Witten spinors on maximal, conformally flat hypersurfaces

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Received 25 May 2011, in final form 31 July 2011
Published 5 September 2011
Online at stacks.iop.org/CQG/28/185004

Abstract

The boundary conditions that exclude zeros of the solutions of the Witten equation (and hence guarantee the existence of a 3-frame satisfying the so-called special orthonormal frame gauge conditions) are investigated. We determine the general form of the conformally invariant boundary conditions for the Witten equation, and find the boundary conditions that characterize the constant and the conformally constant spinor fields among the solutions of the Witten equations on compact domains in extrinsically and intrinsically flat, and on maximal, intrinsically globally conformally flat spacelike hypersurfaces, respectively. We also provide a number of exact solutions of the Witten equation with various boundary conditions (both at infinity and on inner or outer boundaries) that single out nowhere vanishing spinor fields on the flat, non-extreme Reissner–Nordström and Brill–Lindquist data sets. Our examples show that there is an interplay between the boundary conditions, the global topology of the hypersurface and the existence/non-existence of zeros of the solutions of the Witten equation.

PACS number: 04.20.Cv

1. Introduction

It is well known that spinorial techniques, and in particular the Witten equation and the use of two-component spinors, greatly simplified the proof of the positivity of both the ADM and Bondi–Sachs energies, even in the presence of black holes (see e.g. [1, 2]). However, spinors and the Witten equation play only an auxiliary role in the proofs. The only essential point in these investigations was the existence of the solution with given asymptotic behaviour. The spinor fields could have zeros, and the zeros did not have any significance.
A slightly different approach to the proof of the positivity of the gravitational energy was suggested in [3]. That was based on the tetrad formulation of general relativity, and to obtain a simple positivity proof a gauge condition for the orthonormal frame fields had to be imposed [4, 5]. However, since there is a natural correspondence between nowhere vanishing spinor fields on a spacelike hypersurface and non-singular orthogonal vector bases (triads) there [6], one could expect that the orthonormal frame gauge condition could be translated into the language of spinors. Indeed, this gauge condition has been reformulated in this way, and finding its solution is equivalent to finding a solution of the Witten equation with vanishing extrinsic curvature but, in general, with a mass term [7]. Thus, the great advantage of the use of spinors is that while the frame gauge condition in its original form is a system of nonlinear elliptic partial differential equations (p.d.e.), in its spinorial form it is the linear Witten equation.

However, the solution of the Witten equation can have zeros, which can even form a two-codimensional set [8]. Thus, to have a perfectly well-defined frame field on a given domain, the spinor field should have no zero there. In [9], it was argued that while spinor zeros could occur they are not generic, but no general guidance was given there as to how to know when they would or would not occur. Since the Witten (or, more generally, a Dirac type) equation is elliptic, its solution depends also on the boundary condition in an essential way. Hence the frame gauge condition consists not only of the elliptic differential equation on the spacelike hypersurface, but an appropriate boundary condition selecting a non-degenerate solution must also be specified. Thus, in the spinorial language, the question is: How to find the boundary conditions for the Witten equation that ensure the existence of a nowhere vanishing solution on the given domain?

The existence of the global solution of the frame gauge condition could be a very useful tool in various problems in general relativity. In fact, this could provide a geometrically preferred rigid system of frames of reference on an extended domain of a spacelike hypersurface, and the frame would be controlled only by appropriate boundary conditions. Thus, from the point of view of applications, it is desirable to find the conditions that guarantee the existence of globally non-singular solutions of the frame gauge conditions, or, in other words, the existence of nowhere vanishing solutions of the Witten equation.

Unfortunately, the general problem of finding the appropriate boundary conditions appears to be surprisingly difficult. On the other hand, there are still physically important special cases in which there is some hope of being able to clarify the boundary conditions for nowhere vanishing solutions. Such are the maximal, globally conformally flat data sets, which include e.g. the Reissner–Nordström [10], Brill–Lindquist [11] and Bowen–York [12] data sets. These data sets represent finitely many black holes with specified total mass and linear and angular momenta at spatial infinity.

If the spacelike hypersurface is maximal and intrinsically flat, e.g. when the Cauchy data induced on the spacelike hypersurface are flat (such as a spacelike hyperplane in Minkowski spacetime), then the gauge condition is expected to yield a constant spinor field. Thus the boundary condition in the general, curved case must have a form that reduces to the one specifying the constant spinor field in the flat case. In fact, in the flat case it is natural to expect the geometrically preferred orthonormal frame fields to be just the Cartesian ones, which are in a one-to-one correspondence with the constant spinor fields up to a real constant scale factor. This raises the question of whether it is natural to expect that on intrinsically conformally flat hypersurfaces the spinor field should be proportional to a spinor field which is constant with respect to the flat connection, and if that is the case, then what is the appropriate boundary condition?
Here we investigate the question of the boundary conditions for the Witten equation that yield a nowhere vanishing spinor field in the special case when $\Sigma$ is maximal and its intrinsic geometry is globally conformally flat. We determine the boundary conditions that yield the \textit{conformally constant}, and hence nowhere vanishing spinor fields on compact domains in $\Sigma$. Thus, in particular, the frame gauge condition can be satisfied on such domains in maximal, globally conformally flat spacelike hypersurfaces. On the other hand, as the examples of the non-extreme Reissner–Nordström data set show, the boundary conditions on the different connected components of the boundary \textit{at infinity} cannot be chosen independently.

In the next section, we recall the Witten equation (both in its covariant form and in the Geroch–Held–Penrose (GHP) formalism) and clarify its conformal properties and the relationship between the solutions of the Witten equations and certain geometrically distinguished orthogonal vector bases. Then we determine the general form of the conformally invariant boundary conditions for the Witten equation.

In section 3, explicit solutions of the Witten equation are given. First, in subsection 3.1, a simple solution is given which illustrates what kinds of zeros may appear. Then, in subsection 3.2, we determine the fundamental (in some sense spherically symmetric) solutions of the Witten equation on intrinsically and extrinsically flat 3-spaces. We also discuss the potentially reasonable explicitly given boundary conditions both at infinity and at boundaries that are metric spheres of finite radius. We will see that several apparently natural boundary conditions that have already appeared in various problems can yield spinor fields with one or more zeros. Subsection 3.3 is devoted to the solution of the Witten equation on maximal, globally conformally flat spacelike hypersurfaces, namely on the complete, non-extreme Reissner–Nordström and Brill–Lindquist data sets, as well as on a part of the Reissner–Nordström data set with inner boundary. We will see that on the complete Reissner–Nordström data set, the asymptotic values of the nowhere vanishing solutions of the Witten equation cannot be chosen independently on the two infinities.

Finally, in section 4, we determine the boundary conditions on general boundaries in terms of the boundary Dirac operator (built from a Sen-type connection) that single out the \textit{constant} and the \textit{conformally constant} spinor fields from all the possible solutions of the Witten equation on compact domains in flat and in maximal, globally conformally flat spacelike hypersurfaces on which the Hamiltonian constraint equation is satisfied and the weak energy condition holds, respectively. These imply, in particular, that in these situations the orthonormal frame gauge condition can be used as a perfect gauge condition. Our results are summarized in section 5.

Our conventions and notations are mostly those of [13, 14]. In particular, we use the abstract index formalism, and only the boldface and underlined capital Latin indices are concrete name indices. Abstract Lorentzian tensor indices are freely converted to pairs of unprimed and primed spinor indices and back, according to the rule $a \mapsto AA'$. The signature of the spacetime metric is $-2$, and the spacetime Riemann and Ricci tensors, and the scalar curvature are defined by

$$-^4R_{abcd}X^bY^cZ^d := \nabla_Y(\nabla_Z X^a) - \nabla_Z(\nabla_Y X^a) - \nabla_{[Y,Z]}X^a,$$

$$^4R^a := ^4R_{aabc}$$

$$^4R := ^4R_{abc}g^{ab}$$

respectively, and hence Einstein’s equations take the form $^4G_{ab} = -\kappa T_{ab}$, $\kappa > 0$. We use the standard definitions for the GHP spin coefficients and the \textit{edth} operators given in [13].

2. Boundary conditions for the Witten equation

2.1. The Witten equation

2.1.1. The covariant form. Let $\Sigma$ be a smooth spacelike hypersurface of a spacetime $(M, g_{ab})$, $h_{ab}$ the induced (negative definite) metric and $\kappa_{ab}$ the extrinsic curvature on $\Sigma$. Let
$t^a$ be the future pointing unit timelike normal of $\Sigma$, and define \( P^a_b := \delta^a_b - t^a t_b \), the orthogonal projection to $\Sigma$. If $D_a$ denotes the intrinsic Levi-Civita derivative operator on $\Sigma$ determined by $h_{ab}$ and $D_a := P^b_a \nabla_b$, the so-called Sen operator \([15]\), then
\[
D_A \chi^A = D_A \chi^A + \frac{1}{\Omega_1} \kappa_{AAB} \chi^A,
\]
and the Witten equation is $D_A \chi^A = 0$. Here $\chi$ is the trace of the extrinsic curvature.

Next we clarify the behaviour of (2.1) under the conformal rescaling of the spacetime metric $g_{ab}$ by the conformal factor $\Omega$. Suppose that $\lambda_A$ has the conformal weight $w$, i.e., under such a conformal rescaling it transforms as $\lambda_A \mapsto \lambda_A := \Omega^w \lambda_A$. Then, since under such a conformal rescaling the spinor derivative is well known \([13, 14]\) to transform as $\nabla_{\lambda A} \lambda^B = \nabla_{\lambda A} \lambda^B + \delta_{\lambda A}^{\gamma} C_{\gamma AB}$, where $\Gamma_{\gamma} := \nabla_{\lambda} \ln \Omega$, it is easy to deduce that
\[
\Omega^{-w-1} D_A \chi^A = \hat{D}_{A} \hat{\chi}^A + \frac{1}{\Omega_1} \kappa_{AAB} \hat{\chi}^A - (w + \frac{1}{2}) \hat{\chi}^A \hat{D}_{AA} \ln \Omega.
\]
(N.B.: Under the conformal rescaling above, the extrinsic curvature transforms as $\chi_{ab} \mapsto \chi_{ab} + t^{(\nabla \chi)} h_{ab}$, and hence for the transformation of the mean curvature we obtain $\chi \mapsto \hat{\chi} = \chi - \frac{1}{2} \hat{\chi} + 3 \Omega^{-1} t^{(\nabla \chi)} \Gamma_{\chi}$. Therefore, on maximal, intrinsically globally conformally flat hypersurfaces, the solution $\hat{\chi}^A$ of the Witten equation can be recovered as $\hat{\chi}^A = \Omega^2 \chi^A$ from the solution of the intrinsically and extrinsically flat Witten equation $D_A \hat{\chi}^A = 0$. Thus, it is natural to assign the conformal weight $w = -\frac{1}{2}$ to $\lambda_A$. If, for the sake of simplicity, the conformal rescaling is chosen to be purely spatial, i.e., $t^a \nabla_a \chi = 0$ (which will be done in what follows), then the rescaling preserves the maximality of the hypersurface.

