Measuring ultra-large scale effects in the presence of 21cm intensity mapping foregrounds

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ABSTRACT

Hi intensity mapping will provide maps of the large-scale distribution of neutral hydrogen (HI) in the universe. These are prime candidates to be used to constrain primordial non-Gaussianity using the Large Scale Structure of the Universe as well as to provide further tests of Einstein’s theory of Gravity (GR). But HI maps are contaminated by foregrounds, which can be several orders of magnitude above the cosmological signal. Here we quantify how degenerated are the large-scale effects ($f_{NL}$ and GR effects) with the residual foregrounds. We conclude that a joint analysis does not provide a catastrophic degradation of constraints and provides a framework to determine the marginal errors of large scale-effects in the presence of foregrounds. Similarly, we conclude that the macroscopical properties of the foregrounds can be measured with high precision. Notwithstanding, such results are highly dependent on accurate forward modelling of the foregrounds, which incorrectly done catastrophically bias the best fit values of cosmological parameters, foreground parameterisations, and large-scale effects.

Key words: large-scale structure of the Universe, cosmology: miscellaneous

1 INTRODUCTION

Our understanding of the standard cosmological model has incrementally improved in the past decades. Some of the main questions cosmologists have been trying to answer concern the nature of Dark Energy and Dark Matter, with uncertainties on their physical properties becoming constrained to a few percent level. This ΛCDM concordance model is backed by different types of observations (despite some inconsistencies among them) such as the Cosmic Microwave Background (CMB) (Aghanim et al. 2020b), Supernovae Type Ia (for a recent measurement see Riess et al. 2018), and the Large Scale Structure (LSS) of the Universe (see for example the latest SDSS results de Mattia et al. 2021).

Beyond the nature of the dark components of the universe, understanding the very early Universe and the seeds of the LSS still remains interesting open questions. Although the latest Planck results (Aghanim et al. 2020b) give good constraints on the amplitude and spectral index of the primordial curvature fluctuation, these alone are not enough to distinguish between inflationary models (Akrami et al. 2020c). All of these are based on the description that the primordial energy density field is a Gaussian random field sourced by quantum fields during inflation (see Bassett et al. 2006, for a review). Standard single field inflationary models predict the density field to be nearly Gaussian (Maldacena 2003) but a plethora of other models predict deviations from the Gaussian assumption either due to the presence of other fields (see for example the Curvaton scenario Lyth & Wands 2002), inhomogeneous reheating at the end of inflation (Dvali et al. 2004), or non-standard inflationary set-ups. For a review on primordial non-Gaussianity (PNG) from Inflation refer to Bartolo et al. (2004). The latest Planck results already put bounds on primordial non-Gaussian parameters (Akrami et al. 2020b) giving the state-of-the-art measurement of local-type primordial non-Gaussianity $f_{NL}^{\text{local}} = -0.9 \pm 5.1$. Note that CMB and LSS use different normalisations for PNG $f_{NL}^{\text{LSS}} \simeq 1.3f_{NL}^{\text{CMB}}$ (Camera

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Local $f_{NL}$ measures the leading order expansion of the gravitational potential and induces a scale-dependent correction to the bias of Dark Matter tracers (Matarrese & Verde 2008; Dalal et al. 2008). Such non-Gaussian correction to the bias has a $1/k^2$ dependence, being only relevant on very large scales. Similarly, on these scales, other observational effects - so-called “GR effects” - affect how we measure the density fluctuations of biased tracers in the LSS (Yoo et al. 2012; Challinor & Lewis 2011; Bonvin & Durrer 2011). In fact, primordial non-Gaussianity and GR effects on the past light cone are degenerate, and neglecting such effects will bias estimates of $f_{NL}$ using the LSS (Camera et al. 2015). Hence, any analyses intended to measure $f_{NL}$ has to include the GR effects as they also scale with $1/k$ and $1/k^2$. One can immediately see that we require surveys that will cover large volumes of the universe to see effects on super-horizon scales.

Future galaxy surveys such as Euclid (Amendola et al. 2018) and the Vera C. Rubin Observatory (former LSST LSST Dark Energy Science Collaboration 2012) will observe and create catalogues of galaxies in large areas of the sky as well as deeper in redshift. While galaxy surveys are a well-established probe of the LSS, there are other tracers of dark matter. A promising way is to use the emission of 21cm photons of the hyperfine transition of neutral Hydrogen ($\text{Hi}$) and produce maps of the intensity of the signal (Chang et al. 2008; Ansari et al. 2012). After the Epoch of Reionisation, neutral Hydrogen remains only in galaxies while the inter-galactic medium becomes ionised. Hence, measuring the statistical distribution of $\text{Hi}$ intensity we are probing underlying dark matter distribution. Although a weak line, $\text{Hi}$ is by far the largest component of the baryonic matter making such a signal visible. In $\text{Hi}$ intensity mapping (IM) one does not detect individual galaxies, instead one produces CMB-like maps of the $\text{Hi}$ emission. The planned Square Kilometre Array (SKA) will allow $\text{Hi}$ IM tomography in single dish mode (Bacon et al. 2020), with its precursor MeerKAT (Santos et al. 2017) already operational. $\text{Hi}$ IM ideal to study $f_{NL}$ as it permits fast scans of the sky with high redshift resolution covering a wide range of frequencies (Camera et al. 2013; Alonso et al. 2015b).

Despite the great potential of $\text{Hi}$ IM for cosmology, obtaining the clean $\text{Hi}$ signal is a challenge due to the presence of galactic and extragalactic foregrounds that contaminate the cosmological signal. In fact, they are several orders of magnitude above the cosmological signal (Santos et al. 2005) and pose a challenge if one wants to do cosmology with $\text{Hi}$ IM. But there are strategies to deal with the presence of foregrounds which can divide into two main approaches. One such strategy is **foreground avoidance** which, roughly speaking, sets to find an observational window of the power spectrum in $k_{\parallel} - k_{\perp}$ space where the foregrounds do not affect substantially the estimates of the power spectrum (see for example Shaw et al. 2015). The wedge cuts typically renders scales $k_{\parallel} \lesssim 0.01\text{Mpc}^{-1}$ inaccessible due to the smoothness in frequency of the foregrounds. Such an approach is therefore unsuitable for primordial non-Gaussian studies with $\text{Hi}$ IM as no large scales are present in the observational window. The other approach, and extensively studied in the literature, is to do **foreground cleaning**. Most of the foreground removal methods take advantage of their spectral smoothness, while the cosmological $\text{Hi}$ signal is expected to fluctuate in the line-of-sight. Earlier attempts to subtract foregrounds used polynomials of the logarithm of the frequency (Bowman et al. 2009) to capture their contribution to the observed signal. These parametric approaches are not the preferred way of dealing with foregrounds as they put a strong prior to their frequency structure. A much-preferred way of performing foreground removal is by using blind methods that make little assumptions about the foregrounds, except that they are smooth in frequency. These are very similar to what has been done with CMB experiments. Blind methods have been extensively used to recover the LSS power spectrum of $\text{Hi}$ IM including using Independent Component Analyses (ICA) and principal component analyses (PCA) (Alonso et al. 2015a; Zhang et al. 2016; Asorey et al. 2020), GMCA (Generalised Morphological Component Analysis) and other sparse methods in pixel space (Carucci et al. 2020; Cunnington et al. 2021). These methods are able to remove the foreground contamination up to certain scales. Generically they remove the mean of the $\text{Hi}$ signal and suppress the power on large scales where the signal we aim to detect resides. One can try to reconstruct the power on large scales using transfer functions (Switzer et al. 2015; Witzemann et al. 2019) but these fundamentally depend on the input calibration, potentially leading to biased results if the transfer function has been calibrated with the wrong cosmology or with an incomplete setup (Wolz et al. 2014, for example). Therefore to study the effect of primordial non-Gaussianity in the $\text{Hi}$ power spectrum using foreground subtraction, one needs to understand how the blind methods suppress the large scale power (see for example Cunnington et al. 2020).

