Spinor Decomposition of $SU(2)$ Gauge Potential and The Spinor Structures of Chern-Simons and Chern Density

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Abstract

In this paper, the decomposition of $SU(2)$ gauge potential in terms of Pauli spinors is studied. Using this decomposition, the spinor structures of the Chern-Simons form and the Chern density are obtained. Furthermore, by these spinor structures, the knot quantum number of non-Abelian gauge theory is discussed, and the second Chern number is characterized by the Hopf indices and the Brouwer degrees of $\phi$-mapping.

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I. INTRODUCTION

In recent years the decomposition theory of gauge potential is playing a more and more important role in theoretical physics and mathematics. Since the decomposition theory reveals the inner structure of gauge potential, it inputs the topological and other information to the gauge potential (i.e. the connection of principal bundle), and establishes a direct relationship between differential geometry and topology of gauge field. From this viewpoint much progress has been made by other authors [1,2] and by us, such as the decomposition of $U(1)$ gauge potential and the $U(1)$ topological quantum mechanics, the decomposition of $SO(N)$ spin connection and the structure of GBC topological current, and the decomposition of $SU(N)$ connection and the effective theory of $SU(N)$ QCD, etc. [3–7].

The Pauli spinor is the fundamental representation of $SU(2)$ gauge field. In Sect.II of this paper, the spinor decomposition of $SU(2)$ gauge potential is studied. In Sect.III, using this decomposition, we obtain the spinor structure expression of Chern-Simons form, by which the knot quantum number of non-Abelian gauge field is studied. In comparison with the representation of the $SU(2)$ gauge potential, the expression of knot quantum number in terms of spinor is more direct and precise in non-Abelian $SU(2)$ gauge theory. In Sect.IV, the spinor structure of $SU(2)$ Chern density is obtained. By making use of the $\phi$-mapping topological current theory, the Chern density is expressed as $\delta(\phi)$. Therefore, the zero points of $\phi$ field are characterized by the Hopf indices ($\beta_j$) and Brouwer degrees ($\eta_j$) of $\phi$-mapping, and the second Chern number, which is directly related to the Euler characteristic through the top Chern class on 4-dimensional manifold, is characterized by $\beta_j$ and $\eta_j$.

II. THE SPINOR DECOMPOSITION OF $SU(2)$ GAUGE POTENTIAL

It is well known that in $SU(2)$ gauge field theory for spinor representation $\Psi$, the covariant derivative of $\Psi$ is defined as

$$D_\mu \Psi = \partial_\mu \Psi - \frac{1}{2i} A_\mu^a \sigma_a \Psi,$$  \hspace{1cm} (1)
where
\[ A = A_\mu dx^\mu = \frac{1}{2i} A^a_\mu \sigma_a dx^\mu \] is the SU(2) gauge potential (connection), and \( T_a = \frac{1}{2i} \sigma_a \) \((a = 1, 2, 3)\) are the SU(2) generators with \( \sigma_a \) the Pauli matrices. The complex conjugate of \( D_\mu \Psi \) is
\[ D_\mu^\dagger \Psi^\dagger = \partial_\mu \Psi^\dagger + \frac{1}{2i} \Psi^\dagger A^a_\mu \sigma_a. \] The SU(2) gauge field tensor is given by
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \] where
\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad F_{\mu\nu} = \frac{1}{2i} F^a_{\mu\nu} \sigma_a. \] To complete the decomposition of SU(2) gauge potential, multiplying Eq.(1) with \( \Psi^\dagger \sigma_b \) and Eq.(3) with \( \sigma_b \Psi \) respectively and using
\[ \sigma_a \sigma_b + \sigma_b \sigma_a = 2\delta_{ab} I, \] one can easily find
\[ A^a_\mu = \frac{i}{\Psi^\dagger \Psi} (\Psi^\dagger \sigma_a \partial_\mu \Psi - \partial_\mu \Psi^\dagger \sigma_a \Psi) - \frac{i}{\Psi^\dagger \Psi} (\Psi^\dagger \sigma_a D_\mu \Psi - D_\mu^\dagger \Psi^\dagger \sigma_a \Psi). \] Since any 2 \times 2 Hermitian matrix \( X \) can be expressed in terms of Clifford basis \((I, \vec{\sigma})\):
\[ X = \frac{1}{2} Tr(X) I + \frac{1}{2} Tr(X \sigma_a) \sigma_a, \] from Eqs.(2), (7) and (8) we can obtain
\[ A_\mu = a_\mu + b_\mu, \] where
\[ a_\mu = \frac{1}{\Psi^\dagger \Psi} (\partial_\mu \Psi \Psi^\dagger - \Psi \partial_\mu \Psi^\dagger) - \frac{1}{2 \Psi^\dagger \Psi} Tr(\partial_\mu \Psi \Psi^\dagger - \Psi \partial_\mu \Psi^\dagger) I, \]
\[ b_\mu = -\frac{1}{\Psi^\dagger \Psi} (D_\mu \Psi \Psi^\dagger - \Psi D_\mu^\dagger \Psi^\dagger) - \frac{1}{2 \Psi^\dagger \Psi} Tr(D_\mu \Psi \Psi^\dagger - \Psi D_\mu^\dagger \Psi^\dagger) I]. \]
It is easy to prove that $a_\mu$ and $b_\mu$ satisfies the gauge transformation and the vectorial transformation respectively:

\[ a'_\mu = S a_\mu S^\dagger + \partial_\mu SS^\dagger, \]  

\[ b'_\mu = S b_\mu S^\dagger, \]  

where $S^\dagger = S^{-1}$ ($S \in SU(2)$), hence $A_\mu$ satisfies the required $SU(2)$ gauge transformation:

\[ A'_\mu = S A_\mu S^\dagger + \partial_\mu SS^\dagger. \]  

Therefore Eq.(9) with Eqs.(10) and (11) is just the spinor decomposition of $SU(2)$ gauge potential.

**III. THE SPINOR STRUCTURES OF THE CHERN-SIMONS FORM AND THE KNOT QUANTUM NUMBER**

Let $M$ be a compact oriented 4-dimensional manifold, on which there is an open cover \{U, V, W, ...\} with transition function $S_{uv}$ satisfying

\[ S_{uu} = 1, \; S_{uv}^{-1} = S_{vu}, \; S_{uv}S_{vu}S_{wu} = 1. \; (U \cap V \cap W \neq \emptyset) \]  

(15)

On the principal bundle $P(\pi, M, SU(2))$, the Chern-Simons 3-form is defined as:

\[ \Omega = \frac{1}{8\pi^2} Tr(A \wedge dA - \frac{2}{3} A \wedge A \wedge A), \]  

i.e.

\[ \Omega = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\lambda}[A_\mu^a \partial_\nu A_\lambda^a - \frac{1}{3} \epsilon^{abc} A_\mu^a A_\nu^b A_\lambda^c]d^3x. \]  

(17)

This leads to the second Chern class:

\[ c_2(P) = d\Omega, \]  

\[ c_2(P) = \frac{1}{8\pi^2} Tr(F \wedge F) = \rho(x) d^4x, \]  

(19)
where $\rho(x)$ is the $SU(2)$ Chern density.

In the traditional decomposition theory of gauge potential (including Riemann geometry) [3,5], always using the parallel field condition

$$D_{\mu}\Psi = 0,$$

(20)

the solution is then $b_{\mu} = 0$, and $A_{\mu}^{a}$ can be solved in terms of $\Psi$

$$A_{\mu}^{a} = a_{\mu}^{a} = \frac{i}{\Psi^\dagger \Psi} (\Psi^\dagger \sigma_{a} \partial_{\mu} \Psi - \partial_{\mu} \Psi^\dagger \sigma_{a} \Psi).$$

(21)

