Asymptotics of solutions for nonlocal elliptic problems in plane bounded domains

Pavel Gurevich

Abstract

The paper is devoted to the study of asymptotic behavior of solutions for nonlocal elliptic problems in weighted spaces. We deal with the most difficult case when the support of nonlocal terms intersects with boundary of a plane bounded domain. In this situation, a general form of the asymptotics is investigated, and coefficients in the asymptotics are calculated.

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References

1 Introduction

I. This work is devoted to the investigation of asymptotic behavior of solutions for nonlocal elliptic problems. Recently many mathematicians have been studying nonlocal problems. This interest is explained, on the one hand, by a significant theoretical progress in the area and, on the other hand, by a number of important applications arising in plasma theory [3], biophysics, theory of diffusion processes [7,26,30], modern aircraft technology (particularly, in the theory of sandwich shells and plates [26]), etc.

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In the 1-dimensional case the first ones who studied nonlocal problems were A. Sommerfeld [28], Ya.D. Tamarkin [29], M. Picone [19]. In the 2-dimensional case the earliest paper devoted to nonlocal problems is due to T. Carleman [5]. T. Carleman searched for a harmonic function $u$ in a plane bounded domain $G$ subject to a nonlocal condition connecting the values of the unknown function in different points of boundary: $u(x) + bu(\omega(x)) = g(x)$. Here $\omega: \partial G \to \partial G$ is a nondegenerate transformation subject to the restriction $\omega(\omega(x)) \equiv x$ (being referred to as Carleman's condition in the present time). Such a statement of nonlocal problems has originated further research into the area of elliptic problems with nonlocal transformations mapping a boundary onto itself and with abstract boundary conditions [1, 2, 4, 31].

In 1969, A.V. Bitsadze and A.A. Samarskii [3] considered the following nonlocal problem arising in plasma theory: find a harmonic function $u(y_1, y_2)$ in the rectangle $G = \{y \in \mathbb{R}^2 : -1 < y_1 < 1, 0 < y_2 < 1\}$ such that it is continuous in $\overline{G}$ and satisfies the conditions

$$
\begin{align*}
u(y_1, 0) &= f_1(y_1), 
u(y_1, 1) = f_2(y_1), -1 < y_1 < 1, \\
u(-1, y_2) &= f_3(y_2), u(1, y_2) = u(0, y_2), 0 < y_2 < 1,
\end{align*}
$$

where $f_1, f_2, f_3$ are given continuous functions. We notice that this problem principally differs from the one studied by T. Carleman: now the values of the unknown function on the part of the boundary $\partial G$ are connected with the values inside the domain $G$. This problem was solved in [3] by reducing to an integral Fredholm equation and using the maximum principle. In case of an arbitrary domain and general nonlocal transformations, it was formulated as an unsolved one.

The most difficult case turns out to deal with the situation when a part $\Upsilon_1$ of boundary of a domain $G$ is mapped by some nonlocal transformation $\Omega_1$ on $\Omega_1(\Upsilon_1)$ so that $\Omega_1(\Upsilon_1) \cap \partial G \neq \emptyset$. Various versions of such problems were considerer by S.D. Eidelman and N.V. Zhitarashu [6], K.Yu. Kishkis [14], A.K. Gushchin and V.P. Mikhailov [13], etc.

Basis of general theory for elliptic equations of order $2m$ with general nonlocal conditions was founded by A.L. Skubachevskii and his pupils. In a series of works a priori estimates were proved, a right regularizer was constructed, adjoint problems were studied, and properties of index in appropriate spaces were established; spectral properties of some problems were considered [9, 10, 16, 20–25]; asymptotics and smoothness of solutions near some special points were investigated [12, 22]. We remark that the papers [22, 24] were the first ones to deal with the case $\Omega_1(\Upsilon_1) \cap \hat{\Upsilon}_1 \neq \emptyset$, which had not been previously considered even for the Laplace equation with nonlocal conditions in plane domains.

II. In this paper we investigate the most difficult situation mentioned above: the support of nonlocal terms can have a nonempty intersection with boundary of a domain $G$. In that case, power singularities for solutions near some set $K \subset G$ can appear [22, 27]. Therefore it is quite natural to study such problems in special weighted spaces that take into consideration those possible singularities. (The most convenient spaces turned out to be Kondrat’ev’s ones [15]). Thus we arrive at the question of asymptotics of solutions near the set $K$. In the paper [22], A.L. Skubachevskii obtained a general form of an asymptotics of solutions to problems with nonlocal transformations coinciding with a rotation operator near the set $K$. These theorems were applied to investigation of smoothness for generalized solutions of nonlocal elliptic problems (see [22, 27]).

In the present work we generalize the mentioned results of A.L. Skubachevskii and study the case of arbitrary nonlocal transformations, linear near the set $K$. Simultaneously, we get a
formula connecting the indices of one and the same nonlocal problem, but being considered in different weighted spaces.

Moreover, using the results of the paper [12] (which deals with model nonlocal problems in plane angles and in \( \mathbb{R}^2 \setminus \{0\} \)), we get explicit formulas for calculating the coefficients in the asymptotics of solutions. These formulas are given both in terms of eigenvectors and associated vectors of model adjoint problems and in terms of distributions from the kernel of adjoint problem in a bounded domain. The latter shows, in particular, that the values of the coefficients in the asymptotics are the functionals over the right-hand sides of the nonlocal problem under consideration. These functionals depend on the data of the problem in the whole domain, but not only in some neighborhood of the set \( \mathcal{K} \).

We remark that the calculation of the coefficients in the asymptotics is both important itself and has a direct application to the question of smoothness of generalized solutions for nonlocal problems. Roughly speaking, it allows to show that a generalized solution \( u \in W^1_2(G) \) to a nonlocal problem (for an elliptic 2nd order equation) with a right-hand side \( f \in L_2(G) \) is smooth (i.e., \( u \in W^2_2(G) \)) if and only if the function \( f \) satisfies some orthogonality conditions. In a number of cases these conditions can be verified explicitly.

III. The paper is organized as follows. The statement of the problem and some assumptions concerning nonlocal transformations are given in section 2. Most of the assumptions are due to simplify computations throughout the paper. In section 3 we derive an asymptotics (with yet unknown coefficients) for solutions to nonlocal problems. Using the results of section 3 in section 4 we establish a connection between the indices of one and the same problem but being considered in different weighted spaces. In section 5 we obtain an asymptotics of solutions for adjoint nonlocal problems. This allows to get in section 6 explicit formulas for the coefficients in asymptotics of solutions to the original nonlocal problem. In section 7 we consider an example illustrating the results of sections 2–6.

2 Statement of the problem in a bounded domain

Let \( G \in \mathbb{R}^2 \) be a bounded domain with a boundary \( \partial G = \bigcup_{\sigma=1,2} \bar{\Upsilon}_\sigma \), where \( \Upsilon_\sigma \) are open (in the topology of \( \partial G \)) curves of \( C^\infty \) class such that \( \Upsilon_1 \cap \Upsilon_2 = \emptyset, \bar{\Upsilon}_1 \cap \bar{\Upsilon}_2 = \{g_1, h_1\} \). We suppose that in some neighborhoods of the points \( g_1 \) and \( h_1 \) the domain \( G \) coincides with an angle.

We denote by \( \mathbf{P}(y, D_y), B_{\sigma\mu}(y, D_y), T_{\sigma\mu}(y, D_y) \) differential operators of orders \( 2m, m_{\sigma\mu}, m_{\sigma\mu} \) respectively with complex-valued coefficients from \( C^\infty(\mathbb{R}^2) \) (\( m_{\sigma\mu} \leq 2m - 1, \sigma = 1, 2; \mu = 1, \ldots, m \)). Put also \( B_{\sigma}(y, D_y) = \{B_{\sigma\mu}(y, D_y)\}_{\mu=1}^m, T_{\sigma}(y, D_y) = \{T_{\sigma\mu}(y, D_y)\}_{\mu=1}^m \).

Let \( \Omega_\sigma \ (\sigma = 1, 2) \) be an infinitely differentiable nondegenerate transformation mapping some neighborhood \( \bar{\Upsilon}_\sigma \) of \( \Upsilon_\sigma \) onto \( \Omega_\sigma(\bar{\Upsilon}_\sigma) \) such that \( \Omega_\sigma(\Upsilon_\sigma) \subset G \). For definiteness, we consider the case when \( \Omega_1(g_1) = g_2 \in G, \Omega_2(g_1) = g_1, \Omega_1(h_1) = h_1, \Omega_2(h_1) = h_2 \in G \). In this work we also assume that \( g_2 \notin \Omega_2(\Upsilon_2), h_2 \notin \Omega_1(\Upsilon_1) \). The last assumption is made in order to simplify further computations. But, following [22,24], we demand (and it is on principle) that following

\[ \text{If, say, } g_2 \notin \Omega_2(\Upsilon_2), \text{ then either } g_2 = h_2 (\text{in that case, an asymptotics of a solution near the point } g_2 \text{ will} } \]
Figure 2.1: The domain $G$.

condition holds:

**Condition 2.1.** The curves $\Omega_1(T_1)$ and $\Omega_2(T_2)$ are not tangent to the boundary $\partial G$ at the “consistent points” $h_1$ and $g_1$ respectively (see Fig. 2.1).

We also suppose for simplicity that the transformations $\Omega_\sigma(y)$ are linear near the points $g_1$ and $h_1$.

We introduce the set $\mathcal{K} = \{g_1, h_1, g_2, h_2\}$ and consider the nonlocal elliptic problem

$$
\begin{align*}
P(y, D_y)u &= f(y) \quad (y \in G \setminus \mathcal{K}), \\
B_\sigma(y, D_y)u &= B_\sigma(y, D_y)u|_{\Upsilon_\sigma} + (T_\sigma(y, D_y)u)(\Omega_\sigma(y))|_{\Upsilon_\sigma} = f_\sigma(y) \\
(y \in \Upsilon_\sigma; \; \sigma = 1, 2).
\end{align*}
$$

Here $(T_\sigma(y, D_y)u)(\Omega_\sigma(y)) = T_\sigma(y', D_{y'})u(y')|_{y' = \Omega_\sigma(y)}$; $f_\sigma = \{f_{\sigma\mu}\}_{\mu=1}^m$.

**Remark 2.1.** The results of this paper are generalized for the case when the boundary $\partial G$ consists of a finite number of smooth curves $\Upsilon_\sigma, \sigma = 1, \ldots, N$, and nonlocal conditions on each $\Upsilon_\sigma$ contain a finite number of nonlocal terms with different transformations. Moreover, these transformations can map “the consistent points” (which are $g_1$ and $h_1$ in our case) both to the boundary $\partial G$ and inside the domain $G$, forming finite orbits.

We introduce the space $H^l_a(G)$ as a completion of the set $C_0^\infty(\bar{G} \setminus \mathcal{K})$ in the norm

$$
\|u\|_{H^l_a(G)} = \left( \sum_{|\alpha| \leq l} \int_{\bar{G}} \rho^{2(a-\ell+|\alpha|)} |D^\alpha u|^2 dy \right)^{1/2}.
$$

Here $C_0^\infty(\bar{G} \setminus \mathcal{K})$ is the set of infinitely differentiable functions with compact supports contained in $\bar{G} \setminus \mathcal{K}$; $l \geq 0$ is an integer; $a \in \mathbb{R}$; $\rho = \rho(y) = \text{dist}(y, \mathcal{K})$.

If, instead of the domain $G$, one considers an angle with a vertex $g$ or some neighborhood of a point $g$, then one must put $\mathcal{K} = \{g\}$ in the definition of the weighted space.

Influence not only an asymptotics near $g_1$ but near $h_1$ as well) or $g_2 \in \Omega_2(\Upsilon_2)$ (in that case, one must study an asymptotics at the additional point $\Omega_2^{-1}(g_2) \in \Upsilon_2$).
By $H_{a}^{1+1/2}(\Upsilon)$ we denote the space of traces on a smooth curve $\Upsilon \subset \bar{G}$ with the norm
\[ \|\psi\|_{H_{a}^{1+1/2}(\Upsilon)} = \inf \|u\|_{H_{a}^{1}(G)} \quad (u \in H_{a}^{1}(G) : u|_{\Upsilon} = \psi). \]
Let us introduce the operator
\[ L = \{P(y, D_y), B_{\sigma}(y, D_y)\} : \]
\[ H_{a}^{1+2m}(G) \to H_{a}^{1}(G, \Upsilon) \overset{\text{def}}{=} H_{a}^{1}(G) \times \prod_{\sigma=1,2}^{m} \prod_{\mu=1}^{m} H_{a}^{1+2m-\mu\sigma-1/2}(\Upsilon_{\sigma}), \]
which corresponds to nonlocal problem (2.1), (2.2).

Throughout the paper we assume that the operators $P(y, D_y)$ and $B_{\sigma}(y, D_y)$ satisfy the following conditions (see, e.g., [17, Chapter 2, section 1]).

**Condition 2.2.** For all $y \in \bar{G}$ the operator $P(y, D_y)$ is properly elliptic.

**Condition 2.3.** For $\sigma = 1, 2$ and $y \in \bar{\Upsilon}_\sigma$ the system $B_{\sigma}(y, D_y) = \{B_{\sigma\mu}(y, D_y)\}_{\mu=1}^{m}$ covers the operator $P(y, D_y)$.

Remark that we do not impose any restrictions on the nonlocal operators $T_{\sigma\mu}(y, D_y)$ but the natural restriction on their orders.

## 3 Asymptotics of solutions for nonlocal problems

**I.** In this section we obtain an asymptotics of a given solution $u \in H_{a}^{1+2m}(G)$ for problem (2.1), (2.2) with a right–hand side $\{f, f_{\sigma}\} \in H_{a_{1}}^{1}(G, \Upsilon)$, $0 < a - a_{1} < 1$.

Notice that the violation of the inequality $a - a_{1} < 1$ means that $\{f, f_{\sigma}\}$ is “too regular”. In that case, exact results should yield more terms in asymptotics in comparison with our case. This situation can be investigated in the way similar to [22] (see also [15, 18]). Namely, one should consider corresponding equations for a residue in the asymptotics formula and apply to them the results obtained for the case $a - a_{1} < 1$. We are not going to do this here since detailed computations would lead to enormous enlargement of the paper, giving no essentially new results (with respect to the present work and to the paper [22]). The same remark is valid for the case when $u \in H_{a}^{1+2m}(G)$, $\{f, f_{\sigma}\} \in H_{a_{1}}^{1}(G, \Upsilon)$, and $l_{1} \neq l$.

The asymptotics will be found with the help of eigenvalues and corresponding Jordan chains of some holomorphic operator–valued functions. Therefore let us remind some relevant definitions and facts (see [5]).

Suppose $\hat{L}(\lambda) : H_{1} \to H_{2}$ is a holomorphic operator–valued function, $H_{1}$, $H_{2}$ are Hilbert spaces. A holomorphic at a point $\lambda_{0}$ vector–function $\varphi(\lambda)$ with the values in $H_{1}$ is called a root function of the operator $\hat{L}(\lambda)$ at $\lambda_{0}$ if $\varphi(\lambda_{0}) \neq 0$ and the vector–function $\hat{L}(\lambda)\varphi(\lambda)$ is equal to $0$ at $\lambda_{0}$. If $\hat{L}(\lambda)$ has at least one root function at the point $\lambda_{0}$, then $\lambda_{0}$ is called an eigenvalue of $\hat{L}(\lambda)$. Multiplicity of zero for the vector–function $\hat{L}(\lambda)\varphi(\lambda)$ at the point $\lambda_{0}$ is called a multiplicity of the root function $\varphi(\lambda)$; the vector $\varphi^{(0)} = \varphi(\lambda_{0})$ is called an eigenvector
corresponding to the eigenvalue $\lambda_0$. Let $\varphi(\lambda)$ be a root function at the point $\lambda_0$ of multiplicity $\kappa$, and $\varphi(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j \varphi^{(j)}$. Then the vectors $\varphi^{(1)}$, $\ldots$, $\varphi^{(\kappa-1)}$ are called associated with the eigenvector $\varphi_0$, and the ordered set $\varphi^{(0)}$, $\ldots$, $\varphi^{(\kappa-1)}$ is called a Jordan chain corresponding to the eigenvalue $\lambda_0$. Rank of the eigenvector $\varphi^{(0)}$ (rank $\varphi^{(0)}$) is the maximum of multiplicities of all root functions such that $\varphi(\lambda_0) = \varphi^{(0)}$.

Let an eigenvalue $\lambda_0$ of the operator $\tilde{L}(\lambda)$ be isolated, $\dim \ker \tilde{L}(\lambda_0) < \infty$, and rank of $\lambda_0$ finite. Suppose $J = \dim \ker \tilde{L}(\lambda_0)$ and $\varphi^{(0,1)}$, $\ldots$, $\varphi^{(0,J)}$ is a system of linearly independent eigenvectors such that rank $\varphi^{(0,1)}$ is the greatest of ranks of all eigenvectors corresponding to the eigenvalue $\lambda_0$, and rank $\varphi^{(0,j)} (j = 2, \ldots, J)$ is the greatest of ranks of eigenvectors from some orthogonal supplement in $\ker \tilde{L}(\lambda_0)$ to the linear manifold of the vectors $\varphi^{(0,1)}$, $\ldots$, $\varphi^{(0,j-1)}$. The numbers $\kappa_j = \text{rank} \varphi^{(0,j)}$ are called partial multiplicities of the eigenvalue $\lambda_0$, and the sum $\kappa_1 + \cdots + \kappa_J$ is called a (full) multiplicity of $\lambda_0$. If the vectors $\varphi^{(0,j)}$, $\ldots$, $\varphi^{(\kappa_j-1,j)}$ form a Jordan chain for every $j = 1, \ldots, J$, then the set of vectors $\{\varphi^{(0,j)}$, $\ldots$, $\varphi^{(\kappa_j-1,j)} : j = 1, \ldots, J\}$ is called a canonical system of Jordan chains corresponding to the eigenvalue $\lambda_0$.

