STABILITY OF SYZYGY BUNDLES CORRESPONDING TO STABLE VECTOR BUNDLES ON ALGEBRAIC SURFACES

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Abstract. Let \((X, H)\) be a polarized smooth projective algebraic surface and \(E\) is globally generated, stable vector bundle on \(X\). Then the Syzygy bundle \(M_E\) associated to it is defined as the kernel bundle corresponding to the evaluation map. In this article we will study the stability property of \(M_E\) with respect to \(H\).

1. Introduction

The purpose of this paper is to investigate the stability of Syzygy bundles associated to a stable and globally generated vector bundles on a smooth projective algebraic surface.

Let \(X\) be a smooth, irreducible, projective algebraic variety defined over an algebraically closed field \(k\). We fix a very ample divisor \(H\) on \(X\). We refer to the pair \((X, H)\) as a polarized algebraic variety. Let \(E\) be a globally generated vector bundle on \(X\). Then the Syzygy bundle \(M_E\) is defined to be the kernel bundle corresponding to the evaluation map \(\text{ev} : H^0(E) \otimes \mathcal{O}_X \to E \to 0\). Thus we have an exact sequence

\[
0 \to M_E \to H^0(E) \otimes \mathcal{O}_X \to E \to 0.
\]

These vector bundles (and some analogues) arise in a variety of geometric and algebraic problems. For example what are the numerical conditions (optimal) on vector bundles \(E\) and \(F\) on \(X\) such that the natural product map \(H^0(X, E) \times H^0(X, F) \to H^0(X, E \otimes F)\), becomes surjective. The first initiative towards this question was taken by D.C. Butler [2]. He gave an affirmative answer when \(X\) is a smooth projective curve. He approached the question via the vector bundle \(M_E\) associated to a bundle \(E\) generated by global sections. Consequently, there has been considerable interest in trying to establish the stability of \(M_E\) in various settings. When \(X\) is a smooth curve of genus \(g \geq 1\), the semistability was studied by Butler. He proved that when \(E\) is semistable with \(\mu(E) \geq 2g\), then \(M_E\) is semistable.

Our main aim in this paper is to study the (slope) stability of \(M_E\) with respect to \(H\) when \(E\) is stable with respect to \(H\) and \(X\) is a smooth projective surface. In [1] the authors study the stability of \(M_E\) when \(E\) is a very ample line bundle. In fact they showed that, if we take sufficiently large power of \(E\) then the kernel bundle is (slope) stable with respect to \(E\). In this short note we consider the case of \(H\)-stable vector bundles. For any vector bundle \(E\) on \(X\) and \(m > 0\) let \(E(m) := E \otimes \mathcal{O}_X(mH)\). We will prove the following
Theorem 1.1. Let $E$ be a $H$-(slope) stable vector bundle on $X$. There exists $m >> 0$ such that the kernel bundle $M_{E(m)}$ is stable with respect to $H$.

We follow the main proof strategy of [1]. Our method is suitable enhancement of the arguments given in [1]. Two key ingredients of the proof are Mehta-Ramanathan restriction theorem and Butler’s theorem on stability of kernel bundles on curves. We note that our theorem as well as the theorem of [1] are not effective in a sense there is no concrete lower bound of $m$.

2. Proof Of the main theorem

This section is devoted to the proof of the main theorem. First let us fix up some notations. As before, $X$ is a smooth, irreducible projective surface and $H$ a very ample divisor. Let $E$ be a stable vector bundle with respect to $H$ of rank $l$ on $X$. For the entire course of arguments we fix an integer $n >> 0$, sufficiently large, such that $nH - K_X$ is very ample where $K_X$ is the canonical divisor and $H^i(X, E(n)) = 0$, $i = 1, 2$. For any closed point $x$ let $m_x$ denotes the ideal defining the the point $x$.

We observe the following easy Lemma.

Lemma 2.1. If $W \subset H^0(X, E(n))$ is a subspace which generates $E(n)$ then the natural multiplication map

$$H^0(X, O_X((m - n)H) \otimes m_x)) \otimes W \rightarrow H^0(X, E(m) \otimes m_x)$$

is surjective for $m >> 0$.