Here we take a different approach, neither we subtract nor we avoid foregrounds, and try to understand if we can still draw any conclusion of the cosmological model in their presence. Our work is similar to Santos & Cooray (2006) although we focus on $f_{NL}$ and GR effects and in the post-EoR universe. As Camera & Padmanabhan (2020) we look at the degeneracies between $f_{NL}$ and astrophysics. While they focused on the $\text{Hi}$ halo model we focus on the effects of Galactic astrophysics instead. We are also within the same lines of Zaldarriaga et al. (2004) but we consider the foregrounds to be well correlated in frequency and use Santos et al. (2005) to take into account the correlation length of the foregrounds. Instead of just considering the spectral smoothness of the foregrounds, we also want to see if we can learn anything about their angular structure. Therefore we will forward model the foregrounds power spectrum and jointly understand their physical properties together with cosmological parameters in the spirit of Switzer et al. (2019) and Sims & Pober (2019). We will use the standard Fisher formalism to forecast how well one can learn physical properties of the foregrounds and the cosmology, and assess how constraints on $f_{NL}$ and the GR effects are degraded once we marginalise the foreground model parameters. Bear in mind that when we speak about foregrounds, in practice we refer to residual foregrounds, as any calibration method includes models for galactic emission (Wang et al. 2020). One should note that we use simplistic toy models for astrophysical foregrounds, but our main goal here is to understand if such an approach is at all feasible in practice. A more elaborated model should take into account previous modelling and mappings of the sky, such as of de Oliveira-
of several types: cosmological, galactic, satellites, atmospheric effects, etc. For this paper, we focus on galactic and extragalactic foregrounds as they are the dominant contribution to the signal. In addition, there are toy models we can explore and have been used extensively in the literature to simulate the foregrounds. On the other hand, satellites (and other transient effects) are harder to model and only affect some pointings during their passage through the dishes’ field-of-view (FoV). Broadly speaking, atmospheric effects can be dealt with at the calibration level (Wang et al. 2020). Our observed maps can then be expressed as

\[ M(\nu, \mathbf{n}) = \mathcal{S}(\nu, \mathbf{n}) + \mathcal{F}(\nu, \mathbf{n}) + \mathcal{N}(\nu, \mathbf{n}). \]  

In cosmology, it is the statistical structure of density fluctuations that carries the information we wish to measure, therefore we work with contrast maps in frequency shells

\[ \Delta M(\nu, \mathbf{n}) = M(\nu, \mathbf{n}) - \overline{M}(\nu), \]  

where \( \overline{M} \) is the sky average \( \langle \nu \rangle \). We then expand the contrast map into a spherical harmonic basis \( Y_m^\ell(\mathbf{n}) \) as

\[ \Delta M(\nu, \mathbf{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\nu) Y_m^\ell(\mathbf{n}), \]  

where the \( a_{\ell m} \) coefficients only depend on the frequency bin. One can then use the orthonormal properties of the spherical harmonics to compute the \( a_{\ell m} \), which are given by

\[ a_{\ell m}(\nu) = \int d\Omega_\mathbf{n} \Delta M(\nu, \mathbf{n}) Y_m^\ell(\mathbf{n}). \]  

All relevant information about the cosmological signal (and the noise) is as usual captured by the power spectrum

\[ \langle a_{\ell m}(\nu_i), a_{\ell' m'}(\nu_j) \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell}(\nu_i, \nu_j). \]  

Note that the \( C_{\ell} \) of the experimental noise and the signal have a different interpretation than the foregrounds one. While \( a_{\ell m}^N \) is drawn from a Gaussian distribution centred at zero and variance \( C_{\ell}^N \) (same for the instrumental noise), the \( a_{\ell m}^F \) and \( C_{\ell}^F \) represent the angular structure of the foregrounds which does not result from any intrinsic probability distribution. There is a caveat, this may not hold for extragalactic foregrounds as they may be indeed a realisation of the underlying density field. We will assume they have a fixed angular structure as galactic foregrounds do. From the map given by Eq. 1 we get

\[ C_{\ell}^M(\nu_i, \nu_j) = C_{\ell}^S(\nu_i, \nu_j) + C_{\ell}^F(\nu_i, \nu_j) + C_{\ell}^N(\nu_i, \nu_j), \]  

where we considered each component uncorrelated from each other. This is undoubtedly true for the experimental noise. We also don’t expect correlations between the cosmological signal and galactic foregrounds. Extragalactic point sources may indeed correlate with the cosmological signal although we neglect it for now. We expect correlations with extragalactic free-free emission to be negligible as it mainly comes from the intergalactic medium (Santos et al. 2005).

From now on let us suppress the frequency dependence to simplify notation and use the indices \( i, j, p, q \) to refer to frequency/redshift bins. We will therefore follow the notation \( a_{\ell m}(\nu_i) = a_{\ell m,i} \) and \( C_{\ell}(\nu_i, \nu_j) = C_{\ell,i,j} \). Note that the indices \( \ell, m \) will always refer to the spherical harmonic decomposition multipole. Our estimator of the angular power
where the angular resolution is
\[ \theta_{FWHM}(z_i) = 1.22 \frac{\lambda_{\text{obs}}(1 + z_i)}{D_{\text{dish}}} , \]
and \( D_{\text{dish}} \) is the diameter of the dishes. Then the observed angular power spectrum is given by
\[ C_{\ell}^{\text{HI-obs}}(z_i, z_j) = B(z_i) B(z_j) C_{\ell}^{\text{HI}}(z_i, z_j) \]
where \( C_{\ell}^{\text{HI}} \) takes into account the selection and source distribution functions and is given by (Challinor & Lewis 2011)
\[ C_{\ell}^{\text{HI}}(z_i, z_j) = 4\pi \int \Delta W_{\ell}(z, k) \Delta W_{\ell}(z, k) P(k) \, . \]
The weighting does not happen at the power spectrum level itself, it is the transfer functions that are weighted by how we select the data and the distribution of sources. Therefore
\[ \Delta W_{\ell}(z_i, k) = \int dz T^{\text{HI}}(z) W(z, z) \Delta W_{\ell}(z, k) . \]

In the case of HI IM, the distribution of sources is just the HI temperature \( T^{\text{HI}} \). The window function \( W \) is given by how we bin and weight the data in frequency and is a normalised probability distribution function such that \( \int dz W(z, z) = 1 \). The transfer function \( \Delta W_{\ell} \) includes the underlying matter density fluctuations as well as Redshift Space Distortions (RSD) and general relativistic effects that alter the apparent density field. Eq. 15 also relates the observed angular power spectrum with the dimensionless primordial curvature perturbation power spectrum
\[ P(k) = A_s \left( \frac{k}{k_0} \right)^{n_s-1} . \]
where the pivot scale is \( k_0 = 0.05 \, \text{Mpc}^{-1} \), \( A_s \) is the amplitude and \( n_s \) is the spectral index.

