Holographic superconductor developed in BTZ black hole background with backreactions

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Abstract

We develop a holographic superconductor in BTZ black hole background with backreactions. We investigate the influence of the backreaction on the condensation of the scalar hair and the dynamics of perturbation in the background spacetime. When the Breitenlohner-Freedman bound is approached, we argue that only one of two possible operators can reflect the real property of the condensation in the holographic superconductor. This argument is supported by the investigation in dynamics.

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I. INTRODUCTION

Inspired by the anti-de Sitter/conformal field theories (AdS/CFT) correspondence, it has been argued that there exists remarkable connection between the gravitational physics and the condensed matter physics (for reviews, see Refs. [1–3]). It was shown that when the temperature of the black hole drops below the critical value, the bulk AdS black hole becomes unstable and scalar hair condenses on the black hole background. The emergence of the scalar hair in the bulk AdS black hole corresponds to the formation of a charged condensation in the boundary dual CFTs. Due to the potential applications to the condensed matter physics, the condensation in the AdS black hole background has been investigated extensively, see for example [4]–[24]. A lot of studies have been done in the probe approximation. Recently attempts to study the holographic superconductor away from the probe limit by considering the backreaction have been carried out in [25]–[36].

Most holographic superconductors were constructed in (2+1)-dimensions or higher-dimensions. Recently the (1+1)-dimensional holographic superconductor was constructed by using the $\text{AdS}_3/\text{CFT}_2$ correspondence [37]. The Banados-Teitelboim-Zanelli (BTZ) black hole was taken as the gravity background in the construction. This background is interesting, since in this background the AdS/CFT correspondence was first quantitatively proved by finding the quasinormal frequencies of the perturbation in the bulk spacetime in coincidence with the poles of the correlation function in the dual CFT [38]. In [37] distinctive features in both normal and superconducting phases in the (1+1)-dimensional holographic superconductor were disclosed in the probe limit. Since in the $\text{AdS}_3$ spacetime, the full current-current correlation function is analytically solvable at finite temperature and zero chemical potential, the frequency and momentum dependence of the conductivity which describes the linear response to both temporally and spatially oscillating electric field was obtained. It is of interest to generalize the study of the (1+1)-dimensional holographic superconductor away from the probe limit by considering the backreaction. This will be the first task in our work. Furthermore we will study the dynamical properties of the perturbation of charged scalar field in the BTZ background, which can help us further understand the constructed (1+1)-dimensional holographic superconductor.

In studying the holographic superconductor, one has to analyze the asymptotic behavior of a scalar field of mass $m$ in bulk AdS spacetime with the form $\psi \sim \psi_{-} + \psi_{+}$, where $\lambda_{\pm} = (d \pm \sqrt{d^2 + 4m^2})/2$, $d$ is the dimension of the space and $l$ the AdS radius. Usually, normalizability requires that the leading coefficient in $\psi$ must vanish. However, since we have chosen a mass close to the Breitenlohner-Freedman (BF) bound, even...
the leading term in $\psi$ is normalizable. In this case, one has a choice to consider solutions either with $\psi_- = 0$ or $\psi_+ = 0$. In the dual theory, the operator charged under the U(1) is dual to $\psi$. When $m^2$ is close to the BF bound, there are two possible operators depending on how one quantizes $\psi$ in the bulk. Depending on the choice of boundary conditions, we can read off the expectation value of an operator $\langle O_- \rangle \sim \psi_-$, or of an operator $\langle O_+ \rangle \sim \psi_+$. In order to apply the formalism in gravity to study the real condensed matter physics, one may ask which one of the two possible operators can really reflect the properties of the real condensation. In [39], it was found that both operators condensate qualitatively similar to that obtained in BCS theory as observed in many materials when the temperature drops under a critical value, but one of them appears divergence at very low temperature. This divergence was expected to be cured by considering the backreaction of the scalar field on the bulk metric. In [19, 20], it was argued that the condensation gap indicated in the operator marks the ease of the scalar hair to be formed in the AdS black hole background. Whether the easiness of the formation of scalar hair can be reflected consistently in two operators is a question to be asked. In the study of the Gauss-Bonnet effect on the condensation in the probe limit, it was found that two operators cannot reflect the consistent behavior in the condensation influenced by the Gauss-Bonnet factor except that we consider the direct signature of Gauss-Bonnet factor in the scalar mass by selecting the value $m^2 l^2_{eff}$ instead of $m^2 l^2$. It is of interest to examine whether these two operators can reflect the consistent behavior of condensation when the backreaction is taken into account. In this work, we will concentrate our attention on (1+1)-dimensional holographic superconductor and generalize our discussion to higher dimensions.

