ASYMPTOTICALLY COMPATIBLE PIECEWISE QUADRATIC POLYNOMIAL COLLOCATION FOR NONLOCAL MODEL

MINGHUA CHEN*, YINGFAN QI†, AND JIANKANG SHI‡

Abstract. In this paper, we propose and analyze piecewise quadratic polynomial collocation for solving the linear nonlocal diffusion model with the weakly singular kernels. The detailed proof of the convergence analysis for the nonlocal models with the horizon parameter $\delta = O(h^{\beta})$, $\beta \geq 0$ are provided. More concretely, the global error is $O\left(h^{\max\{2.4-2\beta\}}\right)$ if $\delta$ is the grid point, but it shall drop down to $O\left(h^{\min\{2.1+\beta\}}\right)$ if $\delta$ is not the grid point. In particular, the asymptotically compatible scheme are also rigorously proved, which has the global error $O\left(h^{\min\{2.2\beta\}}\right)$ as $\delta, h \to 0$. Finally, numerical experiments are presented to verify the theoretical results.

Key words. piecewise quadratic polynomial collocation, nonlocal model, asymptotically compatible scheme, stability and convergence analysis

AMS subject classifications. 65L60, 65L20

1. Introduction. Nonlocal diffusion problems have been used to model very different scientific phenomena, which can either complement or serve as an alternative to classical partial differential equations (PDEs). The integral formulations of spatial interactions in nonlocal models with nonlocal Dirichlet volume constraint can naturally account for nonlocal interactions effects and allow more singular solution [7]. For example, nonlocal peridynamic (PD) is becoming an attractive emerging tool for the multiscale material simulations of crack nucleation and growth, fracture and failure of composites [13]. Mathematical analysis of PD models and other related nonlocal models, such as nonlocal diffusion, can be found in [7]. In particular, nonlocal models [8] with a finite range $\delta$ of interaction serve as a bridge between fractional PDEs [2] and local PDEs. For $\delta > 0$, compared with classical PDE models, the complexities are introduced by the nonlocal interactions. As $\delta \to 0$, the nonlocal effect vanish and the zero-horizon limit of nonlocal PD models reduce to a classical local PDEs model when the latter is well-defined. Such limiting behavior provides connections and consistencies between nonlocal and local models, and has immense practical significance especially for multiscale modeling and simulations. In particular, if $\delta$ to be proportional to $h$, as $\delta, h \to 0$, the concept of asymptotically compatible (AC) schemes have been introduced in [17, 24].

There has been much recent interest in developing numerical algorithms for nonlocal models, including finite difference [14, 16], finite element [10, 16, 17, 24], collocation method [15, 21, 24], meshfree method [11], and fast multigrid or conjugate gradient method [8, 21]. Among various techniques for solving integral nonlocal problems, collocation methods are the simplest, since they only need a single integration and are much simpler to implement on a computer. However, to seek numerical discretization of the strong form (e.g., collocation, finite difference), it is difficult to show stability of the high-order numerical schemes while trying to keep the asymptotically compatible

*Corresponding author. School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China (Email: chenmh@lzu.edu.cn, chenmh2009@lzu.edu.cn).
† School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China
‡ School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China
property \[15\]. It is well-known that the standard piecewise linear polynomial collocation (PLC) is used to approximate the nonlocal problems, which is not asymptotically compatible \[17, 24\]. To overcome it, the quadrature-based PLC collocation-type finite difference schemes are introduced \[17\] and extended to the multidimensional case \[9\]. Recently, a new corrected quadrature rule based on PLC is derived, which is also asymptotic compatibility \[24\]. How about piecewise quadratic polynomial collocation (PQC) with standard quadrature rule? How about the horizon parameter \(\delta\) is not as a grid point? Since the whole key point of AC schemes is convergent numerical methods which does not require any relation between mesh size/grid points and \(\delta\) \[7\]. Moreover, the horizon of the material is a physical property of the material of the finite bar and should be independent of the computational mesh size \(h\) \[20\]. Based on these observations, we propose and analyze PQC for solving the linear nonlocal diffusion model with the weakly singular kernels. The main contribution of this work as follows. The detailed proof of the convergence analysis for the nonlocal models with the horizon parameter \(\delta = O(h^\beta), \beta \geq 0\) are provided. More concretely, the global error is \(O(h^{\max\{2.4-2\beta\}})\) if \(\delta\) is the grid point, but it shall drop down to \(O(h^{\min\{2.1+\beta\}})\) if \(\delta\) is not the grid point. In particular, the standard PQC asymptotically compatible scheme are also rigorous proved, which has the global error \(O(h^{\min\{2.2\beta\}})\) as \(\delta, h \to 0\), although standard PLC scheme is not asymptotic compatibility.

The paper is organized as follows. In the next section, we provide the discretization schemes for the nonlocal problems by collocation method. In Section \$3\), we analyze the local truncation errors with the general horizon parameter. The global convergence rates for nonlocal models are detailed proved in Section \$4\). To show the effectiveness of the presented schemes, results of numerical experiments are reported in Section \$5\).

2. Collocation method and numerical schemes. Let us consider the nonlocal model with a volumetric constraint domain \[8\]

\[
\begin{cases}
-L_\delta u_\delta(x) = f_\delta(x) & \text{in } \Omega, \\
u_\delta(x) = g(x) & \text{on } \Omega_I.
\end{cases}
\]

For the sake of simplicity, we denote \(u(x)\) as \(u_\delta(x)\). Here

\[
L_\delta u(x) = \int_{B_\delta(x)} \left[ u(y) - u(x) \right] \gamma_\delta(x, y) dy, \quad \forall x \in \Omega
\]

and \(B_\delta(x) = \{ y \in \mathbb{R} : |y - x| < \delta \}\) denotes a neighborhood centered at \(x\) of radius \(\delta\). To keep the expression simple below we assume we are on the unit interval \(\Omega\) with the volumetric constraint domain \(\Omega_I = [-\delta, 0] \cup [1, 1+\delta]\), but everything can be shifted to arbitrary interval \([a, b]\). The specific form of such nonlocal interactions is prescribed by a nonnegative and radial kernel function \(\gamma_\delta = \gamma_\delta(|y - x|)\), which has the finite second order moment of \(\gamma_\delta\), i.e.,

\[
0 < C_\delta = \int_0^\delta z^2 \gamma_\delta(z) dz < \infty,
\]

where \(C_\delta\) is a well-defined elastic modulus or diffusion constant.

It should be noted that if \(u(x)\) is regular enough and \(\delta\) is small enough, from \(2.2\) there exists

\[
L_\delta u(x) = C_\delta u''(x) + O\left( \int_0^\delta z^4 \gamma_\delta(z) dz \right).
\]
2.1. Collocation method for nonlocal integral operator \([22]\). Now, we introduce and discuss the discretization scheme of \((2.1)\). Denote the horizon \(\delta\) and the mesh size \(h\) as
\[
\delta = \begin{cases} 
\bar{\delta} + \delta_0 h, & \delta \geq h, \\
\bar{\delta} - \delta_0 h, & \delta \leq h
\end{cases}
\]
with \(\bar{\delta} = rh, \ 0 \leq \delta_0 < 1\).

Here the ratio \(r = \lfloor \delta/h \rfloor \geq 1\) if \(\delta \geq h\) and \(r = \lceil \delta/h \rceil = 1\) if \(\delta \leq h\). Let the mesh points \(x_i = ih, h = 1/(N + 1), i \in \mathcal{N}\) and
\[
\mathcal{N} = \left\{ -r + \frac{1}{2}, -r + 1, \cdot \cdot \cdot, 0, \frac{1}{2}, \cdot \cdot \cdot, N + \frac{1}{2}, N + 1, \cdot \cdot \cdot, N + r, N + r + 1 \right\},
\]
\[
\mathcal{N}_{in} = \left\{ \frac{1}{2}, 1, \frac{3}{2}, \cdot \cdot \cdot, N - \frac{1}{2}, N, N + \frac{1}{2} \right\}, \ \mathcal{N}_{out} = \mathcal{N} \setminus \mathcal{N}_{in}.
\]

Let the piecewise quadratic basis functions \(\phi_j(x)\) and \(\phi_{j+\frac{1}{2}}(x)\) be given in \([11\) p. 37] and \(u_Q(x)\) be the piecewise Lagrange quadratic interpolant of \(u(x)\), i.e.,
\[
u_{Q}(x) = \sum_{j=-r+1}^{N+r} u(x_j)\phi_j(x) + \sum_{j=-r}^{N+r} u(x_{j+\frac{1}{2}})\phi_{j+\frac{1}{2}}(x).
\]

It should be noted that the piecewise quadratic polynomial \((2.6)\) is used to approximate the nonlocal operator \(L_\delta u(x)\) rather than \(L_\delta u(x)\) in \((2.2)\), which plays a key role in the design of numerical scheme for nonlocal models with the general horizon parameter. Then we have the following piecewise quadratic interpolation of \(L_\delta u(x)\)
\[
L_{\delta,h} u(x_i) = \int_0^{\bar{\delta}} \gamma_3(z) \left\{ \sum_{m=-r}^{r-1} u(x_{i+m})[\phi_m(z) + \phi_m(-z)] + \sum_{m=-r}^{r-1} u(x_{i+m+\frac{1}{2}})[\phi_{m+\frac{1}{2}}(-z) + \phi_{m+\frac{1}{2}}(z)] - 2u_\delta(x_i) \right\} dz
\]
\[
:= \sum_{m=-r}^{r} a_m u(x_{i+m}) + \sum_{m=-r}^{r-1} a_{m+\frac{1}{2}} u(x_{i+m+\frac{1}{2}}), \ i \in \mathcal{N}_{in}.
\]

Here the coefficients in \((2.7)\) are explicitly given by the following
\[
\gamma_3(z) = \begin{cases} 
2 \int_{-1}^{0} \gamma_3(z) dz, & m = 0, \\
\int_{I_m \cup I_{m+1}} \gamma_3(z) \phi_m(z) dz, & m = 1, 2, \cdot \cdot \cdot, r - 1, \\
\int_{I_r} \gamma_3(z) \phi_r(z) dz, & m = r.
\end{cases}
\]

and
\[
a_{m+\frac{1}{2}} = \begin{cases} 
\int_{I_\delta} \gamma_3(z) \phi_{m+\frac{1}{2}}(z) dz, & m = 0, \\
\int_{I_{m+1}} \gamma_3(z) \phi_{m+\frac{1}{2}}(z) dz, & m = 1, 2, \cdot \cdot \cdot, r - 1.
\end{cases}
\]
with \( a_m = a_{-m}, a_{m + \frac{1}{2}} = a_{-(m + \frac{1}{2})} \) and \( I_m = ((m - 1)h, mh) \).

