Dynamical creation of entanglement versus disentanglement in a system of three-level atoms with vacuum-induced coherences

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The dynamics of entanglement between three-level atoms coupled to the common vacuum is investigated. We show that the collective effects such as collective damping, dipole-dipole interaction and the cross coupling between orthogonal dipoles, play a crucial role in the process of creation of entanglement. In particular, the additional cross coupling enhances the production of entanglement. For the specific initial states we find that the effect of delayed sudden birth of entanglement, recently invented by Ficek and Tanaś [Phys. Rev. A 77, 054301(2008)] in the case of two-level atoms, can also be observed in the system. When the initial state is entangled, the process of spontaneous emission causes destruction of correlations and its disentanglement. We show that the robustness of initial entanglement against the noise can be changed by local operations performed on the state.

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I. INTRODUCTION

In a system of coupled multi-level atoms having closely lying energy states and interacting with the vacuum, quantum interference between different radiative transitions can occur, resulting in coherences in a system which are known as vacuum-induced coherences. For example, when the distance between atoms is comparable to the wavelength of the emitted radiation, transitions, resulting in disentanglement of initially entangled pairs of atoms. Both processes crucially depend on the interatomic distance compared to the wavelength of the emitted radiation. For large separation we expect that the collective properties of two atoms are negligible and dissipation causes disentanglement. On the contrary, for small distance the collective effects are so strong that they can partially overcome decoherence. As a result, the system can decay to a stationary state which can be entangled, even if the initial state was separable.

In the present paper, we study the case of distance comparable to the radiation wavelength \( \lambda \). Although the dynamics brings all initial states into the asymptotic state in which both atoms are in their ground states, still there can be some transient entanglement between the atoms. In particular we show that the dynamical creation of entanglement is possible in a system where only one atom is in the excited state. Moreover the production of entanglement is enhanced, when the cross coupling between orthogonal dipoles is present. In the more accessible initial state when the both atoms are excited, and if the cross coupling is absent, the interesting phenomenon of delayed sudden birth of entanglement [11] can be observed: unentangled atoms become entangled after some finite time, despite of the fact that the correlation between the atoms existed earlier. On the other hand, cross coupling causes that the entanglement starts to build up immediately. We consider also the process of disentanglement of initially entangled states in the absence of vacuum induced coherences. Analogously to the case of two-level atoms (see e.g. [12]), there are specific entangled states of our system which decay much slower than the other states. In the limit of small separation, those states decouple from the environment and therefore are stable. They are called (generalized) antisymmetric Dicke states [13] and play the crucial role in characterizing disentanglement properties of given initial state. In particular, the class of maximally entangled states of two-qutrits i.e. (generalized) Bell states [14] can be divided into two subsets. The first set contains those states which have no populations in antisymmetric Dicke states, and they decay rapidly. The remaining states have equal populations in stable Dicke states and decay much slower. Since all Bell states are locally equivalent, local operations performed on the states may change the robustness of entanglement against the noise.
II. MODEL DYNAMICS

We start with a short description of the model studied by Agarwal and Patnaik [3]. Consider two identical three-level atoms (A and B) in the V configuration. The atoms have two near-degenerate excited states |1α⟩, |2α⟩ (α = A, B) and ground states |3α⟩. Assume that the atoms interact with the common vacuum and that transition dipole moments of atom A are parallel to the transition dipole moments of atom B. Due to this interaction, the process of spontaneous emission from two excited levels to the ground state take place in each individual atom but a direct transition between excited levels is not possible. Moreover, the coupling between two atoms can be produced by the exchange of the photons. As it was shown by Agarwal and Patnaik, in such atomic system there is also possible the radiative process in which atom A in the excited state |1A⟩ loses its excitation which in turn excites atom B to the state |2B⟩. This effect manifests by the cross coupling between radiation transitions with orthogonal dipole moments. The evolution this atomic system can be described by the following master equation [3]

\[
\frac{d\rho}{dt} = (L^A + L^B + L^{AB})\rho
\]

