More Parallelism in Dijkstra’s Single-Source Shortest Path Algorithm

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Abstract

Dijkstra’s algorithm for the Single-Source Shortest Path (SSSP) problem is notoriously hard to parallelize in \( o(n) \) depth, \( n \) being the number of vertices in the input graph, without increasing the required parallel work unreasonably. Crauser et al. (1998) presented observations that allow to identify more than a single vertex at a time as correct and correspondingly more edges to be relaxed simultaneously. Their algorithm runs in parallel phases, and for certain random graphs they showed that the number of phases is \( O(n^{1/3}) \) with high probability. A work-efficient CRCW PRAM with this depth was given, but no implementation on a real, parallel system.

In this paper we strengthen the criteria of Crauser et al., and discuss tradeoffs between work and number of phases in their implementation. We present simulation results with a range of common input graphs for the depth that an ideal parallel algorithm that can apply the criteria at no cost and parallelize relaxations without conflicts can achieve. These results show that the number of phases is indeed a small root of \( n \), but still off from the shortest path length lower bound that can also be computed.

We give a shared-memory parallel implementation of the most work-efficient version of a Dijkstra’s algorithm running in parallel phases, which we compare to an own implementation of the well-known \( \Delta \)-stepping algorithm. We can show that the work-efficient SSSP algorithm applying the criteria of Crauser et al. is competitive to and often better than \( \Delta \)-stepping on our chosen input graphs. Despite not providing an \( o(n) \) guarantee on the number of required phases, criteria allowing concurrent relaxation of many correct vertices may be a viable approach to practically fast, parallel SSSP implementations.

1 Introduction

The single-source shortest path (SSSP) problem is one of the most productive problems in computer science. The SSSP problem has so far proven hard to parallelize, and no algorithms with linear, parallel speedup for the general case are so far known. For graphs with non-negative edge costs, Dijkstra’s algorithm \( [6] \) (or variations thereof) is practically and theoretically attractive, but unfortunately also strictly sequential: The vertices of the graph are processed (identified as correct) one after the other, with possible parallelism only in the relaxation step. Better sequential bounds than possible with Dijkstra’s algorithm are known for the RAM model \( [10] \), especially for undirected graphs \( [16] \), and imply that vertices are processed in a different order than that implied by
Dijkstra’s algorithm, but such algorithms are hardly practical. Parallelizations of these algorithms are also not known. Zwick [17] gives an excellent overview of approaches to the SSSP and related problems.

In this paper [11], we explore criteria that allow Dijkstra’s algorithm to settle more than a single vertex at a time, thus providing potentially more parallelism by allowing relaxation of the edges of several vertices in the same parallel phase. Such criteria for identifying whether a vertex is already correct (a shortest path found) were presented explicitly by Crauser et al. [5]. Concretely, they proposed (and combined) two such criteria leading to a version of Dijkstra’s algorithm that runs in parallel phases. In each phase, at least one, but possibly much more vertices are identified as already correct based on the tentative distances computed so far. In a phase, all correct vertices are settled, and all edge relaxations potentially done in parallel. Crauser et al. analyzed the expected number of phases for certain random graphs. For random graphs with \( n \) vertices, their combined criteria reduce the number of phases to \( O(n^{1/3}) \) with high probability. Crauser et al. gave a CRCW PRAM implementation of their algorithm, and discussed simulation results confirming the analytical results. However, they did not give any real implementations.

In this paper, we take up on the work of Crauser et al. We first give much stronger criteria exploiting in each phase more information available in the explored part of the graph as well as in the unexplored part, and discuss the work required in order to decide efficiently whether a vertex is correct. We present a more extensive simulation study with different types of graphs, and compare the number of phases to the lower bound on the number of phases that can also be computed. This study shows that the strengthened criteria improve over Crauser et al.’s criteria by reducing the number of phases to a smaller root of the number of vertices, unfortunately at the drawback of being more expensive to evaluate. We give an implementation for shared-memory multi-core processors of the Crauser et al. criteria, and compare running time and scalability (speed-up) to what can be achieved with \( \Delta \)-stepping [14]. Our benchmarks show that the SSSP algorithm running in phases and using only Crauser’s et al.’s criteria is an attractive alternative to \( \Delta \)-stepping, in many cases providing significantly higher speed-up, and never performing worse. This approach to parallelizing Dijkstra’s SSSP algorithm clearly merits further attention. It is important to point out, though, that the criteria for identifying correct vertices described here (and by Crauser et al.) do not lead to any worst-case guarantees on the number of phases being strictly smaller than the number of vertices or any other theoretical improvements of Dijkstra’s SSSP algorithm, but they do provide much room for engineering the implementations, and as our experiments show for many types of graphs do lead to very significant reductions in the number of phases.

Recently, Garg [8] also took up on the idea of Crauser et. al., and presented improvements and implementation ideas, most of which can be seen as special cases of the extensions we discuss in this paper. Also this paper contains no real implementation, and also no simulation results investigating the strength of the various approaches. The \( \Delta \)-stepping idea is extended by Blelloch et al. [1] with a careful work-depth tradeoff analysis. The paper gives a simulation based experimental analysis of the bounds, but presents no real implementation of the ideas.

2 Preliminaries

Let \( G = (V, E) \) be a directed graph with vertices \( V \) and edges \( E \), with a cost function \( c : E \to \mathbb{R}_{\geq 0} \) assigning a non-negative, real-valued cost to each edge of \( G \). The cost of a path is the sum of the costs.

\[^{1}\text{The paper is based on the Master’s thesis of Michael Kainer, commenced early 2018, completed November 2018.}\]
of the edges along the path, that is \( c(P) = \sum_{e \in P} c(e) \) for path \( P = [(u_0, u_1), (u_1, u_2), \ldots, (u_{p-1}, u_p)] \) consisting of edges \( e_i = (u_i, u_{i+1}), 0 \leq i < p \). The length of a path is the number of edges along the path, that is \( p \) for the path \( P \). Since edge costs are non-negative, for each vertex \( u \) and each vertex \( v \) reachable from \( u \) in \( G \) there is a path having the smallest cost over all possible paths from \( u \) to \( v \). Such a path is called a shortest path from \( u \) to \( v \), and \( \text{dist}(u, v) \) is defined to be the cost of a shortest path from \( u \) to \( v \), with \( \text{dist}(u, u) = 0 \) (empty path); for convenience we also define \( \text{dist}(u, v) = \infty \) if \( v \) is not reachable from \( u \). Let \( s \in V \) be a given source vertex. The Single-Source Shortest Path problem (SSSP) is to compute \( \text{dist}(s, u) \) for each \( u \in V \).

Dijkstra’s algorithm (implicitly) maintains a partition of the vertices of \( G \) into settled vertices \( S \), fringe vertices \( F \), and unexplored vertices \( U \), and works as follows. A tentative distance \( d[u] \) is associated with each vertex \( u \in V \). The settled vertices \( u \in S \) have the property that \( d[u] = \text{dist}(s, u) \). For vertices \( u \in F \), \( d[u] \geq \text{dist}(s, u) \), and additionally \( d[u] \) is the cost of a shortest path from \( s \) to \( u \) passing only through vertices in \( S \). For vertices \( u \in U \), no such path through only vertices in \( S \) exists. To establish the invariants, Dijkstra’s algorithm initially sets \( S = \emptyset, F = \{s\} \) with \( d[s] = 0 \), and \( U = V \setminus F \). The crucial observation is that for a vertex \( u \in F \) with \( d[u] = \min_{v \in F} d[v] \), it holds that \( d[u] = \text{dist}(s, u) \), such that \( u \) can be settled: Vertex \( u \) is moved to the set \( S \). To reestablish the invariants, all outgoing edges incident to \( u \) are relaxed: For each edge \((u, v)\), if \( v \in F \) and \( d[u] + c(u, v) < d[v] \), \( d[v] \) is updated to the shorter path cost \( d[u] + c(u, v) \) obtained by passing through \( u \), and if \( v \in U \), \( d[v] \) is set to \( d[u] + c(u, v) \) and \( v \) is moved to \( F \) from \( U \) (if \( v \in S \), there is nothing to be done, since \( d[u] + c(u, v) \geq d[v] \)).

