Thermal Casimir effect for neutrino and electromagnetic fields in closed Friedmann cosmological model

V. B. Bezerra, 1 V. M. Mostepanenko, 1, 2 H. F. Mota, 1 and C. Romero 1

1 Department of Physics, Federal University of Paraíba, C.P. 5008, 58051-970, João Pessoa, PB, Brazil
2 Noncommercial Partnership “Scientific Instruments”, Tverskaya Street 11, Moscow, 103905, Russia

Abstract

We calculate the total internal energy, total energy density and pressure, and the free energy for the neutrino and electromagnetic fields in Einstein and closed Friedmann cosmological models. The Casimir contributions to all these quantities are separated. The asymptotic expressions for both the total internal energy and free energy, and for the Casimir contributions to them are found in the limiting cases of low and high temperatures. It is shown that the neutrino field does not possess a classical limit at high temperature. As for the electromagnetic field, we demonstrate that the total internal energy has the classical contribution and the Casimir internal energy goes to the classical limit at high temperature. The respective Casimir free energy contains both linear and logarithmic terms with respect to the temperature. The total and Casimir entropies for the neutrino and electromagnetic fields at low temperature are also calculated and shown to be in agreement with the Nernst heat theorem.

PACS numbers: 04.62.+v, 44.40.+a, 98.80.-k
I. INTRODUCTION

The Casimir effect [1] manifests itself as some vacuum polarization energy and force due to material boundaries or nontrivial topology of space. This effect is presently well known owing to its multidisciplinary applications ranging from condensed matter physics and nanotechnology to quantum field theory, gravitation and cosmology (see monographs [2–5]). During the last decade, numerous experiments on measuring the Casimir force have been performed as reported in reviews [6–8]. Their results were used to obtain stronger constraints on the parameters of the Yukawa-type corrections to Newtonian gravitational law predicted in many extensions of the standard model (see the most recent results in Refs. [9–13]). Keeping in mind that measurements of the Casimir force are usually performed at room temperature, the problem of thermal Casimir effect became topical. In this regard it was found that the influence of free charge carriers in the boundary bodies on the thermal correction to the Casimir force is highly nontrivial and leads to complicated problems in quantum statistical physics [5, 6, 8, 14].

In cosmology, the Casimir effect arises not due to the material boundaries, but due to the identification conditions imposed on the wave functions by a nontrivial topology of space. Specifically, the Casimir energy density and pressure for the scalar field in the Einstein and closed Friedmann cosmological models with a topology $S^3 \times R^1$ were found in Refs. [15, 16] and in Ref. [17] in the massless and massive case, respectively. The Casimir density and pressure for the neutrino and electromagnetic fields in the Einstein cosmological model were derived in Ref. [18]. In succeeding years the Casimir effect was investigated in many topologically nontrivial spaces [19–22]. This resulted in the development of new methods in mathematical physics and found prospective application in multidimensional cosmology [23, 24] and in the problem of dark energy [25].

A considerable number of works was devoted to the thermal Casimir effect in cosmological models when not only zero-point photons, but thermal radiation as well was taken into account. Specifically, thermal Green’s functions in Rindler, de Sitter and Schwarzschild spaces were considered in Ref. [26]. In Refs. [27, 28] the thermal stress-energy tensor of the scalar field in the Einstein cosmological model was investigated and the asymptotic behaviors at low and high temperature were found. Similar problem for a cosmological model with a 3-torus topology was solved in Ref. [29]. In Ref. [30] the results of Refs. [27, 28] were
reconsidered using another approach. It was shown that the total thermal stress-energy tensor of the scalar field in Einstein model contains the Casimir contribution which has the same asymptotic behavior at low and high temperature as in the case of two parallel ideal-metal plates. The total thermal stress-energy tensors of the neutrino and electromagnetic fields in Einstein cosmological model were considered in Ref. [31]. The expressions obtained were used [31, 32] to determine the back reaction of the total thermal stress-energy tensor on the space-time by solving the Einstein equations, where this tensor would play the role of a source.

It is common knowledge that at the very early stages the evolution of our universe has been going on at very high temperatures. Because of this, the studies of thermal Casimir effect in cosmological models mentioned above are of great physical significance. Keeping in mind, however, that massless fields of zero spin are not observed in nature, main attention in this respect should be paid to neutrino and electromagnetic fields. Considering that at large scales our universe is spatially homogeneous, here we derive exact expressions for the total thermal stress-energy tensor for these fields in Einstein and closed Friedmann cosmological models. We also find the total and Casimir free energy and the Casimir contributions to the stress-energy tensor (the latter were previously known only at zero temperature). We find the asymptotic behaviors of the obtained expressions in the limiting cases of low and high temperatures. For the neutrino case our results for the total energy and pressure are the same as in Ref. [31]. For the electromagnetic case, however, there is a disagreement between our thermal stress-energy tensor and that obtained in Ref. [31]. We explain this disagreement by the omission in [31] of a nontrivial contribution arising from the zeroth mode in the Poisson (or, equivalently, Abel-Plana) summation formula. The expression for the Casimir free energy and Casimir internal energy for the electromagnetic field obtained by us are in direct analogy with the familiar case of the thermal Casimir effect in an ideal metal spherical shell. For both the neutrino and electromagnetic fields we calculate the Casimir entropy and demonstrate that the third law of thermodynamics (the Nernst heat theorem) is followed.

The structure of the paper is the following. In Sec. II we present general expressions for the free energy, internal energy and stress-energy tensor of the neutrino and electromagnetic fields in the Einstein cosmological model. Special attention is paid to the renormalization procedure and to the definitions of the Casimir contributions to the considered quantities
at nonzero temperature. Section III is devoted to the derivation of the total and Casimir free energy and internal energy for the neutrino field. The total and Casimir pressures are also considered. The asymptotic expressions of the obtained expressions at low and high temperatures are obtained and the validity of the Nernst heat theorem is verified. Similar results for the thermal Casimir effect in the electromagnetic case are presented in Sec. IV. Here, we concentrate on delicate points in the applicability of summation formulas and analogies with the case of electromagnetic field inside an ideal metal spherical shell. Our conclusions and discussions are presented in Sec. V.

For convenience in comparisons with the classical limit (the case of high temperature [33]), below we retain the Planck constant $\hbar$, the velocity of light $c$, and the Boltzmann constant $k_B$ in all equations.

