Finite-size effects and collective vibrations in the inner crust of neutron stars

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We study the linear response of the inner crust of neutron stars within the Random Phase Approximation, employing a Skyrme-type interaction as effective interaction. We adopt the Wigner-Seitz approximation, and consider a single unit cell of the Coulomb lattice which constitutes the inner crust, with a nucleus at its center, surrounded by a sea of free neutrons. With the use of an appropriate operator, it is possible to analyze in detail the properties of the vibrations of the surface of the nucleus and their interaction with the modes of the sea of free neutrons, and to investigate the role of shell effects and of resonant states.

I. INTRODUCTION

In going from the outer crust of a neutron star towards the core of the star, one crosses the so-called inner crust. The baryonic density of the inner crust ranges from about \( \rho = 10^{11} \text{ g cm}^{-3} \) to approximately \( \rho_0/2 \), where \( \rho_0 = 2.8 \times 10^{14} \text{ g cm}^{-3} \) is the density of nuclear matter at saturation. We shall discuss the linear response of this system limiting ourselves to densities smaller than \( \rho_0/10 \). According to the generally accepted theoretical description the inner crust consists, in this region, of a lattice of spherical nuclei immersed in a sea of free neutrons (with a background of electrons) \[1,2\]. At these densities, neutrons are expected to be superfluid, and the inner crust of a rotating star should be threaded by vortex lines.

The presence of nuclei affects a number of thermal and transport properties of the inner crust, and in some astrophysical scenarios may influence the fast cooling of the star \[3,4\]. It can also affect the vortex dynamics, leading to vortex pinning, which might be the explanation for the phenomenon of glitches, sudden changes observed in the rotational period of pulsars \[3,5\].

The interplay between the degrees of freedom of nuclei and those of free neutrons has been considered by different groups at the mean-field level within Hartree-Fock-Bogoliubov theory (HFB) \[6,8,10\]. It is well known, however, that many-body effects can significantly modify the HFB results. Several studies have found that the pairing gap in neutron matter is quenched by spin fluctuations \[11,12\], but results obtained in uniform matter cannot be directly extrapolated to the actual case of the inhomogeneous inner crust. In fact, many-body effects in (isolated) atomic nuclei are strongly dominated by the coupling between single-particle and collective surface-like degrees of freedom. The exchange of surface modes between nucleons moving in time-reversal states close to the Fermi energy gives rise to an attractive interaction, which tends to enhance the pairing gap \[13,16\]. Only exploratory calculations have directly addressed the induced interaction in the inner crust, finding that the density fluctuations associated with the dynamics of the nuclear surface lead to a partial dequenching of the gap and to characteristic changes in the spatial dependence of the gap \[17\].

Within this scenario it is of interest to investigate how the surface response changes from atomic nuclei (negative value of Fermi energy) to the inner crust, where the neutron Fermi energy lies in the continuum.

In the present work we shall investigate the collective response of the nuclear surface by means of a microscopic calculation of the linear response of the system based on the Random Phase Approximation (RPA) theory \[18\]. As in most previous studies, we shall deal with this inhomogeneous system within the Wigner-Seitz approximation (WS), enclosing the system in a spherical unit cell. In this way, we shall neglect the band structure associated with the Coulomb lattice \[19\].

Previous studies \[20,22\] have shown that the response to operators of the type \( r^2 Y_{LM} \) or \( \{ r^2 Y_L \times \sigma \} \) is very close to that found in neutron matter, in keeping with the fact that the nucleus occupies only a small fraction of the volume of the WS cell. However, in the following we shall focus on the modes which are specifically associated with the presence of the nucleus at the center of the WS cell. Therefore, we shall study the response to the operator \( dU/dr Y_{LM} \), where \( U \) is the mean-field potential. In fact, \( dU/dr Y_{LM} \) is the genuine operator associated with collective surface modes, which appears naturally in the particle-vibration coupling Hamiltonian \[23\]. Essentially the same results would be obtained using the density instead of the mean-field potential. In Appendix B we also briefly consider the operator \( r^4 Y_{LM} \).

In this work we shall not consider the effects of superfluidity on the linear response. Pairing correlations can affect considerably the low-lying part of the spectrum, not only because they smear the Fermi surface leading to the partial occupation of single-particle states, but also because they can change the isotopic composition of the
crust

II. OUTLINE OF THE CALCULATIONS

We start from a mean-field Hartree-Fock (HF) calculation, making use of the Skyrme-type SLy4 interaction and working on a mesh with a mesh size of 0.1 fm. Our calculation is in line with other studies previously performed by a number of groups. We impose that the single-particle wavefunctions vanish at the edges of the WS cell. We expect that within the selected density range the results should not be sensitive to the particular choice of the boundary conditions (cf. also Appendix A). For densities larger than those considered in this paper, the WS approximation starts to break down, leading to a dependence on the boundary conditions.

We shall focus on the properties of giant resonances, which are not expected to be sensitive to pairing. Within this context, we shall use in the calculations the favoured number of protons predicted by Negele and Vautherin, predictions which were worked out without pairing. We have anyway indicated in the text where one expects to see the main effects of pairing correlations.

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III. RESULTS

As discussed in the introduction, in the present work we are particularly interested in the study of the dynamics of the nuclear surface. For this reason we shall study the response to the operator $dU/dr Y_{LM}$, for $L=2$ and 3, where $U(r)$ represents the HF potential calculated in the WS cell (cf. Fig. 1). This operator is specifically sensitive to the nuclear surface. In the case of an isolated nucleus it leads to results similar to those obtained with the $r^L Y_{LM}$ operator for both quadrupole and octupole modes. In the following we shall give a detailed discussion of the results obtained for the $^{1364}$Sn WS cell. We shall then present the results associated with the $^{498}$Zr case. Finally, we shall consider the evolution of the response in the Sn isotopes with increasing number of neutrons, going from the closed-shell nucleus $^{132}$Sn to the drip-line nucleus $^{170}$Sn and then into the inner crust to the WS cells $^{506}$Sn and $^{1364}$Sn.

