WICK RAYTION, REGULARIZATION OF PROPAGATORS BY A COMPLEX METRIC AND MULTIDIMENSIONAL COSMOLOGY

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The Wick rotation in quantum field theory is considered in terms of analytical continuation in the signature matrix parameter \( w = \eta_{00} \). Regularization of propagators by a complex metric parameter in most cases preserves (i) the convergence of Feynmann integrals (understood as Lebesgue integrals) if the corresponding integrals of Euclidean theory are convergent; (ii) the regularity of propagators in the coordinate representation if there is regularity in the Euclidean case. The well-known covariant regularization by a complex mass does not in general satisfy these conditions. Theories with a large family of propagators regularized by complex metric were previously considered by the author, and analogues of the Bogoliubov-Parasiuk-Hepp-Zimmermann theorems were proved. This paper shows that in the case of multidimensional cosmology describing the evolution of \( n \) spaces \( M_i, i = 1, \ldots, n \), the Wick rotation in the minisuperspace may be performed by analytical continuation in the dimensions \( N_i = \text{dim} M_i \) or in the dimension of the time submanifold \( M_0 \).

1. Introduction

The divergences occurring in quantum field theory (QFT) and quantum gravity [1–9] (when certain perturbation schemes are considered) may be divided into two groups. The first one is related to the fact that series in coupling constants in QFT usually diverge (they are asymptotical ones [2]). (Recently a new perturbation theory with absolutely convergent series in coupling constants in QFT usually diverge (they are asymptotical ones [2]).) Divergences from the second group are the so-called “infrared” and “ultraviolet” divergences in Feynmann integrals. In renormalizable theory they may be removed by applying certain R-operation procedures [1, 4–7] (with or without intermediate regularization). Some of the latter may be called “pseudo-Euclidean” [14, 15] since they occur in space-times with non-Euclidean signatures. Let us give a simple example. Consider the covariantly regularized propagator of free massive scalar field in \( \mathbb{R}^D \):

\[
\frac{i}{-p_M^2 - m^2 + i\varepsilon}, \quad \varepsilon > 0,
\]

where \( p_M^2 = -p_0^2 + \vec{p}^2 \), \( m^2 > 0 \), then the regularization (1.1) does not guarantee the convergence of Feynmann integrals (understood as Lebesgue integrals) even if the corresponding integrals of Euclidean theory are convergent [14] (see also Subsec. 2.2.4 below). The covariant regularization does not guarantee the propagator regularity in the coordinate representation even if they are regular in the Euclidean case. For \( m^2 > 0, \varepsilon > 0, D \geq 4 \) the Fourier image of (1.1) in the Schwarz space \( S' (\mathbb{R}^D) \) is not a regular distribution [14] (see also Subsec. 2.2.3 below).

There exists an alternative scheme of propagator regularization [13–15] (see also [16–19]). This scheme is free of the above disadvantages and is based on the complex metric regularization. In this scheme we deal with a complex metric on \( \mathbb{R}^D \)

\[
\eta_{ab} (w) dx^a \otimes dx^b = w dt \otimes dt + \sum_{i=1}^{D-1} dx^i \otimes dx^i, \quad (1.2)
\]

\( w \neq 0 \). For non-negative \( w \) the metric (1.2) provides regularization of (singular) Green functions (corresponding to the Minkowski space limit \( w \to -1 + i0 \)). For example, the massive scalar field propagator regularized by the complex metric (1.2) has the form [13]

\[
D(p, w, m) = \frac{w^{-1/2}}{w^{-1}p_0^2 + \vec{p}^2 + m^2}. \quad (1.3)
\]

Thus \( w \) may be considered as a holomorphicity parameter (Wick parameter) for \( w \)-correlators belonging to \( S' (\mathbb{R}^D) \). It should be noted that recently the interest to this problem was greatly stimulated by the paper of J. Greensite [17] (see also [18]).

The aim of this paper is to overview the author’s earlier results concerning the regularization of propagators by a complex metric ([13–15]) and to show a possible application of this scheme to multidimensional quantum cosmology.
The plan of this note is as follows. Sec. 2 considers the complex metric regularization in QFT. Sec. 3 discusses a gravitational model on the manifold $M_0 \times M_1 \times \ldots \times M_n$. The midisuperspace metric depends on the dimensions $N_0 = \dim M_0$ and $N_i = \dim M_i$ and may be used to regularize the cosmological pseudo-Euclidean minisuperspace corresponding to $N_0 = 1$ in two ways: (i) $N_0$ is complex and $N_i$ are real; (ii) $N_0 = 1$ and $N_i$ are complex ($i > 0$).

2. Wick rotation in quantum field theory

2.1. General prescription

We consider the Wick rotation in QFT in terms of analytical continuation of the signature parameter. Let

$$\eta_{ab}(w) = \text{diag}(w, 1, \ldots, 1)$$

be a diagonal $D \times D$-matrix, $a, b = 0, \ldots, D - 1$, $D \geq 2$, where

$$w \in \mathbb{C} \setminus (-\infty, 0] \equiv \Omega$$

is a complex parameter (Wick parameter).

The point $w = 1$ corresponds to Euclidean space (E). As will be shown below, the limits $w \to -1$ or $0$ correspond to the Minkowski spaces $M_{\pm}$ with a “right” or “wrong” direction (arrow) of time, respectively.

