Equitable vertex arboricity of $d$-degenerate graphs

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Abstract

A minimization problem in graph theory so-called the equitable tree-coloring problem can be used to formulate a structure decomposition problem on the communication network with some security considerations. Precisely, an equitable tree-$k$-coloring of a graph is a vertex coloring using $k$ distinct colors such that every color class induces a forest and the sizes of any two color classes differ by at most one. In this paper, we establish some theoretical results on the equitable tree-colorings of graphs by showing that every $d$-degenerate graph with maximum degree at most $\Delta$ is equitably tree-$k$-colorable for every integer $k \geq (\Delta + 1)/2$ provided that $\Delta \geq 10d$.

This generalises the result of Chen et al. [J. Comb. Optim. 34(2) (2017) 426–432] which states that every 5-degenerate graph with maximum degree at most $\Delta$ is equitably tree-$k$-colorable for every integer $k \geq (\Delta + 1)/2$.

Keywords: equitable coloring; vertex arboricity; structure decomposition; degeneracy

1 Introduction

Almost all relationships in the real world can be described by networks, so doing some theoretical researches on the structures of the network is necessary and interesting. Actually, finding tree-like structures in a large network is a fascinating topic that attracts the attentions from many researchers (see [1, 6, 8–10], etc.). When decomposing a large communication network into small pieces, we are sometimes required that each of its small pieces has an “acyclic” property due to some security considerations. This is because that in an acyclic piece we can easily and effectively identify a node failure since the local structure around such a node in this piece is so clear that it can be easily tested using some classic algorithmic tools. Meanwhile, we sometimes do not want have too many small pieces and hope that the scales of any two pieces will not differ a lot. These requirements help us to

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maintain the whole communication network effectively even if we have a low budget. Practically, this structure decomposition problem can be modeled by a minimization problem in graph theory so-called the equitable tree-coloring problem that was introduced by Wu, Zhang and Li [11] in 2013.

From now on, we do not distinguish the word “graph” from the word “network”, and introduce some graph-based definitions and notations. A graph $G$ is $d$-degenerate if every subgraph of $G$ has a vertex of degree at most $d$. The minimum integer so that $G$ is $d$-degenerate is the degeneracy of $G$, which is used to measure the sparseness of the graph. Clearly, the degeneracy of a graph is upper-bounded by its maximum degree. Let $V_1$ and $V_2$ be two subsets of the set $V(G)$ of vertices of a graph $G$. By $e(V_1, V_2)$, we denote the number of edges that have one endvertex in $V_1$ and the other in $V_2$. By $\deg_G(v)$, we denote the degree of $v$ in the graph $G$, where $v \in V(G)$. By $e(G)$, we denote the number of edges in $G$. For other undefined but standard notations, we refer the readers to the book [2].

An equitable tree-$k$-coloring of $G$ is a function $c$ from the vertex set $V(G)$ to $\{1, 2, \cdots, k\}$ so that $c^{-1}(i)$ induces a forest for every $1 \leq i \leq k$, and $|c^{-1}(i)| - |c^{-1}(j)| \leq 1$ for every pair of $i, j$ with $1 \leq i < j \leq k$. The minimum integer $k$ such that $G$ admits an equitable tree-$k$-coloring, denoted by $va_{eq}(G)$, is the equitable vertex arboricity of $G$. A graph that is equitably tree-$k$-colorable may be not equitably tree-$k'$-colorable for some $k' > k$. For example, $va_{eq}(K_{9,9}) = 2$ but $K_{9,9}$ is not equitably tree-3-colorable. In view of this, a new chromatic parameter so-called the equitable vertex arborable threshold is introduced. Precisely, it is the minimum integer $k$, denoted by $va_{eq}^*(G)$, such that $G$ has an equitable tree-$k'$-coloring for any integer $k' \geq k$. Note that those concepts were introduced by Wu, Zhang and Li [11] in 2013, who proposed two conjectures

**Conjecture 1.** [Equitable Vertex Arboricity Conjecture (EVAC)] Every graph with maximum degree at most $\Delta$ is equitably tree-$k$-colorable for any integer $k \geq (\Delta + 1)/2$, i.e, $va_{eq}^*(G) \leq [(\Delta + 1)/2]$.