(a) The case of $^{1364}$Sn

In Fig. 2 we show the HF and RPA strength functions of the operator $dU/dr Y_{LM}$ in the $2^+$ and $3^-$ channels for the $^{1364}$Sn WS cell. The bars indicate the solution of the discrete RPA equations, while the continuous lines are obtained folding the discrete response with a Lorentzian function having a Full Width at Half Maximum (FWHM) obtained folding the discrete response with a Lorentzian function having a Full Width at Half Maximum (FWHM) equal to 1 MeV.

In order to satisfy the Energy Weighted Sum Rule (EWSR) within 93% we had to include single-particle states up to $E_{cut} = 90$ MeV. The $|ph\rangle$ basis is composed by about 8000 states for the quadrupole response and by about 10000 for the octupole response; these values are close to the limits of our computational possibilities. The fraction of the EWSR obtained by integrating the calculated quadrupole response up to a given energy is shown in Fig. 3 as a function of energy. We notice that the main features of the low-lying part of the response can be obtained with a much lower value of $E_{cut}$. This can be seen in Fig. 2(a), where the dotted line shows the RPA response obtained with $E_{cut} = 30$ MeV.

The proton unperturbed response involves deeply bound orbitals. The strongest particle-hole transitions connect orbitals separated by two oscillator shells, and have an energy of approximately 14 MeV, close to the value of $2\hbar\omega_p$ estimated above. There is also a rather strong low-lying transition, $g_{9/2} \rightarrow d_{5/2}$, whose energy is about 5 MeV.

The energies and transition strengths associated with the strongest (unperturbed) neutron particle-hole transitions are collected in Tables I (quadrupole response) and II (octupole response). They involve either bound states $(E_{lj} < E_{cont} = -5.5$ MeV) or states in the continuum $(E_{lj} > E_{cont})$. The latter are found to be resonant states, that is quasibound levels whose wavefunction is concentrated in the interior of the nucleus. To show this we take a potential which is equal to the mean-field potential up to 30 fm and is constant and equal to $E_{cont}$ from 30 fm to infinity. We then compute the resonant states associated to this potential, looking at the wavefunction phase shift that occurs when a particle at a given energy goes past the potential well. The quantum numbers $\{l,j\}$ label a resonance, if the associated phase shift $\delta_{lj}(E)$ reaches the value $(n+1)\pi/2$ ($n = 0, 1, ..$) with a positive slope at the energy $E_{res}$. The width of the resonance is taken to be equal to the full width at half maximum (FWHM) of the derivative of $\delta_{lj}$ [30]. For narrow resonances, the resulting value is almost the same as that obtained making use of the expression

$$\Gamma_{res} = \frac{2}{(d\delta_{lj}/dE)_{res}}.$$  \hspace{1cm} (2)

Comparing Table IV and Table III with Table III one realizes that the quantum numbers of the particle states
FIG. 2. Quadrupole (a) and octupole (b) strength functions calculated in the $^{136}$Sn WS cell. Black histograms refer to the discrete RPA response, while red and blue histograms refer to the discrete HF response for neutron and protons respectively (in units of MeV$^2$ fm$^{-2}$). The solid curve (in units of MeV fm$^{-2}$) is obtained by a convolution of the discrete RPA strength with a Lorentzian curve having a FWHM equal to 1 MeV. The dashed and dash-dotted curve are obtained convoluting the neutron and proton discrete HF strength respectively. The dotted line in (a) shows the quadrupole RPA response obtained with $E_{\text{cut}} = 30$ MeV (see text for details).

FIG. 3. Fraction of the quadrupole EWSR obtained integrating the RPA strength function shown in Fig. 2(a) up to a given energy.

The value of the spin-orbit splitting for these large values of $l$ is close to $\hbar \omega_n$. As a consequence, the intruder state $(l, j+1/2)$ lies rather close to the state $(l-1, j-1/2)$, and the neutron part of the unperturbed response is characterized by strong peaks corresponding to transition energies of the order of $2\hbar \omega_n$ (quadrupole response) or of the order of $3\hbar \omega_n$ (octupole response), with a spreading caused by the width of the resonances. In other words, the strength function is determined to a large extent by the shell structure, as in atomic nuclei. This point will be further discussed below (cf. Figs. 11 and 12).

The RPA $2^+$ response, shown in Fig. 2(a), displays two main peaks, lying close to 3.5 MeV and to 10.0 MeV, with a width of about 3.3 and 4.0 MeV respectively. The $3^+$ response (cf. Fig. 2(b)) shows a strong peak around 3.6 MeV with a width of about 2 MeV and a broader high-energy bump around 18.6 MeV with a width of about 4 MeV. The neutron and proton transition densities associated with the four RPA solutions carrying the largest quadrupole strength are shown in Fig. 3. They are peaked at the surface of the nucleus, and their shape is close to the derivative of the density, as predicted by the semiclassical model for surface collective modes.

The key role played by resonant transitions can be demonstrated by performing a RPA calculation in a WS cell with the same number of protons and with approximately the same asymptotic neutron density, but with a reduced value of $R_{\text{box}} = 25$ fm. The resulting strength, shown in Fig. 3 is very similar to that obtained in $^{136}$Sn.

