A Note on the Hardness of Problems from Cryptographic Group Actions

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Abstract

Given a cryptographic group action, we show that the Group Action Inverse Problem (GAIP) and other related problems cannot be NP-hard unless the Polynomial Hierarchy collapses. We show this via random self-reductions and the design of interactive proofs. Since cryptographic group actions are the building block of many security protocols, this result serves both as an upper bound on the worst-case complexity of some cryptographic assumptions and as proof that the hardness in the worst and in the average case coincide. We also point out the link with Graph Isomorphism and other related NP intermediate problems.

Keywords — hard homogeneous spaces, NP-hardness, random self-reductions

1 Introduction

Group Actions. Cryptographic Group Actions (CGA) were introduced by Almati et al. [3] in 2020 and, under the name of Hard Homogeneous Spaces by Couveignes [11] in 2006. They are a powerful tool to design cryptographic protocols. The purpose of this setting is to generalise some well-known computational problems and assumptions like the Discrete Logarithm Problem (DLP). Their study had an increasing interest with the coming of Isogeny-based cryptography, in particular, among others, CSIDH [10] and Csi-fish [7], but more recently even with some code-based construction [5] and lattices [6].

Computational Complexity Theory. On the other hand, computational complexity has always been a tool in cryptographic proofs, serving as an upper bound on the hardness of certain problems. By its nature, cryptography has to deal with hard-on-average problems and common assumptions embrace this setting. Classical complexity theory initially studied worst-case scenario until Levin, in its work [21], proposed a theory about problems with a distribution associated on their
inputs. This defines a distributional problem and the average-case study of worst-case (believed) intractable problems. A bridge between worst and average-case is built upon random reductions [1, 13, 14] binding a problem \(D\) to a distributional problem \(D'\). In particular, in this work we will use (nonadaptively) random self-reductions from [14], but an analogue construction leading the same results can be done with information hiding schemes introduced in [1]. Another tool to study the worst-case hardness of decision problems is given by the so-called interactive proofs (or protocols). In this setting, two machines, a Prover and a Verifier, exchange messages to decide whether an instance is a solution of a problem.

This work. We introduce some worst-case problems modelling computational assumptions related to group actions from [3]: by their nature, these assumptions cover the average-case view of these problems. We show that, if such problems are NP-hard, then some widely believed complexity assumptions fail. In fact, we prove that GAIP and other problems from regular group actions are random self-reducible, i.e. there exists a reduction from a particular instance to one or many random instances, and this, if they are NP-hard, leads to the collapse of the Polynomial Hierarchy at the third level. This serves as an upper bound on the worst-case complexity of cryptographic group action and proves that the hardness in the average and the worst case is the same. We also study problems from non-transitive group actions and the strong link with the Graph Isomorphism Problem: in particular, the latter is an instance of the more general Decisional Group Inversion Problem (dGAIP). Other NP intermediate problems are shown to be particular instances of dGAIP. We prove that this problem is both in the class \(\text{coAM}\) and at the second level of the low hierarchy \(L_2^P\).

This paper is organised as follows: in Section 1.1 we recall the formal definition of group actions and their cryptographic properties (one-way-ness, weakly unpredictability and weakly pseudorandomness). In the successive Section 1.2 we introduce some tools from complexity theory: worst-to-average-case reductions and interactive protocols. Section 2 concerns our main result, defining some computational problems from regular group actions and analysing them. We deal also with non-transitive actions, their connection with the Graph Isomorphism Problem and their belonging to some complexity classes. Then Section 3 contains conclusions and observations.

1.1 Cryptographic Group Actions

1.1.1 Effective Group Actions

We recall the definition of group action.

Definition 1. Let \(G\) be a multiplicative group and \(X\) a set. We say that \(G\) acts on \(X\) if there exists a map \(\star : G \times X \rightarrow X\) such that:

1. if \(e\) is the identity of \(G\), for every \(x \in X\) we have \(e \star x = x\) and
2. for every \(g_1, g_2 \in G\) and every \(x \in X\), we have \((g_1 g_2) \star x = g_1 \star (g_2 \star x)\).

The group action is denoted with \((G, X, \star)\).

A group action \((G, X, \star)\) can satisfy some particular properties, it can be:

- **Transitive:** for every \(x_1, x_2 \in X\) there exists \(g \in G\) such that \(x_1 = g \star x_2\).
• **Free:** for every \( g \in G \), \( g \) is the identity if and only if there exists \( x \in X \) such that \( g \ast x = x \).

• **Regular:** if \((G, X, \ast)\) is both free and transitive.

We can see that if the action \((G, X, \ast)\) is regular and the group \( G \) is finite, then for every \( x \in X \) the map \( f_x : g \mapsto g \ast x \) is a bijection and \(|G| = |X|\). Given a transitive action, we can define \( \delta(x, y) \) as the element of \( G \) for which \( x = \delta(x, y) \ast y \). If the action is non-transitive, the element \( \delta(x, y) \) could not exist: we will denote this fact saying that \( \delta(x, y) \) is not in \( G \).

To be suitable in cryptography, [3] defines the concept of effective group action.

**Definition 2.** A group action \((G, X, \ast)\) is effective if

- The group \( G \) is finite and there exist probabilistic polynomial-time (PPT) algorithms for
  1. **Membership testing:** decide whether a bit-string represents an element of \( G \).
  2. **Equality testing:** given two bit-strings, decide whether they represent the same element of \( G \).
  3. **Sampling:** given a distribution \( D_G \) on \( G \), sample with respect to \( D_G \).
  4. **Operation:** compute \( g_1 \ast g_2 \) for every \( g_1, g_2 \in G \).
  5. **Inversion:** compute \( g^{-1} \) for every \( g \in G \).

