$D = 7$ $SU(2)$ Gauged Supergravity
From $D = 10$ Supergravity

A. H. Chamseddine and W. A. Sabra

Center for Advanced Mathematical Sciences (CAMS),
and
Physics Department, American University of Beirut, Lebanon

ABSTRACT
The theory of $SU(2)$ gauged seven-dimensional supergravity is obtained by compactifying ten dimensional $N = 1$ supergravity on the group manifold $SU(2)$. 
1 Introduction

Theories of extended supergravity in various dimensions possess rigid symmetries. A subgroup of these symmetries can be gauged by the vector fields present in the theory. For example, in $N = 2$ supergravity in five dimensions, one can gauge the $U(1)$ subgroup of the $SU(2)$ rigid symmetry group of the theory [1]. Gauged supergravity theories exist in higher dimensions in which supersymmetry allows the existence of a cosmological constant. It is well known that a cosmological constant is not allowed in $d = 11$, $d = 10$ and $d = 9$. In $d = 7$, $SU(2)$ and $SO(5) \times SO(5)$ gauged supergravity theories were constructed in [2] and [3] respectively.

The theories of gauged supergravity theories in $d = 4, 5, 7$ are believed to be related to certain compactifications of $d = 10, 11$ supergravity theories. For instance, the four dimensional gauged $N = 8$ supergravity [4] was conjectured to be related to the compactification of the original eleven dimensional supergravity [5] on $S^7$ [6]. This conjecture was proved by de Wit and Nicolai [7] for a different formulation of eleven dimensional supergravity with local $SU(8)$ invariance [8]. The connection to the non linear Kaluza-Klein ansatz of the original $d = 11$ Lagrangian could only be solved in certain sectors. Toroidal compactification of ten dimensional supergravity to four dimensions yields an $N = 4$ supergravity theory with six vector multiplets [9]. The vector and matter fields obtained in four dimensions are in general linear combinations of the internal components of the ten dimensional metric and antisymmetric tensor. The truncation is then performed by identifying the vector components of the ten-dimensional metric with those of the antisymmetric tensor. Another known compactifications of ten dimensional supergravity are due to Scherk and Schwarz [10], in which the internal compactified space is taken to be a group manifold. The maximal group manifold allowed is $S^3 \times S^3$ and a compactification of this particular case for the dual formulation of supergravity was performed in [11]. The resulting four dimensional theory is an $N = 4$ supergravity with a non-compact gauge group containing the factor $SU(2) \times SU(2)$. In order to obtain the Friedman-Schwarz model [12], a compactification of the Sherck-Schwarz type
is needed. This compactification was performed recently in [13]. The crucial point in this analysis is the specific relation between the components of the metric and antisymmetric tensor along the internal dimensions. This was a long standing problem, the main difficulty was finding the right ansatz for the antisymmetric tensor field. It was suggested by the authors of [14] that a more complicated six dimensional internal manifold is needed in order to obtain the Friedman-Schwarz model. This suggestion was motivated by the general result of Friedman, Gibbons and West [15] that non-trivial compactifications of ten-dimensional supergravity are inconsistent. This proved not to be the case as a basic assumption made in [15] regarding the dilaton field needs to be violated.

In recent years, there has been a renewed interest in gauged supergravity theories. This is mainly due to the recently conjectured duality between supergravity and super Yang Mills [16]. Since this conjecture has been made, anti-de Sitter spaces have received a great deal of interest. The purpose of our work here is to demonstrate that \( d = 7 \), \( N = 2 \), \( SU(2) \)-gauged supergravity theory of [2] can be obtained via dimensional reduction of \( N = 1 \) ten dimensional and eleven dimensional supergravity theories.

# 2 Dimensional Reduction and \( D = 7 \) Gauged Supergravity

In this section we will show explicitly how to obtain the seven dimensional gauged supergravity of [2] as a dimensionally reduced ten dimensional \( N = 1 \) supergravity theory. The internal space is taken to be the group manifold \( SU(2) \).