II. At first let us consider an asymptotics of the solution $u$ for problem (2.1), (2.2) near the point $g_2$. In this case we will see that the asymptotics is defined by a model “local” problem in $\mathbb{R}^2 \setminus \{g_2\}$ with a “regular” right–hand side. Such a problem was studied in [12 section 5]. Thereafter we will consider the asymptotics near the point $g_1$. In that case, we will arrive at a model nonlocal problem in some angle $K$ with a right–hand side being a sum of “regular” and “special” functions. The asymptotics of the “special” one will be defined by the asymptotics of the solution $u$ near $g_2$, which is explained by the presence of the nonlocal transformation $\Omega_1$. Then the results of [12] will be applied to this model problem.

Thus we fix a neighborhood $\mathcal{V}(g_2)$ of $g_2$ such that $\overline{\mathcal{V}(g_2)} \cap \partial G = \emptyset$ and $\overline{\mathcal{V}(g_2)} \cap \{h_2\} = \emptyset$. One can see that an asymptotic behavior of $u$ in $\mathcal{V}(g_2)$ does not depend on nonlocal conditions (2.2), but is defined only by the equation

$$
P(y, D_y)u = f(y) \quad (y \in \mathcal{V}(g_2)).$$

(3.1)

Let $\mathcal{P}(D_y)$ be the principal homogeneous part of the operator $P(g_2, D_y)$. Then equation (3.1) can be written in the form

$$
\mathcal{P}(D_y)u(y) = \hat{f}(y) \quad (y \in \mathcal{V}(g_2)),
$$

(3.2)

where $\hat{f}$, by virtue of the condition $0 < a - a_1 < 1$, belongs to the space $H_{a_1}^l (\mathcal{V}(g_2))^2$. We introduce the bounded operator

$$
\mathcal{L}_2 = \mathcal{P}(D_y) : H_{a+2m}^l (\mathbb{R}^2) \to H_0^l (\mathbb{R}^2),
$$

where, defining the weighted spaces, one must put $\mathcal{K} = \{g_2\}$.

We write the operator $\mathcal{P}(D_y)$ in polar coordinates with the pole at the point $g_2$: $\mathcal{P}(D_y) = r^{-2m} \tilde{\mathcal{P}} (\omega, D_\omega, r D_r)$, where $D_\omega = -i \frac{\partial}{\partial \omega}$, $D_r = -i \frac{\partial}{\partial r}$.  

\footnote{To show that $\hat{f} \in H_{a_1}^l (\mathcal{V}(g_2))$, one must estimate the expressions of the two types: 1) $p_a(y) D^\alpha u, |\alpha| \leq 2m - 1$, and 2) $(p_a(y) - p_a(0)) D^\alpha u, |\alpha| = 2m$, where $p_a$ are infinitely differentiable coefficients of $P(y, D_y)$. The 1st one is estimated by direct use of the condition $0 < a - a_1 < 1$, while the 2nd one needs additional application of Lemma 3.3 [13]. Further, in analogous situations, we will omit these explanations.}
Let us introduce the operator–valued function
\[ \tilde{L}_2(\lambda) = \tilde{P}(\omega, D_\omega, \lambda) : W_{2,2\pi}^{l+2m}(0, 2\pi) \to W_{2,2\pi}^{l}(0, 2\pi). \]
Here \( W_{2,2\pi}^{l}(0, 2\pi) \) is the closure of the set of infinitely differentiable \( 2\pi \)-periodic functions in \( W_{2}^{l}(0, 2\pi) \).

The operator \( \tilde{L}_2(\lambda) \) is obtained from the operator \( L_2 \) by passing to polar coordinates, followed by the Mellin transformation with respect to \( r \):
\[ \tilde{u}(\lambda) = (2\pi)^{-1/2} \int_0^\infty r^{-i\lambda-1} u(r) \, dr. \]

From [22, section 1] it follows that there exists a finite–meromorphic operator–valued function \( \tilde{R}_2(\lambda) \) such that its poles (except, maybe, a finite number of them) are located inside a double angle of opening less than \( \pi \) containing the imaginary axis; moreover, if \( \lambda \) is not a pole of \( \tilde{R}_2(\lambda) \), then \( \tilde{R}_2(\lambda) \) is inverse to the operator \( \tilde{L}_2(\lambda) \). Thus a number \( \lambda \) is a pole of \( \tilde{R}_2(\lambda) \) if and only if \( \lambda \) is an eigenvalue (of finite multiplicity) of \( \tilde{L}_2(\lambda) \).

If the line \( \operatorname{Im} \lambda = a + 1 - l - 2m \) contains no eigenvalues of \( \tilde{L}_2(\lambda) \), then, by virtue of [22, section 1], the operator \( L_2 \) is an isomorphism.

In order to formulate a theorem concerning an asymptotics near \( g_2 \), let us introduce some denotation. Suppose \( \lambda_2 \) is an eigenvalue of \( \tilde{L}_2(\lambda) \),
\[ \{ \varphi_2^{(0, \zeta)}, \ldots, \varphi_2^{(\zeta, z-1, \zeta)} : \zeta = 1, \ldots, J_2 \} \]
is a canonical system of Jordan chains of the operator \( \tilde{L}_2(\lambda) \) corresponding to the eigenvalue \( \lambda_2 \).

Consider the vector \( u_2 = \{ u_2^{(k, \zeta)} \} \), where
\[ u_2^{(k, \zeta)}(\omega, r) = r^{i\lambda_2} \sum_{q=0}^{k} \frac{1}{q!} (i \ln r)^q \varphi_2^{(k-q, \zeta)}(\omega), \]
(\( \omega, r \) are polar coordinates with the pole at the point \( g_2 \)).

Notice that (see [12, section 5]) the vector \( u_2 \), the components \( u_2^{(k, \zeta)} \) of which are defined by (3.4), satisfies the relation
\[ L_2 u_2 = 0. \]

**Theorem 3.1.** Let the lines \( \operatorname{Im} \lambda = a + 1 - l - 2m, \operatorname{Im} \lambda = a + 1 - l - 2m \) contain no eigenvalues of \( \tilde{L}_2(\lambda) \) and the strip \( a + 1 - l - 2m < \operatorname{Im} \lambda < a + 1 - l - 2m \) contain the only eigenvalue \( \lambda_2 \) of \( \tilde{L}_2(\lambda) \). Then
\[ u(y) = c_2 u_2(y) + \hat{u}(y) \quad (y \in \mathcal{V}(g_2)). \]

Here \( u_2 = \{ u_2^{(k, \zeta)} \}, u_2^{(k, \zeta)} \) are defined by (3.4); \( c_2 = \{ c_2^{(k, \zeta)} \} \) is a vector of some constants; \( \hat{u} \in H_{a_1}^{l+2m}(\mathcal{V}(g_2)). \)

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\(^3\)In formula (3.6) and further the expressions such as \( c_2 u_2 \) are calculated in the following way: \( c_2 u_2 = \sum_{\zeta=1}^{J_2} \sum_{k=0}^{\zeta} c_2^{(k, \zeta)} u_2^{(k, \zeta)}. \)

\(^4\)The results of this work are evidently generalized for the case when the strip \( a_1 + 1 - l - 2m < \operatorname{Im} \lambda < a + 1 - l - 2m \) contains a finite number of eigenvalues of \( \tilde{L}_2(\lambda) \).
Proof. Introduce the cut–off function $\eta \in C^\infty(\mathbb{R}^2)$ equal to 1 in some neighborhood of the point $g_2$ and vanishing outside $\mathcal{V}(g_2)$. Suppose that the function $\eta u$ is defined in the whole of $\mathbb{R}^2$, being equal to 0 outside $\mathcal{V}(g_2)$. Then from (3.2) and Leibnitz’s formula, it follows that

$$L_2(\eta u) \in H^{l+2m}_{a_1}(\mathbb{R}^2).$$

Now it remains only to apply Theorem 5.1 [12], which establishes the asymptotics of solutions for nonlocal problems in $\mathbb{R}^2 \setminus \{g_2\}$.

\[\boxed{\text{Remark 3.1.} \text{ In fact, the assumption that the line } \text{Im } \lambda = a + 1 - l - 2m \text{ contains no eigenvalues of } \hat{L}_2(\lambda) \text{ is superfluous. Theorem 3.7 remains valid even if it is violated (see Remark 5.1 [12]). But this assumption will be used for studying the adjoint nonlocal problem and for calculating the coefficients } c_3^{(k, \xi)} . \text{ However, this assumption does not lead to the loss in generality. Indeed, one can find an } \varepsilon, 0 < \varepsilon < a - a_1, \text{ such that the strip } a - \varepsilon + 1 - l - 2m \leq \text{Im } \lambda \leq a + 1 - l - 2m \text{ contains no eigenvalues of } \hat{L}_2(\lambda), \text{ and therefore (see [12, section 5]) } u \in H^{l+2m}_{a_{1}-\varepsilon}(\mathcal{V}(g_2)). \text{ Hence we arrive at the situation of Theorem 3.7.}}\]

\[\boxed{\text{Remark 3.2.} \text{ From the results of [12], proof of Theorem 3.7, and Remark 3.1, it follows that if the strip } a_1 + 1 - l - 2m \leq \text{Im } \lambda < a + 1 - l - 2m \text{ has no eigenvalues of } \hat{L}_2(\lambda), \text{ then } u \in H^{l+2m}_{a_1}(\mathcal{V}(g_2)) \text{ for any right–hand side } \{f, f_\sigma \} \in H^{l}_{a_1}(G, \mathcal{Y}).}}\]

### III. Now we consider an asymptotics of the solution $u$ for problem (2.1), (2.2) near the point $g_1$. Fix a neighborhood $\mathcal{V}(g_1)$ of $g_1$ such that

$$\overline{\mathcal{V}(g_1)} \cap \Omega_1(T_1) = \emptyset \text{ and } \overline{\mathcal{V}(g_1)} \cap \{h_2\} = \emptyset. \quad (3.7)$$

Then one can see that an asymptotic behavior of the solution $u$ is defined by the problem

$$P(y, D_y)u = f(y) \quad (y \in \mathcal{V}(g_1) \cap G), \quad (3.8)$$

$$B_1(y, D_y)u|_{\mathcal{V}(g_1) \cap T_1} = f_1(y) - (T_1(y, D_y)u)(\Omega_1(y))|_{\mathcal{V}(g_1) \cap T_1} \quad (y \in \mathcal{V}(g_1) \cap T_1), \quad (3.9)$$

$$B_2(y, D_y)u|_{\mathcal{V}(g_1) \cap T_2} + (T_2(y, D_y)u)(\Omega_2(y))|_{\mathcal{V}(g_1) \cap T_2} = f_2(y) \quad (y \in \mathcal{V}(g_1) \cap T_2).$$

Let $P(D_y)$, $B_\sigma(D_y)$, $T_2(D_y)$ be the principal homogeneous parts of the operators $P(g_1, D_y)$, $B_\sigma(g_1, D_y)$, $T_2(g_1, D_y)$ respectively. Let $T_1(D_y)$ be the principal homogeneous part of the operator $T_1(g_2, D_y)$.

From now on we shall suppose that the origin coincides with the point $g_1$: $g_1 = 0$, and

$$\mathcal{V}(g_1) \cap G = \mathcal{V}(0) \cap K, \quad (3.10)$$

where $K$ is the plane angle: $K = \{y \in \mathbb{R}^2 : r > 0, b_1 < \omega < b_2\}$ with the arms $\gamma_\sigma = \{y \in \mathbb{R}^2 : r > 0, \omega = b_\sigma\}, \sigma = 1, 2$. Here $(\omega, r)$ are polar coordinates with the pole at the point $g_1 = 0$, $0 < b_1 < b_2 < 2\pi$.

\[\text{Notice that earlier, in this section, we denoted by } P(D_\omega) \text{ the principal homogeneous part of the operator } P(g_2, D_y). \text{ To be strict we had to denote these operators by different symbols. But we do not do it since throughout the paper it will always be clear from the context whether we consider the principal homogeneous part of } P(y, D_y) \text{ at } g_1 \text{ or at } g_2.\]
According to the assumptions of section 2, the transformations $\Omega_\sigma(y)$ are linear in $\mathcal{V}(g_1) = \mathcal{V}(0)$. Let $\Omega_1(y)$ ($y \in \mathcal{V}(g_1)$) be a composition of a rotation and an expansion with respect to $g_1$, and the shift by the vector $g_1 g_2$. Let $\Omega_2(y)$ ($y \in \mathcal{V}(g_1)$) coincide with the linear operator $\mathcal{G}_2$ of a rotation by an angle $\omega_2$ ($b_1 < b_2 + \omega_2 < b_2$) and an expansion with a coefficient $\beta_2 > 0$.

Let the neighborhood $\mathcal{V}(g_1) = \mathcal{V}(0)$ be so small that $\{\Omega_1(y) : y \in \mathcal{V}(g_1)\} \subset \mathcal{V}(g_2)$ and relations $\mathcal{G}_2$ are fulfilled with the set $\{\Omega_2(y) : y \in \mathcal{V}(g_1)\}$ substituted for $\mathcal{V}(g_1)$ (which is related to $\mathcal{G}_2$). (Mention that this requirement is automatically fulfilled whenever the expansion coefficients for the transformations $\Omega_\sigma(y)$ near the point $g_1$ are less or equal to 1.)

Now, using asymptotics formula (3.6) for the solution $u$ near $g_2$ and Leibnitz’s formula, we get that problem (3.8), (3.9) in $\mathcal{V}(g_1) \cap G$ is equivalent to the following one in $\mathcal{V}(0) \cap K$:

\[ \mathcal{P}(D_y) u = \hat{f}(y) \quad (y \in \mathcal{V}(0) \cap K), \]
\[ B_1(D_y) u |_{\mathcal{V}(0) \cap \gamma_1} = \hat{f}_1(y) - c_2 f_{12}(y) \quad (y \in \mathcal{V}(0) \cap \gamma_1), \]
\[ B_2(D_y) u |_{\mathcal{V}(0) \cap \gamma_2} + (T_2(D_y) u)(\mathcal{G}_2 y) |_{\mathcal{V}(0) \cap \gamma_2} = \hat{f}_2(y) \quad (y \in \mathcal{V}(0) \cap \gamma_2). \]

Here $\hat{f} = f - (\mathcal{P}(D_y - \mathcal{P}(D_y)))u$,
\[ f_{12} = \left( T_1(D_y) u_{21} \right)(\mathcal{O}_1(y)) |_{\mathcal{V}(0) \cap \gamma_1}, \]

\[ \hat{f}_1 = f_1 - (B_1(D_y - B_1(D_y)))u |_{\mathcal{V}(0) \cap \gamma_1} - (T_1(D_y - T_1(D_y)))u |_{\mathcal{V}(0) \cap \gamma_1} - (T_1(D_y) \hat{u})(\mathcal{O}_1(y)) |_{\mathcal{V}(0) \cap \gamma_1}, \]

\[ \hat{f}_2 = f_2 - (B_2(D_y - B_2(D_y)))u |_{\mathcal{V}(0) \cap \gamma_2} - (T_2(D_y - T_2(D_y)))u(\mathcal{G}_2 y) |_{\mathcal{V}(0) \cap \gamma_2}. \]

Since $T_{1\mu}(D_y)$ is a homogeneous operator of order $m_{1\mu}$, from (3.13) and (3.4) it follows that the components $f_{12}^{(k, \ell)}$ of the vector $f_{12} = \{f_{12}^{(k, \ell)}\}$ are linear combinations of the functions $r^{i\lambda r - m_{1\mu}}(i \ln r)^q, 0 \le q \le k.$

Moreover, by virtue of the condition $0 < a - a_1 < 1$, we have $\hat{f} \in H^{l_1}_{a_1}(\mathcal{V}(0) \cap K), \hat{f}_{\sigma\mu} \in H^{l_1 + 2m - m_{\sigma\mu}, -1/2}_{a_1}(\mathcal{V}(0) \cap \gamma_\sigma).$

Thus we see that (3.11), (3.12) is a model nonlocal problem in $\mathcal{V}(0) \cap K$ with the right-hand side being the sum of the “regular” function $\{\hat{f}, \hat{f}_1, \hat{f}_2\}$ and the “special” function $\{0, -c_2 f_{12}, 0\}$. The asymptotics of the vector $f_{12}$ is defined by the asymptotics of the solution $u$ near the point $g_2$, i.e., by the vector $u_2$ (see (3.4)).