**Conjecture 2.** There exists a constant $k$ such that every planar graph is equitably tree-$k'$-colorable for any integer $k' \geq k$, i.e, $va_{eq}^*(G) \leq k$.

Zhang and Wu [12] proved that every graph with maximum degree at most $\Delta$ has an equitable tree-$[(\Delta + 1)/2]$-coloring provided that $\Delta \geq |G|/2$. Zhang [14] showed that every subcubic graph is equitably tree-$k$-colorable for any integer $k \geq 2$, and thus Conjecture 1 holds for $\Delta \leq 3$. Chen et al [4] verified Conjecture 1 for all 5-degenerate graphs. This implies that Conjecture 1 holds for $\Delta \leq 5$.

In 2015, Esperet, Lemoine and Maffray [5] confirmed Conjecture 2 by showing that $va_{eq}^*(G) \leq 4$ for any planar graph. Since the vertex arboricity of every planar graph is at most 3 [3], it is natural to put forward the following

**Conjecture 3.** Every planar graph is equitably tree-3-colorable, i.e., $va_{eq}^*(G) \leq 3$ for any planar graph $G$.

Conjecture 3 has been verified for some classes of graphs including planar graphs with maximum degree at most 5 [4], planar graphs with girth at least 5 [11], and planar graphs with girth at most 4 and with some conditions on the cycles [13].
For $d$-generate graphs $G$, Esperet, Lemoine and Maffray [5] showed that $\nu_{eq}(G) \leq 3^{d-1}$. This implies the following

**Theorem 4.** Every $d$-degenerate graph with maximum degree at most $\Delta$ is equitably tree-$k$-colorable for any integer $k \geq (\Delta + 1)/2$ provided that $\Delta \geq 2 \cdot 3^{d-1} - 1$.

The aim of this paper is to lower the exponential lower bound for $\Delta$ in the above theorem (for $d \geq 4$) to $10d$. Precisely, we are to prove

**Theorem 5.** Every $d$-degenerate graph with maximum degree at most $\Delta$ is equitably tree-$k$-colorable for any integer $k \geq (\Delta + 1)/2$ provided that $\Delta \geq 10d$.

This theorem not only improves Theorem 4 but also generalizes the result on $5$-degenerate graphs of Chen et al [4] mentioned above. Actually, it verifies Conjecture 1 for all $d$-degenerate graphs with maximum degree at least $10d$.

## 2 The proof of Theorem 5

First of all, we review an useful lemma on the degrees of vertices in a $d$-degenerate graph.

**Lemma 6.** [7, Kostochka and Nakprasit] Let $G$ be a $d$-degenerate graph on $n$ vertices, where $d \geq 2$ is an integer. If $v_1, v_2, \cdots, v_n$ is an order of the vertices of $G$ from the highest degree to the lowest degree, then $\deg_G(v_i) < d(1 + \frac{n}{t})$ for every $i = 1, 2, \ldots, n$.

In what follows, we give the detailed proof of Theorem 5, the idea of which partially comes from the paper [7] contributed by Kostochka and Nakprasit.

**Proof of Theorem 5.** Since 1-degenerate graphs are forests and it is trivial to conclude that every forest admits an equitably tree-$k$-colorable for every integer $k \geq 1$. Therefore we assume $d \geq 2$ in the following.

If $|G| \leq 2$, then the result is trivial. We now proceed the proof by induction on the order (i.e., the number of vertices) of $G$. At the current stage, we are given a $d$-degenerate graph with order $n$, and the induction hypothesis assumes that the theorem holds for every $d$-degenerate graph with order less than $n$.

In the following we assume that $k(t - 1) \leq n < kt$. If $t = 2$ (resp. $t = 1$), then we arbitrarily partition the vertices of $G$ into $k$ subsets such that each of them has either one or two vertices (resp. at most one vertex). Clearly, this partition is an equitable tree-$k$-coloring of $G$. Hence we assume $t \geq 3$.

