Dynamics of Yang-Mills Cosmology Bubbles in Bartnik-McKinnon Spacetimes

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ABSTRACT

We investigate the dynamics of a Yang-Mills cosmology (YMC, the FRW type spacetime) bubble in the Bartnik-McKinnon (BK) spacetimes. Because a BK spacetime can be identified to a YMC spacetime with a finite scale factor in the neighborhood of the origin, we can give a natural initial condition for the YMC bubble. The YMC bubble can smoothly emerge from the origin without an initial singularity. Under a certain condition, the bubble develops continuously and finally replaces the entire BK spacetime, the metric of which is the same as the one of the radiation dominated universe. We also discuss why an initial singularity can be avoided in the present case in spite of the singularity theorems by Hawking and Penrose.

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1 Introduction

A remarkable pioneering work of Israel was writing the Einstein field equation into the form of junction conditions at the boundary between two spacetime regions when the metric is continuous but the curvature and the energy-momentum tensor are not, having some form of singularity [1, 2]. An example is a bubble with a domain wall or a thin shell, on which the singularity of the energy-momentum is of the form of the $\delta$ function, and is allowed without violation of the Einstein field equation. How bubbles develop is also an interesting topic in the context of real cosmology [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. To the best knowledge of the author, there are three reasons for such cosmological interests.

The first is the discussion about child universes. A child universe can develop from a small region by an inflation. If this inflation is due to a “false vacuum” of scalar field which is equivalent to a positive cosmological constant, the false vacuum region is separated from the true vacuum region by a domain wall. The motion of this domain wall just obeys the junction conditions.

The second is the inhomogeneity by local inflations. We can imagine that in the initial universe, there are not only one of the regions entered into the “false vacuum” states of the scalar field. Because such regions will undergo various magnitude of inflations, comparing to the regions which have not undergone any inflation, they will have bigger size and mass densities. Since in different regions, the potential heights of the metastable states are not necessarily equal, the degrees of inflations are also different. Then the development of the inhomogeneity by the phase transitions can be investigated by the consideration of the dynamics of the bubbles.
The third is most intriguing. As the existence of the “false vacuum” of a scalar field is sufficient for an inflation, Farhi and Guth discussed the possibility of realization of an inflation in the present universe, especially by in principle man-made processes. They especially considered a spherically symmetric de Sitter bubble created in, necessarily by Birkhoff theorem, the Schwarzschild spacetime. Since a spherically symmetric bubble should be created classically at \( r = 0 \), the origin of the Schwarzschild coordinates, where a black hole singularity is located, we always have it as an initial singularity of a bubble. Classically, an initial singularity becomes an obstacle because human being does not know how to prepare an initial singularity in the present universe. Such a singularity may be avoided by a quantum tunneling. However, no one could give a complete discussion by an explicit example up to now.

To pursue the idea of Farhi and Guth further is one of the purposes of this paper. We will reconsider the bubble creation in classical sense and examine the possibility to avoid the initial singularity. Let us first analyze the initial singularity which necessarily exists in the de Sitter – Schwarzschild bubble. At first, we find that a de Sitter spacetime expands so fast that such a bubble can always contain an anti-trapped surface. Secondly, the parent spacetime – a Schwarzschild spacetime has a singularity at \( r = 0 \), where the spherically symmetric bubble emerges. For the present purpose, we should consider a bubble in which the spacetime expands less fast that it does not contain an anti-trapped surface in the bubble developing region, and a parent spacetime which is regular everywhere. For the first condition, we may consider a Friedmann-Robertson-Walker (FRW) bubble. Although there
also exist anti-trapped surfaces in a region of the pure FRW spacetime, we shall see in the present paper that the bubble with a non-spacelike shell cannot enter the region to contain an anti-trapped surface. However, if the source of the spacetime is a usual ideal liquid, by Birkhoff theorem, the exterior of the bubble is inevitably a part of the Schwarzschild spacetime too. Then the second condition cannot be satisfied. However, if there exists a long range field in the FRW bubble which can penetrate the shell, we can be free from Birkhoff theorem since the outside is not a vacuum, and have a parent spacetime which is regular everywhere. Previously Yang-Mills cosmology (YMC) solutions were found and carefully discussed [26, 27, 28, 29, 30]. These spacetimes are homogeneous and isotropic, and equivalent to FRW spacetimes with radiation dominant source [26]. Because we, for simplicity, only consider a spherically symmetric bubble, the spacetime outside of the bubble should be static and spherically symmetric, and should be one of the following possibilities: Bartnik-McKinnon (BK) particle-like solutions [19, 20, 22, 23, 25] or Bizon - Kunzle - Masood-ul-Alam black holes [21], depending on the initial spacetime in which a bubble is going to be created. For the case of a Bizon - Kuenzle - Masood-ul-Alam black hole, the situation is similar to that of Schwarzschild spacetime. Thus we shall consider a YMC bubble developing in a Bartnik-McKinnon spacetime. The BK spacetime, which is regular everywhere, is just what we need. We expect that a YMC-BK bubble can be created without an initial singularity.

Homogeneous and isotropic solutions of coupled system of gravity and Yang-Mills field were discussed enormously [26, 27, 28, 29, 30]. These discussions concerned both of Euclidean (whormhole) and Lorentzian signature solutions. The YMC [26] are
Lorentzian ones which are equivalent to all types (closed, open and spatially flat) of FRW spacetimes in the radiation dominant epoch. The solutions are also called hot universes with cold matters. For the convenience of use, we shall give a brief review for the solutions in the subsection \(a\) of the section 2.

The Bartnik-McKinnon solutions [19] are asymptotically flat solutions of coupled system of gravity and Yang-Mills field, which are the spherically symmetric. The solution in the whole spacetime is regular. An important fact is that this spacetime in the neighborhood of the origin can be set to be the same as a YMC (FRW) spacetime due to the equality of the curvatures there. For a YMC bubble created from BK spacetime, if there is no initial singularity, a proper initial condition can/should be given by the equivalence. Then we can imagine a bubble emerging and developing in this spacetime. For the purpose of convenience of the discussion, we shall give a brief review for this kind of solution in the subsection \(b\) of the section 2, there we also give some other new facts and further analysis, especially global ones, for our purposes.

