FOUR - FERMI THEORIES IN FEWER THAN FOUR DIMENSIONS

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Abstract

Four-fermi models in dimensionality $2 < d < 4$ exhibit an ultra-violet stable renormalization group fixed point at a strong value of the coupling constant where chiral symmetry is spontaneously broken. The resulting field theory describes relativistic fermions interacting non-trivially via exchange of scalar bound states. We calculate the $O(1/N_f)$ corrections to this picture, where $N_f$ is the number of fermion species, for a variety of models and confirm their renormalizability to this order. A connection between renormalizability and the hyperscaling relations between the theory’s critical exponents is made explicit. We present results of extensive numerical simulations of the simplest model for $d = 3$, performed using the hybrid Monte Carlo algorithm on lattice sizes ranging from $8^3$ to $24^3$. For $N_f = 12$ species of massless fermions we confirm the existence of a second order phase transition where chiral symmetry is spontaneously broken. Using both direct measurement and finite size scaling arguments we estimate the critical exponents $\beta$, $\gamma$, $\nu$ and $\delta$. We also investigate symmetry restoration at non-zero temperature, and the scalar two-point correlation function in the vicinity of the bulk transition. All our results are in excellent agreement with analytic predictions, and support the contention that the $1/N_f$ expansion is accurate for this class of models.
1. Introduction

The existence of a continuum limit of four-fermi theories is a long standing problem that goes back to the work of Nambu and Jona-Lasinio [1]. The original motivation for studying these models was the fact that they give a qualitatively good description of dynamical chiral symmetry breaking in strong interaction physics. The only degrees of freedom involved are fermions which interact via a short range interaction. As such four-fermi theories do not have a renormalizable perturbation expansion in powers of the coupling constant. However, at strong couplings, scalar bound states appear in the spectrum as a consequence of chiral symmetry breaking and the fermions interact by exchanging these composite scalars. The appearance of the scalars and pseudoscalars is tied to the Nambu-Goldstone realisation of chiral symmetry. By symmetry arguments, it can be seen that the emerging low-energy theory is similar to the linear $\sigma$-model, with scalars coupled to fermions via a Yukawa interaction, which is known to be perturbatively renormalizable. In four dimensions, the equivalence between the two models was noted very early on [2] and has been exploited by many authors [3,4]. Meanwhile, it was discovered by Gross and Neveu that in two dimensions the model is asymptotically free and has a nontrivial continuum limit near the origin [5]. Wilson has argued that four-fermi theories are nontrivial for all $d < 4$ [6]. His arguments are based on the leading order results of the $1/N_f$ expansion, where $N_f$ is the number of fermion species in the model. Many other studies have been done using this expansion technique [7]; a series of articles by Guralnik et al. [3] indicated that below four dimensions, an equivalence between four-fermi and sigma models could be established.

For definiteness, let us introduce the model we shall be dealing with in the bulk of this article. It is often called the Gross-Neveu model in the literature, and is the simplest relativistic theory of interacting fermions. The continuum space-time Lagrangian (we work in Euclidian space throughout) reads,

$$L = \sum_{j=1}^{N_f} \left[ \bar{\psi}_j (i \partial \theta + m) \psi_j - \frac{g^2}{2N_f} (\bar{\psi}_j \psi_j)^2 \right], \quad (1.1)$$

where the index $j$ labels the $N_f$ species, each of which is described by a four component spinor. For purposes of analysis as well as computation, it is useful to introduce an auxiliary field $\sigma$ so that $L$ becomes quadratic in $\psi_j$:

$$L = \sum_{j=1}^{N_f} \left[ \bar{\psi}_j (i \partial \theta + m) \psi_j + \bar{\psi}_j \psi_j + \frac{N_f}{2g^2} \sigma^2 \right]. \quad (1.2)$$

Gaussian functional integration over the $\sigma$ field in (1.2) restores the original Lagrangian (1.1) – however, it also serves as an interpolating field for the scalar bound state governing the fermion interactions described above. In the chiral limit fermion bare mass $m \to 0$ the vacuum expectation value of $\sigma$ becomes a dynamical fermion mass and is a convenient order parameter for any chiral symmetry transition in the theory. Note
that in this limit Eq. (1.2) has a discrete $Z_2$ chiral symmetry: $\sigma \mapsto -\sigma; \, \psi_j \mapsto \gamma_5 \psi_j$; and $\bar{\psi}_j \mapsto -\bar{\psi}_j \gamma_5$. It is this symmetry which is spontaneously broken at strong coupling. The universal critical indices of this critical point have been calculated to $O(1/N_f)$: for $d = 3$, they read

$$
\nu = 1 + \frac{8}{3N_f \pi^2}; \quad \delta = 2 + \frac{8}{N_f \pi^2}; \quad \beta = 1 + O\left(\frac{1}{N_f}\right); \quad \gamma = 1 + \frac{8}{N_f \pi^2}; \quad \eta = 1 - \frac{16}{3N_f \pi^2},
$$

(1.3)
in the conventional notation of classical statistical mechanics [8]. These indices are particularly interesting because they are far from free-field Gaussian model indices. Apparently, the critical field theory describes a strongly interacting fermionic system, complete with composite states, an interesting S-matrix, etc. The critical indices Eq. (1.3) lie at the heart of this article. The deviation of the critical indices from the Gaussian model indices are related to the anomalous dimensions of the various composite fields in the model, as we shall now discuss.

Below four dimensions the Yukawa theory is superrenormalizable with an ultraviolet fixed point at the origin and an infrared fixed point at strong couplings, where we recover scale invariance with scaling governed by non-vanishing anomalous dimensions. At this point the scalar degrees of freedom become composite and the low energy theory is the same as that of the four-fermi theory in the UV region. The compositeness condition is manifested in the vanishing of the scalar kinetic term in the continuum limit. Whereas at weak coupling $(\partial_\mu \sigma)^2$ is marginal irrespective of $d$, in the vicinity of the infrared fixed point the field $\sigma$ scales with an anomalous dimension which, because of unitarity, must be positive. This makes the kinetic term irrelevant at the IR fixed point. Since the IR limit of the Yukawa model now has the same relevant operator content as the UV limit of the four-fermi model, the latter must be renormalizable. Recently, several groups have studied the three dimensional model and demonstrated that Wilson’s ideas survive beyond the leading order in $1/N_f$. In particular, a series of papers by Rosenstein et al [9] have pointed out that the $1/N_f$ expansion around the fixed point in $d = 3$ is renormalizable. This result was extended to general $d \in (2, 4)$ in [10]. Explicit $O(1/N_f)$ calculations appear in references [8,11]. Ref.[12] contains a rigorous proof of renormalizability of the three-dimensional theory.

We should mention the physical meaning of the compositeness condition $Z = 0$, where $Z$ is the wavefunction renormalization factor associated with the scalar kinetic term, and its relationship with the fixed-point condition. Consider the general case of an interacting theory where a bound state $|B >$ with binding energy $E_B = -B < 0$ appears. The wavefunction renormalization constant is defined by

$$
Z = \sum_b | < b|B > |^2
$$

(1.4)
where $|b >$ stands for the bare (elementary particle) states. Using standard techniques, it can be shown that
$Z$ satisfies the following equation [13]

$$1 - Z = \int_0^\infty \frac{d\epsilon}{(\epsilon + B)^2} G^2(\epsilon)$$

with $G^2(\epsilon)$ being the total decay rate of the state $|B >$ proportional to some effective coupling constant. From Eq. (1.4), we see that $Z = 0$ means that the bound state has no projection on the space of bare states. This is the compositeness condition. From (1.5) it is transparent that this condition puts an upper limit on the effective coupling $G(\epsilon)$ thus yielding fixed-point condition.

The renormalizability of a given theory can be argued in several ways. One is the old-fashioned way of constructing a finite theory by adding counterterms and ensuring that all divergences cancel order by order in some kind of expansion. Another is to start with a discretized, and therefore UV finite, theory and look for the region in the bare parameter space where a macroscopic length scale appears. The two methods are equivalent and we will use them both in this paper. The general idea of renormalizability is that the cutoff dependence can be absorbed into a finite set of bare parameters in such a way that the low energy physics which emerges is insensitive to the cutoff. Once this is done, it is possible to find lines of constant physics – renormalization group (RG) trajectories – in the bare parameter space. These lines are uniquely defined no matter which observable is taken. This way, the low-energy quantities depend on each other and not on the cutoff – we will see that this is possible if certain consistency conditions known as hyperscaling relations, governing the scaling of different physical quantities, are obeyed.

