ON VARIETIES OF MAXIMAL ALBANESE DIMENSION

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A smooth projective complex variety \(X\) has maximal Albanese dimension if its Albanese map \(X \to \text{Alb}(X)\) is generically finite onto its image. These varieties have recently attracted a lot of attention and have been shown to have very special geometric properties ([CH1], [CH2], [CH3], [F], [HP], [P], [PP1]).

Assume for example that \(f : X \to Y\) is a surjective morphism between smooth projective varieties of the same dimension. For each positive integer \(m\), denote by \(P_m(X) := h^0(X, \omega_X^{\otimes m})\) the \(m\)-th plurigenus of \(X\). We have \(P_m(X) \geq P_m(Y)\), but it is in general difficult to conclude anything on \(f\) if there is equality. However, when \(Y\) (hence also \(X\)) is of general type and has maximal Albanese dimension, Hacon and Pardini proved in [HP], Theorem 3.2, that if \(P_m(X) = P_m(Y)\) for some \(m \geq 2\), then \(f\) is birational. We give in \(\S\) examples that show that this conclusion does not hold in general.

More generally, if \(X \to I(X)\) and \(Y \to I(Y)\) are the respective Iitaka fibrations of \(X\) and \(Y\), we may assume, taking appropriate birational models, that \(f\) induces a morphism \(I(f) : I(X) \to I(Y)\). When \(Y\) has maximal Albanese dimension, but is not necessarily of general type, Hacon and Pardini proved that if \(P_m(X) = P_m(Y)\) for some \(m \geq 2\), then \(I(f)\) has connected fibers (since \(I(Y)\) is birational to \(Y\) when \(Y\) is of general type, this implies the result quoted above).

But in their proof, Hacon and Pardini actually do not use the assumption that \(Y\) has maximal Albanese dimension; all they need is that \(P_m(X) = P_m(Y) > 0\) and \(I(Y)\) has maximal Albanese dimension (see section 1). However, under their assumption, we prove here a much stronger conclusion.

**Theorem 1** Let \(f : X \to Y\) be a surjective morphism between smooth complex projective varieties of the same dimension. If \(Y\) has maximal Albanese dimension and \(P_m(X) = P_m(Y)\) for some \(m \geq 2\), the induced map \(I(f) : I(X) \dashrightarrow I(Y)\) between the respective Iitaka models of \(X\) and \(Y\) is birational.

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Moreover, $f$ is birationally equivalent to a quotient by a finite abelian group.

For more details on the last statement, we refer to Theorem 2.1.

In another direction, it was shown by Chen and Hacon ([CH1], Theorem 4) that if $X$ is a smooth projective variety of maximal Albanese dimension, the image of the 6-canonical map $\phi_{6K_X}$ has dimension the Kodaira dimension $\kappa(X)$. If $X$ is moreover of general type, $\phi_{6K_X}$ is birational onto its image ([CH3], Corollary 4.3). We prove a common generalization of these results (Theorem 4.1):

**Theorem 2** If $X$ is a smooth complex projective variety with maximal Albanese dimension, $\phi_{5K_X}$ is a model of the Iitaka fibration of $X$.

The proof follows the ideas of [PP2] and is based on a result from [J]. We also prove that $\phi_{3K_X}$ is already a model of the Iitaka fibration of $X$ under a stronger assumption on $X$ (Theorem 4.3).

The article is organized as follows. The first section is devoted to the proof of the birationality of $I(f)$. In the second section, we give a complete structure theorem for $f$ (Theorem 2.1) which shows that the situation is quite restricted. In the third section, we present three examples showing that the conclusion of the above theorem can fail when the varieties do not have maximal Albanese dimension, and in the last section, we prove our results on pluricanonical maps of varieties of maximal Albanese dimension.

We work over the field of complex numbers.

1. **Proof that $I(f)$ is birational**

We begin with a general lemma (we refer to [L], §11, for the definition and properties of the asymptotic multiplier ideal sheaf $\mathcal{J}(||D||)$ associated with a divisor $D$ on a smooth projective variety).

**Lemma 1.1.** Let $f : X \to Y$ be a surjective morphism between smooth projective varieties of the same dimension with $\kappa(Y) \geq 0$. For any $m \geq 2$,

$$f_* (\mathcal{O}_X(mK_X) \otimes \mathcal{J}(||(m-1)K_X||)) \supset \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(||(m-1)K_Y||).$$

**Proof.** Take $N > 0$ and let $\tau_Y : Y' \to Y$ be a log-resolution such that $\tau_Y^*|N(m-1)K_Y| = |L_1| + E_1$, where $L_1$ is a line bundle and $E_1$ is an effective divisor.
where $|L_1|$ is base-point-free and $E_1$ is the fixed divisor. Then we take a log-resolution $\tau_X : X' \to X$ such that we have a commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\tau_X \downarrow & & \downarrow \tau_Y \\
X & \xrightarrow{f} & Y
\end{array}
$$

and $\tau_X^*|N(m-1)K_X| = |L_2| + E_2$ where $|L_2|$ is base-point-free and $E_2$ is the fixed divisor. Let $D \in |(m-1)K_{X/Y}|$. Then $f'^*E_1 + N\tau_X^*D \geq E_2$.

Hence

$$
O_{X'}(K_{X'/X} + m\tau_X^*K_X - \left\lfloor \frac{1}{N}E_2 \right\rfloor) = O_X(mK_X) \otimes \mathcal{I}(\left\lfloor (m-1)K_X \right\rfloor).
$$

By step 2 in the proof of \cite{HP}, Theorem 3.2, we know that $K_{X'/Y'} - \left\lfloor \frac{1}{N}f'^*E_1 \right\rfloor + f'^*\left\lfloor \frac{1}{N}E_1 \right\rfloor$ is an effective divisor, hence

$$
\tau_{X'}(\mathcal{O}_{X'}(K_{X'/X} + m\tau_X^*K_X - \left\lfloor \frac{1}{N}E_2 \right\rfloor)) = \mathcal{O}_{X'}(K_{X'/Y'} - \left\lfloor \frac{1}{N}f'^*E_1 \right\rfloor + f'^*\left\lfloor \frac{1}{N}E_1 \right\rfloor).
$$

This proves the lemma. 

We now prove the first part of Theorem 1, stated in the introduction. We start from a surjective morphism $f : X \to Y$ between smooth projective varieties of the same dimension.