Although we have a static, spherically symmetric spacetime – the BK spacetime – as an initial spacetime of a YMC bubble without initial singularity, we still could not confirm whether such a bubble can develop in a BK spacetime and finally becomes a child universe or not before our investigation of the equation of motion for the shell. In the section 3, we shall discuss how a YMC bubble develops in a BK spacetime quantitatively. There exists a shell trajectory between the two spacetime regions. We shall discuss the equation of motion for the bubble and examine whether the bubble can expand to the extent comparable to the corresponding
FRW universe or not. To discuss a shell in an exact sense, we should discuss a field configuration which asymptotically approaches to two different regions, the YMC and BK spacetimes. Investigating numerically how a BK solution evolves using the perturbation theory, Zhou and Straumann [24] found some shell-like structure with a finite thickness. However, to discuss such an exact shell is very complicated and is impossible analytically. Numerical methods are not effective for the discussions of global properties and the final stage of the bubble. Because we are interested in the cosmological case, the thin shell approximation is always sufficient for it (e.g., see [3, 4, 9]). That is to say, when the shell is very thin comparing to the any other length of interest, the junction condition method is good enough to solve the Einstein field equation with a curvature jump. In the section 4, we found, under some conditions, a open or spatially flat type of bubble can expand infinitely and finally replace the entire BK spacetime.

In fact, from only the viewpoint of a shell between YMC and BK spacetimes, there are some interesting properties which are different from those of de Sitter-Schwarzschild bubble. Since the pressure inside the bubble is always higher than that outside of the shell, except at the $\rho = 0$ where an initial condition makes them equal, the bubble really continuously emerges and pushes into the parent spacetime. Of course it also expands to the direction of the YMC spacetime.

Hawking showed that there is an initial singularity in the standard big bang cosmology model [33]. We should answer why we can create a FRW type bubble without an initial singularity. One might argue such a bubble spacetime is not a maximally extended spacetime to which the singularity theorems [32, 34] apply.
However the total spacetime composed of bubble, outside of the bubble in the parent spacetime and the shell can be regarded as a maximally extended spacetime. In fact, given the energy-momentum tensor in both bubble and parent spacetimes, together with it on the shell, one can always solve the Einstein field equation to get a maximally extended spacetime. In the paper of Farhi and Guth, they already implicated this fact \[10\]. Then what is the reason that we can avoid an initial singularity? After a detailed analysis, we find that there is fundamental difference between a universe created from nothing and a bubble emerging and developing inside another spacetime. Since the universe expands, spatial hypersurfaces may contain anti-trapped surfaces \[33\]. However for the case of a bubble inside the other spacetime, if the bubble is a FRW type spacetime, the bubble only corresponds to a part of a spatial hypersurface in the FRW spacetime. Thus a bubble may or may not contain an anti-trapped surface. In the section 5, we shall show that a non-spacelike shell, as we expected, does not contain an anti-trapped surface and give other detailed discussion about singularity theorems. To make a comparison, we shall also discuss why there exists an initial singularity for a de Sitter-Schwarzschild bubble.

In the section 6, we draw our conclusion and give some discussions.

2 Yang-Mills Cosmology and Bartnik-McKinnon Solutions – Reviews

a. Yang-Mills Cosmology (YMC)
Yang-Mills Cosmologies (YMCs) [28, 29, 27] are spacetimes which are homogeneous and isotropic with the Yang-Mills field as a source and are the same as the FRW universes with radiation dominant perfect liquid.

The metric for the spacetime is assumed to be the form of FRW universe,

\[ ds^2 = -dt^2 + a^2(t)(d\chi^2 + f^2(\chi)d\Omega^2) = a^2(\lambda)(-d\lambda^2 + d\chi^2 + f^2(\chi)d\Omega^2), \] (2.1)

where \( f(\chi) = \sin(\chi), \sinh(\chi), \) and \( \chi \) for closed \((k = 1)\), open \((k = -1)\), and spatially flat \((k = 0)\) Friedmann models, respectively. Here \( d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \), and \( \lambda \) is the conformal time which relates to the cosmic time \( t \) by \( d\lambda/dt = a^{-1} \).

By the Einstein field equation, the line element (2.1) defines for the energy-momentum tensor \( T_{\mu\nu} \) the typical perfect fluid structure:

\[ T^0_0 = -\epsilon(\lambda), \quad T^i_i = p(\lambda)\delta^i_i. \] (2.2)

Since the Yang-Mills field is conformal invariant, one has \( p = \epsilon/3 \). Thus the Yang-Mills cosmology is always radiation dominant. Together with the Bianchi identities, the energy-momentum tensor has to take the form

\[ T^\mu_\nu = \frac{C^2}{2e^2a^4}\text{diag}(-3, 1, 1, 1), \] (2.3)

where \( C^2 \) is some constant, and \( e \) is the coupling constant of the Yang-Mill field.

The Friedmann equation becomes

\[ \left(\frac{da}{d\lambda}\right)^2 + ka^2 = \frac{2\kappa C^2}{e^2}, \] (2.4)

where \( k = 1, -1, 0 \) for closed, open and spatially flat Friedmann models, respectively. In this paper, \( \kappa = 8\pi G \), where \( G \) denotes the gravitational constant.
The line element (2.1) is invariant under actions of the spatial rotations and translations. Thus the gauge field symmetry must be the same. For simplicities, we shall discuss the gauge field with only spatial rotation symmetry at first and then pick out the solution which is also invariant under actions of translations. The well-known Witten ansatz is suitable for our purpose

\[ eA = \omega_0 L_1 d\lambda + \omega_1 L_1 dr - g(L_3 d\theta - L_2 \sin \theta d\phi), \] (2.5)

where \( f \) and \( \omega_i \) are functions of \( \lambda \) and \( \chi \). Here a coordinate dependent gauge group generators \( L_a \), which satisfy the commutative relation \([L_a, L_c] = i\epsilon_{abc} L_c\), are introduced:

\[ L_1 = \sin \theta \cos \phi \sigma_1 \frac{1}{2} + \sin \theta \sin \phi \sigma_2 \frac{1}{2} + \cos \theta \sigma_3 \frac{1}{2}, \]
\[ L_2 = \partial_\theta L_1, \quad L_3 = \frac{1}{\sin \theta} \partial_\phi L_1, \] (2.6)

where \( \sigma_i \) are the Pauli matrices.

