Inhomogeneous cosmological models with homogeneous inner hypersurface geometry

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Abstract

Space–times which allow a slicing into homogeneous spatial hypersurfaces generalize the usual Bianchi models. One knows already that in these models the Bianchi type may change with time. Here we show which of the changes really appear.

To this end we characterize the topological space whose points are the 3-dimensional oriented homogeneous Riemannian manifolds; locally isometric manifolds are considered as same.

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1 INTRODUCTION

In the review ”Physics in an inhomogeneous universe” Krasiński [1] defines a solution of the Einstein equation to be cosmological if it can reproduce the Friedmann–Lemaître–Robertson–Walker metric by taking limiting values of arbitrary constants or functions. In the carefully collected bibliography [2] he classified them according to geometrical properties of the classes of solutions. The main ingredient is the existence of isometries, and it were Collins [3] and Krasiński [4] who taught the relativists to distinguish between ”intrinsic” (internal) and external isometries (a distinction known to mathematicians for
long). He showed that a cosmological model composed of homogeneous and isotropic spatial hypersurfaces need not be a Friedmann model. The essential new feature is that the model can continuously change in time between closed \((k = 1)\) and open \((k \leq 0)\) hypersurfaces (which is impossible for the Friedmann models). In [4a], the Stephani universe was generalized by allowing a changing value \(k\) in the generalized Friedmann model. Analogously, in [4b] the property of spherical symmetry was generalized.

A likewise generalization of the spatially flat \((k = 0)\) Friedmann models is the Szekeres class of solutions [5, 6] of the Einstein equation; all elements of this class possess internally flat spatial hypersurfaces.

In [7] the analogous question (which was considered by Krasiński for homogenous isotropic hypersurfaces) was posed and partially answered for models with homogeneous but not necessarily isotropic hypersurfaces.

It is the aim of the present article to continue the research done in [7]. In [7] one can read (we repeat it because that journal is not available to everybody): “To begin with, we consider the easily tractable case of a change to type I: For each type M there one can find a manifold \(V_4\) such that for \(t \leq 0\) the section \(V_3(t)\) is flat and for each \(t > 0\) it belongs to type M ... Applying this fact twice it becomes obvious that by the help of a flat intermediate all Bianchi types can be matched together. However, if one does not want to use such a flat intermediate the transitions of one Bianchi type to another one become a non-trivial problem.”

Let us repeat the question: A generalized homogenous cosmological model is a space–time possessing a foliation into homogenous spatial hypersurfaces. In which manner the corresponding Bianchi type can change with time? This question can be formulated in several versions; we list five of them. Other versions of the problem can be obtained by restriction to space–times satisfying certain energy conditions; this is already mentioned in the introduction of Ref. 6, but here we do not deal with such versions.

Let us consider continuous one-parameter families \((\Sigma_\lambda, g_\lambda), \lambda \in [0, 1]\), where \(\Sigma_\lambda\) be an orientable smooth three-manifold, and \(g_\lambda\) a homogeneous Riemannian metric on it. Since here we consider the homogeneous three-geometry only locally, we can study it on some open standard manifold, \(\Sigma = \Sigma_\lambda\) for all \(\lambda\). Globally however, \(\Sigma_\lambda\) may change (see Discussion).

Problem 1: Fix a Bianchi type X and a Bianchi type Y. Then, does there exist a one-parameter family of three-geometries \((\Sigma, g_\lambda)\) (continuous in the parameter \(\lambda\)) such that \((\Sigma, g_0)\) is of Bianchi type X and \((\Sigma, g_1)\) is of Bianchi type Y?

Problem 2: Fix a three-geometry \((\Sigma, g_0)\) of Bianchi type X and a three-geometry \((\Sigma, g_1)\) of Bianchi type Y. Then, does there exist a one-parameter
family of three-geometries \((\Sigma, g_\lambda)\) (continuous in the parameter \(\lambda\)) reproducing \((\Sigma, g_0)\) and \((\Sigma, g_1)\) at \(\lambda = 0\) and \(\lambda = 1\), respectively?

Clearly both these problems can be answered "Yes, for all possible \(X\) and \(Y\)" by applying the cited phrase from [7] with \(g_\lambda\) being flat e.g. for \(\frac{1}{3} \leq \lambda \leq \frac{2}{3}\).

Problem 3: Fix a Bianchi type \(X\) and a Bianchi type \(Y\). Then, does there exist a one-parameter family of three-geometries \((\Sigma, g_\lambda)\) (continuous in the parameter \(\lambda\)) such that \((\Sigma, g_0)\) is of Bianchi type \(X\) and \((\Sigma, g_\lambda)\) is of Bianchi type \(Y\) for \(0 \leq \lambda \leq 1\)?

Problem 4: Fix a three-geometry \((\Sigma, g_0)\) of Bianchi type \(X\) and a three-geometry \((\Sigma, g_1)\) of Bianchi type \(Y\). Then, does there exist a one-parameter family of three-geometries \((\Sigma, g_\lambda)\) (continuous in the parameter \(\lambda\)) and some critical parameter \(\lambda_c\) such that \((\Sigma, g_\lambda)\) is of Bianchi type \(X\) for \(0 \leq \lambda \leq \lambda_c\) and \((\Sigma, g_\lambda)\) is of Bianchi type \(Y\) for \(\lambda_c < \lambda \leq 1\) and \(g_0, g_1\) coincide with the prescribed geometries?

Problem 5: Fix a three-geometry \((\Sigma, g_0)\) of Bianchi type \(X\) and a three-geometry \((\Sigma, g_1)\) of Bianchi type \(Y\). Then, does there exist a one-parameter family of three-geometries \((\Sigma, g_\lambda)\) (continuous in the parameter \(\lambda\)) such that \((\Sigma, g_\lambda)\) is of Bianchi type \(X\) for \(\lambda = 0\) and \((\Sigma, g_\lambda)\) is of Bianchi type \(Y\) for \(0 < \lambda \leq 1\) and \(g_0, g_1\) coincide with the prescribed geometries?

It holds: The set of homogeneous Riemannian three-geometries of any fixed Bianchi type is arcwise connected. So Problems 3 and 4 are equivalent. Even more, it turns out that for a pair of Bianchi types \((X, Y)\) solving Problem 3 and 4 the corresponding sets \(S_X\) and \(S_Y\) of homogeneous Riemannian three-geometries are related by \(S_X \subset \partial S_Y\), where \(\partial S_Y\) denotes the boundary of \(S_Y\). Hence Problem 5 is also equivalent to Problems 3 and 4.

This is the problem which is resolved in the following. If the answer with is "Yes" for \((X, Y)\) with \(X \neq Y\), then we also say that there is a transition \(Y \to X\) corresponding to a directed graph (see Appendix B).

## 2 PRELIMINARIES

In this section, we introduce the necessary differential geometric and group theoretic notions, and we shortly review the literature to related topics.

The classical book by Wolf [8] does not only deal with the Riemannian and pseudo-Riemannian spaces of constant curvature, it covers also a lot of results on local and global differential geometry including symmetric and homogeneous manifolds.
2.1 Isometry groups of Riemannian spaces

Karlhede and MacCallum [9] deal with the equivalence problem which consists of the problem to find a procedure for deciding whether two given Riemannian spaces are isometric.

Here and in the following we use the notion “isometry group” in the meaning “the connected component of the unity of the full isometry group”.

Szafron [10] poses the “program to classify cosmological models using intrinsic symmetries”. He mentions the following results:

**Proposition 1**: For Bianchi types II and III, the internal isometry group is always 4-dimensional, for Bianchi types VIII and IX it is sometimes, and for Bianchi types IV and VI \( h \neq 0, 1 \) it is never 4-dimensional.

**Proposition 2**: The conformal flatness of a 3-dimensional Riemannian manifold is equivalent to the vanishing of the Cotton–York tensor. For homogenous spaces this is equivalent to the validity of \( R_{ijkl} = 0 \) where \( [\ ] \) denotes antisymmetrization.

