Picture Fuzzy Soft Robust VIKOR Method and its Applications in Decision-Making

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\section*{ABSTRACT}
This paper introduces the Euclidean, Hamming, and the generalized distance measures for picture fuzzy soft sets and discusses their properties. The numerical examples of decision-making and pattern recognition are focused. We also develop a robust VIKOR method for PFSSs. The relative and precise ideal picture fuzzy values (PFVs), robust factors and ranking indexes are defined. Different algorithmic procedures of robust VIKOR based on the relative and precise ideal PFVs, relative and precise robust factor, precise and picture fuzzy weights and relative and precise ranking indexes are proposed. In the end, the investment problem is solved by using the proposed method.

\section{1. Introduction}
To measure the similarity between any form of data is an important topic. The measures used to find the resemblance between data is called similarity measure. It has different applications in classification, pattern recognition, medical diagnosis, data mining, clustering, decision-making and in image processing.

Fuzziness, as developed in \cite{1}, is a kind of uncertainty which appears often in human decision-making problems. The fuzzy set theory deals with daily life uncertainties successfully. The membership degree is assigned to each element in a fuzzy set. The membership degrees can effectively be taken by fuzzy sets. But in real-life situations, the non-membership degrees should be considered in many cases as well, and it is not necessary that the non-membership degree be equal to the one minus the membership degree. Thus, Atanassov \cite{2} introduced the concept of intuitionistic fuzzy set (IFS) that considers both membership and non-membership degrees. Here the non-membership
degree is not always obtained from a membership degree, which leads to the concept of hesitancy degree. For each element, the sum of its membership \((\xi)\), non-membership \((\nu)\), and hesitancy \((\pi)\) degrees should be equal to one for IFSs, that is, \(\xi + \nu + \pi = 1\). This condition suggests that the region to choose membership and non-membership degrees is a proper subset of \([0, 1] \times [0, 1]\) [3].

Vote for, vote against, and keep neutral are three opinion choices for voters. Fuzzy sets and IFSs have no ability to deal such type of complicated voting situations. To cope these situations, Cuong and Kreinovich [4,5] introduced the ideology of picture fuzzy set (PFS). This is considered as the improvement of fuzzy sets and IFSs because it contains neutrality degree along with membership and non-membership degrees. To provide the better representation of data, this notion is widely used in the literature for real-life problems. Cuong and Hai [6] introduced the interval-valued PFS. The t-norms and t-conorms operators for PFS were introduced by Cuong et al. [7]. Yang et al. [8] defined picture fuzzy soft set (PFSS) and discernibility matrix approach was expanded.

The aggregation operators can be used to fuse the attribute values of alternatives in multiple attribute decision-making (MADM) problems. Different mappings served as an aggregation operators that meet some specific criteria [9]. The linear function based OWA operator was defined by Perez et al. [10]. The weighted average and geometric aggregation operators and their ordered versions for PFSSs were discussed by Wei [11]. The Hamacher operational laws and Hamacher aggregation operators were extended in [12]. The complex MADM problems were handled by Jana et al. [13] by defining Dombi operational laws and Dombi aggregation operators for PFSSs. Wei et al. [14] extended the Heronian mean (HM) function to incorporate the relationship among the attributes in picture fuzzy environment. Garg [15] contemplated aggregation operations on PFSS and applied it to MADM problems. A novel concept of generalised picture fuzzy soft sets (GPFSSs) with their properties was discussed by Khan et al. [16,17]. Moreover, Khan et al. [18,19] put forward a method based on an adjustable weighted soft discernibility matrix to deal with the decision-making problems with GPFSSs.

The uncertain information is significantly measured by distance and similarity measures. The distance and similarity measures are employed to depict the closeness and differences among fuzzy sets and have many applications in real-life situations like medical diagnosis, data mining, decision-making, classification and pattern recognition. To satisfy the axioms of similarity, Khan et al. [20,21] proposed some novel similarity measures based on two parameters for PFSSs. Khan and Kumam [22–24] proposed some novel similarity and distance measures based on the cosine and cotangent function for generalised intuitionistic fuzzy soft sets. Transitivity and monotonicity of fuzzy similarity measures were discussed in [25]. The problem of non-determinism in financial time series forecasting using IFSs were discussed by [26]. The picture fuzzy correlation coefficients were discussed by Ganie [27]. Recently, Ganie and Singh have discussed different innovative distance and similarity measures for PFSSs [28]. Direct operations-based similarity measures for PFSSs were expounded in [29]. The clustering, pattern recognition and MADM problems were solved by defining the power two based similarity measures for PFSSs in [30]. Three constituent degrees-based picture fuzzy similarity measures and their applications were discussed by Luo and Zhang [31]. Strategic decision-making problems were discussed by cosine similarity measures for PFSSs [32]. Picture fuzzy linear programming-based TOPSIS method were discussed by Sindhu et al. [33]. The entropy for PFSSs and their corresponded similarity measures were defined by Thao [34].
The positive ideal (PI) solution is the best available solution in multi-criteria decision-making (MCDM) problems. The compromise solution by VIKOR approach is closest to the PI solution [35]. Thus basic idea in VIKOR method is to find the compromise solution which is closest to the PI solution and any improvement in VIKOR approach that not fulfil the basic idea is not reliable. Recently, Khan et al. [36] discussed the theoretical justifications of empirically successful VIKOR method.

Motivated from the Yang’s model of PFSS [8] and remoteness-based VIKOR method for Pythagorean fuzzy sets by Chen [37]. We diversify this technique to PFSS and apply to MADM problems. We develop robust VIKOR method for PFSS to select priority area for investment. To measure the difference and similarity between PFSSs, different distance and similarity measures for PFSSs are presented.

The aim of this paper is to discuss the priority area for investment for an under-developing country using the robust VIKOR method for PFSS. Also, to define the distance and similarity measures for measuring the difference and similarity between PFSSs. Additionally, to discuss the different algorithmic procedures that incorporate the precise and picture fuzzy weights and precise and relative ideal values.

The paper has following contributions:

1. The Hamming, Euclidean and generalised distance measures are defined for PFSSs and strategic decision-making and pattern recognition problems are debated.
2. The relative and precise ideal values are defined to reach best available solution and avoid worst solution.
3. The relative and precise robust factors are defined for PFSSs.
4. Precise and picture fuzzy (PF) weights are introduced.
5. Different relative and precise ranking indexes are defined for PFSSs to cope with precise and PF weights.
6. The algorithmic procedures of robust VIKOR method are proposed.
7. The problem of selecting a priority area for investment is solved with proposed methods.

The remaining paper is written as follows: Section 2 discuss the basic definitions. The distance and similarity measures and their properties are focused in Section 3. The relative and precise ideals, robust factors, ranking indexes, precise and PF weights, and algorithmic procedures are expounded in Section 4. The application of the propose method in selection of priority area for investment is discussed in Section 5. The comparison and concluding remarks are focused in Sections 6 and 7, respectively.

2. Preliminaries

This segment contains the basic notions of soft set, IFS, PFS and PFSS. Let $\hat{Y} = \{\ell_1, \ell_2, \ldots, \ell_m\}$ represents the universal set throughout the paper which is discrete, finite, non-void discourse set and contains the alternatives, while $\hat{E} = \{j_1, j_2, \ldots, j_n\}$ represents the characteristics or attributes of criteria and called criteria space.

A novel idea of soft set was proposed by Molodtsov [38], where uncertainty deals successfully in the light of parametric point of view. Each member in soft set can be viewed by some characteristic or attributes (criteria).
Definition 2.1 ([38]): For a universal set $\hat{Y}$ and criteria space $\hat{E}$, the soft set is defined by a set valued mapping $\hat{F} : \hat{A} \to P(\hat{Y})$, where $\hat{A} \subset \hat{E}$ and $P(\hat{Y})$ is a power set of $\hat{Y}$ and represented as a pair $(\hat{F}, \hat{A})$.

Atanassov defined the generalisation of fuzzy set by considering the non-membership function. The uncertainty model’s more effectively in IFS.

Definition 2.2 ([2]): The membership function ($\xi_R$) and non-membership function ($\nu_R$) from universal set to unit interval, with a condition $\xi_R(\ell) + \nu_R(\ell) \leq 1$, define the IFS $R$ over a universal set $\hat{Y}$ as follows

$$R = \{(\xi_R(\ell), \nu_R(\ell)) \mid \ell \in \hat{Y}\}.$$

The hesitancy index of the element $\ell \in \hat{Y}$ is defined as $h_R(\ell) = 1 - (\xi_R(\ell) + \nu_R(\ell))$.

Coung defined the generalisation of IFS by including the neutral membership function and called the PFS. This model is important for the situations involves yes, abstain, no and refusal. Voting is a good example for PFS.

Definition 2.3 ([4]): The membership function ($\xi_R$), neutral function ($\eta_R$) and non-membership function ($\nu_R$) from universal set to unit interval, with a condition $\xi_R(\ell) + \eta_R(\ell) + \nu_R(\ell) \leq 1$, defines the PFS $R$ over a universal set $\hat{Y}$ as follows

$$R = \{(\xi_R(\ell), \eta_R(\ell), \nu_R(\ell)) \mid \ell \in \hat{Y}\},$$

The hesitancy index of the element $\ell \in \hat{Y}$ is defined as $h_R(\ell) = 1 - (\xi_R(\ell) + \eta_R(\ell) + \nu_R(\ell))$.

For any $\ell \in \hat{Y}$, the value $(\xi_R(\ell), \eta_R(\ell), \nu_R(\ell))$ is called the picture fuzzy value (PFV) or picture fuzzy number (PFN).

