F\left( \bigoplus_{j=0}^m \tilde{A}_j \otimes \Lambda_m^j \right) = \bigoplus_{p \geq 0} O(p) \otimes_{\Sigma_p} \left( \bigoplus_{j=0}^m \tilde{A}_j \otimes \Lambda_m^j \right)^{\otimes p}
\]

\[
= \bigoplus_{p \geq 0} \bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \ldots + i_p = r} O(p) \otimes (\tilde{A}_{i_1} \otimes \Lambda_m^{i_1}) \otimes \ldots \otimes (\tilde{A}_{i_p} \otimes \Lambda_m^{i_p}) \tag{6}
\]

Observe from (5) that, as we have already done in (6), we can identify \( \tilde{A}_m^{\otimes p} \) with

\[
\bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \ldots + i_p = r} (\tilde{A}_{i_1} \otimes \ldots \otimes \tilde{A}_{i_p}) \otimes (\Lambda_m^{i_1} \otimes \ldots \otimes \Lambda_m^{i_p}) \simeq \bigoplus_{I \in \wp(m)^{\times p}} \tilde{A}_I
\]

In this last expression we are identifying \( I := (I_1, \ldots, I_p) \in \wp(m)^{\times p} \) with the cyclic submodule of \( \Lambda_m^{\otimes p} \) generated by \( \varphi_{I_1} \otimes \ldots \otimes \varphi_{I_p} \); where \( \varphi_J := \varphi_{j_1} \wedge \ldots \wedge \varphi_{j_s} \) whenever \( J = \{j_1, \ldots, j_s\} \) and \( 0 \leq j_1 < \ldots < j_s < m \). Here \( \wp(m)^{\times p} \) is the cartesian product of the powerset of \( \{0, \ldots, m-1\} \) with itself \( p \)-times. We use this set as a set of indexes.
For two modules $A := \bigoplus_{I \in \wp(m)^\times p} A_I$ and $B := \bigoplus_{I \in \wp(m)^\times p} B_I$, indexed by the same set $\wp(m)^\times p$, we take

$$A \otimes B := \bigoplus_{I \in \wp(m)^\times p} A_I \otimes B_I$$

If we call

$$\tilde{O}_m(p) := \bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \ldots + i_p = r} \mathcal{O}(p) \otimes (\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m) \simeq \bigoplus_{I \in \wp(m)^\times p} \tilde{O}_I(p)$$

then, equation (6) can be also written as

$$F_mA \simeq \bigoplus_{p \geq 0} \tilde{O}_m(p) \otimes \tilde{A}_m^{\otimes p}$$

We can now look at the operad structure inherited by $\tilde{O}$. Take $p \geq 0$ and $k_1 + \ldots + k_p = k$. We have to define the operad action $\gamma : \tilde{O}_m(p) \otimes \tilde{O}_m(k_1) \otimes \ldots \otimes \tilde{O}_m(k_p) \to \tilde{O}_m(k)$. We do so in the following way,

$$\gamma(\tilde{O}_I(p) \otimes \tilde{O}_{I_1}(k_1) \otimes \ldots \otimes \tilde{O}_{I_p}(k_p)) :=
\begin{cases}
\gamma(\mathcal{O}(p) \otimes \mathcal{O}(k_1) \otimes \ldots \otimes \mathcal{O}(k_p)) \otimes J & \text{if } I = (\bigcup I_1, \ldots, \bigcup I_p) \\
0 & \text{in any other case}
\end{cases}$$

where $J = (I_1, \ldots, I_p) \in \wp(m)^\times k$. This formula together with multilinearity completely determines $\gamma$.

Since $m \mapsto F_mA$ is actually a simplicial abelian group, we can apply the normalized chain complex functor to pass from a simplicial $\mathbb{Z}$-module to a chain complex. This is to look at the elements in the kernel of all the faces except the last at each level in the simplicial module. The passage from simplicial $\mathbb{Z}$-modules to $\mathbb{Z}$-complexes carries an operad of simplicial $\mathbb{Z}$-modules to an operad of $\mathbb{Z}$-complexes ([9] pp. 36). We are interested in this last operad.

