Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions*

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Abstract
The aim of this paper is to bring into the picture a new phenomenon in the theory of orthogonal matrix polynomials satisfying second order differential equations. The last few years have witnessed some examples of a (fixed) family of orthogonal matrix polynomials whose elements are common eigenfunctions of several linearly independent second order differential operators. We show that the dual situation is also possible: there are examples of different families of matrix polynomials, each family orthogonal with respect to a different weight matrix, whose elements are eigenfunctions of a common second order differential operator.

These examples are constructed by adding a discrete mass at certain point to a weight matrix: \( \tilde{W} = W + \delta_{t_0} M(t_0) \). Our method consists in showing how to choose the discrete mass \( M(t_0) \), the point \( t_0 \) where the mass lives and the weight matrix \( W \) so that the new weight matrix \( \tilde{W} \) inherits some of the symmetric second order differential operators associated with \( W \). It is well known that this situation is not possible for the classical scalar families of Hermite, Laguerre and Jacobi.

1 Introduction

The theory of matrix valued orthogonal polynomials starts with two papers by M. G. Krein in 1949, see [K1, K2]. A sequence of orthonormal matrix polynomials \( (P_n)_n \) can be characterized as solutions of the difference equation

\[
t P_n(t) = A_{n+1} P_{n+1}(t) + B_n P_n(t) + A_n^* P_{n-1}(t), \quad n = 0, 1, \ldots,
\]

where \( A_n \) and \( B_n \) are, respectively, \( N \times N \) nonsingular and Hermitian matrices, and initial conditions \( P_{-1} = 0 \) and \( P_0 \) nonsingular. Each family \( (P_n)_n \) goes along with a weight matrix \( W \) and satisfies \( \int P_n dW P_m^* = \delta_{n,m} I \).

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Nevertheless, more than 50 years have been necessary to produce the first examples of orthogonal matrix polynomials \((P_n)_n\) satisfying second order differential equations of the form

\[
P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0 = \Gamma_n P_n(t), \quad n = 0, 1, \ldots.
\]

Here \(F_2, F_1\) and \(F_0\) are matrix polynomials (which do not depend on \(n\)) of degrees less than or equal to 2, 1 and 0, respectively (see [DG1, GPT2]). Two main methods have been developed in the last five years to produce such examples: solving an appropriate set of differential equations (see [D2, DG1, DG4, DdI]) or coming from the study of matrix valued spherical functions (see [GPT1, GPT3, PT]). These families of orthogonal matrix polynomials are among those that are likely to play in the case of matrix orthogonality the role of the classical families of Hermite, Laguerre and Jacobi in the case of scalar orthogonality. The complexity of the matrix world has however led to an embarrassment of riches if we compare with these very few scalar examples (see, for instance, the papers cited above).

We point out here that the second order differential equation (1.1) for the orthonormal polynomials \((P_n)_n\) is equivalent to the symmetry of the second order differential operator

\[
D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0, \quad \partial = \frac{d}{dt},
\]

with respect to the weight matrix \(W\), as far as the eigenvalues \(\Gamma_n\) are Hermitian (see [D1]). The symmetry of \(D\) with respect to \(W\) is defined by \(\int (PD) dW Q^* = \int P dW (QD)^*\), for any matrix polynomials \(P\) and \(Q\). Here and in the rest of the paper we will follow the notation in [GT] for right-hand side differential operators. In particular, if \(P\) is a matrix polynomial and \(D\) a differential operator as (1.2), by \(PD\) we mean

\[
PD = P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0.
\]

As more families of orthogonal matrix polynomials satisfying second order differential equations become available many new and certainly interesting phenomena are being discovered. These phenomena are absent in the well known scalar theory. One of these phenomena is that the elements of a family of orthogonal polynomials \((P_n)_n\) can be common eigenfunctions of several linearly independent second order differential operators. See [CG, D2, DdI, DL, GdI] for some examples illustrating this phenomenon.

The purpose of this paper is to show what one can call the dual situation to that described in the previous paragraph. It turns out that there also exist second order differential operators as (1.2) having the elements of infinitely many different families of orthogonal matrix polynomials as eigenfunctions. More precisely, we will show examples of a fixed second order differential operator \(D\) as (1.2) for which there exist infinitely many weight matrices \(W_\gamma, \gamma \geq 0\), such that their corresponding monic orthogonal matrix polynomials \((P_{n,\gamma})_n\) are eigenfunctions of \(D\) (with equal eigenvalues)

\[
P_{n,\gamma}D = \Gamma_n P_{n,\gamma}, \quad n = 0, 1, \ldots, \quad \gamma \geq 0,
\]

where \(D\) and \(\Gamma_n\) do not depend on \(\gamma\).
We give a simple but fruitful method to find such annoying examples (Section 2) and show a collection of instructive examples (Section 3).

Our method itself is again a surprise if one compares with the situation in the scalar case. We first take a weight matrix $W$ having several linearly independent symmetric second order differential operators. And then we add to $W$ a Dirac distribution $\delta_0 M(t_0)$, where the real number $t_0$ and the mass $M(t_0)$ (a Hermitian positive semidefinite matrix) are carefully chosen. We show in Section 2 that, fixed $t_0$, we can produce, under certain mild hypothesis, a second order differential operator $D$ symmetric with respect to $W$ and a positive semidefinite matrix $M(t_0)$ such that $D$ is also symmetric with respect to any weight matrix of the form $W + \gamma \delta_0 M(t_0)$, $\gamma > 0$. We illustrate with some examples, in Section 3, that the choice of the point $t_0$ is not related to the support of $W$ but to certain properties of the differential coefficients $F_2(t), F_1(t)$ and $F_0$ evaluated at $t_0$.

