Classical phase fluctuations in $d$-wave superconductors

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We study the effects of low-energy nodal quasiparticles on the classical phase fluctuations in a two-dimensional $d$-wave superconductor. The singularities of the phase-only action at $T \to 0$ are removed in the presence of disorder, which justifies using an extended classical XY-model to describe phase fluctuations at low temperatures.

I. INTRODUCTION

The spectacular successes of the Bardeen-Cooper-Schrieffer (BCS) mean-field theory of superconductivity are based on the fact that the so-called Ginzburg-Levanyuk number $G_i(D)$, which controls the size of fluctuation effects in $D$-dimensional samples, is very small in bulk conventional materials. On the other hand, the order parameter fluctuations become more pronounced, even dominant, in low-dimensional systems with a small Fermi energy, e.g. in quasi-two-dimensional cuprate superconductors. In particular, the fluctuations of the order parameter phase in the underdoped cuprates are enhanced due to a low value of the superfluid density, leading to the large deviations from the BCS picture, including the pseudogap phenomenon. According to Emery and Kivelson, the Cooper pairs survive in underdoped cuprates far above the critical temperature $T_c$, but without global phase coherence, which is destroyed by thermal phase fluctuations through the Berezinskii-Kosterlitz-Thouless mechanism. This idea has been further elaborated by many authors, for a review see, e.g., Refs. 7,9,10. As temperature is lowered, a long-range phase coherence sets in, but the phase fluctuations continue to play important role in the superconducting state, remaining predominantly classical well below $T_c$. One might expect however that quantum phase fluctuations eventually take over at the lowest temperatures. The crossover temperature $T_{cl}$ between the classical and quantum regimes is quite high (of the order of $T_c$) in clean charged systems, but can be significantly reduced in the presence of dissipation. While the estimates of $T_{cl}$ in cuprates obtained by different groups vary considerably, see e.g. Refs. 7,9,10, here we adopt the view that the dissipation is strong enough to make the quantum effects negligible.

In this brief review we develop an effective long-wavelength theory of the classical phase fluctuations in $d$-wave superconductors, both with and without elastic disorder. At the Gaussian level, the lowest-order term in the gradient energy expansion is simply $\rho_s v_s^2/2$, where $\rho_s$ is the superfluid mass density and $v_s$ is the superfluid velocity. We show that the higher-order gradient terms, which contain $\nabla \psi$, are singular in a clean system at low temperatures due to the presence of gap zeros and also discuss the effects of disorder on those singularities. As a by-product of our theory, we address the question whether using the classical XY-model in $d$-wave superconductors can be justified from microscopic theory.

The article is organized as follows. In Sec. III the general field-theoretical description of the bosonic excitations in superconductors and a phase-only effective action are derived in the clean case. In Sec. IV we focus on the case of a two-dimensional neutral $d$-wave superconductor, which is treated in the nodal approximation. The microscopic expressions for the energy of fluctuations are compared to the predictions of the classical XY-model. In Sec. V we derive the $d$-wave phase-only action in the presence of elastic impurity scattering. Sec. VI concludes with a discussion of our results.

II. EFFECTIVE FIELD THEORY: CLEAN CASE

The starting point of our analysis is the tight-binding Hamiltonian

$$H = \sum_{rr'} \xi_{rr'} c_{r\sigma}^\dagger c_{r'\sigma} + \sum_r U_r c_{r\sigma}^\dagger c_{r\sigma} - g \sum_{\langle rr' \rangle} B_{rr'} c_{rr'}^\dagger c_{rr'} + \frac{1}{2} \sum_{rr'} (n_r - n_0) V_{rr'} (n_{r'} - n_0),$$

where $r$ label the sites of a tetragonal lattice. The hopping amplitude $t_{rr'}$ and the chemical potential $\mu$ are combined into the band-dispersion matrix $\xi_{rr'} = -t_{rr'} - \mu \delta_{r,r'}$, which is real and symmetric in the absence of external magnetic field. The second term describes impurity scattering. The third term is the BCS interaction in the $d$-wave channel, $g > 0$ is the coupling constant, and the operator $B_{rr'} = (c_{r\uparrow}^\dagger c_{r\uparrow} - c_{r\downarrow}^\dagger c_{r\downarrow})/\sqrt{2}$ destroys a singlet pair of electrons at the nearest-neighbor sites ($rr'$) in the $xy$ plane. The last term describes the repulsive interaction between electrons, $n_r = c_{r\sigma}^\dagger c_{r\sigma}$ is the particle number density, $n_0 = \langle n_r \rangle$ is the average number of particles per site (which is equal to...
the ionic background density, thus ensuring the overall charge neutrality of the system), and $V_{rr'}$ is the interaction matrix.

Let us first look into the clean case. Setting $U_r = 0$ in Eq. (14), the partition function can be written as a functional integral over Grassmann fields $c_{r\sigma}(\tau)$ and $\bar{c}_{r\sigma}(\tau)$:

$$Z = \text{Tr} e^{-\beta H} = \int \mathcal{D}c \mathcal{D}\bar{c} e^{-S[c,\bar{c}]},$$

where $S = \int_0^\beta d\tau [\sum_r \bar{c}_{r\sigma} \partial_\tau c_{r\sigma} + H(\tau)]$, $\beta = 1/T$ (in our units $k_B = \hbar = 1$). Using the Hubbard-Stratonovich transformation to decouple the interaction terms, we end up with the representation of $Z$ as a functional integral over $c, \bar{c}$ and two bosonic fields: a complex field $\Delta_{rr'}(\tau)$, which describes the superconducting order parameter fluctuations and is non-zero only on the bonds between the nearest neighbors, and a real scalar potential field $\varphi_r(\tau)$. The fermionic part of the action then becomes $S = \text{Tr}(C\bar{G}^{-1}C)$, where

$$C_r = \left( \begin{array}{cc} c_{r\uparrow} \\ \bar{c}_{r\downarrow} \end{array} \right), \quad \bar{C}_r = \left( \begin{array}{cc} \bar{c}_{r\uparrow} \\ c_{r\downarrow} \end{array} \right)$$

are the Nambu spinors, and $\bar{G}^{-1}$ is the inverse Green’s operator:

$$\bar{G}^{-1}(r,\tau; r',\tau') = \delta(\tau - \tau') \begin{pmatrix} \delta_{rr'} [-\partial_\tau + i\varphi_r(\tau)] - \xi_{rr'} & -\Delta_{rr'}(\tau) \\ -\Delta_{rr'}(\tau) & \delta_{rr'} [-\partial_\tau + i\varphi_r(\tau)] + \xi_{rr'} \end{pmatrix}.$$ (3)

We use the notation “Tr” for the full operator trace with respect to both the space-time coordinates and the Nambu matrix indices, reserving “tr” for a 2 × 2 matrix trace in the Nambu space. Integrating out the fermionic fields, we obtain

$$Z = \int \mathcal{D}\Delta^* \mathcal{D}\Delta \mathcal{D}\varphi e^{-S_{eff}[\Delta^*, \Delta, \varphi]},$$

with the effective action

$$S_{eff} = -\text{Tr} \ln \bar{G}^{-1} + \int_0^\beta d\tau \left( \frac{1}{2} \sum_{rr'} |\Delta_{rr'}|^2 + \frac{1}{2} \sum_r \varphi_r V_{rr'}^{-1} \varphi_{r'} - i\hbar n_0 \sum_r \varphi_r \right).$$ (5)

