A Weighted Endpoint Weak-Type Estimate for Multilinear Calderón–Zygmund Operators

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Abstract

Two proofs of a weighted weak-type \((1, \ldots, 1; \frac{1}{m})\) estimate for multilinear Calderón–Zygmund operators are given. The ideas are motivated by different proofs of the classical weak-type \((1, 1)\) estimate for Calderón–Zygmund operators. One proof uses the Calderón–Zygmund decomposition, and the other proof is motivated by ideas of Nazarov, Treil, and Volberg.

Keywords Singular integrals · Multilinear operators · Weak-type estimates · Weighted estimates

Mathematics Subject Classification 42B20

1 Introduction

The following weak-type \((1, 1)\) estimate is essential to the theory of singular integrals.

Theorem 1 Let \(T\) be a Calderón–Zygmund operator. If \(f \in L^1(\mathbb{R}^n)\), then

\[\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{t>0} t |\{|Tf| > t\}| \lesssim \|f\|_{L^1(\mathbb{R}^n)} .\]

The original proof of Theorem 1 uses the Calderón–Zygmund decomposition of \(f \in L^1(\mathbb{R}^n)\), see [5, 6, 20]. The Calderón–Zygmund decomposition method has since become standard for proving endpoint weak-type results for related operators. The Calderón–Zygmund decomposition requires the underlying measure space to possess the doubling property: a Borel measure \(\mu\) has the doubling property if

\[\frac{\mu(B)}{\mu(B')} \leq 2^d \quad \text{for all measurable sets } B \supseteq B' .\]
\[ \mu(B(x, 2r)) \lesssim \mu(B(x, r)) \]

for all \( r > 0 \) and all \( x \) in the space.

Extending the theory to more general settings, Nazarov, Treil, and Volberg gave a new proof of the weak-type \((1, 1)\) estimate for Calderón–Zygmund operators on nonhomogeneous spaces in [16]. A nonhomogeneous space is a metric measure space where the underlying measure \( \mu \) fails to possess the doubling property but instead satisfies the polynomial growth condition:

\[ \mu(B(x, r)) \lesssim r^n \]

for all \( r > 0 \) and all \( x \) in the space. Since Lebesgue measure on \( \mathbb{R}^n \) satisfies the polynomial growth condition, the proof in [16] immediately gives a different proof of Theorem 1.

The Nazarov–Treil–Volberg technique has been studied further. The technique was extended to handle measures with the upper doubling growth condition in [9]. It was shown in [22] that, if one again assumes the doubling condition, crucial steps in the Nazarov–Treil–Volberg proof of Theorem 1 may be bypassed. Also in [22], an adaptation of the Nazarov–Treil–Volberg argument was used to prove a weighted weak-type \((1, 1)\) inequality, and in [23], an adaptation of the argument was used to prove the weak-type \((1, \ldots, 1; \frac{1}{m})\) estimate for multilinear Calderón–Zygmund operators. The linear weighted estimate was previously proved in [18] using the Calderón–Zygmund decomposition, and the multilinear estimate was first proved in [7], also using the Calderón–Zygmund decomposition.

In this paper, we combine both of the previously mentioned settings by proving a weighted weak-type \((1, \ldots, 1; \frac{1}{m})\) estimate for multilinear Calderón–Zygmund operators. Two proofs are given – one uses the Calderón–Zygmund decomposition and the other is inspired by the Nazarov–Treil–Volberg method. See [22] for a comparison between the Calderón–Zygmund decomposition and Nazarov–Treil–Volberg proofs.

We describe the motivating results. For \( 1 \leq p < \infty \), we say that \( w \) is a \( A_p \) weight if \( w \) is locally integrable, positive almost everywhere, and satisfies the \( A_p \) condition:

\[ [w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'}dx \right)^{p-1} < \infty; \]

when \( p = 1 \), the quantity \( \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'}dx \right)^{p-1} \) is interpreted as \((\inf_Q w)^{-1}\). It is well known that \( A_p \) weights are doubling in the sense that if \( w \in A_p \) and \( Q \) is a cube, then

\[ w(2Q) \leq 2^{np}[w]_{A_p}w(Q). \]
**Theorem 2** Let $T$ be a Calderón–Zygmund operator. If $1 \leq p < \infty$, $w \in A_p$, and $f \in L^1(w)$, then

$$
\|T(fw)w^{-1}\|_{L^{1,\infty}(w)} \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} \|f\|_{L^1(w)}.
$$

Theorem 2 was proved by Ombrosi, Pérez, and Recchi in [18] using the Calderón–Zygmund decomposition and by the author in [22] using the Nazarov–Treil–Volberg argument. See [1, 2, 12, 17] for related mixed weak-type inequalities and [4, 8, 10] for sharp quantitative $A_1$ weighted weak-type estimates for Calderón–Zygmund operators.

Much of the Calderón–Zygmund theory was extended to the multilinear setting by Grafakos and Torres in [7]. In particular, they proved the following endpoint weak-type estimate.

**Theorem 3** Let $T$ be a multilinear Calderón–Zygmund operator. If $f_1, \ldots, f_m \in L^1(\mathbb{R}^n)$, then

$$
\|T(f_1, \ldots, f_m)\|_{L^{1/m,\infty}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L^1(\mathbb{R}^n)}.
$$

As in the classical theory, their proof uses the Calderón–Zygmund decomposition. Other proofs, also using the Calderón–Zygmund decomposition, were later given in the bilinear setting by Pérez and Torres in [19], and by Maldonado and Naibo in [15]. Another proof was given by the author and Wick in [23] using a variation of the Nazarov–Treil–Volberg method.

Connecting the weighted and the multilinear settings, Lerner, Ombrosi, Pérez, Torres, and Trujillo-González introduced the classes of multilinear weights in [11]. We use the following notation for multilinear $A_P$ weights: $1 \leq p_1, \ldots, p_m \leq \infty$, $\frac{1}{m} \leq p \leq \infty$ satisfies $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $P = (p_1, \ldots, p_m)$, $w = (w_1, \ldots, w_m)$, and $v_w = \prod_{i=1}^m w_i^{\frac{p}{p_i}}$. We say $w \in A_P$ if

$$
[w]_{A_P} := \sup_Q \left( \frac{1}{|Q|} \int_Q v_w \right)^{\frac{1}{P}} \prod_{i=1}^m \left( \frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{\frac{1}{p_i}} < \infty;
$$

when $p_i = 1$, the factor $\left( \frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{\frac{1}{p_i}}$ is understood as $(\inf_Q w_i)^{-1}$. Note that the quantities $[w]_{A_P}^P$ and $[w]_{A_P}$ coincide when $m = 1$.

We give two proofs of the following theorem.

**Theorem 4** Let $T$ be a multilinear Calderón–Zygmund operator. If $w \in A_{(1, \ldots, 1)}$ and $f \in L^1(w_i)$ for all $i \in \{1, \ldots, m\}$, then

$$
\left\| T \left( f_1 w_1 v_w^{\frac{1-m}{m}}, \ldots, f_m w_m v_w^{\frac{1-m}{m}} \right) v_w^{-1} \right\|_{L^{1/m,\infty}(v_w)} \lesssim [v_w]_{A_1}^{2m^2+2m-2} \prod_{i=1}^m \|f_i\|_{L^1(w_i)}.
$$
The first proof uses the Calderón–Zygmund decomposition and is a weighted version of the proof in [19]; the second proof is inspired by the Nazarov–Treil–Volberg method and is a weighted version of the proof in [23]. See [13] for a related result that is deduced using multilinear extrapolation.

