GEOMETRY OF COLLAPSING AND FREE DEFORMATION RETRACTION

ALEXEY GORELOV

Abstract. We prove that a compact polyhedron \( P \) collapses to a subpolyhedron \( Q \) if and only if there exists a piecewise linear free deformation retraction of \( P \) onto \( Q \).

1. Introduction

A compact polyhedron \( P \) collapses to \( Q \) if there is a triangulation \( T \) of \( P \) in which \( Q \) is triangulated by some subcomplex \( T_Q \) and \( T \) collapses to \( T_Q \) as simplicial complex (see definitions 2.6 and 2.5). One can ask: can we characterize collapsibility in triangulation invariant terms? This could be useful for establishing links between purely topological characteristics of spaces and characteristics defined for some fixed combinatorial structure of them (i.e. triangulation). For example, in \([6]\) Zeeman states the following conjecture:

**Conjecture.** If \( K \) is contractible two-dimensional compact polyhedron then \( K \times I \) is collapsible (to a point).

It is well known that this conjecture implies three-dimensional Poincaré conjecture.

A free deformation retraction \( f : P \times I \to P \) is a deformation retraction with additional “freeness” condition \( f(f(x, t), s) = f(x, \max(t, s)) \). In \([3]\) Isbell proved that two-dimensional compact polyhedron \( P \) is collapsible if and only if there is a free deformation retraction of \( P \) onto a point. Piergallini in \([5]\) proved that this equivalence also holds for three-dimensional manifolds with boundary. Meanwhile, it is known that there exist non-collapsible polyhedrons of dimensions five and more which are freely contractible, see \([1]\).

We prove that this equivalence holds in the general case if we in addition require deformation retraction to be piecewise linear, so our main theorem is

**Theorem.** Let \( P \) be a compact polyhedron and let \( Q \subset P \) be a subpolyhedron. Then the following statements are equivalent:

1. there exists a piecewise linear free deformation retraction of \( P \) onto \( Q \).
2. \( P \) collapses to \( Q \).

It should be noted that this theorem was originally stated in \([5]\), but the reasoning given there is not enough to be a proof, and it is based on the false statement (see comments below Theorem 4.1).
2. Basic definitions

Definition 2.1. A finite affine simplicial complex is a finite nonempty collection \( K \) of simplices in an Euclidean space \( \mathbb{E}^n \) such that:

1. If \( A \in K \) then all the faces of \( A \) are in \( K \).
2. If \( A, B \in K \) then either \( A \cap B = \emptyset \) or \( A \cap B \) is a common face of both \( A \) and \( B \).

From now on by a complex we mean a finite affine simplicial complex.

Definition 2.2. A compact affine polyhedron \( P \) is a underlying point set of some simplicial complex \( K \), namely \( P = |K| \subset \mathbb{E}^n \). The simplicial complex \( K \) is called a triangulation of the polyhedron \( P \).

From now on by a polyhedron we mean a compact affine polyhedron.

Definition 2.3. A piecewise linear map \( f : P \times I \to P \) is called piecewise linear free deformation retraction of a polyhedron \( P \) onto a polyhedron \( Q \subset P \) if

1. \( f : P \times I \to P \) is a strong deformation retraction of \( P \) onto \( Q \), that is

\[
\begin{align*}
|f|_{P \times 0} &= \text{id}_P \\
f(P \times 1) &= Q \\
|f|_{Q \times t} &= \text{id}_Q, \quad \forall t \in I
\end{align*}
\]

2. \( f(f(x, s), t) = f(x, \max(s, t)) \)

Remark. Let \( f_t(x) := f(x, t) \), then freeness condition 2 from definition 2.3 could be rewritten as \( f_t \circ f_s = f_{\max(t, s)} \).

The concept of free deformation retraction was independently introduced by John R. Isbell in [3] and R. Piergallini in [5] (note that R. Piergallini use the term «topological collapsing» for it).

Note that the freeness condition 2 is not sufficient for a piecewise linear map \( f : P \times I \to P \) to be a piecewise linear free deformation retraction. For example, assume that for polyhedrons \( Q \subset P \subset S \) there exists an arbitrary retraction \( g : S \to S \) of \( S \) onto \( P \), that is \( g(S) = P \) and \( g|_P = \text{id}_P \). Assume also that there exists a piecewise linear free deformation retraction \( h : P \times I \to P \). Let \( f : S \times I \to S : (x, t) \mapsto h(g(x), t) \). Clearly the freeness condition 2 holds for \( f \) but \( f \) is not a strong deformation retraction since \( f|_{S \times 0} = g \neq \text{id}_S \).

It is also clear that any piecewise linear free deformation retraction of \( S \) onto \( Q \) cannot be a piecewise linear free deformation retraction of \( S \) onto \( P \) since it doesn’t leaves points of \( P \) fixed.

Obviously, if \( f : P \times I \to P \) is a piecewise linear free deformation retraction of \( P \) onto \( Q \subset P \) then \( f_s(P) = f_t(f_s(P)) \subseteq f_t(P) \) for \( 0 \leq t \leq s \leq 1 \).