In the case of HI intensity mapping the expression of the transfer function in Newtonian gauge is given by (Hall et al. 2013)
\[ \Delta_{\text{HI}}(k) = \left[ (b_{\text{HI}} + \Delta_{\text{HI}}(k)) \delta_k \right] j_k(k) + k_{\text{HI}} \frac{H}{k^2} \]
In the case of HI IM the evolution bias can be written in terms of the temperature

\[ b_{\text{HI}}(z) = -\frac{\partial \ln [P_{\text{HI}}(z)H(z)]}{\partial \ln (1+z)} - 2, \]

which we take to be expressed as (Bacon et al. 2020)

\[ T_{\text{HI}}(z) = 0.056 + 0.232z - 0.024z^2. \]

In Eq. 18 the HI clustering bias receives a scale-dependent correction \( \Delta b_{\text{HI}} \) if there is a non-zero primordial local-type non-Gaussianity. One can show that such correction is well approximated by (Dalal et al. 2008; Matarrese & Verde 2008)

\[ \Delta b_{\text{HI}} = 3f_{\text{NL}} \frac{[b_{\text{HI}}(z) - 1] \Omega_m H_0^2 \delta_c}{D(z)T(k)k^2}, \]

where \( \delta_c \simeq 1.69 \) is the critical matter density contrast for spherical collapse, \( T \) is the matter transfer function (normalised to 1 on large scales) and \( D \) is the growth factor (normalised to 1 at \( z = 0 \)). This correction becomes important on very very large scales (\( k \to 0 \)) where \( T(k) \simeq 1 \) and therefore \( \Delta b_{\text{HI}} \propto f_{\text{NL}}k^{-2} \). We can then see that the effect of primordial non-Gaussianity as a similar scale dependence as the GR light-cone effects. In fact, neglecting such corrections may lead to spurious detections of primordial non-Gaussianity (Camera et al. 2015).

To compute the angular power spectrum we used a modified version of the publicly available code CAMB sources (Challinor & Lewis 2011) to include \( f_{\text{NL}} \) (Camera et al. 2013). We plot three examples of the cosmological signal in cyan in Figure 1.

### 2.2 Foregrounds

Our main goal is to assess how a joint fit to the cosmological parameters and the foregrounds would degrade our constraints. For that we need to establish models for the angular structure of the foregrounds. We follow Santos et al. (2005) and use the generic expression

\[ C^\text{MISOS}_\ell(\nu_i, \nu_j) = A \left( \frac{\nu_i^2}{\nu_j^2} \right)^\alpha \left( \frac{\ell_{\text{ref}}}{\ell} \right)^\beta e^{-\left(\nu_i/\nu_j\right)^2 \xi^2}, \]

for the different foregrounds we consider. Here we will consider four different foreground components: Galactic free-free (GFF) emission, Galactic synchrotron (GS) emission, extragalactic free-free (EFF) and extragalactic point source (EPS).

In Table 1 we specify the fiducial calibrations for the references \( \nu_{\text{ref}} = 130 \) MHz and \( \ell_{\text{ref}} = 1000 \) from Santos et al. (2005).

From the amplitudes of the power spectrum in Table

### Table 1. Calibrations of different foregrounds using \( \nu_{\text{ref}} = 130 \) MHz and \( \ell_{\text{ref}} = 1000 \) from Santos et al. (2005).

| Foreground                  | \( A \) [mK^2] | \( \beta \) | \( \alpha \) | \( \xi \) |
|----------------------------|--------------|------------|------------|--------|
| Extragalactic Point Sources| 57.0         | 1.1        | 2.07       | 1.0    |
| Extragalactic Free-Free    | 0.014        | 1.0        | 2.10       | 35     |
| Galactic Synchrotron      | 700          | 2.4        | 2.80       | 4.0    |
| Galactic Free-Free        | 0.088        | 3.0        | 2.15       | 35     |

Figure 1. Angular power spectrum of each component computed at \( \nu_i = 950 \) MHz (Top), at \( \nu_j = 710 \) MHz (Middle), and considering the cross correlation between the two different frequencies (Bottom). Note that the cross-correlation between frequency bins has no instrumental noise contribution. The cross-correlation of the cosmological signal is plotted with a dashed line if its value is negative.
one can clearly see that synchrotron is expected to be the dominant contributor. As for the signal, we need to convolve the foregrounds power spectra with the telescope beam, i.e.,

$$C_{\ell,i,j} = B_{\ell,i} B_{\ell,j} C_{\ell,i,j}^{\text{MS05}}.$$  (24)

We can find examples of the angular power spectrum of the foregrounds in Figure 1 with Galactic Synchrotron in dotted red, extragalactic point sources in solid blue, Galactic free-free in dot-dashed green and extragalactic free-free in dashed magenta.

### 2.3 Instrumental Noise

Assuming uncorrelated Gaussian instrumental noise we have that

$$C_{\ell,i,j}^N = \frac{4\pi f_{\text{sky}} T_{\text{sys}}^2}{2 N_d \Delta \nu t_{\text{tot}}} \delta_{ij}. $$  (25)

We will take the same system temperature specifications as in (Bacon et al. 2020) for both band 1 and band 2 of SKA1-MID. We will consider a 20 000 deg$^2$ survey over 10 000 hours. We will round the number of dishes that constitute the observatory, taking $N_d = 200$. We will consider fixed size frequency bins of 15 MHz, which correspond to a $\Delta z$ that ranges from 0.018 at low redshifts to 0.22 in the highest redshift bin. In figure 1 we show the instrumental noise in the horizontal dashed black line for two frequencies. Note that we assume that the noise power spectrum is uncorrelated in frequency, hence the bottom panel of 1 has no instrumental noise.

### 2.4 The observed angular power spectrum

To exemplify the importance of each contribution to the total observed angular power spectra is given by Eq. 6, we choose two reference frequencies $\nu_i = 950$ and $\nu_j = 710$ MHz ($z_i = 0.49$ and $z_j = 1.0$). In Figure 1 we plot the cosmological signal and the different foreground components convolved with the telescope beam for the two reference frequencies, as well as the instrumental noise. We also plot the cross-correlation angular spectra between the two frequencies. One can see the hierarchy in contributions to the total signal. Synchrotron emission is overwhelmingly higher than any other, although on large angular scales (small $\ell$) point-sources and galactic free-free are boosted. Fundamentally the cosmological signal is several orders of magnitude below the foregrounds but still above the noise level up to some $\ell$ depending on the observed frequency (as seen in the top and middle plots of Fig 1). Despite the fact that traditional approaches use blind foreground cleaning methods, one can make the question if it would be possible to jointly measure the cosmological signal together with the foregrounds, given that we are in a low noise regime. Especially if we want to measure PNG and GR effects. In addition, the cross-correlations between bins give ample information to constrain the spectral indexes and are not affected by instrumental noise. Note as well that higher multipoles get their power dumped due to the beam of the experiment.

Therefore, the question is how well we need to measure the Hi power spectra on large scale to detect $f_{\text{NL}}$ and the Doppler term (the main GR contributor). Firstly
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Figure 3. Same as Figure 2 but for frequency bins of 20MHz.

Figure 4. Absolute value of the relative difference of the neglect of GR effects on the light-cone for frequency bins of 15MHz. We define the difference \( \Delta C_{f_{NL}}^{GR} \equiv C_f(\epsilon_{GR} = 1) - C_f(\epsilon_{GR} = 0) \).

Figure 5. Same as Figure 4 but for frequency bins of 20MHz.

let us quantify how sensitive is the power spectrum to the large scale effects. In figure 2 we plot relative contribution of PNG to the power spectrum in 15MHz bins for \( f_{NL} = (-100, -50, -10, 10, 50, 100) \) where \( \Delta C_{f_{NL}}^{PNG} \equiv C_f(f_{NL}) - C_f(f_{NL} = 0) \). While for \( f_{NL} = \pm 10 \) the effect is below percent level, for a higher value of \( f_{NL} \) the difference is enormous, especially in the cross-correlations. In the case of the cross-bin power spectra, the effect of PNG can be enormous, even for small values of \( f_{NL} \), as the cross-bin correlations are more sensitive to the large scale effects. Still, the low multipoles are the ones where the relative difference is higher. Roughly speaking, one needs to measure the power spectrum at a sub-percent level to detect the effect of PNG in a single angular power. Besides, this can be relaxed since we have several angular power spectra from which to get cumulative constraints on \( f_{NL} \). In figure 3 we do the same exercise but for thicker bins. One can see that the specific numbers change although the conclusion is similar, the effect is larger on low multipoles with the cross-correlations between frequency bins being the most sensitive.