The mean-field BCS theory corresponds to a stationary and uniform saddle point of the effective action (14), which is found from the equations $\delta S_{eff}/\delta \Delta^* = \delta S_{eff}/\delta \varphi = 0$. The solution describing $d$-wave pairing is given by $\varphi_{0,r} = 0$ and

$$\Delta_{0,rr'} = \left\{ \begin{array}{ll} +\Delta_0, & \text{if} \quad r' = r \pm ax, \\ -\Delta_0, & \text{if} \quad r' = r \pm ay. \end{array} \right.$$ (6)

In the momentum representation, $\Delta_k = \Delta_0(T)\phi_k$, where $\phi_k = 2(\cos k_x a - \cos k_y a)$ is the $d$-wave symmetry factor. The temperature dependence of the gap amplitude $\Delta_0$ is determined by the standard BCS self-consistency equation, generalized to the case of an anisotropic order parameter.$^{12}$

Inverting the operator (3), we find the mean-field matrix Green’s function:

$$G_0(k,\omega_n) = \frac{G_0(k,\omega_n)}{-F_0(k,\omega_n)}, \quad \frac{-F_0(k,\omega_n)}{G_0(-k, -\omega_n)} = \frac{i\omega_n \tau_0 + \xi_k \tau_3 + \Delta_k \tau_1}{\omega^2 + \xi^2_k + \Delta^2_k}.$$ (7)

Here $\omega_n = (2n+1)\pi T$ is the fermionic Matsubara frequency, $G_0$ and $F_0$ are the usual normal and anomalous Gor’kov’s functions of the superconductor,$^{12}$ $\xi_k = \xi_{-k}$ is the band dispersion of free electrons, and $\tau_i$ are Pauli matrices in the Nambu space. The Green’s function (7) determines the single-particle properties in the mean-field approximation. In particular, after its analytical continuation to the real frequency axis, $i\omega_n \rightarrow \omega + i0$, one obtains the energies of elementary fermionic excitations, or the Bogoliubov quasiparticles:

$$E_k = \sqrt{\xi^2_k + \Delta^2_k}.$$ (8)

For the $d$-wave order parameter, the gap in the excitation energy vanishes along the diagonals of the Brillouin zone. These zeros, or the gap “nodes”, are responsible for many peculiar thermodynamical and transport properties in the superconducting state on the mean-field level.$^{12}$ Below we show that the gap nodes also have dramatic effects on the long-wavelength behavior of phase fluctuations.
A. Fluctuations

Deviations from the mean-field solution can be represented in terms of the amplitude and phase fluctuations: \( \Delta_{rr'}(\tau) = [\Delta_{0,rr'} + \delta \Delta_{rr'}(\tau)] e^{i \theta_{rr'}(\tau)} \), where \( \delta \Delta \) and \( \Theta \) are real. We neglect the amplitude fluctuations because they are gapped, see e.g. Ref.\( ^1 \), and therefore make a negligible contribution at low temperatures. Since the number of bonds in a square lattice is twice the number of sites, one needs two on-site phase fields \( \theta_r(\tau) \) and \( \Theta_r(\tau) \) to describe the phase degrees of freedom. One possible parametrization is

\[
\Theta_{rr'} = \begin{cases} 
\theta_{rr}, & \text{if } r' = r + a\hat{x} \\
\theta_{rr} + \Theta_{rr}, & \text{if } r' = r + a\hat{y}.
\end{cases}
\]

The fluctuations of \( \hat{\theta} \), which describe a change in the symmetry of the order parameter from a pure \( d \)-wave to a \( d + is \)-wave, can be neglected.\( ^2 \)

If \( \theta_r \) changes slowly over the lattice constant, then one can make the replacement

\[
\Delta_{0,rr} e^{i \theta_r} \rightarrow \Delta_{0,rr} e^{i (\theta_r + \Theta_r)/2},
\]

(recall that \( r \) and \( r' \) are nearest neighbors). The next step is to perform a gauge transformation to make the off-diagonal elements of \( G^{-1} \) in Eq. (4) real:

\[
U(r, \tau) G^{-1}(r; \tau; r', \tau') U(r', \tau') = \tilde{G}^{-1}(r, \tau; r', \tau'),
\]

where \( U(r, \tau) = \exp[i \tau \theta_r(\tau)/2] \). This transformation leaves the operator trace in the effective action \( ^{5} \) invariant. Although the order parameter \( ^{10} \) is no longer invariant under local phase rotations \( \theta_r \rightarrow \theta_r + 2\pi \), our results are not affected since we consider only small fluctuations of the phase in the low-temperature limit, where the contribution of vortices can be safely neglected.

The operator \( \tilde{G}^{-1} \) can be represented in the form \( \tilde{G}^{-1} = G_0^{-1} - \Sigma \), where \( G_0 \) is the mean-field matrix Green’s function, whose Fourier transform is given by Eq. (7), and

\[
\Sigma(r, \tau; r', \tau') = i \delta(\tau - \tau')\sum_{rr'} \left[ 1 \frac{\partial \theta_r(\tau)}{\partial \tau} + \varphi_r(\tau) \right] + \xi_{rr'} \left[ e^{-i \frac{\pi}{2} \alpha_0(\tau) - \varphi_r(\tau)} - 1 \right]
\]

is the self-energy correction due to fluctuations. At slow temporal and spatial variations of \( \theta \) and small \( \varphi \), one can expand the effective action \( ^{5} \) in powers of \( \Sigma \), keeping only the two lowest orders in the expansion, with the following result:

\[
S_{eff}[\theta, \varphi] = S_0 + \text{Tr}(G_0\Sigma) + \frac{1}{2} \text{Tr}(G_0\Sigma G_0\Sigma) + O(\Sigma^3) + \frac{1}{2} \int_0^\beta d\tau \sum_{rr'} \varphi_r(\tau) V_{rr'}^{-1} \varphi_{r'}(\tau) - i n_0 \int_0^\beta d\tau \sum_r \varphi_r(\tau),
\]

where

\[
S_0 = -\text{Tr} \ln G_0^{-1} + \frac{1}{g} \sum_{rr'} |\Delta_{0,rr'}|^2 = \beta E_0
\]

is the saddle-point action, \( E_0 \) is the total mean-field energy of the superconductor, and \( G_0 \) is the saddle-point Green’s function, see Eq. (4). For non-interacting fluctuations we keep only the terms of the first and second order in \( \Sigma \).