In the following, we say that a vertex \( u \in F \) is correct if \( d[u] = \text{dist}(s, u) \). Dijkstra’s algorithm identifies and settles one correct vertex per iteration.

Now let \( n = |V| \) be the number of vertices and \( m = |E| \) the number of edges of \( G \). It is well-known, but non-trivial that with the right graph representation and data structures, Dijkstra’s algorithm can be implemented to run in \( O(n \log n + m) \) operations \[7\]. An overview of priority queues as needed for Dijkstra’s algorithm can be found in \[3\].

The algorithm is strictly sequential, since correct vertices are identified one after the other, and only possibly the edge relaxation can be done in parallel. Such a straightforward parallelization of Dijkstra’s algorithm can be found in, e.g., \[15\] which gives an EREW PRAM algorithm running in \( O(m/p + n \log n) \) parallel time on \( p \) processors. Using a parallel priority queue with constant time operations, this can be improved to, e.g., \( O(n^2/p + n) \) for dense graphs \[4\]. The number of sequential steps remain the bottleneck.

Our aim in the rest of this paper is to be able identify more correct vertices \( u \in F \) at the same time, and do all relaxations from these vertices in parallel.

### 3 Identifying correct vertices

We use a generic SSSP algorithm that runs in phases in which several vertices can be settled in parallel. Let the partition of \( V \) into \( S, F, \) and \( U \) be as above. Let \( \phi(v) \) be a predicate on vertices \( v \in F \) which we in the following call a criterion. The phased SSSP algorithm initializes \( S, F, U \) as does Dijkstra’s algorithm. At the beginning of a phase, a set \( S' \) consisting of some, all, but at least one vertex \( v \in F \) fulfilling \( \phi(v) \) is computed and removed from \( F \). All \( v \in S' \) are then settled in some order, or in parallel, which means that all outgoing, adjacent edges are relaxed. All vertices of \( S' \) are then moved to \( S \), and data structures needed for the next phase updated. The algorithm terminates when \( F = \emptyset \), or when no vertices \( v \in F \) fulfill \( \phi(v) \).
Definition 1 (Soundness and completeness) A criterion \( \psi(v) \) is called sound if for each \( v \in F \), \( \psi(v) \) implies \( d[v] = \text{dist}(s, v) \), that is \( v \) is correct. A criterion \( \phi(v) \) is called complete if whenever \( F \neq \emptyset \), \( \phi(v) \) holds for at least one vertex \( v \in F \).

A sound criterion guarantees that the generic SSSP algorithm computes only correct distances. A sound and complete criterion ensures that a shortest path to all reachable vertices from \( s \) has indeed been found. Dijkstra’s algorithm is the instance of the generic algorithm with the criterion DIJK\((v)\) defined by \( d[v] = \min_{u \in F} d[u] \), where one vertex satisfying \( \text{DIJK}(v) \) is selected for \( S' \) in each phase. The correctness proof of Dijkstra’s algorithm shows precisely that \( \text{DIJK}(v) \) is sound and complete. It is interesting that any sound criterion in combination with the settling step will maintain the invariant on \( S, F, \) and \( U \), with the important property that edges are relaxed only once. That is, the generic algorithm maintains the label setting property of Dijkstra’s algorithm such that the total work for all relaxations is \( O(m) \). The number of phases incurred by a specific criterion (on given inputs) is a lower bound on the time that a parallel algorithm can achieve, regardless of the number of processors employed.

Let \( \phi \) and \( \psi \) be two criteria on vertices \( v \in F \). We say that \( \psi \) is stronger than \( \phi \) if \( \phi(v) \Rightarrow \psi(v) \), that is, the stronger criterion identifies more vertices as correct. It is then easy to see that if \( \psi \) is sound, then also \( \phi \) is sound, and if \( \phi \) is complete, then also \( \psi \) is complete. Furthermore, a disjunction of criteria is sound if all disjuncts are sound, and complete if at least one of the disjuncts is complete [Chapter 3].

The strongest possible criterion is the clairvoyant oracle criterion \( \text{ORACLE}(v) \) defined as \( d[v] = \text{dist}(s, v) \) which holds as soon as \( d[v] \) happens to be correct. Clearly, \( \text{DIJK}(v) \Rightarrow \text{ORACLE}(v) \). The oracle criterion can be used to determine the smallest number of phases in the generic algorithm and thus the maximum amount of parallelism for a given input graph, but computing it efficiently seems to require knowledge of the distance to all vertices (omniscience, clairvoyance).

We now introduce two new criteria and several weaker variants to be used in our generic algorithm. The criteria considerably strengthens the IN and OUT criteria originally proposed by Crauser et al. [1].

Define the \( \text{IN}(v) \) criterion to hold if

\[
D[v] - \min \left\{ \min_{w \in F} d[w, v] c(w, v) \right\} \leq \min_{u \in F} d[u] \quad (1)
\]

Define the \( \text{OUT}(v) \) criterion to hold if

\[
dx[v] \leq \min \left\{ \min_{u \in F} d[u], c(u, w)
\right\} \quad (2)
\]

Lemma 1 The \( \text{IN}(v) \) criterion defined by Equation (1) is sound and complete.

Proof: We have to prove that whenever \( \text{IN}(v) \) holds, then \( d[v] = \text{dist}(s, v) \) such that \( v \) is correct. First observe that for any incorrect vertex \( v \in F \) with \( d[v] > \text{dist}(s, v) \), there is a shortest path to \( v \) consisting of first a path over vertices in \( S \), an edge between a vertex in \( S \) and a vertex in \( F \), followed by a path of length at least one over vertices in \( F \cup U \). Thus, a shortest path to \( v \) must end with an edge \( (w, v) \) with \( w \in F \cup U \). Now assume that \( v \in F \) is not correct, but that for all \( w \in F \), \( d[w] - c(w, v) \leq \min_{u \in F} d[u] \), that is \( \text{IN}(v) \) holds. Since \( \min_{u \in F} d[u] \leq \text{dist}(s, w) \), it follows that \( d[v] \leq \text{dist}(s, w) + c(w, v) \) for all \( w \in F \), especially the \( w \) for which \( \text{dist}(s, v) = \text{dist}(s, w) + c(w, v) \),
IN(v) = d[v] - \min \left\{ \min_{w \in F, (w,v) \in E} c(w,v) \min_{w \in U, w' \in F \cup U, (w', w) \in E} c(w',w) + c(w,v) \right\} \leq \min_{u \in F} d[u]

Figure 1: Structure of possible shortest paths to vertex v \in F, showing why the IN(v) criterion correctly decides if v is already correct, d[v] = dist(s,v). Vertex v is correct if d[v] is smaller than or equal to the length of each of the shown possible paths from s. The minima InF over all incoming edges from vertices w_1, w_2, w_3, ... \in F and InU over all possible paths of two edges from vertices w_4, w_5, ... \in U that need to be maintained efficiently when implementing the criterion are also illustrated. The crucial observation for the IN(v) criterion is that all edges ending in w_4, w_5, ... \in U must start from vertices w' \in F \cup U; if not, w_4, w_5, ... cannot be in U.
\[ \text{OUT}(v) \equiv d[v] \leq \min \left\{ \min_{u \in F, w \in F, (u, w) \in E} d[u] + c(u, w), \min_{w \in U, w' \in F \cup U, w, w' \in E} d[u] + c(u, w) + c(w, w') \right\} \]

Figure 2: Structure of possible shortest paths to vertex \( v \in F \), showing why the \( \text{OUT}(v) \) criterion correctly decides if \( v \) is already correct, \( d[v] = \text{dist}(s, v) \). Vertex \( v \) is correct if \( d[v] \) is smaller than or equal to each of the possible shortest paths from vertices \( u_1, u_2, \ldots \in F \) shown. The minima \( \text{OutF} \) and \( \text{OutU} \) that need to be maintained efficiently when implementing the criterion are also illustrated. Note that the minima over the outgoing edges of vertices \( w_1, w_2, w_3, \ldots \in U \) for the \( \text{OUT}(v) \) criterion are over edges with endpoint in \( F \cup U \), which is costly to maintain efficiently since they change as vertices adjacent to \( w_1, w_2, w_3, \ldots \) become settled. The \( \text{OUTWEAK}(v) \) criterion therefore uses only statically computed minima over all adjacent edges.
implemented to run in $O(n \log n)$ operations. For completeness, observe that any vertex $v$ with $d[v] = \min_{u \in F} d[u]$ fulfills IN($v$). □

Lemma 2  The OUT($v$) criterion defined by Equation (2) is sound and complete.

