Boundary $K$-matrices and the Lax pair

for 1D open $XYZ$ spin-chain

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Abstract

We analyse the symmetries of the reflection equation for open $XYZ$ model and find their solutions $K^\pm$ case by case. In the general open boundary conditions, the Lax pair for open one-dimensional $XYZ$ spin-chain is given.
1 Introduction

In the formalism of the quantum inverse scattering method (QISM), the integrability of a system with periodic boundary conditions is based on the Yang-Baxter equation. Of central importance in the QISM is the $R$-matrix. Relying on the previous work of Cherednik, Sklyanin generalized the QISM to system on a finite internal with independent boundary conditions on each end. For such a system, the boundary conditions are defined by the operators $K^\pm$, which satisfy the so-called reflection equations (or called the boundary Yang-Baxter equations). The integrability conditions for systems with open boundary conditions are that both the Yang-Baxter equations and the reflection equations must be satisfied. Since Sklyanin’s work, a lot of integrable models with open boundaries, both in quantum field theory and in statistical mechanics, as the Toda chain, nonlinear Schrödinger equation, the $A_2^{(2)}$ spin chain with $U_q[SU(2)]$ symmetry, supersymmetric $t-J$ model, sine-Gordon model, $A_{n-1}$ vertex models, the $XYZ$ model, have been studied extensively.

In this paper, we will present the solutions $K^\pm(u)$ to the reflection equations of open $XYZ$ model. The solutions $K^\pm(u)$ for $XYZ$ model have been studied by de Vega, Gonzalez-Ruiz and Inami. However, we will pursue a new procedure to the problem. Firstly, we analyze the symmetries of the component forms of the reflection equations for $XYZ$ model and determine the equations to be solved. Then we study the chosen equations to get the solutions $K^\pm$ case by case and their corresponding boundary terms in the Hamiltonians. For the case of $trK^+(0) = 0$, we get a new $K(u)$ that is not given in ref. (see (2.18) and (2.25)). Furthermore, we find that several $K$-matrices may correspond to the same boundary terms if they differ by a scaling function satisfied some conditions. We expect our procedure and discussions to be helpful to the understanding of the role played by $K$-matrices in the description of a system. In the formalism of QISM, the systems described by the above Hamiltonians with boundary terms are integrable.

However, in the case of periodic boundary condition, there is an equivalent approach,
which is called the Lax representation [2], to the proof of the integrability of the system. In this formalism, a model is said to be completely integrable if we can find a Lax pair such that the Lax equation is equivalent to the equation of motion of the model. Now, it is natural that one expect a variant of the Lax representation to systems with open boundary conditions. In fact, this has been done, for example, for the one-dimensional (1D) Heisenberg open $XXZ$ chain [12], the Hubbard model [13]. We will construct the Lax pair for the 1D open $XYZ$ spin-chain.

The paper is organized as follows. In section 2, after briefly reviewing the $\mathcal{R}$-matrix for the $XYZ$ model, the corresponding Yang-Baxter equation and the reflection equations, we introduce the symmetries for the reflection equations and find the $K$-matrices case by case. In the second half of this section, we give the corresponding boundary terms for each solution $K^\pm$ and discuss a symmetry of the general Hamiltonian for the open $XYZ$ model. In section 3, we will construct the Lax pair for the $XYZ$ model with general boundary conditions. We will make some discussions in the last section.

## 2 Solutions to the reflection equations for the $XYZ$ model

The $XYZ$ model is defined in terms of the Boltzmann weights given by the elliptic eight-vertex solution of the Yang-Baxter equation:

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u)$$

(2.1)

Here $\mathcal{R}_{12}(u) = \mathcal{R}(u) \otimes 1$, $\mathcal{R}_{23}(u) = 1 \otimes \mathcal{R}(u)$. The $\mathcal{R}$-matrix $\mathcal{R}(u)$ is given as follows:

$$\mathcal{R}(u) = \begin{pmatrix}
\text{sn}(u+\eta) & 0 & 0 & k\text{sn}\eta\text{sn}\eta\text{sn}(u+\eta) \\
0 & \text{sn}u & \text{sn}\eta & 0 \\
0 & \text{sn}\eta & \text{sn}u & 0 \\
k\text{sn}\eta\text{sn}\eta\text{sn}(u+\eta) & 0 & 0 & \text{sn}(u+\eta)
\end{pmatrix}$$

(2.2)
where sn (and cn, dn in the formulas below) is the Jacobi elliptic function of modulus $0 \leq k \leq 1$. For the properties of the $R$-matrix (2.2) see refs. [10, 11].

