RELATIVE COMPACTIFIED JACOBIANS OF LINEAR SYSTEMS ON
ENRIQUES SURFACES

GIULIA SACCÀ

Abstract. Given a general Enriques surface $T$ and a genus $g$ linear system $|C|$ on $T$, we consider
the relative compactified Jacobian $N = \text{Jac}(|C|) \to |C|$. Let $f : S \to T$ be the universal K3 cover
and set $D = f^*C$. We show that $N$ is a smooth $(2g - 1)$-dimensional variety and that it admits an
étale double cover to a Lagrangian subvariety of the relative compactified Jacobian $M = \text{Jac}(|D|)$.
The main results are that, under some technical assumption that can be verified for low values of
$g$, we prove that $\pi_1(N) \cong Z/2$, that $\omega_N \cong \mathcal{O}_N$, and that $h^{p,0}(N) = 0$ for $p \neq 0, 2g - 1$.

Contents

1. Introduction 1
2. Acknowledgements 3
2. Fibrations in compactified Jacobians of curves on Enriques surfaces 3
2.1. Set up and notation 3
2.2. The construction 5
3. Fundamental group of relative compactified Jacobians 12
4. The second Betti number 16
5. The canonical bundle 23
5.1. Degeneration of Hodge bundles 26
6. An example: the threefold case 30
References 36

1. Introduction

Moduli spaces of sheaves on K3 surfaces are among the most studied objects in algebraic geometry.
Part of their interesting features is that they inherit the rich structure coming from the K3 surface
itself. For example, by work of Mukai [Muk84], the symplectic structure on the surface induces
a holomorphic symplectic structure on the smooth locus of the moduli space. When smooth and
projective, these moduli spaces provide examples of compact irreducible hyperkähler manifolds
[Bea83, Huy97, Muk84].

The geometry of hyperkähler manifolds is rich and very few examples of such manifolds are known
[Bea83, O'G99, O'G03]. One of the reasons why hyperkähler manifolds have attracted attention
is that they are, together with Calabi-Yau manifolds and complex tori, the building blocks of Kähler
manifolds with trivial first Chern class [Bea83].
On the other hand, not much work has been done on moduli spaces of sheaves on Enriques surfaces, even though it is natural to expect that their geometry is tightly related to moduli spaces of sheaves on the covering K3 surface. In [Kim98] the author studies the moduli space of stable vector bundles on an Enriques surfaces and shows that it admits a double cover to a Lagrangian subvariety of a hyperkähler manifold, unramified away from the singularities. In [Kim06] he analyzes in more detail the case of rank two vector bundles and Hauzer’s paper [Hau10] improves his results. Recently, and independently from this work, M. Zowislok [Zow12] considers the pullback morphism of pure sheaves (thus not necessarily torsion free) under finite unramified quotients of K3 or abelian surfaces and as in [Kim06] relates the construction to Lagrangian sub varieties of hyperkähler manifolds.

The present paper studies the geometry of moduli spaces of pure dimension one sheaves on a general Enriques surface $T$. If we consider pure dimension one sheaves whose first Chern class is linearly equivalent to a given curve $C$, these moduli spaces may be described as the relative compactified Jacobian of a linear system $|C|$. By a Riemann-Roch calculation, one can see that these spaces are always odd dimensional. In particular, they are not deformation of Hilbert schemes of points on $T$.

In Section 2 we describe these moduli spaces. As in [Kim98] and [Zow12] we relate them to a Lagrangian subvariety of some moduli space of sheaves on the covering K3 surface $S$. To define a moduli space of sheaves on a surface, one needs to consider a polarization, relative to which stability is defined. The main result of the section is that if we choose the polarization to be generic, and if the curve $C$ is primitive, then the pullback morphism from the moduli space of sheaves on the Enriques to the moduli spaces of sheaves on the K3 surface is an étale double cover onto the aforementioned Lagrangian subvariety and, moreover, the moduli space itself is always smooth.

Under some technical conditions (concerning the dimension of the fibers of the relative compactified Jacobian, see Assumption 2.22) in Section 3 we prove that the fundamental group of these moduli spaces is isomorphic to $\mathbb{Z}/(2)$, thus the moduli spaces have the same fundamental group as the surface itself.

In Section 4 we compute the second Betti number for the relative compactified Jacobian of a base point free linear system, by comparing it to the second Betti number of the family of curves itself. It turns out that the behavior is different in the case of hyperelliptic and not hyperelliptic linear systems. This reflects the fact that the moduli space of polarized Enriques surfaces of a given degree is not irreducible.

Finally, in Section 5 we prove that the canonical bundle of the moduli spaces is trivial, under the same technical conditions of Section 3. This is surprising, since the Hilbert scheme of points on $T$ has canonical bundle that is a two-torsion point of the Picard group. We also prove that these moduli spaces, as well as their universal covers, are irreducible Calabi-Yau manifolds, in the sense that all the Hodge number $h^{p,0}$ vanish, except in top and bottom degree.

We end the paper with a section describing the Calabi-Yau threefold that we get in the genus two case (where the condition is automatically satisfied). Where the general theory developed in the previous sections does not apply, we give specific arguments and prove that the Hodge diamond of the relative compactified Jacobian is

$$
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 10 & 0 & \\
1 & 10 & 10 & 1
\end{array}
$$
Acknowledgements. I would like to thank E. Arbarello, D. Huybrechts, J. Kollár, L. Migliorini, M. Popa, V. Shende, C. Voisin and my advisor G. Tian for useful and fun conversations, and I. Dolgachev for a written communication. Part of this work was done while visiting the Beijing International Center for Mathematical Research and the final version of the manuscript was prepared while visiting the Institut de Mathématique de Jussieu. I am grateful to both institutions for the warm hospitality I received and for the good working conditions.

2. Fibrations in compactified Jacobians of curves on Enriques surfaces

2.1. Set up and notation. Let $T$ be an Enriques surface, and let $S$ be its universal covering space, so that $S$ is a K3 surface. We denote by $f : S \to T$ the covering morphism and by $\iota : S \to S$ the covering involution. Let $\omega_T$ be the canonical bundle of $T$. For any sheaf $F$ on $T$, we set

$$F' := F \otimes \omega_T.$$  

If $F = L$ is a line bundle, then $f^* L \cong f^* L'$. For a curve $C \subset T$ we let

$$D := f^{-1}(C) \subset S,$$ 

be its inverse image. If $g \geq 2$, then

$$f : D \to C,$$ 

is a non trivial covering, which is induced by the two-torsion line bundle

$$\eta := \omega_T|_C.$$ 

In this case, the genus of $D$ is given by $h = 2g - 1$. By a simple Riemann-Roch calculation, one can check that if $C^2 \geq 0$ then the line bundle $\mathcal{O}(C) \otimes \omega$ is also effective, and thus isomorphic to $\mathcal{O}(C')$ where $C'$ is a curve of the same genus (cf. [Cos83]).

Remark 2.1. [Nam85] The pullback

$$f^*: \text{NS}(T)) \to \text{NS}(S),$$

is injective so that the image of the Néron-Severi group of $T$ in the Néron-Severi group of $S$ is a primitive sub lattice of rank 10. In particular, if we choose $C$ so that its class is primitive in $\text{NS}(T)$, then so is the class of $D$ in $\text{NS}(S)$. By abuse of notation we say that a curve is primitive if its class in the Néron-Severi group is a primitive element of the lattice.

From now on we let $C \subset T$ be a primitive curve of genus $g \geq 2$, so that $D \subset S$ is a primitive curve of genus $h = 2g - 1$. For any integer $\chi$, we may consider the primitive Mukai vector

$$v = (0, D, \chi) \in H^*(S, \mathbb{Z}).$$

From Lemma 1.2 of [Yos91], it follows that when $\chi \neq 0$ we can find a polarization such that all semi-stable sheaves are in fact stable. Such a polarization is called $v$-generic. More specifically, Yoshioka’s lemma tells us that the locus of ample line bundles that are not $v$-generic is a finite union of hyperplanes in $\text{Amp}(S)$. Such hyperplanes are called walls and they depend explicitly on $v$.

If we let $H$ be a $v$-generic polarization and assume that $\chi \neq 0$, the moduli space

$$M := M_{v,H}(S)$$

of $H$-stable sheaves on $S$ with Mukai vector equal to $v$ is smooth and projective. By [Muk84], $M$ is an irreducible holomorphic symplectic variety of dimension

$$v^2 + 2 = D^2 + 2 = 2h.$$
that is deformation equivalent to the Hilbert scheme of $h$ points on a K3 surface.

Let us now briefly mention a few facts we use about this moduli space. First of all, recall that the points of $M$ correspond to $S$-equivalence classes of sheaves of the form

$$i_* L$$

where $i : \Gamma \subset S$ is the inclusion of a curve in the linear system $|D|$, and $L$ is a pure dimension one sheaf on $\Gamma$ such that $\chi(L) = \chi$ and $c_1(i_* L) = D$. Notice that the condition on the first Chern class, together with the fact that $D$ is primitive, implies that $L$ has rank one on every reduced component of its support.

Moreover, if $\text{Supp}(i_* L)$ is reduced and irreducible, then $i_* L$ is automatically stable with respect to any polarization and hence the $S$-equivalence class coincides with the isomorphism class of the sheaf.

Following [LP93], we consider the natural map

$$\rho: M \to |D|$$

which sends a sheaf to the class of its support. The fiber $M_t$ of $\rho$ over a curve $[D_t]$ in $|D|$ is the Simpson moduli space of degree $d = \chi - 2h - 2$ pure sheaves on $D_t$ that are stable with respect to $\mathcal{O}_{D_t}(H|_{D_t})$. In particular, over the locus of nodal curves the fiber is isomorphic to some degree $d$ compactified Jacobian of the corresponding curve $D_t$ in the sense of [OS79].

**Remark 2.2.** By a theorem of Matsushita [Mat99], $\rho$ is equidimensional and the fibers are Lagrangian with respect to the holomorphic symplectic form on $M$. In particular, $\rho$ is flat.

**Remark 2.3.** We now recall the construction, performed by Mukai in his fundamental paper [Muk84], of the symplectic form on $M$. The symplectic form at a point $[F]$ corresponding to a stable sheaf $F$, is given by the following composition

$$\sigma: \text{Ext}^1(F,F) \times \text{Ext}^1(F,F) \to \text{Ext}^2(F,F) \to H^2(S,\mathcal{O}_S) \cong \mathbb{C},$$

$$\sigma(e,f) \mapsto e \cup f \mapsto \text{tr}(e \cup f)$$

where the identification $H^2(S,\mathcal{O}_S) = H^2(S,\omega_S) \cong \mathbb{C}$ is Serre dual to the isomorphism

$$H^0(S,\omega_S) = \mathbb{C} \sigma \cong \mathbb{C},$$

defined by the choice of a, unique up to scalar, symplectic form $\sigma \in H^0(S,\omega_S)$.

**Remark 2.4.** As we mention above, a rank one torsion free sheaf on a reduced and irreducible curve is stable with respect to any polarization. In particular, the structure sheaf of a reduced and irreducible curve is stable. It follows that when we choose $d = 0$, the fibration $M \to |D|$ has a rational zero-section

$$\sigma: |D| \to M,$$

which is defined on an open subset of $|D|$ containing the locus of reduced and irreducible curves.

**Remark 2.5.** It is well known that by choosing the Enriques surface to be general in moduli, we can ensure that

$$f^*(\text{NS}(T)) = \text{NS}(S),$$

so that $\iota^*$ acts as the identity on $\text{NS}(S)$. In particular, we can choose the $\nu$-generic polarization $H$ to be $\iota^*$-invariant. Notice also that, under these conditions, the ample cones of the two surfaces are identified by means of $f^*$. 

4
From now on we assume that $T$ is general (in the sense of the above remark). This assumption implies that there are no $(-2)$-curves on $T$ or on $S$, because the fact that $f^*(\NS(T)) = \NS(S)$ implies that the square of any class in $\NS(S)$ is divisible by 4. A surface with no smooth rational curves is called unodal. If $T$ is unodal, then any line bundle $L$ on $T$ is ample (respectively nef) as soon as $L^2 > 0$ (respectively $L^2 \geq 0$) and hence, by Proposition 3.1.6 of [CD89], of $L^2 \geq 0$ then $|L|$ has no fixed component.

**Lemma 2.6.** Let $C \subset T$ be a primitive curve of genus $g \geq 2$. If $C$ is irreducible, then so is its preimage $D = f^{-1}(C)$.

**Proof.** By Remark 2.1 we know that $D$ is primitive. Arguing by contradiction, suppose that $D$ has more than one irreducible component. Since every irreducible component of $D$ dominates $C$ and the morphism is $2:1$, it follows that $D$ breaks into two irreducible components $D_1$ and $D_2$. Both components map generically $1:1$ onto $C$ and are interchanged by the involution. However, since $\iota^*$ acts as the identity on $\NS(S)$, it follows that $D_1 \sim D_2$ and hence that $D \sim 2D_i$, contradicting the fact that $C$ and $D$ are primitive. □

2.2. **The construction.** The pullback $\iota^*$ on the space of global sections $H^0(S, O(D))$ induces an action on the projective space $|D|$. The fixed locus of this involution is the union

$$f^*|C| \cup f^*|C'| \subset |D|,$$

of the two disjoint $(g-1)$-dimensional projective spaces $|C|$ and $|C'|$, which we view as subspaces of $|D|$ via the pullback $f^*$ on global sections. Since we are assuming that $T$ is general we can suppose, by Remark 2.5, that the polarization $H$ is $\iota^*$-invariant.

As a consequence, we can define an involution

$$\iota: M \to M$$

$$[F] \mapsto [\iota^*F]$$

by sending the $S$-equivalence class of an $H$-stable sheaf $F$ on $M$ to the class of $H$-stable sheaf $\iota^*F$.

**Lemma 2.7.** The involution defined by (2.11) is anti-symplectic, i.e. if $\omega$ denotes the symplectic form on $M$, then $\iota^*\omega = -\omega$.

Moreover, the fibration $\rho: M \to |D|$ is equivariant with respect to the involutions $\iota^*$ defined above.

**Proof.** Consider $e, f \in \Ext^1(F, F)$. We need to prove that

$$\omega(\iota^*e, \iota^*f)) = -\omega(e, f).$$

By functoriality of the cup product and of the trace map, the following diagram is commutative,

$$(2.12) \quad \Ext^1(F, F) \times \Ext^1(F, F) \xrightarrow{\cup} \Ext^2(F, F) \xrightarrow{\tr} H^2(S, O_S)$$

$$\iota^* \downarrow \quad \iota^* \downarrow \quad \iota^*$$

$$\Ext^1(\iota^*F, \iota^*F) \times \Ext^1(\iota^*F, \iota^*F) \xrightarrow{\cup} \Ext^2(\iota^*F, \iota^*F) \xrightarrow{\tr} H^2(S, O_S).$$

Hence, in order to prove the Lemma, we just need to prove that the identification $H^2(S, O_S) \cong \mathbb{C}$ changes sign if we compose it with $\iota^*$. This follows from the fact that identification (2.9) does change sign once we compose it with $\iota^*$.

As for the second statement, it is a consequence of the very definitions of $\iota^*$ and of $\rho$. □

We are interested in studying the fixed locus $\text{Fix}(\iota)$ of this involution.

**Lemma 2.8.** $\text{Fix}(\iota)$ is a union of isotropic subvarieties of $M$. 5
Proof. First notice that since $\iota$ is an involution on a smooth variety, every connected component of $\text{Fix}(\iota)$ is smooth. The fact that every component is isotropic is a trivial consequence of Lemma 2.7. □

Before getting to the study of $\text{Fix}(\iota)$, we change slightly the point of view, and consider Simpson moduli spaces of stable sheaves on $T$ as well as on $S$. For locally free sheaves, this was done in [Kim98].

Lemma 2.9. Let $F$ be a pure dimension one sheaf on $T$. Let $A$ be an ample line bundle on $T$. If $F$ is $A$-semi-stable, then $f^*F$ is $H := f^*A$-semi-stable on $S$.

Proof. First notice that since $\omega_T$ is a numerically trivial line bundle, tensoring a sheaf by $\omega_T$ does not change the numerical invariants of the sheaf itself. In particular, the operation of tensoring by $\omega_T$ preserves not only the slope, but also stability and semi-stability with respect to any line bundle.