Bona and Coll [11] go a similar way: They look for the isometry groups of 3-dimensional Riemannian metrics, they understand it as part of the program “problem of equivalence of metrics”.

Limits of space–times are considered in [12] using a coordinate–free approach, cf. also the references cited in [12]. In [7, 13 - 16] a topology in the space of Lie algebras was considered which shall be put in relation to the limit of space–times in sect. 3.

A topology in the space of real Lie algebras can be defined as follows: Let \( A \) be a set of \( d \)-dimensional Lie algebras and \( V \) be the \( d^3 \)-dimensional real vector space. We define a subset \( B \subset V \) with \( x \in B \) iff \( x \) represents a set \( C_{ijk} \) of structure constants for an element of \( A \). Then \( A \) is defined to be closed iff \( B \) is closed in the Euclidean topology of \( V \). (For details see appendix A.)

Example: If \( A \) has the commutative algebra as only element, then \( A \) is a closed set, because \( B \) consists of the origin of \( V \) only. On the other hand Segal [16] showed

**Proposition 3**: Compact semisimple Lie algebras represent isolated points in the space of Lie algebras, because of the definiteness of their Cartan metric.

Results of [14] (including the complete topology for the 3-dimensional case) are generalized in [15] (also completing the 4-dimensional case).

There exist several similar constructions in the space of Lie groups. Co- natser [17] considered the Inööii–Wigner contractions of the low-dimensional real Lie algebras. Levy-Nahas [18] and Saletan [19] define contractions and deformations of Lie groups. These concepts are a little more general ones. It holds [18]
**Proposition 4:** There exist exactly two Lie groups which can be contracted to the Poincaré group: The de Sitter group \( SO(4,1) \) and the anti-de Sitter group \( SO(3,2) \). Both of them are semisimple, they have different signatures of the Cartan metric.

It is pointed out in [15] that all these deformations and contractions within the category of finite-dimensional Lie algebras appear in the topology of the resp. space of Lie algebras, but the latter has more general limits, some of which correspond neither to Inönü–Wigner nor to Saletan contractions.

In [20], the Heisenberg group is defined as odd-dimensional simply connected 2–step nilpotent \( ([g, g] \neq 0,\; [g, [g, g]] = 0) \) real Lie group with 1-dimensional center. For every odd dimension, the Heisenberg group is uniquely defined by this condition. In 3 dimensions, its algebra is of Bianchi type II, in dimension \( d > 3 \) its algebra differs from the algebra \( \Pi^\odot(d) \) considered in [14, 15], defined as direct product of II with the \((d-3)\)-dimensional Abelian algebra. \( \Pi^\odot(d) \) is the atom defined in [14]. It has a \((d-2)\)-dimensional dimensional center and is also considered as the generalized 3-dimensional Heisenberg algebra.\)

Kaplan [21] also discusses groups of the Heisenberg type; for 3 dimensions his definition coincides with the usual one, for higher dimensions it differs both from [20] and [14] which can be seen if one compares the dimension of the centers.

The 3-dimensional Lie algebras are classified according to Bianchi. Ref. [22] represents the original citation for the Bianchi types, the book [23] is more widely known and refers to [22]. However, Lie [24], [25] classified them some years earlier; nevertheless, we keep calling them Bianchi types. Cf. also [26] for this classification.

Refs. [27 - 29] and many further ones deal with cosmological models with isometry subgroup belonging to one of the Bianchi types.

### 2.2 Local homogeneity

Let us now come to the condition for homogeneity of a Riemannian manifold. The theorem of Ambrose and Singer [30] is popularized in [31] and gives conditions under which a Riemannian manifold is locally homogeneous. Let \( g \) be the metrical tensor, \( \nabla_L \) the corresponding Levi–Civita connection and \( R \) its curvature tensor. Let \( T \) be the torsion tensor and \( \nabla = \nabla_L - T \). The Ambrose–Singer equations (AS–equations) are the following:

\[
\nabla g = 0 \quad \nabla T = 0 \quad \nabla R = 0
\]

Let us sum up the main ingredients valid for all connected Riemannian manifolds \( V_n \):
Proposition 5 ([30], [31]):

1. If the $V_n$ is complete, simply connected and locally homogeneous, then it is homogeneous.

2. If the $V_n$ is homogenous, then there exists a solution of the AS–equations.

3. If there exists a solution of the AS–equations, then the $V_n$ is locally homogeneous.

As special case one gets the following: In the set of connected simply connected and complete Riemannian manifolds the AS–equations can be solved if and only if the manifold is homogeneous.

(Remark: In Ref. [11], Bona and Coll give (for $n = 3$) also necessary and sufficient conditions under which a Riemannian space is homogeneous. However, their approach is quite different because they do not use the auxiliary torsion as Ambrose and Singer did.)

A further interesting special case is $T = 0$. The AS–equations can be solved with $T = 0$ iff the Riemann tensor is covariantly constant. Such spaces are called locally symmetric. It follows from the AS–theorem, that locally symmetric spaces are locally homogeneous.

In [32] the spatial curvatures of all the Bianchi types are calculated. Refs. [33, 34] deal with homogeneous Riemannian spaces. Milnor [34] gives a review of left–invariant metrics on Lie groups; he writes: “In the 3-dimensional case, the theory is essentially complete.” To our knowledge, it has not been completed yet; more precisely: All particular parts of the puzzle exist, but they have not been put together to a picture. Even the present article will not finish this task.

It is mentioned in [34], that the curvature depends continuously on the structure constants and on the metric. This gives already a hint how the puzzle can be handled.

2.3 Special properties of isometry groups and eigenvalues of the Ricci tensor

A Lie group is unimodular if and only if the left–invariant and the right–invariant Haar measures coincide. Equivalently one can say: a Lie group is unimodular iff the structure constants of the corresponding Lie algebra are trace–free, i.e. $C^n_{ji} = 0$.

In the 3-dimensional case, the 6 unimodular groups are Bianchi types I, II, VI₀, VII₀, VIII, and IX, sometimes also called type A-algebras.
A metric is called *bi–invariant* if it is simultaneously a left–invariant and a right–invariant one. It holds: Only unimodular Lie groups carry bi–invariant metrics. More specifically we have

**Proposition 6** ([34]):
A bi–invariant metric exists on a Lie group if and only if the group is the direct product between a compact and a commutative group.

In the 3-dimensional case, this takes place only for Bianchi types I and IX. For Bianchi type I, every metric is bi–invariant. For Bianchi type IX the metric leading to a space of constant curvature is bi–invariant.

Now we list some results dealing with the signs of the eigenvalues of the Ricci tensor. It holds

**Proposition 7** (Theorem 2.2. of ref. 34): A connected Lie group admits a left invariant metric where all eigenvalues of the Ricci tensor are positive if and only if the group is compact with finite fundamental group. For this case, the metric can be chosen to be bi–invariant, such that, simultaneously, all eigenvalues of the Ricci tensor coincide. In the 3-dimensional case, this takes place for Bianchi type IX only, then, if all eigenvalues of the Ricci tensor coincide, the space is of constant curvature.

**Proposition 8** ([34]): If the Lie group is not unimodular then one of the eigenvalues of the Ricci tensor is negative.

A Lie group is called *nilpotent* if for the corresponding Lie algebra $g$ the sequence

$$g \supset [g, g] \supset [g, [g, g]] \supset \ldots$$

terminates at zero.

In 3 dimensions, Bianchi types I and II are nilpotent. It holds

**Proposition 9** ([34]): Every left–invariant metric on a non–commutative nilpotent Lie group has a positive and a negative eigenvalue of the Ricci tensor.

A Lie group is called *non–trivial* if there exist 3 linearly independent vectors $x, y, z$ of the corresponding Lie algebra such that $[x, y] = z$. (A commutative Lie group is always trivial.) In 3 dimensions, Bianchi type V represents the only trivial non–commutative Lie group. In analogy to proposition 9 we have

**Proposition 10** ([34]): Every non–trivial Lie group possesses a left–invariant metric which has a positive and a negative eigenvalue of the Ricci tensor.

