COMBINATORIAL VS. CLASSICAL DYNAMICS: RECURRENCE

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Abstract. Establishing the existence of periodic orbits is one of the crucial and most intricate topics in the study of dynamical systems, and over the years, many methods have been developed to this end. On the other hand, finding closed orbits in discrete contexts, such as graph theory or in the recently developed field of combinatorial dynamics, is straightforward and computationally feasible. In this paper, we present an approach to study classical dynamical systems as given by semiflows or flows using techniques from combinatorial topological dynamics. More precisely, we present a general existence theorem for periodic orbits of semiflows which is based on suitable phase space decompositions, and indicate how combinatorial techniques can be used to satisfy the necessary assumptions. In this way, one can obtain computer-assisted proofs for the existence of periodic orbits and even certain chaotic behavior.

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1. Introduction

Understanding the asymptotic dynamics of solutions of differential equations is vital in many areas of science. Since in most cases no closed formulas for solutions are available, various other approaches have been invented to get some insight into the behavior of solutions. Qualitative methods based on tools from analysis or topology provide invariants and theorems detecting certain types of solutions. Unfortunately, analytic verification of the assumptions of these theorems in many cases is very arduous or just infeasible. Computer simulations give quick answers, but are not very reliable because of the approximate nature of numerical methods. Computer-assisted proofs in dynamics emerged at the end of the 20th century as a remedy to this situation. They combine qualitative tools with rigorous numerics as a method replacing the arduous analytic computations.

A typical computer-assisted proof requires an a-priori selection of the qualitative method for the particular problem and fine tuning of several parameters, for instance the step of the numerical method, bounds for the area of interest, or the location of a Poincaré section. To properly choose the method and get the parameters right, many numerical experiments have to be performed first. This effort pays for itself in the outcome which often is not only the proof of the existence of the object of interest, but also sharp bounds for its location in space.

Another approach to computer-assisted proofs in dynamics aims at building a global combinatorial model with some formal ties between the model dynamics and the dynamics of interest. This approach turned out to be very successful in the study of the gradient structure of a dynamical system [2] by providing a Morse decomposition of the system together with the associated Conley-Morse graph. In this approach, individual Morse sets in the decomposition are identified on the combinatorial level as strongly connected components in the directed graph representation of the combinatorial dynamics. What we propose in this paper is an extension of the combinatorial approach to get insight into the Morse sets in the case when a Morse set exhibits non-trivial recurrent behavior.

The extension combines a theorem characterizing the existence of periodic orbits for semiflows in terms of Conley index [21] with the recently developed Conley theory for combinatorial multivector fields [20, 26]. The combinatorial multivector field is obtained by constructing a polygonal decomposition of space whose faces are crossed by the flow transversally, a method already applied in [7, 14]. By lifting the problem to combinatorial multivector fields we can not only identify the Morse sets, but also look inside the Morse sets searching for recurrent isolated invariant sets. Once an isolated invariant set with a combinatorial Poincaré section and the Conley index of a hyperbolic periodic orbit is located, we can use the theory presented in this paper to prove the existence of a periodic orbit in the original flow.

The results of the present paper apply to flows and the existence of a periodic orbit. But, they may be extended to semiflows and the existence of chaotic invariant sets. Whenever possible, we have formulated our results for the most general situation. In this way, we will be able to present some applications to semiflows and chaos already in Section 2. However, the full details of the general extension to semiflows, even though they are the same in spirit, are technically more complicated and will be presented elsewhere.

The remainder of this paper is organized as follows. In Section 2 we give a broad overview over potential applications for our main result. For this, we demonstrate how one can establish the existence of periodic orbits in planar systems, show that for a semiflow on a branched
manifold it can be used to establish chaos, and then present a template for an analogous chaos-inducing situation in three dimensions. After we present necessary background definitions and results in Section 3, the following Section 4 is devoted to our main result. Based on the notion of brick decompositions, we can present a general existence result for periodic orbits which is based on Conley index methods. In contrast to earlier results, we do not need the existence of an explicit Poincaré section. Rather, our result relies on a much weaker notion of flow directionality. Finally, Section 5 shows how methods from combinatorial topological dynamics based on multivector fields can be used to establish the assumptions of our main result.

2. Applications of the main results

Before presenting the mathematical theory behind our approach to recurrence, the current section is devoted to illustrating a few applications. This is accomplished in three parts. We begin by demonstrating the basic approach in a simple setting, namely flows generated by ordinary differential equations in the plane. This is followed by an example taken from the sequence of papers [5, 18, 29], where we can use our results to establish the existence of infinitely many different periodic orbits on a branched manifold and with respect to a large class of semiflows. Finally, the third example is concerned with a Lorenz-type situation in three dimensions and which also exhibits chaos.

2.1. Periodic orbits in planar flows. As mentioned in the introduction, our main result for proving the existence of periodic orbits is based on a suitable decomposition of phase space. In this section, we illustrate how this can be used to study periodic orbits in planar flows generated by ordinary differential equations. We would like to emphasize that our examples were primarily chosen to illustrate the main ideas. In these simple cases, the existence of the periodic solutions can easily be obtained by different means. Yet, the approach outlined below can also be used in a higher-dimensional setting.

Our main result, Theorem 4.6 below, relies on a decomposition of a region of interest in phase space into small pieces called bricks. In the simplest case, this decomposition can take the form of a triangulation, even though this will be generalized later on. Our approach for establishing the existence of periodic orbits can then be summarized in the following five steps:

(a) **Numerical determination of a flow transverse triangulation:** After identifying a region of interest in phase space, use local flow or vector field information to construct a triangulation of the region in the following way. Across each edge of the triangulation, the considered flow is transverse, and there are no equilibrium solutions of the flow in any of the triangles.

(b) **Rigorous validation of the triangulation properties:** Once a numerical candidate for the triangulation has been found, rigorous computations based on interval arithmetic have to be used to validate the flow behavior across the triangulation edges and inside the triangles.

(c) **Create a combinatorial multivector field:** Based on the rigorous information from the previous step, one can construct a combinatorial multivector field associated with the flow as described in Proposition 5.25.

(d) **Graph-theoretic analysis of the multivector field:** Finding recurrent sets in the combinatorial setting is now a graph-theoretic problem, which can be solved efficiently
using the concept of strongly connected components in a digraph. If the triangulation constructed in (1) was appropriate, then we can obtain a candidate region which can contain a periodic orbit.

(e) **Existence proof for periodic solutions:** As a last step, the usually large triangulation has to be coarsened in such a way that Theorem 4.6 can be applied to prove the existence of the periodic orbit.

The crucial part of this procedure is clearly the first step, since we need to find an “appropriate triangulation” to apply our results. Nevertheless, a number of options exist, for example the methods described in [7, 14]. For our examples below we will follow a different approach. We would like to point out, however, that the use of a triangulation is not necessary. From a mathematical point of view, the bricks which form the phase space decomposition are characterized by having well-defined entry and exit behavior with respect to the underlying semiflow or flow, and they have to be topologically simple. This will be discussed in more detail in Section 5.

To illustrate the above approach we consider two simple planar ordinary differential equations. The first one is given by

\[
\begin{align*}
\dot{x} &= -y + x \left(x^2 + y^2 - 4\right) \left(x^2 + y^2 - 1\right) \\
\dot{y} &= x + y \left(x^2 + y^2 - 4\right) \left(x^2 + y^2 - 1\right)
\end{align*}
\]  

(1)

and one can easily see that this system has an unstable equilibrium at the origin, as well as two invariant circles which are centered at \((0, 0)\) and have radii 1 and 2, respectively. For our second example, we consider the celebrated van der Pol equation

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \mu y \left(1 - x^2\right) - x
\end{align*}
\]  

(2)

at the parameter value \(\mu = 1\).

For the first example (1), it is straightforward to create a subdivision of the square \([-3, 3] \times [-3, 3]\) into a triangular mesh such that along each edge of the mesh the flow generated by (1) is transverse. While the details of this construction will be described elsewhere, it is based on an adaptive initial meshing procedure, combined with random perturbations of the vertices of the mesh to achieve flow transversality. The resulting mesh is shown in the left diagram of Figure 1 where the flow directions across the mesh edges are indicated by small black line segments which point into the entered mesh triangle. Moreover, we used the interval arithmetic package INTLAB [30] to rigorously verify the flow transversality.

Based on the mesh and the flow directions, one can then create a multivector field on the simplicial complex given by the mesh which encodes the flow information in a combinatorial way. By employing the results of [20], one can then find isolated invariant sets for the combinatorial multivector field. These are shown in the right diagram of Figure 1. In addition to the triangle containing the equilibrium \((0, 0)\), one can also see two regions enclosing the two periodic orbits. Closeups of these two diagrams are contained in Figure 2. Finally, one can divide each of the two annular regions into suitable segments in such a way that an application of Theorem 4.6 establishes a periodic orbit in each region.

For the van der Pol example (2) the first step in the above procedure is somewhat more delicate, due to the strong rotational dynamics exhibited by the system. Nevertheless, in Figure 3 we show a triangulation of an attracting region, which has been verified via interval arithmetic as being flow transverse across triangulation edges. Within this region, the shaded
red region is an isolating neighborhood for the periodic orbit, which is indicated in yellow. This isolating neighborhood was found through the associated combinatorial multivector field as before, and again we can apply Theorem 4.6 after a segmentation of this region.

Our above discussion centered merely on the existence of periodic orbits in planar systems. It is natural to wonder whether one can also establish the existence of connections between such orbits, or between periodic orbits and stationary states, in order to obtain a more global picture of the underlying dynamics. This can actually be accomplished using the notion of connection matrices, which in the case of combinatorial dynamics on Lefschetz spaces were recently introduced in [28]. The detail of this approach will be presented elsewhere. Moreover, while our approach in its present form only provides the existence of a periodic orbit, our method can be used in principle to also obtain lower bounds on the period of the constructed periodic solutions. For this, one can use vector field information to obtain lower bounds on the traversal times of solutions across every cell in the phase space decomposition, which can then be combined with combinatorial periodic solutions in the digraph provided by our multivector field to establish the bound. This will not be pursued further in this paper.

2.2. Infinitely many periodic orbits on a branched manifold. For our second application we consider recent work on the relationship between combinatorial vector fields in the sense of Forman and classical dynamics, as developed in [5, 18, 29]. In these papers, we developed a Conley theory for combinatorial vector fields, and showed that every combinatorial vector field on a simplicial complex gives rise to both a discrete multivalued dynamical system and a semiflow on the underlying polytope which exhibits the same dynamics in the sense of Conley-Morse graphs. It was also shown that in the combinatorial setting one can easily
Figure 2. Closeups of the flow transverse grid in Figure 1 for the flow described by (1). While the right image shows parts of the combinatorial isolated invariant sets containing the periodic orbits, it also shows the transversality directions via small black line segments as in Figure 1. In the left image, we indicate the actual flow direction at the center points of the mesh edges via small red line segments.

Figure 3. Flow transverse triangular grid for the flow described by the van der Pol equation (2) for $\mu = 1$. In the left image, small black line segments indicate in which direction the flow crosses the respective edge, and the actual flow direction at the center points of the mesh edges is given via small red line segments. The right image shows an isolated invariant set which encloses the limit cycle shown in yellow.

Construct systems with complex combinatorial dynamics. For example, it was mentioned in [18] that the combinatorial vector field shown in the left panel of Figure 4 exhibits chaotic behavior in the sense of having infinitely many different periodic orbits, which can easily be
found by following the triangle loops around each of the two eyes in a variety of sequence patterns. In this model, the two triangles $\sigma_1$ and $\sigma_2$ only intersect along the edge $\varphi$, and along this edge they are attached to the triangle $\tau$. Thus, the underlying space is a simplicial complex in the form of a branched two-dimensional manifold $X$.

While the combinatorial vector field in the left panel of Figure 4 exhibits infinitely many periodic orbits, this is not clear when one only considers the associated Conley-Morse graph. It can easily be shown that the combinatorial dynamics has four distinct Morse sets: The union of all triangles is contained in one large Morse set, below which there is a Morse set of index one given by the equilibrium on the bottom edge. Below this equilibrium, there are two stable periodic orbits, which correspond to the interior boundaries of the two eyes.

In [29] we showed that for any combinatorial vector field, there always exists a semiflow on the underlying polytope which has the same Conley-Morse graph. Thus, for the particular vector field in Figure 4 there exists a semiflow with the four Morse sets discussed above. While we do not want to go into the details of the construction in [29], it provides an isolating neighborhood for the top-level Morse set which is indicated in the right panel of Figure 4 through additional shading in violet, orange, and yellow. In fact, the semiflow provided by this construction has the whole boundary of the shaded region as exit set.

But what about the dynamics of this semiflow inside this isolating neighborhood? It is far from clear whether there still are infinitely many periodic orbits in the colored region. Nevertheless, one can prove exactly that. The following theorem relies heavily on the setup of [29], and we will only sketch the proof of this result below.

**Theorem 2.1** (Periodic solutions with prescribed itineraries). Consider the combinatorial vector field shown in the left panel of Figure 4 and let $\varphi$ denote any strongly admissible semiflow on the underlying branched manifold $X$, in the sense described in [29]. Furthermore,
let \( L \) describe the subset of \( X \) colored in violet and yellow in the right panel of Figure 4, and \( R \) the one colored in orange and yellow. Furthermore, consider an arbitrary finite sequence \( S \) using the binary alphabet \( \{ L, R \} \). Then there exists a periodic orbit of \( \varphi \) which infinitely many times traverses the sets \( L \) and \( R \) in clockwise and counter-clockwise manner, respectively, as prescribed by the sequence \( S \).

For example, for the sequence \( S = (R, L) \) which will be discussed in the proof below, the periodic orbit alternately rotates around the right and left loop of \( X \). We would like to point out, however, that we cannot guarantee that during the minimal period of the periodic solution the periodic orbit only completes one \((R, L)\) sequence. It might complete several of these. Nevertheless, one easily obtains the following corollary.

**Corollary 2.2** (Infinitely many periodic orbits on a branched manifold). *In the setting of Theorem 2.1, any strongly admissible semiflow \( \varphi \) on \( X \) contains infinitely many periodic orbits in the union \( L \cup R \).*

As mentioned earlier, we only briefly sketch the proof of Theorem 2.1 in a special case, and leave the details of the general case to the reader.

**Proof:** For the proof, we only consider the specific sequence \( S = (R, L) \). The case of a general finite sequence \( S \) in \( \{L, R\} \) can be discussed analogously.

In order to apply our main Theorem 4.6 in this situation, we cannot consider the space \( X \) directly. Rather, we consider a space \( Z \) which is based on a suitable covering space \( Y \). For our symbol sequence of length two, the covering space \( Y \) will be a double cover, and it is illustrated in the left panel of Figure 5. The space consists of two disjoint copies of \( X \), each of which has been cut along the edge \( \varrho \) shown in the left panel of Figure 4. This results in a total of six copies of this edge, three for each copy of \( X \), and they are identified as shown in the figure. The three blue edges marked \( \varrho_1 \) are identified, as indicated by the shaded blue regions. Similarly, the three red edges marked \( \varrho_2 \) are identified, but in order to keep the sketch simple, this is not separately highlighted by additional shading.

Once the covering space \( Y \) has been created, we can construct the space \( Z \) which is actually used in the proof. The space \( Z \) is obtained from \( Y \) by cutting again along both copies of the edge \( \varrho \), but this time based on the underlying symbol sequence \( S = (R, L) \). At the lower copy of \( X \), we cut off the connection between the left loop and \( \varrho_1 \), since the first symbol in \( S \) is given by \( R \). Similarly, in the upper copy the right loop is severed from \( \varrho_2 \), since the second symbol in \( S \) is \( L \). The remaining edge identifications are shown in shaded orange. This leads to the pink set which now encodes the dynamics prescribed by \( S \).

After these preparations, we finally consider the restriction \( B \subset Z \) of the isolating neighborhood indicated in the right panel of Figure 4, and lifted to \( Z \) in a straightforward way. Notice that this set \( B \) is a proper subset of the pink region in the right panel of Figure 5. Then it follows from the results of [29] that \( B \) is indeed an isolating block for the semiflow \( \varphi \), whose exit set \( B^- \) equals the boundary of \( B \) in \( Z \). Moreover, the set \( B \) has an underlying brick decomposition as required by Theorem 4.6. We would like to point out that the cuts that were performed while passing from \( Y \) to \( Z \) are necessary to deal with the issues identified for semiflows in Proposition 5.22 below. Finally, the Conley index of the isolating block \( B \) is that of an unstable periodic orbit of index one. An application of Theorem 4.6 now easily implies the existence of the desired periodic orbit. For this, one only has to project the periodic orbit in \( Z \) guaranteed by the theorem down to the space \( X \). Since the space \( Z \)
was constructed via a covering space $Y$ which contains two exact copies of both $X$ and the semiflow on $X$, this projection leads to a periodic orbit for the original semiflow $\varphi$ on $X$.  □

We would like to point out that the above two results remain valid for any semiflow on $X$ which is strongly admissible in the sense of [29]. Thus, the trivial combinatorial chaos exhibited by the Forman vector field from Figure 4 gives rise to a robust notion of chaos for a large class of semiflows on the underlying polytope. In other words, the main results of this paper, in combination with purely combinatorial constructions, can be used to create classical semiflows with extremely complicated recurrent dynamics.

One drawback of these results is that we cannot guarantee that the constructed periodic orbits realize the symbol sequence $S$ within their fundamental period. This can, however, be accomplished via an approach based on the Lefschetz fixed point theorem and combinatorial Poincaré maps. This will be presented elsewhere [27].
2.3. Chaos in a Lorenz-type isolating neighborhood. It is well-known that while for planar systems the existence of periodic orbits can also be established using the Poincaré-Bendixson theorem, considering the case of three- or higher-dimensional flows is significantly more involved [15]. Nevertheless, we demonstrate in the present subsection that the main ideas from our branched manifold example can easily be carried over to the three-dimensional situation. Our example is partially motivated by the discussion in [17], and it provides a different approach to establishing chaos in Lorenz-type equations from the approach developed in [22, 23, 25].

Figure 6 presents an example of an isolated invariant set $S$ of a combinatorial multivector field $V$ on a cellular structure $X$ in three dimensions. To improve visibility, only cells contained in $S$ are presented. Each three-dimensional cell is a cube-like polytope with convex, quadrilateral faces. Each multivector $V \in V$ consists of a three-dimensional cell and its faces except the faces indicated by a small arrow. Let $N$ denote the union of the closures of all three-dimensional cells in $S$. Furthermore, let $\varphi$ denote a flow in $\mathbb{R}^3$ which is transversal to all two-dimensional faces. Then one can easily verify that $N$ is an isolating block for $\varphi$.

Note that $N$ is the union of a rectangular cuboid $N_0$ in the middle and two handles attached to it — the sets $N_1$ on the left and $N_2$ on the right. It is easy to check that both $N_0 \cup N_1$ and $N_0 \cup N_2$ are isolating neighborhoods for $\varphi$ with the Conley index of a hyperbolic periodic orbit. Therefore, an argument analogous to the one discussed in the previous section may be
used to show that $\varphi$ has infinitely many periodic orbits in $\mathbb{N}$. A more detailed discussion of this example based on the Lefschetz fixed point theorem and combinatorial Poincaré maps will be presented in [27].

3. Preliminaries

Throughout this paper, we denote the sets of positive integers, integers, rational numbers, reals, and nonnegative reals by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{R}_0^+$, respectively. In addition, given a set $X$ we denote the family of subsets of $X$ by $\mathcal{P}(X)$. Recall that a family $A \subset \mathcal{P}(X)$ is a partition of $X$ if the elements of $A$ are mutually disjoint and their union equals $X$.

