Genus expanded cut-and-join operators and generalized Hurwitz numbers

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(Dedicated to Prof. An-Min Li for his 70th birthday)

Abstract

To distinguish the contributions to the generalized Hurwitz number of the source Riemann surface with different genus, by observing carefully the symplectic surgery and the gluing formulas of the relative GW-invariants, we define the genus expanded cut-and-join operators. Moreover all normalized the genus expanded cut-and-join operators with same degree form a differential algebra, which is isomorphic to the central subalgebra of the symmetric group algebra. As an application, we get some differential equations for the generating functions of the generalized Hurwitz numbers for the source Riemann surface with different genus, thus we can express the generating functions in terms of the genus expanded cut-and-join operators.

Keyword: genus expanded cut-and-join operator; differential algebra; Hurwitz number; generating function

Subject Classification: 14N10, 14N35, 05E10

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1 Introduction

Hurwitz Enumeration Problem has been studied for more than one century, which has been extensively applied to mathematics and physics. It is an interesting project for mathematic physics, geometry, representation theory, integrable systems, Hodge integral, etc. Hurwitz numbers have many different expressions in the different fields, for example, [1], [2], [3], [4], [5], [6], [7], [13], [14], [15]. One of the important geometric tools to deal with Hurwitz numbers is the so-called symplectic surgery: cutting and gluing [8], [9], [10], in the views of algebra and differential equations, which is equivalent to the so-called cut-and-join operators [4], [5], [6], [11], [16]. The standard cut-and-join operators can be used to deal with the almost simple Hurwitz numbers and the almost simple double Hurwitz numbers [4], [5], [6], [11], [16]. A. Mironov, A. Morozov, and S. Natanzon, etc, have defined the generalized cut-and-join operators in terms of the matrix Miwa variable [1], [13], [14]. However, the generating functions obtained by the generalized cut-and-join operators defined by A. Mironov, A. Morozov, and S. Natanzon cannot distinguish the contributions of the source Riemann surface with the different genus, which is obviously interesting and important. In our paper, by observing carefully the symplectic surgery and the gluing formulas of the relative GW-invariants developed by A.M. Li and Y.B. Ruan [10], and by E. Ionel and T. Parker [8], we define the genus expanded cut-and-join operators to distinguish the contributions of the source Riemann surfaces with different genus. A little more precisely, we introduce one parameter $z$ to “mark” the genus, then we associate to every partition $\Delta$ a genus expanded cut-and-join operator $W(\Delta, z)$ as follows:

$$W(\Delta, z) = \sum_{\Gamma, \Gamma' \vdash d} \frac{d!}{|C_{\Gamma'}|} z^{d+l(\Gamma')-l(\Delta)-l(\Gamma)} \mu_0 \mu_{h^+} \mu_{h^-} \frac{\partial}{\partial p_{\Gamma'}}.$$  \hspace{1cm} (1)

For the precise meaning of the notations in the above formula, see the section 2 and the section 3. Moreover, the power of $z$ is exactly the “lost” genus after we have executed the symplectic cutting on the target Riemann surface, referring to Remark 4.3.

One of our main theorems is the followings, referring the theorem 3.5:

**Theorem A** For a given $d$, as operators on the functions of the time-
variables $p = (p_1, p_2, \ldots)$, we have

$$W(\Delta_1, z)W(\Delta_2, z) = \sum_{\Delta_3} z^{d - l(\Delta_1) - l(\Delta_2) + l(\Delta_3)} C_{\Delta_1 \Delta_2}^{\Delta_3} W(\Delta_3, z),$$

(2)

where

$$C_{\Delta_1 \Delta_2}^{\Delta_3} = \frac{d!}{|C_{\Delta_3}|} h.d(\Delta_1, \Delta_2, \Delta_3)$$

(3)

is the structure constants of the center subalgebra of the group algebra $C[S_d]$.

A more interested result in this paper is that all normalized the genus expanded cut-and-join operators with same degree form a differential algebra, which is isomorphic to the central subalgebra of the symmetric group algebra, referring Corollary (3.6).