2.1.2. The GHP form. Next suppose that the spacelike hypersurface $\Sigma$ is foliated by 2-surfaces $S_t$ of spherical topology. Let $v^c$ denote their outward pointing unit spacelike normal tangent to $\Sigma$, and introduce the projection $\Pi^a_b := P^a_b + v^a v_b$ to the surfaces $S_t$. Then we can decompose $D_A \chi^A$ with respect to this foliation as
\[
D_A \chi^A = (\Pi^B_A - v^B v_{AB}) \Gamma_{A} \chi^A - (\Delta \chi^A) v_{AA},
\]
where $\Delta_{S_t} := \Pi^a_b \nabla_a$ is a Sen-type derivative operator on the surfaces $S_t$. (Note that this derivative operator deviates from the intrinsic Levi-Civita derivative operator $\lambda$ by the extrinsic curvature tensor of $S_t$. \([16]\).) The contraction $\Delta_{AA} \chi^A$ is only a part of the derivative $\Delta_{AA} \chi^A$. The remaining part is represented by
\[
\nabla_{\lambda A} \nabla_{\lambda C} := \Delta_{\lambda A} \nabla_{\lambda C} + \frac{1}{\Omega_1} \nabla_{\lambda A} \nabla_{\lambda C},
\]
where $\chi^{A} := \Delta_{\lambda A} \chi^A$ is independent of the actual choice for the two normals $t^a$ and $v^a$; it is completely determined by $S_t$. This $\nabla_{\lambda A} \nabla_{\lambda C}$ is just the covariant form of the 2-surface twistor operator on the 2-surfaces $S_t$, and the 2-surface twistor equation is $T_{\lambda a b} \nabla_{\lambda C} = 0$. This is, in fact, two equations (see below and \([13, 14, 16]\]).

Let us introduce a GHP spin frame field $(\sigma^A, t^A)$ on $\Sigma$ such that
\[
t^{AA} = \frac{1}{2} \sigma^A \sigma^A - t^A t^A,
\]
and define the spinor components in this basis by $\lambda_0 := \lambda_{A} \sigma^A$ and $\lambda_1 := \lambda_{A} t^A$. Then in the GHP formalism, the Witten equation $D_A \chi^A = 0$ takes the form
\[
0 = \delta^A \lambda_0 - v^c \partial_c \lambda_1 + (\rho - \frac{1}{2} v^c (\nabla_{\lambda c}) \sigma^A) \lambda_1 + v^c (\nabla_{\lambda c}) t^A \lambda_0,
\]
\[
0 = \delta \lambda_1 + \frac{1}{2} v^c \partial_c \lambda_1 + (\rho' - \frac{1}{2} v^c (\nabla_{\lambda c}) \sigma^A) \lambda_0 + \frac{1}{2} v^c (\nabla_{\lambda c}) t^A \lambda_1,
\]
where $\delta$ and $\delta'$ are the standard edth operators and $\rho := t^A \sigma^A (\nabla_{\lambda A} \sigma^B) \sigma^B$ and $\rho' := -\sigma^A t^A (\nabla_{\lambda A} \sigma^B) \sigma^B$ are the convergences of the outgoing and incoming future pointing null normals of the surfaces $S_t$ in spacetime, respectively. (For the details, see e.g. \([13, 14]\).)
For later use, let us write down the GHP form of the two-dimensional Sen–Dirac operator \( \Delta_{AA} \lambda^A \) and the 2-surface twistor operator \( T_{AB} \lambda^C \). These are
\[
\bar{\partial}^A \Delta_{AA} \lambda^A = -(\partial^A \lambda^0 + \rho \lambda_1), \quad \bar{\lambda}^A \Delta_{AA} \lambda^A = (\delta \lambda_1 + \rho^0 \lambda_0), \tag{2.7}
\]
\[
\bar{\partial}^A \lambda^A \bar{T}_{AB} C \lambda_C = (\delta^A \lambda^0 + \sigma^0 \lambda_0), \quad \bar{\lambda}^A \partial^A \bar{T}_{AB} C \lambda_C = (\delta \lambda_0 + \sigma \lambda_1), \tag{2.8}
\]
where \( \sigma := \partial^{\hat{A}} (\nabla_{AA} a^B) o^B \) and \( \sigma' := -i^B \partial^{\hat{A}} (\nabla_{AA} a^B) o^B \), the shears of the null normals on \( S \). The 2-surface twistor equation or the equations defining the holomorphic or anti-holomorphic spinor fields are appropriate direct sums of these ‘elementary’ differential operators.

2.2. Witten spinors and geometric triads

In this subsection, we summarize the key results of [6] on the relationship between the solutions of the Witten equation and vector bases on \( \Sigma \) that are orthonormal up to an overall function.

If \( \lambda_{AA} \) is the (e.g., future pointing) unit normal of \( \Sigma \), then \( G_{AA} := \sqrt{2} \lambda_{AA} \) is a positive definite Hermitian metric on the spin spaces, by means of which the primed spinor indices can be converted to unprimed ones according to \( \bar{\lambda}^A \mapsto \lambda^A \) and \( \bar{\lambda}_A \mapsto \lambda_A \). Then \( \rho^2 := G_{AA} \lambda^A \bar{\lambda}_A \), the norm of any non-zero spinor field \( \lambda^A \) on \( \Sigma \), is non-zero and \( \{ \lambda^A, \lambda_i^A \} \) form a basis in the spin spaces. Such a spinor determines a vector basis \( \{ X^a, Y^a, Z^a \} \) on \( \Sigma \) by
\[
\frac{1}{\sqrt{2}} (X^a + iY^a) := \lambda^{(A} \bar{\lambda}^B), \quad \frac{1}{\sqrt{2}} Z^a := \lambda^{i(A} \bar{\lambda}^B). \tag{2.9}
\]

(N.B.: The standard convention \( a = AA' \) of [13, 14] for the Lorentzian tensor and \( SL(2,\mathbb{C}) \) spinor indices yields that a spatial tensor index is identified with a pair of symmetric unprimed \( SU(2) \) spinor indices.) \( \{ X^a, Y^a, Z^a \} \) is a real orthogonal basis, the length of each of these vectors is \( \rho^2 \), and \( \varepsilon_{abc} X^a Y^b Z^c = \rho^3 \), where \( \varepsilon_{abc} \) is the induced volume 3-form on \( \Sigma \). Rewriting the derivatives \( \partial^A \bar{T}_{AB} \lambda^A \) and \( \lambda^B \bar{T}_{AB} \lambda^A \) in terms of these vectors and taking into account that on the domain \( U \subset \Sigma \) where \( \lambda^A \) is not vanishing \( \{ \lambda^A, \lambda_i^A \} \) is a basis in the spin spaces, the Witten equation is equivalent to
\[
D_a X^a = 0, \quad D_a Y^a = 0, \quad D_a Z^a = -\chi \rho^2. \tag{2.10}
\]
\[
Z^a Y^b D_a X_b + (Y^a X^b - X^a Y^b) D_a Z_b = 0. \tag{2.11}
\]
Finally, introducing the orthonormal basis \( \{ E^a_i, E^a_j \} := \rho^{-2} \{ X^a, Y^a, Z^a \} \) on the domain \( U \), if \( \{ \delta^a_i \} \), \( i = 1, 2, 3 \), is the dual 1-form basis on \( U \) and \( \gamma^{ik} := \partial^i E^k_b D_a E^a_j \), the Ricci rotation coefficients, then these conditions can be rewritten as
\[
\gamma^{ij} := -E^a_i D_a \log \rho^2 - \chi \delta^2_i, \quad \gamma_{ik} \delta^{ijk} = 0. \tag{2.12}
\]
Here boldface indices are raised and lowered by the constant negative definite metric \( \eta_{ij} := -\delta_{ij} \), and \( \varepsilon_{ijk} := \varepsilon_{abc} E^a_i E^b_j E^c_k \). The frame gauge condition suggested in [3–5] is equivalent to
\[
\gamma^{ij} := -E^a_i D_a \log \rho^2, \quad \gamma_{ik} \delta^{ijk} = -m \tag{2.13}
\]
for some constant \( m \). For asymptotically Cartesian frames on asymptotically flat 3-geometries \( (\Sigma, h_{ab}) \), this constant is vanishing, but it has a non-zero value for frames e.g. on \( \Sigma \approx S^3 \) [3–5]. Thus, for \( m = 0 \), the frame gauge condition of [3–5] and the Witten gauge condition are the same on maximal hypersurfaces, and hence the present investigations have relevance from the point of view of both.
2.3. The boundary conditions

2.3.1. The general linear, first-order boundary conditions. In typical problems of general relativity, the hypersurface $\Sigma$ is either asymptotically flat/hyperboloidal with or without inner boundary, or compact with outer boundary. In the former case, the solution to the Witten equation is usually assumed to be either asymptotically constant or a solution to the asymptotic twistor equation, depending on whether $\Sigma$ extends to spatial or null infinity. The inner boundary, and, in the latter case, the outer boundary will be denoted by $S$, which is assumed to be a (not necessarily connected) closed orientable spacelike surface.

Since the Witten equation is elliptic, only “half” of the data may be specified on $S$ (see e.g. [17] and references therein). In particular, the most general $\mathbb{C}$-linear, first-order boundary condition for the Witten equation $D_{\lambda A}A^A = 0$ is

$$f = \lambda^A A_A + \nu^i (D_i \lambda^A) v_{AA} B^A,$$

(14)

where $f$ is a given function, while $A_A$ and $B^A$ are given spinor fields on the boundary 2-surface $S$. This is a mixed, inhomogeneous boundary condition, which for vanishing $A_A$ is a purely Neumann, while for vanishing $B^A$ is a purely Dirichlet-type boundary condition. The boundary condition is called homogeneous if $f = 0$. The spinor field $\gamma^A_{\ B}$, introduced in the previous subsection, defines a chirality on the spinor bundle over $S$, and the basis vectors $I^A$ and $\sigma^A$ of a GHP spin frame are right-handed/left-handed spinors with respect to this. (For the details of this notion of chirality, see [16].) The boundary condition is called chiral if the spinor fields $A_A$ and $B^A$ are proportional to the right- or left-handed eigenspinors, $I^A$ or $\sigma^A$, of the chirality operator on $S$.

The advantage of the extra normal vector field $v_{AA}$ in equation (14) (contracted with $B^A$) is that it makes the boundary condition $2+2$ covariant. Indeed, using the decomposition (2.3) of the Witten equation at the points of the boundary $S$, the boundary condition (14) can be rewritten as

$$f = \lambda^A A_A + B^A (\Delta_{AA} \lambda^A),$$

(15)

which is manifestly $2+2$ covariant. Equation (2.3) implies, in particular, that a Neumann-type boundary condition for the Witten equation can always be written as a condition on the derivatives of the spinor field in the directions tangential to $S$. A more general (only $\mathbb{R}$- rather than $\mathbb{C}$-linear) boundary condition is

$$f = \lambda^A A_A + B^A (\Delta_{AA} \lambda^A) + E^A \gamma_{AB} \lambda^B$$

(16)

for some given function $f$ and spinor fields $A_A$, $B^A$ and $E^A$ on $S$.