For the remainder of this section, we collect basic notions from topology and dynamical systems which will be needed later on.

3.1. Basic notions in topology. Given a topological space $X$ and a subset $A \subset X$, we denote by $\text{int} A$, $\text{cl} A$, and $\text{bd} A$ the interior, the closure, and the boundary of $A$, respectively. Furthermore, we recall that the set $A$ is called locally closed, if the difference $\text{mo} A := \text{cl} A \setminus A$ is closed. The set $\text{mo} A$ is called the mouth of $A$.

3.1.1. Pairs of spaces and singular homology. Throughout this paper, we consider topological pairs, i.e., pairs of topological spaces $(X, A)$ for which $A \subset X$ and the topology of $A$ is induced by $X$. In this setting, a single space $X$ is treated as the pair $(X, \emptyset)$. In addition, we use the singular homology $H(X, A)$ of pairs with coefficients in the field of rational numbers $\mathbb{Q}$, as defined for example in [13, 31].

3.1.2. Adjunctions, retracts, and ANRs. For a topological pair $(X, A)$, a topological space $Y$, and a continuous map $f : A \to Y$, the adjunction $X \cup_f Y$ is defined as the quotient space of the disjoint union of $X$ and $Y$ by the identification of $x \in A$ with $f(x) \in Y$. We treat $X \cup_f Y$ as the disjoint union of $X \setminus A$ and $Y$, with a dual treatment of the points in $A$ and $f(A)$.

Let $(X, A)$ be a topological pair. A map $r : X \to A$ is called a retraction if $r(a) = a$ for all $a \in A$. In that case $A$ is called a retract of $X$. A metrizable space $X$ is called an absolute neighborhood retract (which is abbreviated as ANR) if, whenever embedded as a closed subset of a metric space, it is a retract of an ambient neighborhood. If both $X$ and $Y$ are compact ANRs, then the same holds true for the adjunction $X \cup_f Y$, see for example [8, V(2.9)] or [33, Theorem 5.6.1].

3.1.3. Finite topological spaces. We close this discussion of topological prerequisites with a brief review of finite topological spaces. For this, recall that any finite set $X$ which is equipped with a topology is called a finite topological space. While such spaces usually have very weak separation properties, those that satisfy the $T_0$ separation axiom are of particular interest. Recall that $X$ is a $T_0$-space, if for every pair of points $x \neq y$ in $X$ there exists an open set $U \subset X$ which contains exactly one of these two points. It is a consequence of the celebrated Alexandrov Theorem [1] that a $T_0$ topology on a finite set $X$ may be functorially identified with a partial order on $X$. There are two dual ways to make the identification by taking as open sets either the upper or the lower sets with respect to the partial order. In this paper we identify open sets with upper sets, which corresponds to

$$x \leq y \iff x \in \text{cl}\{y\}.$$  

Of particular interest for our applications is the fact that finite collections of subsets of a given set automatically generate finite $T_0$-spaces in the following way. Consider a finite family of
subsets $\mathcal{X} \subset \mathcal{P}(X)$ of the set $X$. Since the inclusion relation on $\mathcal{X}$ defines a partial order, the Alexandrov Theorem induces a unique $T_0$ topology on $\mathcal{X}$. In the case of such an Alexandrov topology, and to avoid potential confusion if $X$ is also a topological space, we denote the interior, closure, and mouth of a subset $V \in \mathcal{X}$ by $\text{Int} V$, $\text{Cl} V$ and $\text{Mo} V$, respectively. Since in the case of the Alexandrov topology the intersection of an arbitrary family of open sets is open, the smallest open set containing a given set $V \in \mathcal{X}$ is open. We denote it by $\text{Opn} V$. Furthermore, if $V = \{\sigma\}$ is a singleton, then we drop the braces and write $\text{Cl} \sigma$ or $\text{Opn} \sigma$ instead of $\text{Cl}\{\sigma\}$ or $\text{Opn}\{\sigma\}$. Finally, observe that

$$\text{Opn} \sigma = \{ \tau \in \mathcal{X} \mid \tau \supset \sigma \}, \quad (3)$$

$$\text{Cl} \sigma = \{ \tau \in \mathcal{X} \mid \tau \subset \sigma \}, \quad (4)$$

$$\tau \in \text{Cl} \sigma \iff \sigma \in \text{Opn} \tau. \quad (5)$$

These characterizations will prove to be useful later on.

Questions of connectedness can easily be addressed for finite topological spaces. It was shown in [3, Proposition 1.2.4] that a finite topological space $X$ is connected if and only if it is path-connected. Moreover, $X$ is connected if and only if between any two points $x, y \in X$ there exists a fence, i.e., there exists a sequence of points $x = x_0, x_1, \ldots, x_n = y$ such that for every $k \in \{0, \ldots, n - 1\}$ we have either $x_k \leq x_{k+1}$ or $x_k \geq x_{k+1}$. We would like to point out that one can view the length of a shortest fence between $x$ and $y$ as a rudimentary measure of distance between points in a connected finite topological space.

3.2. Basic notions in dynamical systems. We now collect some basic definitions and results from the theory of dynamical systems.

3.2.1. Semiflows and flows. Suppose that $X$ is a locally compact metric space. By a semiflow on $X$ we mean a continuous map $\varphi : X \times \mathbb{R}_0^+ \to X$ such that both

$$\varphi(x, 0) = x \quad \text{and} \quad \varphi(\varphi(x, t), s) = \varphi(x, t + s)$$

are satisfied for all $x \in X$ and $s, t \in \mathbb{R}_0^+$. If in the last identity the group $\mathbb{R}$ replaces the semigroup $\mathbb{R}_0^+$, then $\varphi$ is called a flow. In both cases we call $X$ the phase space of $\varphi$. Furthermore, in order to increase readability we sometimes abbreviate $\varphi(x, t)$ by the short-hand $x \cdot t$, as long as $\varphi$ is clear from the context.

Suppose now that $\mathbb{T} = \mathbb{R}_0^+$ or $\mathbb{T} = \mathbb{R}$, and let $\varphi : X \times \mathbb{T} \to X$ denote a semiflow or flow, respectively. Then a solution of $\varphi$ through $x \in X$ is a map $\gamma : I \to X$ such that $I \subset \mathbb{R}$ is a nontrivial interval containing zero, the identity $\gamma(0) = x$ holds, and $\varphi(\gamma(s), t) = \gamma(s + t)$ for all $t \in \mathbb{T}$ and $s \in I$ such that $s + t \in I$. Note that in the case of a flow any two solutions through $x$ coincide on the intersection of their domains. We denote the set of solutions through $x$ by $\text{Sol}(x, \varphi)$. In the special case $I = \mathbb{R}$, we further say that the solution $\gamma$ is full. We say that a solution $\gamma : I \to X$ through $x$ is a backward solution, if we have $\max I = 0$, and the collection of all backward solutions through $x$ is abbreviated by $\text{Sol}^-(x, \varphi)$. Dually, the notion of forward solution is for the case $\min I = 0$ and leads to the set $\text{Sol}^+(x, \varphi)$. Note that every point $x \in X$ admits forward solutions, i.e., we always have $\text{Sol}^+(x, \varphi) \neq \emptyset$. In general, however, backward solutions only exist in the case of a flow. If $\varphi$ is a semiflow there may be points without backward solutions, so-called start points. While such points can exist in many semiflows, the following result rules them out in a number of interesting situations.
Proposition 3.1 (No start points on manifolds). Let \( \varphi : X \times \mathbb{R}_0^+ \to X \) be a semiflow on a metric space \( X \), and suppose that \( x \in X \) has an open neighborhood which is homeomorphic to \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \). Then \( x \) is not a start point, i.e., we have \( \text{Sol}^- (x, \varphi) \neq \emptyset \).

This proposition is originally due to H. Halkin, and its proof can be found for example in [6] Theorem 11.8] or [9] Theorem 4.1. For later use, we also have to introduce the following convenient notation. Suppose that \( \gamma \in \text{Sol}^\pm (x, \varphi) \) is a backward or forward solution through \( x \) with domain \( I \). Then if \( J \subset \mathbb{R} \) is any interval we slightly abuse notation and define

\[
\gamma (J) := \gamma (J \cap I) \quad \text{where} \quad \gamma : I \to X. \tag{6}
\]

This convention will be useful for our definitions of exit and entry sets below.

While the notion of solution incorporates time dependence, the following terms focus on the notion of sections as considered in [21]. For this, let \( \Sigma \) be a section. Then for every \( x \in X \) we can define \( \varphi (x, (0, \varepsilon)) \cap \Sigma \neq \emptyset \). In this case the trajectory through \( x \) is called a periodic trajectory or a periodic orbit. The minimal positive time \( T \) for which \( \varphi (x, T) = x \) is called the basic period of the trajectory. By the \( \omega \)-limit set of \( x \) we mean

\[
\omega (x) := \bigcap_{t > 0} \text{cl} \varphi (x, [t, \infty)).
\]

If the space \( X \) is compact, then \( \omega (x) \neq \emptyset \) for each \( x \in X \). For an arbitrary set \( A \subset X \), the invariant part \( \text{Inv}(A, \varphi) \) of \( A \) is the set of all \( x \in A \) such that there exists a full solution through \( x \) with values in \( A \). If the invariant part of \( A \) is equal to \( A \), then the set \( A \) is called invariant. In particular, each \( \omega \)-limit set is invariant.

3.2.2. Sections. For proving the existence of periodic orbits, we will make extensive use of the notion of sections as considered in [21]. For this, let \( \varepsilon > 0 \), and for \( A \subset X \) define the \( \varepsilon \)-left collar of \( A \) as

\[
L_\varepsilon (A) := \{ x \in X \mid \varphi (x, (0, \varepsilon)) \cap A \neq \emptyset \}. \tag{7}
\]

Then a section for \( \varphi \) is a subset \( \Sigma \subset X \) such that there exists an \( \varepsilon > 0 \) with the following properties:

(i) The \( \varepsilon \)-left collar \( L_\varepsilon (\Sigma) \) of \( \Sigma \) is open, and

(ii) if \( x \in L_\varepsilon (\text{cl} \Sigma) \), then the set \( \varphi (x, (0, \varepsilon)) \cap \text{cl} \Sigma \) consists of exactly one element.

Let \( \Sigma \) be a section. Then for every \( x \in L_\varepsilon (\Sigma) \) we can define \( \xi (x) \in \Sigma \) as the unique point given by property (ii) above. According to [21] Proposition 3.1, the so-defined map \( \xi : L_\varepsilon (\Sigma) \to \Sigma \) is continuous, which immediately implies that an open subset of a section is a section as well.

Let \( N \subset X \) and let \( \Sigma \) be a section. We call \( \Sigma \) a global section in \( N \) (or, as in [21], a Poincaré section in \( N \) if \( N \) is an isolating neighborhood) if for each \( x \in N \) its positive semi-trajectory intersects \( \Sigma \). If \( X = N \), we simply call \( \Sigma \) a global section.

Being able to construct global sections for a semiflow is crucial for proving the existence of periodic orbits. For our results, we make use of an approach due to Fuller [16], which can be
described as follows. Denote the unit circle in the complex plane by $S^1$, and let $f: X \to S^1$ be continuous. For every point $x \in X$ let $F_x: \mathbb{R}_0^+ \to \mathbb{R}$ denote an arbitrary lift of the map
\[
\mathbb{R}_0^+ \ni t \mapsto f(\varphi(x, t)) \in S^1
\]
with respect to the covering map $\exp i(\cdot)$, i.e., suppose that $\exp iF_x(t) = f(\varphi(x, t))$ holds for all $t \in \mathbb{R}_0^+$. Following [16] we then define the map
\[
\Delta f := \Delta \varphi f: X \times \mathbb{R}_0^+ \to \mathbb{R} \quad \text{as} \quad \Delta f(x, t) := F_x(t) - F_x(0).
\]
One can easily see that this map is independent of the choice of lift $F_x$ and that it is continuous. Moreover, we have both $\Delta f(x, 0) = 0$ and $f(\varphi(x, t)) = f(x) \exp i\Delta f(x, t)$ for $t \in \mathbb{R}_0^+$, as well as
\[
\Delta f(x, t + s) = \Delta f(x, t) + \Delta f(\varphi(x, t), s) \quad \text{for all} \quad t, s \in \mathbb{R}_0^+. \quad (7)
\]
The function $\Delta f$ provides the following mechanism for the construction of a global section.

**Proposition 3.2** (Global sections via angular function positivity). Assume that $X$ is compact, and consider a continuous angular function $f: X \to S^1$. If the above-defined function $\Delta f$ satisfies the strict inequality $\Delta f(x, t) > 0$ for every $x \in X$ and $t > 0$, then for each point $z \in S^1$ the inverse image $f^{-1}(z)$ is a global section.

**Proof:** We need to show that the set $\Sigma := f^{-1}(z)$ is a global section. For this, note that in view of (7) and the assumption of the proposition the map $\Delta f(x, \cdot)$ is strictly increasing for every $x \in X$.

We claim first that for every $x \in X$ the positive semi-trajectory of $x$ has to intersect $\Sigma$. Suppose this is not the case. Then the map $\Delta f(x, \cdot)$ has to be bounded, and therefore it converges as $t \to \infty$. But then the set $f(\omega(x))$, which is non-empty due to the assumed compactness of $X$, can consist of only one element, which in fact is equal to $\lim_{t \to \infty} f(x) \exp i\Delta f(x, t)$. This in turn implies that the function $\Delta f(y, \cdot)$ is constant for each $y \in \omega(x)$, contrary to our assumption. This establishes the claim.

We now return to the discussion of $\Sigma$. Due to the compactness of $\Sigma$ there exists an $\varepsilon > 0$ such that
\[
\Delta f(\Sigma, [0, \varepsilon]) \subset [0, \pi). \quad (8)
\]
It follows immediately from [16] that the $\varepsilon$-left collar $L_\varepsilon(\Sigma) = L_\varepsilon(\text{cl} \Sigma)$ satisfies (ii).

For the proof of (i), assume finally that $x \in L_\varepsilon(\Sigma)$. Then the inclusion [8] implies that there exists a neighborhood $U$ of $x$ such that $f(\varphi(y, [0, \varepsilon])) \subset S^1 \setminus \{-z\}$ for every $y \in U$. Consider now the map $\alpha: S^1 \setminus \{-z\} \to (\theta - \pi, \theta + \pi)$ defined as the inverse of $\exp i(\cdot)$, where we use the abbreviation $\alpha(z) = \theta$. Then for each $y \in U$ one has
\[
\Delta f(y, t) = \alpha(f(\varphi(y, t))) - \alpha(f(y)) ,
\]
and therefore $\alpha(f(\varphi(y, \cdot)))$ is strictly increasing on the interval $[0, \varepsilon]$. It follows, in particular, that both $\alpha(f(x)) < \theta$ and $\alpha(f(\varphi(x, \varepsilon))) > \theta$ are satisfied — and the same inequalities hold if $x$ is replaced by $y$ from some neighborhood $V$ of $x$. Therefore, for each $y \in U \cap V$ there exists a $t_y \in (0, \varepsilon)$ such that $f(\varphi(y, t_y)) = z$. This establishes the inclusion $U \cap V \subset L_\varepsilon(\Sigma)$, which proves (i). Thus, the set $\Sigma$ satisfies all the required conditions and the result follows. \[\square\]

The above Proposition 3.2 shows that if the phase space $X$ is compact, then the notion of a surface of section considered in [16] is a particular case of the notion of a global section.
invariant if it equals the invariant part of
3.2.3. Isolated invariant sets and isolating blocks.

Proposition 3.3 (Global sections via positivity at discrete times). Assume that \( X \) is compact, and consider a continuous angular function \( f : X \to S^1 \). If for every \( x \in X \) there exists a time \( t_x > 0 \) such that \( \Delta \varphi f(x, t_x) > 0 \), then there exists a global section for the semiflow \( \varphi \).

3.2.3. Isolated invariant sets and isolating blocks. Recall that a subset \( A \subset X \) is called invariant if it equals the invariant part of \( A \), i.e., if we have \( A = \text{Inv}(A, \varphi) \). In the study of dynamical systems, invariant sets take a prominent role. Unfortunately, however, general invariant sets can be extremely sensitive to perturbations and change dramatically. To remedy this, Conley \( [11] \) introduced the concept of isolated invariant set. For this, let \( N \) be a compact nonempty subset. Then \( \text{Inv}(N, \varphi) \) is called an isolated invariant set. For this, let \( N \subset X \) be a compact nonempty subset. Then \( N \) is called an isolating neighborhood if \( \text{Inv}(N, \varphi) \subset \text{int} N \), and we call \( S = \text{Inv}(N, \varphi) \) an isolated invariant set. In other words, the invariant set \( S \) is maximal in its neighborhood in \( N \). As it turns out, this maximality property makes it much easier to study isolated invariant sets, as one can employ topological tools such as the Conley index \( [11] \).

In general, an isolated invariant set \( S \) can have many associated isolating neighborhoods \( N \). For the purposes of this paper, we will focus our attention to specific isolating neighborhoods called isolating blocks. For this, let \( B \subset X \) be a given compact set and define its weak exit set and weak entry set respectively as

\[
B^{w-} := \{ x \in \text{bd} B \mid \forall \varepsilon > 0 \text{ we have } \varphi(x, (0, \varepsilon)) \cap (X \setminus B) \neq \emptyset \},
\]

\[
B^{w+} := \{ x \in \text{bd} B \mid \forall \gamma \in \text{Sol}^-(x, \varphi) \text{ and } \forall \varepsilon > 0 \text{ we have } \gamma((0, \varepsilon)) \cap (X \setminus B) \neq \emptyset \},
\]

where we use the convention introduced in \( [6] \). We would like to point out that if \( x \in \text{bd} B \) is a start point, then one automatically has \( x \in B^{w+} \) in view of \( \text{Sol}^-(x, \varphi) = \emptyset \). Using these notions, we call a compact set \( B \subset X \) an isolating block, if

\[
\text{bd} B = B^{w-} \cup B^{w+} \quad \text{and} \quad B^{w-} \text{ is closed.} \tag{9}
\]

In order to illustrate this definition, note that \( x \in \text{bd} B \setminus (B^{w-} \cup B^{w+}) \) if and only if there exists a solution \( \gamma : (-\varepsilon_1, \varepsilon_2) \to X \) through \( x \) with \( \gamma((-\varepsilon_1, \varepsilon_2)) \subset B \), i.e., this solution creates an internal semiflow or flow tangency. Thus, the first condition in \( [9] \) rules out such internal tangencies. As for the second condition, consider the exit-time function of a set \( B \) defined as

\[
\tau_B : B \ni x \mapsto \sup\{ t \geq 0 \mid \varphi(x, [0, t]) \subset B \} \in [0, \infty]. \tag{10}
\]

For general sets, the exit time function does not need to be continuous. This changes if we consider isolating blocks, as the following celebrated result shows.

Proposition 3.4 (Ważewski’s theorem). Let \( B \subset X \) be an isolating block for the semiflow \( \varphi \). Then the following hold:

(a) The exit-time function \( \tau_B : B \to [0, \infty] \) is continuous.
(b) We have \( x \in B^{w-} \) if and only if \( \tau_B(x) = 0 \).
(c) For every \( x \in B \) with \( \tau_B(x) < \infty \) one has \( \varphi(x, \tau_B(x)) \in B^{w-} \).
(d) The exit-time function \( \tau_B \) is bounded if and only if we have \( \text{Inv}(B, \varphi) = \emptyset \).

