AN EXPLICIT BOUND ON REDUCIBILITY OF MOD $l$ GALOIS IMAGE FOR DRINFE LD MODULES OF ARBITRARY RANK AND ITS APPLICATION ON THE UNIFORMITY PROBLEM

CHIEN-HUA CHEN

Abstract. Suppose we are given a Drinfeld Module $\phi$ over $\mathbb{F}_q[t]$ of rank $r$ and a prime ideal $l$ of $\mathbb{F}_q[T]$. In this paper, we prove that the reducibility of mod $l$ Galois representation

$$\text{Gal} (\overline{\mathbb{F}}_q(T)^{\text{sep}}/\mathbb{F}_q(T)) \to \text{Aut}(\phi[l]) \cong \text{GL}_r(F_l)$$

gives a bound on the degree of $l$ which depends only on the rank $r$ of Drinfeld module $\phi$ and the minimal degree of place $P$ where $\phi$ has good reduction at $P$. Then, we apply this reducibility bound to study the Drinfeld module analogue of Serre’s uniformity problem.

1. Introduction

In [Ser72], Serre proved the well-known open image theorem for the elliptic curves $E$ over $\mathbb{Q}$ without complex multiplication. Let $E[n]$ be the group of $n$-torsion points in $E(\overline{\mathbb{Q}})$, and let $l$ be a prime number. The open image theorem then implies that there is an integer $N$ depending on $E$ such that the mod $l$ Galois representation

$$\overline{\rho}_{E,l}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(E[l]) \cong \text{GL}_2(F_l)$$

is surjective whenever $l > N$. Serre asked whether the bound $N$ can be made independent of the choice of an elliptic curve, which is the famous “Serre’s uniformity problem” (the bound $N$ is expected to be 37).

Serre’s question is still open but thanks to the works of Serre [Ser72], Mazur [Maz78], Bilu and Parent [BP11], we know that if there is some prime $l > 37$ whose mod $l$ Galois representation is not surjective, then the Galois image must be contained in the normalizer of a non-split Cartan subgroup of $\text{GL}_2(F_l)$.

In this paper, we consider the function field analogue of Serre’s uniformity problem. Let $q = p^e$ be a prime power, $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. In [PR09a], Pink and Rütsche proved the open image theorem for Drinfeld $A$-modules over $F$ of rank $r$ with generic characteristic and without complex multiplication. Thus for every such Drinfeld module $\phi$, there is an integer $N$ depending on $r, q$, and $\phi$ such that for any prime $l$ of $A$ the mod $l$ Galois representation

$$\overline{\rho}_{\phi,l}: \text{Gal}(F^{\text{sep}}/F) \to \text{Aut}(\phi[l]) \cong \text{GL}_r(A/l)$$

is surjective whenever the degree $\deg_T l$ of $l$ is larger than $N$.

In the work of Chen and Lee [CL19], they focused on rank-2 Drinfeld $A$-modules over $F$ and gave an explicit estimation on the bound $N$. Our first result is to apply their strategy to give an explicit bound on the irreducibility of mod $l$ Galois representations for Drinfeld $A$-modules over $F$ of arbitrary rank $r$.

Main Theorem 1. Let $\phi_T = T + g_1 T + \cdots + g_r T^r$ be a Drinfeld $A$-module over $F$ of rank $r$ such that $\phi$ has good reduction at a prime $l$ of $A$ and the mod $l$ Galois representation $\overline{\rho}_{\phi,l}: GF \to \text{Aut}(\phi[l]) \cong \text{GL}_r(A/l)$ is reducible. Let $P$ be a prime of $A$ with minimal degree such that $\phi$ has
good reduction at $\mathcal{P}$. Then
\[
\deg_T I \leq \varphi(\Omega \cdot \Delta \cdot p^{2(r-1)} \cdot n_\phi) \deg_T \mathcal{P},
\]
where $\Omega = \text{l.c.m.}(q-1, q^2 - 1, \cdots, q^r - 1)$, $\Delta = \text{l.c.m.}(|\text{GL}_1(F_q)|, |\text{GL}_2(F_q)|, \cdots, |\text{GL}_{r-1}(F_q)|)$, $n_\phi \leq (q^r - 1)p^{r-1}$ is a constant depending on $\phi$, and $\varphi$ is the Euler $\varphi$-function.

One might propose the following function field analogue of Serre’s uniformity problem:

**Question.** There is an integer $N(q)$ depending only on $q$ such that for any rank-2 Drinfeld $A$-module $\phi$ over $F$ without complex multiplication, the mod $l$ Galois representation
\[
\bar{\rho}_{\phi,l} : \text{Gal}(F_{\text{sep}}/F) \to \text{Aut}(\phi[l]) \cong \text{GL}_2(A/l)
\]
is surjective whenever $\deg_T I > N$.

In 2016, Gekeler [Gek16] studied the Galois image of rank-1 Drinfeld $A$-modules over $F$. We apply his result to show that the above function field analogue of Serre’s uniformity problem might be false if there is no restriction on the degree of bad reduction primes, see counterexample [L3] for more details. Therefore, we modify the uniformity problem for the analogue of Drinfeld modules as the Conjecture below:

**Conjecture.** Let $q = p^e$ an odd prime power, $A = \mathbb{F}_q[T]$, and $F = \mathbb{F}_q(T)$. There is an integer $N(q)$ depending only on $q$ such that for any rank-2 Drinfeld $A$-module $\phi$ over $F$ without complex multiplication, the mod $l$ Galois representation
\[
\bar{\rho}_{\phi,l} : \text{Gal}(F_{\text{sep}}/F) \to \text{Aut}(\phi[l]) \cong \text{GL}_2(\mathbb{F}_l)
\]
has image equal to a subgroup of $\text{GL}_2(\mathbb{F}_l)$ with index $|\text{GL}_2(\mathbb{F}_l) : \bar{\rho}_{\phi,l}(\text{Gal}(F_{\text{sep}}/F))| \leq q - 1$ whenever $\deg_T I > N$.

This conjecture can also be stated for Drinfeld modules over $F_q(T)$ of arbitrary rank, which we refer to Conjecture [H] in section 4.

Our second result is to apply Main theorem [H] and the classification of subgroups in $\text{GL}_2$ over finite fields to show that for a certain infinite family of rank-2 Drinfeld modules over $F$ where the bad reduction primes have bounded degree, we do have a uniform bound for surjective Galois image.