Now we are to apply the results of [12]. Put

\[ \mathcal{B}_1(D_y) u = B_1(D_y) u |_{\gamma_1}, \]
\[ \mathcal{B}_2(D_y) u = B_2(D_y) u |_{\gamma_2} + (T_2(D_y) u)(\mathcal{G}_2 y) |_{\gamma_2} \]

and introduce the bounded operator

\[ \mathcal{L}_1 = \{\mathcal{P}(D_y), \mathcal{B}_\sigma(D_y)\} : \]
\[ H^{l_1 + 2m}_{a_1}(K) \rightarrow H^{l_1}_{a_1}(K, \gamma) \overset{\text{def}}{=} H^{l_1}_{a_1}(K) \times {\prod}_{\sigma=1, 2} \prod_{\mu=1}^m H^{l_1 + 2m - m_{\sigma\mu}, -1/2}_{a_1}(\gamma_\sigma), \]

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which corresponds to the model nonlocal problem in the angle $K$.

Write the operators involved into $\mathcal{L}_1$ in polar coordinates: $\mathcal{P}(D_y) = r^{-2m}\mathcal{P}(\omega, D_\omega; rD_r)$, $B_\sigma(D_y) = \{r^{-m_\sigma}\tilde{B}_\sigma(\omega, D_\omega; rD_r)\}_{\mu=1}^m$, $T_\sigma(D_y) = \{r^{-m_\sigma}\tilde{T}_\sigma(\omega, D_\omega; rD_r)\}_{\mu=1}^m$.

Consider the operator–valued function

$$\tilde{L}_1(\lambda) : W_2^{l+2m}(b_1, b_2) \rightarrow W_2^l[b_1, b_2] \overset{def}{=} W_2^l(b_1, b_2) \times \mathbb{C}^{2m}$$

given by

$$\tilde{L}_1(\lambda)\varphi = \{\mathcal{P}(\omega, D_\omega, \lambda)\varphi, \tilde{B}_1(\omega, D_\omega, \lambda)\varphi(\omega)|_{\omega=b_1}, \tilde{B}_2(\omega, D_\omega, \lambda)\varphi(\omega)|_{\omega=b_2}, \beta_2^{L_\lambda-m_2}\tilde{T}_2(\omega + \omega_2, D_\omega, \lambda)\varphi(\omega + \omega_2)|_{\omega=b_2}\}.$$ 

The operator $\tilde{L}_1(\lambda)$ is obtained from the operator $L_1$ by passing to polar coordinates, followed by the Mellin transformation with respect to $r$.

From Lemmas 2.1, 2.2 [23] it follows that there exists a finite–meromorphic operator–valued function $\mathcal{R}_1(\lambda)$ such that its poles (except, maybe, a finite number of them) are located inside a double angle of opening less than $\pi$ containing the imaginary axis; moreover, if $\lambda$ is not a pole of $\mathcal{R}_1(\lambda)$, then $\mathcal{R}_1(\lambda)$ is inverse to the operator $\tilde{L}_1(\lambda)$. Thus a number $\lambda$ is a pole of $\mathcal{R}_1(\lambda)$ if and only if $\lambda$ is an eigenvalue (of finite multiplicity) of $\tilde{L}_1(\lambda)$.

If the line $\text{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of $\tilde{L}_1(\lambda)$, then, by virtue of Theorem 2.1 [23], the operator $L_1$ is an isomorphism.

In order to formulate a theorem concerning an asymptotics near $g_1$, let us introduce some denotation. Suppose $\lambda_1$ is an eigenvalue of the operator $\tilde{L}_1(\lambda)$ located inside the strip $a_1 + 1 - l - 2m < \text{Im} \lambda < a + 1 - l - 2m$,

$$\{\varphi_1^{(0,\zeta)}, \ldots, \varphi_1^{(\alpha,1-1,\zeta)} : \zeta = 1, \ldots, J_1\} \quad (3.15)$$

is a canonical system of Jordan chains of the operator $\tilde{L}_1(\lambda)$ corresponding to the eigenvalue $\lambda_1$.

Consider the vector $u_1 = \{u_1^{(k,\zeta)}\}$, where

$$u_1^{(k,\zeta)}(\omega, r) = r^{i\lambda_1} \sum_{q=0}^{k} \frac{1}{q!} (i \ln r)^q \varphi_1^{(k-q,\zeta)}(\omega), \quad (3.16)$$

$(\omega, r)$ are polar coordinates with the pole at the point $g_1 = 0$.

Notice that (see Lemma 2.2 [12]) the vector $u_1$, the elements $u_1^{(k,\zeta)}$ of which are defined by (3.16), satisfies the relation

$$L_1u_1 = 0. \quad (3.17)$$

If $\lambda_2$ is an eigenvalue of $\tilde{L}_1(\lambda)$ (i.e., $\lambda_2 = \lambda_1$), then denote by $\kappa(\lambda_2)$ the greatest of partial multiplicities of $\lambda_2$. If $\lambda_2$ is not an eigenvalue of $\tilde{L}_1(\lambda)$ (i.e., $\lambda_2 \neq \lambda_1$), put $\kappa(\lambda_2) = 0$.

**Theorem 3.2.** Let the lines $\text{Im} \lambda = a_1 + 1 - l - 2m$, $\text{Im} \lambda = a + 1 - l - 2m$ contain no eigenvalues of $\tilde{L}_1(\lambda)$ and the strip $a_1 + 1 - l - 2m < \text{Im} \lambda < a + 1 - l - 2m$ contain the only eigenvalue $\lambda_1$ of $\tilde{L}_1(\lambda)$. Then

$$u(y) = c_1u_1(y) + c_2u_{12}(y) + \hat{u}(y) \quad (y \in \mathcal{V}(g_1) \cap G). \quad (3.18)$$
Remark 3.3. Here \(u_1 = \{u_1^{(k, \zeta)}\}\), where \(u_1^{(k, \zeta)}\) is defined by (3.16); \(u_1 = \{u_1^{(k, \zeta)}\}\), where \(u_1^{(k, \zeta)}\) is a linear combination (which will be strictly defined in the proof below) of the functions \(r^{\lambda a}(i \ln r)^q \varphi_{k,\zeta}(\omega)\), \(\varphi_{k,\zeta} \in W_2^{l+2m}(b_1, b_2)\), \(0 \leq q \leq k + x(\lambda_2)\), \((\omega, r)\) are polar coordinates with the pole at \(g_1\); \(c_1 = \{c_1^{(k, \zeta)}\}\) is a vector of some constants; \(c_2\) is the vector of constants appearing in (3.10); \(\hat{u} \in H_1^{l+2m}(\mathcal{V}(g_1) \cap G)\).

Proof. Let \(u_{12} = \{u_{12}^{(k, \zeta)}\}\) be a particular solution (which is defined by Lemma 4.3 [12]) for the problem

\[
\mathcal{P}(D_y)u_{12} = 0 \quad (y \in K), \tag{3.19}
\]
\[
B_1(D_y)u_{12} = -f_{12}, \quad B_2(D_y)u_{12} = 0, \tag{3.20}
\]

Here \(f_{12} = \{f_{12}^{(k, \zeta)}\}\) is defined by (3.13). We remind that each element \(f_{12}^{(k, \zeta)}\) is the linear combination of the functions \(r^{\lambda a}(i \ln r)^q\), \(0 \leq q \leq k\). Therefore, by Lemma 4.3 [12], the particular solution \(u_{12}\) has the form described in the formulation of the theorem. Moreover, each element \(u_{12}^{(k, \zeta)}\) of the vector \(u_{12}\) is uniquely defined if \(\lambda_2\) is not an eigenvalue of \(\tilde{\mathcal{L}}_1(\lambda)\) (i.e., \(\lambda_2 \neq \lambda_1\)). Otherwise (i.e., if \(\lambda_2 = \lambda_1\)) it is defined accurate to an arbitrary linear combination of power solutions (3.16) corresponding to the eigenvalue \(\lambda_2 = \lambda_1\). Later on we shall suppose the particular solution \(u_{12} = \{u_{12}^{(k, \zeta)}\}\) being fixed.

Introduce the cut-off function \(\eta \in C^\infty(\mathbb{R}^2)\) equal to 1 in some neighborhood of the origin and vanishing outside \(\mathcal{V}(0)\). Put \(w = \eta(u - c_2u_{12})\). Since \(u\) is a solution for problem (3.11), (3.12) and \(u_{12}\) is a solution for problem (3.19), (3.20), one can easily check (using Leibnitz’s formula) that \(\tilde{\mathcal{L}}_1w \in H_1^l(K, \gamma)\). Therefore, to conclude the proof, it remains to apply Theorem 2.2 [12], which establishes the asymptotics of solutions for nonlocal problems in angles.

Remark 3.3. In fact, the assumption that the line \(\text{Im } \lambda = a+1-l-2m\) contains no eigenvalues of \(\tilde{\mathcal{L}}_1(\lambda)\) is superfluous. But, using the results of the paper [12], one can show (similarly to Remark 3.1) that this assumption does not lead to the loss in generality. Therefore we remain it since it will be used for studying the adjoint nonlocal problem and for calculating the coefficients \(c_1^{(k, \zeta)}\).

Remark 3.4. If the strip \(a_1 + 1 - l - 2m \leq \text{Im } \lambda < a + 1 - l - 2m\) has no eigenvalues of \(\tilde{\mathcal{L}}_1(\lambda)\), but still has one (say, \(\lambda_2\)) of \(\tilde{\mathcal{L}}_2(\lambda)\), then (3.18) will assume the form \(u(y) = c_2u_{12}(y) + \hat{u}(y)\) \((y \in \mathcal{V}(g_1) \cap G)\). And only if the mentioned strip has neither eigenvalues of \(\tilde{\mathcal{L}}_1(\lambda)\) nor \(\tilde{\mathcal{L}}_2(\lambda)\), the solution \(u\) will be “regular” near the point \(g_1\): \(u \in H_1^{l+2m}(\mathcal{V}(g_1) \cap G)\) for any right-hand side \(\{f, f_\sigma\} \in H_1^l(G, \mathcal{Y})\). (Cf. Remark 3.2.)

Theorem 3.2 shows that the asymptotic behavior of solutions for problem (2.1), (2.2) near the point \(g_1\) depends on the data of the problem both near the point \(g_1\) itself and near the point \(g_2 \in G\), which is connected with \(g_1\): \(g_2 = \Omega_1(g_1)\).

IV. Quite similarly to the above one can study an asymptotics of solutions for problem (2.1), (2.2) near the points \(h_\nu\) in terms of the spectral properties of the operators \(\tilde{\mathcal{L}}'_\nu(\lambda)\) corresponding to the points \(h_\nu\), \(\nu = 1, 2\). The operators \(\tilde{\mathcal{L}}'_\nu(\lambda)\) are introduced similarly to the operators \(\tilde{\mathcal{L}}_\nu(\lambda)\).

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6See footnote 4 on page 7
In order not to repeat the analogous computations, we suppose that the solution \( u \) is “regular” in some neighborhoods \( V(h_\nu) \) of the points \( h_\nu \): \( u \in H^{l+2m}_{a_1} \left(V(h_\nu)\right), \nu = 1, 2. \)

Now we shall formulate the condition that summarize all our assumptions concerning the spectral properties of the operators \( \tilde{L}_\nu(\lambda) \) and \( \tilde{L}'_\nu(\lambda) \).

**Condition 3.1.** Let the lines \( \text{Im}\;\lambda = a_1 + 1 - l - 2m \) and \( \text{Im}\;\lambda = a + 1 - l - 2m \) contain no eigenvalues of the operator–valued functions \( \tilde{L}_\nu(\lambda), \tilde{L}'_\nu(\lambda) \); let the strip \( a + 1 - l - 2m < \text{Im}\;\lambda < a_1 + 1 - l - 2m \) contain the only eigenvalue \( \lambda_\nu \) of \( \tilde{L}_\nu(\lambda) \) and no eigenvalues of \( \tilde{L}'_\nu(\lambda), \nu = 1, 2. \)

We notice that the assumption concerning the absence of eigenvalues of \( \tilde{L}'_\nu(\lambda), \nu = 1, 2 \), in the strip \( a + 1 - l - 2m \leq \text{Im}\;\lambda \leq a_1 + 1 - l - 2m \) guarantees the regularity of solutions in the above sense (see Remarks 2.2 and 3.4).

From now on we suppose Condition 3.1 being fulfilled.

In the sequel it will be convenient to have an asymptotics formula for the solution \( u \in H^{l+2m}_{a_1}(G) \) to problem (2.1), (2.2) in the whole domain \( G \). To write this formula, we introduce infinitely smooth functions \( \eta_\nu \) with the supports in \( V(g_\nu) \) such that \( \eta_\nu(y) = 1 \) in some neighborhoods of the points \( g_\nu, \nu = 1, 2 \). Consider the vector–functions
\[
U_1 = \eta_1 u_1; \quad U_2 = \eta_2 u_2 + \eta_1 u_{12}. \tag{3.21}
\]
The functions \( U_\nu \) are supposed to be defined in the whole domain \( G \), vanishing outside \( V(g_\nu) \), \( \nu = 1, 2 \). Then Theorems 3.1 and 3.2 yield the following asymptotics of \( u \in H^{l+2m}_{a_1}(G) \):
\[
u \equiv \left(c_1 U_1 + c_2 U_2\right) \left(\text{mod}\;H^{l+2m}_{a_1}(G)\right). \tag{3.22}
\]

Let us remark for the sequel that the components \( U_\nu^{(k,\zeta)} \) of the vector \( U_\nu = \{U_\nu^{(k,\zeta)}\} \) are such that
\[
LU_\nu^{(k,\zeta)} \in H^{l}_{a_1}(G, \ U). \tag{3.23}
\]
To prove it, we firstly put \( \{F, F_\sigma\} = LU_1^{(k,\zeta)} \). Since the support of \( U_1^{(k,\zeta)} = \eta_1 u_1^{(k,\zeta)} \) is contained in \( V(g_1) = V(0) \), we have
\[
\begin{align*}
P(y, D_y)\eta_1 u_1^{(k,\zeta)}(y, \gamma_1) &= F(y) \quad (y \in K), \\
B_1(y, D_y)\eta_1 u_1^{(k,\zeta)}(y, \gamma_1) &= F_1(y) \quad (y \in \gamma_1), \\
B_2(y, D_y)\eta_1 u_1^{(k,\zeta)}(y, \gamma_2) + (T_2(y, D_y)\eta_1 u_1^{(k,\zeta)}(y, \gamma_2)) &= F_2(y) \quad (y \in \gamma_2).
\end{align*}
\]
But the vector \( u_1 = \{u_1^{(k,\zeta)}\} \) satisfies (3.17). Therefore, using Leibnitz’s formula, we obtain \( \{F, F_\sigma\} \in H^{l}_{a_1}(G, \ U) \).

Now put \( \{F, F_\sigma\} = LU_2^{(k,\zeta)} \). Similarly to the above we get
\[
\begin{align*}
P(y, D_y)\eta_2 u_2^{(k,\zeta)} + P(y, D_y)\eta_1 u_2^{(k,\zeta)} &= F(y) \quad (y \in K), \\
B_1(y, D_y)\eta_2 u_2^{(k,\zeta)}(y, \gamma_1) + (T_1(y, D_y)\eta_2 u_2^{(k,\zeta)}(y, \gamma_1)) &= F_1(y) \quad (y \in \gamma_1), \\
B_2(y, D_y)\eta_2 u_2^{(k,\zeta)}(y, \gamma_2) + (T_2(y, D_y)\eta_2 u_2^{(k,\zeta)}(y, \gamma_2)) &= F_2(y) \quad (y \in \gamma_2).
\end{align*}
\]
But the vector \( u_2 = \{u_2^{(k,\zeta)}\} \) satisfies (3.5), and the vector \( u_{12} = \{u_{12}^{(k,\zeta)}\} \) satisfies (3.19), (3.20). Therefore, using Leibnitz’s formula, we again obtain \( \{F, F_\sigma\} \in H^{l}_{a_1}(G, \ U) \).

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4 Index of nonlocal problems

In this section we study some properties of the kernel, cokernel, and index of the operator \( L \) corresponding to nonlocal problem (2.1), (2.2). In particular, using the asymptotics formula (3.22), we shall obtain a formula connecting the indices of one and the same problem (2.1), (2.2), but being considered in different weighted spaces.

Let \( \kappa_{\nu} \) be a full multiplicity of the eigenvalue \( \lambda_{\nu} \) of the operator–valued function \( \tilde{L}_{\nu}(\lambda) \):

\[
\kappa_{\nu} = \sum_{\zeta=1}^{J_{\nu}} \kappa_{\zeta,\nu}.
\]

Put \( \kappa = \kappa_{1} + \kappa_{2} \).

Lemma 4.1. Homogeneous problem (2.1), (2.2) can have no more than \( \kappa \) linearly independent modulo \( H^{l+2m}_{a}(G) \) solutions from the space \( H^{l+2m}_{a1}(G) \).

Proof. Put the functions \( U_{\nu}(k,\zeta) \) \( (\nu = 1, 2; \zeta = 1, \ldots, J_{\nu}; k = 0, \ldots, \kappa_{\zeta,\nu} - 1) \) in arbitrary order and denote the elements of the obtained ordered set by \( U_{1}, \ldots, U_{\kappa} \).

