Geometric phase of neutrinos: Differences between Dirac and Majorana neutrinos

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1. Introduction

The mixing and oscillation of neutrinos represent one of the most intriguing phenomena in particle physics. The main issues of neutrino physics are the absolute neutrino mass and the nature of the Dirac vs Majorana neutrino. Many experiments [1–6] have confirmed neutrino oscillations. Such a phenomenon implies that the neutrino has mass. Since neutrinos are neutral particles, they can be Majorana (with neutrinos and antineutrinos being the same particle), or Dirac particles (with neutrinos and antineutrinos being different objects). The Majorana neutrinos allow processes in which the total lepton number is not conserved, such as the neutrinoless double β decay. Such processes are inhibited for Dirac neutrinos. Experiments based on the detection of the double beta decay may contribute to determine the neutrino nature [7], however, at the moment the nature of the neutrino remains an unsolved question.

Here, we show that the study of the total and the geometric phases for neutrino propagation in vacuum and through matter could open a new scenario to investigate neutrino properties. The question how experiments could detect the geometric phase of neutrinos remains open.

In the case of neutrinos traveling through a dense medium, as for example the Sun or the Earth, the neutrino propagation can be affected by interactions with the particles in the medium. Therefore, the oscillation probabilities can be considerably different than the ones due to vacuum propagation. Such an effect is called Mikhaev–Smirnov–Wolfenstein (MSW) effect [8,9]. The phenomenon originates from the fact that electron-neutrinos (and antineutrinos) have different interactions with matter compared to the two other flavor neutrinos. This means that the effective Hamiltonian \( H \), which governs the propagation of neutrinos in matter, has an extra-potential term for the electron neutrinos. In the ordinary matter, due to the coherent forward scattering on electrons, such an extra-potential term is given by \( A_e = \pm \sqrt{2} G_F n_e \), where \( n_e \) is the electron density in the matter. \( G_F \) is the Fermi constant, and the positive (negative) sign applies to electron-neutrino (antineutrinos).

On the other hand, the study of geometric phases [10–23] which appear in the evolution of many physical systems [24–26] have attracted the attention in recent years. A geometric phase arises in the evolution of any state \( \ket{\phi} \), describing a quantum system characterized by a Hamiltonian defined on a parameter space (in the neutrino case the relevant parameters are the mass squared differences \( \Delta m^2 \) and the mixing angles). The geometric phase results from the geometrical properties of such a parameter space and can be used to study the properties of the system [25]. Berry and Berry-like phases have been shown in experimental observations to characterize physical properties of specific quantum systems. In particular, geometric phases turn out to be by themselves physical observables [20–25]. Typical examples are graphene systems [24], semiconductors, superconducting nanocircuits and devices [25]. In interferometric neutron experiments, Berry phases
provided the control on the entanglement detection by means of violation of Bell-like inequalities [27]. Moreover, Berry-like phases and noncyclic invariants associated to particle oscillations (see, for example, Ref. [28] and references therein) have also been studied extensively.

In the present paper we consider the two flavor neutrino mixing case (the discussion can be generalized to the three flavor mixing case). We show that different choices of the Majorana phase $\phi$, violating the charge conjugation–parity (CP) symmetry, in the mixing matrix lead to different values of the total and geometric phases. The oscillation formulas, on the contrary, are independent on the Majorana phase and consequently cannot be used to distinguish between Majorana and Dirac neutrinos [29,30].

Therefore, the oscillation formulas for Majorana neutrinos can be derived by using indifferently any of the mixing matrices, containing the Majorana phase $\phi$, obtained by the rephasing the lepton charge fields in the charged current weak-interaction Lagrangian, (for details see Ref. [29]). For example, one can consider the following mixing matrix

$$U_1 = \begin{pmatrix} \cos \theta & \sin \theta e^{i\phi} \\ -\sin \theta & \cos \theta e^{i\phi} \end{pmatrix},$$

or

$$U_2 = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ -\sin \theta & \cos \theta \end{pmatrix},$$

and achieve the same oscillation formulas. In general, the probabilities of neutrino transitions are indeed invariant under the rephasing $U_{\nu k} \rightarrow e^{i\delta} U_{\nu k} (\alpha = e, \mu; k = 1, 2)$.

