RINGS OF BOUNDED CONTINUOUS FUNCTIONS

YOTAM SVORAY AND AMNON YEKUTIELI

ABSTRACT. We examine several classical concepts from topology and functional analysis, using methods of commutative algebra. We show that these various concepts are all controlled by BC \( \mathbb{R} \)-rings and their maximal spectra.

A BC \( \mathbb{R} \)-ring is a ring \( A \) that is isomorphic to the ring \( F_{bc}(X, \mathbb{R}) \) of bounded continuous \( \mathbb{R} \)-valued functions on some compact topological space \( X \). These rings are not topologized; a morphism of BC \( \mathbb{R} \)-rings is just an \( \mathbb{R} \)-ring homomorphism.

We prove that the category of BC \( \mathbb{R} \)-rings is dual to the category of compact topological spaces. Next we prove that for every topological space \( X \) the ring \( F_{bc}(X, \mathbb{R}) \) is a BC \( \mathbb{R} \)-ring. These theorems combined yield an algebraic construction of the Stone-Čech Compactification of an arbitrary topological space.

There is a similar notion of BC \( \mathbb{C} \)-ring. Every BC \( \mathbb{C} \)-ring \( A \) has a canonical involution. The canonical hermitian subring of \( A \) is a BC \( \mathbb{R} \)-ring, and this is an equivalence of categories from BC \( \mathbb{C} \)-rings to BC \( \mathbb{R} \)-rings.

Let \( K \) be either \( \mathbb{R} \) or \( \mathbb{C} \). We prove that a BC \( K \)-ring \( A \) has a canonical norm on it, making it into a Banach \( K \)-ring. We then prove that the forgetful functor is an equivalence from Banach* \( K \)-rings (better known as commutative unital \( C^* \) \( K \)-algebras) to BC \( K \)-rings. The quasi-inverse of the forgetful functor endows a BC \( K \)-ring with its canonical norm, and the canonical involution when \( K = \mathbb{C} \).

Stone topological spaces, also known as profinite topological spaces, are traditionally related to boolean rings – this is Stone Duality. We give a BC ring characterization of Stone spaces. From that we obtain a very easy proof of the fact that the Stone-Čech Compactification of a discrete space is a Stone space.

Most of the results in this paper are not new. However, most of our proofs seem to be new – and our methods could potentially lead to genuine progress related to these classical topics.

CONTENTS

0. Introduction 2
1. Preliminaries 6
2. Abstract Function Rings 11
3. Rings of Bounded Continuous Real Valued Functions 14
4. Duality for Compact Topological Spaces 16
5. The Stone-Čech Compactification 20
6. Real Banach Rings 25
7. Stone Spaces 27
8. Rings of Bounded Continuous Complex Valued Functions 31
9. Involutive Complex Rings 34
10. Involutive Complex Banach Rings 37
References 39

Date: 19 February 2022.
0. Introduction

In this paper we look at the classical – namely middle 20th century – concepts of compact topological spaces, Stone-Čech Compactification and commutative $\mathbb{C}^*$-algebras from an algebraic geometer’s point of view, using the language of categories and commutative algebra. Our goal in this paper is to clarify some of the ideas involved, and to find an alternative uniform treatment. This is because, in our opinion, the traditional presentation in textbooks on topology and functional analysis tends to obscure some of the interesting aspects of these classical concepts.

Our treatment is based on the unifying concept of BC ring, a concept that we introduce in this paper (see Definition 0.2 below). The algebraic properties of BC rings turn out to control compact topological spaces, Stone-Čech compactification, and commutative $\mathbb{C}^*$ algebras over $\mathbb{R}$ and $\mathbb{C}$. See diagram (0.16) for the web of category equivalences. Of the five categories appearing in this diagram, the category $\text{Rng}_{/\text{bc}}\mathbb{R}$ of BC $\mathbb{R}$-rings, and $\mathbb{R}$-ring homomorphisms between them (these are abstract rings, no continuity is involved), seems to be the easiest to work with.

Let us say a few words about originality. Most of the results in this paper are not new at all. Some of them can even be found in textbooks, and a few are merely rephrasings of classical theorems. Yet most of our proofs are probably new, and could potentially lead to genuine progress.

In order to make our paper accessible to mathematicians working in topology or functional analysis, who might lack sufficient background knowledge of categories and commutative algebra, we have included a recollection of the necessary material in Section 1 with textbook references. Very brief explanations of the terminology are inserted here, in the Introduction, to facilitate reading the main theorems.

We call a topological space $X$ compact if it is Hausdorff and quasi-compact (namely it has the finite open subcovering property). The category of topological spaces, with continuous maps, is denoted by $\text{Top}$, and its full subcategory on the compact spaces is $\text{Top}_{\text{cp}}$.

All rings in this paper are commutative (and unital). The category of rings is denoted by $\text{Rng}$. Throughout the Introduction we let $\mathbb{K}$ denote either of the fields $\mathbb{R}$ or $\mathbb{C}$. By a $\mathbb{K}$-ring we mean a ring $A$ equipped with a ring homomorphism $\mathbb{K} \to A$. (Traditionally $A$ would be called a commutative associative unital $\mathbb{K}$-algebra.) The category of $\mathbb{K}$-rings, with $\mathbb{K}$-ring homomorphisms, is $\text{Rng}/\mathbb{K}$. It is important to emphasize that the objects of the category $\text{Rng}/\mathbb{K}$ do not carry topologies, and hence there is no continuity condition on the morphisms in $\text{Rng}/\mathbb{K}$.

For a topological space $X$ we write $F_{\text{bc}}(X, \mathbb{K})$ for the ring of bounded continuous functions $a : X \to \mathbb{K}$, where the field $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$) is endowed with its standard norm. As $X$ changes we obtain a contravariant functor

\begin{equation}
F_{\text{bc}}(-, \mathbb{K}) : \text{Top} \to \text{Rng}/\mathbb{K}.
\end{equation}

The next definition presents the class of rings that is the focus of our paper.

**Definition 0.2.** Let $\mathbb{K}$ be either of the fields $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-ring $A$ is called a BC $\mathbb{K}$-ring if there is an isomorphism of $\mathbb{K}$-rings $A \cong F_{\text{bc}}(X, \mathbb{K})$ for some compact topological space $X$. The full subcategory of $\text{Rng}/\mathbb{K}$ on the BC rings is denoted by $\text{Rng}_{/\text{bc}}\mathbb{K}$. 
Given a ring $A$, its prime spectrum $\text{Spec}(A)$ is considered here only as a topological space, with the Zariski topology; we ignore the structure sheaf of the affine scheme $\text{Spec}(A)$. The set of maximal ideals of $A$ is $\text{MSpec}(A)$, and it is given the induced subspace topology from $\text{Spec}(A)$.

The prime spectrum is a contravariant functor

\[(0.3) \quad \text{Spec} : \text{Rng}/K \to \text{Top},\]

sending a $K$-ring homomorphism $\phi : A \to B$ to the map

\[(0.4) \quad \text{Spec}(\phi) : \text{Spec}(B) \to \text{Spec}(A), \quad q \mapsto \phi^{-1}(q)\]

in $\text{Top}$. However, $\text{MSpec}$ is not a contravariant functor $\text{Rng}/K \to \text{Top}$; see Example 1.13.

In Sections 3 and 8 we show that for a BC $K$-ring $A$ and a maximal ideal $m \subseteq A$, the residue field is $A/m = K$. This implies that given a homomorphism $\phi : A \to B$ of BC $K$-rings, and a maximal ideal $n \subseteq B$, the ideal $\phi^{-1}(n) \subseteq A$ is maximal. Therefore we get a map

\[(0.5) \quad \text{MSpec}(\phi) : \text{MSpec}(B) \to \text{MSpec}(A), \quad n \mapsto \phi^{-1}(n).\]

We see that there is a contravariant functor

\[(0.6) \quad \text{MSpec} : \text{Rng}_{bc} K \to \text{Top}.\]

Here is the first main result of the paper.

**Theorem 0.7 (Duality).** Let $K$ be either $\mathbb{R}$ or $\mathbb{C}$. The contravariant functor

\[F_{bc}(-, K) : \text{Top}_{cp} \to \text{Rng}_{bc} K\]

is a duality (namely a contravariant equivalence of categories), with quasi-inverse $\text{MSpec}$.

This result is repeated as Theorems 4.16 and 8.11 (for the real and complex cases, respectively), and proved there. A non-categorical phrasing of this result can be found in [Wa, pages 15-16], where it is attributed to Stone. The fact that the functor $F_{bc}(-, \mathbb{R})$ is fully faithful is sometimes called the Gelfand-Kolgomorov Theorem.

We see that BC $K$-rings play a role similar to the role that boolean rings have in the context of Stone spaces; the corresponding duality there is called Stone duality, see [Jo, Corollary II.4.4]. Our Theorem 0.17 below makes the precise connection between BC rings and Stone spaces.

Our second main result says that BC rings occur in much greater generality than Definition 0.2 indicates.

**Theorem 0.8.** Let $K$ be either $\mathbb{R}$ or $\mathbb{C}$. For an arbitrary topological space $X$, the $K$-ring $F_{bc}(X, K)$ is a BC ring.

This theorem is repeated as Theorems 5.9 and 8.5 in the body of the paper, for the case $K = \mathbb{R}$ and $K = \mathbb{C}$ respectively.

A Stone-Čech Compactification (SCC) of a topological space $X$ is a pair $(\bar{X}, c_X)$, consisting of a compact topological space $\bar{X}$ and a continuous map $c_X : X \to \bar{X}$, which is universal for continuous maps from $X$ to compact spaces; see Definition 5.1 for more details. It is clear that an SCC of $X$, if it exists, is unique up to a unique isomorphism. The categorical property of SCC is stated in Proposition 5.3.
are several existence proofs of SCC in the literature, and we mention some of them in Remark 5.21.

Consider a topological space $X$. For a point $x \in X$ and a function $a \in \mathcal{F}_{bc}(X, \mathbb{K})$ the evaluation of $a$ at $x$ is $ev_x(a) := a(x) \in \mathbb{K}$. The algebraic reflection map is the map

$$
\text{refl}_X^{\text{alg}} : X \to \text{MSpec}(\mathcal{F}_{bc}(X, \mathbb{R})), \quad \text{refl}_X^{\text{alg}}(x) := \ker(ev_x)
$$

in Top. Our third main theorem is next.

**Theorem 0.10** (Algebraic SCC). Given a topological space $X$, consider the topological space $\tilde{X}^{\text{alg}} := \text{MSpec}(\mathcal{F}_{bc}(X, \mathbb{R}))$ and the algebraic reflection map $\text{refl}_X^{\text{alg}} : X \to \tilde{X}^{\text{alg}}$. Then the pair $(\tilde{X}^{\text{alg}}, \text{refl}_X^{\text{alg}})$ is a Stone-Čech Compactification of $X$.

This is Theorem 5.19 in the body of the paper. The proof is an easy consequence of Theorems 0.7 and 0.8.

The next theorem is about Banach rings.

**Theorem 0.11** (Canonical Norm). Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Every BC $\mathbb{K}$-ring $A$ admits a unique norm $\|\cdot\|$, called the canonical norm, satisfying the three conditions below.

(i) Given a homomorphism $\phi : A \to B$ in $\text{Rng}_{bc}\mathbb{K}$, and an element $a \in A$, the inequality $\|\phi(a)\| \leq \|a\|$ holds.

(ii) For the $\mathbb{K}$-ring $\mathcal{F}_{bc}(X, \mathbb{K})$ of bounded continuous functions on a topological space $X$, the canonical norm is the sup norm.

(iii) The canonical norm makes $A$ into a Banach $\mathbb{K}$-ring.

This theorem is repeated as Theorem 0.8 for the real case, and as Theorem 10.10 for the complex case.

An involution of a $\mathbb{C}$-ring $A$ is a ring automorphism $(-)^*$ of $A$ such that $(a^*)^* = a$ and $(\lambda \cdot a)^* = \overline{\lambda} \cdot a^*$ for all $a \in A$ and $\lambda \in \mathbb{C}$. Here $\overline{\lambda}$ is the complex conjugate of $\lambda$. The prototypical example is the involution $(-)^*$ of the ring $A := \mathcal{F}_{bc}(X, \mathbb{C})$, where $X$ is some topological space, whose formula is

$$
a^*(x) = \overline{a(x)}
$$

for all $a \in A$ and $x \in X$.

By a Banach*- $\mathbb{C}$-ring we mean a $\mathbb{C}$-ring $A$, equipped with an involution $(-)^*$ and a norm $\|\cdot\|$, such that $A$ is a Banach ring with this norm, and also $\|a \cdot a^*\| = \|a\|^2$ for all $a \in A$. This object is commonly known as a commutative unital $\mathbb{C}^*$ algebra over $\mathbb{C}$.

**Theorem 0.13** (Canonical Involution). Every BC $\mathbb{C}$-ring $A$ admits a unique involution $(-)^*$, called the canonical involution, satisfying the three conditions below.

(i) Given a homomorphism $\phi : A \to B$ in $\text{Rng}_{bc}\mathbb{C}$, and an element $a \in A$, there is equality $\phi(a^*) = \overline{\phi(a)}$ in $B$.

(ii) For the $\mathbb{C}$-ring $\mathcal{F}_{bc}(X, \mathbb{C})$ of bounded continuous functions on a topological space $X$, the canonical involution is the one from equation (0.12).

(iii) The $\mathbb{C}$-ring $A$, equipped with the canonical involution $(-)^*$ and the canonical norm $\|\cdot\|$ from Theorem 0.11, is a Banach*- $\mathbb{C}$-ring.

This theorem is repeated as Theorem 0.5 and part of Theorem 10.10.
For a BC $\mathbb{C}$-ring $A$, with its canonical involution $(-)^*$, the canonical hermitian subring is the $\mathbb{R}$-ring $A_0 := \{ a \in A \mid a^* = a \}$. The next theorem is a combination of Corollary 9.8 and Theorem 9.9 in the body of the paper.

**Theorem 0.14.** There is an equivalence of categories

$$H : \text{Rng}_{bc} \mathbb{C} \to \text{Rng}_{bc} \mathbb{R},$$

which sends a BC $\mathbb{C}$-ring $A$ to its canonical hermitian subring $H(A) = A_0$. The quasi-inverse of $H$ is the induction functor

$$I : \text{Rng}_{bc} \mathbb{R} \to \text{Rng}_{bc} \mathbb{C}, \quad I(A_0) := \mathbb{C} \otimes_{\mathbb{R}} A_0.$$

A Banach* $\mathbb{R}$-ring is an $\mathbb{R}$-ring $A$, equipped with a norm $\| - \|$, such that $A$ is a Banach $\mathbb{R}$-ring with this norm, and for every $a \in A$ there is equality $\| a^2 \| = \| a \|^2$, and the element $1 + a^2$ is invertible in $A$.

**Theorem 0.15.** Let $K$ be either $\mathbb{R}$ or $\mathbb{C}$. There is an equivalence of categories

$$F : \text{BaRng}^*/K \to \text{Rng}_{bc} K,$$

which forgets the norm, and forgets the involution when $K = \mathbb{C}$. The quasi-inverse

$$G : \text{Rng}_{bc} K \to \text{BaRng}^*/K$$

endows a BC $K$-ring with its canonical norm, and with its canonical involution when $K = \mathbb{C}$.

The equivalences of categories from Theorems 0.7, 0.14 and 0.15 are exhibited in the following diagram, which is commutative up to isomorphisms of functors.

![Diagram](image)

The last results to be mentioned in the Introduction concern Stone topological spaces, also known as profinite topological spaces.

Let $X$ be a topological space, with function ring $A := F_{bc}(X, \mathbb{R})$. A function $a \in A$ is called a step function if it takes only finite many values. The subring of $A$ consisting of step functions is denoted by $A_{\text{stp}}$.

**Theorem 0.17.** Let $A$ be a BC $\mathbb{R}$-ring and let $X := \text{MSpec}(A)$. The following two conditions are equivalent:

(i) $X$ is a Stone space.

(ii) The subring $A_{\text{stp}}$ is dense in $A$, with respect to the canonical norm of $A$.

This is repeated as Theorem 7.7 in Section 7 and proved there.
Corollary 0.18. Suppose $X$ is a discrete topological space, with Stone-Čech Compactification $\tilde{X}$. Then $\tilde{X}$ is a Stone topological space.

This fact is well-known of course, but all previous proofs of it that we saw are involved and indirect. Our proof (this is repeated as Corollary [2.8]) is an easy consequence of Theorem [0.17] – a discrete space $X$ admits plenty of step functions.

We end the Introduction with two questions.

Question 0.19. Is there a theory of "BC rings" for commutative nonarchimedean Banach rings? Is there a result analogous to Theorem 0.15 but with $\hat{\mathbb{Q}}_p$ instead of $\mathbb{R}$ or $\mathbb{C}$?

Question 0.20. Is there a useful theory of "noncommutative BC rings", and a noncommutative version of Theorem [0.15] for noncommutative Banach rings?

Acknowledgments. We wish to thank Nicolas Addington, Moshe Kamenski, Assaf Hasson, Eli Shamovich, Francesco Saettone, Shirly Geffen, Ilan Hirshberg and Ken Goodearl for useful discussions.

1. Preliminaries

In this section we review some of the concepts on categories and commutative rings that will be used in our paper, for the benefit of readers who are not familiar with these subjects. This section also serves to introduce notation. Textbook references are provided.

A category $\mathcal{C}$ consists of a set of objects $\text{Ob}(\mathcal{C})$, and for each pair of objects $C_0, C_1 \in \text{Ob}(\mathcal{C})$ a set of morphisms $\text{Hom}_\mathcal{C}(C_0, C_1)$. A morphism $f \in \text{Hom}_\mathcal{C}(C_0, C_1)$ is depicted by $f : C_0 \rightarrow C_1$. Every object $C \in \text{Ob}(\mathcal{C})$ has an identity automorphism $\text{id}_C : C \rightarrow C$. There is an operation of composition

$$
\text{Hom}_\mathcal{C}(C_1, C_2) \times \text{Hom}_\mathcal{C}(C_0, C_1) \rightarrow \text{Hom}_\mathcal{C}(C_0, C_2), \quad (f_2, f_1) \mapsto f_2 \circ f_1
$$

for each triple of objects $C_0, C_1, C_2$. The composition is associative, and the identity automorphisms satisfy $\text{id}_C \circ f = f \circ \text{id}_C$ for all morphisms $f : C_0 \rightarrow C_1$. A morphism $f : C_0 \rightarrow C_1$ is called an isomorphism if there is a morphism $g : C_1 \rightarrow C_0$ such that $g \circ f = \text{id}_{C_0}$ and $f \circ g = \text{id}_{C_1}$. In this case $g$ is unique, it is called the inverse of $f$, and one writes $f^{-1} := g$. As customary, we will use the shorthand $C \in \mathcal{C}$ for an object $C$. We shall ignore set theoretical issues in this paper, and implicitly rely on the concept of Grothendieck universe, as explained in [ML Section I.6] and in [Ye Section 1.1].