We know that a basic element $\sigma \otimes (a_1 \otimes \ldots \otimes a_p) \otimes (x_1 \otimes \ldots \otimes x_p) \in \mathcal{O}(p) \otimes (\hat{A}_{i_1} \otimes \ldots \otimes \hat{A}_{i_p}) \otimes (\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m)$ is in $\bigcap_{r=0}^{m-1} \ker d_r$ if and only if $\sharp x_1 \cup \ldots \cup \sharp x_p = \{0, \ldots, m-1\}$. Then

$$N_m F A \simeq \bigoplus_{p \geq 0} \bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \ldots + i_p = r} \mathcal{O}(p) \otimes (\hat{A}_{i_1} \otimes \ldots \otimes \hat{A}_{i_p}) \otimes N(\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m)$$

where $N(\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m)$ is a shorthand for the $\mathbb{Z}$-submodule of $(\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m)$ generated by the elements $(x_1 \otimes \ldots \otimes x_p)$ with $\sharp x_1 \cup \ldots \cup \sharp x_p = \{0, \ldots, m-1\}$. If we associate $(x_1 \otimes \ldots \otimes x_p) \in (\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m)$ with $(\sharp x_1, \ldots, \sharp x_p) \in \wp(m)^\times p$, we can put a base of $N(\Lambda^{i_1}_m \otimes \ldots \otimes \Lambda^{i_p}_m)$ in a one to one correspondence with the subset $\wp_m(m)^\times p$.
of \( \wp(m)^{\times p} \) whose elements \( I := (I_1, \ldots, I_p) \) are such that \( \bigcup I = \{0, \ldots, m-1\} \). We shall use \( \wp_m(m)^{\times p} \) as index set, and write

\[
\mathbf{N}( \bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \ldots + i_p = r} \mathcal{O}(p) \otimes (\Lambda^i_1 \otimes \ldots \otimes \Lambda^i_m) ) = \bigoplus_{I \in \wp_m(m)^{\times p}} \mathcal{O}[I](p) \quad (9)
\]

**Remark 4** Suppose that \( o \otimes (a_1 \otimes x_1) \otimes \ldots \otimes (x_1 \otimes a_p) \in \mathcal{O}(p) \otimes (\tilde{A}_i \otimes \Lambda^i_1) \otimes \ldots \otimes (\tilde{A}_p \otimes \Lambda^i_m) \simeq \mathcal{O}(p) \otimes (\tilde{A}_i \otimes \cdots \otimes \tilde{A}_i) \otimes (\Lambda^i_1 \otimes \ldots \otimes \Lambda^i_m) \). Then,

\[
d_m(o \otimes (a_1 \otimes x_1) \otimes \ldots \otimes (a_p \otimes x_p)) = o \otimes d_m(a_1 \otimes x_1) \otimes \ldots \otimes d_m(a_p \otimes x_p)
\]

\[
= o \otimes (a_1 \otimes d_m(x_1) + da_1 \otimes \delta_{d_m}(x_1)) \otimes \ldots \otimes (a_p \otimes d_m(x_p)) + da_p \otimes \delta_{d_m}(x_p)
\]

\[
= o \otimes (a_1 \otimes d_m(x_1)) \otimes \ldots \otimes (a_p \otimes d_m(x_p)) + \ldots + o \otimes (da_1 \otimes \delta_{d_m}(x_1)) \otimes \ldots \otimes (da_p \otimes \delta_{d_m}(x_p))
\]

This corresponds to the sum of all the elements of the form

\[
(o \otimes (\varepsilon'_1(x_1) \otimes \ldots \otimes \varepsilon'_p(x_p))) \otimes (\varepsilon''_1(a_1) \otimes \ldots \otimes \varepsilon''_p(a_p))
\]

where \( \varepsilon'_i \) is either \( d_m \) or \( \delta_{d_m} =: \delta_m \) and \( \varepsilon''_i \) is either 1 or \( d \), in accordance with the value of \( \varepsilon'_i \). Since \( \sharp x_1 \cup \ldots \cup \sharp x_p = \{0, \ldots, m-1\} \), the term with all \( \varepsilon'_i = d_m \) is zero.

### 4 Peiffer pairings in \( \mathcal{O} \)-hypercrossed modules

Let us suppose that for all \( m > 0 \),

\[
A_m = \text{Sub}_\mathcal{O}(\sum_{i=0}^{m} s_i(A_{m-1})) \quad (10)
\]

Here \( \text{Sub}_\mathcal{O}(X) \) means the sub-\( \mathcal{O} \)-algebra generated by the subset \( X \). Since the degeneracies are injective \( \mathcal{O} \)-morphisms, we have that \( s_i \mathcal{O}[I_1, \ldots, I_p] \simeq \mathcal{O}[s^*_i I_1, \ldots, s^*_i I_p] \). Hence, condition (10) can also be stated as

\[
\tilde{A}_m = \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O}[I] \otimes \tilde{A}_{i_1} \otimes \ldots \otimes \tilde{A}_{i_I})
\]

\[
= \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O}[I] \otimes (\tilde{A}_{i_1} \otimes I_1) \otimes \ldots \otimes (\tilde{A}_{i_I} \otimes I_{I_I}))
\]

or equivalently, as \( \gamma : \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O} \otimes (\tilde{A}_{i_1} \otimes I_1) \otimes \ldots \otimes (\tilde{A}_{i_I} \otimes I_{I_I})) \to \tilde{A}_m \) being surjective, where \( I = (I_1, \ldots, I_p) \) and \( I_i \neq \emptyset, [m-1] \) for all \( i \). Observe that \( (\tilde{A}_{i_j} \otimes I_j) \subseteq K_{I_j} \). Here we write \( K_I \) for the ideal \( \bigcap_{i \in I} \ker d_i \subseteq A_m \).
Lemma 5 Suppose (11) holds for the simplicial \( \mathcal{O} \)-algebra \( A \). Then, for each \( m \geq 0 \), the following inclusion also holds,

\[
d\tilde{A}_m \subseteq \sum_{\bigcup I = [m-2]} \gamma(\mathcal{O}_{|I|} \otimes K_{I_1} \otimes ... \otimes K_{|I|})
\]

