A NOTE ON A LOCAL ERGODIC THEOREM FOR AN INFINITE TOWER OF COVERINGS.

RYOKICHI TANAKA

ABSTRACT. This is a note on a local ergodic theorem for a symmetric exclusion process defined on an infinite tower of coverings, which is associated with a finitely generated residually finite amenable group.

1. INTRODUCTION

In this note, we study a local ergodic theorem for a particle system on a covering graph of certain amenable group actions. This problem is related to the scaling limit of large scale interacting systems, which is motivated by statistical mechanics (e.g., [5] and [9]). A local ergodic theorem is one of key steps to obtain the hydrodynamic scaling limit of these systems. This theorem for large scale interacting systems is considered to tell us, for example, the following viewpoint of statistical mechanics: We have at least two different time scales, namely, the macroscopic one and the microscopic one. Comparing with the macroscopic time scale, the microscopic one is much longer, so the microscopic system is considered to approach to an equilibrium state locally. On the other hand, there should be many kinds of equilibrium states for such a system. Therefore, there will appear a state which is not uniform, but a state which is pasted together each local equilibrium state. Technically speaking, a local ergodic theorem enables us to replace a local average to a global one and verifies the derivation a hydrodynamic or a macroscopic partial differential equation in the limit.

These kinds of models have been considered on integer lattices, or crystal lattices which admit a free abelian group action. Natural question is to extend these problems on graph with general group actions, for example, amenable groups. Here we formulate the problem on an infinite graph with a finitely generated residually finite amenable group action. This class of groups contains many remarkable ones such as not only $\mathbb{Z}^d$, the Heisenberg group, the Grigorchuk infinite torsion group which has the intermediate volume growth and a certain class of groups generated by finite state automata. ([1], [2] and [8].)

An infinite graph which admits a (quasi-) transitive group action of these groups produces an infinite tower of coverings, by taking quotient finite graphs by finite index normal subgroups. That is an infinite family of Schreier graphs. On each quotient finite graph, we consider particle systems, for example, exclusion processes. By taking the limit in the direction to recover the original infinite covering graph, we will obtain a local ergodic theorem with appropriate time scaling associated with a tower of coverings. The theorem is formulated in terms of local function bundles,
which is introduced in [10], for systematic treatment of systems on different scales. The proof is based on the entropy method in [3] and [6].

At this moment, we do not have any scaling limit results by using our local ergodic theorem. But the scaling limit of these towers of graphs will have rich self-similar structures. (See [1] and Chapter 5 and 6 in [8].) We hope that our technique will be useful to define the limit processes on the scaling limit. Let us mention related works. Jara studies on the hydrodynamic limit on the Sierpinski gasket for zero range process ([4]). He takes an infinite family of finite graphs to approximate the Sierpinski gasket and derive a partial differential equation by verifying the replacement of local average to global one. Cellular Automata on groups are investigated by Ceccherini-Silberstein and Coornaert in [2]. They study the characterization of groups in terms of some dynamics properties of cellular automata on them.

2. Notation and results

2.1. Groups and associated infinite towers of coverings. Let \( \Gamma \) be a finitely generated, residually finite, amenable group. Throughout this note, we assume that \( \Gamma \) is an infinite group. If \( \Gamma \) is a finitely generated group, the residually finiteness is equivalent to the following: there is a descending sequence of finite index normal subgroups, i.e., there exists \( \cdots \subseteq \Gamma_{i+1} \subseteq \Gamma_i \subseteq \cdots \subseteq \Gamma_1 \subseteq \Gamma \) such that \( \Gamma_i \triangleleft \Gamma \) and \( [\Gamma : \Gamma_i] < \infty \) for each \( i \), and \( \bigcap_{i=1}^{\infty} \Gamma_i = \{ \text{id} \} \). Furthermore, if \( \Gamma \) is a finitely generated group, the amenability is equivalent to the existence of a Følner sequence \( \{ F_i \}_{i=1}^{\infty} \), i.e., for each \( i \), \( F_i \) is a finite subset of \( \Gamma \) and \( \lim_{i \to \infty} |\partial_S F_i|/|F_i| = 0 \) where \( S \) is a finite set of generators and \( \partial_S F_i := F_i S \setminus F_i \). We remark that if \( \Gamma \) is not necessarily finitely generated, we have to take nets instead of sequences appearing in the above. From now on, we fix some finite generating set \( S \) in \( \Gamma \) and denote by \( |\cdot|_\Gamma \) the corresponding word norm in \( \Gamma \).

**Examples**

(i) \( \mathbb{Z} \). We can take \( \{ -1, 1 \} \) as a finite set of generators. For each positive integer \( i \), by setting \( \Gamma_i := 2^i \mathbb{Z} \), then \( \{ \Gamma_i \}_{i=1}^{\infty} \) satisfies the condition of residually finiteness. For each positive integer \( i \), by setting \( F_i := [-i, i] \subset \mathbb{Z} \), then \( \{ F_i \}_{i=1}^{\infty} \) is a Følner sequence. In the same way, \( \mathbb{Z}^d (d \geq 1) \) is also included in this class.

(ii) Heisenberg group,

\[
H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.
\]

The residually finiteness follows by taking a natural homomorphism \( H_3(\mathbb{Z}) \to GL_3(\mathbb{Z}/p\mathbb{Z}) \) for each prime number \( p \) and the ball of radius \( i \) forms a Følner sequence.

(iii) Grigorchuck group. Grigorchuck group is residually finite and a subsequence of balls forms a Følner sequence since it has subexponential volume growth. (See, e.g., Chapter 6 in [2].)
(iv) Basilica group. Basilica group is realized as a finitely generated subgroup of automorphism group in the binary tree or generated by a finite state automata, this implies the residually finiteness. The amenability is proved in ([1]).

Remark

We note that there exists a finitely generated group which is not amenable but residually finite, e.g., free group of rank 2, $F_2$, and not residually finite but amenable, e.g., $Alt_5 \wr \mathbb{Z}$, where $Alt_5$ is the alternating group of degree 5.

Let $X = (V, E)$ be an oriented graph such that both oriented edges are included, i.e., if $e \in E$, then its reversed edge $\overline{e} \in E$. Denote the origin $oe$ and the terminus $te$ for an edge $e$. Suppose a finitely generated, residually finite, amenable group $\Gamma$ acts on $X$ freely, and vertex transitively. For example, $X$ is a Cayley graph associated with some finite set of generators of $\Gamma$ and equipped with both oriented edges. The finite quotient graph $X_0 := \Gamma \backslash X$ consists of one vertex and loop edges.

