The impact of renormalization on the observable predictions of inflation

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Abstract. We analyze the generation of primordial perturbations in a (single-field) slow-roll inflationary universe. In momentum space, these (Gaussian) perturbations are characterized by a zero mean and a non-zero variance $\Delta^2(k, t)$. However, in position space the variance diverges in the ultraviolet. The requirement of a finite variance in position space forces one to regularize $\Delta^2(k, t)$. This can be achieved by proper renormalization in an expanding universe in a unique way. This affects the predicted scalar and tensorial power spectra (evaluated when the modes acquire classical properties) for wavelengths that today are at observable scales. As a consequence, the imprint of slow-roll inflation on the CMB anisotropies is significantly altered.

1. Introduction
A sufficiently long period of accelerated expansion in the very early universe is able to solve the questions raised by the standard big bang cosmology [1]. The hot big bang cosmology is an extremely successful theory. It explains the existence of the cosmic microwave background (CMB) and its thermal nature, the observed expansion of the universe, the abundance of light elements and the astrophysical fits for the age of the universe. However, it leaves without answer why our universe appears so homogeneous and nearly flat at large scales. Inflation offers a natural answer to these questions and, at the same time, provides a predictive mechanism to account for the small observed inhomogeneities [2] responsible for the structure formation in the universe and the anisotropies present in the cosmic microwave background (CMB), as first detected by the COBE satellite and further analyzed by the WMAP satellite [3]. Inflation predicts production of primordial density perturbations and relic gravitational waves as amplifications of vacuum fluctuations together with a quantum-to-classical transition at the scale of Hubble sphere crossing. Primordial perturbations leave an imprint in the CMB anisotropies, which are, therefore, of major importance for understanding our universe and its origin. The potential-energy density of a scalar (inflaton) field is assumed to cause the inflationary expansion, and the amplification of its quantum fluctuations and those of the metric are inevitable consequences in an expanding universe [4]. The metric fluctuations provide the initial conditions for the acoustic oscillations of the plasma at the onset of the subsequent radiation-dominated epoch. The detection of the effects of primordial gravitational waves in future high-precision measurements of the CMB anisotropies, as for instance in the PLANCK satellite mission [5], will serve as a highly non-trivial test for inflation. Therefore, it is particularly important to scrutinize, from all points of view, the standard predictions of inflation (see, for instance, [6]) to be tested empirically.
In this respect, it was pointed out in [7] (see also [8]) that quantum field renormalization significantly modifies the amplitude of quantum fluctuations, and hence the corresponding power spectra, in de Sitter inflation. The analysis was further developed in [9, 10] to understand how the fundamental testable predictions of (single-field) slow-roll inflation could be affected by quantum field renormalization. We summarize here our results and the logic of our approach.

2. Reexamining the calculation of the spectrum of perturbations

Let us assume that \( \varphi(\vec{x}, t) \) represents a perturbation obeying a free field wave-equation on the inflationary background \( ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \), where \( a(t) \) is a quasi-exponential expansion factor \( (a(t) \sim e^{Ht}) \). At the quantum level, this field is expanded as

\[
\varphi(\vec{x}, t) = \frac{1}{(2\pi)^3/2} \int d^3k \left[ \varphi_k(t)a_k e^{i\vec{k} \cdot \vec{x}} + \varphi_k^*(t)a_k^\dagger e^{-i\vec{k} \cdot \vec{x}} \right],
\]

where the creation and annihilation operators satisfy the canonical commutation relation \([a_k, a_{k'}^\dagger] = \delta^3(\vec{k} - \vec{k'})\). The mode functions \( \varphi_k(t) \) are required to satisfy the adiabatic condition (see, for instance, [11]). The power spectrum for this perturbation, \( \Delta^2_\varphi(k, t) \), is usually defined in terms of the Fourier transform of the variance of the field

\[
\langle \hat{\varphi}_k(t) \hat{\varphi}_k^\dagger(t) \rangle = \delta^3(\vec{k} - \vec{k'}) \frac{2\pi^2}{k^3} \Delta^2_\varphi(k, t),
\]

where \( \hat{\varphi}_k(t) \equiv \varphi_k(t)a_k^\dagger \). These modes describe a perturbation field characterized, in momentum space, by a zero mean \( \langle \hat{\varphi}_k(t) \rangle = 0 \) and the variance (2). The advantage of working in momentum space resides in the fact that different modes fluctuate independently of each other, as explicitly displayed by the presence of the delta function in (2). This way, the quantum field is regarded as an infinite collection of oscillators, each with a different value of \( \vec{k} \). In position space the perturbation is also characterized by a zero mean \( \langle \varphi(\vec{x}, t) \rangle = 0 \) and a variance (or dispersion)

