On the Mean Curvature Flow for $\sigma_k$-Convex Hypersurfaces

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§1 Introduction

For $n \geq 1$, let $M^n$ be a compact $n$-dimensional manifold without boundary and $F_0 : M^n \to R^{n+1}$ be a smooth immersion of $M^n$ in $R^{n+1}$ as a hypersurface. Recall that $M_0 = F_0(M^n)$ is said to be moved by its mean curvature, if there is a family $F(\cdot, t)$ of smooth immersions of $M^n$ into $R^{n+1}$ with the corresponding hypersurfaces $M_t = F(\cdot, t)(M^n)$ satisfying

$$\frac{\partial F}{\partial t}(p, t) = -H(p, t)\nu(p, t), \quad (p, t) \in M^n \times R_+, \quad (1.1)$$

$$F(p, 0) = F_0(p), \quad p \in M^n \quad (1.2)$$

where $H(p, t)$ and $\nu(p, t)$ are the mean curvature and the outward unitary vector of the hypersurface $M_t$ at $F(p, t)$, respectively. It was proved by Huisken [H1] that there exists a $0 < T = T(M^n) \leq \infty$ such that (1.1)-(1.2) always admits a unique smooth solution for $0 < t < T$ and $\lim_{t \to T} \max_{M_t} |H| = \infty$. When $M_0$ is a convex hypersurface, Huisken [H1] proved that $M_t$ contracts smoothly to a point as $t \to T$.

Without the convexity assumption on $M_0$, Huisken proved in [H2] (Theorem 3.5) that if the singularity is of the Type I, i.e.

$$\max_{M_t} |A|^2 \leq \frac{C}{2(T-t)}, \quad (1.3)$$

then suitable scalings of $M_t$ near the singularity converges smoothly to an immersed nonempty limiting hypersurface $\tilde{M}$, which satisfies the equation

$$H(x) = \langle x, \nu(x) \rangle, \quad (1.4)$$
where \( x \) is the position vector, \( H \) is the mean curvature and \( \nu(x) \) is the outward unit normal vector field. It was also proved by [H2] (Theorem 4.1) that compact manifolds with nonnegative mean curvature satisfying (1.4) are spheres of radius \( \sqrt{n} \). For singularity of Type II, i.e.

\[
\lim_{t \uparrow T} (T - t) \max_{M_t} |A|^2 = \infty. \tag{1.5}
\]

Huisken and Sinestrari proved in [HS1] (Theorem 3.1) and [HS2] (Theorem 1.1) that if \( M_0 \) is mean convex (i.e. \( H \geq 0 \)) then \( \sigma_k \)-curvature (\( 2 \leq k \leq n \)) of \( M_t \) satisfies, for any \( 0 < t < T \),

\[
\sigma_k(M_t(p)) \geq -\epsilon H^k(p, t) - C_{\epsilon, n, k} \tag{1.6}
\]

(see below for the definition of \( \sigma_k \)). Based on this key estimate, they proved in [HS2] (Theorem 4.1) that the so-called essential scaling of \( M_t \) near time \( T \) converges to a smooth mean curvature flow \( \{\tilde{M}_t\}_{t \in \mathbb{R}} \), with \( \tilde{M}_t \) convex hypersurfaces. Moreover, either \( \tilde{M}_t \) is strictly convex translating soliton or (up to rigid motion) \( \tilde{M}_t = R^{n-k} \times \Sigma_t^k \) where \( \Sigma_t^k \) is a lower dimensional strictly convex soliton in \( R^{k+1} \).

For \( n = 2 \), it was shown by [HS1] (Corollary 4.7) that \( \{\Sigma_t^1\}_{t \in \mathbb{R}} \) is the “grim reaper” curve given by \( x = -\ln \cos y + t \).

In contrast with the convex case, it is well known that the mean curvature flow (1.1)-(1.2) can develop singularities before it may shrink to a point. It is a major problem for people to study the nature of its singularity and the asymptotic behavior near the singularity. In this note, we make some effort to try to understand the structure of the singularity set at the first singular time for the initial hypersurface \( M_0 \) belonging to the class consisting of \( \sigma_k \) convex hypersurfaces for some \( 1 \leq k \leq n \). More precisely, we want to understand the limiting set \( M_T \), which is the support of the Radon measure \( \mu_T \) obtained as the limit of Radon measures \( \mu_t \), where \( \mu_t \) is the area measure of \( M_t \) described as below, as \( t \uparrow T \). To better describe our result, we first recall that, in addition to the above classical motion by mean curvatures, Brakke [B] has introduced the motion by its mean curvature for a family of Radon measures \( \{\nu_t\}_{t \in \mathbb{R}} \) in \( R^{n+1} \) (e.g. \( n \)-dimensional
rectifiable varifolds), which satisfies (1.1) in the weak form
\[ \int \phi d\nu_t \leq \int \phi d\nu_s + \int_s^t \int (\phi|H|^2 + H \cdot S^\perp \cdot D\phi) d\nu_t dt, \] (1.7)
for all nonnegative \( \phi \in C_0^1(R^{n+1}) \) and \( 0 \leq s \leq t \). The reader can refer to Ilmanen [I1] for the interpretation of (1.7) and related results.

