Application of a coordinate space method for the evaluation of lattice Feynman diagrams in two dimensions

Dong-Shin Shin

Max-Planck-Institut für Physik
– Werner-Heisenberg-Institut –
Föhringer Ring 6, 80805 Munich, Germany

Abstract

We apply a new coordinate space method for the evaluation of lattice Feynman diagrams suggested by Lüscher and Weisz to field theories in two dimensions. Our work is to be presented for the theories with massless propagators. The main idea is to deal with the integrals in position space by making use of the recursion relation for the free propagator $G(x)$ which allows to compute the propagator recursively by its values around origin. It turns out that the method is very efficient and gives very precise results. We illustrate the technique by evaluating a number of two- and three-loop diagrams explicitly.
1 Introduction

It is generally believed that QCD describes the strong interaction of quarks and gluons. QCD is, however, very complicated to investigate. Therefore, it is useful to study at first similar, but simpler models and apply the related knowledge gained to the real situation. The most frequently studied toy models for this 3+1 dimensional theory are the 1+1 dimensional models, e.g. the non-linear O($n$) $\sigma$ model in two dimensions, which have also found wide applications in the framework of the lattice field theory. They have not only the simple theoretical structure, but also are, due to low dimensionality, simple to simulate numerically. Another high interest in these two-dimensional models is to be found in perturbation theory which plays an important conceptual and practical role in lattice field theory. However, the evaluation of lattice Feynman diagrams is, in general, not easy. On the lattice the integrands become non-trivial function of internal and external momenta, which makes it difficult to apply the standard tools in continuum perturbation theory.

A few years ago, Lüscher and Weisz \[1\] suggested a new position space method for the evaluation of lattice Feynman diagrams in four dimensional Yang-Mills theories, inspired by the observation of Vohwinkel \[2\] that the free lattice propagator can be calculated recursively by its values around origin. Their method concerns a technique of computing Feynman integrals in the coordinate space which allows to obtain very accurate results with little effort and small amount of computer time. In our work on the computation of four-loop $\beta$ function in the two-dimensional non-linear O($n$) $\sigma$ model \[3\], we have made extensive use of this technique. Since we have found it very useful, we would like, in the present paper, to describe the method in the case of two dimensions. This will be done by choosing some typical diagrams and evaluating them explicitly.

To illustrate the basic ideas to be discussed in the present paper, we take the following 3-loop integral into consideration which will be treated in detail in the main text:\[4\]

\[ B_1 = \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2s}{(2\pi)^2} \frac{\sum_{\mu=0}^{1} \hat{q}_\mu \hat{r}_\mu \hat{s}_\mu \hat{t}_\mu}{\hat{q}^2 \hat{r}^2 \hat{s}^2 t^2}, \quad t = -q - r - s, \quad (1.1) \]

where

\[ \hat{p}_\mu = 2 \sin \frac{p_\mu}{2}, \quad \hat{p}^2 = \sum_{\mu=0}^{1} \hat{p}_\mu^2 \quad (1.2) \]

and $p = (p_0, p_1)$ denotes the line momentum.

There are a few known conventional computer programs available to compute this integral. The numerical accuracy which can be achieved by these methods is, however, strongly limited and thus they are not very useful if one requires to have results with high precision. Considering the integral in position space, however, provides us with a new possibility to compute it very accurately with high efficiency.

\[ ^1 \text{Throughout the whole paper, we will work on a two-dimensional square lattice } \Lambda \text{ and keep the lattice constant } a \text{ being equal to 1.} \]
In position space the integral (1.1) has the form

\[ B_1 = \sum_{x \in \Lambda} \sum_{\mu=0}^{1} \left[ \partial_\mu G(x) \right]^4, \]  

where \( \partial_\mu \) denote the lattice derivatives and \( G(x) \) is the position space free massless propagator

\[ G(x) = \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{e^{ipx} - 1}{\hat{p}^2}. \]  

The free propagator \( G(x) \) diverges logarithmically at large \( x \). After differentiation, \( \partial_\mu G(x) \) falls off like \( |x|^{-1} \) and the sum (1.3) is therefore absolutely convergent. In principle, one can then compute \( B_1 \) by summing over all points of \( x \) up to some finite value, e.g. \( |x| \leq 30 \). However, since the convergence is slow, one requires to sum over large range of \( x \) if one would like to have precise result. Taking the limited computing availability into account, this is, of course, not the most efficient way. By studying the large \( x \) behavior of the propagator \( G(x) \), it is however possible to improve on the accuracy systematically until the desired level of precision is reached. It is the aim of the present paper to show how this can be done.

In next section we discuss the properties of the free propagator \( G(x) \) and, in particular, derive a recursion relation which allows us to express the propagator as a linear combination of its values near origin. These results will then be applied to the computation of lattice sums in section 3 where we take various examples to illustrate the procedure. It turns out that the method is quite efficient and gives very accurate results. Finally, in section 4 we present a little more subtle case of evaluating Feynman diagrams with non-zero external momenta, particularly in the continuum limit. The usefulness of the position space method in this case as well is demonstrated by dealing with a number of diagrams explicitly. We conclude the paper by including some detailed computations in a series of appendices to make it more readable.

\section{Investigation of the free propagator and zeta functions}

In this section we prepare for the main work in the next two sections where we show how the position space technique can be applied to the evaluation of Feynman diagrams. This requires the study of large \( x \) behavior of the free propagator. We also need to derive the recursion relation for it which is essential to the computation of the integrals.
2.1 Properties of the propagator $G(x)$

The propagator $G(x)$ is a Green function for $\triangle$, i.e. it satisfies the laplace equation

$$-\triangle G(x) = \delta^{(2)}(x),$$

(2.1)

where $\delta^{(2)}(x)$ is the lattice $\delta$-function and $\triangle$ denotes the lattice laplacian

$$\triangle = \sum_{\mu=0}^{1} \partial^*_{\mu} \partial_{\mu}. \quad (2.2)$$

Here, $\partial_{\mu}$ and $\partial^*_{\mu}$ are the forward and backward lattice derivatives respectively:

$$\partial_{\mu} f(x) = f(x + \hat{\mu}) - f(x), \quad (2.3)$$

$$\partial^*_{\mu} f(x) = f(x) - f(x - \hat{\mu}). \quad (2.4)$$

We note the relation $\partial^*_{\mu} = -\partial^\dagger_{\mu}$ for the backward lattice derivative, where $\partial^\dagger_{\mu}$ is the adjoint of the forward lattice derivative defined with respect to the inner product $(\partial^\dagger_{\mu} f, g) = (f, \partial_{\mu} g)$.

A rigorous derivation of the large $x$ behavior of $G(x)$ shows that in the limit $x \to \infty$ the propagator diverges logarithmically and has the form

$$G^c(x) = -\frac{1}{4\pi}(\ln x^2 + 2\gamma + 3 \ln 2) \quad (2.5)$$

which corresponds to the leading behavior in the continuum limit. By making use of the laplace equation (2.1), one can also work out the subleading terms systematically to obtain the asymptotic expansion. Up to order $|x|^{-6}$, we find

$$G(x) = G^c(x) - \frac{1}{4\pi} \left[ \frac{1}{(x^2)^3} Q_1 + \frac{1}{(x^2)^6} Q_2 + \frac{1}{(x^2)^9} Q_3 + \cdots \right], \quad (2.6)$$

where

$$Q_1 = \frac{1}{2}(x^2)^2 - \frac{2}{3} x^4, \quad (2.7)$$

$$Q_2 = -\frac{371}{120} (x^2)^4 + \frac{47}{5} x^4 (x^2)^2 - \frac{20}{3} (x^4)^2, \quad (2.8)$$

$$Q_3 = \frac{4523}{56} (x^2)^6 - \frac{7657}{21} x^4 (x^2)^4 + \frac{3716}{7} (x^4)^2 (x^2)^2 - \frac{2240}{9} (x^4)^3. \quad (2.9)$$

In the above expansions we have introduced the shorthand notation

$$x^n = \sum_{\mu=0}^{1} (x^\mu)^n. \quad (2.10)$$

We remark that in two dimensions the general relation

$$x^{2n+2} = x^2 x^{2n} - \frac{1}{2} x^{2n-2} [(x^2)^2 - x^4], \quad n \geq 1 \quad (2.11)$$
is valid so that the expansion terms can always be expressed by the powers \(x^2\) and \(x^4\) since higher powers are to be reduced to these two powers (note \(x^0 = 2\)). For example, from Eq. (2.11) we find the identities

\[
x^6 = \frac{1}{2} [3x^2x^4 - (x^2)^3], \quad (2.12)
\]
\[
x^8 = \frac{1}{2} [(x^4)^2 + 2(x^2)^2x^4 - (x^2)^4], \quad (2.13)
\]
\[
x^{10} = \frac{1}{4} x^2 [5(x^4)^2 - (x^2)^4]. \quad (2.14)
\]

The detailed discussion of the asymptotic behavior of \(G(x)\) is presented in Appendix \[A\].

