Analytic calculation of Witten index in $D = 2$ supersymmetric Yang-Mills quantum mechanics

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Abstract

We propose a method for the evaluation of Witten index in $D = 2$ supersymmetric Yang-Mills quantum mechanics. We rederive a known result for the $SU(2)$ gauge group and generalize it to any $SU(N)$ gauge group.

1 Introduction

Witten index[1], denoted by $I_W(T)$, was introduced as a tool to investigate the spontaneous supersymmetry breaking. The quantity

$$I_W(T) = \sum_i \left( e^{-T E_i^{\text{b}} - T E_i^{\text{f}}} - e^{-T E_i^{\text{b}} - T E_i^{\text{f}}} \right)$$

has the advantage of being an example of an topological index. Hence, it may be calculated in perturbation theory in the weak coupling regime and continued to the strong coupling regime. As was argued in the original article by Witten[1], $I_W(T)$ should be a nonzero integer in order for supersymmetry to be unbroken. The argument is straightforward for a system with discrete spectrum. All positive eigenenergies must be paired into supermultiplets and hence do not contribute to the Witten index. Contributions come only from the non-degenerate supersymmetric vacua. Hence, a nonzero value of $I_W(T)$ signifies the existence of such vacua and therefore points out that supersymmetry is not broken. In cases when $I_W(T)$ vanishes such argument is not conclusive. In this work we present an analytic
evaluation of the Witten index of $D = 2$ supersymmetric Yang-Mills quantum mechanics\cite{2}. Supersymmetric Yang-Mills quantum mechanics which are just dimensionally reduced to a single point in space quantum field theories have attracted a lot of attention due to their relations to, among other, the dynamics of supermembranes\cite{3, 4} or the dynamics of D0 branes in M-theory\cite{5}. Up to now, the Witten index was calculated using very refined techniques for the higher dimensional models\cite{6, 7}, namely with $D > 2$, but not for $D = 2$. A numerical Monte Carlo approach for evaluating $I_W(T)$ was recently discussed in Ref.\cite{8}.

In the following we start by briefly describing the $D = 2$ supersymmetric Yang-Mills quantum mechanics and recalling the properties of their spectra. We concentrate on the degeneracy induced by the particle-hole symmetry and on the relation of the eigenenergies to the zeros of Laguerre polynomials. Consequently, in the following section we remind some analytic results concerning the asymptotic distribution of zeros of Laguerre polynomials, which turn out to be needed for the evaluation of $I_W(T)$. The calculation of $I_W(T)$ is presented in the subsequent sections. We analyze separately the models with $SU(2)$ and $SU(3)$ gauge symmetry and show some numerical results which support our approach. Then, we consider the generic models with any $SU(N)$ gauge symmetry and finish with some conclusions.

2 Description of the systems

$D = 2$ supersymmetric Yang-Mills quantum mechanics (SYMQM) were introduced by Claudson and Halpern\cite{2}. They may be thought of as systems obtained through dimensional reduction of quantum $D = 2$ supersymmetric Yang-Mills quantum field theories to a single point in space\cite{9}. They consist of a set of $N^2 - 1$ real scalar fields and a set of $N^2 - 1$ fermion fields, both transforming in the adjoint representation of the $SU(N)$ group. They represent the simplest models of supersymmetric Yang-Mills quantum mechanics.

2.1 Cut Fock basis approach

The cut Fock space approach to supersymmetric Yang-Mills quantum mechanics was proposed by Wosiek\cite{10} and described in details in Ref.\cite{11}. This approach was used to solve analytically the model with $SU(3)$ gauge group\cite{12, 13} and subsequently the generic models with $SU(N)$ gauge symmetry\cite{14}.
The Hilbert space of SYMQM models is composed of states invariant under global $SU(N)$ rotations as a consequence of the imposition of the dimensionally reduced Gauss law. It can be approximated by a subspace spanned by the Fock states having less than $N_{cut}$ quanta. We call $N_{cut}$ the cut-off. For any finite value of $N_{cut}$ the spectra of all quantum systems are discrete. Hence, such cut-off can be used as a regularization of systems possessing continuous spectra. The physical eigenenergies are obtained in the $N_{cut} \to \infty$ limit.

The basis states are constructed using the bosonic and fermionic bricks. For a given $N$ we define $N - 1$ elementary bosonic bricks of the form (using the matrix notation),

$$C_N^\dagger(2) \equiv \text{tr}(a^\dagger a^2), \ C_N^\dagger(3) \equiv \text{tr}(a^\dagger a^3), \ldots, \ C_N^\dagger(N) \equiv \text{tr}(a^\dagger a^N),$$

and any basis state is obtained from the Fock vacuum $|0\rangle$ as

$$|p_2, p_3, \ldots, p_N\rangle = C_N^\dagger(2)^{p_2} C_N^\dagger(3)^{p_3} \cdots C_N^\dagger(N-1)^{p_{N-1}} C_N^\dagger(N)^{p_N} |0\rangle.$$ (2)

Indeed, the states eq.(2) are linearly independent and provide a complete basis of the cut bosonic Hilbert space.

In addition to the elementary bosonic bricks, one also has $d^{n_F}(N)$ composite fermionic bricks in the sector with $n_F$ fermionic quanta. We denote these bricks by $C_N^\dagger(n_B^\alpha, n_F, \alpha)$, where $n_B^\alpha$ corresponds to the number of bosonic creation operators and $n_F$ to the number of fermionic creation operators present in $C_N^\dagger(n_B^\alpha, n_F, \alpha)$. $\alpha, 1 \leq \alpha \leq d^{n_F}(N)$, is an additional index needed to differentiate two fermionic bricks with equal numbers $n_B^\alpha$ and $n_F$.