II. GENERAL EXPRESSIONS FOR THE TOTAL AND CASIMIR FREE ENERGY AND INTERNAL ENERGY FOR NEUTRINO AND ELECTROMAGNETIC FIELDS

We consider the four component neutrino field and the electromagnetic field in thermal equilibrium at some temperature $T$ in the static Einstein cosmological model with a topology $S^3 \times R^1$. The metric of this model is given by

$$ds^2 = c^2 d\tau^2 - a^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\varphi^2)],$$

(1)

where $r$, $\theta$, $\varphi$ are dimensionless coordinates on a three-space of constant curvature $+1$ and $\tau$ is the proper synchronous time. This is a spatially homogeneous model with a finite spatial volume $V = 2\pi^2 a^3$, where $a = \text{const}$ is the scale factor. All the results obtained below are applicable also in the conformally static closed Friedmann cosmological model with the scale factor depending on time (the concept of thermal equilibrium in nonstationary situations is discussed in Ref. [34]). In this case, however, there are additional contributions to the stress-energy tensor due to conformal anomaly and creation of particles [35, 36]. It has long been known that the three Friedmann cosmological models (open, quasi-Euclidean and closed) form the theoretical basis of modern astrophysics and cosmology. Because of this the results obtained below are not of only academic interest but can be used in cosmological applications.
The free energy of the neutrino \((s = 1/2)\) and electromagnetic \((s = 1)\) fields is given by

\[
F^{(s)}(T) = E_0^{(s)} + \Delta F^{(s)}(T),
\]

where \(E_0^{(s)}\) is the zero-point energy at zero temperature and \(\Delta F^{(s)}(T)\) is the thermal correction. For the neutrino field we have \([18, 31]\)

\[
E_0^{(1/2)} = -2\hbar \sum_{n=1}^{\infty} n(n + 1)\omega_n^{(1/2)}, \quad \omega_n^{(1/2)} = \frac{(n + \frac{1}{2})c}{a},
\]

\[
\Delta F^{(1/2)}(T) = -4k_BT \sum_{n=1}^{\infty} n(n + 1)\ln \left[ 1 + e^{-(\hbar\omega_n^{(1/2)}/k_BT)} \right].
\]

For the electromagnetic field the respective expressions are the following \([18, 31]\):

\[
E_0^{(1)} = \hbar \sum_{n=1}^{\infty} (n^2 - 1)\omega_n^{(1)}, \quad \omega_n^{(1)} = \frac{nc}{a},
\]

\[
\Delta F^{(1)}(T) = 2k_BT \sum_{n=1}^{\infty} (n^2 - 1)\ln \left[ 1 - e^{-(\hbar\omega_n^{(1)}/k_BT)} \right].
\]

Note that the zero-point energy \(E_0^{(s)}\) given by Eqs. (3) and (5) is divergent, whereas the thermal correction \(\Delta F^{(s)}(T)\) in Eqs. (4) and (6) is finite. It is notable also that the vacuum energy \(E_0^{(1/2)}\) is negative, as it should be in the spinor case.

The renormalization of the vacuum energy \(E_0^{(s)}\) is conventionally performed by subtracting the terms which are obtained from Eqs. (3) and (5) by replacing the discrete sums with integrations over a continuous variable \([18]\). As a result, the renormalized energies are defined by

\[
E_{0,\text{ren}}^{(1/2)} = -2\hbar \left[ \sum_{n=1}^{\infty} n(n + 1)\omega_n^{(1/2)} - \int_0^{\infty} t(t + 1)\omega_t^{(1/2)} dt \right],
\]

\[
E_{0,\text{ren}}^{(1)} = \hbar \left[ \sum_{n=1}^{\infty} (n^2 - 1)\omega_n^{(1)} - \int_0^{\infty} (t^2 - 1)\omega_t^{(1)} dt \right].
\]

The differences between the sums and the integrals can be simply evaluated by using the following Abel-Plana formulas \([3, 5, 37, 38]\):

\[
\sum_{n=1}^{\infty} \Phi (n) - \int_0^{\infty} \Phi (t) dt = -\frac{1}{2} \Phi (0) + i \int_0^{\infty} \frac{\Phi (it) - \Phi (-it)}{e^{2\pi t} - 1} dt.
\]
\[
\sum_{n=1}^{\infty} \Phi \left( n + \frac{1}{2} \right) - \int_{0}^{\infty} \Phi (t) \, dt = -i \int_{0}^{\infty} \frac{\Phi (it) - \Phi (-it)}{e^{2\pi t} + 1} \, dt.
\]

Equations (9) and (10) are convenient in the cases of boson and fermion fields, respectively. The Abel-Plana formulas (9) and (10) are valid for functions that are analytic in the right-half plane including the imaginary axis.

The application of Eq. (10) to (7) with
\[
\Phi \left( n + \frac{1}{2} \right) = \Phi \left( \frac{1}{2} \right),
\]
leads to \[18, 31\]
\[
E_{0,\text{ren}}^{(1/2)} = \frac{17hc}{480a}
\]

In a similar way, using Eq. (9) in (8) with
\[
\Phi \left( n \right) = \Phi_{1}^{(1)} (n) = n(n^2 - 1) = n^3 - n,
\]
\[
\Phi_{1}^{(1)} (n) = n(n + 1) \left( n + \frac{1}{2} \right)
\]
leads to \[18, 31\]
\[
E_{0,\text{ren}}^{(1)} = \frac{11hc}{120a}
\]

As a result, the total free energy for the neutrino and electromagnetic fields in Einstein cosmological model is given by
\[
F_{\text{tot}}^{(s)} (T) = E_{0,\text{ren}}^{(s)} + \Delta F^{(s)} (T),
\]
where all terms on the right-hand side are defined in Eqs. (4), (6), (12) and (14).

Another important characteristic of the equilibrium state of the field at nonzero temperature is the internal energy which is connected with the expectation value of the stress-energy tensor and the free energy in the following way \[39, 40\]:
\[
U_{\text{tot}}^{(s)} (T) = V \langle T_{0}^{(s)0} \rangle_{\text{tot}} = -T^2 \frac{\partial}{\partial T} \left[ \frac{F_{\text{tot}}^{(s)} (T)}{T} \right].
\]

Using Eqs. (4), (6), (15) and (16), we obtain for the internal energy of the neutrino and electromagnetic fields
\[
U_{\text{tot}}^{(s)} (T) = E_{0,\text{ren}}^{(s)} + \Delta U^{(s)} (T),
\]
where
\[
\Delta U^{(1/2)}(T) = 4\hbar \sum_{n=1}^{\infty} \frac{n(n+1)\omega_n^{(1/2)}}{e^{(\hbar\omega_n^{(1/2)}/k_B T)} + 1},
\]
\[
\Delta U^{(1)}(T) = 2\hbar \sum_{n=1}^{\infty} \frac{(n^2 - 1)\omega_n^{(1)}}{e^{(\hbar\omega_n^{(1)}/k_B T)} - 1}.
\]