As an application of the genus expanded cut-and-join operators, we define the generating function as (referring the section 4)

$$\Phi_g\{z|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\} = \sum_{l_1, \ldots, l_n \geq 0, \Gamma_1, \ldots, \Gamma_k} \frac{z^{2h - 2} \mu^{h.d}(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_k)}{\prod_{j=1}^{n} \prod_{l_j}^{u_j} \prod_{i=1}^{k} p^{(i)}_{\Gamma_i}}$$

(4)

Then we have our another main theorem (referring theorem 4.2):

**Theorem B** For any $i$, we have

$$\frac{\partial \Phi_g\{z|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}}{\partial u_i} = W(\Delta_i, z)\Phi_g\{z|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}.$$  

(5)

Thus we can express the generating functions in terms of the genus expanded cut-and-join operators once we know the initial values. Moreover, the generating functions can be directly expressed by the expansion in terms of the genus of the source Riemann surface, referring to Example 4.5.

Our paper is deeply inspired by A. Mironov, A. Morozov, and S. Natanzon’s works [1], [13], [14], [15].

## 2 Generalized Hurwitz numbers

Let $\Sigma^h$ be a compact (possibly disconnected) Riemann surface of genus $h$, and $\Sigma^g$ a compact connected Riemann surface of genus $g$. For a given
point set \( \{q_1, \ldots, q_k\} \in \Sigma^g \), which is called the set of branch points, we call a holomorphic map \( f : \Sigma^h \to \Sigma^g \) a ramified covering of \( \Sigma^g \) of degree \( d \geq 0 \) by \( \Sigma^h \) with a ramification type \( (\Delta_1, \ldots, \Delta_k) \), if the preimages of \( f^{-1}(q_i) = \{p_{i1}^1, \ldots, p_{im_i}^i\} \) with orders \( \Delta_i = (\delta_{i1}, \ldots, \delta_{im_i}) \) for \( i = 1, \ldots, k \), respectively. Two ramified coverings \( f_1 \) and \( f_2 \) with type \( (\Delta_1, \ldots, \Delta_k) \) are said to be equivalent if there is a homeomorphism \( \pi : \Sigma^h \to \Sigma^h \) such that \( f_1 = f_2 \circ \pi \) and \( \pi \) preserves the preimages and the ramification type of \( f_1 \) and \( f_2 \) at each point \( q_i \in \Sigma^g \). Let

\[
\mu_{g,h}^{h,d}(\Delta_1, \ldots, \Delta_k)
\]

be the number of equivalent covering of \( \Sigma^g \) by \( \Sigma^h \) with ramification type \( (\Delta_1, \ldots, \Delta_k) \), which is called the generalized Hurwitz number. Note that \( \mu_{g,h}^{h,d}(\Delta_1, \ldots, \Delta_k) \) is nonzero only if the Hurwitz formula

\[
(2 - 2g)d - (2 - 2h) = \sum_{i=1}^{k} (d - m_i)
\]

holds, where \( m_i \) is the numbers of the preimage of \( q_i \). How to determine \( \mu_{g,h}^{h,d}(\Delta_1, \ldots, \Delta_k) \) is known as the generalized Hurwitz Enumeration Problem \[7\]. In general, \( \Delta_i \) is not required to be a partition of \( d \), for simplicity, throughout this paper, we will assume it. Denote by \( \Delta_i \vdash d \) if \( \Delta_i \) is a partition of \( d \).

Let \( S_d \) be the symmetric group of \( d \) letters, \( \mathbb{C}[S_d] \) be the group algebra of \( S_d \) and \( Z\mathbb{C}[S_d] \) be the central subalgebra of \( \mathbb{C}[S_d] \), then in the terms of algebra, the story begins from a simple fact as the following formula \[7\],

\[
\mu_{g,h}^{h,d}(\Delta_1, \ldots, \Delta_k) = \frac{1}{d!} [1] \prod_{j=1}^{g} \prod_{a_j, b_j \in S_d} [a_j, b_j] C_{\Delta_1} \cdots C_{\Delta_k},
\]

where \([1]\) means that we take the coefficient of the identity of the product of \( g \)-tuple commutators \( \prod_{a,b \in S_d} [a, b] \) and \( k \)-tuple central elements \( C_{\Delta_1}, \ldots, C_{\Delta_k} \in Z\mathbb{C}[S_d] \) corresponding to the partitions \( \Delta_1, \ldots, \Delta_k \). Thus, according to the theory of representations of the symmetric group \( S_d \), we arrive at the formula \[16\],

