Conjectured lower bound for the clique number of a graph

Clive Elphick*  Pawel Wocjan†

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Abstract

It is well known that $n/(n - \mu)$, where $\mu$ is the spectral radius of a graph with $n$ vertices, is a lower bound for the clique number. We conjecture that $\mu$ can be replaced in this bound with $\sqrt{s^+}$, where $s^+$ is the sum of the squares of the positive eigenvalues. We prove this conjecture for various classes of graphs, including triangle-free graphs, and for almost all graphs.

1 Introduction

Let $G$ be a graph, with no isolated vertices, with $n$ vertices, edge set $E$ with $|E| = m$, average degree $d$, chromatic number $\chi(G)$ and clique number $\omega(G)$. We also let $A$ denote the adjacency matrix of $G$ and let $\mu = \mu_1 \geq \ldots \geq \mu_n$ denote the eigenvalues of $A$. The inertia of $A$ is the ordered triple $(\pi, \nu, \gamma)$ where $\pi$, $\nu$ and $\gamma$ are the numbers counting multiplicities of positive, negative and zero eigenvalues of $A$ respectively. Let

$$s^+ = \sum_{i=1}^{\pi} \mu_i^2 \quad \text{and} \quad s^- = \sum_{i=n-\nu+1}^{n} \mu_i^2.$$  

Note that:

$$\sum_{i=1}^{n} \mu_i^2 = \text{tr}(A^2) = 2m = s^+ + s^-.$$  

2 Replacing $\mu^2$ with $s^+$

Edwards and Elphick [5] proved that

$$\frac{2m}{2m - \mu^2} \leq \chi(G)$$
and Ando and Lin [1] proved a conjecture due to Wocjan and Elphick [15] that 
\[ \frac{2m}{2m-s^+} = 1 + \frac{s^+}{s^-} \leq \chi(G). \]

As another example of replacing \( \mu^2 \) with \( s^+ \), Hong [9] proved for graphs with no isolated vertices that \( \mu^2 \leq 2m - n + 1 \), and Elphick et al [6] proved that for almost all connected graphs \( s^+ \leq 2m-n+1 \). Similarly Favaron et al [7] proved that \( \omega(G) \leq 2m/\mu \) and Wu and Elphick [16] proved the doubly stronger result that \( \chi(G) \leq 2m/\sqrt{s^+} \).

Finally Stanley [12] proved that 
\[ \mu \leq \frac{\sqrt{8m+1} - 1}{2}, \]

and Wu and Elphick [16] proved that 
\[ \sqrt{s^+} \leq \frac{\sqrt{8m+1} - 1}{2}. \]

So in all of these cases we can strengthen known bounds by replacing \( \mu^2 \) with \( s^+ \).

The next section considers the same replacement for a well known lower bound for the clique number.

### 3 Conjectured bound for the clique number

The concise version of Turán’s theorem states that:
\[ \frac{n}{n-d} \leq \omega(G). \]  
(1)

This bound was improved by Caro [4] and Wei [13] using degrees as follows:
\[ \sum_{i=1}^{n} \frac{1}{n-d_i} \leq \omega(G); \]

and by Wilf [14] using the spectral radius as follows:
\[ \frac{n}{n-\mu} \leq \omega(G). \]  
(2)

Bound (2) was strengthened by Nikiforov [11] who proved the following conjecture of Edwards and Elphick [5].
\[ \frac{2m}{2m-\mu^2} \leq \omega(G). \]  
(3)

Note that for regular graphs, all of these bounds equal \( n/(n-d) \). Wocjan and Elphick [15] noted that 
\[ \frac{2m}{2m-s^+} \notin \omega(G). \]
An alternative strengthening of Wilf’s bound is provided by the following conjecture, which we have tested against the thousands of named graphs with up to 40 vertices in the Wolfram Mathematica database, and found no counter-example. Aouchiche [2] has tested this conjecture using his powerful AGX software, and also found no counter-example. Conjecture 1 exceeds \( \frac{n}{n - \sqrt{s^+}} \) for all regular graphs with more than one positive eigenvalue.

**Conjecture 1.** For any graph \( G \)

\[
\frac{n}{n - \sqrt{s^+}} \leq \omega(G).
\]

This conjecture is exact, for example, for complete regular multipartite graphs.

We can prove this conjecture for the following classes of graphs.