The situation is absolutely different to that of the scalar case. When we add a mass point to any of the classical weights of Hermite, Laguerre and Jacobi, the existence of a symmetric second order differential operator automatically disappears. Only when $t_0$ is taken at the endpoints of the support, one eventually gets the symmetry of a fourth (or even larger) order differential operator which, of course, is not symmetric with respect to the original weight. This arises for the particular cases of the Laguerre weight $e^{-t}$ in $(0, +\infty)$, for the Legendre weight $1$ in $(-1, 1)$ and for the special case of the Jacobi weight $(1 - t)^\alpha$ in $(0, 1)$, raising the so called Laguerre type weight $e^{-t} + M \delta_0$, Legendre type weight $1 + M(\delta_{-1} + \delta_1)$ and Jacobi type weight $(1 - t)^\alpha + M \delta_0$, respectively (see [LK]).

For the benefit of the reader, we conclude this Introduction displaying one of our examples. Consider the weight matrix

$$W_a(t) = e^{-t^2} \left( \frac{1 + a^2 t^2}{t} \right), \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}.$$ 

The linear space of differential operators of order at most two having the orthogonal polynomials with respect to $W_a$ as eigenfunctions has dimension five. A basis is formed by the identity and four linearly independent operators of order two (see Section 6 of [CG]). We show in Section 3.1 that the weight matrices $W_{a,\gamma} = W_a + \gamma \delta_0 \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$, $\gamma \geq 0$, share the following symmetric second order differential operator:

$$D_a = \partial^2 \left( \begin{array}{cc} 1 - at & -1 + a^2 t^2 \\ -1 & 1 + at \end{array} \right) + \partial^1 \left( \begin{array}{cc} -2a - 2t & 2a + 2(2 + a^2)t \\ 0 & -2t \end{array} \right) + \partial^0 \left( \begin{array}{c} -1 \\ 4 \\ \frac{2 + a^2}{a^2} \end{array} \right).$$

As we pointed out above, this means that each orthogonal monic family $(P_{n,a,\gamma})_n$, $\gamma \geq 0$, with respect to $W_{a,\gamma}$ satisfies the same second order differential equation, namely

$$P_{n,a,\gamma} D_a = \Gamma_{n,a} P_{n,a,\gamma}, \quad n = 0, 1, \ldots,$$
where
\[
\Gamma_{n,a} = \begin{pmatrix}
-(2n + 1) & \frac{(2 + na^2)(2 + (n + 1)a^2)}{a^2} \\
\frac{4}{a^2} & -2n + 1
\end{pmatrix}.
\]
Notice that neither \(D_a\) nor \(\Gamma_{n,a}\) depend on \(\gamma\).

We will also see in Section 3.1 that the choice of the point \(t_0\) where the discrete mass is added can be located at any real number. There always exists a Hermitian positive semidefinite matrix \(M(t_0)\) such that the weight matrices \(W_a + \gamma \delta(t_0) M(t_0)\), \(\gamma \geq 0\), share a common symmetric second order differential operator (see (3.2)).

2 The main result

This section is devoted to present a set of constraints to guarantee the symmetry of a differential operator of any order with respect to a weight matrix modified by adding a Dirac distribution at an arbitrary point. Before that, we need some definitions and previous results.

Throughout this paper \(W\) will denote an \(N \times N\) weight matrix supported in the real line. Then, we can consider for any pair of matrix polynomials \(P\) and \(Q\) the Hermitian sesquilinear form
\[
\langle P, Q \rangle = \int P(t) dW(t) Q^*(t),
\]
where \(Q^*(t)\) denotes the conjugate transpose of \(Q(t)\).

As far as \(W\) is non-degenerate, we can produce a sequence of orthogonal matrix polynomials \((P_n)_n\) with \(\deg P_n = n\) and nonsingular leading coefficient such that \(\langle P_n, P_m \rangle = \Delta_n \delta_{n,m}\), with \(\Delta_n\) nonsingular. If \(\Delta_n = I\), we say that the sequence \((P_n)_n\) is orthonormal.

The non-degenerateness of \(W\) can be defined in several ways. For instance, by assuming that \(\int P(t) dW(t) P^*(t)\) is nonsingular if the leading coefficient of the matrix polynomial \(P\) is nonsingular. A simple way to check that \(W\) is non-degenerate is the following: consider the absolutely continuous part of \(W\) with respect to the Lebesgue measure and write \(W(t)\) for its Radon-Nikodym derivative. Then \(W\) is non-degenerate if \(W(t)\) is positive definite in an interval of the real line (for a much more complete introduction to matrix orthogonality, see [DG3] and references therein).

If one is considering possible applications of orthogonal matrix polynomials, it is natural to concentrate on those cases where some additional property holds. For instance, in [D1] one of the authors raised the problem of characterizing weight matrices whose orthonormal matrix polynomials are common eigenfunctions of some symmetric right-hand side second order differential operator \(D\) as (1.2) with Hermitian eigenvalues. We say that a differential operator \(D\) is symmetric with respect to a weight matrix \(W\) if
\[
\langle PD, Q \rangle = \langle P, QD \rangle
\]
for any pair of matrix polynomials \(P\) and \(Q\). We recall that we are following the notation in [GT] for right-hand side differential operators. We already mentioned in the Introduction
that in the last few years a large class of families of weight matrices $W$ has been found having symmetric second order differential operators as (1.2).