Note that if \( \{M_t\}_{0 \leq t < T} \) solve (1.1)-(1.2) and we denote \( \mu_t \) as the area measure of \( M_t \) for \( 0 \leq t < T \), then \( \mu_t \) are integral varifolds of multiplicity 1 and satisfy
\[ \mu_t(\phi) - \mu_s(\phi) = \int_s^t \int (-H^2 \phi + D\phi \cdot H) d\mu_t dt, \] (1.8)
for any nonnegative \( \phi \in C_0^1(R^{n+1}) \) and \( 0 \leq s \leq t < T \), where \( \phi_t(\phi) = \int \phi d\mu_t \) and \( H = -H\nu \). In particular, \( \{\mu_t\}_{0 \leq t < T} \) is a motion by mean curvature by Brakke in the sense of (1.7). Moreover, (1.8) implies that there exists a nonnegative measure \( \mu_T \) in \( R^{n+1} \) such that \( \mu_t \to \mu_T \) as convergence of Radon measures in \( R^{n+1} \) as \( t \to T \). Now we extend \( \{\mu_t\}_{0 \leq t \leq T} \) to \( t > T \) such that \( \{\mu_t\}_{0 \leq t \leq \infty} \) are a family of Radon measures moved by its mean curvatures in the sense of (1.7), whose existence is established by Brakke [B]. Our result is concerned with the properties of \( M_T = \text{spt}(\mu_T) \).

One of the most important facts to the family \( \{\mu_t\}_{0 \leq t \leq \infty} \) is the following monotonicity formula, which was first discovered by Huisken [H2] for (1.1) and later obtained by Ilmanen [I2] for Brakke flows (1.7) and has played a key role in the analysis of the singularity for (1.1) and (1.7). The formula says the follows. Let \( \rho(y,s) \) denote the \( n \)-dimensional backward heat kernel centered at \( (y, s) \in R^{n+1} \times R \) defined by
\[ \rho(y,s)(x,t) = (4\pi(s-t))^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right), \quad x \in R^{n+1}, t < s. \]
Let \( \mu = \{(\mu_t, t) : 0 \leq t \leq \infty\} \) and define, for \( 0 < r < s \),
\[ \Theta(\mu, (y, s), r) = \int \rho(y,s)(x, s-r) d\mu_{s-r}(x). \]
Then one has
\[ \int_{s-r_1}^{s-r_2} \int \rho(y,s)(x,t)\|H(x,t)\| + \frac{1}{2(s-t)} S^\perp(x,t) \cdot (x-y)^2 d\mu_t(x) dt \leq \Theta(\mu, (y, s), r_2) - \Theta(\mu, (y, s), r_1). \] (1.9)
for any \((y, s) \in R^{n+1} \times R_+\) and \(0 < r_1 < r_2 < s\), where \(S(x, t) = T_x\mu_t\). A direct consequence of (1.9) is that the density function

\[ \Theta(\mu, (y, s)) \equiv \lim_{r \downarrow 0} \Theta(\mu, (y, s), r) \]

exists for any \((y, s) \in R^{n+1} \times R_+\) and is upper semicontinuous. Using the upper semicontinuity, it is not difficult to prove that \(M_T\) is actually the Hausdorff limit of \(M_t\) as \(t \uparrow T\) (see, Lemma 2.1 below). Our first result is