The proof of parts (a), (b), and (c) can be found in Lemmas 2.1 and 2.2 in \( [32] \), and in (d) it is clear that the boundedness of \( \tau_B \) implies \( \text{Inv}(B, \varphi) = \emptyset \). Moreover, if \( \tau_B(x) = \infty \), then \( \emptyset \neq \omega(x) \subset \text{Inv}(B, \varphi) \) due to the compactness of \( B \) and \( [12] \) Proposition 2.6(iii)].
We note here that if $B$ is an isolating block, then $(B, B^{w-})$ is a very special case of an index pair, a concept used to define the Conley index. In particular, we have the following proposition.

**Proposition 3.5** (Conley index of an isolating block). *If $B \subset X$ is an isolating block, then the homological Conley index of $\text{Inv}(B, \varphi)$ is given by the relative homology $H_*(B, B^{w-})$. □*

Despite the label “weak,” the above notions for exit and entry set are the ones that are usually used to define isolating blocks, see for example [11, 24, 32]. However, for our applications in this paper we need to consider a more restrictive notion of isolating blocks, which is characterized by more regular exit and entry behavior. For this, assume again that $B \subset X$ is a compact set and define the *strong exit* and *strong entry sets* respectively via

\[
B^- := \{ x \in \text{bd} B \mid \exists \varepsilon > 0 \text{ such that } \varphi(x, (0, \varepsilon)) \cap B = \emptyset \}, \\
B^+ := \{ x \in \text{bd} B \mid \forall \varepsilon > 0 \text{ such that } \gamma((-\varepsilon, 0)) \cap B = \emptyset \}.
\]

(11)

While it is clear from the definitions that both $B^- \subset B^{w-}$ and $B^+ \subset B^{w+}$ are satisfied, one can easily see that in general equality does not hold. Moreover, it is not immediately obvious that assuming the identity $\text{bd} B = B^- \cup B^+$ and the closedness of $B^-$ always imply that a compact set $B \subset X$ is an isolating block. Nevertheless, the following proposition shows that this is in fact the case.

**Proposition 3.6** (Properties of stronger exit and entry sets). *Let $B \subset X$ be a compact set. Then the following hold:

(a) If the strong exit set $B^-$ is closed, then we have $B^{w-} = B^-$. \\
(b) If $B^-$ is closed and $\text{bd} B = B^- \cup B^+$, then $B$ is an isolating block.*

**Proof:** It is clear that (b) follows from (a) and the inclusions $B^- \subset B^{w-}$ and $B^+ \subset B^{w+}$. In order to prove (a), note first that we only have to establish the inclusion $B^{w-} \subset B^-$. Suppose therefore that there exists a point $x \in B^{w-} \setminus B^-$. Then $x \notin B^-$ implies that

\[
\varphi(x, (0, \varepsilon)) \cap B \neq \emptyset \quad \text{for all } \varepsilon > 0,
\]

and in view of $x \in B^{w-}$ one has

\[
\varphi(x, (0, \varepsilon)) \cap (X \setminus B) \neq \emptyset \quad \text{for all } \varepsilon > 0.
\]

Thus, for every $n \in \mathbb{N}$ there exist times $0 < a_n < b_n < 1/n$ such that

\[
\varphi(x, a_n) \in B \quad \text{and} \quad \varphi(x, b_n) \notin B.
\]

Let $c_n = \sup\{t \in [a_n, b_n] \mid \varphi(x, t) \in B\}$. Then we have $\varphi(x, c_n) \in B$ since $B$ is closed, and this in turn implies the strict inequality $c_n < b_n$. Moreover, $\varphi(x, (c_n, b_n)) \subset X \setminus B$ due to the properties of the supremum, and therefore $\varphi(x, c_n) \in B^-$ for all $n \in \mathbb{N}$. Since we assumed that $B^-$ is closed, the inequality $0 < c_n < 1/n$ shows that $x = \lim_{n \to \infty} \varphi(x, c_n) \in B^-$, which contradicts our choice of $x$ and proves the result. □

As we will see in the next sections, the classical notion of isolating block does not suffice for our purposes. We therefore make use of the following stronger notion, which is motivated by the above result.

**Definition 3.7** (Strong isolating block). *Let $\varphi$ denote a semiflow on a locally compact metric space $X$, and let $B \subset X$ be a compact set. Then $B$ is called a strong isolating block if

\[
\text{bd} B = B^- \cup B^+ \quad \text{and} \quad B^\pm \text{ are both closed},
\]

\[
\text{Inv}(B, \varphi) \quad \text{and} \quad \text{Sol}(B, \varphi).
\]
where \( B^\pm \) denote the strong entry and exit sets defined in (11).

In view of Proposition 3.6 every strong isolating block is also an isolating block in the classical sense. But, as we will see in the next section, the additional assumptions lead to better-defined exit and entry behavior.

4. Periodic orbits via bricks and the Conley index

In this section we present a method for establishing the existence of periodic orbits in a semiflow or flow. Our approach is based on a decomposition of part of the phase space into smaller pieces, which are specific types of strong isolating blocks whose invariant part is empty. The main result is the subject of Theorem 4.6 where we use this decomposition in combination with the Conley index to obtain a periodic orbit. We would like to point out, however, that a different approach based on transfer maps in homology and the Lefschetz number will be presented in [27].

This section is organized as follows. The first subsection is devoted to a study of the underlying concept of bricks, while the second subsection introduces paths of bricks induced by a semiflow. Finally, the last subsection contains the statement and proof of our main result.

4.1. Bricks and brick paths. The main goal of this subsection is the formulation of combinatorial properties that imply the existence of a periodic orbit. For this, it is essential to have a decomposition of part of the phase space which allows one to track the flow behavior, which is based on the notion of strong isolating blocks defined above.

**Definition 4.1 (Bricks, proper pairs of bricks, and brick paths).** Let \( \varphi \) denote a semiflow on a locally compact metric space \( X \), and let \( B \subset X \). Then \( B \) is called a brick if the following conditions are satisfied:

(i) The set \( B \) is a strong isolating block in the sense of Definition 3.7.

(ii) The invariant part of \( B \) is empty, i.e., we have \( \text{Inv}(B, \varphi) = \emptyset \).

A pair \( (B, \tilde{B}) \) of two bricks \( B \) and \( \tilde{B} \) is called a proper pairs of bricks, if both

\[ B \neq \tilde{B} \quad \text{and} \quad B \cap \tilde{B} = B^- \cap \tilde{B}^+ \neq \emptyset \]

are satisfied. Finally, a finite sequence \( (B_0, \ldots, B_k) \) or an infinite sequence \( (B_0, B_1, \ldots) \) of bricks is called a brick path, if either one has \( k = 0 \) or else the pair \( (B_j, B_{j+1}) \) is a proper pair of bricks for all \( j = 0, \ldots, k-1 \) or all \( j \in \mathbb{N}_0 \), respectively. If, moreover, we have \( B_i \neq B_j \) for all \( i \neq j \), then the path is called injective.

The above notions are central for the results of this paper. Note that in view of Proposition 3.6 every brick \( B \) is an isolating block with \( B^{w^-} = B^- \). Furthermore, Proposition 3.4 shows that the exit-time function \( \tau_B \) is continuous and bounded, and that we have \( x \in \tilde{B}^- \) if and only if \( \tau_B(x) = 0 \). One can also easily show that if \( (B, \tilde{B}) \) is a proper pair of bricks, then one has

\[ B \cap \tilde{B} = B^- \cap \tilde{B}^+ = B \cap \tilde{B}^+ = B^- \cap \tilde{B} \neq \emptyset, \]

since due to the assumed compactness of \( B \) and \( \tilde{B} \) both the inclusions \( B^- \subset B \) and \( \tilde{B}^+ \subset \tilde{B} \) are satisfied.

The concept of bricks is a strengthening of isolating blocks that will prove to be extremely useful. For us, they are the building blocks of a phase space decomposition which allows for
the use of combinatorial techniques in a flow or semiflow setting. More precisely, we consider the following decompositions.

**Definition 4.2 (Brick decomposition).** Let $\varphi$ denote a semiflow on a locally compact metric space $X$, and let $A \subset X$ be a compact flow region of interest which is not necessarily invariant, i.e., its exit set $A^w$ could be nonempty. Then a brick decomposition of $A$ is a finite collection $\mathcal{A}$ of subsets of $A$ such that the following hold:

1. The collection $\mathcal{A}$ covers $A$, i.e., we have $A = \bigcup_{\sigma \in \mathcal{A}} \sigma$.
2. Every set $\sigma \in \mathcal{A}$ is a brick in the sense of Definition 4.1.
3. If the bricks $\sigma_1, \sigma_2 \in \mathcal{A}$ are such that both $\sigma_1 \neq \sigma_2$ and $\sigma_1 \cap \sigma_2 \neq \emptyset$ are satisfied, then either $(\sigma_1, \sigma_2)$ or $(\sigma_2, \sigma_1)$ is a proper pair of bricks, i.e., we have either

   $$\sigma_1 \cap \sigma_2 = \sigma_1^- \cap \sigma_2^+ \quad \text{or} \quad \sigma_1 \cap \sigma_2 = \sigma_1^+ \cap \sigma_2^-,$$

   respectively.

For later use, we note the following consequence of the last of these assumptions.

**Lemma 4.3 (Nontrivial solution pieces are contained in at most one brick).** Let $\mathcal{A}$ be a brick decomposition of the compact set $A \subset X$ in the sense of Definition 4.2. Then for every $x \in A$ and every $t > 0$ with $\varphi(x, [0, t]) \subset A$ there exists at most one brick $\sigma \in \mathcal{A}$ which satisfies the inclusion $\varphi(x, [0, t]) \subset \sigma$.

**Proof:** If $\varphi(x, [0, t])$ is not contained in any brick we are clearly done. Suppose now that there are two bricks $\sigma_1 \neq \sigma_2$ such that both $\varphi(x, [0, t]) \subset \sigma_1$ and $\varphi(x, [0, t]) \subset \sigma_2$ are satisfied. This implies $x \in \sigma_1 \cap \sigma_2$, and Definition 4.2(iii) further yields either $x \in \sigma_1^-$ or $x \in \sigma_2^-$. But then one either has $\varphi(x, (0, \varepsilon)) \subset X \setminus \sigma_1$ or $\varphi(x, (0, \varepsilon)) \subset X \setminus \sigma_2$, respectively, for some $\varepsilon > 0$, which contradicts $\varphi(x, [0, t]) \subset \sigma_1 \cap \sigma_2$. $\square$

We would like to point out that in Definition 4.2 we do not pose any further dynamical requirements on the set $A$. Nevertheless, having a brick decomposition does impose constraints on $A$, as the following result shows.

**Proposition 4.4 (Exit set characterization via a brick decomposition).** Let $\mathcal{A}$ be a brick decomposition of the compact set $A \subset X$ in the sense of Definition 4.2. Then we have:

(a) The weak exit set and the exit set of $A$ coincide, i.e., one has $A^w = A^\tau$.

(b) For every $x \in A$ the inclusion $x \in A^\tau$ is satisfied if and only if we have $\tau_x(x) = 0$ for every $\sigma \in \mathcal{A}$ with $x \in \sigma$.

**Proof:** To prove (a), suppose that $x \in A^w$. If $\sigma \in \mathcal{A}$ is a brick which does not contain $x$, then due to the compactness of $\sigma$ there exists an $\varepsilon_\sigma > 0$ with $\varphi(x, (0, \varepsilon_\sigma)) \cap \sigma = \emptyset$. On the other hand, if $x \in \sigma \in \mathcal{A}$, then $x \in A^w$ and Definition 4.2 imply $x \in \sigma^{-}$, and therefore $x \in \sigma^{-}$ due to Proposition 3.6(a). Thus, also in this case there exists an $\varepsilon_\sigma > 0$ such that $\varphi(x, (0, \varepsilon_\sigma)) \cap \sigma = \emptyset$. If we now define $\varepsilon = \min\{\varepsilon_\sigma \mid \sigma \in \mathcal{A}\} > 0$, then we immediately obtain $\varphi(x, (0, \varepsilon)) \cap A = \cup_{\sigma \in \mathcal{A}}(\varphi(x, (0, \varepsilon)) \cap \sigma) = \emptyset$, i.e., one has $x \in A^\tau$. Together with $A^\tau \subset A^w$, this completes the proof of (a).

As for (b), suppose first that $x \in A^\tau$, and let $\sigma \in \mathcal{A}$ be such that $x \in \sigma$. Then there exists an $\varepsilon > 0$ with $\varphi(x, (0, \varepsilon)) \subset X \setminus A \subset X \setminus \sigma$, and this shows $x \in \sigma^{-}$, as well as $\tau_x(x) = 0$ due to Proposition 3.4(b) and Proposition 3.6(b).

Finally, suppose that $x \in A$ and that $\tau_x(x) = 0$ holds for every $\sigma \in \mathcal{A}$ with $x \in \sigma$. In view of Propositions 3.4(b) and 3.6(b), combined with the finiteness of the collection $\mathcal{A}$, there
exists an $\varepsilon > 0$ such that $\varphi(x,(0,\varepsilon)) \subset X \setminus \sigma$ for every $\sigma$ with $x \in \sigma \in \mathcal{A}$. Moreover, the union $\bigcup_{x \notin \sigma \in \mathcal{A}} \sigma$ is compact, and it does not contain $x$. Hence, one can find an $0 < \varepsilon_0 < \varepsilon$ such that the set $\varphi(x,(0,\varepsilon_0))$ is disjoint from each $\sigma \in \mathcal{A}$, i.e., it is disjoint from $A$. This implies $x \in A^-$, and completes the proof of the proposition.

Despite the implications of Proposition 4.4, in general it is not true that a set $A$ which has a brick decomposition is an isolating block itself. For example, one can easily see that if the bricks create a non-convex corner in $A$, then the exit set $A^-$ might no longer be closed and the semiflow $\varphi$ could have an internal tangency.

4.2. Brick paths induced by a semiflow. Suppose now that we are given a forward solution of a semiflow $\varphi$, which originates at some point in a region with a brick decomposition. As the following result shows, this leads to a well-defined sequence of bricks that are traversed by this solution.

**Proposition 4.5** (Brick paths induced by a brick decomposition). Let $\mathcal{A}$ be a brick decomposition of the compact set $A \subset X$ in the sense of Definition 4.2. Then for every $x \in A \setminus A^-$ exactly one of the following two statements holds:

(a) There exists a unique path $(\sigma_0,\sigma_1,\ldots,\sigma_n)$ of bricks $\sigma_k \in \mathcal{A}$ with $\sigma_k \neq \sigma_{k+1}$ and a finite collection of times $0 = t_0 < t_1 < \ldots < t_{n+1}$ with
\[
\varphi(x,[t_j,t_{j+1}]) \subset \sigma_j \quad \text{and} \quad \varphi(x,t_{j+1}) \in \sigma_j^-(12)
\]
for all $j = 0, 1, \ldots, n$, as well as $\varphi(x,t_{n+1}) \in A^-$ and $\tau_A(x) = t_{n+1}$.

(b) There exists a unique path $(\sigma_0,\sigma_1,\ldots)$ of infinitely many bricks $\sigma_k \in \mathcal{A}$ with $\sigma_k \neq \sigma_{k+1}$ and a sequence of times $0 = t_0 < t_1 < \ldots$ such that (12) is satisfied for all $j \in \mathbb{N}_0$. Furthermore, we have $t_j \to \infty$ as $j \to \infty$, as well as $\tau_A(x) = \infty$.

**Proof:** Due to Proposition 4.4(b) and our choice $x \in A \setminus A^-$ there exists a brick $\sigma_0 \in \mathcal{A}$ with $\tau_{\sigma_0}(x) > 0$. Furthermore, in view of Lemma 4.3 this brick is uniquely defined. We now define $\sigma_0 = 0$ and set $t_1 = \tau_{\sigma_0}(x)$. Then Proposition 3.4(c) and Proposition 3.6 imply the inclusion $\varphi(x,t_1) \subset \sigma_0^-$, and we clearly have $\varphi(x,[0,t_1]) \subset \sigma_0$.

Suppose now that we have constructed times $0 = t_0 < \ldots < t_{n+1}$ and bricks $\sigma_0,\ldots,\sigma_n$ such that (12) holds for all $j = 0,\ldots,n$, where $n \in \mathbb{N}_0$. Let $x_{n+1} = \varphi(x,t_{n+1})$. According to (12) one has $x_{n+1} \in \sigma_n^- \subset A$. If in addition we have $x_{n+1} \in A^-$, then we have established alternative (a) of the proposition.

On the other hand, if $x_{n+1} \notin A^-$, then $x_{n+1} \in A \setminus A^-$, and as above one can construct a unique brick $\sigma_{n+1}$ with $x_{n+1} \in \sigma_{n+1}^-$ and $\tau_{\sigma_{n+1}}(x_{n+1}) > 0$, as well as $\sigma_{n+1} \neq \sigma_n$. If we now define the time $t_{n+2} = t_{n+1} + \tau_{\sigma_{n+1}}(x_{n+1})$, then we have both
\[
\varphi(x,t_{n+2}) = \varphi(\varphi(x,t_{n+1}),\tau_{\sigma_{n+1}}(x_{n+1})) = \varphi(x_{n+1},\tau_{\sigma_{n+1}}(x_{n+1})) \subset \sigma_{n+1}^-
\]
and
\[
\varphi(x,[t_{n+1},t_{n+2}]) = \varphi(x_{n+1},[0,\tau_{\sigma_{n+1}}(x_{n+1})]) \subset \sigma_{n+1}.
\]
Thus, (12) holds for $j = 0,\ldots,n+1$. If one proceeds recursively, this leads to an infinite sequence of bricks and times as in (b), unless of course one has $x_j \in A^-$ for some $j \in \mathbb{N}$ as noted above.

In order to complete the proof of the proposition, it only remains to be shown that if the above procedure leads to an infinite sequence of times, then we have $t_j \to \infty$ as $j \to \infty$. 
Suppose this is not the case, i.e., the increasing sequence \((t_j)_{j \in \mathbb{N}}\) is bounded. Then the limit \(t_\star = \lim_{j \to \infty} t_j\) exists and \(x_\star = \varphi(x, t_\star) = \lim_{j \to \infty} x_j = \lim_{j \to \infty} \varphi(x, t_j)\).

Since the brick collection \(A\) is finite, there exists a brick \(\sigma_\star \in A\) and a subsequence \((t_{k_j})_{j \in \mathbb{N}}\) of times such that \(\sigma_{k_j} = \sigma_\star\) for all \(j \in \mathbb{N}\). Furthermore, according to our above construction we have \(x_{k_j} \in \sigma_{k_j} \cap \sigma_{k_j-1}\) and \(\varphi(x, [t_{k_j-1}, t_{k_j}]) \subset \sigma_{k_j-1}\), which immediately yields \(x_{k_j} \in \sigma_{k_j}^+ = \sigma_\star^+\) in view of Lemma 4.3 and Definition 4.2(iii). Now recall that due to Definition 4.1(i) the entry set \(\sigma_\star^+\) is closed, and therefore \(x_\star = \lim_{j \to \infty} x_{k_j} \in \sigma_\star^+\). Thus, there exists an \(\varepsilon > 0\) such that on the one hand we have

\[ \varphi(x, (t_\star - \varepsilon, t_\star)) \subset X \setminus \sigma_\star, \]

while on the other hand the inclusions

\[ \varphi(x, [t_{k_j}, t_{k_j}+1]) \subset \sigma_{k_j} = \sigma_\star \]

are satisfied for all \(j \in \mathbb{N}\). Together with \(t_\star = \lim_{j \to \infty} t_{k_j}\) this leads to a contradiction and completes the proof of the proposition.

Notice that the above proof heavily relies on the assumed closedness of the entry sets of bricks, even though this closedness is not required in order to ensure that every brick is an isolating block.