**Remark 1** The second proof is actually a weighted version of a simplification of the proof in [23]. Referring to the contents of [23], the current proof shows that the sets $E_{i,j}$ can be constructed as cubes, that the regularity of Lemma 1 is only required for collections of pairwise disjoint cubes, and that Theorem 2 is not necessary for the weak-type estimate.

Section 2 describes the definitions and preliminary results, including a weighted version of the regularity condition first described for bilinear kernels in [19]. Section 3 contains two proofs of the main result, Theorem 4.

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### 2 Preliminaries

We use the notation $A \lesssim B$ if there exists $C > 0$, possibly depending on $n$, $T$, or $m$, such that $A \leq C B$. The Lebesgue measure of $A \subseteq \mathbb{R}^n$ is denoted by $|A|$, while for a weight $w$, $\int_A w(x) dx$ is denoted by $w(A)$. The cube with center $x \in \mathbb{R}^n$ and side length $r$ is denoted by $Q(x, r)$. If $Q$ is a cube, then $rQ$ denotes the cube with the same center as $Q$ and side length equal to $r$ times the side length of $Q$.

Let $m$ be a positive integer. We say $K : \mathbb{R}^{n(m+1)} \setminus \{x = y_i \text{ for all } i \in \{1, \ldots, m\}\} \to \mathbb{C}$ is an $m$-multilinear Calderón–Zygmund kernel if there exists $\delta > 0$ such that the following conditions hold:

1. \textbf{(size)}
   \[
   |K(x, y_1, \ldots, y_m)| \lesssim \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^{nm}}
   \]
   for all $x, y_1, \ldots, y_m \in \mathbb{R}^n$ with $x \neq y_j$ for some $j$,

2. \textbf{(smoothness)}
   \[
   |K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \lesssim \frac{|x - x'|^{\delta}}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}}
   \]
   whenever $|x - x'| \leq \frac{1}{2} \max_{1 \leq i \leq m} |x - y_i|$, and

   \[
   |K(x, y_1, \ldots, y_j, \ldots, y_m) - K(x, y_1, \ldots, y'_j, \ldots, y_m)| \lesssim \frac{|y_j - y'_j|^{\delta}}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}}
   \]
   for each $j \in \{1, \ldots, m\}$ whenever $|y_j - y'_j| \leq \frac{1}{2} \max_{1 \leq i \leq m} |x - y_i|$.  

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Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. We say that a $m$-multilinear operator $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is a multilinear Calderón–Zygmund operator associated to a kernel $K$ if $K$ is a $m$-multilinear Calderón–Zygmund kernel, if $T$ extends to a bounded operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $\frac{1}{m} < q, q_1, \ldots, q_m < \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \cdots dy_m$$

for compactly supported integrable functions $f_i$ and almost every $x \in \mathbb{R}^n \setminus \bigcap_{i=1}^m \operatorname{supp} f_i$. In all instances that follow, $T$ will represent an $m$-multilinear Calderón–Zygmund operator.

The following theorem was proved by Grafakos and Torres in [7].

**Theorem 5** If $1 < p_1, \ldots, p_m < \infty$, $p$ satisfies $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, and $f_i \in L^{p_i}(\mathbb{R}^n)$ for $i \in \{1, \ldots, m\}$, then

$$\|T(f_1, \ldots, f_m)\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

We will use Theorem 5 in the proofs of Theorem 4 when $p_1 = \cdots = p_m = m$ and $p = 1$.

A characterization of the multilinear $A_p$ condition in terms of linear $A_q$ conditions was established in [11].

**Theorem 6** The following conditions are equivalent:

1. $w \in A_p$.
2. $v w \in A_{mp}$ and $w_i^{1-p_i'} \in A_{mp_i'}$ for all $i \in \{1, \ldots, m\}$. When $p_i = 1$, the condition $w_i^{1-p_i'} \in A_{mp_i'}$ is understood as $w_i^{-1} \in A_1$.

**Remark 2** Tracking down the estimates in the proof of Theorem 6 gives the relationships

$$[v w]_{A_{mp}} \leq [w]^p_{A_p},$$

$$[w_i^{1-p_i'}]_{A_{mp_i'}} \leq [w]^{p_i'}_{A_{mp_i'}}$$

when $p_i > 1$,

$$[w_i^{-1}]_{A_1} \leq [w]^{-1}_{A_1}$$

when $p_i = 1$, and

$$[w]_{A_p} \leq [v w]_{A_{mp}} \prod_{i=1}^m [w_i^{1-p_i'}]_{A_{mp_i'}}^{p_i'}.$$

where $[w_i^{1-p_i'}]_{A_{mp_i'}}$ is interpreted as $[w_i^{-1}]_{A_1}$ when $p_i = 1$. 
We will use the following property of $A_1$ weights.

**Lemma 1** If $w \in A_1$ and $0 \leq \gamma \leq 1$, then $w^\gamma \in A_1$ with $[w^\gamma]_{A_1} \leq [w]_{A_1}^\gamma$.

**Proof** The cases when $\gamma = 0$ and $\gamma = 1$ are clear. If $0 < \gamma < 1$, then $\frac{1}{\gamma} > 1$.

Applying Hölder’s inequality and the $A_1$ condition of $w$ gives

$$
\frac{1}{|Q|} \int_Q w(y)^\gamma dy \leq \left( \frac{1}{|Q|} \int_Q w(y) dy \right)^\gamma \left( \frac{1}{|Q|} \int_Q \left( \frac{1}{y} \right)^\gamma' \right)^{\frac{1}{\gamma'}} \leq [w]_{A_1} \left( \inf_Q w^\gamma \right).
$$

$$\square$$

We will use the following maximal function in the second proof of the main theorem in Section 3. Given a weight $w$, define the uncentered maximal function associated to $w$ by

$$
M_w f(x) := \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)| w(y) dy.
$$

**Lemma 2** If $w \in A_p$ and $f \in L^1(w)$, then

$$
\| M_w f \|_{L^1, \infty(w)} \lesssim \| f \|_{L^1(w)}.
$$

The operator norm of $M_w$ does not depend on the $A_p$ characteristic of $w$.

The following lemma is well known and proved in [5, 6, 20].

**Lemma 3** Let $k : [0, \infty) \to [0, \infty)$ be decreasing and continuous except at a finite number of points. If $K(x) = k(|x|)$ is in $L^1(\mathbb{R}^n)$, then for all $f \in L^1_{loc}(\mathbb{R}^n)$,

$$
(\| f \ast K \|_{L^1(\mathbb{R}^n)}) (x) \leq \| K \|_{L^1(\mathbb{R}^n)} M(f)(x),
$$

where $M$ denotes the classical Hardy-Littlewood maximal operator.

