A Class of Linear Quadratic Gaussian
Hybrid Optimal Control Problems with
Realization–Independent Riccati Equations

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Abstract: A class of stochastic linear quadratic hybrid optimal control problems is presented for which
the Hamiltonian boundary conditions appearing in the associated necessary optimality conditions of
the Stochastic Hybrid Minimum Principle and Stochastic Hybrid Dynamic Programming are path-
dependent. Consequently, the linear quadratic Gaussian regulation problem associated with this class
of stochastic hybrid optimal control problems can be solved via (stochastic) Riccati equations which
are independent of the realization of stochastic diffusion terms. An analytic example of a scalar hybrid
system is provided to illustrate the results, and the relation to the deterministic case is discussed.

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1. INTRODUCTION

Linear Quadratic (LQ) problems constitute an extremely im-
portant class of optimal control problems, since they can model
many problems in applications, and more importantly, many
nonlinear control problems can be reasonably approximated by
the LQ problems. Moreover, solutions of LQ problems exhibit
elegant properties due to their simple and nice structures. For
deterministic linear quadratic optimal control problems, one
can employ the elementary method of completion of squares and
obtain an optimal control in a linear state feedback form via
the so-called Riccati equation (see e.g. Yong and Zhou (1999)).
Along this line, the solvability of the Riccati equation leads to
that of the LQ problem. It is interesting to note that both
the Minimum Principle (MP) by Pontryagin et al. (1962) and
Dynamic Programming (DP) by Bellman (1966) can lead to
the Riccati equation, by which one can see more clearly the
relationship between MP and DP.

For stochastic LQ problems, which are also called Linear
Quadratic Gaussian (LQG) problems, the method of comple-
tion of squares, the Stochastic Minimum Principle (SMP), and
Stochastic Dynamic Programming (SDP) all give rise to a
stochastic Riccati equation (see e.g. Yong and Zhou (1999)).
This equation is quite different from the conventional Riccati
equation arising in the deterministic LQ problems. One of the
main differences between the stochastic differential equations
appearing in stochastic optimal control problems and determin-
istic differential equations for deterministic problems is that
“time” cannot be reversed and solvability is interpreted as the
existence of solutions adapted solely to the forward filtration
(see e.g. Ma and Yong (1999)). This requires the introduction of
a notion of forward-backward stochastic differential equations
(FBSDE), first presented by Bismut (1978), and then elabo-
rated more in the optimal control framework by Bensoussan
(1983), Pardoux and Peng (1990), etc., and in the general

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theory of forward-backward stochastic differential equations
by Antonelli, Ma, Protter, Yong, Hu, and Peng (see e.g. Ma
and Yong (1999) and references therein). Stochastic Dynamic
Programming (SDP) including the Stochastic Hamilton-Jacobi-
Bellman (SHJB) equation are generally available only for
the stochastic Riccati equation are generally available only for
the stochastic LQ problem. It is interesting to note that both
the Stochastic Minimum Principle (SMP) are presented by Kushner
(1972); Kushner and Schwartz (1964), Haussmann (1986),
Bismut (1978), Bensoussan (1982), and Peng (1990). It has
been shown using both the SMP and SDP that a stochastic LQ
problem is well-posed if there are solutions to the stochastic
Riccati equation, and an optimal feedback control can then
be obtained via these solutions. Although the stochastic LQ
problem can be reduced to that of solving the stochastic Ric-
cati equation, the existence and uniqueness of the solutions
to the stochastic Riccati equation are generally available only
for certain special cases (see e.g. Yong and Zhou (1999)).

The optimal control of stochastic hybrid systems, i.e. control
systems that involve the interaction of continuous dynamics,
discrete dynamics and stochastic diffusions, has been the sub-
ject of a limited number of studies. Versions of non-classical
stochastic optimal control problems have been studied by Wu
and Zhang (2011), Shi and Wu (2010a,b), etc. but the class of
problems addressed lack many of the key features of hybrid
systems, most notably changes in dynamics and costs. In the
context of Stochastic Hybrid Dynamic Programming (SHDP),
Bensoussan and Menaldi (2000) presents the optimality con-
ditions for infinite horizon problems where optimal controls
are stationary. In the context of the Stochastic Hybrid Hybrid
Minimum Principle (SHMP) Aghayeva and Abushov (2011)
presented the optimality conditions for controlled switchings
only, and Pakniyat and Caines (2016b) presented the SHMP for
a general class of stochastic hybrid systems where autonomous
and controlled state jumps at switching instants are allowed
to be accompanied by changes in the dimension of the state
space. A feature of special interest in this work is the effect of
hard constraints imposed by switching manifolds on diffusion-driven state trajectories.

