Conformal transformation route to gravity’s rainbow

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Abstract

Conformal transformation as a mathematical tool has been used in many areas of gravitational physics. In this paper, we would consider the gravity’s rainbow, in which the metric could be treated as a conformal rescaling of the original metric. By using the conformal transformation technique, we get a specific form of modified Newton’s constant and cosmological constant in gravity’s rainbow, which implies that the total vacuum energy is dependent on probe energy. Moreover, the result shows that the Einstein gravity’s rainbow could be described by an energy-dependent $f(E, \tilde{R})$ gravity. At last, we study the $f(R)$ gravity, when the gravity’s rainbow is considered, it can also be described as another energy-dependent $\tilde{f}(E, \tilde{R})$ gravity.

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I. INTRODUCTION

The classical general relativity has been proved to describe our low energy world quite well. However, recent astronomical and cosmological observations, such as the threshold anomalies of ultrahigh energy cosmic rays and TeV photons [1–6], would cause some puzzles in general relativity. To construct a semi-classical or an effective theory of quantum gravity where the Planck length might play a fundamental role without violating the principle of relativity, deformed or doubly special relativity was proposed [7–10]. This theory leads to a deformed Lorentz symmetry such that the usual energy-momentum relation or dispersion relation in special relativity may be modified with corrections in the order of Planck length.

The deformed special relativity formalism was generalized to curved spacetime by Magueijo and Smolin, whose formalism was called gravity’s rainbow [11]. It shows that there will be an energy-dependent metric. Furthermore, these lead to an energy-dependent connection, curvature and a modification to the Einstein equations. The vacuum energy depending on modified dispersion relations at high energy predicted by gravity’s rainbow was discussed in [12]. As a modified gravity theory, it has been considered in many areas and got many interesting results. Rainbow deformation of various black hole solutions have been performed [13]. In gravity’s rainbow, the uncertainty principle still holds [14, 15], which can transform the test particle energy into the radius of the event horizon. This uncertainty principle relation has been used in the black hole thermodynamics [16–18] and black hole remnants [19, 20]. Gravity’s rainbow has been also considered to study the early universe, such as the nonsingular universes [21–25] or big bounce universes [26]. Furthermore, the gravity’s rainbow was investigated in Gauss-Bonnet gravity [23, 27, 28], massive gravity [29, 30] and $f(R)$ gravity [31]. Recent research shows that the gravity’s rainbow has some connections with other gravity theories and quantum gravity. A connection between
$f(R)$ gravity and gravity’s rainbow has been firstly discussed in [25, 32]. Also, there is a correspondence of gravity’s rainbow with Hořava-Lifshitz gravity [33].

Conformal rescalings and conformal techniques as a mathematical tool have been widely used in general relativity [34], especially in the scalar-tensor theory of gravity. It often maps the equations of motion of physical systems into mathematically equivalent sets of equations that are easier to solve or more convenient to study. This situation emerges mainly in alternative theories of gravity, unified theories in multidimensional spaces. By applying a conformal transformation, problems would move from one conformal frame to another. The Jordan frame and the Einstein frame are those discussed most frequently among many conformal frames. The conformal rescaling to the nonminimal coupling case for the scaler field in Brans-Dicke theory can get the minimal coupling case of the scaler field [35, 36], but the the scaler field may couple to the matter field. Taking into account conformal transformation of Brans-Dicke theory with an electrodynamics Lagrangian, scalar field should couple with electrodynamics in dilaton gravity, which has been discussed in [37]. Physical equivalence between nonlinear gravity and a general-relativistic self-gravitating scalar field was proved by conformal technique as well [38].