PROOF. Apply \( d_m \) on both sides of (11). We get that

\[
d_m(\tilde{A}_m) = d_m \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O}_{|I|} \otimes (\tilde{A}_{I_1} \otimes I_1) \otimes ... \otimes (\tilde{A}_{|I|} \otimes I_{|I|}))
\]

\[
= \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O}_{|I|} \otimes d_m(\tilde{A}_{I_1} \otimes I_1) \otimes ... \otimes d_m(\tilde{A}_{|I|} \otimes I_{|I|}))
\]

(12)

The simplicial identity \( d_k d_m = d_m - 1 d_k \) if \( k < m \), implies \( d_m(\tilde{A}_{i_j} \otimes I_j) \subseteq K_{I_j} \). Hence, from (12) follows that

\[
d_m(\tilde{A}_m) \subseteq \sum_{\bigcup I = [m-2]} \gamma(\mathcal{O}_{|I|} \otimes K_{I_1} \otimes ... \otimes K_{|I|})
\]

The other inclusion was shown in [2] for the case \( \mathcal{O} = Comm \) and in [1] for the case \( \mathcal{O} = Lie \). Essentially the same proof can be adapted for a general \( \mathcal{O} \). We do this in the following

Proposition 6 Let \( A \) be a simplicial \( \mathcal{O} \)-algebra. Let \( I = (I_1, ..., I_p) \), with nonempty \( I_i \)'s and \( \bigcup_{i=1}^p I_i = [m-1] \). Then,

\[
\gamma(\mathcal{O}_p \otimes K_{I_1} \otimes ... \otimes K_{I_p}) \subseteq d\tilde{A}_m
\]

To prove this Proposition, we shall use the following Lemma, whose proof can be found in [5], [2] or [1].

Lemma 7 For a simplicial algebra \( A \), if \( 0 \leq r \leq n \) let \( \overline{NA}_n^{(r)} = \bigcap_{i \neq r} \ker d_i \). Then the map \( \psi : NA_n \rightarrow \overline{NA}_n^{(r)} \), given by

\[
\psi(a) := a - \sum_{k=0}^{n-r-1} s_{r+k} d_n a
\]

is a bijection.

In consequence, \( d_n(A_n) = d_r(\overline{NA}_n^{(r)}) \) for each \( n, r \).

PROOF. (of Proposition 6:) Let \( o \in \mathcal{O}_p \) and \( x_i \in K_{|I_i|}, i = 1, ..., p \). Suppose that \( \bigcup_i I_i = [m-1] \) and \( I_i \neq \emptyset \) for all \( i \). Let \( r \) be the smallest nonzero
element not in $\bigcap_k I_k$, and $i_0$ the first $i$ such that $r \in I_i$. Take $x = \gamma(o \otimes s_r x_1 \otimes ... \otimes s_r x_{i_0} \otimes ... \otimes s_r x_p)$. One obtains that $d_j x = 0$, for $j \neq r$ and

$$
\gamma(o \otimes x_1 \otimes ... \otimes x_{i_0} \otimes ... \otimes x_p) = d_r x \in d_r(\overline{\mathcal{A}_p^{(r)})} = d_n(A_n).
$$

Thus,

$$
\gamma(\mathcal{O}_p \otimes K_{I_1} \otimes ... \otimes K_{I_p}) \subseteq d_n \mathcal{A}_n
$$

We can joint both Lemma 5 and Proposition 6 in

**Theorem 8** Let $A$ be a simplicial $\mathcal{O}$-algebra such that $A_m = \text{Sub}_O(\sum_{i=0}^m s_i(A_{m-1}))$ for every $n$. Then

$$
d_{\tilde{A}_m} = \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O}_{|I|} \otimes K_{I_1} \otimes ... \otimes K_{I_{|I|}})
$$