Let $G \subset f^*F$ be a sub sheaf. Using the projection formula for the finite flat morphism $f$, we see that $f_*G$ is a sub sheaf of the $A$-semi stable sheaf $F \oplus (F \otimes \omega_T)$. Clearly, we have $\chi(G) = \chi(f_*G)$. Moreover, since $G$ is pure of dimension one, $f_*c_1(G) = c_1(f_*G)$. Moreover, $c_1(G) \cdot H = f_*c_1(G) \cdot H'$, so that $\mu_H(G) = \mu_A(f_*G)$. Since $\mu_A(F) = \mu_H(F \oplus F \otimes \omega_T)$, the lemma is proved. □

Lemma 2.10. (cf. [Kim98]) Let $E$ and $G$ be two non isomorphic pure dimension one sheaves on $S$ with the same slope. Suppose they are both $A$-stable and that $f^*E \cong f^*G$. Then,

$$G \cong E \otimes \omega_T.$$ 

Proof. If $f^*E \cong f^*G$, then we also have an isomorphism $E \oplus (E \otimes \omega_T) = f_*f^*E \cong f_*f^*G = G \oplus (G \otimes \omega_T)$. Since all maps from $E$ to $G$ are trivial, it follows that the composition $E \to E \oplus (E \otimes \omega_T) \to G \otimes \omega_T$ is non zero. But $E$ and $G \otimes \omega_T$ are stable of the same slope, so that

$$E \cong G \otimes \omega_T$$

□

Let $F$ be a pure dimension one sheaf on $T$, as for K3 surfaces, we can define the Mukai vector,

$$v(F) = (0, c_1(F), \chi(F)) \in \mathbb{Z} \oplus \text{NS}(T) \oplus \mathbb{Z}.$$ 

If $c_1(F)$ is primitive, then by Remark 2.11, so is $v = v(f^*F)$ in $H^*(S, \mathbb{Z})$.

Lemma 2.11. Let $B$ be a scheme, and let $\mathcal{F}$ on $T \times B$ be a flat family of pure dimension one sheaves on $T$. Then $f_B^*\mathcal{F}$ is a flat family of pure dimension one sheaves on $S$.

Proof. Since $f$ is a flat morphism, we only need to prove that for any pure dimension one sheaf $F$ on $T$, the pullback $f^*F$ is pure of dimension one. Let $Q \subset f^*F$ be a sub sheaf with zero-dimensional support. Since $f_*Q \subset F \oplus F \otimes \omega_T$, it follows that $f_*Q$, and thus $Q$, is the zero sheaf. □

Set $w = (0, [C], \frac{1}{2})$. With this notation and using the above lemmas, there is a well defined pullback map

$$\Phi : M_{w,A}(T) \to M_{v,H}(S),$$

$$[F] \mapsto [f^*F]$$

where $M_{w,A}(T)$ is the moduli space of $A$-stable sheaves on $T$ with Mukai vector $w$, and $H = f^*A$. For simplicity we only consider the connected component of sheaves with support in $|C|$, even

\footnote{Recall that, contrary to what happens for torsion free sheaves, tensoring a pure sheaf of codimension greater or equal to one by a line bundle does not in general preserve stability.}
though the same results hold for those sheaves with support in $|C'|$. As in [Kim98], we can describe the fibers of this map.

**Lemma 2.12.** [Tak73] Let $E$ be a sheaf on $T$. If $E \cong E \otimes \omega_T$, then $f^*E$ is not simple.

**Proof.** We have

$$\text{Hom}(f^*E, f^*E) = \text{Hom}(E, f_*f^*E) = \text{Hom}(E, E) \oplus \text{Hom}(E, E \otimes K_S) \cong \text{Hom}(E, E) \oplus \text{Hom}(E, E).$$

Thus $C \oplus C \subseteq \text{Hom}(f^*E, f^*E)$, and the Lemma is proved. \hfill \Box

If $T$ is general so that $\text{Amp}(S) = f^* \text{Amp}(T)$ we can choose a polarization $A$ that is $w$-generic and such that $H = f^*A$ is $v$-generic, so that on both surfaces every semi-stable sheaf is stable. Under these conditions we have that the pullback of stable sheaves is again stable and not just semi-stable.

**Corollary 2.13.** Let $A$ be such that the strictly semi-stable loci of $M_{w,A}(T)$ and of $M_{v,H}(S)$ are empty. Then for each sheaf $[E] \in M_{w,A}(T)$, we have $E \not\cong E \otimes \omega$.

**Proof.** Let $E$ be a stable sheaf on $T$. By assumption, $f^*E$ is stable and thus simple. It follows from Lemma 2.12 that $E \not\cong E \otimes \omega_T$. \hfill \Box

As in [Kim98], let us now proceed to study the map $\Phi$. First, notice that $\Phi$ factors via the closed embedding $\text{Fix}(\iota) \subset M_{v,H}(S)$.

Let

$$\text{(2.15)} \quad \Phi': M_{w,A}(T) \to \text{Fix}(\iota),$$

be the induced morphism. We only consider the component of $\text{Fix}(\iota)$ that lies over $|C|$. For simplicity we set

$$\text{(2.16)} \quad N = M_{w,A}(T), \quad \text{and} \quad M = M_{v,H}(S).$$

**Proposition 2.14.** The morphism $\Phi'$ defined above is étale.

**Proof.** First, we prove, as in [Kim98], that $\Phi'$ is surjective. Let $F$ be a $H$-stable sheaf on $S$ which is $\iota^*$-invariant, we want to show that,

$$\text{(2.17)} \quad F = f^*G,$$

for some sheaf $G$ on $T$. Since $T$ is a quotient of $S$ by a $\mathbb{Z}/(2)$ action, the descent data translates into the existence of a morphism $\varphi: \iota^*F \to F$, such that the following diagram is commutative

$$\begin{array}{ccc}
\iota^* \iota^*F & \xrightarrow{\iota^*\varphi} & \iota^*F \\
\downarrow & & \downarrow \varphi \\
F & \xrightarrow{id} & F
\end{array}$$

Since $F$ is simple this can always be achieved by multiplying any isomorphism $\iota^*F \to F$ by a suitable scalar.

It follows that there exist a sheaf $G$ such that $f^*G = F$. It is easy to check that $G$ is $A$-stable. Moreover, if $\text{Supp}(F) \in f^*|C|$, then $\text{Supp}(G) \in |C|$ and surjectivity is proved.
We next prove that $\Phi'$ is unramified. Consider a point $[F]$ in $N$. Since $F$ and $f^*F$ are stable sheaves, it is well known (cf. for e.g., Section 4.5 of [HL97]) that the tangent spaces are

$$T_{[F]}N \cong \text{Ext}^1_T(F,F), \quad \text{and} \quad T_{[f^*F]}M \cong \text{Ext}^1_S(f^*F,f^*F).$$

Moreover, the tangent map of $\Phi$ is just the natural morphism

$$d\Phi : \text{Ext}^1_T(F,F) \rightarrow \text{Ext}^1_S(f^*F,f^*F),$$

which is injective since $F$ is not isomorphic to $F \otimes \omega_T$.\footnote{As usual, set $F' = F \otimes \omega_T$, and let $0 \rightarrow F \rightarrow X \rightarrow F \rightarrow 0$ be a non trivial extension of $F$ by $F$. Notice that since $F$ is stable and not isomorphic to $F'$, we can deduce that $\text{Hom}(X,F') = 0$. If the pullback $0 \rightarrow f^*F \rightarrow f^*X \rightarrow f^*f^*F \rightarrow 0$ splits, so does the sequence $0 \rightarrow f_*f^*F \rightarrow f_*f^*X \rightarrow f_*f^*f^*F \rightarrow 0$. However, since $\text{Hom}(X,F') = \text{Hom}(X',F) = 0$, it follows that any morphism $X \oplus X' \rightarrow F \oplus F'$ is diagonal. Hence, any morphism splitting $0 \rightarrow f_*f^*F \rightarrow f_*f^*X \rightarrow f_*f^*F \rightarrow 0$ induces a splitting of $0 \rightarrow F \rightarrow X \rightarrow F \rightarrow 0$ as well.}

Since $d\Phi$ factors via $d\Phi'$, it follows that $d\Phi'$ is also injective. On the other hand, since $\text{Fix}(\iota)$ has the same dimension of $N$ and is smooth, it follows that $d\Phi'$ is an isomorphism. In particular, $\dim T_{[F]}N = \dim N$ so $N$ is smooth at $[F]$, and the claim follows.\hfill \Box

**Corollary 2.15.** If $\text{Fix}(\iota^*)$ non empty, then every component is Lagrangian.

**Proof.** Suppose that $\text{Fix}(\iota^*)$ is non empty. By Proposition 2.14 every components is the image, under a finite pullback map, of a smooth moduli space of $A$-stable sheaves on $T$ with Mukai vector $w = (0, |C|, \frac{1}{2})$. It follows that the dimension of every connected component is equal to $\dim \text{Ext}^1_T(F,F)$. Since we already know that all the components of $\text{Fix}(\iota^*)$ are isotropic, this proves that they are Lagrangian.\hfill \Box

From what was said above, follows that there is a regular fix point free involution

$$\chi = 2\chi' = 2d' - 2g + 2,$$

for some integer $d'$. Let

$$\nu : N \rightarrow |C|,$$

be the support map. By construction, the involution commutes with the projection $\rho : M \rightarrow |D|$ and $\Phi$ is compatible with $\nu$ and $\rho$. Every fiber $M_t$ inherits an involution $\iota_t$ and there is an induced étale double cover

$$N_t \rightarrow \text{Fix}(\iota_t).$$

Notice that for the Jacobian of smooth curves, this is induced by the sequence

$$1 \rightarrow \mathbb{Z}/(2) \rightarrow \text{Jac}(C) \xrightarrow{f^*} \text{Fix}(\iota_t) \subset \text{Jac}(D).$$

By [Mum74], (vi) Section 2 and Corollary 2 Section 3, the fixed locus $\text{Fix}(\iota)$ is exactly $f^*(\text{Jac}_{\iota^*}(C))$. For later use, we highlight the following
Remark 2.16. From [Mum74] it also follows that $\text{Fix}(\iota)$ is equal to $\text{Im}[(1 + \iota) : \text{Pic}^0(D) \to \text{Pic}^0(D)]$. Now, since
$$\text{Pic}^0(D) = H^1(D, \mathcal{O})/H^1(D, \mathbb{Z}), \quad \text{Pic}^0(C) = H^1(C, \mathcal{O})/H^1(C, \mathbb{Z}),$$
and
$$(1 + \iota)H^1(D, \mathcal{O}) = H^1(D, \mathcal{O})^+,$$ it follows that
$$\text{Fix}(\iota) = f^*H^1(C, \mathcal{O})/(\iota^*H^1(C, \mathcal{O})) \cap H^1(D, \mathbb{Z})$$
$$= H^1(D, \mathcal{O})^+/H^1(D, \mathbb{Z}) \cap H^1(D, \mathcal{O})^+,$$
where $H^1(D, \mathcal{O})^+$ denotes the $\iota$-invariant subspace of $H^1(D, \mathcal{O})$.

The following Lemma will be used in Section 3.

Lemma 2.17. Both inclusions
$$(2.24) \quad (1 + \iota)(H^1(D, \mathbb{Z})) \subset f^*H^1(C, \mathbb{Z}), \quad f^*H^1(C, \mathbb{Z}) \subset H^1(D, \mathbb{Z})^+$$
are index 2 sub lattices.

Proof. From the short exact sequence (2.22) and Remark 2.16 it immediately follows that
$$f^*H^1(C, \mathbb{Z}) \subset H^1(D, \mathcal{O})^+$$
is an index two sub lattice.

For the first inclusion, we argue as follows, identifying as we may $H^1(C, \mathbb{Z})$ and $H_1(C, \mathbb{Z})$. The degree 2 unramified cover $f : D \to C$ is induced by an index two sub lattice $K \subset H_1(C, \mathbb{Z})$. Let $\alpha$ in $H_1(C, \mathbb{Z})$ be a primitive element that is not sent to zero under the natural projection onto $H_1(C, \mathbb{Z})/K = \mathbb{Z}/(2)$. Setting $\alpha_1 := \alpha$, it is possible to complete $\alpha$ to a symplectic basis $\{\alpha_i, \beta_i\}$, such that the elements $\alpha_i$, for every $i \neq 1$, and $\beta_i$, for every $i$, lie in $K$. By construction,
$$f^*K \subset (1 + \iota)H^1(D, \mathbb{Z}).$$
This is because the double cover is trivial when restricted to any cycle $\gamma$ representing a class in $K$, so that
$$f^*\gamma = x + \iota x,$$
with $x$ and $\iota x$ disjoint cycles. On the other hand, since $\tilde{\alpha} := f^{-1}(\alpha)$ is connected and $\iota(\tilde{\alpha}) = \tilde{\alpha}$, we have
$$2\tilde{\alpha} = (1 + \iota)\tilde{\alpha}.$$ To conclude, it is sufficient to notice that $\tilde{\alpha}$ is not in $(1 + \iota)H^1(D, \mathbb{Z})$ since we can find a $\tilde{\beta}$ in $H^1(D, \mathbb{Z})$ such that $\iota\tilde{\beta} = \tilde{\beta}$, and $(\tilde{\alpha}, \tilde{\beta}) = 1$.

Notice that from the proof above, it also follows that
$$(2.25) \quad H^1(D, \mathbb{Z})^+/H^1(D, \mathbb{Z}) \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2).$$
Recall that $|C| \cong |C'| \cong \mathbb{P}^{g-1}$, and let $U \subset |C|$ and $U' \subset |C'|$ be the open subsets parametrizing smooth curves.

Let
$$X := X_{v, H, C},$$
be the connected component of $\text{Fix}(\iota)$ containing $\text{Fix}(\iota_t)$, for some $t$ in $U$. We can now sum up the results of this section in the following theorem.

---

3We can choose for $\tilde{\beta}$ one of the connected components of $f^{-1}(\beta_0)$.
Theorem 2.18. The connected component $X$ of $\text{Fix}(\iota)$ is a smooth projective Lagrangian subvariety of $M$. The pullback morphism $\Phi' : N \to X$ of (2.15) is an étale double cover and the restriction of $\rho : M \to |D|$ to $X$ induces a fibration
\begin{equation}
\pi : X \to |C|,
\end{equation}
whose general fiber is isogenous to the Jacobian of a general curve in the linear system $|C|$. If we choose $d = 0$, the fibration $X \to |C|$ has a rational section
\begin{equation}
\sigma : |C| \longrightarrow X,
\end{equation}
derived on the complement of a closed subset strictly contained in the discriminant locus. The analogous statement holds for $X' \to |C'|$.

Proof. First, notice that $\text{Fix}(\iota)$ is non empty because in (2.20) we set $\chi$ to be even. Recall also that $\rho$ is $\iota$-equivariant, so that there is an induced proper morphism $\pi : X \to |C|$, which is surjective because the image contains the open subset $U \subset |C|$ of smooth curves.

For any $t$ in $U \subset |C|$, the intersection $X \cap M_t$ is non empty, and thus equal to $\text{Fix}(\iota_t)$. The fact that $X$ is Lagrangian was shown in Corollary 2.15. To see that $\pi : X \to \mathbb{P}^{g-1}$ has connected fibers, one can argue as follows. Let $X \xrightarrow{a} Z \xrightarrow{b} \mathbb{P}^{g-1}$ be its Stein factorization. The morphism $b$ is finite, but it is also birational since $\pi$ has connected fibers over a dense open subset of $\mathbb{P}^{g-1}$. Since $Z$ is normal and $\mathbb{P}^{g-1}$ is smooth, $b$ is an isomorphism and hence $\pi$ has connected fibers. As for existence of the rational section, it clearly follows from the existence of the rational section for $\rho$ in the case $d = 0$. □

Remark 2.19. Let $X_U$ and $N_U$ be the inverse images of $U$ under $\pi$ and $\nu$ respectively. Then $\pi : X_U \to U$ and $N_U \to U$ are smooth morphisms. Let
\begin{equation}
\Delta \subset |C|,
\end{equation}
be the discriminant locus of the family of curves, and let $X_\Delta$ and $N_\Delta$ be the inverse images. It was proved in [AIK77] that the compactified Jacobian of a reduced and irreducible curve of genus $g$ with planar singularities is irreducible, i.e. any rank one torsion free sheaf can be obtained as the limit of line bundles, and of dimension $g$.

In particular, the two fibrations $\nu$ and $\pi$ are flat when restricted to the locus of irreducible curves.

However, different problems may arise when dealing with non integral curves. First, notice that the compactified Jacobian of a given degree should be replaced by a Simpson moduli space with the appropriate numerical invariants, where the stability is taken with respect to a given polarization. In general, there can exist higher dimensional components of these moduli spaces.

