We derive a set of equations describing the real time dynamics of modes with spatial momentum of order $g^2T$ in a high temperature gauge theory, where $g$ is the coupling constant and $T$ is the temperature. This dynamics is stochastic in nature. Important implications for baryon number violation at high temperature and for the physics at the electroweak phase transition, are discussed.

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It has been known for a long time that perturbative calculations in hot SU(N) gauge theories (QCD) are plagued with an infrared problem. The scale of momentum where the problem emerges is \( g^2 T \) where \( g \) is the gauge coupling and \( T \) is the temperature of the plasma. No reliable way has been suggested for treating these long range modes. The problem can be seen as originating from the fact that thermal fluctuations of the magnetic modes at the scale \( g^2 T \) are large enough for the theory to become essentially non-linear.

The \( g^2 T \) scale is important for understanding a large number of physical problems such as magnetic screening, baryon number transitions in the electroweak theory at high temperature and the determination of the parameters of the electroweak phase transition. In particular, all of these need to be understood for a complete calculation of electroweak baryogenesis. The real-time treatment is appropriate for some problems, especially for computing topological transition rates in the Standard Model.

In this paper we derive a set of equations which describe the real-time dynamics of non-perturbative soft modes in hot gauge theories and we discuss two of their most important applications. The first one is baryon number violation at high \( T \). The physics of the soft modes is stochastic in nature and has the natural time scale \( \sim (g^4 T)^{-1} \). In particular, our equations suggest that baryon number violation at high temperature in the unbroken phase scales as \( \alpha^5 W T^4 \) as the result of a diffusive motion across a topological barrier, rather than \( \alpha^4 W T^4 \) as classical arguments and recent numerical simulations suggest \([1, 2]\). Last but not least, our equations constitute a solid computational tool for the determination of the numerical coefficient of \( \alpha^5 W T^4 \). This would have important consequences for electroweak baryogenesis, and the constraints imposed on it from the structure of the symmetry breaking sector and from the \( CP \) violating sector. As a second application, we introduce a possible path for the real-time resolution of infrared divergences which plague the perturbative computation of the parameters of the electroweak phase transition.

**Stochastic dynamics of soft modes in hot gauge theory**

1. Let us begin with a simple argument showing that the time scale responsible for processes occurring at the space extent of \((g^2 T)^{-1}\) has the natural time scale of \((g^4 T)^{-1}\). To this end we consider thermal fluctuations of the gauge field \( A_\mu \). The power spectrum of these fluctuations can be related to the imaginary part of the gluon propagator by the fluctuation–dissipation theorem. At low frequencies, \( \omega \ll T \), one has,

\[
\langle A_\mu^\dagger(\omega, \mathbf{q}) A_\nu(\omega', \mathbf{q}') \rangle = -\frac{2T}{\omega} \text{Im} D_{\mu\nu}^R(\omega, \mathbf{q}) \cdot (2\pi)^4 \delta(\omega - \omega') \delta(\mathbf{q} - \mathbf{q}')
\]

\( ^1\) For recent attempts to compute the magnetic mass, see \([3]\) and references therein.

\( ^2\) For parallel considerations with an emphasis on cross barrier transitions see \([4]\).
The retarded propagator $D^R$ at small $\omega$ and $q$ acquires important contributions from the hard thermal loops,

$$D^R(\omega, q) \sim \frac{1}{\omega^2 - q^2 - \Pi(\omega, q)}.$$  

For non-perturbative physics, we are interested only in transverse (or magnetic) modes, for which the gluon polarization $\Pi(\omega, q)$ has the following behavior

$$\Pi(\omega, q) = -icg^2T^2q\omega/q,$$

in the regime $\omega \ll q \equiv |q|$, where $c$ is some real constant. Eq. (1) implies, in this regime,

$$\langle A^*(\omega, q)A(\omega', q') \rangle \sim \frac{g^2T^3q}{q^6 + cg^2T^4q^2} \cdot \delta(\omega - \omega')\delta(q - q').$$  

To compute the one-time correlation of the field, we integrate Eq. (2) with respect to $\omega$; this gives

$$\langle A^*(t, q)A(t, q') \rangle \sim \delta(q - q') \int d\omega (-1) \frac{T\mathrm{Im}}{\omega^2 - q^2 + icg^2T^2q^3} \frac{1}{q^2} \delta(q - q') \sim \frac{T}{q^2} \delta(q - q')$$

where the integral is saturated for $\omega \sim g^{-2}T^{-2}q^3$. Substituting $q \sim g^2T$ in this expression implies $\omega \sim g^4T$, which is what we wanted to show. To see that the non-perturbative scale of spatial momenta is $g^2T$, we can use Eq. (3) to find the contribution of the modes with momenta of order $q$ to the correlator of $A$ at coinciding points; we find

$$\langle A^2(t, x) \rangle \sim Tq.$$  

Hence the typical fluctuation size of $A$ is $(Tq)^{1/2}$ if restricted to modes with spatial momentum of order $q$. The nonlinear contribution to the field tensor $\mathbb{g}A^2$ becomes comparable with the linear part $\partial A$ when $q \sim g^2T$. So, we have shown that the non-perturbative physics is associated with modes with spatial momenta of order $g^2T$ and frequencies of order $g^4T$.

