Solitons in the Tonks–Girardeau gas with dipolar interactions

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Abstract
The existence of bright solitons in the model of the Tonks–Girardeau (TG) gas with dipole–dipole (DD) interactions is reported. The governing equation is taken as the quintic nonlinear Schrödinger equation (NLSE) with the nonlocal cubic term accounting for the DD attraction. In different regions of the parameter space (the dipole moment and atom number), matter-wave solitons feature flat-top or compacton-like shapes. For the flat-top states, the NLSE with the local cubic-quintic (CQ) nonlinearity is shown to be a good approximation. Specific dynamical effects are studied assuming that the strength of the DD interactions is ramped up or drops to zero. Generation of dark-soliton pairs in the gas shrinking under the action of the intensifying DD attraction is observed. Dark solitons exhibit particle-like collision behaviour. Peculiarities of dipole solitons in the TG gas are highlighted by comparison with the NLSE including the local CQ terms. Collisions between the bright solitons are also studied. In many cases, the collisions result in the merger of the solitons into a breather, due to a strong attraction between them.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The experimental realization of degenerate Bose gases in tight effectively one-dimensional (1D) traps \cite{1} in the Tonks–Girardeau (TG) regime \cite{2} has rekindled interest in the TG model, which, being well known for a long time, was assumed to have a theoretical value only (for a review see \cite{3}). In the TG state, bosons emulate the Pauli exclusion principle, as a result of the hard-core repulsion, rather than as a manifestation of the quantum statistics. One of the recent trends in this field is the application of mean-field-like approaches to the description of macroscopic dynamics of TG gases \cite{4,5}. The starting point of this approach is the use of a formal analogy between hydrodynamic equations for degenerate Fermi gases and Bose gases in the TG phase. This analogy has led to the derivation of the nonlinear Schrödinger equation (NLSE) with the local quintic repulsive nonlinearity for the ‘wavefunction’ of the TG gas \cite{4}. In the framework of the quintic NLSE, the existence of dark solitons was predicted \cite{4,6}, and the dynamics of dark solitons was investigated \cite{7}. Bright solitons of the gap type in the same model equation including a periodic optical-lattice (OL) potential have also been reported, although in different contexts, such as a phenomenological description of degenerate Fermi gases and BCS superfluids \cite{8–10}. The quasi-mean-field approach relies on the physically plausible assumption that the spatial scale of variations of the gas density is much larger than the healing length. Under this condition, the effective NLSE allows one to approximate collective oscillations of the TG gas in a harmonic trap. Indeed, it has been demonstrated that excitation frequencies derived from the fermionic hydrodynamic equations and from the quintic NLSE agree within a few per cent \cite{11,12}.

A natural extension of this line of research is to consider gases with long-range dipole–dipolar (DD) interactions between atoms. Some bosonic atoms, such as $^{52}$Cr, feature a significant permanent magnetic dipole moment ($\approx 6\mu_B$).
the case of chromium). The creation of the Bose–Einstein condensate (BEC) made of $^{52}$Cr, and various experiments in that quantum gas have been reported [13–15]. A gas of LiCs molecules carrying a permanent electric dipole moment was also recently made available to experiments [16]. In addition to that, atoms may be polarized by an external dc electric field [17].

The nonlocal character of the DD interactions may drastically modify properties of the quantum gas—first of all, changing the character of the collapse in it [14]. Further, stable isotropic [18] and anisotropic [19] 2D solitons have been predicted in the 2D dipolar BEC, whereas such localized states are always unstable in the same model with local interactions [20]. Recently, families of 1D matter-wave solitons in a BEC supported by the competition of contact and dipole–dipole interactions of opposite signs were predicted in [21], and 1D discrete solitons corresponding to the limit case of the dipolar condensate trapped in a very deep OL (i.e., solutions to the discrete NLSE with the long-range DD interactions between lattice sites) were also found [22]. It is relevant to mention that qualitatively similar effects may be induced by the nonlocal nonlinearity in models of optical media [23, 24].

In this work, we study localized structures in the model of a 1D Bose gases supported by competing nonlocal attractive cubic (DD) and local repulsive quintic (contact) nonlinearities, using numerical simulations of the NLSE containing this combination of the nonlinear terms. Besides the cold bosons in the TG regime, this model may apply (at least, at the phenomenological level [9]) to nearly-1D degenerate Fermi gases with DD interactions between atoms.