The hyperscaling hypothesis [14] claims that the only relevant scale in the critical region is the macroscopic correlation length $\xi$. If this hypothesis holds, it is possible to do dimensional analysis using this correlation length as a scale. As a consequence, all dimensionless quantities should be independent of $\xi$. In the context of a simple model of ferromagnetism, the hypothesis can be stated as

$$F_{\text{sing}} = t^{2-\alpha} F(h/t^\Delta) \propto \xi^{-d}$$

where $F_{\text{sing}}$ is the singular piece of the free energy density and $\alpha, \Delta = \beta \delta$ are specific heat and gap exponents respectively. The external symmetry breaking field is $h$ and $t = T - T_c$ is the deviation from the critical temperature. All thermodynamic quantities can be obtained by taking the derivatives of the free energy. In particular, the order parameter (magnetization), defined as $<\phi> = \partial F_{\text{sing}}/\partial h$, satisfies the equation of state (EOS)

$$<\phi> = t^{2-\alpha-\Delta} F'(h/t^\Delta) \equiv \beta F'(h/t^\Delta)$$

with the magnetic exponent defined as $\beta = 2 - \alpha - \Delta$. Similarly, the susceptibility exponent, $\gamma$, is defined via

$$\chi \equiv \frac{\partial <\phi>}{\partial h} = t^{\beta-\Delta} F''(h/t^\Delta) \equiv t^{-\gamma} F''(h/t^\Delta)$$
i.e. \( \gamma = \Delta - \beta \). The behavior of the correlation length in the scaling region is given by

\[
\xi = t^{-\nu} g(h/t^\Delta)
\]  

Using the hyperscaling hypothesis (1.6) we relate \( \alpha \) to \( \nu \): \( d \nu = 2 - \alpha \). The exponents \( \beta \) and \( \nu \) can be related to the scaling dimension of the field \( \phi \). If a field \( \phi \), which develops nonvanishing vacuum expectation value \(< \phi > \neq 0\), has scaling dimension \( d_\phi \equiv \frac{1}{2}(d - 2 + \eta) \), then hyperscaling implies \(< \phi > \sim \xi^{-d_\phi} \). The vanishing of the order parameter is given by \(< \phi > \sim t^\beta \) from where it follows that \( \beta/\nu = d_\phi \). The other scaling relations are derived similarly. For a given channel, the corresponding correlation length (= inverse mass) may be defined by

\[
2d \xi_\phi^2 = \frac{\int_x |x|^2 < \phi(x)\phi(0)>}{\int_x <\phi(x)\phi(0)>}
\]  

Since hyperscaling is an important statement, we should explain its meaning and outline possible implications. It is generally believed that the violation of hyperscaling leads to triviality. This is due to inequalities between certain combinations of critical indices [15]. In the simplest cases like scalar field theories and spin systems, the quantity that measures the violation of hyperscaling is the *dimensionless* renormalized coupling. It is defined through the nonlinear susceptibility \( \chi^{(nl)} \):

\[
g_R = -\frac{\chi^{(nl)}}{\chi^2 \xi^2}, \quad \chi^{(nl)} = \frac{\partial^3 <\phi>}{\partial h^3} \]  

The renormalized coupling is essentially the properly normalized connected four-point function. In terms of the correlation functions, the nonlinear susceptibility is the zero-momentum projection of the four-point function

\[
\chi^{(nl)} = \int_{xyz} <\phi(x)\phi(y)\phi(z)\phi(0)>_{conn} \]  

The normalization factors can be understood by recalling that susceptibility and mass are related to the wave function normalization constant \( Z \) through \( Z = \chi/\xi^2 \). In the numerator we get one power of \( Z^{1/2} \) for each field, which is thus compensated by \( \chi^2 \): the factor of \( \xi^d \) accounts for the fact that there is one extra spatial integration in the numerator of (1.11). Both \( \chi^{(nl)} \) and \( \chi \) can be obtained by differentiating the free energy (1.6): \( \chi = t^{2-\alpha-2\Delta} F''(y) \) and \( \chi^{(nl)} = t^{2-\alpha-4\Delta} F''(y) \). Together with the EOS for the correlation length (1.9), they give the expression for the renormalized coupling

\[
g_R = t^{d_\nu - (2-\alpha)} H(h/t^\Delta), \]  

or, using the definition of the exponents \( \beta \) and \( \gamma \), Eqs. (1.7,8), the alternative form

\[
g_R = t^{-2\Delta + \gamma + d_\nu} H(h/t^\Delta)
\]
In specific (ferromagnetic) models an estimate of the exponent can be made. Since multi-spin correlations cannot extend over a larger range than pair correlations, the following inequality holds [15]:

\[ 2\Delta \leq \gamma + d\nu \]  

(1.14)

We can trade \( t \) with the correlation length in Eq.(1.9) and rewrite Eq.(1.13) in terms of \( \xi \) as

\[ g_R = \xi^{(2\Delta - \gamma - d\nu)/\nu} H(h/t^\Delta) \]  

(1.15)

Since \( g_R \) is dimensionless, hyperscaling implies that it must be independent of \( \xi \) and the exponent must vanish. In this case the renormalized coupling is a function of only one bare variable. Strict inequality in Eq.(1.14) implies triviality. An equivalent way of stating the above inequality is \( d\nu \geq 2 - \alpha \). It implies that the singular part of the free energy vanishes no faster than the prediction of hyperscaling i.e. \( F_{\text{sing}} \geq \xi^{-d} \).

In a similar fashion the scaling of mass ratios can be derived [16]. If hyperscaling is satisfied, for any particular pair of masses that obey Eq.(1.9) e.g. \( M_\pi, M_\sigma \), we have

\[ R(t, h) = \frac{M_\pi^2}{M_\sigma^2} = G(h/t^\Delta) \]  

(1.16)

where \( G(y) \) is a universal function. Comparing it with the expression for the renormalized coupling, we see that both observables depend on the same variable. One of the relations, \( R = G(y) \) or \( g_R = H(y) \) can be inverted to solve for the bare variable e.g. \( y = H^{-1}(g_R) \). This defines an RG trajectory for each value of the renormalized coupling: \( h = H^{-1}(g_R) t^\Delta \). This is then used to obtain the relation between the two observables \( R = R(g_R) \). The same manipulation can be done with two mass ratios. We note that the important point in the inversion is that both observables depend on just one bare variable, so that the inverse relation can always be found, at least in some regions of parameter space. This would be difficult to achieve otherwise, and is a consequence of hyperscaling.

To summarize, the requirement of having a macroscopic correlation length as the only relevant scale leads to relations between the critical exponents. These relations are

\[ 2 - \alpha = d\nu, \quad 2\Delta = \gamma + d\nu, \quad \beta = \frac{\nu}{2}(d - 2 + \eta), \quad \gamma = \nu(2 - \eta) \]  

(1.17)

All the other relations are obtained from Eqs.(1.17) using the definitions of the critical exponents e.g. \( 2\beta\delta - \gamma = d\nu, \ 2\beta + \gamma = d\nu \).

Next we review briefly how these ideas are realized in four-fermi theories, at least to leading order in \( 1/N_f \). We shall see in the next section, where detailed calculations are presented, that subleading corrections
do not change the physics qualitatively. The critical exponents for the model (1.1) have been calculated in Ref. [8]. The physical fermion mass $M$ satisfies a gap equation,

$$M = m - g^2 < \bar{\psi}\psi >, \quad (1.18)$$

In the chiral limit $M$ is proportional to the order parameter $< \bar{\psi}\psi >$. In the large $N_f$ limit, only the simple fermion tadpole contributes to $< \bar{\psi}\psi >$. The exponents $\beta, \delta$ can be obtained from the gap equation. It is convenient to use the definition of the critical coupling $1 = 4g^2 \int_q 1/q^2$, which is cutoff dependent, to rewrite the gap equation in the form

$$tM + m = 4g^2 \int_q \frac{M^3}{q^2(q^2 + M^2)} \quad (1.19)$$

where $t = (g^2 - g_0^2)/g_0^2$. Below four dimensions, the integral is IR divergent in the absence of the fermion mass $M$. It can be evaluated exactly and it is of the order of $M^{d-1}$. Thus, in the limit when either bare mass $m$ or $t$ vanish, we have

$$M \sim t^{1/(d-2)} \quad (m = 0),$$

$$M \sim m^{1/(d-1)} \quad (t = 0). \quad (1.20)$$

Since $< \bar{\psi}\psi > \sim M$, the exponents are: $\beta = 1/(d-2)$, $\delta = d - 1$.

The exponents $\nu, \gamma, \eta$ are obtained from the scalar propagator $D_\sigma$. To leading order in $1/N_f$ it is given by

$$D^{-1}_\sigma(k^2) = 1 + g^2 \Pi(k^2), \quad (1.21)$$

where $\Pi(k^2) = -\text{tr} \int_q S(q + k)S(q)$, and $S$ is the fermi propagator $(i\hat{\sigma} + M)^{-1}$. We associate the scalar mass with the inverse correlation length $1/M^2_\sigma = \xi^2$. The susceptibility and wavefunction renormalization constant for the scalar field are: $\chi^{-1} = 1 + g^2\Pi(0)$, $Z^{-1} = g^2\Pi'(0)$, which using (1.10) leads to

$$\xi^2 = \left( \frac{1}{D^{-1}_\sigma(k^2)} \frac{dD^{-1}_\sigma(k^2)}{dk^2} \right)_{k^2=0} \quad (1.22)$$

The scalar mass and renormalized coupling defined in this way are thus

$$M^2_\sigma = Z\chi^{-1} = 1 + g^2\Pi(0) = \frac{g^2}{g^2\Pi'(0)}, \quad \tilde{g}_R^2 = Zg^2 = \frac{1}{\Pi'(0)}, \quad (1.23)$$

where the renormalized Yukawa coupling $\tilde{g}_R$ is *dimensionful*. It has dimension $(mass)^{(4-d)/2}$ and, since it is a low-energy quantity, it depends on the physical mass $M_\sigma$. The *dimensionless* coupling is thus

$$g^2_R = \frac{\tilde{g}_R^2}{M^4_\sigma}. \quad (1.24)$$

The central question in the study of four-fermi theories is whether the bound states remain composite or become pointlike in the continuum limit. An important consequence of compositeness is the non-triviality
of the theory; i.e. the dimensionless interaction strength remains non-vanishing in the continuum limit. The physical reason in the case of the linear $\sigma$-model is easy to understand. The generic coupling between the fermions and scalars is given by the Goldberger-Treiman relation: $g_{\pi f\bar{f}} = M/f_\pi$, where $M$ is the fermion dynamical mass and $f_\pi$ is the pion decay constant related to the pion radius by $r_\pi \sim 1/f_\pi$. This way, the Yukawa coupling becomes $g_{\pi f\bar{f}} \sim Mr_\pi$ and vanishes in the limit of pointlike pions. In other words, since the scalar mass is not protected from receiving large corrections (of the order of the cutoff scale) via radiative corrections, the effective interactions will always be zero-range in the continuum limit. Such a theory does not have a physical scale and must be trivial above two dimensions. The only way to have a nontrivial limit is if the exchanged particles are composite. We will see later that this requirement is achieved when the bare parameters are tuned so as to make all power-law divergences disappear.

