Model to Describe Fast Shutoff of CoVID-19 Pandemic Spread

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Abstract

Early CoVID-19 growth obeys: \( N(t) = N_0 \exp[+K_o t] \), with \( K_o = [(\ln 2)/(t_{dub})] \), where \( t_{dub} \) is the pandemic growth doubling time. Given \( N(t) \), the daily number of new CoVID-19 cases is \( \rho(t) = dN(t)/dt \). Implementing society-wide Social Distancing increases the \( t_{dub} \) doubling time, and a linear function of time for \( t_{dub} \) was used in our Initial Model:

\[
N_o[t] = 1 \exp[+K_A t/(1 + \gamma_0 t)] \equiv e^{+G_o} \exp(-Z_o[t]),
\]

to describe these changes, with \( G_o \equiv [K_A/\gamma_0] \). However, this equation could not easily model some quickly decreasing \( \rho[t] \) cases, indicating that a second Social Distancing process was involved. This second process is most evident in the initial CoVID-19 data from China, South Korea, and Italy. The Italy data is analyzed here in detail as representative of this second process. Modifying \( Z_o[t] \) to allow exponential cutoffs:

\[
Z_E[t] \equiv +[G_o / (1 + \gamma_o t)] [\exp(-\delta_o t - q_o t^2)] = Z_o[t] \exp(-\delta_o t - q_o t^2),
\]

provides a new Enhanced Initial Model (EIM), which significantly improves data fits, where \( N_E[t] = e^{+G_o} \exp(-Z_E[t]) \). Since large variations are present in \( \rho_{data}[t] \), these models were generalized into an orthogonal function series, to provide additional data fitting parameters:

\[
N(Z) = \sum_{m=M_F}^{m=M} g_m L_m(Z) \exp[-Z].
\]

Its first term can give \( N_o[t] \) or \( N_E[t] \), for \( Z[t] \to Z_o[t] \) or \( Z[t] \to Z_E[t] \). The \( L_m(Z) \) are Laguerre Polynomials, with \( L_0(Z) = 1 \), and \( \{g_m; m = 0, M_F\} \) are constants derived from each dataset. When \( \rho[t] = dN[t]/dt \) gradually decreases, using \( Z_o[t] \) provided good data fits at small \( M_F \) values, but was inadequate if \( \rho[t] \) decreased faster. For those cases, \( Z_E[t] \) was used in the above \( N(Z) \) series to give the most general Enhanced Orthogonal Function (EOF) model developed here. Even with \( M_F = 0, q_o = 0 \), this EOF model fit the Italy CoVID-19 data for \( \rho[t] \equiv dN[t]/dt \) fairly well.

When the \( \rho[t] \) post-peak behavior is not Gaussian, then \( Z_E[t] \) with \( \delta_o \neq 0, q_o = 0 \); which we call \( Z_A[t] \), is also likely to be a sufficient extension of the \( Z_o[t] \) model. The EOF model also can model a gradually decreasing \( \rho[t] \) tail using small \( \{\delta_o, q_o\} \) values [with 6 Figures].

NOTE: This preprint reports new research that has not been certified by peer review and should not be used to guide clinical practice.
1 Introduction

Let \( N(\hat{t}) \) be the total number of CoVID-19 cases in any given locality, with \( \rho(\hat{t}) \) being the predicted number of daily new CoVID-19 cases, so that:

\[
N(\hat{t}) = \int_{t'=0}^{t'=\hat{t}} \rho(t') \, dt',
\]

\[
\rho(\hat{t}) = \frac{dN(\hat{t})}{d\hat{t}}.
\]

Early CoVID-19 growth often obeys \( N(\hat{t}) \approx N_I \exp( + K_o \hat{t} ) \), with \( K_o = \frac{[\ln 2]/t_{dbl} }{ } \), where \( t_{dbl} \) is the pandemic doubling time. The start of society-wide Social Distancing at \( \hat{t} = 0 \) can gradually lengthen \( t_{dbl} \) for \( \hat{t} > 0 \). The \( \hat{t} < 0 \) exponential growth phase is not applicable for estimating Social Distancing effects. For \( \hat{t} > 0 \), an Initial Model for CoVID-19 pandemic shutoff was first developed using a linear function of time to describe the \( t_{dbl} \) changes:

\[
N(\hat{t}) = N_I \exp( + K_o \hat{t} / (1 + \alpha_S \hat{t} ) ).
\]

Given measured \( N_{data}(\hat{t}) \), the data end-points \( \{N_I, N_F\} \) help to set \( \{K_o, \alpha_S\} \). An Orthogonal Function Model [OFM] was developed next, with Eq. [1.2] as the first term of the orthogonal function series. Each new OFM term provides another fitting parameter, to progressively better match \( N_{data}(\hat{t}) \) and \( \rho_{data}(\hat{t}) = (d/d\hat{t}) N_{data}(\hat{t}) \).

The OFM improves on the Initial Model, and it works best with gradually decreasing \( \rho(\hat{t}) \) ["Slow Shutoff"]. In contrast, when \( \rho(\hat{t}) \) decreased quickly ["Fast Shutoff"], the Initial Model was not a good datafit, and a few-term OFM series only gave small improvements. This result indicates there is an inherent limit to what the gradually changing \( t_{dbl} \) doubling time of Eq. [1.2] can model.

For these cases, typified by CoVID-19 pandemic evolution in Italy, data often showed a stage where \( \rho_{data}(\hat{t}) \sim \exp(-\delta_o \hat{t}) \) or \( \rho_{data}(\hat{t}) \sim \exp(-g_o \hat{t}^2) \), which likely represents a second process, independent of the gradually changing \( t_{dbl} \) doubling time. An Enhanced Initial Model (EIM) is developed here to include this second process. The prior OFM methods can then be applied, giving an Enhanced Orthogonal Function (EOF) model for this more general case.

1.1 Review of Prior Models

The Initial Model of Eq. [1.2] is still needed as the first part of the OFM. The Initial Model starts with measured data end-points \( \{N_I, N_F\} \), where \( \hat{t} = (t_F - t_I) \) is the largest data time interval so that \( N(\hat{t} = (t_F - t_I)) = N_F \). Usually \( \alpha_S \) in Eq. [1.2] was chosen first, and \( K_o \) or \( t_{dbl} \) adjusted to match the \( N_F \) data end-point, using an Excel\textsuperscript{TM} Goal-Seek or its equivalent. The final \( \{K_o, \alpha_S\} \) values were the pair with the minimum root-mean-square (rms) error between the given data and the Eq. [1.2] model.

The above \( \{K_o, \alpha_S\} \) also provides a \( t = 0 \) estimate for the pandemic start, and gives \( \{K_A, \gamma_o\} \) as new data fitting parameters:

\[
N_o[t] = 1 \exp( + K_A t / (1 + \gamma_o t) ) \equiv \exp( + G_o \exp(-Z_o[t] ) ,
\]

\[
G_o \equiv ( K_A / \gamma_o ) \ ,
\]

\[
Z_o[t] = + [ G_o / (1 + \gamma_o t) ] ,
\]

\[
O(\hat{t}) = \frac{dN(\hat{t})}{d\hat{t}} ,
\]
\[ \rho_o[t] \equiv dN_o[t] / dt. \]  

Using \( t_F - t_I \), with \( N_o[t = t_I] = N_I \) and \( N_o[t = t_F] = N_F \), sets:

\[ t_I = \ln(N_I) / [K_o + \alpha S \ln(N_I)]. \]  

\[ t_F = \ln(N_F) / [K_o + \alpha S \ln(N_I)] + (t_F - t_I), \]

which determines \( \{K_A, \gamma_o\} \) in \( Z_o[t] \) for Eq. [1.3c]:

\[ \gamma_o = \{[\ln(N_I) / t_I] - [\ln(N_F) / t_F] \} / [\ln(N_F) - \ln(N_I)], \]

\[ K_A = [(1/t_I) - (1/t_F)] / \{[1 / \ln(N_I)] - [1 / \ln(N_F)]\}. \]

The Eq. [1.4a] value for \( t_I \) is what determines the new \( t = 0 \) point, as an extrapolation for when \( N_o[t = 0] = 1 \). In addition:

\[ N_o[t \to \infty, Z_o \to 0] \approx \exp[+K_A / \gamma_o] = \exp[+G_o] \equiv N_{\text{max}}^o, \]

\[ \rho_o[t] = dN_o[t] / dt \approx N_o[t] \left(G_o \gamma_o / (1 + \gamma_o t)^2 \right) \to N_{\text{max}}^o \left(G_o / (\gamma_o t^2)\right), \]

provides an estimate for the total number of cases \((N_{\text{max}}^o)\) at the pandemic end, and determines a function for the \( \rho_o[t] \) long-time tail. The \( \{0 < t < t_I\} \) period prior to the start of Social Distancing, extrapolates what pandemic progress would have looked like, if Social Distancing had begun at \( t = 0 \).

