MEAN PERIODIC SOLUTIONS OF A INHOMOGENEOUS HEAT EQUATION WITH RANDOM COEFFICIENTS

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Abstract. We present conditions ensuring the periodicity of the mathematical expectation of a solution of a scalar linear inhomogeneous heat equation with random coefficients where the coefficient in front of the unknown functions is Gaussian or it is uniformly distributed. The obtained results may be treated as finding a control ensuring the periodicity of the mathematical expectation of a solution of the heat equation.

1. Introduction. Existence conditions of periodic solutions of equation

\[
\frac{dx}{dt} = \varepsilon(t)x + f(t), \quad t \in \mathbb{R}
\]

with deterministic periodic functions \( \varepsilon \) and \( f \) are well known [7].

The problem becomes much more complicated if \( \varepsilon \) and \( f \) are random processes.

The stability of solutions of differential equations with random right-hand side was studied, e.g., in [3].

In [9], [10], the conditions are given under which the mathematical expectation of a solution of equation (1) is periodic.

Many authors have studied the existence of periodic solutions of parabolic equations without randomness with various conditions (see, for instance, [1], [2], [4], [5]).

Consider the Cauchy problem

\[
\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \varepsilon(t)y + f(t, x), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R},
\]

\[y(0, x) = y_0(x),\]

where \( \varepsilon(t) \) and \( f(t, x) \) are independent random processes given by the characteristic functionals \( \varphi_\varepsilon(u) \) and \( \varphi_f(v) \) (the dependence on random events is not indicated).

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A solution of problem (2), (3) is a random process. It is said to be mean periodic with respect to \( t \) if its mathematical expectation is a periodic function with respect to \( t \).

In this paper we will study the mean periodicity of a solution of equation (2) with respect to \( t \in \mathbb{R}_+ \) for two cases.

1) The random process \( \varepsilon(\cdot) \) is Gaussian with the characteristic functional of the form (see, for instance, [8], p. 206)

\[
\varphi_\varepsilon(u) = \exp \left( i \int_{\mathbb{R}_+} E(\varepsilon(s))u(s)\,ds - \frac{1}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} b(s_1, s_2)u(s_1)u(s_2)\,ds_1\,ds_2 \right),
\]

where \( E(\varepsilon(s)) \) is the mathematical expectation, and \( b(s_1, s_2) = E(\varepsilon(s_1)\varepsilon(s_2)) - E(\varepsilon(s_1))E(\varepsilon(s_2)) \) is the covariance function of the random process \( \varepsilon \).

2) The random process \( \varepsilon(\cdot) \) is uniformly distributed with the characteristic functional given by the formula ([8], p. 300)

\[
\varphi_\varepsilon(u) = \frac{\sin \int_{\mathbb{R}_+} a(s)u(s)\,ds}{\int_{\mathbb{R}_+} a(s)u(s)\,ds} \exp \left( i \int_{\mathbb{R}_+} \xi(s)u(s)\,ds \right),
\]

where \( a(s) \geq 0 \) and \( \xi(s) = E(\varepsilon(s)) \) are given continuous functions.

If \( a(s) \equiv 0 \), then we assume that

\[
\varphi_\varepsilon(u) = \exp \left( i \int_{\mathbb{R}_+} \xi(s)u(s)\,ds \right).
\]

2. Auxiliary information. Further the notion of variational derivative is used. We give the corresponding definition. Let \( L_1(\mathbb{R}_+) \) be the space of complex-valued functions integrable on \( \mathbb{R}_+ \) equipped with the norm \( \int_{\mathbb{R}_+} |u(s)|\,ds \), \( \varphi \) be a functional on the space \( L_1(\mathbb{R}_+) \), and let \( h \in L_1(\mathbb{R}_+) \). If

\[
\varphi(u + h) - \varphi(u) = \int_{\mathbb{R}_+} \psi(t, u)h(t)\,dt + o(h),
\]

where the integral is treated in the Lebesgue sense and is a linear functional bounded with respect to \( h \), then \( \psi(t, u) \) is referred to as the variational derivative of the functional \( \varphi \) at the point \( u \).

We will use the function \( \chi(s, t, \tau) = \chi(s, t)(\tau) \), defined on \( \mathbb{R}_+ \) as follows: \( \chi(s, t, \tau) \) is equal to \( \text{sign}(\tau - s) \) for \( \tau \) belonging to the closed interval with endpoints \( s \) and \( t \) and to zero for other \( \tau \).