Proof: A shortest path to $v \in F$ has a first vertex $u$ which is not in $S$. This vertex must be in $F$, and $d[u]$ must be the correct distance to this vertex, $d[u] = \text{dist}(s, u)$. Assume that $v$ is not correct, but that OUT($v$) holds in which case there is at least one edge $(u, w), w \neq v$ on a shortest path to $v$. Then $d[v] \leq d[u] + c(u, w) \leq \text{dist}(s, v)$ for $w \in F$ contradicts that $\text{dist}(s, v) < d[v]$. If $w$ is in $U$, a shortest path to $v$ must have at least one more edge $(w, w') \in E$ with $w' \in F \cup U$, and $d[v] \leq d[u] + (c(u, w) + c(w, w')) \leq \text{dist}(s, v)$ again contradicts the assumption that $d[v] > \text{dist}(s, v)$.

For completeness, for any vertex $u$ leading to the minimum value at the right hand side of Equation (2), OUT($u$) will hold. □

The two criteria are orthogonal. There are vertices $v$ that are correct according to the IN($v$) criterion, but not according to the OUT($v$) criterion, and vice versa. The two criteria can be combined disjunctively to further reduce the number of phases. Note that the completeness argument for the IN($v$) criterion shows that DIJK($v$) $\Rightarrow$ IN($v$). This is not the case for the OUT($v$) criterion which may choose a different vertex than one having minimum tentative distance. The structure of the paths establishing the IN($v$) and OUT($v$) criteria (and the weaker variants discussed in the following) are illustrated in Figures 1 and 2. These figures also illustrate the minima maintained with the data structures described in the algorithms discussed in the following propositions.

We now claim that we can implement the generic SSSP algorithm using the IN($v$) criterion in $O(m \log n)$ operations.

Proposition 1  The generic SSSP algorithm exploiting the IN($v$) criterion of Equation (1) can be implemented to run in $O(n \log n + m \log n)$ operations.

For the implementation, we assume that the input graph is given as an array of adjacency lists of both outgoing and incoming edges for each vertex. Additionally, for each incoming edge $(u, v)$ of $v$, there is a reference to the position of $(u, v)$ in the list of outgoing edges of $u$. Similarly for the outgoing edges. If this is not the case, such a representation can be computed in $O(n + m)$ operations.

Proof: As for Dijkstra’s algorithm, we use a priority queue of tentative distances $d[v]$ for $v \in F$ supporting delete-min, insert, and decrease-key operations.

We associate two heaps (priority queues) $\text{InF}[v], \text{InU}[v]$ supporting find-min, delete, insert and build operations with each vertex $v \in F \cup U$. These heaps will store edge costs corresponding to the two min-terms in the left hand side of the IN criterion, such that the left hand side expression can be computed as $d[v] - \min(\text{find-min}\ \text{InF}[v], \text{find-min}\ \text{InU}[v])$. With the find-min heap operation taking constant time, it can be determined in constant time for any $v \in F$ whether the IN criterion is satisfied. Keeping the values $d[v] - \min(\text{find-min}\ \text{InF}[v], \text{find-min}\ \text{InU}[v])$ in yet another priority
queue InCrit, the vertices $v$ for which the IN$(v)$ criterion are satisfied can be extracted in $O(\log n)$ operations each.

The generic algorithm is instantiated as follows. Initially, for each $v \in U$, for all incoming edges, the costs $(\min_{w' \in F \cup U, w \in U, (w', w) \in E} c(w', w)) + c(w, v)$ are inserted into InU$[v]$ by a bulk build heap operation. The crucial observation here is that for $w \in U$, it will always hold that for all incoming edges $(w', w) \in E$ that $w' \in F \cup U$ (if there were a vertex $w' \in S$, the edge $(w', w)$ would have been relaxed, and $w$ not in $U$). The value $(\min_{w' \in F \cup U, w \in U, (w', w) \in E} c(w', w))$ will therefore not change as long as $w \in U$ and can be precomputed for all vertices in $O(m)$ operations. In other words, the term $\min_{w' \in E \cup U, (w', w) \in E} c(w', w) + c(w, v)$ is approximated by the possibly smaller $\min_{w' \in E \cup U, (w', w) \in E} c(w', w)$ for the OUT$(v)$ criterion.

Let $v \in F$ be a vertex satisfying IN$(v)$. When vertex $v$ is settled and moved to $S$, all outgoing edges $(v, w) \in E$ with $w \in F \cup U$ are relaxed. If $w \in F$, the edge cost $c(w, v)$ is deleted from InF$[w]$. If $w \in U$ and therefore visited for the first time and moved to $F$, all outgoing edges $(w', w)$ with $w' \in F \cup U$ have to be scanned in order to maintain the invariants on the heaps. If $w' \in U$, the cost $(\min_{u \in F \cup U, w' \in U, (w', w) \in E} c(u, w')) + c(w, w')$ is deleted from InU$[w']$. If $w' \in F$, the cost $c(w, w')$ is inserted into the heap InF$[w']$. Since $v$ is moved to $F$ from $U$ only once, there are at most $m$ such heap operations in total. With heap insert and delete operations taking $O(\log n)$ operations, the total number of operations is $O(m \log n)$ as claimed.

The InCrit priority queue has to be updated. This is done by keeping track of all vertices $v$ for which either $d[v]$, InU$[v]$ or InF$[v]$ change in a phase. At the end of the phase, the value in InCrit for these vertices is decreased accordingly. In total, there are at most $m$ such changes over the execution of the algorithm. \(\square\)

For the OUT$(v)$ criterion, the second minimum term in the right hand side of Equation (2) does change throughout the execution of the algorithm. Since the minimum is over edges $(w, w')$ with $w' \in F \cup U$, and $F$ is updated each time a vertex is settled, the minimum can increase. Therefore an OutU$[v]$ heap to decide whether the second minimum term of OUT$(v)$ is satisfied cannot be maintained as for the IN$(v)$ criterion. It does not seem possible to implement the OUT$(v)$ criterion in $O(m \log n)$ operations. However, if we weaken the criterion slightly, the same ideas as in Proposition 1 can be employed.

Define the OUTWEAK$(v)$ criterion to hold if

$$d[v] \leq \min \left\{ \min_{u \in F, w \in F, (u, w) \in E} d[u] + c(u, w), \min_{u \in F, w' \in U, w' \in V, (u, w') \in E} d[u] + c(u, w) + c(w, w') \right\} \quad (3)$$

The difference to the stronger OUT$(v)$ criterion, is that the minimum $(\min_{u \in F, w \in U, w' \in V, (u, w) \in E} d[u] + c(u, w) + c(w, w'))$ is approximated by the possibly smaller $\min_{u \in F, w \in U, (u, w) \in E} d[u] + c(u, w) + \min_{w' \in V, (w', w) \in E} c(w', w')$, such that OUTWEAK$(v) \Rightarrow$ OUT$(v)$

**Proposition 2** The generic SSSP algorithm exploiting the OUTWEAK$(v)$ criterion of Equation (3) can be implemented to run in $O(n \log n + m \log n)$ operations.

