Quantum mechanical potentials related to the prime numbers and Riemann zeros

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Prime numbers are the building blocks of our arithmetic, however, their distribution still poses fundamental questions. Bernhard Riemann showed that the distribution of primes could be given explicitly if one knew the distribution of the non-trivial zeros of the Riemann \( \zeta(s) \) function. According to the Hilbert-Pólya conjecture there exists a Hermitian operator of which the eigenvalues coincide with the real part of the non-trivial zeros of \( \zeta(s) \). This idea encourages physicists to examine the properties of such possible operators, and they have found interesting connections between the distribution of zeros and the distribution of energy eigenvalues of quantum systems. We apply the Marchenko approach to construct potentials with energy eigenvalues equal to the prime numbers and to the zeros of the \( \zeta(s) \) function. We demonstrate the multifractal nature of these potentials by measuring the Rényi dimension of their graphs. Our results offer hope for further analytical progress.

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\section{I. PRIMES, ZEROS AND QUANTA}

The prime numbers are the building blocks for the positive integers, since the fundamental theorem of arithmetic states that every positive integer can be written as a product of primes, and this product is unique up to a rearrangement of the factors. Additionally, not only does the product of prime numbers have this remarkable property, but also their sum. Only a decade ago O. Ramaré proved \[1\] that any positive integer can be written as a sum of no more than six prime numbers, but the Goldbach conjecture \[2\], whether every number is expressible as a sum of two prime numbers, remains unproven.

Based on empirical evidence, many mathematicians conjectured that the prime counting function, \( \pi(x) = |\{p | p \text{ is prime and } p \leq x\}| \), asymptotically behaves as the logarithmic integral \( \text{Li}(x) \). Hadamard \[3\] and de la Vallée Poussin \[4\] independently gave the rigorous proof for this statement. Riemann derived the following exact formula \[5\]

\[\pi(x) = \lim_{x \to \infty} \left( R(x) - \sum_{\rho} R(x^{\rho}) \right) \tag{1}\]

where \( R(x) \) is the so-called Riemann function defined as \( R(x) = \sum_{m=1}^{\infty} \mu(m) \text{Li}(x^{1/m})/m \), and \( \mu(m) \) denotes the Möbius function. The sum in \( \Pi \) is extended over all non-trivial zeros, \( \rho \), of the Riemann \( \zeta(s) \) function, counted with their multiplicities. The latter function \( \zeta(s) \) is defined by the infinite series, \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) for \( s > 1 \), and, otherwise, by its analytic continuation over the complex \( s \) plane.

Exploring the locations of the zeros of \( \zeta(s) \), Riemann made his famous conjecture: all the non-trivial zeros lie on the \( s = \frac{1}{2} + it \) (\( t \) is real) critical line. Proving or disproving the Riemann-hypothesis remains the most tantalising challenge in number theory since David Hilbert nominated it in 1900 \[6\] as the eighth problem on his famous list of compelling problems in mathematics \[27\].

The connection between the Riemann-hypothesis and physics seems to date back to the early years of quantum mechanics. According to the Hilbert-Pólya conjecture, the zeros of \( \zeta(s) \) can be the spectrum of an operator, \( \mathcal{O} = \frac{1}{2} \mathbb{I} + i\mathcal{H} \), where \( \mathcal{H} \) is self-adjoint. This operator \( \mathcal{H} \) might have a physical interpretation as a Hamiltonian of a physical system and, therefore, the key to the proof of the Riemann Hypothesis may have been coded in physics.

The analogy between the properties of \( \zeta(s) \) and the energy eigenvalues of a quantum mechanical system provide us some information about the form of a possible operator \( \mathcal{H} \). One of these similarities, the comparison of the number of \( \zeta(s) \) zeros and the number of energy eigenvalues below a threshold, suggests that the physical system is quasi-one-dimensional. This link is further strengthened by checking different statistics of the zeros, such as the nearest-neighbour spacings, the \( n \)-correlations between the zeros, etc. Montgomery showed that these distributions are all in good agreement with the Gaussian unitary ensemble of random matrix theory \[8\].

