Local rigid cohomology of singular points

David Ouwehand

Humboldt-Universität zu Berlin

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Abstract

We introduce a new equivalence relation for isolated singularities on a scheme over a field of positive characteristic. Our main result is that equivalent singularities have isomorphic local rigid cohomology. As an application we illustrate how this result can be used for the computation of zeta functions of projective hypersurfaces that have weighted homogeneous singularities.

1 Introduction

Let $k$ be a perfect field of characteristic $p > 0$ and take a $k$-scheme $X$ together with a closed point $x \in X$. Now consider the spaces $H^i_{\{x\}, \text{rig}}(X)$, the rigid cohomology of $X$ with support in the closed set $\{x\}$. We call this the local rigid cohomology of $X$ at $x$. Note that this local cohomology will only be interesting if $X$ is singular at $x$. Our goal is to introduce a new notion of equivalence for isolated singularities on $k$-schemes. This equivalence relation has the property that two equivalent singularities $x' \in X'$ and $x \in X$ have isomorphic local rigid cohomology. We briefly explain why this property is important when one wants to compute the zeta function of a hypersurface that has isolated weighted homogeneous singularities.

Assume that $k = \mathbb{F}_q$ is a finite field and that $X$ is an irreducible hypersurface in some projective space $\mathbb{P}_k^n$. The problem of computing the zeta function of $X$ is equivalent to the computation of the zeta function of the complement $U = \mathbb{P}_k^n \setminus X$. In the case where $X$ is smooth, Abbott, Kedlaya and Roe [AKR10] have solved this problem using rigid cohomology. In this case, it is well-known that $H^i_{\text{rig}}(U) = 0$ for $0 < i < n$, that $H^0_{\text{rig}}(U)$ is one-dimensional and that the space $H^n_{\text{rig}}(U)$ has an easily computable basis. The key result of [AKR10] is that in this setting, there exists an efficient algorithm for computing the action of Frobenius on $H^n_{\text{rig}}(U)$. This in turn gives a formula for the zeta function of $U$.

If the hypersurface $X$ has isolated singularities then the problem of computing the zeta function of its complement becomes more complicated. Firstly,
the cohomology space $H_{\text{rig}}^{n-1}(U)$ may be nonzero. Secondly, one still expects $H_{\text{rig}}^i(U)$ to be zero for $0 < i < n - 1$, but this has not been proved in general. The cohomology space $H_{\text{rig}}^n(U)$ also becomes more difficult to compute. In [Klo08], Kloosterman gives a method to compute the action of Frobenius on $H_{\text{rig}}^n(U)$. This is essentially a modified version of the method of [AKR 10].

Let us now take a closer look at the two problems that remain unsolved, namely the computation of $H_{\text{rig}}^{n-1}(U)$ and the claim that $H_{\text{rig}}^i(U) = 0$ for $0 < i < n - 1$. In [Dim90], Dimca gives a method to solve a topological analogue of the first question. More specifically, he shows that if $X \subset \mathbb{P}_\mathbb{C}^n$ is an irreducible hypersurface that has only isolated weighted homogeneous singularities then the Betti cohomology $H^{n-1}(U, \mathbb{C})$ of its complement may be identified with the cokernel of a certain map

$$H^n(U, \mathbb{C}) \longrightarrow H^n_{\Sigma}(X, \mathbb{C})$$

where $\Sigma$ is the singular locus of $X$. Moreover, there is an efficient method to compute a basis for the local cohomology at a singular point of $X$. This basis makes it easy to explicitly compute the map (1.1). We expect that this result can be translated to rigid cohomology. Our strategy for solving the second question is to show that it is equivalent to some properties of the local rigid cohomology at the singular points of $X$. If $X$ only has isolated weighted homogeneous singularities then this local cohomology is computable so that one can check if these properties hold.

The aim of this paper is to introduce a new equivalence relation for isolated singularities. We will call this \textit{étale equivalence}, see definition 2.1 below. Our main result is that \textit{étale equivalent} singularities have isomorphic local rigid cohomology. See theorems 2.3 and 2.6 below. We will prove these theorems in sections 3 and 4. We may use \textit{étale equivalence} to define the notion of an isolated weighted homogeneous singularity on a hypersurface over $k$.

Definition 1.1. Let $X$ be a hypersurface over $k$ and consider an isolated singularity $x \in X$. We say that $x$ is a \textit{weighted homogeneous singularity} if it is \textit{étale equivalent} to the origin of the affine zero locus $Z(g)$ of a weighted homogeneous polynomial $g$. Such a polynomial is called a \textit{normal form} of the singularity.

The motivation for using \textit{étale equivalence} in the definition above is that theorem 2.3 greatly simplifies the computation of the local rigid cohomology of an isolated weighted homogeneous hypersurface singularity: we only need to compute the local cohomology of a normal form. This in turn simplifies the two problems that we described above. In section 5 we illustrate these ideas with an example.
1.1 Conventions and notations

Throughout this paper, $K$ denotes a field of characteristic zero, complete w.r.t. a discrete valuation, with valuation ring $\mathcal{V}$ and perfect residue field $k$ of characteristic $p > 0$.

Any scheme $X$ will be assumed to be reduced, of finite type over $k$ and separated over $k$. Every morphism of schemes will be assumed to be a $k$-morphism. Every closed subset $Z \subset X$ will be equipped with the reduced subscheme structure. In our setting, a singular point $x \in X$ is called isolated if the singular locus $X_{\text{sing}}$ is zero-dimensional at $x$.

Every formal scheme $P$ will be assumed to be separated and topologically of finite type over $\mathcal{V}$. For such a $P$ we may consider the generic fiber $P_K$, which is a quasi-compact separated rigid analytic space over $K$.

Recall that a frame is a series of immersions $(X \subset Y \subset P)$ where $X$ and $Y$ are schemes over $k$ and $P$ is a formal scheme over $\mathcal{V}$. The immersion $X \subset Y$ is required to be open and $Y \subset P$ is assumed to be a closed immersion of $Y$ into the closed fiber of $P$. We let $S$ denote the frame $(\text{Spec } k \subset \text{Spec } k \subset \text{Spf } \mathcal{V})$. In this way, any frame $(X \subset Y \subset P)$ is a frame over $S$. Every morphism of frames will be assumed to be an $S$-morphism. A scheme $X$ is called realizable if there exists a frame $(X \subset Y \subset P)$ with $Y$ proper and $P$ smooth in a neighbourhood of $X$. All quasi-projective schemes are obviously realizable. It is easy to show that for any morphism of realizable schemes $f: X' \rightarrow X$ there exists a morphism of frames

$$
\begin{array}{ccc}
X' & \longrightarrow & Y' \longrightarrow & P' \\
\downarrow f & & \downarrow & \downarrow u \\
X & \longrightarrow & Y \longrightarrow & P
\end{array}
$$

such that $(X' \subset Y' \subset P')$ resp. $(X \subset Y \subset P)$ is a realization of $X'$ resp. of $X$ and such that $u$ is smooth in a neighbourhood of $X'$. Such a morphism of frames is called a realization of $f$. From now on we only consider realizable schemes. For the rest we will use the standard notation and terminology from rigid cohomology. Our main reference for this is \cite{LS07}.

2 Equivalence of singularities

Recall that in the setting of analytic geometry over $\mathbb{C}$, two points on analytic spaces $x' \in X'$ and $x \in X$ are called (contact) equivalent is there exist open neighbourhoods $U_{x'}$ and $U_x$ of $x'$ resp. of $x$ and a diffeomorphism $\varphi: U_{x'} \rightarrow U_x$ such that $\varphi(x') = x$. This is equivalent to saying that there exists an isomorphism $\mathcal{O}_{X,x} \sim \mathcal{O}_{X',x'}$ on local rings.
In the setting of algebraic geometry, one could try to use the same approach by considering an isomorphism $U_{x'} \to U_x : x' \mapsto x$ between Zariski-open neighbourhoods of $x'$ and $x$. The drawback of this is that the Zariski topology is too coarse, and the resulting notion of equivalence (birational equivalence) is also too coarse. There exist several equivalence relations for points on schemes that are much more meaningful. In [GK90] for instance, two points $x' \in X'$ and $x \in X$ are called (contact) equivalent if they have isomorphic infinitesimal neighbourhoods or equivalently, if there exists an isomorphism of completed local rings $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X',x'}$. The drawback of this equivalence relation is that equivalent closed points do not necessarily have isomorphic local rigid cohomology. In order to solve this problem, we propose the following definition for the equivalence of points on $k$-schemes.

**Definition 2.1.** Two points $x' \in X'$ and $x \in X$ are called (étale) equivalent if there exists another scheme $X''$ together with a point $x'' \in X''$ and two morphisms $f' : X'' \to X'$ and $f : X'' \to X$ such that:

i) $f'(x'') = x'$ and $f'$ induces an isomorphism $k(x') \cong k(x'')$ on the residue fields.

ii) $f(x'') = x$ and $f$ induces an isomorphism $k(x) \cong k(x'')$ on the residue fields.

iii) $f'$ and $f$ are étale at $x''$.

We denote this by $(X', x') \sim_{\text{ét}} (X, x)$.

The intuition behind this definition is similar to the intuition behind the definition of [GK90]. Instead of working in the infinitesimal topology, we prefer to work in the étale topology. Indeed, definition 2.1 is equivalent to requiring that the points $x' \in X'$ and $x \in X$ have isomorphic étale neighbourhoods. This observation allows us to prove some basic facts about our equivalence relation.