Remark. We can modify the following argument to extend our results to quasitransitive graphs, i.e., $\Gamma$ acts on $X$ with finite number of orbits. If $\Gamma = \mathbb{Z}^d$, the quasi-transitive graph $X$ is called a $\mathbb{Z}^d$-crystal lattice ([7], [10]).

Fix a descending sequence of normal subgroups $\cdots \subset \Gamma_{i+1} \subset \Gamma_i \subset \cdots \subset \Gamma_1 \subset \Gamma$ such that $\Gamma_i \triangleleft \Gamma$ and $[\Gamma : \Gamma_i] < \infty$ for each $i$, and $\bigcap_{i=1}^{\infty} \Gamma_i = \{\text{id}\}$. For each $i$, define the quotient finite graph $X_i := \Gamma_i \backslash X$. Each $X_i$ is a finite graph since $\Gamma_i$ is a finite index subgroup of $\Gamma$. Here $\Gamma/\Gamma_i$ acts on $X_i$ freely, and vertex transitively, where $\Gamma/\Gamma_i$ is a finite group. Then we have an infinite tower of coverings of a finite graph: $X_i \to X_0$ for $i = 1, 2, \ldots$.

Let us define a particle system on $X$. Define the configuration space by $Z := \{0, 1\}^V$, and denote each configuration by $\eta := \{\eta_x\}_{x \in V}$. The action of $\Gamma$ lifts on $Z$ naturally, by setting $(\sigma \eta)_z := \eta_{\sigma^{-1} z}$ for $\sigma \in \Gamma$, $\eta \in Z$ and $z \in V$. In the same way, for each quotient finite graph $X_m = (V_m, E_m) = \Gamma_m \backslash X$ we define a configuration space $Z_m := \{0, 1\}^{V_m}$. The action of $\Gamma/\Gamma_m$ on $X_m$ lifts on $Z_m$ as above. Here we define the notion of local function bundles, which is defined on the product space of the state space $V$ and the configuration space $Z$. This is used to formulate our local ergodic theorem and originally introduced in [10] to obtain the scaling limit of a particle system on a crystal lattice.

**Definition 2.1.** A $\Gamma$-invariant local function bundle for vertices is a function $f : V \times Z \to \mathbb{R}$ such that:

- There exists $r \geq 0$ such that for any $x \in V$, $f_x : Z \to \mathbb{R}$ depends only on $\{\eta_z\}_{z \in B(x, r)}$. Here $B(x, r)$ denotes the ball in $V$ centered at $x$ with radius $r$ by the graph distance $d$ in $X$.
- For any $\sigma \in \Gamma, x \in V, \eta \in Z$, it holds that $f(\sigma x, \sigma \eta) = f(x, \eta)$.

In the similar way, a $\Gamma$-invariant local function bundle for edges is a function $f : E \times Z \to \mathbb{R}$ such that:

- There exists $r \geq 0$ such that for any $e \in E$, $f_e : Z \to \mathbb{R}$ depends only on $\{\eta_z\}_{z \in B(oe, r) \cup B(te, r)}$. 

• For any $\sigma \in \Gamma, e \in E, \eta \in Z$, it holds that $f(\sigma e, \sigma \eta) = (e, \eta)$.

**Examples**

(i) For $x \in V$, if we define $f_x(\eta) := \eta_x$, then $f$ is a $\Gamma$-invariant local function bundle for vertices. In this case, for each $x \in V$, $f_x$ depends only on a configuration on $x$. The $\Gamma$-invariance follows from the definition of actions of $\Gamma$.

(ii) For $x \in V$, if we define $f_x(\eta) := \prod_{e \in E_x} \eta_{e\gamma}$, where $E_x := \{ e \in E : oe = x \}$, then $f$ is a $\Gamma$-invariant local function bundle for vertices.

(iii) For $e \in E$, if we define $f_e(\eta) := \eta_{oe} + \eta_{te}$, then $f$ is a local function bundle for edges.

(iv) For $e \in E$, if we define $f_e(\eta) := \prod_{e' \in E_e} \eta_{e'\gamma} + \prod_{e' \in E_e} \eta_{e'\gamma} + c$, where $c > 0$, then $f$ is a $\Gamma$-invariant local function bundle for edges and satisfies $f(e, \eta) \geq c$ for all $e \in E, \eta \in Z$.

Let $\{F_i\}_{i=1}^\infty$ be a Følner sequence for $\Gamma$. For a $\Gamma$-invariant local function bundle for vertices $f : V \times Z \to \mathbb{R}$, we define a local average associated with $\{F_i\}_{i=1}^\infty$. For $x \in V$ and $F_i$,

$$\overline{f}_{x,i} := \frac{1}{|F_i|} \sum_{\sigma \in F_i} f_{\sigma x}.$$ 

Note that $\overline{f}_{x,i}$ is again a $\Gamma$-invariant local function bundle.

Let us introduce a local average on controlled Følner set. For a non negative real value $K$, we define

$$b(K) := \max\{ i : F_i \subseteq B(o, K) \},$$

i.e., $b(K)$ is the biggest number $L$ such that all $F_i, i = 1, \ldots, L$ are included in the ball with radius $r$ about $o$. Note that $b(K) \nearrow \infty$ as $K \nearrow \infty$. We also consider a local average of the following type:

$$\overline{f}_{x,b(K)} := \frac{1}{|F_{b(K)}|} \sum_{\sigma \in F_{b(K)}} f_{\sigma x}.$$ 

Fix a distinguished vertex $o \in V$. Now $\Gamma$ acts on $V$ transitively, so any vertex $x$ can be written in the form $x = \gamma o$ for some $\gamma \in \Gamma$. For any $m \geq 1$, denote by the same character $o \in V_m$, the image of $o$ by the covering map $X \to X_m$. For $\sigma \in \Gamma$, denote by $\overline{\sigma} \in \Gamma/\Gamma_m$ the image of $\sigma$ by the canonical surjection. By the $\Gamma$-invariance, $\Gamma$-invariant local function bundles for vertices and edges induce functions on $V_m \times Z_m, E_m \times Z_m$, which are $\Gamma/\Gamma_m$-invariant under the diagonal action, for each $m$. We also use the same character for this induced one.

