CONTINUITY OF THE MACKEY-HIGSON BIJECTION

by

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Abstract. — When $G$ is a real reductive group and $G_0$ is its Cartan motion group, the Mackey-Higson bijection is a natural one-to-one correspondence between all irreducible tempered representations of $G$ and all irreducible unitary representations of $G_0$. In this short note, we collect some known facts about the topology of the tempered dual $\hat{G}$ and that of the unitary dual $\hat{G}_0$, then verify that the Mackey-Higson bijection $\hat{G} \to \hat{G}_0$ is continuous.

1. Introduction

Let $G$ be a real reductive group and $G_0$ be the Cartan motion group attached to $G$ and a choice of maximal compact subgroup $K$: if $g = k \oplus p$ is the Cartan decomposition of the Lie algebra of $G$, then $G_0$ is the semidirect product $K \ltimes p$. Write $\hat{G}$ for the tempered dual of $G$ and $\hat{G}_0$ for the unitary dual of $G_0$.

It has been observed [14, 10, 11, 1] that the parameters necessary to describe $\hat{G}$ and $\hat{G}_0$ are identical: there is a natural, but non-trivial, one-to-one correspondence between $\hat{G}$ and $\hat{G}_0$. We will refer to this correspondence as the Mackey-Higson bijection.

If we bring the Fell topologies of $\hat{G}$ and $\hat{G}_0$ into the picture, then more can be said. Alain Connes and Nigel Higson pointed out in the late 1980s that the Baum-Connes-Kasparov isomorphism for the $K$-theory of the reduced $C^*$-algebra of $G$ can be viewed as a statement that $\hat{G}$ and $\hat{G}_0$, although not homeomorphic, always have in a precise sense the same $K$-theory (see [5])

Building on $C^*$-algebraic methods due to Nigel Higson [10], one of us showed in [2] that the Mackey-Higson bijection is a piecewise homeomorphism, where the homeomorphic pieces are defined through David Vogan’s theory of lowest $K$-types. The pieces are stitched together differently in both duals, but taking $K$-theory somehow blurs out that fact: the Connes-Kasparov isomorphism can be obtained in a rather elementary way from the topological properties of the Mackey-Higson bijection [2].

The purpose of the present short note is to complete this topological information by showing that the correspondence maps $\hat{G}$ continuously onto $\hat{G}_0$, although it is never a homeomorphism.

We will verify this in §4 by focusing on the restriction of the Mackey-Higson bijection to each connected component of $\hat{G}$. The Fell topology on $\hat{G}$ (which is usually non-Hausdorff) has been known quite precisely since the 1980s [7, 20], and the topology of $\hat{G}_0$ (which is also non-Hausdorff) has been described in 1968 [3]; we will use these descriptions and recall the necessary details in §2 and §3.

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2. Topology of the tempered dual

2.1. Harish-Chandra decomposition. — Suppose $G$ is the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$. Fix a Haar measure on $G$, form the reduced $C^*$-algebra $C^*_r(G)$ and the category $\mathcal{M}^t(G) =$ category of continuous nondegenerate $C^*_r(G)$-modules

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of tempered representations of $G$.

Let us first recall how the general shape of Harish-Chandra’s Plancherel formula yields an infinite direct product decomposition of $\mathcal{M}^t(G)$. The presentation is modeled on the $p$-adic case, more precisely, on Schneider and Zink’s tempered version of the Bernstein decomposition of the category of admissible representations [15]. The results of this paragraph are in fact true exactly as stated if $F$ is any local field and $G$ is the group of $F$-points of a connected reductive algebraic group defined over $F$.

Let us call discrete pair any pair $(L, \sigma)$ in which

- $L$ is a Levi subgroup of $G$
- $\sigma$ is a tempered irreducible representation of $L$ that is square-integrable-modulo-center.

The group $G$ acts on the set of discrete pairs; we write $\Omega^t(L)$ for the space of orbits. For every Levi subgroup $L$ of $G$, let us call unramified unitary character of $L$ any unitary character of $L$ that is trivial on every compact subgroup of $L$; we will write $\mathcal{X}_u(L)$ for the set of unramified unitary characters of $L$.

Let us now fix a discrete pair $(L, \sigma)$. The map

$$\Phi_{L, \sigma} : \mathcal{X}_u(L) \to \Omega^t(L)$$

$$\chi \mapsto \text{class of } (L, \sigma \otimes \chi) \text{ in } \Omega^t(L)$$

sends $\mathcal{X}_u(L)$ to a subset $\Theta_{L, \sigma} = \Phi_{L, \sigma}(\mathcal{X}_u(L))$ of $\Omega^t(L)$. Inequivalent pairs $(L, \sigma)$ map to disjoint subsets of $\Omega^t(L)$. We write $B^t(G) = \{ \Theta_{L, \sigma}, [(L, \sigma)] \in \Omega^t(L) \}$ for the set of blocks.

Suppose $\pi$ is an irreducible tempered representation of $G$. Then there exists a discrete pair $(L, \sigma)$ and a parabolic subgroup $P = LN$ of $G$ with the property that $\pi$ is equivalent with one of the irreducible factors of $\text{Ind}_{L,N}^G(\sigma)$. The pair $(L, \sigma)$ determines an element $\Theta_{L, \sigma}$ of $B^t(G)$ which depends only on $\pi$, not on the choice of $(L, \sigma)$: we may and will call it the discrete support of $\pi$. For all this, original sources include [17, 12]; a convenient reference is [6, §5-6].