An Orthogonal Function Model [OFM] was then developed to better model the different observed \( \rho_{\text{data}}[t] \) shapes, as an improvement of the Initial Model:

\[ N(Z) = \sum_{m=0}^{M_F} g_m \, L_m(Z) \exp[-Z], \]  

\[ R(Z) = \sum_{m=0}^{M_F} c_m \, L_m(Z) \exp[-Z], \]

\[ N(Z) = \int_{Z'=-\infty}^{Z'=+\infty} R(Z') \, dZ', \]

\[ c_{M_F-k} = \sum_{m=0}^{k} g_m, \]

with \( L_m(Z) \) being the Laguerre Polynomials, and \( L_m(Z = 0) = L_0(Z) = 1 \). Using \( Z = Z_o[t] \) from Eq. [1.3c] gives \( N(Z) \to N(Z_o) \) and \( R(Z) \to R(Z_o) \).

The \( \{g_m; m = (0, M_F)\} \) constants in Eq. [1.7a] can be arranged in a \( \vec{g} \)-vector form, with comparable constants for \( R(Z) \) from Eq. [1.7b] arranged in a \( \vec{C} \)-vector form. For \( M_F = 2 \), it allows Eq. [1.7d] to be written as:

\[ \vec{C} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \]

Once these \( c_m \) values are determined for \( R(Z) \) in Eqs. [1.7b]-[1.7e], an OFM feature is that \( c_0 \), by itself, becomes the OFM best estimate for the total number of CoVID-19 cases at the pandemic end.

For large enough \( M_F \) values, and a monotonic \( Z \)-function, this OFM can provide successively better approximations to almost any given set of \( N_{\text{data}}[Z] \), with \( Z[t] \to Z_o[t] \) of Eq. [1.3c] being a specific case.

The OFM implicitly uses a Linear Y-axis, so its results differ from the Initial Model datafit on a Logarithmic Y-axis. As an example, compare the Initial Model result of \( N_o(Z_o) = G_o \exp[-Z_o] \) with the Eq. [1.7a] OFM result of \( N(Z_o) = g_0 \exp[-Z_o] \) for \( M_F = 0 \). In the Initial Model, \( G_o \) is fixed so that \( N_o(Z_o) \) exactly matches \( \{N_I, N_F\} \) at the \( \{t_I, t_F\} \) boundaries. In the OFM,
\(g_0 = G_o\) is no longer required, so that the OFM \(N(Z_o)\) best datafit is not constrained to exactly match \(\{N_I, N_F\}\) at \(\{t_I, t_F\}\).

The above \(R(Z)\) and \(Z[t]\) gives \(N[t]\) and \(\rho[t]\) as an explicit functions of time:

\[
N[t] \equiv \int_{t=\min}^{t=\infty} R(Z[t']) \frac{dZ}{dt'} dt' \approx \int_{t=\min}^{t=t} \rho[t'] dt'. \tag{1.9}
\]

Since Eq. [1.6b] gives \(\rho\{t\} \sim [1/t^2]\), the aim here is to model faster decaying functions such as \(\rho\{t\} \sim \exp(-\delta, t)\) or \(\rho\{t\} \sim \exp(-q, t^2)\).

### 1.2 Updated Initial Model Results for Italy

The \(Z_o[t]\) model of Eq. [1.3a] was applied to \(bing.com\) data\(^9\) for Italy, starting with \(N_{data}^{2/23/2020} = 150\) CoVID-19 cases as an early pandemic point, up through June 15, 2020. Here, \(t_I\) is when mandatory Social Distancing was introduced at \(N_{data}^{3/10/2020} = 10,149\); with \(t_F\) being when \(N_{data}^{6/15/2020} = 237,290\). Data prior to Social Distancing \((t < t_I)\) was excluding from this Social Distancing analysis.

**Figure 1** compares the \(\rho_{data}[t]\) results with the updated \(N_o[t]\) and \(\rho_o[t] = dN_o[t]/dt\) predictions using \(Z_o[t]\). The \(N_{data}^{2/23/2020}\) \((Day\ 1)\) to \(N_{data}^{3/10/2020}\) \((Day\ 17)\) interval was examined for estimating a \(t = 0\) pandemic start where \(N_{data}[t = 0] \rightarrow 1\). A best fit value of \(t_{offset} = 9.10055\) \(days\) was found, giving \(t_I = (17 - 9.10055) = 8.89945\) \(days\) for \(N_{data}^{3/10/2020}\) \(\(Day\ 17)\), while \(N_{data}^{6/15/2020}\) \(\(Day\ 114)\) gives \(t_F = (114 - 9.10055) = 104.89945\) \(days\), so that:

\[
N_{data}[t_I] = 8.899 = 10,149; \tag{1.10a}
\]
\[
N_{data}[t_F] = 104.899 = 237,290; \tag{1.10b}
\]
\[
N_{data}^{3/2/2020}[t = 0] = 1; \tag{1.10c}
\]
\[
N[t \to \infty] = 338,165; \tag{1.10d}
\]

with \(\{K_A, \gamma_o\} \approx \{4.2405, 0.33078\}\) and \((t_F - t_I) = 97\) \(days\). In **Fig. 1**, the X-axis uses this \(t = 0\) point where \(N_o[t = 0] \rightarrow 1\), and it shows what Social Distancing effects would have been, if it had been operating throughout the \(t > 0\) period. This \(\rho_o[t]\) prediction still has a much more gradual drop than the data. This discrepancy indicates that a second Social Distancing process is operating, besides just the gradual \(t_{offset}\) lengthening of the Initial Model.

**Figure 2** compares the \(N_o[t]\) predictions for this model, to the measured \(N_{data}[t]\). Systematic deviations are evident, with the net \(rms\) error on a Logarithmic Y-axis being \(rms_{\text{error}} = 0.097828\). To cure these defects, an enhanced \(Z[t]\) model is developed next.

### 2 Developing Enhanced \(Z[t]\) Models

To generalize \(Z[t]\) beyond Eq. [1.3c], it is convenient to use the \(t = \{0^+, \infty^-\}\) domain, and require convergence of \(Z[t] \to Z_o[t]\) in some limit, along with:

\[
\lim_{t \to 0} \{Z[t]\} = G_o \equiv \{K_A/\gamma_o\}; \tag{2.1a}
\]
\[
\lim_{t \to +\infty} \{Z[t]\} = 0; \tag{2.1b}
\]

and that the \(M_F = 0\) case of Eq. [1.7a] remains as:
How to choose an appropriate $Z[t]$, as part of an Enhanced Initial Model (EIM), which also allows a $\rho[t] \sim \exp(-\delta_o t)$ or $\rho[t] \sim \exp(-q_o t^2)$ stage, is examined next. It can be motivated by studying a simple $N_T[t]$ test-case, where $\rho_T[t]$ itself is a pure exponential decay, as a function of time:

$$N_T[t] = [e^{+G_o}] [1 + e^{-G_o}] - \exp(-\delta_o t)],$$

$$\rho_T[t] = \{dN_T[t] / dt\} = [\delta_o e^{+G_o}] \exp(-\delta_o t),$$

while also preserving $N_T[t \to 0] = 1$. Comparing Eq. [2.3a] to the Eq. [2.2], sets $Z_T[t]$ for this test-case:

$$N_T[t] = \exp[+G_o] \exp(-Z_T[t]),$$

$$Z_T[t] = (-1) \ln [(1 + e^{-G_o}) - \exp(-\delta_o t)].$$

At large times, Eq. [2.4b] gives:

$$\lim_{t \to \infty} \{Z_T[t]\} \approx \exp(-\delta_o t),$$

since $(-\ln[1 - x] \approx x)$ for small $x$, which shows that if $\rho_T[t]$ has an exponential tail, then $Z_T[t]$ also has an exponential tail. A simple generalization for $Z[t]$ in Eq. [1.7a]-[1.7d] would be either:

$$Z_E[t] \equiv [g_o / (1 + \gamma_o t)] \exp(-\delta_o t - q_o t^2),$$

$$Z_A[t] \equiv [g_o / (1 + \gamma_o t)] \exp(-\delta_o t) = Z_E[t; q_o \equiv 0].$$

As with Eq. [1.3c], the original CoVID-19 exponential growth factor $K_A$ remains only as part of the $G_o$ scaling factor, while the $\{\delta_o \to 0\}$ limit of Eq. [2.6b] converges back to the Eq. [1.3c] Initial Model.