**Main Theorem 2.** Assuming $q$ is a power of an odd prime $p$ and $q \geq 5$. Let $d \in \mathbb{N}$ a fixed integer. Let $g_2 = T + g_1 T + g_2 T^2$ be a rank-2 Drinfeld $A$-module over $F$ with $g_1, g_2 \in A$ and the following properties:

1. There is a prime $\mathfrak{p}$ of $A$ such that $\mathfrak{p} | g_2$ but $\mathfrak{p} \nmid g_1$.
2. If we write $g_2 = p^h \cdot m$ with $(m, p) = 1$, then $p \nmid \alpha$ and $q - 1 \nmid \alpha$.
3. The primes of $A$ dividing $g_2$ have degree less than or equal to $d$.

Then there is an integer $N(q,d)$ depending only on $q$ and $d$ such that
\[
\bar{\rho}_{\phi,l} : \text{Gal}(F_{\text{sep}}/F) \to \text{Aut}(\phi[l]) \cong \text{GL}_2(A/l)
\]
is surjective whenever $\deg_T I > N$.

In this theorem, the idea for condition (1) and (2) is mainly from Zywina’s construction [Zyw11] on a rank-2 Drinfeld $A$-module over $F$ with surjective adelic Galois representation. The Galois image of inertia group at a stable bad reduction prime can contribute a nontrivial $p$-subgroup. However, unlike the classification of subgroups in $\text{GL}_2(\mathbb{F}_p)$, it is not enough to say a subgroup $\bar{\rho}_{\phi,l}(\text{Gal}(F_{\text{sep}}/F))$ of $\text{GL}_2(A/l)$ contains $\text{SL}_2(A/l)$ by providing only $p | |\bar{\rho}_{\phi,l}(\text{Gal}(F_{\text{sep}}/F))|$. Thus we need condition (2) on a stable bad reduction prime to ensure the $p$-component of the Galois image is large enough.
Remark. In 2010, Pál (Theorem 1.2 in [P10]) proved a function field analogue of Mazur’s result [Maz78] while restricting to rank-2 Drinfeld modules over \( \mathbb{F}_q(T) \). However, it is hard to generalize Pál’s result to Drinfeld modules over \( \mathbb{F}_q(T) \) due to nontrivial technical issues. Besides, there is no function field analogue of Bilu and Parent’s work [BPT1] on the split Cartan case so far.

2. Preliminaries

2.1. Notation

- \( q = p^r \) a prime, \( A := \mathbb{F}_q[T] \), and \( F := \mathbb{F}_q(T) \).
- \( G_F := \text{Gal}(\mathbb{F}_q(T)_{\text{sep}}/\mathbb{F}_q(T)) \).
- For a prime ideal \( \mathfrak{p} \) of \( A \), set \( F_{\mathfrak{p}} \) to be the completion of \( F \) at \( \mathfrak{p} \) and \( A_{\mathfrak{p}} \) to be its valuation ring. We fix a valuation \( \nu_{\mathfrak{p}} \) of \( F_{\mathfrak{p}} \) and denote by \( | \cdot |_{\mathfrak{p}} := q^{-\nu_{\mathfrak{p}}(\cdot)} \) its associated absolute value.
- For a prime ideal \( \mathfrak{l} \) of \( A \), set \( \mathbb{F}_{\mathfrak{l}} = A/\mathfrak{l} \).

3. Main Result

At the beginning, we let \( \phi_T = T + g_1 T + \cdots + g_r T^r \) be a Drinfeld \( A \)-module over \( F \) of rank \( r \) such that the following holds:

- (a) \( \phi \) has good reduction at a prime \( \mathfrak{l} \) of \( A \).
- (b) The mod \( \mathfrak{l} \) Galois representation \( \rho_{\mathfrak{l}} : G_F \to \text{Aut}(\phi[\mathfrak{l}]) \cong \text{GL}_r(\mathbb{F}_q) \) is reducible.

In other words, there is a \( G_F \)-invariant proper submodule \( V \) of the \( \mathbb{F}_q[G_F] \)-module \( \phi[\mathfrak{l}] \). After fixing a basis of \( \phi[\mathfrak{l}] \), the action of \( \sigma \in G_F \) on \( \phi[\mathfrak{l}] \) is a matrix of the form

\[
\begin{pmatrix}
\rho_N(\sigma) & * \\
0 & \rho_{\phi[\mathfrak{l}]/V}(\sigma)
\end{pmatrix}.
\]

Here \( \rho_N(\sigma) \) (resp. \( \rho_{\phi[\mathfrak{l}]/V}(\sigma) \)) we present the action of \( \sigma \in G_F \) on \( V \) (resp. \( \phi[\mathfrak{l}]/V \)).

Our first goal is to prove that there is a certain number \( N \) independent of the choice of \( \mathfrak{l} \) such that either \( \rho_N \) or \( \rho_{\phi[\mathfrak{l}]/V} \) is unramified at any finite place \( \mathfrak{p} \) and tamely ramified at \( \infty \), the place of infinity. For any finite place \( \mathfrak{p} \neq \mathfrak{l} \) where \( \phi \) has good reduction, the inertia group \( I_{\mathfrak{p}} \) acts on \( \phi[\mathfrak{l}] \) trivially. Hence \( I_{\mathfrak{p}} \) acts trivially on both \( V \) and \( \phi[\mathfrak{l}]/V \).

3.1. Ramification at \( \mathfrak{p} = \mathfrak{l} \). Let \( I_l \) be the inertia group at \( \mathfrak{l} \), we may consider the following connected-étale decomposition of \( \mathbb{F}_q[I_{\mathfrak{l}}] \)-modules:

\[
0 \to \phi[\mathfrak{l}]^0 \to \phi[\mathfrak{l}] \to \phi[\mathfrak{l}]^{\text{et}} \to 0.
\]

From [PR99] Proposition 2.7(i) and (ii), we know that \( I_l \) acts on \( \phi[\mathfrak{l}]^{\text{et}} \) trivially and the action of \( I_l \) on \( \phi[\mathfrak{l}]^0 \) is via a fundamental character. Thus \( \phi[\mathfrak{l}]^0 \) is an irreducible \( \mathbb{F}_q[I_{\mathfrak{l}}] \)-module. Since \( V \) is a \( \mathbb{F}_q[G_F] \)-module, we may also view \( V \) as a \( \mathbb{F}_q[I_{\mathfrak{l}}] \)-module. Consider the \( I_l \)-module \( V \cap \phi[\mathfrak{l}]^0 \), we have the following two situations thanks to the irreducibility of \( \phi[\mathfrak{l}]^0 \):

- (1) \( V \supset \phi[\mathfrak{l}]^0 \), this implies the inertia group \( I_l \) acts on \( \phi[\mathfrak{l}]/V \) trivially.
- (2) \( V \cap \phi[\mathfrak{l}]^0 = \{0\} \), this implies \( I_l \) acts on \( V \) trivially.