Definition 2.4 ([4]): For any two PFSs $R$ and $S$ in $\hat{Y}$, the following operations are defined as follows:

1. $R \cap S = \{(\ell, \min\{\xi_R(\ell), \xi_S(\ell)\}, \min\{\eta_R(\ell), \eta_S(\ell)\}, \max\{\nu_R(\ell), \nu_S(\ell)\}) \mid \ell \in \hat{Y}\}$
2. $R \cup S = \{(\ell, \max\{\xi_R(\ell), \xi_S(\ell)\}, \min\{\eta_R(\ell), \eta_S(\ell)\}, \min\{\nu_R(\ell), \nu_S(\ell)\}) \mid \ell \in \hat{Y}\}$
3. $R \subseteq S \iff \xi_R(\ell) \leq \xi_S(\ell), \eta_R(\ell) \leq \eta_S(\ell) \text{ and } \nu_R(\ell) \geq \nu_S(\ell), \forall \ell \in \Theta$
4. $R^c = \{(\ell, \nu_R(\ell), \eta_R(\ell), \xi_R(\ell)) \mid \ell \in \hat{Y}\}$
5. $\lambda(\ell, \xi_R(\ell), \eta_R(\ell), \nu_R(\ell)) = (\ell, 1 - (1 - \xi_R(\ell))^\lambda, \eta_R(\ell), \nu_R(\ell))$

In [8], Yang defined the hybrid structure of PFS and soft set, called PFSS.

Definition 2.5 ([8]): For a universal set $\hat{Y}$ and criteria space $\hat{E}$, the PFSS is defined by a set valued mapping $\hat{F} : \hat{A} \to PF(\hat{Y})$, where $\hat{A} \subset \hat{E}$ and $PF(\hat{Y})$ is the set of all PFSs over $\hat{Y}$, and represented as a pair $(\hat{F}, \hat{A})$. That is, for each element $j \in \hat{A}$, we obtained a PFS $\hat{F}(j)$.

To understand the construction of PFSS, we consider an example of selecting the class representative (CR). Let the class in the university need to select a CR and they agree for voting method. Let three candidates are available for CR and we represent as
Table 1. A PFSS \((\hat{F}, \hat{E})\).

| \(\hat{Y}\) | \(J_1\) | \(J_2\) | \(J_3\) | \(J_4\) | \(J_5\) |
|---|---|---|---|---|---|
| \(\ell_1\) | \(b_{11}\) | \(b_{12}\) | \(b_{13}\) | \(b_{14}\) | \(b_{15}\) |
| \(\ell_2\) | \(b_{21}\) | \(b_{22}\) | \(b_{23}\) | \(b_{24}\) | \(b_{25}\) |
| \(\ell_3\) | \(b_{31}\) | \(b_{32}\) | \(b_{33}\) | \(b_{34}\) | \(b_{35}\) |

\(\hat{Y} = \{\ell_1, \ell_2, \ell_3\}\). The candidates are evaluated based on their negotiation, communication, leadership, problem-solving and team-working skills and represented as \(J_1, J_2, J_3, J_4, J_5\), respectively. Each student of the class give their preference for the candidates against each attributes in the form of yes (\(\xi\)), abstain (\(\eta\)) and no (\(\nu\)). If the number of students of the class are \(L\), then the evaluation of each candidate \(\ell_i\) against each criterion \(J_j\) is calculated as

\[
\hat{b}_{ij} = \frac{\text{Evaluation of all students for a } i\text{th candidate against } j\text{th criterion}}{L}
\]

where each \(\hat{b}_{ij}\) should follow the condition of Definition 2.3. Thus for criteria set \(\hat{E} = \{J_1, J_2, \ldots, J_5\}\), the mapping \(\hat{F} : \hat{E} \to PF(\hat{Y})\) can be define and for each criterion \(J_j\), the \(\hat{F}(J_j)\) is a PFS as follows

\[
\hat{F}(J_j) = \{b_{1j}, b_{2j}, b_{3j}\}, \quad j \in \{1, 2, \ldots, 5\}
\]

This \(\hat{F}(J_j), j \in \{1, 2, \ldots, 5\}\) constitute the PFSS \((\hat{F}, \hat{E})\) and represented in tabular form in Table 1.

**Definition 2.6 ([13]):** The score function \(\delta\) for a PFV \(b = (\xi_b, \eta_b, \nu_b)\) is defined as:

\[
\delta(b) = \frac{1 + \xi_b - \nu_b}{2} \in [0, 1].
\]

### 3. Distance and Similarity Measures

This section contains the Hamming, Euclidean and generalised distance measures for PFSSSs. Additional properties and their applications in decision-making and pattern recognition are discussed here.

**Definition 3.1:** A distance measure between two PFSSs \(\Gamma_1\) and \(\Gamma_2\) is a mapping \(D : PFSS \times PFSS \to [0, 1]\), which satisfies the following properties:

\begin{align*}
(D1) & \quad 0 \leq D(\Gamma_1, \Gamma_2) \leq 1 \\
(D2) & \quad D(\Gamma_1, \Gamma_2) = 0 \iff \Gamma_1 = \Gamma_2 \\
(D3) & \quad D(\Gamma_1, \Gamma_2) = D(\Gamma_2, \Gamma_1) \\
(D4) & \quad \text{If } \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \text{ then } D(\Gamma_1, \Gamma_3) \geq D(\Gamma_1, \Gamma_2) \text{ and } D(\Gamma_1, \Gamma_3) \geq D(\Gamma_2, \Gamma_3).
\end{align*}

**Definition 3.2:** A similarity measure between two PFSSs \(\Gamma_1\) and \(\Gamma_2\) is a mapping \(S : PFSS \times PFSS \to [0, 1]\), which satisfies the following properties:
Definition 3.3: For two PFSSs \( S_1 \) and \( S_2 \), the Hamming distances between \( S_1 \) and \( S_2 \) are defined as follows:

\[
D_h^a(S_1, S_2) = \frac{1}{2mn} \sum_{j=1}^n \sum_{i=1}^m \left[ |\xi_{F(j)}(\ell_i) - \xi_{G(j)}(\ell_i)| + |\eta_{F(j)}(\ell_i) - \eta_{G(j)}(\ell_i)| + |v_{F(j)}(\ell_i) - v_{G(j)}(\ell_i)| \right] \tag{2}
\]

Definition 3.4: Let \( S_1 = (\hat{F}, \hat{A}) \) and \( S_2 = (\hat{G}, \hat{B}) \) be two PFSSs in \( \hat{Y} \), the Euclidean distances between \( S_1 \) and \( S_2 \) are defined as follows:

\[
D_e^a(S_1, S_2) = \left( \frac{1}{2mn} \sum_{j=1}^n \sum_{i=1}^m \left[ (\xi_{F(j)}(\ell_i) - \xi_{G(j)}(\ell_i))^2 + (\eta_{F(j)}(\ell_i) - \eta_{G(j)}(\ell_i))^2 + (v_{F(j)}(\ell_i) - v_{G(j)}(\ell_i))^2 \right] \right)^{\frac{1}{2}} \tag{4}
\]

\[
D_e^b(S_1, S_2) = \left( \frac{1}{2mn} \sum_{j=1}^n \sum_{i=1}^m \left[ (\xi_{F(j)}(\ell_i) - \xi_{G(j)}(\ell_i))^2 + (\eta_{F(j)}(\ell_i) - \eta_{G(j)}(\ell_i))^2 + (h_{F(j)}(\ell_i) - h_{G(j)}(\ell_i))^2 \right] \right)^{\frac{1}{2}} \tag{5}
\]

Definition 3.5: For two PFSSs \( S_1 = (\hat{F}, \hat{A}) \) and \( S_2 = (\hat{G}, \hat{B}) \) in \( \hat{Y} \), the generalised distance measures between \( S_1 \) and \( S_2 \) are defined as follows:

\[
D_p^a(S_1, S_2) = \left( \frac{1}{2mn} \sum_{j=1}^n \sum_{i=1}^m \left[ |\xi_{F(j)}(\ell_i) - \xi_{G(j)}(\ell_i)| + |\eta_{F(j)}(\ell_i) - \eta_{G(j)}(\ell_i)| + |v_{F(j)}(\ell_i) - v_{G(j)}(\ell_i)| \right] \right)^{\frac{1}{p}} \tag{6}
\]
For two PFSSs \( \Gamma_1 = (\hat{F}, \hat{A}) \) and \( \Gamma_2 = (\hat{G}, \hat{B}) \) in \( \hat{Y} \), we find the distance between \( \Gamma_1 \) and \( \Gamma_2 \) by using above-mentioned distance measures.

\[
\Gamma_1 = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} (0.6, 0.1, 0.2) \\ (0.8, 0.0, 0.1) \end{bmatrix} \begin{bmatrix} 0.4, 0.1, 0.4 \\ 0.6, 0.1, 0.2 \end{bmatrix} \begin{bmatrix} 0.2, 0.1, 0.7 \\ 0.4, 0.1, 0.5 \end{bmatrix}
\]

\[
\Gamma_2 = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} (0.4, 0.1, 0.3) \\ (0.5, 0.2, 0.2) \end{bmatrix} \begin{bmatrix} 0.6, 0.1, 0.2 \\ 0.7, 0.1, 0.1 \end{bmatrix} \begin{bmatrix} 0.4, 0.2, 0.3 \end{bmatrix}
\]

\[
D_h^e(\Gamma_1, \Gamma_2) = \frac{1}{12} \sum_{j=1}^{n-m} \sum_{i=1}^{m} \left( |\xi_{\hat{F}(j)}(\ell_i) - \xi_{\hat{G}(j)}(\ell_i)|^p + |\eta_{\hat{F}(j)}(\ell_i) - \eta_{\hat{G}(j)}(\ell_i)|^p \right)
\]

\[
\text{Remark 3.1:} \text{ The generalised distance measures } D_p^e \text{ and } D_p^o \text{ are reduced to Hamming distances } D_h^e \text{ and } D_h^o \text{, respectively, for } p = 1. \text{ Also, the Euclidean distances } D_e^e \text{ and } D_e^o \text{ are obtained from } D_p^e \text{ and } D_p^o, \text{ respectively, for } p = 2.
\]

\[
\text{Example 3.1: Suppose two PFSSs } \Gamma_1 = (\hat{F}, \hat{A}) \text{ and } \Gamma_2 = (\hat{G}, \hat{B}) \text{ in } \hat{Y}. \text{ We find the distance between } \Gamma_1 \text{ and } \Gamma_2 \text{ by using above-mentioned distance measures.}
\]

\[
D_h^o(\Gamma_1, \Gamma_2) = 0.216667, D_e^o(\Gamma_1, \Gamma_2) = 0.225462 \text{ and } D_e^o(\Gamma_1, \Gamma_2) = 0.232737. \text{ The distance measures by using generalised distances for } p = 3 \text{ are } D_p^o(\Gamma_1, \Gamma_2) = 0.257966 \text{ and } D_p^o(\Gamma_1, \Gamma_2) = 0.260446.
\]

\[\text{Definition 3.6: For two PFSSs } \Gamma_1 = (\hat{F}, \hat{A}) \text{ and } \Gamma_2 = (\hat{G}, \hat{B}) \text{ in } \hat{Y}, \text{ the weighted Hamming distances between } \Gamma_1 \text{ and } \Gamma_2 \text{ are defined as follows:}
\]

\[
D_h^{\omega}(\Gamma_1, \Gamma_2) = \frac{1}{2mn} \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j \left[ |\xi_{\hat{F}(j)}(\ell_i) - \xi_{\hat{G}(j)}(\ell_i)| + |\eta_{\hat{F}(j)}(\ell_i) - \eta_{\hat{G}(j)}(\ell_i)| \right]
\]

\[
+ |\nu_{\hat{F}(j)}(\ell_i) - \nu_{\hat{G}(j)}(\ell_i)|
\]
The similarity measures for two PFSSs

