Quantum entanglement under Lorentz boost

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Abstract. To understand the characteristics of quantum entanglement of massive particles under Lorentz boost, we first introduce a relevant relativistic spin observable, and evaluate its expectation values for the Bell states under Lorentz boost. Then we show that maximal violation of the Bell’s inequality can be achieved by properly adjusting the directions of the spin measurement even in a relativistically moving inertial frame. Based on this we infer that the entanglement information is preserved under Lorentz boost as a form of correlation information determined by the transformation characteristic of the Bell state in use.

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1. Introduction

Quantum entanglement is a novel feature of quantum physics when compared with classical physics. It demonstrates the non-local character of quantum mechanics and is the very basis of quantum information processing such as quantum computation and quantum cryptography. Until recently, quantum entanglement was considered only within the non-relativistic regime.
Then, starting with the work of [1] there have been quite a lot of works investigating the effect of quantum entanglement measured in an inertial frame moving with relativistic speed [2]–[15].

Consider two spin-$\frac{1}{2}$ particles with total spin zero moving in opposite directions. Suppose the spin component of each particle is measured in the same direction by two observers in the laboratory (lab) frame. Then the two spin components have opposite values in whichever direction the spin measurements are performed. This is known as the EPR (Einstein–Podolsky–Rosen) correlation and is due to the isotropy of the spin-singlet state. Is the EPR correlation valid even for the two observers sitting in a moving frame which is Lorentz boosted relativistically with respect to the lab frame? This issue has been investigated from various aspects by many people including the above quoted authors. However, the answer to this question has not been clarified so far.

In [1], Czachor considered the spin singlet of two spin-$\frac{1}{2}$ massive particles moving in the same direction. He introduced the concept of a relativistic spin observable which is closely related to the spatial components of the Pauli–Lubanski vector. For two observers in the lab frame measuring the spin component of each particle in the same direction, the expectation value of the joint spin measurement, i.e., the expectation value of the tensor product of the relativistic spin observable of each constituent particle, depends on the boost velocity. Only when the boost speed reaches that of light, or when the direction of the spin measurements is perpendicular to the boost direction, the expectation value becomes $-1$. Thus, only for this limiting case, the results seem to agree with the EPR correlation. Czachor considered only the changes in the spin operator part by defining a new relativistic spin operator. There, the state does not need to be transformed since the observer is at rest.

Starting a couple of years ago there appeared a flurry of papers investigating the effect of Lorentz boost or Wigner rotation on entanglement. Here, we mention some of them that are directly related to the problem discussed in this paper.

Alsing and Milburn [2] considered the entanglement of two particles moving in opposite directions and showed that Wigner rotation under Lorentz boost is a local unitary operation, with which Dirac spinors representing the two particles transform. And they reached a conclusion that the entanglement is Lorentz invariant due to this unitary operation.

Gingrich and Adami [4] investigated the entanglement between the spin and momentum parts of two entangled particles. They concluded that the entanglement of the spin part is carried over to the entanglement of the momentum part under Lorentz boost, although the entanglement of the whole system is Lorentz invariant due to unitarity of the transformation. However, the concept of the reduced density matrix with traced-out momenta used in that work drew some criticism recently [10].

Terashima and Ueda [5] considered the effect of Wigner rotation on the spin singlet and evaluated the Bell observable under Lorentz boost. They concluded that, although the degree of the violation of the Bell’s inequality is decreased under Lorentz boost, the maximal violation of the Bell’s inequality can be obtained by properly adjusting the directions of the spin measurements in the moving frame. They also claimed that the perfect anti-correlation of the spin singlet seen in the EPR correlation is maintained for appropriately chosen spin-measurement directions depending on Lorentz boost, even though the EPR correlation is not maintained when the directions of spin measurements remain the same.

In [5], Terashima and Ueda considered the changes in the states only. Their spin operator has the same form as the non-relativistic spin operator. In this sense their result that the maximal violation of Bell’s inequality can be achieved even in the moving frame was somewhat expected.
due to the unitarity of the state transformation. In fact, [2, 5] considered the changes in the states only, and both reached a similar conclusion that the entanglement can be preserved under Lorentz boost.

However, if one considers the changes of the spin operator under Lorentz transformation as Czachor did in [1], the story becomes different: Bell’s inequality might not be violated, i.e., the entanglement may not be preserved under the transformation.

In a general situation, one has to consider both: the changes in the spin operator and the changes in the states. This was done by Ahn et al [6]. They calculated the Bell observable for the Bell states under Lorentz boost, and showed that the Bell’s inequality is not violated in the relativistic limit. They used the Czachor’s relativistic spin operator and transformed the state under Lorentz boost accordingly. Their result strongly suggested that the entanglement is not preserved under Lorentz boost. They further concluded [15] that quantum entanglement is not invariant under Lorentz boost based on the evaluation of the entanglement fidelity [16].

Here, we would like to note that the spin operator used in [6] is not as general as it should be. This is because the spin operator used in [6] is the same as Czachor’s [1], which is a spin operator for a restricted situation that we call Czachor’s limit in this paper.

We consider the changes in the spin operator under a Lorentz boost in a general situation compared with [6]. We first formulate the relativistic spin observable based on the sameness of the expectation values of one-particle spin measurement evaluated in two relative reference frames, one in the lab frame in which the particle has a velocity $\vec{v}$ and the observer is at rest, the other in the moving frame Lorentz boosted with $\vec{v}$, in which the particle is at rest and the observer is moving with a velocity $-\vec{v}$. Applying the relativistic spin observable for the two-particle spin-singlet state we evaluate the expectation value of the joint spin measurement. Then we calculate the values of the Bell observable for the Bell states. The values of the Bell observable decreased as the boost velocity becomes relativistic. However, we find a new set of spin measurement axes with which the Bell’s inequality is maximally violated. This seems to imply that the information on the correlation due to entanglement is kept even in the moving frame. In fact, under Lorentz boost certain entangled states transform into combinations of different entangled states. However, in certain directions of spin measurements the above combinations of states become eigenstates of these spin operators. In this manner the correlation information in one frame is maintained in other frames.

The paper is organized as follows. In section 2, we formulate the relativistic spin observable. Then in section 3, we evaluate the expectation value of the joint spin measurement for a spin singlet. In section 4, we find a new set of spin measurement axes for a spin-singlet state, with which the Bell’s inequality is maximally violated even under Lorentz boost. In section 5, we show that the same thing can be done for the other Bell states. We conclude with a discussion in section 6.

2. Relativistic spin observable

In this section, we consider a spin measurement of a massive particle viewed from two different inertial reference frames: one in the lab frame where the particle has a certain velocity, the other in the moving frame where the particle is at rest. To make the particle be at rest in the lab frame, the moving frame is Lorentz boosted in the opposite direction of the particle’s velocity in the lab frame just to compensate the particle’s motion in the lab frame.
Since we are just considering the same measurement in the two inertial frames, the respective expectation values of this measurement observed in the two frames should be the same:

\[
\langle \Psi_{\vec{p}} \rvert \frac{\vec{a} \cdot \vec{\sigma}_p}{|\lambda(\vec{a} \cdot \vec{\sigma}_p)|} \rvert \Psi_{\vec{p}} \rangle_{\text{lab}} = \langle \Psi_{\vec{p}=0} \rvert \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \rvert \Psi_{\vec{p}=0} \rangle_{\text{rest}},
\]

where \(\vec{a}\) and \(\vec{p}\) are the spin measurement axis and the momentum of the particle, respectively, in the lab frame, and \(\vec{a}_p\) is an inversely Lorentz-boosted vector of \(\vec{a}\) by \(-\vec{p}\). \(|\Psi_{\vec{p}}\rangle\) and \(|\Psi_{\vec{p}=0}\rangle\) are the wave functions of the particle in the lab and moving frames, respectively.

Now, the two wave functions are related by

\[
|\Psi_{\vec{p}}\rangle = U(L(\vec{p}))(\mathbf{1})|\Psi_{\vec{p}=0}\rangle = U(R(\hat{\vec{p}}))U(L_{\zeta}(\vec{p}))(\mathbf{1})|\Psi_{\vec{p}=0}\rangle,
\]

since for an arbitrary four momentum \(p\) it can be written as \(p = R(\hat{\vec{p}})L_{\zeta}(\vec{p})k\), where \(k = (m, 0, 0, 0)\) is the four momentum of the particle at rest, \(U(L(\vec{p}))\) and \(L_{\zeta}(\vec{p})\) are the Lorentz boosts along \(\vec{p}\) and the \(z\)-axis respectively, and \(R(\hat{\vec{p}})\) is a rotation of \(\hat{\vec{z}}\) to \(\hat{\vec{p}}\).