For every block $\Theta \in B^t(G)$, we write $\mathcal{M}^t(\Theta)$ for the category of continuous nondegenerate $C^*_r(G)$-modules whose irreducible subquotients all have discrete support $\Theta$. Harish-Chandra’s work induces a direct sum decomposition

$$C^*_r(G) = \sum_{\Theta \in B^t(G)} C^*_r(G)_\Theta,$$

where the spectrum of a given component $C^*_r(G)_\Theta$ is

$$\tilde{G}_\Theta = \left\{ \pi \in \tilde{G} \mid \text{whose discrete support is } \Theta \right\};$$

see [6, Proposition 5.17 and Theorem 6.8]. This yields a partition

$$\tilde{G} = \bigsqcup_{\Theta \in B^t(G)} \tilde{G}_\Theta$$

of the tempered dual into disjoint subsets, and a direct product decomposition

$$\mathcal{M}^t(G) = \prod_{\Theta \in B^t(G)} \mathcal{M}^t(\Theta)$$

of the category of tempered representations. We refer to [15] for the parallel with the non-archimedean case.

### 2.2. Connected components of the tempered dual $\tilde{G}$

The following remark is important for what follows. Although it is well-known (see for instance [7, théorème 2.6(ii)]), we will sketch a proof.

**Proposition 2.1.** — The connected components of $\tilde{G}$ are the $\tilde{G}_\Theta$, $\Theta \in B^t(G)$.

**Proof.** — We note first that for every $\Theta$, the subset $\tilde{G}_\Theta$ of $\tilde{G}$ is closed: the decomposition in (2.1) identifies $\tilde{G}_\Theta$ with the set of irreducible representations $C^*_r(G)$ that vanish on the ideal $J_\Theta = \sum_{\Theta' \neq \Theta} C^*_r(G)_{\Theta'}$. That is indeed a closed subset [8, §3.2].

We now check that each $\tilde{G}_\Theta$ is connected. To that end, we will fix an element $\pi_\ast$ of $\tilde{G}_\Theta$ and prove that every element in $\tilde{G}_\Theta$ necessary lies in the same connected component of $\tilde{G}$ as $\pi_\ast$. 

Fix a component $\Theta \in B'(G)$. Among the discrete pairs $(L, \sigma)$ with equivalence class $\Theta$, there is one that has the additional property that if $L = MA$ is the Langlands decomposition of $L$, then the restriction $\sigma|_A$ is trivial; we will assume that our pair has that property. Writing $a$ for the Lie algebra of $A$ and $a^*$ for its vector space dual, recall that an element of $a^*$ is called a-regular when its scalar product with every positive root of $(\mathfrak{g}_C, \mathfrak{a}_C)$ is nonzero. Fix a parabolic subgroup $P = LN$ with Levi factor $L$; this determines an ordering of $a^*$. Fix then an element $\nu_*$ in $a^*$ whose scalar product with every positive root of $(\mathfrak{g}_C, \mathfrak{a}_C)$ is positive (and nonzero), then define $\pi_* = \text{Ind}_{P}^{G} (\sigma \otimes e^{i \nu_*})$. Since $\nu_*$ is a-regular, that representation is irreducible (see e.g. [13, Theorem 14.93]) and it lies in $\widetilde{G}_\Theta$.

For every representation $\pi \in \widetilde{G}_\Theta$, there exists an element $\nu$ in $a^*$ which has the property that the scalar product of $\nu$ with every positive root of $(\mathfrak{g}_C, \mathfrak{a}_C)$ is nonnegative and that $\pi$ is equivalent with one of the irreducible constituents of $\text{Ind}_P^G (\sigma \otimes e^{i \nu})$.

For every $t$ in $[0,1]$, define $\nu_t = t \nu + (1 - t) \nu_*$ and $\pi_t = \text{Ind}_{P}^{G} (\sigma \otimes e^{i \nu_t})$; for $t > 0$, the element $\nu_t$ is a-regular because it has a positive scalar product with every positive root of $(\mathfrak{g}_C, \mathfrak{a}_C)$, so $\pi_t$ is irreducible.

Using the “compact picture” for induced representations, together with the routine criteria for the continuity of parameter integrals, we can exhibit every matrix element of $\pi$ as a limit (in the sense of uniform convergence on compact subset of $G$) of a family of matrix elements of $(\pi_t)_{t > 0}$. This proves that $\pi$ is in the closure of the family $(\pi_t)_{t > 0}$, and thus $\pi = \pi_0$ and $\pi_* = \pi_1$ must lie in the same connected component of $\widetilde{G}$.

2.3. Vogan’s lowest-$K$-type picture for the decomposition. — We now fix a maximal compact subgroup $K$ in $G$ and recall Vogan’s description of the “blocks” $\widetilde{G}_\Theta$ in terms of associate classes of lowest $K$-types (see [19]).