Now we proceed with the definitions of the Casimir free energy and internal energy at nonzero temperature. These definitions go back to the Lifshitz theory [41], where the Casimir free energy of the fluctuating electromagnetic field between two semispaces separated by a gap of width \(a\) is obtained by subtracting the free energy of an unbounded Minkowski space. In other words, not only an (infinite) renormalization of the zero-point energy is done, but the thermal correction \(\Delta F^{(s)}(T)\) undergoes finite renormalization by subtracting the contribution of the black-body radiation [5]. This contribution is proportional to the volume of the gap between semispaces. The same definition for the thermal Casimir effect was used in thermal quantum field theory for the configuration of two parallel ideal metal plates [5, 41]. What is more, for configurations with a finite volume, such as ideal metal rectangular boxes, it was shown [42] that to obtain the Casimir free energy one should subtract from \(\Delta F^{(s)}(T)\) two more terms of quantum nature proportional to the surface area of the box and to the perimeter of its edges. This is in line with Ref. [43], which demonstrated that in the asymptotic limit of high temperature the total free energy in any restricted volume contains the following three types of terms depending on the Planck constant:
\[
\alpha^{(s)}_0 \frac{(k_B T)^4}{(hc)^4}, \quad \alpha^{(s)}_1 \frac{(k_B T)^3}{(hc)^2}, \quad \alpha^{(s)}_2 \frac{(k_B T)^2}{hc},
\]
where the coefficients \(\alpha^{(s)}_0, \alpha^{(s)}_1, \alpha^{(s)}_2\) depend on the spin of the field and are expressed in terms of the heat kernel coefficients [5]. Some of these coefficients may be equal to zero. Specifically, for a conformal massless scalar field in the Einstein cosmological model it is true that [30]
\[
\alpha^{(0)}_0 = -\frac{\pi^2}{90} V, \quad \alpha^{(0)}_1 = 0, \quad \alpha^{(0)}_2 = 0.
\]

It can be seen that in our case the terms [20] in the free energy are contained in the integral
\[
\int_0^{\infty} \Phi^{(s)}_2(t) dt,
\]
\[7\]
where the function $\Phi^{(s)}_2(t)$ for the neutrino and electromagnetic fields in accordance with Eqs. (4) and (6) is given by

$$
\Phi^{(1/2)}_2(t) = -4k_B T \left( t^2 - \frac{1}{4} \right) \ln \left[ 1 + e^{-\left(\hbar c t / ak_B T\right)} \right],
$$

$$
\Phi^{(1)}_2(t) = 2k_B T (t^2 - 1) \ln \left[ 1 - e^{-\left(\hbar c t / ak_B T\right)} \right].
$$

Expanding the logarithms in power series and integrating, one obtains

$$
\int_0^\infty \Phi^{(1/2)}_2(t) dt = -\frac{7\pi^4 a^3 (k_B T)^4}{90 (\hbar c)^3} + \frac{\pi^2 a (k_B T)^2}{12 \hbar c},
$$

$$
\int_0^\infty \Phi^{(1)}_2(t) dt = -\frac{2\pi^4 a^3 (k_B T)^4}{45 (\hbar c)^3} + \frac{\pi^2 a (k_B T)^2}{3 \hbar c}.
$$

As a result, the Casimir free energy is defined as

$$
F^{(s)}_C(T) = E^{(s)}_0, \text{ren} + \Delta F^{(s)}_C(T),
$$

where the Casimir thermal correction is given by

$$
\Delta F^{(s)}_C(T) = \Delta F^{(s)}(T) - \int_0^\infty \Phi^{(s)}_2(t) dt.
$$

The definition (28) generalizes the approach previously followed in the Lifshitz theory and in thermal quantum field theory for ideal metal plates and rectangular boxes. As can be seen in Eq. (25) and (26), for neutrino and electromagnetic fields in the Einstein model we have

$$
\alpha^{(1/2)}_0 = -\frac{7\pi^2}{180} V, \quad \alpha^{(1/2)}_1 = 0, \quad \alpha^{(1/2)}_2 = \frac{\pi^2 a}{12},
$$

$$
\alpha^{(1)}_0 = -\frac{\pi^2}{45} V, \quad \alpha^{(1)}_1 = 0, \quad \alpha^{(1)}_2 = \frac{\pi^2 a}{3}.
$$

In this manner, the definition of the thermal correction in the Casimir free energy presumes the subtraction of not only the contribution of the black-body radiation in free Minkowski space, as it is true for the scalar field [see Eq. (21)], but one more term of quantum nature which is present in the total free energy. In Secs. III and IV it is shown that the asymptotic expressions of the total free energy $F^{(s)}_{\text{tot}}$ at high $T$ do not contain any more power-type terms of quantum nature.
The Casimir contribution to the internal energy (and respective Casimir contribution to the thermal stress-energy tensor) can be defined in a similar way. It is given by

$$U_C^{(s)}(T) = E_{0,\text{ren}}^{(s)} + \Delta U_C^{(s)}(T), \quad (30)$$

where

$$\Delta U_C^{(s)}(T) = \Delta U^{(s)}(T) - \int_0^\infty \Phi_3^{(s)}(t) dt. \quad (31)$$

In accordance with Eqs. (18) and (19), the function $\Phi_3^{(s)}(t)$ for the neutrino and electromagnetic fields is given by

$$\Phi_3^{(1/2)}(t) = \frac{\hbar c}{a} \frac{t(4t^2 - 1)}{e^{(\hbar ct/ak_BT)} + 1},$$

$$\Phi_3^{(1)}(t) = \frac{2\hbar c}{a} \frac{t(t^2 - 1)}{e^{(\hbar ct/ak_BT)} - 1}. \quad (32, 33)$$

The integrals subtracted in Eq. (31) are simply calculated as

$$\int_0^\infty \Phi_3^{(1/2)}(t) dt = \frac{7\pi^4 a^3 (k_BT)^4}{30} \frac{\pi^2 a (k_BT)^2}{12 \hbar c},$$

$$\int_0^\infty \Phi_3^{(1)}(t) dt = \frac{2\pi^4 a^3 (k_BT)^4}{15} \frac{\pi^2 a (k_BT)^2}{3 \hbar c}. \quad (34, 35)$$

It is easily seen that the integrals (25) and (26) subtracted from the total free energy and the respective integrals (33) and (34) subtracted from the total internal energy satisfy the same Eq. (16) as the total quantities.

In the end of this section, we consider other components of the stress-energy tensor different from 00-component defined in Eq. (16). In the spatially homogeneous isotropic metrics of the space-time under consideration the stress-energy tensor is diagonal and has equal spatial components

$$P_{\text{tot}}^{(s)}(T) = - \left< T_i^{(s)i} \right>_{\text{tot}}, \quad (36)$$

where $P_{\text{tot}}^{(s)}(T)$ is the pressure. Both the total and Casimir pressures are expressed using the respective free energies as

$$P_{\text{tot}}^{(s)}(T) = - \frac{\partial F_{\text{tot}}^{(s)}(T)}{\partial V}, \quad P_C^{(s)}(T) = - \frac{\partial F_C^{(s)}(T)}{\partial V}. \quad (37)$$
Considering Eqs. (4), (6), (12), (14), and (15), we obtain for the total pressure defined in Eqs. (37), the following results

\[ P_{\text{tot}}^{(1/2)}(T) = \frac{17hc}{2880\pi^2a^4} + \frac{2h}{3\pi^2a^3} \sum_{n=1}^\infty \frac{n(n+1)\omega_n^{(1/2)}}{e^{(\hbar\omega_n^{(1/2)}/k_BT)} + 1}, \]  