**Theorem 3.1:** The similarity measures for two PFSSs \( \Gamma_1 \) and \( \Gamma_2 \) are obtained from above distance measures by \( S(\Gamma_1, \Gamma_2) = 1 - D(\Gamma_1, \Gamma_2) \).
Table 2. The PFSS.

| $\hat{Y}$ | $\ell_1$ | $\ell_2$ | $\ell_3$ | $\ell_4$ | $y$ |
|-----------|----------|----------|----------|----------|-----|
| $j_1$     | (0.53,0.33,0.09) | (1.00,0.00,0.00) | (0.91,0.03,0.02) | (0.85,0.09,0.05) | (0.90,0.05,0.02) |
| $j_2$     | (0.89,0.08,0.03) | (0.13,0.64,0.21) | (0.07,0.09,0.05) | (0.74,0.16,0.10) | (0.68,0.08,0.21) |
| $j_3$     | (0.42,0.35,0.18) | (0.03,0.82,0.13) | (0.04,0.85,0.10) | (0.02,0.89,0.05) | (0.05,0.87,0.06) |
| $j_4$     | (0.08,0.89,0.02) | (0.73,0.15,0.08) | (0.68,0.26,0.06) | (0.08,0.84,0.06) | (0.13,0.75,0.09) |
| $j_5$     | (0.33,0.51,0.12) | (0.52,0.31,0.16) | (0.15,0.76,0.07) | (0.16,0.71,0.05) | (0.15,0.73,0.08) |
| $j_6$     | (0.17,0.53,0.13) | (0.51,0.24,0.21) | (0.31,0.39,0.25) | (1.00,0.00,0.00) | (0.91,0.03,0.05) |

Table 3. The Distance Between PFSSs.

| $D(\ell_1,y)$ | $D(\ell_2,y)$ | $D(\ell_3,y)$ | $D(\ell_4,y)$ | Rankings |
|---------------|---------------|---------------|---------------|-----------|
| 0.057217      | 0.06675       | 0.0146        | $\ell_4 \succ \ell_1 \succ \ell_3 \succ \ell_2$ |
| 0.145501      | 0.188662      | 0.030419      | $\ell_4 \succ \ell_1 \succ \ell_2 \succ \ell_3$ |
| 0.221269      | 0.279392      | 0.041733      | $\ell_4 \succ \ell_1 \succ \ell_2 \succ \ell_3$ |
| 0.284051      | 0.344331      | 0.053027      | $\ell_4 \succ \ell_1 \succ \ell_2 \succ \ell_3$ |
| 0.335453      | 0.392823      | 0.057047      | $\ell_4 \succ \ell_1 \succ \ell_2 \succ \ell_3$ |
| 0.487938      | 0.526076      | 0.076237      | $\ell_4 \succ \ell_1 \succ \ell_2 \succ \ell_3$ |
| 0.599866      | 0.626869      | 0.09076       | $\ell_4 \succ \ell_2 \succ \ell_1 \succ \ell_3$ |
| 0.680383      | 0.703378      | 0.101805      | $\ell_4 \succ \ell_2 \succ \ell_1 \succ \ell_3$ |
| 0.709566      | 0.731141      | 0.105823      | $\ell_4 \succ \ell_2 \succ \ell_1 \succ \ell_3$ |

3.1. Application in Strategic Decision-Making and Pattern Recognition

In this section, we solve the strategic decision-making and pattern recognition problem adopted from [32,39]. Assume that a firm wants to allocate a plant for making new products. The firm has to decide the standard of new products to obtain the highest benefits and optimal production strategy. After the review of the market, the firm consider four alternatives. Let $\hat{Y} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ represents the four alternative, where $\ell_i$ ($1 \leq i \leq 4$), stands for: product for upper class, upper middle class, lower middle class and for working class, respectively. To make the process of decision-making beneficial and effective, the firm hire the experts from the different fields and constitute a committee to make recommendations for choosing the potential alternative wisely. The committee set up the criteria (attribute) to evaluate the above-mentioned alternatives. Let $\hat{E} = \{j_1, j_2, j_3, j_4, j_5, j_6\}$ represents the six attributes, where $j_j$ ($1 \leq j \leq 6$), stands for: short-term benefits, mid-term benefits, long-term benefits, production strategy risk, potential market and market risk, and industrialisation infrastructure, human resources and financial conditions, respectively.

The committee make assessment of four alternatives against the six attributes and give their preferences in the form of PFSS. The assessment of alternatives is presented in Table 2. Further, the committee decide the unknown production strategy $y$, with data as listed in Table 2. The distance between the each alternatives $\ell_i$ and unknown production strategy $y$ have calculated on the basis of above proposed distance measures.

From Table 3, we have seen that the distance between $\ell_4$ and $y$ is minimum, which shows that the unknown production strategy $y$ belongs to alternative $\ell_4$. We consider the distance measures that includes the hesitancy index in their calculations. There is the slight difference in the ranking for higher values of the parameter $p$ in the distance measures (Table 3). The obtain ranking coincide with the ranking obtained by Wei [32].
4. The Robust VIKOR Method for PFSSs

In this section, the relative and precise ideal values are defined to reach best available solution and avoid worst solution, respectively. Based on the ideal values, the relative and precise robust factors are defined for PFNs. The relative and precise ideal values and robust factors are helpful to define ranking indexes. The algorithmic procedures of robust VIKOR method are proposed.

4.1. The Relative and Precise Ideal Picture Fuzzy Values

The relative positive ideal PFV (rpi-PFV) and relative negative ideal PFV (rni-PFV) are the best available and worst avoidable values, respectively. The rpi-PFV and rni-PFV are based on the available data provided by the decision-makers. If the decision-maker change his/her preferences, then rpi-PFV and rni-PFV influenced. Instead of relative ideal values, the precise ideal values are fixed and not influence by the preferences of the decision-makers. The precise positive ideal PFV (ppi-PFV) and precise negative ideal (pni-PFV) are the best available and worst avoidable values in the domain. These ideal values are useful to obtain the best suitable alternative and to keep away from worst alternative.

As we know that there are two types of criteria, that is, benefit and cost criteria. Let $\hat{E} = \{j_1, j_2, \ldots, j_n\}$ be the criteria space and $\hat{E}_b$ and $\hat{E}_c$ are benefit and cost criterion, respectively. The rpi-PFV and rni-PFV for the PFV decision matrix are defined as follows.

**Definition 4.1:** The rpi-PFV ($b_{+j}$) and rni-PFV ($b_{-j}$) for a PFV decision matrix $b = [b_{ij}]_{m \times n}$ with respect to each attribute $j_j \in \hat{E}$ ($\hat{E} = \hat{E}_b \cup \hat{E}_c$, where $\hat{E}_b \cap \hat{E}_c = \Phi$) are defined as follows:

$$b_{+j} = (\xi_{+j}, \eta_{+j}, \nu_{+j}) = \begin{cases} (\max_{i=1}^{m} \xi_{ij}, \min_{i=1}^{m} \eta_{ij}, \min_{i=1}^{m} \nu_{ij}), & \text{if } j_j \in \hat{E}_b \\ (\min_{i=1}^{m} \xi_{ij}, \min_{i=1}^{m} \eta_{ij}, \max_{i=1}^{m} \nu_{ij}), & \text{if } j_j \in \hat{E}_c \end{cases}$$ (14)

$$b_{-j} = (\xi_{-j}, \eta_{-j}, \nu_{-j}) = \begin{cases} (\min_{i=1}^{m} \xi_{ij}, \min_{i=1}^{m} \eta_{ij}, \max_{i=1}^{m} \nu_{ij}), & \text{if } j_j \in \hat{E}_b \\ (\max_{i=1}^{m} \xi_{ij}, \min_{i=1}^{m} \eta_{ij}, \min_{i=1}^{m} \nu_{ij}), & \text{if } j_j \in \hat{E}_c \end{cases}$$ (15)

The ppi-PFV and pni-PFV for the PFV decision matrix are defined as follows.

**Definition 4.2:** The ppi-PFV ($b_{+j}$) and pni-PFV ($b_{-j}$) for a PFV decision matrix $b = [b_{ij}]_{m \times n}$ with respect to each attribute $j_j \in \hat{E}$ ($\hat{E} = \hat{E}_b \cup \hat{E}_c$, where $\hat{E}_b \cap \hat{E}_c = \Phi$) are defined as follows:

$$b_{+j} = (\xi_{+j}, \eta_{+j}, \nu_{+j}) = \begin{cases} (1, 0, 0), & \text{if } j_j \in \hat{E}_b \\ (0, 0, 1), & \text{if } j_j \in \hat{E}_c \end{cases}$$ (16)

$$b_{-j} = (\xi_{-j}, \eta_{-j}, \nu_{-j}) = \begin{cases} (0, 0, 1), & \text{if } j_j \in \hat{E}_b \\ (1, 0, 0), & \text{if } j_j \in \hat{E}_c \end{cases}$$ (17)

4.2. The Relative and Precise Robust Factors

As we discuss earlier, the ideal values are useful to obtain the best suitable alternative and to keep away from worst alternative. If the separation between each assessed value ($b_{ij}$)
and rpi-PFV \((b_{ij})\) (represented as \(D(b_{ij}, b_{+j})\)) reduces then the affirmative of \(b_{ij}\) with ideal value surges. Since the rpi-PFV is based on the preferences of decision-makers and thus frequently changed among attributes. The separation between rpi-PFV and rni-PFV provides the upper bound of \(D(b_{ij}, b_{+j})\). So, we consider the ratio of \(D(b_{ij}, b_{+j})\) to \(D(b_{+j}, b_{-j})\) instead of considering \(D(b_{ij}, b_{+j})\). But for precise ideal values, the problem of an upper bound is insignificant due to the fact that the separation between ppi-PFV and pni-PFV is one, that is \(D(b_{+j}, b_{-j}) = 1\).