**Proof:** As for Dijkstra’s algorithm, we use a priority queue of tentative distances $d[v]$ for $v \in F$ supporting delete-min, insert, and decrease-key operations.

We associate two heaps (priority queues) OutF$[v]$, OutU$[v]$ supporting find-min, delete, insert and build operations with each vertex $v \in F \cup U$. The OutF$[v]$ heap stores the costs $c(v, w)$ for
\( w \in F, (v, w) \in E \), and the OutU[v] heap the costs \( c(v, w) + \min_{w', v', E, (w, w') \in E} c(w', w') \). The right hand side of Equation (3) can be computed as \( d[u] \) plus the minimum of the two heaps, and we keep this for all \( u \) in a third priority queue OutCrit. At the beginning of a phase, all vertices in the priority queue of tentative distances that are smaller than the minimum value in OutCrit will fulfill OUTWEAK(\( v \)). Values are deleted from OutCrit as the corresponding vertices become settled.

When a vertex \( u \) is settled, the cost \( c(u, w) \) is deleted from the heap OutF[w] for all \( w \in F \). If \( w \in U \) meaning that \( w \) moves to \( F \), the incoming edges \((w', w)\) with \( w' \in F \cup U \) are scanned. If \( w' \in F \), the cost \( c(w', w) \) inserted into OutF[w']. If \( w' \in U \), the cost \( c(w', w) + \min_{v, (w, v) \in E} c(w, v) \) is deleted from OutU[w']. Since a vertex moves to \( F \) once at most, the total number of heap operations is at most \( m \). At the end of the phase, the values in OutCrit are increased for all \( v \) for which the minimum values in OutF[v] or OutU[v] have changed. When a \( d[u] \) value change in some relaxation step, the value in OutCrit is decreased. The OutCrit priority queue must therefore support both decrease-key and increase-key operations efficiently. □

The heaps associated with the vertices can all be eliminated, and the implementation considerably simplified by first presorting the edges in increasing cost order.

**Proposition 3** With a presorting of the incoming and outgoing edges of all vertices in order of increasing cost taking \( O(n + m \log n) \) operations, the generic SSSP algorithm exploiting the IN(\( v \)) and OUTWEAK(\( v \)) criteria can be implemented in \( O(n \log n + m) \) operations.

**Proof:** Four presorting steps are needed, and edge lists need to be maintained as doubly linked lists to support easy deletion of incoming and outgoing edges. For each vertex \( u \in V \), a list of incoming edges in increasing cost order, and a list of outgoing edges in increasing cost order is constructed. For each \( v \), the minima \( M[v] = \min_{(v, w) \in E} c(v, w) \) and \( M'[v] = \min_{(u, v) \in E} c(u, v) \) are precomputed and sorted lists of \( c(v, w) + M[w] \) and \( M'[u] + c(u, v) \) are constructed. Priority queues InCrit and OutCrit are used in addition to a priority queue of tentative distances, and maintained as in the algorithms of Proposition 1 and Proposition 2. With these, the vertices for which the IN(\( v \)) and OUTWEAK(\( v \)) criteria hold can be selected in \( O(\log n) \) time each.

The minima in the right hand sides of the criteria can be found in constant time per vertex, simply by looking at the first element in the corresponding sorted adjacency list. When a vertex \( v \) is settled, the edges \((v, u)\) are removed from the lists of outgoing edges of all \( v \in F \). Also, when the edges \((u, v)\) are relaxed, these edges are removed from the lists of incoming edges for all \( v \). When in the relaxation some vertex \( v \in U \) is seen for the first time and moved to \( F \), the remaining outgoing edges \((v, w)\) are scanned, and the cost \( c(w, v) + M[w] \) removed from the list of these costs of \( w \). Also the cost \( M'[w] + c(w, v) \) is removed from the list of these costs of \( v \).

When the removal of an edge from one of the edge lists causes a minimum to change, the corresponding vertex is marked, and at the end of the phase the values in the InCrit and OutCrit priority queues are adjusted accordingly. □

Weaker, but possibly easier to compute criteria can be derived from these two criteria by taking minima over larger sets and/or over smaller values. All in all, this will lead to smaller minima in the criteria, and thus weaker, because more restrictive, criteria. Such considerations give rise to the following derived criteria.

The weakest criteria we consider use minima over all edges. We call these criteria static because
the minima can be computed in advance and are not changed during the execution of the generic algorithm. The static criteria are the criteria originally introduced by Crauser et al. [5].

Define \( \text{INSTATIC}(v) \) to hold if
\[
d[v] - \min_{w \in V, (w, v) \in E} c(w, v) \leq \min_{u \in F} d[u]
\]

Define \( \text{OUTSTATIC}(v) \) to hold if
\[
d[v] \leq \min_{u \in F, w \in V, (u, w) \in E} d[u] + c(u, w)
\]

Since the minima in both cases do not change during the execution of the algorithm, and can therefore be precomputed in \( O(m) \) operations, the generic, phased SSSP algorithm can be implemented sequentially in \( O(n \log n + m) \) operations with the \( \text{INSTATIC}(v) \) and \( \text{OUTSTATIC}(v) \) criteria.

It might be expected that the minimum terms concerning the unexplored vertices \( U \) in both \( \text{IN}(v) \) and \( \text{OUT}(v) \) criteria will not bring much, since graphs with any expansion properties will quickly lead to \( U = \emptyset \) in the algorithm. For this reason, criteria without these terms are considered as given in Equation (6) and Equation (7). The fact that any predecessor, respectively successor, in \( U \) actually enforces at least two vertices not in \( S \) to be present in a shortest path to \( v \) is not exploited in these criteria. The \( U \) case is simply subsumed under the \( F \) case which considers only a single edge on a shortest path and therefore leads to a potentially smaller minimum. The main advantage of these criteria is again a potentially more efficient implementation as was the case with the \( \text{OUTWEAK}(v) \) criterion. We call these weakenings the \( \text{simple, dynamic} \) criteria. The criteria are called dynamic, since the minima do change over the course of the execution, and thus gradually strengthen of the criteria.

Define \( \text{INSIMPLE}(v) \) to hold if
\[
d[v] - \min_{w \in F \cup U, (w, v) \in E} c(w, v) \leq \min_{u \in F} d[u]
\]

Define \( \text{OUTSIMPLE}(v) \) to hold if
\[
d[v] \leq \min_{u \in F \cup U, w \in V, (u, w) \in E} d[u] + c(u, w)
\]

We have argued that \( \text{DIJK}(v) \Rightarrow \text{INSTATIC}(v) \Rightarrow \text{INSIMPLE}(v) \Rightarrow \text{IN}(v) \), and likewise \( \text{OUTSTATIC}(v) \Rightarrow \text{OUTSIMPLE}(v) \Rightarrow \text{OUTWEAK}(v) \Rightarrow \text{OUT}(v) \).

Finally, we discuss yet another way of instantiating the generic SSSP algorithm which may be attractive to implement. The idea here is to approximate the criteria by recomputing the minima at certain intervals, under certain conditions. We use the same priority queues and basic idea as in the implementation in Proposition 3 with doubly linked lists, but do not presort any edge lists. Edges are eliminated as vertices move to \( S \) and to \( F \), but now exact minima cannot be looked up in constant time. Instead, with each vertex, four approximate minima shall be maintained, namely \( \min_{w \in F} c(w, v) \) and \( \min_{w \in U}(c(w, v) + \min_{w' \in V} c(w', w)) \) for the \( \text{IN}(v) \) criterion, and \( \min_{v \in F} c(v, w) \) and \( \min_{w \in U}(c(v, w) + \min_{w' \in V} c(w, w')) \) for the \( \text{OUTWEAK}(v) \) criterion.