Fix $\Theta \in \widetilde{G}_\Theta$ and a tempered $(L, \sigma)$ with equivalence class $\Theta$. As before, we may assume that if $L = MA$ is the Langlands decomposition of $L$, then $\sigma|_A$ is trivial. Fix a parabolic subgroup $P$ of $G$ with Levi factor $L$, then write $C_\Theta$ for the set of lowest $K$-types of the representation $\text{Ind}_P^G (\sigma)$. The notation is coherent, because all choices of $(L, \sigma)$ with equivalence class $\Theta$ (and the additional restriction on $\sigma$) lead to the same set of lowest $K$-types.

**Proposition 2.2 (Vogan).** — Let $\pi$ be an irreducible tempered representation of $G$. Then $\pi$ lies in the component $\widetilde{G}_\Theta$ if and only if at least one of its lowest $K$-types lies in $C_\Theta$. When that is the case, all the lowest $K$-types of $\pi$ lie in $C_\Theta$.

See [18, Introduction] for connected $G$, and [16, §4] for a class of groups that includes the current one.

2.4. Closure of a subset of $\widetilde{G}$. — These results are well-known; let us mention contributions of Delorme [7] and Miličić (see Vogan [20]).

**Notations 2.3.** — Fix a component $\widetilde{G}_\Theta$ in the tempered dual; assume $\widetilde{G}_\Theta$ is not a single point. Consider

$(L, \sigma):$ a discrete pair corresponding to $\Theta$;

$L = MA:$ the Langlands decomposition of $L$.

We may and will assume that the restriction $\sigma|_A$ is trivial and will identify $\sigma$ with its restriction to $M$ (a genuine discrete series representation).

$P = MAN:$ a parabolic subgroup with Levi factor $L$

$\Delta^+:$ the positive root system for $(\mathfrak{g}_C, \mathfrak{a}_C)$ that corresponds to $N$;

$\mathfrak{a}^+ = \{ \nu \in \mathfrak{a}^*; \forall \alpha \in \Delta^+, \langle \alpha, \nu \rangle \geq 0 \}.$ It is a closed cone in $\mathfrak{a}^*$.

$\mathfrak{a}^*/W =$ the quotient of $\mathfrak{a}^*$ by the Weyl group $W = W(\mathfrak{g}_C, \mathfrak{a}_C)$.

Every class in $\mathfrak{a}^*/W$ has a finite (nonzero) number of representatives in $\mathfrak{a}^*$.

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1. Fix a Hilbert space $V^\sigma$ for $\sigma$; then for each $t \in [0,1]$, the representation $\pi_t$ can be realized on the Hilbert space $\mathcal{H} = \{ f: K \to \ell^2 V^\sigma : \forall u \in (K \cap M), f(ku) = \sigma(u)f(k) \}$. Suppose $\xi$ and $\eta$ are elements of $\mathcal{H}$; then the matrix element $c_{\xi, \eta}: g \mapsto \langle \xi, \pi_t(g)\eta \rangle$ can be expressed using the Iwasawa decomposition $G = MAN$. If any element of $G$ reads $x = \kappa(x)\mu(x)e^{H(x)}\nu(x)$, where $(\kappa(g), \mu(g), H(g), \nu(g)) \in K \times (M \cap \exp(p)) \times \mathfrak{a} \times N$, then we have $c_{\xi, \eta}(g) = \int_K e^{(-i\tau - \rho)H(u^{-1}g)} f(g, u)du$, where $f(g, u) = \langle \xi(u), \sigma(\mu(u^{-1}g))\eta(\kappa(u^{-1}g)) \rangle \nu^\sigma$ and $\rho$ is the usual half-sum of positive roots. See for instance [4, §2.2]. The claim easily follows.
For every irreducible tempered representation \( \pi \) in the component \( \tilde{G}_\Theta \), there is a unique \( \nu = \nu(\pi) \) in \( \mathfrak{a}^*/W \) with the property that \( \pi \) occurs in \( \text{Ind}_{P}^{G}(\sigma \otimes e^{i\nu}) \).

**Proposition 2.4.** Suppose \( B \) is a subset of \( \tilde{G}_\Theta \). Form the set \( \mathcal{V} = \{ \nu(\pi), \pi \in B \} \) of continuous parameters for representations in \( B \), and its closure \( \overline{\mathcal{V}} \) in \( \mathfrak{a}^*/W \). The closure of \( B \) in \( \tilde{G} \) consists of those \( \pi s \) that belong to \( \tilde{G}_\Theta \) and satisfy \( \nu(\pi) \in \overline{\mathcal{V}} \).

**Proof.** Combine [7, Théorème 2.6] or [20, Theorem 3] with the information on connected components from §2.2. \( \Box \)

To obtain the closure of \( B \) in \( \tilde{G} \), we can thus form the set of representations \( \text{Ind}_{P}^{G}(\sigma \otimes \chi) \) for \( \chi \) in the closure of the set of continuous parameters for members of \( B \), then consider all irreducible factors in these.

**Example 2.5.** Suppose \( G = \text{SL}(2, \mathbb{R}) \), \( \text{MAN} \) is the upper triangular subgroup (recall that \( M = \{ \pm I_2 \} \) and \( A \) is the one-parameter subgroup of determinant-one diagonal \( 2 \times 2 \) matrices), \( \sigma \) is the one non-trivial representation of \( M \). The discrete pair \((\text{MAN}, \sigma)\) determines a component \( \Theta \) in \( \tilde{G} \). Consider the subset \( B \) of representations of the form \( \text{Ind}_{\text{MAN}}^{G}(\sigma \otimes \chi) \), where \( \chi \) is a non-trivial unitary character of \( A \) (these representations are all irreducible). To obtain the closure of \( B \), we need to add to \( B \) the two \( \text{"limits of discrete series"} \) whose direct sum is the (reducible) representation \( \text{Ind}_{\text{MAN}}^{G}(\sigma \otimes e^{i0}) \).