Calculating the traces in Eq. (13), we obtain the Gaussian action

\[
S_{eff}[\theta, \varphi] = S_0 + S_1 + S_2,
\]

where

\[
S_1 = \frac{i n_0}{2} \int_0^\beta d\tau \sum_r \frac{\partial \theta_r(\tau)}{\partial \tau} = i \pi n_0 \sum_r \frac{\theta_r(\beta) - \theta_r(0)}{2\pi}
\]

is the topological term containing the phase winding numbers, and

\[
S_2 = \frac{1}{2} \sum_Q \left[ L_{\varphi \varphi}(Q)|\varphi(Q)|^2 + L_{\theta \varphi}(Q)\theta^*(Q)\varphi(Q) + \frac{1}{4} L_{\theta \theta}(Q)|\theta(Q)|^2 \right].
\]
Here we use the shorthand notations \( Q = (q, \nu_m) \) and

\[
\sum_{q} (...) = T \sum_{m} \sum_{q} (...) ,
\]

where \( \nu_m = 2m\pi T \) is the bosonic Matsubara frequency and the momentum summation goes over the first Brillouin zone. The lattice Fourier transforms of the fields are defined by the usual expressions: \( \theta_{ij}(\tau) = N^{-1/2} \sum_{q} \theta(q, \tau)e^{iq\tau} \)

\[
\text{etc, where } N \text{ is the number of lattice sites. Since both } \varphi \text{ and } \theta \text{ are real, they satisfy } \varphi^*(Q) = \varphi(-Q), \theta^*(Q) = \theta(-Q). \]

The coefficients in Eq. (18) are given by

\[
\begin{align*}
L_{\varphi\varphi}(Q) &= V^{-1}(q) + \Pi_0(Q), \\
L_{\theta\varphi}(Q) &= i\nu_m \Pi_0(Q) + q_i \Pi_1^i(Q), \\
L_{\theta\theta}(Q) &= \nu_m^2 \Pi_0(Q) - 2i\nu_m q_i \Pi_1^i(Q) + q_i q_j \Pi_2^{ij}(Q).
\end{align*}
\]

Here \( V(q) \) is the Fourier transform of the interaction matrix \( V_{rr'} \), and

\[
\begin{align*}
\Pi_0(Q) &= -\sum_K \text{tr}[G_0(K + Q)\tau_3 G_0(K)\tau_3], \\
\Pi_1^i(Q) &= \sum_K v_i \text{tr}[G_0(K + Q)\tau_3 G_0(K)\tau_0], \\
\Pi_2^{ij}(Q) &= \sum_K m^{-1}_{ij} \text{tr}[G_0(K)\tau_3] + \sum_K v_i v_j \text{tr}[G_0(K + Q)\tau_0 G_0(K)\tau_0].
\end{align*}
\]

In these expressions, \( K = (k, \omega_n) \),

\[
\sum_{K} (...) = T \sum_{n} \frac{1}{N} \sum_{q} (...) N^{-1} \sum_{n} \Omega \int \frac{d^D k}{(2\pi)^D} (...) ,
\]

\( m^{-1}_{ij}(k) = \partial^2_k \xi_{k} / \partial k_i \partial k_j \) is the inverse effective mass tensor, \( v(k) = \partial \xi_k / \partial k \) is the quasiparticle band velocity, and \( \Omega \) is the unit cell volume (to simplify the notations, below we set \( \Omega = 1 \)). Eqs. (18) are obtained in the limit of small \( q \) from more general expressions, using the gradient expansion \( \xi_{k+q} = \xi_k + v q + (1/2)m^{-1}_{ij} q_i q_j + O(q^3) \).

Integrating out the field \( \varphi \), we finally arrive at the phase-only effective action:

\[
S_{\text{eff}}[\theta] = S_0 + S_1 + \frac{1}{8} \sum_{q} \left[ L_{\theta\theta}(Q) + \frac{L_{\varphi\varphi}(Q)}{L_{\varphi\varphi}(Q)} \right] \theta(Q)^2.
\]

In this article, we focus on the case of classical phase fluctuations and neglect all interactions other than those responsible for the Cooper pairing, which corresponds to neglecting the \( \tau \)-dependence of \( \theta \) and setting \( V(q) = 0 \). Then the topological term vanishes and only the \( \Pi_2^{ij} \) contribution survives in the third term in Eq. (22), so that the effective action becomes

\[
S_{\text{eff}}[\theta] = \beta \mathcal{E}_0 + \beta \mathcal{E}[\theta],
\]

where \( \mathcal{E} \) is the energy of fluctuations in the Gaussian approximation. Calculating the Matsubara sums in Eq. (21) and introducing the superfluid velocity \( v_s = (1/2m) \nabla \theta \), where \( m \) is the electron mass, we obtain

\[
\mathcal{E} = \frac{1}{2} \sum_{q} \mathcal{K}_{ij}(q)v_{s,i}(q)v_{s,j}(q),
\]

with the kernel

\[
\mathcal{K}_{ij}(q) \equiv m^2 \Pi_2^{ij}(q, 0) = \mathcal{K}_{ij}^{(0)} + \mathcal{I}_{ij}(q),
\]

where

\[
\mathcal{K}_{ij}^{(0)} = 2m^2 T \sum_{n} \int \frac{d^D k}{(2\pi)^D} m^{-1}_{ij}(k) G_0(k, \omega_n),
\]

\[
\mathcal{I}_{ij}(q) = -m^2 \int \frac{d^D k}{(2\pi)^D} v_i(k)v_j(k) \left[ C_-(k, q) \tanh \frac{E_{k+q}}{2T} + \tanh \frac{E_k}{2T} + C_+(k, q) \tanh \frac{E_{k+q}}{2T} + \tanh \frac{E_k}{2T} \right],
\]

(27)
and

\[ C_\pm(k,q) = \frac{1}{2} \left( 1 \pm \frac{\xi_k \xi_{k+q} + \Delta_k \Delta_{k+q}}{E_k E_{k+q}} \right) \]  

(28)

are the coherence factors.

An important characteristic of the superconductor is the superfluid density tensor, which is defined as

\[ \rho_{s,ij}(T) = \mathcal{K}_{ij}(0). \]  

(29)

Its temperature dependence can be easily found in two limiting cases. In the normal state, \( \Delta_k = 0 \), and one can use the identity \( \partial G_0/\partial k = v G_0^2 \) in Eq. (21) to obtain \( \rho_{s,ij}(T > T_c) = 0 \). On the other hand, at zero temperature we have \( \rho_{s,ij}(0) = \mathcal{K}_{ij}^{(0)} \), since \( \mathcal{I}_{ij}(0) = 0 \) at \( T = 0 \).

The expressions (25,26,27) are valid for arbitrary band structure and gap symmetry. In a Galilean-invariant system, i.e. for \( \xi_k = k^2/2m - \mu \), the tensor (26) takes a particularly simple form: \( \mathcal{K}_{ij}^{(0)} = \rho_0 \delta_{ij} \), where \( \rho_0 = 2mT \sum_n \int G_0(k,\omega_n) \) is the mass density of electrons. Therefore, the superfluid density tensor at zero temperature is \( \rho_{s,ij}(0) = \rho_0 \delta_{ij} \), i.e. all electrons are superconducting. In general, there is no such simple relation in a crystal.

### III. TWO-DIMENSIONAL CASE

In this section, we apply the general theory developed above to a two-dimensional \( d \)-wave superconductor. In this case, the low-energy physics at \( T \rightarrow 0 \) can be conveniently described using the so-called “nodal approximation” which takes advantage of the fact that the excitation energy (8) for the antinodal approximation near the gap nodes located at \( k_n = k_F \hat{k}_n \) (\( n = 1, 2, 3, 4 \)) on the Fermi surface.