As a simple corollary of the freeness condition 2 we obtain the following. Take a point \( x \in P \) and consider the curve \( \gamma_x := f(x \times I) \). Informally speaking, \( \gamma_x \) is a path of a point \( x \) during retraction. Take a point \( f_t(x) \in \gamma_x \) on this curve. Note that \( f_s(f_t(x)) \) is equal to \( f_t(x) \) when \( s \leq t \) and \( f_s(x) \) when \( s \geq t \). This means that the path of \( f_t(x) \) is
\(\gamma_{f_t(x)} = f(x \times [t, 1])\). In fact, this implies that the curve \(\gamma_x\) has no self-intersections for any point \(x \in P\).

**Definition 2.4.** Let \(K\) be a complex, and let \(A\) and \(B\) be simplices of the complex \(K\) such that

1. \(B\) is a face of \(A\).
2. there is no simplex \(C \in K, C \neq A\) such that \(B\) is a face of \(C\).

Then we say that there is a elementary simplicial collapse of the complex \(K\) on the subcomplex \(K' = K \setminus \{A, B\}\) across the simplex \(A\) from the face \(B\), and write \(K \xrightarrow{B \setminus A} K'\).

**Definition 2.5.** We say that a complex \(K\) simplicially collapses to a complex \(Q\) and write \(K \downarrow Q\) if there is a finite sequence of elementary simplicial collapses going from \(K\) to \(Q\):

\[
K = K_1 \xrightarrow{B_1 \setminus A_1} K_2 \xrightarrow{B_2 \setminus A_2} K_3 \xrightarrow{B_3 \setminus A_3} \cdots B_{n-1} \setminus A_{n-1} K_n = Q
\]

**Definition 2.6.** We say that a polyhedron \(P\) collapses to a subpolyhedron \(Q \subset P\) and write \(P \downarrow Q\) if there is a triangulation \(T\) of \(P\) such that \(Q\) is triangulated by some subcomplex \(T(Q)\) of \(T\) and \(T \setminus T(Q)\).

Further details about collapsing can be found in [7], [2] or [4].

**Definition 2.7.** Let \(N\) be a subset of \(P \times I\).

The shadow of \(N\) is the set

\[
S_-(N) := \{(x, t) \in P \times I : \exists (x, s) \in N, t \leq s\} \subset P \times I
\]

Accordingly, the coshadow of \(N\) is the set

\[
S_+(N) := \{(x, t) \in P \times I : \exists (x, s) \in N, t \geq s\} \subset P \times I
\]

and the total shadow of \(N\) is the set

\[
S(N) := \{(x, t) \in P \times I : \exists (x, s) \in N\} \subset P \times I
\]

**Remark.** Note that we have a natural partially ordered set structure on \(P \times I\): \((x, t_1) \leq (y, t_2)\) if \(x = y\) and \(t_1 \leq t_2\), and the shadow and the coshadow of a set \(N\) are respectively the lower and the upper closure of \(N\) with respect to this natural partial order on \(P \times I\).

**Definition 2.8.** We say that a set \(N \subset P \times I\) is downward closed (upward closed) if \(S_-(N) = N\) (respectively \(S_+(N) = N\)).

Let us give also several useful propositions readily followed from the definitions:

**Proposition 2.9.**

1. The following statements are equivalent:
   (a) \(N\) is downward closed.
   (b) \(((x, t) \in N) \Rightarrow ((x \times [0, t]) \subset N)\)
2. The following statements are equivalent:
   (a) \(N\) is upward closed.
   (b) \(((x, t) \in N) \Rightarrow ((x \times [t, 1]) \subset N)\)
Proposition 2.10. The following is true:

(1) If \( N \) is downward closed then \((P \times I) \setminus N\) is upward closed.
(2) If \( N \) is upward closed then \((P \times I) \setminus N\) is downward closed.

3. CYLINDERWISE COLLAPSING

Definition 3.1. Let \( P \) be a polyhedron.

A triangulation \( T \) of \( P \) is called cylindrical if for any simplex \( A \in T \) its total shadow \( S(A) \) is triangulated by a subcomplex of \( T \).

Definition 3.2. Let \( T \) be a cylindrical triangulation of \( P \times I \) and let \( A \) be a simplex in \( T \). We call a subcomplex \( T_A \) triangulating the total shadow \( S(A) \) of \( A \) the subcylinder of \( A \).

Note that any triangulation \( T \) of \( P \times I \) contains subcomplexes \( T^0 \) and \( T^1 \) triangulating respectively \( P \times 0 \) and \( P \times 1 \). Clearly, \( T^0 \) and \( T^1 \) could be considered as triangulations of \( P \) itself and the inclusions \( i_0, i_1 : P \hookrightarrow P \times I \) of \( P \) as \( P \times 0 \) and \( P \times 1 \) are simplicial with respect to the pairs of triangulations \((T^0, T)\) and \((T^1, T)\) respectively.

Proposition 3.3. The following statements are equivalent:

1. A triangulation \( T \) of \( P \times I \) is cylindrical.
2. \( \text{pr}_P \) is simplicial with respect to a triangulation \( T \) of \( P \times I \) and a triangulation \( T^0 \) of \( P \).
3. \( \text{pr}_P \) is simplicial with respect to a triangulation \( T \) of \( P \times I \) and a triangulation \( T^1 \) of \( P \).

