Variational properties of $\sigma_u$-curvature for closed submanifolds of arbitrary codimension in Riemannian manifolds

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Abstract. The objet of this paper is the study of variations of a functional whose integrant is the $\sigma_u$-curvature of closed submanifolds of arbitrary codimension in Riemannian manifolds.

1. Introduction

The study of Riemannian geometry is based on the analysis of geometric operators like the shape operator, the Ricci tensor, the Schouten operator etc... Some functions built from these operators play a fundamental role in the understanding of this discipline, in particular algebraic invariants such the $r$-th symmetric functions $\sigma_r$ associated with the shape operator and the Newton transformations $T_r$. The following articles can be consulted on this subject ([1], [2], [5], [6], [7], [8], [9], [10], [11]), Reilly (see [9]) has considered the variations where the integrand of the functional is a function of the $r$-th mean curvatures $\sigma_r$. In a recent paper Case (see [5]) introduced and studied the notion of $r$-th weighted curvatures. The aim of this paper is the study of the variations of a functional whose integrand is $\sigma_u$-curvature, $u$ stands for a multi-index, on closed submanifolds of a Riemannian manifold. Some applications to submanifolds of the Euclidian space and the unit round sphere are given.

2. Generalized Newton Transformations

Let $A$ be an endomorphism of a $m$-dimensional real vector space $V$ endowed with the usual inner product. The Newton’s transformations associated with $A$ is a family $T = (T_r)_{r \in \mathbb{N}}$ defined recurrently by:

$$T_0 = id_V$$

$$T_r = \sigma_r id_V - AT_{r-1}.$$ 

Denote by $End(V)$ the space of endomorphisms on $V$. Let $A \in End(V)$ and $A^*$ its adjoint endomorphism. $End(V)$ is then endowed with the inner
product $\langle A, B \rangle = \text{tr}(AB^*)$ where $A, B \in \text{End}(V)$. Borrowing notations from the paper [3], we denote by $N(q)$ the set of all uplets $u = (u_1, \ldots, u_q)$ where $u_j$ stand for positive integers. Let $\text{End}^q(V) = \text{End}(V) \times \ldots \times \text{End}(V)$, $q$-times. For any $A = (A_1, \ldots, A_q) \in \text{End}^q(V)$, $u = (u_1, \ldots, u_q) \in N(q)$, $t = (t_1, \ldots, t_q) \in \mathbb{R}^q$. Define:

$$tA = t_1A_1 + \ldots + t_qA_q$$

and

$$t^u = (t^{u_1}, \ldots, t^{u_q})$$

the Newton polynomial of $A$ is then given by

$$P_A(t) = \det(tA + id_V) = \sum_{|u| \leq p} \sigma_u(A)t^u.$$  

Consider the musical functions $\alpha^\#, \alpha_b : N(q) \rightarrow N(q)$ given by

$$\alpha^\#(u_1, \ldots, u_q) = (u_1, \ldots, u_{\alpha-1}, u_{\alpha} + 1, u_{\alpha+1}, \ldots, u_q)$$

and

$$\alpha_b(u_1, \ldots, u_q) = (u_1, \ldots, u_{\alpha-1}, u_{\alpha} - 1, u_{\alpha+1}, \ldots, u_q).$$

The generalized Newton transformations (in abbreviated form GNT) are defined (see) by: for any curve $t \rightarrow A(t)$ in $\text{End}^q(V)$ such that $A(0) = A$ the GNT of $A$ is a family of endomorphisms $(T_u)_{u \in N(q)}$ given by

$$(1) \quad \frac{d}{dt}\sigma_u(t) \mid_{t=0} = \sum_\alpha \text{tr}(\frac{d}{dt}A_\alpha(t) \mid_{t=0}).T_{\alpha_b(u)}.)$$

Once again we use the notations of ( ). For $i = (i^1, \ldots, i^q) \in N(s, q)$, its weight is defined as $|i| = (|i^1|, \ldots, |i^q|) \in N(q)$ and its length by $||i|| = \sum_{\alpha=1}^q |i^\alpha| = \sum_{\alpha=1, \beta=1}^q i^\alpha_{\alpha, \beta}$. Denote by $I(q, s)$ the subset of $N(q, s)$ of matrices $i$ such that:

- each entry of $i$ is either 0 or 1
- the length of $i$ is $s$
- every column of $i$ contains only one entry equal to 1.

