Abstract. We classify quasi-simple finite groups of essential dimension 3.

In memory of Professor Alfred Lvovich Shmel’kin

1. Introduction

This paper is based on the author’s talk given at the Magadan conference.

Let \( G \) be a finite group, and let \( V \) be a faithful representation of \( G \) regarded as an algebraic variety. A compression is a \( G \)-equivariant dominant rational map \( V \rightarrow X \) of faithful \( G \)-varieties. The essential dimension of \( G \), denoted \( \text{ed}(G) \), is the minimal dimension of all faithful \( G \)-varieties \( X \) appearing in compressions \( V \rightarrow X \). This notion was introduced by J. Buhler and Z. Reichstein [4] in relation to some classical problems in the theory of polynomials. It turns out that the essential dimension depends only on the group \( G \), i.e., it does not depend on the choice of the linear representation \( V \) [4, Theorem 3.1].

The computation of the essential dimension is a challenging problem of algebra and algebraic geometry. The finite groups of essential dimension \( \leq 2 \) have been classified (see [9]). Simple finite groups of essential dimension 3 have been recently determined by A. Beauville [2] (see also [8,18]).

1.1. Theorem. The simple groups of essential dimension 3 are \( A_6 \) and possibly \( \text{PSL}_2(11) \).

The essential dimension of \( p \)-groups was computed by Karpenko and Merkurjev [11].

In this short note, we find all the finite quasi-simple groups of essential dimension 3.

1.2. Definition. A group \( G \) is said to be quasi-simple if \( G \) is perfect, i.e., it equals its commutator subgroup, and the quotient of \( G \) by its center is a simple group.

The main result of this paper is the following.

1.3. Theorem. Let \( G \) be a finite quasi-simple nonsimple group. If \( \text{ed}(G) = 2 \), then \( G \simeq 2.A_5 \). If \( \text{ed}(G) = 3 \), then \( G \simeq 3.A_6 \).

1.4. Notation. Throughout this paper the ground field is supposed to be the field of complex numbers \( \mathbb{C} \). We employ the following standard notation used in group theory:

- \( \mu_n \) denotes the multiplicative group of order \( n \) (in \( \mathbb{C}^* \)),
- \( A_n \) denotes the alternating group of degree \( n \),
- \( \text{SL}_n(q) \) (\( \text{PSL}_n(q) \)) denotes the special linear group (respectively, projective special linear group) over the finite field \( \mathbb{F}_q \),
- \( n.G \) denotes a nonsplit central extension of \( G \) by \( \mu_n \),
- \( z(G) \) ([\( G, G \)]) denotes the center (respectively, the commutator subgroup) of a group \( G \).

All simple groups are supposed to be noncyclic.

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2. Proof of Theorem 1.3

The following assertion is an immediate consequence of the corresponding fact for simple groups [13].

2.1. Proposition. Let $X$ be a three-dimensional rationally connected variety and let $G \subset \text{Bir}(X)$ be a finite quasi-simple nonsimple group. Then $G$ is isomorphic to one of the following groups:

$$\text{SL}_2(7), \; \text{SL}_2(11), \; \text{Sp}_4(3), \; 2.\mathfrak{A}_5, \; n.\mathfrak{A}_6, \; n.\mathfrak{A}_7 \text{ with } n = 2, 3, 6.$$  \hspace{1cm} (2.1.1)

**Proof.** We may assume that $G$ biregularly (and faithfully) acts on $X$. Let $Y := X / z(G)$ and $G_1 := G / z(G)$. Then $Y$ is a three-dimensional rationally connected variety acted on by a finite simple group $G_1$. Then, according to [13], $G_1$ belongs to the following list:

$$\text{PSL}_2(7), \; \text{PSL}_2(8), \; \text{PSL}_2(11), \; \text{PSp}_4(3), \; \mathfrak{A}_5, \; \mathfrak{A}_6, \; \mathfrak{A}_7.$$ \hspace{1cm} (2.1.2)