Suppose \( Z_{t} \in H^{l+2m}_{a1}(G), t = 1, \ldots, d \), are linearly independent modulo \( H^{l+2m}_{a1}(G) \) solutions to homogeneous problem (2.1), (2.2), and \( d > \kappa \). Then by (3.22) we have

\[
Z_{t} \equiv \left( \sum_{k=1}^{\kappa} c_{tk} U_{k} \right) \left( \text{mod } H^{l+2m}_{a1}(G) \right), \quad t = 1, \ldots, d, \quad (4.1)
\]

where \( c_{tk} \) are some constants. Consider the equation for unknown constants \( h_{1}, \ldots, h_{d} \):

\[
\sum_{t=1}^{d} h_{t} Z_{t} = 0 \left( \text{mod } H^{l+2m}_{a1}(G) \right).
\]

By virtue of (4.1) it is equivalent to

\[
\sum_{k=1}^{\kappa} \left( \sum_{t=1}^{d} c_{tk} h_{t} \right) U_{k} = 0 \left( \text{mod } H^{l+2m}_{a1}(G) \right).
\]

Since \( U_{1}, \ldots, U_{\kappa} \) are linearly independent modulo \( H^{l+2m}_{a1}(G) \), the last equation is equivalent to the system

\[
\sum_{t=1}^{d} c_{tk} h_{t} = 0, \quad k = 1, \ldots, \kappa.
\]

By virtue of the inequality \( d > \kappa \), this system necessarily has a nontrivial solution \( (h_{1}, \ldots, h_{d}) \neq 0 \), while we supposed that \( Z_{1}, \ldots, Z_{d} \) were linearly independent modulo \( H^{l+2m}_{a1}(G) \). This contradiction proves the lemma.

Consider the vector \( U = (U_{1}, \ldots, U_{\kappa})^{T} \). Let \( Z = (Z_{1}, \ldots, Z_{d})^{T}, 0 \leq d \leq \kappa \), be a vector, components of which form a maximal set of solutions to homogeneous problem (2.1), (2.2) from the space \( H^{l+2m}_{a1}(G) \), linearly independent modulo the space \( H^{l+2m}_{a1}(G) \) (i.e., a basis
modulo $H_{a_1}^{l+2m}(G)$. By virtue of (4.1), we have $Z \equiv C U \pmod{H_{a_1}^{l+2m}(G)}$, where $C$ is a matrix of order $d \times \infty$. Rank of $C$ equals $d$. Without loss in generality we assume that $C = (C_1, C_2)$, where $C_1$ is a nonsingular $(d \times d)$-matrix. Hence $C_1^{-1} Z \equiv (I, C_1^{-1} C_2) U \pmod{H_{a_1}^{l+2m}(G)}$, where $I$ is the identity $(d \times d)$-matrix. Therefore we can suppose that

$$Z_t \equiv \left( U_t + \sum_{k=d+1}^{\infty} c_{kt} U_k \right) \pmod{H_{a_1}^{l+2m}(G)}, \quad t = 1, \ldots, d. \tag{4.2}$$

We shall say that basis (4.2) is canonical. From now on we fix some canonical basis.

**II.** Along with the operator $L = \{P(y, D_y), B_\sigma(y, D_y)\} : H_{a_1}^{l+2m}(G) \rightarrow H_{a_1}^l(G, \mathcal{Y})$, we consider the adjoint operator $L^* : H_{a_1}^l(G, \mathcal{Y})^* \rightarrow H_{a_1}^{l+2m}(G)^*$ given by

$$< u, L^* \{v, w_\sigma\} > = < P(y, D_y) u, v > + \sum_{\sigma=1}^{m} \sum_{\mu=1}^{m} < B_\sigma(y, D_y) u, w_\sigma > \quad \tag{4.3}$$

for all $u \in H_{a_1}^{l+2m}(G), \{v, w_\sigma\} \in H_{a_1}^l(G, \mathcal{Y})^*$. Here and below $< \cdot, \cdot >$ stands for the sesquilinear form on a pair of corresponding adjoint spaces.

**Lemma 4.2.** Let $d$ be a number of elements in basis (4.2). Then the equation $L^* \{v, w_\sigma\} = 0$ has $\infty - d$ solutions from $H_{a_1}^l(G, \mathcal{Y})^*$, linearly independent modulo $H_{a_1}^{l+2m}(G)^*$.

**Proof.** 1) Let $\{\varphi_t, \psi_{t, \sigma}\}, \ t = 1, \ldots, q$, be some basis modulo $H_{a_1}^l(G, \mathcal{Y})^*$ in the space of solutions from $H_{a_1}^l(G, \mathcal{Y})^*$ for the equation $L^* \{v, w_\sigma\} = 0$.

Suppose $q < \infty - d$. Put $U = c_{d+1} U_{d+1} + \cdots + c_{\infty} U_{\infty}$, where the vector $(c_{d+1}, \ldots, c_{\infty})$ is a nontrivial solution for the $q$ linear algebraic equations

$$< LU, \{\varphi_t, \psi_{t, \sigma}\} > = 0, \quad t = 1, \ldots, q. \tag{4.4}$$

(Notice that, by virtue of (4.3), $LU \in H_{a_1}^l(G, \mathcal{Y})$ and therefore the forms $< LU, \{\varphi_t, \psi_{t, \sigma}\} >$ are well defined). This system does have a nontrivial solution since $q < \infty - d$.

From (4.4) it follows that there exists a solution $\hat{U} \in H_{a_1}^{l+2m}(G)$ for the equation $LU = \hat{U}$. Clearly the function $Z = U - \hat{U} \neq 0$ is a solution from $H_{a_1}^{l+2m}(G)$ for homogeneous problem (2.1), (2.2), which has the asymptotics

$$Z \equiv \left( \sum_{k=d+1}^{\infty} c_k U_k \right) \pmod{H_{a_1}^{l+2m}(G)}. \tag{4.5}$$

We claim that the function $Z$ is linearly independent of $Z_1, \ldots, Z_d$, the elements of basis (4.2) modulo $H_{a_1}^{l+2m}(G)$. Indeed, suppose that

$$Z \equiv \left( \sum_{t=1}^{d} h_t Z_t \right) \pmod{H_{a_1}^{l+2m}(G)};$$

then, by virtue of (4.2), we have

$$Z \equiv \left( \sum_{t=1}^{d} h_t U_t + \sum_{k=d+1}^{\infty} \sum_{t=1}^{d} h_t c_{tk} U_k \right) \pmod{H_{a_1}^{l+2m}(G)}. \tag{4.6}$$
From this, from (4.3), and from the linear independence of the functions \( U_1, \ldots, U_n \) modulo \( H^{l+2m}_{a_1}(G) \), it follows that \( h_1 = \cdots = h_d = 0 \). However, we assumed \( Z_1, \ldots, Z_d \) were the elements of basis \( \{1, 2\} \) modulo \( H^{l+2m}_{a_1}(G) \). This contradiction proves that \( q \geq \kappa - d \).

2) Suppose \( q > \kappa - d \). Denote by \( \{\Phi_h, \Psi_{h,\sigma}\}, h = 1, \ldots, q, \) a system of elements from \( H^l_{a_1}(G, \Upsilon) \) being biortogonal to the system \( \{\Phi_t, \Psi_{t,\sigma}\}, t = 1, \ldots, q, \) and orthogonal to all solutions for the equation \( L^{*}\{v, w_{\sigma}\} = 0 \) from \( H^l_{a_1}(G, \Upsilon)^{*} \). Then there exist solutions \( u_h \in H^{l+2m}_{a_1}(G) \) for the problems \( Lu_h = \{\Phi_h, \Psi_{h,\sigma}\}, h = 1, \ldots, q \). Subtracting from \( u_h \) (if needed) a linear combination of the functions \( Z_1, \ldots, Z_d \) forming basis \( \{1, 2\} \), one can make the relations

\[
u_h = \left( \sum_{k=d+1}^{\kappa} d_{h,k} U_k \right) \mod H^{l+2m}_{a_1}(G), \quad h = 1, \ldots, q. \tag{4.6}
\]

hold.

The functions \( u_1, \ldots, u_q \) are linearly independent modulo \( H^{l+2m}_{a_1}(G) \). Indeed, in the opposite case some linear combination of the functions \( u_h, h = 1, \ldots, q, \) would belong to the space \( H^{l+2m}_{a_1}(G) \). Then the corresponding linear combination of the functions \( Lu_h = \{\Phi_h, \Psi_{h,\sigma}\}, h = 1, \ldots, q, \) would be orthogonal to all the vectors \( \{\phi_t, \psi_{t,\sigma}\}, t = 1, \ldots, q \). This would contradict the choice of the functions \( \{\Phi_h, \Psi_{h,\sigma}\}, h = 1, \ldots, q \). From (4.6) and from the linear independence of the functions \( u_h \), it follows that \( q \leq \kappa - d \). Thus, we necessarily have \( q = \kappa - d \).

III. Consider the operators

\[
L_a = \{P(y, D_y), B_{\sigma}(y, D_y)\} : H^{l+2m}_{a}(G) \to H^{l}_{a}(G, \Upsilon),
\]

\[
L_{a_1} = \{P(y, D_y), B_{\sigma}(y, D_y)\} : H^{l+2m}_{a_1}(G) \to H^{l}_{a_1}(G, \Upsilon).
\]

The operators \( L_a \) and \( L_{a_1} \) correspond to one and the same nonlocal problem \( \{2.1, 2.2\} \), but they act in the spaces with the different weight constants \( (a \text{ and } a_1 \text{ respectively}) \).

**Theorem 4.1.** The operators \( L_a \) and \( L_{a_1} \) are Fredholm, and the following index formula is valid:

\[
\operatorname{ind} L_a = \operatorname{ind} L_{a_1} + \kappa.
\]

**Proof.** By Theorem 3.4 [22], the operators \( L_a \) and \( L_{a_1} \) are Fredholm [4]. By Lemma 4.1, we have \( \dim \ker L_a = \dim \ker L_{a_1} + d \). Then by Lemma 4.2, we have \( \dim \ker L_a^* = \dim \ker L_{a_1}^* - (\kappa - d) \). Hence \( \operatorname{ind} L_a = \dim \ker L_a - \dim \ker L_a^* = \dim \ker L_{a_1} - \dim \ker L_{a_1}^* + \kappa = \operatorname{ind} L_{a_1} + \kappa \). \[\square\]

**Remark 4.1.** Theorem 4.1 remains true without the assumption \( a - a_1 < 1 \), too. Indeed, one can always choose numbers \( a = a^0 > a^1 > \cdots > a^{M} = a_1 \) such that \( 0 < a^i - a^{i+1} < 1 \) and the lines \( \Im \lambda = a^i + 1 - l - 2m \) do not contain eigenvalues of \( \hat{L}_\nu(\lambda), \nu = 1, 2 \). Applying Theorem 4.1 subsequently to the pairs of the operators

\[
L_a = \{P(y, D_y), B_{\sigma}(y, D_y)\} : H^{l+2m}_{a_1}(G) \to H^{l}_{a_1}(G, \Upsilon),
\]

\[
L_{a+1} = \{P(y, D_y), B_{\sigma}(y, D_y)\} : H^{l+2m}_{a_1+1}(G) \to H^{l}_{a_1+1}(G, \Upsilon)
\]

we get the formula \( \operatorname{ind} L_a = \operatorname{ind} L_{a_1} + \kappa \), where \( \kappa \) is the sum of full multiplicities of all eigenvalues of \( \hat{L}_1(\lambda) \) and \( \hat{L}_2(\lambda) \) contained in the strip \( a_1 + 1 - l - 2m < \Im \lambda < a + 1 - l - 2m \).
5 Asymptotics of solutions for adjoint nonlocal problems

In this section we shall obtain an asymptotics near the set $\mathcal{K}$ for solutions to the problem, adjoint to (2.1), (2.2). The results of this section will be applied to calculating the coefficients $c^{(k,j)}_\nu$ in (3.22).

Notice that the approach to the study of adjoint nonlocal problems has been suggested by the author in [9,10,12]. In the papers [9,10], the solvability and smoothness of solutions for model adjoint nonlocal problems in plane and dihedral angles were studied. The paper [12] deals with an asymptotics of solutions for model nonlocal problems in plane angles and in $\mathbb{R}^2 \setminus \{0\}$. In the present work we essentially use both the ideology of the papers [9,10] and the results of the paper [12].

Since we suppose Condition 3.1 being fulfilled, it suffices to obtain appropriate asymptotics formulas only near the points $g_1$ and $g_2$.

Along with formula (4.3) we will use another one for the definition of the adjoint operator. To write this formula, we introduce the following denotation. For any smooth curve $\Upsilon \subset \bar{G}$ and any distribution $w \in H^{k-1/2}_{\alpha_1}(\Upsilon)^*$ we denote by $w \cdot \delta_\Upsilon$ the distribution from $H^k_{\alpha_1}(G)^*$ given by

$$< u, w \cdot \delta_\Upsilon >_G = < u |_\Upsilon, w >_\Upsilon \text{ for all } u \in H^k_{\alpha_1}(G)^*$$

Clearly the support of the distribution $w \cdot \delta_\Upsilon$ is contained in $\bar{\Upsilon}$. Similarly one can define a distribution $w \cdot \delta_\gamma \in H^k_{\alpha_1}(K)^*$, where $\gamma = \{ y \in \mathbb{R}^2 : r > 0, \omega = b \}$ ($b_1 \leq b \leq b_2$) and $w \in H^{k-1/2}_{\alpha_1}(\gamma)^*$.

Denote by $\mathbf{P}^*(y, D_y)$, $B^*_\sigma(y, D_y) = \{B^*_\sigma(y, D_y)\}_{\mu=1}^m$, $T^*_\sigma(y, D_y) = \{T^*_\sigma(y, D_y)\}_{\mu=1}^m$, the operators, formally adjoint to $\mathbf{P}(y, D_y)$, $B_\sigma(y, D_y) = \{B_\sigma(y, D_y)\}_{\mu=1}^m$, $T_\sigma(y, D_y) = \{T_\sigma(y, D_y)\}_{\mu=1}^m$ respectively.

For any distribution $w_{\sigma \mu} \in H^{l+2m-m_\sigma \mu-1/2}_{\alpha_1}(\Omega_{\sigma}(\Upsilon_{\sigma}))^*$ we consider the distribution $w_{\sigma \mu}^{\Omega} \in H^{l+2m-m_\sigma \mu-1/2}_{\alpha_1}(\Omega_{\sigma}(\Upsilon_{\sigma}))^*$ given by

$$< \psi, w_{\sigma \mu}^{\Omega} >_{\Omega_{\sigma}(\Upsilon_{\sigma})} = < \psi(\Omega_{\sigma}(\cdot)), w_{\sigma \mu} >_{\Upsilon_{\sigma}}$$

for all $\psi \in H^{l+2m-m_\sigma \mu-1/2}_{\alpha_1}(\Omega_{\sigma}(\Upsilon_{\sigma}))$.

We claim that the adjoint operator $\mathbf{L}^* : H^l_{\alpha_1}(G, \Upsilon)^* \to H^{l+2m}_{\alpha_1}(G)^*$ can be defined by the formula

$$\mathbf{L}^* \{ v, w_\sigma \} = \mathbf{P}^*(y, D_y)v + \sum_{\sigma=1,2} B^*_\sigma(y, D_y)(w_\sigma \cdot \delta_{\Upsilon_{\sigma}}) + T^*_\sigma(y, D_y)(w_{\sigma}^{\Omega} \cdot \delta_{\Upsilon_{\sigma}}).$$

In this section, for clearness, we denote sesquilinear forms on the pairs of adjoint spaces $H^k_{\alpha_1}(G)^*$ and $H^{k-1/2}_{\alpha_1}(\Upsilon)^*$ by $< \cdot, \cdot >_G$ and $< \cdot, \cdot >_{\Upsilon}$ respectively.
Here and further $w_\sigma = \{w_{\sigma \mu}\}_{\mu=1}^m$, $w_\sigma^\Omega = \{w_{\sigma \mu}^\Omega\}_{\mu=1}^m$.

\[
B_\sigma^*(y, D_y)(w_\sigma \cdot \delta_{\Upsilon_\sigma}) = \sum_{\mu=1}^m B_{\sigma \mu}^*(y, D_y)(w_{\sigma \mu} \cdot \delta_{\Upsilon_\sigma}),
\]

\[
T_\sigma^*(y, D_y)(w_\sigma^\Omega \cdot \delta_{\Omega_\sigma}(\Upsilon_\sigma)) = \sum_{\mu=1}^m T_{\sigma \mu}^*(y, D_y)(w_{\sigma \mu}^\Omega \cdot \delta_{\Omega_\sigma}(\Upsilon_\sigma)).
\]

Indeed, using definition (4.3) of the adjoint operator $L^*$ and then relations (5.2) and (5.1), we get (omitting $(y, D_y)$ for short)

\[
\langle u, L^*\{v, w_\sigma\}\rangle = \langle Pu, v \rangle_G + \sum_{\sigma=1,2} \sum_{\mu=1}^m \left( \langle B_{\sigma \mu} u | \Upsilon_\sigma \rangle, w_{\sigma \mu} >_{\Upsilon_\sigma} + \langle T_{\sigma \mu} u \Omega_\sigma(\cdot) | \Upsilon_\sigma \rangle, w_{\sigma \mu} >_{\Upsilon_\sigma} \right) = \langle Pu, v \rangle_G + \sum_{\sigma=1,2} \sum_{\mu=1}^m \left( \langle B_{\sigma \mu} u, w_{\sigma \mu} \cdot \delta_{\Upsilon_\sigma} \rangle >_G + \langle T_{\sigma \mu} u, w_{\sigma \mu}^\Omega \cdot \delta_{\Omega_\sigma}(\Upsilon_\sigma) \rangle >_G \right)
\]

for all $u \in H^{l+2m}(G)$, which yields (5.3).

