Newman–Tamburino solutions with an aligned Maxwell field

I. De Groote and N Van den Bergh

Department of Mathematical Analysis IW16, Ghent University, Galglaan 2, 9000 Ghent, Belgium

Received 21 February 2008, in final form 19 June 2008
Published 30 July 2008
Online at stacks.iop.org/CQG/25/165007

Abstract

We prove that there exists no aligned Einstein–Maxwell generalization of the spherical class of Newman–Tamburino solutions. The presence of an aligned Maxwell field automatically leads to the cylindrical class.

PACS numbers: 04.20.Jb, 04.40.Nr

1. Introduction

In 1962, Newman and Tamburino published the general empty space solutions for the class of metrics containing hypersurface orthogonal geodesic rays with nonvanishing shear and divergence [1, 2]. Locally their solutions can be subdivided into two classes, called ‘cylindrical’ and ‘spherical’, according to whether respectively \( \rho^2 - \sigma \delta = 0 \) or \( \neq 0 \) holds in an open set of spacetime. In this paper we look at spherical solutions in the presence of an aligned Maxwell field. We will show that no such solutions exist.

To prove this we use the Newman–Penrose (NP) formalism. The choice of tetrad and the main equations are discussed in section 2, while in section 3 we prove our statement. The notations and conventions of [3] are followed throughout.

2. Main equations

Using a complex null tetrad \((m, \bar{m}, l, k)\) a solution of the Einstein–Maxwell equations is described in the NP formalism by means of (a) the 12 spin coefficients, which are independent complex linear combinations of the connection coefficients, (b) the tetrad components of the Weyl tensor, which are expressed in terms of the five complex scalars

\[
\Psi_0 = C_{abcd}k^a l^b k^c m^d, \quad \Psi_1 = C_{abcd}l^a l^b k^c m^d, \\
\Psi_2 = \frac{1}{2}C_{abcd}k^a l^b (k^c l^d - m^c \bar{m}^d), \quad \Psi_3 = C_{abcd}l^a k^b l^c \bar{m}^d, \quad \Psi_4 = C_{abcd}l^a m^b l^c \bar{m}^d
\]

1 The terminology [1] refers to the geometry of the \( u = \text{const.}, r = \text{const.} \) surfaces (using the original Newman–Tamburino coordinates), which admit a single Killing vector in the ‘cylindrical’ class and which resemble distorted spheres in the ‘spherical’ class.
and (c) the components

$$\Phi_0 = F_{ab} k^a m^b, \quad \Phi_1 = \frac{1}{2} F_{ab} (k^a l^b + \bar{m}^a \bar{m}^b), \quad \Phi_2 = F_{ab} \bar{m}^a l^b$$

of the electromagnetic field. The Ricci tensor components are given by

$$\Phi_0^{\alpha \beta} = \kappa_0 \Phi_0^\alpha \Phi_0^\beta, \quad \Phi_1^{\alpha \beta} = \frac{1}{2} \Phi_1^\alpha \Phi_1^\beta, \quad \Phi_2^{\alpha \beta} = \Phi_2^\alpha \Phi_2^\beta$$

where $$\kappa_0$$ is Einstein’s gravitational constant. The spin coefficients and curvature components satisfy the NP and Bianchi equations, which will be referred to as $$NP_1, \ldots, NP_{18}$$ and $$B_1, \ldots, B_{11}$$, following the same ordering as in [3].

The variables $$\Phi_0, \Phi_1, \Phi_2$$ satisfy the Maxwell equations, which will be referred to as $$M_1, \ldots, M_4$$, again following the same order as in [3]. Taking into account the differential equations for $$\Phi_0, \Phi_1, \Phi_2$$, obtained from the Maxwell equations, three of the Bianchi equations ($$B_9, B_{10}, B_{11}$$) become identities.

3. The proof of our main result

The Newman–Tamburino metrics are characterized by the existence of a hypersurface orthogonal and geodesic principal null direction $$k$$, for which the shear and divergence are non-vanishing. In terms of the NP variables this translates into

$$\kappa = \epsilon + \bar{\epsilon} = 0, \quad \Psi_0 = 0, \quad \rho - \bar{\rho} = 0,$$

where equation (2) is the mathematical characterization of the fact that $$k$$ is a principal null vector of the Weyl tensor. The spherical class of Newman–Tamburino metrics, which we consider in the present paper, is further characterized by

$$\rho^2 - \sigma \bar{\sigma} \neq 0.$$  \hspace{1cm} (4)

If this last equation (4) does not hold, we are in the so-called cylindrical class [1]. We suppose that the Maxwell field is aligned, in the sense that $$k$$ is not only a principal null vector of the Weyl tensor (which is expressed by (2)), but also that $$k$$ is a principal null vector of the Maxwell tensor, which means that we can put $$\Phi_0$$ equal to 0 too.