In 3 dimensions on gets furthermore:

**Proposition 11**: A left–invariant metric on a trivial Lie group has never a positive eigenvalue of the Ricci tensor.
Proof: Bianchi type I has vanishing curvature, Bianchi type V leads always to a space of constant negative curvature.

**Proposition 12 ([34]):** Every non–commutative Lie group carries a left–invariant metric with negative Ricci scalar.

Partial results for the curvature invariants of Bianchi types II and VIII have been obtained independently in [35], a systematic classification for the curvature invariants of all Bianchi types can be found in [15].

### 2.4 Homogeneous spaces and triviality

In defining homogeneous manifolds we follow Ziller [37]. A Riemannian or pseudo-Riemannian manifold $M$ is homogeneous if the isometry group acts transitively on it. It holds

**Proposition 13:** Each homogeneous manifold $M$ can be represented as $M = G/H$ where $G$ is the isometry group of $M$ and $H$ its isotropy subgroup.

Of course, sometimes a homogeneous manifold can be represented as $M = \tilde{G}/\tilde{H}$ where $\tilde{G}$ and $\tilde{H}$ are not the isometry resp. isotropy groups.

Slansky [36] gives a review on Lie groups. The exceptional Lie group $G_2$ is the only 14-dimensional compact simple Lie group. One gets the homogeneous sphere $S^6$ by factorizing $G_2/SU(3) = S^6$. This is a reduced quotient as compared to the usual $SO(7)/SO(6) = S^6$, where dim $SO(7) = 21$. In [31] this reads as follows: The homogenous $S^6$ has two different homogeneous structures with different Lie algebras.

Even more remarkable is also the following example: The Heisenberg group (Bianchi type II) has two different homogenous structures, both with the same Lie algebra. (The reason here is: Each left-invariant metric of the Heisenberg group possesses a 4-dimensional isometry group, which is of type $A_{4,10}$ in the classification of [15]. It admits two different, though isomorphic, 3-dimensional transitive subgroups.)

Any $M = G/H$ is defined only with $H$ a closed subgroup of $G$. For a positive definite metric, moreover, $H$ is compact, because it is a closed subgroup of the compact rotation group. For this case one can do the following: Let $g$ be the Lie algebra to $G$, $h$ to $H$, then there exists a subalgebra $p \subset g$ such that $g = h \oplus p$ and $[h, p] \subset p$, so that $p$ can be identified with the tangent space of $M$, cf. [37].

Here we want to emphasize that this identification is not automatically possible for Lorentzian homogeneous metrics. In the following, we only deal
with simply connected complete Riemannian manifolds, so we need not distinguish between homogeneous and locally homogeneous spaces, cf. proposition 5.

Further papers on related topics are [38] for space-time homogeneous manifolds, [39] is a continuation of ref. [11] for Lorentzian metrics, [40] discusses consequences of the fact that the Lorentz group is not compact (in contrast to the compactness of the rotation group). [41] considers left–invariant Lorentz metrics on Lie groups.

A Lie algebra $g$ is trivial if for all $x, y \in g$ the vector $[x, y]$ is a linear combination of $x$ and $y$. (Of course, a Lie group is trivial iff the corresponding Lie algebra is trivial.) Equivalently it holds: $g$ is trivial and non–commutative if there exist a commutative subalgebra $h$ of codimension 1 and a vector $b \in g \setminus h$ such that for all $x \in h$ one has $[b, x] = x$. A further equivalent definition is: $g$ is trivial and non–commutative if there exists a basis $e_1, e_2, \ldots e_d$ such that for all $i, j = 2, \ldots d$ one has $[e_1, e_i] = e_i$ and $[e_i, e_j] = 0$.

Therefore, in each dimension exactly one non-Abelian trivial algebra exists, which is in fact identical to the unique non-Abelian pure vector type algebra of [14,15]. In 3 dimensions it is Bianchi type V. Analogously to proposition 11 it holds

**Proposition 14 ([41]):** Every left–invariant metric on a trivial and non–commutative Lie group is of constant negative curvature.

In 3 dimensions this implies: Bianchi type V represents always a 3-space of constant negative curvature, hence, it possesses a 6-dimensional isometry group.

Concerning left–invariant Lorentz metrics one gets [41]: For trivial Lie groups one gets only spaces of constant curvature. In 3 dimensions it holds: Bianchi types II, VI$_0$ and VII$_0$ admit flat spaces, semi–simple algebras do not admit flat spaces.

### 3 HOMOGENEOUS MANIFOLDS

In subsection 3.1 we deal with 2-dimensional homogeneous manifolds of both possible signatures, in subsection 3.2 we consider the positive definite 3-dimensional homogeneous manifolds.

#### 3.1 Two–dimensional homogeneous manifolds

This subsection repeats only known results but the point of view seems to be new. However, the main reason why we present it here is to show the reader
what style of reasoning is behind subsection 3.2. We do not distinguish between manifolds whose only difference is an overall factor \((-1)\) in front of the metrical tensor. So we have to consider only two possible signatures.

### 3.1.1 Positive definite signature

Let us start with the two 2-dimensional Lie algebras. Every left–invariant metric on the commutative Lie group is flat. This gives the flat Euclidean plane \(\mathbb{R}^2\) with isometry group \(E(2)\) belonging to Bianchi type \(\text{VII}_0\).

The only non–commutative 2-dimensional Lie algebra is trivial (cf. propositions 10, 11 and 14), and so every left–invariant metric gives the \(H^2\), the 2-surface of constant negative curvature. Its isometry group is \(SO(2,1)\) and belongs to Bianchi type \(\text{VIII}\).

What about the sphere \(S^2\), the 2-surface of constant positive curvature? Its isometry group is \(SO(3)\) and it belongs to Bianchi type \(\text{IX}\). In the language of proposition 13 we can write \(S^2 = SO(3)/SO(2)\), and the above consideration shows: The 3-dimensional isometry group of \(S^2\) does not possess a 2-dimensional transitive subgroup, but the 3-dimensional isometry group of \(H^2\) has a 2-dimensional transitive subgroup.

### 3.1.2 Indefinite signature

The commutative Lie group gives the flat 1+1-dimensional Minkowski space–time \(M^2\). The isometry group is the 2-dimensional pseudo-Euclidean group of motions, \(E(1,1)\), with corresponding Bianchi type \(\text{VI}_0\).

The non–commutative Lie group gives a space of constant non–vanishing curvature (sometimes called 2-dimensional de Sitter space–time \(\bar{S}^2\)), the isometry group belongs to Bianchi type \(\text{VIII}\).

Contrary to the case of definite signature, a third type does not exist.

### 3.1.3 The algebras of the isometry groups

Figure 1 represents a subdiagram of the diagram of refs. \([7, 14]\), cf. also appendix A. An arrow denotes convergence in the topology \([14, 16]\), cf. appendix B.

\[
\text{IX} \quad \longrightarrow \quad \text{VII}_0 \]

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Fig. 1 First subdiagram for the Bianchi types

One can see: The Lie algebra $\text{VI}_0$ can be deformed to type VIII only, cf. subsection 3.1.2. The Lie algebra $\text{VII}_0$ can be deformed to types VIII and IX, cf. proposition 4 and subsection 3.1.1. (In accordance with proposition 3 the types VIII and IX cannot be deformed or contracted into each other.) So already at the Lie algebra level the difference between 3.1.1 and 3.1.2 becomes obvious.

3.1.4 The corresponding limits of space–times

Now we come to the scope of this article: models with intrinsically homogeneous hypersurfaces. Here we consider the $2 + 1$-dimensional case with 2-dimensional hypersurfaces.
Fig. 2 Homogeneous 2-surfaces of constant curvature; upper part: definite signature; lower part: indefinite signature.

