TRIANGULAR PRISM EQUATIONS AND CATEGORIFICATION

ZHENGWEI LIU, SEBASTIEN PALCOUX, AND YUNXIANG REN

Abstract. We present the triangular prism equations for fusion categories, demonstrating their equivalence to the pentagon equations in the spherical case, modulo a change of basis. These equations offer valuable insights for managing complexity through localization. Additionally, we establish an enhanced version of Z. Wang’s conjecture regarding the second Frobenius-Schur indicator in pivotal fusion categories. As applications, we introduce new categorification criteria and complete the classification of unitary 1-Frobenius simple integral fusion categories up to rank 8 and up to $\text{FPdim} 20000$.

1. Introduction

The concept of a fusion ring was first introduced by Lusztig in [25], where it is referred to as a based ring; further details can be found in [9]. The categorical counterpart of a fusion ring, known as a fusion category, is comprehensively and systematically explored by Etingof, Nikshych, and Ostrik in [7]. In category theory, the analogue of associativity is encapsulated by the pentagon equations (PE) as described in [26]. The Grothendieck ring of a fusion category is a fusion ring considered categorifiable. Conversely, the categorification (if any) of a fusion ring is represented by a collection of F-symbols (quantum analogues of the 6j-symbols associated with $SU(2)$) that must satisfy the PE (see [5]).

A central problem in this field is determining which fusion rings are categorifiable. While solving the PE theoretically provides a solution, practically it proves to be exceedingly challenging. Typically, we circumvent the direct computation of F-symbols in the study of fusion categories. Most fusion categories are developed through alternative methods, often leaving their F-symbols undetermined.

Nonetheless, explicitly calculating F-symbols is crucial. For instance, from a spherical fusion category $\mathcal{C}$, one can derive a 3-manifold invariant known as the Turaev-Viro 3D topological quantum field theory (TQFT) [35]. In this context, 1-cells correspond to simple objects, 2-cells to morphisms, and the values of 3-cells, or tetrahedra, are given by F-symbols. The pentagon equation corresponds to the 3-cocycle condition. The spherical categorification of a fusion ring is equivalent to constructing this TQFT, which, in turn, equates to solving the PE using variables provided by spherically invariant F-symbols. The PE generally take the following form, where $\star$ denotes an F-symbol, $\sum_s$ sums over all simple objects, and $\sum_m$ sums over a basis of certain hom-spaces.

\[
\sum_m \star^\star = \sum_s \sum_m \star^\star^\star,
\]

The smallest planar trivalent graphs are the tetrahedral and triangular prism graphs, as illustrated below.

In §4.1 and §4.2 we introduce monoidal category versions of these graphs, labeled with morphisms. In §4.3 we demonstrate that they retain their customary symmetries under certain additional conditions, utilizing an oriented graphical calculus (refer to §2.2). Within a fusion category, such a triangular prism can be evaluated using these tetrahedra in two distinct ways, leading to the triangular prism equations (TPE) presented in this paper. This idea has been previously noted in graph theory, as seen in [34, Figure 4] by D. Thurston:

The tetrahedra in these equations represent F-symbols and serve as variables. In §5.2 we establish the following result:

**Theorem 1.1.** In the spherical case, the pentagon equations are equivalent to the triangular prism equations.

The precise formulation of this result, involving a change of basis, is provided in Theorem 5.2. The large number of variables and equations typically makes them challenging to solve. For instance, determining a Gröbner basis for the pentagon equations can already exhibit double exponential growth in complexity. However, the aforementioned equivalence provides valuable insights into managing the localization strategy outlined in §6.1. An initial iteration of this strategy, incorporating insights from the TPE, has been developed into a new criterion as described in §6.2. This led...
beforehand to prove, in §3.3 an enhanced version of a conjecture by Z. Wang [37] Conjecture 4.26] regarding the second Frobenius-Schur indicator $\nu_2$ (see also Remark 3.9):

**Theorem 1.2.** Let $C$ be a pivotal fusion category. Consider objects $X,Y$ in $C$ where $Y$ is simple, $Y \cong Y^*$, and $\text{hom}_C(X^* \otimes X,Y)$ is odd-dimensional. Then $\nu_2(Y) = 1$.

In §3.3 the use of PE with a small spectrum leads to the formulation of two new general categorification criteria (applicable over any field), termed the zero and one spectrum criteria.

In [24], Wu and two authors of this paper applied the Schur product theorem from subfactor theory ([20] Theorem 4.1) as an effective unitary categorification criterion, referred to as the Schur product criterion. For a more recent enhancement, see the primary 3-criterion in [14]. An exhaustive classification of all 1-Frobenius simple integral fusion rings, within certain specified bounds, was presented in [24]. These bounds have since been updated in [2] as follows:

| Rank | FPdim |
|------|-------|
|       | 10^4  | 10^5  | 10^6  | 20000 | 100000 | 50000  | 30000  | 10000  |

There are exactly 505 non-pointed examples, including 8 that are character rings of groups, while 477 are excluded by the primary 3-criterion. Among the remaining 20 fusion rings, only two have a rank of at most 8. In [24 Question 1.1], it is queried whether these two, denoted as $F_{210}$ and $F_{660}$, can be endowed with a unitary categorification. This paper answers negatively, specifically demonstrating in §7 that the localization criterion (from §6.2) excludes any categorification of $F_{210}$ in characteristic 0, and also in positive characteristic in the pivotal case via lifting theory. See also Remark 7.3.

Furthermore, using the small-spectrum criteria (from §6.3), it is shown that $F_{660}$ admits no categorification over any field, irrespective of its characteristic. See also Remark 7.2 for updates on the next open candidate, of rank 9. We conclude:

**Theorem 1.3.** A non-pointed unitary 1-Frobenius simple integral fusion category, up to rank 8 and up to FPdim 20000, is Grothendieck equivalent to $\text{Rep}(\text{PSL}(2,q))$ where $4 \leq q \leq 11$ and $q$ is a prime power.

A conceptual reason why typical criteria cannot exclude $F_{210}$ is worth mentioning. The character table of $\text{Rep}(\text{PSL}(2,q))$ can be understood globally, depending only on $q$ being a prime power (see, for example, [11 §5.2] or [6 §12.5]). This table can be interpolated to any integer $q > 1$, and by applying the Schur orthogonality relations, one obtains an infinite family of simple integral fusion rings (simple for $q \geq 4$), which corresponds to the Grothendieck ring of $\text{Rep}(\text{PSL}(2,q))$ for $q$ a prime power, but are otherwise new and inherit all the favorable arithmetic properties (see [22]).

Kaplansky’s 6th conjecture [19] posits that for every finite-dimensional semisimple Hopf algebra $H$ over $\mathbb{C}$, the integral fusion category $\text{Rep}(H)$ is 1-Frobenius. The extension to all fusion categories over $\mathbb{C}$ remains open [8 Question 1]. If $H$ has a $*$-structure (i.e., it is a Kac algebra), then $\text{Rep}(H)$ is unitary. For an initial step in the proof of this conjecture, see [18 Theorem 2]. We hope that employing TPE will allow for the discovery of a unitary 1-Frobenius simple integral fusion category that is not group-like, particularly within the interpolated family mentioned above.

An upcoming paper [23] utilizes TPE to derive a slight variation of Izumi’s equations for the near-group categories $G+\{G\}$ using a purely categorical approach, including the non-unitary case, and it determines the exact solutions for $C_n+n$ up to $n = 15$.

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   4.1. Monoidal tetrahedron
2. Graphical calculus

The field \( k \) will always be assumed to be algebraically closed when necessary. Let us revisit some fundamental concepts from the theory of monoidal categories, as detailed in [9], along with graphical calculus. According to Mac Lane’s strictness theorem, we can, without loss of generality, assume that the monoidal categories are strict, which generally facilitates graphical calculus.

2.1. **Unoriented graphical calculus.** Consider a monoidal category \( C \) with a unit object \( 1 \). A left dual of an object \( X \) in \( C \) consists of an object \( X^* \) and two maps, namely the evaluation map \( ev_X : X^* \otimes X \to 1 \) and the coevaluation map \( coev_X : 1 \to X \otimes X^* \), which are depicted graphically as follows:

\[
ev_X = \quad \text{and} \quad coev_X = \quad \]

and satisfying the zigzag relations

\[
\begin{align*}
X & \quad = \quad \text{id}_X \quad \text{and} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quoi
The second equality can be proved similarly.

**Lemma 2.2.** Let $\mathcal{C}$ be a monoidal category with left duals. Then $(ev_X)^* = coev_X^*$ and $(coev_X)^* = ev_X^*$. 

**Proof.** Recall that we can take $1^* = 1$ and $(X \otimes Y)^* = Y^* \otimes X^*$, so by a zigzag relation:

$$
(ev_X)^* = 1 \xymatrix{ & |(ev_X) \ar[u] \ar[dr] & \\
X^* \otimes X'' & & X''}
$$

The second equality has a similar proof. □

Observe that if $f^{**} := (f^*)^*$, then $(ev_X)^{**} = ev_X^{**}$ and $(coev_X)^{**} = coev_X^{**}$.

**Definition 2.3 (Pivotal).** A monoidal category $\mathcal{C}$ with left duals is called pivotal if there is an isomorphism of monoidal functors between $id_{\mathcal{C}}$ and $(_\otimes)^{**}$, i.e. there is a collection of isomorphisms (pivotal structure) $a_X : X \rightarrow X^{**}$ such that for all objects $X, Y$ in $\mathcal{C}$ and for all morphism $f$ in $\text{hom}_\mathcal{C}(X, Y)$ then

$$
a_{X \otimes Y} = a_X \otimes a_Y \quad \text{and} \quad f = a_Y^{-1} \circ f^{**} \circ a_X,
$$

which is depicted as follows

![Diagram](image)

Recall that a pivotal monoidal category is rigid, i.e. its objects have left and right duals (where a right dual of an object $X$ in a monoidal category $\mathcal{C}$ is the data of an object $^*X$ together with two maps $ev'_X : X \otimes ^*X \rightarrow 1$ and $coev'_X : 1 \rightarrow ^*X \otimes X$ satisfying the zigzag relations).

**Lemma 2.4.** Let $\mathcal{C}$ be a monoidal category. Let $X, Y$ be objects in $\mathcal{C}$. Let $\alpha$ be a morphism in $\text{hom}_\mathcal{C}(Y, X^*)$ and let $\beta$ be a morphism in $\text{hom}_\mathcal{C}(1, X)$. Then the following equality holds

$$
\begin{align*}
\beta^* \circ \alpha^* & = \beta^{**} \circ \alpha^* \\
\alpha & = \alpha^* \circ \beta^*
\end{align*}
$$

Moreover if $\mathcal{C}$ is pivotal, with pivotal structure $a$, then

![Diagram](image)

**Proof.** By zigzag relation

$$
\begin{align*}
\beta^* \circ \alpha^* & = \beta^{**} \circ \alpha^* \\
\alpha & = \alpha^* \circ \beta^*
\end{align*}
$$

Now if $\mathcal{C}$ is pivotal then $\beta^{**} = a_X \circ \beta$ (using that $a_1 = \text{id}_1$), so

$$
\begin{align*}
\beta^{**} & = a_X \circ \beta \\
\beta & = \beta \circ a_X
\end{align*}
$$

□
Lemma 2.5. Let $C$ be a pivotal monoidal category. Let $a$ be the pivotal structure. Let $X$ be an object in $C$. Then

$$X^* \xrightarrow{a} X^{**} = X^* \xrightarrow{X^* a} X^*$$

and

$$X^{**} \xrightarrow{X^{**} a} X^* = X^{**} \xrightarrow{a^{-1} X^{**} a} X^{**}.$$

Proof. By Lemma 2.2 $ev_X = ev_X^{**}$ and by pivotal structure $ev^{**} \circ a_{X^* \otimes X} = ev_X$ and $a_{X^* \otimes X} = a_{X^*} \otimes a_X$, so:

$$X^* \xrightarrow{a} X^{**} = X^* \xrightarrow{a} X^{**} = X^* \xrightarrow{X^* a} X^*$$

The proof of the second equality is similar. □

Lemma 2.6 ([9], Exercice 4.7.9). Following Lemma 2.5, $a^*_{X} = a_{X}^{-1}$ and $(a_{X}^{-1})^* = a_{X^*}^{-1}$, so $a^*_{X} = a_{X^*}^{-1}$.

Proof. By Lemma 2.5 and a zigzag relation:

$$a^*_{X} = a_{X}^{-1} \xrightarrow{X^* a} a_{X}^{-1} \xrightarrow{X^* a} a_{X}^{-1}.$$

We already observed that if $f$ is an isomorphism then $(f^*)^{-1} = (f^{-1})^*$. So,

$$(a_{X}^{-1})^* = (a_{X}^{-1})^{-1} = a_{X^*}^{-1}.$$

Finally, $a^*_{X} = (a_{X}^{-1})^* = a_{X^*}^{-1}$. □

Definition 2.7 (Trace). Let $C$ be a pivotal monoidal category. Let $a$ be the pivotal structure. Let $X$ be an object in $C$. Let $\alpha$ be a morphism in $\text{hom}_C(X, X)$. The trace of $\alpha$ according to $a$ is defined as follows:

$$\text{tr}_a(\alpha) := \begin{array}{c}
\alpha \\
X^* \\
\alpha \\
X^{**}
\end{array}$$

Definition 2.8 (Dimension). Following Definition 2.7 $\dim_a(X) := \text{tr}_a(\text{id}_X)$.

When no confusion is possible, we will simply write $\text{tr}$ and $\dim$ (without $a$).

Lemma 2.9. Following Definition 2.7, assume that $C$ is $k$-linear, $X$ is simple and $\dim(X)$ is nonzero. Then

$$\alpha = \dim(X)^{-1} \text{tr}(\alpha) \text{id}_X.$$

Proof. By Schur’s lemma, $\text{hom}_C(X, X) = k \text{id}_X$, so there is $k \in k$ such that $\alpha = k \text{id}_X$. Then

$$\text{tr}(\alpha) = k \text{tr}(\text{id}_X) = k \dim(X),$$

and so $k = \dim(X)^{-1} \text{tr}(\alpha)$. □

Lemma 2.10. Following Definition 2.7 the following equality holds:

$$\text{tr}(\alpha^*) = \begin{array}{c}
\alpha \\
X^* \\
\alpha \\
X^{**}
\end{array}$$

Proof. By zigzag relation and Lemma 2.5:

$$\text{tr}(\alpha^*) = \begin{array}{c}
\alpha \\
X^* \\
\alpha \\
X^{**}
\end{array} = \begin{array}{c}
\alpha \\
X^* \\
\alpha \\
X^{**}
\end{array} = \begin{array}{c}
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\end{array} = \begin{array}{c}
\alpha \\
X^* \\
\alpha \\
X^{**}
\end{array}$$
**Definition 2.11 (Spherical).** A pivotal monoidal category $C$ is called spherical if for any object $X$ in $C$, and for any morphism $\alpha$ in $\text{hom}_C(X, X)$, then $\text{tr}(\alpha) = \text{tr}(\alpha^*)$.

**Remark 2.12.** By sphericality, for any object $X$, then $\dim(X) = \dim(X^*)$, since $\text{tr}(\text{id}_X) = \text{tr}(\text{id}_{X^*}) = \text{tr}(\text{id}_X)$. In a tensor category, these equalities are sufficient to get the sphericality, even when assuming that the objects $X$ are simple, as established in [9] Theorem 4.7.15.

**Lemma 2.13.** Let $C$ be a spherical monoidal category, with pivotal structure $\alpha$. Let $X,Y$ be objects in $C$. Let $\alpha$ be a morphism in $\text{hom}_C(1,Y^* \otimes X^*)$. Let $\beta$ be a morphism in $\text{hom}_C(1,X \otimes Y)$. Then

\[
\begin{align*}
\alpha \otimes \beta &= \left( a_X \alpha \right) \circ \left( \beta a_Y^* \right) = \left( a_Y^* \beta \right) \circ \left( \alpha^{-1} a_X \right),
\end{align*}
\]

where $a^2 : \text{id}_C \to (\_)^{****}$ is the natural isomorphism, square of the pivotal structure $\alpha$.