Now we would like to derive a recursion formula for the free propagator \(G(x)\) which makes it possible to determine \(G(x)\) for large \(x\) by its initial values around origin. The key relation for obtaining the formula is the identity, first observed by Vohwinkel [2],

\[
G(x + \hat{\mu}) - G(x - \hat{\mu}) = x_\mu H(x), \quad (2.15)
\]

where \(H(x)\) is independent of \(\mu\) and has the form

\[
H(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ipx} \ln \hat{p}^2. \quad (2.16)
\]

By summing over \(\mu\) and using Eq. (2.1) for \(x \neq 0\), one can eliminate \(G(x + \hat{\mu})\) in Eq. (2.13) and express \(H(x)\) like

\[
H(x) = \frac{2}{\sum_{\mu=0}^{1} x_\mu} \left[2G(x) - \sum_{\mu=0}^{1} G(x - \hat{\mu})\right]. \quad (2.17)
\]

If one inserts this formula into Eq. (2.13), one finally obtains the recursion relation

\[
G(x + \hat{\mu}) = \frac{2x_\mu}{\sum_{\mu=0}^{1} x_\mu} \left[2G(x) - \sum_{\mu=0}^{1} G(x - \hat{\mu})\right] + G(x - \hat{\mu}) \quad (2.18)
\]

which is valid for \(x \neq 0\).

Since the propagator has the properties

\[
G(x_0, x) = G(x_1, x_0), \quad (2.19)
\]
\[
G(x_0, x_1) = G(-x_0, x_1), \quad (2.20)
\]

\(G(x)\) is completely characterized by \(x_0 \geq x_1 \geq 0\). In this sector, Eq. (2.18) is a recursion formula which allows us to express \(G(x)\) as a linear combination of three initial values

\[
G(0, 0) = 0, \quad (2.21)
\]
\[
G(1, 0) = -\frac{1}{4}, \quad (2.22)
\]
\[
G(1, 1) = -\frac{1}{\pi}, \quad (2.23)
\]
i.e. the values of the propagator at the corners of the unit square. Therefore, the propagator has the general form

\[ G(x) = -\frac{1}{4}r_1(x) - \frac{1}{\pi}r_2(x), \quad (2.24) \]

where \( r_1(x) \) and \( r_2(x) \) are rational numbers which are to be computed recursively by Eq. (2.18).

In this place we would like to stress the fact that the analytic knowledge of the initial values (2.21)–(2.23) is crucial to the precise determination of the propagator \( G(x) \) for large \( x \). The reason for this is that \( r_1(x) \) and \( r_2(x) \) increase exponentially with \( x \) while \( G(x) \) grows only logarithmically and thus there is a large cancellation of terms on the r.h.s. of Eq. (2.24).

### 2.2 Properties of the function \( G_2(x) \)

Another function which often appears in the procedure of evaluating diagrams is

\[ G_2(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ipx} - 1 + \frac{1}{\pi} \sum_{\mu=0}^1 \hat{p}_\mu^2 \mu^2 \frac{x^2}{(\hat{p}^2)^2}. \quad (2.25) \]

The laplacian of this function satisfies

\[ \Delta G_2(x) = -G(x). \quad (2.26) \]

One can also compute the function \( G_2(x) \) analytically in the limit of large \( x \) and expand it systematically by using the expansion (2.6) and the laplace equation (2.26). Up to order \( |x|^{-4} \), we find in this way

\[ G_2(x) = \frac{1}{16\pi} \left[ x^2 R_1 + R_2 + \frac{1}{(x^2)^5} R_3 + \frac{1}{(x^2)^8} R_4 + \cdots \right], \quad (2.27) \]

where

\[
\begin{align*}
R_1 & = \ln x^2 + 2\gamma + 3 \ln 2 - 2, \\
R_2 & = -\frac{1}{2} \ln x^2 + \frac{1}{3} \frac{x^4}{(x^2)^2} - \frac{1}{2} (2\gamma + 3 \ln 2), \\
R_3 & = \frac{19}{120} (x^2)^4 - \frac{19}{15} x^4 (x^2)^2 + \frac{4}{3} (x^4)^2, \\
R_4 & = -\frac{10183}{1680} (x^2)^6 + \frac{14041}{420} x^4 (x^2)^4 - \frac{1214}{21} (x^4)^2 (x^2)^2 + \frac{280}{9} (x^4)^3.
\end{align*}
\]

For the explicit derivation of the large \( x \) expansion we refer to Appendix B.

Similarly to the propagator \( G(x) \), one can also derive a recursion relation for \( G_2(x) \). The first step for the derivation is the observation

\[ G_2(x + \hat{\mu}) - G_2(x - \hat{\mu}) = -x_\mu H_2(x), \quad (2.32) \]
where $H_2(x)$ is independent of $\mu$ and has the form

$$H_2(x) = G(x) + \frac{1}{4\pi}.$$  \hfill (2.33)

Eliminating $G_2(x + \hat{\mu})$ in Eq. (2.32) by using Eq. (2.26) gives the formula

$$H_2(x) = \frac{1}{\sum_{\mu=0}^{1} x_{\mu}} \left[ G(x) - 4G_2(x) + 2 \sum_{\mu=0}^{1} G_2(x - \hat{\mu}) \right]$$ \hfill (2.34)

which leads straightforwardly to the recursion relation

$$G_2(x + \hat{\mu}) = -\frac{x_{\mu}}{\sum_{\mu=0}^{1} x_{\mu}} \left[ G(x) - 4G_2(x) + 2 \sum_{\mu=0}^{1} G_2(x - \hat{\mu}) \right] + G_2(x - \hat{\mu})$$ \hfill (2.35)

for $x \neq 0$.

Like the propagator $G(x)$, due to the properties

$$G_2(x_0, x_1) = G_2(x_1, x_0),$$ \hfill (2.36)

$$G_2(x_0, x_1) = G_2(-x_0, x_1),$$ \hfill (2.37)

$G_2(x)$ is completely characterized by $x_0 \geq x_1 \geq 0$. In this sector, the recursion relation (2.33) allows $G_2(x)$ to be expressed in terms of $G(x)$ and a linear combination of its values at the corners of the unit square,

$$G_2(0, 0), \ G_2(1, 0), \ G_2(1, 1).$$ \hfill (2.38)

These initial values can also be computed analytically with the results

$$G_2(0, 0) = 0,$$ \hfill (2.39)

$$G_2(1, 0) = 0,$$ \hfill (2.40)

$$G_2(1, 1) = \frac{1}{8\pi}.$$ \hfill (2.41)

### 2.3 Zeta functions $Z(s, h)$

In the procedure of evaluating the lattice sum like Eq. (1.3), zeta functions appear. As a preparation for the computation of such lattice sums in next section, we discuss the evaluation of the zeta functions $Z(s, h)$ associated with the square lattice $\Lambda$.