If we decompose the particle’s wave functions into their spatial and spin parts, for instance \(|\Psi_{p=0}\rangle = |0\rangle \otimes |\chi_{p=0}\rangle\), using the relation (1) we can obtain the relation between the relativistic spin operator \(\vec{\sigma}_p\) in the lab frame and the non-relativistic spin operator \(\vec{\sigma}\) for the particle at rest frame (the moving frame) in terms of the spin measurement axis \(\vec{a}\) in the lab frame and its Lorentz transformed spin measurement axis \(\vec{a}_p\) in the moving frame:

\[
\langle \chi_{\vec{p}=0} \otimes \vec{p} = 0 \rvert \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \rvert \vec{p} = 0 \rangle \otimes \chi_{\vec{p}=0} \rangle_{\text{rest}} = \langle \Psi_{\vec{p}=0} \rvert U^\dagger(L_{\zeta}(\vec{p})))U^\dagger(R(\hat{\vec{p}})) \left[ \frac{\vec{a}_p \cdot \vec{\sigma}_p}{|\lambda(\vec{a}_p \cdot \vec{\sigma}_p)|} \right] U(R(\hat{\vec{p}}))U(L_{\zeta}(\vec{p}))) \rvert \Psi_{\vec{p}=0} \rangle_{\text{lab}} .
\]

Also, \(U(L_{\zeta}(\vec{p})))|\Psi_{p} = 0\rangle = |p_z\rangle \otimes |\chi_{p_z}\rangle\); thus we have

\[
\langle \vec{p} = 0 | \vec{p} = 0 \rangle \langle \chi_{\vec{p}=0} \rvert \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \rvert \chi_{\vec{p}=0} \rangle_{\text{rest}} = \langle \chi_{p_z} \rangle \otimes \langle p_z \rvert U^\dagger(R(\hat{\vec{p}})) \left[ \frac{\vec{a}_p \cdot \vec{\sigma}_p}{|\lambda(\vec{a}_p \cdot \vec{\sigma}_p)|} \right] U(R(\hat{\vec{p}})) \rvert p_z \rangle \otimes \chi_{p_z} \rangle_{\text{lab}}
\]

\[
\langle \vec{p} | \vec{p} \rangle \langle \chi_{p_z} \rvert U^\dagger(R(\hat{\vec{p}})) \left[ \frac{\vec{a}_p \cdot \vec{\sigma}_p}{|\lambda(\vec{a}_p \cdot \vec{\sigma}_p)|} \right] U(R(\hat{\vec{p}})) \rvert \chi_{p_z} \rangle_{\text{lab}} = \langle \vec{p} | \vec{p} \rangle \langle \chi_{p_z} \rvert U^\dagger(R(\hat{\vec{p}})) \left[ \frac{\vec{a}_p \cdot \vec{\sigma}_p}{|\lambda(\vec{a}_p \cdot \vec{\sigma}_p)|} \right] U(R(\hat{\vec{p}})) \rvert \chi_{p_z} \rangle_{\text{lab}},
\]

\((4)\)
where, we notice that the spin wave function in the particle at rest frame is not affected under an arbitrary Lorentz boost \( L(\vec{p}) \) since the Wigner angle due to the Lorentz boost in this case is zero:

\[
|\Psi_{\vec{p}}\rangle = U(L(\vec{p}))|\Psi_{\vec{p}=0}\rangle = U(L(\vec{p}))|0\rangle \otimes \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = |\vec{p}\rangle \otimes \left( \begin{array}{c} \alpha \\ \beta \end{array} \right).
\]

Namely, \(|x_{\vec{p}=0}\rangle_{\text{rest}} = |x_{p_c}\rangle_{\text{lab}}\), and thus from (3) and (4), we obtain the following relation:

\[
\begin{aligned}
U^\dagger(R(\hat{\vec{p}})) \left[ \frac{\vec{a} \cdot \vec{\sigma}_p}{|\lambda(\vec{a} \cdot \vec{\sigma})|} \right] U(R(\hat{\vec{p}})) &= \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \\
&= \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} |\vec{p}=0\rangle |\vec{p}=0\rangle.
\end{aligned}
\]  

Based on the above observation, we define the relativistic spin observable as

\[
\hat{a} \equiv \frac{\vec{a} \cdot \vec{\sigma}_p}{|\lambda(\vec{a} \cdot \vec{\sigma})|} \equiv U(R(\hat{\vec{p}})) \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} U^\dagger(R(\hat{\vec{p}})),
\]

where \( R(\hat{\vec{p}}) \) is the rotation from the \( z \)-axis to the direction of \( \vec{p} \), which can be written as

\[
R(\hat{\vec{p}}) = R_z(\phi_p)R_y(\theta_p) = \begin{pmatrix}
\cos \phi_p & \cos \theta_p & \sin \phi_p \\
\sin \phi_p & \cos \phi_p & \sin \phi_p \\
- \sin \theta_p & 0 & \cos \theta_p
\end{pmatrix}
\]

and

\[
U(R(\hat{\vec{p}})) = \exp(-i\phi_p\sigma_z/2) \exp(-i\theta_p\sigma_y/2).
\]

The spin measurement axis in the moving frame, \( \vec{a}_p \), is given by the spatial part of \( a_p = [R(\hat{\vec{p}})L_z(|\vec{p}|)]^{-1}a \), where \( p = L_{\hat{\vec{p}}}(|\vec{p}|)k = R(\hat{\vec{p}})L_z(|\vec{p}|)k \) with \( k = (m, 0, 0, 0) \), and the spin measurement axis in the lab frame, \( \vec{a} \), is the spatial part of \( a \). Putting all this together, the relativistic spin observable can be expressed as follows:

\[
\hat{a} = U(R(\hat{\vec{p}})) \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} U^\dagger(R(\hat{\vec{p}})) \]

\[
= \frac{\vec{a}_p}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \cdot \left[ \exp(-i\phi_p\sigma_z/2) \exp(-i\theta_p\sigma_y/2) \exp(i\phi_p\sigma_z/2) \right]
\]

\[
= \vec{a}_p \cdot R(\hat{\vec{p}})\vec{\sigma}/|\lambda(\vec{a}_p \cdot \vec{\sigma})|.
\]

3. Relativistic joint spin measurement for a spin singlet

In this section, we apply the relativistic spin observable defined in the previous section to a spin-singlet state which consists of two massive spin-\( \frac{1}{2} \) particles.

First, we consider a simple case in which the spin measuring device is fixed in the lab frame and both the particles are moving with the same velocity in the lab frame. This is the same set-up
considered by Czachor in his work [1]. The spin measuring axes are in the direction of \( \vec{a} \) for particle 1, and in the direction of \( \vec{b} \) for particle 2. We choose the particles’ moving direction to be the +\( z \) axis. Then the expectation value for joint spin measurement of the particles can be expressed as

\[
\langle \hat{a} \otimes \hat{b} \rangle = \left\langle \Psi \left| \frac{\vec{a} \cdot \vec{\sigma}}{|\lambda(\vec{a} \cdot \vec{\sigma})|} \otimes \frac{\vec{b} \cdot \vec{\sigma}}{|\lambda(\vec{b} \cdot \vec{\sigma})|} \right| \Psi \right\rangle
\]

\[
= \left\langle \Psi \left| \frac{\vec{a}_p \cdot R(\hat{\vec{p}})\vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \otimes \frac{\vec{b}_p \cdot R(\hat{\vec{p}})\vec{\sigma}}{|\lambda(\vec{b}_p \cdot \vec{\sigma})|} \right| \Psi \right\rangle
\]

\[
= \left\langle \Psi \left| \frac{\vec{a}_p \cdot \vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \otimes \frac{\vec{b}_p \cdot \vec{\sigma}}{|\lambda(\vec{b}_p \cdot \vec{\sigma})|} \right| \Psi \right\rangle,
\]

where the state function is given by

\[
|\Psi_1\rangle = \frac{1}{\sqrt{2}} (|\vec{p}, \frac{1}{2}\rangle |\vec{p}, -\frac{1}{2}\rangle - |\vec{p}, -\frac{1}{2}\rangle |\vec{p}, \frac{1}{2}\rangle).
\]