The paper is structured as follows. In section 2 we introduce the model, present typical shapes of solitons expected in the TG dipolar gas, and study their stability in direct simulations. In section 3 we demonstrate, via numerical experiments, dynamics predicted by the model with a variable (time-dependent) strength of the DD interactions, including formation of dark solitons on top of a broad bright soliton, splitting of the soliton, and expansion of the TG gas, in cases when the DD interaction is ramped up, or switched off. In section 4, we report various results of interactions and collisions between two solitons. Section 5 concludes the paper.

2. The model and numerical analysis

In accordance with what was said above, we start with the quintic NLSE introduced in [4], to which we add the nonlocal cubic term accounting for the DD interaction:

$$i\psi_t + \frac{1}{2}\psi_{xx} - \pi^2 N^2 |\psi|^4 \psi + 2N\dd^2 \psi (x, t) \times \int_{-\infty}^{\infty} R(|x - x'|) |\psi(x', t)|^2 \, dx' = 0,$$

(1)

where $N$ is the total number of atoms, and $d$ is the atomic dipole moment, the wavefunction being subject to the normalization condition,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1.$$

Equation (1) is expected to be valid for a large number of atoms $N$, typically $\gg 10$, when oscillations of the matter-wave density along the chain of strongly interacting bosons confined to a tight waveguide are essentially suppressed. On the other hand, the Bose–Fermi mapping, which underlies the solution of the TG model [2], may hold in the present situation provided that the dipolar attraction between atoms is weaker than the contact (hard-core) repulsion. These conditions may be together expressed as $N \gg 2d^2/\xi$, where $\xi$ is the healing length, which is of order 1 in the present notation. We assume that the dipoles are oriented along the $x$ axes, hence the configuration possesses the cylindrical symmetry.

The model based on equation (1) has been studied in detail in some limit cases. With $R(x) = \delta(x)$, the equation reduces to the ordinary cubic-quintic (CQ) NLSE, which supports a fully stable family of bright-soliton solutions in a finite range of the chemical potentials [25]. On the other hand, if the quintic term is absent, equation (1) reduces to the NLSE with nonlocal cubic nonlinearity, whose solutions were studied too, see, e.g., [26]. The degree of non-locality is quantified by the ratio between the soliton’s width and the spatial extent of the nonlocal response function $R(x)$. Depending on this ratio, qualitatively different behaviours can be observed. Namely, in the case of strong nonlocality, i.e., when the characteristic nonlocal response length is greater than the soliton’s width, a sinusoidally ‘breathing’ mode (the so-called ‘accessible soliton’) was predicted [27] and experimentally observed [28]. In the opposite situation, when the width of the response function is small compared to the soliton’s size, the governing equation may be approximated by a modified NLSE [29].

As the kernel in equation (1) we use

$$R(x) = \sqrt{\pi} \exp\left(1 + 2x^2\right) \text{erfc}(x) - 2|x|,$$

(3)

which was derived for the dipolar BEC in the quasi-1D trap [30], assuming that the dipole moments are fixed (by an external magnetic field) along axis $x$, hence the DD interaction is attractive.

As said above, equation (1) with the long-range attraction gives rise to stable solitons in the absence of the quintic repulsive term, a distinctive feature of these solitons being the breathing intrinsic mode that may easily be excited [27]. Effects produced by the quintic term in numerically found solutions increase with $N$, leading to broadening of the localized state, as shown in figure 1, which displays soliton solutions found by means of the imaginary-time propagation method. Indeed, the solution found from the imaginary time propagation method, rather than the solution found from the exact propagation method, gives the constant value $d^2$.
Stationary localized states of equation (1), found by means of numerical integration in imaginary time. The left panel shows the effect of an increasing strength of dipolar interactions at a constant number of atoms \( N = 20 \). The soliton is broad (narrow) at weaker (stronger) dipolar interactions. The middle panel shows the effect of an increasing number of atoms at a constant strength of dipolar interactions \( d = 5 \). The soliton broadens as the number of atoms increases. The right panel shows the chemical potential as a function of the number of atoms for configurations shown in the middle panel.

\[
\mu = -190 \quad \text{for the whole interval of } N \text{ shown on the right panel of figure 1.}
\]

Despite the indefinite character of the VK criterion in this case, the direct analysis demonstrates that, in the model with the competing nonlinearities considered in this work, which correspond to the attractive nonlocal cubic and repulsive local quintic terms, all bright solitons turn out to be stable. To check the stability of the localized stationary solutions to equation (1), we ran simulations in real time, adding spatially random perturbations to the initial state. The solitons were observed to shed off the perturbations, in the form of linear waves, and quickly restored their stationary form, as shown in figure 2, which clearly demonstrates that the localized states are stable.