**Theorem 2.1** ([8], p.297-298). Let \( y_0 \) be a twice continuously differentiable function. If there exist a variational derivative for \( \varphi_\varepsilon(u) \) and continuous mathematical expectation \( E(f(t, x)) \), then the mathematical expectation of a solution of problem (2), (3) is given by the formula

\[
E(y(t, x)) = U_x(t)y_0(x)\varphi_\varepsilon(-i\chi(0, t)) + \int_0^t U_x(t - s)E(f(s, x))\varphi_\varepsilon(-i\chi(s, t))\,ds,
\]

where the semi-group \( U_x(t) \) is defined by the relation

\[
U_x(t)z(\nu, x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} \exp \left( -\frac{\tau^2}{4t} \right) z(\nu, x - \tau)\,d\tau, \quad t \in \mathbb{R}_+,
\]

\( U_x(0) = I \),

\( \nu \in \mathbb{R} \) is a parameter.
As usual, $I$ stands for the identity operator.

3. Operator $W(t,s)$ and its properties. At first we give here a some explanation concerning the inverse operator $U^{-1}_x(t)$. The semi-group $U_x(t)$ transfers the initial value $y(0,x) = y_0(x)$ into a solution of the initial value problem

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}, \quad y(0,x) = y_0(x). \quad (7)$$

It should be noted (see, for instance, [6], p. 196) the uniqueness of a continuous bounded on the whole range of variables solution of problem (7). For fixed $t = t_0$ the function $y(t_0, x)$ is called as the realization of solution at $t = t_0$. The operator $U^{-1}_x(t_0)$ transfers the realization $y(t_0, x)$ into the corresponding initial value $y_0(x)$. Thus, the domain of the linear operator $U^{-1}_x(t)$ with fixed $t$ is a set of realizations of solutions of equation (7) and the range of values of this operator is a set of initial values $\{y_0(x)\}$.

Let $C(L_1(\mathbb{R}_+))$ be the space of continuous bounded functionals on $L_1(\mathbb{R}_+)$ with the norm $||\varphi|| = \sup_{u \in L_1(\mathbb{R}_+)} |\varphi(u)|$.

Introduce the operator $W(t,s)$ by the following way

$$W(t,s)(z(\nu,x)\varphi(u)) = U_x(t)U^{-1}_x(s)z(\nu,x)V(t,s)\varphi(u),$$

where the operator $V(t,s) : C(L_1(\mathbb{R}_+)) \rightarrow C(L_1(\mathbb{R}_+))$ is defined as follows

$$V(t,s)\varphi(u) = \varphi(u - i\chi(s,t)).$$

**Lemma 3.1.** The operator $W(t,s)$ has the following properties:

$$W(t,t) = I, \quad (8)$$

$$W(t,\tau)W(\tau,s) = W(t,s), \quad (9)$$

$$W^{-1}(t,s) = W(s,t). \quad (10)$$

**Proof.** Using the definition of the operator $W(t,s)$, we get (8), namely

$$W(t,t)(z(\nu,x)\varphi(u)) = U_x(t)U^{-1}_x(t)z(\nu,x)V(t,t)\varphi(u)$$

$$= z(\nu,x)\varphi(u - i\chi(t,t))$$

$$= z(\nu,x)\varphi(u).$$

Further we have

$$W(t,\tau)W(\tau,s)(z(\nu,x)\varphi(u))$$

$$= W(t,\tau)U_x(\tau)U^{-1}_x(s)z(\nu,x)V(\tau,s)\varphi(u)$$

$$= U_x(t)U^{-1}_x(\tau)U_x(\tau)U^{-1}_x(s)z(\nu,x)V(t,\tau)\varphi(u - i\chi(s,\tau))$$

$$= U_x(t)U^{-1}_x(s)z(\nu,x)\varphi(u - i\chi(s,\tau) - i\chi(\tau,t))$$

$$= U_x(t)U^{-1}_x(s)z(\nu,x)\varphi(u - i\chi(s,t))$$

$$= U_x(t)U^{-1}_x(s)z(\nu,x)V(t,s)\varphi(u)$$

$$= W(t,s)(z(\nu,x)\varphi(u)).$$

From here (9) follows.