In this work we utilize an inverse scattering formalism and construct potentials of which the energy eigenvalues are the zeros of the Riemann \( \zeta(s) \) function. We also consider the problem when the eigenvalues are taken to be the prime numbers themselves. In Sec. \[11\] we introduce our numerical method and give evidence that it is capable of generating potentials from sets of discrete energy eigenvalues, such as a finite set of \( \zeta \) zeros or a finite set of prime numbers. We calculate the Rényi dimension

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for these potentials. The results anticipate that these potentials have multi-fractal nature. In section III we attempt to further clarify why previous studies presented contradictory results for the fractal dimension.

II. INVERSE SCATTERING FORMALISM

Provided the Hilbert-Pólya conjecture is true, the natural and plausible approach to finding operator $H$ would be to approximate it from a finite number of eigenvalues. We will follow this path assuming the existence of a local potential $V$ whose spectrum is related to the zeros of the Riemann $\zeta(s)$ function or, later, to the prime numbers.

The Marchenko approach aims to reconstruct a symmetric potential using the characteristics of both the bound states (energy eigenvalues and normalisation constants) and the scattering states (reflection coefficient at all energies). The question of the existence and uniqueness of any solution obtained from the inversion procedures is delicate, although if one assumes a one-dimensional, symmetric potential the complete set of eigenvalues uniquely determines the potential [10]. Different, but mathematically equivalent, methods exist [11] for reconstructing the scattering potential in a one-dimensional quantum mechanical problem.

For a given set of energy eigenvalues and reflection coefficient the quantum potential can be calculated from

$$V(x) = -2 \frac{d^2}{dx^2} \ln(\det(I + C)) \tag{5a}$$

where $I$ denotes identity matrix, and

$$C_{m,n} = \frac{c_{m}^n}{\kappa_m + \kappa_n} e^{-(\kappa_m + \kappa_n)x}, \tag{5b}$$

$$\frac{c_{n}^2}{2\kappa_n} = \sum_{m=1}^{N} \left| \frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n} \right| \tag{5c}$$

Choosing a set of eigenvalues, $\{\kappa_n\}$, one may calculate the corresponding normalisation constants, $\{c_n\}$ and matrix $C$ from equations (5b,c). Substituting this matrix into (5a), the desired potential can be calculated.

We note here briefly, that using the matrix identity, $\ln(\det(M)) = \text{Tr}(\ln(M))$, and the power series expansion of $\ln(1 + x)$, one can symbolically derive the following expression for the potential

$$V(x) = 2 \sum_{r=1}^{\infty} \left( \frac{-1}{r} \right) \text{Tr} \left( \frac{d^2}{dx^2} C^r \right) \tag{6}$$

where $\text{Tr}(M)$ denotes the trace of matrix $M$.

The formulae (5a,c) above form the basis of our calculations. Although these expressions may seem simple, the accurate numerical evaluation of the determinant can prove challenging, particularly as the number of eigenvalues increased.

The inversion technique in its present form is numerically not convenient for more than about five hundred eigenvalues, for two reasons: firstly the matrices involved are dense, and secondly, the numerical precision required is demanding, since for medium values of $x$ one has to calculate the exponential functions in [5a] very accurately to have precise cancellation. The transformation of the formulae (5a,c) into a numerically more tractable form is under investigation.

For this reason, for large sets of eigenvalues, we used the dressing-transformation [13] to calculate the potential. We have checked numerically, up to three hundred energy eigenvalues, that Marchenko’s inversion method and the dressing-transformation give identical results within numerical accuracy. The strength of our approach is in the explicit formulae for the potential construction. Using the dressing-transformation one has to recursively solve ordinary differential equations, since the potential is built up by incorporating the energy eigenvalues one by one, so in every step the solution of the previous step is used. Therefore, the applicability of this method to gain general and analytical results is limited. Contrary, in our method all quantities are expressed in terms of the input parameters, viz. the set of energy eigenvalues, offering some hope of analytic progress.
A. Reconstruction of well-known potentials

To illustrate the method, we reconstruct well-known potentials from their spectra, using the triangular and harmonic oscillator potentials. Later the extended Numerov method [14] is used to calculate the energy eigenvalues, starting with one then two, five and finally one hundred eigenvalues. The table shows the first and the last five eigenvalues. The table also serves to correct the validation of the potentials obtained using the Marchenko approach.

