Kaluza-Klein mass spectra on extended dimensional branes

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Exploiting the the Kaluza-Klein decompositions, we study the hierarchy between the Planck and electroweak scales on extended brane models with dimensionalities greater than that of the standard Randall-Sundrum model. Specifically, the extended dimensional four-brane and five-brane are constructed by attaching a circle and a sphere to the three-brane, respectively. Constructing the Kaluza-Klein modes on the extended four-brane and five-brane, we obtain their lightest effective masses which are shown to suppress exponentially with respect to the Planck mass, similar to the standard three-brane case. However, the effective masses in their excited spectra on the four- and five-branes imply intriguing characteristics associated with the quantization of the compact circular and spherical manifolds, respectively.

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I. INTRODUCTION

There have been lots of progresses and discussions on the Randall-Sundrum (RS) model [1, 2]. The RS model on the three-brane has been analyzed in terms of the Kaluza-Klein (KK) decompositions [3, 4] to yield the effective scale of couplings [5, 6]. On the RS brane world, the population evolution and primordial black holes have been investigated to study a modified evaporation law of the black hole [7]. The string excitations in the RS effective model for electroweak symmetry breaking also has been studied in the string theory to investigate the mass of the string states associated with the number of colors. Here it has been shown that there exist strong bounds on the mass of new string states to yield the collider signals [8]. Recently, the RS model has been revisited in which the hierarchy between the Planck and electroweak scales, and the Higgs field confined to a brane at the infrared scale are well defined. Specifically, it has been suggested that in the Standard Model confined to the TeV brane, the KK modes of the graviton in the RS scenario could be detectable in the Large Hadron Collider detector, if the TeV brane energy scale is sufficiently low [9].

On the other hand, the string singularities, which are the string theory version of the Hawking-Penrose singularities [10, 11], have been applied to the early universe with an arbitrary higher dimensionality [12]. In this higher dimensional stringy cosmology, the expansion of the universe has been explained by exploiting Hawking-Penrose type singularity in geodesic surface congruences for the timelike and null strings [13, 14]. Next, the twist and shear have been studied in terms of the expansion of the universe. Moreover, as the early universe evolves with expansion rate, the twist of the stringy congruence decreases exponentially and the initial twist value should be large enough to sustain the rotations of the ensuing universe, while the effects of the shear are negligible to produce the isotropic and homogeneous universe [15]. By exploiting the phantom field, the evolution of cosmology has been also studied in higher dimensional spacetime.

The motivation of this work is to investigate the nature of the universe in terms of the extended dimensional spacetime as in [13, 16], combined with the RS type orbifold. In this paper we will generalize the KK modes in the RS model with two three-branes [3, 6] to the cases of two four-branes and five-branes, to construct the corresponding effective mass spectra of the reduced three-branes. To do this, we will exploit the novel aspects of bulk fields in the KK decompositions in the RS cosmologies, and we will next investigate the hierarchy problems on these extended dimensional branes.

This paper is organized as follows. In section II we introduce the four-brane RS metric to construct the mass spectrum of the KK modes and its implication in the hierarchy problem, and in section III we further increase the dimensionality of the RS model to obtain the five-brane scenario and to find that the lowest effective mass in the five-brane case becomes that on the standard three-brane. Section IV includes summary and discussions.

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II. FOUR-BRANE COSMOLOGY AND EFFECTIVE MASS SPECTRUM

In the RS model \cite{1, 2, 3, 4}, they investigated some interesting aspects originated from the extra dimension to yield successful theoretical predictions. In fact, they used one more extra dimensional model, and in their scenarios there exist the three-brane cosmology connected by this one dimensional manifold. We now extend the dimensionality of the brane to the next larger space, namely the four-brane case. Moreover, the ordinary (4+2)-dimensional cosmology has been shown to possess some intriguing aspects in the stringy early universe \cite{13}. In the limit of the four-brane case in which the total manifold has \( D = 6 \) dimensions, we observed that the strong energy condition in the stringy cosmology

\[
\frac{D - 4}{D - 2} \rho + \frac{D}{D - 2} P \geq 0,
\]  

becomes one of the point particle strong energy conditions

\[
\rho + 3P \geq 0,
\]  

for the massive particle \cite{14}. In that sense, the four-brane cosmology may reveal a critical feature and leads us an interesting physics.

