ON THE STRUCTURE OF FACTOR LIE ALGEBRAS

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Abstract. The Lie algebra analogue of Schur’s result which is proved by Moneyhun in 1994, states that if $L$ is a Lie algebra such that dim$L/Z(L) = n$, then dim$L/Z_2(L) = \frac{1}{2}n(n-1) - s$ for some non-negative integer $s$. In the present paper, we determine the structure of central factor (for $s = 0$) and the factor Lie algebra $L/Z_2(L)$ (for all $s \geq 0$) of a finite dimensional nilpotent Lie algebra $L$, with $n$-dimensional central factor. Furthermore, by using the concept of $n$-isoclinism, we discuss an upper bound for the dimension of $L/Z_n(L)$ in terms of dim$L/(n+1)$, when the factor Lie algebra $L/Z_n(L)$ is finitely generated and $n \geq 1$.

1. Introduction

In 1904, Schur [14] proved that if the center of a group $G$ has finite index, then the derived subgroup $G'$ is also finite. Also, Wiegold [16] showed that if the order of central factor group of $G$ is $p^n$, then $G'$ is a $p$-group of order at most $p^{\frac{1}{2}n(n-1)}$. The structure of a group and its central factor, with regards to the order of derived subgroup, has been already studied by many authors (see [9, 12, 16]). Berkovich [4] studied the structure of $G/Z(G)$, where $G$ is a $p$-group such that $|G/Z(G)| = p^n$ and $|G'| = p^{\frac{1}{2}n(n-1)}$. Later in 2004, Kim [10] characterized the structure of a $p$-group $G$ such that $|G/Z(G)| = p^n$ and $|G'| = p^{\frac{1}{2}n(n-1)-1}$. Also, Hekster [8] showed that if $G$ is a finitely generated group and $n \geq 1$, then $G/Z_n(G)$ is finite if and only if $\gamma_{n+1}(G)$ is finite, where $Z_n(G)$ and $\gamma_{n+1}(G)$ are $n$-th and $(n+1)$-st terms of the upper and lower central series of $G$, respectively.

Throughout of this paper, all Lie algebras are over a fixed field $F$ and $Z_n(L)$ denotes the $n$-th term of the upper central series of a Lie algebra $L$ defined inductively by $Z_1(L) = Z(L)$ and $Z_n(L)/Z_{n-1}(L) = Z(L/Z_{n-1}(L))$ for $n \geq 2$. Also, $L_{(1)} = L$ and $L_{(n)} = [L_{(n-1)}, L]$ for $n \geq 2$, where $[,]$ denotes the Lie bracket. Moreover, $A(n)$ denotes the $n$-dimensional abelian Lie algebra and the Heisenberg Lie algebra $H(m)$ is the $2m + 1$ dimensional real Lie algebra...
with the basis \{a_1, \ldots, a_m, b_1, \ldots, b_m, c\} and the Lie brackets defined by
\[
[a_i, a_j] = [b_i, b_j] = [a_i, c] = [b_i, c] = [c, c] = 0 \quad \text{and} \quad [a_i, b_j] = c \delta_{ij},
\]
where \(\delta_{ij}\) is the Kronecker delta (see [7] for more details).

Nilpotent Lie algebras have played an important role in mathematics over the last 30 years in the classification theory of Lie algebras. Furthermore, the characterization of \(L/Z_n(L)\) \((n \in \mathbb{N})\) has always been one of the most popular problems among Lie algebra experts. The Lie algebra analog of Schur's was proved by Moneyhun [11]: if \(L\) is a Lie algebra such that \(\dim L/Z(L) = n\), then \(\dim L(2) = \frac{1}{2}n(n-1) - s\) for some non-negative integer \(s\). Moreover, Batten et al. [3] proved that if \(L\) is a finite dimensional nilpotent Lie algebra and \(\dim L/Z(L) = n\), then \(\dim L(2) = \frac{1}{2}n(n-1) - s\) and \(\dim (L/Z(L))(2) \leq 1 + s\) for some non-negative integer \(s\). In the present paper, we determine the structure of \(L/Z_2(L)\), where \(L\) is a finite dimensional nilpotent Lie algebra such that \(\dim L/Z(L) = n\) and \(\dim L(2) = \frac{1}{2}n(n-1) - s\) for all non-negative integers \(s\). In particular, we characterize the structure of central factor of \(L\), when \(s = 0\). We show that \(L/Z_2(L)\) must be a nilpotent Lie algebra of dimension not equal to 1. Nilpotent Lie algebras of dimension less than 6, over a field of any characteristic, are classified in [5, 6]. Also, the nilpotent Lie algebras of dimension greater than 7, under the special conditions are characterized in [1, 15].

