Quasi-boundedness of irresolute paratopological groups

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Abstract: Continuing the study of irresolute paratopological groups, our focus in this paper is to define and study the boundedness for irresolute paratopological groups. Premeager property for irresolute paratopological groups is discussed. It is proved that every open subgroup of a quasi-bounded, premeager irresolute paratopological group is premeager. For bounded homomorphisms on an irresolute paratopological group, new notions $nb_q$-quasi bounded and $b_q$-$b_q$-quasi bounded homomorphisms are introduced and discussed.

Subjects: Advanced Mathematics; Foundations & Theorems; History & Philosophy of Mathematics

Keywords: Irresolute paratopological group; quasi-bounded Irresolute paratopological group; $\omega$-quasi-bounded Irresolute paratopological group; Premeager; $nb_q$-quasi bounded; $b_q$-$b_q$-quasi bounded

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1. Introduction

The study of paratopologized groups is not only fascinating but of fundamental importance as well. The notion has been studied extensively by celebrated mathematicians like: Arhangel’skii, Tkachenkov, Banakh, Liu and Ravsky (Arhangel’skii & Reznichenko, 2005; Arhangel’skii & Tkachenko, 2015). In this paper, we shall introduce the relevant concepts of boundedness for irresolute paratopological groups to be able to discuss and develop basic theory.

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PUBLIC INTEREST STATEMENT

Irresolute topological groups were defined and studied by Khan et al. in 2015. Further, in 2016 they introduced four classes of groups namely s-(S-, irresolute-, Irr-) paratopological groups. Each paratopologized group is defined in such a way that the topology is endowed upon a group such that the group operation satisfies certain condition which is either weaker or stronger than continuity. Azar defined the bounded topological group and established some relationships with other topological properties of the space. He proved that if a topological group $G$ is metrizable then $E_G$ is bounded with respect to topology if and only if it is bounded with respect to metric. Lin et al. in 2011 introduced pseudobounded and $\omega$-pseudobounded paratopological groups and showed that when a paratopological group becomes a topological group. In present paper, we shall introduce the relevant concepts of boundedness for irresolute paratopological groups to be able to discuss and develop basic theory.
A paratopological group \( G \) is a group \( G \) with a topology such that the product mapping of \( G \times G \) into \( G \) is jointly continuous and a topological group \( G \) is a paratopological group such that the inverse mapping of \( G \) onto itself associating \( x^{-1} \) with arbitrary \( x \in G \) is continuous. It is well known that paratopological group is a good generalization of topological group.

Irresolute topological groups were defined and studied by Khan, Siab, and Ljubiša (2015). Further, Khan and Noreen (2016) introduced four classes of paratopologized groups which are s- (S-, irresolute-, Irr-) paratopological groups. Each paratopologized group is defined in such a way that the topology \( \tau \) is endowed upon a group \((G, \cdot)\) such that the group operation satisfies certain condition which is either weaker or stronger than continuity. Semi-topological groups with respect to irresolute were defined and studied by Oner et al. independently Oner, burc Kendemir, and Tanay (2013).

Azar (2008) defined the bounded topological group \( G \) and established some relationships with other topological properties of \( G \). He proved that if a topological group \( G \) is metrizable then \( E \subseteq G \) is bounded with respect to topology if and only if it is bounded with respect to metric. Lin & Lin (2011) introduced pseudobounded and \( \omega \)-pseudobounded paratopological groups and showed that when a paratopological group becomes a topological group. In present paper, we shall introduce the relevant concepts of boundedness for irresolute paratopological groups in order to be able to discuss and develop basic theory.

Bounded homomorphisms and their algebraic and topological algebraic structure are of interest for their own right and also for their applications in other area of mathematics. Therefore, it will be of interest to consider different types of bounded homomorphisms on irresolute paratopological groups. So, in this paper, we shall define the new notions \( nb_{q}\)-quasi bounded and \( b_{q}b_{q}\)-quasi bounded of bounded homomorphisms on an irresolute paratopological groups.