In particular, if \( \delta \leq h \), the coefficients in (2.8) and (2.9) reduce to

\[
\alpha_m = \begin{cases} 
2\delta^{-2} \int_0^{\delta} z(2z - 3\delta)\gamma_{\delta}(z)dz, & m = 0, \\
\delta^{-2} \int_0^{\delta} z(2z - \delta)\gamma_{\delta}(z)dz, & m = \pm 1,
\end{cases}
\]

and

\[
\alpha_{m + \frac{1}{2}} = 4\delta^{-2} \int_0^{\delta} z(\delta - z)\gamma_{\delta}(z)dz, \quad m = 0, -1.
\]

**Remark 2.1.** The coefficients \( a_m \) and \( a_{m + \frac{1}{2}} \) in (2.7) also can be derived by

\[
\mathcal{L}_{\delta,h} u(x_i) = \int_{x_{i-r}}^{x_{i+r}} \gamma_{\delta}(x) \left\{ \sum_{j=i-r}^{i+r} u(x_j)\phi_j(y) \right\} dy = \sum_{m=-r}^{r} a_m u(x_{i+m}) + \sum_{m=-r}^{r-1} a_{m + \frac{1}{2}} u(x_{i+m + \frac{1}{2}}), \quad i \in \mathcal{N}_{in}.
\]

### 2.2. Numerical scheme for nonlocal model (2.1)

Let \( u_{\delta,h}(x_i) \) be the approximated value of \( u_\delta(x_i) \) and \( f_{\delta,i} = f_\delta(x_i) \). Then the discretization scheme of (2.1) has the following form

\[
\begin{cases} 
-\mathcal{L}_{\delta,h} u_{\delta,h}(x_i) = f_{\delta,i}, \quad i \in \mathcal{N}_{in}, \\
u_{\delta,h}(x_i) = g_i, \quad i \in \mathcal{N}_{out}.
\end{cases}
\]

For simplicity, we denote \( u_i \) as \( u_{\delta,h}(x_i) \) with the following sketch that characterizes different variables:

\[
\begin{align*}
\underbrace{x_{-r + \frac{1}{2}} \cdots x_0,} & \quad \underbrace{x_{\frac{1}{2}} \cdots x_r,} & \quad \underbrace{x_{r + \frac{1}{2}} \cdots x_{N-r + \frac{1}{2}},} & \quad \underbrace{x_{N-r} \cdots x_{N-r+r+\frac{1}{2}}}, \\
\text{boundary points} & \quad \text{interface points} & \quad \text{internal points} & \quad \text{interface points} & \quad \text{boundary points}
\end{align*}
\]

\[
\begin{align*}
\underbrace{g_{-r + \frac{1}{2}} \cdots g_0,} & \quad \underbrace{u_{\frac{1}{2}} \cdots u_r,} & \quad \underbrace{u_{r + \frac{1}{2}} \cdots u_{N-r + \frac{1}{2}},} & \quad \underbrace{u_{N-r} \cdots u_{N-r+r+\frac{1}{2}},} & \quad \underbrace{g_{N+1} \cdots g_{N+r+\frac{1}{2}}}, \\
\text{boundary values} & \quad \text{interface values} & \quad \text{internal values} & \quad \text{interface values} & \quad \text{boundary values}
\end{align*}
\]

Let

\[
U_{\delta,h} = \begin{bmatrix} u_1, u_2, \ldots, u_N, u_{\frac{1}{2}}, u_{\frac{1}{2}}, \ldots, u_{N+\frac{1}{2}} \end{bmatrix}^T;
\]

\[
F_{\delta,h} = \begin{bmatrix} f_{\delta,1}, f_{\delta,2}, \ldots, f_{\delta,N}, f_{\delta,\frac{1}{2}}, f_{\delta,\frac{1}{2}}, \ldots, f_{\delta,N+\frac{1}{2}} \end{bmatrix}^T.
\]

Let

\[
w_1 = (a_1, a_2, \ldots, a_r)^T, \quad w_2 = (a_\frac{1}{2}, a_2, \ldots, a_{r-\frac{1}{2}})^T, \quad w_3 = (a_2, a_2, \ldots, a_{r-\frac{1}{2}})^T;
\]
Similar to the discussion in [3], we introduce the following auxiliary vector

\[
\begin{align*}
G_1 &= \text{triu} \left( \text{toeplitz} \left( g_0, g_{-1}, \cdots, g_{-r+1} \right) \right), \\
G_2 &= \text{triu} \left( \text{toeplitz} \left( g_{-\frac{1}{2}}, g_{-\frac{3}{2}}, \cdots, g_{-r+\frac{1}{2}} \right) \right), \\
G_3 &= \text{triu} \left( \text{toeplitz} \left( g_{N+1}, g_{N+2}, \cdots, g_{N+r} \right) \right), \\
G_4 &= \text{triu} \left( \text{toeplitz} \left( g_{N+\frac{1}{2}}, g_{N+\frac{3}{2}}, \cdots, g_{N+r-\frac{1}{2}} \right) \right), \\
G_5 &= \text{triu} \left( \text{toeplitz} \left( g_{-\frac{1}{2}}, g_{-\frac{3}{2}}, \cdots, g_{-r+\frac{1}{2}} \right) \right), \\
G_6 &= \text{triu} \left( \text{toeplitz} \left( g_{N+\frac{1}{2}}, g_{N+\frac{3}{2}}, \cdots, g_{N+r+\frac{1}{2}} \right) \right).
\end{align*}
\]

Then the numerical scheme (2.12) can be recast as

\[
(2.13) \quad A_{\delta,h} \tilde{U}_{\delta,h} = F_{\delta,h} + F_{V,L} + F_{V,R},
\]

where the stiffness matrix $A_{\delta,h}$ consists the following four block-structured matrices with Toeplitz-like blocks,

\[
(2.14) \quad A_{\delta,h} = \begin{bmatrix} A & B \\ B^T & \tilde{A} \end{bmatrix}_{(2N+1) \times (2N+1)},
\]

and

\[
\begin{align*}
A_{N \times N} &= -\text{toeplitz}(a_0, a_1, \cdots, a_{r-1}, a_r, (0)_{(N-r-1) \times 1}); \\
\tilde{A}_{(N+1) \times (N+1)} &= -\text{toeplitz}(a_0, a_1, \cdots, a_{r-1}, a_r, (0)_{(N-r) \times 1}); \\
B_{N \times (N+1)} &= -\text{toeplitz} \left( \left[ w_2; (0)_{(N-r) \times 1} \right], \left[ a_\frac{1}{2}; w_2; (0)_{(N-r) \times 1} \right] \right),
\end{align*}
\]

where the coefficients $a_i$ are given in (2.7).

We notice that there are many different choices to prescribe $\gamma_\delta(z)$ for nonlocal problem (2.1). Here, we mainly focus on the weakly singular kernels, e.g.,

\[
(2.15) \quad \gamma_\delta(z) = \frac{3 - \alpha}{\delta^{3-\alpha}} z^{-\alpha} \quad \text{with} \quad \alpha \in (0, 1).
\]

More general kernel types such as the constant kernel, linear kernel, singular kernel, and commonly used kernel can be seen the literatures [5, 10, 16, 22]. Then, we can rewrite (2.2) as

\[
(2.16) \quad \mathcal{L}_\delta u(x) = \frac{3 - \alpha}{\delta^{3-\alpha}} \int_{0}^{\delta} \frac{u(x+z) - 2u(x) + u(x-z)}{z^\alpha} dz, \quad \alpha \in (0, 1).
\]
Moreover, from (2.18), (2.19) and (2.20), it yields

\[
(2.17) \quad a_0 = \eta_{\delta,\alpha}^h \left( 2 + 2\alpha \right) - \frac{6 - 2\alpha}{1 - \alpha} \delta^{-2}, \\
a_m = \eta_{\delta,\alpha}^h \left\{ 4 \left[ (m + 1)^{3-\alpha} - (m - 1)^{3-\alpha} \right] - (3 - \alpha) \left[ (m + 1)^{2-\alpha} + 6m^{2-\alpha} + (m - 1)^{2-\alpha} \right] \right\}, \quad m = 1, 2, \ldots, r - 1, \\
a_r = \eta_{\delta,\alpha}^h \left\{ 4 \left[ r^{3-\alpha} - (r - 1)^{3-\alpha} \right] - (3 - \alpha) \left[ (r - 1)^{2-\alpha} + 3r^{2-\alpha} - (2 - \alpha)r^{1-\alpha} \right] \right\}
\]

with \( \eta_{\delta,\alpha}^h = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)\delta^{\alpha}} \), and

\[
(2.18) \quad a_{\frac{1}{2}} = \eta_{\delta,\alpha}^h 4(1 - \alpha), \\
a_{m+\frac{1}{2}} = \eta_{\delta,\alpha}^h \left[ 8 \left( m^{3-\alpha} - (m + 1)^{3-\alpha} \right) + 4(3 - \alpha) \left( m^{2-\alpha} + (m + 1)^{2-\alpha} \right) \right]
\]

with \( m = 1, 2, \ldots, r - 1 \).