The following lemma is a weighted version of the multilinear geometric Hörmander condition first introduced in the bilinear setting in [19] and generalized in the multilinear setting in [23]. We use the following vector notations $\overrightarrow{y}_{i,k} = (y_i, y_{i+1}, \ldots, y_k)$ and $\overrightarrow{c}_{(i,j_i),(k,j_k)} = (c_{i,j_i}, c_{i+1,j_i+1}, \ldots, c_{k,j_k})$.

**Lemma 4** If $w \in A_1$, $l \in \{1, \ldots, m\}$, and each of $Q_1, \ldots, Q_l$ consists of pairwise disjoint cubes $Q_i = \{Q_{i,1}, Q_{i,2}, \ldots\}$ where $Q_{i,j} = Q(c_{i,j}, r_{i,j})$, then

$$
\sum_{j_1, \ldots, j_l=1}^\infty \prod_{i=1}^l \frac{1}{w^{\frac{1}{m}}(Q_{i,j_i})} \int_{\mathbb{R}^{n-1}} \prod_{i=1}^m w(y_i)^{\frac{1}{m}}
$$

$$
\times \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^{n-l}} |K(x, \overrightarrow{y}_{1,m}) - K(x, \overrightarrow{c}_{(i,j_i),(l,j_l)}, \overrightarrow{y}_{l+1,m})| dxd\overrightarrow{y}_{l+1,m}
$$
\[ \lesssim [w]_{A_1}^{2m-2} \sum_{i=1}^{l} w(\Omega_i) \]

where \( \Omega_i := \bigcup_{j=1}^{\infty} Q_{i,j} \) and \( \Omega_i^* := \bigcup_{j=1}^{\infty} 2^{\sqrt{n}} Q_{i,j} \).

It is not important that the indices of the \( Q_i \) range from 1 to \( l \) – a symmetric proof yields the lemma whenever the set of indices is a nonempty subset of \{1, \ldots, m\}.

**Proof** For \( i = 1, \ldots, l \), fix \( Q_{i,j} \in Q_i \). Use the smoothness condition of \( K \) to see

\[
\sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} |K(x, y) - K(x, y^*)| \, dx \lesssim \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \frac{\sum_{i=1}^{l} |y_i - c_i|}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} \, dx
\]

\[
\lesssim \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \frac{\sum_{i=1}^{l} r_{i,j}^\delta}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} \, dx.
\]

Since for fixed \( y_i \in Q_{i,j}, i = l + 1, \ldots, m \), the function

\[
\int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \left( \sum_{i=1}^{m} |x - y_i| \right)^{nm+\delta} \, dx
\]

is continuous in the variables \( y_i \in Q_{i,j}, i = 1, \ldots, l \), we may write

\[
\sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \frac{\sum_{i=1}^{l} r_{i,j}^\delta}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} \, dx = \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \frac{\sum_{i=1}^{l} r_{i,j}^\delta}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} \, dx,
\]

and

\[
\inf_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \frac{\sum_{i=1}^{l} r_{i,j}^\delta}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} \, dx = \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^{l} \Omega_i^* \right)} \frac{\sum_{i=1}^{l} r_{i,j}^\delta}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} \, dx.
\]
Note that for $x \in \mathbb{R}^n \setminus \bigcup_{i=1}^l \Omega_i^*$, $|x-y_i| \leq \sqrt{n}r_{i,j_i} + |x-y_i^*|$ and $|x-y_i^*| \geq \frac{1}{2}r_{i,j_i}$, so

$$\sum_{i=1}^l |x-y_i| + \sum_{i=l+1}^m |x-y_i| \leq \frac{\sum_{i=1}^l \sqrt{n}r_{i,j_i}}{\sum_{i=1}^l |x-y_i^*| + \sum_{i=l+1}^m |x-y_i|} + 1 \leq 2\sqrt{n} + 1.$$ 

Then

$$\sup_{(y_1,\ldots,y_l)} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \left| K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l), \vec{y}_{l+1,m}}) \right| dx$$

$$\leq \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\sum_{i=1}^l |x-y_i| + \sum_{i=l+1}^m |x-y_i|} dx$$

$$\times \left( \sum_{i=1}^l |x-y_i| + \sum_{i=l+1}^m |x-y_i| \right)^{nm+\delta}$$

$$\leq \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\sum_{i=1}^l |x-y_i| + \sum_{i=l+1}^m |x-y_i|} dx$$

$$\leq \inf_{(y_1,\ldots,y_l)} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \left( \sum_{i=1}^m |x-y_i| \right)^{nm+\delta} dx.$$

Using the previous estimate, Fubini’s theorem, and trivial estimates, we get the bound

$$\sum_{j_1,\ldots,j_l=1}^\infty \prod_{i=1}^l w_i^{\frac{1}{n}}(Q_{i,j_i}) \int_{\mathbb{R}^{n(m-l)}} \prod_{i=l+1}^m w_i(y_i)^{\frac{1}{n}}$$

$$\times \sup_{(y_1,\ldots,y_l)} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \left| K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l), \vec{y}_{l+1,m}}) \right| dx d\vec{y}_{l+1,m}$$

$$\leq \sum_{j_1,\ldots,j_l=1}^\infty \prod_{i=1}^l w_i^{\frac{1}{n}}(Q_{i,j_i}) \int_{\mathbb{R}^{n(m-l)}} \prod_{i=l+1}^m w_i(y_i)^{\frac{1}{n}}$$

$$\times \inf_{(y_1,\ldots,y_l)} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \left( \sum_{i=1}^m |x-y_i| \right)^{nm+\delta} dx d\vec{y}_{l+1,m}$$

$$\leq \sum_{j_1,\ldots,j_l=1}^\infty \int_{\mathbb{R}^{n-m}} \cdots \int_{\mathbb{R}^{n-l}} \prod_{i=1}^l w_i(y_i)^{\frac{1}{n}}$$

$$\times \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^l \Omega_j^*} \left( \sum_{i=1}^m |x-y_i| \right)^{nm+\delta} dx d\vec{y}_{1,m}$$

$$\leq \sum_{k=1}^l \left( \sum_{j_1,\ldots,j_l=1}^\infty \int_{\mathbb{R}^{n(m-l)}} \cdots \int_{\mathbb{R}^{n-l}} \prod_{i=1}^m w_i(y_i)^{\frac{1}{n}} \right) \left( \sum_{i=1}^m |x-y_i| \right)^{nm+\delta} dx d\vec{y}_{1,m}.$$
\[
\times \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^l \Omega_i^* \right)} \left( \sum_{i=1}^l r_{i,j_i}^\delta \right) \left( \sum_{i=1}^m |x - y_i|^{nm+\delta} \right) \cdot dxd\gamma_{n,m-1}^i.
\]

We will control the term of the summation above with \( k = 1 \); the other terms are handled similarly. Using trivial estimates, Fubini’s theorem, and the fact that the \( Q_{i,j_i} \) have disjoint interiors, we obtain

\[
\sum_{j_1, \ldots, j_l=1}^\infty \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^l \Omega_i^* \right)} \left( \sum_{i=1}^l r_{i,j_i}^\delta \right) \left( \sum_{i=1}^m |x - y_i|^{nm+\delta} \right) \cdot dxd\gamma_{n,m-1}^i.
\]

