ON PRESSURE STABILIZATION METHOD FOR NONSTATIONARY NAVIER-STOKES EQUATIONS

TAKAYUKI KUBO
Department of Mathematics, University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571, Japan

RANMARU MATSUI*
Graduate School of Pure and Applied Sciences, University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571, Japan
(Communicated by Hongjie Dong)

Abstract. In this paper, we consider the nonstationary Navier-Stokes equations approximated by the pressure stabilization method. We can obtain the local in time existence theorem for the approximated Navier-Stokes equations. Moreover we can obtain the error estimate between the solution to the usual Navier-Stokes equations and the Navier-Stokes equations approximated by the pressure stabilization method.

1. Introduction. The mathematical description of fluid flow is given by the following Navier-Stokes equations:

\[
\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi &= f, \\
u(0, x) &= a, \\
u(t, x) &= 0,
\end{align*}
\]

where the fluid vector fields \( u = u(t, x) = (u_1(t, x), \ldots, u_n(t, x)) \) and the pressure \( \pi = \pi(t, x) \) are unknown functions, the external force \( f = f(t, x) \) is a given vector function, the initial data \( a \) is a given solenoidal function and \( \Omega \) is some bounded domain (see section 2 for detail). It is well-known that analysis of Navier-Stokes equations (1) is very important in view of both mathematical analysis and engineering, however the problem concerning existence and regularity of solution to (1) is unsolved for a long time. One of the difficulty of analysis for (1) is the pressure term \( \nabla \pi \) and incompressible condition \( \nabla \cdot u = 0 \).

In numerical analysis, some penalty methods (quasi-compressibility methods) are employed as the method to overcome this difficulty. They are methods that eliminate the pressure by using approximated incompressible condition. For example, setting \( \alpha > 0 \) as a perturbation parameter, we use \( \nabla \cdot u = -\pi/\alpha \) in the penalty method, \( \nabla \cdot u = \Delta \pi/\alpha \) in the pressure stabilization method and \( \nabla \cdot u = -\partial_t \pi/\alpha \) in...
the pseudocompressible method. In this paper, we consider the Navier-Stokes equations with incompressible condition approximated by pressure stabilization method. Namely we consider the following equations:

\[
\begin{align*}
\frac{\partial u_\alpha}{\partial t} - \Delta u_\alpha + (u_\alpha \cdot \nabla) u_\alpha + \nabla \pi_\alpha &= f & t \in (0, \infty), x \in \Omega, \\
\nabla \cdot u_\alpha &= \Delta \pi_\alpha/\alpha & t \in (0, \infty), x \in \Omega, \\
u_\alpha(0, x) &= u_0 & x \in \Omega, \\
\partial_\alpha \pi_\alpha(t, x) &= 0 & x \in \partial \Omega.
\end{align*}
\]  

(2) may be considered as a singular perturbation of (1). As \( \alpha \to \infty \), (2) tends to (1) formally and we cancel the Neumann boundary condition for the pressure.

Pressure stabilization method was first introduced by Brezzi and Pitkäranta \[1\]. They considered the approximated stationary Stokes equations which are linearized Navier-Stokes equations with the approximated incompressible condition \( \nabla \cdot u_\alpha = \Delta \pi_\alpha/\alpha \). They obtained the following error estimate by using the energy methods:

\[
\|u_\alpha - u\|_{H^1(\Omega)} + \|\pi_\alpha - \pi\|_{L^2(\Omega)} \leq C_\alpha^{-1/2}\|f\|_{L^2(\Omega)}.
\]  

(3)

Nazarov and Specovius-Neugebauer \[7\] considered the same approximate Stokes problem and derived asymptotically precise estimates for solution to the approximated problem as \( \alpha \to \infty \) by using the parameter-dependent Sobolev norms. Their results are not available by the usually applied energy methods and are optimal results. These results introduced above are concerning the stationary Stokes equations and there are few results concerning the nonstationary Stokes equations and Navier-Stokes equations. As far as the authors know, only the result due to Prohl \[8\] is known as the results concerning the nonstationary problem. In \[8\], Prohl considered the sharp a priori estimate for the pressure stabilization method under some assumptions and showed the following error estimates:

\[
\begin{align*}
\|u_\alpha - u\|_{L^\infty([0, T], L^2(\Omega))} + \|\tau(\pi_\alpha - \pi)\|_{L^\infty([0, T], W^{-1}_2(\Omega))} &\leq C_\alpha^{-1}, \\
\|u_\alpha - u\|_{L^\infty([0, T], W^1_2(\Omega))} + \|\sqrt{\tau}(\pi_\alpha - \pi)\|_{L^\infty([0, T], L^2(\Omega))} &\leq C_\alpha^{-1/2},
\end{align*}
\]  

where \( \tau = \tau(t) = \min(t, 1) \). Since their results are proved based on energy method, all of these estimates are in \( L^2 \) framework for the space. In this paper, we shall use the maximal regularity theorem in order to prove the local in time existence theorem and the error estimate in the \( L^p \) in time and the \( L^q \) in space framework with \( n/2 < q < \infty \) and \( \max\{1, n/q\} < p < \infty \). Here, the maximal regularity theorem means that each term in the abstract Cauchy problem is well-defined and has the same regularity. To be precisely, when we consider the Cauchy problem

\[
\frac{\partial u(t)}{\partial t} + Au(t) = f(t), \quad t > 0, \quad u(0) = 0,
\]  

(4)

where \( X \) be a Banach space, \( A \) be closed linear unbounded operator in \( X \) with dense domain \( D(A) \) and \( f : \mathbb{R}^+ \to X \) is a given function has the maximal regularity, the maximal regularity theorem means for each \( f \in L^p(\mathbb{R}^+, X) \) there exists a unique solution \( u \) to (4) almost everywhere and satisfying \( \partial_t u, Au \in L^p(\mathbb{R}^+, X) \). However it is difficult to analyze equations (2) as it is by using the maximal regularity theorem because the regularity of solution to the first equation is different from the one of the second equations in (2). For this purpose, in order to adjust the regularity of the solution to their equations, we consider the following equations instead of approximated incompressible conditions in (2):

\[
(u_\alpha, \nabla \varphi)_\Omega = \alpha^{-1}(\nabla \pi_\alpha, \nabla \varphi)_\Omega, \quad \varphi \in \tilde{W}^1_q(\Omega)
\]  

(5)
for $1 < q < \infty$. We notice that (5) is a weak form of the approximated incompressible condition $\nabla \cdot u_\alpha = \alpha^{-1} \Delta u_\alpha$. We call (5) approximated weak incompressible condition in this paper. Therefore we consider

$$
\begin{cases}
\partial_t u_\alpha - \Delta u_\alpha + (u_\alpha \cdot \nabla) u_\alpha + \nabla \pi_\alpha = f & t \in (0, \infty), x \in \Omega, \\
u_\alpha(0, x) = a_\alpha & x \in \Omega, \\
u_\alpha(t, x) = 0 & x \in \partial \Omega
\end{cases}
$$

(6)

under the approximated weak incompressible condition (5) in $L^q$-framework $(n/2 < q < \infty)$.

This paper consists of the following five sections. In section 2, we present the main results on local in time unique existence of solution to (6) and certain error estimate between the solutions to (6) and (1) under the weak incompressible condition (Theorem 2.1 and Theorem 2.12). Following the argument due to Shibata and Kubo [10], we can prove the main results by contraction mapping principle with the help of the maximal $L^p-L^q$ regularity theorem. After stating the main results, we present the maximal $L^p-L^q$ regularity theorem for linearized (6) (Theorem 2.2 and Theorem 2.10) and the theorem concerning the existence of $\mathcal{R}$-bounded solution operator for linearized problem (Theorem 2.7). As was seen in Shibata and Shimizu [11], the maximal $L^p-L^q$ regularity theory is direct consequence of Theorem 2.7 concerning the generalized resolvent problem for the linearized equations with the help of Weis’ operator valued Fourier multiplier theorem (Theorem 2.6), so that the main part of this paper is to show Theorem 2.7. Moreover another consequence of Theorem 2.7 is the resolvent estimate (Corollary 2.8), which implies the construct of the semigroup $T_\alpha(t)$ for linearized (6). By real interpolation, we obtain some estimates for $T_\alpha(t)$ (Theorem 2.9 and Theorem 2.11). In section 3, as preliminary, we shall introduce some theorems and lemmas which play important role in this paper. In section 4, we consider the generalized resolvent problem for linearized problem in some bounded domain. For this purpose, we first consider the problem in the whole space case and the half-space case. By using the change of variable with their results, we shall prove the generalized bounded domain cases. In section 5, the following the argument due to Shibata and Kubo [10], we show the local in time existence theorem for (6) and prove the error estimates (Theorem 2.1 and Theorem 2.12).

2. Main results. Before we describe main theorem, we shall introduce some functional spaces and notations throughout this paper. The letter $C$ denotes generic constants and the constant $C_{a,b,...}$ depends on $a,b,\ldots$ The values of constants $C$ and $C_{a,b,...}$ may change from line to line. For $1 < q < \infty$, let $q' = q/(q-1)$. For any two Banach spaces $X$ and $Y$, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from $X$ into $Y$ and we write $\mathcal{L}(X) = \mathcal{L}(X,X)$ for short. $\text{Hol}(U,X)$ denotes the set of all $X$-valued holomorphic functions defined on a complex domain $U$. As the complex domain where a resolvent parameter belongs, we use $\Sigma_\varepsilon = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \varepsilon \}$ and $\Sigma_{\varepsilon,\lambda_0} = \{ \lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0 \}$ for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. For any domain $D$, Banach space $X$ and $1 \leq q \leq \infty$, $L_q(D,X)$ denotes the usual Lebesgue space of $X$-valued functions defined on $D$ and $\| \cdot \|_{L_q(D,X)}$ denotes its norm. We use the notation $L_q(D) = L_q(D,\mathbb{R})$, $\| \cdot \|_{L_q(D)} = \| \cdot \|_{L_q(D,\mathbb{R})}$ and for $a,b,\ldots,c \in L_q(D)$, $\| (a,b,\ldots,c) \|_{L_q(D)} = \| a \|_{L_q(D)} + \| b \|_{L_q(D)} + \cdots + \| c \|_{L_q(D)}$. In a similar way, for $1 \leq q \leq \infty$ and a positive integer $m$, $W^{m,q}_q(D,X)$ denotes the Sobolev spaces of $X$-valued functions of defined on $D$. We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion.
For $1 \leq p, q \leq \infty$, $B_{q,p}^{2(1-1/p)}(D)$ denotes the real interpolation space defined by $B_{q,p}^{2(1-1/p)}(D) = (L_q(D), W_q^2(D))_{1-1/p,p}$. For a Banach space $X$ and some $\gamma_0 \in \mathbb{R}$, we set

$$L_{p,\gamma_0}(\mathbb{R}, X) = \{ f(t) \in L_{p,\text{loc}}(\mathbb{R}, X) \mid \| e^{-\gamma t} f \|_{L_p(\mathbb{R}, X)} < \infty, \ (\gamma \geq \gamma_0) \},$$

$$L_{p,\gamma_0,0}(\mathbb{R}, X) = \{ f(t) \in L_{p,\gamma_0}(\mathbb{R}, X) \mid f(t) = 0 \ (t < 0) \},$$

$$W_{p,\gamma_0,0}^1(\mathbb{R}, X) = \{ f(t) \in L_{p,\gamma_0,0}(\mathbb{R}, X) \mid f'(t) \in L_{p,\gamma_0}(\mathbb{R}, X) \}.$$

In order to deal with the pressure term, we use the following functional spaces:

$$L_{q,\text{loc}}(D) = \{ f \mid f|_K \in L_q(K), \ K \text{ is any compact set in } D \},$$

$$\hat{W}_q^1(D) = \{ \theta \in L_{q,\text{loc}}(D) \mid \nabla \theta \in L_q(D)^n \}.$$

Since our proof is based on Fourier analysis, we next introduce the Fourier transform and the Laplace transform. We define the Fourier transform, its inverse Fourier transform, the Laplace transform and its inverse Laplace transform by

$$\hat{f}(\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

$$\mathcal{F}^{-1}_x[f](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi,$$

$$\mathcal{L}_t[f](\lambda) = \mathcal{F}_t[e^{-\gamma t} f(t)](\tau),$$

$$\mathcal{L}^{-1}_t[f](t) = e^{\gamma t} \mathcal{F}^{-1}_t[f](\tau),$$

respectively, where $x, \xi \in \mathbb{R}^n$, $\lambda = \gamma + i\tau \in \mathbb{C}$ and $x \cdot \xi$ is usual inner product: $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$. Furthermore, we define the Fourier-Laplace transform by

$$\mathcal{L}_t[\mathcal{F}_x[v(t,x)]](\lambda, \xi) = \mathcal{F}_{t,x}[e^{-\gamma t} v(t,x)](\lambda, \xi) = \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^n} e^{-(\lambda t + ix \cdot \xi)} v(t,x) dx \right) dt.$$

By using Fourier transform and Laplace transform, we define $H^s_{p,\gamma_0}(\mathbb{R}, X)$ for a Banach space $X$. For $\lambda = \gamma + i\tau$, we define the operator $\Lambda^s_\gamma$ as

$$(\Lambda^s_\gamma f)(t) = \mathcal{L}^{-1}_t[|\lambda|^s \mathcal{L}_t[f](\lambda)](t) = e^{\gamma t} \mathcal{F}^{-1}_t[(\tau^2 + \gamma^2)^{s/2} \mathcal{F}_t[e^{-\gamma t} f(t)](\tau)](t).$$

For $0 < s < 1$ and $\gamma_0 > 0$, we define the space $H^s_{p,\gamma_0}(\mathbb{R}, X)$ as

$$H^s_{p,\gamma_0}(\mathbb{R}, X) = \{ f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid \| e^{-\gamma t} \Lambda^s_\gamma f \|_{L_p(\mathbb{R}, X)} < \infty (\forall \gamma \geq \gamma_0) \}.$$

In this paper, we assume next assumption for our domain $\Omega$.

**Assumption 2.1.** Let $n/2 < q < \infty$ and $n < r < \infty$. Let $\Omega$ be a uniform $W^{2-1/r}_r$ domain introduced in [5] and $L_q(\Omega)$ has the Helmholtz decomposition.

Here the assumption on a uniformly $W^{2-1/r}_r$ domain is used when we reduce the problem on the bounded domain to one on the bent half-space and on the whole space (see section 4.3 for detail). According to Galdi [6], that “$L_q(\Omega)$ has the Helmholtz decomposition” is equivalent that the following weak Neumann problem is uniquely solvable: for $f \in L_q(\Omega)$,

$$(\nabla \theta, \nabla \varphi) = (f, \nabla \varphi) \quad \varphi \in \hat{W}_q^1(\Omega).$$

The map $P_\Omega$ and $Q_\Omega$ are defined by $Q_\Omega f = \theta$, where $\theta$ is the solution to the above weak Neumann problem and $P_\Omega f = f - \nabla Q_\Omega f$. $P_\Omega$ is called the Helmholtz projection. We remark that if $q = 2$, $L_2(\Omega)$ has the Helmholtz decomposition for any $\Omega$ (see Galdi [6]).