2.3.2. Conformally invariant boundary conditions. Next let us clarify how the general boundary condition (16) changes under a conformal rescaling of the spacetime metric. Suppose that $\hat{\lambda}^A$ has the conformal weight $w$. Then, using $\Pi^A_B = \frac{1}{2} (\delta^A_B \lambda^B + \gamma^A_{\ B} \hat{\lambda}^B)$ and the fact that $\gamma^A_{\ B}$ is trace free, it is easy to derive how $\Delta_{AA} \hat{\lambda}^A$ transforms under spacetime conformal rescalings. We obtain $\Delta_{AA} \hat{\lambda}^A = \Omega^{w-1} (\Delta_{AA} \lambda^A + \gamma_{AA} \lambda^A + (w-1) \gamma_{AB} \epsilon^{AB} \lambda^A)$. Then multiplying equation (16) by $\Omega^{w-1}$ and expressing every field in terms of the conformally rescaled ones, we obtain

$$\Omega^{w-1} f = \hat{\lambda}^A (A_A - \gamma_{AA} \hat{B}^A - (w-1) \gamma_{AB} \epsilon^{AB} \hat{B}^A) + \hat{E}^A (\Delta_{AA} \hat{\lambda}^A) + \Omega^{-1} \hat{E}^A \gamma_{AB} \hat{\lambda}^B.$$

(17)

Thus, the boundary condition (16) for the conformal weight $w$ spinor field $\hat{\lambda}^A$ is conformally covariant if $B^A \mapsto \hat{B}^A := B^A$, $A_A \mapsto \hat{A}_A := A_A - \gamma_{AA} \hat{B}^A - (w-1) \gamma_{AB} \epsilon^{AB} \hat{B}^A$, $E^A \mapsto \hat{E}^A := \Omega^{-1} E^A$ and $f \mapsto \hat{f} := \Omega^{w-1} f$; i.e. in particular, $B^A$ has conformal weight

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zero, $E^A$ has conformal weight $-1$ and $f$ has conformal weight $(w - 1)$. For $w = 1$ and $E^A = 0$, these are precisely the defining transformation properties of local twistors [13, 14], in which case $f$ is just $i$ times the conformally invariant Hermitian scalar product of the local twistors $(\lambda^A, i\Delta_{AB}\lambda^B)$ and $(B^A, -i\lambda^A)$. For $w \neq 1$, the boundary condition is conformally invariant if we impose the additional restriction $f = 0$. Next we determine how the spinor field $A_A$ must be built to have the required transformation property above.

Since $B^A$ has zero conformal weight, $i\Delta_{AA}B^A$ transforms under conformal rescalings as the secondary part of a contravariant twistor, i.e. \( \hat{A}_A + \hat{\Delta}_{AA}\hat{B}^A = A_A + \Delta_{AA}B^A - (w - 1)\gamma_A\Gamma_{AA}B^A \). Thus,

\[
A_A + \hat{\Delta}_{AA}\hat{B}^A = A_A + \Delta_{AA}B^A - (w - 1)\gamma_A\Gamma_{AA}B^A.
\]

The last term on the right-hand side can, however, be compensated by using the (trace of the) spinor form $Q_{AEB}^A := \frac{1}{2}(\Delta_{EE}\gamma^AC)\gamma^C_B$ of the extrinsic curvature tensor of $\Sigma$. $(Q_{AEB}^A$ is in fact the spinor form of the extrinsic curvature tensor, because the GHP convergences and shears can also be given in terms of this spinor as $\rho = \sigma^A\rho^C\sigma^BQ_{AEB}, -\rho' = \eta^A\rho^C\eta^BQ_{AEB}$, \( \sigma = \sigma^A\rho^C\sigma^BQ_{AEB} \) and $-\sigma' = \eta^A\rho^C\eta^BQ_{AEB}$ [16]. The mean curvature vector of $\Sigma$ in the spacetime is $H_{AA} := 2Q_{EAX}^A = -2\rho_A\lambda^A - 2\rho'\lambda^A\gamma_A$, which is real. Since $H'_{AA}H_{AB} = 4\rho\rho'\delta^A_B$, it defines an isomorphism between the spin and complex conjugate spin spaces precisely when $\rho\rho' \neq 0$. Indeed, a simple calculation yields that under a conformal rescaling of the spacetime metric $Q_{EAX}^A \mapsto \hat{Q}_{EAX}^A := Q_{EAX}^A + \gamma_A(\delta^A_F - \Pi^A_F)$, and hence

\[
\hat{Q}_{EAX}^A\hat{B}^A = Q_{EAX}^A\hat{B}^A + \hat{\Delta}_{AA}\hat{B}^A - \Delta_{AA}B^A - \gamma_A\Pi_{AA}B^A.
\]

Expressing the last term from this equation and substituting into the previous one, we obtain that

\[
A_A := C_A - w\Delta_{AA}\hat{B}^A + (w - 1)Q_{EAX}^A\hat{B}^A,
\]

i.e. the expression on the right-hand side has zero conformal weight. Therefore, for any spinor field $C_A$ with zero conformal weight, the spinor field

\[ A_A := C_A - w\Delta_{AA}\hat{B}^A + (w - 1)Q_{EAX}^A\hat{B}^A \tag{2.18} \]

has the desired conformal transformation property. Consequently, the $\mathbb{R}$-linear, conformally invariant first-order boundary condition for the Witten equation must have the form

\[
\lambda^A(C_A + \frac{1}{2}\Delta_{AA}\hat{B}^A - \frac{3}{2}Q_{EAX}^A\hat{B}^A) + \hat{B}^A(\Delta_{A}\lambda^A) + E^A\gamma_{AB}\pi^B = 0. \tag{2.19}
\]

Here $C_A$ and $B^A$ are arbitrary spinor fields with zero conformal weight and $E^A$ is an arbitrary spinor field with conformal weight $-1$ on $\Sigma$.

3. Explicit solutions

3.1. A solution of the Witten equation with one-dimensional zero-sets

In [8], Bär showed that the set of zeros of the solutions of a Dirac equation on an $n$-dimensional Riemannian manifold is of dimension not greater than $(n - 2)$. In this subsection, we illustrate this by a simple solution of the Witten equation in flat 3-space. Its zeros form, in fact, a union of discrete zeros and one-dimensional submanifolds, which may be compact or non-compact (in fact, not bounded); or the set of zeros can consist of purely isolated points.

Suppose that $\Sigma$ is both intrinsically and extrinsically flat, i.e. it is a hyperplane in Minkowski spacetime. Let $x^i = (x, y, z)$, $i = 1, 2, 3$, be a Cartesian coordinate system on $\Sigma \approx \mathbb{R}^3$, i.e. in which $\hbar_{ij} = -\delta_{ij}$, let $\sigma^B_{ij}$ be the $SU(2)$ Pauli matrices (divided by $\sqrt{2}$)
and introduce the notation $D^2 := \delta \bar{\delta} \sigma^A \partial_A$. Then in the Cartesian spin frame adapted to the coordinates above, the Witten equation, $D^2 \lambda^\Sigma = 0$, takes the explicit form

$$\bar{\partial}_\Sigma \lambda^0 + \partial_\Sigma \lambda^1 + i \partial_\Sigma \lambda^0 = 0, \quad \bar{\partial}_\Sigma \lambda^1 - \partial_\Sigma \lambda^0 + i \partial_\Sigma \lambda^0 = 0. \quad (3.1)$$

A particular solution of these equations is

$$\lambda^0 = -(A + B)z^2 + Ax^2 + By^2 - C, \quad \lambda^1 = 2z(Ax - iBy) - D,$$

where $A$, $B$, $C$ and $D$ are constants. If $D = 0$, then for real $A$, $B$ and $C$ the coordinates of its zeros satisfy $z = 0$ and $Ax^2 + By^2 = C$. If the parameters are all positive, then the set of zeros is an ellipse, which is a compact one-dimensional submanifold. However, for $AB < 0$ and $C \neq 0$, the set of zeros is a hyperbola, while for $C = 0$ it is a pair of straight lines crossing each other at the origin and the coordinates of these zeros are given by $(x, \pm x\sqrt{-A/B}, 0)$. These sets are not bounded in $\mathbb{R}^3$. If $B = 0$ and $AC > 0$, then $\lambda^A$ has two isolated zeros at $(\pm \sqrt{C/A}, 0, 0)$. The vector basis corresponding to the constant spinor field $(\lambda^0, \lambda^1) = (-C, 0)$ via (2.9) is just the constant orthonormal triad $E_i^r = -\left(\frac{A}{r}\right)^n$.

3.2. Spherically symmetric solutions in flat 3-space

In this subsection, we determine the fundamental solution of the Witten equation and discuss various explicit boundary conditions on spherically symmetric $S$ that specify, among others, the constant spinor fields.

3.2.1. The fundamental solution. Let $\mathcal{D}$ be the intrinsically and extrinsically flat Sen derivative operator. The surfaces $S_r$ of the foliation of $\Sigma$ will be chosen to be metric spheres of radius $r$. By the vanishing of the extrinsic curvature (indeed by $\chi = 0$), one has $\rho = -1/r$ and $\rho' = 1/2r$. Although the GHP spin frame $\{o^A, i^A\}$ is not constant on $S$ with respect to the flat $\mathcal{D}$, its normal directional derivative is vanishing: $v^\nu \mathcal{D}_\nu o_\lambda = v^\nu \mathcal{D}_\nu i_\lambda = 0$. Substituting these into (2.5) and (2.6), we obtain

$$\delta' \lambda_0 - \frac{1}{r} \lambda_1 - v^\nu \partial_\nu \lambda_1 = 0, \quad \delta \lambda_1 - \frac{1}{2r} \lambda_0 + \frac{1}{2} v^\nu \partial_\nu \lambda_0 = 0. \quad (3.2)$$

Recalling that the spinor components $\lambda_0$ and $\lambda_1$ have spin weights $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, we can expand them in terms of the $\pm \frac{1}{2}$ spin weighted spherical harmonics according to

$$\lambda_0 = \sum_{j=\frac{1}{2}}^\infty \sum_{m=-j}^j c_0^{jm}(r) \frac{s}{4} Y_{jm}, \quad \lambda_1 = \sum_{j=\frac{1}{2}}^\infty \sum_{m=-j}^j c_1^{jm}(r) \frac{s}{4} Y_{jm}. \quad (3.3)$$

Note that $j$ takes the values $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and, for given $j, m = -j, -j+1, \ldots, j$.