**Proposition 2.2.**

i) Two points $x' \in X'$ and $x \in X$ are equivalent if and only if there exists an isomorphism $\mathcal{O}_{X,x}^h \cong \mathcal{O}_{X',x'}^h$ on the Henselizations of the local rings.

ii) The relation $\sim_{\text{ét}}$ from definition 2.1 is indeed an equivalence relation.

iii) Two nonsingular points $x' \in X'$ and $x \in X$ are equivalent if and only if their residue fields are isomorphic and $\dim_{x'} X' = \dim_x X$.

iv) If two points $x' \in X'$ and $x \in X$ are equivalent according to definition 2.1 then they are also equivalent according to the definition of [GK90]. That is, there exists an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X',x'}$ on completed local rings.
Proof. The last three claims follow directly from the first claim. Now recall that the étale local ring of a point $x \in X$ is given by

$$O^h_{X,x} = \lim_{\rightarrow} O_{U,u}$$

where the limit runs over all the étale neighbourhoods of $x$. In our context, an étale neighbourhood of $x \in X$ is an étale map $U \to X: u \mapsto x$ such that the induced morphism $k(x) \to k(u)$ on residue fields is an isomorphism. It is well-known that $O^h_{X,x}$ is the Henselization of $O_{X,x}$. But definition 2.1 states that $x' \in X'$ and $x \in X$ are equivalent if and only if they have isomorphic étale neighbourhoods. This is equivalent to saying that their étale local rings are isomorphic.

The purpose of introducing our new notion of equivalence of points is that equivalent closed points have isomorphic local rigid cohomology.

**Theorem 2.3.** Let $(X', x')$ and $(X, x)$ be two schemes with marked closed points such that $(X', x') \sim_{\text{ét}} (X, x)$. Then for all $i$, there exists a Frobenius-equivariant isomorphism

$$H^i_{\text{rig},\{x\}}(X) \sim H^i_{\text{rig},\{x'\}}(X')$$

on the local rigid cohomology with constant coefficients.

The proof of theorem 2.3 relies on a more general result. For this we will need one more definition.

**Definition 2.4.** Consider two points on $k$-schemes $x' \in X'$ and $x \in X$. We say that $x'$ dominates $x$ if there exists an open neighbourhood $U_{x'}$ of $x'$ and a morphism $f: U_{x'} \to X$ such that:

i) $f(x') = x$ and $f$ induces an isomorphism $k(x) \sim k(x')$ on the residue fields.

ii) $f$ is étale at $x'$.

iii) $f^{-1}(x) = \{x'\}$.

We denote this by $(X', x') \succ (X, x)$.

There is a clear connection between definitions 2.4 and 2.1.

**Proposition 2.5.** Assume that $x' \in X'$ and $x \in X$ are closed points. Then we have that $(X', x') \sim_{\text{ét}} (X, x)$ if and only if there exists a pair $(X'', x'')$ such that $(X'', x'') \succ (X', x')$ and $(X'', x'') \succ (X, x)$.

Proof. First assume that there exists a pair $(X'', x'')$ such that $(X'', x'') \succ (X', x')$ and $(X'', x'') \succ (X, x)$. Then there exist open neighbourhoods $U'_{x'}$ and
of \(x''\) in \(X''\) together with maps \(f': U'' \to X'\) and \(f: U'' \to X\) satisfying the conditions of definition \(2.3\). Then the maps \(f'\) and \(f\) restricted to \(U'' \cap U''\) satisfy the conditions of definition \(2.1\). Conversely, assume that \((X', x') \sim_{\text{ét}} (X, x)\) via some maps \(f': X'' \to X': x'' \to x'\) and \(f: X'' \to X: x'' \to x\). After replacing \(X''\) by an open neighbourhood of \(x''\) we may assume that \(f'\) and \(f\) are étale. It follows that the fibers \((f')^{-1}(x')\) and \(f^{-1}(x)\) consist of a finite union of closed points. We may therefore shrink \(X''\) and assume that \((f')^{-1}(x') = \{x''\}\) and \(f^{-1}(x) = \{x''\}\). Then we have \((X'', x'') \gg (X', x')\) and \((X'', x'') \gg (X, x)\). □

We are now ready to formulate our most important result. It states that if a point \(x' \in X'\) dominates a closed point \(x \in X\) via a morphism \(f: U_{x'} \to X\) étale at \(x'\) then the pullback along \(f\) of an overconvergent \(F\)-isocrystal on \(X\) preserves the local rigid cohomology.

**Theorem 2.6.** Let \((X', x')\) and \((X, x)\) be two schemes with marked closed points such that \((X', x') \gg (X, x)\) via \(f: U_{x'} \to X: x' \to x\). Let \(F \in F-\text{isoc}^+(X/S)\) be a finitely presented overconvergent \(F\)-isocrystal on \(X\). Then for all \(i\), there exists a Frobenius-equivariant isomorphism

\[
H^i_{\text{rig}, \{x\}}(X, F) \sim \rightarrow H^i_{\text{rig}, \{x'\}}(X', f^*F).
\]

Moreover, this isomorphism is functorial in \(F\).

**Remark 2.7.**

i) In the statement of the theorem we have implicitly chosen an extension of \(f^*F\) to all of \(X'\). The choice of the extension is not important, since by [LS07, Proposition 8.2.8] the local rigid cohomology only depends on an open neighbourhood of the support.

ii) In the statement of theorem \(2.6\) it is important that \(x'\) and \(x\) are closed points, otherwise the cohomology with support does not make sense. This is not a problem for our applications. Indeed, the singular locus of a scheme is closed under specialization [Liu02, Lemma 2.4.11.(b)]. So for quasi-compact schemes, every isolated singularity is a closed point.

iii) Recall that in the definition of the rigid cohomology with constant coefficients of a scheme \(X\) one starts by choosing a realization \((X \subset Y \subset P)\). The rigid cohomology \(H^i_{\text{rig}}(X)\) is then defined as the hypercohomology \(\mathbb{H}^i(\{Y_P, j_P^X Omega_{Y/P}^\bullet\})\). In order to show that this definition is independent of the choice of the realization, one can prove that every diagram
with $g$ proper and $u$ smooth in a neighbourhood of $X$ induces an isomorphism on the cohomology. This is done in [LS07, Proposition 6.5.3]. More specifically, it is the base change map that is described in [LS07, Proposition 6.2.6] that induces the isomorphism. Our approach for theorem 2.6 is to prove a local version of this result in the case where $f$ is an étale morphism rather than the identity map on $X$. See theorem 3.3 for more details.

3 Proof of the main theorem

This section contains the proof of our main theorem 2.6. First we recall the definition of the canonical map on sheaves with supports. We show that this map is an isomorphism under certain conditions. In paragraph 3.2 we briefly recall the definition of the base change map of rigid cohomology. After this we reformulate our main theorem 2.6 in terms of base change maps. In the last two paragraphs of this section we then finish the proof of theorem 2.6.

3.1 The canonical map on sheaves with supports

Consider a morphism of frames

$$
\begin{array}{ccc}
X' & \rightarrow & Y' \\
\downarrow{f = \text{Id}_X} & & \downarrow{g} \\
X & \rightarrow & Y
\end{array}
$$

with $u$ smooth and $g$ proper in a neighbourhood of $X$ induces an isomorphism on the cohomology. This is done in [LS07, Proposition 6.5.3]. More specifically, it is the base change map that is described in [LS07, Proposition 6.2.6] that induces the isomorphism. Our approach for theorem 2.6 is to prove a local version of this result in the case where $f$ is an étale morphism rather than the identity map on $X$. See theorem 3.3 for more details.
By the universal property of the kernel we now obtain a canonical morphism
\[ u_K \Gamma^\dagger C j^\dagger X E \longrightarrow \Gamma^\dagger C' j^\dagger X' E. \] (3.1)

In the case where \( E \) is a \( j_X^\dagger \mathcal{O}_{Y|\mu} \)-module, composing with \( j_X^\dagger \) also gives a canonical map
\[ u^\dagger \Gamma^\dagger C E \longrightarrow \Gamma^\dagger C' u^\dagger E. \] (3.2)

The first step towards proving the main theorem 2.6 is to show that the canonical map on sheaves with supports is an isomorphism if the morphism of frames is flat and if the supports are Cartesian.

**Proposition 3.1.** Let

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \longrightarrow P' \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y \longrightarrow P
\end{array}
\]

be a flat morphism of frames. Let \( E \) be a \( j_X^\dagger \mathcal{O}_{Y|\mu} \)-module. Also choose two closed subschemes \( C' \subset X' \) and \( C \subset X \) such that \( C' = C \times_X X' \). Then the canonical map (3.2) is an isomorphism.

**Proof.** We may factor our morphism of frames as

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \longrightarrow P' \\
\downarrow & & \downarrow \\
X'' & \longrightarrow & Y' \longrightarrow P' \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y \longrightarrow P
\end{array}
\]

where \( X'' = X \times_Y Y' \) and where \( X' \to X'' \) is an open immersion. Define \( C'' = C \times_X X'' \). Then the canonical map (3.1) factors as

\[ u_K \Gamma^\dagger C E \longrightarrow \Gamma^\dagger C'' j_{X''}^\dagger u_K^* E \longrightarrow \Gamma^\dagger C' j_{X'}^\dagger u_K^* E \]

where the second map is the canonical morphism for the upper part of the diagram applied to the \( j_{X''}^\dagger \mathcal{O}_{Y'|\mu} \)-module \( j_{X''}^\dagger u_K^* E \). The map (3.2) is obtained...
by applying the functor $j^1_X$ to this composition. Let us first show that the canonical map

$$u_K^*\Gamma^1_{\mathcal{L}^1}E \rightarrow \Gamma^1_{\mathcal{L}^0}j^1_X u^*_K E$$  \hspace{1cm} (3.3)$$

is an isomorphism. Since $X'' = X \times_Y Y'$ we know by [LS07, Corollary 5.3.9] that the canonical map $u_K^* E \rightarrow j^1_X u^*_K E$ is an isomorphism. Now write $U'' = X'' \cap C''$ and $U = X \cap C$. The condition $C'' = C \times_Y X''$ implies that $U'' = U \times_Y X''$. But since also $X'' = X \times_Y Y'$ we have $U'' = U \times_Y Y'$. This means that the canonical map $u^*_K j^1_{U''}E \rightarrow j^1_{U''}u^*_K E$ is also an isomorphism. By [LS07, Corollary 3.3.6] there exists a strict neighbourhood $V$ of $X''$ in $Y'$. By [LS07, Proposition 5.1.13], the pushforward along the inclusion $V \rightarrow Y'$ gives an equivalence of categories between the $j^1_{X''} \mathcal{O}_{U''}$-modules and the $j^1_{U''} \mathcal{O}_{Y''}$-modules. Therefore we may assume $u_K$ to be flat.