We are to study an asymptotics of a given solution $\{v, w_\sigma\} \in H^{l}_{a_1}(G, \Upsilon)^*$ for the problem

\[
L^*\{v, w_\sigma\} = \Psi,
\]

(5.4)
supposing $\Psi \in H^{l+2m}(G)^*$.

For this purpose, parallel to the operator $L^*$, we consider the auxiliary operator

\[
L^*_\Omega : H^{l}_{a_1}(G)^* \times \prod_{\sigma=1,2} \prod_{\mu=1}^m \left( H^{l+2m-m_{\sigma \mu}-1/2}_{a_1}(\Upsilon_\sigma)^* \times H^{l+2m-m_{\sigma \mu}-1/2}_{a_1}(\Omega_\sigma(\Upsilon_\sigma))^* \right) \to H^{l+2m}(G)^*
\]

given by

\[
L^*_\Omega\{v, w_\sigma, w'_\sigma\} = P^*(y, D_y)v + \sum_{\sigma=1,2} \left( B_{\sigma}^*(y, D_y)(w_\sigma \cdot \delta_{\Upsilon_\sigma}) + T_{\sigma}^*(y, D_y)(w'_\sigma \cdot \delta_{\Omega_\sigma}(\Upsilon_\sigma)) \right).
\]

(5.5)

Such an auxiliary operator was used in the papers [10,11] for the study of model nonlocal problems in angles. It also turns out to be very useful in our case. On the one hand, the operator $L^*_\Omega$ is not a nonlocal one since the functions $w_\sigma$ and $w'_\sigma$ are not connected each with another by nonlocal transformations $\Omega_\sigma$ in (5.5). This will allow to use Leibnitz’s formula when necessary. On the other hand, solutions for the problems corresponding to $L^*$ and $L^*_\Omega$ are related in the following way. If $\{v, w_\sigma\}$ is a solution to problem (5.4), then the distribution $\{v, w_\sigma, w_\sigma^\Omega\}$ is a solution to the problem

\[
L^*_\Omega\{v, w_\sigma, w_\sigma^\Omega\} = \Psi,
\]

(5.6)
where \( w^\Omega = \{ w^\Omega_{\sigma} \}_{\mu = 1} \) is defined by (5.2).

So, investigating the asymptotics of the solution \( \{ v, w_\sigma, w^\Omega \} \) for problem (5.6) is equivalent to investigating the asymptotics of the solution \( \{ v, w_\sigma \} \) for problem (5.4).

Our plan is this. At first we will multiply the distribution \( \{ v, w_\sigma, w^\Omega_\sigma \} \) by the cut-off function \( \eta_1 \) and get a corresponding problem near \( g_1 \). Since the operator \( \mathbf{L}^*_\Omega \) is not a nonlocal one, applying Leibnitz’s formula we will show that \( \mathbf{L}^*_\Omega \eta_1 \{ v, w_\sigma, w^\Omega_\sigma \} \in H^{l+2m}_1(G)^* \). Therefore we will arrive at the model adjoint problem in the angle \( K \) with a “regular” right-hand side. Using the results of [12], we will obtain the asymptotics near \( g_1 \). Then we will multiply the distribution \( \{ v, w_\sigma, w^\Omega_\sigma \} \) by \( \eta_2 \) and get a corresponding problem near \( g_2 \). We will arrive at the model adjoint problem in \( \mathbb{R}^2 \). But in this case a right-hand side will be a sum of “regular” and “special” distributions. The asymptotics of the “special” one will be defined by the asymptotics of \( w_1 \) near \( g_1 \), which will have been known from the first step. Further application of the results of [12] will allow to get the asymptotics near \( g_2 \).

Thus let us multiply \( \{ v, w_\sigma, w^\Omega_\sigma \} \) by \( \eta_1 \). Notice that \( \supp \eta_1 \cap \Omega_1(\Gamma_1) = \emptyset \) (see Fig. 2.1) and \( \supp (w_1 \cdot \delta_\Omega_1(\Gamma_1)) \subset \Omega_1(\Gamma_1) \). Therefore \( \eta_1 w^\Omega_1 \cdot \delta_\Omega_1(\Gamma_1) = 0 \). From this and from (5.3), it follows that

\[
\mathbf{L}^*_\Omega \eta_1 \{ v, w_\sigma, w^\Omega_\sigma \} = \mathbf{P}^*(y, D_y) \eta_1 v + B_1^*(y, D_y)(\eta_1 w_1 \cdot \delta_{\Gamma_1}) + B_2^*(y, D_y)(\eta_1 w_2 \cdot \delta_{\Gamma_2}) + T_2^*(y, D_y)(\eta_1 w^\Omega_2 \cdot \delta_{\Omega_2(\Gamma_2)}). \tag{5.7}
\]

Let us show that the distribution \( \eta_1 \{ v, w_\sigma \} \) satisfies the model adjoint problem in the angle \( K \)

\[
\mathcal{L}^*_1 \eta_1 \{ v, w_\sigma \} = \hat{\Psi}, \tag{5.8}
\]

where \( \mathcal{L}^*_1 : H^l_1(K, \gamma)^* \to H^{l+2m}_a(K)^* \) is the operator, adjoint to \( \mathcal{L} : H^{l+2m}_a(K) \to H^l_1(K, \gamma) \); \( \hat{\Psi} \in H^{l+2m}_a(K)^* \).

From (5.6) and Leibnitz’s formula, it follows that, on the one hand,

\[
\mathbf{L}^*_\Omega \eta_1 \{ v, w_\sigma, w^\Omega_\sigma \} = \eta_1 \Psi + \hat{\Psi}_1, \tag{5.9}
\]

where \( \hat{\Psi}_1 \in H^{l+2m}_a(G)^* \) and \( \supp \hat{\Psi}_1 \subset \mathcal{V}(g_1) \). On the other hand, the function \( \eta_1(y) - \eta_1(\Omega_2^{-1}(y)) \) is equal to 0 near \( g_1 \) and has a support inside \( \mathcal{V}(g_1) \) (here we are to suppose \( \supp \eta_1 \) is so small that \( \supp \eta_1(\Omega_2^{-1}(\cdot)) \subset \mathcal{V}(g_1) \)). Hence,

\[
T_2^*(y, D_y)(\eta_1 w^\Omega_2 \cdot \delta_{\Omega_2(\Gamma_2)}) = T_2^*(y, D_y)(\eta_1(\Omega_2^{-1}(\cdot)) w^\Omega_2 \cdot \delta_{\Omega_2(\Gamma_2)}) \in H^{l+2m}_a(G)^*
\]

and has a support inside \( \mathcal{V}(g_1) \). This, (5.7), and (5.9) imply

\[
\mathbf{P}^*(y, D_y) \eta_1 v + B_1^*(y, D_y)(\eta_1 w_1 \cdot \delta_{\Gamma_1}) + B_2^*(y, D_y)(\eta_1 w_2 \cdot \delta_{\Gamma_2}) + T_2^*(y, D_y)(\eta_1(\Omega_2^{-1}(\cdot)) w^\Omega_2 \cdot \delta_{\Omega_2(\Gamma_2)}) = \hat{\Psi}_2,
\]

where \( \hat{\Psi}_2 \in H^{l+2m}_a(G)^* \) and \( \supp \hat{\Psi}_2 \subset \mathcal{V}(g_1) \).

Let \( \mathbf{P}^*(D_y), B_2^*(D_y), T_2^*(D_y) \) be the principal homogeneous parts of the operators \( \mathbf{P}^*(g_1, D_y), B_2^*(g_1, D_y), T_2^*(g_1, D_y) \) respectively. Then, using Leibnitz’s formula, we finally get

\[
\mathbf{P}^*(D_y) \eta_1 v + B_1^*(D_y)(\eta_1 w_1 \cdot \delta_{\Gamma_1}) + B_2^*(D_y)(\eta_1 w_2 \cdot \delta_{\Gamma_2}) + T_2^*(D_y)(\eta_1(\Omega_2^{-1}(\cdot)) w^\Omega_2 \cdot \delta_{\Omega_2(\Gamma_2)}) = \hat{\Psi}, \tag{5.10}
\]
where $\hat{\Psi} \in H^{l+2m}_{a}(K)^*$ and $\text{supp} \hat{\Psi} \subset \mathcal{V}(0)$. Here we also took into account that near the point $g_1 = 0$ the domain $G$ and the curves $\Upsilon_{\sigma}$ coincide with the angle $K$ and the arms $\gamma_{\sigma}$ respectively, while the transformation $\Omega_2$ coincides with the linear operator $\mathcal{G}_2$. But it is easily seen that equality (\ref{equality5.10}) is quite the same as equality (\ref{equality5.8}). Indeed, the only not evident identity one should check is

$$< u, T_2^*(D_y)(\eta_1(\mathcal{G}_2^{-1}) \cdot \delta_{\mathcal{G}_2(\gamma_2)})>_K = <(T_2(D_y)u)(\mathcal{G}_2^*), \eta_1 w_2 >_{\gamma_2},$$

which follows from

$$< u, T_2^*(D_y)(\eta_1(\mathcal{G}_2^{-1}) \cdot \delta_{\mathcal{G}_2(\gamma_2)})>_K = <\eta_1(\mathcal{G}_2^{-1})T_2(D_y)u|_{\mathcal{G}_2(\gamma_2)}, w_2 >_{\mathcal{G}_2(\gamma_2)} = < \eta_1(T_2(D_y)u)(\mathcal{G}_2^*), w_2 >_{\gamma_2}. $$

Here we subsequently used (5.1) and (5.2).

Applying the results of the paper [12] to equality (5.8), we shall now obtain the asymptotics of the distribution $\eta_1 \{v, w_\sigma\}$. Introduce some denotation. Put

$$v_1^{(k, \zeta)} = r_i \bar{\lambda}_1 + 2m - 2 \sum_{q=0}^{k} \frac{1}{q!} (i \ln r)^q \psi_1^{(k-q, \zeta)},$$

$$w_1^{(k, \zeta)} = r_i \bar{\lambda}_1 + m_{\sigma \mu} - 1 \sum_{q=0}^{k} \frac{1}{q!} (i \ln r)^q \chi_1^{(k-q, \zeta)}.$$

Here

$$\{\psi_1^{(0, \zeta)}, \chi_1^{(0, \zeta)}\}, \ldots, \{\psi_1^{(\nu_{\xi,1} - 1, \zeta)}, \chi_1^{(\nu_{\xi,1} - 1, \zeta)}\} : \zeta = 1, \ldots, J_1 \}$$

are Jordan chains of the operator $\hat{\mathcal{L}}_1^*(\lambda)$ (adjoint to $\hat{\mathcal{L}}_1(\lambda)$) corresponding to the eigenvalue $\bar{\lambda}_1$ and forming a canonical system. These chains are supposed (see Lemma 3.2 [12]) to satisfy the following condition of biorthogonality and normalization with respect to the Jordan chains (5.15):

$$\sum_{p=0}^{\nu} \sum_{q=0}^{k} \frac{1}{(\nu + k + 1 - p - q)!} < \partial^{\nu+k+1-p-q}_\lambda \hat{\mathcal{L}}_1(\lambda_1) \varphi_1^{(q, \zeta)}, \psi_1^{(p, \zeta)}, \chi_1^{(p, \zeta)} > = \delta_{\zeta, \zeta} \delta_{\bar{\nu}_{\xi,1} - k, \mu}. \quad (5.12)$$

Here $\zeta, \xi = 1, \ldots, J_1; \nu = 0, \ldots, \nu_{\xi,1} - 1; k = 0, \ldots, \nu_{\xi,1} - 1; \delta_{\zeta, \zeta}$ is the Kronecker symbol.

Analogously to section 3 we introduce the vectors $w_1^{(k, \zeta)} = \{w_1^{(k, \zeta)}\}_{\mu = 1}^{m}$ and $\{v_1, w_1, \sigma\} = \{v_1^{(k, \zeta)}, w_1^{(k, \zeta)}\}.$

We remark that, by Lemma 3.1 [12], the distributions $\{v_1^{(k, \zeta)}, w_1^{(k, \zeta)}\}$ satisfy the homogeneous equation $\hat{\mathcal{L}}_1^*\{v_1^{(k, \zeta)}, w_1^{(k, \zeta)}\} = 0.$

Now from equality (5.8) and Theorem 4.2 [12] we get the following result.

**Theorem 5.1.** Let $\{v, w_\sigma\} \in H^l_{a_1}(G, \Upsilon)^*$ be a solution for equation (3.4) with a right-hand side $\Psi \in H^{l+2m}_{a}(G)^*$. Then the following asymptotics formula is valid:

$$\eta_1 \{v, w_\sigma\} \equiv d_1 \eta_1 \{v_1, w_1, \sigma\} \mod H^l_{a}(G, \Upsilon)^*, \quad (5.13)$$

where $\{v_1, w_1, \sigma\} = \{v_1^{(k, \zeta)}, w_1^{(k, \zeta)}\}$ is defined by (5.11), $d_1 = \{d_1^{(k, \zeta)}\}$ is a vector of some constants.
II. Now let us study the asymptotics of the solution \( \{v,\ w_\sigma\} \) for equation (5.3) near the point \( g_2 \). As mentioned above, in this case we will arrive at the model adjoint problem in \( \mathbb{R}^2 \setminus \{g_2\} \). A right–hand side of the equation obtained will be a sum of “regular” and “special” distributions. The asymptotics of the latter one will be defined by the asymptotics of \( w_\sigma \) near \( g_1 \), which is already known (see Theorem [5.1]).

We multiply \( \{v,\ w_\sigma,\ w_\sigma^\Omega\} \) by \( \eta_2 \). Since the support of \( \eta_2 \) is contained in \( \mathcal{V}(g_2) \) and therefore does not intersect with \( \Upsilon_1, \ Y_2, \) and \( \Omega_2(\Upsilon_2) \) (see Fig. 2.1), we have \( \eta_2w_1 \cdot \delta_{\Upsilon_1} = 0, \eta_2w_2 \cdot \delta_{\Upsilon_2} = 0, \) and \( \eta_2w_2^2 \cdot \delta_{\Upsilon_2(\Upsilon_2)} = 0 \). Combining this with (5.3), we get

\[
L_2^*\eta_2\{v,\ w_\sigma,\ w_\sigma^\Omega\} = P^*(y,\ D_y)\eta_2v + T_1^*(y,\ D_y)(\eta_2w_1^\Omega \cdot \delta_{\Omega_1(\Upsilon_1)}).
\]  

From (5.6) and Leibnitz’s formula, it follows that \( L_2^*\eta_2\{v,\ w_\sigma,\ w_\sigma^\Omega\} = \eta_2\Psi + \tilde{\Psi}_2 \), where \( \tilde{\Psi}_2 \in H_{a_1}^{t+2m}(G)^* \) and supp \( \tilde{\Psi}_2 \subset \mathcal{V}(g_2) \).

Let \( P^*(D_y), \ T_1^*(D_y) \) be the principal homogeneous parts of the operators \( P^*(g_2,\ D_y), \ T_1^*(g_2,\ D_y) \) respectively. Then, analogously to the above, we derive that

\[
P^*(D_y)\eta_2v = -T_1^*(D_y)(\eta_2w_1^\Omega \cdot \delta_{\Omega_1(\Upsilon_1)}) + \tilde{\Psi}_3,
\]

where \( \tilde{\Psi}_3 \in H_{a_2}^{t+2m}(G)^* \) and supp \( \tilde{\Psi}_3 \subset \mathcal{V}(g_2) \).

From definition (5.2) of the distribution \( w_1^\Omega \) and from asymptotics formula (5.13), it follows that \( \eta_2w_1^\Omega \) is a linear combination of the functions \( r^{i\lambda_1 + m_1 - 1}(i\ln r)^q \) modulo \( H_{a_2}^{t+2m-m_2-1/2}(\Upsilon_1) \), where \( r \) is a polar radius of polar coordinates with the pole at \( g_2 \). Since \( T_1^*(D_y) \) is a homogeneous operator of order \( m_1 \), we can write (5.15) (taking into account (5.13)) in the form

\[
L_2^*\eta_2v = -\eta_2d_1\Psi_{21} + \tilde{\Psi}.
\]

Here \( \tilde{\Psi} \in H_{a_2}^{t+2m}(\mathbb{R}^2)^* \) and supp \( \tilde{\Psi} \subset \mathcal{V}(g_2) \); \( \Psi_{21} = \{\Psi_{21}^{(k,\zeta)}\} \), where \( \Psi_{21}^{(k,\zeta)} \) is a linear combination of the distributions \( r^{i\lambda_1 - 2}(i\ln r)^q\Psi_{21q}^{(k,\zeta)}, \ 0 \leq q \leq k \), \( \Psi_{21q}^{(k,\zeta)} \in W_{a_2}^{t+2m}(0,\ 2\pi)^* \), \( r \) is a polar radius of polar coordinates with the pole at \( g_1 \); \( d_1 \) is the vector of constants from (5.13).

Thus we see that (5.16) is a model adjoint problem in \( \mathbb{R}^2 \) with the right–hand side being the sum of the “regular” distribution \( \tilde{\Psi} \) and the “special” distribution \( \Psi_{21} \). The asymptotics of \( \Psi_{21} \) is defined by the asymptotics of the solution \( w_1 \) near the point \( g_1 \), i.e., by the functions \( w_{1,\sigma}^{(k,\mu)} \) (see [5.11]).