Since $G_1$ is perfect, there exists the universal covering group $\tilde{G}_1$, i.e., a central extension of $G_1$ such that for any other central extension $\tilde{G}_1$ there is a unique homomorphism $\tilde{G}_1 \to \tilde{G}_1$ of central extensions (see, e.g., [10, Sec. 11.7, Theorem 7.4]). The kernel of $\tilde{G}_1 \to G_1$ is the Schur multiplier $\text{M}(G_1) = H^2(G_1, \mathbb{C}^*)$ of $G_1$. Thus, $G$ is uniquely (up to isomorphism) determined by $G / z(G)$ and the homomorphism $\text{M}(G_1) \to z(G)$. It is known that $\text{M}(G_1) \cong \mu_2$ in all the cases (2.1.2) except for $\mathfrak{A}_6$ and $\mathfrak{A}_7$, where the Schur multiplier is isomorphic to $\mu_6$, and $\text{PSL}_2(8)$ where the Schur multiplier is trivial (see [10, Sec. 12.3, Theorem 11.4, Sec. 16.3, Theorem 3.2] and [6]). This gives us the list (2.1.1). \hfill \Box

2.1.3. Remark. We do not assert that all the possibilities (2.1.1) occur. Using the technique developed in [13–17] it should be possible to obtain a complete classification of actions of quasi-simple groups on rationally connected threefolds. However, this is much more difficult problem.

2.1.4. Corollary. Let $G$ be a group satisfying the conditions of Proposition 2.1. Then $G$ has an irreducible faithful representation. The minimal dimension of such a representation $V$ is given by the following table [6]:

| $G$       | $2.\mathfrak{A}_5$ | $3.\mathfrak{A}_6$ | $\text{SL}_2(7), \text{Sp}_4(3), \; 2.\mathfrak{A}_6, \; 2.\mathfrak{A}_7$ | $\text{SL}_2(11), \; 6.\mathfrak{A}_6, \; 3.\mathfrak{A}_7, \; 6.\mathfrak{A}_7$ |
|-----------|------------------|-------------------|---------------------------------|---------------------------------|
| $\dim V$  | $2$              | $3$               | $4$                             | $6$                             |

2.2. Construction. Let $G$ be a finite quasi-simple nonsimple group having a faithful irreducible representation $V$. Let $\psi: V \to X$ be a compression with $\dim X = \text{ed}(G)$. Applying an equivariant resolution of singularities (see [1]), we may assume that $X$ is smooth (and projective). Furthermore, consider the compactification $\mathbb{P} := \mathbb{P}(V \oplus \mathbb{C})$ and let $X \xrightarrow{\varphi} Y \xrightarrow{f} \mathbb{P} \supset V$ be an equivariant resolution of $\psi$, where $f$ is a birational morphism and $Y$ is smooth and projective. We also may assume that $f$ is passed through the blowup $\tilde{f}: \tilde{\mathbb{P}} \to \mathbb{P}$ of $0 \in V \subset \mathbb{P}$. Let $E \subset \tilde{V}$ be the $\tilde{f}$-exceptional divisor and let $E \subset Y$ be its proper transform, and let $B := \varphi(E)$. Thus, we have the following $G$-equivariant diagram:

The action of $z(G)$ on $E \cong \mathbb{P}(V)$ and on $E$ is trivial because $V$ is an irreducible representation. Hence, $G / z(G)$ faithfully acts on $E$. By assumption, the action of $z(G)$ on $X$ is faithful. Hence $B \neq X$ and so $B$ is a rationally connected variety of dimension $< \text{ed}(G)$.

2.3. Proposition. Let $G$ be a finite quasi-simple nonsimple group having a faithful irreducible representation. Then $G / z(G)$ acts faithfully of a rationally connected variety of dimension $< \text{ed}(G)$.
Proof. Let $V$ be a faithful irreducible representation of $G$ and let $\psi: V \to X$ be a compression with $\dim X = \text{ed}(G)$. Apply Construction 2.2. Assume that $G$ has a fixed point $P \in X$. Then $G$ has a faithful representation on the tangent space $T_{P,X}$. Let $T_{P,X} = \bigoplus T_i$ be the decomposition in irreducible components. At least one of them, say $T_1$ is nontrivial. Then $G/z(G)$ faithfully acts on $\mathbb{P}(T_1)$, where $\mathbb{P}(T_1) < \dim X = \text{ed}(G)$. Thus, we may assume that $G$ has no fixed points on $X$. By Construction 2.2, the variety $B$ is rationally connected and $\dim B < \text{ed}(G)$. Since $G$ has no fixed points on $B$ and the group $G/z(G)$ is simple, its action on $B$ must be effective. 