*Proof.* By zigzag relations and $a_X \circ a_X^{-1} = \text{id}_X$:

\[
\begin{align*}
\alpha \otimes \beta &= \left( a_X \alpha \right) \circ \left( \beta a_Y^* \right) = \left( a_Y^* \beta \right) \circ \left( \alpha^{-1} a_X \right),
\end{align*}
\]

By sphericality and Lemma 2.10 applied to the component in $\text{hom}_C(X, X)$:

\[
\begin{align*}
\alpha \otimes \beta &= \left( a_X \alpha \right) \circ \left( \beta a_Y^* \right) = \left( a_Y^* \beta \right) \circ \left( \alpha^{-1} a_X \right),
\end{align*}
\]

The first equality follows by applying Lemma 2.5 (several times). We get the third picture with a similar argument. $\square$

Recall that since we are using graphical calculus, the monoidal categories are implicitly assumed to be strict.

**Definition 2.14 ([25], Definition 3.1).** Let $C$ be a $k$-linear pivotal monoidal category. Let $\alpha$ be the pivotal structure. Let $X$ be an object in $C$. Let $E_X^{(n)}$ be the $k$-linear map from $\text{hom}_C(1,X^\otimes n)$ to itself (with $n \geq 1$) defined by

\[
E_X^{(n)}(\alpha) := \underbrace{a_X \alpha \cdots a_X \alpha}_{n \times C}
\]

The $n$-th Frobenius-Schur indicator of $X$ is $\nu_n(X) := \text{Tr}(E_X^{(n)})$, where $\text{Tr}$ is the matrix trace.

Note that if $\text{hom}_C(1,X^\otimes n)$ is one-dimensional then $E_X^{(n)}(\alpha) = \nu_n(X)\alpha$.

**Proposition 2.15 ([25], Theorem 5.1).** Let $C$ be a $k$-linear pivotal monoidal category. Let $X$ be an object in $C$ such that $\text{hom}_C(1,X^\otimes n)$ is one-dimensional. Then $\nu_n(X)^n = 1$.

*Proof.* Let $\alpha$ be the pivotal structure. The idea is to evaluate $A := (E_X^{(n)})^{\otimes n}(\alpha)$ in two different ways, with $\alpha$ a nonzero morphism in $\text{hom}_C(1,X^\otimes n)$. On one hand observe that $A = a_{X^\otimes n}^{-1} \circ \alpha^{**} = \alpha$ by pivotal structure, whereas on the other hand $A = \nu_n(X)^n \alpha$. Then $\alpha = \nu_n(X)^n \alpha$, so $\nu_n(X)^n = 1$. $\square$

**Lemma 2.16.** Let $C$ be a $k$-linear pivotal monoidal category. Let $\alpha$ be the pivotal structure. Let $X$ be an object in $C$ such that $\text{hom}_C(X \otimes X,1)$ is one-dimensional. Then for any morphism $\kappa$ in $\text{hom}_C(X,X^*)$, the following equality holds:

\[
ev_X \circ (a_X \otimes \kappa) = \nu_2(X^*)\ev_X \circ (\kappa \otimes \text{id}_X),
\]

which is depicted as

\[
\begin{align*}
(a_X \otimes \kappa) &= \nu_2(X^*)
\end{align*}
\]

By zigzag relation, it is equivalent to

\[
\kappa \circ a_X = \nu_2(X^*)\kappa.
\]
In particular,
- if $X = X^*$ and $\kappa = \text{id}_X$ then $a_X = \nu_2(X)\text{id}_X$,
- if $X = X^{**}$ and $a_X = \pm \text{id}_X$ then $\kappa^* = \pm \nu_2(X^*)\kappa$.

Proof. Consider $Y := X^*$ and the morphism $\iota := \kappa^*$ in $\text{hom}_C(Y^*, Y)$. Let $\alpha := (\text{id}_Y \otimes \iota) \circ \text{coev}_Y$ in $\text{hom}_C(1, Y \otimes Y)$. Let us evaluate $E_{Y}^{(2)}(\alpha)$ by two ways. On one hand $E_{Y}^{(2)}(\alpha) = \nu_2(Y)\alpha$ because $\text{hom}_C(1, Y \otimes Y)$ is one-dimensional. On the other hand, by zigzag relation

$$E_{Y}^{(2)}(\alpha) = \begin{tikzpicture}[baseline=0.5cm]
    
    \begin{scope}
    \begin{scope}[xshift=1cm]
    \node at (0.5,0) {$\iota$};
    \end{scope}
    \node at (0.5,1) {$\alpha a_Y^{-1}$};
    \node at (1.5,1) {$\kappa$};
    \draw (0,0) -- (0,1) -- (1,1); \node at (1.5,0) {$\alpha^{-1}$};
    \draw (1,0) -- (1,1) -- (0,1);
    \end{scope}
    \node at (0.5,2) {$\iota\alpha^{-1}$};
    \node at (1.5,2) {$\alpha^{-1}$};
    \draw (0,2) -- (1,1); \node at (2,1) {$\nu_2$};
    \draw (1,2) -- (0,1);
    \end{tikzpicture} = (\iota \otimes a_Y^{-1}) \circ \text{coev}_Y = (\iota \otimes a_Y^{-1}) \circ \alpha \circ \kappa = \nu_2(Y)\alpha.$$

Then $(\kappa^* \otimes a_X^{-1}) \circ \text{coev}_X = \nu_2(X^*)(\text{id}_{X^*} \otimes \kappa^*) \circ \text{coev}_X$, and by applying * to this equality, together with Lemmas 2.2 and 2.6, the result follows. □

Remark 2.17. If $a_X = \text{id}_X$, $X \cong X^*$ and $\nu_2(X) = -1$ then $X^* \neq X$, otherwise take $\kappa = \text{id}_X$ in Lemma 2.16 to get a contradiction.

Proposition 2.18. Let $C$ be a pivotal fusion category. Let $a$ be the pivotal structure. Let $X$ be a simple object in $C$ such that $X^* \cong X$. Then $\nu_2(X)^2 = 1$. If moreover $X^* = X$ then $a_X = \nu_2(X)\text{id}_X$; in particular, $a_X^2 = \text{id}_X$, where $a^2 : \text{id}_C \to (\_)^*$ is the natural isomorphism, square of the pivotal structure $a$.

Proof. Clearly, $\text{hom}_C(1, X \otimes X)$ is one-dimensional, so $\nu_2(X)^2 = 1$ by Proposition 2.15. If $X^* = X$, we can apply Lemma 2.16 with $\kappa = \text{id}_X$, to get that $a_X = \nu_2(X)\text{id}_X$. Now, $a_X^2 = a_X \circ a_X$, but $X = X^* = X^{**}$, so $a_X^2 = \nu_2(X)^2\text{id}_X = \text{id}_X$. □

2.2. Oriented graphical calculus. This subsection was inspired by [1]. It will not be used before §4.3. Let $C$ be a pivotal fusion category. Let $a$ be the pivotal structure. Let $X$ be an object in $C$. We will define oriented (co)evaluation maps $(co)e_{v_X, \pm}$. First,

$$(co)e_{v_X, +} := ev_X \in \text{hom}_C(X \otimes X, 1) \quad \text{and} \quad (co)e_{v_X, -} := coev_X \in \text{hom}_C(1, X \otimes X^*)$$
as defined in §2.

Next,

$$(co)e_{v_X, -} := ev_X \circ (a_X \otimes \text{id}_{X^*}) = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture} \in \text{hom}_C(X \otimes X^*, 1),$$

$$(co)e_{v_X, +} := \text{id}_{X^*} \circ a_X^{-1} \circ coev_X = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture} \in \text{hom}_C(1, X^* \otimes X).$$

Let us depict these morphisms as follows

$$(co)e_{v_X, +} = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}, \quad (co)e_{v_X, -} = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}$$

Let us also use the following notations:

$$(x) := \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X^*$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture} \quad \text{and} \quad (x^*) := \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X^*$};
    \node at (1.5,1) {$X$};
    \node at (0.5,0) {$X^*$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X^*$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}.$$

Lemma 2.19 (Oriented zigzag relations). Following the conditions and notations in §2.2

$$(x) = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X^*$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}, \quad \text{and} \quad (x^*) = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X^*$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X^*$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X^*$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}$$

Proof. The first and last equalities are the former zigzag relations (mentioned in §2). About the two other ones:

$$(x) = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X$};
    \node at (1.5,1) {$X$};
    \node at (0.5,0) {$X$};
    \node at (0.5,1) {$X^*$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}, \quad \text{and} \quad (x^*) = \begin{tikzpicture}[baseline=0.5cm]
    \node at (1.5,0) {$X^*$};
    \node at (1.5,1) {$X^*$};
    \node at (0.5,0) {$X^*$};
    \node at (0.5,1) {$X$};
    \draw (0,0) -- (1,1); \node at (1.5,0) {$X^*$};
    \draw (1,0) -- (1,1);
    \end{tikzpicture}.$$
Lemma 2.20 (Pictorial dimension). Following the conditions and notations in §2.2

\[ x = \dim(X) \quad \text{and} \quad x^* = \dim(X^*). \]

Proof. By Definitions 2.7, 2.8 and Lemma 2.10

\[ x = \dim(X) = \text{tr}(\text{id}_X) = \dim(X^*). \]

Lemma 2.21. Following the conditions in §2.2 \((ev_{X,+})^* = \text{coev}_{X^*,-}\) and \((\text{coev}_{X,-})^* = ev_{X^*,+}\).

Proof. The statement with \(+\) everywhere is exactly Lemma 2.2. Next, by Lemma 2.6

\[ (ev_{X,-})^* = (ev_X \circ (a_X \otimes \text{id}_{X^*}))^* = (a_X \otimes \text{id}_{X^*})^* \circ (ev_{X^*,-})^* = (\text{id}_{X^*} \otimes a_X^{-1}) \circ \text{coev}_{X^*,-} = \text{coev}_{X^*,+} \]

\[ (\text{coev}_{X,-})^* = ((\text{id}_X \otimes a_X^{-1}) \circ \text{coev}_{X^*})^* = (\text{coev}_{X^*,+})^* \circ ((\text{id}_X \otimes a_X^{-1}))^* = ev_{X^*,-} \circ (a_X \otimes \text{id}_{X^*}) = ev_{X,-} \]

Remark 2.22. Let \(X\) be a simple object in a pivotal fusion category. By [2] Proposition 4.8.4, \(\dim(X)\) is nonzero. Then, by Lemmas 2.19 and 2.20, the choice of the four above morphisms is unique up to a scalar \(k \in k^*,\) i.e.

\[ k \cdot ev_{X,+}, k \cdot ev_{X,-}, k^{-1} \cdot \text{coev}_{X,+}, k^{-1} \cdot \text{coev}_{X,-}. \]

Lemma 2.23. Following the conditions in §2.2 let \(X\) be a simple object.

- if \(X^* = X\) then \(ev_{X,-} = \nu_2(X)ev_{X,+}\) and \(\text{coev}_{X,-} = \nu_2(X)\text{coev}_{X,+}\).
- if \(X^{**} = X\) and \(a_X = \pm \text{id}_X\) then \(ev_{X,-} = \pm ev_{X^*,-}\) and \(\text{coev}_{X,-} = \pm \text{coev}_{X^*,+}\)

Proof. The first equality of each case follows by Lemma 2.16. Then, apply Lemma 2.21 to get the second equality. \(\square\)

3. Preliminaries on fusion categories

This section establishes several preliminary results on fusion categories related to bilinear forms, the resolution of identity, and the proof of Wang’s conjecture.

Remark 3.1 (Conventions about the unit). Let \(C\) be a fusion category. Recall that \(\text{End}_C(\mathbf{1}) = \text{id}_1\). The morphism \(\text{id}_1\) will sometimes be identified with the scalar \(k \in k^*\). Moreover, any object isomorphic to \(\mathbf{1}\) will be identified with \(\mathbf{1}\) (i.e. \(X \cong \mathbf{1}\) implies \(X = \mathbf{1}\) in this paper), in particular \(\mathbf{1}^* = \mathbf{1}\).

3.1. Bilinear forms.

Lemma 3.2. Let \(C\) be a fusion category. Let \(Z\) be an object in \(C\). Consider the bilinear form

\[ b(\alpha, \beta) := ev_Z \circ (\alpha \otimes \beta) = \begin{array}{c} \alpha \\ z' \\ \beta \\ z \end{array} \]

where \((\alpha, \beta) \in \text{hom}_C(\mathbf{1}, Z^*) \times \text{hom}_C(\mathbf{1}, Z)\). Then there are bases \((e'_i)_{i \in I}\) of \(\text{hom}_C(\mathbf{1}, Z^*)\) and \((e_j)_{j \in J}\) of \(\text{hom}_C(\mathbf{1}, Z)\) such that \(b(e'_i, e_j) = \delta_{i,j}\), and for all \(\alpha, \beta \in \text{hom}_C(\mathbf{1}, Z^*) \times \text{hom}_C(\mathbf{1}, Z)\), \(\alpha = \sum_{i \in I} b(\alpha, e_i)e'_i, \beta = \sum_{j \in J} b(e'_j, \beta)e_j\), and so

\[ b(\alpha, \beta) = \sum_{i \in I} b(\alpha, e_i)b(e'_i, \beta). \]

In particular, the bilinear form \(b\) is non-degenerate.

Proof. First \(C\) is semisimple, so \(Z = \oplus_{i \in I} Z_i\) with \(Z_i\) simple object, but \(C\) is additive thus \(\alpha = \oplus_{j \in J} \alpha_j, \beta = \oplus_{j \in J} \beta_j\) and \(ev_Z = \oplus_{j \in J} ev_{Z_j}\), where \((\alpha_j, \beta_j) \in \text{hom}_C(\mathbf{1}, Z^*_j) \times \text{hom}_C(\mathbf{1}, Z_j)\). Then

\[ b(\alpha, \beta) = \sum_{j \in J} ev_{Z_j} \circ (\alpha_j \otimes \beta_j), \] \[ \text{and so } b(\alpha, \beta) = \delta_{i,j}ev_{Z_j} \circ (\alpha_j \otimes \beta_j), \]

with \((\tilde{\alpha}_i, \tilde{\beta}_j) \in \text{hom}_C(\mathbf{1}, Z^*) \times \text{hom}_C(\mathbf{1}, Z)\) such that \((\tilde{\alpha}_i)_{j} = \delta_{i,j}\alpha_i\) and \((\tilde{\beta}_j)_{i} = \delta_{i,j}\beta_i\). But if \(Z_j\) is not equal to the unit object, then \(\alpha_j = \beta_j = 0\) (by Schur’s lemma). Let \(I\) be the set \(\{j \in J \mid Z_j = \mathbf{1}\}\). Let \(i \in I\) and let \((e'_i, e_j) \in \text{hom}_C(\mathbf{1}, Z^*_j) \times \text{hom}_C(\mathbf{1}, Z_j)\) such that \((e'_i)_{j} = (e_j)_{i} = \delta_{i,j}\text{id}_1\). Now, \(ev_1 \circ (\text{id}_1 \otimes \text{id}_1) = 1\), so for all \(i, j \in I, b(e'_i, e_j) = \delta_{i,j}\). Note that \(|I| = \dim_k(\text{hom}_C(\mathbf{1}, Z^*)) = \dim_k(\text{hom}_C(\mathbf{1}, Z))\), so by bilinearity, \((e'_i)_{i \in I}\) is a basis of
hom\(_C(1, Z^*)\) and \((e_j)\) is a basis of hom\(_C(1, Z)\). Thus for all \((\alpha, \beta) \in \text{hom}_C(1, Z^*) \times \text{hom}_C(1, Z)\), \(\alpha = \sum_{i \in I} b(\alpha, e_i) e_i'\), \(\beta = \sum_{i \in I} b(e_i', \beta) e_i\), and so \(b(\alpha, \beta) = \sum_{i \in I} b(\alpha, e_i) b(e_i', \beta)\).

**Remark 3.3.** In Lemma 3.2, the sequence \((e_i')\) serves as the dual basis to \((e_i)\) with respect to the bilinear form \(b\). Here, dual means in the context of vector spaces, meaning that \(e_i'\) is not the same as \(e_i^*\), which refers to the categorical dual of the morphism \(e_i\).