Let us define a generalized zeta function through

$$Z(s, h) = \sum_{x \in \Lambda} h(x)(x^2)^{-s},$$ \hfill (2.42)

where $h(x)$ is a harmonic homogeneous polynomial in the real variables $x = (x_0, x_1)$ with even degree $d$ and $s$ complex number with $\text{Re} \ s > 1 + d/2$. The primed
summation symbol implies that the point \( x = 0 \) should be omitted in the summation. Note that the sum (2.42) is absolutely convergent in the specified range of \( s \).

The zeta functions can be evaluated as follows. We first introduce the heat kernel

\[
 k(t, h) = \sum_{x \in \Lambda} h(x) e^{-\pi tx^2}.
\]

(2.43)

In terms of the heat kernel, Eq. (2.42) can be rewritten in the form

\[
 Z(s, h) = \pi^s \frac{\Gamma(s)}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} [k(t, h) - h(0)].
\]

(2.44)

Now we observe that the heat kernel has the property

\[
 k(t, h) = (-1)^{d/2} t^{-d-1} k(1/t, h)
\]

(2.45)

which follows from the poisson summation formula

\[
 \sum_{x \in \Lambda} e^{-iqx} e^{-\pi tx^2} = t^{-1} \sum_{x \in \Lambda} e^{-(q+2\pi x)^2/(4\pi t)}
\]

(2.46)

and the condition that \( h(x) \) is a harmonic function.

If we divide the integral (2.44) into two parts and make use of the identity (2.45), we finally obtain the representation

\[
 Z(s, h) = \pi^s \frac{\Gamma(s)}{\Gamma(s)} \left\{ \frac{h(0)}{s(s-1)} + \int_1^\infty dt \left[ t^{s-1} + (-1)^{d/2} t^{d-s} \right] [k(t, h) - h(0)] \right\}.
\]

(2.47)

In the integration range \( 1 \leq t < \infty \), the large \( x \) terms in the series (2.43) are exponentially suppressed. Eq. (2.47) can therefore be computed numerically very precisely with little effort, e.g. by using symbolic language MATHEMATICA.

### 3 Evaluation of lattice sums

By dealing with a number of integrals explicitly, we would now like to show how the results of the last section are to be applied to the evaluation of lattice Feynman diagrams. Concretely, we will consider in the following the two- and three-loop diagrams of figure ❄.

#### 3.1 Two-loop Feynman integrals

As a first example let us consider the integral, encountered in the 3-loop \( \beta \) function computation in the \( O(n) \) \( \sigma \) model,

\[
 A_1 = \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \frac{d^2 l}{(2\pi)^2} \frac{1}{k^2 l^2 s^2} \sum_{\mu=0}^{1} \hat{k}_\mu \hat{l}_\mu s_\mu \cos \frac{s_\mu}{2}, \quad s = k + l
\]

(3.1)
which corresponds to the first diagram in figure [1].

In position space this integral can be written as

\[ A_1 = \frac{1}{4} \sum_{x \in A} \tilde{A}_1(x), \]  

where

\[ \tilde{A}_1(x) = \sum_{\mu=0}^{1} (\partial_\mu - \partial_\mu^*) G(x) \left\{ \left[ (\partial_\mu - \partial_\mu^*) G(x) \right]^2 + \left[ (\partial_\mu + \partial_\mu^*) G(x) \right]^2 \right\}. \]  

\[ G(x) \text{ diverges logarithmically at large } x \text{ and an explicit computation of large } x \text{ behavior of Eq. (3.3) shows that } \tilde{A}_1(x) \text{ falls off like } |x|^{-4}. \]  

The sum (3.2) is therefore absolutely convergent and a first approximation to \( A_1 \) may be obtained by summing over all points \( x \) with (say) \( |x| \leq 30 \). A direct summation in this way gives a result as a first approximation

\[ A_1 = -0.04166(1). \]  

We can now improve on the accuracy by investigating the asymptotic behavior of \( \tilde{A}_1(x) \). From the asymptotic form of \( G(x) \) [Eq. (2.6)], we deduce that

\[ \tilde{A}_1(x) = \left\{ \tilde{A}_1(x) \right\}_\infty + \mathcal{O}(|x|^{-8}), \]  

where

\[ \left\{ \tilde{A}_1(x) \right\}_\infty = -\frac{1}{2\pi^3} \left\{ \frac{1}{(x^2)^2} - 2 \frac{x^4}{(x^2)^4} - \frac{10}{(x^2)^3} + \frac{32}{(x^2)^5} - \frac{24}{(x^2)^7} \right\}. \]  

Taking the asymptotic expansion of Eq. (3.6) into consideration, \( A_1 \) can be rewritten as

\[ A_1 = \frac{1}{4} \left[ A_1' + A_1'' + \tilde{A}_1(0) \right], \]  

where

\[ A_1' = \sum_{x \in A} \left[ \tilde{A}_1(x) - \left\{ \tilde{A}_1(x) \right\}_\infty \right], \]  

\[ A_1'' = \sum_{x \in A} \left\{ \tilde{A}_1(x) \right\}_\infty, \]  

\[ \tilde{A}_1(0) = -\frac{1}{4}. \]
The sum \( A'_1 \) where the asymptotic series has been subtracted converges much more rapidly than the original sum \( A_1 \). This subtracted sum can therefore be computed very accurately by summing over all points \( x \) with \(|x| \leq 30\). As for the term \( A''_1 \) from the asymptotic expansion, it can be computed with help of zeta functions very accurately as well. For the purpose of applying the zeta functions, we transform the asymptotic expansion \((3.6)\) in the following way:

\[
\left\{ \tilde{A}_1(x) \right\}_\infty = \frac{1}{(2\pi)^3} \left\{ \left[ \frac{2}{(x^2)^2} + \frac{1}{(x^2)^3} \right] h_0(x) + \left[ \frac{2}{(x^2)^4} + \frac{4}{(x^2)^5} \right] h_1(x) + \frac{3}{(x^2)^7} h_2(x) \right\}.
\]

(3.11)

Here, \( h_0(x) \), \( h_1(x) \) and \( h_2(x) \) are the harmonic homogeneous polynomials given by

\[
h_0(x) = 1, \quad h_1(x) = 4x^4 - 3(x^2)^2, \quad h_2(x) = 32(x^4)^2 - 48(x^2)^2x^4 + 17(x^2)^4.
\]

(3.12)-(3.14)

In Appendix C we discuss how one computes the harmonic polynomials. The subtraction term \( A''_1 \) can then be written as a sum of zeta functions:

\[
A''_1 = \frac{1}{(2\pi)^3} \left\{ 2Z(2, h_0) + Z(3, h_0) + 2Z(4, h_1) + 4Z(5, h_1) + 3Z(7, h_2) \right\}
\]

(3.15)

to which now the solution \((2.47)\) can be applied.

Summing all three contributions together, we get finally

\[
A_1 = -0.041666666666(1).
\]

(3.16)

It seems very likely that \( A_1 \) is \(-\frac{1}{24}\). Indeed, the analytic computation of \( A_1 \) confirms this hypothesis (see Appendix D). We note that it is straightforward to determine \( A_1 \) more precisely just by including more terms in the large \( x \) expansion of \((3.6)\).

It is interesting to compare our determination with the earlier estimation by Weisz \[4\] who states the value

\[
A_1 = -0.0416666(2).
\]

(3.17)

We clearly see that the result from our method is much more precise than that of Weisz.