**Theorem A.** The \(n\)-dimensional Hausdorff measure of \(M_T\) is finite, i.e. \(H^n(M_T) < \infty\).

By exploring the upper semicontinuity of \(\Theta(\mu, \cdot)\) and extending the idea of the Federer’s dimension reduction argument to the parabolic setting, White [W1] (Theorem 9) has recently obtained the stratification theorem for the support \(\mathcal{M} = \{(\text{spt}(\nu_t), t) : t \geq 0\}\) of Brakke flows for \(k\)-dimensional integral varifolds \(\{\nu_t\}_{t \geq 0}\) in \(R^{n+1}\) \((1 \leq k \leq n)\), which roughly says that the points of \(\mathcal{M}\), for which each tangent flow having its spine dimension (see [W1] for its definition) at most \(l\), is of parabolic Hausdorff dimension at most \(l\) for all \(0 \leq l \leq k + 2\). Inspired by this stratification theorem by White [W1], we shall consider the stratification of the extension set \(M_T\) of the \(\{\mu_t\}_{t \geq 0}\) described as above. First, note that the monotonicity of \(\Theta(\mu, (y, s), \cdot)\) actually implies that \(\Theta(\mu, (y, s))\) is upper semicontinuous with both of its arguments (see, e.g. [W2] theorem 2). Moreover, the uniform upper bound of \(\Theta(\mu, (y, s), \cdot)\) in terms of \(M_0\) and \((y, s) \in \text{spt}(\mu)\) implies that for \(x \in M_T\) if we consider the parabolic blow-up, \(P_{(x,T),\lambda}(\mu)\), defined as

\[ P_{(x,T),\lambda}(\mu)(\phi) = \lambda^{-n} \int \phi((x, T) + (\lambda x, \lambda^2 t)) d\mu_t dt, \forall \phi \in C^1_0(R^{n+1} \times R). \]

Then for any \(\lambda \to 0\) we can extract a subsequence \(\lambda_i \to 0\) and a limiting Brakke flow \(\tilde{\mu} \equiv \{\tilde{\mu}_t\}_{t \in R}\), which is called a tangent flow at \((x, T)\), such that \(P_{(x,T),\lambda_i}(\mu) \to \tilde{\mu}\) as convergence of Radon measures in \(R^{n+1} \times R\). Moreover, as shown by [I1] and [W1] that \(\tilde{\mu}\) is backwardly self-similar, i.e., \(P_{(0,0),\lambda}(\tilde{\mu}|_{t \leq 0}) = \tilde{\mu}|_{t \leq 0}\), and

\[ \Theta(\tilde{\mu}, (0, 0)) = \Theta(\mu, (x, T)) \geq \Theta(\tilde{\mu}, z), \forall z = (y, s) \in R^{n+1} \times R. \]
It was also proved by [W1] that the set \( V(\tilde{\mu}) \equiv \{ x \in R^{n+1} : \Theta(\tilde{\mu}, (x,0)) = \Theta(\tilde{\mu}, (0,0)) \} \) is a vector subspace of \( R^{n+1} \). Denote \( \dim(\tilde{\mu}) = \dim(V(\tilde{\mu})) \). Then, White’s stratification theorem yields

**Proposition B** ([W1]). Assume that the Brakke flow \( \{ \mu_t \}_{t \in R_+} \) is given as above (i.e. \( \{ \mu_t \}_{0 \leq t < T} \) coincides with the smooth mean curvature flow \( \{ M_t \}_{0 \leq t < T} \) of (1.1)-(1.2)). Then

\[
M_T = \Sigma_0 \cup (\Sigma_1 \setminus \Sigma_0) \cup \cdots (\Sigma_n \setminus \Sigma_{n-1}), \tag{1.10}
\]

where

\[
\Sigma_i \equiv \{ x \in M_T : \dim(\tilde{\mu}) \leq i, \text{ for any tangent flow } \tilde{\mu} \text{ at } (x,T) \}
\]

for \( 0 \leq i \leq n \). Moreover, \( \dim_H(\Sigma_i) \leq i \), for \( 0 \leq i \leq n \).

Note that the original proof of White [W1], which is in the nature of parabolic type, can be modified to prove this proposition, we would like to give a slightly different proof of it, which is in the Euclidean nature, in §3.

Now we turn our attention to the \( \sigma_k \) convex case of mean curvature flows. First, we recall the definition of \( \sigma_k \) curvatures for hypersurfaces in \( R^{n+1} \) (see also [HS2]).

**Definition 1.1.** For a closed hypersurface \( M \subset R^{n+1} \), let \( -\infty < \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x) < \infty \) be the principal curvatures of \( M \) at \( x \in M \). For \( 1 \leq k \leq n \), the \( \sigma_k \) curvatures of \( M \) at \( x \) is defined by

\[
\sigma^M_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}(x) \cdots \lambda_{i_k}(x).
\]

Note that the \( \sigma^M_1 \) curvature is nothing but the mean curvature of \( M \), \( \sigma^M_2 \) is the scalar curvature of \( M \), and \( \sigma^M_n \) is the Guassian curvature of \( M \).