If \( \delta \leq h \), (2.20) and (2.21) become the following simply form

\[
(2.19) \quad a_0 = \frac{1}{(2 - \alpha)h^2(2\alpha - 10)}, \quad a_1 = \frac{1}{(2 - \alpha)h^2(1 - \alpha)}, \quad a_{\frac{1}{2}} = \frac{1}{(2 - \alpha)h^2} 4.
\]

Note that from (2.7) and (2.16), we have

\[
(2.20) \quad \mathcal{L}_\delta u(x_i) = \mathcal{L}_{\delta,h} u(x_i) + R_i,
\]

where the local truncation error will be rigorous proved in the next section.

3. Local truncation error for (2.20). We now analyze the local truncation error (LTE) of the piecewise quadratic polynomial collocation for (2.20) with the general horizon parameter.

3.1. Local truncation error for (2.20) with \( \delta \leq h \). We first consider the simply case \( \delta \leq h \).

**Lemma 3.1.** Let \( u(x) \in C^4(\Omega) \) with \( \delta \leq h \). Then for the quadrature rule (2.7), it holds that

\[
|\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| \leq C[h^3 \Theta_1(0, \tilde{\delta}) + h^2 \Theta_2(0, \tilde{\delta}) + \Theta_4(0, \tilde{\delta})]
\]

with \( \Theta_k(a, b) = \int_a^b z^k \gamma_\delta(z)dz \). In particular, there exists

\[
|\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| = \mathcal{O}(h^2).
\]

**Proof.** Define the error function \( e(x) = u(x) - u_Q(x) \). By Taylor series expansion at the point \( x_i \) and using (2.6), we get

\[
e(x_i + z) = [u'(x_i) - u'_Q(x_i)] z + \frac{u''(x_i) - u''_Q(x_i)}{2!} z^2 + \frac{u'''(x_i) - u'''_Q(x_i)}{3!} z^3 + \frac{u^{(4)}(\xi_1)}{4!} z^4,
\]
with \((x_i + z) \in [x_i, x_{i+1}]\) and \(\xi_1 \in [x_i, x_{i+1}]\). Here the first derivative and second derivative of \(u_Q(x)\) is computed by

\[
\begin{align*}
\dot{u}_Q(x_i) &= \frac{-3u(x_i) + 4u(x_{i+\frac{1}{2}}) - u(x_{i+1})}{h}, \\
\ddot{u}_Q(x_i) &= \frac{4u(x_i) - 8u(x_{i+\frac{1}{2}}) + 4u(x_{i+1})}{h^2}.
\end{align*}
\]

Similar, there exists

\[
e(x_i - z) = -\left[u'(x_i) - u'_Q(x_i)\right] z + \frac{u''(x_i) - u''_Q(x_i)}{2!} z^2 - \frac{u^{(3)}(x_i)}{3!} z^3 + \frac{u^{(4)}(\xi_2)}{4!} z^4
\]

with \((x_i - z) \in [x_{i-1}, x_i], \xi_2 \in [x_{i-1}, x_i]\) and

\[
\begin{align*}
\dot{u}_Q(x_i) &= \frac{u(x_{i-1}) - 4u(x_{i-\frac{1}{2}}) + 3u(x_i)}{h}, \\
\ddot{u}_Q(x_i) &= \frac{4u(x_{i-1}) - 8u(x_{i-\frac{1}{2}}) + 4u(x_i)}{h^2}.
\end{align*}
\]

From the above equations and using Taylor expansion of \(u'_Q(x)\) or \(u''_Q(x)\) at the point \(x_i\) again, one has

\[
e(x_i + z) + e(x_i - z) = \left(\frac{u^{(4)}(\xi_3) + u^{(4)}(\xi_4) - u^{(4)}(\xi_5) + u^{(4)}(\xi_6)}{4!} \right) h^3 z + \left(\frac{u^{(4)}(\xi_7) + u^{(4)}(\xi_8) - 2u^{(4)}(\xi_9) + u^{(4)}(\xi_{10})}{4!} \right) h^2 z^2 + \frac{u^{(4)}(\xi_1) + u^{(4)}(\xi_2)}{4!} z^4
\]

with \(\xi_2, \xi_3, \xi_9 \in [x_{i-1}, x_i], \xi_1, \xi_4, \xi_{10} \in [x_i, x_{i+1}]\) and \(\xi_5, \xi_8 \in [x_{i-1/2}, x_i], \xi_6, \xi_7 \in [x_i, x_{i+1/2}]\).

According to the above equations and subtracting (2.7) from (2.2), we have the following local truncation error

\[
e_1 := \mathcal{L}_\tilde{\beta} u(x_i) - \mathcal{L}_{\tilde{\beta}, h} u(x_i) = \int_0^\delta \left[e(x_i + z) + e(x_i - z)\right] \gamma_\tilde{\beta}(z) dz = \epsilon_{11} + \epsilon_{12} + \epsilon_{13}
\]

with

\[
\begin{align*}
\epsilon_{11} &= \int_0^\delta \gamma_\tilde{\beta}(z) \left(\frac{u^{(4)}(\xi_3) + u^{(4)}(\xi_4) - u^{(4)}(\xi_5) + u^{(4)}(\xi_6)}{4!} \right) h^3 z dz; \\
\epsilon_{12} &= \int_0^\delta \gamma_\tilde{\beta}(z) \left(\frac{u^{(4)}(\xi_7) + u^{(4)}(\xi_8) - 2u^{(4)}(\xi_9) + u^{(4)}(\xi_{10})}{4!} \right) h^2 z^2 dz; \\
\epsilon_{13} &= \int_0^\delta \gamma_\tilde{\beta}(z) \frac{u^{(4)}(\xi_1) + u^{(4)}(\xi_2)}{4!} z^4 dz.
\end{align*}
\]

Thus, we get

\[
|e_1| \leq C[h^3 \Theta_1(0, \tilde{\delta}) + h^2 \Theta_2(0, \tilde{\delta}) + \Theta_4(0, \tilde{\delta})].
\]
In particular, using the above inequality and (2.19), it yields

\[ |\epsilon_1| \leq C \left( \frac{h^3}{\delta} + h^2 + \frac{3 - \alpha^2}{5 - \alpha^2} \right) = O(h^2). \]

The proof is completed. \(\square\)

**Lemma 3.2.** Let \(u(x) \in C^4(\Omega)\) with \(\delta \leq h\). Then for the quadrature rule (2.7), it holds that

\[ |\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| = O(h^2). \]

**Proof.** From (2.6) and (3.1) with \(\delta \leq h\), we obtain

\[ \mathcal{L}_\delta u(x) = C_\delta u''(x) + \mathcal{O} \left( \int_0^\delta z^4 \gamma(z) dz \right) = u''(x) + \mathcal{O}(\delta^2) \leq u''(x) + \mathcal{O}(h^2), \]

and

\[ \mathcal{L}_{\delta,h} u(x) = C\bar{\delta} u''(x) + \mathcal{O} \left( \int_0^\bar{\delta} z^4 \gamma(z) dz \right) = u''(x) + \mathcal{O}(h^2). \]

According to the above equations and Lemma 3.1, it implies that

\[ |\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| \leq |\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\bar{\delta},h} u(x_i)| + |\mathcal{L}_{\bar{\delta},h} u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| = O(h^2). \]

The proof is completed. \(\square\)

**3.2. Local truncation error for (2.20) with \(\delta = \bar{\delta} = rh \geq h\).** Let \(u_C(x)\) be the piecewise Lagrange cubic interpolant of \(u(x)\) in \([x_{i-1}, x_i]\), i.e.,

\[
\begin{align*}
  u_C(x) &= -\frac{8(x-x_{i-3/4})(x-x_{i-1/2})(x-x_i)}{h^3} u(x_{i-1}) \\
  &\quad + \frac{64(x-x_{i-1})(x-x_{i-1/2})(x-x_i)}{3h^3} u(x_{i-3/4}) \\
  &\quad - \frac{16(x-x_{i-1})(x-x_{i-3/4})(x-x_i)}{h^3} u(x_{i-1/2}) \\
  &\quad + \frac{8(x-x_i)(x-x_{i-3/4})(x-x_{i-1/2})}{3h^3} u(x_i).
\end{align*}
\]

(3.1)

According to (2.6), (3.1) and Taylor series expansion \cite{1}, p.158, there exists \(\xi_{i-1} \in (x_{i-1}, x_i)\) such that

\[ u_C(x) - u_Q(x) = u_C^{(3)}(\xi_{i-1}) (x-x_{i-1}) (x-x_{i-1/2}) (x-x_i), \quad x \in [x_{i-1}, x_i] \]

with

\[
\frac{u_C^{(3)}(\xi_{i-1})}{3!} = \frac{8}{3h^3} \left[ -3u(x_{i-1}) + 8u(x_{i-3/4}) - 6u(x_{i-1/2}) + u(x_i) \right]
\]

\[ = \frac{u^{(3)}(t_i) + 3u^{(3)}(\xi_i) - u^{(3)}(\zeta_i)}{18}, \quad t_i, \xi_i, \zeta_i \in (x_{i-1}, x_i), \]

\]
where \( \frac{u^{(3)}(\xi_{m-1})}{3!} \) is a constant, since \( u_C(y) - u_Q(y) \) is a cubic polynomial.

**Lemma 3.3.** Let \( u(y) \in C^4(\Omega) \) with \( \delta = rh \) and \( 0 < \alpha < 1 \). Let \( u_C(y) \) and \( u_Q(y) \) be defined by \( 260 \) and \( 321 \), respectively. Then

\[
Q_{j+1} := \int_{x_\frac{i}{2} - 1}^{x_\frac{i}{2}} \frac{u_C(y) - u_Q(y)}{|x_\frac{i}{2} - y|} \, dy = O(h^{5-\alpha})
\]

for \( i \) a positive integer number.

