Conductor and discriminant of Picard curves

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Abstract

We describe normal forms and minimal models of Picard curves, discussing various arithmetic aspects of these. We determine all so-called special Picard curves over $\mathbb{Q}$ with good reduction outside 2 and 3, and use this to determine the smallest possible conductor a special Picard curve may have. We also collect a database of Picard curves over $\mathbb{Q}$ of small conductor.

Introduction

Let $K$ be a field whose characteristic does not equal 3. A Picard curve $Y$ over $K$ is a smooth projective curve of genus 3 over $K$ whose base extension to the algebraic closure $\overline{K}$ of $K$ admits a Galois morphism of degree 3 to $\mathbb{P}^1$, see Definition 1.1. Over $\overline{K}$, the curve $Y$ can be described by an equation of the form

$$Y : y^3 = f(x) = x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

with $f$ separable. A curve $Y$ admitting an equation (§) over $K$ is called superelliptic of exponent 3 over $K$. Our definition is slightly more general than the usual one from [15] in that we do not assume that a Picard curve $Y/K$ admits a superelliptic equation of exponent 3 over $K$.

Picard curves are interesting because they furnish a family of superelliptic curves that are nonhyperelliptic in the smallest genus, namely 3, where such a family exists. As such, they form a logical starting point for generalizing notions developed for hyperelliptic and elliptic curves to superelliptic curves of exponent greater than or equal to 3. Like elliptic curves, Picard curves are also smooth plane curves, as one sees by homogenizing the equation (§). Therefore, one can use tools from invariant theory as, for example, in [20] to study Picard curves. By contrast, hyperelliptic curves are never complete intersections. In this paper, we explore the relations between the superelliptic and planar vantage points, most importantly to study the relation between the conductor and the minimal discriminant of a Picard curve defined over a number field.

We start the paper by studying normal forms for Picard curves. Our approach generalizes that of [15, Appendix] but also works for residue characteristic 2. The first main result (Theorem 1.18) states that every Picard curve over $K$ admits an equation (§) over $K$, that is, superelliptic of exponent 3 over $K$, with one exception. We show that the Picard curves that do not admit such an equation all lie in one isomorphism class over $\overline{K}$, which is characterized by the fact that its automorphism group is as large as possible. We call these special Picard curves. Theorem 4.2.4 states that special Picard curves are superelliptic of exponent 4 over $K$, provided $\text{char}(K) \neq 2$.

In Section 2, we recall the definition of the minimal discriminant $\Delta_{\min}(Y)$ of a Picard curve $Y$ defined over a number field $K$. We show that every nonspecial Picard curve...
admits a long Weierstrass equation (2.2.4) with minimal discriminant. Away from residue characteristic 3 there even exists a short Weierstrass equation (1.11) with minimal discriminant (Theorem 2.2.12). The analogous result for special Picard curves is Theorem 4.5.15.

The primes dividing the minimal discriminant $\Delta_{\text{min}}(Y)$ are exactly the primes of bad reduction of $Y$ (Proposition 2.3.2). We obtain an alternative criterion for good reduction using the point of view of covers of the projective line in Propositions 3.2.1 and 4.3.2. This criterion is formulated in terms of the discriminant of the binary form $f$ from ($\ast$).

From Section 4 onward, we concentrate on special Picard curves. Our main result here is Theorem 5.1.16, which states that the smallest possible value for the conductor of a special Picard curve over $\mathbb{Q}$ is $2^63^6$. This value is attained for the standard special Picard curve $Y_0$ defined by (1.16). To prove this, we apply results from [6] on the computation of the conductor of a superelliptic curves via stable reduction. In Section 5.1, we extend results from [2] for nonspecial Picard curves to special ones, and prove lower bounds on the local conductor $f_p$ at a prime of bad reduction. Our results illustrate how to analyze the effect of twisting on the conductor.

The key ingredient in the proof of Theorem 5.1.16 is Theorem 4.4.54, which classifies all special Picard curves defined over $\mathbb{Q}$ with good reduction outside 2 and 3: There are precisely 800 different $\mathbb{Q}$-isomorphism classes. The proof uses methods in Galois cohomology that generalize beyond this particular case.

We expect $N = 2^63^6$ to be the smallest conductor for any Picard curve defined over $\mathbb{Q}$. Following the strategy for studying this question outlined in [2, Section 5], we have constructed a large database of Picard curves over $\mathbb{Q}$ that have good reduction outside two small primes, and more precisely outside of the pairs $\{2,3\}$, $\{3,5\}$, and $\{3,7\}$. These curves were obtained by methods described in the forthcoming paper by Bouw, Koutsianas, Sijsling and Wewers (summarized in Appendix B) as well as an effective enumeration by Sutherland [38]. Our database gives equations for these Picard curves, as well as their invariants, discriminants, and conductors. Its construction is briefly discussed in the concluding Appendix A.

The database provides evidence for the question, discussed in Section 5.2, whether the conductor of a Picard curve divides its minimal discriminant, as is the case for curves of genus 1 and 2.

Notations and conventions

In this article, a curve is a separated scheme of dimension 1 over a field. Given an affine equation for a curve, we will identify it with the smooth projective curve with the same function field.

1. Picard curves

Let $K$ be a field of characteristic different from 3. We fix an algebraic closure $\overline{K}$ of $K$.

**Definition 1.1.** A Picard curve $Y$ over $K$ is a curve of genus 3 over $K$ such that the base extension $Y_{\overline{K}} = Y \otimes_K \overline{K}$ admits a morphism $\varphi_{\overline{K}} : Y_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ that is a Galois cover of degree 3.

**Remark 1.2.** This definition of a Picard curve is slightly more general than that in [15, Appendix I] and [2], in that we do not require the Galois cover $\varphi_{\overline{K}}$ to be defined over $K$.

Given a Picard curve $Y$ over $K$, a morphism $\varphi_{\overline{K}}$ as in Definition 1.1 corresponds to a subgroup $G \subset \text{Aut}_{\overline{K}}(Y_{\overline{K}})$ of order 3 such that the quotient $Y_{\overline{K}}/G$ has genus 0. We call the group $G$ a distinguished subgroup of automorphisms of $Y_{\overline{K}}$. 
Lemma 1.3. Suppose that $K = \overline{K}$. Let $Y$ be a Picard curve over $K$, and let $G$ be a fixed distinguished subgroup of automorphisms of $Y$. Then there exist affine coordinates $x$ and $y$ on $Y$ that furnish an equation

$$Y : y^3 = f(x) = x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$$

(1.4)

for which $f$ is monic and separable of degree 4 and for which $G$ corresponds to the group of automorphisms generated by

$$\sigma : y \mapsto \zeta_3 y, \quad x \mapsto x.$$

Proof. By definition, $X = Y/G$ is a curve of genus 0. Since $K$ is algebraically closed, we may identify $X$ with $\mathbb{P}^1_K$ by choosing a coordinate $x$ on $X$. Moreover, the Riemann–Hurwitz formula implies that the cover $\varphi : Y \to \mathbb{P}^1_K$ has exactly five branch points. We choose the coordinate $x$ in such a way that $\infty$ is a branch point of $\varphi$.

By Kummer theory, $K(Y) = K(X)[y]/(y^3 - f)$, where $f \in K(X)$ is not a third power. Multiplying $f$ by a suitable third power, we may assume that $f$ is a monic polynomial of the form

$$f = \prod_{i=1}^{4}(x - \alpha_i)^{e_i},$$

(1.6)

with exactly four pairwise distinct roots $\alpha_i$ and $e_i \in \{1, 2\}$. Moreover, since $\varphi$ branches at $\infty$, we have

$$e_1 + e_2 + e_3 + e_4 \equiv 0 \pmod{3}.$$  

(1.7)

Up to reordering the roots of $f$, there are precisely four cases to consider:

$$(e_i)_i = (1, 1, 1, 1), \ (1, 1, 1, 2), \ (1, 2, 2, 2), \ (2, 2, 2, 2).$$

(1.8)

Replacing $y$ by $y^{-1}$, if necessary, we may assume that $(e_i)_i = (1, 1, 1, 1)$ or $(e_i)_i = (1, 1, 1, 2)$. If we are in the second case, then we can apply the change of coordinate $x \mapsto 1/(x - \alpha_4)$ and obtain $(e_i)_i = (1, 1, 1, 1)$.

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Remark 1.9. As the proof of Lemma 1.3 shows, the point at infinity of the model (1.4) is distinguished from the other branch points of $\varphi$. Indeed, as a Galois cover with group $\mathbb{Z}/3\mathbb{Z}$, we have obtained identical local monodromy generators $1 \in \mathbb{Z}/3\mathbb{Z}$ at the finite branch points, so that the point at infinity has local monodromy generator $2 \in \mathbb{Z}/3\mathbb{Z}$.

Definition 1.10. We call (1.4) a short Weierstrass equation for $Y$. For arithmetic reasons that will be described in Theorem 2.2.12, we will also use this name for equations of the more general form

$$Y : by^3 = c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$$

(1.11)

Corollary 1.12. Let $Y$ be a Picard curve over $K$. Then $Y$ is not hyperelliptic.

Proof. Since the polynomial $f$ in (1.4) is separable, the homogenized equation

$$Y : F(y, x, z) = y^3z - x^4 - c_1x^3z - c_2x^2z^2 - c_3xz^3 - c_4z^4 = 0$$

(1.13)

defines a smooth plane quartic model for $Y$ in $\mathbb{P}^2$. Now one concludes either by using the fact that hyperelliptic curves are not complete intersections or by noting that this embedding
corresponds to the canonical morphism defined by the differentials \( x \, dx/y^2, \ y \, dx/y^2, \ dx/y^2 \) of (1.4).

**Remark 1.14.** We have ordered the variables in (1.13) as \( y, x, z \), since this makes a few statements in this paper, particularly Lemma 2.2.1, slightly more elegant.

**Definition 1.15.** We call a Picard curve \( Y \) over \( K \) special if its base extension \( Y_K \) admits more than one distinguished group of automorphisms. Moreover, we call the curve

\[
Y_0 : \ y^3 = x^4 - 1
\]

the standard special Picard curve over \( K \).

**Lemma 1.17.** Suppose that the characteristic of \( K \) also does not equal 2. Then a Picard curve over \( K \) is special if and only if it is isomorphic over \( K \) to the standard special Picard curve (1.16).

**Proof.** This follows from the classification of the automorphism groups of plane curves in [22, Theorem 3.1]. For the exceptional case in loc. cit., which is the Klein quartic in characteristic 7, the statement of the lemma can be verified directly, for example, by using that the relations satisfied by the Dixmier–Ohno invariants [9, 11, 14] of the curves (1.4) are not satisfied. □

Special Picard curves will be considered from Section 4 on; until then, we will exclusively consider nonspecial Picard curves.

**Theorem 1.18.** Let \( Y \) be a nonspecial Picard curve over a field \( K \) with \( \text{char}(K) \neq 3 \). Then \( Y \) admits a short Weierstrass equation (1.4) defined by a monic polynomial \( f \) over \( K \).

**Proof.** Since the distinguished group of automorphisms is defined over \( K \), the ramification divisor of the morphism \( \varphi \) in the proof of Lemma 1.3 is \( K \)-rational. By Remark 1.9, this divisor contains a distinguished point \( P \). Together, these statements imply that \( X = Y/G \) is \( K \)-isomorphic to \( \mathbb{P}^1 \). If we choose a coordinate \( x \) on \( X \) for which \( x(P) = \infty \), then \( \varphi \) ramifies over the divisor of a polynomial \( f \) in \( x \) with \( K \)-rational coefficients. This implies that \( Y \) is given by an equation

\[
Y : \ y^3 = f(x) = c_0 x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4,
\]

where \( f \in K[x] \) is separable of degree 4. A change of coordinates \((x, y) \mapsto (c_0^{-1} x, c_0^{-1} y)\) ensures that \( f \) is monic. □

**Remark 1.20.** Theorem 1.18 can also be proved by the same methods as Theorem 4.2.4.

The isomorphisms between nonspecial Picard curves admit a simple description, as in the case of hyperelliptic curves.

**Lemma 1.21.** Let \( Y_i : y^3 = f_i(x) \) be two nonspecial Picard curves over \( K \).

(a) Let \( \varphi : Y_1 \to Y_2 \) be an isomorphism. Then there exist \( \alpha, \beta, \gamma \in K \), with \( \alpha, \gamma \neq 0 \), such that

\[
\varphi^*(x) = \alpha x + \beta, \quad \varphi^*(y) = \gamma y.
\]
There exists an isomorphism \( \varphi : Y_1 \to Y_2 \) if and only if there exist \( \alpha, \beta, \gamma \in K \) with \( \alpha, \gamma \neq 0 \) such that

\[
f_2(\alpha x + \beta) = \gamma^3 f_1(x),
\]

in which case we can take \( \varphi(x, y) = (\alpha x + \beta, \gamma y) \).

Proof. Let \( G_i \) denote the distinguished automorphism group of the curve \( Y_{i, \bar{K}} \). Since \( Y_1, Y_2 \) are assumed to be nonspecial, we have \( G_2 = \varphi G_1 \varphi^{-1} \), implying that \( \varphi \) induces an isomorphism \( \psi : X_1 \to X_2 \) between the quotients \( X_i = Y_i / G_i \).

The morphism \( \varphi \) maps the ramification (respectively, branch) divisor of \( Y_1 \to X_1 \) onto the ramification (respectively, branch) divisor of \( Y_2 \to X_2 \). We have identified the curves \( X_i \) with the projective line \( \mathbb{P}^1_K \) as in the proof of Theorem 1.18. Therefore, \( \psi \) fixes the distinguished point \( \infty \) in the proof of Theorem 1.18. We get

\[
\varphi^*(x) = \psi^*(x) = \alpha x + \beta,
\]

with \( \alpha, \beta \in K \) and \( \alpha \neq 0 \).

Moreover, \( \varphi \) maps the divisors of zeros and poles of \( y \) to one another. We conclude that \( \varphi^*(y) = \gamma y \) for a constant \( \gamma \in K^\times \). Hence,

\[
\varphi^*(f_2) = \varphi^*(y^3) = \varphi^*(y)^3 = \gamma^3 y^3 = \gamma^3 f_1.
\]

We have shown (a) and the forward direction of (b). The reverse direction of (b) can be verified by a calculation. \( \square \)

Remark 1.26. Alternatively, one notes that any isomorphism between the curves \( Y_1 \) and \( Y_2 \) is induced by an element of the normalizer of the common automorphism group of (1.13), after which one applies [22, Theorem A.1.(iv)].

2. Nonspecial Picard curves as plane curves

In this section, we consider integral equations and reduction properties of a nonspecial Picard curve \( Y \). To this end, we fix a discrete valuation \( v \) on our field \( K \), with uniformizer \( \pi \), valuation ring \( \mathcal{O}_K \), and residue field \( k \). In Section 2.1, we review the discriminant of ternary quartic forms, which gives a way to quantify bad reduction behavior of an integral quartic model of \( Y \). In Section 2.2, we study the integral models of \( Y \) that give rise to the smallest possible discriminant. We show that if \( \text{char}(k) \neq 3 \), we can find such a minimal model that is defined by a short Weierstrass equation. When \( \text{char}(k) = 3 \), we typically need a more general integral equation called the long Weierstrass equation, as defined in (2.2.4), to attain a model of minimal discriminant valuation.

While we limit ourselves to local considerations, our results apply more globally over number rings with trivial class group, as we will occasionally clarify in a remark.

2.1. The discriminant

We refer to [8] and [13, Chapter 13] for a more complete exposition of what follows.

Let \( F \in R[y, x, z] \) be a ternary quartic form over a domain \( R \) whose fraction field does not have characteristic 2. The discriminant \( \Delta(F) = \Delta_{4,4}(F) \) of \( F \) is the element of \( R \) defined as

\[
\Delta(F) = \text{Res}(D_y F, D_x F, D_z F)/2^{14}.
\]

Here \( D_T F \) denotes the partial derivative of \( F \) with respect to the variable \( T \), and \( \text{Res} \) denotes the resultant of three ternary quartics.
Remark 2.1.2. In [20], it is shown that the discriminant $\Delta(F)$ is related to the Dixmier–Ohno invariant $I_{27}(F)$ of $F$ via $\Delta(F) = 2^{40} I_{27}(F)$.

**Proposition 2.1.3.** Let $F \in K[y, x, z]$. The discriminant satisfies the following properties.

(a) $F$ defines a nonsingular curve over $K$ if and only if $\Delta(F) \in K^\times$.

(b) If $F \in \mathcal{O}_K[y, x, z]$ is integral, then the reduction of $F$ modulo $v$ defines a nonsingular curve over $k$ if and only if $v(\Delta(F)) = 0$.

