EXISTENCE AND MULTIPLICITY OF NONTRIVIAL SOLUTIONS FOR KLEIN-GORDON-MAXWELL SYSTEM WITH A PARAMETER

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ABSTRACT. This paper is concerned with the following Klein-Gordon-Maxwell system:
\[
\begin{aligned}
-\Delta u + \lambda V(x)u - (2\omega + \phi)\phi u &= f(x, u), \quad x \in \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]
where \(\omega > 0\) is a constant and \(\lambda\) is the parameter. Under some suitable assumptions on \(V(x)\) and \(f(x, u)\), we establish the existence and multiplicity of nontrivial solutions of the above system via variational methods. Our conditions weaken the Ambrosetti Rabinowitz type condition.

1. Introduction

In this paper, we consider the following Klein-Gordon-Maxwell system:
\[
\begin{aligned}
-\Delta u + \lambda V(x)u - (2\omega + \phi)\phi u &= f(x, u), \quad x \in \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]
where \(\omega > 0\) is a constant, \(\lambda\) is the parameter and \(u, \phi : \mathbb{R}^3 \to \mathbb{R}\).

This system appears as a model describing the nonlinear Klein-Gordon field interacting with the electromagnetic field in the electrostatic field. More specifically, it represents a solitary wave \(\psi(x) = u(x)e^{i\omega t}\) in equilibrium with a purely electrostatic field \(E = -\nabla \phi(x)\) (for more details, see [3, 5, 8, 13] and the references therein). The unknowns of the system are the field \(u\) associated with the particle and the electric potential \(\phi\). The presence of the nonlinear term stimulates the interaction between many particles or external nonlinear perturbations.

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As we know, V. Benci and D. Fortunato [3] are the first to consider the following Klein-Gordon-Maxwell system:

\[
\begin{aligned}
\{-\Delta u + [m_0^2 - (\omega + \phi)^2]u &= f(u), \quad x \in \mathbb{R}^3, \\
\Delta \phi = (\omega + \phi)u^2 &=, \quad x \in \mathbb{R}^3,
\end{aligned}
\tag{1.2}
\]

where \( f(u) = |u|^{q-2}u \) and \( 4 < q < 6 \), and obtained the existence of infinitely many radially symmetric solutions via the variational methods. Azzollini and Pomponio in [2] established the existence of ground state solutions of system (1.2) under the following conditions:

(i) \( 4 \leq q < 6 \) and \( m_0 > \omega \);

(ii) \( 2 < q < 4 \) and \( m_0 \sqrt{q-2} > \omega \sqrt{6-q} \).

In [1], Azzollini et al. improved the existence range of \((m_0, \omega)\) for \( p \in (2, 4) \) as follows:

\[
0 < \omega < m_0 g(p),
\]

and

\[
g(p) = \begin{cases} \sqrt{(p-2)(4-p)} & \text{if } 2 < p < 3, \\ 1 & \text{if } 3 \leq p < 4. \end{cases}
\]

They also considered the limit case that \( m_0 = \omega \). Cassani in [4] considered system (1.2) when \( f(u) = \mu |u|^{p-2}u + |u|^{2^* - 2}u \), where \( \mu \geq 0 \) and \( p \in (4, 6) \). Moreover, he obtained the existence of trivial solution via a Pohožaev-type argument when \( \mu = 0 \) and proved the existence of nontrivial solutions when one of the following conditions is satisfied:

(i) \( p \in (4, 6) \), \( |m| > |\omega| > 0 \) and \( \mu > 0 \);

(ii) \( p = 4 \), \( |m| > |\omega| > 0 \) and \( \mu > 0 \) sufficiently large.

Later, Wang in [15] followed the ideas that appeared in [8] and generalized the result of [4]. He established the existence of at least a radially symmetric nontrivial weak solution of system (1.2) when \( f(u) = \mu |u|^{p-2}u + |u|^{2^* - 2}u \), where \( \mu > 0 \) and one of the following conditions is satisfied:

(i) \( p \in (4, 6) \), \( m > |\omega| > 0 \) and \( \mu > 0 \);

(ii) \( p \in (3, 4) \), \( m > |\omega| > 0 \) and \( \mu > 0 \) sufficiently large;

(iii) \( p \in (2, 3) \), \( m \sqrt{(p-2)(4-p)} > \omega > 0 \) and \( \mu > 0 \) sufficiently large.

Applying the Ekeland’s variational principle and the Mountain Pass Theorem in critical point theory, Xu and Chen in [18] obtained the existence of at least two nontrivial solutions of problem (1.1) with \( \lambda = 1 \) when \( f(x, u) = |u|^{p-1}u + h(x) \), \( p \in (1, 5) \).

In recent paper [10], the authors studied the existence of infinitely many nontrivial solutions of (1.1) with \( \lambda = 1 \) under the following assumptions on \( V(x) \) and \( f(x, u) \):

\( (V_1) \) \( V \in C(\mathbb{R}^3, \mathbb{R}) \) satisfies \( \inf_{x \in \mathbb{R}^3} V(x) \geq a > 0 \), where \( a > 0 \) is a constant.

Moreover, for any \( M > 0 \), \( \text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty \), where \( \text{meas} \) denotes the Lebesgue measure in \( \mathbb{R}^3 \).
Theorem 1.2. Assume conditions (V₁), (f₁'), (f₅) and (f₆) hold. Then there exists \( L_1 > 0 \) such that problem (1.1) has at least one nontrivial solution whenever \( \lambda \geq L_1 \).

Theorem 1.3. Assume conditions (V₁), (f₁'), (f₂), (f₅) and (f₇) hold. Then there exists \( L_2 > 0 \) such that problem (1.1) has at least one nontrivial solution whenever \( \lambda \geq L_2 \).