Comparing the list (2.1.1) with Theorem 3.1, we obtain the following corollary.

2.4. Corollary. Let $G$ be a finite quasi-simple nonsimple group with $\text{ed}(G) \leq 3$. Then for $G$ we have one of the following possibilities:

$$\text{SL}_2(7), n = 2, 3, 6. \quad (2.4.1)$$

Now we consider the possibilities of (2.4.1) case by case.

2.5. Lemma. $\text{ed}(2.A_5) = 2$ and $\text{ed}(3.A_6) = 3$.

Proof. Let us prove, for example, the second equality. Since $3.A_6$ has a faithful three-dimensional representation, $\text{ed}(3.A_6) \leq 3$. On the other hand, $A_6$ cannot effectively act on a rational curve. Hence, by Proposition 2.3 $\text{ed}(3.A_6) \geq 3$.

2.5.1. Lemma. Let $G$ be a quasi-simple nonsimple group. Assume that $G \not= 2.A_5, 3.A_6$. Assume also that $G$ contains a subgroup $H$ such that

(i) $H$ is not Abelian but its image $H \subset G/ z(G)$ is Abelian,

(ii) for any action of $G/ z(G)$ on a rational projective surface the subgroup $H \subset G/ z(G)$ has a fixed point.

Then $\text{ed}(G) \geq 4$.

Proof. Since $H/ (z(G) \cap H) = H$, we have $z(G) \cap H \supset [H, H]$ and $[H, H] \not= \{1\}$ (because $H$ is not Abelian). Assume that $\text{ed}(G) = 3$. Apply Construction 2.2. From the list (2.1.1) one can see that $G/ z(G)$ cannot faithfully act on a rational curve and by Corollary 2.1.4 $G$ has no fixed points on $X$. Hence, $B$ is a (rational) surface. By ii, the group $H$ has a fixed point, say $P$, on $B \subset X$. There is an invariant decomposition $T_{P,X} = T_{P,B} \oplus T_1$, where $\dim T_1 = 1$. The action of $[H, H]$ on $T_{P,B}$ and $T_1$ is trivial. Hence it is trivial on $T_{P,X}$ and $X$, a contradiction.

2.6. Proposition. $\text{ed}(\text{SL}_2(7)) = 4$.

2.6.1. Lemma. Let $S$ be a smooth projective rational surface admitting the action of $\text{PSL}_2(7)$. Let $H \subset \text{PSL}_2(7)$ be a subgroup isomorphic to $\mu_2 \times \mu_2$. Then $H$ has a fixed point on $S$.

Proof. Since $H$ is Abelian, according to [12] it is sufficient to show the existence of a fixed point on some birational model of $S$. By Theorem 3.1, we may assume that $S$ is either $\mathbb{P}^2$ or some special del Pezzo surface of degree 2 (see Theorem 3.1(iii)). In the former case, $\mathbb{P}^2 = \mathbb{P}(W)$, where $W$ is a three-dimensional irreducible representation of $\text{PSL}_2(7)$. Then the Abelian group $H \simeq \mu_2 \times \mu_2$ has a fixed point on $\mathbb{P}^2 = \mathbb{P}(W)$. Thus we assume that $S$ is a del Pezzo surface of degree 2.

Let $\alpha \in H$ be an element of order 2. First assume that $\alpha$ has a curve $C$ of fixed points. The image $\pi(C)$ under the anticanonical double cover $\pi: S \to \mathbb{P}^2$ must be a line (because the action on $\mathbb{P}^2$ is linear). Let $\alpha' \in H, \alpha' \not= \alpha$ be another element of order 2. Then $\alpha'(C)$ is also a curve of $\alpha$-fixed points and $\pi(\alpha'(C))$ is also a line. Hence, $\pi(\alpha'(C)) = \pi(C)$ and $\pi^{-1}(\pi(C))$ contains $\alpha'(C)$ and $C$. Since $\pi^{-1}(\pi(C)) \sim -K_S$ and the fixed point locus of $\alpha$ is smooth, we have $\alpha'(C) = C = \pi^{-1}(\pi(C)) \sim -K_S$ and it is an ample divisor. Note that all the elements of order 2 are conjugate in $\text{PSL}_2(7)$. Hence $\alpha'$ also has a curve of fixed points, say $C'$, and $C' \sim -K_S$. Then the intersection points $C \cap C'$ are fixed by $H = \langle \alpha, \alpha' \rangle$. 