\[
\mu_{g,h}^{h,d}(\Delta_1, \ldots, \Delta_k) = \sum_{\lambda \vdash d} \left( \frac{dim\lambda}{d!} \right)^{2-2g} \phi_{\lambda}(\Delta_1) \cdots \phi_{\lambda}(\Delta_k),
\]

which expresses them through the properly normalized symmetric group characters

\[
\phi_{\lambda}(\Delta) = \frac{1}{dim\lambda} |C_{\Delta}| \chi_{\lambda}(\Delta),
\]

\[4\]
where $\lambda$ is a Young diagram of degree $d$, and $\text{dim}\lambda$ is the dimension of the irreducible representation of the symmetric group $S_d$ corresponding to $\lambda$, $\chi_\lambda(\Delta)$ is the character of a permutation $\sigma \in C_\Delta$ under the irreducible representation $\lambda$, moreover, $|C_\Delta|$ is the number of the permutations of $S_d$ with cyclic type $\Delta$.

Let $p = (p_1, p_2, p_3, \ldots)$ be indeterminantes, which are called the time-variables, and we assume that $\Delta = (\delta_1, \ldots, \delta_n)$ is a partition. We denote

$$l(\Delta) := n,$$
$$m_r(\Delta) := \#\{i | \delta_i = r, \delta_i \in \Delta\},$$
$$\Delta! := \prod_{r \geq 1} m_r(\Delta)!,$$
$$p_\Delta := p_{\delta_1} \cdots p_{\delta_n},$$

and

$$\frac{\partial}{\partial p_\Delta} := \frac{1}{\Delta!} \frac{\partial}{\partial p_{\delta_1}} \cdots \frac{\partial}{\partial p_{\delta_n}}.$$

For any partition $\lambda \vdash d$, it is well known that the symmetric group characters $\phi_\lambda(\Delta)$ are related to the Schur functions $S_\lambda\{p\}$ as following \[13\],

$$S_\lambda\{p\} = \sum_{\Gamma'} \frac{\text{dim}\lambda d!}{|C_{\Delta'}|} \phi_\lambda(\Gamma') p_{\Gamma'}.$$

(10)

Our first important lemma is about the associativity of the generalized Hurwitz number:

**Lemma 2.1** \[15\] For any positive integer $k > l \geq 1$, and $g = g_1 + g_2$, we have

$$\mu_g^{h,d}(\Delta_1, \ldots, \Delta_k) = \sum_{\Delta'} \mu_{g_1}^{h_1,d}(\Delta_1, \ldots, \Delta_l, \Delta') \frac{d!}{|C_{\Delta'}|} \mu_{g_2}^{h_2,d}(\Delta', \Delta_{l+1}, \ldots, \Delta_k).$$

(11)

**Proof** Algebraic proof: by formula(3), we have

$$\text{RHS} = \sum_{\Delta'} \sum_{\lambda} \left( \frac{\text{dim}\lambda}{d!} \right)^{2-2g_1} \phi_\lambda(\Delta_1) \cdots \phi_\lambda(\Delta_l) \phi_\lambda(\Delta') \frac{d!}{|C_{\Delta'}|} \times \sum_{\kappa} \left( \frac{\text{dim}\kappa}{d!} \right)^{2-2g_2} \phi_\kappa(\Delta') \phi_\kappa(\Delta_{l+1}) \cdots \phi_\kappa(\Delta_k).$$

Then the lemma follows from the orthogonal relation of the irreducible characteristic of symmetric group $S_d$:

$$\frac{1}{d!} \sum_{\Delta'} \chi_\lambda(\Delta') |C_{\Delta'}| \chi_\kappa(\Delta') = \delta_{\lambda,\kappa}.$$