### 4.3. Periodic orbits via brick decompositions

The last two subsections have laid the foundation for establishing the existence of periodic orbits using combinatorial techniques. By introducing the notions of brick paths and brick decompositions in Definitions 4.1 and 4.2 we have a purely combinatorial way of encoding potential semiflow or flow solution behavior inside a region of interest. In particular, the notion of proper pair of bricks encodes the direction in which solutions can move between bricks. Needless to say, however, not every combinatorial brick path necessarily leads to a semiflow or flow solution which traverses the bricks in the given order. As the last subsection showed, one can in general only create a unique brick path from a solution, not vice versa.

In the present subsection, we will show that under suitable conditions it is possible to deduce the existence of a periodic orbit for the given semiflow or flow from the existence of an appropriate brick decomposition. Our result is motivated by and based on the work [21]. While the main result in [21] contains a fairly strong assumption concerning the existence of a global section, in our approach we can deduce the existence of this section from a fairly mild combinatorial requirement on brick paths. This condition in turn relies on an appropriate coarsening of the given brick decomposition. More precisely, we have the following main result of this section. For this, recall that \(\mathbb{Z}_p = \{0, 1, \ldots, p-1\}\) denotes the finite cyclic additive group of order \(p\) with respect to addition modulo \(p\).

**Theorem 4.6** (Periodic orbits via brick decompositions). Let \(A\) be a brick decomposition of the compact set \(A \subset X\) in the sense of Definition 4.2. Assume that the following hold:

(a) The set \(A\) is an isolating block for the semiflow or flow \(\varphi\), and both the isolating block \(A\) and its exit set \(A^- = A^w^-\) is an ANR.

(b) For either \(r = 0\) or \(r = 1\) we have

\[ \dim H_{2n+r}(A, A^-) = \dim H_{2n+1+r}(A, A^-) \quad \text{for all} \quad n \in \mathbb{Z}, \]

and not all of these homology groups are trivial.
In addition, we assume that the brick decomposition $\mathcal{A}$ has a coarsening into $p \in \mathbb{N}$ subcollections in the following sense. Suppose that for $i \in \mathbb{Z}_p$ there are subcollections $\emptyset \neq A_i \subset \mathcal{A}$ which are pairwise disjoint, and such that $\bigcup_{i=0}^{p} A_i = \mathcal{A}$. Let $A_i := \bigcup_{\sigma \in A_i} \sigma \subset \mathcal{A}$, then we clearly have $\mathcal{A} = \bigcup_{i=0}^{p} A_i$. In the following, suppose further that whenever $i$ or $j$ are indices of these sets or subcollections, additions involving them are always performed in the cyclic group $\mathbb{Z}_p$. For example, if $i = p - 1$ then $A_{i+1} = A_0$. Then we assume that this coarsening has the following properties:

(c) The number $p$ of subcollections $A_i$ satisfies $p \geq 3$.

(d) We have $A_i \cap A_{i+1} \neq \emptyset$ for all $i \in \mathbb{Z}_p$.

(e) We have $A_i \cap A_j = \emptyset$ for all $i, j \in \mathbb{Z}_p$ with $j \notin \{i + 1, i, i - 1\}$.

(f) For every index $i \in \mathbb{Z}_p$, every brick $\sigma_0 \in A_i$, and every maximal brick path $(\sigma_0, \sigma_1, \ldots)$ in $\mathcal{A}$ there exists a $k \geq 0$ such that either $\sigma_k \subset \mathcal{A}^-$, or one has $\sigma_k \in A_{i+1}$, while for all $\ell = 1, \ldots, k - 1$ we have $\sigma_\ell \notin A_{i+2}$.

Then the isolating block $A$ contains a non-trivial periodic orbit.

**Proof:** As mentioned before the formulation of the theorem, we prove our result by applying Theorem 1.3 in [21]. The latter result guarantees the existence of a periodic orbit as long as (b) is satisfied, an isolating neighborhood of the invariant part of $A$ has a Poincaré section, the phase space is an ANR, and the semiflow has compact attraction. Since (b) is already assumed, in order to verify the other conditions we embed the semiflow $\varphi$ restricted to $A$ into a semiflow on a new compact ANR which has a global section.

For this, we use the following convention. If $w$ and $z$ are points of the unit circle $S^1$, then $(w, z)$ and $[w, z]$ denote the open and closed arc joining $w$ and $z$ in the positive direction, respectively. Furthermore, for $m \in \mathbb{Z}$ we set $z_m := e^{2\pi im/p}$. For every $m = 0, \ldots, p - 1$ we denote by $g_m$ a Urysohn lemma map which transfers the compact set $A_m$ to the arc $[z_m, z_{m+1}] \subset S^1$ such that $g_m|A_{m-1}\cup A_m$ is constant equal to $z_m$, and $g_m|A_m\cap A_{m+1}$ is constant equal to $z_{m+1}$. Note that the existence of $g_m$ follows from (d) and (e). Gluing all maps $g_m$, one then obtains a continuous map $g: A \rightarrow S^1$.

Let $Y := A \cup_{|A^-} S^1$ denote the adjunction. Furthermore, consider the semiflow $\Phi$ on the space $Y$ defined as

$$\Phi(y, t) := \begin{cases} 
\varphi(y, t) & \text{for } y \in A \text{ and } 0 \leq t \leq \tau_A(y), \\
e^{i(t-\tau_A(y))}g(\varphi(y, \tau_A(y))) & \text{for } y \in A \text{ and } t \geq \tau_A(y), \\
e^{it}y & \text{for } y \in S^1 \text{ and } t \geq 0.
\end{cases}$$

One can readily verify that the continuity of $\Phi$ is a consequence of Proposition 3.4(a). Clearly, the adjunction $Y$ is the required new phase space, which is in fact a compact ANR. Moreover, the motion of $\varphi$ inside $A$ is preserved by the semiflow $\Phi$. Thus, it remains to show that $\Phi$ has a global section, since the one restricted to the interior of $A$ then satisfies our requirements.

For this, we extend the map $g$ to a map $G: Y \rightarrow S^1$ given by

$$G(y) := \begin{cases} 
g(y) & \text{for } y \in A, \\
y & \text{for } y \in S^1.
\end{cases}$$

According to Proposition 3.3 the required section exists if for every $y \in Y$ one can find a time $t_y > 0$ which satisfies $\Delta_\Phi G(y, t_y) > 0$. This is immediately clear for $y \in S^1$, since in this case $t_y > 0$ can be chosen arbitrarily.
Consider now a point $y \in A \setminus A^-$ with $\tau_A(y) < \infty$. Then the identity $\Delta_\varphi G(y, t) = \Delta_\varphi g(y, t)$ holds for all $0 \leq t < \tau_A(y)$. If in addition one has the strict inequality $\Delta_\varphi g(y, \tau_A(y)) > 0$, then we can choose $t_y := \tau_A(y)$. On the other hand, if this inequality does not hold, then in view of (7) we can choose any $t_y > \tau_A(y) - \Delta_\varphi g(y, \tau_A(y))$. Either way, also in this case one obtains $\Delta_\varphi G(y, t_y) > 0$, as required.

Finally, consider the remaining case of a point $y \in A \setminus A^-$ with $\tau_A(y) = \infty$. According to our assumption (f) there exists an index $k \in A_{m+1}$, which in turn implies $g(y) = e^{2\pi i T}$ for some $m/p \leq T < (m+1)/p$. In order to prove the existence of a time $t_y$ for which $\Delta_\varphi g(y, t_y) > 0$ we consider the infinite path $(\sigma_0, \sigma_1, \ldots)$ associated to $y$ which is guaranteed by Proposition 4.5(b), with associated transition times $0 = t_0 < t_1 < \ldots$. Moreover, Proposition 4.5(a) guarantees that none of the exit sets $\sigma_j^-$ is contained in $A^-$. Since $y \in \sigma_0$, we either have $\sigma_0 \in A_m$ or $\sigma_0 \in A_{m-1}$. Note that in the latter case one necessarily has $g(y) = z_m$.

We now distinguish two cases, namely $\sigma_0 \in A_m$ and $\sigma_0 \in A_{m-1} \setminus A_m$. To begin with, suppose that $\sigma_0 \in A_m$, which immediately yields

$$\varphi(y, [0, t_1)) \subset [z_m, z_{m+1}].$$

(13)

According to our assumption (f) there exists an index $k$ such that $\sigma_k \in A_{m+1}$, which gives

$$g(\varphi(y, t_k)) \in [z_{m+1}, z_{m+2}],$$

(14)

and none of the bricks $\sigma_1, \ldots, \sigma_{k-1}$ belongs to the collection $A_{m+2}$. Together with (13) this implies

$$g(\varphi(y, [0, t_k)) \cap (z_{m+2}, z_{m+3}) = \emptyset.$$  

(15)

If additionally we select $k$ as the minimal one satisfying (f), then we have $\sigma_{k-1} \in A_m$. Together with (14) this implies $g(\varphi(y, t_k)) = z_{m+1}$. But then (15) shows that

$$\Delta_\varphi g(y, t_k) = 2\pi \left( \frac{m+1}{p} - T \right) > 0,$$

and we therefore can choose $t_y := t_k$ in the first case.

Finally, we consider the second case $\sigma_0 \in A_{m-1} \setminus A_m$. Under this assumption one obtains the inclusion $g(y, [0, t_1)) \subset [z_{m-1}, z_m]$. As before, it follows from (f) that there exists an index $k_*$ such that $\sigma_{k_*} \in A_m$, and one also has

$$g(\varphi(y, [0, t_{k_*}))) \cap (z_{m+1}, z_{m+2}) = \emptyset.$$  

If we again select $k_*$ as minimal, then one further has $g(y_{k_*}) = z_m$ for $y_{k_*} := \varphi(y, t_{k_*})$. Recall that $g(y) = z_m$, which immediately shows that $\Delta_\varphi g(y, t_{k_*}) = 0$. Thus, if we apply the previously considered case to the point $y_{k_*}$, then we have $T = m/p$, and we get the existence of a time $t > 0$ such that $\Delta_\varphi g(y_{k_*}, t) = 2\pi((m+1)/p - T) = 2\pi/p$. Finally, we can now choose the time $t_y := t_{k_*} + t$, for which (7) yields the strict inequality

$$\Delta_\varphi g(y, t_y) = \Delta_\varphi g(y_{k_*}, t) > 0,$$

and this completes the proof of the theorem. 

\[ \square \]

For an illustration of the assumptions of the theorem we refer the reader to Figure 7. Notice that for the theorem to apply, one needs to be able to subdivide the isolating block $A$ into at least three segments $A_0, \ldots, A_{m-1}$ in such a way that there is a well-defined overall semiflow motion in the direction of increasing indices. We would like to point out, however, that the specific assumptions are fairly mild:
Figure 7. Illustration of assumption (f) in Theorem 4.6. It is not necessary that the intersections $A_i \cap A_{i+1}$ of successive segments are flow transverse. There may be brick paths, such as the one shown in red, that first backtrack to a segment $A_j$ with $j < i$ before reaching $A_{i+1}$, as long as it does not loop around to reach $A_{i+2}$ first. Also, brick paths are allowed to reach the exit set $A^-$, such as the one shown in orange.

- While the individual bricks in the collection $\mathcal{A}$ have to be isolating blocks that satisfy additional assumptions, the coarsened segments $A_i$ do not need to be isolating blocks. In particular, the semiflow behavior of $\varphi$ across the intersections $A_i \cap A_{i+1}$ does not have to be unidirectional, and there certainly can be internal flow tangencies to these intersections.

- In order to ensure an overall semiflow direction, one only needs to verify that bricks paths starting in the collection $\mathcal{A}_i$ cannot reach $A_{i+2}$ by looping around in the direction of decreasing indices and thereby avoiding $A_{i+1}$. But they most certainly are allowed to first backtrack before reaching $A_{i+1}$.

In other words, as long as one can generate a sufficiently fine brick decomposition, the coarsening which is required for the application of Theorem 4.6 will be achievable fairly easily.

5. Recurrence in combinatorial topological dynamics

In the last section we presented an approach for proving the existence of periodic orbits in semiflows, which was based on a suitable brick decomposition of part of the phase space. In the present section, we will show that such decompositions can be constructed through combinatorial multivector fields on cellular spaces.

Multivector fields as introduced in [20, 26] constitute a combinatorial counterpart of classical vector fields. They provide an abstract language and methods to design, analyze, and use algorithms facilitating the automated investigation of classical dynamical systems. Traditionally, graph theory is used in this role. Multivector fields may be viewed as directed graphs whose vertices constitute a finite topological space. In this way algorithms based on the methods of topological dynamics may be invented, designed, and tested without the
need of relating them to conventional flows. Classical dynamics is needed only in applications to conventional vector fields: First on input when the combinatorial multivector field is constructed, and then on output when the results are interpreted. But we would like to emphasize that in order to derive the brick decomposition used in the last section, the language of multivector fields proves to be an ideal tool.

The remainder of this section is organized as follows. We begin by introducing the combinatorial framework which provides the link between semiflows and the results of the previous section. For this, Section 5.1 recalls basic definitions and results for cell complexes. This is followed in Section 5.2 by the notion of strongly admissible semiflows, whose behavior can be encoded by studying the flow across cell boundaries. Section 5.3 then describes the concept of combinatorial multivector fields and shows how they can be generated in a canonical way from strongly admissible semiflows. After that, we turn our attention to how the combinatorial multivector field $V$ associated with an admissible semiflow $\varphi$ can be used to establish the existence of periodic orbits for $\varphi$. This is accomplished by first showing in Section 5.4 how isolated invariant sets in the combinatorial setting give rise to associated isolated invariant sets for admissible semiflows, and then introducing combinatorial versions of Poincaré sections and Lyapunov functions in Section 5.5. This last section concludes by showing how the latter concepts can be used to verify the assumptions of our main result.

5.1. **Cellular complexes.** Throughout this subsection, let $X$ denote a topological space. Then a $d$-dimensional cell, or briefly a cell, is a subset $\sigma \subset X$ which is homeomorphic to the closed unit ball in the Euclidean space $\mathbb{R}^d$. In this case, we write $\dim \sigma := d$ for the dimension of the cell. The associated open cell and cellular boundary of the cell $\sigma$ are defined as the image of the open unit ball and of the unit sphere in $\mathbb{R}^d$, respectively, with respect to the same homeomorphism used in the definition of $\sigma$. Since a homeomorphism of a closed ball in $\mathbb{R}^d$ has to preserve the associated open ball and sphere, the open cell and the cellular boundary do not depend on the choice of a particular homeomorphism. We denote the open cell by $\overset{\circ}{\sigma}$, and the cellular boundary by $\partial \sigma$. We note that these concepts have the following straightforward properties, whose proof will be left to the reader.

**Proposition 5.1** (Cell properties). Let $\sigma \subset X$ be a $d$-dimensional cell in a topological space. Then the following identities are satisfied:

\begin{align*}
\overset{\circ}{\sigma} \cup \partial \sigma &= \sigma, \\
\overset{\circ}{\sigma} \cap \partial \sigma &= \emptyset, \\
\partial \sigma &= \sigma \setminus \overset{\circ}{\sigma}, \\
\text{cl} \overset{\circ}{\sigma} &= \sigma.
\end{align*}

Note that the cell boundary $\partial \sigma$ is in general not the topological boundary $\text{bd} \sigma$, and that the open cell $\overset{\circ}{\sigma}$ is not necessarily the topological interior of $\sigma$. □

The notion of cell forms the basic building block for our combinatorial study of dynamical systems. By combining cells in a suitable way, one can arrive at a combinatorial description of certain topological spaces. More precisely, we say that a finite family $\mathcal{X}$ of cells in a topological space $X$ is a cellular decomposition of $X$, if the following two conditions are met:

(CD1) If $\overset{\circ}{\sigma} \cap \tau \neq \emptyset$, then we have the inclusion $\sigma \subset \tau$, for all cells $\sigma, \tau \in \mathcal{X}$. 


(CD2) We have the representation \( \mathcal{X} = \bigcup_{\sigma \in \mathcal{X}} \sigma \).

In the following, we write \( \mathcal{X}_k \) for the subfamily of all at most \( k \)-dimensional cells in \( \mathcal{X} \). We refer to \( \mathcal{X}_k := \bigcup_{\sigma \in \mathcal{X}_k} \sigma \) as the \( k \)-skeleton of \( \mathcal{X} \). Given two cells \( \tau, \sigma \in \mathcal{X} \) such that \( \tau \subset \sigma \) we say that \( \tau \) is a face of \( \sigma \), and that \( \sigma \) is a coface of \( \tau \). The face or coface is called proper, if \( \sigma \neq \tau \). A face \( \tau \) of \( \sigma \) is a facet of \( \sigma \), if \( \dim \sigma = \dim \tau + 1 \). A cell is called top-dimensional, if it is not a proper face of a higher-dimensional cell. We will refer to a top-dimensional cell also as a toplex, and denote the family of all toplexes by \( \mathcal{X}_{\text{top}} \). One can easily see that a space \( X \) which has a cellular decomposition is equal to the union of its toplexes. Finally, the star of a cell \( \sigma \in \mathcal{X} \) is the family of all cofaces of \( \sigma \), and it is denoted by \( \text{Star} \sigma \). We note that the star of a toplex contains only this toplex.

Of course, not every topological space admits a cellular decomposition, and even if it does, a cellular decomposition need not be unique. By a cellular space we mean a metrizable topological space \( X \) which admits a cellular decomposition. Since \( X \) is the union of its toplexes, the finiteness of \( \mathcal{X} \) in turn implies that \( X \) is automatically compact.

A simple induction argument with respect to the cardinality of \( \mathcal{X} \) shows that a cellular space is a finite, regular CW-complex. Vice versa, every finite regular CW-complex is a cellular space [10, Section I.3].

In the sequel we assume that \( X \) is a fixed cellular space and that \( \mathcal{X} \) denotes its fixed cellular decomposition. The following proposition is an immediate consequence of assumption (CD1), and its simple proof is omitted.

**Proposition 5.2 (Open cells form a partition).** Let \( X \) denote a cellular space. Then the collection of open cells of \( X \) forms a partition of \( X \). In particular, for any two cells \( \sigma, \tau \in \mathcal{X} \) the inequality \( \sigma \cap \tau \neq \emptyset \) implies the equality \( \sigma = \tau \). □

It follows immediately from Proposition 5.2 that for every point \( x \in X \) there exists exactly one cell \( \sigma \in \mathcal{X} \) such that \( x \in \sigma \). We denote this unique cell by \( \sigma_x \). Furthermore, as an easy consequence of Proposition 5.2 one obtains the following result, whose straightforward proof is also omitted.

**Proposition 5.3 (Cell representations).** Suppose that \( \mathcal{X} \) is a cellular decomposition of the metric space \( X \). Then we have the representations

\[
\begin{align*}
(\text{i}) & \quad \sigma = \bigcup_{\mathcal{X} \ni \rho \subset \sigma} \rho, \\
(\text{ii}) & \quad \partial \sigma = \bigcup_{\mathcal{X} \ni \rho \subset \sigma} \rho, \\
(\text{iii}) & \quad \sigma \cap \tau = \bigcup_{\mathcal{X} \ni \rho \subset \sigma \cap \tau} \rho,
\end{align*}
\]

where \( \sigma, \tau \in \mathcal{X} \) are arbitrary cells.