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Thus we may assume that any element $\alpha \in H$ of order 2 has only isolated fixed points. The holomorphic Lefschetz fixed point formula shows that the number of these fixed points equals $4\chi(\mathcal{O}_S) = 4$. Then by the topological Lefschetz fixed point formula

$$\text{Tr}_{H^2(S,\mathbb{C})} \alpha^* = 2.$$  

Since $\dim H^2(S,\mathbb{C}) = 8$ and all the eigenvalues of $\alpha^*$ equal $\pm 1$, its determinant must be equal to $-1$ and so we have a non-trivial character of the group $\text{PSL}_2(7)$. This contradicts the fact that $\text{PSL}_2(7)$ is simple.

**Proof of Proposition 2.6.** Consider the subgroup $\hat{H} \subset \text{SL}_2(7)$ generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 5 \\ 1 & -1 \end{pmatrix}.$$  

Let $H$ be its image in $\text{PSL}_2(7)$. It is easy to check that $A^2 = B^2 = -I$ and $[A, B] = -I$. Hence $\hat{H}$ is isomorphic to the quaternion group $Q_8$ and $H \simeq Q_8/\mu_2 \cong \mu_2 \times \mu_2$. By Lemma 2.6.1, the group $\hat{H}$ has a fixed point on $B \subset X$. Hence we can apply Lemma 2.5.1.

**2.7. Proposition.** ed($2\mathfrak{A}_6$) = 4, ed($6\mathfrak{A}_6$) $\geq$ 4.

**Proof.** As above, we are going to apply Lemma 2.5.1. Let $S$ be a projective rational surface acted by $G/\mu_2(G) = \mathfrak{A}_6$. By Theorem 3.1, we may assume that $S \simeq \mathbb{P}^2$. The group $2\mathfrak{A}_6$ is isomorphic to $\text{SL}_2(9)$ (see [6]). As in the proof of Proposition 2.6, take the subgroup $Q_8 \simeq \hat{H} \subset \text{SL}_2(9)$ generated by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 1 & -4 \end{pmatrix}$$  

(2.7.1)

and apply Lemma 2.5.1.

In the case $G = 6\mathfrak{A}_6$, let $Z \subset \mu_2(G)$ be the subgroup of order 3. Then $6\mathfrak{A}_6/Z \simeq 2\mathfrak{A}_6 \simeq \text{SL}_2(9)$. Let $\hat{H}$ be the inverse image of the subgroup of $\text{SL}_2(9)$ generated by $A$ and $B$ from (2.7.1), and let $\hat{H} \subset \hat{H}$ be the Sylow 2-subgroup. Then, as above, $\hat{H} \simeq Q_8$ and we can apply Lemma 2.5.1.

**2.7.2. Remark.** Since $6\mathfrak{A}_6$ has a six-dimensional faithful representation, one has ed($6\mathfrak{A}_6$) $\leq$ 6. However, we are not able to compute the precise value.

**3. Appendix: Simple Subgroups in the Plane Cremona Group**

The following theorem can be easily extracted from the classification [7]. For convenience of the reader, we provide a relatively short and self-contained proof.

**3.1. Theorem ([7]).** Let $G \subset \text{Cr}_2(\mathbb{C})$ be a finite simple subgroup. Then the embedding $G \subset \text{Cr}_2(\mathbb{C})$ is induced by one of the following actions:

(i) $\mathfrak{A}_5$, $\text{PSL}_2(7)$, or $\mathfrak{A}_6$ acting on $\mathbb{P}^2$;

(ii) $\mathfrak{A}_5$ acting on the del Pezzo surface of degree 5;

(iii) $\text{PSL}_2(7)$ acting on some special del Pezzo surface of degree 2 which can be realized as a double cover of $\mathbb{P}^2$ branched in the Klein quartic curve;

(iv) $\mathfrak{A}_5$ acting on $\mathbb{P}^1 \times \mathbb{P}^1$ through the first factor.