Given categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is the data of a function $F_{\text{ob}} : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$, and for every pair of objects $C_0, C_1 \in \text{Ob}(\mathcal{C})$ a function

$$
(1.1) \quad F_{C_0, C_1} : \text{Hom}_\mathcal{C}(C_0, C_1) \rightarrow \text{Hom}_\mathcal{D}(F_{\text{ob}}(C_0), F_{\text{ob}}(C_1)).
$$

The functions $F_{C_0, C_1}$ must respect compositions and identities. When there is no ambiguity, and in order to simplify the notation, we shall use the symbol $F$ both for $F_{\text{ob}}$ and for $F_{C_0, C_1}$. Each category $\mathcal{C}$ has its identity functor $\text{Id}_\mathcal{C}$, which acts identically on objects and morphisms of $\mathcal{C}$. Functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ can be composed in the obvious way, and $\text{Id}_\mathcal{D} \circ F = F \circ \text{Id}_\mathcal{C}$. Sometimes a functor is called a covariant functor.
Suppose \( F, G : C \to D \) are functors. A morphism of functors (also called a natural transformation) \( \eta : F \to G \) is a collection \( \{ \eta_C \}_{C \in \text{Ob}(C)} \) of morphisms \( \eta_C : F(C) \to G(C) \) in \( D \), such that for every morphism \( f : C_0 \to C_1 \) in \( C \) the diagram

\[
\begin{array}{c}
F(C_0) \\
\downarrow^{\eta_{C_0}} \\
G(C_0)
\end{array}
\quad
\begin{array}{c}
\quad F(f) \\
\quad \downarrow^{\eta_{C_1}} \\
\quad G(f)
\end{array}
\]

in \( D \) is commutative. The morphism \( \eta \) is called an isomorphism of functors if for every object \( C \in C \) the morphism \( \eta_C : F(C) \to G(C) \) is an isomorphism.

A functor \( F : C \to D \) is called full (resp. faithful) if for every pair of objects \( C_0, C_1 \in C \) the function \( F_{C_0, C_1} \) in (1.1) is surjective (resp. injective). The functor \( F : C \to D \) is called an equivalence if there exists a functor \( G : D \to C \), and isomorphisms of functors \( \eta : \text{Id}_C \Rightarrow G \circ F \) and \( \zeta : \text{Id}_D \Rightarrow F \circ G \). In this case \( G \) is called a quasi-inverse of \( F \), and it is unique up to a unique isomorphism of functors. It is not hard to show that \( F : C \to D \) is an equivalence if it is both fully faithful (i.e. full and faithful) and essentially surjective on objects (i.e. for every \( D \in D \) there exists some \( C \in C \) with an isomorphism \( F(C) \Rightarrow D \) in \( D \)).

A contravariant functor \( F : C \to D \) is made up of functions \( F_{\text{ob}} \) and \( F_{C_0, C_1} \), but the target sets in (1.1) are reversed; namely for each morphism \( f : C_0 \to C_1 \) in \( C \) there is a morphism \( F(f) : F(C_1) \to F(C_0) \) in \( D \). The contravariant functor \( F \) must respect compositions, in the reversed order, and identities. If \( G : D \to E \) is another contravariant functor, then the composition \( G \circ F : C \to E \) is a (covariant) functor.

A category \( C \) gives rise to the opposite category \( C^{\text{op}} \), which has the same objects as \( C \), but the morphism sets and the compositions are reversed. There is a contravariant functor \( \text{op} : C \to C^{\text{op}} \), which is the identity on object sets, and the identity (in disguise)

\[
\text{op}_{C_0, C_1} : \text{Hom}_C(C_0, C_1) \Rightarrow \text{Hom}_{C^{\text{op}}}(C_1, C_0)
\]

on morphism sets. There is an equality of functors \( \text{op}_{C^{\text{op}}} \circ \text{op} = \text{Id}_C \). Every contravariant functor \( F : C \to D \) can be expressed uniquely as \( F = F^{\text{op}} \circ \text{op} \), where \( F^{\text{op}} : C^{\text{op}} \to D \) is the functor \( F^{\text{op}} := F \circ \text{op} \); see the commutative diagram (1.2). The formula \( F \mapsto F^{\text{op}} \) gives a bijection between the set of contravariant functors \( C \to D \) and the set of functors \( C^{\text{op}} \to D \), and we are going to use this bijection often.

(1.2)

\[
\begin{array}{ccc}
C^{\text{op}} & \xrightarrow{\text{op}_{C^{\text{op}}}} & C \\
\downarrow^{F^{\text{op}}} & & \downarrow^{F} \\
E & \xrightarrow{F} & C
\end{array}
\]

Let \( C \) be a category. A full subcategory \( B \) of \( C \) is a category \( B \) such that \( \text{Ob}(B) \subseteq \text{Ob}(C) \), and for every pair of objects \( C_0, C_1 \in \text{Ob}(B) \) there is equality

\[
\text{Hom}_B(C_0, C_1) = \text{Hom}_C(C_0, C_1).
\]

The identity morphisms and the composition of \( B \) are those of \( C \). Thus the inclusion functor \( \text{inc} : B \to C \) is fully faithful.
For a detailed study of categories and functors in general we recommend the books [HS] and [ML].

Now let us turn attention to a few categories that will play important roles in our paper. First there is the category \( \text{Set} \) of sets, whose objects are the sets (with an implicit bound on size, in terms of Grothendieck universes). The morphisms \( f : S \to T \) in \( \text{Set} \) are the functions.

The category of topological spaces and continuous maps between them is \( \text{Top} \). There is a forgetful functor \( \text{Top} \to \text{Set} \), which sends a topological space to its underlying set, and does nothing to the morphisms.

Recall that a topological space \( - \) is called \textit{Hausdorff} if for every pair of distinct points \( x, y \in X \) there are open subsets \( U, V \subseteq X \) such that \( x \in U, y \in V \) and \( U \cap V = \emptyset \). All metric spaces are Hausdorff, but there are many important Hausdorff topological spaces that are not metrizable (i.e. they do not admit metrics that induce their given topologies).

A topological space \( X \) is called \textit{quasi-compact} if it has the finite open subcovering property. By this we mean that given an open covering \( X = \bigcup_{i \in I} U_i \), indexed by some set \( I \), there exists a finite subset \( I_0 \subseteq I \) such that \( X = \bigcup_{i \in I_0} U_i \). The adjective quasi-compact is prominent in algebraic geometry, but we have not seen it used in classical topology publications.

In our paper will adhere to this convention:

\textbf{Convention 1.3.} A topological space \( X \) is called \textit{compact} if it is both Hausdorff and quasi-compact.

The full subcategory of \( \text{Top} \) on the compact topological spaces is denoted by \( \text{Top}_{cp} \). We will also adhere to the next convention regarding rings:

\textbf{Convention 1.4.} A ring means, by default, a unital commutative ring. A ring homomorphism \( f : A \to B \) must respect units.

The category of (commutative) rings and ring homomorphisms is denoted by \( \text{Rng} \). All rings in Sections 1-5 of this paper are without a topology; exceptions are the fields \( \mathbb{R} \) and \( \mathbb{C} \), which are sometimes seen as topological rings with their standard norm topologies, and when this happens we shall state it explicitly. In Sections 6-10 we shall also deal with Banach rings, and again the presence of a norm will be stated explicitly.

\textbf{Definition 1.5.} Fix a ring \( K \).

1. A \( K \)-ring is a ring \( A \) equipped with a ring homomorphism \( \text{str}_A : K \to A \), called the structural homomorphism.

2. Suppose \( A \) and \( B \) are \( K \)-rings. A \( K \)-ring homomorphism \( \phi : B \to C \) is a ring homomorphism \( \phi \) satisfying \( \phi \circ \text{str}_A = \text{str}_B \).

3. The category of \( K \)-rings and \( K \)-ring homomorphisms is denoted by \( \text{Rng}/K \).

The structural homomorphism \( \text{str}_A \) shall usually remain implicit, and we shall just talk about the \( K \)-ring \( A \). Most textbooks use the expression “unital associative commutative \( K \)-algebra” to mean an \( K \)-ring. Since every ring \( A \) admits a unique ring homomorphism \( \mathbb{Z} \to A \), it follows that there is equality \( \text{Rng}/\mathbb{Z} = \text{Rng} \). In our paper we are going to be mostly concerned with \( \mathbb{R} \)-rings.
The last topic to review is affine schemes. Actually, we won’t require the full structure of the affine scheme \((X, \mathcal{O}_X) = \text{Spec}(A)\) as a locally ringed space. The structure sheaf \(\mathcal{O}_X\) is going to be ignored, and we shall only care about the underlying topological space \(X\), whose definition we now recall. In this definition we introduce some new notation.

**Definition 1.6 (Zariski Topology).** Let \(A\) be a ring.

1. The *prime spectrum* of \(A\) is the set \(\text{Spec}(A)\) of prime ideals of \(A\). Let us write \(X := \text{Spec}(A)\).
2. Given an ideal \(a \subseteq A\), the subset \(\text{Zer}_X(a) := \{p \in X \mid a \subseteq p\} \subseteq X\) is called the closed subset of \(X\) defined by the ideal \(a\).
3. The *Zariski topology* of \(X\) is the topology in which the closed subsets are \(\text{Zer}_X(a)\), as \(a\) runs over all the ideals \(a \subseteq A\).
4. If \(a = (0)\) is a principal ideal, then we write \(\text{Zer}_X(0) = \text{Zer}_X(a)\). This is is called the principal open subset defined by the element \(0\).
5. For an element \(0 \in A\) we write \(\text{NZer}_X(0) = \{p \in X \mid 0 \notin p\} \subseteq X\).

Obviously the principal open subset \(\text{NZer}_X(0)\) is the complement of the principal closed subset \(\text{Zer}_X(0)\). It is easy to verify that

\[
\text{Zer}_X(a) = \bigcap_{p \in \mathfrak{p}} \text{Zer}_X(p).
\]

This formula implies that the principal open subsets of \(X\) form a basis of the Zariski topology; i.e. for every open subset \(U \subseteq X\) and for every point \(x \in U\) there is some element \(a \in A\) such that \(x \in \text{NZer}_X(a) \subseteq U\). It is known that the topological space \(X = \text{Spec}(A)\) is quasi-compact.

Suppose \(\phi : A \to B\) is a ring homomorphism. If \(q \subseteq B\) is a prime ideal, then \(p := \phi^{-1}(q) \subseteq A\) is also a prime ideal. The resulting function

\[
\text{Spec}(\phi) : \text{Spec}(B) \to \text{Spec}(A), \quad \text{Spec}(\phi)(q) := \phi^{-1}(q)
\]

is actually continuous (this is an easy exercise). In this way we obtain a functor

\[
\text{Spec} : \text{Rng}^{\text{op}} \to \text{Top}.
\]

For more information on the Zariski topology of affine schemes see the books [AK] or [Ei].

**Definition 1.10.** Let \(A\) be a ring. The set of maximal ideals of \(A\) is denoted by \(\text{MSpec}(A)\). The set \(\text{MSpec}(A)\) is a subset of \(\text{Spec}(A)\), and we give it the induced Zariski topology.

**Proposition 1.11.** The following are equivalent for a ring \(A\).

1. \(A = 0\).
2. \(\text{MSpec}(A) = \emptyset\).

**Proof.** The implication (i) \(\Rightarrow\) (ii) is trivial. The reverse implication is a standard exercise using Zorn’s Lemma.
Proposition 1.12. Let $A$ be a ring and let $X := \text{MSpec}(A)$. The following are equivalent for an element $a \in A$.

(i) $a$ is invertible in $A$.
(ii) $\text{Zer}_X(a) = \emptyset$.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial. As for the reverse implication let $a \subseteq A$ be the ideal generated by $a$, and let $\bar{A} := A/a$. It is easy to see that there is a canonical bijection $\text{Zer}_X(a) \cong \text{MSpec}(\bar{A})$. Using Proposition 1.11 we see that condition $(ii)$ implies that $\bar{A} = 0$; and this means that $a = A$ and that $a$ is invertible. \hfill \Box

Proposition 1.13. For every ring $A$ the topological space $\text{MSpec}(A)$ is quasi-compact.

Proof. Write $X := \text{Spec}(A)$ and $X_{\text{max}} := \text{MSpec}(A)$. Suppose $X_{\text{max}} = \bigcup_{i \in I} U_i$ is an open covering. We need to prove that this has a finite subcovering.

Since $X_{\text{max}}$ has the subspace topology induced from $X$, for every index $i$ there is some open subset $V_i \subseteq X$ such that $U_i = V_i \cap X_{\text{max}}$. Let $Z$ be the complement in $X$ of the open set $\bigcup_{i \in I} V_i$. Because $X$ has the Zariski topology, there is some ideal $a \subseteq A$ such that $Z = \text{Zer}_X(a)$. Define the ring $\bar{A} := A/a$, so $Z \cong \text{Spec}(\bar{A})$ as topological spaces. Note that $X_{\text{max}} \cap Z = \emptyset$.

Assume, for the sake of contradiction, that $Z \neq \emptyset$. Then the ring $\bar{A}$ is nonzero, and therefore it has some maximal ideal $\mathfrak{m}$. Let $\mathfrak{m} \subseteq A$ be the preimage of $\mathfrak{m}$ under the canonical surjection $A \to \bar{A}$. Then $\mathfrak{m}$ is a maximal ideal of $A$, and it satisfies $\mathfrak{m} \in X_{\text{max}} \cap Z$, which is a contradiction. We conclude that $Z = \emptyset$.

It follows that $\bigcup_{i \in I} V_i = X$. Because the topological space $X$ is quasi-compact, there exists some finite subset $I_0 \subseteq I$ such that $X = \bigcup_{i \in I_0} V_i$. But then $X_{\text{max}} = \bigcup_{i \in I_0} U_i$. \hfill \Box

Remark 1.14. $\text{MSpec}$ is not a functor on $\text{Rng}$. Concretely, given a ring homomorphism $\phi : A \to B$ and a maximal ideal $n \subseteq B$, the prime ideal $p := \phi^{-1}(n) \subseteq A$ is often not maximal; see Example 1.15 below.

In Section 2 of the paper we will study certain categories of rings on which $\text{MSpec}$ is a functor (see Proposition 2.8).

Example 1.15. Consider the polynomial ring $\mathbb{R}[t]$ in the variable $t$ over the field $\mathbb{R}$, and its fraction field $\mathbb{R}(t)$. The inclusion homomorphism is $\phi : \mathbb{R}[t] \to \mathbb{R}(t)$. The ideal $n := (0) \subseteq \mathbb{R}(t)$ is maximal, yet its preimage $p := \phi^{-1}(n) = (0) \subseteq \mathbb{R}[t]$ is not maximal.

We end this section with a discussion of the failure of the Hausdorff property in typical rings that appear in algebraic geometry. This is a prelude to the opposite behavior of $BC \mathbb{R}$-rings, cf. Lemma 4.12(1) and Theorem 4.16.

Example 1.16. Again consider the polynomial ring $A := \mathbb{R}[t]$. Let $X := \text{Spec}(A)$, viewed only as a topological space with its Zariski topology. (In algebraic geometry the affine scheme $X$ is called the affine line over $\mathbb{R}$, with notation $\text{A}^{1}_{\mathbb{R}}$.) Like every affine scheme, the topological space $X$ is quasi-compact. But it is not Hausdorff, because the generic point $p = (0) \in X$ belongs to every nonempty open subset of $X$.

Now we look at the topological subspace $X_{\text{max}} := \text{MSpec}(A)$ of $X = \text{Spec}(A)$. The points of $X_{\text{max}}$ are the maximal ideals $m = (p) \subseteq A$, where $p = p(t)$ is a
monic irreducible polynomial, necessarily of degree 1 or 2. By Proposition 1.13 the topological space \( X_{\text{max}} \) is quasi-compact. We claim it is not Hausdorff. Note that the argument used in the previous paragraph does not apply here.

To demonstrate that \( X_{\text{max}} \) is not Hausdorff, we will prove that every pair \( U, V \subseteq X_{\text{max}} \) of nonempty open subsets satisfies \( U \cap V \neq \emptyset \). To see that this is true, it is enough to look at a pair of nonempty principal open subsets \( U = \text{NZer}_{X_{\text{max}}} (a) \) and \( V = \text{NZer}_{X_{\text{max}}} (b) \) for \( a, b \in A = \mathbb{R}[t] \). Because these open subsets are nonempty, the polynomials \( a \) and \( b \) are nonzero. The product \( a \cdot b \) is then a nonzero polynomial, and it has finitely many irreducible factors. Take an irreducible polynomial \( p \in A = \mathbb{R}[t] \) that does not divide \( a \cdot b \). Then the maximal ideal \( x = m := (p) \) belongs to \( U \cap V \).

2. Abstract Function Rings

In this section we study certain categories of rings on which \( \text{MSpec} \) is functorial.

**Convention 2.1.** Throughout this section \( K \) is some fixed base field.

Recall that for a \( K \)-ring \( A \) we denote the structural homomorphism by \( \text{str}_A : K \to A \). See Definition 1.5.

**Definition 2.2.** Let \( A \) be a \( K \)-ring.

1. A maximal ideal \( m \subseteq A \) is called \( K \)-valued if the structural homomorphism \( \text{str}_{A/m} : K \to A/m \) is bijective.

2. The \( K \)-ring \( A \) is said to be a \( K \)-valued ring if all its maximal ideals are \( K \)-valued.

**Example 2.3.** Assume \( K \) is an algebraically closed field, and let \( A \) be a finitely generated \( K \)-ring. The Hilbert Nullstellensatz says that \( A \) is a \( K \)-valued ring.

**Example 2.4.** Assume \( K = \mathbb{R} \). Consider the polynomial ring \( A := \mathbb{R}[t] \). It is an \( \mathbb{R} \)-ring, but it has maximal ideals that are not \( \mathbb{R} \)-valued, such as \( m := (t^2 + 1) \). So \( A \) is not an \( \mathbb{R} \)-valued ring.

Now let \( S \subseteq A \) be the set of all finite products \( p_1(t) \cdots p_n(t) \), where \( n \geq 0 \), and the \( p_i(t) \) are monic quadratic irreducible polynomials. Let \( B := A_S \), the localization of \( A \) with respect to \( S \). This localization removes all the \( \mathbb{C} \)-valued maximal ideals of \( A \). The maximal ideals of \( B \), those remaining after the localization, are of the form \( m = (t - \lambda) \) for \( \lambda \in \mathbb{R} \). Thus \( B \) is an \( \mathbb{R} \)-valued \( \mathbb{R} \)-ring.

