Variable length universal entanglement concentration
by local operations and
its application to teleportation and dense coding

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Abstract
Using invariance of the $n$-th tensored state w.r.t. the $n$-th symmetric group, we propose
a 'variable length' universal entanglement concentration without classical communication.
Like variable length data compression, arbitrary unknown states are concentrated into
perfect Bell states and not approximate Bell states and the number of Bell states obtained
is equal to the optimal rate asymptotically with the probability 1. One of the point of
our scheme is that we need no classical communication at all. Using this method, we can
construct a universal teleportation and a universal dense coding.

1 Introduction
In quantum systems, we can perform some information processes which do not appear in classical systems. For example, quantum teleportation, dense coding etc. For them it is necessary to share an entangled state between two systems. If the entangled state is the perfect Bell state, its analysis is very easy. Otherwise, it is not easy [1, 2].

We can produce perfect Bell states from arbitrary entangled states by local operations and classical communications (LOCC) and call such an operation an entanglement concentration. As is proved by Bennett et al [3], when we share the $n$-tensor product state $|\phi\rangle\langle\phi|^{\otimes n}$ on the total tensor product system $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$, we can produce, by local operations, $nH(\rho_A)$-qubit perfect Bell states asymptotically with the probability 1, where $\rho_A := \text{Tr}_B |\phi\rangle\langle\phi|$ and $H(\rho_A)$ is the entropy $-\text{Tr} \rho_A \log \rho_A$.

In this paper, we propose a 'variable length' universal entanglement concentration without any classical communication. Like variable length data compression, arbitrary unknown states are concentrated into perfect Bell states and not approximate Bell states, and the number of Bell states obtained is equal to $nH(\rho_A)$ asymptotically with the probability 1. One of the point of our scheme is that we need no classical communication at all.

In §2, we propose a variable length group-invariant entanglement concentration consisting of local operations when the entanglement pure state is invariant w.r.t. the tensor representation on $\mathcal{H}_A \otimes \mathcal{H}_B$ of a group $G$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are equivalent with each other w.r.t. a representation space of $G$. In this method, the final state is always the perfect Bell state and the size is probabilistic. In §3 using invariance of the $n$-th tensored state w.r.t. the $n$-th symmetric group, we construct a variable length universal entanglement concentration (simplified to a universal entanglement concentration), in which, we can, independently of $\rho_A$, produce no less than
$nH(\rho_A)$-qubit perfect Bell states asymptotically with the probability 1. As another method, we can perform an entanglement concentration after the state estimation on $en$ systems. But, if we perform entanglement concentration which depends on the estimated state, the final state is not necessarily the perfect Bell state because the estimated state does not exactly coincide with the true state. As is proved in §4, our universal concentration achieves the optimal failure exponent among universal concentrations which achieve the optimal rate $nH(\rho_A)$ for any state asymptotically with the probability 1.

In the quantum teleportation, we can send a quantum state with LOCC. In such a setting we maximize the number of teleported qubits only with LOCC. As is discussed in §5 to share $R$-qubit perfect Bell state is equivalent with to send $R$-qubit of perfect Bell states only with LOCC. Therefore, we can perform $nH(\rho_A)$ qubits quantum teleportation, under the assumption that we share the $n$-tensor product state $|\phi\rangle\langle\phi|^{\otimes n}$ on the total system $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$. Even if we do not know the density operator $\rho_A$, using our universal entanglement concentration we can perform $nH(\rho_A)$ qubits quantum teleportation asymptotically with the probability 1. In the protocol, it is enough to send the minimum classical communications of the size of $2nH(\rho_A)$ bits.

If entangled states are shared, we can send $R_1$ bits classical message by sending only $R_2(< R_1)$ qubits. This type information process is called (super) dense coding. The number $R_1 - R_2$ signifies the effect of entanglement. Thus, in this setting we can regard the maximum of $R_1 - R_2$ as the capacity. Our setting is different from the usual setting of the dense coding. As is discussed in §6, we can prove that the maximum of $R_1 - R_2$ is asymptotically equal to $nH(\rho)$. Even if we do not know the density $\rho_A$, using our universal entanglement concentration we can send $2nH(\rho)$ bits of classical information by sending only $nH(\rho)$ qubits.

As is pointed out by Keyl and Werner [4], this group invariant method is applicable to the estimation of spectrum. Concerning this topic, we will discuss another paper[5].