where for α = A, B we have

\[
L^α ρ = \sum_{k=1}^{2} \gamma_k(2\sigma_{k3}^α ρ \sigma_{k3}^α - \sigma_{k3}^α ρ \sigma_{k3}^α - \rho \sigma_{k3}^α \sigma_{k3}^α)
\]

and

\[
L^{AB} ρ = \sum_{k=1}^{2} \sum_{α,A,B} \Gamma_{k} \left(2\sigma_{k3}^α ρ \sigma_{k3}^α - \sigma_{k3}^α ρ \sigma_{k3}^α - \rho \sigma_{k3}^α \sigma_{k3}^α\right) + i \sum_{k=1}^{2} \Omega_{k} \left[\sigma_{k3}^α \sigma_{k3}^α - \sigma_{k3}^α \sigma_{k3}^α - \rho \sigma_{k3}^α \sigma_{k3}^α\right] + \Gamma_{vc} \sum_{α,A,B} \left(2\sigma_{32}^α ρ \sigma_{32}^α - \sigma_{32}^α ρ \sigma_{32}^α - \rho \sigma_{32}^α \sigma_{32}^α\right) + i \Omega_{vc} \sum_{α,A,B} \left[\sigma_{23}^α \sigma_{31}^α + \sigma_{32}^α \sigma_{13}^α\right]
\]

In the equations (II.2) and (II.3), -α is A for α = B and B for α = A, σ_{jk}^α is the transition operator from |kα⟩ to |jα⟩ and the coefficient γ_{j3} represents the single atom spontaneous decay rate from the state |j⟩ (j = 1, 2) to the state |3⟩. The coefficients Γ_{j3} and Ω_{j3} are related to the coupling between two atoms and are the collective damping and the dipole-dipole interaction potential, respectively. The coherence terms Γ_{vc} and Ω_{vc} are cross coupling coefficients, which couple a pair of orthogonal dipoles. This cross coupling between two atoms strongly depend on the relative orientation of the atoms. To see this, we put the atom A at the origin of coordinate system and the position of the atom B is give by the vector \( \vec{R} \) which makes an angle \( \phi \) with the x axis and an angle \( \theta \) with the z axis (see FIG. 1). Assume that the dipole moments \( \vec{d}_{13} \) and \( \vec{d}_{23} \) of transitions |1α⟩ → |3α⟩ and |2α⟩ → |3α⟩ are given by

\[
\vec{d}_{13} = \hat{x}d, \quad \vec{d}_{23} = \hat{y}d
\]

Since the states |1α⟩ and |2α⟩ are closely lying, the transition frequencies Ω_{13} and Ω_{23} satisfy

\[
Ω_{13} ≈ Ω_{23} = \omega_0
\]

Similarly, the spontaneous decay rates

\[
γ_{13} ≈ γ_{23} = γ
\]

As was shown in Ref.[3], the coefficients in (II.3) can be written as

\[
Γ_{13} = \frac{3γ}{2} \left(P_i - \sin^2 \theta \cos^2 Φ Q_i\right)
\]

\[
Ω_{13} = \frac{3γ}{2} \left(P_r - \sin^2 \theta \cos^2 Φ Q_r\right)
\]

\[
Γ_{23} = \frac{3γ}{2} \left(P_i - \sin^2 \theta \sin^2 Φ Q_i\right)
\]

\[
Ω_{23} = \frac{3γ}{2} \left(P_r - \sin^2 \theta \sin^2 Φ Q_r\right)
\]

\[
Γ_{vc} = -\frac{3γ}{2} \sin^2 θ \sin \phi \cos \phi Q_i
\]

\[
Ω_{vc} = -\frac{3γ}{2} \sin^2 θ \sin \phi \cos \phi Q_r
\]

where for \( \xi = Rω_0/c \)

\[
P_i = \frac{\sin ξ}{ξ} + \frac{\cos ξ}{ξ^2} - \frac{\sin ξ}{ξ^3}, \quad Q_i = \frac{\sin ξ}{ξ} + \frac{3\cos ξ}{ξ^2} - \frac{3\sin ξ}{ξ^3}
\]

\[
P_r = \frac{\cos ξ}{ξ} - \frac{\sin ξ}{ξ^2} - \frac{\cos ξ}{ξ^3}, \quad Q_r = \frac{\cos ξ}{ξ} - \frac{3\sin ξ}{ξ^2} - \frac{3\cos ξ}{ξ^3}
\]