The minima are maintained conservatively, and might be too small, but this does not invalidate the decisions made by the criteria. During the execution of the algorithm, when either of these minima might change, due to a vertex moving to either \( S \) for \( F \), the corresponding minimum
is recomputed. A potential change to a minimum can be detected by also keeping track of the incoming or outgoing edge corresponding to the minimum value. Recomputation, on the other hand, cannot be afforded at every change (of which there are $O(m)$), so a parameter $k$ is chosen, and minima recalculated at most at every $k$th change.

The resulting, approximate criteria are not comparable to the stricter ones. Since they may not be complete, they need to be combined with DIJK($v$). A particular instance of the approximation is when minima are computed only once as in [8]. This choice leads to criteria at least as strong as INSTATIC($v$) and OUTSTATIC($v$).

4 Simulations

In this section we explore the reduction in the number of phases that can be achieved by applying and combining the criteria introduced in the previous section. We investigate the following combinations of criteria:

- The strongest criteria IN($v$), OUT($v$), and the disjunction IN($v$)$\lor$OUT($v$). These are referred to as the full criteria in the plots.
- The simple, dynamic criteria INSIMPLE($v$), OUTSIMPLE($v$), and the disjunction INSIMPLE($v$)$\lor$OUTSIMPLE($v$)
- The original, static criteria by Crauser et al., INSTATIC($v$), OUTSTATIC($v$), and the disjunction INSTATIC($v$)$\lor$OUTSTATIC($v$)
- The oracle criterion ORACLE($v$)

We measure the number of phases in a generic SSSP algorithm where all vertices fulfilling the criteria prior to a phase are selected and settled. We also estimate the amount of work needed to find vertices in the fringe set $F$ by summing the sizes $|F|$ over all phases. This provides information on the data structure support required for a real, efficient implementation of the generic algorithm.

The first set of simulations was performed on uniformly random graphs which is the same family of graphs used in the paper by Crauser et al.. A uniformly random graph $G(n,p)$ consists of $n$ vertices where the (independent) probability for each edge is $p$. Since there are $n(n-1)$ possible edges the number of edges in the graph is distributed as binom($n(n-1),p$), which has an expected value of $n(n-1)p$, or approximately $n^2p$. The edge weights are uniformly distributed in the range $[0;1]$.

The second set of simulations was performed on Kronecker graphs [12]. Kronecker graphs are generated by repeatedly multiplying a small, so-called (square) initiator matrix of positive real numbers with itself by utilizing the Kronecker product. The result represents the probability for each single possible edge to appear in the sampled graph. The edges are unweighted. For the purposes of these simulations the initiator matrix is $2.5 \begin{pmatrix} 0.57 & 0.19 \\ 0.19 & 0.05 \end{pmatrix}$ as also used in the

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2 The source code of the simulation tool can be downloaded at [https://github.com/kaini/sssp-simulation](https://github.com/kaini/sssp-simulation).

3 The Kronecker product $\otimes$ is defined as $A \otimes B = \begin{pmatrix} a_{1,1}B & \ldots & a_{1,n_A}B \\ \vdots & \ddots & \vdots \\ a_{m_A,1}B & \ldots & a_{m_A,n_A}B \end{pmatrix}$. 

Graph 500 benchmark \([9]\). The multiplication with 2.5 is to control the number of edges of the final graph. In our implementation for sampling Kronecker graphs, the expected number of edges in the resulting graph is \((\sum\text{initiator matrix})^k\) with \(k\) being the Kronecker exponent. If we would use the initiator matrix as it is, this would be 1, i.e., on average a single edge would be generated, no matter the value of \(k\). Of course, the multiplication does not change the structure of the graph in any way, it is just a way to control the number of generated edges. Leskovec et al. claim that Kronecker graphs have many properties that are present in real-world networks, like social networks or citation networks. Furthermore, they provide a fast algorithm to generate such graphs.

Figure 3 shows the mean number of phases required to settle all reachable vertices of a sample of 100 graphs for the given class. The uniform graphs \(G(n,p)\) are generated in such a way that \(m/n = 10\), i.e., care is taken that the expected out-degree of vertices of these graphs stays constant when increasing the number of vertices. The used classes are therefore \(G(100, 0.1010)\), \(G(121, 0.0833)\), \(G(147, 0.0685)\), and so on. The Kronecker graphs were generated by sampling edges from \(\left(\begin{array}{cc} 2.5 & 0.57 \\ 0.19 & 0.05 \end{array}\right)\) ranging from \(k = 7\) to \(k = 16\), i.e., vertex counts from 128 \((2^7)\) to 65536 \((2^{16})\). Edge weights are uniformly distributed in \([0;1]\).

For uniform graphs, INSTATIC\((v)\) ∨ OUTSTATIC\((v)\), i.e., the weakest disjunctive criterion we discuss, already beats all non-disjunctive criteria. This means that it is possible to achieve a reasonably small number of phases with a quite simple criterion and low implementation complexity. INSIMPLE\((v)\) ∨ OUTSIMPLE\((v)\) and to a lesser extend IN\((v)\) ∨ OUT\((v)\) further improves on the number of phases by a factor of about 1.8, respectively 1.2. Nevertheless, ORACLE\((v)\) is still unreached by all criteria. Compared to IN\((v)\) ∨ OUT\((v)\) the oracle only needs a third of the number of phases. For Kronecker graphs the results are similar, except that INSIMPLE\((v)\) and IN\((v)\) are stronger than INSTATIC\((v)\) ∨ OUTSTATIC\((v)\). Again, the ORACLE\((v)\) criterion is unreached.

We performed curve-fitting to obtain numerical estimations of the number of phases for the various criteria. These can be seen in Table 1. Each result was fitted using the functions \(a + b \cdot \log_2(n)\) and \(a + b \cdot n^c\) with the parameters \(a\), \(b\), and \(c\). The most appropriate fit was chosen for the data in Table 1. For simplicity the parameter \(a\) was dropped from the results. A notable result is that for uniform graphs INSTATIC\((v)\), OUTSTATIC\((v)\), INSIMPLE\((v)\), OUTSIMPLE\((v)\), IN\((v)\), and OUT\((v)\) have almost the same exponent. The various criteria only change the multiplicative factor. Only the disjunctive criteria reduce the exponent from about 1/2 down to 1/3 and 1/4. ORACLE\((v)\) only needs a logarithmic number of phases. In other words there is still a considerable gap between our criteria and the oracle. For Kronecker graphs it seems that the family of IN-criteria is stronger than the corresponding family of OUT-criteria, additionally the simple criteria have a higher influence on the exponent than is the case for uniform graphs. Nevertheless, the structure of the results is very similar to uniform graphs.

Figure 4 shows the mean of the sum of the sizes of \(F\) over all phases. This number can be used to estimate the amount of work needed to settle all reachable vertices. As one can see stronger criteria lead to a smaller \(\sum |F|\), i.e., the reduction of number of phases is so powerful that the sum becomes smaller. Again, we used curve-fitting to to obtain the results seen in Table 2. We fitted the function \(a + b \cdot n^c\) with the parameters \(a\), \(b\), and \(c\). Additionally, we tried to fit \(a + b \cdot n \log_2(n)\), which unfortunately did not fit the empirical data well. The sum ranges for the criteria from \(n^{1/2}\) to \(n^{3/4}\) while for the oracle the sum is almost linear with respect to the number of vertices \(n\).