2.2. **Particle systems.** Assume that $Z_m$ and $Z$ are equipped with the prodiscrete topology, i.e., the product topology of discrete topologies. Denote by $\mathcal{P}(Z_m), \mathcal{P}(Z)$ the spaces of Borel probability measures on $Z_m, Z$, respectively. We also define the $(1/2)$-Bernoulli measures $\nu_m$ on $Z_m, \nu$ on $Z$, as the direct product of the $(1/2)$-Bernoulli measure on $\{0,1\}$, respectively.
Let us define the symmetric exclusion process on $X$. For a configuration $\eta \in Z$ and an edge $e \in E$, define by $\eta^{x,y}$ a configuration exchanging states on $x$ and $y$, i.e.,

$$\eta_z^{x,y} := \begin{cases} 
\eta_y & z = x, \\
\eta_x & z = y, \\
\eta_z & \text{otherwise},
\end{cases}$$

for $z \in V$. Furthermore, for a function $F$ on $Z$ and $x, y \in E$, we define $\pi_{x,y}F(\eta) := F(\eta^{x,y}) - F(\eta)$. In particular, for an edge $e \in E$, define $\eta^e := \eta^{o,e,t}$ and $\pi_e F(\eta) := F(\eta^e) - F(\eta)$. We also define the corresponding ones for $\eta \in Z_m$ and $x, y \in E_m$ in the same way. The symmetric exclusion process on $X$ is defined in terms of a $\Gamma$-invariant local function bundle for edges. We call a $\Gamma$-invariant local function bundle $c : E \times Z \to \mathbb{R}$ a jump rate if it the following properties:

- (symmetric) $c(e, \eta) = c(\tau, \eta)$, $c(e, \eta) = c(e, \eta^c)$ for any $e \in E, \eta \in Z$.
- (non-degenerate) There exists a positive constant $c_0$ such that $c_0 \leq c(e, \eta)$ for any $e \in E, \eta \in Z$.

Define the generator of a symmetric exclusion process by the operator $L_m : L^2(Z_m, \nu_m) \to L^2(Z_m, \nu_m)$,

$$L_m F(\eta) := \sum_{e \in E_m} c(e, \eta) \pi_e F(\eta),$$

for $F \in L^2(Z_m, \nu_m)$.

We consider a family of exclusion processes on an infinite tower of coverings. Each process on a covering graph $X_m$ is speeded up by some time scaling factor $t_m$. Suppose $\{t_m\}_{m=1}^{\infty}$ be an increasing sequence of positive numbers: $t_1 < t_2 < \cdots < t_m \to \infty$ with the condition $\sqrt{t_m} \leq 2 \text{diam } X_m$.

Consider the continuous time Markov chain generated by $L_m$ speeded up by $t_m$. Fix an arbitrary time $T > 0$ and denote by $D([0, T], Z_m)$ the space of paths being right continuous and having left limits. For a probability measure $\mu_m$ on $Z_m$, we define by $\mathbb{P}_m$ the distribution on $D([0, T], Z_m)$ of the continuous time Markov chain $\eta_m(t)$ generated by $t_m L_m$ with the initial measure $\mu_m$.

For $0 \leq \rho \leq 1$, denote by $\nu_\rho$ the $\rho$-Bernoulli measure on $Z$, that is the direct product of the $\rho$-Bernoulli measure on $\{0, 1\}$. We define as a global average of a $\Gamma$-invariant local function bundle $f : V \times Z \to \mathbb{R}$ by $\langle f_o \rangle(\rho) := \mathbb{E}_{\nu_\rho}[f_o]$ the expectation of the function $f_o : Z \to \mathbb{R}$ with respect to $\nu_\rho$.

The following estimate enables us to approximate a global average of a local function bundle by a local average of one under a suitable time-space average. This estimate is also referred as a local ergodic theorem, which we show in the super exponential estimate.

**Theorem 2.1.** Fix $T > 0$. For any $\Gamma$-invariant local function bundles $f : V \times Z \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{i \to \infty} \lim_{\varepsilon \to 0} \lim_{m \to \infty} \mathbb{P}_m \left( \frac{1}{[\Gamma : \Gamma_m]} \int_0^T V_{o,\varepsilon,\tau}(\eta(t)) dt \geq \delta \right) = -\infty,$$
\[ V_{o,m,\varepsilon,i}(\eta) = \sum_{\sigma \in \Gamma / \Gamma_m} |\mathcal{J}_{\sigma o,i}(\eta) - \langle f_o \rangle (\overline{\eta}_{\sigma o,b(\varepsilon \sqrt{t_m})})| \].

Here we remark that this theorem is a generalized form of Theorem 4.1 for \( \mathbb{Z}^d \)-crystal lattices in [10].

### 3. Proof of Theorem 2.1

The super exponential estimate for \( \mathbb{P}_m \) is reduced to \( \mathbb{P}^{eq}_m \), which is the distribution of continuous time Markov chain generated by \( t_mL_m \) with the initial measure the \((1/2)\)-Bernoulli measure \( \nu_m \), i.e., an equilibrium measure, since for any measurable set \( A \subset D([0, T], Z_m) \), \( \mathbb{P}_m(A) \leq 2^{[\Gamma : \Gamma_m]} \mathbb{P}^{eq}_m(A) \) and the factor \( 2^{[\Gamma : \Gamma_m]} \) does not contribute in the super exponential estimate. Moreover, the super exponential estimate Theorem 2.1 is reduced to the following theorem.

For \( \mu \in \mathcal{P}(Z_m) \), define the Dirichlet form for \( \varphi := d\mu / d\nu_m \) by

\[ I_m(\mu) := -\int_{Z_m} \sqrt{\varphi} L_m \sqrt{\varphi} d\nu_m, \]

which is also equal to \((1/2) \int_{Z_m} \sum_{e \in E_m} c(e, \eta) (\pi_e \sqrt{\varphi})^2 d\nu_m\). For any \( C > 0 \), we define the subset of \( \mathcal{P}(Z_m) \) by

\[ \mathcal{P}_{m,C} := \left\{ \mu \in \mathcal{P}(Z_m) : \Gamma / \Gamma_m \text{-invariant} \ I_m(\mu) \leq C \frac{[\Gamma : \Gamma_m]}{t_m} \right\}. \]

**Theorem 3.1.** For any \( C > 0 \), it holds that

\[ \lim_{i \to \infty} \limsup_{\varepsilon \to 0} \limsup_{m \to \infty} \sup_{\mu \in \mathcal{P}_{m,C}} \mathbb{E}_\mu \left| \mathcal{J}_{o,i} - \langle f_o \rangle (\overline{\eta}_{o,b(\varepsilon \sqrt{t_m})}) \right| = 0. \]

First, we see how to deduce Theorem 2.1 from Theorem 3.1.

**Proof of Theorem 2.1.** As we observe, it is suffice to prove the required estimate for \( \mathbb{P}^{eq}_m \). Recall that \( V_{o,m,\varepsilon,i}(\eta) = \sum_{\sigma \in \Gamma / \Gamma_m} |\mathcal{J}_{\sigma o,i}(\eta) - \langle f_o \rangle (\overline{\eta}_{\sigma o,b(\varepsilon \sqrt{t_m})})| \).