**Remark 2.5.** Given a \( K \)-ring \( A \), define \( X := \text{Spec}(A) \) and \( X_{\text{max}} := \text{MSpec}(A) \). Here is a standard definition from algebraic geometry: the set of \( K \)-valued points of \( X \) is \( X(K) := \text{Hom}_{\text{Rng}/K}(A, K) \). There is always an inclusion \( X(K) \subseteq X_{\text{max}} \). It is easy to see that the ring \( A \) is \( K \)-valued if and only if \( X(K) = X_{\text{max}} \).

**Lemma 2.6.** Let \( \phi : A \to B \) be a homomorphism in \( \text{Rng}/K \), and let \( n \subseteq B \) be a \( K \)-valued maximal ideal. Then the ideal \( m := \phi^{-1}(n) \subseteq A \) is maximal and \( K \)-valued.

**Proof.** There is an injective \( K \)-ring homomorphism \( \tilde{\phi} : A/m \to B/n \). By assumption the structural homomorphism \( K \to B/n \) is bijective, and this implies that \( \tilde{\phi} \) is surjective. Thus \( \tilde{\phi} \) is an isomorphism, \( A/m \cong K \), and \( m \) is maximal. \( \square \)
Definition 2.7. The full subcategory of $\text{Rng}/K$ on the $K$-valued rings is denoted by $\text{Rng}/val K$.

Proposition 2.8. There is a unique functor

$$\text{MSpec} : (\text{Rng}/val K)^{op} \to \text{Top}$$

sending a ring $A \in \text{Rng}/val K$ to the topological space $\text{MSpec}(A)$, and sending a homomorphism $\phi : A \to B$ in $\text{Rng}/val K$ to the continuous map

$$(\ast) \quad \text{MSpec}(\phi) : \text{MSpec}(B) \to \text{MSpec}(A), \quad n \mapsto \phi^{-1}(n).$$

Proof. Take a homomorphism $A \to B$ in $\text{Rng}/val K$ and some $n \in \text{MSpec}(B)$. By Lemma 2.6 the ideal $\phi^{-1}(n) \subseteq A$ is maximal, so the map of sets $\text{MSpec}(\phi)$ is well-defined. As for continuity of $\text{MSpec}(\phi)$, this is because the map

$$\text{Spec}(\phi) : \text{Spec}(B) \to \text{Spec}(A), \quad q \mapsto \phi^{-1}(q)$$

is continuous for the Zariski topologies of $\text{Spec}(A)$ and $\text{Spec}(B)$, and the spaces $\text{MSpec}(A)$ and $\text{MSpec}(B)$ have the respective induced subspace topologies. We have thus shown that $\text{MSpec}(\phi)$ is a morphism in $\text{Top}$.

Trivially $\text{MSpec}$ respects identity automorphisms. Finally, regarding compositions: given $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ in $\text{Rng}/bc K$, formula $(\ast)$ shows that

$$\text{MSpec}(\phi) \circ \text{MSpec}(\psi) = \text{MSpec}(\psi \circ \phi).$$

$\Box$

Definition 2.9. Let $A$ be a $K$-ring and $m \subseteq A$ a $K$-valued maximal ideal, with canonical surjection $\text{pr}_m : A \to A/m$. Define the evaluation ring homomorphism to be the $K$-ring homomorphism

$$\text{ev}_m := (\text{str}_{A/m})^{-1} \circ \text{pr}_m : A \to K.$$

In other words, for all $a \in A$ there is equality $\text{str}_{A/m}(\text{ev}_m(a)) = a + m \in A/m$. In a commutative diagram it looks like this:

\[
\begin{array}{ccc}
K & \xrightarrow{id_K} & K \\
\downarrow{\text{str}_A} & & \downarrow{\text{str}_{A/m}} \\
A & \xrightarrow{\text{ev}_m} & K \\
\downarrow{\text{pr}_m} & & \downarrow{\text{str}_{A/m}} \\
A/m & = & K
\end{array}
\]

Definition 2.10. Let $X$ be a topological space. We denote by $F(X, K)$ the $K$-ring of all functions $a : X \to K$.

Clearly, as $X$ changes we have a functor

$$F(\cdot, K) : \text{Top}^{op} \to \text{Rng}/K.$$

Definition 2.11. Let $A$ be a $K$-valued $K$-ring. We define the double evaluation ring homomorphism

$$\text{dev}_A : A \to F(\text{MSpec}(A), K)$$

to be the $K$-ring homomorphism with formula $\text{dev}_A(a)(m) := \text{ev}_m(a) \in K$ for $a \in A$ and $m \in \text{MSpec}(A)$. 
Another way to express $\text{dev}_A$ is this: for every $a \in A$ and $m \in \text{MSpec}(A)$ the element $\text{dev}_A(a)(m) \in K$ satisfies

\[(2.12) \quad \text{str}_{A/m}(\text{dev}_A(a)(m)) = a + m \in A/m.\]

The name "double evaluation" is used because the element $\text{dev}_A(a)(m)$ is a function of two arguments: $a$ and $m$.

**Proposition 2.13.** The $K$-ring homomorphism $\text{dev}_A : A \to \text{F}(\text{MSpec}(A), K)$ is functorial in the object $A \in \text{Rng}_{/\text{val}} K$.

**Proof.** Let's use the abbreviations $F := \text{F}(\_, K)$ and $M := \text{MSpec}$. Given a homomorphism $\phi : A \to B$ in $\text{Rng}_{/\text{val}} K$, we need to prove that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{dev}_A} & (F \circ M)(A) \\
\phi \downarrow & & \downarrow (F \circ M)(\phi) \\
B & \xrightarrow{\text{dev}_B} & (F \circ M)(B)
\end{array}
\]

in $\text{Rng}/K$ is commutative.

For the proof we shall require the commutative diagram of ring isomorphisms

\[
\begin{array}{ccc}
K & \xrightarrow{\text{str}_{A/m}} & A/m \xrightarrow{\phi} B/n
\end{array}
\]

where $\bar{\phi}(a + m) = \phi(a) + n$ for $a \in A$.

Take an element $a \in A$ and a maximal ideal $n \in \text{M}(B)$. Let $b := \phi(a) \in B$, $a' := \text{dev}_A(a) \in \text{F}(\text{M}(A))$, $b' := \text{dev}_B(b) \in \text{F}(\text{M}(B))$ and $m := \text{M}(\phi)(n) = \bar{\phi}^{-1}(n) \in \text{M}(A)$. Next let $\mu := a'(m) \in K$ and $\nu := b'(n) \in K$. We have

\[(F \circ M)(\phi)(a)(n) = (F \circ M)(\bar{\phi})(a')(n) = (a' \circ M(\phi))(n) = a'(m) = \mu\]

and

\[(\text{dev}_B \circ \phi)(a)(n) = \text{dev}_B(b)(n) = b'(n) = \nu;\]

and it remains to prove that $\mu = \nu$.

According to formula \[(2.12)\] we know that $\text{str}_{A/m}(\mu) = a + m \in A/m$ and $\text{str}_{B/n}(\nu) = b + n \in A/n$. Since $\bar{\phi}(a + m) = (b + n)$, the commutative diagram of isomorphisms \[(2.15)\] says that $\mu = \nu$. \hfill \Box

The next theorem explains how the definitions in this section behave with respect to finite base change. This theorem will be used in Section 8 of the paper.

**Theorem 2.16.** Let $L$ be a finite extension field of $K$. Let $A$ be a $K$-valued $K$-ring, and define the $L$-ring $B := L \otimes_K A$, with $K$-ring homomorphism $\gamma : A \to B$, $\gamma(a) := 1 \otimes a$. Then:

1. $B$ is an $L$-valued ring.
2. For every $n \in \text{MSpec}(B)$ the ideal $m := A \cap n = \gamma^{-1}(n) \subseteq A$ is maximal.
3. Let $X := \text{MSpec}(A)$ and $Y := \text{MSpec}(B)$. Then the map $f : Y \to X$, $f(n) := \gamma^{-1}(n)$, is a homeomorphism.
The diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B \\
\downarrow{\text{dev}_A} & & \downarrow{\text{dev}_B} \\
F(X, K) & \xrightarrow{F(f, \text{str}_L)} & F(Y, L)
\end{array}
\]

in \text{Rng}/K is commutative.

The homomorphisms \(\text{dev}_A\) and \(\text{dev}_B\) exist because the rings \(A\) and \(B\) are \(K\)-valued and \(L\)-valued, respectively.

**Proof.**

(1, 2) Given a maximal ideal \(n \subseteq B\), define the prime ideal \(m := A \cap n \subseteq A\) and the rings \(L := B/n\) and \(K := A/m\). So \(L\) is a field and \(K \subseteq L\) is a subring. Since the ring homomorphism \(L \otimes_K K \rightarrow L\) is surjective we see that \(L\) is a finitely generated \(K\)-module. According to [AK, Theorem 10.18], \(L\) is an integral extension of \(K\). Next, according to [AK, Lemma 14.1] the ring \(K\) is also a field. Therefore the ideal \(m \subseteq A\) is maximal. Because \(A\) is a \(K\)-valued ring, it follows that \(m \otimes_K L \otimes_K K \otimes_K \). The surjective ring homomorphism \(L \otimes_K K \rightarrow L\) now says that \(L \otimes_K L\).

(3) The map \(f : Y \rightarrow X\) exists by item (2). We shall prove that \(f\) is bijective by producing its inverse. Given a maximal ideal \(m \subseteq A\), there is an exact sequence

\[
0 \rightarrow m \rightarrow A \xrightarrow{\text{ev}_m} K \rightarrow 0.
\]

Applying \(L \otimes_K (-)\) to this sequence, we obtain the exact sequence

\[
0 \rightarrow L \otimes_K m \rightarrow B \xrightarrow{\text{id} \otimes \text{ev}_m} L \rightarrow 0.
\]

Thus \(n := L \otimes_K m\) is a maximal ideal of \(B\). In this way we obtain a function \(g : X \rightarrow Y, m \mapsto n\). An easy calculation shows that the functions \(f\) and \(g\) are mutual inverses; and therefore \(f\) is a bijection.

Let us prove that \(f : Y \rightarrow X\) is a homeomorphism. The map \(f\) is continuous because it is the restriction to maximal spectra of the continuous map \(\text{Spec}(\gamma) : \text{Spec}(B) \rightarrow \text{Spec}(A)\). It remains to prove that \(f\) is an open map. Choose a \(K\)-basis \((\lambda_1, \ldots, \lambda_n)\) of \(L\). Take a principal open set \(V = \text{NZer}_Y(b) \subseteq Y\) for some \(b \in B\). We can expand \(b\) into a sum \(b = \sum_{i=1}^{n} \lambda_i \cdot a_i\) with \(a_i \in A\). Then \(V = \cap_{i=1}^{n} \text{NZer}_Y(a_i)\), and hence \(f(V) = \cap_{i=1}^{n} \text{NZer}_X(a_i)\), which is an open set of \(X\).

(4) This is an easy calculation. \(\square\)

### 3. Rings of Bounded Continuous Real Valued Functions

In the current section we introduce a category of \(\mathbb{R}\)-rings that plays a central role in our paper (and is referred to in the title). These are the BC \(\mathbb{R}\)-rings, see Definition 3.4.

The base field \(\mathbb{R}\) (and later, in Sections 8–10 also the base field \(\mathbb{C}\)) has two distinct incarnations in this paper: sometimes it is just an abstract ring (namely without a topology); and at other times it is a topological ring, with its standard norm topology. To minimize confusion, by default \(\mathbb{R}\) will be considered as an abstract
ring. When $\mathbb{R}$ is considered as a topological ring, this will be stated explicitly (like in Definition 3.1 below). All other rings are abstract rings (with the exception of the Banach rings in Section 10). In particular, morphisms in the category $\text{Rng}/\mathbb{R}$ do not have any continuity condition.

**Definition 3.1.** Let $X$ be a topological space.

1. The $\mathbb{R}$-ring consisting of all continuous functions $a : X \rightarrow \mathbb{R}$ is denoted by $F_c(X, \mathbb{R})$.
2. The $\mathbb{R}$-ring consisting of all bounded continuous functions $a : X \rightarrow \mathbb{R}$ is denoted by $F_{bc}(X, \mathbb{R})$.

Here continuity and boundedness are with respect to the standard norm on the field $\mathbb{R}$.

We repeat: the rings $F_c(X, \mathbb{R})$ and $F_{bc}(X, \mathbb{R})$ are not topologized. Of course if $X$ is a discrete topological space then $F_c(X, \mathbb{R}) = F(X, \mathbb{R})$, and if $X$ is a compact topological space then $F_{bc}(X, \mathbb{R}) = F_c(X, \mathbb{R})$. For the empty topological space the ring $F(\emptyset, \mathbb{R})$ is the zero ring.

**Remark 3.2.** The standard notation for the ring $F_c(X, \mathbb{R})$ is $C(X)$; but this is not good notation for our paper, for three reasons. First, $C(X)$ often refers to the ring $F_c(X, \mathbb{C})$ of continuous $\mathbb{C}$-valued functions, which will show up in Sections 8 and 10 so the notation $C(X)$ would cause unwanted ambiguity. Second, the ring $C(X)$ – in either of its two meanings – is often viewed as a topological ring, with the sup norm, whereas for us it is mostly an abstract ring. And third, in our paper we require more refined notions, and hence also more refined notation.

It is obvious that

$$F_{bc}(\cdot, \mathbb{R}), F_c(\cdot, \mathbb{R}), F(\cdot, \mathbb{R}) : \text{Top}^{\text{op}} \rightarrow \text{Rng}/\mathbb{R}$$

are functors. Furthermore, each of them is a subfunctor of the next one. This means that for every topological space $X$ there are inclusion of rings $F_{bc}(X, \mathbb{R}) \subseteq F_c(X, \mathbb{R}) \subseteq F(X, \mathbb{R})$, and these inclusions are functorial in $X$.

**Definition 3.4.** An $\mathbb{R}$-ring $A$ is called a BC $\mathbb{R}$-ring if it is isomorphic, as an $\mathbb{R}$-ring, to the ring $F_{bc}(X, \mathbb{R})$ for some compact topological space $X$. The full subcategory of $\text{Rng}/\mathbb{R}$ on the BC $\mathbb{R}$-rings is denoted by $\text{Rng}_{bc}/\mathbb{R}$.

Thus the category $\text{Rng}_{bc}/\mathbb{R}$ is the essential image of the functor

$$F_{bc}(\cdot, \mathbb{R}) : (\text{Top}_{cp})^{\text{op}} \rightarrow \text{Rng}/\mathbb{R}.$$ 

The expression "BC" stands for "bounded continuous".

**Remark 3.6.** In Theorem 5.9 we shall see that for every topological space $X$ the ring $F_{bc}(X, \mathbb{R})$ belongs to $\text{Rng}_{bc}/\mathbb{R}$. This fact is closely related to the existence of the Stone-Čech Compactification of $X$.

**Definition 3.7.** Given a topological space $X$ and a point $x \in X$, let

$$\text{ev}_x : F_{bc}(X, \mathbb{R}) \rightarrow \mathbb{R}$$

be the $\mathbb{R}$-ring homomorphism $\text{ev}_x(a) := a(x)$. It is called the evaluation at $x$ homomorphism.
It is clear that \( m := \ker(\text{ev}_x) \) is an \( \mathbb{R} \)-valued maximal ideal of the ring \( A := F_{bc}(X, \mathbb{R}) \).

**Definition 3.8.** For a topological space \( X \) define the map of sets
\[
\text{refl}_X^{\text{alg}} : X \to \text{MSpec}(F_{bc}(X, \mathbb{R})), \quad x \mapsto \ker(\text{ev}_x).
\]
We call it the \textit{algebraic reflection map}.

Here is a general topological analogue of the definition for prime spectra, see Definition 1.6.

**Definition 3.9.** Given a topological space \( X \) and a function \( a \in F_c(X, \mathbb{R}) \), we define the open subset
\[
\text{NZer}_X(a) := \{ x \in X \mid a(x) \neq 0 \} \subseteq X,
\]
and call it the \textit{principal open set} determined by \( a \).

The subset \( \text{NZer}_X(a) \) is open in \( X \) because the function \( a : X \to \mathbb{R} \) is continuous for the norm topology of \( \mathbb{R} \). The relation between Definitions 3.7 and 3.9 is this:
\[
(3.10) \quad \text{NZer}_X(a) = \{ x \in X \mid \text{ev}_x(a) \neq 0 \}
\]

**Proposition 3.11.** Let \( X \) be a topological space, and define the topological space \( \tilde{X} := \text{MSpec}(F_{bc}(X, \mathbb{R})) \).

1. The algebraic reflection map \( \text{refl}_X^{\text{alg}} : X \to \tilde{X} \) is continuous.
2. The image of \( \text{refl}_X^{\text{alg}} \) is dense in \( \tilde{X} \).

**Proof.**

1. Let’s write \( A := F_{bc}(X, \mathbb{R}) \) and \( h := \text{refl}_X^{\text{alg}} \). Recall that \( \tilde{X} = \text{MSpec}(A) \) has the Zariski topology. Thus, to prove continuity of \( h \) it suffices to show that for every principal open set \( \tilde{U} = \text{NZer}_X(a) \subseteq \tilde{X} \), determined by an element \( a \in A \), the preimage \( h^{-1}(\tilde{U}) \subseteq X \) is open. But \( h^{-1}(\tilde{U}) = \text{NZer}_X(a) \), and this is open in \( X \) as explained above.

2. Consider a nonempty principal open set \( \tilde{U} = \text{NZer}_X(a) \subseteq \tilde{X} \) for some \( a \in A \). Since \( \tilde{U} \neq \emptyset \) the element \( a \in A \) is nonzero, so as a function \( a : X \to \mathbb{R} \) must be nonzero. But then there must be some point \( x \in X \) such that \( a(x) \neq 0 \). This \( x \) satisfies \( \text{refl}_X^{\text{alg}}(x) \in \tilde{U} \). \( \square \)

4. Duality for Compact Topological Spaces

In this section we prove Theorem 4.16, which asserts that the categories \( \text{Top}_{\text{cp}} \) and \( \text{Rng}_{/bc} \mathbb{R} \) are dual to each other. Recall that \( \text{Rng}_{/bc} \mathbb{R} \) is the category of BC \( \mathbb{R} \)-rings, without a topology (Definition 3.4), and \( \text{Top}_{\text{cp}} \) is the category of compact topological spaces and continuous maps (see Convention 1.3).