The condition of symmetry for the pair made up of a weight matrix $W$ and a differential operator $D_k$ of order $k$ can be established in terms of a set of difference and differential equations relating $W$ and the coefficients of $D_k$. Indeed, if we write the right-hand side differential operator $D_k$ of order $k$ as

$$D_k = \sum_{i=0}^{k} \partial^i F_i(t), \quad \partial = \frac{d}{dt},$$

where $F_i(t), i = 0, \ldots, k,$ are matrix polynomials of degree less than or equal to $i$,

$$F_i(t) = \sum_{j=0}^{i} t^j F^j_i, \quad F^j_i \in \mathbb{C}^{N \times N},$$

and denote by $\mu_n, n = 0, 1, \ldots$, the moments of the weight matrix $W$, i.e., $\mu_n = \int t^n dW(t)$, then we have the following

**Theorem 2.1.** For a weight matrix $W$ the following two conditions are equivalent:

1. The operator $D_k$ is symmetric with respect to $W$.
2. For $n \geq l$, the following $k + 1$ sets of moments equations hold

$$\sum_{i=0}^{k-l} \binom{k-i}{l} (n-l)_{k-l-i} B^k_n = (-1)^l (B^l_n)^*, \quad l = 0, \ldots, k,$$

where

$$B^l_n = \sum_{i=0}^{l} F^l_{i-\mu_n-i}, \quad l = 0, \ldots, k.$$

Moreover, suppose the weight matrix $W = W(t)dt$ has a smooth density $W(t)$ with respect to the Lebesgue measure which satisfies the boundary conditions that

$$\sum_{i=0}^{p-1} (-1)^{k-i+p-1} \binom{k-i}{l} (F_{k-i} \cdot W)^{(p-1-i)}, \quad p = 1, \ldots, k, \quad l = 0, \ldots, k - p,$$

should have vanishing limits at each of the endpoints of the support of $W$, and the following $k + 1$ matrix differential equations hold

$$\sum_{i=0}^{k-l} (-1)^{k-i} \binom{k-i}{l} (F_{k-i} \cdot W)^{(k-i-l)} = W \cdot F^*_l, \quad l = 0, \ldots, k.$$

Then the differential operator $D_k$ (defined in (2.1)) is symmetric with respect to the weight matrix $W$. 

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(In (2.2) we are using the well known notation for the falling or bounded factorial \((x)_n\) defined by \((x)_n = x(x-1)\cdots (x-n+1)\) for \(n > 0\), \((x)_0 = 1\).)

**Proof:** The first and second part are shown to be equivalent in Proposition 4 in [DdI]. Likewise, last part can be found in Theorem 5 in [DdI], using integration by parts (taking into account the boundary conditions (2.3)).

In particular, the symmetry equations (2.4) for differential operators of order two appeared for the first time in [D1] when \(F_0 W = WF_0^*\) and later in [DG1] and [GPT2]. As we remarked in the Introduction, most of the examples which appeared in the last years have been obtained solving these equations, while some others came from group representation theory. **Moments equations** (2.2) turn out to become a suitable tool in all examples throughout this article.

We now present the result that we will use to generate examples of second order differential operators having infinitely many families of orthogonal matrix polynomials as eigenfunctions. The idea is to find certain constraints which guarantee the symmetry of a differential operator with respect to both \(W\) and a new weight matrix obtained from \(W\) by adding of a Dirac distribution at one point.

Let \(W\) be a weight matrix and consider the following weight matrix

(2.5) \[ \tilde{W}(t) = W(t) + \delta_{t_0}(t)M(t_0), \]

where \(\delta_{t_0}(t) = \delta(t - t_0)\) is the Dirac delta distribution or the “impulse symbol” introduced by P. A. M. Dirac in [D], which we will consider as a measure. The Hermitian positive semidefinite matrix \(M(t_0)\) depends on the point where the Dirac distribution is added.

Weight matrices of the form (2.5) were considered in [YMP1, YMP2] (to study asymptotic properties of the corresponding modified Jacobi matrix) for a weight matrix \(W\) in the Nevai class, i.e., with convergent recurrence coefficients.

The moments of \(\tilde{W}\) are related with the moments of \(W\) by the formula

\[ \tilde{\mu}_n = \int t^n d\tilde{W}(t) = \mu_n + \int t^n \delta_{t_0}(t)M(t_0)dt = \mu_n + t_0^n M(t_0), \quad n = 0, 1, \ldots. \]

Observe that in the special case of \(t_0 = 0\) the only modified moment is the first one \(\tilde{\mu}_0 = \mu_0 + M(0)\), and then \(\tilde{\mu}_n = \mu_n\) for \(n = 1, 2, \ldots\).

The following theorem gives conditions for the symmetry of a differential operator \(D_k\) with respect to the weight matrices \(W\) and \(W + \delta_{t_0}M(t_0)\).