In this paper, we would use the conformal transformation technique to investigate the gravity’s rainbow, in which the metric is a conformal rescaling of the original one. Through the conformal transformation, we get a specific form of modified Newton’s constant and cosmological constant in gravity’s rainbow, which implies an energy-dependent vacuum energy. Furthermore, our result shows that the Einstein gravity’s rainbow could be described as a modified $f(E, \tilde{R})$ gravity, which is energy-dependent; thus, a connection between gravity’s rainbow and $f(R)$ gravity was established. We also get the Friedmann equations in Einstein gravity’s rainbow under the modified $f(E, \tilde{R})$ gravity framework. Motived by this, $f(R)$ gravity was considered and the $f(R)$ gravity’s rainbow can be also treated as an energy-dependent $\tilde{f}(E, \tilde{R})$ gravity.
This paper is organized as follows: Sec.II is a review of the gravity’s rainbow. In Sec.III, The Einstein gravity’s rainbow is investigated by using the conformal transformation. We also consider the Friedmann equation under the modified $f(E, \tilde{R})$ gravity framework. In Sec.IV, we generalize this framework to the $f(R)$ gravity’s rainbow. Conclusions and discussions are given in Sec.V.

II. REVIEW OF GRAVITY’S RAINBOW

Deformed or doubly special relativity is a theory that implies a modified set of principles of special relativity [7, 8]. As a result, the invariant of energy and momentum in general may be modified to

$$E^2 f_1^2(E) - p^2 f_2^2(E) = m_0^2,$$  \hspace{1cm} (1)

where the two general functions $f_1^2(E)$ and $f_2^2(E)$ depend on the energy of probes $E$. The correspondence principle requires that $f_1^2(E), f_2^2(E) \rightarrow 1$ as $E \ll 1$ with the Planck scale $E_{Pl} = 1$. To make sure the contraction between infinitesimal displacement and momentum is a linear invariant [11, 39]

$$dx^\mu p_\mu = dt E + dx^i p_i,$$  \hspace{1cm} (2)

the usual flat metric should be replaced by the rainbow metric defined as

$$ds^2 = -\frac{1}{f_1^2(E)} dt^2 + \frac{1}{f_2^2(E)} dx^2.$$  \hspace{1cm} (3)

The flat rainbow metric indicates that the geometry of spacetime depends on the energy of probes. That is to say, the geometry of spacetime probed by a particle with energy $E$ can be described by an energy-dependent orthonormal frame fields

$$\tilde{e}_0(E) = \frac{1}{f_1(E)} e_0, \quad \tilde{e}_i(E) = \frac{1}{f_2(E)} e_i.$$  \hspace{1cm} (4)
where $e_0$ and $e_i$ represent the energy-independent frame fields. Therefore, the flat rainbow metric can be written as

$$g(E) = \eta^{\mu\nu}\tilde{e}_\mu(E) \otimes \tilde{e}_\nu(E).$$  \quad (5)$$

The rainbow metric formalism can be generalized when the gravity is taken into account. It may lead to the fact that connection $\nabla_\mu(E)$ and curvature tensor $R^\sigma_{\mu\nu\lambda}(E)$ are all energy-dependent. One can also define an energy-dependent energy-momentum tensor $T_{\mu\nu}(E)$. Then the Einstein equations should be replaced by a one parameter family of equations

$$G_{\mu\nu}(E) = 8\pi G(E)T_{\mu\nu}(E) + g_{\mu\nu}(E)\Lambda(E),$$  \quad (6)$$

where the Newton’s constant $G(E)$ and cosmological constant $\Lambda(E)$ is conjectured to be energy-dependent. Furthermore, these are assumed as $G(E) = h_1^2(E)G$ and $\Lambda(E) = h_2^2(E)\Lambda$ from the view point of renormalization group theory [11, 21].

III. EINSTEIN GRAVITY’S RAINBOW

In this section we would consider the relation between Einstein gravity and Einstein gravity’s rainbow. For simplicity, we start with the static spherically symmetric metric. We may write the rainbow metric in either energy-dependent coordinates or energy-independent coordinates. Generally, in energy-dependent coordinates the static spherically symmetric metric takes the form

$$d\tilde{s}^2 = -A(\tilde{r}(E), E)d\tilde{t}(E)^2 + B(\tilde{r}(E), E)d\tilde{r}(E)^2 + \tilde{r}(E)^2d\Omega^2.$$  \quad (7)$$

In terms of energy independent coordinates the general form for a static spherically symmetric metric is [11]

$$d\tilde{s}^2 = -\frac{A(r)}{f_1^2(E)}dt^2 + \frac{B(r)}{f_2^2(E)}dr^2 + \frac{r^2}{f_2^2(E)}d\Omega^2.$$  \quad (8)$$
In the limit of $E \ll 1$, this rainbow metric would reduce to

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega^2 \quad , \tag{9} \]

which is the case in Einstein gravity.