- The group \( X \) is finite and there exist PPT algorithms for
  1. **Membership testing:** decide whether a bit-string represents an element in \( X \).
  2. **Unique representation:** given an element in \( X \) compute a bit-string that canonically represent it.

- There is an efficient algorithm that given \( g \in G \) and \( x \in X \) computes \( g \ast x \).

Another optional property is the existence of a particular set element \( x_0 \in X \) called origin. Given a free action \((G, X, \ast)\), we can construct the regular action \((G, X_0, \ast)\), where \( X_0 = \{g \ast x_0 \mid g \in G\} \).

For simplicity, since we can always construct a regular action from a free one, we will focus on them in Subsection 2.1 and on non-transitive ones in Subsection 2.2. Note that in [11], Hard Homogeneous Spaces are defined from a regular group action.

There is a particular case when we do not assume the Unique representation property for \( X \). In fact, we admit that computing a canonical form for every element \( x \) in \( X \) can be hard. An example of such group action can be seen in Subsection 2.2.

### 1.1.2 Cryptographic Assumptions

Observe that all the requirements in the previous definitions lead to an efficient (and then, effective) use of the action in a protocol. In order to use them in cryptography we need some computational assumptions, as defined in [3]. With \( \mathbf{P}[A] \) we denote the probability of the event \( A \). A function \( f(\lambda) \) is negligible.
in $\lambda$ if there exists an $\lambda_0$ such that for every $\lambda \geq \lambda_0$ we have $f(\lambda) < \frac{1}{x^c}$ for every $c \in \mathbb{N}$.

In [3], they define concepts of one-way function and weakly unpredictable and weakly pseudorandom permutations. To ease the notation, we directly define these concepts in terms of group actions.

**Definition 3.** Let $\lambda$ be a parameter indexing $G$ and $X$. Given $D_G$ and $D_X$ be two distributions over $G$ and $X$ respectively, then the group action $(G,X,\star)$ is $(D_G,D_X)$-one-way if, given the family $\{f_x : G \rightarrow X\}_{x \in X}$, for all PPT adversaries $A$ the function

$$P(\lambda) := \Pr[A(f_x(g)) = f_x(g)]$$

is negligible in the parameter $\lambda$, where $x$ is sampled according to $D_X$ and $g$ according to $D_G$.

For any $g \in G$ we define the map $\pi_g : X \rightarrow X$ where $\pi_g(x) = g \star x$. If the action is regular, $\pi_g$ is a permutation.

**Definition 4.** Let $\lambda$ be a parameter indexing $G$ and $X$. Given $D_G$ and $D_X$ be two distributions over $G$ and $X$ respectively, then the group action $(G,X,\star)$ is $(D_G,D_X)$-weakly unpredictable if, given the family of permutations $\{\pi_g : X \rightarrow X\}_{g \in G}$ and the randomised oracle $\Pi_g$ that, when queried, samples $x$ from $D_X$ and outputs $(x,\pi_g(x))$, for all PPT adversaries $A$ the function

$$P(\lambda) := \Pr[A(\Pi_g(y)) = \pi_g(y)]$$

is negligible in the parameter $\lambda$, where $y$ is sampled according to $D_X$ and $g$ according to $D_G$.

With $S_X$ we denote the set of permutations over the set $X$.

**Definition 5.** Let $\lambda$ be a parameter indexing $G$ and $X$. Given $D_G$ and $D_X$ be two distributions over $G$ and $X$ respectively, then the group action $(G,X,\star)$ is $(D_G,D_X)$-weakly pseudorandom if, given the family of permutations $\{\pi_g : X \rightarrow X\}_{g \in G}$ and two randomised oracles

1. $\Pi_g$ that, when queried samples $x$ from $D_X$, outputs $(x,\pi_g(x))$;
2. $U$ that, when queried samples $x$ from $D_X$ and $\sigma \in S_X$ uniformly at random, outputs $(x,\sigma(x))$;

for all PPT adversaries $A$ the function

$$P(\lambda) := |\Pr[A^{\Pi_g}(1^\lambda) = 1] - \Pr[A^U(1^\lambda) = 1]|$$

is negligible in the parameter $\lambda$, where $y$ is sampled according to $D_X$ and $g$ according to $D_G$.

The flavour of the previous computational definitions is mostly cryptographic but we want to investigate the hardness of problems they implicate. In Section 2 we will focus on worst-case problems directly related to these computational assumptions.

Cryptographic Group Actions, i.e. group actions for which the above properties are assumed, can be used as a tool in the design of a variety of different cryptographic protocols: key exchange, sigma protocols, dual-mode public key encryption, two-message statistically sender-private oblivious transfer and Naor-Reinoldt pseudorandom function. In many of these applications is required an abelian group, however in this work we consider the generic case. An overview is given in [3].
1.2 Tools from Computational Complexity

Let \( \{0,1\}^* \) be the set of binary string of finite length and \( \{0,1\}^\lambda \) the set of binary string of length \( \lambda \). For any \( x \in \{0,1\}^* \), we denote with \( |x| \) its length.

1.2.1 Polynomial Hierarchy and Low Sets

A decision problem is a set \( D \) of binary strings. We ask if, given a string \( x \), it belongs to \( D \). A search problem \( S \) is a set of couples of binary strings \((x,y)\) and we ask, given \( x \), to find \( y \) such that \((x,y) \in S\).