In Figure 4 we plot the absolute value of the relative difference between including and neglecting the GR effects.
While these effects are very small in the auto-bin correlations they become relevant in the cross-bin correlations, well above the percent level. Hence, one may hope that the cumulative signal of cross-bin correlations may allow us to identify the Doppler contribution. Besides, GR effects introduce $\ell$ structure in the cross-correlation which should be a distinctive signal and bigger than the effect of $f_{NL}$. Despite this the importance of the GR effects is dependent on the width of the bin, as we can see in Figure 5. Larger frequency bins reduce the relevance of the so-called GR effects. This comes as no surprise, as the biggest of such effects is the Doppler term (Challinor & Lewis 2011) which is averaged out in thick redshift bins.

3 THE INFORMATION MATRIX

To estimate how well we can jointly measure the cosmological parameters and the foregrounds we will use the information matrix which to leading order is given by (Tegmark et al. 1997)

$$
\mathbf{F}_{\sigma_\theta \sigma_\theta} = \sum_\ell \frac{\partial C^{\text{M}}_{\ell,ij}}{\partial \theta_\phi} \Delta^{-1} \frac{\partial C^{\text{M}}_{\ell,im}}{\partial \theta_\phi},
$$

(26)

for a set of parameters $\{\theta\}$. Then one determines the forecasted marginal error using

$$
\sigma_\theta = \sqrt{(\mathbf{F}^{-1})_{\theta \theta}}
$$

(27)

while the conditional error is

$$
\sigma_{\theta}^{\text{cond}} = 1/\sqrt{\mathbf{F}_{\theta \theta}}.
$$

(28)

In the standard approach $C^{\text{M}}$ is given by Eq. 6 while $\Gamma_\ell$ is given by Eq. 8. In addition to the standard approach we want to compare it with two other scenarios. In the first scenario we want to understand how well we can learn physical and spatial properties of the foregrounds in the absence of cosmology. The goal is to set a base from which one can determine deviations from a foregrounds only scenario. In this no cosmology case, the covariance is simply given by the instrumental noise

$$
(\Gamma_{\ell,\ell'})_{ij,mn}^{\text{N}} = \frac{\delta_{\ell,\ell'}}{(2\ell + 1)f_{\text{sky}}} \left( C_{\ell}^{\text{N},im} C_{\ell'}^{jn} + C_{\ell'}^{\text{N},in} C_{\ell}^{jn,m} \right).
$$

(29)

Similarly the fiducial $C_{\ell}$ to input in Eq. 26 only has contributions from the foregrounds. By the same token, one can only forecast constraints on the foreground parameters but it will give us a “best case scenario” for understanding the foregrounds and how degenerate they are among each other.

In a second scenario, we will no longer consider that the foreground angular power spectrum has no covariance. The foregrounds are not a realisation of some foreground distribution function, they come from gas or charged particles from the galaxy or in the local Universe. Therefore they have no statistical structure. The point sources may be an exception. Still, let us assume that the parameterisation in Eq. 23 is incomplete or the model for the foregrounds has intrinsic small scale fluctuations. We thus model such uncertainty by introducing a foregrounds covariance linearly dependent on the foregrounds themselves, i.e.

$$
(\Gamma_{\ell,\ell'})_{ij,mn} = \frac{\delta_{\ell,\ell'}}{(2\ell + 1)f_{\text{sky}}} \left( C_{\ell}^{\text{GFF},im} C_{\ell'}^{jn} + C_{\ell'}^{\text{GFF},in} C_{\ell}^{jn,m} \right)
$$

(30)

with

$$
C_{\ell,i}^{\text{GFF}} = C_{\ell,i}^{\text{GFF}}(\nu_i, \nu_j) + \epsilon C_{\ell,i}^{\text{F}}(\nu_i, \nu_j) + C_{\ell,i}^{\text{N}}(\nu_i, \nu_j),
$$

(31)

Note that we will still consider each component to be uncorrelated with each other. In a nutshell, we made the covariance larger widening the error bars. One should also note that $\epsilon$ cannot be bigger than the uncertainty from the noise itself ($\epsilon \sim N/A_F$), as that sets the uncertainty in measuring the angular power spectrum of the foregrounds. Looking at Figure 1 this is highly dependent on the scale and may vary between $10^{-12} - 10^{-8}$. In other words, our model uncertainties should not be bigger than the noise itself, otherwise one would change the foregrounds model. An alternative approach would be to insert stochastic nuisance parameters in every $\ell$ in every correlation. In practice, this is unfeasible by the sheer number of stochastic parameters one would need to introduce.

In this paper we will consider both cosmological and foregrounds parameters. For the cosmological parameters we will consider the standard parameters, plus PNG and the GR fudge factor introduced in Eq. 18, i.e.,

$$
\vartheta_{\text{Cosmology}} = \{ A_s, n_s, \Omega_{\text{CDM}}, \Omega_b, w, H_0, f_{NL}, \epsilon_{\text{GR}} \},
$$

together with the biases is each bin $b(z_i)$ as nuisance parameters. We considered the following fiducial values for the cosmological parameters: $A_s = 2.142 \times 10^{-9}$, $n_s = 0.9667$ $\Omega_{\text{CDM}} = 0.26$, $\Omega_b = 0.05$, $w = -1$, $H_0 = 67.74 \text{km/s/Mpc}$. We also took $f_{NL} = 0$ (see Eq. 22) and $\epsilon_{\text{GR}} = 1$ (see Eq. 18). As foreground parameters we will consider the 4 free parameters in the model described by Eq. 23 for each of the 4 foreground components, i.e.,

$$
\vartheta_{\text{Foregrounds}} = \{ \mathbf{A}_{\text{EPS}}, \mathbf{B}_{\text{EPS}}, \mathbf{G}_{\text{EPS}}, \mathbf{G}_{\text{EFF}}, \mathbf{A}_{\text{GRS}}, \mathbf{B}_{\text{GFF}}, \mathbf{G}_{\text{GFF}} \).
$$

The fiducials for the foregrounds are given in Table 1.

4 FORECASTING RESULTS

In all the forecasts presented here we decided to truncate Eq. 26 at $\ell_{\text{max}} = 150$. Although we include the beam, in principle one can consider higher multipoles. This does not add substantial information to cosmological constraints and increases the computation time. We will take $\ell_{\text{min}} = 3$ as the sky area allows it and we are considering the foregrounds in our fit. We also binned the full SKA1-MID into 15MHz slices, this way we ensure that in the lowest redshift bin is thick enough such that we are still in the linear regime along the line-of-sight, while trying to take advantage of the frequency resolution of the SKA1-MID. This gives 63 frequency channels with a total of 2016 independent angular power spectra.