We restrict the discussion to $2 < d < 4$ in what follows to avoid violations of scaling although, regarding the flow of RG trajectories, the conclusions will be the same for $d = 2$ (but not for $d = 4$). Using the gap equation, we get (the quantity $\Pi(k^2)$ will be evaluated in detail in the next section)

$$
\chi^{-1} = \frac{m}{M} + \frac{12}{(d-1)} M^2 Z^{-1}, \quad M_\sigma^2 = \frac{m}{M} Z + \frac{12}{(d-1)} M^2, \quad Z^{-1} = \frac{bg^2}{M^{4-d}}, \tag{1.25}
$$

where $b = 2\Gamma(2-d/2)(d-1)/3(4\pi)^{d/2}$. We measure all the masses in units of the momentum cut-off. Note that the wavefunction renormalization constant $Z$ vanishes in the continuum limit $M \to 0$: this is precisely the compositeness condition [13]. In the chiral limit $m \to 0$ the susceptibility is $\chi^{-1} \propto M^{d-2}$, which implies, using Eq.(1.20), that $\chi^{-1} \propto t$, i.e. $\gamma = 1$. Similarly, the $\sigma$ mass in Eq.(1.25) scales as $M_\sigma \propto M \propto t^{d-2}$ giving $\nu = 1/(d-2)$. The exponent $\eta$ is extracted from the power law decay of the scalar correlation function at the critical point. Simple power counting in Eq.(1.21) gives

$$
\lim_{k^2 \to \infty} D_\sigma(k^2) \propto \frac{1}{k^{d-2}} \propto \frac{1}{k^{2-\eta}}, \tag{1.26}
$$

which gives $\eta = 4 - d$.

To obtain the RG trajectories and relate the low energy observables, we use Eq.(1.25) to calculate the mass ratio $R = M_\sigma^2/M^2$ and the renormalized coupling:

$$
R = \frac{1}{bg^2} \frac{m}{M^{d-1}} + \frac{12}{(d-1)}, \quad g_R^2 = \frac{1}{bR^{2-d/2}}. \tag{1.27}
$$

It is clear from the second relation that the lines of constant mass ratio are also lines of constant renormalized coupling. Regarding the ratio between the scalar and fermion masses we observe two things. In the limit $d \to 4$, since $1/b \propto 4 - d$, $R \to 4$, which is the expected result if the scalar is a weakly bound fermion – anti-fermion state. For arbitrary $d$ in the chiral limit, we obtain the result $R = R_c = 12/(d-1)$. At first
sight this seems to contradict the interpretation of $\sigma$ as a weakly bound state – however, this result is due to our using the definition (1.22) for the scalar mass, together with the fact that the numerical coefficient of $Z$ is far from unity. As we shall confirm in the next section, the scalar propagator has a pole at $k^2 = -4M^2$ in the chiral limit for all $d$. Away from the chiral limit, however, $D_\sigma$ no longer has a pole, which means that the definition (1.22) is more useful from our perspective. Secondly, consider the gap equation;

$$cg^2 M^{d-2} = t + \frac{m}{M},$$  \hspace{1cm} (1.28)

with $c = 8\Gamma(2 - d/2)/(d - 2)(4\pi)^{d/2}$. To leading order in $1/N_f$ this can be regarded as the EOS. From the expression for $R$ and the gap equation we eliminate $M$ to obtain

$$m = t^\Delta \frac{1}{g^2 \beta} \frac{b(R - R_c)}{(c - b(R - R_c))^\Delta},$$  \hspace{1cm} (1.29)

where $\Delta = \beta \delta = (d - 1)/(d - 2)$. This is the expected scaling of the RG trajectories $m \sim t^\Delta$. The lines of constant $R$ fall into two subfamilies. For $t > 0, R - R_c < c/b$ and $t < 0, R - R_c > c/b$. In the chiral limit $R = R_c$. As the binding increases, the mass of the composite, measured in units of the constituent mass, decreases.

We comment about the theory above four dimensions where hyperscaling is violated and where, as a consequence, the continuum limit is trivial. It is easy to see that, for $d \geq 4$ the gap equation (1.19) is regular in the $M \to 0$ limit. As a consequence, the corresponding exponents $\beta, \delta$ are the same as in four dimensions; $\beta = 1/2, \delta = 3$. Again, the reason for this is the IR behavior of the theory. The gap equation in this case is

$$C g^2 \Lambda^{d-4} M^d = t M + m,$$  \hspace{1cm} (1.30)

where we display the explicit cutoff dependence, since the physical mass will turn out not to be the only scale. It is easy to verify that the expression (1.29) for the mass ratio will remain unchanged. However, the renormalized coupling will change;

$$g^2_R = Z M^{d-4} g^2 = \frac{1}{B} \left( \frac{M}{\Lambda} \right)^{d-4}.$$  \hspace{1cm} (1.31)

The wavefunction renormalization constant is obtained as the $k^2$ coefficient in the scalar propagator. To leading order it is

$$Z^{-1} \sim \int_q \frac{1}{(q^2 + M^2)^2},$$  \hspace{1cm} (1.32)

which is clearly finite in the $M \to 0$ limit for $d > 4$. The relations for the remaining critical exponents follow from Eqs.(1.25): $\gamma = 1, \nu = 1/2, \eta = 0$. For $d > 4$ hyperscaling is clearly violated: indeed with
mean field values of the exponents Baker’s inequality (1.14) becomes \( d \geq 4 \). Two things are clear: firstly the trajectories of constant \( R \) and \( g_R \) do not coincide; and secondly in the limit \( M/\Lambda \to 0 \), the renormalized coupling vanishes. Because of the violation of hyperscaling the theory is trivial.

In four dimensions, hyperscaling violations are logarithmic and are not reflected through the violation of the relations between the critical exponents. Explicit calculation to leading order gives

\[
g_R^2 = \frac{8\pi^2}{\ln \left( \frac{\Lambda^2}{m^2} \right)} \tag{1.33}
\]

So, the theory is trivial, but the critical exponents satisfy hyperscaling relations. The reason for this is that the scalars are pointlike: \( Z^{-1} \) is logarithmically divergent in the UV cutoff.

In section 2 of this article we present a calculation of \( 1/N_f \) corrections to the model at length, for arbitrary dimensionality \( 2 < d < 4 \). We will explicitly demonstrate the renormalizability of the \( 1/N_f \) expansion to next-to-leading order in this regime, and also calculate the critical indices and show that they continue to satisfy the hyperscaling relations. We will argue that the physical assumptions underlying renormalizability and hyperscaling are equivalent, and demonstrate this using the chiral symmetry of (1.1,2).

For completeness we also generalize our results to models where the spontaneously broken symmetry is \( U(1) \otimes U(1) \) or \( SU(2) \otimes SU(2) \). Some of these results were presented in briefer form in Ref. [8].

In the rest of the article we shall fix \( d = 3 \) and study the model by computer simulation methods. There are several motivations for doing this. First, we can investigate the validity of the \( 1/N_f \) expansion by a truly non-perturbative numerical scheme. We shall see that standard methods of extracting critical behavior from computer simulations yield results in good agreement with the \( 1/N_f \) expansion when \( N_f \) is chosen large – \( N_f \) will be set to 12 in our numerical work. Second, we can study the model for small \( N_f \), where it may have some relevance to the strongly correlated electron dynamics underlying high \( T_c \) superconductivity, and see if there are dramatic changes as \( N_f \) decreases. Although we have some results at small \( N_f \), this article will concentrate on the first objective and extract a number of results from \( N_f = 12 \) simulations which provide support for the usefulness and reliability of the \( 1/N_f \) expansion. Our main results are a numerical verification of the critical index predictions of Eq.(1.3) to leading order.