In the above text the spinor $\Psi$ is a $2 \times 1$ matrix

$$\Psi = \begin{pmatrix} \phi^0 + i\phi^1 \\ \phi^2 + i\phi^3 \end{pmatrix},$$

(22)

where $\phi^{a}$ ($a = 0, 1, 2, 3$) are real functions, $\phi^{a} \phi^{a} = \|\phi\|^2 = \Psi^\dagger \Psi$. For simplicity, we introduce a unit vector $n^{a}$ ($a = 0, 1, 2, 3$)

$$n^{a} = \frac{\phi^{a}}{\|\phi\|}, \quad n^{a} n^{a} = 1.$$

(23)

Obviously the zero points of $\phi^{a}$ are just the singular points of $n^{a}$. And a normalized spinor $\Psi_{n}$ is introduced:

$$\Psi_{n} = \frac{1}{\sqrt{\Psi^\dagger \Psi}} \Psi = \begin{pmatrix} n^0 + in^1 \\ n^2 + in^3 \end{pmatrix}.$$

(24)

In following, without making mistakes, we can still use the signal ”$\Psi$” instead of ”$\Psi_{n}$” to denote the normalized spinor. Thus Eq.(21) becomes

$$A_{\mu}^{a} = i(\Psi^\dagger \sigma_{a} \partial_{\mu} \Psi - \partial_{\mu} \Psi^\dagger \sigma_{a} \Psi).$$

(25)

Then we can use Eqs.(17) and (25) to study the spinor structure of $\Omega$. Noticing that the Pauli matrix elements satisfy the formulas

$$\sigma_{a}^{\alpha\beta} \sigma_{a}^{\alpha'\beta'} = 2\delta_{\alpha\beta} \delta_{\alpha'\beta'} - \delta_{\alpha\beta'} \delta_{\alpha'\beta},$$

(26)

$$\epsilon_{abc} \sigma_{a}^{\alpha\beta} \sigma_{b}^{\alpha'\beta'} \sigma_{c}^{\alpha''\beta''} = -2i(\delta_{\alpha\beta} \delta_{\alpha'\beta''} \delta_{\alpha''\beta} - \delta_{\alpha\beta'} \delta_{\alpha'\beta'} \delta_{\alpha''\beta}),$$

(27)
we arrive at
\[ \Omega = -\frac{1}{4\pi^2}\Psi^\dagger d\Psi \wedge d\Psi^\dagger \wedge d\Psi. \] (28)

This is just the spinor structure of Chern-Simons 3-form of SU(2) gauge field theory.

The above spinor structure of Chern-Simons form can be applied in studying the knot quantum number of non-Abelian gauge theory. The quantum number of Faddeev-Niemi knot is given by the integration in 3-dimension [13]

\[ Q_{FN} = \frac{1}{32\pi^2} \int \epsilon_{ijk} C_i H_{jk} d^3x, \quad (i,j,k = 1,2,3) \] (29)

where \( H_{ij} \) is an Abelian gauge field tensor

\[ H_{ij} = -\vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) = \partial_i C_j - \partial_j C_i, \quad (\vec{m} \cdot \vec{m} = 1) \] (30)

and \( \vec{m} \) is the non-linear \( \sigma \)-model field; in SU(2) gauge field \( m^a = \Psi^\dagger \sigma^a \Psi \) \( (a = 1,2,3) \). Here \( Q_{FN} \in \pi_3(S^2) \) \( (\pi_3(S^2) = \mathbb{Z}) \), so it is the Hopf invariant.

It is known that [14]

\[ \frac{1}{4} \epsilon_{ijk} C_i H_{jk} = \epsilon_{ijk} Tr(A_i \partial_j A_k - \frac{2}{3} A_i A_j A_k), \] (31)

therefore the Faddeev-Niemi model is related to the non-Abelian SU(2) gauge field theory

\[ Q = \int \Omega = \frac{1}{8\pi^2} \int \epsilon_{ijk} Tr(A_i \partial_j A_k - \frac{2}{3} A_i A_j A_k). \] (32)

and \( Q \in \pi_3(S^3) \) \( (\pi_3(S^3) = \mathbb{Z}) \). Since \( \pi_3(S^3) = \pi_3(S^2) \) [8], Eqs. (28) and (32) give the same knot quantum number, which is just the vacuum number of the SU(2) gauge potential [15–17].