| n   | $\epsilon_{0,n}$ | $\epsilon_{n}$ | $\epsilon_{0,n}$ | $\epsilon_{n}$ |
|-----|------------------|----------------|------------------|----------------|
| 1   | 1.001923         | 1.018793       | 1.015439         | 1.015439       |
| 2   | 3.000020         | 2.338107       | 2.338100         | 2.338100       |
| 3   | 5.000094         | 3.248198       | 3.247152         | 3.247152       |
| 4   | 7.000031         | 4.087949       | 4.087942         | 4.087942       |
| 5   | 9.000069         | 4.820099       | 4.819397         | 4.819397       |

We build up the potentials using a finite number of eigenvalues, starting with one then two, five and finally one hundred eigenvalues from the bottom of the known spectrum. As one may expect, incorporating more and more eigenvalues into the method results in the inverse potential becoming more and more accurate and reproducing the spectrum faithfully. This tendency is clearly captured in Figure (1) for the triangular potential and in Figure (2) for the harmonic potential. Furthermore, the inversion potentials reach their asymptotic value exponentially [15] and this asymptote lies between the last energy eigenvalue used for the inversion and the next eigenvalue. Although the discrepancy between the exact and inversion potentials becomes larger toward the edge of the inversion potential, the energy eigenvalues are still correctly reproduced (see Table I) with tolerable errors.

B. Inversion potential for prime numbers

Using semi-classical arguments, one may show that for a one-dimensional potential the energy eigenvalues cannot increase more rapidly than quadratically, i.e. $\epsilon_n \sim n^2$.

Intuitively this may be seen by noting that in the case of the triangular attractive potential $\epsilon_n$ scales as $n^{2/3}$, while for the harmonic oscillator $\epsilon_n$ goes as $n$, and, as a limiting case, for the infinite-box potential, $\epsilon_n$ varies as $n^2$. A corollary of the Hadamard-Poussin theorem [15] is that the nth prime number is approximately $n \ln(n)$, which is clearly less than $n^2$. We cannot, therefore, rule out the existence of a quantum mechanical potential which has prime numbers as energy eigenvalues.

We now turn to the construction of a semi-classical potential for which the first $n$ energy eigenvalues coincide with the first $n$ prime numbers. There is no theoretical limit on the number of incorporated prime numbers, although numerically the calculation becomes quite cumbersome.

Using the Wentzel-Kramers-Brillouin semi-classical quantisation formula [16] and the leading terms of the prime number counting function, $\pi(E) \approx \frac{E}{\ln(E)}$, one may derive [17] the following implicit equation for a potential of which the eigenvalues are approximately the prime numbers ($E_0 \geq 1$):

$$x(V) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \int_{E_0}^{V} \frac{E^{-1/2}}{\ln(E)\sqrt{V-E}} dE$$

where $E_0$ denotes the reference energy-level. Due to the density of prime numbers, $\rho(E) \approx \ln(E)$, this reference energy cannot be less than 1. Even though the integral, for general $m$, cannot be expressed using elementary functions, one may bound the integral from below and from above such that ($x \gg 1$)

$$x^2 \ln^2(x) < V(x) - E_0 < x^2 \ln^2(x \ln(x)).$$

In Figure 3 we plot the inversion potential calculated from the first two hundred prime numbers and the associated semi-classical potential from equation (7). It is apparent that the inversion potential oscillates around the semi-classical potential except close to the edge of the potential. Similarly, one may solve the Schrödinger equation with the semi-classical potential and with the inversion potential obtained above, comparing how well

| Prime numbers | Riemann $\zeta(s)$ zeros |
|---------------|--------------------------|
| $n$ | $\epsilon_{0,n}$ | $\epsilon_{sc,n}$ | $\epsilon_{n}$ | $\epsilon_{0,n}$ | $\epsilon_{sc,n}$ | $\epsilon_{n}$ |
| 1 | 2 | 0.6895 | 1.6387 | 14.1547 | 13.4690 | 13.0302 |
| 2 | 3 | 2.5316 | 3.0005 | 21.0220 | 23.2274 | 21.0208 |
| 3 | 5 | 5.0674 | 4.7052 | 25.0109 | 29.8790 | 24.7026 |
| 4 | 7 | 7.9717 | 7.0006 | 30.4249 | 36.0644 | 30.4234 |
| 5 | 11 | 11.1201 | 10.702 | 32.9351 | 41.4187 | 32.8091 |