To find boundary terms compatible with integrability, Sklyanin has introduced a pair of boundary $K$-matrices $K^\pm[4]$. $K^\pm$ are the solutions of the following reflection equations:

\[
\mathcal{R}_{12}(u-v)K^-_1(u)\mathcal{R}_{12}(u+v)K^-_2(v) = K^-_2(v)\mathcal{R}_{12}(u+v)K^-_1(u)\mathcal{R}_{12}(u-v) \tag{2.3}
\]
\[
\mathcal{R}_{12}(-u+v)K^{+t_1}_1(u)\mathcal{R}_{12}(-u-v-2\eta)K^{+t_2}_2(v) = K^{+t_2}_2(v)\mathcal{R}_{12}(-u-v-2\eta)K^{+t_1}_1(u)\mathcal{R}_{12}(-u+v) \tag{2.4}
\]

where $K^\pm_1 = K^\pm \otimes 1, K^\pm_2 = 1 \otimes K^\pm$. Here we should note that equations (2.3) and (2.4) only hold for the symmetric $R$-matrix (i.e. $\mathcal{R}$ has both $P$ and $T$ symmetry). The $\mathcal{R}$-matrix given in equation (2.2) is symmetric.

For a solution $K^-$ of the equation (2.3), then

\[
K^+(u) = K^{-t_1}(-u - \eta) \tag{2.5}
\]
gives the solution to the equation (2.4). So it is sufficient to consider the equations (2.3). Let $K^-(u)$ be of the following form:

\[
K^-(u) = \begin{pmatrix} a_1(u) & a_2(u) \\ a_3(u) & a_4(u) \end{pmatrix} \tag{2.6}
\]

For the $\mathcal{R}$-matrix (2.2), the equation (2.3) in component forms is equivalent to the following 12 equations:

\[
\begin{align*}
d_1e_2u_2v_2 + c_1u_3v_2 - c_1u_2v_3 - d_1e_2u_3v_3 &= 0, \\
d_2e_1u_2v_2 + c_2u_3v_2 - c_2u_2v_3 - d_2e_1u_3v_3 &= 0,
\end{align*} \tag{2.7}
\]

\[
\begin{align*}
e_2u_1v_1 + e_1u_4v_1 - e_1u_1v_4 - e_2u_4v_4 &= 0, \\
c_1d_2u_1v_1 + c_2d_1u_4v_1 - c_2d_1u_1v_4 - c_1d_2u_4v_4 &= 0,
\end{align*} \tag{2.8}
\]

\[
\begin{align*}
c_2e_1u_2v_1 - c_1e_2u_2v_1 - d_1u_3v_1 + d_2u_3v_1 - e_1u_1v_2 - e_2u_4v_2 + c_2d_1u_1v_3 + c_1d_2u_4v_3 &= 0, \\
c_2e_1u_2v_4 - c_1e_2u_2v_4 - d_1u_3v_4 + d_2u_3v_4 - e_1u_4v_2 - e_2u_1v_2 + c_2d_1u_4v_3 + c_1d_2u_1v_3 &= 0,
\end{align*} \tag{2.9}
\]
kinds of symmetries for the solutions of equation (2.3). After analyzing the system of the equations (2.7) -(2.12), we find that there are three hand, upon the exchange of $u$ and $v$ with different indices, (2.9), (2.11) are turned into the equations (2.10) and (2.12), respectively. On the other change of $\mu$ is a complex constant, the new matrix

\[
\begin{align*}
&c_2 c_1 u_3 v_1 - c_1 e_2 u_3 v_1 - d_1 u_2 v_1 + d_2 u_2 v_1 - e_1 u_1 v_3 - e_2 u_4 v_3 + c_2 d_1 u_1 v_2 + c_1 d_2 u_4 v_2 = 0, \\
c_2 e_1 u_3 v_4 - c_1 e_2 u_3 v_4 - d_1 u_2 v_4 + d_2 u_2 v_4 - e_1 u_4 v_3 - e_2 u_1 v_3 + c_2 d_1 u_4 v_2 + c_1 d_2 u_1 v_2 = 0, \\
&d_2 e_1 u_2 v_1 + c_2 u_3 v_1 - u_1 v_3 + d_1 d_2 u_1 v_3 + c_1 e_2 u_4 v_3 - d_1 e_2 u_2 v_4 - c_1 u_3 v_4 = 0, \\
d_2 e_1 u_2 v_4 + c_2 u_3 v_4 - u_4 v_3 + d_1 u_4 v_3 + c_1 e_2 u_1 v_3 - d_1 e_2 u_2 v_1 - c_1 u_3 v_1 = 0, \\
d_2 e_1 u_3 v_1 + c_2 u_2 v_1 - u_1 v_2 + d_1 d_2 u_1 v_2 + c_1 e_2 u_4 v_2 - d_1 e_2 u_3 v_4 - c_1 u_2 v_4 = 0, \\
d_2 e_1 u_3 v_4 + c_2 u_2 v_4 - u_4 v_2 + d_1 u_4 v_2 + c_1 e_2 u_1 v_2 - d_1 e_2 u_3 v_1 - c_1 u_2 v_1 = 0,
\end{align*}
\]

where $u_i = a_i(u), v_i = a_i(v) \ (i = 1, 2, 3, 4)$ and

\[
\begin{align*}
c_1 &= \frac{\text{sn}(u+v+\eta)}{\text{sn}(u+v)}, & c_2 &= \frac{\text{sn}(u-v+\eta)}{\text{sn}(u-v)}, \\
d_1 &= \frac{\text{sn}(u+v)}{\text{sn}(u+v+\eta)}, & d_2 &= \frac{\text{sn}(u-v)}{\text{sn}(u-v+\eta)}, \\
e_1 &= k\text{sn}\eta\text{sn}(u+v), & e_2 &= k\text{sn}\text{sn}(u-v).
\end{align*}
\]

After analyzing the system of the equations (2.7) -(2.12), we find that there are three kinds of symmetries for the solutions of equation (2.3).