Fig. 2 is to be read as follows: There exists a sequence of spheres which converge locally to the plane (by blowing them up), analogously there exists a sequence of $H^2$'s which converges to the flat plane. (But a direct continuous deformation between $S^2$ and $H^2$ is not possible.) For indefinite signature it holds analogously: There exists a sequence of $\bar{S}^2$'s which converge locally to the $M^2$.

It should be noticed that Fig. 2 gives the same topological relation as the corresponding Lie group diagram Fig. 1. A 3-dimensional space–time possessing a slicing into 2-dimensional homogeneous positive definite hypersurfaces has the following property: A continuous change of the type of the slices is possible if and only if the corresponding Lie algebras of the internal isometry groups can be contracted into each other.

If one allows complex transformations, the Lie groups IX and VIII on the one hand, and VII$_0$ and VI$_0$ on the other hand, become equivalent, cf. Fig. 1. Analogously, $S^2$, $H^2$, and $\bar{S}^2$ on the one hand, and $R^2$ and $M^2$ on the other hand, are isometric by a complex coordinate transformation, cf. Fig. 2.

The slice is locally uniquely determined by the value of its curvature scalar and the signature of the metric. A continuous change of the slice is possible iff the signature is the same and the corresponding inner curvature scalar changes continuously.

In the next subsection we want to clarify, to which extent analogous results are valid also in the more interesting 3 + 1-dimensional case.

### 3.2 Three–dimensional homogeneous manifolds

We only consider the case of a positive definite metric.
3.2.1 The Kantowski–Sachs model

Let us start with the Kantowski–Sachs model, whose spatial hypersurfaces are $\mathbb{S}^2 \times \mathbb{R}$, the Cartesian product of the sphere and the real line. The isometry group is 4-dimensional, and it does not possess a 3-dimensional transitive subgroup.

The eigenvalues of the Ricci tensor are $(\lambda, \lambda, 0)$ where $\lambda$ is any positive real. It is essential to note that this is the only case (more exactly: a one–parameter set of cases, parametrized by $\lambda$) for a 3-dimensional homogeneous manifold in which no 3-dimensional transitive subgroup of the isometry group exists. A little less known is the following: It is also the only homogeneous 3-dimensional manifold where the Ricci tensor has eigenvalues $(\lambda, \lambda, 0)$ with $\lambda > 0$.

Proof: Because of propositions 8 and 9 we have to check only the unimodular but not nilpotent Lie algebras, i.e. types $\text{VI}_0$, $\text{VII}_0$, VIII and IX. The Ricci scalar equals $2\lambda > 0$, but only type IX allows a positive Ricci scalar. The rest is done by explicit calculus.

Besides this exceptional case one gets all other homogeneous 3–spaces by considering the left–invariant metrics on 3-dimensional Lie groups which are discussed in the next subsection. Nevertheless it is useful to mention the following: $\mathbb{H}^2 \times \mathbb{R}$, the Cartesian product of the 2–surface of constant negative curvature and the real line possesses a 4-dimensional isometry group; it has a transitive subgroup of Bianchi type III (which is the same as $\text{VI}_1$). The Ricci tensor has eigenvalues $(\lambda, \lambda, 0)$ with $\lambda < 0$.

3.2.2 The Bianchi models

Let us consider now 3-dimensional homogeneous spaces different from the Kantowski-Sachs spaces of the last section. These so–called Bianchi models are defined by left–invariant metrics on a 3-dimensional Lie group $G_3$. This is a simply transitive isometry subgroup given by one of the Bianchi types. From subsection 3.2.1, it follows that subsections 3.2.1 and 3.2.2. together cover all 3–dimensional homogeneous Riemannian manifolds.

The different Bianchi types are as follows, cf. also appendix A. The Lie algebra $A_3$ of $G_3$ is determined by the commutators

$$[e_a, e_b] = C^c_{ab} e_c$$

of its generators $e_a$, which may be represented w.r.t. coordinates $\{x^a\}$ as left-invariant vector fields,

$$e_a = e_a^\alpha \frac{\partial}{\partial x^\alpha}.$$ 

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Their duals are the triad 1-forms

\[ e^\alpha = e^\alpha_\alpha dx^\alpha. \] (3.3)

The coordinate components of the first fundamental form \( d^2s = \eta_{\alpha\beta} dx^\alpha dx^\beta \) may be expressed by the anholonomic components w.r.t. the triad, i.e.

\[ \eta_{\alpha\beta} = g_{ab} e^\alpha_a e^\beta_b. \] (3.4)

While \( \eta_{\alpha\beta} \) depend in general on \( x^\alpha \), we can choose the anholonomic representation without loss of generality such that \( g_{ab} \) is constant and diagonal, i.e.

\[
(g_{ab}) = \begin{bmatrix}
    e^x & 0 & 0 \\
    0 & e^y & 0 \\
    0 & 0 & e^z
\end{bmatrix},
\] (3.5)

which corresponds to a positively oriented orthogonal triad frame of reference, where the eigenvalues \( e^x, e^y, e^z \) define the scales of measurement in the 3 orthogonal directions.

In calculations below we also use the parameters

\[
t := x - z; \quad u := y - z; \quad w := y - x.
\] (3.6)

The structure constants can be calculated from the triad and \( d^2s \) by

\[ C_{ijk} = d^2s([e_i, e_j], e_k), \quad C^k_{ij} = C_{ijr} g^{rk}. \] (3.7)

The connection coefficients are given by

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kr}(C_{ijr} + C_{jri} + C_{irj}), \] (3.8)

where

\[ -\Gamma^k_{[ij]} = -\frac{1}{2} C^k_{ij} = e^\mu_i e^\nu_j \partial_{[\mu} e^k_{\nu]} \] (3.9)

is the object of anholonomity.

The curvature operators to the triad basis are defined as

\[ \Re_{ij} := \nabla_{[e_i, e_j]} - (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) \] (3.10)

and the Riemann tensor components are

\[ R_{hijk} := <e_h, \Re_{ij} e_k>. \] (3.11)
The Ricci tensor then is
\[ R_{ij} := R^k_{ikj} = \Gamma^f_{ij} \Gamma^e_{fe} - \Gamma^f_{ie} \Gamma^e_{fj} + \Gamma^e_{ij} C^f_{ej}. \] (3.12)

The Ricci curvature scalar is \( R := R^i_i \), the sum of the squared eigenvalues of the Ricci tensor is
\[ N := R^i_j R^j_i, \] (3.13)
\( S^i_j \) is the trace-free part of the Ricci tensor, we define
\[ S := S^i_j S^j_k S^k_i = R^i_j R^j_k R^k_i - RN + \frac{2}{9} R^3, \] (3.14)
and
\[ Y := \frac{1}{2} (R_{ijk} - R_{ikj}) g^{ii} g^{jm} g^{kn} R_{lmn}. \] (3.15)

These are the four scalar invariants which characterize the local homogeneous space.

This means that the bound "\( p \leq 3 \)" of theorem 6 in ref. 11 can be lowered to "\( p \leq 1 \)" if one restricts to the set of homogeneous spaces. In other words: If one knows these four numbers, then the geometry of the homogeneous space is uniquely determined (up to possible global identifications). If the Bianchi type is given and is neither VIII nor IX, then the three eigenvalues of the Ricci tensor suffice to determine the local geometry completely. Both Bianchi types VIII and IX possess examples where this is not true.

There is a one-to-one correspondence between the ordered triple of numbers \((R, N, S)\) and the non-ordered triple consisting of the three eigenvalues of the Ricci tensor. It depends on the situation which of these triples is more appropriate.

