Infinite norm decompositions of C*-algebras

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Abstract

In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. The norm decomposition is some generalization of Peirce decomposition. It is proved that the infinite norm decomposition of any C*-algebra is a C*-algebra. C*-factors with an infinite and a nonzero finite projection and simple purely infinite C*-algebras are constructed.

Introduction

In the given article the notion of infinite norm decomposition of C*-algebra is investigated. It is known that for any projection $p$ of a unital C*-algebra $A$ the next equality is valid $A = pAp \oplus pA(1-p) \oplus (1-p)A \oplus (1-p)A(1-p)$, where $\oplus$ is a direct sum of spaces. The norm decomposition is some generalization of Peirce decomposition. First such infinite decompositions were introduced in [1] by the author.

In this article a unital C*-algebra $A$ with an infinite orthogonal set \{\(p_i\)\} of equivalent projections such that sup \(i\) \(p_i\) = 1, and the set $\sum_{ij} p_iAp_j = \{\{a_{ij}\} : \text{for any indexes } i, j, a_{ij} \in p_iAp_j, \text{and } \|\sum_{k=1}^{i-1}(a_{ki} + a_{ik}) + a_{ii}\| \to 0 \text{ at } i \to \infty\}$ are considered. Note that all infinite sets like \{\(p_i\)\} are supposed to be countable.

The main results of the given article are the next:
- for any C*-algebra $A$ with an infinite orthogonal set \{\(p_i\)\} of equivalent projections such that sup \(i\) \(p_i\) = 1 the set $\sum_{ij} p_iAp_j$ is a C*-algebra with the component-wise algebraic operations, the associative multiplication and the norm.
- there exist a C*-algebra $A$ and different countable orthogonal sets \{\(e_i\)\} and \{\(f_i\)\} of equivalent projections in $A$ such that sup \(i\) \(e_i\) = 1, sup \(j\) \(f_j\) = 1, $\sum_{ij} e_iAf_j \neq \sum_{ij} f_iAe_j$.
- if $A$ is a W*-factor of type II\(\infty\), then there exists a countable orthogonal set \{\(p_i\)\} of equivalent projections in $A$ such that $\sum_{ij} p_iAp_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij} p_iAp_j$ is not a von Neumann algebra.
- if $A$ is a W*-factor of type III, then for any countable orthogonal set \{\(p_i\)\} of equivalent projections in $A$. The $C^*$-subalgebra $\sum_{ij} p_iAp_j$ is simple and purely infinite. In this case $\sum_{ij} p_iAp_j$ is not a von Neumann algebra.
- there exist a C*-algebra $A$ with an orthogonal set \{\(p_i\)\} of equivalent projections such that $\sum_{ij} p_iAp_j$ is not an ideal of $A$. 
1. Infinite Norm Decompositions

Lemma 1. Let $A$ be a $C^*$-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound 1 in the algebra $A$ and let $\mathcal{A} = \{p_i ap_j : a \in A\}$. Then,

1) the set $\mathcal{A}$ is a vector space with the next componentwise algebraic operations

$$\lambda p_i ap_j = \{ap_j\} \forall \lambda \in \mathbb{C}$$

$p_i ap_j + p_i bp_j = \{p_i(a + b)p_j\}, a, b \in A,$

2) the algebra $\mathcal{A}$ and the vector space $\mathcal{A}$ can be identified in the sense of the next map

$$I : a \in A \rightarrow \{p_i ap_j\} \in \mathcal{A}.$$

Proof. Item 1) of the lemma can be easily proved.

Proof of item 2): We assert that $I$ is a one-to-one map. Indeed, it is clear, that for any $a \in A$ there exists a unique set $\{p_i ap_j\}$, defined by the element $a$.

Suppose that there exist different elements $a$ and $b$ in $A$ such that $p_i ap_j = p_i bp_j$ for all $i, j$, i.e. $I(a) = I(b)$. Then $p_i(a - b)p_j = 0$ for all $i$ and $j$. Observe that $p_i((a - b)p_j(a - b)^*) = ((a - b)p_j(a - b)^*)p_i = 0$ and $((a - b)p_j(a - b)^*) \neq 0$ for all $i, j$. Therefore, the element $(a - b)p_j(a - b)^*$ commutes with every projection in $\{p_i\}$.

We prove $(a - b)p_j(a - b)^* = 0$. Indeed, there exists a maximal commutative subalgebra $A_o$ of the algebra $A$, containing the set $\{p_i\}$ and the element $(a - b)p_j(a - b)^*$. Since $(a - b)p_j(a - b)^*p_i = p_i((a - b)p_j(a - b)^*p_i) = 0$ for all $i, j$, the condition $(a - b)p_j(a - b)^* = 0$ contradicts the equality $\sup_i p_i = 1$.

Indeed, in this case $p_i \leq 1 - 1/((a - b)p_j(a - b)^*)(a - b)p_j(a - b)^*$ for any $i$. Since $(a - b)p_j(a - b)^*p_i = p_i((a - b)p_j(a - b)^*p_i) = 0$ for all $i, j$, we get a contradiction with $\sup_i p_i = 1$. Therefore $(a - b)p_j(a - b)^* = 0$.