(c) $\Delta$ is a homogeneous form of degree 27 in the coefficients of $F$, so that $v(\Delta(\lambda F)) = v(\Delta(F)) + 27\lambda$.

(d) $\Delta$ has weight 36, that is, $v(\Delta(F \circ T)) = v(\Delta(F)) + 36v(\det(T))$ for $T \in \text{GL}_3(K)$.

**Proof.** The first two statements follow from the properties of $\Delta$ described in [8, Section 4], whereas the third results from $\Delta$ being homogeneous of degree 27 in the coefficients of $F$. The final statement follows from the fact that an invariant of ternary quartics of degree 36 has weight 4, as recalled in, for example, [24, (1.11)]. □

The following statement follows from the properties of the resultant in [8, Section 4]. It will be used in many constructions and examples to follow, like Remark 3.1.22.

**Lemma 2.1.4.** Let $f = c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$ be a separable polynomial over $K$ with discriminant $\Delta(f)$, and let $F(y, x, z)$ be a corresponding short Weierstrass equation $(1.11)$. Then

$$\Delta(F) = -3^9b^2c_0^3\Delta(f)^2. \tag{2.1.5}$$

**Definition 2.1.6.** Let $Y$ be a Picard curve over a discretely valued field $(K, v)$. The minimal discriminant exponent of $Y$ at $v$ is the integer

$$e_v(Y) = \min \{v(\Delta(F))\} \in \mathbb{Z}_{\geq 0}. \tag{2.1.7}$$

Here $F$ runs through the quartic forms $F \in \mathcal{O}_K[x, y, z]$ that furnish a model of $Y$. The minimal discriminant of $Y$ at $v$ is the ideal

$$\Delta_{\min}(Y) = (\pi^{e_v(Y)}). \tag{2.1.8}$$

For a Picard curve $Y$ over a number field $K$, we define the minimal discriminant of $Y$ to be the integral ideal

$$\Delta_{\min}(Y) = \prod p^{e_v(Y)}. \tag{2.1.9}$$

Here $p$ runs through the primes of $K$ and $v_p$ is the valuation corresponding to $p$.

**Remark 2.1.10.** In what follows, we will identify the minimal discriminant of a Picard curve over $\mathbb{Q}$ with the positive generator of the integral ideal $(2.1.9)$.

**Remark 2.1.11.** The minimal discriminant was also discussed in [2, Section 4.2]. We take this opportunity to correct a mistake. In [2], above Lemma 4.3, it is claimed that one can find an equation $y^3 = f(x)$ for any Picard curve such that $v_p(\Delta(f)) < 36$. As Elisa Lorenzo García pointed out to us, this is not the case. A counterexample is the family of curves

$$y^3 = f_n(x) = x^4 + x^2 + p^n \in \mathbb{Q}_p[x], \tag{2.1.12}$$

whose discriminant valuation at $p$ gets arbitrarily large as $n$ goes to $\infty$. However, the discriminant of the Weierstrass equation, and hence of $f$, is minimal for all $n$. 


2.2. Discriminant minimization

In this section, we derive a standard model for Picard curves over the discretely valued field $K$. To this end, we first prove a lemma on smooth plane curves.

**Lemma 2.2.1.** Let $X = V(F) \subset \mathbb{P}^3_K$ be a smooth plane curve over $K$ defined by a ternary quartic form $F \in \mathcal{O}_K[y, x, z]$. Let $P \in X(K)$ be a rational point on $X$.

(a) There exists a matrix $T_1 \in \text{GL}_3(\mathcal{O}_K)$ that maps $P$ to $(1 : 0 : 0)$.

(b) Suppose that $P = (1 : 0 : 0)$. Then there exists a matrix $T_2 \in \text{GL}_3(\mathcal{O}_K)$ that fixes $P$ and that maps the tangent line $L$ of $X$ in $P$ to $z = 0$.

(c) Let $X_1$ and $X_2$ be two plane curves with the common rational point $P_1 = P_2 = (1 : 0 : 0)$ and the common tangent line $z = 0$ at these points. Then any isomorphism between $X_1$ and $X_2$ sending $P_1$ to $P_2$ is induced by an upper triangular matrix $T$ in $\text{GL}_3(K)$.

**Proof.** (a) It suffices to construct a matrix $U_1 \in \text{GL}_3(\mathcal{O}_K)$ sending $(1 : 0 : 0)$ to $P$ instead. Suppose, without loss of generality, that the first coordinate of $P$ has trivial valuation. Then we can take $U_1$ to be the matrix whose first column is given by the coordinates of $P$ and whose other columns are the standard basis vectors.

(b) The tangent line $L$ is of the form $\gamma x + \delta z = 0$. We may choose $\gamma$ and $\delta$ to be integral and coprime. Choosing $\alpha$ and $\beta$ integral such that $\alpha \gamma + \beta \delta = 1$, we can take $T_2$ to be the inverse of the matrix corresponding to the transformation $y \mapsto y$, $x \mapsto \alpha x + \beta z$, $z \mapsto \gamma x + \delta z$.

(c) Any isomorphism between smooth plane curves is induced by an invertible matrix $T = (T_{i,j}) \in \text{GL}_3(K)$. Since $T$ has to fix $(1 : 0 : 0)$, we have $T_{2,1} = T_{3,1} = 0$. And since $T$ fixes the common tangent line $z = 0$, we need a corresponding condition on its transpose, yielding $T_{3,1} = T_{3,2} = 0$.

**Remark 2.2.2.** The same considerations apply globally over a number field with trivial class group. For example, the proof of part (a) of the proposition in this more global case uses the fact that given a PID $R$, an inclusion $0 \to R \to R^3$ with torsion-free quotient $Q$ always admits a splitting $Q \to R^3$. Because of this, we can always augment a coprime coordinate vector for $P$ to an invertible matrix over $R$, after which the same argument can be run.

**Lemma 2.2.3.** Let $Y$ be a nonspecial Picard curve over $K$, and let $F \in K[y, x, z]$ be a ternary form defining a curve isomorphic to $Y$. Then after an integral transformation in $\text{GL}_3(\mathcal{O}_K)$, the ternary quartic $F$ yields an equation for $Y$ of the form

\[ Y : (a_0y^3 + a_1(x, z)y^2 + a_2(x, z)y)z = a_4(x, z), \]

where $a_0 \neq 0$, $a_i$ is a homogeneous form of degree $i$ in $x$ and $z$, and where

\[ a_1^2 = 3a_0a_2. \]

Conversely, any equation (2.2.4) satisfying (2.2.5) defines an equation (1.13) of a Picard curve via the substitution $y \mapsto y - (a_1(x, z)/(3a_0))$.

**Proof.** The nonspecial Picard curve $Y$ has a distinguished rational point $P$, corresponding to the point $(1 : 0 : 0)$ in (1.13). By (a) and (b) in Lemma 2.2.1, we may apply an integral transformation to suppose that the point on the curve defined by $F$ corresponding to $P$ is again given by $(1 : 0 : 0)$, with tangent line $z = 0$.

Part (c) of the same lemma then shows that $F$ can be obtained from a short Weierstrass equation by a substitution $y \mapsto \lambda y + \mu(x, z)$. This yields an equation of the form (2.2.4). The relation (2.2.5) is satisfied because $a_0 = \lambda^3$, $a_1 = 3\lambda^2\mu$, $a_2 = 3\lambda\mu^2$. The converse statement can be checked directly.
Definition 2.2.6. Given a Picard curve $Y$, we call an equation for $Y$ of the form (2.2.4) a long Weierstrass equation for $Y$.

Proposition 2.2.7. Let $Y$ be a nonspecial Picard curve over the discretely valued field $(K, v)$. Then $Y$ admits an integral long Weierstrass equation of minimal discriminant exponent.

Proof. One follows the proof of Lemma 2.2.3. By Proposition 2.1.3.(d), the integral transformation in the proof of this lemma preserves the minimality of the discriminant of $F$. □

Remark 2.2.8. An equation with minimal discriminant for a given Picard curve $Y$ over $\mathbb{Q}$ can often be found explicitly by using the algorithms in [10] based on [18]. Our methods then yield a long Weierstrass equation for $Y$. Occasionally, though, Elsenhans’ algorithms will only return equations that are locally minimal in the Bruhat–Tits tree corresponding to the integral models of $Y$.

Remark 2.2.9. Integral long Weierstrass equations with the same discriminant exponent need not be related by an integral transformation, essentially because one may be able to divide out scalars after substitution. For example, for $p \neq 2$ the equation
\begin{equation}
Y_1: py^3z = px^4 + 2x^2z^2 + p^3xz^3 - p^4z^4
\end{equation}
can be transformed into
\begin{equation}
Y_2: y^3z = p^5x^4 + 2x^2z^2 + pxz^3 - z^4
\end{equation}
by substituting $py$ for $y$ and $p^2x$ for $x$. This implies that none of the isomorphisms $Y_1 \to Y_2$ is defined by an element of $GL_3(\mathcal{O}_K)$. Yet both of these equations have discriminant valuation 25 at $p$.

Theorem 2.2.12. Let $Y$ be a nonspecial Picard curve over a discretely valued field $(K, v)$ whose residue characteristic does not equal 3. Then $Y$ admits an integral short Weierstrass equation (1.11) of minimal discriminant exponent.

Proof. Extend $v$ to the Gauss valuation for ternary polynomials, and consider an integral equation (2.2.4) of $Y$ whose discriminant is minimal. First suppose $v(a_0) = 0$. Then the integral coordinate change $y \mapsto y - (a_1(x, z)/(3a_0))$ gives an equation of the form (1.11), and we are done. If $v(a_0) = 1$, then $v(a_1) \geq 1$ by (2.2.5), and the same argument applies. Now suppose $v(a_0) \geq 2$. Then $v(a_1) \geq 1$ by (2.2.5). In this case, substituting $x \mapsto \pi x$ and $z \mapsto \pi z$ yields a ternary form with Gauss valuation at least 3. Since the discriminant has degree 27 and weight 36, dividing out the common factor of the coefficients yields a new integral equation whose discriminant exponent is at least $3 \cdot 27 - 2 \cdot 36$ smaller. This is in contradiction with our minimality assumption. □

Remark 2.2.13. It is not true that if we suppose additionally that the residue characteristic of $\mathcal{O}_K$ does not equal 2, then $Y$ will admit an integral short Weierstrass equation of the more restricted form
\begin{equation}
y^3 = c_0x^4 + c_2x^2 + c_3x + c_4
\end{equation}
whose discriminant exponent is minimal. Note how this contrasts with the behavior of elliptic curves, which always admit a minimal Weierstrass equation $y^2 = x^3 + ax + b$ away from 2 and 3.
For a counterexample, consider the curve
\[ Y_1 : y^3z = px^4 + x^3z + 2x^2z^2 + xz^3 - z^4 \]  
(2.2.15)
for \( p = 17 \) (or large enough). The exponent of the discriminant of \((2.2.15)\) at \( p \) equals 3, which then has to be the minimal discriminant exponent at \( p \) because of (c) and (d) of Proposition 2.1.3. A nonintegral equation of the form \((2.2.14)\) is obtained by substituting \( x \mapsto x - (z/4p) \), which yields
\[ Y_2 : y^3z = px^4 + \frac{269}{2^3p}x^2z^2 - \frac{2177}{2^3p^2}xz^3 + \frac{1275683}{2^8p^3}z^4. \]  
(2.2.16)
Because of Lemma 1.21.(b), all isomorphisms between curves defined by equations of the form \((2.2.14)\) are induced by diagonal matrices. This makes finding an integral equation of the form \((2.2.14)\) with minimal discriminant (among all equations of that particular form) into a problem in linear programming. In this particular case, the minimal discriminant valuation at \( p \) is attained by
\[ Y_3 : y^3z = x^4 - \frac{269}{2^3}x^2z^2 - \frac{2177}{2^3}xz^3 + \frac{1275683}{2^8}z^4. \]  
(2.2.17)
It equals 12, which is strictly larger than the valuation 3 obtained by using the more general class of equations \((1.11)\).

2.3. **Criteria for good reduction**

Let \((K,v)\) be a discretely valued field with \( \text{char}(K) \neq 3 \). In this section, we only deal with questions local to \( v \), and therefore can and will assume that \( K \) is complete with respect to \( v \).

**Definition 2.3.1.** Let \( Y \) be a Picard curve over \( K \). We say that \( Y \) has **good reduction** at \( v \) if there exists a smooth and proper \( \mathcal{O}_K \)-model \( Y \rightarrow \text{Spec} \mathcal{O}_K \) of \( Y \). If not, we say that \( Y \) has **bad reduction**. We say that \( Y \) has **potentially good reduction** if there exists an extension \( L/K \) such that \( Y_L = Y \otimes_K L \) has good reduction.

**Proposition 2.3.2.** A Picard curve \( Y \) over \( K \) has good reduction if and only if \( v(\Delta_{\min}(Y)) = 0 \).

**Proof.** First assume \( v(\Delta_{\min}(Y)) = 0 \). Let \( F \in \mathcal{O}_K[y,x,z] \) be a quartic form with \( v(\Delta(F)) = 0 \). Then \( F \) defines a model \( Y_F \) of \( Y \) over \( \mathcal{O}_K \). Proposition 2.1.3 shows that \( Y_F \) is smooth.

Now assume that \( Y \) has good reduction. Let \( \mathcal{Y} \) be a smooth model of \( Y \) with smooth special fiber \( \overline{Y} \). We show that \( \overline{Y} \) is nonhyperelliptic. Since this property of \( \overline{Y} \), as well as its negation, is stable under base extension, we may assume that \( k = \overline{k} \) is algebraically closed. Then [26, Proposition 10.3.38] shows that \( \text{Aut}_k(\overline{Y}/\overline{k}) \) can be considered as a subgroup of \( \text{Aut}_k(\overline{Y}/\overline{k}) \). More precisely, \( \text{Aut}_k(\overline{Y}/\overline{k}) \) contains a cyclic subgroup \( G \) of order 3 such that \( g(\overline{Y}/G) = 0 \).

If \( \text{char}(k) \neq 3 \), then we conclude that \( \overline{Y} \) is a Picard curve, so that it is nonhyperelliptic by Corollary 1.12. If \( \text{char}(k) = 3 \), then the Riemann–Hurwitz formula implies that there are two possibilities for the cover \( \pi : \overline{Y} \rightarrow \overline{Y}/G \simeq \mathbb{P}^1 \). In the former case, \( \pi \) has a unique branch point. Artin–Schreier theory implies that \( \overline{Y} \) admits an equation \( y^3 - y = x^4 \) over \( \overline{k} \). In the latter case, \( \pi \) has two branch points and over \( \overline{k} \) the curve \( \overline{Y} \) admits an equation of the form \( xy^3 - xy = g(x) \) for some polynomial \( g \) of degree 3. So also in this case \( \overline{Y} \) is nonhyperelliptic.

As at the beginning of [20, §2.1], we now use the relative canonical sheaf of \( \mathcal{Y} \) to obtain a proper \( \mathcal{O}_K \)-morphism \( \varphi : \mathcal{Y} \rightarrow \mathbb{P}^2 \). If \( \varphi \) is not a smooth embedding, then loc. cit. shows that its special fiber is a degree 2 map to a conic over \( k \), so that \( \overline{Y} \) is hyperelliptic. We have just excluded this possibility, so that \( \varphi \) induces an isomorphism of \( \mathcal{Y} \) with its image in \( \mathbb{P}^2 \). By dimension theory, this image is defined by a single ternary quartic form \( F \in \mathcal{O}_K[y,x,z] \). Since
both $F$ and its reduction define a smooth curve, we can once more invoke Proposition 2.1.3 to conclude that $v(\Delta(F)) = 0$ and therefore $v(\Delta_{\text{min}}(Y)) = 0$. □

Remark 2.3.3. An elegant alternative argument to the above case distinction by characteristic was kindly provided by the referee: A smooth curve that admits both a degree-2 and a degree-3 map to $\mathbb{P}^1_k$ is birational to a curve of type $(2,3)$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and therefore of genus at most 2.

Remark 2.3.4. Contrary to Picard curves, a general plane quartic curve over $\mathbb{K}$ with good reduction may not admit a smooth plane quartic model over $\mathcal{O}_K$, as it may have good reduction that is hyperelliptic. Necessary and sufficient conditions for this to happen are given in [20].

Our results also give a criterion for bad reduction that gives an alternative viewpoint of the result in [2, Proposition 3.4].

Proposition 2.3.5. Let $Y$ be a nonspecial Picard curve over a discretely valued field $(\mathbb{K}, v)$ for which 3 has odd valuation. Then $Y$ does not have good reduction.