The bosonic part of \( N = 1 \) supergravity action in ten dimensions is

\[
S_{10} = \int \left( -\frac{1}{4} \hat{R} + \frac{1}{2} \partial_M \hat{\phi} \partial^M \hat{\phi} + \frac{1}{12} e^{-2\hat{\phi}} \hat{H}_{MNP} \hat{H}^{MNP} \right) d^4 x d^6 z
\equiv S_G + S_{\hat{\phi}} + S_{\hat{H}}.
\] (1)

Our notations are as follows. Ten-dimensional quantities are denoted by hatted symbols. Base space and tangent space indices are denoted by late and
early capital Latin letters, respectively. We first comment briefly on the compactification of ten-dimensional supergravity on $S^3 \times S^3$ to four dimensions.

For four-dimensional space-time indices, late and early Greek letters denote base space and tangent space indices, respectively. Similarly, the internal base space and tangent space indices are denoted by late and early Latin letters, respectively.

\[{\{M}\} = \{\mu = 0, \ldots, 3; m = 1, \ldots, 6\}, \quad \{A\} = \{\alpha = 0, \ldots, 3; a = 1, \ldots, 6\}.
\]

The general coordinates $\hat{x}^M$ consist of spacetime coordinates $x^\mu$ and internal coordinates $z^m$. The flat Lorentz metric of the tangent space is chosen to be $(+,-,\ldots,-)$ with the internal dimensions all spacelike. Thus the metric is related to the vielbein by

$$\hat{g}_{MN} = \hat{\eta}_{AB}\hat{e}^A_M\hat{e}^B_N = \eta_{\alpha\beta}\hat{e}_\alpha^A\hat{e}_\beta^B - \delta^{ab}\hat{e}_a^A\hat{e}_b^B,$$

and the antisymmetric tensor field strength is

$$\hat{H}_{MNP} = \partial_M\hat{B}_{NP} + \partial_N\hat{B}_{PM} + \partial_P\hat{B}_{MN}.$$  

The coordinates $z^m$ span the internal compact group space, implying that we have the functions $U^a_m(z)$ satisfying the condition

$$\left(\left(U^{-1}\right)^m_a\right)\left(\left(U^{-1}\right)^n_b\right)\left(\partial_m U^c_n - \partial_n U^c_m\right) = \frac{f_{abc}}{\sqrt{2}},$$

Here $f_{abc}$ are the group structure constants and the internal space volume is $\Omega = \int |U^a_m|dz$. In the maximal case, i.e., $SU(2) \times SU(2)$, each $S^3$ factor admits invariant 1-form $\theta^a = \theta^a_i dz^i$, which satisfies

$$d\theta^a + \frac{1}{2}\epsilon_{abc}\theta^b \wedge \theta^c = 0.$$  

If one chooses

$$U^a_m \equiv U^a_i = -\frac{\sqrt{2}}{g}\theta^a_i$$

where $g$ is a coupling constant, then the structure constants will be given in terms of the coupling constant by $f_{abc} = g\epsilon_{abc}$. For the case where the
coupling constant of one of the $SU(2)$ factors vanishes, the internal space becomes the group manifold $SU(2) \times [U(1)]^3$. In order to get a consistent truncation, one must set six vector multiplets to zero. This is done by identifying the six gauge fields coming from the components of the metric with six vector fields from the components of the antisymmetric tensor. In order to do this identification the vectors coming from the antisymmetric tensor must behave like Yang-Mills gauge fields, and here is the main source of difficulty. It is crucial to have the right ansatz for the antisymmetric tensor to get a consistent truncation.

We now turn to the compactification of ten-dimensional supergravity theory down to seven dimensions. The three dimensional internal space is taken to be the $SU(2)$ group manifold. We shall show that the obtained theory is the $SU(2)$ gauged seven dimensional supergravity theory derived in [2]. Following Scherk and Schwarz [10], we parameterize the vielbein in the following form

$$\tilde{e}^A_M = \left( e^{\frac{1}{2} \hat{\phi}} e^a_\mu(x), \sqrt{2} e^{-\frac{1}{2} \hat{\phi}} A^m_\mu(z) \delta^a_m, e^{-\frac{1}{2} \hat{\phi}} U^a_m(z) \right)$$  \hspace{1cm} (8)

where we have set $\kappa = 1$ and rescaled the gauge fields by $\frac{1}{\sqrt{2}}$. Here $U$ depends on the internal coordinates $(7, 8, 9)$ and $\hat{\phi} = \hat{\phi}(x)$.