In [14, 15] we used the notation

$$w = -a^{-1} = -\exp(-i\varepsilon).$$

Consider a self-interacting scalar field with the action

$$S(\varphi, w) = \int d^D x \sqrt{\eta(w)} \left\{ \frac{1}{2} \eta^{ab}(w) \partial_a \varphi \partial_b \varphi + V(\varphi) \right\},$$

(2.4)

in the complex metric background (2.1), where $V(\varphi) \geq 0$ is a potential,

$$\eta^{ab}(w) = \text{diag}(w^{-1}, 1, \ldots, 1)$$

is the matrix inverse to (2.1) and

$$\eta(w) \equiv \det(\eta_{ab}(w)) = w, \quad w \in \Omega.$$  

(2.5)

(2.6)

The action (2.4) is an analytical continuation of the Euclidean action, (covariantly) defined for $w > 0$, to the domain $\Omega$ (2.2).

From (2.4)-(2.6) we get

$$S(\varphi, w) = \int d^D x \left\{ \frac{1}{2} w^{-1/2}(\partial_0 \varphi)^2 + \frac{1}{2} w^{1/2}(\partial_i \varphi)^2 + w^{1/2}V(\varphi) \right\},$$

(2.7)

$w \in \Omega$. The real part of (2.7) satisfies the relations

$$\text{Re} S(\varphi, w) = \cos \left( \frac{1}{2} \arg w \right) \int d^D x \left\{ \frac{1}{2} |w|^{-1/2}(\partial_0 \varphi)^2 + \frac{1}{2} |w|^{1/2}(\partial_i \varphi)^2 + |w|^{1/2}V(\varphi) \right\} \geq c_1(w)S_E(\varphi),$$

(2.8)

where

$$S_E(\varphi) = \int d^D x \left\{ \frac{1}{2} \delta^{ab} \partial_a \varphi \partial_b \varphi + V(\varphi) \right\}$$

(2.9)

is the Euclidean action,

$$c_1(w) \equiv \cos \left( \frac{1}{2} \arg w \right) \min(|w|^{-1/2}, |w|^{1/2})$$

(2.10)

and $\arg w \in (-\pi, \pi)$ is the argument of the complex number $w$.

The action (2.4) generates a chain of $w$-correlators

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle(w) = \mathcal{D}\mu(\varphi, w) \varphi(x_1) \ldots \varphi(x_n),$$

(2.11)

$$\mathcal{D}\mu(\varphi, w) = \frac{\mathcal{D}\varphi \exp(-S(\varphi, w))}{\int \mathcal{D}\varphi \exp(-S(\varphi, w))},$$

(2.12)

for $w \in \mathbb{R}$ and $w \in \Omega$. In (2.11) and (2.12), the path integrals are understood in the framework of a certain perturbation scheme and $R$-operation.

Remark 1. Due to the relations

$$|e^{-S(\varphi, w)}| = e^{-\text{Re} S(\varphi, w)} \leq e^{-c_1(w)S_E(\varphi)},$$

(2.13)

following from (2.8), one may define for certain $S(\varphi, w)$ the “complex-valued measure” (charge) (2.12) (at least perturbatively) for all $w \in \Omega$ if the corresponding Euclidean measure $\mathcal{D}\varphi \exp(-cS_E(\varphi))$ exists for any $c > 0$.

2.2. Free massive scalar field

To illustrate the approach let us consider in detail the case of a free massive scalar field (vector, spinor, etc. fields may be certainly considered as well [14]), i.e. the potential in (2.4) is

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2, \quad m > 0.$$  

(2.14)

From (2.7), (2.11), (2.12) and (2.14) we get the formal expression for the two-point $w$-correlator

$$\langle \varphi(x) \varphi(0) \rangle(w, m) = \int \frac{d^D p}{(2\pi)^D} e^{-ipx} D(p, w, m)$$

(2.15)

where $w \in \Omega$ and $D(p, w, m)$ is defined in (1.3). Here and below

$$p = (p_a), \quad x = (x^a) \in \mathbb{R}^D, \quad px = p_a x^a.$$  

(2.16)

The relation (2.15) should be understood as

$$\langle \varphi(x) \varphi(0) \rangle(w, m) = \mathcal{F}_p^{-1}[D(p, w, m)](x)$$

(2.17)
where $\mathcal{F}_p^{-1} = \mathcal{F}^{-1}$ is the inverse Fourier transformation in $\mathcal{S}'(\mathbb{R}^D)$, i.e. in the (dual) Schwarz space of generalized functions (tempered distributions). Recall that the mapping $\mathcal{F} : \mathcal{S}'(\mathbb{R}^D) \to \mathcal{S}'(\mathbb{R}^D)$ is the Fourier transformation in the Schwarz space $\mathcal{S}(\mathbb{R}^D)$. The space $\mathcal{S}(\mathbb{R}^D)$ consists of smooth functions $f : \mathbb{R}^D \to \mathbb{C}$ satisfying the relation

$$p_{n,\alpha}(\varphi) = \max\{(1 + x_k^2)^n |D^\alpha \varphi|\} < +\infty$$

(2.20)

for all $n \in \mathbb{N}$ and $\alpha = (\alpha_0, \ldots, \alpha_{D-1}) \in \mathbb{Z}_+^D$. Here $\mathcal{Z}_+ = \{0\} \cup \mathbb{N}$, $D^\alpha = \partial_{\alpha_0} \ldots \partial_{\alpha_{D-1}}$, and $\partial_i = \partial / \partial x^i$. In (2.20) and below

$$x_k^2 = \delta_{ab} x^a x^b, \quad p_k^2 = \delta_{ab} p^a p^b. \quad (2.21)$$

The Schwarz space $\mathcal{S}(\mathbb{R}^D)$ is a locally convex linear topological space over $\mathbb{C}$, with a topology generated by the set of seminorms (2.20) [23].

Now we prove majorizing inequalities for the $w$-correlator that will play a key role in what follows.