\( N = 0 \) appears for the flat space only. For \( N \neq 0 \), the invariant \( N \) is positive and can be used for a homothetic rescaling of the metric,
\[ \hat{g}_{ij} := \sqrt{N} g_{ij}, \] (3.16)
One gets for the Ricci tensor \( \hat{R}^i_j = R^i_j / \sqrt{N} \) and for the scalar invariants
\[ \hat{N} = 1, \quad \hat{R} = R / \sqrt{N}, \quad \hat{S} = S / N^{3/2}, \quad \hat{Y} = Y / N^{3/2}. \] (3.17)

If two eigenvalues of the Ricci tensor are equal, i.e., its matrix reads
\[ Ric = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & ad \end{bmatrix}, \] (3.18)
we obtain for $a > 0$ respectively

$$
\hat{R} = \pm \frac{d + 2}{\sqrt{d^2 + 2}}, \quad \hat{S} = \pm \frac{2}{9} \frac{(d - 1)^3}{(d^2 + 2)^{3/2}}.
$$

(3.19)

For $d \in \mathbb{R} \cup \{\pm \infty\}$ the corresponding values in the $\hat{R}$-$\hat{S}$-plane lie on a double line $L_2$, defined by the range $|\hat{R}| \leq \sqrt{3}$ and the algebraic equation

$$
162\hat{S}^2 = (3 - \hat{R}^2)^3.
$$

(3.20)

cf. Fig. 3. All other algebraically possible points of the $\hat{R}$-$\hat{S}$-plane lie inside the region surrounded by the line $L_2$.

If one eigenvalue of the Ricci tensor equals $R$, i.e., if there exists a pair of eigenvalues of the Ricci tensor $(a, -a)$,

$$
Ric = \begin{bmatrix}
a & 0 & 0 \\
0 & -a & 0 \\
0 & 0 & ad
\end{bmatrix},
$$

(3.21)

we obtain

$$
\hat{R} = \frac{d}{\sqrt{d^2 + 2}}, \quad \hat{S} = \frac{2}{9} \frac{d(d^2 - 3)}{(d^2 + 2)^{3/2}}.
$$

(3.22)

For $d \in \mathbb{R} \cup \{\pm \infty\}$ the corresponding values in the $\hat{R}$-$\hat{S}$-plane lie on a line $L_{+-}$, defined by the range $|\hat{R}| \leq 1$ and the algebraic equation

$$
\hat{S} = \frac{11}{9} \hat{R}^3 - \hat{R}.
$$

(3.23)

Fig. 3 The algebraically possible values of $\hat{R}$ and $\hat{S}$, explanation see text.

In the case that one eigenvalue of the Ricci tensor is zero,

$$
Ric = \begin{bmatrix}
0 & 0 & 0 \\
0 & a & 0 \\
0 & 0 & ad
\end{bmatrix},
$$

(3.24)

we obtain for $a > 0$ respectively

$$
\hat{R} = \pm \frac{d + 1}{\sqrt{d^2 + 1}}, \quad \hat{S} = \pm \frac{1}{9} \frac{(d - 2)(2d - 1)(d + 1)}{(d^2 + 1)^{3/2}}.
$$

(3.25)
For $d \in \mathbb{R} \cup \{\pm \infty\}$ the corresponding values in the $\hat{R}$-$\hat{S}$-plane lie on a line $L_0$, cf. Fig. 3, defined by the range $|\hat{R}| \leq \sqrt{2}$ and the algebraic equation

$$\hat{S} = \frac{\hat{R}}{2}(1 - \frac{5}{9}\hat{R}^2).$$  \hfill (3.26)

Now we calculate the invariants to the homogeneous spaces of the different Bianchi types from their structure constants. Bianchi type I gives only the flat space, but all other ones can be given in their normal form according to eqs. (3.16, 17), and so be inserted into fig. 3. Homogeneous spaces possessing a six–dimensional isometry group are represented by the two tip points at $\hat{R} = \pm \sqrt{3}$. Proposition 6 implies that a subset of Bianchi type IX is at $\hat{R} = \sqrt{3}$. From proposition 14 one obtains that Bianchi type V is always at $\hat{R} = -\sqrt{3}$.

Homogeneous spaces which have a five–dimensional isometry group do not exist.

Homogeneous spaces possessing a four–dimensional isometry group are represented by points on the boundary line $L_2$ except the two tip points. From proposition 1 it follows that Bianchi types II and III belong to this set. (However, there are spaces located on $L_2$, which do not possess a 4-dimensional isometry group.) The eigenvalues of the Ricci tensor for Bianchi type II are $(\lambda, -\lambda, -\lambda)$ with any $\lambda > 0$, i.e.

$$\hat{R}_{\text{II}} = -\frac{1}{\sqrt{3}}, \quad \hat{S}_{\text{II}} = \frac{16\sqrt{3}}{81}$$  \hfill (3.27)

So, in the $\hat{R}$-$\hat{S}$-plane Bianchi type II corresponds to the point $\left(-\frac{1}{\sqrt{3}}, \frac{16\sqrt{3}}{81}\right)$ where the line $L_2$ and the line $L_{+-}$ meet in the quadrant $\hat{R} < 0 < \hat{S}$.

Atoms [14] in the set of Lie algebras are those non–commutative algebras which can be contracted to the commutative algebra only. For every dimension $d \geq 3$ there exist exactly two atoms, for $d = 3$ these are Bianchi types II and V. In other words: A 3–dimensional Lie algebra is an atom iff it is nilpotent or trivial but not commutative. They represent the most simple non–commutative Lie algebras. Together with Bianchi type IV they represent the following figure, cf. appendix A.
For Bianchi type IV we get

\[
R_{\text{IC}IV} = \begin{bmatrix}
-\frac{4e^{w}-1}{2e^{z+w}} & -\frac{1}{e^{z+w}} & 0 \\
-\frac{1}{e^{z+w}} & -\frac{4e^{w}+1}{2e^{z+w}} & 0 \\
0 & 0 & -\frac{4e^{w}+1}{2e^{z+w}} \\
\end{bmatrix} \tag{3.28}
\]

\[
N_{IV} = \frac{48e^{2w} + 16e^{w} + 3}{4e^{2(z+w)}} \tag{3.29}
\]

\[
\hat{R}_{IV} = -\frac{12e^{w} + 1}{\sqrt{48e^{2w} + 16e^{w} + 3}} \tag{3.30}
\]

\[
\hat{S}_{IV} = \frac{8(9e^{w} + 2)}{9(48e^{2w} + 16e^{w} + 3)^{3/2}} \tag{3.31}
\]

Thus, in the $\hat{R}$-$\hat{S}$-plane the Bianchi type IV corresponds to an open line (parametrized by $w$) connecting the edge point $(-\sqrt{3}, 0)$ Bianchi type V (approached by $w \to \infty$) and the Bianchi II point (approached by $w \to -\infty$).

If one removes the parameter $w$ one gets

\[
\hat{S} = \frac{2}{3} (3 - \hat{R})^{2} \frac{5 - \frac{2}{3} \hat{R}^{2} - \hat{R} \sqrt{18 - 5\hat{R}^{2}}}{[\sqrt{18 - 5\hat{R}^{2} - \hat{R}^{3}]^{3}}
\]

a formula which is valid for $-\sqrt{3} < \hat{R} < -1/\sqrt{3}$.

From subsection 3.2.1 we get for the Kantowski–Sachs models always

\[
\hat{R} = \sqrt{2}, \quad \hat{S} = -\frac{\sqrt{2}}{18}
\]

and the mirror point III is part of the Bianchi type III manifolds:

\[
\hat{R} = -\sqrt{2}, \quad \hat{S} = \frac{\sqrt{2}}{18}
\]

(Remark: It follows from proposition 10 that Bianchi type III has also other manifolds.) These are the points where the lines $L_{2}$ and $L_{0}$ intersect in the quadrants $\hat{R} \hat{S} < 0$. 

In the $\hat{R}$-$\hat{S}$-plane, to each of the Bianchi types $\text{VI}_h$ with $h \geq 0$ corresponds one line starting from the point of II (for $w \to \infty$, $w$ from eq. (3.6)).

For $h = 1$ this curve is a segment of $L_2$, for $h \neq 0, 1$ it has no common point with $L_2$. This behaviour follows already from proposition 1.

For $h \to \infty$ the $\text{VI}_h$-curve converges to the IV-curve. The $\text{VI}_0$-curve is part of $L_{++}$.