In the last step, we used \( R(\hat{\vec{p}}) = 1 \) since \( \hat{\vec{p}} = \hat{z} \) in the present case. The measuring axis \( \vec{a}_p \) in the moving frame is given by the spatial part of Lorentz transformed

\[
a_p = \begin{pmatrix}
\cosh \xi & 0 & 0 & -\sinh \xi \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh \xi & 0 & 0 & \cosh \xi
\end{pmatrix}
\begin{pmatrix}
a_x \\
a_y \\
a_z
\end{pmatrix} = \begin{pmatrix}
-az \sinh \xi \\
ax \\
ay \\
az \cosh \xi
\end{pmatrix}.
\]

Since the magnitude of \( \vec{a}_p \) is the same as that of the eigenvalue of \( \vec{a}_p \cdot \vec{\sigma} \), we get

\[
|\lambda_{ap}| = \sqrt{a_x^2 + a_y^2 + a_z^2 \cosh^2 \xi} = \sqrt{1 + a_z^2 \sinh^2 \xi}.
\]

Thus, the relativistic spin observable for particle 1 in the present case is given by

\[
\hat{a} \equiv \frac{\vec{a} \cdot \vec{\sigma}}{|\lambda(\vec{a} \cdot \vec{\sigma})|} = \frac{a_x \sigma_x + a_y \sigma_y + a_z \sigma_z \cosh \xi}{\sqrt{1 + a_z^2 \sinh^2 \xi}}.
\]

The same holds for particle 2. Thus, the expectation value of the joint spin measurement (11) is given by

\[
\langle \hat{a} \otimes \hat{b} \rangle = \left\langle \Psi \left| (a_x \sigma_x + a_y \sigma_y + a_z \sigma_z \cosh \xi) \otimes (b_x \sigma_x + b_y \sigma_y + b_z \sigma_z \cosh \xi) \right| \Psi \right\rangle \\
\sqrt{1 + a_z^2 \sinh^2 \xi} \sqrt{1 + b_z^2 \sinh^2 \xi}
\]

\[
= -\frac{(a_x b_x + a_y b_y + a_z b_z \cosh \xi)}{\sqrt{1 + a_z^2 \sinh^2 \xi} \sqrt{1 + b_z^2 \sinh^2 \xi}},
\]

where we used \( \langle \Psi | \sigma_i \otimes \sigma_j | \Psi \rangle = -\delta_{ij} \) for \( i, j = x, y, z \). Equation (14) agrees with Czachor’s result.
To see whether Bell’s inequality is still maximally violated in this case, we now consider the so-called Bell observable \( C(a, a', b, b') \) defined as [1]

\[
C(a, a', b, b') \equiv \langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle.
\] (15)

For maximal violation, we choose the following set of vectors for spin measurements:

\[
\vec{a} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad \vec{a}' = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),
\]

\[
\vec{b} = (0, 0, 1), \quad \vec{b}' = (0, 1, 0),
\] (16)

which yields \(|C(a, a', b, b')| = 2\sqrt{2}\) in the non-relativistic case. Using (14), we get the Bell observable for the above vector set as

\[
C(a, a', b, b') = -\frac{2(1 + \cosh \xi)}{\sqrt{2 + \sinh^2 \xi}}.
\] (17)

We see that \(|C(a, a', b, b')|\) approaches 2 in the relativistic limit \(\xi \to \infty\), thereby Bell’s inequality is not violated in this relativistic limit.

Next, we consider a more general situation in which the two particles of the spin singlet move in opposite directions in the lab frame and the two observers for particles 1 and 2 are sitting in the moving frame Lorentz-boosted with respect to the lab frame in a direction perpendicular to the particles’ movements. Here, we choose that particles 1 and 2 are moving in the \(+z\) and \(-z\) directions respectively in the lab frame, and the moving frame in which the two observers for particles 1 and 2, Alice and Bob, are sitting is Lorentz-boosted to the \(-x\) direction. Now, the expectation value of the joint spin measurements performed by Alice and Bob can be expressed as

\[
\langle \hat{a} \otimes \hat{b} \rangle = \left\langle \Phi \left| \frac{\vec{a} \cdot \vec{\sigma}_{\Lambda p}}{|\lambda(\vec{a} \cdot \vec{\sigma}_{\Lambda p})|} \otimes \frac{\vec{b} \cdot \vec{\sigma}_{\Lambda p'}}{|\lambda(\vec{b} \cdot \vec{\sigma}_{\Lambda p'})|} \right| \Phi \rightangle
\]

\[
= \left\langle \Phi \left| \frac{\vec{a}_{\Lambda p} \cdot R(\vec{p}_\Lambda)\vec{\sigma}}{|\lambda(\vec{a}_{\Lambda p} \cdot \vec{\sigma})|} \otimes \frac{\vec{b}_{\Lambda p'} \cdot R(\vec{p}_{\Lambda p'})\vec{\sigma}}{|\lambda(\vec{b}_{\Lambda p'} \cdot \vec{\sigma})|} \right| \Phi \rightangle,
\] (18)

where \(|\Phi\rangle = U(\Lambda)|\Psi\rangle\) with \(|\Psi\rangle = \frac{1}{\sqrt{2}}(|\vec{p}, \frac{1}{2}\rangle - |\vec{p}, -\frac{1}{2}\rangle + |\vec{p}, -\frac{1}{2}\rangle - |\vec{p}, \frac{1}{2}\rangle)\), and \(\Lambda\) is the Lorentz boost performed to Alice and Bob (in the moving frame).

In general, the effect of a Lorentz transformation to a state can be expressed as [17]

\[
U(\Lambda)|p, \sigma\rangle = \sum_\sigma D_{\sigma' \sigma}(W(\Lambda, p))|\Lambda p, \sigma'\rangle.
\] (19)

The explicit form for the singlet is given by

\[
U(\Lambda)|\Psi\rangle = \cos \Omega_p |\Psi^-\rangle + \sin \Omega_p |\Phi^+\rangle.
\] (20)
\[ |\Psi_{\Lambda}^{(-)}\rangle = \frac{1}{\sqrt{2}} \left( |\Lambda p, \frac{1}{2}\rangle |\Lambda P p, \frac{-1}{2}\rangle - |\Lambda p, \frac{-1}{2}\rangle |\Lambda P p, \frac{1}{2}\rangle \right), \]
\[ |\Phi_{\Lambda}^{(+)}\rangle = \frac{1}{\sqrt{2}} \left( |\Lambda p, \frac{1}{2}\rangle |\Lambda P p, \frac{1}{2}\rangle + |\Lambda p, \frac{-1}{2}\rangle |\Lambda P p, \frac{-1}{2}\rangle \right), \]
and \( \Omega_p \) is the Wigner angle due to Lorentz boost \( \Lambda \) performed to a particle with momentum \( \vec{p} \) and is given explicitly by
\[ \tan \Omega_p = \frac{\sinh \xi \sinh \chi}{\cosh \xi + \cosh \chi} \text{ with } \tanh \xi = \beta_p \text{ and } \tanh \chi = \beta_{\Lambda}. \tag{21} \]

Here, \( P \) is the space-inversion operator given by
\[ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Pp = \begin{pmatrix} \sqrt{p^2 + m^2} \\ 0 \\ 0 \\ -p \end{pmatrix}. \]