Fermionic basis states can be obtained through the action of appropriate fermionic bricks on the bosonic basis states eq.(2),

$$|p_2, p_3, \ldots, p_N, \alpha\rangle = C_N^\dagger(n_B^\alpha, n_F, \alpha)|p_2, p_3, \ldots, p_N\rangle.$$ (3)

2.2 Spectra at finite cut-off

The Hamiltonian of SYMQM with $SU(N)$ symmetry reduces to a free Hamiltonian in the physical Hilbert space. It must be a $SU(N)$ singlet, so expressed in terms of creation and annihilation operators, it has the form,

$$H = (a^\dagger a) + \frac{N^2 - 1}{4} - \frac{1}{2}(a^\dagger a^\dagger) - \frac{1}{2}(a a).$$ (4)

It was shown in Refs. that the spectra of these models for finite cut-off $N_{cut}$ are given exclusively by the zeros of appropriate generalized
Laguerre polynomials. As it was described there, the set of all solutions can be divided into disjoint subsets called families. These sets can be labeled by the maximal powers of elementary bricks present in the decomposition of energy eigenstates in the Fock basis. Thus, the solutions belonging to the family denoted by \((t_3, t_4, \ldots, t_N)\) contain at most a \(t_3\) power of the \((a^{\dagger 3})\) brick, a \(t_4\) power of the \((a^{\dagger 4})\) brick and so on. The family \((0, 0, \ldots, 0)\) contains solutions build of \((a^{\dagger 2})\) exclusively. In the fermionic sector the families must be labeled by an additional index \(\alpha\), \((t_3, t_4, \ldots, t_N; \alpha)\) which denotes the fermionic brick multiplying the component proportional to the state \(|t_2, t_3, \ldots, t_N\rangle\).

For a finite cut-off, each family contain a finite number of eigen solutions. The corresponding eigenenergies are given by quantization conditions, one per family. All quantization conditions have the same form, namely \(L^\gamma_m(E) = 0\), where \(L^\gamma_m(x)\) is the generalized Laguerre polynomial of index \(\gamma\), order \(m\) and variable \(x\). It was shown \[14\] that the index \(\gamma\) depends on the family labelling in the following way

\[
\gamma = 3t_3 + 4t_4 + \cdots + Nt_N + n_0^B + \frac{1}{2}(N^2 - 1) - 1, \tag{5}
\]

where the integers \(n_0^B\) denote the number of bosonic creation operators in the \(\alpha\)-th fermionic brick.

The complete spectrum \(\{E\}\) of the model with \(SU(N)\) symmetry in the sector with \(n_F\) fermionic quanta can be written in a compact form with the help of a polynomial \(\Theta_{N_{\text{cut}}}^{n_F}(N, E)\), i.e.

\[
\{E\} = \{E : \Theta_{N_{\text{cut}}}^{n_F}(N, E) = 0\}, \tag{6}
\]

where \[14\]

\[
\Theta_{N_{\text{cut}}}^{n_F}(N, E) = \prod_{\alpha=1}^{d^{n_F}(N)} \left\{ \prod_{i=3}^{N} \left[ \prod_{t_i=0}^{N - \left( \sum_{s=3}^{N} s t_s \right) - n_0^B(N)} \frac{1}{2} \left( N_{\text{cut}} - \left( \sum_{s=3}^{N} s t_s \right) - n_0^B(N) \right) \right] \right\}^{1 \frac{1}{2} \left( N_{\text{cut}} - \left( \sum_{s=3}^{N} s t_s \right) - n_0^B(N) \right) + 1 - 1 + n_0^B(N) + \frac{1}{2} \left( N^2 - 1 \right) - 1 + n_0^B(N)} \tag{7}
\]

The product over \(\alpha\) corresponds to the contribution coming from every fermionic brick in the sector with \(n_F\) fermionic quanta. \(d^{n_F}(N)\) describes the number of independent fermionic bricks in sector with \(n_F\) fermionic quanta and obviously must depend on \(N\). Similarly, \(n_0^B(N)\) stands for the number of bosonic creation operators in the \(\alpha\)-th fermionic brick, which depends on
both \( n_F \) and \( N \). The appearance of the integers \( n^A_B(N) \) is crucial for the analysis which follows. Next, there are \( N - 2 \) products over the variables \( t_3, \ldots, t_N \). In fact, there are as many families as there are partitions of the numbers \( 0, 1, 2, \ldots, N_{\text{cut}} - n^A_B \) into the numbers \( 3, 4, \ldots, N \), which is taken into account by the upper limit of these products.

### 3 Particle-hole symmetry

Apart of supersymmetry, the SYMQM models have another symmetry which has important consequences for their spectra, namely the particle-hole symmetry. We describe the latter in this section.