2. To find the effective theory describing these non-perturbative modes, we resort to the recent reformulation of the hard thermal loop Lagrangian in terms of a kinetic Vlasov equation \[5, 6\]. In QCD, the soft dynamics is described by the soft field $A_\mu(t, x)$ and $\delta n^a(t, x, p)$, the deviation from equilibrium of the hard gluon distribution functions. The scale of space and time variation of both $A_\mu(t, x)$ and $\delta n^a(t, x, p)$ is much larger than $T^{-1}$ while $p \sim T$. The time evolution of these two effective variables of the theory is described by a Langevin-type Vlasov equation having the form

$$v^\mu D^b_\mu \delta n^a(t, x, p) + g\mathbb{E} \cdot \mathbb{E} \frac{\partial \delta n}{\partial |p|} = 0$$  

(4)
with
\[ \partial^\mu F_{\mu\nu}^a = 2gN \int \frac{dp}{(2\pi)^3} v_\nu \delta n^a(t, x, p) + \xi^a(t, x) \] (5)

where \( v^\mu = (1, p/|p|) \), and \( D^{ab}_\mu = \delta^{ab} \partial_\mu + gf^{abc} A^c_\mu \) is the covariant derivative. In these equations, \( \bar{n} \) refers to the equilibrium Bose-Einstein distribution. The difference between Eq. (3) and that discussed in Ref. [5] is the presence of the stochastic noise \( \xi^a(x) \) in the field equation. Eqs. (4,5) could be derived using the formalism of Ref. [7] which has recently been implemented in the scalar theory [8]. However, in this letter we will rather use physical arguments to justify the need to include the stochastic noise \( \xi^a(x) \) into the kinetic equation and to find the correlator \( \langle \xi^a_\mu(x) \xi^b_\nu(y) \rangle \). In fact, while the noiseless Vlasov equation describes the time evolution of the main values of the mean field \( A_\mu(x) \) and the distribution function \( \delta n^a(t, x, p) \), there are many instances where spontaneous thermal fluctuations need to be considered (for example, in the problem of baryon number violation in a hot plasma). Because the term \( \xi^a(x) \) in Eq. (6) reflects fluctuations in the current density, one of our main tasks is to find the form of the noise correlator and to simplify Eqs. (4,5) so that they include only the modes we are interested in, i.e., modes with \( \omega \sim g^4 T \) and \( p \sim g^2 T \).

Before proceeding further, let us make one important remark. Eqs. (4,5) have been shown in Ref. [3] to reproduce the HTL (hard thermal loop) effective Lagrangian describing the physics of modes with momentum \( q \sim gT \). One could ask whether the HTL Lagrangian correctly describes the physics at much smaller spatial momentum scale of \( g^2 T \). A diagrammatic analysis shows that beside the HTL diagrams, diagrams with ladder insertions are also important (see [9] and a recent discussion in [10]). However, we argue here that these diagrams do not modify the dynamics at the momentum scale \( g^2 T \). Our first argument is based on the similarity of the ladder diagrams with those considered in [11] in the framework of the scalar theory. In the latter context, it has been shown that the effect of this infinite set of diagrams is the generation of a collision term in the transport equations. One can argue that this statement can be extended also to gauge theories [12]. Therefore, we can take into account all diagrams relevant to the \( g^2 T \) dynamics by simply adding a collision term to the RHS of Eq. (4). However, as will be demonstrated later, the effect of collisions is suppressed at length scales \( (g^2 T)^{-1} \) and starts to becoming important only at length scales of order \( (g^4 T)^{-1} \). As a bonus, we can safely ignore all non-HTL diagrams. A second argument is based on explicit calculations in QED showing that the resumed diagrams do coincide with the naive HTL answer for \( \Pi(\omega, q) \) when \( \omega, q \sim g^2 T \).