Now, we define the relative robust factor (RI\(^d\)) as follows.

**Definition 4.3:** For a distance measure \(D\), the relative robust factor RI\(^d\) of assessed value \((b_{ij})\) is defined as:

\[
RI^d(b_{ij}) = \frac{D(b_{ij}, b_{+j})}{D(b_{-j}, b_{+j})},
\]

**Theorem 4.1:** Let \(b_{ij}, b_{+j}\) and \(b_{-j}\) are the assessment values, rpi-PFV and rni-PFVs, respectively, in the PF decision matrix \(b\). The RI\(^d\) holds the following standards:

1. \(RI^d(b_{ij}) = 0 \iff b_{ij} = b_{+j}\)
2. \(RI^d(b_{ij}) = 1 \iff b_{ij} = b_{-j}\)
3. \(0 \leq RI^d(b_{ij}) \leq 1\)

**Example 4.1:** Consider \(\hat{Y} = \{\ell_1, \ell_2, \ell_3\}\) be the set of available choices to be assessed against the attributes \(\hat{E} = \{j_1, j_2\}\). This is the classical MADM problem, where \(j_1 \in \hat{E}_b\) and \(j_2 \in \hat{E}_c\). Assume that the PF decision matrix is given by

\[
b = [b_{ij}]_{3 \times 2} = \begin{pmatrix}
\ell_1 (0.5, 0.1, 0.1) & \ell_2 (0.4, 0.1, 0.5) \\
\ell_2 (0.5, 0.1, 0.3) & \ell_3 (0.3, 0.1, 0.5) \\
\ell_3 (0.6, 0.2, 0.1) & \ell_2 (0.6, 0.2, 0.2)
\end{pmatrix}
\]

1. According to Definition 4.1, the rpi-PFVs are \(b_{+1} = (0.6, 0.1, 0.1)\) and \(b_{+2} = (0.3, 0.1, 0.5)\). Moreover, the rni-PFVs are \(b_{-1} = (0.5, 0.1, 0.3)\) and \(b_{-2} = (0.6, 0.1, 0.2)\).
2. We calculate the Euclidean distance between two PFVs by using Equation (5) as follows:

\[
D^e_r(b_{ij}, b_{+j}) = \left(\frac{1}{2} \left( |\xi_{b_{ij}} - \xi_{b_{+j}}|^2 + |\eta_{b_{ij}} - \eta_{b_{+j}}|^2 + |\nu_{b_{ij}} - \nu_{b_{+j}}|^2 + |\nu_{b_{ij}} - \nu_{b_{+j}}|^2 \right) \right)^{1/2}
\]

We obtain \(D^e_r(b_{+1}, b_{-1}) = 0.234521\) and \(D^e_r(b_{+2}, b_{-2}) = 0.380789\). The relative robust factors are calculated by using Definition 4.3 as follows: \(RI^d(b_{11}) = D^e_r(b_{11}, b_{+1})/D^e_r(b_{+1}, b_{-1}) = 0.122474/0.234521 = 0.522233\), \(RI^d(b_{21}) = 1\), \(RI^d(b_{31}) = 0.603023\), \(RI^d(b_{12}) = 0.615882\), \(RI^d(b_{22}) = 0.643268\) and \(RI^d(b_{32}) = 0.491304\).

Now, we define the precise robust factor (RI\(^f\)) as follows:
**Definition 4.4:** For a distance measures $D$, the precise robust factor $\text{RI}_f$ of $b_{ij}$ is defined as follows:

$$ \text{RI}_f(b_{ij}) = \frac{D(b_{ij}, b_{+j})}{D(b_{-j}, b_{+j})} = D(b_{ij}, b_{+j}), \quad (20) $$

because $D(b_{-j}, b_{+j}) = 1$ for ppi-PFV and pni-PFVs.

**Theorem 4.2:** Let $b_{ij}$, $b_{+j}$ and $b_{-j}$ be the assessment values, ppi-PFV and pni-PFVs, respectively, in the PF decision matrix $b$. The $\text{RI}_f$ holds the following standards:

1. $\text{RI}_f(b_{ij}) = 0 \iff b_{ij} = b_{+j}$
2. $\text{RI}_f(b_{ij}) = 1 \iff b_{ij} = b_{-j}$
3. $0 \leq \text{RI}_f(b_{ij}) \leq 1$

**Example 4.2:** We continues Example 4.1 for precise ideal and precise robust factors.

1. Since $j_1 \in \hat{E}_b$ and $j_2 \in \hat{E}_c$. Therefore, according to the Definition 4.2, the rpi-PFVs are $b_{+1} = (1, 0, 0)$ and $b_{+2} = (0, 0, 1)$. Moreover, the rni-PFVs are $b_{-1} = (0, 0, 1)$ and $b_{-2} = (1, 0, 0)$.
2. We use Formula (19) for calculating distance between PFVs. The distance between $b_{+j}$ and $b_{-j}$ $(j = 1, 2)$ is 1. The precise robust factor are calculating by using Definition 4.4 as follows: $\text{RI}_f(b_{11}) = D_e(b_{11}, b_{+1})/D_e(b_{+1}, b_{-1}) = 0.484768/1 = 0.484768$, $\text{RI}_f(b_{21}) = 0.484768$, $\text{RI}_f(b_{31}) = 0.374166$, $\text{RI}_f(b_{12}) = 0.583095$, $\text{RI}_f(b_{22}) = 0.561249$ and $\text{RI}_f(b_{32}) = 0.927362$.

### 4.3. Decision-Making Process

The aim of decision-making (DM) process is to choose the favourite option based on experts defined attributes. In DM process, let $\hat{Y} = \{\ell_1, \ell_2, \ldots, \ell_m\}$ be the $m$ options which are evaluated against $n$ attributes (criteria) represented as $\hat{E} = \{j_1, j_2, \ldots, j_n\}$. Each option $\ell_i$ evaluated with respect to each criterion $j_j$ and the evaluated values are saved in the form of PF decision matrix $b = [b_{ij}]_{m \times n}$, where $b_{ij}$ represents the evaluation of $i^{th}$ alternative against $j^{th}$ criterion. The PF decision matrix $b = [b_{ij}]_{m \times n}$ can be represented as follows:

$$ b = [b_{ij}]_{m \times n} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \quad (21) $$

The attributes in real-life scenario are not of equal significance. Some of them have more significance then others. The weights of criteria tackle this issue in DM process. There are two types of weights discuss in this paper, that is, the precise weights and PF weights. The single value of the criterion is assign as precise weight and represented as $\omega = \{\omega_1, \omega_1, \ldots, \omega_n\}_T$
such that $\sum_{j=1}^{n} \omega_j = 1$. While the PF weights contains the importance, neutralness and unimportance degrees of the attribute and represented by $\omega = \{\omega_1, \omega_1, \ldots, \omega_n\}^T$, where $\omega_j = (\omega_1^j, \omega_1^j, \omega_1^j)$. The robust VIKOR method incorporate both type of weights.

The relative and precise ideal values and robust factors, and precise and PF weights are used to define the ranking indexes. First we consider the relative robust factors and precise weights-based ranking indexes.

**Definition 4.5:** The relative robust factor-based group utility index $\hat{S}^d$ of an alternative $\ell_i$ is described as follows:

$$\hat{S}^d(\ell_i) = \sum_{j=1}^{n} \left( R^d(b_{ij}) \cdot \omega_j \right),$$

where $\omega_j$ are the precise weights.

The relative robust factor-based individual regret index $\hat{R}^d$ of $\ell_i$ is defined as follows:

$$\hat{R}^d(\ell_i) = \max_{j=1}^{n} \left\{ R^d(b_{ij}) \cdot \omega_j \right\}.$$

The relative robust factor-based compromise index $\hat{Q}^d$ of $\ell_i$ is defined as follows:

$$\hat{Q}^d(\ell_i) = \lambda \cdot \frac{\hat{S}^d(\ell_i) - \min_{\ell' \in 1} \hat{S}^d(\ell_{\ell'})}{\max_{\ell' \in 1} \hat{S}^d(\ell_{\ell'}) - \min_{\ell' \in 1} \hat{S}^d(\ell_{\ell'})} + (1 - \lambda) \cdot \frac{\hat{R}^d(\ell_i) - \min_{\ell' \in 1} \hat{R}^d(\ell_{\ell'})}{\max_{\ell' \in 1} \hat{R}^d(\ell_{\ell'}) - \min_{\ell' \in 1} \hat{R}^d(\ell_{\ell'})},$$

where $\lambda \in [0, 1]$ is the decision mechanism coefficient.

Now, we consider the precise robust factors and precise weights-based ranking indexes.

**Definition 4.6:** The precise robust factor-based group utility index $\hat{S}^f$ of an alternative $\ell_i$ is described as follows:

$$\hat{S}^f(\ell_i) = \sum_{j=1}^{n} \left( R^f(b_{ij}) \cdot \omega_j \right),$$

where $\omega_j$ are the precise weights.

The precise robust factor-based individual regret index $\hat{R}^f$ of $\ell_i$ is defined as follows:

$$\hat{R}^f(\ell_i) = \max_{j=1}^{n} \left\{ R^f(b_{ij}) \cdot \omega_j \right\}.$$

The precise robust factor-based compromise index $\hat{Q}^f$ of $\ell_i$ is defined as follows:

$$\hat{Q}^f(\ell_i) = \lambda \cdot \frac{\hat{S}^f(\ell_i) - \min_{\ell' \in 1} \hat{S}^f(\ell_{\ell'})}{\max_{\ell' \in 1} \hat{S}^f(\ell_{\ell'}) - \min_{\ell' \in 1} \hat{S}^f(\ell_{\ell'})} + (1 - \lambda) \cdot \frac{\hat{R}^f(\ell_i) - \min_{\ell' \in 1} \hat{R}^f(\ell_{\ell'})}{\max_{\ell' \in 1} \hat{R}^f(\ell_{\ell'}) - \min_{\ell' \in 1} \hat{R}^f(\ell_{\ell'})},$$

where $\lambda \in [0, 1]$ is the decision mechanism coefficient.