Here

\[ \hat{k}_1 = \frac{\hat{x} + \hat{y}}{\sqrt{2}}, \quad \hat{k}_2 = \frac{-\hat{x} + \hat{y}}{\sqrt{2}}, \quad \hat{k}_3 = \frac{-\hat{x} - \hat{y}}{\sqrt{2}}, \quad \hat{k}_4 = \frac{\hat{x} - \hat{y}}{\sqrt{2}}. \]

For the nodal quasiparticles in the vicinity of the \( n \)th node we have \( \mathbf{k} = \mathbf{k}_n + \delta \mathbf{k} \), and

\[ \xi_k = v_F \delta k_\perp, \quad \Delta_k = v_\Delta \delta k_\parallel, \quad E_k = \sqrt{v_F^2 \delta k_\perp^2 + v_\Delta^2 \delta k_\parallel^2}, \]  

(30)

where \( \delta k_\perp \) and \( \delta k_\parallel \) are the momentum components perpendicular and parallel to the Fermi surface, \( v_F \) is the Fermi velocity at the nodes, and \( v_\Delta = |\partial \Delta_k/\partial k_\parallel| \) is the slope of the superconducting gap function near the nodes. Thus, the excitation spectrum near the gap nodes is described by an anisotropic Dirac cone. The anisotropy ratio \( v_F/v_\Delta \) is an important characteristic of the high-\( T_c \) cuprates, which depends on the material and the doping level, e.g. \( v_F/v_\Delta \approx 14 \) and 19 in the optimally doped YBCO and Bi-2212, respectively.

In the nodal approximation, the momentum integration over the whole Brillouin zone is replaced by a sum of four integrals over the small regions in \( k \)-space around the nodes:

\[ \int \frac{d^2 k}{(2\pi)^2} \rightarrow \sum_{n=1}^4 \int \frac{d\delta k_\perp d\delta k_\parallel}{(2\pi)^2} = \frac{1}{2\pi v_F v_\Delta} \sum_{n=1}^4 \int_0^{\epsilon_{max}} d\epsilon \int_0^{2\pi} d\alpha \int_0^{2\pi} d\alpha \]  

(31)

In the last integral, we changed to the polar coordinates: \( v_F \delta k_\perp = \epsilon \cos \alpha, \quad v_\Delta \delta k_\parallel = \epsilon \sin \alpha \), and \( E_k = \epsilon \). The ultraviolet cutoff \( \epsilon_{max} \approx \Delta_0 \) is introduced to make sure that the area of the integration region is equal to the area of the original Brillouin zone. In most calculations in this article this cutoff can be extended to infinity.

#### A. Classical phase fluctuations

We can now calculate the energy of the classical phase fluctuations, see Eq. (24). The nodal approximation cannot be applied to the momentum integral in Eq. (20) because it contains contributions from all electrons, including those far from the Fermi surface. Assuming that the band dispersion can be treated in the effective mass approximation, which amounts to the replacement \( m_{ij}^{-1} \rightarrow (1/m^*) \delta_{ij} \), one obtains

\[ \mathcal{K}_{ij}^{(0)} = \frac{m}{m^*} \rho_0 \delta_{ij}, \]  

(32)
where \( \rho_0 \) is the average mass density of electrons.

In contrast, the second term in the kernel \( K \) can be calculated in the nodal approximation. Using Eqs. (20), we have

\[
\mathcal{I}_{ij}(q) = -\frac{m^2 v_F}{2\pi} \sum_{n=1}^{1} \hat{k}_{n,i} \hat{k}_{n,j} S_n(q),
\]  

where

\[
S_1(q) = S_3(q) = Ts \left( \frac{\gamma}{T} \right), \quad S_2(q) = S_4(q) = Ts \left( \frac{\gamma^2}{T} \right).
\]

Here

\[
\gamma_{1,2}(q) = \frac{1}{\sqrt{2}} \sqrt{v_F^2 (q_x \pm q_y)^2 + v_\Delta^2 (q_x \mp q_y)^2}
\]

are the energies of the nodal quasiparticles with \( \delta k = q \), and the scaling function \( s(x) \) is defined by an integral:

\[
s(x) = \int_0^\infty dy \int_0^{2\pi} \int_0^{2\pi} \frac{d\alpha}{2\pi} [f_+(x, y, \alpha) + f_-(x, y, \alpha)],
\]

where

\[
f_\pm = \frac{1}{2} \left[ 1 \pm \frac{y + x \cos \alpha}{\sqrt{x^2 + y^2 + 2xy \cos \alpha}} \right] \frac{\tanh(\sqrt{x^2 + y^2 + 2xy \cos \alpha}/2)}{\sqrt{x^2 + y^2 + 2xy \cos \alpha}} \pm \tanh( y/2).\]

Note that the cutoff energy \( \epsilon_{\text{max}} \) has been replaced by infinity, due to the rapid convergence of the integrals. One can show that the function \( s(x) \) has the following asymptotics:

\[
s(x) = \begin{cases} 
2 \ln 2 + \frac{x^2}{24} & , \text{at } x \to 0 \\
\frac{\pi x}{8} & , \text{at } x \to \infty.
\end{cases}
\]  

After the summation over the four nodes in Eq. (33), one finally obtains

\[
K_{ij}(q) = \frac{m}{m^*} \rho_0 \delta_{ij} - \frac{m^2 v_F}{2\pi} T \left( F_+(q) F_-(q) - F_-(q) F_+(q) \right)_{ij},
\]

where

\[
F_\pm(q) = s \left[ \frac{\gamma_1(q)}{T} \right] \pm s \left[ \frac{\gamma_2(q)}{T} \right].
\]

The expression (37) is exact in the nodal approximation, i.e. for the conical quasiparticle spectrum. In terms of \( q \), the applicability region of the nodal approximation is \( \gamma_{1,2}(q) \ll \Delta_0 \). At higher energies of quasiparticles, the deviations of the spectrum from the linearized form should be taken into account.

We would like to note that in the nodal approximation, \( \Pi^r_1(q, 0) = 0 \) and therefore \( L_{\theta,\phi}(q, 0) = 0 \), see Eqs. (18), (20). This means that the classical fluctuation energy in two dimensions has the form (24) with the kernel (27), even if the Coulomb interaction is taken into account.

At \( T = 0 \), using the large-\( x \) asymptotics of \( s(x) \) in Eq. (36), the kernel takes the form

\[
K_{ij}(q) = \frac{m}{m^*} \rho_0 \delta_{ij} - \frac{m^2 v_F}{16} \left( \gamma_1(q) + \gamma_2(q) \gamma_1(q) - \gamma_2(q) \gamma_1(q) + \gamma_2(q) \right)_{ij}.
\]

Setting \( q = 0 \) here, we find the superfluid density: \( \rho_s(0) = (m/m^*)^2 \rho_0 \). We also see that the kernel is a non-analytical function of \( q \), which means that no gradient expansion of the energy exists.

