DYNAMICS AND INTEGRABILITY PROPERTY OF THE CHIRAL STRING MODEL

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(5 May 1999, preprint no DARC/99-09)

The effect of fermionic string conductivity by purely right (or purely left) moving “zero modes” is shown to be governed by a simple Lagrangian characterising a certain “chiral” (null current carrying) string model whose dynamical equations of motion turn out to be explicitly integrable in a flat spacetime background.

PACS numbers: 98.80.Cq, 11.27+d

I. INTRODUCTION.

In his epoch making paper entitled “Superconducting Strings”, Witten introduced two essentially different processes that can produce conserved currents on cosmic strings. The first, for which the term “superconducting” is most appropriate, is based on the formation of a mass. (In the case of a “ordinary” cosmic string model, representing a vortex defect of the vacuum, $m$ would be expected to be of the order of magnitude of the relevant Higgs mass scale).

In the Witten model, there is an internal scalar phase field variable $\varphi$ in terms of which the action takes the less trivial form

$$\mathcal{L} = -m^2 - \frac{1}{2} \kappa_0 \gamma^{ij} \varphi_i \varphi_j,$$

for some positive constant $\kappa_0$. In the presence of a Maxwellian gauge covector field $A_\mu$, if the internal field has charge coupling constant $e$, the relevant gauge covariant derivative will have the form $\varphi_i = \varphi_{.,i} - e A_\mu x^\mu_i$.

Soon after its introduction, it was pointed out that for the purpose of treating Witten’s bosonic “superconductivity” mechanism [2], his simple bosonic model [1] would require generalisation [3] to allow for non-linearity. More particularly, it was shown by one of us [5] that Witten’s bosonic “superconductivity” mechanism entailed supersonic wiggle propagation, whereas his own proposed string model [4] was characterised by subsonic wiggle propagation. This discrepancy, which is significant even for very weak currents, has recently been resolved by replacing Witten’s naive model [4] by a more appropriately adapted model [5] whose action is non linearly dependent on the current amplitude.

In so far as Witten’s fermionic “zero-mode” conductivity mechanism is concerned, the question of the adequacy or otherwise of Witten’s naive model [4] has not yet been given much attention. Whereas it is evident that corrections will be needed for cases involving strong currents, what is not yet clear is whether Witten’s model [4] will be invalidated (as in the bosonic case) by non-linear effects even for fermionic currents of very low amplitude.

There is however a special case – the subject of the present article – for which it seems clear in advance that there is no risk of invalidation by non linear effects, because the relevant current amplitude simply vanishes – a condition that will be preserved by the field equations of the model [4] if $e = 0$. The condition of having zero amplitude does not of course mean that the current itself has
to vanish, but merely that it should be null in the sense of being lightlike. Individual fermionic “zero mode” excitations are intrinsically characterised by the condition of being null in this sense, but a generic superposition of “left” and “right” moving excitations will provide a total current that is timelike or spacelike. However, in the special “chiral” case for which only “left” (or equivalently only “right”) moving “zero modes” are excited, the total current will be null, i.e. it will satisfy

$$\gamma_{ij}\varphi_{i}\varphi_{j} = 0. \quad (5)$$

The suggestion that such a “chiral” current might arise naturally from the presence of massive neutrinos in Grand Unified Theories was implicit in Witten’s original article and right-handed neutrino current in $SU(5) \times U(1) \rightarrow SU(5)$ theory with the last symmetry breaking realized by means of a Higgs field in the $\mathbf{126}$ of $SO(10)$, and the question has recently been raised more explicitly by several authors: in particular it has been pointed out that “chiral” (purely left or purely right moving) zero modes are to be expected in strings with half broken supersymmetry in the presence of a Fayet Iliopoulos term. The technical requirements for having lightlike currents flowing along the strings due to fermions are discussed in the appendix.

The likelihood that null currents can arise naturally from “chiral” zero modes has potentially important cosmological implications, in view of the consequent possibility of formation of chiral vortons—which can be expected to be more stable than vortons of other kinds—a perspective that has provided the main motivation for the present investigation.