**Remark 9** Suppose that $\mathcal{O} = \text{Comm}$ and $I = (I_1, ..., I_p)$ with $\bigcup_{i=1}^p I_i = [m-1]$. Recall that $\mathcal{O}_m \simeq \mathbb{Z}$ for all $m$. Composing and using the surjectivity of the product, we get that

$$
\sum_{\bigcup I = [m-1]} \gamma(\mathbb{Z} \otimes K_{I_1} \otimes ... \otimes K_{I_p}) = \sum_{\bigcup I = [m-1]} \gamma((\mathbb{Z} \otimes K_{I'} \otimes K_{I''})
$$

with $I' = \bigcup_{i=1}^p I_i, I'' = \bigcup_{i=q}^p I_i, 1 < q < p$. Hence

$$
\sum_{\bigcup I = [m-1]} \gamma(\mathbb{Z} \otimes K_{I_1} \otimes ... \otimes K_{I_p}) = \sum_{I' \cup I'' = [m-1]} K_{I'} K_{I''}
$$

Compare this last expression with that of [2]. Something similar happens with any quadratic operad.

## 5 Simplicial groups

The use of constructions involving near-rings in the study of simplicial groups is not new [4], even if the use of near-rings we do in this section seems not to appear before in the literature.

We begin by recalling some definitions from [12].

**Definition 10** A right distributive near-ring is a set $N$ together with two binary operations “$+$” and “$\cdot$” such that,

a. $(N, +, 0)$ is a (not necessarily abelian) group,

b. $(N, \cdot)$ is a semigroup,

c. $\forall l, m, n \in N \ (l + m) \cdot n = l \cdot n + m \cdot n$
$N$ is said to be zero-symmetric if $n \cdot 0 = 0$ for all $n$ in $N$. $N$ is unital if $(N, \cdot, 1)$ is a monoid. An element $d \in N$ is said to be distributive if for any $m, n \in N$, $d \cdot (m + n) = d \cdot m + d \cdot n$. A distributive unital zero-symmetric near ring is a ring.

Write $N_d$ for $\{d \in N / d$ is distributive$\}$. $(N_d, \cdot)$ is a sub-semigroup of $N$. We say that $N$ is distributively generated if $(N, +, 0)$ is generated by some subset $D \subseteq N_d$.

Let $X_m := \{\varphi_0, ..., \varphi_{m-1}\}$. Put $(F_m, +, 1)$ for the free monoid generated by $X_m$. Following [12], Definition 6.20, we take $(N_m, +, 0)$ for the free group on $F_m$, and endow it with the product

$$(\sum_i \sigma_i \varphi_i)(\sum_j \sigma_j \varphi_j) := \sum_i \sigma_i(\sum_j \sigma_j \varphi_i \varphi_j)$$

We call $(N_m, +, \cdot, 0, 1)$ the free distributively generated unital near-ring generated by the set $X_m$. Since $\left(\sum_i \sigma_i \varphi_i\right) \cdot 0 = (\sum_i \sigma_i \varphi_i) \cdot (1-1) = \sum_i \sigma_i \varphi_i \cdot (1-1) = \sum_i \sigma_i ((\varphi_i \cdot 1) - (\varphi_i \cdot 1)) = 0$, $N_m$ is also zero-symmetric. Put $(\Lambda(m), +, \cdot, 0, 1)$ for the free distributively generated unital zero-symmetric near-ring generated by the set $X_m$ also satisfying the relations

$$\varphi_i \cdot \varphi_j = -\varphi_j \cdot \varphi_i$$

We can endow $\Lambda(*)$ with a simplicial near-ring structure by formulas (1) and (2), where $+$ is now the not necessarily abelian group operation in $\Lambda(*)$. Note that this group is graded by the length of the word in the $\varphi$'s.

By forgetting the operation $\cdot$ in $\Lambda(*)$, we get a simplicial group $(\Lambda(*), +, 0)$, also written $\Lambda(*)$.

**Definition 11** Let $(G_*, d)$ be a connected chain complex of (not necessarily abelian) groups, and $A_*$ a family of graded groups. We write $G \boxtimes A$ for the sequence of groups $n \mapsto \coprod_{i \geq 0} (G_i \otimes A_n^i)$; where $G_i \otimes A_n^i$ is the group generated by the symbols $g \otimes a$ with $g \in G_i$, $a \in A_n^i$ and subject to the relations

$$g \otimes 0 \approx 1 \otimes a \approx 1 \otimes 0$$
$$g \otimes (a + b) \approx (g \otimes a)(g \otimes b)$$

and $\coprod$ is the coproduct in the category of groups.

We can endow $G \boxtimes \Lambda(*)$ with a simplicial group structure. With the notation of Definition 11, we associate to $\alpha \in \Delta(m, n)$, a morphism $\mathfrak{I}(\alpha) : G \boxtimes \Lambda(n) \to G \boxtimes \Lambda(m)$ by the formula

$$\mathfrak{I}(\alpha)(g \otimes x) := (dg \otimes \check{\alpha}(x))(g \otimes \alpha(x))$$
Here we put \( \bar{\alpha} : \Lambda(n) \to \Lambda(m) \) for the unique group morphism induced by \( \delta_n \); just as in the commutative case.