Example 2.20. Consider a smooth curve $\Gamma'$ of genus $\gamma' \geq 2$, and denote by $\Gamma$ a non reduced scheme obtained by considering a double structure on $\Gamma'$. We denote by $\gamma$ the genus of $\Gamma$. It was shown by Chen and Kass in [CK11], that all components of the Simpson moduli space have dimension $\gamma$, except possibly a $(4\gamma' - 3)$-dimensional component, which exists when $4\gamma' - 3 \geq \gamma$ and parametrizes rank 2 semi-stable sheaves on $\Gamma'$. Let us now suppose that $\Gamma$ and $\Gamma'$ are contained in a smooth surface $Y$, so that the scheme structure defining $\Gamma$ is the one induced by the ideal sheaf $\mathcal{O}(-2\Gamma')$.

By the adjunction formula,
\begin{equation}
\gamma = 4\gamma' - 3 - \deg \omega_{\Gamma'}|^{\Gamma'}.
\end{equation}

It follows that the Simpson moduli space is pure of dimension $\gamma$ if and only if,
\begin{equation}
\deg \omega_{\Gamma'}|^{\Gamma'} = 0.
\end{equation}
We would like to stress on the fact that (2.28) is satisfied for every curve contained in a surface Y, as soon as the canonical bundle of Y is numerically trivial (K3, abelian, Enriques and bi-elliptic surfaces).

Moreover, for linear systems on K3 or abelian surfaces this kind of equidimensionality results are known. In fact, if we let \( v = (0, D, \chi) \) be any primitive Mukai vector and if we choose a \( v \)-generic polarization \( H \), it follows by [Mat99] that the relative compactified Jacobian \( M_{v, H} \) is flat over \( |D| \) (cfr Remark 2.2). The proof, however, relies on the existence of a symplectic structure on these moduli spaces and cannot therefore be applied to moduli spaces of sheaves on other surfaces.

It would be very nice to the answer to the following Question.

**Problem 2.21.** When is the relative compactified Jacobian of a linear system on a smooth projective surface flat? Is it flat for linear systems on Enriques surfaces? Can there exist irreducible components of \( N_\Delta \) which map to codimension \( \geq 2 \) subsets of \( \Delta \)?

The expectation of the author is that the relative compactified Jacobian of a linear system on an Enriques surface should be equidimensional.

For Theorems 3.1 and 5.3 we will make use of the following assumption

**Assumption 2.22.** The linear system \( |C| \) is such that there are no irreducible components of \( N_\Delta \) which map to codimension \( \geq 2 \) subsets of \( \Delta \).

Notice that in some cases of low genus, where the curve of the linear system do not degenerate too much, one can show that the relative compactified Jacobian is equidimensional and thus flat. Examples of linear systems that satisfy equidimensionality are

\[
|e_1 + e_2|, \text{ with } e_1 \cdot e_2 = 1, \quad g(C) = 2, \quad \dim N = 3,
\]

\[
|e + f|, \text{ with } e \cdot f = 2, \quad g(C) = 3, \quad \dim N = 5,
\]

\[
|e_1 + e_2 + e_3| \text{ with } e_i \cdot e_j = 1 \text{ for } i \neq j, \quad g(C) = 4, \quad \dim N = 7.
\]

where \( e_1, e_2, e_3, e \) and \( f \) are primitive elliptic curves. Notice that these linear systems do not contain non-reduced curves.

**Lemma 2.23.** Let \( T \) be a general Enriques surface. Let \( |C| \) be a linear system on \( T \), satisfying Assumption 2.22. For every irreducible component \( \Delta_i \) of the discriminant locus, the divisors \( N_{\Delta_i} \) are irreducible and isomorphic to the pullback \( v^* \Delta_i \).

**Proof.** Under the generality assumption on the Enriques surface, the general points of an irreducible component of the discriminant parametrize either irreducible curves, or the union of an elliptic curve and a smooth hyperelliptic curve meeting in one point. This is because the irregularity of the surface is zero so that any deformation of a curve is linearly equivalent to the curve itself. Hence the loci parameterizing reducible curve are isomorphic to the product of linear systems. Moreover, if \( C = C_1 \cup C_2 \), then \( |C_1| \times |C_2| \) has codimension \( \geq 2 \) in \( |C| \), unless one of the two curves, say \( C_1 \), is a primitive elliptic curve and \( C_1 \cdot C_2 = 1 \) (see Proposition 2.9 of [ASF12]). We here remark that every smooth curve in the linear system is necessarily hyperelliptic, since the pencil \( 2C_1 \) cuts a \( g_2 \) on every curve of \( |C| \). See [CD89] for a more detailed discussion on linear systems whose general member is hyperelliptic. On the other hand, recall that the Jacobian of two smooth curves meeting transversally in one point is smooth and compact, in particular, it is irreducible. From Remark 2.19 it follows that for every \( i \) the general fiber of \( N_{\Delta_i} \) is irreducible. Hence, since we are assuming that no irreducible component of \( N_\Delta \) maps to codimension \( \geq 2 \) subsets of \( \Delta \), \( N_{\Delta_i} \) itself

---

\[^4\]I would like to thank J. Kollár for pointing out to me that it is sufficient to ask for this last assumption and not for flatness.
is irreducible.
Let us consider the case $d' = 0$. In this case, there is a section of $X \to |C|$ that is defined at the general point of every component of the discriminant locus. Indeed, it is defined on the locus of irreducible curves, but also on the locus of curves that are the union of an elliptic curve and a smooth curve meeting in one point. In particular, it is defined on a simply connected open set. Hence it lifts to a rational section (actually two sections, the second of which parametrizes the restriction of $\omega_T$ to the curves) of $\nu$. It follows that all the components of $N_\Delta$ and of $X_\Delta$ are reduced.

As for the case of arbitrary $d'$, observe that the restriction of $X$ (and of $N$) to a general pencil $\mathbb{P}^1 \subset |C|$, always has a section since the corresponding pencil of curves always has a section. It follows that for any $d'$ the components of $N_\Delta$ and of $X_\Delta$ are reduced, and hence

$$X_{\Delta_i} = \pi^* \Delta_i, \quad N_{\Delta_i} = \nu^*(\Delta_i).$$

\[\Box\]

3. Fundamental group of relative compactified Jacobians

This section is devoted to calculating the fundamental group of the relative compactified Jacobian variety constructed in Section 2. For simplicity will do it in the case of degree zero Jacobian varieties, even though the arguments work whenever there is a rational section, defined on an open subset of $|C|$ whose complement has codimension $\geq 2$.

Throughout the section we will assume that $|C|$ is a linear system with the property that the generic line $\mathbb{P}^1 \subset |C|$ is a Lefschetz pencil. In order to prove the main result of this section, we will also need Assumption 2.22.

Notice that the linear systems described in (2.29) can be chosen general in moduli, in order to ensure the existence of Lefschetz pencils.

Let $v$ be the Mukai vector $(0, D, \chi)$ where $\chi = -h + 1$ and let $X = X_{v,H,C}$ be the smooth projective variety of Theorem 2.18. Recall that since $\chi = -h + 1$, there is a rational section (2.27) which is defined on an open set containing the locus of reduced and irreducible curves in $|C|$.

**Theorem 3.1.** Let $X$ and $|C|$ be as above, and suppose that $|C|$ satisfies Assumption 2.22. Then,

$$\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$  

Before proving this theorem, we will make a few preliminary remarks.

The main ingredients in the proof are the following theorem of Leibman, as used in [MT07], and the theory of Picard-Lefschetz (cf. [ACG10], Chapter XI).

**Theorem 3.2** (Leibman, [Lei93]). Let $p: E \to B$ be a surjective morphism of smooth connected manifolds. Assume that $p$ has a section $s$. Let $W \subset B$ be a union of locally closed submanifolds of real codimension at least two. Set $U = B \setminus W$ and $V = p^{-1}(U)$ and assume that $V \to U$ is a locally trivial fibration with fiber $F$. Consider the exact sequence

$$1 \to \pi_1(F) \xrightarrow{j} \pi_1(V) \xrightarrow{s_*} \pi_1(U) \to 1.$$  

Set $H = \ker(\pi_1(U) \to \pi_1(B))$ and view it, via $s_*$, as a subgroup of $\pi_1(V)$. Let $R = [\pi_1(F), H]$ be the commutator subgroup of $\pi_1(F)$ and $H$ in $\pi_1(V)$. Then there is a split exact sequence

$$1 \to R \to \pi_1(F) \xrightarrow{s_*} \pi_1(E) \cong \pi_1(B) \to 1.$$
Notice that, by definition, $R$ is generated by elements of type
\[(3.3) \ c^{-1}\tilde{\lambda}^{-1}c\tilde{\lambda},\]
where $c \in \pi_1(F)$ and $\tilde{\lambda} = s_s(\lambda)$ is a lifting of $\lambda \in \pi_1(U)$ to $\pi_1(V)$.

By a theorem of Zariski, the fundamental group of the complement of a hypersurface in a projective space is generated by the fundamental group of the complement of the intersection of the hypersurface with a general line. We will use this fact in dealing with the fundamental groups of the loci that parametrize smooth curves
\[U = |C| \setminus \Delta \subset |C|, \ V \subset |D|.
\]
Via $f^*$ we view $|C|$ as a subspace of $|D|$. We denote by $\Delta$ the discriminant locus of $|C|$ and by $\Sigma$ the discriminant locus of $|D|$. Notice that
\[\deg \Sigma = 2 \deg \Delta = 2\delta.
\]
Consider a general line $\ell \subset |C|$. Let $t_1, \ldots, t_\delta$ be the points of the intersection $\Delta_\ell = \ell \cap \Delta$. By Lemma 2.3, the curve $D_{t_i}$ is irreducible if and only if $C_{t_i}$ is irreducible. Moreover, since we are assuming that $\ell$ is a Lefschetz pencil, the curve $D_{t_i}$ has two simple nodes and the two nodes are interchanged by the involution. In other words, $\ell$ meets $\Sigma$ in its double locus. More specifically, we have the following

**Lemma 3.3.** At each point $t_i$ of $\Delta_\ell$ such that $D_{t_i}$ is irreducible, there are two branches of $\Sigma$ that meet transversally. Deforming the curve along either branch results in smoothing out one of the two nodes of $D_{t_i}$. Moreover, the two branches are interchanged by the involution.

**Proof.** First we observe that it is enough to check that one can smooth out only one of the two nodes of $D_{t_i}$ at a time. Indeed, acting with $\iota$ interchanges the two nodes, so the involution has to send a family that smoothes out one of the two nodes to a family that smoothes out the other node. It follows that if we can smooth out one node at a time, there have to be two distinct components of the discriminant meeting in $t_i$.

Recall that a linear system on a K3 or Enriques surface is said to be hyperelliptic if it defines a degree two map onto a degree $g - 2$ surface in $\mathbb{P}^{g-1}$. By Proposition 4.5.1 of [CD89], $|C|$ has base points, if and only if it is hyperelliptic. In this case, $|C|$ has two simple base points, and the general member of $|C|$ is a smooth hyperelliptic curve. Since we are assuming $T$ to be general, by Corollary 4.5.1 of [CD89] it follows that
\[C = ne_1 + e_2, \ D = nE_1 + E_2, \ n \geq 1,
\]
where the $e_i$ are primitive elliptic curve and $e_1 \cdot e_2 = 1$ and $E_i = f^{-1}(e_i)$. It is well known [SD74] that $\varphi_{|D|}$ maps $S$ onto a degree $h - 1$ rational normal scroll $R \subset \mathbb{P}^h$. Notice, moreover, that $|E_1| \times |(n - 1)E_1 + E_2|$ is a multiplicity two component of $\Sigma$ and that it parametrizes all the reducible curves in $|D|$. Notice also that
\[\left(|E_1| \times |(n - 1)E_1 + E_2|\right) \cap \Delta = |e_1| \times |(n - 1)e_1 + e_2|.
\]

Going back to the proof, let us assume that linear systems are not hyperelliptic. Under our generality assumption the two surfaces contain no rational curves, and it follows that $|D|$ is actually very ample (see [SD74] Thm 6.1). In particular, the discriminant locus $\Sigma$ is irreducible, and the general point parametrizes curves with just one node. Hence, we can deform $D_{t_i}$ so that only one of the two nodes is smoothed out.

Now, let us assume that $|C|$ is hyperelliptic. In particular, all the singular curves in $|D|$ that don’t belong to $|E_1| \times |(n - 1)E_1 + E_2|$ are irreducible and thus cover a smooth rational curve. We claim that the discriminant locus of $|D|$ is the union of two components, one which we have described
above, and the other one parametrizing irreducible curves. Moreover, we claim that the general curve parametrized by this component has only one node. Indeed, the ramification curve $\Gamma$ of $\varphi$ is smooth so that imposing a tangency condition to $\Gamma$ of a hyperplane section of $R$ is a codimension two linear condition in $|D|$. This second component is thus isomorphic to a $\mathbb{P}^{h-2}$-bundle over $\Gamma$. In particular it is irreducible and the general point parametrizes curve with only one node and no other singularity.

Fix a base point $o \in \ell \setminus \Delta_{\ell}$. For each point $t_1, \ldots, t_\delta$ we can consider a cycle $\lambda_i \in \pi_1(\ell \setminus \Delta_{\ell}, o)$ by considering a small disk around $t_i$ and joining the boundary of the disk with the base point. The set

$$\{\lambda_i\},$$

forms a set of generators of $\pi_1(\ell \setminus \{t_1, \ldots, t_\delta\})$ and thus of $\pi_1(U)$. To each of these $\lambda_i$'s, we can associate a vanishing cycle $\alpha_{\lambda_i}$. The vanishing cycle can be represented by a simple close loop, and the curve $C_{t_i}$ acquires a simple node as a result of the vanishing of $\alpha_{\lambda_i}$. Recall that the Picard-Lefschetz transformation $PL_\lambda$ associated to a path $\lambda$ is, by definition, the monodromy action on the cohomology of $C_o$ induced by the path $\lambda$. Under our assumptions, it is given by the formula,

$$PL_\lambda(c) = -c + (c, \alpha)\alpha, \quad \forall c \in H^1(C_o, \mathbb{Z}).$$

Consider a small deformation $\ell'$ of $\ell$ in $|D|$, and assume that it is transverse to the discriminant $\Sigma$. Suppose that $|C|$ and $|D|$ are hyperelliptic, and let us number the points $t_i$ so that

$$\{t_1, t_2\} = \ell \cap |e_1| \times |(n-1)e_1 + e_2|.$$

Then the monodromy action on the cohomology of $C_o$, induced by the cycles $\lambda_1$ and $\lambda_2$ is trivial, since the curves $C_{t_1}$ and $C_{t_2}$ are reducible and meet in one point (equivalently, because the vanishing cycle that produces the singularity of $C_{t_1} \cup C_{t_2}$ is connected and trivial in homology).

If we let $j : U \to V$ be the inclusion, then from the Picard-Lefschetz formula, it follows that the monodromy action of $j_*\lambda_1$ and $j_*\lambda_2$ is trivial.

**Remark 3.4.** Thus, from the point of view of the monodromy action on the first cohomology of $D_o$ and $C_o$, it thus follows that it is enough to consider the fundamental group of

$$U' = |C| \setminus \Delta', \quad \text{and} \quad V' = |D| \setminus \Sigma',$$

where

$$\Delta' = \begin{cases} \Delta & \text{if } |C| \text{ is not hyperelliptic}, \\ \Delta \setminus |e_1| \times |(n-1)e_1 + e_2| & \text{if } |C| \text{ is hyperelliptic} \end{cases}$$

and analogously for $\Sigma$, $\Delta_{\ell}$ and $\Sigma_{\ell}$.

Set $\ell' \setminus \Sigma' \cap \ell' = \{s_1, s'_1, \ldots, s_\delta, s'_\delta\}$, where $s_i$ and $s'_i$ are the intersections of $\ell'$ with the two branches of $\Sigma'$ that meet in $t_i$. In the same way as above, we can choose, for every $i$, cycles $\eta_i$ and $\eta'_i$ in $V'$ so that the set

$$\{\eta_i, \eta'_i\},$$

generates $\pi_1(\ell' \setminus \Sigma_{\ell})$, and thus $\pi_1(V)$.