The outline of this work is as follows. In section II, we discuss the fully backreacted one-dimensional holographic superconductor and investigate the dependence of condensate on the backreaction. In section III, we study the dynamics of the backreacted BTZ black hole. We will conclude in the last section of our main results.

II. BTZ HOLOGRAPHIC SUPERCONDUCTOR

The general action describing a charged, complex scalar field in the 3-dimensional Einstein-Maxwell action with negative cosmological constant reads [37]

$$S = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-g} (R + \frac{2}{l^2}) + \int d^3 x \sqrt{-g} \left[ -\frac{1}{4} F^{ab} F_{ab} - |\nabla \psi - iq A \psi|^2 - m^2 |\psi|^2 \right], \quad (1)$$
where \( \kappa \) is the three dimensional gravitational constant \( \kappa^2 = 8\pi G_3 \), and \( G_3 \) is the (2+1)-dimensional Newton constant, \( g \) is the determinant of the metric, \( l \) is the AdS radius, \( q \) and \( m \) represent the charge and the mass of the scalar field respectively. In order to consider the effect of the backreaction of the holographic superconductor, we take a metric ansatz as follows

\[
ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2}dx^2.
\]  

(2)

The electromagnetic field and the scalar field can be chosen as

\[
A_t = \phi(r)dt, \quad \psi = \psi(r),
\]

(3)

where \( \psi(r) \) can be taken to be real without loss of generality. The Hawking temperature of this black hole, which will be interpreted as the temperature of the CFT, can be expressed as

\[
T = \frac{f'(r)e^{-\chi(r)/2}}{4\pi} \bigg|_{r=r_h}.
\]

(4)

Considering the ansatz of the metric, the equations of motion can be easily obtained

\[
0 = \psi''(r) + \psi'(r) \left[ \frac{1}{r} + \frac{f'(r)}{f(r)} - \frac{\chi'(r)}{2} \right] + \psi(r) \left[ \frac{q^2 \phi(r)^2 e^{\chi(r)}}{f(r)^2} - \frac{m^2}{f(r)} \right],
\]

\[
0 = \phi''(r) + \phi'(r) \left[ \frac{1}{r} + \frac{\chi'(r)}{2} \right] - \frac{2q^2 \phi(r) \psi(r)^2}{f(r)},
\]

\[
0 = f'(r) + 2\kappa^2 r \left[ \frac{q^2 \phi(r)^2 \psi(r)^2 e^{\chi(r)}}{f(r)} + f(r)\psi'(r)^2 + m^2 \psi(r)^2 + \frac{1}{2} e^{\chi(r)} \phi'(r)^2 \right] - \frac{2r}{l^2},
\]

\[
0 = \chi'(r) + 4\kappa^2 r \left[ \frac{q^2 \phi(r)^2 \psi(r)^2 e^{\chi(r)}}{f(r)^2} + \psi'(r)^2 \right],
\]

(5)

where the prime denotes the derivative with respect to \( r \). There are three useful scaling symmetries to be adopted in the above equations as in [19]

\[
\phi \rightarrow \phi a, \quad \psi \rightarrow \psi a, \quad \kappa^2 \rightarrow \kappa^2 a^{-2}, \quad q \rightarrow qa^{-1},
\]

(6)

\[
r \rightarrow rb, \quad f \rightarrow fb^2, \quad \phi \rightarrow \phi b,
\]

(7)

\[
r \rightarrow rc, \quad l \rightarrow lc, \quad q \rightarrow qc^{-1}, \quad m \rightarrow mc^{-1}.
\]

(8)

We can use the symmetry [3] to set \( q = 1 \), and the symmetries [7], [8] to set \( r_h = 1 \) and \( l = 1 \).