Let $A = (A_1, \ldots, A_q) \in \text{End}^q(V)$ and $i \in I(q, s)$ with $I(q, 0)$ is the set of vector 0, we put (as in)

$$A^i = A^i_1 A^i_2 \ldots A^i_q A^i_1 A^i_2 \ldots A^i_q$$

with

$$A^0 = 1_V.$$

We quote after [3]:

**Proposition 1.** The generalized Newton transformations $(T_u : u \in N(q))$ of $A = (A_1, \ldots, A_q)$ enjoy the following fundamental properties:

1. For every $u \in N(q)$ with $|u| \geq m$, $T_u = 0$
2. Symmetric functions $\sigma_u$ are given by
\[ |u| \sigma_u = \sum_{\alpha} tr(A_{\alpha} T_{\alpha(b)}(u)). \]

3. Variational properties

Consider a one family of parameter \( \psi_t : M^m \to T^n \) of immersions of an \( m \)-dimensional closed manifold \( M^m \) into an \( n \)-Riemannian manifold \( (\overline{M}^n, \langle \cdot, \cdot \rangle) \). We consider on \( M^m \) the Riemannian metric induced by the metric on \( T^m \). If \( \nabla \) stands for the covariant derivative in \( T^m \), for every vector fields \( X,Y \) tangent along \( M^m \) in a neighborhood of a point \( x \), the Gauss formula writes as:

\[ \nabla_X Y = \nabla_X Y + \alpha(X,Y) \]

where

\( \nabla \) is the induced covariant derivative \( M^m \) defined by \( \nabla_X Y = (\nabla_X Y)^\top \)

\( \alpha \) is the second fundamental form given by \( \alpha(X,Y) = (\nabla_X Y)^\perp \).

Similarly if \( \nu \) is a normal vector field along \( M^m \) in a neighborhood of \( x \), we obtain the Weigarten equation:

\[ \nabla_X \nu = -A_\nu(X) + D_X \nu \]

where

\( D \) denotes the covariant derivative on the normal bundle of \( M \), defined by: \( D_X \nu = (\nabla_X \nu)^\perp \)

\( A_\nu \) is the shape operator \( A_\nu(X) = -(\nabla_X \nu)^\top \) which is related to the second fundamental form by:

\[ \langle A_\nu(X), Y \rangle = \langle \alpha(X,Y), \nu \rangle \]
for any vector fields $X$, $Y$ on $M$.

Consider the following variational problem

$$\delta \left( \int_M \sigma_u dV \right) = 0$$

with $u \in I(q, s)$.

**Theorem 1.** With the above notations and assumptions the first variation of the global $\sigma_u$-curvature is given by:

$$\frac{d}{dt} \left( \int_{M^m} \sigma_u dV \right) = \int_{M^m} \left\{ -g^{jk} \sum_{\alpha} R^\mathcal{M}_j (\nu^\alpha, \partial \psi_{\partial x^k}, X) (T_{ab}(u))^i_j + (T_{ab}(u))^j_i \lambda_{\alpha, ij} \\
+ g^{jk} \left( R^\mathcal{M} \right)_i (\partial \psi_{\partial x^k}, X) \frac{d}{dt} (T_{ab}(u)) \\
+ g^{jk} \left( \frac{\partial \psi}{\partial t}, \nu^\alpha \right) (A_{\beta})_i (T_{ab}(u))^j_i - g^{ik} \lambda_{\beta, k} \left( \frac{D_\partial \nu^\alpha}{\partial x^k}, \nu^\beta \right) (T_{ab}(u))^i_j \\
- \lambda_{\beta} g^{jk} \left( D_\partial \nu^\alpha, \nu^\gamma \right) \left( \frac{D_\partial \nu^\beta}{\partial x^k} \right) (T_{ab}(u))^i_j - \left( \lambda_{\beta, \#} (u) \right) (A_{\beta})_i \right\} dV.
$$

By definition of $\sigma_u$, we have

$$\frac{\partial \sigma_u}{\partial t} = \sum_{\alpha} \text{tr} \left( \frac{\partial A_\alpha}{\partial t} T_{ab}(u) \right)$$

with

$$\text{tr} \left( \frac{\partial A_\alpha}{\partial t} T_{ab}(u) \right) = \frac{\partial (A_\alpha)_i^j}{\partial t} (T_{ab}(u))^j_i$$

and

$$(A_\alpha)_i^j = g^{jk} (A_\alpha)_{ik}$$

where

$$(A_\alpha)_{ik} = \left\langle \nabla \frac{\partial \psi}{\partial x^i}, \nu^\alpha \right\rangle = - \left\langle \nabla \frac{\partial \psi}{\partial x^k}, \nu^\alpha, \frac{\partial \psi}{\partial x^i} \right\rangle$$

where $(\nu^1, ..., \nu^k)$ is an orthogonal basis to $M^m$ and $k = n - m$.

