Phase-space Quantization of Field Theory

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1. Introduction

The third complete autonomous formulation of Quantum Mechanics, distinct from conventional Hilbert space or path-integral quantization, is based on Wigner’s phase-space distribution function (WF), which is a special representation of the density matrix. In this formulation, known as deformation quantization, expectation values are computed by integrating mere c-number functions in phase-space, the WF serving as a distribution measure. Such phase-space functions multiply each other through the pivotal $\star$-product, which encodes the noncommutative essence of quantization. The key principle underlying this quantization is the $\star$-product’s operational isomorphism to the conventional Heisenberg operator algebra of quantum mechanics.

Below, we address gauge invariance in phase-space through canonical transformation to and from free systems. Further, we employ the $\star$-unitary evolution operator, a “$\star$-exponential”, to specify the time propagation of Wigner phase-space distribution functions. The answer is known to be remarkably simple for the harmonic oscillator WF, and consists of classical rigid rotation in phase-space for the full quantum system. It serves as the underpinning of the generalization to field theory we consider, in which the dynamics is specified through the evolution of c-number distributions on field phase-space. We start by illustrating the basic concepts in 2d phase-space, without loss of generality; then, in generalizing to field theory, we make the transition to infinite-dimensional phase-space.

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Wigner functions are defined by

\[ f(x, p) = \frac{1}{2\pi} \int dy \psi^*(x - \frac{\hbar}{2} y) e^{-iyp} \psi(x + \frac{\hbar}{2} y). \]  

(1.1)

Even though they amount to spatial auto-correlation functions of Schrödinger wavefunctions \( \psi \), they can be determined without reference to such wavefunctions, in a logically autonomous structure. For instance, when the wavefunction is an energy \( (E) \) eigenfunction, the corresponding WF is time-independent and satisfies the two-sided energy *-genvalue equations

\[ H \star f = f \star H = Ef, \]

(1.2)

where \( \star \) is the essentially unique associative deformation of ordinary products on phase-space,

\[ \star \equiv e^{\frac{\hbar}{2}(\partial_x \partial_p - \partial_p \partial_x)}. \]

(1.3)

It is an exponentiation of the Poisson Bracket (PB) kernel, introduced by Groenewold \(^6\) and developed in two important papers \(^5\). Since it involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments, \( f(x, p) \star g(x, p) = f(x + \frac{\hbar}{2} \partial_p, p - \frac{\hbar}{2} \partial_x) g(x, p) \), to yield pseudodifferential equations.

These WFs are real. They are bounded by the Schwarz inequality \(^8\) to \(-2/\hbar \leq f \leq 2/\hbar\). They can go negative, and, indeed, they do for all but Gaussian configurations, negative values being a ready hallmark of interference. Thus, strictly speaking, they are not probability distributions \(^3\). However, upon integration over \( x \) or \( p \), they yield marginal probability densities in \( p \) and \( x \)-space, respectively. They can also be shown to be orthonormal \(^7\), \(^2\). Unlike in Hilbert space quantum mechanics, naive superposition of solutions of the above does not hold, by virtue of Takabayashi’s \(^9\), \(^8\) fundamental nonlinear projection condition \( f \star f = f/\hbar \).

Anyone beyond the proverbial jaded sophisticate should value the ready intuition on the shape of the WF based on the shape of the underlying configuration \( \psi(x) \). Fix a point \( x_0 \), and reflect \( \psi(x) \) around it, \( \psi(x_0 + (x - x_0)) \mapsto \psi(x_0 - (x - x_0)). \) Then, overlap this with the starting configuration \( \psi(x) \), to survey where the overlap vanishes, and where it is substantial, thereby obtaining a qualitative picture of the support of the WF at \( x_0 \). Thus, eg, it is evident by inspection that the WF \( f(x, p) \) vanishes outside the absolute outer limits of support of the underlying \( \psi(x) \). However, a bimodal \( \psi(x) \) (one consisting of two separated bumps) will evidently yield \( f(x, p) \) with nonvanishing support (“interference”) in the intermediate region between the two bumps in \( \psi(x) \), even though \( \psi(x) \) itself vanishes there.

\section{2. Gauge Systems}

While the question of adapting the WF formalism to gauge systems has been addressed in the literature \(^4\), and interesting proposals have been made about accommodating nontrivial configurations involving Berry’s phase \(^3\), the problem has still not been settled in its full generality. Here, the straightforward solution for
gauge variation limited to a merely 2d phase-space will be provided, by canonical mapping to a free hamiltonian. This solution does not cover the more interesting topologically nontrivial situations which arise in higher dimensions, and which still languish in ambiguity.