The other expressions that appeared above are given by
\[ \Lambda p = \begin{pmatrix} \cosh \chi & \sinh \chi & 0 & 0 \\ \sinh \chi & \cosh \chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{p^2 + m^2} \\ 0 \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} \sqrt{p^2 + m^2} \cosh \chi \\ 0 \\ 0 \\ p \end{pmatrix}, \]
\[ R_y(\hat{\rho}_\Lambda) = \begin{pmatrix} \cos \theta_{\Lambda} & 0 & \sin \theta_{\Lambda} \\ 0 & 1 & 0 \\ -\sin \theta_{\Lambda} & 0 & \cos \theta_{\Lambda} \end{pmatrix}, \quad R_y(\hat{\rho}_{\Lambda P}) = R_y(\pi - \theta_{\Lambda}), \]

where \( E_p = \sqrt{p^2 + m^2} \) and \( \tan \theta_{\Lambda} = (E_p \sinh \chi)/p = (\sinh \chi)/(\tanh \xi) \). Thus,
\[ R_y(\hat{\rho}_\Lambda)\tilde{\sigma} = \begin{pmatrix} \cos \theta_{\Lambda} & 0 & \sin \theta_{\Lambda} \\ 0 & 1 & 0 \\ -\sin \theta_{\Lambda} & 0 & \cos \theta_{\Lambda} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = \begin{pmatrix} \sigma_x \cos \theta_{\Lambda} + \sigma_z \sin \theta_{\Lambda} \\ \sigma_y \\ -\sigma_x \sin \theta_{\Lambda} + \sigma_z \cos \theta_{\Lambda} \end{pmatrix}, \tag{22} \]

and
\[ a_{\Lambda p} = [R_y(\theta_{\Lambda})L_z(\eta)]^{-1} a \]
\[ = \begin{pmatrix} \cosh \eta & 0 & 0 & -\sinh \eta \\ 0 & \cos \theta_{\Lambda} & 0 & -\sin \theta_{\Lambda} \\ 0 & 0 & 1 & 0 \\ -\sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{\Lambda} & 0 & -\sin \theta_{\Lambda} \\ 0 & 0 & 1 & 0 \\ 0 & \sin \theta_{\Lambda} & 0 & \cos \theta_{\Lambda} \end{pmatrix} \begin{pmatrix} 0 \\ a_x \\ a_y \\ a_z \end{pmatrix}. \tag{23} \]
where \( \tanh \eta = |\vec{p}|/E_\Lambda p = \sqrt{(\tanh^2 \xi + \sinh^2 \chi)/\cosh \chi} \). Thus the spatial part of \( a_{\Lambda p} \) and its magnitude are given by

\[
\vec{a}_{\Lambda p} = (a_x \cos \theta_\Lambda - a_z \sin \theta_\Lambda, a_y, \cosh \eta (a_x \sin \theta_\Lambda + a_z \cos \theta_\Lambda)),
\]

\[
|\vec{a}_{\Lambda p}| = \sqrt{1 + \sinh^2 \eta (a_x \sin \theta_\Lambda + a_z \cos \theta_\Lambda)^2}.
\]

(24)

Similarly, from \( b_{\Lambda p} = [R_y(\pi - \theta_\Lambda)B_z(\eta)]^{-1}b \), we get

\[
\vec{b}_{\Lambda p} = (-b_x \cos \theta_\Lambda - b_z \sin \theta_\Lambda, b_y, \cosh \eta (b_x \sin \theta_\Lambda - b_z \cos \theta_\Lambda)),
\]

\[
|\vec{b}_{\Lambda p}| = \sqrt{1 + \sinh^2 \eta (-b_x \sin \theta_\Lambda + b_z \cos \theta_\Lambda)^2},
\]

(25)

and

\[
R_y(\hat{p}_{\Lambda p})\vec{\sigma} = \begin{pmatrix}
-a_x \cos \theta_\Lambda & 0 & \sin \theta_\Lambda \\
0 & 1 & 0 \\
-\sin \theta_\Lambda & 0 & -\cos \theta_\Lambda
\end{pmatrix}
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z
\end{pmatrix} =
\begin{pmatrix}
-\sigma_x \cos \theta_\Lambda + \sigma_z \sin \theta_\Lambda \\
\sigma_y \\
-\sigma_x \sin \theta_\Lambda - \sigma_z \cos \theta_\Lambda
\end{pmatrix}.
\]

(26)

Therefore, the tensor product of relativistic spin observables of particles 1 and 2 for the joint spin measurement can be expressed as

\[
\hat{a} \otimes \hat{b} = \frac{\vec{a}_{\Lambda p} \cdot R(\hat{p}_{\Lambda p})\vec{\sigma}}{|\lambda(\vec{a}_{\Lambda p} \cdot \vec{\sigma})|} \otimes \frac{\vec{b}_{\Lambda p} \cdot R(\hat{p}_{\Lambda p})\vec{\sigma}}{|\lambda(\vec{b}_{\Lambda p} \cdot \vec{\sigma})|} \equiv \frac{\vec{A} \cdot \vec{\sigma} \otimes \vec{B} \cdot \vec{\sigma}}{|\vec{A}_{\Lambda p}||\vec{b}_{\Lambda p}|},
\]

(27)

where

\[
\vec{A} = \begin{pmatrix}
a_x(\cos^2 \theta_\Lambda - \cosh \eta \sin^2 \theta_\Lambda) - a_z(1 + \cosh \eta) \sin \theta_\Lambda \cos \theta_\Lambda \\
0 \\
-a_x(1 + \cosh \eta) \sin \theta_\Lambda \cos \theta_\Lambda - a_z(\sin^2 \theta_\Lambda - \cosh \eta \cos^2 \theta_\Lambda)
\end{pmatrix},
\]

\[
\vec{B} = \begin{pmatrix}
b_x(\cos^2 \theta_\Lambda - \cosh \eta \sin^2 \theta_\Lambda) + b_z(1 + \cosh \eta) \sin \theta_\Lambda \cos \theta_\Lambda \\
0 \\
-b_x(1 + \cosh \eta) \sin \theta_\Lambda \cos \theta_\Lambda - b_z(\sin^2 \theta_\Lambda - \cosh \eta \cos^2 \theta_\Lambda)
\end{pmatrix}.
\]

(28)

Using the following relations,

\[
\sigma_x \otimes \sigma_x |\Phi\rangle = -\cos \Omega_p |\Psi_\Lambda^-\rangle + \sin \Omega_p |\Phi_\Lambda^+\rangle,
\]

\[
\sigma_y \otimes \sigma_y |\Phi\rangle = -\cos \Omega_p |\Psi_\Lambda^-\rangle - \sin \Omega_p |\Phi_\Lambda^+\rangle,
\]

\[
\sigma_z \otimes \sigma_z |\Phi\rangle = -\cos \Omega_p |\Psi_\Lambda^-\rangle + \sin \Omega_p |\Phi_\Lambda^+\rangle,
\]

\[
\sigma_x \otimes \sigma_z |\Phi\rangle = -\cos \Omega_p |\Phi_\Lambda^+\rangle - \sin \Omega_p |\Psi_\Lambda^-\rangle,
\]

\[
\sigma_z \otimes \sigma_x |\Phi\rangle = \cos \Omega_p |\Phi_\Lambda^+\rangle + \sin \Omega_p |\Psi_\Lambda^-\rangle.
\]

(29)
and since the remaining terms do not contribute to the expectation value, we finally get the following expression for the expectation value of the joint spin measurement for the spin singlet:

$$\langle \hat{a} \otimes \hat{b} \rangle = \frac{-1}{|\hat{a}_{\Lambda p}| |\hat{b}_{\Lambda p}|} \left[ (A_x B_x + A_z B_z) \cos 2\Omega_p + A_y B_y + (A_x B_z - A_z B_x) \sin 2\Omega_p \right]$$  \hspace{1cm} (30)

Here, we examine two limiting cases of the above formula:

1. When $\chi \to 0$, $\langle \hat{a} \otimes \hat{b} \rangle \to -\frac{1}{\sqrt{1 + a_x^2 \sinh^2 \chi}} \sqrt{1 + b_x^2 \sinh^2 \chi} (a_x b_x + a_y b_y + a_z b_z)$.

2. When $\xi \to 0$, $\langle \hat{a} \otimes \hat{b} \rangle \to -\frac{1}{\sqrt{1 + a_x^2 \sinh^2 \xi}} \sqrt{1 + b_x^2 \sinh^2 \xi} (a_x b_x \cosh^2 \chi + a_y b_y + a_z b_z)$.

Note that the second case exactly corresponds to the Czachor’s set-up and yields the same result.