### 2 Variable length group-invariant entanglement concentration

For the preparation of our universal entanglement concentration, we construct a entanglement concentration protocol under the group representation-invariance in a non-asymptotic setting. We call this protocol a variable length group-invariant entanglement concentration (simplified to an invariant entanglement concentration). Let $f_A$ and $f_B$ be unitary representations of a group $G$ on finite dimensional spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, which are equivalent with each other. Assume that we share the pure state $|\phi\rangle\langle\phi|$ which is invariant under the tensor representation $f_A \otimes f_B$ on the total system $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e. $f_A(g) \otimes f_B(g)|\phi\rangle = |\phi\rangle, \forall g \in G$.

**Lemma 1** If $f_A$ and $f_B$ are irreducible, the invariant vector $\phi$ is given as

$$|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |e_{j,A}\rangle \otimes |e_{j,B}\rangle,$$

where $\{e_{j,A}\}_{j=1}^{d}$ and $\{e_{j,B}\}_{j=1}^{d}$ are CONSs of $\mathcal{H}_A$ and $\mathcal{H}_B$ such that $f_{is}(e_{j,A}) = e_{j,B}$, where $f_{is}$ is the unique isomorphism map from $\mathcal{H}_A$ to $\mathcal{H}_B$, w.r.t. the representation of $G$. With ambiguity
of constant factor, the vector $|\phi\rangle$ is uniquely defined from the invariance of the representation of $G$.

Then, we call the vector $\phi$ the invariant perfect Bell state on $\mathcal{H}_A \otimes \mathcal{H}_B$.

**Proof** Since $\mathcal{H}_A$ and $\mathcal{H}_B$ are equivalent w.r.t. the representation space of $G$, we can identify the space $\mathcal{H}_A \otimes \mathcal{H}_B \cong \mathcal{H}_A \otimes \mathcal{H}^*_B$ with the set $\mathcal{L}(\mathcal{H})$ of linear transforms on $\mathcal{H}_A$. In this identification, the representation of $G$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is regarded as the adjoint representation on $\mathcal{L}(\mathcal{H})$ because $f_A(g)f_B(g)|\phi_A\rangle \otimes |\phi_B\rangle \cong f_A(g)|\phi_A\rangle \otimes \langle \phi_B|f_B(g)^* \cong f_A(g)|\phi_A\rangle \otimes \langle \phi_B|f_A(g)^{-1}$, $\forall|\phi_A\rangle \in \mathcal{H}_A, \forall|\phi_B\rangle \in \mathcal{H}_B$. Therefore, using Schur’s lemma, we can prove the desired assertion. □

Since the dimension of $\mathcal{H}_A$ is finite, there exists a decomposition into irreducible representations of $\mathcal{H}_A$ as follows:

$$\mathcal{H}_A = \bigoplus_k \left( \bigoplus_{i=1}^{l_k} V_{k,i,A} \right)$$  \hspace{1cm} (1)

$$\mathcal{H}_B = \bigoplus_k \left( \bigoplus_{i=1}^{l_k} V_{k,i,B} \right)$$  \hspace{1cm} (2)

where $V_{k,i}$ and $V_{k,j}$ is equivalent w.r.t. the representation of $G$. Therefore, there are $l_k$ spaces equivalent with $V_{k,1}$ w.r.t. the representation of $G$. Note that the decomposition is not unique, if there is a pair of equivalent subspaces. Let $U_{k,A}$ and $U_{k,B}$ be the vector spaces $\langle e_{k,1,A}, \ldots, e_{k,l_k,A} \rangle$ and $\langle e_{k,1,B}, \ldots, e_{k,l_k,B} \rangle$, and $V_{k,A}$ and $V_{k,B}$ be a vector space equivalent with $V_{k,i,A}$, $V_{k,i,B}$ w.r.t. the representation of $G$. Then we have

$$\mathcal{H}_A = \bigoplus_k U_{k,A} \otimes V_{k,A}$$  \hspace{1cm} (3)

$$\mathcal{H}_B = \bigoplus_k U_{k,B} \otimes V_{k,B}.$$  \hspace{1cm} (4)

**Lemma 2** From the invariance of $f_A \otimes f_B$, we can choose the decomposition (1) and (2) satisfying that

$$|\phi\rangle = \sum_k \sum_{i=1}^{l_k} \sqrt{s_{k,i}} d_k |\phi^P_{k,i}\rangle$$  \hspace{1cm} (5)

where $d_k = \dim V_k$, and the vector $\phi^P_{k,i} \in V_{k,i,A} \otimes V_{k,i,B}$ is the invariant perfect Bell state on $V_{k,i,A} \otimes V_{k,i,B}$.