Hence

$$\frac{\partial (A_\alpha)_i^j}{\partial t} = \frac{\partial g^{jk}}{\partial t} (A_\alpha)_{ik} + g^{jk} \frac{\partial (A_\alpha)_{ik}}{\partial t}.$$ 

Obviously

$$\frac{\partial g^{jk}}{\partial t} = -g^{ji} \frac{\partial g_{pl}}{\partial t} g^{pk}$$

Now, if we consider the calculations in a normal coordinates that is at a point $x \in M$ where the metric tensor fulfills $g_{ij}(x) = \left\langle \frac{\partial \psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right\rangle = \delta_{ij}$ and
\[ \Gamma_{ij}^k (x) = 0, \text{ where } \Gamma_{ij}^k \text{ stand for the Christoffel symbols corresponding to the} \]
\[ \text{metric connection } \nabla \text{ on } M, \text{ we get} \]
\[ \frac{\partial g_{pl}}{\partial t} = \left\langle \nabla_{\frac{\partial}{\partial x_p}} \frac{\partial \psi}{\partial x_l}, \frac{\partial \psi}{\partial x_l} \right\rangle + \left\langle \frac{\partial \psi}{\partial x_p}, \nabla_{\frac{\partial}{\partial x_l}} \frac{\partial \psi}{\partial x_l} \right\rangle \]
\[ = \left\langle \nabla_{\frac{\partial}{\partial x_p}} X, \frac{\partial \psi}{\partial x_l} \right\rangle + \left\langle \frac{\partial \psi}{\partial x_p}, \nabla_{\frac{\partial}{\partial x_l}} X \right\rangle \]
\[ = \left\langle \nabla_{\frac{\partial}{\partial x_p}} \left( \lambda_\beta \nu^\beta + \mu^m \frac{\partial \psi}{\partial x_m} \right), \frac{\partial \psi}{\partial x_l} \right\rangle \]
\[ + \left\langle \frac{\partial \psi}{\partial x_p}, \nabla_{\frac{\partial}{\partial x_l}} \left( \lambda_\beta \nu^\beta + \mu^m \frac{\partial \psi}{\partial x_m} \right) \right\rangle \]
\[ = \mu_{p,l} + \mu_{l,p} - 2 \lambda_\beta (A_{\beta})_{pl}. \]

hence

\[ \frac{\partial g^{jk}}{\partial t} = -g^{jl} g^{pk} \left( \mu_{p,l} + \mu_{l,p} - 2 \lambda_\beta (A_{\beta})_{pl} \right). \tag{10} \]

We have also

\[ \frac{\partial \nu^\alpha}{\partial t} = \left\langle \frac{\partial \nu^\alpha}{\partial t}, \frac{\partial \psi}{\partial x_k} \right\rangle + \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nu^\beta \tag{11} \]
\[ = -\left\langle \nu^\alpha, \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial \psi}{\partial x_k} \right\rangle + \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nu^\beta \]
\[ = -\left\langle \nu^\alpha, \nabla_{\frac{\partial}{\partial x_k}} X \right\rangle + \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nu^\beta \]
\[ = -g^{jk} \left( \lambda_{\alpha,j} + \mu^l (A_{\alpha})_{jl} \right) \frac{\partial \psi}{\partial x_k} + \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nu^\beta. \]

Now we compute

\[ \frac{\partial (A_{\alpha})_{ik}}{\partial t} = -\left\langle \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_k}} \nu^\alpha, \frac{\partial \psi}{\partial x_l} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x_k}} \nu^\alpha, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial \psi}{\partial x_l} \right\rangle \]
\[ = -\left\langle \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_k} \right\rangle \]
\[ \quad - \left\langle \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_k}} \nu^\alpha, \frac{\partial \psi}{\partial x_l} \right\rangle \]
\[ \quad - \frac{\partial}{\partial x_i} \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nu^\beta + \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nabla_{\frac{\partial}{\partial x_i}} \nu^\beta. \]

By formula (3.1), we get

\[ \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_k}} \nu^\alpha = -g^{jm} \left( \lambda_{\alpha,j} + \mu^l (A_{\alpha})_{jl} \right) \frac{\partial \psi}{\partial x_m} \]
\[ - g^{jk} \left( \lambda_{\alpha,j} + \mu^l (A_{\alpha})_{jl} \right) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial \psi}{\partial x_k} \]
\[ + \frac{\partial}{\partial x_k} \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nu^\beta + \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \nabla_{\frac{\partial}{\partial x_i}} \nu^\beta. \]
so
\[
\left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_t} \right\rangle = -g^{jk} \left( \lambda_{\alpha,ji} + \mu_{i}^j (A_\alpha)_{jl} + \mu^l (A_\alpha)_{jl,i} \right) g_{mk} \\
+ \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle \left\langle \nabla_{\partial_{x_i}} \nu^\beta, \frac{\partial \psi}{\partial x_k} \right\rangle
\]
= - \left( \lambda_{\alpha,ki} + \mu_{i}^j (A_\alpha)_{kl} + \mu^l (A_\alpha)_{kl,i} \right) - \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle (A_\beta)_{ik}.
\]
In the same manner, we have
\[
\left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} X \right\rangle = \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \left( \lambda_{\beta,\nu}^\beta + \mu^m \frac{\partial \psi}{\partial x_m} \right) \right\rangle
\]
= \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \lambda_{\beta,\nu}^\beta + \lambda_{\beta} \nabla_{\partial_{x_k}} \nu^\beta + \mu^m \frac{\partial \psi}{\partial x_m} + \mu^m \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_m} \right\rangle
\]
= \lambda_{\beta,k} \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nu^\beta \right\rangle + \lambda_{\beta} \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \nu^\beta \right\rangle - \mu^m_{ik} (A_\alpha)_{im}
+ \mu^m \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_m} \right\rangle.
\]
By noticing that
\[
\left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \nu^\beta \right\rangle = \left\langle \nabla_{\partial_{x_i}} \frac{\partial \psi}{\partial x_j} \right\rangle \left\langle \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_j} \right\rangle + \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nu^\gamma \right\rangle \left\langle \nu^\gamma, \nabla_{\partial_{x_k}} \nu^\beta \right\rangle
\]
= (A_\alpha A_\beta)_{ik} + \left\langle D_{\partial_{x_i}} \nu^\alpha, \nu^\gamma \right\rangle \left\langle D_{\partial_{x_k}} \nu^\beta, \nu^\gamma \right\rangle
\]
where \( D \) stands for the connection on the normal fiber bundle.