All numerical simulations were performed by dint of the split-step fast-Fourier-transform method [33] in a spatial domain of length \( L = 8\pi \) with 1024 modes. The time step was \( \delta t = 0.001 \). To control the accuracy of numerical results, we monitored the accuracy of normalization condition (2). During the entire simulation, it was held to the relative precision better than \( 10^{-3} \). To prevent re-entering of the linear waves emitted by the perturbed soliton into the integration domain, absorbers were installed at domain boundaries. Actually, this numerical procedure is a straightforward extension of that used for simulations of the NLSE in real time [34], and the imaginary-time propagation method for finding ground states in NLSE-based models [35]. In the numerical results presented below we assumed that the ground state of the configuration is attained if the variation of the chemical potential,

\[
\mu = \int_{-\infty}^{+\infty} \left( \frac{1}{2} |\psi_x|^2 + \frac{\pi^2 N^2}{2} |\psi|^6 - 2d^2 N |\psi|^2 \right. \\
\times \left. \int_{-\infty}^{+\infty} R(|x - x'|)|\psi(x', t)|^2 \, dx' \right) \, dx,
\]

in the course of integration becomes less than \( d\mu \sim 10^{-8} \).

Shapes of localized states in figure 1 are determined by the long-range DD forces and short-range contact repulsion for a particular number of atoms in the TG gas. Having analysed a large body of numerical results, we can conclude that, for a moderate size of the dipole moment, \( d \), the soliton develops a ‘flat-top’ shape. In the case of a strong DD interaction, the soliton becomes a ‘compacton’, with very short tails. In fact, the latter case corresponds to the Thomas–Fermi limit, when the kinetic-energy term in equation (1) is negligible, while the integral term in equation (1) plays the role of an effective potential in which the locally repulsive TG gas is trapped. Then, stronger DD interactions mean a tighter trapping potential with steep walls induced by kernel function (3).

It is pertinent to mention that, in contrast to the recently considered model of dipolar BEC with competing cubic local and nonlocal interactions [21], in the present model, which is of the CQ type, parameter \( N \) cannot be eliminated from governing
equation (1) by a rescaling of the wavefunction. This is the key point explaining the existence of arbitrarily broad flat-top solitons in the model. On the other hand, if a broad flat-top bright soliton is available, one may consider it as a background for dark solitons. The advantage of this setting is a possibility of exploring dark solitons and their interactions on top of the uniform background, which is distinct from the usual situation when the quantum gas is confined by a harmonic trap, and hence the background matter-wave density is nonuniform [32].

If the soliton’s width greatly exceeds that of kernel (3), which takes place at \( N \gg d^2 \), equation (1) is reduced to the local NLSE with the CQ nonlinearity. Considering the local counterpart of equation (1), we replace \( R(x) \rightarrow 2\delta(x) \), because \( \int_{-\infty}^{\infty} R(x) dx = 2 \). Then the corresponding local NLSE acquires the form

\[
N_\alpha > \]

\[i\psi_t + \frac{1}{2} \psi_{xx} - \alpha |\psi|^4 \psi + \beta |\psi|^2 \psi = 0, \tag{4}\]

where notations \( \alpha = \pi^2 N^2, \beta = 4d^2 N \) are introduced.

Exact soliton solutions to equation (4) were found in [25]. For the case of the self-focusing cubic (\( \beta > 0 \)) and defocusing quintic (\( \alpha > 0 \)) nonlinearities, the solution is (under normalization condition (2))

\[
\psi(x, t) = \sqrt{\frac{3\beta}{4\alpha}} \tanh(\eta) \exp[i(qx - \mu t)],
\]

where \( q \) and \( \mu \) stand for the wave vector and chemical potential of the soliton. An example of a stationary solution of equation (4) for a particular set of parameters, compared with the solution of original equation (1), is presented in figure 3.

As seen from this figure, the solutions of nonlocal equation (1) with kernel (3), and of the local NLSE are in a qualitative agreement. Therefore, exact solutions (5) of the local equation may be used as appropriate initial conditions for simulations of equation (1).

3. Soliton dynamics under varying dipolar interaction

Properties of solitons can be studied under a varying strength of the DD interaction. In the experiments, it may be varied in time by changing the orientation of dipoles with respect to the axial direction, rotating the external magnetic field [15], or by changing the strength of the external electric field, if the atomic dipole moment is induced by the latter field [17].

Below we present numerical simulations of equation (1) with variable coefficient \( d(t) \). With this objective in mind, we first prepare the ground-state solution of equation (1) by means of the imaginary-time propagation method, as explained in the previous section. Then we insert this solution into equation (1) as the initial condition, and simulate the evolution in real time, with \( d \) substituted by \( d(t) \) of a chosen form.