Essentially the same arguments in the proof of Proposition 2 apply to this case; just take care of terms’ order. Then we have that,

**Proposition 12** Let \( [m] \xrightarrow{\alpha} [n] \xrightarrow{\beta} [p] \in \Delta \), then \( \mathfrak{I}(\beta\alpha) = \mathfrak{I}(\alpha)\mathfrak{I}(\beta) \). In consequence, \( G \boxtimes \Lambda(*) \) is a simplicial group.

Let us take a closest view to faces and degeneracies in \( G \boxtimes \Lambda(*) \). Take \( g \otimes x \in G \boxtimes \Lambda(*) \), and write simply \( s_i \) and \( d_i \) for \( \mathfrak{I}(s_i) \) and \( \mathfrak{I}(d_i) \). Write for a monomial \( x \in \Lambda(n) \), \( \not\exists x := \{i_1, ..., i_r\} \) iff \( x \in \mathbb{Z} \varphi_{i_1}...\varphi_{i_r} \). For \( 0 \leq i \leq n \), we have that \( s_i(g \otimes x) = g \otimes s_i(x) \), for \( 0 \leq i \leq n-1 \), \( d_i(g \otimes x) = g \otimes d_i(x) \), and \( d_n(g \otimes x) = (dg \otimes d_n(x))(g \otimes d_n(x)) \). So, for \( i \neq n \), we can immediately say that \( g \otimes x \in \ker d_i \) if \( i \in \not\exists x \), just as it is the case for abelian groups. Observe that not all element in \( \ker d_i \) has to be of this form; for example, \( [g \otimes \varphi_i, h \otimes \varphi_j] \), is not of this form, although it is in \( \ker d_i \) (and in \( \ker d_j \)).

Let us now recall from [5] the following notation and result. Let \( I = \{i_1, ..., i_r\} \), with \( 0 \leq i_1 < ... < i_r \leq m \), or \( I = \emptyset \). We shall write \( s_I := s_{i_1}...s_{i_r} \) or \( 1 \), respectively, and call them the canonical inclusions. Similarly, we define \( d_I := d_{i_1}...d_{i_r} \) and \( d_\emptyset := 1 \).

Since the group is not necessarily commutative, we write \( \sum_I s_I(x_I) \) for the ordered sum of the \( s_I(x_I) \), according to the inverse lexicographical order.

The central result for us is,

**Proposition 13** Let \( G \) be a simplicial group, and \( NG \) its Moore complex. For every \( n > 1 \) each element \( x \in G_n \) admits a unique expression of the form

\[
x = \sum_{I \in \wp(n)} s_I(x_I) \quad \text{for } x_I \in N_{|I|}G
\]

such that the map

\[
\prod_{I \in \wp(n)} N_{|I|}G \to G_n
\]

given by \( (x_I)_{I \in \wp(n)} \mapsto \sum_{I \in \wp(n)} s_I(x_I) \) is a bijection.

Since \( NG \boxtimes \Lambda(*) \), as defined in 11, is itself a simplicial group, the results just enounced apply to it. Observe that \( g \in N_nG \) if and only if \( g \otimes \varphi_0...\varphi_{n-1} \in \mathbb{N}(NG \boxtimes \Lambda(*)) \), although not all the elements of \( \mathbb{N}(NG \boxtimes \Lambda(*)) \) are of this form. Take \( s_I(g_I) := s_I(g \otimes \varphi_0...\varphi_{n-1}) = g \otimes s_I(\varphi_0...\varphi_{n-1}) = \sum_i g \otimes \varphi^{(i)} \). The i-th term of this ordered sum is in \( \bigcap_{j \in \wp(n)} \ker d_i \in G \boxtimes \Lambda(n+|I|) \). On the other hand, any \( \varphi_J \), with \( J \subseteq [m-1] \), can be written as \( \varphi_J = \sum_i \varepsilon_i s_{j_i}(\varphi_{j_i}) \) for some \( 0 \leq j_i \leq m-1 \), \( J_i \subseteq [m-2] \) and \( \varepsilon_i = \pm 1 \). Indeed, the following Proposition
suppose

Now, take t > j / J out that J in I. We do induction on PROOF. In this case we can identify I with the complement of J. This construction of J allows us to identify I with a subset of [m − 2], with certain order. The construction of I is based on the following observations. If j = m − 2 then 

\[ \varphi_J = s_{m-1} \varphi_{[m-2]} \]

if j = m − 2 − 1 then 

\[ \varphi_J = -s_{m-1} \varphi_{[m-2]} + s_{m-1-1} \varphi_{[m-2]} \]

and in general, if j = m − 2 − q \(\in\) I then 

\[ \varphi_J = -\varphi_J + s_{m-1-q} \varphi_{[m-2]} \]

where \( J' = [m-2] - \{m-2-q+1\} \). In this way we get an effective recursive procedure to find the appropriate I. Observe that this procedure does not affect those \( \varphi_k \) with \( k < j \).