2. Preliminaries

In 1963, Levine defined semi-open sets in topological spaces. Since then many mathematicians explored different concepts and generalized them using these sets (see Anderson & Jensen, 1967; Crossley & Hildebrand, 1971a; Nour, 1998; Piatrowski, 1979). A subset \( A \) of a topological space \( X \) is said to be semi-open, if there exists an open set \( U \) in \( X \) such that \( U \subset A \subset \text{Cl}(U) \), or equivalently if \( A \subset \text{Cl}(\text{Int}(A)) \). \( \text{SO}(X) \) denotes the collection of all semi-open sets in \( X \) and \( \text{SO}(X,x) \) is the collection of all semi-open sets containing \( x \). \( \text{si}(A) \) represents the semi-interior of \( A \), which is union of all semi-open sets contained in \( A \). The complement of a semi-open set is said to be semi-closed; the semi-closure of \( A \subset X \), denoted by \( \text{sCl}(A) \), is the intersection of all semi-closed subsets of \( X \) containing \( A \) (Crossley & Hildebrand, 1971a, 1971b). Let us mention that \( x \in \text{sCl}(A) \) if and only if for any semi-open set \( U \) containing \( x \), \( U \cap A \neq \emptyset \).

Clearly, every open (closed) set is semi-open (semi-closed). It is known that the union of any collection of semi-open sets is again a semi-open set, while the intersection of two semi-open sets need not be semi-open. The intersection of an open set and a semi-open set is semi-open. If \( A \subset X \) and \( B \subset Y \) are semi-open in spaces \( X \) and \( Y \), then \( A \times B \) is semi-open in the product space \( X \times Y \). Basic properties of semi-open sets are given in Levine (1963), and of semi-closed sets and the semi-closure in (Crossley & Hildebrand, 1971a, 1971b). A set \( U \subset X \) is a semi-neighbourhood of a point \( x \in X \) if there exists \( A \subset \text{SO}(X) \) such that \( x \in A \subset U \). A set \( A \subset X \) is semi-open in \( X \) if and only if \( A \) is a semi-neighbourhood of each of its points. If a semi-neighbourhood \( U \) of a point \( x \) is a semi-open set, we say that \( U \) is a semi-open neighbourhood of \( x \). A topological space \((X, \tau)\), is said to be: semi-compact (Carnahan, 1973; Gupta & Noiri, 2006; Sarak, 2009), if every semi-open cover of \( X \) has a finite subcover, locally semi-compact if and only if every point \( x \) has a semi-open neighbourhood \( U \) whose semi-closure is semi-compact or it is semi-compact in the small semi-open neighbourhoods. A subset \( D \) of a topological space \((G, \tau)\) is said to be dense if \( \text{sCl}(D) = X \) (Modak, 2011). In topological space \((X, \tau)\), a set which cannot be expressed as the union of two semi-separated sets is said to be a semi-connected set. The topological space \((X, \tau)\) is said to be semi-connected if and only if \( X \) is semi-connected.
Definition 2.1  Levine (1963) Let $X$ and $Y$ be topological spaces. A mapping $f: X \to Y$ is semi-continuous if for each open set $V$ in $Y$, $f^{-1}(V) \in SO(X)$.

Clearly, continuity implies semi-continuity; the converse need not be true. Notice that a mapping $f: X \to Y$ is semi-continuous if and only if for each $x \in X$ and each neighbourhood $V$ of $f(x)$, there is a semi-open neighbourhood $U$ of $x$ with $f(U) \subseteq V$. Let $X = Y = [0, 1]$. Let $f: X \to Y$ be defined as follows: $f(x) = 1$ if $0 \leq x \leq \frac{1}{2}$ and $f(x) = 0$, if $\frac{1}{2} < x \leq 1$. Then $f$ is semi-continuous but not continuous.

In Kempisty (1932), Kempisty defined quasi-continuous mappings: a mapping $f: X \to Y$ is said to be quasi-continuous at a point $x \in X$, if for each open neighbourhood $U$ of $x$ and each open neighbourhood $W$ of $f(x)$, there is a non-empty open set $V \subseteq U$ such that $f(V) \subseteq W$; $f$ is quasi-continuous, if it is quasi-continuous at each point (see also Marcus, 1961). Neubrunnová (1973) proved that semi-continuity and quasi-continuity coincide.