Using Eq. [2.6a] for $Z_E[t]$ in the Eq. [2.2] $N(Z_E)$ example gives:

$$\rho[t] = \{dN[t] / dt\} = g_o \frac{d}{dt} \exp[-Z_E] = -g_o \exp[-Z_E] \frac{d}{dt} Z[t] = \{[+G_o\gamma_o] + \frac{G_o}{(1 + \gamma_o t)} + \frac{G_o}{(1 + \gamma_o t)} [\delta_o + 2q_o t]\},$$

which exhibits the following variety of long-time limits:

$$\rho[t; \delta_o = 0, q_o = 0] \to N(Z_E) \{[+G_o\gamma_o] + \frac{G_o}{(1 + \gamma_o t)}\},$$

$$\rho[t; q_o = 0] \to N(Z_E) \exp(-\delta_o t) \{[+G_o\gamma_o] \},$$

$$\rho[t; \delta_o = 0] \to N(Z_E) \exp(-q_o t^2) \{[2G_o\gamma_o] + \frac{G_o}{(1 + \gamma_o t)}\},$$

$$\rho[t] \to N(Z_E) \exp(-q_o t^2) \{[2G_o\gamma_o] + \frac{G_o}{(1 + \gamma_o t)}\} \frac{2G_o q_o}{\gamma_o} \exp[+\frac{1}{2} \delta_o^2 / q_o].$$

Here, any $q_o \neq 0$ Gaussian component in $Z_E[t]$ gives a $\rho[t]$ tail that is also a pure Gaussian. An exponential component ($q_o \equiv 0$) in $Z_E[t]$ gives a time-modified exponential $\rho[t]$ tail, while having $\{q \equiv 0, \delta_o \equiv 0\}$ in $Z_E[t]$ gives the prior $\rho[t]^{-1 / t^2}$ result of Eq. [1.6b].

3 Pandemic Fast vs Slow Shutoffs

Each $Z_E[t]$ function modifies $N[t]$ predictions for the pandemic start, pandemic end, and the mid-range where $\rho[t]$ has its pandemic peak. The Eq. [2.6a] $Z_E[t]$ function especially alters the calculated CoVID-19 pandemic tail. For either $q_o \neq 0$ or $\delta_o \neq 0$, Eqs. [2.6a]-[2.6b] gives a pandemic Fast Shutoff, compared to the gradually decreasing $Z_o[t]$ of Eq. [1.3c] in the Initial Model, which is a pandemic Slow Shutoff. However $Z_o[t]$ from the Initial Model, and $Z_A[t]$ from Eq. [2.6b] both gave long-term $\rho[t]$ tails that decay much slower than the $q_o \neq 0$ Eq. [2.6a] Gaussian.
If data does not show evidence of a Gaussian pandemic Fast Shutoff, assuming the post-peak \( \rho[t] \) data will be Gaussian is likely to provide optimistically inaccurate \( N[t] \) predictions for CoVID-19 pandemic evolution. Apparently, this is exactly what was done by the University of Washington IHME (Institute of Health Metrics and Evaluation) in their widely publicized initial preprint\(^3\) of 27 March 2020, with this Gaussian model continuing throughout their subsequent updates\(^4-6\) up through 29 April 2020.

IHME changed everything in their 4 May 2020\(^7-8\) update. They no longer used \( \rho[t] \) Gaussian tails, and it doubled or tripled their predicted CoVID-19 pandemic death rates. Thus, unless the post-peak \( \rho[t] \) exhibits Gaussian behavior, the \( Z_A[t] \) with \( \delta_o \neq 0 \) is likely the most important modification to \( Z_o[t] \), which is the pandemic Fast Shutoff model used here.

Using \( Z_A[t] \), the pandemic Fast Shutoff can be extrapolated to calculate a pandemic start point where \( N[t=0] = 1 \). We then examine if this \( Z_A[t] \) must also carry over to the \( \rho[t] \) long-term tail,

Since the long-term low \( \rho_{data}[t] \) tail may differ among localities, and is not well known, the \( \delta_o \neq 0 \) case of Eq. [2.6b] could end with a Slow Shutoff, giving:

\[
Z_{B}[t] = +[G_o / (1 + \gamma_o t)] \{\exp[-\delta_o t / (1 + \lambda_o t)]\}, \quad \text{[3.1a]}
\]

\[
\lim_{t \to \infty} \{ Z_{B}[t]\} \approx +[G_o / (1 + \gamma_o t)] \{\exp[-\delta_o / \lambda_o ]\}, \quad \text{[3.1b]}
\]

with \( G_o \) as in Eq. [2.1a]. This Eq. [3.1a] \( Z_{B}[t] \) function has the \( \{ \delta_o, \lambda_o \} \) Mitigation Measure operating at the start of Social Distancing, but reverting to the Initial Model in the long-time limit. Combining Eq. [2.6b] and Eq. [3.1a] cases gives this Enhanced Initial Model (EIM) equation:

\[
Z_{B}[t] = +[G_o / (1 + \gamma_o t)] \{\exp[-\delta_o t / (1 + \kappa \gamma_o t)]\}, \quad \text{[3.2]}
\]

where \( \kappa = 0 \) is a pure exponential, and \( \kappa = 1 \) has a modified tail that includes its own long-term shutoff. Comparing \( \kappa = \{0, 1\} \) in Eq. [3.2] provides a simple test for which model matches CoVID-19 data better in any locality.

Any other \( \kappa > 0 \) value then recovers the more general \( \lambda_o \equiv \kappa \gamma_o \) case. This Eq. [3.2] \( Z_{B}[t] \) replaces \( Z_o[t] \) of Eq. [1.3c], and its EIM companion \( N_{B}[t] \) is:

\[
N_{B}[t] = [e^{G_o}] \{\exp(-Z_{B}[t])\}, \quad \text{[3.3]}
\]

while using \( Z[t] \rightarrow Z_{B}[t] \) in Eqs. [1.7a]-[1.7d] gives an Enhanced Orthogonal Function (EOF) model.