Hamiltonian eq. (4) is a particular case of an operator invariant under the following transformation (see also Ref.[15])

\[
x \rightarrow -x, \quad p \rightarrow -p, \quad f \rightarrow f^\dagger, \quad f^\dagger \rightarrow f.
\]

By the same transformation the supercharges corresponding to eq. (4) transform under (8) as \( Q \rightarrow Q^\dagger \) and \( Q^\dagger \rightarrow Q \). The canonical commutation and anticommutation relations remain unchanged,

\[
[x_a, p_b] \rightarrow [-x_a, -p_b] = i\delta_{a,b}, \quad \{f_a, f_b^\dagger\} \rightarrow \{f_a^\dagger, f_b\} = \delta_{a,b}.
\]

Therefore, there exist an unitary operator \( U \) which realizes such transformation in the Hilbert space. Obviously, \( U^2 = 1 \), and hence, \( U^\dagger = U \). We have,

\[
U f_a U^\dagger = f_a, \quad U f_a^\dagger U = f_a^\dagger, \quad U x_a U = -x_a, \quad U p_a U = -p_a.
\]

One can check that the image of the Fock vacuum under \( U \) is an eigenstate of the fermion occupation number operator \( \text{tr}(f^\dagger f) \). Indeed,

\[
\text{tr}(f^\dagger f)U|0\rangle = f_a^\dagger f_a U|0\rangle = U f_a U U f_a^\dagger U U|0\rangle = U f_a f_a^\dagger |0\rangle = (N^2 - 1)U|0\rangle,
\]

where we used the fact that \( a = 1, \ldots, N^2 - 1 \). Thus, the state \( U|0\rangle \) is an eigenstate of \( \text{tr}(f^\dagger f) \) to the eigenvalue \( N^2 - 1 \) which is the maximal value allowed by the Pauli exclusion principle. We denote such state by \( |1\rangle \equiv U|0\rangle \).

A generic bosonic state \( |E\rangle_0 \) can be written as

\[
|E\rangle_0 = \sum_{n_B=0}^{\infty} f_{n_B}(a^\dagger; E)|0\rangle,
\]

where

\[
f_{n_B}(a^\dagger; E) = \int \prod_{j=1}^{n_B} \frac{d\phi_j}{2\pi} e^{\frac{i}{2} \sum_{j<k} \phi_j \phi_k} e^{\frac{i}{2} \sum_{j<k} \phi_j \phi_k} e^{-\phi_j^2}.
\]
with the coefficients $f_{n_B}(a^\dagger; E)$ being operators constructed with $n_B$-th power of the $a^\dagger$ operator, whereas numerical factors are chosen so that $H|E\rangle_0 = |E\rangle_0$. Therefore,

$$|E\rangle_{N^2-1} \equiv U|E\rangle_0 = \sum_{n_B=0}^{\infty} (-1)^{n_B} f_{n_B}(a^\dagger; E)|1\rangle. \quad (13)$$

A simple generalization of the above observation to the sector with $n_F$ fermionic quanta is the following. For a state $|E\rangle_{n_F}$ we have

$$|E\rangle_{n_F} = \sum_{n_B=0}^{\infty} f_{n_B,n_F}(a^\dagger, f^\dagger; E)|0\rangle, \quad (14)$$
Figure 1: Schematic structure of supermultiplets in models with different $SU(N)$ gauge groups. Marks on the horizontal axis denote fermionic sectors with consecutive number of fermionic quanta. For the $SU(2)$ model (figure 1(a)) $n_F = 0, \ldots, 3$, whereas for the $SU(3)$ model (figure 1(b)) $n_F = 0, \ldots, 8$. Figures 1(c) and 1(d) show a generic situation for $N$ even and odd. Horizontal intervals connecting neighboring fermionic sectors represent possible supermultiplets constructed with states from these sectors. Note the degeneracy of the spectrum in the middle sector (with $n_F = \frac{1}{2}(N^2-1)$) in models with $N$ odd. The $SU(2)$ model is somewhat special, since parity forbids the connection of states coming from sectors with $n_F = 1$ and $n_F = 2$.

and the image of this state under $U$ has the following decomposition,

$$|E\rangle_{N^2-1-k_F} \equiv U|E\rangle_{k_F} = \sum_{n_B=0}^{\infty} (-1)^{n_B} f_{n_B,k_F}(a^\dagger, f; E)|1\rangle.$$

(15)
If $|E\rangle_{nF}$ is an energy eigenstate to the eigenvalue $E$, $H|E\rangle_{nF} = E|E\rangle_{nF}$, the energy of the state $|E\rangle_{N^2-1-nF}$ is,

$$H|E\rangle_{N^2-1-nF} = HU|E\rangle_{nF} = EU|E\rangle_{nF} = E|E\rangle_{N^2-1-nF},$$

(16)

Hence, the particle-hole symmetry generates a double degeneracy of the spectrum. To each eigenenergy coming from the sector with $n_F$ fermionic quanta, $n_F \leq \frac{1}{2}(N^2 - 1)$, corresponds an equal eigenenergy in the sector with $N^2 - 1 - n_F$ fermionic quanta.

It follows that for the models with $SU(N)$ gauge groups with $N$ odd the spectrum in the middle sector (with $n_F = \frac{1}{2}(N^2 - 1)$) has a double degeneracy. The states from this sector form supermultiplets with both the left neighboring sector (with $n_F = \frac{1}{2}(N^2 - 1) - 1$) and the right neighboring sector (with $n_F = \frac{1}{2}(N^2 - 1) + 1$). The particle-hole symmetry requires that the spectra of the latter two sectors were identical. Hence, the spectrum of the sector with $n_F = \frac{1}{2}(N^2 - 1)$ is doubly degenerate. There is no such effect for the models with $N$ even. A schematic figure [11] depicts the above arguments. The difference in the structure of supermultiplets in models with $N$ even and $N$ odd can be seen by comparing figures [1(a) and 1(c)] with [1(b) and 1(d)].