Recalling (see e.g. [13]) that the action of $\delta$ and $\delta'$ on the spin $s$ weighted spherical harmonics $s Y_{jm}$ is

$$\delta s Y_{jm} = -\frac{1}{\sqrt{2r}} \sqrt{(j + s + 1)(j - s)} s_{s+1} Y_{jm}, \quad \delta' s Y_{jm} = \frac{1}{\sqrt{2r}} \sqrt{(j + s + 1)(j + s)} s_{s-1} Y_{jm}, \quad (3.4)$$

(3.2) reduces to the system of ordinary differential equations

$$\frac{dc_0^{jm}}{dr} + \frac{1}{r} c_0^{jm} - \frac{\sqrt{2}}{r} \left( j + \frac{1}{2} \right) c_1^{jm} = 0, \quad \frac{dc_1^{jm}}{dr} + \frac{1}{r} c_1^{jm} - \frac{1}{\sqrt{2r}} \left( j + \frac{1}{2} \right) c_0^{jm} = 0. \quad (3.5)$$
These equations yield the second-order equation
\[ c'' + \frac{3}{r} c' + \frac{1}{r^2} \left[ 1 - \left( j + \frac{1}{2} \right)^2 \right] c = 0 \] (3.6)
both for \( c = c_0^m \) and \( c_1^m \), where the prime denotes differentiation with respect to \( r \). Multiplying this with \( r^k \) for some real \( k \), we obtain
\[
(r^k c)^{''} + \frac{3 - 2k}{r}(r^k c)^{'} + \frac{1}{r^2} \left[ k^2 - 2k + 1 - \left( j + \frac{1}{2} \right)^2 \right] (r^k c) = 0. \] (3.7)
Thus, to simplify this equation, let us choose \( k \) to make the last term vanish, i.e. let \( k = 1 \pm (j + \frac{1}{2}) \). Then (3.7) can be integrated directly: \( c(r) = C r^{k-2} \), i.e.
\[ c^m_j (r) = \pm C^m_{j} r^{-1 \pm (j + \frac{1}{2})}; \] (3.8)
where \( \pm C^m_{j} \) are constants. Substituting this both for \( c_0^m \) and \( c_1^m \) back into the first-order equations (3.5), we find that the sign \( \pm \) in the exponent in (3.8) for \( c_0^m \) and \( c_1^m \) is the same; furthermore \( \sqrt{2} c_1^m = \pm C_0^m \). Therefore, in the solution only the coefficients \( \pm C_0^m \) will appear, and in the rest of this paper we use the notation \( A^m := \pm C_0^m \), \( B^m := \pm C_0^m \). Thus,
\[
\lambda_0 = \sum_{j,m} A^m r^{-1+(j+\frac{1}{2})}Y_{jm} + \sum_{j,m} B^m r^{-1-(j+\frac{1}{2})}Y_{jm}, \]
\[
\sqrt{2} \lambda_1 = \sum_{j,m} A^m r^{-1+(j+\frac{1}{2})^-}Y_{jm} - \sum_{j,m} B^m r^{-1-(j+\frac{1}{2})^-}Y_{jm}. \] (3.10)
This solution is analogous to the fundamental solution of the flat space Laplace equation with centre \( r = 0 \); thus, we may call it the fundamental solution of the Witten equation. Its general solution is a superposition of such fundamental solutions with different centres, and it is the boundary conditions that specify the actual solutions that we are interested in.

For example, if \( \Sigma \) is isometric with \( \mathbb{R}^3 \) and we want that the solutions be bounded at infinity, then by (3.9) and (3.10) all the constants \( A^m \) must be zero for \( j \geq \frac{3}{2} \). Then \( \lambda_A \) tends to a constant spinor field at infinity, and the asymptotic value of \( \lambda_A \) is fixed by \( A^\pm \frac{1}{2} \). Hence, the form of the spinor components is given by
\[
\lambda_0 = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A^m + \frac{B^m}{r^2} \right)_{\pm} Y_{jm} + \sum_{j=\frac{1}{2}}^{\infty} \sum_{m=-j}^{j} \frac{B^m}{r^{1+(j+\frac{1}{2})}} Y_{jm}, \] (3.11)
\[
\sqrt{2} \lambda_1 = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A^m - \frac{B^m}{r^2} \right)_{\pm} Y_{jm} - \sum_{j=\frac{1}{2}}^{\infty} \sum_{m=-j}^{j} \frac{B^m}{r^{1+(j+\frac{1}{2})}} Y_{jm}. \] (3.12)
On the other hand, these can be regular at the origin precisely when \( B^m = 0 \) for all \( j \), in which case \( \lambda_A \) is constant on \( \Sigma \). There are two such linearly independent spinor fields, which are parametrized by the constants \( A^\pm \frac{1}{2} \). Similarly, if \( \Sigma \) is isometric to \( \mathbb{R}^3 \setminus \{0\} \), then the solution is bounded precisely when it is constant.
3.2.2. Boundary conditions: the compact case. Suppose that \( \Sigma \) is isometric to the solid ball \( B \subset \mathbb{R}^3 \) of radius \( R \), i.e. \( \Sigma \) is compact with boundary \( \partial \Sigma = \Sigma_R \). Then, to ensure the regularity of the spinor field (3.9) and (3.10) at \( r = 0 \), all the coefficients \( B^{im} \) must be zero, and hence

\[
\lambda_0 = \sum_{j=0}^{\frac{1}{2}} \sum_{m=-j}^{j} A^{im} r^{j-\frac{1}{2}} Y_{jm}, \quad \sqrt{2} \lambda_1 = \sum_{j=0}^{\frac{1}{2}} \sum_{m=-j}^{j} A^{im} r^{j-\frac{1}{2}} Y_{jm}. \quad (3.13)
\]

This spinor field is completely fixed by one of the freely specifiable spinor components \( \lambda_0, \lambda_1, \nabla^s \partial \lambda_0 \) or \( \nabla^s \partial \lambda_1 \), or at least by a combination of them, on \( \Sigma_R \). Next we discuss a few special cases.

First, since the spin weighted spherical harmonics \( s Y_{jm} \) form a basis in the space of the spin \( s \) weighted functions on \( \Sigma_R \), any homogeneous, chiral Dirichlet-type boundary condition, e.g. \( \lambda_1|_{\Sigma_R} = 0 \), yields an identically zero spinor field.

Next, it is easy to see that the inhomogeneous chiral boundary condition \( \lambda_1|_{\Sigma_R} = c Y_{1,1} + c^{+\frac{1}{2}} Y_{1,-1} \) yields a constant spinor field on \( \Sigma \), where \( c \) and \( c^{+\frac{1}{2}} \) are complex constants. Such spinor fields can also be characterized e.g. by \( \delta \lambda_1|_{\Sigma_R} = 0 \) (though the actual constant spinor field is not specified explicitly by this condition). This equation is just half of the equations defining the holomorphic spinor fields [18], which equation appears in the 2-surface twistor equation too (see the first expression in (2.8)). The constant spinor fields on \( \Sigma \) can also be characterized on \( \Sigma_R \) by conditions on the other component of the spinor field, e.g. by \( \lambda_0|_{\Sigma_R} = d Y_{1,1} + d^{+\frac{1}{2}} Y_{1,-1} \), where \( d \) and \( d^{+\frac{1}{2}} \) are complex constants. This \( \lambda_0 \) on \( \Sigma_R \) is just the general solution of \( \delta \lambda_0|_{\Sigma_R} = 0 \), which is half of the equations defining the anti-holomorphic spinor fields on \( \Sigma_R \), as well as a half of the 2-surface twistor equation (see the second expression in (2.8)).

On the other hand, by (3.13) for fixed \( j \geq \frac{1}{2} \), the inhomogeneous chiral boundary condition \( \lambda_1|_{\Sigma_R} = \sum_{m=-j}^{j} c^m Y_{jm} \) yields a non-constant spinor field, which vanishes at the origin as \( r^{-\frac{1}{2}} \).

Instead of chiral homogeneous boundary conditions involving only \( \lambda_0 \) or \( \lambda_1 \), we can consider the more general condition \( (\delta \lambda_0 + \rho \lambda_1)|_{\Sigma_R} = 0 \), where actually \( \rho = -1/R \). Substituting (3.13) here, we obtain that \( A^{im} = 0 \) for all \( j \geq \frac{1}{2} \). Thus, only \( A^{+\frac{1}{2}} \) may be non-zero, and hence the solution that this boundary condition singles out is constant. This boundary condition is just one-half of the defining equation of the holomorphic spinor fields on \( \Sigma_R \), and can also be written as \( \delta \lambda_A \Delta A \lambda_A = 0 \) (see (2.7)). Therefore, by the general considerations of section 2.3.1 (in particular by (2.15)), or more explicitly by (3.2), this is just the chiral, homogeneous Neumann boundary condition \( \nabla^s (D \lambda_A)^A_A = 0 \). Similarly, we can impose \( (\delta \lambda_1 + \rho' \lambda_0)|_{\Sigma_R} = 0 \), where \( \rho' = 1/2 R \). This specifies the constant solutions too, which boundary condition is just one-half of the defining equation of the anti-holomorphic spinor fields on \( \Sigma_R \). This can also be written as \( \bar{-\lambda} \Delta \lambda_A = 0 \) (see (2.7)), or, equivalently, as the chiral, homogeneous Neumann boundary condition \( \nabla^s (D \lambda_A)^A_A = 0 \).

Thus, there are several mathematically inequivalent ways to single out the constant spinor fields, but all these boundary conditions can be considered as weakening of the conditions defining the constant spinor fields on the boundary \( \Sigma_R \). (For further possibilities, see [19] and the appendix of [20].)

3.2.3. Boundary conditions: the non-compact case. Now suppose that \( \Sigma \) is isometric to \( \mathbb{R}^3 - B \), where \( B \subset \mathbb{R}^3 \) is the solid ball of radius \( R > 0 \) and the overline denotes topological closure in \( \mathbb{R}^3 \). Then \( r \) is defined for \( [R, \infty) \), and the inner boundary is \( \Sigma_R \). By (3.11) and (3.12) for given boundary conditions at infinity yielding asymptotically constant spinor fields (i.e. for
fixed \( A^{±\frac{1}{2}} \), the solution \( \lambda_A \) is completely determined by the coefficients \( B^{jm} \). In particular, \( \lambda_A \) is fixed by one of the freely specifiable spinor components \( \lambda_0, \lambda_1, v' \partial_v \lambda_0 \) or \( v' \partial_v \lambda_1 \), or at least by specifying a combination of them, on \( \mathcal{S}_R \). Next we discuss some particular cases.

First consider \( \lambda_0|_{\mathcal{S}_S} = 0 \), which is a homogeneous chiral Dirichlet-type boundary condition. This was used in the proof of the positivity of the total (ADM and Bondi–Sachs) energy in the presence of black holes [1, 2]. Then, by the completeness of the spherical harmonics \( Y_{jm} \) in the space of the functions with spin weight \( s \) on \( \mathcal{S}_R \), (3.11) implies that \( B^{jm} = 0 \) for \( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \), and that \( B^{±m} = -R^2 A^{±m} \). Therefore,

\[
\lambda_0 = \left( 1 - \frac{R^2}{r^2} \right) \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} A^{±m} \frac{r}{r^2} Y_{jm}, \quad \sqrt{2} \lambda_1 = \left( 1 + \frac{R^2}{r^2} \right) \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} A^{±m} \frac{-r}{r^2} Y_{jm},
\]

which is a uniquely determined non-constant solution. The solution space is two dimensional, and can be coordinatized by \( A^{±\frac{1}{2}} \), or, equivalently, by the components of the spinor field at infinity. Recalling that the \( s = \pm \frac{1}{2} \) spin weighted spherical harmonics for \( j = \frac{1}{2} \) in the standard complex stereographic coordinates \( \zeta := \exp(i \phi) \cot \frac{\theta}{2} \) take the form

\[
\begin{align*}
\frac{i}{\sqrt{2}} Y_{\frac{1}{2}} & = \frac{i}{\sqrt{2 \pi}} \frac{\zeta}{\sqrt{1 + \zeta}}, & \frac{i}{\sqrt{2}} Y_{-\frac{1}{2}} & = \frac{i}{\sqrt{2 \pi}} \frac{1}{\sqrt{1 + \zeta}}, \\
\frac{i}{\sqrt{2}} Y_{\frac{3}{2}} & = \frac{i}{\sqrt{2 \pi}} \frac{1}{\sqrt{1 + \zeta}}, & \frac{i}{\sqrt{2}} Y_{-\frac{3}{2}} & = -\frac{i}{\sqrt{2 \pi}} \frac{\zeta}{\sqrt{1 + \zeta}},
\end{align*}
\]

by \( \lambda_0|_{\mathcal{S}_S} = 0 \) the spinor field \( \lambda_A \) has a zero on \( \mathcal{S}_R \). Indeed, if we write \( \lambda_1|_{\mathcal{S}_S} = a_{-\frac{1}{2}} Y_{\frac{1}{2}} \) \( + b_{-\frac{1}{2}} Y_{-\frac{1}{2}} \), then \( \lambda_1|_{\mathcal{S}_S} \) is vanishing at \( \zeta = \bar{a} \) \( b \) for non-zero \( b \), while for \( b = 0 \) it is vanishing at the ‘north pole’ \( \zeta = \infty \) of \( \mathcal{S}_R \).

