The solitons redistribution in Bose-Einstein condensate in
quasiperiodic optical lattice

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Abstract

We numerically study the dynamical excitations in Bose-Einstein condensate (BEC) placed in
periodic and quasi-periodic 2D optical lattice (OL). In case of the repulsive mean-field interaction
the BEC quantum tunnelling leads to a progressive soliton’s splitting and generating of secondary
solitons, which migrate to closest trapping potential minima. A nontrivial soliton dynamics appears
when a series of π-pulses (phase kicks) are applied to the optical lattice. Such sudden perturbation
produces a dynamic redistribution of the secondary solitons, leading to a formation of an artificial
solitonic superlattice. Different geometries of OL are analyzed.
I. INTRODUCTION

Different types of solitons in Bose-Einstein condensates (BECs) have been intensively studied both theoretically and experimentally [1]-[22]. Recently, solitons supported by weak attractive interactions between atoms were created in the condensate of $^7\text{Li}$ trapped in strongly elongated traps [23], [24], [25]. Ref. [26] details a set of experiments showing evidence of the existence of BEC of polaritons. Above a critical density, the authors observed massive occupation of the ground state developing from a polariton gas at thermal equilibrium.

Also, a possibility to design 1D and 2D quantum systems by ”freezing out” one or two degrees of freedom by adding a 1D or 2D optical lattice to the magnetic trap has been demonstrated experimentally [27], [28]. However the soliton’s dynamics in two dimensions is much more involved and cannot be reduced to the one-dimensional case.

It is well known that solitonic solutions appearing in integrable models, such as, for instance, 1D spatial solitons in the cubic nonlinear Schrödinger equation (NSE) [29], reveal particle-like behavior. In particular, they remain unperturbed after collisions, completely preserving their identities and form. However, there are no loosely bound stable solitons in 2D (see e.g. [30]).

The properties of solitons are well studied for a long optical lattice case (gap solitons). In the repulsive condensate gap solitons (GSs) can be created as a result of the interplay of the self-defocusing nonlinearity and periodic potential induced by an optical lattice (OL) - an interference pattern created by counterpropagating laser beams illuminating the condensate, [31]-[35].

The dynamics of BEC in 2D optical lattice are currently a subject of intensive theoretical and experimental studies [36]-[41]. An interesting direction consists in using trapping quasiperiodic potentials for generation of several nonlinear excitations. The authors [42] have constructed some families of solitons for one- and two-dimensional Gross-Pitaevskii equations with a repulsive nonlinearity and a potential of quasicrystallic type (in the 2D case, the potential corresponds to a fivefold optical lattice). Some stable 2D and 3D solitons in the self-attractive Gross-Pitaevskii equation were found in [36] by using the variational approximation and direct simulations in real and imaginary time. Also, investigation of stability of localized vortices in an effectively 2D (“pancake-shaped”) trapped BEC with
negative scattering length was performed in [39].

Usually, in experiments on solitons generation in BEC it is possible to obtain a unique soliton which is stabilized in gap zones of a periodic OL. However from the point of view of quantum-information technologies the possibility of generation of nonlinear excitations with complex structure, similarly to multisoliton state is more important. Several nonlinear regimes of interaction of such excitations have been studied in the literature. Nevertheless, the problem of experimental generation of multisolitonic states in BEC is still far from being completely solved.

In this paper we explore an intermediate case when the size of BEC is about ten (or less) spatial periods of the optical lattice. In this situation the approximation of a perfect potential periodicity, $V(x + L) = V(x)$ (necessary for the soliton stabilization) strictly speaking, is not valid, and a more realistic model should be constructed. In such configurations the dynamics of a single soliton are defined by the structure of the trapping potential in the vicinity of the soliton, rather by the far OL order. One can say that a soliton is subjected to a local spatially non-uniform nonlinear force field, which is defined by OL configuration, and by the excitation’s amplitude. Such a field drastically changes the soliton dynamics: the soliton can tunnel into the closest minima (in the case of repulsive interaction), or turn to a self-compression dynamics (in the case of attractive interaction).

We focus on the repulsive interaction case. Then, if the initial soliton position does not coincide with the minima of OL, the shape of such excitation will be split into fragments tending to flow into the closest minima of trapping potential, and the soliton dynamics essentially depends on a local spatial structure of OL.