Now, we propose two algorithmic procedures for robust VIKOR method. These algorithmic procedures for PFVs are based on the precise and relative ideals, precise weights, robust factors and ranking indexes.
Algorithm 1 Scenario 1: relative ideals, precise weights and PF decision matrix

1. Let the options (alternatives) set and attribute set are represented by \( Y = \{ \ell_1, \ell_2, ..., \ell_m \} \) and \( \hat{E} = \{ j_1, j_2, ..., j_n \} \), respectively.

2. The PF decision matrix \( b = [b_{ij}]_{m \times n} \) is obtained when each option \( \ell_i, 1 \leq i \leq m \) is assessed against each attribute \( j_j, 1 \leq j \leq n \). The precise weights can be acquired by decision-makers or choosing the suitable linguistics variables.

3. Equations (14) and (15) are used to formulate the rpi-PFV and rni-PFV for each attribute, respectively.

4. The separations between each assessed value \( b_{ij} \) and rpi-PFV, and rpi-PFV and rni-PFV are calculated and represented as \( D(b_{ij}^-, b_{ij}^+) \) and \( D(b_{ij}, b_{ij}^+) \), respectively.

5. The relative robust factor of each PFV \( b_{ij} \), that is, \( RI^d(b_{ij}) \) is calculated by Equation (18).

6. The separations between each assessed value \( b_{ij} \) and rpi-PFV, and rpi-PFV and rni-PFV are calculated and represented as \( D(b_{ij}^-, b_{ij}^+) \) and \( D(b_{ij}, b_{ij}^+) \), respectively. Then the relative robust factor-based compromise index \( \hat{Q}^d \) for each alternative is computed by Equation (24).

7. The computations of \( \hat{S}^d, \hat{R}^d \) and \( \hat{Q}^d \) provide the three ranking lists of the alternatives.

8. The minimum value in the ranking list of \( \hat{Q}^d \) serves as the compromise solution \( \ell' \) if the following standards holds:

   C1. Acceptable advantage: \( \hat{Q}^d(\ell'') - \hat{Q}^d(\ell') \geq \frac{1}{m-1} \), where \( \ell'' \) is the second minimum alternative in the ranking list of \( \hat{Q}^d \).

   C2. Acceptable stability: The ranking lists of \( \hat{S}^d \) and \( \hat{R}^d \) also have the minimum value for alternative \( \ell' \). The violation of any above-mentioned standards will lead to the set of the ultimate compromise solution, which consists of:

   a. The violation of second standard C2 leads to the compromise solution that contains \( \ell' \) and \( \ell'' \).

   b. While the violation of the first standard C1 leads to the compromise solution that contains \( \ell', \ell'', ..., \ell^p \), where \( p \) is the maximum value for which \( \hat{Q}^d(\ell^p) - \hat{Q}^d(\ell') < \frac{1}{m-1} \).

The PF weights and relative robust factor-based ranking indexes are defined for alternatives.

**Definition 4.7:** The relative robust factor-based group utility index \( S^d \) of an alternative \( \ell_i \) with a set of PF weights \( \sigma_j = (\sigma_j^\xi, \sigma_j^\eta, \sigma_j^\nu) \) for all \( j_j \in \hat{E} \) is defined as follows:

\[
S^d(\ell_i) = \sum_{j=1}^{n} \delta \left( RI^d(b_{ij}) \cdot \sigma_j \right)
= \sum_{j=1}^{n} \delta \left( 1 - (1 - \sigma_j^\xi) RI^d(b_{ij}), (\sigma_j^\eta) RI^d(b_{ij}), (\sigma_j^\nu) RI^d(b_{ij}) \right)
= \sum_{j=1}^{n} \left[ \frac{1}{2} \left( 2 - (1 - \sigma_j^\xi) RI^d(b_{ij}) - (\sigma_j^\nu) RI^d(b_{ij}) \right) \right], \tag{28}
\]
Algorithm 2 Scenario 2: precise ideals, precise weights and PF decision matrix

1. Steps 1 and 2 are same as Algorithm 1.
2. Equations (16) and (17) are used to formulate the ppi-PFV and pni-PFV for each attribute, respectively.
3. The separations between each assessed value $b_{ij}$ and ppi-PFV are calculated and represented as $D(b_{ij}, \bar{b}_j)$.
4. The precise robust factor of each PFV $b_{ij}$, that is, $\text{RI}^f(b_{ij})$ is calculated by Equation (20).
5. The precise robust factor-based group utility index $\hat{S}^f$ and individual regret index $\hat{R}^f$ are formulated by Equations (25) and (26), respectively. Then the precise robust factor-based compromise index $\hat{Q}^f$ for each alternative is computed by Equation (27).
6. The computations of $\hat{S}^f$, $\hat{R}^f$ and $\hat{Q}^f$ provide the three ranking lists of the alternatives.
7. The minimum value in the ranking list of $\hat{Q}^f$ serves as the compromise solution $\ell'^f$ if the following standards holds:
   C1. Acceptable advantage: $\hat{Q}^f(\ell''^f) - \hat{Q}^f(\ell'^f) \geq \frac{1}{m_T}$, where $\ell''^f$ is the second minimum alternative in the ranking list of $\hat{Q}^f$.
   C2. Acceptable stability: The ranking lists of $\hat{S}^f$ and $\hat{R}^f$ also have the minimum value for alternative $\ell'^f$. The violation of any above-mentioned standards will lead to the set of the ultimate compromise solution, which consists of:
   a. The violation of second standard C2 leads to the compromise solution that contains $\ell'^f$ and $\ell''^f$.
   b. While the violation of the first standard C1 leads to the compromise solution that contains $\ell'^f, \ell''^f, \ldots, \ell^p$, where $p$ is the maximum value for which $\hat{Q}^f(\ell^p) - \hat{Q}^f(\ell'^f) < \frac{1}{m_T}$.

where $\delta$ and $b_{ij} \in [b_{ij}]_{m \times n}$ are score function and PFVs, respectively. Additionally, the multiplication of PFV with scalar in Equation (28) is defined in Definition 2.4.

The relative robust factor and PF weights-based individual regret index $R^d$ of $\ell_i$ is defined as follows:

$$R^d(\ell_i) = \max_{j=1}^n \delta \left( \text{RI}^d(b_{ij}) \cdot \varpi_j \right)$$

$$= \max_{j=1}^n \left\{ \frac{1}{2} \left( 2 - (1 - \varpi_j^x) \text{RI}^d(b_{ij}) - (\varpi_j^y) \text{RI}^d(b_{ij}) \right) \right\}. \quad (29)$$

The relative robust factor and PF weights compromise index $Q^d$ of $\ell_i$ is defined as follows:

$$Q^d(\ell_i) = \lambda \cdot \frac{S^d(\ell_i) - \min_{\ell=1}^m S^d(\ell_r)}{\max_{\ell=1}^m S^d(\ell_r) - \min_{\ell=1}^m S^d(\ell_r)} + (1 - \lambda) \cdot \frac{R^d(\ell_i) - \min_{\ell=1}^m R^d(\ell_r)}{\max_{\ell=1}^m R^d(\ell_r) - \min_{\ell=1}^m R^d(\ell_r)}.$$$ \quad (30)$$

The PF weights and precise robust factor-based ranking indexes are defined for alternatives.
Definition 4.8: The precise robust factor-based group utility index $S^f$ of an alternative $\ell_i$ with a set of PF weights $\varpi_j = (\varpi_j^x, \varpi_j^n, \varpi_j^v)$ for all $j \in \hat{E}$ is defined as follows:

$$S^f(\ell_i) = \sum_{j=1}^{n} \delta \left( Rl^f(b_{ij}) \cdot \varpi_j \right)$$

$$= \sum_{j=1}^{n} \delta \left( 1 - (1 - \varpi_j^x)Rl^f(b_{ij}), (\varpi_j^n)Rl^f(b_{ij}), (\varpi_j^v)Rl^f(b_{ij}) \right)$$

$$= \sum_{j=1}^{n} \left[ \frac{1}{2} \left( 2 - (1 - \varpi_j^x)Rl^f(b_{ij}) - (\varpi_j^v)Rl^f(b_{ij}) \right) \right]$$

(31)

where $\delta$ and $b_{ij} \in [b_{ij}]_{m \times n}$ are score function and PFVs, respectively. Additionally, the multiplication of PFV with scalar in Equation (31) has defined in Definition 2.4.

The precise robust factor and PF weights-based individual regret index $R^f$ of $\ell_i$ is defined as follows:

$$R^f(\ell_i) = \max_{j=1}^{n} \delta \left( Rl^f(b_{ij}) \cdot \varpi_j \right)$$

$$= \max_{j=1}^{n} \left\{ \frac{1}{2} \left( 2 - (1 - \varpi_j^x)Rl^f(b_{ij}) - (\varpi_j^v)Rl^f(b_{ij}) \right) \right\}$$

(32)

The precise robust factor and PF weights-based compromise index $Q^f$ of $\ell_i$ is defined as follows:

$$Q^f(\ell_i) = \lambda . \frac{S^f(\ell_i) - \min_{\ell_f=1}^{m} S^f(\ell_f)}{\max_{\ell_f=1}^{m} S^f(\ell_f) - \min_{\ell_f=1}^{m} S^f(\ell_f)} + (1 - \lambda) . \frac{R^f(\ell_i) - \min_{\ell_f=1}^{m} R^f(\ell_f)}{\max_{\ell_f=1}^{m} R^f(\ell_f) - \min_{\ell_f=1}^{m} R^f(\ell_f)}$$

(33)

Again we propose two algorithmic procedures for VIKOR method. These algorithmic procedures represent the robust VIKOR methods for PFVs based on PF weights. This VIKOR method is based on precise and relative ideals, precise and relative indexes, PF weights and precise and relative ranking indexes.