This means that the functor $u^*_K$ is exact and we obtain the following diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & u^*_K \Gamma^1_{\mathcal{L}^1}E & \rightarrow & u^*_K E & \rightarrow & u^*_K j^1_X E & \rightarrow & 0 \\
& & \parallel & \parallel & \equiv & \equiv & \\
0 & \rightarrow & \Gamma^1_{\mathcal{L}^0}j^1_X u^*_K E & \rightarrow & j^1_{X''} u^*_K E & \rightarrow & j^1_{U''} u^*_K E & \rightarrow & 0 \\
\end{array}
$$

The induced morphism on the kernels, which is equal to (3.3), is indeed an isomorphism. It remains to show that the map

$$j^1_X \Gamma^3_{\mathcal{L}^3}j^1_X u^*_K E \rightarrow \Gamma^3_{\mathcal{L}^0}j^1_X u^*_K E$$  \hspace{1cm} (3.4)$$

is an isomorphism. For this we consider the following commutative diagram with exact rows in which $U' = X' \setminus C'$:

$$
\begin{array}{cccccc}
0 & \rightarrow & \Gamma^1_{\mathcal{L}^0}j^1_{X''} u^*_K E & \rightarrow & j^1_{X''} u^*_K E & \rightarrow & j^1_{U''} u^*_K E & \rightarrow & 0 \\
\parallel & & \parallel & & \equiv & & \equiv & & \\
0 & \rightarrow & \Gamma^1_{\mathcal{L}^0}j^1_X u^*_K E & \rightarrow & j^1_{X'} u^*_K E & \rightarrow & j^1_{U'} u^*_K E & \rightarrow & 0 \\
\end{array}
$$

By definition, the canonical map

$$\Gamma^1_{\mathcal{L}^0}j^1_{X''} u^*_K E \rightarrow \Gamma^1_{\mathcal{L}^0}j^1_{X'} u^*_K E$$

is the induced map on the kernels. Recall that the functor $j^1_X$ is exact, hence preserving kernels. So the canonical map (3.4) may be computed by applying $j^1_X$ to the diagram above and then taking the induced morphism on the kernels. In order to prove our last claim, it is therefore sufficient to show that the canonical map

$$j^1_X j^1_{U''} u^*_K E \rightarrow j^1_{U'} u^*_K E$$
is an isomorphism. Since \( C' = C \times_X X' \) we have that \( C' = C'' \times_{X''} X' \). This means that \( U' = U'' \times_{X''} X' \) or in other words, \( U' = U'' \cap X' \). This implies that \( j_{X'}^1 j_{U''}^1 = j_{U'}^1 \).

### 3.2 Base change maps

We recall some facts about base change maps in the context of rigid cohomology. Consider a commutative diagram of rigid analytic spaces

\[
\begin{array}{ccc}
V' & \xrightarrow{\alpha'} & W' \\
\downarrow{\beta'} & & \downarrow{\beta} \\
V & \xrightarrow{\alpha} & W
\end{array}
\]

and let \( E \) be an \( \mathcal{O}_V \)-module. Then there is a canonical base change map

\[
\beta^* \alpha_* E \rightarrow \alpha'_*(\beta')^* E.
\]

By definition, this map is obtained by adjunction from the canonical morphism

\[
\alpha_* E \rightarrow \beta_* \alpha'_*(\beta')^* E = \alpha_* \beta'_*(\beta')^* E
\]

that is obtained by applying the functor \( \alpha_* \) to the adjunction unit \( E \rightarrow \beta'_*(\beta')^* E \). There is also a base change map

\[
\beta^* (\mathbb{R}\alpha_*) E \rightarrow (\mathbb{R}\alpha'_*)(\beta')^* E
\]

in the derived category \( D^+(\mathcal{O}_V \text{-Mod}) \). We refer to paragraphs XII.4 and XVII.2 of \([SGA4]\) for more details. This construction is the starting point for the definition of the base change map of rigid cohomology. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow{u} & & \downarrow{w} \\
X & \xrightarrow{u} & Y \xrightarrow{u} P
\end{array}
\]

be a morphism of frames. Also choose two closed subschemes \( C' \subset X' \) and \( C \subset X \) such that \( f^{-1}(C) \subset C' \). Let \( E \) be a \( j_X^1 \mathcal{O}_{Y|P} \)-module with an integrable connection over \( K \). By applying the base change map of the diagram

\[
\begin{array}{ccc}
]Y'[P' & \xrightarrow{u_K} & ]Y'[P \\
\downarrow{u_K} & & \downarrow{\text{Id}} \\
]Y[P & \xrightarrow{\text{Id}} & ]Y[P
\end{array}
\]

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to the de Rham complex $\Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K}$ we obtain a morphism

$$(u^*)_1: \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \to (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right).$$

Then the canonical map $\text{Id} \to j_X^\dagger$ gives us another morphism

$$(u^*)_2: (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right) \to (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right).$$

After this, the canonical morphism $u_K^*: \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \to u_K^* \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K}$ gives us another map

$$(u^*)_3: (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right) \to (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right).$$

Finally, we may use the canonical map $u_K^* \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \to \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K}$ to obtain a morphism

$$(u^*)_4: (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right) \to (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right).$$

**Definition 3.2.** The canonical map

$$u^*: \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \to (\mathbb{R}u_K^*) u_K^* \left( \Gamma^\dagger C \otimes \Omega^\bullet_{Y[p]/K} \right)$$

that is given by the composition

$$u^* = (u^*)_4 \circ (u^*)_3 \circ (u^*)_2 \circ (u^*)_1$$

is called the base change map of rigid cohomology.

See [LS07, Proposition 6.2.6] for a more general construction of the base change map of rigid cohomology.

### 3.3 Part I of the proof: Reformulation

As we mentioned before, the key to proving our main theorem 2.6 is to generalize [LS07, Proposition 6.5.3]. The big difference is that in our generalized setting we can only obtain a local result. More specifically, let $x' \in X'$ and $x \in X$ be closed points such that $(X', x') \succ (X, x)$ via some map $f: U_{x'} \to X$. Since the local cohomology at $x'$ only depends on an open neighbourhood of $x'$, we may assume that $U_{x'} = X'$ and that $f$ is étale. Our main result for
this section is that for the realization of such an \( f \), the base change map with \( C' = \{ x' \} \) and \( C = \{ x \} \) is an isomorphism.

**Theorem 3.3.** Let

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow \phi & \quad & \downarrow \psi \\
X & \longrightarrow & Y \\
\end{array}
\]

be a proper smooth morphism of smooth \( S \)-frames. Also assume that \( f \) is \( \acute{e} \)tale. Let \( E \) be a coherent \( j^! _X \mathcal{O}_{Y[p]} \)-module with an integrable connection over \( K \). Choose two closed points \( x' \in X' \) and \( x \in X \) such that \( f^{-1}(x) = \{ x' \} \) and such that \( f \) induces an isomorphism \( k(x) \overset{\sim}{\longrightarrow} k(x') \) on the residue fields. Then the base change map

\[
u^*: \Gamma^! \{ x \} E \otimes \mathcal{O}_{Y[p]} \mathcal{O}^*_{Y[p]/K} \longrightarrow (\mathbb{R}p_{K*}) \Gamma^! \{ x' \} u^! E \otimes \mathcal{O}_{Y'[p']} \mathcal{O}^*_{Y''[p']/K}
\]

is an isomorphism in the derived category \( D^+ (\mathcal{O}_{Y[p]} \text{-Mod}) \).

The proof of theorem 3.3 will be covered in the next two paragraphs. In the remainder this paragraph we show that our main theorem 2.6 is a simple consequence of the theorem above.