Proof. Let us prove that the statements (1) and (2) are equivalent, the proof of the equivalence of (1) and (3) is the same.

Assume that \( \text{pr}_P \) is simplicial with respect to triangulations \( T \) and \( T^0 \). Then \( i_0 \circ \text{pr}_P : P \times I \rightarrow P \times I \) is also simplicial with respect to the triangulation \( T \). Hence it is clear that for any simplex \( A \in T \) the subcomplex \((i_0 \circ \text{pr}_P)^{-1}((i_0 \circ \text{pr}_P)(A))\) triangulates \( S(A) \), so \( T \) is cylindrical.

Conversely, assume that a triangulation \( T \) of \( P \times I \) is cylindrical. Note that if \( i_0 \circ \text{pr}_P \) is simplicial with respect to \( T \) then \( \text{pr}_P = i_0^{-1} \circ i_0 \circ \text{pr}_P \) is simplicial with respect to \( T \) and \( T^0 \).

Take a simplex \( A \in T \). By assumption, \( S(A) \) is triangulated by a subcomplex \( T_A \) of \( T \). In \( T_A \) there is a subcomplex \( T^0_A := T^0 \cap T_A \) triangulating \( S(A) \cap (P \times 0) \). Note that \( i_0 \circ \text{pr}_P(A) = S(A) \cap (P \times 0) \), so \( T^0_A \) triangulating the image of \( A \) under \( i_0 \circ \text{pr}_P \). We shall prove that \( T^0_A \) is a simplex and its vertices are images of vertices of \( A \).

Assume that there is a vertex \( p \) of the complex \( T^0_A \) that is not an image of a vertex of \( A \) under \( i_0 \circ \text{pr}_P \). Since \( T \) is cylindrical there is a one dimensional subcomplex of \( T \) triangulating \( S(p) \). It intersects \( A \) along some subcomplex \( T^A_p \) and its vertices are vertices of \( A \). Clearly, these vertices maps to \( p \) under \( i_0 \circ \text{pr}_P \), a contradiction.
Now assume that \( p \) is a vertex of the simplex \( A \). Then we have a one dimensional subcomplex \( T_p \) triangulating \( S(p) \), and it contains a subcomplex \( T_p^0 \) triangulating \( S(p) \cap (P \times 0) \) which is also a subcomplex of \( T_A^0 \). Obviously, \( T_p^0 \) consists of one vertex which is exactly an image of \( p \) under \( i_0 \circ \text{pr}_P \). So vertices of \( A \) map to vertices of \( T_A^0 \) under \( i_0 \circ \text{pr}_P \).

Let us prove that \( T_A^0 \) is a simplex by induction on \( \dim A \). For \( n = 0 \) it is obvious. Assume that \( \dim A > 0 \). Take a vertex \( p \) of \( A \) and its opposite face \( B \). By the induction hypothesis, the subcomplex \( T_B^0 \) triangulating \( S(B) \cap (P \times I) \) is a simplex. We have two cases:

1. \( T_p^0 \) is one of the vertices of \( T_B^0 \), then \( T_A^0 = T_B^0 \) and \( T_A^0 \) is a simplex.
2. \( p_0 := T_p^0 \) is a vertex of \( T_0 \) and \( p \notin T_B^0 \). Note that for any face \( C_0 \) of the simplex \( T_B^0 \) there exists a face \( C \) of \( B \) such that \( (i_0 \circ \text{pr}_P)(C) = C_0 \). Then \( (i_0 \circ \text{pr}_P)(pC) = p_0 C_0 \) and \( p_0 C_0 \) is a simplex by the induction hypothesis.

Therefore, for any faces \( C_1 \) and \( C_2 \) of the simplex \( T_B^0 \) there are simplices \( p_0 C_1 \) and \( p_0 C_2 \) and they intersect along a common face \( p_0 D \) where \( D \) is a common face of \( C_1 \) and \( C_2 \) in \( T_B^0 \). Recall that the a vertex of \( T_A^0 \) is either \( p_0 \) or a vertex of \( T_B^0 \). Using this, it is easy to see that \( T_A^0 = p_0 T_B^0 \) is a simplex and simplices of the form \( p_0 C_i \) are its faces.

Thus, we just prove that the map \( i_0 \circ \text{pr}_P \) is simplicial with respect to the triangulation \( T \). As we noted above, it follows that \( \text{pr}_P \) is simplicial with respect to the triangulations \( T \) and \( T^0 \).

\[ \square \]

**Proposition 3.4.** Let \( A \) be a simplex, let \( E \) be a Euclidean space, and let \( F : A \to E \) be a linear map.

Then there are only the following two possibilities:

1. \( F \) is injective and, consequently, \( F : A \to F(A) \) is a homeomorphism.
2. \( F(A) = F(\mathring{A}) \) and, moreover, for any codimension one face \( B \) of \( A \) we have \( F(A) = F(\mathring{A} \setminus \mathring{B}) \).

Note that \( F(A) \) is not is not necessarily be a simplex in the second case.