In order to study in more detail the resonant neutron particle-hole transitions in the RPA response, it is useful to distinguish their contributions to the strength from those involving the other, ‘continuum’ states [31]. We shall limit ourselves to a simple analysis and consider a single-particle state in the WS cell to be ‘resonant’ if its quantum numbers $(l, j)$ coincide with one of those listed in Table III and if its energy lies in the range $E_{l^\nu}^{\text{res}} \pm \Gamma_{l^\nu}^{\text{res}}$. The strength of multipolarity $L$ associated with a given phonon $|\nu>$ can be schematically written

$$S^\nu_L = \sum_{ph} |X^\nu_{ph} + (-1)^L Y^\nu_{ph}|^2$$

$$<jp|Y_L|jh> \sim R_p dU/dv |R_h|^2,$$  \hspace{1cm} (3)

where $X^\nu_{ph}$ and $Y^\nu_{ph}$ denote the forwarsgoing and back-
wardsgoing amplitudes associated with each RPA root, while \( R_p, R_h \) denote the single-particle radial wavefunctions. In the following, we shall single out the 'resonant' contributions to the computed strength function, restricting the sum in Eq. (3) to transitions between two bound states (including both protons and neutrons), and to 'resonant' neutron transitions, namely those for which at least one of the two states has a 'resonant' character. In Fig. 4 we compare the full \( 2^+ \) and \( 3^- \) strengths with the correspondent resonant contributions. It is seen that the latter reproduce quite well the shape of the full response.

It is also interesting to perform a new RPA calculation excluding the transitions between bound states and the resonant transitions from the diagonalization. This produces a featureless response which accounts for about 5% of the EWSR \( 2^+ \) in the range 0-20 MeV (cf. the dotted lines in the insets of Fig. 3). Performing instead a new RPA calculation including in the diagonalization only the transitions between bound states and the resonant transitions one finds that the resulting quadrupole strength displays a very collective peak at \( E \approx 12 \) MeV with a narrow width of the order of 2 MeV, which accounts for about 30% of the EWSR (cf. the dashed line in the inset of Fig. 3(a)). A small peak around 4.5 MeV is also visible. This originates from low-energy transitions between the well-bound proton states. The octupole strength instead displays two peaks of comparable strength located at about 6 MeV and 20 MeV. The coupling with continuum states increases the strength of the peaks and lowers their energy, producing the full strength: in macroscopic terms, this can be associated with a decrease of the surface tension, taking place when the nucleus is immersed in the neutron fluid, as well as with the increase of the mass of the fluid which takes part in the motion [32].

From the analysis described above, we conclude that the collective response of the system is largely driven by the transitions involving the deeply bound protons and

\[
\begin{array}{cccccc}
 n_h & l_h & j_h & E_h & n_p & l_p & j_p & E_p & E_{ph} & T \text{[MeV}}^2 \text{fm}^{-2}] \\
 1 & 5 & 9/2 & -9.3 & 5 & 7 & 13/2 & 8.2 & 17.5 & 83.5 \\
 1 & 5 & 9/2 & -9.3 & 6 & 7 & 13/2 & 10.3 & 19.6 & 60.8 \\
 1 & 6 & 13/2 & -5.8 & 5 & 8 & 17/2 & 9.5 & 15.3 & 154.4 \\
 1 & 6 & 13/2 & -5.8 & 6 & 8 & 17/2 & 11.4 & 17.2 & 88.7 \\
 2 & 6 & 11/2 & 0.1 & 7 & 8 & 15/2 & 17.0 & 16.9 & 69.0 \\
 2 & 6 & 11/2 & 0.1 & 8 & 8 & 15/2 & 20.6 & 20.5 & 67.4 \\
 3 & 7 & 15/2 & 2.2 & 7 & 9 & 19/2 & 18.1 & 15.9 & 126.4 \\
 3 & 7 & 15/2 & 2.2 & 8 & 9 & 19/2 & 21.6 & 19.4 & 79.4 \\
\end{array}
\]

TABLE I. List of the eight unperturbed neutron particle-hole transitions in the \(^{136}\text{Sn}\) WS cell and associated with the largest transition strengths calculated with the operator \( dU/drY_{2M} \).

In the first four columns we give principal quantum number, the orbital angular momentum, the total angular momentum and the energies \( n_h, l_h, j_h \) and \( E_h \) of the hole; in the next four columns, the corresponding quantities \( n_p, l_p, j_p \) and \( E_p \) for the particle. In the last two columns, we give the energy \( E_{ph} = E_p - E_h \) and the transition strength \( T \) associated with the transition. All energies are in MeV.

\[
\begin{array}{cccccc}
 n_h & l_h & j_h & E_h & n_p & l_p & j_p & E_p & E_{ph} & T \text{[MeV}}^2 \text{fm}^{-2}] \\
 1 & 5 & 11/2 & -14.1 & 5 & 8 & 17/2 & 9.5 & 24.6 & 139.0 \\
 1 & 5 & 11/2 & -14.1 & 6 & 8 & 17/2 & 11.4 & 25.5 & 82.2 \\
 1 & 5 & 9/2 & -9.3 & 8 & 8 & 15/2 & 20.6 & 29.9 & 67.4 \\
 1 & 6 & 13/2 & -5.8 & 7 & 9 & 19/2 & 18.1 & 23.8 & 133.7 \\
 2 & 6 & 11/2 & 0.1 & 9 & 9 & 17/2 & 28.3 & 28.2 & 71.8 \\
 3 & 7 & 15/2 & 2.2 & 5 & 8 & 17/2 & 9.5 & 7.3 & 114.5 \\
 3 & 7 & 15/2 & 2.2 & 8 & 10 & 21/2 & 25.2 & 22.9 & 101.2 \\
 3 & 7 & 15/2 & 2.2 & 9 & 10 & 21/2 & 29.3 & 27.1 & 108.7 \\
\end{array}
\]

TABLE II. The same as in Table I for the operator \( dU/drY_{3M} \).