As we explained in Section 3.1.3, we will consider \( \mathcal{X} \) as an Alexandrov topological space with the Alexandrov topology induced by the inclusion relation on \( \mathcal{X} \). We would like to point out that the star of a cell \( \sigma \in \mathcal{X} \) coincides with \( \text{Opn} \sigma \) in the Alexandrov topology. In particular, a singleton which consists of a toplex is open in this topology. It can easily be seen that the family of all stars of cells in \( \mathcal{X} \) forms a basis of the Alexandrov topology. For a family \( \mathcal{A} \subset \mathcal{X} \) of cells we define its support by

\[ |\mathcal{A}| := \bigcup_{\sigma \in \mathcal{A}} \sigma. \]

In this case we say that \( \mathcal{A} \) is a combinatorial representation of \( |\mathcal{A}| \). Moreover, a set \( A \subset X \) is called representable, if there exists a family \( \mathcal{A} \subset \mathcal{X} \) such that \( A = |\mathcal{A}| \). Note that if such
a family exists, it is uniquely determined by the set $A$. As an immediate consequence of Proposition 5.2, we obtain the following result, which summarizes the properties of support and representable sets.

**Proposition 5.4** (Characterization of representable sets). Suppose that $X$ is a cellular decomposition of the metric space $X$. Then a subset $A \subset X$ is representable if and only if

$$\forall \sigma \in \mathcal{X} : \overset{\circ}{\sigma} \cap A \neq \emptyset \Rightarrow \overset{\circ}{\sigma} \subset A.$$  

(20)

**Proof:** Assume first that the set $A$ is representable. Then we have $A = \bigcup_{\tau \in A} \overset{\circ}{\tau}$ for some family $A \subset \mathcal{X}$. Let $\sigma \in \mathcal{X}$ be such that $\overset{\circ}{\sigma} \cap A \neq \emptyset$. We claim that $\sigma \in A$. Indeed, if not, then by Proposition 5.2 we get $\overset{\circ}{\sigma} \cap \overset{\circ}{\tau} = \emptyset$ for every $\tau \in A$, and therefore

$$\overset{\circ}{\sigma} \cap A = \bigcup_{\tau \in A} \left( \overset{\circ}{\sigma} \cap \overset{\circ}{\tau} \right) = \emptyset,$$

which is a contradiction. Hence, $\sigma \in A$ and, in consequence, $\overset{\circ}{\sigma} \subset \bigcup_{\tau \in A} \overset{\circ}{\tau} = A$.

Suppose now that (20) holds. Let $A := \{ \tau \in \mathcal{X} \mid \overset{\circ}{\tau} \cap A \neq \emptyset \}$. From (20) one immediately obtains $|A| \subset A$. To verify the opposite inclusion, consider an arbitrary point $x \in A$. Then in view of $x \in \overset{\circ}{\sigma}_x$ we have $\overset{\circ}{\sigma}_x \cap A \neq \emptyset$. Hence, $\sigma_x \in A$ and $x \in |A|$.

The following simple result shows how set representations are affected by set operations. Its simple proof is omitted.

**Proposition 5.5** (Properties of set representations). Suppose that $X$ is a cellular decomposition of the metric space $X$, and that $A, B \subset X$ are families of cells. Then

$$|A \cup B| = |A| \cup |B|,$$

(21)

$$|A \cap B| = |A| \cap |B|,$$

(22)

$$|A| \setminus |B| = |A \setminus B|,$$

(23)

$$X = |\mathcal{X}|.$$  

(24)

In particular, the space $X$ is representable, and the union, intersection and difference of representable sets is again representable.

**Corollary 5.6** (Representability of cell components). Suppose that $X$ is a cellular decomposition of the metric space $X$. Then the sets $\sigma$, $\overset{\circ}{\sigma}$, and $\partial \sigma$ are representable for each $\sigma \in \mathcal{X}$.

**Proof:** The representability of $\sigma$ and $\overset{\circ}{\sigma}$ follows immediately from (CD1) and Proposition 5.4. Hence, the representability of $\partial \sigma$ follows from Proposition 5.5.

In our combinatorial approach to the study of dynamics the toplexes in a cellular decomposition play a special role, since — in some sense — they pass on the orbits amongst each other. This “passing along” is realized along the intersections of toplexes, which can be described using the following notion. We set

$$\text{Fr}(X) := |\mathcal{X} \setminus \mathcal{X}_{\text{top}}|$$

and refer to $\text{Fr}(X)$ as the frame of $X$. Then one can readily see that the frame contains the union of all toplex intersections.

We close this subsection on cell complexes and cellular decompositions with a sequence of results, which will be useful in subsequent sections.
Proposition 5.7 (Openness of the star). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$. Then for every $\sigma \in \mathcal{X}$ the support of its star is open in $X$.

Proof: Let $\sigma \in \mathcal{X}$. It suffices to prove that $A := X \setminus \text{Star} \sigma$ is closed. Hence, assume that $x \in \overline{A}$. Let $(x_n)$ be a sequence of points in $A$ converging to $x$. Set $\sigma_n := \sigma_{x_n}$. Then we clearly have $\sigma_n \notin \text{Star} \sigma$. Since $\mathcal{X}$ is finite, without loss of generality we may assume that the identity $\sigma_n = \tau$ holds for some $\tau \in \mathcal{X}$ and all $n \in \mathbb{N}$. It follows that $x_n \in \tau$ for all $n \in \mathbb{N}$, and by (19) we get $x \in \overline{\tau} = \tau$. It follows that $x \in \tau \cap \sigma_x$. Therefore, property (CD1) of a cellular decomposition implies that $\sigma_x \subset \tau$. Since $\tau \notin \text{Star} \sigma$, we get $\sigma_x \notin \text{Star} \sigma$. It follows that $x \notin |\text{Star} \sigma|$, which means that $x \in A$. Hence, $A$ is closed and $|\text{Star} \sigma|$ is open. \hfill $\square$

Proposition 5.8 (Support of the combinatorial closure of a cell). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$. Then for every $\sigma \in \mathcal{X}$

$$|\overline{\text{Cl}} \sigma| = \sigma.$$ 

Proof: Consider an arbitrary point $x \in |\overline{\text{Cl}} \sigma|$. Then $x \in \overset{\circ}{\sigma}$ for some $\tau \in \overline{\text{Cl}} \sigma$. It follows that $x \in \overset{\circ}{\tau} \subset \tau \subset \sigma$. To see the opposite inclusion, consider now a point $x \in \sigma$. Let $\tau \in \mathcal{X}$ be such that $x \in \overset{\circ}{\tau}$. Then one has $\overset{\circ}{\tau} \cap \sigma \neq \emptyset$, which in turn implies $\tau \subset \sigma$. Consequently, we have both $\tau \in \overline{\text{Cl}} \sigma$ and $x \in |\overline{\text{Cl}} \sigma|$. \hfill $\square$

Proposition 5.9 (Topological properties of toplex components). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$. Let $\sigma, \tau \in \mathcal{X}_{\text{top}}$ be such that $\sigma \neq \tau$. Then we have:

$$\overset{\circ}{\sigma} \subset \text{int} \sigma,$$

$$\partial \sigma \subset \partial \sigma,$$ 

$$\overset{\circ}{\sigma} \cap \partial \tau = \emptyset,$$

$$\partial \sigma \cap \partial \tau = \partial \sigma \cap \partial \tau.$$ 

Proof: Since $\sigma \in \mathcal{X}_{\text{top}}$, we have $\text{Star} \sigma = \{\sigma\}$. Hence $\overset{\circ}{\sigma} = |\text{Star} \sigma|$ is open in view of Proposition 5.7 and it follows that $\overset{\circ}{\sigma} \subset \text{int} \sigma$, thereby establishing (25). Property (26) follows immediately from (25) and (18). To verify (27), suppose that $\overset{\circ}{\sigma} \cap \partial \tau \neq \emptyset$. According to Corollary 5.6 the set $\tau$ is representable, and Proposition 5.4 implies the inclusion $\sigma \subset \tau$. Since $\sigma \in \mathcal{X}_{\text{top}}$, this in turn yields $\sigma = \tau$, which contradicts the choice of $\tau$ and $\sigma$, and therefore proves (27). Finally, in order to prove (28), note that the choice of $\tau$ and $\sigma$, in combination with (CD1), implies the identity $\overset{\circ}{\sigma} \cap \sigma = \emptyset$, which in turn gives $\overline{\sigma} \subset \overline{\text{Cl}}(X \setminus \sigma)$. Hence, property (19) implies that $\partial \sigma \cap \partial \tau \subset \sigma \cap \tau = \sigma \cap \overline{\text{Cl}} \overset{\circ}{\sigma} \subset \sigma \cap \overline{\text{Cl}}(X \setminus \sigma) = \partial \sigma$. A symmetric argument proves that $\partial \sigma \cap \partial \tau \subset \partial \sigma$. Therefore, one has $\partial \sigma \cap \partial \tau \subset \partial \sigma \cap \partial \tau$. The opposite inclusion follows immediately from (26). This completes the proof. \hfill $\square$

Proposition 5.10 (The point-to-cell mapping). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$, and consider the point-to-cell map

$$\Sigma : X \ni x \mapsto \sigma_x \in \mathcal{X}.$$ 

Then the following hold:

(i) For every $A \subset \mathcal{X}$ we have $\Sigma^{-1}(A) = |A|$.

(ii) For every $A \subset \mathcal{X}$ we have $\Sigma(|A|) = A$. 


(iii) The point-to-cell map $\Sigma$ is a continuous, open surjection.

**Proof:** Properties (i), (ii), and the surjectivity of $\Sigma$ follow immediately from the definitions. From Proposition 5.7 and property (i) we see that $\Sigma^{-1}(\text{Star } \sigma)$ is open in $X$ for every $\sigma \in \mathcal{X}$. Since the stars of cells in $\mathcal{X}$ form a basis of the topology in $\mathcal{X}$, this proves the continuity of the point-to-cell map $\Sigma$.

To prove that $\Sigma$ is an open map, consider an open set $U \subset X$. We have to prove that $\Sigma(U)$ is open in $X$. Thus, consider a cell $\tau \in \Sigma(U)$. We will be done if we prove that

$$\text{Star } \tau \subset \Sigma(U).$$

(29)

Choose a point $x \in U$ such that $\sigma_x = \tau$ is satisfied. Then we clearly have $x \in \overset{0}{\sigma}$. In order to verify (29), consider also a cell $\varrho \in \text{Star } \tau$. Since $\tau \subset \varrho = \text{cl } \overset{0}{\varrho}$, and since the set $U$ is open and contains $x$, there exists a point $x' \in U \cap \overset{0}{\varrho}$. But this inclusion immediately implies $\varrho = \Sigma(x')$, as well as $\Sigma(x') \in \Sigma(U)$. Combined, they yield $\varrho \in \Sigma(U)$, which in turn proves (29). □

As an immediate consequence of Proposition 5.10 and property (23) of Proposition 5.5 we obtain the following corollary.

**Corollary 5.11** (Topological properties of the support). Suppose that $X$ is a cellular decomposition of the metric space $X$, and let $A \subset X$ denote a family of cells. Then the support $|A|$ is open (respectively closed) in $X$ if and only if $A$ is open (respectively closed) in $X$. In other words, a representable set is open (respectively closed) in $X$ if and only if its representation is open (respectively closed) in $X$. □

Needless to say, while on the level of representable sets openness and closedness can be verified either in the metric space $X$ or in the Alexandrov space $\mathcal{X}$, in general the topology of the former space is significantly larger.

**Proposition 5.12** (Generating cellular decompositions). Suppose that $X$ is a cellular decomposition of the metric space $X$. Then the following hold:

(i) $A$ is a cellular decomposition of $|A|$ for every closed $A \subset \mathcal{X}$.

(ii) $X \setminus \overset{0}{\sigma}$ is a cellular space with $\mathcal{X} \setminus \{\sigma\}$ as a cellular decomposition for every $\sigma \in \mathcal{X}_{\text{top}}$.

(iii) $\text{Fr}(X)$ is a cellular space with $\mathcal{X} \setminus \mathcal{X}_{\text{top}}$ as a cellular decomposition.

(iv) $X_k$, the $k$-skeleton of $X$, is a cellular space with $X_k$ as a cellular decomposition.

**Proof:** Observe that for every $\sigma \in A$ we have $\overset{0}{\sigma} \subset |A|$, which together with (19) implies the inclusion $\sigma \subset \text{cl } |A| = |A|$, since $|A|$ is closed in $X$ by Corollary 5.11. Hence, the collection $A$ is a family of cells not only in $X$ but also in $|A|$. To see that $A$ is a cellular decomposition of $|A|$, observe that (CD1) is inherited from $\mathcal{X}$ and (CD2) follows immediately from the definition of support. This proves (i). The remaining three assertions are straightforward consequences of (i). □

As an immediate consequence of Proposition 5.12 and the characterization of ANRs contained in [8, Theorem IV.6.1], we obtain the following proposition.

**Proposition 5.13** (Cellular decompositions as ANRs). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$. If $A \subset \mathcal{X}$ is closed, then $|A|$ is a compact ANR. □

**Proposition 5.14** (Acyclicity of the star). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$ and $\tau \in \mathcal{X}$. Then the set $|\text{Star } \tau| \subset X$ is acyclic.
Proof: Let $n$ denote the cardinality of $S := \text{Star} \tau$. We establish the result by induction on $n$. If $n = 1$, then we have $|\text{Star} \tau| = 0$. Hence, it is acyclic as an open ball. Now, assume that $n > 1$ and the claim holds for stars of cardinality less than $n$. Select a $\sigma \in \text{Star} \tau$ of maximal dimension. Since $n > 1$, we have $\sigma \neq \tau$. By Proposition 5.12, the family $\mathcal{Y} := \mathcal{X} \setminus \{\sigma\}$ is a cellular decomposition of cellular space $Y := |\mathcal{Y}|$, and the family $Z := \text{Bd} \sigma \setminus \{\sigma\}$ is a cellular decomposition of cellular space $Z := |Z| = \partial \sigma$. Clearly, $\tau \in \mathcal{Y} \cap Z$. Let $S_Y$ denote the support of the star of $\tau$ in $\mathcal{Y}$, and let $S_Z$ denote the support of the star of $\tau$ in $Z$. Without loss of generality we may assume that $\sigma$ is a unit ball centered at $o \in \sigma$. Let $p : \sigma \setminus \{o\} \to \partial \sigma$ denote the radial projection and set $V_0 := p^{-1}(S_Z)$. Then one can easily verify that $U := \sigma$ and $V := S_Y \cup V_0$ are both open in $S$. Moreover, $p|_{V_0} : V_0 \to S_Z$ is a deformation retraction of $V_0$ onto $S_Z$ and $\text{id}_{S_Y} \cup p|_{V_0} : V \to S_Y$ is a deformation retraction of $V$ onto $S_Y$. Since $S_Y$ is the support of the star of $\tau$ in $\mathcal{Y}$ and $S_Z$ is the support of the star of $\tau$ in $Z$, both of these supports are acyclic by our induction assumption. Hence, also $V$ and $U \cap V = V_0$ are acyclic. Clearly, $U = \sigma$ is acyclic as an open ball. Therefore, the conclusion follows by applying the Mayer-Vietoris theorem to $U$ and $V$, since one easily verifies that $S = U \cup V$. \hfill \Box

Proposition 5.15 (Point-to-cell mapping induces an isomorphism in homology). Suppose that $\mathcal{X}$ is a cellular decomposition of the metric space $X$ and $B \subset A$ are closed sets in $\mathcal{X}$. Then the restriction of the map $\Sigma$ defined in Proposition 5.10 and given by

$$\Sigma_{|A|} : (|A|, |B|) \ni x \mapsto \sigma_x \in (A, B)$$

is a well-defined and continuous map of pairs. Moreover, the map induced in singular homology and given by

$$H_*(\Sigma_{|A|}) : H_*(|A|, |B|) \to H_*(A, B)$$

is an isomorphism.

Proof: The well-definedness of the definition and the continuity of $\Sigma_{|A|}$ follow immediately from Proposition 5.10. To prove that $\Sigma_{|A|}$ induces an isomorphism in singular homology assume first that $B = \emptyset$. In this case, we need to verify that $H_*(\Sigma_{|A|}) : H_*(|A|) \to H_*(A)$ is an isomorphism. Note that for each $\sigma \in A$ we have $\Sigma^{-1}(\text{Star} \sigma) = |\text{Star} \sigma|$ by Proposition 5.10(i). Hence, $\Sigma^{-1}(\text{Star} \sigma) = \Sigma^{-1}(\text{Opn} \sigma)$ is acyclic by Proposition 5.14. Therefore, the conclusion for $B = \emptyset$ follows from [4, Theorem 2.2]. The case of general $B$ then follows easily by applying the five lemma. \hfill \Box

5.2. Strongly admissible semiflows. Equipped with our toolset for studying cellular spaces, we can now turn our attention to the study of semiflows on such spaces. For this, it will be crucial to study the sequence of cells that is traversed by solutions of the semiflow, and for this, the following notions will be extremely useful.

Definition 5.16 (Immediate future and past). Let $X$ denote a cellular space with cellular decomposition $\mathcal{X}$, and let $\varphi : X \times \mathbb{R}_+^\circ \to X$ denote a semiflow on $X$.

- We say that a cell $\sigma \in \mathcal{X}_\text{top}$ is the immediate future of a point $x \in X$, if we have

$$\varphi(x, (0, \varepsilon)) \subset \overset{\circ}{\sigma} \text{ for some } \varepsilon > 0.$$

In this case, we write $\text{ift}(x) = \sigma$. 


We say that \( \sigma \in \mathcal{X}_{\text{top}} \) is the immediate past of a backward solution \( \gamma \in \text{Sol}^- (x, \varphi) \) through the point \( x \in X \), if
\[
\gamma ((-\varepsilon, 0)) \subset \overset{\circ}{\sigma} \quad \text{for some} \quad \varepsilon > 0,
\]
and we write \( \text{ipt}(\gamma) = \sigma \).

Note that in the case of a flow \( \varphi : X \times \mathbb{R} \to X \) the immediate past of a solution through \( x \) depends only on the point \( x \). Hence, since in this case every point admits a backward solution, we can define the immediate past of a point \( x \) as the immediate past of any of its backward solutions, and we denote it by \( \text{ipt}(x) \). Observe also that we have the implication
\[
\sigma = \text{ift} x \quad \Rightarrow \quad x \in \sigma.
\] (30)

Finally, we would like to point out that not every point \( x \) has to have an immediate future, since the solution starting at \( x \) might oscillate between two neighboring cells if \( x \) lies in their intersection. Thus, the existence of an immediate future does impose certain regularity assumptions on a semiflow, and similarly the existence of the immediate past. This is formalized in the following definition.

**Definition 5.17 (Admissible and strongly admissible semiflows).** Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and let \( \varphi : X \times \mathbb{R}_0^+ \to X \) denote a semiflow on \( X \). We say that \( \varphi \) is admissible with respect to \( \mathcal{X} \), if the following three conditions are satisfied.

(A1) The immediate future of every point in \( X \) and the immediate past of every backward solution are well-defined.

(A2) For every point \( x \in \text{Fr}(X) \) the immediate future of \( x \) is different from the immediate past of any backward solution through \( x \).

(A3) For every \( \sigma \in \mathcal{X}_{\text{top}} \) the strong exit and entry sets \( \sigma^- \) and \( \sigma^+ \) are closed, where these sets are defined in (11).

In addition, if for every cell \( \sigma \) its invariant part is empty unless \( H_s(\sigma, \sigma^-) \) is non-trivial, then we say that \( \varphi \) is strongly admissible.

The above notions are motivated by our earlier work [29], where we used combinatorial vector fields to construct strongly admissible semiflows on simplicial complexes. Note, however, that the underlying phase space decomposition used in [29] is not always a cellular decomposition.

The conditions required by Definition 5.17 impose a number of constraints on the semiflow behavior. These allow us to draw several useful conclusions. To begin with, we have the following characterizations of strong exit and entry sets \( \sigma^- \) and \( \sigma^+ \), respectively, for every toplex \( \sigma \in \mathcal{X}_{\text{top}} \).