The property of compact topological spaces that is important for us is stated in the next theorem. This is a classical result, usually appearing as two distinct theorems, as the proof shows.
Theorem 4.1 (Separation by Continuous Functions). Let $X$ be a compact topological space, and let $Y_0$ and $Y_1$ be disjoint closed subsets of $X$. Let $Z := [0, 1]$, the closed interval in $\mathbb{R}$, with its usual norm topology. Then there is a continuous function $f : X \to Z$ such that $f(Y_0) \subseteq \{0\}$ and $f(Y_1) \subseteq \{1\}$.

Proof. According to [Mn, Theorem 32.3] or [Ke, Theorem 5.9] the space $X$ is normal, and according to [Mn, Theorem 33.1] or [Ke, Lemma 4.4] (the Urysohn Lemma) such a function $a$ exists.

Lemma 4.2. Let $X$ be a compact topological space.

1. The elements of $F_{bc}(X, \mathbb{R})$ separate the points of $X$. Namely, given two distinct points $x, y \in X$, there exists a function $a \in F_{bc}(X, \mathbb{R})$ such that $a(x) \neq a(y)$.

2. The principal open sets form a basis of the topology of $X$. By this we mean that for every open set $U \subseteq X$ and every point $x \in U$ there exists some $a \in F_{bc}(X, \mathbb{R})$ such that $x \in \text{NZer}_X(a) \subseteq U$.

Proof. (1) Consider the disjoint closed subsets $\{x\}$ and $\{y\}$ of $X$. By the Theorem 4.1 there is a bounded continuous function $a : X \to \mathbb{R}$ such that $a(x) = 1$ and $a(y) = 0$.

(2) Consider the disjoint closed subsets $\{x\}$ and $Z := X - U$ of $X$. By Theorem 4.1 there is a bounded continuous function $a : X \to \mathbb{R}$ such that $a(x) = 1$ and $a(z) = 0$ for all $z \in Z$. For such a $a$ we have $x \in \text{NZer}_X(a) \subseteq U$. □

Given a ring $A$, its maximal spectrum $\text{MSpec}(A)$ is equipped with the Zariski topology (recalled in Section 1).

Lemma 4.3. Let $X$ be a compact topological space, and let $A := F_{bc}(X, \mathbb{R})$.

1. Given $m \in \text{MSpec}(A)$, there is a point $x \in X$ such that $m = \text{Ker}(ev_x) = \text{refl}_{X}^{alg}(x)$.

2. The algebraic reflection map $\text{refl}_{X}^{alg} : X \to \text{MSpec}(A)$ is a homeomorphism.

Proof. These assertions are not new, cf. [AK, Exercise 14.26]. Item (1) has a full proof in [Wa, Proposition 1.22]. For the convenience of the reader we are going to provide a proof of item (2).

Let’s write $\tilde{X} := \text{MSpec}(A)$. Lemma 4.2(1) implies that the map $\text{refl}_{X}^{alg} : X \to \tilde{X}$ is injective. Indeed, if $x, y \in X$ are distinct, and $a \in A$ is such that $a(x) = 1$ and $a(y) = 0$, then $a \in \text{refl}_{X}^{alg}(y)$ but $a \notin \text{refl}_{X}^{alg}(x)$, so these maximal ideals are distinct. Item (1) above shows that the map $\text{refl}_{X}^{alg}$ is surjective. We see that $\text{refl}_{X}^{alg} : X \to \tilde{X}$ is bijective.

It remains to prove that $\text{refl}_{X}^{alg} : X \to \tilde{X}$ is a homeomorphism. Given an element $a \in A$, consider the principal open sets $U := \text{NZer}_X(a) \subseteq X$ and $\tilde{U} := \text{NZer}_\tilde{X}(a) \subseteq \tilde{X}$. An easy calculation shows that $\text{refl}_{X}^{alg}(U) = \tilde{U}$. We know (see Section 1) that the principal open sets $U'$ form a basis of the Zariski topology of $\tilde{X}$, and according to Lemma 4.2(2) the principal open sets $U$ form a basis of the given topology of $X$. Therefore $\text{refl}_{X}^{alg}$ is a homeomorphism. □

Lemma 4.4. If $A$ is a BC $\mathbb{R}$-ring then it is $\mathbb{R}$-valued.
Proof. We can assume that \( A = F_{bc}(X, \mathbb{R}) \) for some compact topological space \( X \). Let \( m \) be a maximal ideal of \( A \). By Lemma 4.3 there is a point \( x \in X \) such that \( m = \text{Ker}(ev_x) \). But \( ev_x \) is an \( \mathbb{R} \)-ring homomorphism \( ev_x : A \to \mathbb{R} \), so it induces an \( \mathbb{R} \)-ring isomorphism \( A/m \xrightarrow{\approx} \mathbb{R} \).

\[ \square \]

Remark 4.5. Lemma 4.4 can be understood as an analogy of the Hilbert Nullstellensatz, see Example 2.3. However a BC \( \mathbb{R} \)-ring \( A \) is almost always infinitely generated as an \( \mathbb{R} \)-ring, and almost never noetherian, so the analogy to classical algebraic geometry breaks down fast.

**Proposition 4.6.** The maximal spectrum is a functor

\[ \text{MSpec} : (\text{Rng}_{bc} \mathbb{R})^{\text{op}} \to \text{Top} \]

is sends a homomorphism \( \phi : A \to B \) in \( \text{Rng}_{bc} \mathbb{R} \) to the continuous map \( \text{MSpec}(\phi) : \text{MSpec}(B) \to \text{MSpec}(A) \), \( \text{MSpec}(\phi)(n) = \phi^{-1}(n) \).

**Proof.** Use Lemma 4.4 and Proposition 2.8

By definition there is a functor

\[ F_{bc}(\cdotp, \mathbb{R}) : (\text{Top}_{cp})^{\text{op}} \to \text{Rng}_{bc} \mathbb{R}. \]

In view of Proposition 4.6 there is a composed functor

\[ \text{MSpec} \circ F_{bc}(\cdotp, \mathbb{R}) : \text{Top}_{cp} \to \text{Top}. \]

There is also the inclusion functor \( \text{Inc} : \text{Top}_{cp} \to \text{Top} \).

**Lemma 4.7.** The algebraic reflection map

\[ \text{refl}^\text{alg}_X : X \to \text{MSpec}(F_{bc}(X, \mathbb{R})) \]

in \( \text{Top} \) is functorial in the object \( X \in \text{Top}_{cp} \). In other words, as \( X \) changes there is a morphism

\[ \text{refl}^\text{alg} : \text{Inc} \to \text{MSpec} \circ F_{bc}(\cdotp, \mathbb{R}) \]

of functors \( \text{Top}_{cp} \to \text{Top} \).

**Proof.** We shall use the abbreviations \( M := \text{MSpec} \) and \( F_{bc} := F_{bc}(\cdotp, \mathbb{R}) \). Given a map \( f : Y \to X \) in \( \text{Top}_{cp} \), we must prove that the diagram

\[ Y \xrightarrow{\text{refl}^\text{alg}_Y} (M \circ F_{bc})(Y) \]

\[ \downarrow f \]

\[ X \xrightarrow{\text{refl}^\text{alg}_X} (M \circ F_{bc})(X) \]

in \( \text{Top} \) is commutative.

Write \( A := F_{bc}(X) \), \( B := F_{bc}(Y) \) and \( \phi := F_{bc}(f) : A \to B \). Take a point \( y \in Y \), and define \( x := f(y) \in X \), \( n := \text{refl}^\text{alg}_y(y) \in M(B) \) and \( m := M(\phi)(n) = \phi^{-1}(n) \in M(A) \).

We need to prove that \( m = \text{refl}^\text{alg}_X(x) \). By definition we have \( n = \text{Ker}(ev_y) \subseteq B \), and also \( \phi(a)(y) = a(f(y)) = a(x) \). Now we calculate:

\[ m = \phi^{-1}(n) = \phi^{-1}(\text{Ker}(ev_y)) = \text{Ker}(ev_y \circ \phi) = \{ a \in A \mid \phi(a)(y) = 0 \} \]

\[ = \{ a \in A \mid a(x) = 0 \} = \text{Ker}(ev_x) = \text{refl}^\text{alg}_X(x) \]
as required. □

Lemma 4.9. Let $X$ be a compact topological space, and define $A := F_{bc}(X, \mathbb{R})$.

1. Take $a \in A$ and $x \in X$. Let $m := \text{refl}^\text{alg}_X(x) \in \text{MSpec}(A)$. Then there is equality $\text{dev}_A(a)(m) = a(x)$ in $\mathbb{R}$.

2. For every $a \in A$ the function $\text{dev}_A(a) : \text{MSpec}(A) \to \mathbb{R}$ is continuous (for the norm topology of $\mathbb{R}$) and bounded; i.e. $\text{dev}_A(a) \in F_{bc}(\text{MSpec}(A), \mathbb{R})$.

3. The $\mathbb{R}$-ring homomorphism $\text{dev}_A : A \to F_{bc}(\text{MSpec}(A), \mathbb{R})$ is bijective.

Proof. (1) Let $\mu := \text{ev}_m(a) = \text{dev}_A(a)(m) \in \mathbb{R}$ and $\nu := \text{ev}_x(a) = a(x) \in \mathbb{R}$. We need to prove that $\mu = \nu$.

Recall that $m = \text{Ker}(\text{ev}_x)$, so there is a unique $\mathbb{R}$-ring isomorphism $\psi : A/m \cong \mathbb{R}$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{pr}_m} & A/m \\
\downarrow{\psi} & & \downarrow{\text{ev}_x} \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]

in $\text{Rng}/\mathbb{R}$ is commutative. Then $\psi$ satisfies $\psi(a + m) = \text{ev}_x(a) = \nu$. But by formula (2.12) we know that $\text{str}_{A/m}(\mu) = a + m \in A/m$, so $\psi(a + m) = (\psi \circ \text{str}_{A/m})(\mu) = \mu$.

(2) Item (1) says that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{refl}^\text{alg}_X} & \text{MSpec}(A) \\
\downarrow{a} & & \downarrow{\text{dev}_A(a)} \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]

in $\text{Set}$ is commutative. Therefore $\text{dev}_A(a) = a \circ (\text{refl}^\text{alg}_X)^{-1}$. By Lemma 4.3(2) the map $(\text{refl}^\text{alg}_X)^{-1}$ is continuous, and by definition $a$ is bounded continuous. Hence $\text{dev}_A(a)$ is bounded continuous.

(3) The commutative diagram (4.11) shows that the function $\text{dev}_A$ is bijective, with inverse $\tilde{a} \mapsto \text{refl}^\text{alg}_X \circ \tilde{a}$. □

Lemma 4.12. Let $A \in \text{Rng}_{bc}\mathbb{R}$ and $X := \text{MSpec}(A)$.

1. The topological space $X$ is compact.

2. For every $a \in A$ the function $\text{dev}_A(a) : X \to \mathbb{R}$ is continuous (for the norm topology of $\mathbb{R}$) and bounded.

3. The $\mathbb{R}$-ring homomorphism $\text{dev}_A : A \to F_{bc}(X, \mathbb{R})$ is bijective.

Proof. (1) We can assume that $A = F_{bc}(Y, \mathbb{R})$ for some compact topological space $Y$. By Lemma 4.3(2) there is a homeomorphism $\text{refl}^\text{alg}_Y : Y \cong X = \text{MSpec}(A)$. So $X$ is compact.

(2, 3) Let’s introduce the abbreviations $M := \text{MSpec}$, $F := F(-, \mathbb{R})$ and $F_{bc} := F_{bc}(-, \mathbb{R})$. Choose a ring isomorphism $\phi : A \cong B$, where $B = F_{bc}(Y)$ for some compact topological space $Y$. By Proposition 4.6 we know that $M : (\text{Rng}_{bc}\mathbb{R})^{op} \to$
Top is a functor. There are also the functors $F_{bc}, F : \text{Top}^{\text{op}} \to \text{Rng/R}$, and $F_{bc} \subseteq F$ is a subfunctor. According to Proposition 2.13 there is a morphism $\text{dev} : \text{Inc} \to F \circ M$ of functors $\text{Rng/}_{bc}\text{R} \to \text{Rng/R}$, where $\text{Inc} : \text{Rng/}_{bc}\text{R} \to \text{Rng/R}$ is the inclusion functor. Therefore we have this solid commutative diagram

\[ (\text{F}_{bc} \circ M)(A) \xrightarrow{\text{inc}} (\text{F} \circ M)(A) \xrightarrow{(\text{F} \circ M)(\phi)} (\text{F} \circ M)(B) \xrightarrow{\text{inc}} (\text{F}_{bc} \circ M)(B) \]

in $\text{Rng/R}$. The arrows marked inc are inclusions of rings. Lemma 4.9 says that the homomorphism $\text{dev}_{B}$ on the dashed arrow going down from $B$ exists, and it is an isomorphism. An easy diagram chase shows the homomorphism $\text{dev}_{A}$ on the dashed arrow going down from $A$ also exists, it is an isomorphism, and the whole diagram (4.13) is commutative. □

Remark 4.14. The continuity in Lemmas 4.9(2) and 4.12(2) seems to be quite rare. When $A$ is an arbitrary $\mathbb{R}$-valued $\mathbb{R}$-ring, there is no reason for the functions $\text{dev}_{A}(a) : \text{MSpec}(A) \to \mathbb{R}$, with $a \in A$, to be continuous (for the Zariski topology on $\text{MSpec}(A)$ and the norm topology on $\mathbb{R}$). Below is a counterexample.

Example 4.15. Take the $\mathbb{R}$-ring $B$ from Example 2.4. We showed there that $B$ is an $\mathbb{R}$-valued ring. The element $t \in B$ gives a bijection of sets $\text{dev}_{B}(t) : \text{MSpec}(B) \to \mathbb{R}$, but the map $\text{dev}_{B}(t)$ is not continuous.

Theorem 4.16 (Duality). The functor

\[ F_{bc}(-, \mathbb{R}) : (\text{Top}_{cp})^{\text{op}} \to \text{Rng/}_{bc}\text{R} \]

is an equivalence of categories, with quasi-inverse $\text{MSpec}$.

Proof. Let’s use the abbreviations $M := \text{MSpec}$ and $F_{bc} := F_{bc}(-, \mathbb{R})$. Lemma 4.7 gives a functorial map $\text{refl}_{X} : X \to (M \circ F_{bc})(X)$ in $\text{Top}$ for every $X \in \text{Top}_{cp}$, and by Lemma 4.3(2) this is a homeomorphism, i.e. an isomorphism in $\text{Top}$.

In the reverse direction, the functor $M : (\text{Rng/}_{bc}\text{R})^{\text{op}} \to \text{Top}_{cp}$ exists by Proposition 4.6 and Lemma 4.12(1). For every $A \in \text{Rng/}_{bc}\text{R}$, Lemma 4.12(3) gives a ring isomorphism $\text{dev}_{A} : A \to (F_{bc} \circ M)(A)$, and by Proposition 2.13 this isomorphism is functorial in $A$. □

5. THE STONE-ČECH COMPACTIFICATION

There are several distinct definitions of Stone-Čech Compactification (SCC) in the literature. We prefer Definition 5.1 below, which is the most general; it is stated, with a proof of existence, in [SP, Lemma tag = 0908]. Weaker definitions and results can be found in the books [CI], [Wa] and [Ke]. See Remark 5.21 for a brief discussion.
The uniqueness of SCC, as it is defined in Definition 5.1, is trivial, and so is its functoriality (Proposition 5.3). The interesting aspect is proving existence. In this section we give an algebraic proof of the existence of the SCC (Theorem 5.19). We also prove Theorem 5.9 regarding the functor $F_{bc}(-, \mathbb{R})$.

Recall that in this paper a topological space $X$ is called compact if it is Hausdorff and quasi-compact. The category of topological spaces and continuous maps is denoted by $\text{Top}$, and its full subcategory on the compact spaces is denoted by $\text{Top}_{cp}$.

**Definition 5.1.** Let $X$ be a topological space. A **Stone-Čech Compactification** of $X$ is a pair $(\bar{X}, c_X)$, consisting of a compact topological space $\bar{X}$ and a continuous map $c_X : X \to \bar{X}$, such that the following universal property holds:

(C) Given a continuous map $f : X \to Y$, where the target $Y$ is a compact topological space, there is a unique continuous map $\bar{f} : \bar{X} \to Y$ such that $f = \bar{f} \circ c_X$.

The commutative diagram below, in the category $\text{Top}$, illustrates the universal property (C) in Definition 5.1.

\begin{equation}
\begin{array}{ccc}
X & \xrightarrow{c_X} & \bar{X} \\
\downarrow{f} & & \downarrow{\bar{f}} \\
Y & & \\
\end{array}
\end{equation}

As we mentioned above, it is clear that if a Stone-Čech Compactification $(\bar{X}, c_X)$ of a space $X$ exists, then it is unique up to a unique isomorphism. The next proposition is also immediate from the definition.

**Proposition 5.3.** Assume that every topological space $X$ admits a Stone-Čech Compactification $(\bar{X}, c_X)$. Then:

1. There is a functor $\text{SCC} : \text{Top} \to \text{Top}_{cp}$, sending an object $X \in \text{Top}$ to its Stone-Čech Compactification $\text{SCC}(X) := \bar{X}$, and sending a map $f : X \to Y$ in $\text{Top}$ to the unique map $\text{SCC}(f) : \text{SCC}(X) \to \text{SCC}(Y)$ in $\text{Top}_{cp}$ such that the diagram

\begin{equation}
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{c_X} & & \downarrow{c_Y} \\
\text{SCC}(X) & \xrightarrow{\text{SCC}(f)} & \text{SCC}(Y) = \bar{Y} \\
\end{array}
\end{equation}

in $\text{Top}$ is commutative.

2. A topological space $X$ is compact if and only if $c_X : X \to \bar{X}$ is a homeomorphism.

Recall that for a topological space $X$ we denote by $F_{bc}(X, \mathbb{R})$ the ring of bounded continuous functions $a : X \to \mathbb{R}$, where $\mathbb{R}$ has its usual norm topology. As $X$ changes this becomes a functor

\begin{equation}
F_{bc}(-, \mathbb{R}) : \text{Top}^{op} \to \text{Rng}/\mathbb{R}.
\end{equation}

The next definition is very similar to material in [Ke, page 152].
Definition 5.5. Let $X$ be some nonempty topological space, and let $A := F_{bc}(X, \mathbb{R})$.

(1) Put on $\mathbb{R}$ its usual norm topology. For each $a \in A$ let $Z_a$ be the smallest closed interval in $\mathbb{R}$ containing the set $a(X)$. Give $Z_a$ the induced subspace topology from $\mathbb{R}$.

(2) Define the topological space $\tilde{X}^{\text{top}} := \prod_{a \in A} Z_a$, with the product topology.