The above discussion has immediate consequences for the Witten index. The double degeneracy of the spectrum implies that $I_W(T)$ vanishes for any $N$ even. In these cases the degenerate sectors with $n_F$ and $N^2 - 1 - n_F$ fermions have opposite parities under $(-1)^{n_F}$. For $N$ odd, one can define the restricted Witten index, denoted by $I_W(T)_R$, which is the sum over a single copy of eigenenergies. Thus, $I_W(T)_R = \frac{1}{2}I_W(T)$ for $N$ odd. Note that the case of $N = 2$ is somehow special, since parity forbids the connection between sectors $n_F = 1$ and $n_F = 2$. Hence, $I_W(T)_R$ can be also defined for this model and indeed is nontrivial [16]. The introduction of the cut-off does not break the particle-hole symmetry; therefore $I_W(T)_R$ is a well defined quantity also at any finite cut-off.

We now proceed with the evaluation of $I_W(T)_R$ for any $N$ odd. However, before we present explicit computations we remind some properties of the distributions of zeros of Laguerre polynomials.
4 Moments of the zeros distribution of Laguerre polynomials

The distribution of zeros of Laguerre polynomial of index $\gamma$ can be defined for a polynomial of any order $N$ as

$$\rho_N^\gamma(E) = \frac{1}{N} \sum_{i=1}^{N} \delta(E - E_i(N)),$$  \hspace{1cm} (17)

where $L_N^\gamma(E_i(N)) = 0$. Let us also introduce the moments of $\rho_N^\gamma(E)$ as

$$\mu_n^\gamma(N) = \frac{1}{N} \sum_{i=1}^{N} (E_i(N))^n = \int_{-\infty}^{\infty} E^n d\rho_N^\gamma(E).$$  \hspace{1cm} (18)

Such moments can be computed recursively\[17\]

$$\mu_0^\gamma(N) = 1$$
$$\mu_1^\gamma(N) = N + \gamma$$
$$\mu_2^\gamma(N) = (N + \gamma)(2N + \gamma - 1)$$  \hspace{1cm} (19)

with the general term given by

$$\mu_n^\gamma(N) = (2N + \gamma - n + 1)\mu_{n-1}^\gamma(N) + N \sum_{t=1}^{n-2} \mu_{n-1-t}^\gamma(N)\mu_t^\gamma(N)$$  \hspace{1cm} (20)

For large $N$ the moments $\mu_n^\gamma(N)$ can be approximated as

$$\mu_n^\gamma(N) = x_n N^n + \gamma y_n N^{n-1} + O(N^{n-1}) + \gamma O(N^{n-2}),$$  \hspace{1cm} (21)

with $x_0 = 1$ and $y_0 = 0$, $y_1 = 1$. Inserting this into the recursion relation eq.(20) we get two coupled recursions for $x_n$ and $y_n$,

$$x_n = 2x_{n-1} + \sum_{t=1}^{n-2} x_{n-1-t}x_t,$$  \hspace{1cm} (22)

and

$$y_n = 2y_{n-1} + x_{n-1} + 2 \sum_{t=1}^{n-2} x_{n-1-t}y_t.$$  \hspace{1cm} (23)
Eq. (22) can be rewritten using the fact that \( x_0 = 1 \) as
\[
x_n = \sum_{t=0}^{n-1} x_{n-1-t} x_t,
\]
which is the recursion relation for the Catalan numbers,
\[
x_n = \frac{1}{n+1} \binom{2n}{n},
\]
in agreement with an earlier result [18]. The recursion relation for \( y_n \) then becomes
\[
y_n = \frac{1}{n} \binom{2n-2}{n-1} + 2 \sum_{t=0}^{n-1} \frac{1}{n-t} \binom{2n-2t-2}{n-1-t} y_t.
\]
The solution for \( y_n \) can be guessed to be
\[
y_n = \binom{2n-1}{n-1}.
\]
To prove this, one can use the identity (5.62) from Ref.[19], which reads for \( n \) integer and all real \( r,s \) and \( t \),
\[
\sum_k \binom{tk + r}{k} \binom{tn - tk + s}{n - k} \frac{r}{tk + r} = \binom{tn + r + s}{n}.
\]
Hence, we have
\[
\mu_n^\gamma(\mathcal{N}) = \frac{1}{n+1} \binom{2n}{n} \mathcal{N}^n + \gamma \binom{2n-1}{n-1} \mathcal{N}^{n-1} + \mathcal{O}(\mathcal{N}^{n-2}) + \gamma \mathcal{O}(\mathcal{N}^{n-2}).
\]
We will use this result in the following sections.

## 5 Evaluation of the Witten index

The evaluation of the Witten index of SYMQM systems is nontrivial since their spectra in all sectors are continuous. Therefore, one needs to introduce a regularization in order to define the Witten index in a mathematically correct way. In Ref.[16] the model with \( SU(2) \) symmetry was considered and the regularization was done by putting the system in a ball of radius \( R \). At the end of the calculations the limit of \( R \to \infty \) was taken. The regularization proposed in this note is motivated by the Fock space approach [11] and it
is done automatically by the cut-off. The regulator is the maximal number of quanta contained in the basis states, which we denoted by $N_{\text{cut}}$, and we should eventually take the limit of $N_{\text{cut}} \to \infty$. Note that, as was observed in [20], such cut-off provides a infrared and ultraviolet regularization. We have