(38)

\[ P_{\text{tot}}^{(1)}(T) = \frac{11hc}{720\pi^2a^4} + \frac{h}{3\pi^2a^3} \sum_{n=1}^\infty \frac{(n^2 - 1)\omega_n^{(1)}}{e^{(\hbar\omega_n^{(1)}/k_BT)} - 1}. \]  

(39)

Comparing Eq. (38) with Eqs. (16)–(18) and Eq. (39) with Eqs. (16), (17) and (19), we get the equation of state

\[ P_{\text{tot}}^{(s)}(T) = \frac{1}{3}\varepsilon_{\text{tot}}^{(s)}(T), \]  

(40)

where the total energy density is just given by

\[ \varepsilon_{\text{tot}}^{(s)}(T) = \left\langle T^{(s)0}_0 \right\rangle_{\text{tot}} = \frac{U^{(s)}(T)}{V}, \]  

(41)

in accordance with Eq. (16). Using Eqs. (25), (26), (34) and (35), it can be easily verified that the same equation of state is satisfied for the Casimir quantities defined in Eqs. (27), (28), (30), (31) and (37):

\[ P_{C}^{(s)}(T) = \frac{1}{3}\varepsilon_{C}^{(s)}(T). \]  

(42)

Here, the Casimir energy density is the 00-component of the diagonal Casimir stress-energy tensor

\[ \varepsilon_{C}^{(s)}(T) = \left\langle T_{00}^{(s)} \right\rangle_{C} = \frac{U_{C}^{(s)}(T)}{V}. \]  

(43)

The other components of this tensor are given by \(-P_{C}^{(s)}(T)\).

III. CALCULATION OF THE TOTAL AND CASIMIR FREE ENERGY AND INTERNAL ENERGY FOR NEUTRINO FIELD

Here, we calculate the free energy and internal energy for the neutrino field in Einstein and Friedmann cosmological models. We begin from the representation of the Casimir free energy \( F_{C}^{(1/2)}(T) \) as given by Eq. (27) with \( s = 1/2 \), where the energy at zero temperature is expressed by Eq. (12) and the Casimir thermal correction by Eqs. (4), (23) and (28).
It is convenient to calculate the difference between the sum in Eq. (4) and the integral in Eq. (28) by using the Abel-Plana formula (10) with \( \Phi(t) = \Phi_2^{(1/2)}(t) \) defined in Eq. (23). This function is analytic in the right-half plane including the imaginary frequency axis. Expanding the logarithm in power series, one obtains

\[
\Phi_2^{(1/2)}(t) = -4k_B T \left( t^2 - \frac{1}{4} \right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{\hbar c n t}{ak_B T}}
\]  

(44)

and

\[
\Phi_2^{(1/2)}(it) - \Phi_2^{(1/2)}(-it) = -8i k_B T \left( t^2 + \frac{1}{4} \right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{\hbar c n t}{ak_B T}.
\]  

(45)

Then, calculating the integral on the right-hand side of Eq. (10), we arrive at the representation of the Casimir free energy in terms of the double sum:

\[
F_C^{(1/2)}(T) = \frac{17\hbar c}{480a} + \frac{16\hbar c}{a} \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{m=1}^{\infty} (-1)^{m+1} \left\{ \frac{(\hbar c n/ak_B T)^2 - 3(2\pi m)^2}{[(\hbar c n/ak_B T)^2 + (2\pi m)^2]^3} \right\}.
\]  

(46)

In Eq. (46) one can carry out the summation either in \( m \) or in \( n \). The resulting representations for the Casimir free energy are, strictly speaking, equivalent, but convenient for obtaining the asymptotic limits in the case of low and high temperature, respectively. The summation in \( m \) can be performed by using the formula

\[
\sum_{m=1}^{\infty} (-1)^{m+1} \left[ \frac{x^2 - 3y^2 m^2}{(x^2 + y^2 m^2)^3} - \frac{1}{8(x^2 + y^2 m^2)} \right] = \frac{1}{8x^4} \left\{ 4 - \pi^3 x^3 [3 + \cosh(2\pi x/y)] \text{csch}^3(\pi x/y) \right\}
\]  

\[
- \frac{y - \pi x \text{csch}(\pi x/y)}{16x^2 y},
\]  

(47)

where \( x = \hbar c n/ak_B T \) and \( y = 2\pi \). By performing the summation in \( m \) in Eq. (46) with the help of Eq. (47), one arrives at

\[
F_C^{(1/2)}(T) = \frac{17\hbar c}{480a} + \frac{16\hbar c}{90} \left( \frac{k_B T}{\hbar c} \right)^4 + \frac{\pi^2 a (k_B T)^2}{12 \hbar c}
\]  

\[- k_B T \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n \sinh^3(\hbar c n/2ak_B T)}.
\]  

(48)

From this equation, in the low temperature limit, where \( \hbar c / ak_B T \to \infty \), we have

\[
F_C^{(1/2)}(T) = \frac{17\hbar c}{480a} + \frac{7\pi^4 a^3 (k_B T)^4}{90 (\hbar c)^3} - \frac{\pi^2 a (k_B T)^2}{12 \hbar c}
\]  

\[- 8k_B T e^{-\frac{3\hbar c}{2ak_B T}}.
\]  

(49)
This result is similar to the low-temperature behavior of the Casimir free energy in the case of two parallel ideal metal plates \[3, 41\].

From Eq. (48) it is easy to obtain the exact expression for the total free energy. Using Eqs. (15), (25) and (28) one has all power-type contributions in \(T\) canceled, and then the total free energy is given by

\[
F_{\text{tot}}^{(1/2)}(T) = \frac{17hc}{480a} - k_B T \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1}{\sinh^3(hcn/2ak_BT)}.
\]  

(50)

Thus, the thermal correction in the total free energy at low temperature is exponentially small:

\[
F_{\text{tot}}^{(1/2)}(T) = \frac{17hc}{480a} \left( 8 - k_B Te^{-3hc/(2ak_BT)} \right).
\]  

(51)

To obtain the asymptotic behavior of the Casimir and total free energy at high temperature, we perform the summation in \(n\) in Eq. (46) first. This leads to the following representation for the Casimir free energy:

\[
F_C^{(1/2)}(T) = \frac{17hc}{480a} + \frac{a^2}{16\pi^4} \frac{(k_BT)^3}{(hc)^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^4 \sinh^3(2\pi^2mak_BT/hc)}
\times \left\{ 4\pi^2m \left[ 4\pi^2m^2 + \frac{(hc)^2}{(ak_BT)^2} \left( 2 + \pi^2m^2 \right) \right] \cosh \left( \frac{4\pi^2mak_BT}{hc} \right) 
+ \frac{(hc)^3}{(ak_BT)^3} \left( 6 + \pi^2m^2 \right) \sinh \left( \frac{2\pi^2mak_BT}{hc} \right) 
+ \frac{(hc)^2}{(ak_BT)^2} \left[ 12m^2\pi^4 - \frac{(hc)^2}{(ak_BT)^2} \left( 2 + \pi^2m^2 \right) + 4\pi^2 \frac{hc}{ak_BT} m \sinh \left( \frac{4\pi^2mak_BT}{hc} \right) \right] 
- \frac{(hc)^3}{(ak_BT)^2} \left( 6 + \pi^2m^2 \right) \sinh \left( \frac{6\pi^2mak_BT}{hc} \right) \right\}.
\]  