This observation suggests the exploration the soliton evolution in BEC when the spatial OL configuration is rapidly changed (which can be easily achieved experimentally). We have found that remarkable dynamics occur in situation when a soliton, steady in an initial OL configuration, becomes nonequilibrium in a new OL. This establishes the possibility to drastically increase the speed of forming the secondary solitons with such OL switching. Below we demonstrate that as a result of such a dynamics an artificial solitonic superlattice in BEC is formed.

In this paper we analyze dynamics of a soliton placed in periodic or quasi-periodic optical lattices in 2D Bose-Einstein condensate (BEC) in the intermediate case (when the spatial scale of the condensate area is about 10 or less OL wavelengths), when the OL is suddenly
switched after applying periodical $\pi$ phase shifts (phase kick). It will be shown that if the period of switching is sufficiently large (with respect to the tunneling time) secondary solitons are formed in OL minima. At later (successive) $\pi$-switching such secondary solitons pass through series of metastable states and migrate from the center to the periphery. In a suitable moment this process can be interrupted, which allows a free interaction of generated nonlinear excitations in agreement with the fundamental $2D$ nonlinear Schrödinger equation.

The paper is organized as follows. In Section II we discuss basic equations for BEC atoms placed into OL. In Section III we present numerical studies of the soliton dynamics in the repulsive mean-field interaction for $2D$ case with periodical $\pi$-phase OL shifts. Various geometries of OL, including periodic and quasiperiodic, are analyzed. We also discuss some peculiarities of the evolution in the attractive interaction case. In the last Section we summarize our results.

II. BASIC EQUATIONS

In order to trap a Bose-Einstein condensate in a (quasi)-periodic potential, it is sufficient to exploit the interference pattern created by two or more overlapping laser beams and the light force exerted on the condensate atoms. We mainly focus on the physical situation when the number of atoms is sufficiently large, so that atomic number fluctuations are negligible and the mean-field approximation can be applied. In this approach an anisotropic BEC cloud, loaded into a two-dimensional optical lattice potential $V(\mathbf{r})$, is described by the Gross-Pitaevskii (GP) equation ($T = 0$)

$$i \frac{\partial \Phi}{\partial \tau} = [-\nabla_\perp^2 + V(\mathbf{r}) + G |\Phi|^2] \Phi,$$  

(1)

where $\mathbf{r} = \{x, y\}$, $\nabla_\perp = \{\partial_x, \partial_y\}$, $\Phi$ is the condensate wave function, $G = +1, -1$ for repulsive and attractive interaction correspondingly. This equation is obtained by assuming a tight confinement in the direction perpendicular to the lattice ("pancake" trapping geometry) and the standard dimensionality reduction procedure (see e.g. [8], [17]). According to Eq. (1), a particle with the (condensate) wave function $\Phi(\mathbf{r}, t)$ evolves in the external potential $V(\mathbf{r})$ induced by OL plus the mean-field potential created by the remaining particles.

It is made dimensionless by using the characteristic length $h_0 = p/\sqrt{8\pi a_S}$, energy $E_0 = g/p^2 = \hbar/t_0$, and time $t_0 = \hbar p^2 / g = mp^2/4\pi \hbar a_S$, here and below $m$ and $N_0$ are the mass
and number of the trapped atoms respectively, $g = 4\pi\hbar^2 a_s/m$ is the interaction strength, and $a_s$ is the $s$-wave scattering length. The wave function is made dimensionless as $\Phi \to p\Phi$, where the factor $p$ is specified by the normalization conditions $N_0 = \int |p\Phi|^2 dV_0$ (i.e., $n_0(r,\tau) = |p\Phi(r,\tau)|^2$), where $V_0$ is the volume of the condensate.

We consider a quasiperiodic time-dependent trapping potential $V$ of the following form,

$$V = V(x, y, \tau) = V_c + \epsilon \sum_{n=1}^{N} \cos(k^{(n)} r + \pi \theta(\tau)) = V_c + \epsilon (1 - 2\theta(\tau)) \sum_{n=1}^{N} \cos(k^{(n)} r),$$  \hspace{1cm} \text{(2)}$$

where $k^{(n)} = \{\cos(2\pi(n - 1)/N), \sin(2\pi(n - 1)/N)\}$, $V_c = \text{const}$, and the time-dependence appears through the pulse function $\theta(\tau) = 1$ for $(2n + 1)T_t < \tau \leq (2n + 2)T_t$ and $\theta(\tau) = 0$ otherwise, $T_t$ is a time period, $n = 0, 1, 2, \ldots$. For stationary OL ($T_t \to \infty$) and $N = 5$ such a potential case was studied in [42]. Some typical time independent potentials, $V(x, y, 0)$, are shown in Figs.1(b),(c),(d) for the $N = 4, 5, 7$ cases correspondingly. Such quasiperiodic optical lattices can be created in the 2D case, as a combination of $N = 4, 5, 7$ cases. The band-gap spectrum of 2D photonic crystals of the PT type has been studied in [43], [44], [45]. Recently the interesting properties of such lattices (e.g., Penrose lattices), have been discovered for quasiperiodic pinning arrays [46].