### 3.1 Proof for triangle-free graphs

**Proof.** Let \( t \) denote the number of triangles in a graph. It is well known that:

\[
\sum_{i=1}^{n} \mu_i^3 = \text{tr}(A^3) = 6t,
\]

so for triangle-free graphs

\[
\sum_{i=1}^{\pi} \mu_i^3 = \sum_{i=n-\nu+1}^{n} \mu_i^3.
\]

Therefore, using that \( \mu \geq |\mu_n| \)

\[
s^- \geq \sum_{i=n-\nu+1}^{n} \frac{\mu_i^3}{\mu_n} \geq \sum_{i=1}^{\pi} \frac{\mu_i^3}{|\mu_n|} \geq \frac{\mu^3}{|\mu_n|} \geq \mu^2.
\]

Therefore, using the lower bound on the largest eigenvalue \( \mu \geq 2m/n \), the equality \( \frac{1}{2}(s^+ + s^-) = m \) combined with the arithmetic-geometric-mean inequality, and the above lower bound on \( s^- \), we obtain

\[
\sqrt{s^+} \leq \mu \frac{n}{2m} \sqrt{s^+} \leq \frac{n}{2m} \sqrt{s^-} \sqrt{s^+} \leq \frac{n}{2m} \sqrt{2m} = \frac{n}{2}.
\]

\[\square\]

### 3.2 Proof for weakly perfect graphs

**Proof.** Weakly perfect graphs have \( \omega(G) = \chi(G) \). Therefore using the result due to Ando and Lin [1] discussed above and that \( \mu \geq 2m/n \):

\[
\frac{n}{n - \sqrt{s^+}} \leq \frac{2m}{2m - s^+} \leq \chi(G) = \omega(G).
\]

\[\square\]
3.3 Proof for some strongly regular graphs

We do not know how to prove this conjecture for all strongly regular graphs. However we can prove the conjecture for the subset of strongly regular graphs which are Kneser graphs. The Kneser graph $KG_{p,k}$ is the graph whose vertices correspond to the $k$-element subset of a set of $p$ elements, and where two vertices are joined if and only if the corresponding sets are disjoint. The Kneser graphs with $k = 2$ are strongly regular, with only three distinct eigenvalues. For these graphs

$$n = \binom{p}{2}, \omega = \left\lfloor \frac{p}{2} \right\rfloor, 2m = \binom{p}{2} \left( \frac{p-2}{2} \right) \quad \text{and} \quad p \geq 2k = 4.$$

The eigenvalues (see Godsil and Royle [8]) are:

$$(-1)^i \binom{p-2-i}{2-i} \text{ with multiplicity } \binom{p}{i} - \binom{p}{i-1}, \text{ for } i = 0, 1, 2.$$

We are seeking to prove that

$$\frac{n}{n - \sqrt{s^+}} \leq \frac{p - 1}{2} \leq \frac{p}{2} = \omega(KG_{p,2}),$$

which re-arranges to

$$s^+ = 2m - s^- \leq \frac{n^2(p-3)^2}{(p-1)^2} = \frac{p^2(p-3)^2}{4}.$$

Inserting the negative eigenvalues this becomes

$$\binom{p}{2} \left( \frac{p-2}{2} \right) - (p-1) \binom{p-3}{1}^2 \leq \frac{p^2(p-3)^2}{4}.$$

Simple algebra reduces this to

$$2p^2 - 9p + 6 \geq 0$$

which is true for all $p \geq 4$.

3.4 Proof for almost all graphs

Proof. We use the Erdos-Renyi random graph model $G_n(p)$, which consists of all graphs with $n$ vertices in which edges are chosen independently with probability $p$. Bollobás and Erdos [3] proved that the clique number is almost always $x$ or $x+1$ where

$$x = \frac{2 \log n}{\log (1/p)} + O(\log \log n).$$

Since almost all graphs have all degrees very close to $n/2$ we let $p = 0.5$. Therefore

$$s^+ \leq s^+ + s^- = 2m \approx n \frac{n}{2}.$$

So for almost all graphs
\[ \frac{n}{n - \sqrt{s^+}} \leq \frac{n}{n - n/\sqrt{2}} \approx 3.4 < \frac{2 \log n}{\log 2} \approx \omega(G). \]

\[ \square \]

4 Conclusion

Lower bounds for the clique number are often proved using the Motzkin-Straus [10] inequality, which can be expressed as follows. For any adjacent vertices \( i \) and \( j \) such that \( i < j \) we write \( i \sim j \). Then for any vector \((p_1, \ldots, p_n)\) with \( p_i \geq 0 \) for all \( i \) and \( \sum_{i=1}^{n} p_i = 1 \):

\[ \sum_{i \sim j} p_ip_j \leq \frac{\omega - 1}{2\omega}. \]

It is however not evident how to use this approach in the context of Conjecture 1, where the number of positive eigenvalues varies greatly between graphs with \( n \) vertices.

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