For notational simplicity, take $\hbar = 1$ in this section. The area element in phase-space is preserved by canonical transformations

$$ (x, p) \mapsto (X(x, p), P(x, p)),$$

which yield trivial Jacobians ($dXdP = dxdp \{X, P\}$) by preserving the PBs,

$$\{u, v\}_{xp} \equiv \frac{\partial u}{\partial x} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial x}. $$

They thus preserve the “canonical invariants” of their functions, $\{X, P\}_{xp} = 1$, hence $\{x, p\}_{XP} = 1$. Equivalently,

$$\{x, p\} = \{X, P\},$$

in any basis. Motion being a canonical transformation, Hamilton’s classical equations of motion are preserved, for $\mathcal{H}(X, P) \equiv H(x, p)$, as well. What happens upon quantization?

Since, in deformation quantization, the hamiltonian is a c-number function, and so transforms “classically”, $\mathcal{H}(X, P) \equiv H(x, p)$, the effects of a canonical transformation on the quantum theory will be carried by a “covariantly” transformed Wigner function, not the classical transform of it that is frequently demonstrated to be unworkable in the literature. Naturally, the answer can be deduced from Dirac’s quantum transformation theory.

Consider the classical canonical transformations specified by an arbitrary generating function $F(x, X)$:

$$p = \frac{\partial F(x, X)}{\partial x}, \quad P = - \frac{\partial F(x, X)}{\partial X}. $$

Following Dirac’s celebrated exponentiation of such a generator, Schrödinger’s energy eigenfunctions transform canonically through a generalization of the “representation-changing” Fourier transform. Namely,

$$\psi_E(x) = N_E \int dX e^{iF(x, X)} \psi_E(X).$$

Thus, the pair of Wigner functions in the respective canonical variables, $f(x, p)$ and

$$\mathcal{F}(X, P) = \frac{1}{2\pi} \int dY \Psi^* (X - \frac{Y}{2}) e^{-iYP} \Psi (X + \frac{Y}{2}),$$

are connected through a convolution by a transformation functional,

$$f(x, p) = \int dX \int dP \mathcal{T}(x, p; X, P) \mathcal{F}(X, P).$$
(In proof, it has come to our attention that this equation has already been given in Ref.\(^\text{14}\)). In Curtright et al.\(^\text{2}\), this transformation functional was evaluated to be

\[
T(x, p; X, P) = |N|^2 \frac{1}{2\pi} \int dYdy \exp \left( -ipy + iYP - iF^*(x - \frac{y}{2}, X - \frac{Y}{2}) + iF(x + \frac{y}{2}, X + \frac{Y}{2}) \right). \tag{2.8}
\]

Now consider the succession of two canonical transformations with generating functionals \(F_1\) and \(F_2\), respectively,\(^\text{12}\), useful for generating all other classes of canonical transformations,

\[
(x, p) \mapsto (X, P) \mapsto (\chi, \pi). \tag{2.9}
\]

By inspection of (2.7, 2.8), it follows that the corresponding transformation functional \(T(x, p; \chi, \pi)\) is

\[
T(x, p; \chi, \pi) = \int dXdP T_1(x, p; X, P) T_2(X, P; \chi, \pi) \tag{2.10}
\]

\[
= \frac{|N_1 N_2|^2}{(2\pi)^2} \int dX dP dY dy dw dW \exp (-ipy + iYP - iwP + iW\pi
\]
\[ -iF_1^*(x - \frac{y}{2}, X - \frac{Y}{2}) + iF_1(x + \frac{y}{2}, X + \frac{Y}{2}) - iF_2^*(X - \frac{w}{2}, \chi - \frac{W}{2}) + iF_2(X + \frac{w}{2}, \chi + \frac{W}{2}) ),
\]

and, performing the trivial \(P\), whence \(w\) integrations,

\[
= \frac{|N|^2}{2\pi} \int dy dW \exp \left( -ipy + iW\pi - iF^*(x - \frac{y}{2}, \chi - \frac{W}{2}) + iF(x + \frac{y}{2}, \chi + \frac{W}{2}) \right),
\]

where the effective generator \(F\) is specified by Dirac’s integral expression:

\[
\exp(iF(x, \chi)) = \int dX \exp(iF_1(x, X) + iF_2(X, \chi)). \tag{2.11}
\]

The classical result for the concatenation of two canonical transformations would require that the exponent of the integrand does not depend on \(X\), so that the momenta \(P\) at the intermediate phase space point coincide. Instead, quantum mechanically, Dirac’s celebrated result dictates integration over all intermediate points, as seen.