As a second example, we consider the following integral which also appears in the 3-loop \( \beta \) function computation (see figure 1)

\[
A_2 = \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \frac{\hat{s}^2 - \hat{k}^2 - \hat{l}^2 - \hat{s}^4}{\hat{k}^2\hat{l}^2 (\hat{s}^2)^2}, \quad s = k + l.
\]

(3.18)

Here, we made use of the notation of Eq. (2.10).
The evaluation of this integral proceeds in exactly the same way as that of $A_1$. In terms of the propagator $G(x)$ and the function $G_2(x)$ [Eq. (2.25)], in the position space the integral can be expressed by

$$A_2 = -\frac{1}{2} \sum_{x \in \Lambda} \tilde{A}_2(x), \quad (3.19)$$

where

$$\tilde{A}_2(x) = \left\{ \sum_{\mu=0}^1 \partial^*_\mu \partial_\mu \partial^*_\mu \partial_\mu G_2(x) \right\} \left\{ \sum_{\mu=0}^1 \left[ \left( (\partial_\mu - \partial^*_\mu) G(x) \right)^2 + \left( (\partial_\mu + \partial^*_\mu) G(x) \right)^2 \right] \right\}. \quad (3.20)$$

The terms in this sum fall off like $|x|^{-4}$ and a direct evaluation without subtraction gives

$$A_2 = -0.18465(1). \quad (3.21)$$

The convergence can be improved by subtracting the asymptotic form of $\tilde{A}_2(x)$ which can be deduced from the asymptotic forms of $G(x)$ and $G_2(x)$. Up to order $|x|^{-6}$, we find for the large $x$ expansion of $\tilde{A}_2(x)$

$$\tilde{A}_2(x) = \{ \tilde{A}_2(x) \}_{\infty} + \mathcal{O}(|x|^{-8}), \quad (3.22)$$

where

$$\{ \tilde{A}_2(x) \}_{\infty} = \frac{1}{16\pi^3} \left\{ \frac{5}{(x^2)^3} h_0(x) + \left[ \frac{8}{(x^2)^4} + \frac{2}{(x^2)^5} \right] h_1(x) + \frac{25}{(x^2)^7} h_2(x) \right\}. \quad (3.23)$$

The subtracted sum

$$A'_2 = \sum_{x \in \Lambda} \left[ \tilde{A}_2(x) - \{ \tilde{A}_2(x) \}_{\infty} \right] \quad (3.24)$$

is now rapidly convergent and may therefore be computed accurately. On the other hand, the subtraction term

$$A''_2 = \sum_{x \in \Lambda} \{ \tilde{A}_2(x) \}_{\infty} \quad (3.25)$$

can be expressed as a sum of zeta functions:

$$A''_2 = \frac{1}{16\pi^4} \left\{ 5Z(3, h_0) + 8Z(4, h_1) + 2Z(5, h_1) + 25Z(7, h_2) \right\} \quad (3.26)$$

which can be computed very accurately, too.

Since

$$A_2 = -\frac{1}{2} \left[ A'_2 + A''_2 + \tilde{A}_2(0) \right], \quad (3.27)$$

we obtain finally

$$A_2 = -0.1846545169(1). \quad (3.28)$$

This result is also to be compared with the earlier estimation by Weisz [4] who states

$$A_2 = -0.1846544(1). \quad (3.29)$$
3.2 Three-loop Feynman integrals

Now we come to the 3-loop integral of Eq. (1.1) which we have already discussed in Introduction (see figure [1]). The summand of $B_1$ from Eq. (1.3)

$$\tilde{B}_1(x) = \sum_{\mu=0}^{1} \left[ \partial_\mu G(x) \right]^4$$  \hspace{1cm} (3.30)

falls off like $|x|^{-4}$ at large $x$ and a direct summation without subtraction gives

$$B_1 = 0.0169611(1).$$  \hspace{1cm} (3.31)

For the acceleration of the convergence, we expand Eq. (3.30) for large $x$. Up to order $|x|^{-6}$, we find

$$\tilde{B}_1(x) = \{ \tilde{B}_1(x) \}_\infty + \mathcal{O}(|x|^{-8}),$$  \hspace{1cm} (3.32)

where

$$\{ \tilde{B}_1(x) \}_\infty = \frac{1}{32\pi^4} \left\{ 2 \frac{x^4}{(x^2)^4} - \frac{3}{(x^2)^3} - 6 \frac{x^4}{(x^2)^5} + 16 \frac{(x^4)^2}{(x^2)^7} \right\}.$$  \hspace{1cm} (3.33)

From the identity

$$B_1 = B'_1 + B''_1 + \tilde{B}_1(0)$$  \hspace{1cm} (3.34)

with

$$B'_1 = \sum_{x \in A} ' \left[ \tilde{B}_1(x) - \{ \tilde{B}_1(x) \}_\infty \right],$$  \hspace{1cm} (3.35)

$$B''_1 = \sum_{x \in A} ' \{ \tilde{B}_1(x) \}_\infty,$$  \hspace{1cm} (3.36)

$$\tilde{B}_1(0) = \frac{1}{128}$$  \hspace{1cm} (3.37)

and the zeta function representation

$$B''_1 = \frac{1}{64\pi^4} \left\{ 3Z(2, h_0) + 4Z(3, h_0) + Z(4, h_1) + 9Z(5, h_1) + Z(7, h_2) \right\},$$  \hspace{1cm} (3.38)

it is straightforward to compute $B_1$ to get a more precise result

$$B_1 = 0.016961078576(1).$$  \hspace{1cm} (3.39)

We remark that this integral has been computed by Caracciolo and Pelissetto [4] who state the value

$$B_1 = 0.016961.$$  \hspace{1cm} (3.40)

As another example, we consider the 3-loop integral of the form

$$B_2 = \int_{-\pi}^{\pi} d^2 q \int_{-\pi}^{\pi} d^2 r \int_{-\pi}^{\pi} d^2 s \left( \sum_{\mu=0}^{1} \hat{q}_\mu \hat{r}_\mu \hat{s}_\mu \hat{t}_\mu \right)^2, \quad t = -q - r - s.$$  \hspace{1cm} (3.41)
In terms of the position space propagator $G(x)$, the integral can be expressed by

$$B_2 = \sum_{x \in \Lambda} \tilde{B}_2(x), \quad (3.42)$$

where

$$\tilde{B}_2(x) = \sum_{\mu=0}^{1} \sum_{\nu=0}^{1} \left[ \partial_\mu \partial_\nu G(x) \right]^4. \quad (3.43)$$

The terms in this sum fall off like $|x|^{-8}$ and converge very well. Therefore, the direct summation already gives a quite precise result:

$$B_2 = 0.13661977236(1). \quad (3.44)$$

Through exactly the same procedure as $B_1$, we obtain the following identity for $B_2$ which should give more accurate result:

$$B_2 = \sum_{x \in \Lambda} \left[ \tilde{B}_2(x) - \{ \tilde{B}_2(x) \}_\infty \right] + \frac{1}{32\pi^4} \left\{ 3Z(4, h_0) + Z(8, h_2) \right\} + \tilde{B}_2(0) \quad (3.45)$$

with

$$\{ \tilde{B}_2(x) \}_\infty = \frac{1}{8\pi^4} \left\{ \frac{5}{(x^2)^4} - 12 \frac{x^4}{(x^2)^6} + 8 \frac{(x^4)^2}{(x^2)^8} \right\}, \quad (3.46)$$

$$\tilde{B}_2(0) = \frac{34}{\pi^4} - \frac{36}{\pi^3} + \frac{15}{\pi^2} - \frac{3}{\pi} + \frac{1}{4}. \quad (3.47)$$

The numerical evaluation of Eq. (3.45) then yields a more precise value

$$B_2 = 0.136619772367581(1). \quad (3.48)$$

This number can also be compared with that by Caracciolo and Pelissetto [5]:

$$B_2 = 0.1366198. \quad (3.49)$$

Finally, we consider the 3-loop integral of the form

$$B_3 = \int_{\pi}^{\pi} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2s}{(2\pi)^2} \frac{[(\hat{q} + \hat{r})^2 - \hat{q}^2 - \hat{r}^2] [\hat{s} + \hat{t}]^2 - s^2 - t^2}{\hat{q}^2 \hat{r}^2 \hat{s}^2 \hat{t}^2}, \quad (3.50)$$

$$t = -q - r - s. \quad (3.50)$$

The position space representation of this integral is given by

$$B_3 = \frac{1}{4} \sum_{x \in \Lambda} \tilde{B}_3(x), \quad (3.51)$$

where

$$\tilde{B}_3(x) = \left\{ \sum_{\mu=0}^{1} \left[ (\partial_\mu + \partial^*_\mu) G(x) \right]^2 + \left[ (\partial_\mu + \partial^*_\mu) G(x) \right]^2 \right\}^2. \quad (3.52)$$
The evaluation of the sum (3.51) runs in exactly the same way as above and we get the result
\[ B_3 = 0.09588764425(1), \]
where we have evaluated the sum by subtracting the asymptotic terms from the summand in Eq. (3.51). Finally, this result is to be compared with that in Ref. [5]:
\[ B_3 = 0.095887. \]  

4 Evaluation of diagrams with external momenta

In this section, we consider the Feynman diagrams in figure 2 with external momenta \( p \). We would like to evaluate the integrals in the continuum limit which corresponds to taking \( p \) to zero since we are using lattice units and there are no mass parameters. In other words, we have to work out the leading terms of the integrals in an asymptotic expansion for \( p \to 0 \).

\[ \text{Figure 2: One-, two- and three-loop diagrams with non-zero external momenta} \]

For the evaluation of diagrams with external momenta \( p \), it is necessary to solve the \( p \)-dependent zeta functions. As a preparation for the actual treatment of such diagrams, we therefore discuss at first the zeta functions which are depending on the external momenta.