In contrast to the stochastic case, the optimal control of deterministic hybrid systems has been the subject of numerous studies. The generalization of the fundamental Pontryagin Minimum Principle (PMP) results in the Hybrid Minimum Principle (HMP). The formulation by Clarke and Vinter (1989a,b), referred to by them as “Optimal Multiprocesses”, provides a Minimum Principle for hybrid systems of a very general nature in which switching conditions are regarded as constraints in the form of set inclusions and the dynamics of the constituent processes are governed by (possibly nonsmooth) differential inclusions. A similar philosophy is followed by Sussmann (1999a,b) where a nonsmooth Minimum Principle is presented for hybrid systems possessing a general class of switching structures. Due to the generality of the results by Clarke and Vinter (1989a,b); Sussmann (1999a,b), degeneracy is not precluded and therefore, additional hypotheses need to be imposed to make the HMP results significantly informative (see e.g. Caines et al. (2006) for more discussion); such hypotheses (typically of a controllability nature) are usually too restrictive to cover many practical problems of engineering interest. An alternative philosophy, followed by Shaikh and Caines (2007), Garavello and Piccoli (2005), Taringoo and Caines (2011, 2013), and Pakniyat and Caines (2017b) is to ensure the validity of the HMP in a non-degenerate form by introducing hypotheses on the dynamics, transitions and switching events. To name a few other versions of the HMP in its appearances within the development of optimal control theory one cites the work of Lygeros et al. (2007); Schöllig et al. (2007) and Shaikh and Caines (2009), In Pakniyat and Caines (2016b) the deterministic framework is to ensure the validity of the HMP in the work of Hedlund and Rantzer (2002), Caines et al. (1998), Barles et al. (2010); Dharmatti and Ramaswamy (2012) for more discussion); such hypotheses (typically of a retic philosophy, followed by Shaikh and Caines (2017b) is to ensure the validity of the HMP in

In this paper, a class of Linear Quadratic Gaussian Hybrid Optimal Control Problems (LQG-HOCP) is presented for which the Hamiltonian boundary conditions are path-independent and therefore, the corresponding stochastic Riccati equations are independent from the realization of stochastic diffusion terms. An analytic example is provided to illustrate the results, and the relation to the deterministic case is discussed.

2. A CLASS OF LINEAR QUADRATIC GAUSSIAN HYBRID OPTIMAL CONTROL PROBLEMS

Let \((\Omega, \mathcal{F}, P)\) be a probability space with filtration \(\mathcal{F}_t\), and \(w(\cdot)\) be a standard Wiener process. Consider a class of LQG-HOCP with completely observed states, i.e. \(\mathcal{F}_t^t = \sigma\{w(s) : 0 \leq s \leq t\}\), which is the natural filtration associated with the sigma-algebra generated by the Wiener process.

Consider a hybrid system possessing linear vector fields in the form of

\[
dx_q(t) = (A_q x_q + B_q u_q) dt + G_q dw, \quad t \in [t_i, t_{i+1}),
\]

where \(q_i \in Q, x_q \in \mathbb{R}^{n_q}, u_q \in \mathbb{R}^{m_q}, A_q, B_q \in \mathbb{R}^{n_q \times n_q}, G_q \in \mathbb{R}^{n_q}, 0 \leq i \leq L, t_{i+1} = t_{i+1}.\) The initial condition \((q(x)(t_0) = (q_0, x_0))\) is assumed to be deterministically known at the initial time \(t_0.\) In this paper, we only consider prearranged controlled switchings, which result in a fixed sequence of discrete states \(q_1, q_2, \ldots, q_L,\) and at the switching instances \(t_j, 1 \leq j \leq L,\) which are decision variables, i.e. not a priori determined, the state jump maps are provided as

\[
x_{q_j}(t_j) = \Psi_{q_j, x_{q_{j-1}}(t_j^-)}(t_j^-) \equiv \Psi_{q_{j-1}, q_j}(x_{q_{j-1}}(t_j^-)).
\]