A succession of several intermediate points \((X_i, P_i)\) integrated over naturally generalizes to entire paths of canonical transformations; their effective generators entering in (2.8) are given by the usual Dirac path integral expressions generalizing the above. For motion, all generators are the same function, \(F = S(x = x(t_i), X = x(t_f))\), the action increment. Thus, the generalization yields the familiar Dirac/Feynman functional integral propagator. Consequently, each wavefunction in (2.6) propagates independently by its respective Feynman effective action.

Finally, consider the canonical transformation from the free hamiltonian \(H = \frac{p^2}{2m}\) to the one for a minimally coupled particle in a “magnetic” field, \(H = (\pi - eA(\chi))^2 / 2m\), in one space dimension. In the formalism outlined, this is effected by a succession of two canonical transformations,

\[
(x, p) \mapsto (X = -p, P = x) \mapsto (\chi = x, \pi = p + eA(x)), \tag{2.12}
\]
generated by \( F_1 = -x X \) and \( F_2 = X \chi - e \int dz \ A(z) \), respectively.

Consequently, the effective exponentiated generator \((2.11)\) is

\[
\exp(iF(x, \chi)) = 2\pi \delta(x - \chi) \exp \left( -ie \int^x dz A(z) \right). \tag{2.13}
\]

Thus, the gauge-invariant eigenfunction \( \psi \) of the free hamiltonian transforms to the gauge-variant eigenfunction \( \Psi \) of the EM hamiltonian,

\[
\psi(x) = e^{-ie \int^x dz A(z)} \Psi(x). \tag{2.14}
\]

Hence, the transformation functional \((2.8)\) reduces to

\[
T(x, p; X, P) = \frac{1}{2\pi} \int dY \exp \left( -iY(p - P) - ie \int_{x - Y/2}^{x + Y/2} dz A(z) \right) \delta(x - X). \tag{2.15}
\]

As a result, the free-particle WF canonically transforms to the gauge-invariant one for a particle in the presence of a vector potential, already considered by \(^{[10]}\) (with \( \hbar \) reinstated for completeness):

\[
f(x, p) = \frac{1}{2\pi} \int dy \Psi^*(x - \frac{\hbar}{2}y) e^{-ipy - (ie/\hbar) \int_{x - \hbar y/2}^{x + \hbar y/2} dz A(z)} \Psi(x + \frac{\hbar}{2}y). \tag{2.16}
\]

Of course, this gauge-invariant WF is controlled by the Moyal equation (detailed in the next section) driven by the free \( H \), not \( \mathcal{H} \), which controls \( \mathcal{F} \) instead.

The above discussion is only a starting point of setting up gauge invariance in phase space. By contrast to this discussion, in more than one space dimensions, the line integral of the vector potential is ambiguous, and straight paths from every point \( x \) to all endpoints (involving \( y \), integrated over) are inequivalent, in general, to alternate paths in nontrivial field settings. One should then not expect these problems to be canonically equivalent to free ones. The motivated reader is referred to Pati\(^{[11]}\).

§3. Time Evolution

Time-dependence for WFs was succinctly specified as flows in phase-space by Moyal through the commutator bracket\(^{[4]}\) bearing his name,

\[
i\hbar \frac{\partial}{\partial t} f(x, p; t) = H \star f(x, p; t) - f(x, p; t) \star H. \tag{3.1}
\]

This turns out to be the essentially unique associative generalization\(^{[5]}\) of the PB, to which it reduces as \( \hbar \to 0 \), yielding Liouville’s theorem of classical mechanics, \( \partial_t f + \{ f, H \} = 0 \).

For the evolution of the fundamental phase-space variables \( x \) and \( p \), time evolution is simply the convective part of Moyal’s equation, so the apparent sign is

\(^{*}\) The question was asked at the talk what generalizes the \( \star \)- product when redundant variables are present, notably for Dirac Brackets (DB); but it has not been answered. Eg, on a hypersphere \( S^n \), if the redundant variables are eliminated, one deals with simple coordinate changes for PB, as usual.
reversed, while the Moyal Bracket actually reduces to the PB. That is, the $h$-dependence drops out, and these variables, in fact, evolve simply by the classical Hamilton’s equations of motion, $\dot{x} = \partial_p H$, $\dot{p} = -\partial_x H$.