3. Let us now derive the correlator of \( \xi^a_\mu(x) \). We begin by considering in detail a
simpler case, namely, hot QED, for which the Vlasov equation has the form,

$$ \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) \delta n(t, x, p) + eE \frac{\partial \bar{n}(|p|)}{\partial |p|} = 0 $$

(6)

$$ \partial^\mu F_{\mu\nu} = 4e \int \frac{dp}{(2\pi)^3} v_\nu \delta n(t, x, p) + \xi_\nu(t, x). $$

(7)

Here $\delta n(t, x, p)$ is the deviation from equilibrium of the distribution function of fermions with a given spin (we assume that the distribution functions are diagonal in the spin space), and $\bar{n}$ is the equilibrium Fermi distribution. The factor 4 is due to the fact that the plasma consists of particles and anti-particles, each having 2 spin degrees of freedom. The dynamics in QED is simplified, since the Vlasov equation is linear, but still, it is useful to consider it before turning to the more complex case of QCD.

To see the physical origin of the stochastic noise $\xi_\nu$ in Eq. (7), let us imagine the plasma as a collection of fermions and anti-fermions, with, for distribution functions, $n_+(t, x, p)$ and $n_-(t, x, p)$, respectively. In the absence of an external field, $n_\pm(t, x, p)$ are equal to the thermal distribution $\bar{n}(p)$ and the resulting current density in the plasma vanishes

$$ j_\mu(t, x) = e \int \frac{dp}{(2\pi)^3} v_\mu \sum_\alpha (n_+^\alpha(t, x, p) - n_-^\alpha(t, x, p)) = 0, $$

(8)

(the index $\alpha$ labels the spin degrees of freedom). However, Eq. (8) represents only the average of the current density: there are always thermal fluctuations of $j_\mu(t, x)$ due to fluctuations of the distribution functions $n_\pm(t, x, p)$ about the Bose-Einstein main value. These fluctuations can be divided into a long-distance part and a short-distance part. The former is $\delta n(t, x, p)$ which enters the Vlasov equation (Eq. (6)). These fluctuations have for typical length scale $\sim (g^2 T)^{-1}$. The short-distance part fluctuates over much shorter distance scales (up to $T^{-1}$). These short-distance fluctuations are essential for keeping the soft modes in thermal equilibrium. A Vlasov equation without the stochastic noise predicts that modes with $\omega < q$ go away after a sufficiently large amount of time due to Landau damping: this is not the case of fluctuations in a realistic plasma. Let us denote the short-distance fluctuations of the distribution function by $\Delta n_\alpha(t, x, p)$. Then $\xi_\nu(x)$, in Eq. (7), is the fluctuation of the current density

$$ \xi_\nu(t, x) = e \int \frac{dp}{(2\pi)^3} v_\nu \sum_\alpha (\Delta n_+^\alpha - \Delta n_-^\alpha). $$

3The distribution function of anti-particles deviates from equilibrium by the amount $-\delta n(t, x, p)$. 
One property of $\xi_\nu$ follows from charge conservation, $\partial_\nu \xi^\nu = 0$. To find the correlation of $\xi_\nu(x)$, we first observe that the correlation of the thermal fluctuations $\Delta n_\pm(t, x, p)$ at coinciding times is given by

$$\langle \Delta n_+^\alpha(t, x, p) \Delta n_-^\alpha(t, x', p') \rangle = (2\pi)^3 \delta(p - p')\delta(x - x')\delta^{\alpha\alpha'} \bar{n}(|p|)(1 - \bar{n}(|p|)) \quad (9)$$

(the correlator $\langle \Delta n_+ \Delta n_- \rangle$ vanishes). The spatial delta function $\delta(x - x')$ reflects the short-distance nature of the fluctuations included in $\Delta n_\pm(t, x, p)$. Since we are interested in the correlation of $\xi_\nu(x)$ at different times, we also have to know the time evolution of $\Delta n_\pm(t, x, p)$. It is obtained by solving the following transport equation\(^4\)

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) \Delta n(t, x, p) = 0. \quad (10)$$

Its solution is $\Delta n(t, x, p) = \Delta n(0, x - vt, p)$. From this we infer that the noise correlator is

$$\langle \xi_\mu(t, x) \xi_\nu(t', y) \rangle = 4e^2 \int \frac{dp}{(2\pi)^3} v_\mu v_\nu \delta(x - y - v(t - t')) \bar{n}(|p|)(1 - \bar{n}(|p|)) \quad (11)$$

The chain of arguments which lead to Eq. (11) is somewhat intuitive. We can, however, verify this result in various ways. One way is to compute the correlator directly in terms of the fundamental fermionic field operator, $\psi$,

$$\langle \xi_\mu(x) \xi_\nu(y) \rangle = e^2 \langle \bar{\psi}(x) \gamma_\mu \psi(x) \cdot \bar{\psi}(y) \gamma_\nu \psi(y) \rangle \quad (12)$$

The RHS of Eq. (12) is easily evaluated in the momentum representation. It is, in the approximation of free fermions,

$$\langle \xi_\mu(\omega, q) \xi_\nu(\omega', q') \rangle = 4e^2 \int \frac{dp}{(2\pi)^3} v_\mu v_\nu \delta(\omega - \omega - vq) \bar{n}(|p|)(1 - \bar{n}(|p|)) \times (2\pi)^4 \delta(\omega - \omega')\delta(q - q') \quad (13)$$

It is simple to show that Eq. (13) is just the Fourier transform of Eq. (11).