From the Einstein and the Yang-Mills field equations, we obtain

\[ \omega_0 = -\frac{f \sqrt{1 - kf^2}}{1 + f^2(\psi^2 - k)} \frac{d\psi}{d\lambda}, \quad \omega_1 = \frac{\psi f^2 (\psi^2 - 1)}{1 + f^2(\psi^2 - k)}, \] (2.7)

and

\[ g = 1 - \sqrt{1 + f^2(\psi^2 - k)}. \] (2.8)

Here \( \psi \) satisfies the equation

\[ \left( \frac{d\psi}{d\lambda} \right)^2 + (\psi^2 - k)^2 = C^2. \] (2.9)

By checking directly, we can find that the Yang-Mills field with \( \omega_i \) and \( f \) given by the equations (2.7) and (2.8) give a homogeneous energy-momentum tensor as
the source of the FRW type of spacetime. That is to say, The solutions are also invariant under the action of translations.

b. Bartnik-McKinnon (BK) Solutions

The static spherically symmetric (SSS) solutions of Einstein-Yang-Mills (EYM) equations have been paid much attention since the last decade after an explicit series of such solutions was found by Bartnik and McKinnon [19, 20, 22, 23, 25]. Before that the existence of such kind of solutions was doubted because neither the vacuum Einstein nor the pure Yang-Mills equations have non-trivial regular SSS solutions with finite energy and there is no such a solution for EYM equation in (2+1)-dimensions.

All of the solutions found by Bartnik and McKinnon are numerical. What is interesting is that the solutions were shown to be analogues of sphalerons in the pure Yang-Mills theory or the Weinberg-Salam model [20, 21, 22, 23, 25], which may be called EYM sphalerons.

The Yang-Mills connection 1-form ansatz for the solutions is

\begin{equation}
\text{e} \mathbf{A} = w \sigma_1 d\theta + (\cot \theta \sigma_3 + w \sigma_2) \sin \theta d\phi,
\end{equation}

where \( w \) is an unknown function which depends only on \( r \). The metric is

\begin{equation}
ds^2 = -T^{-2} dt^2 + R^2 dr^2 + r^2 d\Omega^2,
\end{equation}

where \( T \) and \( R \) are some functions of \( r \).

Rescaling the variable \( r \rightarrow \frac{\sqrt{\kappa}}{e} r \) removes the dependence on \( \kappa \) and \( e \) from the Einstein and Yang-Mills (EYM) equations. Introducing a convenient variable \( m(r) \)
by writing \( R = (1 - 2m(r)/r)^{-1/2} \), the static spherically symmetric EYM equations read

\[
m' = (1 - \frac{2m}{r})w'^2 + \frac{1}{2}\left(\frac{1 - w^2}{r^2}\right),
\]

(2.12)

\[
r^2(1 - \frac{2m}{r})w'' + [2m - \left(\frac{1 - w^2}{r}\right)]w' + (1 - w^2)w = 0,
\]

(2.13)

\[
2r(1 - \frac{2m}{r})\frac{T'}{T} = \left(\frac{1 - w^2}{r^2}\right) - 2(1 - \frac{2m}{r})w'^2 - \frac{2m}{r},
\]

(2.14)

where a prime denotes \( \partial/\partial r \).

The obtained power series form of solutions is \(^{19}\)

\[
2m = 4b^2r^3 + \frac{16}{5}b^3r^5 + O(r^7),
\]

(2.15)

\[
w = 1 + br^2 + \left(\frac{4}{5}b^3 + \frac{3}{10}b^2\right)r^4 + O(r^6),
\]

(2.16)

where \( b \) is a parameter, and some discrete set of values are taken for \( b \) in the range \(-0.706 < b < 0\). The solutions (2.15) and (2.16) are regular in the whole range of \( r \), i.e., \( 0 \leq r < \infty \). The solutions can be classified by the zeroes \( n \) of \( w_n \), where \( n \) can take all the odd positive integers.

It is easy to show that the density \( T_{00} \) is regular even at \( r = 0 \), which is different from the Schwarzschild solution and the Reissner-Norstrom solutions. The spacetimes are asymptotically flat. The spacetimes have no event horizon at all. This fact by no means violates Birkhoff theorem, because here the solutions are not for the vacuum Einstein equation.

By the equations (2.13) and (2.16), we obtain the energy-momentum tensor

\[
T^0_0 = -6b^2\kappa^{-2} + O(r^2),
\]

(2.17)
and

\[ T^i_k = 2b^2 e^2 \kappa^{-2} \delta^i_k + O(r^2), \quad \text{for} \quad i, k = 1, 2, 3, \quad (2.18) \]

for small \( r \). Here we have already written the energy-momentum tensor \((2.17)\) and \((2.18)\) in terms of the unscaled \( r \). It is worthy of noticing that this energy-momentum tensor approaches that of perfect liquid with \( p = \epsilon/3 = 2b^2 e^2 \kappa^{-2} \) in the FRW spacetime as \( r \) approaches zero. Thus the metric can also be set into the same as the metric of a FRW spacetime near the spatial origin approximately. This is a very important fact for our purposes.

To understand the global properties, we should give the Penrose diagram of the BK spacetime. For this purpose, it is sufficient to consider the two-dimensional metric,

\[ ds^2 = -T^{-2} dt^2 + R^2 dr^2. \quad (2.19) \]

The null geodesic equation is

\[ 0 = g_{\mu\nu} k^\mu k^\nu = -T^{-2} \dot{t}^2 + R^2 \dot{r}^2. \quad (2.20) \]

Thus, the radial null geodesic of BK spacetime satisfies

\[ t = \pm r_\ast + \text{constant}, \quad (2.21) \]

where

\[ r_\ast = \int^r R(r')T(r')dr'. \quad (2.22) \]

We define the null coordinates \( u, v \) by

\[ u = t - r_\ast, \quad (2.23) \]
and

\[ v = t + r_*. \] (2.24)

In terms of the null coordinates, the metric (2.19) can be written as

\[ ds^2 = -T^{-2}(u, v)dudv. \] (2.25)

Because both of the function \( T^{-2}(u, v) \) and the transformations (2.23) and (2.24) are always regular, the metric (2.25) is conformal to those of the Minkowski spacetime suppressing the metric for the two-sphere. The Penrose diagram is exactly the same as that of the Minkowski spacetime.

## 3 Dynamics of Bubbles between BK and YMC Spacetimes

We shall discuss the dynamics of a bubble between BK and YMC spacetimes and examine if a bubble can be created without a singularity. This is intuitively possible because the BK spacetimes, unlike the Schwarzschild spacetime with spacelike singularity at \( r = 0 \) in which the bubble necessarily has an initial singularity [10, 13], is regular in the whole spacetime even at \( r = 0 \). However, we further ask what are the properties of the bubble. Especially, can it expand large enough to become a child universe? In this section we shall give a quantitative discussion.