Changing the notation from the introduction, we let $V$ and $W$ be the respective Iitaka models of $X$ and $Y$, and we may assume, taking appropriate birational models, that we have a commutative diagram of
morphisms

\[
\begin{align*}
X \xrightarrow{f} Y & \xrightarrow{a_Y} A \\
V \xrightarrow{g} W & \xrightarrow{a_W} A/K
\end{align*}
\]

where \(h_X\) and \(h_Y\) are the respective Iitaka fibrations of \(X\) and \(Y\), \(a_Y\) and \(a_W\) are the respective Albanese morphisms of \(Y\) and \(W\), and \(K\) is an abelian subvariety of \(A := \text{Alb}(Y)\) (see [HP], §2.1). We set

\[
\begin{align*}
\mathcal{H}_X := h_{X*}(\mathcal{O}_X(mK_X) \otimes \mathcal{J}(|(m-1)K_X|)) & & \mathcal{F}_X := a_{W*}g_*\mathcal{H}_X \\
\mathcal{H}_Y := h_{Y*}(\mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(|(m-1)K_Y|)) & & \mathcal{F}_Y := a_{W*}\mathcal{H}_Y.
\end{align*}
\]

When \(m \geq 2\), we have \(\mathcal{F}_Y \subset \mathcal{F}_X\) by Lemma 1.1 and we denote by \(\mathcal{Q}\) the quotient sheaf \(\mathcal{F}_X/\mathcal{F}_Y\) on \(A/K\).

Assume now \(P_m(X) = P_m(Y) = M > 0\). By Theorem 11.1.8 and Proposition 11.2.10 in [L], we have

\[
P_m(Y) = h^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(|mK_Y|)) = h^0(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(|(m-1)K_Y|)) = h^0(W, \mathcal{H}_Y) = h^0(A/K, \mathcal{F}_Y).
\]

Similarly,

\[
P_m(X) = h^0(V, \mathcal{H}_X) = h^0(A/K, \mathcal{F}_X).
\]

Thus \(\mathcal{H}_Y \subset h_{Y*}(\mathcal{O}_Y(mK_Y))\) is a nonzero torsion-free sheaf. Since \(h_Y\) is a model of the Iitaka fibration of \(Y\) whose general fibers are birationally isomorphic to abelian varieties ([HP], Proposition 2.1), the latter sheaf has rank 1. So the rank of \(\mathcal{H}_Y\) is also 1. We have the same situation for \(h_X\), hence the rank of \(\mathcal{H}_X\) is again 1. On the other hand, we claim the following.

**Claim:** \(\mathcal{Q} = 0\), hence \(\mathcal{F}_Y = \mathcal{F}_X\).

In order to prove the Claim, we want to apply Proposition 2.3 in [HP]. Namely, it is enough to prove \(h^j(A/K, \mathcal{F}_X \otimes P) = h^j(A/K, \mathcal{F}_Y \otimes P)\) for all \(j \geq 0\) and all \(P \in \text{Pic}^0(A/K)\). We will first prove that when \(j \geq 1\). By Lemma 2.1 in [J], we have

\[
(2) \quad H^j(W, \mathcal{H}_Y \otimes a_W^*P) = H^j(W, g_*\mathcal{H}_X \otimes a_W^*P) = 0,
\]

for all \(P \in \text{Pic}^0(A/K)\) and all \(i \geq 1\). We now prove

\[
(3) \quad R^j a_{W*}\mathcal{H}_Y = R^j a_{W*}(g_*\mathcal{H}_X) = 0,
\]

for all \(j \geq 1\), as follows.
First we take a very ample line bundle $H$ on $A/K$ such that, for all $k \geq 1$ and $j \geq 0$,

\[(4) \quad H^k(A/K, R^j a_{W^*} \mathcal{F}_Y \otimes H) = H^k(A/K, R^j a_{W^*} (g_* \mathcal{F}_X) \otimes H) = 0\]

and $R^j a_{W^*} \mathcal{F}_Y \otimes H$ and $R^j a_{W^*} (g_* \mathcal{F}_X) \otimes H$ are globally generated. Again by Lemma 2.1 in [J],

\[H^j(W, \mathcal{H}_Y \otimes a_{W^*} H) = H^j(W, g_* \mathcal{H}_X \otimes a_{W^*} H) = 0,\]

for all $j \geq 1$. Therefore, by Leray’s spectral sequence and (4), we conclude that

\[H^0(A/K, R^j a_{W^*} \mathcal{F}_Y \otimes H) = H^0(A/K, R^j a_{W^*} (g_* \mathcal{F}_X) \otimes H) = 0\]

for $j \geq 1$. Since $R^j a_{W^*} \mathcal{H}_Y \otimes H$ and $R^j a_{W^*} (g_* \mathcal{H}_X) \otimes H$ are globally generated, we deduce that $R^j a_{W^*} \mathcal{H}_Y = R^j a_{W^*} (g_* \mathcal{H}_X) = 0$, for all $j \geq 1$.

Applying the Leray spectral sequence to (2), we get, by (3), for all $i \geq 1$ and $P \in \text{Pic}^0(A/K),$

\[H^i(A/K, \mathcal{F}_Y \otimes P) = H^i(W, \mathcal{H}_Y \otimes g^* P) = 0,\]

and

\[H^i(A/K, \mathcal{F}_X \otimes P) = H^i(W, g_* \mathcal{H}_X \otimes g^* P) = 0.\]

Finally, for all $P \in \text{Pic}^0(A/K),$

\[h^0(A/K, \mathcal{F}_Y \otimes P) = \chi(A/K, \mathcal{F}_Y \otimes P) = \chi(A/K, \mathcal{F}_Y) = h^0(A/K, \mathcal{F}_Y) = M,\]

and similarly,

\[h^0(A/K, \mathcal{F}_X \otimes P) = h^0(A/K, \mathcal{F}_X) = M.\]

We have finished the proof of the Claim.