Conformal rescalings and conformal techniques as a mathematical tool have been widely used in general relativity for a long time [34]. It often maps the equations of motion of physical systems into mathematically equivalent sets of equations that are more easily solved or more convenient to study. A well-known example is that the nonlinear gravity theories in Jordan frame could be equivalent to the Einstein frame with a scalar field under the conformal transformation [38, 40]. For the gravity’s rainbow, an interesting case of the rainbow function is [7]

\[ f_1 = f_2 = \frac{1}{1 + \lambda E} \quad , \tag{10} \]

which would not produce a varying speed of light $c$. In this case, the rainbow metric is just a conformal rescaling by an energy-dependent function

\[ ds^2 = \frac{1}{f_2^2(E)} ds^2 \quad . \tag{11} \]

To consider the Einstein gravity’s rainbow, let us start with the Einstein-Hilbert action

\[ S = \frac{1}{16\pi G} \int \sqrt{-g}(R + 2\Lambda)d^4x + S_M \quad , \tag{12} \]

where $\Lambda$ is the cosmological constant and $S_M$ represents the matter field action. It is possible to derive an action in gravity’s rainbow framework under the conformal transformation

\[ \tilde{g}_{\mu\nu} = f_2^{-2}(E)g_{\mu\nu} \quad , \tag{13} \]

where $\tilde{g}_{\mu\nu}$ represents the rainbow metric and $g_{\mu\nu}$ is the original metric. As $f_2(E)$ is independent of the coordinates $\{t, r, \theta, \phi\}$, one can get the following relations

\[ R = f_2^{-2}(E)\tilde{R}, \quad \sqrt{-g} = f_2^4(E)\sqrt{-\tilde{g}} \quad . \tag{14} \]
Then one can get the action in gravity’s rainbow framework

\[
\bar{S} = \frac{1}{16\pi G} \int \sqrt{-\bar{g}} f_2^4(E)(f_2^{-2}(E)\bar{R} + 2\Lambda)d^4x + \bar{S}_M \quad .
\]

(15)

Varying this action with respect to \( \bar{g}_{\mu\nu} \), one obtains

\[
\bar{R}_{\mu\nu}(E) - \frac{1}{2} \bar{R}(E)\bar{g}_{\mu\nu} = f_2^2(E)\Lambda\bar{g}_{\mu\nu} + 8\pi G f_2^{-2}(E)\bar{T}_{\mu\nu} \quad .
\]

(16)

This equation would be consistent with eq.(6), if we set

\[
G(E) = G f_2^{-2}(E), \quad \Lambda(E) = \Lambda f_2^2(E) \quad .
\]

(17)

This result agrees with the previous assumption about \( G(E) \) and \( \Lambda(E) \) [11, 21].

Moreover, the energy-dependent Newton’s constant and cosmological constant would imply an energy-dependent total vacuum energy \( \tilde{E}_{\text{vac}}(E) \) rather than a constant one [13]. The vacuum energy including a dependence on gravity’s rainbow has been discussed in [12]. In fact, giving a rainbow metric as eq.(8), we note that in general the volume of a spatial region with fixed size \( r \) is energy-dependent as \( \tilde{V} \sim f_2^{-3}(E)V \). Then the total vacuum energy should be

\[
\tilde{E}_{\text{vac}} \sim \tilde{\rho}_{\text{vac}}\tilde{V} \sim \frac{\Lambda(E)}{8\pi G(E)} f_2^{-3}(E)V \sim f_2(E)E_{\text{vac}} \quad .
\]

(18)

In parallel, we would like to consider the gravity’s rainbow from another side. As the modified Einstein equations eq.(16) could be obtained by varying the action of Einstein gravity’s rainbow eq.(15). The Einstein gravity’s rainbow could be treated as a modified \( f(E, \bar{R}) \) gravity which is energy-dependent, if we set \( f(E, \bar{R}) = f_2^4(E)(f_2^{-2}(E)\bar{R} + 2\Lambda) \). In the model of \( f(R) \) gravity, the field equation could be written as [41, 42]