At finite temperatures and small \( q \), such that \( \gamma_{1,2}(q) \ll T \), the small-\( x \) asymptotics of \( s(x) \) yields

\[
K_{ij}(q) = \left( \frac{m}{m^*} \rho_0 - \frac{2 \ln 2}{\pi} \frac{v_F}{v_\Delta} \frac{m^2}{T} \right) \delta_{ij} - \frac{m^2 v_F}{48\pi} \frac{1}{v_\Delta} \left( (v_F^2 + v_\Delta^2)(q_x^2 + q_y^2) - 2(v_F^2 - v_\Delta^2)q_x q_y \right)_{ij}.
\]
The first term describes the depletion of the superfluid density due to the thermal excitation of quasiparticles:

$$\rho_s(T) = \rho_s(0) - \frac{2 \ln 2}{\pi} \frac{v_F}{v_\Delta} m^2 T,$$

(40)

see also Refs.\textsuperscript{46,49}, which explains a linear in $T$ increase of the magnetic penetration depth $\lambda(T)$ at low temperatures\textsuperscript{20,21} observed in high-$T_c$ cuprates.\textsuperscript{22}

The quadratic $q$-dependence of the expression \textsuperscript{[39]} implies that the kernel $K_{ij}(R)$ in real space is proportional to $\exp(-|R|/\xi)$, with the length $\xi$ given by

$$\xi(T) = \left( \frac{m^2 v_F^2}{48 \pi v_\Delta \rho_s} \right)^{1/2} \left( \frac{T_c}{T} \right)^{1/2} \xi_0,$$

(41)

where $\xi_0 = v_F/2\pi T_c$ is the BCS coherence length (we assumed that $v_F \gg v_\Delta$). This behavior is similar to that of the electromagnetic response function in conventional s-wave superconductors, see e.g. Ref.\textsuperscript{23}. An important difference however is that the characteristic length $\xi$ is temperature-dependent: $\xi(T) \sim T^{-1/2}$ at $T \to 0$. It is because of the divergence of $\xi$ that the gradient expansion of the classical energy of fluctuations breaks down at $T \to 0$. Note also that the length $\xi$ is different from other characteristic lengths discussed in the literature: $\xi_0$ – the coherence length, or the correlation length of the gap amplitude fluctuations, which remains constant at $T \to 0$, and $\xi_{\text{pair}}$ – the size of a Cooper pair, which is infinite in the $d$-wave case.\textsuperscript{14} In a conventional s-wave superconductor, all three lengths are of the same order.

The physical interpretation of our findings is the same as that of the non-local Meissner effect\textsuperscript{24}, since the gap function $\Delta_k$ has nodes on the Fermi surface, then the anisotropic coherence length $v_F/|\Delta_k|$ exceeds the London penetration depth $\lambda_L$ close to the nodes, and the local electrodynamics breaks down.

One can expect that the nodal quasiparticles also affect the non-Gaussian terms in the effective action \textsuperscript{[19].} As shown in the Appendix, indeed no expansion of the classical fluctuation energy in powers of $v_s$ exists at $T = 0$.

B. Failure of the classical XY-model

The effects of the phase fluctuations in superconductors are most often studied using either the classical or the quantum versions of the XY-model. The energy of the classical XY-model in the absence of external fields has the form

$$\mathcal{E}_{XY} = \sum_{RR'} J_{RR'} (1 - \cos(\theta_R - \theta_{R'})),$$

(42)

where $\theta_R$ is the phase of the order parameter at site $R$ of a coarse-grained square lattice, whose lattice spacing $d$ is of the order of the superconducting correlation length $\xi$. The summation goes over all bonds in the lattice, and the coupling constants $J_{RR'} = J(\rho)$, where $\rho = R - R'$, are called the phase stiffness coefficients. While in the Gaussian approximation, see below, the coupling constants are temperature-independent, the interaction of fluctuations leads to a thermal renormalization of the $J$s, and eventually to a phase transition into the disordered state.

The XY-model is believed to provide a correct description of any system with broken U(1) symmetry if the amplitude fluctuations of the order parameter are negligible. Typical examples are classical Heisenberg magnets, superfluids and superconductors. The experimental systems to which the lattice model \textsuperscript{[19]} has been applied include granular superconductors and fabricated arrays of Josephson junctions, see, e.g., Ref.\textsuperscript{26} and the references therein. In those cases $d$ is given by the distance between grains. Although the simplicity of the XY-model is physically appealing, its rigorous microscopic derivation for homogeneous high-$T_c$ superconductors does not exist. The usual way of justification, see, e.g., Ref.\textsuperscript{22}, involves expanding the cosine in Eq. \textsuperscript{[12]} and matching the expansion coefficients with those in the Gaussian phase-only action. That the microscopic theory fails to reproduce the quantum generalization of the XY-model has already been noticed in Ref.\textsuperscript{29}. Here we show, following Ref.\textsuperscript{12}, that even for the classical phase fluctuations in a $d$-wave superconductor at low temperatures, the long-wavelength limit of the microscopic theory is not consistent with Eq. \textsuperscript{[12]}.

For slow variations of the phase, the energy \textsuperscript{[12]} takes a Gaussian form:

$$\mathcal{E}_{XY} = \frac{1}{2} \sum_{RR'} J_{RR'} (\theta_R - \theta_{R'})^2 = \frac{1}{2} \sum_q |\theta(q)|^2 \sum_{\rho} J(\rho)(1 - \cos q\rho).$$
In terms of the superfluid velocity, we have
\[
\mathcal{E}_{XY} = \frac{1}{2} \sum_q K_{ij}^{XY}(q) v^*_s,i(q) v_s,j(q).
\] (43)

The kernel here has a well-defined Taylor expansion in powers of \( q \):
\[
K_{ij}^{XY}(q) = \rho_{s,ij}^{XY} + \Lambda_{ij,kl} q_k q_l + O(q^4),
\] (44)
where
\[
\rho_{s,ij}^{XY} = 2m^2 \sum_{\rho} J(\rho) \rho_i \rho_j
\] (45)
is the superfluid mass density tensor (for example, if the only non-zero coupling is between the nearest-neighbor sites, then \( \rho_{s,ij}^{XY} = 8m^2 d^2 J \delta_{ij} \)), and
\[
\Lambda_{ij,kl} = -\frac{m^2}{6} \sum_{\rho} J(\rho) \rho_i \rho_j \rho_k \rho_l.
\] (46)

Comparing Eqs. (44) and (38), we see that the effective long-wavelength theory of the classical phase fluctuations at \( T = 0 \) does not have the form of the \( XY \)-model, since the momentum dependence of the two energies is clearly different. At \( T > 0 \), although the expression (39) is quadratic in \( q \), the coefficients diverge as \( T \to 0 \), which is not the case for \( \Lambda_{ij,kl} \) above. Thus, the microscopic theory fails to reproduce the long-wavelength structure of the classical \( XY \)-model in a clean \( d \)-wave superconductor.

IV. DISORDERED CASE

In the presence of impurities, a full effective field theory for the disordered interacting system described by the Hamiltonian would include the fluctuations of the order parameter and of the scalar potential coupled with the disorder-induced soft modes (the diffusons and the Cooperons). Such theories, usually having the form of a non-linear \( \sigma \)-model, have been developed, see, e.g. Ref. and the references therein, to study the effects that are beyond the scope of the present work, for instance the suppression of \( T_c \) due to the interplay of disorder and interactions in \( s \)-wave superconductors. Our goal here is to check if the elastic impurity scattering removes the divergencies in the gradient expansion of the classical phase-only action discussed above. The disorder is treated essentially in the saddle-point approximation and the Coulomb interaction is neglected.