**Definition 1.2.** For \( 1 \leq k \leq n \). A closed hypersurface \( M \subset R^{n+1} \) is called \( \sigma_k \) convex (respectively, strictly \( \sigma_k \) convex) if \( \min_{x \in M} \sigma^M_k(x) \geq 0 \) (respectively, \( \min_{x \in M} \sigma^M_k(x) > 0 \)). In particular, the mean convexity of \( M \) is equivalent to the \( \sigma_1 \) convexity of \( M \).
For mean curvature flows of mean curvature sets \( \{ F_t(K) \}_{t \geq 0} \) (e.g. \( K_0 = F_0(\partial K) \) is a mean convex smooth hypersurface), a striking and difficult theorem by White [W3] (theorem 1) claimed that the singular set of \( K = \{(x, t) : x \in F_t(\partial K), t \geq 0 \} \), \( \text{sing}(K) \), has parabolic Hausdorff dimension at most \( n-1 \). Here the singular set is defined to the complement of these regular points near where \( K \) is a smooth manifold and has its tangent plane non-horizontal. As a direct consequence of this regularity theorem of White, one knows that the top dimensional subset \( \Sigma_n \setminus \Sigma_{n-1} \subset M_T \) defined as in the proposition B is regular set of \( M_T \), namely near each point in \( \Sigma_n \setminus \Sigma_{n-1} \) \( M_T \) is a smooth \( n \)-dimensional manifold.

By exploring the estimates (1.6) on \( \sigma_k \) curvatures for the mean curvature flow of mean convex hypersurfaces (1.1)-(1.2) obtained by [HS2] and the partial regularity theorem for mean convex flows of [W3] (i.e. the singular set has parabolic Hausdorff dimension at most \( n-1 \)), we obtain the following result.

**Theorem C.** Assume that the Brakke flow \( \{ u_t \}_{t \in R^+} \) is given as same as that in Proposition B. For any \( 2 \leq k \leq n-1 \), if the initial closed hypersurface \( M_0 \) is \( \sigma_k \) convex. Then

\[
\Sigma_{n-1} \setminus \Sigma_{n-2} = \cdots = \Sigma_{n-k+1} \setminus \Sigma_{n-k} = \emptyset. \tag{1.11}
\]

This note is written as follows. In §2, we prove Theorem A; In §3, we give a proof of Proposition B; In §4, we prove Theorem C.

§2 Proof of Theorem A

This section is devoted to the proof of theorem A. First, we show

**Lemma 2.1.** \( M_t \) converges to \( M_T \) in the Hausdorff distance sense, as \( t \uparrow T \).

**Proof.** For any \( t \uparrow T \), we can extract a subsequence \( t_i \uparrow T \) and a closed subset \( A \subset R^{n+1} \) such that \( M_{t_i} \) converges to \( A \) in the Hausdorff distance. Now we want to show \( A = M_T \). Suppose \( x_0 \not\in A \). Then there exists \( r_0 > 0 \) such that \( B_{r_0}(x_0) \cap M_{t_i} = \emptyset \) for \( i \) sufficiently large. In particular, \( H^n(M_{t_i} \cap B_{r_0}(x_0)) = 0 \). Hence, \( \mu_T(B_{r_0}(x_0)) = 0 \) and \( x_0 \not\in M_T \). This gives that \( M_T \subset A \). To prove
A ⊂ M_T, we argue by contradiction. Suppose that there exists x_0 ∈ A \ M_T. Then there exists r_0 > 0 such that \( \mu_T(B_{r_0}(x_0)) = 0 \). On the other hand, there exist x_i ∈ M_{t_i} such that x_i → x_0. Therefore, by the upper semicontinuity of \( \Theta(\mu, \cdot) \) and the fact that \( \Theta(\mu, (x_i, t_i)) = 1 \) (since \( x_i \in M_{t_i} \) and \( M_{t_i} \) is smooth), we have

\[
\Theta(\mu, (x_0, T)) \geq \limsup_{i \to \infty} \Theta(\mu, (x_i, t_i)) = 1.
\]

This implies that \( x_0 \in M_T \) (for otherwise \( \Theta(\mu, (x_0, T)) = 0 \)). We get the desired contradiction.