(52)

The same expression can be equivalently rewritten in the form

\[
F_C^{(1/2)}(T) = \frac{17hc}{480a} + \frac{a^2}{4\pi^4} \frac{(k_BT)^3}{(hc)^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^4}
\times \left\{ -\frac{(hc)^3}{(ak_BT)^3} \left( 6 + \pi^2m^2 \right) + 2m\pi^2 \text{csch} \left( \frac{2\pi^2mak_BT}{hc} \right) 
\times \left[ \frac{\pi^2 hc m}{ak_BT} \left( \frac{hc m}{ak_BT} + 4\coth \left( \frac{2\pi^2mak_BT}{hc} \right) \right) + 2 \left( \frac{(hc)^2}{(ak_BT)^2} + 2m^2\pi^4 \right) 
4m^2\pi^4 \text{csch}^2 \left( \frac{2\pi^2mak_BT}{hc} \right) \right] \right\}.
\]  

(53)

From Eqs. (52) or (53) in the limit of high temperature one obtains

\[
F_C(T) = 4\pi^2 a^2 \frac{(k_BT)^3}{(hc)^2} e^{-2\pi^2 mak_BT/hc}.
\]  

(54)
Note that in the limit of high $T$ the zero-temperature contribution to the Casimir free energy was canceled by the corresponding term in the thermal correction. We emphasize that the Casimir free energy at high $T$ is exponentially small. It does not contain the classical term proportional to $k_B T$ (see Section IV), as is expected for a spinor field.

The total free energy is obtained by adding Eq. (25) to Eqs. (52) or (53). The expressions obtained for $F_{tot}^{(1/2)}(T)$ are equivalent to the more compact Eq. (50). The asymptotic behavior of the total free energy at high $T$ is obtained by adding Eqs. (25) and (54).

A quantity closely related to the free energy is the entropy. The total and the Casimir entropies for both the neutrino and electromagnetic fields are defined as

$$S^{(s)}_{tot}(T) = -\frac{\partial F^{(s)}_{tot}(T)}{\partial T}, \quad S^{(s)}_C(T) = -\frac{\partial F^{(s)}_C(T)}{\partial T}. \quad (55)$$

The exact expression for the Casimir entropy of the neutrino field can be obtained from Eq. (48). It is given by

$$S^{(1/2)}_C(T) = \pi^2 k_B \frac{a k_B T}{3 \hbar c} \left[ \frac{1}{2} - \frac{14\pi^2 (ak_B T)^2}{15 (\hbar c)^2} \right]$$

$$+ k_B \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[ \text{csch}^2(\hbar c n/(2ak_B T)) + \frac{3\hbar c n}{2ak_B T} \cosh \left( \frac{\hbar c n}{2ak_B T} \right) \right] \times \text{csch}^4(\hbar c n/(2ak_B T)). \quad (56)$$

When the temperature vanishes, the asymptotic expression for the Casimir entropy is given by

$$S^{(1/2)}_C(T) = \pi^2 k_B \frac{a k_B T}{3 \hbar c} \left[ \frac{1}{2} - \frac{14\pi^2 (ak_B T)^2}{15 (\hbar c)^2} \right]$$

$$+ \frac{12\hbar c}{a T} e^{-(3\hbar c/2ak_B T)}. \quad (57)$$

As is seen from Eq. (57), the Casimir entropy goes to zero when $T$ goes to zero, i.e., the third law of thermodynamics (the Nernst heat theorem) is satisfied [39, 40]. The total entropy is obtained from Eqs. (50) and (55). It is given by Eq. (56) with the first term on the right-hand side omitted. The asymptotic behavior of the total entropy at low $T$ is given by the exponentially small term in Eq. (57). Thus, the Nernst heat theorem also holds for the total entropy of neutrino field.

Now we turn to the consideration of the internal energy. The total internal energy for the neutrino field is given by Eqs. (17) and (18). The Casimir internal energy can be calculated
by Eqs. (30) and (31) with the function \( \Phi(t) \) defined in Eq. (32). The calculation follows the same steps as shown above for the free energy and uses the Abel-Plana formula (10). The same results can also be obtained from the respective expressions for the free energy by using Eq. (16) and similar equations for the Casimir quantities. Thus, from Eqs. (50) and (16), we find the following result for the total internal energy:

\[
U^{(1/2)}_{\text{tot}}(T) = \frac{17hc}{480a} + \frac{3hc}{2a} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cosh(hcn/2ak_BT)}{\sinh^4(hcn/2ak_BT)}.
\]  
(58)

In the limit of low temperature, from Eq. (58), one obtains

\[
U^{(1/2)}_{\text{tot}}(T) = \frac{17hc}{480a} + \frac{12hc}{a} e^{-(3hc/2ak_BT)}.
\]  
(59)

In the limit of high temperature the asymptotic behavior of the total internal energy is

\[
U^{(1/2)}_{\text{tot}}(T) = \frac{7\pi^4 a^3 (k_BT)^4}{30 \frac{(hc)^3}{(hc)^3}} - \frac{\pi^2 a (k_BT)^2}{12 \frac{hc}{(hc)^3}} e^{-(2\pi^2 ak_BT/hc)} + 8\pi^4 a^3 \frac{(k_BT)^4}{(hc)^3} e^{-(2\pi^2 ak_BT/hc)}.
\]  
(60)

Dividing both sides of Eqs. (59) and (60) by the spatial volume \( V \), we get

\[
\varepsilon^{(1/2)}_{\text{tot}}(T) = \frac{17hc}{960\pi^2 a^4} + \frac{6hc}{\pi^2 a^4} e^{-(3hc/2ak_BT)},
\]  
(61)

\[
\varepsilon^{(1/2)}_{\text{tot}}(T) = \frac{7\pi^2 (k_BT)^4}{60 \frac{(hc)^3}{(hc)^3}} - \frac{1}{24a^2} \frac{(k_BT)^2}{hc} e^{-(2\pi^2 ak_BT/hc)} + 4\pi^2 \frac{(k_BT)^4}{(hc)^3} e^{-(2\pi^2 ak_BT/hc)}
\]  
(62)

at low and high temperature, respectively. These coincide with the asymptotic behavior of the energy density for the neutrino field found in Ref. [31] (note that the exponentially small terms were omitted in [31]).