**Proof.** We may assume that simplex \( A \) is embedded in an affine space of dimension \( \dim A \). Note that \( A \) in the affine space is defined by the system of \( n + 1 \) linear inequalities \( g_0(x) \leq 0, g_1(x) \leq 0, g_2(x) \leq 0, \ldots, g_n(x) \leq 0 \) and each of them corresponds to a codimension one face of \( A \), or more precisely to an affine hyperplane containing a codimension one face of \( A \).

If \( F \) is injective then clearly \( F \) is a homeomorphism onto its image and the statement holds.

Assume that \( F \) is not injective. In this case for any point \( p \in F(A) \) its preimage is an intersection of some affine plane \( V \) (\( \dim V = \dim \ker F \)) with \( A \), namely a convex polyhedron in \( V \) defined by some of the inequalities \( g_i(x) \leq 0 \). Each of these inequalities corresponds to the intersection of a convex polyhedron with a codimension one face of \( A \). It is clear that more then one inequality is needed to define a convex polyhedron,
so $V$ intersects at least two codimension one faces of $A$. Hence any point $p \in F(A)$ has preimages in at least two codimension one faces of $A$ and the statement \ref{2} holds.

\textbf{Definition 3.5.} Let $P$ be a polyhedron and let $T$ be a cylindrical triangulation of $P \times I$.

Since $pr_P$ is linear, from Proposition \ref{3.4} it follows that for any simplex $A \in T$ there are two possibilities:

1. $pr_P|_A$ is a homeomorphism onto the image and then we call the simplex $A$ horizontal
2. $pr_P(A) = pr_P(A)$ and then we call the simplex $A$ vertical

\textbf{Definition 3.6.} Let $T$ be a cylindrical triangulation and let $A$ be a simplex in $T$. Denote the subcylinder of $A$ by $T_A$ (see Definition \ref{3.2}). We call a simplex $B \in T_A$ a main simplex of subcylinder $T_A$ if $pr_P(B) = pr_P(A)$.

Let $A$ be a simplex of a cylindrical triangulation $T$. Consider the simplices of the subcylinder $T_A$ of $A$. Note that for any interior point $p$ of the simplex $pr_P(A)$ the segment $p \times I$ intersects a main simplex $B$ of $T_A$ either at a point if $B$ is horizontal or along a subsegment if $B$ is vertical, see Figure \ref{1}.

Hence we have a partition of the segment $p \times I$. Note that $p \times I$ is a chain in $P \times I$ with the natural partial order, so we have the total order on $p \times I$, and it induces the total order on the set of main simplices of $T_A$. Clearly it doesn’t depend on the choice of an interior point $p$. Denote this total order by $\prec$.

It is clear that the following holds: $B \preceq C$ if and only if for any pair of comparable points $(x, t) \in B$ and $(x, s) \in C$ we have $t \leq s$.

Note that if a main simplex $B_1$ covers another main simplex $B_2$ with respect to the total order on main simplices (that is there is no main simplex $C$ such that $B_2 < C < B_1$) then it is necessary for one of them to be vertical and for another one to be horizontal.
**Theorem 3.7** (Cylinderwise collapsing). Let $P$ be a polyhedron and let $M_1$ and $M_2$ be a pair of downward closed (resp. upward closed) subpolyhedrons of $P \times I$ such that $M_1 \subset M_2 \subset P \times I$ and $P \times 0 \subset M_1$ (resp. $P \times 1 \subset M_1$).

Also let $T$ be a cylindrical triangulation of $P \times I$, and assume that $M_1$ and $M_2$ are triangulated by subcomplexes $T(M_1)$ and $T(M_2)$ of $T$ respectively. Then $T(M_2) \searrow T(M_1)$ and the following holds:

1. Elementary collapses are performed across vertical simplices from horizontal simplices, and both of them are main simplices of the same subcylinder.
2. Let $T_{k-1}^{B \searrow_A} T_k$ be the $k$-th elementary collapse. Then $B \succ A$ and if $C \succ B$ then $C \notin T_{k-1}$ (resp. $B \prec A$ and if $C \prec B$ then $C \neq T_{k-1}$).

In fact Theorem 3.7 was proved in [7, chapter 7], we shall provide the more detailed proof for the sake of the completeness.

**Lemma 3.8.** Consider the subcomplexes $T(M_1) \cup T(M_2)$ from the statement of Theorem 3.7. Let $T_A$ be a subcylinder of a simplex $A \in T^0$.

Then there exists a main horizontal simplex $B_1 \in T_A \cap T(M_1)$ of the subcylinder $T_A$ such that

\[(C \preceq B_1) \iff (C \text{ is a main simplex of } T_A \text{ lying in } T(M_1)) \tag{1}\]

Also there exists a main horizontal simplex $B_2 \in T_A \cap T(M_2)$ of the subcylinder $T_A$ such that

\[(C \preceq B_2) \iff (C \text{ is a main simplex of } T_2 \text{ lying in } T(M_2)) \tag{2}\]

and $B_1 \preceq B_2$.