Proof. Suppose that $Y$ has good reduction. As in the proof of Proposition 2.3.2, we obtain an integral equation (2.2.4) whose discriminant valuation equals 0 and whose reduction modulo 3 is nonsingular. We have $v(a_0) = 0$ by nonsingularity of the reduction. Equation (2.2.5) then implies $v(a_1) > 0$, and because 3 has odd valuation, this in turn implies $v(a_2) > 0$. This implies that the reduction is defined by an equation of the form $y^3 = f(x)$, which is a contradiction since such an equation does not define a smooth curve in characteristic 3. □

3. Nonspecial Picard curves as superelliptic curves

Instead of minimizing the ternary form $F$ defining the Picard curve in (1.13), we can also reduce the binary form $f$ figuring in (1.4). Indeed, Theorem 2.2.12 states the existence of an integral short Weierstrass equation with minimal discriminant, but does not provide an easy-to-use criterion for recognizing such minimal equations.

In this section, we follow an alternative approach that modifies the binary form $f$ from (1.13) rather than the whole equation. While this approach does not necessarily find an equation with minimal discriminant, the algorithm is far simpler, and suffices for the purpose of recognizing Picard curves with good reduction, see Section 3.2. Moreover, an extension of this approach can be used to calculate the stable reduction in the case of bad reduction. This approach works in principle for all superelliptic curves, and does not use that the curve is a complete intersection. We refer to [6] for more details.

3.1. Reducing binary quartics

Let $K$ be a field and let $Y_1$ and $Y_2$ be two nonspecial Picard curves over $K$. Lemma 1.21 implies that isomorphisms over $K$ between $Y_1$ and $Y_2$ induce isomorphisms between the branch divisors $\tilde{D}_i$ of the degree-3 quotient maps $\varphi : Y_1 \to X_1 = \mathbb{P}^1_k$. One of the branch points is distinguished, see Remark 1.9. We call $X$ together with a (4,1)-divisor a (4,1)-marked projective line over $K$.

In what follows we choose a coordinate on $X$ such that the distinguished point is $x = \infty$. Write $D$ for the finite part of the divisor $\tilde{D}$. Then there exists a unique monic quartic $f \in K[x]$ such that $D = (f)_0$ is the divisor of zeros of $f$. To study these, we make the following definitions.

Definition 3.1.1. (a) A quartic over $K$ is a monic and separable quartic polynomial $f = x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4 \in K[x]$.
(b) Two quartics $f_1, f_2 \in K[x]$ are equivalent if there exists a matrix
\[
A = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix},
\]
(3.1.2)
such that
\[
f_2 = f_1 \circ A := \alpha^{-4} f_1(\alpha x + \beta).
\]
(3.1.3)

The factor $\alpha^{-4}$ is inserted in (3.1.3) to ensure that $f_1 \circ A$ is again monic.

Now again let $v$ be a discrete valuation on $K$ with valuation ring $\mathcal{O}_K$ and residue field $k$.

**Definition 3.1.4.** A quartic $f \in K[x]$ is called reduced with respect to $v$ if the following holds.

(a) $f \in \mathcal{O}_K[x]$.
(b) If $g \in \mathcal{O}_K[x]$ is equivalent to $f$, then $v(\Delta(f)) \leq v(\Delta(g))$.

For a quartic $f \in K[x]$, we shall write $-\lambda(f) \in \mathbb{Q}$ for the largest slope of the Newton polygon of $f$ with respect to $v$. In other words,
\[
\lambda(x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4) = \min_{1 \leq i \leq 4} \frac{v(c_i)}{i}.
\]
(3.1.5)
We have that $\lambda(f)$ is the minimum of the valuations of the roots of $f$ in $\overline{K}$, and that $f \in \mathcal{O}_K[x]$ if and only if $\lambda(f) \geq 0$.

**Lemma 3.1.6.** A quartic $f \in K[x]$ is reduced if both of the following conditions hold.

(a) $0 \leq \lambda(f) < 1$.
(b) If $\lambda(f) = 0$, then the image of $f$ in $k[x]$ is not a fourth power.

**Proof.** We assume that $f$ is not reduced. Then there exist $\alpha, \beta \in K$, $\alpha \neq 0$, such that
\[
g = \alpha^{-4} f(\alpha x + \beta) \in \mathcal{O}_K[x], \quad v(\Delta(g)) = v(\Delta(f)) - 12v(\alpha) < v(\Delta(f)).
\]
(3.1.7)
From this we obtain that $v(\alpha) \geq 1$ and $v(\beta) \geq 0$. Let $L/K$ be the splitting field of $f$ and let $w$ be an extension of $v$ to $L$. Write
\[
f = \prod_{i=1}^{4} (x - \xi_i).
\]
(3.1.8)
Then
\[
g = \prod_{i=1}^{4} \left( x - \frac{\xi_i - \beta}{\alpha} \right).
\]
(3.1.9)
Since $f$ and $g$ are both integral, we have
\[
w(\xi_i) \geq 0, \quad w\left( \frac{\xi_i - \beta}{\alpha} \right) = w(\xi_i - \beta) - v(\alpha) \geq 0,
\]
(3.1.10)
for all $i$. Using $v(\alpha) > 0$, we conclude that for all $i$ we also have
\[
w(\xi_i - \beta) > 0.
\]
(3.1.11)
We have to show that either (a) or (b) is false. Assume that (a) is true, that is,
\[
\lambda(f) = \min_i w(\xi_i) < 1.
\]
(3.1.12)
Then there exists some \( j \in \{1, \ldots, 4\} \) such that \( w(\xi_j) < 1 \). Since \( w(\beta) \geq 0 \) and \( w(\beta) = v(\beta) \in \mathbb{Z} \), the strict triangle inequality, applied to (3.1.11), shows that \( w(\xi_j) = w(\beta) = 0 \). But then (3.1.11) also implies that \( w(\xi_i) = 0 \) for all \( i \). We conclude that
\[
\lambda(f) = 0 \quad \text{and} \quad \mathcal{I}(f) = (x - \beta)^4.
\] (3.1.13)

So (b) is false, and the lemma is proved. \( \square \)

**Remark 3.1.14.** Conditions (a) and (b) are sufficient but not necessary for \( f \) to be reduced. For example, the quartic \( f = (x + 1)^4 + p \) over \( \mathbb{Q} \) is reduced with respect to the \( p \)-adic valuation \( v_p \), but (b) does not hold.

Algorithmically, a given quartic can be reduced as follows.

**Algorithm 3.1.15.** Let \( f \) be a quartic polynomial over \( K \). The following algorithm returns a reduced quartic equivalent to \( f \).

(a) Set \( n = \lfloor \lambda(f) \rfloor \) and replace \( f \) by \( \pi^{-4n} f(\pi^n x) \). Now \( 0 \leq \lambda(f) < 1 \).

(b) Suppose \( \lambda(f) = 0 \). Let \( \mathcal{I} f \in k[x] \) denote the image of \( f \). If \( \mathcal{I} f = (x - \pi)^4 \), with \( \pi \in k^\times \), replace \( f \) by \( f(x + a) \), for some lift \( a \in \mathcal{O}_K \) of \( \pi \), and go back to (a).

(c) Now \( f \) is reduced, by Lemma 3.1.6.

Note that Algorithm 3.1.15 terminates after finitely many rounds because \( v(\Delta(f)) \) gets strictly smaller each time we repeat Step (b).

The next proposition relates reduced quartics to good reduction.

**Definition 3.1.16.** We say that a \((4,1)\)-marked projective line \((X, \infty, D)\) over \( K \) has good reduction with respect to \( v \) if there exists a smooth and proper \( \mathcal{O}_K \)-model \( \mathcal{X} \to \text{Spec} \mathcal{O}_K \) of \( X \) such that the scheme-theoretic closure \( \mathcal{D} \subset \mathcal{X} \) of the divisor \( \mathcal{D} = D \cup \{\infty\} \) is étale over \( \text{Spec} \mathcal{O}_K \).

**Proposition 3.1.17.** Let \( f \in K[x] \) be a quartic and \((X, \infty, D)\) the corresponding \((4,1)\)-marked projective line. Then the following are equivalent.

(a) \((X, \infty, D)\) has good reduction with respect to \( v \).

(b) The discriminant of the reduced quartic equivalent to \( f \) is a unit in \( \mathcal{O}_K \).

**Proof.** Let us assume that \((X, \infty, D)\) has good reduction with respect to \( v \), and let \( \mathcal{X} \) be an \( \mathcal{O}_K \)-model as in Definition 3.1.16. Then there exists an \( \mathcal{O}_K \)-isomorphism \( \mathcal{X} \cong \mathbb{P}^1_{\mathcal{O}_K} \), which sends \( \infty \) to \( \infty \). Let \( g \) be the quartic (equivalent to \( f \)) whose divisor of zeros is the image of \( D \) in \( \mathbb{P}^1_{K} \) under this isomorphism. Then \( g \in \mathcal{O}_K[x] \) (because the closure of \( \infty \) in \( \mathcal{X} \) is disjoint from the closure of \( D \)) and the image \( \mathcal{I} g \in k[x] \) of \( g \) is separable (because the closure of \( D \) is étale over \( \text{Spec} \mathcal{O}_K \)). It follows that \( \Delta(g) \in \mathcal{O}_K^\times \), which implies (b).

Conversely, if (b) holds, then there exists a \( g \in \mathcal{O}_K[x] \), equivalent to \( f \), such that \( \Delta(g) \in \mathcal{O}_K^\times \). Let \( X \cong \mathbb{P}^1_K \) be the isomorphism that sends \( D \) to the zero divisor of \( g \). It extends to an isomorphism \( \mathcal{X} \cong \mathbb{P}^1_{\mathcal{O}_K} \), for a unique smooth \( \mathcal{O}_K \)-model \( \mathcal{X} \) of \( X \). The assumptions on \( g \) now imply that the closure of \( D \cup \{\infty\} \subset X \) in \( \mathcal{X} \) is étale over \( \text{Spec} \mathcal{O}_K \), and so (a) holds. \( \square \)

Proposition 3.1.17 gives us a second way to optimize a short Weierstrass equation (1.4) of a given Picard curve.
Corollary 3.1.18. Let $(K,v)$ be a discretely valued field and assume that both\char(K) \neq 3 and char(k) \neq 3. Every nonspecial Picard curve $Y/K$ admits an equation  
\[ Y : y^3 = cf_0(x), \]  
where $0 \leq v(c) \leq 2$ and $f_0 \in \mathcal{O}_K[x]$ is a reduced quartic.

Proof. Let $y^3 = f(x)$ with $f(x) \in K[x]$ be a short Weierstrass equation for $Y$, which exists by Theorem 1.18. Scaling the coordinates if necessary, we may assume that $f = \gamma g$, where \( \gamma \in \mathcal{O}_K \) and where $g$ is monic.

Let $f_0$ be the reduced quartic equivalent to $g$. We have  
\[ g(\alpha x + \beta) = \gamma \alpha^{-4} f_0. \]

Now $\delta = \gamma \alpha^{-4}$ is integral since $v(\alpha) \leq 0$. Let $e = \lfloor v(\delta)/3 \rfloor$. Set $c = \delta \pi^{-3n}$. By Lemma 1.21, $(x,y) \mapsto (\alpha x + \beta, \pi^n y)$ defines an isomorphism between $Y$ and the curve defined by (3.1.19) satisfying the requirement of the statement. \( \square \)

Definition 3.1.21. We call an equation for $Y$ as in Corollary 3.1.18 a reduced short Weierstrass equation for $Y$.

Remark 3.1.22. Reduced short Weierstrass equations are not always minimal, as is shown by the example  
\[ Y_1 : y^3 = 7(x^4 - 9x^2 - 10x - 9) \]  
for $p = 7$. This reduced equation has discriminant valuation 19. However, the short Weierstrass equation  
\[ Y_2 : y^3 = 49x^4 - 56x^3 + 15x^2 + 2x - 1 \]  
defines an isomorphic curve and has discriminant valuation 10.

3.2. Criteria for good reduction

We can characterize good reduction of nonspecial Picard curves by their reduced short Weierstrass model as long as the residue characteristic does not equal 3.

Proposition 3.2.1. Let $(K,v)$ be a complete discretely valued field with char$(K) \neq 3$ whose residue field does not have characteristic 3. Let $Y$ be a nonspecial Picard curve over $K$ with reduced short Weierstrass equation  
\[ Y : y^3 = cf. \]  
Then $Y$ has good reduction modulo $v$ if and only if $v(c) = 0$ and $v(\Delta(f)) = 0$.

Proof. For the proof, we may replace $K$ by an unramified extension and therefore we may assume that $K$ contains a primitive third root of unity $\zeta_3$. Assume that the hypotheses on $c$ and $f$ hold. Let $\mathcal{Y} \subset \mathbb{P}^2_{\mathcal{O}_K}$ be the plane quartic over $\mathcal{O}_K$ obtained by homogenizing (3.2.2). Using Lemma 2.1.4 and Proposition 2.3.2, we see that $\mathcal{Y}$ is smooth over $\mathcal{O}_K$, so that $Y$ has good reduction.

For the converse, let $y^3 = cf(x)$ be a reduced short Weierstrass equation for $Y$. Let $G$ be the distinguished group of automorphisms of $Y$ generated by the automorphism $\sigma : y \mapsto \zeta_3 y$. Let $L$ be a minimal Galois extension of $K$ over which $Y$ has semistable reduction. Then by [2, Section 4.1], the curve $Y_L$ has a unique stable $\mathcal{O}_L$-model $\mathcal{Y}$. Moreover, the action of $G$ extends to $\mathcal{Y}$ and the quotient scheme $\mathcal{X} = \mathcal{Y}/G$ is a semistable model of $Y_L/G = \mathbb{P}^1_L$. If $D \subset \mathcal{X}$ denotes...
the closure of the branch divisor $\hat{D} = (f)_{\emptyset} \cup \{\infty\}$, then $(X, \hat{D})$ is the stably marked model of $(\mathbb{P}^1_K, D)$. Let us denote by $\overline{Y}$ and $\overline{X}$ the special fibers of $Y$ and $X$. The induced map $\overline{Y} \to \overline{X}$ is called the stable reduction of the cover $Y \to \mathbb{P}^1_K$. Lemma 4.1 and Theorem 4.2 of [2] give a precise description of this map, distinguishing five cases (a)–(e). Case (a) is the only case where $\overline{Y}$ is smooth. In this case, $\overline{X}$ is smooth as well.

Assume that $X$ has good reduction over $K$. Then $Y$ has semistable reduction over $L = K$, and its stable model $\mathcal{Y}$ is smooth over $\mathcal{O}_K$. Therefore, we are in Case (a) of [2, Lemma 4.1, Theorem 4.2]. It follows that $X = \mathcal{Y}/G$ is a smooth model of $\mathbb{P}^1_K$ such that the closure $\hat{D}$ in $\mathcal{X}$ of the branch divisor $\hat{D}$ is étale over $\text{Spec} \mathcal{O}_K$. By Proposition 3.1.17, this implies that the discriminant of $f$ is a unit in $\mathcal{O}_K$, that is, that the reduction $\overline{f} \in k[x]$ of $f$ is separable. It also follows from the proof of [2, Theorem 4.1] that $v_K(c) = 0$; a more detailed argument in a general setting is given in the proof of [6, Proposition 4.5]. □

3.3. Invariants and twists

Let $K$ be a field with $\text{char}(K) \neq 3$, and let $\overline{K}$ be an algebraic closure of $K$. In general, the isomorphism classes of Picard curves over $\overline{K}$ can be classified by using the Dixmier–Ohno invariants [20]. The definition of these invariants uses classical invariants of ternary quartics and is relatively complicated. If we suppose additionally that $\text{char}(K) \neq 2$, as we do throughout this section, then we can describe the isomorphism classes of nonspecial Picard curves over $\overline{K}$ and $K$ by using a smaller and more elementary set of invariants than the full set of Dixmier–Ohno invariants.

**Proposition 3.3.1.** Let $Y$ be a nonspecial Picard curve over $K$. Then $Y$ admits an equation

$$Y : y^3 = x^4 + c_2x^2 + c_3x + c_4.$$  \hspace{1cm} (3.3.2)

The isomorphism class of $Y$ over $K$ is determined by the point $(c_2 : c_3 : c_4)$ in the weighted projective space $\mathbb{P}(6 : 9 : 12)(K)$.

*Proof. B* y Theorem 1.18, $Y$ admits a short Weierstrass equation (1.4) over $K$, from which we can obtain (3.3.2) via a Tschirnhausen transformation.