The $\text{VI}_h$ ($0 < h < 1$)-curves cover that part of the areas A2 and A10 which lie above the dotted line.

The $\text{VI}_h$ ($h > 1$)-curves cover all points above the IV-curve which are either in area A2 and below $L_2$ or in A1 and below or at the dotted line.

A 1-parameter subset of $\text{VI}_h$ (defined by $w = 0$) corresponds to a curve (which is dotted in fig. 5) starting from the point $(-1, -\frac{2}{9})$ (which is the point where the three lines $L_0$, $L_{+-}$ and $L_2$ intersect at $\hat{R} < 0$, i.e. the other end of the VI$_0$ line, shortly called ”point of VI$_0”$), passing the point of III for $h = 1$ and approaching the point of V for $h \to \infty$. One gets

$$\hat{R} = -\sqrt{3 - \frac{2}{h^2 + 1}}, \quad \hat{S} = \frac{18h^2 - 2}{9[(h^2 + 1)(3h^2 + 1)]^{3/2}}$$

Example: $h = \frac{1}{3}$ gives $\hat{S} = 0$ and $\hat{R} = -\sqrt{6/5}$. In the diagram fig. 5 this line can be inserted by eliminating the parameter $h$ to give

$$\hat{S} = -(3 - \hat{R}^2)^2 \frac{5\hat{R}^2 - 6}{18\hat{R}^3}$$

which makes sense for $-\sqrt{3} < \hat{R} < -1$.

For Bianchi type $\text{VII}_h$, the expressions for the scalar invariants $N$, $\hat{R}$, and $\hat{S}$ are symmetric under the parameter reflection $w \leftrightarrow -w$. For $w = 0$, $h > 0$ we obtain $(\hat{R}_{\text{VII}}, \hat{S}_{\text{VII}}) = (-\sqrt{3}, 0)$, which is the same as $(\hat{R}_V, \hat{S}_V)$. This reflects the fact that the 6-dimensional isometry group of Bianchi type V possesses 3-dimensional transitive subgroups belonging to Bianchi type VII$_h$ for every value $h > 0$.

In the $\hat{R}$-$\hat{S}$-plane, to each of the Bianchi types $\text{VII}_h$ with fixed $h > 0$ there corresponds an open line (parametrized by $w$) between the points of V (for $w = 0$) and II (for $|w| \to \infty$). These lines approach the line of Bianchi type IV for $h \to \infty$ with the same endpoints. For $h \to 0$ they approach the open piece of the line $L_{+-}$ which lies between II and the origin (0,0). This line segment is the locus of the Bianchi type VII$_0$. The lines of the types VII$_h$, $h > 0$, lie in the interior of a region bounded by the lines of VII$_0$, IV and the axis $\hat{S} = 0$. The explicit expressions for the eigenvalues of the Ricci
tensor for types \( VI_h \) and \( VII_h \) are complicated (see e.g. [15] or [32]), so we
give them for \( VII_0 \) only:

\[
Ric \ \forall VII_0 \sim (1 - e^{2w}, \ e^{2w} - 1, \ -(1 - e^w)^2)
\]

One peculiarity must be mentioned: For positive \( w \) the corresponding points
in the \( \hat{R} \)-\( \hat{S} \)-diagram converge to the origin as \( w \) tends to zero, but for \( w = 0 \)
the resulting space is flat and cannot be represented in that diagram. (The
latter is a consequence of the fact that the 6–dimensional isometry group of
the flat space possesses a 3–dimensional transitive subgroup of Bianchi type
\( VII_0 \).) II, \( VI_0 \) and \( VII_0 \) together fill the \( \hat{R} < 0 \)-part of the curve \( L_{++} \).

The 2-parameter expressions for the scalar invariants \( N, \hat{R}, \) and \( \hat{S} \) for
Bianchi type \( VIII \) and \( IX \) are symmetric under \( u \leftrightarrow t \).  

\[
Ric \ \forall VIII \sim (1 + 2e^{t+u} - e^{2u} - e^{2t}, \ e^{2t} - 1 - e^{2u} - 2e^u, \ e^{2u} - 1 - e^{2t} - 2e^t)
\]

\[
Ric \ \forall IX \sim (1 + 2e^{t+u} - e^{2u} - e^{2t}, \ e^{2t} - 1 - e^{2u} + 2e^u, \ e^{2u} - 1 - e^{2t} + 2e^t)
\]

In the \( \hat{R} \)-\( \hat{S} \)-plane, the boundary of the regions of Bianchi types \( VIII \) and
\( IX \) is given by the lines \( L_2, \ L_0 \) and that of \( VII_0 \), which is the common
boundary of \( VIII \) and \( IX \). This boundary can be reached from each region
as an asymptotic limit. Bianchi type \( VIII \) lies in the region to the left w.r.t.
\( VII_0 \), while Bianchi \( IX \) in the region to the right w.r.t. \( VII_0 \).

The points of \( II \) and \( (0,0) \), the endpoints of the line \( VII_0 \), are corner
points to both \( VIII \) and \( IX \). They can be reached from each of them as
asymptotic limit points. The point \( III \) is a corner point of the region \( VIII \).
There, one endpoint of \( L_0 \) hits on \( L_2 \).

The point \( (\sqrt{2}, \frac{1}{\sqrt{2}}) \) is a tip of the region \( IX \) which does not belong
to \( IX \) itself but can be reached as a limit of from points in \( IX \), e.g. for
\( (t,u) \to (\infty, \infty) \). This point corresponds to the Kantowski-Sachs spaces (KS
in Fig. 5 below), cf. subsection 3.2.1.

In the \( \hat{R} \)-\( \hat{S} \)-plane the region of \( IX \) is connected. In its point \( (1, \frac{3}{2}) \) (which,
by the way, is the mirror point to the \( VI_0 \) point) the lines \( L_2 \) and \( L_0 \) intersect.
If this point in the \( \hat{R} \)-\( \hat{S} \)-plane were missing, \( IX \) would be disconnected. Note
that this point in the \( \hat{R} \)-\( \hat{S} \)-plane actually corresponds to a 1-dimensional
line in the connected 2-dimensional space of homogeneous spaces modulo
absolute scale. It is then a special effect of the projection to the \( \hat{R} \)-\( \hat{S} \)-plane
that that 1-dimensional line degenerates to a point in this plane. Here the
fourth curvature invariant \( Y \) eq. (3.15) is necessary.

According to [11], if a maximal 4-dimensional isometry group exists, then
it is either \( IX \oplus R, \ VIII \oplus R \) or \( A_{4,10} \). This nicely agrees with possible
transitions \( IX \oplus R \to A_{4,10} \) and \( VIII \oplus R \to A_{4,10} \).
Fig. 5 The homogeneous 3–spaces, a) with description b) only the essential lines and the 16 areas

We conclude this section with Fig. 5 depicting the locus of local homogeneous 3–spaces with positive definite metric in the $\hat{R}$-$\hat{S}$–plane defined by fig. 3. The regions covered by type VIII and IX are horizontally hatched, the region covered by lines VII$_h$ for $h > 0$ is vertically hatched. The outer boundary line $L_2$ (double eigenvalue), the inner boundary line $L_0$ (some zero eigenvalue) and the line $L_{+-}$ (pair of eigenvalues differing only in their sign) are drawn as full lines. The dotted line is part of the boundary of the region covered by VI$_h$, $h > 0$, and IV is a full line.

4 ORIENTATION

If we prescribe an orientation, then both a Lie algebra and a homogeneous manifold get additional structure. For the 3–dimensional Lie algebras one can say the following: Bianchi types I, V, and VI$_h$ (arbitrary $h$) are self–dual, i.e., there exists an orientation–reversing Lie algebra isomorphism. All other Bianchi types lead to pairs of dual algebras if the orientation is prescribed, cf. e.g. [14].

The analogous question for the corresponding manifolds is not completely answered yet. But for cosmological models it should be required that the spatial orientation keeps always the same. To go further, one should find out, which of the homogeneous 3–manifolds possess an orientation–reversing isometry; we call such manifolds also self–dual.