Applying the results of the paper [12] to equality (5.16), we shall now obtain the asymptotics of the distribution \( \eta_2v \). Introduce some notation. Put

\[
v_{2(q,\zeta)} = r^{i\lambda_2 + 2m - 2} \sum_{q=0}^k \frac{1}{q!}(i\ln r)^q\Psi_{21q}^{(k-q,\zeta)}.
\]

The distribution \( \eta_2r^{i\lambda_1 - 2}(i\ln r)^q\Psi_{21q}^{(k,\zeta)} \in H_{a_1}^{t+2m}(\mathbb{R}^2)^* \) is given by

\[
< u, \ r^{i\lambda_1 - 2}(i\ln r)^q\Psi_{21q}^{(k,\zeta)} > = \int_0^\infty < \eta_2(\cdot, r)u(\cdot, r), \ \Psi_{21q}^{(k,\zeta)} > (0,\ 2\pi) \frac{r^{i\lambda_1 - 2}(i\ln r)^q}{r}dr
\]

for all \( u \in H_{a_1}^{t+2m}(\mathbb{R}^2) \).
Here
\[
\left\{ \psi_2^{(0, \zeta)} , \ldots , \psi_2^{(\kappa_2-1, \zeta)} : \zeta = 1, \ldots , J_2 \right\}
\]
are Jordan chains of the operator \( \tilde{L}_2^*(\lambda) \) (adjoint to \( \tilde{L}_2(\lambda) \)) corresponding to the eigenvalue \( \tilde{\lambda}_2 \) and forming a canonical system. These chains are supposed (see [12]) to satisfy the following condition of biorthogonality and normalization with respect to the Jordan chains (3.3):
\[
\sum_{p=0}^{\nu} \sum_{q=0}^{k} \frac{1}{(\nu + k + 1 - p - q)!} < \partial^\nu+k+1-p-q \tilde{L}_2(\lambda_2) \psi_2^{(q, \xi)},
\]
\[
\psi_2^{(p, \zeta)} = \delta_{\zeta, \xi} \delta_{\kappa_2, 2-k-1, p}. \quad (5.18)
\]
Here \( \zeta, \xi = 1, \ldots , J_2; \nu = 0, \ldots , \kappa_2-1; k = 0, \ldots , \kappa_2-1. \)

Analogously to section 3, we introduce the vector \( v_2 = \{ v_2^{(k, \zeta)} \} \).

We remark that, according to [12, section 5], the distributions \( v_2^{(k, \zeta)} \) satisfy the homogeneous equation \( \tilde{L}_2 v_2^{(k, \zeta)} = 0. \)

If \( \tilde{\lambda}_1 \) is an eigenvalue of \( \tilde{L}_2^*(\lambda) \) (i.e., \( \tilde{\lambda}_1 = \tilde{\lambda}_2 \)), then denote by \( \varpi(\tilde{\lambda}_1) \) the greatest of partial multiplicities of \( \tilde{\lambda}_1. \) If \( \tilde{\lambda}_1 \) is not an eigenvalue of \( \tilde{L}_2^*(\lambda) \) (i.e., \( \tilde{\lambda}_1 \neq \tilde{\lambda}_2 \)), put \( \varpi(\tilde{\lambda}_1) = 0. \)

**Theorem 5.2.** Let \( \{ v, w_0 \} \in H^1_{3a}(G, \ U) \) be a solution for equation (5.4) with a right-hand side \( \Psi \in H^{1+2m}_{4a}(G)^* \). Then the following asymptotics formula is valid:
\[
\eta_2 v \equiv \left( d_2 \eta_2 v_2 + d_1 \eta_2 v_{21} \right) (\mod H^1_{3a}(G, \ U)^*). \quad (5.19)
\]

Here \( v_2 = \{ v_2^{(k, \zeta)} \} \) is defined by (5.17), \( d_2 = \{ d_2^{(k, \zeta)} \} \) is a vector of some constants; \( v_{21} = \{ v_{21}^{(k, \zeta)} \} \), where \( v_{21}^{(k, \zeta)} \) is a linear combination of the functions \( r^{i \lambda_1-2(i \ln r)} \Psi_{21q}^{(k, \zeta)} \), \( 0 \leq q \leq k \), \( \Psi_{21q}^{(k, \zeta)} \in W^1_{2,2a}(0, 2\pi)^*; \) \( d_1 \) is the vector of constants appearing in (5.15).

**Proof.** Let \( v_{21} = \{ v_{21}^{(k, \zeta)} \} \) be a particular solution (which is defined by Lemma 5.2 [12]) for the problem
\[
\tilde{L}_2^{*} v_{21} = -\Psi_{21}, \quad (5.20)
\]
where \( \Psi_{21} = \{ \Psi_{21}^{(k, \zeta)} \} \) is a “special” distribution appearing in (5.16). We remind that each element \( \Psi_{21}^{(k, \zeta)} \) is a linear combination of the distributions \( r^{i \lambda_1-2(i \ln r)} \Psi_{21q}^{(k, \zeta)} \), \( 0 \leq q \leq k \). Therefore, by Lemma 5.2 [12], the particular solution \( v_{21} \) has the form described in the formulation of the theorem. Moreover, each component \( v_{21}^{(k, \zeta)} \) of the vector \( v_{21} \) is uniquely defined if \( \tilde{\lambda}_1 \) is not an eigenvalue of \( \tilde{L}_2^*(\lambda) \) (i.e., \( \tilde{\lambda}_1 \neq \tilde{\lambda}_2 \)). Otherwise (i.e., if \( \tilde{\lambda}_1 = \tilde{\lambda}_2 \)) it is defined accurate to an arbitrary linear combination of power solutions (5.17) corresponding to the eigenvalue \( \tilde{\lambda}_1 = \tilde{\lambda}_2. \)

From now on we shall suppose a particular solution \( v_{21} = \{ v_{21}^{(k, \zeta)} \} \) being fixed.

Combining (5.16) with (5.20) and using Leibnitz’s formula, one easily checks that \( \tilde{L}_2^*(\eta_2 v - d_1 \eta_2 v_{21}) \in H^{1+2m}_{4a}(\mathbb{R}^2)^*. \) Now the asymptotics (5.19) is resulted from Theorem 5.3 [12], which establishes the asymptotics of solutions for adjoint problems in \( \mathbb{R}^2. \)

**Theorem 5.2** shows that the asymptotic behavior of solutions for adjoint nonlocal problem (5.4) near the point \( g_2 \) depends on the data of the problem both near the point \( g_2 \) itself and near the point \( g_1, \) which is connected with \( g_2: g_1 = \Omega_1^{-1}(g_2). \)
IV. Let us write the asymptotics formula for the solution \( \{v, w_\sigma\} \in H^1_{a_1}(G, \Upsilon)^* \) for adjoint nonlocal problem (5.4) in the whole domain \( G \). Put (cf. (3.21))

\[
\{V_2, W_{2,\sigma}\} = \eta_2\{v_2, 0\}; \quad \{V_1, W_{1,\sigma}\} = \eta_1\{v_1, w_{1,\sigma}\} + \eta_2\{v_2, 0\}.
\]

(5.21)

Now Theorems 5.1 and 5.2 yield the following asymptotics of \( \{v, w_\sigma\} \in H^1_{a_1}(G, \Upsilon)^* \) (cf. formula (3.22)):

\[
\{v, w_\sigma\} \equiv (d_1\{V_1, W_{1,\sigma}\} + d_2\{V_2, W_{2,\sigma}\}) \left( \bmod H^{l+2m}_{a_1}(G, \Upsilon)^* \right).
\]

(5.22)

6 Calculation of the coefficients in the asymptotics formulas

I. In this section we will calculate the coefficients \( c^{(k,\zeta)}_\nu \) appearing in asymptotics (3.22).

To begin with, let us remark that the coefficients can be calculated in the following way. At first one should find \( c^{(k,\zeta)}_1 \). Since in the neighborhood \( V(g_2) \) of the point \( g_2 \) the function \( u \) has asymptotics (3.6), by Theorem 5.2 [12] we have

\[
c^{(k,\zeta)}_2 = \langle L_2\eta_2 u, iv^{(\kappa,1-k-1,\zeta)}_2 \rangle,
\]

(6.1)

where \( v^{(k,\zeta)}_2 \) is defined in (5.17). Further, by Theorem 3.2 the function \( u' = u - c_2u_{12} \) (where \( c_2 \) is calculated in (6.1), \( u_{12} \) is defined in the proof of Theorem 3.2) has the following asymptotics in the neighborhood \( V(g_1) \) of the point \( g_1 \):

\[
u'(y) = c_1u_1(y) + \hat{u}(y) \quad (y \in V(g_1) \cap G).
\]

(6.2)

Here \( u_1 \) is defined by (3.16); \( c_1 \) is to be found; \( \hat{u} \in H^{l+2m}_{a_1}(V(g_1) \cap G) \). From asymptotics (6.2) and Theorem 4.1 [12], it follows that

\[
c^{(k,\zeta)}_1 = \langle L_1\eta_1 u', iv^{(\kappa,1-k-1,\zeta)}_1 w^{(\kappa,1-k-1,\zeta)}_1 \rangle,
\]

(6.3)

where \( \{v^{(k,\zeta)}_1, w^{(k,\zeta)}_1\} \) is defined in (5.11).

Formulas (6.1) and (6.3) show that the value of \( c_1 \) (as well as the general form of the asymptotics near \( g_1 \)) depends not only on the data of the problem near the point \( g_1 \) but also from the data near \( g_2 = \Omega_1(g_1) \).

We remark that similarly to (6.1) and (6.3) one can calculate the coefficients \( c_\nu \) with the help of the Green formula and so-called formally adjoint problems generated by the Green formula. The corresponding technique is developed in [10, 12]. We will not recall the Green formula here, but only mention that the corresponding formulas for \( c_\nu \) are immediately obtained if we use Theorems 5.4 [12] and 4.3 [12] instead of Theorems 5.2 [12] and 4.1 [12] respectively.

\[\text{In this case, additionally to Conditions 2.2 and 2.3 one must demand the system } \{B_{\sigma\nu}(D_y)\}_{\nu=1}^m \text{ to be normal on } \gamma_\sigma \ (\sigma = 1, 2), \text{ where } B_{\sigma\nu}(D_y) \text{ is the principle homogeneous part of } B_{\sigma\nu}(g_1, D_y).\]
Formally adjoint problems have the advantage that they are considered in “original” spaces, but not in adjoint ones (spaces of distributions). Therefore corresponding eigenvectors and associated vectors can be found explicitly in a number of cases.

But, anyway, both adjoint problem— and formally adjoint problem–based formulas for $c_\nu$ involve the solution $u$ itself. Further we are to get formulas allowing to calculate the coefficients $c_\nu$ only in terms of a right–hand side $\{f, f_\nu\}$ of problem (2.1), (2.2).

II. We are supposed to calculate $c_\nu$ with the help of some special distributions from the kernel of the operator $L^*: H^{l+2m}_a(G, \gamma) \to H^{l+2m}_a(G)^*$. To begin with, assume that $\{v, w_\sigma\} \in H^l_a(G, \gamma)$ is an arbitrary distribution from the kernel of $L^*$.

Let us calculate the value of the expression $\langle L u, i\{v, w_\sigma\}\rangle$.

We suppose that the following consistent condition is fulfilled. If the vector $c_\nu$ contains $c_\nu^{(k, \zeta)}$ in its $k$th position, then the vector $d_\nu$ has $d_\nu^{(k, \zeta, \nu-k-\zeta, \nu)}$ in its $\nu$th position. The same is true for all the other vectors related to the adjoint problem $\{\{v_1, w_{1,\sigma}\}, v_2, \text{etc.}\}$.

Besides, we keep assuming that the Jordan chains corresponding to the eigenvalues $\lambda_\nu$ and $\tilde{\lambda}_\nu$ of the operators $L_\nu(\lambda)$ and $L_\nu(\lambda^*)$ respectively satisfy conditions of biorthogonality and normalization (5.12) (for $\nu = 1$) and (5.13) (for $\nu = 2$).

By virtue of (3.22), we have $L u = L c_1 U_1 + L c_2 U_2 + L \hat{u}$, where $\hat{u} \in H^{l+2m}_a(G)$. Since $\{v, w_\sigma\}$ belongs to the kernel of $L^*: H^{l+2m}_a(G, \gamma) \to H^{l+2m}_a(G)^*$, we get $\langle L \hat{u}, i\{v, w_\sigma\}\rangle = 0$. Therefore, by virtue of (3.23), we can write

$$\langle L u, i\{v, w_\sigma\}\rangle = \langle L c_1 U_1, i\{v, w_\sigma\}\rangle + \langle L c_2 U_2, i\{v, w_\sigma\}\rangle. \quad (6.4)$$

Let $\eta_\nu(\omega, r)$ be the function $\eta_\nu$ written in polar coordinates with the pole at $g_\nu$ ($\nu = 1, 2$). For $\varepsilon > 0$ we introduce the functions $\eta_{\nu, \varepsilon}(\omega, r) = \eta_\nu(\omega, r/\varepsilon)$.

At first let us consider the 1st term in the right–hand side of (6.4). Since the difference $\eta_1 - \eta_{1,\varepsilon}$ vanishes near $g_1$, we have $(\eta_1 - \eta_{1,\varepsilon}) c_1 u_1 \in H^{l+2m}_a(G)$. It follows from this and from (3.21) that

$$\langle L c_1 U_1, i\{v, w_\sigma\}\rangle = \langle L c_1 \eta_{1,\varepsilon} u_1, i\{v, w_\sigma\}\rangle. \quad (6.5)$$

Put for short $U_{1,\varepsilon} = c_1 \eta_{1,\varepsilon} u_1$. Since $\text{supp } U_{1,\varepsilon} \subset V(g_1) \cap G = V(0) \cap K$, we have

$$L U_{1,\varepsilon} = L U_{1,\varepsilon} + \{P(y, D_y) - P(D_y), B_{\sigma}(y, D_y) - B_{\sigma}(D_y)\} U_{1,\varepsilon}.$$

Here $P(D_y)$ is the principal homogeneous part of $P(g_1, D_y)$; $B_{\sigma}(D_y)$ is defined by (3.14). This and Theorem 5.1 imply that the right–hand side of (6.5) has the form

$$\langle L_1 U_{1,\varepsilon}, id_{1}\{v_1, w_{1,\sigma}\}\rangle + \langle L_1 U_{1,\varepsilon}, i\{F, G_\sigma\}\rangle + \langle \{P(y, D_y) - P(D_y), B_{\sigma}(y, D_y) - B_{\sigma}(D_y)\} U_{1,\varepsilon}, i(d_{1}\{v_1, w_{1,\sigma}\} + \{F, G_\sigma\}\rangle, \quad (6.6)$$

where $\{v_1, w_{1,\sigma}\}$ is defined by (5.11), $\{F, G_\sigma\} \in H^l_a(K, \gamma)^*$. By Theorem 4.1 [12], the 1st term in (6.6) is equal to $(c_1, d_1)$ [11]. The 2nd term in (6.6) is majorized by

$$c \|U_{1,\varepsilon}\|_{H^{l+2m}_a(K)} \|\{F, G_\sigma\}\|_{H^l_a(K, \gamma)^*} = O(1),$$

\[11\] Here $(c_1, d_1)$ (and further $(c_2, d_2)$, etc) stands for the inner product of the corresponding complex vectors:

$$(c_1, d_1) = \sum_{k, \zeta} c_1^{(k, \zeta)} d_1^{(k, \zeta, \nu-k-\zeta, \nu)}.$$
where we use the Hardy–Littlewood symbol “O” with its usual interpretation (O(1) tends to 0 as \( \varepsilon \to 0 \)).

By virtue of the boundedness of the imbedding operator of \( H^{l+2m}_{a_1+1}(K) \) into \( H^{l+2m-1}_{a_1}(K) \), Lemma 3.3' [15], and the inequality \( a < a_1 + 1 \), the last term in (6.6) is majorized by

\[
 c\|u_{1,\varepsilon}\|_{H^{l+2m}_{a_1+1}(K)}\|d_1\{v_1, \ w_{1,\sigma}\} + \{F, \ G_\sigma\}\|_{H_{a_1}(K, \ \gamma)} \leq c'\|u_{1,\varepsilon}\|_{H^{l+2m}_{a_1}(K)} = O(1).
\]

Thus, as \( \varepsilon \) tends to 0, we get

\[
 < Lc_1U_1, \ i\{v, \ w_\sigma\} >= (c_1, \ d_1).
\] (6.7)

Now let us consider the 2nd term in the right–hand side of (6.4). Since the functions \((\eta_2 - \eta_{2,\varepsilon})c_2u_2\) and \((\eta_1 - \eta_{1,\varepsilon})c_2u_{12}\) belong to the space \( H^{l+2m}_{a_1}(G) \), we obtain from (3.21) that

\[
< Lc_2U_2, \ i\{v, \ w_\sigma\} >= < Lc_2(\eta_2,\varepsilon,u_2 + \eta_{1,\varepsilon}u_{12}), \ i\{v, \ w_\sigma\} > .
\] (6.8)

Put for short \( U_{2,\varepsilon} = c_2\eta_2,\varepsilon,u_2, \ U_{12,\varepsilon} = c_2\eta_{1,\varepsilon}u_{12} \). Using Theorems 5.1 and 5.2, write the right–hand side of (6.8) in the form

\[
< Lc_2U_{2,\varepsilon}, \ id_2v_2 > + < Lc_2U_{2,\varepsilon}, \ id_1v_{21} > + < \mathcal{P}(D_y)U_{12,\varepsilon}, \ id_1v_1 > + < \mathcal{B}_1(D_y)U_{12,\varepsilon} + (T_1(D_y)U_{2,\varepsilon})(\Omega_1(y))|_{\gamma_1}, \ id_1w_{1,1} > + < \mathcal{B}_2(D_y)U_{12,\varepsilon}id_1w_{1,2} > + O(1). \] (6.9)

Here \( \mathcal{P}(D_y), \ T_1(D_y) \) are the principal homogeneous parts of \( \mathcal{P}(g_1, \ D_y), \ T(g_2, \ D_y) \) respectively; \( \mathcal{B}_\sigma(D_y), \ \sigma = 1, \ 2, \) are defined by (3.14).