**Proof of Theorem 2.6.** First note that by [LS07, Proposition 8.2.8] the local cohomology of \( X' \) at \( x' \) only depends on an open neighbourhood of \( x' \). We may therefore assume that \( U_{x'} = X' \) and that \( f \) is \( \acute{e} \)tale. Now choose a realization of \( f \):

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow \phi & \quad & \downarrow \psi \\
X & \longrightarrow & Y \\
\end{array}
\]

Then let \( E \) be the realization of \( F \) on the frame \( (X \subset Y \subset P) \). Note that by definition, \( u^! E \) is the realization of \( f^* F \) on \( (X' \subset Y' \subset P') \). Let \( p: (X \subset Y \subset P) \rightarrow S \) denote the structural morphism of the \( S \)-frame \( (X \subset Y \subset P) \). Then we construct an isomorphism

\[
H^\bullet_{\text{rig}, \{ x \}} (X, F) \overset{\sim}{\longrightarrow} H^\bullet_{\text{rig}, \{ x' \}} (X', f^* F)
\]

by applying the functor \( \mathbb{R}p_{K*} \) to the isomorphism of theorem 3.3. The fact that this isomorphism is functorial in \( F \) is an easy consequence of the fact that base change maps behave in a functorial way. Our construction is moreover Frobenius-equivariant. These last two facts even remain true if one carries out the construction for a general \( f \) that is not necessarily \( \acute{e} \)tale. \( \square \)
Let us briefly recall why the map $H^\bullet_{rig,\{x\}}(X, F) \to H^\bullet_{rig,\{x'\}}(X', f^*F)$ is Frobenius-equivariant for any morphism $f: X' \to X$. We will denote the structural morphism of $X'$ (resp. of $X$) by $p'$ (resp. by $p$). Let $F_{X'}$ (resp. $F_X$) denote the absolute Frobenius on $X'$ (resp. on $X$). Also fix an isometric lifting $\sigma: K \to K$ of the Frobenius on $k$. Let $F^\bullet_{X'}$ (resp. $F^\bullet_X$) denote the pullback of an isocrystal on $X'$ (resp. on $X$) relative to this lifting. Also let $\Phi: F^\bullet_X F \xrightarrow{\sim} F$ denote the Frobenius structure on $F^\bullet_X$. Now consider the following diagram:

The rows of this diagram describe the Frobenius actions on $R^p_{rig,\{x\}}F$ and on $R^p'_{rig,\{x'\}}f^* F$. The vertical arrows all come from the base change map of the pullback along $f$. We have to check that this diagram is commutative. The leftmost square commutes because of the fact that $f \circ F_{X'} = F_X \circ f$. The square on the right commutes by functoriality of the base change map.

3.4 Part II of the proof: The quasi-compact étale case

The aim of this paragraph is to prove theorem 3.3 in the case of an étale morphism of frames such that the induced morphism on tubes $u_K: ]Y'[p \to Y[\mathcal{P}$ is quasi-compact. For such a morphism of frames and for $E = j_X^! \mathcal{O}|_{Y[\mathcal{P}}$, it has already been shown in the appendix of [Ber97] that there exists a canonical isomorphism

$$\Gamma^!_{\{x\}}E \otimes \mathcal{O}|_{Y[\mathcal{P}} \xrightarrow{\sim} (R^pu_K)_!! \left( \Gamma^!_{\{x'\}}f^! \otimes \mathcal{O}|_{Y'|[\mathcal{P}} \right) \xrightarrow{\sim} \left( \mathcal{O}|_{Y'|[\mathcal{P}} \right)^{\bullet'}|_{[\mathcal{P}]}.$$ 

However, it is not very clear from this proof that the isomorphism is given by the base change map of rigid cohomology. In other words, it is not clear that the resulting isomorphism on the local cohomology is Frobenius-equivariant.

We will use similar techniques as in [Ber97, Proposition A.10] to derive a more precise result that moreover holds for any $j_X^! \mathcal{O}|_{Y[\mathcal{P}}$-module with an integrable connection.

**Theorem 3.4.** Let

\[ X' \xrightarrow{f} Y' \xrightarrow{u} P' \]

\[ X \xrightarrow{f} Y \xrightarrow{u} P \]

be an étale morphism of smooth $S$-frames such that the induced morphism on tubes $u_K: ]Y'[p \to Y[\mathcal{P}$ is quasi-compact. Choose two closed points $x' \in X'$
and $x \in X$ such that $f^{-1}(x) = \{x'\}$ and such that $f$ induces an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on the residue fields. Let $E$ be a $j^\dagger_X \mathcal{O}_{Y|p}$-module with an integrable connection over $K$. Then the base change map

$$u^\ast: \lim_{\longrightarrow \eta} j^\dagger \mathcal{O}_{Y|p} \mathcal{O}_{Y|p/K} \rightarrow (\mathbb{R}u_K) \lim_{\longrightarrow \eta} j^\dagger \mathcal{O}_{Y|p} \mathcal{O}_{Y'|p'/K}$$

is an isomorphism in the derived category $D^+ (\mathcal{O}_{Y|p} \text{-Mod})$.

Note that we require $u$ to be étale in a neighbourhood of $X'$, not just in a neighbourhood of $x'$ as is the case in [Ber97, Proposition A.10]. As far as we can see, this stronger condition is necessary to ensure that the base change map is an isomorphism, even in the case where $E = j^\dagger_X \mathcal{O}_{Y|p}$. Before we can give the proof of this result, we will need to introduce modified versions of the functors $j^\dagger_X$ and $\Gamma^\dagger_{C|p}$ that also appear in [Ber97].

**Definition 3.5.** Consider a frame $(X \subset Y \subset P)$ and let $C \subset Y$ be a closed subset. For $\eta < 1$, we denote by $j_\eta$ the immersion

$$j_\eta: Y[p \setminus Y \setminus X]_{p,\eta} \hookrightarrow Y[p]$$

and by $i_\eta$ the immersion

$$i_\eta: Y[p \setminus C]_{p,\eta} \hookrightarrow Y[p] .$$

If $E$ is an $\mathcal{O}_{Y|p}$-module then we define

$$j^\dagger_{X,\eta} E = j_\eta^\ast j_\eta^\ast E$$

and

$$\Gamma^\dagger_{C|p,\eta} E = \text{Ker} \left( E \rightarrow i_\eta^\ast i_\eta^\ast E \right) .$$

We state some basic properties of the functors $j^\dagger_{X,\eta}$ and $\Gamma^\dagger_{C|p,\eta}$, some of which are already shown in [Ber97].

**Proposition 3.6.** Let $(X \subset Y \subset P)$ be a frame. Take a closed subset $C \subset Y$ and write $C = \overline{C} \cap X$. Also write $U = X \setminus C$. Let $E$ be an $\mathcal{O}_{Y|p}$-module. Then the following properties hold.

i) There are canonical isomorphisms

$$\lim_{\longrightarrow \eta} j^\dagger_{X,\eta} E \xrightarrow{\sim} j^\dagger_X E \quad (3.5)$$

and

$$\lim_{\longrightarrow \eta} \Gamma^\dagger_{C|p,\eta} E \xrightarrow{\sim} \Gamma^\dagger_{C|p} E . \quad (3.6)$$
ii) Write $W = \overline{C}_P$ and let $i_W : W \hookrightarrow Y_P$ denote the inclusion map. Choose any $\eta < 1$. Then the base change map

$$\Gamma^\dagger_{C[P],\eta} E \to (\mathbb{R}i_{W*})i^!_W \Gamma^\dagger_{C[P],\eta} E$$  \hspace{1cm} (3.7)$$

that is associated to the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{i_W} & Y_P \\
\downarrow i_W & & \downarrow \text{Id} \\
Y_P & \xrightarrow{\text{Id}} & Y_P
\end{array}
$$

is an isomorphism in the derived category $D^+(\mathcal{O}_{Y_P}-\text{Mod})$.

iii) For any $\eta < 1$, we have a short exact sequence

$$0 \to \Gamma_{C[P],\eta} j^\dagger_{X,\eta} E \to j^\dagger_{X,\eta} E \to j^\dagger_{U,\eta} E \to 0$$  \hspace{1cm} (3.8)$$

In particular, $\Gamma^\dagger_{C[P],\eta} j^\dagger_{X,\eta} E$ only depends on $C$ and not on $\overline{C}$. Therefore we will write $\Gamma^\dagger_{C,\eta} j^\dagger_{X,\eta} E$ instead of $\Gamma^\dagger_{C[P],\eta} j^\dagger_{X,\eta} E$.

iv) There is a canonical isomorphism

$$\lim_{\eta} \Gamma^\dagger_{C,\eta} j^\dagger_{X,\eta} E \sim \Gamma^\dagger_{C} j^\dagger_{X} E.$$

Proof. The first two claims are proved in [Ber97 A.9.1] and [Ber97 A.9.2]. Property iii) is analogous to [LS07 Proposition 5.2.9]. The last claim is a direct consequence of i) and iii) and of the fact that filtered colimits of $\mathcal{O}_{Y_P}$-modules commute with finite limits.

Next we investigate how the functors $j^\dagger_{X,\eta}$ and $\Gamma^\dagger_{C,\eta} j^\dagger_{X,\eta}$ behave w.r.t. morphisms of frames.

Proposition 3.7. Let

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow u & & \downarrow u \\
X & \xrightarrow{u} & Y
\end{array}
$$

be a morphism of frames. Choose closed subsets $C' \subset X'$ and $C \subset X$ such that $f^{-1}(C) \subset C'$. Let $E$ be an $\mathcal{O}_{Y_P}$-module. Then the following properties hold.
i) There is a canonical map $u_K^* j_{X, \eta}^! E \to j_{X', \eta}^! u_K^* E$ that is an isomorphism when the morphism of frames is Cartesian.

ii) There is a canonical map

$$u_K^* \Gamma_{C, \eta} j_{X, \eta}^! E \to \Gamma_{C', \eta} j_{X', \eta}^! u_K^* E.$$  \hspace{1cm} (3.10)

iii) Assume that $E$ is a $j_X^! \mathcal{O}_{|Y|_p}$-module. If the morphism of frames is flat and if $C' = C \times_X X'$ then the morphism

$$j_{X', \eta}^! u_K^* \Gamma_{C, \eta} E \to \Gamma_{C', \eta} j_{X', \eta}^! u_K^* E$$  \hspace{1cm} (3.11)

that is obtained by applying $j_{X', \eta}^!$ to (3.10) is an isomorphism.

iv) For any $\eta < 1$ we have a commutative diagram

$$u_K^* \Gamma_{C, \eta} j_{X, \eta}^! E \quad \xrightarrow{u_K^* \Gamma_{C', \eta} j_{X', \eta}^!} \quad \Gamma_{C', \eta} j_{X', \eta}^! u_K^* E$$

$$u_K^* \Gamma_{C} j_{X}^! E \quad \xrightarrow{\Gamma_{C'} j_{X'}^!} \quad \Gamma_{C'} j_{X'}^! u_K^* E$$

in which the horizontal arrows are given by (3.10) and (3.1) and where the vertical arrows come from the isomorphism (3.9). Moreover, if $E$ is a $j_X^! \mathcal{O}_{|Y|_p}$-module then for any $\eta \leq \lambda < 1$ we obtain a commutative diagram

$$j_{X', \lambda}^! u_K^* \Gamma_{C, \eta} E \to \Gamma_{C', \eta} j_{X', \lambda}^! u_K^* E$$  \hspace{1cm} (3.12)

by using the isomorphism (3.9).