**Proposition 5.18 (Exit and entry set characterizations).** Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and assume that \( \varphi \) is an admissible semiflow with respect to \( \mathcal{X} \). Furthermore, let \( \sigma \in \mathcal{X}_{\text{top}} \) be arbitrary. Then
\[
\sigma^- = \{ x \in \text{bd} \sigma \mid \text{ift}(x) \neq \sigma \}
\] (31)
and
\[
\sigma^+ = \{ x \in \text{bd} \sigma \mid \forall \gamma \in \text{Sol}^- (x, \varphi) : \text{ipt}(\gamma) \neq \sigma \}.
\] (32)

Moreover, if \( \varphi \) is an admissible flow, then we have
\[
\sigma^+ = \{ x \in \text{bd} \sigma \mid \text{ipt}(x) \neq \sigma \}.
\] (33)
Proof: Assume first that \( x \in \sigma^- \). Then the inclusion \( x \in \partial \sigma \) is satisfied, and there exists an \( \varepsilon > 0 \) such that \( \varphi(x,(0,\varepsilon)) \cap \sigma = \emptyset \). This in turn implies \( \text{ift}(x) \neq \sigma \). Suppose now conversely that \( x \in \partial \sigma \) and that \( \tau := \text{ift}(x) \neq \sigma \). Then one has \( \varphi(x,(0,\varepsilon)) \subset \tau^\circ \) for some constant \( \varepsilon > 0 \). It follows from properties (26) and (27) that then \( \varphi(x,(0,\varepsilon)) \cap \sigma = \emptyset \), which means that \( x \in \sigma^- \). This proves (31).

In order to verify (32), we first assume that \( \varepsilon > 0 \) exists such that \( z \in \text{ift}((x,\varphi),\varepsilon) \cap \sigma = \emptyset \). This yields \( \text{ipt}(\gamma) \neq \sigma \) and proves that \( x \) belongs to the right-hand side of (32). Now suppose conversely that \( x \) belongs to the right-hand side of (32). Let \( \gamma \in \text{Sol}^-(x,\varphi) \) and assume that \( \tau := \text{ipt}(\gamma) \neq \sigma \). Then \( \gamma((-\varepsilon,0)) \subset \tau^\circ \) for some constant \( \varepsilon > 0 \). As above, it follows from properties (26) and (27) that \( \gamma((-\varepsilon,0)) \cap \sigma = \emptyset \), which means \( x \in \sigma^- \). This proves (32), and (33) follows immediately. \( \square \)

In addition to the straightforward exit and entry set characterizations derived above, the notion of admissible semiflow also shows that toplexes have to be strong isolating blocks. This is the subject of the next result.

Proposition 5.19 (Toplexes are strong isolating blocks). Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and assume that \( \varphi \) is an admissible semiflow with respect to \( \mathcal{X} \). Then every \( \sigma \in \mathcal{X}_{\text{top}} \) is a strong isolating block for \( \varphi \) in the sense of Definition 3.7.

Proof: Condition (A3) of admissibility assures that both \( \sigma^- \) and \( \sigma^+ \) are closed, and according to their definition in (11) we have \( \sigma^- \cup \sigma^+ \subset \partial \sigma \). In order to see the reverse inclusion, suppose that \( x \in \partial \sigma \) and that \( x \notin \sigma^- \). Then \( \text{ift}(x) = \sigma \) in view of (31). But then it follows from (A2) that for every backward solution \( \gamma \) through \( x \) we have to have the inequality \( \text{ipt}(\gamma) \neq \text{ift}(x) = \sigma \). Hence, by (33), we finally obtain \( x \in \sigma^+ \). This implies the desired identity \( \sigma^- \cup \sigma^+ = \partial \sigma \), and the result follows. \( \square \)

Our next result shows that as long as we stay in the interior of a cell, the immediate future of a point cannot change. Note that this result holds for any cell, not just top-dimensional ones.

Proposition 5.20 (Immediate future is constant on cell interiors). Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and assume again that \( \varphi \) is an admissible semiflow with respect to \( \mathcal{X} \). Then for every cell \( \sigma \in \mathcal{X} \) and any two points \( x,y \in \sigma \) we have \( \text{ift}(x) = \text{ift}(y) \).

Proof: If \( \sigma \in \mathcal{X}_{\text{top}} \), then property (25) immediately implies \( \text{ift}(x) = \sigma = \text{ift}(y) \). Consider now the case \( \sigma \notin \mathcal{X}_{\text{top}} \), and assume that \( \text{ift}(x) \neq \text{ift}(y) \). Let \( \sigma_1 := \text{ift}(x) \) and \( \sigma_2 := \text{ift}(y) \). Then we have both \( x \in \partial \sigma_1 \) and \( y \in \partial \sigma_2 \). Since \( \mathcal{X} \) is a cellular decomposition, we further have \( \sigma \subset \sigma_1 \cap \sigma_2 \) due to (CD1). Clearly, the inclusions \( x \in \sigma_2^- \) and \( y \in \sigma_1^- \) are satisfied, since both \( \text{ift}(x) = \sigma_1 \neq \sigma_2 \) and \( \text{ift}(y) = \sigma_2 \neq \sigma_1 \).

Now recall that \( \sigma \) is path connected, as the homeomorphic image of an open ball in Euclidean space. We therefore can consider a path \( \mu : [0,1] \rightarrow \sigma \) which satisfies both \( \mu(0) = x \) and \( \mu(1) = y \). By shortening the path and potentially renaming the point \( y \), if necessary, we may assume without loss of generality that \( \{ \text{ift}(\mu(s)) \mid s \in [0,1] \} \subset \{ \sigma_1, \sigma_2 \} \). Now consider the argument \( s_* := \sup \{ s \in [0,1] \mid \text{ift}(\mu(s)) = \sigma_1 \} \) and let \( z := \mu(s_*) \). Then \( z \in \partial \sigma_2 = \sigma_2^{-} \), as well as \( z \in \partial \sigma_1 = \sigma_1^{-} \). It follows that we then have \( \text{ift}(z) \neq \sigma_2 \) and \( \text{ift}(z) \neq \sigma_1 \) — which in turn contradicts the inclusion \( \text{ift}(z) \in \{ \sigma_1, \sigma_2 \} \) and proves the result. \( \square \)
Thus, Proposition 5.19 establishes the identities have $x \in \sigma \cap \tau$ belongs to the left-hand side, but not to the right-hand side. Consider first the case that we have $\sigma \cap \tau = \emptyset$, which gives again a contradiction. If on the other hand $x \in \sigma \cap \tau$, an analogous argument based on (33) shows that $\sigma = \text{ipt}(x) = \tau$, which gives again a contradiction.

Proposition 5.21 (Intersections of toplexes for admissible flows). Let $X$ denote a cellular space with cellular decomposition $\mathcal{X}$, and assume that $\varphi$ is an admissible flow with respect to $\mathcal{X}$. Then for every pair of cells $\sigma, \tau \in \mathcal{X}_{\text{top}}$ such that $\sigma \neq \tau$ we have

$$\sigma \cap \tau = (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-).$$

Proof: In view of $\tau \neq \sigma$, we deduce from Proposition 5.2 that $\hat{\sigma} \cap \hat{\tau} = \emptyset$. Moreover, property (27) implies both $\hat{\tau} \cap \partial \sigma = \emptyset$ and $\hat{\sigma} \cap \partial \tau = \emptyset$. Therefore, we have

$$\sigma \cap \tau = (\hat{\sigma} \cup \partial \sigma) \cap (\hat{\tau} \cup \partial \tau) = \partial \sigma \cap \partial \tau,$$

and together with property (28) this further implies

$$\sigma \cap \tau = \text{bd} \sigma \cap \text{bd} \tau.$$

Thus, Proposition 5.19 establishes the identities

$$\sigma \cap \tau = \text{bd} \sigma \cap \text{bd} \tau = (\sigma^- \cup \sigma^+) \cap (\tau^- \cup \tau^+) = (\sigma^- \cap \tau^-) \cup (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-) \cup (\sigma^+ \cap \tau^+).$$

By the very definition of entry and exit sets we get

$$(\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-) \subset \sigma \cap \tau.$$

Hence, to complete the proof it suffices to verify the inclusion

$$(\sigma^- \cap \tau^-) \cup (\sigma^+ \cap \tau^+) \subset (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-).$$

For this, suppose that the inclusion does not hold. Then there exists a point $x \in X$ which belongs to the left-hand side, but not to the right-hand side. Consider first the case that we have $x \notin \sigma^- \cap \tau^-$. Since $x \notin (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-)$, we then get $x \notin \tau^+$ and $x \notin \sigma^+$. It follows from (33) that $\text{ipt}(x) = \sigma$ and $\text{ipt}(x) = \tau$, and therefore $\sigma = \tau$, which is a contradiction. If on the other hand $x \in \sigma^+ \cap \tau^+$, an analogous argument based on (31) shows that $\sigma = \text{ift}(x) = \tau$, which gives again a contradiction.

As the above result shows, in the flow case a nontrivial intersection of two toplexes has to be the union of $\sigma^- \cap \tau^+$ and $\sigma^+ \cap \tau^-$. We say that $\sigma$ and $\tau$ have a circular intersection, if their intersection $\sigma \cap \tau$ is non-empty and does not equal just $\sigma^- \cap \tau^+$ or $\sigma^+ \cap \tau^-$. One can easily see that in general circular intersections are possible. However, in practical situations of interest they are very unlikely to appear and their absence may easily be verified algorithmically.

Unfortunately, the situation is different in the semiflow case. As demonstrated in the two examples shown in Figure 8, it is not true in general that $\sigma \cap \tau \subset (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-)$. Nevertheless, even in the semiflow case one can fairly easily determine whether two specifically given toplexes satisfy this inclusion or not, since the next result identifies the exceptional situation.
Figure 8. Counterexamples to Proposition 5.21 in the semiflow case. In both cases, the point $x$ is not contained in either $\sigma^+ \cap \tau^+$ or $\sigma^- \cap \tau^-$. Note, however, that one can use Proposition 5.22 to show that both $(\sigma, \varrho)$ and $(\tau, \varrho)$ are proper brick pairs. While this is immediate in the left figure, in the right image the point $y$ is the unique point in $\sigma^- \cap \varrho^-$ or $\tau^- \cap \varrho^-$, respectively, but there is no backward solution inside $\varrho$ in either case.

Proposition 5.22 (Intersection of toplexes for admissible semiflows). Let $X$ denote a cellular space with cellular decomposition $\mathcal{X}$, and assume that $\varphi$ is an admissible semiflow with respect to $\mathcal{X}$. Then for every pair of cells $\sigma, \tau \in X_{\text{top}}$ such that $\sigma \neq \tau$ exactly one of the following two alternatives holds:

(i) The equality $\sigma \cap \tau = (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-)$ is satisfied.

(ii) There exists a point $x \in \sigma^- \cap \tau^-$ and a toplex $\varrho \in X_{\text{top}}$ which is different from $\sigma$ and $\tau$ such that $\text{lift}(x) = \varrho$, and there are backward solutions $\gamma, \eta \in \text{Sol}^-(x, \varphi)$ such that both $\gamma((\varepsilon, 0)) \subset \sigma$ and $\eta((\varepsilon, 0)) \subset \tau$ are satisfied for some $\varepsilon > 0$.

Proof: As in the proof of Proposition 5.21 one can show that

$$\sigma \cap \tau = (\sigma^- \cap \tau^-) \cup (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-) \cup (\sigma^+ \cap \tau^+),$$

and the same argument used in that proof can still be used to establish

$$\sigma^+ \cap \tau^+ \subset (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-).$$

Thus, the only way that the inclusion in (i) cannot be satisfied is if there exists a point $x \in X$ which belongs to $(\sigma^- \cap \tau^-) \setminus (\sigma^- \cap \tau^+) \cup (\sigma^+ \cap \tau^-)$. But such a point has to have the properties stated in (ii). \qed

5.3. Combinatorial multivector fields induced by admissible semiflows. In this subsection we demonstrate that every admissible semiflow on a cellular space gives rise to a well-defined combinatorial multivector field on $\mathcal{X}$. This construction will be used in the next subsection to show that, conversely, periodic orbits in the combinatorial setting give rise to corresponding periodic orbits in the underlying semiflow under suitable conditions.

To set the stage, we recall some basic definitions from [20]. Suppose we are given a cellular space $X$ with cellular decomposition $\mathcal{X}$, which will be considered as a finite Alexandrov space. Then a combinatorial multivector field on the finite topological space $\mathcal{X}$ is a partition $\mathcal{V}$ of $\mathcal{X}$ into locally closed, non-empty subsets which we call multivectors. A multivector $V \in \mathcal{V}$ is regular if $H_*(\text{Cl}V, \text{Mo}V) = 0$, otherwise it is critical. This distinction is motivated by the
Walawski’s Theorem (Proposition 3.4). A subset \( A \subset X \) is \( \mathcal{V} \)-compatible if \( V \cap S \neq \emptyset \) for some multivector \( V \in \mathcal{V} \) implies \( V \subset S \). Note that given a \( \mathcal{V} \)-compatible subset \( A \subset X \), there is an induced multivector field \( \mathcal{V}_A := \{ V \in \mathcal{V} \mid V \subset A \} \) on \( A \).

With a multivector field \( \mathcal{V} \) we associate the combinatorial dynamical system generated by the multivalued map \( F_V : X \rightarrow X \) whose value for the cell \( \sigma \in X \) is defined as
\[
F_V(\sigma) := \text{Cl} \sigma \cap [\sigma]_V.
\]
(34)
In this formula, \( \text{Cl} \sigma \) denotes the closure of \( \sigma \) in the finite topological space \( X \), and \([\sigma]_V\) denotes the unique multivector in \( \mathcal{V} \) which contains \( \sigma \).

The combinatorial dynamics associated with the combinatorial multivector field \( \mathcal{V} \) is given by the dynamics of the multivalued map \( F_V \). Thus, a partial map \( \gamma : \mathbb{Z} \rightarrow X \) is called a solution of \( F_V : X \rightarrow X \), if the inclusion \( \gamma(i+1) \in F_V(\gamma(i)) \) is satisfied for all \( i, i+1 \in \text{dom} \gamma \). A solution \( \gamma \) through \( \sigma \in X \) is a solution such that \( 0 \in \text{dom} \gamma \) and \( \gamma(0) = \sigma \). A solution \( \gamma \) in \( A \subset X \) is a solution with values in the set \( A \). As in the classical case, a solution \( \gamma \) is called full, if we have \( \text{dom} \gamma = \mathbb{Z} \). Finally, a solution \( \gamma \) is called a path, if its domain \( \text{dom} \gamma \) is a bounded interval in \( \mathbb{Z} \). In this case, the endpoints of the path are the values of the path at the endpoints of the domain. Formula (34) for the multivalued map \( F_V \) induced by the multivector field \( \mathcal{V} \) has one drawback: Every cell \( \sigma \in X \) admits a stationary (constant) solution, because one has \( \sigma \in \text{Cl} \sigma \). To overcome this, we say that a full solution \( \gamma : \mathbb{Z} \rightarrow X \) is essential, if for every \( t \in \mathbb{Z} \) for which the multivector \( V := [\gamma(t)]_V \) is regular, there exist positive offsets \( k, k' > 0 \) such that \( \gamma(t+k) \) and \( \gamma(t-k') \) are not contained in \( V \).

We say that a full solution \( \varrho : \mathbb{Z} \rightarrow X \) is recurrent, if \( \varrho^{-1}(x) \) is unbounded from both the left and the right for every \( x \in \text{im} \varrho \). We say that a multivector field \( \mathcal{V} \) is gradient-like, if for every recurrent solution \( \varrho : \mathbb{Z} \rightarrow X \) there exists a \( V \in \mathcal{V} \) such that \( \text{im} \varrho \subset V \).

The combinatorial setting also has notions of invariance. We first define the invariant part of a subset \( \mathcal{N} \subset X \), denoted by Inv\( \mathcal{N} \), as the set of cells \( \sigma \in \mathcal{N} \) such that there exists an essential solution through \( \sigma \) in \( \mathcal{N} \). We say that a subset \( A \subset X \) is invariant, if \( A = \text{Inv} A \). A closed set \( \mathcal{N} \subset X \) is called an isolating set (or invariant set) \( \mathcal{S} \), if every path in \( \mathcal{N} \) with endpoints in \( \mathcal{S} \) is a solution in \( \mathcal{S} \). If such an isolating set exists, then the set \( \mathcal{S} \) is called an isolated invariant set.

For later reference, we recall the following two properties of isolated invariant sets in the combinatorial setting. Their proof can be found in [20] Propositions 4.10 and 4.12.

**Proposition 5.23** (Properties of isolated invariant sets). Assume that \( \mathcal{S} \) is an isolated invariant set for a combinatorial multivector field \( \mathcal{V} \). Then
\begin{enumerate}
\item \( \mathcal{S} \) is locally closed, and
\item \( \mathcal{S} \) is \( \mathcal{V} \)-compatible.
\end{enumerate}
\( \square \)

The above notions form the foundation for a dynamical theory of combinatorial multivector fields on finite topological spaces, which includes concepts such as index pair, the Conley index, and Morse decompositions. As in the classical case, the homological Conley index is defined as the homology of an index pair. However, unlike the classical case, there is a very special index pair, namely \((\text{Cl} \mathcal{S}, \text{Mo} \mathcal{S})\). This leads to the following proposition.

**Proposition 5.24** (Conley index of an isolated invariant set). The Conley index of an isolated invariant set \( \mathcal{S} \) in a combinatorial vector field \( \mathcal{V} \) is \( H_\ast(\text{Cl} \mathcal{S}, \text{Mo} \mathcal{S}) \).
\( \square \)

For more details, we refer the reader again to [20]. For us, however, combinatorial multivector fields will be a useful language for determining periodic orbits in semiflows, more
precisely, admissible semiflows on the cellular space $X$. This in turn makes it necessary to have a canonical way of creating a combinatorial multivector field from a given admissible semiflow — and this will be described in the remainder of this subsection.

Suppose now that $\varphi$ is an admissible semiflow on the cellular space $X$ with cellular decomposition $\mathcal{X}$. Then we say that $\sigma \in \mathcal{X}$ is the immediate future of a cell $\tau \in \mathcal{X}$, if we have $\text{ift}(x) = \sigma$ for every $x \in \tau$. In this case, we write $\text{ift}(\tau) = \sigma$. Notice that in view of Proposition 5.20 the immediate future $\text{ift}(\tau)$ is well-defined for every $\tau \in \mathcal{X}$. With this concept, we can now assign a combinatorial multivector field on $\mathcal{X}$ to the semiflow $\varphi$.

**Proposition 5.25** (Combinatorial multivector field generated by an admissible semiflow). Let $X$ denote a cellular space with cellular decomposition $\mathcal{X}$, and assume that $\varphi$ is an admissible semiflow with respect to $\mathcal{X}$. For every $\sigma \in \mathcal{X}_{\text{top}}$ we define

$$V_\sigma := \{ \tau \in \mathcal{X} \mid \text{ift}(\tau) = \sigma \}.$$  

Then the collection

$$\mathcal{V} := \{ V_\sigma \mid \sigma \in \mathcal{X}_{\text{top}} \}$$

is a combinatorial multivector field on $\mathcal{X}$. We call it the combinatorial multivector field $\mathcal{V}$ associated with the semiflow $\varphi$.

**Proof:** We begin by showing that $\mathcal{V}$ is a partition of the cellular decomposition $\mathcal{X}$. To see that the elements of $\mathcal{V}$ are mutually disjoint, assume that $\sigma, \sigma' \in \mathcal{X}_{\text{top}}$ are toplexes such that there exists a cell $\tau \in V_\sigma \cap V_{\sigma'}$. In view of the definition of the immediate future of a cell this implies $\sigma = \text{ift}(\tau) = \sigma'$, which in turn proves that $V_\sigma = V_{\sigma'}$. Furthermore, it is clear from the definitions that every $\tau \in \mathcal{X}$ belongs to $V_{\text{ift}(\tau)}$. Hence, one has $\mathcal{X} = \bigcup_{\sigma \in \mathcal{X}_{\text{top}}} V_\sigma$, and this shows that $\mathcal{V}$ is a partition of $\mathcal{X}$.

