Kernels of splitting homomorphisms

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Abstract
Lei and Wu have given a description of the second homotopy group of a closed orientable 3-manifold in terms of the kernels of the epimorphisms from the fundamental group of a Heegaard splitting surface onto the fundamental groups of the two handlebody sides. In this note, we give a geometric derivation of this result and collect some observations about the relation between the various groups and the topology of the 3-manifold and the Heegaard splitting.

1 Introduction
The role of the fundamental group in 3-dimensional topology is central. In [17], Stallings put forth an approach to the 3-dimensional Poincaré conjecture by way of the group theory associated to a Heegaard splitting of a homotopy 3-sphere. This approach was furthered by Jaco [6] who proved that several group-theoretic statements involving free groups and surface groups were equivalent to the Poincaré conjecture.

Let $M$ be a compact orientable 3-manifold and let $M = H_\alpha \cup H_\beta$ be a Heegaard splitting of $M$ with $\Sigma = H_\alpha \cap H_\beta$ an orientable surface of genus $g$. Fix a point $*$ in $\Sigma$ and consider the associated pushout diagram of groups, with morphisms induced by inclusions.

\[
\begin{array}{ccc}
\pi_1(\Sigma,*) & \longrightarrow & \pi_1(H_\alpha,*) \\
\downarrow & & \downarrow \\
\pi_1(H_\beta,*) & \longrightarrow & \pi_1(M,*)
\end{array}
\]

The maps from the surface group to the free groups of the handlebodies are surjective as any curve in a handlebody can be made to miss the spine of the handlebody and thereby can be homotoped to the boundary. The map $\phi: \pi_1(\Sigma) \rightarrow \pi_1(H_\alpha) \times \pi_1(H_\beta)$ is the splitting homomorphism associated to the Heegaard surface $\Sigma$. Jaco showed how all of the topology of $M$ is encoded in the splitting homomorphism [6]. From here out, we will not generally mention the basepoints explicitly.

Building on the approach of Stallings and Jaco, Hempel [5] showed the condition in the following theorem is equivalent to the Poincaré conjecture and therefore, by the work of Perelman [14], [15], it follows:

**Theorem 1.** (Perelman)

For all integers $g \geq 0$ and any pair of surjective homomorphisms $\phi_1, \phi_2 : \pi_1(\Sigma) \rightarrow F_g \times F_g$ where $\Sigma$ is a genus $g$ closed orientable surface, and $F_g$ is a rank $g$ free group, there is an isomorphism

$$
\pi_1(\Sigma) \xrightarrow{\cong} \pi_1(\Sigma)
$$
and a pair of isomorphisms

\[ F_g \cong F_g \]

such that the following commutes

\[
\begin{array}{ccc}
\pi_1(\Sigma) & \xrightarrow{\phi_1} & F_g \times F_g \\
\cong & & \cong \\
\pi_1(\Sigma) & \xrightarrow{\phi_2} & F_g \times F_g
\end{array}
\]

In other words, there is a unique surjective homomorphism \( \pi_1(\Sigma) \to F_g \times F_g \) up to pre- and post-composition with automorphisms (where post-composition is by automorphisms that are a product of automorphisms on the \( F_g \) factors).

More recently, a similar group-theoretic statement equivalent to the smooth 4-dimensional Poincaré conjecture has been given by Abrams, Gay, and Kirby [1]. In considering these connections, we wanted to understand how the basic topological invariants of a space can be seen from the perspective of splitting homomorphisms. Let

\[ K_\alpha = \ker(\pi_1(\Sigma) \to \pi_1(H_\alpha)) \]

and

\[ K_\beta = \ker(\pi_1(\Sigma) \to \pi_1(H_\beta)) \].

If the handlebodies \( H_\alpha \) and \( H_\beta \) are described using a Heegaard diagram, then \( K_\alpha \) and \( K_\beta \) are normally generated by the curves describing \( H_\alpha \) and \( H_\beta \), respectively. In this note, we give a geometric proof of a result of Lei and Wu [10] that shows how to compute \( \pi_2(M) \) in terms of \( K_\alpha \) and \( K_\beta \) (see Theorem 3).