(3) Define the map of sets $\text{refl}^{\text{top}}_X : X \to \tilde{X}^{\text{top}}$ to be $\text{refl}^{\text{top}}_X(a) := \{a(x)\}_{a \in A}$.

(4) Define the topological subspace $\tilde{X}^{\text{top}} \subseteq \tilde{X}^{\text{top}}$ to be the closure of $\text{refl}^{\text{top}}_X(X)$, with its induced subspace topology.

(5) Let $\text{refl}^{\text{top}}_X : X \to \tilde{X}^{\text{top}}$ be the induced map of sets. We call it the topological reflection map.

Observe the similarity between the topological reflection map $\text{refl}^{\text{top}}_X$ defined above and the algebraic reflection map $\text{refl}^{\text{alg}}_X$ from Definition 3.8.

Lemma 5.6. Let $X$ be a nonempty topological space.

(1) The topological space $\tilde{X}^{\text{top}}$ is compact.

(2) The topological reflection map $\text{refl}^{\text{top}}_X : X \to \tilde{X}^{\text{top}}$ is continuous.

(3) The subset $\text{refl}^{\text{top}}_X(X) \subseteq \tilde{X}^{\text{top}}$ is dense.

Proof. (1) Since $\tilde{X}^{\text{top}}$ is a product of compact topological spaces, it is compact by the Tychonoff Theorem (see [Ke, Theorem 5.13]). Therefore the closed subspace $\tilde{X}^{\text{top}} \subseteq \tilde{X}^{\text{top}}$ is also compact.

(2) For every $a \in A$ let $p_a : \tilde{X}^{\text{top}} \to Z_a$ be the projection on the factor indexed by $a$. Then $p_a \circ \text{refl}^{\text{top}}_X = a : X \to \mathbb{R}$ is continuous. By the universal property of the product topology the map $\text{refl}^{\text{top}}_X : X \to \tilde{X}^{\text{top}}$ is continuous. But $\tilde{X}^{\text{top}}$ has the subspace topology induced from $\tilde{X}^{\text{top}}$, and therefore the map $\text{refl}^{\text{top}}_X : X \to \tilde{X}^{\text{top}}$ is continuous.

(3) This immediate from the definition of $\tilde{X}^{\text{top}}$. \qed

Here is the key lemma of this section.

Lemma 5.7. For every nonempty topological space $X$, the $\mathbb{R}$-ring homomorphism

$$F_{bc}(\text{refl}^{\text{top}}_X, \mathbb{R}) : F_{bc}(\tilde{X}^{\text{top}}, \mathbb{R}) \to F_{bc}(X, \mathbb{R})$$

is an isomorphism.

The lemma makes sense because $\text{refl}^{\text{top}}_X : X \to \tilde{X}^{\text{top}}$ is a map in $\text{Top}$, and $F_{bc}(-, \mathbb{R})$ is a functor, see formula (5.4). See Remark [5.21] regarding the similarity between proof of this lemma and other proofs of existence of SCC.

Proof. We shall use the abbreviations $A := F_{bc}(X, \mathbb{R})$, $\tilde{A} := F_{bc}(\tilde{X}^{\text{top}}, \mathbb{R})$ and $\phi := F_{bc}(\text{dev}^{\text{top}}_X, \mathbb{R}) : \tilde{A} \to A$. We need to prove that the $\mathbb{R}$-ring homomorphism $\phi : \tilde{A} \to A$ is bijective.

First we’ll prove that $\phi$ is injective. Suppose $\tilde{a}_1, \tilde{a}_2 \in \tilde{A}$ satisfy $\phi(\tilde{a}_1) = \phi(\tilde{a}_2)$ in $\mathbb{R}$. This means that their restrictions to the subspace $Y := \text{refl}^{\text{top}}_X(X)$ of $\tilde{X}^{\text{top}}$ satisfy $\tilde{a}_1|_Y = \tilde{a}_2|_Y$ as continuous functions $Y \to \mathbb{R}$, where $\mathbb{R}$ if given its standard norm topology. By Lemma 5.7 the space $Y$ is dense in $\tilde{X}^{\text{top}}$, and therefore $\tilde{a}_1 = \tilde{a}_2$.\]
Now we will prove that $\phi$ is surjective. Take some function $a \in A$. Let $p_a : \tilde{X}^{\text{top}} \to Z_a$ be the projection on the factor indexed by $a$. Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{ref}^\text{top}_X} & \tilde{X}^{\text{top}} \\
\downarrow \alpha & & \downarrow \beta \\
\mathbb{R} & \xleftarrow{g} & Z_a \\
\end{array}
\]

(5.8)

in $\text{Top}$, where $\mathbb{R}$ has its standard norm topology, $f$ and $g$ are the inclusions, and $\tilde{a} := g \circ p_a \circ f$. The square subdiagram is obviously commutative. The definition of $\text{ref}^\text{top}_X$ directly implies that the two outer paths from $X$ to $\mathbb{R}$ are equal. Hence the whole diagram is commutative. We see that $\tilde{a} \circ \text{ref}^\text{top}_X = a$. But $\tilde{a} \circ \text{ref}^\text{top}_X = \phi(\tilde{a})$. \hfill \square

Recall from Definition 3.4 that a BC $\mathbb{R}$-ring is an $\mathbb{R}$-ring that is isomorphic to $F_{bc}(\tilde{X}^{\text{top}}, \mathbb{R})$ for some compact topological space $\tilde{X}$. The next theorem shows that these rings occur in much greater generality.

**Theorem 5.9.** For an arbitrary topological space $X$, the ring $F_{bc}(X, \mathbb{R})$ is a BC $\mathbb{R}$-ring.

**Proof.** We may assume $X$ is a nonempty topological space. According to Lemma 5.7 there is an $\mathbb{R}$-ring isomorphism $F_{bc}(\tilde{X}^{\text{top}}, \mathbb{R}) \cong F_{bc}(X, \mathbb{R})$, and by Lemma 5.6(1) the topological space $\tilde{X}^{\text{top}}$ is compact. Therefore $F_{bc}(X, \mathbb{R})$ is a BC $\mathbb{R}$-ring. \hfill \square

The category $\text{Rng}/_{bc} \mathbb{R}$ is the full subcategory of $\text{Rng}/\mathbb{R}$ on the BC rings. Proposition 4.6 says that there is a functor

\[
M\text{Spec} : (\text{Rng}/_{bc} \mathbb{R})^{\text{op}} \to \text{Top}.
\]

According to Theorem 4.16 the functor

\[
F_{bc}(\cdot, \mathbb{R}) : (\text{Top}_{\text{op}})^{\text{op}} \to \text{Rng}/_{bc} \mathbb{R}
\]

is an equivalence of categories, with quasi-inverse $M\text{Spec}$. An immediate consequence of Theorem 5.9 is:

**Corollary 5.12.** The functor $F_{bc}(\cdot, \mathbb{R})$ from formula (5.4) factors through, and the functor $F_{bc}(\cdot, \mathbb{R})$ from formula (5.11) extends to, a functor

\[
F_{bc}(\cdot, \mathbb{R}) : \text{Top}^{\text{op}} \to \text{Rng}/_{bc} \mathbb{R}.
\]

**Corollary 5.13.** Let $A$ be an $\mathbb{R}$-ring, with $X := M\text{Spec}(A)$. The following two conditions are equivalent:

(i) $A$ is a BC $\mathbb{R}$-ring.

(ii) $A$ is an $\mathbb{R}$-valued ring, for every $a \in A$ the function $\text{dev}_A(a) : X \to \mathbb{R}$ is continuous, and the double evaluation $\mathbb{R}$-ring homomorphism $\text{dev}_A : A \to F_c(X, \mathbb{R})$ is bijective.

**Proof.**

(i) ⇒ (ii): By Lemma 4.12(1) the topological space $X$ is compact, and therefore $F_{bc}(X, \mathbb{R}) = F_c(X, \mathbb{R})$. Lemma 4.4 says that $A$ is an $\mathbb{R}$-valued ring. According to Lemma 4.12(2), for every $a \in A$ the function $\text{dev}_A(a) : X \to \mathbb{R}$ is continuous. Finally, by Lemma 4.12(3) the ring homomorphism $\text{dev}_A : A \to F_{bc}(X, \mathbb{R}) = F_c(X, \mathbb{R})$ is bijective.
(ii) \( \Rightarrow \) (i): Proposition 1.13 says that the topological space \( X \) is quasi-compact. This implies that \( \mathcal{F}_{bc}(X, \mathbb{R}) = \mathcal{F}_c(X, \mathbb{R}) \), and hence there is an isomorphism \( \text{dev}_A : A \xrightarrow{\sim} \mathcal{F}_{bc}(X, \mathbb{R}) \) of \( \mathbb{R} \)-rings. According to Theorem 5.9, \( A \) is a BC \( \mathbb{R} \)-ring. □

Example 5.14. Take the \( \mathbb{R} \)-ring \( B \) from Examples 2.4 and 4.15. It is an \( \mathbb{R} \)-valued ring. Yet for the element \( t \in B \) the map \( \text{dev}_B(t) : \text{MSpec}(B) \to \mathbb{R} \) is not continuous. Therefore \( B \) is not a BC \( \mathbb{R} \)-ring.

Let us denote by \( \text{Inc} : \text{Top}_{cp} \to \text{Top} \) the inclusion functor. The algebraic reflection map

\[
(5.15) \quad \text{refl}_{X}^{\text{alg}} : X \to \text{MSpec}(\mathcal{F}_{bc}(X, \mathbb{R}))
\]

for an arbitrary topological space \( X \), was introduced in Definition 3.8. By Lemma 4.7, as \( X \) moves in \( \text{Top}_{cp} \), this becomes a morphism

\[
(5.16) \quad \text{refl}_{X}^{\text{alg}} : \text{Inc} \to \text{MSpec} \circ \mathcal{F}_{bc}(\cdot, \mathbb{R})
\]

of functors \( \text{Top}_{cp} \to \text{Top} \). There is also the identity functor \( \text{Id} : \text{Top} \to \text{Top} \). By Corollary 5.12 and Proposition 4.6 there is a functor

\[
(5.17) \quad \text{MSpec} \circ \mathcal{F}_{bc}(\cdot, \mathbb{R}) : \text{Top} \to \text{Top}.
\]

Lemma 5.18. The algebraic reflection map \( \text{refl}_{X}^{\text{alg}} \) from formula (5.15) is functorial in \( X \in \text{Top} \). Therefore the morphism of functors \( \text{refl}_{X}^{\text{alg}} \) in formula (5.16) extends to a morphism

\[
\text{refl}_{X}^{\text{alg}} : \text{Id} \to \text{MSpec} \circ \mathcal{F}_{bc}(\cdot, \mathbb{R})
\]

of functors \( \text{Top} \to \text{Top} \).

Proof. Given a map \( f : Y \to X \) in \( \text{Top} \), we must prove that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{refl}^{\text{alg}}_X} & (\text{M} \circ \mathcal{F}_{bc})(Y) \\
\downarrow f & & \downarrow (\text{M} \circ \mathcal{F}_{bc})(f) \\
X & \xrightarrow{\text{refl}^{\text{alg}}_X} & (\text{M} \circ \mathcal{F}_{bc})(X)
\end{array}
\]

in \( \text{Top} \) is commutative. The proof of Lemma 4.7 works here without any modification, except that now we know that \( \text{M} \circ \mathcal{F}_{bc} \) is a functor from \( \text{Top} \) to itself (i.e. \( X \) and \( Y \) do not need to be compact). □

The functor \( \mathcal{F}_{bc}(\cdot, \mathbb{R}) \) in Corollary 5.12 is not an equivalence, but it does provide an algebraic construction of the SCC.

Theorem 5.19 (Algebraic SCC). Given a topological space \( X \), consider the topological space \( \tilde{X}^{\text{alg}} := \text{MSpec}(\mathcal{F}_{bc}(X, \mathbb{R})) \) and the algebraic reflection map \( \text{refl}^{\text{alg}}_X : X \to \tilde{X}^{\text{alg}} \) from Definition 3.8. Then the pair (\( \tilde{X}^{\text{alg}} \), \( \text{refl}^{\text{alg}}_X \)) is a Stone-Čech Compactification of \( X \).

Proof. We will use the abbreviations \( \text{M} := \text{MSpec} \) and \( \text{F}_{bc} := \mathcal{F}_{bc}(\cdot, \mathbb{R}) \). We may assume \( X \) is nonempty.

By Proposition 3.11(1) the map \( \text{refl}^{\text{alg}}_X : X \to \tilde{X}^{\text{alg}} \) is continuous. By Theorem 5.9 the ring \( A := \mathcal{F}_{bc}(X) \) belongs to \( \text{Rng}^{bc} \mathbb{R} \), and therefore by Lemma 4.12(1) the topological space \( \tilde{X}^{\text{alg}} = \text{M}(A) \) is compact.
We need to verify that the pair \((\tilde{X}^{\text{alg}}, \text{refl}_{\tilde{X}}^{\text{alg}})\) has the universal property (C) from Definition 5.1. Suppose \(f : X \to Y\) is a map in \(\text{Top}\) with compact target \(Y\). Due to Proposition 2.8 and Corollary 5.12 the map

\[(M \circ F_{bc})(f) : (M \circ F_{bc})(X) \to (M \circ F_{bc})(Y)\]

exists in \(\text{Top}\), see formula (5.17). Consider the following solid diagram in \(\text{Top}\).

\[
\begin{array}{ccc}
X & \xrightarrow{\text{refl}_{\tilde{X}}^{\text{alg}}} & \tilde{X}^{\text{alg}} = (M \circ F_{bc})(X) \\
\downarrow f & & \downarrow (M \circ F_{bc})(f) \\
Y & \xleftarrow{\text{refl}_{\tilde{X}}^{\text{alg}}} & (M \circ F_{bc})(Y)
\end{array}
\]

By Lemma 5.18 this is a commutative diagram, and by Lemma 4.3 (2) the map \(\text{refl}_{Y}^{\text{alg}}\) is an isomorphism. The map

\[\tilde{f} := (\text{refl}_{Y}^{\text{alg}})^{-1} \circ (M \circ F_{bc})(f) : \tilde{X}^{\text{alg}} \to Y\]

in \(\text{Top}\) satisfies \(\tilde{f} \circ \text{refl}_{\tilde{X}}^{\text{alg}} = f\). Such a map \(\tilde{f}\) is unique because \(\text{refl}_{X}^{\text{alg}}(X)\) is dense in \(\tilde{X}^{\text{alg}}\), see Proposition 3.11 (2), and \(Y\) is Hausdorff. \(\square\)

**Remark 5.20.** It is possible to give a more abstract (but more complicated) proof that \((\tilde{X}^{\text{alg}}, \text{refl}_{\tilde{X}}^{\text{alg}})\) has the universal property (C) from Definition 5.1. One first shows that the pair \((M \circ F_{bc}, \text{refl}_{\tilde{X}}^{\text{alg}})\) is a projector, also called an *idempotent pointed functor* and an *idempotent monad*. The idempotence is proved just like in the proof of Theorem 5.19. Then the categorical properties of projectors can be used. See [KS, Section 4.1] or [VY, Section 2] for details on this abstract approach.

**Remark 5.21.** The existence of an SCC in the strongest sense, namely the one in Definition 5.1 is proved in [SP, Lemma tag=0908]. Textbooks on topology, such as [GJ], [Wa] and [Ke], provide proofs of special cases only; for example [Ke, Theorem 5.24] only considers the SCC of a Tychonoff space.

All proofs we are aware of, namely those mentioned above, rely on variations of the same idea; in our paper this idea occurs in the proof of the Lemma 5.7.

6. **Real Banach Rings**

In this section we study *Banach* \(\mathbb{R}\)-*rings*, better known as *commutative unital* \(C^\ast\) \(\mathbb{R}\)-*algebras*. In Corollary 6.10 we prove that the forgetful functor is an equivalence from the category of *Banach* \(\mathbb{R}\)-*rings* to the category of BC \(\mathbb{R}\)-*rings*. Our source for the material here is the book [Go].

By a *norm* on an \(\mathbb{R}\)-module \(M\) we mean a function \(\| - \| : M \to \mathbb{R}\) satisfying the following conditions for all \(m, n \in M\) and \(\lambda \in \mathbb{R}\): nonnegativity, \(\| m \| \geq 0\); nondegeneracy, \(\| m \| = 0\) iff \(m = 0\); the triangle inequality, \(\| m + n \| \leq \| m \| + \| n \|\); and the homothety equality, \(\| \lambda \cdot m \| = |\lambda| \cdot \| m \|\). The norm induces a metric on \(M\) in the obvious way. A normed \(\mathbb{R}\)-module \(M\) is called a *Banach \mathbb{R}\-module* if \(M\) is complete as a metric space. Of course all textbooks will say "\(\mathbb{R}\)-vector space" whenever we say "\(\mathbb{R}\)-module", and "\(\mathbb{R}\)-algebra" whenever we say "\(\mathbb{R}\)-ring".
Definition 6.1. A commutative Banach $\mathbb{R}$-ring is a commutative $\mathbb{R}$-ring $A$, equipped with a norm $\|\cdot\| : A \to \mathbb{R}$, making $A$ into a Banach $\mathbb{R}$-module. Moreover, the norm must satisfy these extra conditions: $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$, and $\|1_A\| = 1$, where $1_A \in A$ is the unit element.

Definition 6.2. A commutative Banach $^\ast \mathbb{R}$-ring is a commutative Banach $\mathbb{R}$-ring $A$, such that for every $a \in A$ there is equality $\|a^2\| = \|a\|^2$, and the element $1 + a^2$ is invertible in $A$.

Definition 6.3. Let $A$ and $B$ be commutative Banach $\mathbb{R}$-rings. A Banach $\mathbb{R}$-ring homomorphism is an $\mathbb{R}$-ring homomorphism $\phi : A \to B$ such that $\|\phi(a)\| \leq \|a\|$ for every element $a \in A$.

The category of commutative Banach $\mathbb{R}$-rings, with Banach $\mathbb{R}$-ring homomorphisms, is denoted by $\mathbf{BaRng}/\mathbb{R}$. The full subcategory of $\mathbf{BaRng}/\mathbb{R}$ on the Banach $^\ast \mathbb{R}$-rings is denoted by $\mathbf{BaRng}^\ast/\mathbb{R}$.

Remark 6.4. Definition 6.2 is a slight modification of the definition of a commutative real $C^\ast$ algebra from [Go, Chapter 8]. The definition in [Go] allows the ring $A$ to have an involution $\gamma$; here the involution $\gamma$ is trivial. We think that the case of a nontrivial involution, and the corresponding Arens-Kaplansky Theorem (see [Go, Chapter 12]), should be treated as a noncommutative case.