Lemma 1.21 implies that the only possible isomorphisms between two curves $Y$ and $Y'$ of the form (3.3.2) are of the form $(x, y) \mapsto (\lambda x, \mu y)$, with $\lambda^4 = \mu^3$. Writing out the conditions for this to be an isomorphism, we end up with

$$(c_2', c_3', c_4') = (\lambda^2 \mu^{-3} c_2, \lambda \mu^{-3} c_3, \lambda \mu^{-3} c_4) = (\lambda^{-2} c_2, \lambda^{-3} c_3, \lambda^{-4} c_4).$$  \hspace{1cm} (3.3.3)

Since we need $\lambda^4 = \mu^3$, we see that $\lambda = (\mu/\lambda)^3$ has to be a third power. If we write $\nu = \mu/\lambda$, then we have

$$(c_2', c_3', c_4') = (\nu^{-6} c_2, \nu^{-9} c_3, \nu^{-12} c_4)$$  \hspace{1cm} (3.3.4)

Conversely, if such a $\nu$ exists, then we can take $\lambda = \nu^3$, $\mu = \nu^4$. □

**Remark 3.3.5.** Proposition 3.3.1 implies that we can represent isomorphism classes of nonspecial Picard curves over both $\overline{K}$ and $K$ by the unique weighted representatives of points in weighted projective space that were defined in [21, Section 1]. We have used these representatives for the nonspecial curves in our database described in Appendix A, as they give us an effective criterion for the isomorphism of a given curve with one in the database.

**Remark 3.3.6.** Let $(K, v)$ be a discretely valued field whose residue field $k$ has characteristic not equal to 2 or 3. We can then apply the Tschirnhausen transformation to both the curve
Y and its reduction modulo $v$. Proposition 3.2.1 then yields the following simple criterion for good reduction. Let $\Delta$ be the discriminant of the polynomial $x^4 + c_2x^2 + c_3x + c_4$ in (3.3.2). Then $Y$ has good reduction (respectively, potentially good reduction) if and only if the point $(c_2, c_3, c_4, \Delta)$ admits an integral representative in $\mathbb{P}(6:9:12:36)(K)$ (respectively, in $\mathbb{P}(6:9:12:36)(\overline{K})$) that reduces to a point whose final coordinate is nonzero. A complete criterion that includes the cases of residue characteristic 2 and 3 is given in [20].

**Remark 3.3.7.** Proposition 3.3.1 allows us to give a complete description of the twists of nonspecial Picard curves, which were already classified in work by Lorenzo García [12]. They are as follows.

Again let $K$ be a field of characteristic $\neq 2, 3$, and let $Y : y^3 = x^4 + c_2x^2 + c_3x + c_4$ be a nonspecial Picard curve over $K$. Generically, $\text{Aut}_K(Y_K) \simeq \mathbb{Z}/3\mathbb{Z}$, in which case the isomorphism classes of twists of $Y$ correspond bijectively to the elements of the quotient $K^*/(K^*)^3$ by assigning to $\lambda \in K^*$ the twist $Y_\lambda : y^3 = x^4 + \lambda c_2x^2 + \lambda^2c_3x + \lambda^3c_4$. (3.3.8)

We have $\text{Aut}_K(Y_K) \simeq \mathbb{Z}/6\mathbb{Z}$ if and only if $c_3 = 0$ and $c_2 \neq 0$. In this case, the twists of $Y$ correspond bijectively to $K^*/(K^*)^6$ via

$$Y_\lambda : y^3 = x^4 + \lambda c_2x^2 + \lambda^2c_4.$$ (3.3.9)

Finally, we have $\text{Aut}_K(Y_K) \simeq \mathbb{Z}/9\mathbb{Z}$ if and only if $c_2 = c_4 = 0$. In this case, the twists of $Y$ correspond bijectively to $K^*/(K^*)^9$ via

$$Y_\lambda : y^3 = x^4 + \lambda c_3x.$$ (3.3.10)

Given an integral polynomial $f$ defining a nonspecial curve $Y$ over a number field $K$, along with a finite set $S$ of primes of $K$, we can quickly determine the twists of $Y$ with good reduction outside $S$, since for this we need only consider classes represented by $\lambda$ having trivial valuation outside the primes in $S$ and the primes dividing the discriminant of $f$. Indeed, considering the twists above, an invocation of Proposition 3.2.1 shows that nontrivially twisting a Picard curve at a prime where it has good reduction yields curves that no longer have good reduction at this prime.

Twists of the standard special Picard curve and their reduction properties will be discussed in Section 4.4. Special Picard curves form a single $\overline{K}$-isomorphism class, which corresponds to the point $(0:0:1)$ in Proposition 3.3.1.

### 4. Special Picard curves

In this section, we turn our attention to special Picard curves (Definition 1.15) and extend most of our previous results for nonspecial curves to them. Moreover, we give a complete list of all special Picard curves over $\mathbb{Q}$ with good reduction outside the primes $p = 2, 3$.

The situation is more complex than for nonspecial Picard curves, due to the larger automorphism group. Special Picard curves can always be written as superelliptic curves of exponent 4, that is, they admit an equation of the form

$$x^4 = g(y),$$ (4.0.1)

where $g$ is a quartic polynomial which is $\text{PGL}_2$-equivalent, over the algebraic closure of $K$, to the cubic polynomial $y^3 + 1$. Although it is easy to write down a versal family of such polynomials, writing down the finite list of all twists with good reduction outside a finite set of primes turns out to be a surprisingly subtle problem.
Throughout, $K$ denotes a field of characteristic $\neq 2, 3$. We choose an algebraic closure $\overline{K}/K$ and set $\Gamma = \text{Gal}(\overline{K}/K)$. We also choose a primitive 12th root of unity $\zeta_{12} \in \overline{K}$ and set $\zeta_4 := \zeta_{12}^3$ and $\zeta_3 := \zeta_4^2$.

4.1. The automorphism group of $Y_0$

Recall from Lemma 1.17 that there is a unique special Picard curve $Y_0$ over $\overline{K}$. We write it as a superelliptic curve of exponent 4:

$$Y_0 : x^4 = y^3 + 1.$$  \hspace{1cm} (4.1.1)

We let $G := \text{Aut}_K(Y_0)$ denote its automorphism group. It contains the three elements $\sigma, \tau, \rho \in G$ defined as follows:

$$\sigma(x, y) = (x, \zeta_3 y), \quad \tau(x, y) = (\zeta_4 x, y), \quad \rho(x, y) = \left( \frac{\sqrt[3]{3} x}{y + 1}, \frac{-y + 2}{y + 1} \right).$$  \hspace{1cm} (4.1.2)

Here $\sqrt[3]{3}$ is chosen such that $\zeta_4 \sqrt[3]{3} = 2 \zeta_3 + 1$.

Let

$$\psi_0 : Y_0 \to \mathbb{P}^1_{\overline{K}}$$  \hspace{1cm} (4.1.3)

denote the morphism defined by the element $y$ of the function field of $Y_0$. Then $\psi_0$ is a cyclic Galois cover of degree 4, with Galois group generated by $\tau$. The branch locus is the divisor

$$D_0 = (y^3 + 1)_0 \cup \{\infty\} \subset \mathbb{P}^1_{\overline{K}}, \quad \text{so} \quad D_0(\overline{K}) = \{\infty, -1, -\zeta_3, -\zeta_3^2\},$$  \hspace{1cm} (4.1.4)

and $\psi_0$ is totally branched over each of these four points. Note that the four monodromy generators are all equal (that is, $\psi_0$ has type $(1,1,1,1)$, in the notation of Remark 1.9).

**Lemma 4.1.5.** (a) The group $G$ has order 48 and is generated by the three elements $\sigma, \tau, \rho$ defined in (4.1.2). (It has label $[48; 33]$ in GAP’s database of small groups.)

(b) The center $Z(G)$ of $G$ is cyclic of order 4, generated by $\tau$. The quotient $\overline{G} := G/Z(G)$ can be identified, via the map $\psi_0$, with the subgroup of $\text{Aut}_{\overline{K}}(\mathbb{P}^1_{\overline{K}}) = \text{PGL}_2(\overline{K})$ that fixes the branch divisor $D_0$. After labeling the points of $D_0$, the induced permutation representation of $\overline{G}$ on $D_0$ induces an isomorphism $\overline{G} \cong A_4$.

(c) There are precisely four subgroups of $G$ of order 3. They are all conjugate to the subgroup $\langle \sigma \rangle$, and each of them fixes exactly one of the four fixed points of $\tau$.

**Proof.** Statements (a) and (b) are consequences of [22, Theorem 6.1]. Statement (c) can be shown by a calculation. \hfill $\square$

The group $G$ has a natural linear action on $V := H^0(Y_0, \Omega^1_{Y_0})$. To describe this action, we choose a basis

$$\omega_1 = \frac{x \, dx}{y^2}, \quad \omega_2 = \frac{dx}{y}, \quad \omega_3 = \frac{dx}{y^2}$$  \hspace{1cm} (4.1.6)

of $V$ as $\overline{K}$-vector space. This representation decomposes into a direct sum of two irreducible subrepresentations, as follows:

$$V = V_1 \oplus V_2, \quad \text{where} \quad V_1 = \langle \omega_1 \rangle_{\overline{K}} \quad \text{and} \quad V_2 = \langle \omega_2, \omega_3 \rangle_{\overline{K}}.$$  \hspace{1cm} (4.1.7)

Indeed, the center $Z(G) = \langle \tau \rangle$ acts via a character of order 2 on $V_1$ and via a character of order 4 on $V_2$. Note that $G$ acts faithfully on $V_2$. 

The canonical embedding of $Y_0$ is described by $Y_0 \hookrightarrow \mathbb{P}(V^*)$, where $V^*$ is the vector space dual to $V$. The composition
\[
y_0 \hookrightarrow \mathbb{P}(V^*) \longrightarrow \mathbb{P}(V_2^*),
\]
with the natural projection on $\mathbb{P}(V_2^*)$ can be identified with the map $\psi_0 : Y_0 \rightarrow \mathbb{P}^1_K$ corresponding to $y$. Indeed, we have $y \equiv \omega_2/\omega_3$. In the following, we will often identify $G$ with its image under the embedding
\[
G \hookrightarrow \text{GL}(V_2^*) \simeq \text{GL}_2(K)
\]
obtained by the representation $V_2^*$. Then we obtain an embedding of two short exact sequences:
\[
\begin{array}{ccccc}
1 & \longrightarrow & Z(G) & \longrightarrow & G \\
& & \downarrow & & \\
1 & \longrightarrow & K^* & \longrightarrow & \text{GL}_2(K) & \longrightarrow & \text{PGL}_2(K) & \longrightarrow & 1
\end{array}
\]
(4.1.10)
Here the embedding $\iota : G \hookrightarrow \text{PGL}_2(K)$ comes from the action on the quotient curve $Y_0/Z(G) \cong \mathbb{P}^1_K$, see Lemma 4.1.5.(b).

Remark 4.1.11. For $g \in G$, one may compute the characteristic polynomial of its image $\iota(g) \in \text{GL}_2(K)$, either by a direct computation, or by using the character table of $G$. One finds that $\det(\iota(g)) \in \{\pm 1\}$. One checks that the index-2 subgroup $G_1 := G \cap \text{SL}_2(K)$ is isomorphic to $A_4$. Here $A_4$ is the unique nontrivial central extension of $A_4$ by $\{\pm 1\}$.

The image $\overline{\rho}$ of the element $\rho$ in $G \simeq A_4$ corresponds to a permutation with cycle type $(2, 2)$. It interchanges $-1$ with $\infty$ and $-\zeta_3$ with $-\zeta_3^2$. The four lifts $\tau^i \rho$ of $\overline{\rho}$ to $G$ have order 2 if $i \equiv 0 \pmod{2}$ and 4 if $i \equiv 1 \pmod{2}$. The lifts of order 4 are in $G_1$, and the lifts of order 2 are not.

4.2. Descent for special Picard curves

We let $Y_{0,K}$ denote the $K$-model of $Y_0$ given by equation (4.1.1). This is called the standard model of $Y_0$. There is a natural identification $Y_0 = Y_{0,K} \otimes_K K$, which induces a semilinear action of $\Gamma$ on $Y_0$. This action may be regarded as a section $s_0 : \Gamma \rightarrow \text{Aut}_K(Y_0)$ of the short exact sequence
\[
1 \rightarrow G \rightarrow \text{Aut}_K(Y_0) \rightarrow \Gamma \rightarrow 1.
\]
(4.2.1)
The section $s_0$ defines a continuous action of $\Gamma$ on $G$, by conjugation. We call it the standard action, and we will henceforth regard the group $G$ as a $\Gamma$-group in the sense of [32, §5.1], using the standard action.

Recall from Lemma 1.17 that any special Picard curve $Y$ over $K$ admits an isomorphism
\[
Y \otimes_K K \cong Y_0.
\]
(4.2.2)
The $\Gamma$-action on $Y_0$ induced by such an isomorphism corresponds to another section $s : \Gamma \rightarrow \text{Aut}_K(Y_0)$ of (4.2.1). The ‘difference’ between $s$ and $s_0$ defines a 1-cocycle
\[
(A_\gamma)_{\gamma \in \Gamma} \subset Z^1(\Gamma, G), \quad \text{where} \quad A_\gamma = s(\gamma) \circ s_0(\gamma)^{-1}.
\]
(4.2.3)
Its class in $H^1(\Gamma, G)$ only depends on the $K$-model $Y$ of $Y_0$. This correspondence yields a bijection between the set of isomorphism classes of special Picard curves over $K$ and the Galois cohomology set $H^1(\Gamma, G)$, see, for example, [35, Chapter V, §4].

As a first application of descent theory we prove that every special Picard curve can be written as a superelliptic curve of exponent 4, with some kind of normal form. A slightly different approach to this question can be found in [12, Section 5].
Theorem 4.2.4. Let $Y$ be a special Picard curve over a field $K$ of characteristic $\neq 2, 3$.

(a) The curve $Y$ admits a defining equation

$$Y : x^4 = ag(y), \quad g(y) = y^4 + by^2 + cy + d \in K[y]$$

with $a \in K^\times$ and $\Delta(g) \neq 0$.

(b) Equation (4.2.5) as in (a) defines a special Picard curve if and only if

$$b^2 + 12d = 0.$$
where \( D_0 = \{ \infty, -1, -\zeta_3, -\zeta_3^2 \} \) is the branch divisor of \( \psi_0 \) and \( C \) is the matrix from (4.2.7), considered as an element of \( \text{PGL}_2(\mathbb{P}^1_K) \).

To prove (b), we note that a superelliptic curve \( Y \) given by an equation \( x^4 = y^4 + 6by^2 + cy - 3b^2 \in K[y] \) is a twist of \( Y_0 \) if and only if there exists \( C \in \text{PGL}_2(\mathbb{K}) \) such that \( D = C(D_0) \), where \( D \) is the set of roots of \( g \). It is well known that this is the case if and only if the \( j \)-invariants of \( D \) and \( D_0 \) are equal. Here the \( j \)-invariant of a divisor \( D \) is defined as

\[
 j(D) := 256(\lambda^2 - \lambda + 1)^3 \frac{\lambda^2(\lambda - 1)^2}{\lambda^2},
\]

where \( \lambda \in \mathbb{K} \) is the cross ratio of the four \( \mathbb{K} \)-points of \( D \). Note that \( j(D_0) = 0 \). A straightforward computation shows that \( j(D) = j(D_0) \) if and only if \( b^2 + 12d = 0 \). This concludes the proof of the theorem. \( \square \)

**Definition 4.2.14.** A special polynomial is a polynomial of the form

\[
g(y) = y^4 + 6by^2 + cy - 3b^2 \in K[y] \quad \text{with} \quad \Delta(g) \neq 0. \tag{4.2.15}
\]

**Remark 4.2.16.** The proof of Theorem 4.2.4 also yields a description of the isomorphisms between special Picard curves, analogous to the description in Lemma 1.21 in the nonspecial case. In fact, we see that two special Picard curves \( Y : x^4 = ag(y) \) and \( Y' : x^4 = a'g'(y) \) are isomorphic over \( K \) if and only if there exists

\[
 A = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{GL}_2(K)
\]

and \( \epsilon \in K^\times \) such that

\[
a'g' = \epsilon^4ag(\frac{\alpha x + \beta}{\gamma x + \delta})(\gamma x + \delta)^4. \tag{4.2.18}
\]

The image of a special polynomial under an element of \( \text{PGL}_2(K) \) need not be special. However, each equivalence class is represented by a special polynomial. Identifying a special polynomial with the divisor corresponding to its roots allows us to interpret \( g \circ A \) also if \( A(\infty) \) is a root of \( g \).

**Lemma 4.2.19.** Let \( K \) be a field of characteristic \( \neq 2, 3 \). Let \( g(y) = y^4 + 6by^2 + cy - 3b^2 \in K[y] \) be a special polynomial and \( L/K \) be a Galois extension that contains the splitting field of \( g \). Then \( \gamma \in \text{Gal}(L/K) \) acts as an odd permutation on the roots of \( g \) if and only if \( \gamma(\zeta_3) = \zeta_3^2 \).