### 4 Finding \( \{K_A, \gamma_o, \delta_o\} \) for \( Z_{B}[t] \) from Data

If \( \delta_o = 0 \), the prior Eqs. [1.4a]-[1.5b] for \( \{K_A, \gamma_o, \delta_o = 0\} \) and \( \{t_f, t_F\} \) can be used, with the initial best-fit \( \{K_o, \alpha_S\} \) values determined by minimizing the rms error between Eq. [1.2] and the measured data on a Logarithmic Y-axis. Unfortunately, Eqs. [1.4a]-[1.5b] cannot be used when \( \delta_o \neq 0 \), although finding a good \( \{K_A, \gamma_o, \delta_o\} \) starting point is still needed for the EIM:

\[
Z_{B}[t] = +[G_o / (1 + \gamma_o t)] \{\exp[-\delta_o t / (1 + \kappa \gamma_o t)]\}, \quad \text{[4.1a]}
\]

prior to any EOF analysis. The \( \kappa = 1 \) case also has this special symmetry:

\[
+[\delta_o t / (1 + \gamma_o t)] = (\delta_o / \gamma_o) [1 - 1 / (1 + \gamma_o t)], \quad \text{[4.2a]}
\]

\[
\exp[-\delta_o t / (1 + \gamma_o t)] = e^{-(\delta_o / \gamma_o)} \exp[\delta_o / \gamma_o] (1 / (1 + \gamma_o t)), \quad \text{[4.2b]}
\]

6
\[ Z_B[t] = +\left[ G_o \left( \frac{\delta_o}{1 + \gamma_o t} \right) \right] e^{-\left( \frac{\delta_o}{\gamma_o} \right) \exp[-(\delta_o / \gamma_o) \left( \frac{1}{1 + \gamma_o t} \right)]}, \]
\[ Z_B[t] = +\left[ G_o \left( \frac{\delta_o}{1 + \gamma_o t} \right) \right] \exp[-(\delta_o / \gamma_o) \left( \frac{1}{1 + \gamma_o t} \right)], \]
which can be re-written as:
\[ W(X[t]) \equiv X[t] \exp(\pm X[t]), \]
\[ X[t] = \left[ (\delta_o / \gamma_o) / (1 + \gamma_o t) \right], \]
\[ X[0] = \delta_o / \gamma_o, \]
\[ Z_B[t]_{\kappa=1} = +G_o W(X[t]) / W(X[0]). \]
\[ N_B(Z_B) = [e^{+G_o}] \exp(-Z_B[t]_{\kappa=1}), \]
while the \( \kappa = 0 \) case is:
\[ Z_A[t] = +[G_o / (1 + \gamma_o t)] \exp(-\delta_o t), \]
\[ N_A(Z_A) = [e^{+G_o}] \exp(-Z_B[t]_{\kappa=0}) = [e^{+G_o}] \exp(-Z_A[t]). \]

For \( \kappa = \{0, 1\} \), the \( t = 0 \) point, \( G_o \) from Eq. \[2.1a\], and the \( N_I(t_I) \) and \( N_F(t_F) \) initial and final points, give these equations to help set \{\( K_A, \gamma_o, \delta_o \)\}:
\[ N_I[t = 0] = 1 = G_o \exp(-Z_B[t = 0]), \]
\[ N_I[t_I] = N_I = G_o \exp(-Z_B[t_I]), \]
\[ N_F[t_F] = N_F = G_o \exp(-Z_B[t_F]), \]

for the EIM. Minimizing the \textit{rms} error between the Eq. \[3.3\] \{\( K_A, \gamma_o, \delta_o, t_I \)\} functions and measured data on a Logarithmic Y-axis can be done as follows. Start with estimated values for \{\( \hat{K}_A, \hat{\gamma}_o, \hat{\delta}_o, \hat{t}_{offset} \)\} in:
\[ \hat{Z}[t] = +[\hat{G}_o / (1 + \hat{\gamma}_o t)] \{\exp(-\hat{\delta}_o t / (1 + \hat{\gamma}_o t))\}, \]
\[ \hat{N}[t] = [e^{+\hat{G}_o}] \exp(-\hat{Z}[t]), \]
\[ \hat{G}_o = \left( \hat{K}_A / \hat{\gamma}_o \right), \]
\[ t = (t_{data} - \hat{t}_{offset}). \]

where \( t_{data} \) is the data start time. Set a preliminary value for \( \hat{t}_{offset} \) first, to fix the time scale for the \( N_{data}[t] \) measured values:
\[ N_I \equiv N_{data}[t_{data} - \hat{t}_{offset}], \]
\[ N_F \equiv N_{data}[t_{data} - \hat{t}_{offset}], \]
\[ (t_F - t_I) \equiv (t_{data} - t_{offset}). \]

Next, pick values for \{\( \tilde{\gamma}_o, \tilde{\delta}_o \)\} for \( \tilde{N}[t] \) in Eq. \[4.6b\], allowing direct comparison between \( N[t] \) and \( \tilde{N}[t] \) at each data point:
\[ \tilde{N}[t] \equiv \tilde{N} = (t_{data} - \tilde{t}_{offset}), \]
\[ N_{data}[t] \equiv N_{data}[t = (t_{data} - \tilde{t}_{offset})]. \]

The resulting calculated values for both \{\( \tilde{N}[t_I], \tilde{N}[t_F] \)\} can often be much too high or low, compared to the \{\( N_I, N_F \)\} measured data, but those values can be renormalized to:
\[ \bar{N}[t] = \tilde{N}[t] \left( N_I / \tilde{N}[t_I] \right) \equiv S_I \tilde{N}[t], \]
\[ \bar{N}[t_F] = \tilde{N}[t_F] \left( N_I / \tilde{N}[t_I] \right) \equiv S_I \tilde{N}[t_F]. \]

Here, \( S_I \) is the renormalization coefficient, and \( \bar{N}[t] \) allows easy comparison to the measured \( N_{data}[t] \) since \( \bar{N}[t_I] \equiv N_I \). Given \{\( \tilde{\gamma}_o, \tilde{\delta}_o \)\}, the \( \hat{K}_A(\tilde{\gamma}_o) \) value that is needed to obey \( \bar{N}[t_F] \rightarrow N_F \) can be set by using \textit{Excel} \textit{Goal-Seek} or its equivalent, which also sets a particular \( S_I \) value. Next, the \( \tilde{\gamma}_o \) value is adjusted to find the specific \{\( K_A, \gamma_o \)\} parameter pair that gives \( S_I = 1 \). This process is needed because these \( \delta_o \neq 0 \) cases do not allow easy determination of \( t_I \) as in Eq. \[1.4a\], or for \{\( K_A, \gamma_o \)\}, given \( t_I \), as in Eqs. \[1.5a\]-\[1.5b\].
The \textit{rms} error on a \textit{Logarithmic Y-axis}, between this $\overline{N}[t ; K_A, \gamma_o]$ and the $N_{\text{data}}[t]$ is one of many $\{K_A, \gamma_o, \delta_o, \hat{t}_{\text{offset}}; S_I = 1\}$ choices. The minimum \textit{rms} error among all these $S_I = 1$ cases and the $N_{\text{data}}[t]$, when varying $\{\delta_o, \hat{t}_{\text{offset}}\}$ gives the best $\{K_A, \gamma_o, \delta_o, t_{\text{offset}}\}$ values for Eqs. [4.3a]-[4.3e].

5 \textit{Enhanced Initial Model [EIM]} Results for Italy

The Eqs.[1.3a] \textit{Initial Model} results were shown in Figs. 1-2. Mandatory \textit{Social Distancing} was introduced at $N_{\text{data}}^{3/10/2020} = 10,149$ which is the $t_I$ data point, with $N_{\text{data}}^{5/15/2020} = 237, 290$ being the $t_F$ data point. The \textit{EIM} was then applied to the same data, to highlight the improvements that can be obtained from using the \textit{EIM} of $Z_A[t]$ and $N_A(Z_A)$, in place of $Z_o[t]$ and $N_o(Z_o)$.

\textbf{Figures 3-4} show the resulting \textit{EIM} best-…ts for $Z_A[t]$ and $N_A(Z_A)$, along with using a $\rho_A[t]$ tail that is a pure exponential decay.