\(^4\)This equation is appropriate for describing the propagation of fluctuations on a time scale much smaller than the transport mean free time.
We can also verify Eqs. (11,13) using the fluctuation-dissipation theorem. Schematically, we can solve Eq. (6,7) to relate the power spectrum of the transverse part of $A_\mu$ with that of $\xi_\mu$ by the very simple formula,

$$\langle A^*(\omega, q)A(\omega', q') \rangle = \frac{\langle |\xi(\omega, q)|^2 \rangle}{|\omega^2 - p^2 - \Pi(\omega, q)|^2}.$$  \hspace{1cm} (14)

We did check that Eq. (14) leads to the spectrum of $A_\mu$-fluctuations satisfying the fluctuation-dissipation theorem, Eq. (1).

Having obtained the correlator, we now proceed to developing an effective dynamics for the long range modes of hot QED. We can solve the Vlasov equation Eq. (6) for $\delta n(t, x, p)$, and use it to obtain an equation for the induced current responding to the external field $A_\mu$. We obtain

$$\delta n(t, x, p) = -e \int_0^\infty du v E(t - u, x - uv) \frac{\partial \bar{n}|p|}{\partial |p|},$$  \hspace{1cm} (15)

and

$$j(t, x) = 4e \int \frac{dp}{(2\pi)^3} v \delta n(t, x, p) = -\frac{e^2 T^2}{12\pi} \int dy \frac{x - y}{|x - y|^4} [(x - y) \cdot E(t - |x - y|, y)].$$  \hspace{1cm} (16)

Until now, our only assumption has been that $\omega$ and $k$ are much smaller than $T$. In the regime $\omega \sim g^4 T, k \sim g^2 T$, additional simplification applies. In Eq. (14) we can replace $t - |x - y|$ with $t$, since the typical time scale is $(g^4 T)^{-1}$ and $x - y \sim (g^2 T)^{-1} \ll (g^4 T)^{-1}$. Substituting (14) into Eq. (7) and recalling that $\mathbf{E} = \mathbf{A}$, is negligible in comparison to $\nabla \times \mathbf{B}$, where $\mathbf{B} \equiv \nabla \times \mathbf{A}$ is the magnetic field, we obtain an equation which evolves long range fields in hot QED,

$$\nabla \times [\nabla \times A(t, x)] = \frac{e^2 T^2}{12\pi} \int dy \frac{x - y}{|x - y|^4} [(x - y) \cdot \dot{A}(t, y)] + \xi(t, x).$$  \hspace{1cm} (17)

Finally, we observe from Eq. (11) that the noise correlator is only non-vanishing on the light cone, $t = \pm |x|$. Since $x$ is typically much smaller(by a factor of $g^2$) than the time scale we are interested in, we can approximate the time dependence of the noise correlator with a delta function. To find the spatial dependence of $\xi_\nu$, we integrate the RHS of Eq. (14) with respect to $t$ and find,

$$\langle \xi_i(t, x)\xi_j(t', y) \rangle = \frac{e^2 T^3}{6\pi} \frac{(x - y)_i(x - y)_j}{|x - y|^4} \delta(t - t').$$  \hspace{1cm} (18)

5Since the important modes are magnetic ones, we can limit ourselves to the three-vector $\mathbf{j}$. 6
Eqs. (17,18) are the final equations describing the evolution of the soft modes in hot QED. Before going further, we need to justify the neglect of the collision term in the Vlasov equation. The typical time scale for scattering to significantly affect particle distribution functions, is determined by the transport cross section and is of order \((g^4T)^{-1}\). Therefore one would think that a scattering term should be included in the Vlasov equation (Eq. (5)) since this time scale is of the same order as the time scale under investigation. However, this is not so as we now explain. According to Eq. (16), the current density \(j_\mu(x)\) depends on the background field \(A_\mu(y)\) at time moments \(y_0\) so that \(x_0 - y_0 \sim (g^4T)^{-1}\). This means that, on the time scale of \((g^4T)^{-1}\), the plasma responds almost instantaneously to a change in the soft background, and the slow relaxation due to particle scattering has only a negligible effect. We can verify this statement by including a simple collision term to the Vlasov equation,

\[
\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \delta n(t, x, p) + gE_v \frac{\partial \bar{n}(p)}{\partial p} = -\frac{1}{\tau} \delta n(t, x, p),
\]

where \(\tau\) is the transport mean free time \(\sim (g^4T)^{-1}\), and observe that its sole effect is the smearing of the delta-function over \(\omega - vq \sim g^4T\) in Eq. (13). This has a negligible effect on the final equation (17) and the noise correlator (18) (the effect becomes large only when considering much smaller spatial momentum scale, \(g^4T\)). As a result, we can neglect the effect of scattering on the behavior of the plasma since the scale of spatial momentum responsible for non-perturbative physics in hot QCD has been shown to be \(g^2 T\).