Hence, since $A$ is a $C^*$-algebra, than $\|(a - b)p_j(a - b)^*\| = \|(a - b)p_j(a - b)^*p_i\| = \|(a - b)p_j\| \|((a - b)p_j(a - b)^*)\| = \|(a - b)p_j\|^2 = 0$ for any $i$. Therefore $(a - b)p_j = 0, (a - b)^*p_j = 0$ for all $j$. Hence the elements $a - b, (a - b)^*$ commute with every projection in $\{p_i\}$. Then there exists a maximal commutative subalgebra $A_o$ of the algebra $A$, containing the set $\{p_i\}$ and the element $(a - b)(a - b)^*$. Since $p_i(a - b)(a - b)^*p_i = p_i((a - b)(a - b)^*p_i) = 0$ for all $i, j$, then the condition $(a - b)(a - b)^* = 0$ contradicts the equality $\sup_i p_i = 1$.

Therefore, $(a - b)(a - b)^* = 0, a - b = 0$, i.e. $a = b$. Thus the map $I$ is one-to-one.

Lemma 2. Let $A$ be a $C^*$-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound 1 in the algebra $A$ and $a \in A$. Then, if $p_i ap_j = 0$ for all $i, j$, then $a = 0$.

Proof. Let $p \in \{p_i\}$. Observe that $p_i ap_j a^* = p_i \{ap_j a^*\} = ap_j a^* p_i = \{ap_j a^*\} p_i = 0$ for all $i, j$ and $ap_j a^* = ap_j a^* = (ap_j a^*) (p_i a) (ap_j a^*) \geq 0$. Therefore, the element $ap_j a^*$ commutes with all projections of the set $\{p_i\}$.

We prove $ap_j a^* = 0$. Indeed, there exists a maximal commutative subalgebra $A_o$ of the algebra $A$, containing the set $\{p_i\}$ and the element $ap_j a^*$. Since $p_i(ap_j a^*) = (ap_j a^*) p_i = 0$ for any $i$, then the condition $ap_j a^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of lemma 1). Hence $ap_j a^* = 0$.

Hence, since $A$ is a $C^*$-algebra, then $\|ap_j a^*\| = \|(ap_j a^*)\| = \|ap_j\| \|a^*\| = \|ap_j\|^2 = 0$ for any $j$. Therefore $ap_j = 0, p_j a = 0$ for any $j$. Analogously we have $p_j a = 0, a^* p_j = 0$ for any $j$. Hence the elements $a, a^*$ commute with all projections.
of the set \( \{p_i\} \). Then there exists a maximal commutative subalgebra \( A_o \) of the algebra \( A \), containing the set \( \{p_i\} \) and the element \( aa^* \). Since \( p_i aa^* = aa^* p_j = 0 \) for any \( i \), then the condition \( aa^* \neq 0 \) contradicts the equality \( \sup_i p_i = 1 \) (see the proof of lemma 1). Hence \( aa^* = 0 \) and \( a = 0 \). \( \square \)

**Lemma 3.** Let \( A \) be a \( C^* \)-algebra of bounded linear operators on a Hilbert space \( H \), \( \{p_i\} \) be an infinite orthogonal set of projections in \( A \) with the least upper bound 1 in the algebra \( B(H) \). Then \( a \geq 0 \) if and only if for any finite subset \( \{p_k\}_{k=1}^n \subset \{p_i\} \) the inequality \( pap \geq 0 \) holds, where \( p = \sum_{k=1}^n p_k \).

**Proof.** By positivity of the operator \( T : a \to bab, a \in A \) for any \( b \in A \), if \( a \geq 0 \), then for any finite subset \( \{p_k\}_{k=1}^n \subset \{p_i\} \) the inequality \( pap \geq 0 \) holds.

Conversely, let \( a \in A \). Suppose that for any finite subset \( \{p_k\}_{k=1}^n \subset \{p_i\} \) the inequality \( pap \geq 0 \) holds, where \( p = \sum_{k=1}^n p_k \).

Let \( a = c + id \), for some nonzero self-adjoint elements \( c, d \) in \( A \). Then \( (p_i + p_j)(c + id)(p_i + p_j) = (p_i + p_j)c(p_i + p_j) + i(p_i + p_j)d(p_i + p_j) \geq 0 \) for all \( i, j \). In this case the elements \( (p_i + p_j)c(p_i + p_j) \) and \( (p_i + p_j)d(p_i + p_j) \) are self-adjoint. Then \( (p_i + p_j)d(p_i + p_j) = 0 \) and \( p_i dp_j = 0 \) for all \( i, j \). Hence by lemma 2 we have \( d = 0 \). Therefore \( a = c = c^* = a^* \), i.e. \( a \in A_{sa} \). Hence, \( a \) is a nonzero self-adjoint element in \( A \). Let \( b_n^a = \sum_{k=1}^n p_k^a = \sum_{i} p_i^a \) for all natural numbers \( n \) and finite subsets \( \{p_i\}_{k=1}^n \subset \{p_i\} \). Then the set \( \{b_n^a\} \) ultraweakly converges to the element \( a \).