Notice that we can set up things in such a way that $\iota_*\eta_i = \eta'_i$ and that,

$$j_*\lambda_i = \eta_i \circ \eta'_i.$$

For more details we refer the reader to [ASP12].
Moreover, if $\lambda_n$ and $\lambda_n'$ are the vanishing cycles associated to $\eta_i$ and $\eta_i'$, then
\begin{equation}
\iota(\lambda_n) = \lambda_n'.
\end{equation}

**Lemma 3.5.** Suppose that $E \to B$ is the family of relative Jacobian varieties of a family $Z \to B$ of curves. Suppose that $E \to B$ satisfies the assumptions of Theorem 3.2 and let $R$ be as in the theorem. Then $R$ is generated by vanishing cycles.

**Proof.** By unravelling the definition of Picard-Lefschetz transformation\footnote{I am grateful to E. Arbarello for teaching this to me.} and of the commutator $R$ of the theorem, one can check (cf. \cite{ASF12}) that, passing as we may to the additive notation, we have
\begin{equation}
c^{-1}\tilde{\lambda}^{-1}c\tilde{\lambda} = -c + PL_\lambda(c).
\end{equation}
It follows that $R$ is generated by
\[-c + c + (c, \alpha_\lambda)\alpha_\lambda = (c, \alpha_\lambda)\alpha_\lambda,
\]
with $\alpha_\lambda$ vanishing cycle. \hfill \Box

Let us now come to the proof itself.

As usual, given a point $t \in |D|$ we denote by $M_t$ the fiber of $\rho$ over $t$. The first step is to apply Theorem 3.2 to $\rho : M \to \mathbb{P}^h = |D|$. This yields the following

**Lemma 3.6.** Consider a point $t$ in $U$, and let $D_t$ be the corresponding curve. Recall that $M_t \cong H^1(D_t, \mathcal{O})/H^1(D_t, \mathbb{Z})$. Under the isomorphism $\pi_1(M_t) \cong H_1(M_t, \mathbb{Z}) \cong H^1(D_t, \mathbb{Z})$, the first cohomology group of $M_t$ is generated by vanishing cycles.

**Proof.** We apply Theorem 3.2 in the following setting: let $B \subset \mathbb{P}^h$ be the locus of irreducible curves, $E = \rho^{-1}(B)$ and $U \subset B$ the locus of smooth curves. Since $\text{codim}(\mathbb{P}^h \setminus B, \mathbb{P}^h) \geq 2$ it follows that $\pi_1(B) = \{1\}$. Moreover, the fibration is equidimensional so that $\text{codim}(M \setminus E, M) \geq 2$ so that $\pi_1(E) \cong \pi_1(M) = 1$. Equation 3.2 then yields
\begin{equation}
1 \to R \to \pi_1(M_t) \to \pi_1(E) \cong 1,
\end{equation}
and thus that $\pi_1(M_t) \cong R$. To conclude the proof it is sufficient to remark from Lemma 3.5 we know that $R$ is generated by vanishing cycles. \hfill \Box

Now we apply Theorem 3.2 to $X \to \mathbb{P}^{g-1}$. As in the previous case, we let $B' \subset |C|$ be the locus of the irreducible curves\footnote{In the case $|C|$ is hyperelliptic, we can consider the union of the locus of irreducible curve with the open subset $|C| \setminus \Delta'$.}, and we set $E' := \pi^{-1}(B') \subset X$. Recall that we have denoted by $U$ the locus parametrizing smooth curves. Under Assumption 2.22 the codimension of $X \setminus E'$ is $\geq 2$ and hence there is an isomorphism $\pi_1(E') \cong \pi_1(X)$.

Consider the following commutative diagram of abelian groups,
\begin{equation}
\begin{array}{ccccccc}
1 & \longrightarrow & R & \longrightarrow & \pi_1(M_t) & \longrightarrow & 1 \\
1 & \longrightarrow & R' & \longrightarrow & \pi_1(X_t) & \longrightarrow & \pi_1(X) \longrightarrow 1,
\end{array}
\end{equation}
and notice that the first two vertical maps are injections. To prove the theorem, it is sufficient to show that
$$\pi_1(X_t)/R' \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2).$$
Since $R = [\pi_1(M_t), \pi_1(V)]$ and $R' = [\pi_1(X_t), \pi_1(U)]$, the inclusion $R' \subset R$ is determined by the inclusion
\begin{equation}
\pi_1(X_t) \subset \pi_1(M_t),
\end{equation}
and by the pushforward
\begin{equation}
j_*: \pi_1(U) \to \pi_1(V),
\end{equation}
induced by the inclusion $j: U \hookrightarrow V$.

By Remark 2.16 it follows that the inclusion (3.11) is
\begin{equation}
(3.12)
\end{equation}
and by the pushforward
\begin{equation}
(3.13)
\end{equation}
\begin{equation}
(3.14)
\end{equation}
Recall that there is an étale double cover
\begin{equation}
\Phi': N \to X.
\end{equation}
Let us define a torsion line bundle $L$ on $X$, by setting
\begin{equation}
(3.16)
\end{equation}
Notice that \( L^{\otimes 2} \cong \mathcal{O}_X \), and that \( L \) generates the kernel of \( \Phi^* : \text{Pic}(X) \to \text{Pic}(N) \).

**Proposition 3.8.** The double cover \( N \to X \) is induced by the index two subgroup

\[
(3.17) \quad f^* H^1(C, \mathbb{Z})/(1 + i)H^1(D, \mathbb{Z}) \subset \pi_1(X).
\]

**Proof.** Let \( u : J \to X \) be the cyclic double cover induce by \((3.17)\), and let \( L' \) be the associated two-torsion line bundle. We will prove that \( L \) and \( L' \) are isomorphic. First of all, we prove that for every \( t \in U \), the induced coverings \( N_t \to X_t \) and \( J_t \to X_t \) are isomorphic, so that we find an isomorphism

\[
L_t \cong L'_t, \quad \forall t \in U.
\]

Indeed, \( N_t \to X_t \) is induced by the surjection of the exact sequence

\[
0 \to H^1(C_t, \mathbb{Z}) \overset{f^*}{\to} H^1(D_t, \mathbb{Z})^+ \to \mathbb{Z}/(2) \to 0.
\]

On the other hand, by definition, \( J_t \to X_t \) is induced by the composition of \( \pi_1(X_t) \to \pi_1(X) \) with the projection \( \pi_1(X) \to \pi_1(X)/(f^* H^1(C, \mathbb{Z})/(1 + i)H^1(D, \mathbb{Z})) \) which is exactly

\[
H^1(D_t, \mathbb{Z})^+ \to H^1(D_t, \mathbb{Z})^*/f^* H^1(C, \mathbb{Z}).
\]

Now let \( L_U \) and \( L'_U \) be the restrictions of the two line bundles to \( X_U \). Since \( L_U \) and \( L'_U \) agree on every fiber, we can write

\[
L_U \otimes L'_U \cong \pi^* \mathcal{I},
\]

for some \( \mathcal{I} \) in \( \text{Pic}(U) \). However, since the locus \( Z \) where the section \( \sigma \) is defined is simply connected, the restrictions \( L_{|\sigma(B)} \) and \( L'_{|\sigma(B)} \) to \( \sigma(B) \) are trivial, so that \( \sigma^*(L_U \otimes L'_U) = \mathcal{O}_U \) and hence \( \mathcal{I} \cong \mathcal{O}_U \). It follows that \( L \otimes L' \) is a linear combination of the boundary divisors. However, under our Assumption \( \ref{assumption2.22} \) we can use Lemma \( \ref{lemma2.23} \) to ensure that the irreducible components \( X_{\Delta_i} \) of \( X_\Delta \) are pullback of divisors on \( \mathbb{P}^{g-1} \). It follows that a linear combination of boundary divisor is not numerically trivial unless it actually trivial, and we may conclude that

\[
L \otimes L' \cong \mathcal{O}_X.
\]

\[\square\]

As consequence of this proposition, we find that

\[
(3.18) \quad \pi_1(N) \cong f^* H^1(C, \mathbb{Z})/(1 + i)H_1(D, \mathbb{Z}) \cong H_1(C, \mathbb{Z})/\text{fiber}H_1(D, \mathbb{Z}).
\]

Now let

\[
\Psi : Y \to N,
\]

be the universal covering space.

Using the so called Norm map, we can get a geometric interpretation of the covering \( Y_U \to N_U \), at least in the case of degree zero Jacobians\footnote{I am grateful C. Voisin for suggesting this nice geometric interpretation to me.} Let \([C_i] \subset [C]\) be a smooth curve, and set \( D_t = f^{-1}(C_t) \).

The Norm map is a morphism of abelian varieties \( \text{Norm} : \text{Pic}^0(D_t) \to \text{Pic}^0(C_t) \) defined by

\[
\text{Norm}_t : L \mapsto \det(f_*L) \otimes \det(f_*\mathcal{O}_D)^Y.
\]

Let \( M_U \) be the restriction of \( M = M_{\text{et},H}(S) \) to \( U \). We can consider a relative norm map

\[
\text{Norm}_U : M_U \to N_U,
\]

which is surjective. Over each curve \([C_i] \subset U\) the kernel of \( \text{Norm}_t \) has two connected components (cf. \[\text{Mum74}\]). It follows that \( \text{Norm}_t \) does not have connected fibers, and we can consider the Stein factorization,

\[
M_U \to Q_U \to N_U.
\]
We claim that the double cover $Q_U \to N_U$ is isomorphic to the double cover $Y_U \to N_U$. Indeed, the covering $Q_t \to N_t$ corresponds to the inclusion $f_*H_1(D_t, \mathbb{Z}) \subset H_1(C_t, \mathbb{Z})$, so that we can conclude with an argument just like that in the proof of Proposition 3.17.

Compactifications of the identity component of the kernel of Norm$_U$ are studied in [ASF12].

In the following proposition we analyze the induced covering space of the general singular fiber.

**Proposition 3.9.** Let $C_0 \in |C|$ be an irreducible curve with one single node and no other singularity, and let $N_0$ be its compactified Jacobian. Then,

$$Y_0 := \Psi^{-1}(N_0),$$

is irreducible.

**Proof.** Recall from [OST9] that if $0 \in |C|$ is a point corresponding to an irreducible curve with a single node, then the Jacobian $N_0$ of $C_0$ is l.c.i and has the following properties. The normalization $\tilde{N}_0$ of $N_0$ is smooth and isomorphic to a $\mathbb{P}^1$-bundle over the Jacobian of the normalization $\tilde{C}_0$ of $C_0$. Moreover, the normalization morphism $n : \tilde{N}_0 \to N_0$ identifies, up to a twist on the base, two section of the $\mathbb{P}^1$-bundle. We claim that in order to prove that $Y_0$ is irreducible it is sufficient to prove that the covering $\Psi^{-1}(\text{Sing}(N_0))$ is non trivial. Let us assume this. Since $\text{Sing}(Y_0) \cong \Psi^{-1}(\text{Sing}(N_0))$, it then follows that $\text{Sing}(Y_0)$ is connected and isogenous to $\text{Jac}(\tilde{C}_0)$.

To prove that, under this hypothesis $Y_0$ is irreducible, we proceed by contradiction and suppose that $Y_0$ is the union $Z_1 \cup Z_2$ of two irreducible components. We then have $Z_1 \cap Z_2 \cong \text{Sing}(Y_0)$. Moreover, since the morphism $Z_1 \cup Z_2 = Y_0 \to N_0$ is étale, the two branches meeting in $\text{Sing}(Y_0)$ are smooth. Hence both $Z_1$ and $Z_2$ are smooth and thus the induced finite birational morphism $Z_i \to N_0$, $i = 1, 2$, factors through the normalization of $N_0$ via a lifting $Z_i \to \tilde{N}_0$. Under this last morphism the image of the connected set $Z_1 \cap Z_2$ should be the disjoint union of two sections of the $\mathbb{P}^1$-bundle $\tilde{N}_0$, which is absurd.

Let us now prove that the induced covering of $\text{Sing}(N_0) \cong \text{Jac}(\tilde{C}_0) \subset N_0$ is non trivial. Recall that there is an embedding $i_0 : C_0 \to N_0$, and that this embedding is such that the node of $C_0$ is exactly the intersection of $C_0$ with the singular locus of $N_0$. Moreover, one can see that $n^{-1}(C_0) \subset \tilde{N}_0$ is the normalization $\tilde{C}_0$ of $C_0$.

Notice that the image of $\pi_1(\text{Sing}(N_0))$ in $\pi_1(N_0)$ coincides with the image

$$n_* (\pi_1(\tilde{N}_0)) \subset \pi_1(N_0).$$

Since there is an isomorphism $H_1(\tilde{N}_0, \mathbb{Z}) \cong H_1(\text{Jac}(\tilde{C}_0, \mathbb{Z}), \mathbb{Z}) \cong H_1(\tilde{C}_0, \mathbb{Z})$ that is compatible with the normalization morphisms $\tilde{N}_0 \to N_0$ and $\tilde{C}_0 \to C_0$ and the inclusion $C_0 \to N_0$, to prove that the covering of $\text{Sing}(N_0)$ is non trivial it is sufficient to prove that the composition

$$H_1(\tilde{C}_0, \mathbb{Z}) \xrightarrow{n_*} H_1(C_0, \mathbb{Z}) \xrightarrow{i} H_1(N, \mathbb{Z}) \cong \mathbb{Z}/(2),$$

is non trivial, where $i$ is the embedding $C_0 \to N$.

Choose a base point $t \in U$ in a neighborhood of $0$, and let $C_t$ and $D_t$ be the corresponding curves. We have seen (cf. (3.18)) that

$$H_1(N, \mathbb{Z}) \cong H_1(C_t, \mathbb{Z})/f_*H_1(D_t, \mathbb{Z}),$$

where the isomorphism is induced by the embeddings $C_t \subset N_t \subset N$.

First, we claim that $i_*$ can be identified with the surjection

$$p : H_1(C_0, \mathbb{Z}) \to H_1(C_0, \mathbb{Z})/f_*(D_0, \mathbb{Z}).$$

In fact, if $t$ is close enough to $0$, we can consider retractions $r : C_t \to C_0$ and $s : D_t \to D_0$. These retractions induce the following commutative diagram
The last thing that we need to prove is that $\nu^* \Delta_i$ is irreducible and that it satisfies Assumption 2.22. Let us

\begin{equation}
\text{(3.20)}
\end{equation}

Thus $p$ and $i^*$ are identified and we have proved the claim.

The last thing that we need to prove is that

\begin{equation}
\text{(3.20)}
\end{equation}

is non zero. Let $m : \tilde{D}_0 \to D_0$ be the normalization of $D_0$, and let $\tilde{f} : \tilde{D}_0 \to \tilde{C}_0$ be the induced double cover. Since $D_0$ and $\tilde{D}_0$ are irreducible, it is not hard to see that there is a commutative diagram

\[
\begin{array}{c}
H_1(D, Z) \xrightarrow{s_*} H_1(D_0, Z) \xrightarrow{f_*} H_1(C, Z) \\
\downarrow \quad \downarrow \\
Z \xrightarrow{i_*} H_1(C, Z) \xrightarrow{r_*} H_1(C_0, Z) \xrightarrow{p} 0 \\
\end{array}
\]

where the kernels of $s_*$ and of $r_*$ are generated by the vanishing cycles of $D_0$ and $C_0$ respectively. Thus $p$ and $i^*$ are identified and we have proved the claim.

The last thing that we need to prove is that

\begin{equation}
\text{(3.20)}
\end{equation}

is non zero. Let $m : \tilde{D}_0 \to D_0$ be the normalization of $D_0$, and let $\tilde{f} : \tilde{D}_0 \to \tilde{C}_0$ be the induced double cover. Since $D_0$ and $\tilde{D}_0$ are irreducible, it is not hard to see that there is a commutative diagram

\[
\begin{array}{c}
H_1(D, Z) \xrightarrow{m_*} H_1(D_0, Z) \xrightarrow{f_*} H_1(C, Z) \xrightarrow{n_*} H_1(C_0, Z) \xrightarrow{\nu^*} 0 \\
\end{array}
\]

Since $\text{coker } f_* \cong \text{coker } \tilde{f}_*$, it follows that $n_* H_1(C_0, Z)$ is not contained in $f_* H_1(D_0, Z)$, and thus \text{(3.20)} is non zero, and the proof is complete.

\begin{corollary}
Set $y = \nu \circ \Psi$. Then $y : Y \to |C|$ is a fibration in abelian varieties. Moreover, for every irreducible component $\Delta_i$ of the discriminant locus, the divisors $Y_{\Delta_i}$ are irreducible and isomorphic to the pullback $\nu^* \Delta_i$.
\end{corollary}

\section{The second Betti number}

This section is devoted to calculating the second Betti number of the relative compactified Jacobian varieties of Section 2.