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{f_R} \left( T_{\mu\nu}^{\text{curv}} + 8\pi G T_{\mu\nu} \right) \quad ,
\]

(19)

where \( f_R = df/dR \) and \( T_{\mu\nu}^{\text{curv}} \) is curvature energy-momentum tensor defined as

\[
T_{\mu\nu}^{\text{curv}} = \frac{1}{2}g_{\mu\nu}(f - Rf_R) + (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma})\nabla^\rho\nabla^\sigma f_R \quad .
\]

(20)
Similarly, in the Einstein gravity’s rainbow, one can get $f_R = f_2^2(E)$ and $\tilde{T}^\text{curv}_{\mu\nu} = \Lambda f_2^4(E)\tilde{g}_{\mu\nu}$. Then one can get the field equation

$$
\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = \frac{1}{f_2^2(E)}\left(\Lambda f_2^4(E)\tilde{g}_{\mu\nu} + 8\pi G\tilde{T}_{\mu\nu}\right),
$$

(21)

which is the same as eq.(16). However, there is no need to introduce an energy-dependent Newton’s constant and cosmological constant in this framework.

Therefore, in the case of $f_1(E) = f_2(E)$, the Einstein gravity’s rainbow is just a conformal rescale for a static spherically symmetric metric, and it can be described by an energy-dependent $f(E, R)$ gravity. In addition, we will show that this result also holds for the case of $f_1(E) \neq f_2(E)$.

Considering the static spherically symmetric metric eq.(8), if we introduce an energy-dependent time coordinate

$$
\hat{t}(E) = \frac{f_2(E)}{f_1(E)}t,
$$

(22)

eq(8) could be written as

$$
d\hat{s}^2 = \frac{1}{f_2^2(E)}\left[-A(r)d\hat{t}^2 + B(r)dr^2 + r^2d\Omega^2\right] = \frac{1}{f_2^2(E)}d\hat{s}^2.
$$

(23)

It is also a conformal transformation of the original metric with the coordinates $\{\hat{t}, r, \theta, \phi\}$. We should point out that the energy-dependent time $\hat{t}$ won’t change the time-like killing vector, as $f_2(E)/f_1(E)$ is independent with coordinates [11]. Then the line element $d\hat{s}^2$ also satisfies the Einstein equations. In fact, when $f_1(E) \neq f_2(E)$, the speed of light $c(E) = f_2(E)/f_1(E)$ is energy-dependent [43]. It is naturally to introduce $\hat{t} = c(E)t$. Once $f_1(E) = f_2(E)$, we have $\hat{t} = t$, which would reduce to the previous case. Therefore, the above discussion also holds for $f_1(E) \neq f_2(E)$, and the price is to introduce an energy-dependent time coordinate $\hat{t}$. We would like to point out that the original time coordinate should be substituted back after all the calculations, especially for the cosmological time.
As an example, we consider the rainbow universe under modified $f(E, R)$ gravity framework. The modified flat FRW metric for the gravity’s rainbow could be expressed as [11, 21, 26]

$$ds^2 = \frac{1}{f_2^2(E)} \left[ -d\tilde{t}^2 + a^2(\tilde{t}) (dr^2 + r^2 d\Omega^2) \right].$$

(24)

With the substitution $\tilde{t} = f_2^{-1}(E) \tilde{t}$ and $\tilde{a}(\tilde{t}) = f_2^{-1}(E) a(\tilde{t})$, the modified flat FRW metric is just

$$ds^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t}) (dr^2 + r^2 d\Omega^2),$$

(25)

which should satisfy the Friedmann equations in a modified $f(E, \tilde{R})$ gravity. Generally, the Friedmann equations for $f(R)$ gravity are [40]

$$H^2 = \frac{Rf_R - f}{6f_R} - H \frac{\dot{f}_R}{f_R} + \frac{8\pi G}{3f_R} \rho,$$

(26)

$$\dot{H} = -\frac{\dot{f}_R}{2f_R} + H \frac{\dot{f}_R}{2f_R} - \frac{4\pi G}{f_R} (\rho + p).$$