The exact expression for the Casimir internal energy is obtained by subtracting Eq. (34) from Eq. (58). As a result, at low and high temperature we have, respectively,

\[
U^{(1/2)}_{C}(T) = \frac{17hc}{480a} - \frac{7\pi^4 a^3 (k_BT)^4}{30 \frac{(hc)^3}{(hc)^3}} + \frac{\pi^2 a (k_BT)^2}{12 \frac{hc}{(hc)^3}} + \frac{12hc}{a} e^{-(3hc/2ak_BT)},
\]  
(63)

\[
U^{(1/2)}_{C}(T) = 8\pi^4 a^3 \frac{(k_BT)^4}{(hc)^3} e^{-(2\pi^2 ak_BT/hc)}.
\]  
(64)
Thus, for the neutrino field the Casimir internal energy at high $T$ is exponentially small. The limiting values for the Casimir energy density are obtained from Eqs. (63) and (64) after the division by $V$. As to the total and Casimir pressures, they can be obtained from the equations of state (40) and (42).

IV. CALCULATION OF THE FREE ENERGY AND INTERNAL ENERGY FOR ELECTROMAGNETIC FIELD

The case of the electromagnetic field is more complicated because the function $Φ_2^{(1)}(t)$ defined in Eq. (24) goes to infinity when $t$ goes to zero. This prevents direct application of the Abel-Plana formula (9) for the calculation of the Casimir free energy of the electromagnetic field defined in Eqs. (27) and (28). A similar problem arises in the calculation of the total and Casimir internal energies given by Eqs. (17), (19) and (30), (31), respectively. The point is that the function $Φ_3^{(1)}(t)$ in Eq. (33), determining the thermal correction (19) in the internal energy, has poles along the imaginary frequency axis. This also prevents the application of the Abel-Plana formula in its simplest form (9).

We start from the total free energy (2), (6), and expand the logarithm into the power series to obtain

$$F^{(1)}_{\text{tot}}(T) = \frac{11hc}{120a} - 2k_B T \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} (n^2 - 1) e^{-\frac{(hc m/a k_B T)}{}}.$$  (65)

Here, the sum with respect to $n$ can be calculated leading to

$$F^{(1)}_{\text{tot}}(T) = \frac{11hc}{120a} - 2k_B T \sum_{m=1}^{\infty} \frac{3e^{hc m/a k_B T} - 1}{m(e^{hc m/a k_B T} - 1)^{3/2}}.$$  (66)

In accordance with Eq. (28) the Casimir free energy for the electromagnetic field is obtained subtraction of Eq. (26) from Eq. (66):

$$F^{(1)}_{\text{C}}(T) = \frac{11hc}{120a} + \frac{2\pi^4 a^3 (k_B T)^4}{45 (hc)^3} - \frac{\pi^2 a (k_B T)^2}{3} - \frac{2k_B T \sum_{m=1}^{\infty} 3e^{hc m/a k_B T} - 1}{m(e^{hc m/a k_B T} - 1)^{3/2}}.$$  (67)

Equations (66) and (67) are convenient for obtaining the asymptotic expressions at low temperature. For the total free energy one obtains from Eq. (66)

$$F^{(1)}_{\text{tot}}(T) = \frac{11hc}{120a} - 6k_B T e^{-\frac{2hc}{a k_B T}}.$$  (68)
It is seen that total correction to the energy at zero temperature is exponentially small. The low temperature behavior of the Casimir free energy is given by the difference between Eqs. (68) and (26). In this case, there are power-type corrections to the energy at zero temperature.

Before considering the asymptotic behavior of the free energy at high temperature, we obtain the exact expression for the internal energy and its asymptotic behaviors at low and high temperatures. Using the expansion in power series in Eq. (19), we can rewrite Eq. (17) in the form

\[ U^{(1)}_{\text{tot}}(T) = \frac{11\hbar c}{120a} + \frac{2\hbar c}{a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n(n^2 - 1) e^{-\left(\frac{\hbar cmn}{ak_B T}\right)}. \]  

(69)

By performing the summation in \( n \) we obtain the following exact expression:

\[ U^{(1)}_{\text{tot}}(T) = \frac{11\hbar c}{120a} + \frac{12\hbar c}{a} \sum_{m=1}^{\infty} e^{\frac{2\hbar cm}{ak_B T}} \left( e^{\frac{\hbar cm}{ak_B T} - 1} \right)^4. \]  

(70)

The Casimir internal energy is given by Eqs. (30) and (31). Taking into account Eq. (35) we have

\[ U^{(1)}_{\text{C}}(T) = \frac{11\hbar c}{120a} - \frac{2\pi^4 a^3 (k_B T)^4}{15 (\hbar c)^3} + \frac{\pi^2 a (k_B T)^2}{3 \hbar c} + \frac{12\hbar c}{a} \sum_{m=1}^{\infty} e^{\frac{2\hbar cm}{ak_B T}} \left( e^{\frac{\hbar cm}{ak_B T} - 1} \right)^4. \]  

(71)

Equations (70) and (71) are convenient for obtaining the asymptotic expressions at low temperature. Thus, from Eq. (70) it follows

\[ U^{(1)}_{\text{tot}}(T) = \frac{11\hbar c}{120a} + \frac{12\hbar c}{a} e^{-\left(\frac{2\hbar cm}{ak_B T}\right)}. \]  

(72)

After subtraction of Eq. (35) from Eq. (72), the asymptotic expression for the Casimir internal energy at low temperature is obtained. All these expressions for the internal energy are connected with respective expressions for the free energy, obtained above, by Eq. (16) and a similar equation for the Casimir quantities.

Now we turn to the Casimir internal energy in the form of Eqs. (30) and (31), and calculate it in a different way, which allows us to obtain the high temperature limit. The poles of the function \( \Phi^{(1)}_3(t) \) defined in Eq. (33) are located at the points \( t = it_l = 2\pi ilak_B T/\hbar c \), where \( l = \pm 1, \pm 2, \ldots \). In addition, at \( t = 0 \) this function takes a nonzero value \( \Phi^{(1)}_3(0) = -2k_B T \). The presence of the poles along the imaginary frequency axis makes the standard
Abel-Plana formula inapplicable. There is, however, a generalization of this formula for the case when there are poles on the imaginary frequency axis. It reads

\[ \sum_{n=1}^{\infty} \Phi(n) - \int_0^{\infty} \Phi(t) dt = -\frac{1}{2} \Phi(0) - \pi \sum_{l=it} \text{Res} \left[ \frac{\Phi(t)e^{i\pi t}}{\sin(\pi t)} \right] + i \int_0^{\infty} \frac{\Phi(it) - \Phi(-it)}{e^{2\pi t} - 1} dt. \]  

(73)

Here, it is assumed that summation is done only over the positive \( l \) and the function \( \Phi(t) \) satisfies the condition

\[ \Phi(t) = \Phi(-t) + o \left( \frac{1}{(t-it)} \right) \text{ when } t \to it. \]  