By the Chebychev inequality, for any \( a > 0 \) and any \( \delta > 0 \),

\[ \mathbb{P}^{eq}_m \left( \int_0^T V_{o,m,\varepsilon,i}(\eta(t)) dt \geq \delta \right) \leq \mathbb{E}^{eq}_m \exp \left( a \int_0^T V_{o,m,\varepsilon,i}(\eta(t)) dt - a\delta \frac{[\Gamma : \Gamma_m]}{t_m} \right). \]

For any \( a > 0 \), we consider the operator

\[ t_mL_m + aV_{o,m,\varepsilon,i} : L^2(Z_m, \nu_m) \to L^2(Z_m, \nu_m), \]

which is self-adjoint for all \( a > 0 \) by the definition of \( L_m \). Denote by \( \lambda_{o,m,\varepsilon,i}(a) \) the largest eigenvalue of \( t_mL_m + aV_{o,m,\varepsilon,i} \). By the Feynmann-Kac formula,

\[ \mathbb{E}^{eq}_m \exp \left( a \int_0^T V_{o,m,\varepsilon,i} dt \right) \leq \exp T \lambda_{o,m,\varepsilon,i}(a). \]

Therefore, it is suffice to show that for any \( a > 0 \),

\[ \lim_{i \to \infty} \limsup_{\varepsilon \to 0} \limsup_{m \to \infty} \frac{1}{[\Gamma : \Gamma_m]} \lambda_{o,m,\varepsilon,i}(a) = 0. \]
In fact, by using (3.1), we have that
\[ \lim_{i \to \infty} \lim_{m \to \infty} \limsup_{\epsilon \to 0} \frac{1}{[\Gamma : \Gamma_m]} \log \mathbb{P}_m^\eta \left( \frac{1}{[\Gamma : \Gamma_m]} \int_0^T V_{o,m,\epsilon,i}(\eta(t))dt \geq \delta \right) \leq -a\delta. \]
Taking \( a \to \infty \), we obtain Theorem 2.1.

It remains to prove (3.1). By the variational principle, the largest eigenvalue of this operator can be expressed in the following form:
\[ \lambda_{o,m,\epsilon,i}(a) := \sup_{\mu \in \mathbb{P}(Z_m)} \left\{ a \int_{Z_m} V_{o,m,\epsilon,i}d\mu - t_m I_m(\mu) \right\}. \]

It is enough to consider only the case where \( a \int_{Z_m} V_{o,m,\epsilon,i}d\mu \geq t_m I_m(\mu) \).

For \( \mu \in \mathbb{P}(Z_m) \), we deduce by \( \overline{\mu} \) the average of \( \mu \) by the \( \Gamma/\Gamma_m \)-action, that is,
\[ \overline{\mu} := \frac{1}{[\Gamma : \Gamma_m]} \sum_{a \in \Gamma/\Gamma_m} \mu \circ \sigma. \]

Here \( \overline{\mu} \) is a \( \Gamma/\Gamma_m \)-invariant probability measure. Now we have
\[ \frac{1}{[\Gamma : \Gamma_m]} \int_{Z_m} V_{o,m,\epsilon,i}d\mu = \mathbb{E}_{\overline{\mu}} \left| \mathcal{F}_{o,i} - \langle f_o \rangle (\overline{\eta}_{a,b(\epsilon \sqrt{t_m})}) \right|. \]

Since there exists a constant \( C(f) > 0 \) depending only on \( f \) such that \( aC(f)[\Gamma : \Gamma_m] \geq a \int_{Z_m} V_{o,m,\epsilon,i}d\mu \). By the convexity of \( I_m, I_m(\overline{\mu}) \leq (1/[\Gamma : \Gamma_m]) \sum_{a \in \Gamma/\Gamma_m} I_m(\mu \circ \sigma) \) and thus \( I_m(\overline{\mu}) \leq aC(f)[\Gamma : \Gamma_m]/t_m \). We define by \( \mu \in \mathbb{P}_{m,aC(f)} \) the set of probability measures \( \mu \) satisfying the \( \Gamma/\Gamma_m \)-invariance and \( I_m(\mu) \leq aC(f)/t_m \), then
\[ \frac{1}{[\Gamma : \Gamma_m]} \lambda_{o,m,\epsilon,i}(a) \leq a \sup_{\mu \in \mathbb{P}_{m,aC(f)}} \mathbb{E}_\mu \left| \mathcal{F}_{o,i} - \langle f_o \rangle (\overline{\eta}_{a,b(\epsilon \sqrt{t_m})}) \right|. \]

Here (3.1) follows from Theorem 3.1. \( \square \)

In this article, we often identify a probability measure on \( Z_m \) with one on \( Z \) by the periodic extension as in the following way: Let \( \pi_m : V \to V_m \) be the covering map induced by the \( \Gamma \)-action. Define a periodic inclusion \( \iota_m : Z_m \to Z \) by \( (\iota_m \eta)_z := \eta_{\pi_m(z)}, \eta \in Z_m, z \in V \). We identify \( \mu \) on \( Z_m \) with its push forward by \( \iota_m \), which is a periodic extension of \( \mu \) on \( Z \). Conversely, we identify a \( \Gamma_m \)-invariant probability measure on \( Z \) with one on \( Z_m \) in the natural way.

Theorem 3.1 follows the one-block estimate (Theorem 3.2) and the two-blocks estimate (Theorem 3.3). First, we prove the one-block estimate.

**Theorem 3.2** (The one-block estimate). For any \( \Gamma \)-invariant local function bundle \( f : V \times Z \to \mathbb{R} \) and for any \( C > 0 \), it holds that
\[ \lim_{i \to \infty} \limsup_{m \to \infty} \sup_{\mu \in \mathbb{P}_{m,C}} \mathbb{E}_\mu \left| \mathcal{F}_{o,i} - \langle f_o \rangle (\overline{\eta}_{a,i}) \right| = 0. \]