**Proof.** The statement of the lemma holds for the branch divisor \( D_0 \) of \( Y_{0,K} \).

Let \( g(y) \in K[y] \) be an arbitrary special polynomial and \( D \) its divisor of roots. Any element \( \gamma \in \text{Gal}(L/K) \) acts both on \( D \) and on \( D_0 \). The permutation representation of \( \rho \) acting on \( D \) differs from that on \( D_0 \) by \( C^{-1}\gamma(C) \) for some \( C \in \text{PGL}_2(L) \) by the proof of Theorem 4.2.4.(a). The statement of the lemma follows, since \( C^{-1}\gamma(C) \in \mathbb{C} \simeq A_4 \), by Lemma 4.1.5.(b). \( \square \)

### 4.3. Reduction of special Picard curves

In this section, we characterize the special Picard curves with good reduction to characteristic \( p \). The stable reduction of the special Picard curve \( Y_0/Q \) (1.16) has been computed in [1, Section 5.1.3]. (See also [2, Example 5.6].) This result implies that every special Picard curve \( Y \) has bad reduction to characteristic \( 2 \) over any field extension of \( K \) and has potentially good reduction to characteristic \( p \neq 2 \).

We assume that \( (K, v) \) is a discretely valued field of mixed characteristic zero. For the results of this section, it is no restriction to assume that its residue field \( k \) is algebraically closed. We
write $O_K$ for the valuation ring and $\pi$ for the uniformizer. The following proposition treats the cases of residue characteristic $p = 2, 3$. The result for $p = 3$ extends Proposition 2.3.5 to special Picard curves. The proof given here follows that from [2, Proposition 3.4].

**Proposition 4.3.1.** Let $Y$ be a special Picard curve over a discretely valued field $(K, v)$ of mixed characteristic zero.

(a) If 3 has odd valuation for $v$, then $Y$ has bad reduction at $v$.
(b) If the residue characteristic is $p = 2$, then $Y$ has bad reduction over any extension of $K$.

**Proof.** We choose equation (4.2.5) for $Y/K$. Let $L/K$ be a Galois extension such that $Y$ has good reduction over $L$. After possibly extending $L$, we may assume that $L$ contains the splitting field of $g$ and a primitive third root of unity $\zeta_3$. Let $Y$ denote the stable model of $Y_{\overline{K}}$.

Define $W = Y/\langle \tau \rangle$ and write $\overline{Y}$ and $\overline{W}$ for the special fibers of $Y$ and $W$, respectively.

The branch divisor $D$ of $\psi : Y \to Y/\langle \tau \rangle = W \cong \mathbb{P}^1_6$ extends to an étale divisor over $W$. The image $\overline{D}$ of $D$ in $\overline{W}$ therefore consists of four distinct $k$-rational points. Let $\gamma \in \text{Gal}(L/K)$ be such that $\gamma(\zeta_3) = \gamma^3_3$. Such an element exists, by our assumption on $K$. Lemma 4.2.19 implies that $\gamma$ acts nontrivially on $\overline{D}$ and hence on $\overline{Y}$. As in the proof of [2, Proposition 3.4], we conclude that there does not exist a smooth model of $Y$ over $K$.

Statement (b) follows from [1, Section 5.1.3].

**Proposition 4.3.2.** Let $Y/K$ be a special Picard curve given by an equation $x^4 = ag(y)$. Assume that the residue characteristic is different from 2. Then $Y$ has good reduction at $v$ if and only if

(a) $v(a) \equiv 0 \pmod{4}$, and
(b) the splitting field of $g$ is unramified at $v$.

Note that Proposition 4.3.1.(a) implies that the conditions in Proposition 4.3.2 are never satisfied in the case that $K/Q_3$ is unramified. We refer to Example 5.1.9 for a closer consideration of such a case of bad reduction.

**Proof.** Assume that Conditions (a) and (b) are satisfied. Since we assume that $K$ is complete with respect to $v$ and that the residue field $k$ is algebraically closed [6, Corollary 4.6] implies that there exists a stable model of $Y$ over $\mathcal{O}_K$. Since $Y$ has potentially good reduction, it has good reduction over $K$.

Assume that Condition (a) or (b) is not satisfied, and let $L/K$ be a Galois extension that contains both a 4/gcd($v(a), 4$)-th root of $a$ and the splitting field of $g$. Then $Y_{\overline{K}}$ has good reduction over $L$. It follows from [6, Section 5] that the Galois group $\text{Gal}(L/K)$ acts nontrivially on the reduction $\overline{Y}$ of $Y_{\overline{K}}$. (This is similar to the argument in the proof of Proposition 4.3.1.) We conclude that $Y$ does not have good reduction over $K$.

4.4. Good reduction outside $p = 2, 3$

We will determine all special Picard curves over $\mathbb{Q}$ with good reduction outside $p = 2, 3$ in Theorem 4.4.54. It follows from Proposition 4.3.1 that this is the smallest possible set of primes of bad reduction of a special Picard curves over $\mathbb{Q}$. The method we use is more general and can be applied over any number field using any finite set of primes. The crucial finiteness result one needs is that there are only finitely many number fields of given degree that are unramified outside a given set of primes. This result follows directly from Hermite’s Theorem (see [28, Chapter III, Theorem (2.16)]). In our situation, the relevant finite list of number fields can be obtained from the database of number fields by Jones and Roberts [16].
By Theorem 4.2.4, any special Picard curve over \( \mathbb{Q} \) is given by an equation
\[
Y : x^4 = ag(y), \quad g = y^4 + 6by^2 + cy - 3b^2, \tag{4.4.1}
\]
where \( a, b, c \in \mathbb{Z}, a \neq 0 \), and \( 0 \neq \Delta(g) = -3^3(64b^3 + c^2)^2 \). We may further assume that \( a \) has no fourth power as a nontrivial divisor. Then Proposition 4.3.2 states that \( Y \) has good reduction outside \( p = 2, 3 \) if and only if

(i) the splitting field of \( g \) is unramified outside 2, 3, and
(ii) \( a = \pm 2^\mu 3^\nu \), with \( 0 \leq \mu, \nu \leq 3 \).

We will first find all special polynomials \( g \) as in (i), up to isomorphism (Proposition 4.4.33). Most of the proof is formulated in a more general setup, which is introduced below. The main result (Theorem 4.4.54) is then a direct consequence of Proposition 4.4.33.

We return to the general assumption of this section. So \( K \) is a field of characteristic \( \neq 2, 3 \) with algebraic closure \( \overline{K} \), and \( \Gamma := \text{Gal}(\overline{K}/K) \). In addition, we fix a subfield \( L \subset \overline{K} \) which is a Galois extension of \( K \) containing the third root of unity \( \zeta_3 \). (In our final application, \( K = \mathbb{Q} \) and \( L/\mathbb{Q} \) is the maximal extension unramified outside 2, 3.)

We recall the setup from Sections 4.1 and 4.2. We denote by \( Y_0 \) the special Picard curve over \( K \) defined by (4.1.1) and \( Y_0, K \) for the \( K \)-model of \( Y_0 \) defined by the same equation. The choice of this model is determined by an action of \( \Gamma \) on \( Y_0 \), which we call the standard action.

The branch divisor of \( \psi_0 : Y_0 \to Y_0/\langle \tau \rangle \simeq \mathbb{P}^1 \) is denoted by \( D_0 \). Note that \( D_0 \) splits over \( L \). We order the points of \( D_0(\mathbb{Q}) \) as follows:
\[
\alpha_1 := \infty, \quad \alpha_2 := -1, \quad \alpha_3 := -\zeta_3, \quad \alpha_4 := -\zeta_3^2. \tag{4.4.2}
\]
This identifies \( \mathcal{G} = \text{Aut}_K(D_0) \subset \text{PGL}_2(\overline{K}) \) with \( A_4 \). The exact sequence (4.2.1) induces a diagram with exact rows, where the two lower left vertical arrows describe the actions on the set of roots:
\[
\begin{array}{cccccc}
1 & \to & G & \to & \text{Aut}_K(Y_0) & \to & \Gamma & \to & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mathcal{G} & \to & \text{Aut}_K(D_0) & \to & \Gamma & \to & 1 \\
& & \rotatebox{90}{$\simeq$} & \downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow \\
1 & \to & A_4 & \to & S_4 & \simeq & \langle \pm 1 \rangle & \to & 1 \\
\end{array} \tag{4.4.3}
\]
Lemma 4.2.19 implies that \( \chi : \Gamma \to \{ \pm 1 \} \) is the character given by
\[
\gamma(\zeta_3) = \zeta_3^{\chi(\gamma)}, \quad \text{for } \gamma \in \Gamma. \tag{4.4.4}
\]
We write \( s_0 : \Gamma \to \text{Aut}_K(Y_0) \) for the section of \( \text{Aut}_K(Y_0) \to \Gamma \) corresponding to the standard action of \( \Gamma \) on \( Y_0 \). It induces a section \( s_0 : \Gamma \to \text{Aut}_K(D_0) \).

We are interested in describing the set
\[
S := \{ D \subset \mathbb{P}^1_K \mid \text{there exists } C \in \text{PGL}_2(\overline{K}) \text{ with } D = C(D_0) \text{ such that } D \text{ splits over } L \}/ \sim,
\]
where \( \sim \) means \textit{modulo the PGL}_2(K)-action. By Theorem 4.2.4, every element of \( S \) can be represented by a divisor \( D = (g)_0 \), where \( g \in K[y] \) is a special polynomial.

Let \( D = (g)_0 \in S \) be given. Let \( Y \) be the special Picard curve over \( K \) given by \( Y : x^4 = g(y) \); it is a \( K \)-twist of \( Y_0, K \) and hence corresponds to a section \( s : \Gamma \to \text{Aut}_K(Y_0) \), up to conjugation by \( G \). Let \( \pi : \Gamma \to \text{Aut}_K(D_0) \) be the induced section. Composition of \( \pi \) with the
map \( \text{Aut}_K(D_0) \to S_4 \) from Diagram (4.4.3) yields a homomorphism \( \rho : \Gamma \to S_4 \) which satisfies the following two conditions:

\[
\text{sgn} \circ \rho = \chi, \quad \text{and} \quad \rho \text{ factors over } \text{Gal}(L/K).
\] (4.4.6)

We call \( \rho \) the homomorphism induced by \( D \). It has the following concrete interpretation, which does not involve the curve \( Y \). By definition, there exists an element \( C \in \text{Aut}_K(\mathbb{P}^1_K) = \text{PGL}_2(K) \) such that \( D = C(D_0) \). In particular, we obtain a bijection between the geometric points of \( D_0 \) and those of \( D \) and therefore, via (4.4.2), a numbering of the four geometric points of \( D \). The homomorphism \( \rho \) corresponds to the action of \( \Gamma \) on these points, with respect to this numbering. We note that the choice of \( C \) is unique up to an element of \( \overline{G} \cong A_4 \). It follows that \( \rho \) is uniquely determined by \( D \in S \), up to conjugation by an even permutation.

We reverse this construction. Let \( \rho : \Gamma \to S_4 \) be a homomorphism satisfying (4.4.6). Chasing diagram (4.4.3) shows that given \( \gamma \in \Gamma \), there is a unique lift \( \alpha \) of \( \gamma \) to \( \text{Aut}_K(D_0) \) with \( \varphi(\alpha) = \rho(\gamma) \). This implies that \( \rho \) comes from a unique section \( \tilde{\alpha} : \Gamma \to \text{Aut}_K(D_0) \). We thus obtain a cocycle

\[
(A_\gamma)_{\gamma \in \Gamma} \in H^1(\Gamma, \overline{G}), \quad A_\gamma := s(\gamma) \circ s_0(\gamma)^{-1}.
\] (4.4.7)

Let \( c(\rho) \in H^1(\Gamma, \text{PGL}_2(\overline{K})) \) denote the image of this cocycle under the natural map

\[
H^1(\Gamma, \overline{G}) \to H^1(\Gamma, \text{PGL}_2(\overline{K})).
\] (4.4.8)

We call \( c(\rho) \) the obstruction class of \( \rho \). Recall also that the short exact sequence

\[
1 \to \overline{K}^\times \to \text{GL}_2(\overline{K}) \to \text{PGL}_2(\overline{K}) \to 1
\] (4.4.9)
gives rise to an injection

\[
1 = H^1(\Gamma, \text{GL}_2(\overline{K})) \to H^1(\Gamma, \text{PGL}_2(\overline{K})) \to H^2(\Gamma, \overline{K}^\times).
\] (4.4.10)

The equality follows from Hilbert 90 [33, §X.1, Proposition 3]. We therefore may consider the obstruction class \( c(\rho) \) as element of the Brauer group \( \text{Br}(K) = H^2(\Gamma, \overline{K}^\times) \) of the field \( K \). A calculation as in Remark 4.1.11 shows that \( c(\rho) \) even lies in \( \text{Br}(K)[2] = H^2(\Gamma, \{\pm 1\}) \).

The following proposition states that a homomorphism \( \rho \) is induced from an element \( D \in S \) if and only if the obstruction class \( c(\rho) \) vanishes. In what follows, we identify \( \rho \) with its equivalence class under conjugacy with \( A_4 \). The theory of Weil descent then implies the following.

**Proposition 4.4.11.** There is a bijection between the set \( S \) and the set of homomorphisms \( \rho : \Gamma \to S_4 \) satisfying (4.4.6) and such that \( c(\rho) \) is trivial, modulo the action of \( A_4 \) by conjugation. This bijection depends only on the choice of the element \( D_0 \in S \).

Our next goal is to make the obstruction class \( c(\rho) \) more explicit. For this we need some preparation. Recall from [34, §1.5] that there exists a unique central extension

\[
1 \to \{\pm 1\} \to \tilde{S}_4 \to S_4 \to 1
\] (4.4.12)
of \( S_4 \) with the property that transpositions lift to elements of order 2 in \( \tilde{S}_4 \) and \( (2,2) \)-cycles lift to elements of order 4. The extension (4.4.12) corresponds to a certain class

\[
s_4 \in H^2(S_4, \{\pm 1\}).
\] (4.4.13)

For a homomorphism \( \rho : \Gamma \to S_4 \), we denote by \( \rho^* s_4 \) the element of \( H^2(\Gamma, \{\pm 1\}) \) obtained via restriction along \( \rho \). The short exact sequence

\[
1 \to \{\pm 1\} \to \overline{K}^\times \to \overline{K}^\times \to 1
\] (4.4.14)
induces an injection

\[ 1 = H^1(\Gamma, \mathbb{K}^\times) \rightarrow H^2(\Gamma, \{\pm 1\}) \rightarrow H^2(\Gamma, \mathbb{K}^\times). \tag{4.4.15} \]

So for a given homomorphism \( \rho : \Gamma \rightarrow S_4 \), we may consider both \( \rho^* s_4 \) and the obstruction class \( c(\rho) \) as elements of \( \text{Br}(K) \).

**Lemma 4.4.16.** Let \( \rho : \Gamma \rightarrow S_4 \) be a homomorphism satisfying (4.4.6), and let \( c(\rho) \in \text{Br}(K) \) be the corresponding obstruction class. Then

\[ c(\rho) = \rho^* s_4. \tag{4.4.17} \]

**Proof.** Let \( \tilde{A}_4 \subset \tilde{S}_4 \) denote the inverse image of \( A_4 \subset S_4 \). These groups fit into the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \{\pm 1\} & \rightarrow & \tilde{A}_4 & \rightarrow & A_4 & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \{\pm 1\} & \rightarrow & \tilde{S}_4 & \rightarrow & S_4 & \rightarrow & 1 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\{\pm 1\} & \overset{\pi}{\rightarrow} & \{\pm 1\} & & & & & & 1 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
1 & & & & 1 & & & & 1
\end{array}
\tag{4.4.18}
\]

Considering the groups in (4.4.18) as \( \Gamma \)-groups with respect to the trivial action of \( \Gamma \), we obtain a commutative diagram of nonabelian cohomology sets:

\[
\begin{array}{ccc}
H^1(\Gamma, \tilde{A}_4) & \rightarrow & H^1(\Gamma, A_4) \\
\downarrow & & \downarrow \\
H^1(\Gamma, \tilde{S}_4) & \rightarrow & H^1(\Gamma, S_4) \\
\downarrow & & \downarrow \\
H^1(\Gamma, \{\pm 1\}) & \overset{\Delta}{\rightarrow} & H^1(\Gamma, \{\pm 1\})
\end{array}
\tag{4.4.19}
\]

The rows and columns of (4.4.19) are exact as sequences of pointed sets. We remark that an element of \( H^1(\Gamma, H) \) for a group \( H \) with trivial \( \Gamma \)-action is simply a homomorphism \( \Gamma \rightarrow H \). In particular, a homomorphism \( \rho : \Gamma \rightarrow S_4 \) satisfying (4.4.6) is simply an element of the fiber \( \pi^{-1}(\chi) \subset H^1(\Gamma, S_4) \). The definitions of the maps in [32, Chapter I, §5] imply

\[ \Delta(\rho) = \rho^* s_4. \tag{4.4.20} \]

We have \( \Delta(\rho_0) = \rho_0^* s_4 = 1 \), where \( \rho_0 : \Gamma \rightarrow S_4 \) corresponds to the standard divisor \( D_0 \in \mathcal{S} \) defined in (4.1.4). Indeed, the image of \( \rho_0 \) is trivial if \( \zeta_3 \in K \) and generated by a transposition otherwise, and the extension \( \tilde{S}_4 \rightarrow S_4 \) has the property that transpositions lift to elements of order 2 in \( \tilde{S}_4 \).