**Theorem 2.2.** Let \(D_k\) be a differential operator of order \(k\) as in (2.1). Let \(W\) be a weight matrix. Assume that associated with the real point \(t_0 \in \mathbb{R}\) there exists a Hermitian positive semidefinite matrix \(M(t_0)\) satisfying

(2.6) \[ F_j(t_0)M(t_0) = 0, \quad j = 1, \ldots, k, \]

\[ F_0 M(t_0) = M(t_0)F_0^*. \]

Then the operator \(D_k\) is symmetric with respect to \(W\) if and only if it is symmetric with respect to \(\tilde{W} = W + \delta_{t_0}M(t_0)\).
Proof: Recalling definitions around (2.2) for \( \tilde{W} = W + \delta t_0 M(t_0) \), we produce
\[
\bar{B}_n^l = \sum_{i=0}^{l} F_{l-i} \mu_{n-i} = B_n^l + t_0^{n-l} F_l(t_0) M(t_0), \quad l = 0, \ldots, k.
\]
Using conditions (2.6) for \( j = 1, \ldots, k \), we obtain
\[
\bar{B}_n^0 = B_n^0 + t_0^0 F_0 M(t_0), \quad \bar{B}_n^l = B_n^l, \quad l = 1, \ldots, k.
\]
Consequently, this shows that equations (2.2), \( l = 1, \ldots, k \), are just the same for \( W \) and \( \tilde{W} \). For \( l = 0 \), equations (2.2) for \( W \) and \( \tilde{W} \) are, respectively:
\[
\sum_{i=0}^{k-1} (n)_{k-i} B_n^{k-i} + B_n^0 = (B_n^0)^*,
\]
\[
\sum_{i=0}^{k-1} (n)_{k-i} B_n^{k-i} + t_0^n F_0 M(t_0) = (B_n^0)^* + t_0^n M(t_0) F_0^*.
\]
The last condition in (2.6) shows again that those equations are the same for \( W \) and \( \tilde{W} \).

We emphasize that Theorem 2.2 for \( N = 1 \), i.e., for scalar orthogonality, implies that either \( M = 0 \) or there exists a common zero for all coefficients of the differential operator. For instance, for \( k = 2 \), the existence of that common zero is not possible for the classical families of Hermite, Laguerre and Jacobi.

Let us notice that once we generate \( W, D_k, t_0 \) and \( M(t_0) \) satisfying constraints (2.6) of the theorem above, we can produce not only one single weight matrix for which \( D_k \) is symmetric, but also infinitely many different weight matrices for which \( D_k \) is symmetric as well. If Theorem 2.2 fulfills for \( W + \delta t_0 M(t_0) \) then automatically also fulfills for \( W + \gamma \delta t_0 M(t_0) \) where \( \gamma \) is any positive real number.

For a weight matrix \( W \), the constraints (2.6) mean that by adding a Dirac distribution to \( W \), the chance that \( W \) and \( W + \delta t_0 M(t_0) \) share a symmetric differential operator of order \( k \) increases with the number of linearly independent symmetric differential operators of order \( k \) for \( W \). In fact, all the examples we show in the next section are built from a weight matrix \( W \) having several linearly independent second order differential operators. Sometimes we can add the Dirac distribution at any real number: in our examples, this happens when the number of linearly independent symmetric second order differential operators is four. Sometimes we lose this freedom and we have to add the Dirac distribution at an specific real number: in our examples this happens when the number of linearly independent symmetric second order differential operators is just two. In this case, we have to locate the mass at one of the endpoints of the support. Both kind of examples reveal that the choice of the point \( t_0 \) where the Dirac distribution is located depends more on the matrices \( F_2(t_0), F_1(t_0) \) and \( F_0 \) than on the support of \( W \).
3 Examples

In this section we exhibit a collection of instructive examples. The first three examples of $(2 \times 2)$ weight matrices $W$ (supported in $(-\infty, +\infty)$, $(0, +\infty)$ and $(0, 1)$, respectively) have the property that they provide four linearly independent second order differential operators having a fixed family of orthogonal matrix polynomials with respect to $W$ as eigenfunctions. As we mentioned above, for any real number $t_0$ we can find a positive semidefinite matrix $M(t_0)$ and a symmetric second order differential operator $D$ as in (1.2) satisfying constraints (2.6). According to Theorem 2.2, the operator $D$ will be also symmetric with respect to $W + \gamma \delta_{t_0} M(t_0)$, $\gamma \geq 0$.

The last two examples deal with $(N \times N)$ weight matrices $W$ (supported in $(0, +\infty)$ and $(0, 1)$, respectively) with at least two linearly independent symmetric second order differential operators. This forces us to locate $t_0$ at one of the endpoints of the support of $W$ with a mass $M$ carefully chosen.

More examples will appear in the PhD dissertation of one of the authors [dI].

3.1 $W_a(t) = e^{-t^2} e^{At} e^{A^* t}$ with $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $a \in \mathbb{R} \setminus \{0\}$

This weight matrix was introduced for the first time in Section 5.1 in [DG1] (for the general size $N \times N$). It was deeply explored in [DG2] and a set of generators of second order differential operators can be found in Section 6 in [CG].