Now, let us evaluate the Bell observable for a set of measurement vectors which yield the maximal violation of Bell’s inequality in the non-relativistic case:

$$\tilde{a} = \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad \tilde{a}' = \left( 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$\tilde{b} = (0, 0, 1), \quad \tilde{b}' = (0, 1, 0).$$  \hspace{1cm} (31)

With this set of measurement vectors, the Bell observable $C(a, a', b, b')$ is given by

$$C(a, a', b, b') = \langle \hat{a} \otimes \hat{b} \rangle + \langle \hat{a} \otimes \hat{b}' \rangle + \langle \hat{a}' \otimes \hat{b} \rangle - \langle \hat{a}' \otimes \hat{b}' \rangle$$

$$= \frac{2}{\sqrt{1 + \sin^2 \theta_\Lambda + \cosh^2 \eta \cos^2 \theta_\Lambda}}$$

$$+ \frac{2}{\sqrt{1 + \sin^2 \theta_\Lambda + \cosh^2 \eta \cos^2 \theta_\Lambda \sin^2 \theta_\Lambda + \cosh^2 \eta \cos^2 \theta_\Lambda}} \left[ (\cosh \eta \cos^2 \theta_\Lambda - \sin^2 \theta_\Lambda)^2 - (1 + \cosh \eta)^2 \sin^2 \theta_\Lambda \cos^2 \theta_\Lambda \cos 2\Omega_p \right]$$

$$- (1 + \cosh \eta) (\cosh \eta \cos^2 \theta_\Lambda - \sin^2 \theta_\Lambda) \sin 2\theta_\Lambda \sin 2\Omega_p \right].$$  \hspace{1cm} (32)

Here also, we consider two limiting cases of the above formula:

1. When $\chi \to 0$, $\theta_\Lambda \to 0$, $\eta \to \xi$, we get $|C(a, a', b, b')| \to 2(1 + \cosh \xi)/\sqrt{2 + \sinh^2 \xi}$.

2. When $\xi \to 0$, $\theta_\Lambda \to \pi/2$, $\eta \to \chi$, we get $|C(a, a', b, b')| \to 2\sqrt{2}$.

The first case is similar to the Czachor’s set-up in the sense that the observers are at rest and only the particles are moving in opposite directions with the same speed. The result is the same.
as the one that we can infer from Czachor’s result. The second case corresponds to a Czachor’s case in which the spin measurement directions are perpendicular to the particle’s movement. The obtained result agrees with Czachor’s result.

What happens if the two particles have different velocities, but not in opposite directions? In order to make the discussion simple, we consider the case when the observer is at rest in the lab frame. Let \( p \) and \( q \) be the momentum of particles 1 and 2, respectively. In this case, the Wigner angle \( \Omega_p \) is zero. Then, the expectation value of the joint spin measurement is given by

\[
\langle \hat{a} \otimes \hat{b} \rangle = \langle \Psi \left| \frac{\vec{a} \cdot R(\vec{p})\vec{\sigma}}{|\lambda(\vec{a}_p \cdot \vec{\sigma})|} \otimes \frac{\vec{b} \cdot R(\vec{q})\vec{\sigma}}{|\lambda(\vec{b}_q \cdot \vec{\sigma})|} \right| \Psi \rangle = \langle \Psi \left| \frac{\vec{A} \cdot \vec{\sigma}}{|\vec{a}|} \otimes \frac{\vec{B} \cdot \vec{\sigma}}{|\vec{b}|} \right| \Psi \rangle,
\]

(33)

where

\[
\vec{A} = \begin{pmatrix}
a_x (\cos^2 \theta_p - \cosh \xi_p \sin^2 \theta_p) - a_z (1 + \cosh \xi_p) \sin \theta_p \cos \theta_p \\
- a_y (1 + \cosh \xi_p) \sin \theta_p \cos \theta_p - a_z (\sin^2 \theta_p - \cosh \xi_p \cos^2 \theta_p)
\end{pmatrix},
\]

\[
\vec{B} = \begin{pmatrix}
b_x (\cos^2 \theta_q - \cosh \xi_q \sin^2 \theta_q) - b_z (1 + \cosh \xi_q) \sin \theta_q \cos \theta_q \\
- b_y (1 + \cosh \xi_q) \sin \theta_q \cos \theta_q - b_z (\sin^2 \theta_q - \cosh \xi_q \cos^2 \theta_q)
\end{pmatrix},
\]

(34)

and its value becomes

\[
\langle \hat{a} \otimes \hat{b} \rangle = -\frac{\vec{A} \cdot \vec{B}}{|\vec{a}| |\vec{b}|}.
\]

(35)

For the same set of measurement vectors as in (31), the Bell observable is given by

\[
C(a, a', b, b') = -\frac{2}{\sqrt{2 + \sinh^2 \xi_p \cos^2 \theta_p}} - \frac{2}{\sqrt{2 + \sinh^2 \xi_q \cos^2 \theta_q}} \sqrt{1 + \sinh^2 \xi_p \cos^2 \theta_p} \sqrt{1 + \sinh^2 \xi_q \cos^2 \theta_q} \times \{(1 + \cosh \xi_p)(1 + \cosh \xi_q) \sin \theta_p \cos \theta_p \sin \theta_q \cos \theta_p \\
+ (\sin^2 \theta_p - \cosh \xi_p \cos^2 \theta_p)(\sin^2 \theta_q - \cosh \xi_q \cos^2 \theta_q)\}.
\]

(36)

One can check that this result reduces to the one from (32), if \( \vec{p} \) and \( \vec{q} \) are in the opposite directions with the same magnitude. In the next section, we shall see that we can find the corrected vector set for the maximal violation of Bell’s inequality in this case also.

### 4. Corrected Bell observable for a spin singlet

In this section, we will show that by appropriately choosing the vector set for spin measurements, the maximal violation of Bell’s inequality can be achieved even in a relativistically moving inertial frame. For the non-relativistic case, it is known that a fully entangled state, such as the
spin singlet, maximally violates Bell’s inequality giving the value \(2\sqrt{2}\) for the Bell observable. For the spin singlet case, the vector set inducing the maximal violation may be chosen as

\[
\vec{a} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \vec{a}' = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),
\]

\[
\vec{b} = (0, 0, 1), \quad \vec{b}' = (0, 1, 0).
\]  \(37\)

In the non-relativistic case, the expectation value of the joint spin measurement for the spin singlet is given by

\[
\langle \hat{\sigma} \otimes \hat{\sigma} \rangle = -\vec{a} \cdot \vec{b}
\]  \(38\)

for a set of measurement vectors, \(\vec{a}\) for particle 1 and \(\vec{b}\) for particle 2, where \(\hat{\sigma} = \vec{a} \cdot \vec{\sigma}, \hat{\sigma} = \vec{b} \cdot \vec{\sigma}\). However, when the movement of the particles or the observers become relativistic the above expectation value is not maintained as Czachor and others have shown [1, 5, 6].

Following the reasoning of Terashima and Ueda [5], here we investigate whether we can find a set of spin measurement directions which preserve the non-relativistic expectation value (38) even under relativistic situations. To do this, we consider the case in which a new set of spin measurement directions \(\vec{a}_c, \vec{b}_c\) in a Lorentz-boosted frame yields the relation

\[
\langle \hat{a}_c \otimes \hat{b}_c \rangle = -\vec{a} \cdot \vec{b}
\]  \(39\)

with the previously chosen vector set \(\vec{a}, \vec{b}\) in the non-relativistic lab frame. The existence of the new vector set \(\vec{a}_c, \vec{b}_c\) implies that in the new frame the correlation between the two entangled particles can be seen when the measurement is performed along these new direction vectors and not along the previously given directions \(\vec{a}, \vec{b}\) in the lab frame. Hence we will attempt to determine \(\vec{a}_c, \vec{b}_c\) satisfying (39) for a simple case of Czachor and then for a more general case.