Instead of specifying \( \lambda_0 \), we can prescribe only its tangential derivatives. By (3.11)

\[
\partial' \lambda_0 = \frac{1}{\sqrt{2}r} \left( \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A^{±m} + \frac{r}{r^2} B^{±m} \right) - \frac{1}{r} Y_{jm} + \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \sum_{j=-\frac{1}{2}}^{\frac{1}{2}} \left( j + \frac{1}{2} \right) \frac{B^{jm}}{r^{j+\frac{1}{2}}} \right) Y_{jm},
\]

and hence if \( \partial' \lambda_0|_{\mathcal{S}_S} = 0 \), then \( B^{jm} = 0 \) for \( j = \frac{3}{2}, \frac{5}{2}, \ldots \) and \( B^{±m} = -R^2 A^{±m} \). Substituting these into (3.11) and (3.12), we obtain (3.14) above, i.e. in particular, \( \lambda_0|_{\mathcal{S}_S} = 0 \). Indeed, general theorems (see e.g. [13]) on the dimension of the kernel of the \( \partial \) operators guarantee that \( \partial' \lambda_0 = 0 \) implies the vanishing of \( \lambda_0 \) itself on \( \mathcal{S}_R \).

Again by (3.11),

\[
\delta \lambda_0 = -\frac{1}{\sqrt{2}r} \sum_{j=-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( j + \frac{3}{2} \right) \left( j + \frac{1}{2} \right) \frac{B^{jm}}{r^{j+\frac{1}{2}}} \frac{r}{r^2} Y_{jm}.
\]

Thus, if our boundary condition is \( \delta \lambda_0|_{\mathcal{S}_S} = 0 \), which is one-half of the 2-surface twistor equation (see the second expression in (2.8)), then \( B^{jm} = 0 \) for \( j = \frac{3}{2}, \frac{5}{2}, \ldots \). Hence,

\[
\lambda_0 = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A^{±m} + \frac{r}{r^2} B^{±m} \right) \frac{r}{r^2} Y_{jm}, \quad \sqrt{2} \lambda_1 = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A^{±m} - \frac{r}{r^2} B^{±m} \right) \frac{-r}{r^2} Y_{jm}.
\]

This boundary condition on \( \mathcal{S}_R \) is equivalent to an inhomogeneous chiral Dirichlet-type boundary condition of the form \( \lambda_0|_{\mathcal{S}_S} = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} c^m Y_{jm} \) with complex constants \( c^m \). The space of the spinor fields (3.19) is four dimensional. Clearly, the investigation of the
boundary condition $\delta^g \lambda|_{\partial \Sigma} = 0$ for any given $n \in \mathbb{N}$ can be carried out similarly. For example, $\delta^g \lambda|_{\partial \Sigma} = 0$ is equivalent to an inhomogeneous one on $\delta_0$, and yields eight complex dimensional space of solutions of the Witten equation.

Finally, instead of equations for $\lambda_0$ or $\lambda_1$ on $S_R$, we can impose the boundary condition $(\delta \lambda_0 + \rho \lambda_1)|_{\partial \Sigma} = 0$. Then by (3.11) and (3.12), this implies that $B^{jm} = 0$ for all $j$, and hence the spinor field $\lambda_A$ is constant on $\Sigma$. Similarly, $(\delta \lambda_1 + \rho \lambda_0)|_{\partial \Sigma} = 0$ also yields the constant spinor fields. As in the compact case, these are equivalent to chiral, homogeneous Neumann boundary conditions.

3.3. Solutions on maximal, intrinsically conformally flat hypersurfaces

Suppose that $\Sigma$ is maximal (i.e. $\chi = 0$), the intrinsic metric $h_{ab}$ is conformally flat, and related to a flat metric $\tilde{h}_{ab}$ by a globally defined conformal factor: $h_{ab} = \Omega^{-2} \tilde{h}_{ab}$. The $t = \text{const}$ hypersurfaces in the Reissner–Nordström spacetime, or the Brill–Lindquist [11] and Bowen–York data sets [12], are such hypersurfaces. Here we solve the Witten equation on these data sets.

3.3.1. Asymptotically constant solutions on complete Reissner–Nordström data sets.

The base manifold of the (maximally extended) Reissner–Nordström data set is $\Sigma \approx \mathbb{R}^3 - \{0\}$ with the standard Cartesian coordinates $\{x^1, x^2, x^3\}$, and the corresponding spherical polar coordinates $(r, \theta, \phi)$. The conformal factor is given by $\Omega^{-1}(r) := (1 + \frac{2}{\lambda_1})^\frac{1}{2} - (\frac{\lambda_1}{r})^\frac{1}{2}$, where $m > |e|$. The extrinsic curvature of $\Sigma$ in the spacetime vanishes. The surface $r = \frac{1}{2}\sqrt{m^2 - e^2}$ is a minimal surface in $(\Sigma, h_{ab})$, whose points are just the fixed points of the discrete isometry $I : x^i \mapsto \frac{1}{2}(m^2 - e^2)^{\frac{1}{2}}$ (see [10]). This surface represents the black hole event horizon in $\Sigma$, while the $r \to \infty$ and $r \to 0$ regimes are the two asymptotically flat ends.

Let $\tilde{\lambda}_A$ be a solution of the flat Witten equation on $(\Sigma, \tilde{h}_{ab})$. Then, by equation (2.2), $\lambda_A = \Omega^{\frac{1}{2}} \tilde{\lambda}_A$ is a solution of the Witten equation on $(\Sigma, h_{ab})$. The conformal rescaling $\varepsilon_{AB} = \Omega^{-1} \delta_{AB}$ implies the rescaling $o^A = \Omega^{1-\frac{k}{2}} \tilde{o}^A$, $i^A = \Omega^{\frac{k}{2}} \tilde{i}^A$ of the normalized spin frame with undetermined $k \in \mathbb{R}$. We choose $k = \frac{1}{2}$ (the symmetric rescaling), so that

$$\lambda_A o^A = \Omega^{\frac{1}{2}} \tilde{\lambda}_A \tilde{o}^A, \quad \lambda_A i^A = \Omega^{\frac{1}{2}} \tilde{\lambda}_A \tilde{i}^A.$$ 

Since $\Omega \to 1$ if $r \to \infty$, the spinor field $\lambda_A$ can be non-singular on $\Sigma$ and bounded in this limit only if the components of $\tilde{\lambda}_A$ in the spin frame $\{\tilde{o}^A, \tilde{i}^A\}$ are given by (3.11) and (3.12). However, since in the $r \to 0$ limit $\Omega$ tends to zero as $r^2$, the solution $\lambda_A$ is bounded on the other asymptotic end precisely when $B^{jm} = 0$ for all $j \geq \frac{3}{2}$, so that the spinor field is asymptotically constant there too. It is given explicitly by

$$\lambda_A o^A = \frac{4r^2}{(2r + m)^2 - e^2} \sum_{m=-\frac{3}{2}}^{\frac{3}{2}} \left( A^{\frac{1}{2}m} + B^{\frac{1}{2}m} \right) \frac{1}{2} Y^{\frac{1}{2}m},$$

$$\sqrt{2} \lambda_A i^A = \frac{4r^2}{(2r + m)^2 - e^2} \sum_{m=-\frac{3}{2}}^{\frac{3}{2}} \left( A^{\frac{1}{2}m} - B^{\frac{1}{2}m} \right) \frac{1}{2} Y^{\frac{1}{2}m}.$$ 

Thus its asymptotic values are determined by $A^{\frac{1}{2}m}$ and $B^{\frac{1}{2}m}$. This solution can be considered as the sum of two spinor fields, one with $A^{\frac{1}{2}m} = 0$ and the other with $B^{\frac{1}{2}m} = 0$, and each is proportional to some constant spinor field with respect to some flat connection. However, their factors of proportionality are different, yielding different asymptotic properties: while
one spinor field tends to a non-zero constant spinor at one asymptotic end and tends to zero at the other end, the other spinor field behaves in just the opposite way.

By (3.15) and (3.16), the spinor field $\lambda_A$ vanishes at the point $(r, \zeta, \bar{\zeta})$ precisely when

$$
\zeta \left( r^2 A^\frac{-1}{2} + B^\frac{1}{2} \right) + \left( r^2 A^\frac{-1}{2} + B^\frac{-1}{2} \right) = 0,
$$

$$
\left( r^2 A^\frac{1}{2} - B^\frac{1}{2} \right) - \overline{\zeta} \left( r^2 A^\frac{-1}{2} - B^\frac{-1}{2} \right) = 0.
$$

These equations yield

$$
\zeta = -\frac{r^2 A^\frac{-1}{2} + B^\frac{-1}{2}}{r^2 A^\frac{1}{2} + B^\frac{1}{2}},
$$

from which it follows that

$$
r^4 \left( |A| A^\frac{1}{2} + |A| B^\frac{-1}{2} \right) - r^2 \left( A^\frac{1}{2} B^\frac{1}{2} + A^\frac{-1}{2} B^\frac{-1}{2} - A^\frac{1}{2} B^\frac{-1}{2} - A^\frac{-1}{2} B^\frac{1}{2} \right)
$$

$$
- \left( |B| A^\frac{-1}{2} + |B| B^\frac{1}{2} \right) = 0.
$$

(3.22)

However, the coefficient of $r^2$ is purely imaginary while all the other terms are real. Thus, to have a real solution for $r^2$, this imaginary coefficient must vanish, yielding one real condition for the coefficients $A^\frac{1}{2}$ and $B^\frac{1}{2}$. Parameterizing these coefficients as

$$
A^\frac{1}{2} := |A| \cos \chi \ e^{i\alpha_1}, \quad A^\frac{-1}{2} := |A| \sin \chi \ e^{i\alpha_2},
$$

$$
B^\frac{1}{2} := |B| \cos \psi \ e^{i\beta_1}, \quad B^\frac{-1}{2} := |B| \sin \psi \ e^{i\beta_2},
$$

the condition that the coefficient of $r^2$ in (3.22) must be vanishing is

$$
\cos^2 \psi = \frac{\sin^2 \chi \sin^2(\alpha_2 - \beta_2)}{\cos^2 \chi \sin^2(\alpha_1 - \beta_1) + \sin^2 \chi \sin^2(\alpha_2 - \beta_2)}.
$$

(3.23)

Then for the coordinates of the zero, we obtain

$$
r^2 = \frac{|B|}{|A|}, \quad \zeta = -\frac{\sin \chi \ e^{i\alpha_2} + \sin \psi \ e^{i\beta_2}}{\cos \chi \ e^{i\alpha_1} + \cos \psi \ e^{i\beta_1}}.
$$

(3.24)

Therefore, the solution $\lambda_A$ does not have any zero if the parameters $A^\frac{1}{2}$ and $B^\frac{1}{2}$ do not satisfy (3.23). If (3.23) holds, then by the first of (3.24), the solution $\lambda_A$ has a zero for $r \in (0, \infty)$ precisely when $A^\frac{1}{2} \neq 0$ and $B^\frac{1}{2} \neq 0$, while the zero is at $r = \infty$ for $A^\frac{1}{2} = 0$ and at $r = 0$ for $B^\frac{1}{2} = 0$. Moreover, even for fixed (non-zero) $A^\frac{1}{2}$ and point $p = (r, \zeta, \bar{\zeta})$, there is a parameter $B^\frac{1}{2}$ such that the corresponding spinor field has a zero at $p$. These solutions are not conformal to a spinor field which would be constant with respect to some flat connection. Criterion (3.23) for the existence of the zeros determines a seven-dimensional submanifold in the space of the parameters ($|A|, \chi, \alpha_1, \alpha_2; |B|, \psi, \beta_1, \beta_2$). Therefore, the solutions of the Witten equation on the non-extreme Reissner–Nordström data set that are constant at the two infinities form a four-dimensional complex vector space, in which there is a seven-dimensional real submanifold of solutions with a zero.