Suppose that $v_1, v_2, \cdots, v_n$ is an order of the vertices of $G$ from the highest degree to the lowest degree with the property that

$$\deg_G(v_1) \geq \cdots \geq \deg_G(v_\mu) \geq \lambda(t)d > \deg_G(v_{\mu+1}) \geq \cdots \geq \deg_G(v_n),$$

where $\lambda(t) = 1$ for $t = 1, 2$, and $\lambda(t) = t$ for $t \geq 3$. Then it follows from Lemma 6 that $\deg_G(v_i) < d(1 + \frac{n}{t})$ for every $i = 1, 2, \ldots, n$. Therefore, we can find a tree-$k$-coloring of $G$ with $k \geq (\Delta + 1)/2$ provided that $\Delta \geq 10d$. This completes the proof.
where

$$\lambda := \lambda(t) = 1 + \frac{t}{s(t)}$$  \hspace{1cm} (2.1)

and

$$s := s(t) = \left\{ \begin{array}{ll} 1, & \text{if } 3 \leq t \leq 7; \\ \left\lceil \frac{t}{5} \right\rceil, & \text{if } t \geq 8. \end{array} \right.$$  \hspace{1cm} (2.2)

Note that

$$4 \leq \lambda = 1 + \frac{t}{s(t)} \leq 8 \text{ if } 3 \leq t \leq 7,$$

$$4.3 < 6 - \frac{20}{8 + 4} \leq 6 - \frac{20}{t + 4} = 1 + \frac{t}{\frac{t + 4}{2}} \leq 1 + \frac{t}{\left\lceil \frac{t}{5} \right\rceil} \leq 1 + \frac{t}{7} = 6 \text{ if } t \geq 8.$$  \hspace{1cm} (2.3)

$$4.3 < 6 - \frac{20}{8 + 4} \leq 6 - \frac{20}{t + 4} = 1 + \frac{t}{\frac{t + 4}{2}} \leq 1 + \frac{t}{\left\lceil \frac{t}{5} \right\rceil} \leq 1 + \frac{t}{7} = 6 \text{ if } t \geq 8.$$  \hspace{1cm} (2.4)

Since \( \deg_{G}(v_{\mu}) < d(1 + \frac{n}{\mu}) \) by Lemma 6, we conclude \( \mu < \frac{n}{\lambda - 1} < n \) by (2.3) and (2.4). This implies, by the induction hypothesis, that the graph induced by \( V' = \{v_1, v_2, \cdots, v_\mu\} \) has an equitable tree-\( k \)-coloring \( c' \), and

$$\text{every color class of } c' \text{ contains at most } \left\lfloor \frac{\mu}{k} \right\rfloor \leq \left\lfloor \frac{n}{\lambda - 1} \right\rfloor \leq s \text{ vertices in } V'.$$  \hspace{1cm} (2.5)

We color \( v_{\mu+1}, v_{\mu+2}, \cdots, v_n \) one by one in such an order. Precisely, in the \( i \)-th step \( (i = \mu + 1, \mu + 2, \cdots, n) \), we attempt to color the vertex \( v_i \) with a color from \( \{1, 2, \cdots, k\} \), and meanwhile recolor (if necessary) previously colored vertices so that the resulting coloring satisfies

(a) each color class induces a forest,

(b) each color appears on at most \( t \) vertices among \( \{v_1, \cdots, v_\mu, v_{\mu+1}, \cdots, v_i\} \), and

(c) no vertex among \( \{v_1, \cdots, v_\mu\} \) is recolored.

If we success, then turn to the next step unless we have completed the last step, in which case the algorithm returns TRUE, and if we fail to do such a task, the algorithm returns FALSE and ends.

In the following, we prove that the above algorithm will return TRUE. Therefore, every vertex of \( G \) has been colored, and by (a) and (b), the resulting coloring is an equitable tree-\( k \)-coloring.