Let \((\tilde{A}_4)_{\rho_0}\) and \((A_4)_{\rho_0}\) be twisted \( \Gamma \)-groups, as defined in [32, Chapter I, §5.3]. The theory in loc. cit. shows that the fiber \( \pi^{-1}(\chi) \subset H^1(\Gamma, S_4) \) to which \( \rho \) and \( \rho_0 \) belong is in
canonical bijection with \( H^1(\Gamma, (A_4)_{\rho_0}) \). Moreover, it follows from Diagram (4.4.18) that we have isomorphisms of \( \Gamma \)-groups

\[
G_1 \cong (\tilde{A}_4)_{\rho_0}, \quad \mathcal{G} \cong (A_4)_{\rho_0},
\]

(4.4.21)

compatible with the natural maps \( G_1 \to \mathcal{G} \) and \( \tilde{A}_4 \to A_4 \). From this, we obtain canonical bijections

\[
\pi^{-1}(\chi) = H^1(\Gamma, (A_4)_{\rho_0}) = H^1(\Gamma, \mathcal{G}), \quad \tilde{\pi}^{-1}(\chi) = H^1(\Gamma, (\tilde{A}_4)_{\rho_0}) = H^1(\Gamma, G_1).
\]

(4.4.22)

Since \( \Delta(\rho_0) = 0 \), [32, Proposition 44] implies that the restriction of the map \( \Delta \) to \( \pi^{-1}(\chi) = H^1(\Gamma, \mathcal{G}) \) equals the map

\[
H^1(\Gamma, \mathcal{G}) \to H^2(\Gamma, \{\pm 1\})
\]

(4.4.23)

induced from the sequence \( 1 \to G_1 \to \mathcal{G} \to 1 \).

Equation (4.1.10) together with Remark 4.1.11 implies that the map \( G_1 \to \mathcal{G} \) is compatible with \( \text{GL}_2(K) \to \text{PGL}_2(K) \). It follows that the image of \( \rho \) under the map

\[
\pi^{-1}(\chi) = H^1(\Gamma, \mathcal{G}) \xrightarrow{\Delta} H^2(\Gamma, \{\pm 1\})
\]

(4.4.24)

is equal to the obstruction class \( c(\rho) \). Together with (4.4.20), this proves the lemma. \( \square \)

**Example 4.4.25.** Let \( K \) be a field not containing \( \zeta_3 \). We discuss the case that the image of \( \rho: \Gamma \to S_4 \) is an elementary abelian group with four elements that is not contained in \( A_4 \) (the biquadratic case). Up to conjugation by an element of \( A_4 \) we may assume that the image is

\[
\rho(\Gamma) = \langle (1 2), (3 4) \rangle.
\]

(4.4.26)

By Galois theory, we obtain a pair \( (M_1, M_2) \) of quadratic subextensions of \( L/K \), namely \( M_1 = M^\langle (3 4) \rangle \) and \( M_2 = M^\langle (1 2) \rangle \). By Kummer theory, we can write \( M_i = K[\sqrt{d_i}] \) for some \( d_i \in K^\times \setminus (K^\times)^2 \). Condition (4.4.6) implies that the subextension of \( L \) corresponding to \( (1 2)(3 4) \) is \( K(\zeta_3) \). This means that

\[
d_1d_2 \equiv -3 \pmod{(K^\times)^2}.
\]

(4.4.27)

We now calculate the descent obstruction \( c(\rho) \). While this can be done explicitly, we instead use Lemma 4.4.16 and calculate \( \rho^*s_4 \) instead. It follows from the proof of [34, Lemma 2, p. 661] that

\[
\rho^*s_4 = (d_1, d_2) \in \text{Br}(K).
\]

(4.4.28)

Namely, the displayed formula right above (18) in loc. cit. states that \( \rho^*s_4 = \rho_1^*s_2 + \rho_2^*s_2 + (d_1, d_2) \). Here \( \rho_i: \Gamma \to S_2 \) is the homomorphism corresponding to \( M_i/K \) and \( s_2 \in H^2(\Gamma, \{\pm 1\}) \) is defined in [34, § 1.5]. In the middle of p. 654 in loc. cit. it is shown that \( s_2 = 0 \). This implies (4.4.28).

Therefore, Proposition 4.4.11 states that \( \rho \) is induced by a divisor \( D \in \mathcal{S} \) if and only if the quadratic form

\[
d_1x^2 + d_2y^2 - z^2 = 0
\]

(4.4.29)

has a nontrivial zero in \( K \).

Once again let \( D \in \mathcal{S} \). After replacing \( D \) by an equivalent one, we may write as \( D = (g)_0 \) for a special polynomial \( g \). The group \( H_D := \text{Aut}_K(D) \subset \text{PGL}_2(K) \) is conjugate to \( \mathcal{G} = \text{Aut}_K(D_0) \) in \( \text{PGL}_2(K) \). The stabilizer in \( \text{Aut}_K(D) \) of each root \( \alpha \) of \( g \) is a cyclic group of order 3. Denote
by $\beta = \beta(\alpha) \in \mathbb{P}^1_K$ the unique point different from $\alpha$ with the same stabilizer. We obtain a divisor $D'$ as the sum of the $\beta$ for $\alpha$ running through the roots of $g$. If no $\beta$ is equal to $\infty$, then

$$D' = (g')_0, \quad \text{with } g' = \prod_{\beta}(y - \beta) \in K[y]$$

(4.4.30)

Otherwise, $D' = (g')_0 + \infty$, for a unique monic cubic polynomial $g' \in K[y]$. In either case, we call the polynomial $g'$ the shadow of $g$.

**Lemma 4.4.31.** Let $g \in K[x]$ be a special polynomial with shadow $g'$, and let $\rho : \Gamma \to S_4$ (respectively, $\rho'$) be the homomorphism corresponding to $D = (g)_0$ (respectively, $D' = (g')_0$). Then $\rho'$ can be obtained by composing $\rho$ with an inner automorphism of $S_4$ given by conjugation with an odd element of $S_4$.

In particular, given an extension $M$ of $K$, there are at most two nonequivalent special polynomials over $K$ whose splitting field equals $M$.

**Proof.** The divisor $D' = (g')_0$ is stabilized by $H_D$ by construction. Let $N_D$ be the normalizer of $H_D$ in $\text{PGL}_2(K)$. Then $N_D$ is isomorphic to $S_4$. Let $B$ be an element of $N_D \setminus H_D$. Then $B$ flips the two divisors $D$ and $D'$. In particular, the divisor $D \cup D'$ has automorphism group $N_D \cong S_4$ over $\overline{K}$.

Note that it suffices to check the claims above for the divisor $D_0$, where

$$D'_0 = (y^4 - 8y)_0 = \{0, 2, 2\zeta_3, 2\zeta_3^2\} \quad \text{and } B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$ 

(4.4.32)

The statement of the lemma follows from the above, the construction of the homomorphism $\rho$ induced by $D$, and the fact that all automorphisms of $S_4$ are inner.

**Proposition 4.4.33.** There are exactly 26 nonequivalent special polynomials over $\mathbb{Q}$ with good reduction away from 2,3:

$$x^4 + x,$$ 

(4.4.34)

$$x^4 + 2x, \quad x^3 - 2,$$ 

(4.4.35)

$$x^4 + 3x, \quad x^3 - 3,$$ 

(4.4.36)

$$x^4 + 6x, \quad x^3 - 6,$$ 

(4.4.37)

$$x^4 + 12x, \quad x^3 - 12,$$ 

(4.4.38)

$$x^4 - 6x^2 - 3, \quad x^4 + 6x^2 - 3,$$ 

(4.4.39)

$$x^4 - 12x^2 - 12, \quad x^4 + 12x^2 - 12,$$ 

(4.4.40)

$$x^4 + 6x^2 + 8x - 3, \quad x^4 + 4x^3 - 6x^2 - 4x - 7,$$ 

(4.4.41)

$$x^4 - 24x^2 + 32x - 48, \quad x^4 - 4x^3 + 24x^2 - 16x - 32,$$ 

(4.4.42)

$$x^4 + 12x^2 - 8x - 12, \quad x^4 - 2x^3 - 12x^2 + 4x - 14,$$ 

(4.4.43)

$$x^4 - 36x^2 + 96x - 108, \quad x^4 - 8x^3 + 36x^2 - 48x - 12,$$ 

(4.4.44)

$$x^4 + 12x^2 + 64x - 12, \quad x^4 + 16x^3 - 12x^2 - 32x - 140,$$ 

(4.4.45)

$$x^4 + 12x^2 - 16x - 12, \quad x^4 - 4x^3 - 12x^2 + 8x - 20,$$ 

(4.4.46)

$$x^4 - 12x^2 + 32x - 12.$$ 

(4.4.47)

The polynomial on the right is equivalent to the shadow of the polynomial on the left.
Proof. We apply Proposition 4.4.11 to the case where $K = \mathbb{Q}$ and $L/\mathbb{Q}$ is the maximal algebraic extension unramified outside 2,3. We start by listing the corresponding classes of homomorphisms $\rho: \Gamma \to S_4$ satisfying (4.4.6) up to conjugation by $S_4$. Such a homomorphism corresponds to a tuple $(M_1, \ldots, M_r)$ of finite extensions $M_i/\mathbb{Q}$ that are unramified away from 2,3 and such that $\sum_i[M_i : \mathbb{Q}] = 4$.

Equation (4.4.4) implies that Condition (4.4.6) is equivalent to

$$\prod_i d(M_i) \equiv -3 \pmod{(\mathbb{Q}^\times)^2},$$

(4.4.48)

where $d(M)$ denotes the discriminant of a field $M$. In this setup, the Galois group of the composite extension $M := M_1 \cdots M_r/\mathbb{Q}$ is identified with a subgroup of $S_4$. Condition (4.4.48) implies that the field of invariants of $M$ under $\text{Gal}(M/\mathbb{Q}) \cap A_4$ is $\mathbb{Q}(\zeta_3)$.

Of course, we may ignore the occurrences of $M_i = \mathbb{Q}$. Ordering the subfields $M_i$ by their degrees, we either obtain one field $M/\mathbb{Q}$ of degree 2,3,4, or a pair $(M_1, M_2)$ of quadratic number fields.

A consultation of the Database of Number Fields [16] shows that there are precisely eleven possibilities in the former case of a single number field $M$. More precisely, there is exactly one quadratic extension $M/\mathbb{Q}$ unramified outside 2,3 such that $d(M) \equiv -3$, namely

$$M = K[\zeta_3] \quad \text{ (with generating polynomial } x^2 + x + 1).$$

(4.4.49)

There are precisely four cubic number fields with this property, given by the generating polynomials

$$x^3 - 2, \quad x^3 - 3, \quad x^3 - 6, \quad x^3 - 12,$$

(4.4.50)

and finally there are eight quartic fields, described by

$$x^4 - 6x^2 - 3, \quad x^4 - 12x^2 - 12, \quad x^4 + 6x^2 + 8x - 3, \quad x^4 - 24x^2 + 32x - 48,$$

$$x^4 + 12x^2 - 8x - 12, \quad x^4 - 36x^2 + 96x - 108, \quad x^4 + 12x^2 + 64x - 12,$$

$$x^4 + 12x^2 - 16x - 12.$$  

(4.4.51)

Lemma 4.4.31 implies that each of these options corresponds to at most two special polynomials defining elements of $S$. If there are two, then one is the shadow of the other and conversely. A simple search yields one pair $(g, g')$ for each of the possibilities in (4.4.49)–(4.4.51). These polynomials are listed as (4.4.34)–(4.4.46) above. As can be verified using the methods in [23], the polynomial in (4.4.34) is the only polynomial from this list that is $\text{PGL}_2(K)$-equivalent to its own shadow. It follows from Proposition 4.4.11 and Lemma 4.4.31 that every element of the set $S$ that is either irreducible or has at least one rational point is $\text{PGL}_2(K)$-equivalent to a set of roots of a polynomial in this list.

In the latter case of a pair of quadratic number fields $(M_1, M_2)$, the possibilities are

$$(x^2 + 1, x^2 - 3), \quad (x^2 - 2, x^2 + 6), \quad (x^2 + 2, x^2 - 6).$$

(4.4.52)

In the notation of Example 4.4.25, these pairs correspond to the Hilbert symbols

$$(d_1, d_2) \in \{(-1, 3), (2, -6), (-2, 6)\}.$$  

(4.4.53)

A simple calculation with Hilbert symbols shows that the elements $(-1, 3), (2, -6) \in \text{Br}(\mathbb{Q})$ are nontrivial, whereas $(-2, 6) = 0$ in $\text{Br}(\mathbb{Q})$. We conclude that only the pair $(-2, 6)$ can occur. Again a search yields the polynomial $g$ in line (4.4.47) with splitting field $\mathbb{Q}(\sqrt{-2}, \sqrt{6})$. The shadow polynomial $g'$ is equivalent to $g$, and is therefore not listed. Now our list is complete. □
Theorem 4.4.54. Up to isomorphism, there are 800 nonisomorphic special Picard curves over \( \mathbb{Q} \) with good reduction outside \( p = 2, 3 \).

Proof. Let \( Y \) be a special Picard curve. The discussion above shows that \( Y \) admits an equation

\[
Y : x^4 = ag(y),
\]

where \( g \) is one of the polynomials in Proposition 4.4.33 and where \( a = \pm 2^\mu 3^\nu \), with \( 0 \leq \mu, \nu \leq 3 \). By the same proposition, the polynomial \( g \) is uniquely determined by \( Y \).

It remains to see, given such a special polynomial \( g \), when \( Y \) is equivalent to another curve of the form \( Y' : x^4 = a'g(y) \) for \( a = \pm 2^\mu 3^\nu \). For this to happen with \( \mu \neq \mu' \) or \( \nu \neq \nu' \), the class of \( g \) needs to admit nontrivial \( K \)-automorphisms, as described in Remark 4.2.16. Using [23], there turns out to be a single nontrivial automorphism for the polynomials (4.4.34), (4.4.39), (4.4.40), and (4.4.47). In the cases (4.4.39) and (4.4.40), this automorphism scales the polynomial \( g \) itself by a fourth power, so that we still cannot have \( Y \cong Y' \) unless \( a = a' \). However, in the cases (4.4.34) and (4.4.47), the nontrivial automorphism multiplies \( g \) by a factor in \( 9(\mathbb{Q}^*)^4 \). Therefore, \( Y \) and \( Y' \) are isomorphic if and only if the quotient of \( a \) and \( a' \) is in either \( (\mathbb{Q}^*)^4 \) or \( 9(\mathbb{Q}^*)^4 \).

This means that in 24 of the 26 cases of Proposition 4.4.33, we get 32 distinct curves for the values \( a = \pm 2^\mu 3^\nu \), whereas in two cases, we obtain only sixteen distinct curves. All in all we obtain \( 24 \cdot 32 + 2 \cdot 16 = 800 \) twists. \( \square \)

4.5. Discriminant minimization

In this section, we briefly mention discriminant minimization of equations of special Picard curves over a field \( K \), as a variant of the considerations in Section 2.2.

Definition 4.5.1. Let \( Y \) be a special Picard curve over \( K \). We call an equation for \( Y \) of the form

\[
Y : bx^4 = c_0y^4 + c_1y^3 + c_2y^2 + c_3y + c_4
\]

a short Weierstrass equation for \( Y \).

Remark 4.5.3. The discriminant of the binary form corresponding \( F = bx^4 - (c_0y^4 + c_1y^3 + c_2y^2 + c_3y + c_4) \) corresponding to equation (4.5.2) is

\[
\Delta(F) = -2^{16}b^9\Delta^3(f),
\]

where \( \Delta(f) \) is the discriminant of the univariate polynomial \( f = c_0y^4 + c_1y^3 + c_2y^2 + c_3y + c_4 \).

By Theorem 4.2.4, every special Picard curve over \( K \) admits a short Weierstrass equation. Conversely, by part (b) of the same theorem, a nonsingular curve over \( K \) with defining equation (4.5.2) is a special Picard curve if and only if the binary quartic invariant

\[
I = 12c_0c_4 - 3c_1c_3 + c_2^2
\]

vanishes [7].