3.2. Ramification at place \( \mathfrak{p} \) where \( \phi \) has stable bad reduction. Suppose that \( \phi \) has stable bad reduction at \( \mathfrak{p} \) of rank \( 1 \leq d \leq r - 1 \). From Tate uniformization, we obtain a Drinfeld module \( \psi \) over \( A_{\mathfrak{p}} \) of rank \( d \) with good reduction and an \( A \)-lattice \( \psi \Lambda_{\mathfrak{p}} \) of rank \( r - d \). And we have a \( G_{F_{\mathfrak{p}}} \)-equivariant short exact sequence of \( A \)-modules

\[
0 \to \psi[\mathfrak{l}] \to \phi[\mathfrak{l}] \xrightarrow{\psi_{\mathfrak{l}}} \Lambda_{\mathfrak{p}}/I\Lambda_{\mathfrak{p}} \to 0.
\]

Therefore, the action of \( \sigma \in I_{\mathfrak{p}} \) on \( \phi[\mathfrak{l}] \) is via matrices of the type

\[
\begin{pmatrix}
I_d & * \\
0 & \rho(\sigma)
\end{pmatrix}.
\]

Here \( I_d \) is the \( d \times d \) identity matrix and \( \rho(\sigma) \) represents the action of \( \sigma \) on \( \Lambda_{\mathfrak{p}}/I\Lambda_{\mathfrak{p}} \).
To study the action of $\sigma \in I_p$ on $\Lambda_p$, we apply the norm created by Gardeyn \cite{Gar02}. First of all, it is known that the valuation $v_p(\lambda)$ is negative for any $0 \neq \lambda \in \Lambda_p$ and that
\[
 v_p(a \cdot \lambda) = v_p(\psi_a(\lambda)) = |a|^d v_p(\lambda).
\]
We define the following $\text{Gal}(F_p^{\text{sep}}/F_p)$-invariant norm $\| \cdot \|_p := (-v_p(\cdot))^{1/d}$, see p.243 in \cite{Gar02} for more details. Then we have
\[
\|a \cdot \lambda\|_p = |a|_\infty \|\lambda\|_p.
\]
Following the terminology in \cite{Gek19}, we let $B := \{\lambda_1, \lambda_2, \cdots, \lambda_{r-d}\}$ be a successive minimum basis of the lattice $\Lambda_p$. It is defined as below:

For each $1 \leq i \leq r - d$, $\lambda_i$ has minimum absolute value $\|\lambda_i\|_p$ among $\lambda \in \Lambda_p \setminus \sum_{1 \leq j < i} A \lambda_j$.

We collect basis vectors in $B$ of equal lengths to get packets $B_m$ with $|B_m| = r_m$ and order basis vectors in $B$ by absolute values of basis vectors:

\[
\|\lambda_1\|_p = \cdots = \|\lambda_{r_1}\|_p < \|\lambda_{r_1+1}\|_p = \cdots = \|\lambda_{r_1+r_2}\|_p < \cdots = \|\lambda_{r-d}\|_p.
\]

Thus we have $B = \bigcup_{1 \leq m \leq t} B_m$. Moreover, we set
\[
B_m = B_1 \cup \cdots \cup B_m \text{ and } \Lambda_m = \sum_{\lambda \in B_m} A \lambda \text{ for } 1 \leq m \leq t.
\]
The sequence of sublattices $\Lambda_0 := \{0\} \subset \Lambda_1 \subset \cdots \subset \Lambda_t = \Lambda_p$ is called the spectral filtration on $\Lambda_p$. Now we consider the norm of a linear of lattice elements. We have
\[
\| \sum_{i=1}^{r-d} a_i \lambda_i \|_p = \max_{1 \leq i \leq r-d} \{\|a_i \cdot \lambda\|_p\} = \max_{1 \leq i \leq r-d} \{|a_i|_\infty \|\lambda_i\|_p\}.
\]

By our definition of $\| \cdot \|_p$, the Galois group $\text{Gal}(F_p^{\text{sep}}/F_p)$ acts $A$-linearly on $\Lambda_p$ and also preserves the norm $\| \cdot \|_p$. Hence we have $\sigma \in \text{Gal}(F_p^{\text{sep}}/F_p)$ satisfies
\[
\sigma(\lambda_j) = \sum_{1 \leq i \leq r-d} a_{i,j} \cdot \lambda_i \text{ for } 1 \leq j \leq r - d, \text{ where } a_{i,j} \in A \text{ with } |a_{i,j}|_\infty \leq \|\lambda_j/\lambda_i\|_p.
\]

Therefore, the action of $\sigma \in \text{Gal}(F_p^{\text{sep}}/F_p)$ on $\Lambda_p$ is represented by $(r - d) \times (r - d)$ matrix of the form
\[
\begin{pmatrix}
B_1 & B_2 & \psi_{a_{1,j}} \\
0 & \ddots & \ddots \\
& & B_t
\end{pmatrix}.
\]
Here $B_m \in \text{GL}_{r_m}(\mathbb{F}_q)$ we present the action of $\sigma$ on $\Lambda_m/\Lambda_{m-1}$, the entries in upper triangular part $a_{i,j} \in A$ with $|a_{i,j}|_\infty \leq \|\lambda_i/\lambda_j\|_p$, and we write $\psi_{a_{i,j}}$ in upper triangular entries to emphasize the action of $a_{i,j}$ acting on lattice elements via $\psi$. 

\[
\begin{pmatrix}
B_1 & B_2 & \psi_{a_{1,j}} \\
0 & \ddots & \ddots \\
& & B_t
\end{pmatrix}.
\]
Let $\Delta = \operatorname{l.c.m.}([\operatorname{GL}_1(\mathbb{F}_q)], [\operatorname{GL}_2(\mathbb{F}_q)], \cdots, [\operatorname{GL}_{r-1}(\mathbb{F}_q)])$, we have $\sigma^\Delta$ acts on $\Lambda_\mathfrak{p}$ via matrix of the type
\[
\begin{pmatrix}
I_{r_1} & & \\
& I_{r_2} & & \\
& & & \ast \\
0 & & & I_{r_t}
\end{pmatrix},
\]
where $I_{r_m}$ is the $r_m \times r_m$ identity matrix for $1 \leq m \leq t$.