Repeatedly use Lemma 3 first with \( K = \frac{1}{(\sum_{i=1}^m |x - y_i|^{nm})^m} \), second with \( K = \frac{1}{(\sum_{i=1}^m |x - y_i|^{nm})^m} \), etcetera, and the fact that \( M \left( \frac{w^{1/m}}{m} \right)(x) \leq \left[ w^{1/m} \right]_{A_1} w(x)^{1/m} \) (which is true by Lemma 1) to control the above expression by

\[
\sum_{j_1=1}^\infty \int_{\Omega_{j_1}} w(y_1)^{1/m} \int_{\mathbb{R}^n \setminus \left( \bigcup_{i=1}^l \Omega_i^* \right)} \left( r_{1,j_1}^\delta M \left( \frac{w^{1/m}}{m} \right)(x) \right) |x - y_1|^{nm} \cdot dxdy_1.
\]
Similarly, for 

This completes the proof. \( \square \)
3 Main Results

We give two proofs of the main result. The first proof uses the Calderón–Zygmund decomposition, and the second proof is inspired by the Nazarov–Treil–Volberg method. Recall that, for a measure $\mu$, the quasinorm $\| \cdot \|_{L^p(\mu)}$ is given by $\|f\|_{L^p(\mu)}^p := \sup_{t > 0} t^p \mu(\{|f| > t\})$.

**Theorem 4** If $w \in A_{(1, \ldots, 1)}$ and $f \in L^1(w_i)$ for all $i \in \{1, \ldots, m\}$, then

$$
\|T \left( \frac{f_1 w_1 v_1^{-\frac{1}{m}}, \ldots, f_m w_m v_m^{-\frac{1}{m}}}{v_w^{-1}} \right) \|_{L^{\frac{m}{m-\infty}}(v_w)} \lesssim \left[ v_w \right]_{A_1}^{2m^2 + 2m - 2} \prod_{i=1}^m \| f_i \|_{L^1(w_i)}.
$$

**Proof** Let $t > 0$ be given. We will show that

$$
v_w \left( \left\{ \left| T \left( \frac{f_1 w_1 v_1^{-\frac{1}{m}}, \ldots, f_m w_m v_m^{-\frac{1}{m}}}{v_w^{-1}} \right) \right| v_w^{-1} > t \right\} \right) \lesssim \left[ v_w \right]_{A_1}^{2m + 2m - 2} \prod_{i=1}^m \| f_i \|_{L^1(w_i)}.
$$

Without loss of generality, assume that $f_1, \ldots, f_m$ are continuous functions with compact support and that $\| f_i \|_{L^1(w_i)} = \cdots = \| f_m \|_{L^1(w_m)} = 1$. Apply the Calderón–Zygmund decomposition to $f_i w_i v_w^{-1}$ at height $t \frac{1}{m}$ with respect to $v_w dx$ to write

$$
f_i w_i v_w^{-1} = g_i + b_i = g_i + \sum_{j=1}^{\infty} b_{i,j}
$$

where the following properties hold:

1. $\| g_i \|_{L^\infty(\mathbb{R}^n)} \lesssim [v_w]_{A_1} t^\frac{1}{m}$ and $\| g_i \|_{L^1(v_w)} \lesssim \| f_i \|_{L^1(w_i)}$,
2. the $b_{i,j}$ are supported on pairwise disjoint cubes $Q_{i,j}$ satisfying $\sum_{j=1}^{\infty} v_w(Q_{i,j}) \leq t^{\frac{1}{m}} \| f_i \|_{L^1(w_i)}$,
3. $\int_{Q_{i,j}} b_{i,j}(x) v_w(x) dx = 0$,
4. $\| b_{i,j} \|_{L^1(v_w)} \lesssim [v_w]_{A_1} t^{\frac{1}{m}} v_w(Q_{i,j})$, and
5. $\| b_i \|_{L^1(v_w)} \lesssim \| f_i \|_{L^1(w_i)}$.

Set

$$
S_1 := \left\{ \left| T \left( \frac{g_1 v_1^{\frac{1}{m}}, \ldots, g_m v_m^{\frac{1}{m}}}{v_w^{-1}} \right) \right| v_w^{-1} > t^{\frac{1}{2m}} \right\},
$$

$$
S_2 := \left\{ \left| T \left( \frac{b_1 v_1^{\frac{1}{m}}, \ldots, b_m v_m^{\frac{1}{m}}}{v_w^{-1}} \right) \right| v_w^{-1} > t^{\frac{1}{2m}} \right\},
$$

$$
S_3 := \left\{ \left| T \left( \frac{g_1 v_1^{\frac{1}{m}}, b_2 v_2^{\frac{1}{m}}, \ldots, g_m v_m^{\frac{1}{m}}}{v_w^{-1}} \right) \right| v_w^{-1} > t^{\frac{1}{2m}} \right\},
$$

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where each $S_{s} = \left\{ T \left( h_{1} v_{w}^{1/m}, \ldots, h_{m} v_{w}^{1/m} \right) \mid v_{w}^{-1} > \frac{t}{2^m} \right\}$ with $h_{i} \in \{ b_{i}, g_{i} \}$ and all the sets $S_{s}$ are distinct. Since

\[
 v_{w} \left( \left\{ T \left( f_{1} w_{1} v_{w}^{1-m}, \ldots, f_{m} w_{m} v_{w}^{1-m} \right) \mid v_{w}^{-1} > t \right\} \right) \leq \sum_{s=1}^{2^m} v_{w}(S_{s}),
\]

it suffices to control each $v_{w}(S_{s})$.

Use Chebyshev’s inequality, the boundedness of $T$ from $(L^{m}(\mathbb{R}^{n}))^{m}$ to $L^{1}(\mathbb{R}^{n})$ (which holds by Theorem 5), and property (1) to see

\[
v_{w} \left( S_{1} \right) \leq t^{-1} \int_{\mathbb{R}^{n}} \left| T \left( g_{1} v_{w}^{1/m}, \ldots, g_{m} v_{w}^{1/m} \right) (x) \right| dx \\
\leq t^{-1} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^{n}} |g_{i}(x)|^{m} v_{w}(x) dx \right)^{\frac{1}{m}} \\
\leq t^{-1} \prod_{i=1}^{m} \|g_{i}\|_{L^{1}(v_{w})}^{\frac{1}{m}} \\
\leq t^{-\frac{1}{m}}.
\]

Consider the set $S_{s}$ for a fixed $2 \leq s \leq 2^m$. Suppose that there are $l$ functions of the form $b_{i}$ and $m - l$ functions of the form $g_{i}$ appearing as entries in the $T \left( h_{1} v_{w}^{1/m}, \ldots, h_{m} v_{w}^{1/m} \right)$ involved in the definition of $S_{s}$. By symmetry, we may assume that the $b_{i}$ are in the first $l$ entries and the $g_{i}$ are in the remaining $m - l$ entries. Let $Q_{i,j}^{*} := 2 \sqrt{\eta} Q_{i,j}, \Omega_{i}^{*} := \bigcup_{j=1}^{\infty} Q_{i,j}^{*}$, and $\Omega^{*} := \bigcup_{i=1}^{m} \Omega_{i}^{*}$.