Definition 2.2  A mapping $f: X \to Y$ between topological spaces $X$ and $Y$ is called:

1. semi-open Biswas (1969), if for every open set $A$ of $X$, the set $f(A)$ is semi-open in $Y$;
2. quasi-open Lin and Lin (2011) if we have $Int(f(U)) \neq \emptyset$ for each non-empty open subset $U$ of $X$.
3. irresolute Crossley and Hildebrand (1971a), if for every semi-open set $B$ in $Y$, the set $f^{-1}(B)$ is semi-open in $X$. Equivalently $f$ is irresolute if and only if for every $x \in X$ and every semi-open set $V \subseteq Y$ containing $f(x)$, there exists a semi-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$. It is known that $f$ is irresolute if and only if, for all $B \subseteq Y$, $sCl(f^{-1}(B)) \subseteq f^{-1}(sCl(B))$.

Definition 2.3  Khan et al. (2015) A triple $(G, +, \tau)$ is an irresolute paratopological group (resp. irresolute paratopological group Khan and Noreen (2016)) with a group $(G, +)$ and a topology $\tau$ such that for each $x, y \in G$ and for each semi-open neighbourhood $W$ of $x + y$ (resp. $x + y$), there are semi-open neighbourhoods $U$ of $x$ and $V$ of $y$ such that $U + V^{-1} \subseteq W$ (resp. $U + V \subseteq W$). The topological groups and irresolute topological groups are independent. (If $(G, +, \tau)$ is an irresolute paratopological group, then the multiplication mapping $m: G \times G \to G$ and the inverse mapping $i: G \to G$ are irresolute.)

Definition 2.4  Let $G$ be a paratopological group and $E \subseteq G$. We say that $E$ is an pseudobounded Azar (2008) (resp. $\omega$-pseudobounded Lin & Lin, 2011) subset of $G$, if for every open neighbourhood $U$ of the identity element $e$ of $G$, there exists a natural number $n$ such that $E \subseteq U^n$ (resp. $E \subseteq_{\omega p} U$). If $G$ is an pseudobounded (resp. $\omega$-pseudobounded) subset of $G$, then we say that $G$ is pseudobounded (resp. $\omega$-pseudobounded).

Definition 2.5  A set $U$ in a topological space is called nowhere dense, if $Int(Cl(U)) = \emptyset$.

Definition 2.6  Lin and Lin (2011) Let $G$ be a paratopological group. $G$ is called premeager if, for any nowhere dense subset $E$ of $G$, we have $E^n \neq G$ for each $n \in \mathbb{N}$.

Lemma 2.1

1. Khan, Noreen, and Bosan (2016) let $(G, +, \tau)$ be an extremally disconnected irresolute topological group, $H$ is its invariant subgroup. Then $(G/H, +, \tau_\alpha)$ is an irresolute topological group. $(s\tau_\alpha)$ is semi-quotient topology. For detail see Khan et al., 2016.
2. Pipitonee and Russo (1975) let $X$ be a topological space, $X_\alpha$ a subspace of $X$. If $A \subseteq SO(X_\alpha)$, then $A = B \cap X_\alpha$ for some $B \in SO(X)$.
3. Noiri and Ahmad (1982) if $X_\alpha \in P_\omega(X_\beta)$, then $B \cap X_\alpha \subseteq SO(X_\beta)$ for all $B \in SO(X_\beta)$
4. Mehshewari and Prasad (1975) in a topological space $(X, \tau)$ the following conditions are equivalent
(a) $X$ is s-regular.

(b) For every point $x \in X$ and every open set $U$ containing $x$, there is a semi-open set $V$ such that $x \in V \subset sCl(V) \subset U$

(5) Khan and Noreen (2016) every open subgroup $H$ of an

(a) $s$-paratopological group is also an $s$-paratopological group (called $s$-paratopological subgroup of $G$).

(b) irresolute paratopological group $(G, +, *)$ is also an irresolute paratopological group (called irresolute paratopological subgroup of $G$).

(6) Khan and Bosan (2014) let $(G, +, *)$ be an irresolute topological group. Then for every subset $A$ of $(G, +, *)$ and every open neighbourhood $U$ of the neutral element $e$, $sCl(A) \subset A + U$.

(7) Crossley and Hildebrand (1972) $sInt(sCl(A)) = \emptyset$ if and only if $A$ is nowhere dense.

3. Quasi-bounded and $\omega$-Quasi-bounded irresolute paratopological Groups

In this section, we will define quasi-bounded and $\omega$-quasi-bounded irresolute paratopological groups by following Lin and Lin (2011). Note that “bounded” in Azar (2008) was called “pseudobounded” in Lin and Lin (2011) since bounded has another meaning as well in topological algebra.

