RAPID CALCULATION OF EQUATORIAL ROTATION CURVES

C.S. Kochanek
Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138, ckochanek@cfa.harvard.edu

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ABSTRACT

We derive a simple, fast one-dimensional integral for the equatorial rotation curve of a thin disk with surface density $\Sigma(R)$ modeled as a spheroid with axis ratio $q$. The result is simpler than standard expressions even in the limit of an infinitely thin disk ($q \to 0$).

Subject headings: galaxies: kinematics and dynamics

1. INTRODUCTION

Existing expressions for the calculation of the equatorial rotation curves of thin disks have undesirable features. Derivations using elliptic integrals are complicated and contain a singularity in the plane which is usually removed by evaluating the rotation curve slightly above the plane (e.g. Binney & Tremaine 1984). Derivations in terms of Hankel (Toomre 1962) or Stieltjes (Evans & de Zeeuw 1992) transforms are poorly suited for numerical calculation. Standard expressions for finite thickness disks require intrinsically two-dimensional numerical integrals (e.g. Casertano 1983, Cuddeford 1993) or multidimensional tables (e.g. Dehnen & Binney 1998).

In conducting some experiments on the compatibility of central dark matter cusps with galaxy rotation curves (see, e.g., Moore 2001), we needed a faster means of computing equatorial rotation curves including the effects of a finite disk thickness. By faster we mean a simple, non-singular, non-oscillatory, one-dimensional numerical integral for the rotation curve of a disk with an arbitrary surface density profile $\Sigma(R)$. The solution was found by considering the equatorial rotation curves of flattened spheroids. Modeling disks as flat spheroids is little used, although the mathematical approach was developed by Brandt (1960), Brandt & Belton (1962), Mestel (1963), Lynden-Bell & Pineault (1978), Lynden-Bell (1980) and Cuddeford (1993). These studies show that the equatorial rotation curve can be expressed as an integral over the Abel transform of a spheroid, since it is the rotation curve of a spheroid ($\rho(R^2 + z^2/q^2)$) defined by the surface density $\Sigma(R)$ of a disk.

2. THE EQUATORIAL ROTATION CURVE OF A SPHEROIDAL DISK

We develop our result following the procedures outlined in Binney & Tremaine (1984) and Cuddeford (1993), although we focus only on the equatorial rotation curve rather than the global potential. For a density distribution $\rho(m^2)$ with $m^2 = R^2 + z^2/q^2$, the equatorial rotation curve is simply

$$v_c^2(R) = 4\pi Gq \int_0^{R} \frac{\rho(m^2)dm}{(R^2 -(1-q^2)m^2)^{1/2}}$$

(1)

where the density distribution is the Abel transform of the surface density

$$\rho(m^2) = -\frac{1}{\pi q} \int_{m^2}^{\infty} \frac{\Sigma(R) dR}{(R^2 -(1-q^2)m^2)^{1/2}}$$

(2)

for a sufficiently regular surface density. For example, the Abel transform of an exponential disk, $\Sigma = \Sigma_0 \exp(-R/R_d)$, is

$$\rho(m^2) = \frac{\Sigma_0}{\pi q R_d} K_0(m/R_d)$$

(3)

where $K_0(x)$ is a modified Bessel function, and the rotation curve is

$$v_c^2 = 4G \Sigma_0 R_d \int_0^{R/R_d} \frac{K_0(u)u^2du}{(R^2/R_d^2 -(1-q^2)u^2)^{1/2}}$$

(4)

Cuddeford (1993). For $q = 0$ we obtain the standard analytic result by Freeman (1970).

Our extension to these results is that we can reduce the calculation to a one-dimensional integral by reversing the order of integration to find that

$$v_c^2(R) = \frac{4GR}{1-q^2} \int_0^{\infty} du \frac{d\Sigma}{du} [E(\phi, \alpha) - F(\phi, \alpha)]$$

(5)

where $\alpha^2 = (1-q^2)u^2/R^2$ and $\sin \phi = \min(1, R/u)$ \footnote{The special functions $F(\phi, \alpha) = \int_0^{\phi} d\theta (1-\alpha^2 \sin^2 \theta)^{-1/2}$ and $E(\phi, \alpha) = \int_0^{\phi} d\theta (1-\alpha^2 \sin^2 \theta)^{1/2}$ are elliptic integrals of the first and second kind respectively.} The integrand has an integrable, logarithmic singularity at $\alpha = 1$ ($u = R$) for an infinitely thin disk ($q \to 0$). The result simplifies further to

$$v_c^2 = \frac{4G}{3R} \int_0^{\infty} du \frac{d\Sigma}{du} u^2 \sin^3 \phi R_D (\cos^2 \phi, 1 - \alpha^2 \sin^2 \phi, 1)$$

(6)
if we use the Carlson forms for elliptic integrals\(^2\) rather than the Legendre forms. The function \(R_D(x, y, z)\) has a logarithmic singularity when \(x = y = 0\), again corresponding to \(\alpha = 1\) \((u = R)\) with \(q = 0\). Integrating by parts, to convert \(d\Sigma/du\) into \(\Sigma\), and taking the limit \(q \to 0\) leads to the standard expression for the circular velocity in terms of elliptic integrals evaluated in the disk plane (e.g. Binney & Tremaine [1984]). Integrands based on \(d\Sigma/dR\), like eqns. (5) and (6), have a weaker singularity than those based on \(\Sigma(R)\), and the singularity is easily removed by giving the disk a non-zero axis ratio \(q\).

3. SUMMARY

Our expressions for the equatorial rotation curves of spheroidal disks of surface density \(\Sigma(R)\) and axis ratio \(q\) are simple (even compared to standard expressions for infinitely thin disks) and can be computed very rapidly due to the excellent numerical properties of the Carlson elliptic integrals. The one disadvantage of the result is that real disks (e.g. de Grijs [1998] and references therein) are better modeled as separable functions of \(R\) and \(z\) (e.g. \(\Sigma(R) \exp(-|z|/H)\)) than as spheroids \((\rho(R^2 + z^2/q^2))\) with axis ratio \(q\). This distinction will be important in some circumstances (e.g. detailed studies of vertical equilibrium in stellar disks), but should matter little for the global rotation curve if the axis ratio is adjusted to roughly match the central disk scale height. In any case, the simplicity of our final expression makes it useful even for infinitely thin disks.

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\(^2\)Carlson’s elliptic integral of the second kind is defined by

\[
R_D(x, y, z) = \frac{1}{4} \int_0^\infty \frac{t^2}{(\lambda + x)^{1/2}(\lambda + y)^{1/2}(\lambda + z)^{1/2}} dt.
\]