2 Preliminary lemmas and general remarks

We say a surface \( F \) is of finite type if \( \pi_1(F) \) is finitely generated, otherwise we say that \( F \) has infinite type. Surfaces of finite type are up to homeomorphism determined by their genus, number of boundary components, and number of punctures. Surfaces of infinite type also admit a classification in terms of their genus, number of boundary components, and space of ends [9], [16].

For a surface of finite type \( F \) with genus greater than or equal to 2, together with a choice of hyperbolic metric on \( F \), there are at most finitely many many closed geodesics of length less than a given constant \( L \in \mathbb{R} \).

**Theorem 2.** If \( g \geq 2 \), then \( K_\alpha \cap K_\beta \) is a not-finitely-generated free group.

**Proof.** In [7], it is shown that \( K_\alpha \cap K_\beta \neq 1 \). Let \( \tilde{\Sigma} \) be the cover of \( \Sigma \) corresponding to \( K_\alpha \cap K_\beta \). Then \( \tilde{\Sigma} \) is noncompact since \( K_\alpha \cap K_\beta \) has infinite index in \( \pi_1(\Sigma) \). Therefore, \( K_\alpha \cap K_\beta \) is free since noncompact surfaces have free fundamental groups [8]. Since \( g \geq 2 \), the surface \( \tilde{\Sigma} \) obtains a hyperbolic metric by pulling back a hyperbolic metric on \( \Sigma \). Since \( K_\alpha \cap K_\beta \) is normal, the deck translations of the covering act on \( \tilde{\Sigma} \) as isometries. Let \( \gamma \) be a closed geodesic in \( \tilde{\Sigma} \). Then all of the infinite translates of \( \gamma \) have the same length as \( \gamma \). Thus, \( \tilde{\Sigma} \) is of infinite type, by the preceding discussion, and therefore \( K_\alpha \cap K_\beta \) is not finitely generated.

Recall that a 3-manifold \( M \) is called reducible if there is an embedded sphere in \( M \) that does not bound a 3-ball in \( M \), and irreducible otherwise. Equivalently, by the Sphere Theorem [13], \( M \) is irreducible if and only if \( \pi_2(M) = 0 \). A Heegaard splitting \( M = H_\alpha \cup_\Sigma H_\beta \) is called reducible if there is an essential simple closed curve in \( \Sigma \) that bounds embedded disks in both \( H_\alpha \) and \( H_\beta \); and irreducible otherwise. Haken’s lemma [4] asserts that any Heegaard splitting of a reducible 3-manifold is reducible.
Figure 1: This figure shows a genus 2 example. The basepoint is assumed to be on the boundary of the surface. The curves in red are the $a_i$ curves and the curves in blue are the $b_j$ curves. Note here that $A$ and $B$ are assumed to be normal subgroups, so the lack of a basepoint for the curves is irrelevant – i.e., if some choice of arcs to the basepoint results in based curves that are in $A$ and $B$, respectively, then so too does any other choice of arcs.

**Proposition 1.** The Heegaard splitting $M = H_\alpha \cup_\Sigma H_\beta$ is reducible if and only if the subgroup $K_\alpha \cap K_\beta$ contains a nontrivial element that can be represented by an embedded curve. If $M$ is reducible, then $K_\alpha \cap K_\beta$ contains a nontrivial element that can be represented by an embedded curve.

*Proof.* The first follows immediately from the definitions. The second part follows from Haken’s lemma.

Note that given any irreducible Heegaard splitting of an 3-manifold, the group $K_\alpha \cap K_\beta$ is a nontrivial subgroup of a surface group that, by Proposition 1, cannot contain any elements that can be represented by embedded curves. Other examples of this phenomenon, in fact finite index examples, are known (see [12], [11]).

The following lemma is immediate upon considering the “4g-gon with a hole” picture of a genus $g$ surface with one boundary component.

**Lemma 1.** Let $X$ be a topological space with basepoint $p \in X$ and let $A, B \leq \pi_1(X, p)$ be two normal subgroups. An element $\gamma \in \pi_1(X, p)$ is in $[A, B]$ if and only if there is a continuous map $f: (S, \partial S) \to (X, \gamma)$ where $S$ is a genus $g$ orientable surface with one boundary component such that the images of the curves $a_i$ and $b_j$ in Figure 1 are in the normal subgroups $A$ and $B$ respectively for $1 \leq i, j \leq g$.