**Proposition 1.** For any $m > 0$, $w \in \Omega$ and $p = (p_n) \in \mathbb{R}^D$

$$c_1^{-1}(w) \leq |D(p, w, m)| \leq c_2^{-1}(w), \quad (2.22)$$

where $c_1(w)$ is defined in (2.10) and

$$c_2(w) = \max(|w|^{-1/2}, |w|^{1/2}). \quad (2.23)$$

**Proof.** Denote

$$X = (D(p, w, m))^{-1} = w^{-1/2}\rho_0^2 + w^{1/2}(\rho^2 + m^2). \quad (2.24)$$

It is clear that

$$|X| \leq c_2(w)(\rho_0^2 + m^2). \quad (2.25)$$

We also get:

$$|X| \geq \text{Re} X = \cos\left(\frac{\pi}{4}\arg w\right)[|w|^{-1/2}\rho_0^2 + |w|^{1/2}(\rho^2 + m^2)] \geq c_1(w)(\rho_0^2 + m^2). \quad (2.26)$$

The proposition is proved.

We denote by $\mathcal{F}_s(\mathbb{R}^D, \mathbb{C})$ the vector space of functions $f : \mathbb{R}^D \to \mathbb{C}$, such that the Lebesgue integral

$$\int d^D p \, f(p)(\rho_0^2 + 1)^{-n}$$

(2.27)

exists for some $n \in \mathbb{N}$ (in this case the function $f$ is measurable). The function $f \in \mathcal{F}_s(\mathbb{R}^D, \mathbb{C})$ generates the “slowly increasing” measure $d\mu(p) = |f(p)| d^D p \quad [23]$. Let

$$I : \mathcal{F}_s(\mathbb{R}^D, \mathbb{C}) \longrightarrow \mathcal{S}'(\mathbb{R}^D) \quad (2.28)$$

be a canonical embedding defined by the relation

$$\langle I(f), \varphi \rangle = \int d^D p \, f(p)\varphi(p). \quad (2.29)$$

We call the image of the map (2.28) the subspace of regular tempered distributions (generalized functions) and denote

$$\text{Im}I = I(\mathcal{F}_s(\mathbb{R}^D, \mathbb{C})) \equiv \text{reg} \mathcal{S}'(\mathbb{R}^D). \quad (2.30)$$

**Proposition 2.** For any $w \in \Omega, m > 0$,

$$D(\cdot, w, m) \in \mathcal{F}_s(\mathbb{R}^D, \mathbb{C}) \quad (2.31)$$

(see (1.3)) and the corresponding regular distribution

$$D(w, m) \equiv I(D(\cdot, w, m)) \in \text{reg} \mathcal{S}'(\mathbb{R}^D) \quad (2.32)$$

is (weakly) holomorphic with respect to $w$ in $\Omega$ (for fixed $m > 0$), i.e. $\langle D(w, m), \varphi \rangle$ is holomorphic with respect to $w$ in $\Omega$ for any $\varphi \in \mathcal{S}(\mathbb{R}^D)$ and fixed $m > 0$.

**Proof.** For fixed $m > 0$ and $w \in \Omega$ the function $D(\cdot, w, m)$ is smooth on $\mathbb{R}^D$ and hence measurable. The relation (2.31) follows from the right inequality in (2.22). The holomorphic behaviour of the integral

$$\langle D(w, m), \varphi \rangle = \int d^D p \, D(p, w, m)\varphi(p). \quad (2.33)$$

follows from the relation

$$\frac{\partial}{\partial w} \langle D(w, m), \varphi \rangle = \int d^D p \left(\frac{\partial}{\partial w} D(p, w, m)\right)\varphi(p) \quad (2.34)$$

($\varphi \in \mathcal{S}(\mathbb{R}^D)$) that can be easily verified by a straightforward calculation using certain uniform estimates in the sectors

$$\Omega_{r, R, \delta} = \{w : r < |w| < R, \quad |\arg w| < \pi - \delta\}, \quad 0 < r < R, \quad 0 < \delta < \pi, \quad (2.35)$$

following from Proposition 1. In this case

$$\left(\partial / \partial w\right) D(\cdot, w, m) \in \mathcal{F}_s(\mathbb{R}^D, \mathbb{C}) \quad (2.36)$$

and in $\mathcal{S}'(\mathbb{R}^D)$

$$\frac{\partial}{\partial w} D(w, m) = I \left(\frac{\partial}{\partial w} D(\cdot, w, m)\right) \quad (2.36)$$

for $w \in \Omega, m > 0$. 

Wick Rotation, Regularization of Propagators by a Complex Metric and Multidimensional Cosmology
Coordinate representation

The calculation of the inverse Fourier transformation (2.17) gives us the following expression for the two-point $w$-correlator:

$$
\langle \phi(x)\phi(0)\rangle(w,m) = (2\pi)^{-D/2} \left[ \frac{m^2}{x^2(w)} \right]^{\nu/2} K_\nu \left( m\sqrt{x^2(w)} \right)
$$

(2.37)

where $\nu = \nu(D) = D/2 - 1$, $m^2 > 0$, $w \in \Omega$ and

$$
x^2(w) \equiv \eta_{ab}(w)x^a x^b = w(x^0)^2 + \vec{x}^2.
$$

(2.38)

For any $w \in \Omega$ and $m > 0$

$$
\tilde{D}(w,m) \equiv \mathcal{F}^{-1}(D(w,m)) = I(\langle \phi(.)\phi(0)\rangle(w,m)) \in \text{reg } S'(\mathbb{R}^D)
$$

(2.39)

is a regular distribution, generated by the function of slowly increasing measure

$$
\langle \phi(.)\phi(0)\rangle(w,m) \in \mathcal{F}_s(\mathbb{R}^D, \mathcal{C}).
$$

(2.40)