$$I_W(T) = \lim_{N_{\text{cut}} \to \infty} \int_0^\infty e^{-ET} \left( \delta^{\text{bosonic}}(N_{\text{cut}}) \rho_{N_{\text{cut}}}^{\text{bosonic}}(E) + \right.$$

$$\left. - \delta^{\text{fermionic}}(N_{\text{cut}}) \rho_{N_{\text{cut}}}^{\text{fermionic}}(E) \right) dE, \quad (30)$$

where $\rho_{N_{\text{cut}}}^{\text{bosonic}}(E)$ and $\rho_{N_{\text{cut}}}^{\text{fermionic}}(E)$ are the densities of bosonic and fermionic eigenenergies respectively at cut-off $N_{\text{cut}}$, whereas $\delta^{\text{bosonic}}(N_{\text{cut}})$ and $\delta^{\text{fermionic}}(N_{\text{cut}})$ denote the number of bosonic and fermionic states, respectively.

5.1 Model with $SU(2)$ gauge symmetry

In this section we consider the model with $SU(2)$ gauge group. We present the analytic treatment and describe some numerical results supporting our approach.

5.1.1 Analytic calculation

We start with the simplest situation, namely of the restricted Witten index in the $SU(2)$ model. Although $N = 2$ is even, one can define $I_W(T)_R$ for this model (see section 3). There are only two sectors that need to be considered, so

$$I_{N_{\text{cut}}}^W(T)_R = \sum_i e^{-E_i^{n_F=0}T} - e^{-E_i^{n_F=1}T} \quad (31)$$

According to eq.(7) the eigenenergies in the bosonic sector are given by the zeros of $L_{\lceil \frac{1}{2}N_{\text{cut}} \rceil +1}^1 (E^{n_F=0})$, whereas the eigenenergies in the sector with one fermionic quantum are given by the zeros of $L_{\lceil \frac{1}{2}(N_{\text{cut}}-1) \rceil +1}^2 (E^{n_F=1})$.

The idea of the approach is to work at given finite cut-off and expand the exponent into a Taylor series. For simplicity we assume $N_{\text{cut}}$ to be odd, $N_{\text{cut}} = 2M - 1$, so that $\lceil \frac{1}{2}N_{\text{cut}} \rceil + 1 = M$ and $\lceil \frac{1}{2}(N_{\text{cut}} - 1) \rceil + 1 = M$. We get

$$I_{N_{\text{cut}}}^W(T)_R = \sum_{n=0}^\infty \frac{(-1)^n}{n!} T^n M \left( \int_0^\infty E^n \rho_M^1(E) dE - \int_0^\infty E^n \rho_M^2(E) dE \right), \quad (32)$$

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The integrals in eq. (32) are nothing but the moments of the zeros distribution of Laguerre polynomials with an appropriate $\gamma$ index, namely $\gamma = \frac{1}{2}$ for $n_F = 0$ and $\gamma = \frac{3}{2}$ for $n_F = 1$. Hence,

$$I_{W}^{\text{Ncut}}(T)R = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n M \left( \mu_{\frac{1}{2}}(M) - \mu_{\frac{3}{2}}(M) \right),$$

where the sum starts at $n = 1$ since the zeroth term vanishes, $\mu_{\frac{1}{2}}(M) - \mu_{\frac{3}{2}}(M) = 1 - 1 = 0$. Inserting the expression for the moments eq. (29) all terms which are not proportional to $\gamma$ cancel. The leading non-vanishing term in $M$ is therefore proportional to the index of Laguerre polynomials. We get, for large $M$,

$$I_{W}^{\text{Ncut}}(T)R = \left( \frac{1}{2} - \frac{3}{2} \right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n M^n \left( \frac{2n - 1}{n - 1} \right) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n M^n \left( \frac{2n - 1}{n - 1} \right).$$

The sum can be performed to yield

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \left( \frac{2n - 1}{n - 1} \right) = \frac{1}{2} e^{-2x} I_0(2x),$$

where $I_0(x)$ is the modified Bessel function of the first kind. The zeroth term is nontrivial and can be evaluated as

$$\lim_{n \to 0} \frac{1}{n!} \left( \frac{2n - 1}{n - 1} \right) = \lim_{n \to 0} \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n+1)^2} = \lim_{n \to 0} \frac{2^{2n} \frac{1}{2}^{2n}}{\sqrt{2\pi} \frac{1}{2}^{n+1}} = \frac{1}{2}.$$

Hence,

$$I_{W}^{\text{Ncut}}(T)R = \frac{1}{2} - \frac{1}{2} e^{-(N_{\text{cut}}+1)T} I_0((N_{\text{cut}} + 1)T).$$

One can exploit the asymptotic form of $I_0(x)$ for a large argument,

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + O\left( \frac{1}{x} \right)), \quad \text{as} \ x \to \infty,$$

to finally get

$$I_{W}^{\text{Ncut}}(T)R = \frac{1}{2} - \frac{1}{2\sqrt{2\pi} (N_{\text{cut}}T)^{\frac{3}{2}}} + O\left( \frac{1}{(N_{\text{cut}}T)^{\frac{5}{2}}} \right).$$
Figure 2: Comparison at finite cut-off of $I_{W}^{N_{\text{cut}}}(T)$ evaluated by explicitly calculating all eigenenergies (solid line) and $I_{W}^{N_{\text{cut}}}(T)$ evaluated with approximated moments (dashed line) for four different cut-offs: $N_{\text{cut}} = 101$ for the lower pair of lines, $N_{\text{cut}} = 201, 401$ and $N_{\text{cut}} = 601$ for the upper pair of lines.