(A) The Scaling Symmetry. Multiplication of the solution $K^{-}(u)$ by an arbitrary function $f(u)$ is still a solution of equation (2.3).

(B) Symmetry of Spectral Parameter. If we take a new spectral parameter $\bar{u} = \mu u$ where $\mu$ is a complex constant, the new matrix $K^{-}(\bar{u})$ is still a solution of (2.3). This symmetry is useful when we consider the rational limit of the matrix $K^{-}(u)$.

(C) Symmetry of Interchanging Variables $u$ and $v$ with Different Indices. Under the exchange of $u_2 \leftrightarrow u_3$ and $v_2 \leftrightarrow v_3$, the equations (2.7) are invariant, while the equations (2.9), (2.11) are turned into the equations (2.10) and (2.12), respectively. On the other hand, upon the exchange of $u_1 \leftrightarrow u_4$ and $v_1 \leftrightarrow v_4$, the equations (2.8) are invariant, while for each set of equations (2.9)-(2.12), one equation is changed into another.
In view of the relation among $c_i, d_i, e_i (i = 1, 2)$

$$\frac{d_1 e_2}{c_1} = \frac{d_2 e_1}{c_2}, \quad (2.14)$$
two equations in (2.7) and (2.8) are equivalent, respectively. Because of this fact and the above symmetries, we only need choose four equations from equations (2.7)-(2.12) in order to solve the reflection equation (2.3). In the following, we will take the first equation of (2.7), (2.8), (2.9) and (2.11), respectively. Of the four chosen equations, three equations are necessary to determine the $K^-$ due to the symmetry (A). The fourth equation may be as the consistency condition for the solutions of the three equations.

The solutions can be classified according to whether $a_1$ and $a_2$ equal to zero or not. If assuming $a_2 \neq 0$, we differentiate both sides of the first equation of (2.7) with respect to the spectral variable $v$ and set $u = v$, then we get the following equation

$$\frac{d}{du} \left( \frac{u_3}{u_2} \right) = -k \text{sn}(2u)[1 - \left( \frac{u_3}{u_2} \right)^2]. \quad (2.15)$$

Similarly, assuming $a_1 \neq 0$, one also has

$$\frac{d}{du} \left( \frac{u_4}{u_1} \right) = -\frac{1}{\text{sn}(2u)}[1 - \left( \frac{u_4}{u_1} \right)^2]. \quad (2.16)$$

if we differentiate both sides of the first equation of (2.8) with respect to $v$ and set $u = v$. Now we solve the four equations chosen above case by case. The procedure we adopt below is the same as that in refs.[10, 11]. We give the main results in the following.

Case (a): $a_1 = 0$. Using the first equation of (2.8), we have $a_4 = 0$. The nontrivial cases correspond to $a_2 \neq 0$ and $a_3 \neq 0$ which come from the first equation of (2.7). It has the following two subcases.

Subcase (a1): $(\frac{u_3}{u_2})^2 = 1$. From equation (2.15) and the symmetry (A), we obtain

$$K^-(u) = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} \quad (2.17)$$
Subcase (a2): \((u_3 / u_2)^2 \neq 1\). Integrating equation (2.15) and taking account of the symmetry (A), we have

\[
K^{-}(u) = \begin{pmatrix}
0 & \lambda_-(1 - k\sin^2 u) + 1 + k\sin^2 u \\
\lambda_-(1 - k\sin^2 u) - 1 - k\sin^2 u & 0
\end{pmatrix}
\]  
(2.18)

here \(\lambda_-\) is an arbitrary constant.

Case (b): \(a_2 = 0\). From the first equation of (2.7), we have \(a_3 = 0\), so the nontrivial cases are those of \(a_1 \neq 0\) and \(a_4 \neq 0\) from the first equation of (2.8). Here we also have two subcases.

Subcase (b1): \((u_3 / u_1)^2 = 1\). From equation (2.16) and the symmetry (A), we get

\[
K^{-}(u) = \begin{pmatrix}
1 & 0 \\
0 & \pm 1
\end{pmatrix}
\]  
(2.19)

Subcase (b2): \((u_3 / u_1)^2 \neq 1\). Considering the symmetry (A) and integrating equation (2.16), we have

\[
K^{-}(u) = \frac{1}{\sin \xi_-} \begin{pmatrix}
\sin(\xi_- + u) & 0 \\
0 & \sin(\xi_- - u)
\end{pmatrix}
\]  
(2.20)

here \(\xi_-\) is an arbitrary constant.

Case (c): \(a_1 \neq 0, a_2 \neq 0\). Using the first equation of (2.7) and (2.8), respectively, we know that \(a_3 \neq 0, a_4 \neq 0\). For this case, we may have following four subcases.