Consider the LQG-HOCP for the hybrid cost

\[
J = \frac{1}{2} \mathbb{E} \left\{ \sum_{i=0}^{L-1} \int_{t_i}^{t_{i+1}} \left( \|x_q(t)\|^2_{q_i} + \|u_q(t)\|^2_{R_q} \right) \, dt + \|x_{q_L}(t_f)\|^2_{H_{q_L}} \right\},
\]

where \(0 \leq t_i \leq L_q, x_q \in \mathbb{R}^{n_q}, u_q \in \mathbb{R}^{m_q}, 0 < R_q \leq R_q, 0 \leq H_q \leq H_q, x_q \in \mathbb{R}^{n_q}, u_q \in \mathbb{R}^{m_q}.
\]

It is further assumed that

\[
G_{q_k} = \Psi_{q_k-1, q_k} G_{q_{k+1}},
\]

for all \(1 \leq k \leq L,\) which implies equivalent diffusion fields before and after switching events.

3. NECESSARY OPTIMALITY CONDITIONS OF THE STOCHASTIC HYBRID MINIMUM PRINCIPLE

In order to determine the necessary optimality conditions of the Stochastic Hybrid Minimum Principle (SHMP) established in Pakniyat and Caines (2016b), we form the family of system Hamiltonians as

\[
H_q(x_q, u_q, \lambda_q, K_q) = \frac{1}{2} \left( \|x_q\|^2_{L_q} + \|u_q\|^2_{R_q} \right) + \lambda_q^T (A_q x_q + B_q u_q) + K_q^T G_q,
\]

where \(\lambda_q, x_q, u_q, K_q, L_q, R_q, H_q\) are the adjoint process, the state, the control, the costate, the Riccati matrix, the linearization parameter, and the Hamiltonian, respectively.
with \( \lambda_q \in \mathbb{R}^{n_q}, K_q \in \mathbb{R}^{n_q} \). Then, according to the SHMP, for the optimal input \( u^o \) and the corresponding trajectory \( x^o \) there exists \( \lambda^o, K^o : \mathcal{S}^T \) adapted, such that
\[
H_q(x^o_q, u^o_q, \lambda^o_q, K^o_q) \leq H_q(x^o_q, v, \lambda^o_q, K^o_q),
\]
almost everywhere \( t \in [t_0, t_f] \), almost surely for all \( v : \mathcal{S}^T \) measurable random variables in \( \mathbb{R}^{n_q} \), that is to say, the Hamiltonian is minimized with respect to the control input, which determines the optimal continuous (valued) input as
\[
u^o_q = -R_q^{-1}B_q^T \lambda^o_q,
\]
and further, the pairs of states and adjoint processes satisfy the following stochastic Hamiltonian canonical equations
\[
dx_q^o = \frac{\partial H_q}{\partial \lambda_q} (x^o_q, u^o_q, \lambda^o_q, K^o_q) dt + \frac{\partial H_q}{\partial K_q}(x^o_q, u^o_q, \lambda^o_q, K^o_q) dW, \tag{7}
\]
\[
d\lambda^o_q = -\frac{\partial H_q}{\partial x_q}(x^o_q, u^o_q, \lambda^o_q, K^o_q) dt + K^o_q dw, \tag{8}
\]
almost everywhere \( t \in [t_0, t_f] \), subject to
\[
x^o_q(t_0) = x_0, \tag{9}
\]
\[
x^o_q(t_f) = \Psi_{q_{t_f-1}} \Psi_{q_{t_f-2}} \cdots \Psi_{q_{t_f-j}} x^o_q (t_{j-1}), \tag{10}
\]
\[
\lambda^o_q(t_f) = H_q(x^o_q(t_f), \lambda^o_q(t_f)), \tag{11}
\]
\[
\lambda^o_{q_{t_f-1}} (t_{j-1}) = \left[ \Psi_{q_{t_f-1-j}} \right]^T \lambda^o_q (t_f). \tag{12}
\]
Moreover, at a switching time \( t \), the Hamiltonian satisfies
\[
H_{q_{t_f-1}} (x_{q_{t_f-1}}, u_{q_{t_f-1}}, \lambda^o_{q_{t_f-1}}, K^o_{q_{t_f-1}}) - K^o_{q_{t_f-1}} G_{q_{t_f-1}} |_{t_f-1} = H_{q_{t_f-1}} (x^o_{q_{t_f-1}}, u^o_{q_{t_f-1}}, \lambda^o_{q_{t_f-1}}, K^o_{q_{t_f-1}}) - K^o_{q_{t_f-1}} G_{q_{t_f-1}} |_{t_f-1}. \tag{14}
\]