**Proof** Similarly to Lemma 1, using Schur’s lemma, we can prove the desired assertion. □

The constant factor $s_k$ satisfies that

$$\rho_A = \sum_k \sum_{i=1}^{l_k} s_{k,i} V_{k,i,A},$$
where we identify the subspace of $\mathcal{H}_A$ with its projection and $\rho_A := \text{Tr}_B |\phi\rangle\langle \phi|$. We cannot choose the decompositions (1) and (2) satisfying (3) from the invariance of $f_A \otimes f_B$. But, can uniquely construct the decompositions (3) and (4) from the invariance of $f_A \otimes f_B$.

Let us construct the invariant entanglement concentration. First, we perform the projection measurements $\{U_{k,A} \otimes V_{k,B}\}_k$ and $\{U_{k,B} \otimes V_{k,B}\}_k$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, i.e. we perform the projection measurement $\{U_{k,A,A} \otimes U_{k,B,B} \otimes V_{k,B,B}\}_{k,a,k,b}$ on the total system $\mathcal{H}_A \otimes \mathcal{H}_B$. It follows from (3) that the event $k_A \neq k_B$ happens with the probability 0 and the event $k_A = k_B = k$ happens with the probability $c_k =: d_k \sum_{i=1}^{l_k} s_{k,i}$. If the measured value $k_A = k_B$ is $k$, the state on $U_{k,A,A} \otimes U_{k,B,B} \otimes V_{k,B,B}$ is written by

$$\frac{1}{\sqrt{c_k}} \sum_{i=1}^{l_k} \sqrt{s_{k,i} d_k} |\phi_{k,i}^P\rangle.$$

Next, we take the partial trace on $U_{k,A} \otimes U_{k,B}$. Then the final state is the invariant perfect Bell state on $U_{k,A} \otimes U_{k,B}$, whose size is $d_k = \text{dim} V_k$. Using this protocol, we can get the perfect Bell state with the dim $V_k$ in the probability $c_k =: \text{dim} V_k \sum_{i=1}^{l_k} s_{k,i} = \text{Tr} \rho_A U_k \otimes V_k$.

### 3 Universal entanglement concentration

It is well-known that the tensor product state is invariant under the representation of $n$-th symmetric group. Applying the invariant entanglement concentration to this case, we can construct a universal entanglement concentration. Let $d$ be the maximum of dim $\mathcal{H}_A$ and dim $\mathcal{H}_B$. We add some vectors so that the relation $d = \text{dim} \mathcal{H}_A = \text{dim} \mathcal{H}_B$ holds.

We assume that the state on the tensored total system $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ is written by $n$-tensored state $|\phi\rangle\langle \phi|^{\otimes n}$, where $|\phi\rangle\langle \phi|$ is a pure state on the single total system $\mathcal{H}_A \otimes \mathcal{H}_B$. Define the subscript $n$ by

$$n := (n_1, \ldots, n_d), \quad \sum_{i=1}^{d} n_i = n, \quad n_i \geq n_{i+1}.$$  

The subscript $n$ uniquely corresponds to the unitary irreducible representation $V_n$ of the $n$-th symmetric group $S_n$ and the unitary irreducible representation $U_n$ of the special unitary group $SU(d)$. The tensored space $\mathcal{H}_A^{\otimes n}$ is decomposed as (3) by

$$\mathcal{H}_A^{\otimes n} = \bigoplus_n W_n, \quad W_n := U_n \otimes V_n.$$  

For the detail, see Weyl [4], Goodman-Wallach [7], Iwahori [8]. The density $\rho_A^{\otimes n}$ is invariant w.r.t. the representation of the $n$-th symmetric group $S_n$ on the tensored space $\mathcal{H}_A^{\otimes n}$. This type decomposition does not depends on $\rho_A$ and $|\phi\rangle\langle \phi|$ and depends on the group representation invariance. But the type of (4) depends on $\rho_A$. Now, we perform the above invariant entanglement concentration w.r.t. the subscript $n$. In this case, when we get measured value $n$, the final state is the perfect Bell state with the size dim $V_n$. Its probability is $\text{Tr} W_n \rho_A^{\otimes n}$. 