After describing the lattice formulation of the model and the hybrid Monte Carlo algorithm in Sec. 3, we turn to numerical results. Most of our simulations are performed directly in the chiral limit \( m = 0 \). In Sec. 4 we consider the model for \( N_f = 12 \) on symmetric lattices ranging in size from \( 8^3 \) to \( 20^3 \). Measurements of the vacuum expectation value \( \Sigma_0 \equiv \langle \sigma \rangle \) clearly show a chiral transition in the lattice model at an inverse coupling \( 1/g^2 \) near unity. Measurement of \( \Sigma_0 \) and its susceptibility \( \chi \) as a function of the coupling \( 1/g^2 \) allow
us to measure the critical indices $\beta$ and $\gamma$ directly. The leading order terms in Eq. (1.3), $\beta = \gamma = 1$, will be confirmed with good precision. The size dependence of the critical point will also provide a rough estimate of $\nu$, using finite size scaling arguments, but more quantitative results will be presented in later sections of this article. In particular, the model will be considered at nonzero temperature in Sec. 5 by simulating it on asymmetric lattices, $N_x \times N^2$ with $N = 36$ lattice spacings and $N_x$ ranging from 2 through 12. We shall find a second order phase transition at a critical temperature in good agreement with large $1/N_f$ calculations. A measurement of the critical index $\nu$ will follow. In Sec. 6 we introduce a small bare fermion mass $m$ into the theory’s action and measure the critical index $\delta$ at the zero temperature critical point as well as at the finite temperature transition. The first set of measurements, done on $8^3$, $16^3$ and $24^3$ lattices, are nicely consistent with the leading order term in Eq.(1.3), $\delta = 2$. The second set of measurements uses a $12 \times 36^2$ lattice and gives $\delta_T = 3.0$, the expected Gaussian model index for the temperature-driven phase transition [17]. In Sec. 7 we turn to measurements of the two-point scalar correlation function, and our attempts to extract the scalar mass using the formalism developed in Sec. 2. As we shall see, this requires some novel fitting techniques, and the results presented here are probably only a first step towards a complete understanding. Our results are summarized in Sec. 8.

2. Calculations of $O(1/N_f)$ Corrections

In this section we will calculate the lowest non-trivial corrections, i.e. those of $O(1/N_f)$, to the model in the vicinity of the strongly-coupled fixed point at $g = g_c$. We will work in dimensionality $d$ where $2 < d < 4$, and have chosen to follow the treatment of Gat et al [11], and define the Lagrangian involving the auxiliary scalar field in the chiral limit as follows:

$$L = Z_\psi \bar{\psi}_i \partial_\mu \psi_i + \frac{g}{\sqrt{N_f}} Z_\psi Z_\sigma \bar{\sigma} \psi_i \psi_i + \frac{1}{2} Z_\sigma \sigma^2,$$

where a sum on $i = 1, \ldots, N_f$ is understood. Notice that in this section of the paper the $\sigma$ field is normalized differently to that of Eq.(1.2); since $\sigma$ is auxiliary this has no physical consequences. The constants $Z_\psi$, $Z_\sigma$, and $g$ are cutoff-dependent, and must be adjusted at each order of the $1/N_f$ expansion to keep Green functions finite, the relevant ones being the fermion propagator $S_F$, the scalar-fermion vertex $\Gamma_{\bar{\psi}\psi\psi}$, and the scalar propagator $D_\sigma$. It should be noted that the constants $Z_\psi$, $Z_\sigma$ are chosen dimensionless for convenience and are not directly related to the constant $Z$ of the previous section, which has dimension $(mass)^2$. For additional simplicity we work for the most part in the chiral limit $m = 0$. In the broken phase $t > 0$ we will choose renormalization conditions so as to keep the physical fermion mass, that is, the pole of the
renormalized fermion propagator, a finite constant independent of $1/N_f$.

Our Feynman rules are thus:

\[
S_F^{(0)-1} = Z_\psi(i\not{\partial} + \Sigma_0); \\
\Gamma^{(0)}_{\sigma \psi \psi} = -\frac{g}{\sqrt{N_f}} Z_\psi Z_\sigma^\dagger;
\]  \hspace{1cm} (2.2)

where the dynamical fermion mass $\Sigma_0$ appearing in the propagator is defined by the expectation value of the scalar field (this is equivalent to the definition of $\Sigma_0$ used elsewhere in the paper):

\[
\Sigma_0 = \frac{gZ_\sigma^\dagger}{\sqrt{N_f}} < \sigma >.
\]  \hspace{1cm} (2.3)

We determine $\Sigma_0$ from the equation of motion $\delta \mathcal{L}/\delta \sigma = 0$, yielding the gap equation. To leading order,

\[
\frac{\Sigma_0}{g^2} = -\frac{Z_\psi}{N_f} < \bar{\psi}\psi_i > = \text{tr} \int_\mathbb{P} \frac{1}{i\not{\partial} + \Sigma_0},
\]  \hspace{1cm} (2.4)

which immediately leads to an expression for $g$ in terms of $\Sigma_0$ and the cutoff $\Lambda$:

\[
\frac{1}{g^2} = \frac{8}{(4\pi)^{\frac{d}{2}}(d-2)} \left[ \frac{\Lambda^{d-2}}{\Gamma\left(\frac{d}{2}\right)} - \Sigma_0^{d-2} \Gamma\left(2 - \frac{d}{2}\right) \right].
\]  \hspace{1cm} (2.5)

Details of how $\int_\mathbb{P}$ is defined for $d \in (2,4)$ are given in an appendix. The limit $\Sigma_0 \to 0$ of expression (2.5) defines the critical coupling $g_c$: for $d = 3$ we obtain $g_c = 2\Lambda/\pi^2$. To the same leading order the scalar propagator $D_\sigma$ is now given by the sum over all numbers of chained fermion - antifermion bubbles $\Pi(k^2)$:

\[
D_\sigma^{-1}(k^2) = Z_\sigma - \Pi(k^2)
\]

\[= Z_\sigma g^2 \text{tr} \int_q \frac{1}{i\not{q} + \Sigma_0} \left( \frac{1}{\Sigma_0} + \frac{1}{i(\not{q} + \not{\Sigma})} + \Sigma_0 \right) \]

\[= Z_\sigma g^2 \frac{2^{\frac{d}{2}-1} \Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^\frac{d}{2}} (k^2 + 4\Sigma_0^2)^\frac{d}{2} - 1 \int_0^{(1+4\Sigma_0^2/k^2)^{-1}} dx x^{-\frac{d}{2}} (1-x)^{\frac{d}{2}-2},
\]  \hspace{1cm} (2.6)

where use has been made of relation (2.4). The integral over $x$ in (2.6) defines an incomplete Beta function, which in turn may be expressed in terms of a hypergeometric function. The result is

\[
D_\sigma^{-1}(k^2) = Z_\sigma g^2 \frac{2\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^\frac{d}{2}} \frac{(k^2 + 4\Sigma_0^2)}{\Sigma_0^{d-2}} F\left(1, 2 - \frac{d}{2}, \frac{3}{2}, -\frac{k^2}{4\Sigma_0^2}\right).
\]  \hspace{1cm} (2.7)

For $d = 3$ we recover the usual form [9]

\[
D_\sigma^{-1}(k^2) = Z_\sigma g^2 \frac{(k^2 + 4\Sigma_0^2) \tan^{-1}(\sqrt{k^2/2\Sigma_0})}{2\pi \sqrt{k^2}}.
\]  \hspace{1cm} (2.8)

Notice that for $k^2 \ll \Sigma_0^2$ $D_\sigma$ resembles the canonical form for a boson field: $(k^2 + 4\Sigma_0^2)^{-1}$; indeed, in the chiral limit it has poles at $k^2 = -4\Sigma_0^2$. However, for $k^2 \gg \Sigma_0^2$, the relevant limit for the calculation of $O(1/N_f)$ loop corrections, the asymptotic form of (2.7) is quite different:

\[
\lim_{k^2 \to \infty} D_\sigma(k^2) = \frac{1}{Z_\sigma g^2} \left( \frac{A_d}{(k^2)^{\frac{d}{2}-1}} \right).
\]  \hspace{1cm} (2.9a)
where
\[ a_d = \frac{(4\pi)^{\frac{d}{2}}}{4\Gamma(2 - \frac{d}{2})B(\frac{d}{2}, \frac{d}{2} - 1)}. \] (2.96)

To renormalize the model at leading order, we may take \( Z_\psi = 1 \), and the physical mass \( M \) to be equal to \( \Sigma_0 \). At higher orders this will not be the case. Finally, to render \( D^{-1}_\sigma \) finite we define
\[ Z_\sigma g^2 = \frac{1}{M^{d-2}}. \] (2.10)

The exact value of the right hand side is unimportant provided it is finite, since it cancels from all calculations of physical quantities. In effect we have specified \( \sigma \) to have dimension \( [\frac{d}{2}] \), but since it is an auxiliary field this is purely a matter of convention.

The simplest corrections at \( O(1/N_f) \) are one loop corrections to \( S_F \) and \( \Gamma_{\sigma \bar{\psi} \psi} \), as shown in Fig. 1 (in fact, there is a two loop contribution to \( \Gamma_{\sigma \bar{\psi} \psi} \) at this order [9]: since, however, it involves a closed fermion loop with an odd number of legs, it is finite and can be ignored). The fermion self-energy \( \Sigma(p) \) is given by
\[ \Sigma(p) = \Sigma_0 - \frac{g^2}{N_f} Z_\psi^2 Z_\sigma \times \left( \frac{1}{Z_\psi} \times \int \frac{1}{i(p + \#)} + \Sigma_0 \right) D_\sigma(q^2). \] (2.11)
The integral over \( q \) is logarithmically divergent but straightforward, and leads to the result for the full inverse propagator
\[ S_F^{-1} = Z_\psi \left[ i\bar{\psi} \left( 1 + \frac{(d - 2)}{2d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M} \right) + \Sigma_0 \left( 1 - \frac{C_d}{2N_f} \ln \frac{\Lambda}{M} \right) \right], \] (2.12)
where we have used the fact that \( M = \Sigma_0 \) to leading order, and the numerical constant \( C_d \), which we will find to be ubiquitous, is given by
\[ C_d = \frac{1}{\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2})B(\frac{d}{2}, \frac{d}{2} - 1)}. \] (2.13)

In \( d = 3 \), \( C_d = 4/\pi^2 \). For finiteness we thus require
\[ Z_\psi = 1 - \frac{(d - 2)}{2d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M} \] (2.14)
and
\[ \Sigma_0 \equiv Z_M M = M \left( 1 + \frac{(d - 1)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M} \right). \] (2.15)

Similarly, at vanishing external momentum we find the vertex to \( O(1/N_f) \):
\[ \Gamma_{\sigma \bar{\psi} \psi} = -\frac{g}{\sqrt{N_f}} Z_\psi Z_\sigma^\frac{1}{2} \left( 1 - \frac{C_d}{2N_f} \ln \frac{\Lambda}{M} \right). \] (2.16)

The requirement that this expression be finite gives a condition on the combination \( Z_\sigma g^2 \):
\[ g Z_\sigma^\frac{1}{2} \times Z_\psi^{-1} \left( 1 + \frac{C_d}{2N_f} \ln \frac{\Lambda}{M} \right); \]
\[ Z_\sigma g^2 = \frac{1}{M^{d-2}} \left( 1 + \frac{2(d-1)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M} \right). \] (2.17)

As we saw in Eq.(2.7), there is a second condition on \( Z_\sigma g^2 \) at leading order from the requirement that the scalar propagator be finite. This continues to be the case at higher orders, as we shall see: consistency of the two conditions at each order is vital for the renormalizability of the model.

