On a perturbation treatment of a model for MHD viscous flow

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Abstract

We discuss the solution of a nonlinear ordinary differential equation that appears in a model for MHD viscous flow caused by a shrinking sheet. We propose an accurate numerical solution and derive simple analytical expressions. Our results suggest that a recent perturbation treatment of the same problem exhibits a pathological behaviour and conjecture its probable cause.

1 Introduction

There has recently been enormous interest in the so called homotopy perturbation methods. In particular there is an open controversy between the developers of homotopy perturbation method (HPM) and homotopy analysis method (HAM)\cite{1,2}. One of the controversial points is related to the appearance of some “secular” terms in the perturbation solutions. Whereas He\cite{1} proposed their removal, Liao\cite{2} argued that they are harmless because they vanish as the variable increases towards infinity.

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Sajit and Hayat\cite{3} have recently discussed the MHD viscous flow due to a shrinking sheet. They converted the model partial differential equations into a nonlinear ordinary differential equation which they solved by means of HAM. The authors discussed the convergence of the perturbation approach and obtained apparently accurate results for several values of the model parameters. In fact, they concluded that “The obtained HAM solution is valid for all values of the suction parameter and Hartman number.”

The purpose of this paper is to analyze the accuracy of those HAM results because we believe that they may cast light on the abovementioned discussion about the “secular” terms. In Sec. 2 we consider an accurate numerical calculation, in Sec. 3 we derive simple approximate analytical expressions, and in Sec. 4 we discuss the results for a particular choice of the model parameters and draw conclusions.

2 Accurate numerical calculation

Since we are not interested in the validity and usefulness of the model we just concentrate on the equation that the authors solved approximately by means of HAM. By means of an appropriate transformation Sajit and Hayat\cite{3} converted the model partial differential equations into the ordinary nonlinear differential equation

\[
\begin{align*}
    f'''(\eta) - M^2 f'(\eta) - f'(\eta)^2 + mf(\eta)f''(\eta) &= 0 \\
    f(0) &= s, \quad f'(0) = -1 \\
    \lim_{\eta \to \infty} f'(\eta) &= 0
\end{align*}
\]

(1)

where a prime indicates differentiation with respect to the variable $\eta$ and $M$, $m$ and $s$ are model parameters. The main problem is to obtain the value of $f''(0)$ that is consistent with the condition at infinity. Once we have it, then we can resort to any numerical integration routine to obtain the solution for
all \( \eta > 0 \).

In order to determine \( f''(0) = \alpha \) accurately we apply the Hankel–Padé method developed some time ago that proved successful for the treatment of two-point boundary value problems [4,5,6]. It consists of expanding the solution in a Taylor series about \( \eta = 0 \)

\[
f(\eta) = \sum_{j=0}^{\infty} f_j(\alpha) \eta^j
\]

and then calculating the roots of the Hankel determinant \( H_D^D(\alpha) \) with matrix elements \( f_{i+j+d}(\alpha) \), \( i, j = 1, 2, \ldots, D \). The calculation is straightforward because the matrix elements and, consequently, the Hankel determinant are polynomial functions of the unknown parameter \( \alpha \). As shown in earlier applications of the method [4,5,6] one expects to find a sequence of roots \( \alpha_D, D = 2, 3, \ldots \), that converges towards the appropriate value of \( f''(0) \). As said above, once we have a sufficiently accurate value of this parameter then we can apply any numerical integration algorithm and obtain \( f(\eta) \) for all \( \eta > 0 \). Alternatively, in some cases the Padé approximants, on which the method is based, give sufficiently accurate results [5].

3 Approximate analytical expressions

Approximate analytical solutions to the equations of a physical model commonly provide greater insight into the nature of the phenomenon under investigation. The analytical expressions provided by HAM [3] appear to be so complicated that one can only use them within a computer algebra system. In this sense this kind of solution is not much different from the results provided by the numerical integration routines that are also built in most such software packages. The purpose of this section is to provide simple analytical expressions for the straightforward discussion of the MHD model.
As $\eta$ increases $f'(\eta)^2$ is expected to be smaller than the other terms and therefore $f(\eta)$ will behave approximately as the solution of $f'''(\eta) - M^2 f'(\eta) + m f_\infty f''(\eta) = 0$, where $f_\infty = f(\eta \to \infty)$. In other words, we expect that $f'(\eta) \approx b e^{-\beta \eta}$ for sufficiently large $\eta$. Therefore, it seems reasonable to try the ansatz

$$f^{[N]}(\eta) = \sum_{j=0}^{N} b_j e^{-\beta j \eta}, \quad N = 1, 2, \ldots$$

If we substitute it into the differential equation (I) we obtain an expression of the form

$$\sum_{j=1}^{2N} R_j(b_1, b_2, \ldots, b_N, \beta) e^{-\beta j \eta} = 0$$

Therefore, the optimal values of the adjustable parameters $b_j$ and $\beta$ should be solutions to

$$f^{[N]}(0) = s, \quad f^{[N]}(0) = -1$$
$$R_j(b_1, b_2, \ldots, b_N, \beta) = 0, \quad j = 1, 2, \ldots, N$$