Theorem 3 The probabilities are evaluated as

\[
\lim_{n \to \infty} \frac{-1}{n} \log \sum \{ \operatorname{Tr} W_n \rho_A^{\otimes n} | \dim V_n \leq 2^{nR} \} = \sup \{ D(q \| p) | H(q) \leq R \} = \sup_{s \geq 1} \frac{(1 - s)R - \psi(s)}{s} \quad \text{if } R \leq H(\rho)
\]

\[
\lim_{n \to \infty} \frac{-1}{n} \log \sum \{ \operatorname{Tr} W_n \rho_A^{\otimes n} | \dim V_n \geq 2^{nR} \} = \sup \{ D(q \| p) | H(q) \geq R \} = \sup_{0 < s \leq 1} \frac{(1 - s)R - \psi(s)}{s} \quad \text{if } R \geq H(\rho),
\]

where \( \psi(s) := \log \operatorname{Tr} \rho_A^s \) and the vector \( p = (p_1, \ldots, p_d) \) is the set of eigenvalues of \( \rho_A \) satisfying \( p_1 \geq p_2 \geq \ldots \geq p_d \). Thus, when \( R < H(\rho_A) \), \( \sup_{s \geq 1} \frac{(1 - s)R - \psi(s)}{s} > 0 \).

This theorem implies that this protocol achieve the bound with the probability which goes to 1. The above theorem follows from the following lemmas proved in Appendix.

Lemma 4 There exists a constant number \( C \) such that

\[
\left| \frac{1}{n} \log \dim V_n - H \left( \frac{n}{n} \right) \right| \leq \frac{2d^2 + d}{2n} \log(n + d) + \frac{C}{n}, \quad \forall n.
\]  \hspace{1cm} (6)

Lemma 5 For any state \( \rho_A \) on \( \mathcal{H}_A \) and any set \( R \subset R_+ := \{ p | p_1 \geq p_2 \geq \ldots \geq p_d \geq 0, \sum_{i=1}^d p_i = 1 \} \subset \mathbb{R}^d \) and any \( \epsilon > 0 \) there exists \( N \) such that

\[
\lim_{n \to \infty} \frac{-1}{n} \log \sum_{n \in R} \operatorname{Tr} W_n \rho_A^{\otimes n} \geq D(\overline{R} \| p) := \inf_{q \in \overline{R}} D(q \| p),
\]  \hspace{1cm} (7)

where \( \overline{R} \) is the closure of \( R \).

4 Optimal exponent of universal entanglement concentration

We prove that our universal entanglement concentration is optimal among universal entanglement concentrations which achieving the optimal rate \( nH(\rho_A) \) for any state. We call a decomposition \( C = \{ C(\omega) \}_\omega \) by CP maps of a trace preserving CP map an instrument. We discuss only local operations in this section. A sequence \( \{ (C_n = \{ C_n(\omega) \}_\omega, H_n) \} \) pairs of an instrument consisting of local operations on \( \{ \mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n} \} \) and function \( H_n : \omega \mapsto H_n(\omega) \) is called an approximately entanglement concentration of \( \langle \phi \| \phi \rangle \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \) if

\[
\sum_\omega \operatorname{Tr} C_n(\omega)(\langle \phi \| \phi \rangle^{\otimes n}) \| \frac{C_n(\omega)(\langle \phi \| \phi \rangle^{\otimes n})}{\operatorname{Tr} C_n(\omega)(\langle \phi \| \phi \rangle^{\otimes n})} - | \phi_{H_n(\omega)} \rangle \langle \phi_{H_n(\omega)} | \| \rightarrow 0, \hspace{1cm} (8)
\]
where $|\phi_{H_n(\omega)}\rangle\langle\phi_{H_n(\omega)}|$ is the perfect Bell state with the size $H_n(\omega)$. From the monotonicity of the infimum of the relative entropy $D(|\phi\rangle, \rho)$ among non-entanglement states $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, an approximately entanglement concentration $\{(C_n, H_n)\}$ of $|\phi\rangle\langle\phi|$ satisfies that
\[
\lim_{n \to \infty} \sum_\omega \text{Tr} C_n(\omega)(|\phi\rangle\langle\phi|^{\otimes n}) \frac{H_n(\omega)}{n} \leq H(\rho_A). \tag{9}
\]
A sequence $\{(C_n = \{C_n(\omega)\}_\omega, H_n)\}$ is called an approximately universal entanglement concentration of a state family $\mathcal{S} := \{\theta \in \Theta | |\phi_\theta\rangle\langle\phi_\theta|\}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ if it is an approximately entanglement concentration of any state $|\phi_\theta\rangle\langle\phi_\theta|$ and satisfies that
\[
\lim_{n \to \infty} \sum_\omega \left\{ \text{Tr} C_n(\omega)(|\phi_\theta\rangle\langle\phi_\theta|^{\otimes n}) |H_n(\omega)| \geq n(H(\rho_{A,\theta}) - \epsilon) \right\} = 1, \quad \forall \epsilon > 0, \forall \theta \in \Theta, \tag{10}
\]
where $\rho_{A,\theta} := \text{Tr}_B |\phi_\theta\rangle\langle\phi_\theta|$. From (9) and (10), the equation
\[
\sum_\omega \left\{ \text{Tr} C_n(\omega)(|\phi_\theta\rangle\langle\phi_\theta|^{\otimes n}) \left| \frac{H_n(\omega)}{n} - H(\rho_{A,\theta}) \right| > \epsilon \right\} = 1, \quad \forall \epsilon > 0, \forall \theta \in \Theta. \tag{11}
\]
Thus, we can regard the function $\frac{H_n(\omega)}{n}$ as a consistent estimator of the parameter $H(\rho_{A,\theta})$ on the state family $\{\theta \in \Theta | \rho_{A,\theta}\}$. Therefore, we have the following theorem.