To get the existence of infinitely many solutions for the problem (1.1), the assumption (f₅) is not needed. Instead, we need another assumption (V₂), but it is not very restrictive.
There exist $d > 0$ and $R_0 > 0$ such that the set $\{x \in \mathbb{R}^3 : V(x) \leq d\}$ is nonempty and $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq d\} = 0$, where $B_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$.

**Theorem 1.4.** Assume conditions $(V_1)$, $(V_2)$, $(f'_1)$, $(f_4)$ and $(f_6)$ hold. Then there exists $\Lambda_3 > 0$ such that problem (1.1) has infinitely many nontrivial weak solutions whenever $\lambda \geq \Lambda_3$.

**Theorem 1.5.** Assume conditions $(V_1)$, $(V_2)$, $(f'_1)$, $(f_2)$, $(f_4)$ and $(f_7)$ hold. Then there exists $\Lambda_4 > 0$ such that problem (1.1) has infinitely many nontrivial weak solutions whenever $\lambda \geq \Lambda_4$.

**Notation 1.1.** Throughout this paper, we shall denote by $|\cdot|_r$ the $L^r$-norm and $C$ various positive generic constants, which may vary from line to line. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it $\{u_n\}$ again.

The remainder of this paper is as follows. In Section 2, we mainly consider the existence of at least one nontrivial solution. In Section 3, the existence of infinitely many nontrivial solutions is discussed.

## 2. Existence of nontrivial solutions

In this section, we consider the existence of nontrivial solutions for problem (1.1).

Define the space

$$E_\lambda := \{u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} \lambda V(x) u^2 < +\infty\}$$

with the inner product and norm

$$\langle u, v \rangle_{E_\lambda} = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x) uv) dx, \quad ||u||_{E_\lambda} = (u, u)_{X}^{\frac{1}{2}}.$$

Moreover, by Lemma 3.4 in [19], we know that under the assumption $(V_1)$, the embedding $E_\lambda \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for $2 \leq r \leq 6$ and $E_\lambda \hookrightarrow L^r(\mathbb{R}^3)$ is compact for $2 \leq r < 6$, i.e., there exists constants $\tau_r$ such that

$$||u||_r \leq \tau_r ||u||_{E_\lambda}.$$  \hspace{1cm} (2.1)

Note that problem (1.1) has a variational structure and its solution can be regarded as critical point of the energy functional defined on the space $E_\lambda$ by

$$J(u, \phi) = \frac{1}{2} ||u||_{E_\lambda}^2 - \frac{1}{2} \int_{\mathbb{R}^3} |
abla \phi|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} (2\omega + \phi) \phi u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx.$$  \hspace{1cm} (2.1)

Under the assumptions $(V_1)$ and $(f'_1)$, the functional $J$ belongs to $C^1(E_\lambda, \mathbb{R})$ and also exists a strong indefiniteness. To avoid the indefiniteness, we can apply a reduction method described in [6, 18], by which we are led to study a one variable functional that does not present such a strong indefinite nature.
Lemma 2.1 ([8, 18]). For every \( u \in E_\lambda \) there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) which solves \( \Delta \phi = (\omega + \phi)u^2 \). Furthermore

(i) In the set \( \{ x : u(x) \neq 0 \} \), we have \(-\omega \leq \phi_u \leq 0 \) for \( \omega > 0 \).

(ii) If \( u \) is radially symmetric, \( \phi_u \) is radial too.

According to Lemma 2.1, we can consider the functional \( I : E_\lambda \to \mathbb{R} \) defined by \( I(u) = J(u, \phi_u) \). After multiplying \( \Delta \phi_u = (\omega + \phi_u)u^2 \) by \( \phi_u \) and integration by parts, we obtain

\[
(2.2) \quad \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx = -\int_{\mathbb{R}^3} (\phi_u)^2 u^2 dx - \int_{\mathbb{R}^3} \omega \phi_u u^2 dx.
\]

Therefore, the reduced functional takes the form

\[
(2.3) \quad I(u) = \frac{1}{2} ||u||_{E_\lambda}^2 - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx.
\]

Moreover, \( I \) is \( C^1 \) and we have for any \( u, v \in E_\lambda \),

\[
(2.4) \quad \langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda uv - (2\omega + \phi_u)uv) dx - \int_{\mathbb{R}^3} f(x, u)v dx.
\]

Remark 2.1. By (2.2) and Hölder inequality, we have

\[
||\phi_u||_{D^{1,2}(\mathbb{R}^3)}^2 \leq \int_{\mathbb{R}^3} \omega|\phi_u| u^2 dx \leq \omega||\phi_u||_{E_\lambda} ||u||_{E_\lambda}^2,
\]

then

\[
(2.5) \quad ||\phi_u||_{D^{1,2}(\mathbb{R}^3)} \leq C||u||_{E_\lambda}^2 \cdot \int_{\mathbb{R}^3} \omega|\phi_u| u^2 dx
\]

\[
\leq C||\phi_u||_{D^{1,2}(\mathbb{R}^3)}^2 \leq C||u||_{E_\lambda}^4 \leq C||u||_{E_\lambda}^2.
\]

Now we can apply Lemma 2.2 of [7] or Lemma 2.3 of [18] to our functional \( I \) and obtain:

Lemma 2.2 ([7, 18]). The following statements are equivalent:

(i) \( (u, \phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3) \) is a critical point of \( I \) (i.e., \( (u, \phi) \) is a solution of problem (1.1)).

(ii) \( u \) is a critical point of \( I \) and \( \phi = \phi_u \).