Additionally, we simulated four graphs from the SNAP dataset \([13]\).
Figure 3: The number of phases required by the criteria on uniform graphs with an expected out-degree of 10 for each vertex and Kronecker graphs with the initiator matrix 2.5 (0.57 0.19; 0.19 0.05).
Table 1: The number of phases required by various criteria to settle all vertices in uniform and Kronecker graphs. The numbers were obtained by curve-fitting.

| Criterion                  | Uniform Graphs | Kronecker Graphs |
|----------------------------|----------------|------------------|
| \( \text{OUTSTATIC}(v) \) | \( 2.48 \cdot n^{0.5} \) | \( 1.79 \cdot n^{0.51} \) |
| \( \text{INSTATIC}(v) \)   | \( 2.28 \cdot n^{0.5} \) | \( 2.17 \cdot n^{0.43} \) |
| \( \text{OUTSTATIC}(v) \lor \text{INSTATIC}(v) \) | \( 3.97 \cdot n^{0.34} \) | \( 3.49 \cdot n^{0.31} \) |
| \( \text{OUTSIMPLE}(v) \)  | \( 1.66 \cdot n^{0.5} \) | \( 1.68 \cdot n^{0.42} \) |
| \( \text{INSIMPLE}(v) \)   | \( 1.43 \cdot n^{0.46} \) | \( 3.01 \cdot n^{0.32} \) |
| \( \text{OUTSIMPLE}(v) \lor \text{INSIMPLE}(v) \) | \( 3.75 \cdot n^{0.29} \) | \( 4.03 \cdot n^{0.24} \) |
| \( \text{OUT}(v) \)        | \( 1.62 \cdot n^{0.48} \) | \( 1.54 \cdot n^{0.43} \) |
| \( \text{IN}(v) \)         | \( 1.47 \cdot n^{0.43} \) | \( 2.83 \cdot n^{0.3} \) |
| \( \text{OUT}(v) \lor \text{IN}(v) \) | \( 4.60 \cdot n^{0.26} \) | \( 3.65 \cdot n^{0.24} \) |
| \( \text{ORACLE}(v) \)     | \( 1.69 \cdot \log_2(n) \) | \( 1.17 \cdot \log_2(n) \) |

Table 2: Sum of the sizes of \( F \) over all phases for each the criteria on uniform graphs with an expected out-degree of 10 for each vertex and Kronecker graphs with the initiator matrix \( (2.5 \ 0.57 \ 0.19; \ 0.19 \ 0.05) \).

| Criterion                  | Uniform Graphs | Kronecker Graphs |
|----------------------------|----------------|------------------|
| \( \text{OUTSTATIC}(v) \) | \( 0.79 \cdot n^{1.50} \) | \( 0.48 \cdot n^{1.53} \) |
| \( \text{INSTATIC}(v) \)   | \( 0.73 \cdot n^{1.50} \) | \( 0.5 \cdot n^{1.42} \) |
| \( \text{OUTSIMPLE}(v) \lor \text{INSTATIC}(v) \) | \( 1.17 \cdot n^{1.33} \) | \( 0.7 \cdot n^{1.31} \) |
| \( \text{OUTSIMPLE}(v) \)  | \( 0.64 \cdot n^{1.50} \) | \( 0.52 \cdot n^{1.45} \) |
| \( \text{INSIMPLE}(v) \)   | \( 0.39 \cdot n^{1.49} \) | \( 0.46 \cdot n^{1.33} \) |
| \( \text{OUTSIMPLE}(v) \lor \text{INSIMPLE}(v) \) | \( 0.95 \cdot n^{1.31} \) | \( 0.63 \cdot n^{1.26} \) |
| \( \text{OUT}(v) \)        | \( 0.58 \cdot n^{1.49} \) | \( 0.49 \cdot n^{1.36} \) |
| \( \text{IN}(v) \)         | \( 0.35 \cdot n^{1.49} \) | \( 0.41 \cdot n^{1.34} \) |
| \( \text{OUT}(v) \lor \text{IN}(v) \) | \( 0.96 \cdot n^{1.29} \) | \( 0.55 \cdot n^{1.27} \) |
| \( \text{ORACLE}(v) \)     | \( 2.49 \cdot n^{1.05} \) | \( 1.16 \cdot n^{1.08} \) |
Figure 4: Sum of the sizes $|F|$ over all phases for each the criteria on uniform graphs with an expected out-degree of 10 for each vertex and Kronecker graphs with the initiator matrix $2.5 \times (0.57 \ 0.19; 0.19 \ 0.05)$. 
Table 3: Number of phases required for four simulated SNAP graphs: Web Graph Berk Stan, Web Graph Notre Dame, Road Network Texas, and Road Network Pennsylvania.

| Criterion                | Berk Stan | Notre Dame | TX         | PA         |
|--------------------------|-----------|------------|------------|------------|
| OUTSTATIC(v)             | 6165      | 2350       | 32948      | 25938      |
| INSTATIC(v)              | 5029      | 2224       | 32904      | 26027      |
| OUTSTATIC(v) ∨ INSTATIC(v) | 2289   | 875        | 28938      | 22811      |
| OUTSIMPLE(v)             | 3114      | 1643       | 20046      | 15784      |
| INSIMPLE(v)              | 3762      | 1358       | 31710      | 25062      |
| OUTSIMPLE(v) ∨ INSIMPLE(v) | 1622   | 692        | 18930      | 14903      |
| OUT(v)                   | 2454      | 1183       | 16261      | 12798      |
| IN(v)                    | 2341      | 1040       | 27962      | 22118      |
| OUT(v) ∨ IN(v)           | 1365      | 601        | 16092      | 12740      |
| ORACLE(v)                | 582       | 53         | 898        | 716        |

million edges.

**Web Graph “Notre Dame”** A directed web-graph consisting of 325 thousand vertices and 1.5 million edges.

**Road Network Texas** An undirected graph representing the road network of Texas, with 1.3 million vertices and 1.9 million edges.

**Road Network Pennsylvania** An undirected graph representing the road network of Pennsylvania, with 1 million vertices and 1.5 million edges.

Since our implementation is only able to work with directed graphs, we preprocessed the two road networks in such a way that for each edge \((a, b)\) an additional edge \((b, a)\) was inserted into the input file. This means that the number of edges for the two road networks has been doubled. Additionally, since the input graphs are unweighted, we added a uniformly random edge weight between 0 and 1 for each edge. Using unweighted graphs would trivialize the SSSP.

Figure 5 and Table 3 show the results for the two web graphs. One can see that neither \(IN(v)\) nor \(OUT(v)\) alone are able to realize the full potential of our approach in reducing the number of phases. Only the combination \(IN(v) \lor OUT(v)\) manages to do so. Different than for the road networks, for these two graphs the difference between the static, simple and full variations of our criteria is not as pronounced, especially in the case of \(IN(v) \lor OUT(v)\). ORACLE(v) still performed an order of magnitude better than the strongest of our criteria.

Figure 6 and Table 3 show the results for the two road networks. One can see that for these two road networks \(IN(v)\) is a quite weak criterion compared to \(OUT(v)\). The combination of these two is not much stronger than \(OUT(v)\) alone. \(OUT(v)\) and \(OUTSIMPLE(v)\) manages to increase the potential parallelism compared to \(OUTSTATIC(v)\), nevertheless the settled vertices per phase are still low with about 100. The theoretical optimum is much better than all our criteria. It manages to settle about 3000 to 4000 thousand nodes per phase, which implies that in these graphs there is still a lot of untapped potential for much stronger criteria.

One can also see that the settling patterns are vastly different between the two types of graphs: The two road networks have a steadily growing number of settled nodes, which after reach-
Figure 5: Number of vertices settled per phase for the two web graphs. The number of phases required can be seen by looking at the end of each line. The lines are smoothed in order to be able to display the data without heavy overplotting. The squiggly line at the end of ORACLE(\(v\)) is an artifact of this. ORACLE(\(v\)) reaches up to 13000 for Berk Stan and up to 30000 for Notre Dame and had to be cut off in order to keep the graph legible.
Figure 6: Number of vertices settled per phase for the two road networks. The number of phases required can be seen by looking at the end of each line. The lines are smoothed in order to be able to display the data without heavy overplotting. ORACLE(v) reaches up to 3000 for PA and up to 4000 for TX and had to be cut off in order to keep the graph legible.
ing its peak steadily declines, i.e., there is potential for parallelism in almost all phases. The two web graphs have a sharp increase in the first few phases followed by a long tail of phases where only very few vertices are settled. In this tail there is hardly any potential for parallelism. This can especially be seen in the Berk Stan graph.