For the \textit{EIM}, a new best estimate of $t_{\text{offset}} = 2.866 \text{days}$ was found within the $N_{\text{data}}^{2/25/2020}$ (Day 1) to $N_{\text{data}}^{3/10/2020}$ (Day 17) data, setting the $EIM$ $t = 0$ point. Then $t_I = (17 - 2.866) = 14.134 \text{days}$, while $N_{\text{data}}^{5/15/2020}$ (Day 114) gives $t_F = (114 - 2.866) = 111.134 \text{days}$, along with:

\begin{align*}
N_{\text{data}}^{3/10/2020}[t_I] &= 14.134 = 10,149; \quad [5.1a] \\
N_{\text{data}}^{5/15/2020}[t_F] &= 111.134 = 237, 290; \quad [5.1b] \\
N_{\text{data}}^{2/25/2020}[t = 0] &= 1; \quad [5.1c] \\
(t_F - t_I) &= 97 \text{days}. \quad [5.1d]
\end{align*}

This X-axis $t = 0$ point is a hypothetical \textit{EIM} pandemic starting point, if \textit{Social Distancing} had been operating throughout the initial \textit{CoVID-19} period. \textbf{Figure 3} has a predicted \textit{CoVID-19} pandemic peak of $\sim 5,217 / \text{day}$ at $t = 29,134 \text{days}$ on 3/25/2020, with $\sim 243,100$ total cases at the pandemic end. This datafit has $> 4X$ error reduction over the $\delta_o = 0$ case, as summarized next:

\begin{align*}
K_A & \quad \gamma_o & \quad t_{\text{dbl}, \text{days}} & \quad N[t \rightarrow \infty] & \quad \delta_o & \quad t_I, \text{days} & \quad \textit{rms}_{ERR} \\
1.285436 & .101062 & .553060 & \sim 243,100 & .0336 & 14.134 & .023755 \\
4.240513 & .33078 & .163458 & \sim 338,165 & 0 & 7.899 & .097828
\end{align*}

[5.2]

The \textit{EIM} with $Z_A[t]$ and $\delta_o = 0.0336$ gives a $\rho_A[t]$ curve that is in excellent agreement with the \textbf{Fig. 3} measured $\rho_{\text{data}}[t]$ data. Comparing $N_A[t]$ and $N_{\text{data}}[t]$ in \textbf{Fig. 4} also shows an excellent match over the whole \textit{Logarithmic Y-axis} range, used in the \textit{rms} error minimization.

The \textbf{Figs. 3-4} $\kappa = 0$ results were then compared to the $\kappa = 1$ case, using $Z_B[t]$ and $N_B(Z_B)$ of Eqs. [3.2]-[3.3]. The $\kappa = 1$ case has a \textit{Social Distancing} factor that gradually turns off the \textit{EIM} exponential decay. The resulting \textit{rms} error best fits converged to $\delta_o \rightarrow 0$, as follows:

\begin{align*}
\gamma_o & \quad t_{\text{dbl}} & \quad \textit{rms}_{ERR} & \quad \delta_o & \quad t_{\text{offset}, \text{days}} & \quad t_I, \text{days} & \quad t_F, \text{days} \\
.07146 & .775863 & .2223783 & .2000 & 10.0 & 7 :: 104 \\
.18623 & .295372 & .1751635 & .2000 & 11.0 & 6 :: 103 \\
.30360 & .181417 & .2102839 & .2000 & 12.0 & 5 :: 102
\end{align*}
6 Enhanced Orthogonal Functions for Italy

Any monotonic $Z[t]$ can convert measured $N_{\text{data}}[t]$ data into $N_{\text{data}}(Z)$. Using the Eq. [2.6b] $Z_A[t] \rightarrow Z[t]$ in Eqs. [1.7a]-[1.7d] extends the EIM into an EOF model, where Eq. [1.9] gives:

$$\rho[t] = R(Z_A[t]) \frac{dZ_A}{dt} = Z_A[t] R(Z_A[t]) \{ \delta_0 + \frac{\gamma_0}{(1 + \gamma_0 t)} \}, \quad [6.1a]$$

$$Z_A[t] = \left[ G_0 / (1 + \gamma_0 t) \right] \exp(-\delta_0 t). \quad [6.1b]$$

The Eq. [5.2] $\delta_0 \neq 0$ entries and Eq. [5.1a]-[5.1d] boundary conditions give:

$$Z_A^{\text{min}}[t_F = 111.134] = 0.024226510, \quad [6.2a]$$

$$Z_A^{\text{max}}[t_I = 14.134] = 3.176125728. \quad [6.2b]$$

These data-driven $Z_A[t]$ limits are used next, along with $Z[t] \rightarrow Z_A[t]$ in:

$$N(Z) = \sum_{m=0}^{M_F} g_m L_m(Z) \exp[-Z], \quad [6.3a]$$

$$g_m = \sum_{m=0}^{M_F} g_m \int_{Z=0}^{Z_{A}^{\text{max}}} I_n(Z) L_m(Z) \exp[-Z] dZ = \int_{Z=Z_A^{\text{min}}}^{Z_A^{\text{max}}} I_n(Z) N_{\text{data}}(Z) dZ + \int_{Z=Z_A^{\text{max}}}^{+\infty} I_n(Z) N(Z) dZ, \quad [6.3b]$$

where $m = \{0, M_F\}$ sets how many terms are in the Eq. [6.3a] series. Generally $M_F = 2$ is used here. The $L_m(Z)$ are the Laguerre Polynomials, with the first few $L_m(Z)$ being:

$$L_{-1}(Z) \equiv 0, \quad [6.4a]$$

$$L_0(Z) \equiv 1 = L_m(Z = 0), \quad [6.4b]$$

$$L_1(Z) \equiv (1 - Z), \quad [6.4c]$$

$$L_2(Z) \equiv (1 - 2 Z + \frac{1}{2} Z^2), \quad [6.4d]$$

$$L_3(Z) \equiv (1 - 3 Z + \frac{3}{2} Z^2 - \frac{1}{4} Z^3), \quad [6.4e]$$

$$L_4(Z) \equiv (1 - 4 Z + 3 Z^2 - \frac{9}{4} Z^3 + \frac{1}{8} Z^4). \quad [6.4f]$$

Some important properties of the Laguerre Polynomials are:

$$\int_{Z=0}^{+\infty} L_m(Z) L_n(Z) \exp(-Z) dZ = \delta_{m,n}, \quad [6.5a]$$
\( \delta_{m,n} = \begin{cases} 1 & \text{for } m=n, \\ 0 & \text{otherwise} \end{cases} \) \tag{6.5b}

\[
\int_{Z'=+\infty}^{+\infty} L_m(Z') \exp(-Z') dZ' = [L_m(Z) - L_{m-1}(Z)] \exp(-Z), \tag{6.5c}
\]

\[
L_m(Z) \exp(-Z) = \frac{1}{m!} \frac{d^m}{dZ^m} [Z^m e^{-Z}] = e^{-Z} \sum_{k=0}^{k=m} (-1)^k \frac{n!}{k! (m-k)!} \left[ \frac{Z^k}{k!} \right], \tag{6.5d}
\]

\[
L_{m>2}(Z) = [2 - (Z+1\frac{m}{m})] L_{m-1}(Z) - [1 - \frac{1}{m}] L_{m-2}(Z). \tag{6.5e}
\]

Here Eq. [6.5a] defines an orthogonal function set. The "\( n! \)" (n-factorial) in Eq. [6.3b], for \( n \) an integer, is defined as the product:

\[
n! \equiv (n)(n-1)(n-2)(n-3)...(3)(2)(1), \tag{6.6a}
\]

\[
1! \equiv 0! \equiv 1, \tag{6.6b}
\]

where factorials with negative integers are not allowed. For \( M_F > 2 \), the following equations developed by Watson, and improved by Gillis and Weiss, helps in evaluating Eq. [6.3b]:

\[
L_r(Z) L_s(Z) = \sum_{t=r-s}^{t=r+s} C_{rst} L_t(Z), \tag{6.7a}
\]

\[
C_{rst} = \int_{X=+\infty}^{X=+\infty} L_r(X) L_s(X) L_t(X) \exp(-X) dX, \tag{6.7b}
\]

\[
C_{rst} \equiv \frac{(-1)^p}{2^p} \sum_{n=0}^{n=(r+s)} \frac{(2^n)}{(r-n)! (s-n)! (2n-p)! (p-n)!} \tag{6.7c}
\]

where all terms in the Eq. [6.7c] sum for \( n = \{0, (r+s)\} \) have an implicit requirement that all negative factorials arguments are excluded. Since the \( N(Z) \) of Eq. [6.3a] has \( \{g_m; m = 0, M_F\} \), and \( N(Z) \) also appears in each \( g_m \)-equation of Eq. [6.3b], how to determine each \( g_m \) by itself, can be done as follows. First define:

\[
Q_n = \int_{Z=Z_{A}^{max}}^{Z=Z_{A}^{min}} L_n(Z) N_{data}(Z) dZ, \tag{6.8a}
\]

\[
K_{m,n} = \int_{Z=Z_{A}^{min}}^{Z=Z_{A}^{max}} L_m(Z) L_n(Z) \exp(-Z) dZ = K_{n,m}. \tag{6.8b}
\]