Since it is our purpose, for the reasons already given above, to keep the results as simple as possible we just consider the first two approximation orders explicitly. For $N = 1$ and $N = 2$ we easily obtain

$$b_0 = s - \frac{1}{\beta}, \quad b_1 = \frac{1}{\beta}, \quad \beta = \frac{\sqrt{4M^2 + m^2 s^2 - 4m + ms}}{2}$$

and

$$b_0 = \frac{\beta^2 - M^2}{m \beta}, \quad b_1 = \frac{2(M^2 - \beta^2) + m(2\beta s - 1)}{m \beta}, \quad b_2 = \frac{\beta^2 - M^2 + m(1 - \beta s)}{m \beta}$$

$$4\beta^4 - 4\beta^3 ms(2 - m) - 2\beta^2 [2m(m^2 s^2 - ms^2 - 1) - M^2 (3m - 4)]$$
$$- 2\beta ms [M^2 (5m - 4) - 2m(m - 1)]$$
$$- 2M^2 (3m - 2) - 2M^2 m(2 - 3m) - m^2 (m - 1) = 0$$

respectively.
It is probable that this approach, or somewhat similar to it, had been used in the past. However, it is useful for our purposes and we are not aware that it had been applied to the problem discussed by Sajid and Hayat\[3\].

4 Results and discussion

Sajid and Hayat\[3\] analyzed the form of \(f'(\eta)\) for several values of the model parameters. Here we simply consider the case \(M = m = 2\) for the largest values of \(s\) for which \(f'(\eta)\) appears to exhibit a maximum according their Figure 2.

For \(s = 1.8\) a sequence of roots \(\alpha_D\) of the Hankel determinant \(H^d_D(\alpha)\) with \(d = 1\) and \(D = 2, \ldots, 30\) suggests that \(\alpha = 4.20411340\) (the reader may find examples of the rate of convergence of the method elsewhere\[4,5,6\]). Sajid and Hayat\[3\] only showed values of \(f''(0)\) for \(M = 2, s = 1\) and \(m = 1, 2\); therefore, their result for \(s = 1.8\) is not available for comparison. The correct behaviour of the numerical Runge–Kutta solution for large values of \(\eta\) provides an additional confirmation of the accuracy of that value of \(f''(0)\). Fig. 1 shows such numerical results and also those given by the analytical expressions (6) and (7). Our simple analytical expressions already provide satisfactory results for all values of \(\eta\) as well as the following acceptable estimates of \(f''(0)\):

\[\alpha^{[1]} = \sqrt{131/5 + 9/5} \approx 4.1\] and \[\alpha^{[2]} \approx 4.198\]. We appreciate that the accuracy increases with \(N\) (at least for the first two approximations). Besides, the fact that \(b_1 \approx 0.238 \gg b_2 \approx 0.00309\) for \(N = 2\) suggests a remarkable convergence rate.

Fig. 1 clearly shows that neither the approximate analytical expressions nor the accurate numerical results exhibit a maximum. Therefore, we conclude that the maxima found by Sajid and Hayat\[3\] when increasing the suction parameter are merely artifacts of the HAM. In our opinion, such spurious
maxima are probably caused by the “secular” terms of the form $\eta^k e^{-\eta}$ shown in Eq. (24) of Ref. [3] and discussed by He an Liao[12]. Although those terms certainly vanish as $\eta \to \infty$ they may have some non–negligible undesirable effect for moderate values of $\eta$.

Summarizing: we have verified once more that the Hankel–Padé method is useful for the treatment of two–point boundary value problems by obtaining an accurate value of the unknown parameter appearing in the nonlinear equation for a MHD viscous flow model[3]. It is necessary for a successful application of any numerical integration routine. We have also derived simple accurate analytical expressions that may be useful for the discussion of the physics of the problem. Both the analytical expressions and the numerical results have revealed a pathological behaviour of the results produced by the much more elaborate approach called HAM[3]. In this way we hope to have settled the argument about the “secular” terms in the HAM expressions[12]. Perhaps, the HAM users may want to verify our conjecture and throw some more light on the subject.

References

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Fig. 1. Numerical results (circles), first–order (dashed line) and second–order (solid line) analytical expressions for $f'(^\eta)$.