The inclusion (2.39) follows from the asymptotical relations

$$
\langle \phi(x)\phi(0)\rangle(w,m) \sim A_D [x^2(w)]^{-\nu}, \quad D > 2,
$$

(2.41)

$$
\sim A_2 \ln[m^2 x^2(w)], \quad D = 2
$$

(2.42)

as $x_E^2 \to +0$, and

$$
\langle \phi(x)\phi(0)\rangle(w,m) \sim B_D [x^2(w)]^a \exp[-m\sqrt{x^2(w)}],
$$

(2.43)

as $x_E^2 \to +\infty$, and the inequalities

$$
c_1(w^{-1})x_E^2 \leq |x^2(w)| \leq c_2(w^{-1})x_E^2.
$$

(2.44)

Here $A_D$, $B_D$ and $a = a_D$ are constants and $c_i(w) = c_i(w^{-1})$, $i = 1, 2$ are defined in (2.10) and (2.23).

The relation (2.37) may be obtained by the following three steps. For $w = 1$ (i.e. in the Euclidean case) we use the well-known Euclidean formula

$$
\langle \phi(x)\phi(0)\rangle(w,m) = \frac{1}{(2\pi)^{D/2}} \left[ \frac{m^2}{x_E^2} \right]^{\nu/2} K_\nu \left( m\sqrt{x_E^2} \right).
$$

(2.45)

Then, performing the dilation $x^0 \to w^{1/2} x^0$, we get the correlator (2.37) for $w > 0$. (Note that this dilation generates well-defined endomorphisms of the linear topological spaces $S(\mathbb{R}^D)$ and $S'(\mathbb{R}^D)$.) The distribution $\tilde{D}(w,m)$ holomorphically depends on the parameter $w$ in the domain $\Omega$. It follows from the holomorphic $w$-dependence of $D(w,m)$ in $\Omega$ and the fact that the Fourier transformations $\mathcal{F}$ and $\mathcal{F}^{-1}$ in $S'(\mathbb{R}^D)$ preserve weak holomorphicity. The right-hand side of Eq. (2.37) defines for $w \in \Omega$ a regular distribution from $S'(\mathbb{R}^D)$ holomorphically depending on $w$ in $\Omega$. This implies that the relation (2.37) may be extended from $\mathbb{R}_+$ to the domain $\Omega$.

Proper-time representation

For the $w$-correlator (1.3) we get the following proper-time representation (a-representation $[1,4,10]$):

$$
D(p,w,m) = \int_0^{+\infty} d\alpha e^{-\alpha w^{1/2} [p^2(w) + m^2]},
$$

(2.46)

for $w \in \Omega$, $m > 0$ and $p \in \mathbb{R}^D$. Here

$$
p^2[w] \equiv \eta^{ab}(w)p_ap_b = w^{-1}(p^0)^2 + (\vec{p})^2.
$$

(2.47)

Just as in [1,10], we may also consider some general class of the so-called proper $w$-correlators. A proper $w$-correlator in the momentum representation has the form

$$
\langle \Phi_i \Phi_j \rangle(p,w,m) = P_{ij}(p,w,m)
$$

$$
\times \int_0^{+\infty} d\alpha f(\alpha w^{1/2}) e^{-\alpha w^{1/2} [p^2(w) + m^2]},
$$

(2.48)

Here $w \in \Omega$, $m > 0$, $p \in \mathbb{R}^D$ and $i,j = 1,\ldots,N$ are indices (e.g. vector, spinor, etc). Besides,

(A) The function $f(\alpha)$ is holomorphic in the domain $\{ \Re \alpha > 0 \}$ and continuous in $\{ \Re \alpha \geq 0 \} \setminus \{ 0 \};$

(B) $f(\alpha) = O(\alpha^T)$ for $\alpha \to \infty$, $\Re \alpha \geq 0;$

(C) there exists $s > -1$ such that, for all $\delta > 0$, $f(\alpha) = O(\alpha^{s-\delta})$ for $\alpha \to 0;$

(D) all $P_{ij}(p,w,m)$ are polynomials in momenta $p = (p_i)$ with coefficients holomorphically depending on $w$ in $\Omega$ (and $P_{ij}(p,w,m)$ are “$w$-covariant”).

The proper $w$-correlators extended to the case $m \geq 0$ form a rather wide class of $w$-correlators that occur in QFT.

The Minkowski space limit, $w \to -1 \pm i0$

For the distributions $D(w,m) \in \text{reg } S'(\mathbb{R}^D)$ from (2.32) the limits $w \to -1 \pm i0$ exist and are covariant distributions from $S'(\mathbb{R}^D)$ defined by the relation

$$
D(-1 \pm i0,m) = \mp \frac{i}{2m^2} [2 - \hat{A}L_{\pm}(m)],
$$

(2.49)

where

$$
\hat{A} = p_a \partial / \partial p_a,
$$

(2.50)

is a continuous operator in $S'(\mathbb{R}^D)$ and

$$
L_{\pm}(m) \in \text{reg } S'(\mathbb{R}^D)
$$

(2.51)

are regular distributions generated by the functions from $\mathcal{F}_s(\mathbb{R}^D, \mathcal{C})$

$$
L_{\pm}(p,m) = \ln |p_M^2 + m^2| \mp i\pi\theta(-p_M^2 - m^2)
$$

$$
= \ln((-1 \mp i0)p_M^2 + \vec{p}^2 + m^2)
$$

$$
= \ln|p_M^2 + m^2 \mp i0|,
$$

(2.52)
Here

\( p^2_M + m^2 \neq 0 \). Here

\[ p^2_M \equiv \eta^{ab}(1)p_ap_b = -p^2 + \vec{p}^2. \tag{2.53} \]

(For \( p^2_M + m^2 = 0 \) we put \( L_\pm(p, m) = 0 \).)