Suppose, to the contrary, that this algorithm returns FALSE at the \( i \)-th step with \( \mu + 1 \leq i \leq n \), that is, \( v_i \) cannot be colored (even possibly with some previous colored vertices recolored) so that (a), (b) and (c) hold. By \( G_{i-1} \), we denote the graph induced by \( \{v_1, \cdots, v_\mu, v_{\mu+1}, \cdots, v_{i-1}\} \). In the following, we search for contradictions by dividing the arguments into two cases.

**Case 1.** \( \mu + 1 \leq i < \frac{3}{7} n \).

Let \( c \) be an equitable tree-\( k \)-coloring of \( G_{i-1} \) satisfying (a), (b) and (c). A color class of \( c \) is **big** if it contains exactly \( t \) vertices, and is **small** otherwise. If there is a small color class of \( c \) that contains at most one neighbor of \( v_i \) in \( G \), then \( v_i \) can be moved into this class (i.e., \( v_i \) can be colored with the color of this class), and the algorithm will be turn to the \( (i + 1) \)-th step. Therefore, every small color class of \( c \) contains at least two neighbors of \( v_i \) in \( G \). This implies that there is at most \( \left\lfloor \frac{\deg_{G}(v_i)}{2} \right\rfloor \) small
classes in \( c \), and thus at least \( k - \lfloor \frac{\deg_G(v_i)}{2} \rfloor \) big classes in \( c \). In other words, at least \( (k - \lfloor \frac{\deg_G(v_i)}{2} \rfloor)t \) vertices have been colored under \( c \). Hence by (2.3) and (2.4), we have (note that \( \deg_G(v_i) < \lambda d \) by the choice of \( \mu \), and \( k \geq \frac{\Delta + 1}{2} \geq \frac{10d + 1}{2} > 5d \))

\[
i > \left(k - \left\lfloor \frac{\deg_G(v_i)}{2} \right\rfloor\right) \cdot t > \left(k - \frac{\deg_G(v_i)}{2}\right) \cdot t
\]

\[
> \left(k - \frac{\lambda d}{2}\right) t \geq \begin{cases} (k - 4d) t > \left(k - \frac{4}{5} k\right) t = \frac{1}{5} k t > \frac{1}{5} n, \text{ if } t \geq 4; \\ (k - 2d) t > \left(k - \frac{2}{3} k\right) t = \frac{2}{3} k t > \frac{2}{3} n, \text{ if } t = 3. \end{cases}
\]

From (2.7), one immediately conclude that \( i > \frac{3}{2} n \) if \( t = 3 \), which contradicts the fact that \( i < \frac{3}{2} n \). Therefore, \( t \geq 4 \).

By (2.6),

\[
\deg_G(v_i) \cdot t > 2(kt - i) > 2(n - i).
\]

On the other hand, \( \deg_G(v_i) \cdot t < dt(1 + \frac{2}{5}) \) by Lemma 6. This implies

\[
2(n - i) < dt(1 + \frac{n}{i}) \leq d\left(\frac{n + k}{k}\right)\left(\frac{n + i}{i}\right).
\]

Since \( n \geq k(t - 1) \) and \( t \geq 4 \),

\[
n + k \leq \frac{t}{t - 1} n \leq \frac{4}{3} n.
\]

Since \( k > 5d \) and \( \Delta \leq 2k - 1 \), by (2.8) and (2.9), we conclude

\[
\frac{\Delta}{d} \leq \frac{2k - 1}{d} < \frac{2k}{d} \leq \frac{(n + k)(n + i)}{(n - i)i} \leq \frac{4}{3} \frac{n(n + i)}{i(n - i)} = \frac{4}{3} \frac{1 + \frac{i}{n}}{1 - \frac{i}{n}}.
\]

Let \( \alpha = \frac{i}{n} \) and \( f(\alpha) = \frac{4 - \alpha}{3(1 - \alpha^2)} \). Since \( \frac{1}{5} n < i < \frac{3}{5} n \) by (2.7) and the condition of this case, we have \( \frac{1}{5} < \alpha < \frac{3}{5} \). Since

\[
f'(\alpha) = \frac{4 (1 + \alpha)^2 - 2}{3 \alpha^2(1 - \alpha^2)},
\]

\( f(\alpha) \) decreases if \( \alpha < \sqrt{2} - 1 \) and increase if \( \alpha > \sqrt{2} - 1 \). Hence

\[
f(\alpha) \leq \max\left\{f\left(\frac{1}{5}\right), f\left(\frac{3}{5}\right)\right\} = \max\left\{10, \frac{80}{9}\right\} = 10,
\]

which implies \( \Delta < 10d \) by (2.10), contradicting the fact that \( \Delta \geq 10d \).