4.1 Zeta functions \( \mathcal{Z}(p, s) \)

Let us consider the \( p \)-dependent zeta functions of the kind \( (s \geq 1) \)
\[
\mathcal{Z}(p, s) = \sum_{x \in \Lambda} \frac{e^{-ipx}}{(x^2)^s}.
\]

We rewrite Eq. (4.1) in the form
\[
\mathcal{Z}(p, s) = \frac{\pi^s}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \left( \sum_{x \in \Lambda} e^{-ipx} e^{-\pi tx^2} - 1 \right).
\]
We are interested in the evaluation of this integral in the limit \( p \to 0 \). For this purpose, we consider the integral by dividing the integration range into two parts

\[
Z(p, s) = Z'(p, s) + Z''(p, s),
\]

where

\[
Z'(p, s) = \frac{\pi^s}{\Gamma(s)} \int_1^\infty dt \, t^{s-1} \left( \sum_{x \in \Lambda} e^{-ipx} e^{-\pi tx^2} - 1 \right), \quad (4.4)
\]

\[
Z''(p, s) = \frac{\pi^s}{\Gamma(s)} \int_0^1 dt \, t^{s-1} \left( \sum_{x \in \Lambda} e^{-ipx} e^{-\pi tx^2} - 1 \right). \quad (4.5)
\]

Now, note that \( Z'(p, s) \) is well-defined for all \( s \geq 1 \). We can therefore set \( p \) to zero in that function. Then, \( Z'(0, s) \) can be evaluated numerically very accurately since the large \( x \) terms in the series of the function are exponentially suppressed.

On the other hand, \( Z''(p, s) \) contains singular term for small \( p \). For the evaluation of this function, we first apply the Poisson summation formula [Eq. (2.46)] to rewrite

\[
Z''(p, s) = \frac{\pi^s}{\Gamma(s)} \int_0^1 dt \, t^{s-1} \left[ t^{-1} \sum_{x \in \Lambda} e^{-(p+2\pi x)^2/(4\pi t)} - 1 \right]. \quad (4.6)
\]

In the series of the integrals appearing in Eq. (4.6), the only dangerous term for \( p \to 0 \) is that with \( x = 0 \). Hence, we split the integral \( Z''(p, s) \) into the regular and singular parts

\[
Z''(p, s) = Z''_r(p, s) + Z''_s(p, s),
\]

where

\[
Z''_r(p, s) = \frac{\pi^s}{\Gamma(s)} \sum_{x \in \Lambda} \int_1^\infty dt \, t^{-1} e^{-(p+2\pi x)^2/(4\pi t)}, \quad (4.8)
\]

\[
Z''_s(p, s) = \frac{\pi^s}{\Gamma(s)} \int_0^1 dt \, t^{s-1} \left[ t^{-1} e^{-p^2/(4\pi t)} - 1 \right]. \quad (4.9)
\]

After having set \( p \) to zero in the well-defined function \( Z''_r(p, s) \), the integral \( Z''_r(0, s) \) can also be calculated numerically with high precision. As for \( Z''_s(p, s) \), this function can easily be evaluated analytically in the continuum limit by appropriate separation of the singularity.

Summarizing the whole terms, we get finally for \( p \to 0 \)

\[
Z(p, s) = \frac{\pi^s}{\Gamma(s)} \left\{ \sum_{x \in \Lambda} \int_1^\infty dt \, t^{s-1} e^{-\pi x^2 t} + \sum_{x \in \Lambda} \int_1^\infty dt \, t^{-1} e^{-\pi x^2 t} - \frac{1}{s} \right\}
\]

\[
+ \rho^{1-s} \left[ \int_0^1 dz \, z^{s-2} e^{-1/z} + \int_1^0 dz \, z^{s-2} (e^{-1/z} - 1) + \int_1^\rho dz \, z^{s-2} \right] \]

with \( \rho = 4\pi/p^2 \).
4.2 Scalar one-loop integral

We consider the integral

\[ D_1(p) = \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{p^2 - \hat{k}^2}{\hat{k}^2 \hat{s}^2} \hat{s}^2, \quad s = k + p \]  

(4.11)

which corresponds to the first diagram in figure 2. By applying the coordinate space method, we would like to show that \( D_1(p) \) has the form

\[ D_1(p) = c_1 \ln p^2 + c_2 + O(p^2) \]  

(4.12)

and determine the coefficients \( c_1 \) and \( c_2 \).

In position space, Eq. (4.11) has the form

\[ D_1(p) = -\sum_{x \in \Lambda} e^{-ipx} \Delta G(x)^2. \]  

(4.13)

For large \( x \), the function \( \Delta G(x)^2 \) has the asymptotic behavior

\[ \Delta G(x)^2 = \{ \Delta G(x)^2 \}^\infty + O(|x|^{-6}) \]  

(4.14)

with

\[ \{ \Delta G(x)^2 \}^\infty = \frac{1}{2\pi^2} \frac{1}{x^2} \left[ 1 - \frac{7}{2} \frac{1}{x^2} + 5 \frac{x^4}{(x^2)^3} \right]. \]  

(4.15)

This suggests the splitting of \( D_1(p) \) into four parts:

\[ D_1(p) = -[D'_1(p) + D''_1(p) + D'''_1(p) + \Delta G(x)^2|_{x=0}], \]  

(4.16)

where

\[ D'_1(p) = \sum_{x \in \Lambda} e^{-ipx} \left[ \Delta G(x)^2 - \{ \Delta G(x)^2 \}^\infty \right], \]  

(4.17)

\[ D''_1(p) = \sum_{x \in \Lambda} e^{-ipx} \left[ \{ \Delta G(x)^2 \}^\infty - \frac{1}{2\pi^2} \frac{1}{x^2} \right], \]  

(4.18)

\[ D'''_1(p) = \frac{1}{2\pi^2} \sum_{x \in \Lambda} e^{-ipx} \frac{1}{x^2}, \]  

(4.19)

\[ \Delta G(x)^2|_{x=0} = \frac{1}{4}. \]  

(4.20)

The functions \( D'_1(p) \) and \( D''_1(p) \) are well-defined and therefore the limit \( p \to 0 \) exists. After having set \( p \) to zero, \( D'_1(0) \) can be computed numerically with high precision by applying the position space method. We receive the result

\[ D'_1(0) = -0.14215315037(1). \]  

(4.21)

\footnote{In this section \( O(p^n) \) stands for a remainder \( R(p) \) such that \( \lim_{p \to 0} R(p)/|p|^{n-\epsilon} = 0 \) for all \( \epsilon > 0 \).}
\( D''_1(0) \) can also be computed very accurately by making use of the solution of zeta functions [Eq. (2.47)]. A straightforward calculation gives
\[
D''_1(0) = 0.275883297410294(1).
\]
The function \( D''_1(p) \), however, diverges in the continuum limit, which makes it necessary to apply Eq. (4.10). After some algebra and numerical evaluations, we find
\[
D''_1(p) = -\frac{1}{2\pi} \ln p^2 + 0.1678588533415551(1). \tag{4.23}
\]
Summing all four contributions together, we confirm that \( D_1(p) \) has indeed the form of Eq. (4.12), and for the constants \( c_1 \) and \( c_2 \) we obtain
\[
c_1 = \frac{1}{2\pi}, \tag{4.24}
\]
\[
c_2 = -0.55158900038(1). \tag{4.25}
\]

### 4.3 Scalar two-loop integral

Now let us consider an example of two-loop diagrams. Concretely, we would like to discuss the following integral shown in the second diagram of figure 2:
\[
D_2(p) = \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \frac{d^2 l}{(2\pi)^2} \frac{d^2 s}{(2\pi)^2} \frac{(k + l)^2}{k^2 l^2} \frac{1}{s^2} \left( \delta_{p+k+s,0} - \delta_{p+k,t,0} \right). \tag{4.26}
\]
We have used here the shorthand notation \( \delta_{p,0} = (2\pi)^2 \delta^{(2)}(p) \).