| \( l \) | \( j \) | \( E_{ij} \) [MeV] | \( \Gamma_{ij}^{++} \) [MeV] |
|---------|---------|----------------|----------------|
| 4       | 9/2     | -2.1           | 0.08           |
| 4       | 7/2     | -0.2           | 1.0            |
| 6       | 11/2    | 0.2            | 0.04           |
| 7       | 15/2    | 2.4            | 0.08           |
| 7       | 13/2    | 8.7            | 2.5            |
| 8       | 17/2    | 9.9            | 1.7            |
| 8       | 15/2    | 20.3           | 8.0            |
| 9       | 19/2    | 18.3           | 4.5            |
| 10      | 21/2    | 28.7           | 9.0            |

TABLE III. Total and orbital angular momentum \( l, j \), energies \( E_{res} \) and widths \( \Gamma_{res} \) of the resonant neutron single particle states calculated extrapolating the HF potential of the \(^{136}\text{Sn}\) WS cell up to infinity. The continuum spectrum starts at \( E_{cont} = -3.5 \) MeV.

The resonant neutron levels. However, the coupling with the continuum shifts the energy of the peaks to lower energy and increases considerably the width of the states, as well as the strength of the low-energy peaks. We remark that the low-lying part of the spectrum would be modified by neutron pair correlations, not taken into account in the present study, although there are no resonant single-particle states lying very close to the Fermi energy \( E_F = 5.5 \) MeV, cf. Table III.
FIG. 4. Neutron (left panel) and proton (right) transition densities associated with the four strongest RPA transitions calculated in the response to the operator $dU/dr_{2M}$ for the $^{1364}\text{Sn}$ WS cell, shown in Fig. 2(a). The energies of the transitions are listed in the legends of the figures.

FIG. 5. The HF (dashed line) and RPA (solid line) quadrupole strength functions calculated in $^{1364}\text{Sn}$, are compared to those calculated in a WS cell with 50 protons and 654 neutrons and a radius $R_{\text{box}}=25$ fm, corresponding to about the same asymptotic neutron density (dashed and dash-dotted lines).

FIG. 6. (a) The RPA quadrupole strength function calculated for the $^{1364}\text{Sn}$ WS cell (solid line, cf. Fig. 2(a)), is compared with the strength obtained including only the amplitudes associated with proton and resonant neutron particle-hole transitions (dashed line, cf. text for details). In the inset, the full RPA quadrupole strength function is instead compared with the RPA strength calculated using a restricted particle-hole basis limited either to proton states and to bound and resonant neutron states (dashed line), or to neutron continuum states (dotted line) (b). The same as (a), for the operator $dU/dr_{3M}$. 
TABLE IV. List of the eight unperturbed neutron particle-hole transitions calculated in the $^{498}$Zr WS cell and associated with the largest transition strengths with the operator $dU/drY_{ZM}$. In the first four columns we give the orbital angular momentum, the total angular momentum and the energies $l_h, j_h$ and $E_h$ of the hole; in the next four columns, the corresponding quantities $l_p, j_p$ and $E_p$ for the particle. In the last two columns, we give the energy of the particle-hole jump $E_{ph} = E_p - E_h$ and its transition strength $T$. All energies are in MeV.

| $n_h$ | $l_h$ | $j_h$ | $E_h$ | $n_p$ | $l_p$ | $j_p$ | $E_p$ | $E_{ph}$ | $T [\text{MeV}^{2}\text{fm}^{-4}]$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-----------|------------------|
| 1     | 4     | 7/2   | -12.3 | 6     | 6     | 11/2  | 8.0   | 20.3      | 58.2             |
| 1     | 4     | 7/2   | -12.3 | 7     | 6     | 11/2  | 9.0   | 21.3      | 79.8             |
| 1     | 5     | 11/2  | -7.6  | 7     | 7     | 15/2  | 10.1  | 17.7      | 198.6            |
| 1     | 5     | 9/2   | -1.3  | 10    | 7     | 13/2  | 18.6  | 19.9      | 69.7             |
| 1     | 5     | 9/2   | -1.3  | 11    | 7     | 13/2  | 21.4  | 22.7      | 59.2             |
| 2     | 6     | 13/2  | 1.5   | 9     | 8     | 17/2  | 17.3  | 15.8      | 101.4            |
| 2     | 6     | 13/2  | 1.5   | 8     | 8     | 17/2  | 19.4  | 17.9      | 120.9            |
| 2     | 6     | 13/2  | 1.5   | 8     | 8     | 17/2  | 22.2  | 20.7      | 68.1             |

TABLE V. The same as in Table IV for the operator $dU/drY_{SM}$.

| $l$   | $j$   | $E_{res}$ [MeV] | $\Gamma_{res}$ [MeV] |
|-------|-------|-----------------|----------------------|
| 4     | 9/2   | 4.7             | 2.0                  |
| 6     | 13/2  | 1.6             | $5 \times 10^{-3}$   |
| 6     | 11/2  | 8.8             | 1.2                  |
| 7     | 15/2  | 10.1            | 1.0                  |
| 7     | 13/2  | 21.0            | 6.5                  |
| 8     | 17/2  | 19.0            | 3.7                  |
| 9     | 19/2  | 30.3            | 8.5                  |

TABLE VI. Total and orbital angular momentum $lj$, energies $E_{res}$ and widths $\Gamma_{res}$ of the resonant neutron single particle states calculated in the $^{498}$Zr WS cell. The continuum spectrum starts at $E_{cont} = -0.5$ MeV.