Proof. Note that it is enough to prove that the map

$$H^0(X, O_X((m - n)H))) \otimes W \rightarrow H^0(X, E(m))$$

is surjective for some $m >> 0$. By definition of $W$ we have a surjection $W \otimes O_X(-n) \rightarrow E$. Let $K$ be the kernel of this surjection. Then we have the following exact sequence

$$0 \rightarrow K \rightarrow W \otimes O_X(-n) \rightarrow E \rightarrow 0$$

We choose $m >> 0$ such that $H^1(X, K(m)) = 0$. After tensoring the above exact sequence by $O_X(m)$ and passing to the corresponding long exact sequence we get

$$H^0(X, O_X((m - n)H))) \otimes W \rightarrow H^0(X, E(m))$$

is surjective.

We now choose a $m >> 0$ independent of $n$ such that

1. $O_X(mH)$ and $O((m - n)H)$ are very ample.
2. If $W \subset H^0(X, E(n))$ is a subspace which generates $E(n)$ then the natural multiplication map

$$H^0(X, O(m - n)H \otimes m_x)) \otimes W \rightarrow H^0(X, E(m) \otimes m_x)$$

is surjective.
We aim to show that there exists a $m >> 0$ such that $M_{E(m)}$ is $H$-stable. We first analyze if for some $m > 0$, $M_{E(m)}$ is not $H$-stable. In this case there exists a saturated locally free subsheaf $F_m \subset M_{E(m)}$ such that

$$\frac{c_1(F_m).H}{rk(F_m)} \geq \frac{c_1(M_{E(m)}).H}{rk(M_{E(m)})}$$

Our goal is to show that for sufficiently large $m >> 0$ no such $F_m$ can exists.

Pick a smooth and irreducible curve $C_m \in |(m - n)H|$ through a fixed point $x \in X$. We may also assume that $M_{E(m)}/F_m$ is locally free along $C_m$. Observe that

$$\mu_H(F_m) = \frac{c_1(F_m).H}{rk(F_m)} = \frac{1}{(m-n)}\mu(F_m|_{C_m}),$$

Similarly,

$$\mu_H(M_{E(m)}) = \frac{1}{(m-n)}\mu(M_{E(m)}|_{C_m})$$

Thus we have

$$(2.1) \quad \mu(F_m|_{C_m}) \geq \mu(M_{E(m)}|_{C_m})$$

Since, $H^1(X, E(n)) = 0$ we have the following exact sequence

$$(2.2) \quad 0 \to \mathcal{O}_c^{h_0(E(n))} \to M_{E(m)}|_{C_m} \to \overline{M}_m \to 0$$

where $\overline{M}_m$ is the kernel bundle corresponding to $E(m)|_{C_m}$.

As $E$ is stable with respect to $H$, $E(m)$ is stable with respect to $H$. Now if we choose $m$ so that $(m - n) >> 0$ then by Mehta-Ramanathan restriction theorem ([3, Theorem 7.2.8]) $E(m)|_{C_m}$ is stable on $C_m$. Now we have

$$\deg(E(m)|_{C_m}) = \deg(E|_{C_m}) + r\deg(\mathcal{O}_x(m)|_{C_m})$$

Write

$$mH = K_x + mH - nH + Q$$

where $Q = nH - K_x$. By our assumption $Q$ is very ample. Since, $C_m \in |(m - n)H|$ by adjunction formula we get $K_{C_m} = (K_x + (m - n)H)|_{C_m}$. Thus $\deg(\mathcal{O}_x(m)|_{C_m}) = \deg(K_{C_m}) + Q.C_m$. As $Q.C_m \geq 3$

$$(2.3) \quad \deg(K_{C_m}) + Q.C_m \geq 2g_{C_m} + 1$$

We have

$$\deg(E(m)|_{C_m}) = \deg(E|_{C_m}) + r\deg(\mathcal{O}_x(m)|_{C_m}) \geq \deg(E|_{C_m}) + r(2g_{C_m} + 1)$$