**Lemma 3.4.** Let \(C\) be a fusion category. For any object \(Z\) in \(C\), let \(b_Z\) be the bilinear form and let \((e_i, Z)\) be the basis of \(\text{hom}_C(1, Z)\), mentioned in Lemma 3.2. For all object \(X\) in \(C\) and for all \(i, j\), we have
\[
\begin{align*}
\Delta(x, x', e_j, x) = \sum_{\gamma} b(\epsilon, e_j) & b(\gamma, x) = \delta_{i, j}, \tag{3.3} \end{align*}
\]
where \(I_X\) and \(I_{X^*}\) are identified.

**Proof.** In the proof of Lemma 3.2, the chosen bases \((e_{i, X^*})\) does not depend on whether \(Z = X\) or \(X^*\).

**Lemma 3.5.** Let \(C\) be a pivotal fusion category, with pivotal structure \(a\). Let \(X, Y\) be objects in \(C\). Let \(Z\) be \(X \otimes Y\). Consider the \(k\)-linear maps \(f : \text{hom}_C(1, Z^*) \rightarrow \text{hom}_C(X, Y^*)\) and \(g : \text{hom}_C(1, Z) \rightarrow \text{hom}_C(Y^*, X)\) defined by:
\[
\begin{align*}
f(\alpha) &= \begin{array}{c} \alpha \\ \uparrow X \end{array} \quad \text{and} \quad g(\beta) &= \begin{array}{c} \beta \\ \downarrow X^* \end{array} \tag{3.4} \end{align*}
\]
Then \(\text{tr}(f(\alpha) \circ g(\beta)) = \text{tr}(g(\beta) \circ f(\alpha)) = b(\alpha, \beta)\), with \(b\) from Lemma 3.2.

**Proof.** By Definition 2.7 Lemma 2.5 and then pivotality \((a_Z^{-1} \circ \beta^{**} = \beta)\):
\[
\begin{align*}
\text{tr}(f(\alpha) \circ g(\beta)) &= \begin{array}{c} \beta \\ \downarrow Y^* \\ \uparrow Y \end{array} = \begin{array}{c} \alpha \\ \downarrow X \end{array} = b(\alpha, \beta), \tag{3.5} \\
\text{tr}(g(\beta) \circ f(\alpha)) &= \begin{array}{c} \alpha \\ \downarrow X \end{array} = \begin{array}{c} \beta \\ \downarrow Y \end{array} = b(\alpha, \beta). \tag{3.6} \end{align*}
\]

**Lemma 3.6.** Following Lemma 3.5, consider the bases \((e_i')\) of \(\text{hom}_C(1, Z^*)\) and \((e_i)\) of \(\text{hom}_C(1, Z)\), from Lemma 3.2. Let \(e'_i := f(e_i)\) and \(e_i := g(e_i)\). Then \(\text{tr}(e'_i \circ e_j) = \text{tr}(e_j \circ e'_i) = \delta_{i, j}\), \((e_i')\) is a basis of \(\text{hom}_C(X, Y^*)\) and \((e_i)\) is a basis of \(\text{hom}_C(Y^*, X)\).

**Proof.** Immediate from Lemmas 3.2 and 3.5 and the fact that \(f\) and \(g\) are bijective \(k\)-linear maps.

### 3.2 Resolution of identity

The subsequent results (Lemma 3.7 and Proposition 3.8) are commonly acknowledged within the field. However, for the sake of completeness and to establish our notation, we will provide their proofs here.

**Lemma 3.7.** Let \(A\) be an algebra over a field \(k\), let \(O\) be a finite set, and for each \(x \in O\), let \(I_x\) be a finite set. Consider the elements \((\tau_{x, i, j})_{x \in O, i, j \in I_x}\), which are nonzero elements in \(A\), satisfying the condition
\[
\tau_{x, i, j} \tau_{x', k, \ell} = \delta_{x, x'} \delta_{j, k} d_x \tau_{x, i, \ell},
\]
where \(d_x\) is a nonzero element in \(k\). Then the elements \((\tau_{x, i, j})_{x \in O, i, j \in I_x}\) are linearly independent and generate a \(k\)-subalgebra isomorphic to \(\bigoplus_{x \in O} M_{n_x}(k)\), where \(n_x := |I_x|\).
Proof. Assume that

$$0 = \sum_{x \in \mathcal{O}, i,j \in I_x} \lambda_{x,i,j} \tau_{x,i,j}$$

with $\lambda_{x,i,j} \in \mathbb{k}$. Notice that $\tau_{x',k,\ell} \tau_{x,i,j} \tau_{x',k,\ell} = d_x^2 \delta_{x,x'} \delta_{k,k} \tau_{x',k,\ell}$, thus

$$0 = \tau_{x',k,\ell} \left( \sum_{x \in \mathcal{O}, i,j \in I_x} \lambda_{x,i,j} \tau_{x,i,j} \right) \tau_{x',k,\ell} = \sum_{x \in \mathcal{O}, i,j \in I_x} \lambda_{x,i,j} \tau_{x',k,\ell} \tau_{x,i,j} \tau_{x',k,\ell} = d_x^2 \lambda_{x',k,\ell} \tau_{x',k,\ell},$$

implying $\lambda_{x',k,\ell} = 0$ for all $x' \in \mathcal{O}, k, \ell \in I_{x'}$. The result follows. □

Recall from [9, Proposition 4.8.4] that for any simple object $X$ in a pivotal fusion category, $\dim(X)$ is nonzero.

**Proposition 3.8** (Resolution of identity). Let $\mathcal{C}$ be a pivotal fusion category with pivotal structure $a$. Let $X, T$ be objects in $\mathcal{C}$ with $X$ being simple. Let $\mathcal{O}(T)$ denote the set of simple subobjects of $T$, up to isomorphism. There exist bases $(b_{X,i})_{i \in I_X}$ for $\text{hom}_\mathcal{C}(T, X)$ and $(b'_{X,i})_{i \in I_X}$ for $\text{hom}_\mathcal{C}(X, T)$ such that $\text{tr}(b'_{X,i} \circ b_{X,j}) = \text{tr}(b_{X,j} \circ b'_{X,i}) = \delta_{i,j}$. Define $\tau_{X,i,j} := b'_{X,i} \circ b_{X,j}$, ensuring $\text{tr}(\tau_{X,i,j}) = \delta_{i,j}$. Then

$$\tau_{X,i,j} \circ \tau_{X',k,\ell} = \delta_{X,X'} \delta_{j,k} \dim(X)^{-1} \tau_{X,i,j},$$

and the set $(\tau_{X,i,j})_{X \in \mathcal{O}(T), i,j \in I_X}$ forms a basis of $\text{hom}_\mathcal{C}(T, T)$. For any $\gamma \in \text{hom}_\mathcal{C}(T, T)$, we have:

$$\gamma = \sum_{X \in \mathcal{O}(T)} \sum_{i,j \in I_X} \dim(X) \text{tr}(\gamma \circ \tau_{X,i,j}) \tau_{X,i,j}.$$  

In particular (resolution of identity):

$$\text{id}_T = \sum_{X \in \mathcal{O}(T)} \sum_{i,j \in I_X} \dim(X) \tau_{X,i,i}.$$  

Proof. The bases mentioned initially are ensured by Lemma [3.6] with $Y = *T$. Due to associativity, we have

$$\tau_{X,i,j} \circ \tau_{X',k,\ell} = b'_{X,i} \circ (b_{X,j} \circ b'_{X',k}) \circ b_{X',\ell}.$$

Here, $b_{X,j} \circ b'_{X',k}$ belongs to $\text{hom}_\mathcal{C}(X', X)$, which is $\delta_{X,X'}$-dimensional by Schur’s lemma. Thus, by Lemma [2.9]

$$b_{X,j} \circ b'_{X',k} = \delta_{X,X'} \dim(X)^{-1} \text{tr}(b_{X,j} \circ b'_{X',k}) \text{id}_X = \delta_{X,X'} \dim(X)^{-1} \delta_{j,k} \text{id}_X.$$

Hence,

$$\tau_{X,i,j} \circ \tau_{X',k,\ell} = \delta_{X,X'} \delta_{j,k} \dim(X)^{-1} \tau_{X,i,j}.$$

Since $\tau_{X,i,j}$ is nonzero (because $\tau_{X,i,j} \circ \tau_{X,i,j} = \dim(X)^{-1} \tau_{X,i,j}$ and $\text{tr}(\tau_{X,i,j}) = 1$), Lemma [3.7] confirms that the elements in the set $(\tau_{X,i,j})_{X \in \mathcal{O}(T), i,j \in I_X}$ are linearly independent. Moreover,

$$\dim_k(\text{hom}_\mathcal{C}(T, T)) = \sum_{X \in \mathcal{O}(T)} \dim_k(\text{hom}_\mathcal{C}(T, X))^2 = \sum_{X \in \mathcal{O}(T)} |I_X|^2,$$

which equals the cardinality of the set, thereby establishing it as a basis for $\text{hom}_\mathcal{C}(T, T)$.

For any morphism $\gamma$ in $\text{hom}_\mathcal{C}(T, T)$, there exist coefficients $\lambda_{X,i,j}$ in $\mathbb{k}$ such that

$$\gamma = \sum_{X \in \mathcal{O}(T), i,j \in I_X} \lambda_{X,i,j} \tau_{X,i,j}.$$  

Thus, for all $X' \in \mathcal{O}(T), k, \ell \in I_{X'}$, we have

$$\text{tr}(\gamma \circ \tau_{X',k,\ell}) = \sum_{X \in \mathcal{O}(T), i,j \in I_X} \lambda_{X,i,j} \text{tr}(\tau_{X,i,j} \circ \tau_{X',k,\ell}) = \lambda_{X',k,\ell} \dim(X')^{-1}.$$

Hence, $\lambda_{X,i,j} = \dim(X) \text{tr}(\gamma \circ \tau_{X,i,j})$. Finally,

$$\text{tr}((\text{id}_T \circ \tau_{X,i,j}) = \text{tr}(\tau_{X,i,j}) = \delta_{i,j}.$$  

The result follows. □
3.3. **Proof of Wang’s conjecture.** This subsection is devoted to proving Theorem 1.2 which supports the odd case of a conjecture proposed by Z. Wang [37, Conjecture 4.26]. Contrarily, the even case has been disproven; specifically, Rep(G) where \( G = \text{PSU}(3,2) \) serves as a counterexample (details in \[8,1\]. Remarkably, this is the smallest counterexample emerging from finite group theory, with \(|G| = 72\).

**Remark 3.9.** The following two points address related literature. First, [10, Theorem page L257] discusses a weaker version of Theorem 1.2, applicable only when \( X \) is simple and \( C \) is braided. However, under certain conditions, the braiding requirement in [10] may be unnecessary. Second, our counterexample for the even case was also independently identified in [30, and earlier, 27] had provided another counterexample (of order 128) where \( G = C_2^4 \times Q_8 \) operates faithfully by conjugation.

**Proposition 3.10.** Consider a \( k \)-linear pivotal monoidal category, \( C \), with a pivotal structure denoted by \( a \). Let \( Y \) be an object in \( C \) such that \( \text{hom}_C(Y \otimes Y, 1) \) is one-dimensional. For any nonzero morphism \( \kappa \) in \( \text{hom}_C(Y,Y^*) \), and for any object \( X \) in \( C \), consider the following bilinear map from \( \text{hom}_C(1, X \otimes Y \otimes X^*)^2 \) to \( \text{End}_C(1) \):

\[
\omega(\alpha, \beta) := \put(250,300){\begin{array}{c}
| & \alpha & \beta \\
\alpha & X & Y \\
\beta & Y^* & X^* \\
\end{array}}
\]

Then \( \omega(\alpha, \beta) = \nu_2(Y^*)\omega(\beta, \alpha) \).

**Proof.** By Lemma 2.4 and the equality \( a_{X \otimes Y \otimes X^*} = a_X \otimes a_Y \otimes a_{X^*} \) (from the pivotal structure),

\[
\omega(\alpha, \beta) = \put(250,300){\begin{array}{c}
| & \beta & \alpha \\
\beta & X & Y \\
\alpha & Y^* & X^* \\
\end{array}}
\]

Now by Lemma 2.2 \( (ev_X)^* = ev_{X^*} \), and by pivotal structure, \( (ev_X)^* \circ a_{X \otimes Y} = ev_X \), so \( ev_{X^*} \circ a_{X \otimes Y} = ev_X \). Moreover by Lemma 2.16 \( ev_{Y^*} \circ (a_Y \otimes \kappa) = \nu_2(Y^*)ev_Y \circ (\kappa \otimes \text{id}_Y) \). It follows that

\[
\omega(\alpha, \beta) = \nu_2(Y^*) = \nu_2(Y^*)\omega(\beta, \alpha). \quad \square
\]

**Corollary 3.11.** Let \( C \) be a \( k \)-linear pivotal monoidal category with \( \text{End}_C(1) \simeq k \). Suppose \( X \) and \( Y \) are objects in \( C \) such that \( \text{hom}_C(Y \otimes Y, 1) \) is one-dimensional, and the bilinear form \( \omega \) from Proposition 3.10 is non-degenerate. If \( \text{dim}_C(1, X \otimes Y \otimes X^*) \) has an odd dimension, then \( \nu_2(Y^*) = 1 \).

**Proof.** Initially, if the field \( k \) has characteristic two, then by Proposition 2.15 \( \nu_2(Y^*) = 1 \) since \(-1 \equiv 1 \pmod{2} \). Therefore, we can assume that \( k \) does not have characteristic two. Given that \( \text{End}_C(1) \simeq k \), the map \( \omega \) is a bilinear form. If \( \nu_2(Y^*) = -1 \), then \( \omega \) corresponds to a skew-symmetric matrix \( M \) (i.e., \( M^T = -M \)). However, since \( \det(M^T) = \det(M) \) and \( \det(-M) = (-1)^n \det(M) \), where \( n \) is the assumed odd dimension of \( \text{hom}_C(1, X \otimes Y \otimes X^*) \), it follows that \( \det(M) = -\det(M) \), leading to \( 2 \det(M) = 0 \). Yet, by the non-degeneracy condition, \( \det(M) \) is a nonzero element in \( k \), implying \( 2 = 0 \), which contradicts our assumption that \( k \) is not of characteristic two. Hence, \( \nu_2(Y^*) \neq -1 \), implying \( \nu_2(Y^*) = 1 \) by Proposition 2.15. \( \square \)

**Remark 3.12.** The proof of Corollary 3.11 reaffirms the well-established result that a symplectic vector space over a field with a characteristic not equal to two must be of even dimension. This result is noted without proof in [10, §6.9].

We thus arrive at the proof of Theorem 1.2.

**Proof.** By the natural adjunction isomorphism [9, Proposition 2.10.8], we have \( \dim_k(\text{hom}_C(Y \otimes Y, 1)) = \dim_k(\text{hom}_C(Y, Y^*)) = 1 \) because \( Y^* \simeq Y \) is simple, and \( \dim_k(\text{hom}_C(X^* \otimes X, Y)) = \dim_k(\text{hom}_C(1, X \otimes Y \otimes X^*)) \) as \( X^* \simeq X \). Furthermore, \( \omega(\alpha, \beta) = b(f \circ \alpha, \beta) \), where \( f = a_X \otimes \kappa \otimes \text{id}_{X^*} \) and \( b \) is the bilinear form from Lemma 3.2 with \( Z = X \otimes Y \otimes X^* \), making \( \omega \) non-degenerate by Lemma 3.2 and the fact that \( f \) is an isomorphism. The result follows from Corollary 3.11. \( \square \)
In Theorem 1.2, the object $X$ is not assumed to be simple, which might be advantageous if the Grothendieck ring is noncommutative.

4. MONOIDAL TETRAHEDRON AND TRIANGULAR PRISM

4.1. Monoidal tetrahedron. This subsection introduces the concept of a tetrahedron within a monoidal category and establishes some fundamental results related to the cyclic permutations $(2, 3, 4)$ and $(3, 2, 1)$. These permutations generate the orientation-preserving symmetry group $A_4$ of a standard tetrahedron, which is utilized in §4.3.