5 Implementations

We now describe our parallel, shared-memory implementations of the SSSP algorithm running in phases and utilizing the \textsc{INSTATIC}(v) and \textsc{OUTSTATIC}(v) criteria, and our implementation of the \(\Delta\)-stepping algorithm. The implementations use a standard, adjacency-array representation of the input graphs. A graph is stored as an array of vertices and an array of edges. The array of edges is grouped by the source vertex of each edge, such that all outgoing edges of a vertex are stored consecutively in memory. Each vertex is identified by its index in the vertex array and consists of a pointer to the group of its outgoing edges and the number of outgoing edges (outdegree). For the implementations here, the incoming edges of each vertex are not strictly needed and storing them would roughly double the space requirements. Incoming edges are therefore not stored.

The static criteria have been implemented as proposed by Crauser et al.. The tentative distances \(d[v]\) are kept in a priority queue as in Dijkstra’s algorithm. For the \textsc{INSTATIC}(v) criterion an additional priority queue is maintained that stores \(d[v] - \min_{w \in V, (w,v) \in E} c(w,v)\) for each vertex, while for the \textsc{OUTSTATIC}(v) criterion an additional priority queue is used that stores \(d[v] + \min_{w \in V, (w,v) \in E} c(v,w)\) for each vertex. The initial minima are computed for each vertex at the beginning of the implementation. This takes \(O(m)\) time and is included in the time measurements of the next section to provide a fair comparison. These priority queues are then used to quickly identify the vertices for which Equation (6) (as long as \(d[v] - \min_{w \in V, (w,v) \in E} c(w,v) \leq d[u]\) is true, the equation holds for \(v\)), respectively Equation (7) (as long as \(d[u] \leq d[v] + \min_{w \in V, (w,v) \in E} c(v,w)\) is true, the equation holds for \(u\)), hold. For the combination of the two criteria it suffices to check for both conditions disjunctively.

Once a vertex has been identified by either criteria, it is removed from all priority queues and the check is repeated. When no criteria identifies a vertex, all vertices that will be settled in the phase have been collected. A sequential implementation would now just iterate over the set of identified vertices and settle each of them as done in Dijkstra’s algorithm. Such an implementation would not have any advantage over Dijkstra’s algorithm; it would probably be slower because of the multiple priority queues that have to be maintained instead of the single queue in Dijkstra’s algorithm.

The parallel implementation is written in the C++ programming language and uses native C++ threads. Additionally, we needed to implement three primitives not provided by the C++ standard library: a reduction operation, an atomic-min operation, and a barrier.

The reduction, given a starting value \(s\), each processor’s contribution \(c\) and an operation \(\oplus\), is implemented by utilizing a shared atomic variable \(v\). Each processor reads the shared variable, calculates \(c \oplus v\) and tries to store the result back into \(v\) utilizing a compare-and-exchange operation. If this fails, the whole process is retried. While this is a naïve implementation, the time required by the reduction operations is completely irrelevant, and therefore does not warrant implementing a more complex algorithm.

\footnote{The source code can be downloaded at \url{https://github.com/kaini/sssp-shm}.}
The atomic-min operation is implemented similarly, with the small optimization that the compare-and-exchange does not have to be retried if the own contribution is already greater or equal to the read value.

The barrier implementation is heavily inspired by Boost’s barrier [2], but was reimplemented without using locks (mutex’es) and condition-variables. A barrier consists of an atomic integer \( w \) (waiting) and an atomic boolean \( g \) (generation). Once a processor enters the barrier it performs an atomic fetch-and-increment operation on \( w \) and fetches \( g \). If after this operation \( w \neq p \), the processor busy-waits until the value of \( g \) is flipped. If after this operation \( w = p \), the processor flips \( g \) and executes an atomic subtraction \( w := w - p \).

As a datastructure for all priority queues we utilized Boost’s Fibonacci heap implementation [2] paired with a custom allocator that avoids the negative performance impact of repeatedly calling the system allocator on such an allocation-heavy datastructure. We also tried to use Pairing heaps, but they turned out to be slower than using Fibonacci heaps.

The implementation can be split into three separate phases:

1. Preprocessing: Calculate the minimum outgoing/incoming edge for each vertex.
2. Per-phase identification: Identify the vertices to be settled in this phase according to the criteria.
3. Per-phase settling: Settle the identified vertices, relax tentative distances and update the priority queues.

In our parallel implementation, each of these phases are parallelized. We assume we have a set of \( p \) processors (cores), each running a thread. Our implementation statically partitions the set of vertices over the processors such that each processor (thread) is responsible for a statically assigned subset of vertices. Each processor performs all operations related to these vertices, and no processor performs update operations on vertices assigned to another processor. In our current implementation the assignment is not randomized, that is processor \( i \) is assigned vertices \( v \) for which \( v/p = i \).

Preprocessing: Calculating the cheapest outgoing edge for each vertex is trivial: Each processor just iterates over the outgoing edges of vertices it is responsible for. Calculating the cheapest incoming edge for each vertex is a bit more involved since processors do not know the incoming edges of their vertices. We use a global array of \( n \) atomic doubles, initialized to \( \infty \). Each processing unit iterates over the outgoing edges of the vertices it is responsible for and uses an atomic-min operation to update the cell corresponding to the target vertex of the edge with the cost of the edge. When all processors have finished doing so, each can read the cost of the cheapest incoming edge from this array for the vertices it is responsible for.

Identification: To calculate the set of vertices identified by \( \text{INSTATIC}(v) \) we use a priority queue ordered by \( d[v] - \min_{w\in V, (w,v)\in E} c(w,v) \). Since each processor is only aware of the vertices it is responsible for, each manages an independent priority queue containing only vertices it is responsible for. To find the minimum required to decide \( \text{INSTATIC}(v) \), each processor first finds the minimum from its own priority queue. Second, a reducing operation is executed across all processors using these minima to obtain a global minimum tentative distance. Once each processor
knows the global minimum, they can now independently identify the vertices that can be settled among these they are responsible for. The vertices satisfying \( \text{OUTSTATIC}(v) \) are found for each processor in the same way.

**Settling:** Once the vertices to be settled in the current phase are identified, the relaxation is executed for each outgoing edge of these vertices, with each processor relaxing vertices it is responsible for. For each relaxation it is decided whether it is local, meaning that the target vertex is belonging to the same processor, or whether it is remote, meaning that the target vertex belongs to some other processor. Local relaxations are executed immediately. For global relaxations, the target vertex and new tentative distance is buffered in an array owned by the destination processor. Once all processors are done iterating over all outgoing edges of the vertices to be settled, they iterate over all buffered relaxations they received in their array and execute them. Settling is complete when all processors have finished this step. In order to keep the update work small, remote relaxations are buffered only if the can potentially improve the tentative distance of the target vertex. To this end an approximate set of tentative distances is stored in an array and updated with an atomic store by processors whose updates improve the previously stored value. It does not matter that this array is not accurate, therefore expensive atomic operations can be avoided here.

The buffers for incoming relaxations are implemented by using an array of chunks and a counter variable. Each chunk contains \( 1024^2 \) (about one million) items. The array of chunks is large enough that the theoretically maximum needed number of chunks can be allocated. Initially, the array of chunks contains only null-pointers and the counter is 0. If a processor wants to place something into a buffer it first executes an atomic fetch-and-increment operation on the counter variable to obtain the index the item may be placed in. The first chunk is responsible for indices 0 to \( 1024^2 - 1 \), the second for \( 1024^2 \) to \( 2 \cdot 1024^2 - 1 \), and so on. Therefore, once a processor obtained the index it knows the chunk and the index in the chunk. If the target chunk is still null, the processor allocates memory for the chunk and places the pointer into the array of chunks using an atomic compare-and-exchange operation. If another processor allocated the memory first, the memory allocated by the other processor is used. Once the chunk is allocated, it suffices for all processors to just place their item at the index obtained by the fetch-and-increment operation, i.e., almost always an insertion into such a buffer consists of a single atomic fetch-and-increment operation and a simple (non-atomic) write into an array.