Quite apart from the consideration that it will be applicable exactly in such cases, the chirality condition has also frequently been advocated for use as an approximation even for bosonic currents. The first occasion this was done was in the seminal article in which the question of vorton states was originally raised by Davis and Shellard, and the approximation has in fact been systematically employed in the more recent work. In the particular case of circular string loops the solution are characterised by a certain Bernoulli ratio, $b$, that would need to be close to unity for the chiral approximation to be tenable. However even when such a condition is satisfied the use of such an approximation is dangerous, since it artificially suppresses potentially unstable degrees of freedom that are present when the generic non linear model is used.

Although its use as an approximation for the treatment of string currents that are actually spacelike or timelike is rather questionable, there can be no such doubts about the validity of the chiral string model in cases when the current is physically constrained to be null, as will be the case when it arises from fermionic zero modes of the chiral type discussed above. Chiral solutions are obtainable as solutions for string models of the generic non-linear elastic type including the Witten limit case (in the absence of electromagnetic coupling, i.e. for $e = 0$) subject to the nullity constraint. One of the first points to be emphasised here is that the chiral string model can also conveniently be characterised directly in its own right by a Lagrangian of its own, which can be taken to have the form

$$\mathcal{L} = -m^2 - \frac{1}{2} \psi^2 \gamma_{ij} \varphi_i \varphi_j. \quad (6)$$

This chiral string Lagrangian is obtained by replacing the positive constant $\kappa_0$ in the Witten model by the square of an auxiliary Lagrange multiplier field $\psi$. For the purpose of applying the variation principle, the auxiliary amplitude $\psi$ and the phase scalar $\varphi$ are to be treated on the same footing as independent internal field variables on the string worldsheet, whose spacetime location is of course also to be considered to be freely variable.

**II. DYNAMICAL EQUATIONS OF THE CHIRAL STRING MODEL.**

It is evident that the requirement of invariance of the integral with respect to free variations of $\psi$ in leads back immediately to the nullity condition. In particular, since the worldsheet is only 2-dimensional, this has the well known corollary that the field $\phi$ will be harmonic, i.e.

$$\varphi^2_{ij} = 0, \quad (7)$$

using a semicolon to indicate covariant differentiation with respect to the worldsheet metric $\gamma_{ij}$. Furthermore, by differentiation of (5), it can be seen the phase gradient must satisfy the geodicity condition

$$\varphi_{ij} \varphi^{ij} = 0. \quad (8)$$

The only other internal field equation on the world sheet, namely the one obtained from the requirement of invariance of the integral with respect to free variations of $\phi$ in (6), will evidently take the form

$$\left( \psi^2 \varphi^{ij} \right)_{;i} = 0. \quad (9)$$

In view of (6), the latter reduces simply to the condition

$$\varphi_{ij} \psi_{;i} = 0, \quad (10)$$

to the effect that $\psi$ is constant along the world sheet null lines on which $\varphi$ is constant, or in other words that $\psi$ is a function only of $\varphi$.

For the purpose of dealing with the extrinsic equations of motion governing the evolution of the worldsheet it is convenient to work in terms of tensors specified with respect to the background spacetime coordinates $x^\mu$, starting with the fundamental tensor $\gamma^{\mu \nu}$ that is defined as the spacetime projection of the contravariant inverse $\gamma_{ij}$ of the induced metric, i.e. 2}
\[ \gamma^{\mu \nu} = \gamma^{ij} x^\mu_i x^\nu_j, \]  
(whose mixed form \( \gamma^{\mu \nu} \) is the tangential projection tensor) and the tangent vector  
\[ \nu^\mu = \psi \partial_i x^\mu. \]  
The latter will be equivalently expressible in terms of the tangentially projected covariant differentiation operator  
\[ \nabla_\mu = \gamma^{\mu \nu} \nabla_\nu, \]  
(\( \nabla_\nu \) is the usual operator of covariant differentiation with respect to \( g_{\mu \nu} \)) in the form  
\[ \nu^\mu = \psi \nabla_\mu \psi. \]  
(14)