The relations (2.49) follow from the identity

\[
D(p, w, m) = \frac{w^{-1/2}}{2m^2} \left[ 2 - \mu \frac{\partial}{\partial \mu} \ln(\eta^{ab}(w)p_ap_b + m^2) \right],
\]

\[ w \in \Omega, (m > 0) \quad p \in \mathbb{R}^D. \]

Consider the function from \( F_\alpha(\mathbb{R}^D, \mathbb{C}) \)

\[
D^\text{cov}_\pm(p, \varepsilon, m) = \mp \frac{i}{p^2_M + m^2 + i\varepsilon}, \tag{2.55}
\]

\( \varepsilon > 0, m > 0 \). Using the representation

\[
D^\text{cov}_\pm(p, \varepsilon, m) = \mp \frac{i}{2(m^2 + i\varepsilon)} \left[ 2 - \mu \frac{\partial}{\partial \mu} \ln(p^2_M + m^2 + i\varepsilon) \right], \tag{2.56}
\]

(\( \varepsilon > 0 \)), we see that the covariant regular distributions

\[ D^\text{cov}_\pm(\varepsilon, m) \in \text{reg} S'(\mathbb{R}^D) \]

corresponding to (2.56) have limits as \( \varepsilon \to +0 \), coinciding with (2.49):

\[ D^\text{cov}_\pm(0, m) = D(-1 \pm i0, m). \tag{2.57} \]

**Remark 2.** Performing differentiation in (2.49) and using the well-known relations in \( S'(\mathbb{R}) \) (see e.g. [22]

\[
(\ln|x|)' = \frac{1}{x}, \quad x \frac{1}{x} = 1, \quad \theta'(x) = \delta(x) = \delta(-x),
\]

we obtain:

\[
D(p, -1 \pm i0, m) = \mp \frac{i}{p^2_M + m^2 + i0} = \mp \left[ \frac{1}{p^2_M + m^2} \pm i\pi\delta(p^2_M + m^2) \right], \tag{2.58}
\]

that agrees with the well-known Sokhotski relation.

In the coordinate representation the free massive scalar field propagator has the following form:

\[
\langle 0|T_+(\varphi(x)\varphi(0))|0\rangle(m) = F_p^{-1}\left[ \frac{-i}{p^2_M + m^2 - i0} \right](x). \tag{2.59}
\]

Analogously,

\[
\langle 0|T_-(\varphi(x)\varphi(0))|0\rangle(m) = F_p^{-1}\left[ \frac{i}{p^2_M + m^2 + i0} \right](x). \tag{2.60}
\]

Here \( T_+(\ldots) \) and \( T_-(\ldots) \) are chronologically and antichronologically ordered operator products, respectively \( (F_p^{-1} = F^{-1}) \) is the inverse Fourier transformation in \( S'\left(\mathbb{R}^D\right)\). \( \forall \)From (2.17), (2.57), (2.59), (2.60) we get:

\[
\langle 0|T_\pm(\varphi(x)\varphi(0))|0\rangle(m) = \langle \varphi(x)\varphi(0) \rangle(-1 \pm i0, m). \tag{2.61}
\]

Thus the limits \( w = -1 \pm i0 \) in 2-point \( w \)-correlators correspond to propagators in Minkowski space with the “right” and “wrong” time directions.

In Ref. [3] we discussed a Feynmann integral corresponding to an arbitrary connected diagram (graph) for the theory with the proper \( w \)-correlator (2.48) (with \( w = -e^{-i\varepsilon} \)).

Analogues of the Bogoliubov-Parasiuk-Hepp-Zimmermann theorems were proved, namely:

(i) the Feynman integral corresponding to an arbitrary connected diagram renormalized in the \( \alpha \)-representation exists (as a Lebesgue integral) for all \( 0 < \varepsilon < 2\pi \) and

(ii) the corresponding generalized function (distribution) of external momenta has a limit as \( \varepsilon \to +0 \) in the appropriate Schwartz space. This limit is a covariant distribution.

**Covariant regularization**

For comparison let us consider the covariant regularization of the propagator (2.55) \( D^\text{cov}_\pm(p, \varepsilon, m) \). This regularization has some disadvantages as compared with the complex metric regularization (1.3). First, the covariant regularization does not guarantee the existence of Feynmann integrals (understood as Lebesgue integrals) even if the corresponding Euclidean integrals do exist. The simplest example is: \( D = 3 \),

\[
\int \frac{d^3p}{[(p + q)^2 + m^2 - i\varepsilon][p^2_M + m^2 - i\varepsilon]} \tag{2.62}
\]

The Lebesgue integral (2.62) does not exist whatever \( q \in \mathbb{R}^3, \varepsilon > 0, m > 0 \) [4]. The corresponding Euclidean integral

\[
\int \frac{d^3p}{[(p + q)^2 + m^2][p^2_M + m^2]} \tag{2.63}
\]

exists (as a Lebesgue integral) for all \( q \in \mathbb{R}^3 (m > 0) \).