In table 2 present our forecasts for the foreground parameters. We consider 5 different cases.
Different foregrounds and the instrumental noise power spectra in different scenarios. We chose the value of \( \epsilon \) to be the ratio of the uncertainty of the angular power spectrum has only instrumental noise. In this case we pretend to create a baseline for how well one we would constrain the foregrounds with our experiment and survey in the absence of cosmological signal. We called this case “without” cosmology;

(ii) the second case is the standard case where we include the Hi power spectrum in the covariance as computed in Appendix A. We therefore called this case “with cosmology”;

(iii) the further 3 cases include uncertainties in the modelling of the foregrounds by adding a contribution to the estimator covariance modulated by the parameter \( \epsilon \) (see Eq. 31). We chose the value of \( \epsilon \) to be roughly the ratio of the foregrounds and the instrumental noise power spectra in different \( \ell \)-scales. We chose this ratio as any uncertainties in the foregrounds above the noise would be detectable. In practice such contribution increase the uncertainty budget.

| \( \mathcal{A}_{\text{GS}} \) | \( \beta_{\text{GS}} \) | \( \alpha_{\text{GS}} \) | \( \xi_{\text{GS}} \) |
|----------------|--------|--------|--------|
| w/o cosmology | 7.3e-9 | 5.5e-10 | 5.4e-10 | 9.5e-10 |
| w/ cosmology  | 2.1e-05| 1.7e-08| 1.5e-08| 3.3e-08|
| log \( \epsilon \) = -12 | 4.3e-06| 4.6e-07| 4.1e-07| 4.4e-07|
| log \( \epsilon \) = -9 | 4.5e-06| 4.8e-07| 4.3e-07| 4.5e-07|
| log \( \epsilon \) = -6 | 9.3e-04| 1.4e-04| 3.6e-05| 2.6e-05|

| \( \mathcal{A}_{\text{GFF}} \) | \( \beta_{\text{GFF}} \) | \( \alpha_{\text{GFF}} \) | \( \xi_{\text{GFF}} \) |
|----------------|--------|--------|--------|
| w/o cosmology | 6.6 e-07| 2.0e-08| 2.5e-08| 2.4e-08|
| w/ cosmology  | 2.2e-05| 6.4e-07| 9.0e-07| 8.8e-07|
| log \( \epsilon \) = -12 | 7.0e-04| 2.6e-05| 2.9e-05| 2.8e-05|
| log \( \epsilon \) = -9 | 7.2e-04| 2.7e-05| 3.0e-05| 2.8e-05|
| log \( \epsilon \) = -6 | 5.4e-02| 3.3e-03| 2.8e-03| 2.4e-03|

| \( \mathcal{A}_{\text{EPS}} \) | \( \beta_{\text{EPS}} \) | \( \alpha_{\text{EPS}} \) | \( \xi_{\text{EPS}} \) |
|----------------|--------|--------|--------|
| w/o cosmology | 8.2e-05| 3.4e-06| 9.3e-07| 1.8e-05|
| w/ cosmology  | 4.0e-03| 1.7 e-04| 3.8e-05| 8.7e-04|
| log \( \epsilon \) = -12 | 4.8e-03| 2.0e-04| 7.7e-05| 1.1e-03|
| log \( \epsilon \) = -9 | 5.6e-03| 2.3e-04| 8.6e-05| 1.3e-03|
| log \( \epsilon \) = -6 | 1.7e-02| 1.7e-03| 9.1e-04| 3.8e-03|

| \( \mathcal{A}_{\text{EFF}} \) | \( \beta_{\text{EFF}} \) | \( \alpha_{\text{EFF}} \) | \( \xi_{\text{EFF}} \) |
|----------------|--------|--------|--------|
| w/o cosmology | 3.3e-01| 2.0e-02| 5.7e-03| 9.7e-02|
| w/ cosmology  | 1.6e01| 9.4e-01| 2.7e-01| 4.8e-00|
| log \( \epsilon \) = -12 | 1.9e01| 1.5e00| 4.6e-01| 5.7e00|
| log \( \epsilon \) = -9 | 2.2e01| 1.7e00| 5.1e-01| 6.7e00|
| log \( \epsilon \) = -6 | 1.4e02| 4.0e01| 1.1e01| 1.9e01|

Table 2. Forecasted relative 1-\( \sigma \) marginal errors (\( \sigma_{\ell}/\bar{\ell} \)) in percentage (%) for each foreground parameter under different assumptions.

| \( A_{\text{SN}} \) | \( n_s \) | \( H_0 \) | \( \Omega_{\text{CDM}} \) | \( \Omega_b \) | \( w \) |
|----------------|--------|--------|--------|--------|--------|
| w/o foregrounds | 3.57    | 1.42   | 2.31   | 2.32   | 4.05   | 2.41   |
| w/ foregrounds  | 3.60    | 1.43   | 2.33   | 2.38   | 4.11   | 2.52   |
| log \( \epsilon \) = -12 | 3.61    | 1.43   | 2.33   | 2.38   | 4.11   | 2.53   |
| log \( \epsilon \) = -9 | 3.64    | 1.44   | 2.34   | 2.41   | 4.14   | 2.57   |
| log \( \epsilon \) = -6 | 3.73    | 1.47   | 2.40   | 2.48   | 4.26   | 2.66   |

Table 3. Forecasted relative 1-\( \sigma \) marginal errors (\( \sigma_{\ell}/\bar{\ell} \)) in percentage (%) for the standard cosmological parameters under different assumptions.

Generically one can constrain very well the foregrounds in all scenarios one considers. The exception is extragalactic free-free emission. In the normal scenario, one can only constrain the amplitude of EFF with 16% and its other parameters around percent level. In general, one can conclude that we will be able to learn the foregrounds with high accuracy even when we include foreground model uncertainties. This result may come as no surprise, we input well-defined models of the foregrounds in an extremely low noise experiment therefore we expect to measure them very well! This is true even when we assumed model uncertainties to be present. The results in table 2 are not the crucial conclusion to take home, instead one should look for extensive and realistic modelling of the foregrounds, as they can be statistically tested with high precision. Not only they would complement the information about dust and Synchrotron emission from higher frequencies (Akrami et al. 2020a) but also improve the constraints available. While for Planck there were 9 frequency channels available, the SKA1-MID is only limited by RFI flagging. Here we considered 63 which is almost an order of magnitude increase in the band available.

The results for the standard cosmological parameters are presented in table 3. As for table 2, the different lines correspond to the different cases we want to investigate (although we now do not consider the case with noise only). The most important conclusion to take is that the marginal constraints on the standard cosmological parameters are fairly independent of the foregrounds even when we increase the covariance. This is in agreement with previous results (see for example Wolf et al. 2014), as the foreground cleaning methods do not affect smaller scales. Note as well that we considered the bias as nuisance parameters but we assumed the Hi temperature to be known which can degrade considerably the constraints on the cosmological parameters. In principle one can assume that other summary statistic like the 3D power spectrum, or its multipoles, with foreground cleaning are used for constraining the standard cosmological model, which then one can take as “known” when studying large scale effects.

Our main point for this paper was to understand how the foregrounds would affect \( f_{\text{NL}} \). Let’s first start with recapping the forecasts without foregrounds. The first three lines of table 4 present the forecasts of the conditional error, the marginal error when we neglect the presence of foregrounds, and the case when we fix the bias. The marginal only degrades by a quarter and is not very sensitive to the knowledge of the bias. Note that the results presented here
are different (and worse) from the results of Alonso et al. (2015b). Here we include the beam and have a more stringent cut on the \( \ell_{\text{max}} \) (150 instead of 500), as well as using a smaller area and having more bias nuisance parameters. More than comparing forecasts we are interested in assessing how marginalising over the foregrounds degrades constraints. While one would hint that such an approach would render impossible any constraints, this is not the case. Although the inclusion of the foregrounds degrades the constraints by \( \sim 75\% \), this is not catastrophic. In particular, one can devise a strategy where we use other methods to determine the cosmological parameters and only use the approach presented in this paper for the large scale effects and foreground parameterisations. Such a strategy would produce a marginally better \( \sigma_{f_{\text{NL}}} = 6.6 \), which although far from the target \( \lesssim 1 \), is comparable with other LSS experiments. Once we start making the covariance bigger due to foreground model uncertainties the forecasted error worsens. Similar conclusions can be taken for the GR effects. Still, it one clearly concludes that a joint fit is not unrealistic.