We assume that $c_1(E)$ is effective then $c_1(E).C_m \geq 0$. Then we have

$$\deg(E|_{C_m}) + r(2g_{C_m} + 1) \geq r(2g_{C_m} + 1)$$

Therefore, $\deg(E(m)|_{C_m}) \geq r(2g_{C_m} + 1)$ and hence $\mu(E(m)|_{C_m}) \geq (2g_{C_m} + 1)$. By Butler’s theorem ([2, Theorem 1.2]) we have $\overline{M}_m$ is stable.
Let $K_m$ be the kernel of $F_m \hookrightarrow M_{E(m)}|_{c_m} \rightarrow \overline{M}_m$, and $N_m$ be the image. As $\overline{M}_m$ is stable $K_m \neq 0$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & K_m \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_{c_m}^{h_0(E(n))} \\
\downarrow & & \downarrow \\
0 & \rightarrow & M_{E(m)}|_{c_m} \\
\downarrow & & \downarrow \\
0 & \rightarrow & \overline{M}_m \\
\end{array}
$$

(2.4)

We complete the proof following two crucial lemmas. The proofs of these two lemmas follow the exact line of arguments of [11] Lemma 1.1, Lemma 1.2] with some modifications. However, for completeness sake we will provide the proofs.

**Lemma 2.2.** For any $x \in X$, $\text{rank}(F_m) \geq h^0(X, \mathcal{O}_X((m-n)H) \otimes m_z) = h^0(X, \mathcal{O}_X((m-n)H)) - 1$.

**Proof.** For any vector space $V$ on $X$, $\mathbb{P}_{\text{sub}}(V)$ denotes the projective space associated to $V$ consisting 1 dimensional linear subspace of $V$. Multiplication of sections gives rise to a finite morphism

$$
\mu_m : \mathbb{P}_{\text{sub}}(H^0(X, \mathcal{O}_X((m-n)H) \otimes m_z) \times \mathbb{P}_{\text{sub}}(H^0(X, E(n))) \rightarrow \mathbb{P}_{\text{sub}}(H^0(X, E(m) \otimes m_x))
$$

which sends

$$
([s_1], [s_2]) \mapsto s_1 \otimes s_2.
$$

Note that this morphism is composition of the Segre embedding

$$
\mathbb{P}_{\text{sub}}(H^0(X, \mathcal{O}_X((m-n)H), \otimes m_z) \times \mathbb{P}_{\text{sub}}(H^0(X, E(n))) \rightarrow \mathbb{P}_{\text{sub}}(H^0((X, \mathcal{O}_X((m-n)H) \otimes m_z)) \otimes H^0(X, E(n)),
$$

followed by the rational map

$$
\mathbb{P}_{\text{sub}}(H^0((X, \mathcal{O}_X((m-n)H) \otimes m_z)) \otimes H^0(X, E(n)) \rightarrow \mathbb{P}_{\text{sub}}(H^0(X, E(m) \otimes m_x))
$$

For any $x \in X$ we have $F_m(x) \subseteq M_m(x) = H^0(X, E(m) \otimes m_x)$. Let $Z := \mu_m^{-1}(\mathbb{P}_{\text{sub}}(F_m(x))$. Then $\mu_m : Z \rightarrow \mathbb{P}_{\text{sub}}(F_m(x))$ is finite morphism. Therefore,

$$
\dim(Z) \leq \dim(F_m(x)) - 1
$$

(2.5)

Note that for any $s \in H^0(X, \mathcal{O}_X((m-n)H) \otimes m_z)$ the map $H^0(X, E(n)) \rightarrow H^0(X, E(m) \otimes m_x)$ which sends $\varphi \mapsto s \otimes \varphi$ is injective. Thus for any $[s] \in \mathbb{P}_{\text{sub}}(H^0(X, \mathcal{O}_X((m-n)H)

$$
[s] \times K_m(x) \subseteq Z$. Therefore, $\pi_1 : Z \rightarrow \mathbb{P}_{\text{sub}}H^0(X, \mathcal{O}_X((m-n)H)$ is dominant where $\pi_1 : \mathbb{P}_{\text{sub}}(H^0(X, \mathcal{O}_X((m-n)H) \otimes m_z)) \times \mathbb{P}_{\text{sub}}(H^0(X, E(n))) \rightarrow \mathbb{P}_{\text{sub}}H^0(X, \mathcal{O}_X((m-n)H)$ is the first projection. Consequently, we have

$$
\dim(Z) \geq h^0(\mathcal{O}((m-n)H \otimes m_z)) - 1
$$

(2.6)