(12)
Geometric proof: it is well known [11] that we can interpret the generalized Hurwitz numbers \( \mu_{h,d}^{g}(\Delta_1, \cdots, \Delta_k) \) as the relative Gromov-Witten invariants defined by A.M. Li and Y.B. Ruan [10], and by E. Ionel and T. Parker [8], thus we only need to execute symplectic surgery [9], [10] by cutting the Riemannian surface \( \Sigma_g \) into two parts: one part has the branch point \( P_1, \cdots, P_l \) with genus \( g_1 \), and another part has the rest branch points \( P_{l+1}, \cdots, P_k \) with genus \( g_2 \), moreover each part has one more branch point \( P' \), which corresponds to the infinity end [10], with the ramification type (or tangent multiplicities [10]) \( \Delta' \). Note that \( g = g_1 + g_2 \). Thus we have to consider two relative GW-invariants: \( \mu_{h_1,d}^{g_1}(\Delta_1, \cdots, \Delta_l, \Delta') \) and \( \mu_{h_2,d}^{g_2}(\Delta', \Delta_{l+1}, \cdots, \Delta_k) \), which are nonzero only if

\[
(2 - 2g_1)d - (2 - 2h_1) = \sum_{i=1}^{l} (d - m_i) + (d - l(\Delta')) , \quad (13)
\]

or

\[
(2 - 2g_2)d - (2 - 2h_2) = (d - l(\Delta')) + \sum_{i=l+1}^{k} (d - m_i) , \quad (14)
\]

respectively. The factor \( \frac{d!}{\left| C_{\Delta'} \right|} \) in formula (11) comes from the tangent multiplicities and the automorphisms at the infinity end [10]. Then applying the gluing formula [10], mimicking what we have done in [11], we can derive the above lemma.

3 Genus expanded cut-and-join operators

Observing the symplectic surgery and the gluing formula [LR], we can associate to every partition \( \Delta \vdash d \) a genus expanded cut-and-join operator \( W(\Delta, z) \) as follows:

\[
W(\Delta, z) = \sum_{\Gamma, \Gamma' \vdash d} \frac{d!}{\left| C_{\Gamma'} \right|} z^{d+l(\Gamma')-l(\Delta)-l(\Gamma)} \mu_0^{h_+,d}(\Gamma', \Delta, \Gamma) \partial_{\Gamma'} \partial_{\Gamma} , \quad (15)
\]

where genus \( h_+ \) is determined by the Hurwitz formula:

\[
2d - (2 - 2h_+) = (d - l(\Gamma')) + (d - l(\Delta)) + (d - l(\Gamma)) . \quad (16)
\]

Remark 3.1 The power \( d + l(\Gamma') - l(\Delta) - l(\Gamma) \) of \( z \) in the formula (15) is the “lost” genus after we have executed the symplectic surgery on the
Riemann surface, referring to Remark 4.3, which is the reason for us to define the genus expanded cut-and-join operator like formula (15), rather than the normalized genus expanded cut-and-join operator as formula (24).

Remark 3.2 A. Mironov, A. Morozov, and S. Natanzon, etc, have defined the cut-and-join operators in terms of the matrix Miwa variable \([1], [13], [14]\), but it is easy to check that these two definitions coincide if we take \(z = 1\) and keep the degree \(d\) invariant.

To find out the eigenfunctions of the genus expanded cut-and-join operator \(W(\Delta, z)\), we define the “genus expanded” Schur functions \(S_\lambda\{p, z\}(\lambda \vdash d)\) similar to formula (10) as follows:

\[
S_\lambda\{p, z\} := \sum_{\Gamma'} \frac{z^{-d-\ell(\Gamma')} \dim \lambda}{d!} \phi_\lambda(\Gamma') p_{\Gamma'}.
\]  

First of all, for any \(\Delta, \Delta' \vdash d\), we note that

\[
\frac{\partial}{\partial p_{\Delta}} p_{\Delta'} = \delta_{\Delta, \Delta'}.
\]  

Then by formulas \([15], [17]\), we have

Lemma 3.3 For any partition \(\Gamma' \vdash d\), the following equality holds:

\[
W(\Delta, z)p_{\Gamma'} = \sum_{\Gamma \vdash d} \frac{d!}{C_{\Gamma'}} z^{d+\ell(\Gamma')-\ell(\Delta)-\ell(\Gamma)} \mu_0^{h+}(\Gamma', \Delta, \Gamma)p_{\Gamma}.
\]  

Moreover \(W(\Delta, z)\) have the genus expanded Schur function \(S_\lambda\{p, z\}\) as their eigenfunctions and \(z^{d-\ell(\Delta)}\phi_\lambda(\Delta)\) as the corresponding eigenvalues:

\[
W(\Delta, z)S_\lambda\{p, z\} = z^{d-\ell(\Delta)} \phi_\lambda(\Delta)S_\lambda\{p, z\}.
\]

Proof By straightforward calculation, we omit it.

