On the number of hinges defined by a point set in $\mathbb{R}^2$

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Abstract

This note strengthens, modulo log $n$ factor, the Guth-Katz estimate for the number of pair-wise incidences of lines in $\mathbb{R}^3$, arising in the context of the plane Erdős distinct distance problem to a second moment bound. This enables one to show that the number of distinct types of three-point hinges, defined by a plane set of $n$ points is $\gg n^2 \log^{-3} n$, where a hinge is identified by fixing two pair-wise distances in a point triple.

Given an $n$-point set $P \subset \mathbb{R}^2$, define a hinge as an equivalence class $h = [p, q, r]$ of $(p, q, r) \in P^3$, identified by a pair of two fixed distances $\|p - q\|, \|q - r\|$. Let $H = H(P)$ be the set of hinges and $r_H(h)$ the number of realisations of a hinge $h \in H(P)$, that is the number of triples $(p, q, r)$ in the equivalence class $h$.

Define

$$E = E_H(P) := \sum_{h \in H} r_H^2(h).$$

Two triples $(p, q, r)$ and $(p', q', r')$ are in the same equivalence class if and only if simultaneously

$$\|p - q\| = \|p' - q'\|, \quad \|r - q\| = \|r' - q'\|. \quad (1)$$

Consider the Elekes-Sharir map (see [1], and [4] for some generalisations) $\phi : P^2 \to K(\mathbb{R}^3)$, acting as $(p, p') \rightarrow l_{pp'}$, where $K(\mathbb{R}^3)$ is the Klein quadric, the set of lines in $\mathbb{R}P^3$. Explicitly, in Plücker coordinates, with $p = (p_1, p_2)$, etc., one has

$$l_{pp'} = \left(\frac{p_2' - p_2}{2} : \frac{p_1 - p_1'}{2} : 1 : \frac{p_2' + p_2}{2} : \frac{p_1 + p_1'}{2} : \frac{\|p\|^2 - \|p'\|^2}{4}\right). \quad (2)$$

It is well known (2, and [4] for some generalisations) that the set of $n^2$ lines $L := \{l_{pp'}\}_{(p, p') \in P^2}$ has the property that just $O(n)$ may be concurrent, coplanar or lie in a regulus.

The hinge condition (1) holds if and only if simultaneously

$$l_{pp'} \cap l_{qq'} \neq \emptyset, \quad l_{rr'} \cap l_{qq'} \neq \emptyset.$$

It follows that, $\nu(l)$ denoting the number of other lines in $L$, meeting some $l \in L$,

$$E_H(P) \ll \sum_{l \in L} \nu^2(l).$$

This note uses the standard $\ll, \gg, \sim$ notations to subsume absolute constants. All point/line sets involved are finite, of cardinality $|\cdot|$. 

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Theorem 1 One has $H(P) \ll n^4 \log^3 n$, hence $|H| \gg n^2 \log^{-3} n$, for any $P \subset \mathbb{R}^2$, with $|P| = n$.

The proof is an application of the Guth-Katz type incidence bound for lines in $\mathbb{R}^3$ from the celebrated paper [2], plus some of its more recent developments.

Theorem 2 (Guth-Katz) Let $L$ be a set of lines in $\mathbb{R}^3$ with at most $O(n)$ lying in a plane or regulus. For $k \geq 2$, the number of points where at least $k$ lines meet is

$$O\left(\frac{|L|^{3/2}}{k^2} + \frac{|L|n}{k^3} + \frac{|L|}{k}\right).$$

Theorem 2 implies that a typical line from $n^2$ lines $l_{pp'} \in L$ from [2] meets $O(n \log n)$ other lines. Theorem 1 extends this to a second moment bound: the average, over lines, of the square of the number of lines a line meets, is $O(n^2 \log^3 n)$, for the price of a $\log n$ factor.

Theorem 1 gives a positive answer (up to $\log n$) to the question asked by Iosevich. By taking $P = [1, \ldots, N] \times [1, \ldots, N]$, so $n = N^2$, it’s easy to see from [2] that a typical line in $L$ meets $\sim n \log n$ lines, and no line meets more than $O(n \log n)$ other lines. So the estimate of Theorem 1 may be off by $\log n$, because $n = N^2$.

To establish Theorem 1 one uses two somewhat more elaborate variants of Theorem 2, as follows. The first one is a “bipartite” (with two line sets involved not necessarily disjoint) version for $k = O(1)$, which can be found in/extracted from, respectively, [6, 3, Proof of Theorem 12].