Example 6.5. Suppose $X$ is a compact topological space. Then the ring $A := F_c(X, \mathbb{R})$, endowed with the sup norm, is a Banach $^\ast \mathbb{R}$-ring.

The key classical result on Banach $^\ast \mathbb{R}$-rings is the next one. It is part of the more general Arens-Kaplansky Theorem, see [Go, Theorem 12.5].

Theorem 6.6 ([Go, Theorem 11.5]). Let $A$ be a commutative Banach $^\ast \mathbb{R}$-ring.

1. The set $X$ of complex characters of $A$, suitably topologized, is compact.
2. The Gelfand transform $A \to F_c(X, \mathbb{R})$ is an isomorphism of Banach $^\ast \mathbb{R}$-rings.

Corollary 6.7. Let $A$ be a commutative Banach $^\ast \mathbb{R}$-ring. Then, after forgetting the norm, $A$ is a BC $\mathbb{R}$-ring.

Proof. This is immediate from Theorem 6.6. □

Theorem 6.8. Every BC $\mathbb{R}$-ring $A$ admits a unique norm $\|\cdot\|$, called the canonical norm, satisfying the three conditions below.

(i) Given a homomorphism $\phi : A \to B$ in $\mathbf{Rng}/_{bc} \mathbb{R}$, and an element $a \in A$, the inequality $\|\phi(a)\| \leq \|a\|$ holds.
(ii) For the $\mathbb{R}$-ring $F_{bc}(X, \mathbb{R})$ of bounded continuous functions on a topological space $X$, the canonical norm is the sup norm.
(iii) The canonical norm makes $A$ into a Banach $^\ast \mathbb{R}$-ring.

Note that $F_{bc}(X, \mathbb{R})$ is indeed a BC $\mathbb{R}$-ring, by Theorem 5.9.

Proof. Given an arbitrary BC $\mathbb{R}$-ring $A$, let $X := \text{MSpec}(A)$ and $B := F_{bc}(X, \mathbb{R})$. By Lemma 4.12(3) the $\mathbb{R}$-ring homomorphism $\text{dev}_A : A \to F_{bc}(X, \mathbb{R}) = B$ is bijective. The ring $B$ has the norm imposed by condition (ii), and it is a Banach $^\ast \mathbb{R}$-ring; cf. Example 6.5. The canonical norm of $A$ is defined to be the unique norm such that
dev\_A : A \to B is an isomorphism of Banach\* R-rings. Conditions (i) and (ii) force this norm of A to be unique.

We need to prove functoriality, i.e. that condition (i) holds for an arbitrary homomorphism \( \phi : A \to B \) in \( \text{Rng}_{bc}^\bullet \). Define the topological spaces \( X := \text{MSpec}(A) \) and \( Y := \text{MSpec}(B) \). In view of Lemma 4.4 and Proposition 2.8 there is an induced map \( f := \text{MSpec}(\phi) : Y \to X \) in \( \text{Top} \). Consider the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow_{\text{dev}_A} & & \downarrow_{\text{dev}_B} \\
\text{F}_{bc}(X, \mathbb{R}) & \xrightarrow{\text{F}_{bc}(f, \mathbb{R})} & \text{F}_{bc}(Y, \mathbb{R}) \\
\downarrow_{\epsilon_X} & & \downarrow_{\epsilon_Y} \\
\text{F}(X, \mathbb{R}) & \xrightarrow{\text{F}(f, \mathbb{R})} & \text{F}(Y, \mathbb{R})
\end{array}
\]

in \( \text{Rng}/\mathbb{R} \). The vertical arrows marked \( \epsilon_X \) and \( \epsilon_Y \) are the inclusions, and the bottom square is commutative, since \( \text{F}_{bc}(-, \mathbb{R}) \) is a subfunctor of \( \text{F}(-, \mathbb{R}) \). The two outer paths from \( A \) to \( \text{F}(Y, \mathbb{R}) \) are equal, by Proposition 2.13. Because \( \epsilon_Y \) is an injection, the top square is commutative too: \( \text{F}_{bc}(f, \mathbb{R}) \circ \text{dev}_A = \text{dev}_B \circ \phi \). The \( \mathbb{R} \)-ring homomorphism \( \text{F}_{bc}(f, \mathbb{R}) \) respects the sup norms on the function rings, and by construction the \( \mathbb{R} \)-ring isomorphisms \( \text{dev}_A : A \xrightarrow{\sim} \text{F}_{bc}(X, \mathbb{R}) \) and \( \text{dev}_B : B \xrightarrow{\sim} \text{F}_{bc}(Y, \mathbb{R}) \) respect the canonical norms on these rings. It follows that \( \phi : A \to B \) respects the canonical norms. \( \square \)

**Corollary 6.10.** The forgetful functor

\[
F : \text{BaRng}\^\bullet /\mathbb{R} \to \text{Rng}_{bc}^\bullet
\]

is an equivalence of categories.

**Proof.** Given a BC R-ring \( B \), the canonical norm of \( B \) makes it into a Banach\* R-ring, which we denote by \( G(B) \). Condition (i) in Theorem 6.8 says that

\[
G : \text{Rng}_{bc}^\bullet \to \text{BaRng}\^\bullet /\mathbb{R}
\]

is a functor. It is clear that \( F \circ G = \text{id} \). A short calculation using the conditions in Theorem 6.8 shows that \( G \circ F = \text{id} \) too. \( \square \)

### 7. Stone Spaces

Recall that a topological space \( X \) is called a Stone space if it is compact and totally disconnected (i.e. the only connected subsets of \( X \) are the singletons). It is known that the category of Stone spaces is dual to the category of boolean rings. This is called Stone duality, see [Jo, Corollary II.4.4], and it is very similar to Theorem 4.16 above. In this section we are going to study Stone spaces via their rings of continuous \( \mathbb{R} \)-valued functions.

A topological space \( X \) called a profinite space if \( X \cong \lim_{\rightarrow} X_i \), where \( \{X_i\}_{i \in I} \) is an inverse system of finite discrete spaces, and the inverse limit is taken in \( \text{Top} \).

First we need to quote the next classical theorem. For proofs see [BI, Theorem 3.4.7] or [SP, Lemma tag = 08ZY].
Theorem 7.1. The following conditions are equivalent for a topological space $X$.

(i) $X$ is a Stone space.

(ii) $X$ is a profinite space.

Suppose $X$ is a topological space and $Y \subseteq X$ is a subset. The indicator function of $Y$ is the function $1_Y : X \to \mathbb{R}$ given by the rule $1_Y(x) := 1$ if $x \in Y$ and $1_Y(x) := 0$ if $x \notin Y$.

Lemma 7.2. Let $X$ be a topological space and $e : X \to \mathbb{R}$ a function. The following conditions are equivalent.

(i) $e$ is an idempotent, i.e. $e^2 = e$, and it is continuous.

(ii) $e = 1_Y$ for some open-closed subset $Y \subseteq X$.

Proof. (i) $\Rightarrow$ (ii): An idempotent $\mathbb{R}$-valued function must take the values 1 and 0 only. Letting $Y := \{x \in X \mid e(x) = 1\}$ we see that $e = 1_Y$. Since $e$ and $1_X - e$ are continuous, the set $Y = \text{NZer}_X(e) = \text{Zer}_X(1_X - e)$ is open-closed.

(ii) $\Rightarrow$ (i): The function $1_Y \in A$ is idempotent and continuous. \qed

Definition 7.3. Let $X$ be a topological space and $A := \mathbb{F}_{bc}(X, \mathbb{R})$.

(1) A function $a \in A$ is called a step function if $a$ takes only finitely many values.

(2) The set of step functions in $A$ is denoted by $A_{\text{stp}}$.

It is easy to see that $A_{\text{stp}}$ is an $\mathbb{R}$-subring of $A$.

Lemma 7.4. Let $X$ be a topological space and $A := \mathbb{F}_{bc}(X, \mathbb{R})$. The following conditions are equivalent for $a \in A$.

(i) $a$ is a step function.

(ii) $a = \sum \lambda_i \cdot e_i$, a finite sum, with $e_i \in A$ idempotents and $\lambda_i \in \mathbb{R}$.

(iii) $a$ factors through a finite discrete topological space $Z$.

Proof. (i) $\Rightarrow$ (iii): Let $Z := a(x) \subseteq \mathbb{R}$. Then $Z$, with the induced topology from $\mathbb{R}$, is a finite discrete space.

(iii) $\Rightarrow$ (ii): Suppose $a = b \circ f$ with $f : X \to Z$ continuous. Letting $U_z := f^{-1}(z) \subseteq X$, we obtain a finite covering $X = \bigcup_{z \in Z} U_z$ by pairwise disjoint open-closed subsets. The function $e_z := 1_{U_z}$ is an idempotent element of $A$ by Lemma 7.2 and $a = \sum_{z \in Z} b(z) \cdot e_z$.

(ii) $\Rightarrow$ (i): The image of $a$ is contained in the finite set $\{\lambda_i\} \subseteq \mathbb{R}$. \qed

Definition 7.5. Let $X$ be a topological space, and let $f : X \to Z$ be a continuous map to a finite discrete topological space $Z$. For every $z \in Z$ let $V_z := f^{-1}(z)$. Then $V := \{V_z\}_{z \in Z}$ is an open covering of $Z$, and we call it the covering induced by $f : X \to Z$.

Observe that an open covering $\{V_z\}_{z \in Z}$ of $X$ induced by a map to a finite discrete space $Z$ is the same as a finite covering $\{V_i\}_{i \in I}$ of $X$ such that each $V_i$ is open and closed, and such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Such coverings are considered in [SP].

Lemma tag = [08ZZ].
Let $X$ be a topological space, and let $\mathbf{U} = \{U_i\}_{i \in I}$ and $\mathbf{V} = \{V_i\}_{i \in I}$ be open coverings of $X$. We say that $\mathbf{V}$ is a refinement of $\mathbf{U}$ if there is a function $\rho : I \to I$ such that $V_j \subseteq U_{\rho(j)}$ for every $j \in J$.

**Lemma 7.6.** Let $X$ be a Stone topological space, and let $\mathbf{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. Then there is an open covering $\mathbf{V}$ of $X$ that refines $\mathbf{U}$, and $\mathbf{V}$ is induced by a continuous map $\mathbf{X} \to \mathbf{Z}$ to a finite discrete topological space $\mathbf{Z}$.

**Proof.** Our proof is a slight modification of the proof of [SP Lemma tag=08Z2]. According to Theorem 7.1 there is a homeomorphism $X \cong \lim_{\to} X_k$, where $\{X_k\}_{k \in K}$ is an inverse system of finite discrete spaces. Let $f_k : X \to X_k$ be corresponding maps. We will prove that for some $k_0 \in K$ the covering induced by $f_{k_0} : X \to X_{k_0}$ refines $\mathbf{U}$.

Take some point $x \in X$. Because the space $X$ is homeomorphic to a closed subspace of the product $\prod_{k \in K} X_k$, there is a finite subset $K_x \subseteq K$ and an index $i_x \in I$ such that $x \in \bigcap_{k \in K_x} f_{k_i}^{-1}(f_k(x)) \subseteq U_{i_x}$. But $K$ is a directed set, so there is some $k_x \in K$ that dominates the finite set $K_x$. This means that $x \in f_{k_x}^{-1}(f_k(x)) \subseteq U_{i_x}$.

We see that the collection $\{f_{k_x}^{-1}(f_k(x))\}_{x \in X}$ is an open covering of $X$ that refines $\mathbf{U}$. Because $X$ is compact, we can pass to a subcovering $\{f_{k_x}^{-1}(f_k(x))\}_{x \in X_0}$ indexed by some finite subset $X_0 \subseteq X$.

The finite set $\{k_x\}_{x \in X_0}$ is dominated by some $k_0 \in K$. For every $x \in X_0$ there is equality $f_{k_x}^{-1}(f_k(x)) = \bigcup_{y \in f_{k_0}^{-1}(f_{k_0}(x))} f_{k_0}^{-1}(y) \subseteq X$, where $f_{k_0}/k_x : X_{k_0} \to X_{k_x}$ is the corresponding map. This means that $f_{k_0}(X) \subseteq \bigcup_{x \in X_0} f_{k_0}^{-1}(f_{k_x}(x))$. And that $\{f_{k_0}^{-1}(y)\}_{y \in f_{k_0}(X)}$ is a covering of $X$ refines $\mathbf{U}$. But this covering is precisely the covering induced by $f_{k_0} : X \to X_{k_0}$.

**Theorem 7.7.** Let $A$ be a BC $\mathbf{R}$-ring and $X := \text{MSpec}(A)$. The following two conditions are equivalent:

(i) $X$ is a Stone space.
(ii) The subring $A_\text{stp}$ is dense in $A$, with respect to the canonical norm of $A$.

**Proof.** The topological space $X$ is compact by Theorem 4.16 (or, more precisely, by Lemma 4.17). For convenience we identify the ring $A$ with the ring $F_{bc}(X, \mathbf{R})$, via the $\mathbf{R}$-ring isomorphism $\text{dev}_A$, see Lemma 4.12(3). Thus every element $a \in A$ is seen as a continuous function $a : X \to \mathbf{R}$. The canonical norm on the Banach ring $A$ coincides with the sup norm on the compact space $X$.

(ii) $\Rightarrow$ (i): We already noted that $X$ is compact. It remains to prove that $X$ is totally disconnected. In other words, given a connected subset $Y$ of $X$, we must prove that $Y$ is a singleton.

If $a \in A$ is a step function then $a(X)$ is a finite discrete subspace of $\mathbf{R}$. Because $Y$ is connected, the set $a(Y) \subseteq \mathbf{R}$ has to be a singleton. We conclude that the function $a|_Y$ is constant.

Next take an arbitrary function $a \in A$. Given $\epsilon > 0$, condition (ii) says that there is a step function $a'$ such that $\|a - a'\| < \epsilon$. We already know that $a'|_Y$ is constant. It follows that $a|_Y$ is constant.
By Theorem 4.1, the elements of $A$ separate the points of $X$. We have just shown that every $a \in A$ is constant on $Y$. It follows that $Y$ has only one element.

(i) $\Rightarrow$ (ii): Take an arbitrary $a \in A$ and $e > 0$. We need to produce a step function $a' \in A$ such that $\|a - a'\| < e$.

The function $a$ is bounded. Let $[\lambda_0, \lambda_1]$ be a finite closed interval in $\mathbb{R}$ that contains $a(X)$. Write $\lambda := \lambda_1 - \lambda_0$. Choose $k \in \mathbb{N}$ large enough such that $2^{-k} \cdot \lambda < 2^{-1} \cdot e$. Define the numbers $\mu_i := \lambda_0 + i \cdot 2^{-k} \cdot \lambda \in \mathbb{R}$, so $\mu_0 = \lambda_0$ and $\mu_{2k} = \lambda_1$. Next define the indexing set $I := \{0, \ldots, 2^k\}$. For every $i \in I$ define the open interval $W_i := (\mu_{i-1}, \mu_{i+1})$ in $\mathbb{R}$. Then $[\lambda_0, \lambda_1] \subseteq \bigcup_i W_i$, and the length of $W_i$ is $< e$.

We now move to the space $X$. For every $i \in I$ let $U_i := a^{-1}(W_i) \subseteq X$. The collection of open sets $U := \{U_i\}_{i \in I}$ is an open covering of $X$. According to Lemma 7.6 there is a continuous map $f : X \rightarrow Z$ to a finite discrete space $Z$, such that associated open covering $V := \{V_z\}_{z \in Z}$, where $V_z := f^{-1}(z)$, refines $U$. So there is a function $\rho : Z \rightarrow I$ such that $V_z \subseteq U_{\rho(z)}$ for all $z \in Z$. Let $b : Z \rightarrow \mathbb{R}$ be the function $b(z) := \mu_{\rho(z)}$. Finally let $a' := b \circ f : X \rightarrow \mathbb{R}$. Clearly $a'$ is a step function. For every point $x \in V_z$ we have $a(x), a'(x) \in U_{\rho(z)}$, so $|a(x) - a'(x)| < e$. But $X = \bigcup_{z \in Z} V_z$, so $\|a - a'\| < e$, as required.

The next corollary is classical. However, all the proofs we found in the literature are indirect and quite difficult. We give a straightforward proof, using Theorem 7.7.

**Corollary 7.8.** Suppose $X$ is a discrete topological space, with Stone-Čech Compactification $\hat{X}$. Then $\hat{X}$ is a Stone topological space.

**Proof.** Let’s write $A := F_{bc}(X, \mathbb{R})$. According to Theorem 5.19 we may assume that $\hat{X} = \text{MSpec}(A)$, and the compactification map $c_X : X \rightarrow \hat{X}$ is reflexive. Let us write $\hat{A} := F_{bc}(\hat{X}, \mathbb{R})$. By Theorem 7.7 it is enough to prove that $\hat{A}_{\text{stp}}$ is dense in $\hat{A}$ for its canonical norm, which is the sup norm.

The $\mathbb{R}$-ring homomorphism $\text{dev}_A : A \rightarrow \hat{A}$ is bijective, see Lemma 4.123. Since $c_X(X)$ is dense inside $\hat{X}$, the sup norm of a function $a \in A$, when calculated on $X$, equals the sup norm of $\tilde{a} = \text{dev}_A(a) \in \hat{A}$, when calculated on $\hat{X}$. We conclude that it suffices to prove that $A_{\text{stp}}$ is dense in $A$ for the sup norm on $X$.

Take some function $a \in A$ and some $e > 0$. We need to produce a step function $a'$ such that $\|a - a'\| < e$ in the sup norm on $X$. Now the set $a(X)$ is contained in some bounded closed interval $Z = [\lambda_0, \lambda_1] \subseteq \mathbb{R}$ of length $\lambda := \lambda_1 - \lambda_0$. Take $k \in \mathbb{N}$ sufficiently large such that $2^{-k} \cdot \lambda < e$. Define the numbers $\mu_i := \lambda_0 + i \cdot 2^{-k} \cdot \lambda$, and the intervals $Z_i := [\mu_{i-1}, \mu_i]$ and $Z_i := (\mu_{i-1}, \mu_i)$ for $2 \leq i \leq 2^k$, so $\{Z_i\}_{1 \leq i \leq 2^k}$ is a partition of $Z$ into intervals of length $< e$. Let $a' : X \rightarrow \mathbb{R}$ be the step function that has the value $\mu_i$ on the subset $a^{-1}(Z_i) \subseteq X$. Then $\|a - a'\| < e$ as required.