To fully specify the renormalization constants to \( O(1/N_f) \) we must calculate \( g \) as a function of physical mass \( M \) using the two-loop gap equation (Fig. 2). The \( O(1/N_f) \) contribution is:

\[
\frac{1}{g^2} - \frac{1}{g^{(0)2}} = -\frac{4}{N_f} \int \int (p^2 + \Sigma_0^2)(p^2 + \Sigma_0^2)^2 \times D_\sigma(q^2) - \frac{4(d-1)\Lambda^{d-2}}{N_f(4\pi)^\frac{d}{2}(d-2)} \left( 1 - \frac{2d-3(d-1)}{N_f} \right). \] (2.18)

using expression (2.9) for \( D_\sigma \). The full expression for \( g \) is thus:

\[
\frac{1}{g^2} = \frac{8\Lambda^{d-2}}{(4\pi)^\frac{d}{2}(d-2)} \left( 1 - \frac{(d-1)}{2N_f} \right) - \frac{8\Sigma_0^{d-2}}{(4\pi)^\frac{d}{2}(d-2)} \left( 1 - \frac{2d-3(d-1)}{N_f} \right). \] (2.19)

Once again, we extract the critical coupling in the limit \( \Sigma_0 \to 0 ):

\[
\frac{1}{g_c^2} = \frac{8\Lambda^{d-2}}{(4\pi)^\frac{d}{2}(d-2)} \left( 1 - \frac{(d-1)}{2N_f} \right). \] (2.20)

We have now used up all our freedom in adjusting the constants \( g, Z_\psi \) and \( Z_\sigma \) by absorbing divergences in the fermion self-energy and vertex corrections, and by relating quantities to a physical scale via the gap equation. Power counting shows, however, that we have not yet exhausted the list of divergent graphs at \( O(1/N_f) \): there remain the two-loop contributions to the scalar propagator shown in Fig. 3. If the model is renormalizable, then the \( O(1/N_f) \) expression for \( Z_\sigma g^2 \) already derived in (2.17) together with the definition of \( g \) via (2.19) must suffice to cancel the UV divergences in \( D_\sigma \). We will show this by explicit calculation in the symmetric phase, where the massless nature of the fermion makes the loop integrals much simpler. Of course, the UV divergence structure of a field theory should be insensitive to the particular phase in which it resides. We begin by following the renormalization convention of [10]: the leading order expression for the inverse scalar propagator in the symmetric phase is now (Cf. (2.6))

\[
D_\sigma^{-1}(k^2) = Z_\sigma g^2 \left[ \frac{1}{g^2} + \frac{1}{i\not{q}} \frac{1}{i\not{k} + \not{q}} \right]. \] (2.21)

The integral over \( q \) is easily done, to yield

\[
D_\sigma^{-1}(k^2) = Z_\sigma g^2 \left[ \frac{1}{g^2} - \frac{8\Lambda^{d-2}}{(4\pi)^\frac{d}{2}(d-2)(\frac{d}{2})} \times \left( \frac{k^2}{A_d} \right)^\frac{d-1}{2} \right]. \] (2.22)
where $A_d$ is given in (2.9). We now impose the off-shell renormalization condition

$$D_{\sigma}^{-1}(\mu^2) \equiv \frac{2}{A_d},$$  \hspace{1cm} (2.23)

where $\mu$ is some arbitrary scale parameter. The power-law divergence in (2.22) can now be removed, first by specifying

$$Z_{\sigma} g^2 = \mu^{2-d};$$  \hspace{1cm} (2.24)

then, substituting (2.22) into (2.23) we find

$$\frac{1}{g^2} = \frac{8\Lambda^{d-2}}{(4\pi)^{\frac{d}{2}}(d-2)\Gamma\left(\frac{d}{2}\right)} + \frac{\mu^{d-2}}{A_d}.$$  \hspace{1cm} (2.25)

With $Z_{\sigma}$ and $g^2$ both fixed, $D_{\sigma}^{-1}$ is completely finite:

$$D_{\sigma}^{-1}(k^2) = \frac{1}{A_d \mu^{d-2}} \left( (k^2)^{\frac{d}{2}-1} + \mu^{d-2} \right).$$  \hspace{1cm} (2.26)

We see that in the symmetric phase $D_{\sigma}(k^2)$ has no poles on the physical sheet, and so $\mu$ cannot be interpreted as a scalar mass: indeed the $\sigma$ field in this case does not interpolate a stable particle. Instead, $\mu$ can be regarded as the width of an unstable resonance, but as shown later, it can still serve as an inverse correlation length in the statistical mechanics sense. The critical coupling $g_c$ is obtained in the limit $\mu \to 0$, whereupon we find the same result as for the $\Sigma_0 \to 0$ limit of (2.5). Note that in (2.25) $g < g_c$, which is consistent with being in the symmetric phase. To leading order no further divergences are present, so as before $Z_{\psi} = 1$.

It is important to note that the asymptotic form of (2.26) matches that of the propagator in the broken phase (2.9), as it must do. It is therefore no surprise that we obtain almost identical results for the fermion wavefunction and vertex corrections at $O(1/N_f)$ as for the broken phase (Fig. 1), the only difference being that now no mass renormalization is required, due to the chiral symmetry of the model. We obtain two conditions for the renormalization constants essentially identical to Eqs.(2.14,17):

$$Z_{\psi} = 1 - \frac{(d - 2) \, C_d \, \ln \frac{\Lambda}{\mu}}{2d \, N_f};$$  \hspace{1cm} (2.27)

$$Z_{\sigma} g^2 = \frac{1}{\mu^{d-2}} \left( 1 + \frac{2(d - 1) \, C_d \, \ln \frac{\Lambda}{\mu}}{d \, N_f} \right),$$  \hspace{1cm} (2.28)

with $C_d$ given by (2.13).

Since there are no non-trivial solutions to the gap equation for $g < g_c$, the renormalization of the model can only be completed by extracting $g(\Lambda, \mu)$ from the full inverse scalar propagator to $O(1/N_f)$, which necessitates a two-loop calculation. The relevant contributions are shown diagramatically in Fig. 3 (once
again, there is a finite three-loop contribution which we ignore \([9]\)). For external momentum \(k\), the two-loop contribution to \(D^{-1}\sigma(k^2)\) is

\[
\Pi^{(1)}(k^2) = -\frac{Z_\sigma g^2}{N_f} \int_q \frac{A_d}{q^2} \left[ \frac{1}{i\not{p} i(\not{p} + \not{k}) i(\not{p} + \not{k} + \not{q}) i(\not{p} + \not{q})} \right] + 2tr \left( \frac{1}{i\not{p} i(\not{p} + \not{q}) i\not{p} i(\not{p} + \not{k})} \right).
\]

(2.29)

We will spare the reader the full details of the calculation, and merely remark that the integral over \(q\) is considerably simplified by our choice of momentum routing through the internal scalar line. The divergences are generically of two kinds: a power-law form

\[
(i) \sim \frac{\Lambda^{d-2}}{(d-2)} - \mu^{d-2} \ln \frac{\Lambda}{\mu};
\]

(2.30)

and a momentum-dependent logarithm

\[
(ii) \sim (k^2)^{\frac{d}{2}-1} \ln \frac{\Lambda}{\{\mu|k\}},
\]

(2.31)

where the symbol \(\{\mu|k\}\) denotes a function of both \(\mu\) and \(k\) which simplifies in three limits of interest:

\[
\begin{align*}
\lim_{\mu \to 0} \{\mu|k\} &= \sqrt{k^2}; \\
\lim_{k \to 0} \{\mu|k\} &= \mu; \\
\lim_{\mu^2 = k^2} \{\mu|k\} &= \mu.
\end{align*}
\]

(2.32)

Further details on how forms \((i)\) and \((ii)\) are extracted from (2.29) are given in the appendix.