By the doubling property of $v_{w} dx$, the fact that the $Q_{i,j}$ are pairwise disjoint, and property (2), we have

\[
v_{w}(\Omega^{*}) \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} v_{w}(Q_{i,j}^{*}) \leq [v_{w}]_{A_{1}} \sum_{i=1}^{m} \sum_{j=1}^{\infty} v_{w}(Q_{i,j}) \leq [v_{w}]_{A_{1}} t^{-\frac{1}{m}}.
\]

Therefore,

\[
v_{w}(S_{s}) \leq v_{w}(\Omega^{*}) + v_{w} \left( \left\{ \mathbb{R}^{n} \setminus \Omega^{*} : T \left( b_{1} v_{w}^{1/m}, \ldots, b_{l} v_{w}^{1/m}, g_{l+1} v_{w}^{1/m}, \ldots, g_{m} v_{w}^{1/m} \right) \mid v_{w}^{-1} > \frac{t}{2^m} \right\} \right) \\
\leq [v_{w}]_{A_{1}} t^{-\frac{1}{m}} + v_{w} \left( \left\{ \mathbb{R}^{n} \setminus \Omega^{*} : T \left( b_{1} v_{w}^{1/m}, \ldots, b_{l} v_{w}^{1/m}, g_{l+1} v_{w}^{1/m}, \ldots, g_{m} v_{w}^{1/m} \right) \mid v_{w}^{-1} > \frac{t}{2^m} \right\} \right).
\]
Now use Chebyshev’s inequality, the fact that $\int_{Q_{i,j_1}} b_{i,j_1}(x) v_w(x) dx = 0$, and trivial bounds to estimate

\[
v_w \left( \left\{ \mathbb{R}^n \setminus \Omega^* : \left| T \left( b_{1,1} v_{w_1}^m, \ldots, b_{l,1} v_{w_l}^m, g_{l+1} v_{w_l}^m, \ldots, g_m v_{w_m}^m \right) \right| v_{w-1} > \frac{t}{2m} \right\} \right) \\
\lesssim t^{-1} \int_{\mathbb{R}^n \setminus \Omega^*} \left| T \left( b_{1,1} v_{w_1}^m, \ldots, b_{l,1} v_{w_l}^m, g_{l+1} v_{w_l}^m, \ldots, g_m v_{w_m}^m \right) (x) \right| dx \\
\lesssim t^{-1} \sum_{j_1, \ldots, j_l = 1}^\infty \int_{\mathbb{R}^n \setminus \Omega^*} \int_{\mathbb{R}^n \setminus \Omega^*} \cdots \int_{\mathbb{R}^n \setminus \Omega^*} K(x, y_{1,m}) \\
\times \left( \prod_{i=1}^l b_{i,j_1}(y_i) \right) \left( \prod_{i=1}^l g_i(y_i) \right) \left( \prod_{i=1}^m v_w(y_i)^{\frac{1}{m}} \right) d y_{1,m} \ dx \\
\leq t^{-1} \sum_{j_1, \ldots, j_l = 1}^\infty \left( \prod_{i=1}^m \left| g_i(y_i) \right| v_w(y_i)^{\frac{1}{m}} \right) \int_{\mathbb{R}^n \setminus \Omega^*} \left| K(x, y_{1,m}) - K(x, y_{1,m}) \right| dx \ dx \\
\leq t^{-1} \sum_{j_1, \ldots, j_l = 1}^\infty \left( \prod_{i=1}^m \left| b_{i,j_1}(y_i) \right| v_w(y_i)^{\frac{1}{m}} \right) \\
\times \left( \prod_{i=1}^l \left| b_{i,j_1}(y_i) \right| v_w(y_i)^{\frac{1}{m}} \right) \int_{\mathbb{R}^n \setminus \Omega^*} \left| K(x, y_{1,m}) - K(x, y_{1,m}) \right| dx \ dx \\
\times \left( \prod_{i=1}^m v_w(y_i)^{\frac{1}{m}} \right) d y_{l+1,m} \ dx \\
\leq \left[ v_w \right]^{n+l}_{A_1} \sum_{j_1, \ldots, j_l = 1}^\infty \left( \prod_{i=1}^l \left( \inf_{Q_{i,j_i}} v_w \right)^{\frac{1-m}{m}} \right) \left( \prod_{i=1}^l \left| Q_{i,j_i} \right| \left( \inf_{Q_{i,j_i}} v_w \right) \right) \\
\times \int_{\mathbb{R}^n \setminus \Omega^*} \left( \prod_{i=1}^m v_w(y_i)^{\frac{1}{m}} \right) d y_{l+1,m} \ dx \\
\leq \left[ v_w \right]^{n+l}_{A_1} \sum_{j_1, \ldots, j_l = 1}^\infty \left( \prod_{i=1}^l \left( \inf_{Q_{i,j_i}} v_w \right)^{\frac{1-m}{m}} \right) \left( \prod_{i=1}^l \left| Q_{i,j_i} \right| \left( \inf_{Q_{i,j_i}} v_w \right) \right)
\]
\begin{align*}
&\times \left( \prod_{i=l+1}^{m} w_i(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx \\
\leq & \ [v_w]_{A_1}^{2m} \sum_{j_1, \ldots, j_l = 1}^{\infty} \left( \prod_{i=1}^{l} \{Q_{i,j_i} \} \left( \inf \frac{1}{Q_{i,j_i}} \right) \right) \\
&\times \left( \prod_{i=l+1}^{m} w_i(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx \\
&\leq \ [v_w]_{A_1}^{2m} \sum_{j_1, \ldots, j_l = 1}^{\infty} \left( \prod_{i=1}^{l} w_i(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx.
\end{align*}

By Lemma 4 and property (2), the above expression is controlled by a constant times:

\begin{align*}
[v_w]_{A_1}^{2m+\frac{2m-2}{m}} \sum_{i=1}^{l} v_w \left( \bigcup_{j=1}^{\infty} Q_{i,j} \right) &\lesssim [v_w]_{A_1}^{2m+\frac{2m-2}{m}} t^{-\frac{1}{m}}.
\end{align*}

Therefore

\begin{align*}
v_w(S_x) &\leq [v_w]_{A_1} t^{-\frac{1}{m}} + [v_w]_{A_1}^{2m+\frac{2m-2}{m}} t^{-\frac{1}{m}} \lesssim [v_w]_{A_1}^{2m+\frac{2m-2}{m}} t^{-\frac{1}{m}}.
\end{align*}

Putting the previous estimates together gives

\begin{align*}
v_w \left( \left\{ \left| T \left( f_1 w_1^{m+1} v_w^{-1}, \ldots, f_m w_m^{m+1} v_w^{-1} \right) \right| v_w^{-1} > t \right\} \right) &\lesssim t^{-\frac{1}{m}} + \sum_{s=2}^{m} [v_w]_{A_1}^{2m+\frac{2m-2}{m}} t^{-\frac{1}{m}} \\
&\lesssim [v_w]_{A_1}^{2m+\frac{2m-2}{m}} t^{-\frac{1}{m}}.
\end{align*}

\hfill \Box
Proof Let $t > 0$ be given. We will show that

$$
v_w \left( \left\{ T \left( f_1 w_1 v_w^{\frac{1-m}{m}}, \ldots, f_m w_m v_w^{\frac{1-m}{m}} \right) \bigg| v_w^{\frac{1}{w}} > t \right\} \right) \lesssim [v_w]_{A_1}^{2m+\frac{2m-2}{m}} t^{-\frac{1}{m}} \prod_{i=1}^{m} \| f_i \|_{L^1(w_i)}.
$$