There are \( p \) such buffers, one for each processor where all other processors place the incoming relaxations. Therefore, iterating over all incoming relaxations is very simple. Each processor just has to iterate over its own buffer.

Our implementation of \( \Delta \)-stepping \[14\] follows the same principles: Each processor maintains its own set of buckets, i.e., each a bucket for vertices whose tentative distance is between 0 and \( \Delta \), between \( \Delta \) and \( 2\Delta \), and so on. To find the bucket that has to be emptied in any given iteration each processor proposes its first non-empty local bucket. By reduction operation over the proposed buckets, the globally first non-empty bucket is identified. Each processor then empties this bucket and performs all relaxations for light edges (edges whose cost is less than \( \Delta \)). If the bucket is non-empty after this step it is repeated. Finally, once all processors have completed all light edge relaxations, in potentially several repetitions of the previous step, the remaining heavy edges (edges whose cost is larger than \( \Delta \)) are relaxed. Similarly to our implementation of Crauser’s algorithm, local relaxations are executed immediately, while remote relaxations are buffered in an array for the destination processor. This means that all processors have to wait for each other after each phase,
and after each iteration concerning the light edges. Once all buckets are empty, the algorithm is finished. This can be easily detected by the reduction operation that determines the globally first non-empty bucket.

6 Experimental results

The first set of benchmarks was performed on a shared memory system “mars” with eight Intel Xeon E7-8850 processors. Each processor has 10 cores and is capable of running 20 threads in parallel. Each core has a base frequency of 2 GHz. The system has about 1 TiB of main memory, but since our benchmarks do not need a lot of memory, this does not matter.

All measurements were repeated at least 10 times, and the median of these run-times was taken as basis for the following results. The run-time of a single repetition was obtained by using the maximum thread run-time. In the case of uniformly random and Kronecker graphs, each repetition used a different seed, i.e., a different graph instance. Nevertheless, care was taken that all criteria got the same set of different seeds to ensure that the comparison stays absolutely fair.

Each benchmark was run in two configurations: First, the criteria were implemented using Fibonacci-heaps as described in the previous section. Second, each criterion was implemented using a single plain array that is scanned linearly instead of utilizing priority queue data-structures. On one hand, this means that instead of find-min operations, a simple linear scan was performed to find the minimum. On the other hand, this also means that the performance overhead of maintaining the priority queues goes away.

Each criteria is compared with an efficient sequential implementation of Dijkstra’s algorithm, i.e., the graphs show the absolute speedup. Our implementation of Dijkstra’s algorithm is made efficient by the fact that we utilize Fibonacci-heaps with a hand-written custom allocator that avoids the performance overhead of a heavily allocating data-structure such as Fibonacci-heaps. Our implementation of Dijkstra’s algorithm is included in the source-code repository linked in the previous section.

The first benchmark was performed on uniformly random graphs $G(1000000, 0.0001)$ with uniformly random edge weights between 0 and 1. Therefore, each graph instance has exactly one million vertices and about 100 outgoing edges per vertex, i.e., about 100 million edges in total. Figure 7 shows that the static criteria as defined by Crauser et al. are indeed highly competitive compared to $\Delta$-stepping. We achieve an absolute speedup of up to 15 when utilizing the combination of $\text{INSTATIC}(v)$ and $\text{OUTSTATIC}(v)$ for identifying correct vertices. Unfortunately, the algorithm seems to stop scaling for more than 40 threads on this system.

For Kronecker graphs (Figure 8) with uniformly random edge weights between 0 and 1 the performance is much worse, as we only reach an absolute speedup just shy of 4.5 with 80 threads. This seems to imply that the structure of Kronecker graphs is indeed vastly different than the structure of uniformly random graphs, in such a way that it is much harder to achieve good speedups with our algorithm.

According to the previous results using $\text{INSTATIC}(v) \lor \text{OUTSTATIC}(v)$ without priority queues but with plain arrays instead is the fastest implementation. Using this implementation, we performed benchmarks utilizing the four graphs from the SNAP dataset introduced in the simulation, i.e., the two web-graphs “Berk Stan” and “Notre Dame” and the two (preprocessed) road networks for Texas and Pennsylvania, all with uniformly random edge weights between 0 and 1. The results can be seen in Figure 9. Unfortunately, these instances do not scale very well. We believe that this
Figure 7: Absolute speedup of the INSTATIC(ν) and OUTSTATIC(ν) criteria, and Δ-Stepping compared with an efficient implementation of Dijkstra’s algorithm. The input graph is $G(1000000, 0.0001)$ with uniformly random edge weights in $[0; 1]$. The system used is mars.
Figure 8: Absolute speedup of the INSTATIC(v) and OUTSTATIC(v) criteria, and Δ-Stepping compared with an efficient implementation of Dijkstra’s algorithm. The input graph is a Kronecker graph based on \((2.5 \times (0.57 \times 0.19; 0.19 \times 0.05))^2\) with uniformly random edge weights in \([0; 1]\). The system used is mars.
is due to the small size of the input graphs (the edge counts range from 1.5 to 7.5 million).

Additionally, we ran the first benchmark, i.e., random graphs $G(1000000, 0.0001)$ with uniformly random edge weights in $[0; 1]$ on a different system “nebula.” This system consists of two AMD EPYC 7551 CPUs, with each a base clock speed of 2 GHz and 32 cores/64 threads. The system has 256 GiB of main memory. The results can be seen in Figure 10. The results are similar to those of the system mars, with the notable exception that on nebula the algorithms do not see a sharp decline in absolute speedup for utilizing a high amount of threads.

We were not able to find an efficient implementation of the stronger criteria discussed in Section 3. A simple implementation of $\text{INSIMPLE}(v) \lor \text{OUTSIMPLE}(v)$ led to speedups worse than these of $\Delta$-stepping and is not usable in practice. Nevertheless, it is included in the source-code package linked in the previous section.

7 Concluding remarks

This paper discussed parallelization of Dijkstra’s algorithm based on criteria for detecting more than a single, correct candidate vertex for relaxation at a time. We strengthened criteria previously introduced by Crauser et al. [5], and discussed various ideas that can be used for practical implementation of these. Simulation results show that for random and Kronecker graphs, as often used in such studies, the (stronger) criteria can indeed reduce the number of phases and therefore the parallel depth significantly to a small root of the number of vertices in the input graph. Stronger criteria indeed lead to stronger reduction in the number of phases. We implemented a generic, parallel Dijkstra algorithm running in phases, exploiting the two static criteria also proposed by Crauser et al., and showed that for random graphs this implementation can indeed be more than competitive with the $\Delta$-stepping approach which is often claimed to be the fastest and most efficient practical parallel SSSP implementation.

The implementations and discussions provided here leave much room for further (practical) improvements, e.g., on the need for complex data structures (priority queues), tradeoffs in the implementations between criteria accuracy and overhead, etc. Based on the encouraging speed-up results also in comparison to $\Delta$-stepping, we believe that this is worthwhile.
Figure 9: Absolute speedup of the $\text{INSTATIC}(v) \lor \text{OUTSTATIC}(v)$ criterion using arrays, compared with an efficient implementation of Dijkstra’s algorithm. The input graphs are from the SNAP dataset [13] with uniformly random edge weights between 0 and 1. The system used is mars.
Figure 10: Absolute speedup of the INSTATIC(v) and OUTSTATIC(v) criteria, and Δ-Stepping compared with an efficient implementation of Dijkstra’s algorithm. The input graph is $G(1000000, 0.0001)$ with uniformly random edge weights in [0; 1]. The system used is nebul.
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