Definition 4.1. Let $C$ be a monoidal category with left duals. Let $F$, $G$ and $H$ be functors from $C^3$ to Set defined as the composition of usual functors such that $F(X,Y,Z) = \text{hom}_C(1,X \otimes Y \otimes Z)$, $G(X,Y,Z) = \text{hom}_C(1,Y \otimes Z \otimes X^{**})$ and $H(X,Y,Z) = \text{hom}_C(1,X \otimes Y \otimes Z^{**})$, for all objects $X,Y,Z$ in $C$. Let $\rho : F \to G$ and $\rho' : G \to F$ be natural transformations defined by

$$\rho(\alpha) = \begin{array}{c}
\alpha \\
\end{array} \quad \text{and} \quad \rho'(\beta) = \begin{array}{c}
\beta \\
\end{array}$$

for all $\alpha \in \text{hom}_C(1,X \otimes Y \otimes Z)$ and for all $\beta \in \text{hom}_C(1,Y \otimes Z \otimes X^{**})$. Next, assuming the existence of a pivotal structure $a$, let $\sigma : F \to H$ and $\sigma' : H \to F$ be natural transformations defined by

$$\sigma(\alpha) = (\text{id}_{X \otimes Y} \otimes a_Z) \circ \alpha \quad \text{and} \quad \sigma'(\gamma) = (\text{id}_{X \otimes Y} \otimes a_Z^{-1}) \circ \gamma,$$

for all $\alpha \in \text{hom}_C(1,X \otimes Y \otimes Z)$ and for all $\gamma \in \text{hom}_C(1,X \otimes Y \otimes Z^{**})$.

Lemma 4.2. Following Definition 4.1, $\rho$ and $\sigma$ are natural isomorphisms, with $\rho^{-1} = \rho'$ and $\sigma^{-1} = \sigma'$.

Proof. By zigzag relations

$$\langle \rho' \circ \rho \rangle(\alpha) = \begin{array}{c}
\alpha \\
\end{array} = \begin{array}{c}
\alpha \\
\end{array} = \begin{array}{c}
\alpha \\
\end{array} = \alpha.$$

So $\rho' \circ \rho = \text{id}_F$. Similarly, $\rho \circ \rho' = \text{id}_G$. Finally, we trivially have $\sigma' \circ \sigma = \text{id}_F$ and $\sigma \circ \sigma' = \text{id}_H$. \qed

Remark 4.3. If $C$ is $k$-linear pivotal, and $X = Y = Z$, then $(\sigma^{-1} \circ \rho)(\alpha) = E_X^{(3)}(\alpha)$. So, if moreover $\text{hom}_C(1,X^{\otimes 3})$ is one-dimensional, then $\rho(\alpha) = v_3(X)\sigma(\alpha)$ with $v_3(X)^3 = 1$, by Proposition 2.15.

Definition 4.4 (Monoidal tetrahedron). Let $C$ be a monoidal category with left duals. Let $(X_i)_{i=1,\ldots,6}$ be objects in $C$. Consider morphisms $\alpha \in \text{hom}_C(1,X_1 \otimes X_2 \otimes X_3)$, $\beta \in \text{hom}_C(1,X_1^* \otimes X_3^* \otimes X_2^*)$, $\gamma \in \text{hom}_C(1,X_5 \otimes X_2 \otimes X_6^*)$ and $\delta \in \text{hom}_C(1,X_6 \otimes X_1^* \otimes X_4)$. Let us define the (labeled) monoidal tetrahedron as follows:

$$T(\alpha,\beta,\gamma,\delta) :=$$

$$\begin{array}{c}
\beta \\
\end{array} \quad \begin{array}{c}
\gamma \\
\end{array} \quad \begin{array}{c}
\delta \\
\end{array}$$

Proposition 4.5. Following Definition 4.4

$$T(\alpha,\beta,\gamma,\delta) = T(\rho(\alpha),\delta^{**},\beta,\gamma) = T(\rho^2(\alpha),\gamma^{**},\delta^{**},\beta) = T(\rho^3(\alpha),\beta^{**},\gamma^{**},\delta^{**}) = T(\alpha^{**},\beta^{**},\gamma^{**},\delta^{**}),$$

where $\rho$ is the natural transformation from Definition 4.1.

Proof. We only need to prove the first equality (about the last one, observe that $\rho^3(\alpha) = \alpha^{**}$). Apply Lemma 2.4 to $T(\alpha,\beta,\gamma,\delta)$ with $X = X_6 \otimes X_1^* \otimes X_4$ and $Y = 1$. Then
Note that the equality $T(\alpha, \beta, \gamma, \delta) = T(\alpha^{**}, \beta^{**}, \gamma^{**}, \delta^{**})$ can be proved directly in the pivotal case.

**Proposition 4.6.** Following Definition 4.4, if $\mathcal{C}$ is spherical, then

$$T(\alpha, \beta, \gamma, \delta) = T(\rho^{-3}(\beta), \rho^{-1}(\sigma^2(\gamma)), \rho(\alpha), \rho(\delta))$$

where $\rho$ and $\sigma$ are the natural transformations from Definition 4.4.

**Proof.** By applying Lemma 2.13 with $(X, Y) = (X_6, X_2 \otimes X_3 \otimes X_4)$ we get:

Next, by some zigzag relations:

and by Lemma 2.4 (together with some zigzag relations):

Finally, observe that $^{**}\beta = \rho^{-3}(\beta)$. $\square$
Remark 4.7. The order in which the morphisms appear in the RHS of the first equalities in Propositions 4.5 and 4.6 correspond to the cyclic permutations (2,3,4) and (3,2,1), which generate the alternating group $A_4$:

```gap
g:=SymmetricGroup(4);;
u:=Subgroup(g,[(2,3,4),(3,2,1)]);
u:=AlternatingGroup(4);
```

true

4.2. Monoidal triangular prism and equation. This subsection introduces the concept of a triangular prism within a monoidal category, following the initial representation described in Remark 4.16 of the triangular prism graph. It derives an equation, referred to as the Triangular Prism Equation (TPE), specifically for a fusion category, by evaluating it in two distinct ways.

Definition 4.8 (Monoidal triangular prism). Let $C$ be a monoidal category with left duals. Let $(X_i)_{i=1,\ldots,9}$ be objects in $C$. Consider morphisms $\alpha_i \in \text{hom}_C(1, W_i)$, with $W_1 = X_4 \otimes X_1^* \otimes X_6^{**}$, $W_2 = X_5 \otimes X_2^* \otimes X_4^*$, $W_3 = X_6 \otimes X_3^* \otimes X_5^*$, $W_4 = X_9 \otimes X_1 \otimes X_7^*$, $W_5 = X_7 \otimes X_2 \otimes X_8^*$, $W_6 = X_8 \otimes X_3 \otimes X_9^{**}$. The (labeled) monoidal triangular prism is defined as:

\[
TP(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) := [Diagram]
\]

Remark 4.9. Without going into details (useless here), in the spherical case we can prove some equalities as in §4.1, corresponding to the permutations (1,2,3)(4,5,6) and (1,4)(2,5)(3,6) generating the (orientation-preserving) symmetry group $C_6$ of the triangular prism.

Theorem 4.10 (TPE). Let $C$ be a pivotal fusion category. Let $a$ be the pivotal structure. Let $(X_i)_{i=1,\ldots,9}$ and $(\alpha_i)_{i=1,\ldots,6}$ be objects and morphisms in $C$ as in Definition 4.8. Then

\[
\sum_{\beta_0 \in B_0} \sum_{\beta_i \in B_i} \dim(X) T \left( \rho^{-2}(\alpha_2), \rho(\alpha_3), \rho^{-1}(\alpha_1), \beta_0 \right) T \left( \rho^{-1}(\alpha_5), \beta'_0, \rho(\alpha_4), \rho^{-1}(\alpha_6) \right) = \\
\sum_{X \in \mathcal{O}} \sum_{\beta_i \in B_i} \dim(X) T \left( \rho^{-2}(\sigma(\beta'_0)), \rho(\alpha_3), \rho^2(\beta_1), \rho^{-1}(\alpha_0) \right) T \left( \rho^{-1}(\alpha_4), \alpha_1, \beta'_1, \beta_2 \right) T \left( \rho^{-1}(\alpha_5), \alpha_2, \beta'_2, \beta_3 \right),
\]

where $B_i$ and $B'_i$ are the bases of $\text{hom}_C(1, Z_i)$ and $\text{hom}_C(1, Z'_i)$ from Lemma 3.2 ($x \mapsto x'$ denotes the bijection from $B_i$ to $B'_i$), with $Z_0 := X_1 \otimes X_2 \otimes X_3$, and for $i = 1, 2, 3$, $Z_i := X \otimes Y_i$, where $Y_1 := X_9 \otimes X_6^*$, $Y_2 := X_7 \otimes X_4$, $Y_3 := X_8 \otimes X_5$, and $\mathcal{O}$ is the set of simple subobjects of both $Y_1^*$, $Y_2^*$ and $Y_3^*$ (up to isomorphism). As before, $\rho$ and $\sigma$ are the natural transformations from Definition 4.1.

Proof. The idea is to evaluate $TP := TP(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ with two different ways, providing the LHS and RHS of above equation. On one hand, observe that we can apply Lemma 3.2 (with $Z = Z_0$) to $TP$, and then

\[
TP = \sum_{\beta_0 \in B_0} \sum_{\beta_i \in B_i} \dim(X) T \left( \rho^{-2}(\alpha_2), \rho(\alpha_3), \rho^{-1}(\alpha_1), \beta_0 \right) T \left( \rho^{-1}(\alpha_5), \beta'_0, \rho(\alpha_4), \rho^{-1}(\alpha_6) \right).
\]
so by some zigzag relations

\[ TP = \sum_{\beta_0 \in B_0} \alpha_3 \beta_0 \alpha_1 \beta_0' \alpha_4 \alpha_6 \]

again by some zigzag relations we get that

\[ TP = \sum_{\beta_0 \in B_0} T (\rho^{-2}(\alpha_2), \rho(\alpha_3), \rho^{-1}(\alpha_1), \beta_0) T (\rho^{-1}(\alpha_5), \beta_0', \rho(\alpha_4), \rho^{-1}(\alpha_6)) = LHS \]

On the other hand, we can apply Proposition 3.8 (resolution of identity) on \( TP \) three times, with \( T = T_j = Y_j^* \), \( j = 1, 2, 3 \), \( O(T_j) \) the set of simple subobjects of \( T_j \) (up to isomorphism), and the bases from Lemma 3.6. Then

\[ TP = \sum_{S_i \in O(T_j), \beta_i \in B_i} \prod_{i=1}^{3} \dim(S_i) \]

Observe that above picture is of the form
where \((S_i)_{i=1,2,3}\) are simple objects. By Schur’s lemma, it is nonzero if (up to isomorphism) \(S_1 = S_2 = S_3 =: X \in \mathcal{O}\), in which case, it is equal to the following by Lemma 2.9 (applied two times):

\[
\dim(X)^{-2} \begin{vmatrix} A & X \\ X & D \end{vmatrix} \text{tr}(B)\text{tr}(C)
\]

Then

\[
TP = \sum_{X \in \mathcal{O}} \sum_{\beta_i \in B_i, i=1,2,3} \dim(X) 
\]

After applying some zigzag relations, Lemma 2.5 and pivotality, we get the original form of monoidal tetrahedra:

\[
TP = \sum_{X \in \mathcal{O}} \sum_{\beta_i \in B_i, i=1,2,3} \dim(X) 
\]

and so

\[
TP = \sum_{X \in \mathcal{O}} \sum_{\beta_i \in B_i, i=1,2,3} \dim(X) T \left( \rho^{-2}(\sigma(\beta'_3)), \rho(\alpha_3), \rho^2(\beta_1), \rho^{-1}(\alpha_6) \right) T \left( \rho^{-1}(\alpha_4), \alpha_1, \beta'_1, \beta_2 \right) T \left( \rho^{-1}(\alpha_5), \alpha_2, \beta'_2, \beta_3 \right) = \text{RHS}. \]

4.3. Oriented monoidal tetrahedron and spherical TPE. In this subsection, we introduce a novel pictorial notation for the monoidal tetrahedron in the spherical case, utilizing the oriented notations from §2.2, along with specific rules. This new approach aligns precisely with the familiar geometric regular tetrahedron and its orientation-preserving symmetry group, \(A_4\). Subsequently, we reinterpret TPE using this notation within the context of a spherical fusion category.
Definition 4.11. Following Definition 4.11,

\[
\begin{align*}
\theta_1 X & = X \theta_1 = X \theta_1 = X \rho(\theta) = X \rho^{-1}(\theta) \\
\theta_2 X & = X \theta_2 = X \theta_2 = X \sigma^{-2}(\rho(\theta)) = X \rho^{-1}(\sigma^2(\theta))
\end{align*}
\]

Observe that the two rules in each line are in fact equivalent (inverse each other). We justify these rules as follows:

Lemma 4.12. Following Definition 4.11 and above rules in the spherical case
Proof. By Proposition 4.5, \( T(\alpha, \beta, \gamma, \delta) = T(\rho(\alpha), \delta^{**}, \beta, \gamma) \), but by Definition 4.11:

\[
T(\rho(\alpha), \delta^{**}, \beta, \gamma) = X^\alpha_6 \delta^{**} X^\gamma_{12} X^\beta_2 X^\gamma_3 X^\alpha_1 \rho(\alpha).
\]

Now \( \delta^{**} = \rho^3(\delta) \), and we can apply (4.1) three times at its vertex, then one time at the vertex labeled by \( \rho(\alpha) \); the first equality follows. Similarly, by Proposition 4.6, \( T(\alpha, \beta, \gamma, \delta) = T(\rho^{-3}(\beta), \rho^{-1}(\sigma^2(\gamma)), \rho(\alpha), \rho(\delta)) \), depict this last using Definition 4.11, then apply one time \( (4.1) \) at \( \rho(\alpha) \), three times \( (4.1) \) at \( \rho^{-3}(\beta) \), one time \( (4.2) \) at \( \rho^{-1}(\sigma^2(\gamma)) \), and lastly one time \( (4.1) \) at \( \rho(\delta) \); the second equality follows. \( \square \)

Remark 4.13. Observe that the three pictures in Lemma 4.12 use exactly the same labels, what changing are just the ordering, the orientation of some edges and the corner of the label of some vertices. This result can (similarly) be extended into an equality between 12 such pictures. These pictures make a set on which the (orientation-preserving) symmetry group of the regular tetrahedron (i.e. the alternating group \( A_4 \)) acts transitively. The two equalities of Lemma 4.12 correspond to two generators of the group \( A_4 \) (see Remark 4.7). This action can be made precise by encoding the data of such a picture on a (3-dimensional) regular tetrahedron, for so we just need to replace the choice of a corner to label a vertex by the choice of a face containing this vertex. An inside corner corresponds to the face containing it, whereas an outside corner corresponds to the hidden face (a tetrahedron has four faces, but our planar representation shows only three ones, the remaining one is hidden).

Let us restate into the following proposition:

**Proposition 4.14.** Consider the action of the alternating group \( A_4 \) on the set of labeled oriented tetrahedral graphs, as described in Remark 4.13. Consider the orbit of the one in Definition 4.11. Any element of this orbit can reach the reference pattern (of Definition 4.11) by applying (possibly several times) above rules, and its categorical interpretation equals that of the initial one, by applying (possibly several times) Propositions 4.5 and 4.6.

Now let us reformulate the TPE in the spherical case using these new notations.

**Theorem 4.15 (Spherical TPE).** Following Theorem 4.10, assume that \( C \) is spherical. Then

\[
\sum_{\beta_0 \in B_0} \beta_0' \in B_1 \sum_{i=1, 2, 3} \dim(X_i) \rho^2(\beta_i) = \sum_{X \in O_i} \sum_{\beta_i \in B_i, i=1, 2, 3} \dim(X) \rho^2(\beta_i').
\]
Proof. It is just a reformulation of the equality in Theorem \ref{thm:tpe}. By Definition \ref{def:iso}, Lemma \ref{lem:iso}, and \ref{lem:iso2}.

\[
T \left( \rho^{-2}(\alpha_2), \rho(\alpha_3), \rho^{-1}(\alpha_1), \beta_0 \right) = \rho^{-1}(\alpha_1) = \cdots = \rho^{-2}(\alpha_2) \quad \rho(\alpha_3) \quad \beta_0 \quad \alpha_2 \quad x_5 \quad x_4 \quad x_3 \quad x_2 \quad x_1
\]

The proof of the reformulation of the other tetrahedra in Theorem \ref{thm:tpe} is similar. \qed

Remark 4.16. Here are three representations of the triangular prism graph,

\[
\begin{array}{c}
\begin{array}{c}
\text{the first is the one we used in Definition} \ref{def:triang}, \text{the second is the usual one (already mentioned in \S1), and the last is the one we will use in Definition} \ref{def:tpe}.
\end{array}
\end{array}
\]

Without going into details (useless here), following Remark \ref{rem:iso}, we can realize the (orientation-preserving) symmetry group \( C_6 \) of the uniform triangular prism, as for the regular tetrahedron in Remark \ref{rem:iso3}. The monoidal triangular prism in Definition \ref{def:triang} can be noted (as for Definition \ref{def:iso} using Definition \ref{def:tpe}.