5 USING THE WRONG MODELS

The conclusions we arrived at here can be reached just by looking at figure 1 without any calculation. If one neglects a contribution somewhere, it necessarily needs to be absorbed by other free parameters as it disappeared from the signal, irrespective of the volume of available data. Although all foregrounds included in our toy model were known, the results hint that any miss-modelling would be noticed straight away. In addition, the inclusion of fudge foreground component can be a test of unmodelled components. Hence a relevant question, which we have not considered yet, is whether a poor or incomplete modelling of the foreground components leads to significant biases in the final results. Considering how large is the foreground contribution, compared to the signal, this is a potentially crucial issue to investigate. In some sense including extra contributions to the covariance of the \( C_\ell \) was an attempt to include issues with bad modelling, but they only degrade the precision of the measurement. This is quite clear in the results of the previous section. Irrespective of the precision, how is the accuracy of our measurement affected by using wrong models?

To answer the question of how well we need to model the foregrounds, let us consider the case of nested model selection (Heavens et al. 2007). Let us say that a bigger model \( M \) has \( \psi_i \) parameters and that the nested model \( M' \) has \( \theta_j \), meaning that \( \phi_k \) remaining parameters are in the bigger model but not in the smaller. This means that \( M' \subset M \) and \( \{ \psi_i \} \cup \{ \phi_k \} \). If we fix the \( \phi_k \) parameters at their correct values then the peak of the likelihood of the subset remains unchanged (although the shape of the posterior changes). On the other hand, if we fix them at an incorrect value \( \Delta \phi_k \) away from the correct one, one can show (see Appendix B) that we then bias the best fit value of the nested sample by

\[
\Delta \theta_j = -(H^{-1})_{\theta_j \phi_k} F_{\theta_k \phi_k} \Delta \phi_k ,
\]

where \( H \) is the fisher matrix of the simpler model while \( F \) is the fisher matrix of the bigger model. Note that this expression is only valid for perturbations around the best fit model. In this paper, we will grossly extrapolate its validity to gain insight into how badly one can bias the best fit values.

The first error one can make is to assume an incorrect number of foregrounds in the forward model. As expected such information would spill into the other components. Let us take for example that we have neglected extragalactic free-free, as it is the smallest of the foregrounds component. That means \( \Delta \phi_k = \Delta A_{\text{EFF}} = -0.014 \). If we had done so, we would have biased our best fit parameter of the GR corrections and \( f_{\text{NL}} \) by

\[
\Delta f_{\text{NL}} = 9687 , \quad \Delta \phi_k = 13460 .
\]

These numbers are slightly non-sensical in the sense that if we had measured them in reality one would have immediately understood that something had gone wrong. Still, the amplitude of a neglected component needs to be absorbed by any other free parameter. In this test case, not only the large scale effects are biased but all other cosmological parameters. Although the foreground parameters are biased they are minimally so.

Another possible “error” would be to assume that we only have galactic foregrounds. If this was the case the biasing of the super horizon scale effects would be even more severe

\[
\Delta f_{\text{NL}} \sim \Delta \phi_k \sim -10^9 .
\]

Note that the exact number is not needed as our approach has already broken down. But it does mean that missing foreground components will bias any fit of the cosmological parameters. In this scenario it is the galactic foregrounds that change substantially, even absorbing the cosmological signal,

\[
\Delta A_{\phi_k} = 12.4 , \quad \Delta A_{\phi_k} = 0.11 .
\]

This would make the angular power of galactic free-free to more than double in amplitude. One could also assume that the foregrounds are either perfectly correlated or uncorrelated in frequency. If they are perfectly correlated, \( \xi \to \infty \). In practice one cannot do this but we can make \( \xi \) be large. One can estimate the value of \( \xi \) that makes them correlated at least of in 1 in a thousand within the frequency range of the experiment [350,1420] MHz. That requires a value of \( \xi = 31.3 \) but let’s make it equal to 35 for simplicity. Then \( \Delta \phi_k = (35 - \xi_{\text{EPS}}) = (35 - \xi_{\text{EFF}}) = (35 - \xi_{\text{GS}}) = 0.31) \).
The results would be catastrophic for \( f_{NL} \) and \( \epsilon_{GR} \), as well as for the fits to the foregrounds themselves. On the other hand, if we consider the foregrounds uncorrelated in frequency then \( \xi \to 0 \). In practice one cannot do this but one can take a small value of \( \xi \), like \( \xi = 1 \). Then \( \Delta \phi = 1 - \left( C_{EE}^{fid} + C_{FF}^{fid} - 2C_{EF}^{fid} \right) = (0.34, 3.34) \) would bias the best fit parameters of all others parameters.

### 6 Discussion

The main goal of this paper was to estimate how the presence of foregrounds degrades forecasted constraints on \( f_{NL} \) and GR effects using Hi IM. In this paper, we took an alternative approach to what is commonly found in the literature. Instead of using foreground cleaning methods, which highly suppresses information on very large scales, we use toy models of foregrounds to estimate which results can be obtained from a joint fit of foreground parameterisations, cosmological parameters, and large scale effects. We, therefore, started by reviewing maps of intensity, and the summary statistics we use in this paper, the angular power spectra \( C_\ell \). We reviewed the theoretical covariance of the power spectra and argued that it should be independent of the foregrounds. We then modelled each component: the cosmological signal of HI, the foregrounds, and the instrumental noise. We also exemplified how the foregrounds and the cosmological signal compare with each other. Despite the wide amplitude differences between each contribution, i.e., the cosmological signal being subdominant with respect to any of the foreground components, the instrumental noise (plus the cosmology) sets the uncertainty one can measure the total angular power. We then exemplified the effect of large scale effects in the power spectrum to gain insight on how well one needs to measure the cosmological contributions in the total observed power spectra.

We then reviewed the fisher matrix formalism and set up the parameters of interest as well as their fiducial values. We considered 3 “experimental setups”: one in which only the noise is relevant in the covariance, the traditional one where the variance of the angular power is given by the instrumental noise and cosmic variance, and a third where we model Gaussian fluctuations of the foregrounds proportional to its amplitude. Each case was considered in our SKA1-MID experimental setup. One of the main conclusions is that irrespective of the case one should be able to learn very well the angular structure and tomographic structure of the foregrounds in frequency. This is fundamentally due to the low experimental noise and a high number of correlations possible. Although including foreground model uncertainties degrades the constraints on the model parameters, these are limited. Fundamentally such variations can be detectable in such low noise experiments. Also, being able to understand so well the foregrounds does not affect the forecasted constraints on the standard cosmological parameters.

On the other hand, the large scale effects get degraded substantially in the presence of foregrounds. Although the forecasted error degrades around 75% in the case of \( f_{NL} \), the constraints on GR effects is only degraded by 30%. Even though the foregrounds degrade the constraints, these are neither catastrophic neither represent a substantial source of degeneracy. If we compare the conditional error with a marginal error when the cosmology is known, the foregrounds only represent a 50% degradation. In any case, the survey specs we used of 20000deg\(^2\) can provide Planck-level constraints even if marginalised over the foregrounds - although strong priors on the cosmological parameters are required. In the absence of foregrounds our forecasted constraints on \( f_{NL} \) are of same order of magnitude of what was previously found by Alonso et al. (2015b), still far from a desirable constraining power of \( \sigma_{NL} \lesssim 1 \). We therefore require more futuristic experiments, cross-correlate with optical galaxy surveys (Fonseca et al. 2015) or add bispectra information (Karagiannis et al. 2018).

Hence, it seems potentially feasible to jointly fit the cosmology, foreground parameters, and primordial non-Gaussianity. Such “feasibility” can be dismissive. We used toy models that are a strong theoretical prior. We, therefore, tried to gain insight into how biased one would be if our understanding of the foregrounds is wrong. We used nested models to quantify the biases. We, therefore, imagined what would happen if we neglected one or several components to the power spectrum. We concluded that it would be fundamentally catastrophic, not only for the large scale effects but also to what one can learn about the foregrounds. As one expects, if we do not model a contribution it needs to be absorbed by other components. A similar conclusion is obtained when we assume wrong correlations in frequency. What this means is that any unaccounted contribution to the power spectra will jeopardise our results.