We will use the shooting method to solve the equations of motion numerically with appropriate boundary conditions. There are two different boundaries we need to consider. At the black hole horizon \( r_h \) which is the root of \( f(r_h) = 0 \), the solutions of the gauge and the scalar fields have to be regular

\[
\phi(r_h) = 0, \quad \psi'(r_h) = \frac{m^2}{f'(r_h)} \psi(r_h),
\]

(9)
and the coefficients in the metric ansatz obey

\begin{align*}
    f'(r_h) &= \frac{2r_h}{l^2} - 2\kappa^2 r_h \left[ m^2 \psi(r_h)^2 + \frac{1}{2} \epsilon \chi(r_h) \phi'(r_h)^2 \right], \\
    \chi'(r_h) &= -4\kappa^2 r_h \left[ q^2 \phi'(r_h)^2 \psi(r_h)^2 \epsilon \chi(r_h) \phi'(r_h)^2 + \psi'(r_h)^2 \right].
\end{align*}

(10)

At the spatial infinity, the asymptotic behaviors of the solutions are

\begin{align*}
    \chi &\to 0, \quad f(r) \sim \frac{r^2}{l^2}, \\
    \phi(r) &\sim \rho + \mu \ln(r), \quad \psi(r) \sim \frac{\psi_-}{r^{\lambda_-}} + \frac{\psi_+}{r^{\lambda_+}},
\end{align*}

(11)

where \( \lambda_{\pm} = 1 \pm \sqrt{1 + l^2 m^2} \). \( \mu \) is the chemical potential. When \( \lambda_+ - \lambda_- = 2n \) (\( n = 1, 2, \ldots \)), there appears a logarithmic term in the asymptotic behavior at infinity. For \(-1 \leq m^2 < 0\), both of these falloffs in \( \psi(r) \) are normalizable, so one can impose the boundary condition that either one vanishes. After imposing the condition that either \( \psi_- \) or \( \psi_+ \) vanishes we have a one parameter family of solutions. For \( m^2 \geq 0 \), we can only relate the scalar operator in the field theory dual to the branch \( \psi_+ \) to describe the condensation.

In order to find the effect of backreaction on the scalar condensation, we need to count on numerical calculations. In the following we list our numerical results in solving Eq. (5) for different strength of the backreaction \( \kappa^2 \) with \( m^2 = 0 \) and \( m^2 = -1 \) respectively.

When choosing the mass of the scalar field \( m^2 = 0 \), the asymptotic behavior of the scalar field at infinity takes the form

\begin{equation}
    \psi(r) = \psi_- + \frac{\psi_+}{r^2}.
\end{equation}

(12)

Only \( \psi_+ \) can be chosen dual to the scalar operator in the field theory to describe the condensate. In Table 1, we list the critical temperature for the operator starts to condense for different strength of the backreaction. We found that with the increase of the backreaction, the critical temperature decreases. Thus the effect of the backreaction is to make it harder for scalar hair to form. This property can also be seen from the condensation as shown in Fig. 1. The condensation of the operator \( \langle O_+ \rangle \) can start when gap becomes higher for the stronger backreaction, which means that the scalar hair can be formed more difficult when the backreaction is stronger. We can fit these data near the critical point and find that \( \langle O_+ \rangle \sim (T_c - T)^{1/2} \) as expected from mean field theory. The exponent 1/2 implies that the phase transition is of the second order. The order of the phase transition is not changed when the backreaction is taken into account. When the temperature tends to zero, the condensate tends to a finite value, which is qualitatively similar to that observed in the BCS theory.
TABLE I: The dependence of the critical temperature $T_c$ on the backreaction $\kappa^2$ for $m^2 = 0$. Obviously, the critical temperature decreases as the backreaction grows.

| $\kappa^2$ | 0   | 0.05 | 0.1  | 0.15 | 0.2  |
|------------|-----|------|------|------|------|
| $T_c/\mu$  | 0.0460 | 0.0368 | 0.0295 | 0.0236 | 0.0189 |

FIG. 1: Plot of condensate as a function of temperature with $m^2 = 0$. Lines from bottom to top correspond to $\kappa^2 = 0, 0.05, 0.1, 0.15$ and $0.2$ respectively.