In terms of this worldsheet tangential vector \( n^\mu \), the Lagrangian (4) of the chiral string model can be rewritten in the form  
\[ \mathcal{L} = -m^2 - \frac{1}{2} \nu^\mu \nu_\mu, \]  
(15) and it can be seen that the ensuing internal field equations as obtained above will be expressible as the nullity condition  
\[ \nu^\mu \nu_\mu = 0, \]  
(16) and the worldsheet current conservation law  
\[ \nabla_\mu \nu^\mu = 0. \]  
(17)

The standard procedure [11] for obtaining the extrinsic equations of motion starts from the evaluation of the relevant surface stress momentum energy density tensor as defined by the general formula  
\[ \mathbf{T}^{\mu \nu} = 2 \frac{\partial \mathcal{L}}{\partial g_{\mu \nu}} + \mathcal{L} \gamma^{\mu \nu}, \]  
(18) which in the present application (13) simply gives  
\[ \mathbf{T}^{\mu \nu} = \nu^\mu \nu_\nu - m^2 \gamma^{\mu \nu}, \]  
(19) when the on-shell condition (14) is satisfied. In terms of this tensor, the dynamical condition for the invariance of the action (4) with respect to arbitrary infinitesimal displacements is equivalent to the condition of vanishing external force  
\[ f^\mu = 0, \]  
(20) where the force density is given by an expression of the usual form  
\[ f^\mu = \nabla_\nu \mathbf{T}^{\nu \mu}. \]  
(21) as the condition for the invariance of the action (4) with respect to arbitrary infinitesimal displacements.

As usual the tangential projection of (20) will be satisfied as a Noether identity, i.e. we shall automatically have  
\[ \gamma^{\mu \nu} f^\mu = 0, \]  
(22) when the relevant internal equations, in this case (10) and (17), are satisfied. This means that the evolution of the worldsheet will be given just by the orthogonal projection of (17), i.e. its contraction  
\[ \perp_\mu f^\mu = 0, \]  
(23) where the orthogonal projection tensor \( \perp_\mu \) is defined as the complement of the fundamental tensor \( \gamma^{\mu \nu} \), i.e.  
\[ \perp_\mu = g^\mu - \gamma^{\mu \nu}. \]  
(24)

By the usual integration by parts operation this orthogonally projected force density on the worldsheet is obtained in the standard form  
\[ \perp_\mu f^\mu = \mathbf{T}^{\mu \nu} K_{\mu \nu \rho}, \]  
(25) where the \( K_{\mu \nu \rho} \) is the second fundamental tensor as defined (3) in terms of tangential differentiation of the first fundamental tensor \( \gamma^{\rho \sigma} \) by  
\[ K_{\mu \nu \rho} = \gamma^{\rho \sigma} \nabla_\mu \gamma_\sigma. \]  
(26) It evidently follows that in the particular case of the chiral model, as characterised by (14), the extrinsic dynamical equation (23) will take the form  
\[ m^2 K^\mu = \nu^\rho \nu_\rho K_{\mu \nu \rho}, \]  
(27) where the extrinsic curvature vector \( K^\mu \) is the trace of the second fundamental tensor, i.e.  
\[ K^\mu = K^\mu_{\mu \rho}, \]  
(28) (whose vanishing expresses the equation of motion for the Nambu Goto model, as obtained in the limit for which the null current vector \( \nu^\mu \) vanishes so that the right hand side of (27) drops out).

### III. Characteristic Formulation of the Dynamical Equations.

In order to proceed it is convenient to introduce a conventionally normalised null tangent vector diad on the worldsheet, taking one member to be the conserved null current vector \( \nu^\mu \) that has already been introduced, which can without loss of generality (if necessary by adjusting the sign convention for \( \nu \) or \( \psi \)) be assumed to be oriented towards the future. Subject to the convention that it too should be oriented towards the future the other null tangent vector, \( o^\mu \) say, will be characterised by  
\[ o^\mu o_\mu = 0 \quad o^\mu \nu_\mu = -1, \]  
(29)
so that the fundamental tensor will take the form
\[ \gamma_{\mu\nu} = -2\nu^\rho \chi_{\rho \mu} \]

using round brackets to denote index symmetrisation. The worldsheet stress momentum energy density tensor \([13]\) can then be given an expression of the analogous form
\[ T_{\mu\nu} = \nu^\rho \beta_{\rho \mu} \]

in which \(\beta^\rho\) is a future directed timelike worldsheet tangent vector given by
\[ \beta^\mu = \nu^\mu + 2m^2\sigma^\mu \].