As a bookkeeping device to obtain an effective action for the order parameter fluctuations, we use the replica trick: \( \langle \ln Z \rangle = \lim_{n \to 0} (\langle Z^n \rangle - 1)/n \) (the angular brackets denote averaging with respect to disorder). From Eq. (41) we have
\[
Z^n = \int \prod_{a=1}^n Dc^a D\bar{c}^a e^{-S[\bar{c}, c]},
\] (47)
where \( S = S_0 + S_{int} \),
\[
S_0 = \sum_a \int_0^\beta d\tau \left( \sum_r \bar{c}^a_{r\tau} \partial_\tau c^a_{r\tau} + \sum_{rr'} \xi_{rr'} \bar{c}^a_{r\tau} c^a_{r'\tau} + \sum_r U_r \bar{c}^a_{r\tau} c^a_{r\tau} \right),
\] (48)
\[
S_{int} = -g \sum_a \int_0^\beta d\tau \sum_{\langle rr' \rangle} B^a_{rr'}, B^a_{rr'}. \] (49)

The impurity potential here is assumed to be Gaussian-distributed, with zero mean and the correlator
\[
\langle U_r U_{r'} \rangle = \frac{1}{2\pi N_F \tau} \delta_{rr'},
\]
where \( N_F \) is the density of states at the Fermi level, and \( \tau \) is the electron mean-free time due to elastic scattering. The next step is to use an incomplete Hubbard-Stratonovich transformation to decouple only the interaction terms \( S_{int} \) in each replica, before disorder averaging. Proceeding as in Sec. III we have

\[
\langle Z^n \rangle = \int \prod_{a=1}^{n} D\Delta^a \, D\Delta^a \, e^{-S_{eff}[\Delta^a, \Delta]},
\]

with the effective action

\[
S_{eff} = -\ln(\det G^{-1}) + \frac{1}{g} \sum_a \int dt \sum_{rr'} |\Delta^a_{rr'}|^2,
\]

instead of Eq. (54). Here

\[
G^{-1}_{ab}(r, \tau; r', \tau') = \delta_{ab} \delta(\tau - \tau') \begin{pmatrix} \delta_{rr'}(-\partial_\tau - U_r) - \xi_{rr'} & -\Delta^a_{rr'}(\tau) \\ -\Delta^a_{rr'}(\tau) & \delta_{rr'}(-\partial_\tau + U_r) + \xi_{rr'} \end{pmatrix},
\]

and “Det” stands for the full operator determinant with respect to the space-time coordinates and the Nambu and the replica indices. The disorder averaging in the first term in \( S_{eff} \) generates effective coupling between different replicas, resulting in a non-linear bosonic field theory.

While the saddle point of the effective action (54) has the same structure as in the clean case: \( \Delta^a_{0,rr'}(\tau) = \Delta_{0,rr'} \), see Eq. (6), the temperature dependence of the gap amplitude is different, in particular, the critical temperature is suppressed by impurities. The mean-field Green’s function is the unity matrix in the replica space: \( G_{0,ab} = \delta_{ab} \), where

\[
G_0^{-1}(r, r'; \omega_n) = \begin{pmatrix} \delta_{rr'}(-i\omega_n - U_r) - \xi_{rr'} & -\Delta_{0,rr'} \\ -\Delta_{0,rr'} & \delta_{rr'}(i\omega_n + U_r) + \xi_{rr'} \end{pmatrix},
\]

in the Matsubara frequency representation. The disorder averaging can be done using the standard diagram technique. In the so-called self-consistent Born approximation, which assumes a sufficiently weak impurity scattering and also neglects the diagrams with crossed impurity lines, the average mean-field Green’s function has the form

\[
\langle G_0(k, \omega_n) \rangle = -\frac{i\omega_n \tau_0 + \xi_k \tau_3 + \Delta_k \tau_1}{\omega^2_n + \xi^2_k + \Delta^2_k},
\]

where \( \omega_n \) satisfies the equation \( i\omega_n = i\omega_n + (1/2\tau)(i\omega_n/\sqrt{\omega_n^2 + \Delta_k^2})k \).

Assuming replica-symmetric phase fluctuations, the order parameter can be written in the form \( \Delta^a_{rr'}(\tau) = \Delta_{0,rr'} e^{[\theta_r(\tau) + \theta_r(\tau)]/2} \), see Eq. (10). Performing the gauge transformation in each replica, we have \( G_{ab}^{-1} = \delta_{ab}(G_0^{-1} - \Sigma) \), where the self-energy operator has the form with \( \varphi_r(\tau) = 0 \). Therefore,

\[
\ln(\det G^{-1}) = \ln(\det G_0^{-1}) = n(\langle \text{Tr} \ln G_0^{-1} \rangle + n(\text{Tr} \ln(1 - G_0 \Sigma)) + O(n^2),
\]

in the limit \( n \to 0 \). Substituting this expansion in Eq. (54), we see that the replica index can be omitted, and the effective action gets replaced by its disorder average:

\[
S_{eff}[\theta] = \beta E_0 - \langle \text{Tr} \ln(1 - G_0 \Sigma) \rangle,
\]

where \( E_0 \) is the mean-field energy of a disordered superconductor. Considering only the static fluctuations and expanding the operator trace in powers of the phase gradients, we obtain the effective action in the form, where the energy of fluctuations is now given by

\[
E = \frac{1}{2} \sum_q \langle K_{ij}(q) \rangle v_{s,i}^*(q) v_{s,j}(q),
\]

with

\[
\langle K_{ij}(q) \rangle = \langle K_{ij}(0) \rangle + \langle I_{ij}(q) \rangle.
\]
Before proceeding with the calculation of the disorder averages, we would like to note that Eq. (56) could also be derived using a less formal approach, without introducing replicas. Assuming that electrons move in the presence of the random potential and a given order parameter field \( \Delta_{rr'} = \Delta_{0,rr'} e^{i(\theta_r + \theta_{r'})}/2 \), one can define the energy functional of phase fluctuations:

\[
\mathcal{E}[\theta] = -\frac{1}{\beta} \langle \ln Z[\theta] \rangle + \frac{1}{\beta} \langle \ln Z[0] \rangle,
\]

where \( Z[\theta] = \text{Det} (\mathcal{G}_0^{-1} - \Sigma) \) is the partition function. Expanding \( \ln Z[\theta] \) in powers of the phase gradients, followed by averaging each expansion coefficient with respect to disorder, we arrive at Eq. (56).

The disorder averaging of the first term in Eq. (57) is straightforward:

\[
\langle \mathcal{K}_{ij}(0) \rangle = 2m^2 T \sum_n \int \frac{d^D k}{(2\pi)^D} \frac{m_{ij}^{-1}(k)}{m} \langle G_0(k, \omega_n) \rangle = \frac{m}{m^*} \rho_0 \delta_{ij},
\]

where we used the effective mass approximation, as in Eq. (62).