By expanding the exponential, we find that

$$W_a(t) = e^{-t^2} e^{At} e^{A^* t} = e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}.$$

Using (2.4) (for $k = 2$), we find an expression for the 5-dimensional (real) linear space of symmetric differential operators of order at most two with respect to $W_a$. Then, for a fixed real number $t_0$, we solve the equations (2.6). In this case, we find that the following differential operator

$$D_{a,t_0} = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0,$$

where

$$F_2(t) = \begin{pmatrix} -\xi_{\alpha,t_0}^{\pm} + at_0 - at & -1 - (a^2 t_0^2 + a^2 t^2) \\ -1 & -\xi_{\alpha,t_0}^{\pm} + at \end{pmatrix},$$

$$F_1(t) = \begin{pmatrix} -2a + 2\xi_{\alpha,t_0}^{\pm} & -2t_0 - 2a \xi_{\alpha,t_0}^{\pm} + 2(2 + a^2)t \\ 2t_0 & 2(\xi_{\alpha,t_0}^{\pm} - at_0)t \end{pmatrix},$$

$$F_0 = \begin{pmatrix} \xi_{\alpha,t_0}^{\pm} + \frac{2t_0}{a} & \frac{2 + a^2}{a^2} \\ \frac{2}{a^2} & -\xi_{\alpha,t_0}^{\pm} - \frac{2t_0}{a} \end{pmatrix}. $$
and the Hermitian positive semidefinite matrix $M(t_0)$

$$M(t_0) = M(a, t_0) = \begin{pmatrix} (\xi_{a,t_0}^\pm)^2 & \xi_{t_0,a}^\pm \\ \xi_{t_0,a}^\pm & 1 \end{pmatrix},$$

where

$$\xi_{a,t_0}^\pm = \frac{at_0 \pm \sqrt{4 + a^2t_0^2}}{2},$$

satisfy the constraints (2.6).

This differential operator can be obtained by a linear combination of the second order differential operators introduced in [CG] ($D_i$, $i = 1, 2, 3, 4$), namely

$$D_{a,t_0} = \left(-\xi_{a,t_0}^\mp \frac{2t_0}{a} \right) I - \xi_{a,t_0}^\mp D_1 - \frac{4t_0}{a} D_2 + \frac{4}{a^2} D_4.$$

It is easy to verify, using $\xi_{a,t_0}^+ \xi_{a,t_0}^- + 1 = 0$, that the coefficients of $D_{a,t_0}$ evaluated at $t_0$ satisfy the conditions (2.6). Since $D_{a,t_0}$ has been chosen to be symmetric with respect to the weight matrix (3.1), Theorem 2.2 implies that $D_{a,t_0}$ is also symmetric with respect to any of the following weight matrices:

$$W_{a,t_0,\gamma}(t) = W_a(t) + \gamma \delta_{t_0}(t) M(a, t_0), \quad \gamma \geq 0.$$

As a consequence, the sets of monic orthogonal matrix polynomials $(P_{n,a,t_0,\gamma})_n$ with respect to $W_{a,t_0,\gamma}$, $\gamma \geq 0$, are eigenfunctions of a common second order differential operator, namely $D_{a,t_0}$, defined in (3.2). That is,

$$P_{n,a,t_0,\gamma} D_{a,t_0} = \Gamma_{n,a,t_0} P_{n,a,t_0,\gamma}, \quad n = 0, 1, \ldots,$$

where

$$\Gamma_{n,a,t_0} = \begin{pmatrix} \xi_{a,t_0}^\mp (2n+1) & \frac{2t_0}{a} (2 + (n+1)a^2) \\ \frac{4}{a^2} & \xi_{a,t_0}^\mp (2n-1) - 2nat_0 - \frac{2t_0}{a} \end{pmatrix}.$$  

Notice that neither $D_{a,t_0}$ nor $\Gamma_{n,a,t_0}$ depend on $\gamma$.

We would like to make an special mention to the weight matrix $W(t) = e^{-t^2} e^{At} e^{A^*t}$ where $A$ is the $3 \times 3$ matrix

$$A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b \in \mathbb{R} \setminus \{0\}.$$

Assuming that

$$a^2b^2 = 4(b^2 - a^2),$$

the dimension of the linear space of differential operators of order at most two having the orthogonal polynomials with respect to $W$ as eigenfunctions is just three (if (3.3) does not fulfill, the dimension is two, while for the weight matrix $W_a$ in (3.1) is five). A basis is formed by the identity and two symmetric second order linear operators (see [D2]). Our method, however, does not provide any common symmetric second order operator for $W$ and any modification of $W$ by adding a Dirac distribution.
3.2 \( W_{a,\alpha}(t) = t^\alpha e^{-tB}tB^* \) with \( B = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \ a \in \mathbb{R} \setminus \{0\} \)

This weight matrix was introduced for the first time in Section 6.2 in [DG1] (for the general size \( N \times N \)) and it was extensively studied in [DL]. Unlike the first example, its algebra of differential operators has not been studied in depth, but as we mentioned at the beginning of this section, it turns out that there are four linearly independent symmetric second order differential operators and the behavior of the algebra is quite similar to the one introduced in Section 3.1.