4. By now, we have accumulated all the tools to finally address the dynamics of the soft modes in a non-Abelian plasma in the non-perturbative regime of interest. We will consider an SU(\(N\)) pure gauge theory. The distribution function of gluons in a plasma is a matrix \(N_{ab}(t, x, p)\) in color space. The mean value of \(N_{ab}\) is the equilibrium distribution \(\bar{n}(|p|)\delta_{ab}\), and, as before, fluctuations of \(N_{ab}\) can be divided into soft and hard components. The soft component can be written in the form \(-if^{abc}\delta n^c\) where \(\delta n\) is the parameter entering the Vlasov equations (4, 5). The hard component \(\Delta N^{ab}\) gives rise to the stochastic source in the RHS of Eq. (5),

\[
\xi^a_\mu(t, x) = ig \int \frac{dp}{(2\pi)^3} v_\mu f^{abc} \sum_\alpha \Delta N^{bc}_\alpha(t, x, p)
\]

(\(\alpha\) denotes gluon polarization). The one-time correlator of \(\Delta N^{ab}\) is given by a trivial

\[\text{We assume that the distribution is trivial with respect to gluon polarization, see, for example, [13].}\]
generalization of Eq. (19),
\[
\langle \Delta N_{ab}(t, x, p) \Delta N_{a'b'}(t, x', p') \rangle = (2\pi)^3 \delta(p - p') \delta(x - x') \delta^{aa'} \delta^{bb'} \delta^{\alpha\alpha'} \bar{n}(|p|)(1 + \bar{n}(|p|)).
\]
Furthermore, \(\Delta N_{ab}\) satisfies the transport equation,
\[
v^\mu D_\mu \Delta N = 0 \quad (21)
\]
where 
\[
D_\mu = \partial_\mu + [A_\mu, \ldots], \quad A_\mu^{ab} = g f^{abc} A_c^\mu.
\]
The substitution of the covariant derivative to the space derivative \(\partial_\mu\) in Eq. (21) reflects the fact that hard gluon precesses in color space when propagating on the soft background. Remembering that the field strength \(A\) is typically of order \(gT\), we observe that \(\partial_\mu\) is of the same order as \(gA \sim g^2T\), in which case, the \(gA\) in the covariant derivative term cannot be dropped.\(^7\)

Combining Eqs. (19), (20) and (21), we find for the noise correlator,
\[
\langle \xi^a_\mu(t, x) \xi^{b\nu}_\nu(t', y) \rangle = 2g^2 N \int \frac{d^3p}{(2\pi)^3} v^\nu v_\nu \delta_n^a(\bar{n}(|p|))(1 + \bar{n}(|p|)).
\]

\(U(x, y)\) is the Wilson line connecting \(x\) and \(y\) in the adjoint representation
\[
U(x, y) = T \exp \left( - \int_y^x dz^\mu A_\mu(z) \right).
\]
The presence of the Wilson line renders Eq. (22) explicitly gauge covariant.

The generalization of Eqs. (17, 18) is straightforward. First, we solve the Vlasov equation Eq. (3) with respect to \(\delta n^a(t, x, p)\),
\[
\delta n^a(t, x, p) = -g \frac{\partial \bar{n}(|p|)}{\partial |p|} \int_0^\infty du U^{ab}(x, x - uv) v \cdot E^b(x - uv).
\]
We then insert the expression obtained in the expression for the induced current which appears on the RHS of the field equation Eq. (5),
\[
\sigma_\nu^a(t, x) = 2gN \int \frac{d^3p}{(2\pi)^3} v^\nu \delta n^a(t, x, p).
\]
\(^7\)In more physical terms, the color orientation of a hard particle changes essentially when moving over distances of order \((g^2T)^{-1}\). In contrast, the deviation of the particle trajectory from the straight line on the scale of \((g^2T)^{-1}\) is negligible. In fact, the magnetic field \(B \sim \partial A \sim g^3T^2\) corresponds to the curvature radius of the trajectory of hard particles of order \(T/gB \sim (g^4T)^{-1}\) which is much larger than the scale we are interested in.
The equation obtained is non-local in space and time. However, we have already argued that, in the regime \( \omega \sim g^4 T, k \sim g^2 T \), the non-locality in time is confined to a time interval of order \( \sim (g^2 T)^{-1} \), much smaller than the typical time scale of variation of the field \( \sim (g^4 T)^{-1} \). Hence, in the limit \( g \ll 1 \), we can ignore the non-locality in time.