Indeed, we have \( A \subseteq B(H) \). Let \( \{q_\xi\} \) be a maximal orthogonal set of minimal projections of the algebra \( B(H) \) such that \( p_i = \sup_{q_\eta} q_\eta \), for some subset \( \{q_\eta\} \subset \{q_\xi\} \), for any \( i \). For arbitrary projections \( q \) and \( p \) in \( \{q_\xi\} \) there exists a number \( \lambda \in \mathbb{C} \) such that \( qap = \lambda u \), where \( u \) is an isometry in \( B(H) \), satisfying the conditions \( q = uu^*, p = u^*u \). Let \( q_\xi \xi = q_\xi \), \( q_\eta \eta = \sum_{\eta} \lambda_{\xi\eta} \xi_\eta \) be such element that \( q_\xi = q_\xi \eta_\xi^* \), \( q_\eta = q_\eta \xi_\eta \) for all different \( \xi \) and \( \eta \). Then, let \( \{\lambda_{\xi\eta}\} \) be a set of numbers such that \( q_\xi q_\eta = \lambda_{\xi\eta} q_\xi q_\eta \) for all \( \xi, \eta \). In this case, since \( q_\xi a a^* q_\eta = q_\xi (\sum_{\eta} \lambda_{\xi\eta} \xi_\eta^*) q_\xi < \infty \), the quantity of nonzero numbers of the set \( \{\lambda_{\xi\eta}\} \) \( (\xi, \eta) \)-string of the infinite dimensional matrix \( \{\xi_\xi^*\}_{\xi \eta} \) is not greater then the countable cardinal number and the sequence \( (\lambda_{\xi\eta})_\eta \) of these nonzero numbers converges to zero. Let \( v_{q_\xi} \) be a vector of the Hilbert space \( H \), which generates the minimal projection \( q_\xi \). Then the set \( \{v_{q_\xi}\} \) forms a complete orthonormal system of the space \( H \). Let \( v \) be an arbitrary vector of the space \( H \) and \( \mu_\xi \) be a coefficient of Fourier of the vector \( v \), corresponding to \( v_{q_\xi} \), in relative to the complete orthonormal system \( \{v_{q_\xi}\} \). Then, since \( \sum_{\xi} \mu_\xi \mu_\xi < \infty \), then the quantity of all nonzero elements of the set \( \{\mu_\xi\} \) is not greater then the countable cardinal number and the sequence \( (\mu_\eta) \) of all these nonzero numbers converges to zero.

Let \( v_{\xi} \) be the \( \xi \)-th coefficient of Fourier (corresponding to \( v_{q_\xi} \)) of the vector \( a(v) \in H \) in relative to the complete orthonormal system \( \{v_{q_\xi}\} \). Then \( \nu_\xi = \sum_{\eta} \lambda_{\xi\eta} \mu_\eta \) and the scalar product \( \langle a(v), v \rangle \) is equal to the sum \( \sum x \nu_\xi \mu_\xi \). Since the element \( a(v) \) belongs to \( H \) we have the quantity of all nonzero elements in the set \( \{\nu_\xi\} \) is not greater then the countable cardinal number and the sequence \( (\nu_\eta) \) of all these nonzero numbers converges to zero.

Let \( \varepsilon \) be an arbitrary positive number. Then, since quantity of nonzero numbers of the sets \( \{\mu_\xi\} \) and \( \{\nu_\xi\} \) \( \xi \) is not greater then the countable cardinal number, and \( \sum_\xi \nu_\xi \nu_\xi < \infty \), \( \sum_\xi \nu_\xi \mu_\xi < \infty \), then there exists \( \{f_\xi\}_{k=1}^n \subset \{p_i\} \) such, that for the
set of indexes $\Omega_1 = \{ \xi : \exists p \in \{ f_k \}_{k=1}^1, q_\xi \leq p \}$ we have

$$\left| \sum_{\xi} \nu_\xi \mu_\xi - \sum_{\xi \in \Omega_1} \nu_\xi \mu_\xi \right| < \varepsilon.$$ 

Then, since quantity of nonzero numbers of the sets $\{ \mu_\xi \}_\xi$ and $\{ \lambda_\xi \}_\eta$ is not greater than the countable cardinal number, and $\sum_\eta \lambda_\xi \eta \lambda_\xi < \infty$, $\sum_\xi \mu_\xi \mu_\xi < \infty$, then there exists $\{ e_k \}_{k=1}^m \subset \{ p_i \}$ such, that for the set of indexes $\Omega_2 = \{ \xi : \exists p \in \{ e_k \}_{k=1}^m, q_\xi \leq p \}$ we have

$$\left| \sum_{\eta} \lambda_\xi \eta \mu_\eta - \sum_{\eta \in \Omega_2} \lambda_\xi \eta \mu_\eta \right| < \varepsilon.$$ 