\textbf{Proof of Theorem A.} Since for any \( 0 \leq t < T \) and \( y \in M_t \), one has \( (\Theta, (y, t)) = 1 \). The upper semicontinuity implies that \( \Theta(\mu, (x_0, T)) \geq 1 \) for any \( x_0 \in M_T \). For any \( \epsilon > 0 \), the estimate of Cheng [C] implies that there exists sufficiently large \( K_\epsilon > 0 \) such that

\[
G_{(x_0, T)}(x, T - r^2) \leq r^{-n}, \quad \forall x \in \mathbb{R}^{n+1},
\]

\[
\leq \epsilon G_{(x_0, T+\epsilon r^2)}(x, T - r^2), \forall |x - x_0| \geq K_\epsilon r.
\]

Therefore, we have, for any \( x_0 \in M_T \),

\[
1 \leq \Theta(\mu, (x_0, T))
\]

\[
\leq Cr^{-n}H^n(M_{T-r^2} \cap B_{K_\epsilon r}(x_0)) + \epsilon \int_{M_{T-r^2}} G_{(x_0, T+\epsilon r^2)}(x, T - r^2).
\]

(2.1)

The monotonicity for \( \Theta(\mu, (x_0, T + r^2), \cdot) \) implies that

\[
\int_{M_{T-r^2}} G_{(x_0, T+\epsilon r^2)}(x, T - r^2)
\]

\[
= \int_{M_{T+\epsilon r^2-2r^2}} G_{(x_0, T+\epsilon r^2)}(x, T + r^2 - 2r^2)
\]

\[
\leq \int_{M_{T+\epsilon r^2-2r^2}} G_{(x_0, T+\epsilon r^2)}(x, T + r^2 - R^2)
\]

\[
\leq CR^{-n}H^n(M_{T+\epsilon r^2-2r^2}) \leq CR^{-n}H^n(M_0)
\]

for any \( \sqrt{2r} \leq R \leq \sqrt{T+r^2} \). In particular, for sufficiently small \( r \), by choosing \( R^2 = \frac{T}{2} \), we have

\[
\int_{M_{T-r^2}} G_{(x_0, T+\epsilon r^2)}(x, T - r^2) \leq CT^{-\frac{n}{2}}H^n(M_0).
\]
Hence, by choosing $\epsilon = \epsilon(M_0, T) > 0$ sufficiently small, we have
\[
r^{n} \leq CH^{n}(M_{T-r^2} \cap B_{K,r}(x_0)). \tag{2.2}
\]
Observe that the family $\mathcal{F} = \{B_{K,r}(x) : x \in M_T\}$ covers $M_T$ so that the Vitali’s covering Lemma implies that there exists a disjoint subfamily $\{B_{K,r}(x_i) : x_i \in M_T\}_{i=1}^{\infty}$ such that
\[
M_T \subset \bigcup_i B_{5K,r}(x_i),
\]
so that
\[
H^{n}(M_{T}) \leq (5K\epsilon)^n \sum_i r^n \\
\leq (5K\epsilon)^n \sum_i H^{n}(M_{T-r^2} \cap B_{K,r}(x_i)) \\
= (5K\epsilon)^n H^{n}(M_{T-r^2} \cap (\bigcup_i B_{K,r}(x_i))) \\
\leq (5K\epsilon)^n H^{n}(M_{T-r^2}) \leq C(\epsilon, M_0) < \infty.
\]
This finishes the proof for Theorem A.

§3 Proof of Proposition B

In this section, we give a slightly different and also easier proof of Proposition B, which is essentially due to White [W1].

For any $0 \leq i \leq n$, it follows from the definition of $\Sigma_i$ that for any $x_0 \in \Sigma_i$ and each tangent flow $\tilde{\mu}$ of $\mu$ at $(x_0,T)$, which is a backwardly self-similar Brakke flow, $V(\tilde{\mu} = \{x \in R^{n+1} : \Theta(\tilde{\mu},(x,0)) = \Theta(\tilde{\mu},(0,0))\}$ is a vector space of dimension at most $i$. Moreover, it follows from the argument in [W1] (see also theorem 3 [W2]) that if we let $W(\tilde{\mu}) \equiv \{(x,t) \in R^{n+1} \times R_- : \Theta(\tilde{\mu},(x,t)) = \Theta(\tilde{\mu},(0,0))\}$ then either $W(\tilde{\mu}) = V(\tilde{\mu}) \times R_-$ and there exists a minimal hypercone $C^{n-d} \in R^{n+1-d}$ such that
\[
\tilde{\mu}|_{t \in R_-} = (R^d \times C^{n-d}) \times R_-,
\tag{3.1}
\]
or $W(\tilde{\mu}) = V(\tilde{\mu})$ and there exists a backwardly self-similar $(n-d)$ dimensional Brakke flow $\nu = \{\nu_t\}_{t \in R}$ in $R^{n+1-d}$ such that
\[
\tilde{\mu}|_{t \in R_-} = V(\tilde{\mu}) \times \nu|_{t \in R_-}, \tag{3.2}
\]

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here \( d = \dim(V(\tilde{\mu})) \). Define
\[
\eta_{x,\rho}(y) = \rho^{-1}(y - x), \forall y \in \mathbb{R}^{n+1}
\]
Now we claim