**Proof.** Denote by $L$ the totally ordered set of main simplices of $T_A$. Let us consider $L \cap T(M_1)$. First, $L \cap T(M_1) \neq \emptyset$ since $A \in L \cap T(M_1)$. Second, $L \cap T(M_1)$ is downward closed in $L$ since $M_1$ is downward closed in $P$. Let $B_1$ be the maximal element of $L \cap T(M_1)$. If $B_1$ is vertical then obviously it has a face $C$ such that $C \succ B_1$ and $C \in T(M_1)$. Thus $B_1$ is horizontal. Since $L \cap T(M_1)$ is downward closed [1] holds for $B_1$.

Similarly we obtain the main horizontal simplex $B_2 \in T_A \cap T(M_2)$ for which [2] holds.

Next, we have $M_1 \subset M_2$ so $T(M_1) \subset T(M_2)$ and, consequently, $B_1 \in L \cap T(M_2)$. From this and from [1] it follows that $B_1 \preceq B_2$. \qed

**Lemma 3.9.** Let $T_A$ be a subcylinder of a simplex $A \in T^0$ and let $B$ be a main horizontal simplex of $T_A$. Then if $B$ is a codimension one face of a simplex $C$, there are only the following two possibilities:

1. $C$ is a main vertical simplex of the subcylinder $T_A$
2. $C$ is a main horizontal simplex of a subcylinder $T_D$ where $D \in T^0$ and $A$ is a face of $D$

**Proof.** Let $B$ be a codimension one face of a simplex $C$, that is $\dim C = \dim B + 1$.

Then one of the following holds:

1. $i_0 \circ \text{pr}_P(C) = A$, and then $C$ clearly is a main vertical simplex of $T_A$. 


(2) $i_0 \circ \text{pr}_P (C) = D$ and $A$ is a face of $D$. Then $C$ is a main simplex of $T_D$ and $C$ is clearly necessarily be horizontal since $\dim C = \dim B + 1 = \dim A + 1 = \dim D$.

Proof of Theorem 3.7. Let $A_1, A_2, \ldots, A_n$ be the simplices of $T^0$ in non-increasing order of dimension. In each subcylinder $T_{A_i}$ we can find the maximal main simplex lying in $M_1$, denote it by $B^1_i$. Similarly, denote by $B^2_i$ the maximal main simplex of $T_{A_i}$ lying in $M_2$. Then the set of main simplices of $T_{A_i}$ lying in $M_1$ (resp. $M_2$) is nothing but the lower set of $B^1_i$ (resp. $B^2_i$) by Lemma 3.8.

Let us perform the collapsing in subcylinders $T_{A_i}$ in the order described above: in each subcylinder we sequentially perform elementary collapses from horizontal main simplices across vertical main simplices from greatest (with respect to the total order on $T_{A_i}$) to $B^1_i$.

Note that a horizontal simplex $B$ from that we perform an elementary collapse is a free face at each step because

1. It is the maximal main simplex of the remaining main simplices of its subcylinder, so $B$ is a free face of only one vertical main simplex of the same subcylinder.
2. It cannot be a free face of some horizontal main simplex of another subcylinder of bigger dimension since the subcylinders of bigger dimension were collapsed before.
3. It cannot be a free face of other simplices by Lemma 3.9.

so each elementary collapse is legal.

It remains to show that the collapsing described above induces the collapsing of $T(M_2)$ to $T(M_1)$. To do this, first note that for any elementary collapse $T_{k-1} \rightarrow_T T_k$ in the collapsing we have $A < B$ and, obviously, there is no $C$ such that $A < C < B$, so

1. if $B > B^2_i$ then $A > B^2_i$ and $A, B \notin T(M_2)$
2. if $B \leq B^2_i$ then $A < B^2_i$ and $A, B \in T(M_2)$

It is clear that in the first case the elementary collapse doesn’t affect simplices in $T(M_2)$, and in the second case it induces the elementary collapse in $T(M_2)$ since $B$ is a free face of $A$ in the remaining part of $T(M_2)$.

4. MAIN RESULT

Theorem 4.1. Let $P$ be a polyhedron and let $Q \subset P$ be a subpolyhedron. Then the following statements are equivalent:

1. there exists a piecewise linear free deformation retraction of $P$ onto $Q$.
2. $P \searrow Q$.

It should be noted that the statement of Theorem 4.1 is formulated in [5], but, in our opinion, the reasoning given there is not a proof, because it unreasonably uses the existence of a pair of triangulations $(T, T_P)$ of the cylinder $P \times I$ and the polyhedron $P$ with respect to which both the projection $\text{pr}_P$ and the piecewise linear free deformation
retraction $f$ are simplicial. Note that in a common case for two piecewise linear maps $g : P \to Q$ and $h : P \to S$ there are no triangulations of $P$, $Q$ and $S$ with respect to which both maps are simplicial, see Example 1 to Theorem 1 in [7].

First we shall prove that the existence of a piecewise linear free deformation retraction implies the collapsibility. For this let us introduce some notation.