We shall need a consequence of the above results, taken from [20, Theorem 3], that says what happens when one considers the continuous parameters for a convergent sequence of irreducible tempered representations.

**Corollary 2.6.** Suppose \( \pi \) is an irreducible tempered representation of \( G \) in the component \( \tilde{G}_\Theta \), and \( (\pi_n)_{n \in \mathbb{N}} \) is a sequence of irreducible tempered representations that admits \( \pi \) as a limit point. There exists a subsequence \( (\pi_{n_j})_{j \in \mathbb{N}} \) of representations that all lie in \( \tilde{G}_\Theta \) and have the property that as \( j \) goes to infinity, \( \nu(\pi_{n_j}) \) goes to \( \nu(\pi) \) in \( \mathfrak{a}^*/W \).

3. Topology of the motion group dual and remarks on the Mackey-Higson bijection

3.1. Mackey parameters for \( \tilde{G}_0 \) and \( \tilde{G} \).

Suppose \( \sigma \) is a unitary irreducible representation of the motion group \( G_0 = K \rtimes \mathfrak{p} \) of §1. Given a pair \((\chi, \mu)\) in which \( \chi \) is an element of \( \mathfrak{p}^* \) and \( \mu \) is an irreducible representation of the stabilizer \( K_\chi \) of \( \chi \) in \( K \), we shall say that \((\chi, \mu)\) is a Mackey parameter for \( \sigma \) when \( \sigma \) is equivalent, as a representation of \( G_0 \), with \( \text{Ind}_{K_\chi \rtimes \mathfrak{p}}^{G_0}(\mu \otimes e^{i\chi}) \). Every unitary irreducible representation \( \sigma \) of \( G_0 \) admits at least one Mackey parameter, and two different Mackey parameters for a given \( \sigma \) must be conjugate under \( K \).

The information we will need about the Mackey-Higson bijection \( \mathcal{M} : \tilde{G} \to \tilde{G}_0 \) can be phrased as follows.

If \( \pi \) is an irreducible tempered representation of the reductive group \( G \), we will say that \((\chi, \mu)\) is a Mackey parameter for \( \pi \) when it is a Mackey parameter for the representation \( \mathcal{M}(\pi) \) of \( G_0 \).

**Lemma 3.1.** Let \( \pi \) be an irreducible tempered representation of \( G \). Suppose

- \( P = LN = \text{MAN} \) is a parabolic subgroup of \( G \),
- \( \mu \) is an irreducible representation of \((K \cap L)\) and \( V_L(\mu) \) is the unique irreducible tempered representation of \( L \) that has real infinitesimal character and lowest \((K \cap L)\)-type \( \mu \),
- \( \chi \) is an a-regular element of \( \mathfrak{a}^* \).

If \( \pi \) is equivalent with the irreducible representation \( \text{Ind}_{P}^{G}(V_L(\mu) \otimes e^{i\chi}) \), then \((\chi, \mu)\) is a Mackey parameter for \( \pi \).

This comes from the construction of the correspondence in [1, §3]; for a description of the correspondence \( \mu \leftrightarrow V_L(\mu) \), see for instance §3.1 there. \( \Box \)
3.2. Discontinuity of $\mathcal{M}^{-1} : \widehat{G_0} \to \widehat{G}$. — Before we consider the topology of $\widehat{G_0}$ in more detail, let us remark that we already know enough about $\widehat{G}$ to verify that the Mackey bijection cannot be a homeomorphism.

**Proposition 3.2.** — The bijection $\mathcal{M}^{-1} : \widehat{G_0} \to \widehat{G}$ is never continuous.

We shall see, for instance, that the trivial representation of $G_0$ is always a discontinuity point of $\mathcal{M}^{-1}$.

**Proof.** — Suppose $\chi$ is a regular element of $p^*$. Consider the induced representation $\sigma = \text{Ind}_{M \rtimes p}^{G_0}(1 \otimes e^{i\chi})$, where $M = Z(K)\chi$ is the centralizer of $\chi$ in $p^*$.

For every $\alpha > 0$, let us consider the representation $\sigma _\alpha = \text{Ind}_{M \rtimes p}^{G_0}(1 \otimes e^{i\alpha \chi})$, and observe what happens as $\alpha$ goes to infinity. If $\widehat{K}_M$ is the set of those $\lambda \in \widehat{K}$ that occur in $L^2(K/M)$, and if we identify the elements of $\widehat{K}_M$ with representations of $G_0$ in which $p$ acts trivially, then we will check that $B = \{\sigma _\alpha, \alpha \in [1, +\infty[\} \cup \widehat{K}_M$ is a connected subset of $\widehat{G_0}$.

Recall that we can realize $\sigma _\alpha$ as a representation that acts on $L^2(K/M)$, in which $K$ acts through the left regular representation $L$ and $v \in p$ acts through $f \mapsto [u \mapsto \chi(k^{-1}v)f(k)]$.