**Case 2.** \( \frac{3}{5} n \leq i \leq n \).

Let \( e \) be the current equitable tree-\( k \)-coloring of \( G_{i-1} \). Define an auxiliary digraph \( \mathcal{H} := \mathcal{H}(e) \) on the color classes of \( e \) by \( XY \in E(\mathcal{H}) \) if and only if some vertex \( x \in X \setminus V' \) has at most one neighbor in \( Y \). In this case we say that \( x \) witnesses \( XY \). If \( P := M_1 M_{l-1} \cdots M_1 M_0 \) is a path in \( \mathcal{H} \) and \( x_i (l \geq i \geq 1) \)
is a vertex in $M_l$ such that $x_i$ witnesses $M,M_{l-1}$, then switching witnesses along $P$ means moving $x_i$ to $M_{l-1}$ for every $l \geq i \geq 1$. This operation decreases $|M_l|$ by one and increases $|M_0|$ by one, while leaving the sizes of the interior vertices of $\mathcal{H}$ (color classes) unchanged.

Let $Y_0$ be the set of small color classes of $c$, i.e., color classes containing less than $t$ vertices. By $Y_i$ ($i \geq 1$), we denote the set of color classes of $c$ such that

(i) $Y_i \cap \bigcup_{j=1}^{i-1} Y_j = \emptyset$, and

(ii) for any color class $M_l \in Y_i$ there exists a color class $M_{l-1} \in Y_{i-1}$ so that $M_i,M_{l-1} \in \mathcal{H}$.

Let $\mathcal{Y} = \bigcup Y_j$ and let $y = |\mathcal{Y}|$. If $v_i$ has at most one neighbor in some color class $M_j \in Y_j \in \mathcal{Y}$, then move $v_i$ into $M_j$ and switching witnesses along $P = M_j M_{j-1} \cdots M_0$, where $M_l \in Y_i$ ($j \geq t \geq 0$). At this moment, $v_i$ has been colored with some previously colored vertices recolored so that (a), (b) and (c) hold, a contradiction. Therefore,

\[ v_i \text{ has at least two neighbors in every color class of } \mathcal{Y}, \text{ implying } \deg_C(v_i) \geq 2y. \tag{2.11} \]

If there is a vertex $v_j \notin \bigcup_{M \in \mathcal{Y}} M$ with $\mu + 1 \leq j \leq i - 1$, then

\[ v_j \text{ has at least two neighbors in every color class of } \mathcal{Y}, \tag{2.12} \]

otherwise $v_j$ has at most one neighbor in some color class $M_a \in Y_a \in \mathcal{Y}$ and thus $v_j$ is actually included in some color class $M_{a+1} \in Y_{a+1} \in \mathcal{Y}$, a contradiction to the choice of $v_j$. Note that $v_j \notin V'$.

Divide $V(G_{i-1}) \setminus \bigcup_{M \in \mathcal{Y}} M$ into two subsets $A$ and $B$, where $A \subset V'$ and $B \cap V' = \emptyset$. By (2.12), every vertex in $B$ has at least two neighbors in every color class of $\mathcal{Y}$, which implies

\[ e\left(B, \bigcup_{M \in \mathcal{Y}} M\right) \geq 2y|B|. \tag{2.13} \]

By the choice of $V'$, $\deg_C(w) \geq \lambda d$ if $w \in V'$. Therefore, the number of edges that are incident with $A$ is exactly

\[ \sum_{w \in A} \deg_C(w) - e(A) \geq \lambda d|A| - e(A) > \lambda d|A| - d|A| = (\lambda - 1)d|A|. \tag{2.14} \]

Note that the graph induced by $A$ is $d$-degenerate and thus $e(A) < d|A|$.