(27)

For the Einstein gravity’s rainbow, we should set $f(E, \tilde{R}) = f_2^4(E)(f_2^{-2}(E)\tilde{R} + 2\Lambda)$. The energy-momentum tensor has a perfect fluid form

$$\tilde{T}_{\mu\nu} = \tilde{\rho} u_\mu u_\nu + \tilde{p} (\tilde{g}_{\mu\nu} + u_\mu u_\nu),$$

(28)

where $u_\mu$ depends on $E$ and is defined as $u_\mu = (f_2^{-1}(E), 0, 0, 0)$ with the time coordinate $\tilde{t}$, such that $\tilde{g}^{\mu\nu} u_\mu u_\nu = -1$ [11]. Thus, the modified Friedmann equations could be

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{f_2^2(E)\Lambda}{3} + \frac{8\pi G}{3f_2^2(E)} \tilde{\rho},$$

(29)

$$\frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{f_2^2(E)} (\tilde{\rho} + \tilde{p}).$$

(30)

These would produce a conservation equation

$$\dot{\tilde{\rho}} = -3\frac{\dot{a}}{a} (\tilde{\rho} + \tilde{p}).$$

(31)
If we substitute back to the cosmological time and scale factor $a$, one can get the Friedmann equations

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{f_2^2(E)\Lambda}{3f_1^2(E)} + \frac{8\pi G}{3f_1^2(E)f_2^2(E)} \tilde{\rho},$$  \hspace{0.5cm} (32)

$$\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 = -\frac{4\pi G}{f_1^2(E)f_2^2(E)}(\tilde{\rho} + \tilde{p}),$$  \hspace{0.5cm} (33)

which are consistent with the Friedmann equations derived by other methods [11, 21, 26]. They could get some interesting results with some specific rainbow functions, like no singularity cosmological solution [21] and big bounce universe [26].

Thus, the Einstein gravity's rainbow could be treated as an energy-dependent $f(E, \tilde{R}) = f_2^4(E)(f_2^{-2}(E)\tilde{R} + 2\Lambda)$ gravity model through the conformal transformation technique. This may give a convenient route to study the gravity’s rainbow.

**IV. $f(R)$ GRAVITY’S RAINBOW**

For more general cases, we consider the $f(R)$ gravity’s rainbow. Let us start with the action in $f(R)$ gravity

$$S = \frac{1}{16\pi G} \int \sqrt{-g} f(R)d^4x + S_M,$$  \hspace{0.5cm} (34)

where $S_M$ is the action of matter fields. The field equation can be derived by varying the action with respect to $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{f_R} (T_{\mu\nu}^{\text{curv}} + 8\pi G T_{\mu\nu})$$  \hspace{0.5cm} (35)

where $T_{\mu\nu}^{\text{curv}}$ is curvature energy-momentum tensor defined as eq.(20). When the gravity’s rainbow is considered, it may lead to a conformal transformation $\tilde{g}_{\mu\nu} = f_2^{-2}g_{\mu\nu}$ like the Einstein gravity’s rainbow. Noting the relation eq.(17), then the action becomes

$$\tilde{S} = \frac{1}{16\pi G} \int \sqrt{-\tilde{g}} f_2^4(E)f_2^{-2}(E)\tilde{R}d^4x + \tilde{S}_M.$$  \hspace{0.5cm} (36)
Varying this action with respect to $\tilde{g}_{\mu\nu}$, one obtains

$$
\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} = \frac{1}{\tilde{f}_2^2(E)f_{R}(\tilde{f}_2^{-2}(E)\tilde{R})} \left( \tilde{T}^{\text{curv}}_{\mu\nu} + 8\pi G \tilde{T}_{\mu\nu} \right),
$$