First main result is concerned with the local in time existence theorem for (6) with approximated weak incompressible condition (5).
Theorem 2.1. Let $n \geq 2$, $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Let $\alpha > 0$ and $T_0 \in (0, \infty)$. For any $M > 0$, assume that the initial data $u_0 \in B^{2(1-1/p)}_{q,p}(\Omega)$ and the external force $f \in L^p((0,T_0),L^q(\Omega)^n)$ satisfy
\[
\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L^p((0,T_0),L^q(\Omega)^n)} \leq M.
\] (7)
Then, there exists $T^* \in (0,T_0)$ depending on only $M$ such that (6) under (5) has a unique solution $(u_\alpha, \pi_\alpha)$ of the following class:
\[
u \in W^1_p((0,T^*),L^q(\Omega)^n) \cap L^p((0,T^*),W_q^2(\Omega)^n), \quad \pi_\alpha \in L^p((0,T^*),\hat{W}_q^1(\Omega)).
\]
Moreover the following estimate holds:
\[
\|u_\alpha\|_{L^\infty((0,T^*),L^q(\Omega))} + \|\nabla u_\alpha, \nabla^2 u_\alpha, \nabla \pi_\alpha\|_{L^p((0,T^*),L^q(\Omega))} + \|
abla u_\alpha\|_{L^r((0,T^*),L^q(\Omega))} \leq C_{n,p,q,T^*}
\]
for $1/p - 1/r \leq 1/2$.

Here we state the outline of the proof of main theorem (Theorem 2.1). We show Theorem 2.1 by using the contraction mapping principle with two type maximal regularity theorems (Theorem 2.2 and Theorem 2.9). In order to prove Theorem 2.2, we use the Weis’ operator valued Fourier multiplier theorem. For this purpose, we have to show the existence of $R$-bounded solution operator to the generalizd resolvent problem of (6) (see Theorem 2.7 for detail). In order to prove Theorem 2.9, we need the some estimate of semigroup $T_\alpha(t)$ for linearized problem of (6). For this purpose, we have to show the resolvent estimate (Corollary 2.8), which is a corollary of Theorem 2.7. Therefore our main task is to show Theorem 2.7.

We shall explain the proof of Theorem 2.1 in more detail. In order to prove Theorem 2.1, we use the contraction mapping principle and maximal $L_p$-$L_q$ regularity theorem for the following linearized problems corresponding to (6):
\[
\begin{cases}
\partial_t u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f & t > 0, x \in \Omega, \\
u_\alpha(t,x) = 0 & x \in \partial \Omega, \quad x \in \Omega, \\
u_\alpha(0,x) = u_0 & x \in \Omega.
\end{cases}
\] (8)
under the approximated weak incompressible condition
\[
(u_\alpha, \nabla \varphi)_\Omega = \alpha^{-1}(\nabla \pi_\alpha, \nabla \varphi)_\Omega + (g, \nabla \varphi)_\Omega \quad \varphi \in \hat{W}_q^1(\Omega).
\] (9)
First result is concerned with the maximal $L_p$-$L_q$ regularity theorem for (8) under (9) with $u_\alpha = 0$.

Theorem 2.2. Let $1 < p, q < \infty$ and $\alpha > 0$. Then there exists a positive number $\gamma_0$ such that the following assertion holds: for any $f, g \in L_{p,\gamma_0}(\mathbb{R}, L_q(\Omega))$, (8) under (9) with $u_\alpha = 0$ has a unique solution:
\[
u_\alpha \in L_{p,\gamma_0}(\mathbb{R},W_q^2(\Omega)) \cap W^1_{p,\gamma_0}(\mathbb{R}, L_q(\Omega)), \quad \nabla \pi_\alpha \in L_{p,\gamma_0}(\mathbb{R},\hat{W}_q^1(\Omega)).
\]
Moreover, the following estimate holds:
\[
\|e^{-\gamma t}(\partial_t u_\alpha, \gamma u_\alpha, \Lambda^\frac{1}{2}_0 \nabla u_\alpha, \Lambda^{1/2}_\gamma \nabla \pi_\alpha)\|_{L^p(R,L^q(\Omega))} \leq C_{n,p,q}\|e^{-\gamma t}(f, \alpha g)\|_{L^p(R,L^q(\Omega))}
\]
for any $\gamma \geq \gamma_0$.

Remark 2.3. By the property of Helmholtz decomposition, we can solve (9) for $u_\alpha, g \in L_q(\Omega)$ and we see $\pi_\alpha = \alpha Q\Omega(u_\alpha - g)$.
In order to prove Theorem 2.2, we use the operator valued Fourier multiplier theorem due to Weis [12]. This theorem needs $\mathcal{R}$-boundedness of solution operator. To this end, we first introduce the definition of $\mathcal{R}$-boundedness.

**Definition 2.4.** The family of the operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $f_j \in X$ ($j = 1,\ldots,N$) and for all sequences $\left\{\gamma_j(u)\right\}_{j=1}^N$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, there holds the inequality:

$$
\int_0^1 \left\| \sum_{j=1}^N \gamma_j(u)T_jf_j \right\|_Y^p \, du \leq C \int_0^1 \left\| \sum_{j=1}^N \gamma_j(u)f_j \right\|_X^p \, du.
$$

The smallest such $C$ is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, which is denoted by $\mathcal{R}(\mathcal{T})$.

**Remark 2.5.** According to [3], the following properties concerning $\mathcal{R}$-boundedness is known. From Definition 2.4, $\mathcal{R}$-boundedness of the family of operators implies uniform boundedness.

$$
\|T\|_{\mathcal{L}(X,Y)}^p = \sup_{\|x\|=1} \|T(x)\|_Y^p \leq \mathcal{R}(\mathcal{T}).
$$

Moreover it is well-known that $\mathcal{R}$-bounds behave like norms. Namely, the following properties hold.

(i) Let $X, Y$ be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X,Y)$ be $\mathcal{R}$-bounded. Then $\mathcal{T} + \mathcal{S} = \{T+S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S})$.

(ii) Let $X, Y, Z$ be Banach spaces and $\mathcal{T} \subset \mathcal{L}(X,Y)$ and $\mathcal{S} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{ST} = \{\mathcal{S}\mathcal{T} \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{ST}) \leq \mathcal{R}(\mathcal{S})\mathcal{R}(\mathcal{T})$.

The following theorem is the operator valued Fourier multiplier theorem proved by Weis [5] for $X = Y = L_q(\Omega)$.

**Theorem 2.6.** Let $1 < p, q < \infty$ and $M(\tau) \in C^1(\mathbb{R}\setminus\{0\}, \mathcal{L}(X,Y))$ be satisfy

$$
\mathcal{R}(\{M(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\}) = c_0 < \infty, \quad \mathcal{R}(\{|\tau|\partial_\tau M(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\}) = c_1 < \infty.
$$

Then, $T_M$ defined by $T_Mf(t) = \mathcal{F}_x^{-1}[M(\tau)\mathcal{F}_x(f)(\tau)](t)(f \in \mathcal{S}(\mathbb{R},X))$ is the bounded operator from $L_p(\mathbb{R},X)$ to $L_p(\mathbb{R},Y)$. Moreover, the following estimate holds:

$$
\|T_Mf\|_{L_p(\mathbb{R},Y)} \leq C(c_0 + c_1)\|f\|_{L_p(\mathbb{R},X)} \quad (f \in L_p(\mathbb{R},X)),
$$

where $C$ is a positive constant depending on $p, X$.

In order to prove the maximal $L_p$-$L_q$ regularity theorem with the help of Theorem 2.6, we need the $\mathcal{R}$-boundedness for solution operator to the following generalized resolvent problem

$$
\begin{cases}
\lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f & \text{in } \Omega, \\
u_\alpha = 0 & \text{on } \partial \Omega
\end{cases}
$$

(10)

under the approximated weak incompressible condition (9), where the resolvent parameter $\lambda$ varies in $\Sigma_{\varepsilon,\lambda_0}$ ($0 < \varepsilon < \pi/2, \lambda_0 > 0$).

We can show the existence of the $\mathcal{R}$-boundedness operator to (10) under (9) as follows:

**Theorem 2.7.** Let $\alpha > 0, 1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\Omega) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\Omega)\}$, then there exist a $\lambda_0 > 0$ and operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with

$$
\mathcal{U}(\lambda) \in \text{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), W_q^2(\Omega)^n)), \quad \mathcal{P}(\lambda) \in \text{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), \tilde{W}_q^1(\Omega)))
$$
such that for any \( f, g \in L_q(\Omega) \) and \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \), \((u_{\alpha}, \pi_{\alpha}) = (U(\lambda)F, P(\lambda)F)\), where \( F = (f, ag) \), is a unique solution to (10) under (9) and \((U(\lambda), P(\lambda))\) satisfies the following estimates:

\[
R_{L(Y_0), L_q(\Omega)^n}(\{(\tau \partial_t)^l(G_{\lambda, \alpha} U(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}\}) \leq C (\ell = 0, 1),
\]

\[
R_{L(X_\varepsilon), L_q(\Omega)^n}(\{(\tau \partial_t)^l(\nabla P(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}\}) \leq C (\ell = 0, 1)
\]

for \( G_{\lambda, \alpha} = (u_{\alpha}, \lambda^{1/2} \nabla u_{\alpha}, \nabla^2 u_{\alpha}, (\lambda + \alpha)^{1/2}(\nabla \cdot u_{\alpha})) \) and \( \tilde{N} = n + n^2 + n^3 \).

By Remark 2.5, we can prove the resolvent estimate for (10) under (9).

**Corollary 2.8.** Let \( \alpha > 0 \), \( 1 < q < \infty \) and \( 0 < \varepsilon < \pi/2 \). Let \( \lambda > 0 \) be a number obtained in Theorem 2.7. For \( f, g \in L_q(\Omega) \) and \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \), there exists a unique solution \((u_{\alpha}, \pi_{\alpha})\) to (10) under (9) which satisfies the following inequality:

\[
\|((u_{\alpha}, \lambda^{1/2} \nabla u_{\alpha}, \nabla^2 u_{\alpha}, (\lambda + \alpha)^{1/2}(\nabla \cdot u_{\alpha}), \nabla \pi_{\alpha}))\|_{L_q(\Omega)} \leq C \|(f, g)\|_{L_q(\Omega)}.
\]

Let \( \mathcal{A}_\alpha \) be the linear operator defined by \( \mathcal{A}_\alpha u_{\alpha} = \Delta u_{\alpha} - \alpha \nabla Q_{\Omega} u_{\alpha} \) and \( \mathcal{P}(\mathcal{A}_\alpha) = \{u \in W_2^3(\Omega)^n \mid u|_{\partial \Omega} = 0\}. \) By Corollary 2.8 with \( g = 0 \), we see that \( \mathcal{A}_\alpha \) generates the semigroup \( \{T_{\alpha}(t)\}_{t \geq 0} \) on \( L_p(\Omega)^n \). Moreover there exists a positive constant \( C > 0 \) such that for any \( a_{\alpha} \in L_q(\Omega)^n \), \( u_{\alpha}(t) = T_{\alpha}(t)a_{\alpha} \) satisfies

\[
\|(u_{\alpha}, t^{1/2} \nabla u_{\alpha}, t \nabla^2 u_{\alpha}, t \partial_t u_{\alpha})\|_{L_q(\Omega)} \leq C e^{\lambda_{aT}} \|a_{\alpha}\|_{L_q(\Omega)} \quad (t \geq 0).
\]

By the equations (8), we have

\[
\|
\nabla \pi_{\alpha}\|_{L_q(\Omega)} \leq \|\partial_t u_{\alpha}\|_{L_q(\Omega)} + \|\Delta u_{\alpha}\|_{L_q(\Omega)} \leq C t^{-1/2} e^{\lambda_{aT}} \|a_{\alpha}\|_{L_q(\Omega)}.
\]

On the other hand, since \( \pi_{\alpha} = \alpha \nabla Q_{\Omega} u_{\alpha} \) is the pressure associated with \( u_{\alpha} = T_{\alpha}(t)a_{\alpha} \) and \( \nabla \pi_{\alpha} = \alpha(u_{\alpha} - P_{\Omega} u_{\alpha}) \), \((u_{\alpha}, \pi_{\alpha})\) enjoys (8) under (9) and \( \nabla \pi_{\alpha} \) satisfies the following estimate:

\[
\|\nabla \pi_{\alpha}\|_{L_q((0,T), L_q(\Omega))} \leq C e^{\lambda_{aT}} \|a_{\alpha}\|_{L_q(\Omega)},
\]

which implies \( \|\nabla \pi_{\alpha}\|_{L_q((0,T), L_q(\Omega))} \leq C e^{\lambda_{aT}} \|a_{\alpha}\|_{L_q(\Omega)} \). This is the effect of the pressure stabilization method.

By real interpolation, we can see the following maximal \( L_p-L_q \) regularity theorem for (8) with \( f = g = 0 \).

**Theorem 2.9.** Let \( \alpha > 0 \) and \( 1 < p, q < \infty \). Let \( \lambda > 0 \) be a number obtained in Theorem 2.7. For \( a_{\alpha} \in B_2^{(1-1/p)}(\Omega) \), \( u_{\alpha} = T_{\alpha}(t)a_{\alpha} \) satisfy

\[
\|e^{-\lambda_{aT}}(\partial_t u_{\alpha}, \nabla^2 u_{\alpha})\|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \|a_{\alpha}\|_{B_2^{(1-1/p)}(\Omega)},
\]

\[
(\gamma - \lambda_0)^{1/p} \|e^{-\gamma t} u_{\alpha}\|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \|a_{\alpha}\|_{L_q(\Omega)},
\]

\[
(\gamma - \lambda_0)^{1/(2p)} \|e^{-\gamma t} \nabla u_{\alpha}\|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \|a_{\alpha}\|_{B_2^{(1-1/p)}(\Omega)}.
\]

for any \( \gamma > \lambda_0 \). Moreover \( \pi_{\alpha} = \alpha Q_{\Omega} u_{\alpha} \) satisfy

\[
\|e^{-\lambda_{aT}} \nabla \pi_{\alpha}\|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \|a_{\alpha}\|_{B_2^{(1-1/p)}(\Omega)},
\]

\[
\|\nabla \pi_{\alpha}\|_{L_q((0,T), L_q(\Omega))} \leq C_{n,p,q} e^{\lambda_{aT}} \|a_{\alpha}\|_{L_q(\Omega)}
\]

for any \( T > 0 \).

Next we consider the error estimate between the solution \((u, \pi)\) to (1) under the weak incompressible condition \((u, \nabla \varphi)|_{\Omega} = 0 \) for \( \varphi \in \tilde{W}_q^1(\Omega) \) and solution \((u_{\alpha}, \pi_{\alpha})\) to (6) under (5). To this end, setting \( u_E = u - u_{\alpha} \) and \( \pi_E = \pi - \pi_{\alpha} \), we see that \((u_E, \pi_E)\) enjoys that
for any \( \gamma \geq \lambda \). For \( a_{E, \alpha} \) given, if we consider (8) under (13) with \( a_{E} = 0 \) has a unique solution:

\[ u_{E} = L_{p,q,\gamma,E,0}(\Omega), W_{p,q,\gamma,E,0}(\Omega) \]

Moreover, the following estimate holds.

\[
\| e^{-\gamma t}(\partial_t u_E, \alpha u_E, \nabla u_E, \nabla^2 u_E, \Lambda_{\gamma+\alpha}^{1/2} (\nabla \cdot u_E), \nabla \pi_E) \|_{L_p(R, L_q(\Omega))} \leq C_{n,p,q} \| e^{-\gamma t} \|_{L_p(R, L_q(\Omega))}
\]

for any \( \gamma \geq \gamma_E \).