The amplitude of transitions between different flavors are instead dependent on the choice of the mixing matrix $U$; see Eq. (22) of [29], here reported for reader’s convenience: $\langle \nu^a | U | \psi(t) \rangle = \langle \nu_1 | U | \psi(t) \rangle = A_{\nu 1} \left( U_{\nu 1} U_{\nu 1}^* e^{-iE_1 t} + U_{\nu 2} U_{\nu 2}^* e^{-iE_2 t} \right)$. Obviously, when squaring the amplitudes, the phases disappear and the resulting oscillation formulas do not depend on the phase.

As we will show below, this total phase (the dynamical and the geometric one) depends on the transition amplitudes and, in the case of transition between different flavors, it depends on the choice of the matrix $U$. In these transitions, different choices of $U$ lead thus to different values of the total phases.

In our computations we consider the matrices corresponding to $U_1$ and $U_2$, for the case of neutrinos traveling through a dense medium.

We show that, by using the matrix corresponding to $U_2$ in Eq. (2), the geometric and the total phase due to the transition between different flavors are different for Majorana and for Dirac neutrinos. On the other hand, by using the matrix corresponding to $U_1$, all the phases are independent from $\phi$ and, consequently, Majorana neutrinos cannot be distinguished from the Dirac ones.

Thus, the total and geometric phase provide a tool to determine the choice of the mixing matrix $U$. Similar results are obtained in the case of a propagation in the vacuum.

2. Majorana and Dirac neutrino

A Majorana field has a physical phase $\phi$ which violates the CP symmetry. In this case, assuming for simplicity only two flavor eigenstates, $|\nu_1\rangle$, $|\nu_2\rangle$, by considering the mixing matrix $U_2$, one can write

$$|\nu_1\rangle = \cos \theta |\nu_1\rangle + e^{-i\phi} \sin \theta |\nu_2\rangle,$$

$$|\nu_2\rangle = -e^{i\phi} \sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle,$$

where $\phi$ is the mixing angle and $|\nu_1\rangle$, $|\nu_2\rangle$ are the eigenstates of the free Hamiltonian. The phase $\phi$ cannot be eliminated since the Lagrangian of Majorana neutrinos is not invariant under $U(1)$ global transformation and the rephasing is not possible (see below).

In this case, only left-handed components of the neutrino fields $\nu_1 = \frac{1+i}{2} \nu_1^e$, $\nu_2 = \nu_1^c$ and right-handed components of the antineutrino fields $\bar{\nu}_1^e = \frac{1+i}{2} \bar{\nu}_1^e$, $\bar{\nu}_2^c = \frac{1+i}{2} \bar{\nu}_2^c$ appear in the Hamiltonian. Here $\gamma^1 = \frac{1}{\sqrt{2}} \gamma^\mu \gamma^\nu \gamma^\rho$, $\gamma^5 = \epsilon^{\mu\nu\rho\sigma}$, with $\gamma^1$ Dirac matrices [31], $\gamma_5 = \epsilon_{\mu\nu\rho\sigma}$ is the charge conjugated spinor and the matrix $C$ satisfies the relations: $C^T C = 1$, $C \gamma^L \gamma^C = 1 = -\gamma_\alpha$, $C^T = -C$. For Majorana neutrinos the interaction Hamiltonian, which does not conserve the lepton numbers, has the form

$$H = m_e \bar{\nu}_e^C \nu_e + m_\mu \bar{\nu}_\mu^C \nu_\mu + m_\tau \bar{\nu}_\tau^C \nu_\tau + \frac{\sqrt{2}}{G_F} \left( \bar{\nu}_e^C \nu_\mu + \bar{\nu}_\mu^C \nu_e \right) + \text{h.c.},$$

where the parameters $m_e$, $m_\mu$, $m_\tau$ have the dimensions of a mass.