As is well known, a suitable tool for the research of the evolution of a thin shell is what is called the junction conditions on the shell which correspond to the Einstein field equation together with conservation law of the energy-momentum tensor. We shall summarize the results in the subsection a [9, 12].
Before we can solve the junction condition equations, we should know the form of energy-momentum tensor on the shell. Provided that the shell is sufficiently thin, we can simply assume that it has a δ function distribution across the shell. In the subsection b, we shall derive the equation of motion of the thin shell. Considering that the shell is made of the Yang-Mills field, rather than scalar fields, the properties of it should greatly differ from the case of a scalar shell, we shall discuss it in detail in the subsection c near the creation, and the subsection d for the final stage, respectively.

a. The Einstein field equation on the thin shell and Junction conditions

To describe the behavior of the shell which divides the spacetime into two regions $V^+$ and $V^-$, it is simplest to introduce a Gaussian normal coordinate system in the neighborhood of the shell. If we denote by $\Sigma$ the (2+1)-dimensional spacetime hypersurface in which the shell lies, we can introduce (2+1)-dimensional coordinates $x^i \equiv (\tau, x^2, x^3) (i = 1, 2, 3)$ on $\Sigma$. For the timelike coordinate we use the proper-time variable $\tau$ that would be measured by an observer comoving with the shell. In the Gaussian normal coordinate system, a geodesic in a neighborhood of $\Sigma$ which is orthogonal to $\Sigma$ is taken as the third spatial coordinate denoted by $\eta$. Thus, the full set of coordinates is given by $x^\mu \equiv (x^i, \eta)$. We can take $x^2 = \theta$ and $x^3 = \phi$ in the case of spherically symmetric shell. Then the metric can be written as

$$ds^2 = d\eta^2 + g_{ij} dx^i dx^j. \quad (3.1)$$

We introduce an unit vector field $\xi^\mu(x)$ which is normal to $\eta = \text{const.}$ hypersurfaces and pointing from the YMC to the BK spacetime. The induced metric on the
hypersurface $\Sigma$ can then be written as

$$h_{\mu\nu} = g_{\mu\nu} - \xi_{\mu}\xi_{\nu}. \quad (3.2)$$

In the Gaussian normal coordinates,

$$\xi^\mu(x) = \xi_\mu(x) = (0, 0, 0, 1) \quad (3.3)$$

The Gaussian normal coordinate system is suitable for the Gauss-Codazzi formalism in the neighborhood of $\Sigma$ with the coordinate $\eta$ orthogonal to the slices near $\Sigma$. The extrinsic curvature is given by

$$K_{ij} = -\Gamma^\eta_{ij} = \frac{1}{2} \partial_\eta g_{ij}. \quad (3.4)$$

Assuming that the energy-momentum tensor $T_{\mu\nu}$ is of the form

$$T_{\mu\nu} = S_{\mu\nu}\delta(\eta) + (\text{regular terms}), \quad (3.5)$$

where $S_{\mu\nu}$ is the surface stress-energy tensor, we can rewrite the Einstein field equation as

$$[K^i_j] = -\kappa(S^i_j - \frac{1}{2} \delta^i_j S), \quad (3.6)$$

$$S_{ij} = -[T^\eta_j], \quad (3.7)$$

and

$$\{K^i_j\} S^i_j = [T^\eta_j], \quad (3.8)$$

where $\{K_{ij}\} = \frac{1}{2}(K^+_{ij} + K^-_{ij})$, and $[\Psi] = \Psi^+ - \Psi^-$, hereafter for any quantity $\Psi$.

By the symmetry analysis and the energy-momentum conservation [9], the surface stress-energy tensor on the shell should be

$$S^{\mu\nu} = \sigma(\tau)U^\mu U^\nu - \zeta(\tau)(h^{\mu\nu} + U^\mu U^\nu), \quad (3.9)$$
where $U^\mu = (1, 0, 0, 0)$ is the four-velocity of the shell. Here $\sigma$ is the surface energy density, and $\zeta$ is the surface tensor of the shell. $\sigma$ and $\zeta$ should be constrained by the Einstein field equation or conservation equation (see the equation (3.14) below) and determined by the Yang-Mills equation.

b. The dynamics of the bubble between BK and YMC spacetimes

For the sake of simplicity, we shall only consider a spherically symmetric shell. In the coordinates of BK spacetime, the metric is described by

$$ds^2 = -T^{-2}dt^2 + R^2dq^2 + r^2(q,t)d\Omega_2^2.$$  \hfill (3.10)

The coordinate $q$ is defined so that it originates from the center of the bubble to the outside. The induced metric on the shell is determined by

$$ds^2|_\Sigma = -d\tau^2 + \rho^2(\tau)d\Omega_2^2,$$  \hfill (3.11)

where $\tau$ is the proper time of the shell.

Inside the bubble is the domain of the YMC (FRW) spacetime with the metric

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + d\Omega^2).$$  \hfill (3.12)

The matter source is the Yang-Mills field in the YMC.

To the observer in this coordinate system, the shell corresponds to $\rho = a \cdot f(\chi)$, $U^0 = U_0 = 1$, and $U^1 = U_1 = 0$. The explicit form of the extrinsic curvature of the shell in this case was already given in Ref. [11] and [12]. We just take advantage of results there;

$$K_2^2 = -\frac{\Sigma}{\rho}(\rho^2 + 1 - \frac{\kappa}{3}\epsilon\rho^2)^{1/2},$$  \hfill (3.13)
where $\Sigma = 1$ for the case of the increasing shell radius and $\Sigma = -1$ for the case of the decreasing shell radius in the direction of the outer normal. Here, by the results in the section 2, $\epsilon = \frac{3C^2}{2\epsilon^2 a^4}$.

For the spherically symmetric bubble, the equation (3.7) will give a relation

$$\dot{\sigma} = -2(\sigma - \zeta) \frac{\dot{\rho}}{\rho} - g\rho \dot{\sigma},$$

(3.14)

where $g = \frac{\epsilon}{\sqrt{\kappa}}$ and the overdot denotes a derivative with respect to $\tau$. Here we used the equations (A17) and (A18) in the Appendix, where the form of $[T_{k\eta}]$ has been discussed for general spherically symmetric Yang-Mills shell configurations.

For the shell, by the equations (2.15) and (2.16) and the relation $\rho = r$ on the shell, the 2-2 component of the extrinsic curvature is obtained as

$$K_{22}^2 = -\frac{\Sigma}{\rho} (\dot{\rho}^2 + 1 - \frac{2m(\rho)}{\rho})^{1/2}.$$  

(3.15)

in the coordinates of the BK spacetime.