**Remark 7.9.** Stone spaces have become more interesting recently, since they form the background upon which condensed mathematics is built; see the notes by Scholze. In this theory the Stone-Čech compactifications of discrete spaces play an important role.
8. RINGS OF BOUNDED CONTINUOUS COMPLEX VALUED FUNCTIONS

Here we introduce BC \textbf{C}-rings, which are the complex variant of BC \textbf{R}-rings. The main result is Theorem 9.5: it states that BC \textbf{C}-rings admit canonical involutions. Theorem 9.9 then says that taking canonical hermitian subrings is an equivalence from the category of BC \textbf{C}-rings to the category of BC \textbf{R}-rings.

In analogy to Definition 3.1, given a topological space \( X \), the following \textbf{C}-rings exist: the ring \( F(X, \mathbb{C}) \) of all functions \( a : X \to \mathbb{C} \), the ring \( F_c(X, \mathbb{C}) \) of continuous functions \( a : X \to \mathbb{C} \), and the ring \( F_{bc}(X, \mathbb{C}) \) of bounded continuous functions \( a : X \to \mathbb{C} \). Here continuity and boundedness are with respect to the standard norm on the field \( \mathbb{C} \). As in the real case, the rings \( F(X, \mathbb{C}) \), \( F_c(X, \mathbb{C}) \) and \( F_{bc}(X, \mathbb{C}) \) are not topologized. If \( X \) is a discrete topological space then \( F_c(X, \mathbb{C}) = F(X, \mathbb{C}) \), and if \( X \) is a compact topological space then \( F_{bc}(X, \mathbb{C}) = F_c(X, \mathbb{C}) \).

**Definition 8.1.** A \textbf{C}-ring \( A \) is called a BC \textbf{C}-ring if it is isomorphic, as a \textbf{C}-ring, to the ring \( F_{bc}(X, \mathbb{C}) \) for some compact topological space \( X \). The full subcategory of \( \text{Rng/} \mathbb{C} \) on the BC \textbf{C}-rings is denoted by \( \text{Rng}_{bc} \mathbb{C} \).

Thus, like in the real case, the category \( \text{Rng}_{bc} \mathbb{C} \) is the essential image of the functor \( F_{bc}(-, \mathbb{C}) : (\text{Top}_{\mathbb{C}})^{op} \to \text{Rng/} \mathbb{C} \).

The next definition is the complex analogue of Definitions 3.7 and 3.8.

**Definition 8.2.** Let \( X \) be a topological space.

1. For a point \( x \in X \), let \( \text{ev}_x : F_{bc}(X, \mathbb{C}) \to \mathbb{C} \) be the \textbf{C}-ring homomorphism \( \text{ev}_x(a) := a(x) \). It is called the evaluation homomorphism.

2. Define the map of sets \( \text{refl}^{\text{alg}}_X : X \to \text{MSpec}(F_{bc}(X, \mathbb{C})) \) to be \( x \mapsto \text{Ker}(\text{ev}_x) \). This map is called the algebraic reflection map.

Here is the complex analogue of Lemma 4.3.

**Lemma 8.3.** Let \( X \) be a compact topological space, and let \( A := F_{bc}(X, \mathbb{C}) \).

1. Given \( m \in \text{MSpec}(A) \), there is a point \( x \in X \) such that \( m = \text{Ker}(\text{ev}_x) = \text{refl}^{\text{alg}}_X(x) \).

2. The algebraic reflection map \( \text{refl}^{\text{alg}}_X : X \to \text{MSpec}(A) \) is a homeomorphism.

**Proof.**

1. Unlike the real case, which is well-known, we could not find a reference for the complex case, so here is the proof.

Assume for the sake of contradiction that no such point \( x \) exists. This means that for every \( x \in X \) there is some element \( a \in m \) such that \( a(x) \neq 0 \). By continuity of \( a \), there is an open neighborhood \( U \) of \( x \) such that \( a(x') \neq 0 \) for all \( x' \in U \). Therefore we can find an open covering \( X = \bigcup_{i \in I} U_i \) and elements \( a_i \in m \), such that \( U_i \subseteq \text{NZer}_X(a_i) \) for all \( i \). The compactness of \( X \) implies that there is a subcovering \( X = \bigcup_{i \in I_0} U_i \) indexed by some finite subset \( I_0 \subseteq I \). For every \( i \in I_0 \) the conjugate function \( a_i^* \), namely \( a_i^*(x) := \overline{a_i(x)} \), belongs to \( A \). Therefore the function \( b := \sum_{i \in I_0} a_i^* \cdot a_i \) belongs to the maximal ideal \( m \). The function \( a_i^* \cdot a_i \) is positive on \( U_i \), and hence the function \( b \) is positive on all of \( X \), and thus nonzero. The multiplicative inverse \( b^{-1} : X \to \mathbb{C} \) is continuous, and by compactness it is
bounded. Therefore the function \( b \) is invertible in the ring \( A \). This contradicts the fact that \( b \in m \).

(2) Let’s write \( \bar{X} := \text{MSpec}(A) \). Because \( F_{bc}(X, \mathbb{R}) \subseteq A \), Lemma 4.2(1) implies that the map \( \text{refl}_{X}^{\text{alg}} : X \rightarrow \bar{X} \) is injective. Item (1) above shows that the map \( \text{refl}_{X}^{\text{alg}} \) is surjective. We see that \( \text{refl}_{X}^{\text{alg}} : X \rightarrow \bar{X} \) is bijective.

It remains to prove that \( \text{refl}_{X}^{\text{alg}} : X \rightarrow \bar{X} \) is a homeomorphism. Given an element \( a \in A \), consider the principal open sets \( U := \text{NZer}_{X}(a) \subseteq X \) and \( \bar{U} := \text{NZer}_{X}(a) \subseteq \bar{X} \). An easy calculation shows that \( \text{refl}_{X}^{\text{alg}}(U) = \bar{U} \). We know (see Section 1) that the principal open sets \( \bar{U} \) form a basis of the Zariski topology of \( \bar{X} \). On the other hand, since \( F_{bc}(X, \mathbb{R}) \subseteq A \), Lemma 4.2(2) implies that principal open sets form a basis of the given topology of \( X \). Therefore \( \text{refl}_{X}^{\text{alg}} \) is a homeomorphism. \( \square \)

**Lemma 8.4.** If \( A \) is a BC \( \mathbb{C} \)-ring then it is \( \mathbb{C} \)-valued.

**Proof.** We can assume that \( A = F_{bc}(X, \mathbb{C}) \) for some compact topological space \( X \). Let \( m \) be a maximal ideal of \( A \). By Lemma 8.3(1) there is a point \( x \in X \) such that \( m = \text{Ker}(ev_{x}) \). But \( ev_{x} \) is a \( \mathbb{C} \)-ring homomorphism \( ev_{x} : A \rightarrow \mathbb{C} \), so it induces a \( \mathbb{C} \)-ring isomorphism \( A/m \cong \mathbb{C} \). \( \square \)

Next is the complex analogue of Theorem 5.9.

**Theorem 8.5.** For an arbitrary topological space \( X \), the ring \( F_{bc}(X, \mathbb{C}) \) is a BC \( \mathbb{C} \)-ring.

**Proof.** We may assume \( X \) is a nonempty topological space. Consider the Stone-
Čech Compactification \((\bar{X}, c_{\bar{X}})\) of \( X \) (see Definition 5.1 and Theorem 5.19), and the \( \mathbb{C} \)-ring homomorphism

\[
F_{bc}(X, c_{X}) : F_{bc}(\bar{X}, \mathbb{C}) \rightarrow F_{bc}(X, \mathbb{C}).
\]

Given a function \( a \in F_{bc}(X, \mathbb{C}) \), let \( Z \subseteq \mathbb{C} \) be a closed disc in \( \mathbb{C} \) that contains \( a(X) \). Because \( Z \) is a compact topological space, the universal property of the SCC says that \( a \) extends uniquely to a continuous function \( \bar{a} : \bar{X} \rightarrow Z \). The assignment \( a \mapsto \bar{a} \) is an inverse of the ring homomorphism \( F_{bc}(X, c_{X}) \), so the latter is an isomorphism. But the topological space \( \bar{X} \) is compact, and hence \( F_{bc}(\bar{X}, \mathbb{C}) \) is a BC \( \mathbb{C} \)-ring. \( \square \)

**Lemma 8.6.** Let \( A \in \text{Rng}_{bc} \mathbb{C} \) and \( X := \text{MSpec}(A) \).

1. The topological space \( X \) is compact.
2. There is a BC \( \mathbb{R} \)-ring \( A_{0} \), with an isomorphism of \( \mathbb{C} \)-rings \( \mathbb{C} \otimes_{\mathbb{R}} A_{0} \cong A \).
3. For every \( a \in A \), the function \( \text{dev}_{A}(a) : X \rightarrow \mathbb{C} \) is continuous and bounded with respect to the standard norm of \( \mathbb{C} \).
4. The \( \mathbb{C} \)-ring homomorphism \( \text{dev}_{A} : A \rightarrow F_{bc}(X, \mathbb{C}) \) is bijective.

**Proof.**

1. This is like the proof of Lemma 4.12(1), but now we use Lemma 8.3(2) instead of Lemma 4.3(2).

2. By definition there is a \( \mathbb{C} \)-ring isomorphism \( \psi : A \xrightarrow{\cong} B \) where \( B := F_{bc}(Y, \mathbb{C}) \) for some compact topological space \( Y \). The ring \( B_{0} := F_{bc}(Y, \mathbb{R}) \) is a BC \( \mathbb{R} \)-ring, and the inclusion \( \gamma_{B} : B_{0} \rightarrow B \) induces a \( \mathbb{C} \)-ring isomorphism \( \mathbb{C} \otimes_{\mathbb{R}} B_{0} \xrightarrow{\cong} B \). Then the
subring $A_0 := \psi^{-1}(B_0) \subseteq A$ is also a BC $\mathbb{R}$-ring, and the inclusion $\gamma_A : A_0 \to A$ induces a $\mathbb{C}$-ring isomorphism $\mathbb{C} \otimes_{\mathbb{R}} A_0 \cong A$.

(3) We continue with $Y$, $B$, $B_0$ and $A_0$ as above. According to Theorem 4.16 there is a homeomorphism $\text{refl}_Y^f : Y \cong \text{MSpec}(B_0)$. Define $X_0 := \text{MSpec}(A_0)$. Theorem 4.16 says that the isomorphism $\psi_0 : A_0 \cong B_0$ in $\text{Rng}_{/bc} \mathbb{R}$ induces a homeomorphism $f := \text{MSpec}(\psi_0) : Y \to X_0$. By Theorem 2.16(3) there is a homeomorphism $p = \text{MSpec}(\gamma_A) : X \to X_0$. By Proposition 2.13 the homomorphisms $\text{dev}_{(-)}$ are functorial on $\text{Rng}_{/bc} \mathbb{C}$, so we may as well identify $X$ and $X_0$ via $p$. By Theorem 2.16(4) there is this commutative diagram

$$
\begin{array}{ccc}
A_0 & \xrightarrow{\gamma_A} & A \\
\text{dev}_{A_0} \downarrow & & \downarrow \text{dev}_A \\
\text{F}(X, \mathbb{R}) & \xrightarrow{\text{F}(X, \gamma_\mathbb{C})} & \text{F}(X, \mathbb{C})
\end{array}
$$

in $\text{Rng}/\mathbb{R}$. Here $\gamma_\mathbb{C} : \mathbb{R} \to \mathbb{C}$ is the inclusion.

Now the element $a \in A$ can be expressed as $a = a_0 + b_0 \cdot i$ with $a_0, b_0 \in A_0$. The commutative diagram (8.7) says that

$$
\text{dev}_{A}(a) = \text{dev}_{A_0}(a_0) + \text{dev}_{A_0}(b_0) \cdot i \in \text{F}(X, \mathbb{C}).
$$

By Lemma 4.12(3) the functions $\text{dev}_{A_0}(a_0), \text{dev}_{A_0}(b_0) : X \to \mathbb{R}$ are continuous and bounded. It follows that $\text{dev}_{A}(a)$ is continuous and bounded.

(4) It remains to prove that $\text{dev}_{A} : A \to F_{bc}(X, \mathbb{C})$ is bijective. By item (3) we know that there is the commutative diagram (8.7) in $\text{Rng}/\mathbb{R}$, and the the left vertical arrow in it is a bijection. We now pass to the induced commutative diagram in $\text{Rng}/\mathbb{C}$

$$
\begin{array}{ccc}
\mathbb{C} \otimes_{\mathbb{R}} A_0 & \xrightarrow{\text{id}_{\mathbb{C}} \otimes \gamma_A} & A \\
\text{id}_{\mathbb{C}} \otimes \text{dev}_{A_0} \downarrow & & \downarrow \text{dev}_A \\
\mathbb{C} \otimes_{\mathbb{R}} F_{bc}(X, \mathbb{R}) & \xrightarrow{\text{id}_{\mathbb{C}} \otimes F_{bc}(X, \gamma_\mathbb{C})} & F_{bc}(X, \mathbb{C})
\end{array}
$$

We see that the right vertical arrow here is bijective. □

Next is the complex version of Corollary 5.13

**Theorem 8.9.** Let $A$ be a $\mathbb{C}$-ring, with $X := \text{MSpec}(A)$. The following two conditions are equivalent:

(i) $A$ is a BC $\mathbb{C}$-ring.

(ii) $A$ is a $\mathbb{C}$-valued ring, for every $a \in A$ the function $\text{dev}_{A}(a) : X \to \mathbb{C}$ is continuous, and the $\mathbb{C}$-ring homomorphism $\text{dev}_{A} : A \to F_{c}(X, \mathbb{C})$ is bijective.

**Proof.**

(i) ⇒ (ii): By Lemma 8.6(1) the topological space $X$ is compact, and therefore $F_{bc}(X, \mathbb{C}) = F_{c}(X, \mathbb{C})$. According to Lemma 8.6(3), for every $a \in A$ the function $\text{dev}_{A}(a) : X \to \mathbb{C}$ is continuous. Finally, by Lemma 8.6(4) the homomorphism $\text{dev}_{A} : A \to F_{bc}(X, \mathbb{C}) = F_{c}(X, \mathbb{C})$ is bijective.
(ii) ⇒ (i): Proposition 1.13 says that the topological space \( X \) is quasi-compact. This implies that \( F_{bc}(X, \mathbb{C}) = F_c(X, \mathbb{C}) \), and hence there is an isomorphism of \( \mathbb{C} \)-rings \( \text{dev}_A : A \rightarrow F_{bc}(X, \mathbb{C}) \). According to Theorem 8.5, \( A \) is a BC \( \mathbb{C} \)-ring. □

Example 8.10. Take the \( \mathbb{C} \)-ring \( \mathbb{C} = \mathbb{C}[t] \), the polynomial ring in one variable over \( \mathbb{C} \), and let \( X := \text{MSpec}(A) \). The ring \( A \) is a \( \mathbb{C} \)-valued ring. The function \( \text{dev}_{A(t)} : X \rightarrow \mathbb{C} \) is a bijection, but it is not continuous. Theorem 8.9 says that \( A \) is not a BC \( \mathbb{C} \)-ring.

Here is the complex version of Theorem 4.16.

Theorem 8.11 (Duality). The functor
\[
F_{bc}(-, \mathbb{C}) : (\text{Top}_{cp})^{\text{op}} \rightarrow \text{Rng}/_{bc} \mathbb{C}
\]
is an equivalence of categories, with quasi-inverse \( \text{MSpec} \).

Proof. The proof is almost identical to the proof of Theorem 4.16. By Lemma 8.3(2) the map \( \text{ref}_{X}^{\text{alg}} : X \rightarrow \text{MSpec}(A) \) is a homeomorphism, and the complex version of Lemma 4.7 shows that \( \text{ref}_{X}^{\text{alg}} \) is functorial in \( X \).

In the reverse direction, Lemma 8.6(4) shows that the \( \mathbb{C} \)-ring homomorphism \( \text{dev}_A : A \rightarrow F_{bc}(X, \mathbb{C}) \) is bijective. By Proposition 2.13 the homomorphism \( \text{dev}_A \) is functorial in \( A \). □

9. Involutive Complex Rings

In this section we prove that every BC \( \mathbb{C} \)-ring \( A \) admits a canonical involution. This is Theorem 9.5. The canonical involution is used to prove Theorem 9.9, which states that the categories \( \text{Rng}/_{bc} \mathbb{C} \) and \( \text{Rng}/_{bc} \mathbb{R} \) are equivalent.

Definition 9.1 (Involutive \( \mathbb{C} \)-Rings).

1. An involution of a \( \mathbb{C} \)-ring \( A \) is an \( \mathbb{R} \)-ring automorphism \( (\cdot)^* : A \rightarrow A \), satisfying \((a^*)^* = a\) and \((\lambda \cdot a)^* = \bar{\lambda} \cdot a^*\) for all \( a \in A \) and \( \lambda \in \mathbb{C} \). Here \( \bar{\lambda} \) is the complex conjugate of \( \lambda \).
2. An involutive \( \mathbb{C} \)-ring is a \( \mathbb{C} \)-ring \( A \) equipped with an involution \((\cdot)^*\).
3. Suppose \( A \) and \( B \) are involutive \( \mathbb{C} \)-rings. A homomorphism of involutive \( \mathbb{C} \)-rings \( \phi : A \rightarrow B \) is a \( \mathbb{C} \)-ring homomorphism \( \phi \) satisfying \( \phi(a^*) = \phi(a)^* \) for every element \( a \in A \).
4. The category of commutative involutive \( \mathbb{C} \)-rings, with involutive \( \mathbb{C} \)-ring homomorphisms, is denoted by \( \text{Rng}^{*}/\mathbb{C} \).

In functional analysis books, e.g. [Co], the names for the notions in items (2) and (3) above are \( * \)-algebra and \( * \)-homomorphism, respectively. (If \( A \) is not commutative, then an involution has to be an anti-automorphism, i.e. \((a \cdot b)^* = b^* \cdot a^*\).)

Definition 9.2. Suppose \( A \) is an involutive commutative \( \mathbb{C} \)-ring. Define the subset of hermitian elements of \( A \) to be
\[
A_0 := \{ a \in A \mid a^* = a \} \subseteq A.
\]
This is an \( \mathbb{R} \)-subring of \( A \).
**Example 9.3.** Let $A_0$ be an $\mathbb{R}$-ring, and define the $\mathbb{C}$-ring $A := \mathbb{C} \otimes_{\mathbb{R}} A_0$. The ring $A$ has an involution $(-)^*$, with formula $(\lambda \otimes a_0)^* := \bar{\lambda} \otimes a_0$ for $\lambda \in \mathbb{C}$ and $a_0 \in A_0$. The hermitian subring of $A$ is $A_0$.