Proof. The canonical map from i) is just the base change map for the diagram below, applied to the sheaf $j_{X, \eta}^! E$.

$$|Y''[P'] \setminus |Y' \setminus X'|_{P', \eta} \to |Y'|_{P'}$$

$$\xrightarrow{j_{n}}$$

$$|Y'[P] \setminus |Y \setminus X|_{P, \eta} \to |Y|_{P}$$

$$\xrightarrow{u_K}$$

We may then use the short exact sequence (3.8) in order to define a canonical map on sheaves with supports, just as we did in paragraph 3.1. The proof of iii) is then analogous to the proof of proposition 3.1. The only subtle point is that we may still use proposition [LS07, Proposition 5.1.13] to reduce to the
case where \( u_K \) is flat. This is because \( E \) is assumed to be a \( j_X^! \mathcal{O}_{Y'[P']} \)-module, not just a \( j_{X', \eta}^! \mathcal{O}_{Y[P']} \)-module. For the proof of the last claim, use the fact that filtered colimits commute with finite limits and that \( u_K^* \) is a left adjoint, hence preserving colimits.

We now use the results about the functors \( j_X^! \) and \( \bigoplus_{\eta} j_{X', \eta}^! \) to prove a weak version of theorem 3.3.

**Proposition 3.8.** Consider an étale morphism of frames

\[
X' \xrightarrow{f} Y' \xrightarrow{u} P', \quad X \xrightarrow{\ } Y \xrightarrow{u} P
\]

such that the induced morphism on tubes \( u_K : [Y'[P'] \to Y[P] \) is quasi-compact. Choose two closed points \( x' \in X' \) and \( x \in X \) such that \( f^{-1}(x) = \{x'\} \) and such that \( f \) induces an isomorphism \( k(x) \to k(x') \) on the residue fields. Let \( E \) be a \( j_X^! \mathcal{O}_{Y[P]} \)-module. Then the canonical map

\[
\bigoplus_{\{x\}} E \xrightarrow{\sim} (\mathcal{R}u_K)_* u_K^! \bigoplus_{\{x\}} E \xrightarrow{\sim} (\mathcal{R}u_K)_* u_{\{x\}}^! \bigoplus_{\{x\}} E \quad (3.13)
\]

that is defined in a similar way as the composition \( (u_*)_2 \circ (u_*)_1 \) from paragraph 2.3 is an isomorphism.

**Proof.** Define \( W' = [\{x'\}]_{P'} \) and \( W = [\{x\}]_P \). Then we have a commutative diagram

\[
\begin{array}{ccc}
W' & \xrightarrow{i_{W'}} & Y'[P'] \\
| & v & | u_K \\
W & \xrightarrow{i_W} & Y[P]
\end{array}
\]

where \( v \) is the restriction of \( u_K \) and \( i_{W'}, i_W \) are open immersions. The fact that \( f \) induces an isomorphism \( k(x) \to k(x') \) on residue fields means that the restriction \( f : \text{Spec} k(x') \to \text{Spec} k(x) \) is an isomorphism. Since \( u_K \) is étale in a neighbourhood of \( x' \) it then follows from [LS07, Proposition 2.3.15] that \( v \) is an isomorphism. Now observe that we have \( \bigoplus_{\{x\}, \eta} E = \bigoplus_{\{x\}, \eta} E \) for any \( \eta < 1 \) because \( \{x\} \) is closed in \( X \) and in \( Y \). The isomorphism (3.7) then tells us that

\[
\bigoplus_{\{x\}, \eta} E \xrightarrow{\sim} (\mathcal{R}i_{W'})^* i_{W}^! \bigoplus_{\{x\}, \eta} E. \quad (3.14)
\]
A similar remark applies to $\Gamma_{\{x\}}^! \eta, \eta \gamma_{X, \eta, u_K^* E}$. We now apply [SGA4, Proposition XII.4.4] to the diagram

\[
\begin{array}{ccc}
W' & \xrightarrow{\iota_{W'}} & Y' \xrightarrow{[P', u_K]} Y' \xrightarrow{[\text{Id}]} Y' \\
v & & \downarrow \text{Id} \\
W & \xrightarrow{i_W} & Y' \xrightarrow{u_K} Y' \xrightarrow{[\text{Id}]} Y' \\
\end{array}
\]

(3.15)

and to the sheaf $i_W^* \Gamma_{\{x\}}^! \eta, \eta E$. By making use of the isomorphism (3.14) we obtain a commutative diagram

\[
\begin{array}{ccc}
\Gamma_{\{x\}}^! \eta, \eta E & \xrightarrow{a_1} & (\mathbb{R}u_{K*}) u_K^* \Gamma_{\{x\}}^! \eta, \eta E \\
& & \downarrow \text{Id} \\
\mathbb{R}(u_{K*} i_{W*}) v^* i_W^* \Gamma_{\{x\}}^! \eta, \eta E & \xrightarrow{a_2} & \mathbb{R}(u_{K*} i_{W*}) ^* \Gamma_{\{x\}}^! \eta, \eta E \\
\end{array}
\]

The horizontal arrow $a_1$ is the base change map coming from the rightmost square of (3.15), applied to the sheaf $(\mathbb{R}i_{W*}) i_W^* \Gamma_{\{x\}}^! \eta, \eta E$. The map $a_2$ is obtained by applying $\mathbb{R}u_{K*}$ to the base change map coming from the leftmost square of (3.15). By using [SGA4, Proposition XII.4.4] again we see that $a_2$ is equal to the morphism that one gets after applying $\mathbb{R}u_{K*}$ to the canonical morphism

\[
u_K^* \Gamma_{\{x\}}^! \eta, \eta E \longrightarrow (\mathbb{R}i_{W*}) i_W^* u_K^* \Gamma_{\{x\}}^! \eta, \eta E.
\]

The arrow $a_3$ is obtained by taking the base change map of the total diagram (3.15). Note that $a_3$ is equal to the morphism that one gets by applying $\mathbb{R}i_{W*}$ to the canonical map

\[
i_W^* \Gamma_{\{x\}}^! \eta, \eta E \longrightarrow (\mathbb{R}v*) v^* i_W^* \Gamma_{\{x\}}^! \eta, \eta E.
\]

Since $v$ is an isomorphism it follows that $a_3$ is an isomorphism. Let us now look at the diagram

\[
\begin{array}{ccc}
(\mathbb{R}u_{K*}) u_K^* \Gamma_{\{x\}}^! \eta, \eta E & \xrightarrow{b_1} & (\mathbb{R}u_{K*}) j_{X', \eta}^! u_K^* \Gamma_{\{x\}}^! \eta, \eta E \\
& & \downarrow b_2 \\
(\mathbb{R}u_{K*} i_{W*}) i_W^* u_K^* \Gamma_{\{x\}}^! \eta, \eta E & \xrightarrow{b_3} & (\mathbb{R}u_{K*}) j_{X', \eta}^! (\mathbb{R}i_{W*}) i_W^* u_K^* \Gamma_{\{x\}}^! \eta, \eta E \\
\end{array}
\]

in which the horizontal arrows are obtained by applying $\mathbb{R}u_{K*}$ to the canonical map $\text{Id} \rightarrow j_{X', \eta}^!$. It is obvious that $b_3$ is an isomorphism. Indeed, the functor $j_{X', \eta}^!$ is exact so that $j_{X', \eta}^! (\mathbb{R}i_{W*}) = \mathbb{R}(j_{X', \eta}^! i_{W*})$. On the other hand, we have that $j_{X', \eta}^! i_{W*} = i_{W*}$. We may now construct another diagram
where 
\[ A = j_{X',\eta}^! u_K^* \Gamma_{\{x\},\eta}^! E \]
and 
\[ B = \Gamma_{\{x'\},\eta}^! j_{X',\eta}^! u_K^* E. \]

Here we have implicitly used the fact that 
\[ i^{*}_{W'} = i^{*}_{W} j_{X',\eta}^! \]
The arrows \( c_1 \) and \( c_3 \) are obtained from the canonical map \( (3.11) \). Since our morphism of frames is flat and since \( f^{-1}(x) = \{x'\} \) we know by proposition \( 3.7 \) that \( c_1 \) and \( c_3 \) are isomorphisms. Also, \( c_2 \) is an isomorphism by \( (3.7) \). We now have the identity
\[ c_2 \circ c_1 \circ b_1 \circ a_1 = c_3 \circ b_3 \circ a_3 \]
and we have shown that \( a_3, b_3, c_1, c_2 \) and \( c_3 \) are isomorphisms. It follows that the composition \( b_1 \circ a_1 \) is an isomorphism. For any \( \eta < 1 \) we now construct a commutative diagram
\[
\begin{array}{ccc}
\Gamma_{\{x\},\eta}^! E & \xrightarrow{d_1} & \mathbb{R}u_K^* j_{X',\eta}^! u_K^* \Gamma_{\{x\},\eta}^! E \\
& \downarrow{d_6} & \downarrow{d_3} \\
\Gamma_{\{x'\},\eta}^! j_{X',\eta}^! u_K^* E & \xrightarrow{d_5} & \mathbb{R}u_K^* \Gamma_{\{x'\},\eta}^! j_{X',\eta}^! u_K^* E
\end{array}
\]
as follows: The rightmost square is defined by applying \( \mathbb{R}u_K^* \) to the diagram \( (3.12) \) with \( \lambda = \eta \). The leftmost square is built by applying the canonical map \( \{u*\}_1 \) to the morphism \( \Gamma_{\{x\},\eta}^! E \rightarrow \Gamma_{\{x\}}^! E \) coming from \( (3.6) \) and then using the isomorphism \( (3.5) \). This means that \( d_1 \) is equal to \( b_1 \circ a_1 \) and \( d_5 \) is equal to the map \( (3.13) \). In particular, \( d_1 \) is an isomorphism. The maps \( d_2 \) and \( d_4 \) are also isomorphisms by proposition \( 3.7 \). Now observe that since \( u_K \) is quasi-compact, the functor \( \lim_{\eta \rightarrow \eta} \) commutes with \( \mathbb{R}u_K^* \) by \( [Ber96, 0.1.8] \). So if we take the limit \( \eta \rightarrow 1 \) then by \( (3.6) \) and \( (3.9) \) the maps \( d_6 \) and \( d_3 \) will become isomorphisms. Hence \( d_5 \) is an isomorphism. Since \( d_5 \) is equal to the map \( (3.13) \) this concludes the proof.