Let $F$ be a surface in a 3-manifold $M$. By *adding a tube* to $F$, we mean creating a new surface $F'$, that is obtained from $F$ by taking the symmetric difference of $F$ with the boundary of an embedded $(D^2 \times [0, 1], D^2 \times \{0, 1\}) \to (M, F)$ such that the image of $D^2 \times (0, 1)$ is disjoint from $F$.

**Lemma 2.** Let $F$ be a compact surface in $S^3$. There is a surface $F'$ obtained from adding tubes to $F$ such that $F'$ is isotopic to the standard genus $g$ Heegaard splitting surface of $S^3$. Moreover, if $F$ is a surface in $B^3$ with $F \cap \partial B^3$ connected, then we can add tubes to $F$ to obtain a standard surface as in Figure 2.
Figure 2: This figure shows a genus 3 example of the trivial surface in a 3-ball. The boundary of the surface is the equator and the surface is the result of stabilizing (as with Heegaard splittings) the equitorial disk three times.

**Proof.** Recall that any 3-manifold $M$ with nonempty boundary has a handle decomposition with a 0-handle, 1-handles, and then 2-handles – one way of seeing this is that $M$ admits a Morse function that is constant on the boundary and that has increasing index critical points. By carving out the 2-handles, we obtain a handlebody and therefore, from every 3-manifold with boundary, we can obtain a handlebody if we carve out enough tubes. Now let $F$ be a surface in $S^3$. Note that carving tubes out of one side of $F$ corresponds to adding 1-handles to the other side, and notice that adding 1-handles to a handlebody produces another handlebody (with higher genus). Therefore, by adding tubes to one side of $F$, we can obtain a new surface that bounds a handlebody on one side, and by adding tubes to the other side, we can obtain a surface that bounds handlebodies on both sides. The moreover statement is obtained from this result by adding a ball with a standard equator disk to the $B^3$ containing $F$, with the boundary of the disk glued to the boundary of $F$, and performing the above argument with all of the tubes not intersecting this trivial disk-ball pair.

**Lemma 3.** Let $\Sigma$ be a Heegaard splitting surface of a closed orientable 3-manifold $M$, $F$ be a compact connected surface, and $f : F \rightarrow M$ be a continuous map. Then $f$ can be homotoped so that $f^{-1}(\Sigma)$ is connected.

**Proof.** First, homotope $f$ such that it is an immersion transverse to $\Sigma$. Let $\gamma_1$ and $\gamma_2$ be two distinct connected components of $f^{-1}(\Sigma)$ such that there is an arc $a \subset F$ with endpoints in $\gamma_1$ and $\gamma_2$ respectively and such that $f(a)$ is entirely contained in $H_\alpha$ or $H_\beta$. Note that any properly embedded arc in a handlebody can be homotoped relative to its endpoints so as to be contained in the boundary of the handlebody. Therefore, by homotoping $f$ close to $a$ following this homotopy of $a$ into $\Sigma$ we obtain a new map with one fewer connected component in the inverse image of $\Sigma$. Repeating this process then gives the result.

**3 $\pi_2(M)$ from splitting homomorphisms**

The following result is given by Lei and Wu in [10], where it is stated that it follows from general methods in [2]. Here we provide a hands-on geometric argument.
Suppose that $\gamma$ homotoping $H$ by our construction of $\phi$ that is contained in $H$ as in Figure 2, then, by Lemma 1, we have that $\gamma \in (K_\alpha \cap K_\beta)/[K_\alpha, K_\beta]$. With respect to this action, the above isomorphism is a $\mathbb{Z}_\pi_1(M)$-module isomorphism.

Proof. We first define a map $\phi: K_\alpha \cap K_\beta \rightarrow \pi_2(M)$. Notice that, given a curve $\gamma \in K_\alpha \cap K_\beta$ then by the definition of $K_\alpha$ and $K_\beta$, there exist disks $D_\alpha \subset H_\alpha$ and $D_\beta \subset H_\beta$ such that $\gamma = \partial D_\alpha = \partial D_\beta$. If $D'_\alpha$ and $D'_\beta$ are other such disks, then the spheres $D_\alpha \cup \gamma D_\beta$ and $D'_\alpha \cup \gamma D'_\beta$ are homotopic with the homotopy fixing the basepoint, since $\pi_2$ of a handlebody is trivial and so the union of the disks can be extended to a map from a 3-ball. Therefore, we have a map $\phi$ which is seen to be a group homomorphism.