We assume that $|C|$ is base point free and such that the discriminant locus $\Delta$ is irreducible. By Proposition 4.5.1 of [CD89], our assumption excludes the case that $C$ is hyperelliptic but is satisfied whenever $|C|$ is very ample. In particular, $g \geq 3$. In the last section we do the $g = 2$ case which can be appropriately generalized to all the hyperelliptic linear systems.

\begin{theorem}
Assume that $|C|$ is base point free, and that it satisfies Assumption \text{[2.22]}. Let us moreover assume that the discriminant locus of $|C|$ is irreducible. Let $w = (0, |C|, \frac{1}{2}, \nu^* \Delta)$, $v = (0, [D], \chi)$ and let $A$ be a general ample line bundle in $T$. Set $X = X_{v,H,C}$ and $N = M_{w,A}(T)$. Then

\[
h^2(X) = h^2(N) = 11.
\]

As usual, we denote by $\mathcal{C} \subset T \times \mathbb{P}^{g-1}$ the universal family of curves. Since the linear system $|C|$ is base point free, the second projection

\begin{equation}
\text{(4.1)}
\end{equation}

$q : \mathcal{C} \to T$
is a fibration in \((g-2)\)-dimensional projective spaces, the Leray spectral sequence degenerates and there is an isomorphism

\[(4.2)\quad H^2(C, \mathbb{C}) \cong H^2(T, \mathbb{C}) \oplus H^2(\mathbb{P}^{g-2}, \mathbb{C}) \cong \mathbb{C}^{11}.
\]

Notice, also, that

\[(4.3)\quad H^1(C, \mathcal{O}_C) = H^2(C, \mathcal{O}_C) = 0,
\]
since \(H^{2,0}(T) = H^{1,0}(T) = 0\) and by base change in cohomology \(R^i q_* \mathcal{O}_C\) is trivial if \(i\) is not equal to 0.

We will prove Theorem \[4.1\] by comparing the second cohomology of \(X\) and of \(N\) with that of the universal family \(C\).

Let \(U\) the open subset of \(|C|\) parametrizing smooth curves. The first step is to compare the second cohomology of \(X_U\), \(N_U\) and \(C_U\). In all three cases, the Leray Spectral sequence relative to the morphism to \(U\) degenerates so that there are isomorphisms

\[(4.4)\quad H^2(X_U, \mathbb{C}) = H^2(U, \mathbb{C}) \oplus H^1(U, R^1 \pi_* \mathcal{C}_X) \oplus H^0(U, R^2 \pi_* \mathcal{C}_X),
\]

\[H^2(N_U, \mathbb{C}) = H^2(U, \mathbb{C}) \oplus H^1(U, R^1 \nu_* \mathcal{C}_N) \oplus H^0(U, R^2 \nu_* \mathcal{C}_N),\]

and

\[H^2(C_U, \mathbb{C}) = H^2(U, \mathbb{C}) \oplus H^1(U, R^1 p_* \mathcal{C}_C) \oplus H^0(U, R^2 p_* \mathcal{C}_C).\]

The double cover \(\Phi: N \to X\) induces an isomorphism of local systems\[8\]

\[(4.5)\quad R^i \pi_* \mathcal{C}_{X_U} \sim R^i \pi_* (\Phi_* \mathcal{C}_{N_U}) \cong R^i \nu_* \mathcal{C}_{N_U},\]

This follows by proper base change and the fact that the stalk of the natural morphism between the local systems is just the isomorphism in complex cohomology induced by the isogeny \(2.22\). The well known isomorphisms

\[R^i \nu_* \mathcal{C}_{N_U} \cong \wedge^i R^3 p_* \mathcal{C}_U,\]

imply that for every \(i\) there are isomorphisms

\[(4.6)\quad \wedge^i R^1 \pi_* \mathcal{C}_{X_U} \sim R^i p_* \mathcal{C}_U.
\]

Notice that in the same way, one can see that the natural maps

\[(4.7)\quad \varphi^+: R^i \pi_* \mathcal{O}_{X_U} \to R^i \nu_* \mathcal{O}_{N_U},\]

are also isomorphisms.

**Lemma 4.2.** The vector spaces \(H^0(U, R^2 \pi_* \mathcal{C}_X)\), \(H^0(U, R^2 \nu_* \mathcal{C}_N)\) and \(H^0(U, R^2 p_* \mathcal{C}_C)\) are one dimensional.

**Proof.** Since \(R^i \pi_* \mathcal{C}_{X_U} \cong R^i \nu_* \mathcal{C}_{N_U}\), it is enough to prove that \(H^0(U, R^2 \nu_* \mathcal{C}_N) \cong H^0(U, R^2 p_* \mathcal{C}_C) \cong \mathbb{C}\).

Recall that the global sections of a local system are identified with the monodromy invariant elements of a fiber of the local system itself. Clearly, the monodromy acts trivially on the top cohomology \(H^2(C_t, \mathbb{C})\), and hence the space \(H^0(U, R^2 p_* \mathcal{C}_C)\) is one dimensional.

As for \(H^0(U, R^2 \pi_* \mathcal{C}_N)\), we argue as follows. Notice that

\[(4.8)\quad H^0(U, R^2 \pi_* \mathcal{C}_N) \cong H^2(N_t, \mathbb{C})^{\text{inv}} \cong (\wedge^2 H^1(C_t, \mathbb{C}))^{\text{inv}}.
\]

Now let

\[(4.9)\quad \rho: \pi_1(U) \to \text{Aut}(H^1(C_t, \mathbb{C}))\]
be the monodromy representation. Since $\rho$ preserves the symplectic pairing $(\cdot, \cdot)$, the isomorphism $w : H^1(C_t, \mathbb{C}) \cong H_1(C_t, \mathbb{C})$ induced by $(\cdot, \cdot)$ is $\rho$-equivariant. By composing with $w$, it follows that any $\rho$-invariant element of $\wedge^2 H^1(C_t, \mathbb{C})$ can be thought of as a $\rho$-invariant morphism

$$\varphi : H^1(C_t, \mathbb{C}) \to H^1(C_t, \mathbb{C}).$$

By Theorem 3.4 of [Voi03], if the discriminant locus is irreducible then the restriction of the monodromy representation to the vanishing cycles is irreducible. Recall, however, that $H^1(C_t, \mathbb{C}) = H^1(N_t, \mathbb{C})$ is generated by vanishing cycles. It follows then by Schur’s Lemma that

$$\varphi = \lambda \text{id},$$

for some $\lambda$ in $\mathbb{C}$, and hence that

$$\wedge^2 H^1(C_t, \mathbb{C})^{\text{inv}} \cong \mathbb{C}.$$

Notice that the invariants are actually defined over $\mathbb{Z}$. In fact, the theta divisor of $\text{Jac}(C_t)$ is always invariant under the monodromy, since it can be identified with the isomorphism $w : H^1(C_t, \mathbb{Z}) \cong H_1(C_t, \mathbb{Z})$. Since the group $H^2(N, \mathbb{Z})^{\text{inv}}$ is one-dimensional, it is generated by the theta divisor of $C_t$.

Recall that the cohomology groups of a smooth quasi-projective variety are endowed with a canonical mixed Hodge structure (MHS) [Del71].

**Lemma 4.3.** There are isomorphisms of MHS

$$H^2(X_U, \mathbb{C}) \cong H^2(N_U, \mathbb{C}) \cong H^2(U, \mathbb{C}).$$

**Proof.** By Theorem 5.1 of [Ara10], there is a functorial MHS on the cohomology of local systems that arise from families of smooth projective varieties. These MHS are compatible with the MHS on the cohomology of the total space of the family, in the sense (cf. point (3) of Theorem 5.1 of cit.) that they are subquotients of the canonical MHS on the cohomology of the total space.

On the other hand, consider a smooth projective morphism of smooth varieties $j : Y \to Z$. By the decomposition theorem for a smooth projective morphism of smooth varieties (see for e.g. Theorem 5.2.2 of [ICM09]), it follows that

$$j_* \mathcal{Q}_Y \cong \bigoplus_{i \geq 0} R^i j_* \mathcal{Q}_Y[-i].$$

Taking cohomologies on both sides, it follows that the MHS on the cohomology of $Y$ is actually a direct sum of the MHS on the cohomologies of the local systems.

Using the isomorphism of local systems [145], it follows that $H^2(N_U, \mathbb{C}) \cong H^2(X_U, \mathbb{C})$ as MHS. Let us now prove that $H^2(N_U, \mathbb{C}) \cong H^2(U, \mathbb{C})$.

First assume $d' = g - 1$, and let $N'$ be the degree $d'$ relative compacted Jacobian of $|C|$. The Abel Jacobi map

$$a : \mathcal{C}_U \hookrightarrow N'_U,$$

induces in cohomology a morphism $a^* : H^2(N_U, \mathbb{C}) \to H^2(U, \mathbb{C})$ of MHS which is an isomorphism because it is compatible with the decomposition coming from the Leray spectral sequence.

To end the proof of the claim it is now sufficient to notice that the MHS on the cohomology of the total space of a family of degree $d$ Jacobian varieties does not depend on $d$. Indeed, the monodromy

\[\text{is actually split.}\]
representation on the rational cohomology of a relative Jacobian does not depend on its degree. Hence, the local systems and thus their cohomologies are all isomorphic and we conclude applying the decomposition theorem once again.

We will now compare the long exact sequence for the relative cohomology of the pairs \((N, N_U), (X, X_U)\) and \((C, C_U)\).

By Poincaré-Lefschetz duality (cf. [Spa66] Chapter 6, Section §2 Thm 17 or also [PS08] Thm B.28),

\[(4.14) \quad H_i(N, N_U) \cong H^{2n-i}(N_\Delta).\]

Let \(S(N_\Delta)\) be the singular locus of \(N_\Delta\). Taking \(i = 2\), since the real codimension of \(S(N_\Delta)\) in \(N_\Delta\) is greater or equal to 2, we have

\[(4.15) \quad H^{2n-2}(N_\Delta) \cong H^{2n-2}(N_\Delta, S(N_\Delta)) \cong H_0(N_\Delta \setminus S(N_\Delta)) \cong \mathbb{C},\]

where the second to last isomorphism is again given by Poincaré-Lefschetz duality\(^{10}\) and the last isomorphism holds because by our assumptions \(N_\Delta\) is irreducible and hence \(N_\Delta \setminus S(N_\Delta)\) is connected.

Notice also that \(H^1(N_U) = 0\). In fact, \(H^0(U, R^1\pi_* \mathbb{C})\) is trivial because by the invariant cycle theorem [Del71] this space is isomorphic to \(\text{im}[H^1(N) \to H^1(N_t)]\), and by Theorem 3.1 we know that \(H^1(N, \mathbb{C}) = 0\); moreover \(H^1(U) = 0\) since \(U\) is the complement of an irreducible hypersurface ([Dim92] Proposition (1.3)).

It follows that there is an exact sequence

\[(4.16) \quad 0 \to \mathbb{C} \to H^2(N, \mathbb{C}) \to H^2(N_U, \mathbb{C}),\]

where the inclusion \(\mathbb{C} \subset H^2(N, \mathbb{C})\) is identified with the pullback

\[(4.17) \quad \pi^*: H^2(\mathbb{P}^{g-1}) \to H^2(N).\]

Let \(W_2(H^2(N_U, \mathbb{C})) \subset H^2(N_U, \mathbb{C})\) be the lowest weight piece of the weight filtration \(W_\bullet\) defined on \(N_U\). By Cor 3.2.17 of [Del71]

\[(4.18) \quad \text{Im}[H^2(N, \mathbb{C}) \to H^2(N_U, \mathbb{C})] = W_2(H^2(N_U, \mathbb{C})).\]

Clearly, the analogues of (4.16) and of (4.18) hold for \(C\) and for \(X\),

\[(4.19) \quad 0 \to \mathbb{C} \to H^2(C, \mathbb{C}) \to H^2(C_U, \mathbb{C}),\]

\[(4.19) \quad 0 \to \mathbb{C} \to H^2(X, \mathbb{C}) \to H^2(X_U, \mathbb{C}).\]

By Lemma 4.3, there is an isomorphism between the lowest weight part

\[(4.20) \quad \text{Im}[H^2(N, \mathbb{C}) \to H^2(N_U, \mathbb{C})] = W_2(H^2(N_U, \mathbb{C})) \cong W_2(H^2(C_U, \mathbb{C})) = \text{Im}[H^2(C, \mathbb{C}) \to H^2(C_U, \mathbb{C})].\]

We hence deduce an isomorphism of Hodge structures

\[(4.21) \quad H^2(C, \mathbb{C}) \cong H^2(N, \mathbb{C}) \cong H^2(C, \mathbb{C}),\]

which ends the proof of Theorem 4.1.

This fact, together with (4.3) also yields

\[(4.22) \quad H^{2,0}(X) = H^{2,0}(N) = 0.\]

\(^{10}\)cit. Spanier, Thm 19 plus the fact the the pair \((N_\Delta, S(N_\Delta))\) is taut in \(N\).
Remark 4.4. We can argue in the same way and prove that $H^2(Y, \mathbb{C}) \cong H^2(N, \mathbb{C})$. Indeed, the two families $N_U$ and $Y_U$ have isomorphic local systems and by Corollary 3.10 we can argue as in formula (4.11) and show that there is an exact sequence $0 \to \mathcal{C} \to H^2(Y, \mathbb{C}) \to H^2(Y_U, \mathbb{C})$. We then proceed in the same way as above.

5. The canonical bundle

In this section we prove that the canonical bundles of $X$, $N$ and $Y$ are trivial. We will need Assumption 2.22 and also that the general line $\mathbb{P}^1 \subset |C|$ is a Lefschetz pencil.

Before starting, let us emphasize that

Lemma 5.1. The canonical bundle of $X$ is trivial if and only if the canonical bundle of $N$ is trivial.

Proof. The only if part is clear, since $\Phi' : N \to X$ is étale, so we only need to address the if part. Recall the definition of $L$ from formula (3.16). If $\omega_N \cong \omega_N$, then $\Phi^* \omega_X \cong \omega_X$, implying either that $\omega_X$ is trivial, or that it is isomorphic to $L$. To conclude the proof, we make the following two claims. Consider a point $t \in U$ and denote as usual by $X_t$ the fiber over $t$. The first claim is that $L_{X_t}$ is not trivial, which follows immediately from (2.22). Whilst the second claim is that $(\omega_X|_{X_t})$ is trivial. This follows immediately from the fact that $\omega_{X_t}$ and $N_{X_t}|X$ are trivial. Hence, $\omega_X$ cannot be isomorphic to $L$, and the Lemma is proved.

Remark 5.2. The same argument applies to show that the canonical bundle of $Y$ is trivial if and only if the canonical bundle of $N$ is trivial. In fact, the only thing that we use here is that on each fiber the double cover $Y_t \to N_t$ is non trivial. This is a consequence of formula (3.18). The fact that the universal cover of $N$ induces a non trivial cover of each fiber, establishes a difference with Enriques surfaces: any Enriques surface $T$ admits an elliptic fibration $T \to \mathbb{P}^1$ which has exactly two multiple fibers. The canonical bundle is the difference of the two half fibers; moreover, the universal cover $S \to T$ induces a trivial cover of every reduced fiber; indeed, if $e$ is a primitive elliptic curve in $T$, then for any reduced curve $\Gamma \in \mathbb{P}^1$, $f^{-1}(e)$ is the disjoint union of two curve in the linear system $|f^*e|$. In fact, the covering $S \to T$ is induced by base change via a degree two morphism $\mathbb{P}^1 \to \mathbb{P}^1$ ramified at the two points corresponding to the non reduced fibers; as we have already mentioned, in the case of $\nu : N \to |C|$, the restriction of the universal cover to the fibers of $\nu$ is non trivial.

The rest of the section is devoted to prove that $\omega_N$ is trivial.