Lemma 2.3 ([11], Mountain Pass Theorem). Let \( E \) be a real Banach space with its dual space \( E^* \), and suppose that \( I \in C^1(E, \mathbb{R}) \) satisfies

\[
\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{||u||=\rho} I(u)
\]

for some \( \mu < \eta, \rho > 0 \) and \( e \in E \) with \( ||e|| > \rho \). Let \( c \geq \eta \) be characterized by

\[
e = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau))
\]

where \( \Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e \} \) is the set of continuous paths joining \( 0 \) and \( e \), then there exists a sequence \( \{ u_n \} \subset E \) such that

\[
I(u_n) \to c \quad \text{and} \quad (1 + ||u_n||)||I'(u_n)|| \to 0, \quad n \to \infty.
\]
Lemma 2.4 ([9, Theorem A.2]). Let $\Omega$ be an open set in $\mathbb{R}^N$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ be a function such that $|f(x,u)| \leq c(|u|^r + |u|^*)$ for some $c > 0$ and $1 \leq r < s < \infty$. Suppose that $s \leq p < \infty$, $r \leq t < \infty$, $t > 1$, $\{u_n\}$ is a bounded sequence in $L^p(\Omega) \cap L^s(\Omega)$, $u_n \rightharpoonup u$ a.e. in $\Omega$ and in $L^p(\Omega \cap B_R) \cap L^s(\Omega \cap B_R)$ for all $R > 0$. Then, passing to a subsequence, there exists a sequence $\{v_n\}$ such that

$$v_n \rightharpoonup u \quad \text{in} \quad L^p(\Omega) \cap L^s(\Omega)$$

and

$$f(x,u_n) - f(x,u_n - v_n) - f(x,u) \to 0, \quad \text{in} \quad L^{s/r}(\Omega) + L^{s/s}(\Omega),$$

where $v_n(x) = \chi(2|x|/R_n)u(x)$, $\chi \in C^\infty(\mathbb{R}, [0,1])$ be such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t \geq 2$, $R_n > 0$ is a sequence of constants with $R_n \to \infty$ as $n \to \infty$, the space $L^p(\Omega) \cap L^s(\Omega)$ has the norm $||u||_{\mu,t} := ||u||_p + ||u||_t$ and the space $L^p(\Omega) + L^s(\Omega)$ with the norm

$$||u||_{\mu,t} := \inf \{||v||_p + ||w||_t : v \in L^p(\Omega), w \in L^s(\Omega), u = v + w\}.$$

Lemma 2.5. Assume that $(V_1)$, $(f_1')$ and $(f_0)$ hold. Then there exists $\Lambda > 0$ such that $I$ satisfies the $(PS)_c$ condition for all $\lambda \geq \Lambda$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. Firstly, we prove that $\{u_n\}$ is bounded in $E_\Lambda$ for $\lambda > 0$ large enough. Arguing by contradiction, we can assume that $||u_n||_{E_\Lambda} \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{||u_n||}$. Then $||v_n|| = 1$ and $||v_n||_r \leq \tau_r ||v_n||_{E_\Lambda} = \tau_r$ for $2 \leq r \leq 6$. Set

$$h(t) := F(x,t^{-1}z) t^\mu, \quad \forall \ t \in [1, +\infty) \text{ and } (x,z) \in \mathbb{R}^3 \times \mathbb{R}.$$

Then by $(f_0)$, we have

$$h'(t) = f(x,t^{-1}z)(-\frac{z}{t}) t^\mu + F(x,t^{-1}z) \mu t^{\mu-1} = t^{\mu-1} [\mu F(x,t^{-1}z) - t^{-1} z f(x,t^{-1}z)] \leq C_0 t^{\mu-3} |z|^2,$$

where $|z| \geq r_0$ and $t \in [1, \frac{1}{r_0^\mu}]$. Then

$$h(\frac{|z|}{r_0}) - h(1) = \int_1^{\frac{|z|}{r_0}} h'(t)dt \leq \int_1^{\frac{|z|}{r_0}} C_0 t^{\mu-3} |z|^2 dt = \frac{C_0 |z|^\mu}{(\mu - 2)r_0^{\mu-2}} - \frac{C_0 |z|^2}{\mu - 2}.$$

Therefore, we have

$$F(x,z) = h(1) \geq h(\frac{|z|}{r_0}) - \frac{C_0 |z|^\mu}{(\mu - 2)r_0^{\mu-2}} \geq (\frac{\beta}{r_0^\mu} - \frac{C_0}{(\mu - 2)r_0^{\mu-2}}) |z|^\mu.$$

Thus $\frac{\beta}{r_0^\mu} - \frac{C_0}{(\mu - 2)r_0^{\mu-2}} > 0$ for $C_0 < \frac{\beta(\mu-2)}{r_0^\mu}$. Since $\mu > 4$, then there exists a constant $4 < \theta < 6$ such that $\theta < \mu$, and hence

$$\lim_{|u| \to \infty} \frac{F(x,u)}{|u|^\theta} = +\infty.$$
In particularly, we have

\begin{equation}
\lim_{|u| \to \infty} \frac{F(x, u)}{|u|^4} = +\infty.
\end{equation}

From (f′_1), we have

\begin{equation}
F(x, u) \leq \frac{c_1}{2} |u|^2 + \frac{c_2}{p} |u|^p.
\end{equation}

It follows from (2.6) and (2.8) that for any \( M > 0 \), there exists a constant \( C(M) > 0 \) such that

\begin{equation}
F(x, u) \geq M |u|^\theta - C(M) |u|^2.
\end{equation}

Furthermore, we have

\begin{equation}
I(u_n) = \frac{1}{2} \frac{||u_n||^2_{E_\lambda}}{||u||^2_{E_\lambda}} - \frac{1}{2} \int_{\mathbb{R}^3} \frac{\omega \phi u^2}{||u||^\theta_{E_\lambda}} - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u||^\theta_{E_\lambda}} dx.
\end{equation}

Then by (2.5) and \( \theta > 4 \), we deduce that

\begin{equation}
\lim_{n \to +\infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u||^\theta_{E_\lambda}} dx = 0.
\end{equation}