**Claim 3.1.** For any \( x_0 \in \Sigma_i \) and each \( \delta > 0 \) there exists an \( \epsilon > 0 \) (depending on \( \mu, x_0, \delta \)) such that for each \( \rho \in (0, \epsilon] \)
\[
\eta_{x_0,\rho} \{ x \in B_\rho(x_0) : \Theta(\mu, (x, T)) \geq \Theta(\mu, (x_0, T)) - \epsilon \}
\subset \text{the } \delta - \text{neighbourhood of } L_{x_0,\rho}
\]
(3.3)
for some \( i \)-dimensional subspace \( L_{x_0,\rho} \) of \( \mathbb{R}^{n+1} \).

**Proof.** If this is false, there exist \( \delta > 0 \) and \( x_0 \in \Sigma_i \) and \( \rho_k \downarrow 0 \) and \( \epsilon_k \downarrow 0 \) such that
\[
\{ x \in B_1(0) : \Theta(P_{(x_0, T),\rho_k}(\mu), (x_0, 0)) \geq \Theta(\mu, (x_0, T)) - \epsilon_k \}
\subset \not\subset \delta - \text{neighbourhood of } L,
\]
(3.4)
for any \( i \)-dimensional subspace \( L \) of \( \mathbb{R}^{n+1} \). But \( P_{(x_0, T),\rho_k}(\mu) \rightarrow \tilde{\mu} \), a tangent flow of \( \mu \) at \( (x_0, T) \), and \( \Theta(\tilde{\mu}, (0, 0)) = \Theta(\mu, (x_0, T)) \). Since \( x_0 \in \Sigma_i \), we have \( \dim(V(\tilde{\mu})) \leq i \), there is a \( i \)-dimensional subspace \( L_0 \subset \mathbb{R}^{n+1} \) such that \( V(\tilde{\mu}) \subset L_0 \). Moreover, the upper semicontinuity of \( \Theta(\tilde{\mu}, \cdot) \) implies that there is a \( \alpha > 0 \) such that
\[
\Theta(\tilde{\mu}, (x, 0)) < \Theta(\tilde{\mu}, (0, 0)) - \alpha, \forall x \in B_1(0) \text{ with } \text{dist}(x, L_0) \geq \delta.
\]
(3.5)
Then the upper semicontinuity of \( \Theta(\mu, \cdot) \) for convergence of both of its variables implies that we must have, for \( k' \) sufficiently large,
\[
\Theta(P_{(x_0, T),\rho_{k'}}(\mu), (x, 0)) < \Theta(\tilde{\mu}, (0, 0)) - \alpha, \forall x \in B_1(0) \text{ with } \text{dist}(x, L_0) \geq \delta.
\]
(3.6)
This clearly contradicts with (3.4). The claim is proven. \[\blacksquare\]

**Completion of Proof of Proposition B.** We decompose \( \Sigma_i = \bigcup_{l=1}^{\infty} \Sigma^l_i \), where \( \Sigma^l_i \) denotes the points \( x \in \Sigma_i \) such that the claim 3.1 holds for \( \epsilon = l^{-1} \). Now we
decompose $\Sigma_i^l = \bigcup_{q=1}^{\infty} \Sigma_{i}^{l,q}$, where $\Sigma_{i}^{l,q} = \{x \in \Sigma_i^l : \frac{q-1}{l} \leq \Theta(\mu, x) \leq \frac{q}{l}\}$. Hence, claim 3.1 implies that for $A = \Sigma_{i}^{l,q}$,

$$\eta_{x,\rho}(A \cap B_\rho(x)) \subset \delta - \text{neighbourhood of } L_{x,\rho}, \forall x \in A, \rho < l^{-1}. \quad (3.7)$$

for some $i$-dimensional subspace $L_{x,\rho} \subset \mathbb{R}^{n+1}$. The proof is completed if we apply the following Lemma, whose proof can be found in the Lecture 2.4 of Simon [S].

**Lemma 3.2.** There is a $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t \to 0} \beta(t) = 0$ such that if $\delta > 0$ and if $A \in \mathbb{R}^{n+1}$ satisfying the property (3.7) above, then $H^{i+\beta(\delta)}(A) = 0$.

§4 Proof of Theorem C

In this section, we outline the proof of the theorem C. But, first, we gather together needed key estimates by [HS2] on the $\sigma_k$ curvature under the mean curvature flow (1.1)-(1.2) with the initial $M_0$ being mean convex.