(37)

where

$$
\tilde{T}^{\text{curv}}_{\mu\nu} = \frac{1}{2} \tilde{f}_2^2(E)\tilde{g}_{\mu\nu} [\tilde{f}_2^2(E)f(\tilde{f}_2^{-2}(E)\tilde{R}) - \tilde{R}f_{R}(\tilde{f}_2^{-2}(E)\tilde{R})]
$$

$$
+ \tilde{f}_2^2(E)(\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} - \tilde{g}_{\mu\nu}\tilde{g}_{\rho\sigma}) \nabla^\sigma \nabla^\rho f_{R}(\tilde{f}_2^{-2}(E)\tilde{R})
$$

(38)

It ends up with an energy-dependent modified $\tilde{f}(E, \tilde{R})$ gravity. Obviously, if $f(R) = R + 2\Lambda$, it reduces to the Einstein gravity’s rainbow, and the curvature energy-momentum tensor $\tilde{T}^{\text{curv}}_{\mu\nu}$ would lead to $\Lambda(E) = \frac{f_2^2(E)}{f_1(E)}\Lambda$.

Thus, from the view point of action, the $f(R)$ gravity’s rainbow can be treated as another energy-dependent $\tilde{f}(E, \tilde{R})$ gravity, where $\tilde{f}(E, \tilde{R}) = \frac{f_2^2(E)}{f_1(E)}f(\tilde{f}_2^{-2}(E)\tilde{R})$. We should point out that the time coordinate should be $\hat{t} = \frac{f_2(E)}{f_1(E)}t$ for the case of $f_1(E) \neq f_2(E)$. So when a $f(R)$ gravity’s rainbow was considered, it is convenient to turn to a corresponding $\tilde{f}(E, \tilde{R})$ gravity. This would open up a door to study the $f(R)$ gravity’s rainbow. As an interesting subject, $f(R)$ theories have been applied to dark energy [44, 45]. In this way, when the $\tilde{f}(E, \tilde{R})$ gravity is considered, it may lead to an energy-dependent dark energy which might explain the late-time cosmic acceleration [46].

V. CONCLUSIONS AND DISCUSSIONS

In summary, we investigated the gravity’s rainbow which was proposed by Magueijo and Smolin, and found that the rainbow metric is just a conformal rescaling of the the original one for the case of $f_1(E) = f_2(E)$. Motivated by this, we introduced the conformal transformation technique to study the gravity’s rainbow. For the Einstein gravity’s rainbow, by using the conformal transformation
technique, we give the action of Einstein gravity’s rainbow. Varying this action, we got a specific form of modified Newton’s constant and cosmological constant which agrees with the renormalization group theory. And these would imply an energy-dependent total vacuum energy rather than a constant one. In addition, for the case of $f_1(E) \neq f_2(E)$, there also exists a conformal transformation with an energy-dependent time coordinate $\hat{t} = f_2(E)/f_1(E)t$, and the above results also hold.

From the viewpoint of action, we found that the Einstein gravity’s rainbow could be described by an energy-dependent $f(E, \tilde{R})$ gravity, where $f(E, \tilde{R}) = f_2^4(E)(f_2^{-2}(E)\tilde{R} + 2\Lambda)$. The $f(E, \tilde{R})$ gravity might provide an effective route to research the gravity’s rainbow. A connection between gravity’s rainbow and $f(R)$ theory was established. Moreover, under $f(E, \tilde{R})$ gravity framework, we investigated the dynamic equations of cosmology. For the FRW cosmology, the same Friedmann equations were obtained and these equations could get some interesting cosmology solutions.

For a more general case, we considered the $f(R)$ gravity’s rainbow and found that it can also be described by an energy-dependent $\tilde{f}(E, \tilde{R})$ gravity where $\tilde{f}(E, \tilde{R}) = f^4_2(E)f(f_2^{-2}(E)\tilde{R})$. When $f(R) = R + 2\Lambda$, it can reduce to the Einstein gravity’s rainbow and the curvature energy-momentum tensor of $f(E, \tilde{R})$ leads to the modified cosmological constant. As many dark energy models were presented in $f(R)$ gravity, when the gravity’s rainbow is considered, the dark energy or vacuum energy might depend on the energy of probe [12]. These would be left for further research in our later works.
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[1] D. Colladay and V.A. Kostelecký. Lorentz-violating extension of the standard model. *Phys. Rev. D*, 58:116002, Oct 1998.