(b) The case of $^{498}$Zr

In this section we shall discuss the results of a study analogous to that carried out for $^{1364}$Sn, but this time for the $^{498}$Zr WS cell. The HF and RPA quadrupole and octupole strength functions are shown in Fig. 4. The main differences as compared to $^{1364}$Sn are found in the low-energy part of the response, where one notices the absence of the low-lying peak in the quadrupole strength and the presence of a strong, very low-energy peak in the octupole response. Both these features are related to the change occurring in the proton single-particle spectrum going from $Z = 50$ to $Z = 40$. In fact in $^{1364}$Sn one finds a gap of about 3 MeV between the last occupied proton orbit $1g_{9/2}$ and the first unoccupied orbit $1g_{7/2}$. Low-lying quadrupole transitions are then possible between the $1g_{9/2}$ orbital and the $1g_{7/2}$ or $2d_{5/2}$ orbitals. On the other hand, the lowest octupole transitions have an energy of about 6 MeV ($1h_{11/2}$). In $^{498}$Zr the $1g_{9/2}$ orbital is not occupied and low-energy octupole transitions are possible ($p_{3/2} \rightarrow g_{9/2}$ with an energy of 2.7 MeV), and only $2h_{11/2}$ quadrupole transitions are possible. The application of these results to the actual case of the inner crust is model dependent. As we indicated above, detailed calculations of the isotopic composition of the crust predict different values for the favoured proton number, typical values being $Z \approx 40$ or $Z \approx 50$. In Figs. 11(e),(f) below we show the strength functions calculated for $^{506}$Sn ($Z = 50$). As expected, in this case one recovers the low-lying peak in the quadrupole response while the low-lying peak in the octupole response disappears.

Proton pairing correlations would lead to the partial occupation of the $1g_{9/2}$ orbital, allowing the simultaneous presence of quadrupole and octupole low energy transitions. The low-lying part of the spectrum might also be affected by neutron pairing correlations.

The analysis of the neutron transitions leads to the same conclusions presented above in the case of $^{1364}$Sn. On the one hand, the transition densities associated with the strongest quadrupole phonons (cf. Fig. 5) have a very clear surface character. On the other hand, the strongest neutron particle-hole transitions (listed in Tables IV and V) are again associated with the resonant single-particle states identified by the phase shifts and listed in Table VI.

Furthermore, the analysis of the strength functions (cf. Fig. 9) shows that the shape of the response is well reproduced including only the amplitudes over proton and resonant states. Similarly to $^{1364}$Sn, also in $^{498}$Zr a RPA calculation including only proton and resonant states produces a narrow collective peak in the quadrupole strength, close to 13 MeV (cf. the dashed curve in the inset of Fig. 9). Compared to the $^{1364}$Sn, however, the coupling to the neutron continuum produces a smaller shift of the peak and a smaller increase of the width, due to the reduced density of the system.
FIG. 7. Quadrupole (a) and octupole (b) strength functions calculated in the $^{498}$Zr WS cell. Black histograms refer to the discrete RPA response, while red and blue histograms refer to the discrete HF response for neutron and protons respectively (in units of MeV$^2$ fm$^{-2}$). The solid curve (in units of MeV fm$^{-2}$) is obtained by a convolution of the discrete RPA strength with a Lorentzian function having a FWHM equal to 1 MeV. The dashed and dash-dotted curve are obtained convoluting the neutron and proton discrete HF strength respectively.

FIG. 8. Neutron (left panel) and proton (right) transition densities associated with the four strongest RPA transitions calculated in the response to the operator $dU/drY_{2M}$ for the $^{498}$Zr WS cell, shown in Fig. 7(a). The energies of the transitions are listed in the legends of the figures.

FIG. 9. (a) The RPA quadrupole strength function calculated for the $^{498}$Zr WS cell (solid line, cf. Fig. 7(a)), is compared with the strength obtained including only the amplitudes associated with proton and resonant neutron particle-hole transitions (dashed line, cf. text for details). In the inset, the RPA quadrupole strength function is instead compared with the RPA strength calculated using a restricted particle-hole basis limited either to proton states and to resonant neutron states (dashed line) or to continuum neutron states (dotted line). (b) The same as (a), for the operator $dU/drY_{3M}$. 
Having seen that the resonant single-particle states largely determine the $2^+$ and $3^-$ response to the $dU/drY_{LM}$ operator, one may expect that the general features of the response to the $dU/dr$ operator should evolve in a reasonably continuous fashion going from the atomic nucleus into the inner crust. In Fig. 12 we compare in a schematic way the quadrupole and octupole neutron particle-hole transitions calculated $^{136}_{50}$Sn and $^{49}_{20}$Zr with those calculated in the closed-shell, neutron-rich nucleus $^{132}_{50}$Sn. The horizontal lines indicate the energies of the neutron bound levels or of the resonant levels (cf. Tables III and VI for $^{136}_{50}$Sn and $^{49}_{20}$Zr). The solid and dashed lines indicate the strongest quadrupole and octupole transitions connecting bound or resonant levels (cf. Tables I and II for $^{136}_{50}$Sn and Tables IV and V for $^{49}_{20}$Zr). One can clearly recognize that the shell structure plays a similar role in the three cases, leading to unperturbed particle-hole transitions characterized by an energy $2\hbar\omega_n$ (for quadrupole transitions) or $1\hbar\omega_n$ and $3\hbar\omega_n$ (for octupole transitions).

In Fig. 11 we compare the the $2^+$ and $3^-$ strength functions in the inner crust with those of atomic nuclei. We keep the same number of protons ($Z = 50$) and increase the number of neutrons, going from the neutron-rich, closed-shell nucleus $^{132}_{50}$Sn (panels (a),(b)) to the drip-line, closed-shell nucleus $^{176}_{50}$Sn ((c),(d)) and then to the WS cell $^{506}_{50}$Sn ((e),(f)) and finally to $^{1364}_{50}$Sn ((g),(h)). The calculation of $^{506}_{50}$Sn is very similar to that discussed above for $^{49}_{20}$Zr, the only difference being the number of protons. The proton and neutron mean-field potentials are compared in Fig. 10.