Let (see Figure 2)

- $f : P \times I \to P$ be a piecewise linear free deformation retraction of $P$ onto $Q$ (we also use the notation $f_t$ for the map $P \to P : x \mapsto f(x,t)$).
- $F : P \times I \to P \times I : (x,t) \mapsto (f(x,t), t)$
- $M := F(P \times I)$
- $S := \text{Fr} M \cup (Q \times 1)$ (here $\text{Fr} M$ is the topological boundary of $M$ in $P \times I$)

Clearly, $F$ is piecewise linear and both $M$ and $S$ are subpolyhedrons of $P \times I$. Note that $M$ and $S$ could be considered as the subpolyhedrons in $P \times I$ as the domain of $F$.

**Lemma 4.2.** The following holds:

1. $P \times 0 \subset M$
2. $Q \times I \subset M$
3. $M \cap (P \times 1) = Q \times 1$

**Proof.** It follows directly from the definition of $M$. \qed

**Lemma 4.3.** $F^{-1}(P \times t) \subset P \times t$

**Proof.** It follows directly from the definition of $F$. \qed

**Lemma 4.4.** $F$ is a retraction of $P \times I$ onto $M$. 
Proof. In short, we want to prove that $F|_M = \text{id}_M$.

Let $(x, t) \in M$. By the definition of $M$ it means that $\exists (y, t) \in P \times I : F(y, t) = (x, t)$ or, in other words, $f_t(y) = x$. Hence $f_t(x) = f_t(f_t(y)) = f_t(y) = x$, so $F(x, t) = (x, t)$. □

**Lemma 4.5.** $M$ is downward closed.

**Proof.** It is sufficient to prove that for any $(x, t) \in M$ we have $x \times [0, t] \in M$.

Let $(x, t) \in M$. Lemma 4.4 implies that $F(x, t) = (x, t)$ or, in other words, $f_t(x) = x$. Take an arbitrary $s \in [0, t]$. We have $f_s(x) = f_s(f_t(x)) = f_t(x) = x$, so $F(x, s) = (x, s)$ and $(x, s) \in M$, $\forall s \in [0, t]$.

**Remark.** From the previous lemma it follows that the set $(P \times I) \setminus \hat{M}$ is upward closed.

Clearly the set $((P \times I) \setminus \hat{M}) \cup (Q \times 1)$ is also upward closed. Note that $\text{Fr}(((P \times I) \setminus \hat{M}) \cup (Q \times 1)) = S$.

**Lemma 4.6.** $F$ is injective on $F^{-1}(\hat{M})$.

**Proof.** We argue by contradiction.

Obviously $F^{-1}(y, 0) = (y, 0)$ since $F^{-1}(P \times 0) = P \times 0$ and $F|_{P \times 0} = \text{id}_P$.

Assume that $\exists (y, t) \in \hat{M} : |F^{-1}(y, t)| > 1$ and $t > 0$. Lemma 4.4 implies that $(y, t) \in F^{-1}(y, t)$, so $\exists x \neq y$ such that $F(x, t) = (y, t)$.

Consider $J := F(x \times [0, t])$. Note that $(y, t) \in J$ by construction. Let $t_0 := \inf \{s : (y, s) \in J\}$ and let $J_0 := F(x \times [0, t_0])$ (see Figure 3). We clearly have $F(x, t_0) = (y, t_0)$ and $J_0 \cap (y \times [0, t_0]) = (y, t_0)$ as $F$ is continuous. It is also clear that $(y, t_0) \in \hat{M}$ since $0 < t_0 < t$.

Now let us suppose that $P$ is embedded in a Euclidean space $\mathbb{R}^n$ of appropriate dimension and, therefore, $P \times I$ is embedded in $\mathbb{R}^{n+1}$. Hence we can take a neighborhood of $(y, t_0)$
of the form \( B := V \times [t_0 - \epsilon, t_0 + \epsilon] \subset \hat{M} \) where \( V \subset \mathbb{R}^n \) and \( \epsilon < t_0 \). The curve \( J_0 \) must intersect the boundary of \( B \) at some point \((z, s)\). It is clear that \( s < t_0 \) and \( z \neq y \).

For the point \((z, t_0)\) we have \( f_{t_0}(z) = f_{t_0}(f_s(x)) = f_{t_0}(x) = y \) or, in other words, \( F(z, t_0) = (y, t_0) \).

Recall that \((z, t_0) \in B \subset M\), so \( F(z, t_0) = (z, t_0) \neq (y, t_0) \) by Lemma 4.4 we have arrived at a contradiction.

**Lemma 4.7.** \( F^{-1}(\hat{M}) = \hat{M} \)

**Proof.** Lemma 4.4 implies that any point \( x \in \hat{M} \) is a preimage of itself under \( F \) and from Lemma 4.6 it follows that \( x \) has no other preimages.

Let \( pr_P \) be the projection of \( P \times I \) onto \( P \). Note that \( f = pr_P \circ F : P \times I \to P \).

Let us construct the suitable triangulations of the domain and the codomain of \( F \) (recall that they are both \( P \times I \)) as follows.

First, take triangulations of the domain and the codomain such that \( F \) is simplicial with respect to them and both \( M \) and \( S \) are triangulated by some subcomplexes of the both triangulations. Second, take a subdivision of the triangulation of the domain with respect to which the map \( i_0 \circ pr_P \) is simplicial. Denote the resulting triangulations of the domain and the codomain by \( T_d \) and \( T_c \) respectively. It is clear that \( pr_P \) is simplicial with respect to \( T_d \) and \( T_d^0 \) (\( T_d^0 \) is the subcomplex of \( T_d \) triangulating \( P \times 0 \)).