**Theorem 2.11.** Let \( 1 < p, q < \infty \) and \( \alpha > 0 \). Let \( \lambda_0 \) be a number obtained in Theorem 2.7. For \( a_E \in B_{q,p}^{2(1-1/p)}(\Omega) \), \( u_E = T_{\alpha}(t)a_E \) and \( \pi_E = \alpha Q_{\Omega} u_E - \pi \) satisfy

\[
\| e^{-\lambda_0 t}(\partial_t u_E, \nabla^2 u_E, \nabla \pi_E) \|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \| a_E \|_{B_{q,p}^{2(1-1/p)}(\Omega)},
\]

\[
(\gamma - \lambda_0)^{1/p} \| e^{-\gamma t} u_E \|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \| a_E \|_{L_q(\Omega)},
\]

\[
(\gamma - \lambda_0)^{1/(2p)} \| e^{-\gamma t} \nabla u_E \|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} \| a_E \|_{B_{q,p}^{2(1-1/p)}(\Omega)}
\]

for any \( \gamma > \lambda_0 \). If \( \pi \in L_{\infty}((0,\infty), \tilde{W}_{q}^{1}(\Omega)) \), \( \pi_E \) satisfies

\[
\| e^{-\gamma_0 t} \nabla \pi_E \|_{L_{\infty}((0,T), L_q(\Omega))} \leq C_{\alpha} \| a_E \|_{L_q(\Omega)} + \| \nabla \pi \|_{L_{\infty}((0,\infty), L_q(\Omega))}
\]

for any \( T > 0 \).

By above two theorems, we can obtain the following theorem concerned with the error estimates.

**Theorem 2.12.** Let \( n \geq 2 \), \( n/2 < q < \infty \), \( \max\{1,n/q\} < p < \infty \) and \( \alpha > 0 \). Let \( T^* \) be a positive constant obtained in Theorem 2.1 and \((u_{E, \alpha}, \pi_{E, \alpha})\) be a solution obtained in Theorem 2.1. For any \( M > 0 \), assume that \( a_E \in B_{q,p}^{2(1-1/p)}(\Omega) \) satisfies

\[
\| a_E \|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq M \alpha^{-1}.
\]

Then there exists \( T^b \in (0, T^*) \) such that (12) has a unique solution \((u_E, \pi_E)\) which satisfies

\[
\| u_E \|_{L_p((0,T^b), L_q(\Omega))} + \| \nabla u_E \|_{L_p((0,T^b), L_q(\Omega))} \leq C_{n,p,q,T^b, \alpha^{-1}}
\]

for \( 1/p - 1/r \leq 1/2 \).
Remark 2.13. (1) Theorem 2.10 and Theorem 2.11 with \( \gamma = \lambda_0 + 1 \) means that the error estimate for the Stokes equations. Namely the error estimate is given by
\[
\|e^{-\gamma t}(u - u_\alpha)\|_{L_p((0,T),L_q(\Omega))} \leq \frac{C}{\alpha} \|e^{-\gamma t}\nabla \pi\|_{L_p((0,T),L_q(\Omega))} + Ce^{(\lambda_0+1)T}\|a_E\|_{L_q(\Omega)}
\]
for any \( T > 0 \). If \( T < \infty \) and \( \|e^{-\gamma t}\nabla \pi\|_{L_\infty((0,T),L_q(\Omega))} < \infty \), we see
\[
\|e^{-\gamma t}(u - u_\alpha)\|_{L_\infty((0,T),L_q(\Omega))} = \lim_{p \to \infty} \|e^{-\gamma t}(u - u_\alpha)\|_{L_p((0,T),L_q(\Omega))}
\]
\[
\leq \frac{C}{\alpha} \|e^{-\gamma t}\nabla \pi\|_{L_\infty((0,T),L_q(\Omega))} + Ce^{(\lambda_0+1)T}\|a_E\|_{L_q(\Omega)}.
\]
Under assumption (14) in Theorem 2.12, we see that there exists a positive constant \( C \) depending on \( T,M \) and \( \|\nabla \pi\|_{L_\infty((0,T),L_q(\Omega))} \) such that
\[
\|u - u_\alpha\|_{L_\infty((0,T),L_q(\Omega))} \leq C\alpha^{-1}, \quad \|\nabla(\pi - \pi_\alpha)\|_{L_\infty((0,T),L_q(\Omega))} \leq C
\]
for any \( T > 0 \).

(2) (15) means the following error estimates for the Navier-Stokes equations:
\[
\|u - u_\alpha\|_{L_\infty((0,T^\ast),L_q(\Omega))} \leq C\alpha^{-1},
\]
\[
\|\nabla^2(u - u_\alpha), \partial_t(u - u_\alpha), \nabla(\pi - \pi_\alpha)\|_{L_p((0,T^\ast),L_q(\Omega))} \leq C\alpha^{-1},
\]
In a similar way to (1), we obtain
\[
\|\nabla^2(u - u_\alpha), \partial_t(u - u_\alpha), \nabla(\pi - \pi_\alpha)\|_{L_\infty((0,T^\ast),L_q(\Omega))} \leq C\alpha^{-1}.
\]
In comparison with the result due to Prohl [8], we can extend \( L_2 \) framework to \( L_q \) framework with respect to the error estimate.

3. Preliminary. In this section, we shall introduce some lemmas and definitions, which plays important role for our proof. Before we describe some propositions and lemmas, we introduce the notation of symbols. Set
\[
r = |\xi'|, \quad \omega_\lambda = \sqrt{\lambda + r^2}, \quad \omega = \sqrt{\lambda + \alpha + r^2},
\]
\[
E(z) = e^{-z(x_n + y_n)}, \quad M(a,b,x_n) = \frac{e^{-ax_n} - e^{-bx_n}}{a-b},
\]
(16)
where \( \xi' = (\xi_1, \ldots, \xi_{n-1}) \). Here \( E(\omega_\lambda) \) is the symbol corresponding to heat equation and \( M(\omega_\lambda, r, x_n) \) is the symbol corresponding to Stokes equations.

We next introduce some lemmas. In order to apply the operator-valued Fourier multiplier theorem proved by Weis [12], we need the \( \mathcal{R} \)-boundedness of solution operator to (8). However since it is difficult to prove \( \mathcal{R} \)-boundedness directly from its definition, we first introduce the following sufficient condition for showing \( \mathcal{R} \)-boundedness of solution operator given in Theorem 3.3 in Enomoto and Shibata [4].

Theorem 3.1. Let \( 1 < q < \infty \) and \( 0 < \varepsilon < \pi/2 \). Let \( m(\lambda, \xi) \) be a function defined on \( \Sigma_\varepsilon \times (\mathbb{R}^n \setminus \{0\}) \) such that for any multi-index \( \beta \in \mathbb{N}_0^n (\mathbb{N}_0 = \mathbb{N} \cup \{0\}) \) there exists a constant \( C_\beta \) depending on \( \beta \) and \( \lambda \) such that
\[
|\partial_\xi^\beta m(\lambda, \xi)| \leq C_\beta |\xi|^{-|\beta|}
\]
for any \((\lambda, \xi) \in \Sigma_\varepsilon \times (\mathbb{R}^n \setminus \{0\})\). Let \(K_\lambda\) be an operator defined by \([K_\lambda f](x) = \mathcal{F}_\xi^{-1}[m(\lambda, \xi)\mathcal{F}_x[f](\xi')](x)\). Then the set \(\{K_\lambda \mid \lambda \in \Sigma_\varepsilon\}\) is \(\mathcal{R}\)-bounded on \(\mathcal{L}(L_q(\mathbb{R}^n))\) and

\[
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^n))}(\{K_\lambda \mid \lambda \in \Sigma_\varepsilon\}) \leq C \max_{|\beta| \leq n+2} C_\beta
\]

with some constant \(C\) that depends solely on \(q\) and \(n\).

To prove the \(\mathcal{R}\)-boundedness of the solution operator in \(\mathbb{R}^n_+\), we use the following lemma proved by Shibata and Shimizu [11] (see Lemma 5.4 in [11]).

**Lemma 3.2.** Let \(0 < \varepsilon < \pi/2, 1 < q < \infty\). Let \(m(\lambda, \xi')\) be a function defined on \(\Sigma_\varepsilon\) such that for any multi-index \(\delta' \in \mathbb{N}^{n-1}_0\) there exists a constant \(C_{\delta'}\) depending on \(\delta', \varepsilon\) and \(N\) such that

\[
|\partial^\delta_{\xi'} m(\lambda, \xi')| \leq C_{\delta'} r^{-|\delta'|}.
\]

Let \(K_j(\lambda, m) (j = 1, \ldots, 5)\) be the operators defined by

\[
K_1(\lambda, m)g(x) = \int_0^\infty \mathcal{F}_\xi^{-1}[m(\lambda, \xi')r\mathcal{E}(\omega, r, x_n + y_n)\tilde{g}(\xi', y_n)](x')dy_n,
\]

\[
K_2(\lambda, m)g(x) = \int_0^\infty \mathcal{F}_\xi^{-1}[m(\lambda, \xi')r^2\mathcal{M}(\omega, r, x_n + y_n)\tilde{g}(\xi', y_n)](x')dy_n,
\]

\[
K_3(\lambda, m)g(x) = \int_0^\infty \mathcal{F}_\xi^{-1}[m(\lambda, \xi')|\lambda|^{1/2}r\mathcal{M}(\omega, r, x_n + y_n)\tilde{g}(\xi', y_n)](x')dy_n,
\]

\[
K_4(\lambda, m)g(x) = \int_0^\infty \mathcal{F}_\xi^{-1}[m(\lambda, \xi')\omega r\mathcal{M}(\omega, \omega, x_n + y_n)\tilde{g}(\xi', y_n)](x')dy_n,
\]

\[
K_5(\lambda, m)g(x) = \int_0^\infty \mathcal{F}_\xi^{-1}[m(\lambda, \xi')|\lambda|^{1/2}r\mathcal{M}(\omega, \omega, x_n + y_n)\tilde{g}(\xi', y_n)](x')dy_n.
\]

Then, the sets \(\{(\partial^\delta_{\xi'} K_j(\lambda, m) \mid \lambda \in \Sigma_\varepsilon\} (j = 1, \ldots, 5, \ell = 0, 1)\) are \(\mathcal{R}\)-bounded families in \(\mathcal{L}(L_q(\mathbb{R}^n_+))\). Moreover, there exists a constant \(C_{n,q,A}\) such that

\[
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^n_+))}(\{(\partial^\delta_{\xi'} K_j(\lambda, m) \mid \lambda \in \Sigma_\varepsilon\}) \leq C_{n,q,A} \quad (j = 1, \ldots, 5, \ell = 0, 1).
\]

This lemma is proved in a similar way to Lemma 5.4 in [11] with the following lemma.

**Lemma 3.3.** For \(0 < \varepsilon < \pi/2\), let \(\lambda \in \Sigma_\varepsilon\).

(i) There exist positive constants \(C_1, C_2\) and \(C_3\) depending on \(\varepsilon\) such that the following inequalities hold:

\[
|\omega_\lambda| \geq C_1(|\lambda|^{1/2} + r), \quad C_2(\alpha^{1/2} + |\lambda|^{1/2} + r) \leq \text{Re } \omega \leq C_3(\alpha^{1/2} + |\lambda|^{1/2} + r).
\]

(ii) There exist positive constants \(C\) such that the following inequalities hold:

\[
|D^\delta_{\xi'} r^\gamma| \leq C r^{|\delta'| - |\gamma'|},
\]

\[
|D^\delta_{\xi'} \omega^\gamma| \leq C (|\lambda|^{1/2} + r)^{|\gamma'| - |\delta'|},
\]

\[
|D^\delta_{\xi'} \omega^\gamma| \leq C (|\lambda|^{1/2} + r)^{|\gamma'| - |\delta'|},
\]

\[
|D^\delta_{\xi'} (r + \omega)^\gamma| \leq C (|\lambda|^{1/2} + r)^{|\gamma'| - |\delta'|},
\]

\[
|D^\delta_{\xi'} (\omega + \omega)^\gamma| \leq C (|\lambda|^{1/2} + \alpha^{1/2} + r)^{|\gamma'| - |\delta'|}.
\]

for any \(s \in \mathbb{R}\) and multi-index \(\delta\).
(iii) There exist positive constants $C$ such that the following inequalities hold:
\[
|D_{\xi}^\ell \{ (\tau \partial_r e^{-\tau x_n} ) \} | \leq C\epsilon^{-|\delta'|/2} r x_n, \\
|D_{\xi}^\ell \{ (\tau \partial_r e^{-\omega x_n} ) \} | \leq C(\lambda^{1/2} + r)^{-|\delta'|} e^{-d(\lambda^{1/2} + r)x_n}, \\
|D_{\xi}^\ell \{ (\tau \partial_r e^{-\omega x_n} ) \} | \leq C(\alpha^{1/2} + |\lambda|^{1/2} + r)^{-|\delta'|} e^{-d(\alpha^{1/2} + |\lambda|^{1/2} + r)x_n}, \\
|D_{\xi}^\ell \{ (\tau \partial_r e^{-\omega x_n} ) \} | \leq C(x_n \text{ or } |\lambda|^{-1/2}) e^{-d r x_n r^{-|\delta'|}}, \\
|D_{\xi}^\ell \{ (\tau \partial_r e^{-\omega x_n} ) \} | \leq C(x_n \text{ or } |\lambda|^{-1/2}) e^{-d(\lambda^{1/2} + r)x_n(\lambda^{1/2} + r)^{-|\delta'|}} \quad (19)
\]
for $\ell = 0, 1$ and any multi-index $\delta'$ and $(\xi', x_n) \in (\mathbb{R}^{n-1} \setminus \{0\}) \times (0, \infty)$, where $d$ is a positive constant independent of $\epsilon$ and $\delta'$.

Proof.

(i) (17) are proved by elementary calculation.

(ii) Let $f(t) = t^{r/2}$. By Bell formula, we see
\[
D_{\xi}^\ell e^s = \sum_{\ell=1}^{|\delta|} f^{(\ell)}(r^2) \sum_{\delta_1 + \cdots + \delta_\ell = |\delta|, \delta_1 \geq 1} \Gamma_{\delta_1, \ldots, \delta_\ell}(D_{\xi}^{\delta_1} r^2) \cdots (D_{\xi}^{\delta_\ell} r^2),
\]
where $\Gamma_{\delta_1, \ldots, \delta_\ell}$ is some constant and $f^{(\ell)}(t) = d^\ell f(t)/dt^\ell$. Since $|D_{\xi}^\ell r^2| \leq 2r^2 - |\delta'|$, we can obtain the first estimate. We can prove the other estimates in a similar way to the first estimate taking the elementary estimate: $|\lambda + |\xi||^2 \geq (\sin \epsilon(|\lambda| + |\xi|^2)) (0 < \epsilon < \pi/2, \xi \in \mathbb{R}^n)$ into account.