Definition 4.17. Following Definition \ref{def:triang},

\[
:= TP(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6).
\]

Remark 4.18. Generally, we anticipate deriving equations, similar to TPE, from every planar trivalent graph that reflects the symmetries of its regular geometric realization, when such a realization exists.

Remark 4.19. According to \cite{13}, any fusion category \( \mathcal{C} \) can be equivalently represented as a strictified skeletal category where \((\_)^{**} = \text{id}_\mathcal{C}\) applies to objects and \((\_)^{****} = \text{id}_\mathcal{C}\) applies to morphisms. Alternatively, it is easy to show by \cite{28} Theorem 2.2 that a spherical fusion category is equivalent, as a spherical fusion category, to a strictly-pivotal spherical fusion category, meaning \(X^{**} = X\) and \(a_X = \text{id}_X\) for any object \(X\) in \(\mathcal{C}\). Within this framework, even though \(X^*\) and \(X\) may be isomorphic, they are not identical. However, utilizing Lemma \ref{lem:iso} if \(X\) is a simple object and \(\kappa : X \to X^*\) is an isomorphism, then \(\kappa^* = \nu_2(X)\kappa\).

When addressing the categorification problem, we can, without loss of generality, focus on the cases outlined in Remark \ref{rem:iso}. This simplifies the formulation of TPE. To further simplify, we have refined the definition of certain morphisms \((\alpha_i\) and \(\beta_i\)) in Theorem \ref{thm:tpe2}—note that they differ slightly from those in Theorem \ref{thm:tpe1}. Such simplifications would be unattainable without additional assumptions at the outset, as seen in Theorem \ref{thm:tpe}. The modifications to the morphisms \((\alpha_i)\) permits to get the oriented labeled triangular prism graph, which varies in the orientation of \(X_i\) for \(i = 1, 2, 3, 7, 8, 9\), as depicted in Figure \ref{fig:triang}.

Theorem 4.20 (Simplified TPE). Let \(\mathcal{C}\) be a spherical fusion category. Let \(a\) be the spherical structure. Assume (without loss of generality by Remark \ref{rem:iso}) that \(X^{**} = X\) and \(a_X^2 = \text{id}_X\) for any object \(X\) in \(\mathcal{C}\). Let \(X_1, \ldots, X_9\) be objects in \(\mathcal{C}\). Consider morphisms \(\alpha_1 \in \text{hom}_\mathcal{C}(1, X_1 \otimes X_2 \otimes X_3^*), \alpha_2 \in \text{hom}_\mathcal{C}(1, X_5 \otimes X_2 \otimes X_4^*), \alpha_3 \in \text{hom}_\mathcal{C}(1, X_6 \otimes X_3 \otimes X_5^*), \alpha_4 \in \text{hom}_\mathcal{C}(1, X_8 \otimes X_5^* \otimes X_7), \alpha_5 \in \text{hom}_\mathcal{C}(1, X_7 \otimes X_5^* \otimes X_8)\) and \(a_6 \in \text{hom}_\mathcal{C}(1, X_6 \otimes X_3 \otimes X_9)\). Then the following TPE
where $d_X = \dim(X)$; $O$ is the set of simple subobjects of both $X_4 \otimes X_7$, $X_5 \otimes X_8$ and $X_6 \otimes X_9$ (up to isomorphism); $B_i$ is a basis of $\text{hom}_C(1, Z_i)$ with $Z_0 = X_3 \otimes X_2 \otimes X_1$, $Z_1 = X_4^* \otimes X \otimes X_7^*$, $Z_2 = X_5^* \otimes X \otimes X_8^*$ and $Z_3 = X_6^* \otimes X \otimes X_9^*$; $\beta'_i \in B'_i$ the dual basis of $B_i$ according to the bilinear form in Lemma 3.2; and $\beta_i \mapsto \beta'_i$ is the usual bijection.

**Lemma 4.21** (Rotation eigenvalue). Following Remark 4.3 if $\text{hom}_C(1, X^{\otimes 3})$ is one-dimensional with generator (say) $\theta$, if moreover $X^{**} = X$ and $a_X = \pm \text{id}_X$, then

$$X \theta X = \omega_X,$$

where $\omega_X := \nu_3(X)$, so that $\omega_X^3 = 1$.

**Proof.** We will apply Rule 4.1, Remark 4.3 and finally Lemma 2.23 (together with $\sigma(\theta) = \pm \theta$, as $a_X = \pm \text{id}_X$):

$$X^* \rho(\theta) X = \nu_3(X) = \nu_3(X) \sigma(\theta) X = \nu_3(X).$$

**Remark 4.22** (More simplifications). Let $C$ be a spherical fusion category, and let $X_1, \ldots, X_9$ be simple objects in $C$. By Proposition 2.18, the assumptions in Theorem 4.20 are always satisfied if $X = X^*$ for all simple object $X$ in $C$. If moreover $\nu_2(X) = 1$ for all simple object $X$, then by Lemma 2.23, there is no need to orientate the edges:
Alternatively, the morphism label of a one-dimensional hom-space can be replaced by a bullet $\bullet$, so if every involved hom-space is one-dimensional (in particular, if the Grothendieck ring is of multiplicity one) then:

$$X_6 \bullet X_8 = \sum_{X \in O} d_X X_4 X_7$$

Finally, if it is of multiplicity one, with $X^* = X$, $\nu_2(X) = \nu_3(X) = 1$ for all simple object $X$, then (by Lemma 4.21)

$$X_6 \bullet X_8 = \sum_{X \in O} d_X X_4 X_7$$

Let us conclude this section with some remarks on non-planar trivalent graphs and braiding. Besides the tetrahedral and triangular prism graphs, there exists a third trivalent graph with no more than six vertices, which is non-planar: the complete bipartite graph $K_{3,3}$. Below are three representations of this graph:

The first representation is the conventional one, the second is crafted to resemble the typical depiction of the triangular prism graph, and the third is used to derive the following monoidal category version, assuming the presence of a braiding:

This illustration closely resembles the one in Definition 4.8 with the primary difference being the inclusion of braiding. It suggests a braided version of Theorem 4.10 (TPE). In general, we might derive such equations from any trivalent graph.

5. Pentagon and Triangular Prism Equations

5.1. Pentagon equations. This subsection recalls the explicit way to write the Pentagon Equations (see [5, 37]). Let us keep the notations from §6.2. The chosen basis of $\text{hom}_C(X_i \otimes X_j, X_k)$ will be denoted $B(i, j; k)$, and a morphism in there, represented as

$$(5.1)$$
For indices \(i_1, i_2, \ldots, i_6 \in I\) and nonzero morphisms \(\mu_1 \in B(i_1, i_2; i_3), \mu_2 \in B(i_3, i_4; i_5), \mu_3 \in B(i_2, i_4; i_6)\), and \(\mu_4 \in B(i_1, i_6; i_5)\), the F-symbol \(\left( \begin{array}{ccc} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_4 & \end{array} \right)\) is defined as follows (however, it is zero if any \(\mu_i\) is zero):

\[
\mu_2(\mu_1 \otimes \text{id}_{i_4}) = \sum_{\mu_3, \mu_4} \left( \begin{array}{ccc} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_4 & \end{array} \right) \mu_4(\text{id}_{i_1} \otimes \mu_3),
\]

with \(i_6, \mu_3\) and \(\mu_4\) summing over their respective set. Pictorially,

\[
(5.2)
\]

The F-symbols satisfy the *Pentagon Equations* (PE) written below, with a pictorial interpretation in Figure 2.

\[
\sum_{\mu_6} \left( \begin{array}{ccc} i_2 & i_7 & i_8 \\ i_9 & i_3 & i_1 \\ \mu_5 & \mu_6 & \mu_0 \\ \mu_4 & \mu_0 & \end{array} \right) \left( \begin{array}{ccc} i_5 & i_4 & i_2 \\ i_1 & i_3 & i_6 \\ \mu_2 & \mu_0 & \mu_3 \\ \mu_1 & \end{array} \right)
= \sum_{\mu_7, \mu_8, \mu_9} \left( \begin{array}{ccc} i_5 & i_4 & i_2 \\ i_7 & i_8 & i_0 \\ \mu_2 & \mu_5 & \mu_7 \\ \mu_6 & \mu_8 & \mu_3 \\ \mu_9 & \mu_3 & \mu_1 \\ \mu_4 & \end{array} \right)
\]

for \(i_0, i_1, i_2, \ldots, i_9 \in I\) and morphisms \(\mu_1 \in B(i_4, i_3; i_6), \mu_2 \in B(i_5, i_4; i_2), \mu_3 \in B(i_5, i_6; i_3), \mu_4 \in B(i_7, i_6; i_1), \mu_5 \in B(i_2, i_7; i_8), \mu_6 \in B(i_8, i_9; i_3), \mu_7 \in B(i_4, i_7; i_0), \mu_8 \in B(i_5, i_0; i_6), \mu_9 \in B(i_1, i_9; i_6), \mu_0 \in B(i_2, i_1; i_3)\), with \(i_0\) and \(\mu_k\) summing over their respective set. We will see in §5.2 that the PE can be interpreted as the TPE of a TP with a specific configuration (see Figure 3).

5.2. TPE versus PE. This section, following Notation 6.1, shows that in the spherical case, TPE equals PE, up to a change of basis. Note that it is not used in this paper, in particular §6.3 and §7 are independent of it, but it is added for information, because it should be useful for future work.

**Definition 5.1.** Let \(\mathcal{C}\) be a monoidal category with left duals. Let \(F\) and \(G\) be functors from \(\mathcal{C}^3\) to \(\text{Set}\) defined as the composition of usual functors such that \(F(X, Y, Z) = \text{hom}_\mathcal{C}(X \otimes Y, Z)\) and \(G(X, Y, Z) = \text{hom}_\mathcal{C}(1, Z \otimes Y^* \otimes X^*)\), for all...
objects $X, Y, Z$ in $C$. Consider the natural transformation $\mu \mapsto \tilde{\mu}$ from $F$ to $G$ defined by

$$\tilde{\mu} = \begin{array}{c} \mu \\ \downarrow \gamma \\ X^* \end{array}$$

It is a natural isomorphism by applying natural adjunction isomorphisms [9, Proposition 2.10.8], and

$$\mu = \begin{array}{c} \tilde{\mu} \\ \downarrow \gamma \\ Y \end{array}$$

Now let us reformulate (5.2) using above natural isomorphism:

$$\sum_{i_5} \sum_{\tilde{\mu}_2, \tilde{\mu}_1, \tilde{\mu}_3} i_1 i_2 i_3 i_4 = \sum_{i_6} \sum_{\mu_5, \mu_4} (i_1 i_2 i_3 i_4 | \mu_1 \mu_2 \mu_3 \mu_4) \cdot \tilde{\mu}_4 \tilde{\mu}_3 \tilde{\mu}_4$$

Next, we eliminate all the terms on the RHS except one by composing with morphisms from the right dual bases (denoted by the mapping $\alpha \mapsto \check{\alpha}$ for the usual bijection) and by applying the corresponding bilinear form (as for 4.3). We get:

Then, by some zigzag relations and the fact that $\text{hom}_C(1, X_{i_6} \otimes X_{i_6}^*) = \operatorname{kcoev}_{X_{i_6}}$:

$$T(\rho^{-1}(\tilde{\mu}_3), \tilde{\mu}_2, \tilde{\mu}_1, \tilde{\mu}_4) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ i_4 & i_5 & i_6 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = d_{i_6} T(\rho^{-1}(\tilde{\mu}_3), \tilde{\mu}_2, \tilde{\mu}_1, \tilde{\mu}_4).$$

It follows that:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ i_4 & i_5 & i_6 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = d_{i_6} T(\rho^{-1}(\tilde{\mu}_3), \tilde{\mu}_2, \tilde{\mu}_1, \tilde{\mu}_4).$$

Now we can reformulate PE (5.3) as follows:

$$\sum_{\tilde{\mu}_0} T(\rho^{-1}(\tilde{\mu}_4), \tilde{\mu}_6, \tilde{\mu}_5, \tilde{\mu}_0) T(\rho^{-1}(\tilde{\mu}_1), \tilde{\mu}_2, \tilde{\mu}_3) = \sum_{i_0} \sum_{\mu_7, \mu_8, \mu_9} d_{i_0} T(\rho^{-1}(\tilde{\mu}_7), \tilde{\mu}_5, \tilde{\mu}_2, \tilde{\mu}_8) T(\rho^{-1}(\tilde{\mu}_9), \tilde{\mu}_6, \tilde{\mu}_8, \tilde{\mu}_3) T(\rho^{-1}(\tilde{\mu}_4), \tilde{\mu}_9, \tilde{\mu}_7, \tilde{\mu}_1)$$

Then following Definition 4.11 we get:

$$\sum_{\mu_0} \sum_{\tilde{\mu}_0} \tilde{\mu}_5 \tilde{\mu}_6 = \sum_{i_0} \sum_{\mu_7, \mu_8, \mu_9} d_{i_0} \tilde{\mu}_5 \tilde{\mu}_6 \tilde{\mu}_7 \tilde{\mu}_8 \tilde{\mu}_9 \tilde{\mu}_1$$

and by Rule 4.1:

$$\sum_{\mu_0} \sum_{\tilde{\mu}_0} \tilde{\mu}_5 \tilde{\mu}_6 = \sum_{i_0} \sum_{\mu_7, \mu_8, \mu_9} d_{i_0} \tilde{\mu}_5 \tilde{\mu}_6 \tilde{\mu}_7 \tilde{\mu}_8 \tilde{\mu}_9 \tilde{\mu}_1$$

Then by applying Proposition 4.14 we can get the following form:
Next, by applying Rules 4.1 and 4.2 several times, we can put the labels in the same corners than in Theorem 4.20.

\[
\sum_{\mu_0} \rho^{-1}(\tilde{\mu}_2) = \sum_{i_0} \sum_{\mu_2, \mu_8, \mu_9} d_{i_0}
\]

Observe that \((\alpha)' = (\alpha') = \alpha\), \((\alpha)'' = \alpha'\) and \(**(\alpha')' = \alpha'.\) Then
\[
\rho(\tilde{\mu}_7) = \rho^{-2}(\tilde{\mu}_7) = \rho^{-2}((\tilde{\mu}_7)^{*}) = \rho^{-2}(\tilde{\mu}_7).
\]

Now, under the assumptions of Theorem 4.20, \(\sigma^2 = \text{id}\), so by Lemmas 2.13 and 2.5
\[
(\rho^{-1}(\alpha))' = \sigma^{-2} \circ \rho(\alpha') = \rho(\alpha').
\]

Then by applying the rules:

Finally, by adjusting the orientation using the natural isomorphisms \(\sigma_i\) \((i = 1, 2, 3)\) defined as for \(\sigma\) but for the \(i\)th leg (so that \(\sigma = \sigma_3\)), we recover the TPE from Theorem 4.20.

**Theorem 5.2 (PE-TPE, Change of Basis).** Following Theorem 4.20, its TPE is exactly PE (5.3) under the following change of basis:

\[
\begin{align*}
\alpha_1 &= \rho(\tilde{\mu}_3), & \alpha_2 &= \sigma_1^{-1}(\tilde{\mu}_1), & \alpha_3 &= \rho^{-1}(\tilde{\mu}_2), & \alpha_4 &= \sigma_1(\sigma_3^{-1}(\rho^{-1}(\tilde{\mu}_6))), & \alpha_5 &= \sigma_1(\sigma_2(\sigma_3^{-1}(\rho(\tilde{\mu}_4)))), & \alpha_6 &= \sigma_1(\rho(\tilde{\mu}_5)), \\
\beta_0 &= \tilde{\mu}_0, & \beta_1 &= \rho^{-1}(\tilde{\mu}_9), & \beta_2 &= \rho^{-1}(\tilde{\mu}_7), & \beta_3 &= \tilde{\mu}_8,
\end{align*}
\]

\[
X_{i_0} = X_1, \ X_1 = X_{i_1}^*, \ X_2 = X_{i_2}, \ X_3 = X_{i_3}, \ X_4 = X_{i_4}, \ X_5 = X_{i_5}, \ X_6 = X_{i_6}, \ X_7 = X_{i_7}^*, \ X_8 = X_{i_8}, \ X_9 = X_{i_9}.
\]

The change of basis in Theorem 5.2 can be depicted as a TP with a specific (PE) configuration, see Figure 3 (which is rotated for a better matching with Figure 1).