From the formulas (II.4) and (II.5) it follows that the coupling coefficients are small for large distance between the atoms and
tend to zero for $R \to \infty$. On the other hand, when $R \to 0$, $\Omega_{13}, \Omega_{23}$ and $\Omega_{vc}$ diverge, whereas

$$\Gamma_{13}, \Gamma_{23} \to \gamma \quad \text{and} \quad \Gamma_{vc} \to 0$$

In the following we will consider two special configurations of atomic system.

**Configuration I:** $\theta = \pi$ i.e. both atoms lie along the $z$ axis and $\phi = \pi/4$. In that case

$$\Gamma_{13} = \Gamma_{23}, \quad \Omega_{13} = \Omega_{23}$$

and the coherence terms $\Gamma_{vc} = \Omega_{vc} = 0$.

**Configuration II:** $\theta = \pi/2$ i.e. both atoms lie on the $xy$ plane and $\phi = \pi/4$. In that case

$$\Gamma_{13} = \Gamma_{23}, \quad \Omega_{13} = \Omega_{23}$$

and the coherence terms $\Gamma_{vc} \neq 0, \Omega_{vc} \neq 0$.

The time evolution of the initial state of two - atomic system is given by the semi - group $\{T_t\}_{t\geq0}$ of completely positive mappings acting on density matrices, generated by $L^A + L^B + L^{AB}$. The properties of this semi - group crucially depend on the distance between the two atoms and the geometry of the system. It can be shown by a direct calculation, that irrespective to the geometry, when the distance is large (compared to the radiation wavelength $\lambda$), the semi - group $\{T_t\}_{t\geq0}$ is uniquely relaxing with the asymptotic state $|3_A\rangle \otimes |3_B\rangle$. Thus, for any initial state, its entanglement approaches 0 when $t \to \infty$. But still there can be some transient entanglement between the atoms. In the following, we study in details time evolution of some classes of initial states and show how the creation of entanglement as well as the process of disentanglement are sensitive to the geometry of the system.

### III. NEGATIVITY

To describe the process of creation or destruction of entanglement between the atoms, we need the effective measure of mixed - state entanglement. For such a measure we take a computable measure of entanglement proposed in [13]. The measure is based on the trace norm of the partial transposition $\rho^{PT}$ of the state $\rho$. From the Peres - Horodecki criterion of separability [16][17], it follows that if $\rho^{PT}$ is not positive, then $\rho$ is entangled and one defines the negativity of the state $\rho$ as

$$N(\rho) = \frac{||\rho^{PT}|| - 1}{2} \quad \text{(III.1)}$$

$N(\rho)$ is equal to the absolute value of the sum of the negative eigenvalues of $\rho^{PT}$ and is an entanglement monotone, but it cannot detect bound entangled states [18].

Although negativity of a given state is easy to compute numerically, the analytical formulas for general mixed states of two qutrits can be only obtained for some limited classes of states. The density matrix $\rho$ which we consider to compute negativity is defined on the space $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\rho$ is written in the basis of product states

$$|j_A\rangle \otimes |k_B\rangle, \quad j, k = 1, 2, 3 \quad \text{(III.2)}$$

taken in the lexicographic order. In particular, for the states of the form

$$\rho = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{33} & 0 & \rho_{36} & \rho_{37} & \rho_{38} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{63} & 0 & \rho_{66} & \rho_{67} & \rho_{68} & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_{73} & 0 & \rho_{76} & \rho_{77} & \rho_{78} & 0 \\
0 & 0 & \rho_{83} & 0 & \rho_{86} & \rho_{87} & \rho_{88} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{99}
\end{pmatrix} \quad \text{(III.3)}$$

the negativity is given by

$$N(\rho) = \frac{1}{2} \left[ \sqrt{4(|\rho_{37}|^2 + |\rho_{38}|^2 + |\rho_{67}|^2 + |\rho_{68}|^2) + \rho_{99}^2 - \rho_{99}} \right] \quad \text{(III.4)}$$