When the \( N_{data}(Z) \) is comprised of \( j = \{1, 2, ... J\} \) discrete values between \( \{Z_{A}^{min}, Z_{A}^{max}\} \), with each \( Z_j \) having an \( N^{(j)}(Z_j) \) value, the Eq. [6.8a] integral needs to be replaced by a sum. Let \( Z_0 = Z_1 \) and \( Z_{J+1} = Z_J \), the \( Q_n \) replacement for Eq. [6.8a] is then:

\[
Q_n \equiv \sum_{j=1}^{J} L_n(Z_j) N^{(j)}(Z_j) \Delta_j, \tag{6.9a}
\]

\[
\Delta_j = \frac{1}{2}[Z_{j+1} - Z_{j-1}]. \tag{6.9b}
\]

Eq. [6.3b] can then be re-written as a \( 3 \times 3 \) matrix \( \mathbf{M}_3 \), which relates a data-driven \( Q_3 \)-vector to a resultant \( \mathbf{g}_3 \)-vector:

\[
\mathbf{Q}_3 = \mathbf{M}_3 \mathbf{g}_3, \tag{6.10a}
\]

\[
\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} K_{0,0} & K_{0,1} & K_{0,2} \\ K_{1,0} & K_{1,1} & K_{1,2} \\ K_{2,0} & K_{2,1} & K_{2,2} \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{6.10b}
\]
\[
(M_3)^{-1} \overline{Q}_3 \equiv \overline{g}_3, \quad \text{[6.10c]}
\]

where \((M_3)^{-1}\) is the matrix inverse of \(M_3\). When \(\{Z_A^{\text{min}}, Z_A^{\text{max}}\} \rightarrow \{0, +\infty\}\), this \(M_3\) becomes the Identity Matrix. The following \(k_{m,n}(Z)\) integrals set \(K_{m,n}\):

\[
k_{m,n}(Z) = \int_{Z'=0}^{Z'=+\infty} L_m(Z') L_n(Z'') \exp(-Z'') dZ'' = k_{m,n}(Z), \quad \text{[6.11a]}
\]

\[
K_{m,n} \equiv k_{m,n}(Z_A^{\text{min}}) - k_{m,n}(Z_A^{\text{max}}) = K_{m,n}. \quad \text{[6.11b]}
\]

The \(k_{m,n}(Z)\) integrals can be determined using Eq. [6.5c], which gives:

\[
k_{0,0}(Z) = 1 \exp(-Z), \quad \text{[6.12a]}
\]

\[
k_{1,1}(Z) = \{1 + Z^2\} \exp(-Z), \quad \text{[6.12b]}
\]

\[
k_{2,2}(Z) = \{1 + 2Z^2 - Z^3 + \frac{1}{2}Z^4\} \exp(-Z), \quad \text{[6.12c]}
\]

\[
k_{0,1}(Z) = (-Z) \exp(-Z), \quad \text{[6.12d]}
\]

\[
k_{0,2}(Z) = (-Z) \{1 - \frac{1}{2}Z\} \exp(-Z), \quad \text{[6.12e]}
\]

\[
k_{1,2}(Z) = (-Z) \{1 - Z + \frac{1}{2}Z^2\} \exp(-Z). \quad \text{[6.12f]}
\]

To extract \(\{g_0, g_1, g_2\}\), the 3 \times 3 symmetric \(M_3\) matrix needs inversion:

\[
M = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}, \quad \text{[6.13a]}
\]

\[
\det[M] \equiv [abc - ae^2 - bdf^2 - cde^2 + 2def], \quad \text{[6.13b]}
\]

\[
\det[M] (M)^{-1} \equiv \begin{pmatrix} \frac{\bc - e^2}{b} & -(\cde - ef) & -(b\de - de) \\ -(\cde - ef) & \frac{\ac - f^2}{a} & -(a\de - df) \\ -(b\de - de) & -(a\de - df) & \frac{\ab - d^2}{d} \end{pmatrix}, \quad \text{[6.13c]}
\]

which determines \(\{g_0, g_1, g_2\}\) from the \(\{Q_0, Q_1, Q_2\}\) data. A best-fit \(N(Z)\) for \(Z = \{0^+, \infty, -\}\) results, along with an equivalent fit for \(R(Z)\) using Eq. [1.7d].

Instead of having to find the best \(\{g_0, g_1, g_2\}\) triplet, one could find the best \(\{g_0', g_1'\}\) by just using \(\{Q_0, Q_1\}\) and an \(M_2\) sub-matrix; or one could find the best \(\{g_0^+, g_1^+\}\) by itself by just using \(\{Q_0\}\) and an \(M_1\) sub-matrix:

\[
\overline{Q}_2 = M_2 \overline{g}_2, \quad \text{[6.14a]}
\]

\[
\begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} = \begin{pmatrix} K_{0,0} & K_{0,1} \\ K_{1,0} & K_{1,1} \end{pmatrix} \begin{pmatrix} g_0' \\ g_1' \end{pmatrix}, \quad \text{[6.14b]}
\]

\[
\begin{pmatrix} g_0' \\ g_1' \end{pmatrix} = \begin{pmatrix} K_{0,0} & K_{0,1} \\ K_{1,0} & K_{1,1} \end{pmatrix}^{-1} \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ K_{1,1} - K_{0,1} \end{pmatrix} \begin{pmatrix} K_{1,1} & K_{1,0} \\ K_{0,1} & K_{0,0} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix}, \quad \text{[6.14c]}
\]

\[
\overline{Q}_1 = M_1 \overline{g}_1, \quad \text{[6.14d]}
\]

\[
\begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} = \begin{pmatrix} K_{0,0} \\ K_{1,0} \end{pmatrix} \begin{pmatrix} g_0^+ \\ g_1^+ \end{pmatrix}, \quad \text{[6.14e]}
\]

\[
\begin{pmatrix} g_0^+ \\ g_1^+ \end{pmatrix} = \begin{pmatrix} K_{0,0} \\ K_{1,0} \end{pmatrix}^{-1} \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix}. \quad \text{[6.14f]}
\]

Once the \(\{g_m; m = (0, M_F)\}\) constants are found and used in Eq. [1.8], its \(c_0 \) value provides the new \(EOM\) estimate for the predicted total number of CoVID-19 cases at the pandemic end, refining the initial Eq. [5.2] \(N[t \rightarrow \infty] EIM\) value.
7 EOF Model Results for Italy

The EOF model starts with the EIM of Eq. [2.6b] using \( Z_A[t] \), and the bin.com Italy data\(^9 \), which gives:

\[
\begin{align*}
N_{data}[t_f = 14.134] &= 10, 149; \quad [7.1a] \\
N_{data}[t_F = 111.134] &= 237, 290; \quad [7.1b] \\
N_A[t \to \infty] &= 243, 109; \quad [7.1c] \\
Z_A^{min}[t_F = 111.134] &= 0.024226510; \quad [7.1d] \\
Z_A^{max}[t_I = 14.134] &= 3.176125728; \quad [7.1e]
\end{align*}
\]

via Eqs. [5.1a]-[5.1d], [5.2], and [6.2a]-[6.2b]. For these \( \{Z_A^{min}, Z_A^{max}\} \) values, with \((t_F - t_I) = 97 \) days, the \( M_3 \) matrix of \( K_{m,n} \) entries, via Eq. [6.8b], is:

\[
M_3 = \begin{pmatrix}
K_{0,0} & K_{0,1} & K_{0,2} \\
K_{1,0} & K_{1,1} & K_{1,2} \\
K_{2,0} & K_{2,1} & K_{2,2}
\end{pmatrix} = \begin{pmatrix}
0.93432 & 0.10895 & -0.10133 \\
0.10895 & 0.51376 & 0.35717 \\
-0.10133 & 0.35717 & 0.36868
\end{pmatrix}. \quad [7.2]
\]