With proposition \( 3.8 \) in place the proof of theorem \( 3.4 \) becomes relatively straightforward.

**Proof of Theorem 3.4.** Like we did in paragraph \( 3.2 \) we divide the base change map \( u* \) into several parts \( (u*)_i \) for \( i = 1, \ldots, 4 \). It follows from proposition \( 3.1 \) that \( (u*)_4 \) is an isomorphism. By \( [LS07, Corollary 3.3.6] \) we know that the
map \( u_K : [Y]'[p_→]Y[p] \) is étale on some strict neighbourhood \( V' \) of \( [X]'[p_\rightarrow]Y'[p_\rightarrow] \). It then follows from propositions 5.1.13 and 6.2.2 in [LS07] that \((u\ast)\) is an isomorphism if and only if the corresponding map for the restriction \( u_K : V' \rightarrow Y[p] \) is an isomorphism. Therefore we may assume \( u_K \) to be étale, which implies that \( \Omega^\bullet_{[Y'[p_\rightarrow]Y[p]/K} = 0 \). Since our frames are smooth, we may also assume that \( [Y]'[p_\rightarrow]Y[p] \) are smooth. In this situation it is very easy to compute the Gauss-Manin filtration on the de Rham complex of any \( O_{[Y'[p_\rightarrow]Y[p]} \) module with an integrable connection and it immediately follows that \((u\ast)\) is an isomorphism. It remains to show that the composition \((u\ast)_2 \circ (u\ast)_1 \) is an isomorphism. For this we can use proposition 3.8. Indeed, proposition 3.8 is equivalent to saying that for every \( j^!_{x\mid X'} \) \( O_{[Y'[p_\rightarrow]Y[p]} \)-module \( E \), the map

\[
\Gamma^!_{(x)} E \longrightarrow u_{K\ast} u^! \Gamma^!_{(x)} E
\]

is an isomorphism and

\[
(R^iu_{K\ast}) u^! \Gamma^!_{(x)} E = 0
\]

for all \( i > 0 \). Note that by Proposition 5.3.2 and Corollary 5.3.3 in the spectral sequence of hypercohomology

\[
E^{m,n}_1 = (R^mu_{K\ast}) u^! \Gamma^!_{(x)} E \otimes O_{[Y'[p_\rightarrow]Y[p]/K} \Omega^\bullet_{Y[p]/K} \Rightarrow (R^{m+n}u_{K\ast}) u^! \Gamma^!_{(x)} E \otimes O_{[Y'[p_\rightarrow]Y[p]/K} \Omega^\bullet_{Y[p]/K}
\]

we deduce from (3.17) that

\[
(Ru_{K\ast}) u^! \Gamma^!_{(x)} E \otimes O_{[Y'[p_\rightarrow]Y[p]} \Omega^\bullet_{Y[p]/K} = u_{K\ast} u^! \Gamma^!_{(x)} E \otimes O_{[Y'[p_\rightarrow]Y[p]} \Omega^\bullet_{Y[p]/K}
\]

This means that \((u\ast)_2 \circ (u\ast)_1 \) is equal to the canonical map

\[
\Gamma^!_{(x)} E \otimes O_{[Y'[p_\rightarrow]Y[p]} \Omega^\bullet_{Y[p]/K} \longrightarrow u_{K\ast} u^! \Gamma^!_{(x)} E \otimes O_{[Y'[p_\rightarrow]Y[p]} \Omega^\bullet_{Y[p]/K}
\]

But this map can be computed term by term. From the fact that (3.16) is an isomorphism it then follows that (3.18) is an isomorphism as well. We have now shown that the maps \((u\ast)_4, (u\ast)_3\) and \((u\ast)_2 \circ (u\ast)_1\) are isomorphisms. This finishes the proof.

### 3.5 Part III of the proof: The general case

In this paragraph we finish the proof of theorem 3.3. First we improve theorem 3.4 by removing the condition that the induced map on tubes \( u_K : [Y]'[p_→]Y'[p_→] \) should be quasi-compact.

**Theorem 3.9.** Let
be an étale morphism of smooth \( S \)-frames. Also assume that \( f \) is étale. Let \( E \) be a coherent \( f_X^!\mathcal{O}_{Y’/\mathcal{P}} \)-module with an integrable connection over \( K \). Choose two closed points \( x’ \in X’ \) and \( x \in X \) such that \( f^{-1}(x) = \{x’\} \) and such that \( f \) defines an isomorphism \( k(x) \sim k(x’) \) on the residue fields. Then the base change map

\[
u^\star: \Gamma(x’^\circ E \otimes_{\mathcal{O}_{Y’/\mathcal{P}}} \Omega^!_{Y’/\mathcal{P}/K} \rightarrow (\mathbb{R}u_{K*}) \sum_{[x’]} u^! E \otimes_{\mathcal{O}_{Y’/\mathcal{P}}} \Omega^!_{Y’/\mathcal{P}/K}
\]

is an isomorphism in the derived category \( D^+(\mathcal{O}_{Y’/\mathcal{P}}\text{-Mod}) \).

Proof. Let \( Q’ \subset \mathcal{P}’ \) be an open neighbourhood of \( X’ \) such that the restriction \( u_{Q’} \) is étale. Then define \( X’’ = X \times_{\mathcal{P}} Q’ \) and \( Y’’ = Y \times_{\mathcal{P}} P’ \). We claim that the canonical map \( X’ \rightarrow Y’’ \) is an open immersion. Since this morphism factors through \( X’’ \) and since it is easy to see that the canonical map \( X’’ \rightarrow Y’’ \) is an open immersion, it suffices to show that the canonical morphism \( \alpha: X’ \rightarrow X’’ \) is an open immersion. It is clear that \( \alpha \) is an immersion. Now consider the projection morphism \( \beta: X’’ \rightarrow X \). Since \( f = \beta \circ \alpha \) and since \( \beta \) is étale by construction, it follows that \( \alpha \) is étale as well. This shows that \( \alpha \) is indeed an open immersion, proving our claim. The fact that \( X’ \rightarrow Y’’ \) is an open immersion allows us to factor our morphism of frames as follows:

Recall that the rigid analytic spaces \( P’_K \) and \( P_K \) are quasi-compact and quasi-separated. Hence the morphism \( u_K: P’_K \rightarrow P_K \) is quasi-compact. Moreover, we have that \( u_{K}^{-1}(\mathcal{P}[\mathcal{P}]) = \mathcal{P}[\mathcal{P}] \) according to [LS07, Proposition 2.2.6]. It follows that the induced morphism on tubes \( u_K: \mathcal{P}[\mathcal{P}] \rightarrow \mathcal{P}[\mathcal{P}] \) is quasi-compact as well. This allows us to apply theorem 3.4 to the lower part of the diagram. Now observe that the morphism \( Y’ \rightarrow Y’’ \) is a closed immersion, hence proper. So according to [LS07, Proposition 6.5.3], the base change map that is associated to the upper part of the diagram is an isomorphism as well.
The key idea for the proof of theorem 3.3 is to show that an étale map \( f: X' \to X \) has an étale realization, at least after shrinking \( X' \) and \( X \). In this way one reduces the problem to theorem 3.9. The proof of this fact relies on a number of geometric results that we discuss below.