**Figure 3. PE configuration**

Let \(\mathcal{F}\) be a fusion ring with basis \((b_i)_{i \in I}\) and fusion coefficients \(N_{k,j}^i\). In §6.2, we employed the TPE to establish criteria for the categorification of \(\mathcal{F}\). Applying the results of Theorem 5.2 and [3] Proposition 3.7, the TPE can indeed be used to categorify \(\mathcal{F}\), assuming it is the Grothendieck ring of a spherical fusion category. It is important to note that the proof of Theorem 5.2 hinges on the property of sphericality; therefore, the categorification requires not only the verification of all TPE but also additional assumptions to ensure sphericality. The TP are considered up to \(A_4\)-symmetry, meaning they are spherically invariant. Recall that if \(X^* = X\) and \(a_X^2 = \text{id}_X\) (as assumed in Theorem 4.20), then \(a_X = \pm \text{id}_X\). Thus, we define \(\epsilon_i = \pm 1\) for each \(i \in I\) to represent the pivotal structure, ensuring that \(a_{X_i} = \epsilon_i \text{id}_{X_i}\) and that \(\epsilon_{i*} = \epsilon_{i}\), in accordance with Lemma 2.6 and [5] Equation (35) regarding the TP. Consequently, we have \(\epsilon_{i} = \pm 1\) (Lemma 2.23), which defines the TP up to the following equalities on the edges:

\[
\frac{i}{i} = \epsilon_i
\]

Lastly, according to [9] Proposition 4.7.12, we may assume that the dimension function \(i \mapsto d_i\) is a character of the fusion ring. The relation \(d_{i*} = d_i\) ensures sphericality, as further explained in Remark 2.12.
6. Localization

6.1. Localization strategy. We introduce a localization strategy for solving F-symbols, which involves the following steps:

(a) Establish a complexity measure for F-symbols and PE, where known F-symbols are considered less complex.
(b) Identify a small set $V$ of indeterminate F-symbols with low complexity to use as variables, and construct an overdetermined system $E$ of PE that includes only the variables in $V$.
(c) Compute the Gröbner basis of $E$.
(d) Resolve $V$ using the Gröbner basis of $E$:
   - If $E$ has solutions, assign the resolved values to the F-symbols in $V$ as constants and repeat the process from step (a).
   - If no solutions exist, then the fusion ring cannot be categorized.

To implement this localization strategy through a computer program, we must first devise an effective complexity measure. The program can then incrementally generate $V$ and $E$ based on this complexity. The sizes of $V$ and $E$ should be small enough to allow the computer to compute the Gröbner basis in step (c) within a reasonable timeframe. In practice, it is often more efficient to resolve $V$ using the Gröbner basis of $E$ rather than solving $E$ directly, which justifies the retention of step (c) even though it is not strictly necessary. Different complexities can be applied to solve different subsets of F-symbols, and these solutions can be integrated. Partial solutions of $V$ in step (d) may also prove beneficial. Theoretically, this localization strategy can be adapted for families of fusion rings with arbitrarily large rank, a topic we will explore in future publications.

The monoidal triangular prism can be viewed as the trace of the multiplication of three 'I'-shaped diagrams within the algebroid of hom-spaces $\text{hom}_C(X \otimes Y, Z \otimes T)$, as illustrated in the initial representation in Remark 4.16. The 'I'-shaped diagrams represent matrix units of these hom-spaces. A 90° rotation, termed the string Fourier transform (see, for example, [18]), interchanges the 'I'-shaped and 'T'-shaped diagrams. Consequently, the tetrahedra can be regarded as matrix entries of the string Fourier transform, relative to the 'I'-shaped diagrammatic basis of the hom-spaces.

We can execute the localization strategy by designing triangular prism configurations and TPEs where the LHS are known (or less complex) and the RHS comprises unknown variables represented by the matrix entries of the string Fourier transform. For instance, if $X$ is a self-dual simple object of $\mathcal{C}$, and $\text{hom}_C(X \otimes X, X \otimes X)$ is $n$-dimensional with a basis $B$, the string Fourier transform will have $n^2$ entries, considered as $n^2$ variables of TPE. There are $n^3$ standard triangular prism configurations (Figure 1), such that $X_1 = X_2 = \cdots = X_n = X$. Accounting for the $C_3$ rotational symmetry, we derive approximately $n^3/3$ TPEs with $n^2$ variables, aiming to solve these variables locally through an overdetermined set of equations.

6.2. Categorification criteria from localization. This subsection introduces a categorification criterion that serves as an initial iteration in the localization strategy outlined in §6.1. The primary focus is on choosing a set of TPEs that not only produce meaningful equations but also minimize the number of variables (F-symbols) involved. The objective is to create a small yet significant subsystem for which the Gröbner basis can be readily calculated. Following this, the strategy involves building upon the solutions obtained and incrementally expanding the subsystem through an inductive process until the entire system is addressed.

Notation 6.1. In a fusion category, the simple objects will be represented as $(X_i)_{i \in I}$, considered up to isomorphism, with $X_1 = 1$. The dimension of the hom space $\text{hom}_C(X_i \otimes X_j, X_k)$ will be denoted by $N^k_{i,j}$. An object $X$ is termed self-dual if $X = X^*$. Any label associated with an edge, represented by an object $X_i$, will be simplified to a label by its index $i$. Lastly, let $d_i := d_{X_i} = \text{dim}(X_i)$ denote the dimension of $X_i$.

Theorem 6.2. Let $\mathcal{C}$ be a spherical fusion category. Consider $X_k$, a self-dual simple object within $\mathcal{C}$, such that for every simple object $X_a$, the condition $N^a_{k,k} \leq 1$ holds. Assume that if $N^a_{k,k} = 1$, then $X_a$ is self-dual. Define $S_k$ as the set of such indices $a$. Assume further that $k \in S_k$. Let $S'_k$ be a subset of $S_k$. There exist variables $x(i, b, c)$ and $y(i, b)$, with $i \in S_k$ and $b, c \in S'_k$, satisfying the following equations:

\begin{align}
(6.1) & \quad x(a, b, c) = \sum_{i \in S_k} d_i y(i, a) y(i, b) y(i, c), \\
(6.2) & \quad y(a, b) y(a, c) = \sum_{i \in S_k} d_i y(i, a) x(i, b, c),
\end{align}

with $x(a, k, k) = y(a, k)^2$, $x(a, b, c) = 0$ if $N^b_{a,c} = 0$, $y(a, b) = y(b, a)$, $y(1, b) = d_k^{-1}$, $x(1, b, c) = \delta_{b,c} (d_b d_k)^{-1}$.

Proof. Let $a, b, c \in S'_k$. Consider the following two types of triangular prisms (TP), labeled respectively as $(a, b, c, k, k, k, k, k, k)$ and $(k, k, a, b, k, k, c, k, k, k)$, developed according to Theorem 1.20 and Remark 1.22. Since the objects are self-dual and the multiplicities do not exceed 1, we can label the morphisms simply with a bullet in the TP (though not always in the
TPE, as seen in Equations 6.3 and 6.4. According to Theorem 1.2, for all \(a \in S_k\), we have \(\nu_2(X_a) = 1\) (since 1 is odd), eliminating the need for orienting the edges.

Let \(i \in S_k\), let \(b, c \in S_k'\), and consider the following two types of variables:

\[
x(i, b, c) := \sum_{\beta \in B} c_{\beta} b \cdot k \cdot a = \sum_{i \in S_k} d_i \sum_{\beta \in B} c_{\beta} b \cdot i \cdot k \cdot a,
\]

\[
y(i, b) := \sum_{\beta \in B} c_{\beta} b \cdot i \cdot k
\]

We still need to reformulate Equations (6.3) and (6.4) using the variables introduced above. The RHS of these equations are already satisfactory; our focus is on addressing the rotation eigenvalues and the \(A_4\) symmetry for the LHS. For \(a \in S_k'\), sphericity, along with Rule (4.1) and Lemma 4.21, implies that:

\[
x(a, b, c) := \sum_{\beta \in B} c_{\beta} b \cdot k \cdot k \cdot a = \cdots = \omega_k^\beta \delta_{k,b,c} \omega_k^2 \delta_{k,b,c} \omega_k^\beta \delta_{k,b,c} \sum_{\beta_0 \in B_0} c_{\beta_0} b \cdot \beta_0 \cdot a \cdot c
\]
where \( \beta_0 = \rho^{-1}(\beta), \beta'_0 = \rho(\beta'), \) and \( \omega_k^{\delta_k,b} \omega_k^{2\delta_k,c} \omega_k^{2\delta_k,b} \omega_k^{\delta_k,c} = \omega_k^{3\delta_k,b} \omega_k^{3\delta_k,c} = 1. \) In fact, the use of rotation eigenvalues can be avoided by using specific morphism labels \( \rho^s(\alpha_t) \) for \( s \in \{-1, 0, 1\} \) and \( t \in \{a, b, c\} \), in conjunction with Rule [4.1].

Similarly,

\[
y(a, b)y(a, c) = \begin{array}{ccc}
\bullet & k & \bullet \\
\bullet & a & \bullet \\
\bullet & k & \bullet \\
k & b & k
\end{array} \quad \begin{array}{ccc}
\bullet & k & \bullet \\
\bullet & c & \bullet \\
\bullet & k & \bullet \\
k & k & k
\end{array},
\]

\( x(a, k, k) = y(a, k)^2 \) and \( y(a, b) = y(b, a) \). The result follows. \( \square \)

**Corollary 6.3.** Following Theorem [6.2], there are variables \( x(i, b) \) and \( y(i, b) \) with \( (i, b) \in S_k \times S'_k \) such that for all \( a, b \in S'_k \)

\[
(6.5) \quad \delta_{a,b} = d_b \sum_{i \in S_k} d_i y(i, a)y(i, b),
\]

\[
(6.6) \quad x(a, b) = \sum_{i \in S_k} d_i y(i, a)y(i, b)^2,
\]

\[
(6.7) \quad y(a, b)^2 = \sum_{i \in S_k} d_i y(i, a)x(i, b),
\]

with \( x(a, k) = y(a, k)^2; x(a, b) = 0 \) if \( N^a_{b,b} = 0; \ y(a, b) = y(b, a); \ y(1, b) = d_k^{-1}; \ x(1, b) = (d_b d_k)^{-1}. \)

**Proof.** Apply Theorem [6.2] with \( x(i, b) = x(i, b, b) \), Equation [6.1] with \( c = 1 \), Equation [6.1] with \( b = c \), and Equation [6.2] with \( b = c. \) \( \square \)

**Remark 6.4.** Note that we can restrict Equation [6.5] to \( a \neq 1 \) and \( a \geq b \) (after fixing an order on \( S_k \)), Equation [6.6] to \( a, b \neq 1 \), and Equation [6.7] to \( b \neq 1, k. \)

**Theorem 6.5.** Following Theorem [6.2], consider \( k, S_k \) and \( S'_k \), and let \( E_k \) be the subsystem given by Corollary [6.3]. Let \( l \in S'_k \) with \( l \neq k \), and \( S_l, S'_l, E_l \) be as above. Then there is an extra equation linking the subsystems \( E_k \) and \( E_l \):

\[
x_k(l, l) = \sum_{i \in S_k \cap S_l} d_i y(i, l)x_k(i, l)
\]

**Proof.** Consider the TPE labeled with \( (k, k, l, l, l, k, k, l) \):

\[
(6.8) \quad \begin{array}{ccc}
\bullet & l & \bullet \\
\bullet & k & \bullet \\
\bullet & l & \bullet \\
l & k & l
\end{array} = \sum_{i \in S_k \cap S_l} d_i y(i, l)x_k(i, l)
\]

The result follows by sphericality and Lemma [1.21] using the notation of the variables as in Corollary [6.3] indexed by \( k \) (resp. \( l \)) for \( E_k \) (resp. \( E_l \)). \( \square \)

The results proved in this section will be used in §7.1.
6.3. Zero and one spectrum criteria. This subsection retains the notation introduced in \S 5.1 and assumes that $X^{**} = X$ for every object $X$, which establishes a bijection $i \mapsto i^*$ on the set $I$, ensuring that $X_i = X_{i^*}$. The morphisms $e_{i, i}$ and $coe_{i, i}$ are denoted by $\cup_i$ and $\cap_i$, respectively. According to (5.2), a summand on the RHS of the PE (5.3) is non-zero only when $i_0 \in I_s := \{i \in I : N_{i_1, i_2}^k, N_{i_3, i_4}^k, N_{i_5, i_6}^k > 0\}$, which we refer to as the spectrum of this PE. The cardinality $|I_s|$ is considered a measure of this PE’s complexity. We will provide two categorification criteria related to the existence of equations in the form $xy = 0$ or $0 = xyz$, where $x, y, z \neq 0$, corresponding to $|I_s| = 0$ or $1$.

**Notation 6.6.** We will use the standard bijection $\alpha \mapsto \alpha'$ from $B(i, j; k)$ to its dual basis in $\text{hom}_C(X_k, X_i \otimes X_j)$, as defined by the bilinear form in Lemma 3.2, following certain natural adjunction isomorphisms \cite[Proposition 2.10.8]{[Ref]}.

**Lemma 6.7 (One-dimensional trick).** Consider non-zero morphisms $\mu_1 \in B(i_1, i_2; i_3), \mu_2 \in B(i_3, i_4; i_5), \mu_3 \in B(i_2, i_4; i_6)$ and $\mu_4 \in B(i_1, i_6; i_5).$ If

$$\sum_k N_{i_2, i_6}^k N_{i_3, i_4}^k = 1 \text{ or } \sum_k N_{i_1, i_2}^k N_{i_5, i_6}^k = 1 \text{ or } \sum_k N_{i_3, i_5}^k N_{i_4, i_6}^k = 1,$$

then

$$\left(\begin{array}{cccc} i_1 & i_2 & i_3 & \mu_1 \\ i_4 & i_5 & i_6 & \mu_2 \\ i_1 & i_2 & i_3 & \mu_1 \\ i_4 & i_5 & i_6 & \mu_2 \end{array}\right) \neq 0.$$

**Proof.** Take $\mu_5$ and $\mu_4'$ from Notation 6.6, then

$$\mu_2(\mu_1 \otimes \id_{i_4})(\id_{i_1} \otimes \mu_3')\mu_4' = \left(\begin{array}{cccc} i_1 & i_2 & i_3 & \mu_1 \\ i_4 & i_5 & i_6 & \mu_2 \\ i_1 & i_2 & i_3 & \mu_1 \\ i_4 & i_5 & i_6 & \mu_2 \end{array}\right) \neq 0,$$

To show

$$\left(\begin{array}{cccc} i_1 & i_2 & i_3 & \mu_1 \\ i_4 & i_5 & i_6 & \mu_2 \\ i_1 & i_2 & i_3 & \mu_1 \\ i_4 & i_5 & i_6 & \mu_2 \end{array}\right) \neq 0,$$

it is enough to show that $\mu_2(\mu_1 \otimes \id_{i_4})(\id_{i_1} \otimes \mu_3')\mu_4' \neq 0$. The sums stated in the assumption correspond to the dimensions of hom-spaces. The sums presented below are identical to those in the assumption (up to certain natural adjunction isomorphisms of the hom-spaces).