(iii) It is sufficient to prove the last estimate with $\ell = 0$ in (19), since we can prove the other estimates similarly. Since $\mathcal{M}(\omega\lambda, \omega, x_n) = -x_n \int_0^1 e^{-(1-\theta)\omega x_n + \theta x_n} d\theta$, by Bell formula, we have
\[
|D_{\xi}^\ell e^{-(1-\theta)\omega x_n + \theta x_n}| \\
\leq C_{\delta'} \sum_{\ell=1}^{\delta'} x_n^\ell e^{-(c_1(1-\theta)(|\lambda|^{1/2} + r) + c_2\theta(\alpha^{1/2} + |\lambda|^{1/2} + r))x_n} \\
\times \prod_{j=1}^{\ell} (1 - \theta)(|\lambda|^{1/2} + r)^{1-|\delta'|} + \theta(\alpha^{1/2} + |\lambda|^{1/2} + r)^{1-|\delta'|}),
\]
where we used $|e^{-(1-\theta)\omega x_n + \theta x_n}| = e^{-(1-\theta)\Re \omega x_n + \theta \Re \omega x_n}$. Setting $c = \min(c_1, c_2)$, we see
\[
|D_{\xi}^\ell e^{-(1-\theta)\omega x_n + \theta x_n}| \\
\leq C_{\delta'} e^{-(c/2)((1-\theta)(|\lambda|^{1/2} + r) + \theta(\alpha^{1/2} + |\lambda|^{1/2} + r))x_n} (|\lambda|^{1/2} + r)^{-|\delta'|},
\]
which implies
\[
|D_{\xi}^\ell \mathcal{M}(\omega\lambda, \omega, x_n)| \\
\leq C_{\delta'} \int_0^1 e^{-(c/2)((1-\theta)(|\lambda|^{1/2} + r) + \theta(\alpha^{1/2} + |\lambda|^{1/2} + r))x_n} d\theta x_n (|\lambda|^{1/2} + r)^{-|\delta'|} \\
= C_{\delta'} \int_0^1 e^{-(c/2)(|\lambda|^{1/2} + r)x_n} e^{-\theta(\alpha^{1/2} + |\lambda|^{1/2} + r)x_n} d\theta x_n (|\lambda|^{1/2} + r)^{-|\delta'|}.
\]
By integrating this right hand side, we have
\[
|D_{\xi}^\ell \mathcal{M}(\omega\lambda, \omega, x_n)| \leq C_{\delta'} (c/2)^{-1} \alpha^{-1/2} e^{-(c/2)(|\lambda|^{1/2} + r)x_n (|\lambda|^{1/2} + r)^{-|\delta'|}}. \quad (20)
\]
On the other hands, by \( e^{-\theta(c/2)\alpha^{1/2}x_n} \leq 1 \), we have
\[
|D^{3'}_{\xi'} M(\omega_x, \omega, x_n)| \leq C \theta(c/2)(|\lambda|^{1/2} + r)x_n(|\lambda|^{1/2} + r)^{-|\delta'|}.
\] (21)
Therefore, we obtain the last estimate with \( \ell = 0 \) in (19).

\[\square\]

4. \textbf{Boundedness of the solution operator to resolvent problem.} Goal of this section is to prove the \( R \)-boundedness of the solution operator to the following resolvent problem (10) in \( \Omega \):
\[
\begin{cases}
\lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha &= f \quad \text{in } \Omega, \\
u_\alpha &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\] (10)
where \( \lambda \in \Sigma_{e, \lambda_0}(0 < \varepsilon < \pi/2, \lambda_0 > 0) \) under the approximated weak incompressible condition (9). Our method is based on cut-off technique. For this purpose, we shall first prove the whole space case. Secondly we shall prove the half-space case by using the result for the whole space case and some lemma introduced in section 3. Next we shall prove the bent half-space case by reducing to the result for the half-space case with the change of variable. Finally we shall prove the bounded domain case by using the result for the whole space and the bent half-space case with cut-off technique.

4.1. \textbf{Problem in the whole space.} In this subsection, we shall prove the following theorem:

\textbf{Theorem 4.1.} Let \( \alpha > 0 \), \( 1 < q < \infty \) and \( 0 < \varepsilon < \pi/2 \). Set \( X_\varepsilon(\mathbb{R}^n) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\mathbb{R}^n)\} \). Then, there exist operator families \( \mathcal{U}(\lambda) \) and \( \mathcal{P}(\lambda) \) with
\[\mathcal{U}(\lambda) \in \text{Hol}(\Sigma_e, \mathcal{L}(X_\varepsilon(\mathbb{R}^n), W^2_q(\mathbb{R}^n))), \quad \mathcal{P}(\lambda) \in \text{Hol}(\Sigma_e, \mathcal{L}(X_q(\mathbb{R}^n), \tilde{W}^3_q(\mathbb{R}^n)))\]
such that for any \( f, g \in L_q(\mathbb{R}^n) \) and \( \lambda \in \Sigma_e, \) \( (u_\alpha, \pi_\alpha) = \mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F \), where \( F = (f, \alpha g) \), is a unique solution to (10) under (9) for the case \( \Omega = \mathbb{R}^n \) and \( (\mathcal{U}(\lambda), \mathcal{P}(\lambda)) \) satisfies the following estimates:
\[
\begin{align*}
\mathcal{R}_{\mathcal{L}(X_\varepsilon(\mathbb{R}^n), L_q(\mathbb{R}^n))}(\lambda) &\leq C \quad (\ell = 0, 1), \\
\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n))}(\lambda) &\leq C \quad (\ell = 0, 1)
\end{align*}
\]
for \( G_{\lambda, \alpha, u} = (\lambda u, \lambda^2 \nabla u, \nabla^2 u, (\lambda + \alpha)^{1/2}(\nabla \cdot u)) \) and \( \tilde{N} = 1 + n + n^2 + n^3 \).

\textbf{Proof.} In order to prove the \( \mathcal{R} \)-boundedness of solution operator by using Theorem 3.1, we shall obtain the solution formula to (10) under (9) by using Fourier transform. By the property of Helmholtz projection, we know \( \nabla \pi_\alpha = \alpha \nabla Q_{\mathbb{R}^n}(u_\alpha - g) \) and \( F[\nabla Q_{\mathbb{R}^n}v] = |\xi|^{-2}(\xi \cdot \tilde{v}) \). Applying the Fourier transform to (10), we obtain the following solution formula:
\[
u_\alpha,j(x) = u_j(x) + \tilde{u}_\alpha,j(x) \quad \text{and} \quad \pi_\alpha(x) = \pi(x) + \tilde{\pi}_\alpha(x),
\]
where \( (u, \pi) \) is the solution to Stokes equations given by
\[
\begin{align*}
u_j(x) &= F^{-1}_{\xi} \left[ \frac{1}{\lambda + \xi^2} \tilde{f}_j(\xi) \right](x) - \sum_{k=1}^{n} F^{-1}_{\xi} \left[ \frac{\xi_k \xi_k}{(\lambda + \xi^2)\xi^2} \tilde{f}_k(\xi) \right](x), \\
\pi(x) &= -i \sum_{k=1}^{n} F^{-1}_{\xi} \left[ \frac{\xi_k \xi_k}{\xi^2} \tilde{f}_k(\xi) \right](x)
\end{align*}
\] (22) (23)
for $j = 1, \ldots, n$ and the error term $(u^E_\alpha, \pi^E_\alpha)$ given by

$$ u^E_{\alpha,j} = \sum_{k=1}^{n} F^{-1}_X \left[ \frac{\xi_j \xi_k (f_k(\xi) - \alpha \hat{g}_k)}{\|\xi\|^2(\lambda + \alpha + |\xi|^2)} \right] (x), $$

$$ \pi^E_\alpha = i \sum_{k=1}^{n} F^{-1}_X \left[ \frac{\xi_k (\lambda + |\xi|^2)(f_k(\xi) - \alpha \hat{g}_k)}{\|\xi\|^2(\lambda + \alpha + |\xi|^2)} \right] (x) \tag{24} $$

for $j = 1, \ldots, n$. Since in the whole space case, it is well-known that the solution operator to Stokes equations is $\mathcal{R}$-bounded (\cite{11} for detail), we consider the only error term $(u^E_\alpha, \pi^E_\alpha)$. By Leibniz rule, for $\ell = 0, 1$, we obtain

$$ |(\tau \partial_\tau)^\ell D^\xi \frac{\xi_j \xi_k}{\|\xi\|^2(\lambda + \alpha + |\xi|^2)}| \leq C_{\epsilon, \delta} |\xi|^{-|\ell|}, $$

$$ |(\tau \partial_\tau)^\ell D^\xi \frac{\xi_m \xi_n \xi_j \xi_k}{\|\xi\|^2(\lambda + \alpha + |\xi|^2)}| \leq C_{\epsilon, \delta} |\xi|^{-|\ell|}, $$

$$ |(\tau \partial_\tau)^\ell D^\xi \frac{\xi_j \xi_k (\lambda + |\xi|^2)}{\|\xi\|^2(\lambda + \alpha + |\xi|^2)}| \leq C_{\epsilon, \delta} |\xi|^{-|\ell|}, $$

which implies from Theorem 3.1

$$ \mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n))^N}(\{(\tau \partial_\tau)^\ell (G_{\lambda, \alpha} \mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq C \ (\ell = 0, 1), $$

$$ \mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n))^N}(\{(\tau \partial_\tau)^\ell (\nabla \mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq C \ (\ell = 0, 1). $$

This completes the proof of Theorem 4.1.

\[ \square \]

**Remark 4.2.** By Theorem 4.1, we see that the existence of the solution $(u_\alpha, \pi_\alpha)$ to the resolvent problem (10). Moreover by Theorem 2.6 and Remark 2.5, $(u_\alpha, \pi_\alpha)$ satisfies the following resolvent estimate:

$$ \|(\lambda u_\alpha, \lambda^{1/2} \nabla u_\alpha, \nabla^2 u_\alpha, (\lambda + \alpha)^{1/2}(\nabla \cdot u_\alpha), \nabla \pi_\alpha)\|_{L_q(\mathbb{R}^n)} \leq C_{n, q, \epsilon, \delta} \|(f, \alpha g)\|_{L_q(\mathbb{R}^n)}.$$

### 4.2. Problem in the half-space.

In this section we shall prove the following theorem:

**Theorem 4.3.** Let $\alpha > 0$, $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set $X_q(\mathbb{R}^n_+^+) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\mathbb{R}^n_+)\}$. Then, there exist operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with

$$ \mathcal{U}(\lambda) \in \text{Hol}(\Sigma_\epsilon, \mathcal{L}(X_q(\mathbb{R}^n_+)), W_q^2(\mathbb{R}^n_+)^n), \quad \mathcal{P}(\lambda) \in \text{Hol}(\Sigma_\epsilon, \mathcal{L}(X_q(\mathbb{R}^n_+)), \tilde{W}_q^2(\mathbb{R}^n_+)),$$

such that for any $f, g \in L_q(\mathbb{R}^n_+)^n$ and $\lambda \in \Sigma_\epsilon$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda) F, \mathcal{P}(\lambda) F)$, where $F = (f, \alpha g)$, is a unique solution to (10) under (9) and $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates:

$$ \mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\mathbb{R}^n_+))^N}(\{(\tau \partial_\tau)^\ell (G_{\lambda, \alpha} \mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq C \ (\ell = 0, 1), $$

$$ \mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\mathbb{R}^n_+))^N}(\{(\tau \partial_\tau)^\ell (\nabla \mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\epsilon\}) \leq C \ (\ell = 0, 1) $$

for $G_{\lambda, \alpha} u = (\lambda u, \lambda^{1/2} \nabla u, \nabla^2 u, (\lambda + \alpha)^{1/2}(\nabla \cdot u))$ and $\tilde{N} = 1 + n + n^2 + n^3$.

In order to prove Theorem 4.3 by Lemma 3.2, we shall obtain the solution formula to (10) under (9). By density argument, we may let $f, g \in C_0^\infty(\mathbb{R}_+^n)$. In this case,
equation (10) under (9) is equivalent to the following equations:
\[
\begin{array}{l}
\lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f, \quad \nabla \cdot u_\alpha - \alpha^{-1} \Delta \pi_\alpha = \nabla \cdot g \quad \text{in } \mathbb{R}_+^n, \\
u|_{\partial \mathbb{R}_+^n} = 0, \quad \partial_\alpha \pi_\alpha|_{\partial \mathbb{R}_+^n} = 0.
\end{array} \tag{26}
\]
We shall obtain the solution formula to (26). For this purpose, we extend the external force \( f \) and \( g \) to the whole space. For \( f = (f_1, \ldots, f_n) \) and \( g = (g_1, \ldots, g_n) \), let \( F = (f_1^r, \ldots, f_{n-1}^r, f_n^r) \) and \( G = (g_1^r, \ldots, g_{n-1}^r, g_n^r) \), where
\[
f_j^r(x) = \begin{cases} f_j(x', x_n) & (x_n > 0) \\
f_j(x', -x_n) & (x_n < 0) \end{cases}, \quad f_n^r(x) = \begin{cases} f_n(x', x_n) & (x_n > 0) \\
-f_n(x', -x_n) & (x_n < 0) \end{cases},
\]
where \( x' = (x_1, \ldots, x_{n-1}) \). We consider the resolvent problem with \( F \) and \( G \):
\[
\lambda U_\alpha - \Delta U_\alpha + \nabla \Pi_\alpha = F, \quad \nabla \cdot U_\alpha = \alpha^{-1} \Delta \Pi_\alpha + \nabla \cdot G \quad \text{in } \mathbb{R}_+^n. \tag{27}
\]
Here we remark that from the definition of our extension, \((U_\alpha, \Pi_\alpha)\) enjoys the boundary condition
\[
U_\alpha(x', 0) = 0, \quad \partial_\alpha \Pi_\alpha(x', 0) = 0. \tag{28}
\]
By the result for the whole space and the definition of our extension, the following estimates hold:
\[
\|{(\lambda U_\alpha, \lambda^{1/2} \nabla U_\alpha, \Delta^{1/2} U_\alpha, (\lambda + \alpha)^{1/2} (\nabla \cdot U_\alpha, \nabla \Pi_\alpha)}\|_{L_2(\mathbb{R}_+^n)} \leq C \|{(F, aG)}\|_{L_2(\mathbb{R}_+^n)}. \tag{29}
\]
Setting \( u_\alpha = w_\alpha + U_\alpha \) and \( \pi_\alpha = \rho_\alpha + \Pi_\alpha \), we see that to solve (26) is equivalent to solve
\[
\begin{array}{l}
\lambda w_\alpha - \Delta w_\alpha + \nabla \rho_\alpha = 0, \quad \nabla \cdot w_\alpha = \Delta \rho_\alpha / \alpha \\
(w_\alpha)_j|_{x_n=0} = h_j|_{x_n=0}, \quad \partial_\alpha \rho_\alpha|_{x_n=0} = 0,
\end{array} \tag{30}
\]
where \( h_j = -(U_\alpha)_j \) for \( j = 1, \ldots, n-1 \) and \( h_n = 0 \). Applying div and \((\lambda + \alpha - \Delta)\Delta\) to the first equation in (30), we obtain
\[
(\lambda + \alpha - \Delta) \Delta \rho_\alpha = 0, \quad (\lambda + \alpha - \Delta)(\lambda - \Delta) \Delta w_\alpha = 0. \tag{31}
\]
By applying the partial Fourier transform defined by
\[
\tilde{g}(\xi', x_n) = \int_{\mathbb{R}_+^n} e^{-ix' \cdot \xi'} g(x', x_n) dx'
\]
to (30) and (31) , we have
\[
\begin{align*}
\lambda \tilde{w}_\alpha)_{j} + r^2 \tilde{w}_\alpha)_{j} - \partial_\alpha^2 \tilde{w}_\alpha)_{j} + (i \xi_j) \tilde{\rho}_\alpha = 0, \\
\lambda \tilde{w}_\alpha)_{n} + r^2 \tilde{w}_\alpha)_{n} - \partial_\alpha^2 \tilde{w}_\alpha)_{n} + \partial_\alpha \tilde{\rho}_\alpha = 0, \\
i \xi' \cdot \tilde{w}_\alpha' + \partial_\alpha (\tilde{w}_\alpha)_{n} = \alpha^{-1} (-r^2 \tilde{\rho}_\alpha + \partial_\alpha^2 \tilde{\rho}_\alpha), \\
(\tilde{w}_\alpha)_{j}(\xi', 0) = \tilde{h}_j(\xi', 0), \quad (\tilde{w}_\alpha)_{n}(\xi', 0) = 0, \quad \partial_\alpha \tilde{\rho}_\alpha(\xi', 0) = 0
\end{align*} \tag{32}
\]
and
\[
\begin{align*}
(\lambda + \alpha + r^2 - D_n^2)(r^2 - D_n^2) \tilde{\rho}_\alpha = 0, \\
(\lambda + \alpha + r^2 - D_n^2)(\lambda + r^2 - D_n^2)(r^2 - D_n^2) \tilde{w}_\alpha = 0.
\end{align*} \tag{33}
\]
where \( i \xi' \cdot \tilde{w}_\alpha' = \sum_{j=1}^{n-1} (i \xi_j)(\tilde{w}_\alpha)_{j} \). Since from (33), we see the solution \((\tilde{w}_\alpha, \tilde{\rho}_\alpha)\) can be expressed by
\[
\tilde{\rho}_\alpha = pe^{-rx_n} + q e^{-wx_n}, \quad (\tilde{w}_\alpha)_{j} = a_j e^{-rx_n} + b_j e^{-wx_n} + c_j e^{-\omega x_n}. \tag{34}
\]
for \( j = 1, \ldots, n \), we shall find the solution to (32) having the form (34). By substituting (34) to (32), we see