\[
\langle \varphi^2(\vec{x}, t) \rangle = \frac{1}{(2\pi)^3} \int d^3kd^3k' \langle \hat{\varphi}_k(t) \hat{\varphi}_k^\dagger(t) \rangle e^{i(\vec{k} - \vec{k'}) \cdot \vec{x}},
\]

which, due to spatial homogeneity, turns out to be independent of \( \vec{x} \). This variance is formally related to the power spectrum by

\[
\langle \varphi^2(\vec{x}, t) \rangle = \int_0^\infty \frac{dk}{k} \Delta^2_\varphi(k, t).
\]

As is well-known in quantum field theory, the above expectation value quadratic in the field \( \varphi \) is divergent. It suffers from quadratic and logarithmic ultraviolet divergences

\[
\langle \varphi^2(\vec{x}, t) \rangle \sim \frac{1}{4\pi^2} \int_0^\infty dk \left( \frac{k}{a^2} + \frac{\dot{a}^2}{a^2k} + \ldots \right).
\]

The first term corresponds to the usual contribution from vacuum fluctuations in Minkowski space. This contribution can be eliminated by standard renormalization in flat space. The logarithmic divergence, however, is characteristic of vacuum fluctuations in a curved background. Because the different \( k \)-modes fluctuate independently of each other, one could be tempted to get rid of this logarithmic ultraviolet divergence by simply eliminating the modes with \( k > aH \) and leaving the rest unaffected (see, for instance, [12]). If one eliminates this divergence using a window function in this way, as is usual for random fields, then one obtains \( \Delta^2_\varphi(k) \approx H^2/4\pi^2 \), where \( \Delta^2_\varphi(k) \) is defined by the quantity \( \Delta^2_\varphi(k, t) \) evaluated a few Hubble times after the “horizon
crossing time $t_k (a(t_k)/k = H(t_k))$, since this is the time scale at which the modes behave as classical perturbations. However, one should take into account that the field fluctuations are quantum in nature and, therefore, one should consider the subtle points of quantum field theory (QFT) regarding the ultraviolet divergences.

Even though free quantum field theory is usually regarded as an infinite set of independent harmonic oscillators (one for each $k$-mode), there are fundamental holistic aspects of QFT that cannot be properly understood in terms of independent modes. Renormalization is the hallmark of the holistic aspect of QFT. This is clear in the fact that, although the renormalization schemes in QFT in curved spacetimes are based on the ultraviolet behavior of the theory, the infrared sector is also affected by renormalization, leading potentially to observable consequences. This can be explicitly displayed by considering, for instance, the Casimir effect. The energy density between the two conducting plates obtained by proper renormalization provides the well-known and experimentally tested expression. However, a naive subtraction obtained by introducing a high-frequency cut-off in the integrals in momentum space (i.e., treating the $k$-modes as being independent) produces a quite different result (see the discussion of [13]).

Taking this into account, we see that the logarithmic divergence in (5) should be dealt with by renormalization and one can not rule out, a priori, the possibility that the treatment of the divergences at very high values of $k$ may produce some impact at lower momenta. Therefore, we propose that in the standard definitions of the spectrum $\Delta^2_\varphi(k,t)$, as given in (2) and (4), one should replace the unrenormalized $\langle \varphi^2(\vec{x},t) \rangle$ by the renormalized variance, $\langle \varphi^2(\vec{x},t) \rangle_{\text{ren}}$. Writing $\Delta^2_\varphi(k,t)$ for the spectrum defined in this way, the definition in (4) (and similarly in (2)) is replaced by the corresponding renormalized expression

$$\langle \varphi^2(\vec{x},t) \rangle_{\text{ren}} = \int_0^\infty \frac{dk}{k} \Delta^2_\varphi(k,t) .$$

This way, the physical variance $\langle \varphi^2(\vec{x},t) \rangle_{\text{ren}}$ remains a well-defined quantity, in the same way as one could obtain a finite expression for the expectation values of the quantum stress-energy tensor. To complete the physical consistency of this approach, it would be desirable to define a unique expression for the necessary subtractions required to produce a consistent $\langle \varphi^2(\vec{x},t) \rangle_{\text{ren}}$.