5. Selection of Priority Area for Investment in Under-Developing Countries

Mostly under-developing countries are facing the problems of corruption. Therefore, the economic situations of many countries are going down day by day. Some countries are taking some decisions against the corrupt elements. The economy, environment and budget should be focused while making an investments by under-developing countries. The short- and long-term benefits, operational costs, job creation, maintenance, revenue generated, yield, reliability and minimum effect on environment and peoples are important parameters. A good sector should focus on job opportunity for the peoples. Therefore, the suitable area for investment for under-developing country should be choosen wisely.

In this part, we study the problems of selecting an area for investment for under-developing countries. Let $\hat{Y} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ represents the set of different sectors or areas for investment (alternative), where $\ell_1$, $\ell_2$, $\ell_3$, $\ell_4$, $\ell_5$ and $\ell_6$ stands for food
Algorithm 3: Scenario 3: relative ideals, PF weights and PF decision matrix

1. Let the options (alternatives) set and attribute set are represented by $\hat{Y} = \{\ell_1, \ell_2, \ldots, \ell_m\}$ and $\hat{E} = \{j_1, j_2, \ldots, j_n\}$, respectively.
2. The PF decision matrix $b = [b_{ij}]_{m \times n}$ is obtained when each option $\ell_i$, $1 \leq i \leq m$ is assessed against each attribute $j_j$, $1 \leq j \leq n$. The PF weights can be acquired by decision-makers or choosing the suitable linguistics variables.
3. Steps 3-5 are same as Algorithm 1.
4. The relative robust factor-based group utility index $S^d$ and individual regret index $R^d$ are formulated by Equations (28) and (29), respectively. Then the relative robust factor-based compromise index $Q^d$ for each alternative is computed by Equation (30).
5. The computations of $S^d$, $R^d$ and $Q^d$ provide the three ranking lists of the alternatives.
6. The minimum value in the ranking list of $Q^d$ serves as the compromise solution $\ell'$ if the following standards holds:
   
   **C1. Acceptable advantage:** $Q^d(\ell'') - Q^d(\ell') \geq \frac{1}{m-1}$, where $\ell''$ is the second minimum alternative in the ranking list of $Q^d$.

   **C2. Acceptable stability:** The ranking lists of $S^d$ and $R^d$ also have the minimum value for alternative $\ell'$. The violation of any above-mentioned standards will lead to the set of the ultimate compromise solution, which consists of:
   
   a. The violation of second standard C2 leads to the compromise solution that contains $\ell'$ and $\ell''$.
   
   b. While the violation of the first standard C1 leads to the compromise solution that contains $\ell', \ell'', \ldots, \ell^p$, where $p$ is the maximum value for which $Q^d(\ell^p) - Q^d(\ell') < \frac{1}{m-1}$.

processing, textile, logistics, automobiles, IT & ITes, power sector, respectively. Let $\hat{E} = \{j_1, j_2, j_3, j_4, j_5, j_6\}$ represents the set of criteria (attributes), where $j_1, j_2, j_3, j_4, j_5$ and $j_6$ stands for short term benefits, long-term benefits, operational costs, job creation, revenue generated and reliability, respectively. The shortlisted areas are evaluated against the six parameters (criteria).

To solve this problem, a committee of different personals established that consist of economists, decision-makers, managers, governments servants and some other policy makers. The committee is responsible for assessment of the available alternatives against predefined criteria. It is possible for a committee to change the criteria or characteristics to assess the options. So, the committee evaluate the alternatives against criteria and propose their preferences in the form of PFV. Their preferences generates the PF decision matrix of six rows and six columns and represented as $b = [b_{ij}]_{6 \times 6}$, where $b_{ij}$ shows the evaluation of $i^{th}$ alternative against $j^{th}$ criterion. The PF decision matrix for this problem is displayed in Table 4.

The PF decision matrix is normalised, because $j_3$ is the cost criterion, by following equation:

$$b_{ij} = \begin{cases} (\xi_{bij}, \eta_{bij}, \nu_{bij}) & \text{if } j_j \in \hat{E}_b \\ (v_{bij}, \eta_{bij}, \xi_{bij}) & \text{if } j_j \in \hat{E}_c \end{cases}$$

(34)
Algorithm 4 Scenario 4: precise ideals, PF weights and PF decision matrix

1. Steps 1 and 2 are same as Algorithm 3.
3. Steps 3-5 are same as Algorithm 2.
6. The relative robust factor-based group utility index $S_f$ and individual regret index $R_f$ are formulated by Equations (31) and (32), respectively. Then the relative robust factor-based compromise index $Q_f^\prime$ for each alternative is computed by Equation (33).
7. The computations of $S_f$, $R_f$ and $Q_f^\prime$ provide the three ranking lists of the alternatives.
8. The minimum value in the ranking list of $Q_f^\prime$ serves as the compromise solution $\ell'$ if the following standards holds:

C1. Acceptable advantage: $Q_f^\prime(\ell'') - Q_f^\prime(\ell') \geq \frac{1}{m-1}$, where $\ell''$ is the second minimum alternative in the ranking list of $Q_f^\prime$.

C2. Acceptable stability: The ranking lists of $S_f$ and $R_f$ also have the minimum value for alternative $\ell'$. The violation of any above-mentioned standards will lead to the set of the ultimate compromise solution, which consists of:

a. The violation of second standard C2 leads to the compromise solution that contains $\ell'$ and $\ell''$.

b. While the violation of the first standard C1 leads to the compromise solution that contains $\ell', \ell'', ..., \ell_p$, where $p$ is the maximum value for which $Q_f^\prime(\ell_p) - Q_f^\prime(\ell') < \frac{1}{m-1}$.

---

Table 4. The PFSS $\Gamma = (\hat{F}, \hat{A})$

| $\hat{Y}$ | $j_1$        | $j_2$        | $j_3$        | $j_4$        | $j_5$        | $j_6$        |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|
| $\ell_1$  | (0.1,0.3,0.5)| (0.2,0.3,0.4)| (0.5,0.2,0.2)| (0.3,0.1,0.5)| (0.4,0.1,0.4)| (0.3,0.2,0.5)|
| $\ell_2$  | (0.5,0.1,0.3)| (0.3,0.1,0.3)| (0.4,0.1,0.3)| (0.3,0.2,0.4)| (0.3,0.2,0.5)| (0.4,0.1,0.4)|
| $\ell_3$  | (0.2,0.4,0.3)| (0.2,0.2,0.5)| (0.1,0.3,0.5)| (0.2,0.1,0.6)| (0.5,0.1,0.3)| (0.4,0.2,0.3)|
| $\ell_4$  | (0.6,0.1,0.2)| (0.1,0.2,0.4)| (0.5,0.1,0.2)| (0.7,0.1,0.1)| (0.2,0.2,0.5)| (0.2,0.4,0.3)|
| $\ell_5$  | (0.2,0.2,0.5)| (0.3,0.1,0.4)| (0.5,0.1,0.2)| (0.5,0.2,0.2)| (0.6,0.1,0.2)| (0.2,0.2,0.4)|
| $\ell_6$  | (0.5,0.1,0.3)| (0.2,0.2,0.5)| (0.4,0.2,0.1)| (0.4,0.1,0.3)| (0.3,0.1,0.5)| (0.5,0.1,0.3)|

Table 5. The PFSS $\Gamma = (\hat{F}', \hat{A})$

| $\hat{Y}$ | $j_1$        | $j_2$        | $j_3$        | $j_4$        | $j_5$        | $j_6$        |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|
| $\ell_1$  | (0.1,0.3,0.5)| (0.2,0.3,0.4)| (0.2,0.2,0.5)| (0.3,0.1,0.5)| (0.4,0.1,0.4)| (0.3,0.2,0.5)|
| $\ell_2$  | (0.5,0.1,0.3)| (0.3,0.1,0.3)| (0.3,0.1,0.4)| (0.3,0.2,0.4)| (0.3,0.2,0.5)| (0.4,0.1,0.4)|
| $\ell_3$  | (0.2,0.4,0.3)| (0.2,0.2,0.5)| (0.3,0.1,0.4)| (0.2,0.1,0.6)| (0.5,0.1,0.3)| (0.4,0.2,0.3)|
| $\ell_4$  | (0.6,0.1,0.2)| (0.1,0.2,0.4)| (0.5,0.1,0.1)| (0.2,0.2,0.5)| (0.2,0.4,0.3)|
| $\ell_5$  | (0.2,0.2,0.5)| (0.3,0.1,0.4)| (0.2,0.1,0.5)| (0.5,0.2,0.2)| (0.6,0.1,0.2)| (0.2,0.2,0.4)|
| $\ell_6$  | (0.5,0.1,0.3)| (0.2,0.2,0.5)| (0.1,0.2,0.4)| (0.4,0.1,0.3)| (0.3,0.1,0.5)| (0.5,0.1,0.3)|

All the criterion are treated as the benefit type in the normalised PF decision matrix shown in Table 5.