3.1. Contraction of the TG gas by strengthening the dipolar interaction

The stationary solitons in the model of the dipolar TG gas are formed due to the balance between the contact repulsion and long-range attraction between atoms. When the strength of one of these forces is varied in time, solitons naturally shrink or expand. The most significant observation following from the numerical simulations is that when the strength of the DD interaction is swiftly ramped up, dark soliton–antisoliton pairs are created, as shown in figure 4 (here, ‘solitons’ and ‘antisolitons’ are defined as patterns with opposite signs of the phase gradient across the density depression, see the left panel in figure 6).

The number of the generated dark soliton–antisoliton pairs depends on the speed at which the dipolar interaction is ramped up. Reaching the edge of the flat-top background, the dark soliton is reflected back towards the centre. This effect can be seen even after the original soliton splits (see the right panel in figure 4). Colliding at the centre, two dark solitons interact repulsively and bounce back. This is a manifestation of the particle-like nature of dark matter-wave solitons, which was experimentally observed in BEC [38].

In order to highlight distinctive features of solitons in the dipolar TG gas, we have performed numerical simulations, similar to those shown in figure 4, also with the local counterpart of the original equation (1), i.e., equation (4), as shown in figure 5. In the case of local NLSE, the varying dipole moment \( d(t) \) is emulated by varying the coefficient of the cubic nonlinearity, \( \beta(t) \), in equation (4). From comparing figures 4 and 5, one can observe drastically different behaviours. The most prominent distinction concerns the generation of long-lived oscillating dark solitons on top of the flat-top soliton in the nonlocal model. The other difference concerns the emission of linear waves under the varying coefficient of the cubic nonlinearity—\( d(t) \) or \( \beta(t) \), respectively. Namely, the soliton of the local NLSE, equation (4), strongly radiates, while the soliton of nonlocal equation (1) shows almost no radiation. The mode of splitting of these two types of the solitons, when the coefficient in front of the cubic nonlinear term is rapidly varied, is also very different (see the middle and right panels in figures 4 and 5).
Figure 4. Evolution of the wavefunction according to equation (1) with kernel (3), when the strength of the dipolar interaction is ramped up as $d(t) = 2[1 + \tanh(\gamma t)]$. In the left panel, when the strength of the dipolar interaction is slowly raised with $\gamma = 0.1$, dark solitons do not emerge, while the TG gas performs contracting-expanding oscillations, preserving its integrity. In the middle panel, at a moderately fast raise of $d(t)$, with $\gamma = 0.3$, a dark soliton–antisoliton pair is generated, making the quasi-particle bouncing of the dark solitons evident. In the right panel, swiftly ramping up $d(t)$ with $\gamma = 0.5$ gives rise to multiple generation of dark solitons and splitting of the original bright soliton. In time interval $2 < t < 3$, which precedes the breakup of the bright soliton, repeated dark-soliton collisions are observed. Oscillating dark solitons remain in the split parts of the original bright one.

Figure 5. Numerical simulations similar to those displayed in the previous figure, but for local equation (4). The coefficient in front of the cubic nonlinearity is varied in time according to $\beta(t) = 4N[d(t)]^2$, where $d(t)$ is the same as in the previous figure. The initial state is the flat-top soliton of local cubic-quintic equation (4), which is shown in figure 3 by the solid red line. In contrast to the situation observed in the simulations of the nonlocal equation, dark solitons do not appear on top of the flat-top soliton.

In experiments performed so far, dark solitons were created by optically imprinting a phase gradient onto a cigar-shaped BEC [39]. The present approach, based on ramping up the strength of the DD interactions, may be considered as an alternative approach to the controllable creation of dark solitons in quantum gases.

3.2. Free expansion of the TG gas

Expansion of a quantum gas released from the potential trap bears important information about correlations in the system. For instance, the momentum distribution of bosonic atoms, which expands after a sudden removal of the trapping potential, has been the key evidence showing that the atoms exhibited a pronounced fermionic behaviour, i.e., the TG regime has been achieved [1]. During the expansion of a 1D gas of hard-core bosons, the momentum-distribution function becomes equal to that of the equivalent non-interacting fermions. This phenomenon, known as the dynamical fermionization, is the most interesting manifestation of the Bose–Fermi duality [40]. Quantum correlations in the dynamically evolving TG gas in an OL were studied in [41].

Bright solitons in the TG gas with attractive DD interactions can offer additional possibilities in exploring properties of quantum gases. Specifically, since in the present setting the strong repulsion between bosonic atoms is balanced by the long-range DD attraction, the momentum distribution of a freely expanding gas, achieved by suddenly switching off or decreasing the strength of the DD interaction, develops from the initial spatial distribution specific to bright solitons (in contrast to that corresponding to the harmonic potential or the box-shaped trap in previously studied settings). In the experiment, the DD interaction can be turned off by the rotation of the external magnetic field, or by eliminating the external dc electric field responsible for the induced electric dipole moment.