**Theorem 3.1** If $U \in SO(G)$, then the set $L = \bigcup_{n=1}^{\infty} U^n$ is a semi-open set in an irresolute paratopological group $(G, +, *)$.

**Proof** Let $U$ be a semi-open set in an irresolute paratopological group $(G, +, *)$. Then by (Theorem 3.19 (5), Khan & Noreen, 2016), $U \cdot U = U^2 \in SO(G)$. $U^2 \cdot U = U^3 \in SO(G)$. Similarly each $U^n$, $U^2$, ... is semi-open set in $G$. Thus the set $L = \bigcup_{n=1}^{\infty} U^n$ being the union of semi-open sets is a semi-open set. \hfill \square

**Theorem 3.2** Suppose that $G$ is an irresolute paratopological group, and $U$ any semi-open neighbourhood of the neutral element $e$ in $G$. Then $sCl(M) \subset MU^{-1}$, for each subset $M$ of $G$.

**Proof** Put $F = G \setminus (gUg \in G, gU \cap M = \emptyset)$. Then, clearly, $F$ is a semi-closed subset of $G$ and $M \subset F$. Take any $y \in F$. Then $y \not\in U$ or $y \not\in M$, that is, $y = mh^{-1}$, for some $h \in U$ and $m \in M$. Hence, $y = mh^{-1} \in MU^{-1}$. Thus, $F \subset MU^{-1}$. Since $M \subset F$, it follows that $sCl(M) \subset MU^{-1}$. \hfill \square

**Theorem 3.3** Suppose that $G$ is an irresolute paratopological group and not an irresolute topological group. Then there exists a semi-open neighbourhood $U$ of the neutral element $e$ of $G$ such that $U \cap U^{-1}$ is semi- nowhere dense in $G$, that is, the semi-interior of the semi-closure of $U \cap U^{-1}$ is empty.

**Proof** By Theorem 3.19 (2) Khan and Noreen (2016), the multiplication mapping $m : G \times G \rightarrow G$ is an irresolute mapping, whereas the inverse operation in $G$ is not irresolute. Therefore, by Theorem 3.25 Khan and Noreen (2016), the inverse mapping is not irresolute at $e$, and we can choose a semi-open neighbourhood $W$ of $e$ such that $e \not\in sint(W^{-1})$. Since the multiplication in $G$ is irresolute, we can find a semi-open neighbourhood $U$ of $e$ such that $U \subset W$. Assume the contrary. Then there exists a nonempty semi-open set $V$ such that $V \subset sCl(U \cup U^{-1})$. By Theorem 3.2, it follows that $V \subset sCl(U \cup U^{-1}) \subset (U \cup U^{-1}) \subset U^{-1}$. Then $VU^{-1} \subset U \subset W^{-1}$. Clearly, $V \cap U \not\subset \emptyset$, and the set $VU^{-1}$ is semi-open in $G$ by Theorem 3.19 (5) Khan and Noreen (2016). Therefore, $e \in VU^{-1} \subset sInt(W^{-1})$, a contradiction. \hfill \square

**Definition 3.1** Let $G$ be an irresolute paratopological group and $E \subset G$. We say that $E$ is quasi-bounded subset of $G$, if for every semi-open neighbourhood $U$ of the identity element $e$ of $G$, there is a natural number $n$, such that $E \subset U^n$. If $G$ is itself quasi-bounded, then we say that $G$ is quasi-bounded.

**Definition 3.2** Let $G$ be an irresolute paratopological group and $E \subset G$. We say that $E$ is a $\omega$- quasi-
bounded subset of $G$, if for every semi-open neighbourhood $U$ of the identity element $e$ of $G$, we have $E \subseteq \bigcup_{n \in \mathbb{N}} U^n$. If $G$ is an $\omega$-quasi-bounded subset of $G$, then we say that $G$ is $\omega$-quasi-bounded.

**Theorem 3.4** If an irresolute paratopological group $(G, \tau)$ contains a quasi-bounded ( $\omega$-quasi-bounded) dense subgroup, then $(G, \tau)$ is quasi-bounded ( $\omega$-quasi-bounded).