- If $\sum_k N_{i_2, i_6}^k N_{i_3, i_4}^k = 1$, then hom$_C(X_{i_1} \otimes X_{i_6}, X_{i_3} \otimes X_{i_4})$ is one-dimensional. But $(\mu_1 \otimes \id_{i_4})(\id_{i_1} \otimes \mu_3')$ and $\mu_2^\prime \mu_4$ are non-zero morphisms in this hom-space, so $(\mu_1 \otimes \id_{i_4})(\id_{i_1} \otimes \mu_3') = \lambda \mu_2^\prime \mu_4$, for some non-zero $\lambda$. Therefore,

$$\mu_2(\mu_1 \otimes \id_{i_4})(\id_{i_1} \otimes \mu_3')\mu_4' = \mu_2^\prime \mu_4 \neq 0.$$

- If $\sum_k N_{i_1, i_2}^k N_{i_5, i_6}^k = 1$, then hom$_C(X_{i_1} \otimes X_{i_2} \otimes X_{i_3}, X_{i_5} \otimes X_{i_6})$ is one-dimensional. But $\mu_2(\mu_1 \otimes \id_{i_4})$ and $\mu_4(\id_{i_1} \otimes \mu_3)$ are non-zero morphisms in this hom-space, so $\mu_2(\mu_1 \otimes \id_{i_4}) = \lambda \mu_4(\id_{i_1} \otimes \mu_3)$, for some non-zero $\lambda$. Therefore,

$$\mu_2(\mu_1 \otimes \id_{i_4})(\id_{i_1} \otimes \mu_3')\mu_4' = \mu_4(\id_{i_1} \otimes \mu_3) \neq 0.$$

- If $\sum_k N_{i_3, i_5}^k N_{i_4, i_6}^k = 1$, then hom$_C(X_{i_3} \otimes X_{i_4} \otimes X_{i_5}, X_{i_6} \otimes X_{i_4})$ is one-dimensional. But $((\mu_1 \otimes \id_{i_5})(\id_{i_1} \otimes \cap_{i_2})) \otimes \id_{i_6})\mu_4'$ and $(\id_{i_3} \otimes \zeta')\mu_2'$ are non-zero morphisms in this hom-space, where $\zeta := (\cup_{i_2} \otimes \id_{i_5})(\id_{i_5} \otimes \mu_3')$ is non-zero in hom$_C(X_{i_3} \otimes X_{i_5}, X_{i_4}),$ and $\zeta'$ is given by Notation 6.6. So

$$(\mu_1 \otimes \id_{i_5})(\id_{i_1} \otimes \cap_{i_2})) \otimes \id_{i_6})\mu_4' = \lambda(\id_{i_3} \otimes \zeta')\mu_2',$$

for some non-zero $\lambda$. Therefore,

$$\mu_2(\mu_1 \otimes \id_{i_5})(\id_{i_1} \otimes \mu_3')\mu_4' = \mu_2(\mu_1 \otimes \zeta)((\mu_1 \otimes \id_{i_5})(\id_{i_1} \otimes \cap_{i_2})) \otimes \id_{i_6})\mu_4'$$

$$= \mu_2(\id_{i_3} \otimes \zeta')(\id_{i_3} \otimes \zeta')\mu_2' \neq 0.$$  \hfill \Box

**Theorem 6.8 (Zero spectrum criterion).** For a fusion ring $R$, if there exist indices $i_j \in I$ for $1 \leq j \leq 9$, such that the fusion coefficients $N_{i_4, i_1}^k, N_{i_5, i_4}^k, N_{i_6, i_5}^k, N_{i_7, i_6}^k, N_{i_8, i_7}^k, N_{i_9, i_8}^k$, and $N_{i_1, i_9}^k$ are non-zero, and if the following conditions hold:

\begin{align*}
(6.9) & \quad \sum_k N_{i_4, i_1}^k N_{i_5, i_4}^k N_{i_6, i_5}^k = 0, \\
(6.10) & \quad N_{i_1, i_9}^k = 1, \\
(6.11) & \quad \sum_k N_{i_5, i_6}^k N_{i_9, i_8}^k = 1 \text{ or } \sum_k N_{i_5, i_4}^k N_{i_3, i_1}^k = 1 \text{ or } \sum_k N_{i_1, i_9}^k N_{i_3, i_4}^k = 1, \\
(6.12) & \quad \sum_k N_{i_9, i_8}^k N_{i_8, i_9}^k = 1 \text{ or } \sum_k N_{i_2, i_7}^k N_{i_7, i_2}^k = 1 \text{ or } \sum_k N_{i_8, i_9}^k N_{i_9, i_8}^k = 1,
\end{align*}

then $R$ cannot be categorified, meaning that it is not the Grothendieck ring of a fusion category, over any field.
Proof. Assume that $\mathcal{C}$ is a categorification of $R$. Given that the fusion coefficients $N_{i_4,i_1}^{i_6,i_3}, N_{i_5,i_3}^{i_7,i_9}, N_{i_1,i_7}^{i_3,i_9}, N_{i_8,i_8}^{i_9,i_9}, N_{i_1,i_7}^{i_3,i_9},$ and $N_{i_8,i_8}^{i_9,i_9}$ are non-zero, we can select non-zero morphisms $\mu_1, \mu_2, \ldots, \mu_6$ in the corresponding hom-spaces of $\mathcal{C}$. According to (6.9), the spectrum of the PE (5.3) is empty, which implies that its RHS is zero. On the other hand, by (6.10), its LHS is given by the following product:

$$\left( \begin{array}{cccccc} i_2 & i_7 & i_8 & \mu_5 & \mu_6 \\
9 & 3 & 0 & \mu_4 & \mu_0 \end{array} \right) \left( \begin{array}{cccccc} i_5 & i_4 & \mu_2 & \mu_0 \\
i_1 & 3 & 6 & \mu_1 & \mu_3 \end{array} \right),$$

where $\mu_0$ is a non-zero morphism in the hom-space $\text{hom}_\mathcal{C}(X_{i_2} \otimes X_{i_3}, X_{i_3})$ which is one-dimensional by (6.10).

However, Equations (6.11) and (6.12), together with Lemma 6.7, indicate that each F-symbol on the LHS is non-zero. This result contradicts the fact that the RHS is zero, thereby concluding that $R$ cannot be a categorified.

**Theorem 6.9 (One spectrum criterion).** For a fusion ring $R$, if there are indices $i_j \in I$, $0 \leq j \leq 9$, such that the fusion coefficients $N_{i_4,i_1}^{i_6,i_3}, N_{i_5,i_3}^{i_7,i_9}, N_{i_1,i_7}^{i_3,i_9}, N_{i_8,i_8}^{i_9,i_9}, N_{i_1,i_7}^{i_3,i_9}, N_{i_8,i_8}^{i_9,i_9}$ are non-zero, and

$$\sum_k N_k^{k} N_{i_4,i_7}^{k} N_{i_8,i_7}^{k} = 1,$$

$$N_{i_4,i_7}^{i_6,i_7} = N_{i_5,i_8}^{i_9,i_8} = N_{i_6,i_7}^{i_9,i_7} = 1,$$

$$N_{i_4,i_7}^{i_8,i_7} = 0,$$

$$\sum_k N_k^{k} N_{i_5,i_7}^{k} N_{i_8,i_7}^{k} = 1 \text{ or } \sum_k N_k^{k} N_{i_4,i_7}^{k} N_{i_8,i_7}^{k} = 1,$$

$$\sum_k N_k^{k} N_{i_5,i_7}^{k} N_{i_8,i_7}^{k} = 1 \text{ or } \sum_k N_k^{k} N_{i_4,i_7}^{k} N_{i_8,i_7}^{k} = 1,$$

then $R$ cannot be categorified.

Proof. Assume that $\mathcal{C}$ is the categorification of $R$. As $N_{i_4,i_1}^{i_6,i_3}, N_{i_5,i_3}^{i_7,i_9}, N_{i_1,i_7}^{i_3,i_9}, N_{i_8,i_8}^{i_9,i_9}, N_{i_1,i_7}^{i_3,i_9}, N_{i_8,i_8}^{i_9,i_9}$ are non-zero, we can take non-zero morphisms $\mu_1, \mu_2, \ldots, \mu_6$ in the corresponding hom-spaces. By (6.13), the LHS of PE (5.3) is zero. By (6.14), this $k$ is precisely $i_0$, so the single element of the spectrum of the PE (5.3). Thus the RHS of the PE is

$$\left( \begin{array}{cccccc} i_5 & i_4 & \mu_2 & \mu_5 \\
i_7 & i_8 & 0 & \mu_7 & \mu_8 \end{array} \right) \left( \begin{array}{cccccc} i_5 & i_4 & \mu_2 & \mu_5 \\
i_9 & i_3 & i_6 & \mu_9 & \mu_3 \end{array} \right) \left( \begin{array}{cccccc} i_4 & i_7 & i_0 & \mu_7 & \mu_9 \\
i_9 & i_6 & i_1 & \mu_4 & \mu_1 \end{array} \right)$$

for some non-zero morphisms $\mu_7, \mu_8, \mu_9$ in the corresponding hom-spaces which are one-dimensional by (6.14). By Equations (6.16), (6.17), (6.18) and Lemma 6.7 the F-symbols in the RHS of the PE are non-zero, contradiction.

The criteria proved in this section will be applied in §7.2.

**Proposition 6.10.** The zero and one spectrum criteria are not equivalent; furthermore, neither implies the other.

Proof. Consider the following two rank-6 fusion data drawn from the dataset presented in [33]:

| i_1 | i_2 | i_3 | i_4 | i_5 | i_6 | i_7 | i_8 | i_9 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |
| 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   |
| 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |

The first one is ruled out by Theorem 6.8 with $(i_1, i_2, \ldots, i_9) = (4, 5, 6, 3, 5, 4, 5, 6)$, but not by Theorem 6.9 (computer-assisted checking), and vice versa for the second one with $(i_0, i_1, \ldots, i_9) = (1, 5, 5, 3, 4, 5, 3, 4, 5)$.

**7. Applications**

In this section, we leverage the findings from earlier sections to demonstrate that certain fusion rings cannot be categorified. Specifically, we use the results from 6.2 and 6.3 to rule out the categorification of $\mathcal{F}_{210}$ and $\mathcal{F}_{650}$, respectively. Consequently, the classification presented in Theorem 1.3 is established.
Our efforts to classify all potential induction matrices revealed 17,843,535 solutions for the square component (involving equations with 34 variables. Paul Breiding recently found a solution using homotopy continuation type as Theorem 7.1.

Definition 7.4

Let us label (and order) the simple objects by $1, 5_1, 5_2, 5_3, 6_1, 7_1, 7_2$, with FPdim $1, 5, 5, 6, 7, 7$ (respectively). Observe that all the fusion matrices are self-adjoint, so the simple objects are self-dual.

**Theorem 7.1.** The fusion ring $F_{210}$ cannot be categorified in characteristic zero.

**Proof.** We can assume the field to be $\mathbb{C}$. Suppose the existence of a fusion category $C$ over $\mathbb{C}$ whose Grothendieck ring is $F_{210}$. Since $C$ is integral, it is pseudo-unitary and then spherical by [3] Propositions 9.6.5 and 9.5.1. We can apply Corollary 6.3 with $k = 5_1, 5_2 = \{1, 5_1, 5_3, 7_1, 7_2\}$, and $S'_k = \{1, 5_1, 5_2\}$. We get the following subsystem $E_k$ of 10 variables and 12 equations:

$$
\begin{align*}
5u_0 + 7u_1 + 7u_2 - 4/25 &= 0, \\
5v_0 + 5v_1 + 7v_3 + 7v_5 + 1/5 &= 0, \\
25v_0^2 + 25v_1^2 + 35v_2^2 + 35v_3^2 - 4/5 &= 0, \\
5v_0 + 5v_1 + 7v_3 + 7v_5 - v_0^2 + 1/125 &= 0, \\
5u_0v_1 + v_1^2 + 7u_1v_3 + 7u_2v_5 + 1/125 &= 0, \\
5u_1 + 5v_2 + 7v_4 + 7v_6 + 1/5 &= 0, \\
25v_0v_1 + 25v_1v_2 + 35v_3v_4 + 35v_5v_6 + 1/5 &= 0, \\
5v_0^2v_1 + 5v_1^2v_2 + 7v_2^2v_4 + 7v_3^2v_6 - v_1^2 + 1/125 &= 0, \\
25v_0^2 + 25v_1^2 + 35v_2^2 + 35v_3^2 - 4/5 &= 0, \\
5v_1 + 5v_1^2 + 7v_3^2 + 7v_4^2 - u_0 + 1/125 &= 0, \\
5u_0v_2 - v_2^2 + 7u_1v_4 + 7u_2v_6 + 1/125 &= 0
\end{align*}
$$

where $u_0 = x_k(5_3, 5_1), u_1 = x_k(7_1, 5_3), u_2 = x_k(7_2, 5_3), v_0 = y_k(5_1, 5_1), v_1 = y_k(5_3, 5_1), v_2 = y_k(5_3, 5_3), v_3 = y_k(7_1, 5_1), v_4 = y_k(7_1, 5_3), v_5 = y_k(7_2, 5_1), v_6 = y_k(7_2, 5_3)$. This subsystem admits 14 solutions in characteristic 0, which can be expressed as a Gröbner basis (see 8.2).

Next we can apply Theorem 6.5 with $k, S_k, S'_k$ as above, together with $l = 5_3$, $S_l = \{1, 5_2, 5_3, 7_1, 7_2\}$ and $S'_l = \{1, 5_2, 5_3\}$. We get a subsystem $E_l$ (equivalent to $E_k$, see 8.2) with $w_0 = x_l(5_2, 5_2), w_1 = x_l(7_1, 5_2), w_2 = x_l(7_2, 5_2), z_0 = y_l(5_2, 5_2), z_1 = y_l(5_3, 5_2), z_2 = y_l(5_3, 5_3), z_3 = y_l(7_1, 5_2), z_4 = y_l(7_1, 5_3), z_5 = y_l(7_2, 5_2), z_6 = y_l(7_2, 5_3)$; together with the following extra equation

$$(7.1) \quad 5u_0z_2 + 7u_1z_4 + 7u_2z_6 - u_0 + 1/125 = 0.$$ 

It remains to show that for all solutions of $E_k$ and of $E_l$, Equation (7.1) is never satisfied. That can be done formally and quickly (less than 1min) as follows: compute a Gröbner basis for $E_k$, for $E_l$, put them together with Equation (7.1), then you get a system with a trivial Gröbner basis (see the code for TwoParallel in Subsection 8.2).

sage: %time TwoParallel(0)
CPU times: user 48.5 s, sys: 0 ns, total: 48.5 s
Wall time: 48.5 s

[1]

**Remark 7.2.** The next open candidate, mentioned in the introduction, has rank 9, FPdim 504 and shares the same type as $\text{Rep}(\text{PSL}(2, 8))$, albeit with slightly different fusion data. Applying Corollary 6.3 to it yields a subsystem of 45 equations with 34 variables. Paul Breiding recently found a solution using homotopy continuation [3].

**Remark 7.3.** We are exploring alternative methods to prove Theorem 7.1 possibly by leveraging the Drinfeld center. Our efforts to classify all potential induction matrices revealed 17,843,535 solutions for the square component (involving $I(1)$), suggesting that this approach might also present significant challenges.

The rest of this section will focus on exclusions in positive characteristic. As preliminaries, we will discuss pseudo-unitary fusion categories, character tables, formal codegrees, and the dual-Burnside property over the complex field.

**Definition 7.4** ([9], Definition 9.4.4). A fusion category $C$ over $\mathbb{C}$ is termed pseudo-unitary if $\dim(C) = \text{FPdim}(C)$.
According to [3] Proposition 9.5.1, a pseudo-unitary fusion category $C$ possesses a spherical structure such that \( \dim(X) = \FPdim(X) \) for every simple object $X$ in $C$.

**Definition 7.5.** Let $F$ be a commutative fusion ring. Denote its fusion matrices by $M_1, \ldots, M_r$, and let $D_i = \text{diag}(\lambda_{i,j})$, $i = 1, \ldots, r$ be their simultaneous diagonalization (which is well-defined because $M_i^* M_i = M_i M_i^*$). The character table of $F$ consists of the entries $(\lambda_{i,j})$. Conventionally, we take $\lambda_{i,1} = \|M_i\|$, which represents the Frobenius-Perron dimension of the corresponding simple object.