Therefore, in the limit of infinite cut-off we obtain a $T$-independent value $\frac{1}{2}$, in agreement with the result of [16],

$$I_{W}^{\infty}(T)_{R} = \frac{1}{2}.$$  \hspace{1cm} (40)

Eq. (39) indicates also that the results converge to the limiting value in a rather slow way, namely as $\frac{1}{\sqrt{N_{\text{cut}}}}$.

It also follows from the above discussion that for the restricted Witten index defined in the sectors $n_{F} = 2$ and $n_{F} = 3$ we get $I_{W}(T)_{R} = -\frac{1}{2}$. Therefore, summing the two contributions we get

$$I_{W}^{\infty}(T) = 0,$$  \hspace{1cm} (41)
which is in agreement with our expectation based on the particle-hole symmetry.

5.1.2 Numerical evidence

On Figure 2 we plot $I_W(T)_R$ obtained for four different odd values of the cut-off, $N_{cut} = 101, 201, 401$ and $N_{cut} = 601$. The solid lines represent $I_W(T)_R$ calculated by a numerical solution of the quantization conditions $L_{\lfloor \frac{1}{2}N_{cut} \rfloor+1}^{\frac{1}{2}}(E_{nF=0}) = 0$ and $L_{\lfloor \frac{1}{2}(N_{cut}-1) \rfloor+1}^{\frac{1}{2}}(E_{nF=1}) = 0$, taking the exponent and summing all contributions. The dashed lines correspond to expression eq. (37).

Three remarks can be made:

• both sets of curves to converge to the analytic prediction $\frac{1}{2}$;
• the curves present less and less dependence on $T$ as $N_{cut}$ increases;
• the discrepancy between the exact result (solid lines) and approximated one (dashed lines) is decreasing as we go to higher cut-offs.

The vanishing of $I_W^{N_{cut}}(T)_R$ at $T \to 0$ is related to the fact that $I_W^{N_{cut}}(0)_R$ is equal to the difference in the number of states in sectors with $n_F = 0$ and $n_F = 1$ which is 0 for $N_{cut}$ odd. For $N_{cut}$ even $I_W^{N_{cut}}(0)_R = 1$, since there is one additional bosonic state. In the large $N_{cut}$ limit there should be no difference whether for $N_{cut}$ is odd and even, apart the $T = 0$ point.

The results shown on Figure 2 confirm that the approximations made in order to derive expression eq. (37) are correct.

5.2 Model with $SU(3)$ gauge symmetry

In this section we present the generalization of the discussion presented above to the case of the model with $SU(3)$ symmetry.

5.2.1 Analytic calculation

Although the spectra in all fermionic sectors of the $SU(3)$ model are also given by the zeros of the generalized Laguerre polynomials [13], their structure turns out to be much more complicated than that of the $SU(2)$ model (see section 2). As was already mentioned, the eigensolutions can be grouped into disjoint sets called families. To each family corresponds a single quantization condition. Hence, the spectrum in each sector is given by the zeros of
a set of several Laguerre polynomials with different indices and of different orders. Therefore, we can write the Witten index as

\[ I_\infty^W(T) = \lim_{N_{\text{cut}} \to \infty} \left( \sum_{\eta \in \text{bosonic families}} \sum_i e^{-E_i^\eta(N_{\text{cut}})T} + \sum_{\eta' \in \text{fermionic families}} \sum_i e^{-E_i^\eta'(N_{\text{cut}})T} \right) = \lim_{N_{\text{cut}} \to \infty} \int_0^\infty e^{-ET} \sum_{\eta \in \text{bosonic families}} \sum_{\eta' \in \text{fermionic families}} \left( \delta^\eta(N_{\text{cut}}) \rho^\eta_{N_{\text{cut}}}(E) + \delta^\eta'(N_{\text{cut}}) \rho^{\eta'}_{N_{\text{cut}}}(E) \right) dE, \quad (42) \]

where \( \eta, \eta' \) are some multi-indices labeling the families present in this model, whereas \( \delta^\eta(N_{\text{cut}}) \) and \( \delta^\eta'(N_{\text{cut}}) \) are the numbers of solutions belonging to the family \( \eta \) and \( \eta' \), respectively, and are functions of \( N_{\text{cut}} \). Their exact form will be presented below.

For the \( SU(3) \) model the families can be labeled by two integers: \((p, \beta)\) (see also eq.(5)). We now consider the contribution to \( I_\infty^W(T) \) coming from two families characterized by \((p, \beta)\) and \((q, \alpha)\),

\[ I_{q,\alpha}^{N_{\text{cut}}} = \int_0^\infty e^{-ET} \left[ \delta^{(q,\alpha)}(N_{\text{cut}}) \rho^{(q,\alpha)}_{N_{\text{cut}}}(E) - \delta^{(p,\beta)}(N_{\text{cut}}) \rho^{(p,\beta)}_{N_{\text{cut}}}(E) \right] dE. \quad (43) \]

It was shown in Ref.[13] that (see also eq.(5) and eq.(7))

\[ \delta^{(q,\alpha)}(N_{\text{cut}}) = \left\lfloor \frac{1}{2} (N_{\text{cut}} - 3q - n_B^\alpha) + 1 \right\rfloor, \]

\[ \rho^{(q,\alpha)}_{N_{\text{cut}}}(E) = \rho^{\left\lfloor \frac{1}{2} (N_{\text{cut}} - 3q - n_B^\alpha) + 1 \right\rfloor}(E). \]