Subcase (c1): \((u_3 / u_2)^2 = 1, (u_4 / u_1)^2 = 1\), i.e. \(u_3 = \varepsilon_2 u_2, u_4 = \varepsilon_1 u_1, \varepsilon_1^2 = \varepsilon_2^2 = 1\). Using the first equation of (2.9), we get two \(K^{-}\) solutions:

\[
K^{-}_A(u) = \begin{pmatrix}
1 + \varepsilon_2 k\sin^2 u & \alpha_- \sin u \\
\varepsilon_2 \alpha_- \sin u & 1 + \varepsilon_2 k\sin^2 u
\end{pmatrix}
\]  
(for \(\varepsilon_1 = 1\))  
(2.21)

\[
K^{-}_B(u) = \begin{pmatrix}
1 + \varepsilon_2 k\sin^2 u & \beta_- \cnu \\
\varepsilon_2 \beta_- \cnu & 1 - \varepsilon_2 k\sin^2 u
\end{pmatrix}
\]  
(for \(\varepsilon_1 = -1\))  
(2.22)

where \(\alpha_-,\beta_-\) are two arbitrary constants.

Subcase (c2): \((u_4 / u_2)^2 = 1, (u_4 / u_1)^2 \neq 1\), then \(u_3 = \varepsilon_2 u_2, \varepsilon_2^2 = 1\). Solving the first equation of
(2.9), we have the following solution,

\[
K^-(u) = \frac{1}{\sn \xi_-} \left( \begin{array}{cc}
\sn(\xi_- + u)(1 + \varepsilon_2 \ksn^2 u) & \frac{\alpha_- \sn \cn \dn}{1 - k^2 \sn^2 \xi_-, \sn^2 u} \\
\frac{\varepsilon_2 \alpha_- \sn \cn \dn}{1 - k^2 \sn^2 \xi_-, \sn^2 u} & \sn(\xi_- - u)(1 + \varepsilon_2 \ksn^2 u)
\end{array} \right)
\]  

(2.23)

here \( \xi_- \) and \( \alpha_- \) are two arbitrary constants.

**Subcase (c3):** \( \left( \frac{m_2}{u_2} \right)^2 \neq 1, \left( \frac{m_1}{u_1} \right)^2 = 1 \). Taking \( u_4 = \varepsilon_1 u_1 \) and solving the first equation of (2.9), we get two \( K^- \) solutions,

\[
K^-_A(u) = \left( \begin{array}{cc}
cn \dn \nu & \mu_- \sn(2u)[(1 - \ksn^2 u)\lambda_- + 1 + \ksn^2 u] \\
\mu_- \sn(2u)[(1 - \ksn^2 u)\lambda_- - 1 - \ksn^2 u] & cn \dn \nu
\end{array} \right) 
\]  

(for \( \varepsilon_1 = 1 \))  

(2.24)

\[
K^-_B(u) = \left( \begin{array}{cc}
\sn \nu & \mu_- \sn(2u)[(1 - \ksn^2 u)\lambda_- + 1 + \ksn^2 u] \\
\mu_- \sn(2u)[(1 - \ksn^2 u)\lambda_- - 1 - \ksn^2 u] & -\sn \nu
\end{array} \right) 
\]  

(for \( \varepsilon_1 = -1 \))  

(2.25)

here \( \mu_-, \lambda_- \) are two arbitrary constants.

**Subcase (c4):** \( \left( \frac{m_2}{u_2} \right)^2 \neq 1, \left( \frac{m_1}{u_1} \right)^2 \neq 1 \). This is the most general case. Now we first have to integrate equations (2.15) and (2.16), then insert the results into the first equation of (2.9). Upon the use of the symmetry (A), we find that \( K^- \) has the following form:

\[
K^- = \frac{1}{\sn \xi_-} \left( \begin{array}{cc}
\sn(\xi_- + u) & \mu_- \sn(2u)[(1 - \ksn^2 u)\lambda_- + 1 + \ksn^2 u] \\
\mu_- \sn(2u)[(1 - \ksn^2 u)\lambda_- - 1 - \ksn^2 u] & \sn(\xi_- - u)
\end{array} \right)
\]  

(2.26)

here \( \xi_-, \mu_-, \lambda_- \) are arbitrary constants.

Note that the subcases (a1),(b1) and (c1) are discussed in ref.[10], while the subcase (c4) is studied in ref.[11].