In this paper, using the spinor expression of Chern-Simons form (28), from Eq. (32) the knot quantum number can be directly expressed as

\[ Q = \int \Omega = -\frac{1}{4\pi^2} \int \epsilon_{ijk} \Psi^\dagger \partial_i \Psi \partial_j \Psi \partial_k \Psi d^3x. \] (33)

In comparison with Eq. (32), the expression of knot quantum number \( Q \) in terms of spinor is obviously more direct and precise in non-Abelian SU(2) gauge field theory.
IV. THE SPINOR STRUCTURE OF CHERN DENSITY AND THE INNER STRUCTURE OF THE SECOND CHERN NUMBER

From Eqs. (18) and (28) we can obtain the spinor structure of the second Chern class:

\[ c_2(P) = -\frac{1}{4\pi^2} d\Psi^\dagger \wedge d\Psi \wedge d\Psi^\dagger \wedge d\Psi, \]

and the spinor structure of \( SU(2) \) Chern density:

\[ \rho(x) = -\frac{1}{4\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \Psi^\dagger \partial_\nu \Psi \partial_\lambda \Psi^\dagger \partial_\rho \Psi. \]

In terms of the unit vector \( n^a \), the Chern density \( \rho(x) \) (Eq.(35)) can be expressed as \[ ]

\[ \rho(x) = \frac{1}{12\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d. \]

By making use of our \( \phi \)-mapping topological current theory \[ \], we can use Eq.(23) and the Green function relation in \( \phi \)-space

\[ \frac{\partial^2}{\partial \phi^a \partial \phi^a} \left( \frac{1}{||\phi||^2} \right) = -4\pi^2 \delta^4(\vec{\phi}) \]

(37)

to reexpress Chern density \( \rho(x) \) in a \( \delta \)-function form \[ \]

\[ \rho(x) = \delta^4(\vec{\phi}) D(\frac{\phi}{x}), \]

(38)

where \( D(\phi/x) \) is the Jacobi determinant

\[ \epsilon^{abcd} D(\frac{\phi}{x}) = \epsilon^{\mu\nu\lambda\rho} \partial_\mu \phi^a \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d. \]

(39)

The implicit function theory shows that \[ \], under the regular condition \( D(\phi/x) \neq 0 \), the general solutions of

\[ \phi^a(x^0, x^1, x^2, x^3) = 0 \ (a = 0, 1, 2, 3) \]

(40)

can be expressed as \( N \) isolated points

\[ x^\mu = x_j^\mu. \ (\mu = 0, 1, 2, 3; \ j = 1, ..., N) \]

(41)
In δ-function theory [21], one can prove

\[
\delta^4(\vec{\phi}) = \sum_{j=1}^{N} \frac{\beta_j \delta^4(x^\mu - x^\mu_{j})}{|D(\phi/x)|_{x^\mu_j}},
\]

where the positive integer \( \beta_j \) is the Hopf index of \( \phi \)-mapping. In topology it means that when the point \( x^\mu \) covers the neighborhood of the zero point \( x^\mu_{j} \) once, the vector field \( \phi^a \) covers the corresponding region in \( \phi \)-space \( \beta_j \) times. Introducing the Brouwer degree of \( \phi \)-mapping

\[
\eta_j = \frac{D(\phi/x)}{|D(\phi/x)|_{x^\mu_j}} = \text{sign}[D(\phi/x)]_{x^\mu_j} = \pm 1,
\]

Eq. (43) can be expressed as

\[
\rho(x) = \sum_{j=1}^{N} \beta_j \eta_j \delta^4(x^\mu - x^\mu_{j}).
\] (44)

Eq. (44) directly shows that the Chern density does not vanish only at the \( N \) 4-dimensional zero points of \( \phi^a \), i.e. the singular points of \( n^a \), which are characterized by the Hopf indices \( \beta_j \) and the Brouwer degrees \( \eta_j \) of \( \phi \)-mapping.