One can see from Fig. 11 that the main features of the giant resonances are the same in the four cases. The centroids of the main peaks, associated with $2\hbar\omega$ transitions in the quadrupole response and with $1\hbar\omega$ and $3\hbar\omega$ in the octupole response, are lowered going from $^{132}_{50}$Sn to $^{1364}_{50}$Sn. This is in part due to the shift of the unperturbed response, associated with the larger radii of the potentials, particularly in the case of protons, and in part is caused by the coupling with the neutron continuum, as we discussed above. The coupling also increases the width of the peaks. A more complete study should include the coupling to $2p - 2h$ configurations.

In Fig. 11(a),(b) we show also the strength functions obtained with the operators $r^2Y_{2M}$ and $r^3Y_{3M}$ in the case of $^{132}_{50}$Sn. Their shape essentially coincide with those calculated making use of the operator $dU/drY_{LM}$; this is not the case for the calculations performed in the inner crust, as discussed in Appendix B.
FIG. 10. Neutron (a) and proton (b) mean-field potentials calculated for the nuclei $^{132}\text{Sn}$ and $^{176}\text{Sn}$ and for the WS cells $^{506}\text{Sn}$ and $^{1364}\text{Sn}$. 
FIG. 11. Evolution of the quadrupole and octupole response going from the neutron-rich nucleus $^{132}$Sn to the drip line and into the inner crust. We show the strength functions of the operators $dU/drY^2_M$ and $dU/drY^3_M$ calculated in $^{132}$Sn (panels (a) and (b)), $^{176}$Sn (panels (c) and (d)), $^{506}$Sn (panels (e) and (f)) and in $^{1364}$Sn (panels (g) and (h), already shown in Fig. 2). In all the panels, the black histograms refer to the discrete RPA response while the red and blue histograms refer to the discrete HF response for neutrons and protons respectively (in units of MeV fm$^{-2}$). The red dashed, blue dash-dotted and solid curves (in units of MeV fm$^{-2}$) are obtained convoluting the neutron and proton HF responses and the RPA response with a Lorentzian function having a FHWM equal to 1 MeV. In panels (a) and (b) we also show the quadrupole and octupole responses associated with the operators $r^2Y^2_M$ (in units of fm$^{-4}$ and multiplied by 4) and $r^3Y^3_M$ in $^{132}$Sn (in units of fm$^{-6}$ and multiplied by 120).
FIG. 12. The strongest neutron particle-hole transitions in $^{132}$Sn (a), in $^{498}$Zr (b) and in $^{1364}$Sn (c), see text for details. Single-particle levels of even and odd parity are drawn by solid and dashed lines. Solid (red) and dashed (blue) arrows refer to $2^+$ and to $3^-$ transitions. The Fermi energy is represented by the thick dashed line.
IV. CONCLUSIONS

As Negele and Vautherin noticed in their seminal paper, 'the degree to which the nuclei in the free neutron regime resemble ordinary nuclei' is striking, and 'this similarity is also manifested in the behaviour of the single-particle energies' [23]. In this work we have shown that the persistence of the shell structure influences the linear response of the Wigner-Seitz cell in the inner crust to a large extent. We have found it useful to study the ear response of the Wigner-Seitz cell in the inner crust the persistence of the shell structure influences the lin-

utrations, the surface fluctuations studied in this paper are important to examine the coupling of neutrons with lattice pinning [5, 6, 17]. In this respect, it would be also im-

portant to focus on the effects associated with those of ordinary atomic nuclei. However, the interaction of the bound nucleons with the neutron sea, has an important effect, leading to a decrease of the energy of the peaks, and to an increase of their width.

The existence of vibrational modes associated with the nuclear surface can have important consequences on the neutron superfluidity of the system in the $1S_0$ channel. While medium effects tend to suppress pairing correlations in neutron matter, being dominated by spin fluctuations, the surface fluctuations studied in this paper lead to an attractive contribution. Even if the global effects may not be large, due to small volume occupied by the nucleus, they can affect the spatial dependence of the pairing gap, and have important consequences on vortex pinning [5, 6, 17]. In this respect, it would be also im-

portant to examine the coupling of neutrons with lattice vibrations [32, 36].

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VI. APPENDIX A

In this Appendix we shall compare the results of numerical calculations of the linear response in a WS cell without protons with analytical results which are available in uniform matter in the case of Skyrme interactions 33. We shall consider the excitation of either density or spin modes, computing the response function to the external fields

$$V_{ext}(\vec{r}) = e^{i\vec{q}\cdot\vec{r}}\Theta^{(\alpha)},$$  (4)

where $\alpha = 0, 1$ distinguishes the two possible channels $S = 0$ and $S = 1$: $\Theta^{0} = 1$, $\Theta^{1} = \sigma_{z}$ according to whether density or spin modes are considered.

In neutron matter, the RPA strength function can be written as

$$S^{(\alpha)}(q, E) = -\frac{1}{\pi}Im\Pi^{(\alpha)}(q, E) =$$

$$-\frac{1}{\pi} \int d^{3}r d^{3}r' V_{ext}(\vec{r}')V_{ext}(\vec{r})Im\Pi(\vec{r}, \vec{r}', E),$$  (5)

where $\Pi(\vec{r}, \vec{r}', E)$ is the polarization function 34.