\[
\begin{align*}
\lambda a_j + (i\xi_j)p &= 0, \\
-\alpha c_j + (i\xi_j)q &= 0, \\
\lambda a_n - rp &= 0, \\
n\xi' \cdot b' - r\alpha a_n &= 0, \\
i\xi' \cdot c' - \omega \alpha n &= \alpha^{-1}(\alpha + \lambda)q, \\
\end{align*}
\]

for \( j = 1, \ldots, n - 1 \). Setting \( A = \lambda(\omega\lambda - r^2) \) and \( B = \alpha\omega(\omega\lambda - r) \), we see

\[
p = -\frac{\alpha\lambda\omega}{r(A + B)}\xi' \cdot \tilde{h}', \quad q = -\frac{r}{\omega}p,
\]

\[
a_j = -\frac{i\xi_j}{\lambda}p, \\
b_j = \tilde{h}_j + \frac{\xi_j}{\lambda}p + \frac{\xi_j r}{\alpha\omega}p, \\
c_j = -\frac{i\xi_j r}{\alpha\omega}p,
\]

\[
a_n = \frac{r}{\lambda}p, \\
b_n = -\frac{r}{\lambda}p - \frac{r}{\alpha}p, \\
c_n = \frac{r}{\alpha}p.
\]

Therefore, we obtain the solution formula \((\tilde{w}_{\alpha})_j = \tilde{w}_j + \tilde{w}_{\alpha E} \) and \( \tilde{\rho}_{\alpha} = \tilde{\rho} + \tilde{\rho}_{E} \),

where \((\tilde{w}, \tilde{w}_{\alpha E}, \tilde{\rho}, \tilde{\rho}_{E})\) is given

\[
\tilde{w}_j = \tilde{h}_j e^{-\omega\lambda x_n} + \frac{\xi_j}{r}\xi' \cdot \tilde{h}'(\omega\lambda, r, x_n),
\]

\[
\tilde{w}_{\alpha E} = -\frac{\xi_j}{\lambda}p \frac{A}{A + B} \xi' \cdot \tilde{h}'(\omega\lambda, r, x_n) - \frac{\xi_j}{\omega\lambda + r} \frac{\alpha\lambda}{A + B} \xi' \cdot \tilde{h}'(\omega, \omega\lambda, x_n),
\]

\[
\tilde{w}_n = i\xi' \cdot \tilde{h}'(\omega\lambda, r, x_n),
\]

\[
\tilde{w}_{\alpha n} = \frac{B}{A + B} \frac{\alpha\omega}{\omega + \lambda} \xi' \cdot \tilde{h}'(\omega, \omega\lambda, x_n) - \frac{\alpha\omega}{\omega + \lambda} \frac{\alpha\lambda}{A + B} i\xi' \cdot \tilde{h}'(\omega, \omega\lambda, x_n),
\]

\[
\tilde{\rho} = -\frac{\omega\lambda + r}{r} i\xi' \cdot \tilde{h}'(\omega, \omega\lambda, x_n),
\]

\[
\tilde{\rho}_{E} = \frac{\omega\lambda + r}{A + B} \frac{\alpha\lambda}{A + B} i\xi' \cdot \tilde{h}'(\omega, \omega\lambda, x_n).
\]

Since the symbol \( M(a, b, x_n) \) defined by (16) has the following properties:

\[
\begin{align*}
\partial_a M(a, b, x_n) &= -e^{-ax_n} - b M(a, b, x_n), \\
\partial^2_a M(a, b, x_n) &= (a + b) e^{-ax_n} + b^2 M(a, b, x_n)
\end{align*}
\]

and by \( g(0) = -\int_{0}^{\infty} \partial_n g(y_n) dy_n \), we have

\[
\tilde{h}(\xi', 0)e^{-ax_n} = \int_{0}^{\infty} E(a) (a - D_n) \tilde{h}(\xi', y_n) dy_n,
\]

\[
\tilde{h}(\xi', 0) M(a, b, x_n) = \int_{0}^{\infty} \{ E(a) \tilde{h}(y_n) + M(a, b, x_n + y_n) (b - D_n) \tilde{h}(\xi', y_n) \} dy_n,
\]

where \( E(z) \) is defined by (16). Therefore, setting \( \xi_j = \xi_j / r \), we obtain

\[
w_j(x) = \int_{0}^{\infty} F_{\xi' \cdot c'}^{-1} E(\omega\lambda) (\omega\lambda - D_n) \tilde{h}_j(\xi', y_n) d\lambda
\]

\[
+ \sum_{k=1}^{n-1} \int_{0}^{\infty} F_{\xi' \cdot c'}^{-1} \tilde{\xi}_k E(\omega\lambda) r \tilde{h}_k(\xi', y_n)
\]

\[
+ M(\omega\lambda, r, x_n + y_n) (r - D_n) \tilde{h}_k(\xi', y_n) d\lambda.
\]
\( (w_\alpha)^E(x) = -\sum_{k=1}^{n-1} \int_0^\infty F_{\xi_i}^{-1}[\xi_k] \xi_k \frac{A}{A+B} (\mathcal{E}(\omega) r \tilde{h}_k(\xi', y_n) + M(\omega, \lambda, r, x_n + y_n)(r - D_n) r \tilde{h}_k(\xi', y_n))(x') dy_n \)
+ \frac{r \xi_j \xi_k}{\omega \lambda + r} \frac{\alpha \lambda}{A + B} (\mathcal{E}(\omega) r \tilde{h}_k(\xi', y_n))
+ M(\omega, \lambda, \omega, x_n + y_n)(\omega - D_n) r \tilde{h}_k(\xi', y_n))(x') dy_n,
\]
\( w_n(x) = \sum_{k=1}^{n-1} \int_0^\infty F_{\xi_i}^{-1}[\xi_k] (\mathcal{E}(\omega) r \tilde{h}_k(\xi', y_n)) + M(\omega, \lambda, r, x_n + y_n)(r - D_n) r \tilde{h}_k(\xi', y_n))(x') dy_n, \)
\( (w_\alpha)^E(x) = \sum_{k=1}^{n-1} \int_0^\infty F_{\xi_i}^{-1}[\xi_k] \frac{i B}{A + B} (\mathcal{E}(\omega) r \tilde{h}_k(\xi', y_n)) + M(\omega, \lambda, r, x_n + y_n)(r - D_n) r \tilde{h}_k(\xi', y_n))(x') dy_n, \)
\( \rho(x) = -\sum_{k=1}^{n-1} \int_0^\infty F_{\xi_i}^{-1}[\xi_k] \frac{\omega \lambda + r}{r} \mathcal{E}(r)(r - D_n) r \xi_k \tilde{h}_k(\xi', y_n))(x') dy_n, \)
\( (\rho_\alpha)^E(x) = \sum_{k=1}^{n-1} \int_0^\infty F_{\xi_i}^{-1}[\xi_k] \frac{\omega \lambda + r}{r} \frac{A}{A + B} i \mathcal{E}(r)(r - D_n) r \tilde{h}_k(\xi', y_n))(x') dy_n + \frac{\alpha \lambda}{A + B} \frac{\omega \lambda + r}{r} i \mathcal{E}(\omega)(\omega - D_n) r \tilde{h}_k(\xi', y_n))(x') dy_n. \) \tag{35}

We remark that \((w, \rho)\) is the solution to the usual Stokes equations and \((w^E, \rho^E)\) is the error between the solution to Stokes equations and Stokes equations approximated by pressure stabilization. Since Shibata and Shimizu \cite{ShibataShimizu} proved \(R\)-boundedness of solution operator to Stokes equations, it is sufficient to consider \((w_n^E, \rho_n^E)\) only. For this purpose, we prepare the following lemma.

**Lemma 4.4.** Let \(0 < \varepsilon < \pi/2\) and \(\alpha > 0\). For any multi-index \(\delta'\) and \((\lambda, \xi', x_n) \in \Sigma_{\varepsilon} \times (\mathbb{R}^{n-1} \setminus \{0\}) \times (0, \infty), m(\lambda, \xi') = r(\omega \lambda + r)^{-1}, \omega(\omega \lambda + \omega)^{-1}, A(A + B)^{-1}, B(A + B)^{-1} and \alpha \lambda(A + B)^{-1} enjoy
\[
|\partial^\delta_{\xi'} m(\lambda, \xi')| \leq C r^{-|\delta'|}, \tag{36}
\]
where \(C\) is a positive constant which is dependent of \(\varepsilon\) and \(\delta'\).

**Proof.** We first show that \(m(\lambda, \xi') = r(\omega \lambda + r)^{-1}\) and \(\omega(\omega \lambda + \omega)^{-1}\) enjoy (36). By Leibniz rule with (18), we see
\[
\left| \frac{D^\xi_{\xi'} r}{\omega \lambda + r} \right| \leq C \sum_{\delta = \delta_1 + \delta_2} r^{1 - |\delta'_1|} \frac{r^{-|\delta'_2|}}{|\lambda|^{1/2} + r} \leq C r^{-|\delta'|},
\]
Proof of Theorem 4.3.

Set \(\xi' = \xi + \varepsilon\). In order to prove \(m(\lambda, \xi') = A(A + B)^{-1}B(A + B)^{-1} \) and \(\alpha \lambda (A + B)^{-1}\), we shall consider \(D_{\xi'}^j(A + B)\). Since

\[
A + B = (\lambda + \alpha)\omega(\omega \lambda - r) + \lambda r(\omega - r) = \frac{\lambda(\lambda + \alpha)\omega}{\omega \lambda + r} + \frac{\lambda(\lambda + \alpha)r}{\omega + r},
\]

we have

\[
\left| D_{\xi'}^j(A + B) \right| \leq C|\lambda|(|\lambda| + \alpha) \left\{ \frac{|\lambda|^{1/2} + \alpha^{1/2} + r}{|\lambda|^{1/2} + \alpha^{1/2} + r} \right\} r^{-|\delta'|} \leq C|\lambda|(|\lambda|^{1/2} + \alpha^{1/2})^2(\lambda|^{1/2} + \alpha^{1/2} + r)\lambda|^{1/2} + \alpha^{1/2} + r) \right)^{-1} - \delta'}. \tag{37}
\]

Since \(|\arg[\omega(\omega + r)/(\omega \lambda + r)]| < \pi - \varepsilon\), we know \(\omega r^{-1}(\omega + r)(\omega \lambda + r)^{-1} \in \Sigma_\varepsilon\), which implies that

\[
|A + B| = |\lambda + \alpha||\lambda| \left\{ \frac{r}{\omega + r} \right\} \left\{ \frac{\omega}{\omega \lambda + r} \cdot \frac{\omega + r}{r} + 1 \right\} \geq C(|\lambda|^{1/2} + \alpha^{1/2})^2|\lambda| r^{1/2} + \alpha^{1/2} + r)^{-1} \left\{ \frac{\omega}{\omega \lambda + r} \cdot \frac{\omega + r}{r} + 1 \right\} \geq C(|\lambda|^{1/2} + \alpha^{1/2})^2|\lambda|^{1/2} + \alpha^{1/2} + r)^{-1} \right)^{-1}.
\]

By Bell’s formula with (37), we obtain

\[
\left| D_{\xi'}^j(A + B)^{-1} \right| \leq C|\lambda|^{-1}(|\lambda|^{1/2} + \alpha^{1/2})^2(|\lambda|^{1/2} + \alpha^{1/2} + r)^{-1}(|\lambda|^{1/2} + \alpha^{1/2} + r)^{-1}\right)^{-1},
\]

which implies (36) for \(m(\lambda, \xi') = A(A + B)^{-1}B(A + B)^{-1} \) and \(\alpha \lambda (A + B)^{-1}\).