**Lemma 4.1.** (a). For any closed hypersurface $M$ in $\mathbb{R}^{n+1}$. For any $1 \leq k \leq n$, if $M$ is $\sigma_k$ convex (or $\sigma_k$ strictly convex, respectively) then $M$ is also $\sigma_l$ convex (or $\sigma_l$ strictly convex, respectively) for all $1 \leq l \leq k$. In particular, $M$ is mean convex.

(b). For any $1 \leq k \leq n$. Assume that $M_0$ is a $\sigma_k$ convex closed hypersurface and $\{M_t\}_{0 \leq t < T}$ is the mean curvature flow (1.1)-(1.2). Then, for any $0 < t < T$, $M_t$ is $\sigma_k$ strictly convex. In particular, $M_t$ is $\sigma_l$ strictly convex for $0 < t < T$ and all $1 \leq l \leq k$.

**Proof.** The reader can find the details of the proof in Proposition 3.3 (i) (ii) in [HS2].

It follows from Lemma Lemma 4.1 that for $1 \leq k \leq n$ if $M_0$ is a $\sigma_k$ convex closed hypersurface then for any $0 < t < T$ there exists a $\epsilon = \epsilon_t > 0$ such that

$$\sigma_l^{M_t}(x) \geq \epsilon H^l(x, t), \forall x \in M_t, 1 \leq l \leq k. \quad (4.1)$$

Proposition 3.4 of [HS2] then asserts that the inequality (4.1) is kept under the mean curvature flow (1.1)-(1.2). More precisely, we have
Lemma 4.2. For $1 \leq k \leq n$. Let $\{M_t\}_{0 \leq t < T}$ be the mean curvature flow (1.1)-(1.2), with $M_0$ being a $\sigma_k$ convex closed hypersurface. Then (3.1) holds with $\epsilon = \epsilon_t > 0$ for all $s \in [t, T)$ and all $1 \leq l \leq k$.

Proof of Theorem C. Suppose that the conclusion fails. Then there exists a $2 \leq i \leq k$ such that $\Sigma_{n-i+1} \setminus \Sigma_{n-i} \neq \emptyset$. Pick a point $x_0 \in \Sigma_{n-i+1} \setminus \Sigma_{n-i}$. Then it follows from the definition that any tangent flow of $\mu$ at $(x_0, T)$ has no more than $n-i+1$ dimensions of spatial translating invariant directions, and there exists at least one $\lambda_m \downarrow 0$ and a vector subspace $V \subset \mathbb{R}^{n+1}$, with $\dim(V) = n-i+1$, such that either (3.1) or (3.2) holds, namely either (a): there exists a minimal hypercone $C^{i-1} \subset \mathbb{R}^i$ such that

$$P(x_0, T), \lambda_m(\mu)|_{t \in R_-} \to (V \times C^{i-1}) \times R_-,$$

or (b): there exists a backwardly self-similar $(i-1)$ dimensional Brakke flow $\nu = \{\nu_t\}_{t \in R}$ in $\mathbb{R}^i$ such that

$$P(x_0, T), \lambda_m(\mu)|_{t \in R_-} \to V \times \nu|_{t \in R_-}.$$

As a special case of the partial regularity theorem of White [W1], we know that the singular set of both $C^{i-1}$ (in the case (a)) and $\mathcal{M}_{-1} = \text{spt}(V \times \nu_{-1})$ (in the case (b)) has Hausdorff dimension at most $i-4$, here the singular set of a subset $A \subset \mathbb{R}^i$ is the complement of regular points in $A$ and the regular points are points near where $A$ is a smooth manifold. For, otherwise, the singular set of the corresponding tangent flow has parabolic Hausdorff dimension larger than $(i-4) + (n-i+1) + 2 = n-1$, which contradicts with White’s theorem. Since case (a) can be handled similarly to case (b), we want to discuss case (b) only. Let $N \subset \mathcal{M}_{-1}$ denotes the singular set. Then, for any $x \in \mathcal{M}_{-1} \setminus N$, $V \times \nu$ is smooth in a spacetime neighbourhood $U$ of $(x, -1)$. Moreover, by the Brakke (unit density) regularity theorem ([B]), we can assume that $\mu_m \equiv P(x_0, T), \lambda_m(\mu)$ converge smoothly to $V \times \nu$ in the neighbourhood $U$. We now claim

Claim 4.3. The second fundamental form $A$ of $\mathcal{M}_{-1}$ vanishes everywhere on $\mathcal{M}_{-1} \setminus N$. 