We set in (9) at first $s = t$, $\tau = s$ and then $t = s$, $\tau = t$. Taking into account (8) we get

$$W(t,s)W(s,t) = W(t,t) = I,$$

$$W(s,t)W(t,s) = W(s,s) = I.$$
The property (10) follows from the two last equalities.

Lemma 3.2. If \( \varepsilon(\cdot) \) is the Gaussian random process and in (4) \( E(\varepsilon(\cdot)) \) is an \( \omega \)-periodic function, \( b(\cdot, \cdot) \) is an \( \omega \)-periodic function of both variables or \( \varepsilon(\cdot) \) is the uniformly distributed process and in (5) \( \alpha(\cdot), \xi(\cdot) \) are \( \omega \)-periodic functions, then the following assertions hold.

\[
W(t + \omega, 0)(z(\nu, x)\varphi_\varepsilon(u)) = W(\omega, 0)W(t, 0)(z(\nu, x)\varphi_\varepsilon(u))
\]

\[
=W(t, 0)W(\omega, 0)(z(\nu, x)\varphi_\varepsilon(u)),
\]

(11)

\[
W(0, t + \omega)(z(\nu, x)\varphi_\varepsilon(u)) = W(0, \omega)W(0, t)(z(\nu, x)\varphi_\varepsilon(u))
\]

\[
=W(0, \omega)W(0, t)(z(\nu, x)\varphi_\varepsilon(u)).
\]

(12)

Proof. Let \( \varepsilon(\cdot) \) have characteristic functional (4). We will use the properties of the semi-group \( U_x(t) \) and the equality \( \int_{t}^{t+\omega} a(s)ds = \int_{0}^{\omega} a(s)ds \) which is correct for each \( t \in \mathbb{R} \) and any \( \omega \)-periodic function \( a(\cdot) \). Taking into account the symmetry of the function \( b(s_1, s_2) \) and the Fubini theorem we have

\[
W(t + \omega, 0)(z(\nu, x)\varphi_\varepsilon(u))
\]

\[
=U_x(t + \omega)U_x^{-1}(0)z(\nu, x)V(t + \omega, 0)\varphi_\varepsilon(u)
\]

\[
=U_x(t)U_x(\omega)(z(\nu, x)\varphi_\varepsilon(u - i\chi(0, t + \omega))
\]

\[
=U_x(t)U_x(\omega)(z(\nu, x)\exp \left(i \int_{\mathbb{R}_+} E(\varepsilon(s))u(s)ds + \int_{0}^{t} E(\varepsilon(s))ds + \int_{t}^{t+\omega} E(\varepsilon(s))ds
\right)
\]

\[
- \frac{1}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} b(s_1, s_2)u(s_1)u(s_2)ds_1ds_2 + \int_{\mathbb{R}_+} b(s_1, s_2)u(s_1)ds_2ds_1
\]

\[
+ i \int_{\mathbb{R}_+} \int_{t}^{t+\omega} b(s_1, s_2)u(s_1)ds_2ds_1 + \frac{1}{2} \int_{t}^{t+\omega} b(s_1, s_2)ds_2ds_1
\]

\[
+ \frac{1}{2} \int_{t}^{t+\omega} b(s_1, s_2)ds_1ds_2 + \frac{1}{2} \int_{t}^{t+\omega} b(s_1, s_2)ds_1ds_2
\]

\[
+ \frac{1}{2} \int_{t}^{t+\omega} b(s_1, s_2)ds_1ds_2
\]

(12)

Next, using the properties of the semi-group \( U_x(t) \), we have

\[
W(\omega, 0)W(t, 0)(z(\nu, x)\varphi_\varepsilon(u))
\]

\[
=U_x(\omega)U_x^{-1}(0)z(\nu, x)\varphi_\varepsilon(u - i\chi(0, t)).
\]
This coincides with the above-obtained expression. Thus the first assertion \( (11) \)

is proved.

The second assertion \( (12) \) can be proved in a similar way.