Consequently
\[
\left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} X \right\rangle = \lambda_{\beta,k} \left\langle D_{\partial_{x_i}} \nu^\alpha, \nu^\beta \right\rangle + \lambda_{\beta} \left\langle D_{\partial_{x_i}} \nu^\alpha, \nu^\gamma \right\rangle \left\langle D_{\partial_{x_k}} \nu^\beta, \nu^\gamma \right\rangle
+ \lambda_{\beta} \left\langle D_{\partial_{x_i}} \nu^\alpha, \nu^\gamma \right\rangle \left\langle D_{\partial_{x_k}} \nu^\beta, \nu^\gamma \right\rangle
- \mu^m_{ik} (A_\alpha)_{im} + \mu^m \left\langle D_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_m} \right\rangle.
\]
We get
\[
g^{jk} \frac{\partial (A_\alpha)_{ik}}{\partial t} = -g^{jk} R^{\mu\nu}_{\alpha \beta \gamma} \left( \nabla_{\partial_{x_i}} \nu^\mu, \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_t} \right) + g^{jk} \left( \lambda_{\alpha,ki} + \mu_{i}^j (A_\alpha)_{kl} + \mu^l (A_\alpha)_{kl,i} \right) X
+ g^{jk} \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle (A_\beta)_{ik} - g^{ik} \lambda_{\beta,\nu}^\beta \left\langle D_{\partial_{x_i}} \nu^\alpha, \nu^\beta \right\rangle - g^{jk} \lambda_{\beta} \left\langle D_{\partial_{x_i}} \nu^\alpha, \nu^\gamma \right\rangle
- \lambda_{\beta} g^{jk} \left\langle \nabla_{\partial_{x_i}} \nu^\alpha, \nu^\gamma \right\rangle \left\langle D_{\partial_{x_k}} \nu^\beta, \nu^\gamma \right\rangle
+ g^{ik} \mu^m_{ik} (A_\alpha)_{im} - \mu^m g^{jk} \left\langle D_{\partial_{x_i}} \nu^\alpha, \nabla_{\partial_{x_k}} \frac{\partial \psi}{\partial x_m} \right\rangle.
\]
Taking into account formula (10), we get
\[ \frac{\partial (A_{\alpha})^j_i}{\partial t} = -g^{ik} R_{\mu \nu} (A_{\alpha}, \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_k}, X) + g^{jk} \left( \lambda_{\alpha, ki} + \mu_i (A_{\alpha})_{kij} \right) \]
\[ + g^{jk} \left\langle \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right\rangle (A_{\beta})_{ik} - g^{jk} \lambda_{\beta, k} \left\langle D_{\frac{\partial}{\partial x_k}}, \nu^\alpha, \nu^\beta \right\rangle \]
\[ - \lambda_{\beta} g^{jk} \left\langle D_{\frac{\partial}{\partial x_k}}, \nu^\alpha, \nu^\gamma \right\rangle \right\rangle \left\langle D_{\frac{\partial}{\partial x_k}}, \nu^\beta, \nu^\gamma \right\rangle \]
\[ - \mu^m g^{jk} \left\langle D_{\frac{\partial}{\partial x_k}}, \nu^\alpha, \nabla_{\frac{\partial}{\partial x_m}} \psi \right\rangle + \lambda_{\beta} (A_{\alpha} A_{\beta})^j_i. \]

To compute \( \frac{\partial \sigma}{\partial t} \), we multiply both sides of (3.1) by \((T_{ab}(u))^i_j\) and sum.

For the continuation of the calculations we will need the following lemma which is a form of the property (4) of the proposition (1).