In view of the general theorem \([11]\) to the effect that the characteristic covectors \(\chi_{\mu}\) (i.e. gradients of scalars that are constant along admissible directions of propagation of infinitesimal discontinuity) of the extrinsic motion are specified as solutions of the quadratic characteristic equation
\[ T_{\mu\nu} \chi_{\mu} \chi_{\nu} = 0 \]

it is evident from \([31]\) that the null vector \(\nu^\mu\) defined by \([12]\) and the timelike vector \(\beta^\mu\) defined by \([32]\) lie along the corresponding bicharacteristic directions (i.e. admissible directions of propagation of infinitesimal discontinuity) of the extrinsic motion. It is also evident that as far as the internal dynamical equations \([16]\) and \([17]\) are concerned there is only a single bicharacteristic direction, which coincides with the extrinsic bicharacteristic direction specified by \(\nu^\mu\).

The preceding conclusion means that the dynamical behaviour of chiral string model will be much simpler than that of a generic elastic string model \([11]\) or that of the Witten model \([1]\) in particular, for which there are two independent internal (sonic type) bicharacteristic directions, neither of which coincides with either of the two extrinsic bicharacteristic directions. The case of the chiral string model is intermediate between that of the special transonic string model \([12]\) for which there are two independent internal bicharacteristic directions which exactly coincide with the two extrinsic bicharacteristic directions, and that of the simple Nambu Goto model, which has no internal degrees of freedom, so that it has no internal bicharacteristic directions at all.

Following the example of the transonic case \([13]\), the coincidence of the only internal bicharacteristic direction of the chiral string model with one of its extrinsic bicharacteristic directions can be exploited for the purpose of converting its dynamical equations to a particularly convenient characteristic form. To start with, as is possible for any string model \([11]\), the characteristic expression \([31]\) for the surface stress tensor can be substituted into the generic force equation \([21]\) from which, using the worldsheet integrability condition
\[ \nabla_\mu (\beta^\rho \nabla_\rho \nu^\mu - \nu^\nu \nabla_\nu \beta^\mu) = 0 \]

one obtains the extrinsic (orthogonally projected) part of the force equation in the convenient form
\[ \nabla^\rho \nu^\mu \nabla_\rho \beta^\mu = 0 \]

In the particular case of the chiral model, the internal dynamical equation \([16]\) evidently implies \(\nu^\mu \nabla_\mu \nu^\nu = 0\), from which by \((29)\) it can be seen that the other internal dynamical equation \([17]\) will be expressible in either of what, by \((24)\), are the equivalent forms
\[ \nu^\mu \nabla_\mu \nu^\rho = 0 \]

By further use of \((16)\), \((29)\) and \((30)\), it can be seen that the preceeding relations are equivalent to
\[ \gamma^\rho \mu \nu^\nu \nabla_\nu \nu^\mu = 0 \]

which by \((32)\) will also be equivalently expressible as
\[ \gamma^\rho \mu \nu^\nu \nabla_\nu \beta^\mu = 0 \]

Using the analogous formula \((33)\) it can now be seen that this internal dynamical equation \((18)\) can be amalgamated with the extrinsic dynamical equation \((23)\) to give a combined dynamical equation of the simple characteristic form
\[ \nu^\nu \nabla_\nu \beta^\mu = 0 \]

which effectively states that the timelike characteristic vector \(\beta^\mu\) is subject to transport by the null characteristic vector \(\nu^\mu\). This transport equation evidently preserves the normalisation conditions
\[ \beta^\mu \nu_\mu = -2m^2, \quad \beta^\mu \beta_\mu = -4m^2 \]

that follow from the defining relation \((33)\).