(74)

We apply Eq. (73) to the function \( \Phi(t) = \Phi_3^{(1)}(t) \) defined in Eq. (33). In doing so, we take into account that

\[ -\pi \sum_{l=1}^{\infty} \text{Res} \left[ \frac{\Phi_3^{(1)}(t)e^{i\pi t}}{\sin(\pi t)} \right] = \frac{32\pi^4 a^3 (k_B T)^4}{(\hbar c)^3} \sum_{l=1}^{\infty} \frac{l^3}{e^{4\pi^2 a (k_B T)^2 (\hbar c) / (\hbar c)^3} - 1} \]

\[ + \frac{8\pi^2 a (k_B T)^2}{\hbar c} \sum_{l=1}^{\infty} \frac{l}{e^{4\pi^2 a (k_B T)^2 (\hbar c) / (\hbar c)^3} - 1} \]  

(75)

and

\[ \Phi_3^{(1)}(it) - \Phi_3^{(1)}(-it) = \frac{2\hbar c}{a} (it^2 + 1). \]  

(76)

Calculating the integral on the right-hand side of Eq. (73),

\[ i \int_0^{\infty} \frac{\Phi_3^{(1)}(it) - \Phi_3^{(1)}(-it)}{e^{2\pi t} - 1} dt = -\frac{11}{120} \frac{\hbar c}{a}, \]  

(77)

we find that it cancels the contributions of the Casimir energy at zero temperature \( E_{0,\text{ren}}^{(1)} \) defined in Eq. (14). Then from Eqs. (30) and (31) one obtains

\[ U_C^{(1)}(T) = k_B T + \frac{32\pi^4 a^3 (k_B T)^4}{(\hbar c)^3} \sum_{l=1}^{\infty} \frac{l^3}{e^{4\pi^2 a (k_B T)^2 (\hbar c) / (\hbar c)^3} - 1} \]

\[ + \frac{8\pi^2 a (k_B T)^2}{\hbar c} \sum_{l=1}^{\infty} \frac{l}{e^{4\pi^2 a (k_B T)^2 (\hbar c) / (\hbar c)^3} - 1}. \]  

(78)

This is an alternative exact expression for the Casimir internal energy. Numerical computations show that it gives the same values of \( U_C^{(1)}(T) \) as Eq. (71) over the entire range of temperatures. The total internal energy \( U^{(1)}_{\text{tot}}(T) \) is obtained by adding Eq. (35) to Eq. (78).
Equation (78) is convenient for obtaining the asymptotic limit at high temperature. Thus, from this equation the asymptotic expression for the Casimir internal energy at high $T$ is given by

$$U_C^{(1)}(T) = k_B T + \frac{32\pi^4 a^3 (k_B T)^4}{(\hbar c)^3} e^{-(4\pi^2 a k_B T / \hbar c)}.$$  (79)

The asymptotic expression at high $T$ for the total internal energy is obtained from Eq. (79) by adding Eq. (35):

$$U_{\text{tot}}^{(1)}(T) = \frac{2\pi^4 a^3 (k_B T)^4}{15 (\hbar c)^3} - \frac{\pi^2 a (k_B T)^2}{3 \hbar c} + k_B T$$

$$+ \frac{32\pi^4 a^3 (k_B T)^4}{(\hbar c)^3} e^{-(4\pi^2 a k_B T / \hbar c)}.$$  (80)

From Eq. (79) it is seen that the electromagnetic Casimir internal energy at high temperature has a classical limit [33], which does not depend on $\hbar$ and $c$. The total internal energy (80) also contains a classical term equal to $k_B T$. The respective asymptotic expression for the total internal energy density is

$$\varepsilon_{\text{tot}}^{(1)}(T) = \frac{U_{\text{tot}}^{(1)}(T)}{V} = \frac{\pi^2 (k_B T)^4}{15 (\hbar c)^3} - \frac{1}{6a^2} \frac{(k_B T)^2}{\hbar c} + \frac{k_B T}{2\pi^2 a^3}$$

$$+ 16\pi^2 \frac{(k_B T)^4}{(\hbar c)^3} e^{-(4\pi^2 a k_B T / \hbar c)}.$$  (81)

Note that the high temperature behavior of $\varepsilon_{\text{tot}}^{(1)}$ obtained in Ref. [31] does not contain the classical term. This is explained by the omission of the term $\Phi_{3}^{(1)}(0) / 2 = -k_B T$ in the Poisson summation formula (Eq. (4) of Ref. [31]), which is equivalent to the Abel-Plana formula used here.

The obtained expression (79) for the Casimir internal energy at high $T$ can be used for the determination of the high temperature behavior of the Casimir free energy, which is still unknown. Using the thermodynamic connection (16) between the free energy and internal energy, we have

$$F_C^{(1)}(T) = -T \left[ \int \frac{U_C^{(1)}(T)}{T^2} dT + k_B R \right],$$  (82)

where $R$ is an arbitrary dimensionless constant independent of $T$. Substituting Eq. (79) in Eq. (82) and keeping only the high order terms, one arrives at

$$F_C^{(1)}(T) = -k_B T \ln \frac{a k_B T}{\hbar c} - R k_B T$$

$$+ 8\pi^2 a^2 \frac{(k_B T)^3}{(\hbar c)^2} e^{-(4\pi^2 a k_B T / \hbar c)}.$$  (83)
The value of the constant $R$ can be determined from the exact expression for $F_C^{(1)}(T)$ [see Eq. (67)]. Computations show that the exact expression (67) leads to the same values of the Casimir free energy as the asymptotic expression (83) up to six significant figures for the values of parameter $ak_B T/\hbar c \geq 1$ when $R = 1.77698$. Thus, the Casimir free energy of the electromagnetic field at low $T$ contains not only the classical (entropic) term, but the logarithmic contribution as well. The same holds for the Casimir free energy inside an ideal metal spherical shell [5, 45, 46] (at zero temperature the Casimir energy for fields of different spins in spherically symmetric cavities was considered in Refs. [47, 48]). The low temperature behavior of the total free energy is obtained from Eq. (83) by adding Eq. (26).

In the end of this section we consider the total and Casimir entropy for the electromagnetic field in Einstein and closed Friedmann cosmological model. At low temperature from Eqs. (55) and (68) one obtains the following main contribution to the total entropy:

$$S_{\text{tot}}^{(1)}(T) = \frac{12\hbar c}{aT} e^{-(2\hbar c/ak_B T)}. \tag{84}$$

For the Casimir entropy at low $T$ using Eq. (26) we arrive at

$$S_C^{(1)}(T) = 2\pi^2 k_B^2 \frac{ak_B T}{3\hbar c} \left[ 1 - \frac{4\pi^2}{15} \left( \frac{ak_B T}{\hbar c} \right)^2 \right] + \frac{12\hbar c}{aT} e^{-(2\hbar c/ak_B T)}. \tag{85}$$

As is seen from Eqs. (84) and (85), both the total and Casimir entropies go to zero when the temperature vanishes, in accordance with the Nernst heat theorem.