In position space, the integral (4.26) can be written in the form
\[
D_2(p) = -\frac{1}{2} \sum_{x \in \Lambda} e^{ipx} \tilde{D}_2(x) \tag{4.27}
\]
with
\[
\tilde{D}_2(x) = G(x) \sum_{\mu=0}^1 \left\{ \left( \partial_{\mu} - \partial^*_{\mu} \right) G(x) \right\}^2 + \left\{ \left( \partial_{\mu} + \partial^*_{\mu} \right) G(x) \right\}^2. \tag{4.28}
\]
For the evaluation of the sum (4.27), we first observe the expansion terms of \( \tilde{D}_2(x) \) for large \( x \). Up to order \( |x|^{-4} \), we find
\[
\tilde{D}_2(x) = \left\{ \tilde{D}_2(x) \right\}_\infty + \mathcal{O}(|x|^{-6}). \tag{4.29}
\]
where
\[
\left\{ \tilde{D}_2(x) \right\}_\infty = -\frac{1}{4\pi^3} \left( \frac{\ln x^2}{x^2} + \frac{2\gamma + 3 \ln 2}{x^2} \right.
- \frac{7}{2} \frac{1}{(x^2)^2} \left[ \ln x^2 + 3 \ln 2 + 2\gamma - \frac{1}{7} \right]
+ 5 \frac{x^4}{(x^2)^4} \left[ \ln x^2 + 2\gamma + 3 \ln 2 - \frac{2}{15} \right]. \tag{4.30}
\]
Now we separate the singular part in Eq. (4.27) and rewrite it as

\[ D_2(p) = -\frac{1}{2} \left[ D'_2(p) + D''_2(p) + D'''_2(p) \right], \] (4.31)

where

\[ D'_2(p) = \sum_{x \in \Lambda} e^{ipx} \left[ \tilde{D}_2(x) - \{ \tilde{D}_2(x) \}_{\infty} \right], \] (4.32)

\[ D''_2(p) = \sum_{x \in \Lambda} e^{ipx} \left[ \{ \tilde{D}_2(x) \}_{\infty} + \frac{1}{4\pi^3} \left( \frac{\ln x^2}{x^2} + \frac{2\gamma + 3 \ln 2}{x^2} \right) \right], \] (4.33)

\[ D'''_2(p) = -\frac{1}{4\pi^3} \sum_{x \in \Lambda} e^{ipx} \left( \frac{\ln x^2}{x^2} + \frac{2\gamma + 3 \ln 2}{x^2} \right). \] (4.34)

We note that \( \tilde{D}_2(0) = 0 \).

The function \( D'_2(p) \) is well-defined in the limit \( p \to 0 \) and can thus be computed numerically in this limit by applying the position space method. We obtain

\[ D'_2(0) = 0.0705019591(1). \] (4.35)

\( D''_2(0) \) can also be evaluated very accurately by using the solution of zeta functions [Eq. (2.47)]:

\[ D''_2(0) = -0.13850144234234(1). \] (4.36)

The function \( D'''_2(p) \), however, diverges in the continuum limit. To this divergent function we can again apply the solution (4.10) as in the case of the one-loop integral. The only difference here is the appearence of a logarithmic term in zeta function which can, however, be evaluated by differentiating the appropriate zeta function with respect to \( s \). In this way we obtain

\[ D'''_2(p) = -\frac{1}{8\pi^2} (\ln p^2)^2 + \frac{5\ln 2}{(2\pi)^2} \ln p^2 - 0.0841257673993860(1). \] (4.37)

Summing all three terms together, we finally get for \( D_2(p) \)

\[ D_2(p) = \frac{1}{(4\pi)^2} (\ln p^2)^2 - \frac{5\ln 2}{8\pi^2} \ln p^2 + \kappa_1 \] (4.38)

with \( \kappa_1 \) given by

\[ \kappa_1 = 0.0760626064(1). \] (4.39)

### 4.4 Scalar three-loop integral

As the last example, we consider the 3-loop integral of the form (see figure 2)

\[ D_3(p) = \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \frac{d^2s}{(2\pi)^2} \frac{d^2t}{(2\pi)^2} \frac{(k+l)^2 - \bar{k}^2 - \bar{l}^2}{k^2\bar{l}^2\bar{s}^2\bar{t}^2} \cdot (\delta_{p+k+l+s+t,0} - \delta_{p+k+l+s,0} - \delta_{p+k+l+t,0} + \delta_{p+k+l,t,0}). \] (4.40)
In position space this integral can be expressed by

\[ D_3(p) = -\frac{1}{2} \sum_{x \in \Lambda} e^{ipx} \tilde{D}_3(x), \] (4.41)

where

\[ \tilde{D}_3(x) = G(x)^2 \sum_{\mu=0}^1 \left\{ \left[ (\partial_\mu - \partial_\mu^*) G(x) \right]^2 + \left[ (\partial_\mu + \partial_\mu^*) G(x) \right]^2 \right\}. \] (4.42)

Up to order \(|x|^{-4}\), we obtain for the large \(x\) expansion of the function \(\tilde{D}_3(x)\)

\[ \tilde{D}_3(x) = \left\{ \tilde{D}_3(x) \right\}_\infty + \mathcal{O}(|x|^{-6}), \] (4.43)

where

\[ \left\{ \tilde{D}_3(x) \right\}_\infty = \frac{1}{16\pi^4} \frac{1}{x^2} \left\{ (\ln x^2)^2 + 2Q_0 \ln x^2 + Q_0^2 \right. \\
+ \frac{1}{x^2} \left[ -\frac{7}{2} (\ln x^2)^2 + (1 - 7Q_0) \ln x^2 - \frac{7}{2} Q_0^2 + Q_0 \right] \\
+ \left. \frac{x^4}{(x^2)^3} \left[ 5(\ln x^2)^2 + (10Q_0 - \frac{4}{3}) \ln x^2 + 5Q_0^2 - \frac{4}{3} Q_0 \right] \right\} \] (4.44)

with \(Q_0 = 2\gamma + 3 \ln 2\).