As in the familiar example of the Nambu Goto case (for which both of the extrinsic characteristic directions are null) it is useful for many purposes to use coordinates that are constant along the respective characteristic directions. As far as the null characteristic direction is concerned, the scalar \(\nu\) introduced above can evidently serve for this purpose. With respect to worldsheet coordinates consisting of \(\varphi\) and another characteristic coordinate, \(\sigma\) say, chosen to be constant along the timelike characteristic direction specified by \(\beta^\mu\), it can be seen from \((13)\) and \((10)\) that, whatever the precise convention used to fix the normalisation of \(\varphi\) and \(\sigma\), the timelike characteristic vector will be given by
\[ \beta^\mu = -2m^2 \nabla_\nu \varphi \]
IV. INTEGRABILITY IN A FLAT BACKGROUND.

The preceding equations are applicable in an arbitrarily curved cosmological background spacetime. Let us now restrict our attention to the case of a flat background, and more specifically let us restrict the coordinates to be of ordinary Minkowski type, \(x^\mu \leftrightarrow \{x^\rho, x^\sigma\}\), \(a = 1, 2, 3\) so that the metric is given by

\[ g_{\mu\nu} dx^\mu dx^\nu = (dx^a)^2 - \delta_{ab} dx^a dx^b \]  

where \(\delta_{ab}\) is the Kronecker unit matrix. Since the Christoffel connection will vanish in such a reference system, the characteristic dynamical equation (33) will reduce to the simple form

\[ \frac{\partial}{\partial \sigma} \beta^\mu = 0. \]  

Since by (10) we know that \(\psi\) depends only on \(\varphi\), it can be seen from (11) that (13) is equivalent to

\[ \frac{\partial^2 x^\mu}{\partial \sigma \partial \varphi} = 0, \]  

whose general solution is given as the sum of a pair of generating curves \(\zeta^\mu\{\sigma\}, \xi^\mu\{\varphi\}\) parametrised by the characteristic coordinates \(\sigma\) and \(\varphi\) respectively, i.e.

\[ x^\mu\{\sigma, \varphi\} = \zeta^\mu\{\sigma\} + \xi^\mu\{\varphi\}. \]  

In order for such a solution to represent a physically admissible configuration for the chiral string model, the generating curves can not be chosen in an entirely arbitrary manner: in order to satisfy the condition (16) that \(\nu^\mu\) should be null, the tangent to the first generating curve \(\zeta^\mu\{\sigma\}\) must be null, while in order for \(\beta^\mu\) to be timelike the same must apply to the tangent to the second generating curve \(\xi^\mu\{\varphi\}\), i.e. using the notation

\[ \zeta^\mu, \quad \dot{\zeta}^\mu = \frac{d\zeta^\mu}{d\sigma}, \quad \xi^\mu, \quad \dot{\xi}^\mu = \frac{d\xi^\mu}{d\varphi}. \]  

we must construct the generating curves so as to satisfy the equality and inequality

\[ \zeta^\mu \dot{\zeta}_\nu = 0, \quad \dot{\xi}^\mu \xi^\nu < 0. \]  

The chiral string worldsheet solution obtained in this way can be seen to differ from its well known analogue for the Nambu Goto case [12] only by the condition that in the Nambu Goto case both generating curves are restricted to be null, whereas in the chiral case this restriction applies to only one of them. The transonic string model is also solved by an ansatz of the form [13], but in this case [13] neither of the generating curves is restricted to be null.