**Lemma 1.** For any \( \lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m \), the symmetric functions \( \sigma_u \) fulfill the following recurrence relation:
\[ \sum_{\alpha, \beta} \lambda_{\beta} \text{tr}(A_{\alpha} A_{\beta} T_{\beta, \alpha}(u)) = -\langle \lambda, u \rangle \sigma_u + \sum_{\beta} \lambda_{\beta} \text{tr}(A_{\beta}) \sigma_{\beta, \alpha}(u) \]
where \( \langle \lambda, u \rangle = \sum_{\alpha} \lambda_{\alpha} u_{\alpha}. \)

**Proof.** First, we have
\[ \sum_{\beta} \lambda_{\alpha} A_{\alpha} T_{\beta, \alpha}(u) = \sum_{\beta} \lambda_{\alpha} A_{\beta} (\sigma_{\beta, \alpha}(u)) 1 - \sum_{\alpha} A_{\alpha} T_{\alpha, \beta}(u) \]
\[ = \sum_{\beta} \lambda_{\alpha} \sigma_{\beta, \alpha}(u) A_{\beta} - \sum_{\alpha, \beta} \lambda_{\beta} A_{\alpha} A_{\beta} T_{\alpha, \beta}(u). \]

By passing to the traces,
\[ \sum_{\alpha, \beta} \lambda_{\beta} \text{tr} \left( A_{\beta} A_{\alpha} T_{\alpha, \beta}(u) \right) = \sum_{\beta} \lambda_{\beta} \text{tr}(A_{\beta}) \sigma_{\beta, \alpha}(u) - \sum_{\beta} \lambda_{\beta} \text{tr}(A_{\beta} T_{\beta, \alpha}(u)). \]

It remains to compute the last term of the right hand side of the above equality. To do so, we consider the curve \( A(\tau) = (1 + \lambda \tau) A \). We get \( A(0) = A \) and \( \frac{dA(\tau)}{d\tau} |_{\tau = 0} = \lambda A \). The expanding of the polynomial \( P_{A(\tau)} (t) = \det(1 + t A(\tau)) = \sum u_{\alpha} \sigma_{\alpha}(u) t^u \) with \( \sigma_{\alpha}(\tau) = (1 + \lambda \tau)^u \sigma_u. \) We have
\[ \left. \frac{d\sigma_u (\tau)}{d\tau} \right|_{\tau = 0} = \sum_{\beta} u_{\beta} \lambda_{\beta} \sigma_u = \langle \lambda, u \rangle \sigma_u \]
and
\[ \sum_{\beta} \text{tr} \left( \frac{d}{dt} A_{\beta}(\tau) \right) |_{\tau = 0} A_{\beta}(u) = \sum_{\beta} \lambda_{\beta} \text{tr}(A_{\beta} T_{\alpha, \beta}(u)). \]

By the definition of the (GTN) Newton transformations we obtain
\[ \sum_{\beta} \lambda_{\beta} \text{tr}(A_{\beta} T_{\alpha, \beta}(u)) = \langle \lambda, u \rangle \sigma_u. \]
First, by Lemma (1), we have

$$\sum_{\alpha,\beta} \lambda_{\beta} (A_\alpha A_{\beta})^j_i (T_{\alpha(b)})^i_j = \sum_{\alpha,\beta} \lambda_{\beta} \text{tr} (A_\alpha A_{\beta} T_{\alpha(b)})$$

$$= \sum_{\alpha,\beta} \lambda_{\beta} \text{tr} (A_\alpha A_{\beta} T_{\beta(u)})$$

$$= \sum_{\alpha,\beta} \lambda_{\beta} \text{tr} (A_\alpha A_{\beta} T_{\beta(u)} \#(u))$$

$$= - \langle \lambda, \beta \#(u) \rangle \sigma \#(u) + \sum_{\beta} \lambda_{\beta} \text{tr} (A_{\beta}) \cdot \sigma_u$$

we also write,

$$g^{jk} \sum_{\alpha} \lambda_{\alpha \cdot k i} (T_{\alpha(b)})^i_j = \sum_{\alpha} \lambda_{\alpha \cdot k i} (T_{\alpha(b)})^{k i}$$

By the Codazzi formula, we get

$$g^{jk} \mu^l \sum_{\alpha} (A_\alpha)_{k l \cdot i} (T_{\alpha(b)})^i_j = g^{jk} \mu^l \sum_{\alpha} (A_\alpha)_{k l \cdot i} (T_{\alpha(b)})^i_j$$

$$+ g^{jk} \left( R^M \right)_{\parallel \left( \frac{\partial \psi}{\partial x_i}, X \right) \frac{\partial \psi}{\partial x_k} (T_{\alpha(b)})^i_j}$$

and by equation (1), we infer that

$$g^{jk} \mu^l \sum_{\alpha} (A_\alpha)_{k l \cdot i} (T_{\alpha(b)})^i_j = \sum_{\alpha} \mu^l \text{tr} ((A_\alpha) \cdot (T_{\alpha(b)}))$$