**Proof** Let $H$ be a quasi-bounded dense subgroup of an irresolute paratopological $(G, \tau)$. Take a semi-open neighbourhood $U$ of the identity $e$ in $(G, \tau)$. Since $H$ is a quasi-bounded subset of $(G, \tau)$, there exists $n \in \mathbb{N}$ such that $H \subseteq U^n$, equivalently, $H \subseteq U^{n-1}$. Hence, using Theorem 3.2, $G = sCl(H) \subseteq HU^{-1} \subseteq U^{n-1}$, hence, $G = U^{n+1}$. Using a similar argument, we can prove that if $H$ is a quasi-bounded dense subgroup of an irresolute paratopological group $G$, then $(G, \tau)$ is $\omega$-quasi-bounded.

**Theorem 3.5** Suppose that $H$ is a discrete invariant subgroup of an $\omega$-quasi-bounded irresolute topological group $G$. Then each element of $H$ commutes with each element of $G$, that is, $H$ is contained in the centre of the group $G$.

**Proof** If $H = \{ e \}$, there is nothing to prove. Assume that $H$ is a non-trivial subgroup of $G$. Choose an arbitrary point $x \in H \setminus \{ e \}$. Since $H$ is discrete, there exists an open neighbourhood $U$ of $x$ in $G$ such that $U \cap H = \{ x \}$. It follows from the definition, there exists $V \in SO(G, e)$, such that $V \cdot x \cdot V^{-1} \subseteq U$.

Claim: For each $y \in V$, we have $x \cdot y = y \cdot x$. Indeed, for each $y \in V$, since $H$ is an invariant subgroup, we have $y \cdot x \cdot y^{-1} \in H$. Moreover, we have $y \cdot x \cdot y^{-1} \in V \cdot x \cdot V^{-1} \subseteq U$. Thus $y \cdot x \cdot y^{-1} \in H \cap U = \{ x \}$, that is, $x \cdot y = y \cdot x \cdot y^{-1} = x$. Since $G$ is $\omega$-quasi-bounded irresolute topological group, we have $G = \bigcup_{n=1}^{\infty} V^n$. For each $g \in G$, there exists an $n \in \mathbb{N}$ such that $g \in V^n$, that is, the element $g$ can be written in the form $g = y_1 \cdots y_n$, where $y_1, \ldots, y_n \in V$. Since $x$ commutes with each element of $V$ by claim, we have $y_1 \cdots y_n \cdot x = y_1 \cdots y_n \cdot x = y_1 \cdots y_n \cdot y_1 \cdots y_n = x \cdot y_1 \cdots y_n = x \cdot g$. Therefore, the element $x \in H$ is in the centre of the group $G$. Because $x$ is an arbitrary element in $H$, we conclude that the centre of $G$ contains $H$.

**Theorem 3.6** Let $G$ and $H$ be irresolute paratopological groups and suppose that $f : G \rightarrow H$ is group isomorphism. If $f$ is irresolute and $E \subseteq G$ is quasi-bounded subset of $G$, then $f(E)$ is quasi-bounded subset of $H$.

**Proof** Let $V$ be a semi-open neighbourhood of $e \in H$. Then $f^{-1}(V)$ is a semi-open neighbourhood of $e$. Since $E$ is quasi-bounded subset of $G$, there is a natural number $n$, such that $E \subseteq f^{-1}(V)^n \subseteq f^{-1}(V)$, because $f$ is group isomorphism. This implies that $f(E) \subseteq V^n$. Thus $f(E)$ is quasi-bounded subset of $H$.

**Theorem 3.7** Let $f : G \rightarrow H$ be an irresolute homomorphism from an irresolute topological group $G$ onto the irresolute topological group $H$. If $G$ is quasi-bounded ( $\omega$-quasi-bounded), then $H$ is quasi-bounded ( $\omega$-quasi-bounded).

**Proof** Suppose that $(G, \tau)$ is $\omega$-quasi-bounded. Take a semi-open neighbourhood $U$ of the identity in $H$. Put $U = f^{-1}(V)$. Since $f$ is an irresolute homomorphism, $U$ is a semi-open neighbourhood of the identity in $G$. By hypothesis, $G \subseteq \bigcup_{n \in \mathbb{N}} U^n$. We conclude, $H = \bigcup_{n \in \mathbb{N}} f(U^n) = \bigcup_{n \in \mathbb{N}} (f(U))^n = \bigcup_{n \in \mathbb{N}} V^n$, so $H$ is $\omega$-quasi-bounded. The proof of the quasi-bounded case is similar.