Example 3.4  (1) In the simplest case \(\Delta = (2) \vdash 2\), we have

\[
W(\Delta, z) = \frac{1}{2} z^2 p_2 \frac{\partial^2}{\partial p_1 \partial p_1} + p_1 p_1 \frac{\partial}{\partial p_2},
\]

which is the standard cut-and-join operator \([3], [2], [2]\).
Corollary 3.6 to the associativity of the generalized Hurwitz numbers as in Lemma 1.1.

Twice; or from algebraic viewpoint, the third and fifth equality are equivalent

From geometric viewpoint, the third and fifth equality imply that we can

which is equivalent to formula (21), and where $h, h_1, h_2, h_3, h', h$ are the genus.

From geometric viewpoint, the third and fifth equalities imply that we can obtain the same results if we execute the symplectic surgery once instead of twice; or from algebraic viewpoint, the third and fifth equality are equivalent to the associativity of the generalized Hurwitz numbers as in Lemma 1.1.

Theorem 3.5 For a given $d$, as operators on the functions of the time-variables $p = (p_1, p_2, \ldots)$, we have

$$W(\Delta_1, z)W(\Delta_2, z) = \sum_{\Delta_3} z^{d-l(\Delta_1)-l(\Delta_2)+l(\Delta_3)} C^{\Delta_3}_{\Delta_1\Delta_2} W(\Delta_3, z),$$

where

$$C^{\Delta_3}_{\Delta_1\Delta_2} = \frac{d!}{|C_{\Delta_3}|} h.d(\Delta_1, \Delta_2, \Delta_3)$$

is the structure constants of the center subalgebra of the group algebra $C[S_d]$.

Proof For any $\Gamma' \vdash d$, by formula (19), we have

$$W(\Delta_1, z) W(\Delta_2, z) p_{\Gamma'} = W(\Delta_1, z) \sum_{\Gamma} \frac{d!}{|C_{\Gamma'}|} z^{d+l(\Gamma')-l(\Delta_2)-l(\Gamma)} h^2, (\Gamma', \Delta_2, \Delta) p_{\Gamma'} \mu_0^h(\Delta_1, \Delta, \Delta) p_{\Delta}$$

which is equivalent to formula (21), and where $h_1, h_2, h_3, h', h$ are the genus.

From geometric viewpoint, the third and fifth equalities imply that we can obtain the same results if we execute the symplectic surgery once instead of twice; or from algebraic viewpoint, the third and fifth equality are equivalent to the associativity of the generalized Hurwitz numbers as in Lemma 1.1.
then for a given $d$, as operators on the space of functions in time-variables $p = (p_1, p_2, \cdots)$, all genus expanded cut-and-join operators $\hat{W}(\Delta, z)$ for $\Delta \vdash d$ form a commutative associative algebra, denoted by $W_d$,

$$\hat{W}(\Delta_1, z)\hat{W}(\Delta_2, z) = \sum_{\Delta_3} C_{\Delta_1\Delta_2}^{\Delta_3} \hat{W}(\Delta_3, z)$$  \hspace{1cm} (25)

i.e., we have an algebraic isomorphism:

$$W_d \cong Z(C[S_d])$$

$$\hat{W}(\Delta, z) \mapsto C_{\Delta}.$$  \hspace{1cm} (26)

Moreover, by formula (20), $\hat{W}(\Delta, z)$ have the genus expanded Schur function $S_{\lambda}\{p, z\}$ as their eigenfunctions and $\phi_{\lambda}(\Delta)$ as the corresponding eigenvalues:

$$\hat{W}(\Delta, z)S_{\lambda}\{p, z\} = \phi_{\lambda}(\Delta)S_{\lambda}\{p, z\}.$$  \hspace{1cm} (27)

Proof By straightforward calculation, we omit it.