[2] S. Coleman and S.L. Glashow. High-energy tests of Lorentz invariance. *Phys. Rev. D*, 59:116008, Apr 1999.

[3] G. Amelino-Camelia and T. Piran. Planck-scale deformation of Lorentz symmetry as a solution to the ultrahigh energy cosmic ray and the TeV-photon paradoxes. *Phys. Rev. D*, 64:036005, Jul 2001.

[4] T. Jacobson, S. Liberati, and D. Mattingly. TeV astrophysics constraints on Planck scale Lorentz violation. *Phys. Rev. D*, 66:081302, Oct 2002.

[5] R.C. Myers and M. Pospelov. Ultraviolet Modifications of Dispersion Relations in Effective Field Theory. *Phys. Rev. Lett.*, 90:211601, May 2003.

[6] T. Jacobson, S. Liberati, D. Mattingly, and F.W. Stecker. New Limits on Planck Scale Lorentz Violation in QED. *Phys. Rev. Lett.*, 93:021101, Jul 2004.

[7] J. Magueijo and L. Smolin. Lorentz Invariance with an Invariant Energy Scale. *Phys. Rev. Lett.*, 88:190403, Apr 2002.

[8] J. Magueijo and L. Smolin. Generalized Lorentz invariance with an invariant energy scale. *Phys. Rev. D*, 67:044017, Feb 2003.

[9] G. Amelino-Camelia. Testable scenario for relativity with minimum length. *Phys. Lett. B*, 510(1):255 – 263, 2001.
[10] G. Amelino-camelia. Relativity in spacetimes with short-distance structure governed by an observer-independent (Planckian) length scale. *Int. J. Mod. Phys. D*, 11(01):35–59, 2002.

[11] J. Magueijo and L. Smolin. Gravity’s rainbow. *Class. Quant. Grav.*, 21(7):1725, 2004.

[12] R. Garattini and G. Mandanici. Modified dispersion relations lead to a finite zero point gravitational energy. *Phys. Rev. D*, 83:084021, Apr 2011.

[13] H. Li, Y. Ling, and X. Han. Modified (A)dS Schwarzschild black holes in rainbow spacetime. *Class. Quantum Gravit.*, 26(6):065004, 2009.

[14] S. Gangopadhyay, A. Dutta, and M. Faizal. Constraints on the generalized uncertainty principle from black-hole thermodynamics. *EPL*, 112(2):20006, 2015.

[15] G. Amelino-Camelia, M. Arzano, Y. Ling, and G. Mandanici. Black-hole thermodynamics with modified dispersion relations and generalized uncertainty principles. *Class. Quantum Gravit.*, 23(7):2585, 2006.

[16] Y. Ling, X. Li, and H. Zhang. Thermodynamics of modified black holes from gravity’s rainbow. *Mod. Phys. Lett. A*, 22(36):2749–2756, 2007.

[17] P. Galán and G.A.M. Marugán. Entropy and temperature of black holes in a gravity’s rainbow. *Phys. Rev. D*, 74:044035, Aug 2006.

[18] S.H. Hendi, M. Faizal, B.E. Panah, and S. Panahiyan. Charged dilatonic black holes in gravity’s rainbow. *Eur. Phys. J. C*, 76(5):296, May 2016.

[19] A.F. Ali. Black hole remnant from gravity’s rainbow. *Phys. Rev. D*, 89:104040, May 2014.

[20] Ronald J. Adler, Pisin Chen, and David I. Santiago. The generalized uncertainty principle and black hole remnants. *Gen. Relativ. Gravit.*, 33(12):2101–2108, Dec 2001.

[21] Y. Ling. Rainbow universe. *JCAP*, 2007(08):017, 2007.

[22] A. Awad, A.F. Ali, and B. Majumder. Nonsingular rainbow universes. *JCAP*, 2013(10):052, 2013.
[23] S.H. Hendi, M. Momennia, B.E. Panah, and M. Faizal. Nonsingular universes in Gauss-Bonnet gravity’s rainbow. *ApJ*, 827(2):153, 2016.

[24] B. Majumder. Singularity free rainbow universe. *Int. J. Mod. Phys. D*, 22(12):1342021, 2013.