Consider the morphism $\nu : N \to \mathbb{P}^{g-1}$ of smooth projective varieties. Then,

\begin{equation}
\omega_N = \nu^* \omega_{\mathbb{P}^{g-1}} \otimes \omega_{\nu},
\end{equation}

where $\omega_{\nu}$ is the relative canonical sheaf of $\nu$. Let $\omega_{\nu|U}$ be the restriction of $\omega_{\nu}$ to $N_U$. Since $\nu : N_U \to U$ is smooth and $\omega_{\nu|N_t} \cong \mathcal{O}_{N_t}$ for every $t \in U$, it follows that $\nu_* \omega_{\nu|U}$ is a line bundle on $U$ and that the natural map

\begin{equation}
\omega_{\nu_U} \to \nu^* \nu_* \omega_{\nu_U},
\end{equation}

is an isomorphism. Under our Assumption 2.22 we can use Lemma 2.23 to ensure that the irreducible components $N_{\Delta_i}$ of $N_{\Delta}$ are such that $N_{\Delta_i} \cong \nu^* \Delta_i$. Consider the following commutative
diagram

\[
\begin{array}{c}
\sum \mathbb{Z}N_{\Delta_i} \longrightarrow \text{Pic}(N) \longrightarrow \text{Pic}(N_U) \longrightarrow 0 \\
\sum \mathbb{Z}\Delta_i \longrightarrow \sum \text{Pic}(\mathbb{P}^{g-1}) \longrightarrow \text{Pic}(U) \longrightarrow 0.
\end{array}
\]

Since the rows are exact in the middle and to the right, we get

\[
\omega_N \cong \nu^* \mathcal{O}_{\mathbb{P}^{g-1}}(k),
\]

for some \(k\) in \(\mathbb{Z}\) to be determined. The rest of this Section is devoted to show that

\[
k = g,
\]

thus proving the following Theorem.

**Theorem 5.3.** Under Assumption 2.22, the canonical bundle of \(N\) is trivial, i.e.,

\[
\omega_N \cong \mathcal{O}_N.
\]

**Corollary 5.4.** Under Assumption 2.22,

\[
\chi(\mathcal{O}_X) = \chi(\mathcal{O}_N) = 0.
\]

**Proof.** It is enough to prove that \(\chi(\mathcal{O}_X) = 0\). This follows from \[Bea11\] which implies, since \(\dim M_{v,H}(S) \equiv 2 (\text{mod } 4)\), that

\[
\int_X \text{Todd}(X) = 0.
\]

\[\square\]

The main ingredients in the proof of (5.5) are the following. First of all, the simple observation that to compute \(k\) we can restrict the family \(N\) to a generic line in \(\mathbb{P}^{g-1}\). Second, the theory of degeneration of Hodge bundles, as explained in [Zuc84] and Section 2 of [Kol86b], to understand the direct image of the relative canonical sheaf of \(N \to \mathbb{P}^{g-1}\).

Let \(\mathbb{P}^1 \subset \mathbb{P}^{g-1}\) be a generic line, set

\[
\nu' : N' = \nu^{-1}(\mathbb{P}^1) \to \mathbb{P}^1,
\]

and set

\[
V = U \cap \mathbb{P}^1.
\]

We can choose the \(\mathbb{P}^1\) so that \(N'\) is smooth and so that it parametrizes a Lefschetz pencil of curves. Notice that under this assumption, the discriminant locus \(\Delta' := \Delta \cap \mathbb{P}^1\) of \(\nu\) and of \(p\) is a set of reduced points \(t_1, \ldots, t_{\delta}\), where \(\delta = \deg \Delta\). Clearly, \(\nu'\) is flat because it is equidimensional over a smooth curve.

**Lemma 5.5.** With the above notation, we have

\[
\omega_{\nu'} \cong \omega_{\nu'|_{N'}}\quad \text{i.e.} \quad \omega_{\nu'} \cong \nu'^* \mathcal{O}_{\mathbb{P}^1}(k).
\]

**Proof.** Notice that \(\nu'^* \mathcal{N}_{p^1|_{\mathbb{P}^{g-1}}} \cong \mathcal{N}_{N'|_{\mathbb{P}^1}}\). Since \(\det \mathcal{N}_{p^1|_{\mathbb{P}^{g-1}}} \cong \mathcal{O}_{\mathbb{P}^1}(g-2)\), it follows by adjunction that

\[
\omega_{N'} \cong (\omega_{\nu'} \otimes \nu'^* \mathcal{O}_{\mathbb{P}^{g-1}}(-g))|_{N'} \otimes \nu'^* \mathcal{O}_{\mathbb{P}^1}(g-2).
\]

\[\text{The natural surjective base change map between the conormal bundles } \nu'^* \mathcal{N}_{p^1|_{\mathbb{P}^{g-1}}} \to \mathcal{N}_{N'|_{\mathbb{P}^1}} \text{ is an isomorphism because the two bundles have the same rank.}\]
and comparing this to \( \omega_{N'} \cong \nu'^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \omega_{\nu'} \) we get,
\begin{equation}
\omega_{\nu'|N'} \cong \omega_{\nu'}.
\end{equation}
\[
\]
This allows us to restrict our attention to \( N' \to \mathbb{P}^1 \), where the situation is simpler. Indeed, since \( \mathbb{P}^1 \subset |C| \) parametrizes a Lefschetz pencil, the restriction \( \nu' : C' \to \mathbb{P}^1 \) of the universal family \( C \to \mathbb{P}^{g-1} \) to the line is just the blow up of the surface \( S \) at the \( 2g - 2 \) simple base points of the pencil. Moreover, all the curves parametrized by the \( \mathbb{P}^1 \) are irreducible and have a simple node as the only singularity (except in the hyperelliptic case, in which case there are exactly two reducible fibers that are the union of two smooth curves meeting in one point).

We have now the following proposition, whose proof will be given at the end of the Section after a brief detour on degeneration of Hodge bundles.

**Proposition 5.6.** With the above notation. Then
\begin{equation}
\nu'_* \omega_{\nu'} \cong (\det R^1 p_* \mathcal{O}_{C'})^\vee.
\end{equation}

The right hand side of \( (5.10) \) is readily calculated. By Theorem 2.11 of [Ste77] the sheaf \( R^1 p_* \mathcal{O}_{C'} \) is locally free of rank \( g \). Hence, there exist \( g \) integers \( a_1, \ldots, a_g \), such that
\begin{equation}
R^1 p_* \mathcal{O}_{C'} \cong \bigoplus_{i=1}^g \mathcal{O}_{\mathbb{P}^1}(a_i).
\end{equation}

Since the base is one dimensional, the Leray spectral sequence degenerates and we can use the Hodge numbers of \( C' \) to calculate the \( a_i \)’s. From
\begin{equation}
0 = H^1(C', \mathcal{O}) \cong H^1(\mathbb{P}^1, \mathcal{O}) \oplus H^0(\mathbb{P}^1, R^1 p_* \mathcal{O}_{C'}), \quad \text{and}
0 = H^2(C', \mathcal{O}) \cong H^2(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, R^1 p_* \mathcal{O}_{C'}) \oplus H^0(\mathbb{P}^1, R^2 p_* \mathcal{O}_{C'}),
\end{equation}
we deduce that
\begin{equation}
a_i < 0, \quad \text{and} \quad a_i > -2.
\end{equation}

We conclude that \( a_i = -1 \) for every \( i \), so that
\begin{equation}
\nu'_* \omega_{\nu'} \cong \mathcal{O}_{\mathbb{P}^1}(g).
\end{equation}

By relative Serre duality applied to \( \nu' : N' \to \mathbb{P}^1 \), we get an isomorphism
\begin{equation}
\nu'_* \omega_{\nu'} \cong (R^0 \nu'_* \mathcal{O}_{N'})^\vee.
\end{equation}

Moreover, if we let \( V \subset \mathbb{P}^1 \) be the locus where \( \nu' \) is smooth,
\begin{equation}
R^1 \nu'_* \mathcal{O}_{N'}|_V \cong R^1 p'_* \mathcal{O}_{C'}|_V, \quad \text{and} \quad R^0 \nu'_* \mathcal{O}_{N'}|_V \cong \land^g R^1 p'_* \mathcal{O}_{N'}|_V.
\end{equation}

In order to prove the Proposition we wish to extend these isomorphisms to the whole \( \mathbb{P}^1 \) to get an isomorphism
\begin{equation}
R^0 \nu'_* \mathcal{O}_{N'} \cong \land^g R^1 p'_* \mathcal{O}_{C'}.
\end{equation}

Before talking about variation of Hodge structure and Hodge bundles, we wish to make the following remark.

\[\text{To see that the first isomorphism holds one can use (5.14) and then argue, for example, as follows. The family of curves } C' \to \mathbb{P}^1 \text{ has a section, which gives a closed immersion } C'_V \subset N_V \text{ in the relative Jacobian of any degree. Moreover, the induced map between the first higher direct images of the structure sheaves of the two families is an isomorphism because it is such when restricted to any smooth fiber. As for the second isomorphism, it holds because the higher weight variations of Hodge structures associated to a family of abelian varieties are induced by the weight one variation of HS.}\]
Remark 5.7. From our assumption on the existence of Lefschetz pencils and classical results of Oda and Seshadri [OS79], the fiber of $\nu'$ and of $p'$ over a point $t_i$ in $\Delta'$ is a reduced and irreducible divisor with normal crossings. Moreover, the local monodromy of each family around a point $t_i$ is unipotent. By the theory of Lefschetz pencils we know this is the case for the family of curves, and hence the claim follows also for the families of abelian varieties.

5.1. Degeneration of Hodge bundles. Following [Zuc84], [Kat71], and [Kol86b] we start by briefly recalling some facts about variations of Hodge structures and Hodge bundles.

Let $B$ be a smooth curve, and consider the degree $i$ variation of Hodge structures (VHS for short) induced by a smooth projective morphism $f : Z \to B$. The local system underlying this VHS is $R^i f_* \mathbb{C}$. By [Del70], Prop. I.2.28, the locally free sheaf

$$H^i := R^i f_* \mathbb{C} \otimes \mathcal{O}_B$$

is isomorphic to the hypercohomology sheaf

$$R^i f_* \Omega^*_{Z|B},$$

where $\Omega^*_{Z|B}$ is the complex of relative differentials of the family. Let

$$\nabla : H^i \to H^i \otimes \Omega^1_B$$

be the Gauss-Manin connection of the family $Z \to B$.

Consider the so called filtration bête of the complex

$$\mathcal{F}^p \Omega^*_{Z|B} = \Omega^*_{Z|B}^{\geq p}.$$

The spectral sequence in hypercohomology associated to this filtration has $E_1^{p,q}$ term equal to

$$R^q f_* \Omega^p_{Z|B},$$

and abuts to (the associated graded pieces of)

$$R^i f_* \Omega^*_{Z|B} = H^i.$$

Since $f$ is smooth these sheaves are locally free. Moreover, the spectral sequence degenerates at $E_1$ and the maps

$$R^i f_* \Omega^*_{Z|B}^{\geq p} \to R^i f_* \Omega^*_{Z|B}$$

are injective. In particular, the filtration induced on $H^i$ is

$$\mathcal{F}^p H^i = R^i f_* \Omega^*_{Z|B}^{\geq p} \subset H^i,$$

and the associated graded pieces of $(H^i, \mathcal{F}^p(H^i))$ are the sheaves

$$R^q f_* \Omega^p_{Z|B}, \quad \text{with} \quad p + q = i.$$

The filtration (5.23) is precisely the filtration of the variation of Hodge structures of the family $Z \to B$.

Now consider a smooth projective compactification

$$f : \overline{Z} \to \overline{B}$$

and suppose that $W := f^{-1}(\overline{B} \setminus B)$ is a reduced divisor with normal crossing. We let $\Omega^*_{Z|\overline{B}}(\log W)$ be the complex of relative logarithmic differentials.

By a classical theorem (Thm 7.9 of [Del70]), it is known that $(H^i, \nabla)$ is an algebraic differential equation with regular singular points, in the sense that $\nabla$ has logarithmic poles at every point $b$
in $\overline{B} \setminus B$. That is to say, there exists a vector bundle extension $\overline{H}^i$ of $\mathcal{H}^i$ to all of $\overline{B}$, such that the connection $\nabla$ extends to a morphism

$$\nabla : \overline{H}^i \to \overline{H}^i \otimes \Omega^1_{\overline{B}|B}(\log W).$$

By definition, the residue $\text{Res}_b(\nabla)$ of $\nabla$ at a point $b$ in $\overline{B} \setminus B$ is the endomorphism of $\overline{H}^i_b$ induced by taking the residue in $b$. In other words, if we fix a local trivialization of $\overline{H}^i$ around $b$ and if $z$ is a local coordinate on $\overline{B}$, the residue is defined by $\nabla = dz \otimes (\text{Res}_b(\nabla))_1/z + \ldots$.

By [Ste77], Thm 2.11, (cf. also the exposition in [Zuc84]) the sheaves $R^i f_* \Omega^\bullet_{Z|B}(\log W)$ and $R^q f_* \Omega^p_{Z|B}(\log W)$, $p + q = i$, are locally free extensions of (5.21) and (5.20) respectively. Moreover, Katz proved in [Kat71] that there is a morphism

$$\nabla : R^i f_* \Omega^\bullet_{Z|B}(\log W) \to R^i f_* \Omega^\bullet_{Z|B}(\log W) \otimes \Omega^1_{X|B}(\log W),$$

which extends the Gauss-Manin connection. It follows that we can set

$$R^i f_* \Omega^\bullet_{Z|B}(\log W) \cong R^i f_* \Omega^\bullet_{B|B}(\log W).$$

On the other hand, the filtration (5.19) extends to a filtration

$$\mathcal{F}^p \Omega^\bullet_{Z|B}(\log W) := \Omega^\geq_p \Omega^\bullet_{Z|B}(\log W),$$

which gives a spectral sequence whose $E^p,q_1$ terms are

$$R^i f_* \Omega^\bullet_{Z|B}(\log W).$$

Since these sheaves are locally free and the differential is generically zero, the differential is identically zero. Hence the spectral sequence degenerates at $E_1$ and abuts to the associated graded pieces of

$$R^i f_* \Omega^\bullet_{Z|B}(\log W).$$

Since the graded pieces $R^i f_* \Omega^p_{Z|B}(\log W)$ are locally free, the extensions

$$\mathcal{F}^p(\overline{H}^i) := R^i f_* \Omega^\geq_p \Omega^\bullet_{X|B}(\log W)$$

of the sheaves $\mathcal{F}^p(\mathcal{H}^i)$ are actually extension as vector sub-bundles of $\overline{H}^i$. This is a particular case of Schmid’s Nilpotent Orbit Theorem.

The last ingredient is the following classical theorem

**Theorem 5.8** (Manin, [Del70], Prop. 5.4). Let $(\mathcal{H}, \nabla)$ be an algebraic differential equation with regular singular points on the pointed disc $\Delta^*$. Then $(\mathcal{H}, \nabla)$ admits a unique locally free extension $(\overline{\mathcal{H}}, \overline{\nabla})$ to the disc $\Delta$, which satisfies that following two properties

1. $\overline{\nabla} : \overline{\mathcal{H}} \to \overline{\mathcal{H}} \otimes \Omega^1_{\overline{B}|B}(\log 0)$ has logarithmic poles;
2. The eigenvalues $\lambda$ of $\text{Res}_0(\overline{\nabla})$ satisfy $0 \leq \text{Re} (\lambda) < 1$. 

27
An extension of an algebraic differential equation $\mathcal{H}$ as in the Theorem is called the canonical extension.

Recall that we are assuming that $W \subset Z$ is a reduced divisor with normal crossing. By the regularity theorem of Katz [Kat71] (see [SZ85] for the case of normal crossing and not just simple normal crossing; (5.1) for the notation and (5.3) for the result), it follows that

$$R^0 f_* \Omega^*_Z | \mathring{B} (\log W),$$

satisfies the assumptions of Theorem 5.8 with the eigenvalues of the residue $\text{Res}_b(\nabla)$ of the Gauss-Manin connection equal to zero for every $b \in \overline{B} \setminus B$.

Under this circumstance we say, by abuse of notation, that $R^0 f_* \Omega^*_Z | \mathring{B} (\log W)$ and $R^0 f_* \Omega^*_Z | \mathring{B} (\log W)$ are the canonical extensions of $R^0 f_* \Omega^*_Z | \mathring{B}$ and $R^0 f_* \Omega^*_Z | \mathring{B}$ respectively.

Notice that we can say “canonical extension” also for $F^p(\overline{\mathcal{H}})$ and for $\text{Gr}^p(\mathcal{H})$ because they are unique. In fact, let $j : B \to \overline{B}$ be the open immersion, let $\mathcal{H}$ be a vector bundle on $B$ and let $E \subset \mathcal{H}$ be a sub-bundle. Suppose we are given a vector bundle extension $\overline{\mathcal{H}}$ of $\mathcal{H}$ on the whole of $\overline{B}$. Any extension of $E$ to $\overline{B}$ as a sub-bundle of $\overline{\mathcal{H}}$ is always contained in the saturation of $\overline{\mathcal{H}} \cap j_* E$ in $\overline{\mathcal{H}}$, and thus has to be isomorphic to the saturation itself. In particular, since the extension of $E$ as a vector sub-bundle of $\overline{\mathcal{H}}$ is unique, so is the extension of the quotient $\mathcal{H}/E$.