Hence foe the finite set $\{ p_k \}_{k=1}^n = \{ f_k \}_{k=1}^n \cup \{ e_k \}_{k=1}^m$ and the set $\Omega = \{ \xi : \exists p \in \{ p_k \}_{k=1}^n, q_\xi \leq p \}$ of indexes we have

$$\left| \sum_{\xi} \nu_\xi \mu_\xi - \sum_{\xi \in \Omega} (\sum_{\eta \in \Omega} \lambda_\xi \eta \mu_\eta) \mu_\xi \right| < \varepsilon.$$ 

At the same time, $\langle (\sum_{k=1}^n p_k a p_i)(v), v \rangle = \sum_{\xi \in \Omega} (\sum_{\eta \in \Omega} \lambda_\xi \eta \mu_\eta) \mu_\xi$. Therefore,

$$| \langle a(v), v \rangle - \langle (\sum_{k=1}^n p_k a p_i)(v), v \rangle | < \varepsilon.$$ 

Hence, since the vector $v$ and the number $\varepsilon$ are chosen arbitrarily, we have the net $(b_n^a)$ ultraweakly converges to the element $a$.

We have there exists a maximal orthogonal set $\{ e_\xi \}$ of minimal projections of the algebra $B(H)$ of all bounded linear operators on $H$, such that the element $a$ and the set $\{ e_\xi \}$ belong to some maximal commutative subalgebra $A_\eta$ of the algebra $B(H)$. We have for any finite subset $\{ p_k \}_{k=1}^n \subset \{ p_i \}$ and $e \in \{ e_\xi \}$ the inequality $e(\sum_{k=1}^n p_k a p_i)e \geq 0$ holds by the positivity of the operator $T : b \to e b e, b \in A$.

By the previous part of the proof the net $(e_\xi b_n^a e_\xi)_{\alpha n}$ ultraweakly converges to the element $e_\xi a e_\xi$ for any index $\xi$. Then we have $e_\xi b_n^a e_\xi \geq 0$ for all $n$ and $\alpha$. Therefore, the ultraweak limit $e_\xi a e_\xi$ of the net $(e_\xi b_n^a e_\xi)_{\alpha n}$ is a nonnegative element. Hence, $e_\xi a e_\xi \geq 0$. Therefore, since $e_\xi$ is chosen arbitrarily then $a \geq 0$.

**Lemma 4.** Let $A$ be a $C^\ast$-algebra of bounded linear operators on a Hilbert space $H$, $\{ p_i \}$ be an infinite orthogonal set of projections in $A$ with the least upper bound 1 in the algebra $B(H)$ a $\in A$. Then

$$\| a \| = \sup \{ \| \sum_{k=1}^n p_k a p_i \| : n \in N, \{ p_k \}_{k=1}^n \subset \{ p_i \} \}.$$ 

**Proof.** The inequality $-\| a \| 1 \leq a \leq \| a \| 1$ holds. Then $-\| a \| p \leq p a p \leq \| a \| p$ for all natural number $n$ and finite subset $\{ p_k^a \}_{k=1}^n \subset \{ p_i \}$, where $p = \sum_{k=1}^n p_k$. Therefore

$$\| a \| \geq \sup \{ \| \sum_{k=1}^n p_k a p_i \| : n \in N, \{ p_k \}_{k=1}^n \subset \{ p_i \} \}.$$ 

At the same time, since the finite subset $\{ p_k \}_{k=1}^n$ of $\{ p_i \}$ is chosen arbitrarily and by lemma 6 we have

$$\| a \| = \sup \{ \| \sum_{k=1}^n p_k a p_i \| : n \in N, \{ p_k \}_{k=1}^n \subset \{ p_i \} \}.$$
Otherwise, if

$$\|a\| > \lambda = \sup\{\| \sum_{kl}^n p_k a_{p_l} \| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\},$$

then by lemma 3 $-\lambda 1 \leq a \leq \lambda 1$. But the last inequality is a contradiction. □

**Lemma 5.** Let $A$ be a $C^*$-algebra of bounded linear operators on a Hilbert space $H$, $\{p_i\}$ be an infinite orthogonal set of projections in $A$ with the least upper bound 1 in the algebra $B(H)$, and let $\mathcal{A} = \{p_i a p_j : a \in A\}$. Then,

1) the vector space $\mathcal{A}$ is a unit order space respect to the order $\{p_i a p_j\} \geq 0$ if for any finite subset $\{p_k\}_{k=1}^n \subseteq \{p_i\}$ the inequality $p_i a p_j \geq 0$ holds, where $p = \sum_{k=1}^n p_k$, and the norm

$$\|\{p_i a p_j\}\| = \sup\{\| \sum_{kl}^n p_k a_{p_l} \| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\}.$$

2) the algebra $\mathcal{A}$ and the unit order space $\mathcal{A}$ can be identified as unit order spaces in the sense of the map

$$\mathcal{I} : a \in A \to \{p_i a p_j\} \in \mathcal{A}.$$

**Proof.** This lemma follows by lemmas 1, 3 and 4. □

**Remark.** Observe that by lemma 4 the order and the norm in the unit order space $\mathcal{A} = \{p_i a p_j : a \in A\}$ can be defined as follows to: $\{p_i a p_j\} \geq 0$, if $a \geq 0$; $\|\{p_i a p_j\}\| = \|a\|$. By lemmas 3 and 4 they are equivalent to the order and the norm, defined in lemma 5, correspondingly.