Moreover, Salemkar and Mirzaei [13] proved a Lie algebra version of the Hekster's result and showed that if \(L\) is a finitely generated Lie algebra, then \(L/Z_n(L)\) is finite dimensional if and only if \(L_{(n+1)}\) is finite dimensional. In the last section, we give an upper bound for the dimension of \(L/Z_n(L)\), when \(L_{(n+1)}\) is finite dimensional and \(L/Z_n(L)\) is finitely generated. We show that
\[
\dim L/Z_n(L) \leq d^n \cdot \dim L_{(n+1)},
\]
where \(d\) is the minimal number of generators of \(L/Z_n(L)\) and \(n \geq 1\). Note that the first author and Saeedi proved the above result for \(n = 1\) in [2]. Here, we use the idea of \(n\)-isoclinism discussed in [13], which gives us a different method from the technique applied in [2].

2. Preliminary results

In this section, we discuss some preliminary results, which will be used in the proof of the main theorems.

**Lemma 2.1** ([3, 11]). Let \(L\) be a Lie algebra such that \(\dim L/Z(L) = n\). Then \(\dim L(2) \leq \frac{1}{2}n(n-1) - s\) for some non-negative integer \(s\). Moreover, if \(L\) is a finite dimensional nilpotent Lie algebra, then \(\dim (L/Z(L))(2) \leq 1 + s\).

**Definition 2.2.** A Lie algebra \(L\) is called capable, if there exists a Lie algebra \(H\) such that \(L \cong H/Z(H)\).
The following result for capable Lie algebras is proved by the first author and Saeedi in [2], which has an important role in the proof of Theorems 3.1 and 3.2.

**Theorem 2.3** ([2]). Let $L$ be a capable Lie algebra such that $\dim L_{(2)} = m$. Then $\dim L/Z(L) \leq 2m^2$.

All nilpotent Lie algebras of dimension at most 5 are classified over an arbitrary field $F$.

**Theorem 2.4** ([6]). Let $L$ be a finite dimensional nilpotent Lie algebra. Then

(a) $L$ is 1-dimensional if and only if $L \cong A(1)$.
(b) $L$ is 2-dimensional if and only if $L \cong A(2)$.
(c) $L$ is 3-dimensional if and only if $L \cong A(3)$ or $H(1)$.
(d) $L$ is 4-dimensional if and only if $L \cong A(4), H(1) \oplus A(1)$ or
text continues...
and note that \( \dim Z(L/Z(L)) = n - 2m \), from Equation (1). If \( D < 2 \), then this implies that \( m < 1 \), which contradicts \( \dim K = 1 \). Hence we must have \( D = 2 \) and \( m = 1 \).

**Theorem 3.2.** Let \( L \) be a finite dimensional nilpotent Lie algebra such that \( \dim L/Z(L) = n \) and \( \dim L^{(2)} = \frac{4}{9}n(n - 1) - s \) for some integer \( s \geq 1 \). Then one of the following holds:

(a) \( L/Z(L) \) is an \( n \)-dimensional abelian Lie algebra.

(b) \( L/Z(L) \cong H(1) \oplus A(n - 3) \).

(c) \( L/Z(L^2) \) is an \( i \)-dimensional nilpotent Lie algebra, where \( 2 \leq i \leq 2 \left(1 + \frac{s}{2}\right)^2 \), and \( \dim Z(L) = n - i \).

**Proof.** By Lemma 2.1, we have \( \dim(L/Z(L))^{(2)} \leq 1 + s \). If \( \dim(L/Z(L))^{(2)} \leq 1 \), then \( L/Z(L) \) is abelian or by Theorem 3.1, \( L/Z(L) \) is isomorphic to \( H(1) \oplus A(n - 3) \). Now fix \( m := \dim(L/Z(L))^{(2)} \). Then by Theorem 2.3

\[
\dim \frac{L}{Z(L)} = \dim \frac{L/Z(L)}{Z(L)/Z(L)} = \dim \frac{L/Z(L)}{Z(L/Z(L))} \leq 2m^2.
\]

Now from Equation (2), we get

\[
\dim L/Z(L) - \dim Z(L)/Z(L) = \dim L/Z(L) = i,
\]

where \( 2 \leq i \leq 2m^2 \). Thus, we have \( \dim Z(L)/Z(L) = n - i \).

**4. An upper bound for the dimension of** \( L/Z_n(L) \)**

First, we present the notion of \( n \)-isoclinism, which is the key part of our method in the proof of last main theorem. The \( n \)-isoclinism with respect to the variety of all nilpotent Lie algebras \( L \) for which \( L = Z_n(L) \).

**Definition 4.1.** Let \( L \) and \( H \) be Lie algebras. Then an \( n \)-isoclinism \( (n \geq 1) \) between \( L \) and \( H \) is a pair of isomorphisms \( (\alpha, \beta) \) with \( \alpha : L/Z_n(L) \rightarrow H/Z_n(H) \) and \( \beta : L^{(n+1)} \rightarrow H^{(n+1)} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
L/Z_n(L) \oplus \cdots \oplus L/Z_n(L) & \xrightarrow{\alpha^{n+1}} & L^{(n+1)} \\
H/Z_n(H) \oplus \cdots \oplus H/Z_n(H) & \xrightarrow{\beta} & H^{(n+1)}
\end{array}
\]

where horizontal maps are defined by

\[
(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}) \rightarrow [\cdots [x_1, x_2], \ldots, x_{n+1}]
\]

such that \( \bar{x}_i = x_i + Z_n(L) \) and \( \bar{x}_i = x_i + Z_n(H) \) in the top and bottom horizontal maps, respectively (see [8, 13] for more details). If there exists such an \( n \)-isoclinism, we say that \( L \) is \( n \)-isoclinic to \( H \) and write \( L \sim_n H \).