**3.2. Remark.** It is known (see [7, Sec. 8] and [5, Sec. B]) that the above listed actions are not conjugate in $\text{Cr}_2(\mathbb{C})$.

**Proof.** Applying the standard arguments (see, e.g., [7, Sec. 3]) we may assume that $G$ faithfully acts on a smooth projective rational surface $X$ which is either a del Pezzo surface or an equivariant conic bundle. Moreover, one has rk Pic($X$) = 1 (rk Pic($X$) = 2) in the del Pezzo (respectively, conic bundle) case.

First, consider the case where $X$ has an equivariant conic bundle structure $\pi: X \to B \simeq \mathbb{P}^1$. Since the group $G$ is simple, it acts faithfully either on the base $B$ or on the general fiber. Hence $G$ is embeddable.
This implies that the representation of Lefschetz fixed point formula, the action of $G$ only possibility is $\Lambda$. The intersection form induces an even quadratic form on $\Lambda$ and, therefore, it induces a quadratic form on $\mathcal{O}(X)$.

By making elementary transformations in $\text{GL}(F, \mathcal{O}(X))$, we may replace $\mathcal{O}(X)$ with $\mathcal{O}(X)$, where $n' = 0$ or 1. If $n' = 1$, then contracting the negative section we obtain the action on $\mathcal{O}(X)$ with a fixed point. This is impossible for $G \simeq \mathfrak{A}_5$. Hence, we may assume that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Additional elementary transformations allow to trivialize the action on the second factor. This is the case (iv). See [5, Sec. B] for details.

From now on we assume that $X$ is a del Pezzo surface$^1$ with $\text{rk Pic}(X)^G = 1$. We consider the possibilities according to the degree $d = K^2_X$.

Case $d = 1$. This case cannot occur, as $|−K_X|$ has one base point $P$, and $G$ has to act on $\mathcal{O}(X)$ effectively. Hence $G \subset \text{GL}(\mathcal{O}(X))$. However, there are no simple finite subgroups in $\text{GL}(\mathbb{C}, \mathcal{O}(X))$, a contradiction.

Case $d = 2$. Then the anticanonical map $X \to \mathbb{P}^2$ is a double cover whose branch divisor $B \subset \mathbb{P}^2$ is a smooth quartic. The action of $G$ on $X$ descends to $\mathbb{P}^2$ so that $B$ is $G$-stable. Therefore, $G \subset \text{Aut}(B)$. According to the Hurwitz bound $|G| \leq 168$. Moreover, $\text{Aut}(B)$ contains no elements of order 5. Then the only possibility is $G \simeq \text{PSL}_2(7)$ and $B = \{x_1^2 + x_2^2x_3 + x_1^3x_1 = 0\}$. We get the case (iii).

Case $d = 3$. Then $X$ is a cubic surface in $\mathbb{P}^3$. The action of $G$ on the lattice $\Lambda := K_X^2 \subset \text{Pic}(X)$ is faithful. Hence our group $G$ has a representation on the vector space $\Lambda/2\Lambda = (\mathcal{O}(X))^6$ over the field $\mathbb{F}_2$. The intersection form induces an even quadratic form on $\Lambda$ and, therefore, it induces a quadratic form on $\Lambda/2\Lambda$

$$q(x) := \frac{1}{2}(x, x) \mod 2.$$ 

Take a standard basis $h, e_1, \ldots, e_6$ in $\text{Pic}(X)$ with $(h, h) = 1$, $(h, e_i) = 0$, $(e_i, e_j) = −\delta_{ij}^3$. Then using the basis $e_1 − e_2, e_2 − e_3, e_4 − e_5, e_5 − e_6, h − e_1 − e_2 − e_3, h − e_4 − e_5 − e_6$ of $\Lambda$, we can write $q(x)$ in the following form:

$$q(x) = x_1^2 + x_2^2 + x_1x_2 + x_3^2 + x_2^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6.$$ 