By means of a rotation, a boost and a null rotation of the null tetrad $$(m, \bar{m}, l, k)$$, we can make sure that the tetrad is parallelly propagated along the geodesic null congruence $$k$$,

$$\kappa = \epsilon = \pi = 0.$$ \hspace{1cm} (5)

The restrictions so far obtained on the spin coefficients are invariant under boosts $$k \rightarrow A k$$ satisfying $$D A = 0$$. Under such a boost the spin coefficients $$\tau, \alpha, \beta$$ transform as follows:

$$\tau \rightarrow \tau, \quad \alpha \rightarrow \alpha + \frac{1}{2} \Delta (\ln A), \quad \beta \rightarrow \beta + \frac{1}{2} \delta (\ln A).$$

We therefore can make

$$\tau = \bar{\alpha} + \beta,$$ \hspace{1cm} (6)

provided that the boost parameter $$A$$ satisfies the equation $$\delta A = (\tau - \beta - \bar{\alpha}) A.$$ Using the $$(\delta, D)$$ and $$[\bar{\delta}, \delta]$$ commutators the integrability conditions for this system of pde’s read

$$D(\tau - \bar{\alpha} - \beta) = (\tau - \bar{\alpha} - \beta) \rho + (\bar{\tau} - \alpha - \bar{\beta}) \sigma,$$

$$\delta(\tau - \alpha - \bar{\beta}) - \bar{\delta}(\tau - \bar{\alpha} - \beta) = \tau \bar{\alpha} - 2\bar{\alpha} \bar{\beta} - \bar{\tau} \beta - \tau \alpha + 2\alpha \beta + \tau \bar{\beta},$$

which are identically satisfied under the equations $$NP_3, NP_4, NP_5$$ and $$NP_{12}, NP_{17}, NP_{18}$$, respectively. So henceforth we will assume that (6) holds.
From the NP equations it also follows that $D(\sigma/\dot{\sigma}) = 0$ and so a final rotation $\mathbf{m} \to e^{i\theta} \mathbf{m}$ exists such that

$$\dot{\sigma} = \sigma.$$  \hspace{1cm} (7)

The remaining tetrad freedom consists now of boosts $A$ and null rotations $B$ about $\mathbf{k}$, satisfying

$$DA = \delta A = \delta B = 0 = DB.$$  

Taking into account the above transformations (5), (6) and (7), we can rewrite the NP equations determining the evolution along the geodesic rays as follows:

$$D\sigma = 2\rho\sigma,$$

$$D\rho = \rho^2 + \sigma^2,$$

$$D\alpha = \rho\alpha + \beta\sigma,$$

$$D\lambda = \rho\lambda + \mu\sigma,$$

$$D\mu = \mu\rho + \sigma\lambda + \Psi_2 + \frac{1}{12} R,$$

$$D\nu = (\alpha + \beta)\mu + (\bar{\alpha} + \beta)\lambda + \Psi_3 + \Phi_2 \Phi_1,$$

$$D\beta = \rho\beta + \sigma\alpha + \Psi_1,$$

$$D\gamma = \alpha\bar{\alpha} + 2\alpha\beta + \beta\bar{\beta} + \Psi_2 - \frac{1}{24} R + \Phi_1 \Phi_1.$$  

We can also rewrite the Maxwell equations $M_1$ and $M_3$,

$$D\Phi_1 = 2\rho\Phi_1,$$  \hspace{1cm} (8)

$$\delta\Phi_1 = 2(\bar{\alpha} + \beta)\Phi_1 - \sigma\Phi_2,$$

and the Bianchi equations $B_1$ and $B_2$,

$$D\Psi_1 = 4\rho\Psi_1,$$  \hspace{1cm} (9)

$$\delta\Psi_1 = 2(2\bar{\alpha} + 3\beta)\Psi_1 - 3\sigma\Psi_2 - 2\sigma\Phi_1 \Phi_1.$$  \hspace{1cm} (10)