We discuss about a restricted region in \( X \) and define the corresponding Dirichlet form and so on. Suppose \( \Lambda = (V_\Lambda, E_\Lambda) \) be a subgraph of \( X \). Define the restricted configuration space by \( Z_\Lambda := \{0, 1\}^{V_\Lambda} \), and the \((1/2)\)-Bernoulli measure \( \nu_\Lambda \) on \( Z_\Lambda \) by the direct product of the \((1/2)\)-Bernoulli measure on \( \{0, 1\} \).
In our setting, $\Gamma$ acts on $X$ vertex transitively, thus $o \in V$ is a fundamental domain in $V$. We can choose a fundamental domain $E^0$ in $E$ such that for every $e \in E$, oe or $te \in E^0$. We use the same notation $E^0$ for the image of $E^0$ on $X_m$ by the covering map. Define the operator on $L^2(Z_\Lambda, \nu_\Lambda)$ by

$$L^o_\Lambda := \frac{1}{2} \sum_{e \in E_\Lambda} \pi_e.$$ 

For $\mu \in \mathcal{P}(Z)$, we denote by $\mu|_\Lambda$ the restriction of $\mu$ on $Z_\Lambda$, that is, defined by taking the average outside of $\Lambda$. Set $\varphi_\Lambda := d\mu|_\Lambda/d\nu_\Lambda$ the density of $\mu|_\Lambda$. The corresponding Dirichlet form of $\sqrt{\varphi_\Lambda}$ is

$$I^o_\Lambda(\mu) := -\int_{Z_\Lambda} \sqrt{\varphi_\Lambda} L^o_\Lambda \sqrt{\varphi_\Lambda} d\nu_\Lambda.$$ 

Proof of Theorem 3.2. Define a subgraph $\Lambda$ of $X$ by setting $V_\Lambda := B(o, K)$, $E_\Lambda := \{e \in E : oe, te \in V_\Lambda\}$. For any $\mu_m \in \mathcal{P}_{m,C}$, by the convexity of the Dirichlet form and the $\Gamma/\Gamma_m$-invariance of $\mu$ and $\nu_m$, we have

$$I^o_\Lambda(\mu) = \frac{1}{2} \sum_{e \in E_\Lambda} \int_{Z_\Lambda} (\pi_e \sqrt{\varphi_\Lambda})^2 d\nu_\Lambda$$

$$\leq \frac{1}{2} \sum_{e \in E_\Lambda} \int_{Z_m} (\pi_e \sqrt{\varphi})^2 d\nu_m$$

$$= \frac{1}{2} \sum_{e \in \Gamma/\Gamma_m, |e|_\Gamma \leq K} \sum_{e \in E^0} \int_{Z_m} (\pi_e \sqrt{\varphi})^2 d\nu_m$$

$$= \frac{1}{2} |B_\Gamma(K)| \sum_{e \in E^0} \int_{Z_m} (\pi_e \sqrt{\varphi})^2 d\nu_m,$$

where in the last term $B_\Gamma(K)$ denotes the ball of radius $K$ in $\Gamma$ about id in $\Gamma$ by the word norm $| \cdot |_\Gamma$. On the other hand in the same way, we have $I_m(\mu) = [\Gamma : \Gamma_m](1/2) \sum_{e \in E^0} \int_{Z_m} c(e, \eta) (\pi_e \sqrt{\varphi})^2 d\nu_m$. By the uniformly boundedness of $c(e, \eta) \geq c_0 > 0$ (non-degeneracy of $c(\cdot, \cdot)$), it holds that

$$I^o_\Lambda(\mu) \leq \frac{|B_\Gamma(K)|}{c_0 [\Gamma : \Gamma_m]} I_m(\mu).$$

Since $\mu$ satisfies $I_m(\mu) \leq C |V_m|/t_m$, we have that $I^o_\Lambda(\mu) \leq C_K/t_m \to 0$ as $m \to \infty$, where $C_K$ is a constant depending only on $K$. The space of probability measures $\mathcal{P}(Z)$ is compact by the weak topology. Thus, any sequence $\{\mu_i\}_{i=1}^\infty$ in $\mathcal{P}(Z)$ has a convergence subsequence. Let $\mathcal{A} \subset \mathcal{P}(Z)$ be the set of all limit points of $\{\mu_i\}_{i=1}^\infty$ in $\mathcal{P}(Z)$. By the above argument, we see that $I^o_\Lambda(\mu) = 0$ for any $\mu \in \mathcal{A}$. We obtain that $\mu|_\Lambda(\eta^e) = \mu|_\Lambda(\eta)$ for any $e \in E_\Lambda$ and any $\eta \in Z_\Lambda$ since $I^o_\Lambda(\mu) = (1/2) \sum_{e \in E_\Lambda} \sum_{\eta \in Z_\Lambda} (\pi_e \sqrt{\mu|_\Lambda(\eta)})^2 = 0$. This implies that random variables $\{\eta_e\}_{e \in V}$ are exchangeable under $\mu$. By the de Finetti theorem, there exists a probability measure $\lambda$ on $[0,1]$ such that $\mu = \int_0^1 \nu_\rho \lambda(d\rho)$, where $\nu_\rho$ is the $\rho$-Bernoulli
measure on $Z$. Then we have
\[
\limsup_{m \to \infty} \sup_{\mu \in \mathcal{P}_{m,C}} \mathbb{E}_\mu \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right| \leq \sup_{\mu \in \mathcal{A}} \mathbb{E}_\mu \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right|
\]
\[
\leq \sup_{\rho \in [0,1]} \mathbb{E}_{\nu\rho} \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right|.
\]
Therefore, it is enough to show that $\lim_{i \to \infty} \sup_{\rho \in [0,1]} \mathbb{E}_{\nu\rho} \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right| = 0$. Since $f$ is a $\Gamma$-invariant local function bundle, there exists $L \geq 0$ such that $f_o : Z \to \mathbb{R}$ depends only on $\{\eta_z : d(o,z) \leq L\}$ and there exists a constant $C(f) > 0$ depending only on $f$ such that $\mathbb{E}_{\nu\rho} \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right|^2 \leq C(f)C_L/|F_i| \to 0$ as $i \to \infty$, where $C_L$ is a constant depending only on $L$. Since $\langle f_o \rangle(\rho)$ is a polynomial with respect to $\rho$, in particular, uniformly continuous on $[0,1]$, $\sup_{\rho \in [0,1]} \mathbb{E}_{\nu\rho} \left| \langle f_o \rangle(\pi_{o,i}) - \langle f_o \rangle(\rho) \right| \to 0$, $i \to \infty$. By the triangular inequality,
\[
\sup_{\rho \in [0,1]} \mathbb{E}_{\nu\rho} \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right| \\
\leq \sup_{\rho \in [0,1]} \mathbb{E}_{\nu\rho} \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\rho) \right| + \sup_{\rho \in [0,1]} \mathbb{E}_{\nu\rho} \left| \langle f_o \rangle(\rho) - \langle f_o \rangle(\pi_{o,i}) \right| \to 0, i \to \infty.
\]
This concludes the theorem $\lim_{i \to \infty} \limsup_{m \to \infty} \sup_{\mu \in \mathcal{P}_{m,C}} \mathbb{E}_\mu \left| \mathcal{F}_{o,i} - \langle f_o \rangle(\pi_{o,i}) \right| = 0$.

Next, we prove the 2-blocks estimate.