Inserting Eq. (24) in the field equation (Eq. (5)), and using the fact that \( \dot{\mathbf{E}} \) can be neglected in comparison to \( \mathbf{D} \times \mathbf{A} \), we find, after little algebra,

\[
[D \times [D \times \mathbf{A}(t, \mathbf{x})]]^a = \frac{g^2 T^2 N}{12\pi} \int d\mathbf{y} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} U^{ab}(t, \mathbf{x}, \mathbf{y}) \dot{\mathbf{A}}^b(t, \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) + \xi^a(t, \mathbf{x}) \quad (25)
\]

with the noise correlator (22) taking the form,

\[
\langle \xi^a(t, \mathbf{x}) \xi^b(t', \mathbf{y}) \rangle = \frac{g^2 T^3 N}{6\pi} U^{ab}(t, \mathbf{x}, \mathbf{y}) \frac{(x - y)_i (x - y)_j \delta(t - t')}{|\mathbf{x} - \mathbf{y}|^4}. \quad (26)
\]

It can be checked that Eq. (26) is consistent with the transversality condition \( D_i \xi_i = 0 \).

Equations (25) and (26), constitute the highlight of our paper.

5. In summary, we have derived a Langevin-type equation, Eq. (25), which describes the non-perturbative behavior of very soft modes in hot gauge theories. They are non-local in space, but local in time. Equation (25) is a first-order differential equation with respect to \( t \), and \( \dot{\mathbf{A}} \) appears in the equation in a linear fashion (there is no terms like \( \dot{\mathbf{A}}^2 \)). The noise correlator is local (white) in time, but has a non-trivial spatial dependence. Eq. (25) can be used for simulating the dynamics of soft modes in a variety of contexts. To do so, one generates, at any given time, a noise \( \xi \), transverse and distributed according to Eq. (26), then runs the system to the next time-slice using Eq. (25); one repeats these steps until the system reaches a stationary state. The benefit of our formalism is that one can concentrate on relevant modes of spatial momenta of order \( g^2 T \) only, since all higher modes have already been integrated out in deriving Eqs. (25, 26). Therefore, one can use a lattice with the lattice cutoff \( a^{-1} \) of order \( g^2 T \), but not \( T \) or \( gT \) as in most of the methods suggested so far (the hard modes have been taken into account analytically), which leads to the possibility of doing calculations with a much larger lattice spacing. However, to actually use Eq. (25) for simulating the dynamics of soft modes, one has to invert the linear operator acting on \( \dot{\mathbf{A}} \) on the RHS of Eq. (25). This seems to be a potentially non-trivial problem for numerical calculations, and we hope that this obstacle can be overcome.

\[\text{8This condition is required for Eq. (25) to have solutions with respect to } \dot{\mathbf{A}}.\]
Application I: Baryon number violation at high $T$

1. Our main motivation for developing an effective theory for soft modes is their involvement in numerous important phenomena taking place in a high temperature environment, one of which being the mechanism of baryon number violation at high $T$. Existing numerical simulations, and in particular the most recent ones [1, 2], suggest that, $\Gamma_w$, the rate of baryon number violation per unit volume, is $\kappa \alpha_w^4 T^4$ with $\kappa \sim 1$. This corroborates the most naive dimensional estimates which suggest that $\Gamma$ scales as $\sim \xi^4$ where $\xi \sim (g^2 T)^{-1}$, is a natural length scale of the problem. Two reasons suggest caution:

- 1. First, the above naive dimensional analysis does not recognize that the relevant time scale is $(g^4 T)^{-1}$ rather than $(g^2 T)^{-1}$.

- 2. Second, existing numerical simulations incorporate high momenta modes classically. This method is potentially sensitive to an ultraviolet cutoff and needs to be implemented with great care in order not to induce lattice artifacts[10].

The effective theory developed in the previous section, has the virtue of incorporating the physics of the hard modes into the physics of the soft modes, hence, it alleviates the sensitivity on very short distance fluctuations. The outcome is a Langevin-type equation (one derivative in time, two derivatives in space) which suggests a cross-barrier transition of a diffusive type with natural length scale and time scale of order $(g^2 T)^{-1}$ and $(g^4 T)^{-1}$, respectively.