Let \( \varepsilon (\cdot) \) have characteristic functional \( (5) \), then in view of \( (5) \) and properties of

the semi-group \( U_x(t) \) we have

\[
W(t + \omega, 0)(z(\nu, x) \varphi_\varepsilon (u)) = U_x(t + \omega)U_x^{-1}(0)z(\nu, x)V(t + \omega, 0)\varphi_\varepsilon (u)
\]

\[
= U_x(t)U_x(\omega)z(\nu, x)V(t, 0)V(\omega, 0)\varphi_\varepsilon (u)
\]

\[
= W(t, 0)W(\omega, 0)(z(\nu, x)\varphi_\varepsilon (u))
\]

\[
= U_x(t)U_x(\omega)z(\nu, x)V(t, 0)\exp \left( i \int_{\mathbb{R}_+} E(\varepsilon(s))u(s) \, ds + \int_0^\omega E(\varepsilon(s)) \, ds \right)
\]

\[
- \frac{1}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} b(s_1, s_2)u(s_1)u(s_2) \, ds_1 \, ds_2 + i \int_{\mathbb{R}_+} \int_0^\omega b(s_1, s_2)u(s_1) \, ds_1 \, ds_2
\]

\[
+ \frac{1}{2} \int_0^\omega \int_0^\omega b(s_1, s_2) \, ds_1 \, ds_2
\]

\[
= U_x(t)U_x(\omega)z(\nu, x)\exp \left( i \int_{\mathbb{R}_+} E(\varepsilon(s))u(s) \, ds + \int_0^t E(\varepsilon(s)) \, ds + \int_0^\omega E(\varepsilon(s)) \, ds \right)
\]

\[
- \frac{1}{2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} b(s_1, s_2)u(s_1)u(s_2) \, ds_1 \, ds_2 + i \int_{\mathbb{R}_+} \int_0^t b(s_1, s_2)u(s_1) \, ds_1 \, ds_2
\]

\[
+ \frac{1}{2} \int_0^t \int_0^\omega b(s_1, s_2) \, ds_1 \, ds_2 + \frac{1}{2} \int_0^\omega \int_0^t b(s_1, s_2) \, ds_1 \, ds_2
\]

\[
+ \frac{1}{2} \int_0^\omega \int_0^\omega b(s_1, s_2) \, ds_1 \, ds_2
\]

This coincides with the above-obtained expression. Thus the first assertion \( (11) \)

is proved.
We rewrite formula (6) with the use of the operator \( W(t, s) \) in the form

\[
E(y(t, x)) = [W(t, 0)(W(0, \omega) - I)^{-1} \int_{t}^{t+\omega} W(0, s)(E(f(s, x))\varphi\varepsilon(u))ds]|_{u=0} \tag{13}
\]

is the \( \omega \)-periodic with respect to \( t \in \mathbb{R}_+ \) mathematical expectation of a solution of equation (2).

**Proof.** We rewrite formula (6) with the use of the operator \( W(t, s) \) in the form

\[
E(y(t, x)) = [W(t, 0)(y_0(x)\varphi\varepsilon(u)) + \int_{0}^{t} W(t, s)(E(f(s, x))\varphi\varepsilon(u))ds]|_{u=0}. \tag{14}
\]

If \( E(y(t, x)) \) is an \( \omega \)-periodic function of \( t \), then

\[
E(y(0, x)) = y_0(x) = E(y(\omega, x)).
\]

We write out the last relation applying (14)

\[
y_0(x) = [W(\omega, 0)(y_0(x)\varphi\varepsilon(u)) + \int_{0}^{\omega} W(\omega, s)(E(f(s, x))\varphi\varepsilon(u))ds]|_{u=0}.
\]

From here, we obtain the following equation for the initial condition:

\[
[(I - W(\omega, 0))(y_0(x)\varphi\varepsilon(u)) - \int_{0}^{\omega} W(\omega, s)(E(f(s, x))\varphi\varepsilon(u))ds]|_{u=0} = 0.
\]

In view of the assumptions of the theorem, \( (I - W(\omega, 0))^{-1} \) exists, therefore,

\[
y_0(x) = [(I - W(\omega, 0))^{-1} \int_{0}^{\omega} W(\omega, s)(E(f(s, x))\varphi\varepsilon(u))ds]|_{u=0}.
\]
By substituting this expression into relation (14) and by taking into account the properties of the operator $W(t, s)$, we obtain the representation

$$E(y(t, x)) = [W(t, 0) - I]^{-1} \int_0^\omega W(\omega, s)(E(f(s, x))\varphi_x(u)) ds$$

$$+ \int_0^t W(0, s)(E(f(s, x))\varphi_x(u)) ds = [W(t, 0)(W(0, \omega) - I)^{-1}(\int_0^\omega W(0, s)(E(f(s, x))\varphi_x(u)) ds)$$