Next, we calculate $[\delta, D]\Psi_1$, which by (9), (10) and the commutator relation

$$[\delta D - D\delta) = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa \Delta - \sigma \bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta,$$  \hspace{1cm} (11)

leads to the expression:

$$\left(\delta\rho - \bar{\alpha}\rho - 3\sigma\alpha - \bar{\beta}\rho - \bar{\beta}\sigma + \frac{1}{2}\Psi_1\right)\Psi_1 + \sigma\bar{\delta}\Psi_1 + 2\sigma\rho\Phi_1 \Phi_1 = 0.$$  \hspace{1cm} (12)

Applying the $D$ operator to this equation gives:

$$\sigma D\delta\Psi_1 + \Psi_1 D\delta\rho + 2\sigma\rho\delta\Psi_1 + 4\rho\Psi_1\delta\rho + (2\sigma^2 + 14\rho^2) \sigma\Phi_1 \Phi_1$$

$$- (6(\bar{\alpha} + \beta)\rho^2 + (22\sigma\alpha + 8\bar{\beta}\sigma + 13\Psi_1)\rho + \sigma\Psi_1 + 2(\bar{\alpha} + \beta)\sigma^2)\Psi_1$$

$$= 0.$$  \hspace{1cm} (13)

The two expressions (12) and (13) enable us to simplify $[\bar{\delta}, D](\rho\Psi_1)$:

$$(5(\alpha + \bar{\beta})\rho^2 - (\bar{\alpha} + \beta)\sigma + 5\delta\rho + 2\delta\sigma - \Psi_1\rho)\Psi_1 + \frac{5}{2} \frac{\Psi_1^2 \rho^2}{\sigma} + 3\sigma^2 \Psi_2 \rho = 0.$$  \hspace{1cm} (14)

Eliminating $\delta\rho$ by $NP_{11}$ allows us to solve for $\delta\sigma$:

$$\delta\sigma = \frac{5}{14} \frac{\rho\Psi_1}{\sigma} + \frac{3}{7} \frac{\sigma^2 \Psi_2}{\Psi_1^2} + 2\sigma\bar{\alpha} - \frac{6}{7} \beta\sigma + \frac{6}{7} \Psi_1.$$  \hspace{1cm} (15)

Note that $\Psi_1$ has to be non-zero, but as we are assuming nonvanishing shear, this condition is fulfilled automatically: if $\Psi_1 = 0$, then $\Phi_1 = 0$ (by (12)) and hence also $\Phi_2 = 0$ by (8).
If we substitute (15) into (14) we can solve for \( \delta \rho \), and by substituting the result into (12) we get an equation for \( \delta \Psi_1 \):

\[
\delta \rho = \frac{3\sigma^2\Psi_2}{7\Psi_1} + \frac{5\Psi_1\rho}{14\sigma} - \sigma\alpha + \frac{1}{7}B\sigma - \frac{1}{7}\Psi_1 + \rho\beta + \rho\alpha.
\]

\[
\delta \Psi_1 = \left( \frac{23}{14\sigma} - \frac{3\sigma\Psi_2}{7\Psi_1} + 4\alpha + \frac{6}{7}\beta - \frac{5}{14}\sigma^2 \right)\Psi_1 - 2\Phi_1\rho\Phi_1.
\]

The next step in our calculation is to look at \( B_1 \), which by now can be solved for \( D\Psi_2 \):

\[
D\Psi_2 = -\frac{5}{14}\sigma^2 - \frac{3\sigma\Psi_2}{7\Psi_1} + 2\alpha\Psi_1 + \frac{6}{7}\Psi_1\beta + \frac{23}{14}\Psi_1^2 + 3\rho\Psi_2.
\]

Substituting all the above into \( [\delta, D]\rho \), we find an expression for \( \beta \):

\[
\beta = \frac{1}{8}4\Psi_2\sigma^3 + \Psi_1\sigma\Psi_1 + \Psi_1^2\rho.
\]

This almost finishes our proof. The next steps are to look at \( [\delta, D]\Phi_1 \) and Maxwell’s equation \( M_2 \) which can be solved for \( \delta \Phi_1 \) and \( D\Phi_2 \) after which \( [\delta, D]\Phi_1 \) yields an expression for \( \Phi_2 \):

\[
\delta \Phi_1 = -\frac{1}{4}\Phi_1\Psi_1\rho + 2\Phi_1\alpha + \frac{5\Phi_1\Psi_1}{4\sigma},
\]

\[
D\Phi_2 = \rho\Phi_2 - \frac{1}{4}\Phi_1\Psi_1\rho + 2\Phi_1\alpha + \frac{5\Phi_1\Psi_1}{4\sigma},
\]

\[
\Phi_2 = \frac{1}{2}\Phi_1(\sigma\Psi_1 + \rho + 2\Psi_2\sigma^3).
\]