**Theorem 6** An approximately universal entanglement concentration $\{(C_n, H_n)\}$ of a state family $\mathcal{S} := \{\theta \in \Theta | |\phi_\theta\rangle\langle\phi_\theta|\}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfies that
\[
\limsup_{n \to \infty} \frac{-1}{n} \log \sum_{\frac{H_n(\omega)}{n} \in R} \text{Tr} C_n(\omega)(|\phi_0\rangle\langle\phi_0|^{\otimes n}) \leq \inf_{H(\rho_{A,\theta}) \in R} D(\rho_{A,\theta}||\rho_{A,0}) \tag{12}
\]
for any open set $R \in \mathbb{R}$ and any state $|\phi_0\rangle\langle\phi_0|$.

From this theorem, we can see that our universal entanglement concentration achieves the optimal failure exponent for the state family of all pure states on the total system $\mathcal{H}_A \otimes \mathcal{H}_B$.

**Proof** We define two probabilities $p_n := \sum_{\frac{H_n(\omega)}{n} \in R} \text{Tr} C_n(\omega)(|\phi_0\rangle\langle\phi_0|^{\otimes n})$ and $q_n := \sum_{\frac{H_n(\omega)}{n} \in R} \text{Tr} C_n(\omega)(|\phi_\theta\rangle\langle\phi_\theta|^{\otimes n})$ for any state $\rho_{A,\theta}$ satisfying $H(\rho_{A,\theta}) \in R$. Since we can regard $C_n$ as a POVM on $\mathcal{H}_A^{\otimes n}$, using the monotonicity of relative entropy we have
\[
nD(\rho_{A,\theta}||\rho_{A,0}) \geq q_n \log \frac{q_n}{p_n} + (1 - q_n) \log \frac{1 - q_n}{1 - p_n}. \tag{13}
\]
Since it follows from (11) that $q_n \to 1$, we have
\[
\limsup_{n \to \infty} \frac{-1}{n} \log p_n \leq D(\rho_{A,\theta}||\rho_A). \tag{14}
\]
We obtain the desired assertion. \qed
5 Teleportation

If we perform \( R \)-qubits teleportation, we can make \( R \) qubits perfect Bell state by LOCC. Conversely, if we make \( R \) qubits perfect bell state by LOCC, we are possible to perform \( R \)-qubits teleportation. In the above setting, the bound of the number of qubit of teleportation is \( nH(\rho_A) \).

Next, we discuss how many classical bits we need to perform \( nH(\rho_A) \) qubits quantum teleportation in the above setting. It is clear that we need \( 2nH(\rho) \) bits classical information. Using our universal entanglement concentration, we can perform it with \( 2nH(\rho) \) bits classical information. From this point of view, our universal entanglement concentration is effective for the teleportation.