The algorithmic procedures proposed above are used to solve the problem of selecting an area for investment for under-developing countries. This provides the more liberty to choose the any procedure according to their sources and interest.
5.1. Formulation by Algorithm 1

The normalised PF decision matrix is already generated in Table 5. Equations (14) and (15) are used to formulate the rpi-PFVs \(b_{+j}\) and rni-PFVs \(b_{-j}\) as follows:

\[
\begin{align*}
\text{rpi - PFV } \rightarrow b_{+j} &= \left\{ 
\begin{array}{l}
b_{+1} = (0.6, 0.3, 0.2) \\
b_{+2} = (0.3, 0.3, 0.3) \\
b_{+3} = (0.5, 0.3, 0.1) \\
b_{+4} = (0.7, 0.2, 0.1) \\
b_{+5} = (0.6, 0.2, 0.2) \\
b_{+6} = (0.5, 0.4, 0.3)
\end{array}
\right. \\
\text{rni - PFV } \rightarrow b_{-j} &= \left\{ 
\begin{array}{l}
b_{-1} = (0.1, 0.1, 0.5) \\
b_{-2} = (0.1, 0.1, 0.5) \\
b_{-3} = (0.1, 0.1, 0.5) \\
b_{-4} = (0.2, 0.1, 0.6) \\
b_{-5} = (0.2, 0.1, 0.5) \\
b_{-6} = (0.2, 0.1, 0.5)
\end{array}
\right.
\end{align*}
\]  

Equation (19) is used to calculate the separation between rpi-PFVs and rni-PFVs. We use this equation to formulate the separation between two PFVs. The results summarised in Equation (37).

\[
\begin{align*}
D^e_{\circ}(b_{+j}, b_{-j}) &= \left\{ 
\begin{array}{l}
D^e_{\circ}(b_{+1}, b_{-1}) = 0.43589 \\
D^e_{\circ}(b_{+2}, b_{-2}) = 0.2 \\
D^e_{\circ}(b_{+3}, b_{-3}) = 0.40000 \\
D^e_{\circ}(b_{+4}, b_{-4}) = 0.5 \\
D^e_{\circ}(b_{+5}, b_{-5}) = 0.36056 \\
D^e_{\circ}(b_{+6}, b_{-6}) = 0.26458
\end{array}
\right.
\end{align*}
\]  

The separation between each PFVs \(b_{ij}\) and the rpi-PFVs \(b_{+j}\) is calculated by Equation (19) and represented as \(D^e_{\circ}(b_{ij}, b_{+j}) = c_{ij}\). The results are displaced in Equation (38).

\[
\begin{align*}
c_{11} & 0.43589 & 0.223607 & 0.387298 & 0.4 & 0.2 & 0.223607 \\
c_{12} & 0.1 & 0. & 0.264575 & 0.360555 & 0.316228 & 0.1 \\
c_{13} & 0.360555 & 0.223607 & 0.2 & 0.5 & 0.1 & 0.1 \\
c_{14} & 0. & 0.173205 & 0.360555 & 0. & 0.360555 & 0.3 \\
c_{15} & 0.360555 & 0.1 & 0.360555 & 0.173205 & 0. & 0.244949 \\
c_{16} & 0.1 & 0.2236 & 0.360555 & 0.264575 & 0.3 & 0.
\end{align*}
\]  

The relative robust factor \(R^d_{\circ}(b_{ij})\) are calculated by using Definition 4.3. The \(R^d_{\circ}(b_{ij})\) are multiplying by precise weights \(\omega_j (j = \{1, 2, \ldots, 6\})\) and represented as \(R^d_{\circ}(b_{ij}) \cdot \omega_j = d_{ij}\). If the weight vector is \(\omega = \{0.1780, 0.1644, 0.1507, 0.1918, 0.1918, 0.1918\}\), then the results are presented in Equation (39).

\[
\begin{align*}
d_{11} & 0.178 & 0.1838 & 0.1459 & 0.1534 & 0.1064 & 0.1621 \\
d_{12} & 0.0408 & 0. & 0.0997 & 0.1383 & 0.1682 & 0.0725 \\
d_{13} & 0.1472 & 0.1838 & 0.0754 & 0.1918 & 0.0532 & 0.0725 \\
d_{14} & 0. & 0.1424 & 0.1358 & 0. & 0.1918 & 0.2175 \\
d_{15} & 0.1472 & 0.0822 & 0.1358 & 0.0664 & 0. & 0.1776 \\
d_{16} & 0.0408 & 0.1838 & 0.1358 & 0.1015 & 0.1596 & 0.
\end{align*}
\]  

Equations (22), (23) and (24) are used to calculate the relative robust factor and precise weights-based group utility index \(\hat{S}^d_{\circ}\), individual regret index \(\hat{R}^d_{\circ}\) and the compromise index
\[ \hat{Q}^d, \text{ respectively. All the calculations are displayed in Equation (40).} \]

\[
\begin{pmatrix}
\hat{S}^d(\ell_i) \\
\hat{R}^d(\ell_i) \\
\hat{Q}^d(\ell_i)
\end{pmatrix} = 
\begin{pmatrix}
0.9297 & 0.5195 & 0.7239 & 0.6875 & 0.6093 & 0.6216 \\
0.1838 & 0.1682 & 0.1918 & 0.2175 & 0.1776 & 0.1838 \\
0.6582 & 0.4885 & 0.7048 & 0.2044 & 0.2826 & 0.2044 & 0.2826
\end{pmatrix}
\]

\[ (40) \]

\[
\begin{align*}
\hat{S}^d(\ell_i) & > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \\
\hat{R}^d(\ell_i) & > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \\
\hat{Q}^d(\ell_i) & > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \\
\end{align*}
\]

From Equation (41), the three ranking lists \( \ell_2 > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \), \( \ell_2 > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \) and \( \ell_2 > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \) are obtained by sorting each \( \hat{S}^d(\ell_i) \), \( \hat{R}^d(\ell_i) \) and \( \hat{Q}^d(\ell_i) \) value in ascending order, respectively. The textile sector among the different available sectors, i.e. \( \ell_2 \) is the finest choice among three ranking lists. Moreover,

\[
\hat{Q}^d(\ell'') - \hat{Q}^d(\ell') = \hat{Q}^d(\ell_2) - \hat{Q}^d(\ell_5) = 0.2044 > \frac{1}{6-1} = \frac{1}{5} = 0.2
\]

Thus both conditions (standards C1 & C2) in step 8 of Algorithm 1 are satisfied for textile sector. Therefore, textile sector is the compromise solution for investment problem. The order of the investment options is \( \ell_2 > \ell_5 > \ell_6 > \ell_4 > \ell_3 > \ell_1 \).

### 5.2. Formulation by Algorithm 2

The PFVs \((1, 0, 0)\) and \((0, 0, 1)\) serve as the ppi-PFV and the pni-PFV for all criteria due to normalisation in Table 5. The separation between \( b_{ij} \) and \( b_{i+} \) is formulated by Equation (19) and represented as \( D^e(b_{ij}, b_{i+}) = C_{ij} \). The separation results are presented in Equation (42).

\[
\begin{pmatrix}
C_{i1} \\
C_{i2} \\
C_{i3} \\
C_{i4} \\
C_{i5} \\
C_{i6}
\end{pmatrix} = 
\begin{pmatrix}
0.7616 & 0.6708 & 0.6856 & 0.6164 & 0.5196 & 0.6245 \\
0.4243 & 0.5831 & 0.5916 & 0.5916 & 0.6245 & 0.5196 \\
0.6708 & 0.6856 & 0.4243 & 0.7141 & 0.4243 & 0.5 \\
0.3317 & 0.755 & 0.6856 & 0.2449 & 0.6856 & 0.6708 \\
0.6856 & 0.5916 & 0.6856 & 0.4123 & 0.3317 & 0.6633 \\
0.4243 & 0.6856 & 0.7416 & 0.5 & 0.6164 & 0.4243
\end{pmatrix}
\]

\[ (42) \]

The precise robust factor of \( b_{ij} \) is equal to the distance between \( b_{ij} \) and \( b_{i+} \), that is, \( R_l^f(b_{ij}) = D^e(b_{ij}, b_{i+}) \) (due to Definition 4.4). The \( R_l^f(b_{ij}) \) are multiplying by precise weights \( \omega_j \) \((j = \{1, 2, \ldots, 6\})\) and represented as \( R_l^f(b_{ij}), \omega_j = D_{ij} \). If the precise weight vector is
\( \omega = \{0.15, 0.2, 0.15, 0.175, 0.2, 0.125\} \), then the results are presented in Equation (43).

\[
\begin{pmatrix}
\ell_1 \\
\ell_2 \\
\ell_3 \\
\ell_4 \\
\ell_5 \\
\ell_6 \\
\end{pmatrix} = 
\begin{pmatrix}
0.1142 & 0.1342 & 0.1028 & 0.1079 & 0.1039 & 0.0781 \\
0.0636 & 0.1166 & 0.0887 & 0.1035 & 0.1249 & 0.0650 \\
0.1006 & 0.1371 & 0.0636 & 0.1250 & 0.0849 & 0.0625 \\
0.0498 & 0.1510 & 0.1028 & 0.0429 & 0.1371 & 0.0839 \\
0.1028 & 0.1183 & 0.1028 & 0.0722 & 0.0849 & 0.0625 \\
0.0636 & 0.1371 & 0.1112 & 0.0875 & 0.1233 & 0.0530 \\
\end{pmatrix}
\]

(43)

The precise robust factor and precise weights based group utility index \( \hat{S}'_f \), individual regret index \( \hat{R}'_f \) and compromise index \( \hat{Q}'_f \) are formulating by Equations (25), (26) and (27).

\[
\begin{pmatrix}
\ell_1 \\
\ell_2 \\
\ell_3 \\
\ell_4 \\
\ell_5 \\
\ell_6 \\
\end{pmatrix} = 
\begin{pmatrix}
0.6411 & 0.5624 & 0.5737 & 0.5674 & 0.5454 & 0.5758 \\
0.1342 & 0.1249 & 0.1371 & 0.1510 & 0.1183 & 0.1371 \\
0.7424 & 0.1894 & 0.4355 & 0.6150 & 0.4465 & \\
\end{pmatrix}
\]

(44)

Rankings

\[
\begin{pmatrix}
\ell_1 \\
\ell_2 \\
\ell_3 \\
\ell_4 \\
\ell_5 \\
\ell_6 \\
\end{pmatrix} = 
\begin{pmatrix}
\ell_5 > \ell_2 > \ell_4 > \ell_3 > \ell_6 > \ell_1 \\
\ell_5 > \ell_2 > \ell_1 > \{\ell_3, \ell_6\} > \ell_4 \\
\ell_5 > \ell_2 > \ell_3 > \ell_6 > \ell_4 > \ell_1 \\
\end{pmatrix}
\]

(45)

By sorting the \( \hat{S}'(\ell_i) \), \( \hat{R}'(\ell_i) \) and \( \hat{Q}'(\ell_i) \) values in ascending order, the three ranking lists \( \ell_5 > \ell_2 > \ell_4 > \ell_3 > \ell_6 > \ell_1 \), \( \ell_5 > \ell_2 > \ell_1 > \{\ell_3, \ell_6\} > \ell_4 \) and \( \ell_5 > \ell_2 > \ell_3 > \ell_6 > \ell_4 > \ell_1 \), respectively, are obtained from Equation (45). All ranking lists mark IT & ITes sector is the best among different options and the standard C1 holds well. Moreover,

\[
\hat{Q}'(\ell''') - \hat{Q}'(\ell') = \hat{Q}'(\ell_2) - \hat{Q}'(\ell_5) = 0.1894 < \frac{1}{6 - 1} = \frac{1}{5} = 0.20
\]

Thus the first standard (acceptable advantage) in Step 8 of Algorithm 2 is not satisfied. Therefore, the ultimate compromise solution is proposed. The IT & ITes and textile are the ultimate compromise solutions of the problem. The order of the investment problem is \( \{\ell_5, \ell_2\} > \ell_3 > \ell_6 > \ell_4 > \ell_1 \).