We are confident that our results will become a tool to further study the rate of baryon number violation at high $T$. The qualitative and quantitative understanding of the latter has important consequences, some of which are illustrated below.

2. Let us consider the outset of electroweak baryogenesis. The most recent computation of the baryon asymmetry produced at the electroweak era, in the framework of some supersymmetric theories[14], concluded that in a region of parameter space where all squarks are heavy and nearly degenerate (> 100 GeV), the baryon asymmetry scales as $\sin \theta_{cp} \Gamma_w/\Gamma_s$. Here, $\Gamma_s$ is the rate per unit volume of violation of the axial charge due to strong anomalous QCD processes and $\theta_{cp}$ is a CP violating phase. In contrast, in a regime of non-degeneracy, with, say, the super-partners of the third generation of quarks lighter than $T$ and lighter than the other squarks, the baryon asymmetry scales as $\sin \theta_{cp} \Gamma_w$.

In the speculative situation where $\Gamma_{w,s}$ scales as $\alpha_{w,s}^5$ rather than as $\alpha_{w,s}^4$, the result of these computations is offset in respect to those quoted in Ref. [14], by a factor of $\alpha_w/\alpha_s \sim 1/3$ in the degenerate case and a factor of $\alpha_w \sim 1/30$, in the non-degenerate case. In the latter case, these corrections are very significant as they correspondingly shift
the quoted lower bounds on the $CP$ violating phase $\theta_{cp}$ from a range $10^{-4} - 10^{-2}$ to a range $\sim 30$ larger, bringing these bounds to a “dangerous” overlap with the quoted lower bounds obtained from electric dipole moment experiments [15].

3. Finally, let us briefly speculate on another possible interesting consequence for electroweak baryogenesis. Another source of significant constraint on electroweak baryogenesis is the preservation of the baryon asymmetry at times subsequent to its production. This requires that baryon number violating processes be suppressed in the broken phase after the transition. Currently, this requirement yields very strong constraints on the parameters of the Higgs sector. It remains to be explored whether the above considerations will affect the determination of these constraints.

**Application II: Parameters of the electroweak phase transition**

Let us now turn to a second important application of our formulation of long range physics in a non-Abelian plasma: the determination of the parameters of the electroweak phase transition. We refer, more specifically, to the order of the transition, and, if first order, to the nucleation rate, and the dynamical properties of the phase interface such as final Higgs expectation value, shape, profile and velocity. The knowledge of these quantities is required for a quantitative study of electroweak baryogenesis. One of the computational tools often used in their calculation is the equilibrium free energy, $V(\phi, T)$, of the system. The perturbative determination of the latter is known to be plagued with infrared divergences. Those arising from the physics at scale $gT$ can be accounted for perturbatively by resummation of hard thermal loops [16, 17]. Others, arising at scale $g^2T$ in the magnetic components of the gauge sector, remain unaccounted for perturbatively; their presence is an obstacle to a complete understanding of the outcome of the electroweak phase transition.

The stochastic dynamics of soft modes presented above sheds new light on these issues and provides a new tool for resolving them. To illustrate how, we begin by briefly presenting a real-time calculation of the free energy $V(\phi, T)$.

1. We consider a gas of gauge bosons with mass $m = g\phi/2$, in the presence of a space-dependent Higgs background $\phi(x)$ which interpolates smoothly between phases of broken and unbroken electroweak symmetry: $\phi(-\infty) = 0$, $\phi(+\infty) = \phi_0$ and $\partial_x\phi(\pm\infty) = 0$. Focusing on the gauge components of the plasma and their interactions with the kink, we can simultaneously describe the dynamics of the kink and of the particles distributions $n^a(x,p)$ with the following equations [18].
\[
\frac{d}{dx} n^a(x, p) - \frac{1}{2\varepsilon_p} \frac{d}{dx} \frac{d n^2(x)}{dx} \frac{d}{dx} n^a(x, p) = C(\tau)
\]

\[
-\partial^2_x \phi(x) = -\frac{\partial V(\phi)}{\partial \phi} + \frac{dm^2(x)}{dx} \int \frac{dp}{(2\pi)^3} \frac{1}{\varepsilon_p} n^a(x, p),
\]

where \(\varepsilon_p\) is the space-dependent energy of a particle with momentum \(p\) at point \(x\). The force term in the first Vlasov equation above is derived from energy conservation, which, in the kink frame, implies \(p^2 + m^2(x) = \text{constant}\). We assume the existence of a stationary situation. This is guaranteed dynamically by the presence of the collision term \(C(\tau)\). Without the latter, the kink would continuously accelerate leading to an unpleasant situation \([18, 17]\).