By (2.11) and Lemma 6, we have

\[ 2y \leq \deg_C(v_i) < d\left(1 + \frac{n}{i}\right) \leq d\left(1 + \frac{5}{3}\right) < (\lambda - 1)d \tag{2.15} \]

since $\frac{5}{3}n \leq i \leq n$ and $\lambda \geq 4$ by (2.3) and (2.4).

Note that $A \cup B = V(G_{i-1}) \setminus \bigcup_{M \in \mathcal{Y}} M$ and thus it is made up of the vertices from the $(k-y)$ color classes of $c$ that are not involved in $\mathcal{Y}$. By the choice of $Y_0$, those $(k-y)$ color classes are big, i.e., each of them contains exactly $t$ vertices. This concludes

\[ |A| + |B| = t(k-y). \tag{2.16} \]
By (2.13), (2.14) and (2.16), we have
\[ e(G) \geq 2y|B| + (\lambda - 1)d|A| > 2y(|B| + |A|) = 2yt(k - y). \] 
(2.17)

On the other hand, since \( G \) is degenerate, \( e(G) < dn < dkt. \) It follows from (2.17) that
\[ \varphi(y) := 2y^2 - 2ky + kd > 0. \]

By (2.15), we have \( y \leq \frac{4}{3}d. \) Since \( k \geq \frac{\Delta + 1}{2} \geq \frac{10d + 1}{2} > 5d, \)
\[ \varphi\left(\frac{4}{3}d\right) = \frac{32}{9}d^2 - \frac{5}{3}kd < \frac{32}{9}d^2 - \frac{25}{3}d^2 < 0 \]
\[ \varphi(0.6d) = 0.72d^2 - 0.2kd < 0.72d^2 - d^2 < 0. \]

This implies that
\[ y < 0.6d. \] 
(2.18)

By (2.5) and by the condition (c) that no vertex among \{v_1, \cdots, v_\mu\} is recolored in any step, every color class of \( c \) contains at most \( s \) vertices in \( V' \). Since \( A \subset V' \), we conclude
\[ |A| \leq s(k - y), \]
which implies
\[ |B| \geq (t - s)(k - y). \] 
(2.19)

by (2.16).

Since \( n < kt \) and there are \( k \) color classes in \( c \), there is at least one color class containing less than \( t \) vertices, which implies that \( Y_0 \neq \emptyset \). Choose \( M_0 \in Y_0 \). It follows that \( |M_0| \leq t - 1. \) Let \( l \) be the number of vertices in \( V' \) that is contained in \( M_0 \). By (2.5), \( l \leq s. \) Since \( M_0 \) contains \( |M_0| - l \) vertices that are not in \( V' \), which have degree less than \( \lambda d \), the number of neighbors of \( M_0 \) is at most \( l\Delta + (|M_0| - l)\lambda d \leq s\Delta + (|M_0| - s)\lambda d \leq s\Delta + (t - 1 - s)\lambda d, \) which implies
\[ e(M_0, B) \leq s\Delta + (t - 1 - s)\lambda d. \] 
(2.20)

Note that \( \Delta - \lambda d \geq 10d - 8d = 2d > 0 \) by (2.3) and (2.1).

On the other hand, by (2.12), every vertex in \( B \) has at least two neighbors in \( M_0 \), which implies
\[ e(M_0, B) \geq 2|B| \geq 2(t - s)(k - y). \]

by (2.19). Combine it with (2.18) and (2.20), we immediately have
\[ 2(t - s)\left(\frac{\Delta + 1}{2} - 0.6d\right) \leq 2(t - s)(k - 0.6d) < s\Delta + (t - 1 - s)\lambda d, \]
which implies
\[(t - 2s)\Delta + (t - s) < \left(1.2(t - s) + (t - 1 - s)\lambda\right)d\] (2.21)

We now divide the proofs into two subcases according to the value of \(t\).