**Theorem 3.8** Let $G$ be an extremally disconnected irresolute topological group and let $H$ be a clopen invariant subgroup of $G$. If $H$ and $G / H$ are $\omega$-quasi-bounded, then $G$ is $\omega$-quasi-bounded.

**Proof** Let $U$ be a semi-open neighbourhood of $e$ in $G$. By Lemma 2.1 (3), the set $V = U \cap H$ is a semi-open neighbourhood of $e$ in $H$. Since $G / H$ and $H$ are $\omega$-quasi-bounded, we have $\bigcup_{n \in \mathbb{N}} V^n = H$ and $\bigcup_{n \in \mathbb{N}} (U/H)^n = G/H$. We claim that $\bigcup_{n \in \mathbb{N}} U^n = G$. In fact, let $x \in G$. Case 1: $x \in H$. Then $x \in H \cap V^n \subseteq U^n \subseteq \bigcup_{n \in \mathbb{N}} U^n$. Case 2: $x \notin H$. Then we have $xH \subseteq G/H$,
and hence there exists an \( m \in \mathbb{N} \) such that \( xH = (U/H)^m \). Therefore, there exist points \( x_1, \ldots, x_r \in U \) such that \( xH = x_1 \ldots x_r H \). Hence, there exist an \( h \in H \) and an \( n \in \mathbb{N} \) such that \( xH = U^n \) and \( h \in V^i \).

It follows that \( x \in U^n H = U^n V^i \cup U^n U^i = \bigcup_{n=1}^{\infty} U^n \). Therefore, we have \( \bigcup_{n=1}^{\infty} U^n = G \), that is, \( G \) is \( \omega \)-quasi-bounded.

\[ \square \]

### 4. Premeager irresolute paratopological groups

In this section, we will define and discuss the premeager property for irresolute paratopological groups.

**Definition 4.1** Let \((G, \cdot, \tau)\) be an irresolute paratopological group. \( G \) is called premeager if, for any its semi-nowhere dense subset \( A \) of \( G \), we have \( A^n \neq G \) for each \( n \in \mathbb{N} \).

The Sorgenfrey line \( X (X = \mathbb{R}) \) does not have the premeager property. In particular, the Euclidean line does not have the premeager property.

**Proof** Let \( C \) be the usual Cantor set in \([0, 1] \). It is well known that \( C \) is nowhere dense in \( X \). By [Khurshidzhev (2004), Lemma A1], we have \( C + C = [0, 2] \), where ‘+’ is the usual addition. Let \( A = \bigcup (2n + C) \), where \( \mathbb{Z} \) is the set of integer. Then \( A \) is semi-nowhere dense in \( X \), but \( A + A = X \) since \( C + C = [0, 2] \).

**Definition 4.2** A map \( f: X \to Y \) is called quasi-semi-open if we have \( s\text{Int}(f(U)) \neq \phi \) for each non-empty semi-open subset \( U \) of \( X \).

**Theorem 4.1** Let \( f: G \to H \) be an irresolute quasi-semi-open homomorphism, where \( G, H \) are irresolute paratopological groups. If \( G \) is premeager, then \( H \) is also premeager.

**Proof** Let \( A \) be any semi-nowhere dense subset of \( H \). Suppose that there exists some \( n \in \mathbb{N} \) such that \( A^n = H \). Therefore, \( (f^{-1}(A))^n = f^{-1}(A^n) = f^{-1}(H) = G \). Since \( G \) is premeager, the set \( f^{-1}(A) \) is a non-semi-nowhere dense subset of \( X \). Hence there is a non-empty semi-open subset \( U \) of \( X \) such that \( U \subset s\text{Cl}(f^{-1}(A)) \). It follows from definition (3), \( U \subset s\text{Cl}(f^{-1}(A)) \subset f^{-1}(s\text{Cl}(A)) \) that \( f(U) \subset s\text{Cl}(A) \). Since \( f \) is quasi-open, we have \( \phi \neq s\text{Int}(f(U)) \subset f(U) \subset s\text{Cl}(A) \), which is a contradiction.

\[ \square \]

Since open maps are quasi-open maps and quasi-open maps are quasi-semi-open maps, so, we have a corollary.