If $\xi, \xi'$ are vectors in $L^2(K/M)$ that both lie in the isotypical subspace of $L^2(K/M)$ for a class $\lambda \in \widehat{K}$, then the associated matrix element – a complex-valued function on $K \times p$ – reads $(k, v) \mapsto e^{i(\xi(k)\lambda \xi'(k))}$, which is a matrix element for the representation of $G_0$ that extends $\lambda \in \widehat{K}_M$. This proves that $B$ is connected.

Now, the image of $B$ under $\mathcal{M}_{G_0} \to \widehat{G}$ is not connected: the image of $\{\sigma _\alpha, \alpha \in [1, +\infty[\} \cup \{\text{triv}_K\}$ is the spherical principal series, which is by itself a connected component (2) of $\widehat{G}$. The images of those elements of $\widehat{K}_M$ that are nontrivial are thus contained in other connected components of $\widehat{G}$, and that would not be possible if $\mathcal{M}^{-1} : \widehat{G_0} \to \widehat{G}$ were continuous.

3.3. Baggett’s description of the topology of $\widehat{G_0}$. —

**Theorem 3.3 (Baggett [3]).** — Let $\{\sigma_n\}_{n \in \mathbb{N}}$ be a sequence of unitary irreducible representations of $G_0$, and $\sigma_\infty$ be a unitary irreducible representation of $G_0$. For every $n$ in $\mathbb{N} \cup \{\infty\}$, fix a Mackey parameter $(\chi_n, \mu_n)$ for $\sigma_n$.

The representation $\sigma_\infty$ is a limit point of $\{\sigma_n\}_{n \in \mathbb{N}}$ in $\widehat{G_0}$ if and only if $\{\sigma_n\}_{n \in \mathbb{N}}$ admits a subsequence $\{\sigma_{n_j}\}_{j \in \mathbb{N}}$ that satisfies the following conditions:

(i) as $j$ goes to infinity, $\chi_{n_j}$ goes to $\chi_\infty$ in $p^*$;

(ii) the sequence $(K_{\chi_{n_j}}, \mu_{n_j})$ $j \in \mathbb{N}$ is actually a constant $(K_{\lim}, \mu_{\lim})$ in which $K_{\lim}$ is a subgroup of $K_{\chi_\infty}$;

(iii) the induced representation $\text{Ind}_{K_{\lim}}^{K_{\chi_\infty}}(\mu_{\lim})$ contains $\mu_\infty$.

**Remark on the statement.** — Baggett works with the more general setting of a semidirect product $H \rtimes N$ where $H$ is a compact group and $N$ a locally compact abelian group, and his main result does not look exactly like the above: there is the anecdotal point that he needs to use nets rather than sequences, and the more serious fact that he calls in a topology (introduced by Fell [9]) on the set $A(H)$ of pairs $(J, \tau)$, where $J$ is a closed subgroup of $H$ and $\tau$ is an irreducible representation of $J$. Let us first give a statement closer in spirit to [3, Theorem 6.2-A]. For the Cartan motion group $G_0 = K \rtimes p$, using the notations of Theorem 3.3, Baggett proves that $\sigma_\infty$ is a limit point of $\{\sigma_n\}_{n \in \mathbb{N}}$ if and only if there is a subsequence $\{\sigma_{n_j}\}_{j \in \mathbb{N}}$ such that:

(a) as $j$ goes to infinity, $\chi_{n_j}$ goes to $\chi_\infty$;

(b) $(K_{\chi_{n_j}}, \mu_{n_j})_{j \in \mathbb{N}}$ goes, in the Fell space $A(K)$, to a pair $(K_{\lim}, \mu_{\lim})$ where $K_{\lim}$ is a subgroup of $K_{\chi_\infty}$;

(c) the induced representation $\text{Ind}_{K_{\lim}}^{K_{\chi_\infty}}(\mu_{\lim})$ contains $\mu_\infty$.

If we fix a minimal parabolic subgroup $P_{\min} = M_{\min}A_{\min}N_{\min}$, we can choose the Mackey parameters $(\chi_n, \mu_n)$ in such a way that $\chi_n$ always belongs to the closed Weyl chamber $a^+_{\min}$ that comes with $P_{\min}$. But then, since there is only a finite number of subgroups of $K$ that can arise as the stabilizer of some $\chi$ in $a^+_{\min}$, there is only a finite number of subgroups that can arise as $K_{\chi_n}$ for some $n$. In Baggett’s statement above, we can thus replace (b) with

2. It is the connected component associated with a discrete pair of the form $(L_{\min}, 1)$ where $L_{\min}$ is the Levi factor for a Borel subgroup of $G$ and $1$ is the trivial representation of $L$. 
(b') the sequence \( \left( (K_{n_j}, \mu_{n_j}) \right) \) is eventually constant and \( K_{n_j} \) is actually a subgroup of \( K_{\infty} \).

This leads to the statement in Theorem 3.3. \( \square \)

4. Continuity of the Mackey-Higson bijection

\textbf{Theorem 4.1.} — The bijection \( \mathcal{M} : \tilde{G} \to \tilde{G}_0 \) is continuous.