Since in the spherically symmetric case $A_{22} = A_{33}$ for any second-rank tensor $A^{\mu\nu}$, we need only consider the 2-2 component of the junction condition equation (3.6)

$$-\frac{\Sigma_{\text{out}}}{\rho} (\dot{\rho}^2 + 1 - \frac{2m(\rho)}{\rho})^{1/2} + \frac{\Sigma_{\text{in}}}{\rho} (\dot{\rho}^2 + 1 - \frac{\kappa\epsilon}{3}\rho^2)^{1/2} = \frac{\kappa}{2}\sigma.$$  

(3.16)

It is easy to show that the independent components of the Einstein field equation and the conservation equation are the two equations (3.14) and (3.16). Now we engage ourself in solving these equations.

We should have an initial condition for the shell. In the neighborhood of the origin of the BK spacetime, the curvature approaches that of YMC (FRW) spacetime.
(see Fig. 1). If we suitably choose the initial scale factor $a_0$ and the “energy” $C^2$ of the Yang-Mills field in YMC spacetime so that

$$a_0 = \left(\frac{\kappa^2 C^2}{4e^4 b^2}\right)^{1/4},$$  \hspace{1cm} (3.17)

we can get a YMC bubble smoothly created from the BK spacetime, because of the continuity of curvatures for the two spacetimes on the shell. The meaning of it is that, for a fixed BK spacetime namely for a given $b$, a small part of the YMC spacetime near the origin satisfying the condition (3.17) may smoothly emerge out of the BK spacetime and develop into a bubble. For the later development of the bubble, we should solve the equations (3.14) and (3.16) with the initial condition (3.17).

However we still have too many unknown functions $\sigma$, $\zeta$ and $\rho$, while we have only two independent equations. We need an equation of state of the shell from the Yang-Mills field equation. However for thin shells, the simplest class of the equation of state may be of the form $\zeta = (1 - \alpha)\sigma$. Under this condition, we can easily solve Eq. (3.14)

$$\sigma = C'(\frac{\rho}{1 + g\rho})^{-2\alpha},$$  \hspace{1cm} (3.18)

and

$$\zeta = C'(1 - \alpha)(\frac{\rho}{1 + g\rho})^{-2\alpha}.$$  \hspace{1cm} (3.19)

Here $C'$ is a constant with unit $[L]^{2\alpha-3}$. For the case without initial singularity, we should have $\alpha < 0$, because at initial point the YMC bubble is a part of the BK spacetime near $r = 0$ and there is no shell at all there.
Substituting the surface energy $\sigma$ given by Eq. (3.18) into Eq. (3.16), we obtain

$$- \Sigma_{\text{out}} (\dot{\rho}^2 + 1 - \frac{2m(\rho)}{\rho})^{1/2} + \Sigma_{\text{in}} (\dot{\rho}^2 + 1 - \frac{\kappa\epsilon}{3} \rho^2)^{1/2} = \frac{\kappa C'}{2} \rho^{1-2\alpha} \frac{\rho^{1-2\alpha}}{(1 + g\rho)^{-2\alpha}}. \quad (3.20)$$

Twice squaring the equation (3.20), we get the equation of motion for the shell,

$$\dot{\rho}^2 + V(\rho) = 0, \quad (3.21)$$

where

$$V(\rho) = \frac{(1 + g\rho)^{-4\alpha}}{(\kappa C')^2 \rho^{2-4\alpha}} [4(1 - \frac{2m(\rho)}{\rho})(1 - \frac{\kappa\epsilon}{3} \rho^2) - (2 - \frac{2m(\rho)}{\rho}) - \frac{\kappa\epsilon}{3} \rho^2 \nonumber - \frac{(\kappa C')^2}{2} \rho^{2-4\alpha} \frac{\rho^{2-4\alpha}}{(1 + g\rho)^{-4\alpha})}] \quad (3.22)$$

Then the equation of motion for the shell is identical to that of a classical particle moving in one-dimension under the influence of the potential (3.22), with the energy zero.

For the convenience of the later discussions, we write the potential (3.22) as

$$V(\rho) = 1 - \frac{2m(\rho)}{\rho} - \frac{1}{4B} (A - B)^2, \quad (3.23)$$

where

$$A = \frac{2m(\rho)}{\rho} - \frac{\kappa\epsilon}{3} \rho^2, \quad (3.24)$$

and

$$B = \frac{(\kappa C')^2}{2} \frac{\rho^{2-4\alpha}}{(1 + g\rho)^{-4\alpha}}. \quad (3.25)$$

c. The properties near the creation

We are especially interested in the case without an initial singularity. In the language of the potential (3.22), a good behavior of it is necessary at the point near
\( \rho = 0 \). However, it is perhaps impossible to analytically solve the equation of motion with the potential (3.23). This is not only because of the complexity of the potential but also because of the proper time dependence of \( \epsilon(\tau) \) of the potential. We shall give a numerical analysis in the forthcoming paper. Here let us analyze some special cases. We are especially interested in the case of the very small bubble because we want to know whether the initial singularity can be avoided or not. If we only consider a bubble developing in a short duration comparing with the whole life of the YMC universe, we can approximately take \( \epsilon \) as a constant. For our purpose it is sufficient to consider a special case \( \alpha = -1/2 \). The potential (3.23) is equal to

\[
V(\rho) = 1 - \frac{2m(\rho)}{\rho} - \frac{\beta^2}{4},
\]

(3.26)

where \( \beta \equiv \frac{A}{B^{1/2}} \) is a constant because as \( \rho \to 0, A^2, B \propto \rho^4 \), and \( \epsilon \to \frac{3\kappa^2}{2\epsilon^2 v_0^5} \). In the equation of motion (3.20), we have two independent arbitrary constants \( C' \) and \( b \) (see equation (2.13)). The constant \( \beta \) can be written in terms of \( C' \) and \( b \),

\[
\beta = \frac{8e^2b^2 - 2/3\kappa^2\epsilon}{\kappa^2C'},
\]

(3.27)

with \( \alpha = -1/2 \). If the shell satisfies the condition

\[
\beta^2 \geq 4,
\]

(3.28)

remarkably, the potential is finite at \( \rho = 0 \), and a shell is always classically possible. This finiteness is what is necessary. Otherwise, there is an inconsistency if there is no initial singularity.