The class \( \mathcal{P} \) consists of decision problems that can be solved in (deterministic) polynomial-time, while the class \( \mathcal{NP} \) contains decision problems solvable by a non-deterministic polynomial-time algorithm (or, equivalently, by a nondeterministic Turing Machine). Alternatively, \( \mathcal{NP} \) can be defined as all the problems \( D \) such that for every instance \( x \), if \( x \) is in \( D \), then there exists a “witness” \( y_x \) such that \( x \in D \) can be verified in polynomial time thanks to \( y_x \). A problem is said \( \mathcal{NP} \)-hard if every problem in \( \mathcal{NP} \) can be reduced to it in polynomial time. More generally, given a class \( C \), we say that a problem is \( C \)-hard if every problem in that class can be reduced to it. A problem is \( C \)-complete if it \( C \)-hard and it is in \( C \).

The complement of a decision problem \( D \) is denoted with \( \overline{D} \) and consists of all the strings \( x \) that are not in \( D \). Given a class \( C \), we define the class \( \text{co}C \), containing all the complements of problems in \( C \).

The Polynomial Hierarchy (PH) [27] is a chain of classes \( \Sigma_i^P \) and \( \Pi_i^P \) indexed by non-negative integers \( i \), where \( \Sigma_0^P = \text{co} \Pi_0^P \). At lower levels we have \( \Sigma_0^P = \Pi_0^P = \mathcal{P} \) and \( \Sigma_1^P = \Pi_1^P = \mathcal{NP} \). We define \( \text{PH} = \bigcup_{i \geq 0} \Sigma_i^P \), or equivalently \( \text{PH} = \bigcup_{i \geq 0} \Pi_i^P \). It is known that each level \( i \) is contained in the \((i+1)\)-th one, but it is not known if these inclusions are proper. If there exists an \( i \) such that \( \Sigma_i^P = \Sigma_{i+1}^P \), we have that \( \text{PH} = \Sigma_i^P \) [27] and we say that the Polynomial Hierarchy collapses at the \( i \)-th level.

It is believed that \( \text{PH} \) does not collapses at any level.

We can construct another hierarchy called Low Hierarchy [24]. Let \( P(D) \) be the set of decision problems solved by a polynomial-time machine with access to the oracle solving the problem \( D \). Analogously let \( \mathcal{NP}(D) \) be the set of decision problems solved by a non-deterministic polynomial-time machine with access to the oracle solving the problem \( D \). We can extend the definitions above to oracles solving a class \( C \) of problems taking the union \( \mathcal{NP}(C) = \bigcup_{D \in C} \mathcal{NP}(D) \). Now define \( \Sigma_0(C) = P(C) \) and \( \Sigma_k^P(C) = \mathcal{NP}(\Sigma_{k-1}^P(C)) \). A decision problem \( D \) is said low for the level \( k \), in symbols \( D \in L_k^P \), if \( \Sigma_k^P(D) = \Sigma_k^P \).

1.2.2 Random Self-Reductions

We observe that in [14] most of the definitions are based on decision problems. In this work, we deal both with decision and search problems, and the core concept of random self-reduction, introduced in [14], is adapted to our case.

For a binary string \( x \), set \( |x| = \lambda \) and with \( \tau \) we denote the randomness used in the following reduction. The length of \( \tau \) is a polynomially bounded function \( \omega \) of \( \lambda \), that is \( |	au| = \omega(\lambda) \). Also \( k = k(\lambda) \) is a polynomially bounded function.

**Definition 6.** Let \( L \) be a search or decision problem. If \( x \) is an instance for \( L \), we denote with \( F_L(x) \) the function solving \( L \) on \( x \). The problem \( L \) is said nonadaptively \( k \)-random self-reducible if there are polynomial computable functions \( \sigma \) and \( \phi \) such that
- for any \( \lambda \) and \( x \in \{0, 1\}^\lambda \), the reduction is correct with a good probability:

\[
F_L(x) = \phi(x, r, F_L(\sigma(1, x, r)), \ldots, F_L(\sigma(k, x, r))
\]

for at least \( \frac{3}{4} \) of all \( r \) in \( \{0, 1\}^{\omega(\lambda)} \);

- for any \( \lambda \) and any pair of instances \( x_1, x_2 \in \{0, 1\}^\lambda \), if \( r \) is chosen uniformly at random, then \( \sigma(i, x_1, r) \) and \( \sigma(i, x_2, r) \) are instances of \( L \) for every \( 1 \leq i \leq k \).

Moreover, they are identically distributed for every \( 1 \leq i \leq k \).

We briefly comment this definition: on input the round \( i \) and the instance \( x \), \( \sigma \) generates another instance based on the randomness given. The new instances and the randomness \( r \) are given to the reduction \( \phi \) that, using an oracle on these random instaces and the initial one \( x \), returns the answer for \( x \).

The following result is proven in [14] for decision problem.

**Theorem 7.** If a decision problem \( L \) is complete for \( \Sigma^P_i \) or \( \Pi^P_i \) and it is non-adaptively \( k \)-random self-reducible, then the Polynomial Hierarchy collapses at level \( i + 2 \), that is \( \text{PH} = \Sigma^P_{i+2} \).

The class \( \text{NP} \) corresponds to \( \Sigma^P_1 \) and we have the following corollary, originally stated in [14] for decision problem, but adapted in [13] for generic \( \text{NP} \)-hard problems.