Let $A$ be a $C^*$-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in $A$ such that $\sum p_i = 1$ and

$$\sum_{ij} a_{ij} = \{a_{ij} : \text{for any indexes } i, j, a_{ij} \in p_i a p_j, \text{ and }$$

$$\| \sum_{k=1, \ldots, i-1} (a_{ki} + a_{ik}) + a_{ii} \| \to 0 \text{ at } i \to \infty \}. $$

If we introduce the componentwise algebraic operations in this set, then $\sum_{ij} p_i a p_j$ becomes a vector space. Also, note that $\sum_{ij} p_i a p_j$ is a vector subspace of $\mathcal{A}$. Observe that $\sum_{ij} p_i a p_j$ is a normed subspace of the algebra $\mathcal{A}$ and $\|\sum_{i,j=1}^n a_{ij} - \sum_{i,j=1}^{n+1} a_{ij}\| \to 0$ at $n \to \infty$ for any $\{a_{ij}\} \in \sum_{ij} p_i a p_j$.

Let $\sum_{ij} a_{ij} = \lim_{n \to \infty} \sum_{i,j=1}^n a_{ij}$, for any $\{a_{ij}\} \in \sum_{ij} p_i a p_j$, and

$$C^*:\{p_i a p_j\}_{ij} = \{a_{ij} : a_{ij} \in \sum_{ij} p_i a p_j\}.$$

Then $C^*\{p_i a p_j\}_{ij} \subseteq \mathcal{A}$. By lemma 5 $A$ and $\mathcal{A}$ can be identified. We observe that, the normed spaces $\sum_{ij}^\alpha p_i a p_j$ and $C^*\{p_i a p_j\}_{ij}$ can also be identified. Further, without loss of generality we will use these identifications.

**Theorem 6.** Let $A$ be a unital $C^*$-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in $A$ and $\sup p_i = 1$. Then $\sum_{ij}^\alpha p_i a p_j$ is a $C^*$-subalgebra of $A$ with the componentwise algebraic operations, the associative multiplication and the norm.
Corollary 7. Let \( H \) be a unital \( C^* \)-algebra of bounded linear operators in a Hilbert space \( H \), \( \{ p_i \} \) be a countable orthogonal set of equivalent projections in \( A \) and \( \sup p_i = 1 \). Let \( \{ q_i \} \) be a countable orthogonal set of equivalent projections in \( B(H) \) defined by the set \( \{ p_i \} \) in \( B(H) \). Then \( \sum_{ij} q_i A q_j \) is a \( C^* \)-subalgebra of the algebra \( A \).

Proof. Let \( \{ \{ p^j_i \} \} \) be the set of infinite subsets of \( \{ p_i \} \) such that for all distinct \( \xi \) and \( \eta \) \( \{ p^\xi_j \} \cap \{ p^\eta_j \} = \emptyset \), \( \| p^\xi_j \| = \| p^\eta_j \| \) and \( \{ p_i \} = \cup_i \{ p^j_i \} \). Then let \( q_i = \sup_j p^j_i \) in \( B(H) \), for all \( i \). Then \( \sup q_i = 1 \) and \( \{ q_i \} \) be a countable orthogonal set of equivalent projections. Then we say that the countable orthogonal set \( \{ q_i \} \) of equivalent projections is defined by the set \( \{ p_i \} \) in \( B(H) \). We have the next corollary.