We now go back to our situation applying these remarks to the families $C' \to \mathbb{P}^1$ and $N' \to \mathbb{P}^1$.

Proof of Proposition 5.6. Recall that our aim is to prove that there is an isomorphism $R^p \nu_* \mathcal{O}_N \cong \wedge^p R^1 p'_* \mathcal{O}_{C'}$ extending that of (5.15). As should be clear by now, we want to prove this isomorphism by showing that the sheaves in question are both isomorphic to canonical extensions.

Since $\mathbb{P}^1 \subset |C|$ is a Lefschetz pencil, the families $C' \to \mathbb{P}^1$ and $N' \to \mathbb{P}^1$ have singular fibers that are reduced and have normal crossing. The sheaves

$$\overline{\mathcal{H}}_{N'} = R^1 \nu_* \Omega^*_N | \mathbb{P}^1(\log(N'_\Delta))$$

$$\overline{\mathcal{H}}_{C'} = R^1 p'_* \Omega^*_C | \mathbb{P}^1(\log(C'_\Delta))$$

extend $R^1 \nu_* \mathcal{O} \otimes \mathcal{O}_{N' | V} \cong R^1 p_* \mathcal{O} \otimes \mathcal{O}_{C' | V}$, and are isomorphic to the canonical extension. Hence, there is an isomorphism

$$\overline{\mathcal{H}}_{N'} \cong \overline{\mathcal{H}}_{C'},$$

which extends the isomorphism over $V$. As a consequence, we get an isomorphism of the canonical extension of $F^1(\overline{\mathcal{H}}_{N'}) \cong F^1(\overline{\mathcal{H}}_{C'})$, i.e.,

$$F^1(\overline{\mathcal{H}}_{N'}) \cong F^1(\overline{\mathcal{H}}_{C'}),$$

which in turn implies that there is an isomorphism of the first graded pieces

$$\text{Gr}^0(\overline{\mathcal{H}}_{N'}) \cong \text{Gr}^0(\overline{\mathcal{H}}_{C'}),$$

i.e., an isomorphism

$$R^1 \nu_* \mathcal{O}_{N'} \cong R^1 p_* \mathcal{O}_{C'}.$$
of the induced connection on $\wedge^j \mathcal{H}$. In other words, the sheaf $\wedge^j \mathcal{H}$ is the canonical extension. We deduce that for any $i$,
\begin{equation}
\wedge^i \mathcal{H}_{N'} \cong \wedge^i \mathcal{H}_C^1.
\end{equation}
Moreover, since $N'_U$ is a family of abelian varieties, we have an isomorphism of VHS $\wedge^j \mathcal{H}_{N'} \cong \mathcal{H}_{N'}^g$. Using again the result of Katz, we known that
\begin{equation}
R^q \nu'_* \Omega^\bullet_{N''|\mathbb{P}^1}(\log N'\Delta')
\end{equation}
is also the canonical extension. It follows that,
\begin{equation}
\wedge^g \mathcal{H}_C^1 \cong \mathcal{H}_{N''}^g.
\end{equation}
This induces an isomorphism of the canonical extension of the respective filtrations, and thus of the respective graded pieces. Since $\mathrm{Gr}^g(\wedge^g \mathcal{H}_C^1) \cong \wedge^g \mathrm{Gr}^g(\mathcal{H}_C^1)$ we conclude
\begin{equation}
R^g \pi'_* \mathcal{O}_{N''} \cong \wedge^g R^1 p'_* \mathcal{O}_C',
\end{equation}
and the proposition is proved. $\square$

Remark 5.9. We want to highlight a difference in behavior between other type of moduli spaces of sheaves on Enriques surfaces. In [OS11], Oguiso and Schröer prove that the Hilbert scheme of $n$ points on a given Enriques surface $T$ has the property that the canonical bundle is not trivial, but twice the canonical bundle is trivial. It would be interesting to know if this is a general phenomenon, that is to say if there is a different behavior of the canonical bundle of odd and even dimensional moduli spaces of sheaves on Enriques surfaces.

For example, given a genus $g$ linear system $|C|$, it would be interesting to study the geometry of the rational Abel-Jacobi maps
\begin{equation}
T^{[g-1]} \longrightarrow \mathcal{N}^{g-1}, \quad \text{and} \quad \mathcal{N}^g \longrightarrow T^{[g]},
\end{equation}
where we have denoted by $\mathcal{N}^d$ the degree $d$ relative compactified Jacobian of $|C|$.

We end the section with the following theorem.

Theorem 5.10. Under Assumption 2.22, the relative compactified Jacobian $\mathcal{N}$ is an irreducible Calabi-Yau manifold, in the sense that
\begin{equation}
H^p(\mathcal{N}, \mathcal{O}_\mathcal{N}) \cong \begin{cases} 
\mathbb{C} & \text{if } p = 0, 2g - 1, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Proof. The main step in proving the theorem is the following claim:
\begin{equation}
R^q \nu_* \mathcal{O}_\mathcal{N} \cong \wedge^q \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^g-1}(-1),
\end{equation}
which is a generalization of what we already proved for the restriction of $N$ to a general Lefschetz pencil. The claim uses a theorem of Matsushita [Mat05] on the higher direct images of the morphism $M \to |D|$. Assuming the claim, the spectral sequence to calculate $H^p(\mathcal{N}, \mathcal{O}_\mathcal{N})$ degenerates since
\begin{equation}
H^k(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^g-1}(-p + k)) \cong \begin{cases} 
\mathbb{C} & (k, p) = (0, 0) \text{ or } (k, p) = (g - 1, 2g - 1) \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
and the theorem easily follows. To prove the claim first notice that since $\omega_\mathcal{N} \cong \mathcal{O}_\mathcal{N}$, by Theorem 2.1 in [Kol86a] the sheaves $R^i \nu_* \mathcal{O}_\mathcal{N}$ are torsion free and by Corollary 3.9 in [Kol86b] they are reflexive. The same argument applies to $R^q \pi_* \mathcal{O}_X$. From what we proved in the last subsection we can consider an open subset $W \subset \mathbb{P}^{g-1}$, whose complement has codimension greater or equal to two, that has the property that
\begin{equation}
(R^i \nu_* \mathcal{O}_\mathcal{N})|_W \cong (R^i \pi_* \mathcal{O}_X)|_W, \quad \text{and} \quad (R^q \nu_* \mathcal{O}_\mathcal{N})|_W \cong (\wedge^q R^1 \nu_* \mathcal{O}_\mathcal{N})|_W.
\end{equation}
Here $W$ can be taken to be the locus of irreducible curves with one simple node and no other singularities or, in the hyperelliptic case, the locus of irreducible curves with at worst one node and no other singularities union the open subset of the discriminant that parametrizes two smooth curves meeting transversally in one point. Since all the above sheaves are reflexive, these isomorphisms extends to global isomorphisms on the whole $\mathbb{P}^{g-1}$. We are thus reduced to prove that

$$R^1\pi_*\mathcal{O}_X \cong \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^{g-1}}(-1).$$

To see this, we argue as follows. Let $\mathcal{I}$ denote the ideal sheaf of $|C|$ in $|D|$. The following exact sequence,

$$0 \to \mathcal{I}/\mathcal{I}^2 \to (\Omega^1_{|D|})|_C \to \Omega^1_{|C|} \to 0,$$

is split, and the involution $\iota^*$ acts on it. The sheaf $\mathcal{I}/\mathcal{I}^2 \cong \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^{g-1}}(-1)$ is the $\iota^*$-anti invariant part of $(\Omega^1_{|D|})|_C$, whereas the sheaf $\Omega^1_{|C|}$ is the invariant part. By a Theorem 1.3 of [Mat05], there is an isomorphism

$$(5.41) \quad \Omega^1_{|D|} \cong R^1\rho_*\mathcal{O}_M,$$

which is induced by the symplectic form $\omega$ of $M$. The natural morphism

$$(R^1\rho_*\mathcal{O}_M)|_V \to (R^1\pi_*\mathcal{O}_X)|_V,$$

is surjective, and $(R^1\pi_*\mathcal{O}_X)|_V$ is exactly the $\iota^*$-invariant part of $(R^1\rho_*\mathcal{O}_M)|_V$. Since $\iota^*\omega = -\omega$, isomorphism [5.41] interchanges the $\iota^*$-invariant and the $\iota^*$-anti invariant parts of $\Omega^1_{|D|} \cong R^1\rho_*\mathcal{O}_M$. We deduce that there is an isomorphism

$$(R^1\pi_*\mathcal{O}_X)|_V \cong \mathcal{I}/\mathcal{I}^2|_V,$$

which, using the fact that these sheaves are reflexive, extends to an isomorphism

$$R^1\pi_*\mathcal{O}_X \cong \mathcal{I}/\mathcal{I}^2 \cong \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^{g-1}}(-1).$$

□

**Remark 5.11.** In the same way we can show that the higher direct images of $\mathcal{O}_N$ and of $\mathcal{O}_Y$ are isomorphic. Hence,

$$H^p(Y, \mathcal{O}_Y) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, 2g - 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $Y$ is an irreducible Calabi-Yau manifold.

6. An example: the threefold case

In this section we work out in more detail the geometry of the relative compactified Jacobian of a genus two linear system on a general Enriques surface $T$.

In this case, the morphisms $N \to |C|$ and $X \to |C|$ are automatically flat, and Assumption 2.22 is satisfied.

We start with describing the well known and beautiful construction ([Cos83], [CD89]) of non-special genus two pencils. We will see that by putting ourselves in a general setting we can assume that $|C|$ is a Lefschetz pencil. Despite the presence of reducible fibers in codimension one, Theorems 3.1 and 5.3 apply in this case too. It follows that the relative compactified Jacobian $N$ of a non-special genus two pencil is a Calabi-Yau 3-fold.
Finally, we compute the second Betti number. The technique is the same as in Section 4 with the slight difference due to the presence of base points and reducible fibers in codimension one. We show that the Euler characteristic is zero, and compute the Hodge diamond.

Let $e_1, e_2 \subset T$ be two primitive elliptic curves such that $e_1 \cdot e_2 = 1$. By [Cos83] the general curve $C \in |e_1 + e_2|$ is a smooth genus 2 curve. The linear system $|e_1 + e_2|$ is a pencil with exactly two simple base points, $p = e_1 \cap e_2$ and $q = e_1' \cap e_2$, where the curves $e_i'$ (notation as in (2.1)) are also smooth and elliptic. Recall also that, \[ |e_i| = |e_i'| = \{pt\}, \quad \text{and} \quad |2e_i| = |2e_i'| = \mathbb{P}^1. \]

A genus two pencil is called non special, if it can be written as $|e_1 + e_2|$ for two primitive elliptic curves meeting in one point. Following Cossec, we let \[ \varphi = (\varphi_C, \varphi_{C'}) : T \to \mathbb{P}^1 \times \mathbb{P}^1, \] be the map associated to the linear systems $|C|$ and $|C'|$. It is a rational, generically 2:1, map defined on the complement of the four base points. The four elliptic curves are contracted by $\varphi$.

**Lemma 6.1** ([Cos83]). The ramification divisor of the map $\varphi : T \to \mathbb{P}^1 \times \mathbb{P}^1$ consists in four lines $r_1, r_1', r_2, r_2'$ forming a square and a curve $\Gamma$ of bidegree $(4,4)$ having simple nodes at each of the corners of the square. The branches of the nodes are not tangent to the sides of the square. If $T$ is a general Enriques surface, then $\Gamma$ is smooth outside the four nodes.

**Proof.** Let $\tilde{T} \to T$ be the blow up of $T$ at the four base points, and let $\tilde{\varphi} : \tilde{T} \to \mathbb{P}^1 \times \mathbb{P}^1$ the induced morphism. Then $\tilde{\varphi}$ contracts the proper transforms of the elliptic curves, and is ramified along the four exceptional divisors. The images of the exceptional divisors form a square in $\mathbb{P}^1 \times \mathbb{P}^1$. The curves in either of the linear systems double cover the lines of either of the two rulings. Since the generic curve is a smooth genus two curve, the intersection number of the lines in each pencil with the ramification curve has to be equal to six. We already know that the map $\tilde{\varphi}$ ramifies along the sides of the square, so the remaining part $\Sigma$ of the ramification divisor has to be of type $(4,4)$.

Now let $P$ be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the corners of the square, let $l_1, \ldots, l_4$ be the four exceptional divisors, let $R_1, R_1', R_2, R_2'$ be the proper transforms of the sides of the square, and finally let $\Gamma$ be the proper transform of $\Delta$. Since $\tilde{\varphi}$ contracts the four elliptic curves to the four corners of the square, the induced rational map $\psi : \tilde{T} \to B$ is a generically 2:1 morphism, ramified over the disjoint union of $\Gamma$ and $R_1, R_1', R_2, R_2'$. Hence $\psi$ induces a double cover of the proper transform of the four elliptic curves onto the exceptional divisors. Notice also that such double covers are ramified at four distinct points, which are the intersection points of the exceptional divisor with the ramification curve. Since all of the exceptional divisors meets transversally two sides of the square, it follows that $\Gamma$ has to meet each exceptional divisor in two distinct points, and thus $\Sigma$ has a simple node at each corner of the square and that the branches of the nodes are not tangent to the sides of the square.

If we suppose that the only reducible members of the linear system $|C|$ are the curves $e_1 + e_2$ and $e_1' + e_2'$ (property that is achieved, for example, for a general $T$) then the morphism \[ \psi : \tilde{T} \to P, \] is finite, and since it is a degree two cover between two smooth surfaces, the ramification curve has to be a disconnected union of smooth curves. It follows that in the general case $\Gamma$ is smooth, and the four nodes of $\Sigma$ are its only singularities. \[\square\]

The following Lemma reverses the construction.
Lemma 6.2. The general non special genus two pencil is obtained by reversing the construction of Lemma 6.1. Moreover, one can choose the ramification curve so that the singular curves in the linear system have exactly one node.

Proof. Consider a square of lines \( r_1, r'_1, r_2, r'_2 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). In the linear system \(|O_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4)|\) consider the dimension 12 linear subspace \( W \) of curves having a node at the four corners of the square. The general curve in \( W \) is smooth outside the four nodes and, moreover, the branches of the nodes are not tangent to the sides of the square. Let \( \eta : P \to \mathbb{P}^1 \times \mathbb{P}^1 \) be the blowup of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at the four nodes of the square, let \( l_1, \ldots, l_4 \) be the four exceptional divisors, let \( R_1, R'_1, R_2, R'_2 \) be the proper transforms of the sides of the square, and finally let \( \Gamma \) be the proper transform of a curve \( \Gamma' \in W \) that is smooth outside the four nodes. Now consider the double cover \( \xi : Y \to B \), ramified along \( \Gamma \cup R_1 \cup R'_1 \cup R_2 \cup R'_2 \). Then,

\[
2K_Y = \xi^*(2K_B + \Gamma + R_1 + R'_1 + R_2 + R'_2).
\]

Since

\[
K_B = \eta^*\mathcal{O}(-2, -2) + l_1 + l'_1 + l_2 + l'_2,
\]

and

\[
\Gamma = \eta^*\mathcal{O}(4, 4) - 2(l_1 + l'_1 + l_2 + l'_2),
\]

we have

\[
2K_Y = \xi^*(R_1 + R'_1 + R_2 + R'_2).
\]

For \( i = 1, \ldots, 4 \), the curves \( T_i, T'_i \) are smooth rational \((-1\)-)curves, and have the property that \( 2T_i = \xi^*(R_i) \), \( 2T'_i = \xi^*(R'_i) \). Let \( \epsilon : Y \to T' \) be the contraction of these four rational curves. Then \( K_Y = \epsilon^*\omega_T + T_1 + T'_1 + T_2 + T'_2 \), implies

\[
2\epsilon^*\omega_T = \mathcal{O}_Y.
\]

Thus \( 2\omega_{T'} = \mathcal{O}_{T'} \) and then it is readily checked that \( T' \) is an Enriques surface.