Since \( ||v_n||_{E_\lambda} = 1 \), going if necessary to a subsequence, we can assume that \( v_n \rightharpoonup v \) in \( E_\lambda \), \( v_n \to v \) in \( L^r(\mathbb{R}^3) \) for \( 2 \leq r < 6 \) and \( v_n \to v \) a.e. in \( \mathbb{R}^3 \). Set \( \Omega = \{ x \in \mathbb{R}^3 : v(x) \neq 0 \} \). If \( \text{meas}(\Omega) > 0 \), then \( \int_{\Omega} |v|^\theta dx > 0 \). By (2.9), we have

\begin{equation}
\int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u||^\theta_{E_\lambda}} dx \geq M ||v_n||^\theta_{E_\lambda} - C(M) ||v_n||_{E_\lambda}^2.
\end{equation}

Therefore

\begin{equation}
0 = \liminf_{n \to +\infty} \left( \int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u||^\theta_{E_\lambda}} dx + C(M) \frac{||v_n||_2^2}{||v_n||^\theta_{E_\lambda}} \right)
\end{equation}

\begin{equation}
\geq \liminf_{n \to +\infty} M ||v_n||^\theta_{E_\lambda} \geq M \int_{\Omega} |v|^\theta dx > 0,
\end{equation}

which is a contradiction, then \( \text{meas}(\Omega) = 0 \), and as a result \( v = 0 \) a.e. in \( \mathbb{R}^3 \).

Therefore, from (V1), we have

\begin{equation}
||v_n||^2 = \int_{V(x) \geq 1} |v_n|^2 dx + \int_{V(x) < 1} |v_n|^2 dx \leq \frac{1}{\lambda} ||v_n||^2_{E_\lambda} + o(1) \leq \frac{2}{\lambda}
\end{equation}

for \( n \) large enough. It follows from (f′_1) and (f_6) that there exists a constant \( c > 0 \) such that

\begin{equation}
\mu F(x, u) - uf(x, u) \leq c|u|^2
\end{equation}
for all \((x, u) \in \mathbb{R}^3 \times \mathbb{R}\). Therefore, by Lemma 2.1 and \(\mu \in (4, 6)\), we have
\[
0 \leq \frac{1}{\|u_n\|_{E_\lambda}^2} [\mu I(u_n) - \langle I'(u_n), u_n \rangle]
\]
\[
= \frac{1}{\|u_n\|_{E_\lambda}^2} \left[ \frac{\mu - 2}{2} \|u_n\|_{E_\lambda}^2 + \frac{4 - \mu}{2} \int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 dx \right.
+ \left. \int_{\mathbb{R}^3} (f(x, u_n) u_n - \mu F(x, u_n)) dx + \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx \right]
\]
\[
\geq \frac{\mu - 2}{2} - \epsilon \int_{\mathbb{R}^3} |v_n|^2 dx
\]
\[
\geq \frac{\mu - 2}{2} - \frac{2\epsilon}{\lambda}.
\]
Let \(\lambda > 0\) so large that the term \(\frac{\mu - 2}{2} - \frac{2\epsilon}{\lambda} > 0\), then we get a contradiction.
Hence, \(\{u_n\}\) is bounded in \(E_\lambda\) for large \(\lambda\). Therefore, going if necessary to a subsequence, there exists \(u \in E_\lambda\) such that
\[
u n \rightharpoonup u, \text{ in } E_\lambda.
\]
\[
u n \rightarrow u, \text{ in } L^r(\mathbb{R}^3), \quad 2 \leq r < 6.
\]
\[
u n \rightarrow u, \text{ a.e. in } \mathbb{R}^3.
\]
Take \(v_n(x) = \chi_{\left(\frac{2|x|}{R_n}\right)}(u(x))\), where \(R_n > 0\) is a sequence of constants with \(R_n \rightarrow +\infty\) as \(n \rightarrow +\infty\). We claim that \(v_n \rightarrow u\) in \(E_\lambda\). Indeed, \(u \in E_\lambda\) implies that for any \(\epsilon > 0\), there exists a \(\rho = \rho(\epsilon)\) such that
\[
\int_{\mathbb{R}^3 \setminus B(0)} |\nabla u_n|^2 dx \leq \epsilon \quad \text{and} \quad \int_{\mathbb{R}^3 \setminus B(0)} \lambda V(x)|u_n|^2 dx \leq \epsilon.
\]
Hence, by (2.13), we have
\[
\|v_n - u\|_{E_\lambda}^2 = \int_{\mathbb{R}^3} |\nabla (v_n - u)|^2 dx + \int_{\mathbb{R}^3} \lambda V(x)|v_n - u|^2 dx
\]
\[
= \int_{\mathbb{R}^3} |\nabla (\chi_{\left(\frac{2|x|}{R_n}\right)}(u - u))|^2 dx + \int_{\mathbb{R}^3} \lambda V(x)|\chi_{\left(\frac{2|x|}{R_n}\right)}(u - u)|^2 dx
\]
\[
\leq \int_{\mathbb{R}^3} |\chi_{\left(\frac{2|x|}{R_n}\right)}(u - u)|^2 dx + \frac{2}{R_n} \cdot 2 \int_{\mathbb{R}^3} |\chi_{\left(\frac{2|x|}{R_n}\right)}(u)|^2 dx
+ \int_{\mathbb{R}^3} \lambda V(x)|\chi_{\left(\frac{2|x|}{R_n}\right)}|\lambda^2 dx
\]
\[
= \int_{B(0)} |\chi_{\left(\frac{2|x|}{R_n}\right)}(u - u)|^2 dx + \frac{2}{R_n^2} \int_{\mathbb{R}^3} |\chi_{\left(\frac{2|x|}{R_n}\right)}(u)|^2 dx
+ \int_{B(0)} \lambda V(x)|\chi_{\left(\frac{2|x|}{R_n}\right)} - 1|^2 |u|^2 dx + \epsilon.
\]
Furthermore, by the Hölder inequality, we have

\[ \|u_n - v_n\|_{E_\lambda}^2 - \int_{\mathbb{R}^3} (f(x, u_n) - f(x, v_n))(u_n - v_n)\,dx \]