**Proof of Theorem 4.2.** We shall prove Theorem 4.3 by Lemma 3.2 with Lemma 4.4.

Set \((w_n)^E_{j,k,\ell}(x)(k = 1, \ldots, n - 1, \ell = 1, \ldots, 6)\) as follows

\[
(w_n)^E_{j,k,1}(x) = \int_0^\infty e_{\xi'}(x) \left( c_{j,k} \frac{A}{A + B} E(\omega \lambda) r \hat{h}_{\xi'}(\xi', y_n) \right) dy_n,
\]

\[
(w_n)^E_{j,k,2}(x) = \int_0^\infty e_{\xi'}(x) \left( c_{j,k} \frac{A}{A + B} M(\omega \lambda, r, x_n + y_n, r^2 \hat{h}_{\xi'}(\xi', y_n) \right) dy_n,
\]

\[
(w_n)^E_{j,k,3}(x) = \int_0^\infty e_{\xi'}(x) \left( c_{j,k} \frac{A}{A + B} M(\omega \lambda, r, x_n + y_n, r D_n \hat{h}_{\xi'}(\xi', y_n) \right) dy_n,
\]

\[
(w_n)^E_{j,k,4}(x) = \int_0^\infty e_{\xi'}(x) \left( r c_{j,k} \frac{\alpha}{\omega + r A + B} E(\omega \lambda) r \hat{h}_{\xi'}(\xi', y_n) \right) dy_n,
\]

\[
(w_n)^E_{j,k,5}(x) = \int_0^\infty e_{\xi'}(x) \left( r c_{j,k} \frac{\alpha}{\omega + r A + B} M(\omega \lambda, \omega, x_n + y_n, r \omega \hat{h}_{\xi'}(\xi', y_n) \right) dy_n,
\]

\[
(w_n)^E_{j,k,6}(x) = \int_0^\infty e_{\xi'}(x) \left( r c_{j,k} \frac{\alpha}{\omega + r A + B} M(\omega \lambda, \omega, x_n + y_n, r D_n \hat{h}_{\xi'}(\xi', y_n) \right) dy_n.
\]

Setting \(K_{\alpha,\ell,\xi}(h_k) = (w_n)^E_{j,k,\ell}(x)\) for \(\ell = 1, 2, 4, 5, 6\), by Lemma 3.2, Lemma 4.4 and (29), we see that \(K_{\alpha,\ell,\xi}\) is \(\mathcal{R}\)-bounded. Since \(h_k = -(U_n)_{\xi,k}\), \(U_n = U_{\xi,n} \langle \lambda \rangle F\), where \(U_{\xi,n} \langle \lambda \rangle\) is the solution operator in \(\mathbb{R}^n\) and \(F = (f, \alpha g)\), setting \(V_{j,k,\ell}(\lambda)F = K_{\alpha,\ell,\xi}(U_{\xi,k} \langle \lambda \rangle F)\), we see that \(G_{\lambda,\alpha} V_{j,k,\ell}(\lambda)F = K_{\alpha,\ell,\xi}(G_{\lambda,\alpha} U_{\xi,k} \langle \lambda \rangle F)\) is \(\mathcal{R}\)-bounded by Remark 2.5.
Since Lemma 3.2 and Lemma 4.4 and the relation:
\[ \lambda(w_\alpha)^{E}_{\Phi}(x) = \int_0^\infty F_\xi^{-1}\tilde{\xi}k - A + B^{\alpha}M(\omega_\lambda, r, x_n + y_n) \times r|\lambda|^{1/2}\frac{\lambda}{|\lambda|}(|\lambda|^{1/2}D_n\tilde{h}_k(\xi', y_n)))(x')dy_n, \]
we see there exists a \( \mathcal{R} \)-bounded operator \( K_{\alpha,3,j} \) such that \( K_{\alpha,3,j}(|\lambda|^{1/2}D_nh_k) = \lambda(w_\alpha)^{E}_{\Phi}(x) \). Setting \( \lambda V_{j,k,3}(\lambda)F = K_{\alpha,3,j}(|\lambda|^{1/2}D_nh_k) \), we see \( \lambda V_{j,k,3}(\lambda)F \) is \( \mathcal{R} \)-bounded. In a similar way, we can show that \( G_{\alpha,\Phi}V_{j,k,\ell}(\lambda)F(\ell = 3, 6) \) is \( \mathcal{R} \)-bounded. Summing up, setting \( \mathcal{U}(\lambda)(F) = \sum_{k,\ell}V_{j,k,\ell}(\lambda)F \) and \( \mathcal{U}(\lambda)F = ((\mathcal{U}(\lambda)F)_{j=1,...,n} \), we see \( \mathcal{U}(\lambda)F \) is the solution operator in \( \mathbb{R}_+^n \) and \( G_{\alpha,\Phi} \mathcal{U}(\lambda)F \) is \( \mathcal{R} \)-bounded.

In the same way, we obtain the results for \( (w_\alpha)^{E}_{\Phi}(x) \) from the results for \( (w_\alpha)^{E}_{\Phi}(x) \) and the results for \( (\rho_\alpha)^{E}(x) \) from the equations (10) and the results for \( (w_\alpha)^{E}_{\Phi}(x) \) and \( (\rho_\alpha)^{E}(x) \).

4.3. Problem in the bent half-space and the bounded domain. Before we describe the theorem for bent half-space, we shall introduce some notations. Let \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) be a bijection of \( C^1 \) class and let \( \Phi^{-1} \) be its inverse map. Writing \( \nabla \Phi = A + B(x) \) and \( \nabla \Phi^{-1} = A_1 + B_1(x) \), we assume that \( A \) and \( A_1 \) are orthogonal matrices with constant coefficients and \( B(x) \) and \( B_1(x) \) are matrices of functions in \( W^{1,\infty}(\mathbb{R}^n) \) with \( n < r < \infty \) such that
\[ \|B, B_1\|_{L_r(\mathbb{R}^n)} \leq M_1, \quad \|\nabla(B, B_1)\|_{L_r(\mathbb{R}^n)} \leq M_2. \] (38)

We shall choose \( M_1 \) small enough later, so that we may assume that \( 0 < M_1 \leq 1 \leq M_2 \).

Let \( \Omega_+ = \Phi(\mathbb{R}^n_0) \) and \( \partial \Omega_+ = \Phi(\mathbb{R}^n_0) \).

Theorem 4.5. Let \( \alpha > 0 \), \( 1 < q < \infty \) and \( 0 < \varepsilon < \pi/2 \). Set \( X_q(\Omega^+ \cap \mathbb{R}^n_0) = \{(f_1, f_2) \mid F_1, F_2 \in L_{q}(\Omega_+^+)\} \). Then there exist \( M_1 \in (0, 1), \lambda_0 \geq 1 \) and solution operator families \( \mathcal{U}(\lambda) \) and \( \mathcal{P}(\lambda) \) with
\[ \mathcal{U}(\lambda) \in \text{Hol}(\lambda^\alpha, \mathcal{L}(X_q(\Omega^+_+), W^{2,\infty}_q(\Omega_+))), \]
\[ \mathcal{P}(\lambda) \in \text{Hol}(\lambda^\alpha, \mathcal{L}(X_q(\Omega^+_+), W^{1,\infty}_q(\Omega_+))), \] (39)

such that for any \( (f, \alpha) \in X_q(\Omega^+_+) \) and \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \), \( (u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F) \), where \( F = (f, \alpha) \), is a unique solution to problem (10) under (9). Moreover \( (\mathcal{U}(\lambda), \mathcal{P}(\lambda)) \) satisfies the following estimates:
\[ \mathcal{R}_{\ell}(\lambda \alpha, \lambda^\alpha, \mathcal{L}(X_q(\Omega^+_+), L_q(\Omega^+_+))) \subseteq C \quad (\ell = 0, 1), \]
\[ \mathcal{R}_{\ell}(\lambda \alpha, \lambda^\alpha, \mathcal{L}(X_q(\Omega^+_+), L_q(\Omega^+_+))) \subseteq C \quad (\ell = 0, 1) \]
for \( \mathcal{L}_{\alpha,\alpha} = (\lambda u, \lambda^{1/2}du, \sqrt{2}u, (\lambda + \alpha)^{1/2}(\nabla \cdot u)) \) and \( \tilde{N} = 1 + n + n^2 + n^3 \).

Proof. In order to prove Theorem 4.5, we transfer (10) and (9) into a problem in \( \mathbb{R}^n_+ \) by the change of variable \( x = \Phi^{-1}(y) \) with \( y \in \Omega_+ \) and \( x \in \mathbb{R}^n_+ \) and by the change of unknowns: \( v(x) = A_{-1}(u_\alpha(\Phi(x)) \), \( \rho = A_{-1}(\pi_\alpha(\Phi(x))) \) and \( \psi(x) = A_{-1}(\rho(\Phi(x))) \). Since \( \partial_y = \sum_{\ell=1}^n(A_{\ell,j} + B_{\ell,j})\partial_{x_j} \), employing the same argument as Shibata [9], we have the following equations
\[ \begin{cases} \lambda v - \Delta v + \nabla \rho = f_+ + \mathcal{F}(v, \rho) & x \in \mathbb{R}^n_+, \\ v = 0 & x \in \partial \mathbb{R}^n_+ \end{cases} \] (40)
under

\[(v, \nabla \psi)_{\mathbb{R}^n_{+}} = \alpha^{-1}(\nabla \rho, \nabla \psi)_{\mathbb{R}^n_{+}} + (g_+, \nabla \psi)_{\mathbb{R}^n_{+}} + (G(v, \rho), \nabla \psi)_{\mathbb{R}^n_{+}}, \quad \psi \in \mathbb{W}^1_q(\mathbb{R}^n_{+}), \]

(41)

where \(f_+(x) = A_{-1}(f(\Phi(x)))\) and \(g_+(x) = A_{-1}(g(\Phi(x))) + M_4 g\). Moreover \(\mathcal{F}(v, \rho)\) and \(G(v, \rho)\) have the following forms:

\[\mathcal{F}(v, \rho) = M_1 \nabla^2 v + M_2 \nabla v + M_3 \nabla \rho, \quad G(v, \rho) = \alpha^{-1}(M_4 v + M_5 \nabla \rho)\]

(42)

with some matrices of functions \(M_k (k = 1, \ldots, 5)\) possessing the estimates

\[\|M_j\|_{L_{\infty}(\mathbb{R}^n_{+})} \leq CM_1, \quad \|(M_2, \nabla M_j)\|_{L_{\infty}(\mathbb{R}^n_{+})} \leq CM_2\]

(43)

for \(j = 1, 3, 4, 5\) and \(n < r < \infty\). Setting \(\mathcal{F}(\lambda) F = \mathcal{F}(U_{\mathbb{R}^n_{+}}(\lambda)) F\) and \(\mathcal{G}(\lambda) F = \mathcal{G}(U_{\mathbb{R}^n_{+}}(\lambda)) F\), where \(F = (f_+, \alpha g_+)\) and \((U_{\mathbb{R}^n_{+}}(\lambda), P_{\mathbb{R}^n_{+}}(\lambda))\) is the solution operator in \(\mathbb{R}^n_{+}\), we can obtain, for \(\ell = 0, 1\),

\[\mathcal{R}_{L(X_\lambda, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell \mathcal{F}(\lambda) \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq \left\{C(\sigma + M_1) + C_{\sigma} \lambda_0^{-1/2}\right\} \kappa_0,\]

\[\mathcal{R}_{L(X_\lambda, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell \alpha \mathcal{G}(\lambda) \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq \left\{C(\sigma + M_1) + C_{\sigma} \lambda_0^{-1/2}\right\} \kappa_0,\]

where \(\kappa_0\) is the \(\mathcal{R}\)-bound of the half-space case and \(\sigma > 0\), by the method due to Shibata [9]. We choose \(\sigma\) and \(M_1\) so small that \(C(\sigma + M_1) \kappa_0 \leq 1/8\) and \(\lambda_0 \geq 1\) so large that \(C_{\sigma} \lambda_0^{-1/2} \kappa_0 \leq 1/8\). Thus, we have

\[\mathcal{R}_{L(X_{\lambda}, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell \mathcal{F}(\lambda) \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq 1/4\]

(\(\ell = 0, 1\)),

\[\mathcal{R}_{L(X_{\lambda}, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell \alpha \mathcal{G}(\lambda) \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq 1/4\]

(\(\ell = 0, 1\)).

Since \(\mathcal{R}\)-boundedness implies the usual boundedness (see Remark 2.5), we have

\[\|\left(\mathcal{F}(\lambda) F, \alpha \mathcal{G}(\lambda) F\right)\|_{L_2(\mathbb{R}^n_{+})} \leq 2^{-1} \|F\|_{L_2(\mathbb{R}^n_{+})},\]

where \(F = (f, \alpha g)\) for \(\lambda \in \Gamma_{\varepsilon, \lambda_0}\). Therefore \(\mathcal{R}(\lambda) F = (\mathcal{F}(\lambda) F, \alpha \mathcal{G}(\lambda) F)\) is a contraction map from \(X_\lambda_{\mathbb{R}^n_{+}}\) into itself, so that for each \(\lambda \in \Gamma_{\varepsilon, \lambda_0}\), \((I + \mathcal{R}(\lambda))^{-1}\) exists and \(\|\left(I + \mathcal{R}(\lambda)\right)^{-1}\|_{L(X_{\lambda}, L_2(\mathbb{R}^n_{+}))} \leq 2\). If we define \(v\) and \(\rho\) by \(v = U_{\mathbb{R}^n_{+}}(\lambda)(I + \mathcal{R}(\lambda))^{-1} F\) and \(\rho = P_{\mathbb{R}^n_{+}}(\lambda)(I + \mathcal{R}(\lambda))^{-1} F\), where \(F = (f, \alpha g)\), then \((v, \rho)\) is a unique solution to (40) under (41). Moreover we have

\[\mathcal{R}_{L(X_{\lambda}, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell (1 + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq 2\]

(\(\ell = 0, 1\)),

which implies

\[\mathcal{R}_{L(X_{\lambda}, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell \mathcal{G} \lambda^\alpha U_{\mathbb{R}^n_{+}}(\lambda)(I + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq 2 \kappa_0, \quad (\ell = 0, 1),\]

\[\mathcal{R}_{L(X_{\lambda}, L_2)}\left(\left\{\left(\tau \partial_\tau\right)_A^\ell \nabla P_{\mathbb{R}^n_{+}}(\lambda)(I + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_\varepsilon, \lambda_0\right\}\right) \leq 2 \kappa_0, \quad (\ell = 0, 1).\]

By the change of variable \(y = \Phi(x)\) transfer (10) under (9) in the half-space case into the bent half-space case, we see that \(u_\alpha(y) = T_{\mathcal{A}_{-1}}(v(\Phi^{-1}(y)))\) and \(\pi_\alpha = T_{\mathcal{A}_{-1}}(\rho(\Phi^{-1}(y)))\) is a unique solution to (10) under (9) in the bent half-space and we construct an \(\mathcal{R}\)-bounded solution operator. This completes the proof of Theorem 4.5.