When the mass of the scalar field is chosen near the BF bound (we show for example $m^2 = -1$ below), the asymptotic behavior of the scalar field exhibits

$$\psi(r) = \frac{\psi_-}{r} \ln(r) + \frac{\psi_+}{r}.$$ (13)

We have a choice to consider solutions either with $\psi_+ = 0$ or $\psi_- = 0$.

TABLE II: The dependence of the critical temperature $T_c$ on the backreaction $\kappa^2$ for $m^2 = -1$ with $\psi_- = 0$. Obviously, the critical temperature decreases as the backreaction grows.

| $\kappa^2$ | 0   | 0.05 | 0.1  | 0.15 | 0.2  |
|------------|-----|------|------|------|------|
| $T_c/\mu$  | 0.136 | 0.133 | 0.131 | 0.128 | 0.126 |

The dependances of the critical temperature on the backreaction for the operators start to condense in the field theory are shown in Tables II and III for choosing $\psi_- = 0$ and $\psi_+ = 0$, respectively. Combining with Table III, we find that no matter how one quantizes $\psi$ in the bulk, the critical temperatures consistently drop when the backreaction grows, which shows that both operators condense more difficult when the backreaction becomes stronger. This property will not be altered when we change the mass of the scalar field.

In Fig. 2, we showed the condensations by plotting the expectation values of operators $\langle O_- \rangle \sim \psi_-$ and $\langle O_+ \rangle \sim \psi_+$. We find that the operator $\langle O_- \rangle$ can reflect the condensation when the backreaction is taken
TABLE III: The dependence of the critical temperature $T_c$ on the backreaction $\kappa^2$ for $m^2 = -1$ with $\psi_+ = 0$. Obviously, the critical temperature decreases as the backreaction grows.

| $\kappa^2$ | 0   | 0.05 | 0.1  | 0.15 | 0.2  |
|-----------|-----|------|------|------|------|
| $T_c/\mu$ | 0.050 | 0.043 | 0.038 | 0.034 | 0.030 |

into account. The gap of the condensation becomes higher when the backreaction becomes stronger, which indicates the consistent picture shown in $T_c$ that the backreaction makes the condensation to be formed harder. When the BF bound is approached, the operator $\langle \mathcal{O}_+ \rangle$ shows drastically different behavior from that exhibited in Fig.1. In the probe approximation, the operator appears divergent at very low temperature as observed in higher dimensions [25]. This divergence can be cured by considering the backreaction as expected. Furthermore, when the BF bound is approached, the backreaction effect makes the gap of condensation lower in the operator $\langle \mathcal{O}_+ \rangle$, which shows different condensation behavior as illustrated in Fig. 1. This result disagrees with the behavior exhibited in the critical temperature and cannot correctly reflect the ease of the scalar hair to be formed in the AdS black hole background when the backreaction is considered. This phenomenon does not only appear in (1+1)-dimensional superconductor. Actually different effects due to the backreaction on condensations were also shown in two different operators in (2+1)-dimensional superconductors, see Fig. 1 in [25]. When the BF bound is approached, one branch in the asymptotic behavior of the scalar field behaves as $\sim 1/r$ at the spatial infinity. If we relate the expectation value of the operator to this branch, like $\langle \mathcal{O}_+ \rangle$ above and $\langle \mathcal{O}_1 \rangle$ in [25], in addition to the divergence appears at very low temperature in the operator in the probe limit, the condensation gap indicated in the operator cannot reflect the correct extent of the scalar hair to be formed. This shows that not both of the operators can reflect correctly of the condensation. We need to discard one of them when we are close to the BF bound. In the following section, we will further show this result from the study of the dynamical properties.