The covariant regularization (2.55) does not guarantee the propagator regularity in the coordinate representation. Indeed, performing the Fourier transformation in (2.59)–(2.60), we obtain:

\[
F_p^{-1}\left[ \frac{-i}{p^2_M + m^2 - i0} \right](x) = \frac{1}{(2\pi)^D/2} \frac{m^2 + i\varepsilon}{x^2_M + i0} K_{D/2} \left( \sqrt{(m^2 + i\varepsilon)(x^2_M + i0)} \right) \tag{2.64}
\]

where \( \nu = \nu(D) = D/2 - 1 \). The relation (2.64) may be obtained from (2.37) by performing the replacement \( m^2 \leftrightarrow m^2 + i\varepsilon \) and the limit \( w \to -1 \pm i0 \). Here we used that fact that the limit (2.64) in \( S'\left(\mathbb{R}^D\right) \) is unchanged if the replacement \( (-1 \pm i0)(x^2)^2 \leftrightarrow x^2 \to \)
\(x_M^2 \pm i0\) is performed. The “most singular part” in (2.64) is
\[\text{const } (x_M^2 \pm i0)^{-\nu}.\] (2.65)

For \(D \geq 4\) we have \(\nu \geq 1\) and the distribution (2.65) is singular. Hence (2.64) is singular too.

In Ref. [13] the complex metric regularization with \(w = -1 \pm ie\), \(\epsilon > 0\) was considered in the context of the 3-dimensional \(\sigma\)-model from [21] (\(n\)-field) and convergence theorems for \(\epsilon > 0\) were proved. The regularization of propagators with \(w = -1 \pm i\epsilon\) is closely related to regularization method for pseudo-Euclidean singularities suggested by W. Zimmermann in Ref. [3].

\[p_M^2 \rightarrow p_M^2 - \epsilon \bar{p}^2.\]

3. Multidimensional cosmology with \(n\) Einstein spaces

3.1. Gravitational model

Consider the manifold
\[M = M_0 \times M_1 \times \ldots \times M_n,\] (3.1)
with the metric
\[g = e^{2\gamma(x)}g^{(0)} + \sum_{i=1}^{n} e^{2\phi^i(x)}g^{(i)},\] (3.2)
\[g^{(0)} = g^{(0)}_{\mu\nu}(x) \ dx^\mu \otimes dx^\nu\] (3.3)
is a metric on the manifold \(M_0\), and \(g^{(i)}\) is a metric on the manifold \(M_i\) satisfying
\[R_{\mu_1\mu_2\nu_1\nu_2}(g^{(i)}) = \lambda_i \delta g^{(i)}_{\mu_1\mu_2\nu_1\nu_2},\] (3.4)
m, \(n_i = 1, \ldots, N_i\); \(\lambda_i = \text{const}, i = 1, \ldots, n_i\). Thus \((M_i, g^{(i)})\) are Einstein spaces. In (3.2) we denote by \(\hat{g}^{(\alpha)} = \hat{p}^{(\alpha)}_{n} g^{(\alpha)}\) the pullback of the metric \(g^{(\alpha)}\) to the manifold \(M\) by the canonical projection: \(p_\alpha : M \rightarrow M_\alpha, \alpha = 0, \ldots, n\). The functions \(\gamma, \phi^i : M_0 \rightarrow \mathbb{R}\) are smooth, \(\gamma = 0, \ldots, n\).

Consider the gravitational action
\[S = S_g = \frac{1}{2} \int_M d^Dx \sqrt{|g|} \{R[g] - 2\Lambda\} + S_{\text{GH}},\] (3.5)
where \(|g| = \text{det}(g_{\mu\nu})\), \(S_{\text{GH}}\) is the standard Gibbons-Hawking boundary term [39] and \(\Lambda\) is the cosmological constant.

The field equations for the action (3.5) (Einstein equations)
\[R_{MN}[g] - \frac{1}{2} g_{MN} R[g] = -\Lambda g_{MN}\] (3.6)
are (for \(D \neq 2\)) equivalent to
\[R_{MN}[g] = 2\Lambda g_{MN} / (D - 2),\] (3.7)
where \(D = \sum_{k=0}^{n} N_k = \text{dim } M\) is the dimension of the manifold (3.1), \(N_k = \text{dim } M_k, k = 0, \ldots, n\).

Although the cosmological case \(N_0 = 1\) will be our main subject, we shall need the more general non-exceptional case \(N_0 \neq 2\). In this case we put, just as in [24, 52],
\[\gamma = \gamma_0(\phi) = \frac{1}{2} \left(\frac{n}{N_0}\right) \sum_{i=1}^{n} N_i \phi^i.\] (3.8)

It may be shown (see [24, 26, 52]) that Eqs. (3.6), (3.7) for the metric (3.2) with \(\gamma\) from (3.8) are equivalent to the equations of motion for the \(\sigma\)-model
\[S_0[g^{(0)}, \phi] = \frac{1}{2} \int_{M_0} d^Dx \sqrt{|g^{(0)}|} \left\{ R[g^{(0)}] \right\} - G_{ij} g^{(0)\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - 2V(\phi) \] (3.9)
where \(|g^{(0)}| = \text{det}(g^{(0)\mu\nu})\),
\[G_{ij} = N_i \delta_{ij} + N_i N_j / N_0 - 2\] (3.10)
are components of the “midisuperspace” (or target space) metric on \(\mathbb{R}^n\)
\[G = G_{ij} d\phi^i \otimes d\phi^j\] (3.11)
and
\[V(\phi) = \Lambda e^{2\gamma_0(\phi)} - \frac{1}{2} \sum_{i=1}^{n} \lambda_i N_i e^{-2\phi^i + 2\gamma_0(\phi)}\] (3.12)
is the potential. (In [26] authors start with \(\gamma = 0\).)
We note that \(G_{ij}\) is not determinate (3.13)
\[\text{det}(G_{ij}) = N_1 \ldots N_n \frac{2 - D - 2 - N_0}{2 - N_0} \neq 0.\]

3.2. Multidimensional cosmology

Now consider the cosmological case
\[N_0 = 1, \quad g^{(0)} = -dt \otimes dt.\] (3.14)