Theorem 3 Let $L, L'$ be two sets of lines in $\mathbb{R}^3$ with at most $O(1)$ lines from either set being concurrent and at most $O(n)$ lying in a plane or regulus. Suppose, $|L'| \leq |L|$. The number of points where two distinct lines $l, l'$, with $l \in L$, $l' \in L'$ meet is $O(\sqrt{|L|} \sqrt{|L'|} + n|L'|)$.

We will also take advantage of a (quite powerful) generalisation of the Guth-Katz incidence bound, due to Sharir and Solomon [5].

Theorem 4 Let $P$ be a set of points and $L$ a set of lines in $\mathbb{R}^3$, with at most $O(n)$ lines lying in a plane. Suppose, $L$ is contained in a zero set of a polynomial of degree $d$. The number $I(P, L)$ of incidences between $P$ and $L$ satisfies the bound

$$I(P, L) \ll |P|^{1/2} |L|^{1/2} d^{1/2} + |P|^{2/3} d^{2/3} n^{1/3} + |P| + |L|.$$

Note that any set of lines $L$ can be included into a zero set of a polynomial of degree $O(\sqrt{|L|})$. This gives the “generic” case of the latter bound, implicit in the Guth-Katz paper [2].

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1The regulus constraint matters only in the case $k = 2$.

2Two lines $(a : b : 1 : c : -d : -ac + bd)$ and $(a' : b' : 1 : c' : -d' : -a'c' + b'd')$ in Plücker coordinates meet iff $ac' - bd' + a'c - b'd = ac - bd + a'c' - b'd'$, with $a, b, c, d$ half-integers $O(N)$, and after rearranging and changing variables one sees that for a typical $(a, b, c, d)$, the number of quadruples $(a', b', c', d')$ satisfying the latter equation is roughly the number of quadruples of natural numbers $n_1/n_2 = n_3/n_4$, with $n_i = O(N)$, which is known and easily seen to be $\sim N^2 \log N$. 

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Proof. [Proof of Theorem 1] Partition, for dyadic \( k = 2, 4, 8, \ldots, \leq n \) the set \( P \) of pair-wise intersections of lines in \( L \) into sets \( P_k \), consisting of intersection points where some number of lines in the interval \([k, 2k)\) lines meet.

Theorem 2, with \(|L| = n^2\), \( k \leq n \) yields
\[
|P_k| \ll \frac{n^3}{k^2}. \tag{3}
\]

Given \( k \), for dyadic \( t = 2, 4, \ldots, \ll \frac{n^2}{k} \), let \( L_{k,t} \) be the set of \( t \)-rich lines, relative to points \( P_k \), that is the set of lines in \( L \), supporting some number of points in \( P_k \) in the interval \([t, 2t)\).

Thus each line \( l \in L \) can belong to \( O(\log n) \) sets \( L_{k,t} \). If \( l \) supports \( \nu(l) \) line intersections altogether, then partitioning these intersections by sets \( P_k \), with \( t_k \) intersection points lying in \( P_k \), yields
\[
\nu^2(l) \ll \log n \sum_k (kt_k)^2.
\]

It follows that Theorem 1 can be violated only if there is some pair \((k, t)\), such that
\[
|L_{k,t}| (kt)^2 \gg C n^4, \tag{4}
\]
for a sufficiently large \( C \).

Since trivially \(|L_{k,t}| \leq n^2\), (4) may possibly take place only for \( kt \geq n\sqrt{C} \).

Let us show that (4) cannot hold.

First, consider separately the case \( k = O(1) \) by applying Theorem 3 to \( L \) and \( L' = L_{k,t} \). It follows that
\[
t|L_{k,t}| \ll n^2 \sqrt{|L_{k,t}|} + n|L_{k,t}|.
\]
If the second term in the latter inequality dominates, then \( t = O(n) \), and since \( k = O(1) \), (4) cannot hold. If the first term dominates, then
\[
|L_{k,t}| \ll n^4/t^2,
\]
and once again (4) cannot hold.

From now on we may assume that \( k \) is sufficiently large, relative to absolute constants, hidden in the \( \ll, \gg \) symbols.

Restrict \( P_k \) to only its points supported on lines in \( L_{k,t} \). Apply the generic case of Theorem 4, that is setting \( d = \sqrt{|L_{k,t}|} \), to estimate the number of incidences \( I(P_k, L_{k,t}) \):
\[
t|L_{k,t}| \leq |P_k|^{1/2} |L_{k,t}|^{3/4} + n^{1/3} |P_k|^{2/3} |L_{k,t}|^{1/3} + |P_k|, \tag{5}
\]
There are three cases to consider.