Notice that (III.4) is equal to zero when the coherences $\rho_{37}, \rho_{38}, \rho_{67}, \rho_{68}$ are all equal to zero, and is greater then zero when at least one of them is nonzero. Similarly for the states

$$\rho = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{33} & 0 & 0 & \rho_{37} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_{66} & 0 & \rho_{68} & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_{73} & 0 & 0 & \rho_{77} & 0 \\
0 & 0 & 0 & \rho_{86} & 0 & \rho_{88} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{99}
\end{pmatrix}, \quad \text{(III.5)}$$

the negativity can be computed from the formula

$$N(\rho) = \frac{1}{2} \left[ \sqrt{4(|\rho_{37}|^2 + |\rho_{68}|^2) + \rho_{99}^2 - \rho_{99}} \right]. \quad \text{(III.6)}$$

On the other hand, for the states

$$\rho = \begin{pmatrix}
\rho_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{33} & 0 & 0 & \rho_{37} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{99}
\end{pmatrix}, \quad \text{(III.7)}$$

their negativity

$$N(\rho) = \max \left( 0, \tilde{N}(\rho) \right) \quad \text{(III.8)}$$

where

$$\tilde{N}(\rho) = \frac{1}{2} \left[ \sqrt{(\rho_{11} - \rho_{99})^2 + 4|\rho_{37}|^2 - \rho_{11} - \rho_{99}} \right]. \quad \text{(III.9)}$$
can be zero, even if the coherence $\rho_{37}$ is not zero. There is a threshold for the coherence at which two atoms become entangled.

**IV. CREATION OF ENTANGLEMENT**

In this section we study the process of creation of transient entanglement between atoms prepared in separable initial states. We fix the distance between the atoms and solve numerically the master equation (II.1) in two cases of configurations of the system.

### A. Initial states $|1_A\rangle \otimes |3_B\rangle$ and $|1_A\rangle \otimes |2_B\rangle$

When the system is prepared in the pure state $|1_A\rangle \otimes |3_B\rangle$ (atom A in the excited state and atom B in the ground state) and both atoms lie along the $\gamma$ axis (Configuration I), so

$$
\Gamma_{12} = \Gamma_{23}, \quad \Omega_{13} = \Omega_{23} \quad \text{and} \quad \Gamma_{vc} = \Omega_{vc} = 0,
$$

one can check that the density matrix at time $t$ takes the form

$$
\rho(t) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_{33}(t) & 0 & 0 & 0 & \rho_{37}(t) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_{73}(t) & 0 & 0 & 0 & \rho_{77}(t) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{99}(t) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{99}(t)
\end{pmatrix}
$$

(IV.1)

and by (III.4)

$$
N(t) = \frac{1}{2} \left[ \sqrt{4|\rho_{37}(t)|^2 + \rho_{99}(t)} - \rho_{99}(t) \right]
$$

(IV.2)

Since the process of the photon exchange produces coherence between the states $|1_A\rangle \otimes |3_B\rangle$ and $|3_A\rangle \otimes |1_B\rangle$, the value of $|\rho_{37}|$ starts to grow and the system becomes entangled (FIG. 2).

FIG. 2: The time evolution of $|\rho_{37}|$ for the initial state $|1_A\rangle \otimes |3_B\rangle$ when $\Gamma_{vc} = \Omega_{vc} = 0$ and $R/\lambda = 0.2$.

When both atoms lie on the $xy$ plane (Configuration II), the cross coupling coefficients $\Gamma_{vc}$ and $\Omega_{vc}$ are nonzero and the dynamics of the system is changed significantly. The additional coupling between orthogonal dipoles produces new coherences $\rho_{36}, \rho_{38}, \rho_{67}, \rho_{68}$ and $\rho_{78}$, so the state at time $t$ has the form (III.3). In particular, the values of $|\rho_{37}|, |\rho_{38}|, |\rho_{67}|$ and $|\rho_{68}|$ become nonzero (FIG. 3), so the negativity of the state can be computed from the formula (III.4). In FIG. 4 we plot the time evolution of negativity of initial state $|1_A\rangle \otimes |3_B\rangle$ in two cases: $\Gamma_{vc} \neq 0, \Omega_{vc} \neq 0$ ($N_{vc}$) and $\Gamma_{vc} = \Omega_{vc} = 0$ ($N$). In both cases $R/\lambda = 0.2$.