$$+ g^{jk} \sum_{\alpha} \left( R^M \right)_{\parallel \left( \frac{\partial \psi}{\partial x_i}, X \right) \frac{\partial \psi}{\partial x_k} (T_{\alpha(b)})^i_j}$$

$$= \mu^l \sigma_{u,l} + g^{jk} \left( R^M \right)_{\parallel \left( \frac{\partial \psi}{\partial x_i}, X \right) \frac{\partial \psi}{\partial x_k} (T_{\alpha(b)})^i_j}. $$
Hence
\[
\frac{\partial \sigma_u}{\partial t} = -g^{jk} \sum_{\alpha} R^M (\nu^\alpha, \frac{\partial \psi}{\partial x_k}, \frac{\partial \psi}{\partial x_i}, X) (T_{\alpha b}(u))^i_j + \sum_{\alpha} (T_{\alpha b}(u))^i_j \lambda_{\alpha ij}
\]
\[+ \mu' \sigma_{u,l} + g^{jk} \left( R^M \right)^{\perp} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_k} (T_{\alpha b}(u))^i_j
\]
\[+ g^{jk} \left( \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right) (A_\beta)_{ik} (T_{\alpha b}(u))^i_j - g^{jk} \lambda_{\beta, k} \left( D_{\frac{\partial }{\partial x_k}} \nu^\alpha, \nu^\beta \right) (T_{\alpha b}(u))^i_j
\]
\[- \lambda_{\beta} g^{jk} \left( D_{\frac{\partial }{\partial x_k}} \nu^\alpha, \nu^\gamma \right) \left( D_{\frac{\partial }{\partial x_i}} \nu^\beta, \nu^\gamma \right) (T_{\alpha b}(u))^i_j - \lambda u \sum_{\beta} \lambda_{\beta} \sigma_{\beta#(u)}
\]
\[+ \sum_{\beta} \lambda_{\beta} \text{tr} (A_\beta) \sigma_u.
\]

The expression of \( \frac{\partial dV}{\partial t} \) is standard and it is given by:
\[
(18) \quad \frac{\partial dV}{\partial t} = \left( -\lambda_{\alpha} \text{tr}(A_{\alpha}) + \mu_{l}^l \right) dV.
\]

By expressions (3.1) and (18), we infer that:
\[
\frac{\partial \sigma_u}{\partial t} + \sigma_u \left( -\lambda_{\alpha} \text{tr}(A_{\alpha}) + \mu_{l}^l \right) =
\]
\[-g^{jk} \sum_{\alpha} R^M (\nu^\alpha, \frac{\partial \psi}{\partial x_k}, \frac{\partial \psi}{\partial x_i}, X) (T_{\alpha b}(u))^i_j + \sum_{\alpha} \text{tr} (T_{\alpha b}(u) \text{hess}(\lambda_{\alpha}))
\]
\[+ g^{jk} \left( R^M \right)^{\perp} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_k} (T_{\alpha b}(u))^i_j
\]
\[+ g^{jk} \left( \frac{\partial \nu^\alpha}{\partial t}, \nu^\beta \right) (A_\beta)_{ik} (T_{\alpha b}(u))^i_j - g^{jk} \lambda_{\beta, k} \left( D_{\frac{\partial }{\partial x_k}} \nu^\alpha, \nu^\beta \right) (T_{\alpha b}(u))^i_j
\]
\[- \lambda_{\beta} g^{jk} \left( D_{\frac{\partial }{\partial x_k}} \nu^\alpha, \nu^\gamma \right) \left( D_{\frac{\partial }{\partial x_i}} \nu^\beta, \nu^\gamma \right) (T_{\alpha b}(u))^i_j - \lambda u \sum_{\beta} \lambda_{\beta} \sigma_{\beta#(u)}
\]
\[+ \left( \sigma_u \mu_{l}^l \right)_{,l}.
\]

where hess is the hessian of \( \lambda_{\alpha} \).

4. Special cases

To simplify the expression of the integrand in Theorem 1, we consider submanifolds with flat normal bundle. First we recall the following facts: for \( x \in M \) the tangent space \( T_x (\mathbb{M}^n) \) splits as:
\[ T_x \left( \mathcal{M}^n \right) = T_x \left( M^m \right) \oplus N_x \left( M^m \right) \]

where \( T_x \left( M^m \right) \) is the tangent space of \( M^m \) at \( x \) and \( N_x \left( M^m \right) = T_x \left( M^m \right) ^\perp \) the normal space of \( M^m \) at \( x \).