Assume moreover that $W$ has maximal Albanese dimension, so that $a_W$ is generically finite onto its image $Z$, the rank of $\mathcal{H}_Y$ is 1, and the rank of $\mathcal{F}_X = \mathcal{F}_Y = a_{W^*} \mathcal{H}_Y$ on $Z$ is $\deg(a_W)$. Consider the Stein factorization

\[g : V \xrightarrow{p} U \xrightarrow{q} W,\]

where $p$ is an algebraic fiber space and $q$ is surjective and finite. Because $h^0(U, p_* \mathcal{H}_X) = h^0(V, \mathcal{H}_X) = M > 0$, the nonzero torsion-free sheaf $p_* \mathcal{H}_X$ has rank $\geq 1$. We can write

\[\mathcal{F}_X = a_{W^*} g_* \mathcal{H}_X = a_{W^*} q_* (p_* \mathcal{H}_X),\]

and conclude that the rank of $\mathcal{F}_X$ on $Z$ is $\geq \deg(q) \cdot \deg(a_W)$. This implies $\deg(q) = 1$ hence $g$ has connected fibers. Essentially, this is Hacon and Pardini’s proof of [HP], Theorem 3.2.
Assume finally that $Y$ has maximal Albanese dimension. We just saw that $g \circ h_X$ is an algebraic fiber space and we denote by $X_w$ a general fiber. The main ingredient is the following lemma.

**Lemma 1.2.** In the above situation, the sheaf 

$$g_* \mathcal{H}_X = (g \circ h_X)_*(\mathcal{O}_X(mK_X) \otimes \mathcal{J} (||(m-1)K_X||))$$

has rank $P_m(X_w) > 0$.

This lemma will be proved later. We first use it to finish the proof of the first part of Theorem 1.

Assume that $g$ is not birational. Since it is an algebraic fiber space, we have $\dim(W) < \dim(V)$. Hence by the easy addition formula ([M], Corollary 1.7), we have $\dim(V) = \kappa(X) \leq \kappa(X_w) + \dim(W)$, hence $\kappa(X_w) \geq 1$. Since $X$ is of maximal Albanese dimension, $X_w$ is also of maximal Albanese dimension, hence $P_m(X_w) \geq 2$ by Chen and Hacon’s characterization of abelian varieties ([CH4], Theorem 3.2). Then, by Lemma 1.2 the rank of $\mathcal{H}_X$ on $Z$ is $\deg(a_W) \cdot P_m(X_w) \geq 2 \deg(a_W))$, which is a contradiction. This concludes the proof.

In order to prove Lemma 1.2, we begin with an easy lemma.

**Lemma 1.3.** Let $X$ be a smooth projective variety, let $D_1$ be a divisor on $X$ with nonnegative Iitaka dimension, and let $D_2$ be an effective divisor on $X$. We have an inclusion

$$\mathcal{I} (||D_1 + D_2||) \supset \mathcal{I} (||D_1||) \otimes \mathcal{O}_X(-D_2).$$

**Proof.** Take $N > 0$ such that $|ND_1| \neq \emptyset$. Choose a log-resolution

$$\mu : X' \to X$$

for $ND_1$, $ND_2$, and $N(D_1 + D_2)$. Write

$$\mu^*(||ND_1||) = |W_1| + E_1$$
$$\mu^*(||ND_2||) = |W_2| + E_2$$
$$\mu^*(|N(D_1 + D_2)||) = |W_3| + E_3,$$

where $E_1$, $E_2$, and $E_3$ are the fixed divisors and $|W_1|$, $|W_2|$, and $|W_3|$ are free linear series. We have

$$N \mu^*D_2 \supseteq E_2 \quad \text{and} \quad E_1 + E_2 \supseteq E_3,$$
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hence

\[ \mu_*(K_{X'/X} - \left\lfloor \frac{1}{N} E_3 \right\rfloor) \supset \mu_*(K_{X'/X} - \left\lfloor \frac{1}{N} (E_1 + E_2) \right\rfloor) \supset \mu_*(K_{X'/X} - \left\lfloor \frac{1}{N} (E_1 + N \mu^* D_2) \right\rfloor) = \mu_*(K_{X'/X} - \left\lfloor \frac{1}{N} E_1 \right\rfloor) \otimes O_X(-D_2). \]

By the definition of asymptotic multiplier ideal sheaves, this proves Lemma 1.3. \( \Box \)

Proof of Lemma 1.2. We will reduce Lemma 1.2 to Proposition 3.6 in [J]. Since \( Y \) is of maximal Albanese dimension and \( h_Y \) is a model of the Iitaka fibration of \( Y \), by a theorem of Kawamata (see also Theorem 3.2 in [J]), there exists an étale cover \( \pi_Y : \tilde{Y} \to Y \) induced by an étale cover of \( A \) and a commutative diagram:

\[ \begin{array}{ccc} \tilde{Y} & & Y \\ \downarrow \pi_Y & & \downarrow h_Y \\ \hat{W} & & W, \end{array} \]

where \( \hat{W} \) is a smooth projective variety of general type, the rational map \( h_{\hat{W}} \) is a model of the Iitaka fibration of \( \tilde{Y} \), and \( b_{\hat{W}} \) is generically finite and surjective.

Let \( \tilde{X} \) be a connected component of \( X \times_Y \tilde{Y} \), denote by \( \pi_{\tilde{X}} \) the induced morphism \( \tilde{X} \to X \), and denote by \( f_{\tilde{X}} \) the induced morphism \( \tilde{X} \to \tilde{Y} \). Denote by \( k \) and \( k_{\tilde{X}} \) respectively the morphism \( g \circ k_{\tilde{X}} = h_Y \circ f \) and the map \( h_{\hat{W}} \circ f_{\tilde{X}} \). After birational modifications of \( \tilde{X} \), we may suppose that \( k_{\tilde{X}} \) is a morphism such that \( k_{\tilde{X}}(E) \) is a proper subvariety of \( \hat{W} \), where \( E \) is the \( \pi_{\tilde{X}} \)-exceptional divisor. All in all, we have the commutative diagram:

\[ \begin{array}{ccc} \tilde{X} & & X \\ \downarrow k_{\tilde{X}} & & \downarrow f \\ \tilde{Y} & & Y \\ \downarrow \pi_{\tilde{X}} & & \downarrow \pi_Y \\ \hat{W} & & W, \end{array} \]
We then take the Stein factorization:

$$k_\tilde{X} : \tilde{X} \xrightarrow{k_1} W_1 \xrightarrow{b_{W_1}} \hat{W}.$$  

The important point is that $W_1$ is still of general type. Again by taking birational modifications of $\tilde{X}$ and $W_1$, we may assume that $k_1 : \tilde{X} \to W_1$ is an algebraic fiber space between smooth projective varieties. We can apply Proposition 3.6 in [J] to the following diagram:

It follows that the sheaf

$$k_*(\mathcal{O}_X(mK_X) \otimes \mathcal{J}(||(|(m-1)K_X/W + k^*K_W||)) \otimes \mathcal{O}_W(-(m-2)K_W))$$

has rank $P_m(X_w)$. By Lemma 3.4 in [J], the line bundle $(m-1)K_X/W + k^*K_W$ has nonnegative Iitaka dimension. By Lemma 1.3,

$$\mathcal{J}(||(|m-1)K_X||) \supset \mathcal{J}(||(|(m-1)K_X/W + k^*K_W||) \otimes \mathcal{O}_X(-(m-2)k^*K_W)).$$

Therefore,

$$k_*(\mathcal{O}_X(mK_X)) \supset k_*(\mathcal{O}_X(mK_X) \otimes \mathcal{J}(||(|m-1)K_X||)) \supset k_*(\mathcal{O}_X(mK_X) \otimes \mathcal{J}(||(|(m-1)K_X/W + k^*K_W||)) \otimes \mathcal{O}_X(-(m-2)k^*K_W)).$$

Since the rank of the first and the third sheaf are both $P_m(X_w)$, so is the rank of the second. □

2. A COMPLETE DESCRIPTION OF $f : X \to Y$

By using Kawamata’s Theorem 13 in [K] (see also Theorem 3.2 in [J]), we obtain the following complete description of $f$.

**Theorem 2.1.** Let $f : X \to Y$ be a surjective morphism of smooth projective varieties of the same dimension, with $Y$ of maximal Albanese dimension.

If $P_m(X) = P_m(Y)$ for some $m \geq 2$, there exist

- a normal projective variety $V_X$ of general type,
- an abelian variety $A_X$,
- a finite abelian group $G$ which acts faithfully on $V_X$ and on $A_X$ by translations,
- a subgroup $G_2$ of $G$,

such that
• $X$ is birational to $(A_X \times V_X)/G$, where $G$ acts diagonally on $A_X \times V_X$,
• $Y$ is birational to $(A_Y \times V_Y)/G_1$, where
  \* $V_Y = V_X/G_2$ and $A_Y = A_X/G_2$,
  \* $G_1 := G/G_2$ acts diagonally on $A_Y \times V_Y$,
• $f$ is birational to the quotient morphism $(A_X \times V_X)/G \to (A_Y \times V_Y)/G_1$.

Proof. In the diagram (1), we already know that $g : V \to W$ is birational so we may assume that $V = W$ and $g$ is the identity. We then consider the diagram:

(5)\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow h_X & & \downarrow h_Y \\
V & \to & A/K \\
\end{array}
\]

Taking the Stein factorizations for $f$ and $a_Y$, we may assume that $X$ and $Y$ are normal and $f$ and $a_Y$ are finite. Similarly we take the Stein factorization for $Y \to A \to A/K$ and may assume that $V$ is normal and $a_V$ is finite.

By Poincaré reducibility, there exists an isogeny $B \to A/K$ such that $A \times_{A/K} B \simeq K \times B$. We denote by $H$ the kernel of this isogeny. Apply the étale base change $B \to A/K$ to diagram (5) and get

(6)\[
\begin{array}{ccc}
\overline{X} & \to & \overline{Y} \\
\downarrow h_{\overline{X}} & & \downarrow h_{\overline{Y}} \\
\overline{V} & \to & B \\
\end{array}
\]

where
• $\overline{V} = V \times_{A/K} B$ and $\overline{Y} = Y \times_V \overline{V}$ (which are connected because $a_Y$ and $a_V$ are the Albanese maps),
• $\overline{X} = X \times_Y \overline{Y}$ (which is also connected because $\overline{X} = X \times_Y \overline{Y} = X \times_Y (Y \times_V \overline{V}) = X \times_V \overline{V}$),
• $h_X : X \to V$ is an algebraic fiber space.

Let $A_X$ and $A_Y$ be the respective general fibers of $h_{\overline{X}}$ and $h_{\overline{Y}}$. We have the following induced diagram from (6):

\[
\begin{array}{ccc}
A_X & \to & A_Y \\
\beta & & \alpha_Y \\
\end{array}
\]

By Proposition 2.1 in [HP], $A_X$ and $A_Y$ are birational to abelian varieties. Hence the morphisms $\alpha_X$ and $\alpha_Y$ are birationally equivalent
to étale covers. Since $a_{Y}$ and $a_{Y} \circ f$ are finite, $\alpha_{X}$ and $\alpha_{Y}$ are also finite. Thus $\alpha_{X}$ and $\alpha_{Y}$ are isogenies of abelian varieties by Zariski's Main Theorem. We denote by $\tilde{G}$, $\tilde{G}_{1}$, and $\tilde{G}_{2}$ the abelian groups $\text{Ker}(A_{X} \to K)$, $\text{Ker}(A_{Y} \to K)$, and $\text{Ker}(A_{X} \to A_{Y})$ respectively. Then $\tilde{G}_{1} = \tilde{G}/\tilde{G}_{2}$ and $A_{Y} = A_{X}/\tilde{G}_{2}$. Let $k \in K$ be a general point, let $V_{Y}$ be the normal variety $a_{Y}^{-1}(k \times B)$, and let $V_{X}$ be the normal variety $f^{-1}a_{Y}^{-1}(k \times B)$.

We know that $A_{X}$ and $A_{Y}$ respectively act on $X$ and $Y$ in such a way that $f$ is equivariant for the $A_{X}$-action on $X$ and the $A_{Y}$-action on $Y$. Furthermore, the actions induce a faithful $\tilde{G}$-action on $V_{X}$ and a faithful $\tilde{G}_{1}$-action on $V_{Y}$, and we have an $A_{X}$-equivariant isomorphism $X \simeq (A_{X} \times V_{X})/\tilde{G}$ and an $A_{Y}$-equivariant isomorphism $Y \simeq (A_{Y} \times V_{Y})/\tilde{G}_{1}$, where $\tilde{G}$ acts on $A_{X} \times V_{X}$ diagonally and $\tilde{G}_{1}$ acts on $A_{Y} \times V_{Y}$ diagonally.