This work seems to disagree with Liu & Tegmark (2011) on the possibility of constraining foreground properties. We argue that there is a substantial amount we can learn about the foregrounds’ macroscopical properties. There are differences between this work and their work. Firstly we parametrise the summary statistics of the angular and frequency structure of foregrounds while Liu & Tegmark (2011) works in pixel space using physical descriptions of foreground emission. Secondly, we focused on the SKA1-MID at higher frequencies in a wider range. This means that we assume we have more data available at a lower instrumental noise. Our work was not intended to replicate theirs and a proper comparison would need to be done with their parameterisations and the instrumental and survey specifications of SKA1-LOW. We leave this to future work.

What we concluded here is that one can indeed jointly measure foreground parameters and large scale effects in HI IM. Although the degeneracies between foregrounds and \( f_{NL} \) degrade the constraining power this is not catastrophic. But one needs to carefully include the proper foregrounds model and components. This means that although we show that we can measure foregrounds and large-scale effects in their presence we should take this result with a slight pinch of salt. Firstly we considered toy models for the foregrounds and neglected systematics like 1/f noise, polarisation leakage, and beam asymmetries. We also considered well defined spectral indices for each foreground instead of taking their power spectrum. Neither we consider a proper map of the foregrounds with mask cuts. As we showed here, any incorrect modelling will jeopardise any attempt not only of understanding the foregrounds but more interestingly primordial non-Gaussianity. Still, the results here indicate that, as in CMB studies (Aghanim et al. 2020a; Efstathiou & Gratton 2019), it is worth resorting to template fitting for specific
applications which require accurate reconstructions of the large angular scales in the survey. This will allow us to learn properties of the foregrounds and constrain $f_{NL}$.

As a summary, we explored the potential of intensity mapping surveys to constrain primordial local-type non-Gaussianity. We find that, with realistic settings for forthcoming experiments the expected constraints are at the level of those already achievable by Planck. This can of course be improved with more futuristic settings. However, the main message of our work is that the constraints are not significantly degraded by foreground contamination, provided the templates are accurate enough. Therefore, a template fitting approach is a worth pursuing methodology on the large scales required for $f_{NL}$ and GR effects studies.

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DATA AVAILABILITY  

Data and codes are available on request. The version of CAMB used is already available online https://github.com/ZeFon/CAMB_sources_MT_ZF.

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APPENDIX A: THE COVARIANCE OF THE ANGULAR POWER SPECTRUM IN FULL SKY

Here we review the calculation of the covariance of the observed angular power spectrum. As in Eq. 7 we consider the full sky estimator or the angular power spectrum to be

\[ \hat{C}_{\ell} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \left[ C_{\ell m} a_{\ell m} + \bar{a}_{\ell m} \right]. \]  

(A1)

From Eqs. 1 and 4 it follows that

\[ a_{\ell m} = a_{\ell m}^S + a_{\ell m}^F + a_{\ell m}^N, \]  

(A2)

and assuming the components are independent, i.e.,

\[ \langle a_{\ell m}^S a_{\ell' m'}^F \rangle = \delta_{\ell \ell'} \delta_{mm'} C_{\ell}^{\text{A}} \]  

(A3)

Then the expected value of our estimator is given by

\[ \langle \hat{C}_{\ell} \rangle = C_{\ell}^{\text{A}} = C_{\ell}^{\text{S}} + C_{\ell}^{\text{F}} + C_{\ell}^{\text{N}}, \]  

(A4)

which can be seen as a biased estimator of the signal with the bias being the foregrounds. Note that, since the different components are independent, the total angular power spectrum is just the sum of the angular power spectrum of each component.

The covariance of our angular power spectrum estimator is defined as

\[ \text{Cov} \left[ \hat{C}_{\ell,ij}, \hat{C}_{\ell',pq} \right] = \langle \Gamma_{\ell,\ell'} \rangle_{ij,pq}, \]  

\[ \equiv \langle \hat{C}_{\ell,ij} \hat{C}_{\ell',pq} \rangle - \langle \hat{C}_{\ell,ij} \rangle \langle \hat{C}_{\ell',pq} \rangle . \]

The expected value of the angular power spectrum has already been given by Eq. A4. The first term becomes

\[ \langle \hat{C}_{\ell,ij} \hat{C}_{\ell',pq} \rangle = \frac{1}{(2\ell + 1)(2\ell' + 1)} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell'} \frac{1}{4} \times \left\{ C_{\ell m}^{M} a_{\ell m}^{M^*} + a_{\ell m}^{M^*} C_{\ell m}^{M} \times \left[ C_{\ell' m'}^{M} a_{\ell' m'}^{M^*} + a_{\ell' m'}^{M^*} C_{\ell' m'}^{M} \right] \right\}. \]  

(A6)

It is a summation over 4 similar 4-point functions. For simplicity let us just analyse the first one

\[ \langle a_{\ell m}^M a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \]

One would be tempted to use Wick’s theorem immediately and just expand this 4-point function. But Wick’s theorem in its simplest form is only valid for quantities that follow some distribution function. The noise is usually taken to be Gaussian, i.e., follows as Gaussian distribution with zero mean and covariance given by \( C_{\ell}^{\text{G}} \). The same is true for cosmological perturbations, it is a Gaussian random variable with covariance \( C_{\ell}^{\text{G}} \). On the other hand, the foregrounds are not a realisation of any intrinsic spatial distribution function. They are simply a local offset and \( C_{\ell}^{\text{G}} \) is just a characterisation of the spatial structure of the foregrounds in spherical harmonic space. Technically speaking, \( a_{\ell m}^F \) gets out of the expected value brackets. Noting that Gaussian distributions have zero value odd-point functions we write

\[
\langle a_{\ell m}^M a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle = \begin{cases} 
\langle a_{\ell m}^M a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \\
+ \langle a_{\ell m}^M a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \\
+ \langle a_{\ell m}^M a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \\
+ \langle a_{\ell m}^M a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle 
\end{cases}
\]

(A7)

where we defined the Gaussian part as \( a_{\ell m}^G \equiv a_{\ell m}^G + a_{\ell m}^N \). Hence when we do the summation in \( m \) and \( m' \) in Eq. A6 we get for the last 3 terms of Eq. A7

\[
\begin{align*}
\sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell'} \frac{1}{(2\ell + 1)(2\ell' + 1)} \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle &= C_{\ell}^{G} C_{\ell'}^{\text{G}} \\
\sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell'} \frac{1}{(2\ell + 1)(2\ell' + 1)} \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle &= C_{\ell}^{G} C_{\ell'}^{\text{G}} \\
\sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell'} \frac{1}{(2\ell + 1)(2\ell' + 1)} \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle &= C_{\ell}^{G} C_{\ell'}^{\text{G}} 
\end{align*}
\]

(A8)

The factors of 1/4 vanish as well since we have 4 combinations of the \( a_{\ell m} \) in Eq. A6. We can now expand the Gaussian part of the 4 point function using Wick’s theorem as

\[
\begin{align*}
\langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle &= \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \\
&\quad + \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \\
&\quad + \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle \\
&\quad + \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle
\end{align*}
\]

(A9)