5.3. Formulation by Algorithm 3

Algorithm 3 is employed to solve this problem by PF weights. The ideal values rpi-PFVs \( b_{ij}^+ \) and rni-PFVs \( b_{ij}^- \) have calculated in Equations (35) and (36), respectively. The separation between \( b_{ij}^+ \) and \( b_{ij}^- \) have calculated in Equation (37). The relative robust factor \( R_{ij}'(b_{ij}) \) is calculated by using Definition 4.3. Let the PF importance weights are \( \varpi_1 = (0.7, 0.1, 0.2) \), \( \varpi_2 = (0.5, 0.2, 0.3) \), \( \varpi_3 = (0.6, 0.1, 0.3) \), \( \varpi_4 = (0.6, 0.2, 0.2) \), \( \varpi_5 = (0.7, 0.1, 0.1) \), and \( \varpi_6 = (0.3, 0.0, 0.7) \).
The relative robust factor-based group utility index $S^d(\ell_i)$, individual regret index $R^d(\ell_i)$ and the compromise index $Q^d(\ell_i)$ with a set of PF importance weights $\varpi_j = (\varpi_j^f, \varpi_j^n, \varpi_j^r)$ are calculated by using Equations (28)–(30), respectively. The results are summarised in Equation (46).

\[
S^d(\ell_i) = \begin{pmatrix} 3.514 & 2.248 & 2.937 & 2.294 & 2.299 & 2.748 \\ 0.75 & 0.7597 & 0.7 & 0.8 & 0.6832 & 0.7428 \\ 0.7859 & 0.3275 & 0.3438 & 0.5184 & 0.0202 & 0.4526 \end{pmatrix}
\]

(46)

\[
R^d(\ell_i) = \begin{pmatrix} \ell_2 > \ell_4 > \ell_5 > \ell_6 > \ell_3 > \ell_1 \\ \ell_5 > \ell_3 > \ell_6 > \ell_1 > \ell_2 > \ell_4 \end{pmatrix}
\]

(47)

From Equation (46), three ranking lists $\ell_2 > \ell_4 > \ell_5 > \ell_6 > \ell_3 > \ell_1$, $\ell_5 > \ell_3 > \ell_6 > \ell_1 > \ell_2 > \ell_4$ and $\ell_5 > \ell_2 > \ell_3 > \ell_6 > \ell_4 > \ell_1$ are obtained by sorting each $S^d$, $R^d$ and $Q^d$ value in ascending order, respectively.

5.4. Formulation by Algorithm 4

Algorithm 4 is employed to solve this problem by PF weights. The PFVs $(1, 0, 0)$ and $(0, 0, 1)$ serve as the ppi-PFV and pni-PFV for all criteria due to normalisation in Table 5. The separation between $b_{ij}$ and $b_{i,j}$ is formulated by Equation (42). The precise robust factor of $b_{ij}$ is equal to the separations between $b_{ij}$ and $b_{i,j}$, that is, $R^f(b_{ij}) = D^\zeta(b_{ij}, b_{i,j})$ (Due to Definition 4.4).

The precise robust factor and PF weights-based group utility index $S^f$, individual regret index $R^f$ and the compromise index $Q^f$ are calculating by Equations (31), (32) and (33), respectively, and the results are presented in Equation (48).

\[
S^f(\ell_i) = \begin{pmatrix} 2.942 & 2.661 & 2.695 & 2.433 & 2.594 & 2.701 \\ 0.6534 & 0.6456 & 0.6072 & 0.6778 & 0.6151 & 0.641 \\ 0.8267 & 0.4953 & 0.2574 & 0.5 & 0.2144 & 0.5027 \end{pmatrix}
\]

(48)

\[
R^f(\ell_i) = \begin{pmatrix} \ell_4 > \ell_5 > \ell_2 > \ell_3 > \ell_6 > \ell_1 \\ \ell_3 > \ell_5 > \ell_6 > \ell_2 > \ell_1 > \ell_4 \end{pmatrix}
\]

(49)

\[
Q^f(\ell_i) = \begin{pmatrix} \ell_5 > \ell_3 > \ell_2 > \ell_4 > \ell_6 > \ell_1 \end{pmatrix}
\]
From Equation (49), the three ranking lists $\ell_4 \succ \ell_5 \succ \ell_2 \succ \ell_3 \succ \ell_6 \succ \ell_1$, $\ell_3 \succ \ell_5 \succ \ell_6 \succ \ell_1$, $\ell_2 \succ \ell_1 \succ \ell_4$ and $\ell_5 \succ \ell_3 \succ \ell_2 \succ \ell_4 \succ \ell_6 \succ \ell_1$ are obtained by sorting each $S_i^f(\ell_i)$, $R_i^f(\ell_i)$ and $Q_i^f(\ell_i)$ value in ascending order, respectively. The conditions of acceptable advantage and acceptable stability are not satisfied. The ultimate solution and ranking of order is $\{\ell_5, \ell_3\} \succ \ell_2 \succ \ell_4 \succ \ell_6 \succ \ell_1$.

6. Comparison Analysis

A comparison of the suggested robust VIKOR method with the existing MCDM methods of PFSs is made in this section.

Remark 6.1: In order to compare our proposed MCDM method, we solve the problem studied in Section 5 by existing MCDM methods. The rankings of alternatives are slightly different depending on the methodology selected. The summary of the comparison with existing methods is displayed in Table 6.

Table 6. Comparison with existing MCDM methods.

| Reference | Method | Ranking |
|-----------|--------|---------|
| Wei [11]  | PFWA operator | $\ell_4 \succ \ell_5 \succ \ell_3 \succ \ell_2 \succ \ell_6 \succ \ell_1$ |
| Wei [11]  | PFWG operator | $\ell_5 \succ \ell_2 \succ \ell_3 \succ \ell_4 \succ \ell_6 \succ \ell_1$ |
| Thao [34] | Similarity measure | $\ell_6 \succ \ell_2 \succ \ell_3 \succ \ell_4 \succ \ell_3 \succ \ell_1$ |
| Wang et al. [40] | Projection-based VIKOR | $\{\ell_6, \ell_5\} \succ \ell_2 \succ \ell_4 \succ \ell_1 \succ \ell_3$ |
| Zhang et al. [41] | EDAS method | $\ell_5 \succ \ell_3 \succ \ell_4 \succ \ell_6 \succ \ell_2$ |
| Proposed  | Algorithm 1 | $\ell_2 \succ \ell_5 \succ \ell_6 \succ \ell_4 \succ \ell_3 \succ \ell_1$ |
| Proposed  | Algorithm 2 | $\{\ell_5, \ell_2\} \succ \ell_3 \succ \ell_6 \succ \ell_4 \succ \ell_1$ |
| Proposed  | Algorithm 3 | $\{\ell_5, \ell_3\} \succ \ell_3 \succ \ell_6 \succ \ell_4 \succ \ell_1$ |
| Proposed  | Algorithm 4 | $\{\ell_5, \ell_3\} \succ \ell_2 \succ \ell_4 \succ \ell_6 \succ \ell_1$ |

From Section 3.1, the strategic decision-making and pattern recognition problem from [32] are discussed and obtain the same ranking as [32].

In Section 3.1, the strategic decision-making and pattern recognition problem from [32] are discussed and obtain the same ranking as [32].

Finally, in this section, we apply our robust VIKOR method on different existing real-life MCDM problems. There are minor differences in the conclusions, if we compare the ranking that we obtain with the solutions provided by the literature. Our proposed method is reliable and accurate because it is based on axiomatically supported distance measures.

Example 6.1: We adopted the problem of implementing the enterprise resource planning system form [11]. Wei solve this problem with two methods and obtain rankings: $\ell_3 \succ \ell_2 \succ \ell_1 \succ \ell_5 \succ \ell_4$ and $\ell_3 \succ \ell_1 \succ \ell_2 \succ \ell_5 \succ \ell_4$.

When we solve this problem by our proposed methods, we obtain slightly different ranking. The Algorithm 1 have ranking $\{\ell_2, \ell_3\} \succ \ell_5 \succ \ell_1 \succ \ell_4$ and the ranking obtained by fixed ideal values, i.e. the Algorithm 2 is $\{\ell_2, \ell_3\} \succ \ell_5 \succ \ell_1 \succ \ell_4$. We obtain the ultimate solution of this problem.

Example 6.2: In [42], the beef supply chain case example is solved by picture fuzzy-ordered weighted distance VIKOR model. The optimal alternative obtained by Meksavang et al. [42] is the same as we obtained by using Algorithm 1. The ranking obtained in our proposed method is $\ell_{10} \succ \ell_4 \succ \ell_6 \succ \ell_8 \succ \ell_5 \succ \ell_3 \succ \ell_2 \succ \ell_5 \succ \ell_7 \succ \ell_9$ which is slightly different from [42] approach.
7. Conclusion

The Euclidean, Hamming and the generalised distance measures for PFSSs have introduced. Additional properties of the distance measures and their applications in decision-making and pattern recognition have focused. We have developed the robust VIKOR method for PFSSs. The relative and precise ideal PFVs, relative and precise robust factors, and relative and precise ranking indexes have defined. Different algorithmic procedures of robust VIKOR based on the relative and precise ideal PFVs, the relative and precise robust factor, precise and PF weights and relative and precise ranking indexes have proposed. In the end, investment problems have solved by using different algorithmic procedures of the robust VIKOR method. In the future, we will find other distance and similarity measures for PFSSs and use them to define the MCDM methods. Additionally, we will focus on distance and similarity measures for GPFSSs [16], linear Diophantine fuzzy set [43,44], bipolar fuzzy sets [45,46] and temporal IFSs [47].

Disclosure statement

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