Moreover, when we raise $\sigma \in \operatorname{Gal}(F_\mathfrak{p}^{\text{sep}}/F_\mathfrak{p})$ to the power $\sigma^\Delta p^{r-1}$, it acts trivially on $\Lambda_\mathfrak{p}$. Thus for $\sigma \in I_\mathfrak{p}$, we get that $\sigma^\Delta p^{r-1}$ acts trivially on $\Lambda_\mathfrak{p}/I_\mathfrak{p}$. As a result, $\sigma^\Delta p^{r-1} \in I_\mathfrak{p}$ acts on $\phi[I]$ via matrix of the form \[
\begin{pmatrix}
I_d & & \\
& 0 & & \\
& & & I_{d-1}
\end{pmatrix}.
\]
After raising to $p^{r-1}$ power again, the action of $\sigma^\Delta p^{2(r-1)}$ on $\phi[I]$ is trivial. Therefore, both representations $\rho_V^\Delta p^{2(r-1)}$ and $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ are unramified at $\mathfrak{p}$.

We summarize this subsection by the following Proposition:

**Proposition 3.1.** Suppose that $\phi$ has stable bad reduction at a place $\mathfrak{p}$ of rank $1 \leq d \leq r-1$. Let $\Delta = \operatorname{l.c.m.}([\operatorname{GL}_1(\mathbb{F}_q)], [\operatorname{GL}_2(\mathbb{F}_q)], \cdots, [\operatorname{GL}_{r-1}(\mathbb{F}_q)])$. Both representations $\rho_V^\Delta p^{2(r-1)}$ and $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ are unramified at $\mathfrak{p}$.

### 3.3. Ramification at place $\mathfrak{p}$ where $\phi$ is not semistable.

Recall that $\phi_T = T + g_1\tau_1 + \cdots + g_r\tau^r$ is a Drinfeld $A$-module over $F$. Let $p$ be a uniformizer of $F_\mathfrak{p}$. Let $e_\phi = \min_{1 \leq i \leq r} \frac{v_p(g_i)}{q-1}$.

Consider the field extension $F'_\mathfrak{p} = F_\mathfrak{p}(\pi^{e_\phi})$ of $F_\mathfrak{p}$, then $\phi$ has semistable reduction over $F'_\mathfrak{p}$. Let $F_{\mathfrak{p}}^{\text{nr}}$ be the maximal unramified extension of $F_\mathfrak{p}$, we have the following two properties so far:

1. The field extension $F'_\mathfrak{p}/F_{\mathfrak{p}}$ is tamely ramified.
2. The index $[F_{\mathfrak{p}}^{\text{nr}} : F'_\mathfrak{p} : F_{\mathfrak{p}}^{\text{nr}}]$ divides $\Omega := \operatorname{l.c.m.}(q-1, q^2-1, \cdots, q^r-1)$.

Let $\mathfrak{p}'$ be the place of $F'_\mathfrak{p}$ lying above the place $\mathfrak{p}$. Since $\phi$ has semistable reduction at $\mathfrak{p}'$, we know $\phi$ has either good reduction or stable bad reduction at $\mathfrak{p}'$. After going through the same process as in subsection 3.2, we can get the same result as in Proposition 3.1 with $\mathfrak{p}$ replaced by $\mathfrak{p}'$. In other words, we have both $\rho_V^\Delta p^{2(r-1)}$ and $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ are unramified when restrict to $G_{F_{\mathfrak{p}'}}$.

From the properties above, we have that
\[
\rho_V^\Delta p^{2(r-1)} \quad \text{and} \quad \rho_{\phi[I]/V}^\Delta p^{2(r-1)}
\]
are unramified when restrict to $G_{F_{\mathfrak{p}'}}$.

This is equivalent to saying that both $\rho_V^\Delta p^{2(r-1)}$ and $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ are unramified at $\mathfrak{p}$.

### 3.4. Ramification at infinity.

From subsection 3.1 to 3.3, we know that for the mod $I$ Galois representation $\rho_{\phi,I} : G_F \to \operatorname{Aut}(\phi[I]) \cong \operatorname{GL}_r(\mathbb{F}_I)$ of $\phi$, the image of $\sigma \in G_F$ is of the form
\[
\begin{pmatrix}
\rho_V(\sigma) & & \\
& & \\
0 & & \rho_{\phi[I]/V}(\sigma)
\end{pmatrix}.
\]
The representations $\rho_V$ and $\rho_{\phi[I]/V}$ satisfy the followings:

1. $\rho_V^\Delta p^{2(r-1)}$ and $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ are unramified at any finite place $\mathfrak{p} \neq I$.

2. either $\rho_V^\Delta p^{2(r-1)}$ or $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ is unramified at $I$.

In this subsection, our goal is to prove the following Proposition:

**Proposition 3.2.** The ramification of both $\rho_V^\Delta p^{2(r-1)}$ and $\rho_{\phi[I]/V}^\Delta p^{2(r-1)}$ at $\infty$ are tame.
Proof. Let $L_V$ (resp. $L_{\phi[I]/V}$) be the field extension of $F$ “cut off” by $\rho_V^{\Omega \Delta p^{2(r-1)}}$ (resp. $\rho_{\phi[I]/V}^{\Omega \Delta p^{2(r-1)}}$), i.e. the representation $\rho_V^{\Omega \Delta p^{2(r-1)}}$ (resp. $\rho_{\phi[I]/V}^{\Omega \Delta p^{2(r-1)}}$) factors through $\text{Gal}(L_V/F)$ (resp. $\text{Gal}(L_{\phi[I]/V}/F)$).

We claim that both $[L_V : F]$ and $[L_{\phi[I]/V} : F]$ are prime to $p$, then the proposition follows. To prove the claim, we focus on the extension $L_V/F$. The proof for $L_{\phi[I]/V}/F$ follows the same procedure.