**Proof.** Based on the idea of Lemma 3.2 in [4], the desired results are obtained. \( \square \)

**Lemma 3.4.** Let \( u(y) \in C^4(\Omega) \) with \( \delta = rh \) and \( 0 < \alpha < 1 \), \( u_Q(y) \) and \( u_C(y) \) be defined by \( 260 \) and \( 321 \), respectively. Then

\[
Q_i := \int_{x_\frac{i}{2} - r}^{x_\frac{i}{2} - 1} \frac{u_C(y) - u_Q(y)}{|x_\frac{i}{2} - y|} \, dy
\]

\[
= -h^{4-\alpha} \cdot \frac{u^{(3)}(x)}{6} \sum_{m=1}^{r-1} \int_0^1 \frac{(t - \frac{1}{2}) (t - 1)}{m - \frac{1}{2} + t} \, dt + O(h^4)\delta^{1-\alpha} + O(h^{5-\alpha}) + J_4,
\]

where \( i \) is a positive integer number and

\[
J_4 = \frac{u^{(3)}(\xi_{i+\frac{1}{2} - r})}{6} \int_{x_\frac{i}{2} - r}^{x_\frac{i}{2} - 1} \frac{(y - x_{i+\frac{1}{2} - r}) (y - x_{i+\frac{1}{2} - r}) (y - x_{i+1 - r})}{(x_\frac{i}{2} - y)^{\alpha}} \, dy.
\]

**Proof.** From \( 324 \), there exists \( \xi_m \in (x_{m-1}, x_m) \) such that

\[
u_C(y) - u_Q(y) = \frac{u^{(3)}(\xi_m)}{3!} \left( y - x_{m-1} \right) \left( y - x_{\frac{m}{2}} \right) \left( y - x_m \right), \ y \in [x_{m-1}, x_m].
\]

For the sake of simplicity, we take \( w(\xi_m) = \frac{u^{(3)}(\xi_m)}{3!} \) and

\[
w(\xi_m) = [w(\xi_m) - \bar{w}(x_m)] + \bar{w}(x_\frac{i}{2}) + \left[ \bar{w}(x_m) - \bar{w}(x_\frac{i}{2}) \right] \text{ with } \bar{w}(x) = \frac{u^{(3)}(x)}{6}.
\]

Then

\[
Q_i = \sum_{m=\lceil \frac{i}{2} \rceil - 1}^{\lceil \frac{i}{2} \rceil - r + 1} w(\xi_m) \int_{x_{m-1}}^{x_m} \frac{(y - x_{m-1}) (y - x_{\frac{m}{2}}) (y - x_m)}{(x_\frac{i}{2} - y)^{\alpha}} \, dy
\]

\[
+ w(\xi_{i+\frac{1}{2} - r}) \int_{x_\frac{i}{2} - r}^{x_{i+1 - r}} \frac{(y - x_{i+\frac{1}{2} - r}) (y - x_{\frac{i}{2} - r}) (y - x_{i+1 - r})}{(x_\frac{i}{2} - y)^{\alpha}} \, dy
\]

\[
:= J_1 + J_2 + J_3 + J_4
\]
with
\[ J_1 = \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \left[ w(\xi_m) - \bar{w}(x_m) \right] \int_{x_{m-1}}^{x_m} \frac{(y-x_{m-1})\left(y-x_{m-\frac{1}{r}}\right)(y-x_m)}{(x_{\frac{1}{r}}-y)^\alpha} \, dy; \]
\[ J_2 = \bar{w}(x_{\frac{1}{r}}) \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_{x_{m-1}}^{x_m} \frac{(y-x_{m-1})\left(y-x_{m-\frac{1}{r}}\right)(y-x_m)}{(x_{\frac{1}{r}}-y)^\alpha} \, dy; \]
\[ J_3 = \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \left[ \bar{w}(x_m) - \bar{w}(x_{\frac{1}{r}}) \right] \int_{x_{m-1}}^{x_m} \frac{(y-x_{m-1})\left(y-x_{\frac{1}{r}}\right)(y-x_m)}{(x_{\frac{1}{r}}-y)^\alpha} \, dy; \]
\[ J_4 = \bar{w}(\xi_{\frac{1}{r}+r}) \int_{x_{\frac{1}{r}}}^{x_{\frac{1}{r}+r}} \frac{(y-x_{\frac{1}{r}})(y-x_{\frac{1}{r}+r})(y-x_m)}{(x_{\frac{1}{r}}-y)^\alpha} \, dy. \]

Using integration by parts and \( \int_0^1 \tau \left( \tau - \frac{1}{2} \right) (\tau - 1) \, d\tau = 0 \), it yields
\[ \int_{x_{m-1}}^{x_m} \frac{(y-x_{m-1})\left(y-x_{m-\frac{1}{r}}\right)(y-x_m)}{(x_{\frac{1}{r}}-y)^\alpha} \, dy = \alpha h^{4-\alpha} \int_0^1 \frac{t \left( t - \frac{1}{2} \right) (t - 1)}{(\frac{1}{2} - m + 1 - t)^{1+\alpha}} \, dt \]
\[ = -\alpha h^{4-\alpha} \int_0^1 \frac{t \left( t - \frac{1}{2} \right) (t - 1)}{(\frac{1}{2} - m + t)^{1+\alpha}} \, dt = -\alpha h^{4-\alpha} \int_0^1 \frac{t \left( \tau - \frac{1}{2} \right) (\tau - 1) \, d\tau}{(\frac{1}{2} - m + t)^{1+\alpha}}. \]

Moreover, we have
\[ \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \left| \int_0^t \frac{\tau \left( \tau - \frac{1}{2} \right) (\tau - 1) \, d\tau}{(\frac{1}{2} - m + t)^{1+\alpha}} \right| \, dt \leq \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \frac{1}{(\frac{1}{2} - m + t)^{1+\alpha}} \, dt \]
\[ = \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \frac{1}{(m - \frac{1}{2} + t)^{1+\alpha}} \, dt = \frac{1}{\alpha} \left( 2^\alpha - \left( \frac{r-\frac{1}{2}}{2} \right)^{-\alpha} \right) \quad \text{if odd,} \]
and
\[ \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \left| \int_0^t \frac{\tau \left( \tau - \frac{1}{2} \right) (\tau - 1) \, d\tau}{(\frac{1}{2} - m + t)^{1+\alpha}} \right| \, dt \leq \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \frac{1}{(\frac{1}{2} - m + t)^{1+\alpha}} \, dt \]
\[ = \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \frac{1}{(m + t)^{1+\alpha}} \, dt = \frac{1}{\alpha} \left( 1 - \frac{1}{r^\alpha} \right) \quad \text{if even.} \]

From (3.3), (3.4), it leads to
\[ |J_1| \leq \frac{4}{18} h^{5-\alpha} 2^\alpha = O(h^{5-\alpha}) \quad \text{with} \quad L = \max_{x_m \in [1]} |w'(x_m)|, \]
and
\[ J_2 = \begin{cases} -h^{4-\alpha} \cdot \bar{w}(x_{\frac{1}{r}}) \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \frac{t \left( t - \frac{1}{2} \right) (t - 1)}{(m - \frac{1}{2} + t)^{\alpha}} \, dt & \text{if odd,} \\
-h^{4-\alpha} \cdot \bar{w}(x_{\frac{1}{r}}) \sum_{m=\left\lfloor \frac{i}{r} \right\rfloor - r+1}^{\left\lfloor \frac{i}{r} \right\rfloor - 1} \int_0^1 \frac{t \left( t - \frac{1}{2} \right) (t - 1)}{(m + t)^{\alpha}} \, dt & \text{if even.} \end{cases} \]
Next we estimate the error term $J_3$. From (2.6)–(2.8), we have

$$J_3 \leq \begin{cases} Lah^{\alpha} \sum_{m=1}^{r-1} (m - \frac{1}{2}) \int_0^1 \frac{1}{(m - \frac{1}{2} + t)^{1+\alpha}} dt \leq Lah^{\alpha} \sum_{m=1}^{r-1} (m - \frac{1}{2})^{-\alpha}, \\
Lah^{\alpha} \sum_{m=1}^{r-1} m \int_0^1 \frac{1}{(m + t)^{1+\alpha}} dt \leq Lah^{\alpha} \sum_{m=1}^{r-1} m^{-\alpha}. \end{cases}$$

We can check

$$\sum_{m=1}^{r-1} m^{-\alpha} = \sum_{m=1}^{r-1} \int_{m-1}^m \frac{1}{x^\alpha} dx \leq \sum_{m=1}^{r-1} \int_{m-1}^m \frac{1}{x^\alpha} dx = \int_0^{r-1} \frac{1}{x^\alpha} dx = \frac{1}{1-\alpha} (r-1)^{1-\alpha} \leq \frac{h^{\alpha-1}}{1-\alpha} \delta^{1-\alpha}, \quad \text{i is even.}$$

On the other hand, if i is odd, we have

$$\sum_{m=1}^{r-1} (m - \frac{1}{2})^{-\alpha} = \left(\frac{1}{2}\right)^{-\alpha} + \sum_{m=2}^{r-1} \int_{m-1}^m \frac{1}{(m - \frac{1}{2})^\alpha} dx \leq 2^\alpha \int_1^{r-1} \frac{1}{(x - \frac{1}{2})^\alpha} dx$$

It implies that

$$|J_3| \leq L^{\alpha}_\frac{h^4}{1-\alpha} \delta^{1-\alpha} = \mathcal{O}(h^4) \delta^{1-\alpha}.$$ 

The proof is completed. 