Therefore, by the Lebesgue dominated convergence theorem, we have

\[ \|u_n - u\|_{E_\lambda} \to 0 \quad \text{as} \quad n \to +\infty. \quad (2.14) \]

Since \( u_n \to u \) in \( E_\lambda \) and \( I'(u_n) \to 0 \), we have \( (I'(u_n) - I'(u), u_n - u) \to 0 \) as \( n \to +\infty \). By \( \|u_n - u\|_{E_\lambda} \to 0 \), \( I \in C^1(E_\lambda, \mathbb{R}) \) and the boundedness of \( \{u_n\} \) in \( E_\lambda \), we have

\[ \|I'(u_n) - I'(v_n), u_n - v_n)\| \leq \|I'(u_n) - I'(u), u_n - v_n)\| + \|I'(u) - I'(v_n), u_n - v_n)\| \]

\[ \to 0 \quad \text{as} \quad n \to +\infty. \quad (2.16) \]

Meanwhile, by (2.5), (2.11), (2.14) and Lemma 2.1, we have

\[ |2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_{v_n} v_n)(u_n - v_n)\,dx| \]

\[ = |2\omega \int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n - v_n)\,dx - 2\omega \int_{\mathbb{R}^3} \phi_{v_n} v_n(u_n - v_n)\,dx| \]

\[ \leq 2\omega \|\phi_{u_n} u_n\|_2 \|u_n - u\|_2 + 2\omega \|\phi_{v_n} v_n\|_2 \|v_n - v\|_2 \]

\[ + 2\omega \|\phi_{v_n} v_n\|_2 \|u_n - u\|_2 + 2\omega \|\phi_{v_n} v_n\|_2 \|v_n - v\|_2 \]

\[ \leq C \|\phi_{u_n}\|_6 \|u_n\|_6 \left(\|u_n - u\|_2 + \|v_n - v\|_2\right) \]

\[ + C \|\phi_{v_n}\|_6 \|v_n\|_6 \left(\|u_n - u\|_2 + \|v_n - v\|_2\right) \]

\[ \to 0 \quad \text{as} \quad n \to \infty, \quad (2.17) \]

and

\[ |\int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_{v_n}^2 v_n)(u_n - v_n)\,dx| \]

\[ = |\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n(u_n - v_n)\,dx - \int_{\mathbb{R}^3} \phi_{v_n}^2 v_n(u_n - v_n)\,dx| \]

\[ \leq \left( \int_{\mathbb{R}^3} \phi_{u_n}^6 \,dx \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} |u_n - u|^{\frac{6}{5}} |u_n|^{\frac{6}{5}} \,dx \right)^{\frac{5}{6}} \]

\[ \to 0 \quad \text{as} \quad n \to \infty. \quad (2.18) \]
As a consequence, we have
\[
\omega_n \rightarrow \omega \quad \text{as} \quad n \rightarrow +\infty.
\]
Take \(\rho = 1, s = p - 1\). It follows from Lemma 2.4 that
\[
g_n(x) \rightarrow 0, \quad \text{in} \quad L^2(\mathbb{R}^3) + L^{\frac{6}{2-p}}(\mathbb{R}^3),
\]
where \(g_n(x) = f(x, u_n) - f(x, u) - f(x, u_n - v_n)\). Then
\[
\int_{\mathbb{R}^3} |f(x, u_n) - f(x, u) - f(x, u_n - v_n)||u_n - v_n|dx \leq \|g_n\|_{2\vee p'} \|u_n - v_n\|_{2\vee p} \rightarrow 0,
\]
as \(n \rightarrow +\infty\), where \(p' = \frac{6}{2-p}\). Take \(u_n = v_n\) for all \(n > 0\) in Lemma 2.4. Then
\[
f(x, v_n) - f(x, u) \rightarrow 0, \quad \text{in} \quad L^2(\mathbb{R}^3) + L^{\frac{6}{2-p}}(\mathbb{R}^3).
\]
Consequently, we have \(\int_{\mathbb{R}^3} |f(x, v_n) - f(x, u)||u_n - v_n|dx \rightarrow 0\) as \(n \rightarrow +\infty\). Then one has
\[
\int_{\mathbb{R}^3} |f(x, u_n) - f(x, u) - f(x, u_n - v_n)||u_n - v_n|dx
\]
\[
\leq \int_{\mathbb{R}^3} |f(x, u_n) - f(x, u) - f(x, u_n - v_n)||u_n - v_n|dx
\]
\[
+ \int_{\mathbb{R}^3} |f(x, u) - f(x, u)||u_n - v_n|dx
\]
\[
\rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.
\]
Set \(\omega_n = u_n - v_n\). Then by (V\text{I}) and \(\omega_n \rightarrow 0\), we have
\[
|\omega_n|^2 = \int_{V(x) \geq 1} |\omega_n|^2dx + \int_{V(x) < 1} |\omega_n|^2dx \leq \frac{1}{\lambda}||\omega_n||_{E_\lambda}^2 + o(1)
\]
for \(n\) large enough. Take \(0 < \alpha < \min\{1, \frac{6-p}{2}\}\). Then \(2 < \frac{2(p-\alpha)}{2-p} < 6\). By
\[
|\omega_n|^p \leq ||\omega_n||_{2\vee p}^p ||\omega_n||_{2\vee p}^{p-\alpha} \leq C(\lambda)^{-\frac{\alpha}{2p}} ||\omega_n||_{E_\lambda}^p + o(1)
\]
for \(n\) large enough. Consequently, by (2.11), (2.21) and (f\text{I}), one has
\[
|\int_{\mathbb{R}^3} f(x, \omega_n)\omega_n dx| \leq c_1 ||\omega_n||_2^2 + c_2 ||\omega_n||_p^p \leq \frac{c_1}{\lambda}||\omega_n||_{E_\lambda}^2 + \frac{c_2}{(\lambda)^2} ||\omega_n||_{E_\lambda}^p + o(1)
\]
for \( n \) large enough.