We approximate the factors \( \delta^{(q,\alpha)}(N_{\text{cut}}) \) and \( \delta^{(p,\beta)}(N_{\text{cut}}) \) as

\[ \left\lfloor \frac{1}{2} (N_{\text{cut}} - 3q - n_B^\alpha) \right\rfloor + 1 \approx \frac{1}{2} N_{\text{cut}}, \quad \left\lfloor \frac{1}{2} (N_{\text{cut}} - 3p - n_B^\beta) \right\rfloor + 1 \approx \frac{1}{2} N_{\text{cut}}, \quad (44) \]

which is justified in the leading order of large \( N_{\text{cut}} \). Introducing the moments
of the distributions we get, keeping only the leading terms in $N_{\text{cut}}$,

$$I_{q, \alpha; p, \beta}(T) \approx \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} T^n \frac{1}{2} N_{\text{cut}} \left[ \frac{n}{n+1} \left( \frac{1}{2} (-3q - n_B^\alpha) \right) + \right.$$

$$\left. - \left( \frac{1}{2} (-3p - n_B^\beta) \right) \right] \binom{2n}{n} + \left( 3q + n_B^\alpha - 3p - n_B^\beta \right) \binom{2n - 1}{n - 1} \left( \frac{1}{2} N_{\text{cut}} \right)^{n-1}. \right.$$  (45)

We perform the sums to get

$$I_{q, \alpha; p, \beta}(T) = \frac{3p + n_B^\beta - 3q - n_B^\alpha}{2} \left( 1 - e^{-N_{\text{cut}} T} I_0(N_{\text{cut}} T) \right)$$

$$- \left( \frac{1}{2} (-3q - n_B^\alpha) - \frac{1}{2} (-3p - n_B^\beta) \right) e^{-N_{\text{cut}} T} I_1(N_{\text{cut}} T), \quad (46)$$

where in the $N_{\text{cut}} \to \infty$ limit yields

$$I_{q, \alpha; p, \beta}(T) = \frac{3p + n_B^\beta - 3q - n_B^\alpha}{2}. \quad (47)$$

In order to evaluate $I_{W}(T)$ one has to:

1. check that there are exactly as many bosonic families as there are fermionic ones,

2. sum the contributions from all such pairs.

The first point can be shown as follows. We will discuss it for a general $SU(N)$ SYMQM model. The number of singlet basis states with given number of bosonic and fermionic quanta, denoted by $D_{n_B, n_F}$, can be obtained, following [9], from the generating function

$$G(a, b) = \sum_{n_B=0}^{\infty} \sum_{n_F=0}^{\infty} D_{n_B, n_F} a^{n_B} (-b)^{n_F}. \quad (48)$$

It can be shown[9] that $G(a, b)$ has a convenient integral representation, namely

$$G(a, b) = \frac{1}{N!} \left( \frac{1 - b}{1 - a} \right)^{N-1} \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\alpha_i}{2\pi} \prod_{i \neq j} (1 - e^{i(\alpha_i - \alpha_j)}) \frac{1 - be^{i(\alpha_i - \alpha_j)}}{1 - ae^{i(\alpha_i - \alpha_j)}}. \quad (49)$$
Moreover, it turns out the $G(a, b)$ can be also written as \[ G(a, b) = \left( \prod_{k=2}^{N} \frac{1}{(1-a^{k})} \right) \sum_{n_{F}=0}^{N^{2}-1} (-b)^{n_{F}} c_{n_{F}}(a). \] (50)

Setting $b = 1$ we obtain the difference of the sum over the bosonic sectors and the sum over the fermionic sectors,

\[ \left( \prod_{k=2}^{N} (1-a^{k}) \right) G(a, b) \big|_{b=1} = \sum_{n_{F} \text{ even}} c_{n_{F}}(a) - \sum_{n_{F} \text{ odd}} c_{n_{F}}(a). \] (51)

The polynomials $c_{n_{F}}(a) \equiv \sum_{n=0}^{\infty} \chi_{n}(n_{F}) a^{n}$ contain all the information about the numbers $n_{B}^{\alpha}$. Namely, there are $\chi_{n}(n_{F})$ fermionic bricks with $n$ bosonic creation operators in the sector with $n_{F}$ fermionic quanta. From eq.\((49)\) one immediately sees that the left-hand side of eq.\((51)\) vanishes,

\[ \sum_{n=0}^{\infty} \left( \sum_{n_{F} \text{ even}} \chi_{n}(n_{F}) - \sum_{n_{F} \text{ odd}} \chi_{n}(n_{F}) \right) a^{n} = 0. \] (52)

Hence,

\[ \sum_{n_{F} \text{ even}} \chi_{n}(n_{F}) = \sum_{n_{F} \text{ odd}} \chi_{n}(n_{F}). \] (53)

This means that there is an equal number of fermionic bricks with $n$ bosonic creation operators in the sectors with $n_{F}$ even as there are such operators in the sectors with $n_{F}$ odd. It follows from this that the set of numbers $n_{B}^{\alpha}$ from the sectors with $n_{F}$ even is equal to the set from the sectors with $n_{F}$ odd,

\[ \left\{ n_{B}^{\alpha} \right\}_{n_{F} \text{ even}} = \left\{ n_{B}^{\alpha} \right\}_{n_{F} \text{ odd}} \] (54)

This can be directly checked with explicit values for the $n_{B}^{\alpha}$ since for the $SU(3)$ model they are known explicitly, i.e. in Ref.\([11]\) all fermionic bricks for this model were presented. In table \[\text{I}\] we just summarize the resulting values of $n_{B}^{\alpha}$ in different fermionic sectors.