4 Generating functions and its differential equations

For any given genus $g \geq 0$, any given degree $d$ and partitions $\Delta_1, \cdots, \Delta_n \vdash d$, to apply the genus expanded cut-and-join operators (15), we define the generating function as

$$\Phi_g}\{z|(u_1, \Delta_1), \cdots, (u_n, \Delta_n)|p^{(1)}, \cdots, p^{(k)}, p\}$$

$$= \sum_{l_1, \cdots, l_n \geq 0} \sum_{\Gamma, \Gamma_1, \cdots, \Gamma_k} z^{2h - 2g} \mu_d^{k, d}(\Delta_1, \cdots, \Delta_n, \cdots, \Delta_n, \Gamma_1, \cdots, \Gamma_k)\prod_{j=1}^{n} \frac{u_{l_j}}{l_j!}\prod_{i=1}^{k} \frac{p_{l_i}}{l_i!}\Gamma$$

$$= \sum_{l_1, \cdots, l_n \geq 0} \sum_{\Gamma, \Gamma_1, \cdots, \Gamma_k} z^{2h - 2g} \frac{\dim \lambda}{(-d)!^{2-2g}} \prod_{j=1}^{n} (\phi_{\lambda}(\Delta_j))^{l_j} \prod_{i=1}^{k} \phi_{\lambda}(\Gamma_i)\prod_{i=1}^{k} \phi_{\lambda}(\Gamma_i)p_{l_i}^{(i)}\Gamma,$$  \hspace{1cm} (28)

where $z, u_1, \cdots, u_n$ are indeterminate variables, $p, p^{(1)}, \cdots, p^{(k)}$ are time-variables, and $2h - 2$ is determined by the Hurwitz formula:

$$(2 - 2g)d - (2 - 2h) = \sum_{j=1}^{n} l_j(d - l(\Delta_j)) + \sum_{j=1}^{k} (d - l(\Gamma_j)) + (d - l(\Gamma)).$$  \hspace{1cm} (29)

Moreover, we have some special initial values:
\[
\Phi_0\{z||p\} = \sum_\lambda \sum_\Delta z^{-d-l(\Delta)} \left( \frac{\dim \lambda}{d!} \right)^2 \phi(\Delta)p_{\Delta}
\]
\[
= \sum_\lambda \frac{\dim \lambda}{d!} S_\lambda\{p, z\}
\]
\[
= z^{-2d} p_{1!}^d. \tag{30}
\]
\[
\Phi_0\{z||p^{(1)}, p\} = \sum_\lambda \sum_{\Delta_1, \Delta_2} z^{-l(\Delta_1)-l(\Delta_2)} \left( \frac{\dim \lambda}{d!} \right)^2 \phi(\Delta_1)\phi(\Delta_2)p^{(1)}_{\Delta_1}p_{\Delta_2}
\]
\[
= \sum_\lambda z^{2d} S_\lambda\{p^{(1)}, z\} S_\lambda\{p, z\}
\]
\[
= \sum_\Delta z^{-2l(\Delta)} \left( \frac{|C_\Delta|}{d!} \right) p^{(1)}_{\Delta} p_{\Delta}. \tag{31}
\]

Remark 4.1 The formula (30) can be regarded as the genus and degree expansion of hook formula \([12],[13]\) and the formula (31) as the genus and degree expansion of the Cauchy-Littlewood identity \([12],[13]\). Moreover one side of both of the formulas is algebraic expression of Hurwitz numbers and the another side is geometric expression of the Hurwitz numbers.