The sum (4.41) can now be evaluated in exactly the same way as \(D_2(p)\). The only difference in this case is that, due to the quadratically logarithmic term in the expansion (4.44), it is necessary to differentiate zeta functions two times with respect to \(s\). After a straightforward computation, we get for \(D_3(p)\)

\[ D_3(p) = \frac{1}{32\pi^4} \left\{ \frac{1}{3} (\ln p^2)^3 - 5 \ln 2 (\ln p^2)^2 + (5 \ln 2)^2 \ln p^2 \right\} + \kappa_2 \] (4.45)

with \(\kappa_2\) given by

\[ \kappa_2 = -0.01802345(1). \] (4.46)

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**A Large \(x\) expansion of \(G(x)\)**

In this appendix we derive for the free propagator \(G(x)\) given in Eq. (1.4) the large \(x\) expansion of Eqs. (2.3) and (2.6).
As a first step, we use Cauchy’s integral formula to do the $p_0$ integration and for simplicity set $x = (x_0, 0)$. Then\(^3\)

$$G(x_0) = \int_0^\pi \frac{dp}{\pi} \frac{e^{x_0 \ln \frac{1}{2} [p^2 + \sqrt{p^2 (p^2 + 4)]}} - 1}{\sqrt{p^2 (p^2 + 4)}}.$$  \hspace{1cm} (A.1)

Change variable $y = \sin p/2$, then $\hat{p} = 2y$, $dp = 2dy/\sqrt{1 - y^2}$ and Eq. (A.1) becomes

$$G(x_0) = \frac{1}{2\pi} \int_0^1 \frac{dy}{y} \frac{e^{x_0 f(y)} - 1}{\sqrt{1 - y^4}},$$  \hspace{1cm} (A.2)

where $f(y)$ is defined by

$$f(y) = \ln \left(1 + 2y^2 - 2y \sqrt{1 + y^2}\right).$$  \hspace{1cm} (A.3)

Now, divide the integral of Eq. (A.2) into two parts

$$G(x_0) = \frac{1}{2\pi} \left[\bar{G}(x_0) + \tilde{G}(x_0)\right],$$  \hspace{1cm} (A.4)

where

$$\bar{G}(x_0) = \int_0^1 \frac{dy}{y} \left[\frac{e^{x_0 f(y)}}{\sqrt{1 - y^4}} - 1\right],$$  \hspace{1cm} (A.5)

$$\tilde{G}(x_0) = \int_0^1 \frac{dy}{y} \left(1 - \frac{1}{\sqrt{1 - y^4}}\right).$$  \hspace{1cm} (A.6)

We are interested in the evaluation of $G(x_0)$ in the limit $x_0 \to \infty$. For this purpose, we perform another variable change $t = x_0 y$ in $\bar{G}(x_0)$. In the continuum limit ($x_0 \to \infty$), Eq. (A.5) then reduces to

$$\bar{G}(x_0) \sim \int_0^{x_0} dt \frac{e^{-2t} - 1}{t}.$$  \hspace{1cm} (A.7)

The integrals of Eqs. (A.6) and (A.7) can now be computed analytically and we finally get for the leading terms in the continuum limit

$$G^c(x_0) = -\frac{1}{4\pi} (2 \ln x_0 + 2\gamma + 3 \ln 2).$$  \hspace{1cm} (A.8)

Using rotation invariance in the continuum limit, the leading behavior for $G(x)$ is then equal to

$$G^c(x) = -\frac{1}{4\pi} (\ln x^2 + 2\gamma + 3 \ln 2).$$  \hspace{1cm} (A.9)

\(^3\)For a given function $F(x)$, the simplified notation $F(x_0)$ stands for the exact $F(x)|_{x = (x_0, 0)}$.\(^3\)
The subleading terms can also be computed systematically if we make use of the laplace equation (2.1). We derive, at first, the first subleading term s through the ansatz

\[ G(x) = G^c(x) + G^{(1)}(x) + \mathcal{O}(|x|^{-4}) \]  
(A.10)

with

\[ G^{(1)}(x) = -\frac{1}{4\pi} \left[ c_1 \frac{1}{x^2} + c_2 \frac{x^4}{(x^2)^3} \right]. \]  
(A.11)

For determination of \( c_1 \) and \( c_2 \), we evaluate the lattice laplacian of \( G(x) \) in Eq. (A.10) to get

\[ \triangle G(x) = -\frac{1}{4\pi} \left\{ \frac{6}{(x^2)^2} - 8 \frac{x^4}{(x^2)^4} + c_1 \frac{4}{(x^2)^2} + 12c_2 \frac{1}{(x^2)^2} - \frac{x^4}{(x^2)^4} \right\} + \mathcal{O}(|x|^{-6}). \]  
(A.12)

From the condition that the r.h.s. of Eq. (A.12) should be zero, we obtain

\[ c_1 = \frac{1}{2}, \quad c_2 = -\frac{2}{3}. \]  
(A.13)

Now, we compute the next subleading terms and try with the ansatz

\[ G(x) = G^c(x) + G^{(1)}(x) + G^{(2)}(x) + \mathcal{O}(|x|^{-6}), \]  
(A.14)

where

\[ G^{(2)}(x) = -\frac{1}{4\pi} \left[ c_3 \frac{1}{(x^2)^2} + c_4 \frac{x^4}{(x^2)^4} + c_5 \frac{(x^4)^2}{(x^2)^6} \right]. \]  
(A.15)

The lattice laplacian of this function is given by

\[ \triangle G(x) = -\frac{1}{4\pi} \left\{ -\frac{170}{(x^2)^3} + 480 \frac{x^4}{(x^2)^5} - \frac{x^4}{(x^2)^7} + c_3 \frac{16}{(x^2)^3} \\
+ c_4 \frac{12}{(x^2)^3} - c_5 \left[ \frac{16}{(x^2)^3} - 72 \frac{x^4}{(x^2)^5} + 48 \frac{(x^4)^2}{(x^2)^7} \right] \right\} + \mathcal{O}(|x|^{-8}). \]  
(A.16)

Since the r.h.s. has to vanish, we obtain

\[ 24c_3 + 18c_4 - 95 = 0, \quad c_5 = -\frac{20}{3}. \]  
(A.17)

Unfortunately, the condition from the laplace equation alone is not enough to determine the coefficients completely and we need another constraint for them. It can be obtained by computing the subleading terms of \( G(x_0) \) in Eq. (A.11) to this order. After some algebra, we find

\[ G(x_0) = G^c(x_0) + \frac{1}{24\pi \ x_0^2} \left( 1 + \frac{43}{20 \ x_0^2} \right) + \mathcal{O}(x_0^{-6}). \]  
(A.18)
By comparing this with Eq. (A.14) for \( x = (x_0, 0) \), we get the relation
\[
c_3 + c_4 + c_5 = -\frac{43}{120}.
\] (A.19)

From Eqs. (A.17) and (A.19), the coefficients \( c_3 \) and \( c_4 \) can now be determined:
\[
c_3 = -\frac{371}{120}, \quad c_4 = \frac{47}{5}.
\] (A.20)

The computation of the third subleading terms proceeds completely analogously.

After a lengthy, but straightforward computation, we find
\[
G(x) = G^{c}(x) + G^{(1)}(x) + G^{(2)}(x) + G^{(3)}(x) + \mathcal{O}(|x|^{-8})
\] (A.21)

with
\[
G^{(3)}(x) = -\frac{1}{4\pi} \left[ \frac{4523}{56} \frac{1}{(x^2)^3} - \frac{7657}{21} \frac{x^4}{(x^2)^5} + \frac{3716}{7} \frac{(x^4)^2}{(x^2)^7} - \frac{2240}{9} \frac{(x^4)^3}{(x^2)^9} \right].
\] (A.22)

### B Large \( x \) expansion of \( G_2(x) \)

We derive here the large \( x \) expansion of Eq. (2.27) for the function \( G_2(x) \) given in Eq. (2.25).

We first do the \( p_0 \) integration and for simplicity set \( x = (x_0, 0) \). Then
\[
G_2(x_0) = \frac{1}{16\pi} \int_0^\pi \frac{dp}{\pi} \left\{ \frac{(x_0 + 1)e^{x_0 \ln \frac{1}{2}(\hat{p}^2 + 2 - P)} - 1}{\hat{p}^2} - \frac{x_0^2}{4}(\hat{p}^2 - P) \right\}
\]

\[
+ \frac{(\hat{p}^2 + 2 - P)[e^{x_0 \ln \frac{1}{2}(\hat{p}^2 + 2 - P)} - 1] - \frac{x_0^2}{4}(\hat{p}^2 - P)^2}{P^3}
\]

with \( P = \sqrt{\hat{p}^2(\hat{p}^2 + 4)} \). By changing variable \( y = \sin p/2 \), Eq. (B.1) transforms to
\[
G_2(x_0) = \frac{1}{16\pi} \int_0^1 \frac{dy}{y^3} \left\{ \frac{e^{x_0 f(y)}(2y^2 + 2x_0 y\sqrt{1 + y^2} + 1) - [1 + 2(1 - x_0^2)y^2]}{1 + y^2}\sqrt{1 - y^2} \right\}
\] (B.2)

with \( f(y) \) given in Eq. (A.3).