**Corollary 8.** If an \( \text{NP} \)-hard problem is nonadaptively \( k \)-random-self-reducible, then the Polynomial Hierarchy collapses at the third level: \( \text{PH} = \Sigma^P_3 \).

Since it is widely believed that the Polynomial Hierarchy does not collapse at any level, no complete problems that are nonadaptively \( k \)-random-self-reducible should exist.

### 1.2.3 Interactive Proofs and Arthur-Merlin Games

Now we deal with interactive protocols and proofs. We recall the definition of the class \( \text{IP} \), introduced in [16].

**Definition 9.** Let \( f \) be an integer function. A decision problem \( D \) is in \( \text{IP}[f(n)] \) if there exist two algorithms, \text{PROVER} and \text{VERIFIER}, such that on common input \( x \):

1. \text{PROVER} and \text{VERIFIER} exchange \( f(|x|) \) messages;
2. \text{VERIFIER} is probabilistic and polynomial-time in \( |x| \);
3. \text{PROVER} is computationally unbounded;
4. if \( x \) is in \( D \), then \text{VERIFIER} accepts with a probability of at least \( \frac{3}{4} \);
5. if \( x \) is not in \( D \), then \text{VERIFIER} accepts with probability at most \( \frac{1}{4} \).

The class \( \text{IP} \) is defined as \( \text{IP} = \bigcup_{k \geq 0} \text{IP}[n^k] \).

We can see that \( \text{NP} \) is contained in \( \text{IP}[1] \), interactive proofs with only one round of communication.

Another useful class consists in Arthur-Merlin games (or protocols) [4]. In brief, an \( \text{AM} \) protocol is an interactive protocol where \text{PROVER} sends only random strings. We define \( \text{AM} = \text{AM}[2] \) and the reason will be given in the next theorem. The main difference between \( \text{IP} \) and \( \text{AM} \) is that in the latter, the randomness produced by \( \text{VERIFIER} \) is public, while for \( \text{IP} \) this is not needed. The next result synthesises some facts about \( \text{IP} \) and \( \text{AM} \) [4, 17].
Theorem 10. For every integer function $f$:

- $\text{AM}[k] = \text{AM}$ for every $k \geq 2$;
- $\text{IP}[f(n)] \subseteq \text{AM}[f(n) + 2]$ for every positive integer $n$.

Now we present results from [8, 25], dealing with the collapse of $\text{PH}$.

Theorem 11. We have that $\text{NP} \cap \text{coAM} \subseteq L^{P^2}$. Moreover, if $\text{NP} \subseteq \text{coAM}$, then $\text{PH}$ collapses to the second level.

In the following sections, we will show that some problems cannot be $\text{NP}$-hard unless $\text{PH} = \Sigma^P_2$ or $\text{PH} = \Sigma^P_3$.

2 Problems from Group Actions

We consider group actions $(G, X, \star)$ where the length of the bit-string representation of $X$ (or $G$) is polynomial in the length of the representation of $G$ (or $X$, respectively). Moreover, in this section we assume that, in the definition of effective group action, all the operations involving $G$ and $X$ can be performed in polynomial time (instead of probabilistic polynomial time, see Definition 2).

2.1 Regular Group Actions

From now on, we refer to $\lambda$ as the complexity parameter indexing $G$ and $X$. For instance, $\lambda$ can be the length of the bit-string representation of $G$ and $X$. We define the following computational problems.

**Definition 12.** Let $(X, G, \star)$ be a group action. Given $x$ and $y$ in $X$, find, if there exists, an element $g \in G$ such that $x = g \star y$. This problem is called **Group Action Inverse Problem** (GAIP).

We can see that the GAIP models in the worst-case the cryptographic concept of one-way group action. In fact, if there exists a polynomial-time algorithm that solves GAIP, then it will break the One-way group action assumption.

**Definition 13.** Let $(X, G, \star)$ be a group action and let $q$ be an integer polynomial. Given the set of pairs $\{(x_i, y_i)\}_{i=1}^q$, find, if there exists, an element $g \in G$ such that $x_i = g \star y_i$ for every $i$. This problem is called **Multiple Group Action Inverse Problem** (mGAIP).

Even in this case, mGAIP models the worst-case scenario of a $(D_G, D_X)$-weakly unpredictable group action.

**Definition 14.** Let $(X, G, \star)$ be a group action and let $q$ be an integer polynomial. Let $L_g$ be the set indexed by $g \in G$ containing all the pairs $(x, g \star x)$. Given the set of pairs $\{(x_i, y_i)\}_{i=1}^q$, decide if it is a subset of $L_g$ for some $g$ or it is sampled uniformly from $X \times X$. This problem is called **Pseudorandom Group Action Inverse Problem** (pGAIP).

Finally, pGAIP is the worst-case scenario of a $(D_G, D_X)$-weakly pseudorandom group action. Observe that both mGAIP and pGAIP are polynomially reducible to GAIP.
Theorem 15. Let \((G, X, \ast)\) be a regular effective group action. Then

1. Group Action Inverse Problem,
2. Multiple Group Action Inverse Problem,
3. Pseudorandom Group Action Inverse Problem

are nonadaptively 1-random self-reducible.

Proof. 1. We denote with \(F_{\text{GAIP}}\) the function that solves the GAIP problem: on input \((x, y)\), \(F_{\text{GAIP}}(x, y)\) is an element of \(G\) such that \(x = F_{\text{GAIP}}(x, y) \ast y\). We show that there exist polynomial computable functions \(\sigma\) and \(\phi\) that satisfy Definition 6.