| $n$ | $\epsilon_{0,n}$ | $\epsilon_{sc,n}$ | $\epsilon_{n}$ | $\epsilon_{0,n}$ | $\epsilon_{sc,n}$ | $\epsilon_{n}$ |
| 96 | 503 | 513.8440 | 503.0088 | 229.3374 | 284.3914 | 229.3354 |
| 97 | 509 | 520.3027 | 508.7052 | 231.2502 | 287.1530 | 231.2259 |
| 98 | 521 | 526.7728 | 520.9981 | 231.9872 | 289.9657 | 231.9865 |
| 99 | 523 | 533.2544 | 522.9371 | 233.6934 | 292.8088 | 233.6322 |
| 100 | 541 | 539.7472 | 540.9843 | 236.5242 | 295.7020 | 236.5215 |
FIG. 1: The reference potential, \( V_0(x) = x \) (dashed line), and the inversion potentials (solid lines), \( V(x) \), are shown using (a) one (b) two and (c) five energy eigenvalues indicated with horizontal dashed lines. The lower figure depicts the reference and inversion potential derived using the first fifty energy eigenvalues; the inset illustrates the difference between \( V_0(x) \) and \( V(x) \).

FIG. 2: Figures show the reference potential, \( V_0(x) = x^2 \), (dashed line) and the inversion potential, \( V(x) \), (solid line) using (a) one (b) two and (b) five energy eigenvalues which are indicated with horizontal dashed lines. In the lower panel, the same quantities are presented with the first one hundred energy eigenvalues utilised. The inset depicts \( V_0(x) - V(x) \).

they reconstruct the original set of eigenvalues. Table II comprises a selection from the original set of eigenvalues (\( \epsilon_{0,n} \)), labeled by \( n \), and the energy eigenvalues of the semi-classical (\( \epsilon_{sc,n} \)) and inversion potential (\( \epsilon_n \)) with the same quantum number. This also served as a numerical check of our method. The semi-classical energy eigenvalues capture the trend, but – as expected – those derived from the inversion potentials are much better approximations to the exact eigenvalues. One may notice that the agreement between \( \epsilon_{0,n} \) and \( \epsilon_n \) is consistently much better for even values of \( n \). The same tendency can also be seen in Table II for the triangular and harmonic oscillator potential, although on an order of magnitude smaller scale. The underlying reason for this effect is the subject of ongoing investigation.

C. Inversion potential for \( \zeta(s) \) zeros

Similarly to the semi-classical approximation derived for the prime numbers, one may calculate a potential corresponding to the Riemann \( \zeta(s) \) zeros using the fact that the number of zeros \[15]\[
N(E) = \frac{1}{2\pi} E \ln(E) - \frac{1 + \ln(2\pi)}{2\pi} E + \frac{7}{8} + \mathcal{O}(\ln(E)) \tag{9}
\]
Calculating the average density of the \( \zeta(s) \) zeros from the expression above restricts the choice of the otherwise arbitrary reference energy level to \( E_0 \geq 2\pi \). Inserting the density into the Wentzel-Kramers-Brillouin semi-classical
quantisation formula, we obtain (see 2.727.5 in [18])

\[ x(V) = \frac{1}{\pi} \left[ \sqrt{V - E_0} \ln \left( \frac{E_0}{2\pi e^2} \right) + \sqrt{V} \ln \left( \sqrt{V} + \sqrt{V - E_0} \right) \right], \tag{10} \]

which is identical to that given by Wu and Sprung [19]. The structure of the semi-classical potential close to the origin depends on the choice of the reference energy level. If \( E_0 > 2\pi \) then \( V(x) - E_0 \sim x^2 \), but in case of \( E_0 = 2\pi \) the potential grows as \( V(x) - E_0 \sim x^{2/3} \). As \( |x| \) approaches infinity the potential becomes independent of \( E_0 \) and expression (10) can be inverted to obtain the following asymptotic

\[ V(x) \approx \frac{\pi^2 x^2}{4} \left[ W \left( \frac{\sqrt{\pi} |x|}{e} \right) \right]^{-2}, \tag{11} \]

where \( W(z) \) denotes the Lambert-W function.