The average of the product of two Green’s functions in the second term includes the impurity vertex corrections. However, since we are interested in the behavior of the kernel in the long-wavelength limit \( q \to 0 \) in a sufficiently clean system (the precise criterion will be discussed below), the vertex corrections are negligible. Thus

\[
\langle \mathcal{I}_{ij}(q) \rangle = m^2 T \sum_n \int \frac{d^D k}{(2\pi)^D} v_i(k) v_j(k) \text{tr}\{[G_0(k + q, \omega_n)]\tau_0 [G_0(k, \omega_n)]\tau_0 \}.
\]

In order to calculate the Matsubara sum we use the spectral representation

\[
\mathcal{G}_0(k, \omega_n) = \int \frac{d\epsilon}{\pi} \frac{\text{Im} \mathcal{G}_0^R(k, \epsilon)}{\epsilon - i\omega_n},
\]

where the retarded matrix Green’s function is obtained from Eq. (54) by analytical continuation:

\[
\mathcal{G}_0^R(k, \epsilon) = \frac{t(\epsilon) \tau_0 + \xi_k \tau_3 + \Delta_k \tau_1}{t^2(\epsilon) - \xi_k^2 - \Delta_k^2},
\]

and \( t(\epsilon) = (i\omega_n)_{i\omega_n \to \epsilon + i0} \) satisfies the equation \( t = \epsilon + (i/2\pi)(t/\sqrt{t^2 - \Delta_k^2})_k \). Instead of finding the exact energy dependence of \( t(\epsilon) \) for the \( d \)-wave gap, we use below a simplified expression

\[
t(\epsilon) = \epsilon + \Gamma,
\]

which captures the essential qualitative effects of the impurity broadening of the single-electron states, with \( \Gamma \) being an energy-independent effective scattering rate. The exact solution shows that the scattering rate indeed approaches a constant value at \( \epsilon \to 0 \), i.e. in the so-called universal limit \( \frac{10}{28} \).

Inserting the representation (61) in Eq. (60), one obtains

\[
\langle \mathcal{I}_{ij}(q) \rangle = -\frac{m^2}{2} \int_{-\infty}^{\infty} \frac{d\epsilon_1}{\pi} \int_{-\infty}^{\infty} \frac{d\epsilon_2}{\pi} \frac{\tanh \frac{\epsilon_1}{2T} - \tanh \frac{\epsilon_2}{2T}}{\epsilon_1 - \epsilon_2} \int \frac{d^D k}{(2\pi)^D} v_i(k) v_j(k)
\]

\[
\times \left\{ C_-(k, q)[d_+(k + q, \epsilon_1) d_-(k, \epsilon_2) + d_-(k + q, \epsilon_1) d_+(k, \epsilon_2)]
\right.
\]

\[
\left. + C_+(k, q)[d_+(k + q, \epsilon_1) d_+(k, \epsilon_2) + d_-(k + q, \epsilon_1) d_-(k, \epsilon_2)] \right\},
\]

where

\[
d_{\pm}(k, \epsilon) = \frac{1}{\pi} \frac{\Gamma}{(\epsilon \pm E_k)^2 + \Gamma^2},
\]

and \( C_{\pm} \) are the coherence factors (28). The expressions (27) are recovered from Eq. (61) in the clean limit \( \Gamma \to 0 \), when \( d_\pm(k, \epsilon) \to \delta(\epsilon \pm E_k) \).

We focus on the case of zero temperature, when the kernel is a non-analytical function of \( q \) in the absence of impurities, see Eq. (38). At \( T = 0 \), the integrals over \( \epsilon_{1,2} \) can be calculated, giving

\[
\langle \mathcal{I}_{ij}(q) \rangle = -\frac{2m^2}{\pi} \int \frac{d^D k}{(2\pi)^D} v_i(k) v_j(k) \left[ C_-(k, q) \frac{\arctan \frac{E_{k+q}}{E_{k+q} + E_k}}{E_{k+q} + E_k} + C_+(k, q) \frac{\arctan \frac{E_{k+q}}{E_{k+q} - E_k}}{E_{k+q} - E_k} \right].
\]
Comparing this to Eq. (27), we see that the energy of classical phase fluctuations in the disordered case at zero temperature has exactly the same form as in the clean case at a finite temperature, if one formally replaces \( \tanh(E/2T) \rightarrow (2/\pi) \arctan(E/\Gamma) \). Therefore one can expect that the disorder will affect the phase fluctuations in the same way as temperature does, i.e. the singularities of the effective action will be washed out.

To check this conclusion quantitatively, let us evaluate the long-wavelength asymptotics of \( \langle I_{ij}(q) \rangle \) in the nodal approximation. Using Eqs. (30,31), we have

\[
\langle I_{ij}(q) \rangle = -\frac{m^2 v_F}{2\pi v_\Delta} \sum_{n=1}^{4} \hat{k}_{n,j} \hat{S}_n(q),
\]

where

\[
\hat{S}_1(q) = \hat{S}_3(q) = \Gamma \hat{s} \left( \frac{q_1}{\Gamma} \right), \quad \hat{S}_2(q) = \hat{S}_4(q) = \Gamma \hat{s} \left( \frac{q_2}{\Gamma} \right),
\]

and \( \gamma_{1,2}(q) \) are given by Eq. (34). The function \( \hat{s}(x) \) is defined by

\[
\hat{s}(x) = \frac{2}{\pi} \int_0^{y_{\text{max}}} \frac{d\alpha}{2\pi} \int_0^x dy [\tilde{f}_+(x,y,\alpha) + \tilde{f}_-(x,y,\alpha)],
\]

where \( y_{\text{max}} = \epsilon_{\text{max}}/\Gamma \), and

\[
\tilde{f}_\pm = \frac{1}{\pi} \left( 1 \pm \frac{y + x \cos \alpha}{\sqrt{x^2 + y^2 + 2xy \cos \alpha}} \right) \arctan(\sqrt{x^2 + y^2 + 2xy \cos \alpha} / y). \]

In contrast to the clean case, see Eq. (35), the energy cutoff here cannot be extended to infinity, due to the logarithmic divergence of the integrals.

The clean-case expression (35) is recovered from Eq. (67) at \( x \gg 1 \), i.e. at \( \Gamma \ll \gamma_{1,2}(q) \ll \epsilon_{\text{max}} \). In the opposite limit \( x \ll 1 \), one finds

\[
\hat{s}(x) = \frac{2}{\pi} \ln y_{\text{max}} + \frac{1}{6\pi} x^2,
\]

which is valid at \( y_{\text{max}} \gg 1 \). Inserting this into Eq. (66), we finally obtain

\[
\langle K_{ij}(q) \rangle = \left( \frac{m}{m^*} \rho_0 \frac{2 v_F}{\pi^2 v_\Delta^2} m^2 \Gamma \ln \frac{\epsilon_{\text{max}}}{\Gamma} \right) \delta_{ij} \frac{m^2 v_F \Gamma}{12 \pi^2 v_\Delta^2} \left( \frac{v_F^2 + v_\Delta^2}{2(\frac{v_F^2}{\Gamma} - \frac{v_\Delta^2}{\Gamma})} q_x q_y \right). \quad (69)
\]

The first term describes the depletion of the superfluid density at zero temperature due to the impurity scattering:

\[
\rho_s(T = 0, \Gamma) = \rho_s(T = 0, \Gamma) - \frac{2 v_F}{\pi^2 v_\Delta^2} m^2 \Gamma \ln \frac{\Delta_0}{\Gamma},
\]

see also Ref.32 (here we used \( \epsilon_{\text{max}} \approx \Delta_0 \)). Upon increasing \( \Gamma \), the superfluid density decreases and eventually vanishes at some critical disorder strength. One cannot reach the critical point using the expression (70), since its validity is limited to the case of weak impurity scattering \( \Gamma \ll \Delta_0 \).