Our ansatz in terms of the metric components is thus given by

$$\hat{g}_{\mu\nu} = e^{\frac{1}{2} \hat{\phi}} g_{\mu\nu} - 2 A^a_\mu A^a_\nu e^{-\hat{\phi}}$$
$$\hat{g}_{mn} = -\sqrt{2} U^a_m A^a_\mu e^{-\hat{\phi}}$$
$$\hat{g}_{mn} = -U^a_m U^a_n e^{-\hat{\phi}}$$  \hspace{1cm} (9)

Using equation (38) of [10] for the reduction of $S_G$, one obtains the reduced Lagrangian, which reads

$$\mathcal{L}_G = -\frac{1}{4} R - \frac{1}{8} e^{-\hat{\phi}} F_{\mu\nu} F^{\mu\nu} + \frac{3}{10} g^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} + \frac{3g^2}{16} e^{\frac{1}{2} \hat{\phi}}$$  \hspace{1cm} (10)

where

$$F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu$$  \hspace{1cm} (11)

\footnote{When using this formula, we rescale the gauge fields}
For the antisymmetric tensor our ansatz is

\[ \hat{B}_{\mu\nu} = B_{\mu\nu}, \quad \hat{B}_{\mu m} = -\frac{1}{\sqrt{2}} A_{\mu}^a U_a^m, \quad \hat{B}_{mn} = \tilde{B}_{mn} \]

(12)

where \( B_{\mu\nu} \) is a function of \( x \) and \( \tilde{B}_{mn} \) is a function of the internal coordinates \( z \) only. The field strengths are given by

\[
\begin{align*}
\hat{H}_{\mu\nu\rho} &= H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \\
\hat{H}_{\mu m} &= -\frac{1}{\sqrt{2}} (\partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a) U_m^a \\
\hat{H}_{\mu mn} &= \frac{1}{2} f_{abc} A_{\mu}^a U_m^b U_n^c \\
\hat{H}_{mnp} &= \partial_m \tilde{B}_{np} + \partial_n \tilde{B}_{pm} + \partial_p \tilde{B}_{mn}
\end{align*}
\]

(13)

where we also require that

\[
\hat{H}_{mnp} = \frac{1}{2\sqrt{2}} f_{abc} U_m U_n U_p^c
\]

(14)

The reduction of \( S_{\hat{B}} \) gives the following

\[
\mathcal{L}_B = \left( \frac{1}{12} e^{-\frac{14}{5} \hat{\phi}} H_{\mu\nu\rho}' H_{\mu\nu\rho}' - \frac{1}{8} e^{-\frac{8}{5} \hat{\phi}} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{g^2}{16} e^{-\frac{1}{5} \hat{\phi}} \right)
\]

(15)

where

\[
\begin{align*}
H_{\mu\nu\rho}' &= H_{\mu\nu\rho} - \omega_{\mu\nu\rho} \\
\omega_{\mu\nu\rho} &= -6(A_{\mu}^a \partial_\nu A_{\rho}^a + \frac{1}{3} f_{abc} A_{\mu}^a A_{\nu}^b A_{\rho}^c)
\end{align*}
\]

(16)

In deriving equation (15) care must be taken to include the off-diagonal components of the metric. This can be effectively done by first defining \( \hat{H}_{ABC} = e_A^M e_B^N e_C^p \hat{H}_{MNP} \) and then writing

\[
\begin{align*}
\hat{H}_{\alpha\beta\gamma} &= e^{-\frac{14}{5} \hat{\phi}} e_{\alpha}^\mu e_{\beta}^\nu e_{\gamma}^\rho H_{\mu\nu\rho}', \\
\hat{H}_{\alpha\beta\gamma} &= e^{-\frac{14}{10} \hat{\phi}} e_{\alpha}^\mu e_{\beta}^\nu F_{\mu\nu}^a, \\
\hat{H}_{\alpha\beta\gamma} &= 0, \\
\hat{H}_{abc} &= e^{\frac{3}{2} \hat{\phi}} \frac{g}{2\sqrt{2}} \epsilon_{abc}.
\end{align*}
\]
The appearance of the Chern-Simons term $\omega_{\mu\nu\rho}$ in $H'_{\mu\nu\rho}$ is a crucial test of the consistency of the ansatz.