Theorem 4.2 For any \(i\), we have
\[
\frac{\partial \Phi_g\{z|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}}{\partial u_i}
\]
\[
= W(\Delta_i, z) \Phi_g\{z|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}. \tag{32}
\]

Proof Obviously, we have
\[
\frac{\partial \Phi_g\{z|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}}{\partial u_i}
\]
\[
= \sum_{l_1, \ldots, l_n \geq 0} \mu_g^{(h,d)}(\underbrace{\Delta_1, \ldots, \Delta_1}_{l_1}, \ldots, \underbrace{\Delta_n, \ldots, \Delta_n}_{l_n}, \underbrace{\Gamma_1, \ldots, \Gamma_k, \Gamma}_{l_n}) \frac{(u_i)^{l_i-1}}{(l_i - 1)!} \prod_{j=1, j \neq i}^n \frac{(u_j)^{l_j}}{l_j!} \prod_{j=1}^k p_{\Gamma_j}^{(j)} p_{\Gamma}. \]

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We can write RHS of equation (32) as
\[
\text{RHS} = \sum_{l_1 \geq 1, \ldots, l_n \geq 0} \sum_{\Gamma_1, \ldots, \Gamma_k, \Gamma'} z^{2h^i - 2} \\
\times \mu_g^{h^i, d}(\Delta_1, \ldots, \Delta_1, \ldots, \Delta_i, \ldots, \Delta_i, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_k, \Gamma') \\
\times \left( \frac{(u_j)^{l_{i-1}}}{(l_i - 1)!} \prod_{j=1, j \neq i}^n \frac{(u_j)^{l_j}}{l_j!} \prod_{j=1}^k p_{\Gamma_j}^{(j)} \right) W(\Delta_i, z)p_{\Gamma'},
\]
where \( \hat{l}_i \) means that we omit \( l_i \), and \( 2h^i - 2 \) is also determined by the Hurwitz formula:
\[
(2 - 2g)d - (2 - 2h^-) = \sum_{j=1}^n l_j(d - l(\Delta_j)) - (d - l(\Gamma)) + \sum_{j=1}^k (d - l(\Gamma'_j)) + (d - l(\Gamma')) - 2
\]
(33)

Moreover we have the following facts:

- **Fact (1):** By lemma 2.1,
  \[
  W(\Delta_i, z)p_{\Gamma'} = \frac{d!}{|\Gamma'|} \sum_{\Gamma} z^{d+l(\Gamma')-l(\Delta_i)-l(\Gamma')} \mu_0^{h^i, d}(\Gamma', \Delta_i, \Gamma)p_{\Gamma'};
  \]

- **Fact (2):** \( \mu_0^{h^i, d}(\Gamma', \Delta_i, \Gamma) \neq 0 \) only if
  \[
  2d - (2 - 2h^-) = (d - l(\Gamma')) + (d - l(\Gamma)) + (d - l(\Gamma'));
  \]
(34)

- **Fact (3):** by the formula (29), (33), (34), we have
  \[
  h = h^i + h^\perp + l(\Gamma') - 1.
  \]

Then the proposition follows from Lemma 1.1.

**Remark 4.3** We note an interesting phenomenon in the above proof, i.e., we “lost” the genus after we have executed the symplectic cutting:
\[
(2h - 2) - (2h^\perp - 2) = (2h^\perp - 2) + 2l(\Gamma') = d + l(\Gamma') - l(\Delta_i) - l(\Gamma),
\]
(35)

which is the key observation to define the genus expanded cut-and-join operators.

Immediately, we have
Corollary 4.4

\[ \Phi_g \{ z \mid (u_1, \Delta_1), \ldots, (u_n, \Delta_n) \} p^{(1)}, \ldots, p^{(k)} \} = \left[ \prod_{i=1}^{n} \exp(u_i W(\Delta_i, z)) \right] \Phi_g \{ z \mid p^{(1)}, \ldots, p^{(k)} \} \]

In particular, if we take \( g = 0, k = 0, 1 \), we have

\[ \Phi_0 \{ z \mid (u_1, \Delta_1), \ldots, (u_n, \Delta_n) \} p \} = \left[ \prod_{i=1}^{n} \exp(u_i W(\Delta_i, z)) \right] (z^{2d} \prod_{i} \partial_{p_i}^2); \] (36)

\[ \Phi_0 \{ z \mid (u_1, \Delta_1), \ldots, (u_n, \Delta_n) \} q, p \} = \left[ \prod_{i=1}^{n} \exp(u_i W(\Delta_i, z)) \right] \left( \sum_{\Delta} z^{-2l(\Delta)} \frac{|C_{\Delta}|}{d!} q_{\Delta} p_{\Delta} \right). \]