Furthermore, when integrating the second Chern class, one obtains the second Chern number:

\[
C_2 = \int c_2(P) = \int \rho(x) d^4x = \sum_{j=1}^{N} \beta_j \eta_j.
\] (45)

Since the base manifold \( M \) is 4-dimensional, the second Chern class \( c_2(P) \) is just the top Chern class on \( P \); on the other hand, there is a direct relation between the top Chern class and the Euler class [22]

\[
c_2(P) = e(E),
\] (46)

where \( E \) is a real vector bundle which is the real counterpart of complex vector bundle \( P \), and \( e(E) \) is the Euler class on \( E \). Therefore the Euler characteristic, which is just the sum of indices of zero points of vector field \( \phi^a \) on \( M \), is obtained through the Gauss-Bonnet theorem.
\[ \chi(M) = \int e(E) = \sum_{j=1}^{N} \beta_j \eta_j. \] (47)

So the indices of zero points of \( \phi^a \) field can be composed of the topological numbers \( \beta_j \) and \( \eta_j \).

At last there are two points which should be stressed. Firstly, besides in this paper, the spinor decomposition of \( SU(2) \) gauge potential can also be applied in studying the \( U(1) \) field tensor in \( SU(2) \) gauge field. Secondly, when the self-dual condition

\[ F^*_{\mu\nu} = F_{\mu\nu} \quad (F^*_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}) \] (48)

is satisfied, the corresponding zero points of \( \phi^a \) field on \( \mathbb{R}^4 \) are just the instantons, so their topological numbers can also be studied. These two points will be detailed in our other papers.

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REFERENCES

[1] L. Faddeev and A. J. Niemi, Phys. Rev. Lett. 82 (1999) 1624; ibid. Phys. Lett. B 449 (1999) 214; ibid. Phys. Lett. B 464 (1999) 90

[2] Y. M. Cho, H. Lee and D. G. Pak, hep-th/9905215; Y. M. Cho, hep-th/9906198

[3] Y. S. Duan, M. L. Ge, Sci. Sin. 11 (1979) 1072; Y. S. Duan and X. H. Meng, J. Math. Phys. 34 (1993) 1149

[4] Y. S. Duan, H. Zhang and S. Li, Phys. Rev. B 58 (1998) 125; Y. S. Duan and H. Zhang, Eur. Phys. J. D 5 (1999) 47

[5] Y. S. Duan, S. Li and G. H. Yang, Nucl. Phys. B 514 (1998) 705

[6] Y. S. Duan and L. B. Fu, J. Math. Phys. 39 (1998) 4343

[7] S. Li, Y. Zhang and Z. Y. Zhu, Phys. Lett. B 487 (2000) 201

[8] S. Nash and S. Sen, Topology and Geometry of Physicists (Academic Press INC, 1983)

[9] A. Schwarz, Topology For Physicist (Springer-Verlag Press, 1994)

[10] S. S. Chern and J. Simon, Ann. Math. 99 (1974) 48

[11] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (NY), 140, 372 (1982), (E) 185, 406 (1985).

[12] S. S. Chern, Ann. Math. 45 (1944) 747; ibid. 46 (1945) 674

[13] L. Faddeev and A. Niemi, Nature 387 (1997) 58; R. A. Battye and P. M. Sutcliffe, hep-th/98081029

[14] E. Langmann and A. J. Niemi, Phys. Lett. B 463 (1999) 252

[15] Y. M. Cho, Phys. Lett. 81 B (1979) 25

[16] Y. M. Cho, hep-th/0110076; ibid. cond-mat/0112325
[17] P. V. Baal and A. Wipf, Phys. Lett. B 515 (2001) 181

[18] Y. S. Duan, L. B. Fu and G. Jia, J. Math. Phys. 41 (2000) 4379

[19] Y. S. Duan, SLAC-PUB-3301 (1984)

[20] É. Goursat, A Course in Mathematical Analysis, vol. I (translated by Earle Raymond Hedrick, 1904)

[21] J. A. Schouten, Tensor Analysis for Physicists (Clarendon, Oxford, 1951)

[22] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rep. 66, No. 6 (1980) 213