**Proposition 3.10.**

i) Consider a proper morphism of frames

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow f & & \downarrow \\
X & \longrightarrow & Y \\
\end{array}
\]

where \( f \) is quasi-projective. Then we can blow up a closed subvariety of \( Y' \) outside \( X' \) in \( P' \) to obtain a diagram

\[
\begin{array}{ccc}
\tilde{Y'} & \longrightarrow & \tilde{P}' \\
\downarrow & & \downarrow \\
X' \bigcup \mathbb{V} & \longrightarrow & Y' \bigcup \mathbb{V} \\
\downarrow f & & \downarrow \\
X & \longrightarrow & Y \\
\end{array}
\]

where the composition \( \tilde{Y'} \to Y \) is projective.

ii) Consider a strict morphism of frames

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow u \\
X & \longrightarrow & Y \\
\end{array}
\]

where \( u \) is a formal blowing up. Then the map \( u_K: |Y'|_P \to |Y|_P \) is an isomorphism. Moreover, any admissible open neighbourhood \( V \) of \(|Y|_P \) in \(|Y'|_P \) if and only if \( u_K^{-1}(V) \) is a strict neighbourhood of \(|X'|_P \) in \(|X|_P \). That is, giving a \( j^{\frac{1}{k}}_X \mathcal{O}_{|Y'|_P} \)-module amounts to the same thing as giving a \( j^{\frac{1}{k}}_X \mathcal{O}_{|Y|_P} \)-module.

iii) Consider a frame \((X \subset Y \subset P)\) together with a diagram
where the map $X' \to Y'$ is an open immersion, $f$ is a local complete intersection morphism and $g$ is projective. Then locally on $X'$, there exists a closed subscheme $Y'' \subset Y'$ containing $X'$ such that the map $g|_{Y''}$ extends to a proper étale morphism of frames

$$
\begin{array}{c}
X' & \longrightarrow & Y' \\
| & f & \downarrow g \\
X & \longrightarrow & Y & \longrightarrow & P
\end{array}
$$

Proof.

i) Apply [RG71, Corollaire 5.7.14] to the morphism $Y' \to Y$ and the open subset $X' \subset Y'$, which is quasi-projective over $Y$.

ii) This follows from [LS07, Corollary 2.2.7] and [LS07, Proposition 3.1.13].

iii) The composition $Y' \to Y \to P$ can be factored through a closed immersion $Y' \to \mathbb{P}^N_P$ for some $N$. It now suffices to show that the morphism $i$ in the diagram below is a regular immersion. The rest of the proof is analogous to [LS07, Lemma 6.5.1].

First note that $i$ is an immersion, since the maps $X' \to \mathbb{P}^N_P$ and $\mathbb{P}^N_P \times_P X \to \mathbb{P}^N_P$ are immersions. Also, $\mathbb{P}^N_P \times_P X \to X$ is smooth since it is obtained by base extension from a smooth morphism. It follows from [Liu02, Corollary 6.3.22] that $i$ is indeed regular.

With all the preliminary work, the proof of theorem \ref{thm:main} becomes very similar to the proof of [LS07, Proposition 6.5.3].
Proof of Theorem 3.3. First note that we can always replace $X'$ and $X$ by open neighbourhoods $U_{x'}$, $U_x$ of $x'$ resp. of $x$ such that $f(U_{x'}) \subset U_x$. Indeed, the base change map coming from the diagram

$$
\begin{array}{ccc}
U_x & \rightarrow & P \\
X & \rightarrow & Y
\end{array}
$$

is simply the canonical map $\Gamma_{\{x\}} \rightarrow \Gamma_{\{x\}} \cdot U$ applied to the de Rham complex of $E$. This is an isomorphism by [LS07, Proposition 5.2.12]. A similar argument holds for the inclusion $U_{x'} \hookrightarrow X'$ and the $j_{X'} \cdot O_{Y'|\{x'\}}$-module $u^! E$. We may also replace $Y'$ by a closed subscheme that contains $X'$. By [LS07, Proposition 6.5.3] this does not alter the base change map either. We will refer to a combination of these two operations as a shrinking of the data. The fact that a shrinking of the data does not alter the base change map can be used to reduce the problem to the case where $f$ has an étale realization. Indeed, after replacing $X'$ and $X$ by open neighbourhoods of $x'$ resp. of $x$ we may assume that $f$ is an affine morphism, hence quasi-projective. By the first two points of proposition 3.10 we then reduce to the case where $g$ is projective. After some more shrinking of $X'$ and $Y'$ we may use the third point of proposition 3.10 to obtain an étale morphism of frames

$$
\begin{array}{ccc}
X' & \rightarrow & P'' \\
Y' & \rightarrow & P'' \\
X & \rightarrow & Y & \rightarrow & P
\end{array}
$$

Now consider the diagonal embedding $Y' \hookrightarrow P''' = P' \times_P P''$ and let $p_1 : P''' \rightarrow P'$ and $p_2 : P''' \rightarrow P''$ denote the projection maps. By construction we have that

$$
v \circ p_1 = u \circ p_2.
$$

(3.19)

Also, $p_1$ and $p_2$ are smooth since they are obtained by base extension from $v$ resp. from $u$. By the identity (3.19) it is now sufficient to prove that the base change maps that are associated to $v$ and to the diagrams

$$
\begin{array}{ccc}
X' & \rightarrow & Y' \\
& \nearrow_{p_1 \cdot p_2} & \\
& & P'''
\end{array}
$$

and

$$
\begin{array}{ccc}
& & P''' \\
X' & \rightarrow & Y' \\
& \searrow_{p_1 \cdot p_2} & \\
& & P''
\end{array}
$$
are isomorphisms. In theorem 3.9 we have already proved that the base change map for the étale morphism of frames \(v\) is an isomorphism. For the two morphisms of frames \(p_1\) and \(p_2\) it follows directly from [LS07, Proposition 6.5.3].

4 Cohomology with constant coefficients

Now that we have proved the main theorem 2.6 we discuss theorem 2.3, which deals with the local cohomology with constant coefficients. It is a direct consequence of theorem 2.6 and of the following proposition.

**Proposition 4.1.** Consider any morphism \(f : X' \to X\). Also choose a closed point \(x' \in X'\). Let \(\mathcal{O}_{X'/K}, \mathcal{O}_{X/K}\) be the constant overconvergent \(F\)-isocrystals on \(X'\) resp. on \(X\). There exists a canonical morphism \(f^*\mathcal{O}_{X/K} \to \mathcal{O}_{X'/K}\) such that the induced map

\[
H^i_{\{x'\}}(X', f^*\mathcal{O}_{X/K}) \longrightarrow H^i_{\{x'\}}(X', \mathcal{O}_{X'/K})
\]

is an isomorphism for every \(i\).

The canonical map \(f^*\mathcal{O}_{X/K} \to \mathcal{O}_{X'/K}\) is easy to define. First consider any realization of \(f\):

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \longrightarrow P' \\
\downarrow f & & \downarrow u \\
X & \longrightarrow & Y \longrightarrow P
\end{array}
\]

On this realization, the morphism is defined as the composition of the canonical map \(u_K j_X^!\mathcal{O}_{Y|p} \to j_X^! u_K^* \mathcal{O}_{Y|p}\) with the isomorphism \(j_X^! u_K^* \mathcal{O}_{Y|p} \cong j_Y^! \mathcal{O}_{Y'|p}\).

**Proof of Proposition 4.1.** We start by choosing a realization of \(f\) as above. We can now factor this realization as an open immersion followed by a Cartesian morphism. More precisely, we have a diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \longrightarrow P' \\
\downarrow f_1 & & \downarrow u \\
X'' & \longrightarrow & Y' \longrightarrow P' \\
\downarrow f_2 & & \downarrow u \\
X & \longrightarrow & Y \longrightarrow P
\end{array}
\]

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where \( X'' = X \times_Y Y' \), \( f_1 \) is an open immersion and \( f = f_2 \circ f_1 \). It is clear that the canonical map \( f^* \mathcal{O}_{X/K} \to \mathcal{O}_{X'/K} \) factors as
\[
f_1^* f_2^* \mathcal{O}_{X/K} \longrightarrow f_1^* \mathcal{O}_{X''/K} \longrightarrow \mathcal{O}_{X'/K}.
\]
Also, the map \( f_2^* \mathcal{O}_{X/K} \to \mathcal{O}_{X''/K} \) is an isomorphism since the canonical map
\[
u^*_K j^\dagger_{X'} \mathcal{O}_Y |_{P'} \longrightarrow \Gamma^\dagger_{\{x'\}} j^\dagger_X \mathcal{O}_Y |_{P'}
\]
is an isomorphism. So it suffices to show that \( f_1^* \mathcal{O}_{X''/K} \to \mathcal{O}_{X'/K} \) induces an isomorphism on the local cohomology. For this it is sufficient to know that the canonical map
\[
\Gamma^\dagger_{\{x'\}} j^\dagger_X \mathcal{O}_Y |_{P'} \longrightarrow \Gamma^\dagger_{\{x'\}} j^\dagger_X \mathcal{O}_Y |_{P'}
\]
is an isomorphism. This is proved in [LS07, Proposition 5.2.12].

**Proof of Theorem 2.3.** This is a direct consequence of proposition 2.5, theorem 2.6 and of proposition 4.1.

5 Applications and examples

In this final section we will apply our results to a threefold hypersurface called Schoen’s quintic, which is an example of a nodal Dwork hypersurface. It is defined by the equation
\[
X: \sum_{i=0}^{4} x_i^5 - 5 \prod_{i=0}^{4} x_i = 0
\]
in \( \mathbb{P}^4_k \). We will take \( k = \mathbb{F}_p \) where \( p \) is a prime number satisfying \( p \equiv 1(20) \).

Let \( U = \mathbb{P}^4_k \setminus X \) denote the complement. Our goal is to use the methods that we have sketched in the introduction in order to compute \( H^3_{\text{rig}}(U) \) and to show that \( H^i_{\text{rig}}(U) = 0 \) for \( i \in \{1, 2\} \). We hope that this will convince the reader of the usefulness of theorem 2.3.