Let \( D \) denote the normal covariant derivative on a \( m \)-dimensional submanifold \( M^n \) of a Riemannian manifold \( \overline{M}^n \) and consider the curvature tensor of the normal bundle

\[
R_D(X,Y)\nu = D_X D_Y \nu - D_Y D_X \nu - D_{[X,Y]} \nu.
\]

The normal bundle \( N(M^m) \) of \( M^n \) in \( M^n \) is said flat if and only if \( R_D(x) = 0 \) for any \( x \in M^n \) and \( M^n \) is called submanifold with flat normal bundle. The normal connection is called flat if the normal bundle of \( M^n \) is flat. It is well known in this case there is in each point \( y \) of \( M^n \) an orthonormal basis \((\nu_1, ..., \nu_{n-m})\) of \( N(M^m) \) such that each vector field \( \nu_\alpha \) is parallel in \( N(M^m) \) that is to say \( \nabla_X \nu_\alpha = 0 \) for each \( \nu_\alpha \in N(M^m) \) and \( X \in T(M^m) \). If the ambient manifold \( \overline{M}^n \) has a constant curvature \( c \) then for any vector fields \( X, Y, Z, \)

\[
R_{MN}(X,Y)Z = c \left( \langle Z,Y \rangle X - \langle Z,X \rangle Y \right)
\]

so for \( X, Y \) tangent and \( \nu \) normal to \( M^n \)

(20) \[ R_D(X,Y)\nu = 0. \]

As a consequence of formula (20), we have

**Theorem 2.** Let \( M^m \) be an \( m \)-dimensional closed submanifold of an \( n \)-dimensional space \( \overline{M}^n \) of constant sectional curvature \( c \). The first variation of the global \( \sigma_u \)-curvature is given by:

\[
\frac{d}{dt} \left( \int_{M^m} \sigma_u dV \right) = \int_{M^m} \left. \left( - \langle \lambda, \beta^\# \left( u \right) \rangle \sigma_{\beta^\#(u)} + c \left( m + 1 - |u| \right) \sum_\alpha \lambda_\alpha \sigma_{\alpha(b)} \right) \right) dV
\]

**Proof.** Indeed, we have:

\[
g^{jk} \sum_\alpha R^{\overline{M}}(\nu^\alpha, \frac{\partial \psi}{\partial x_k}, \frac{\partial \psi}{\partial x_i}, X) (T_{ab}(u))_i^j = -c \left( \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_k} \right) \sum_\alpha \langle X, \nu^\alpha \rangle (T_{ab}(u))_i^j
\]

\[
= -cg^{jk}g^{i\ell} \sum_{\alpha,\beta} \lambda_\beta \left( \nu^\beta, \nu^{\alpha} \right) (T_{ab}(u))_j^i
\]

\[
= -c \left( m + 1 - |u| \right) \sum_\alpha \lambda_\alpha \sigma_{\alpha(b)(u)}. \]

On the other hand since the normal connection of \( M^n \) is flat, for every \( x \in M^n \) there exist an orthonormal vector fields \( \nu_1, ..., \nu_{n-m} \) in an open neighborhood \( U \) of \( x \) such \( D_Y \nu = 0 \) in \( U \) where \( Y \in T_x M \). Let \( c_s \) be a curve on
VARIATIONAL PROPERTIES OF $\sigma_u$–CURVATURE FOR CLOSED SUBMANIFOLDS OF ARBITRARY CODIMENSION IN RIEMANNIAN MANIFOLDS

Consider a manifold $M^n$ such that $\nu^\alpha = \frac{\partial}{\partial s}|_{s=0} c_s$. Then $\frac{\partial \nu^\alpha}{\partial t}|_{t=0} = \frac{\partial}{\partial t}|_{t=0} c_s (\psi_t) = \nabla X \nu^\alpha$ and $\langle \frac{\partial \nu^\alpha}{\partial t}|_{t=0}, \nu^\beta \rangle = (\nabla X \nu^\alpha, \nu^\beta)$. So since the integral of a differentiable form $\omega$ on a manifold $M$ is defined as the sum of integrals of this form multiplied by an element $\rho_i$ of a partition $(\rho_i)_{i \in I}$ subordinated to an open cover $(U_j)_{j \in J}$ of $M$ over $U_j(i)$ which contains the support of $\rho_i$; it follows that the integrals of all terms containing the normal covariant derivatives in the expression (3.1) cancel. The same is also true for $\lambda_{\alpha,ij}$ since in an open neighborhood $U$ of each point $x \in M^n$, we have

$$\lambda_{\alpha,ij} = \langle \nabla \frac{\partial}{\partial x_i} X^\perp, \nu^\alpha \rangle + \langle X^\perp, \nabla \frac{\partial}{\partial x_i} \nu^\alpha \rangle = 0$$

hence $\lambda_{\alpha,ij} = 0$ in $U$. Consequently the integral on $M^n$ of $(T_{\alpha(u)})^{ij} \lambda_{\alpha,ij}$ cancels also.

### 4.1. Submanifolds of Euclidean space.

**Definition 1.** A submanifold $M^m$ of an Euclidean space $E^n$ is said $\sigma_u$-minimal if $\sigma_v$ vanishes identically where $v \in N(n - m)$ is a multi-index with length $|v| = |u| + 1$.