It has a rather small \( \text{det}(M_3) = 0.04024218 \) value, with an inverse of:

\[
(M_3)^{-1} = \begin{pmatrix}
1.536745 & -1.897502 & 2.260643 \\
-1.897502 & 8.304570 & -8.566823 \\
2.260643 & -8.566823 & 11.633092
\end{pmatrix}. \quad [7.3]
\]

A convolution of \( L_m(Z_A) \) functions with the measured \( \overline{Q}_3 \) dataset vector of Eqs. [6.9a]-[6.9b], along with the above \((M_3)^{-1}\), gives this final \( g \)-vector\(^{12} \):

\[
(M_3)^{-1} \overline{Q}_3 \equiv (M_3)^{-1} \begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix} = (M_3)^{-1} \begin{pmatrix}
+226, 767 \\
+26, 978 \\
-22, 399
\end{pmatrix} \equiv
\]

\[
\overline{g}_3 = \begin{pmatrix}
g_0 \\
g_1 \\
g_2
\end{pmatrix} = \begin{pmatrix}
+246, 656 \\
-14, 362 \\
+20, 954
\end{pmatrix}, \quad [7.4]
\]

determining the constants for \( N(Z_A) \) in Eq. [1.7a]. The coefficients for \( R(Z_A) \), which sets the predicted number of daily new CoVID-19 cases, are:

\[
\overline{C}_3 = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} \overline{g}_3 = \begin{pmatrix}
c_0 \\
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix}
+253, 248 \\
+6, 592 \\
+20, 954
\end{pmatrix}, \quad [7.5]
\]

determining the constants needed for \( R(Z_A) \) in Eq. [1.7b]. Using these \( \{g_0, g_1, g_2\} \) values along with Eq. [1.8] gives:

\[
N(Z_A \to 0) \equiv N[t \to \infty] = c_0 \equiv \{253, 248\}, \quad [7.6]
\]

as a new predicted total number of CoVID-19 cases at the pandemic end for the EOF model, which is a \(~4.17\%\) or 10, 139 increase in the number of cases, compared to the EIM value of Eq. [7.1c].

Using Eq. [6.1b] for \( Z_A[t] \), and substituting the Eq. [7.3] \( \overline{C}_3 \) values into Eq. [1.7b] gives \( R(Z_A) \). The \( \rho[t] \) in Eq. [6.1a] is derived from \( R(Z_A) \) using Eq. [1.7b], with the resulting EOF \( \rho[t] \) plotted in Figure 5, along with the \( t > t_I \)

raw data for the daily new CoVID-19 cases.

The **Figure 5** EOF model also gives a \( t < t_I \) extrapolation, which shows what the combination of processes would look like, if they all had been operating continuously from the CoVID-19 pandemic start. The companion \( N[t] \) analytic
result, along with the $t > t_1$ raw data for the total number of CoVID-19 cases is show in Figure 6.

Comparing the size and timing of the $\rho[t]$ pandemic peak, and its Day 200 value, between the EIM (Figs. 3-4) and EOF model (Figs. 5-6), gives:

\[
\begin{pmatrix}
N[t \to \infty] & 243,109 & 253,248 \\
\max\{\rho[t_p]\} & 5,217 / \text{day} & 5,142 / \text{day} \\
\frac{t_p}{\text{Date}} & 3/25/2020 & 3/27/2020 \\
\frac{\rho_{\text{day}}}{\rho[t]} & 6.67 / \text{day} & 7.59 / \text{day}
\end{pmatrix}, \quad [7.7]
\]

showing the EOF model predicts more cases total and more daily new CoVID-19 cases at Day 200, as well as modifying the pandemic peak predictions.

While the above analysis used $M_F = 2$, with the Eq. [7.4] $\vec{\gamma}_3$ setting the best $\{g_0, g_1, g_2\}$ values, this EOF model also provides estimates for the simpler $M_F = \{0, 1\}$ cases, as outlined by Eqs. [6.14a]-[6.14f]. For $M_F = 1$, the best two $\{g_0, g_1\}$ values were gotten by only using \{Q_0, Q_1\} and an M_2 sub-matrix of M. For $M_F = 0$, the best $\{g_0^+\}$ by itself is derived by using \{Q_0\} and the M_1 sub-matrix. These alternative estimates give:

\[
\vec{\gamma}_2 = \begin{pmatrix} g_0^+ \\ g_1^+ \end{pmatrix} = \begin{pmatrix} .93432 \\ .10895 \end{pmatrix}^{-1} \begin{pmatrix} 226,767 \\ 26,978 \end{pmatrix} = \begin{pmatrix} 242,584 \\ 1,069 \end{pmatrix}
\]

\[
\vec{\gamma}_1 = \begin{pmatrix} g_0^+ \end{pmatrix} = 0.93432^{-1} \begin{pmatrix} +226,767 \end{pmatrix} = \begin{pmatrix} 242,709 \end{pmatrix}. \quad [7.8a]-[7.8b]
\]

These additional calculations give the following progression of estimates for $N[t \to \infty]$, which is the final number of CoVID-19 cases at the pandemic end:

\[
\begin{pmatrix}
N_A[t \to \infty]; \text{EIM} \\
N_F[t \to \infty]; \text{EOF} \\
N_M[t \to \infty]; \text{EOF}
\end{pmatrix} = \begin{pmatrix} 243,109 \\ 243,709 \\ 243,653 \end{pmatrix}, \quad [7.9]
\]

based on increasing the number of data fitting parameters used with the original data. This summary shows the $N[t \to \infty]$ projections are fairly stable, with an average and 1σ standard deviation:

\[
< N[t \to \infty] > = 245,680 \pm 5,060; \quad [7.10]
\]

among these different calculations, where 1σ is ~2.06% of the overall average.

Comparing the results among Figs. 1-6 also highlights these items:

(a) All $\rho[t]$ functions have a sharp rise, and slower decreasing tail. The fastest changing $\rho_{\text{data}}[t]$ tail, as in the Italy CoVID-19 (Fast Shutoff) data, was successfully modeled by adding in an exponential term, as in Eq. [6.1b].

(b) The data fits in Fig. 4 and Fig. 6 show that the extra parameters in the EIM and EOF model fits the $\rho_{\text{data}}[t]$ shape progressively better.

(c) The EOF model shows only relatively small changes of ~2.06% in the $N[t \to \infty]$ limits (Eq. [7.10]), as an estimate of uncertainty in the EIM.

(d) The Enhanced Initial Model (EIM) function captures much of the progression to a pandemic Fast Shutoff, as seen in the Italy data.

The $\rho[t]$ tail may still differ from these predictions, due to factors such as:

(i) The CoVID-19 dynamics may change in the long-term low $\rho[t]$ regime;

(ii) A "second wave" or multiple waves of $\rho[t]$ resurgence may occur, which are beyond the scope of this CoVID-19 pandemic modeling.
8 Summary and Conclusions

The early stages of the CoVID-19 coronavirus pandemic starts off with a nearly exponential rise in the number of infections with time. Defining $N[t]$ as the expected total number of CoVID-19 cases vs time, this basic function:

$$N_o[t] = 1 \exp[+K_A t/(1 + \gamma_o t)] = \exp[+G_o \exp[-Z_o]], \quad [8.1a]$$

$$Z_o[t] = +[G_o/(1 + \gamma_o t)], \quad G_o \equiv [K_A / \gamma_o], \quad [8.1b]$$

$$\rho_o[t] \equiv dN_o[t]/dt, \quad [8.1c]$$

models Social Distancing effects as a gradual lengthening of the pandemic growth doubling time, which enables pandemic shut off with only a small population of infected persons. The Eq. [8.1b] $Z_o[t]$ was our Initial Model\(^1\), and gives a CoVID-19 Slow Shut off with a long-term $\rho_o[t]^{-1/t^2}$ tail. Previously we showed\(^1\)–\(^2\) that this $Z_o[t]$ model fits many $N_{data}[t]$ and $\rho_{data}[t]$ cases.