Now, take t > 1 and suppose that \( j_1 < \ldots < j_t \) are all the elements in the complement of J. Suppose that we have already built up \( I' \) such that 

\[ \varphi_{J'} = \sum_{I \subseteq [m]} \varepsilon_I s_I(\varphi_{[r-1]}) \]

with \( J' = [m-2] - \{j_1, \ldots, j_{t-1}\} \). Now we do apply the procedure firstly described to get 

\[ \varphi_J = \sum_{i \in I} \varepsilon_i s_i(\varphi_J) \]

We can do so, since this procedure is blind to the \( j \in J \) with \( j < j_t \). Finally, we get 

\[ \varphi_J = \sum_{i \in I} \varepsilon_i s_i(\sum_{I \subseteq [m]} \varepsilon_I s_I(\varphi_{[r-1]})) = \sum_{i \in I} \sum_{I \subseteq [m]} \varepsilon_i s_i(\varphi_{[r-1]}) \]

\[ = \sum_{I \subseteq [m]} \varepsilon_I s_I(\varphi_{[r-1]}) \]

Remark 15 The following observations, although trivial, may be useful.

Let \( I, J \subseteq [m] \), and G a simplicial group. Suppose that \( x, y \in G_m \) are such that \( x \in \cap_{i \in I} \ker d_i \) and \( y \in \cap_{j \in J} \ker d_j \). Then \( [x, y] \in \cap_{i \in I, j \in J} \ker d_i \).

On the other hand, 

\[ s_I(\varphi_{[r-1]}) = \sum_{l \in s_I^{-1}(0) \times \ldots \times s_I^{-1}(r-1)} \varphi_{\#l} \]

with \( s_I^{-1}(0) \times \ldots \times s_I^{-1}(r-1) \) lexicographically ordered, and \( \#l := \{l_0, ..., l_{r-1}\} \), whenever \( l = (l_0, ..., l_{r-1}) \).

Remark 16 (from [10]) Let \( x \in \mathbb{N}_n G \) and \( y \in G_{n-1} \). Take \( \theta_y(x) := s_{n-1}(y) x s_{n-1}(y^{-1}) : \mathbb{N}_n G \to G_n \). Since \( d_i \theta_y(x) = 1 \) for \( 0 \leq i \leq n-1 \),
θ_γ(x) ∈ N_nG. Furthermore, d_nθ_γ(x) = yd_n(x)y^{-1}, and in consequence, yd(x)y^{-1} ∈ d(N_nG). Hence, d(N_nG) is a normal subgroup of G_{n-1}.

We are now prompt to relate the construction in [5] with ours. We construct a morphism of groups Φ : NG ⊠ Λ → G. We do it degreewise. Φ_0 : N_0G → G_0 is simply the identity. Let us denote by Zϕ_I the cyclic subgroup generated by ϕ_I. The restriction of Φ_m to N_mG ⊠ Zϕ_{[m-1]} is the obvious isomorphism with N_mG ⊆ G_m. On the other hand, for J ⊂ [m-1], we have seen in Proposition 14 that ϕ_J = ∑_{I ∈ I} ε_I s_I(ϕ_{[r]}); then we define

Φ_m(g ⊗ ϕ_J) := ∑_{I ∈ I} ε_I s_I(g)

Since Φ_m is defined on each NG ⊠ Zϕ_I, Definition 11 and the universal property of the coproduct, allow us to extend it in a unique way to all of NG ⊠ Λ(m).

Lemma 17 The homomorphism Φ_m defined above is onto.

PROOF. Immediate from Proposition 13.

In fact, we have that

Proposition 18 Φ : NG ⊠ Λ → G is a surjective morphism of simplicial groups.

PROOF. The same definition of Φ guarantees it to commute with the degeneracies. So we must just verify it also commutes with the faces; that is to say, that for 0 ≤ i ≤ m,

\[ d_i \Phi_m = \Phi_{m-1} d_i \]

Since the elements of the form g ⊗ s_I(ϕ_{[r-1]}), with g ∈ N_rG generate G ⊠ Λ(m), it will suffice to see that (13) holds when evaluating at this elements. Suppose \( i \neq m \). Then,

\[ d_i \Phi_m(g ⊗ s_I(ϕ_{[r-1]})) = d_i s_I(g) \]

On the other hand,

\[ \Phi_{m-1} d_i(g ⊗ s_I(ϕ_{[r-1]})) = \Phi_{m-1}(g ⊗ d_i s_I(ϕ_{[r-1]})) = d_i s_I(g) \]

Hence they agree.