It holds: If the space is locally symmetric (cf. the text after proposition 5) or if a hypersurface–orthogonal Killing vector field exists, then such an isometry exists. Therefore, all manifolds of Bianchi types I, II, III, V and the KS–spaces, and some of the Bianchi IX manifolds are self–dual.

A further partial answer is as follows: If all eigenvectors of the Ricci tensor are different and the underlying Lie algebra of isometries is not self–dual, then also the manifold is not self–dual. Therefore, all Bianchi type IV, some of Bianchi type VII, some of VIII and some of type IX manifolds fail to be self–dual.

5 DISCUSSION

The answer to Problem 3 (see the end of the Introduction) is "Yes" if and only if $(X, Y)$ is one of the following pairs: $(X, Y)$ for $X = Y$, $(I, Y)$ for arbitrary $Y$, $(II, Y)$ for $Y \not\in \{I, V\}$, $(V, IV)$, $(VI_0, VIII)$, $(VII_0, Y)$ for $Y \in \{VIII, IX\}$.
Note that \( \lim_{h \to \infty} \text{VI}_h = \text{IV} = \lim_{h \to \infty} \text{VII}_h \), hence Problems 1 and 2 have solutions by a continuous change of the parameter \( h \) in every subset of \( \{ \text{IV}, \text{VI}_h, \text{VII}_h \} \) also without going through the flat intermediate.

If we generalize Problems 3, 4, 5 for Bianchi types to the analogous Problems 3', 4', 5' for Bianchi and Kantowski-Sachs types then 3', 4', 5' are again equivalent. This problem has additional solutions for following pairs: (KS, IX), (X, KS) for \( X \in \{ \text{I}, \text{KS} \} \).

Generalized homogeneous cosmological models enjoy a renewed interest, see e.g. [1, 2, 42].

The review [43] entitled "Cosmic Topology" contains many valuable facts on related questions. E.g. Thurston's homogeneous three-geometries (appearing as universal covering spaces of some compact homogeneous three-geometries as quotients w.r.t. some discrete isometry subgroup acting simply transitive on its orbits) correspond to some characteristic Bianchi or Kantowski-Sachs types each. (The reverse is not true.) According to [43], \( R^3, S^3, H^3, S^2 \times R, H^2 \times R, \text{SL}(2, R) \) correspond to type I, IX, V, KS, III, VIII, respectively. We add: the remaining Thurston types, \( \text{Nil} \) and \( \text{Sol} \), correspond to Bianchi types II and VI respectively. So it is obvious, as mentioned in the Introduction, that certain transitions of the local geometry to another Bianchi or Kantowski-Sachs type determine changes of the global topology of the homogeneous three-manifolds. Hence, although we concentrated here on a classification problem of transitions in local geometry, it also helps to classify related transitions of the global geometry.

In the present paper we concentrated on the question, how the Bianchi type can change with time in those models, and how this is related to the topology in the space of 3-dimensional Lie algebras. One should mention that the definition for the generalized homogeneous cosmological models differs in different papers. The main difference is the following: "There exists a system of reference such that the spatial slices \( t = \text{const.} \) have homogeneous inner geometry." versus "There exists a synchronized system of reference such that the spatial slices \( t = \text{const.} \) have homogeneous inner geometry." (compare also Refs. [3, 4]). For our purpose this distinction is not essential, but if one solves the Einstein equation for models of this kind the distinction becomes essential.

A systematic consideration of homogeneous models is also useful for a canonical quantization. In [32] the Wheeler–de Witt equation was considered for the Bianchi models with 2 or 3 minisuperspace dimensions.

What has to be done yet? The analogous question should be considered for Lorentz signature spaces. The set of manifolds is no more known to be a Hausdorff space, the calibration to \( \hat{N} = 1 \) is no longer possible in general, and more subcases have to be distinguished. Also for positive definite signature,
some questions remain open yet, e.g.: "If the four invariants \( R, N, S, \) and \( Y \) (eqs. (3.13-15)) of a sequence of homogeneous 3–spaces converge, then the corresponding manifolds also converge." This statement is stronger than "Equality of these 4 invariants implies isometry."

The main result of the present paper are figures 3 and 5. The 4 lines \( L_0, L_{+-}, \) axes \( \hat{R} = 0, \) and \( \hat{S} = 0 \) divide the surface surrounded by \( L_2 \) of fig. 3 into \( 2^4 = 16 \) areas. We denote the upper 8 areas from left to right by \( A_1, \ldots, A_8 \) and the lower 8 areas from left to right by \( A_9, \ldots, A_{16} \). (So, area \( A_m \) and area \( A(17 - m) \) are centrally mirrored to each other.) We have shown at the beginning of sect. 3.2.2. that all these 16 areas represent algebraically possible values for \( \hat{R}, \hat{S} \).

Now we can add: The interior of the following 7 areas does not represent a homogeneous space: \( A_6, A_7, A_9, A_{12}, A_{13}, A_{14}, A_{15} \). Exactly four points of the boundaries of these areas represent homogeneous manifolds: KS, Bianchi type V, the VI\(_0\) point and that type IX space with eigenvalues \((0, 0, 1)\). No point lying above the dotted line in region A1 represents a homogeneous space. This enumeration is complete: All other points in the diagram do really represent a homogeneous space. (Up to now, one can find in the literature only examples like: "There does not exist a homogeneous \( V_3 \) whose eigenvalues of the Ricci tensor are \((\lambda, -\lambda, 0)\) with \( \lambda \neq 0 \)." This corresponds to the value \( \hat{R} = \hat{S} = 0 \).) The interior of the 4 areas \( A_4, A_5, A_8, A_{16} \) is fully covered by Bianchi type IX spaces. Only the 5 areas \( A_1, A_2, A_3, A_9, A_{10} \) possess a more detailed structure which becomes clear from sect. 3.

Section 3.1. figs. 1 and 2 can be summarized as follows: If a sequence of 2-dimensional homogeneous manifolds converges, then the corresponding Lie algebras of the (always 3–dimensional!) isometry groups also converge in the Segal topology. Vice versa: If the Lie algebras converge in the Segal topology, then one can find examples of correspondingly converging manifolds.

Section 3.2. figs. 3 - 5 does not possess such a simple summary. The reason is that the dimension of the isometry groups for 3–dimensional homogeneous manifolds varies between 3, 4 and 6. But for some subsets one can state analogous results, let us pick up fig. 4, presenting Bianchi types I, II, V, and IV. It holds within this set: If a sequence of manifolds converges, then the corresponding Lie algebras converge in the Segal topology, and analogous vice versa as above. One can illustrate this situation in a plane where the origin represents type I, the positive \( x \)–axis type V, the positive \( y \)–axis type II, and the region \( x > 0, y > 0 \) represents all type IV spaces. In other words: If a sequence of manifolds of type IV converges to a homogeneous manifold of another type, then it is of Bianchi type I, II, or V. Every manifold of type I, II, and V appears as limit of such a sequence.
APPENDIX A. THE TOPOLOGICAL SPACE OF LIE ALGEBRAS

The bracket \([ \cdot, \cdot \] \) defines a Lie algebra of dimension \(n\), iff the structure constants satisfy the \(n\{ \binom{n}{2} + \binom{n}{1} \}\) antisymmetry conditions \(C_{[ij]}^k = 0\), and the \(n \cdot \binom{n}{3}\) quadratic compatibility constraints \(C_{[ij}^l C_{kl]}^m = 0\), called Jacobi identity.