Now we consider the boundary terms corresponding to the \( K^- \)-matrices given in equations (2.17)-(2.26). For the case of \( \text{tr} K^+(0) \neq 0 \), the Hamiltonians with boundary terms are obtained from the first derivative of the transfer matrix [4, 10, 11]

\[
H = 2r(\eta) \left( \sum_{n=1}^{N-1} H_{n,n+1} + \frac{1}{2} K^-_1(0)^{-1} K^-_1(0) + \frac{\text{tr}_{\text{tr} K^+(0)} K^+(0) H_{N0}}{\text{tr} K^+(0)} \right)
\]  

(2.27)
where $K(0)^{-1}$ in the second term is introduced for the case when $K(0) \neq 1$, $r(\eta) = \sin \eta$ and the two-site Hamiltonian $H_{n,n+1}$ is given by

$$H_{n,n+1} = \frac{1}{r(\eta)} P_{n,n+1} R'_{n,n+1}(0)$$  \hspace{1cm} (2.28)

$P_{i,j}$ is the permutation operator acting on $V_i \otimes V_j$ with $V_i = V_j \cong V$, i.e. $P(x \otimes y) = y \otimes x$ ($x, y \in V$). If

$$trK^+(0) = 0,$$  \hspace{1cm} (2.29)

then as pointed out in ref.[10], we will not have a well defined Hamiltonian (2.27) from the first derivative of the transfer matrix. In this case, we may obtain a Hamiltonian from the second derivative of the transfer matrix. Only when the condition

$$tr_0[K_0^+(0)H_{N_0}(0)] \propto 1$$  \hspace{1cm} (2.30)

holds for the $K$-matrix, will such a Hamiltonian be that with nearest-neighbor interactions. Otherwise, it will contain terms that couple every pair of sites in the bulk with the boundary, i.e. the Hamiltonian is non-local [10, 14]. The condition (2.29) holds for the $K$-matrices given in equations (2.17), (2.18), (2.19), (2.22) and (2.25), but the condition (2.30) does not hold for these $K$-matrices. Due to the above reason, therefore, we will not discuss the Hamiltonians corresponding to these $K$-matrices below.

Now we focus on the case of $trK^+(0) \neq 0$. Using the equations (2.2) and (2.8), the first term in the equation (2.27), which describes the bulk properties of the $XYZ$ model, is given as follows,

$$H_{\text{bulk}} = \sum_{n=1}^{N-1} [(1 + \Gamma) \sigma_n^x \sigma_{n+1}^x + (1 - \Gamma) \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z],$$  \hspace{1cm} (2.31)

where

$$\Gamma = k \sin^2 \eta, \quad \Delta = c \eta \delta \eta.$$  \hspace{1cm} (2.32)

and $\sigma_n^x, \sigma_n^y, \sigma_n^z$ are the Pauli matrices. The second and the third terms in the equations (2.27) represent the left and right boundary terms, respectively. Using the equations (2.2)
and (2.28), the boundary terms for the $K^-$-matrices (2.20), (2.21), (2.23), (2.24), (2.26)
and their corresponding $K^+$-matrices are of the following form,

$$H_{\text{boundary}} = \text{sn}(A_-\sigma_1^z + B_-\sigma_1^+ + C_-\sigma_1^- + A_+\sigma_N^z + B_+\sigma_N^+ + C_+\sigma_N^-) \quad (2.33)$$

with coefficients $A_{\pm}, B_{\pm}, C_{\pm}$ being as follows, respectively,

$$A_{\pm} = \frac{\text{cn} \xi_{\pm} \text{dn} \xi_{\pm}}{\text{sn} \xi_{\pm}}, \quad B_{\pm} = C_{\pm} = 0; \quad (\text{for subcase (b2)}) \quad (2.34)$$

$$A_+ = 0, \quad B_+ = \varepsilon_2\alpha_+, \quad B_- = \alpha_-, \quad C_+ = \alpha_+, \quad C_- = \varepsilon_2\alpha_-; \quad (\text{for subcase (c1)}) \quad (2.35)$$

$$A_{\pm} = \frac{\text{cn} \xi_{\pm} \text{dn} \xi_{\pm}}{\text{sn} \xi_{\pm}}, \quad B_+ = \frac{-\varepsilon_2\alpha_+}{\text{sn} \xi_{\pm}}, \quad B_- = \frac{\alpha_-}{\text{sn} \xi_{\pm}}, \quad C_+ = \frac{\alpha_+}{\text{sn} \xi_{\pm}}, \quad C_- = \frac{-\varepsilon_2\alpha_-}{\text{sn} \xi_{\pm}}; \quad (\text{for subcase (c2)}) \quad (2.36)$$

$$A_{\pm} = 0, \quad B_\pm = \mp2\mu_{\pm}(\lambda_{\pm}+1), \quad C_\pm = \mp2\mu_{\pm}(\lambda_{\pm}-1); \quad (\text{for subcase (c3)}) \quad (2.37)$$

$$A_{\pm} = \frac{\text{cn} \xi_{\pm} \text{dn} \xi_{\pm}}{\text{sn} \xi_{\pm}}, \quad B_\pm = \frac{2\mu_{\pm}(\lambda_{\pm}+1)}{\text{sn} \xi_{\pm}}, \quad C_\pm = \frac{2\mu_{\pm}(\lambda_{\pm}-1)}{\text{sn} \xi_{\pm}}; \quad (\text{for subcase (c4)}) \quad (2.38)$$

It is easy to see that the equation (2.20) is the special case $\mu_- = 0$ of the equation (2.26).