6 Dense coding

We formulate the effect of dense coding as follows. We assume that the state on the tensored total system \( \mathcal{H}_A^\otimes n \otimes \mathcal{H}_B^\otimes n \) is written by \( n \)-tensored state \( |\phi\rangle \langle \phi|^{\otimes n} \), where \( |\phi\rangle \langle \phi| \) is a pure state on the single total system \( \mathcal{H}_A \otimes \mathcal{H}_B \). We call the quadruple \( \Phi^n = (M_n, N_n, C_i^{(n)}, X^{(n)}) \) a code for \( |\phi\rangle \langle \phi|^{\otimes n} \) when it consists of a natural number \( M_n \) (the size of sent classical information), a natural number \( N_n \) (the size of sending quantum state), a POVM (decoding) \( X^{(n)} = \{X_i^{(n)}\}_{i=1}^{M_n} \) and a mapping (encoding) \( C_i^{(n)} : \{1, \ldots, M_n\} \ni i \rightarrow C_i^{(n)} \), where \( C_i^{(n)} \) is a CP map from \( \mathcal{S}(\mathcal{H}_A^\otimes n) \) to \( \mathcal{S}(\mathcal{C}^N) \) and \( \mathcal{S}(\mathcal{H}) \) denotes the set of densities on \( \mathcal{H} \). Therefore, the effect of entanglement is characterized by the quantity \( \log \frac{M_n}{N_n} \). For a code \( \Phi^{(n)} = (M_n, N_n, C_i^{(n)}, X^{(n)}) \), the average error probability is represented by

\[
E[\Phi^{(n)}] = \frac{1}{M_n} \sum_i \text{Tr}(C_i^{(n)} \otimes I)(|\phi\rangle \langle \phi|^{\otimes n})(I - X_i^{(n)}).
\]

Thus, we focus the following quantity

\[
C(|\phi\rangle \langle \phi|) := \sup \left\{ \liminf_{n \to \infty} \frac{1}{n} \log \frac{M_n}{N_n} \mid \exists \{\Phi^{(n)} = (M_n, N_n, C_i^{(n)}, X^{(n)})\} \text{s.t. } E[\Phi^{(n)}] \to 0 \right\}.
\]

We have the following theorem.

**Theorem 7**

\[
C(|\phi\rangle \langle \phi|) = H(\rho_B) = H(\rho_A),
\]

where \( \rho_A := \text{Tr}_B |\phi\rangle \langle \phi| \) and \( \rho_B := \text{Tr}_A |\phi\rangle \langle \phi| \).

**Proof** Define the following quantities:

\[
I(P, \rho_\bullet, X) := \sum_i P_i \sum_j \text{Tr} X_i \rho_j \log \frac{\text{Tr} X_i \rho_j}{\text{Tr} \rho_j}
\]

\[
I(P, \rho_\bullet) := \sum_i P_i D(\rho_j \| \rho) = H(\rho) - \sum_i P_i H(\rho_i),
\]

\[
I(P, \rho_\bullet, X) := \sum_i P_i \sum_j \text{Tr} X_i \rho_j \log \frac{\text{Tr} X_i \rho_j}{\text{Tr} \rho_j}.
\]
where \( \bar{\rho} := \sum_j P_j \rho_j \) and \( D(\rho||\sigma) := \text{Tr} \rho (\log \rho - \log \sigma) \). According to Barenco-Ekert [3], there exists the set \( \{U_i\}_i \) of unitaries on \( \mathcal{H}_A \) and the probability \( P \) on it such that

\[
I(P, U_{\bullet} \otimes I(|\phi\rangle\langle\phi|)) = H(\rho_B) + \text{dim} \mathcal{H}_A. \tag{15}
\]

Using the quantum channel coding theorem in the pure state case [10], we can prove that there exists a code achieving the bound \( H(\rho_B) \).

Conversely, we can prove that there does not exists a code exceeding the bound \( H(\rho_B) \) as follows. For any density \( \sigma \) the relations

\[
\sum_i P_i D(\rho_j||\sigma) = - \sum_i P_i H(\rho_i) - \text{Tr} \bar{\rho} \log \sigma \\
= - \sum_i P_i H(\rho_i) + H(\bar{\rho}) + D(\bar{\rho}||\sigma) \\
\geq - \sum_i P_i H(\rho_i) + H(\bar{\rho}) = I(P, \rho_{\bullet})
\]

hold. Letting \( P_i^{(n)} := \frac{1}{M_n} \), from Fano’s inequality, we have

\[
- \log 2 + (1 - E[\Phi^{(n)}]) \log M_n \\
\leq I(P^{(n)}, C_i^{(n)} \otimes I(|\phi\rangle\langle\phi|^{\otimes n}), X^{(n)}) \\
\leq I(P^{(n)}, C_i^{(n)} \otimes I(|\phi\rangle\langle\phi|^{\otimes n})) \\
\leq \sum_i P_i^{(n)} D(C_i^{(n)} \otimes I(|\phi\rangle\langle\phi|^{\otimes n}))[\frac{1}{N_n}I \otimes \rho_B^{\otimes n}] \\
= - \sum_i P_i^{(n)} H(C_i^{(n)} \otimes I(|\phi\rangle\langle\phi|^{\otimes n})) + \log N_n + H(\rho_B^{\otimes n}).
\]