In Czachor’s set-up in which both the particles are moving in the +\(z\) direction, the relation (39) is satisfied if

\[
\frac{\vec{a}_c \cdot \vec{\sigma}}{|\vec{a}_c|} = \vec{a} \cdot \vec{\sigma}.
\]

Let us denote \(\vec{a}_c = (a_{cx}, a_{cy}, a_{cz})\), then from the relation \(L_z(-\xi)a_c = a_{cp}\) and \(\tanh \xi = \beta_p\), we have

\[
\begin{pmatrix}
\cosh \xi & 0 & -\sinh \xi \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
a_{cx} \\
a_{cy}
\end{pmatrix} =
\begin{pmatrix}
-a_{cz} \sinh \xi \\
a_{cx} \\
a_{cy}
\end{pmatrix}
\]

\(40\)

Thus we can write \(\vec{a}_{cp} = (a_{cx}, a_{cy}, a_{cz} \cosh \xi)\), and we get the following equation for the corrected vector \(\vec{a}_c\):

\[
\frac{1}{\sqrt{1 + a_{cz}^2 \sinh^2 \xi}} (a_{cx}, a_{cy}, a_{cz} \cosh \xi) = (a_x, a_y, a_z).
\]  \(41\)

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Solving the above equation, we get
\[ a_{cx} = \frac{a_x}{\sqrt{1 - a_z^2 \tanh^2 \xi}}, \quad a_{cy} = \frac{a_y}{\sqrt{1 - a_z^2 \tanh^2 \xi}}, \quad a_{cz} = \frac{a_z}{\cosh \xi \sqrt{1 - a_z^2 \tanh^2 \xi}}. \] (42)

Similarly, we get the same expression for the remaining corrected vector \( \vec{b}_c \) of the new frame. The expectation value of the joint spin measurement for the corrected vector set is now given by (14),
\[ \langle \hat{a}_c \otimes \hat{b}_c \rangle = -(a_x b_x + a_y b_y + a_z b_z) \cosh^2 \xi \frac{\sqrt{1 + a_z^2 \sinh^2 \xi}}{\sqrt{1 + b_z^2 \sinh^2 \xi}}, \] (43)
which satisfies our requirement.

To determine the Bell observable, we first evaluate the corrected vectors for \( \vec{a}_c, \vec{a}'_c \) given by (37) using (42) and similarly for \( \vec{b}_c, \vec{b}'_c \) given by (37):
\[ \vec{a}_c = \left(0, \frac{1}{\sqrt{2 - \tanh^2 \xi}}, \frac{1}{\cosh \xi \sqrt{2 - \tanh^2 \xi}}\right), \]
\[ \vec{a}'_c = \left(0, -\frac{1}{\sqrt{2 - \tanh^2 \xi}}, \frac{1}{\cosh \xi \sqrt{2 - \tanh^2 \xi}}\right), \]
\[ \vec{b}_c = (0, 0, 1), \quad \vec{b}'_c = (0, 1, 0). \] (44)

Using (43), the Bell observable with the above corrected vector set is now evaluated as
\[ C(a_c, a'_c, b_c, b'_c) = \langle \hat{a}_c \otimes \hat{b}_c \rangle + \langle \hat{a}_c \otimes \hat{b}'_c \rangle + \langle \hat{a}'_c \otimes \hat{b}_c \rangle - \langle \hat{a}'_c \otimes \hat{b}'_c \rangle \]
\[ = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\sqrt{2} \] (45)
retrieving the value of the maximal violation of the Bell’s inequality in the non-relativistic case.

Next, we consider a more general case as in section 3. The two particles of the spin singlet are moving in the \(+z\) and \(-z\) directions respectively in the lab frame, and the observers, Alice and Bob, are sitting in a moving frame which is Lorentz boosted towards the \(-x\) direction. In this case, the expectation value of the joint spin measurement for a corrected vector set is given by (30),
\[ \langle \hat{a}_c \otimes \hat{b}_c \rangle = \frac{-1}{|\vec{a}_{c\Lambda p}| |\vec{b}_{c\Lambda p}|} [(A_{cx} B_{cx} + A_{cz} B_{cz}) \cos 2\Omega_p + A_{cy} B_{cy} + (A_{cx} B_{cz} - A_{cz} B_{cx}) \sin 2\Omega_p], \] (46)
where \( \vec{A}_c, \vec{B}_c \) correspond to \( \vec{A}, \vec{B} \) of (28) in which \( \vec{a}, \vec{b} \) are replaced with \( \vec{a}_c, \vec{b}_c \), and other notations such as \( \Omega_p, \xi, \chi, \eta, \theta_\Lambda \), etc are the same as in section 3.
Here again, we will get the corrected vector set of spin measurement directions if we find \( a_c, b_c \) which give the expectation value (46) to be \(-\vec{a} \cdot \vec{b}\). Namely, we want to find \( a_c, b_c \) that satisfy the following equation:

\[
\frac{-1}{|a_{c\Lambda p}|} [(A_{cx}B_{cx} + A_{cz}B_{cz}) \cos 2\Omega_p + A_{cy}B_{cy} + (A_{cx}B_{cx} - A_{cz}B_{cz}) \sin 2\Omega_p] = -\vec{a} \cdot \vec{b},
\]

where \((a_{c\Lambda p}, b_{c\Lambda p})\) correspond to \((a_{\Lambda p}, b_{\Lambda p})\) of the previous section as above. One can check that the equation (47) is satisfied if the following relation is satisfied:

\[
\frac{A_{ci}}{|a_{c\Lambda p}|} = \bar{a}_i, \quad \frac{B_{ci}}{|b_{c\Lambda p}|} = \bar{b}_i \quad \text{for } i = (x, y, z),
\]

where

\[
\bar{a}_i \equiv R_y(\Omega_p) a_i, \quad \bar{b}_i \equiv R_y(-\Omega_p) b_i \quad \text{with} \quad R_y(\Omega_p) = \begin{pmatrix} \cos \Omega_p & 0 & \sin \Omega_p \\ 0 & 1 & 0 \\ -\sin \Omega_p & 0 & \cos \Omega_p \end{pmatrix}.
\]

Solving (48), we obtain \( a_c, b_c \) in terms of \( \bar{a}, \bar{b} \):

\[
\begin{align*}
a_{cz} &= \frac{\bar{a}_z}{\sqrt{[F_a(1 + \cosh \eta) \sin \theta_\Lambda \cos \theta_\Lambda - (\sin^2 \theta_\Lambda - \cosh \eta \cos^2 \theta_\Lambda)]^2 - \bar{a}_z^2 \sinh^2 \eta (F_a \sin \theta_\Lambda + \cos \theta_\Lambda)^2}}, \\
\end{align*}
\]

\[
\begin{align*}
a_{cx} &= \bar{a}_x \sqrt{1 + a_{cz}^2 \sinh^2 \eta (F_a \sin \theta_\Lambda + \cos \theta_\Lambda)^2}, \\
\end{align*}
\]

\[
\begin{align*}
a_{cy} &= a_y \sqrt{1 + a_{cz}^2 \sinh^2 \eta (F_a \sin \theta_\Lambda + \cos \theta_\Lambda)^2}, \\
\end{align*}
\]

\[
\begin{align*}
b_{cz} &= \frac{\bar{b}_z}{\sqrt{[F_b(1 + \cosh \eta) \sin \theta_\Lambda \cos \theta_\Lambda - (\sin^2 \theta_\Lambda - \cosh \eta \cos^2 \theta_\Lambda)]^2 - \bar{b}_z^2 \sinh^2 \eta (F_b \sin \theta_\Lambda - \cos \theta_\Lambda)^2}}, \\
\end{align*}
\]

\[
\begin{align*}
b_{cx} &= \bar{b}_x \sqrt{1 + b_{cz}^2 \sinh^2 \eta (F_b \sin \theta_\Lambda - \cos \theta_\Lambda)^2}, \\
\end{align*}
\]

\[
\begin{align*}
b_{cy} &= b_y \sqrt{1 + b_{cz}^2 \sinh^2 \eta (F_b \sin \theta_\Lambda - \cos \theta_\Lambda)^2}, \\
\end{align*}
\]

where \( a_i, b_i \) for \( i = x, y, z \) are given by (49), and

\[
\begin{align*}
F_a &= \frac{(1 + \cosh \eta) \tan \theta_\Lambda - f_a (\tan^2 \theta_\Lambda - \cosh \eta)}{(1 - \cosh \eta \tan^2 \theta_\Lambda) - f_a (1 + \cosh \eta) \tan \theta_\Lambda}, \\
F_b &= -\frac{(1 + \cosh \eta) \tan \theta_\Lambda + f_b (\tan^2 \theta_\Lambda - \cosh \eta)}{(1 - \cosh \eta \tan^2 \theta_\Lambda) + f_b (1 + \cosh \eta) \tan \theta_\Lambda}, \\
\end{align*}
\]

with \( f_a \equiv \frac{\bar{a}_x}{\bar{a}_z}, \quad f_b \equiv \frac{\bar{b}_x}{\bar{b}_z} \).
And thus $|\vec{a}_{\Lambda,p}|, |\vec{b}_{\Lambda,p}|$ are given by
\[
|\vec{a}_{\Lambda,p}| = \sqrt{1 + a_{cz}^2 \sinh^2 \eta \left(F_a \sin \theta_{\Lambda} + \cos \theta_{\Lambda}\right)^2},
\]
\[
|\vec{b}_{\Lambda,p}| = \sqrt{1 + b_{cz}^2 \sinh^2 \eta \left(F_b \sin \theta_{\Lambda} - \cos \theta_{\Lambda}\right)^2}.
\]

Now, we consider how the correlation due to entanglement is changed by Lorentz boost. For the spin singlet, the two spins of particles 1 and 2 are always antiparallel in the non-relativistic case. Thus we would like to see how the corrected vector set in the Lorentz-boosted moving frame shows the correlation between the two spins of the entangled particles that exists in the non-relativistic lab frame.