FIG. 2: Plot of the condensates as functions of temperature for $m^2 = -1$ with $\psi_+ = 0$ (the left panel) and $\psi_- = 0$ (the right panel), respectively. In the left panel five lines from bottom to top correspond to $\kappa^2 = 0 , 0.05 , 0.1 , 0.15 , 0.2$ respectively, but lines in the right panel show different order of strength of the backreaction.
III. DYNAMICS OF THE BACKREACTION BTZ BLACK HOLE

In this part, we begin to investigate the dynamics of a backreacted BTZ black hole when the black hole approaches marginally stable from the high temperature. We will consider the minimally coupled, charged scalar perturbation, \( \psi, q(r) e^{-i(\omega t + kx)} \), with mass \( m \) obeying the wave equation

\[
0 = \psi''(r) + \psi'(r) \left[ \frac{1}{r} + \frac{f'(r)}{f(r)} \right] + \psi(r) \left\{ \frac{[\omega + q\phi(r)]^2 e^{\chi(r)}}{f(r)^2} - \frac{m^2}{f(r)} - \frac{l^2 k^2}{f(r)r^2} \right\},
\]

where the prime denotes derivative with respect to \( r \). The analytical solutions of the above equations are expressed as

\[
\chi(r) = C, \quad \phi(r) = \mu \ln \left( \frac{r}{r_h} \right), \quad f(r) = -\kappa^2 e^{\chi(r)} \mu^2 \ln \left( \frac{r}{r_h} \right) + \frac{r^2 - r_h^2}{l^2},
\]

where \( C \) is an integral constant.

In fact, we can use three similar scaling symmetries as expressed in Eqs. (6) - (8) to set \( q = 1, r_h = 1 \) and \( l = 1 \) as well. The difference is that we have to use the following symmetry to replace (7)

\[
r \rightarrow rb, f \rightarrow fb^2, \phi \rightarrow \phi b, \omega \rightarrow \omega b^{-1}, k \rightarrow kb^{-1}.
\]

Eq. (14) can be rewritten as

\[
0 = \psi''(u) + \psi'(u) \left[ \frac{1}{u} + \frac{g'(u)}{g(u)} \right] + \psi(u) \left\{ \frac{[\varpi + \Pi(u)]^2 u^{2} e^{\chi(u)}}{g(u)^2} - \frac{m^2}{g(u)} - \frac{l^2 q^2}{g(u)u^2} \right\},
\]

with

\[
g(u) = -\kappa^2 \sigma^2 \ln(u) + \frac{u^2 - 1}{l^2},
\]

\[
\sigma = \frac{\mu}{r_h},
\]

\[
\varpi = \frac{\omega}{r_h},
\]

\[
q = \frac{k}{r_h},
\]

\[
\Pi(u) = q \sigma \ln(u),
\]

where \( u = r/r_h \) and the prime here denotes the derivative with respect to \( u \). Now \( u = \infty \) is the AdS boundary and \( u = 1 \) is the location of the horizon. The Hawking temperature of the black hole is

\[
T = \frac{2 - \kappa^2 \sigma^2}{4 \pi \sigma \mu}.
\]
Near the AdS boundary \( u \sim \infty \), Eq. (20) becomes
\[
\psi_{\mp, q}''(u) + \frac{3}{u} \psi_{\mp, q}'(u) - \frac{m^2}{u^2} \psi_{\mp, q}(u) = 0 ,
\]
and \( \psi_{\mp, q}(u) \) has a fall-off behavior as
\[
\psi_{\mp, q}(u) \sim \frac{\psi_{\mp, q}}{u^{\lambda_-}} + \frac{\psi_{\mp, q}}{u^{\lambda_+}},
\]
where \( \lambda_{\pm} = 1 \pm \sqrt{1 + m^2 l^2} \) are characteristic exponents of the perturbation equation.

Following the AdS/CFT dictionary, the order parameter expectation value \( \langle \mathcal{O}_{\mp, q} \rangle \) corresponds to \( \psi^+_{\mp, q} \) while the source term is \( \psi^-_{\mp, q} \), so the response function can be defined by
\[
\chi_{\mp, q} := \frac{\delta \langle \mathcal{O}_{\mp, q} \rangle}{\delta \psi_{\mp, q}} \bigg|_{\psi_{\mp, q} \rightarrow 0} \propto \frac{\psi^+_{\mp, q}}{\psi^-_{\mp, q}}.
\]
Also we can choose the other way round, the expectation value \( \langle \mathcal{O}_{\mp, q} \rangle \) corresponds to \( \psi^-_{\mp, q} \) while the source term is \( \psi^+_{\mp, q} \), so that \( \chi_{\pm, q} \propto \frac{\psi^-_{\mp, q}}{\psi^+_{\mp, q}} \).