Now, in order to investigate the bulk fields in the (4 + 2)-dimensional RS model, we introduce the metric of the specific form

\[
dx^2_{4+2} = e^{-2kr_c|\alpha|} (\eta_{\mu\nu} dx^\mu dx^\nu - R^2 d\phi^2) - r_c^2 d\alpha^2.
\]  

Here \( x^\mu (\mu = 0, 1, 2, 3) \) are the coordinates on the (3 + 1)-dimensional Lorentzian spacetime \( \mathcal{M} \). The \( \phi \) is the coordinate of a compact manifold defined by \( 0 \leq \phi \leq 2\pi \). We also have the \( S^1(\alpha)/\mathbb{Z}_2 \) orbifold constraint to yield \( (x^\mu, \phi, \alpha) = (x^\mu, \phi, -\alpha) \). The four-branes are imposed to be located at \( \alpha = 0 \) and \( \alpha = \pi \). The \( r_c \) and \( k \) are the size of the extra dimension in the \( \alpha \)-direction and the order of Planck scale, and \( R \) is the radius of the compact manifold \( S^1(\phi) \) defined on the coordinate \( \phi \). We then have an action for a free scalar field in the bulk as follows

\[
S = \frac{1}{2} \int_{\mathcal{M}} d^4x \int_0^{2\pi} d\phi \int_{-\pi}^\pi d\alpha \sqrt{|g|} (g^{AB} \partial_A \Psi \partial_B \Psi - m^2 \Psi^2),
\]  

where \( g_{AB} (A, B = \mu, \phi, \alpha) \) are given by

\[
g_{AB} = \text{diag} (e^{-2kr_c|\alpha|} \eta_{\mu\nu}, -e^{-2kr_c|\alpha|} R^2, -r_c^2),
\]  

from which we obtain \( \sqrt{|g|} = e^{-5\sigma} R r_c \) with \( \sigma = kr_c|\alpha| \). After some algebra, we arrive at

\[
S = \frac{1}{2} \int_{\mathcal{M}} d^4x \int_0^{2\pi} d\phi \int_{-\pi}^\pi d\alpha R r_c \left( e^{-3\sigma} \eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi - \frac{e^{-3\sigma}}{R^2} (\partial_\phi \Psi)^2 + \frac{1}{r_c^2} \Psi \partial_\alpha (e^{-5\sigma} \partial_\alpha \Psi) - m^2 e^{-5\sigma} \Psi^2 \right).
\]  

Next, we consider the KK decompositions of \( \Psi(x, \phi, \alpha) \) along the compact \( \alpha \) manifold in this four-brane case by introducing

\[
\Psi(x, \phi, \alpha) = \sum_n \psi_n(x, \phi) \frac{y_n(\alpha)}{\sqrt{r_c}}, \quad (n = 1, 2, 3, \cdots)
\]  

where the fields \( y_n(\alpha) \) satisfy the normalization condition

\[
\int_{-\pi}^\pi d\alpha \ e^{-3\sigma} y_n(\alpha)y_m(\alpha) = \delta_{nm}
\]  

and the differential equation related to the Bessel functions

\[
-\frac{1}{r_c^2} \frac{d}{d\alpha} \left( e^{-5\sigma} \frac{dy_n}{d\alpha} \right) + m^2 e^{-5\sigma} y_n = m_n^2 e^{-3\sigma} y_n.
\]  

Introducing new variables

\[
z_n = \frac{m_n}{k} e^{-\sigma}, \quad f_n = e^{-5\sigma/2} y_n,
\]  

\[
\int_{-\pi}^\pi d\alpha \ e^{-3\sigma} f_n f_m = \delta_{nm}.
\]  

(2.10)
we rewrite (2.9) for $\alpha \neq 0, \pm \pi$ as

$$z_n \frac{d^2 f_n}{dz_n^2} + z_n \frac{df_n}{dz_n} + \left(z_n^2 - \left(\frac{5}{2}\right)^2 - \frac{m^2}{k^2}\right)f_n = 0,$$

(2.11)

whose solutions are given by the Bessel functions

$$f_n(z_n) = \frac{1}{N_n} \left(J_\nu(z_n) + b_{\nu} Y_\nu(z_n)\right),$$

(2.12)

where $N_n$ is a normalization factor to be fixed and $J_\nu$ and $Y_\nu$ are the first- and second-kind Bessel functions of order

$$\nu = \left(\frac{5}{2} + \frac{m^2}{k^2}\right)^{1/2}.$$ 

(2.13)