From now on we will concentrate on a repulsive interaction case, $G = 1$. First of all, we note that the generalized momentum density is not conserved because the trapping potential $V(x, y)$ induces a spatial inhomogeneity in a system [49]. Let us estimate the force acting on the condensate particles. It is well known that the GP equation (1) can be generated from the Lagrangian density

$$L = \frac{i}{2} [\Phi^* \Phi_r - \Phi \Phi^*_r] - |\nabla \perp \Phi|^2 - V(r) |\Phi|^2 - \frac{G}{2} |\Phi|^4,$$  \hspace{1cm} \text{(3)}$$

and the corresponding full condensate energy has the form

$$E = \int dr [|\nabla \perp \Phi|^2 + V(r) |\Phi|^2 + \frac{G}{2} |\Phi|^4] = \text{const.}$$  \hspace{1cm} \text{(4)}$$

Observe that the constant part of the trapping potential, $V_c$, can be removed from the Lagrangian [3] by renormalizing the wave function as follows,
\[ \Phi \to \Phi \exp(-iV_c \tau). \] (5)

Thus, without loss of generality we suppose that the potential \( V(x,y) \) in (1) and (3) can be renormalized such that \( V_{\text{min}}(x,y) = 0 \) and \( V(x,y) \geq 0 \). Note that in the case of such renormalization the energy \( (4) \) is shifted by \( V_c N_0 \), which allows us to associate \( V_c \) with the chemical potential \( \mu = dE/dN_0 \).

The Lagrangian (3) generates the following force field acting on the soliton,

\[ F(x,y) = \frac{\delta L}{\delta \Phi^*} = \frac{\partial L}{\partial \Phi^*} - \nabla \frac{\partial L}{\partial \nabla \Phi^*} = F_V + F_S + F_N, \] (6)

where

\[ F_V = -V(x,y,\tau)\Phi, \quad F_N = -G|\Phi|^2 \Phi, \quad F_S = \nabla^2 \Phi. \] (7)
One can observe that $F_V = 0$ in the vicinity of $V_{\min} = 0$ and therefore in the potential minima the soliton evolution is governed by the standard nonlinear Schrödinger model dynamics, i.e. only subjected to the forces $F_S$ and $F_N$. This, in particular, leads to a very low tunneling rate even for an intermediate depth of OL. Thus, as we have numerically corroborated, a sufficiently narrow soliton placed in one of the potential minima is practically a steady-state.

It is clear that a deviation from $V_{\min} = 0$ generates a nonzero $F_V$, which essentially modifies the soliton behavior. The effect of $F_V$ becomes predominant in the vicinity of the maximum of the potential. Thus, after applying a phase kick (when the minima of the potential are suddenly converted into maxima), the initial ”steady-state” soliton starts to rapidly evolve in the direction of the closest minima of the new potential. Besides, in the vicinity of the soliton maximum (where $\Phi$ is real and positive), the forces $F_N$ and $F_S$ are all negative. These forces stretch and deform the initial packet leading to the soliton splitting. It is worth noting that due to the energy conservation (4), the decay of some part of the BEC soliton amplitude $|\Phi|$ must be compensated by sharp gradients of some other parts.