Suppose that $p||\text{Gal}(L_V/F)$, then there is an element $\sigma \in \text{Gal}(L_V/F)$ whose order is equal to $p$ by Cauchy’s theorem. Thus there is an element $\sigma' \in G_F$ such that the order of $\rho_V(\sigma')$ is equal to $\Omega \Delta p^{2r-1}$. After raising to the power $\Omega \Delta$, we get an element $\tilde{\sigma} \in G_F$ such that $\rho_V(\tilde{\sigma})$ has order equal to $p^{2r-1}$. As $\rho_V(\tilde{\sigma})$ lies in $\text{GL}_s(F_1)$ with $1 \leq s = \dim_{F_1}(V) < r$, we know $\rho_V(\tilde{\sigma})$ lies in the Sylow $p$-subgroup $\left\{ \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \right\}$ of $\text{GL}_s(F_1)$ after suitable conjugation. Since elements in the Sylow $p$-subgroup of $\text{GL}_s(F_1)$ have order less than or equal to $p^s$, it is impossible to have such an element $\tilde{\sigma} \in G_F$ whose order is $p^{2r-1} > p^s$. Hence we have a contradiction, this completes the proof of claim.

As a summary, we understand the ramification at all places of either $L_V/F$ or $L_{\phi[I]/V}/F$.

$(\ast)$. Either $L_V/F$ or $L_{\phi[I]/V}/F$ is unramified at all finite places and tamely ramified at $\infty$.

3.5. An application of Hurwitz formula. In this subsection, we apply Hurwitz genus formula to prove that either $L_V/F$ or $L_{\phi[I]/V}/F$ is a constant extension. From $(\ast)$, we may assume that $L_V/F$ is unramified at all finite places and tamely ramified at infinity. The proofs for the case of $L_{\phi[I]/V}/F$ follows the same procedure. Let $M$ be the constant field of $L_V$, then $L_V/M$ is a finite separable geometric extension.

Proposition 3.3. $L_V/F$ is a constant extension, i.e. $L_V = M$.

Proof. Let $g_{L_V}$ (resp. $g_M$) be the genus of $L_V$ (resp. $M$). We get from Hurwitz genus formula that

$$2g_{L_V} - 2 = [L_V : M](2g_M - 2) + \sum_{i=1}^{t} (e_i - 1)f_i.$$ 

Here $e_i$ denotes the ramification index of places $\infty_i$ of $L_V$ lying above of $\infty$ of $M$ and $f_i$ denotes the inertia degree of $\infty_i$ over $\infty$. Since $M/F$ is a constant extension, its genus $g_M$ is zero. Combining with the fact $\sum_{i=1}^{t} e_if_i = [L_V : M]$, we have

$$0 \leq 2g_{L_V} = 2 - [L_V : M] - \sum_{i=1}^{t} f_i.$$ 

Thus we have $[L_V : M] = 1$, this means $L_V = M$ is a constant extension over $F$.

Now we try to bound the index $[L_V : F_q]$. Since $L_V/F$ is a constant extension, we have

$$[L_V : F_q] \leq [F_{F(V)} : F_q] \leq [F_{F(\phi[I])} : F_q].$$

Here $F_{F(V)}$ is the constant field of $F(V)$, the field extension of $F$ by joining elements in $V$, and $F_{F(\phi[I])}$ is the constant field of $F(\phi[I])$, the splitting field of the polynomial $\phi[I](x)$. Since $[F_{F(\phi[I])} : F_q] = [F_{F_{\infty}(\phi[I])} : F_q]$ and Remark 4.2(ii) of [Gek19], we have

$$[F_{F(\phi[I])} : F_q] = [F_{F_{\infty}(\phi[I])} : F_q] \leq [F_{F_{\infty}(\text{tor}(\phi))} : F_{F_{\infty}}] \leq (q^r - 1)p^{r-1}.$$
Thus we have 
\[ \rho \Delta \cdot p^{2(r-1) - n_\phi} = 1, \] where \( n_\phi \leq [L_V : F] \leq (q^r - 1)p^{r-1} \) is a constant depending on \( \phi \).

As a summary of this subsection, we have the following proposition:

**Proposition 3.4.** Either \( \rho \Delta \cdot p^{2(r-1) - n_\phi} = 1 \) or \( \Omega \cdot \cdot p^{2(r-1) - n_\phi} = 1 \), where \( n_\phi \leq [L_V : F] \leq (q^r - 1)p^{r-1} \) is a constant depending on the Drinfeld module \( \phi \).

### 3.6. Proof of the main theorem 1

From Proposition 3.4 we may assume \( \rho \Delta \cdot p^{2(r-1) - n_\phi} = 1 \).

The proof for the other case is the same.

**Proof.** We follow the settings in the main theorem, let \( \mathcal{P} \) be a prime of \( A \) with minimal degree such that it has good reduction at \( \mathcal{P} \). Consider the characteristic polynomial \( P_{\phi, \mathcal{P}}(x) \in A[x] \) of \( \text{Frob}_\mathcal{P} \) acting on the Tate module \( T_1(\phi) \). We know that \( P_{\phi, \mathcal{P}}(x) \equiv P_{\phi, \mathcal{P}}(x) \mod \mathcal{P} \) is the characteristic polynomial of \( \bar{\rho}_{\phi, \mathcal{P}}(\text{Frob}_\mathcal{P}) \). On the other hand, let \( \mathcal{P} \) be the order of \( \rho \phi(\text{Frob}_\mathcal{P}) \).

As \( \rho \Delta \cdot p^{2(r-1) - n_\phi} = 1 \), we know the followings:

1. \( d | \Omega \cdot \Delta \cdot p^{2(r-1) - n_\phi} \).
2. Let \( S_d(x) \in \mathbb{F}_q[x] \) be the \( d \)-th cyclotomic polynomial over \( \mathbb{F}_q \), the minimal polynomial of \( \rho \phi(\text{Frob}_\mathcal{P}) \) divides both \( S_d(x) \) and \( \bar{P}_{\phi, \mathcal{P}}(x) \).