Remark 4.5.6. A given plane quartic curve \( Y \) over \( K \) is a special Picard curve if and only if its Dixmier–Ohno invariants coincide with those of the standard special Picard curve (1.16). This yields an effective algorithm to decide whether \( Y \) is special. To find an equation of the form (4.5.2) for \( Y \), one calculates the automorphism group of \( Y \) and splits the ambient projective plane, or more canonically \( \mathbb{P}^2\mathcal{H}^0(Y, \Omega_Y) \), into the eigenspaces (4.1.7) to obtain coordinates \( x \) and \( y, z \).
As mentioned in Remark 4.2.16, given two equations of special Picard curves $Y_i : x^4 = g_i(y)$ over $K$, finding isomorphisms between $Y_1$ and $Y_2$ reduces to determining equivalences of binary forms (up to scalars) between $g_1$ and $g_2$. For this question, too, effective algorithms exist [23].

To obtain long Weierstrass equations, we have to work slightly harder this time around. First we define the appropriate notion.

**Lemma 4.5.7.** Let $Y$ be a plane curve over $K$ defined by a nonsingular equation of the form

$$Y : a_0 x^4 + a_1(z)x^3 + a_2(z)x^2 + a_3(z)x = a_4(y, z),$$

(4.5.8)

where $a_i$ is a homogeneous form of degree $i$ in $y$ and $z$. Suppose that $Y$ is obtained from an equation of the form (4.5.2). Then the equations

$$8a_0a_2 = 3a_1^2,$$

(4.5.9)

$$16a_2^2a_3 = a_1^3$$

(4.5.10)

are satisfied.

**Proof.** This follows from a direct calculation. □

**Definition 4.5.11.** Given a special Picard curve $Y$ over $K$, we call an equation for $Y$ of the form (4.5.8) a long Weierstrass equation for $Y$.

**Proposition 4.5.12.** Let $Y$ be a special Picard curve over a discretely valued field $(K, v)$. Then $Y$ admits an integral long Weierstrass equation of minimal discriminant exponent.

**Proof.** Consider an integral short Weierstrass equation (4.5.2) for $Y$ in $\mathbb{P}^2(x : y : z)$. Let $F = bx^4 - c_0y^4 - c_1y^3z - c_2y^2z^2 - c_3yz^3 - c_4z^4$ the corresponding ternary form. We use the basis $x, y, z$ to identify $K^3$ with $Kx \oplus Ky \oplus Kz$. Let $F_0 \in O_K[x, y, z]$ be an integral form of minimal discriminant defining a curve that is $K$-isomorphic to $Y$, and let $T_0 \in M_3(O_K)$ be such that up to a scalar $F \cdot T_0 = F_0$ under the right action of $M_3(O_K)$.

**Claim 1.** There exists a matrix $T_1 \in \text{GL}_3(O_K)$ such that $T_0T_1$ maps the subspace $Ky \oplus Kz$ to itself.

**Proof.** Consider the intersection $M = (V_2 \cdot T_0) \cap O_K^3$. Then $M$ is a torsion-free $O_K$-submodule of $O_K^3$ of rank 2. Because $O_K$ is a principal ideal domain, we can find a complementary submodule of $M$ inside $O_K^3$. On the level of matrices, this comes down to saying that there exists $U_1 \in \text{GL}_3(O_K)$ whose second and third rows generate $M$. We can then take $T_1 = U_1^{-1}$ to prove the claim.

The matrix $T_0T_1$ now has the form

$$T_0T_1 = \begin{pmatrix} * & b & a \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

(4.5.13)

**Claim 2.** There exists a matrix $T_2 \in \text{GL}_3(O_K)$ such that $T_0T_1T_2$ is of the form (4.5.13) with $b = 0$.

**Proof.** If $b = 0$, we are done, and if $a = 0$, then we can take $T_2$ to be the matrix corresponding to the transformation sending $(x, y, z)$ to $(x, z, y)$. For the same reason, we
may otherwise suppose that \( v(b) > v(a) \). But in that case we can take \( T_2 \) to correspond to \((x, y, z) \mapsto (x, y, (-b/a)y + z)\). The claim is proved.

To conclude, let \( U = T_0 T_1 T_2 \). Since \( T_1 \) and \( T_2 \) are in \( \text{GL}_3(\mathcal{O}_K) \), the matrix \( U \) still has the property that a scalar multiple \( G \) of \( F \cdot U \) has minimal discriminant. Because of the form of \( U \), the ternary quartic \( G \) yields equation (4.5.8). □

**Remark 4.5.14.** As in Remark 2.2.2, the same considerations apply globally over a number field with trivial class group: the step in Claim 2 can then be replaced by repeated division with remainder.

**Theorem 4.5.15.** Let \( Y \) be special Picard curve over a discretely valued field \((K, v)\) whose residue characteristic does not equal 2. Then \( Y \) admits an integral short Weierstrass equation (4.5.2) of minimal discriminant exponent.

**Proof.** Extend \( v \) to the Gauss valuation for ternary polynomials, and consider an integral equation (4.5.8) of \( Y \) whose discriminant is minimal. First suppose \( v(a_0) = 0 \). Then the integral coordinate change \( y \mapsto y - (a_1/(4a_0)) \) gives an equation of the form (1.11), and we are done. If \( v(a_0) = 1 \), then \( v(a_1) \geq 1 \) by (4.5.10), and the same argument applies. If \( v(a_0) = 2 \), then similarly \( v(a_1) \geq 2 \) by (4.5.10).

Finally, suppose \( v(a_0) \geq 3 \). Then \( v(a_1) \geq 2 \) by (4.5.10), which also yields \( v(a_0) \leq (3/2)v(a_1) \). If \( v(a_2) = 0 \), then (4.5.9) yields \( v(a_0) \geq 2v(a_1) \), a contradiction since \( v(a_1) > 0 \). This implies \( v(a_2) \geq 1 \). Knowing this, we can apply the same argument as at the end of the proof of Theorem 2.2.12. □

**Example 4.5.16.** Let \( Y_0 \) be the standard special Picard curve defined by (1.16). This model has discriminant \( 2^{16}3^9 \) and is minimal except for \( p = 2 \). A long Weierstrass equation (4.5.8) which is minimal for all \( p \) is
\[
2x^4 - 4x^3 + 3x^2 - x = y^3,
\]
which one also recognizes as a short Weierstrass equation of the previously considered form (1.11). It has discriminant \( 2^73^9 \).

5. **Comparing the conductor and the discriminant**

In this section, we strengthen some results from [2] on the exponent of \( p \) in the conductor of a Picard curve. Our main result Theorem 5.1.16 states that the standard special Picard curve \( Y_{0,\mathbb{Q}} \) has the smallest conductor among all special Picard curves defined over \( \mathbb{Q} \). Our method is based on the results of [6] on computing the conductor of a superelliptic curve via stable reduction. The results of this section illustrate that this method allows us to analyze the effect of twisting the curve on the conductor. In Section 5.2, we discuss the question of comparing the conductor with the minimal discriminant of a superelliptic curve.

5.1. **Calculating the conductor via stable reduction**

We recall from [6, Section 2] some facts on the conductor of a curve over a number field and its relation to stable reduction. More details and references to the literature can be found there.

Let \( K \) be a number field and \( Y/K \) a Picard curve. The conductor of \( Y/K \) is an ideal
\[
c = \prod_p p^{f_p},
\]
(5.1.1)
where the product runs over the prime ideals of $\mathcal{O}_K$. The conductor exponent $f_p$ is trivial if $Y$ has good reduction at $p$.

Denote by $K_{\text{nr}}^u$ the maximal unramified extension of the completion of $K$ at $p$. The conductor exponent $f_p$ measures the ramification of the representation of $I_p := \text{Gal}(\overline{K}_p/K_{\text{nr}}^u)$ acting on the étale cohomology group $H^1_{\text{et}}(Y_{\overline{K}_K}, \mathbb{Q}_\ell)$ for some auxiliary prime $\ell$ different from the characteristic $p$ of $k = \mathcal{O}_K/p$. In the rest of this section, we consider $Y$ as a curve over $K_{\text{nr}}^u$. We drop $p$ from the notation and write $K$ instead of $K_{\text{nr}}^u$.

Let $L/K$ be a Galois extension such that $Y_L$ has stable reduction over $L$ and let $Y$ be the stable model of $Y_L$ over $\text{Spec}(\mathcal{O}_L)$. Define $\Gamma := \text{Gal}(L/K)$. For $u \geq 0$, we denote by $\Gamma^u$ the higher ramification groups in the upper numbering. Since we assume that $k$ is algebraically closed, we have that $\Gamma = \Gamma^0$, and the reduction $\overline{Y}$ is a stable curve over $k$ on which the arithmetic Galois group $\Gamma$ acts $k$-linearly.

The quotient curve $\overline{Y}^u := \overline{Y}/\Gamma^u$ is again semistable. The following proposition is [2, Proposition 2.3] and follows from [5, Corollary 2.14]; [6, Theorem 2.9].

**Proposition 5.1.2.** The conductor exponent of the curve $Y/K$ is given by

$$f_p = \epsilon + \delta,$$

where

$$\epsilon := 6 - \dim H^1_{\text{et}}(\overline{Y}^0, \mathbb{Q}_\ell)$$

and

$$\delta := \int_0^{\infty} \left(6 - 2g(\overline{Y}^u)\right) du.$$  

In particular, $\delta = 0$ if and only if $Y$ acquires stable reduction over a tamely ramified extension.

We say that the stable reduction is of *compact type* if its dual graph is a tree. This is equivalent to the Jacobian of $Y$ having potentially good reduction. In the case that $\overline{Y}^u$ is a semistable curve of compact type, we have that $\dim H^1_{\text{et}}(\overline{Y}^u, \mathbb{Q}_\ell) = 2g(\overline{Y}^u)$, and $g(\overline{Y}^u)$ is the sum of the genera of the irreducible components of the normalization of $\overline{Y}^u$. In the general case, one needs to add the number of loops of the dual graph of $\overline{Y}^u$. We refer to [2, Section 2.2] for a precise formula.

**Remark 5.1.6.** Let $Y/Q_{3}^{\text{nr}}$ be a Picard curve given as superelliptic curve of exponent 3 (1.4). In [2, Theorem 3.2], the possibilities for the stable reduction are determined and [2, Theorem 3.6] yields a lower bound on $f_3$ for each of the possibilities. The statement is

(i) $f_3 \geq 6$ if $Y$ has potentially good reduction (Case (a));
(ii) $f_3 \geq 4$ if $Y$ does not have potentially good reduction, and $\overline{Y}$ is of compact type (Cases (b) and (c));
(iii) $f_3 \geq 5$ if the stable reduction of $\overline{Y}$ has loops (Cases (d) and (e)).

Here the cases are as in [2, Theorem 3.2]. All lower bounds are attained. One may check that these lower bounds also apply for special Picard curves defined by a superelliptic equation of exponent 3 (1.4) over $Q_{3}^{\text{nr}}$.

In [2, Theorem 3.6.(a)] it is mistakenly claimed that if $f_3 \leq 6$, then $Y$ achieves stable reduction over a tamely ramified extension. A counterexample is given by the curve

$$Y : y^3 = x^4 + x^3 + 3^2x^2 + 3^5x.$$ (5.1.7)

For this curve we find $\epsilon = 4$ and $\delta = 1$, and hence $f_3 = 4 + 1 = 5$. The stable reduction of $Y$ is in Case (b) of [2, Theorem 3.2].
This example illustrates that the statement of [2, Theorem 3.6.(c)] needs to be modified, as well. The correct statement is that if \( \delta = 5 \), then the stable reduction at \( p = 3 \) contains loops (Cases (d) or (e)). If \( f_3 = 5 \), then \( (\epsilon, \delta) \in \{(5,0),(4,1)\} \), and both possibilities occur.

In the rest of this section, we consider special Picard curves. We start by treating the case of residue characteristic \( p = 3 \).

**Proposition 5.1.8.** Let \( Y/K := \mathbb{Q}_3^{nr} \) be a special Picard curve given by an equation \( x^4 = ag(y) \) as in (4.2.5). We have that \( \epsilon \in \{4,6\} \) and \( f_3 \geq 4 \).

**Proof.** Let \( L/K \) be a Galois extension such that \( Y \) admits a stable model over \( \mathcal{O}_L \). Assume that \( \zeta_3 \in L \). Let \( Y \rightarrow \text{Spec}(\mathcal{O}_L) \) be the stable model of \( Y_L \) and \( Y \) its special fiber. As in Section 4.3, we write \( \mathcal{W} = Y/\langle \gamma \rangle \) and \( \mathcal{W} \) for the special fiber of \( \mathcal{W} \). Since \( Y \) has potentially good reduction by [1, Section 5.1.3], \( \mathcal{Y} \) and \( \mathcal{W} \) are smooth curves of genus 3 and 0, respectively.

We write \( \Gamma = \text{Gal}(L/K) \) and \( \mathcal{Y}^0 = Y/\Gamma \). Proposition 5.1.2 implies \( f_3 \geq \epsilon \geq 6 - 2g(\mathcal{Y}^0) \). We claim that \( g(\mathcal{Y}^0) \leq 1 \). This implies \( \epsilon \in \{4,6\} \).

Since \( \zeta_3 \notin K \), Lemma 4.2.19 implies that there exists an element \( \gamma \in \Gamma \) that acts on \( \mathcal{D} \) as an odd permutation. One calculates the genus of the quotient \( Y/\langle \gamma \rangle \) for all possibilities for the action of \( \gamma \) on \( Y \). Since \( \gamma \) acts as an element of \( \text{Aut}_k(Y) \), this may be done on the reduction \( Y_0 \) of the standard special Picard curve \( Y_0 \) (1.16). It follows that the only possibility for \( g(\mathcal{Y}/\langle \gamma \rangle) \) to be positive is that \( \gamma \) acts as a 4-cycle on \( \mathcal{D} \) and fixed-point free on \( \mathcal{Y} \), but \( \gamma^2 \) fixes exactly four points. In this case, \( g(\mathcal{Y}/\langle \gamma \rangle) = 1 \) and hence \( g(\mathcal{Y}^0) \leq 1 \). In all other cases, we have that \( g(\mathcal{Y}^0) = 0 \). The statement of the proposition follows. \( \square \)

The following example shows that the lower bound in Proposition 5.1.8 is sharp.

**Example 5.1.9.** We consider the special Picard curve over \( \mathbb{Q} \) defined by

\[ Y_a : x^4 = ag(y) \]  
with \( g(y) = y^4 + 6by^2 + 3cy - 3b^2 \) such that \( 3 \nmid b \) and \( a \neq 0 \). \( (5.1.10) \)

It is no restriction to assume that \( 0 \leq i := v_3(a) \leq 3 \). Let \( K = \mathbb{Q}_3^{nr} \) and \( L = K[\pi] \) with \( \pi^4 = 3 \). Substituting \( x = \pi^{i+1}x_1 \) and \( y = \pi y_1 \) and dividing both sides of the equation by \( \pi^4 \) yields a smooth \( \mathcal{O}_L \)-model of \( Y_L \). Its special fiber is a smooth projective curve of genus 3 with affine equation

\[ \mathcal{Y} : x_1^4 = a_4(y_1^4 - b_1^2), \]

where \( b \) is the reduction of \( b \) and \( \pi \) the reduction of \( a/3^i \) modulo \( \pi \).

The arithmetic Galois group \( \text{Gal}(L/K) \) is generated by \( \gamma(\pi) = \zeta_4 \pi. \) The \( k \)-linear automorphism on \( \mathcal{Y} \) induced by \( \gamma \) satisfies \( (x_1,y_1) \mapsto (\zeta_4 x_1, \zeta^i_4 y_1) \) and acts as a 4-cycle on the reduction \( \mathcal{D} \) of the branch locus of \( \psi \).

One computes that \( g(\mathcal{Y}/\Gamma) = 1 \) if and only if \( v_p(a) \in \{1,2\} \) and \( g(\mathcal{Y}/\Gamma) = 0 \) otherwise. Since \( L/K \) is tamely ramified, \( f_3 = \epsilon \). We conclude that \( Y_a \) has conductor exponent \( f_3 = 4 \) if and only if \( v_p(a) \in \{1,2\} \). In the case that \( v_p(a) \in \{0,3\} \), one finds similarly that \( f_3 = 6 \).

The next proposition treats the case of residue characteristic \( p \neq 3 \). The corresponding result for Picard curves given by a superelliptic equation of exponent 3 (1.4) is [2, Theorem 4.4]. In general, it is not true that \( f_p = 0 \) if and only if \( Y \) has good reduction to characteristic \( p \): in the case that the curve \( Y \) has bad reduction but its Jacobian variety has good reduction to characteristic \( p \), we also have that \( f_p = 0 \). An example is given in [2, Example 5.5]. However, this does not occur for special Picard curves.
Proposition 5.1.12. Let $Y/K = \mathbb{Q}_p^{nr}$ be a special Picard curve.