Therefore, it follows from (2.15), (2.16), (2.17), (2.18), (2.22) and the boundedness of \( \{\omega_n\} \) that

\[
o(1) \geq \|u_n - v_n\|^2_{E_\lambda} - \int_{\mathbb{R}^3} f(x, u_n - v_n)(u_n - v_n)dx \\
\geq (1 - \frac{c_1}{\lambda} - \frac{cc_2}{\lambda^2})\|\omega_n\|_{E_\lambda}^{p-1})\|\omega_n\|_{E_\lambda}^2.
\]

Let \( \Lambda > 0 \) be so large that the term in the brackets above is positive when \( \lambda \geq \Lambda \), thus we get \( \omega_n \to 0 \) as \( n \to +\infty \) in \( E_\lambda \). Since \( \omega_n = u_n - v_n \) and \( v_n \to u \) in \( E_\lambda \), then we have \( u_n \to u \) in \( E_\lambda \). The proof is complete. \( \square \)

**Proof of Theorem 1.2.** For any \( 0 < \varepsilon < \frac{1}{\tau^2} \), it follows from (\( f'_1 \)) and (2.6) that there exists \( c(\varepsilon) > 0 \) such that

\[
|F(x, u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{\varepsilon}{p}|u|^p.
\]

Therefore, for small \( \rho > 0 \),

\[
I(u) = \frac{1}{2}\|u\|^2_{E_\lambda} - \frac{1}{2}\int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u)dx \\
\geq \frac{1}{2}(\|u\|^2_{E_\lambda} - \varepsilon \tau^2_2\|u\|^2_{E_\lambda}) - \frac{\varepsilon}{p}\tau p\|u\|^p_{E_\lambda} \\
\geq \frac{1}{4}(\|u\|^2_{E_\lambda} - \varepsilon \tau^2_2\|u\|^2_{E_\lambda})
\]

for all \( u \in \overline{B}_\rho \), where \( B_\rho = \{u \in E_\lambda : \|u\|_{E_\lambda} < \rho \} \). Hence,

\[
I|_{\partial B_\rho} \geq \frac{1}{4}(1 - \varepsilon \tau^2_2)\rho^2 := \eta > 0.
\]

Take \( 0 \neq u \in E_\lambda \). It follows from (\( f'_1 \)) and (2.6) that for any \( M > 0 \), there exists \( C(M) > 0 \) such that

\[
F(x, u) \geq M|u|^4 - C(M)|u|^2.
\]

Then by Lemma 2.1, one has

\[
I(tu) = \frac{t^2}{2}\|u\|^2_{E_\lambda} - \frac{t^2}{2}\int_{\mathbb{R}^3} \omega \phi_{tu} u^2 dx - \int_{\mathbb{R}^3} F(x, tu)dx \\
\leq \frac{t^2}{2}\|u\|^2_{E_\lambda} + \frac{t^2}{2}\int_{\mathbb{R}^3} \omega u^2 dx + C(M)t^2 \int_{\mathbb{R}^3} u^2 dx - Mt^4 \int_{\mathbb{R}^3} u^4 dx \\
\to -\infty
\]

as \( t \to +\infty \). Therefore, there exists a point \( e \in E_\lambda \setminus \overline{B}_\rho \) such that \( I(e) \leq 0 \). By Lemma (2.3), \( I \) satisfies the \((PS)_c \) condition for large \( \lambda > 0 \). Furthermore, it is obvious that \( I(0) = 0 \). Hence \( I \) possesses a critical value \( c \geq \eta \) by Lemma 2.3, i.e., problem (1.1) has a nontrivial weak solution in \( E_\lambda \). The proof is complete. \( \square \)
Proof of Theorem 1.3. From the proof of Theorem 1.2, we know that there exist constants \( \rho > 0 \) and \( \eta > 0 \) such that \( I|_{\partial B_r} \geq \eta > 0 \) and there is a point \( e \in E_\lambda \setminus \overline{B} \) such that \( I(e) \leq 0 \). Now we prove that \( I \) satisfies the \((PS)_c\) condition for large \( n \). We need to prove that \( \{u_n\} \) is bounded in \( E_\lambda \). If \( \{u_n\} \) is unbounded in \( E_\lambda \), we can assume that \( ||u_n||_{E_\lambda} \to +\infty \) as \( n \to \infty \). Let \( v_n = \frac{u_n}{||u_n||} \). Then \( ||v_n|| = 1 \) and \( ||v_n||_c \leq \tau r ||v_n||_{E_\lambda} = \tau r \) for \( 2 \leq r \leq 6 \).