The reduction of the scalar part of ten dimensional supergravity, $S_{\hat{\phi}}$ gives the following contribution to the seven dimensional Lagrangian

$$L_S = \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi}$$

(17)

Therefore, combining $L_G$, $L_B$ and $L_S$, the seven dimensional theory is described by the Lagrangian

$$L_7 = -\frac{1}{4} R - \frac{1}{4} e^{-8\hat{\phi}} F_{\mu\nu} F^{\mu\nu} + \frac{4}{5} g^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} + \frac{1}{12} e^{-16\hat{\phi}} H'_{\mu\nu\rho} H'_{\mu\nu\rho} + \frac{g^2}{8} e^{8\hat{\phi}}$$

(18)

The $N = 2$ $SU(2)$ gauged $d = 7$ supergravity which was constructed in [2] is given by

$$e^{-1} L = -\frac{1}{2} R - \frac{e}{48} \sigma^{-4} F_{\mu\nu\rho} F^{\mu\nu\rho} - \frac{\sigma^2}{4} F_{\mu i} F^{\mu i} - \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi}$$

$$+ \frac{i}{48\sqrt{2}} \epsilon^{\mu\nu\rho\sigma\kappa\lambda} F_{\mu\nu\rho} \partial_\sigma A_{\kappa\lambda i}^j + \alpha^2 \sigma^{-2}$$

(19)

where $\sigma = e^{-\sqrt{5} \phi}$. The Lagrangian in (19) can be seen to be identical to (18), after multiplying equation (18) by an overall factor of 2 and under the following identifications

$$\hat{\phi} = \frac{\sqrt{5}}{4} \phi,$$

(20)

$$g = 2\alpha,$$

(21)

$$A_{\mu i}^j = -A_{\mu}^a (\tau^a)_i^j.$$  

(22)

The Lagrangian (19) contains a three-form $A_{\mu\nu\rho}$ instead of the two-form $B_{\mu\nu}$ appearing in (18). These forms are, however, related by a duality transformation. To see this we add to the seven dimensional compactified action in (18) the term

$$-\frac{1}{36} \int d^7 x \epsilon^{\mu\nu\rho\sigma\kappa\lambda} H_{\mu\nu\rho} \partial_\sigma A_{\kappa\lambda}$$

and assume that $H_{\mu\nu\rho}$ is not the field strength of $B_{\mu\nu}$ but an independent field. The equation of motion of $A_{\kappa\lambda}$ then implies that $H_{\mu\nu\rho}$ is the field strength of
a two form $B_{\mu \nu}$. On the other hand, now the $H_{\mu \nu \rho}$ appears quadratically and linearly in the Lagrangian, the Gaussian integration of $H_{\mu \nu \rho}$ can be carried out resulting in the terms

$$\int d^7 x \left( \frac{e}{96} e^{\frac{1}{16} \phi} F_{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma} - \frac{1}{144 \sqrt{2}} \epsilon^{\mu \nu \rho \sigma \kappa \lambda \eta} F_{\mu \nu \rho \sigma} \omega_{\kappa \lambda \eta} \right).$$

Integrating the last term by parts and using the identity

$$\partial_{[\mu} \omega_{\nu \rho \sigma]} = -\frac{3}{2} F_{\mu \nu} F_{\rho \sigma},$$

one obtains the Chern-Simons term

$$\frac{1}{24 \sqrt{2}} \int d^7 x e^{\mu \nu \rho \sigma \kappa \lambda \eta} F_{\mu \nu} F_{\rho \sigma} A_{\kappa \lambda \eta}$$

and this is seen to agree with the Lagrangian in (19) after integrating by parts.