Let \(x, y \in X\), since the action is regular, there exists \(g\) for which \(g \ast x = y\). Let \(r\) be the randomness string of length polynomial in \(\lambda = |x|\). Set \(\sigma\) as the function whose, on inputs \((i, (x, y), \tau)\), deterministically generates elements \(g_{x,i}\) and \(g_{y,i}\) from \(\tau\) and \(i\), and outputs \((g_{x,i} \ast x, g_{y,i} \ast y)\). By construction \(\sigma(i, (x, y), \tau)\) is an instance of GAIP. Then define the function \(\phi\) that on inputs \(((x, y), \tau, F_{\text{GAIP}}(\sigma(1, (x, y), \tau)))\), generates \(g_{x,1}\) and \(g_{y,1}\) from \(\tau\), and returns \((g_{x,1}^{-1} \ast F_{\text{GAIP}}(\sigma(1, (x, y), \tau)) \ast g_{y,1})\). If we denote with \(x_{1}, y_{1}\) the output of \(\sigma(1, (x, y), \tau)\), we can verify

\[
\begin{align*}
x &= g_{x,1}^{-1} \ast x_{1} \\
&= g_{x,1}^{-1} \ast (F_{\text{GAIP}}(x_{1}, y_{1}) \ast y_{1}) \\
&= (g_{x,1}^{-1} \ast F_{\text{GAIP}}(x_{1}, y_{1})) \ast (g_{y,1} \ast y)
\end{align*}
\]

and the reduction is correct with probability 1. By construction, if \(r\) is chosen uniformly at random, then the random variables \((\tilde{x}, \tilde{y}) = \sigma(1, (x, y), \tau)\) and \((\tilde{x}', \tilde{y}') = \sigma(1, (x', y'), \tau)\) are identically distributed. Indeed let \(z, w \in X\), then

\[
P[\tilde{x} = z, \tilde{y} = w] = P[g_{x,i} = \delta(z, x), g_{y,i} = \delta(w, y)]
\]

\[
= \frac{1}{|G|^{2}} = P[g'_{x,i} = \delta(z, x'), g'_{y,i} = \delta(w, y')]
\]

\[
= P[\tilde{x}' = z, \tilde{y}' = w],
\]

where the second and fourth equalities use the fact that \((g_{x,i}, g_{y,i})\) and \((g'_{x,i}, g'_{y,i})\) are picked independently from \((x, y)\) and \((x', y')\). This implies that GAIP is nonadaptively 1-random self-reducible.

2. Suppose \(F_{\text{mGAIP}}\) is the function that solves mGAIP. This reduction is very similar to the one for GAIP. Given \(Q = \{(x_{j}, y_{j})\}_{j=1}^{q(\lambda)}\), where \(q\) is a polynomial, on input \((i, Q, \tau)\), \(\sigma\) generates \(g_{Q,i}\) and sets \(\tilde{x}_{j} = g_{Q,i} \ast x_{j}\) and \(\tilde{y}_{j} = g_{Q,i} \ast y_{j}\) for each \(j = 1, \ldots, q(\lambda)\), then outputs \(\{\tilde{x}_{j}, \tilde{y}_{j}\}_{j=1}^{q(\lambda)}\). By construction \(\sigma(i, Q, \tau)\) is an instance of mGAIP. The function \(\phi\), on inputs \((Q, \tau, F_{\text{mGAIP}}(\sigma(1, Q, \tau)))\), retrieves \(g_{Q,1}\) and outputs \((g_{Q,1}^{-1} \ast F_{\text{mGAIP}}(\sigma(1, Q, \tau)) \ast g_{Q,1})\). Let \(Q = \sigma(1, Q, \tau) = \{\tilde{x}_{j}, \tilde{y}_{j}\}_{j=1}^{q(\lambda)}\), then

\[
(g_{Q,1}^{-1} \ast F_{\text{mGAIP}}(\tilde{Q})) \ast y_{i} = (g_{Q,1}^{-1} \ast F_{\text{mGAIP}}(\tilde{Q})) \ast (g_{Q,1} \ast y_{j})
\]

\[
= g_{Q,1}^{-1} \ast (F_{\text{mGAIP}}(\tilde{Q})) \ast (g_{Q,1} \ast y_{j})
\]

\[
= g_{Q,1}^{-1} \ast (g_{Q,1} \ast x_{j}) = x_{j}
\]
for every $j = 1, \ldots, q(\lambda)$. Given $Q_1$ and $Q_2$ of the same form of $Q$ above, if $\tau$ is chosen uniformly random, then $\sigma(i, Q_1, \tau)$ and $\sigma(i, Q_2, \tau)$ are identically distributed: let $z, w \in X$, then for every $(x_j, y_j)$ in $Q_1$ and $(x'_j, y'_j)$ in $Q_2$ we have

\[
P[\tilde{x}_j = z, \tilde{y}_j = w] = P[g_{Q_1,i} = \delta(z, x_j), g_{Q_2,i} = \delta(w, y_j)]
= P[g_{Q_2,i} = \delta(z, x'_j), g_{Q_2,i} = \delta(w, y'_j)]
= P[\tilde{x}'_j = z, \tilde{y}'_j = w],
\]

where we identify with $\{\tilde{x}_j, \tilde{y}_j\}_{j=1}^{q(\lambda)}$ and $\{\tilde{x}'_j, \tilde{y}'_j\}_{j=1}^{q(\lambda)}$ the output of $\sigma(i, Q_1, \tau)$ and $\sigma(i, Q_2, \tau)$, respectively.