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Proof of Claim 4.3. Since $M_0$ is assumed to be $\sigma_k$ convex, it follows from part (b) of Lemma 4.1 that without loss of generality we can assume that $M_0$ is in fact $\sigma_k$ strictly convex so that Lemma 4.2 implies that there exists a $\epsilon > 0$ such that

$$\sigma^M_l(x) \geq \epsilon H_l(x, t) > 0, \forall x \in M_l, 0 \leq t < T, 1 \leq l \leq k. \quad (4.3)$$

Note that (4.3), combined with scalings, implies that, for any $1 \leq l \leq k$, $t \in [-\lambda_m^{-2}T_0, 0)$, and $x \in \text{spt}(\mu_m(t)) = \text{spt}(\mu_m \cap \{t\})$,

$$\sigma^\mu_l(t)(x) = \lambda_m^l \sigma^M_{T_0 + \lambda_m t}(x_0 + \lambda_mx) \geq \epsilon \lambda_m^l H^l(x_0 + \lambda_mx, T_0 + \lambda_m t) = \epsilon (\sigma^\mu_l(t)(x))^l, \quad (4.4)$$

$$H(x, t) > 0, \forall x \in \text{spt}(\mu_m(t)), t \in [\lambda_m^{-2}T_0, 0). \quad (4.5)$$

This, combined with the smooth convergence fact at $M_{-1} \setminus N$ as above, implies, for any $1 \leq l \leq k$

$$\sigma^M_{l-1}(x) \geq \epsilon H_l(x, -1) \geq 0, \forall x \in M_{-1} \setminus N. \quad (4.6)$$

On the other hand, since $M_{-1} = V \times \text{spt}(\nu_{-1})$ and $\dim(V) = n - i + 1$, it follows from the definition 1.1 that

$$\sigma^M_{l-1}(x) = \cdots = \sigma^M_{k-1}(x) = 0, \forall x \in M_{-1} \setminus N. \quad (4.7)$$

(4.6) and (4.7) imply that

$$H(x, -1) = 0, \forall x \in M_{-1} \setminus N. \quad (4.8)$$

Therefore, for any $x \in M_{-1} \setminus N$,

$$\sigma^M_{2-1}(x) = \frac{1}{2}(H^2(x, -1) - |A|^2(x, -1))$$

$$= -\frac{1}{2}|A|^2(x, -1) \leq 0.$$

This, combined with (4.6), implies that

$$\sigma^M_{2-1}(x) = |A|(x, -1) = 0, \forall x \in M_{-1} \setminus N. \quad (4.9)$$
This finishes the proof of Claim 4.3. It follows from the Claim 4.3 that $\mathcal{M}_{-1} \setminus N$ is flat. Note also that $V \times \nu_{i+1}$ is also a backwardly self-similar Brakke flow. Therefore, we know that $\mathcal{M}_{-1}$ is also a minimal hypercone in $\mathbb{R}^{n+1}$. Hence $\mathcal{M}_{-1}$ is a hyperplane in $\mathbb{R}^{n+1}$. This clearly contradicts with the fact that $2 \leq i \leq k$ and $x_0 \in \Sigma_{n-i+1} \setminus \Sigma_{n-i}$. Therefore, the proof is complete.

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**REFERENCES**

[B] K. Brakke, The motion of a surface by its mean curvature. Princeton U. Press, 1978.

[C] X. Cheng, Estimate of the singular set of the evolution problem for harmonic maps. J. Diff. Geom. 34 (1991), no. 1, 169-174.

[H1] G. Huisken, Flow by mean curvature of convex surface into spheres. J. Diff. Geom., 20 (1984), 237-266.

[H2] G. Huisken, Asymptotic behaviour for singularities of the mean curvature flow. J. Diff. Geom. 31 (1990), 285-299.

[HS1] G. Huisken & C. Sinestrari, Mean curvature flow singularities for mean convex surfaces. Calc. Var. Partial Differential Equations, 8 (1999), 1-14.

[HS2] G. Huisken & C. Sinestrari, Convexity estimates for mean convex flow and singularities of mean convex surfaces.

[I1] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature. Memoris of AMS, 108 (1994).

[I2] T. Ilmanen, Singularities of mean curvature flow of surfaces. Preprint.

[S] L. Simon, Singularities of geometric variational problems. Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), 185-223. IAS/Park City Math. Ser., 2, Amer. Math. Soc., Providence, RI, 1996.
[W1] B. White, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*. J. Reine angew. Math. 488 (1997), 1-35.

[W2] B. White, *Partial regularity theorem for mean curvature flows*. International Math. Research Notices 4 (1994), 185-192.

[W3] B. White, *The size of the singular set in mean curvature flow of mean convex surfaces*. J. Amer. Math. Soc. 13 (2000), no. 3, 665-695.