Assume that $f_1, \ldots, f_m$ are nonnegative, continuous functions with compact support and that $\| f_1 \|_{L^1(w_1)} = \cdots = \| f_m \|_{L^1(w_m)} = 1$. Assume that $v_w(\mathbb{R}^n) > t^{-\frac{1}{m}}$ (otherwise there is nothing to prove). Set

$$\Omega_i := \left\{ M_{v_w} \left( f_i w_i v_w^{-1} \right) > t^{\frac{1}{m}} \right\} \quad \text{and} \quad \Omega := \bigcup_{i=1}^{m} \Omega_i.$$

Apply a Whitney decomposition to write

$$\Omega_i = \bigcup_{j=1}^{\infty} Q_{i,j},$$

a disjoint union of dyadic cubes where

$$2\text{diam}(Q_{i,j}) \leq d(Q_{i,j}, \mathbb{R}^n \setminus \Omega_i) \leq 8\text{diam}(Q_{i,j}).$$

Put

$$g_i := f_i w_i v_w^{-1} \mathbb{1}_{\mathbb{R}^n \setminus \Omega_i}, \quad b_i := f_i w_i v_w^{-1} \mathbb{1}_{\Omega_i}, \quad \text{and} \quad b_{i,j} := f_i w_i v_w^{-1} \mathbb{1}_{Q_{i,j}}.$$

Then

$$f_i w_i v_w^{-1} = g_i + b_i = g_i + \sum_{j=1}^{\infty} b_{i,j},$$

where

1. $\| g_i \|_{L^\infty(\mathbb{R}^n)} \leq t^{\frac{1}{m}}$ and $\| g_i \|_{L^1(v_w)} \leq \| f_i \|_{L^1(w_i)}$,
2. the $b_{i,j}$ are supported on pairwise disjoint cubes $Q_{i,j}$ satisfying

$$\sum_{j=1}^{\infty} v_w(Q_{i,j}) \lesssim t^{-\frac{1}{m}} \| f_i \|_{L^1(w_i)},$$

3. $\| b_{i,j} \|_{L^1(v_w)} \leq (17\sqrt{n})^n [v_w]_{A_1} t^{\frac{1}{m}} v_w(Q_{i,j})$, and
4. $\| b_i \|_{L^1(v_w)} \leq \| f_i \|_{L^1(w_i)}$.

To justify the above properties, since

$$f_i(x) w_i(x) v_w^{-1}(x) \leq M_{v_w} \left( f_i w_i v_w^{-1} \right)(x) \leq t^{\frac{1}{m}}$$
for almost every \( x \in \mathbb{R}^n \setminus \Omega_i \), it is true that \( \| g_i \|_{L^\infty(\mathbb{R}^n)} \leq t^{\frac{1}{m}} \). Noticing that \( g_i \) is a restriction of \( f_i w_i v_w^{-1} \), we have \( \| g_i \|_{L^1(w)} \leq \| f_i \|_{L^1(w)} \), so property (1) holds. We obtain property (2) using Lemma 2 as follows:

\[
\sum_{j=1}^{\infty} v_w(Q_{i,j}) = v_w(\Omega_i) \lesssim t^{-\frac{1}{m}}.
\]

Addressing (3), for a fixed \( Q_{i,j} \), let \( Q_{i,j}^* = 17 \sqrt{n} Q_{i,j} \). Then \( Q_{i,j}^* \cap (\mathbb{R}^n \setminus \Omega_i) \neq \emptyset \), so there is a point \( x \in Q_{i,j}^* \) such that \( M_w\left( f_i w_i v_w^{-1}\right)(x) \leq t^{\frac{1}{m}} \). In particular, \( \int_{Q_{i,j}^*} f_i(y) w_i(y) dy \leq t^{\frac{1}{m}} v_w(Q_{i,j}^*) \). Since \( v_w(Q_{i,j}^*) \leq (17 \sqrt{n})^{n} [v_w]_{A_1} v_w(Q_{i,j}) \), we have

\[
\| b_{i,j} \|_{L^1(w)} = \int_{Q_{i,j}} f_i(y) w_i(y) dy \leq \int_{Q_{i,j}^*} f_i(y) w_i(y) dy \\
\leq t^{\frac{1}{m}} v_w(Q_{i,j}^*) \leq (17 \sqrt{n})^{n} [v_w]_{A_1} t^{\frac{1}{m}} v_w(Q_{i,j}),
\]

proving (3). Property (4) follows since \( b_i \) is a restriction of \( f_i w_i v_w^{-1} \).

Set

\[
S_1 := \left\{ T \left( g_1 v_w^\frac{1}{m}, g_2 v_w^\frac{1}{m}, \ldots, g_m v_w^\frac{1}{m} \right) \bigg| v_w^{-1} > \frac{t}{2^m} \right\},
\]

\[
S_2 := \left\{ T \left( b_1 v_w^\frac{1}{m}, g_2 v_w^\frac{1}{m}, \ldots, g_m v_w^\frac{1}{m} \right) \bigg| v_w^{-1} > \frac{t}{2^m} \right\},
\]

\[
S_3 := \left\{ T \left( g_1 v_w^\frac{1}{m}, b_2 v_w^\frac{1}{m}, \ldots, g_m v_w^\frac{1}{m} \right) \bigg| v_w^{-1} > \frac{t}{2^m} \right\},
\]

\[
\ldots
\]

\[
S_{2^m} := \left\{ T \left( b_1 v_w^\frac{1}{m}, b_2 v_w^\frac{1}{m}, \ldots, b_m v_w^\frac{1}{m} \right) \bigg| v_w^{-1} > \frac{t}{2^m} \right\};
\]

where each \( S_s = \left\{ T \left( h_1 v_w^\frac{1}{m}, \ldots, h_m v_w^\frac{1}{m} \right) \bigg| v_w^{-1} > \frac{t}{2^m} \right\} \) with \( h_i \in \{ b_i, g_i \} \) and all the sets \( S_s \) are distinct. Since

\[
v_w \left( \left\{ T \left( f_1 w_1 v_w^{\frac{1-m}{m}}, \ldots, f_m w_m v_w^{\frac{1-m}{m}} \right) \bigg| v_w^{-1} > \frac{t}{2^m} \right\} \right) \leq \sum_{s=1}^{2^m} v_w(S_s),
\]

it suffices to control each \( v_w(S_s) \).

Use Chebyshev’s inequality, the boundedness of \( T \) from \( L^m(\mathbb{R}^n)^m \) to \( L^1(\mathbb{R}^n) \) (which holds by Theorem 5), and property (1) to see
\[
\begin{align*}
\nu_w(S_s) & \lesssim t^{-1} \int_{\mathbb{R}^n} \left| T \left( g_1 \nu_w^{1/m}, \ldots, g_m \nu_w^{1/m} \right) (x) \right| \, dx \\
& \lesssim t^{-1} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \left| g_i(x) \right|^m \nu_w(x) \, dx \right)^{1/m} \\
& \leq t^{-\frac{1}{m}} \prod_{i=1}^{m} \| g_i \|_{L^1(v_w)} \\
& \leq t^{-\frac{1}{m}}.
\end{align*}
\]

Consider the set \( S_s \) for a fixed \( 2 \leq s \leq 2^m \). Suppose that there are \( l \) functions of the form \( b_i \) and \( m - l \) functions of the form \( g_i \) appearing as entries in the \( T \left( h_1 \nu_w^{1/m}, \ldots, h_m \nu_w^{1/m} \right) \) involved in the definition of \( S_s \). By symmetry, we may assume that the \( b_i \) are in the first \( l \) entries and the \( g_i \) are in the remaining \( m - l \) entries.