As expressed in (47), the corrected vector set for the spin singlet is defined to satisfy
\[
\langle \hat{a}_c \otimes \hat{b}_c \rangle = -\vec{a} \cdot \vec{b}.
\]
In other words, when the two measurement directions for particles 1 and 2 are the same, $\vec{a} = \vec{b}$, in the non-relativistic lab frame, the expectation value of the joint spin measurement with the new directions, $(\hat{a}_c, \hat{b}_c)$, in the Lorentz-boosted moving frame should be $-1$. Here, we would like to see what these corrected spin measurement directions are and consider the meaning of these new directions when $\vec{a} = \vec{b} = (0, 0, 1)$. In this case, from (49) $\vec{a}_i, \vec{b}_i$ for $i = x, y, z$ are given by
\[
\{\vec{a}_i\} = (\sin \Omega_p, 0, \cos \Omega_p), \quad \{\vec{b}_i\} = (-\sin \Omega_p, 0, \cos \Omega_p),
\]
and thus from (50) the corrected vectors are given as follows:
\[
a_{cx} = \cos \Omega_p \sqrt{D_a}, \quad a_{cz} = \sin \Omega_p \sqrt{1 + \cos^2 \Omega_p \sinh^2 \eta \left(F_a \sin \theta_{\Lambda} + \cos \theta_{\Lambda}\right)^2}, \quad a_{cy} = 0,
\]
\[
b_{cx} = \cos \Omega_p \sqrt{D_b}, \quad b_{cz} = -\sin \Omega_p \sqrt{1 + \cos^2 \Omega_p \sinh^2 \eta \left(F_a \sin \theta_{\Lambda} + \cos \theta_{\Lambda}\right)^2}, \quad b_{cy} = 0,
\]
where
\[
D_a = \left[F_a(1 + \cosh \eta) \sin \theta_{\Lambda} \cos \theta_{\Lambda} - (\sin^2 \theta_{\Lambda} - \cosh \eta \cos^2 \theta_{\Lambda})\right]^2
- \cos^2 \Omega_p \sinh^2 \eta \left(F_a \sin \theta_{\Lambda} + \cos \theta_{\Lambda}\right)^2,
\]
\[
D_b = \left[F_b(1 + \cosh \eta) \sin \theta_{\Lambda} \cos \theta_{\Lambda} + (\sin^2 \theta_{\Lambda} - \cosh \eta \cos^2 \theta_{\Lambda})\right]^2
- \cos^2 \Omega_p \sinh^2 \eta \left(F_a \sin \theta_{\Lambda} + \cos \theta_{\Lambda}\right)^2,
\]
and
\[
F_a = \frac{(1 + \cosh \eta) \tan \theta_{\Lambda} - \tan \Omega_p (\tan^2 \theta_{\Lambda} - \cosh \eta)}{(1 - \cosh \eta \tan^2 \theta_{\Lambda}) - \tan \Omega_p (1 + \cosh \eta) \tan \theta_{\Lambda}},
\]
\[
F_b = \frac{(1 + \cosh \eta) \tan \theta_{\Lambda} + \tan \Omega_p (\sin^2 \theta_{\Lambda} - \cosh \eta)}{(1 - \cosh \eta \tan^2 \theta_{\Lambda}) + \tan \Omega_p (1 + \cosh \eta) \tan \theta_{\Lambda}}.
\]
with \( \tanh \xi = \beta_p \), \( \tanh \chi = \beta_{\Lambda} \), \( \cosh \eta = \cosh \xi \cosh \chi \), \( \tan \theta_{\Lambda} = \sinh \chi / \tanh \xi \), \( \tan \Omega_p = \sinh \xi \sinh \chi / (\cosh \xi + \cosh \chi) \).

In the limit, when \( \xi \to \infty, \chi \to \infty \), the above result yields after some numerical calculation

\[
F_a \to 0, \quad a_{cz} \to 0, \quad a_{cx} \to 1, \\
F_b \to 0, \quad b_{cz} \to 0, \quad b_{cx} \to -1.
\]

This result tells us that in the highly relativistic limit, when the boost speed reaches the speed of light, both spins become parallel and not anti-parallel since the two spin measurement directions should be opposite in order to maintain the same expectation value \( -1 \) for the joint spin measurement in the moving frame. This agrees with what we expected for spin rotation under Lorentz boost.

Finally, we consider the case that we dealt with at the end of the last section in which the two particles are not moving in opposite directions. Namely, the two particles have arbitrary momenta \( p \) and \( q \) and the observer is at rest in the lab frame. Following the same argument, the corrected vector set should satisfy the following condition:

\[
\langle \hat{a_c} \otimes \hat{b_c} \rangle = -\frac{\vec{A}_c \cdot \vec{B}_c}{|\vec{a}_{cp}| |\vec{b}_{cq}|} = -\vec{a} \cdot \vec{b}.
\]

Here, \( \vec{A}_c, \vec{B}_c \) are given by \( \vec{A}, \vec{B} \) in (34) with \( \vec{a}, \vec{b} \) replaced with \( \vec{a}_c, \vec{b}_c \), respectively. This condition can be split into two conditions

\[
\frac{\vec{A}_c}{|\vec{a}_{cp}|} = \vec{a}, \quad \frac{\vec{B}_c}{|\vec{b}_{cq}|} = \vec{b},
\]

and the result is given for \( \vec{a}_c \) to be

\[
a_{cz} = \frac{a_z}{\sqrt{D_a}},
\]

where

\[
D_a = \left[ F_a (1 + \cosh \xi_p) \sin \theta_p \cos \theta_p - (\sin^2 \theta_p - \cosh \xi_p \cos^2 \theta_p) \right]^2
\]

\[
- a_z^2 \sinh^2 \xi_p (F_a \sin \theta_p + \cos \theta_p)^2,
\]

\[
F_a = \frac{(1 + \cosh \xi_p) \tan \theta_p - (a_z/a_z)(\tan^2 \theta_p - \cosh \xi_p)}{(1 - \cosh \xi_p \tan^2 \theta_p) - (a_z/a_z)(1 + \cosh \xi_p) \tan \theta_p},
\]

and

\[
a_{cx} = a_x \sqrt{1 + a_{cz}^2 \sinh^2 \xi_p (F_a \sin \theta_p + \cos \theta_p)^2},
\]

\[
a_{cy} = a_y \sqrt{1 + a_{cz}^2 \sinh^2 \xi_p (F_a \sin \theta_p + \cos \theta_p)^2} \ldots .
\]

In the case of \( \vec{b}_c, b_{cz} = b_z / \sqrt{D_b} \), and \( \vec{a} \) and \( \vec{p} \) are replaced with \( \vec{b} \) and \( \vec{q} \), respectively, in the expression for \( \vec{a}_c \). One can check that this result reduces to the one from (50) with \( \Omega_p = 0, \eta \to \xi, \tan \theta_{\Lambda} = 0 \), if \( \vec{p} \) and \( \vec{q} \) are in opposite directions with the same magnitude.
5. Corrected Bell observable for the Bell states

In this section, we will find corrected vector sets of spin measurement directions for the remaining Bell states.