The induced morphism

\[
\begin{array}{ccc}
V_{X} & \xrightarrow{f|_{V_{X}}} & V_{Y} \\
\downarrow h_{X} & & \downarrow h_{Y} \\
V & \xrightarrow{\text{id}} & V
\end{array}
\]

is equivariant for the $\tilde{G}$-action on $V_{X}$ and the $\tilde{G}_{1}$-action on $V_{Y}$. Thus $V_{Y} = V_{X}/\tilde{G}_{2}$ and $f|_{V_{X}}$ is the quotient morphism.

Thus we obtain $A_{Y} = A_{X}/\tilde{G}_{2}$ and $V_{Y} = V_{X}/\tilde{G}_{2}$, and $f : X \to Y$ is the quotient morphism $(A_{X} \times V_{X})/\tilde{G} \to (A_{Y} \times V_{Y})/\tilde{G}_{1}$, so $f : X = (A_{X} \times V_{X})/\tilde{G} \to Y = (A_{Y} \times V_{Y})/\tilde{G}_{1}$ is also the quotient morphism.

Let $G = \text{Ker}(A_{X} \times B \to A)$ and $G_{1} = \text{Ker}(A_{Y} \times B \to A)$. We have exact sequences of groups

\[1 \to \tilde{G} \to G \to H \to 1 \quad \text{and} \quad 1 \to \tilde{G}_{1} \to G_{1} \to H \to 1.\]

Then $X = (A_{X} \times V_{X})/G$ and $Y = (A_{Y} \times V_{Y})/G_{1}$, and $f$ is the quotient map. This proves Theorem 2.1 with $G_{2} = \tilde{G}_{2} \subset G$.

\[\square\]

3. Examples

In the next two examples, we see that the conclusion of our theorem does not hold in general, even for surfaces of general type.

**Example 3.1.** Let $C_{1}$ and $C_{2}$ be smooth projective curves of genus 2 with respective hyperelliptic involutions $i_{1}$ and $i_{2}$. Define $Y$ to be
the minimal resolution of singularities of \((C_1 \times C_2)/(i_1, i_2)\). Let \(X\) be the blow-up of \(C_1 \times C_2\) at the 36 fixed points of \((i_1, i_2)\). There is a 2-to-1 morphism \(f : X \to Y\). We have \(K_Y^2 = \frac{1}{2}K_{C \times C}^2 = 4\) and \(c_2(Y) = \frac{1}{2}(c_2(C_1 \times C_2) - 36) + 72 = 56\). Since \(Y\) is a minimal surface, we have \(P^2(Y) = K_Y^2 + \frac{1}{12}(K_Y^2 + c_2(Y)) = 9\). We also have \(P^2(X) = P^2(C_1 \times C_2) = 3 \times 3 = 9\). Hence we have a nonbirational morphism \(f : X \to Y\) between smooth projective surfaces of general type with \(q(Y) = 0\) (so that \(Y\) does not have maximal Albanese dimension!) and \(P^2(X) = P^2(Y) = 9\).

**Remark 3.2.** It turns out that the situation in the case of surfaces of general type can be completely worked out. More precisely, one can show that if \(f : S \to T\) is a nonbirational morphism between smooth projective surfaces of general type such that \(P^m(S) = P^m(T)\) for some \(m \geq 2\), then \(m = 2\) and one of the following occurs:

1) either \(S\) is birational to the product of two smooth projective curves of genus 2, and \(f\) is birationally equivalent to the quotient by the diagonal hyperelliptic involution (see Example 3.1 above);

2) or \(S\) is birational to the theta divisor of the Jacobian of a smooth projective curve of genus 3 and \(f\) is birationally equivalent to the bicanonical map of \(S\);

3) \(S\) is birational to a double cover of a principally polarized abelian surface branched along a divisor in \(|2\Theta|\) having at most double points and \(f\) is birationally equivalent to the bicanonical map of \(S\).

In higher dimensions, we have many more examples.

**Example 3.3.** (Compare with [Ko1], Proposition 8.6.1) We denote by \(P(a_0^s, \ldots, a_k^s)\) the weighted projective space with \(s_j\) coordinates of weight \(a_i\) (see [D]). For any integer \(k \geq 3\), denote by \(P_X\) the weighted projective space \(P(1, (2k)^{4k+5}, (2k + 1)^{4k-3})\) with coordinates \(x_i\) and by \(P_Y\) the weighted projective space \(P(2, (2k)^{4k+5}, (2k + 1)^{4k-3})\) with coordinates \(y_i\). As in the proof of Proposition 8.6.1 in [Ko1], one can check that \(P_X\) and \(P_Y\) both have canonical singularities. There is a natural degree-2 morphism \(\varepsilon : P_X \to P_Y\) defined by \(y_0 = x_0^2\) and \(y_i = x_i\) for \(i \geq 1\).

Let \(Y'\) be a general hypersurface of weighted degree \(d = 16k^2 + 8k\) in \(P_Y\) and let \(X'\) be the pull-back by \(\varepsilon\) of \(Y'\). Since \(2k(2k + 1)d\) and \(Y'\) is general, \(X'\) is also general and both \(X'\) and \(Y'\) have canonical singularities. Take resolutions \(X \to X'\) and \(Y \to Y'\) such that \(\varepsilon\) induces a degree-2 morphism \(f : X \to Y\). The canonical sheaves are
\[ \omega_{X'} = \mathcal{O}_{X'}(2) \text{ and } \omega_{Y'} = \mathcal{O}_{Y'}(1). \] Since both \( X' \) and \( Y' \) have canonical singularities, we have, for any integer \( m \geq 0, \)

\[ P_m(X) = h^0(X', \mathcal{O}_{X'}(2m)) \text{ and } P_m(Y) = h^0(Y', \mathcal{O}_{Y'}(m)). \]

It follows from Theorem 1.4.1 in \([D]\) that for \( m \) even and \( < 2k \), we have \( P_m(X) = P_m(Y) = 1. \) By Theorem 4.2.2 and Corollary 2.3.6 in \([D]\), \( q(X) = q(Y) = 0. \)

Under the assumptions of our theorem, one might expect that \( f : X \to Y \) be birational to an étale morphism. However the example below (see also Example 1 in \([CH5]\)) shows that this is not the case in general.