As in the well known Goto Nambu example, it will be convenient for many purposes to use a background time parametrisation for the generating curves, i.e. to take

\[ \sigma = \xi^\circ, \quad \varphi = \xi^\circ, \]  

in which case (47) will be expressible as the condition that the space vector \(\xi^a\) must lie on the “Kibble Turok” unit sphere, while \(\dot{\xi}^a\) must lie inside the unit sphere, i.e.

\[ \delta_{ab} \xi^a \xi^b = 1, \quad \delta_{ab} \dot{\xi}^a \dot{\xi}^b < 1. \]  

Whatever parametrisation convention is used, it can be seen from (11) that the null and timelike generators \(\nu^\mu\) and \(\beta^\mu\) will be respectively given by

\[ \nu^\mu = \psi (\dot{\zeta}^\nu \zeta^\rho)^{-1} \zeta^\mu, \quad \beta^\mu = -\frac{m^2}{\psi} \dot{\xi}^\mu, \]  

in which, in order to satisfy the normalisation condition (14), the auxiliary field amplitude \(\psi\) will be given by

\[ \psi^2 = -m^2 \dot{\xi}^\mu \xi^\mu. \]  

The corresponding explicit expression for the surface stress tensor of the chiral string can thus be seen from (14) to be given by

\[ T^{\mu\nu} = -2m^2 (\dot{\xi}^\rho \zeta^\mu)^{-1} \zeta^{(\mu} \dot{\xi}^{\rho)}. \]  

V. CONCLUSIONS.

Having just one internal degree of freedom – corresponding to longitudinal modes propagating at the speed of light in just one (forward but not backward) direction – the chiral string model can be described as being intermediate between the Nambu Goto string model which has no internal degrees of freedom at all, and the generic elastic models [5] (of which the Witten [1] model (4) was the original prototype) which have two internal degrees of freedom corresponding to longitudinal propagating in both left and right directions, with a “sound” speed that is equal to that of light in the case of the naive Witten model but that will be slower for more realistic models. There is a general theorem [14] to the effect that vortex equilibrium states will be classically stable whenever the relevant propagation speed of extrinsic “wiggles” is subsonic or transonic. The condition of subsonicity is always satisfied for the Witten model [1], which is precisely the reason why this model is physically unacceptable for the representation of bosonic string “superconductivity” in which, at least for weak currents, the wiggles propagation speed is predicted to be supersonic, with the implication that the corresponding vortex stability problem will be rather complicated [14][13]. It is to be similarly remarked that the condition of transonicity is automatically satisfied for the chiral string model [1], with the implication that this model too is potentially misleading, in that it can give a false impression of stability, when used [1] as an approximation for the
representation of string currents of the bosonic “superconducting” type. This caveat needs to be particularly emphasized in view of the very convenient integrability property established in the preceding section, which reinforces the temptation to use the chiral string model for applications beyond its range of legitimate validity.

Although potentially misleading when used as an approximation in the bosonic case, the use of the chiral model and the implication that the corresponding vorton states will always be classically stable – is physically justifiable as a valid description in cases for which the current under consideration arises from fermionic zero modes of chiral type such as are expected to arise in some grand unified theories or certain scenarios involving incomplete supersymmetry breaking.

APPENDIX: ELIMINATION OF FERMIONIC CURRENT ANOMALIES.

The purpose of this appendix is to show how the possibility of having conserved – anomaly free – chiral currents on the string depends on the requirement that they consist of axial fermions that (as was assumed in the preceding work) are not coupled to any long range – electromagnetic or other – gauge field. We start by using the Nielsen-Olesen string solution to check that in the absence of long range field coupling the two-dimensional theory in the string background will be anomaly free even though the background itself is nontrivial. We then go on to consider a prototypical four-dimensional theory admitting a Nielsen-Olesen string solution that contains also fermionic carriers coupled to both the strings and some extra long range – electromagnetic type – gauge fields.

The essential point is that unless they are entirely decoupled from the axial fermions constituting the string current, such fields would give rise to anomalies whose cancellation would require the introduction of other fermions, at the cost of spoiling the chiral nullity condition in the manner illustrated in figure 1.