Here is the character table of $F_{210}$:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & -1 & -\zeta_7 & -\zeta_7^2 & -\zeta_7^2 & -\zeta_7 & 0 & 0 \\
5 & -1 & -\zeta_7^2 & -\zeta_7^2 & -\zeta_7^2 & -\zeta_7 & 0 & 0 \\
5 & -1 & -\zeta_7 & -\zeta_7^2 & -\zeta_7^2 & -\zeta_7 & 0 & 0 \\
6 & 0 & -1 & -1 & 1 & 1 & 1 & 1 \\
7 & 1 & 0 & 0 & 0 & 0 & \zeta_7 + \zeta_7^4 & \zeta_7^2 + \zeta_7^3 \\
7 & 1 & 0 & 0 & 0 & 0 & \zeta_7^2 + \zeta_7^3 & \zeta_7^5 + \zeta_7^6
\end{bmatrix}
\]

**Lemma 7.6.** Let $F$ be a commutative fusion ring of basis $\{b_1, \ldots, b_r\}$. A map $\chi : F \to C$ is a linear character of $F$ if and only if it is given by a column of its character table, i.e., there is $j$ such that $\chi(b_i) = \lambda_{i,j}$.

**Proof.** A linear character $\chi$ is a ring homomorphism, so $\chi(b_i)\chi(b_j) = \sum_k N_{i,j}^k \chi(b_k)$; in other words, $M_i\chi = \chi(b_i)v$ where $v$ is the vector $((\chi(b_j)))_s$, which turns out to be a common eigenvector for every fusion matrix $M_i$, with eigenvalue $\chi(b_i)$. So by definition of the character table, there is $j$ such that $\chi(b_i) = \lambda_{i,j}$. Conversely, let $v$ be a nonzero vector such that $M_i v = \lambda_{i,t} v$, for all $i$. By associativity, $M_i M_j v = \sum_k N_{i,j}^k M_k v$, so $\lambda_{i,t} \lambda_{j,t} = \sum_k N_{i,j}^k \lambda_{k,t}$. It follows that $\chi_j : b_i \to \lambda_{i,j}$ induces a linear character ($M_1$ being identity, $\chi_j(b_1) = 1$).

**Definition 7.7** [29], §2.3. Let $F$ be a commutative fusion ring of basis $\{b_1, \ldots, b_r\}$. The formal codegree of a linear character $\chi$ is $n_\chi := \sum_i |\chi(b_i)| \lambda_{i,j}$.

Following Lemma 7.6 and Definition 7.7, the formal degrees of a commutative fusion ring with character table $(\lambda_{i,j})$ are $(n_j) = (\sum_i |\lambda_{i,j}|)$. Note that the formal codegrees are the eigenvalues of the left multiplication matrix of $\sum_i b_i b_j^*$. A map $\chi : F \to C$ is a linear character of $F$ if and only if it is given by a column of its character table, i.e., there is $j$ such that $\chi(b_i) = \lambda_{i,j}$. Conventionally, we take $\lambda_{i,1} = \|M_i\|$, which represents the Frobenius-Perron dimension of the corresponding simple object.

**Definition 7.8** [3]. A commutative fusion ring $F$ is called dual-Burnside if for all linear character $\chi$, the following are equivalent:

- $\chi$ is non-vanishing, meaning that for all basic element $b$ then $\chi(b)$ is nonzero,
- the formal codegree $n_\chi = \FPdim(F)$.

Recall that a linear character of the character ring $\text{ch}(G)$ of a finite group $G$ corresponds to a conjugacy class $C$, and its formal codegree is the order of the centralizer subgroup $C_G(g)$, with $g \in G$. The equality between the formal codegree and the global $\FPdim$ (which is the order of $G$) means that $g$ is central. In particular, if $G$ is centerless, then $\text{ch}(G)$ is dual-Burnside if and only if all the columns of the character table, the first excepted, have a zero entry. The character ring of every non-abelian and non-alternating finite simple group is dual-Burnside except Mathieu groups $M_{23}$ and $M_{24}$.

**Lemma 7.9.** A pivotal categorification $C$ over $C$ of a dual-Burnside commutative fusion ring is pseudo-unitary.

**Proof.** By [3] Proposition 4.7.12 (involving a pivotal structure), the dimension function $\dim$ for the objects of $C$ induces a linear character $\chi$ on its Grothendieck ring, and [3] Definition 7.21.3) the categorical dimension $\dim(C)$ equals the formal codegree $n_\chi$. By [3] Proposition 4.8.4, $\dim(X)$ is nonzero for all simple object $X$ in $C$. Thus $\chi(b)$ is nonzero for all basic element $b$, and by dual-Burnside assumption, $\dim(C) = n_\chi = \FPdim(F)$, meaning pseudo-unitary.

Now, we are ready to consider the positive characteristic.

**Theorem 7.10.** Let $F$ be a fusion ring with a pivotal categorification $C$ over $F_p$. Then there is a non-vanishing (Definition 7.8) linear character $\chi : F \to F_p$. Moreover, if all the formal codegrees (Definition 7.7) are integers coprime with $p$, then $F$ admits a categorification over $C$.

**Proof.** By [3] Proposition 4.7.12 (again), the dimension function $\dim$ for $C$ induces a linear character $\chi : F \to F_p$. By [3] Proposition 4.8.4, $\dim(X)$ is nonzero for all simple object $X$ in $C$, meaning that $\chi$ is non-vanishing. Let $r$ be the rank of $F$. The usual ring epimorphism $n \mapsto [n]_p = n + pz$ from $Z$ to $Z/pZ = F_p$ induces a ring epimorphism $M \mapsto [M]_p$ from $M_r(Z)$ to $M_r(F_p)$, and a ring epimorphism $P \mapsto [P]_p$ from $Z[X]$ to $F_p[X]$. The formal codegrees are the eigenvalues of the left multiplication matrix $M \in M_r(Z)$ of $\sum_i b_i b_i^*$. But $\dim(C) = \chi(\sum_i b_i b_i^*)$ is an eigenvalue of $[M]_p$. Let $P$ be the characteristic polynomial of $M$, then $[P]_p$ is the characteristic polynomial of $[M]_p$. Let $(n_i)$ be the eigenvalues of $M$ then $P(X) = \prod_i (X - n_i)$. If the numbers $(n_i)$ are all integers, then $[P]_p(X) = \prod_i (X - [n_i]_p)$, meaning that $(\{\langle n\rangle\}_p)$ are the eigenvalues of $[M]_p$. If moreover the numbers $(n_i)$ are all coprime with $p$ then the numbers $(\{\langle n\rangle\}_p)$ are all nonzero, in
particular, dim(C) is nonzero. Thus, by [9] Theorem 9.16.1, C lifts to characteristic zero (with the same Grothendieck ring).

\[\Box\]

Corollary 7.11. The fusion ring \( \mathcal{F}_{210} \) admits no pivotal categorification in positive characteristic.

\textbf{Proof.} The formal degrees of \( \mathcal{F}_{210} \) are \((5, 6, 7, 7, 7, 210)\) as computed below:

```python
sage: M=[
    \ldots: [1,0,0,0,0,0,0], [0,1,0,0,0,0,0], [0,0,1,0,0,0,0], [0,0,0,1,0,0,0], [0,0,0,0,1,0,0], [0,0,0,0,0,1,0], [0,0,0,0,0,0,1],
    \ldots: [1,0,1,0,0,0,0], [0,1,1,0,0,0,0], [0,0,1,1,0,0,0], [0,0,0,1,1,0,0], [0,0,0,0,1,1,0], [0,0,0,0,0,1,1], [0,0,0,0,0,0,1],
    \ldots: [0,1,0,0,0,0,0], [1,1,0,0,0,0,0], [0,0,1,0,0,0,0], [0,0,0,1,0,0,0], [0,0,0,0,1,0,0], [0,0,0,0,0,1,0], [0,0,0,0,0,0,1],
    \ldots: [0,0,1,0,0,0,0], [0,1,1,0,0,0,0], [0,0,1,1,0,0,0], [0,0,0,1,1,0,0], [0,0,0,0,1,1,0], [0,0,0,0,0,1,1], [0,0,0,0,0,0,1],
    \ldots: [0,0,0,1,0,0,0], [0,0,0,1,1,0,0], [0,0,0,0,1,1,0], [0,0,0,0,0,1,1], [0,0,0,0,0,0,1], [0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,1],
    \ldots: [0,0,0,0,1,0,0], [0,0,0,0,1,1,0], [0,0,0,0,0,1,1], [0,0,0,0,0,0,1], [0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,1],
    \ldots: [0,0,0,0,0,1,0], [0,0,0,0,0,1,1], [0,0,0,0,0,0,1], [0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,1],
    \ldots: [0,0,0,0,0,0,1], [0,0,0,0,0,0,1,1], [0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,0,1],
    \ldots: [0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,1,1], [0,0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,0,1], [0,0,0,0,0,0,0,0,0,0,1]]
```

```
sage: MM=sum(matrix(n)*(matrix(m).transpose())) for m in M)
sage: MM.eigenvalues()
```

\( \{210, 6, 5, 5, 7, 7, 7\} \)

But \(210 = 2^1 3^1 5^1 7^1\), thus the formal degrees are integers coprime with \(p \notin \{2, 3, 5, 7\}\). So by Theorems 7.1 and 7.10 we are reduce to consider \(p \in \{2, 3, 5, 7\}\). But in these cases, the following computation shows that for all linear characters \( \chi : F \rightarrow \mathbb{F}_p \) there is a basic element \( b \) such that \( \chi(b) \) is zero.

```python
sage: for p in [2,3,5,7]:
    \ldots: F=GF(p); n=len(M)
    \ldots: R = PolynomialRing(F, n, 'd'); dim = R.gens()
    \ldots: Eq=[\text{dim}[i]*\text{dim}[j]-\text{sum}(M[i][j][k]*\text{dim}[k] for k in range(n)) for i in range(n) for j in range(n)]
    \ldots: \text{Eq.append(dim[0]-1)}
    \ldots: Id=R.ideal(Eq); G=Id.groebner_basis()
    \ldots: FF=Id.variety(F.algebraic_closure())
    \ldots: print(p,[prod((f[d] for d in dim)) for f in FF])
```

\(2 [0, 0, 0, 0, 0, 0, 0, 0]
3 [0, 0, 0, 0, 0, 0, 0, 0]
5 [0, 0, 0, 0, 0, 0, 0, 0]
7 [0, 0, 0, 0, 0, 0, 0, 0]

The result follows by Theorem 7.10.

\[\Box\]

Observe that the number of linear characters in characteristic \(p \in \{2, 3, 5, 7\}\) is less than the rank of \( \mathcal{F}_{210} \), suggesting that the ring \( \mathcal{F}_{210} \), induced by \( \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) is not semisimple.

Question 7.12. Is there a non-pivotal and/or non-semisimple categorification of \( \mathcal{F}_{210} \) in positive characteristic?

The fusion ring \( \mathcal{F}_{210} \) belongs to the family of interpolated simple integral fusion rings as described in [22], specifically corresponding to \( q = 6 \). We are interested in whether Theorem 7.1 can be generalized to all non-prime-power values of \( q \). This remains an open question for any \( q \neq 6 \). Should the theorem hold for these values, Corollary 7.11 would likely extend as well. Applying Corollary 6.3 to the case where \( q = 10 \), having rank 11, FDim 990, and type \([1, 1], [9, 5], [10, 1], [11, 4]\), results in five overdetermined subsystems. Each of these subsystems comprises 59 variables and 69 equations.

7.2. Fusion ring \( \mathcal{F}_{660} \). In this section, we discuss the simple integral fusion ring \( \mathcal{F}_{660} \). It is of rank 8, FDim 660 and type \([1, 1], [5, 2], [10, 2], [11, 1], [12, 2]\), with fusion matrices as follows,

```latex
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
```

Note that this fusion ring passed all former categorification criteria.

\textbf{Theorem 7.13.} The fusion ring \( \mathcal{F}_{660} \) cannot be categorized.

\textbf{Proof.} Apply Theorem 6.8 with \( (i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9) = (1, 3, 4, 1, 1, 3, 4, 2, 2) \).

\[\Box\]

In our extensive database [30], which contains thousands of fusion rings characterized as either simple, perfect integral, or of small rank and multiplicity, \( \mathcal{F}_{660} \) stands out as the only fusion ring excluded from categorification by Theorem 6.8 or 6.9 among those excluded from unitary categorification by the Schur product criterion [24].
8. Appendix

8.1. Counterexamples to Wang’s conjecture in even case. Let us check by GAP [12] what we stated about $\text{PSU}(3, 2)$ in §3.3.

```gap
gap> G:=PSU(3,2);
gap> Indicator(CharacterTable(G),2);
[ 1, 1, 1, -1, 1 ]
gap> M:=RepGroupFusionRing(G);
gap> M[6][6];
[ 1, 1, 1, 1, 2, 7 ]
```

The function RepGroupFusionRing computes the fusion matrices of the representation ring of a finite group.

RepsGroupFusionRing:=function(g)
local irr,n,M;
irr:=Irr(g); n:=Size(irr);
M:=List([1..n],i->List([1..n],j->List([1..n],k->ScalarProduct(irr[i]*irr[j],irr[k]))));
return M;
end;;

There are exactly three other such counterexamples up to order 128, given by

```gap
gap> G:=SmallGroup(128,n);
```

with $n = 764, 801, 802$ (the first is the one from [27]). Moreover, the smallest counterexample among the finite simple groups is $\text{PSU}(3, 5)$, of order 126000. They can all be checked as above.

8.2. TwoParallel code. Here is the SageMath code [32] of the function TwoParallel (used in the proof of Theorem 7.1):

```python
def TwoParallel(p):
    if p==0:
        F=QQ
    else:
        F=GF(p)
    R1.<u0,u1,u2,v0,v1,v2,v3,v4,v5,v6>=PolynomialRing(F,10)
    E1=[u0+7/F(5)*u1+7/F(5)*u2-4/F(125),
        5*v0+5*v1+3+7*v5+1/F(5),
        25*v0^2+25*v1^2+35*v3^2+35*v5^2-2-4/F(5),
        5*v0^3+5*v1^3+3+7*v5^3-v0^2+1/F(125),
        5*v0*v1^2+5*v1*v2^2+7*v3^2+7*v5^2+1/F(125),
        5*u0*v1-v1^2+7*u1*v3+7*u2*v5+1/F(125),
        5*v0+5*v1^2+7*v3^2+7*v6^2+1/F(125),
        5*u1+5*v2+7*v4^2+7*v6^2+1/F(5),
        25*v0*v1+25*v1^2+35*v4^2+35*v5^2+1/F(5),
        5*v0^2+5*v1^2+3+7*v5^2-4/F(5),
        5*v1^3+5*v2^3+7*v4^3+7*v6^3-v2^2+1/F(125),
        5*u0*v2-v2^2+7*u1*v4+7*u2*v6+1/F(125)]
    Id1=R1.ideal(E1)
    G1=Id1.groebner_basis() #list of solutions: print(Id1.variety(F.algebraic_closure()))
    C1=[g for g in G1] #explicit Groebner basis: print(C1)
    R2.<w0,w1,w2,z0,z1,z2,z3,z4,z5,z6>=PolynomialRing(F,10)
    E2=[w0+7/F(5)*w1+7/F(5)*w2-4/F(125),
        5*z0+5*z1+3+7*z3+7*z5+1/F(5),
        25*z0^2+25*z1^2+35*z3^2+35*z5^2-2-4/F(5),
        5*z0^3+5*z1^3+3+7*z5^3-w0+1/F(125),
        5*w0*z0-z0^2+7*w1*z3+7*w2*z5+1/F(125),
        5*z0*z1^2+5*z1*z2^2+7*z3^2+7*z5^2-z1^2+1/F(125),
        5*z1+5*z2^2+7*z4^2+7*z6^2+1/F(5),
        25*z0*z1+25*z1^2+35*z3^2+35*z5^2+1/F(5),
        5*z0^2+5*z1^2+3+7*z5^2-4/F(5),
        5*z1^3+5*z2^3+7*z4^3+7*z6^3-z2^2+1/F(125)]
    Id2=R2.ideal(E2)
    C2=[g for g in G2] #explicit Groebner basis: print(C2)
    R.<u0,u1,u2,v0,v1,v2,v3,v4,v5,v6,w0,w1,w2,z0,z1,z2,z3,z4,z5,z6>=PolynomialRing(F,20)
    C=C1+C2+[5*u0*z2 + 7*u1*z4 + 7*u2*z6 - u0 + 1/F(125)]
    Id=R.ideal(C)
    G=Id.groebner_basis()
    return G
```