Since the power spectrum is defined in momentum space, the natural scheme is renormalization in momentum space, so we define

$$\langle \varphi^2(\vec{x},t) \rangle_{\text{ren}} = \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk (|\varphi_k(t)|^2 - C_k(t)) , \quad (7)$$

where $C_k(t)$ represents the expected counterterms. Therefore, we have to apply well established renormalization techniques in QFT in curved spacetimes to properly remove these quantum divergences. As we will see all schemes of renormalization leads to the same counterterms.

3. Renormalization in momentum space

While in particle physics experiments the absolute value of the vacuum energy is unobservable, in the presence of gravity the absolute value of the vacuum energy matters. This is why normal ordering suffices to extract relevant physical quantities for free fields in Minkowski space but requires a refinement when the spacetime background departs from Minkowski space. A particularly intuitive method of renormalization was proposed by Bunch-Parker [14] in the late seventies. It was proposed aiming at extending to curved space the standard momentum-space methods of perturbation theory for interacting fields in Minkowsky space. This renormalization scheme introduce a local momentum-space representation of two-point functions which allows to remove the ultraviolet divergences of quantized fields in a general curved background. This
way the standard Minkowskian propagator of a scalar free field in momentum space

\[ \frac{1}{k^2 + m^2} \]  

is replaced by a series with leading terms

\[ \frac{1}{k^2 + m^2} + \left( \frac{1}{6} - \xi \right) R \frac{1}{(k^2 + m^2)^2} + \ldots \]  

where \( m \) is the mass and \( \xi \) is the coupling curvature parameter of the corresponding wave equation

\[ \left[ \nabla_\mu \nabla^\mu - m^2 - \xi R \right] \varphi(x) = 0 \]  

The Fourier transform leading to momentum space is crucially performed with respect to normal Riemann coordinates around the reference point \( \vec{x} \). The presence of the curvature introduce additional divergent terms which are well counted by the above corrections to the Minkowskian propagator. For a massive field the above expansion, when translated to configuration space, turns out to be equivalent \[14\] to the DeWitt-Schwinger propertime representation of two-point functions \[15, 16\] (see also \[11, 17\]). However, in contrast to the DeWitt-Schwinger renormalization scheme, which requires \( m \neq 0 \) since it is based on an expansion in inverse power of the mass, the above momentum-space series can be extended to the massless limit. Following \[14\], the renormalized two-point function is given, in general, by the following subtraction

\[ \langle \varphi(x) \varphi(x') \rangle_{\text{ren}} = \langle \varphi(x) \varphi(x') \rangle - i \frac{g^{-1/4}(x)}{(2\pi)^4} \int d^4 k e^{i k \xi y} \left[ \frac{1}{p^2 + m^2} + \left( \frac{1}{6} - \xi \right) R \frac{1}{(p^2 + m^2)^2} \right] , \]  

where \( g^\mu \) are the Riemann normal coordinates around the reference point \( x' \).

We will now restrict now attention to a massless field and a cosmological metric of the form \( ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \). At \( t = t' \), and for a generic expansion factor \( a(t) \), we have \( y^\mu = 0 \) and \( \vec{y} = a(t)(\vec{x} - \vec{x}') \). Performing then the \( k_0 \) integration we get for the subtraction term

\[ -i \frac{g^{-1/4}(x)}{(2\pi)^4} \int d^4 k e^{i k \xi y} \left[ \frac{1}{p^2} + \left( \frac{1}{6} - \xi \right) \frac{R}{p^4} \right] = \frac{1}{2(2\pi)^3 a^3} \int d^3 k \left[ \frac{1}{w_k} + \frac{1}{2w_k'} \right] e^{i k(\vec{x} - \vec{x}')} , \]  

where \( w_k = k/a(t) \). A further angular integration yields to the following subtraction counterterm

\[ \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \left( \frac{a}{k} + (1 - 6\xi) \frac{a^3}{2k^3} \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] \right) \frac{\sin k|\vec{x} - \vec{x}'|}{k|\vec{x} - \vec{x}'|} . \]  