Let \( H \) be an infinite dimensional Hilbert space, \( B(H) \) be the algebra of all bounded linear operators. Let \( \{ p_i \} \) be a countable orthogonal set of equivalent projections in \( B(H) \) and \( \sup p_i = 1 \). Let \( \{ p^j_i \} \) be the set of infinite subsets of \( \{ p_i \} \) such that for all distinct \( \xi \) and \( \eta \) \( \{ p^\xi_j \} \cap \{ p^\eta_j \} = \emptyset \), \( \| p^\xi_j \| = \| p^\eta_j \| \) and \( \{ p_i \} = \cup_i \{ p^j_i \} \). Then let \( q_i = \sup_j p^j_i \) in \( B(H) \), for all \( i \). Then \( \sup q_i = 1 \) and \( \{ q_i \} \) be a countable orthogonal set of equivalent projections. Then we say that the countable orthogonal set \( \{ q_i \} \) of equivalent projections is defined by the set \( \{ p_i \} \) in \( B(H) \). We have the next corollary.
Example 1. Let $M$ be the closure on the norm of the inductive limit $M_n$ of the inductive system
\[ C \rightarrow M_2(C) \rightarrow M_3(C) \rightarrow M_4(C) \rightarrow \ldots, \]
where $M_n(C)$ is mapped into the upper left corner of $M_{n+1}(C)$. Then $M$ is a C*-algebra (1). The algebra $M$ contains the minimal projections of the form $e_{ii}$, where $e_{ij}$ is an infinite dimensional matrix, whose $(i, i)$-th component is 1 and the rest components are zeros. These projections form the countable orthogonal set \( \{ e_{ii} \}_{i=1}^{\infty} \) of minimal projections. Let
\[ M^o_n(\mathcal{C}) = \{ \sum_{ij} \lambda_{ij} e_{ij} : \text{for any indexes } i, j, \lambda_{ij} \in \mathcal{C}, \text{ and} \]
\[ \| \sum_{k=1}^{\infty} (\lambda_{jk} e_{kj} + \lambda_{ik} e_{ik}) + \lambda_{ii} e_{ii} \| \rightarrow 0 \text{ at } i \rightarrow \infty \}. \]

Then $\mathcal{C} \cdot 1 + M^o_n(\mathcal{C}) = \mathcal{M}$ (see [2]) and by theorem 6 $M^o_n(\mathcal{C})$ is a simple C*-algebra. Note that there exists a mistake in the formulation of theorem 3 in [2].

C. I + $M^o_n(\mathcal{C})$ is a C*-algebra. But the algebra $\mathcal{C} \cdot 1 + M^o_n(\mathcal{C})$ is not simple. Because $\mathcal{C} \cdot 1 + M^o_n(\mathcal{C}) \neq M^o_n(\mathcal{C})$ and $M^o_n(\mathcal{C})$ is an ideal of the algebra $\mathcal{C} \cdot 1 + M^o_n(\mathcal{C})$, i.e. $(\mathcal{C} \cdot 1 + M^o_n(\mathcal{C})) \cdot M^o_n(\mathcal{C}) \subseteq M^o_n(\mathcal{C})$.

2. There exist a C*-algebra $A$ and different countable orthogonal sets $\{ e_i \}$ and $\{ f_i \}$ of equivalent projections in $A$ such that $\text{sup}_i e_i = 1$, $\text{sup}_i f_i = 1$, $\sum\nolimits_{ij} e_{ij} A e_{ij} \neq \sum\nolimits_{ij} f_{ij} A f_{ij}$. Indeed, let $H$ be an infinite dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{ p_i \}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\text{sup}_i p_i = 1$. Then $\sum\nolimits_{i} p_i B(H) p_j \subset B(H)$.

Let $\{ \{ p^o_{ij} \} \}_{i,j}$ be the set of infinite subsets of $\{ p_i \}$ such that for all distinct $i$ and $j$ and $\eta \{ p^o_{ij} \}_j \cap \{ p^o_{ij} \}_j = \emptyset$, $|\{ p^o_{ij} \}_j| = |\{ p^o_{ij} \}_j|$ and $\{ p_i \} = \bigcup_i \{ p^o_{ij} \}_j$. Then let $q_i = \text{sup}_j p^o_{ij}$ for all $i$. Then $\text{sup}_i q_i = 1$ and $\{ q_i \}$ be a countable orthogonal set of equivalent projections. We assert that $\sum\nolimits_{ij} p_i B(H) p_j \neq \sum\nolimits_{ij} q_i B(H) q_j$. Indeed, let $\{ x_{ij} \}$ be a set of matrix units constructed by the infinite set $\{ p^o_{ij} \}_j$, i.e. for all $i, j, x_{ij} x_{ij} = p^o_{ij}, x_{ij} x_{ij} = p^o_{ij}, x_{ii} = p^o_{ij}$. Then the von Neumann algebra $\mathcal{N}$ generated by the set $\{ x_{ij} \}$ is isometrically isomorphic to $B(H)$ for some Hilbert space $H$. We note that $\mathcal{N}$ is not subset of $\sum\nolimits_{ij} p_i B(H) p_j$. At the same time, $\mathcal{N} \subseteq \sum\nolimits_{ij} q_i B(H) q_j$ and $\sum\nolimits_{ij} p_i N p^o_{ij} \subseteq \sum\nolimits_{ij} p_i B(H) p_j$.

Theorem 8. Let $A$ be a unital simple C*-algebra of bounded linear operators in a Hilbert space $H$, $\{ p_i \}$ be a countable orthogonal set of equivalent projections in $A$ and $\text{sup}_i p_i = 1$. Let $\{ q_i \}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{ p_i \}$ in $B(H)$. Then $\sum\nolimits_{ij} q_i A p_j$ is a simple C*-algebra.