By Theorem 5.2 [12], the 1st term in (6.9) is equal to \((c_2, \ d_2)\).

Since \( Lc_2u_2 = 0 \) (see (3.5)), the 2nd term in (6.9) is equal to

\[
< [L_2, \ \eta_{2,\varepsilon}]c_2u_2, \ id_1v_{21} >,
\] (6.10)

where \([, \ · \ ]\) is the commutator. Using the condition \( 0 < a - a_1 < 1 \), one can easily check that (6.10) is equal to

\[
(\tilde{A}_{12}(\varepsilon)c_2, \ d_1),
\] (6.11)

where \( \tilde{A}_{12}(\varepsilon) \) is a matrix of the corresponding order, the elements of which are linear combinations of the functions \( \varepsilon^{\lambda_2-\lambda_1}(i \ln \varepsilon)^q \).

Further, let us recall that the function \( u_{12} \) is a solution for problem (3.19), (3.20). Hence the sum of the 3rd, 4th, and 5th terms in (6.9) is equal to

\[
< [\mathcal{P}(D_y), \ \eta_{1,\varepsilon}]c_2u_{12}, \ id_1v_1 > + < [\mathcal{B}_1(D_y), \ \eta_{1,\varepsilon}]c_2u_{12} + ([T_1(D_y), \ \eta_{1,\varepsilon}]c_2u_2)(\Omega_1(y))|_{\gamma_1}, \ id_1w_{1,1} > + < [\mathcal{B}_2(D_y), \ \eta_{1,\varepsilon}]c_2u_{12}, \ id_1w_{1,2} >
\] (6.12)

and therefore is of the form (6.11). Thus we see that

\[
< Lc_2U_2, \ i\{v, \ w_\sigma\} >= (c_2, \ d_2) + (A_{12}(\varepsilon)c_2, \ d_1) + O(1),
\] (6.13)
where $A_{12}(\varepsilon)$ is a matrix of the corresponding order, the elements of which are linear combinations of the functions $\varepsilon^{\lambda_2-\lambda_1}(i \ln \varepsilon)^q$.

From equations (6.4), (6.7), and (6.13), it follows that

$$\langle Lu, i\{v, w_\sigma\}\rangle = (c_2, d_2) + (c_1 + A_{12}(\varepsilon)c_2, d_1) + O(1), \quad (6.14)$$

III. Keeping denotation of section 3 we will denote by $\mathcal{U}_1, \ldots, \mathcal{U}_\kappa$ the ordered set of functions $U^{(k, \zeta)}_\nu$, which are the elements of the vectors $U_\nu$, $\nu = 1, 2$, defined by (3.21).

Denote by $\mathcal{V}_1, \ldots, \mathcal{V}_\kappa$ the set of functions $\{V^{(k, \zeta)}_\nu, W^{(k, \zeta)}_{\nu, \sigma}\}$, which are the elements of the vectors $\{V_\nu, W_{\nu, \sigma}\}$, $\nu = 1, 2$, defined by (3.21).

Suppose the sets $\{\mathcal{U}_1, \ldots, \mathcal{U}_\kappa\}$ and $\{\mathcal{V}_1, \ldots, \mathcal{V}_\kappa\}$ are ordered consistently, i.e., the equality $U_t = U^{(k, \zeta)}_\nu$ is fulfilled simultaneously with the equality $V_t = V^{(k, \zeta)}_\nu, W^{(k, \zeta)}_{\nu, \sigma}$.

In this work we restrict ourselves to the case when $d = 0$ in Lemma 4.2. This means that any solution to homogeneous problem (2.1), (2.2) from the space $A$ has the form

$$U^{(k, \zeta)}_\nu, \quad \nu = 1, 2,$$

and will not be considered here. We shall describe below) and will not be considered here.

So, suppose $d = 0$. Then, by virtue of Lemma 4.2, there exist solutions $\mathcal{Y}_1, \ldots, \mathcal{Y}_\kappa \in H^l(G, \mathcal{Y})^*$ for the equation $Lu = 0$, linearly independent modulo $H^l(G, \mathcal{Y})^*$. By (5.22) we have $\mathcal{Y}_t = \sum_{k=1}^\kappa d_{tk} \mathcal{Y}_k \mod H^l(G, \mathcal{Y})^*$, $t = 1, \ldots, \kappa$. Since $\mathcal{Y}_1, \ldots, \mathcal{Y}_\kappa$ are linearly independent modulo $H^l(G, \mathcal{Y})^*$, the matrix $\|d_{tk}\|$ is nonsingular. Hence, without loss in generality, we can assume that

$$\mathcal{Y}_t \equiv \mathcal{V}_t \mod H^l(G, \mathcal{Y})^*, \quad t = 1, \ldots, \kappa. \quad (6.15)$$

Now let us prove that the elements of the matrix $A_{12}(\varepsilon)$ appearing in (6.14) have finite limits as $\varepsilon \to 0$. This limit will be denoted by $A_{12}$:

$$A_{12} = \lim_{\varepsilon \to 0} A_{12}(\varepsilon).$$

Let $l_\nu$ be the length of the vector $c_\nu$ (or $d_\nu$, which is the same), $\nu = 1, 2$. Clearly, $l_2 + l_1 = \kappa$. Suppose for definiteness that the first $l_2$ elements in the ordered set $\{\mathcal{U}_1, \ldots, \mathcal{U}_\kappa\}$ ($\{\mathcal{V}_1, \ldots, \mathcal{V}_\kappa\}$) are components of the vector $U_2$ ($\{V_2, W_{2, \sigma}\}$) and the last $l_1$ ones are components of the vector $U_1$ ($\{V_1, W_{1, \sigma}\}$):

$$\{\mathcal{U}_1, \ldots, \mathcal{U}_\kappa\} = \left\{U_2, \begin{array}{c} U_1 \end{array} \right\}_{l_2}^{l_1}, \quad \{\mathcal{V}_1, \ldots, \mathcal{V}_\kappa\} = \left\{V_2, \begin{array}{c} V_1 \end{array} \right\}_{l_2}^{l_1}.$$

Now fix an arbitrary $t$ from the set $\{1, \ldots, l_2\}$ and an arbitrary $k$ from the set $\{l_2 + 1, \ldots, \kappa\}$. Substituting in (6.14) $u = \mathcal{U}_t$ (which is a component of the vector $U_2$) and
\{v, w_\sigma\} = \mathcal{Y}_k (\text{which is, by (6.15)}, a \text{ component of the vector } \{V_1, W_{1,\sigma}\} \text{ modulo } H^1_0(G, \Upsilon)^*), we get \(c_1 = 0, d_2 = 0,\) and therefore,

\[ < LU_t, i\mathcal{Y}_k > = a_{tk}(\varepsilon) + O(1). \quad (6.16) \]

Here \(a_{tk}(\varepsilon)\) is the corresponding element of the matrix \(A_{12}(\varepsilon).\) The left–hand side of (6.16) does not depend on \(\varepsilon.\) Therefore \(a_{tk}(\varepsilon)\) has a finite limit as \(\varepsilon \to 0.\)

Thus, passing in (6.14) to the limit as \(\varepsilon \to 0,\) we get

\[ < Lu, i\{v, w_\sigma\} > = (c_2, d_2) + (c_1 + A_{12} c_2, d_1). \quad (6.17) \]

**Theorem 6.1.** Let \(u \in H^{l+2m}_a(G)\) be a solution for problem (2.1), (2.2) with a right–hand side \(\{f, f_\sigma\} \in H^l_1(G, \Upsilon).\) Then \(u\) has the asymptotics

\[ u \equiv \left( \sum_{t=1}^\infty c_t U_t \right) \left( \text{mod } H^{l+2m}_a(G) \right). \quad (6.18) \]

The constants \(c_t\) \((t = 1, \ldots, \infty)\) can be calculated by the formulas

\[ c_t = < \{f, f_\sigma\}, i\mathcal{Y}_t > \quad (6.19) \]

if \(t \leq l_2\) \((\text{i.e., } c_t \text{ coincides with a component of the vector } \{c_2^{(k, \zeta)}\});

\[ c_t = < \{f, f_\sigma\}, i\mathcal{Y}_t - i \left[ A_{12}(\mathcal{Y}_1, \ldots, \mathcal{Y}_{l_2})^T \right]_{t-l_2} > \quad (6.20) \]

if \(l_2 < t \leq \infty\) \((\text{i.e., } c_t \text{ coincides with a component of the vector } \{c_1^{(k, \zeta)}\}).\) Here \([\cdot]_j\) stands for the \(j\)th component of a vector.

**Proof.** Substituting \(\{v, w_\sigma\} = \mathcal{Y}_1, \ldots, \{v, w_\sigma\} = \mathcal{Y}_\infty\) subsequently in (6.17), we obtain formulas (6.19) and (6.20).

Theorem 6.1 shows that the values of the coefficients \(c_\nu^{(k, \zeta)}\) are the functionals over the right–hand sides \(\{f, f_\sigma\}\) of problem (2.1), (2.2). These functionals depend on the data of the problem in the whole domain \(G,\) but not only in the neighborhoods \(\mathcal{V}(g_1)\) and \(\mathcal{V}(g_2).\)

**Remark 6.1.** We remind that the elements of the matrix \(A_{12}(\varepsilon)\) are linear combinations of the functions \(\varepsilon^{\lambda_2 - \lambda_1}(i \ln \varepsilon)^q.\) Hence, if \(\lambda_1 \neq \lambda_2,\) then \(A_{12} = 0.\)

7 Example

1. In this section we consider an example illustrating the results of sections 2–6.
Keeping denotation and assumptions of sections 2 and 3 we consider the following nonlocal problem

\[ P(y, D_y) \equiv \sum_{|\alpha| \leq 2} p_\alpha(y) \frac{\partial^{|\alpha|} u}{\partial y_1^\alpha \partial y_2^\alpha} = f(y) \quad (y \in G \setminus K), \]  

(7.1)

\[ B_\sigma u \equiv u(y)|_{\Gamma_\sigma} + e_\sigma u(\Omega_\sigma(y))|_{\Gamma_\sigma} = f_\sigma(y) \quad (y \in \Gamma_\sigma; \ \sigma = 1, 2). \]  

(7.2)

Here \( P(y, D_y) \) is a 2nd order differential operator, properly elliptic in \( \bar{G} \), with infinitely smooth coefficients \( p_\alpha(y) \); \( e_\sigma \in \mathbb{C} \). For clearness we assume

\[ \sum_{|\alpha| \leq 2} p_\alpha(g_\nu) \frac{\partial^{|\alpha|} u}{\partial y_1^\alpha \partial y_2^\alpha} = \Delta u, \quad \nu = 1, 2. \]  

(7.3)

Let us obtain the asymptotics of a solution \( u \in H^2_a(G) \) for problem (7.1), (7.2) with a right–hand side \( \{f, f_\sigma\} \in H^0_{a_1}(G, \ U) \) de    \( H^3/2_{a_1}(\Upsilon_\sigma) \), assuming \( 0 < a_1 < 1 \).

At first, according to section 3, we consider the asymptotics of the solution \( u \) in the neighborhood \( V(g_2) \) of the point \( g_2 \). For this purpose one must write the model equation in \( \mathbb{R}^2 \setminus \{g_2\} \).

Taking into account (7.3), we obtain

\[ \Delta u = \hat{f}(y) \quad (y \in \mathbb{R}^2 \setminus \{g_2\}), \]  

(7.4)

where \( \hat{f} \in H^0_{a_1}(\mathcal{V}(g_2)) \).

Write equation (7.4) in polar coordinates with the pole at \( g_2 \):

\[ r \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial \omega^2} = r^2 f(\omega, r) \quad (0 < \omega < 2\pi, \ r > 0). \]

Applying formally the Mellin transformation, we get

\[ \frac{d^2 \tilde{u}}{d\omega^2} - \lambda^2 \tilde{u} = \tilde{F}(\lambda, \omega) \quad (0 < \omega < 2\pi), \]

where \( \tilde{u} \) and \( \tilde{F} \) are the Mellin transforms of \( u \) and \( r^2 f \) with respect to \( r \).

Introduce the corresponding operator–valued function

\[ \tilde{L}_2(\lambda) = \frac{d^2}{d\omega^2} - \lambda^2 : W^2_{2\pi}(0, 2\pi) \to L_2(0, 2\pi). \]

Let us suppose, additionally to Condition 3.1 that there is the only eigenvector \( \varphi_2(\omega) \) corresponding to the eigenvalue \( \lambda_2 \ (a_1 - 1 < \text{Im} \lambda_2 < a - 1) \) of \( \tilde{L}_2(\lambda) \) and there are no associated vectors.

Then, by Theorem 3.1 we have

\[ u(y) = c_2 u_2(y) + \hat{u}(y) \quad (y \in \mathcal{V}(g_2)). \]  

(7.5)

Here \( c_2 \) is a scalar constant, \( u_2 = r^{i\lambda_2} \varphi_2(\omega) \) is a power solution for homogeneous equation (7.4); \( (\omega, r) \) are polar coordinates with the pole at \( g_2 \) and the polar axis being, for definiteness, tangent to the curve \( \Omega_1(\Upsilon_1) \) at \( g_2 \); \( \hat{u} \in H^2_{a_1}(\mathcal{V}(g_2)) \).
Now we consider the asymptotics of the solution \( u \) for problem (7.1), (7.2) in the neighborhood \( V(g_1) \) of the point \( g_1 \). Let \( \Omega_1(y) (y \in V(g_1)) \) be a rotation with respect to \( g_1 \) (with no expansion for simplicity) and the shift by the vector \( \overrightarrow{g_1 g_2} \). Let \( \Omega_2(y) (y \in V(g_1)) \) coincide with the operator \( \mathcal{G}_2 \) of a rotation by an angle \( \omega_2 \) (\( b_1 < b_2 + \omega_2 < b_2 \)) and an expansion with a coefficient \( \beta_2 > 0 \).

According to section 3 and assumption (7.3), the asymptotics of a solution for the problem

\[
\Delta u = \hat{f}(y) \quad (y \in V(0) \cap K),
\]

\[
u|_{V(0) \cap \gamma_1} = \hat{f}_1 - c_2 f_{12} \quad (y \in V(0) \cap \gamma_1),
\]

\[
u|_{V(0) \cap \gamma_2} + e_2 u(\mathcal{G}_2 y)|_{V(0) \cap \gamma_2} = f_2 \quad (y \in V(0) \cap \gamma_2).
\]

Here \( \hat{f} \in H^0_{\alpha_1}(V(0) \cap K) \),

\[
\hat{f}_1 = f_1 - e_1 \hat{u}(\Omega_1(y))|_{V(0) \cap \gamma_1} \in H^3_{\alpha_1}(V(0) \cap \gamma_1),
\]

\[
f_{12} = e_1 u_2(\Omega_1(y))|_{V(0) \cap \gamma_1} = e_1 r^{i\lambda_2} \varphi_2(0)^{12}
\]

Similarly to the above we obtain the corresponding operator–valued function \( \tilde{L}_1(\lambda) : W^2_2(b_1, b_2) \to L_2(b_1, b_2) \times \mathbb{C}^2 \) given by

\[
\tilde{L}_1(\lambda)\varphi = \left\{ \frac{d^2 \varphi}{d\omega^2} - \lambda^2 \varphi, \varphi(\omega)|_{\omega=b_1}, \varphi(\omega)|_{\omega=b_2} + e_2 e^{i\lambda \ln \beta_2} \varphi(\omega + \omega_2)|_{\omega=b_2} \right\}.
\]

Let us suppose, additionally to Condition 3.1, that there is the only eigenvector \( \varphi_1(\omega) \) corresponding to the eigenvalue \( \lambda_1 \) (\( a_1 - 1 < \text{Im} \lambda_1 < a_1 - 1 \)) of \( \tilde{L}_1(\lambda) \) and there are no associated vectors.

Then, by Theorem 3.18 we have

\[
u(y) = c_1 u_1(y) + c_2 u_{12}(y) + \hat{u}(y) \quad (y \in V(g_1)).
\]

Here \( c_1 \) is some scalar constant, \( c_2 \) is the constant appearing in (7.5); \( u_1 = r^{i\lambda_1} \varphi_1(\omega) \ (\varphi_1 \in W^2_2(b_1, b_2)) \) is a particular solution for the following problem in the angle \( K \) with the “special” right–hand side (cf. (3.19), (3.20)):

\[
\Delta u = 0 \quad (y \in K),
\]

\[
u|_{\gamma_1} = -f_{12}, \quad \nu|_{\gamma_2} + e_2 u(\mathcal{G}_2 y)|_{\gamma_2} = 0;
\]

\( u_1 = r^{i\lambda_1} \varphi_1(\omega) \) is a solution for homogeneous problem (7.10), (7.11); \( (\omega, r) \) are polar coordinates with the pole at the point \( g_1 = 0 \); \( \hat{u} \in H^2_{\alpha_1}(V(g_1)) \).