Moreover, if we take \( \Delta = (2, 1, \cdots, 1) \vdash d \), then we obtain the generating functions for the classical almost simply Hurwitz numbers and the almost simply double Hurwitz numbers [4], [5], [6], [11], [16]:

\[ \Phi_0 \{ z \mid (u, \Delta) \} p \} = \exp(u W((2, 1, \cdots, 1), z)) (z^{2d} \prod_{i} \partial_{p_i}^2); \] (37)

\[ \Phi_0 \{ z \mid (u, \Delta) \} q, p \} = \exp(u W((2, 1, \cdots, 1), z)) \left( \sum_{\Delta} z^{-2l(\Delta)} \frac{|C_{\Delta}|}{d!} q_{\Delta} p_{\Delta} \right). \] (38)

Example 4.5 Assume \( d = 3 \) and \( \Delta = (2, 1) \), then we have

\[ W(\Delta, z) = z^2 \left[ 2p_1 \frac{\partial^2}{\partial p_1 \partial p_2} + \frac{1}{2} p_1 p_2 \frac{\partial^3}{\partial p_1^2 \partial p_1 \partial p_2} + 3p_1 p_2 \frac{\partial^3}{\partial p_1 \partial p_2 \partial p_3} \right]. \]

Thus we get the generating function(with any genus) from (37) for \( d = 3 \) (which also can be checked by direct calculation):

\[ \Phi_0 \{ z \mid (u, \Delta) \} \} = \frac{1}{6} p_1^3 z^{-6} + \frac{1}{2} u p_1 p_2 z^{-4} + \frac{1}{2} u^2 p_3 z^{-4} + \frac{1}{4} u^2 p_1^3 z^{-4} + \frac{3}{4} u^3 p_1 p_2 z^{-2} + \cdots, \]

where we omit the higher terms of \( u \).

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References

[1] A. Alexandrov, A. Mironov, A. Morozov, and S. Natanzon, On KP-integrable Hurwitz functions, J. High Energy Phys. 2014, no. 11, 080.

[2] X.-M. Ding, Y.P. Li, and L.X. Meng, From r-Spin Intersection Numbers to Hodge Integrals, \url{arXiv:1507.04093}.

[3] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, Hurwitz Numbers and Intersections on Moduli Spaces of Curves, Invent. Math. 146 (2001) 297–327.

[4] I.P. Goulden and D.M. Jackson, A proof of a conjecture for the number of ramified covering of the sphere by the torus, J. Combin. Theory Ser. A 88 (1999) 246–258.

[5] I.P. Goulden and D.M. Jackson, The number of ramified covering of the sphere by the double torus, and a general form for higher genera, J. Combin. Theory Ser. A 88 (1999) 259–275.

[6] I.P. Goulden and D.M. Jackson, Transitive factorisations into transpositions and holomorphic mapping on the sphere, Proc. Amer. Math. Soc. 125 (1997) 51–60.

[7] A. Hurwitz, Ueber Riemann’sche Flächen mit gegebenen Verzweigungspunkten, Math. Ann. 39 (1891) 1–60.

[8] E. Ionel and T. Parker, Gromov-Witten invariants of symplectic sums, Math. Res. Lett. 5 (1998) 563–576.

[9] E. Lerman, Symplectic cuts, Math. Res. Lett. 2 (1995) 247–258.

[10] A.M. Li and Y.B. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145 (2001) 151–218.

[11] A.M. Li, G.S. Zhao, and Q. Zheng, The number of ramified covering of a Riemann surface by Riemann surface, Comm. Math. Phys. 213 (2000) 685–696.
[12] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[13] A. Mironov, A. Morozov, and S. Natanzon, Complete Set of Cut-and-Join Operators in Hurwitz-Kontsevich Theory, arXiv:0904.4227.

[14] A. Mironov, A. Morozov, and S. Natanzon, Universal Algebras of Hurwitz Numbers, arXiv:0909.1164.

[15] A. Mironov, A. Morozov, and S. Natanzon, Algebra of differential operators associated with Young diagrams, J. Geom. Phys. 62 (2012) 148–155.

[16] A. Okounkov and R. Pandharipande, Gromov-Witten theory, Hurwitz theory, and completed cycles, Ann. of Math. 163 (2006) 517–560.