We then evaluate \( [\delta, [\delta, \Phi_1] \), from which we eliminate \( \delta\alpha \) by \( [\delta, \delta]\Phi_1 \). This results in the expression (16). We also write the equation for \( [\delta, [\delta, \Phi_1] \), after eliminating from it \( \delta\alpha \) by \( [\delta, \delta]\sigma \) (namely (17)):

\[
3\rho\sigma\Psi_1^2\Psi_1 - 4(\sigma^2 + \rho^2)\Psi_1^2\Psi_1^2 + \rho\sigma\Psi_1^2\Psi_1^3 - 24\alpha\sigma^3\Psi_1^3\Psi_1
\]

\[
- 8\alpha\sigma\Psi_1^2\Psi_1^2 + 12\rho\sigma^3\Psi_2\Psi_1^3 + 8(\sigma^2 - 4\rho^2)\sigma^2\Phi_1\Phi_1\Psi_1^2\Psi_1
\]

\[
+ 4(\Psi_2 - 2\Phi_1)\rho\alpha\Psi_1^2\Psi_1^2 + 3(2\alpha\Psi_1 - \rho\Psi_2)\sigma^3\Phi_1\Phi_1\Psi_1 = 0,
\]

\[
- \rho\sigma\Psi_1^2\Psi_1^2 + 4(\sigma^2 + 2\rho^2)\Psi_1^3\Psi_1^2 - 3\rho\sigma\Psi_1^2\Psi_1^3 + 8\alpha\sigma^3\Psi_1^3\Psi_1
\]

\[
+ 24\alpha^3\Psi_1^2\Psi_1^2 - 4\rho\alpha\Psi_1^3\Psi_1^3 + 8(4\rho^2 - 2\sigma^2)\sigma^2\Phi_1\Phi_1\Psi_1^3\Psi_1
\]

\[
- 3(3\Psi_2 + 2\Phi_1\Phi_1)\rho\alpha^3\Psi_1^2\Psi_1^2 + 32(2\alpha\Psi_1 - \rho\Psi_2)\sigma^3\Phi_1\Phi_1\Psi_1 = 0.
\]

Note that we can solve these equations for \( \alpha \) and \( \alpha \),

\[
\alpha = \frac{1 - 2\Psi_1^2\rho^2 + 4\Psi_2\rho\sigma^3 - \Psi_1^2\sigma^2 + \rho\Psi_1\sigma\Psi_1}{8\Psi_1^4},
\]

\[
\alpha = \frac{1 - 2\Psi_1^2\rho^2 + 4\Psi_2\rho\sigma^3 - \Psi_1^2\sigma^2 + \rho\Psi_1\sigma\Psi_1}{8\Psi_1^4} + \frac{2\Phi_1\Phi_1(\sigma^2 - \rho^2)}{\Psi_1\sigma},
\]

but, as can easily be seen from the above results, this implies

\[
\frac{2\Phi_1\Phi_1(\sigma^2 - \rho^2)}{\Psi_1\sigma} = 0.
\]

It is obvious now that \( \Phi_1 \) cannot be zero. To prove this, it is sufficient to look at expression (8). From this expression we see that if \( \Phi_1 \) were zero then also \( \Phi_2 \) would be zero and we would no longer have a non-null Maxwell field. The above equation can thus—for a non-null Maxwell field—only be satisfied if \( \sigma^2 = \rho^2 \), which means we are in the cylindrical class, and thus proves our statement.
4. Conclusion

We have shown that in the presence of an aligned non-null Maxwell field no generalization of the ‘spherical’ Newman–Tamburino solutions exists. As the final equation shows, assuming the existence of such a Maxwell field forces us into the cylindrical class. The next step in our investigation is to integrate the latter. Although the calculations are rather more lengthy and complicated than in the vacuum, we can show this class is fully integrable.

Acknowledgments

NVdB expresses his thanks to Robert Debever and Jules Leroy (Université Libre de Bruxelles), whose questions about the Newman–Tamburino metrics have led to the present investigation.

References

[1] Newman E and Tamburino L 1962 J. Math. Phys. 3 902–7
[2] Carmeli M 1977 Group Theory and General Relativity (New York: McGraw-Hill)
[3] Kramer D, Stephani H, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press)