Case (i). Suppose \( t|L_{k,t}| \ll |P_k|^{1/2} |L_{k,t}|^{3/4} \). It follows that
\[
|L_{k,t}| (kt)^2 \ll \frac{|P_k|^2}{t^4} (kt)^2 \ll \frac{n^6}{(kt)^2} \leq n^4,
\]
using (3) and that implicit in (4), \( kt \geq n \). Thus (4) does not hold in Case (i).
Case (ii). Now suppose the third term in (5) dominates, namely
\[ |L_{k,t}| \ll \frac{|P_k|}{t}. \] (6)
Hence, using (3),
\[ |L_{k,t}|(kt)^2 \ll n^3 t, \]
therefore (4) may possibly be true for \( t \gg Cn \) only.

We proceed by putting the set \( L_{k,t} \) in a zero set \( Z \) of a polynomial of degree \( d \ll \sqrt{|P_k|} \), so \( P_k \subset Z \), considering incidences between the set of lines \( L \) and \( P_k \) (recall that \( P_k \) has been restricted to points lying on lines in \( L_{k,t} \)). Let us partition \( L \) into \( L^\perp \) and \( L^\parallel \), where members of \( L^\perp \) do not lie in \( Z \), while those of \( L^\parallel \) do.

By Bezout theorem one has
\[ k|P_k| \leq I(P_k, L) \ll n^2 d + I(P_k, L^\parallel). \] (7)
If the first term in the estimate dominates, then by the estimate on \( d \) and (6) one gets
\[ t \ll \frac{n^4}{k^2 |P_k|} \ll \frac{n^4}{k^2 t |L_{k,t}|}, \]
and therefore (4) cannot hold.

Otherwise assume
\[ n^2 d \ll I(P_k, L^\parallel). \] (8)

To bound \( I(P_k, L^\parallel) \) we use Theorem (4). This, since \( k \) is sufficiently large, yields
\[ I(P_k, L^\parallel) \ll |P_k|^{1/2} d^{1/2} n + |P_k|^{2/3} d^{2/3} n^{1/3} + n^2. \] (9)
If the first term in the latter estimate dominates, then by (8) \( d \ll |P_k|/n^2 \), and hence, from (7), \( k|P_k| \ll |P_k| \). This is a contradiction, for \( k \) is meant to be sufficiently large.

If the third term dominates, then \( |P_k| \ll \frac{n^2}{k^3} \) and from (6)
\[ L_{k,t}(kt)^2 \ll n^2 (kt) \leq n^4, \] (10)
since clearly \( kt \leq n^2 \).

Thus it remains to consider the dominance of the second term in estimate (9). If so, we would have
\[ k|P_k|, n^2 d \ll |P_k|^{2/3} d^{2/3} n^{1/3}. \]
From the second inequality
\[ d^{1/3} \ll n^{-5/3} |P_k|^{2/3}, \]
so
\[ k|P_k| \ll n^{-3} |P_k|^2, \]
thus \( |P_k| \gg n^3 k \), which contradicts (3). Thus (4) does not hold in Case (ii).
Case (iii). The term $n^{1/3}|P_k|^{2/3}|L_{k,t}|^{1/3}$ dominates estimate (5), that is

$$|L_{k,t}| \ll |P_k|n^{1/2}t^{-3/2} \ll \frac{n^{7/2}}{k^2t^{3/2}}.$$  \hspace{1cm} (11)

Thus (11) can possibly hold only if $t \gg C^2 n$.

We repeat the analysis from Case (ii), now with $d \ll |P_k|^{1/2}n^{1/4}t^{-3/4}$. Returning to (7), if the first term in the right-hand side dominates we have

$$k|P_k| \ll n^{2}d \ll |P_k|^{1/2}n^{9/4}t^{-3/4}.$$  

Hence

$$|P_k|^{2/3} \ll n^{3}t^{-1}k^{-4/3}, \quad t|L_{k,t}|^{2/3} \ll n^{10/3}t^{-1}k^{-4/3},$$

the latter by (5). Thus

$$|L_{k,t}|(kt)^{2} \ll \frac{n^{5}}{t} \leq n^{4},$$

given that $t \geq n$.

The rest of the analysis repeats what has already been done in Case (ii) apropos of relations (7)–(9). The only change is that dominance of the term $n^2$ in estimate (9) would mean, once again, $|P_k| \ll \frac{n^2}{k^2}$, and thus by (11), instead of estimate (10) one has

$$|L_{k,t}|(kt)^{2} \ll n^{5/2}(kt)t^{-1/2} \leq n^{4},$$

since $t \geq n$ and $kt \leq n^2$.

Thus (11) cannot hold in Case (iii) either, which concludes the proof of Theorem 1. \hfill \Box

References

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