Therefore, it follows that

\[
H(\rho_B) \geq \frac{1}{n} I(H(\rho_B^{\otimes n}) - \sum_i P_i^{(n)} H(C_i \otimes I(|\phi\rangle\langle\phi|^{\otimes n})) \\
\geq \frac{\log M_n - \log N_n}{n} - \frac{1}{n} (\log 2 + E[\Phi^{(n)}] \log M_n).
\]

Since \( E[\Phi^{(n)}] \rightarrow 0 \), we have the converse inequality. \( \square \)

Using our universal entanglement concentration, we make the perfect Bell state with the size \( \text{dim} V_n \). With the probability 1, we can send classical information with the size \( \text{dim} V_n^2 \) by sending the quantum state with the size \( \text{dim} V_n \). In this case \( M_n = \text{dim} V_n^2 \) and \( N_n = \text{dim} V_n \). If \( R \leq H(\rho_A) \), the probability of the relation \( \frac{M_n}{N_n} = \text{dim} V_n \leq 2^n R \) goes to 0 with the exponent \( \sup_{s \geq 1} \frac{(1-s)R - \psi(s)}{s} \). This is another proof of the direct part of Theorem 4.

Next, we compare its exponent with another protocol. The Burnashev-Holevo [11] random coding exponent of the pair \( (\{U_i\}_i, P) \) satisfying \( \sup_{s \geq 1} \frac{(1-s)R - \psi(s)}{s} \) is \( \sup_{s \geq 1} (1-s)R - \psi(s) \), which is better than \( \sup_{s \geq 1} \frac{(1-s)R - \psi(s)}{s} \) when \( R < H(\rho_A) \) is large enough.
A Proof of Lemma 4

According Weyl [6], Iwahori [8], the dimension of $V_n$ is written by

$$\dim V_n = \frac{n!}{(n_1 + d - 1)! (n_2 + d - 2)! \ldots n_d!} \prod_{j > i} (n_i - n_j - i + j).$$

Then the equation

$$\frac{1}{n} \log \dim V_n = \sum_{i=1}^{d} \frac{n_i - 1}{n} \log \frac{n_i}{n} + \frac{1}{n} \sum_{i=1}^{d} \log \frac{n_i!}{(n_i + d - i)!} + \frac{1}{2n} \log \frac{n_1 n_2 \ldots n_d}{n}$$

$$+ \frac{1}{n} \sum_{j > i} \log(n_i - n_j - i + j) + \frac{1}{n} (\delta_n - \sum_{i=1}^{d} \delta_{n_i})$$

holds, where $\delta_n$ is defined as $n! = e^{\delta_n} n^{n+\frac{1}{2}} e^{-n}$ and converges to the constant $\frac{1}{2} \log 2\pi$. Then, we choose the constant $C$ as $C := d \sup_n \delta_n$. Since

$$\left| \frac{1}{n} \left( \sum_{i=1}^{d} \log \frac{n_i!}{(n_i + d - i)!} + \frac{1}{2} \log \frac{n_1! n_2! \ldots n_d}{n} + \sum_{j > i} \log(n_i - n_j - i + j) \right) \right|$$

$$\leq \frac{1}{n} \log(n + d)(\frac{d(d - 1)}{2} + \frac{d}{2} + d(d - 1)) = \frac{3d^2 - 2d}{2n} \log(n + d),$$

the inequality (6) holds.

B Proof of Lemma 5

Define the vectors $v^l, g v$ and $d$ by

$$v^l := (v^l_1, v^l_2, \ldots, v^l_d)$$

$$g v := (v_{g(1)}, v_{g(2)}, \ldots, v_{g(d)})$$

$$d := (d - 1, d - 2, \ldots, 0).$$

for any $g \in S_d$. According to Weyl [6], Iwahori [8], the probability $\text{Tr} W_n \rho_A^{\otimes n}$ is written by

$$\text{Tr} W_n \rho_A^{\otimes n} = \dim V_n \det(p_{n_1 + d - 1}, p_{n_2 + d - 2}, \ldots, p_{n_d}) \prod_{j > i} (p_i - p_j).$$