Then it is easy to see that the Arf invariant of $q(x)$ equals 1. The group preserves the intersection form and the quadratic form $q(x)$. Therefore, there is a natural embedding $G \hookrightarrow O_6(\mathbb{F}_2)^−$. Since $G$ is simple, $G \subset [O_6(\mathbb{F}_2)^−, O_6(\mathbb{F}_2)^−]$. It is known that $[O_6(\mathbb{F}_2)^−, O_6(\mathbb{F}_2)^−] \simeq \text{PSp}_4(3)$ (see, e.g., [6]). Moreover, $G$ is isomorphic to one of the following groups: $\text{PSp}_4(3)$, $\mathfrak{A}_6$ or $\mathfrak{A}_5$. On the other hand, $G$ faithfully acts on $H^0(X, −K_X) \simeq \mathbb{C}^4$. Then the only possibility is $G \simeq \mathfrak{A}_5$. The defining equation $\psi(z) = 0$ of $X \subset \mathbb{P}^3$ is a cubic invariant on $H^0(X, −K_X)$. Hence the representation on $H^0(X, −K_X)$ is the standard irreducible representation of $\mathfrak{A}_5$. Then in a suitable basis we have $\psi = z_1^3 + \cdots + z_4^3 − (z_1 + \cdots + z_4)^3$. The fixed point locus of an element $α \in G$ of order two is a union of a line and three isolated points. By the topological Lefschetz fixed point formula, the action of $α$ on $\Lambda$ is diagonalizable as follows: $α = \text{diag}(1, 1, 1, 1, −1, −1)$. This implies that the representation of $G$ on $\Lambda$ is the sum of the irreducible four-dimensional representation and the trivial one. This contradicts the minimality assumption $\text{rk Pic}(X)^G = 1$.

Case $d = 4$. Then $X = X_{2, 2} = Q' ∩ Q'' \subset \mathbb{P}^4$ is an intersection of two quadrics. The group $G$ acts on the pencil of quadrics $⟨Q', Q''⟩$ leaving invariant the subset of five singular elements. It is easy to see that in this case the action on $⟨Q', Q''⟩$ must be trivial. Hence, $G$ fixes vertices $P_1, \ldots, P_5$ of five

$^1$More generally, actions of simple groups on del Pezzo surfaces with log terminal singularities were studied in [3].
G-stable quadratic cones $Q_i \in \langle Q', Q'' \rangle$. Since these points $P_1, \ldots, P_5$ generate $\mathbb{P}^5$, $G$ must be Abelian, a contradiction.

Case $d = 5$. A del Pezzo surface of degree 5 is unique up to isomorphism. Consider the (faithful) action of $G$ on the space $\text{Pic}(X) \otimes \mathbb{C}$ and on the orthogonal complement $K_X^1 \subset \text{Pic}(X) \otimes \mathbb{C}$. The intersection form induces a non-degenerate quadratic form on $K_X^1$. Hence $G$ faithfully acts on a two-dimensional quadric in $\mathbb{P}^3$. Then the only possibility is $G \simeq \mathfrak{A}_5$. One can see that $\text{Aut}(X)$ isomorphic to the symmetric group $\mathfrak{S}_5$ and so a del Pezzo surface of degree 5 admits an action of $\mathfrak{A}_5$. We get the case (ii).

Case $6 \leq d \leq 8$. Then the action of $G$ on $\text{Pic}(X) \simeq \mathbb{Z}^{10-d}$ must be trivial. This contradicts the minimality assumption $\text{rk Pic}(X)^G = 1$.

Case $d = 9$. Then $X = \mathbb{P}^2$. So $G \subset \text{PGL}_3(\mathbb{C})$, and by the classification of finite subgroups in $\text{PGL}_2(\mathbb{C})$ we get the case (i).

Theorem 3.1 is proved. □

3.3. Theorem ([7, 19]). Let $G \subset \text{Cr}_2(\mathbb{C})$ be a finite quasi-simple nonsimple subgroup. Then $G \simeq 2.\mathfrak{A}_5$.