To interpret this result in the language of the geometric triads of subsection 2.2, let us recall that a spinor determines an orthonormal triad only up to an overall real scale factor. Thus, the boundary conditions for the triads at the two asymptotic ends form a real $(3+3)$-dimensional manifold (corresponding to the parameters $(\chi, \alpha_1, \alpha_2)$ and $(\psi, \beta_1, \beta_2)$ above). Since, however, the ‘modulus’ of the spinor fields, $|A|$ and $|B|$, are not involved in (3.23), it defines a five-dimensional submanifold of boundary conditions for those frames which are singular somewhere inside the Reissner–Nordström initial data hypersurface. Thus, we do not have complete freedom (i.e. constant rotations) to choose the frame at the two infinities any way we want.
3.3.2. Solutions on Reissner–Nordström data sets with internal boundary. Let $\Sigma$ be the subset $\{| x | | \delta x^A x^A \geq R^2 > 0 \}$ of the complete Reissner–Nordström data set of the previous subsection, whose inner boundary $\mathcal{S}$ is the 2-sphere with coordinate radius $R$. Consider the solutions $\lambda_{\alpha}$ of the Witten equation that are asymptotically constant at the ‘outer’ infinity $r \to \infty$. Then, by the discussion in the first half of the second paragraph of subsection 3.2.3, the components of $\lambda_{\alpha}$ are

$$
\lambda_0 = \frac{4r^2}{(2r + m)^2 - e^2} \left( \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A_{\lambda}^{\frac{1}{2} m} + \frac{B_{\lambda}^{\frac{1}{2} m}}{r^2} \right) \tilde{y}_{\lambda m} + \sum_{j=2}^{\infty} \sum_{m=-j}^{j} \frac{B_{\lambda}^{\frac{1}{2} m}}{r^{1+(j+1)\frac{1}{2}}} \hat{y}_{jm} \right),
$$

$$
\sqrt{2} \lambda_1 = \frac{4r^2}{(2r + m)^2 - e^2} \left( \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \left( A_{\lambda}^{\frac{1}{2} m} - \frac{B_{\lambda}^{\frac{1}{2} m}}{r^2} \right) \tilde{y}_{\lambda m} - \sum_{j=2}^{\infty} \sum_{m=-j}^{j} \frac{B_{\lambda}^{\frac{1}{2} m}}{r^{1+(j+1)\frac{1}{2}}} \hat{y}_{jm} \right).
$$

(3.25)

(3.26)

The areal radius of the spheres $S_\nu$ of coordinate radius $r$, defined by $(\frac{1}{4\pi} \text{Area}(S_\nu))^2$, is $r \Omega^{-1} = \frac{1}{4}(2r + m)^2 - e^2)$. Moreover, if the sign of the normal of $S_\nu$ is chosen such that $\hat{n} = (\frac{\delta}{\partial r})$, which points inward on the hypersurface $\Sigma$, then for the mean curvature of the 2-spheres $S_\nu$, we obtain

$$
\nu = -\frac{8r}{((2r + m)^2 - e^2) (4r^2 - (m^2 - e^2))}.
$$

(3.27)

Then, since the Reissner–Nordström data set is extrinsically flat, the outgoing and incoming null convergences on the 2-spheres $S_\nu$ are $\rho = -\frac{1}{4} \nu$ and $\rho' = \frac{1}{4} \nu$, respectively. Note that these are vanishing on the minimal surface $2r = \sqrt{m^2 - e^2}$. Next we discuss a few explicit boundary conditions on $\mathcal{S}$.

First, as in subsection 3.2.3, any of the homogeneous, chiral Dirichlet boundary conditions, $\lambda_0|_{\mathcal{S}} = 0$ or $\lambda_1|_{\mathcal{S}} = 0$, specifies a solution on $\Sigma$ which is asymptotically constant at infinity but vanishing somewhere on $\mathcal{S}$. This solution is conformal to that given by (3.14). Also as in subsection 3.2.3, this solution can be characterized by the boundary condition $\delta \lambda_0 = 0$ or $\delta \lambda_1 = 0$, respectively.

The purely right-handed or left-handed parts of the 2-surface twistor equation, $\delta \lambda_0 = 0$ or $\delta' \lambda_1 = 0$, specify solutions conformal to the non-constant spinor fields (in the flat 3-space) given by (3.19).

However, the non-triviality of the conformal factor gives new possibilities. Clearly,

$$
- \partial^A \Delta_{A} \lambda^A = \delta \lambda_0 + \rho \lambda_1 = \Omega (\Omega \tilde{\lambda}_0 + \frac{1}{2} r \nu \hat{\lambda}_1).
$$

(3.28)

$$
\iota^A \Delta_{A} \lambda^A = \delta \lambda_1 + \rho' \lambda_0 = \Omega (\Omega \tilde{\lambda}_1 + \frac{1}{2} r \nu \hat{\lambda}_0).
$$

(3.29)

where the hat refers to the flat 3-space: $\tilde{\lambda}$ is the flat space edth operator and $\hat{\lambda} = 1/r$ and $\rho^j = 1/2r$ are the flat spacetime convergences of subsection 3.2.3, while the spinor components $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ are given by (3.11) and (3.12), respectively. Then an apparently obvious choice for the boundary condition would be e.g. $\delta^A \Delta_{A} \lambda^A = 0$, as in subsection 3.2.3. Then by expression (3.17) for $\delta \lambda_0$ this boundary condition yields $B_{jm} = 0$ for all $j \geq \frac{3}{2}$ and $R^2 A_{jm} + B_{jm} + N (R^2 A_{jm} - B_{jm}) = 0$, $m = \pm \frac{1}{2}$, where

$$
N := \frac{1}{2} \frac{4R^2 - (m^2 - e^2)}{(2R^2 + m)^2 - e^2}.
$$
which is proportional to the mean curvature $\nu$ of the boundary (see equation (3.27)), and takes its values between $-\frac{1}{2}$ and $\frac{1}{2}$. Thus, finally, the solution is given by

$$
\lambda_0 = \Omega \left( 1 - \frac{1 + N R^2}{1 - N r^2} \right) \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} A_m^m Y_m^{\frac{1}{2}} Y_m^m, \quad \sqrt{2}\lambda_1 = \Omega \left( 1 + \frac{1 + N R^2}{1 - N r^2} \right) \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} A_m^{-m} Y_m^{-m}
$$

(3.30)

for $r \geq R$. If $N > 0$, i.e. if $\Sigma$ does not contain the minimal surface, then $\frac{1 + N}{1 - N R^2} > 1$, and hence there is a value $r = r_0 > R$ for which $\lambda_0$ is vanishing. Then by (3.15) and (3.16), the other spinor component is zero for some $(\zeta_0, \tilde{\zeta}_0)$, and hence at the point $(r_0, \zeta_0, \tilde{\zeta}_0)$ the spinor field $\hat{\lambda}_A$ is vanishing. If $N = 0$, i.e. if the boundary $\Sigma$ is just the minimal surface, then $\lambda_0 = 0$ there, and this case reduces to that of the homogeneous chiral Dirichlet boundary condition above. If $N < 0$, i.e. when the minimal surface is contained in the interior of $\Sigma$, then the spinor field $\hat{\lambda}_A$ does not have any zero. It might be worth noting that in the limit $R \to 0$ (i.e. when the boundary $\Sigma$ is ‘pushed out’ to the other infinity), this spinor field tends to one of the two conformally constant spinor fields in (3.20) and (3.21).

Nevertheless, there is another ‘natural’ choice for the boundary condition: $\tilde{\delta}^\nu \lambda_0 + \rho \tilde{\lambda}_1 = 0$, i.e. (by the results of subsection 3.2.3) the condition that the spinor field $\hat{\lambda}_A$ be constant in the geometry of the flat 3-space. Then $\tilde{\delta} \hat{\lambda}_1 + \rho \hat{\lambda}_0 = 0$ also holds, and hence a direct calculation yields that

$$
-\tilde{\delta}^\nu A_A^A \hat{\lambda}^A = \Omega \left( -\nu \hat{\rho} \hat{\lambda}_1 + \frac{1}{2} r \nu \hat{\lambda}_1 \right) = -\frac{1}{r} \left( \frac{1}{2} r \nu - \Omega \right) \hat{\lambda}_1 = -\tilde{\delta}^\nu (v^\nu (D_\nu \ln \Omega) v_{A A} \lambda^A)
$$

(3.31)

$$
\tilde{\nu}^\nu A_A^A \hat{\lambda}^A = \Omega \left( -\nu \hat{\rho} \hat{\lambda}_0 + \frac{1}{2} r \nu \hat{\lambda}_0 \right) = \frac{1}{2r} \left( \frac{1}{2} r \nu - \Omega \right) \hat{\lambda}_0 = \tilde{\nu}^\nu (v^\nu (D_\nu \ln \Omega) v_{A A} \lambda^A).
$$

(3.32)

Thus,

$$
A_A^A \lambda^A = v^\nu (D_\nu \ln \Omega) v_{A A} \lambda^A,
$$

(3.33)

i.e. $A_A^A \lambda^A$ is proportional to the spinor field itself and the factor of proportionality is the normal directional derivative of the logarithm of the conformal factor. In the next section, we will see that the boundary condition in the non-spherically symmetric case is a direct generalization of this.

Conversely, any of (3.31) and (3.32) as a boundary condition implies that $\hat{\lambda}_A$ is conformal to a spinor field which is constant in the flat 3-space. For example, $0 = \tilde{\delta}^\nu (v^\nu (D_\nu \ln \Omega) v_{A A} \lambda^A - A_A^A \lambda^A) = \Omega^2 (\hat{\delta} \hat{\lambda}_0 - \frac{1}{2} \hat{\lambda}_1)$, yielding the boundary condition for the constant spinor fields in flat 3-space.