The surjectivity of $\phi$ is the content of Lemma [3] in the case where $F$ is a sphere.

We will now show that the map $\phi$ descends to the quotient $K_\alpha \cap K_\beta/[K_\alpha, K_\beta]$. As a preliminary remark, observe that the commutator $[K_\alpha, K_\beta]$ is a subgroup of the intersection $K_\alpha \cap K_\beta$, because the kernels $K_\alpha, K_\beta$ are normal subgroups of $\pi_1(\Sigma)$. To see that $[K_\alpha, K_\beta] \leq \ker(\phi)$ note that by Lemma [4], there exists a surface $S$ as in Figure [1] and a continuous map $f: (\Sigma, \partial S) \rightarrow ([K_\alpha, K_\beta])$ such that $f(a_i) \in K_\alpha$ and $f(b_j) \in K_\beta$. Therefore, we have immersed disks $D_{\alpha}^a, D_{\beta}^b$ with $\partial D_{\alpha}^a = f(a_i)$ and $\partial D_{\beta}^b = f(b_j)$. We thus have a capped surface (i.e., the result of adding to the $a_i$ and $b_j$ curves in Figure [1]) mapping into $M$ with the boundary mapping to $\gamma$ -- call the image $\hat{S}$. Now consider a neighborhood of $\hat{S}$ which is topologically a ball. Thus, we obtain a map of a 2-sphere whose equator maps to $\gamma$ and such that the two disks bounding the equator map to disks $D_\alpha$ and $D_\beta$ in $H_\alpha$ and $H_\beta$, respectively. Then $\phi(\gamma) = 0$ since $D_\alpha \cup \gamma D_\beta$ bounds a ball.

We now prove injectivity of the resulting map

$$\phi: (K_\alpha \cap K_\beta)/[K_\alpha, K_\beta] \rightarrow \pi_2(M)$$

Suppose that $\gamma \in \ker(\phi)$ and let $g: B^3 \rightarrow M$ with $g(\partial B^3) = D_\alpha \cup \gamma D_\beta = \phi(\gamma)$. If $g^{-1}(\Sigma)$ is standard as in Figure [2] then, by Lemma [5], we have that $\gamma \in [K_\alpha, K_\beta]$. If $g^{-1}(\Sigma)$ is not standard, then, by homotoping $g$, we can add tubes to $g^{-1}(\Sigma)$, since such a tube maps to a thickened arc in $M$ that is either contained in $H_\alpha$ or $H_\beta$ (where as in the proof of Lemma [2] every arc can be homotoped to lie in the boundary). Therefore, by Lemma [6] we can make $g^{-1}(\Sigma)$ standard. Thus $\ker(\phi) \leq [K_\alpha, K_\beta]$ and therefore $\phi$ induces an isomorphism of abelian groups $K_\alpha \cap K_\beta/[K_\alpha, K_\beta] \cong \pi_2(M)$.

Now, consider the action of $\pi_1(M)$ on $(K_\alpha \cap K_\beta)/[K_\alpha, K_\beta]$ as in the statement of the result. By our construction of $\phi$ and the definition of the action of $\pi_1(M)$ on $\pi_2(M)$, we see that $\phi$ is in fact an isomorphism of $\mathbb{Z}[\pi_1(M)]$-modules, thus proving the result.

Remark 1. We have not been discussing the subgroup $K_\alpha K_\beta \leq \pi_1(\Sigma)$, which is another subgroup of interest (here $K_\alpha K_\beta$ denoted the join of $K_\alpha$ and $K_\beta$ -- i.e., the smallest subgroup containing both of them). Note that $\pi_1(\Sigma)/K_\alpha K_\beta \cong \pi_1(M)$ (see [7]) and the cover of $\Sigma$ corresponding to $K_\alpha K_\beta$ is the preimage of $\Sigma$ in the universal cover for $M$. It then follows, as in Theorem [3] that either

1. $\pi_1(M)$ is finite and $K_\alpha K_\beta$ is isomorphic to the fundamental group of a closed orientable surface, or

2. $\pi_1(M)$ is infinite and $K_\alpha K_\beta$ is an infinitely generated free group.
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