In this case (3.8) corresponds to the harmonic-time gauge and the minisuperspace metric
\[\hat{G}_{ij} = N_i \delta_{ij} - N_i N_j\] (3.15)
has the pseudo-Euclidean signature \((- , + , \ldots , +)\). The equations of motion for the cosmological model under consideration are equivalent to the Lagrange equations for the Lagrangian
\[L = \frac{1}{2} \hat{G}_{ij} \dot{\phi}^i \dot{\phi}^j - V\] (3.16)
with the energy constraint \(\dot{V}\)
\[E = \frac{1}{2} \hat{G}_{ij} \dot{\phi}^i \dot{\phi}^j + V = 0.\] (3.17)
(For exact solutions of the Einstein equations see e.g. [27, 38].)
3.3. The quantum case

The quantization of the zero-energy constraint (3.17) leads to the Wheeler-DeWitt (WDW) equation in the harmonic time gauge [28] (see also [36, 35])

$$\hat{H}\Psi = \left[ -\frac{1}{2} \hat{G}^{ij} \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^j} + V \right] \Psi = 0,$$

(3.18)

where

$$\hat{G}^{ij} = \frac{\delta_{ij}}{N_i} + \frac{1}{2-D}$$

(3.19)

are components of the matrix inverse to $\left(\hat{G}_{ij}\right)$ in (3.15).

Third quantized model. The WDW equation (3.18) corresponds to the action of so-called ”third-quantized” cosmology [12-13] (see also [14-11, 30])

$$S = \frac{1}{2} \int d^{n+1}z \Psi \hat{H} \Psi.$$

(3.20)

We may quantize (3.20) and study the processes of “creation” and “interaction” of “multidimensional universes” [15-19].

Diagonalization. The operator $\hat{H}$ (3.18) may be diagonalized by the linear transformation

$$\phi^a = S_i^a \phi^i,$$

(3.21)

where

$$S_i^a \delta_{ab} S_j^b = G_{ij},$$

(3.22)

and the vectors $\vec{u}_b^{(i)} \in \mathbb{R}^{n-1}$ satisfy the relations

$$\vec{u}_b^{(i)} \vec{u}_b^{(j)} = 4 \left( \frac{\delta_{ij}}{N_i} + \frac{1}{1-D} \right).$$

(3.30)

The operator $\hat{H}$ in the new variables reads

$$\hat{H} = -\eta^{ab} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^b} + V(\phi),$$

(3.31)

where $V = V(\phi)$ is defined in (3.28) and

$$\eta^{ab} = \eta_{ab} = \eta^{ab}(-1) = \text{diag}(-1,1,\ldots,1).$$

(3.32)

Complex dimensions. Consider the simplest case $V = 0$. Then the third quantized cosmological model is equivalent to the theory of a free massive field in $\mathbb{R}^{n-1}$ with the Minkowski metric (3.32). There are at least two possibilities of complex metric regularization for (3.15).

First, we may consider the gravitational model (3.1)-(3.12) with the time manifold $M_0$ of dimension $N_0 > 2$ (when the midisuperspace metric is Euclidean), then perform the analytical continuation to the region $N_0 < 2$ and consider the limit

$$N_0 \to 1 - i0.$$

(3.33)

(see (3.13).)

The second possibility [24] is as follows: we put $N_0 = 1$ and consider (formally) the range of “small” dimensions $N_i$

$$N_i > 0, \quad \sum_{i=1}^{n} N_i < 1,$$

(3.34)

where the midisupermetric (3.15) is Euclidean, and then perform the analytical continuation to the original $N_i$ considering the limits

$$N_i - i0, \quad i = 1,\ldots,n.$$  

(3.35)

In this context it is worth noting that there already exist studies of multidimensional models with dimension “dynamics” [53]. In this model the midisupermetric signature may change (if “small” dimensions are considered).

4. Concluding remarks

In the cosmological model under study the minisupermetric does not depend on the signatures of the manifolds $(M_i, g^i)$. There exist models where such a dependence takes place. In [53] a gravitational model with several forms and dilatonic fields was considered.
It generalizes the “pure gravitational” model of Subsec. 3.1. It turns out that the part of the midisuperspace metric corresponding to the forms crucially depends on the signatures of the manifolds \( (M_i, g^i) \), \( i = 1, \ldots, n \). It will of interest to consider the Wick rotation in this model.

Another problem of interest is connected with some cosmological models with spinors (e.g. supersymmetric models) were the Wick rotation (both in space and in midisuperspace) may be non-trivial.