### 3.3.3. On solutions on the vacuum Brill–Lindquist/Bowen–York data sets.

The base manifold $\Sigma$ of the Brill–Lindquist data set is $\mathbb{R}^3$ from which the points with the Cartesian coordinates $x^1, \ldots, x^N$ have been removed. The conformal factor is $\Omega^{-1} = (1 + \sum_{m=1}^{N} \frac{m_i m_j}{x^i x^j})^2$, where $m_1, \ldots, m_N$ are positive constants, while the extrinsic curvature is vanishing. Under these conditions, the vacuum constraint equations are satisfied [11]. (This data set is generalized by the Bowen–York data set, in which the extrinsic curvature is only traceless and is chosen to give a prescribed linear or angular momentum at spatial infinity [12]. Since what is important for us is the conformal flatness of the 3-metric and the vanishing of the trace of the extrinsic
curvature, our analysis can be extended to this more general case without any difficulty.) This data set represents $N + 1$ asymptotically flat ends at $r \to \infty$ and at the (missing) points $x_1^i, \ldots, x_N^i$, and $N$ black holes with apparent horizons (in the form of minimal surfaces) surrounding the ‘internal’ asymptotic ends $x_1^i, \ldots, x_N^i$ (see also [21]).

If $\lambda_A$ is a solution of the Witten equation on $(\Sigma, h_{ab})$ such that it is bounded at the ‘outer’ asymptotic end, i.e. when $r \to \infty$, then $\lambda_A = \Omega^2 \hat{\lambda}_A$, where $\hat{\lambda}_A$ is the sum of $N$ spinor fields $\tilde{\lambda}_A^1, \ldots, \tilde{\lambda}_A^N$ whose components in the rescaled spin frame $[\tilde{\alpha}^A, \tilde{t}^A]$ are given by (3.11) and (3.12) with the centres at $x_1^i, \ldots, x_N^i$, respectively. On the other hand, since $\Omega(x^i) \to 0$ as $|x^i - x_i^1| \to 0$ when $x^i \to x_i^1$, the spinor field $\lambda_A$ is bounded near the ‘internal’ asymptotic ends too precisely when it is asymptotically constant there. Such a spinor field is parametrized by $2(N + 1)$ complex constants, but the solutions parametrized by constants belonging to a proper submanifold of $C^{2(N+1)}$ have zeros.

4. Boundary conditions for the constant and conformally constant spinor fields

4.1. Boundary conditions for the constant spinors

Let $\Sigma$ be a subset of a spacelike hyperplane in Minkowski spacetime with a (not necessarily connected) smooth boundary $S$. Clearly, there are precisely two linearly independent spinor fields on $\Sigma$ which are constant in the sense that $D_A \lambda^A = 0$. Then the spinor fields are constant on $S$ with respect to $\Delta_c$, and hence, in particular, they satisfy $\Delta_{\Sigma} \lambda^A = 0$ too. Now we show that the converse is also true. Namely, a solution $\lambda_A$ of the Witten equation is constant if and only if its restriction to the boundary satisfies the boundary condition $\Delta_{\Sigma} \lambda^A = 0$.

The key ingredient is the (Reula–Tod form of the) Sen–Witten identity [2],

\[- h^{ij} (D_i \lambda_A)(D_j \lambda_A) \chi^{A'} = \frac{1}{2} t^{A'B'} G_{ab} \lambda^A \lambda^{B'} = 2 t^{A'B'} (D_{A'B'} \lambda^B) \]

\[- D_0 (t^{A'B'} \lambda^B D_{A'B'} \lambda^A) = 0, \]

whose integral on $\Sigma$ can be written as

\[ \| \lambda_A \|^2 := \int_{\Sigma} \left( - h^{ij} (D_i \lambda_A)(D_j \lambda_A) \chi^{A'} - \frac{1}{2} t^{A'B'} G_{ab} \lambda^A \lambda^{B'} \right) d\Sigma \]

\[ = 2 \int_{\Sigma} t^{A'B'} (D_{A'B'} \lambda^B) d\Sigma + \oint_{S} \tilde{\lambda}^{A'} \gamma_{A^{1}}^{B'} \Delta_{\Sigma} \lambda^{B} dS. \]  

(Here $^4G_{ab}$ is the spacetime Einstein tensor, which, actually, vanishes by assumption.) The first line of (4.2), which is essentially a Sobolev norm, is an integral of pointwise non-negative expressions (even if the data induced on $\Sigma$ were not flat, but satisfied the dominant energy condition), while the volume integral in the second line is vanishing by the Witten equation. Thus, if $\lambda_A$ is a solution of the Witten equation satisfying the boundary condition $\Delta_{\Sigma} \lambda^A = 0$, then by (4.2) $\| \lambda_A \| = 0$, i.e. $D_0 \lambda_A = 0$ follows.

If $\lambda_B$ is constant with respect to $D_0$ on $\Sigma$, then its restriction to the boundary satisfies the 2-surface twistor equation as well: $T_{A'AB'} c_c = 0$. Although in the special cases considered in subsection 3.2 any of the two equations $T_{A'AB'} c_c = 0$ appears to be an appropriate boundary condition to single out the constant spinor fields, we do not have an equation analogous to (4.2) by means of which we could show a direct relationship between the $D_0 \lambda_A$ derivative on $\Sigma$ and the 2-surface twistor derivative of $\lambda_A$ on $S$. Thus, we formulate our boundary condition in terms of the 2-surface Dirac rather than the 2-surface twistor operator (though the conformal invariance of the latter could have suggested to use it, especially in the conformally flat spaces).
However, the boundary condition $\Delta_{A'A} \lambda^A = 0$ appears to contradict the general theory of boundary value problems for elliptic systems: it is too strong, as it represents two rather than only one restriction on the spinor field. Although this boundary condition can be imposed in the special case of flat geometries, we should be able to reformulate it in a way that is compatible with the general theory of elliptic boundary value problems. In fact, the previous theorem can be proven with the following weaker, $\mathbb{R}$-linear (rather than $\mathbb{C}$-linear) boundary condition: for some complex function $\alpha : \mathcal{S} \to \mathbb{C}$, which may depend on the spinor field, the spinor field satisfies $\Delta_{A'A} \lambda^A = \alpha \hat{\gamma}_{A'B} \hat{\lambda}^B$. Or, in other words, it is required that the $\hat{\gamma}_{A'B} \hat{\lambda}^B$-component of the derivative of the spinor field vanishes: $\hat{\lambda}^B \hat{\gamma}_{A'B} \Delta_{A'A} \lambda^A = 0$. This provides the correct number of boundary conditions, and can be rewritten as $\tilde{B}^A (\Delta_{A'A} \lambda^A - \alpha \hat{\gamma}_{A'B} \hat{\lambda}^B) = 0$ for any spinor field $B^A$ and some $\alpha$ on $\mathcal{S}$.

4.2. Conformally constant spinor fields on maximal, conformally flat hypersurfaces

By the results of subsections 2.1.1 and 4.1, the solution $\lambda_A$ of the Witten equation (with conformal weight $w = -\frac{1}{2}$) is conformally constant in the conformally flat 3-space precisely when $\hat{\lambda}^A := \Omega^{-2} \lambda^A$ satisfies the boundary condition $\Delta_{A'A} \hat{\lambda}^A = 0$. Then by the conformal rescaling formula and $\nu^\nu \nabla, \Omega = 0$, this is equivalent to

$$\Delta_{A'A} \lambda^A = \left(\frac{4}{\Omega} \delta_{A'A} + 4 \nu_{A'B} D_b \ln \Omega \right) \lambda^A;$$

i.e. the derivative $\Delta_{A'A} \lambda^A$ is proportional to the spinor field itself where the factor of proportionality is built from the derivatives of the conformal factor. (N.B.: On functions the derivative operator $\Delta_x$ coincides with the intrinsic Levi-Civita derivative operator $\delta_x$.) Contracting (4.3) with any given spinor field $B^A$ and denoting the coefficient of $\lambda^A$ in the resulting formula by $-A_A$, it has the form (2.15) with $f = 0$. Moreover, assigning zero conformal weight to $B^A$, it is straightforward to check that under a conformal rescaling $h_{ab} = \Omega^{-2} \tilde{h}_{ab} \mapsto \alpha^2 h_{ab} = \alpha^2 \Omega^{-2} \tilde{h}_{ab}$ of the physical metric, the spinor field $\lambda_A$ transforms just in the way required in the conformally invariant boundary condition (2.19). Indeed, since the concept of conformally constant spinor fields is conformally invariant, the boundary condition that specifies these should also be conformally invariant.

Our aim is to characterize the conformally constant spinor fields among the solutions of the Witten equation by appropriate boundary conditions in the physical, conformally flat (rather than in the flat, rescaled) geometry of the hypersurface. By (4.3) the boundary condition must be searched for in the form

$$\Delta_{A'A} \lambda^A = \left(\frac{4}{\Omega} \delta_{A'A} + \nu_{A'B} \right) \lambda^A$$

for some functions $\alpha, \beta : \mathcal{S} \to \mathbb{R}$. Clearly, for $\alpha = \text{const}$, $\beta = 0$, this reduces to the boundary condition $\Delta_{A'A} \lambda^A = 0$ for the constant spinors found in the previous subsection. However, these functions cannot be arbitrary, because the (globally conformally flat) geometry $(\Sigma, h_{ab})$ determines the conformal factor by means of which the geometry can be rescaled to be flat. On the other hand, there can be different flat 3-spaces that are globally conformal to each other (with a non-trivial conformal factor). Hence $(\Sigma, h_{ab})$ does not determine the conformal factor completely. There can be some ambiguity in $\Omega$. Next we determine what kind of conditions should $\ln \Omega$ and $\nu^\nu \nabla, \ln \Omega$ satisfy, and hence what kind of conditions should $\alpha$ and $\beta$ satisfy on $\mathcal{S}$.