By using the cut-off technique with Theorem 4.5, we shall prove Theorem 2.7.
Proof of Theorem 2.7. We set $\mathcal{H}_j^1 = \Phi_1^1(\mathbb{R}_+^n)$, $\partial \mathcal{H}_j^1 = \Phi_1^1(\partial \mathbb{R}_+^n)$ and $\mathcal{H}_j^2 = \mathbb{R}^n$ and set $\xi_j^i$ as the cut-off function enjoys $0 \leq \xi_j^i \leq 1$ and $\text{supp}\xi_j^i \subseteq B_{d_j}(x_j) = \{x \in \Omega \mid |x-x_j| < d_j\}$. Let $f, g \in L_{\eta}(\Omega)$. We consider the two equations

$$\begin{cases}
\lambda u_j^1 - \Delta u_j^1 + \nabla \pi_j^1 = \xi_j^1 f & x \in \mathcal{H}_j^1, \\
u_j = 0 & x \in \partial \mathcal{H}_j^1
\end{cases} \tag{44}$$

under

$$(u_j^1, \nabla \varphi)_{\mathcal{H}_j^1} = \alpha^{-1}((\nabla \pi_j^1, \nabla \varphi)_{\mathcal{H}_j^1} + (\xi_j^1 g, \nabla \varphi)_{\mathcal{H}_j^1}) \quad \varphi \in \hat{W}_q^1(\mathcal{H}_j^1) \tag{45}$$

and

$$\lambda u_j^2 - \Delta u_j^2 + \nabla \pi_j^2 = \xi_j^2 f \quad x \in \mathcal{H}_j^2 \tag{46}$$

under

$$(u_j^2, \nabla \varphi)_{\mathcal{H}_j^2} = \alpha^{-1}((\nabla \pi_j^2, \nabla \varphi)_{\mathcal{H}_j^2} + (\xi_j^2 g, \nabla \varphi)_{\mathcal{H}_j^2}) \quad \varphi \in \hat{W}_q^1(\mathcal{H}_j^2). \tag{47}$$

By Theorem 4.1 and Theorem 4.5, there exist operator families

$$\mathcal{U}_j^k(\lambda), \mathcal{P}_j^k(\lambda) \quad (k = 1, 2)$$

with

$$\mathcal{U}_j^k(\lambda) \in \text{Hol}(\Gamma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{H}_j^k, W_{\eta}^2(\mathcal{H}_j^k))),$$

$$\mathcal{P}_j^k(\lambda) \in \text{Hol}(\Gamma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{H}_j^k, \hat{W}_q^1(\mathcal{H}_j^k)))$$

such that $(u_j^k, \pi_j^k) = (\mathcal{U}_j^k(\lambda)(\xi_j^k f, \alpha \xi_j^k g), \mathcal{P}_j^k(\lambda)(\xi_j^k f, \alpha \xi_j^k g))$ uniquely solves the problem (44) under (45) and the problem (46) under (47), respectively. Moreover we see

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathcal{H}_j^k), L_{\eta}(\mathcal{H}_j^k))}((\tau \partial_x)^{\ell} G_{\lambda_0} \mathcal{U}_j^k(\lambda) | \lambda \in \Gamma_{\epsilon, \lambda_0}) \leq \kappa_2,$$

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathcal{H}_j^k), L_{\eta}(\mathcal{H}_j^k))}((\tau \partial_x)^{\ell} \mathcal{P}_j^k(\lambda) | \lambda \in \Gamma_{\epsilon, \lambda_0}) \leq \kappa_2 \tag{48}$$

with some constant $\kappa_2$ independent of $j \in \mathbb{N}$. By (48), we obtain

$$\|\lambda u_j^k - \lambda^{1/2} \nabla u_j^k, \nabla^2 u_j^k, \lambda (\alpha + 1)^{1/2} \nabla \cdot u_j^k, \nabla \pi_j^k\|_{L_{\eta}(\mathcal{H}_j^k)} \leq \kappa_2 \|((\xi_j^k f, \alpha \xi_j^k g))\|_{L_{\eta}(\mathcal{H}_j^k)}.$$  

For $f, g \in L_{\eta}(\Omega)$, we set

$$U(\lambda)(f, \alpha g) = \sum_{j=1}^{\infty} \xi_j^1 u_j^1 + \sum_{j=1}^{\infty} \xi_j^2 u_j^2, \quad P(\lambda)(f, \alpha g) = \sum_{j=1}^{\infty} \xi_j^1 \pi_j^1 + \sum_{j=1}^{\infty} \xi_j^2 \pi_j^2.$$  

Inserting $(v, \pi) = (U(\lambda)(f, \alpha g), P(\lambda)(f, \alpha g))$ into (10) and (9), we have

$$\begin{cases}
\lambda v - \Delta v + \nabla \pi = f - \mathcal{V}_1(\lambda)(f, \alpha g), & x \in \Omega, \\
v = 0 & x \in \partial \Omega
\end{cases} \tag{49}$$

under

$$(v, \nabla \varphi)_{\Omega} = \alpha^{-1}((\nabla \pi, \nabla \varphi)_{\Omega} + (g, \nabla \varphi)_{\Omega} + \mathcal{V}_2(\lambda)(f, \alpha g), \nabla \varphi)_{\Omega}$$

with

$$\mathcal{V}_1(\lambda)(f, \alpha g) = \sum_{j=1}^{\infty} \{2(\nabla \xi_j^1) \cdot (\nabla u_j^1) + (\Delta \xi_j^1) u_j^1 - (\nabla \xi_j^1) \pi_j^1$$

$$+ 2(\nabla \xi_j^2) \cdot (\nabla u_j^2) + (\Delta \xi_j^2) u_j^2 - (\nabla \xi_j^2) \pi_j^2 \} ,$$

$$\mathcal{V}_2(\lambda)(f, \alpha g) = \alpha^{-1} \sum_{j=1}^{\infty} \{ (\nabla \xi_j^1) \pi_j^1 + (\nabla \xi_j^2) \pi_j^2 \}.$$
Since by Poincare inequality we obtain
\[ \| (\nabla \xi_j^k) \pi_j^k \|_{L_q(\Omega)} \leq C \| \nabla \pi_j^k \|_{L_q(\Omega)} \leq C \alpha \| u \|_{L_q(\Omega)} \]
and \( \pi = \alpha Q u \), we have \( V^1(\lambda)(f, \alpha g), V^2(\lambda)(f, \alpha g) \in L_q(\Omega) \) and
\[ \| (V^1(\lambda)(f, \alpha g), \alpha V^2(\lambda)(f, \alpha g)) \|_{L_q(\Omega)} \leq C \lambda_0^{-1/2} (1 + \lambda_0^{-1/2} + \alpha \lambda_0^{-1/2}) \| (f, \alpha g) \|_{L_q(\Omega)}. \]

Choosing \( \lambda_0 \geq 1 \) so large that \( C \lambda_0^{-1/2} (1 + \lambda_0^{-1/2} + \alpha \lambda_0^{-1/2}) \leq 1/2 \) and setting \( V(\lambda)F = (V^1(\lambda)F, V^2(\lambda)F) \), where \( F = (f, \alpha g) \), we see that \( (I - V(\lambda))^{-1} \in \mathcal{L}(X_q(\Omega)) \) exists and \( (u, \pi) = (U(\lambda)(I - V(\lambda))^{-1}F, P(\lambda)(I - V(\lambda))^{-1}F) \) is a unique solution to problem (10) under (9).

Finally we shall show the \( \mathcal{R} \)-boundedness of solution operator. Let
\[
U(\lambda)F = \sum_{j=1}^{\infty} \xi_j^1 U_j^1(\lambda)F + \sum_{j=1}^{\infty} \xi_j^2 U_j^2(\lambda)F, \\
P(\lambda)F = \sum_{j=1}^{\infty} \xi_j^1 P_j^1(\lambda)F + \sum_{j=1}^{\infty} \xi_j^2 P_j^2(\lambda)F
\]
and
\[
V^1(\lambda)F = \sum_{j=1}^{\infty} \{ 2(\nabla \xi_j^1) \cdot (\nabla U_j^1(\lambda)F) + (\Delta \xi_j^1) U_j^1(\lambda)F - (\nabla \xi_j^1) P_j^1(\lambda)F \\ + 2(\nabla \xi_j^2) \cdot (\nabla U_j^2(\lambda)F) + (\Delta \xi_j^2) U_j^2(\lambda)F - (\nabla \xi_j^2) P_j^2(\lambda)F \}, \\
V^2(\lambda)(f, \alpha g) = \alpha^{-1} \sum_{j=1}^{\infty} \{ (\nabla \xi_j^1) P_j^1(\lambda)F + (\nabla \xi_j^2) P_j^2(\lambda)F \},
\]
where \( F = (f, \alpha g) \). We see that \( U(\lambda) \in \text{Hol}(\Gamma_{\varepsilon, \lambda_0}, \mathcal{L}(X_q(\Omega), W_q^2(\Omega))) \) and \( P(\lambda) \in \text{Hol}(\Gamma_{\varepsilon, \lambda_0}, \mathcal{L}(X_q(\Omega), W_q^1(\Omega))) \) and \( (v, \pi) = (U(\lambda)F, P(\lambda)F) \), where \( F = (f, \alpha g) \) satisfies
\[
\begin{cases} 
\lambda v - \Delta v + \nabla \pi = f - V^1(\lambda)(f, \alpha g) & x \in \Omega, \\
v = 0 & x \in \partial \Omega
\end{cases}
\]
under
\[
(v, \nabla \varphi)_\Omega = \alpha^{-1}(\nabla \pi, \nabla \varphi)_\Omega + (g, \nabla \varphi)_\Omega + (V^2(\lambda)(f, g), \nabla \varphi)_\Omega.
\]

Since
\[
\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{ (\tau \partial_r)^j V^1(\lambda) \mid \lambda \in \Gamma_{\varepsilon, \lambda_0} \}) \leq C \lambda_0^{-1/2} (1 + \lambda_0^{-1/2} + \alpha \lambda_0^{-1/2}) \kappa_2, \\
\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{ (\tau \partial_r)^j \alpha V^2(\lambda) \mid \lambda \in \Gamma_{\varepsilon, \lambda_0} \}) \leq C \lambda_0^{-1/2} (1 + \lambda_0^{-1/2} + \alpha \lambda_0^{-1/2}) \kappa_2,
\]
Choosing \( \lambda_0 \geq 1 \) so large that \( C \lambda_0^{-1/2} (1 + \lambda_0^{-1/2} + \alpha \lambda_0^{-1/2}) \leq 1/2 \), we have
\[
\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{ (\tau \partial_r)^j (I - V(\lambda))^{-1} \mid \lambda \in \Gamma_{\varepsilon, \lambda_0} \}) \leq 2.
\]
Therefore we obtain
\[
\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{ (\tau \partial_r)^j G_{\lambda, \alpha} U(\lambda) \mid \lambda \in \Gamma_{\varepsilon, \lambda_0} \}) \leq C, \\
\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{ (\tau \partial_r)^j \nabla P(\lambda) \mid \lambda \in \Gamma_{\varepsilon, \lambda_0} \}) \leq C.
\]
We see that \( U(\lambda)(I - V(\lambda))^{-1} \) is a required \( \mathcal{R} \)-bounded solution operator to (10) under (9). This completes the proof of Theorem 2.7. \( \Box \)
5. Application to the approximated Navier-Stokes equations. In this section, we shall prove the local in time existence theorem for (2) and (12) (Theorem 2.1 and Theorem 2.12) by the method due to Shibata-Kubo [10]. Before we prove these theorems, we shall describe some facts shown by using maximal $L_p$-$L_q$ regularity theorem (Theorem 2.2).

Let $(w, \tau) = M_T (f)$ be the solution to

\[
\begin{aligned}
\partial_t w - \Delta w + \nabla \tau &= f & \quad & x \in \Omega, \ t \in (0, T), \\
w(0, x) &= 0 & \quad & x \in \Omega, \\
w(t, x) &= 0 & \quad & x \in \partial \Omega
\end{aligned}
\] (49)

under the approximated weak incompressible condition (5)

For $f \in \mathcal{L}_p((0, T), L_q(\Omega))$, let $f_0(t) = f(t)$ ($0 < t < T$) and $f_0(t) = 0$ ($t \notin (0, T)$). Then, letting $(w, \tau)$ be the solution to Stokes equation for $f = f_0$ on $t \in (0, \infty)$, $(w, \tau)$ can define on $\mathbb{R}$. Moreover, this solution satisfies $w(t) = \tau (t) = 0$ ($t \leq 0$) and (49) on $t \in (0, T)$. Furthermore, by Theorem 2.2, the following estimate holds: for $0 < S \leq T$, we have

\[
\|\partial_t w\|_{L_p((0, S), L_q(\Omega))} \leq e^{\gamma S} \|e^{-\gamma t} \partial_t w\|_{L_p((0, T), L_q(\Omega))} \leq C_{n, p, q} e^{\gamma S} \|f\|_{L_p((0, T), L_q(\Omega))}.
\] (50)

Similarly we have

\[
\|\nabla^2 w\|_{L_p((0, S), L_q(\Omega))} + \|\nabla \tau\|_{L_p((0, S), L_q(\Omega))} \leq C_{n, p, q} e^{\gamma S} \|f\|_{L_p((0, T), L_q(\Omega))}.
\] (51)

Moreover taking into account the fact about Bessel potential space:

\[
\|e^{-\gamma t} u\|_{L_q(\mathbb{R}, X)} \leq C\|e^{-\gamma \Lambda_1^a} u\|_{L_p(\mathbb{R}, X)} \leq C\gamma^{-2} \|e^{-\gamma \Lambda_2^\beta} u\|_{L_p(\mathbb{R}, X)}
\] (52)

for Banach space $X$, $1 < p < q < \infty$, $\alpha = 1/p - 1/q$, $\alpha < \beta < \infty$ and $\gamma \geq 0$ and the estimate:

\[
\|e^{-\gamma t} u\|_{L_z(\mathbb{R}, X)} \leq C\|e^{-\gamma \Lambda_3^r} u\|_{L_p(\mathbb{R}, X)}
\] for $0 < \alpha - 1/p < 1$ and $1 < p < \infty$ (see [2]), by Theorem 2.2 we obtain

\[
\begin{aligned}
\|\nabla w\|_{L_r((0, S), L_q(\Omega))} + \|w\|_{L_\infty((0, S), L_q(\Omega))} &
\leq Ce^{\gamma S}\|e^{-\gamma \Lambda_1^\beta} \nabla w\|_{L_p(\mathbb{R}, L_q(\Omega))} + Ce^{\gamma S}\|e^{-\gamma \Lambda_2^\beta} w\|_{L_p(\mathbb{R}, L_q(\Omega))} \\
&\leq Ce^{\gamma S}\|e^{-\gamma \Lambda_3^{r/2}} \nabla w\|_{L_p(\mathbb{R}, L_q(\Omega))} + Ce^{\gamma S}\|e^{-\gamma \Lambda_3^{r/2}} w\|_{L_p(\mathbb{R}, L_q(\Omega))} \\
&\leq Ce^{\gamma S}\|f\|_{L_p((0, T), L_q(\Omega))},
\end{aligned}
\] (53)

where $1/p - 1/r \leq 1/2$.