Let \( \varphi : T' \to \mathbb{P}^1 \times \mathbb{P}^1 \) be the induced rational morphism. Then \( C = \varphi^*\mathcal{O}(1, 0) \) is a linear system of genus two, and the generic curve is a smooth hyperelliptic curve because it is a double cover of a line, ramified at 4 distinct points. Moreover, notice that we can choose the curve \( \Gamma \) so that the lines in \( \mathcal{O}(1,0) \) are neither bitangent nor flex lines to \( \Gamma \) itself (this can be achieved because one can easily find degenerate curves with this property and, since it is an open condition, one can also find a curve satisfying this requirement and such that the only nodes are at the four corners of the square). If we choose such a curve \( \Gamma \), then the curves in the linear system \(|C|\) have at most one simple node. Since the automorphism group of \( \mathbb{P}^1 \times \mathbb{P}^1 \) fixing a square of lines is two dimensional and acts on the 12-dimensional sublinear system \( W \), it follows that this construction depends on 10 parameters. \( \square \)

Corollary 6.3. Let \( p : C \to \mathbb{P}^1 \) be the universal family of a general genus two pencil on a general Enriques surface \( T \). Then \( p \) has exactly 18 singular fibers, 16 of which are irreducible with one node, and 2 of which are reducible, and consist of two elliptic curve meeting tranversely at one point.

Proof. It is sufficient to apply a classical cut and paste argument. \( \square \)

From now on we let \(|C|\) be a general non special genus two pencil on an Enriques surface \( T \), and we let \( C \to \mathbb{P}^1 \) be the universal curve. Let \( t_1, \ldots, t_{16} \) be the points in \( \mathbb{P}^1 \) parameterizing the irreducible fibers. Since the Lefschetz pencil \( C \to \mathbb{P}^1 \) has a section, the relative compactified Jacobians of
Theorem 6.6. Next, we have the computation of the second cohomology group. As we said, the strategy to compute this cohomology group is the same as in the general case. However, in this case, in fact, the universal family \( \mathcal{N} \) is obtained by compactifying the \( \mathbb{C}^* \)-bundle over \( \tilde{f}^* \text{Pic}^0(\mathcal{C}) \) to a \( \mathbb{P}^1 \)-bundle, and glueing appropriately the zero and infinity sections.

Proposition 6.5. The Euler characteristics of \( N \) and of \( X \) are trivial. Proof. This follows from the above Lemma, since a rank one degeneration of an \( n \geq 2 \) abelian variety has trivial Euler number. By cutting and pasting, it follows that

\[
\chi_{\text{top}}(X) = 0.
\]

It would be interesting to compute the Euler characteristic of \( N \) in the higher genus case.

We recall that since the Jacobian variety of a union of two smooth curves meetings transversally in one point is smooth and compact, we only need to prove the Lemma for the fibers over the nodal curves. Moreover, as we have already mentioned in the proof of Proposition 3.9 the compactified Jacobian of an irreducible curve with one node is a rank one degeneration of an abelian variety, we only need to prove the result for \( X \). Let \( C \subseteq T \) be an irreducible curve with one node, and let \( D \rightarrow C \) be the induced double cover and let \( \tilde{f} : \tilde{D} \rightarrow \tilde{C} \) be the induced double cover between the normalizations. Applying \( \iota^* \) to the exact sequence

\[
1 \rightarrow \mathbb{C}^* \times \mathbb{C}^* \rightarrow \text{Pic}^0(D) \rightarrow \text{Pic}^0(\tilde{D}) \rightarrow 1,
\]

we get

\[
1 \rightarrow \mathbb{C}^* \rightarrow f^* \text{Pic}^0(C) \rightarrow \tilde{f}^* \text{Pic}^0(\tilde{C}) \rightarrow 1,
\]

and it is not hard to see that \( \text{Fix}(\iota^*) \subset \text{Jac}^0(D) \) is obtained by compactifying the \( \mathbb{C}^* \)-bundle over \( \tilde{f}^* \text{Pic}^0(\tilde{C}) \) to a \( \mathbb{P}^1 \)-bundle, and glueing appropriately the zero and infinity sections.

Proposition 6.5. The Euler characteristics of \( N \) and of \( X \) are trivial.

Proof. This follows from the above Lemma, since a rank one degeneration of an \( n \geq 2 \) abelian variety has trivial Euler number. By cutting and pasting, it follows that

\[
\chi_{\text{top}}(X) = 0.
\]

It would be interesting to compute the Euler characteristic of \( N \) in the higher genus case.

Since \( |C| \) is a Lefschetz pencil, we may apply Theorem 3.11 to get,

\[
\pi_1(X) = \mathbb{Z}/(2) \times \mathbb{Z}/(2), \quad \text{and} \quad \pi_1(N) = \mathbb{Z}/(2).
\]

Theorem 6.6. \( H^2(X, \mathcal{C}) \cong H^2(N, \mathcal{C}) \cong \mathbb{C}^{10} \).

As we said, the strategy to compute this cohomology group is the same as in the general case. However, in this case, in fact, the universal family \( \mathcal{C} \) is isomorphic to the blow up of \( T \) in the two base points of the linear system \( |C| \). Hence,

\[
H^2(\mathcal{C}) = H^2(T) \oplus \mathbb{C}^2 = \mathbb{C}^{12}.
\]

Notive, nevertheless, that the argument to prove that \( H^2(N, \mathcal{C}) = H^2(X, \mathcal{C}) = H^2(Y, \mathcal{C}) \) (Section 3) applies verbatim in this case. It is thus enough to prove that \( H^2(N, \mathcal{C}) \cong \mathbb{C}^{10} \).

Let \( V \subset \mathbb{P}^1 \) be the open set where \( \mathcal{C} \rightarrow \mathbb{P}^1 \) is smooth, and as usual let \( U \subset \mathbb{P}^1 \) be the open set where \( N \rightarrow \mathbb{P}^1 \) is smooth. The exact sequences (4.16) and (4.19) are here replaced by

\[
0 \rightarrow H^1(N_V, \mathcal{C}) \rightarrow H^2(N, N_V) \rightarrow H^2(N, \mathcal{C}) \rightarrow H^2(N_V),
\]

By rank one degeneration of an abelian variety of dimension \( n \) we mean a variety obtained by glueing (up to a translation in the base) the zero and infinity sections of a \( \mathbb{P}^1 \)-bundle over an \((n-1)\)-dimensional abelian variety.
and
\[ (6.10) \quad 0 \to H^1(C_V, \mathbb{C}) \to H^2(C, C_V) \to H^2(C, \mathbb{C}) \to H^2(C_V, \mathbb{C}). \]

The Leray spectral sequence of both fibrations degenerates because the base is 1-dimensional (cf. [PS08]) so that
\[ H^1(N_V, \mathbb{C}) \cong H^1(C_V, \mathbb{C}) \cong H^1(V, \mathbb{C}) = \mathbb{C}^{17}. \]

Using Poincaré-Lefschetz duality as in the general case (cf. equation (4.15)), we have
\[ H^2(N, N_V) \cong H^0(N_\Delta \setminus S(N_\Delta)) \cong \mathbb{C}^{18} \]
and \[ H^2(C, C_V, \mathbb{C}) \cong \mathbb{C}^{20}. \] It follows that we have two exact sequences
\[ (6.12) \quad \mathbb{C} \longrightarrow H^2(N, \mathbb{C}) \xrightarrow{r} H^2(N_V, \mathbb{C}) \]
\[ \quad \mathbb{C}^3 \longrightarrow H^2(C, \mathbb{C}) \xrightarrow{s} H^2(C_V, \mathbb{C}). \]

The second vertical arrow is induced by the inclusion \( j : C_V \subset N_V \) which in turn is induced by a section \( C \to \mathbb{P}^1 \) of the family of curves.

It is enough to prove that the images of the maps \( r \) and \( s \) are isomorphic and we will achieve this by proving that the restriction morphism \( j^* \) induces an isomorphism of MHS
\[ (6.13) \quad H^2(N_V, \mathbb{C}) \cong H^2(C_V, \mathbb{C}). \]

Since \( j^* \) is a morphism of MHS, it is enough to prove that the graded pieces of the Leray spectral sequence are isomorphic. However, compared to the previous case, we cannot apply directly the argument that proved that \( H^0(V, R^2n_*\mathbb{C}_N) = \mathbb{C} \), because we used the irreducibility of the discriminant locus, whereas there is no problem for the cohomology of the other local systems. The following proposition ends the proof.

**Proposition 6.7.** Let \( |C| \) be a general genus two linear system on a general Enriques surface \( T \). The monodromy action on the first cohomology of the fibers of the Jacobian fibration \( N \) is irreducible.

**Proof.** The generality assumptions on the linear system allow us to assume that the singular fibers of \( |C| \) are as in Corollary 6.3. Fix a point \( t \) in \( V \), and let
\[ \rho : \pi_1(V) \to \text{Aut}(H^1(C_t, \mathbb{C})) \]
be the monodromy representation. Since the monodromy action preserves the symplectic pairing, we can identify \( H^1(C_t, \mathbb{C}) \) and \( H^1(C_t, \mathbb{C}) \). To prove that there are no invariant subspaces, we argue by contradiction. Suppose therefore that there is a non trivial invariant subspace
\[ (6.14) \quad F \subset H_1(C_t, \mathbb{C}). \]

We first check that \( F \) is a symplectic subspace. Applying Theorem 3.12 to \( N \to \mathbb{P}^1 \), recalling Lemma 3.5 and using the fact that \( \pi_1(N) \cong \mathbb{Z}/(2) \), we know that the vanishing cycles \( \alpha_i' \) associated to a set of generators \( \{ \lambda_i' \} \) of \( \pi_1(V) \), generate over \( \mathbb{Z} \) an index two sub lattice \( H' \subset H_1(C_t, \mathbb{Z}) \). In particular, the set \( \{ \alpha_i' \} \) generates the complex vector space \( H_1(C_t, \mathbb{C}) \). Since the intersection pairing is non degenerate, it follows that for any \( \beta \) in \( F \) we can find a \( i(\beta) \) such that the vanishing cycle \( \alpha_i' \) satisfies
\[ (\beta, \alpha_i'(i(\beta))) \neq 0. \]
As $F$ is an invariant subspace, it follows that $PL_{\lambda_{i(\beta)}}(\beta) = -\beta + (\beta, \alpha'_{i(\beta)})\alpha'_{i(\beta)}$ lies in $F$, and thus that
\begin{equation}
\alpha'_{i(\beta)} \in F,
\end{equation}
and that $F$ is a symplectic subspace. Since the action preserves the symplectic pairing the orthogonal complement $F^\perp$ is also invariant and, by the same argument as above, symplectic. It follows that the invariant subspaces $F$ and $F^\perp$ are two dimensional and irreducible.

The above argument also shows that every vanishing cycles $\alpha_i'$ lies either in $F$ or in $F^\perp$. In particular, we can decompose $R' = R_F \oplus R_{F^\perp}$, where $R_F$ and $R_{F^\perp}$ are the sublattices generated by the vanishing cycles that lie in $F$ and $F^\perp$ respectively. Since $R \subset H^1(C_t, \mathbb{Z})$ has index two, it follows that the determinant of the intersection matrix for $R$ is 4. Moreover, $R_F$ and $R_{F^\perp}$ are orthogonal with respect to the symplectic pairing so that, up to switching $F$ and $F^\perp$, we can find a basis $B_F$ of $R_F$ and a basis $B_{F^\perp}$ of $R_{F^\perp}$ such that the intersection matrix of $R$ with respect to $(B_F, B_{F^\perp})$ is of the form
\begin{equation}
\begin{pmatrix}
0 & 2 & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\end{equation}
In particular, we can assume that $R_{F^\perp}$ is primitive. To simplify notations, set $E := F^\perp$, $E_Z := R_{F^\perp}$, and let $F_Z$ be the orthogonal complement of $E_Z$ in $H^1(C_t, \mathbb{Z})$. Then $F_Z$ is primitive and by Corollary (1.2.6) of [BHPvdV04] we have
\begin{equation}
H^1(C_t, \mathbb{Z}) = F_Z \oplus E_Z.
\end{equation}

On the other hand, we have the following commutative diagram,
\begin{equation}
\begin{array}{ccc}
H^1(C_t, \mathbb{Z}) & \xrightarrow{j} & H^1(C_t, \mathcal{O}_{C_t}) \\
\downarrow{j} & & \downarrow{D} \\
H^1_{dR}(C_t, \mathbb{C}) & \xrightarrow{p} & H^0_{\sigma}(C_t).
\end{array}
\end{equation}
The left hand side vertical arrow is the composition of the base change inclusion $H^1(C_t, \mathbb{Z}) \subset H^1(C_t, \mathbb{C})$, and the De Rham isomorphism. The top horizontal arrow is given by the exponential sequence, the right hand side vertical arrow is the Dolbeaut isomorphism, and the bottom arrow is the projection onto the Dolbeaut group. The two spaces $p(F)$ and $p(E)$ contain the lattices $Dj(F_Z)$ and $Dj(E_Z)$ respectively, and hence are both 1 dimensional subspaces. Set $F' = D^{-1}p(F)$ and $E' = D^{-1}p(E)$. It follows that
\begin{equation}
F_t := F'/j(F_Z),
\end{equation}
and
\begin{equation}
E_t := E'/j(E_Z),
\end{equation}
are two smooth elliptic curves and that, since $N_t \cong H^1(C_t, \mathcal{O}_{C_t})/H^1(C_t, \mathbb{Z})$, we have
\begin{equation}
N_t \cong E_t \times F_t.
\end{equation}
Since $E_Z$ and $F_Z$ are orthogonal and primitive, the intersection product on $H^1(C_t, \mathbb{Z})$ breaks into the sum of the intersection products on the two factors. Recall now that the intersection product on $H^1(C_t, \mathbb{Z})$, viewed as an element in $H^2(N_t, \mathbb{Z}) \cong \Lambda^2 H^1(C_t, \mathbb{Z})$, corresponds to the theta divisor. It follows that the isomorphism in (6.21) is an isomorphism of principally polarized abelian varieties.
Since $N_t$ is the Jacobian of a smooth genus 2 curve, it cannot break in the product of two principally polarized abelian subvarieties. This gives a contradiction and we have proved the Proposition. □

By Proposition 3.9, we can use the same argument and show that $H^2(N) \cong H^2(Y)$.

**Corollary 6.8.** The Hodge diamond of $Y$, $N$ and $X$ is

\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 10 & 0 & \\
1 & 10 & 10 & 1
\end{array}
\]

**Proof.** Notice that from the computation of $H^2(N)$, it follows that $H^{2,0}(N) = 0$ since $H^2(N)$ injects into $H^2(C)$. The corollary this follows from the fact that $\chi(N) = 0$. □

**References**

[ACG10] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Vol. II, with a contribution by J. D. Harris*, volume 268 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, New York, 2010.

[AIK77] A. B. Altman, A. Iarrobino, and S. L. Kleiman. Irreducibility of the compactified Jacobian. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 1–12. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[ASF12] E. Arbarello, G. Saccà, and A. Ferretti. The relative Prym variety associated to the double cover of an Enriques surface. *(to appear on arXiv)*, 2012

[Ara10] D. Arapura. Mixed Hodge structures associated to geometric variations. In *Cycles, motives and Shimura varieties*, Tata Inst. Fund. Res. Stud. Math., pages 1–34. Tata Inst. Fund. Res., Mumbai, 2010.

[Bea83] A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Diff. Geom.*, 18:755–782, 1983.

[Bea11] D. Beauville. Antisymplectic involutions of holomorphic symplectic manifolds. *J. Topol.*, 4(2):300–304, 2011.

[BHPvdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. van de Ven. *Compact complex surfaces*. Number 4 in A Series of Modern Surveys in Mathematics. Springer, 2004.

[CD89] F. R. Cossec, I. V. Dolgachev. *Enriques surfaces. I*, volume 76 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1989.

[CK11] D. Chen, J. L. Kass. Moduli of generalized line bundles on a ribbon, arXiv:1106.5441

[Cos83] F. R. Cossec. Projective models of enriques surfaces. *Mathematische Annalen*, 265:283–334, 1983. 10.1007/BF01456021.

[dCM09] A. de Cataldo and L. Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. *Bull. Amer. Math. Soc. (N.S.)*, 46(4):535–633, 2009.

[Del70] P. Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970.

[Del71] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Dim92] A. Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992.

[Han10] M. Hauzer. On moduli spaces of semistable sheaves on Enriques surfaces arXiv: 1003.5857

[HL97] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Number E 31 in Aspects of Mathematics. Vieweg, 1997.

[Huy97] D. Huybrechts. Birational symplectic manifolds and their deformations. *J. Diff. Geom.*, 45:488–513, 1997.

[Kat71] N. M. Katz. The regularity theorem in algebraic geometry. In *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 1, pages 437–443. Gauthier-Villars, Paris, 1971.

[Kim98] H. Kim. Moduli spaces of stable vector bundles on Enriques surfaces. *Nagoya Math. J.*, 150:85–94, 1998.

[Kim06] H. Kim. Stable vector bundles of rank two on Enriques surfaces. *J. Korean Math. Soc.*, 43(4):765–782, 2006.