**Lemma 3.5.** Let $u(y) \in C^4(\Omega)$ with $\delta = r\gamma$ and $0 < \alpha < 1$, $u_Q(y)$ and $u_C(y)$ be defined by (2.20) and (3.21), respectively. Then

$$Q_r := \int_{x_{j+1}^a}^{x_{j+1}^b} \frac{u_C(y) - u_Q(y)}{|x_{j+1}^a - y|^\alpha} dy$$

$$= h^{4-\alpha} \cdot \frac{u^{(3)}(x_{j+1}^a)}{6} \sum_{m=1}^{r-1} \int_0^1 \frac{t (t - \frac{1}{2}) (t - 1)}{(m - \frac{1}{2} + t)^\alpha} dt + \mathcal{O}(h^4) \delta^{1-\alpha} + \mathcal{O}(h^{5-\alpha}) + \tilde{J}_4,$$

where i is a positive integer number and

$$\tilde{J}_4 = \frac{u^{(3)}_C(x_{j+1}^a + r)}{6} \int_{x_{j+1}^a}^{x_{j+1}^b} \frac{(y - x_{j+1}^a - r) (y - x_{j+1}^a + r)}{|y - x_{j+1}^a|^\alpha} dy.$$ 

**Proof.** The similar arguments can be performed as Lemma 3.4, we omit it here.

**Lemma 3.6.** Let $J_4$ and $\tilde{J}_4$ be defined in Lemmas 3.3 and 3.5, respectively. Then

$$|J_4 + \tilde{J}_4| = \mathcal{O}(h^4) \delta^{1-\alpha}.$$
Proof. From Lemmas 3.4 and 3.5 we get
\[ J_4 + \tilde{J}_4 = \frac{u_C^{(3)}}{6} \left[ \int_{x_{\tilde{r}} - r}^{x_{\tilde{r} + r}} \frac{y - x_{\tilde{r} - r}}{(x_{\tilde{r}} - y)^\alpha} \left( y - x_{\tilde{r} + r} \right) dy \right] \]
\[ + \frac{u_C^{(3)}}{6} \left[ \int_{x_{\tilde{r}} - r}^{x_{\tilde{r} + r}} \frac{y - x_{\tilde{r} - r}}{(x_{\tilde{r}} - y)^\alpha} \left( y - x_{\tilde{r} + r} \right) dy \right]. \]

If \( i \) is even, it leads to \( J_4 + \tilde{J}_4 = 0 \). If \( i \) is odd, since
\[ \int_{x_{\tilde{r}} - r}^{x_{\tilde{r} + r}} \frac{y - x_{\tilde{r} - r}}{(x_{\tilde{r}} - y)^\alpha} \left( y - x_{\tilde{r} + r} \right) dy = h^{4-\alpha} \int_0^\frac{1}{2} \frac{(t + \frac{1}{2}) t (t - \frac{1}{2})}{(r - t)^\alpha} dt \]
\[ \int_{x_{\tilde{r}} - r}^{x_{\tilde{r} + r}} \frac{y - x_{\tilde{r} - r}}{(x_{\tilde{r}} - y)^\alpha} \left( y - x_{\tilde{r} + r} \right) dy = -h^{4-\alpha} \int_0^\frac{1}{2} \frac{(t + \frac{1}{2}) t (t - \frac{1}{2})}{(r - t)^\alpha} dt. \]

Then we have
\[ \left| J_4 + \tilde{J}_4 \right| \leq \left| \frac{u_C^{(3)}}{6} \left( \frac{x_{\tilde{r} + r}}{x_{\tilde{r} - r}} \right) - \frac{u_C^{(3)}}{6} \left( \frac{x_{\tilde{r} + r}}{x_{\tilde{r} - r}} \right) \right| h^{4-\alpha} \int_0^\frac{1}{2} \frac{(t + \frac{1}{2}) t (t - \frac{1}{2})}{(r - t)^\alpha} dt \]
\[ \leq L\delta h^{4-\alpha} \int_0^\frac{1}{2} \frac{1}{(r - t)^\alpha} dt \leq L2\delta h^{4-\alpha} \frac{1}{2} \frac{1}{(r - \frac{1}{2})^\alpha} \leq L\delta h^{4-\alpha} \frac{2^\alpha}{(2r - 1)^\alpha} \]
\[ \leq L\delta h^{4-\alpha} 2^\alpha r^{-\alpha} = L\delta h^{4-\alpha} 2\delta^{-\alpha} h^\alpha = L2^\alpha h^\delta = \mathcal{O}(h^\delta). \]

The proof is completed. □

**Lemma 3.7.** Let \( u(y) \in C^4(\Omega) \) with \( \delta = rh \) and \( 0 < \alpha < 1 \). Let \( u_Q(y) \) and \( u_C(y) \) be defined by (246) and (331), respectively. Then
\[ \int_{x_{\tilde{r}} - \delta}^{x_{\tilde{r}} + \delta} \frac{u_C(y) - u_Q(y)}{|x_{\tilde{r}} - y|^\alpha} dy = \mathcal{O}(h^\delta) \delta^{-\alpha} + \mathcal{O}(h^{5-\alpha}). \]

Proof. From \( \delta = rh \), we have
\[ \int_{x_{\tilde{r}} - \delta}^{x_{\tilde{r}} + \delta} \frac{u_C(y) - u_Q(y)}{|x_{\tilde{r}} - y|^\alpha} dy = \int_{x_{\tilde{r}} - r}^{x_{\tilde{r}} + r} \frac{u_C(y) - u_Q(y)}{|x_{\tilde{r}} - y|^\alpha} dy = Q_1 + Q_2 + Q_r \]
with
\[ Q_1 := \int_{x_{\tilde{r}} - r}^{x_{\tilde{r}} + r} \frac{u_C(y) - u_Q(y)}{|x_{\tilde{r}} - y|^\alpha} dy, \]
\[ Q_2 := \int_{x_{\tilde{r}} - \delta}^{x_{\tilde{r}} + \delta} \frac{u_C(y) - u_Q(y)}{|x_{\tilde{r}} - y|^\alpha} dy, \]
\[ Q_r := \int_{x_{\tilde{r}} + \delta}^{x_{\tilde{r}} + r} \frac{u_C(y) - u_Q(y)}{|x_{\tilde{r}} - y|^\alpha} dy. \]
According to Lemmas 3.8 and 3.9, we obtain the remainder
\[ R_i = \int_{x_i \pm \delta}^{x_i} \frac{u(y) - u_Q(y)}{x_i - y} dy = O(h^4) \delta^{1-\alpha} + O(h^{5-\alpha}). \]

The proof is completed. \( \square \)

**Lemma 3.8.** Let \( u(y) \in C^4[a, b] \) with \( \delta = rh \) and \( u_C(y) \) be defined by (3.1). Then
\[ \int_{x_i \pm \delta}^{x_i} \frac{u(y) - u_C(y)}{x_i - y} dy = O(h^4)\delta^{1-\alpha}, \ 0 < \alpha < 1. \]

**Proof.** According to (3.1) and Taylor series expansion [1, p. 158], we have
\[ u(y) - u_C(y) = \frac{u^{(4)}(\xi)}{4!} (y - x_{j-1}) \left(y - x_{j-3/4}\right)(y - x_j), \ y \in [x_{j-1}, x_j], \]
for some \( \xi \in (x_{j-1}, x_j) \) depending on \( y \). It yields
\[ \left| \int_{x_i \pm \delta}^{x_i} \frac{u(y) - u_C(y)}{x_i - y} dy \right| \leq \frac{h^4}{24} \max_{\xi \in [x_i]} \left| u^{(4)}(\xi) \right| \int_{x_i \pm \delta}^{x_i} \frac{1}{(x_i - y)^\alpha} dy = O(h^4)\delta^{1-\alpha}. \]

The proof is completed. \( \square \)

**Lemma 3.9.** Let \( u(y) \in C^4(\Omega) \) with \( \tilde{\delta} = rh \) and \( u_Q(y) \) be defined by (2.10). Then
\[ |L_{\tilde{\delta}}u(x_i) - L_{\tilde{\delta}, h}u(x_i)| = \frac{3 - \alpha}{\tilde{\delta}^{3-\alpha}} \int_{x_i \pm \tilde{\delta}}^{x_i} \frac{u(y) - u_Q(y)}{x_i - y} dy = O(h^4)\tilde{\delta}^{-2}, \ 0 < \alpha < 1. \]

**Proof.** According to Lemmas 3.7-3.8 and the triangle inequality, the desired result is obtained. \( \square \)

**Theorem 3.10.** Let \( u(y) \in C^4(\Omega) \) with \( \delta = O(h^\beta), \ \beta \geq 0 \) and \( u_Q(y) \) be defined by (2.10). If \( \delta \) is the grid point, then
\[ |L_\delta u(x_i) - L_{\tilde{\delta}, h}u(x_i)| = O\left(h^{\max(2.4-2\beta)}\right), \ \beta \geq 0. \]

**Proof.** From Lemmas 3.8 and 3.9, the desired results are obtained. \( \square \)

3.3. **Local truncation error for (2.20) with** \( \delta = \tilde{\delta} \pm \delta_0 h, \ 0 < \delta_0 < 1. \) Since the whole key point of AC schemes is that convergent numerical methods which do not require any relation between mesh size/grid points and \( \delta [7]. \) Moreover, the horizon of the material is a physical property of the material of the finite bar and should be independent of the computational mesh size \( h [20]. \) Based on this observation, we shall consider the nonlocal model with the general horizon parameter in (2.4).

**Theorem 3.11.** Let \( u(y) \in C^4(\Omega) \) with \( \delta = O(h^\beta), \ \beta \geq 0 \) and \( u_Q(y) \) be defined by (2.10). If \( \delta \) is not the grid point, then
\[ |L_\delta u(x_i) - L_{\tilde{\delta}, h}u(x_i)| = O\left(h^{\min(2.1+\beta)}\right), \ \beta \geq 0. \]
Proof. We can rewrite the general horizon parameter $\delta$ in (2.15) as the general form $\delta = ch^\beta$, $\beta \geq 0$, $c > 0$. We prove the desired results via the following three cases.