We note in passing that the seven dimensional Lagrangian can also be obtained by compactifying eleven dimensional supergravity and then truncating. This can be seen by embedding the ten-dimensional supergravity into eleven-dimensional supergravity after truncating half the degrees of freedom. One has the following identifications for the eleven dimensional fields:

$$e^A_M = e^{-\frac{1}{48} \phi} e^A_M$$
$$e^{11}_{11} = e^{\frac{1}{4} \phi}$$
$$e^{\bar{M}}_{11} = 0$$
$$e^A_{11} = 0$$

$$A_{M N \bar{1} 1} = \hat{B}_{M N}$$
$$A_{M N P} = 0$$

This identification is important in lifting special solutions from seven to ten and eleven dimensions.

---

2The different signs for the kinetic terms as well as the i factor appearing with the epsilon tensor in (19) are due to the choice of the metric (+ + + + + + +) in 2
We now comment on related work that appeared in the literature. First in the work of Duff, Townsend and van Niewenhuizen [14] a compactification of ten dimensional supergravity on $S^3$ was given, but this breaks all supersymmetries. Recently, a consistent truncation of eleven dimensional supergravity to $N = 4$ seven dimensional supergravity has been given by Nastase, Vaman and van Niewenhuizen [17]. They gave the complete non linear Kaluza-Klein reduction on $AdS_7 \times S_4$. Lu and Pope [18] gave an ansatz for the reduction and truncation of eleven dimensional supergravity to seven dimensions. The Lagrangian they obtained has two coupling constants $g$ and $m$ where $m$ is the coefficient of the topological term

$$m e^{\mu\nu\rho\kappa\lambda\eta} F_{\mu\nu\rho\sigma} A_{\kappa\lambda\eta}.$$

which is only possible in the $A_{\mu\nu\rho}$ formulation of $d = 7$ supergravity but not in the $A_{\mu\nu}$ formulation. The case considered in this paper corresponds to setting $m = 0$ in [18] and is only obtained from their work as a singular limit.

In this work we have not given the ansatz for the reduction of the fermionic parts. This should be straightforward but tedious. The consistency of the ansatz for the fermionic sector has been checked for the reduction of the $N = 1$ ten dimensional supergravity on $S^3 \times S^3$ to four dimensions [18]. The reduction considered here is very similar to that case and the consistency check should follow the same steps.

Acknowledgments W. Sabra would like to thank N. N. Khuri and his group for hospitality at Rockefeller University where most of this work was done.

References

[1] E. Cremmer, in Supergravity and superspac, eds. S. Hawking and M. Rocek, Cambridge University Press, (1981)
[2] P. K. Townsend and P. Van Nieuwenhuizen, *Phys. Lett.* **125B** (1983) 41.

[3] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Phys. Lett.* **143B** (1984) 103.

[4] B. de Wit and H. Nicolai, *Nucl. Phys.* **B208** (1982) 323.

[5] E. Cremmer, B. Julia and J. Scherck, *Phys. Lett.* **76B** (1978) 61.

[6] M. J. Duff, B. E. Nilsson and C. Pope, *Phys. Rep.* **130** (1986) 1.

[7] B. de Wit and H. Nicolai, *Nucl. Phys.* **B281** (1987) 211.

[8] B. de Wit and H. Nicolai, *Nucl. Phys.* **B274** (1986) 363.

[9] A. H. Chamseddine, *Nucl. Phys.* **B185** (1981) 403.

[10] J. Scherk, J. H. Schwarz, *Nucl. Phys.* **B153** (1979) 61.

[11] A. H. Chamseddine, *Phys. Rev.* **D24** (1981) 3065.

[12] D. Z. Freedman, J. H. Schwarz, *Nucl. Phys.* **B137** 333 (1978).

[13] A. H. Chamseddine and M. S. Volkov, *Phys. Rev. Lett.* **79** 3343 (1997); *Phys. Rev.* **D57** (1998) 6242.

[14] M. J. Duff, P. K. Townsend and P. van Niewenhuizen, *Phys. Lett.* **122B** (1983) 232.

[15] D. Z. Freedman, G. W. Gibbons and P. C. West, *Phys. Lett.* **124B** (1984) 491.

[16] J. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett.* **428B** (1998) 105; E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253.

[17] H. Nastase, D. Vaman and P. van Nieuwenhuizen, [hep-th/9905075](http://arxiv.org/abs/hep-th/9905075).

[18] H. Lu and C. N. Pope, [hep-th/9906168](http://arxiv.org/abs/hep-th/9906168).