First we compute the local cohomology of \( X \) at the singular points. It is easy to verify that the singular locus of \( X \) is given by
\[
\left\{ (\zeta^{a_0}, \zeta^{a_1}, \ldots, \zeta^{a_4}) \mid \sum_{i=0}^{4} a_i \equiv 0 (5) \right\}
\]
where \( \zeta \in k \) is a primitive 5th root of unity. For each of these singular points, there is an automorphism of \( X \) that maps it to \((1: 1: 1: 1: 1)\), so it is sufficient to compute the local cohomology of \( X \) at this point. Let us now introduce affine coordinates \( y_i = \frac{x_i}{a_i} - 1 \) for \( i = 1, \ldots, 4 \). This transformation translates our problem to the computation of the local cohomology of the affine
hypersurface

\[ 1 + \sum_{i=1}^{4} (y_i + 1)^5 - 5 \prod_{i=1}^{4} (y_i + 1) = 0 \]

at the origin. A suitable linear change of coordinates brings this equation into the more manageable form

\[ Y: \sum_{i=1}^{4} y_i^2 \left(1 + c_i(y)\right) = 0 \]

where the \( c_i(y) \) are polynomials without constant terms in the variables \( y_1, \ldots, y_4 \). We now claim that the origin of \( Y \) is an ordinary double point. That is, the origin of \( Y \) is equivalent to the origin of \( Y' \):

\[ \sum_{i=1}^{4} y_i^2 = 0. \]

In order to prove this, we introduce variables \((\underline{u}; \underline{v}; \underline{w})\) for \( \mathbb{A}^{12}_k \) where \( \underline{u} = (u_1, \ldots, u_4) \), \( \underline{v} = (v_1, \ldots, v_4) \), and \( \underline{w} = (w_1, \ldots, w_4) \). Now look at the variety

\[ Y'': \begin{cases} w_i^2 = 1 + c_i(\underline{v}) \text{ for } i = 1, \ldots, 4 \\ u_i w_i = v_i \left(1 + c_i(\underline{v})\right) \text{ for } i = 1, \ldots, 4 \\ \sum_{i=1}^{4} v_i^2 \left(1 + c_i(\underline{v})\right) = 0 \end{cases} \]

in \( \mathbb{A}^{12}_k \). We also introduce the notation \( Q := (0; 0; 1) \in Y'' \). Now let \( V \) denote the open subset \( \bigcap_{i=1}^{4} D \left(1 + c_i(\underline{v})\right) \) of \( Y'' \). We can then define two morphisms

\[ f: V \longrightarrow Y: (\underline{u}; \underline{v}; \underline{w}) \mapsto \underline{v} \]

and

\[ f': V \longrightarrow Y': (\underline{u}; \underline{v}; \underline{w}) \mapsto \underline{w}. \]

We want show that \( f \) and \( f' \) are étale at \( Q \). Let us carry out the computation for \( f \). Writing down the Jacobian matrix for \( f \), we obtain:

\[
\begin{pmatrix}
    u_1 w_1 - v_1 \left(1 + c_1(\underline{v})\right) & u_2 w_1 & \cdots & u_4 w_1 \\
    \cdots & \cdots & \cdots & \cdots \\
    u_4 w_4 - v_4 \left(1 + c_4(\underline{v})\right) & w_4 & \cdots & u_4 \\
    w_1^2 - 1 - c_1(\underline{v}) & 2w_1 & \cdots & \cdots \\
    w_4^2 - 1 - c_4(\underline{v}) & \cdots & \cdots & 2w_4
\end{pmatrix}
\]

The determinant of this matrix evaluated at \( Q \) is nonzero. It follows that \( f \) is étale at \( Q \). A similar computation shows that \( f' \) is étale at \( Q \). We now
have that \((Y, 0) \sim_{\text{ét}} (Y', 0)\) according to definition \(2.1\). So by theorem \(2.3\) it remains to compute the local cohomology of \(Y'\) at the origin. Note that since \(p \equiv 1(4)\), \(Y'\) is isomorphic to the affine cone of \(\mathbb{P}^1_k \times \mathbb{P}^1_k\). This fact can be used to compute the following table for the local cohomology of \(Y'\):

\[
\begin{array}{c|ccccccc}
 i & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \dim H^i_{\{0\}}(Y') & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 \text{Frobenius on } H^i_{\{0\}}(Y') & 0 & 0 & 0 & p & p^2 & 0 & p^3 \\
\end{array}
\]

In order to show that \(H^i_{\text{rig}}(U) = 0\) for \(i \in \{1, 2\}\), we use the following observation.

**Proposition 5.1.** Let \(X \subset \mathbb{P}^4_k\) be an irreducible hypersurface with only isolated singularities. Denote its complement by \(U\). Call \(\Sigma\) the singular locus of \(X\). Then the following are equivalent:

i) \(H^i_{\text{rig}}(U) = 0\) for \(i \in \{1, 2\}\).

ii) The local rigid cohomology of \(X\) at the singular points satisfies the following conditions:

- \(H^5_{\text{rig}, \Sigma}(X) = 0\).
- \(\dim H^6_{\text{rig}, \Sigma}(X) = |\Sigma|\).

**Proof.** Throughout the proof we will denote the Betti numbers of rigid cohomology by \(h^i\). We begin by reformulating the statements from i). From the fact that \(X\) is irreducible it easily follows that \(h^6(X) = 1\). Now consider the long exact sequence with compact supports:

\[
\ldots \longrightarrow H^i_{\text{rig}, c}(U) \longrightarrow H^i_{\text{rig}, c}(\mathbb{P}^4_k) \longrightarrow H^i_{\text{rig}}(X) \longrightarrow \ldots
\]

By using Poincaré duality on \(U\) and the fact that \(h^6(X) = 1\) we obtain the equalities \(h^1(U) = 0\) and \(h^2(U) = h^5(X)\). So it remains to show that the statements from ii) hold if and only if \(h^5(X) = 0\). Now look at the following long exact sequence with compact supports:

\[
\ldots \longrightarrow H^i_{\text{rig}, c}(X \setminus \Sigma) \longrightarrow H^i_{\text{rig}}(X) \longrightarrow H^i_{\text{rig}}(\Sigma) \longrightarrow \ldots
\]

By using the Lefschetz hyperplane theorem for \(X \subset \mathbb{P}^4_k\) and Poincaré duality on \(X \setminus \Sigma\) we can deduce that \(h^4(X \setminus \Sigma) = 1\), \(h^5(X \setminus \Sigma) = |\Sigma| - 1\) and \(h^6(X \setminus \Sigma) = 0\). Let us now prove the implication ii) \(\Rightarrow\) i). If \(h^5_\Sigma(X) = 0\) then we have an exact sequence

\[
0 \longrightarrow H^5_{\text{rig}}(X) \longrightarrow H^5_{\text{rig}}(X \setminus \Sigma) \longrightarrow H^6_{\text{rig}}(X) \longrightarrow H^6_{\text{rig}}(X) \longrightarrow 0
\]

By using the assumption that \(h^6_\Sigma(X) = |\Sigma|\) we see that \(h^5(X) = 0\) as required. Conversely, assume that \(h^5(X) = 0\). We again make use of the long exact sequence

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\[ \ldots \rightarrow H^i_{\text{rig}, X}(X) \rightarrow H^i_{\text{rig}}(X) \rightarrow H^i_{\text{rig}}(X \setminus \Sigma) \rightarrow \ldots \]

It follows directly that \( h^5_\Sigma(X) = |\Sigma| \). It is also easy to see that the map \( \alpha: H^4_{\text{rig}}(X) \rightarrow H^4_{\text{rig}}(X \setminus \Sigma) \) is nonzero. Since \( h^4(X \setminus \Sigma) = 1 \) it follows that \( \alpha \) is surjective. The fact that \( h^5(X) = 0 \) tells us that \( H^5_{\text{rig}, \Sigma}(X) = \text{Coker}(\alpha) = 0 \). This proves that the statements from ii) hold.

We have just computed that the conditions of the proposition hold for Schoen’s quintic. It follows that \( H^i_{\text{rig}}(U) = 0 \) for \( i \in \{1, 2\} \).

Finally, we compute the cohomology space \( H^3_{\text{rig}}(U) \). For this we mimic the method from [Dim90]. Let \( X \subset \mathbb{P}^4_C \) be a hypersurface with singular locus \( \Sigma \). In this context there exists a map \( \beta: S_5 \rightarrow H^4(U, C) \), where \( S_5 \) denotes the space of homogeneous polynomials of degree 5 in \( C[x_0, \ldots, x_4] \). This map has the property that the composition with (1.1) does not alter the cokernel. If \( X \) only has ordinary double points then the space \( H^4_\Sigma(X, C) \) consists of \( |\Sigma| \) copies of a one-dimensional space \( W \). It is shown in [Dim90] that for a certain choice of lifts \( \tilde{P} \in C^5 \) of the points \( P \in \Sigma \) there exists a basis \( \{b\} \) for \( W \) such that the composition of \( \beta \) with the map (1.1) can be explicitly computed by the formula:

\[
S_5 \rightarrow H^4_\Sigma(X, C) = \bigoplus_{P \in \Sigma} W \\
g \mapsto \bigoplus_{P \in \Sigma} g(\tilde{P}) \cdot b
\]  

(5.1)

This formula can be used to compute the cokernel of (1.1), which is isomorphic to \( H^3(U, C) \).

The map (1.1) is constructed from the standard exact sequences of Betti cohomology. By using the rigid counterparts of these sequences one can show that \( H^3_{\text{rig}}(U) \) is isomorphic to the cokernel of a certain canonical map

\[
H^4_{\text{rig}}(U) \rightarrow H^4_{\text{rig}, \Sigma}(X).
\]  

(5.2)

We conjecture that if \( X \) only has weighted homogeneous singularities then the cokernel of (5.2) may be computed in a similar way as in [Dim90]. In particular, we can make a good guess about the cokernel of (5.2) in the case where \( X \) is equal to Schoen’s quintic. Note that the image of (5.1) is independent of the basis vector \( b \). Therefore we may repeat the computation above with the Betti cohomology replaced by rigid cohomology and with \( S_5 \) replaced by the space of homogeneous polynomials of degree 5 in \( Q_p[x_0, \ldots, x_4] \). By carrying out the computation in this way we have found that the only eigenvalue of the Frobenius action on \( H^3_{\text{rig}}(U) \) is \( p^2 \), with multiplicity 24. This result is consistent with [CdIORV03], in which the zeta function of \( U \) is computed in a way that is very different from our methods.
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