In fact the value of the \( \beta \) can be determined uniquely by the initial velocity of the shell. For a shell with a zero velocity (or infinitesimal velocity), we have \( \beta = 4 \)
(or $\beta \to 4$). A typical potential is shown in Fig. 2 corresponding to a BK spacetime of $b = -0.45371627277$, and $\beta^2 = 4$.

d. The properties of the final stage

In general case, of course, $\epsilon$ is time dependent, because it is the energy density in the YMC (FRW) spacetime. Although we cannot solve the equation of motion \((3.21)\) analytically, we can show an important property for the trajectory of shell. Let us consider the potential \((3.23)\). By the properties of the BK solution, one can find $2m'(\rho) < \frac{6m(\rho)}{\rho}$. This fact means that $\frac{m(\rho)}{\rho^2}$ monotonically decreases with $\rho$ increasing. Thus the last term in the potential \((3.23)\) monotonically decreases with $\rho$ increasing. As we known, when $\rho \to 0$, the last term approaches a constant. Since the constant is smaller or equal to 1 and the second term is definitely negative, the potential is definitely negative for all $\rho > 0$. This means that the velocity of the shell cannot change the sign in the evolution. The shell always increases since the initially it increases. Thus the bubble cannot be the type of $k = 1$, since the bubble would collapse finally. This fact depends on the special type of shell with $\alpha = -1/2$. Though both of $k = 0$ and $k = -1$ types of bubbles are possible (be determined by the initial spatial curvature), here we only discuss the case of $k = -1$ in detail. The discussion for the case of $k = 0$ is similar.

Finally $\rho$ will become very large, and the potential \((3.23)\) can be written as

$$V \simeq -\frac{2m(\rho)}{\rho} - \frac{(\kappa C')2}{16} \frac{\rho^4}{(1 + g \rho)^2},$$

\((3.29)\)

approximately. Thus with the increasing of $\rho$, $V$ will decrease to negative infinity,
which makes $\dot{\rho} \to \infty$. In terms of $\dot{\rho}$, we can describe the velocity of the shell as

$$\frac{d\rho}{dt}_{BK} = \frac{\dot{\rho}}{T^2\sqrt{1 + R^2\dot{\rho}^2}},$$

(3.30)

with respect to the observer inside the BK spacetime, and

$$\frac{d\chi}{d\lambda}_{YMC} = \frac{\dot{\rho}}{\sqrt{1 + \dot{\rho}^2}},$$

(3.31)

with respect to the observer inside the YMC spacetime. The trajectory of the shell approaches null with respect to both of the observers, as the shell becomes infinity.

We can understand the fact better by the Penrose diagrams. Since the scale factor $a^2(\lambda)$ is a conformal factor in the metric (2.1), which will not show up in the Penrose diagram, $\dot{\rho}$ is just equivalent to $\dot{\chi}$ in terms of the conformal metric $ds^2 = -d\lambda^2 + d\chi^2 + f^2(\chi)d\Omega^2$. However, by the equation (3.29), $\rho$ always increases at the later stage. Thus the shell will always increase in the $\chi$ parameter space of spatial hypersurface $H^3$. The trajectory of the shell approaches null and finally intersects $I^+$. The situation is shown in the Fig. 3.

For an observer in the region of BK spacetime, because $\rho$ corresponds the usual radial coordinate, the shell always increases too (see Fig. 3).

We also give a combined figure as Fig. 4.

The situation described above is totally different from the case of a de Sitter bubble in the Schwarzschild spacetime [9, 10, 13].

### 4 Relations to Singularity Theorems

It is well known that there are a series of theorems which state the existence of a singularity in a spacetime under some physically reasonable conditions [31, 32, 21, 22].
Why can we avoid an initial singularity for a YMC bubble inside a BK spacetime? In the introduction we already gave an intuitive discussion about this. Now we shall give some further discussions relevant to the singularity theorems.

At first, we should examine the energy conditions because our spacetime contains a special form of matter – shell, although the regions of the BK and YMC spacetimes satisfy the strong energy condition. Let us consider a general shell with the surface stress-energy tensor (3.9). Since

\[ T_{\mu\nu} k_\mu k_\nu = \sigma (k_\tau)^2 \delta(\eta) \geq 0, \]  

for any null vector \( k^\mu \), the very weak energy condition \[10\] is satisfied. To check the strong energy condition, we may consider the quantity,

\[ T^\mu\nu U_\mu U_\nu + \frac{1}{2} T = \frac{1}{2} (\sigma - 2\zeta) \delta(\eta), \]  

for the 4-velocity \( U_\mu \) of the shell. We find the strong energy condition fails on the shell for a repulsive shell, \( \sigma - 2\zeta \leq 0 \), in which we are interested.

We then find no contradiction between our result and the singularity theorem of Hawking and Penrose \[33\], since the theorem needs the strong energy condition. However, the singularity theorem of Penrose needs only very weak energy condition which is satisfied on the shell. We should answer why we can avoid this theorem too. The Penrose theorem tells us that

spacetime \((\mathcal{M}, g)\) cannot be past-null geodesically complete if

(1) \( R^\mu\nu k_\mu k_\nu \geq 0 \) for all null \( k_\mu \),

(2) there is a noncompact Cauchy surface in \( \mathcal{M} \), and
(3) there is an anti-trapped surface in $\mathcal{M}$.

Here we have expressed the original theorem slightly differently. Namely we replace the trapped surface by the anti-trapped surface, for the purpose of the discussion of an initial singularity.

In the paper of Farhi and Guth [10], the authors argued that an initial singularity necessarily exists because the de Sitter bubble inside the Schwarzschild spacetime has anti-trapped surfaces. Although there exists an initial singularity in that case, we should notice at least two points. First, we cannot apply the Penrose theorem only to the de Sitter bubble region because there is no well-defined Cauchy surface. Nor can we use its modification in which the second condition is replaced by the stable causality and the generic condition [35], because the de Sitter spacetime does not satisfy the latter one. To define a Cauchy surface, we need to consider the combination of the bubble and parent spacetimes. For example, a Cauchy surface is shown in the Fig. 5 in the case of a de Sitter-Schwarzschild bubble. Second, the bubble itself is not an inextendible spacetime. However a total combination of bubble, parent spacetimes and the shell between them can be considered as an inextendible spacetime to which the singularity theorem applies. Thus we can apply the Penrose theorem to this combination since it is inextendible and permitted to define a Cauchy surface.

In the proof of his theorem, Penrose defined a closed trapped surface [31]. A significant feature of a trapped surface $T$ arises from the fact that the null geodesics meeting it orthogonally are the generators of horismos $E^+(T)$ (or $E^-(T)$) [32]. On the surface, these null geodesics start out by converging. If the weak energy condition
and null completeness are assumed, the Raychaudhuri equation for the null geodesic tells us that they must continue to converge until encountering focal points finally. The geodesic segments joining $T$ to the focal point must sweep out a compact set. By this means $T$ is called future (or past) trapped set [32].