Proof. By theorem 6 $\sum\nolimits_{ij} p_i A p_j$ is a C*-algebra. Let $\{ \{ p^o_{ij} \} \}_j$ be the set of infinite subsets of $\{ p_i \}$ such that for all distinct $i$ and $j$ and $\eta \{ p^o_{ij} \}_j \cap \{ p^o_{ij} \}_j = \emptyset$, $|\{ p^o_{ij} \}_j| = |\{ p^o_{ij} \}_j|$ and $\{ p_i \} = \bigcup_i \{ p^o_{ij} \}_j$. Then let $q_i = \text{sup}_j p^o_{ij}$ in $B(H)$, for all $i$. Then we have for all $i$ and $j$ $q_i A q_j = \{ \{ p^a_{ij} \}_j : a \in A \}$. Hence $q_i A q_j \subset A$ for all $i$ and $j$. By corollary 7 $\sum\nolimits_{ij} q_i A q_j$ is a C*-algebra.

Since projections of the set $\{ p_i \}$ pairwise equivalent then the projection $q_i$ is equivalent to $1 \in A$ for any $i$. Hence $q_i A q_i \cong A$ and $q_i A q_i$ is a simple C*-algebra for any $i$. 

Let $q$ be an arbitrary projection in $\{q_i\}$. Then $qAq$ is a subalgebra of $\sum_i q_i Aq_i$. Let $I$ be a closed ideal of the algebra $\sum_i q_i Aq_i$. Then $IQAq \subseteq I$ and $IQqAq \subseteq IQ$. Therefore $IQqAq \subseteq qIQ$, that is $IQq$ is a closed ideal of the subalgebra $qAq$. Since $qAq$ is simple then $IQq = qAq$.

Let $q_1$, $q_2$ be arbitrary projections in $\{q_i\}$. We assert that $q_1 Iq_2 = q_1 Aq_2$ and $q_2 Iq_1 = q_2 Aq_1$. Indeed, we have the projection $q_1 + q_2$ equivalent to $1 \in A$. Let $e = q_1 + q_2$. Then $e A e \cong A$ and $e A e$ is simple $C^*$-algebra. At the same time we have $e A e$ is a subalgebra of $\sum_i q_i Aq_i$ and $I$ is an ideal of $\sum_i q_i Aq_i$. Hence $I e A e \subseteq I$ and $I e A e \subseteq I e$. Therefore $e I e A e \subseteq e I e$, that is $e I e$ is a closed ideal of the subalgebra $e A e$. Since $e A e$ is simple then $e I e = e A e$. Hence $q_1 Iq_2 = q_1 Aq_2$ and $q_2 Iq_1 = q_2 Aq_1$. Therefore $q_i Iq_j = q_i Aq_j$ for all $i$ and $j$. We have $I$ is norm closed. Hence $I = \sum_i q_i Aq_i$, i.e. $\sum_i q_i Aq_i$ is a simple $C^*$-algebra.

2. Applications

**Theorem 9.** Let $\mathcal{N}$ be a $W^*$-factor of type $\text{II}_\infty$ of bounded linear operators in a Hilbert space $H$, $\{p_i\}$ be a countable orthogonal set of equivalent projections in $\mathcal{N}$ and $\sum_i p_i = 1$. Then for any countable orthogonal set $\{q_i\}$ of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$ the $C^*$-algebra $\sum_i q_i N q_j$ is a $C^*$-factor with a nonzero finite and an infinite projection. In this case $\sum_i q_i N q_j$ is not a von Neumann algebra.

**Proof.** By the definition of the set $\{q_i\}$ we have $\sup_i q_i = 1$ and $\{q_i\}$ is a countable orthogonal set of equivalent infinite projections. By theorem 6 we have $\sum_i q_i N p_j$ is a $C^*$-subalgebra of $\mathcal{N}$. Let $q$ be a nonzero finite projection of $\mathcal{N}$. Then there exists a projection $p \in \{q_i\}$ such that $qp \neq 0$. We have $q N q$ is a finite von Neumann algebra. Let $x = pq$. Then $x N x^*$ is a weakly closed $C^*$-subalgebra. Note that the algebra $x N x^*$ has a center-valued faithful trace. Let $e$ be a nonzero projection of the algebra $x N x^*$. Then $e p = e$ and $e \in p N p$. Hence $e \in \sum_i q_i N q_j$. We have the weak closure of $\sum_i q_i N q_j$ in the algebra $\mathcal{N}$ coincides with this algebra $\mathcal{N}$. Then by the weak continuity of the multiplication $\sum_i q_i N q_j$ is a factor. Note since $1 \notin \sum_i q_i N q_j$ then $\sum_i q_i N q_j$ is not weakly closed in $\mathcal{N}$. Hence the $C^*$-factor $\sum_i q_i N q_j$ is not a von Neumann algebra.

**Remark.** Note that, in the article [3] a simple $C^*$-algebra with an infinite and a nonzero finite projection have been constructed by M.Rørdam. In the next corollary we construct a simple purely infinite $C^*$-algebra. Note that simple purely infinite $C^*$-algebras are considered and investigated, in particular, in [4] and [5].