What we will actually check is that if \( \pi_{\infty} \) is an element of \( \tilde{G} \) and if \( (\pi_n)_{n \in \mathbb{N}} \) is a sequence of irreducible tempered representations of \( G \) that admits \( \pi_{\infty} \) as a limit point, then there exists a subsequence of \( (\pi_n)_{n \in \mathbb{N}} \) whose image under \( \mathcal{M} \) admits \( \mathcal{M}(\pi_{\infty}) \) as a limit point.

4.1. Preliminaries on the reductive side. — Write \( \tilde{G}_\emptyset \) for the connected component of \( \tilde{G} \) that contains \( \pi_{\infty} \).

There is a subsequence of \( (\pi_n)_{n \in \mathbb{N}} \) whose terms all lie in \( \tilde{G}_\emptyset \); since we will have a handful of successive extractions to do hereafter, we will replace \( (\pi_n)_{n \in \mathbb{N}} \) by such a subsequence without changing the notation.

We now take up the setting of Notations 2.3 and consider a parabolic subgroup \( LN \), a “discrete series” representation \( \sigma \) of \( L \), and parameters \( (\nu_k)_{k \in \mathbb{N} \cup \{ \infty \}} \) in \( \mathfrak{a}^+ \), in such a way that \( \pi_{\infty} \) occurs in \( \text{Ind}^G_L(\sigma \otimes e^{i\nu_\infty}) \) and for each \( n \), \( \pi_n \) occurs in \( \text{Ind}^G_L(\sigma \otimes e^{i\nu_n}) \).

By Corollary 2.6, after passing to a subsequence if necessary, we may assume that \( \nu(\pi_n) \) goes to \( \nu(\pi_\infty) \) in \( \tilde{a}/W \). Since the Weyl group \( W \) is finite and \( \nu_n \) (resp. \( \nu_\infty \)) is a representative of \( \nu(\pi_n) \) (resp. \( \nu(\pi_\infty) \)) in \( \mathfrak{a}^+ \), we may assume, perhaps after another passage to subsequences, that \( \nu_n \) in fact goes to \( \nu_\infty \) in \( \mathfrak{a}^+ \).

4.1.1. Strata in the Weyl chamber. — For each \( n \) in \( \mathbb{N} \cup \{ \infty \} \), let us consider \( S_n = \{ \alpha \in \Delta^+ \mid \langle \alpha, \nu_n \rangle > 0 \} \). That subset of \( \Delta^+ \) keeps track of the “\( \alpha \)-regularity of \( \nu_n \)”: when \( S_n = \emptyset \) we have \( \nu_n = 0 \), whereas \( \nu_n = \Delta^+ \) if and only if \( \nu_n \) is \( \alpha \)-regular.

\textbf{Observation 4.2.} — Suppose \( \Sigma \) is a subset of \( \Delta^+ \). If \( \{ n \in \mathbb{N} \mid S_n = \Sigma \} \) is infinite, then \( \Sigma \) contains \( S_\infty \).

This is because when \( \alpha \in \Delta^+ \) lies in \( S_\infty \), we have \( \langle \alpha, \nu_\infty \rangle > 0 \), so that eventually \( \langle \alpha, \nu_n \rangle > 0 \) and \( \alpha \in S_n \). \( \square \)

Let us now fix a subset \( \Sigma \subset \Delta^+ \) such that \( \{ n \in \mathbb{N} \mid S_n = \Sigma \} \) is infinite. After passing again to a subsequence if necessary, we may assume that \( S_n = \Sigma \) for all \( n \). We now observe that the centralizer \( Z_G(\nu_n) \) of \( \nu_n \) in \( G \) does not depend on \( n \). See for instance the proof of Theorem 3.2(a) in [1]: if \( L_{\min} = M_{\min}A_{\min} \) is a minimal Levi subgroup of \( G \) contained in \( L \), then \( L_{\nu_n} \) is generated by \( M_{\min} \) and the root subgroups for roots \( \alpha \) that lie in \( S_n = \Sigma \). These do not depend on \( n \).

We will henceforth write \( L_{\text{seq}} \) for the common centralizer of all \( \nu_n \), \( n \in \mathbb{N} \). Given Observation 4.2, if we write \( L_\infty \) for the centralizer of \( \nu_\infty \) in \( G \), then

\[ L \subset L_{\text{seq}} \subset L_\infty \]

and if we form the Langlands decompositions \( L_{\text{seq}} = M_{\text{seq}}A_{\text{seq}} \) and \( L_\infty = M_\infty A_\infty \), then

\[ A_\infty \subset A_{\text{seq}} \subset A. \]

We remark, following the proof of [1, Theorem 3.2], that \( L \) is in fact a Levi subgroup in the reductive group \( L_{\text{seq}} \), and that \( L_{\text{seq}} \) is itself a Levi subgroup of \( L_\infty \). With apologies for the clumsy notation, let us write \( \tilde{N} \) for a subgroup of \( L_{\text{seq}} \) such that \( L\tilde{N} \) is a parabolic subgroup of \( L_{\text{seq}} \) with Levi factor \( L \), and \( N_{\text{seq}} \) for a subgroup of \( L_\infty \) such that \( L_{\text{seq}}N_{\text{seq}} \) is a parabolic subgroup of \( L_\infty \) with Levi factor \( L_{\text{seq}} \). We will also need the maximal compact subgroups

\[ K_{\text{seq}} = K \cap L_{\text{seq}} \quad \text{and} \quad K_\infty = K \cap L_\infty. \]

4.1.2. Mackey parameters. — The above observations make it easy to pin down the Mackey parameters for each of the \( \pi_n \)s that remain at this stage (after the extractions already performed).