III. NUMERICAL RESULTS

A. Repulsive interaction

Previous arguments have only heuristic meaning and do not provide a quantitative description of the soliton dynamics in the 2D case, especially for time-dependent OL potential. The deeper insight into the BEC dynamics in a general OL depth case can be reached only by numerical methods. We have investigated two-dimensional dynamics for various initial conditions and different parameters of the trapping potential. The results are shown in Figs.2-6. For our numerical experiments we used the combination of known splitting methods [50] generalized to the nonlinear case. We have numerically solved Eq.(1) in domain $200 \times 200$ with zero boundary conditions for times $\tau < 200$. Greater temporal intervals normally require a spatial grid with greater size. Our calculations were done in the complex $\Phi$ plane to observe the evolution of both amplitude and phase of the soliton. Both the norm of the wave function and energy (4) were conserved with good accuracy (less than 1%). We have studied the evolution of the initial Gaussian packet, Fig.1(a),
\[ \Phi(x, y, 0) = \Phi_0 \exp\left[-q_x (x - x_0)^2 - q_y (y - y_0)^2\right] \] (8)

for different factors \( q_{x,y}, \) and the amplitude \( \Phi_0, \) which is a function of the total number of trapped atoms \( N_0. \) We have numerically analyzed the packet (8) evolution initially centered at different points of the plane \((x_0, y_0),\) corresponding either to minima \( (V_{min}) \) and maxima \( (V_{max}) \) of the potential.

Let us start with a periodic trapping potential Eq.(2), corresponding to \( N = 4 \) (see Fig.1 (b)) and for \( G = 1. \) We have used the parameters: \( q_x = q_y = 0.004, \epsilon = 0.7, \Phi_0 = 1, \) and the initial packet was placed at the central minimum, Fig.1(a). In the case of a stationary potential the initial packet evolves in a well known way by splitting into fragments and slowly tunneling into the closest minima of the optical lattice. Nontrivial dynamics arise after sudden (short with respect to the scale \( t_0 \)) shift by \( \pi \) of the optical lattice phase, when OL minima are switched with their maxima. Being placed in these new maxima the soliton flow out into new closest minima and after a certain transition time the secondary solitons are generated in those minima. In our simulations the period of the phase shift, \( T_t = 50, \) is chosen larger than the above mentioned transition time (proportional to the characteristic time of quantum tunneling), so that the details of the initial state are washed out and the secondary "solitons" acquire well pronounced shape.

These dynamics are shown in Fig.2. In Fig.2 (a) the soliton state at time \( \tau = 40 \) before the first phase switching is shown. One can observe the beginning of slow condensate tunneling into closest potential minima. In Fig.2 (b) the condensate state at \( \tau = 80 \) (after first phase switching) is presented. One can appreciate an essential reduction and widening of pulse amplitude. We observe that such dynamics become significantly faster with respect to the stationary case. At \( \tau = 120 \) (after second phase switching), Fig.2 (c), the state acquires a more complex shape with quite pronounced secondary solitonic peaks at the corners. The final state at \( \tau = 160 \) (after third phase switching) is shown in Fig.2 (d). We observe already a well formed artificial solitonic superlattice with the peaks located in the trapping potential minima. It is clear that the details of this new soliton structure depend mainly on the local vicinity of OL.

Due to a recent interest to quasiperiodic structures, we also analyze the BEC evolution in an optical structure with a lower symmetry trapping potential. Such dynamic is shown
FIG. 2: Evolution of BEC soliton for period $\pi$-phase shift $T = 50$ and different times: (a) $\tau = 40$; (b) $\tau = 80$; (c) $\tau = 120$; (d) $\tau = 160$. See details in text.

in Fig. 3 for potential (2) with $N = 5$ and in Fig. 4 for potential (2) with $N = 7$ (see Fig. 1 (c) and (d) accordingly). The case $N = 5$ is essentially a two-dimensional Penrose-tiled lattice, where the tiles are two kinds of rhombus: a thin tile (with vertex angles of $36^\circ$ and $144^\circ$) and a fat tile ($72^\circ$ and $108^\circ$) [47]. We observe from Fig. 3 and Fig. 4 that for quasiperiodic potentials the system evolves in a less symmetric way. The condensate is distributed on closed smooth cells without strongly pronounced solitonic peaks. Such cells appear more smooth in the 5-fold case, see Fig. 3. This effect is similar to 2D Penrose lattices in superconductors, where the pinning of vortices is related to matching conditions between the vortex lattice and the quasiperiodic lattice of pinning centers [16]. We notice again that the final structure of a condensate still has the form of localized states distributed on the minima of the potential.

It is representative to illustrate the state evolution as the time behavior of the amplitude $|\Phi|$ of the maximal peak, as shown in Fig. 5 (a). It can be observed that the amplitude after
some period of growing (which is accompanied by the stretching of the initial distribution due to non-linear effects) starts to diminish and reaches its minimum (quasistationary value) at \( \tau = 130 \), although the splitting is still continuing.