### 4. HYBRID STOCHASTIC RICCATI EQUATIONS

We conjecture that \( x^q_i \) and \( \lambda^o_i \), \( 0 \leq i \leq L \), are related by
\[
\lambda^o_i(t) = \Pi_i(t) x^q_i(t), \tag{15}
\]
with \( \Pi_i(t) \in \mathcal{C}^1([t_f, t_{f+1}], \mathbb{R}^{n_q \times n_q}) \). Applying Ito’s formula to (15) (see e.g. Yong and Zhou (1999)) with appropriate substitution of (8) to (13) (see also Pakniyat and Caines (2016a)) one obtains
\[
K^o_i(t) = \Pi_i(t) G_{q_i}, \tag{16}
\]
and
\[
\Pi_i = \Pi_i B_{q_i} R_{q_i}^{-1} B_{q_i}^T \Pi_i - \Pi_i A_{q_i} - A_{q_i}^T \Pi_i - L_{q_i}, \tag{17}
\]
subject to
\[
\Pi_{q_{t_f}} (t_f) = H_{q_{t_f}}, \tag{18}
\]
\[
\Pi_{q_{t_f-1}} (t_{j-1}) = \Psi_{q_{t_f-1-j}} \Pi_{q_{t_f-1}} (t_{j-1}), \tag{19}
\]
and from (14) we obtain
\[
\lambda^o_{q_{t_f-1}} (t_{j-1}) = \left[ \Psi_{q_{t_f-1-j}} \right]^T \lambda^o_{q_{t_f}} (t_f). \tag{20}
\]

### 5. ILLUSTRATIVE EXAMPLE

#### 5.1 Problem Formulation

Consider the following scalar hybrid system for which the continuous state is governed by the following stochastic differential equations:
\[
dx_1 = \left( \frac{31}{16} x_1 + u_1 \right) dt + g_1 dw, \tag{22}
\]
\[
dx_2 = \left( \frac{3}{8} x_2 + u_2 \right) dt + g_2 dw, \tag{23}
\]
with \( g_1 = 1, g_2 = \sqrt{2} g_1 = \sqrt{2} \), and the performance measure is given as
\[
J(t_0, t_f, h_0, L, h_L) := \mathbb{E} \left[ \frac{1}{2} \int_{t_0}^{t_f} \left( (u_1(t))^2 + \frac{1}{2} (x_1(t))^2 \right) dt \right.
\]
\[
+ \left. \frac{1}{2} \int_{t_0}^{t_f} \left( (u_2(t))^2 + \frac{1}{4} (x_2(t))^2 \right) dt + \frac{1}{2} \times 6 (x_2(t_f))^2 \right], \tag{24}
\]
where \( t_f \) indicates the time of a controlled switching with the jump map \( x_2(t_f) = \sqrt{2} x_1(t_f) \).