To write the asymptotics in the whole domain \( G \), we introduce the functions \( U_1 = \eta_1 u_1, U_2 = \eta_2 u_2 + \eta_1 u_{12} \). Then (7.5) and (7.9) imply:

\[
\text{Let } u \in H^2_{\alpha}(G) \text{ be a solution for problem (7.1), (7.2) with a right–hand side } \{ f, f_\sigma \} \in H^0_{\alpha_1}(G, Y), 0 < a - a_1 < 1. \text{ Then we have}
\]

\[
u \equiv (c_1 U_1 + c_2 U_2) \left( \mod H^2_{\alpha_1}(G) \right),
\]

\(^{12}\)We calculate \( \varphi_2(\omega) \) for \( \omega = 0 \) because of the special choice of polar coordinates (see above).
where \( c_1, c_2 \) are some scalar constants.

**II.** From asymptotics formula \((7.12)\) and Theorem 4.1 we can derive the connection between
the indices of the operators

\[
L_a = \{P(y, D_y), B_\sigma\} : H^2_a(G) \rightarrow H^0_a(G, \mathcal{Y}),
\]

\[
L_{a_1} = \{P(y, D_y), B_\sigma\} : H^2_{a_1}(G) \rightarrow H^0_{a_1}(G, \mathcal{Y})
\]

corresponding to problem \((7.1), (7.2)\), but acting in different weighted spaces. Since the sum
of full multiplicities of eigenvalues \( \lambda_1 \) and \( \lambda_2 \) is equal to 2 in our case, the connection between
the indices is as follows:

\[
\text{ind } L_a = \text{ind } L_{a_1} + 2.
\]

**III.** To calculate the coefficients \( c_\nu \) in formula \((7.12)\), we will study the asymptotics of
solutions for the adjoint nonlocal problem.

Consider the operator \( L^* : H^0_{a_1}(G, \mathcal{Y})^* \rightarrow H^2_{a_1}(G)^* \),
adjoint to \( L = \{P(y, D_y), B_\sigma\} : H^2_{a_1}(G) \rightarrow H^0_{a_1}(G, \mathcal{Y}) \). The operator \( L^* \) is given by

\[
< u, L^* \{v, w_\sigma\} > = < P(y, D_y)u, v >_G + \sum_{\sigma=1,2} < B_\sigma u, w_\sigma >_{\mathcal{Y}_\sigma},
\]

where \( \{v, w_\sigma\} \in H^0_{a_1}(G, \mathcal{Y})^* \), \( u \in H^2_{a_1}(G) \).

Let us study the asymptotics of a solution \( \{v, w_\sigma\} \in H^0_{a_1}(G, \mathcal{Y})^* \) for the problem

\[
L^* \{v, w_\sigma\} = \Psi, \quad (7.13)
\]

where \( \Psi \in H^2_a(G)^* \).

In [12] it is shown that \( \bar{\lambda}_\nu \) is an eigenvalue of the operator \( \tilde{\mathcal{L}}^*_{\nu}(\lambda) \),
adjoint to \( \tilde{\mathcal{L}}_{\nu}(\bar{\lambda}) \). Denote by \( \{\psi_1, \chi_{1,\sigma}\} \in L_2(b_1, b_2) \times \mathbb{C}^2 \) \( (\psi_2 \in L_2(0, 2\pi)) \)
the eigenvector of \( \tilde{\mathcal{L}}^*_1(\lambda) \) \( (\tilde{\mathcal{L}}^*_2(\lambda)) \) corresponding
to the eigenvalue \( \bar{\lambda}_1 \) \( (\bar{\lambda}_2) \). Conditions of biorthogonality and normalization \((5.12)\)
and \((5.18)\) assume the form

\[
< -2\nu \varphi_\nu, \psi_\nu > = 1^{13}\quad (7.14)
\]

Put \( \{v_1, w_{1,\sigma}\} = \{r^{i\bar{\lambda}_1}\psi_1, r^{i\bar{\lambda}_1-1}\chi_{1,\sigma}\} \) \( (v_2 = r^{i\bar{\lambda}_2}\psi_2) \),
where \( (\omega, r) \) are polar coordinates with
the pole at \( g_1 \) \( (\omega, r) \) \( (g_2) \) \( (g_2) \) \( (g_2) \)
and the polar axis being tangent to the curve \( \Omega_1(\mathcal{Y}_1) \)
at \( g_2 \).

Further, by Theorem 5.1 we have

\[
\eta_1 \{v, w_\sigma\} \equiv d_1 \eta_1 \{v_1, w_{1,\sigma}\} \left( \text{mod } H^0_a(G, \mathcal{Y})^* \right), \quad (7.15)
\]

where \( d_1 \) is some scalar constant. By Theorem 5.2 we have

\[
\eta_2 v \equiv \left( d_2 \eta_2 v_2 + d_1 \eta_2 v_{21} \right) \left( \text{mod } H^0_a(G)^* \right). \quad (7.16)
\]

Here \( d_2 \) is some scalar constant, \( d_1 \) is the constant appearing in \((7.15)\).
\( v_{21} = r^{i\bar{\lambda}_1}\psi_{21} \) \( (\omega, r) \) \( (\omega, r) \) \( (\omega, r) \)
are polar coordinates with the pole at \( g_2 \) and the polar axis being tangent to the curve \( \Omega_1(\mathcal{Y}_1) \)

\^[13]One can show that \( \lambda_\nu \neq 0 \) whenever there are no associated vectors corresponding to the eigenvalue \( \lambda_\nu \).
Hence there always exist vectors \( \{\psi_1, \chi_{1,\sigma}\} \) and \( \psi_2 \) satisfying \((7.14)\).
at \( g_2; \psi_{21} \in L^2(0, 2\pi) \). Moreover, the distribution \( \psi_{21} \) is a particular solution for the following adjoint equation in \( \mathbb{R}^2 \setminus \{g_2\} \) with the “special” right-hand side (cf. (5.20)):

\[
\int_{\mathbb{R}^2} \Delta u \cdot \bar{v} \, dy = \int_0^\infty u(0, r) \cdot (-\bar{e}_1 \chi_{1,1} r^{1/\lambda_1 - 1}) \, dr \quad \text{for all } u \in C^\infty_0(\mathbb{R}^2 \setminus \{g_2\})^{14}
\]

Put \( \{V_2, W_{2,\sigma}\} = \eta_2 \{v_2, 0\}, \{V_1, W_{1,\sigma}\} = \eta_1 \{v_1, w_{1,\sigma}\} + \eta_2 \{v_{21}, 0\} \).

Then (7.15) and (7.16) imply:

Let \( \{v, w_{\sigma}\} \in H^0_{a_1}(G, \Upsilon)^* \) be a solution for problem (7.13) with a right-hand side \( \Psi \in H^2_a(G)^* \). Then we have

\[
\{v, w_{\sigma}\} \equiv (d_1 \{V_1, W_{1,\sigma}\} + d_2 \{V_2, W_{2,\sigma}\}) \mod H^2_a(G, \Upsilon)^*, \quad (7.17)
\]

where \( d_1, d_2 \) are some constants.

**IV.** Now let us calculate the coefficients \( c_\nu \) appearing in (7.12). Formulas (6.1) and (6.3) assume the form

\[
c_2 = \langle \Delta (\eta_2 u), i v_2 \rangle_{\mathbb{R}^2}, \quad c_1 = \langle \Delta u', u'|_{\gamma_1}, u'|_{\gamma_2} + e_2 u'(G_2 y)|_{\gamma_2} \rangle_i \{v_1, w_{1,\sigma}\},
\]

where \( u' = \eta_1(u - c_2 u_{12}) \).

Now let us write the formulas allowing to calculate the coefficients \( c_\nu \) only in terms of a right-hand side \( \{f, f_{\sigma}\} \) of problem (7.1), (7.2) (i.e., independent of a solution \( u \)).

Following section 6, we assume for simplicity that any solution to homogeneous problem (7.1), (7.2) from the space \( H^2_a(G) \) necessarily belongs to the space \( H^2_a(G) \). Then there exist solutions \( \mathcal{V}_1, \mathcal{V}_2 \in H^0_{a_1}(G, \Upsilon)^* \) for the equation \( L^* \mathcal{V} = 0 \), linearly independent modulo \( H^0_a(G, \Upsilon)^* \) such that

\[
\mathcal{V}_\nu \equiv \{V_\nu, W_{\nu,\sigma}\} \mod H^0_a(G, \Upsilon)^*, \quad \nu = 1, 2.
\]

Let \( \mathcal{V}_{\nu,\varepsilon} \) be the functions defined in section 6.

Then from Theorem 6.1 we obtain the following result.

Let \( u \in L^2_a(G) \) be a solution for problem (7.1), (7.2) with a right-hand side \( \{f, f_{\sigma}\} \in H^0_{a_1}(G, \Upsilon) \). Then the function \( u \in H^2_a(G) \) has asymptotics (7.18). The constants \( c_\nu (\nu = 1, 2) \) are calculated by the formulas

\[
c_2 = \langle \{f, f_{\sigma}\}, i \mathcal{V}_1 \rangle, \quad c_1 = \langle \{f, f_{\sigma}\}, i (\mathcal{V}_1 - A_{12} \mathcal{V}_2) \rangle.
\]

Here \( A_{12} \) is a scalar constant given by

\[
A_{12} = \lim_{\varepsilon \to 0} \langle \Delta (\eta_{2,\varepsilon} u_2), i v_{21} \rangle + \langle \Delta (\eta_{1,\varepsilon} u_{12}), \eta_{1,\varepsilon} u_{12}|_{\gamma_1}, \eta_{1,\varepsilon} f_{12}|_{\gamma_1}, \eta_{1,\varepsilon} u_{12}|_{\gamma_2} + e_2 (\eta_{1,\varepsilon} u_{12}) (G_2 y)|_{\gamma_2} \rangle_i \{v_1, w_{1,\sigma}\}, \quad (7.18)
\]

where the limit does exist.

\[14\text{We calculate } u(\omega, r) \text{ for } \omega = 0 \text{ because of the special choice of polar coordinates (see above).} \]
Remark 7.1. The function \( u_2 (u_{12}) \) is a solution for homogeneous equation (7.4) (a solution for problem (7.10), (7.11) with the special right-hand side \( \{ 0, -f_{12}, 0 \} \)). Therefore, similarly to section 6, one can easily check that \( A_{12} = \lim_{\varepsilon \to 0} \text{const} \cdot \varepsilon^{i(\lambda_2 - \lambda_1)} \). From this and from the existence of the limit in (7.18), it follows that \( A_{12} = 0 \) whenever \( \lambda_1 \neq \lambda_2 \).

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References

[1] A.B. Antonevich, The index and the normal solvability of a general elliptic boundary value problem with a finite group of translations on the boundary, Differentsial’nye Uravneniya, 8 (1972), 309-317; English transl. in Differential Equations, 8 (1974).

[2] R. Beals, Nonlocal elliptic boundary value problems, Bull. Amer. Math. Soc., 70 (1964), 693-696.

[3] A.V. Bitsadze and A.A. Samarskii, On some simple generalizations of linear elliptic boundary value problems, Dokl. Akad. Nauk SSSR, 185 (1969), 739-740; English transl. in Soviet Math. Dokl., 10 (1969).

[4] F. Browder, Non-local elliptic boundary value problems, Amer. J. Math., 86 (1964), 735-750.

[5] T. Carleman, Sur la théorie des equations integrales et ses applications, Verhandlungen des Internat. Math. Kongr. Zürich, 1 (1932), 132-151.

[6] S.D. Eidel’man and N.V. Zhitarashu, Nonlocal boundary value problems for elliptic equations, Mat. Issled., 6 (1971), 63-73 (Russian).

[7] W. Feller, Diffusion processes in one dimension, Trans. Amer. Math. Soc., 77 (1954), 1-30.

[8] I.C. Gohberg and E.I. Sigal, An operator generalization of the logarithmic residue theorem and the theorem of Rouché, Mat. Sb., 84 (126) (1971), 607-629; English transl. in Math. USSR Sb., 13 (1971).

[9] P.L. Gurevich, Nonlocal elliptic problems in dihedral angles and the Green formula Dokl. Akad. Nauk, 379 (2001), 735-738; English transl. in Russian Acad. Sci. Dokl. Math., (2001).

[10] P.L. Gurevich, Nonlocal problems for elliptic equations in dihedral angles and the Green formula, Mitteilungen aus dem Math. Seminar Giessen, Math. Inst. Univ. Giessen, Germany, 247 (2001), 1-74.

[11] P.L. Gurevich, Solvability of nonlocal elliptic problems in dihedral angles, Mat. Zametki, 72 (2002), 178-197; English transl. in Math. Notes, 72 (2002).
[12] P.L. Gurevich, Asymptotics of solutions for nonlocal elliptic problems in plane angles, *Tr. semin. im. I.G. Petrovskogo*, 23 (2003); English transl. in *J. Math. Sci., New York* (2004).

[13] A.K. Gushchin and V.P. Mikhailov, On solvability of nonlocal problems for elliptic equations of second order, *Mat. sb.*, 185 (1994), 121-160; English transl. in *Math. Sb.*, (1994).

[14] K.Yu. Kishkis, The index of a Bitsadze–Samarskii problem for harmonic functions, *Differentsial’nye Uravneniya*, 24 (1988), 105-110; English transl. in *Differential Equations*, 24 (1988), 83–87.

[15] V.A. Kondrat’ev, Boundary value problems for elliptic equations in domains with conical or angular points, *Trudy Moskov. Mat. Obshch.*, 16 (1967), 209-292; English transl. in *Trans. Moscow Math. Soc.*, 16 (1967).

[16] O.A. Kovaleva and A.L. Skubachevskii, Solvability of nonlocal elliptic problems in weighted spaces, *Mat. Zametki*, 67 (2000), 882-898; English transl. in *Math. Notes*, 67 (2000).

[17] J.L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications, Vol. I*, Springer, Berlin, 1972.

[18] S.A. Nazarov and B.A. Plamenevskii, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, Moscow: Nauka, 1991; English transl. in De Gruyter Expositions in Mathematics, 13. Walter de Gruyter Publichers, Berlin – New York, 1994.

[19] M. Picone, Equazione integrale traducente il più generale problema lineare per le equazioni differenziali lineari ordinarie di qualsivoglia ordine, *Academia nazionale dei Lincei. Atti dei convegni*, 15 (1932), 942-948.

[20] V.V. Podiapolskii, Completeness and basisness by Abel of a system of root functions of some nonlocal problem, *Differentsial’nye Uravneniya*, 35 (1999), 568-569; English transl. in *Differential Equations*, 35 (1999).

[21] A.L. Skubachevskii, Nonlocal elliptic problems with a parameter, *Mat. Sb.*, 121 (163) (1983), 201-210; English transl. in *Math. USSR Sb.*, 49 (1984).

[22] A.L. Skubachevskii, Elliptic problems with nonlocal conditions near the boundary, *Mat. Sb.*, 129 (171) (1986), 279-302; English transl. in *Math. USSR Sb.*, 57 (1987).

[23] A.L. Skubachevskii, Model nonlocal problems for elliptic equations in dihedral angles, *Differentsial’nye Uravneniya*, 26 (1990), 120-131; English transl. in *Differential Equations*, 26 (1990).

[24] A.L. Skubachevskii, Truncation–function method in the theory of nonlocal problems, *Differentsial’nye Uravneniya*, 27 (1991), 128-139; English transl. in *Differential Equations*, 27 (1991).

[25] A.L. Skubachevskii, On the stability of index of nonlocal elliptic problems, *Journal of Mathematical Analysis and Applications*, 160 (1991), 323-341.
[26] A.L. Skubachevskii, *Elliptic Functional Differential Equations and Applications*, Basel–Boston–Berlin, Birkhäuser, 1997.

[27] A.L. Skubachevskii, Regularity of solutions for some nonlocal elliptic problem, *Russian J. of Mathematical Physics*, 8 (2001), 365-374.

[28] A. Sommerfeld, Ein Beitrag zur hydrodinamischen Erklärung der turbulenten Flüssigkeitsbewegungen, *Proc. Intern. Congr. Math., Rome, 1908, Reale Accad. Lincei. Roma.*, 3 (1909), 116-124.

[29] J.D. Tamarkin, *Some General Problems of the Theory of Ordinary Linear Differential Equations and Expansion of an Arbitrary Function in Series of Fundamental Functions*, Petrograd, 1917; adridged English transl. in *Math. Z.*, 27 (1928), 1-54.

[30] A.D. Ventsel’, On boundary conditions for multidimensional diffusion processes, *Teoriya Veroyatn. i ee Primen.*, 4 (1959), 172-185; English transl. in *Theory Prob. and its Appl.*, 4 (1959).

[31] M.I. Vishik, On general boundary value problems for elliptic differential equations, *Trudy Moskov. Mat. Obshch.*, 1 (1952), 187-246; English transl. in *Amer. Math. Soc. Transl.* (2), 24 (1963).