To evaluate quantitatively the behavior of BEC shape we have studied the behavior of the soliton ”center of mass” \( R(\tau) \) defined as

\[
R(\tau) = \int (r - r_0) |\Phi(r, \tau)|^2 \, dr,
\]

where \( r_0 \) indicates the initial soliton position. The dynamics of \( R(\tau) = |R(\tau)| \) are shown in Fig.5(b). One can observe that \( R(\tau) \) grows in a very different way for periodic and quasiperiodic potentials, which is a reflection of the different local structure of OL. On the other hand, while the minima are progressively filling up with the condensate the quantity \( R(\tau) \) tends to saturation in both cases.
FIG. 4: The same as in Fig.2 except $N = 7$ in trapping potential $V(x, y, \tau)$ Eq.(2).

B. Attractive interaction.

Although the main goal of this paper is studying the repulsive interaction, for completeness, we also discuss the dynamics of the initial Gaussian pulse (8) in the attractive interaction case ($G = -1$). Let us recall that for the 2D nonlinear Schrödinger model (without potential) the dynamics of the average radius of an excitation are (see e.g. Chap. VIII in Ref.30) $\langle R(\tau) \rangle^2 = \int \rho^2 |\Phi|^2 d^2r \approx (E/N_0)\tau^2 + R_0^2$ (see also[51] and references therein). So, if the energy (4) is negative, $E < 0$, then such an excitation undergoes a self-compressing. For the initial Gaussian (8) this yields the inequality $\Phi_0^2 > q_{x,y}$, which is satisfied in our case. However a similar analytical condition for 2D BEC with OL is not established yet in the literature. Therefore we have studied the soliton evolution for the 2D attractive interaction numerically for the initial state Fig.1 (a) and different OL geometries (Fig.1b,c,d)), similar to the repulsive case.
FIG. 5: Time dynamics in case of repulsive interaction ($G = 1$). (a) $|\Phi_{\text{max}}(\tau)| / |\Phi_{\text{max}}(0)|$, and (b) mass center $R(\tau)$, see (9). Solid line, dash line and dash-dots line correspond to $N = 4$, $N = 5$ and $N = 7$ cases accordingly for trapping potential $V(x, y, \tau)$ in Eq.(2). Here $\tau \equiv n$ ($n$ is the time step index).

As it can be appreciated from Fig.6 the soliton dynamics completely change in the attractive case $G = -1$. It is well known that for the 2D attractive interaction the collapse dynamics takes place and there are no loosely bound stable states. We have found that for our initial state the collapse dynamics take place and there are no loosely bound stable states in two dimensions. Nevertheless, it turns out that such instabilities can be exploited in order to prepare BEC solitons[17]. We have investigated such a dynamics for various geometries of the trapping potential. In contrast to the repulsive interaction case we found that the self-compressing behavior is practically insensible to changing OL symmetry. We observe from Fig.6 that just a single pulse is generated from the initial packet, which evolves without any splitting or visible deformation. Such self-compression leads to a singularity.
and therefore, the local structure of OL in the vicinity of the soliton does not notably affect its evolution. Obviously, the tunneling processes quickly become practically negligible in this case. We have also found that small shifts of the soliton center from the minimum of the potential lead to the same dynamics, and in particular, do not change the single-peak structure of the final state. Generating of two and more peaks as well as stabilization of self-compressing can be reached via a specific initial state and/or OL geometry. However in general this question remains open and will be studied elsewhere.

IV. CONCLUSION

We have studied the dynamics of soliton in Bose-Einstein condensate (BEC) placed in periodic or quasi-periodic 2D optical lattices (OL). In the case of the repulsive mean-field
interaction the BEC tunneling leads to progressive deformation and splitting of solitons. It results in the generation of secondary solitons migrating to closest trapping potential minima. We found that nontrivial dynamics arise when a series of \( \pi \)-phase shifts (phase kicks) are applied to the OL, so that the minima and maxima of OL are periodically exchanged. This results in a progressive migration of the solitons from the initial position to the periphery with the forming of the artificial solitonic superlattice. We have found that the geometry of forming solitonic superlattice is rather sensitive to the geometry of the trapping potential.

Such an effect suggests an effective and simple method for splitting of the initial BEC soliton and generation of multisolitonic nonlinear excitations in the experiment, and thus, can be, in principle, applied in quantum information technology, since it allows a creation and manipulation of solitons as bits (qubits) in BEC quantum structures.

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