After some work we arrive at the full expression for \(D^{-1}\sigma(k^2)\) to \(O(1/N_f)\):

\[
D^{-1}\sigma(k^2) = Z_\sigma g^2 \left[ \frac{1}{g^2} - \frac{8\Lambda^{d-2}}{(4\pi)^{\frac{d}{2}}(d-2)\Gamma(d/2)} + \frac{(k^2)^{\frac{d}{2}-1}}{A_d} \right.
\]

\[
+ \frac{8B(\frac{d}{2} - 1, \frac{d}{2} - 1)\Gamma(2 - \frac{d}{2}) A_d}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \left( \frac{A_d}{N_f} \ln \frac{\Lambda}{\mu} - \frac{2\ln \Lambda}{\mu} \right) - \frac{2}{d}(k^2)^{\frac{d}{2}-1} \ln \frac{\Lambda}{\{\mu|k\}} \right]
\]

(2.33)

where \(g_c\) is identical to the result for the broken phase, Eq.(2.20). Once again, we renormalize using condition (2.23), plus the expression (2.28) for \(Z_\sigma g^2\) coming from vertex renormalization. We find an expression for \(g\):

\[
\frac{1}{g^2} = \frac{1}{g_c^2} + \frac{\mu^{d-2}}{A_d} \left( 1 + \frac{(d-1)(d-2)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{\mu} \right).
\]

(2.34)
Substituting back into (2.33), we find the renormalized inverse scalar propagator:

\[
D^{-1}_\sigma(k^2) = Z_\sigma g^2 \left[ \frac{\mu^{d-2}}{A_d} \left( 1 - \frac{2(d-1)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{\mu} \right) + \frac{(k^2)^{\frac{d}{2}-1}}{A_d} \left( 1 - \frac{2(d-1)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{\mu|k|} \right) \right]
\]

(2.35)

\[
= \frac{1}{A_d \mu^{d-2}} \left[ \mu^{d-2} + (k^2)^{\frac{d}{2}-1} \left( 1 + \frac{2(d-1)}{d} \frac{C_d}{N_f} \ln \frac{\mu|k|}{\mu} \right) \right].
\]

This expression is manifestly UV finite both for \( k^2 = 0 \), and in the critical limit \( \mu \to 0 \), although in the latter case there is the usual IR divergence associated with massless particles.

Since we have been able to adjust \( Z_\psi, Z_\sigma \) and \( g \) to eliminate all divergences in Green functions to \( O(1/N_f) \), at least in the symmetric phase, we have succeeded in our aim of showing the model is renormalizable to this order. In effect the fact that the combination \( Z_\sigma g^2 \) serves both to correct the fermion-scalar vertex and the inverse scalar propagator is really a statement that the renormalized four-point fermion Green function is finite to this order (Fig. 4). If we recall that the underlying model we started from was one with an explicit four-fermi interaction, we can now understand why \( Z_\sigma g^2 \) is apparently overdetermined: in the four-fermi model there can only be two adjustable parameters: the unphysical \( Z_\psi \); and the strength of the four-point coupling. Another consequence is that for \( k^2 \gg M^2, \mu^2 \), logarithmic contributions to the four-point scattering of the form \( \ln(k/M) \) must also cancel, which means that in this asymptotic limit the renormalized four-fermion scattering amplitude assumes a universal form \( A_d/N_f k^{d-2} \). This receives no \( 1/N_f \) corrections if the model is renormalizable.

We now consider the model from the viewpoint of statistical mechanics, by calculating the critical exponents associated with the chiral symmetry breaking transition, and demonstrating a relation between renormalizability and hyperscaling. All essential information about a continuous phase transition is encoded in its critical exponents, and it is straightforward to calculate these to \( O(1/N_f) \) from the gap equation. The order parameter is of course the condensate \( \langle \bar{\psi} \psi \rangle \), but from (2.4) we can equally well consider \( \Sigma_0 \), since it shares the same non-analytic behavior as \( g \to g_c, \Lambda/M \to \infty \). Parameterizing the distance from criticality by \( t \equiv g^2(g^2 - g_c^2) \), we find from (2.19) the critical exponent \( \beta \):

\[
\beta = \frac{1}{d-2} + O\left( \frac{1}{N_f} \right),
\]

(2.36)

whence

\[
\beta = \frac{1}{d-2} + O\left( \frac{1}{N_f} \right). \tag{2.37}
\]

To determine the other critical exponents describing the critical behavior of the order parameter, we need to consider the effects of introducing an explicit bare mass term \( Z_\psi \bar{\psi} \psi \) into the Lagrangian (2.1).
To leading order $M = \Sigma_0 + m$, and we obtain an inverse scalar propagator which now has a contribution diverging as a power of $\Lambda$ at leading order:

$$
D^{-1}_\sigma(k^2; m) = Z_\sigma g^2 \text{tr} \int \frac{1}{q + i\Lambda} \left( \frac{1}{\Sigma_0} + \frac{1}{i(q + \Lambda)} + M \right)
$$

$$
= Z_\sigma g^2 \left\{ \frac{2\Gamma(2 - \frac{d}{2}) (k^2 + 4M^2)}{M^{d-2}} F(1, 2 - \frac{d}{2}; \frac{3}{2}; -\frac{k^2}{4M^2}) + \frac{2}{(4\pi)^{\frac{d}{2}}(d-2)\Sigma_0} \left[ \Lambda^{d-2} - M^{d-2}\Gamma(2 - \frac{d}{2}) \right] \right\};
$$

$$
\lim_{k^2 \to \infty} D^{-1}_\sigma(k^2; m) = Z_\sigma g^2 \left\{ \frac{(k^2)^{\frac{d}{2}-1}}{A_d} + \frac{2}{(4\pi)^{\frac{d}{2}}(d-2)\Sigma_0} \left[ \Lambda^{d-2} - M^{d-2}\Gamma(2 - \frac{d}{2}) \right] \right\}. \tag{2.38a}
$$

It is interesting to note that away from the chiral limit $D_\sigma$ no longer has poles in the complex $k$-plane – therefore to define a “$\sigma$ mass” we are forced to use the definition (1.10,22). If we substitute the asymptotic form (2.38b) into the two-loop gap equation (2.18), with $M$ replacing $\Sigma_0$ in the fermi propagators, we obtain the full gap equation in the presence of external mass:

$$
\frac{\Sigma_0}{g^2} = M - M \left[ \frac{8M^{d-2}\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}(d-2)} - \frac{8(d-1)}{(4\pi)^{\frac{d}{2}}(d-2)^2\Gamma(\frac{d}{2})} N_f \right] \ln \left( 1 + \frac{(d-2)C_d \Sigma_0}{2m} \right). \tag{2.39}
$$

Setting $g = g_c$ and taking the limit $m \ll \Sigma_0 \sim M$ we find

$$
\frac{m}{g_c^2} \left[ \frac{1 + (d-1) C_d}{(d-2) N_f} \ln \frac{\Sigma_0}{m} \right] = \frac{8\Sigma_0^{d-1}\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}(d-2)}, \tag{2.40}
$$

ie.

$$
\delta = (d-1) \left[ 1 + \frac{C_d}{N_f} \right], \tag{2.41}
$$

where we have assumed that the logarithm exponentiates beyond $O(1/N_f)$. To extract $\gamma$, we must return to the full expression (2.38a) for $D_\sigma(k^2; m)$, substitute it into (2.18) and evaluate $\partial/\partial m|_{m=0}$ on the resulting gap equation. After some work we find

$$
\frac{1}{g^2} \left[ 1 + (d-1) \frac{C_d}{N_f} \ln \frac{\Lambda}{\Sigma_0} \right] = (1 + \chi)\Sigma_0^{d-2} \frac{8\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}}. \tag{2.42}
$$

Near criticality $\chi \gg 1$, from which we can infer

$$
\chi \propto \Sigma_0^{-\left[ d-2 + (d-1)C_d/N_f \right]} \propto t^{-\left[ 1 + (d-1)C_d/(d-2)N_f \right]} \tag{2.43}
$$

using (2.36), ie:

$$
\gamma = 1 + \frac{(d-1) C_d}{(d-2) N_f}. \tag{2.44}
$$

We now see that our derived values for $\beta$, $\delta$ and $\gamma$ are consistent to $O(1/N_f)$ with the scaling relation

$$
\gamma = \beta(\delta - 1), \tag{2.45}
$$

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which raises the issue of whether the exponents calculated in this way obey scaling or hyperscaling relations. To develop this picture we need to calculate further critical exponents not directly determined by the behavior of the order parameter, and hence need to supply some physical insight. The obvious choice for correlation length is the inverse physical fermion mass $M^{-1}$. Using Eqs. (2.15) and (2.36) we find

$$ t \propto \xi^{-\frac{1}{\nu}} \propto M^{(d-2)[1-(d-1)C_d/dN_f]}, $$

leading to a prediction for the exponent $\nu$:

$$ \nu = \frac{1}{d-2} \left[ 1 + \frac{(d-1)C_d}{dN_f} \right]. $$

We can calculate the final exponent $\eta$ in two different ways. Firstly and more directly, consider the renormalized scalar propagator (ie. two-point correlator) (2.35): at criticality we expect the scaling form

$$ D^{-1}_\sigma \propto k^{2-\eta}. $$

From the $\mu \rightarrow 0$ limit of (2.35) we deduce

$$ \eta = 4 - d - \frac{2(d-1)C_d}{dN_f}. $$

Of course, strictly we should consider correlations of the scalar composite $\bar{\psi}\psi$, which defines the true order parameter. Following the treatment of [10], we can renormalize this local operator, and use (2.4) and (2.15), to write

$$ M = \frac{Z_{\bar{\psi}\psi}Z_{\bar{\psi}\psi}^{-1}g^2}{N_f} < \bar{\psi}_i \psi_i > = \frac{Z_{\bar{\psi}\psi}Z_{\bar{\psi}\psi}^{-1}g^2}{N_f} < (\bar{\psi}_i \psi_i)_R >, $$

where the constant $Z_{\bar{\psi}\psi}$ has been introduced to renormalize $\bar{\psi}_i \psi_i$. Now, both sides of (2.50) are cutoff-independent (since $M$ is a physical mass), as is $(\bar{\psi}_i \psi_i)_R$ itself; note also that a factor of $Z_{\psi}$ is absorbed in taking the trace to calculate $< \bar{\psi}_i \psi_i >$. We conclude that

$$ \Lambda \frac{\partial}{\partial \Lambda} \left( Z_{\bar{\psi}\psi}^{-1}g^2Z_{\bar{\psi}\psi} \right) = 0, $$

ie.