Recall from Vogan’s work that for every irreducible representation \( \mu \) of \( K_{\text{seq}} \), there exists a unique irreducible tempered representation of \( L_{\text{seq}} \) that has real infinitesimal character and lowest \( K_{\text{seq}} \)-type \( \mu \). Write \( V_{L_{\text{seq}}} (\mu) \) for that representation.

3. For the coadjoint action, where we view \( \nu_n \in \mathfrak{a}^* \) as an element of \( g^* \).
Lemma 4.3. — For every $n$ in $\mathbb{N}$, there exists $\mu_n$ in $\hat{K}_{\text{seq}}$ such that $\pi_n$ is equivalent with the irreducible representation $\text{Ind}_{L_{\text{seq}}}^G(\mathbf{V}_{L_{\text{seq}}} (\mu_n) \otimes e^{i\nu_n})$. Furthermore, the representation $\mu_n$ is uniquely determined by the set of lowest $K$-types of $\pi_n$.

Proof. — Recall that $\pi_n$ occurs in $\text{Ind}_{L_N}^G (\sigma \otimes e^{i\nu_n})$ and remark, as in the proof of [1, Theorem 3.2(c)], that if $L_{\text{seq}}N'$ is a parabolic subgroup of $G$ with Levi factor $L_{\text{seq}}$, then

$$\text{Ind}_{L_{\text{seq}}}^G (\sigma \otimes e^{i\nu_n}) \simeq \text{Ind}_{L_{\text{seq}}}^G \left( \text{Ind}_{L_N}^G (\sigma \otimes 1) \otimes e^{i\nu_n} \right).$$

(4.4)

The representation $\varpi = \text{Ind}_{L_N}^G (\sigma \otimes 1)$ is tempered, has real infinitesimal character and a finite number of irreducible components. Since $\nu_n$ is $a_{\text{seq}}$-regular, for every irreducible component $\tau$ of $\varpi$, the representation $\text{Ind}_{L_{\text{seq}}N} (\tau \otimes e^{i\nu_n})$ is in fact irreducible — and its lowest $K$-types are entirely determined by $\tau$. Thus, the lowest $K$-types of $\pi_n$ make it possible to pin down the one irreducible constituent $\tau_n$ for which $\pi_n \simeq \text{Ind}_{L_{\text{seq}}N} (\tau_n \otimes e^{i\nu_n})$. Writing $\mu_n$ for the unique lowest $K_{\text{seq}}$-type of $\tau_n$, we obtain $\tau_n \simeq \mathbf{V}_{L_{\text{seq}}} (\mu_n)$, as desired.

In the above statement, $\mu_n$ may depend on $n$; but it determines the set of lowest $K$-types of $\pi_n$, which is a subset of the class $C_n$ described in §2.2, and different values for $\mu_n$ lead to different sets of lowest $K$-types [1, Lemma 4.2]. So there are only a finite number of possibilities for the element $\mu_n$ in $\hat{K}_{\text{seq}}$. After a new extraction, we obtain the following result:

Lemma 4.4. — There exists a subsequence $(\pi_{n_j})_{j \in \mathbb{N}}$ of $(\pi_n)_{n \in \mathbb{N}}$ with the property that $\mu_{n_j}$ does not depend on $j$, so that there exists $\mu_{\infty} \in \hat{K}_{\text{seq}}$ such that: $\forall j \in \mathbb{N}, \pi_{n_j} \simeq \text{Ind}_{L_{\text{seq}}}^G (\mathbf{V}_{L_{\text{seq}}} (\mu_{\infty}) \otimes e^{i\nu_{n_j}})$.

If we replace $(\pi_{n})_{n \in \mathbb{N}}$ by the above subsequence and take up the notations of §3.1, we now know from Lemma 3.1 that for each $n$, a Mackey parameter for (the new) $\pi_n$ is $(\nu_n, \mu_{\infty})$.

4.1.3. Remark on the lowest $K$-types of $\pi_{\infty}$. — At this stage, since each $\nu_n$ is $a_{\text{seq}}$-regular, we know that the set of lowest $K$-types of $\pi_n$ does not depend on $n$; it is determined by the pair $(\hat{K}_{\text{seq}}, \mu_{\infty})$. Example 2.5 shows that $\pi_{\infty}$ does not necessarily have the exact same set of lowest $K$-types as the $\pi_n$: if $G = \text{SL}(2, \mathbb{R})$ and $\pi_{\infty}$ is a limit of discrete series, then $\pi_{\infty}$ has a unique lowest $SO(2)$-type, but $\pi_{\infty}$ is a limit point of the nonspherical principal series, which consists of representations having two distinct lowest $SO(2)$-types (one of which is that of $\pi_{\infty}$).

Lemma 4.5. — Suppose $C_{\text{seq}} \subset \hat{K}$ is the set of lowest $K$-types common to all $\pi_n$, $n \in \mathbb{N}$. Then the set of lowest $K$-types of $\pi_{\infty}$ is contained in $C_{\text{seq}}$.