The second term in Eq. (69) shows that, in contrast to the clean case, the energy is a non-singular function of momentum, even at \( T = 0 \). Comparing it to the expression (13), we see that the microscopic theory, at least at the Gaussian level, has the same long-wavelength behavior as the classical XY-model. The non-zero elements of the tensor \( \Lambda \) are given by

\[
\Lambda_{xx,xx} = \Lambda_{yy,yy} = -\frac{m^2 v_F}{12 \pi^2 v_\Delta^2} \left( \frac{v_F^2 + v_\Delta^2}{\Gamma} \right),
\]

\[
\Lambda_{xy,xy} = \frac{-m^2 v_F}{12 \pi^2 v_\Delta^2} \left( \frac{v_F^2 - v_\Delta^2}{\Gamma} \right),
\]

with other elements obtained by symmetry. One can see that keeping only the nearest-neighbor phase stiffness coefficient in Eq. (10) is not sufficient to reproduce the tensor structure (71), therefore one has to consider an extended XY-model, in which the couplings between next-nearest neighbors etc. are also taken into account. This was first noticed in Ref.32 for an s-wave superconductor.
V. CONCLUSIONS

To summarize, we have shown that the effective field theory for classical phase fluctuations in a clean $d$-wave superconductor suffers from singularities which make the gradient expansion of the fluctuation energy impossible. This means, in particular, that the physics of classical phase fluctuations cannot be described by the XY-model at low temperatures. In the presence of disorder, a well-defined gradient expansion of the Gaussian phase-only action is restored, and has the same form as that of an extended classical XY-model.

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APPENDIX A: CONDENSATE ENERGY AND NON-LINEAR MEISSNER EFFECT

In this Appendix we calculate the higher-order terms in the expansion of the effective action

$$S_{\text{eff}} = S_0 - \text{Tr} \ln \tilde{G}^{-1} + \text{Tr} \ln G^{-1},$$

in powers of the phase gradients. We consider only the limit of uniformly-moving condensate, when the superfluid velocity $v_s$ is constant, so that $\theta_r - \theta_{r'} = 2m v_s(r - r')$. In the absence of the scalar potential, the self-energy (12) becomes translationally invariant:

$$\Sigma(r, \tau; r', \tau') = \delta(\tau - \tau') \tau^3 \xi_{rr'} \left[ e^{-i\tau^3 m v_s(r - r')} - 1 \right].$$

The gauge-transformed Green’s function (11) becomes diagonal in the momentum space:

$$\tilde{G}^{-1}(k, \omega_n) = \left( i\omega_n - \xi_+ - \Delta_k \quad -\Delta_k \right),$$

where $\xi^\pm_k = \xi_k \pm m v_s$, which allows one to calculate the operator traces:

$$\text{Tr} \ln \tilde{G}^{-1} = \ln \text{Det} \tilde{G}^{-1} = \beta V \sum_n \int \frac{d^D k}{(2\pi)^D} \ln \det \tilde{G}^{-1}(k, \omega_n)$$

where $V$ is the system volume and “det” denotes a $2 \times 2$ matrix determinant in the electron-hole space. Inserting this in Eq. (A1), we obtain $S_{\text{eff}} = S_0 + \beta \mathcal{E}$, where

$$\mathcal{E} = -V T \sum_n \int \frac{d^D k}{(2\pi)^D} \ln \frac{(i\omega_n - \tilde{E}_{k,+})(i\omega_n + \tilde{E}_{k,-})}{(i\omega_n - \tilde{E}_{k})(i\omega_n + \tilde{E}_{k})}$$

has the meaning of the kinetic energy of uniformly moving condensate, and

$$\tilde{E}_{k,\pm} = \pm \frac{\xi^+ - \xi^- + \sqrt{(\xi^+ + \xi^-)^2 + \Delta_k^2}}{2}$$

are the quasiparticle energies affected by the superflow. Using the identity

$$T \sum_n \ln \frac{i\omega_n - \tilde{E}}{i\omega_n - E} = T \ln \frac{\cosh \frac{\tilde{E}}{2T}}{\cosh \frac{E}{2T}},$$

the energy density can be written in the form

$$\frac{\mathcal{E}}{V} = -T \int \frac{d^D k}{(2\pi)^D} \ln \frac{\cosh \frac{\tilde{E}_{k,+}}{2T} \cosh \frac{\tilde{E}_{k,-}}{2T}}{\cosh^2 \frac{\tilde{E}_k}{2T}}.$$
At $T = 0$, this becomes

$$
\frac{\mathcal{E}}{\mathcal{V}} = -\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \left( |\tilde{E}_{k,+}| + |\tilde{E}_{k,+} - 2E_k| \right).
$$

(A6)

We focus now on the case of a two-dimensional $d$-wave superconductor. Assuming for simplicity a Galilean-invariant case, characterized by the parabolic dispersion $\xi_k = k^2/2m - \mu$, with the effective mass equal to the bare electron mass $m$, we have

$$
\tilde{E}_{k,\pm} = \sqrt{(\xi_k + \zeta)^2 + \Delta_k^2} \pm kv_s = E_{\zeta,k} \pm kv_s,
$$

where $kv_s$ is the so-called Doppler shift of the quasiparticle energy in the presence of moving condensate, and $\zeta = mv_s^2/2$. The expression (A6) can then be written as $\mathcal{E}/\mathcal{V} = (\mathcal{E}/\mathcal{V})_{reg} + (\mathcal{E}/\mathcal{V})_{Dopp}$. The first term,

$$
\left( \frac{\mathcal{E}}{\mathcal{V}} \right)_{reg} = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left( E_{\zeta,k} - E_k \right) - \zeta \int \frac{d^2k}{(2\pi)^2} \frac{\xi_k}{E_k} + O(\zeta^2) = \frac{\rho_0 v_s^2}{2} + O(v_s^4),
$$

(A7)

has a well-defined Taylor expansion in powers of $v_s$. Note that the nodal approximation cannot be used here because of the contributions from the regions far from the Fermi surface, see Sec. III A. In contrast, the second term,

$$
\left( \frac{\mathcal{E}}{\mathcal{V}} \right)_{Dopp} = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ E_{\zeta,k} + \eta + |E_{\zeta,k} - \eta| - 2E_{\zeta,k} \right],
$$

(A8)

can be calculated in the nodal approximation. Using Eqs. (30, 31), we obtain

$$
\left( \frac{\mathcal{E}}{\mathcal{V}} \right)_{Dopp} = -\frac{1}{12\pi v_F v_{\Delta}} \sum_{n=1}^4 |k_n v_s|^3,
$$

which is a non-analytical function of $v_s$. Putting all pieces together, we arrive at the following result:

$$
\frac{\mathcal{E}}{\mathcal{V}} = \frac{\rho_0 v_s^2}{2} - \frac{m^3 v_F^2}{12\sqrt{2}v_{\Delta}} \left(|v_{s,x} + v_{s,y}|^3 + |v_{s,x} - v_{s,y}|^3\right) + O(v_s^4).
$$

(A9)

We see that the nodal quasiparticles make the kinetic energy of the condensate a non-analytical function of the superfluid velocity. This singular behavior is closely related to the so-called non-linear Meissner effect: the screening supercurrent acts as a pair-breaker in $d$-wave superconductors, creating a finite density of normal excitations even at $T = 0$. The depletion of the supercurrent by those excitations leads to a non-analytical dependence of the electromagnetic response functions on $v_s$ and therefore the external magnetic field.

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