We then obtain the action of the form

$$S = \frac{1}{2} \sum_n \int_M d^4 x \int_0^{2\pi} d\phi \left(\eta^{\mu\nu} \partial_\mu \psi_n \partial_\nu \psi_n - \frac{1}{R^2} \left(\partial_\phi \psi_n\right)^2 - m_{np}^2 \psi_n^2\right),$$

(2.14)

Here one notes that $m$ defined in the six-dimensional spacetime and the lightest mass $m_{n=1}$ of the KK modes in the five-dimensional spacetime are related to each other as follows

$$m_{n=1} \sim m e^{-kr_c \pi}.$$ 

(2.15)

After some algebra, one can readily observe that the equations (2.8), (2.9) and (2.11) hold in the four-brane case, as expected. We again perform further the KK decompositions of $\psi_n(x, \theta)$ along the $\theta$ direction by exploiting the following ansatz for a given $n$

$$\psi_n(x, \phi) = \sum_p X_{np}(x) e^{ip\phi} \sqrt{2\pi R}.$$ 

(2.16)

The action (2.14) is then rewritten as

$$S = \frac{1}{2} \sum_{n,p} \int_M d^4 x \left(\eta^{\mu\nu} \partial_\mu X_{np} \partial_\nu X_{np} - m_{np}^2 X_{np}^2\right),$$

(2.17)

where the effective mass $m_{np}$ on the reduced three-brane is given by

$$m_{np} = \left(m_n^2 + \frac{p^2}{R^2}\right)^{1/2}, \quad (n = 1, 2, 3, \cdots; \ p = 0, \pm 1, \pm 2, \cdots).$$ 

(2.18)

It is interesting to note that, in the large $R$ limit with $R$ being the radius of the extended dimension of the four-brane, $m_{np}$ on the three-brane reduces to the effective mass $m_n$ on the four-brane. Assuming that $m$ is of order of the Planck scale mass $M_{\text{Planck}}$, from (2.15) and (2.18) we observe that the mass $m_{n=1,p=1}$ on the reduced three-brane is suppressed with respect to the $M_{\text{Planck}}$ as follows

$$m_{n=1,p=1} \sim M_{\text{Planck}} e^{-kr_c \pi},$$

(2.19)

whose characteristics will be discussed in the Conclusions, together with those of the five-brane case.

III. FIVE-BRANE AND KALUZA-KLEIN DECOMPOSITIONS

Next, we consider the five-brane cosmology whose metric is given by

$$d\xi^2 = e^{-2k r_c |\alpha|} \left(\eta_{\mu\nu} dx^\mu dx^\nu - R^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right) - r_c^2 d\alpha^2$$

(3.1)

to yield

$$g_{AB} = \text{diag} \left(e^{-2k r_c |\alpha|} \eta_{\mu\nu}, -e^{-2k r_c |\alpha|} R^2, -e^{-2k r_c |\alpha|} R^2 \sin^2 \theta, -r_c^2\right).$$ 

(3.2)
and $\sqrt{|g|} = e^{-6\sigma} R \sin \theta r_c$. On the five-brane, the action \((2.3)\) for the four-brane case is now modified as

$$S = \frac{1}{2} \int_{\mathcal{M}} d^4x \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\alpha \sqrt{|g|} \left( g^{AB} \partial_A \Psi \partial_B \Psi - m^2 \Psi^2 \right). \quad (3.3)$$

After some algebra, the above action has the following form

$$S = \frac{1}{2} \int_{\mathcal{M}} d^4x \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\pi}^{\pi} d\alpha R^2 \sin \theta r_c \left( e^{-4\sigma} \eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi - \frac{e^{-4\sigma}}{R^2} (\partial_\phi \Psi)^2 - \frac{1}{r_c^2} \Psi \partial_\alpha (e^{-6\sigma} \partial_\alpha \Psi) - m^2 e^{-6\sigma} \Psi^2 \right). \quad (3.4)$$