Comparing the equation (2.34) with the equation (2.38), we know that the off-diagonal elements of the $K$-matrix are responsible for the $B_{\pm}, C_{\pm}$ terms in the $H_{\text{boundary}}$.

From the symmetry (A) and the equation (2.27), we know that if the scaling factor $f(u)$ satisfies the conditions: $f'(0) = 0$ and $f(0) = \text{const.}$, then $f(u)$ will not change the boundary terms. $f(u) = \text{cn} \mu \text{dn} u/(1-k^2 \text{sn}^2 u)$ is such an example. When we take $\xi_- = K(K$ is the half-period magnitude of the Jacobi elliptic function) in the equation (2.23), then the equation (2.23) reduces to the equation (2.21) up to a factor $\text{cn} \mu \text{dn} u/(1-k^2 \text{sn}^2 u)$; Similarly, the equation (2.26) reduces to the equation (2.24) up to a factor $1/(1-k^2 \text{sn}^2 u)$.
These two factors satisfies the conditions for the scaling factor \( f(u) \) above. The equations (2.21) and (2.23), therefore, correspond to the same boundary terms upon taking \( \xi_- = K, \xi_+ = -K \); the same holds for the equations (2.24) and (2.26).

In closing this section, we make a remark on the symmetry of the Hamiltonian (2.27). It follows from the equations (2.31) and (2.33) that the Hamiltonian is transposition invariant if we make any replacement given below,

(1) \( B_- \rightarrow C_-, \ B_+ \rightarrow C_+; \) (2.39)

(2) \( a) \sigma_n \rightarrow \sigma_{N-n+1}, \ \text{in particular,} \ \sigma_1 \rightarrow \sigma_N, \ \sigma_2 \rightarrow \sigma_{N-1}; \) \( b) \ A_- \rightarrow A_+, \ B_- \rightarrow C_+, \ C_- \rightarrow B_+. \) (2.40)

In the next section, we will find that the second replacement is very useful for the construction of the Lax pair of the open XYZ spin-chain.

### 3 The Lax pair for the open XYZ spin-chain

In section 2, we have derived the Hamiltonian of the open \( XYZ \) spin-\( \frac{1}{2} \) chain. For the 1D system with \( N \) sites, the general form of the the Hamiltonian is given by the equations (2.27), (2.31) and (2.33) [11],

\[
H = -\frac{1}{2} \sum_{n=1}^{N-1} (J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z) + \text{sn}(A_- \sigma_1^+ + B_- \sigma_1^+ + C_- \sigma_1^- + A_+ \sigma_N^+ + B_+ \sigma_N^+ + C_+ \sigma_N^-),
\]

(3.1)

where

\[
J_x = -2(1 + k \text{sn}^2 \eta), \ J_y = -2(1 - k \text{sn}^2 \eta), \ J_z = -2 \text{cn} \eta \text{dn} \eta,
\]

(3.2)

The equations of motion for the system (3.1) are given as follows,

\[
\frac{d}{dt} \sigma_n^x = -J_y \sigma_n^x (\sigma_{n+1}^y + \sigma_{n-1}^y) + J_x \sigma_n^x (\sigma_{n+1}^z + \sigma_{n-1}^z),
\]

(3.3)

\[
\frac{d}{dt} \sigma_n^y = -J_z \sigma_n^y (\sigma_{n+1}^z + \sigma_{n-1}^z) + J_x \sigma_n^y (\sigma_{n+1}^x + \sigma_{n-1}^x),
\]

(3.4)
where \( u \) and \( Q \) of periodic boundary condition. For the
conditions. The consistency conditions for the equations (3.12)-(3.15) are the following

In order to rewrite equations above in the Lax form, we consider the following operator
version of an auxiliary linear problem,

\[
\frac{d}{dt} \sigma_n^x = -J_x \sigma_n^y (\sigma_{n+1}^x + \sigma_{n-1}^x) + J_y \sigma_n^x (\sigma_{n+1}^y + \sigma_{n-1}^y), \quad (n = 2, \cdots, N - 1)
\]

where \( u \) is the spectral parameter which does not depend on the time \( t \), \( L_n(u), M_n(u), Q_1(u) \) and \( Q_N(u) \) are called the Lax pair. \( Q_1(u) \) and \( Q_N(u) \) are responsible for the boundary conditions. The consistency conditions for the equations (3.12)-(3.15) are the following Lax equations:

\[
\frac{d}{dt} L_n(u) = M_{n+1}(u)L_n(u) - L_n(u)M_n(u), \quad (n = 2, \cdots, N - 1)
\]