On the contrary, for Dirac neutrino, the Lagrangian is invariant under $U(1)$ global transformation and the phase $\phi$ can be removed. The Hamiltonian interaction in this case is

$$H = m_e \bar{\nu}_e v_e + m_\mu \bar{\nu}_\mu v_\mu + m_\tau \bar{\nu}_\tau v_\tau,$$

where $v_\sigma$, with $\sigma = e, \mu$ are the antineutrino fields.

The phase $\phi$ does not affect the oscillation formulas for neutrino propagating in the vacuum and through matter. Therefore, neglecting the dissipation [32], the oscillation formulas cannot reveal the nature of neutrinos [33].

3. Neutrinos propagating through the matter

In the case of neutrinos propagating through the matter, the evolution equation in the flavor basis is described by the Schrödinger equation

$$i \frac{d}{dt} \begin{pmatrix} |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = H_f \begin{pmatrix} |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix},$$

where the Wolfenstein effective Hamiltonian [9] for flavored neutrinos can be written as (for a formal derivation see also Ref. [34])

$$H_f = \left( p + \frac{m_1^2 + m_2^2}{4E} + \frac{\sqrt{2}}{2} G_F (n_e - n_\mu) \right) I + H_1. \quad (4)$$

Here, $p$ is the neutrino momentum, $m_i$ are the masses of the free neutrinos $\nu_i$ ($i = 1, 2$) for relativistic neutrinos $E_i \approx p + \frac{m_i^2}{2}$, $n_e$ is the electron density, $n_\mu$ is the neutron number density, $G_F$ the Fermi weak coupling constant and $H_1$ for Majorana neutrinos is [35]

$$H_1 = \begin{pmatrix} \frac{G_F n_e}{\sqrt{2}} - \frac{\Delta m^2}{4E} \cos 2\theta & \frac{\Delta m^2}{4E} e^{-i\phi} \sin 2\theta \\ -\frac{\Delta m^2}{4E} e^{i\phi} \sin 2\theta & \frac{G_F n_e}{\sqrt{2}} + \frac{\Delta m^2}{4E} \cos 2\theta \end{pmatrix} \quad (5)$$

$I$ is the $2 \times 2$ identity matrix, $\phi$ the CP violating phase (which can be put equal to zero in the Dirac neutrino case), $E$ the neutrino energy and $\Delta m^2 = m_2^2 - m_1^2$. The first term of $H_1$ (the one proportional to the identity matrix) is responsible only of an overall phase factor in the neutrino evolution, therefore we will neglected it in the computation of the geometric phase and focus on $H_1$.

By defining $\Delta m^2 = \Delta m^2 R$, $\sin 2\theta = \sin 2\theta / R$, with

$$R = \left( \sqrt{2} G_F n_e E \frac{\sin^2 2\theta}{\Delta m^2} \right)^2 + \sin^2 2\theta,$$
$H_I$ in Eq. (5) can be written as

$$H_I = \frac{\Delta m^2}{4E} \left( -\cos 2\theta_m - e^{-i\phi} \sin 2\theta_m \right).$$

(7)

The eigenvalues of $H_I$ are $\lambda_\pm = \pm \frac{\Delta m^2}{2E}$. Moreover, $H_I$ can be diagonalized by means of the matrix $U_m$.

$$U_m = \left( \begin{array}{cc} \cos \theta_m & -e^{-i\phi} \sin \theta_m \\ -e^{i\phi} \sin \theta_m & \cos \theta_m \end{array} \right).$$