What is the time-evolution of a WF like? This is the first question answered through the isomorphism of $\ast$-multiplication associative combinatorics to the parallel algebraic manipulations of quantum mechanical operators, which are familiar. Eqn (3.1) can be solved for the time-trajectories of the WF, which turn out to be notably simple. By virtue of the $\ast$-unitary evolution operator, a “$\ast$-exponential”

$$U_\ast(x,p; t) = e^{itH/\hbar_\ast} \equiv 1 + \frac{(it/\hbar_\ast)^2}{2!} H \ast H + \frac{(it/\hbar_\ast)^3}{3!} H \ast H \ast H + \ldots,$$

the time-evolved Wigner function is obtainable formally in terms of the Wigner function at $t = 0$ through associative combinatoric operations completely analogous to the conventional formulation of quantum mechanics of operators in Hilbert space. Specifically,

$$f(x,p; t) = U_\ast^{-1}(x,p; t) \ast f(x,p; 0) \ast U_\ast(x,p; t).$$

As indicated, the dynamical variables evolve classically,

$$\frac{dx}{dt} = x \ast H - H \ast x \bigg/ \frac{i\hbar}{\hbar_\ast} = \partial_p H,$$

and

$$\frac{dp}{dt} = p \ast H - H \ast p \bigg/ \frac{i\hbar}{\hbar_\ast} = -\partial_x H.$$

Consequently, by associativity, the quantum evolution,

$$x(t) = U_\ast \ast x \ast U_\ast^{-1},$$

$$p(t) = U_\ast \ast p \ast U_\ast^{-1},$$

turns out to flow along classical trajectories.

What can be said about this formal time-evolution expression? If the WF can be written as a $\ast$-function, i.e., a sum of $\ast$-products of the phase-space variables, then associativity will permit application of the above $\ast$-similarity transformation throughout the WFs.

Any WF in phase-space, upon Fourier transformation resolves to

$$f(x,p) = \int dadb \tilde{f}(a,b) e^{i\alpha x} e^{ibp}.$$

But if, instead, the redundant variables are retained, subject to the constraint $x_0^2 + x_1^2 + x_2^2 + \ldots + x_n^2 = 1$, and hence $x_0p_0 + x_1p_1 + x_2p_2 + \ldots + x_np_n = 0$, the PBs are supplanted by the DBs suitable to this isospin hypersphere,

$$\{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} - x_i x_j, \quad \{p_i, p_j\} = x_j p_i - x_i p_j.$$

The kernel realizing these brackets is

$$\ast \equiv \hat{\partial}_x \cdot \hat{\partial}_p - \hat{\partial}_x \cdot x \cdot \hat{\partial}_p - \hat{\partial}_p \cdot x \cdot \hat{\partial}_x + \hat{\partial}_p \cdot x \cdot x \cdot \hat{\partial}_x + \hat{\partial}_p \cdot \hat{\partial}_p \cdot x \cdot x \cdot \hat{\partial}_p - \hat{\partial}_p \cdot \hat{\partial}_p \cdot x \cdot \hat{\partial}_p \cdot \hat{\partial}_p.$$

All exponentiation assignments of this kernel examined so far have not yielded an associative modification of the $\ast$-product. For a different approach, see Antonsen\cite{17}.\footnote{\cite{17}}.
However, note that exponentials of individual functions of $x$ and $p$ are also $\star$-exponentials of the same functions, or $\star$-versions of these functions, since the $\star$-product trivializes in the absence of a conjugate variable, so that

$$e^{iax} e^{ibp} = e^{i\star ax} e^{i\star bp}. \quad (3.9)$$

Moreover, this is proportional to a $\star$-product, since

$$e^{i\star ax} \star e^{i\star bp} = e^{i\star a(x + i\hbar \partial_p/2)} e^{i\star bp} = e^{i\star ax} e^{i\star bp} e^{-i\hbar a/2}. \quad (3.10)$$

Consequently, any Wigner function can be written as

$$f(x,p) = \int dadb \tilde{f}(a,b) e^{i\star ax} \star e^{i\star bp}. \quad (3.11)$$

It follows then, that, by insertion of $U_\star \star U_\star^{-1}$ pairs at every $\star$-multiplication, in general,

$$f(x,p; t) = \int dadb \tilde{f}(a,b) e^{i\hbar a/2} e^{i\star ax} \star U_\star \star e^{i\star bp} \star U_\star^{-1}. \quad (3.12)$$