Near the horizon \( u \sim 1 \), we should impose the incoming wave boundary condition
\[
\psi_{\mp, q}(u) \sim (u - 1)^{-i \pi r_H}.
\]
Introducing a new variable \( \varphi \) as \( \psi_{\mp, q}(u) = \Re(u) \varphi_{\mp, q}(u) \) and choosing \( \Re(u) = \exp[-i \int_1^u \frac{\pi + i \mathcal{N}(u)}{g(u)}] \), which asymptotically approaches Eq. (23) at the horizon, one can express the boundary condition at the horizon as \( \varphi_{\mp, q} \bigg|_{u=1} = \text{const.} \), and Eq. (20) becomes
\[
0 = \varphi''_{\mp, q}(u) + B_1(u) \varphi'_{\mp, q}(u) + B_2(u) \varphi_{\mp, q}(u) ,
\]
with
\[
B_1(u) = \frac{g'(u)}{g(u)} + \frac{2i[\mathcal{N}(u) + \overline{\mathcal{N}}]}{g(u)} + \frac{1}{u} , \quad B_2(u) = -\frac{m^2}{g(u)} - \frac{l^2 q^2}{u^2 g(u)} - \frac{i[\mathcal{N}(u) + \overline{\mathcal{N}} + u \mathcal{N}'(u)]}{g(u)} .
\]

Near the AdS boundary \( u \sim \infty \), \( \varphi_{\mp, q} \) behaves as
\[
\varphi_{\mp, q}(u) \sim \frac{\varphi^-_{\mp, q}}{u^{\lambda_-}} + \frac{\varphi^+_{\mp, q}}{u^{\lambda_+}}.
\]
The boundary conditions at the horizon are now given by
\[
\varphi_{\mp, q} \bigg|_{u=1} = 1 , \quad \varphi'_{\mp, q} \bigg|_{u=1} = -\frac{B_2(u)}{B_1(u)} \bigg|_{u=1} .
\]
Eq. (27) is a linear equation and $\varphi_{\omega, q}(u)$ must be regular at the horizon. Since we do not concentrate on the amplitude of $\varphi_{\omega, q}(u)$, we can set $\varphi_{\omega, q}|_{u=1} = 1$.

Eq. (27) has to be solved numerically under the boundary conditions (30). We will examine the behavior of the charged scalar field perturbation which can present us an objective picture on how the black hole approaches the marginally stable mode when the temperature drops and the backreaction becomes stronger. Without loss of generality, hereafter we will consider $m^2 l^2 = -1$ as an explicit example in our calculation.

The dimensionless parameter $\sigma$ determines the phase structure and its critical value $\sigma_c$ can be calculated numerically. When $\kappa^2 = 0$, the background configuration goes back to the probe limit and the corresponding critical temperature $T_c$ should be consistent with the result obtained in [37].

Taking $m^2 l^2 = -1$, near the AdS boundary, $\varphi_{\omega, q}(u)$ behaves as

$$\varphi_{0, 0}(u) \sim \frac{\varphi^-_{0, 0}}{u} \ln(u) + \frac{\varphi^+_{0, 0}}{u}.$$  \hspace{1cm} (31)

Choosing $\varphi^-_{0, 0} = 0$, we get the critical point $\sigma_c$ and the critical temperature $T_c$ for $m^2 = -1$ with different values of the backreaction $\kappa^2$, i.e., $\kappa^2 = 0, 0.05, 0.1, 0.15$ and 0.2, which have been presented in Table III. It is clear that the results in Table IV are consistent with those in Table II.