From the eqs. (3.3)-(3.5) and (3.16), we see that they are the same as those in the case of periodic boundary condition. For the \( XYZ \) model with periodic boundary condition,
the Lax pair $L_n, M_n$ has been given by Sogo and Wadati\[15\]. The operators $L_n$ and $M_n$ are of the following forms,

$$L_n = \begin{pmatrix} w_4 + w_3\sigma^z_n & w_1\sigma^x_n - iw_2\sigma^y_n \\ w_1\sigma^x_n + iw_2\sigma^y_n & w_4 - w_3\sigma^z_n \end{pmatrix} \quad (3.19)$$

$$M_n = \begin{pmatrix} M^0_n + M^3_n & M^1_n - iM^2_n \\ M^1_n + iM^2_n & M^0_n - M^3_n \end{pmatrix} \quad (3.20)$$

where

$$M^0_n = \sum_{i=1}^{3} F^i_n \sigma^i_{n-1},$$

$$M^k_n = G^k_n (\sigma^k_n + \sigma^k_{n-1}) + \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon^{ijk} H^j_n \sigma^i_n \sigma^j_{n-1},$$

$$w_1(u) + w_2(u) = sn\eta, \quad w_1(u) - w_2(u) = ksnpsnusu(u + \eta),$$

$$w_3(u) + w_4 = sn(u + \eta), \quad w_4(u) - w_3 = snu,$$

and $\sigma^0 = 1, \sigma^1, \sigma^2, \sigma^3 = \sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices. For the explicit expressions of $F^i, G^i, H^i$ see ref. [15]. So, for the open $XYZ$ model, we only have to construct operators $Q_1(u)$ and $Q_N(u)$. If we write $Q_1(u)$ and $Q_N(u)$ in the following forms:

$$Q_1(u) = (a^{(-)}_{ij})_{2\times 2}, \quad Q_N(u) = (a^{(+)}_{ij})_{2\times 2} \quad (3.23)$$

and

$$a^{(-)}_{ij} = \sum_{l=0}^{3} a^{(-)l}_{ij} \sigma^l_1, \quad a^{(+)}_{ij} = \sum_{l=0}^{3} a^{(+l)}_{ij} \sigma^l_N. \quad (3.24)$$

Substituting the equations (3.23) and (3.24) into the equations (3.17) and (3.18), we obtain 32 equations for $a^{(\pm)l}_{ij}(i, j = 1, 2, l = 0, 1, 2, 3)$. Due to the following lemma, we only have to solve 16 equations, for example, for $a^{(+l)}_{ij}(i, j = 1, 2, l = 0, 1, 2, 3)$.

**Lemma:** If we make the replacement (2.40), then we have

$$a^{(-)}_{ij} \rightarrow a^{(+)}_{ij} \quad (3.25)$$

provided that the following correspondences hold

$$\begin{cases} \frac{d}{dt} \sigma^x_1 \rightarrow - \frac{d}{dt} \sigma^x_N, \\ \frac{d}{dt} \sigma^y_1 \rightarrow \frac{d}{dt} \sigma^y_N, \\ \frac{d}{dt} \sigma^z_1 \rightarrow - \frac{d}{dt} \sigma^z_N. \end{cases} \quad (3.26)$$
under the transposition. Here the superscript \( t \) denotes the transposition of a matrix.

Note that \( (\sigma^x)^t = \sigma^x, (\sigma^y)^t = -\sigma^y, (\sigma^z)^t = \sigma^z \), then from the equations (3.6)-(3.11) and the expressions for \( M_1^i, M_2^i (i = 1, 2, 3) \), it is straightforward to show that the conditions (3.26) and (3.27) hold. So if we get the solutions for \( a_{ij}^{(+)} \), using (3.25) the expressions for \( a_{ij}^{(-)} \) are also obtained. After a lengthy calculations, the expressions for \( a_{ij}^{(\pm)} (i, j = 1, 2, l = 0, 1, 2, 3) \) are as follows,

\[
\begin{align*}
(a_{11}^{(+0)}) &= \frac{4i\sin A\eta}{D}a(u)b(u); & (3.28) \\
(a_{11}^{(+1)}) &= \frac{i\sin A\eta}{D}\{\pm(B_+ - C_+)c(u)(-2e(u) + f(u) + g(u)) \\
&- (B_+ + C_+)(b^2(u) + c^2(u) - f^2(u))\}; & (3.29) \\
(a_{11}^{(+2)}) &= \pm\frac{\sin A\eta}{D}\{\pm(B_+ - C_+)(a^2(u) + c^2(u) - e^2(u)) \\
&+ (B_+ + C_+)c(u)(f(u) - g(u))\}; & (3.30) \\
(a_{11}^{(+3)}) &= G_3 - \frac{i\sin A\eta}{D}d(u); & (3.31) \\
(a_{12}^{(+0)}) &= \frac{2i\sin A\eta}{D}c(u)\{\pm(B_+ - C_+)a(u) + (B_+ + C_+)b(u)\}; & (3.32) \\
(a_{12}^{(+1)}) &= G_1 \mp \frac{2i\sin A\eta}{D}b(u)(-2e(u) + f(u) + g(u)); & (3.33) \\
(a_{12}^{(+2)}) &= -iG_2 \mp \frac{2i\sin A\eta}{D}a(u)(-f(u) + g(u)); & (3.34) \\
(a_{12}^{(+3)}) &= \mp \frac{i\sin A\eta}{D}(f(u) + g(u))\{\pm(B_+ - C_+)a(u) + (B_+ + C_+)b(u)\}; & (3.35) \\
(a_{21}^{(+0)}) &= \frac{2i\sin A\eta}{D}c(u)\{-(B_+ - C_+)a(u) + (B_+ + C_+)b(u)\}; & (3.36) \\
(a_{21}^{(+1)}) &= G_1 \mp \frac{2i\sin A\eta}{D}b(u)(-2e(u) + f(u) + g(u)); & (3.37) \\
(a_{21}^{(+2)}) &= iG_2 \mp \frac{2i\sin A\eta}{D}a(u)(-f(u) + g(u)); & (3.38) \\
(a_{21}^{(+3)}) &= \frac{i\sin A\eta}{D}(f(u) + g(u))\{-(B_+ - C_+)a(u) + (B_+ + C_+)b(u)\}; & (3.39) \\
(a_{22}^{(+0)}) &= -a_{11}^{(+0)}; & (3.40)
\end{align*}
\]
where