We conclude that $m\text{GAIP}$ is nonadaptively 1-random self-reducible.

3. Suppose $F_{p\text{GAIP}}$ is the function that decides $p\text{GAIP}$. Given $Q = \{(x_j, y_j)\}_{j=1}^{q(\lambda)}$, we have $F_{p\text{GAIP}}(Q) = 1$ if there exists $g \in G$ such that $x_j = g \cdot y_j$ for every $j = 1, \ldots, q(\lambda)$, while $F_{p\text{GAIP}}(Q) = 0$ if $x_j$ and $y_j$ are sampled from the uniform distribution over $X$.

Given the instance $Q = \{(x_j, y_j)\}_{j=1}^{q(\lambda)}$, on input $(i, Q, \tau)$, $\sigma$ generates $g_{Q,i}$ and sets $\tilde{x}_j = g_{Q,i} \cdot x_j$ and $\tilde{y}_j = g_{Q,i} \cdot y_j$ for each $j = 1, \ldots, q(\lambda)$, then outputs $\{(\tilde{x}_j, \tilde{y}_j)\}_{j=1}^{q(\lambda)}$. We can see that $\sigma(i, Q, \tau)$ is an instance of $p\text{GAIP}$: if $F_{p\text{GAIP}}(Q) = 1$, then it fits in the definition. Otherwise when $F_{p\text{GAIP}}(Q) = 0$, then the output of $\sigma$ is a set of uniformly picked couple of elements in $X$. Let $z, w \in X$, then since $\pi_{g_{Q,i}}$ is a permutation,

\[
P[\tilde{x}_j = z, \tilde{y}_j = w] = P[g_{Q,i} \cdot x_j = z, g_{Q,i} \cdot y_j = w] = \frac{1}{|X|},
\]

The function $\phi$, on inputs $(Q, \tau, F_{p\text{GAIP}}(\sigma(1, Q, \tau)))$, simply outputs $F_{p\text{GAIP}}(\sigma(1, Q, \tau))$. We prove that the reduction is correct: let

\[
\tilde{Q} = \sigma(1, Q, \tau) = \{\tilde{x}_j, \tilde{y}_j\}_{j=1}^{q(\lambda)},
\]

then

- if $F_{p\text{GAIP}}(\tilde{Q}) = 1$, then there exists $\tilde{g}$ such that $\tilde{x}_j = \tilde{g} \cdot \tilde{y}_j$ for every $j$. Then we have that $x_j = (g_{Q}^{-1} \cdot \tilde{g}) \cdot y_j$ and this implies $F_{p\text{GAIP}}(Q) = 1$.

- If $F_{p\text{GAIP}}(\tilde{Q}) = 0$, then $\tilde{x}_j, \tilde{y}_j$ are sampled from the uniform distribution over $X$, and so are $g_{Q}^{-1} \cdot \tilde{x}_j = x_j$ and $g_{Q}^{-1} \cdot \tilde{y}_j = y_j$. This implies $F_{p\text{GAIP}}(Q) = 0$.

Given $Q_1$ and $Q_2$ of the same form of $Q$ above, if $\tau$ is chosen uniformly random, then $\sigma(1, Q_1, \tau)$ and $\sigma(1, Q_2, \tau)$ are identically distributed due to equations (1) and (2). Then $p\text{GAIP}$ is nonadaptively 1-random self-reducible.

We point out that these problems fit also in the definition of information hiding schemes (ihs) from [1], in fact, both $\text{GAIP}$, $m\text{GAIP}$ and $p\text{GAIP}$ are 1-ihs using the proof of Theorem 15.

Due to the previous result and to Corollary 8, we can show the following fact.

**Corollary 16.** If $\text{GAIP}$, $m\text{GAIP}$ or $p\text{GAIP}$ are NP-hard, the Polynomial Hierarchy collapses at the third level.
2.2 Non-transitive Group Actions

Now we consider non-transitive group actions where, given \( x, y \) in \( X \), we do not know if there exists \( \delta(x, y) \). We want to analyse the complexity of a decisional variant of \( \text{GAIP} \), using the same technique of [23] for Code Nonequivalence Problem and [15] for Graph Nonisomorphism Problem.

**Definition 17.** Let \( (G, X, \star) \) be an effective non-transitive group action. Given \( x \) and \( y \) in \( X \) decide whether \( \delta(x, y) \) exists. This problem is called **Decisional Group Inversion Problem** (dGAIP).

It is easy to see that dGAIP is in \( \text{NP} \): if \( (x, y) \) is a yes-instance, the element \( \delta(x, y) \) is a witness.

This problem embraces many decision problems present in literature.

1. Let \( G = S_{\{1, \ldots, n\}} \) be the set of permutations of \( n \) elements, and let \( X = G_n \) be the set of graphs with \( n \) vertices. The action of \( G \) over \( X \) is defined as the permutation of the vertices of the graph. The dGAIP for the above group action is the **Graph Isomorphism Problem**. Many other isomorphism problems fall in this framework, for example the **Ring Isomorphism Problem** [18] and the **Polynomial Isomorphism Problem** [12].