(a) If $p = 2$, then $f_2 \geq 6$.
(b) If $p \geq 5$, then $f_p \in \{0, 4, 6\}$. Moreover, $f_p = 0$ if and only if $Y$ has good reduction at $p$.

Proof. Assume that we are given a defining equation $x^4 = ag(y)$ for $Y$ of the form (4.2.5). Let $g'$ be the shadow polynomial of $g$ as defined in Section 4.4. Let $\alpha$ be a root of $g$. Lemma 4.1.5.(c) implies that there is a unique subgroup $H_\alpha \subset \text{Aut}_K(Y)$ of order 3 with $P_\alpha := (0, \alpha)$ as fixed point. We denote by $y = \beta$ the second fixed point of $\sigma$ considered as automorphism of $Y/\langle \tau \rangle = \mathbb{P}^1_y$. In other words, $\beta$ is the root of the shadow polynomial $g'$ of $g$ corresponding to $\alpha$, see Section 4.4. The ramification locus $\mathcal{R}_\varphi$ of $\varphi : Y \to Y/H_\alpha$ is precisely the inverse image of $y = \beta$ on $Y$, together with $P_\alpha$. Here $P_\alpha$ is the distinguished point described in Remark 1.9.

We apply the strategy of [6, Sections 4+5] for computing the stable reduction of $Y$. The stable reduction of the standard special Picard curve $Y_{0,K}$ is computed in [1, Section 5.1.3]. That calculation implies that to find all irreducible components of the stable reduction $\overline{Y}$ of the twist $Y$ of $Y_{0,K}$ it suffices to separate the points of $\mathcal{R}_\varphi$. We refer to [6, Sections 4.2 and 4.3] for an explanation of the procedure of separating the points. The structure of the stable reduction $\overline{Y}$ is as follows.

Here the dots indicate the specialization of $\mathcal{R}_\varphi$. The dot marked $\infty$ is the specialization of the distinguished point $P_\alpha$. All three irreducible components are smooth curves of genus 1.

We first consider the case that $g$ has a $K$-rational root $\alpha$. In this case, the subgroup $H_\alpha$, considered as subgroup of $\text{GL}_2(K)$ as in (4.1.9), is $K$-rational. Arguing as in the proof of Theorem 1.18, we conclude that $Y/K$ admits a defining equation (1.4) as superelliptic curve of exponent 3. Since $Y$ is special it even follows that $Y$ admits an equation

$$y^3 = ax^4 + d.$$  

(5.1.13)

Arguing as in [1, Section 5.1.3], we find that any Galois extension $L/K$ such that $Y_L$ has semistable reduction contains $\zeta_4$ and $\sqrt{2}$. Moreover, any element $\gamma \in \text{Gal}(L/K)$ with $\gamma(\zeta_4) = -\zeta_4$ acts nontrivially on at least one of the two irreducible components $\overline{Y}_2, \overline{Y}_3$. Similarly, any element $\gamma \in \text{Gal}(L/K)$ with $\gamma(\sqrt{2}) \neq \sqrt{2}$ acts nontrivially on either $\overline{Y}_1$, or on $\overline{Y}_2$ and $\overline{Y}_3$, or on all three irreducible components. It follows therefore from Proposition 5.1.2 that

$$\epsilon \geq 4, \quad \delta \geq 2$$  

(5.1.14)

and hence that $f_2 = \delta + \epsilon \geq 6$. This proves statement (a) in this case.

Assume that none of the roots of $g$ is $\mathbb{Q}_2^{nr}$-rational. It follows from the definition of $g'$ that none of the roots of $g'$ is $\mathbb{Q}_2^{nr}$-rational, as well. One computes using the explicit stable model of $Y_0$ from [1, Section 5.1.3] that all sixteen points whose $y$-coordinate is a root of $g'$ specialize to pairwise distinct points of the stable reduction of $\overline{Y}$; exactly half specialize to $\overline{Y}_2$ and $\overline{Y}_3$, respectively. The automorphism group of these sixteen points is exactly the group $\tilde{A}_4$ discussed in Section 4.4. The subgroup $\tilde{A}_4$ of index 2 stabilizes each of the two components $\overline{Y}_2$ and $\overline{Y}_3$.

Let $L/K$ be a Galois extension over which $Y$ admits stable reduction. It is no restriction to assume that all sixteen points whose $y$-coordinate is a root of $g'$ are defined over $L$. Let $\gamma \in \text{Gal}(L/K)$ be an element that fixes none of the zeros $(g')_0$. It follows from the description
of $\tilde{A}_4$ in Remark 4.1.11 that the quotient of $Y_2 \cup Y_3$ by $\gamma$ has genus 0. Here we use that $\gamma$ does not lift to an automorphism of order 2 and hence does not just permute the two components $Y_2$ and $Y_3$. Proposition 5.1.2 implies in this situation that
\[ \epsilon \geq 4, \quad \delta \geq 4. \quad (5.1.15) \]

As in the first case, we conclude that $f_2 = \delta + \epsilon \geq 4 + 4 \geq 6$ and Statement (a) is proved.

We prove (b). The curve $Y$ has potentially good reduction to characteristic $p \geq 5$. It follows that $Y^0 = Y/\Gamma$ is a smooth curve, and hence that $\epsilon = 2g(Y) - 2g(Y^0) \leq 6$ is even. Lemma 4.1.5 implies $\text{Aut}_k(Y) = \text{Aut}_\Gamma(Y) = G_{18}$. A calculation using the description of this group in Lemma 4.1.5.(a) implies that $g(Y/H) \leq 1$ for all nontrivial subgroups $H < \text{Aut}_k(Y)$. We conclude that $\epsilon \in \{0, 4, 6\}$.

It follows from [6, Corollary 4.6] that the stable model of $Y$ may be defined over a tame extension of $K = \mathbb{Q}_p^{nr}$. This implies that $\delta = 0$ and hence that $f_p = \epsilon \in \{0, 4, 6\}$. This finishes the proof. \qed

The standard special Picard curve $Y_{0, 2}$ defined by (1.16) has conductor $N = 2^63^6$. This is the smallest value in our database [4]. In [2, Problem 5.7], it was asked whether there exists a Picard curve with strictly smaller conductor. As a consequence of the results of Sections 4 and 5.1, we can at least answer this question for special Picard curves.

**Theorem 5.1.16.** Let $Y/\mathbb{Q}$ be a special Picard curve. Then $N_Y \geq 2^63^6$.

**Proof.** Let $Y/\mathbb{Q}$ be a special Picard curve and assume $N_Y < 2^63^6$. Propositions 5.1.8 and 5.1.12.(a) imply that $N_Y \geq 2^63^4$. Therefore, Proposition 5.1.12.(b) implies that $Y$ has good reduction at all primes $p \geq 5$. Moreover, it follows from Proposition 5.1.8 that $f_3 < 6$ and hence that the invariant $\epsilon_3$ equals 4.

Let $x^4 = ag(y)$ be an equation for $Y$ of the form (4.2.5) and let $L_0/\mathbb{Q}$ be the splitting field of $g$. Since $\epsilon_3 = 4 < 6$ the proof of Proposition 5.1.8 implies that the inertia group $I_3$ at 3 of the extension $L_0/\mathbb{Q}$ acts as a 4-cycle on the divisor of zeros $D = (g)$. The only special polynomials found in Proposition 4.4.33 that satisfy this condition are
\[ g \in \{y^4 \pm 12y^2 - 12, y^4 \pm 6y^2 - 3\}. \quad (5.1.17) \]

It follows from Example 5.1.9 that $v_3(a) \in \{1, 2\}$. In all eight cases, we have indeed that $f_3 = \epsilon_3 = 4 < 6$.

We consider the conductor exponent at 2. The considerations at 3 imply that $g$ is one of the polynomials in (5.1.17). One checks that $g$ is irreducible over $\mathbb{Q}_2^{nr}$. The proof of Proposition 5.1.12.(a) therefore implies that the splitting field of $g'$, and hence of $g$, over $\mathbb{Q}_2^{nr}$ is contained in any Galois extension $L/\mathbb{Q}_2$ over which $Y$ acquires good reduction. The Galois group of the splitting field of $g$ is $\tilde{A}_4$ for $g \in \{y^4 \pm 6y^2 - 3\}$ and $\tilde{S}_4$ for $g \in \{y^4 \pm 12y^2 - 12\}$. Computing the jumps in the filtration of the higher ramification groups of the splitting fields and applying Proposition 5.1.2, one finds that $f_2 \geq 12$ in all cases. This contradicts the assumption that $N < 2^63^6$ and the proof is finished. \qed

**Example 5.1.18.** The curve
\[ Y : x^4 = 12(y^4 + 6y^2 - 3) \quad (5.1.19) \]
has conductor $N = 2^{12}3^4$. 
5.2. An upper bound for the conductor exponent?

Let \((K, v)\) be a complete discrete valuation field of mixed characteristic whose residue field \(k\) is a perfect field of characteristic \(p\). Sutherland asked us whether for smooth plane curves \(Y/K\) it is true that
\[
f_p \leq v_p(\Delta_{\text{min}}) .
\]

The corresponding formula holds for elliptic curves (where it follows from Ogg’s formula \([30]\)), and (with a suitable notion of minimal discriminant) for curves of genus 2, by work of Liu \([25]\).

The inequality (5.2.1) would imply that plane curves with small discriminant also have small conductor. This is useful when collecting curves of fixed genus with small conductor, as was done in \([38]\) for the case of plane quartics. Building on our results on discriminant minimization, Roman Kohls has proved that (5.2.1) holds for all Picard curves and for residue characteristic \(\neq 2,3\). See \([17]\). Moreover, as detailed in Appendix A, our computations of explicit examples give strong evidence that the result remains true for \(p = 2,3\) as well.

To elaborate somewhat, recall that every nonspecial Picard curve over \(K\) has a short Weierstrass equation (1.11) by Theorem 1.18. Moreover, if \(p \neq 3\), then we may assume that the corresponding homogenous equation has minimal discriminant exponent (Theorem 2.2.12).

Similarly, a special Picard curve can be given by a short Weierstrass equation (4.5.2) by Theorem 4.2.4, and for \(p \neq 2\) we may once more assume that the corresponding homogenous equation has minimal discriminant exponent (Theorem 4.5.15). In each case, the curve \(Y\) can be given as a superelliptic curve of exponent prime to \(p\). In his PhD thesis \([17]\), Kohls proves upper bounds for the conductor exponent at \(p\) of general superelliptic curves of exponent prime to \(p\). Applied to integral short Weierstrass equations of the form (1.11) and (4.5.2), these bounds prove the inequality (5.2.1), for all Picard curves and for \(p \neq 2,3\).

The inequalities proved in \([17]\) are closely related to results of Srinivasan \([37]\) and Obus and Srinivasan \([29]\) for hyperelliptic curves. In the latter paper, the authors prove the inequality
\[
v_p(-\text{Art}(Y/K)) \leq v_p(\Delta_{\text{min}})
\]
for all hyperelliptic curves when the residue characteristic of \(K\) is odd. Here \(\text{Art}(Y/K)\) denotes the Artin conductor of \(Y\). Although the Artin conductor is related to what we call the conductor of \(Y\), it is not equal, and the result of \([29]\) do not directly give a bound on \(f_p\).

Appendix A. A description of the database

We have made a database of Picard curves available at \([4]\). At the moment of publication, this database contained 6866 isomorphism classes of Picard curves over \(\mathbb{Q}\) that have bad reduction at only two primes in \(\{2,3,5,7\}\). Of these curves, 800 are the special Picard curves described in Theorem 4.4.54. The other curves were constructed from input furnished by \([27]\) and \([1]\), from computations in the forthcoming paper by Bouw, Koutsianas, Sijssing, and Wewers that we describe in Appendix B, and from an exhaustive search conducted by Andrew Sutherland \([38]\).

We represent a Picard curve \(Y\) in the database by a reduced polynomial (3.1.19), but the database also gives minimal long and short long Weierstrass equations over \(\mathbb{Z}\). It further includes the invariants of \(Y\) over \(\mathbb{Q}\) and \(\overline{\mathbb{Q}}\) (Proposition 3.3.1), the factorization of its discriminant (Definition 2.1.6), and finally, in many cases, its conductor along with its reduction type at bad primes (Section 5).

Because determining the geometric and arithmetic invariants of a given Picard curve over \(\mathbb{Q}\) via Proposition 3.3.1 is an efficient operation, it is possible to quickly look up whether a nonspecial curve over \(\mathbb{Q}\) given by the user is in the database. Similarly, for special curves, isomorphisms are found by using fast algorithms exploiting the special configuration of
hyperflexes described in Remark 4.5.6. Finally, using $\mathbb{Q}$-invariants allows us to quickly look up all the twists of a given curve that are in the database.

For details of the implementation which is written in MAGMA [3] and SAGEMath [39], we refer to the file README.md at [4]. For 3516 of the curves in the database, we were able to compute the conductor using the MCLF package in SAGEMath [31]. In all these cases, the conductor exponent is bounded by the exponent of the minimal discriminant at all primes, including 2 and 3. For all but 224 of the other curves, we still managed to calculate the conductor exponents away from the prime 3 (for nonspecial curves) or 2 (for special curves).

Appendix B. Computations

Let $S$ be a finite set of primes containing 3. By Proposition 3.2.1, we can compute the set of all nonspecial Picard curves with good reduction outside $S$ by computing the set of equivalence classes of binary quartic forms whose discriminant is a $S$-unit. A method to find these binary quartic forms was developed by Smart [36]; it reduces this question to solving of $S$-unit equations. In this section, we briefly explain how one can modify and improve Smart’s method in order to compute Picard curves with good reduction outside $S$ for $S = \{2, 3\}$, $\{3, 5\}$ and $\{3, 7\}$. We will give a detailed exposition of the method in the forthcoming paper by Bouw, Koutsianas, Sijslings, and Wewers.

Let $Y/\mathbb{Q}$ be a nonspecial Picard curve with good reduction outside $S$. By Corollary 3.1.18, the curve $Y$ admits a reduced short Weierstrass equation

$$y^3 = cf(x), \quad (B.1)$$

where $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and $c, a_i \in \mathbb{Z}$. Let $K/\mathbb{Q}$ be the splitting field of $f$, let $\alpha_1, \ldots, \alpha_4$ be the roots of $f$ in $K$, and let $G = \text{Gal}(K/\mathbb{Q})$. Let $S_K$ be the set of prime ideals over $S$, and let $\mathcal{O}_{K,S_K}^* \subset K^*$ the $S$-unit group associated to $S_K$.

By Proposition 3.2.1, we have $v_p(c) = v_p(\Delta(f)) = 0$ for all $p$ outside $S$. Since $\Delta(f) = \prod_{1 \leq i < j \leq 4}(\alpha_i - \alpha_j)^2$, we obtain

$$\alpha_i - \alpha_j \in \mathcal{O}_{K,S_K}^*. \quad (B.2)$$

Given $i, j, k \in \{1, 2, 3, 4\}$ with $i \neq j, k$ and $j \neq k$, we define the cross ratio

$$\lambda_{i,j,k} = \frac{\alpha_j - \alpha_i}{\alpha_k - \alpha_i}. \quad (B.3)$$

Then for all $i, j, k$ as above we have that $\lambda_{i,j,k} \in \mathcal{O}_{K,S_K}^*$ and $1 - \lambda_{i,j,k} \in \mathcal{O}_{K,S_K}^*$. This means that $\lambda_{i,j,k}$ is a solution of the $S$-unit equation

$$\lambda + \mu = 1, \quad (B.4)$$

where $\lambda, \mu \in \mathcal{O}_{K,S_K}^*$.

Given the values $\lambda_{i,j,k}$, the roots $\alpha_i$ can be calculated using an argument involving Hilbert’s Theorem 90. This gives us the $\mathbb{Q}$-isomorphism class of $Y$, from which we can determine all models (3.1.19) with good reduction outside $S$ by using the invariants from Section 3.3 along with the description of twists of nonspecial Picard curves in [12].

Methods for solving $S$-equations were developed by De Weger, Tzanakis, Smart, and others [36, 40]. The complexity of their algorithms increases exponentially with the rank of $\mathcal{O}_{K,S_K}^*$; we typically cut it down by restricting to a subgroup of $\mathcal{O}_{K,S_K}^*$, as in [19]. Moreover, this complexity depends on the representation of the Galois group $G$ on the roots of $f$. Currently, our database contains partial results for $S$ equal to $\{2, 3\}$, $\{3, 5\}$, and $\{3, 7\}$ and $G$ acting either trivially or as $\langle(1, 2)\rangle$ or $\langle(1, 2, 3)\rangle$. 
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