| $\kappa^2$ | $0$ | $0.05$ | $0.1$ | $0.15$ | $0.2$ |
|------------|-----|-------|------|------|-----|
| $\sigma_c/\mu$ | 3.165 | 2.903 | 2.688 | 2.508 | 2.357 |
| $T_c/\mu$ | 0.050 | 0.043 | 0.038 | 0.034 | 0.030 |

Now we report the influence of the backreaction on the scalar perturbation behavior. We concentrate on the lowest quasinormal frequency which gives the relaxation time [41, 42]. We can obtain the quasinormal frequencies by solving Eq. (27) based on the boundary conditions (30) at the horizon and $\varphi^-_{\omega, q=0} = 0$ at the AdS boundary. We deviate $\sigma$ away from the critical value $\sigma_c$ and denote the deviation by $\varepsilon_\sigma, \varepsilon_\sigma = 1 - \sigma/\sigma_c = 10^{-5} n, n = 1, 2, \cdots, 20$. We see that all the imaginary parts of the quasinormal frequencies are negative, which shows that the black hole spacetime is stable. For the larger backreaction effect, the imaginary part of the lowest quasinormal frequency has larger deviation from zero. This implies that the stronger backreaction can ensure the system to be more stable and can slow down the process to make the high temperature black hole phase become marginally stable. With the decrease of the black hole temperature, we see that the lowest quasinormal frequency approaches the origin and vanishes when the temperature of the system reaches the
critical value, which indicates that the system approaches marginally stable. We show the object picture in Fig. 3 The lowest quasinormal frequency approaches the origin with equal spacing and we fit the results for different strength of the backreaction in polynomials as below

\[
\begin{align*}
\kappa^2 &= 0, \quad \omega_{QN} \sim (1.58 - 1.19i) \times 10^{-13} + (0.803 - 0.489i) \varepsilon_\sigma - (0.48 + 0.29i) \varepsilon_\sigma^2, \\
\kappa^2 &= 0.05, \quad \omega_{QN} \sim (3.60 - 2.25i) \times 10^{-12} + (0.791 - 0.493i) \varepsilon_\sigma + (0.50 - 0.89i) \varepsilon_\sigma^2, \\
\kappa^2 &= 0.1, \quad \omega_{QN} \sim (6.05 - 0.18i) \times 10^{-15} + (0.781 - 0.497i) \varepsilon_\sigma - (0.46 + 0.30i) \varepsilon_\sigma^2, \\
\kappa^2 &= 0.15, \quad \omega_{QN} \sim (-1.47 + 0.95i) \times 10^{-12} + (0.770 - 0.501i) \varepsilon_\sigma - (0.45 + 0.30i) \varepsilon_\sigma^2, \\
\kappa^2 &= 0.2, \quad \omega_{QN} \sim (3.89 - 2.62i) \times 10^{-13} + (0.760 - 0.505i) \varepsilon_\sigma - (0.44 + 0.31i) \varepsilon_\sigma^2. 
\end{align*}
\]  (32)

The effect of the backreaction on the scalar perturbation behavior will not be altered when we set \( \varphi^+, \varphi = 0 \) at the AdS boundary, see the right panel in Fig. 3

![Fig. 3](image)

FIG. 3: (Color online) The trajectories of the imaginary parts of the lowest quasinormal frequency for different values of \( \kappa^2 \) with \( \varphi^+, \varphi = 0 \) (the left panel) and \( \varphi^+, \varphi = 0 \) (the right panel), respectively. The blue line corresponds to \( \kappa^2 = 0 \) while the pink one is \( \kappa^2 = 0.2 \). While the temperature drops to the critical point, the system approaches the marginally stable mode.