At coincidence \( \vec{x} \to \vec{x}' \) we have

\[ (4\pi^2 a(t)^3)^{-1} \int_0^\infty k^2 dw_k [w_k(t)^{-1} + (W_k(t)^{-1})^{(2)}] \]  

where the counterterms \( w_k(t)^{-1} + (W_k(t)^{-1})^{(2)} \) are

\[ w_k^{-1} + (W_k^{-1})^{(2)} = \frac{a}{k} + (1 - 6\xi) \frac{a^3}{2k^3} \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] . \]
As pointed out in [10] the same counterterms are exactly obtained, in the massless limit, by the method of adiabatic renormalization [18], and this is why we have used the notation of adiabatic renormalization for naming the momentum-space counterterms. The normalized modes are usually expressed in momentum space as \(2(2\pi a(t))^3)^{-1/2}e^{ik\xi}\bar{\phi}(t)\). Then the renormalized variance \(\langle \phi^2(\vec{x}, t)\rangle_{\text{ren}}\) is finally given by the expression

\[
\langle \phi^2(\vec{x}, t)\rangle_{\text{ren}} = (4\pi^2 a(t)^3)^{-1} \int_0^{\infty} k^2 dk (|\bar{\phi}(t)|^2 - w(t)^{-1} - (W(t)^{-1})^2).
\]

(16)

Therefore, for the field perturbations arising from inflation the counterterms \(C_k(t)\) in (7) are univocally constructed thus defining a unique expression for the spectrum \(\tilde{\Delta}_n^2(k, t)\). The holistic nature of QFT is then explicitly realized through (6). Although the counterterms are fully determined by the ultraviolet behavior of the modes, the long wavelength sector, and hence the new \(\tilde{\Delta}_n^2(k, t)\), is significantly affected by the subtractions.

4. New predictions for the power spectra

To evaluate the physical power spectrum we have to specify when the classical perturbations are created. It happens when the decaying mode becomes very small relative to the dominant one [19] and this corresponds a few \(n\) efolds after the Hubble radius exit time \(t_k\) (we note that \(n\) is related to the squeezing parameter). In the slow-roll scenario, when \(H\) slowly decreases with time, the effects of renormalization have a non-trivial impact on \(\tilde{\Delta}_n^2(k, t)\) when this quantity is evaluated a few Hubble times after the time of horizon crossing \(t_k\). We then obtain for the tensorial and scalar spectra

\[
\tilde{\Delta}_n^2(k, n) \approx \frac{2}{M_p^2} \left( \frac{H(t_k)}{2\pi} \right)^2 \epsilon(t_k)(2n - 3/2)
\]

\[
\tilde{\Delta}_n^2(k, n) \approx \frac{1}{2M_p^2\epsilon(t_k)} \left( \frac{H(t_k)}{2\pi} \right)^2 (3\epsilon(t_k) - \eta(t_k))(2n - 3/2),
\]

(17)

where \(\epsilon, \eta\) are the standard slow-roll parameters. Note that the parameter \(n\) enters in the power spectra parameterizing the (unknown) time at which the modes exhibit classical behavior.

However, since both tensorial and scalar spectra have the same dependence on \(n\), the tensor-to-scalar ratio \(r\) is not sensitive to the unknown parameter \(n\), in the same way as it is insensitive to the scale of inflation \(H(t_k)\). As a consequence of (17), the imprint of slow-roll inflation on the CMB anisotropies is significantly altered. Note that, in contrast with the standard (unrenormalized) approach, the tensorial amplitude does not uniquely depend on the scale of inflation. Moreover, one can also obtain a non-trivial change in the consistency condition that relates the tensor-to-scalar ratio \(r\) to the spectral indices [10] for single-field inflation

\[
r = 4(1 - n_s - n_t) + \frac{4n'_t}{n_t^4 - 2n_t^2} \left(1 - n_s - \sqrt{2n_t' + (1 - n_s)^2 - n_t^2}\right).
\]

(18)

The tensor-to-scalar ratio is now related with the spectral indices \(n_t\), \(1 - n_s\), and also \(n_t' \equiv d\ln n_t/d\ln k\). This modification has far reaching consequences. For instance, since the observations from WMAP [3](with BAO+SN) strongly suggest that \((1 - n_s) \approx 0.030\pm0.015\) (with \(r < 0.22\)), expression (18) allows for an exact scale invariant tensorial power spectrum, \(n_t = 0\), while being compatible with a non-zero ratio \(r \approx 0.12 \pm 0.06\), which is forbidden by the standard prediction \((r = -8n_t)\). We also note that the new predictions remain in agreement with observation for the simplest forms of inflation \((\phi^2\text{ and }\phi^4\text{ potentials})\). The influence of relic gravitational waves on the CMB may soon come within the range of planned measurements, offering a non-trivial test of the new predictions.
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