Unfortunately, in general, the above steps cannot be simply reversed to yield an integrand of the form $f(x,p) e^{i\star ax(-t)} e^{i\star bp(-t)}$. But, in some limited fortuitous circumstances, they can, and in this case the evolution of the Wigner function reduces to merely backward evolution of its arguments $x, p$ along classical trajectories, while its functional form itself remains unchanged:

$$f(x,p; t) = f(x(-t), p(-t); 0). \quad (3.13)$$

To illustrate this, consider the simple linear harmonic oscillator (taking $m = 1$, $\omega = 1$),

$$H = \frac{p^2 + x^2}{2} = \frac{x - ip}{\sqrt{2}} \star \frac{x + ip}{\sqrt{2}} + \hbar/2. \quad (3.14)$$

It is easily seen that

$$i\hbar \dot{x} = x \star H - H \star x = i\hbar p, \quad i\hbar \dot{p} = p \star H - H \star p = -i\hbar x, \quad (3.15)$$

and thus the canonical variables indeed evolve classically:

$$X \equiv x(t) = U_\star \star x \star U_\star^{-1} = x \cos t + p \sin t,$$

$$P \equiv p(t) = U_\star \star p \star U_\star^{-1} = p \cos t - x \sin t. \quad (3.16)$$

This also checks against the explicitly evaluated $\star$-exponential for the SHO, $e^{iH/\hbar} = \frac{1}{\cos(t/2)} \exp(2t \tan(t/2) H/\hbar)$.

Now, recall the degenerate reduction of the Baker-Campbell-Hausdorff combinatoric identity for any two operators with constant commutator with respect to
any associative multiplication, thus for any phase-space functions \( \xi \) and \( \eta \) under \( \star \)-multiplication. If
\[
\xi \star \eta - \eta \star \xi = c,
\]
then,
\[
e^{\xi \star \eta} = e^{\xi} e^{\eta^{c/2}}.
\]
Application of this identity as well as (3.10) and (3.9) yields directly
\[
e^{iax(-t)} \star e^{ibp(-t)} e^{i\hbar t/2} = e^{i(a \cos t + b \sin t)x + i(b \cos t - a \sin t)p} e^{i\hbar (a \cos t + b \sin t)(b \cos t - a \sin t)/2} \]
\[
ed \star e^{i(b \cos t - a \sin t)p} = e^{i(a \cos t + b \sin t)x} e^{i(b \cos t - a \sin t)p}.
\]
Consequently,
\[
f(x, p; t) = \int da db \tilde{f}(a, b) e^{iax(-t)} e^{ibp(-t)},
\]
and hence the reverse convective flow (3.13) indeed obtains for the SHO.

The result for the SHO is the preservation of the functional form of the Wigner distribution function along classical phase-space trajectories:
\[
f(x, p; t) = f(x \cos t - p \sin t, p \cos t + x \sin t; 0).
\]
What this means is that any Wigner distribution rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete quantum mechanical description.
Naturally, this rigid rotation in phase-space preserves areas, and thus illustrates the uncertainty principle. By contrast, in general, in the conventional formulation of quantum mechanics, this result is deprived of intuitive import, or, at the very least, simplicity: upon integration in $x$ (or $p$) to yield usual probability densities, the rotation induces apparent complicated shape variations of the oscillating probability density profile, such as wavepacket spreading (as evident in the shadow projections on the $x$ and $p$ axes of the figure). Only when (as is the case for coherent states) a Wigner function configuration has an additional axial $x-p$ symmetry around its own center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density. (In the figure, a rectangle is taken to represent a generic configuration, and a small circle to represent such a coherent state.)

The result (3.21), of course, is not new. It was clearly recognized by Wigner. It follows directly from (3.1) for (3.14) that

$$ (\partial_t + p\partial_x - x\partial_p) f(x, p; t) = 0. $$

(3.22)

The characteristics of this partial differential equation correspond to the above uniform rotation in phase space, so it is easily seen to be solved by (3.21). The result was given explicitly in [6] and also [20], following different derivations. Lesche [21], has also reached this result in an elegant fifth derivation, by noting that for quadratic Hamiltonians such as this one, the linear rotation of the dynamical variables (3.16) leaves the symplectic quadratic form invariant, and thus the $\star$-product invariant. That is, the gradients in the $\star$-product may also be taken to be with respect to the time-evolved canonical variables (3.16), $X$ and $P$; hence, after inserting $U_{\star} \star U_{\star}^{-1}$ in the $\star$-functional form of $f$, the $\star$-products may be taken to be with respect to $X$ and $P$, and the functional form of $f$ is preserved. (3.13). This only holds for quadratic Hamiltonians, which thus generate linear canonical transformations.