Let \( c_{i,j} \) denote the center of \( Q_{i,j} \) and let \( a_{i,j} = \| b_{i,j} \|_{L^1(v_w)} (17\sqrt{n})^{-n} [\nu_w^{1/A_1}] \). Set

\[
E_{i,j} := Q(c_{i,j}, r_{i,j}),
\]

where \( r_{i,j} > 0 \) is chosen so that \( \nu_w(E_{i,j}) = a_{i,j} t^{-\frac{1}{m}} \). Note that such \( E_{i,j} \) exist since the function \( r \mapsto \nu_w(Q(x, r)) \) increases to \( \nu_w(\mathbb{R}^n) > t^{-\frac{1}{m}} \) as \( r \to \infty \), approaches 0 as \( r \to 0 \), and is continuous from the right for almost every \( x \in \mathbb{R}^n \). Using property (3), we see

\[
\nu_w(E_{i,j}) = a_{i,j} t^{-\frac{1}{m}} \leq \nu_w(Q_{i,j}).
\]

Since \( E_{i,j} \) is a cube with the same center as \( Q_{i,j} \) and since \( \nu_w(E_{i,j}) \leq \nu_w(Q_{i,j}) \), it is true that \( E_{i,j} \subset Q_{i,j} \). Define

\[
E_i := \bigcup_{j=1}^{\infty} E_{i,j}.
\]

For \( k = 0, \ldots, l \), define

\[
\sigma_k := T \left( (17\sqrt{n})^{n} [\nu_w]_{A_1} t^{\frac{1}{m}} \xrightarrow{\nu_w^{1/A_1}} \nu_w^{1/2} b_{k+1, j} \nu_w^{1/2} \xrightarrow{\nu_w^{1/A_1}} \nu_w^{1/m} \right),
\]

Then, by adding and subtracting \( \sigma_k \) for \( 1 \leq k \leq l \), we have

\[
\begin{align*}
\nu_w(S_s) & \leq \sum_{k=1}^{l} \nu_w \left( \left\{ |\sigma_{k-1} - \sigma_k| > \frac{t}{(l+1)2^m} \right\} \right) + \nu_w \left( \left\{ |\sigma_l| > \frac{t}{(l+1)2^m} \right\} \right) \\
& \lesssim \nu_w(\Omega) + \sum_{k=1}^{l} \nu_w \left( \left\{ \mathbb{R}^n \setminus \Omega : |\sigma_{k-1} - \sigma_k| > \frac{t}{(l+1)2^m} \right\} \right).
\end{align*}
\]
\[ + v_w \left( \left\{ |\sigma| > \frac{t}{(l+1)2^m} \right\} \right). \]

Using property (2), we have
\[ v_w(\Omega) \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} v_w(Q_{i,j}) \lesssim t^{-\frac{1}{m}}, \]
therefore,
\[ v_w(S_S) \lesssim t^{-\frac{1}{m}} + \sum_{k=1}^{l} v_w(P_k) + v_w(P), \]
where
\[ P_k := \left\{ \mathbb{R}^n \setminus \Omega : |\sigma_{k-1} - \sigma_k| > \frac{t}{(l+1)2^m} \right\}, \quad \text{and} \]
\[ P := \left\{ |\sigma| > \frac{t}{(l+1)2^m} \right\}. \]

We will first estimate \( v_w(P_k) \) for \( k \in \{1, \ldots, l\} \). Notice that
\[
\sigma_{k-1}(x) - \sigma_k(x) = T \left( (17\sqrt{n})^n [v_w]_{A_1} t^{\frac{1}{m}} \mathbb{I}_{E_{i,k-1}} \mathbb{I}_{v_w} \right),
\]
\[
\left( b_k - (17\sqrt{n})^n [v_w]_{A_1} t^{\frac{1}{m}} \mathbb{I}_{E_k} \right) v_w \mathbb{I}_{b_{k+1,l}} v_w \mathbb{I}_{g_{l+1,m} v_w} (x). \]

Begin by using Chebyshev’s inequality, the fact that
\[ \int_{\mathbb{R}^n} \left( b_{k,jk}(x) - (17\sqrt{n})^n [v_w]_{A_1} t^{\frac{1}{m}} \mathbb{I}_{E_{k,jk}(x)} \right) v_w(x) dx = 0, \]
and trivial estimates to see
\[ v_w(P_k) \lesssim t^{-1} \int_{\mathbb{R}^n \setminus \Omega} |\sigma_{k-1}(x) - \sigma_k(x)| dx \]
\[ \leq t^{-1} \sum_{j_1, \ldots, j_{l-1}} \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} K(x, y_{1,m}) \right) \right) \right) \right) \right) \right) \]
\[ \times \left( \prod_{i=1}^{k-1} (17\sqrt{n})^n [v_w]_{A_1} t^{\frac{1}{m}} \mathbb{I}_{E_{i,j,i,j_i}(y_i)} v_w(y_i) t^{\frac{1}{m}} \right) \]
\[ \times \left( b_{k,jk}(y_k) - (17\sqrt{n})^n [v_w]_{A_1} t^{\frac{1}{m}} \mathbb{I}_{E_{k,jk}(y_k)} v_w(y_k) t^{\frac{1}{m}} \right) \]
\[
\times \left( \prod_{i=k+1}^{l} b_{i,i,j_i}(y_i) v_{w}(y_i)^{1/m} \right) \left( \prod_{i=l+1}^{m} g_i(y_i) v_{w}(y_i)^{1/m} \right) d \nrightarrow 1,m \bigg| dx
\]
\[
\leq t^{-\frac{k-m-1}{m}} \sum_{j_1,\ldots,j_l=1}^{\infty} \left( \prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{w_{m}}^{1-m} \right) \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} \cdots \int_{E_{j_1,j_l}} \left| K(x, \nrightarrow 1,m) - K(x, c^*(1,j_1),(l,j_l), \nrightarrow l+1,m) \right| \left( \prod_{i=1}^{k-1} v_{w}(y_i) \right) \\
\times \left( b_{k,j_k}(y_k) - (17\sqrt{n})^m [v_{w}]_{A_1} t^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}}(y_k) \right) \left( \prod_{i=k+1}^{l} b_{i,i,j_i}(y_i) v_{w}(y_i) \right) \\
\times \left( \prod_{i=l+1}^{m} g_i(y_i) v_{w}(y_i)^{1/m} \right) d \nrightarrow 1,m dx.
\]