In order to establish $\mathcal{V}$ as a combinatorial multivector field, we still need to verify that the sets $V_\sigma$ are locally closed for all $\sigma \in \mathcal{X}_{\text{top}}$. For this, we first verify that

$$V_\sigma \subset \text{Cl} \sigma.$$  

Let $\tau \in V_\sigma$. Then, by (30), for any $x \in \tau$ we have $x \in \sigma$. Hence, assumption (CD1) implies that $\tau \subset \sigma$. According to the definition of the Alexandrov topology this implies $\tau \in \text{Cl} \sigma$, and establishes the validity of (35).

Using (35), one immediately obtains $\text{Cl} V_\sigma \subset \text{Cl} \sigma$. Moreover, due to the definition of the immediate future we always have $\sigma \in V_\sigma$, which in turn implies the inclusion $\text{Cl} \sigma \subset \text{Cl} V_\sigma$. It follows that $\text{Cl} V_\sigma = \text{Cl} \sigma$, as well as $\text{Mo} V_\sigma = \text{Cl} \sigma \setminus V_\sigma$. We will now verify that

$$|\text{Mo} V_\sigma| = \{ x \in \sigma \mid \text{ift}(x) \neq \sigma \} = \{ x \in \text{bd} \sigma \mid \text{ift}(x) \neq \sigma \}.$$  

To this end, consider a point $x \in X$ and let $\tau = \sigma_x$. Assume first that $x \in |\text{Mo} V_\sigma|$. Then we have both $\tau \in \text{Cl} \sigma$ and $\tau \not\subset V_\sigma$. It follows that $\tau \subset \sigma$ and $\text{ift}(\tau) \neq \sigma$, which implies that $x$ belongs to the middle set of (36). Assume now, conversely, that the point $x$ belongs to the middle set of (36). Then $x \in \tau \cap \sigma$, and we obtain from property (CD1) of a cellular decomposition that $\tau \subset \sigma$. Hence, we have $\tau \in \text{Cl} \sigma$, and since $\text{ift}(\tau) = \text{ift}(x) \neq \sigma$, we also get $\tau \not\subset V_\sigma$. In consequence, $\tau \in \text{Mo} V_\sigma$ and $x \in |\text{Mo} V_\sigma|$. This proves (36), since the second equality in this equation is immediate.

If we finally combine (36) with assumption (A3) in Definition 5.17 and with Proposition 5.18, then one immediately sees that the set $|\text{Mo} V_\sigma|$ is closed in $X$. Therefore, by
Corollary 5.11, the set \( \text{Mo} V_\sigma \) is closed in the finite topological space \( \mathcal{X} \), which means that \( V_\sigma \) is locally closed. This completes the proof of the proposition. \( \square \)

To illustrate Proposition 5.25 consider the fragment of cellular space \( X \) presented in the left part of Figure 9. Its cellular decomposition consists of squares as toplexes, edges, and vertices. A sample admissible flow, representing a saddle, is visualized in the figure by a collection of its trajectories. The resulting multivector field is shown on the right. All its multivectors are regular except the one in the middle, which is critical. Note that the unique critical multivector covers the unique stationary point of the flow. For all other multivectors the trajectories pass through them.

In the sequel, in the setting of the combinatorial multivector field \( V_\varphi \), by a regular toplex we mean a toplex \( \sigma \in \mathcal{X}_{\text{top}} \) such that the associated multivector \( V_\sigma \) is regular.

5.4. Isolating blocks for admissible semiflows via combinatorial dynamics. We begin with two straightforward propositions concerning the multivector field \( V \) associated with an admissible semiflow \( \varphi \).

Proposition 5.26 (Multivector closures in the induced multivector field). Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and assume that \( \varphi \) is an admissible semiflow with respect to \( \mathcal{X} \). Furthermore, let \( V \) denote the associated multivector field introduced in Proposition 5.25. Then for every \( \sigma \in \mathcal{X}_{\text{top}} \) we have

\[
\text{Cl} V_\sigma = \text{Cl} \sigma.
\]

Proof: Consider a cell \( \tau \in \text{Cl} V_\sigma \). Then we have \( \tau \subset \varrho \) for some \( \varrho \in V_\sigma \) such that \( \text{itf}(\varrho) = \sigma \). In view of (35) one obtains \( \varrho \subset \sigma \), which in turn implies \( \tau \subset \varrho \) and gives \( \tau \in \text{Cl} \sigma \). This proves the inclusion \( \text{Cl} V_\sigma \subset \text{Cl} \sigma \). In order to see the opposite inclusion, observe that \( \sigma \in V_\sigma \), which immediately yields \( \text{Cl} \sigma \subset \text{Cl} V_\sigma \). \( \square \)
Proposition 5.27 (Representation of $\mathcal{V}$-compatible sets). Let $X$ denote a cellular space with cellular decomposition $\mathcal{X}$, and assume that $\varphi$ is an admissible semiflow with respect to $\mathcal{X}$. Furthermore, let $\mathcal{V}$ denote the associated multivector field introduced in Proposition 5.25, and assume that $S \subset X$ is $\mathcal{V}$-compatible. Then we have

$$\left| \text{Cl} S \right| = \bigcup_{\sigma \in S \cap \mathcal{X}_{\text{top}}} \sigma.$$ 

Proof: Since $S$ is $\mathcal{V}$-compatible, we have

$$S = \bigcup_{V \subset S} V_{\sigma}.$$ 

Therefore, by Proposition 5.26

$$\text{Cl} S = \bigcup_{V \subset S} \text{Cl} V_{\sigma} = \bigcup_{V \subset S} \text{Cl} \sigma,$$

and Proposition 5.8 further implies

$$\left| \text{Cl} S \right| = \bigcup_{V \subset S} \left| \text{Cl} \sigma \right| = \bigcup_{\sigma \in S \cap \mathcal{X}_{\text{top}}} \sigma,$$

because clearly one has $V_{\sigma} \subset S$ if and only if $\sigma \in S \cap \mathcal{X}_{\text{top}}$. 

The next proposition shows that every isolated invariant set of a multivector field $\mathcal{V}$ associated with an admissible flow $\varphi$ induces an isolating block for $\varphi$ with the same Conley index as $\mathcal{V}$.

Theorem 5.28 (Isolating blocks and Conley index via combinatorics). Let $X$ denote a cellular space with cellular decomposition $\mathcal{X}$, and assume that $\varphi$ is an admissible flow with respect to $\mathcal{X}$. Furthermore, let $\mathcal{V}$ denote the associated multivector field introduced in Proposition 5.25. Finally, assume that $S$ is an isolated invariant set for $\mathcal{V}$ and let $N := \left| \text{Cl} S \right|$. Then the following holds:

(i) The exit set of $N$ is given by $N^{-} = \left| \text{Mo} S \right|$.

(ii) The set $N$ is an isolating block for $\varphi$.

(iii) The Conley indices of $S$ and $\text{Inv}(N, \varphi)$ coincide.

Proof: It follows from Proposition 5.27 that $N$ is compact as a finite union of compact sets. To prove (i), consider first an $x \in N^{-}$. Then $x \in N$, which implies that $\tau := \sigma_x \in \text{Cl} S$. Hence, there is a $\varrho \in S$ such that $\tau \subset \varrho$. Let $\sigma \in \mathcal{X}_{\text{top}}$ be such that $\varrho \in V_{\sigma}$. Since $S$ is $\mathcal{V}$-compatible by Proposition 5.23(ii) and $\varrho \in S$, we see that also $\sigma \in S$. Let $\omega := \text{ift}(x) = \text{ift}(\tau)$. Then one has $\omega \notin S$, since otherwise $x \notin N^{-}$. From the definition of the multivector $V_{\omega}$ we get $\tau \in V_{\omega}$. Since $\omega \notin S$ and $\omega \in V_{\omega}$, again by $\mathcal{V}$-compatibility of $S$ one obtains $\tau \notin S$. Therefore, $\tau \in \text{Mo} S$ and $x \in \left| \text{Mo} S \right|$. This establishes $N^{-} \subset \left| \text{Mo} S \right|$. To see the opposite inclusion, consider an $x \in \left| \text{Mo} S \right|$. Let $\tau := \sigma_x$. Then $\tau \in \text{Cl} S \setminus S$. Hence, $\tau \notin \mathcal{X}_{\text{top}}$, and therefore $x \in \text{Fr}(X)$. Let $\omega := \text{ift}(x)$. Since $\tau \notin S$, and $\tau, \omega \in V_{\omega}$, we get $\omega \notin S$. It follows that $\omega \cap N = \emptyset$, which finally implies $x \notin N^{-}$.

We now turn our attention to (ii). In view of (i) and Corollary 5.11 the exit set $N^{-}$ is closed. Thus, Proposition 3.6(b) immediately implies (ii), as long as we can decompose the boundary of $N$ in the form $\text{bd} N = N^{+} \cup N^{-}$. Assume that this is false. Then there exists a point $x \in \text{bd} N \setminus (N^{+} \cup N^{-})$. Let $\sigma := \text{ift}(x)$, $\tau := \text{ipt}(x)$, and $\omega := \sigma_x$. We have $\sigma \in S$,
because otherwise \( x \in N^+ \). Similarly, one has to have \( \tau \in S \), since otherwise \( x \in N^+ \). Due to the inclusion \( x \in \mathring{\omega} \cap \tau \cap \sigma \), we further get \( \omega \subset \tau \cap \sigma \). Moreover, since \( \text{ift}(x) = \text{ift}(x) = \sigma \), we see that \( \omega \subset V_\sigma \subset \sigma \). Hence, the \( \mathcal{V} \)-compatibility of \( S \) implies that \( \omega \in S \), because \( \sigma \in S \). The set \( U := | \text{Star} \omega | \) is open in \( X \) according to Proposition 5.7. Thus, the continuity of \( \varphi \) shows that we can find an open neighborhood \( V \) of \( x \) and an \( \varepsilon > 0 \) such that \( \varphi(V, [-\varepsilon, \varepsilon]) \subset U \). By possibly decreasing the size of the neighborhood \( V \) that we can find an open neighborhood \( U \) of \( x \), since also some neighborhood of \( v \) is disjoint from \( N \). But then we have to have \( \mu \notin S \), because otherwise \( v \in N \). According to our construction, one also has \( \varphi(v, -\varepsilon) \in \mathring{\omega} \subset N \). Therefore, there has to be a point \( z \in \varphi(v, (-\varepsilon, 0)) \cap \text{bd} N \) such that \( \text{ift}(z) = \mu \). Let \( \nu := \sigma_z \). Then we have the inclusion \( \nu \in V_\mu \), and since \( \mu \notin S \), the \( \mathcal{V} \)-compatibility of \( S \) implies \( \nu \notin S \). In addition, Proposition 5.27 in combination with \( \nu \in V_\mu \), and since \( \mu \notin S \), the \( \mathcal{V} \)-compatibility of \( S \) implies \( \nu \notin S \). Thus, the continuity of \( \varphi \) shows that we have \( z \in \omega \) for some \( \omega \in S \cap X_{\text{top}} \). It follows from (CD1) that \( \nu \subset \omega \). Since we also have
\[
z \in \varphi(v, (-\varepsilon, 0)) \subset \varphi(V, [-\varepsilon, \varepsilon]) \subset U = | \text{Star} \omega |,
\]
we immediately obtain \( \nu \in \text{Star} \omega \), and therefore \( \omega \subset \nu \). Since \( \omega, \rho \in S \) and \( \rho \) is locally closed by Proposition 5.23, we finally deduce \( \nu \in S \), which is a contradiction. Altogether, we have therefore established the equality \( \text{bd} N = N^+ \cup N^- \), as well as (ii).

In order to complete the proof of the theorem, we still have to verify the validity of (iii). By Proposition 5.15 we have the isomorphism
\[
H(|\text{Cl} S|) : H(|\text{Cl} S|, |\text{Mo} S|) \to H(\text{Cl} S, \text{Mo} S).
\]
In addition, \( H(|\text{Cl} S|, |\text{Mo} S|) = H(N, N^-) \) is the Conley index of the isolated invariant set \( \text{Inv}(N, \varphi) \) due to Proposition 3.5 and \( H(\text{Cl} S, \text{Mo} S) \) is the Conley index of \( S \) in view of Proposition 5.24. This completes the proof. \( \square \)

5.5. Periodic solutions via combinatorial Poincaré sections. Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and assume that \( \varphi \) is an admissible flow with respect to \( \mathcal{X} \). Theorem 5.28 facilitates the search for interesting isolated invariant sets for \( \varphi \) via an algorithmic study of the associated multivector field \( \mathcal{V} \). Note that due to Proposition 5.19, every toplex \( \sigma \in X_{\text{top}} \) is a strong isolating block. Hence, if \( \sigma \) is also regular, then \( \sigma \) is a brick in the sense of Definition 4.1. Therefore, it is natural to expect that the assumptions of Theorem 4.6 may in some cases be verified algorithmically. We first make an observation on a correspondence between certain brick paths and certain solutions of \( \mathcal{V} \). This requires a few more concepts.

We say that a proper pair of bricks \( (B, \tilde{B}) \) is sharp, if there exists a point \( x \in B \cap \tilde{B} \) and a time \( t > 0 \) such that \( \varphi(x, [0, t]) \subset \tilde{B} \). Moreover, we say that a brick path is sharp, if each two consecutive bricks in the path form a sharp pair. It is not difficult to verify that assumption (f) in Theorem 4.6 may be weakened by considering only sharp brick paths. This matters in the context of strongly admissible flows, because we have the following proposition.

**Proposition 5.29** (Sharp brick pairs and combinatorial paths). Let \( X \) denote a cellular space with cellular decomposition \( \mathcal{X} \), and assume that \( \varphi \) is a strongly admissible semiflow with respect to \( \mathcal{X} \). Furthermore, let \( \mathcal{V} \) denote the associated multivector field introduced in Proposition 5.25. Finally, assume that \( \tau, \sigma \in X_{\text{top}} \) are regular toplexes. Then they are bricks for \( \varphi \). Moreover, if \( (\tau, \sigma) \) is a sharp pair of bricks, then there is an \( \alpha \in X \) such that \( \tau, \alpha, \sigma \)}
is a path in $\mathcal{V}$. The reverse implication holds true if additionally the intersection of $\tau$ and $\sigma$ is not circular.

**Proof:** It follows from Proposition 5.19 and the definition of strong admissibility that $\tau$ and $\sigma$ are bricks. Suppose now that $(\tau, \sigma)$ is a sharp pair of bricks. Let $x \in \tau^{-} \cap \sigma^{+}$ and choose the time $t > 0$ such that $\varphi(x, (0, t)) \subset \sigma$. Let $g := \text{ift}(x)$. Then we have $g \cap \sigma \neq \emptyset$. This in turn implies $g \subset \sigma$, and since $g$ is also a toplex, we immediately obtain $g = \sigma$. Now consider the cell $\alpha := \sigma_x$. Since $x \in \tau^{-}$, we get $\alpha \subset \tau^{-} \subset \tau$ which yields $\alpha \in \text{Cl}\tau \subset F_{\mathcal{V}}(\tau)$. We also have $\sigma = g \in [\alpha]_{\mathcal{V}} \subset F_{\mathcal{V}}(\alpha)$, and this proves that $\tau, \alpha, \sigma$ is indeed a path in $\mathcal{V}$.

Assume now that $\tau, \alpha, \sigma$ is a path in $\mathcal{V}$, and that the intersection of $\tau$ and $\sigma$ is not circular. Then $(\tau, \sigma)$ is a proper pair of bricks. To see that it is even a sharp pair, we observe that in view of $\sigma \in F_{\mathcal{V}}(\alpha)$ one has the inclusion $\sigma \in [\alpha]_{\mathcal{V}}$, because $\sigma$ is a toplex. Therefore, $\alpha \in V_{\sigma}$. Now select an $x \in \tilde{\alpha}$. Then $\text{ift}(x) = \text{ift}(\alpha) = \sigma$, which proves that the pair $(\tau, \sigma)$ is sharp. □

Proposition 5.29 has a straightforward extension to sharp paths of bricks. To formulate it, we need one more final definition. We say that a solution $V$ of $\mathcal{A}$ is a *toplex*. Then if $t > 0$ such that $\varphi(x, (0, t)) \subset \sigma$. Let $g := \text{ift}(x)$. Then we have $g \subset \sigma$, and since $g$ is also a toplex, we immediately obtain $g = \sigma$. Now consider the cell $\alpha := \sigma_x$. Since $x \in \tau^{-}$, we get $\alpha \subset \tau^{-} \subset \tau$ which yields $\alpha \in \text{Cl}\tau \subset F_{\mathcal{V}}(\tau)$. We also have $\sigma = g \in [\alpha]_{\mathcal{V}} \subset F_{\mathcal{V}}(\alpha)$, and this proves that $\tau, \alpha, \sigma$ is indeed a path in $\mathcal{V}$.

Assume now that $\tau, \alpha, \sigma$ is a path in $\mathcal{V}$, and that the intersection of $\tau$ and $\sigma$ is not circular. Then $(\tau, \sigma)$ is a proper pair of bricks. To see that it is even a sharp pair, we observe that in view of $\sigma \in F_{\mathcal{V}}(\alpha)$ one has the inclusion $\sigma \in [\alpha]_{\mathcal{V}}$, because $\sigma$ is a toplex. Therefore, $\alpha \in V_{\sigma}$. Now select an $x \in \tilde{\alpha}$. Then $\text{ift}(x) = \text{ift}(\alpha) = \sigma$, which proves that the pair $(\tau, \sigma)$ is sharp. □

**Proposition 5.30** (Correspondence between brick paths and combinatorial solutions). In the situation of Proposition 5.29, assume that $\mathcal{A} \subset \mathcal{X}$ is a $\mathcal{V}$-compatible collection, which contains only regular multivectors and is free of circular pairs. Then there is a one-to-one correspondence between sharp paths of bricks in $|\mathcal{A}|$ and combinatorial solutions of $\mathcal{V}$ in $\mathcal{A}$ in the following sense. For every sharp path of bricks in $|\mathcal{A}|$ there exists a reduced solution of $\mathcal{V}$ in $\mathcal{A}$ such that the sequence of toplices along the solution is exactly the path of bricks, and the sequence of toplices along a reduced solution of $\mathcal{V}$ in $\mathcal{A}$ is a sharp path of bricks in $|\mathcal{A}|$. □

We now turn our attention to establishing the assumptions of the main Theorem 4.6. Clearly, there are many ways in which the brick decomposition $\mathcal{A}$ of $\mathcal{A}$ can be decomposed into subcollections $\mathcal{A}_i$. Note, however, that at the very least one has to make sure that none of the collections $\mathcal{A}_i$ contain a closed brick path. Hence, by Proposition 5.30 it cannot contain a periodic solution of $\mathcal{V}$. Thus, any choice of brick decomposition considered in Theorem 4.6 should start with the selection of a family $\mathcal{A}_0$ which breaks all periodic solutions of $\mathcal{V}$. From the point of view of pure graph theory this is the feedback vertex set problem [19] in the directed graph whose vertices are cells in $\mathcal{X}$ and there is a directed edge from $\tau \in \mathcal{X}$ to $\sigma \in \mathcal{X}$ whenever $\sigma \in F_{\mathcal{V}}(\tau)$, although the problem of constructing a minimal feedback vertex set is NP-complete in general [19], in our setting it is not too difficult, because one can take as a candidate for $\mathcal{A}$ the minimal locally closed, $\mathcal{V}$-compatible superset of the support of a nontrivial singular cohomology cocycle in $\text{Cl}\mathcal{S}$. To construct such a cocycle it suffices to find it in a small retract of $\text{Cl}\mathcal{S}$. Typically, both finding a small retract and transferring the cocycle in the retract back to the original space may be achieved in linear time. This suggests the following procedure which, if successful, leads to an appropriate coarsening of the brick decomposition of the isolating neighborhood considered in Theorem 4.6.