Letting $\beta = n/(2q)$ and $\ell_k (k = 1, 2, 3)$ are the positive constants satisfying

\[
0 < \frac{1}{p} - \frac{1}{\beta p \ell_1} \leq \frac{1}{2}, \quad 0 < \frac{1}{p} - \frac{1}{(1-\beta)p \ell_2} \leq \frac{1}{2}, \quad \beta + \frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} = 1
\]

and setting

\[
\gamma = 1/(\ell_3 p), \quad r_1 = \beta p \ell_1, \quad r_2 = (1-\beta)p \ell_2,
\] (54)

by Sobolev embedding theorem and Holder’s inequality, we obtain

\[
\|(v \cdot \nabla)v\|_{L_p((0, S), L_q(\Omega))} \leq S^\gamma \|v\|_{L_\infty((0, S), L_q(\Omega))} \|
\begin{aligned}
\|v\|_{L_\infty((0, S), L_q(\Omega))} \|
\n\end{aligned}
\|\nabla v\|_{L_p((0, T), L_q(\Omega))}^\gamma \|
\begin{aligned}
\|\nabla v\|_{L_p((0, T), L_q(\Omega))}^\gamma \|
\n\end{aligned}
\|\nabla \nabla v\|_{L_p((0, T), L_q(\Omega))}^\gamma
\] (55)

for any $v, w \in W^1_p((0, T), L_q(\Omega)) \cap L_p((0, T), W^2_q(\Omega))$ and $0 < S \leq T$. 

Proof of Theorem 2.1. Setting \( u^* = T_\alpha(t)u_\alpha \) and \( \pi^* = \alpha Q_1u_\alpha \), by Theorem 2.9 and (11), \((u^*, \pi^*)\) is the solution to (8) under (9) and satisfies

\[
\|e^{-\lambda_0 t}(\partial_t u^*, \nabla^2 u^*, \nabla \pi^*)\|_{L_p((0, \infty), L_q(\Omega))} \leq C_{n,p,q} \|a_0\|_{B_{p,q}^{2(1-1/p)}(\Omega)} \leq CM, \tag{56}
\]

where \( 1 < p, q < \infty \) and \( \lambda_0 \) is a positive number obtained in Theorem 2.7. Setting \( v_\alpha = u_\alpha - u^* \), and \( \rho_\alpha = \pi_\alpha - \pi^* \), we see that what \((u_\alpha, \pi_\alpha)\) is the solution to (6) under (9) is equivalent to what \((v_\alpha, \rho_\alpha)\) is the solution to

\[
\begin{cases}
\partial_t v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = f - N_1(v_\alpha) - N_2(u^*) & t \in (0, T), x \in \Omega, \\
v_\alpha(0, x) = 0 & x \in \Omega, \\
v_\alpha(t, x) = 0 & t \in (0, T), x \in \partial \Omega \tag{57}
\end{cases}
\]

under the approximated weak incompressible condition (5), where

\[
N_1(v_\alpha, u^*) = (v_\alpha \cdot \nabla)v_\alpha + (u^* \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)u^*, \quad N_2(u^*) = (u^* \cdot \nabla)u^*.
\]

In order to prove Theorem 2.1, we consider (57) under (5). For this purpose, we set

\[
\langle \langle w, \tau \rangle \rangle_T := \|\partial_t w\|_{L_p((0, T), L_q(\Omega))} + \|\nabla^2 w\|_{L_p((0, T), L_q(\Omega))} + \|\nabla \tau\|_{L_p((0, T), L_q(\Omega))} + \|w\|_{L_\infty((0, T), L_q(\Omega))} + \|\nabla w\|_{L_{r_1}(0, T), L_q(\Omega)} + \|\nabla w\|_{L_{r_2}(0, T), L_q(\Omega)} \tag{58}
\]

with \( r_1, r_2 \) is defined by (54). By (7), (50), (51) and (53), we have

\[
\langle \langle M_T^*(f) \rangle \rangle_T \leq C_{n,p,q} e^{\lambda_0 T^*} \|f\|_{L_p((0, T^*), L_q(\Omega))} \leq C_{n,p,q} e^{\lambda_0 T^*} M. \tag{59}
\]

Set \( L = C_{n,p,q} e^{\lambda_0 T^*} M \). To prove Theorem 2.1 by contraction mapping principle, we shall define the underlying space \( X_{T,L} \) as follows:

\[
X_{T,L} = \{ (w, \tau) \in W^1_p((0, T), L_q(\Omega)^n) \cap L_p((0, T), W^2_q(\Omega)^n)) \times L_\infty((0, T), \tilde{W}^1_q(\Omega)) \mid \|w\| = 0, \langle \langle w, \tau \rangle \rangle_T \leq 2L \}. \tag{60}
\]

Here the constant \( T \) is determined later as the sufficiently small constant. We define the map \( \Phi \) as

\[
\Phi(w, \theta) = M_T(f) - M_T(N_1(v_\alpha, u^*)) - M_T(N_2(u^*)),
\]

where \( M_T \) is the solution operator to (49) under (5). We shall prove that \( \Phi \) is the contraction mapping on \( X_{T,L} \). By (55) and (56) we have

\[
\|N_2(u^*)\|_{L_p((0, S), L_q(\Omega))} \leq \|(u^* \cdot \nabla)u^*\|_{L_p((0, S), L_q(\Omega))} \leq CS_\gamma e^{2\lambda_0 S} M^2
\]

for \( 1 < p \leq \infty \) and \( n/2 < q < \infty \). By (50) the following inequality holds:

\[
\langle \langle M_T(N_2(u^*)) \rangle \rangle_T \leq C_{n,p,q} e^{2\lambda_0 T^*} \|N_2(u^*)\|_{L_p((0, T^*), L_q(\Omega))} \leq C_{n,p,q} e^{2\lambda_0 T^*} M^2 \tag{61}
\]

for \( 0 < T^* \leq T_0 \). In a similar way, for \((v_\alpha, \rho_\alpha) \in X_{T^*,L} \) we obtain

\[
\|N_1(v_\alpha, u^*)\|_{L_p((0, S), L_q(\Omega))} \leq Ce^{\lambda_0 T^*} S_\gamma ML,
\]

which implies

\[
\langle \langle M_T(N_1(v_\alpha, u^*)) \rangle \rangle_T \leq C_{n,p,q} \|N_1(v_\alpha, u^*)\|_{L_p((0, T^*), L_q(\Omega))} \leq C(T^*) e^{\lambda_0 T^*} ML. \tag{62}
\]

Therefore there exists a constant \( C = C_{n,p,q, T_0} \) such that

\[
\langle \langle \Phi(w, \theta) \rangle \rangle_T \leq L + C(T^*) e^{\lambda_0 T^*} (e^{2\lambda_0 T^*} M^2 + e^{\lambda_0 T^*} ML)
\]
for \((v_\alpha, \rho_\alpha) \in X_{T^*}\). Taking the time \(T^*(\leq T_0)\) sufficiently small such that
\[ C(T^*) e^{\lambda_0 T^*} M \leq 1/2 \] and \(C(T^*) e^{2\lambda_0 T^*} M^2 \leq L/2\), we have \(\langle \Phi(w, \tau) \rangle_{T^*} \leq 2L\).
Therefore, \(\Phi\) is the mapping on \(X_{T^*,L}\). Moreover taking into account the facts:
\[ \Phi(w_1, \tau_1) - \Phi(w_2, \tau_2) = M_{T^*}(N_1(w_2, u^*) - N_1(w_1, u^*)) \]
and
\[ N_1(w_2, u^*) - N_1(w_1, u^*) = ((w_2 - w_1) \cdot \nabla) u^* + (u^* \cdot \nabla)(w_2 - w_1) \]
for \((w_i, \tau_i) \in X_{T^*,L} (i = 1, 2)\), by (55), (56) and (60), we can show the following inequality holds:
\[ \|N_1(w_2) - N_1(w_1)\|_{L_p((0,T^*), L_q)} \leq C_{n,p,q,T_0}(T^*) e^{\lambda_0 T^*} M \langle ((w_2, \tau_2) - (w_1, \tau_1))_{T^*} \rangle, \]
which implies
\[ \langle \Phi(w_1, \tau_1) - \Phi(w_2, \tau_2) \rangle_{T^*} \leq C_{n,p,q,T_0}(T^*) e^{\lambda_0 T^*} M \langle ((w_2, \tau_2) - (w_1, \tau_1))_{T^*} \rangle. \]
Taking \(T^*\) sufficiently small such that \(C(T^*) e^{\lambda_0 T^*} M \leq 1/2\) if necessary, we obtain
\[ \langle \Phi(w_1, \tau_1) - \Phi(w_2, \tau_2) \rangle_{T^*} \leq (1/2) \langle ((w_1, \tau_1) - (w_2, \tau_2))_{T^*} \rangle. \]
Therefore, we see that \(\Phi\) is the contraction mapping on \(X_{T^*}\). By the contraction mapping principle, we see that \(\Phi\) has fixed point \((v_\alpha, \rho_\alpha)\). Satisfying \(\Phi(v_\alpha, \rho_\alpha) = (v_\alpha, \rho_\alpha)\), by (61), we see that \((u_\alpha, \pi_\alpha) = (u^* + v_\alpha, \pi^* + \rho_\alpha)\) is the unique solution for (6) under (5). Therefore we obtain Theorem 2.1. \(\Box\)

**Proof of Theorem 2.12.** Let \((u^*, \pi^*)\) be a solution to (8) with \(f = g = 0\) and \(a_\alpha = a_E\). By Theorem 2.9, the following estimates hold.
\[ \| e^{-\lambda_0 t} (\partial_t u^*, \nabla^2 u^* \nabla \pi^*) \|_{L_p((0,\infty), L_q(\Omega))} \leq C_{n,p,q} a_E \|a_E\|_{B^{2(1-1/p)}_p(\Omega)} \leq C M a^{-1}, \]
where \(1 < p, q < \infty\). In order to look for the solution \((v_\alpha, \rho_\alpha)\) of (12) as \(v_\alpha = u_E - u^*\) and \(\rho_\alpha = \pi_E - \pi^*\), we shall obtain the solution to
\[ \begin{cases}
\partial_t v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = -N_1(v_\alpha, u^*) - N_2(u^*, u_\alpha) & t \in (0, \infty), x \in \Omega, \\
v_\alpha(0, x) = 0 & x \in \Omega, \\
v_\alpha(t, x) = 0, & x \in \partial \Omega, \\
\end{cases} \]
under the approximated weak incompressible condition (13), where
\[ N_1(v_\alpha, u^*) = (v_\alpha \cdot \nabla)v_\alpha + ((u^* + u_\alpha) \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)(u^* + u_\alpha), \]
\[ N_2(u^*, u_\alpha) = (u^* \cdot \nabla)(u^* + u_\alpha) + (u_\alpha \cdot \nabla)u^*. \]
In a similar way to Theorem 2.1, we shall define underlying space \(X_{T,L_E}\) as follows:
\[ X_{T,L_E} = \{ (w, \tau) \in (W^1_p((0,T), L_q(\Omega))^n) \cap L_p((0,T), W^2_q(\Omega))^n) \times L_p((0,T), \tilde{W}^1_q(\Omega)) | \| w \|_{t=0} = 0, \alpha((w, \tau))_T \leq L_E \}, \]
where \(\langle (w, \tau) \rangle_T\) is defined in (58). Setting the map \(\Phi\) defined by
\[ \Phi(w, \theta) = -M_{T^*}(N_1(v_\alpha, u^*)) - M_{T^*}(N_2(u^*, u_\alpha)), \]
where \(M_T(f)\) is a solution operator to (49) under (13), we shall estimate \(N_1(v_\alpha, u^*)\) and \(N_2(u^*, u_\alpha)\) in a similar way to Theorem 2.1. Setting \(\beta, \xi_k (k = 1, 2, 3), \gamma, r_i (i = 1, 2)\) as the same positive constant in proof of Theorem 2.1, we see
\[ \| N_1(v_\alpha, u^*) \|_{L_p((0,S), L_q(\Omega))} \leq \frac{CS^\gamma}{\alpha} \left( \frac{1}{\alpha} L^2_E + \frac{1}{\alpha} e^{\lambda_0 T^*} M L_E + LL_E \right) \]
and
\[ \|N_2(u^*, u_\alpha)\|_{L_p((0,T),L_q(\Omega))} \leq CS^{\gamma} \left( \frac{1}{\alpha} e^{2\lambda_0 T^*} M^2 + e^{\lambda_0 T^*} M L \right) \]
for \( 1 < p < \infty \), by (14), (50) for \( 0 < T^* \leq T^* \), the following inequality holds:
\[ \alpha \langle M_{T^*}(N_1(v_\alpha, u^*) + N_2(u^*, u_\alpha)) \rangle \leq C_{n,p,q,M,L,L_E}(T^*)^{\gamma}. \]
In a similar way to Theorem 2.1, taking \( T^* \) sufficiently small if necessary, we can prove that \( \Phi \) is the contraction mapping on \( X_{T^*,L_E} \). Therefore we obtain Theorem 2.12.

Acknowledgement. The authors are grateful to the referee who made many comments which were crucial in improving the presentation of this paper, both mathematically and stylistically. This work was supported by the Grant-in-Aid for Scientific Research (C)[grant number 15K04946].

REFERENCES

[1] F. Brezzi and J. Pitkäranta, On the stabilization of finite element approximations of the Stokes equations, in W. Hackbush, editor, Efficient Solutions of Elliptic Systems, Note on Numerical Fluid Mechanics, Braunschweig, 10 1984.
[2] A. P. Calderon, Lebesgue spaces of differentiable functions and distributions, Proc. Symp. in Pure Math, 4 (1961), 33–49.
[3] R. Denk, M. Hieber and J. Prüss, \( \mathcal{R} \)-boundedness Fourier multipliers and problems of elliptic and parabolic type, Memories of the American Mathematical Society, 788 (2003).
[4] Y. Enomoto and Y. Shibata, On the \( \mathcal{R} \)-sectoriality and the initial boundary value problem for the viscous compressible fluid flow, Funkcialaj Ekvacioj, (2013), 441–505.
[5] Y. Enomoto, L. v. Below and Y. Shibata, On some free boundary problem for a compressible barotropic viscous fluid flow, Ann Univ. Ferrara, 60 (2014), 55–89.
[6] G. P. Galdi, An Introduction to The Mathematical Theory of The Navier-Stokes Equations, Vol.I: Linear Steady Problems, Vol.II: Nonlinear Steady Problems, Springer Tracts in Natural Philosophy, Springer Verlag New York, 38, 39 (1994), 2nd edition (1998).
[7] S. A. Nazarov and M. Specovius-Neugebauer, Optimal results for the Brezzi-Pitkäranta approximation of viscous flow problems, Differential and Integral Equations, 17 (2004), 1359–1394.
[8] A. Prohl, Projection and Quasi-Compressibility Methods for Solving The Incompressible Navier-Stokes Equations, Advances in Numerical Mathematics, 1997.
[9] Y. Shibata, Generalized resolvent estimates of the Stokes equations with first order boundary condition in a general domain, Journal of Mathematical Fluid Mechanics, (2013), 1–40.
[10] Y. Shibata and T. Kubo, (Japanese) [Nonlinear partial differential equations] Asakura Shoten, 2012.
[11] Y. Shibata and S. Shimizu, On the maximal \( L_p−L_q \) regularity of the Stokes problem with first order boundary condition: model problems, The Mathematical Society of Japan, 64 (2012), 561–626.
[12] L. Weis, Operator-valued Fourier multiplier theorems and maximal \( L_p \)-regularity, Math. Ann., 319 (2001), 735–758.

Received May 2017; revised January 2018.

E-mail address: tkubo@math.tsukuba.ac.jp
E-mail address: rmatsui@math.tsukuba.ac.jp