FIG. 3: The time evolution of $|\rho_{38}|, |\rho_{67}|, |\rho_{68}|$ for the initial state $|1_A\rangle \otimes |3_B\rangle$ in the case when $\Gamma_{vc} \neq 0, \Omega_{vc} \neq 0$ and $R/\lambda = 0.2$.

FIG. 4: The time evolution of negativity of initial state $|1_A\rangle \otimes |3_B\rangle$ in two cases: $\Gamma_{vc} \neq 0, \Omega_{vc} \neq 0$ ($N_{vc}$) and $\Gamma_{vc} = \Omega_{vc} = 0$ ($N$). In both cases $R/\lambda = 0.2$. As we see, the cross coupling between the atoms enhances the production of entanglement.
The same behaviour of negativity can be observed for initial state \(|2_A⟩ \otimes |3_B⟩\).

On the other hand, when the system is prepared in the initial state \(|1_A⟩ \otimes |2_B⟩\) (both atoms in excited states) and the cross coupling is absent, the entanglement production is due to the coherences \(\rho_{37}\) and \(\rho_{68}\). As in the previous case, the presence of cross coupling enhances the production of entanglement, but the maximal value of negativity is much less then in the case of initial state \(|1_A⟩ \otimes |3_B⟩\) (FIG. 5).

![FIG. 5: The time evolution of negativity of initial state \(|1_A⟩ \otimes |2_B⟩\) in two cases: \(\Gamma_{vc} \neq 0, \Omega_{vc} \neq 0 (N_{vc})\) and \(\Gamma_{vc} = \Omega_{vc} = 0 (N)\). In both cases \(R/\lambda = 0.2\).](image)

B. Initial state \(|1_A⟩ \otimes |1_B⟩\) and delayed sudden birth of entanglement

If the system is prepared in the state \(|1_A⟩ \otimes |1_B⟩\) (both atoms are in the same excited state), and the cross coupling is absent, the state at time \(t\) takes the form \(|\Psi⟩\). As in the case of initial state \(|1_A⟩ \otimes |3_B⟩\), the entanglement production is due to the creation of coherence \(\rho_{37}\), but in the present case, the nonzero coherence is only the necessary condition for entanglement. As it follows from (III.8) and (III.9), there is a threshold for \(|\rho_{37}|\) at which the negativity becomes nonzero. A detailed numerical analysis shows that there is no entanglement at earlier times, and suddenly at some time the entanglement starts to build up (FIG. 6). This is the example of phenomenon of delayed sudden birth of entanglement, studied by Ficek and Tanaš [11] in the case of two-level atoms. To get some insight into the process of creation of entanglement in this case, consider the initial state

\[|\Psi⟩ = \cos \phi |1_A⟩ \otimes |1_B⟩ + \sin \phi |1_A⟩ \otimes |3_B⟩, \quad \phi \in [0, \pi/2]\]

(IV.3)

FIG. 7 shows that the evolution of initial state (IV.3) crucially depends on the superposition angle \(\phi\). The smaller is the prob-

![FIG. 7: The time evolution of negativity of initial state (IV.3) for different values of \(\phi\) \((\Gamma_{vc} = \Omega_{vc} = 0, R/\lambda = 0.2)\).](image)

FIG. 8: The time of birth of entanglement as a function of \(\phi\).

FIG. 8: The time of birth of entanglement as a function of \(\phi\).
ability that the system is prepared in the state $|1_A\rangle \otimes |1_B\rangle$, the earlier the atoms become entangled. In FIG. 8 we plot the time of the birth of entanglement as the function of the superposition angle. This time is maximal for $\phi = 0$ (i.e. $|\Psi\rangle = |1_A\rangle \otimes |1_B\rangle$) and is equal to zero for $\phi = \pi/2$.