**Example 3.4.** Let \( G = \mathbb{Z}_{rs} \) and let \( G_2 = s\mathbb{Z}_{rs} \) be the subgroup of \( G \) generated by \( s \), with \( s \geq 2 \) and \( r \geq 2. \) Let \( G_1 = G/G_2 \simeq \mathbb{Z}_s \). Consider an elliptic curve \( E \), let \( B_1 \) and \( B_2 \) be two points on \( E \), and let \( L \) be a line bundle of degree 1 such that \( B = (rs - a)B_1 + aB_2 \in |tmL| \) with \( 1 \leq a \leq m - 2 \) and \( (a, rs) = 1. \) Taking the normalization of the \((rs)\)-th root of \( B \), we get a smooth curve \( C \) and a Galois cover \( \pi : C \to E \) with Galois group \( G \). By construction, \( \pi \) ramifies at two points, \( B_1 \) and \( B_2 \).

Following \([B]\) §VI.12, we have \( h^0(C, \omega_C^2) = 2. \)

Let \( L^{(i)} \) be \( L(-\lfloor \frac{i}{rs} \rfloor) \) and denote \( (L^{(i)})^{-1} \) by \( L^{-(i)} \). Then, by Proposition 9.8 in \([Ko2]\),

\[ \pi_* \mathcal{O}_C = \bigoplus_{i=0}^{rs-1} L^{-(i)}. \]

Let \( C_1 \) be the curve

\[ \text{Spec}(\bigoplus_{i=0}^{s-1} L^{-(ri)}), \]

where \( \bigoplus_{i=0}^{s-1} L^{-(ri)} \) has the subalgebra structure of \( \pi_* \mathcal{O}_C \). Consider the Stein factorization

\[ \pi : C \twoheadrightarrow C_1 \xrightarrow{\pi_1} E. \]

Then \( C_1 = C/G_2 \) and \( \pi_1 \) is a Galois cover with Galois group \( G_1 \) which also ramifies only at \( B_1 \) and \( B_2 \). Hence we again have

\[ h^0(C_1, \omega_{C_1}^2) = 2. \]

Finally we take an abelian variety \( K \) such that \( G \) acts freely on \( K \) by translations and set \( K_1 = K/G_2. \) Let

\[ \tilde{X} = C \times K \quad \text{and} \quad \tilde{Y} = C_1 \times K \]

and

\[ X = \tilde{X}/G = (C \times K)/G \quad \text{and} \quad Y = \tilde{Y}/G = (C_1 \times K_1)/G_1, \]
where $G$ and $G_1$ act diagonally. Hence $\tilde{X}$ and $\tilde{Y}$ are étale covers of $X$ and $Y$ respectively. There is a natural finite dominant morphism $f : X \to Y$ of degree $r$. Since its lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ is not étale, $f$ is not étale.

Since $H^0(X, \omega_X^2) \simeq H^0(\tilde{X}, \omega_{\tilde{X}}^2)^G \simeq H^0(C, \omega_C^2)^G$ and $H^0(Y, \omega_Y^2) \simeq H^0(\tilde{Y}, \omega_{\tilde{Y}}^2)^{G_1} \simeq H^0(C_1, \omega_{C_1}^2)^{G_1}$, we have

$$P_2(X) = P_2(Y) = 2.$$  

4. PLURICANONICAL MAPS OF VARIETIES OF MAXIMAL ALBANESE DIMENSION

Let $X$ be a smooth projective variety of maximal Albanese dimension. As mentioned in the introduction, Chen and Hacon proved in [CH1] and [CH3] that $\phi_{6K_X}(X)$ has dimension $\kappa(X)$; if $X$ is moreover of general type, $\phi_{6K_X}$ is birational onto its image. They also showed that if $\chi(\omega_X) > 0$, the map $\phi_{3K_X}$ is already birational onto its image. Pareschi and Popa provided in [PP2], §6, a conceptual approach to these theorems based on their regularity and vanishing theorems.

We prove a unifying statement for varieties with maximal Albanese dimension which are not necessarily of general type. The proof is parallel to that of Pareschi and Popa.

In this section, we will always assume $f : X \to Y$ is a birational model of the Iitaka fibration of $X$.

**Theorem 4.1.** If $X$ is a smooth projective variety with maximal Albanese dimension, the linear system $|\mathcal{O}_X(5K_X) \otimes f^*P|$ induces the Iitaka fibration. In particular, $\phi_{5K_X}$ is a model of the Iitaka fibration of $X$.

**Proof.** We may as in ([I]) assume that we have a diagram

$$\begin{array}{ccc}
X & \longrightarrow & \text{Alb}(X) \\
\downarrow f & & \downarrow f_* \\
Y & \longrightarrow & \text{Alb}(Y)
\end{array}$$

where $f$ is the Iitaka fibration of $X$ and $a_X$ and $a_Y$ are the respective Albanese morphisms of $X$ and $Y$.

Since $f$ is a model of the Iitaka fibration of $X$, $f_*(\omega_X^2 \otimes \mathcal{J}(\|K_X\|))$ is a torsion-free rank 1 sheaf on $Y$. We now use the following lemma ([I], Lemma 2.1):
Lemma 4.2. Suppose that $f : X \to Y$ is a surjective morphism between smooth projective varieties, $L$ is a $\mathbb{Q}$-divisor on $X$, and the Iitaka model of $(X, L)$ dominates $Y$. Assume that $D$ is a nef $\mathbb{Q}$-divisor on $Y$ such that $L + f^*D$ is a divisor on $X$. Then we have

$$H^i(Y, R^j f_*(\mathcal{O}_X(K_X + L + f^*D) \otimes \mathcal{I}(||L|| \otimes Q))) = 0,$$

for all $i \geq 1$, $j \geq 0$, and all $Q \in \text{Pic}^0(X)$.

By Lemma 4.2, we have

$$H^i(Y, f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))) = 0$$

for all $i \geq 1$ and $Q \in \text{Pic}^0(X)$. As in Lemma 2.6 in [J], $R^j a_Y_*(f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))) = 0$ for all $j \geq 1$. Hence

$$H^i(\text{Alb}(Y), a_Y_*(f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q)))) = H^i(Y, f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))) = 0,$$

for all $i \geq 1$ and $Q \in \text{Pic}^0(X)$. Thus for any $Q \in V(\omega_X) \subset \text{Pic}^0(X)$, $a_Y_*(f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))$ is a nonzero IT-sheaf of index 0 and in particular, it is $M$-regular. By [PP2, Corollary 5.3], $a_Y_*(f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))$ is continuously globally generated. Since $a_Y$ is generically finite, the exceptional locus $Z_1$ of $a_Y$ is a proper closed subset of $Y$. Then $f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))$ is continuously globally generated away from $Z_1$. By definition, this means that for any open subset $V \subset \text{Pic}^0(Y)$, the evaluation map

$$\bigoplus_{P \in V} H^0(Y, f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q)) \otimes P) \otimes P^{-1} \to f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X|| \otimes Q))$$

is surjective away from $Z_1$.