Proof. As in the proof of Theorem 3.1 we may assume that $G$ faithfully acts on a smooth projective rational surface $X$ which is either a del Pezzo surface with $\text{rk Pic}(X)^G = 1$ or an equivariant conic bundle with $\text{rk Pic}(X)^G = 2$. Let $\tilde{G} := G/z(G)$. Then $\tilde{G}$ is a simple group acting on a rational surface $X/\pi z(G)$. Hence $\tilde{G}$ is embeddable to $\text{Cr}_2(\mathbb{C})$ and by Theorem 3.1 we have $G \simeq \mathfrak{A}_5$, $\mathfrak{A}_6$, or $\text{PSL}_2(7)$. Therefore, as in the proof of Proposition 2.1 we have one of the following possibilities: $G \simeq 2.\mathfrak{A}_5$, $\text{SL}_2(7)$, or $n.\mathfrak{A}_6$ for $n = 2, 3, 6$. If $X$ has an equivariant conic bundle structure $\pi : X \to B \simeq \mathbb{P}^1$, then $G$ nontrivially acts either on the base $B$ or on the general fiber. This is possible only if $G \simeq 2.\mathfrak{A}_5$.

Assume that $X$ is a del Pezzo surface with $\text{rk Pic}(X)^G = 1$. Let $Z \subset z(G)$ be a cyclic subgroup of prime order $p$ and let $\pi : X \to Y := X/Z$ be the quotient. The surface $Y$ is rational and $G := G/Z$ faithfully acts on $Y$.

First, consider the case where $Z$ has only isolated fixed points. If $p = 2$, then by the holomorphic Lefschetz formula the number of fixed points equals 4. These points cannot be permuted by $G$, so they are fixed by $G$. Similarly, in the case $p = 3$ denote by $n_0 (n_1, n_2)$ the number of fixed points with action of type $\frac{1}{3}(1, -1)$ (respectively, $\frac{1}{3}(1, 1), \frac{1}{3}(-1, -1)$). Then again by the holomorphic Lefschetz formula $n_1 = n_2, n_0 + n_1 = 3$. Hence there are at most three points of each type and so, as above, all these points are fixed by $G$. Since the groups $\text{SL}_2(7)$ and $n.\mathfrak{A}_6$ cannot act faithfully on the tangent space to a fixed point, the only possibility is $G \simeq 2.\mathfrak{A}_5$.

Now consider the case where the fixed point locus $X^Z$ of $Z$ is one-dimensional. Let $C$ be the union of all the curves in $X^Z$. Note that $C$ is smooth because it is a union of components of the fixed point locus. Clearly, $C$ is $G$-invariant. Then the class of $C$ must be proportional to $-K_X$ in $\text{Pic}(X)$. Hence $C$ is ample and connected. Since $C$ is smooth, it is irreducible. The group $G/Z$ nontrivially acts on $C$. Hence $C$ cannot be an elliptic curve. Moreover, if $C$ is rational, then $G/Z \simeq \mathfrak{A}_5$ and we are done. Thus we can write $C \sim -aK_X$ with $a > 1$. If $-K_X$ is very ample, then $C$ is contained in a hyperplane section and $a = 1$, a contradiction. Thus, it remains to consider only two possibilities: $K_X^2 = 1$ and $K_X^3 = 2$. If $K_X^2 = 1$, then the anticanonical linear system has a unique base point, say $O$. Since, the representation of $G$ in the tangent space $T_{O, X}$ is faithful, the only possibility is $G \simeq 2.\mathfrak{A}_5$. Finally, assume that $K_X^2 = 2$. Then the anticanonical map is a double cover $\Phi_{-K_X} : X \to \mathbb{P}^2$. The action of $Z$ of $\mathbb{P}^2$ must be trivial (otherwise $a = 1$). Hence $p = 2$, $Z$ is generated by the Geiser involution $\gamma$, and $C$ is the ramification curve of $\Phi_{-K_X}$. On the other hand, there exists the homomorphism

$$\lambda : \text{Aut}(X) \hookrightarrow \text{GL}(\text{Pic}(X)) = \text{GL}_8(\mathbb{Z}) \xrightarrow{\text{det}} \{\pm 1\},$$

where $\lambda(\gamma) = -1$. Since our group $G$ is perfect, $\gamma \notin G$, a contradiction. □

3.4. Remark. In the same manner, one can describe the actions of $G = 2.\mathfrak{A}_5$ on rational surfaces, i.e., the embeddings $2.\mathfrak{A}_5 \hookrightarrow \text{Cr}_2(\mathbb{C})$ (see [19] for details).
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