$$+ \int_0^t W(0, \omega)W(0, s)(E(f(s, x))\varphi_x(u)) ds$$

$$- \int_0^t W(0, s)(E(f(s, x))\varphi_x(u)) ds]_{u=0}. \tag{15}$$

$E(f(t, x))$ is an $\omega$-periodic function of $t$, hence

$$\int_0^\omega W(0, \omega)W(0, s)(E(f(s, x))\varphi_x(u)) ds$$

$$= \int_0^t W(0, \omega + s)(E(f(s, x))\varphi_x(u)) ds$$

$$= \int_0^{t+\omega} W(0, s)(E(f(s - \omega, x))\varphi_x(u)) ds$$

$$= \int_0^{t+\omega} W(0, s)(E(f(s, x))\varphi_x(u)) ds.$$

Substituting this expression into relation (15) and combining integrals, we obtain

$$E(y(t, x)) = [W(t + \omega, 0)(W(0, \omega) - I)^{-1} \int_{t+\omega}^{t+2\omega} W(0, s)(E(f(s, x))\varphi_x(u)) ds]_{u=0}$$

$$=[W(t, 0)W(0, \omega) - I]^{-1} \int_t^{t+2\omega} W(0, s)(E(f(s, x))\varphi_x(u)) ds]_{u=0}$$

$$=[W(t, 0)(W(0, \omega) - I)^{-1}] W(\omega, 0) \int_t^{t+\omega} W(0, s)(E(f(s, x))\varphi_x(u)) ds]_{u=0}$$

$$=[W(t, 0)(W(0, \omega) - I)^{-1} W(\omega, 0) \int_t^{t+\omega} W(0, \omega)W(0, s)(E(f(s, x))\varphi_x(u)) ds]_{u=0}$$

$$=[W(t, 0)(W(0, \omega) - I)^{-1} \int_t^{t+\omega} W(0, s)(E(f(s, x))\varphi_x(u)) ds]_{u=0}$$

$$=E(y(t, x)).$$

The proof of the theorem is complete. \qed

**Remark 1.** The invertibility of the operator $I - W(\omega, 0)$ implies that the homogeneous equation corresponding to (2) has no non-zero mean $\omega$-periodic solutions.
**Corollary 1.** If a random process $\varepsilon(\cdot)$ is given by the characteristic functional (4) and the assumptions of Theorem 4.1 are satisfied, then

$$E(y(t,x)) = [(W(0,\omega) - I)^{-1} \int_{t}^{t+\omega} U_x(t)U_x^{-1}(s)(E(f(s,x)))$$

$$\cdot \exp \left( i \int_{\mathbb{R}^+} E(\varepsilon(s))u(s)ds - \int_{t}^{s} E(\varepsilon(s))ds \right)$$

$$- \frac{1}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} b(s_1,s_2)u(s_1)u(s_2)ds_1ds_2 - i \int_{\mathbb{R}^+} \int_{t}^{s} b(s_1,s_2)u(s_1)ds_1ds_2$$

$$+ \frac{1}{2} \int_{t}^{s} \int_{t}^{s} b(s_1,s_2)ds_1ds_2 \right) ds \bigg|_{u=0}$$

is the $\omega$-periodic with respect to $t \in \mathbb{R}^+$ mathematical expectation of a solution of equation (2).

**Corollary 2.** If a random process $\varepsilon(\cdot)$ be given by the characteristic functional (5) and the assumption of Theorem 4.1 are satisfied, then

$$E(y(t,x)) = [(W(0,\omega) - I)^{-1} \int_{t}^{t+\omega} U_x(t)U_x^{-1}(s)(E(f(s,x)))$$

$$\cdot \sin \left( \int_{\mathbb{R}^+} a(s)u(s)ds + i \int_{t}^{s} a(s)ds \right)$$

$$\int_{\mathbb{R}^+} a(s)u(s)ds + i \int_{t}^{s} a(s)ds$$

$$\cdot \exp \left( i \int_{\mathbb{R}^+} \xi(s)u(s)ds - \int_{t}^{s} \xi(s)ds \right) ds \bigg|_{u=0}$$

is the $\omega$-periodic with respect to $t \in \mathbb{R}^+$ mathematical expectation of a solution of equation (2).

**Remark 2.** We can consider the function $f(t,x)$ in (2) as a control function. Then the obtained results may be treated as finding a control ensuring the periodicity of the mathematical expectation of a solution of heat equation (2).

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