Applying the formulae (10c) we calculated a number of potentials supporting the first \( n \) zeros of the \( \zeta(s) \) as energy eigenvalues, utilising a tabulated form of the low-lying zeros [20]. As an example, we have plotted the potential for \( N = 200 \) in Figure 4 and in Table I one can compare how well the energy eigenvalues of the inversion potential coincide with the original eigenvalues, i.e. the zeros of the Riemann \( \zeta(s) \) function. In this case the agreement is even better than it was for the prime numbers. This can be explained by the much slower increase of the potential than that for prime numbers, as \( x \) approaches infinity. Similar effects are seen for the two pedagogical examples in Table I. The energy eigenvalues for the triangular potential are, at least, an order of magnitude more accurate than those for the harmonic potential.

![FIG. 3: The main figure depicts the semi-classical potential (dashed line), \( V_0(x) \), and the inversion potential (solid line), \( V(x) \), derived from using Marchenko’s method with the first two hundred prime numbers. The inset shows the difference of \( V_0(x) \) and \( V(x) \).](image1)

![FIG. 4: Main figure shows the semi-classical potential (dashed line), \( V_0(x) \), and the inversion potential (solid line), \( V(x) \), derived from the inverse scattering method using the first two hundred energy eigenvalues. The inset depicts the difference of \( V_0(x) \) and \( V(x) \).](image2)

**III. COMPARISON WITH EARLIER RESULTS**

Both variational and dressing-transformation techniques, have already been applied to construct quantum mechanical potentials for which the energy eigenvalues are either the zeros of the Riemann \( \zeta(s) \) function [13, 19, 21], or the prime numbers [21]. The common feature of these methods is that the potential is built up in recursion by incorporating more and more eigenvalues into the spectrum.

Previous works [19, 21] estimated the box-counting dimension of the potentials belonging to the prime numbers to be 1.8, and for the Riemann \( \zeta(s) \) zeros to be 1.5, where the number of eigenvalues used ranged from 100 to 32000. Our measurements broadly support these values (see \( D_0 \) in Figure 5).

In order to reproduce these findings we treat the graph of the potential as a signal. To measuring the fractal dimension we de-trend the signal, i.e. subtract the actual inversion potential from the semi-classical potential, \( \xi(x) = V(x) - V_{sc}(x) \). Moreover, we limit ourselves to the spatial range of \([0, 10]\) to eliminate any boundary effect arising from the fact that both Marchenko’s method and the dressing-transformation produce a potential with a constant asymptotic value for large spatial coordinates.

We have measured the Rényi-dimension of the potentials [3, 22], defined as

\[ D_\alpha(X) = \frac{1}{\alpha - 1} \lim_{\epsilon \to 0^+} \left[ \ln \left( \frac{\sum_{i=1}^{N} p_i^\alpha}{\ln(\epsilon)} \right) \right] \tag{12} \]

where \( p_i \) is the probability that the discrete random variable \( X \) falls into a box centered at \( x \), with side \( \epsilon \). This probability can be approximated using the relative frequencies obtained by dividing the embedding two-
dimensional \((x, V)\) space into a finite number of bins and counted how many times the potential takes its value in the given box. Contrary to the general box-counting method, which treats every box equally regardless of how many points of the fractal a given box contains, if \(\alpha > 0\) then boxes with higher relative frequencies will dominate the summand, therefore determining \(D_\alpha(X)\). On the other hand, if \(\alpha < 0\) then the formula weighs the less dense boxes more and measures their scaling properties. As a special case, \(\alpha = 0\) associates equal weights to every box, and therefore \(D_0\) should reproduce the box-counting dimension. It can also be shown \([9]\) that for \(\alpha \to 1\) the numerator including the pre-factor, converges to the Shannon-entropy, defined to be \(-\sum_i p_i \ln(p_i)\). Concluding, by calculating the generalised Rényi dimension one can “scan” the fractal by its “density”, and measure its heterogeneity. If \(D_\alpha\) varies with \(\alpha\) then the fractal is a multi-fractal, since its subsets scale differently.

One may derive two statistics for \(\xi\) based on \([12]\): (a) using a two-dimensional grid and approximating the two-dimensional conditional probability distribution with the relative frequencies of the individual boxes, or (b) calculating the generalised dimension for the marginal probability distribution of \(\xi\).