Now we claim that there exists an open dense subset $U \subset Y - Z_1$ such that the sheaf

$$a_Y_*(\mathcal{I}_y \otimes f_*(\mathcal{O}_X(3K_X) \otimes \mathcal{I}(||2K_X||)))$$

is $M$-regular for any $y \in U$.

We first assume the claim and finish the proof of the theorem.

We conclude by the claim that $\mathcal{I}_y \otimes f_*(\mathcal{O}_X(3K_X) \otimes \mathcal{I}(||2K_X||))$ is continuously globally generated away from $Z_1$. Denote respectively by $\mathcal{L}$ and $\mathcal{L}_1$ the rank-1 torsion-free sheaves $f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{I}(||K_X||))$ and $f_*(\mathcal{O}_X(3K_X) \otimes \mathcal{I}(||2K_X||))$. Let $U_1$ be a dense open subset of $Y - Z_1$ such that $\mathcal{L}$ and $\mathcal{L}_1$ are locally free on $U_1$. Then by [PP3, Proposition 2.12], $\mathcal{L}_1 \otimes \mathcal{L}$ is very ample over $U \cap U_1$. We have $\mathcal{L} \otimes \mathcal{L}_1 \hookrightarrow \mathcal{I}_y \otimes f_*(\mathcal{O}_X(3K_X) \otimes \mathcal{I}(||2K_X||))$.
Thus, \( f_*(\mathcal{O}_X(5K_X)) \) is very ample on a dense open subset of \( Y \). This concludes the proof of the theorem.

For the claim, let

\[
U \subset U_1 \bigcap (Y - \bigcup T_i \cap Q) \cap \text{Bs}(|f_*(\omega_X \otimes Q)|)
\]

be any dense open subset of \( Y \), where \( T_i \) runs through all the components of \( V_0(\omega_X) \) and \( \text{Bs}(|f_*(\omega_X \otimes Q)|) \) denotes the locus where the evaluation map

\[
H^0(Y, f_*(\omega_X \otimes Q)) \otimes \mathcal{O}_Y \to f_*(\omega_X \otimes Q)
\]

is not surjective. For each component \( T_i \) of \( V_0(\omega_X) \), we may write

\[
T_i = P_i + f^*S_i,
\]

where \( S_i \) is a subtorus of \( \text{Pic}^0(Y) \) and \( P_i \in \text{Pic}^0(X) \) (see \([GL]\)).

Again, by Lemma 4.2, we have

\[
H^i(Y, L_1 \otimes Q) = 0
\]

for all \( i \geq 1 \) and any \( Q \in \text{Pic}^0(Y) \). For \( y \in U \), consider the exact sequence

\[
0 \to \mathcal{I}_y \otimes L_1 \to L_1 \to \mathcal{C}_y \to 0.
\]

We push forward this short sequence to \( \text{Alb}(Y) \). Since \( y \in U \), we have

\[
0 \to a_{Y*}(\mathcal{I}_y \otimes L_1) \to a_{Y*}L_1 \to \mathcal{C}_{aY(y)} \to 0.
\]

Hence \( H^i(Y, a_{Y*}(\mathcal{I}_y \otimes L_1) \otimes Q) = 0 \) for any \( i \geq 2 \) and \( Q \in \text{Pic}^0(Y) \).

We now assume that \( a_{Y*}(\mathcal{I}_y \otimes L_1) \) is not \( M \)-regular. Then by definition of \( M \)-regularity, we have

\[
\text{codim}_{\text{Pic}^0(Y)} V_1(a_{Y*}(\mathcal{I}_y \otimes L_1)) \leq 1.
\]

Hence \( y \) is a base-point of all sections in \( H^0(Y, L_1 \otimes P) \), for all \( s \in V_1(a_{Y*}(\mathcal{I}_y \otimes L_1)) \).

On the other hand, by \([J]\) Lemma 2.2,

\[
\dim H^0(X, \mathcal{O}_X(3K_X) \otimes f^*P)
\]

is constant for \( P \in \text{Pic}^0(Y) \). Then,

\[
\dim H^0(Y, L_1 \otimes P) = \dim H^0(Y, L_1) = \dim H^0(X, \mathcal{O}_X(3K_X)) = \dim H^0(X, \mathcal{O}_X(3K_X) \otimes f^*P).
\]

Hence the inclusion

\[
H^0(X, \mathcal{O}_X(3K_X) \otimes \mathcal{I}(|2K_X|) \otimes f^*P) \to H^0(X, \mathcal{O}_X(3K_X) \otimes f^*P)
\]

is an isomorphism. Therefore, \( y \in \text{Bs}(|f_*(\mathcal{O}_X(3K_X) \otimes P)|) \), for all \( s \in V_1(a_{Y*}(\mathcal{I}_y \otimes L_1)) \).
Since $y \in Y - \bigcup_{i, \cap Q \not\in T_i} \text{Bs}(|f_*(\omega_X \otimes Q)|)$, let $V_i \subset S_i$ be a dense open subset such that $y \notin \text{Bs}(|f_*(\omega_X \otimes Q)|)$, for any $Q \in P_i + f^*V_i$.

We may shrink $U$ so that $f_*(\mathcal{O}_X(KX) \otimes P_i)$ and $f_*(\mathcal{O}_X(2KX) \otimes P_i^{-1})$ are locally free on $U$ for all $i$. Moreover, we can require that, for each $i$, the multiplication

$$f_*(\mathcal{O}_X(KX) \otimes P_i) \otimes f_*(\mathcal{O}_X(2KX) \otimes P_i^{-1}) \rightarrow f_*(\mathcal{O}_X(3KX))$$

is an isomorphism on $U$, since both sheaves are of rank 1.