$$ \tilde{\gamma}_{\bar{\psi}\psi} = \Lambda \frac{\partial}{\partial \Lambda} Z_{\bar{\psi}\psi} = g^2 \Lambda \frac{\partial}{\partial \Lambda} \left( \frac{Z_{\bar{\psi}\psi}}{g^2} \right) = (d-2)\frac{g^2}{g_c^2} + \frac{(d-1)C_d}{dN_f}, $$

where we have used (2.5,15), and the quantity denoted by $\tilde{\gamma}_{\bar{\psi}\psi}$ is the anomalous dimension of the operator $\bar{\psi}\psi$ defined in the conventional field-theoretic sense [18]. At the fixed point $g = g_c$, we can use the resulting $\tilde{\gamma}_{\bar{\psi}\psi} = (d-2) + (d-1)C_d/dN_f$ to derive $\eta$. We expect the renormalized two-point correlation function $< (\bar{\psi}\psi(0))_R(\bar{\psi}\psi(x))_R >$ to scale asymptotically in momentum space as

$$ \Gamma^{(0):2}_R(k^2) \propto k^{-[d-2\tilde{\gamma}_{\bar{\psi}\psi}+2\tilde{\gamma}_{\bar{\psi}\psi}]} \propto k^{-[2-d+2\tilde{\gamma}_{\bar{\psi}\psi}]}, $$
where we have introduced the canonical (i.e., free-field) scaling dimension for $\bar{\psi}\psi$:

$$d^{(0)}_{\bar{\psi}\psi} = d - 1.$$  (2.54)

or in other words $\eta = d - 2\bar{\gamma}\bar{\psi}\psi$ using (2.48). The value for $\eta$ thus derived is identical to (2.49), demonstrating that these rather formal arguments about operator renormalization are valid, and that for all practical purposes $\sigma$ and $\bar{\psi}\psi$ can be considered identical.

The calculation of the critical exponents is now complete: to $O(1/N_f)$ the values we have computed for $\beta, \delta, \gamma, \nu$ and $\eta$ satisfy the hyperscaling relations (1.17):

$$2\beta + \gamma = d\nu; \ 2\beta\delta - \gamma = d\nu; \ \beta = \frac{1}{2}\nu(d - 2 + \eta); \ \gamma = \nu(2 - \eta).$$  (2.55)

In the realistic case $d = 3$ discussed in the rest of this paper, the values coincide with those given in Eq.(1.3). It is also interesting to examine the limits $d \to 4_-$ and $d \to 2_+$. Noting that $C_d$ vanishes in each case, as $(4 - d)$ and as $(d - 2)$ respectively, we find for $d \to 4$ we recover mean-field exponents:

$$\beta = \frac{1}{2}; \ \delta = 3; \ \gamma = 1; \ \nu = \frac{1}{2}; \ \eta = 0,$$  (2.56)

while for $d \to 2$ we find

$$\beta = \frac{1}{(d - 2)}; \ \delta = 1; \ \gamma = 1 + \frac{1}{2N_f}; \ \nu = \frac{1}{(d - 2)}; \ \eta = 2.$$  (2.57)

It is interesting to note that we could have inferred the renormalizability of the model merely by considering the exponents, without performing the integral (2.29). Suppose we had assumed hyperscaling to derive $\eta$ from the gap equation exponents $\beta, \delta, \gamma$, and made no further assumptions about operator renormalization, then we could have used relations (2.48) to predict the divergence structure of the scalar propagator at criticality:

$$D_{\sigma}^{-1}(k^2)_{g = g_c} \propto k^{2 - \eta} \sim k^{d - 2} \left[ 1 - \frac{2(d - 1)}{d} C_d \frac{\ln \Lambda}{N_f} \right].$$  (2.58)

The argument of the logarithm is determined on dimensional grounds, since the cutoff is the only other scale left in the problem. Similarly we could use an alternative definition of susceptibility, together with the derived values of $\gamma$ and $\nu$, to deduce the momentum-independent divergence:

$$\chi^{-1} \equiv D_{\sigma}^{-1}(k^2 = 0) \propto t^\gamma \propto \xi^\gamma \sim M^{d - 2} \left[ 1 - \frac{2(d - 1)}{d} C_d \frac{\ln \Lambda}{M} \right].$$  (2.59)

In both cases we can check that $Z_\sigma g^2$ given in (2.17) eliminates all dependence on the cutoff $\Lambda$ and renders $D_{\sigma}^{-1}$ finite. Hence we can infer renormalizability at this order without ever doing a two-loop calculation.
(although the gap equation has two loops, it has one fewer fermi propagator, and no external momentum dependence, and so is much simpler!).

There would appear to be an intimate connection between hyperscaling and renormalizability which we can demonstrate to all orders in $1/N_f$ without doing any detailed calculations. Consider the effect of shifting the $\sigma$ field in the Lagrangian (2.1) so that the fermions pick up a mass term $Z_\psi \Sigma_0 \bar{\psi}_i \psi_i$. It is then straightforward to derive the following functional identity [19]:

$$\frac{\partial}{\partial \Sigma_0} \frac{\delta^2 \Gamma}{\delta \psi \delta \bar{\psi}} = \frac{\sqrt{N_f}}{gZ_\sigma^2} \frac{\delta^3 \Gamma}{\delta \psi \delta \bar{\psi} \delta \sigma'},$$

(2.60)

where $\sigma'$ is the shifted scalar field and $\Gamma = \Gamma[\psi, \bar{\psi}, \sigma]$ is the effective action. We can use this relation in the broken phase to relate the fermion self-energy $\Sigma(k^2)$ to the full fermion-scalar vertex [9]:

$$\Sigma(0) = -\Sigma_0 \frac{\sqrt{N_f}}{gZ_\sigma Z_\psi} \Gamma_{\sigma \bar{\psi} \psi}(0),$$

(2.61)

The chiral symmetry of the original Lagrangian implies $\Sigma(0)$ is proportional to $\Sigma_0$, and power counting shows that $\Gamma_{\sigma \bar{\psi} \psi}$ and hence $\Sigma$ are at most logarithmically divergent functions of $\Lambda$. Using the definitions implicit in (2.11-16), we deduce

$$Z_\psi^{-1} Z_M^{-1} \Sigma_0 \propto -\frac{\Sigma_0}{g} Z_\sigma^{-\frac{1}{2}} Z_\psi^{-1},$$

ie.

$$Z_M^2 \propto Z_\sigma g^2.$$ (2.62)

The constant of proportionality is $O(M^{2-d})$ from (2.10). Now, we know that $Z_M$ has the form

$$Z_M = 1 + a \ln \frac{\Lambda}{M},$$

(2.63)

where we assume $a \sim O(1/N_f)$ is small, so that $\Sigma_0 \propto M^{1-a}$. By definition, we also have $t \propto \Sigma_0^{\frac{1}{2}} \propto M^{\frac{1-a}{2}}$, which enables us to relate $a$ to $\beta$ and $\nu$ assuming that higher powers of the logarithm exponentiate as required by the hypothesis of power-law scaling near a critical point:

$$1 - a = \frac{\beta}{\nu},$$

(2.64)

so from (2.63,64) we deduce

$$Z_\sigma g^2 \propto 1 + 2 \left(1 - \frac{\beta}{\nu}\right) \ln \frac{\Lambda}{M}.$$ (2.65)

From previous we know that if the model is renormalizable $Z_\sigma g^2$ must suffice to render the inverse scalar propagator $D_\sigma^{-1}$ finite, by cancelling both $k$-dependent and $k$-independent logarithmic divergences, once
power-law divergences have been removed by tuning $g^2$ to its critical value (Cf. Eq.(2.33)). Consider each case separately. At criticality, we must have on dimensional grounds

$$D^{-1}_\sigma(k^2) \propto k^{2-\eta} \sim k^{d-2} \left[ 1 - (4 - d - \eta) \ln \frac{\Lambda}{k} \right]. \tag{2.66}$$

If the logarithmic divergence in (2.66) is to be cancelled by the $Z_\sigma g^2$ of (2.65), then

$$2 \left( 1 - \frac{\beta}{\nu} \right) = 4 - d - \eta,$$

ie.

$$\beta = \frac{1}{2\nu} (d - 2 + \eta), \tag{2.67}$$

which is one of the hyperscaling relations (2.55). Similarly in the limit $k^2 \to 0$ we have

$$D^{-1}_\sigma(0) \propto M^{2-\eta} \sim M^{d-2} \left[ 1 - \left( 2 - d + \frac{\gamma}{\nu} \right) \ln \frac{\Lambda}{M} \right], \tag{2.68}$$

so that renormalizability implies

$$2\beta + \gamma = d\nu, \tag{2.69}$$

which is another hyperscaling relation. Hence with a few simple assumptions about the form of divergences at $O(1/N_f)$ and beyond, we can see the equivalence of hyperscaling and renormalizability. This puts the renormalizability of the model on a very physical footing; indeed, the main hypothesis behind both properties is the assumption that there is only one important physical length scale (correlation length) whose divergence in cutoff units at the critical point controls the scaling of all other physical quantities. In this model the physical length scale is the renormalized fermion mass $M$ in the broken phase and the scalar width $\mu$ in the symmetric phase: both ratios $\Lambda/M$ and $\Lambda/\mu$ diverge with the same critical exponent $\nu$ in the chiral limit at the phase transition, which defines the continuum limit. The continuity of $\nu$ is demanded on physical grounds [20], since otherwise the free energy of the system would be discontinuous across the phase transition.