(8)

so that $H_I = U_m H_0 U_m^{-1}$, with $H_0 = \text{diag}(\lambda_+, \lambda_-)$. The matrix $U_m$ relates the flavor states $|\nu_e(z)\rangle$ and $|\nu_\mu(z)\rangle$ with the energy eigenstates $|\nu_1(z)\rangle$ and $|\nu_2(z)\rangle$. Here we consider, in natural units, the approximation, $z \equiv t$, where $t$ is the neutrino propagation time and $z$ the distance traveled by neutrinos. Explicitly one has

$$|\nu_e(z)\rangle = \cos \theta_m e^{i\frac{\Delta m^2}{2E}t} |\nu_1\rangle + e^{-i\phi} \sin \theta_m e^{-i\frac{\Delta m^2}{2E}t} |\nu_2\rangle,$$

$$|\nu_\mu(z)\rangle = -e^{i\phi} \sin \theta_m e^{i\frac{\Delta m^2}{2E}t} |\nu_1\rangle + \cos \theta_m e^{-i\frac{\Delta m^2}{2E}t} |\nu_2\rangle,$$

(9)

with $\langle \nu_e(z) | \nu_e(z) \rangle = \langle \nu_\mu(z) | \nu_\mu(z) \rangle = 1$ and $\langle \nu_e(z) | \nu_\mu(z) \rangle = 0$. In the case of the propagation in vacuum, $\Delta = 1$, then $\Delta m^2 \rightarrow \Delta m^2$, $\theta_m \rightarrow \theta$ and $U_m$ in Eq. (8) coincides with $U_2$ of Eq. (2).

4. Geometric phases for neutrinos in matter

In the following we consider the states given by Eq. (9). To compute the geometric and the total phases generated by the states (9), we use the definition of Mukunda-Simon geometric phase [16], which generalizes the Berry phase to the non-adiabatic and non-cyclic case. Such a phase, derived within a kinematical approach, is associated to any open curve of unit vectors in Hilbert space. For a quantum system whose state vector $|\psi(s)\rangle$ belongs to a curve $\Gamma$, with the real parameter $s$ such that $s \in [s_1, s_2]$, the Mukunda-Simon phase is defined as the difference between the total and the dynamic phase:

$$\Phi^g(\Gamma) = \Phi^\text{tot}_{\psi}(s) - \Phi^\text{dyn}_{\psi}(s)$$

$$= \text{Arg}(\psi(s))|_{s_1}^{s_2} \Theta_{s_1}^s_{s_2} |\psi(s)\rangle \langle \psi(s)\rangle ds .$$

(10)

Here the dot denotes the derivative with respect to the real parameter $s$. In Eq. (10), $\text{Arg}(\psi(s))|_{s_1}^{s_2}$ represents the total phase, and $\Theta_{s_1}^s_{s_2} |\psi(s)\rangle \langle \psi(s)\rangle$ is the dynamical one.

In the specific case of neutrinos, the geometric phase of electron neutrino, for an initial state $|\nu_e\rangle$, is

$$\Phi^g_{\nu_e}(z) = \Phi^\text{tot}_{\nu_e}(z) - \Phi^\text{dyn}_{\nu_e}(z)$$

$$= \text{Arg}(\langle \nu_e(0) | \nu_e(z)\rangle) - \int_z^0 \langle \nu_e(z') | \nu_e(z')\rangle |dz'| .$$

(11)

and explicitly one obtains

$$\Phi^g_{\nu_e}(z) = \text{Arg} \left[ \cos \left( \frac{\Delta m^2 z}{4E} \right) + i \cos 2\theta_m \sin \left( \frac{\Delta m^2 z}{4E} \right) \right]$$

$$- \frac{\Delta m^2 z}{4E} \cos 2\theta_m .$$

(12)

In a similar way, the geometric phase for the muon neutrino, for an initial state $|\nu_\mu\rangle$, is given by $\Phi^g_{\nu_\mu}(z) = \text{Arg}(\langle \nu_\mu(0) | \nu_\mu(z)\rangle) - \int_0^z \langle \nu_\mu(z') | \nu_\mu(z')\rangle |dz'|$. One has $\Phi^g_{\nu_\mu}(z) = -\Phi^g_{\nu_e}(z)$. Eq. (12) does not depend on the $CP$ violating phase, thus it holds both for Majorana and for Dirac neutrinos.