We then conclude that $y$ is a base point of all sections of

$$H^0(Y, f_*(\mathcal{O}_X(2KX) \otimes P_i^{-1}) \otimes Q')$$

where $Q' \in V_i(a_{Y*}(\mathcal{I}_y \otimes \mathcal{L}_1)) - V_i$.

We may further shrink $U$ so that

$$f_*(\mathcal{O}_X(2KX) \otimes \mathcal{J}(||KX||) \otimes P_i^{-1})|_U = f_*(\mathcal{O}_X(2KX) \otimes P_i^{-1})|_U$$

is locally free for each $i$. Then $y \in U$ belongs to

$$\text{Bs}|f_*(\mathcal{O}_X(2KX) \otimes \mathcal{J}(||KX||) \otimes P_i^{-1}) \otimes Q'|$$

for each $Q' \in V_i(a_{Y*}(\mathcal{I}_y \otimes \mathcal{L}_1)) - V_i$.

By [CH1], Theorem 1, the union of all the $S_i$ generates $\text{Pic}^0(Y)$. Hence by [K], for some $i$, $V_i(a_{Y*}(\mathcal{I}_y \otimes \mathcal{L}_1)) - V_i$ contains an open subset of $\text{Pic}^0(Y)$ and this contradicts the fact that $f_*(\mathcal{O}_X(2KX) \otimes \mathcal{J}(||KX||) \otimes P_i^{-1})$ is continuous globally generated away from $Z_1$. This concludes the proof of the claim. 

Our Theorem 4.1 is just an analog of Theorem 6.7 in [PP2]. The main point is just that $a_{Y*}f_*(\mathcal{O}_X(2KX) \otimes \mathcal{J}(||KX||))$ is $M$-regular. On the other hand, if $X$ is of general type, of maximal Albanese dimension, and if moreover $a_X(X)$ is not ruled by tori, Pareschi and Popa proved that $a_{X*}\omega_X$ is $M$-regular, which is the main ingredient of the proof of Theorem 6.1 in [PP2]. If $X$ is not of general type, $a_X(X)$ is always ruled by tori of dimension $n - \kappa(X)$. But we still have:

**Theorem 4.3.** If $X$ is a smooth projective variety with maximal Albanese dimension $n$, and if its Albanese image $a_X(X)$ is not ruled by tori of dimension $> n - \kappa(X)$, the map $\phi_{3KX}$ is a model of the Iitaka fibration of $X$.

**Proof.** We just need to show that under our assumptions, and with the notation of the proof of Theorem 4.1, $a_{Y*}f_*(\omega_X)$ is $M$-regular. The rest is the same as the proof of Theorem 4.1. By Kawamata’s theorem [K, Theorem 13], we have the following commutative diagram:
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\[
\begin{array}{cccccc}
\tilde{Y} \times \tilde{K} & \xrightarrow{\mu} & \tilde{X} & \xrightarrow{\pi_X} & X & \xrightarrow{a_X} & \text{Alb}(X) \\
\downarrow \text{pr}_1 & & \downarrow \bar{f} & & \downarrow f & & \downarrow f_* \\
\tilde{Y} & \xrightarrow{b_Y} & \tilde{Y} & \xrightarrow{a_Y} & Y & \xrightarrow{a_Y} & \text{Alb}(Y),
\end{array}
\]

where \(\pi_X\) is birationally equivalent to a finite étale cover of \(X\) induced by isogeny of \(\text{Alb}(X)\), \(\mu\) is a birational morphism, \(\tilde{K}\) is an abelian variety isogenous to \(\ker f_*\), \(\tilde{X}\) is a smooth projective variety of general type, and \(b_Y\) is generically finite. We set \(g_Y = a_Y \circ b_Y\).

Since \(a_X(X)\) is not ruled by tori of dimension \(> n - \kappa(X)\), we conclude that \(g_Y(\tilde{Y}) = a_Y(Y)\) is not ruled by tori. We make the following:

**Claim:** \(g_Y \ast \omega_{\tilde{Y}}\) is \(M\)-regular.

We first see how the Claim implies Theorem 4.3. Since \(\tilde{K}\) is an abelian variety, we have obviously \(\text{pr}_1 \ast \omega_{\tilde{Y} \times \tilde{K}} = \omega_{\tilde{Y}}\). Hence

\[
g_Y \ast \text{pr}_1 \ast \omega_{\tilde{Y} \times \tilde{K}} = g_Y \ast \text{pr}_1 \ast \mu \ast \omega_{\tilde{X}} = a_Y \ast f_* \pi_X \ast \omega_{\tilde{X}}
\]

is \(M\)-regular on \(\text{Alb}(Y)\). On the other hand, \(\omega_X\) is a direct summand of \(\pi_X \ast \omega_{\tilde{X}}\) since \(\pi_X\) is birationally equivalent to an étale cover. Therefore, \(a_Y \ast f_* \omega_X\) is a direct summand of \(g_Y \ast \text{pr}_1 \ast \omega_{\tilde{Y} \times \tilde{K}}\) and hence is \(M\)-regular.

We now prove the Claim.

We first define the following subset of \(\text{Pic}^0(Y)\) for any \(i \geq 0:\)

\[
V_i(\tilde{Y}, \text{Pic}^0(Y)) := \{ P \in \text{Pic}^0(Y) : H^i(\tilde{Y}, \omega_{\tilde{Y}} \otimes g_Y^* P) \neq 0 \}.
\]

Since the image of \(g_Y : \tilde{Y} \to \text{Alb}(Y)\) is not ruled by tori, the same argument in the last part of the proof of Theorem 3 in [EL] shows that \(\text{codim}_{\text{Pic}^0(Y)} V_i(\tilde{Y}, \text{Pic}^0(Y)) > i\) for any \(i \geq 1\). On the other hand, by Grauert-Riemenschneider vanishing, \(R^i g_Y \ast \omega_{\tilde{Y}} = 0\) for any \(i \neq 0\). Thus

\[
H^i(\tilde{Y}, \omega_{\tilde{Y}} \otimes g_Y^* P) \simeq H^i(\text{Alb}(Y), g_Y \ast \omega_{\tilde{Y}} \otimes P).
\]

Hence we have \(V_i(g_Y \ast \omega_{\tilde{Y}}) = V_i(\tilde{Y}, \text{Pic}^0(Y))\) as subset of \(\text{Pic}^0(Y)\). This finishes the proof of the Claim. \(\square\)

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