Now, divide the integral of Eq. (B.2) into two parts:
\[
G_2(x_0) = \frac{1}{16\pi} \left[ \bar{G}_2(x_0) + \tilde{G}_2(x_0) \right],
\] (B.3)

where \( [Y = (1 + y^2)\sqrt{1 - y^4}] \)
\[
\bar{G}_2(x_0) = \int_0^1 \frac{dy}{y^3} \left\{ \frac{e^{x_0 f(y)}(2y^2 + 2x_0 y\sqrt{1 + y^2} + 1)}{Y} - [1 + (1 - 2x_0^2)y^2] \right\}, \quad (B.4)
\]
\[
\tilde{G}_2(x_0) = \int_0^1 \frac{dy}{y^3} \left\{ 1 + y^2 - \frac{1 + 2y^2}{Y} \right\} - 2x_0^2 \int_0^1 \frac{dy}{y} \left( 1 - \frac{1}{Y} \right).
\] (B.5)
We are interested in the evaluation of \( G^2(x_0) \) in the continuum limit \( x_0 \to \infty \). For this purpose, we perform the variable change \( t = x_0 y \). In the continuum limit, Eq. (B.3) then reduces to

\[
\bar{G}^2(x_0) \sim \int_0^{x_0} dt \frac{e^{-2t}[(1 + 2t)x_0^2 + t^2]}{t^3} - \int_0^{x_0} dt \frac{e^{-2t}(1 + 2t)}{t^3}.
\]  

In the limit of infinitely large \( x_0 \), these integrals can be evaluated analytically to yield

\[
\bar{G}^2(x_0) \sim x_0^2 (2 \ln x_0 + 2 \ln 2 + 2\gamma - 1) - \ln x_0 - \ln 2 - \gamma + \frac{1}{3}.
\]  

As for \( \tilde{G}^2(x_0) \), we can also compute the function analytically:

\[
\tilde{G}^2(x_0) = x_0^2 (\ln 2 - 1) - \frac{1}{2} \ln 2.
\]  

Summing two contributions together, we have finally for the leading terms in the continuum limit

\[
G^c_2(x_0) = \frac{1}{16\pi} \left[ x_0^2 (2 \ln x_0 + 3 \ln 2 + 2\gamma - 2) - \ln x_0 - \frac{3}{2} \ln 2 - \gamma + \frac{1}{3} \right].
\]  

Unfortunately, from Eq. (B.9) alone one cannot derive the corresponding function \( G^c_2(x) \) unambiguously since \( G^2_2(x) \) in the continuum limit is not rotation invariant. We therefore apply the laplace equation (2.26) from which we obtain for \( G^2_2(x) \) the following leading terms in the infinitely large \( x \) limit:

\[
G^c_2(x) = \frac{1}{16\pi} \left[ x^2 \ln x^2 + (3 \ln 2 + 2\gamma - 2)x^2 + \kappa_1 \ln x^2 + \frac{1}{3} \frac{x^4}{(x^2)^2} + \kappa_2 \right].
\]  

Comparison of this result for \( x = (x_0, 0) \) with Eq. (B.3) gives finally

\[
\kappa_1 = -\frac{1}{2}, \quad \kappa_2 = -\frac{1}{2}(3 \ln 2 + 2\gamma).
\]  

The subleading terms for \( G^2_2(x) \) can also be worked out systematically by applying the laplace equation (2.26) and the asymptotic expansion for \( G^2_2(x_0) \) of Eq. (B.3). We do not present here the detailed computations leading to the result of Eq. (2.27) since they are too long and also proceed very similarly to the case of \( G(x) \).

C Derivation of harmonic functions

We discuss here how one computes the harmonic homogeneous polynomials appearing in the main text.
Let us consider the following polynomial functions of degree 4

\[ h_n(x) = \sum_{r=0}^{n} C_r^{(n)} (x^4)^r (x^2)^{2(n-r)} \]  

with \( C_r^{(n)} \neq 0 \). We would like to derive the conditions for the coefficients \( C_r^{(n)} \) so that \( h_n(x) \) are harmonic functions.

For determination of the conditions, we first note the relation

\[ \Delta_c \left[ (x^4)^r (x^2)^s \right] = 4 \left[ -2r(r-1)(x^4)^{r-2}(x^2)^{s+3} + 3r(2r-1)(x^4)^{r-1}(x^2)^{s+1} + s(4r+s)(x^4)^r(x^2)^{s-1} \right] \]  

where \( \Delta_c \) denotes the continuum laplacian \( \sum_{\mu=0}^{1} \partial^2 / \partial x_\mu^2 \). In deriving the relation above we made use of the identity (2.12). By using Eq. (C.2) we obtain

\[ \Delta_c h_n(x) = 4 \sum_{r=0}^{n} \left[ 4C_r^{(n)}(n^2 - r^2) + 3C_{r+1}^{(n)}(r+1)(2r+1)\theta(n-1-r) - 2C_{r+1}^{(n)}(r+1)(r+2)\theta(n-2-r) \right] (x^4)^r(x^2)^{2n-2r-1} \]  

where \( \theta(t) \) is a step function defined by

\[ \theta(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \]

For \( h_n(x) \) to be harmonic, the r.h.s. of Eq. (C.3) should be zero, which leads to the conditions for the coefficients \( C_r^{(n)} \) to satisfy. Now, we investigate these conditions by dividing the \( n + 1 \) terms in the summation contributing to \( \Delta_c h_n(x) \) into the following three cases:

1. For \( r = n \): the contribution to \( \Delta_c h_n(x) \) is identically zero and there is therefore no constraint on the coefficients \( C_r^{(n)} \).

2. For \( r = n-1 \): the contribution to \( \Delta_c h_n(x) \) is zero if

\[ 3nC_n^{(n)} + 4C_{n-1}^{(n)} = 0. \]  

3. For \( r \leq n-2 \): the contribution to \( \Delta_c h_n(x) \) is zero if

\[ 4C_r^{(n)}(n^2 - r^2) + 3C_{r+1}^{(n)}(r+1)(2r+1) - 2C_{r+2}^{(n)}(r+1)(r+2) = 0. \]  

It is now straightforward to derive the harmonic homogeneous polynomials including those of Eqs. (3.12)–(3.14) by solving the recursion relations (C.4) and (C.5) up to an arbitrary overall normalization factor.
D Analytic computation of $A_1$

This appendix where we evaluate the two-loop integral $A_1$ given in Eq. (3.1) analytically is due to communication with A. Pelissetto.

We first use the identity $\hat{s}_\mu \cos \frac{s_\mu}{2} = \sin s_\mu$ and the permutation symmetry between $k$, $l$ and $(-s)$ to transform the given integral to

$$A_1 = \frac{1}{3} \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \frac{1}{k^2 l^2 s^2} \sum_{\mu=0}^{1} \hat{k}_\mu \hat{l}_\mu \hat{s}_\mu (\sin s_\mu - \sin k_\mu - \sin l_\mu). \tag{D.1}$$

Now, we use the following trigonometric identity being valid for $\alpha + \beta + \gamma = 0$

$$\sin \alpha + \sin \beta + \sin \gamma = -4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \tag{D.2}$$

to rewrite

$$A_1 = -\frac{4}{3} \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \frac{1}{k^2 l^2 s^2} \sum_{\mu=0}^{1} \hat{k}_\mu \hat{l}_\mu \hat{s}_\mu \sin \frac{k_\mu}{2} \sin \frac{l_\mu}{2} \sin \frac{s_\mu}{2}. \tag{D.3}$$

Noting the relation $\sin \frac{s_\mu}{2} = \frac{1}{2} \hat{s}_\mu$, we see that this integral is, apart from a factor, same as $A^{(3)}$ in Appendix A of Ref. [6] which is known analytically. From this, we obtain

$$A_1 = -\frac{1}{24}. \tag{D.4}$$

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