The space of all sets \(\{C_{ij}^k\}\) satisfying the antisymmetry condition and the Jacobi identity can be considered as a subvariety \(W^n \subset \mathbb{R}^{n^3}\) of dimension

\[
\dim W^n \leq n^3 - \frac{n^2(n+1)}{2} = \frac{n^2(n-1)}{2}.
\]

For \(n = 3\) the structure constants can be written as

\[
C_{ij}^k = \epsilon_{ijl}(n^{lk} + \epsilon^{lkm}a_m),
\]

where \(n^{ij}\) is symmetric and \(\epsilon_{ijk} = \epsilon_{ijk}\) totally antisymmetric with \(\epsilon_{123} = 1\). The Jacobi identity is equivalent to \(n^{lm}a_m = 0\). Without loss of generality one gets \(a_m = (h, 0, 0)\) and \(n^{lk} = \text{diag}(n_1, n_2, n_3)\). The quadruples \((h, n_1, n_2, n_3)\) for the Bianchi types are \((h \geq 0\) but arbitrary): I(0, 0, 0, 0); II(0, 1, 0, 0); III=VI; IV(1, 0, 0, 1); V(1, 0, 0, 0); VI\(_h\)(h, 0, 1, -1); VII\(_h\)(h, 0, 1, 1); VIII(0, 1, 1, -1) and IX(0, 1, 1, 1).

Throughout the following, we will need the separation axioms from topology.

**Axiom** \(T_0\): For each pair of different points there is an open set containing only one of both.

**Axiom** \(T_1\): Each one–point set is closed.

**Axiom** \(T_2\) (HAUSDORFF): Each pair of different points has a pair of disjoint open neighbourhoods.

It holds: \(T_2\) implies \(T_1\), \(T_1\) implies \(T_0\).

\(GL(n)\) basis transformations induce \(GL(n)\) tensor transformations between equivalent structure constants.

\[
C_{ij}^k \sim (A^{-1})_h^k C_{fg}^h A_i^f A_j^g \quad \forall A \in GL(n),
\]

where \(\sim\) denotes the equivalence relation. This induces the space \(K^n = W^n/GL(n)\) of equivalence classes w.r.t. the nonlinear action of \(GL(n)\) on \(W^n\).

The \(GL(n)\) action on \(W^n\) is not free in general. It holds:

\[
\dim W^n > \dim K^n \geq \dim W^n - n^2.
\]
The first inequality is a strict one, because the multiples of the unit matrix in $GL(n)$ give rise to equivalent points of $K^n$.

Let $\phi : W^n \to K^n$ be the canonical map for the equivalence relation $\sim$ defined by the action of $GL(n)$ in $W^n$. The natural topology $\kappa^n$ of $K^n$ is given as the quotient topology of the induced topology of $W^n \subset \mathbb{R}^n$ w.r.t. $GL(n)$ equivalence. This means: A sequence of Lie algebras converges to a limit algebra if there exists a basis for each algebra such that the corresponding structure constants converge to each other as real numbers. For $n \geq 2$, the topology $\kappa^n$ is a $T_0$ but not a $T_1$–space.

Figure 6 shows the topological relation between the Bianchi types, the relation between the graph and the topology is outlined in appendix B.

**APPENDIX B. DIRECTED GRAPHS AND FINITE TOPOLOGICAL SPACES**

This appendix shall help reading the diagrams. First we give the intuitive idea and second we outline the mathematical apparatus behind it.

First, look at Fig. 4 and imagine that "I" is represented by the origin of a Cartesian coordinate system. "V" shall denote all points of the positive $x$–axis, "II" the positive $y$–axis, and "IV" is the region of points with $x > 0$ and $y > 0$. Then it is clear what the arrows mean: the possible convergence of respective representatives. There is no extra arrow directly from "IV" to "I" (seen e.g. by representatives at the line $x = y$), because the transitivity property is know to be satisfied, and we want to have a minimal number of arrows.

Second, let $X$ be a finite nonvoid set. A directed graph in $X$ is a subset $\Gamma \subset X \times X; (x, y) \in \Gamma$ means: there is a directed edge from vertex $x$ to vertex $y$. The directed graph $\Gamma$ is called transitively closed, if $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$ imply $(x, z) \in \Gamma$ and $(x, x) \in \Gamma$ for all $x \in X$. The transitive closure $Cl \Gamma$ of the directed graph $\Gamma$ is the smallest transitively closed directed graph containing $\Gamma$. Each directed graph $\Gamma$ defines a topology in $X$ as follows: $A \subset X$ is closed iff $x \in A$ and $(x, y) \in \Gamma$ imply $y \in A$.

Proof: $\emptyset$ and $X$ are closed by this definition. $X$ is finite, so it suffices to show that both union and intersection of two closed subsets are closed. $\square$

It holds: Two directed graphs $\Gamma, \bar{\Gamma}$ define the same topology iff $Cl \Gamma = Cl \bar{\Gamma}$. The topology defined by $\Gamma$ is finer than the topology defined by $\bar{\Gamma}$ iff $Cl \Gamma \subset Cl \bar{\Gamma}$.

Proof: For a directed graph, we define the closure-operation $cl_\Gamma$ as follows: For $A \subset X$, $cl_\Gamma(A) := A \cup \{y | x \in A, (x, y) \in \Gamma\}$

$cl_\Gamma$ is idempotent (i.e., $cl_\Gamma^2 \equiv cl_\Gamma \circ cl_\Gamma = cl_\Gamma$) iff $\Gamma \cup \{(x, x) | x \in X\}$ is
transitively closed. Let \( \text{card } X = n \), \( A \subset X \) and \( \text{cl } A \) be the closure of \( A \) in the topology defined by \( \Gamma \), then it holds \( \text{cl } A = c_{\Gamma}^n A \). \( \square \)

Examples: The discrete topology in \( X \) is defined from the empty graph \( \Gamma = \emptyset \).
The trivial topology in \( X \) is defined from the complete graph \( \tilde{\Gamma} = X \times X \).
The inverse graph to \( \Gamma \) is defined by
\[
\Gamma^{-1} := \{(y, x) \mid (x, y) \in \Gamma\}
\]

It holds: The topologies defined by \( \Gamma \) and \( \Gamma^{-1} \) are inverse to each other, i.e., \( A \subset X \) is closed in the topology defined by \( \Gamma \) iff it is open in the topology defined by \( \Gamma^{-1} \).

Definition: A closed path in \( \Gamma \) is a sequence of edges
\[
(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1) \in \Gamma
\]
connecting at least two vertices (i.e., \( \text{card } \{x_1, \ldots, x_n\} \geq 2 \)). It holds: The topology defined by \( \Gamma \) is \( T_0 \) iff \( \Gamma \) does not contain a closed path.

Let us now go the other direction: Let a topology in \( X \) be given, \( \text{cl} \) denotes the closure with respect to it. We define a directed graph \( \Gamma \) by \( (x, y) \in \Gamma \) iff \( y \in \text{cl } \{x\} \). It holds: This graph \( \Gamma \) is transitively closed, and the topology defined from it coincides with the initial topology.

Conclusion: There is a one-to-one correspondence between all topologies and all transitively closed directed graphs in the finite set \( X \).

For visualizing a finite topological space, however, the transitively closed graphs are not best suited, one should prefer a graph with less edges. To find the best suited graph we define

Definition: A directed graph \( \Gamma \) is called minimal if \( \tilde{\Gamma} \subset \Gamma \) and \( \text{Cl } \tilde{\Gamma} = \text{Cl } \Gamma \) imply \( \tilde{\Gamma} = \Gamma \).

Each directed graph \( \Gamma \) contains a minimal subgraph \( \tilde{\Gamma} \) with \( \text{Cl } \Gamma = \text{Cl } \tilde{\Gamma} \) and therefore, each finite topological space can be visualized by a minimal graph. For card \( X = 2 \) this minimal graph is unique.

In general, however, there exists more than one minimal graph for one topology. For card \( X \geq 3 \), \{\((x_1, x_2), (x_2, x_3), (x_3, x_1)\)\} and \{\((x_1, x_3), (x_3, x_2), (x_2, x_1)\)\} represent two different minimal graphs with same transitive closure. It holds: The minimal graph representing a finite topological space \( X \) is unique iff all the connected components of \( X \) possessing more than two points are \( T_0 \)-spaces.

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