The above equations supplemented with appropriate boundary conditions for the kink and with \(\varepsilon_p(-\infty) = p^2\), not only provide a unique solution to the propagation of a phase interface (in a regime where the curvature can be ignored) such as its velocity relative to the plasma\([10]\) and its profile, but also provide us with a powerful scheme for computing \(V(\phi, T)\). A useful quantity to look at, is the space variation of \(T_{\mu\nu}(x)\), the stress energy tensor of the system. The leading order term of its expansion in \(v_k\), the relative kink-plasma velocity, yields the free energy of the system \(T_{xx}(x) - T_{xx}(-\infty) = V(0, T) - V(\phi(x), T) + v_k L(\phi(x), T) + O(v_k^2)\). Because the “drag force” \(L(\phi(x), T)\) vanishes as the collision term in the first Vlasov equation vanishes, the effective potential is the solution of Eq. (28) in the collisionless limit. The solution is \(\varepsilon^2_p(x) = p^2 + m^2(x)\) and, \(V(\phi, T)\) given by the so-called “one-loop” effective potential\([18]\). Its minimum, \(\phi_0\), scales as \(\phi_0 \sim gT m_H^2/m_W^2\) which, in turn, fixes the size of the kink to be of the order of \(L_k \sim \int d\phi/\sqrt{V(\phi, T)} \sim 1/g^2 T \) \((m_H/m_W)\). This approximation of the free energy has been known for a long time to be inadequate as it does not incorporate any of the long range plasma physics. In particular, in the limit \(\phi \to 0\), there is an uncontrollable number of soft quanta \(\sim T/g\phi\) whose interactions (with strength \(g^2\)) among themselves and with the hard modes yield uncontrollably large corrections to the free energy. The remaining of this section addresses that question.

2. Inclusion of the \(gT\)-scale physics. Plasma physics at the scale \(\sim gT\) (“plasmon physics”) becomes relevant when \(g\phi \sim gT\), that is for \(\phi < T\). From our estimate of \(\phi_0\) above, we see that this physics affects the determination of the free energy for all values of \(\phi \leq \phi_0\), i.e., across the whole kink. The resolution of these difficulties is known \([16, 17]\). One has to incorporate hard thermal loops in the calculation of \(V(\phi, T)\). In the above real time framework, this operation can be performed very simply by noting that plasmon physics is

\[^9\text{Eqs. (28) were used in Ref. [18] to compute the relative motion of the kink in the plasma.}\]
contained within a distance $\sim 1/gT < L_k \sim 1/g^2T$: the plasmon physics is “local” inside the kink and is in a large extent independent on the properties of the latter. One can then describe the kink as coupled to a gas of free bosons with longitudinal modes replaced with plasmons. The dispersion relation of a plasmon is, to a sufficient approximation: $\varepsilon_p^2 = p^2 + \Pi(p = 0)$, where $\sqrt{\Pi(p = 0)}$ is the plasmon frequency, $\sim gT$. Incorporating this modified dispersion relation in the boundary conditions at $x \to -\infty$, yields an expression for the source $s(x)$, which, after simple algebra, reproduces a form of the free energy known in the literature as the “one loop improved” effective potential\cite{footnote}. This is a significant improvement because gauge interactions among plasmons are computable perturbatively, their effective strength scales according to $g^2 T/\sqrt{\Pi(p = 0)} \sim g < 1$. There remains an infrared behavior in the transverse components of the gauge sector.

3. Inclusion of the $g^2T$-scale physics. It is transparent from the above analysis that the calculation of $V(\phi, T)$ involves the natural scale $L_k \sim 1/g^2T$ which is of the order of the one characteristic of the non-linearity in a hot gauge plasma. Any determination of $V(\phi, T)$ which claims to incorporate the latter physics, is doomed to be intertwined with the physics of the $g^2T$ modes, in contrast with the situation we encountered with the $gT$ modes. A path for a resolution is provided by evolving simultaneously the plasma in a Higgs background and in a soft gauge background. The equations consist in a generalization of Eqs. (25) and (26) to the presence of a space-dependent background Higgs field, that is, a Vlasov equation supplemented with the gauge field equations and the Higgs equation of motion. In contrast with the situation with the $g^2T$ modes, these last two equations do not decouple from each other because the typical lengths of variation of both backgrounds are comparable, $\sim (g^2T)^{-1}$. In particular, the space derivative on the LHS of Eq. (28) is to be substituted with a covariant derivative. These equations, whose study is beyond the scope of this letter, constitute a natural real-time framework to incorporate the soft gauge physics in the calculation of the thermodynamics of the electroweak phase transition.

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\footnote{This scale is not an artifact of our scheme of computation, not only does this scale set the size of the interface between two phases but it also fixes the size of the critical radius in nucleation theory.}
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