[25] G.J. Olmo. Palatini actions and quantum gravity phenomenology. *JCAP*, 2011(10):018, 2011.

[26] Y. Ling and Q. Wu. The big bounce in rainbow universe. *Phys. Lett. B*, 687(2):103 – 109, 2010.

[27] S.H. Hendi and M. Faizal. Black holes in Gauss-Bonnet gravity’s rainbow. *Phys. Rev. D*, 92:044027, Aug 2015.

[28] S.H. Hendi, S. Panahiyan, B.E. Panah, M. Faizal, and M. Momennia. Critical behavior of charged black holes in Gauss-Bonnet gravity’s rainbow. *Phys. Rev. D*, 94:024028, Jul 2016.

[29] S.H. Hendi, B.E. Panah, and S. Panahiyan. Topological charged black holes in massive gravity’s rainbow and their thermodynamical analysis through various approaches. *Phys. Lett. B*, B769:191–201, 2017.

[30] S.H. Hendi, S. Panahiyan, S. Upadhyay, and B.E. Panah. Charged BTZ black holes in the context of massive gravity’s rainbow. *Phys. Rev. D*, 95:084036, Apr 2017.

[31] S.H. Hendi, B.E. Panah, S. Panahiyan, and M. Momennia. $F(R)$ gravity’s rainbow and its Einstein counterpart. *Adv. High Energy Phys.*, 2016:9813582, 2016.

[32] R. Garattini. Distorting general relativity: gravity’s rainbow and $f(R)$ theories at work. *JCAP*, 2013(06):017, 2013.

[33] R. Garattini and E.N. Saridakis. Gravity’s Rainbow: a bridge towards Hořava-Lifshitz gravity. *Eur. Phys. J. C*, 75(7):343, Jul 2015.

[34] V. Faraoni, E. Gunzig, and P. Nardone. Conformal transformations in classical gravitational theories and in cosmology. *Fund. Cosmic Phys.*, 20:121, 1999.
[35] R.H. Dicke. Mach’s Principle and Invariance under Transformation of Units. *Phys. Rev.*, 125:2163–2167, Mar 1962.

[36] M.P. Dabrowski, J. Garecki, and D.B. Blaschke. Conformal transformations and conformal invariance in gravitation. *Ann. der Phys.*, 18(1):13–32, 2009.

[37] S.H. Hendi and M.S. Talezadeh. Nonlinearly charged dilatonic black holes and their Brans-Dicke counterpart: energy dependent spacetime. *Gen. Relativ. Gravit.*, 49(1):12, Dec 2016.

[38] G. Magnano and L.M. Sokołowski. Physical equivalence between nonlinear gravity theories and a general-relativistic self-gravitating scalar field. *Phys. Rev. D*, 50:5039–5059, Oct 1994.

[39] D. Kimberly, J. Magueijo, and J. Medeiros. Nonlinear relativity in position space. *Phys. Rev. D*, 70:084007, Oct 2004.

[40] A. De Felice and S. Tsujikawa. $f(R)$ Theories. *Living Rev. Rel.*, 13(1):3, 2010.

[41] S. Capozziello and A. De Felice. $f(R)$ cosmology from Noether’s symmetry. *JCAP*, 2008(08):016, 2008.

[42] T.P. Sotiriou and V. Faraoni. $f(R)$ theories of gravity. *Rev. Mod. Phys.*, 82:451–497, Mar 2010.

[43] S. Alexander and J. Magueijo. Non-commutative geometry as a realization of varying speed of light cosmology. *arXiv preprint hep-th/0104093*, 2001.

[44] L. Amendola, D. Polarski, and S. Tsujikawa. Are $f(R)$ Dark Energy Models Cosmologically Viable? *Phys. Rev. Lett.*, 98:131302, Mar 2007.

[45] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa. Conditions for the cosmological viability of $f(R)$ dark energy models. *Phys. Rev. D*, 75:083504, Apr 2007.

[46] S.M. Carroll, V. Duvvuri, M. Trodden, and M.S. Turner. Is cosmic speed-up due to new gravitational physics? *Phys. Rev. D*, 70:043528, Aug 2004.