Salemkar and Mirzaei investigated the \( n \)-isoclinism of Lie algebras in [13].
Lemma 4.2 ([13]). If $L$ is a Lie algebra with a subalgebra $H$ such that $L = H + Z_n(L)$, then $L \simeq_n H$. Conversely, if the factor Lie algebra $L/Z_n(L)$ is finite dimensional and $L \simeq_n H$, then $L = H + Z_n(L)$.

Proposition 4.3 ([13]). Let $L$ be a finitely generated Lie algebra. Then the following statements are equivalent.

(i) $L/Z_n(L)$ is finite dimensional.
(ii) $L_{(n+1)}$ is finite dimensional.
(iii) $(L/Z(L))_{(n)}$ is finite dimensional.

Now, we prove the last theorem. Recall that in [2], the following result is proved for $n = 1$.

Theorem 4.4. Let $L$ be a Lie algebra such that $L_{(n+1)}$ is finite dimensional and $L/Z_n(L)$ is finitely generated. Then

$$\dim L/Z_n(L) \leq d^n \cdot \dim L_{(n+1)},$$

where $d$ is the minimal number of generators of $L/Z_n(L)$.

Proof. We proceed inductively. Suppose that $n = 1$. Fix $x_1, \ldots, x_d \in L$ such that $\{x_1 + Z(L), \ldots, x_d + Z(L)\}$ generates $L/Z(L)$. Let $H = \langle x_1, \ldots, x_d \rangle$, and so $L = H + Z(L)$. By Lemma 4.2, we have $L \simeq_1 H$, and hence $L/Z(L) \cong H/Z(H)$ and $L_{(2)} \cong H_{(2)}$. Therefore, we may replace $L$ by $H$. Let $y \in H$ and define

$$f: \bigoplus_{i=1}^d C_H(x_i) \to H_{(2)} \oplus H_{(2)} \oplus \cdots \oplus H_{(2)},$$

where $C_H(x_i)$ is the centralizer of $x_i$ in $H$. The definition of $\cap_{i=1}^d C_H(x_i)$ implies that $f$ is well-defined and one may easily check that it is an injective linear transformation. Thus

$$\dim L/Z(L) = \dim H/Z(H) = \dim H/\cap_{i=1}^d C_H(x_i) \leq d \cdot \dim L_{(2)}.$$

Now, assume that the claim holds for $(n - 1)$, i.e., $\dim L/Z_{n-1}(L) \leq d^{n-1} \cdot \dim L_{(n)}$. Fix $x_1, \ldots, x_d \in L$ such that $\{x_1 + Z_n(L), \ldots, x_d + Z_n(L)\}$ generates $L/Z_n(L)$. Let $H = \langle x_1, \ldots, x_d \rangle$, and so $L = H + Z_n(L)$. Then Lemma 4.2 implies that $H$ is $n$-isoclinic to $L$. Trivially, $H/Z(H)$ is finitely generated. It follows from Proposition 4.3 and finiteness of $\dim H_{(n+1)}$ that $\dim (H/Z(H))_{(n)}$ is finite. Therefore, $H/Z(H)$ satisfies the induction hypothesis. Hence

$$\dim H/Z_n(H) = \dim \frac{H/Z(H)}{Z_{n-1}(H/Z(H))} \leq d^{n-1} \cdot \dim (H/Z(H))_{(n)}.$$

On the other hand, by the second isomorphism theorem, we have

$$(H/Z(H))_{(n)} \cong \frac{H_{(n)}}{H_{(n)} \cap Z(H)} \cong \frac{H_{(n)}}{\cap_{i=1}^d C_{H_{(n)}}(x_i)}.$$
The above isomorphisms and defining an analogous map to \( f \), imply that

\[
\dim(H/Z(H))_{(n)} = \frac{H_{(n)}}{\bigcap^{n}_{i=1} C_{H_{(n)}}(x_i)} \leq d \cdot \dim H_{(n+1)},
\]

and since \( L \sim_{n} H \), we have

\[
\dim L/Z_{n}(L) = \dim H/Z_{n}(H) \leq (d \cdot \dim H_{(n+1)}) \cdot d^{n-1} = d^{n} \cdot \dim H_{(n+1)} = d^{n} \cdot \dim L_{(n+1)},
\]

which completes the proof. \( \square \)

**Acknowledgments.** The authors would like to thank the referee for his/her valuable comments which helped to improve the article.

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