Combining (2.5) and (2.6) we get the Lemma. \hfill \Box

As $F_m \not\subseteq M_{E(m)}$ and $\text{rank}(M_{E(m)}) = h^0(E(m)) - l$. Therefore, $\text{rank}(F_m) < h^0(E(m)) - l$. Note that by Riemann Roch formula and the above lemma, we have $\text{rank}(F_m) \geq \frac{m^2}{2} + O(m)$, where $O(m)$ is a linear function of $m$. Thus we have,

$$
\text{rank}(F_m) = f(m), \quad \text{where } f(m) \text{ is a function of } m \text{ such that } \frac{f(m)}{m^2} = r, \quad \frac{1}{2} \leq r \leq \frac{l}{2}, \quad \text{as } m \rightarrow \infty
$$

(2.7)
Lemma 2.3. \( \text{rank}(K_m) \geq r h^0(E(n)) \) for large \( m \gg 0 \).

Proof. From the equation 2.4 we get

\[
\mu(F_m|c_m) = \frac{\text{deg}(K_m) + \text{deg}(N_m)}{\text{rank}(F_m)} \leq \frac{\text{deg}(N_m)}{\text{rank}(F_m)} = \mu(N_m) \frac{\text{rank}(N_m)}{\text{rank}(F_m)}.
\]

Since \( M_m \) is stable we get

\[
\mu(N_m) \frac{\text{rank}(N_m)}{\text{rank}(F_m)} \leq \mu(M_m) \frac{\text{rank}(N_m)}{\text{rank}(F_m)} = \mu(M_m) \left(1 - \frac{\text{rank}(K_m)}{\text{rank}(F_m)}\right).
\]

Now from 2.3 we have \( \text{deg}(M_m|c_m) = \text{deg}(M_m) \), and since \( \mu(M_m|c_m) \leq \mu(F_m|c_m) \)

from equation 2.9 we get

\[
\frac{\text{deg}(M_m|c_m)}{h^0(E(n)) + \text{rank}(M_m)} < \frac{\text{deg}(M_m|c_m)}{\text{rank}(M_m)} \left(1 - \frac{\text{rank}(K_m)}{\text{rank}(F_m)}\right).
\]

Noting the fact \( \text{deg}(M_m|c_m) < 0 \) we get

\[
\frac{1}{h^0(E(n)) + \text{rank}(M_m)} > \frac{1}{\text{rank}(M_m)} \left(1 - \frac{\text{rank}(K_m)}{\text{rank}(F_m)}\right).
\]

i.e.,

\[
\frac{\text{rank}(M_m)}{h^0(E(n)) + \text{rank}(M_m)} > \left(1 - \frac{\text{rank}(K_m)}{\text{rank}(F_m)}\right)
\]

Thus

\[
\frac{\text{rank}(K_m)}{\text{rank}(F_m)} > 1 - \frac{\text{rank}(M_m)}{h^0(E(n)) + \text{rank}(M_m)} = \frac{h^0(E(n))}{h^0(E(n)) + \text{rank}(M_m)}
\]

Therefore,

\[
\text{rank}(K_m) > h^0(E(n)) \frac{\text{rank}(F_m)}{\text{rank}(M_m)}
\]

By Lemma 2.2 and since \( \text{rank}(M_m) = h^0(E(m)) - l \) we have

\[
\text{rank}(K_m) > h^0(E(n)) \frac{rm^2 + q(m)}{h^0(E(m)) - l}
\]

We have \( \frac{rm^2 + q(m)}{h^0(E(m)) - l} = r - \epsilon(m) \) where \( \epsilon(m) \to 0 \) as \( m \to \infty \). Thus for large \( m \gg 0 \) we get

the inequality

\[
\text{rank}(K_m) \geq r h^0(E(n)).
\]
Let $G$ be the subsheaf of $E(n)$ generated by $K_m(x)$. We claim that $\text{rank}(G) > 1$. Note that as $G$ is generated by $K_m(x)$ we have $K_m(x) \subseteq H^0(X, G)$ in other words

\[(2.13) \quad \text{rank}(K_m(x)) \leq h^0(G). \]