When the cross coupling coefficients $\Gamma_{vc}$ and $\Omega_{vc}$ are not zero the numerical analysis shows that the time delayed creation of entanglement does not occur. In that case, even for the initial state $|1_A\rangle \otimes |1_B\rangle$ (or $|2_A\rangle \otimes |2_B\rangle$), the entanglement starts to build up immediately after the atoms begin to interact with the vacuum (FIG. 9).

V. DISENTANGLEMENT

Apart from the effect of creation of entanglement, the quantum evolution given by the master equation (II.1) may cause also destruction of correlations, resulting in disentanglement of initially entangled states. In this section we study the process of disentanglement for some entangled pure initial states. We start with the characterization of maximally entangled states of two three - level systems (two qutrits).

### A. Maximally entangled states of two qutrits and generalized Dicke states

The basis $|\Psi_\alpha\rangle$, $\alpha = 1, \ldots, 9$ of the space $C^3 \otimes C^3$ consisting of maximally entangled Bell - like states was constructed in [14] (see also [19]). The states $|\Psi_\alpha\rangle$ can be written as follows:

\[
|\Psi_1\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |1_B\rangle + |2_A\rangle \otimes |2_B\rangle + |3_A\rangle \otimes |3_B\rangle ) \\
|\Psi_2\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |2_B\rangle + |2_A\rangle \otimes |3_B\rangle + |3_A\rangle \otimes |1_B\rangle ) \\
|\Psi_3\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |3_B\rangle + |2_A\rangle \otimes |1_B\rangle + |3_A\rangle \otimes |2_B\rangle ) \\
|\Psi_4\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |1_B\rangle + w|2_A\rangle \otimes |2_B\rangle + \overline{w}|3_A\rangle \otimes |3_B\rangle ) \\
|\Psi_5\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |2_B\rangle + w|2_A\rangle \otimes |3_B\rangle + \overline{w}|3_A\rangle \otimes |1_B\rangle ) \\
|\Psi_6\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |3_B\rangle + w|2_A\rangle \otimes |1_B\rangle + \overline{w}|3_A\rangle \otimes |2_B\rangle ) \\
|\Psi_7\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |1_B\rangle + \overline{w}|2_A\rangle \otimes |2_B\rangle + w|3_A\rangle \otimes |3_B\rangle ) \\
|\Psi_8\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |2_B\rangle + \overline{w}|2_A\rangle \otimes |3_B\rangle + w|3_A\rangle \otimes |1_B\rangle ) \\
|\Psi_9\rangle = \frac{1}{\sqrt{3}} (|1_A\rangle \otimes |3_B\rangle + \overline{w}|2_A\rangle \otimes |1_B\rangle + w|3_A\rangle \otimes |2_B\rangle )
\]

(V.1)

where

\[w = e^{2\pi i/3}\]

One can check that the states (V.1) have maximal negativity and that they are locally equivalent.

There is another class of pure entangled states of two qutrits which are very important for the analysis of the dynamics of coupled three - level atoms. The generalized symmetric and antisymmetric Dicke states (see e.g. [13]), defined by the formulas

\[
|s_{kl}\rangle = \frac{1}{\sqrt{2}} ((|k_A\rangle \otimes |l_B\rangle + |l_A\rangle \otimes |k_B\rangle ) \\
|a_{kl}\rangle = \frac{1}{\sqrt{2}} ((|k_A\rangle \otimes |l_B\rangle − |l_A\rangle \otimes |k_B\rangle )
\]