Thus the resultant \( R(P_{\phi, \mathcal{P}}(x), S_d(x)) \subset A \) is divisible by \( \mathcal{P} \). Now we write down the resultant explicitly:

\[ R(P_{\phi, \mathcal{P}}(x), S_d(x)) = \prod \frac{1}{(x_i - \zeta)(x_i - \zeta)}(x_i - \zeta), \]

where \( x_i \) are roots of \( P_{\phi, \mathcal{P}}(x) \) and \( \zeta \) runs through primitive \( d \)-th roots of unity. We then compute their valuation at infinity:

For \( 1 \leq i \leq r \), we have \( |x_i|_\infty := q^{-v_\infty(x_i)} = q^{\frac{1}{d} \deg_T \mathcal{P}} \). And \( |\zeta|_\infty = 1 \) for any primitive \( d \)-th roots of unity \( \zeta \). Thus we can deduce

\[ 0 < |R|_\infty \leq \max\{ q^{\frac{1}{d} \deg_T \mathcal{P}}, 1 \} \end{equation}

As \( d | \Omega \cdot \bar{\Delta} \cdot p^{2(r-1) - n_\phi}, \) we have \( \varphi(d) \leq \varphi(\Omega \cdot \bar{\Delta} \cdot p^{2(r-1) - n_\phi}). \)

Combining with the property \( 1 | R \), we get

\[ \deg_T (1) \ll \varphi(\Omega \cdot \bar{\Delta} \cdot p^{2(r-1) - n_\phi}) \deg_T \mathcal{P}. \]

**Remark 3.5.** From Proposition 3.4, the constant \( n_\phi \leq (q^r - 1)p^{r-1} \) is bounded by a number that only depends on the rank of Drinfeld modules. Hence the term \( \varphi(\Omega \cdot \bar{\Delta} \cdot p^{2(r-1) - n_\phi}) \) can be bounded by a constant which depends only on \( r \) and \( q \).

From Theorem 1 and the Remark above, we can get a uniform bound on irreducibility of \( \mod I \) Galois representation for Drinfeld modules over \( F \) with good reduction everywhere.

**Corollary 3.6.** Let \( \phi \) be a Drinfeld \( A \)-module over \( F \) of rank \( r \) with good reduction everywhere, then there is an integer \( N \) depending on \( r \) and \( q \) such that the \( \mod I \) Galois representation

\[ \bar{\rho}_{\phi, I} : G_F \rightarrow \text{GL}_r(\overline{\mathbb{F}}_I) \]

is irreducible whenever \( \deg_T (1) > N \).

For rank-2 Drinfeld \( A \)-modules over \( \mathbb{F}_2(T) \) with good reduction everywhere, Pál actually proved a stronger bound for irreducibility of \( \mod I \) Galois representations:
4. Function field analogue of Serre’s uniformity problem

From now on, we focus on rank-2 Drinfeld \( A \)-module \( \phi_T = T + g_1 \tau + g_2 \tau^2 \) over \( F \). Up to isomorphisms over \( F \), we may assume \( g_1, g_2 \in A \). Since we have adelic openness for Drinfeld modules by Pink and Rütsche [PR09a], it’s reasonable for us to state the uniformity problem on Galois image associated to Drinfeld modules, which is an analogue of the Serre’s uniformity problem for elliptic curves over \( Q \).

**Question 4.1.** Let \( q = p^e \) an odd prime power, \( A = F_q[T] \), and \( F = F_q(T) \). There is an integer \( N(q) \) depending only on \( q \) such that for any rank-2 Drinfeld \( A \)-module \( \phi \) over \( F \) with complex multiplication, the mod \( 1 \) Galois representation

\[
\tilde{\rho}_{\phi, 1} : \text{Gal}(F^{\text{sep}}/F) \to \text{Aut}(\phi[1]) \cong \GL_2(F_1)
\]

is surjective whenever \( \deg_T I > N \).

Our goal at here is to give a partial result on the question above.

Before providing a proof of Main Theorem 2, We would like to justify condition (3) in our main theorem by showing that a given bound on degree of bad reduction primes is necessary to achieve a uniform bound for surjective Galois image.

The following theorem is from Gekeler’s work on Galois image of twisted Carlitz modules.

**Theorem 4.2** (Theorem 3.13 of [Gek16]). Given \( \phi_T = T + \Delta \tau \), we may assume \( \Delta \in A \) with \( \Delta = c^{k_0} \cdot p_1^{k_1} \cdots p_s^{k_s} \), where \( 0 \leq k_0 < q - 1 \), and \( 0 < k_i < q - 1 \) for \( 1 \leq i \leq s \). Here \( c \in F_q^\ast \) is a primitive \((q - 1)\)-th root of unity, and \( s \geq 0 \) different monic primes \( p_i \) of degree \( d_i \). Set \( d = \deg \Delta = \sum_{1 \leq i \leq s} k_i d_i \).

Let further \( N \) be a non-constant element of \( A \). Suppose \( p_i | N \) for \( 1 \leq i \leq r \), and \( p_i \nmid N \) for \( r < i \leq s \). Then the Galois image \( \tilde{\rho}_{\phi, N}(G_F) \) is a subgroup of \((A/N)^\ast\) with index equal to

\[
g.c.d.(d - 1, q - 1, k_0^*, k_{r+1}, \cdots, k_s),
\]

where

\[
k_0^* = \begin{cases} 
  k_0, & \text{if } q \text{ or } d = \deg \Delta \text{ is even} \\
  \text{the unique } k \equiv k_0 + (q - 1)/2 \mod q - 1 \text{ with } 0 \leq k < q - 1, & \text{otherwise}
\end{cases}
\]

In particular, we have the following corollary for the case \( N = 1 \) is a prime of \( A \).