Assume that $\mathcal{A}$ is a connected isolated invariant set for a combinatorial multivector field $\mathcal{V}$ associated with a strongly admissible flow $\varphi$. Furthermore, consider a non-empty, locally closed, and $\mathcal{V}$-compatible subset $\mathcal{P} \subset \mathcal{A}$. Set $\mathcal{H} := \text{Cl}(\mathcal{A} \cap \text{Mo}\mathcal{P})$, let $\mathcal{R} := \text{Cl}\mathcal{A} \setminus \mathcal{H}$, and
define $\mathcal{P} := \text{Cl}_\mathcal{R} \mathcal{P}$. Suppose further that
\begin{align*}
\mathcal{H} &\text{ is non-empty and } \mathcal{R} \text{ is connected,} \\
\text{Every maximal solution in } \text{Cl} \mathcal{A} \text{ reaches } \mathcal{H} \cup \text{Mo} \mathcal{A}, \text{ and} \\
\mathcal{H} \cap \text{Cl} \text{Bd}_\mathcal{R} \mathcal{P} &\neq \emptyset.
\end{align*}
We then say that $\mathcal{P}$ is a \textit{combinatorial Poincaré section} for $\mathcal{A}$.

For a pair of cells $\sigma, \tau \in \mathcal{R}$, denote by $d(\sigma, \tau)$ the length of the shortest fence in $\mathcal{R}$ joining $\tau$ and $\sigma$. Such a fence exists, because of assumption (37). For a cell $\sigma \in \mathcal{R}$ we further set $d(\sigma, \mathcal{P}) := \min \{ d(\sigma, \tau) \mid \tau \in \mathcal{P} \}$, while for a path $\gamma$ in $\mathcal{R}$ we define $L(\gamma) := \max \{ d(\sigma, \mathcal{P}) \mid \sigma \in \text{im} \gamma \}$.

Then $L : \mathcal{R} \to \mathbb{N}$ is a function which decreases along solutions of $\mathcal{V}$ in $\mathcal{R}$, i.e., the function $L$ is a Lyapunov function. More precisely, we have the following straightforward proposition.

\textbf{Proposition 5.31} ($L$ is a Lyapunov function). \textit{For any pair of cells $\sigma, \tau \in \mathcal{R}$ with $\sigma \in F \mathcal{V}(\tau)$ we have $L(\sigma) \leq L(\tau)$.} \hfill $\square$

Since $\mathcal{H} \neq \emptyset$ by assumption (37), also $\mathcal{A} \cap \text{Mo} \mathcal{P} \neq \emptyset$. Therefore, since both $\mathcal{P}$ and $\mathcal{A}$ are $\mathcal{V}$-compatible, we see that $\bar{n} := \min \{ L(\tau) \mid \tau \in \text{Opn} \mathcal{H} \setminus \text{Cl} \mathcal{P} \}$ is well-defined. Clearly, since $L(\tau) > 0$ for all $\tau \notin \mathcal{P}$, we also have $\bar{n} > 0$.

Using the Lyapunov function $L$, we can now recursively define a finite sequence $(L_k)_{k=0}^{k_{\text{max}}}$ of subsets of $\mathcal{R}$ as follows. First, let $L_0 := \emptyset$, as well as $L_1 := \mathcal{P}$. Now consider $1 < k < k_{\text{max}}$, and assume that the collection $L_{k-1} \subset \mathcal{R}$ has already been defined and satisfies $L_{k-1} \neq \emptyset$. Then $B_k := \{ \sigma \in \mathcal{R} \setminus L_{k-1} \mid L_{k-1} \cap \text{Cl} \sigma \neq \emptyset \}$ is non-empty, and $n_k := \max \{ L(\sigma) \mid \sigma \in B_k \}$ is well-defined. Now, if we have $n_k \geq \bar{n}$, then we define $L_k := \mathcal{R}$, let $k_{\text{max}} := k$, and stop the recursion. Otherwise $L^{-1}([0, n_k]) \neq \mathcal{R}$ and we continue recursion by setting $L_k := L^{-1}([0, n_k])$.

A straightforward argument based on Proposition 5.31 can be used to prove the following proposition.

\textbf{Proposition 5.32} (Properties of the sets $L_k$). \textit{For every $0 \leq k < k_{\text{max}}$ the above-defined set $L_k$ is closed in $\mathcal{R}$ and $\mathcal{V}$-compatible.} \hfill $\square$

Clearly, we have the inclusions $B_k \subset L_k \setminus L_{k-1}$ and $L_{k-1} \subset L_k$. From the definitions of $\bar{n}$ and $k_{\text{max}}$ one further obtains
\begin{equation}
L_{k_{\text{max}}-1} \cap \text{Opn} \mathcal{H} \subset \text{Cl} \mathcal{P}.
\end{equation}
For $i = 0, 1, \ldots, k_{\text{max}} - 1$ we now define the sets
\begin{equation*}
\mathcal{A}_i := L_{i+1} \setminus \text{Int}_\mathcal{R} L_i,
\end{equation*}
and we refer to the sequence \( (A_i)_{i=0}^{k_{\text{max}}-1} \) as the sequence of shifts of \( \mathcal{P} \). These shifts are clearly non-empty and they are ordered in the following way.

**Proposition 5.33** (Ordering of the shifts of \( \mathcal{P} \)). *In the above situation, assume that the inequality \( A_i \cap A_j \neq \emptyset \) holds for some \( 0 \leq i < j < k_{\text{max}} \). Then one has to have \( j = i + 1 \).*

**Proof:** First observe that

\[ A_i \cap A_j = (L_{i+1} \setminus \text{Int}_\mathcal{R} L_i) \cap (L_{j+1} \setminus \text{Int}_\mathcal{R} L_j) = L_{i+1} \setminus \text{Int}_\mathcal{R} L_j. \]

Now consider a cell \( \tau \in A_i \cap A_j \). Since \( \tau \notin \text{Int}_\mathcal{R} L_j \), we can select a \( \sigma \in \mathcal{R} \cap \text{Opn} \tau \setminus L_j \). In view of the inequality \( i + 1 \leq j \), we have \( L_{i+1} \subset L_j \). Hence, we also have \( \sigma \notin L_{i+1} \). Together with the inclusion \( \tau \in L_{i+1} \cap \text{Cl} \sigma \), one further obtains \( \sigma \in B_{i+2} \subset L_{i+2} \). Finally, since \( \sigma \notin L_j \), this yields \( j < i + 2 \), i.e., we have \( j = i + 1 \).

We also have the following two propositions.

**Proposition 5.34** (Further properties of the \( L_k \)). *For arbitrary \( 0 < k \leq k_{\text{max}} \) we have the inclusion \( L_{k-1} \subset \text{Int}_\mathcal{R} L_k \).*

**Proof:** Assume to the contrary that there exists a cell \( \tau \in L_{k-1} \setminus \text{Int}_\mathcal{R} L_k \). Then there also has to be a cell \( \rho \in \text{Opn} \tau \setminus L_k \subset \text{Opn} \tau \setminus L_{k-1} \). It follows that \( \rho \in B_k \subset L_k \), which is a contradiction.

**Proposition 5.35** (Properties of the \( A_i \)). *If \( 0 < i < k_{\text{max}} - 1 \), then \( \mathcal{H} \cap \text{Cl} A_i = \emptyset \).*

**Proof:** Assume to the contrary that there exists a cell \( \tau \in \mathcal{H} \cap \text{Cl} A_i \). Choose a \( \sigma \in A_i \) such that \( \tau \in \text{Cl} \sigma \). Then \( \sigma \in \text{Opn} \tau \subset \text{Opn} \mathcal{H} \) and

\[ \sigma \in L_{i+1} \setminus \text{Int}_\mathcal{R} L_i \subset \mathcal{R} \cap L_{k_{\text{max}}-1} \setminus \text{Int}_\mathcal{R} L_1. \]

Hence, from (40) we get \( \sigma \in \mathcal{R} \cap \text{Cl} \mathcal{P} = \overline{\mathcal{P}} \). Since \( L_1 = \overline{\mathcal{P}} \) and \( \sigma \notin \text{Int}_\mathcal{R} L_1 \), one can easily see that \( \text{Opn}_\mathcal{R} \sigma \subset \mathcal{P} \). Hence, \( \sigma \in \text{Bd}_\mathcal{R} \mathcal{P} \) and \( \tau \in \mathcal{H} \cap \text{Cl} \text{Bd}_\mathcal{R} \mathcal{P} \), which contradicts (39).

**Proposition 5.36** (Further properties of the \( A_i \)). *For arbitrary \( i = 0, 1, \ldots, k_{\text{max}} - 1 \) we have to have \( \text{Cl} A_i \cap \text{Cl} A_{i+1} \neq \emptyset \), where \( i + 1 \) is taken modulo \( k_{\text{max}} \), i.e., we assume \( A_{k_{\text{max}}} = A_0 \).*

**Proof:** Assume first that the inequality \( i < k_{\text{max}} - 1 \) is satisfied, and select a cell \( \sigma \in B_{i+2} \). Then \( \sigma \notin L_{i+1} \), but we do have \( L_{i+1} \cap \text{Cl} \sigma \neq \emptyset \). Hence, we can choose a \( \tau \in L_{i+1} \cap \text{Cl} \sigma \). We have \( \tau \in \text{Cl} \sigma \subset B_{i+2} \subset \text{Cl} A_{i+1} \), yet one cannot have \( \tau \in \text{Int}_\mathcal{R} L_i \), because then one would obtain \( \sigma \in \text{Opn} L_i \subset L_i \subset L_{i+1} \), which is a contradiction. But this in turn implies that we have \( \tau \in L_{i+1} \setminus \text{Int}_\mathcal{R} L_i = A_i \subset \text{Cl} A_i \), which yields \( \text{Cl} A_i \cap \text{Cl} A_{i+1} \neq \emptyset \) for \( i < k_{\text{max}} - 1 \).

Consider now the remaining case \( i = k_{\text{max}} - 1 \). Then we have both \( A_{i+1} = A_0 = L_1 = \overline{\mathcal{P}} \) and \( A_i = A_{k_{\text{max}}-1} = \mathcal{R} \setminus \text{Int}_\mathcal{R} L_{k_{\text{max}}-1} \). If one selects a cell \( \sigma \in \text{Opn} \mathcal{H} \setminus \text{Cl} \mathcal{P} \), then \( \sigma \notin L_{k_{\text{max}}-1} \), because clearly \( n_{k_{\text{max}}-1} < n_i \). It follows that \( \sigma \in \mathcal{R} \setminus \text{Int}_\mathcal{R} L_{k_{\text{max}}-1} = A_{k_{\text{max}}-1} \). Since we also have \( \sigma \in \text{Opn} \mathcal{H} \), we can choose a \( \tau \in \mathcal{H} \) such that \( \sigma \in \text{Opn} \tau \). Then \( \tau \in \text{Cl} \sigma \subset \text{Cl} A_{k_{\text{max}}-1} \), but also \( \tau \in \mathcal{H} \subset \text{Cl} \mathcal{P} = \overline{\mathcal{P}} = \text{Cl} A_0 \). Therefore, one obtains \( \text{Cl} A_0 \cap \text{Cl} A_{k_{\text{max}}-1} \neq \emptyset \), which gives the conclusion also for \( i = k_{\text{max}} - 1 \).

After these preparations we can now finally establish the following link between the combinatorial setting and Theorem [4.6].
Theorem 5.37 (Periodic orbits via combinatorics). Assume that X is a cellular space with cellular decomposition \( \mathcal{X} \), and that \( \varphi \) is a strongly admissible flow on X with no cyclic intersection of toplexes. Let \( \mathcal{A} \) be a connected isolated invariant set for the combinatorial multivector field \( \mathcal{V} \) associated with \( \varphi \) such that \( \mathcal{A} \) contains only regular multivectors, and such that for either \( r = 0 \) or \( r = 1 \) we have

\[
\dim H_{2n+r}(\text{Cl}\mathcal{A}, \text{Mo}\mathcal{A}) = \dim H_{2n+1+r}(\text{Cl}\mathcal{A}, \text{Mo}\mathcal{A}) \quad \text{for all} \quad n \in \mathbb{Z},
\]

where not all of these homology groups are trivial. Assume further that \( \mathcal{P} \subset \mathcal{A} \) is a combinatorial Poincaré section for \( \mathcal{A} \), and that the associated sequence of shifts of \( \mathcal{P} \) is given by \((\mathcal{A}_i)_{i=0}^{k_{\text{max}}-1}\) with \( k_{\text{max}} \geq 3 \). Then the collection \( \mathcal{A}^{\text{top}} := \mathcal{A} \cap \mathcal{X}^{\text{top}} \) is a brick decomposition of the isolating block \( \mathcal{A} := |\text{Cl}\mathcal{A}| \) for \( \varphi \), with a coarsening \( \overline{\mathcal{A}} := \{ \mathcal{A}_i^{\text{top}} \}_{i=1}^{k_{\text{max}}-1} \), where we define \( \mathcal{A}_i^{\text{top}} := \mathcal{A}_i^{k_{\text{max}}-1-i} \cap \mathcal{X}_i^{\text{top}} \). Moreover, all the assumptions of Theorem 4.6 are satisfied. In consequence, the set \( \mathcal{A} \) contains a non-trivial periodic orbit of \( \varphi \).

Proof: We get from Theorem 5.28 (i-ii) that \( \mathcal{A} \) is indeed an isolating block for \( \varphi \). As for the assumptions of Theorem 4.6, note that (a) follows from Proposition 5.13, while (b) is a consequence of Theorem 5.28 (iii). According to Propositions 5.29 and 5.27 the set \( \mathcal{A}^{\text{top}} \) is a brick decomposition of \( \mathcal{A} \) with coarsening \( \overline{\mathcal{A}} \). Then assumption (c) follows from the fact that we suppose \( k_{\text{max}} \geq 3 \). To verify (e), we define \( A_i := \bigcup_{\sigma \in A_i^{\text{top}}} \sigma = |\text{Cl}\mathcal{A}_i| \), and assume that the inequality \( A_i \cap A_j \neq \emptyset \) is satisfied for some \( 0 \leq i < j < k_{\text{max}} \). Let \( x \in A_i \cap A_j \) and consider the cell \( \tau := \sigma_x \). If we have \( \tau \in \mathcal{R} \), then \( \tau \in \text{Cl}_\mathcal{R} A_i \cap \text{Cl}_\mathcal{R} A_j = A_i \cap A_j \), because \( A_i \) and \( A_j \) are closed in \( \mathcal{R} \). Therefore, we get from Proposition 5.33 that \( j = i + 1 \). If \( \tau \not\in \mathcal{R} \), then we have the inclusion \( \tau \in \mathcal{H} \), because clearly \( \tau \in \text{Cl}\mathcal{A} \). By Proposition 5.35 this is only possible if \( i = 0 \) and \( j = k_{\text{max}} - 1 \). Hence, in both of these cases (e) follows. Assumption (d) follows easily from Proposition 5.36 and the definition of the sets \( \mathcal{A}_i \).

Finally, to verify property (f) consider a maximal brick path \((\sigma_0, \sigma_1, \ldots)\). As we mentioned before Proposition 5.29 we may assume that the path is sharp. Hence, by Proposition 5.30 there is a reduced solution \( \gamma \) of \( \mathcal{V} \) such that the toplexes along this path are precisely the toplexes \((\sigma_0, \sigma_1, \ldots)\), and clearly we may assume that \( \gamma(0) = \sigma_0 \in \mathcal{A}^{\text{top}} \). Then there exists an index \( k \in \{0, 1, \ldots, k_{\text{max}} - 1\} \) such that \( \sigma_0 \in \mathcal{A}_k^{\text{top}} \subset \mathcal{A}_k \subset \mathcal{L}_{k+1} \). Obviously, the solution \( \gamma \) is also a maximal solution. Therefore, by \( \{38\} \) we may choose a \( j \) such that \( \gamma(j) \in \mathcal{H} \cup \text{Mo}\mathcal{A} \). If \( \gamma(j) \in \text{Mo}\mathcal{A} \), then clearly the corresponding toplex satisfies \( \sigma^{-1} \subset \mathcal{A} \). Hence, we now assume \( \gamma(j) \in \mathcal{H} \). Without loss of generality, we may further suppose that \( j \) is chosen as the first index with this property. Then one has \( \gamma([0, j - 1]) \subset \mathcal{R} \), and since \( \gamma(0) = \sigma_0 \in \mathcal{L}_{k+1} \), we also get from Proposition 5.31 the inclusion \( \gamma([0, j - 1]) \subset \mathcal{L}_{k+1} \). Now let \( i \) denote the first index such that \( \gamma(i) \not\in \mathcal{A}_k \). Such an index exists and satisfies \( i \leq j \), because both the inclusion \( \gamma(j) \in \mathcal{H} \) and the inclusion \( \mathcal{H} \cap \mathcal{A}_k \subset \mathcal{H} \cap \mathcal{R} = \emptyset \) are satisfied. It suffices to verify that \( \gamma(i) \in \mathcal{A}_{k-1} \) holds. If \( k = 0 \), then we have \( i = j \), as well as \( \gamma(i) \in \mathcal{H} \subset \mathcal{A}_{k_{\text{max}}-1} \). Hence, we can now assume \( k > 0 \). In this case one has \( \gamma(i) \in \mathcal{V}(\gamma(i-1)) = \text{Cl}\gamma(i-1) \cup [\gamma(i-1)]_\mathcal{V} \). We cannot have \( \gamma(i) \in \text{Cl}\gamma(i-1) \), because \( \gamma(i-1) \in \mathcal{A}_k \) and then \( \gamma(i) \in \text{Cl}_\mathcal{R} \mathcal{A}_k = \mathcal{A}_k \), a contradiction. Therefore, \( \gamma(i) \in [\gamma(i-1)]_\mathcal{V} \). Note that \( \gamma(i-1) \in \mathcal{A}_k \) if \( \mathcal{L}_{k+1} \setminus \text{Int}_\mathcal{R} \mathcal{L}_k \). In particular, \( \gamma(i-1) \in \mathcal{L}_{k+1} \) and by Proposition 5.32 also \( \gamma(i) \in \mathcal{L}_{k+1} \). But, \( \gamma(i) \not\in \mathcal{A}_k \). Therefore, \( \gamma(i) \in \text{Int}_\mathcal{R} \mathcal{L}_k \subset \mathcal{L}_k \). We will show that \( \gamma(i) \in \mathcal{A}_{k-1} \). Indeed, if \( \gamma(i) \not\in \mathcal{A}_{k-1} \), then \( \gamma(i) \in \text{Int}_\mathcal{R} \mathcal{L}_{k-1} \), because \( \gamma(i) \in \mathcal{L}_k \). Then, by \( \mathcal{V} \)-compatibility of \( \mathcal{L}_{k-1} \) we get \( \gamma(i-1) \in \mathcal{L}_k \) and by Proposition 5.34 we obtain \( \gamma(i-1) \in \text{Int}_\mathcal{R} \mathcal{L}_k \). Therefore, \( \gamma(i-1) \not\in \mathcal{A}_k \), which is a contradiction and proves that \( \gamma(i) \in \mathcal{A}_{k-1} \). \( \square \)
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