(V.2)

where $k,l = 1, 2, 3; k < l$, are not maximally entangled (their negativity is equal to 1/2) but have a remarkable properties. As it was shown in our previous paper [10], in the limit of small separation between the atoms, the process of photon exchange produces such correlations that the dynamics is not ergodic and there are nontrivial asymptotic stationary states. In that case, the symmetric Dicke states $|s_{kl}\rangle$ decay to the ground state $|3_A\rangle \otimes |3_B\rangle$ whereas antisymmetric states $|a_{13}\rangle$ and $|a_{23}\rangle$ decouple from the environment and therefore are stable. Moreover, the state $|a_{12}\rangle$ is not stable, but is asymptotically nontrivial. So for the distances comparable to the radiation wavelength $\lambda$, symmetric and antisymmetric Dicke states will decay with significantly different rates and the populations in the antisymmetric states can be used to characterize disentanglement properties of given initial state.
B. Time evolution of Dicke states

Now we consider the evolution of the antisymmetric Dicke state
\[ |a_{13}⟩ = \frac{1}{\sqrt{2}} (|1⟩_A ⊗ |3⟩_B - |3⟩_A ⊗ |1⟩_B) \]  
(V.3)
in the case when the distance between the atoms is comparable to \( λ \). If \( Γ_{vc} = Ω_{vc} = 0 \), the state at time \( t \) takes the form (V.4), so the degree of its entanglement is determined by the coherence \( ρ_{37}(t) \). The same is true for the symmetric state
\[ |s_{13}⟩ = \frac{1}{\sqrt{2}} (|1⟩_A ⊗ |3⟩_B + |3⟩_A ⊗ |1⟩_B) \]  
(V.4)
which decays to the ground state even in the limit of small separation. As we show numerically, time evolution of \( ρ_{37} \) for the symmetric state (V.4) differs significantly from that for antisymmetric state (V.3) (see FIG. 10) and the latter disentangle much slower than the former (FIG. 11).

![FIG. 10: The time evolution of \( |ρ_{37}| \) for the symmetric and antisymmetric Dicke states in the case \( Γ_{vc} = Ω_{vc} = 0 \) and \( R/λ = 0.2 \).](image)

When the cross coupling coefficients are not zero, the dynamics of Dicke states is more complicated, and at a given time \( t \) they have the form (III.3). Detailed analysis of the evolution of negativity indicates that for the antisymmetric state \( |a_{13}⟩ \), additional coupling between transitions with orthogonal dipole moments, slow down the process of disentanglement (FIG. 12). On the other hand, this coupling does not influence rapid disentanglement of the symmetric state.

The antisymmetric state \( |a_{12}⟩ \) is not stable in the limit of small separation between the atoms, but it is asymptotically nontrivial [10]. It can be shown that \( |a_{12}⟩ \) evolves to the asymptotic state \( ρ \) which has the form (III.5) with
\[ ρ_{22} = ρ_{99} = 0, \quad ρ_{33} = ρ_{66} = ρ_{88} = \frac{1}{4}, \quad ρ_{37} = ρ_{68} = -\frac{1}{4} \]

so by (III.6), the asymptotic negativity of \( |a_{12}⟩ \) has the value \( \sqrt{2}/4 \). For the atom separation comparable with \( λ \), this state disentangle quicker than \( |a_{13}⟩ \).

![FIG. 11: Disentanglement of Dicke states \( |s_{13}⟩ \) and \( |a_{13}⟩ \) (\( Γ_{vc} = Ω_{vc} = 0, R/λ = 0.2 \)).](image)

C. Disentanglement of Bell states

As it was stated before, in the case of small separation between the atoms, the antisymmetric Dicke states \( |a_{12}⟩, |a_{23}⟩ \) are stable, and the state \( |a_{12}⟩ \) has a nontrivial asymptotic limit. For that reason, the initial states which have the property of trapping populations in \( |a_{13}⟩, |a_{23}⟩ \) and \( |a_{12}⟩ \), decay to an entangled asymptotic states. In the process of evolution, the initial populations in \( |a_{13}⟩ \) and \( |a_{23}⟩ \) are conserved, whereas the population in \( |a_{12}⟩ \) can be transformed into \( |a_{13}⟩ \) and \( |a_{23}⟩ \), giving the enlargement of initial populations.
Consider now the Bell states \(|\Psi_1\rangle\text{,} |\Psi_2\rangle\text{.} One checks that the states \(|\Psi_1\rangle\text{,} |\Psi_4\rangle\text{ and} |\Psi_7\rangle\) have zero populations in \(|a_{13}\rangle\text{,} |a_{23}\rangle\text{ and} |a_{12}\rangle\rangle\text{, so they decay to the separable asymptotic state. In fact, the limiting state in this case is the ground state} |3_A\rangle \otimes |3_B\rangle\text{.} On the other hand, the remaining Bell states \(|\Psi_2\rangle\text{,} |\Psi_3\rangle\text{,} |\Psi_5\rangle\text{,} |\Psi_6\rangle\text{,} |\Psi_8\rangle\text{ and} |\Psi_9\rangle\) have equal populations in antisymmetric Dicke states, so they have the same asymptotic entanglement. Take for example the state \(|\Psi_2\rangle\). Since the corresponding populations are equal 1/6, by the general result of \([10]\) the asymptotic state is of the form