**Theorem 3.3** (The two-blocks estimate). For any $C > 0$, it holds that
\[
\lim_{i \to \infty} \limsup_{\varepsilon \to \infty} \sup_{L \to \infty} \limsup_{m \to \infty} \sup_{\sigma \in \Gamma} \sup_{m \to \infty} \sup_{\mu \in \mathcal{P}_{m,C}} \mathbb{E}_\mu \left| \mathcal{F}_{o,i} - \pi_{o,i} \right| = 0.
\]

We introduce the following notions: Denote by $\mathcal{P}(Z \times Z)$ the space of probability measures on $Z \times Z$. For $\sigma \in \Gamma$, define $\hat{\sigma} : Z \to Z \times Z$ by $\hat{\sigma}(\eta) := (\eta, \sigma^{-1}\eta)$. For $\mu \in \mathcal{P}(Z)$, denote by $\hat{\sigma} \mu \in \mathcal{P}(Z \times Z)$ the push forward by $\hat{\sigma}$, i.e., $\hat{\sigma} \mu := \mu \circ \hat{\sigma}^{-1}$. We define the subset $\mathcal{A}_{e,L}$ in $\mathcal{P}(Z \times Z)$ as the set of all limit points of $\{\hat{\sigma} \mu : L < |\sigma| \leq \varepsilon \sqrt{m}, \mu \in \mathcal{P}_{m,C}\}$ as $m \to \infty$, and the subset $\mathcal{A}_e$ in $\mathcal{P}(Z \times Z)$ as the set of all limit points of $\mathcal{A}_{e,L}$ as $L \to \infty$.

The required estimate is reduced to estimate the left hand side in the following:
\[
\limsup_{L \to \infty} \limsup_{m \to \infty} \sup_{\sigma \in \Gamma, \sigma \leq \varepsilon \sqrt{m}} \sup_{\mu \in \mathcal{P}_{m,C}} \int_{Z \times Z} \left| \mathcal{F}_{o,i} - \pi_{o,i} \right| (\hat{\sigma} \mu)(d\eta d\eta') \\
\leq \sup_{\eta \in \mathcal{A}_e} \int_{Z \times Z} \left| \mathcal{F}_{o,i} - \pi_{o,i} \right| \mu(d\eta d\eta').
\]

As in the proof of the one block estimate, we introduce a subgraph $\Lambda$ in $X$, by setting $V_\Lambda := B(o, K)$, $E_\Lambda := \{e \in E : oe, te \in V_\Lambda\}$ and define the generator $L_\Lambda^o$ on $L^2(Z, \nu)$ by
\[
L_\Lambda^o := \frac{1}{2} \sum_{e \in E_\Lambda} \pi_e.
\]
Then let us define the two generators on $L^2(Z \times Z, \nu \otimes \nu)$ and the corresponding Dirichlet forms. First, we define the generator on $L^2(Z \times Z, \nu \otimes \nu)$ by $L^0_\Lambda \otimes 1 + 1 \otimes L^0_\Lambda$. For $\mu \in \mathcal{P}(Z \times Z)$, denote by $\mu|_{\Lambda \times \Lambda}$ the restriction of $\mu$ on $Z_\Lambda \times Z_\Lambda$, i.e., $\mu|_{\Lambda \times \Lambda}(\eta, \eta') := \mu(\{(\tilde{\eta}, \tilde{\eta}') : \tilde{\eta}|_{\Lambda} = \eta, \tilde{\eta}'|_{\Lambda} = \eta'\})$ for $(\eta, \eta') \in Z_\Lambda \times Z_\Lambda$.

The corresponding Dirichlet form of $\sqrt{\varphi_{\Lambda \times \Lambda}}$, where $\varphi_{\Lambda \times \Lambda} := d\mu|_{\Lambda \times \Lambda}/d\nu_\Lambda \otimes \nu_\Lambda$ is defined by

$\int_{Z_\Lambda \times Z_\Lambda} \sqrt{\varphi_{\Lambda \times \Lambda}}(L^0_\Lambda \otimes 1 + 1 \otimes L^0_\Lambda)\sqrt{\varphi_{\Lambda \times \Lambda}}d\nu_\Lambda \otimes \nu_\Lambda.$

Second, we define the generator on $L^2(Z \times Z, \nu \otimes \nu)$ in the following way: For $(x, y) \in V \times V$ and $(\eta, \eta') \in Z \times Z$, we make a new configuration $(\eta, \eta')^{(x, y)} \in Z \times Z$ by setting $(\eta'_y, \eta_x)$ on $(x, y)$, $(\eta'_x, \eta_y)$ on $(y, x)$ and keeping unchanged otherwise. For $F \in L^2(Z \times Z, \nu \otimes \nu)$, $(x, y) \in V \times V$, define $\sigma_x F((\eta, \eta')) = F((\eta, \eta')^{(x, y)}) - F((\eta, \eta'))$. Define the generator $L^0_{o,o}$ on $L^2(Z \times Z, \nu \otimes \nu)$ by

$L^0_{o,o} := \tilde{\sigma}_{o,o},$

and the corresponding Dirichlet form of $\sqrt{\varphi_{\Lambda \times \Lambda}}$ by

$I^0_{\Lambda \times \Lambda}(\mu) := \int_{Z_\Lambda \times Z_\Lambda} \sqrt{\varphi_{\Lambda \times \Lambda}}L^0_{o,o}\sqrt{\varphi_{\Lambda \times \Lambda}}d\nu_\Lambda \otimes \nu_\Lambda.$

Then we use the following lemma. The proof appears in Lemma 4.2 in [10], so we omit it.

**Lemma 3.1.** For any $F \in L^2(Z_m, \nu_m)$ and any $\sigma \in \Gamma/\Gamma_m$, it holds that

$\int_{Z_m} (\pi_{o,\sigma} F)^2 d\nu_m \leq 4d(o, \sigma)^2 \sum_{e \in E_0} \int_{Z_m} (\pi_e F)^2 d\nu_m.$

For any constant $\tilde{C} > 0$, we define

$A_{\varepsilon, \tilde{C}} := \{\mu \in \mathcal{P}(Z \times Z) : I^0_{\Lambda \times \Lambda}(\mu) = 0, I^0_{\Lambda \times \Lambda}(\mu) \leq \tilde{C}\varepsilon^2\}.$

Then we have the following lemma.