However, some data had a CoVID-19 Fast Shut off, with a $\rho_{data}[t] \sim [\exp(-\delta_o t)]$ exponential tail, such as in Italy\(^9\), where a Gaussian tail $\rho[t] \sim [\exp(-q_0 t^2)]$ would have decreased too quickly. An Enhanced Initial Model (EIM) was developed here, using this $Z_A[t]$ function:

$$N_A[t] \approx 1 \exp[+G_o \exp(-Z_A[t])], \quad [8.2a]$$

$$Z_A[t] \equiv +[G_o / (1 + \gamma_o t)] \exp(-\delta_o t), \quad [8.2b]$$

$$G_o \equiv [K_A / \gamma_o], \quad [8.2c]$$

$$\rho_A[t] \equiv dN_A[t]/dt. \quad [8.2d]$$

We also examined if the $\exp(-\delta_o t)$ exponential decay could also be subject to a Slow Shut off, giving $\exp[-\delta_o t / (1 + \gamma_o t)]$ instead of $\exp(-\delta_o t)$, but that did not match the Italy data. To allow more data fitting parameters beyond just $\{K_A, \gamma_o, \delta_o\}$, an orthogonal function method was developed\(^2\):

$$N(Z) = \sum_{m=0}^{M_F} g_m L_m(Z) \exp[-Z], \quad [8.3a]$$

$$R(Z) = \sum_{m=0}^{M_F} c_m L_m(Z) \exp[-Z], \quad [8.3b]$$

$$N(Z) \equiv \int_{Z'=Z}^{Z'=+\infty} R(Z') \, dZ', \quad [8.3c]$$

$$c_{M_F-k} = \sum_{m=0}^{k} g_m, \quad [8.3d]$$

which is applicable to a generic $Z[t]$ function, with $N[t] = N(Z[t])$, where $Z[t] \rightarrow Z_o[t]$ and $Z[t] \rightarrow Z_A[t]$ are special cases. Larger $M_F$ with more $\{L_m(Z); m = (0, +M_F)\}$ terms can match almost any arbitrary function, enabling fits to a variety of $N[t]$ and $\rho[t]$ shapes. The $\{g_m; m = (0, +M_F)\}$ are constants determined from each dataset. The $L_m(Z)$ are the Laguerre Polynomials, with several important properties given in Eqs. [6.4a]-[6.5e].

Using $Z_A[t]$ in Eqs. [8.3a]-[8.3d] results in this Enhanced Orthogonal Function [EOF] model, which is applicable to both Slow or Fast Shut off CoVID-19 pandemic data. The $\rho[t]$ expected number of daily new CoVID-19 cases is:

$$N[t] \equiv \int_{t'=(-1/\gamma_o)}^{t'=t} \rho[t'] \, dt', \quad [8.4a]$$
\begin{equation}
\rho[t] = R(Z_A[t]) \frac{dZ_A}{dt} = R(Z_A[t]) \left[ \frac{g_m}{(1+\gamma_o t)} \right] \delta_o + \left[ \frac{\gamma_o}{(1+\gamma_o t)} \right] \exp(-\delta_o t). \quad [8.4b]
\end{equation}

Methods were developed to derive the \( \{K_A, \gamma_o, \delta_o\}\) values, and to determine the \( \{g_m; m = (0,+M_F)\}\) and \( \{c_m; m = (0,+M_F)\}\) constants from data. Whereas our \textit{Initial Model} and \textit{EIM} were \( M_F = 0 \) cases, the \( M_F = 2 \) case was used here to examine the Italy CoVID-19 data, as an \textit{EOF} model example.

The \textit{bing.com} data for Italy up to \(~6/15/2020\) was then analyzed, with \textbf{Figures 3-6} giving the new Italy results. Both the \textit{EIM} and the \textit{EOF} model provided good data fits, giving similar \( N[t \to \infty] \) results for the final number of CoVID-19 pandemic cases, differing by only \(~2\%\) at the \( 1\sigma \) level.

The \( \rho[t] \) post-peak behavior best indicates if a \( \delta_o \neq 0 \) model (CoVID-19 pandemic \textit{Fast Shutoff}) is applicable. The \( \delta_o \neq 0 \) case likely is a second \textit{Social Distancing} process, that operates along with, but is independent of the gradual pandemic \textit{doubling time} changes. That \textit{doubling time} change gives rise to a CoVID-19 pandemic \textit{Slow Shutoff} (\( \gamma_o \neq 0 \)), and that process still operates concurrently with the \( \delta_o \neq 0 \) CoVID-19 pandemic \textit{Fast Shutoff}.

This analysis shows a wide variety of CoVID-19 data can be modeled using \( \{K_A, \gamma_o, \delta_o, f_{off set}\} \) as parameters, covering: (I) an exponential rise at CoVID-19 pandemic start; (II) a gradual lengthening of \textit{doubling times} for a pandemic \textit{Slow Shutoff}; and (III) an exponential decay for pandemic \textit{Fast Shutoffs}.

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10 References

1. https://www.MedRxiv.org/content/10.1101/2020.05.04.20091207v1, https://doi.org/10.1101/2020.05.04.20091207, "Initial Model for the Impact of Social Distancing on COVId-19 Spread", Genghmun Eng.

2. https://www.MedRxiv.org/content/10.1101/2020.06.30.20143149v1, https://doi.org/10.1101/2020.06.30.20143149, "Orthogonal Functions for Evaluating Social Distancing Impact on Covid-19 Spread", Genghmun Eng.

3. https://www.medrxiv.org/content/10.1101/2020.03.27.20043752v1, "Forecasting COVID-19 impact on hospital bed-days, ICU-days, ventilator-days and deaths by US state in the next 4 months", IHME COVID-19 Health Service Utilization Forecasting Team.

4. https://www.geekwire.com/2020/ univ-washington-epidemiologists-predict-80000-covid-19-deaths-u-s-july/ "Univ. of Washington researchers predict 80,000 COVID-19 deaths in U.S. by July", Alan Boyle, GeekWire, March 26, 2020.

5. https://www.yahoo.com/finance/news/coronavirus-modelers-raise-projected-u-041641553.html, "Coronavirus modelers raise projected U.S. death toll and lengthen state-by-state recovery timeline", Alan Boyle, GeekWire, April 27, 2020.

6. https://covid19.healthdata.org, update of 29 April 2020.

7. http://www.healthdata.org/covid/updates "COVID-19: What's New for May 4, 2020: Updated IHME COVID-19 projections: Predicting the Next Phase of the Epidemic", IHME COVID-19 Health Service Utilization Forecasting Team.

8. https://finance.yahoo.com/news/ pandemic-projection-puts-u-death-220824741.html, "New pandemic projection puts U.S. death toll at nearly 135,000, due to less social distancing", Alan Boyle, GeekWire, May 4, 2020.

9. www.bing.com/covid: 'Bing COVID-19 Tracker', and https://www.bing.com/covid?form=CPVD07.

10. G. N. Watson, "A Note of the Polynomials of Hermite and Laguerre", Journal of the London Mathematical Society, 13(1938), pp. 29-32.

11. J. Gillis and G. Weiss, "Products of Laguerre Polynomials", Math. Comput., 14(69), Jan. 1960, pp. 60-63

12. For correct results, calculations done by swap55.exe requires no commas in the integer matrix or vector entries, when using: ScientificWorkplace™ _Compute_EvaluateNumerically. It is due to swap55.exe also accepting European notation where 10,149 ⇔ 10.149.
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Enhanced Orthog.Func.[EOF] Model: ITALY CoVID-19 Pandemic, 6/15/20 Update

Early May 2020 bing.com revision changed all data back to pandemic start

Total Number of CoVID-19 Cases

Days from 2/25/2020, Social-Distancing started on 3/10/2020

N1(t) Tot.Cases: Orthog.Func.Fit  ▶ Revised bing.com Data

3/10/20 = 10,149 Cases; 6/15/20 = 237,290 Cases