Using the above results, the expressions for the total and Casimir pressures of electromagnetic field and their behaviors at low and high temperature can be obtained from Eqs. (40) and (41).

V. CONCLUSIONS AND DISCUSSION

In the foregoing we have investigated the thermal Casimir effect for the neutrino and electromagnetic fields in the Einstein cosmological model. The results obtained are also valid in the closed Friedmann cosmological model where they should be considered as complementary to the terms describing the vacuum polarization and particle creation caused by the nonstationary regime of the metric. For both fields under consideration we found general expressions for the total internal energy, energy density and pressure (considered
earlier) and for the total free energy. In all cases, we separated the Casimir contributions from the obtained quantities by means of an additional subtraction procedure.

Both the total and Casimir internal energy and free energy were represented in the form of single sums. The asymptotic expressions for these sums were found in the limiting cases of low and high temperature. For the neutrino field, our results for the total internal energy (energy density) are in agreement with those obtained in Ref. [31]. We have also found the exponentially small corrections to the results of Ref. [31]. Our results for the Casimir free energy of neutrino field are similar to those obtained for configurations with material boundaries. Specifically, we have shown that the Casimir free energy of neutrino field does not possess a classical limit at high temperature and the Casimir entropy satisfies the Nernst heat theorem.

For the electromagnetic field, our calculation result for the total internal energy differs from the result of Ref. [31] by a classical term which arises from the contribution of zero arguments in the Poisson and Abel-Plana summation formulas. We have shown that for the electromagnetic field at high temperature the total internal energy includes not only the Planck-type terms, as it was believed previously, but also the linear in temperature classical term. For the Casimir free energy of the electromagnetic field in Einstein cosmological model we have proved that there are both classical and logarithmic in temperature terms in the limit of high temperature, as it holds for the thermal Casimir effect inside an ideal metal sphere. Both the total and Casimir entropies were demonstrated to be in agreement with the Nernst heat theorem.

In future, it would be interesting to determine the influence of the revealed classical term on the cosmological evolution, where the total internal energy plays the role of a source, and consider multi-dimensional generalizations of the obtained results for application in brane-world scenarios. Specifically, it seems advantageous to generalize the results of Refs. [23, 24] to the case of nonzero temperature.

Acknowledgments

V.B.B., V.M.M. and C.R. were supported by CNPq (Brazil). H.F.M. was supported by CAPES (Brazil). The authors are grateful to G. L. Klimchitskaya and A. A. Saharian for numerous helpful discussions. V.M.M. is also grateful to Federal University of Paraiba,
where this work was done, for kind hospitality.

[1] H. B. G. Casimir, Proc. Kon. Ned. Akad. Wet. B 51, 793 (1948).
[2] M. Krech, The Casimir Effect in Critical Systems (World Scientific, Singapore, 1994).
[3] V. M. Mostepanenko and N. N. Trunov, The Casimir Effect and its Applications (Oxford University Press, Oxford, 1997).
[4] K. A. Milton, The Casimir Effect: Physical Manifestation of Zero-Point Energy (World Scientific, Singapore, 2001).
[5] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Advances in the Casimir Effect (Oxford University Press, Oxford, 2009).
[6] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. 81, 1827 (2009).
[7] A. W. Rodriguez, F. Capasso, and S. G. Johnson, Nature Photon. 5, 211 (2011).
[8] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Int. J. Mod. Phys. B 25, 171 (2011).
[9] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Phys. Rev. D 81, 055003 (2010).
[10] G. L. Klimchitskaya and C. Romero, Phys. Rev. D 82, 115005 (2010).
[11] E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Eur. Phys. J. C 68, 223 (2010).
[12] Ph. Brax, C. Van de Bruck, D. F. Mota, N. J. Nunes, and H. A. Winter, Phys. Rev. D 82, 083503 (2010).
[13] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Phys. Rev. D 83, 075004 (2011).
[14] V. M. Mostepanenko and G. L. Klimchitskaya, Int. J. Mod. Phys. A 25, 2302 (2010).
[15] L. H. Ford, Phys. Rev. D 11, 3370 (1975).
[16] J. S. Dowker and R. Critchley, J. Phys. A 9, 535 (1976).
[17] S. G. Mamayev, V. M. Mostepanenko, and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 70, 1577 (1976) [Sov. Phys. JETP 43, 823 (1976)].
[18] L. H. Ford, Phys. Rev. D 14, 3304 (1976).
[19] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).

[20] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions* (Springer, Berlin, 1995).

[21] A. A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. **266**, 1 (1996).

[22] E. Elizalde, J. Phys. A **39**, 6299 (2006).

[23] E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **67**, 063515 (2003).

[24] E. Elizalde, S. D. Odintsov, and A. A. Saharian, Phys. Rev. D **79**, 065023 (2009).

[25] E. Elizalde, S. Nojiri, S. D. Odintsov, and P. Wang, Phys. Rev. D **71**, 103504 (2005).

[26] J. S. Dowker, Phys. Rev. D **18**, 1856 (1978).

[27] J. S. Dowker and R. Critchley, Phys. Rev. D **15**, 1484 (1977).

[28] A. Zhuk and H. Kleinert, Theor. Math. Phys. **109**, 1483 (1996).

[29] H. Kleinert and A. Zhuk, Theor. Math. Phys. **108**, 1236 (1996).

[30] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Phys. Rev. D **83**, 104042 (2011).

[31] M. B. Altaie and J. S. Dowker, Phys. Rev. D **18**, 3557 (1978).

[32] B. L. Hu, Phys. Lett. **103B**, 331 (1981).

[33] J. Feinberg, A. Mann, and M. Revzen, Ann. Phys. (N.Y.) **288**, 103 (2001).

[34] G. Kennedy, J. Phys. A **11**, L77 (1978).

[35] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[36] A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory Publishing, St.Petersburg, 1994).

[37] A. A. Saharian, *The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect* (Yerevan State University Publishing House, Yerevan, 2008); arXiv: 0708.1187.

[38] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (Kriger, New York, 1981), Vol. 1.

[39] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, Oxford, 1980), Part I.

[40] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Dower, New York, 1987).

[41] L. S. Brown and G. J. Maclay, Phys. Rev. **184**, 1272 (1969).

[42] B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, Eur. Phys. J. C **57**, 823 (2008).
[43] J. S. Dowker and G. Kennedy, J. Phys. A 11, 895 (1978).

[44] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York, 1980).

[45] R. Balian and B. Duplantier, Ann. Phys. (N.Y.) 104, 300 (1977).

[46] R. Balian and B. Duplantier, Ann. Phys. (N.Y.) 112, 165 (1978).

[47] G. Cognola, E. Elizalde, and K. Kirsten, J. Phys. A 34, 7311 (2001).

[48] E. Elizalde, M. Bordag, and K. Kirsten, J. Phys. A 31, 1743 (1998).