**Lemma 3.2.** There exists a constant $\tilde{C} > 0$ such that $A_{\varepsilon} \subset A_{\varepsilon, \tilde{C}}$.

**Proof.** Define a subgraph $\Lambda = (V_\Lambda, E_\Lambda)$ of $X$ as in the proof of the one-block estimate: $V_\Lambda := B(o, K)$, $E_\Lambda := \{e \in E : oe, te \in V_\Lambda\}$. Here we regard $X_m$ as a suitable fundamental domain in $X$ by $\Gamma_m$-action. Take large enough $L, m$ for the diameter of $\Lambda$ and $K$ so that for any $\sigma$ with $L < |\sigma|, V_\Lambda \cap \sigma V_\Lambda = \emptyset$, for any $\sigma$ with $L < |\sigma| \leq \varepsilon \sqrt{t_m}$, $V_\Lambda \cap \sigma V_\Lambda \subset X_m$.

For given $(\eta, \eta') \in Z_\Lambda \times Z_\Lambda$, we define $\tilde{\eta} \in Z_{\Lambda \cup o} \Lambda$ by $\tilde{\eta}|_{\Lambda} = \eta, \sigma^{-1}\tilde{\eta}|_{\Lambda} = \eta'$. Then $(\tilde{\sigma} \mu)|_{\Lambda \times \Lambda}(\eta, \eta') = \mu|_{\Lambda \cup o} \Lambda(\tilde{\eta})$. By using the generator $L^0_{\Lambda \cup o \Lambda}$ on $L^2(Z, \nu)$, we have the identity $I^0_{\Lambda \times \Lambda}(\tilde{\sigma} \mu) = I^0_{\Lambda \cup o \Lambda}(\mu)$ for $\mu \in \mathcal{P}_{m,C}$. For any $\mu \in \mathcal{P}_{m,C}$ as in the previous section, by the convexity of the Dirichlet form and the $\Gamma/\Gamma_m$-invariance of $\mu$ and $\nu_m$,

$I^0_{\Lambda \times \Lambda}(\tilde{\sigma} \mu) = I^0_{\Lambda \cup o \Lambda}(\mu) \leq \frac{|B_t(K)|}{2c_o|\Gamma : \Gamma_m|} I_m(\mu) \leq C \frac{|B_t(K)|}{2c_o t_m} \to 0,$
as \( m \to \infty \). Therefore, for any \( \mu_L \in \mathcal{A}_{\varepsilon,L} \), \( I_{\varepsilon}^o(\mu_L) = 0 \) and for any \( \tilde{\mu} \in \mathcal{A}_\varepsilon \), \( I_{\varepsilon}^o(\tilde{\mu}) = 0 \) by the continuity of \( I_{\varepsilon}^o \). Furthermore, by the convexity of the Dirichlet form,

\[
I_{\varepsilon}^{(o,o)}(\tilde{\sigma} \mu) = \frac{1}{2} \int_{Z_{\varepsilon}^{o,o}} \left( \pi_{o,o} \sqrt{d\mu_{|A_o}} \right)^2 d\nu_{A_o} \leq \frac{1}{2} \int_{Z_m} \left( \pi_{o,o} \sqrt{d\mu_{|A_o}} \right)^2 d\nu_m.
\]

By Lemma 3.1 and the \( \Gamma/\Gamma_m \)-invariance of \( \mu \in \mathcal{P}_{m,C} \),

\[
\int_{Z_m} \left( \pi_{o,o} \sqrt{d\mu_{|A_o}} \right)^2 d\nu_m \leq 4d(o,o)^2 \sum_{\varepsilon \in E_o} \int_{Z_m} \left( \pi_{\varepsilon} \sqrt{d\mu_{|A_o}} \right)^2 d\nu_m \\
\leq 4d(o,o)^2 \frac{1}{c_0[\Gamma : \Gamma_m]} I_m(\mu).
\]

There exists a constant \( A > 0 \) independent of \( \sigma \in \Gamma \) such that \( d(o,o) \leq A|\sigma|_\Gamma \), and thus, we have for \( \sigma \in \Gamma \) with \( L < |\sigma| \leq \varepsilon \sqrt{t_m} \),

\[
I_{\varepsilon}^{(o,o)}(\tilde{\sigma} \mu) \leq \frac{1}{2} \cdot 4(A\varepsilon \sqrt{t_m})^2 \frac{1}{[\Gamma : \Gamma_m]} \frac{C[\Gamma : \Gamma_m]}{c_0} = \frac{2C}{c_0} \varepsilon^2 A^2.
\]

Define \( \bar{C} := 2CA^2/c_0 \). By the continuity of \( I_{\varepsilon}^{(o,o)} \), for any \( \tilde{\mu} \in \mathcal{A}_\varepsilon \), \( I_{\varepsilon}^{(o,o)}(\tilde{\mu}) \leq \bar{C} \varepsilon^2 \). This implies that \( \mathcal{A}_\varepsilon \subset \mathcal{A}_{\varepsilon,\bar{C}} \). \( \Box \)

**Proof of Theorem 3.3.** Define \( \mathcal{A}_0 \), the set of all limit points of \( \mathcal{A}_{\varepsilon,\bar{C}} \) as \( \varepsilon \to 0 \). For any \( \tilde{\mu}_0 \in \mathcal{A}_0 \), it holds that \( I_{\varepsilon}^{(o,o)}(\tilde{\mu}_0) = 0 = I_{\varepsilon}^{(o,o)}(\tilde{\mu}_0) \). Since this shows that for any \( (x,y) \in V_{\varepsilon} \), \( \pi_{x,y}(\tilde{\mu}_0_{|A_{\varepsilon}}) = 0 \), the random variables \( \{ (\eta_x, \eta'_y) \}_{x,y \in V} \) are exchangeable on \( Z \times Z \) under \( \tilde{\mu}_0 \). By the de Finetti theorem, there exists a probability measure \( \lambda \) on \([0,1]\) such that

\[
\tilde{\mu}_0 = \int_{[0,1]} \nu_\rho \otimes \nu_\rho \lambda(d\rho).
\]

In the proof of the one-block estimate, we have

\[
\lim_{\varepsilon \to 0} \sup_{\rho \in [0,1]} \mathbb{E}_{\nu_\rho} \left| \tilde{\eta}_{o,i} - \rho \right|^2 = 0.
\]