The Bell states are defined by [18]

\[
|\Phi^{(+)\,p}\rangle = \frac{1}{\sqrt{2}}(|p, 1/2\rangle - p, 1/2\rangle + |p, -1/2\rangle - p, -1/2\rangle),
\]

\[
|\Phi^{(-)\,p}\rangle = \frac{1}{\sqrt{2}}(|p, 1/2\rangle - p, 1/2\rangle - |p, -1/2\rangle - p, -1/2\rangle),
\]

(54)

\[
|\Psi^{(+)\,p}\rangle = \frac{1}{\sqrt{2}}(|p, 1/2\rangle - p, -1/2\rangle + |p, -1/2\rangle - p, 1/2\rangle),
\]

\[
|\Psi^{(-)\,p}\rangle = \frac{1}{\sqrt{2}}(|p, 1/2\rangle - p, -1/2\rangle - |p, -1/2\rangle - p, 1/2\rangle),
\]

and transform under Lorentz boost as

\[
U(\Lambda)|\Phi^{(+)\,p}\rangle = \cos \Omega_p |\Phi^{(+)\,\Lambda_p}\rangle - \sin \Omega_p |\Psi^{(-)\,\Lambda_p}\rangle,
\]

\[
U(\Lambda)|\Phi^{(-)\,p}\rangle = |\Phi^{(-)\,\Lambda_p}\rangle,
\]

(55)

\[
U(\Lambda)|\Psi^{(+)\,p}\rangle = |\Psi^{(+)\,\Lambda_p}\rangle,
\]

\[
U(\Lambda)|\Psi^{(-)\,p}\rangle = \cos \Omega_p |\Psi^{(-)\,\Lambda_p}\rangle + \sin \Omega_p |\Phi^{(+)\,\Lambda_p}\rangle,
\]

where \(\Omega_p\) is the Wigner angle due to the Lorentz boost \(\Lambda\) performed to a particle with momentum \(\vec{p}\) and is given by (21).

In the non-relativistic case, the expectation values of the joint spin measurement for the Bell states are given by

\[
\langle \Phi^{(+)\,p} | \vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma} | \Phi^{(+)\,p} \rangle = a_x b_x - a_y b_y + a_z b_z,
\]

\[
\langle \Phi^{(-)\,p} | \vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma} | \Phi^{(-)\,p} \rangle = a_x b_x - a_y b_y + a_z b_z,
\]

\[
\langle \Psi^{(+)\,p} | \vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma} | \Psi^{(+)\,p} \rangle = a_x b_x + a_y b_y - a_z b_z,
\]

\[
\langle \Psi^{(-)\,p} | \vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma} | \Psi^{(-)\,p} \rangle = -a_x b_x - a_y b_y - a_z b_z.
\]

(56)

In the relativistic case, we consider the general case that we studied in the previous sections in which particles 1 and 2 are moving in the \(+z\) and \(−z\) directions respectively in the lab frame, and the two observers for particles 1 and 2, Alice and Bob, are Lorentz-boosted to the \(−x\) direction, the expectation values of the joint spin measurement are given by use of the
The corrected vector sets for maximal violation of Bell’s inequality as we have done for the singlet case. First we notice that for the states \(\left| 3/\Phi_{1}^{+}\right\rangle,\ \left| 3/\Psi_{1}^{-}\right\rangle\) the corrected vector set \((\vec{a}_{c},\ \vec{b}_{c})\) should satisfy
\[
\vec{A}_{c}/|\vec{a}_{c}\Lambda_{p}| = R_{y}(\Omega_{p})\vec{a}, \quad \vec{B}_{c}/|\vec{b}_{c}\Lambda_{p}| = R_{y}(-\Omega_{p})\vec{b},
\]
(58) to get the same expectation values of the non-relativistic case such that
\[
\langle \hat{a}_{c} \otimes \hat{b}_{c} \rangle = \begin{cases} 
 a_{x}b_{x} - a_{y}b_{y} + a_{z}b_{z} & \text{for } \left| 3/\Phi_{1}^{+}\right\rangle, \\
 -a_{x}b_{x} - a_{y}b_{y} - a_{z}b_{z} & \text{for } \left| 3/\Psi_{1}^{-}\right\rangle.
\end{cases}
\]
(59)

Next, for the states \(\left| 3/\Phi_{1}^{-}\right\rangle,\ \left| 3/\Psi_{1}^{+}\right\rangle\) the corrected vector set \((\vec{a}_{c},\ \vec{b}_{c})\) should satisfy
\[
\vec{A}_{c}/|\vec{a}_{c}\Lambda_{p}| = \vec{a}, \quad \vec{B}_{c}/|\vec{b}_{c}\Lambda_{p}| = \vec{b},
\]
(60) such that
\[
\langle \hat{a}_{c} \otimes \hat{b}_{c} \rangle = \begin{cases} 
 a_{x}b_{x} - a_{y}b_{y} + a_{z}b_{z} & \text{for } \left| 3/\Phi_{1}^{-}\right\rangle, \\
 a_{x}b_{x} + a_{y}b_{y} - a_{z}b_{z} & \text{for } \left| 3/\Psi_{1}^{+}\right\rangle.
\end{cases}
\]
(61)

Here, \((\vec{A}_{c},\ \vec{B}_{c})\) are the ones with \((\vec{a}_{c},\ \vec{b}_{c})\) instead of \((\vec{a},\ \vec{b})\) in (28).

In this manner, we can find the corrected vector sets for the other Bell states once the original vector sets which induce the maximal violation of Bell’s inequality for each Bell state in the non-relativistic case are given. Once we find the solutions for the equations (58) and (60), then we have the corrected vector sets for the joint spin measurement, which will induce the maximal violation of Bell’s inequality for the remaining Bell states.
6. Conclusions

In this paper, we show that by appropriate rotations of the directions of spin measurement one can achieve maximal violation of Bell’s inequality even in a relativistically moving frame if the state is fully entangled in a non-relativistic lab frame. To do this, we first define the relativistic spin observable which we use for the spin measurement in an arbitrary Lorentz boosted inertial frame. With this relativistic spin observable we evaluate the expectation values of the joint spin measurement for the Bell states in a Lorentz-boosted frame. For the spin-singlet case, the expectation value evaluated in the lab frame in which the two particles are moving in the same direction agrees exactly with Czachor’s result.

To measure the degree of violation of Bell’s inequality, we then evaluate the so-called Bell observable for the Bell states. The degree of violation decreases under Lorentz boost. However, this is the case when one evaluates the Bell observable with the same spin measurement directions as in the non-relativistic lab frame. In fact, we show that Bell’s inequality is still maximally violated in a Lorentz-boosted frame, if we properly choose new set of spin measurement directions. We show this following the reasoning of Terashima and Ueda [5] for all the Bell states.

In the non-relativistic case, maximal violation of Bell’s inequality implies full entanglement of a given state. Thus we may infer that the restoration of maximal violation of Bell’s inequality in a Lorentz-boosted frame indicates the preservation of the entanglement information in a certain form even under Lorentz boost.

We check this idea by investigating how the EPR correlation of the spin singlet whose spins are up and down in the \( z \)-direction changes under a ultra-relativistic Lorentz boost that reaches the speed of light in the \( x \)-direction. As we discussed in section 4, the new spin measurement directions which give the maximal violation of Bell’s inequality, i.e., which preserve the expectation value of the joint spin measurement, become the \(+x\) and \(−x\) directions when the original directions for the joint spin measurement in the lab frame are both in the \(+z\) direction. Specifically, the perfect anti-correlation of the spin singlet becomes the perfect correlation under an ultra-relativistic Lorentz boost perpendicular to the original spin directions.

However, one can also check that for the unchanged Bell states under Lorentz boost such as \( |\Phi^−⟩ \), \( |\Psi^+⟩ \) in (54), the form of correlation does not change.

We thus conclude that the entanglement information is preserved as a form of correlation information determined by the transformation characteristic of the Bell state in use.

Finally, we would like to note that even though the Lorentz boost is not unitary, our relativistic spin observable (7) in section 2 is determined by the rotation from the \( z \)-axis to the direction of the particle’s momentum \( \vec{p} \). From this and the fact that the spin state is transformed by Wigner rotation which is unitary, we can infer that the restoration of the maximal violation of Bell’s inequality by adjusting the measurement axes might be expected from the unitarity of the rotation. This is in a sense similar to the cases of [2, 5] where the unitarity of the spin-state transformation given by Wigner rotation ensures the same result, since their spin observables are not changed.

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