Using Eq. A3 and the property \( a_{\ell m} = a_{\ell - m} \), we get

\[
\begin{align*}
\langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle &= C_{\ell}^{G} C_{\ell'}^{\text{G}} \\
&\quad + C_{\ell}^{G} C_{\ell'}^{\text{G}} \delta_{\ell \ell'} \delta_{mm'} \\
&\quad + C_{\ell}^{G} C_{\ell'}^{\text{G}} \delta_{\ell \ell'} \delta_{mm'} \\
&\quad + C_{\ell}^{G} C_{\ell'}^{\text{G}} \delta_{\ell \ell'} \delta_{mm'} \\
\end{align*}
\]

(A10)

We can then make the summation in \( m \) and \( m' \)

\[
\sum_{m,-\ell}^{\ell} \sum_{m'=-\ell}^{\ell'} \frac{1}{(2\ell + 1)(2\ell' + 1)} \langle a_{\ell m}^G a_{\ell m}^{M^*} a_{\ell' m'}^M a_{\ell' m'}^{M^*} \rangle = C_{\ell}^{G} C_{\ell'}^{\text{G}} \\
+ \delta_{\ell \ell'} \left[ C_{\ell}^{G} C_{\ell'}^{\text{G}} \right] \\
+ C_{\ell}^{G} C_{\ell'}^{\text{G}} \delta_{\ell \ell'} \delta_{mm'} \\
+ C_{\ell}^{G} C_{\ell'}^{\text{G}} \delta_{\ell \ell'} \delta_{mm'}
\]

(A11)

Therefore the 2 point function of the angular power spectrum is

\[
\begin{align*}
\langle \hat{C}_{\ell,ij} \hat{C}_{\ell',pq} \rangle &= C_{\ell,ij}^{\text{G}} C_{\ell',pq}^{\text{G}} \\
&\quad + C_{\ell,ij}^{\text{G}} C_{\ell',pq}^{\text{G}} \\
&\quad + C_{\ell,ij}^{\text{G}} C_{\ell',pq}^{\text{G}} \\
&\quad + C_{\ell,ij}^{\text{G}} C_{\ell',pq}^{\text{G}}
\end{align*}
\]

(A12)

The first two lines are just \( \langle \hat{C}_{\ell,ij} \rangle \langle \hat{C}_{\ell',pq} \rangle \), then we arrive to the covariance of our estimator to be

\[
\langle \Gamma_{\ell,\ell'} \rangle_{ij,pq} = \frac{\delta_{\ell \ell'}}{(2\ell + 1)} \left[ C_{\ell}^{G} C_{\ell'}^{\text{G}} + C_{\ell}^{G} C_{\ell'}^{\text{G}} \right] \\
\]

(A13)
where \( C_{m,ij} = C_{e,ij} + C_{N,ij} \). Therefore the covariance only depends on the angular power spectra of the cosmological signal and the instrumental noise and independent of the foregrounds.

**APPENDIX B: THE BIAS ON PARAMETERS**

Let us assume we have a \( n \)-sized data vector \( \mathbf{X} \) (or a \( n \times 1 \) matrix) with covariance \( \Sigma \), with a Gaussian likelihood given some parameters \( m \) parameters array \( \vartheta \) is

\[
\mathcal{L}(\mathbf{X}|\vartheta) = \frac{1}{(2\pi \det \Sigma)^{n/2}} e^{-\frac{1}{2} (\mathbf{X} - \bar{\mathbf{X}})^T \Sigma^{-1} (\mathbf{X} - \bar{\mathbf{X}})},
\]

with \( \mathbf{X} \) the values of \( \mathbf{X} \) that maximise the likelihood. Note that although omitted both \( \mathbf{X} \) and \( \Sigma \) depend on \( \vartheta \). Let us also assume that the posterior probability of the parameters given the data is Gaussian and expressed as

\[
\mathbf{P}(\vartheta|\mathbf{X}) = \frac{1}{(2\pi \det \mathbf{F}^{-1})^{m/2}} e^{-\frac{1}{2} (\vartheta - \bar{\vartheta})^T \mathbf{F} (\vartheta - \bar{\vartheta})},
\]

where \( \mathbf{F} \) is the inverse of the covariance of the parameters we which to measure. One can also see that

\[
\mathbf{F}_{\vartheta_i,\vartheta_j} = -\frac{\partial^2 \ln \mathcal{L}(\mathbf{X}|\vartheta)}{\partial \vartheta_i \partial \vartheta_j} \bigg|_{\vartheta_i = \bar{\vartheta}_i, \vartheta_j = \bar{\vartheta}_j}.
\]

In fact even if we had not assumed the Gaussian case, one could still define the information matrix assuming that a maximum of the posterior exists at \( \bar{\vartheta} \). Bayes theorem states that

Posterior probability \( \propto \) Likelihood \( \times \) Prior.

Then, in the absence of priors, one can write the information matrix as it is commonly found in the literature

\[
\mathbf{F}_{\vartheta_i,\vartheta_j} = -\frac{\partial^2 \ln \mathcal{L}(\mathbf{X}|\vartheta)}{\partial \vartheta_i \partial \vartheta_j} \bigg|_{\vartheta_i = \bar{\vartheta}_i, \vartheta_j = \bar{\vartheta}_j}.
\]

For this paper our data points are the observed angular power spectra, i.e., \( \mathbf{X} = C(\ell, z) \). From this one gets Eq. (26) assuming that the 4 point function \( \Gamma \) is weakly dependent on the cosmological parameters. Note that all assumes the existence of a local maximum of the posterior \( (\partial_{\vartheta} \mathbf{P})_{|\vartheta = 0} = 0 \), not necessarily that the distributions are Gaussian.

Now let us assume that we have a model with parameters \( \psi \) and a subset of those, \( \varphi \) that we will fix while the remaining \( \vartheta \) parameters we wish to fit. We know that at the maximum

\[
\partial_{\vartheta_m} \ln \mathcal{L}|_{\bar{\vartheta}_m} = 0, \quad \text{(B6)}
\]

differentiate at

\[
\partial_{\vartheta_i} \ln \mathcal{L}(\mathbf{X}|\vartheta)|_{\bar{\vartheta}_j} = 0. \quad \text{(B7)}
\]

If one expands the likelihood around the maximum

\[
\ln \mathcal{L}(\vartheta|\varphi) = \ln \mathcal{L}(\bar{\vartheta}|\varphi) + \delta \vartheta_i \partial_{\vartheta_i} \ln \mathcal{L}(\bar{\vartheta}|\varphi) |_{\bar{\vartheta}, \varphi} + \delta \varphi_m \partial_{\varphi_m} \ln \mathcal{L}(\bar{\vartheta}|\varphi) |_{\bar{\vartheta}, \varphi}, \quad \text{(B8)}
\]

Then taking the derivative with respect to \( \vartheta_j \), and using Eq. B7 we get

\[
0 = \partial_{\vartheta_i} \partial_{\vartheta_j} \partial_{\vartheta_i} \ln \mathcal{L}(\bar{\vartheta}|\varphi) |_{\bar{\vartheta}, \varphi} + \delta \varphi_m \partial_{\vartheta_j} \partial_{\varphi_m} \ln \mathcal{L}(\bar{\vartheta}|\varphi) |_{\bar{\vartheta}, \varphi},
\]

\[
= \partial_{\vartheta_j} \ln \mathbf{F}_{\varphi_m \varphi_m} + \delta \varphi_m \mathbf{F}_{\varphi_m \varphi_j}, \quad \text{(B10)}
\]

where we defined the information matrix of the smaller set of parameters \( \{ \vartheta \} \) as \( \mathbf{H} \). In fact \( \mathbf{H} \) is a subset of \( \mathbf{F} \). Then we have that the bias on the best fit model by fixing a larger model with the wrong parameters is given by

\[
\delta \vartheta_i = -\delta \varphi_m \mathbf{F}_{\varphi_m \varphi_i} \mathbf{H}_{\varphi_i \varphi_j}^{-1} \delta \varphi_j. \quad \text{(B11)}
\]

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