However, the relation between trapped surface and trapped set becomes subtle if a shell exists. It is instructive to analyze again the reason why the de Sitter-Schwarzschild bubble has an initial singularity. A null geodesic starting from a trapped surface (if exists in one region of the spacetime) will penetrate the shell and enter into another region generally. Then we can determine whether this trapped surface is a future trapped set or not by looking at the whole spacetime, but cannot determine by only looking at the bubble or the parent spacetimes separately. For a de Sitter bubble inside the Schwarzschild spacetime, we can easily find an anti-trapped surface. One of the anti-trapped surfaces is shown in Fig. 5. The outer directed $E^-(T)$ passes through the shell and go into the region of Schwarzschild spacetime. However, this $E^-(T)$ will also converge in the region of white hole and form a focal point, if completeness of null geodesic is assumed. Thus this anti-trapped surface is a past trapped set. Similarly to the proof of the Penrose theorem, this completeness would make contradiction between the compactness of horismos $E^-$ and the non-compactness of the Cauchy surface, and then there must be a singularity – null geodesics should not be complete. Here we should notice that for an initial singularity we should have an anti-trapped surface, and for a final or collapsing singularity, we should have a trapped surface.

A YMC (FRW) bubble created inside of the BK spacetime looks similar to that
of the de Sitter case. However, a fundamental difference is that the anti-trapped surface region is in a different area in the corresponding Penrose diagrams. In a FRW universe (e.g. $k = -1$), there always exists an anti-trapped surface if

$$\frac{d}{dt}(a^2(t)\sinh^2(\chi)) > 0$$

holds for both families of inward and outward null geodesics [33]. For the convenience of our discussions, we should know the region in the Penrose diagram where anti-trapped surfaces exist. To a spacetime which is invariant under action of translations, e.g., a whole FRW universe, the inequation (4.3) does not serve to give a specific spatial region of anti-trapped surfaces. However, since we are discussing a bubble which develops from an infinitesimal bubble at the origin of the BK spacetime, we can specify an anti-trapped region with respect to a point which corresponds to the origin of the BK spacetime. In terms of conformal time $\lambda$, we can write the equation (4.3) as

$$\frac{1}{\tanh(\lambda_0)} > \pm \frac{1}{\tanh(\chi_0)},$$

(4.4)

where the subscript 0 corresponds to a two-sphere. Here we have used the condition that the radius of the bubble $\chi_0 = 0$ at the time $\lambda_0 = 0$. This inequality is equivalent to

$$\lambda_0 < \chi_0,$$

(4.5)

where $\lambda_0 = \chi_0$ corresponds to the event horizon with respect to the observer at the center of the bubble (Although the apparent horizon is typically a null or spacelike surface which lies inside or coincides with the event horizon assuming cosmic censorship, we cannot use this result directly because the cosmic censorship does not
apply for the child universe. However, by the calculation above, we showed that this result is also correct for our case even without assuming cosmic censorship). Thus an anti-trapped surface always exists beyond the event horizon (to the observer at origin) of the YMC (FRW) spacetime.

To summarize, as shown in the Penrose diagrams (see Fig. 5), anti-trapped surface region is III and trapped surface region is IV for the de Sitter spacetime, while there is no trapped surface and the anti-trapped region is II for the open FRW spacetime (see Fig. 3).

It is easy to see that any de Sitter bubble with a non-spacelike shell can enter the region with an anti-trapped surface. Then there must be an initial singularity, if other conditions of the Penrose theorem are also satisfied.

In the case of the YMC bubble in a BK spacetime, a bubble can be parameterized by $\sinh(\chi)$ in a spacetime with metric $ds^2 = -d\lambda^2 + d\chi^2 + \sinh^2(\chi)d\Omega^2$ which is conformal to YMC spacetime. The developing of the bubble corresponds the change of radius $\chi_s$ of the shell. A bubble may or may not contain an anti-trapped surface. If the motion of a shell is non-spacelike, a bubble which develops from the origin of the BK spacetime can never enter the region with an anti-trapped surface. In fact, any past directed null geodesic starting from point inside the bubble will go to null infinity $I^-$ in the BK spacetime after penetrating the shell. Thus there exists no two-surface which is a past trapped set. This is the reason of the absence of initial singularity.

Because there is no anti-trapped surface, there is also no contradiction to the singularity theorems with generalized strong energy conditions [36, 37].
5 Conclusions and Discussions

In this paper, we have discussed the properties of YMC-BK bubble and given an example to show how a YMC bubble emerges from a BK spacetime without an initial singularity. By contrasting to the case of a de Sitter bubble created from a Schwarzschild spacetime, we have also given an intuitive picture as depicted in Fig. 6 which shows snapshots of the four characteristic epochs. Comparing to the de-Sitter-Schwarzschild bubble, a fundamental difference is that the bubble can never be separated from the parent spacetime.

Coincidently our results can just describe the instability of the outside BK spacetime. At early stage of the bubble, our results qualitatively coincide to that of “mini star” case which was discussed by Zhou and Straumann numerically [24]. However, their discussion could not be extended to the later stage. The reason is that, with the evolution, the Yang-Mills field configuration of the “mini star” die away more and more rapidly as increasing of $r$. While the numerical method is not suitable, thin shell method is still effective to such a case. The thin shell formalism could give an analytical discussion from which we could give the global properties of the YMC-BK bubble which corresponds to a “mini star” spacetime at early stage.

There are two kinds of YMC bubbles which can develop inside the BK spacetime. The shells are exact same even when the inside spacetimes are different. The type of the bubble is further constrained by the constant $C^2$. If $0 \leq C^2 \leq 1$, the bubble should uniquely be a part of the YMC spacetime with $k = 0$. No $k = -1$ type of FRW solution if $0 \leq C^2 \leq 1$ [20]. If $C^2 > 1$, both $k = 0$ and $k = -1$ types of bubble are possible.
Because the shells expand not only toward the YMC but also toward the BK spacetimes, comparing to a de Sitter bubble in a Schwarzschild spacetime, the developing of a YMC bubble in a BK spacetime is less ambiguous because there is not any paradox here. The observer on the side of YMC spacetime near the shell attributes the expansion of the shell to the expansion of the YMC spacetime and high pressure comparing to the outside. The outer observer near the shell attributes this to that the pressure is lower than the YMC bubble.

**Appendix: The Form of \([T^\eta_i]\) on Spherical Yang-Mills field Shells**

Because of the existence of the global time coordinate both for inside and outside regions, it is easier to calculate the term \([T^\eta_i]\) in the case of a shell of a finite thickness than the one in the previous infinite thin shell case. We shall first consider the shell configuration exactly. Then by taking a limit of zero thickness, we can get \([T^\eta_i]\) for the thin shell. In this Appendix, we shall give the general form of \([T^\eta_i]\) for a spherically symmetric, Yang-Mills field shell.