The RPA strength function calculated in neutron matter with a Skyrme effective interaction reads

$$\Pi^{(\alpha)}(\nu, k) = 2\Pi_{0}[1 - W^{(\alpha)}_{2}\Pi_{0}]$$

$$-2W^{(\alpha)}_{2}k_{F}^{2}\left(k^{2} - \frac{\nu^{2}}{1 -(m_{k}k_{F}/\nu)^{2}W_{2}^{(\alpha)}}\right)\Pi_{0}$$

$$+ \left(W^{(\alpha)}_{2}k_{F}^{2}\right)^{2}(\Pi_{2} - \Pi_{0}\Pi_{4} + 4k_{F}^{2}\nu^{2}\Pi_{0}^{2} - \frac{2m_{k}k_{F}}{\nu}k_{2}^{2}\Pi_{0})]^{-1}$$

$$+ 2W^{(\alpha)}_{2}k_{F}^{2}(2k_{2}\Pi_{0} - \Pi_{2})$$  (6)

where $\Pi_{2}$ e $\Pi_{4}$ are generalized Lindhard functions given in [33], and we have introduced the adimensional variables $k = \frac{k}{k_{F}}$ and $\nu = \frac{m_{k}E}{k_{F}^{2}q^{2}}$, expressed in terms of the Fermi momentum $k_{F}$ and of the effective mass $m_{k}$. The coefficients $W^{(\alpha)}_{i}$ can be expressed in terms of the usual coefficients defining the Skyrme interactions. In neutron matter one finds

$$W^{0}_{1} = t_{0}(1 - x_{0}) + \frac{1}{4}t_{1}q^{2}(1 - x_{1}) - \frac{3}{4}t_{2}q^{2}(1 + x_{2})$$

$$+ \frac{t_{3}}{6}\rho g(1 - x_{3})\left(\gamma + 1\right)\left(\gamma + 2\right)\frac{2}{\nu}$$

$$W^{1}_{1} = -t_{0}(1 - x_{0}) - \frac{1}{4}t_{1}g^{2}(1 - x_{1}) - \frac{1}{4}t_{2}g^{2}(1 + x_{2})$$

$$- \frac{t_{3}}{6}\rho g(1 - x_{3})$$

$$W^{0}_{2} = \frac{1}{4}t_{1}(1 + x_{1}) + \frac{3}{4}t_{2}(1 + x_{2})$$

$$W^{1}_{2} = -\frac{1}{4}t_{1}(1 - x_{1}) + \frac{1}{4}t_{2}(1 + x_{2})$$  (7)

The unperturbed response is obtained substituting the Lindhard function $\Pi_{0}$ in place of $\Pi(q, E)$.

We shall compare these analytic results with the numerical calculations performed putting 498 neutrons in a WS cell of radius $R_{WS} = 42.2$ fm without protons. In the latter case the response function reads

$$S^{(\alpha)}_{WS}(q, E) = \sum_{n=0}^{\infty} |n\rangle \langle n| e^{i\vec{q}\cdot\vec{r}}\Theta^{(\alpha)} | 0 >^{2} L(E, E_{n}),$$  (8)

where $|0\rangle$ is the ground-state, $|n\rangle$ are the (mean-field or RPA) excited states of the system, and the computed discrete response has been convoluted with a Lorentzian function $L(E, E_{n})$. 
Exploiting the spherical symmetry of the system, one can introduce the multipole decompositon of the external operator

$$e^{i\vec{q}\cdot\vec{r}} = 4\pi \sum_{L=0}^{\infty} \sum_{M=-L}^{+L} i^L j_L(qr) Y_{LM}^*(\Omega_q) Y_{LM}(\Omega_r),$$

where $j_L(qr)$ is a spherical Bessel function, leading to an analogous decomposition of the response in the WS cell:

$$s_{WS,L}^{(n)}(q, E) = \sum_{L=0}^{\infty} s_{WS,L}(q, E).$$

The RPA excited states are given by

$$|n\rangle \equiv |\nu_{JM}\rangle = \sum_{n\text{ph}} \left( X_{\nu_{ph}}(J)|p(h)^{-1}\rangle + Y_{\nu_{ph}}(J)(-)^{J+M}|(p)^{-1}h\rangle \right),$$

where the $X$ and $Y$ denote the forward and backward amplitudes calculated in RPA, and one obtains for density modes ($S = 0, J = L$)

$$s_{WS,L}^{(0)}(q, E) = 4\pi(2L + 1) \sum_{\nu=0}^{\infty} \sum_{j_{\nu} j_{\nu}} \left( X_{\nu_{ph}}(L) + \right.$$

$$\left. (-)^{L} Y_{\nu_{ph}}^*(L) \right) \langle \nu_{ph}||j^L_{Y_{\nu_{ph}}}||R_{\nu_{ph}}||j_{\nu_{ph}}(qr)||R_{\nu_{ph}}\rangle \right) \right)^2 L(E, \nu).$$

The contributions to the $S = 0$ response in the cell associated with several multipoarities are shown in Fig. 13 for the momentum transfer $q = 0.51 \text{ fm}^{-1}$. It is seen that the contributions from the different multipoarities increase with $L$ reaching the maximum for $L = 15$, and then rapidly decrease, becoming negligible for $L$ larger than about 20. This is in keeping with the simple estimate of the maximum angular momentum which can be transferred by the external field, given by $L_{\text{max}} \sim q R_{\text{WS}} \sim 21$.