However, in addition to the phases $\Phi^g_{\nu_e}(z)$ and $\Phi^g_{\nu_\mu}(z)$, one also has the following phases due to the neutrino transitions between different flavors,

$$\Phi^g_{\nu_e \rightarrow \nu_\mu}(z) = \text{Arg} \left[ \langle \nu_e(0) | \nu_\mu(z)\rangle - \int_z^0 \langle \nu_e(z') | \nu_\mu(z')\rangle |dz'| \right] .$$

(13)

$$\Phi^g_{\nu_\mu \rightarrow \nu_e}(z) = \text{Arg} \left[ \langle \nu_\mu(0) | \nu_e(z)\rangle - \int_z^0 \langle \nu_\mu(z') | \nu_e(z')\rangle |dz'| \right] .$$

(14)

In this case, by using the states in Eq. (9) for Majorana neutrinos, since $\phi$ cannot be removed, one obtains

$$\Phi^g_{\nu_e \rightarrow \nu_\mu}(z) = \text{Arg} \left[ \sin 2\theta_m \sin \left( \frac{\Delta m^2 z}{4E} \right) \left( \sin \phi - i \cos \phi \right) \right]$$

$$+ \left( \frac{\Delta m^2}{4E} \sin 2\theta_m \cos \phi \right) z,$$

(15)

which holds provided there is mixing ($\Delta m^2 \neq 0$ and $\theta \neq 0$). Thus, we have

$$\Phi^g_{\nu_e \rightarrow \nu_\mu}(z) = \frac{3\pi}{2} + \phi + \left( \frac{\Delta m^2}{4E} \sin 2\theta_m \cos \phi \right) z .$$

(16)

and in similar way

$$\Phi^g_{\nu_\mu \rightarrow \nu_e}(z) = \frac{3\pi}{2} - \phi + \left( \frac{\Delta m^2}{4E} \sin 2\theta_m \cos \phi \right) z .$$

(17)

Then, $\Phi^g_{\nu_e \rightarrow \nu_\mu} \neq \Phi^g_{\nu_\mu \rightarrow \nu_e}$, which reveals an asymmetry between the transitions $\nu_e \rightarrow \nu_\mu$ and $\nu_\mu \rightarrow \nu_e$ due to the presence of $\phi$. Notice that, also the total phases (which are the relevant ones for oscillation experiments), reveal such an asymmetry. Indeed, one has $\Phi^g_{\nu_e \rightarrow \nu_\mu} = \frac{3\pi}{2} + \phi$ and $\Phi^g_{\nu_\mu \rightarrow \nu_e} = \frac{3\pi}{2} - \phi$. On the contrary, in the case of Dirac neutrinos, the phase $\phi$ can be removed by means of $U(1)$ global transformation, then, the phases of Eqs. (13) and (14) (or equivalently Eqs. (16) and (17)) become equivalent and reduce to

$$\Phi^g_{\nu_e \rightarrow \nu_\mu}(z) = \Phi^g_{\nu_\mu \rightarrow \nu_e}(z)$$

$$= \frac{3\pi}{2} + \left( \frac{\Delta m^2}{4E} \sin 2\theta_m \right) z .$$

(18)

In a similar way, the total phases reduce to $\Phi^\text{tot}_{\nu_e \rightarrow \nu_\mu}(z) = \Phi^\text{tot}_{\nu_\mu \rightarrow \nu_e}(z) = \frac{3\pi}{2}$. Let us now observe that the result of Eq. (12) is independent on the choice of the mixing matrix. The same result is indeed obtained by using, for example, the mixing matrix corresponding to $U_1$ which diagonalize the interaction Hamiltonian $H^I_F$, derived by $H_I$ in Eq. (7) by setting $\phi = 0$.

On the contrary, the phases defined in Eqs. (13) and (14) are dependent on the choice of the mixing matrix. Indeed, the result of Eq. (18) is obtained also for Majorana neutrinos when one considers the mixing matrix corresponding to $U_1$. We therefore conclude that the phases $\Phi^g_{\nu_e \rightarrow \nu_\mu}$ and $\Phi^g_{\nu_\mu \rightarrow \nu_e}$ and the total phases $\Phi^\text{tot}_{\nu_e \rightarrow \nu_\mu}$ and $\Phi^\text{tot}_{\nu_\mu \rightarrow \nu_e}$ discriminate between the two matrices $U_1$ and $U_2$.