By computing \( t^B \) for \( B = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \), we find that

\[
W_{a,\alpha}(t) = t^\alpha e^{-tB}tB^* = t^\alpha e^{-t} \begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}, \ t \in (0, +\infty), \ \alpha > -1.
\]

Proceeding as in Section 3.1, for a fixed real number \( t_0 \), we find that the following differential operator (symmetric with respect to \( W_{a,\alpha} \))

\[
D_{a,\alpha,t_0} = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0,
\]

where

\[
F_2 = \begin{pmatrix} at_0 & a^2t_0 \\ -t_0 & -at_0 \end{pmatrix} + t \begin{pmatrix} \zeta^+ - \frac{1+(\alpha+t_0)(1+a^2)}{a} & -(\alpha + t_0 + 1)(1 + a^2) \\ 0 & \zeta^+ + a \end{pmatrix},
\]

\[
F_1 = \begin{pmatrix} -a(3t_0 - 2 + (\alpha + 1)^2) & 2a\zeta^+ - (1 + a^2)(2t_0 + 3\alpha) \\ + (\alpha + 3)(\zeta^+ - \frac{t_0}{1+a+1}) & \alpha(2t_0 + 3\alpha - t_0 + \alpha + 3) \end{pmatrix} + t \begin{pmatrix} -\zeta^+ + a(t_0 - 1) + \frac{t_0 + \alpha + 1}{a} & (1 + a^2)(\alpha^2 + 3\alpha - t_0\alpha + 2) + 2(\alpha - 1) \\ 0 & -\zeta^+ + a\alpha \end{pmatrix},
\]

\[
F_0 = \begin{pmatrix} -\zeta^+ & \frac{a(t_0 - 1)}{2} + \frac{(\alpha + 2)(t_0 + \alpha + 1)}{2a} - \frac{a(1+\alpha)}{1+a^2} - \frac{1 + a\alpha\zeta^+ - \alpha(t_0 - 1)(a^2 + \alpha + 2) + \frac{1+\alpha}{1+a^2}}{2} \\ \frac{1+a}{1+a^2} & a(t_0 - 1) - \frac{(\alpha + 2)(t_0 + \alpha + 1)}{2a} + \frac{a(1+\alpha)}{1+a^2} \end{pmatrix},
\]

and the Hermitian positive semidefinite matrix \( M(t_0) \)

\[
M(t_0) = M(a, \alpha, t_0) = \begin{pmatrix} (\zeta^+)^2 & \zeta^+ \\ \zeta^+ & 1 \end{pmatrix},
\]

where

\[
\zeta^\pm = \zeta^\pm_{a,\alpha,t_0} = \frac{1}{2} \frac{(a^2 + 1)(t_0 + \alpha) - a^2 + 1 \pm \sqrt{(a^2 + 1)(a^2(t_0 - \alpha - 1)^2 + (t_0 + \alpha + 1)^2)}}{a},
\]

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satisfy the constraints (2.6).

Thus, Theorem 2.2 implies that $D_{a,a,t_0}$ is symmetric with respect to any of the following weight matrices:

$$W_{a,a,t_0,γ}(t) = W_{a,a}(t)\chi_{(0,+,∞)}(t) + γ\delta_{t_0}(t)M(a,a,t_0), \quad t \in \mathbb{R}, \quad α > -1, \quad γ ≥ 0.$$ 

Note that the discrete mass on the Delta distribution can be located in or out of the support of the original weight matrix (3.4).

### 3.3 An example supported in $(0,1)$

The following weight matrix is a modification of the one introduced in [PT] (see also (3.8) below in this paper):

$$W_{α,β,k}(t) = t^α(1-t)^β\begin{pmatrix} kt^2 + β - k + 1 & (β - k + 1)(1-t) \\ (β - k + 1)(1-t) & (β - k + 1)(1-t)^2 \end{pmatrix}, \quad t ∈ (0,1),$$

where $α, β > -1$ and $0 < k < β + 1$.

The difference between $W_{α,β,k}$ and the weight matrix introduced in [PT] ($N = 2$), is that $W_{α,β,k}$ enjoys four linearly independent symmetric second order differential operators while that in [PT] enjoys only two. The example in [PT] is related to a group-theoretical situation but this is not the case of our $W_{α,β,k}$, as far as we know.

Like in the previous examples, for a fixed real number $t_0$ (except for $t_0 = -\frac{1}{k}(α+β-k+2)$ and $t_0 = 1$), no matter if it is located in or out of the support of the weight matrix $W_{α,β,k}$, we find a symmetric second order differential operator $D_{α,β,k,t_0}$ with respect to $W_{α,β,k}$ and a Hermitian positive semidefinite matrix $M(t_0)$ satisfying the constrains (2.6). Hence, this operator $D_{α,β,k,t_0}$ is also symmetric for any of the weight matrices $W_{α,β,k} + γδ_{t_0}M(t_0), γ ≥ 0$.

Since the formulas for arbitrary $α, β$ and $k$ are unpleasant, we show here only one concrete example: $α = 0, β = 0$ and $k = 1/2$:

$$F_2 = \begin{pmatrix} (1-t)(-t_0 + t(2t_0 - 3)) + \frac{t(1-t)(1-t_0)}{φ^±} & 2t + t_0 - 2t_0t - t_0^2 \\ -t_0(1-t)^2 & (t-1)(t-t_0) + \frac{t(1-t)(1-t_0)}{φ^±} \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 12t - 10 + 8t_0(1-t) + \frac{(t_0 - 1)(4t - 3)}{φ^±} & 9 - t - 4t_0(t + 2) + \frac{2(t_0 - 1)}{φ^±} \\ (1 - 4t_0)(t - 1) & 4(t-t_0) + \frac{(t_0 - 1)(4t - 1)}{φ^±} \end{pmatrix},$$

$$F_0 = \begin{pmatrix} \frac{(1-t_0)(1 + 2φ^±)}{2φ^±} & -t_0 - 3 + \frac{1 - t_0}{2φ^±} \\ -t_0 + \frac{1 - t_0}{2φ^±} & \frac{(1-t_0)(1 + 2φ^±)}{2φ^±} \end{pmatrix},$$

and

$$M(t_0) = \begin{pmatrix} 1 & φ^± \\ φ^± & (φ^±)^2 \end{pmatrix},$$

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where
\[ \varphi^{\pm}(t_0) = \frac{2 - t_0 \pm \sqrt{2t_0^2 - 2t_0 + 1}}{t_0 + 3}. \]

We need to impose \( t_0 \neq -3 \) to avoid singularities in \( M(t_0) \) and \( t_0 \neq 1 \) in \( \varphi^- \) (because \( \varphi^-(1) = 0 \)) to avoid singularities in the differential coefficients \( F_2, F_1 \) and \( F_0 \).