Since \( ||v_n||_{E_\lambda} = 1 \), going if necessary to a subsequence, we can assume that \( v_n \to v \) in \( E_\lambda \), \( v_n \to v \) in \( L^r(\mathbb{R}^3) \) for \( 2 \leq r < 6 \) and \( v_n \to v \) a.e. in \( \mathbb{R}^3 \). Set \( \Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\} \). If \( \text{meas}(\Omega) > 0 \), then \( \int_{\Omega} |v|^4 dx > 0 \). It follows from \((f_1')\) and \((f_2)\) that for any \( M > \frac{C_0}{2} \), there exists a constant \( C_0(M) > 0 \) such that

\[
(2.23) \quad F(x, u) \geq M|u|^4 - C_0(M)|u|^2,
\]

where

\[
C_0 = \sup_{u \in E_\lambda \setminus \{0\}} \frac{\int_{\mathbb{R}^3} \omega|\phi_u||u|^2 dx}{||u||_{E_\lambda}^2}.
\]

Furthermore, we have

\[
\frac{I(u_n)}{||u_n||_{E_\lambda}^2} = \frac{1}{2||u_n||_{E_\lambda}^2} - \frac{1}{2||u_n||_{E_\lambda}^4} \int_{\mathbb{R}^3} \omega\phi_{u_n} v_n^2 dx - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u_n||_{E_\lambda}^4} dx.
\]

Then by \((2.5)\), we deduce that

\[
\lim_{n \to \infty} \liminf_{\tau \to \infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u_n||_{E_\lambda}^4} dx \leq \frac{C_0}{2}.
\]

By \((2.23)\), we have

\[
\int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u_n||_{E_\lambda}^4} dx \geq M||v_n||_c^4 - C_0(M) ||v_n||_{E_\lambda}^2.
\]

Therefore

\[
\frac{C_0}{2} \geq \lim_{n \to \infty} \liminf_{\tau \to \infty} \left( \int_{\mathbb{R}^3} \frac{F(x, u_n)}{||u_n||_{E_\lambda}^4} dx + C_0(M) ||v_n||_{E_\lambda}^2 \right)
\]

\[
\geq \lim_{n \to \infty} \liminf_{\tau \to \infty} M||v_n||_c^4 \geq M \int_{\Omega} |v|^4 dx > \frac{C_0}{2},
\]

which is a contradiction, then \( \text{meas}(\Omega) = 0 \), and as a result \( v = 0 \) a.e. in \( \mathbb{R}^3 \).

Therefore, from \((V_1)\), we have

\[
||v_n||_c^2 = \int_{V(x) \geq 1} |v_n|^2 dx + \int_{V(x) < 1} |v_n|^2 dx \leq \frac{1}{\lambda} ||v_n||_{E_\lambda}^2 + o(1) \leq \frac{2}{\lambda}
\]

for \( n \) large enough. It follows from \((f_1)\) and \((f_7)\) that there exists a constant \( c > 0 \) such that

\[
4F(x, u) - uf(x, u) \leq c|u|^2
\]
for all \((x, u) \in \mathbb{R}^3 \times \mathbb{R}\). Therefore, by Lemma 2.1, one has
\[
0 \leftarrow \frac{1}{\|u_n\|_{E_\lambda}^2} \left[ 4I(u_n) - \langle I'(u_n), u_n \rangle \right] \\
= \frac{1}{\|u_n\|_{E_\lambda}^2} \left[ \|u_n\|_{E_\lambda}^2 + \int_{\mathbb{R}^3} (f(x, u_n)u_n - 4F(x, u_n)dx) + \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx \right] \\
\geq 1 - c \int_{\mathbb{R}^3} |v_n|^2 dx \\
\geq 1 - \frac{2c}{\lambda} \text{ as } n \to \infty.
\]
Let \(\lambda > 0\) be so large that the term \(1 - \frac{2c}{\lambda} > 0\), then we get a contradiction. Hence \(\{u_n\}\) is bounded in \(E_\lambda\) for large \(\lambda\). Therefore, \(I\) possesses a critical value by Lemma 2.3, i.e., problem (1.1) has at least one nontrivial solution. The proof is complete. \(\square\)

3. Existence of infinitely many nontrivial solutions

In this section, we consider the existence of infinitely many solutions of problem (1.1). We will give the proofs of Theorem 1.4 and Theorem 1.5. To complete the proof, we need the following results.

**Lemma 3.1** ([17, Lemma 2.2]). Let \(X\) be an infinitely dimensional Banach space and let \(I \in C^1(X, \mathbb{R})\) be even, satisfy \((PS)_c\) condition, and \(I(0) = 0\). If \(X = \mathcal{Y} \oplus \mathcal{Z}\), where \(\mathcal{Y}\) is finite dimensional and \(I\) satisfies
(i) There exists constants \(\rho, \alpha > 0\) such that \(I|_{\partial B_\rho \cap \mathcal{Z}} \geq \alpha\);
(ii) For any finite dimensional subspace \(\tilde{X} \subset X\), there is \(R = R(\tilde{X}) > 0\) such that \(I(u) \leq 0\) on \(\tilde{X} \setminus B_R\).

Then \(I\) possesses an unbounded sequence of critical values.

Let \(\{e_j\}\) be a total orthonormal basis of \(L^2(B_{R_0})\) \((B_{R_0}\) appears in \((V_2)\)) and define \(X_j = \mathbb{R}e_j\), \(j \in \mathbb{N}\), \(Y_k = \bigoplus_{j=1}^k X_j\), \(Z_k = \bigoplus_{j=k+1}^{\infty} X_j\), \(k \in \mathbb{N}\).

Set
\[
E_\lambda(B_{R_0}) := \{ u \in H^1(B_{R_0}) | \int_{B_{R_0}} \lambda V(x) u^2 dx < +\infty \}
\]
with the norm
\[
\|u\|_{E_\lambda(B_{R_0})} = \left( \int_{B_{R_0}} (\|\nabla u\|^2 + \lambda V(x) u^2) dx \right)^{\frac{1}{2}}.
\]

**Lemma 3.2.** Suppose that \((V_1)\) is satisfied. Then for \(2 \leq r < 6\)
\[
\beta_k := \sup_{u \in Z_k, \|u\|_{E_\lambda(B_{R_0})} = 1} |u|_{L^r(B_{R_0})} \to 0 \text{ as } k \to +\infty.
\]