In Figure 5 both set of statistics are demonstrated showing the generalised dimension for the two-dimensional probability distribution and the insets depicting \(D_\alpha\) for the marginal distribution. The box-counting dimensions, \(D_0\), are also indicated on the graphs. Although both curves in the main figures have an overall “S” shape, their structure is different. In the case of the potential generated from the prime numbers, both the positive and negative \(\alpha\) branches of the curve show monotonic decrease towards the right. However, for the potential designed from the zeros of the Riemann \(\zeta(s)\) function, the negative \(\alpha\) branch of the curve remains nearly constant.

These results suggest that the potentials calculated for the prime numbers and for the zeros of the Riemann \(\zeta(s)\) function are indeed multi-fractals \([23]\). The steep middle part of the curves also explains why earlier studies \([13, 19, 21, 24]\) differed in the box-dimension. The number of incorporated energy eigenvalues strongly influence the conditional probability associated to one box and eventually shift \(D_0\).

Finally, we mention another conjectured property of the quantum system supposed to possess the zeros of the Riemann \(\zeta(s)\) function as energy eigenvalues, namely that \(\mathcal{H}\) may violate the time-reversal symmetry.

Similarly to the prime counting function, \(\pi(x)\), one may define a function, \(N(t)\), which counts the zeros of the Riemann \(\zeta(s)\) function, i.e. a function jumping by unity whenever \(t\) passes over of the zeros, \(\rho_n\). It is proven \([15]\) that the function \(N\) can be decomposed into a smooth and fluctuation part: \(N(t) = N(t) + N_{\text{osc}}(t)\), where \(N(t)\) has been given in \([10]\) explicitly. The fluctuating term has remarkably similar structure to Gutzwiller’s trace formula \([23]\) giving the density of states of a quantum system. The comparison of the two formulae led to the hypothesis \([20]\) that a quantum system with the zeros of the Riemann \(\zeta(s)\) function as energy eigenvalues, does not possess time-reversal symmetry. The approach presented in this paper creates a symmetric, one-dimensional, although multi-fractal potential, \(V(x)\), for which the corresponding energy eigenvalues coincide with the first \(n\) non-trivial zeros of the Riemann \(\zeta(s)\) function. The reflection symmetry of the potential, \(V(x) = V(-x)\), guarantees time-reversal symmetry of the Hamiltonian, \(\mathcal{H} = p^2/2m + V(x)\). This result, therefore, allows us to assume the existence of a quantum system having the \(\zeta\) zeros as energy eigenvalue and obeying the time-reversal symmetry simultaneously.
IV. CONCLUSION

In the present paper we used Marchenko’s method, one of the inverse scattering methods, to construct one-dimensional, symmetric quantum potentials, the energy eigenvalues of which coincide with either the prime numbers or the zeros of the Riemann \(\zeta(s)\) function. We have demonstrated the accuracy and usefulness of this method on two pedagogical examples, the triangular and harmonic potentials. For both cases we showed the reconstructed potentials and calculated the energy eigenvalues, which agreed with the predescribed values very well. Later, we applied the same technique and numerically calculated potentials for the prime numbers and zeros of \(\zeta(s)\). We found that the outcome of the Marchenko method is identical to that of the dressing-transform used previously. At the present stage, the latter method is numerically preferable to the Marchenko’s method. Using the dressing-transform, we created potentials, to high accuracy, from the first one-hundred thousand prime numbers and also for the same number of zeros of the Riemann \(\zeta(s)\) function. Looking at the graphs of these potentials as signals one can analyse their statistical properties. After de-trending these signals we calculated the Rényi-dimension, which is a generalised fractal dimension. Our results suggest that inversion potentials are multi-fractals for both the prime numbers and for the zeros of \(\zeta(s)\). The specific values of the generalised dimension for the prime numbers \(D_0 = 1.808\), and for the \(\zeta(s)\) zeros, \(D_0 = 1.458\), agree well with [21].

Even though Marchenko’s method is not yet able to compete with the dressing-transform in the number of eigenvalues incorporated into the potential, it gives explicit formulae for how one can build up such potentials without recursion. Looking at formula (5a) one may see that the determinant of the matrix \(I + C\) is a polynomial of the entries, i.e. of exponential functions. Taking the natural logarithm and differentiating twice with respect to the spatial variable will not change the fact the potential is a rational function of a finite number of exponentials. This fact is more apparent in formula (9). This gives us hope to be able to explore analytically the properties of a quantum system which possess the zeros of the Riemann \(\zeta(s)\) function as energy eigenvalues.

Simplification of formulae (5m-c) is under investigation.

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