Note that

1. \( F \) is linear on the simplices of \( T_d \)
2. \( f \) is also linear on the simplices of \( T_d \) since \( f \) is and the projection \( pr_P \) is a linear map.
3. the preimage of a subcomplex of \( T_c \) under \( F \) is a subcomplex of \( T_d \).
4. both \( M \) and \( S \) are triangulated by subcomplexes of both \( T_d \) and \( T_c \).
5. the triangulation \( T_d \) is cylindrical by Proposition 3.3.

Let \( N := ((P \times I) \setminus \hat{M}) \cup (Q \times 1) \). Obviously \( N \) is triangulated by some subcomplex \( K_0 \) of \( T_0 \). It is also clear that \( N \) is upward closed (see the remark for Lemma 4.5) and \( P \times 1 \subset N \). Thus \( T_0 \) collapses cylinderwise to \( K_0 \) by Theorem 3.7. Note that Lemma 4.7 implies that \( F(K_0) = S \). It is also clear that \( f(K_0) = P \) since \( pr_P(S) = P \).

Also note that \( K_0 \) collapses cylinderwise to the subcomplex triangulating \( P \times 1 \), so there is a sequence of elementary collapses

\[
K_0 \setminus \lambda A_1 K_2 \setminus \lambda A_2 K_2 \setminus \lambda A_3 \cdots B_n \setminus \lambda A_n K_n
\]

where by \( K_n \) we denote the subcomplex triangulating \( P \times 1 \). Note that \( f(|K_0|) = P \) and \( f(|K_n|) = Q \). Thus if we show that \( f(|K_i|) \setminus f(|K_{i+1}|) \) then by concatenating these collapses we obtain the desired collapsing \( P \setminus \lambda Q \).

**Lemma 4.8.** Let \( K_i B_{i+1} \setminus \lambda A_{i+1} K_{i+1} \) and let \( D := f(|K_i|) \setminus f(|K_{i+1}|) \).

Assume that \( D \neq \emptyset \). Let us consider the set \( L := f^{-1}(D) \cap |K_i| = f_{|K_i|}^{-1}(D) \) as the subset of the domain of \( f \).
Then $f(S_-(L)) \cap f(|K_{i+1}|) = \emptyset$.

Proof. Clearly $L \subset |K_i| \setminus |K_{i+1}| = \hat{A}_{i+1} \cup \hat{B}_{i+1}$ and $f(L) = D$.

Take an arbitrary point $(x, t_1) \in L$. It is sufficient to prove that $f(x \times [0, t_1]) \cap f(|K_{i+1}|) = \emptyset$.

First note that $f(x, t_1) \in D$ so $f(x, t_1) \notin f(|K_{i+1}|)$.

Assume the converse, that is $\exists t_0 \in [0, t_1) : f(x, t_0) \in f(|K_{i+1}|)$. This means that $\exists (y, s) \in |K_{i+1}|$ such that $f(y, s) = f(x, t_0)$ or, in other words, $f_s(y) = f(x, t_0)$.

Let us consider two cases:

1. if $s > t_1$ then $f(x, t_1) = f_{t_1}(x) = f_{t_1}(f_{t_0}(x)) = f_{t_1}(f_s(y)) = f_s(y) = f(y, s) \in f(|K_{i+1}|)$, but we have $f(x, t_1) \notin f(|K_{i+1}|)$, contradiction.
2. if $s \leq t_1$ then $f(x, t_1) = f_{t_1}(x) = f_{t_1}(f_{t_0}(x)) = f_{t_1}(f_s(y)) = f_{t_1}(y) = f(y, t_1) \in f(|K_{i+1}|)$. Note that $(y, s) \in |K_{i+1}|$ and $|K_{i+1}|$ is upward closed since the collapsing (3) is cylinderwise, so $(y, t_1) \in |K_{i+1}|$ and we have arrived at a contradiction.

\[ \square \]

Lemma 4.9. In collapsing (3) we have $f(|K_i|) \setminus f(|K_{i+1}|)$.

Proof. There is nothing to prove for $f(|K_i|) = f(|K_{i+1}|)$.

Assume that $\emptyset \neq D := f(|K_i|) \setminus f(|K_{i+1}|)$. Note that $|K_i| \setminus |K_{i+1}| = \hat{A} \cup \hat{B}$ where $A$ and $B$ are simplices respectively across and from which the elementary collapse $K_i \to K_{i+1}$ is performed. Hence we have $f(A) \neq f(\hat{A} \cup \hat{B})$ since $A \setminus (\hat{A} \cup \hat{B}) = \hat{A} \setminus \hat{B}$, and Proposition 3.4 implies that $f|_A : A \to f(A)$ is a homeomorphism.