**Proof.** The proof is similar to Lemma 3.2 of [12] or Lemma 3.2 of [16], so we omit it here. \(\square\)
By Lemma 3.2, we can choose an integer $m \geq 1$ such that
\begin{equation}
\int_{B_{R_0}} u^2 dx \leq \frac{1}{2c_1} \int_{B_{R_0}} (|\nabla u|^2 + \lambda V(x)u^2) dx, \quad \forall \ u \in Z_m \cap E_{\lambda}(B_{R_0}),
\end{equation}
where $c_1$ appears in $(f'_1)$. Let $\gamma(x) = 0$ if $|x| \leq R_0$ and $\gamma(x) = 1$ if $|x| \geq R_0$. Define
\begin{equation}
Y = \{(1 - \gamma)u : u \in E_{\lambda}, (1 - \gamma)u \in Y_m\}
\end{equation}
and
\begin{equation}
Z = \{(1 - \gamma)u : u \in E_{\lambda}, (1 - \gamma)u \in Z_m\} \cup \{\gamma v : v \in E_{\lambda}\}.
\end{equation}
Then $Y$ and $Z$ are subspaces of $E_{\lambda}$, and $E_{\lambda} = Y \oplus Z$.

**Lemma 3.3.** Suppose that $(V_1)$, $(V_2)$ and $(f'_1)$ are satisfied. Then there exist constants $\rho$, $\alpha > 0$ such that $I_{|\partial B_{\gamma} \cap Z} \geq \alpha$ for large $\lambda$.

**Proof.** It follows from (3.1), (3.3) and $(V_2)$ that
\begin{equation}
\|u\|^2 = \int_{|x| < R_0} |u|^2 dx + \int_{|x| \geq R_0} |u|^2 dx
\end{equation}
\begin{equation}
\leq \frac{1}{2c_1} \|u\|^2_{E_{\lambda}} + \frac{1}{\lambda d} \int_{\{x \in \mathbb{R}^3 : V(x) > d\}} \lambda V(x) |u|^2 dx
\end{equation}
\begin{equation}
\leq \frac{1}{2c_1} \|u\|^2_{E_{\lambda}} + \frac{1}{\lambda d} \|u\|^2_{E_{\lambda}}, \quad \forall \ u \in Z.
\end{equation}
Therefore, by (2.1), (2.8) and (3.4), we have
\begin{equation}
I(u) = \frac{1}{2} \|u\|^2_{E_{\lambda}} - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx
\end{equation}
\begin{equation}
\geq \frac{1}{2} \|u\|^2_{E_{\lambda}} - c_1 \|u\|^2_{E_{\lambda}} - \frac{c_2}{p} \|u\|^p_{E_{\lambda}}
\end{equation}
\begin{equation}
\geq \frac{1}{4} \|u\|^2_{E_{\lambda}} - \frac{c_1}{2\lambda} \|u\|^2_{E_{\lambda}} - \frac{c_2}{p} \|u\|^p_{E_{\lambda}}
\end{equation}
\begin{equation}
\geq \frac{1}{8} \|u\|^2_{E_{\lambda}} - \frac{c_2}{p} \|u\|^p_{E_{\lambda}}
\end{equation}
for $n$ large enough. Since $2 < p < 6$, then there exist constants $\rho$, $\alpha > 0$ such that $I_{|\partial B_{\gamma} \cap Z} \geq \alpha$. The proof is complete. 

**Lemma 3.4.** Suppose that $(f'_1)$ and $(f_2)$ are satisfied. Then for any finite dimensional subspace $\tilde{E}_{\lambda} \subset E_{\lambda}$, there is $R = R(\tilde{E}_{\lambda}) > 0$ such that $I(u) \leq 0$ on $\tilde{E}_{\lambda} \setminus B_R$.

**Proof.** For any finite dimensional subspace $\tilde{E}_{\lambda} \subset E_{\lambda}$, by the equivalence of norms in the finite dimensional space, there is a constant $C(4) > 0$ such that
\begin{equation}
\|u\|^4 \geq C(4) \|u\|^4_{E_{\lambda}}, \quad \forall \ u \in \tilde{E}_{\lambda}.
\end{equation}
It follows from \((f'_1)\) and (2.7) that for any \(M > \frac{C_0}{2C(4)}\) (where \(C_0\) appears in (2.23)), there exists a constant \(C(M) > 0\) such that
\[
F(x, u) \geq M|u|^4 - C(M)|u|^2, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}.
\]
Then
\[
I(u) = \frac{1}{2}|u|^2_{E_\lambda} - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx
\leq \frac{1}{2}|u|^2_{E_\lambda} + \frac{C_0}{2}||u||^4_{E_\lambda} + C(M)||u||^2_{E_\lambda} - M||u||^4_{E_\lambda}
\leq \left( \frac{1}{2} + C(M)\tau_2^2 \right)||u||^2_{E_\lambda} - (MC(4) - \frac{C_0}{2})||u||^4_{E_\lambda}
\]
for all \(u \in \tilde{E}_\lambda\). Hence, there is a large \(R = R(\tilde{E}_\lambda) > 0\) such that \(I(u) \leq 0\) on \(E_\lambda \setminus B_R\). The proof is complete. \(\square\)

**Proof of Theorem 1.4.** Let \(X = E_\lambda, Y\) and \(Z\) be defined by (3.2) and (3.3), respectively. From \((f_6)\), Lemma 2.5, Lemma 3.3, Lemma 3.4 and \(I(0) = 0\), we know that \(I\) satisfies all the conditions of Lemma 3.1. Therefore, problem (1.1) has infinitely many nontrivial weak solutions. The proof is complete. \(\square\)

**Proof of Theorem 1.5.** Let \(X = E_\lambda, Y\) and \(Z\) be defined by (3.2) and (3.3), respectively. From the proof of Theorem 1.3 and Theorem 1.4, we know that \(I\) satisfies all the conditions of Lemma 3.1. Therefore, problem (1.1) has infinitely many nontrivial weak solutions. The proof is complete. \(\square\)

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