As $E$ is rank $l$ slope stable vector bundle with respect to $H$, for any proper subsheaf $F \subset E$ i.e., $0 < \text{rank}(F) < l$, we have , for $d >> 0$,

\[
\chi(F(d)) \leq \frac{\chi(E(d))}{l}
\]

where $\chi$ is the Euler characteristic of the respective sheaves. Considering $F := G(-n)$ we get, from the above equation,

\[
\frac{\chi(G)}{\text{rk}(G)} < \frac{\chi(E(n))}{l}
\]

Now we choose $n$, sufficiently large, such that $\chi(G) = h^0(G)$ and $\chi(E(n)) = h^0(E(n))$. Therefore, we have

\[(2.14) \quad \frac{h^0(G)}{\text{rk}(G)} < \frac{h^0(E(n))}{l}. \]

Combining equation (2.13) and (2.14) we get $\frac{\text{rank}(K_m(x))}{\text{rk}(G)} < \frac{h^0(E(n))}{l}$. On the other hand by Lemma 2.3 $\frac{\text{rank}(K_m(x))}{\text{rk}(G)} \leq \frac{\text{rank}(K_m(x))}{\text{rk}(G)}$ where $\frac{1}{2} < r \leq \frac{1}{2}$. This immediately implies that $\text{rk}(G) > lr \geq 1$ since $r > \frac{1}{2}$ and $l \geq 2$. We have $h^0(G(m - n) \otimes m_x) = h^0(G(m - n)) - \text{rk}(G) = \frac{\text{rk}(G)}{2}m^2 + p(m)$ where $p \in \mathbb{Q}[n][m]$ and linear in $m$. Let $t = \frac{\text{rk}(G)}{2}$ then $t \geq 1$.

**Case I :** $r < 1$.

As $t \geq 1$ by equation (2.7) for large $m >> 0$

\[(2.15) \quad \text{rank}(F_m(x)) < h^0(G(m - n) \otimes m_x). \]

For any $x \in X$ from the inclusion $G \hookrightarrow E(n)$ we obtain $H^0(G((m - n) \otimes m_x) \hookrightarrow H^0(E(m) \otimes m_x)$ and by (2.15) we get $\text{Im}(F_m(x)) \neq \text{Im}(H^0(G((m - n) \otimes m_x))$ inside $H^0(E(m) \otimes m_x)$. Let $B_m(x) = \text{Im}(F_m(x)) \cap \text{Im}(H^0(G((m - n) \otimes m_x))$. Then $B_m(x) \subset H^0(G((m - n) \otimes m_x))$ For any section $s \in H^0(O_X((m - n)) \otimes m_x)$ multiplication by $s$ maps

\[H^0(G) \rightarrow H^0(G(m - n) \otimes m_x)\]

and

\[K_m(x) \rightarrow B_m(x).\]

Thus from the commutative diagram (2.12) we get
Since $K_m(x)$ generates $G$ by (2) we conclude that

$$H^0(G) \to H^0(G(m - n) \otimes m_x)$$

is surjective. This leads to a contradiction as we vary sections over an open set of $H^0(\mathcal{O}_X((m - n)H))$ the images of $K_m(x)$ span the whole vector space $H^0(G(m - n) \otimes m_x)$. But from the above commutative diagram every image lies on the proper fixed subspace $B_m(x)$. Therefore, $E(m)$ is $H$-stable.

Case II: $r \geq 1$.
In this case by Lemma 2.3 rank of $K_m = h^0(E(n))$. In other words, $G = E(n)$. Thus the Theorem follows by similar arguments as in Case I.

REFERENCES

[1] Lawrence Ein; Lazarsfield R; and Mustopa Y. Stability of syzygy bundles on an algebraic surface. Mathematical Research Letters, Vol-20 (2013) no.1, 73-80.
[2] Butler D.C. Normal generation of vector bundles over a curve, Journal of Differential Geometry, 39(1994); no.1, 1-34.
[3] Huybrechts, D; Lehn, M. The geometry of moduli spaces of sheaves, Cambridge University Press, 2010.

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