**Corollary 4.3.** For each prime \( I \) of \( A \), we consider the Drinfeld module \( \phi_T = T + \Delta \tau \) with \( \Delta = c^{k_0} \cdot p_1^{k_1} \cdots p_s^{k_s} \), where \( 0 \leq k_0 < q - 1 \), and \( 0 < k_i < q - 1 \) for \( 1 \leq i \leq s \). Here \( c \in F_q^\ast \) is a primitive \((q - 1)\)-th root of unity, and \( s \geq 0 \) different monic primes \( p_i \) of degree \( d_i \). Set \( d = \deg \Delta = \sum_{1 \leq i \leq s} k_i d_i \). Then the Galois image \( \tilde{\rho}_{\phi, I}(G_F) \) is a subgroup of \( F_1^\ast \) with index equal to

\[
g.c.d.(d - 1, q - 1, k_0^*, k_2, \cdots, k_s),
\]

where

\[
k_0^* = \begin{cases} 
  k_0, & \text{if } q \text{ or } d = \deg \Delta \text{ is even} \\
  \text{the unique } k \equiv k_0 + (q - 1)/2 \mod q - 1 \text{ with } 0 \leq k < q - 1, & \text{otherwise}
\end{cases}
\]
Counterexample 4.4. Suppose $q = p^e$ is an odd prime power with $4 \mid q - 1$, and let $l$ be any prime of $A$ with odd degree. We consider the Drinfeld module $\varphi_T = T + \tau r$. Applying Corollary 4.3 with $\Delta = l$, we know the Galois image $\bar{\rho}_{\varphi,l}(G_F)$ is a subgroup of $\mathbb{F}_l^*$ of index equal to $g.c.d.((q-1)/2, l^e)$. Since $k_0 = 0$ from our choice of $\Delta$ and both $q$ and $\deg \Delta = \deg l$ are odd, we have $k_0^e = (q-1)/2$. Now we know $\deg l - 1, q - 1$, and $(q-1)/2$ are all even, the index of $\bar{\rho}_{\varphi,l}(G_F)$ in $\mathbb{F}_l^*$ is at least 2.

Therefore, let $\psi_T = T + \tau - l r^2$ be a rank-2 Drinfeld $A$-module over $F$. By Proposition 7.1 of [Val04], the image of its mod $l$ Galois representation has determinant $\det \circ \bar{\rho}_{\varphi,l}(G_F) \cong \bar{\rho}_{\varphi,l}(G_F)$ equal to a subgroup of $\mathbb{F}_l^*$ of index at least 2. This implies the mod $l$ Galois representation $\bar{\rho}_{\varphi,l}$ is not surjective.

In conclusion, given $q = p^e$ odd prime power with $4 \mid q - 1$, and any positive integer $N$. There is an odd degree prime $l$ of $A$ with $\deg l > N$. We then consider the rank-2 Drinfeld module $\psi_T = T + \tau - l r^2$. The Galois image $\bar{\rho}_{\varphi,l}(G_F)$ has index at least 2 in $GL_2(\mathbb{F}_l)$. This implies Question 4.1 would fail without any restriction on the degree of bad reduction primes.

According to the counterexample, we modify Question 4.1 as follows:

**Definition 4.5.** Let $\phi$ be a rank-$r$ Drinfeld $A$-module over $F$ and $l$ be a prime ideal of $A$. We say the mod $l$ Galois representation

$$\bar{\rho}_{\phi,l} : \text{Gal}(F^{sep}/F) \to \text{Aut}(\phi[l]) \cong GL_r(\mathbb{F}_l)$$

is full if the image of $\bar{\rho}_{\phi,l}$ in $GL_r(\mathbb{F}_l)$ has index $[\text{GL}_r(\mathbb{F}_l) : \bar{\rho}_{\phi,l}(\text{Gal}(F^{sep}/F))] \leq q - 1$

**Conjecture 1.** Let $q = p^e$ an odd prime power, $A = \mathbb{F}_q[T]$, and $F = \mathbb{F}_q(T)$. There is an integer $N(q)$ depending only on $q$ such that for any rank-$r$ Drinfeld $A$-module $\phi$ over $F$ without complex multiplication, the mod $l$ Galois representation

$$\bar{\rho}_{\phi,l} : \text{Gal}(F^{sep}/F) \to \text{Aut}(\phi[l]) \cong GL_r(\mathbb{F}_l)$$

is full whenever $\deg_F l > N$.

The conjecture is now valid for $r = 1$ by Gekeler’s work [Gek16], and is open for $r \geq 2$.

4.1. **Subgroups of $GL_2(\mathbb{F}_l)$**. In the following subsections, we concentrate ourselves on rank-2 Drinfeld modules described in main theorem. Namely, we are under the following setting:

- (i) $q = p^e$ is a prime power with $q \geq 5$, and $d \in \mathbb{N}$ is a fixed integer.
- (ii) $\phi_T = T + g_1 \tau + g_2 \tau^2$ is a rank-2 Drinfeld $A$-module over $F$ with $g_1, g_2 \in A$.
- (iii) There is a prime $p$ of $A$ such that $p \nmid g_2$ but $p \mid g_1$.
- (iv) Write $g_2 = p^\alpha \cdot m$ with $(m, p) = 1$, we have $p \mid \alpha$ and $q - 1 \mid \alpha$.
- (v) Primes of $A$ dividing $g_2$ have degree less than or equal to $d$.

Let $l$ be a prime of $A$. Suppose that the image of mod $l$ Galois representation $\bar{\rho}_{\phi,l} : G_F \to GL_2(\mathbb{F}_l)$ is not surjective. We denote the image of $\bar{\rho}_{\phi,l}$ by $\Gamma_l$, then $\Gamma_l$ is a proper subgroup of $GL_2(\mathbb{F}_l)$ and here are some things we can say about such a subgroup $\Gamma_l$ due to [Zyw11] Lemma A.1 and [Lan95] Theorem 2.3.

- (1) If $|\Gamma_l|$ divides $|\Gamma_1|$, then either $\Gamma_1$ lies in a Borel subgroup or $\Gamma_1 \supset SL_2(\mathbb{F}_l)$.
  - If $p \mid |\Gamma_1|$, then $\Gamma_1$ is classified as follows:
    - (2) $\Gamma_1$ is contained in a Cartan group
    - (3) $\Gamma_1$ is contained in the normalizer of a Cartan group but not in the Cartan subgroup itself
    - (4) Denote $P \Gamma_1$ by the image of $\Gamma_1 \subset GL_2(\mathbb{F}_l)$ in $PGL_2(\mathbb{F}_l)$, then $P \Gamma_1$ is isomorphic to $A_4$, $S_4$, or $A_5$, respectively.
As our choice of Drinfeld module has a stable bad reduction at a prime $p$, from Proposition 4.1 of [Zyw11] we know that for $\deg l > d$, there is a basis of the $\mathbb{F}_r$-vector space $\phi[\bar{t}]$ such that
\[
\rho_{\phi,1}(I_p) \subset \left\{ \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \mid b \in \mathbb{F}_r, c \in \mathbb{F}_q^* \right\}.
\]
Moreover, let $e_\phi$ be the order of $\frac{v_p(j_\phi)}{(q-1)|\bar{t}|} + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ where $v_p$ denotes the valuation of $F$ at $p$, then $|\rho_{\phi,1}(I_p)| \geq e_\phi$.