For a developing shell, the Yang-Mills field and spacetime are time dependent. The metric for the spacetime with such a shell can be written as

\[
\text{d}s^2 = -e^{2\psi}\text{d}t^2 + e^{2\lambda}\text{d}r^2 + r^2\text{d}\Omega^2, \tag{A1}
\]

where \(\psi\) and \(\lambda\) are the functions of \(r\) and \(t\). The Yang-Mills connection 1-form is assumed to be \([19]\)

\[
e\text{A} = u\sigma_3\text{d}t + w\sigma_1\text{d}\theta + (\cot\theta\sigma_3 + w\sigma_2)\sin\theta\text{d}\phi, \tag{A2}
\]
where $u$ and $w$ depend only on $r$ and $t$. By the scale transformation $r \rightarrow \frac{\sqrt{\kappa}}{e^\lambda} r$, the Einstein and Yang-Mills equations can be transformed into $e$ and $\kappa$ independent for convenience of calculations.

The Yang-Mills field components are

$$B_T^2 = \frac{e^{-2\lambda}}{r^2} (u')^2, \quad B_L^2 = \left(1 - \frac{w^2}{r^4}\right), \quad (A3)$$

and

$$E_K^2 = \frac{e^{-2\psi}}{r^2} u^2, \quad E_T^2 = e^{-2(\psi+\lambda)} u^2, \quad E_L^2 = \frac{e^{-2\psi}}{r^2} u^2 w^2. \quad (A4)$$

Here the prime denotes partial derivative with respect to $r$. In this Appendix, we always denotes $\partial/\partial t$ by an overdot, which is different in previous sections.

The non-vanishing energy-momentum tensor components for the Yang-Mills field are

$$T_0^0 = -[B_T^2 + \frac{1}{2} B_L^2 + E_K^2 + \frac{1}{2} E_T^2 + E_L^2], \quad (A5)$$

$$T_1^1 = B_T^2 - \frac{1}{2} B_L^2 + E_K^2 - \frac{1}{2} E_T^2 + E_L^2, \quad (A6)$$

and

$$T_0^1 = \frac{2ww'}{r^2} e^{-2\lambda}. \quad (A7)$$

Zhou and Straumann found the numerical solution for this system [24]. In terms of $m$ introduced by the relation

$$e^{-2\lambda} = 1 - \frac{2m(r,t)}{r}, \quad (A8)$$

similarly to the case in BK solution, these solution can be written as

$$m(r,t) = 2b(t)^2 r^3 + O(r^5), \quad (A9)$$
and
\[ w(r, t) = 1 + b(t)r^2 + O(r^4), \] (A10)
near the spatial origin. It is worthy of noticing that this solution becomes the BK solution if \( b(t) = b \) is a constant.

By the equations (A9) and (A10), the energy-momentum tensor components (A5), (A6) and (A7) can be written as
\[ T_{00} = -6b(t)^2 + O(r^2), \] (A11)
\[ T_{ki} = 2b(t)^2\delta_i^k + O(r^2), \] (A12)
and
\[ T_{01} = 2(b(t)^2)r + O(r^3), \] (A13)
near the origin. It is easy to find that the spacetime is still the YMC (FRW) type at the origin.

Because we are discussing a thick shell configuration which connects the YMC spacetime near the origin to the outside BK spacetime, we have
\[ [T_{01}] \equiv T_{01}(r \to r_{BK}) - T_{01}(r \to r_{YMC}) = -2((b(t)^2)r_{YMC}, \] (A14)
because \( T_{01}(r \to r_{BK}) = 0 \). Here \( r_{YMC} \) is the corresponding radius of the YMC spacetimes where the shell configuration connects to.

Now let us return to the thin shell case. By the definition (7)
\[ S_k^i = \lim_{\delta \to 0} \int d\eta T_k^i = \sigma \delta(\eta)\delta_k^i, \] (A15)
we have \( \sigma = 2b^2(t) - (2b^2 + O(r^2))_{BK} \), since the region \( r < r_{YMC} \) can be regarded as the YMC spacetime region. Here \( \delta = 2\epsilon \) is the thickness of the shell. Considering
the time independence of the terms \((2b^2 + O(r^2))_{BK}\) in \(\sigma\), we have

\[
[T_0^1] = -\dot{\sigma} r. \tag{A16}
\]

Taking the approximation that the thickness of the shell approaches zero and denoting in terms of the proper time of the shell, we finally obtain

\[
[T_\theta^\eta] = [T_\phi^\eta] = 0, \tag{A17}
\]

and

\[
[T_\tau^\eta] = -\frac{e}{\sqrt{\kappa}} \rho \frac{d\sigma}{d\tau}, \tag{A18}
\]

where \(\eta\) is the Cauchy normal coordinate. Notice that we have already written the equation \((A18)\) in a form including the constants \(e\) and \(\kappa\) by the transformation \(r \to \frac{e}{\sqrt{\kappa}}r\).

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Figures Captions

Fig. 1  The BK spacetime is equivalent to a YMC spacetime near the origin.

Fig. 2  The potential of the shell for the case $b = -0.45371627277$, and $\beta^2 = 4$.

Fig. 3  The Penrose diagrams for the case of a YMC-BK bubble. The one for the YMC ($k = -1$) spacetime is the same to those of FRW spacetime in the radiation dominant epoch. The one for the BK spacetime is the same to those of the Minkowski spacetime. The bubble cannot enter into the region II, including an anti-trapped surface, since the motion of the shell is non-spacelike. The solid line shows the trajectory of the shell.

Fig. 4  The Penrose diagram with the inside and outside of the bubble sewn together. The trajectory of the shell approaches lightlike as the bubble expands to $I^+$.

Fig. 5  The Penrose diagrams for the case of a de Sitter-Schwarzschild bubble. There are anti-trapped surfaces in the region III. $T$ is one of anti-trapped surfaces whose past horismos $E^-(T)$ is compact if the null completeness were supposed. A Cauchy surface is also shown.

Fig. 6  Four characteristic times for the evolution of the bubble. The picture $a$ is a BK spacetime.
Fig. 1
potential

\begin{align*}
b &= 0.45371627277
\end{align*}
Fig. 3

- Thin shell
- Inside of bubble
- Outside of bubble
Fig. 4
inside of bubble

anti-trapped

domain wall

outside of bubble

Cauchy surface
Fig. 6