#### 5.2 Analytical Solution of the Riccati Equations

The associated stochastic Riccati equations in Section 4 are written as
\[
\Pi_1 = \Pi_1^2 - \frac{31}{8} \Pi_1 - \frac{1}{2} = (\Pi_1 - 4) \left( \Pi_1 + \frac{1}{8} \right), \tag{25}
\]
\[
\Pi_2 = \Pi_2^2 - \frac{3}{4} \Pi_2 - \frac{1}{4} = (\Pi_2 - 1) \left( \Pi_2 + \frac{1}{4} \right), \tag{26}
\]
which are subject to the terminal and boundary conditions
\[
\Pi_2 (t_f) = 6, \tag{27}
\]
\[
\Pi_1 (t_f) = \left( \sqrt{2} \right)^2 \Pi_2 (t_f) = 2 \Pi_2 (t_f), \tag{28}
\]
\[
(\Pi_1 (t_f))^2 - \frac{31}{8} \Pi_1 (t_f) - \frac{1}{2} = 2 \left( (\Pi_2 (t_f))^2 - \frac{3}{4} \Pi_2 (t_f) - \frac{1}{4} \right). \tag{29}
\]

The above equations possess path-independent solutions in the form of
\[
\Pi_2 (t) = \frac{k_2 e^{x^2 t} + \frac{1}{2}}{k_2 e^{x^2 t} - 1}, \tag{30}
\]
\[
\Pi_1 (t) = \frac{k_2 e^{x^2 t} + \frac{1}{2}}{k_2 e^{x^2 t} - 1}, \tag{31}
\]
Fig. 1. A sample path for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2, g_1 = 1, g_2 = \sqrt{2}$.

Fig. 2. Ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2, g_1 = 0.1, g_2 = 0.1\sqrt{2}$.

Fig. 3. Ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians in the example with $t_f = 1$ and $x_0 = 2, g_1 = 1, g_2 = \sqrt{2}$.

Fig. 4. The corresponding deterministic trajectories ($g_1 = g_2 = 0$) for the example with $t_f = 1$ and $x_0 = 2$. 
where

\[
k_2 = \frac{5}{4} e^{\frac{3t_f}{4}},
\]

\[
t_f = \frac{4}{5} \ln \left( \frac{17}{27} \right),
\]

\[
k_1 = \frac{17}{6} e^{\frac{3t_f}{6}}.
\]

5.3 Simulations

For the values of \( t_f = 1, x_0 = 2, g_1 = 1, g_2 = \sqrt{2} \), the simulations for a sample path of continuous states, adjoint processes, continuous inputs and Hamiltonians are illustrated in Figure 1. A collection of ten sample paths for continuous states, adjoint processes, continuous inputs and Hamiltonians for the same values is presented in Figure 2. For smaller values of diffusion coefficients \( g_1, g_2 \), trajectories are less influenced by diffusion terms and the results more closely resemble those of the deterministic LQ problem. This is illustrated in Figure 3 for the values of \( g_1 = 0.1, g_2 = 0.1\sqrt{2} \) and in Figure 4 for the corresponding deterministic case with \( g_1 = g_2 = 0 \). It is observed in these figures that, in contrast to the deterministic case, Hamiltonian functions are not constants for stochastic hybrid optimal control problems.

6. CONCLUDING REMARKS

The linear quadratic Gaussian hybrid optimal control problems studied in this paper constitute a class of LQG-HOCP whose associated Riccati equations are independent form realizations of stochastic diffusions. In this paper we derive the (hybrid) stochastic Riccati equations using the Stochastic Hybrid Minimum Principle (SHMP). As proved in the case of deterministic hybrid optimal control problems (see e.g. Pakniyat and Caines (2014b, 2016a)), the adjoint process in the HMP and the gradient of the value function in Hybrid Dynamic Programming (HDP) are identical to each other almost everywhere. This intrinsic relation becomes an essential equivalence in the case of LQ-HOCPs. Due to the existence of the same relationship between the Stochastic Minimum Principle (SMP) and Stochastic Dynamic Programming (SDP) and the same equivalence in the LQG case (see e.g. Yong and Zhou (1999)), it is natural to expect the adjoint process in the SHMP and the gradient of the value function in Stochastic HDP to be identical almost everywhere. Indeed, the formulation of SHDP, the investigation of its relationship to the SHMP, and the demonstration of the equivalence of the SHMP and SHDP in the LQG-HOCP case is the subject of another study expected to be presented in a consecutive paper.

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