**Theorem 10.** Let $\mathcal{N}$ be a $W^*$-factor of type $\text{III}$ of bounded linear operators in a Hilbert space $H$. Then for any countable orthogonal set $\{p_i\}$ of equivalent projections in $\mathcal{N}$ such that $\sup_i p_i = 1$, $\sum_i p_i N p_j$ is a simple purely infinite $C^*$-algebra. In this case $\sum_i p_i N p_j$ is not a von Neumann algebra.

**Proof.** Let $p_i$, be a projection in $\{p_i\}$. We have the projection $p_i$, can be exhibited as a least upper bound of a countable orthogonal set $\{p'_i\}_j$ of equivalent projections in $\mathcal{N}$. Then for any $i$ the projection $p_i$ has a countable orthogonal set $\{p'_i\}_j$ of equivalent projections in $\mathcal{N}$ such that the set $\bigcup_i \{p'_i\}_j$ is a countable orthogonal set of equivalent projections in $\mathcal{N}$. Hence the set $\{p_i\}$ is defined by the set $\bigcup_i \{p'_i\}_j$ in $B(H)$ (in $\mathcal{N}$). Hence by theorem 8 $\sum_i p_i N p_j$ is a simple $C^*$-algebra. Note since
1 \notin \sum_i^\infty p_i N p_i \text{ then } \sum_i^\infty p_i N p_i \text{ is not weakly closed in } N. \text{ Hence } \sum_i^\infty p_i N p_i \text{ is not a von Neumann algebra.}

Suppose there exists a nonzero finite projection } q \text{ in } \sum_i^\infty p_i N p_i. \text{ Then there exists a projection } p \in \{p_i\} \text{ such that } qp \neq 0. \text{ We have } q(\sum_i^\infty p_i N p_i)q \text{ is a finite C*-algebra. Let } x = pq. \text{ Then } x N x^* \text{ is a C*-subalgebra. Moreover } x N x^* \text{ is weakly closed and } x N x^* \subset p N p. \text{ Hence } x N x^* \text{ has a center-valued faithful trace. Then } x N x^* \text{ is a finite von Neumann algebra with a center-valued faithful normal trace. Let } e \text{ be a nonzero projection of the algebra } x N x^*. \text{ Then } ep = e \text{ and } e \in p N p. \text{ Hence } e \in N. \text{ This is a contradiction.}

\text{□}

Example. Let } H \text{ be a separable Hilbert space and } B(H) \text{ the algebra of all bounded linear operators on } H. \text{ Let } \{q_i\} \text{ be a maximal orthogonal set of equivalent minimal projections in } B(H). \text{ Then } \sum_i q_i B(H)q_i \text{ is a two sided closed ideal of the algebra } B(H). \text{ Using the set } \{q_i\} \text{ we construct a countable orthogonal set } \{p_i\} \text{ of equivalent infinite projections such that } \text{sup}_i p_i = 1. \text{ Let } \{(q_i^j)_j\}_i \text{ be the countable set of countable subsets of } \{q_i\} \text{ such that for all distinct } i_1 \text{ and } i_2 \text{ we have } (q_i^j)_{i_1} \cap (p_j^j)_{i_2} = \emptyset \text{ and } \{q_i\} = \cup_i (q_i^j)_j. \text{ Then let } p_i = \text{sup}_j q_i^j \text{ for all } i. \text{ Then } \text{sup}_i p_i = 1 \text{ and } \{p_i\} \text{ is a countable orthogonal set of equivalent infinite projections in } B(H) \text{ defined by } \{q_i\} \text{ in } B(H).

Let } \{q_{nm}^{ij}\} \text{ be the set of matrix units constructed by the set } \{(q_i^j)_i\}, \text{ i.e. } q_{nm}^{ij} q_{nm}^{ij} = q_{nm}^{ij} q_{nm}^{ij} = q_{nm}^{ij} = q_{nm}^{ij} \text{ for all } i, j, n, m. \text{ Then let } a = \{a_{nm}^{ij} q_{nm}^{ij}\} \text{ be the decomposition of the element } a \in B(H), \text{ where the components } a_{nm}^{ij} \text{ are defined as follows}

\[ a_{11}^{11} = \lambda, a_{12}^{12} = \lambda, a_{13}^{13} = \lambda, \ldots, a_{1n}^{1n} = \lambda, \ldots, \]

and the rest components } a_{nm}^{ij} \text{ are zero, i.e. } a_{nm}^{ij} = 0. \text{ Then } p_i a = a. \text{ Then since } a \notin \sum_i^\infty p_i B(H)p_j \text{ and } p_i \in \sum_j^\infty p_j B(H)p_j \text{ then } \sum_i^\infty p_i B(H)p_j \text{ is not an ideal of } B(H). \text{ But by theorem } 6 \sum_i^\infty p_i B(H)p_j \text{ is a C*-algebra. Hence there exists a C*-algebra } A \text{ with an orthogonal set } \{p_i\} \text{ of equivalent projections such that } \sum_i^\infty p_i A p_j \text{ is not an ideal of } A.

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