Now by condition (iv), the valuation $v_p(j_\phi)$ is prime to $|\bar{t}|$ implies $|\rho_{\phi,1}(I_p)| \geq |\bar{t}|$. Next we compute the determinant $\bar{\rho}_{\phi,1}(I_p)$. The condition $\left( q-1 \mid v_p(g_2) \right)$ in (iv) and Proposition 7.1 of [vdH04] implies that
\[
\det \bar{\rho}_{\phi,1} \cong \bar{\rho}_{\psi,1}.
\]
the mod $l$ representation of a twisted Carlitz module $\psi_T = T - g_2T$ which has good reduction at $p$.

Therefore, we have $\det \bar{\rho}_{\phi,1}(I_p) = 1$. Combining with $|\bar{\rho}_{\phi,1}(I_p)| \geq |\bar{t}|$ we get
\[
\bar{\rho}_{\phi,1}(I_p) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_r \right\}.
\]
Therefore, we can conclude the following proposition:

**Proposition 4.6.** For $\deg l > d$, the Galois image $\Gamma_l$ is either contained in a Borel subgroup or $\Gamma_l \supset SL_2(\mathbb{F}_l)$.

**Proof.** Since $\bar{\rho}_{\phi,1}(I_p) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_r \right\}$, we know that $|\Gamma_l|$ is divisible by $|\bar{t}|$. The result then follows from (1) of our descriptions on $\Gamma_l$ in the previous page. \(\square\)

### 4.2. Proof of main theorem 2.

**Proof.** From main theorem [1] we may choose the integer
\[
N(q, d) = \varphi((q^2 - 1) \cdot (q^2 - 1)(q^2 - q) \cdot p^2 \cdot (q^2 - 1)p) \cdot (d + 1).
\]
Then for prime $l$ of $A$ with $\deg_T l > N$, the Galois image $\Gamma_l$ is irreducible. Thus Proposition [1] implies $\Gamma_l \supset SL_2(\mathbb{F}_l)$. Furthermore, $\deg_T l \geq N > d$ means $l$ is coprime with the leading coefficient $g_2$ of our Drinfeld module $\phi$. Thus $\det \bar{\rho}_{\phi,1} \cong \bar{\rho}_{\psi,1}$, where $\psi$ is the twisted Carlitz module $\psi_T = T - g_2T$.

By [Gek16] example 3.15 and the fact $1 \nmid g_2$, we have $\det \bar{\rho}_{\phi,1} \cong \bar{\rho}_{\psi,1} \cong \mathbb{F}_r^*$. Therefore, the Galois image $\Gamma_l = \text{GL}_2(\mathbb{F}_l)$ whenever $\deg_T l > N$. \(\square\)

We end this paper by providing the uniformity problem for Drinfeld modules with good reduction everywhere.

**Conjecture 2.** Let $q = p^e$ an odd prime power with $q > 5$, $A = \mathbb{F}_q[T]$, and $F = \mathbb{F}_q(T)$. There is an integer $N(q)$ depending only on $q$ such that for any rank-2 Drinfeld $A$-module $\phi$ over $F$ without complex multiplication and $\phi$ has good reduction everywhere, the mod $l$ Galois representation
\[
\bar{\rho}_{\phi,1} : \text{Gal}(F_{\text{sep}}/F) \to \text{Aut}(\phi[\bar{t}]) \cong \text{GL}_2(\mathbb{F}_l)
\]
is surjective whenever $\deg_T l > N$.

For rank-2 Drinfeld module $\phi$ with good reduction everywhere, the mod $l$ Galois representation has determinant $\det \bar{\rho}_{\phi,1} = \mathbb{F}_r^*$ for any prime $l$ of $A$ by [Gek16] example 3.15. Thus we do not have counterexample [1,3] happened in this case.

From the proof of main theorem [2] we observe the difficulty for this conjecture is that we don’t have a stable bad reduction prime to control the $p$-component of the Galois image $\Gamma_l$. On the
other hand, since $\phi$ has no complex multiplication, the adelic openness theorem from [PR09a] implies $\Gamma_1 = \operatorname{GL}_2(\mathbb{F}_l)$ once the degree of $l$ is large enough.

Acknowledgements

The author would like to thank Professor Mihran Papikian for reading the early drafts of this paper and helpful discussions.

References

[BP11] Yuri Bilu and Pierre Parent. Serre’s uniformity problem in the split Cartan case. Ann. of Math. (2), 173(1):569–584, 2011.

[CL19] Imin Chen and Yoonjin Lee. Explicit surjectivity results for Drinfeld modules of rank 2. Nagoya Math. J., 234:17–45, 2019.

[Gar02] Francis Gardeyn. Une borne pour l’action de l’inertie sauvage sur la torsion d’un module de Drinfeld. Arch. Math. (Basel), 79(4):241–251, 2002.

[Gek16] Ernst-Ulrich Gekeler. The Galois image of twisted Carlitz modules. J. Number Theory, 163:316–330, 2016.

[Gek19] Ernst-Ulrich Gekeler. On the field generated by the periods of a Drinfeld module. Arch. Math. (Basel), 113(6):581–591, 2019.

[Lan95] Serge Lang. Introduction to modular forms, volume 222 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1995. With appendices by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.

[Maz78] B. Mazur. Rational isogenies of prime degree (with an appendix by D. Goldfeld). Invent. Math., 44(2):129–162, 1978.

[Pí10] Ambrus Pál. On the torsion of Drinfeld modules of rank two. J. Reine Angew. Math., 640:1–45, 2010.

[PR09a] Richard Pink and Egon Rütsche. Adelic openness for Drinfeld modules in generic characteristic. J. Number Theory, 129(4):882–907, 2009.

[PR09b] Richard Pink and Egon Rütsche. Image of the group ring of the Galois representation associated to Drinfeld modules. J. Number Theory, 129(4):866–881, 2009.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. Invent. Math., 15(4):259–331, 1972.

[vdH04] Gert-Jan van der Heiden. Weil pairing for Drinfeld modules. Monatsh. Math., 143(2):115–143, 2004.

[Zyw11] David Zywina. Drinfeld modules with maximal Galois action on their torsion points. arXiv e-prints, page arXiv:1110.4365, October 2011.