By the conformal rescaling formulae for the three-dimensional Ricci tensor,

$$0 = \tilde{R}_{ab} = R_{ab} + D_a D_b \ln \Omega - (D_a \ln \Omega)(D_b \ln \Omega) + h_{ab}(D_c D^c \ln \Omega + (D_c \ln \Omega)(D^c \ln \Omega)),$$

(4.5)
it is straightforward to calculate the three-dimensional Einstein tensor. For its ‘constraint parts’ on the boundary surface, \( G_{ab} v^a v^b \) and \( G_{bc} v^b \Pi^c_\nu \), we obtain
\[
\delta_\nu \delta^\nu \ln \Omega - v(v^a D_a \ln \Omega) - (v^a D_a \ln \Omega)^2 = -G_{ab} v^a v^b, \tag{4.6}
\]
\[
\delta_\nu (v^a D_a \ln \Omega) - (v^a D_a \ln \Omega) \delta_\nu \ln \Omega - v^a \delta_\nu \ln \Omega = -G_{bc} v^b \Pi^c_\nu. \tag{4.7}
\]
Here \( v_{ab} := \Pi^a_{\nu} \Pi^b_{\nu} D_a v_{bc} \), the extrinsic curvature of \( S \) in \( \Sigma \), and \( v \) is its trace. Therefore, as we claimed, \( \ln \Omega \) and \( v^a (D_a \ln \Omega) \) in (4.3) are restricted by the geometry \((\Sigma, h_{ab})\) via (4.6) and (4.7). On the other hand, we show that once \( \ln \Omega \) or \( v^a (D_a \ln \Omega) \) is given on \( S \), then the conformal factor on \( \Sigma \) is already completely determined, provided that Einstein’s equations hold, \( \delta_\nu G_{ab} = -\kappa T_{ab}, \) and the weak energy condition \( T_{ab} v^a v^b \geq 0 \) is satisfied. For, first observe that by the maximality of the hypersurface and the Hamiltonian constraint part of Einstein’s equation \( R = \chi_{ab} \chi^{ab} - \chi^2 + 2\kappa T_{ab} v^a v^b \geq 0 \), i.e. the scalar curvature is non-negative. Second, by taking the trace of (4.5), it is straightforward to derive the linear equation
\[
D_a D^a \sqrt{\Omega} + \frac{1}{2} R \sqrt{\Omega} = 0. \tag{4.8}
\]
This equation is also known as the Lichnerowicz or Yamabe equation. Next suppose that \( \Omega' \) is another solution of (4.8), and define \( u := \sqrt{\Omega} - \sqrt{\Omega'} \). Then \( u \) also satisfies (4.8). Multiplying this by \( u \) and integrating on \( \Sigma \), by the Gauss theorem we obtain
\[
\int_S (\sqrt{\Omega} - \sqrt{\Omega'}) v^a D_a (\sqrt{\Omega} - \sqrt{\Omega'}) dS = \int_{\Sigma} \left( -(D_a u)(D^a u) + \frac{1}{8} Ru^2 \right) d\Sigma,
\]
where both terms in the integrand on the right-hand side are pointwise non-negative. Thus, if either \( \Omega \) and \( \Omega' \), or \( v^a D_a \sqrt{\Omega} \) and \( v^a D_a \sqrt{\Omega'} \) coincide on \( S \), then \( \Omega' = \Omega \) on the whole \( \Sigma \), too. Therefore, \( \Omega \) is, in fact, completely determined e.g. by its own value on the boundary, and the ambiguity in \( \Omega \) corresponds to the non-uniqueness of the solution of (4.6) and (4.7).

To determine this ambiguity, let us suppose that both \( \Omega \) and \( \Omega' := \omega^{-1} \Omega \) are solutions of (4.5). Then the difference of this equation for \( \Omega \) and for \( \Omega' \) is a differential equation, whose trace (multiplied by \( \Omega \)) and trace-free part (multiplied by \( 2\Omega^{-\frac{3}{2}} \)) yield the system of equations
\[
D_{(a}(\Omega D^{a} \omega^{-\frac{3}{2}}) = 0, \tag{4.9}
\]
\[
D_{(a}(\Omega^{-2} D_{b) \omega} + D_{b}(\Omega^{-2} D_{a} \omega) - \frac{2}{3} h_{ab} D_{c}(\Omega^{-2} D^{c} \omega) = 0. \tag{4.10}
\]
However, the second is just the conformal Killing equation for the hypersurface–orthogonal vector field \( K_a := \Omega^{-2} D_a \omega \). Thus, by determining those conformal Killing vectors \( K_a \) for which \( \Omega^2 K_a \) is a gradient, we obtain a class of functions \( \omega \), and the ambiguity in the conformal factor is represented by those of these functions that solve (4.9), too. It might be worth noting that in the Cartesian coordinates \( \{x^i\} \) of the flat 3-space \((\Sigma, \tilde{h}_{ab})\), the solution \( \omega \) can be given explicitly. In fact, in terms of the rescaled (flat) metric, (4.9) and (4.10) take the form
\[
\tilde{D}_a \tilde{D}^{a} \omega^{-\frac{3}{2}} = 0, \quad \tilde{D}_a \tilde{D}_{b} \omega - \frac{1}{3} \tilde{h}_{ab} \tilde{D}_c \tilde{D}^{c} \omega = 0.
\]
Then the solution of these equations is \( \omega = C \delta_i (x^i + C^i)(x^j + C^j) \) for some constants \( C \) and \( C^i \).

Thus, to summarize, the boundary condition that specifies the conformally constant spinor fields among the solutions of the Witten equation is (4.4), where the functions \( \alpha \) and \( \beta \) are solutions of the differential equations
\[
\delta_v \delta^\nu \alpha - v^a \beta - \beta^2 = -G_{ab} v^a v^b, \tag{4.11}
\]
\[
\delta_v \beta - \delta_\nu \alpha - v^a \delta_\nu v^a = -G_{bc} v^b \Pi^c_\alpha. \tag{4.12}
\]
Since \( \ln \Omega \) and \( \omega'(D_e \ln \Omega) \) solve these, we know that such \( \alpha \) and \( \beta \) exist, i.e. our boundary condition can always be imposed. The solution determines a conformal factor on \( \Sigma \) in a unique way such that the corresponding conformal rescaling yields a flat 3-space and a constant spinor field. Although the solution \((\alpha, \beta)\) of (4.11) and (4.12) is not unique, the ambiguity is completely controlled by the function \( \omega \), which solves (4.9) and (4.10). The meaning of this \( \omega \) is, however, only a ‘pure gauge’, telling us which flat 3-metric is chosen to be the ‘reference’ with respect to which the physical metric is conformally flat.

5. Summary and conclusions

In certain physical problems and in the study of the structure of the field equations, it is useful to reduce the gauge freedom of the theory by some appropriate gauge condition. On a spacelike hypersurface, this could be the use of the orthonormal triad field coming from the spinor fields solving the Witten equation, or the triad field satisfying the so-called special orthonormal frame gauge condition. These conditions take the form of some elliptic p.d.e.; thus, their solutions can be controlled by the boundary condition. However, for general boundary conditions the triad fields (either built from the solution of the Witten equation or satisfying the frame gauge condition) can be degenerate. Thus, the proper gauge condition should consist of the elliptic p.d.e. and the boundary conditions selecting the globally non-singular ones from the infinitely many solutions.

In this paper, these boundary conditions were investigated on maximal, globally intrinsically conformally flat spacelike hypersurfaces. (Such hypersurfaces are e.g. the Reissner–Nordström, the Brill–Lindquist and the Bowen–York initial data sets for finitely many black holes in asymptotically flat spacetimes.) On such hypersurfaces the two gauge conditions above are equivalent, and hence can be studied simultaneously. We determined the boundary conditions that characterize (1) the constant spinor fields on compact domains in intrinsically and extrinsically flat hypersurfaces, and (2) the conformally constant spinor fields on compact domains in maximal, intrinsically globally conformally flat spacelike hypersurfaces (provided the Hamiltonian constraint holds and the weak energy condition is satisfied). Thus, in particular, the special orthonormal frame gauge condition can always be satisfied by globally non-degenerate frames on arbitrary compact domains in such hypersurfaces.

The exact solutions of subsections 3.2 and 3.3 show that many of the ‘natural’ boundary conditions (appearing in various special problems) yield a degenerate triad field. In addition, the example of the Reissner–Nordström (and the more general Brill–Lindquist and Bowen–York) data sets shows that there is a (still not quite well understood) interplay between the boundary conditions, the global topology of the hypersurface and the existence/non-existence of zeros of the solutions of the Witten equation: the boundary conditions on the different connected components of the boundary cannot be chosen independently.

Acknowledgments

JF is grateful to the Research Institute for Particle and Nuclear Physics, Budapest, and to the Albert Einstein Institute, Golm, for kind hospitality while this work was finished. The work of JMN was supported by the National Science Council of the ROC under the grant number NSC-99-2112-M-008-004 and in part by the Taiwan National Center of Theoretical Sciences (NCTS). LBSz is grateful to the Center for Mathematics and Theoretical Physics, National Central University, Chung-Li, Taiwan, for hospitality during the preparation of a preliminary version of this work. This work was partially supported by the Marsden Fund.
of the Royal Society of New Zealand under the grant number UOO0922 and the Hungarian Scientific Research Fund (OTKA) grant K67790.

References

[1] Gibbons G W, Hawking S W, Horowitz G T and Perry M J 1983 Positive mass theorem for black holes Commun. Math. Phys. 88 295–308
[2] Reula O and Tod K P 1984 Positivity of the Bondi energy J. Math. Phys. 25 1004–8
[3] Nester J M 1989 A positive gravitational energy proof Phys. Lett. A 139 112–4
[4] Nester J M 1989 A gauge condition for orthonormal three-frames J. Math. Phys. 30 624–6
[5] Nester J M 1992 Special orthonormal frames J. Math. Phys. 33 910–3
[6] Frauendiener J 1991 Triads and the Witten equation Class. Quantum Grav. 8 1881–7
[7] Dimakis A and Müller-Hoissen F 1989 On a gauge condition for orthonormal three-frames Phys. Lett. A 142 73–4
[8] Bár C 1997 On nodal sets for Dirac and Laplace operators Commun. Math. Phys. 188 709–21
[9] Nester J M 2007 On the zeros of spinor fields and an orthonormal frame gauge condition Proc. 11th Marcel Grossmann Meeting on General Relativity (Berlin, Germany, 23–29 July 2006) ed H Kleinert and R T Jantzen (Singapore: World Scientific) pp 1332–4
[10] Gibbons G W 1984 The isoperimetric and Bogomolny inequalities for black holes Global Riemannian Geometry (Ellis Horwood Series in Mathematics and Its Applications) ed T J Willmore and N J Hitchin (Chichester: Ellis Horwood) pp 194–202
[11] Brill D R and Lindquist R W 1963 Interaction energy in geometrostatics Phys. Rev. 131 471–6
[12] Bowen J M and York J W 1980 Time-asymmetric initial data for black holes and black hole collisions Phys. Rev. D 21 2047–56
[13] Penrose R and Rindler W 1984 Spinors and Spacetime vol 1 (Cambridge: Cambridge University Press) Penrose R and Rindler W 1986 Spinors and Spacetime vol 2 (Cambridge: Cambridge University Press)
[14] Hugget S A and Tod K P 1985 An Introduction to Twistor Theory (London Mathematical Society Student Texts vol 4) (Cambridge: Cambridge University Press)
[15] Sen A 1981 On the existence of neutrino ‘zero-modes’ in vacuum spacetimes J. Math. Phys. 22 1781–6
[16] Szabados L B 1994 Two-dimensional Sen connections in general relativity Class. Quantum Grav. 11 1833–46
[17] Dain S 2006 Elliptic systems Analytical and Numerical Approaches to Mathematical Relativity (Lecture Notes in Physics vol 692) ed J Frauendiener, D J W Giulini and V Perlick (Berlin: Springer) pp 117–39 (arXiv:gr-qc/0411081)
[18] Dougan A J and Mason L J 1991 Quasilocal mass constructions with positive energy Phys. Rev. Lett. 67 2119–22
[19] Szabados L B 1994 Two-dimensional Sen connections and quasi-local energy-momentum Class. Quantum Grav. 11 1847–66
[20] Szabados L B 2002 On certain quasi-local spin-angular momentum expressions for large spheres near the null infinity Class. Quantum Grav. 18 5487–510
[21] Gibbons G W 1972 The time symmetric initial value problem for black holes Commun. Math. Phys. 27 87–102