**Corollary 4.1** Let \( f: G \to H \) be an open and irresolute homomorphism, where \( G, H \) are irresolute paratopological groups. If \( G \) is premeager, then \( H \) is also premeager.

**Theorem 4.2** Let \((G, \cdot, \tau)\) be a quasi-bounded and premeager irresolute paratopological group. Then every open subgroup of \((G, \cdot, \tau)\) is premeager.

**Proof** Let \( H \) be an open subgroup of \( G \). Suppose that \( H \) is non-premeager. Then there exists a semi-nowhere dense subset \( A \) of \( H \) and an \( n \in \mathbb{N} \) such that \( A^n = H \). Since \( G \) is quasi-bounded, it follows that there is an \( m \in \mathbb{N} \) such that \( H^m = G \). Hence \( (A^n)^m = H^m = G = A^{nm} \). However, the set \( A \) is a semi-nowhere dense subset of \( G \), which is a contradiction.

\[ \square \]

### 5. \( nb_q \)-quasi bounded and \( b_q b_q \)-quasi bounded homomorphism

Bounded homomorphisms and their algebraic and topological algebraic structure are of interest for their own right and also for their applications in other area of mathematics. Therefore, it will be of interest to consider different types of bounded homomorphisms on irresolute paratopological groups. So, in this section, we will define new notions \( nb_q \)-quasi bounded and \( b_q b_q \)-quasi bounded of bounded homomorphisms on an irresolute paratopological group.

**Definition 5.1** Let \( G \) and \( H \) be two irresolute paratopological groups. A homomorphism \( \varphi: G \to H \) is said to be
(1) $\text{nb}_q$-quasi bounded if there exists a semi-open neighbourhood $U$ of $e_H$ such that $\varphi(U)$ is quasi bounded in $H$.

(2) $b_q$-quasi bounded if for every quasi bounded set $B \subset G$, $\varphi(B)$ is quasi bounded in $H$.

The set of all $\text{nb}_q$-quasi bounded ($b_q$-quasi bounded) homomorphisms from an irresolute topological group $G$ to an irresolute topological group $H$ is denoted by $\text{Hom}_{\text{nb}_q}(G,H)$ ($\text{Hom}_{b_q}(G,H)$). We write $\text{Hom}(G)$ instead of $\text{Hom}(G,G)$.

**Theorem 5.1** For irresolute paratopological groups $G$ and $H$ the following holds:

$$\text{Hom}_{\text{nb}_q}(G,H) \subset \text{Hom}_{b_q}(G,H)$$

**Proof** Let $\varphi: G \to H$ be an $\text{nb}_q$-quasi bounded homomorphism. Then it is $b_q$-quasi bounded. For, suppose $B \subset G$ is a quasi bounded set. Since $\varphi$ is $\text{nb}_q$-quasi bounded there is a semi-open neighbourhood $U$ of $e_H$ such that $\varphi(U)$ is quasi bounded in $H$. Quasi boundedness of $B$ implies $B \subset U^n$ for some natural number $n$. We prove that $\varphi(B)$ is quasi bounded in $H$. Let $V$ be a semi-open neighbourhood of $e_H$. Quasi boundedness of $\varphi(U)$ implies that there is $m \in \mathbb{N}$ such that $\varphi(U) \subset V^m$. Then

$$\varphi(B) \subset \varphi(U^n) = (\varphi(U))^n \subset V^mm$$

i.e. $\varphi(B)$ is quasi bounded in $H$. \qed

**Theorem 5.2** Each irresolute homomorphism between two irresolute paratopological groups is $b_q$-quasi bounded.

**Proof** Let $G$ and $H$ be irresolute paratopological groups, $\varphi: G \to H$ be an irresolute homomorphism, and $B$ a quasi bounded subset of $G$. Suppose a semi-open neighbourhood $V$ of $e_H$ is given. There exists a semi-open neighbourhood $U$ of $e_H$ such that $\varphi(U) \subset V$. Also, since $B$ is quasi bounded in $G$, there is an $n \in \mathbb{N}$ with $B \subset U^n$. Thus,

$$\varphi(B) \subset \varphi(U^n) = (\varphi(U))^n \subset V^nm$$

i.e. $\varphi(B)$ is quasi bounded in $H$. \qed

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