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References

[1] N.N. Bogoliubov and V.D. Shirkov, "Introduction to the Theory of Quantized Fields", Nauka, Moscow, 1984 (in Russian).
[2] N.N. Bogoliubov, A.A. Logunov, A.I. Oksak and I.T. Todorov, "General Principles of Quantum Field Theory", Nauka, Moscow, 1984 (in Russian).
[3] O.M. Zav’yalov, “Renormalized Feynmann Integrals”, Nauka, Moscow, 1979 (in Russian).
[4] W. Zimmermann, Commun. Math. Phys. 15, 208 (1969).
[5] K. Hepp, “Renormalization Theory” Nauka, Moscow, 1974 (in Russian).
[6] G. ’t Hooft and M. Veltman, Nucl. Phys. 44, 189 (1972).
[7] J. Glimm and A. Jaffe, "Quantum Physics: A functional Integral Point of View", Springer. New York - Heidelberg - Berlin. 1981.
[8] B.S. DeWitt, "Dynamical Theory of Groups and Fields", Gordon and Breach, New York, 1965.
[9] V.V. Belokurov, E.T. Shavgulidze and Yu.P. Solovyov, “Perturbation Theory with Absolutely Convergent Series”. I. “Toy Models”; II. “Functional Integrals in Hilbert Space”. Leipzig University. Preprint 22/95, 1995.
[10] O.G. Smolianov and E.T. Shavgulidze, “Continual Integrals”, Moscow University Press, 1990 (in Russian).
[11] K.P. Staniukovich and V.N. Melnikov, "Hydrodynamics, Fields and Constants in Gravitation Theory", Energoatomizdat, Moscow, 1983 (in Russian).
[12] V.D. Ivashchuk, "(1/N)-theory of Perturbations in the Space with Regularized Minkowski Metric", 1982 (unpublished).
[13] V.D. Ivashchuk, Izv. Akad. Nauk Mold. SSR, Ser. Fiz.-Tekhn. i Math. Nauk 3, 8 (1987) (in Russian).
[14] V.D. Ivashchuk, Izv. Akad. Nauk Mold. SSR, Ser. Fiz.-Tekhn. i Math. Nauk 1, 10 (1988) (in Russian).
[15] I.L. Buchbinder and E.N. Kirillova, Izv. Vuzov, Fizika No 6, 20 (1988) (in Russian).
[16] J. Greensite, Phys. Lett. B300, 34 (1993).
[17] A. Carlini and J. Greensite, Phys. Rev. D49, 34 (1994).
[18] E. Elizalde, S.D. Odintsov and A. Romero, Class. Quantum Grav. 11, L61 (1994).
[19] I.Ya. Aref’eva, Dr. Sci. Dissertation, Steklov Mathematical Institut, Moscow, 1982 (in Russian).
[20] I.M. Gel’fand and G.E. Shilov, "Generalized Functions. VI. Generalized Functions an Actions with Them" Fizmatgiz, Moscow, 1971 (in Russian).
[21] V.S. Vladimirov, "Generalized Functions in Mathematical Physics", Nauka, Moscow, 1976 (in Russian).
[22] K. Isida, "Functional Analysis", Mir, Moscow, 1967 (in Russian).
[23] V.A. Berezin, G. Domenech, M.L. Levinas, C.O. Lousto and N.D. Umerez, Gen. Relativ. Grav. 21, 1177 (1989).
[24] M. Szyd/lowski, Acta Cosmologica 18, 85 (1992).
[25] M. Rainer and A. Zhuk, Phys. Rev. D 54, 6186 (1996).
[26] V.D. Ivashchuk and V.N. Melnikov, Phys. Lett. A136, 465 (1989).
[27] V.D. Ivashchuk, V.N. Melnikov and A.I. Zhuk, Nuovo Cim. B104, 575 (1989).
[28] M. Deminaisky and A. Polnarev, Phys. Rev. D41, 3003 (1990).
[29] U. Bleyer, D.-E. Liebscher and A.G. Polnarev, Class. Quant. Grav. 8, 477 (1991).
[30] V.D.Ivashchuk, Phys. Lett. A170, 16 (1992).
[31] V.D. Ivashchuk and V.N. Melnikov, Theor. Math. Phys. 98, 312 (1994) (in Russian).
[32] V.D. Ivashchuk and V.N. Melnikov, Int. J. Mod. Phys. D3, No 4, 795 (1994).
[33] V.D. Ivashchuk and V.N. Melnikov, Class. Quantum Grav. 11, 1793 (1994).
[34] V.D. Ivashchuk and V.N. Melnikov, Nucl. Phys. B429, 177 (1994).
[35] V.D. Ivashchuk and V.N. Melnikov, Grav. and Cosmol. 1, No. 3, 204 (1995).
[36] V.R. Gavrilov, V.D. Ivashchuk and V.N. Melnikov, J. Math. Phys. 36, 5829 (1995).
[37] V.R. Gavrilov, V.D. Ivashchuk and V.N. Melnikov, Class. Quantum Grav. 13, 3039 (1996).
[39] G.W. Gibbons and S. Hawking, *Phys. Rev.* **D15**, 2752 (1977).

[40] N. Birrell and P. Davies, “Quantized Fields in Curved Space-Time”, Cambridge University Press, Cambridge, 1980.

[41] A.A. Grib, S.G. Mamaev and V.M. Mostepanenko, "Vacuum Quantum Effects in Strong Fields", Friedmann Laboratory Publishing, St. Petersburg, 1994.

[42] V.A. Rubakov, *Phys. Lett.* **B214**, 503 (1988).

[43] S. Giddings and A. Strominger, *Nucl. Phys.* **B321**, 481 (1989).

[44] Y. Peleg, *Class. Quantum Grav.* **8**, 827 (1991).

[45] Y. Peleg, *Mod. Phys. Lett.* **A8**, 1849 (1993).

[46] E.I. Guendelman and A.B. Kaganovich, *Phys. Lett.* **B301**, 15 (1993).

[47] T. Horigushi, *Mod. Phys. Lett.* **A8**, 777 (1993).

[48] A. Zhuk, *Class. Quant. Grav.* **9**, 202 (1992).

[49] A.I. Zhuk, *Sov. Journ. Nucl. Phys.* **58**, No 11, (1995).

[50] J.J. Halliwell, *Phys. Rev.* **D38**, 2468 (1988).

[51] J.B. Hartle and S.W. Hawking, *Phys. Rev.* **D28**, 2960 (1983).

[52] V.D. Ivashchuk and V.N. Melnikov, *Grav. and Cosmol.* **2** No 3, 177 (1996).

[53] V.D. Ivashchuk and V.N. Melnikov, *Grav. and Cosmol.* **2** No 4, 295 (1996).

[54] P. van Nieuwenhuizen and A. Waldron, “A Continuous Wick Rotation for Spinor Fields and Supersymmetry in Euclidean Space”, hep-th/9611043

[55] U. Bleyer, M. Mohazzab and M. Rainer, *Astron. Nachr.* **317**, 3 (1996).