### 3.4 An example of general size

We consider here the weight matrix defined by

\begin{equation}
W(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}} J t^{\frac{1}{2}} J^* e^{A^* t}, \quad t \in (0, \infty), \quad \alpha > -1,
\end{equation}

where

\begin{equation}
J = \sum_{i=1}^{N} (N - i) E_{ii}, \quad A = \sum_{i=1}^{N-1} \nu_i E_{i,i+1}, \quad \nu_i \in \mathbb{R} \setminus \{0\}, \quad i = 1, \ldots, N - 1.
\end{equation}

Here we are using \( E_{ij} \) to denote the matrix with entry \((i, j)\) equal to 1 and 0 otherwise.

This weight matrix was introduced for the first time in [DdI]. This example enjoys the special property of having symmetric odd order differential operators, a phenomenon that is not possible in the classical scalar theory. For more details, the reader should consult [DdI].

It is proved in [DdI] that (3.6) has always a symmetric second order differential operator given by

\[ D_1 = \partial^2 t I + \partial^1 [(\alpha + 1) I + J + t(A - I)] + \partial^0 [(J + \alpha I) A - J], \]

where \( A \) and \( J \) are defined in (3.7). Note that there are \( N - 1 \) free parameters in \( D_1 \).

Assuming the following conditions on the parameters \( \nu_1, \ldots, \nu_{N-2} \):

\[ i(N - i) \nu_{N-1}^2 = (N - 1) \nu_i^2 + (N - i - 1) \nu_i^2 \nu_{N-1}^2, \quad i = 1, \ldots, N - 2, \]

the weight matrix \( W \) has another symmetric second order differential operator

\[ D_2 = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0, \]

where the coefficients are given by

\[ F_2 = t(J - At), \]
\[ F_1 = ((1 + \alpha) I + J) J + Y - t(J + (\alpha + 2) A + Y^* - AY + YA), \]
\[ F_0 = \frac{N - 1}{\nu_{N-1}^2} [J - (\alpha I + J) A], \]

\( A \) and \( J \) are defined in (3.7), and \( Y = \sum_{i=1}^{N-1} \frac{i(N - i)}{\nu_i} E_{i+1,1} \). Note that now the only free parameter is \( \nu_{N-1} \).
Let us call $M = M(\alpha, \nu_1, \ldots, \nu_{N-1})$ the Hermitian positive semidefinite matrix (and singular)

$$M = \sum_{i,j=1}^{N} \left( \prod_{k=\min(i,j)}^{\max(i,j)-1} \frac{\nu_k(\alpha + N - k)}{N - k} \right) \left( \prod_{k=1}^{N-\max(i,j)} \frac{\nu_{N-k}(\alpha + k)}{k} \right)^2 E_{ij}.$$ 

Let $D$ be the following differential operator

$$D = -(N-1)D_1 + D_2.$$ 

Then, evaluating the differential coefficients $\tilde{F}_i(t)$, $i = 0, 1, 2$, of $D$ at $t = 0$, we observe that

$$\tilde{F}_2(0) = 0,$$

$$\tilde{F}_1(0) = Y - \sum_{i=1}^{N} (i-1)(\alpha + N - i + 1)E_{ii},$$

$$\tilde{F}_0 = \frac{(N-1)(1 + \nu_{N-1}^2)}{\nu_{N-1}^2} [J - (\alpha I + J)A].$$

Writing down the expressions and considering the bidiagonal structure of $\tilde{F}_1(0)$ and $\tilde{F}_0$, it is easy to conclude that the conditions (2.6) are satisfied. Also, the second order differential operator $D$ is a linear combination of symmetric differential operators with respect to the weight matrix (3.6), so it is symmetric. Hence, Theorem 2.2 implies that $D$ is symmetric with respect to any of the following weight matrices:

$$W_\gamma(t) = W(t) + \gamma \delta(t)M, \quad t \in (0, +\infty), \quad \alpha > -1, \quad \gamma \geq 0.$$ 

We illustrate the case of size $2 \times 2$ ($\nu_1 = a$). The weight matrix $W_\gamma$ is

$$W_\gamma(t) = t^\alpha e^{-t} \left( t(1 + a^2) \at \at \at 1 \right) + \gamma \left( \frac{a^2(a+1)^2}{a(a+1)} \at \at 1 \right) \delta(t),$$

while the second order differential operator $D$ is

$$D = \partial^2 \left( \begin{array}{cc} 0 & -at^2 \\ 0 & -t \end{array} \right) + \partial^1 \left( \begin{array}{cc} t & -(1 + a^2(\alpha + 3))t \\ 1 & a(\alpha + 1) \end{array} \right) + \partial^0 \left( \begin{array}{cc} a^2 + 1 & -(1 + a^2)(\alpha + 1) \\ a^2 & a + (\alpha + 1) \end{array} \right).$$