Now we start to investigate the correlation length of the system. The critical behavior of the system near the critical point is determined by the large-scale fluctuations. The correlation length is the scale parameter that exists in the system near the phase transition point. It increases while the temperature approaches its critical value and becomes infinite at the moment of the phase transition. Now we check the influence imposed by the backreaction on the correlation length \( \xi \), which is defined by \( \xi^2 := -q^{-2} \). We consider a perturbation with \( \varphi = 0 \) for different \( \kappa^2 \), and solve Eq. (27) with \( \varphi = 0 \) under boundary conditions: Eq. (30) at the horizon and \( \varphi^+, \varphi = 0, q = 0 \) at the AdS boundary first. The results for the correlation length \( \xi \) can be fitted by
polynomials as
\[ \kappa^2 = 0.1, \quad \xi^{-2} \sim 3.07 \times 10^{-11} + 9.08 \varepsilon_\sigma, \]
\[ \kappa^2 = 0.15, \quad \xi^{-2} \sim 6.84 \times 10^{-12} + 9.13 \varepsilon_\sigma, \]
\[ \kappa^2 = 0.2, \quad \xi^{-2} \sim 1.44 \times 10^{-11} + 9.24 \varepsilon_\sigma. \] (33)

It is obvious that the correlation length \( \xi \) depends on the backreacting parameter \( \kappa^2 \). For the smaller backreaction, the correlation length is bigger for the same deviation from the critical point of the system, which means that it is easier for the system to approach the phase transition point when the backreaction is smaller. This result is consistent with the influence of the backreaction on the critical temperature and agrees to the property exhibited in the perturbation behavior.

When we set \( \phi_{\overline{\omega}}, q=0 \) at the AdS boundary, the condensation is expressed by the operator \( \langle \mathcal{O}_+ \rangle \). The results for the correlation length \( \xi \) are fitted by polynomials as listed below
\[ \kappa^2 = 0.1, \quad \xi^{-2} \sim 4.27 \times 10^{-13} + 2.374 \varepsilon_\sigma, \]
\[ \kappa^2 = 0.15, \quad \xi^{-2} \sim 4.12 \times 10^{-12} + 2.360 \varepsilon_\sigma, \]
\[ \kappa^2 = 0.2, \quad \xi^{-2} \sim 1.59 \times 10^{-12} + 2.347 \varepsilon_\sigma. \] (34)

This result implies that for the same deviation from the critical point of the system the correlation length is bigger when the backreaction is bigger, which means that it is easier for the system to approach the phase transition point when the backreaction is stronger. This is in contrary to the influence of the backreaction disclosed in the lowest quasinormal frequency and the critical temperature, which supports the argument in the study of condensation that \( \langle \mathcal{O}_+ \rangle \) is not an appropriate operator.

\section*{IV. CONCLUSIONS AND DISCUSSIONS}

We developed the holographic superconductors in the BTZ black hole background with backreactions. From the critical temperature and the dynamical perturbation properties, we observed that the stronger backreaction makes it more difficult for the scalar hair to condensate.

In the holographic superconductor, we have two operators to describe the condensation. It is of interest to ask which one can really reflect the properties of the condensation. In the constructed (1+1)-dimensional holographic superconductor, we observed that the condensation read from expectation values of different operators reflects different influence of the backreaction on the formation of the scalar hair. This phenomenon was also
observed in the study of the effect of the backreaction in the (2+1)-dimensional holographic superconductor [25].

The operator \( \langle O_- \rangle \) in our paper is similar to \( \langle O_2 \rangle \) [25]. The gaps of condensation increase with the increase of the backreaction for these two operators as reported in our manuscript and in [25]. \( \langle O_+ \rangle \) in our paper is similar to \( \langle O_1 \rangle \) in [25], where these two gaps of the condensation increase when the backreaction decreases.

In [25], \( \langle O_1 \rangle \) is of mass dimension one operator, which is related to the asymptotic behavior \( \frac{\psi(1)}{r} \) at infinity. In that case, the mass was not equal to the BF bound. In our case, when the BF bound is approached, \( \langle O_+ \rangle \) can have the similar asymptotic behavior \( \frac{\psi(1)}{r} \).

The condensation gap indicated in the operator marks the ease of the scalar hair to be formed [19,20]. It should reflect the consistent influence of the backreaction as that shown in the critical temperature and dynamical perturbation. However, we found that one of the two operators cannot exhibit the consistency. We argued that this inconsistency tells us that this operator is not appropriate to describe the condensation. We observed that this operator is usually associated with the branch with asymptotic behavior of the scalar field \( \sim 1/r \) at the spatial infinity. This argument is supported by the property of the correlation length disclosed in studying the dynamics of the system.

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