Suppose now that \( i = m \). On one hand, we have that

\[ d_m \Phi_m(g ⊗ s_I(ϕ_{[r-1]})) = d_m s_I(g) = s_I s_{r-1}(dg)s_I(g) \]
(see for example [5], pp. 123 or compute it). On the other hand,

\[
\Phi_{m-1} d_m (g \otimes s_I (\varphi_{[r-1]})) = \Phi_{m-1} (dg \otimes \bar{s}_I (\varphi_{[r-1]})) \Phi_{m-1} (g \otimes d_m s_I (\varphi_{[r-1]})) \\
= s_I s_{r-1} (dg) s_I (g)
\]

This finishes the proof.

**Remark 19** Let us consider the construction of Definition 11, for the case \(A = \Lambda\). We would like to get back the construction of Section 2 in the abelian case, even though there is nothing like a "distributivity on the left" for \(\otimes\) in definition 11. This situation can be amended by asking for new identities involving elements of the form \(gh \otimes a\); at least when \(A = \Lambda\). The problem with this approach is that we did not find a small nice set of such identities implying them all, although it is possible to describe them.

In the rest of this remark, we use the notation of Definition 11. It holds, in each \(G \otimes \Lambda(n)\), that \(gh \otimes \varphi_{[n-1]} = (g \otimes \varphi_{[n-1]}) (h \otimes \varphi_{[n-1]})\). Once we know this identity to hold, we have a procedure to express \(gh \otimes \varphi_I\) when \(I < [n-1]\) by using Proposition 14. We shall illustrate it with an example.

Let \(g, h \in G_1\), and consider \(g \otimes \varphi_i, h \otimes \varphi_i\), with \(i = 0, 1\), in \(G \otimes \Lambda(2)\). We want to calculate \(gh \otimes \varphi_0\) and \(gh \otimes \varphi_1\). First, observe that in \(G \otimes \Lambda(1)\) we have \(gh \otimes \varphi_0 = (g \otimes \varphi_0)(g \otimes \varphi_0)\). Then,

\[
gh \otimes \varphi_0 = s_1(gh \otimes \varphi_0) = s_1((g \otimes \varphi_0)(g \otimes \varphi_0)) = (g \otimes \varphi_0)(g \otimes \varphi_0)
\]

On the other hand,

\[
s_0(gh \otimes \varphi_0) = gh \otimes (\varphi_0 + \varphi_1) = (gh \otimes \varphi_0)(gh \otimes \varphi_1) = (h \otimes -\varphi_0)(g \otimes -\varphi_0)(gh \otimes \varphi_1)
\]

and

\[
s_0((g \otimes \varphi_0)(g \otimes \varphi_0)) = (g \otimes \varphi_0 + \varphi_1)(g \otimes \varphi_0 + \varphi_1) = (g \otimes \varphi_0)(g \otimes \varphi_1)(h \otimes \varphi_0)(h \otimes \varphi_1)
\]

Comparing last expressions we deduce that

\[
(gh \otimes \varphi_1) = (h \otimes -\varphi_0)(g \otimes -\varphi_0)(g \otimes \varphi_0)(g \otimes \varphi_1)(h \otimes \varphi_0)(h \otimes \varphi_1) \\
= (h \otimes -\varphi_0)(g \otimes \varphi_1)(h \otimes \varphi_0)(h \otimes \varphi_1) \\
= (g \otimes \varphi_1) (h \otimes \varphi_0) (h \otimes \varphi_1)
\]

Unfortunately, although relations for \(n > 2\), can be find in essentially the same way, they are not so neat as those just founded. Despite of this fact, they all reduces to "left distributivity" after abelianization.
6 Peiffer pairings in hypercrossed groups

It was shown in [10], Prop. 2.3.7 (see also [11]) that,

**Lemma 20** Let \( G \) be a simplicial group. If \( n \geq 2 \) and \( I, J \subseteq [n-1] \) with \( I \cup J = [n-1] \), we have that,

\[
[\bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j] \subseteq d(N_nG)
\]

We refer the interested reader to [10] for a proof of this Lemma. We shall concern in this section in proving the following

**Lemma 21** Let \( G \) be a simplicial group. Let \( D_n \) be the subgroup of \( G_n \) generated by the degenerate elements. If \( G_n = D_n \) for \( n \geq 2 \), then we have that

\[
d(N_nG) \subseteq \prod_{I \cup J = [n-1]} [\bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j]
\]

Before proving this lemma, we shall make a couple of remarks that make more clear the relationship between \( NG \boxtimes \Lambda \) and \( G \).