\[
\rho_{as} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{8} & 0 & -\frac{11}{12} & \frac{1}{8} & \frac{1}{12} & \frac{1}{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{12} & \frac{1}{8} & \frac{1}{12} \\
0 & 0 & -\frac{1}{12} & 0 & 0 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{12} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} \\
0 & 0 & \frac{1}{12} & 0 & 0 & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
0 & 0 & -\frac{1}{12} & 0 & 0 & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12}
\end{pmatrix}
\] (V.5)

Its negativity can be computed numerically and one obtains that \(N(\rho_{as}) \simeq 0.0968\).

If the distance \(R\) is comparable with the wavelength \(\lambda\) and the cross coupling is absent, the populations in \(|a_{13}\rangle\) and \(|a_{23}\rangle\) are no longer conserved but increase at the beginning (since the population in \(|a_{12}\rangle\) decreases) and then decay much slower than the populations of the remaining states (FIG. 13). In this way, the entanglement of state \(|\Psi_2\rangle\) is more robust against the noise than the entanglement of \(|\Psi_1\rangle\) (FIG. 14). The same is true for the other states from this class. So all Bell states can be divided into two classes. The states from the first class containing \(|\Psi_1\rangle\text{,} |\Psi_4\rangle\text{ and} |\Psi_7\rangle\) decay rapidly, whereas remaining

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13.png}
\caption{Time evolution of population in antisymmetric \(|a_{13}\rangle\text{,} symmetric \(|s_{13}\rangle\text{ and} antisymmetric \(|a_{12}\rangle\) for the initial Bell state \(|\Psi_2\rangle\). We take here \(\Gamma_{vc} = \Omega_{vc} = 0\) (NON VC) and \(R/\lambda = 0.08\).
\end{figure}

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{Disentanglement of the Bell state \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\). We take here \(\Gamma_{vc} = \Omega_{vc} = 0\) and \(R/\lambda = 0.08\).
\end{figure}

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15.png}
\caption{Disentanglement of the Bell state \(|\Psi_2\rangle\) in two cases: \(\Gamma_{vc} = \Omega_{vc} = 0\) (NON VC) and \(\Gamma_{vc} = \Omega_{vc} = 0\) (VC). In both cases \(R/\lambda = 0.08\).
\end{figure}

VI. CONCLUSIONS

We have studied the dynamics of entanglement in the system of three - level atoms in the V configuration, coupled to the common vacuum and separated by a distance comparable to the radiation wavelength. In this case only some transient entanglement between the atoms can exist but the dy-
namical generation of such entanglement is possible. It happens for example, when the cross coupling between orthogonal dipoles is absent and initially only one atom is excited. Additional coupling enhances the production of entanglement and causes that entanglement can be produced also in the case when two atoms are excited. Initial states with two atoms excited lead also to the interesting phenomenon of delayed sudden birth of entanglement. The process of disentanglement of initially entangled states is less sensitive to cross coupling between the atoms. We have shown this for the maximally entangled Bell states. On the other hand, the rate of disentanglement of Bell states crucially depends on populations of initial state in the antisymmetric Dicke states, which are more robust against the noise then the Bell states. We have demonstrated that those Bell states which have no populations in the antisymmetric Dicke states rapidly disentangle, whereas remaining Bell states disentangle much slower.

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