Notice that all the above phases vanish in the limit of zero neutrino masses where the mixing is absent and Dirac and Majorana neutrinos are equivalent.
Although the measurements of the geometric phase for neutrinos is out of the present experimental possibilities, in the following, in order to clarify the connection of our analysis to experimental parameter values, we present a numerical analysis and we plot in Figs. 1 and 2 the total and geometric phases by using the values used in experiments like RENO [2] and T2K [4].

5. Numerical analysis

In our analysis, for the total and geometric phases associated with the evolution of νe, we consider the energy of neutrinos produced in nuclear reactors $E \in [2-8] \text{MeV}$, the electron density $n_e = 10^{24} \text{cm}^{-3}$, $\Delta m^2 = 7.6 \times 10^{-4} \text{eV}^2$ and a distance $z = 100 \text{km}$. Such values are of the same order of the ones of RENO experiment [2] and can be consistently used in the formulas of our results. The plots of the total and the geometric phases so obtained are shown in Fig. 1. Similar results can be found considering energies of few GeV, which are characteristic of neutrino beams produced at particle accelerators, and distances of several hundred of km, as in long base line experiments. For the analysis of geometric phases due to the transition between different flavors (Eqs. (13) and (14)), we consider energies $E \sim 1 \text{GeV}$ and a distance $z = 300 \text{km}$, which are of the same order of the ones in T2K experiment [4]. Moreover we consider $\phi = 0.3$, and the values of $n_e$ and $\Delta m^2$ considered above. The plots of the phases in Eqs. (16), (17) and (18) are reported in Fig. 2.

6. Conclusions

We have shown that the total and geometric phase, due to a transition between different neutrino flavors, take different values depending on the representation of the mixing matrix and on the nature of neutrinos. Different values for $\Phi^g_{\nu_e \rightarrow \nu_x}$ and $\Phi^g_{\nu_x \rightarrow \nu_y}$ (and for $\Phi^g_{\nu_x \rightarrow \nu_x}$ and $\Phi^g_{\nu_y \rightarrow \nu_y}$) are due the Majorana phase $\phi$. Such a difference implies that the mixing matrix used to be is of type U2, which then removes the ambiguity in the use of $U_1$ and $U_2$. On the other hand, if $\Phi^g_{\nu_x \rightarrow \nu_x} = \Phi^g_{\nu_y \rightarrow \nu_y}$ (and $\Phi^g_{\nu_x \rightarrow \nu_x} = \Phi^g_{\nu_y \rightarrow \nu_y}$) then the ambiguity between $U_1$ and $U_2$ remains and nothing can be said on the nature (Dirac or Majorana) of neutrinos.

We have plotted the total and the geometric phases by using the parameters entering in the RENO and T2K experiments. Although the question of how the geometric phase of neutrinos could be detected remains open, our theoretical results could open interesting scenarios in neutrino research.

The theoretical aspects of particle mixing have been studied extensively in the contexts of quantum mechanics (QM) [36–39] and of quantum field theory (QFT) [40–42] where corrections to the QM oscillation formulas in vacuum have been derived (see also Refs. [43,44]). In the discussion here presented these corrections can be safely neglected [41].

Acknowledgements

A.C. and G.V. acknowledge partial financial support from MIUR and INFN. A.C. also acknowledges the COST Action CA1511 Cosmology and Astrophysics Network for Theoretical Advances and Training Actions (CANTATA). S.M.G. financial support from the Ministry of Science, Technology and Innovation of Brazil and B.C.H. Austrian Science Fund (FWFP26783).

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Note that dissipative effects in neutrino oscillation in matter can be mostly neglected since they are of the order of $E^2/M_p$, with $E$ neutrino energy and $M_p$ Plank Mass. For more details see Ref. [32].