Then

\[
\sup_{\tilde{\mu}_0 \in \mathcal{A}_0} \mathbb{E}_{\tilde{\mu}_0} \left| \tilde{\eta}_{o,i} - \tilde{\eta}_{o,i} \right| \leq \sup_{\rho \in [0,1]} \mathbb{E}_{\nu_\rho \otimes \nu_\rho} \left| \tilde{\eta}_{o,i} - \tilde{\eta}_{o,i} \right| \leq 2 \sup_{\rho \in [0,1]} \mathbb{E}_{\nu_\rho} \left| \tilde{\eta} - \rho \right| \to 0,
\]
as $i \to \infty$. Here we used the triangular inequality in the last inequality. By Lemma 3.2, $A_{\varepsilon} \subset A_{\varepsilon, \tilde{C}}$ for some $\tilde{C} > 0$,

$$
\limsup_{\varepsilon \to 0} \limsup_{L \to \infty} \limsup_{m \to \infty} \sup_{\sigma \in \Gamma s.t. L < |\sigma| \leq \varepsilon \sqrt{t_m}} \sup_{\mu \in \mathcal{P}_{m,C}} \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{\sigma o,i} \right|
$$

$$
\leq \limsup_{\varepsilon \to 0} \sup_{\mu \in A_{\varepsilon}} \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}'_{o,i} \right|
$$

$$
\leq \limsup_{\varepsilon \to 0} \sup_{\mu \in A_{\varepsilon, \tilde{C}}} \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}'_{o,i} \right|
$$

$$
\leq \sup_{\mu \in A_0} \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}'_{o,i} \right| \to 0,
$$
as $i \to \infty$. This proves the theorem

$$
\lim_{i \to \infty} \lim_{\varepsilon \to 0} \lim_{L \to \infty} \limsup_{m \to \infty} \sup \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{\sigma o,i} \right| = 0.
$$

\begin{proof}[Proof of Theorem 3.1] For $\eta \in Z$, the following uniform upper bound holds:

$$
\left| \bar{\eta}_{o,b(\varepsilon \sqrt{t_m})} - \frac{1}{|F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)|} \sum_{\sigma \in F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)} \bar{\eta}_{\sigma o,i} \right|
$$

$$
\leq \left| \frac{1}{|F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)|} \sum_{\sigma \in F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)} \eta_{\sigma o} + \frac{1}{|F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)|} \sum_{\sigma \in F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)} \eta_{\sigma o} \right|
$$

$$
\quad - \left| \frac{1}{|F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)|} \sum_{\sigma \in F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)} \bar{\eta}_{\sigma o,i} \right|
$$

$$
\leq 2|F_{b(\varepsilon \sqrt{t_m})}| \left( |\partial F_{b(\varepsilon \sqrt{t_m})}| + |\partial F_{L}| \right) \left| \frac{1}{|F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)|} \right| \left( 1 - \frac{|F_{b(\varepsilon \sqrt{t_m})} \setminus F_{L}|}{|F_{b(\varepsilon \sqrt{t_m})}|} \right) \frac{|F_{L}|}{|F_{b(\varepsilon \sqrt{t_m})}|} \to 0,
$$
as $m \to \infty$, since for a Følner sequence $|\partial S F_{b(\varepsilon \sqrt{t_m})}|/|F_{b(\varepsilon \sqrt{t_m})}| \to 0$ as $m \to \infty$.

For a $\Gamma$-invariant $\mu$ on $Z$,

$$
\mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{o,b(\varepsilon \sqrt{t_m})} \right| \leq \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \frac{1}{|F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)|} \sum_{\sigma \in F_{b(\varepsilon \sqrt{t_m})} \setminus B_\varepsilon(L)} \bar{\eta}_{\sigma o,i} \right| + o_m, \varepsilon, i.
$$

The first term in the right hand side is bounded by

$$
\mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{o,b(\varepsilon \sqrt{t_m})} \right| \leq \sup_{\sigma \in F_{b(\varepsilon \sqrt{t_m})} \setminus F_{L}} \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{\sigma o,i} \right|
$$

$$
\leq \sup_{\sigma \in \Gamma s.t. L < |\sigma| \leq \varepsilon \sqrt{t_m}} \sup_{\mu \in \mathcal{P}_{m,C}} \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{\sigma o,i} \right|.
$$

By Theorem 3.3,

$$
\lim_{i \to \infty} \lim_{\varepsilon \to 0} \limsup_{m \to \infty} \sup \mathbb{E}_\mu \left| \bar{\eta}_{o,i} - \bar{\eta}_{o,b(\varepsilon \sqrt{t_m})} \right| = 0.
$$

\end{proof}
Since for every $\Gamma$-periodic local function bundles $f : V \times Z \to \mathbb{R}$, $\langle f_0 \rangle (\cdot)$ is uniformly continuous on $[0, 1]$,

$$
\lim_{i \to \infty} \limsup_{\epsilon \to 0} \limsup_{m \to \infty} \sup_{\mu \in \mathcal{P}_{m,C}} E_\mu \left| \langle f_0 \rangle (\eta_{o,i}) - \langle f_0 \rangle (\eta_{o,b(\epsilon \sqrt{t_m})}) \right| = 0.
$$

By Theorem 3.2 and the triangular inequality, we conclude that

$$
\lim_{i \to \infty} \limsup_{\epsilon \to 0} \limsup_{m \to \infty} \sup_{\mu \in \mathcal{P}_{m,C}} E_\mu \left| f_{o,i} - \langle f_0 \rangle (\eta_{o,b(\epsilon \sqrt{t_m})}) \right| = 0.
$$

□

References

[1] Bartholdi, L., Virág, B.: Amenability via random walks. Duke Math. J. 130, No. 1, 39-56 (2005)
[2] Ceccherini-Silberstein, T., Coornaert, M.: Cellular Automata and Groups. Springer Monographs in Mathematics, Springer Verlag (2010)
[3] Guo, M.Z., Papanicolaou, G.C., Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. Comm. Math. Phys. 118, 31-59 (1988)
[4] Jara, M.: Hydrodynamic limit for a zero-range process in the Sierpinski gasket. Comm. Math. Phys. 288, 773-797 (2009)
[5] Kipnis, C., Landim, C.: Scaling limits of interacting particle systems. Grundlehren der mathematischen Wissenschaften 320, Springer-Verlag, Berlin (1999)
[6] Kipnis, C., Olla, S., Varadhan, S. R. S.: Hydrodynamics and large deviation for simple exclusion processes. Comm. Pure Appl. Math., Vol. XLII, 115-137 (1989)
[7] Kotani, M., Sunada, T.: Albanese maps and off diagonal long time asymptotics for the heat kernel. Comm. Math. Phys. 209, 633-670 (2000)
[8] Nekrashevych, V.: Self-Similar Groups. Mathematical Surveys and Monographs 117, American Mathematical Society (2005)
[9] Spohn, H.: Large Scale Dynamics of Interacting Particles, Texts and Monograph in Physics, Springer Verlag, Heidelberg (1991)
[10] Tanaka, R.: Hydrodynamic limit for weakly asymmetric simple exclusion processes in crystal lattices. Comm. Math. Phys. 315, 603-641 (2012)

Advanced Institute for Materials Research and Mathematical Institute, Tohoku University, 2-1-1 Katahira, Aoba-ku, Sendai, 980-8577, Japan

E-mail address: rtanaka@wpi-aimr.tohoku.ac.jp