The total response $s_{WS}(q, E)$ to the external fields is compared in Fig. 14 with the analytic result, in uniform neutron matter at the same density ($\rho \sim 0.01 \rho_0$) and for the same value of $q = 0.51 \text{ fm}^{-1}$, both in the unperturbed and in the RPA case. We find reasonable agreement with the calculations in the WS cell. The strength functions computed in the WS cell display a tail, which is not present in the uniform system, where the excitation energy of the system has a maximum value, imposed by energy and momentum conservation and equal to

$$\hbar \omega = \frac{\hbar^2}{2m_k}(q^2 + 2qk_f),$$

which, in the present case, corresponds to about 13.5 MeV, taking into account that the neutron effective mass very close is very close to the bare mass ($m_k = 0.99 \text{ m}$) at the present density. The analytic Energy Weighted Sum Rule (EWSR) associated with the unperturbed response function $s(q, E)$ is given by

$$\left( \frac{EWSR_{HF}}{V} \right)_{\text{uni}} = \rho \frac{\hbar^2 q^2}{2m_k},$$

while the RPA response function in the $S = 0$ channel is given by

$$\left( \frac{EWSR_{RPA}^{S=0}}{V} \right)_{\text{uni}} = \rho \frac{\hbar^2 q^2}{2m}. \quad (14)$$

FIG. 13. Response function per unit volume as a function of the energy in the $S = 0$ channel, computed in a WS cell of 42.2 fm radius containing 498 neutrons. The contributions from several multipoarities are shown. The linear momentum transferred by the external field is $q = 0.51 \text{ fm}^{-1}$.

FIG. 14. Response functions per unit volume as a function of the energy in the $S = 0$ and $S = 1$ channel, computed in a WS cell of 42.2 fm radius cell containing 498 neutrons (dashed lines) and in uniform neutron matter (solid lines) for the linear momentum $q = 0.51 \text{ fm}^{-1}$ transferred from the external field. Both systems correspond to a Fermi momentum $k_F = 0.39 \text{ fm}^{-1}$.

The Energy Weighted Sum Rule (EWSR) in the WS cell is numerically calculated as the sum of the EWSR associated to different multipoarities $J$:

$$EWSR(e^{i\vec{q}\cdot\vec{r}}) = 4\pi \sum_L EWSR(\nu_{\nu_{ph}}(qr)). \quad (15)$$

The contribution calculated for each $L$ can be compared to the value obtained evaluating the classical EWSR on the HF ground state:

$$EWSR_{\text{classical}}(\nu_{\nu_{ph}}(qr)) =$$

$$\hbar^2 \frac{2\lambda + 1}{4\pi} N < 0 \left( \left( \frac{d}{dr} j_L(qr) \right)^2 + L(L+1) \left( \frac{2j_L(qr)}{r} \right)^2 \right) |0>, \quad (16)$$
The cumulative EWSR is shown in Fig. 15 for the RPA calculation. The total value and turns out to be only a few percent larger than the analytic value.

![Graph showing cumulative EWSR](image)

**FIG. 15.** EWSR per unit volume calculated in the $S = 0$ channel for a WS cell of 42.2 fm radius containing 508 neutrons, which corresponds to a Fermi energy $k_F = 0.39$ fm$^{-1}$. The squares show the cumulative EWSR obtained from the RPA strength function as a function of the orbital angular momentum $L$, while the dots are obtained using Eq. (16). The arrow indicates the analytic value for the uniform system (cf. Eq. (14)).

We conclude that the numerically computed response in the WS cell without protons is quite close to the response calculated analytically in uniform neutron matter at the same density. This gives us confidence in our numerical approach, which is applied in the main text to the case of a WS cell with a nucleus in its center.

**VII. APPENDIX B**

In ref. 21, 22, the strength associated with the operator $r^L$ was calculated in the inner crust in RPA and in QRPA (see also 20). Usually the operator $r^L$ is used as a convenient approximation to the operator $e^{i\vec{q}\cdot\vec{r}}$ in the limit of long wavelengths. In the inner crust one is dealing with boxes with of the order of 20 fm and with Fermi energies of the order of several MeV, so that the produce $k_F R$ is not small and the choice of this operator does not seem to be well motivated.

Nevertheless, we have calculated the strength function associated with $r^2 Y_{2m}$ for the cells $^{136}S_{n}$ and $^{498}Z_{r}$ in order to compare with the results obtained in 22. The strength functions are shown in Fig. 16 where we also show calculations for two cells without protons and with a similar number of nucleons, indicated as $^{131}X$ and $^{508}X$. In the present case, a cut off $E_{cut} = 30$ MeV is sufficient to exhaust the EWSR.

Our results coincide with those calculated by the Orsay group for the same cells 37, and the strength function show the features discussed in refs. 21 and 22, displaying a strong bump at low energy (‘named supergiant resonance’ in refs. 21, 22). The presence of the nucleus does not change the energy dependence of the strength, except for the fine details of the fragmentation of the discrete peaks produced by Landau damping. The main effect of the nucleus is the decrease of the absolute value of the EWSR: the EWSR for $^{131}X$ is about 20% smaller than for $^{136}S_{n}$, due to the fact that in the latter about 150 neutrons are bound in the nucleus, and couple to the $r^2$ operator much less efficiently than the unbound neutrons. The neutron transition densities associated with the vibrations building the main peak are shown in Fig. 17 for the $^{131}X$ and for the $^{136}S_{n}$ cell. In both cases they show large values in the middle of the cell. The nucleus in the $^{136}S_{n}$ case only induces a modest increase of the transition densities for $r \approx 10$ fm for two of the phonons. These results are obviously in keeping with the fact that the $r^2$ operator weights heavily the region far from the nucleus, at variance from the $dU/dr Y_{LM}$ operator considered in the main text.
FIG. 16. (top) HF and RPA quadrupole strength functions calculated in the cells $^{1364}\text{Sn}$ (a) and $^{1314}\text{X}$ (b), in units of MeV$^{-1}$ fm$^4$. (bottom) The same but for the cell $^{498}\text{Zr}$ (c) and $^{508}\text{X}$ (d). The transition strengths of the discrete states calculated in RPA are also shown in histogram form (in units of fm$^4$).

FIG. 17. Neutron transition densities associated with the strongest phonons calculated in $^{1314}\text{X}$ (a) and in $^{1364}\text{Sn}$ (b). The energy of the phonons is indicated.
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