In figure 7 we illustrate the free expansion of the TG gas from the compacton-like soliton state shown in figure 1. Investigation of the scaling law governing the expansion of
the TG gas of dipolar atoms, as well as of its momentum distribution, may be interesting topics for subsequent studies.

4. Interactions and collisions of dipole solitons

The wave–particle duality of solitons can be most clearly observed in their interactions and collisions. While ‘genuine’ solitons in integrable models collide strictly elastically, solitons of non-integrable models show complex collision behaviours, ranging from almost elastic to fully destructive. Inelasticity of collisions show up as significant emission of linear waves resulting from the collision, merger of colliding solitons into a single dynamically evolving wave packet (including a possible transition to the wave collapse), splitting of solitons and multiple generation of secondary solitons.

Collisions of matter-wave solitons composed of cold atoms with contact interactions have extensively been studied, both in numerical simulations [42] and in real experiments [43]. In particular, inelasticity in collisions of solitons described by the local NLSE with self-focusing cubic and quintic terms, the latter generated by the deviation of the effective equation from the one-dimensionality, was analysed in [44].

The collision of matter-wave solitons in BEC with competing cubic local and dipolar interactions has recently been studied by means of numerical simulations in [21]. Here, we explore collisions between dipole solitons in the TG-gas model. Since the objective is to reveal features introduced by the long-range dipolar forces, we focus on the interactions and collisions between compacton-like solitons, whose properties are dominated by dipolar forces.

There are essential differences in the present model in comparison with the work of [21]: here the self-defocusing nonlinear term is quintic, and the arrangement of interacting solitons is different. Namely, in our case one soliton (the ‘target’) is at rest at the origin ($x = 0$), while the other (the ‘missile’) is set in motion towards the target. This setting allows us to explicitly investigate the momentum exchange between the colliding solitons.

An important conclusion suggested by the numerical experiments is that the long-range attraction is the dominant force between the interacting solitons. This nonlocal force is phase independent and is much stronger than the usual short-range interaction forces which depend on the phase shift between the solitons.

In the first set of the numerical experiments, we set two quiescent (zero-velocity) dipole solitons at distance $\delta x = 4\pi$ from each other, and observed their evolution. In either case of in-phase (see figure 8 (a)) and out-of-phase (not shown) pairs of the solitons, they attracted each other and merged into a breather. The fact that the merger time in both cases was almost the same ($t \simeq 6$) indicates that the phase-dependent short-range interaction force has played no tangible role in the dynamics.

Next, we consider collisions between moving solitons. With this objective in mind, we prepared the initial state with one-soliton set at the origin ($x = 0$) with zero velocity ($v = 0$), and the other one placed at $x = 4\pi$ with finite velocity $v_0$. General features of soliton collisions can be summarized as follows (see figures 8(b), (c) and (d)): at moderately small velocities, e.g., $v_0 = -2$, solitons merge into a single breather. In fact, the dipolar attraction between the solitons facilitates the merger. At greater velocities (such as $v_0 = -4$), the initially moving soliton passes through the quiescent one, transferring...
to it a small velocity in the same direction. In this case, although the solitons separate after the collision, the dipolar attraction between them overcomes the trend in the separation and the solitons again merge into a single breather. At still greater velocities (e.g., \( v_0 = -6 \)), the two solitons separate and fly apart, overcoming the dipolar attraction. The conservation of the total momentum of the colliding solitons can be clearly observed in these numerical experiments.

5. Conclusions

In this work, our aim was to study the existence, stability and basic dynamical properties of bright one-dimensional solitons based on the balance between the local (hard-core) repulsion and long-range dipole–dipole (DD) attraction between atoms in the model of the TG gas. It was found that, depending on the number of atoms and strength of the DD interaction, the solitons assume flat-top or compacton-like shapes. For solitons of the former type, the solution of the local cubic-quintic NLSE with competing nonlinearities is found to be a good approximation. Numerical simulations of the underlying nonlocal equation with a variable strength of the DD interaction have revealed various dynamical regimes, including the formation of dark solitons on top of a bright one (of the flat-top type), particle-like collisions between them, splitting of the flat-top solitons, and self-similar ballistic expansion of the gas after dropping the DD attraction. Collisions between bright solitons of the compacton type have been investigated in different regimes. The strong dipolar attraction between the solitons explains that, in many cases, the colliding solitons merge into a breather.

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