Next use the fact that \( v_{w}(E_{i,j_i}) \lesssim t^{-\frac{1}{m}} \|b_{i,j_i}\|_{L^1(v_{w})} \), Fubini’s theorem, trivial estimates, the fact that
\[
\left\| b_{k,j_k} - (17\sqrt{n})^m [v_{w}]_{A_1} t^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}} \right\|_{L^1(v_{w})} \lesssim \left\| b_{k,j_k} \right\|_{L^1(v_{w})},
\]
and property (1) to control
\[
v_{w}(P_k) \lesssim t^{-\frac{l}{m}} \sum_{j_1,\ldots,j_l=1}^{\infty} \left( \prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{w_{m}}^{1-m} \right) \\
\left( \prod_{i=1}^{k-1} \|b_{i,j_i}\|_{L^1(v_{w})} \right) \left( \prod_{i=l+1}^{m} \|g_i\|_{L^\infty(\mathbb{R}^n)} \right) \int_{\mathbb{R}^n \setminus \Omega} \cdots \int_{\mathbb{R}^n \setminus \Omega} \left| K(x, \nrightarrow 1,m) - K(x, c^*(1,j_1),(l,j_l), \nrightarrow l+1,m) \right| \\
\times \left( \prod_{i=l+1}^{m} v_{w}(y_i)^{1/m} \right) dx d \nrightarrow l+1,m \\
\lesssim t^{-\frac{l}{m}} \sum_{j_1,\ldots,j_l=1}^{\infty} \left( \prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{w_{m}}^{1-m} \right) \\
\left( \prod_{i=1}^{k-1} \|b_{i,j_i}\|_{L^1(v_{w})} \right) \\
\times \int_{\mathbb{R}^n \setminus \Omega} \cdots \int_{\mathbb{R}^n \setminus \Omega} \left| K(x, \nrightarrow 1,m) - K(x, c^*(1,j_1),(l,j_l), \nrightarrow l+1,m) \right| \\
\times \left( \prod_{i=l+1}^{m} v_{w}(y_i)^{1/m} \right) dx d \nrightarrow l+1,m.
\]
Use property (3), the fact that \( \left( \sup_{Q_{i,j}} v_{w}^{1-m} \right) = \left( \inf_{Q_{i,j}} v_{w}^{1-m} \right) \), the \( A_1 \) condition of \( v_w \), and trivial estimates to estimate \( v_w(P_k) \)

\[
v_w(P_k) \lesssim [v_w]_{A_1} \sum_{j_1, \ldots, j_l=1}^{\infty} \left( \prod_{i=1}^{l} \left( \sup_{Q_{i,j_i}} v_{w}^{1-m} \right) \right) \left( \prod_{i=1}^{l} v_w(Q_{i,j_i}) \right)
\]

\[
\times \int_{\mathbb{R}^{n(m-l)}} \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \rightarrow y_{1,m}) - K(x, \rightarrow c_{(1,j_1)}, \rightarrow y_{l+1,m})| \cdot \left( \prod_{i=l+1}^{m} v_w(y_{i})^{\frac{1}{m}} \right) dx dy_{l+1,m}
\]

\[
\leq [v_w]_{A_1} \sum_{j_1, \ldots, j_l=1}^{\infty} \left( \prod_{i=1}^{l} \left( \inf_{Q_{i,j_i}} v_{w}^{1-m} \right) \right) \left( \prod_{i=1}^{l} v_w(Q_{i,j_i}) \right)
\]

\[
\times \int_{\mathbb{R}^{n(m-l)}} \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \rightarrow y_{1,m}) - K(x, \rightarrow c_{(1,j_1)}, \rightarrow y_{l+1,m})| \cdot \left( \prod_{i=l+1}^{m} v_w(y_{i})^{\frac{1}{m}} \right) dx dy_{l+1,m}
\]

\[
\leq [v_w]_{A_1}^{2l} \sum_{j_1, \ldots, j_l=1}^{\infty} \left( \prod_{i=1}^{l} |Q_{i,j_i}| \left( \inf_{Q_{i,j_i}} v_{w}^{1-m} \right) \right)
\]

\[
\times \int_{\mathbb{R}^{n(m-l)}} \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \rightarrow y_{1,m}) - K(x, \rightarrow c_{(1,j_1)}, \rightarrow y_{l+1,m})| \cdot \left( \prod_{i=l+1}^{m} v_w(y_{i})^{\frac{1}{m}} \right) dx dy_{l+1,m}
\]

\[
\leq [v_w]_{A_1}^{2m} \sum_{j_1, \ldots, j_l=1}^{\infty} \left( \prod_{i=1}^{l} v_{w}^{\frac{1}{m}}(Q_{i,j_i}) \right)
\]

\[
\times \int_{\mathbb{R}^{n(m-l)}} \sup_{(y_1, \ldots, y_l)} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \rightarrow y_{1,m}) - K(x, \rightarrow c_{(1,j_1)}, \rightarrow y_{l+1,m})| \cdot \left( \prod_{i=l+1}^{m} v_w(y_{i})^{\frac{1}{m}} \right) dx dy_{l+1,m}.
\]
Use Lemma 4 and property (2) to finish the estimate

\[ v_w(P_k) \lesssim \left[ v_w \right]_{A_1}^{2m+2m-2 \over m} \sum_{i=1}^l v_w(\Omega) \lesssim \left[ v_w \right]_{A_1}^{2m+2m-2 \over m} + t^{-1 \over m}. \]

The control of \( v_w(P) \) follows from Chebyshev’s inequality, construction of the sets \( E_i \), property (1), and property (4):

\[ v_w(P) \lesssim t^{-1} \int_{\mathbb{R}^n} |\sigma_l(x)| \, dx \]
\[ \lesssim t^{-1+{1 \over m}} \left( \prod_{i=1}^l v_w(E_i) \right) \left( \prod_{i=l+1}^m \left( \int_{\mathbb{R}^n} g_i(x)^m v_w(x) \, dx \right)^{1 \over m} \right) \]
\[ \lesssim t^{-1 \over m} \left( \prod_{i=1}^l \|b_i\|_{L^1(v_w)}^{1 \over m} \right) \left( \prod_{i=l+1}^m \|g_i\|_{L^1(v_w)}^{1 \over m} \right) \]
\[ \leq t^{-1 \over m}. \]

Put the estimates of \( v_w(P_k) \) and \( v_w(P) \) together to get

\[ v_w(S_s) \lesssim \left[ v_w \right]_{A_1} t^{-1 \over m} + \sum_{k=1}^l \left[ v_w \right]_{A_1}^{2m+2m-2 \over m} + t^{-1 \over m} + t^{-1 \over m} \lesssim \left[ v_w \right]_{A_1}^{2m+2m-2 \over m} + t^{-1 \over m}. \]

Finally, use the estimates of \( v_w(S_s) \), \( 1 \leq s \leq 2^m \) to complete the proof

\[ v_w \left( \left\{ \left| T \left( f_1 w_1 v_w^{1-m} \right), \ldots, f_m w_m v_w^{1-m} \right) \right| v_w^{-1} > t \right\} \leq |S_1| + \sum_{s=2}^{2^m} |S_s| \]
\[ \lesssim t^{-1 \over m} + \sum_{s=2}^{2^m} \left[ v_w \right]_{A_1}^{2m+2m-2 \over m} + t^{-1 \over m} \lesssim \left[ v_w \right]_{A_1}^{2m+2m-2 \over m} + t^{-1 \over m} . \]

\( \square \)

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