Let us consider the set $L := f|_{|K_i|}^{-1}(D) = f^{-1}(D) \cap |K_i|$ in the domain of $f$. It is clear that $L \subset \hat{A} \cup \hat{B}$. Denote the shadow $S_-(L)$ by $L_-$. Note that $L_- \setminus (\hat{A} \cup \hat{B})$ lies in main simplices of the subcylinder of $A$ smaller then $B$ with respect to the natural order on main simplices. Because the collapsing (3) is cylinderwise, all simplices smaller then $B$ are already collapsed so $L_- \cap |K_i| \subset \hat{A} \cup \hat{B}$. This means that $L_- \cap |K_i| = L$.

Note that $M := |K_i| \setminus L = |K_{i+1}| \cup f|_{|K_i|}^{-1}(f(|K_{i+1}|))$ is an upward closed subpolyhedron of the upward closed polyhedron $|K_{i+1}|$ and $P \times 1 \subset M$. Thus $|K_i|$ collapses to $M$ cylinderwise by Theorem 3.7.

We have $|K_i| \setminus M = L$ and $f(M) = f(|K_{i+1}|)$ by the definition. Since $L \subset A$ and $f|_A$ is a homeomorphism, the collapsing $|K_i| \setminus M$ induces the collapsing $f(|K_i|) \setminus f(|K_{i+1}|)$ in the image of $f$.

\[ \square \]

This completes the proof that the existence of a piecewise linear free deformation retraction implies the collapsibility. Now let us prove the reverse implication.

Lemma 4.10. Let $P$ be a polyhedron, let $Q \subset P$ be a subpolyhedron of $P$ and let $S \subset Q$ be a subpolyhedron of $Q$. Also let $f : P \times I \to P$ be a piecewise linear free deformation.
retraction of $P$ onto $Q$ and let $g : Q \times I \rightarrow Q$ be a piecewise linear free deformation retraction of $Q$ onto $S$.

Then there exists a piecewise linear free deformation retraction $h : P \times I \rightarrow P$ of $P$ onto $S$.

**Proof.** Let $h : P \times I \rightarrow P : (x, t) \mapsto \begin{cases} f(x, 2t), & t \leq \frac{1}{2}; \\ g(f(x, 1), 2t - 1), & t \geq \frac{1}{2}. \end{cases}$

Obviously $h$ is piecewise linear. Since $f|_Q = \text{pr}_Q$ it is also clear that $h$ is a strong deformation retraction. It remains to check the freeness condition, that is $h_s \circ h_t = h_{\text{max}(s,t)}$. For the cases $s, t \leq \frac{1}{2}$ and $s, t \geq \frac{1}{2}$ this follows directly from the freeness of $f$ and $g$. Let $s \leq \frac{1}{2} \leq t$. Then we have

\[
\begin{align*}
h_s(h_t(x)) &= h(h(x,t), s) = f(g(f(x, 1), 2t - 1), 2s) = g(f(x, 1), 2t - 1) = h_t(x) \\
h_t(h_s(x)) &= h(h(x,s), t) = g(f(x, 2s), 1), 2t - 1) = g(f(x, 1), 2t - 1) = h_t(x)
\end{align*}
\]

Thus it is sufficient to prove that if there is an elementary collapse $P^{B \setminus x_A}$ $Q$ then there exists a piecewise linear free deformation retraction of $P$ onto $Q$.

**Lemma 4.11.** Let $P$ be a polyhedron, let $Q \subset P$ be a subpolyhedron.

Then if there is an elementary collapse $P^{B \setminus x_A}$ $Q$ then there exists a piecewise linear free deformation retraction of $P$ onto $Q$.

**Proof.** Let us construct such a retraction $h : P \times I \rightarrow P$.

As $h$ should be a strong deformation retraction, we clearly put $h|_{Q \times I} = \text{id}_{Q \times I}$, so we should define $h$ only on $P \setminus Q = \hat{A} \cup \hat{B}$.

Denote by $a$ the vertex of $A$ opposite to the face $B$. Assume that $A$ is embedded in an Euclidean space $\mathbb{R}^n$ of dimension $\dim A$ in such a way that $a$ goes to the point $(1, 0, 0, \ldots, 0)$ and the face $B$ lies in the affine hyperplane $x_1 = 0$. Then the cylinder $A \times I$ is naturally embedded in $\mathbb{R}^{n+1}$. Denote by $A_0$ and $A_1$ the simplices $A \times 0$ and $A \times 1$ in $\mathbb{R}^{n+1}$ respectively. Let $V$ be a hyperplane through the point $a_1$ and the simplex $B_0$.

Clearly, $V$ cuts the cylinder $A \times I$ into two parts: the first («lower») one contains the simplex $A \times 0$ and the second («upper») one contains the simplex $A \times 1$. Define $h$ on the first part as the projection $\text{pr}_A$ and on the second part as the composition $\text{pr}_A \circ \pi$ where $\pi$ is the projection parallel to the $x_1$ axis.

Thus, $h$ is a piecewise linear strong deformation retraction of $P$ onto $Q$ by definition, and it is clear that the freeness condition holds for $h$. \qed

Therefore, we complete the proof of the reverse implication and hence Theorem 4.1.
Figure 4. Left figure: the retraction $h$ for two-dimensional $A$; right figure: the projection $\text{pr}_A \circ \pi$ onto the boundary of the second part of $A \times I$ on the section $A \times t$, the image of projection is red.

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