**Example 9.4.** Let $X$ be a topological space, and define the $\mathbb{C}$-ring $A := F_{bc}(X, \mathbb{C})$. The ring $A$ has an involution $(-)^*$, with formula $a^*(x) := \overline{a(x)}$ for $a \in A$ and $x \in X$. The hermitian subring of $A$ is $A_0 := F_{bc}(X, \mathbb{R})$.

**Theorem 9.5.** Every BC $\mathbb{C}$-ring $A$ admits a unique involution $(-)^*$, called the canonical involution, satisfying the two conditions below.

1. **Functoriality**: Given a homomorphism $\phi : A \to B$ in $\text{Rng}_{bc}/\mathbb{C}$, and an element $a \in A$, there is equality $\phi(a^*) = \phi(a)^*$ in $B$.
2. **Normalization**: For the $\mathbb{C}$-ring $F_{bc}(X, \mathbb{C})$ of bounded continuous functions on a topological space $X$, the canonical involution is the one described in Example 9.4.

Note that $F_{bc}(X, \mathbb{C})$ is indeed a BC $\mathbb{C}$-ring, by Theorem 8.5.

**Proof.** Given an arbitrary BC $\mathbb{C}$-ring $A$, let $X := \text{MSpec}(A)$ and $B := F_{bc}(X, \mathbb{C})$. By Lemma 8.6(4) the $\mathbb{C}$-ring homomorphism $\text{dev}_A : A \to F_{bc}(X, \mathbb{C}) = B$ is bijective. The ring $B$ has the involution imposed by condition (ii). The canonical involution of $A$ is defined to be the unique involution such that $\text{dev}_A : A \to B$ is an isomorphism of involutive $\mathbb{C}$-rings. Conditions (i) and (ii) force this involution of $A$ to be unique.

We need to prove functoriality, i.e. that condition (i) holds for an arbitrary homomorphism $\phi : A \to B$ in $\text{Rng}_{bc}/\mathbb{C}$. Define the topological spaces $X := \text{MSpec}(A)$ and $Y := \text{MSpec}(B)$. In view of Lemma 8.4 and Proposition 2.8 there is an induced map $f := \text{MSpec}(\phi) : Y \to X$ in $\text{Top}$. Consider the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\text{dev}_A} & = & \downarrow{\text{dev}_B} \\
F_{bc}(X, \mathbb{C}) & \xrightarrow{F_{bc}(f, \mathbb{C})} & F_{bc}(Y, \mathbb{C}) \\
\epsilon_X & = & \epsilon_Y \\
F(X, \mathbb{C}) & \xrightarrow{F(f, \mathbb{C})} & F(Y, \mathbb{C})
\end{array}
\]

in $\text{Rng}/\mathbb{C}$ (which is similar to diagram 4.13). The vertical arrows marked $\epsilon_X$ and $\epsilon_Y$ are the inclusions, and the bottom square is commutative, since $F_{bc}(-, \mathbb{C})$ is a subfunctor of $F(-, \mathbb{C})$. The two outer paths from $A$ to $F(Y, \mathbb{C})$ are equal, by Proposition 2.13. Because $\epsilon_Y$ is an injection, the top square is commutative too: $F_{bc}(f, \mathbb{C}) \circ \text{dev}_A = \text{dev}_B \circ \phi$. The $\mathbb{C}$-ring homomorphism $F_{bc}(f, \mathbb{C})$ respects the canonical involutions on the function rings, and by construction the $\mathbb{C}$-ring isomorphisms $\text{dev}_A : A \xrightarrow{\cong} F_{bc}(X, \mathbb{C})$ and $\text{dev}_B : B \xrightarrow{\cong} F_{bc}(Y, \mathbb{C})$ respect the canonical involutions on these rings. It follows that $\phi : A \to B$ respects the canonical involutions. \hfill \square

**Corollary 9.6.** The procedure of equipping a BC $\mathbb{C}$-ring $A$ with its canonical involution $(-)^*$ is a fully faithful functor

\[G : \text{Rng}_{bc}/\mathbb{C} \to \text{Rng}^*/\mathbb{C}.\]
Proof. Clear from condition (i) in Theorem 9.5. □

Definition 9.7. Given a BC \( \mathbb{C} \)-ring \( A \), the hermitian subring of \( A \), with respect to its canonical involution, is called the **canonical hermitian subring** of \( A \).

**Corollary 9.8.** Let \( A \) be a BC \( \mathbb{C} \)-ring

1. The canonical hermitian subring \( A_0 \) of \( A \) is a BC \( \mathbb{R} \)-ring.
2. The ring homomorphism \( \mathbb{C} \otimes_{\mathbb{R}} A_0 \rightarrow A, \lambda \otimes a_0 \mapsto \lambda \cdot a_0 \), is an isomorphism of involutive \( \mathbb{C} \)-rings. Here \( A \) has the canonical involution, and \( \mathbb{C} \otimes_{\mathbb{R}} A_0 \) has the involution from Example 9.3.

**Proof.**

(1) Let \( X := \text{MSpec}(A) \) and \( B := \text{F}_{bc}(X, \mathbb{C}) \). Put on \( A \) and \( B \) their canonical involutions. By Lemma 8.6 and Theorem 9.5 the homomorphism dev\(_A : A \rightarrow B \) is an isomorphism of \( \mathbb{C} \)-rings. Condition (i) of Theorem 9.5 says that \( \phi \) is an isomorphism of involutive \( \mathbb{C} \)-rings. The hermitian subring of \( B \) is \( B_0 := \text{F}_{bc}(X, \mathbb{R}) \), which is a BC \( \mathbb{R} \)-ring. Since dev\(_A \) induces an \( \mathbb{R} \)-ring isomorphism \( A_0 \rightarrow B_0 \), we see that \( A_0 \) is a BC \( \mathbb{R} \)-ring.

(2) For \( B = \text{F}_{bc}(X, \mathbb{C}) \) as above the homomorphism

\[
\mathbb{C} \otimes_{\mathbb{R}} B_0 = \mathbb{C} \otimes_{\mathbb{R}} \text{F}_{bc}(X, \mathbb{R}) \xrightarrow{id_c \otimes \text{F}_{bc}(X, \mathbb{C})} \text{F}_{bc}(X, \mathbb{C}) = B
\]

is an isomorphism of involutive \( \mathbb{C} \)-rings. Because dev\(_A \) : \( A \rightarrow B \) is an isomorphism of involutive \( \mathbb{C} \)-rings, it follows that \( \mathbb{C} \otimes_{\mathbb{R}} A_0 \rightarrow A \) is also an isomorphism of involutive \( \mathbb{C} \)-rings. \( \square \)

**Theorem 9.9.** The procedure of sending a BC \( \mathbb{C} \)-ring \( A \) to its canonical hermitian subring \( H(A) \) is an equivalence of categories

\[ H : \text{Rng}_{bc} \mathbb{C} \rightarrow \text{Rng}_{bc} \mathbb{R}. \]

The quasi-inverse of \( H \) is the functor

\[ I : \text{Rng}_{bc} \mathbb{R} \rightarrow \text{Rng}_{bc} \mathbb{C}, \quad I(A_0) := \mathbb{C} \otimes_{\mathbb{R}} A_0. \]

**Proof.**

Step 1. Let us show that \( H \) is a functor. Consider a homomorphism \( \phi : A \rightarrow B \) in \( \text{Rng}_{bc} \mathbb{C} \). By Corollary 9.6 there is a homomorphism \( G(\phi) : G(A) \rightarrow G(B) \) in the category \( \text{Rng}^*_{/\mathbb{C}} \); this just means that \( \phi \) respects the canonical involutions on \( A \) and \( B \). Passing to hermitian subrings we obtain an \( \mathbb{R} \)-ring homomorphism \( H(\phi) : H(A) \rightarrow H(B) \). By Corollary 9.8(1) the \( \mathbb{R} \)-rings \( A_0 := H(A) \) and \( B_0 := H(B) \) are BC \( \mathbb{R} \)-rings. We conclude that \( H : \text{Rng}_{bc} \mathbb{C} \rightarrow \text{Rng}_{bc} \mathbb{R} \) is a functor.

Step 2. Obviously the procedure \( I(A_0) := \mathbb{C} \otimes_{\mathbb{R}} A_0 \) and \( I(\phi_0) := \text{id}_c \otimes \phi_0 \) is a functor \( I : \text{Rng}_{bc} \mathbb{R} \rightarrow \text{Rng}_{bc} \mathbb{C} \). If \( A_0 \) is a BC \( \mathbb{R} \)-ring, then, taking \( X := \text{MSpec}(A_0) \), we have a \( \mathbb{C} \)-ring isomorphism

\[
\mathbb{C} \otimes_{\mathbb{R}} A_0 \cong \mathbb{C} \otimes_{\mathbb{R}} \text{F}_{bc}(X, \mathbb{R}) \xrightarrow{id_c \otimes \text{F}_{bc}(X, \mathbb{C})} \text{F}_{bc}(X, \mathbb{C}).
\]

In view of Theorem 8.5 we see that \( I(A_0) \) is a BC \( \mathbb{C} \)-ring. Therefore we obtain a functor \( I : \text{Rng}_{bc} \mathbb{R} \rightarrow \text{Rng}_{bc} \mathbb{C} \).
Step 3. Take some \( A_0 \in \text{Rng}_{/bc} \mathbb{R} \). The \( \mathbb{C} \)-ring isomorphism (9.10), with conditions (i) and (ii) of Theorem 9.5, say that the canonical involution on \( I(A_0) = \mathbb{C} \otimes_{\mathbb{R}} A_0 \) coincides with that of Example 9.3. Therefore the homomorphism \( A_0 \to \mathbb{C} \otimes_{\mathbb{R}} A_0, \quad a_0 \mapsto 1_\mathbb{C} \otimes a_0 \), induces an \( \mathbb{R} \)-ring isomorphism

\[
\zeta_{A_0} : A_0 \xrightarrow{\cong} (H \circ I)(A_0).
\]

This isomorphism is functorial in \( A_0 \).

Step 4. Now take some \( A \in \text{Rng}_{/bc} \mathbb{C} \). Corollary 9.8(2) gives a \( \mathbb{C} \)-ring isomorphism

\[
\eta_A : (I \circ H)(A) \xrightarrow{\cong} A.
\]

It is easy to verify that \( \eta_A \) is functorial in \( A \).

10. Involutive Complex Banach Rings

In this final section of the paper we relate the results from the previous sections of the paper to the theory of complex \( \mathbb{C}^* \)-algebras from functional analysis. Our source for material on functional analysis is [Co].

As before, all rings in this section are commutative unital, and all ring homomorphisms preserve units.

The notions of a Banach \( \mathbb{R} \)-ring, and a Banach \( \mathbb{R} \)-ring homomorphism, were recalled in Definitions 6.1 and 6.3. The complex variants are the same, just with \( \mathbb{C} \) instead of \( \mathbb{R} \). The conventional name is "commutative unital Banach algebra", see [Co, Definition VII.1.1].

**Definition 10.1.** The category of commutative Banach \( \mathbb{C} \)-rings, with Banach \( \mathbb{C} \)-ring homomorphisms, is denoted by \( \text{BaRng}_{/\mathbb{C}} \).

The next definition is a rephrasing of [Co, Definition VIII.1.1].

**Definition 10.2.** A commutative Banach \( \mathbb{C} \)-ring, commonly known as a commutative unital \( \mathbb{C}^* \)-algebra over \( \mathbb{C} \), is a commutative Banach \( \mathbb{C} \)-ring \( A \), with a norm \( \| - \| \) and an involution \( ( - )^* \), satisfying the extra condition \( \| a^* \cdot a \| = \| a \|^2 \) for all \( a \in A \).

The category of commutative Banach \( \mathbb{C} \)-rings, with involutive Banach \( \mathbb{C} \)-ring homomorphisms, as in Definitions 6.3 and 9.1(3), is denoted by \( \text{BaRng}^*_{/\mathbb{C}} \).

There are two obvious forgetful functors: the functor \( \text{BaRng}^*_{/\mathbb{C}} \to \text{BaRng}_{/\mathbb{C}} \) that forgets the involutions, and the functor \( \text{BaRng}^*_{/\mathbb{C}} \to \text{Rng}^*_{/\mathbb{C}} \) that forgets the norms.

Here is an easy to prove classical fact:

**Proposition 10.3** ([Co, Proposition VIII.1.11(d)]). Let \( A \) and \( B \) be involutive commutative Banach \( \mathbb{C} \)-rings, and let \( \phi : A \to B \) be a homomorphism of involutive \( \mathbb{C} \)-rings. Then \( \| \phi(a) \| \leq \| a \| \) for all \( a \in A \). Thus \( \phi \) is a Banach \( \mathbb{C} \)-ring homomorphism.

**Corollary 10.4.** The forgetful functor \( \text{BaRng}^*_{/\mathbb{C}} \to \text{Rng}^*_{/\mathbb{C}} \) is fully faithful.

A much stronger result is our Corollary 10.11 below.

Here are some examples of involutive commutative Banach \( \mathbb{C} \)-rings.
Example 10.5. The field \( \mathbb{C} \) is a Banach \( \mathbb{C} \)-ring, with norm \( \|\lambda\| := |\lambda| \) and involution \( \lambda^* := \bar{\lambda} \).

Let \( n \) be a positive integer, and let \( A \) be the \( \mathbb{C} \)-ring \( A := \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C} \), with componentwise ring operations. Put on \( \mathbb{C} \) called the evaluation homomorphism with its Zariski topology, and that the Gelfand transform coincides with the double weak \( \text{Remark 10.9.} \)

Let \( \mathbb{C} \) instead of Lemma 4.4. Also diagram (6.9) is now in the category \( \mathbb{C} \), minor changes: we now use Example 10.6 instead of Example 6.5, and Lemma 8.4 in the proof of Theorem 6.8 is valid also in the complex case, except for a few

Proof. The proof of Theorem 10.8 is valid also in the complex case, except for a few minor changes: we now use Example 10.6 instead of Example 6.5, and Lemma 8.4 instead of Lemma 4.4. Also diagram (6.9) is now in the category \( \mathbb{C} \). \( \square \)

Corollary 10.11. The functor

\[ F : \text{BaRng}^\times / \mathbb{C} \to \text{Rng}_{/bc} \mathbb{C} \]

that forgets the norms and the involutions is an equivalence of categories.
Proof. Take $A \in \text{BaRng}^*/\mathbb{C}$. According to Theorem 10.8 there is a $\mathbb{C}$-ring isomorphism $F(A) \cong F_c(X, \mathbb{C})$, where $X$ is a compact topological space. By Definition 8.1 the $\mathbb{C}$-ring $F(A)$ is a BC $\mathbb{C}$-ring.

The quasi-inverse of $F$ is the functor $G : \text{Rng}_{bc}/\mathbb{C} \to \text{BaRng}^*/\mathbb{C}$, which puts on a BC $\mathbb{C}$-ring $A_0$ the canonical involution from Theorem 9.5 and the canonical norm from Theorem 10.10.

Example 10.12. Consider the ring $A := \mathbb{C}[t]$, the polynomial ring in one variable over $\mathbb{C}$. In Example 8.10 we saw that $A$ is not a BC $\mathbb{C}$-ring. Corollary 10.11 implies that $A$ does not admit a structure of Banach* $\mathbb{C}$-ring.

Let $A$ and $B$ be commutative involutive Banach $\mathbb{C}$-rings, and let $\phi : A \to B$ be a $\mathbb{C}$-ring homomorphism. The classical Proposition 10.3 says that $\phi$ is a homomorphism of involutive Banach $\mathbb{C}$-rings (Definition 10.2) iff $\phi$ respects the involutions; namely the norms are automatically respected. Our final result says that much more is true: the involutions are also automatically respected.

Corollary 10.13. Let $A$ and $B$ be commutative Banach* $\mathbb{C}$-rings, and let $\phi : A \to B$ be a $\mathbb{C}$-ring homomorphism. Then $\phi$ is a homomorphism of Banach* $\mathbb{C}$-rings, i.e. $\phi$ respects the involutions and the norms.

Proof. Corollary 10.11 implies that the forgetful functor $F : \text{BaRng}^*/\mathbb{C} \to \text{Rng}/\mathbb{C}$ is fully faithful. In plain terms this says that for $A, B \in \text{BaRng}^*/\mathbb{C}$ the function $F : \text{Hom}_{\text{BaRng}^*/\mathbb{C}}(A, B) \to \text{Hom}_{\text{Rng}/\mathbb{C}}(A, B)$ is bijective. Thus the homomorphism $\phi \in \text{Hom}_{\text{Rng}/\mathbb{C}}(A, B)$ actually belongs to $\text{Hom}_{\text{BaRng}^*/\mathbb{C}}(A, B)$.

References

[AK] A. Altman and S. Kleiman, “A Term of Commutative Algebra”, free online at https://www.researchgate.net/publication/325591008_A_term_of_Commutative_Algebra
[BJ] F. Borceux and G. Janelidze, “Galois Theories”, Cambridge University Press, 2001.
[Co] J.B. Conway, “A Course in Functional Analysis”, Second Edition, Springer, 1990.
[Ei] D. Eisenbud, “Commutative Algebra”, Springer, 1994.
[GJ] L. Gillman and M. Jerison, “Rings of Continuous Functions”, Van Nostrand, 1960.
[Go] K.R. Goodearl, “Notes on real and complex C*-algebras”, Shiva, 1982, online https://www.worldcat.org/oclc/1150943391.
[HS] P.J. Hilton and U. Stambach, “A Course in Homological Algebra”, Springer, 1971.
[Jo] P.T. Johnstone, “Stone spaces”, Cambridge University Press, 1982.
[Ke] J. Kelley, “General topology”, Springer, 1975 (reprint).
[KS] M. Kashiwara and P. Schapira, “Categories and Sheaves”, Springer, 2006.
[ML] S. Mac Lane, “Categories for the Working Mathematician”, Springer, 1978.
[Mn] J.R. Munkres, “Topology”, 1975, Prentice Hall.
[Sc] P. Scholze, Lectures on Condensed Mathematics, preprint 2019, https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.
[SP] The Stacks Project, J.A. de Jong (Editor), http://stacks.math.columbia.edu
[Wa] R.C. Walker, “The Stone-Čech Compactification”, Springer, 1974.
[VY] R. Vyas and A. Yekutieli, Weak proregularity, weak stability, and the noncommutative MGM equivalence, J. Algebra 513 (2018), 265-325.
A. Yekutieli, “Derived Categories”, Cambridge Studies in Advanced Mathematics 183, Cambridge University Press, 2019. Prepublication version

Svoray: Department of Mathematics, Ben Gurion University, Be’er Sheva 84105, Israel
Email address: ysavorai@post.bgu.ac.il

Yekutieli: Department of Mathematics, Ben Gurion University, Be’er Sheva 84105, Israel
Email address: amyekut@math.bgu.ac.il