In order to show that there are as many bosonic families as fermionic ones we can now consider the following quantity, denoted by $W$,

\[ W = \sum_{n_{F} \text{ even}} \sum_{\alpha=1}^{d_{n_{F}}} \left[ \frac{1}{4} (N_{\text{cut}} - n_{B}^{\alpha}) \right] - \sum_{n_{F} \text{ odd}} \sum_{\beta=1}^{d_{n_{F}}} \left[ \frac{1}{4} (N_{\text{cut}} - n_{B}^{\beta}) \right] \] (55)
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$n_F$ & $n_B^1$ & $n_B^2$ & $n_B^3$ \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 \\
2 & 1 & 2 & 3 \\
3 & 0 & 1 & 2 \\
4 & 1 & 2 & 3 \\
5 & 0 & 1 & 2 \\
6 & 1 & 2 & 3 \\
7 & 0 & 1 & 2 \\
8 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{The values of $n_B^\alpha$ for the $SU(3)$ group.}
\end{center}

Since the sets of numbers \{n_B^\alpha\}_{n_F \text{ even}} and \{n_B^\alpha\}_{n_F \text{ odd}} are equal, therefore $W = 0$. Hence, at any finite cut-off, there is exactly as many bosonic families of solutions as there are fermionic ones.

As far as the second point is concerned we have

$$I_{\infty}^W(T) = 0,$$ \hfill (57)

and

$$I_{\infty}^W(T)_R = 0.$$

\hfill (58)

### 5.2.2 Numerical evidence

The results given by eqs. (57) and (58) were confirmed numerically. The spectra at given cut-off were calculated using the cut Fock basis approach and a recursive algorithm to speed up the calculations. We indeed obtained $I_{W_{\text{cut}}}^{N_{\text{cut}}}(T) = 0$ and $I_{W_{\text{cut}}}^{N_{\text{cut}}}(T)_R = 0$ for any $N_{\text{cut}}$.

### 6 Higher groups

The structure of solutions, namely the grouping into families having similar characteristics, for example the index of the Laguerre polynomials, which
was observed for the model with $SU(3)$ symmetry [13], persists also for higher $N > 3$[14]. Hence, again, the spectrum consists of eigenenergies coming from the quantization conditions of all families, as is also indicated by eq. (7). Thus, we can write the Witten index as

$$I_\infty^W(T) = \lim_{N_{\text{cut}} \to \infty} \int_0^\infty e^{-ET} \sum_{\eta \in \text{bosonic families}} \left( \delta^\eta(N_{\text{cut}}) \rho^\eta_{N_{\text{cut}}}(E) + \right. $$

$$\left. - \delta^{\eta'}(N_{\text{cut}}) \rho^{\eta'}_{N_{\text{cut}}}(E) \right) dE.$$  (59)

This time, the multi-index $\eta$ is even more complicated, as the generic families of the $SU(N)$ model are labeled by a set of integers $t_k$, $3 \leq k \leq N$, and additionally by the index $\alpha$ denoting the fermionic brick, which multiplies the basis state with maximal number of bosonic quanta[14].

The equality (54) was derived for models with any gauge group $SU(N)$. Hence, one can now easily show that in the sum in eq.(59) there are as many families of solutions in the sectors with $n_F$ even as there are families in the sectors with $n_F$ odd by generalizing the argument around eq.(55). Therefore, we can sum the contributions to $I_\infty^W(T)$ coming from all pairs of families (taking account of eq.(5)) and we get

$$I_\infty^W(T) = \frac{1}{2} \sum_{n_F \text{ even}} d^{n_F} \sum_{\alpha=1}^{N} \left( \left( \sum_{i=3}^{N} it_i \right) + n_\alpha^F(n_F) \right)$$

$$- \frac{1}{2} \sum_{n_F \text{ odd}} d^{n_F} \sum_{\alpha=1}^{N} \left( \left( \sum_{i=3}^{N} it_i \right) + n_\alpha^F(n_F) \right),$$  (60)

from which, using again eq.(54), we immediately get

$$I_\infty^W(T) = 0,$$  (61)

and

$$I_\infty^W(T)_R = 0.$$  (62)

7 Conclusions

In this paper we calculated the Witten index as well as the restricted Witten index for the $D = 2$ supersymmetric Yang-Mills quantum mechanics with any $SU(N)$ gauge group. $I_\infty^W(T)$ vanishes for all, even and odd, $N$. We
evaluated the restricted Witten index as well, which also vanishes for all \( N \) odd, except of the \( SU(2) \) model for which we recovered the known result \( I_W(T)_R = \frac{1}{2} \).

The vanishing of the Witten index does not imply that the supersymmetry is broken in this model. By studying the structure of the solutions \[14\] one can persuade himself that there exist several nondegenerate states with zero energy in each studied models. They appear in nonadjacent sectors, therefore cannot be linked by the action of supercharges. The vanishing \( I_W(T) \) signifies that there is as many supersymmetric vacua in sectors with \( n_F \) even as there such states in sectors with \( n_F \) odd, which could have been already suspected from the results of Ref.[14] and Ref.[21]. However, since the spectra of these models are continuous, the confirmation of this fact by the explicit evaluation of the Witten index is nontrivial.

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