According to Weyl [6], Iwahori [8], the dimension of $\dim V_n$ has another form as

$$\dim V_n = \sum_{g \in S_d} \text{sgn}(g) C(n + d - gd),$$
where $C_n$ is defined as

$$C(n) := \begin{cases} \frac{n!}{n_1!n_2!...n_d!} & \text{if } n_i \geq 0, \sum_{i=1}^d n_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the above formula, we can calculate the probability as

$$\text{Tr } W_n \rho_A^{\otimes n} = \frac{1}{\prod_{i>j}(p_i - p_j)} \sum_{g,g' \in S_d} \sgn(gg') \prod_{i=1}^d p_i^{n_{g(i)}+d-g(i)} C(n+d-gd)$$

$$= \frac{1}{\prod_{i>j}(p_i - p_j)} \sum_{g,g' \in S_d} \sgn(gg') \prod_{i=1}^d p_i^{d-gg'(i)} \text{Mul}(p, gn - gg'd + gd)$$

where we denote the Multinomial distribution of $p$ by $\text{Mul}(p, \bullet)$. For any $\epsilon_1 > 0$, there exists an integer $N$ such that

$$\frac{gn - gg'd + gd}{n} \in U((R^c_\epsilon))_{\epsilon_1}.$$  

$U(R)_{\epsilon_1} := \cup_{q \in R} U_{q,\epsilon_1}$ and $U_{q,\epsilon_1}$ is $\epsilon_1$-neighborhood of $q$. It follows from Sanov’s theorem that for any $\epsilon_2 > 0$ there exists $N$ such that the inequalities

$$\sum_{\mathbf{p} \in R} \text{Mul}(p, gn - gg'd + gd) \leq \sum_{\mathbf{p} \in U((R^c_\epsilon))_{\epsilon_1}} \text{Mul}(p, \mathbf{n}) \leq e^{-nD(U(U(R^c_\epsilon))_{\epsilon_1}\|p)}$$

hold for any $g' \in S_d$, any non-identical element $g \in S_d$ and any $n \geq N$. For any $\epsilon_3 > 0$, there exists an integer $N$ such that

$$\frac{n - gd + d}{n} \in U(R)_{\epsilon_3}, \forall \mathbf{n} \in R, \forall n \geq N.$$  

From Sanov’s Theorem, for any $\epsilon_4 > 0$ there exists $N$ such that

$$\sum_{\mathbf{n} \in R} \text{Mul}(p, \mathbf{n} - gd + d) \leq e^{-nD(U(R)_{\epsilon_3}\|p)}, \forall n \geq N.$$  

Letting $D(\mathbf{P}) := \frac{1}{\prod_{i>j}(p_i - p_j)} \sum_{g,g' \in S_d} \sgn(gg') \prod_{i=1}^d p_i^{d-gg'(i)}$, we have

$$\sum_{\mathbf{p} \in R} \text{Tr } W_n \rho_A^{\otimes n} \leq \frac{1}{\prod_{i>j}(p_i - p_j)} \sum_{g,g' \in S_d} \sgn(gg') \prod_{i=1}^d p_i^{d-gg'(i)} \sum_{\mathbf{n} \in R} \text{Mul}(p, \mathbf{n} - gd + d)$$

$$+ \frac{1}{\prod_{i>j}(p_i - p_j)} \sum_{g,g' \in S_d} \sgn(gg') \prod_{i=1}^d p_i^{d-gg'(i)} \sum_{\mathbf{p} \in R} \text{Mul}(p, gn - gg'd + gd)$$

$$\leq D(\mathbf{P}) \left( e^{-nD(U(U(R)_{\epsilon_3}))_{\epsilon_4}\|p} + e^{-nD(U(U(R^c_\epsilon))_{\epsilon_1}\|p)} \right).$$
Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{\frac{a}{n} \in R} \text{Tr} W_n \rho_A^{\otimes n} \leq -\min\{D(U(U(R^c_+))_{\epsilon_1}\|p), D((U(U(R))_{\epsilon_4}\|p)\}.
\]
From the arbitrarity of \(\epsilon_1, \epsilon_2, \epsilon_3 \epsilon_4 > 0\), we have
\[
\lim_{n \to \infty} \frac{-1}{n} \log \sum_{\frac{a}{n} \in R} \text{Tr} W_n \rho_A^{\otimes n} \geq D(\overline{R}\|p),
\]
where \(\overline{R}\) is the closure of \(R\).

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