Dirac’s interaction representation may then be based on this property, for a general Hamiltonian consisting of a basic SHO part, $H_0 = (p^2 + x^2)/2$, supplemented by an interaction part,

$$ H = H_0 + H_I. $$

(3.23)

Now, upon defining

$$ w \equiv e^{iH_0/\hbar} \star f \star e^{-iH_0/\hbar}, $$

(3.24)

it follows that Moyal’s evolution equation reads,

$$ i\hbar \frac{\partial}{\partial t} w(x, p; t) = \mathcal{H}_I \star w(x, p; t) - w(x, p; t) \star \mathcal{H}_I, $$

(3.25)

where $\mathcal{H}_I \equiv e^{iH_0/\hbar} \star H_I \star e^{-iH_0/\hbar}$. Expressing $H_I$ as a $\star$-function leads to a simplification.

In terms of the convective variables (3.16), $X, P$, $\mathcal{H}_I(x, p) = H_I(X, P)$, and $w(x, p; t) = f(X, P; t)$, while $\star$ may refer to these convective variables as well. Finally, then,

$$ i\hbar \frac{\partial}{\partial t} f(X, P; t) = H_I(X, P) \star f(X, P; t) - f(X, P; t) \star H_I(X, P). $$

(3.26)
In the uniformly rotating frame of the convective variables, the WF time-evolves according to the interaction Hamiltonian—while, for vanishing interaction Hamiltonian, \( f(X, P; t) \) is constant and yields (3.21). Below, in generalizing to field theory, this provides the basis of the interaction picture of perturbation theory, where the canonical fields evolve classically as above.

§4. Scalar Field Theory in Phase Space

To produce Wigner functionals in scalar field theory, one may start from the standard, noncovariant, formulation of field theory in Hilbert space, in terms of Schrödinger wave-functionals.

For a free field Hamiltonian, the energy eigen-functionals are Gaussian in form. For instance, without loss of generality, in two dimensions (\( x \) is a spatial coordinate, and \( t = 0 \) in all fields), the ground state functional is

\[
\Psi[\phi] = \exp \left( -\frac{1}{2\hbar} \int dx \phi(x) \sqrt{m^2 - \nabla_x^2} \phi(x) \right),
\]  

Boundary conditions are assumed such that the \( \sqrt{m^2 - \nabla_x^2} \) kernel in the exponent is naively self-adjoint. “Integrating by parts” one of the \( \sqrt{m^2 - \nabla_x^2} \) kernels, functional derivation \( \frac{\delta}{\delta \phi(z)} \Psi[\phi] = -\left( \sqrt{m^2 - \nabla_x^2} \phi(z) \right) \Psi[\phi] \),

\[
\hbar \frac{\delta}{\delta \phi(z)} \Psi[\phi] = -\left( \sqrt{m^2 - \nabla_x^2} \phi(z) \right) \Psi[\phi],
\]

\( \hbar^2 \frac{\delta^2}{\delta \phi(w) \delta \phi(z)} \Psi[\phi] =
\]

\[
= \left( \sqrt{m^2 - \nabla_w^2} \phi(w) \right) \left( \sqrt{m^2 - \nabla_z^2} \phi(z) \right) \Psi[\phi] - \hbar \sqrt{m^2 - \nabla_w^2} \delta(w - z) \Psi[\phi].
\]

Note that the divergent zero-point energy density,

\[
E_0 = \frac{\hbar}{2} \lim_{w \to z} \sqrt{m^2 - \nabla_z^2} \delta(w - z),
\]  

may be handled rigorously using \( \zeta \)-function regularization.

