1 Introduction

The purpose of this note is to prove a version of Woodin’s HOD dichotomy [4] from a strongly compact cardinal. A cardinal \( \kappa \) is strongly compact if every \( \kappa \)-complete filter extends to a \( \kappa \)-complete ultrafilter. The class of hereditarily ordinal definable sets, denoted HOD, is the minimum class \( M \) that contains all the ordinals and is definably closed: if \( S \subseteq M \) is definable (in the universe of sets \( V \)) by a set theoretic formula with parameters in \( M \), then \( S \in M \). HOD is a proper class transitive model of ZFC.

**Theorem 1.1.** Suppose \( \kappa \) is strongly compact. Then one of the following holds:

1. For every singular strong limit cardinal \( \lambda \geq \kappa \), \( \lambda \) is singular in HOD and \( \lambda^{+\text{HOD}} = \lambda^+ \).

2. All sufficiently large regular cardinals are measurable in HOD.

If (1) holds, (2) cannot (since there are arbitrarily large successor cardinals that are not inaccessible in HOD), so Theorem 1.1 is truly a dichotomy.

We will actually prove a stronger dichotomy. An inner model \( M \) is said to have the \( \lambda \)-cover property if every set of ordinals of cardinality less than \( \lambda \) is covered by (that is, included in) a set of ordinals in \( M \) of cardinality less than \( \lambda \).

**Theorem 1.2.** Suppose \( \kappa \) is strongly compact. Then one of the following holds:

1. For every strong limit cardinal \( \lambda \geq \kappa \), HOD has the \( \lambda \)-cover property.

2. All sufficiently large regular cardinals are \( \omega \)-strongly measurable in HOD.

The definition of \( \omega \)-strong measurability is deferred until the beginning of the next section, but let us note here that any cardinal that is \( \omega \)-strongly measurable in HOD is indeed measurable in HOD. Note that Theorem 1.2(1) strengthens Theorem 1.1(1) while Theorem 1.2(2) strengthens Theorem 1.1(2).

Assuming instead that \( \kappa \) is HOD-supercompact, Woodin [4] proved an even stronger dichotomy theorem, for example establishing that if (either) condition (1) above holds, then \( \kappa \) is supercompact in HOD. On the other hand, Cheng-Friedman-Hankins [1] produce a model of ZFC with a supercompact cardinal \( \kappa \) in which (1) holds, yet \( \kappa \) is not weakly compact in HOD, which shows that Woodin’s theorem cannot be proved from the hypotheses we assume here.
2 The proof

Suppose $\delta$ is a regular cardinal and $S \subseteq \delta$ is a stationary set. We say an inner model $M$ splits $S$ if $S \in M$ and for all $\gamma$ such that $(2^\gamma)^M < \delta$, there is a partition of $S$ into $\gamma$-many stationary sets. A cardinal is $\omega$-strongly measurable in HOD if HOD does not split the set of ordinals less than $\delta$ with countable cofinality, which is denoted by $S^\delta_\omega$.

**Theorem 2.1** (Woodin, [3] Lemma 3.37). If $S$ is an ordinal definable stationary subset of a regular cardinal $\delta$ and HOD does not split $S$, then $(C_\delta \mid S) \cap$ HOD is atomic in HOD. In particular, if $\delta$ is $\omega$-strongly measurable in HOD, then $\delta$ is measurable in HOD and $\delta$ contains an $\omega$-club of cardinals inaccessible in HOD.

The last sentence is supposed to highlight that the existence of $\omega$-strongly measurable cardinals entails a massive failure of the cover property even for countable sets. Actually there is an $\omega_1$-club of cardinals measurable in HOD below any cardinal that is $\omega_1$-strongly measurable in HOD in the natural sense.

**Proposition 2.2.** Suppose $\kappa$ is strongly compact, $\delta \geq \kappa$ is regular, and HOD splits $S^\delta_\omega$. Then for any $\gamma$ such that $(2^\gamma)^{\text{HOD}} < \delta$, there is a $\kappa$-complete fine ultrafilter $U$ on $P_\kappa(\gamma)$ such that $U$ concentrates on $P_\kappa(\gamma) \cap$ HOD.

**Proof.** Let $j : V \rightarrow M$ be an elementary embedding from the universe into an inner model such that crit($j$) = $\kappa$ and $j[\bar{S}]$ is contained in some set $S \in M$ with $|S|^M < j(\kappa)$. In particular, the ordinal $\delta_\kappa = \sup j[\bar{S}]$ has cofinality less than $j(\kappa)$ in $M$. Let $C \subseteq \delta_\kappa$ be a closed unbounded set of ordertype cf($\delta_\kappa$).

Fix a cardinal $\gamma$ such that $(2^\gamma)^{\text{HOD}} < \delta$ and let $(S_\alpha)_{\alpha < \gamma}$ witness that HOD splits $S^\delta_\omega$. Then let

$$\sigma = \{\xi < j(\gamma) : T_\xi \cap \delta_\kappa \text{ is stationary in } M\}$$

where $\bar{T} = j(\bar{S})$. Thus $\sigma \in \text{HOD}^M$. Notice that $j[\gamma] \subseteq \sigma$: $j[S_\xi] \subseteq T_{j(\xi)}$, so in fact $T_{j(\xi)} \cap \delta_\kappa$ is truly stationary (not just in $M$). For all $\xi \in \sigma$, $T_\xi \cap C \neq \emptyset$, so let $f(\xi) = \min(T_\xi \cap C)$. Then $f : \sigma \rightarrow C$ is an injection. So $|\sigma| = \text{cf}^M(\delta_\kappa)$. In particular, $\sigma \in j(P_\kappa(\gamma))$.