Case 1: $\beta > 1$. It corresponds to $\delta = \tilde{\delta} - \delta_0 h \leq h$ in (2.5). From Lemma 3.2, we have

$$|\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| = \mathcal{O}(h^2).$$

We next consider the case $\beta \in [0,1]$. Since $\delta = ch^\beta$, $0 \leq \beta \leq 1$, $c > 0$, it implies that $\delta = \delta - \delta_0 h = \mathcal{O}(h^\beta)$. From Lemma 3.2, it yields

(3.6) $$|\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| = \mathcal{O}(h^{1-\beta}).$$

On the other hand, from (2.2), we have

(3.7) $$\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i) = e_1 + e_2$$

with

$$e_1 = \int_0^{\tilde{\delta}} [\gamma_3(z) - \gamma_3(z)] [u(x_i + z) - 2u(x_i) + u(x_i - z)] dz,$$

and

$$e_2 = \int_0^{\tilde{\delta}} \gamma_3(z) [u(x_i + z) - 2u(x_i) + u(x_i - z)] dz.$$

Case 2: $\beta \in (0,1]$. According to (3.7), (2.15), Taylor series expansion and integral mean-value theorem, there exist

$$e_1 = \frac{\tilde{\delta}^{3-\alpha} - \delta^{3-\alpha}}{5 - \alpha} u''(x_i) + \frac{3 - \alpha}{24} u^{(4)}(\xi_1) + \frac{3 - \alpha}{24} u^{(4)}(\xi_2) \tilde{\delta}^{3-\alpha} - \delta^{3-\alpha} \tilde{\delta}^2$$

and

$$e_2 = -\frac{\tilde{\delta}^{3-\alpha} - \delta^{3-\alpha}}{5 - \alpha} u''(x_i) + \frac{3 - \alpha}{24} u^{(4)}(\xi_3) + \frac{3 - \alpha}{24} u^{(4)}(\xi_4) \delta^{5-\alpha} - \tilde{\delta}^{5-\alpha} \delta^{3-\alpha}$$

Thus, using (3.6), (3.7) and triangle inequality, we have

$$|\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| \leq |\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| + |\mathcal{L}_{\delta,h} u(x_i) - \mathcal{L}_{\delta,h} u(x_i)| = \mathcal{O}(h^{1+\beta})$, $\beta \in (0,1]$.\)

Case 3: $\beta = 0$. For this case, we just require $u(y) \in C^3(\Omega)$. From (3.7), (2.15) and mean value theorem with $\xi_5 \in (x_i, x_i + z)$ and $\xi_6 \in (x_i - z, x_i)$, one has

$$e_1 = (3 - \alpha) \left( \frac{1}{\tilde{\delta}^{3-\alpha}} - \frac{1}{\delta^{3-\alpha}} \right) \int_0^{\tilde{\delta}} [u'(\xi_5) - u'(\xi_6)] z^{1-\alpha} dz = \mathcal{O}(h).$$
since
\[ \frac{\dot{\delta}^{3-\alpha} - \delta^{3-\alpha}}{\delta^{3-\alpha}} = 1 - \left(1 + \frac{\delta h}{\delta}\right)^{3-\alpha} = O(h). \]

Moreover, there exist \( \xi_7 \in (x_i, x_i + z) \) and \( \xi_8 \in (x_i - z, x_i) \) such that
\[ e_2 = 3 - \frac{\alpha}{\delta^{3-\alpha}} \int_\delta^\delta |u'(\xi_7) - u'(\xi_8)| z^{1-\alpha} \, dz \leq 3 - \frac{\alpha}{\delta^{3-\alpha}} 2 \max_{\xi \in \Omega} |u'(\xi)| \delta^{1-\alpha} \int_\delta^\delta 1 \, dz = O(h). \]
Thus, using (3.6), (3.7) and triangle inequality, we have
\[ |L_\delta u(x_i) - L_{\delta,h} u(x_i)| \leq |L_\delta u(x_i) - L_{\delta,h} u(x_i)| + |L_{\delta,h} u(x_i) - L_{\delta,h} u(x_i)| = O(h), \delta = c. \]
The proof is completed. \( \Box \)

4. Stability and convergence analysis. In this section, the detailed proof of the global error for the nonlocal models (2.1) with a general horizon parameter are provided. We first introduce some lemmas, which will be used latter.

Lemma 4.1. Let matrix \( A^\delta_{\delta,h} \) be defined by (2.14). Then \( A^\delta_{\delta,h} \) is a diagonally dominant symmetric matrix with positive entries on the diagonal and nonpositive off-diagonal entries.

Proof. From (2.8) and (2.15), we have
\[ a_0 = 2h^{-2} \int_0^h z(2z - 3h) \gamma_\delta(z) \, dz - 2 \int_0^h \gamma_\delta(z) \, dz < 0, \]
and
\[ \int_{j_m} \gamma_\delta(z) \phi_m(z) \, dz = \eta^h_{\delta,\alpha} a_m^-, \; m = 1, 2, \ldots, r - 1 \]
with
\[ a_m^- = 4 \left[ m^{3-\alpha} - (m - 1)^{3-\alpha} \right] - (3 - \alpha) \left[ (m - 1)^{2-\alpha} + 3m^{2-\alpha} - (2 - \alpha)m^{1-\alpha} \right]. \]
Using Taylor expansion, there exists
\[ (1 + z)^a = 1 + az + \frac{a(a - 1)}{2!} z^2 + \frac{a(a - 1)(a - 2)}{3!} z^3 + \ldots \]
\[ = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{a + 1 - k}{n!} z^n, \; |z| \leq 1, a > 0. \]
Therefore, we get
\[ a_m^- = 4m^{3-\alpha} \left[ 1 - \left(1 - \frac{1}{m}\right)^{3-\alpha} \right] - (3 - \alpha)m^{2-\alpha} \left[ \left(1 - \frac{1}{m}\right)^{2-\alpha} + 3 \right] \]
\[ + (3 - \alpha)(2 - \alpha)m^{1-\alpha} \]
\[ = 4m^{3-\alpha} \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{(4 - \alpha - k)(-1)^{n+1}}{n!} \frac{1}{m^n} \]
\[ - (3 - \alpha)m^{2-\alpha} \left( 4 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{(3 - \alpha - k)(-1)^{n+1}}{n!} \frac{1}{m^n} \right) + (3 - \alpha)(2 - \alpha)m^{1-\alpha}. \]
Moreover, a calculation shows that

\[
a_m = \frac{(3 - \alpha)(2 - \alpha)(1 - \alpha)}{m^{\alpha}} \left( \frac{1}{6} - \sum_{n=2}^{\infty} \prod_{k=1}^{n} \frac{(\alpha + k - 1)(n + 7)}{(n + 3)!} \frac{1}{m^n} \right) > 0,
\]

since

\[
\sum_{n=2}^{\infty} \prod_{k=1}^{n} \frac{(\alpha + k - 1)(n + 7)}{(n + 3)!} \frac{1}{m^n} \leq \sum_{n=2}^{\infty} \frac{n!(n + 7)}{(n + 3)!m^n} = \sum_{n=2}^{\infty} \frac{1}{(n + 2)(n + 1)m^n} + \sum_{n=2}^{\infty} \frac{4}{(n + 3)(n + 2)(n + 1)m^n} 
\leq 2 \sum_{n=2}^{\infty} \frac{1}{(n + 2)(n + 1)^2} \leq \frac{1}{4 \times 3 \times 2} + \frac{1}{4} \sum_{n=3}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + 2} \right) < \frac{1}{6}.
\]

Similar, we can prove

\[
\int_{I_{m+1}} \gamma(z) \phi_m(z) \, dz = \eta_{h, \alpha}^h a^+_m > 0, \quad m = 1, 2, \ldots, r - 1
\]

with

\[
a_m = 4 \left[ (m + 1)^{3-\alpha} - m^{3-\alpha} \right] - (3 - \alpha) \left[ (m + 1)^{2-\alpha} + 3m^{2-\alpha} + (2 - \alpha)m^{1-\alpha} \right].
\]

It yields \(a_m = \eta_{h, \alpha}^h (a^+_m + a^-_m) > 0, m = 1, 2, \ldots, r - 1\) and \(a_r = \eta_{h, \alpha}^h a^-_r > 0\). Moreover, from (2.11), (2.15) and the definition of the piecewise quadratic basis function \(\phi_{j+\frac{1}{2}}(x)\) in [1, p. 37], it yields \(a^{m+\frac{1}{2}}_m > 0\).

Then \(A_{h,h}^z\) is a symmetric matrix with positive entries on the diagonal and nonpositive off-diagonal entries; and \(A_{h,\alpha}^z\) is diagonally dominant symmetric matrix, since taking \(u(x) \equiv 1\) in (2.7), there exists

\[
a_0 + 2 \sum_{m=1}^{r} a_m + 2 \sum_{m=0}^{r-1} a_{m+\frac{1}{2}} = 0.
\]

The proof is completed. \(\square\)

Note that Lemma 4.1 does not guarantee that the matrix \(A_{h,h}^z\) is neither nonsingular nor positive definite, since \(A_{h,h}^z\) is reducible. Hence, we need to prove it positive definite in further.

**Lemma 4.2.** Let matrix \(A_{h,h}^z\) be defined by (2.11). Then \(A_{h,h}^z\) is a symmetric positive definite matrix with positive entries on the diagonal and nonpositive off-diagonal entries. Moreover, \(A_{h,h}^z\) is a positive matrix.

**Proof.** Let \(L_N = \text{tridiag}(-1, 2, -1)\) be the \(N \times N\) one dimensional discrete Laplacian. Let

\[
A_{\text{res}} = A_{h,h}^z - a_1 A_{\text{main}} \quad \text{with} \quad A_{\text{main}} = \begin{bmatrix} L_N & O \\ O & L_{N+1} \end{bmatrix}.
\]

From (4.1) and (4.2), it shows that \(A_{\text{res}}\) is a diagonally dominant symmetric matrix with positive entries on the diagonal and nonpositive off-diagonal entries. Thus, \(A_{\text{res}}\) is a semi-positive matrix by Gerschgorin circle theorem [19, p. 21].
On the other hand, the matrix $A_{\text{main}}$ is a positive matrix, since the determinant $\det A = (N + 1)(N + 2) > 0$. The proof is completed.