3.5 The one step example

This weight matrix was introduced for the first time in [PT]. An associated second order differential operator comes from the Casimir operator acting on matrix valued functions on the group SU($n + 1$). The reader is advised to consult, apart from [PT], either [GPT3] or even better [GPT1] for a full account.
The weight matrix, for \( t \in (0, 1) \), \( \alpha, \beta > -1 \) and \( 0 < k < \beta + 1 \) has the following structure:

\[ W_{\alpha,\beta,k}(t) = t^{\alpha}(1-t)^{\beta}Z(t), \]

where

\[ Z(t) = \sum_{i,j=1}^{N} \left( \sum_{r=1}^{N} \left( \begin{array}{c} r - 1 \\ i - 1 \end{array} \right) \left( \begin{array}{c} r - 1 \\ j - 1 \end{array} \right) \right) \left( \frac{N + k - r - 1}{N - r} \right)(1 - t)^{i+j-2}t^{N-r}E_{ij}. \]

Likewise, as in Section 3.4, we have an expression of two symmetric second order differential operators. A first one \( D_1 \) is given in [PT] (Theorem 3.4) and a second one \( D_2 \) is given in [PR] (Theorem 3.1). Notice that in [PT, PR], left-hand side differential operators are considered.

Let us call \( M = M(\alpha, \beta, k) \) the Hermitian positive semidefinite matrix (and singular)

\[ M = \sum_{i,j=1}^{N} \left( \begin{array}{c} N - 1 \\ i - 1 \end{array} \right) \left( \begin{array}{c} N - 1 \\ j - 1 \end{array} \right) \frac{(\alpha + \beta - k + i + 1)(N-i)(\alpha + \beta - k + j + 1)(N-j)}{(\beta - k + i)(N-i)(\beta - k + j)(N-j)}E_{ij}, \]

where the number \((x)^{(n)}\) denotes the rising or upper factorial defined by \((x)^{(n)} = x(x+1)\cdots(x+n-1)\) for \( n > 0 \), \((x)^{(0)} = 1\).

Let \( D \) be the following differential operator

\[ D = -(\alpha - N + 1)D_1 + D_2. \]

Thus, evaluating the coefficients \( \tilde{F}_i(t) \), \( i = 0, 1, 2 \), of \( D \) at \( t = 0 \) we observe that

\[ \tilde{F}_2(0) = 0, \]
\[ \tilde{F}_1(0) = \sum_{i=1}^{N-1} 3i(\alpha + \beta - k + i + 1)E_{i,i+1} - \sum_{i=2}^{N} 3(\beta - k + i - 1)(N-i+1)E_{i,i-1} \]
\[ + \sum_{i=1}^{N} 3[i - 1](\alpha + \beta - k + i) + (i - 4)(\beta - k + i)]E_{ii}, \]
\[ \tilde{F}_0 = -3(N + k - 1)\sum_{i=1}^{N} (i - 1)(\alpha + \beta - k + i)E_{ii} + \sum_{i=2}^{N} (N - i + 1)(\beta - k + i - 1)E_{i,i-1}. \]

Writing down the expressions and considering the tridiagonal structure of \( \tilde{F}_1(0) \) and \( \tilde{F}_0 \), it is easy to conclude that the conditions (2.6) are satisfied. Also, the second order differential operator \( D \) is a linear combination of symmetric differential operators with respect to the weight matrix (3.8), so it is symmetric. Then, Theorem 2.2 implies that \( D \) is symmetric with respect to any of the following weight matrices:

\[ W_{\alpha,\beta,k,\gamma}(t) = W_{\alpha,\beta,k}(t) + \gamma\delta(t)M(\alpha, \beta, k), \quad t \in (0, 1), \quad \alpha, \beta > -1, \quad 0 < k < \beta + 1, \quad \gamma \geq 0. \]
Like before, we illustrate the case of size $2 \times 2$. The weight matrix $W_{\gamma} = W_{\alpha,\beta,k,\gamma}$ is

$$W_{\gamma}(t) = t^\alpha(1-t)^\beta \begin{pmatrix} kt + \beta - k + 1 & (1-t)(\beta - k + 1) \\ (1-t)(\beta - k + 1) & (1-t)^2(\beta - k + 1) \end{pmatrix} + \gamma \begin{pmatrix} \frac{(\alpha + \beta - k + 2)^2}{(\beta - k + 1)^2} & \frac{\alpha + \beta - k + 2}{\beta - k + 1} \\ \frac{\alpha + \beta - k + 2}{\beta - k + 1} & 1 \end{pmatrix} \delta(t),$$

while the second order differential operator $D$ is

$$D = \partial^2 \begin{pmatrix} 0 & 0 \\ t & t(1-t) \end{pmatrix} + \partial^1 \begin{pmatrix} -\beta + k - 1 & (\beta - k + 1)(t - 1) \\ \alpha + \beta - k + 2 & (\alpha + \beta + 4)(1-t) - (k + 2) \end{pmatrix} + \partial^0 \begin{pmatrix} 0 & (1+k)(\beta - k + 1) \\ 0 & (1+k)(\alpha + \beta - k + 2) \end{pmatrix}.$$

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