2. Let \( F_2 \) be the field with two elements. Set \( G = \text{GL}_k(F_2) \times S_{\{1, \ldots, n\}} \), where \( \text{GL}_k(F_2) \) is the set of \( k \times k \) invertible matrices over the field of two elements and \( S_{\{1, \ldots, n\}} \) be the set of permutation of \( n \) elements represented by binary \( n \times n \) matrices. Let \( X = C_{n,k}(F_2) \) be the set of generator matrices of linear codes over \( F_2 \) having length \( n \) and dimension \( k \). The action of \( G \) over \( X \) is defined as

\[
\star : \quad G \times X \rightarrow X \\
((S, P), A) \mapsto (S, P) \star A = SAP.
\]

The dGAIP for the above group action coincides with the **Permutation Code Equivalence Problem**.

3. If \( G \) is a graph with \( n \) vertices and \( \pi \in S_{\{1, \ldots, n\}} \), we denote with \( \pi(G) \) the graph with vertices permuted by \( \pi \). Given a graph \( G \) with vertices \( \{1, \ldots, n\} \), the graph \( G^{(1)} \) is the subgraph with vertices \( \{1, \ldots, n\} \setminus \{i\} \). A **deck** of \( G \) is a \( n \)-tuple \( D(G) = (G^{(1)}, \ldots, G^{(n)}) \) and we say that \( G \) is a **reconstruction** of \( D(G) \). Given a graph \( G \) and a sequence of graphs \( D = (G_1, \ldots, G_n) \), the problem of deciding whether \( G \) is a reconstruction of \( D \) is called **Deck Checking Problem** [19, 20]. Now let \( G = (S_{\{1, \ldots, n\}})^{n+1} \) and \( X = (G_{n-1})^n \). Define \( \star \) as follows

\[
\star : \quad G \times X \rightarrow X \\
((\sigma, \pi_1, \ldots, \pi_n), (G_1, \ldots, G_n)) \mapsto (\pi_1(G_{\sigma(1)}), \ldots, \pi_n(G_{\sigma(n)})).
\]

We can see the Deck Checking Problem as the instance \( (D(G), D) \) of dGAIP for the previous group action. This is not surprising since Deck Checking can be reduced to Graph Isomorphism.

4. Now we admit that the canonical form for \( X \) can be computed having access to an oracle for \( \text{NP} \). Let \( G = S_{\{1, \ldots, n\}} \) and \( X \) be the set of boolean functions on \( n \) variables, the action is defined as the permutation of variables \( x_1, \ldots, x_n \). The dGAIP for this group action is called **Boolean Isomorphism Problem** and
Parameters: \((G, X, \star)\)

Input: \((y_0, y_1) \in X \times X\)

**Verifier**

- Picks \(b \in \{0, 1\}\) and \(g \in G\) at random.

\[
x = g \star y_0
\]

- Computes \(b'\), where \(\delta(x, y_0) \in G\).

**Prover**

Accepts if \(b = b'\).

Figure 1: Interactive protocol for \(dGAIP^\perp\)

It is in \(\Sigma^P_2\) and \(\text{coNP}\)-hard [9], but unlikely to be \(\Sigma^P_2\)-complete, unless \(PH = \Sigma^P_3\) [2].

Due to the similarity to such known problems, we can prove the following theorem.

**Proposition 18.** Let \((G, X, \star)\) be an effective non-transitive group action. Then \(dGAIP\) is in \(\text{coAM}\).

**Proof.** This proof uses the same technique used in Section III of [23]. We design an interactive protocol for \(\overline{dGAIP}\), the complement of \(dGAIP\).

We can show that the protocol in Figure 1 satisfies requirements in Definition 9. Given a couple \(y_0, y_1\) such that they belong to \(\overline{dGAIP}\), **Verifier** will always accepts. Conversely, if \((y_0, y_1)\) does not belong to \(\overline{dGAIP}\), then there exists \(h\) such that \(y_0 = h \star y_1\) and then **Prover** can only guess \(b'\) since both \(\delta(x, y_0)\) and \(\delta(x, y_1)\) are in \(G\). Then \(b = b'\) happens with probability \(\frac{1}{2}\). Repeating the protocol a polynomial number of time gives us a probability at most \(\frac{1}{4}\) that **Verifier** accepts.

Then \(\overline{dGAIP}\) is in \(\text{IP}[2] \subseteq \text{AM}[4] \subseteq \text{AM}\). This implies that \(dGAIP\) is in \(\text{coAM}\).

Using Theorem 11, we can then state the following result.

**Corollary 19.** Let \((G, X, \star)\) be an effective non-transitive group action. If \(dGAIP\) is \(\text{NP}\)-complete, the Polynomial Hierarchy collapses at the second level.

Another clue that \(dGAIP\) is not \(\text{NP}\)-complete comes from the fact that it is in \(\text{L}^P_2\), in fact, since \(dGAIP\) is in both \(\text{NP}\) and \(\text{coAM}\), using Theorem 11, we have the following result.

**Corollary 20.** Let \((G, X, \star)\) be an effective non-transitive group action. Then \(dGAIP\) is in \(\text{L}^P_2\), i.e. the second class of the low hierarchy.

We point out that Corollary 20 has the same consequences of Proposition 18: in case \(dGAIP\) is \(\text{NP}\)-complete we have \(PH = \Sigma^P_2\) using a result from [25].
3 Conclusions

We showed how some problems deriving from standard computation assumptions for group actions cannot be NP-hard under widely believed conjectures of Complexity Theory. This should not be seen as a completely negative result and their use in cryptography could still be secure: these results concern the worst-case scenario and do not influence the validity of security assumptions used in cryptographic proofs. The fact that a problem is NP-hard is not a certificate of security: for instance, we can think about the Merkle–Hellman knapsack cryptosystem [22] based on the NP-complete Subset Sum Problem but broken in polynomial time [26]. Another consideration about implications of Theorem 15 is that GAIP, mGAIP and pGAIP are solvable with the same effort both on hard and random instances. This implies that the hardness of these problems in the worst-case equals the hardness in the average-case.