The four-dimensional model we shall have in mind consists of a gauged Abelian Higgs model characterised by a Lagrangian containing a bosonic contribution that is coupled via a Yukawa term to a set of fermionic fields denoted by Ψ in a total of the form \[ L = L_{A,H}(H, B_\mu) + L_Y(H, \Psi) + \sum \bar{\Psi} \gamma^\mu D_\mu \Psi, \] in which the final gauge interaction term involves a sum running over all the fermions that couple to the strings (i.e. all those having non trivial zero mode solutions in a string background). For each individual fermion, the couplings with the gauge fields have the form \[ D_\mu \Psi = (\nabla_\mu - ieQ A_\mu - igR B_\mu) \Psi, \] in which \( Q \) and \( R \) are the (hyper)charges of the fermion in question, and in which \( e \) and \( g \) the coupling constants, where \( A_\mu \) is the long range electromagnetic type gauge vector, while \( B_\mu \) is the gauge vector associated with the symmetry whose spontaneous breaking is responsible for the existence of the strings. In order for the string solutions of \( L_{A,H} \) to admit zero modes, the fermionic couplings with \( B_\mu \) must be different for the left-handed and the right-handed ones, thereby potentially generating the so-called \( QQR \) and \( QRR \) anomalous terms.

The possibility of having purely chiral, let us say leftwards moving, currents in the string depends on the condition that the relevant fermionic fields should be of the corresponding, let us say left handed, chiral type in the four dimensional sense, since their Yukawa coupling to the string-forming Higgs field \( H \) will give rise to zero modes propagating in only one direction, with anti-particles propagating in the same direction, as illustrated in figure 2.

Since a chirally opposite, i.e. right handed, fermion field would similarly couple with the Higgs field conjugate...
$H^*$ (so that the vortex appears as an anti-vortex) the corresponding modes would propagate in the opposite direction, i.e. they would be rightwards movers. In the generic case for which both kinds of mode are present, their simultaneous occupation gives rise to currents that need no longer be null but can be timelike, as illustrated in figure 3.

Provisionally setting aside the possibility of coupling with an external field $A_\mu$, by setting $e = 0$, let us consider the possibility of anomalies in the string itself, considered as a two-dimensional worldsheet admitting zero modes of the relevant fermionic model in a given gauge background. For such a model, anomalies might arise at the one-loop level because of the coupling between the fermions and the string-forming gauge field $B_\mu$, whose presence implies a nonvanishing background $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. In such a theory one finds that in general the axial symmetry can be broken by generic background gauge curvature field $B_{\mu\nu}$. More particularly, the string worldsheet will contain an axial current with internal components $j^5_{i\mu} = \overline{\Psi} \gamma^5 \gamma^\mu \Psi$ that would acquire a corresponding divergence of the form $\epsilon^{ij}_{\mu\nu} (\overline{\Psi} \gamma^5 \gamma^\mu \gamma^\nu \Psi)$, where the pull back of the gauge curvature onto the string worldsheet is defined by $G_{ij} = G_{\mu\nu} x^\mu_i x^\nu_j$. In practice however, in the particular case of a Nielsen-Olesen background, there will be no such contribution because in a configuration of this type the associated gauge curvature will only have components in directions orthogonal to the world sheet, i.e. it will have vanishing tangentially projected components, $G_{ij} = 0$.

Let us now consider what would happen if the fermions were coupled to an independent long range gauge field $A_\mu$ associated with an electromagnetic type curvature field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which can give rise to chiral anomalies proportional to $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ in the four dimensional theory.

What one finds for fermions having a definite handedness [like right-handed neutrinos in the $SO(10) \rightarrow SU(5)$ model] is that they contribute to the $QQR$ and $QRR$ triangle anomalies with an amount that depends on their $Q$-charges in such a way that the theory is anomaly-free if and only if

$$\sum_l Q_l^2 = \sum_r Q_r^2,$$

where the sums are taken over the left and right handed particles respectively. It follows that a theory involving only, let us say, left handed particles – so that the right hand side of this equation automatically vanishes – can be anomaly free only if the left hand side of this equation also vanishes, i.e. only if all the relevant coupling charges are zero. The absence of charge coupling is therefore essential for a theory of the kind providing purely chiral fermionic string currents, as illustrated in figure 2.

ACKNOWLEDGEMENTS.

The authors wish to thank Anne Davis, Stephen Davis, and Warren Perkins for helpful conversations.

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