From Lemma 4.2, we have the following discrete maximum principle.

**Proposition 4.3.** The collocation schemes (2.12) satisfy the discrete maximum principle:

\[ f(x) \leq 0, \text{ for } x \in \Omega \Rightarrow \max_{i \in \mathcal{N}_{\text{in}}} u_i \leq \max_{i \in \mathcal{N}_{\text{out}}} u_i; \]

\[ f(x) \geq 0, \text{ for } x \in \Omega \Rightarrow \min_{i \in \mathcal{N}_{\text{in}}} u_i \geq \min_{i \in \mathcal{N}_{\text{out}}} u_i. \]

**Lemma 4.4.** Let the discrete operator $\mathcal{L}_{\delta,h}$ be defined by (2.7). Then

\[
\|(\mathcal{L}_{\delta,h})^{-1}\|_{\infty} \leq \frac{1 + 4\delta(1 + \delta)}{8C_{\delta}} \text{ with } C_{\delta} > 0 \text{ in (2.3)}.
\]

**Proof.** Let

\[ v_\delta(x) = \frac{x(1-x)+\delta(1+\delta)}{2C_{\delta}}. \]

From (2.10), we have

\[-\mathcal{L}_{\delta} v_\delta(x) = -\int_{0}^{x} \gamma_\delta(z) [v_\delta(x+z) - 2v_\delta(x) + v_\delta(x-z)] \, dz = \frac{1}{C_{\delta}} \int_{0}^{x} z^2 \gamma_\delta(z) \, dz = 1. \]

Let

\[ v_{\delta,h}(x) = \left[v_\delta(x_1), v_\delta(x_2), \ldots, v_\delta(x_N), v_\delta(x_{1+2}), v_\delta(x_{2+2}), \ldots, v_\delta(x_{N+1})\right]^T, \]

and

\[ g_\delta(x) = [1, 1, \ldots, 1, 1, 1, \ldots, 1]^T. \]

According to the above equations and the proving process of Lemmas 3.2 and 3.3, we can see that the local truncation error is zero since $v_\delta(x)$ is a quadratic function, i.e.,

\[-\mathcal{L}_{\delta,h} v_{\delta,h}(x) = -\mathcal{L}_{\delta} v_\delta(x) = g_\delta(x). \]

Using Lemma 4.1 and the discrete maximum principle in Proposition 4.3 with $v_\delta(x_j) \geq 0$ for all $j \in \mathcal{N}_{\text{out}}$, we obtain

\[
\|(\mathcal{L}_{\delta,h})^{-1}||_{\infty} = \|(\mathcal{L}_{\delta,h})^{-1} g_\delta\|_{\infty} = \|v_{\delta,h}\|_{\infty} \leq \|v_\delta\|_{\infty} \leq \frac{1 + 4\delta(1 + \delta)}{8C_{\delta}}.
\]

The proof is completed.

**4.1. Error analysis of collocation method.** From Lemma 4.4 and stability definition in [12, p.19], the stability of the discrete scheme (2.13) can be established immediately. We now show the convergence behavior and error estimates with the general horizon parameter $\delta$.

**Theorem 4.5.** Let $u_\delta(x) \in C^4(\Omega)$ with $\delta = \mathcal{O}(h^\beta)$, $\beta \geq 0$. Let $u_{\delta,h}(x_i)$ be the approximate solution of $u_\delta(x_i)$ computed by the discretization scheme (2.13). If $\delta$ is the grid point, then

\[
\|u_{\delta,h}(x_i) - u_\delta(x_i)\|_{\infty} = \mathcal{O}\left(h^{\max(2,4-2\beta)}\right), \quad \beta \geq 0.
\]
If \( \delta \) is not the grid point, then
\[
\| u_{\delta,h}(x_i) - u_\delta(x_i) \|_\infty = O\left(h^{\min\{2.1+\beta\}}\right), \quad \beta \geq 0.
\]

**Proof.** Subtracting (4.12) from (4.11), it yields
\[
-\mathcal{L}_{\delta,h}[u_{\delta,h}(x_i) - u_\delta(x_i)] = \mathcal{L}_{\delta,h} u_\delta(x_i) - \mathcal{L}_{\delta,h} u_\delta(x_i).
\]
Thus, we have
\[
\| u_{\delta,h}(x_i) - u_\delta(x_i) \|_\infty \leq \|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \| \mathcal{L}_{\delta,h} u_\delta(x_i) - \mathcal{L}_{\delta,h} u_\delta(x_i) \|_\infty.
\]
According to Lemma 4.4 and Theorems 3.11, 3.10, the desired results are obtained.

**4.2. Error analysis of AC scheme.** To connect the nonlocal problem (2.1) with its local limit, we also require that
\[
C_{\delta} \to C_0, \quad f_{\delta} \to f, \quad \text{as} \quad \delta \to 0.
\]
As \( \delta \to 0 \), the solution of nonlocal problems (2.1) converges to the solution of the two-point boundary value problem [8]
\[
(4.3)
\]
which is the classic diffusion problem.

Let us study the asymptotic compatibility of the collocation scheme (2.12), i.e.,
\[
(4.4)
\]

**Definition 1.** \([17, 24]\) A family of convergent approximations \( \{u_{\delta,h}\} \) defined by (4.4) is said to be asymptotically compatible to the solution \( u_0 \) defined by (4.3) if both \( \delta \to 0 \) and \( h \to 0 \), we have \( u_{\delta,h} \to u_0 \).

**Lemma 4.6.** Let \( \delta \leq h \) with \( \delta, h \to 0 \). Let \( u_{\delta,h} \) and \( u_0 \) be the solution of (4.3) and (4.4), respectively. Then it holds that
\[
\| u_{\delta,h} - u_0 \|_\infty = O(h^2) \quad \text{as} \quad \delta, h \to 0.
\]

**Proof.** According to (2.7), (2.19) and Taylor series expansion, it yields
\[
\mathcal{L}_{\delta,h} u_0(x_i) = a_1 [u_0(x_{i-1}) + u_0(x_{i+1})] + a_0 u_0(x_i) + a_{\frac{1}{2}} \left[ u_0(x_{i-\frac{1}{2}}) + u_0(x_{i+\frac{1}{2}}) \right]
\]
\[
= u_0''(x_i) + \left[ 2a_1 \sum_{l=1}^{\infty} \frac{h^{2l+2}}{(2l+2)!} + 2a_{\frac{1}{2}} \sum_{l=1}^{\infty} \frac{(\frac{h}{2})^{2l+2}}{(2l+2)!} \right] u_0^{(2l+2)}(x_i)
\]
\[
= u_0''(x_i) + O(h^2) \quad \text{as} \quad \delta, h \to 0,
\]
and \( C_{\delta} \to C_0 = 1 \). From (4.3), (4.5) and Lemma 4.4, we have
\[
\| u_{\delta,h} - u_0 \|_\infty \leq \|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \| f + \mathcal{L}_{\delta,h} u_0 \|_\infty \leq \|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \| \mathcal{L}_{\delta,h} u_0 - u_0'' \|_\infty.
\]
The proof is completed.

**Lemma 4.7.** Let \( \delta \geq h \) with \( \delta, h \to 0 \). Let \( u_{\delta,h} \) and \( u_0 \) be the solution of (4.5) and (4.4), respectively. Then it holds that
\[
\|u_{\delta,h} - u_0\|_\infty = O(\delta^2) \quad \text{as} \quad \delta, h \to 0.
\]

**Proof.** From (2.7), (2.17), (2.18), (4.11) and Taylor's series expansion, it yields
\[
\mathcal{L}_{\delta,h}u_0(x_i) = \sum_{i=0}^{\infty} C_i' h u_0^{(2i+2)}(x_i) + \left( 2 \sum_{m=1}^{r} a_{m} + 2 \sum_{m=0}^{r-1} a_{m+\frac{1}{2}} + a_{0} \right) u(x_i)
\]
(4.6)
\[
= \sum_{i=0}^{\infty} C_i' h u_0^{(2i+2)}(x_i) = C_0'(h) u_0'(x_i) + \sum_{i=1}^{\infty} C_i' h u_0^{(2i+2)}(x_i)
\]
with
\[
C_i' h = 2 \sum_{m=1}^{r} a_{m} \left( \frac{m h}{2l + 2} \right)^{2l+2} + 2 \sum_{m=0}^{r-1} a_{m+\frac{1}{2}} \left( \frac{m + \frac{1}{2}}{2l + 2} \right)^{2l+2}.
\]

We next prove \( C_0'(h) = C_\delta \) in (4.3). Choosing \( u_0(z) = z^2 \) with \( x = 0 \) in (2.16), we get
\[
\mathcal{L}_{\delta,h}u_0(0) = \mathcal{L}_\delta u_0(x)|_{x=0} = 2 \int_0^\delta z^2 \gamma_\delta(z) dz = 2C_\delta = 2 \int_0^\delta z^2 \gamma_\delta(z) dz = 2C_\delta.
\]
Similarly, taking \( x_i = 0 \) in (2.7), it leads to
\[
\mathcal{L}_{\delta,h}u_0(0) = \mathcal{L}_{\delta,h} u_0(x_i)|_{x_i=0} = 2 \sum_{m=1}^{r} a_{m} (m h)^2 + 2 \sum_{m=0}^{r-1} a_{m+\frac{1}{2}} \left( \frac{m + \frac{1}{2}}{2} \right)^2 \right] = 2C_0'(h).
\]

Using the above equations and the proving process of Lemmas 3.2 and 3.9, it implies that the local truncation error is zero since \( u_0(x) \) is a quadratic function, i.e.,
\[
C_0'(h) = C_\delta = 1.
\]