As in the paper of Reilly (see [9]) we will express the minimality of a submanifold of an Euclidean space $E^n$ in terms of partial differential equations. Let $\psi = (\psi_1, ..., \psi_{n+1})$ be the position vector of the submanifold $M^m$ and $\psi_{,ij} = (\psi_{1,ij}, ..., \psi_{n,ij})$ the second covariant derivative of $x$ on $M^m$.

$$\psi_{,ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} - d\psi \left( \nabla \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$$

$$= \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \nabla \psi \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$$

$$= \langle \frac{\partial^2 \psi}{\partial x_i \partial x_j}, N \rangle \ N$$

$$= \sum_\alpha \lambda_\alpha \ (A_\alpha)_{ij} \ N$$

where $N$ denotes a normal vector field to $M^m$, $\lambda_\alpha = \langle N, \nu^\alpha \rangle$ and $(\nu^1, ..., \nu^{n-m})$ an orthonormal basis to $M^m$.

Hence

$$\psi_{,ij} \ (T_u)^{ij} = \sum_\alpha \lambda_\alpha \ (A_\alpha)_{ij} \ (T_{\alpha(u)})^{ij} \ N = \langle \lambda, \alpha^\#(u) \rangle \sigma_{\alpha^\#(u)} \ N.$$  

### 4.2. Submanifolds of the unit round sphere.

**Definition 2.** A submanifold in the unit round sphere is said $\sigma_u$-minimal with if

$$\langle \lambda, \beta^\#(u) \rangle \sigma_{\beta^\#(u)} - (m + 1 - |u|) \sum_\alpha \lambda_\alpha \sigma_{\alpha(u)} = 0$$
Let $\psi = (\psi_1, ..., \psi_n)$ be the position vector of the hypersurface $M^m$ in the unit round sphere $S^n$ and $\psi_{ij} = (\psi_{1,ij}, ..., \psi_{n,ij})$ the second covariant derivative of $x$ on $M^m$. If $\nabla$ denotes the covariant derivative on $M^n$ induced by the covariants derivative $\nabla^{S_n}$ on the unit round sphere. We have

$$
\psi_{ij} = \nabla_{x_j} \nabla_{x_i} \psi = \left( \nabla_{x_j} \frac{\partial}{\partial x_i} \psi \right) \left( \frac{\partial}{\partial x_i} \right) \\
= \nabla_{x_j} \frac{\partial}{\partial x_i} \psi \left( \frac{\partial}{\partial x_i} \right) - d\psi \left( \nabla_{x_j} \frac{\partial}{\partial x_i} \right) \left( \frac{\partial}{\partial x_i} \right) \\
= \left\langle \nabla^{S_n}_{x_j} \frac{\partial}{\partial x_i} \psi, N \right\rangle N \\
= \sum_{\alpha} \lambda_\alpha \left\langle \nabla^{S_n}_{x_j} \frac{\partial}{\partial x_i} \psi, \nu^\alpha \right\rangle N \\
= \sum_{\alpha} \lambda_\alpha (A_\alpha)_{ij} N.
$$

where $\nu^\alpha$, $\alpha = 1, ..., n-m$ is a normal orthonormal basis to $M^m$, $N$ a normal vector field to $M^m$ (as submanifold of $S^n$) and $\lambda_\alpha = \langle N, \nu^\alpha \rangle$. In order to characterize the $\sigma_u$-minimality of submanifolds of the unit sphere, in case $\langle \lambda, \beta_b(u) \rangle \neq 0$, we multiply both sides of (4.1) by $T^{ij}_u - (m - |u| + 1) \sum_\alpha \lambda_\alpha \langle \lambda, \beta_b(u) \rangle T^{ij} \delta_\alpha^\beta \beta_b(u)$ and sum to infer

$$
T^{ij}_u \psi_{ij} = \sum_{\alpha} \lambda_\alpha T^{ij}_{\alpha_\beta \alpha \#(u)} (A_\alpha)_{ij} - (m - |u| + 1) \sum_{\alpha, \beta} T^{ij}_{\alpha_\beta \beta_b(u)} \lambda_\alpha \langle \lambda, \beta_b(u) \rangle (A_\alpha)_{ij} \\
= \left\langle \lambda, \beta^\#(u) \right\rangle \sigma^\beta^\#(u) - (m - |u| + 1) \sum_{\alpha} \lambda_\alpha \sigma_{\alpha_b(u)}.
$$

if $\langle \lambda, \beta_b(u) \rangle = \lambda_1 u_1 + ... + \lambda_{\beta-1} u_{\beta-1} + \lambda_\beta (u_{\beta} - 1) + ... + \lambda_m m$ for any multi-index $u$ with length $|u| \geq 2$ then $\lambda_\alpha = 0$ and $M^m$ is a totally geodesic submanifold of $S^n$; if $|u| = 1$ necessarily $u_{\beta} = 1$ and $M^m$ is $\sigma_{(0, ..., 0, 1, 0, ..., 0)}$-minimal submanifold of $S^n$.

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