Leaving this zero-point energy present leads to the standard energy eigenvalue equation, again through integration by parts,

\[
\frac{1}{2} \int dz \left( -\hbar^2 \frac{\delta^2}{\delta \phi(z)^2} + \phi(z) \left( m^2 - \nabla_z^2 \right) \phi(z) \right) \Psi[\phi] = E_0 \Psi[\phi].
\]

A natural adaptation to the corresponding Wigner functional is the following. For a functional measure \( [d\eta/2\pi] = \prod_x d\eta(x)/2\pi \), one obtains

\[
W[\phi, \pi] = \int \left[ \frac{d\eta}{2\pi} \right] \Psi^* \left[ \phi - \frac{\hbar}{2\eta} \right] e^{-i \int dx \eta(x) \pi(x)} \Psi \left[ \phi + \frac{\hbar}{2\eta} \right],
\]  

\( \ast \) One may note alternate discussions of field theoretic interaction representations in phase-space, which do not appear coincident with the one to be presented here.
where \( \pi (x) \) is to be understood as the local field variable canonically conjugate to \( \phi (x) \). However, in this expression, both \( \phi \) and \( \pi \) are classical variables, not operator-valued fields, in full analogy to the phase-space quantum mechanics already discussed.

For the Gaussian ground-state wavefunctional, this evaluates to

\[
W [\phi, \pi] = \int \left[ \frac{d\eta}{2\pi} \right] \exp \left( -\frac{1}{2\hbar} \int dx \left( \phi (x) - \frac{\hbar}{2} \eta (x) \right) \sqrt{m^2 - \nabla_x^2} \left( \phi (x) - \frac{\hbar}{2} \eta (x) \right) \right) \times \\
\times e^{-i \int dx \eta(x) \pi(x)} \exp \left( -\frac{1}{2\hbar} \int dx \left( \phi (x) + \frac{\hbar}{2} \eta (x) \right) \sqrt{m^2 - \nabla_x^2} \left( \phi (x) + \frac{\hbar}{2} \eta (x) \right) \right)
\]

\[
= \exp \left( -\frac{1}{\hbar} \int dx \phi (x) \sqrt{m^2 - \nabla_x^2} \phi (x) \right) \times \\
\times \left( \int \left[ \frac{d\eta}{2\pi} \right] e^{-i \int dx \eta(x) \pi(x)} \exp \left( -\frac{\hbar}{4} \int dx \eta (x) \sqrt{m^2 - \nabla_x^2} \eta (x) \right) \right).
\]

So

\[
W[\phi, \pi] = \mathcal{N} \exp \left( -\frac{1}{\hbar} \int dx \left( \phi (x) \sqrt{m^2 - \nabla_x^2} \phi (x) \right) + \left( \pi (x) \left( \sqrt{m^2 - \nabla_x^2} \right)^{-1} \pi (x) \right) \right),
\]

where \( \mathcal{N} \) is a normalization factor. It is the expected collection of harmonic oscillators.

This Wigner functional is, of course\(^3\), an energy *-genfunctional, also checked directly. For

\[
H_0[\phi, \pi] \equiv \frac{1}{2} \int dx \left( \pi (x)^2 + \phi (x) \left( m^2 - \nabla_x^2 \right) \phi (x) \right),
\]

and the predictable infinite dimensional phase-space generalization\(^3\)

\[
\star \equiv \exp \left( \frac{i\hbar}{2} \int dx \left( \frac{\partial}{\partial \phi (x)} \frac{\partial}{\partial \pi (x)} - \frac{\partial}{\partial \pi (x)} \frac{\partial}{\partial \phi (x)} \right) \right),
\]

it follows that

\[
H_0 \star W =
\]

\[
= \int \frac{dx}{2} \left( \left( \pi (x) - \frac{i\hbar}{2} \frac{\partial}{\partial \pi (x)} \right)^2 + \left( \phi (x) + \frac{i\hbar}{2} \frac{\partial}{\partial \phi (x)} \right) \left( m^2 - \nabla_x^2 \right) \left( \phi (x) + \frac{i\hbar}{2} \frac{\partial}{\partial \phi (x)} \right) \right) W [\phi, \pi]
\]

\[
= \int \frac{dx}{2} \left( \pi (x)^2 - \frac{\hbar^2}{4} \frac{\partial}{\partial \pi (x)} \left( m^2 - \nabla_x^2 \right) \frac{\partial}{\partial \pi (x)} + \phi (x) \left( m^2 - \nabla_x^2 \right) \phi (x) - \frac{\hbar^2}{4} \frac{\partial^2}{\partial \phi (x)^2} \right) W [\phi, \pi]
\]

\[
= E_0 W [\phi, \pi].
\]

This is indeed the ground-state Wigner energy*-genfunctional. The *-genvalue is again the zero-point energy, which could have been removed by point-splitting the energy density, as indicated earlier. There does not seem to be a simple point-splitting