Finally let $U$ be the ultrafilter on $P_\kappa(\gamma)$ derived from $j$ using $\sigma$. That is, let $U = \{A \subseteq P_\kappa(\gamma) : \sigma \subseteq j(A)\}$. Since $\kappa \in \text{HOD}^M$, $\sigma \in j(P_\kappa(\gamma) \cap \text{HOD})$, and hence HOD $\cap P_\kappa(\gamma) \cap \text{HOD}$. Since $j[\gamma] \subseteq \sigma$, $U$ is fine. Since crit($j$) = $\kappa$, $U$ is $\kappa$-complete.

The main observation involved in the proof above is that the stationary splitting argument from [3] (which Woodin calls “Solovay’s Lemma” although it is a bit different from the related lemma in [4]) can be adapted to strongly compact cardinals. Usuba [3] made the same observation independently and earlier.

**Lemma 2.3.** Suppose $\kappa$ is strongly compact. Then one of the following holds:

1. HOD has the $\kappa$-cover property.
2. All sufficiently large regular cardinals are $\omega$-strongly measurable in in HOD.

**Proof.** Assume that there are arbitrarily large regular cardinals $\delta$ that are not $\omega$-strongly measurable in HOD, or in other words, HOD splits $S^\delta_\omega$. Applying Proposition 2.2 to sufficiently large such $\delta$, for all $\gamma \geq \kappa$, there is a $\kappa$-complete fine ultrafilter $U$ on $P_\kappa(\gamma)$ such that $P_\kappa(\gamma) \cap \text{HOD} \in U$. For each $\sigma \in P_\kappa(\gamma)$, $\{\tau \in P_\kappa(\gamma) : \sigma \subseteq \tau\} \in U$. Since $U$ is a filter, it follows that $\{\tau \in P_\kappa(\gamma) : \sigma \subseteq \tau\} \in U$, and in particular this set is nonempty. This yields $\tau \in P_\kappa(\gamma) \cap \text{HOD}$ covering $\sigma$, as desired. \qed
We now extend the cover property of HOD to all strong limit cardinals greater than or equal to the first strongly compact cardinal.

Proof of Theorem 1.2. Suppose (2) fails, so by Lemma 2.3, HOD has the $\kappa$-cover property. Given this, it suffices to show that for all $\delta \geq \kappa$, for some $\delta' < \beth_\omega(\delta)$, every set of ordinals of cardinality at most $\delta$ is covered by a set of ordinals in HOD of cardinality at most $\delta'$.

Let $\mathcal{U}$ be a fine $\kappa$-complete ultrafilter on $P_\kappa(\delta)$. Fix $A \subseteq \kappa$ such that $V_\kappa \subseteq \text{HOD}_A$.

We will show that for every set $S$ of ordinals of cardinality at most $\delta$, there is a set $T$ of cardinality at most $2^\delta$ such that $S \subseteq T$. By Vopenka’s theorem, $\text{HOD}_{A,\mathcal{U}}$ is a forcing extension of HOD by a forcing in HOD of cardinality less than $\beth_\omega(\delta)$, so the theorem follows.

Note that $\text{HOD}_{A,\mathcal{U}}$ is closed under $<\kappa$-sequences since $\text{HOD}_{A,\mathcal{U}}$ has the $\kappa$-cover property and $V_\kappa \subseteq \text{HOD}_{A,\mathcal{U}}$. As a consequence, $M_\mathcal{U}$ satisfies that $j_\mathcal{U}(\text{HOD}_{A,\mathcal{U}})$ is closed under $<j_\mathcal{U}(\kappa)$-sequences. Also $j_\mathcal{U}(\text{HOD}_{A,\mathcal{U}}) \subseteq \text{HOD}_{A,\mathcal{U}}$: there is a wellorder of $j_\mathcal{U}(\text{HOD}_{A,\mathcal{U}})$ definable from $A$ and $\mathcal{U}$.

Finally, suppose $S$ is a set of ordinals of cardinality at most $\delta$. Then since $\mathcal{U}$ is fine, there is some $T$ in $M_\mathcal{U}$ such that $S \subseteq T$ and $|T|^{M_\mathcal{U}} < j_\mathcal{U}(\delta)$. Therefore $T \in j_\mathcal{U}(\text{HOD}_{A,\mathcal{U}}) \subseteq \text{HOD}_{A,\mathcal{U}}$. Moreover $|T| \leq |j_\mathcal{U}(\delta)| \leq 2^\delta$. This completes the proof. 

It may be unclear where we used that $\kappa \neq \omega$ in the proof above. In fact, we used the countable completeness of $\mathcal{U}$ to establish that there is a wellorder of $j_\mathcal{U}(\text{HOD}_{A,\mathcal{U}})$ definable from $A$ and $\mathcal{U}$. (If $\mathcal{U}$ is countably incomplete, the canonical wellorder of $j_\mathcal{U}(\text{HOD}_{A,\mathcal{U}})$ as computed in $M_\mathcal{U}$ is not a wellorder at all.)

Using these theorems, we prove the dichotomy involving successors of singulares.

Proof of Theorem 1.1. By Lemma 2.3 (and Theorem 2.1), we can assume that HOD has the $\kappa$-cover property. Fix a singular strong limit cardinal $\lambda > \kappa$. Theorem 1.2 easily implies that $\lambda$ is singular in HOD. Let $\gamma = \lambda^+_{\text{HOD}}$. Since $\gamma$ is regular in HOD, Theorem 1.2 implies $\text{cf}(\gamma) \geq \lambda$. Since $\lambda$ is singular, $\text{cf}(\gamma) > \lambda$, and so $\gamma = \lambda^+$.

References

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