Now we introduce Kluza-Klein modes along the $\alpha$ and $\phi$ directions by exploiting

$$\Psi(x, \theta, \phi, \alpha) = \sum_{n, p} X_{np}(x, \theta) \frac{\eta^{\mu\nu} y_n(\alpha)}{\sqrt{2\pi R r_c}}, \quad (n = 1, 2, 3, \ldots; \; p = 0, \pm 1, \pm 2, \cdots)$$

to arrive at

$$S = \frac{1}{2} \sum_{n, p} \int_{\mathcal{M}} d^4x \int_0^\pi d\theta R \sin \theta \left( \eta^{\mu\nu} \partial_\mu X_{np} \partial_\nu X_{np} - \frac{1}{R^2} (\partial_\theta X_{np})^2 - \frac{p^2}{R^2 \sin^2 \theta} X_{np}^2 - m^2 X_{np}^2 \right). \quad (3.5)$$

We again perform the Kluza-Klein decompositions along the compact $\theta$ coordinate with the following ansatz

$$X_{np}(x, \theta) = \sum_l R_{np l}(x) \frac{N_{lp}}{\sqrt{R}} P_l^p(\theta), \quad (l = 0, 1, 2, 3, \cdots) \quad (3.6)$$

where $P_l^p(\theta)$ are the associated Legendre functions satisfying the identities

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP_l^p}{d\theta} \right) = -\frac{p^2}{\sin^2 \theta} P_l^p = -l(l+1) P_l^p,$$

$$\int_0^{\pi} d\theta \sin \theta P_l^p(\theta) P_l^{p'}(\theta) = N_{lp}^{-1} \delta_{p', p''}, \quad (3.7)$$

with

$$N_{lp} = \left( \frac{2l+1}{2} \cdot \frac{(l-p)!}{(l+p)!} \right)^{1/2}. \quad (3.8)$$

Here, for any given $l$, there are $(2l + 1)$ possible values of $p$: $p = -l, -l + 1, \cdots, -1, 0, 1, \cdots, l - 1, l$.

Using the above identities \((3.8)\), we then rewrite the action \((3.6)\) in the form

$$S = \frac{1}{2} \sum_{n, p, l} \int_{\mathcal{M}} d^4x \left( \eta^{\mu\nu} \partial_\mu R_{np l} \partial_\nu R_{np l} - m_{nl}^2 R_{np l}^2 \right) \quad (3.9)$$

where the effective mass spectrum on the reduced three-brane is given by

$$m_{nl} = \left( m_n^2 + \frac{l(l+1)}{R^2} \right)^{1/2}, \quad (n = 1, 2, 3, \cdots; \; l = 0, 1, 2, \cdots). \quad (3.10)$$

It is interesting to see that, even though we have enlarged the dimensionality by two with respect to the three-brane RS model, the effective mass spectrum in \((3.11)\) in the five-brane case at hand has the form similar to the four-brane result \((2.18)\). This phenomena originate from the fact that the harmonic spectrum of $S^2$ manifold is described in term of only one quantum number $l \; (l = 0, 1, 2, 3, \cdots)$ in \((3.11)\). Following the algorithms in the previous section, from \((2.15)\) and \((3.11)\) we obtain that the mass $m_{n=1, l=0}$ on the reduced three-brane is suppressed exponentially again with respect to the Planck mass $M_{\text{Planck}}$

$$m_{n=1, l=0} \cong M_{\text{Planck}} e^{-k r_c \pi}, \quad (3.12)$$

which is the same as the three-brane RS model result \((3)\).
IV. CONCLUSIONS

In conclusion, we have revisited the hierarchy between the Planck and electroweak scales on higher dimensional brane cosmologies whose dimensionalities are greater than that of the standard Randall-Sundrum model. As examples, we have considered the three-brane and four-brane, which are constructed by attaching the circle and the sphere to the three-brane, respectively. We have used the KK decompositions on the three-brane and four-brane to construct the mass spectra on their brane worlds. In the four- and five-brane cases, the lightest effective masses on the reduced these branes have been shown to suppress exponentially relative to the Planck mass, similar to the standard three-brane case.

Now, we have comments to address. In (2.18) and (3.11), we recognize that the effective masses of the excited spectrum of the four-brane and five-brane depend on the radii of the additional circular and spherical manifolds included in their branes. These characteristics originate from the quantizations of the compact circular and spherical spaces in the four-brane and five-brane, respectively. Moreover, it is intriguing to observe that the $R$ dependence in (2.18) and (3.11) implies that, as $R$ gets smaller, its suppression effect on the reduced three-brane masses becomes weaker.

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