\[
a(u) = w_2w_3 + w_1w_4, \quad b(u) = w_1w_3 + w_2w_4,
\]

\[
c(u) = w_1w_2 + w_3w_4,
\]

\[
d(u) = (w_2 - w_1)^2\{(w_3 - w_4)^2 - (w_2 + w_1)^2\} + (w_2 + w_1)^2\{(w_3 + w_4)^2 - (w_2 - w_1)^2\},
\]

\[
e(u) = w_2^2 - w_3^2, \quad f(u) = w_2^2 - w_3^2, \quad g(u) = w_1^2 - w_4^2.
\]

The equations (3.28)-(3.43) combined with the operators \(L_n, M_n\) are the Lax pair for the open \(XYZ\) spin-chain. In the trigonometric limit \(k \to 0\), where \(\text{sn} u \to \sin u\) and taking \(J_x = J_y = 1, J_z = \cos \eta, B_\pm = C_\pm = 0\), we recover the result in the case of open \(XXZ\) model given in the ref. [12] (up to a replacement of \(\eta\) by \(2\eta\)).

4 Remarks and discussions

In this paper, based on the analysis of the symmetries of the reflection equations, we present their solutions for the open \(XYZ\) model case by case. We realised that both diagonal and off-diagonal elements of the \(K\)-matrices have contributions to the boundary terms of the Hamiltonian, and a specific scaling factor does not affect the properties of the system. Furthermore, we construct the Lax pair for the open \(XYZ\) model to show its integrability in another way. We believe that our procedure to the solutions of the reflection equations can be applied to other problems. Recently, the classification of six-vertex, eight-vertex solutions of the coloured Yang-Baxter equation has been given[16, 17].
the Yang-Baxter equation with dynamical parameters (or called the Gervais-Neveu-Felder equation) has been studied\cite{18}, we expect to investigate their corresponding reflection equations on the basis of analysing their symmetries.

References

[1] L. D. Faddeev, in: *Recent Advances in Field Theory and Statistical Mechanics, Les Houches XXXIX*, eds. J. B. Zuber and R. Stora, North-Holland Publ., 1984, pp 561-608;

P. P. Kulish and E. K. Sklyanin, in: *Integrable Quantum Field Theories*, Lecture Notes in Physics **151**, Springer-Verlag, 1982, pp 61-119;

L. D. Faddeev, *How Algebraic Bethe Ansatz Works for Integrable Model*, preprint, hep-th/9605187.

[2] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.

[3] I. Cherednik, *Teor. Mat. Fiz.*, **61**(1984)35.

[4] E. K. Sklyanin, *J. Phys.*, **A 21**(1988)2375.

[5] L. Mezincescu and R. I. Nepomechie, *Int. J. Mod. Phys.*, **A 6**(1991)5231; **A 7**(1992)5657.

[6] A. Foerster and M. Karowski, *Nucl. Phys.*, **B 408**(1993)512.

[7] A. González-Ruiz, *Nucl. Phys.*, **B 424**(1994)468.

[8] S. Ghoshal and A. Zamolodchikov, *Int. J. Mod. Phys.*, **A 9**(1994)3841; S. Ghoshal, *Int. J. Mod. Phys.*, **A 9**(1994)4801.

[9] H J. de Vega and A. González-Ruiz, *J. Phys.*, **A 26**(1993)L519.

[10] H J. de Vega and A. González-Ruiz, *J. Phys.*, **A 27**(1994)6129.
[11] T. Inami and H. Konno, *J. Phys.*, **A 27**(1994)L913.

[12] H. Q. Zhou, *J. Phys.*, **A 29**(1996)L489.

[13] X. W. Guan, M. S. Wang and S. D. Yang, *Nucl. Phys.*, **B 485** (1997)685.

[14] R. Cuerno and A. González-Ruiz, *J. Phys.*, **A 26** (1993)L605.

[15] K. Sogo and M. Wadati, *Prog. Theor. Phys.*, **68**(1982)85.

[16] X. D. Sun, S. K. Wang and K. Wu, *J. Math. Phys.*, **36** (1995)6043.

[17] S. K. Wang, *J. Phys.*, **A 29**(1996)2259.

[18] J. Avan, O. Babelon and E. Billey, *Commun. Math. Phys.*, **178**(1996)281.