**Subcase 2.1.** \(3 \leq t \leq 7\).

In this case \(s = 1\) by (2.2) and \(\lambda = 1 + t\) by (2.1). Since \(\Delta \geq 10d\), we can deduce from (2.21) that
\[10(t - 2) < 1.2(t - 1) + (t - 2)(t + 1),\]
which implies \(t \geq 8\), a contradiction.

**Subcase 2.2.** \(t \geq 8\).

In this case \(\frac{t}{s} = \lambda - 1\) by (2.1). Since \(\Delta \geq 10d\), we can deduce from (2.21) that
\[10 \leq \frac{\Delta}{d} < \frac{t - s}{t - 2s}(1.2 + \lambda) = \frac{\frac{t}{s} - 1}{\frac{t}{s} - 2}(1.2 + \lambda) = \frac{(\lambda - 2)(\lambda + 1.2)}{\lambda - 3},\]
which implies \(\lambda > 6.6\) or \(\lambda < 4.2\), contradicting (2.4).

\[\Box\]

### 3 More about \(d\)-degenerate graphs

As we have seen in Section 1, Esperet, Lemoine and Maffray [5] showed that \(\text{va}_{eq}^*(G) \leq 3^{d-1}\) for every \(d\)-degenerate graph. Actually, it is very interesting to find an \(o(d)\) upper bound for the equitable vertex aborable threshold of \(d\)-degenerate graphs. Recently, the first and the last author of this paper proved that \(\text{va}_{eq}^*(G) \leq d\) for every graph \(G\) with treewidth at most \(d\) in an unpublished paper, and showed that the upper bound \(d\) for the equitable vertex aborable threshold there is sharp.

For the completeness, we use some words from that paper to show why the upper bound \(d\) is sharp. Let \(H\) be a complete graph on \(d\) vertices, and let \(S\) be a set of independent vertices, each of which is adjacent to every vertex of \(H\). By \(G\), we denote the resulting graph, which clearly has treewidth at most \(d\). We next prove that \(G\) admits no equitable tree-\((d - 1)\)-colorings. Suppose otherwise that \(c\) is an equitable tree-(\(d - 1\))-coloring of \(G\). Since there are \(d - 1\) colors in \(c\) and \(H\) has \(d\) vertices, at least two vertices of \(H\) are monochromatic. But every color class in \(c\) induces a forest, so every color in \(c\) appears on \(H\) at most twice. Therefore, there is a color in \(c\) appearing on \(H\) exactly twice, and moreover, this color cannot be used by any vertex in \(S\) (otherwise a monochromatic triangle occurs). This is to say that there is a color in \(c\) that are used by \(G\) exactly twice. Since \(c\) is equitable, every color in \(c\) appears on \(G\) at most three times, which implies that the number of colored vertices is at most \(2 + 3(d - 2) = 3d - 4\). Choose \(|S|\) to be at least \(2d - 3\), we conclude a required contradiction.

Since every graph with treewidth at most \(d\) is clearly \(d\)-degenerate, we naturally release the following conjecture.

**Conjecture 7.** Every \(d\)-degenerate graph \(G\) is equitably tree-\(d'\)-colorable for any integer \(d' \geq d\), i.e, \(\text{va}_{eq}^*(G) \leq d\).
4 Problems for further research

One interesting problem is to lower the bound 10 for $\Delta/d$ in Theorem 5. If it can be improved to 1, then the Equitable Vertex Arboricity Conjecture (Conjecture 1) will be verified. This can be easily seen from a trivial fact that every graph with maximum degree at most $\Delta$ is $\Delta$-degenerate. Another interesting problem is to consider Conjecture 3 for planar graphs — special 5-degenerate graphs. Instead of proving Conjecture 3 directly, we may investigate the following problem at first, that is, to determine a constant $\beta \leq 1$ (the larger the better) such that every $n$-vertex planar graph with maximum degree at most $\Delta$ is equitably tree-3-colorable provided that $\Delta \leq \beta n$. Clearly, if $\beta = 1$, then Conjecture 3 is verified.

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