The strong link seen in Section 2.2 between dGAIP and other central problems in Complexity Theory, like the Graph Isomorphism Problem, suggests that problems from group actions could be situated in the gap between the set of NP-hard ones and problems solvable in polynomial time.

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References

[1] Abadi, M., Feigenbaum, J., Kilian, J.: On hiding information from an oracle. Journal of computer and system sciences 39(1), 21–50 (1989)

[2] Agrawal, M., Thierauf, T.: The boolean isomorphism problem. In: Proceedings of 37th Conference on Foundations of Computer Science, pp. 422–430. IEEE (1996)

[3] Alamati, N., De Feo, L., Montgomery, H., Patranabis, S.: Cryptographic group actions and applications. In: International Conference on the Theory and Application of Cryptology and Information Security, pp. 411–439. Springer (2020)

[4] Babai, L., Moran, S.: Arthur-merlin games: a randomized proof system, and a hierarchy of complexity classes. Journal of Computer and System Sciences 36(2), 254–276 (1988)

[5] Barenghi, A., Biasse, J.F., Persichetti, E., Santini, P.: Less-fm: Fine-tuning signatures from a code-based cryptographic group action. IACR Cryptol. ePrint Arch. 2021, 396 (2021)

[6] Beullens, W., Katsumata, S., Pintore, F.: Calamari and falaf: Logarithmic (linkable) ring signatures from isogenies and lattices. In: International Conference on the Theory and Application of Cryptology and Information Security, pp. 464–492. Springer (2020)
[7] Beullens, W., Kleinjung, T., Vercauteren, F.: Csi-fish: Efficient isogeny based signatures through class group computations. In: International Conference on the Theory and Application of Cryptology and Information Security, pp. 227–247. Springer (2019)

[8] Boppana, R.B., Hastad, J., Zachos, S.: Does co-np have short interactive proofs? Information Processing Letters 25(2), 127–132 (1987)

[9] Borchert, B., Ranjan, D., Stephan, F.: On the computational complexity of some classical equivalence relations on boolean functions. Theory of Computing Systems 31(6), 679–693 (1998)

[10] Castryck, W., Lange, T., Martindale, C., Panny, L., Renes, J.: Csidh: an efficient post-quantum commutative group action. In: International Conference on the Theory and Application of Cryptology and Information Security, pp. 395–427. Springer (2018)

[11] Couveignes, J.M.: Hard homogeneous spaces. IACR Cryptol. ePrint Arch. 2006, 291 (2006)

[12] Faugere, J.C., Perret, L.: Polynomial equivalence problems: Algorithmic and theoretical aspects. In: Annual International Conference on the Theory and Applications of Cryptographic Techniques, pp. 30–47. Springer (2006)

[13] Feigenbaum, J.: Locally random reductions in interactive complexity theory. In: Advances in Computational Complexity Theory, pp. 73–98 (1990)

[14] Feigenbaum, J., Fortnow, L.: Random-self-reducibility of complete sets. SIAM Journal on Computing 22(5), 994–1005 (1993)

[15] Goldreich, O., Micali, S., Wigderson, A.: Proofs that yield nothing but their validity or all languages in np have zero-knowledge proof systems. Journal of the ACM (JACM) 38(3), 690–728 (1991)

[16] Goldwasser, S., Micali, S., Rackoff, C.: The knowledge complexity of interactive proof systems. SIAM Journal on Computing 18(1), 186–208 (1989)

[17] Goldwasser, S., Sipser, M.: Private coins versus public coins in interactive proof systems. In: Proceedings of the eighteenth annual ACM symposium on Theory of computing, pp. 59–68 (1986)

[18] Kayal, N., Saxena, N.: On the ring isomorphism & automorphism problems. In: 20th Annual IEEE Conference on Computational Complexity (CCC’05), pp. 2–12. IEEE (2005)

[19] Köbler, J., Schöning, U., Torán, J.: Graph isomorphism is low for pp. Computational Complexity 2(4), 301–330 (1992)

[20] Kratsch, D., Heusenpandla, L.A.: On the complexity of graph reconstruction. Mathematical Systems Theory 27(3), 257–273 (1994)

[21] Levin, L.A.: Average case complete problems. SIAM Journal on Computing 15(1), 285–286 (1986)

[22] Merkle, R., Hellman, M.: Hiding information and signatures in trapdoor knapsacks. IEEE transactions on Information Theory 24(5), 525–530 (1978)
[23] Petrank, E., Roth, R.M.: Is code equivalence easy to decide? IEEE Transactions on Information Theory 43(5), 1602–1604 (1997)

[24] Schöning, U.: A low and a high hierarchy within np. Journal of Computer and System Sciences 27(1), 14–28 (1983)

[25] Schöning, U.: Graph isomorphism is in the low hierarchy. Journal of Computer and System Sciences 37(3), 312–323 (1988)

[26] Shamir, A.: A polynomial time algorithm for breaking the basic merkle-hellman cryptosystem. In: 23rd Annual Symposium on Foundations of Computer Science (sfcs 1982), pp. 145–152. IEEE (1982)

[27] Stockmeyer, L.J.: The polynomial-time hierarchy. Theoretical Computer Science 3(1), 1–22 (1976)