**Proposition 22** Let \( G \) be the simplicial group \( NH \boxtimes \Lambda \) for some \( H \in Grp^{\Lambda^{op}} \). We have the equality \( N_nG = N_n C_n \), where \( N_n \) is the normal subgroup of \( G_n \) generated by \( N_nH \otimes \mathbb{Z}\varphi[n-1] \), \( C_n = \tilde{C}_n \cap N_nG \) and \( \tilde{C}_n \) is the normal subgroup of \( G_n \) generated by \([N_{|I|}H \otimes \varphi, N_{|I|}H \otimes \varphi, J] \) with \( I, J \subset [n-1] \).

**PROOF.** Let us take \( x \in N_nG \). Each element of \( G_n \) is a product of the form \( x = x_1...x_r \), with \( x_i = g_i \otimes \varphi_i \) and \( I \subseteq [n-1] \). Since \( N_n \) is normal in \( N_nG \), \( N_nG/N_n \) is a group. Then \( \bar{x} = \bar{x}_1...\bar{x}_r \) with each \( \bar{x}_i \in N_nG/N_n \), the image of some \( x_i \) not in \( N_n \). For any \( 0 \leq j \leq n-1 \), we have that \( d_j(\bar{x}_1...\bar{x}_r) = 1 \) and as \( \bar{x}_i \notin N_n \), there exists \( 0 \leq k \leq n-1 \) such that \( d_k(\bar{x}_i) \neq 1 \).

Take \( i \) such that \( d_i(\bar{x}_i) \neq 1 \), and call \( y_i \) the elements of \( \{\bar{x}_1, ..., \bar{x}_r\} \) not in the kernel of \( d_i \). This set is not void because we have, for example, \( y_1 = \bar{x}_1 \). Modulo commutators, we have that \( \bar{x} = y_1...y_q \). Since \( d_i(\bar{x}) = 1 \), we deduce that \( y_1...y_q = 1 \), and hence, \( \bar{x}_1...\bar{x}_{i_q} = 1 \) modulo commutators. Then \( \bar{x} \in C_n \).

**Proposition 23** With the notation of Proposition 22. We have that \( N_n \cap D_n = 1 \).

**PROOF.** Write \( G_n = \bigsqcup_{I \subseteq [n-1]} N_{|I|}H \otimes \mathbb{Z}\varphi I = (N_nH \otimes \mathbb{Z}\varphi[n-1]) \bigsqcup \bigsqcup_{I \subseteq [n-1]} N_{|I|}H \otimes \mathbb{Z}\varphi I \). The proposition follows from the freeness of the product.
and the fact that $D_n = \prod_{I \subseteq [n-1]} N_{[n]} H \otimes Z \varphi_I$.

**Remark 24** Observe that the condition $G_n = D_n$ may be written as a condition on $NG \otimes \Lambda$. Indeed, $G_n = D_n$ if and only if for every $x \in N_n$ there exists $y \in C_n$ such that $\Phi_n(x) = \Phi_n(y)$, where $\Phi$ is the morphism of Proposition 18.

**PROOF.** (of Lemma 21): Suppose that $g \in N_n G$. Then $g = \Phi_n (g \otimes \varphi_{[n-1]})$. By Remark 24, there is an $x \in C_n$ such that $\Phi_n(x) = \Phi_n (g \otimes \varphi_{[n-1]}) = g$. Since $\Phi$ is a morphism of simplicial groups we have that $d_n(g) = d_n(\Phi_n(x)) = \Phi_n^{-1}(d_n(x))$. Since $x \in C_n$, $x = x_1...x_p$ with $x_i = [y_i, z_i]$ for $1 \leq i \leq p$, where $y_i \in K_{I_i}$, $z_i \in K_{J_i}$, $I_i \cup J_i = [n-1]$ and $I_i, J_i \neq [n-1]$. Then

$$d_n(g) = \Phi_n^{-1}(d_n x) = \Phi_n^{-1}(d_n x_1)\ldots \Phi_n^{-1}(d_n x_p) = \Phi_n^{-1}(d_n x[y_1, z_1])\ldots \Phi_n^{-1}(d_n y_p, z_p) = [\Phi_n^{-1}(d_n y_1), \Phi_n^{-1}(d_n z_1)]\ldots [\Phi_n^{-1}(d_n y_p), \Phi_n^{-1}(d_n z_p)].$$

Since $d_j d_n = d_{n-1} d_j$ if $j < n$, we conclude that $\Phi_n^{-1}(d_n y_i) \in K_{I_i}$ and $\Phi_n^{-1}(d_n z_i) \in K_{J_i}$ for every $i$. Hence $d_n(g) \in \prod_{I\cup J = [n-1]} [K_I, K_J]$.

We can collect previous results in the following

**Theorem 25** Let $G$ be a simplicial group with Moore complex $NG$ in which $G_n = D_n$, is the normal subgroup of $G_n$ generated by the degenerate elements in dimension $n$, then

$$d(N_n G) = \prod_{I, J \subseteq [n-1]} \ker d_i \cap \ker d_j$$

for $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$.

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