\(^3\) Recall that \( x \) is now a labelling parameter, not a phase-space variable.
derivatives, and the field equations evolved backwards in time context, one may simply check this solution again using the chain rule for functional

\[-i\hbar \partial_t \phi = H \ast \phi - \phi \ast H, \quad -i\hbar \partial_t \pi = H \ast \pi - \pi \ast H. \quad (4.12)\]

For $H_0$, these equations are the classical evolution equations for free fields,

\[
\partial_t \phi (x, t) = \pi (x, t), \quad \partial_t \pi (x, t) = - \left( m^2 - \nabla^2 \right) \phi (x, t). \quad (4.13)
\]

Formally, the solutions are represented as

\[
\phi (x, t) = \cos \left( t \sqrt{m^2 - \nabla^2} \right) \phi (x, 0) + \sin \left( t \sqrt{m^2 - \nabla^2} \right) \frac{1}{\sqrt{m^2 - \nabla^2}} \pi (x, 0)
\]

\[
\pi (x, t) = - \sin \left( t \sqrt{m^2 - \nabla^2} \right) \sqrt{m^2 - \nabla^2} \phi (x, 0) + \cos \left( t \sqrt{m^2 - \nabla^2} \right) \pi (x, 0).
\quad (4.14)
\]

From these, it follows by the functional chain rule that

\[
\int dx \left( \pi (x, 0) \frac{\delta}{\delta \phi (x, 0)} - \left( m^2 - \nabla^2 \right) \phi (x, 0) \right) \frac{\delta}{\delta \pi (x, 0)}
\]

\[
= \int dx \left( \pi (x, t) \frac{\delta}{\delta \phi (x, t)} - \left( m^2 - \nabla^2 \right) \phi (x, t) \right) \frac{\delta}{\delta \pi (x, t)}.
\quad (4.16)
\]

for any time $t$.

Consider the free-field Moyal evolution equation for a generic (not necessarily energy-$\ast$-genfunctional) WF, corresponding to \((3.22), \)

\[
\partial_t W = - \int dx \left( \pi (x) \frac{\delta}{\delta \phi (x)} - \phi (x) \left( m^2 - \nabla^2 \right) \frac{\delta}{\delta \pi (x)} \right) W.
\quad (4.17)
\]

The solution is an infinite-dimensional version of \((3.13), \)

\[
W [\phi, \pi; t] = W [\phi (-t), \pi (-t); 0].
\quad (4.18)
\]

Adapting the method of characteristics for first-order equations to a functional context, one may simply check this solution again using the chain rule for functional derivatives, and the field equations evolved backwards in time as specified:

\[
\partial_t W [\phi, \pi; t] = \partial_t W [\phi (-t), \pi (-t); 0]
\]

\[
= \int dx \left( \partial_t \phi (x, -t) \frac{\delta}{\delta \phi (x, -t)} + \partial_t \pi (x, -t) \frac{\delta}{\delta \pi (x, -t)} \right) W [\phi (-t), \pi (-t); 0]
\]

\[
= \int dx \left( (-\pi (x, -t)) \frac{\delta}{\delta \phi (x, -t)} + \left( m^2 - \nabla^2 \right) \phi (x, -t) \frac{\delta}{\delta \pi (x, -t)} \right) W [\phi (-t), \pi (-t); 0]
\]

\[
= \int dx \left( (-\pi (x, -t)) \frac{\delta}{\delta \phi (x, -t)} + \left( m^2 - \nabla^2 \right) \phi (x, -t) \frac{\delta}{\delta \pi (x, -t)} \right) W [\phi, \pi; t]
\]

\[
= - \int dx \left( \pi (x) \frac{\delta}{\delta \phi (x)} - \left( m^2 - \nabla^2 \right) \phi (x) \frac{\delta}{\delta \pi (x)} \right) W [\phi, \pi; t].
\quad (4.19)
\]
The quantum Wigner Functional for free fields time-evolves along classical field configurations (which propagate noncovariantly according to (4.14,4.15)). In complete analogy to the interaction representation for single particle quantum mechanics, the perturbative series in the interaction Hamiltonian (written as a $\ast$-function of fields) is then defined in terms of the convective free field variables $\Phi, \Pi$ propagating classically. Specifically,

$$i\hbar \frac{\partial}{\partial t} W[\Phi, \Pi; t] = H_I[\Phi, \Pi] \ast W[\Phi, \Pi; t] - W[\Phi, \Pi; t] \ast H_I[\Phi, \Pi].$$

(4.20)

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