On the other hand, there exists
\[
|C_i'(h)| \leq \frac{2}{(2l + 2)!} \left\{ \sum_{m=1}^{r} a_{m} (m h)^{2l} + 2 \sum_{m=0}^{r-1} a_{m+\frac{1}{2}} (m h)^{2l} + \left( \frac{m + \frac{1}{2}}{2} \right)^2 \right\}
\]
\[
= \frac{2(rh)^{2l}}{(2l + 2)!} C_0'(h) \leq \frac{2\delta^{2l}}{(2l + 2)!} C_0'(h),
\]
which yields
\[
\sum_{i=1}^{\infty} C_i'(h) u_0^{(2i+2)}(x_i) = O(\delta^2) \quad \text{as} \quad \delta, h \to 0.
\]

From the above equations, we can rewrite (4.6) as
\[
\mathcal{L}_{\delta,h}u_0(x_i) = u_0''(x_i) + O(\delta^2) \quad \text{as} \quad \delta, h \to 0.
\]
and $C_\delta = C_0 = 1$. Using (4.4), (4.5) and Lemma (4.4), we obtain

$$\|u_{\delta,h} - u_0\|_\infty \leq \|(L_{\delta,h})^{-1}\|_\infty \|f + L_{\delta,h}u_0\|_\infty \leq \|(L_{\delta,h})^{-1}\|_\infty \|L_{\delta,h}u_0 - u_0''\|_\infty.$$  

The proof is completed.

**Theorem 4.8.** Let $u_\delta(x) \in C^4(\Omega)$ with $\delta = O(h^\beta)$, $\beta \geq 0$. Let $u_{\delta,h}$ and $u_0$ be the solution of (4.5) and (4.4), respectively. Then it holds that

$$\|u_{\delta,h} - u_0\|_\infty = O(h^{\min\{2,2\beta}\)),$$  

as $\delta, h \to 0$.

**Proof.** From Lemmas 4.6 and 4.7, the desired result is obtained.

### 5. Numerical experiments

We now report results of numerical experiments which substantiate the analysis given earlier. The numerical errors are measured by the $l_\infty$ (maximum) norm.

#### 5.1. Numerical result with general horizon parameter

In order to get simpler benchmark solutions, we choose the exact solution of the nonlocal diffusion problem (2.1) as $u_\delta(x) = x^2(1 - x^2)$. This naturally leads to a $\delta$-dependent right-hand side $f_\delta(x) = 12x^2 - 2 + \frac{(3-\alpha)\delta^3}{3-\alpha}$. [Table 5.1]

| $\delta$ | $\alpha = 0.3$ | $\alpha = 0.7$ |
|---------|----------------|----------------|
| $h$     | Rate           | $\delta = 5h$ Rate | $\delta = 0.1$ Rate | $\delta = 5h$ Rate |
| $1/10$  | 1.08e-07       | 6.73e-08        | 1.19e-06           | 7.87e-07           |
| $1/20$  | 5.88e-09       | 4.19            | 1.58e-08           | 3.97              | 1.86e-07           | 2.07 |
| $1/40$  | 3.32e-10       | 4.14            | 3.86e-09           | 2.04              | 4.78e-09           | 3.98 |
| $1/80$  | 9.4e-11        | 4.09            | 9.49e-10           | 2.02              | 2.98e-10           | 4.00 |
| $1/160$ | 1.94e-11       | 4.09            | 9.49e-10           | 2.02              | 2.98e-10           | 4.00 |

| $\delta = \sqrt{h}$ | Rate       | $\delta = h^\beta$ Rate | $\delta = \sqrt{h}$ Rate | $\delta = h^\beta$ Rate |
|----------------------|------------|--------------------------|---------------------------|--------------------------|
| $1/10$               | 6.74e-05   | 8.70e-03                 | 5.88e-04                  | 6.30e-3                  |
| $1/20$               | 7.70e-07   | 3.22                      | 5.51e-04                  | 3.06                      | 4.14e-04             | 1.96 |
| $1/40$               | 9.42e-09   | 3.17                      | 3.42e-05                  | 2.00                      | 1.21e-07             | 3.05 |
| $1/80$               | 1.25e-10   | 3.11                      | 2.13e-06                  | 2.00                      | 1.80e-09             | 3.03 |

Tables 5.1 and 5.2 show that the quadric polynomial collocation method (2.12) has the convergence rate $O\left( h^{\max\{2,4-2\beta\}} \right)$ if $\delta$ is the grid point and $O\left( h^{\min\{2,1+\beta\}} \right)$ if $\delta$ is not the grid point, which is in agree Theorem 4.5.

#### 5.2. Numerical result with asymptotic compatibility

We now choose the exact solution of the local diffusion problem (4.4) as $u_0(x) = x^2(1 - x^2)$ and we can find the right-hand side $f(x) = 12x^2 - 2$. Table 5.3 shows that the quadric polynomial collocation method (2.12) has the convergence rate $O\left( h^{\min\{2,1+\beta\}} \right)$, which is in agree Theorem 4.8.

### 6. Conclusions

There are already some theoretical convergence results for nonlocal model when the horizon parameter $\delta$ is not a grid point. We notice that the proofs are mainly focusing on asymptotically compatible schemes as $\delta \to 0$. However, in the nonlocal system, the horizon of the material is a physical property of the material which does not require any relation between mesh size/grid points and $\delta$. 

Table 5.2: Convergence results of $\|u_{\delta,h} - u_{\delta}\|_\infty$ with $\delta$ not as grid point

| $h$  | $\delta = \frac{1}{4}$ | Rate | $\delta = \frac{10}{4}h$ | Rate | $\delta = \frac{1}{4}$ | Rate | $\delta = \frac{10}{4}h$ | Rate |
|------|------------------------|------|--------------------------|------|------------------------|------|--------------------------|------|
| 1/10 | 1.1e-03                | 2.01e-04 | 9.93e-04 | 1.88e-04 |
| 1/20 | 2.82e-04               | 0.98  | 1.20e-05 | 2.03  | 2.56e-04               | 0.97  | 1.13e-05 | 2.02  |
| 1/40 | 7.12e-05               | 0.99  | 7.43e-07 | 2.00  | 6.45e-05               | 0.99  | 6.99e-07 | 2.00  |
| 1/80 | 1.78e-05               | 0.99  | 4.64e-08 | 2.00  | 1.61e-05               | 0.99  | 4.32e-08 | 2.00  |

| $h$  | $\delta = \sqrt{h}$ | Rate | $\delta = h^2$ | Rate | $\delta = \sqrt{h}$ | Rate | $\delta = h^2$ | Rate |
|------|---------------------|------|-----------------|------|---------------------|------|-----------------|------|
| 1/10 | 3.96e-04            | 2.18e-05 | 3.66e-04 | 1.65e-05 |
| 1/20 | 4.60e-05            | 1.55  | 1.36e-06 | 2.00  | 4.26e-05            | 1.55  | 1.03e-06 | 2.00  |
| 1/40 | 4.96e-06            | 1.60  | 8.53e-08 | 2.00  | 4.61e-06            | 1.60  | 6.45e-08 | 2.00  |
| 1/80 | 4.39e-07            | 1.74  | 5.95e-09 | 1.91  | 4.08e-07            | 1.74  | 4.03e-09 | 1.99  |

Table 5.3: Convergence results of $\|u_{\delta,h} - u_0\|_\infty$ with asymptotic compatibility.

| $h$  | $\delta = \frac{1}{4}h$ | Rate | $\delta = 5h$ | Rate | $\delta = \frac{1}{4}h$ | Rate | $\delta = 5h$ | Rate |
|------|------------------------|------|-----------------|------|------------------------|------|-----------------|------|
| 1/10 | 8.12e-06               | 2.27e-05 | 7.54e-06 | 2.11e-05 |
| 1/20 | 2.11e-05               | 2.00  | 5.64e-06 | 2.08  | 1.88e-06               | 2.00  | 5.24e-06 | 2.00  |
| 1/40 | 5.05e-07               | 2.00  | 1.40e-06 | 2.00  | 4.69e-07               | 2.00  | 1.30e-06 | 2.03  |
| 1/80 | 1.26e-07               | 2.00  | 3.51e-07 | 2.00  | 1.17e-07               | 1.99  | 3.27e-07 | 2.00  |

| $h$  | $\delta = \sqrt{h}$ | Rate | $\delta = h^2$ | Rate | $\delta = \sqrt{h}$ | Rate | $\delta = h^2$ | Rate |
|------|---------------------|------|-----------------|------|---------------------|------|-----------------|------|
| 1/10 | 1.6e-03             | 2.18e-05 | 1.5e-04  | 1.65e-05 |
| 1/20 | 4.29e-04            | 0.94  | 1.36e-06 | 2.00  | 3.97e-04            | 0.95  | 1.03e-06 | 2.00  |
| 1/40 | 1.10e-04            | 0.97  | 8.53e-08 | 2.00  | 1.02e-04            | 0.97  | 6.45e-08 | 2.00  |
| 1/80 | 2.80e-05            | 0.99  | 5.95e-09 | 1.91  | 2.60e-05            | 0.98  | 4.03e-09 | 1.99  |

In this work, we design the efficient PQC scheme to solving the nonlocal model with the general horizon parameter $\delta$ including a fixed $\delta > 0$. The global errors and superconvergence orders are rigorous proved with the weakly singular kernel. Although to seek numerical discretization of the strong form (e.g., collocation, finite difference) with the hypersingular kernel, it is still difficult to show stability of the high-order numerical schemes while trying to keep the discrete maximum principle [18]. However, numerical results shows that the inverse operator of $-L_{\delta,h}$ is still bounded with the hypersingular kernel by Hadamard finite-part integral, which shows the stability in our framework; In the future, we will try to find the way to verify the bounded of the inverse operator.

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