Proof. — We first recall that if $\lambda$ is a class in $\hat{K}$ and $\chi_\lambda : K \to \mathbb{C}$ is its character, there is a simple criterion for ascertaining that an irreducible tempered representation $\pi$ of $G$ does not contain $\lambda$ upon restriction to $K$: one needs only check that for every matrix element $c : G \to \mathbb{C}$ of $\pi$, the convolution product $\chi_\lambda \ast (c|_K)$ vanishes.

We also recall that the partial ordering on $\hat{K}$ used to define the notion of lowest $K$-type can be built from a positive-valued function $\| \cdot \|_{\hat{K}}$ on $\hat{K}$. All classes in $\lambda$ in $C_{\text{seq}}$ have the same norm $\| \chi_\lambda \|_{\hat{K}}$; write $\| C_{\text{seq}} \|_{\hat{K}}$ for the common value.

Consider then

- a $K$-type $\lambda$ such that $\| \chi_\lambda \|_{\hat{K}} \leq \| C_{\text{seq}} \|_{\hat{K}}$, but which does not belong to $C_{\text{seq}}$,

- and a matrix element $c$ of $\pi_{\infty}$;

let us inspect the convolution $\chi_\lambda \ast (c|_K)$. By definition of the Fell topology, there exists a sequence $(c_n)_{n \in \mathbb{N}}$ of complex-valued functions on $G$, in which each map $c_n$ is a matrix element of $\pi_n$, and which as $n$ goes to infinity goes to $c$ uniformly on compact subsets of $G$. Since the $K$-type $\lambda$ appears in none of the $\pi_n$, we have $\chi_\lambda \ast ((c_n)|_K) = 0$ for each $n$. But $(c_n)|_K$ goes to $c|_K$ uniformly on $K$; so $\chi_\lambda \ast ((c_n)|_K)$ does pointwise converge to the $\chi_\lambda (c|_K)$. Thus the latter function must be zero.

We conclude that a $K$-type whose norm does not exceed $\| C_{\text{seq}} \|_{\hat{K}}$ can appear in $\pi_{\infty}$ only if it belongs to $C_{\text{seq}}$. Besides, $\pi_{\infty}$ lies in $\hat{G}_{\emptyset}$, so we already know (by Proposition 2.2) that its lowest $K$-types all have norm $\| C_{\text{seq}} \|_{\hat{K}}$; this proves the lemma.

4. This is the convolution product over $K$: the function $k \mapsto \int_K \chi_\lambda (ku^{-1})c(u)du$ from $K$ to $\mathbb{C}$. 

□
4.2. Verification of Baggett’s criterion. — We can now prove that $\mathcal{M}(\pi_\infty)$ is a limit point of $(\mathcal{M}(\pi_n))_{n \in \mathbb{N}}$ in the unitary dual $\hat{G}_0$. In §4.1.2, we showed that a Mackey parameter for $\pi_n$ is $(\nu_n, \mu_{\text{seq}})$ — recall that in the present context, the centralizer of $\nu_n$ in $K$ is equal to $K_{\text{seq}}$ (and independent of $n$).

By the argument already used for Lemma 4.3, we also know that there exists a parabolic subgroup $P_\infty$ with Levi factor $L_\infty$, and an irreducible representation $\mu_\infty \in \hat{K}_\infty$, such that $\pi_\infty$ is equivalent with the irreducible representation $\text{Ind}_{P_\infty}^{G_\infty}(V_{L_\infty}(\mu_\infty) \otimes e^{i\nu_\infty})$; the representation $\pi_\infty$ then admits $(\nu_\infty, \mu_\infty)$ as a Mackey parameter.

Given Baggett’s criterion 3.3, to complete the proof of Theorem 4.1, we need only verify the following fact:

**Lemma 4.6.** The representation $\mu_\infty$ of $K_\infty$ is contained in $\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_{\text{seq}})$.

**Proof.** Suppose the Lemma is false. Let us induce to $K$ and compare the $K$-modules

$$\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_\infty) \quad \text{and} \quad \text{Ind}_{K_{\text{seq}}}^{K_\infty}(\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_{\text{seq}})) \simeq \text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_{\text{seq}}); \quad (4.5)$$

more precisely, let us inspect their lowest $K$-types. If $\tilde{\mu}$ is an irreducible representation of $K_\infty$ that appears in $\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_\infty)$, then under our assumption that the Lemma is false, we have $\mu_\infty \neq \tilde{\mu}$; but in that situation, we know [1, Lemma 4.2] that the representations $\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_\infty)$ and $\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\tilde{\mu})$ have no lowest $K$-type in common. The second $K$-module in $(4.5)$ is a direct sum of $K$-modules that each read $\text{Ind}_{K_{\text{seq}}}^{K_\infty}(\mu_\infty)$ for some $\mu_\infty \neq \mu_\infty$; so the second $K$-modules to be compared in $(4.5)$ actually have no lowest $K$-type in common.

To see that this is impossible, we only have to point out that the two $K$-modules in $(4.5)$ are the restrictions to $K$ of $\pi_\infty$ and $\pi_n$ (see [1], Remark 2.3). Lemma 4.5 shows that each lowest $K$-type in $\pi_\infty$ is also a lowest $K$-type in $\pi_n$, so the set of lowest $K$-types of the first module of $(4.5)$ is contained in the set of lowest $K$-types of the second. The Lemma follows.

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