Suppose we had found that in the broken phase the value of $\nu$ demanded by hyperscaling differed from the value $\nu_M$ derived by assuming that the renormalized fermion mass $M$ is indeed an inverse correlation length. We would then have been forced to conclude that either the interaction strength tended to zero as $g \to g_c \ (\nu_M > \nu)$, so that the model described free massive fermions, or that the fermions became very massive and decoupled ($\nu_M < \nu$), leaving a theory of non-interacting scalar bound states. In either case the model would have had a trivial continuum limit. However, this scenario is impossible in the present model, since we know from the identity (2.60,61) that the definition of renormalized fermion mass is tied to the renormalization constant $Z_\sigma g^2$, which in turn cancels out divergences in the inverse scalar propagator if and
only if hyperscaling is obeyed. Therefore a model in which \( \nu_M \neq \nu \) would be non-renormalizable. Thus in this model, and as we shall see, in similar four-fermi models, the issues of hyperscaling, renormalizability, and non-triviality are inseparable.

Finally, we note that in addition to references [9,11], there have been other calculations beyond leading order in the Gross-Neveu model for \( 2 < d < 4 \). Our result for \( Z_\phi \) is consistent with the calculation of Hikami and Muta [21], modulo different definitions of the trace over the Dirac algebra. We also agree with the results of [9,11,22] for \( d = 3 \). Gracey [23] has also computed the exponent \( \eta \) (in our notation) and has actually obtained \( Z_\psi \) to \( O(1/N_f^2) \); his sophisticated methods rely on being exactly at the critical point and hence cannot be used to calculate the other exponents. Other recent calculations beyond leading order for general \( d \) have appeared in [24].

Finally in this section, for completeness, we consider extensions of the Gross-Neveu model to cases where the spontaneously broken symmetry is a continuous one [5]. There are two interesting cases; the symmetries in the Lagrangian are either \( U(1)_L \otimes U(1)_R \) (\( \equiv U(1)_V \otimes U(1)_A \)):

\[
\mathcal{L} = \bar{\psi}_i \slashed{D} \psi_i - \frac{g^2}{2N_f} \left[ (\bar{\psi}_i \gamma_5 \psi_i)^2 - (\bar{\psi}_i \gamma_5 \psi_i)^2 \right];
\]  
(2.70a)

or \( SU(2)_L \otimes SU(2)_R \):

\[
\mathcal{L} = \bar{\psi}_p \slashed{D} \psi_p - \frac{g^2}{2N_f} \left[ (\bar{\psi}_p \gamma_5 \psi_p)^2 - (\bar{\psi}_p \gamma_5 \psi_p)^2 \right],
\]  
(2.70b)

where \( \vec{\tau} \) are the Pauli matrices, normalised to \( \text{tr}(\tau^\alpha \tau^\beta) = 2 \delta^{\alpha\beta} \), and we show the indices \( p, q \) running from 1 to 2 explicitly. In \( 2 < d < 4 \) dimensions we define \( \gamma_5 \) thus: \( \gamma_5^2 = 1 \), \( \{ \gamma_5, \gamma_\mu \} = 0 \), \( \text{tr}(\gamma_5 \gamma_\mu \cdots \gamma_\mu) = 0 \). We can immediately bosonize the models, and introduce renormalization constants as in (2.1):

\[
\mathcal{L} = Z_\psi \bar{\psi}_i \slashed{D} \psi_i + \frac{g}{\sqrt{N_f}} Z_\phi \frac{1}{2} \left[ \sigma \bar{\psi}_i \psi_i + i \pi \bar{\psi}_i \gamma_5 \psi_i \right] + \frac{1}{2} Z_\phi (\sigma^2 + \pi^2);
\]  
(2.71a)

\[
\mathcal{L} = Z_\psi \bar{\psi}_i \slashed{D} \psi_i + \frac{g}{\sqrt{N_f}} Z_\phi \frac{1}{2} \left[ \sigma \bar{\psi}_i \psi_i + i \vec{\sigma} \bar{\psi}_i \gamma_5 \vec{\tau} \psi_i \right] + \frac{1}{2} Z_\phi (\sigma^2 + \pi^2),
\]  
(2.71b)

where \( \sigma \) is an auxiliary scalar and \( \pi \) an auxiliary pseudoscalar. Note that in each case the combination \( \phi = \sigma + i \pi (\sigma + i \vec{\pi}) \) is proportional to an element of the chiral group, so that eg. in the \( SU(2)_L \otimes SU(2)_R \) case we have \( \psi_L \mapsto U \psi_L; \psi_R \mapsto V \psi_R; \phi \mapsto V \phi U^{-1} \), where \( \psi_{L,R} = (1 \pm \gamma_5) \psi/2 \), and \( U \) and \( V \) are \( SU(2) \) matrices. This property does not in general extend to higher flavor groups such as \( U(n) \). Model (2.71b) closely resembles the original Gell-Mann - Lévy \( \sigma \)-model [25]. Notice also that the same renormalization constant \( Z_\phi \) serves for both \( \sigma \) and \( \pi \) fields; this is the result of a Ward identity which we discuss below.

To leading order in \( 1/N_f \) model (2.70a) has an identical gap equation to (2.5) and expression for \( D_\sigma \) (2.7). The only new feature at this order is the pion propagator, which in the broken phase is given by

\[
D_\pi^{-1}(k^2) = Z_\phi g^2 \frac{2 \Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \frac{k^2}{\Sigma_0^4-d} F(1, 2 - \frac{d}{2}, \frac{3}{2}; -\frac{k^2}{4\Sigma_0^4}).
\]  
(2.72)
The pole at \( k^2 = 0 \) explicitly shows that the \( \pi \) field has a Goldstone mode in the broken phase. In the symmetric phase we find as before (2.24,26):

\[
D_{\pi}^{-1}(k^2) = D_{\pi}^{-1}(k^2) = \frac{Z_g g^2}{A_d} \left( (k^2)^{\frac{d}{2} - 1} + \mu^{d-2} \right).
\]

The SU(2)\( \otimes \)SU(2) model goes through in very similar fashion apart from an overall factor of 2 in the gap equation from the trace over Pauli indices, and a corresponding factor of \( \frac{1}{2} \) in \( D_{\sigma} \) and \( D_{\pi} \), the latter of course now carrying an implicit \( \delta^{\alpha\beta} \).

The calculation of \( O(1/N_f) \) corrections proceeds as before, the only new feature being the appearance of a logarithmic divergence in the \( O(1/N_f) \) correction to the gap equation containing an internal \( \pi \) line. The renormalization constants are found to be, for U(1)\( \otimes \)U(1):

\[
Z_\psi = 1 - \frac{(d - 2)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M};
\]

\[
Z_M = 1 + \frac{(d - 2)}{N_f} \frac{C_d}{d} \ln \frac{\Lambda}{M};
\]

\[
Z_g g^2 \propto 1 + \frac{2(d - 2)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M},
\]

with \( g^2 \) given by the gap equation:

\[
\frac{1}{g^2} = \frac{8 \Lambda^{d-2}}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})(d-2)} \left[ 1 - \frac{(d - 2)}{N_f} \right] - \frac{8 \Sigma_0^{d-2} \Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} (d-2)} \left[ 1 - \frac{2^{d-2} (d - 2)}{N_f} \right] + \frac{(d - 2) C_d}{N_f} \ln \frac{\Lambda}{\Sigma_0}. \tag{2.74}
\]

Similarly for SU(2)\( \otimes \)SU(2) we find:

\[
Z_\psi = 1 - \frac{(d - 2)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M};
\]

\[
Z_M = 1 + \frac{(d - 4)}{2d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M};
\]

\[
Z_g g^2 \propto 1 + \frac{(d - 4)}{d} \frac{C_d}{N_f} \ln \frac{\Lambda}{M},
\]

and the gap equation:

\[
\frac{1}{g^2} = \frac{16 \Lambda^{d-2}}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})(d-2)} \left[ 1 - \frac{2(d - 5)}{2N_f} \right] - \frac{16 \Sigma_0^{d-2} \Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} (d-2)} \left[ 1 - \frac{2^{d-2} (2d - 5)}{2N_f} \right] + \frac{3}{2} \frac{(d - 2) C_d}{N_f} \ln \frac{\Lambda}{\Sigma_0}. \tag{2.77}
\]

Once again, we find that the combination \( Z_g g^2 \) derived via vertex renormalization is also sufficient to cancel divergences in the two-loop scalar and pseudoscalar propagator corrections. The calculation of critical indices is straightforward, and the results for these, plus the critical coupling \( g_c^2 \), are tabulated in Table I. It is easily checked that the exponents satisfy the hyperscaling relations (2.55) to \( O(1/N_f) \).
The arguments (2.60 - 69) showing the relation between renormalizability and hyperscaling go through exactly as before, except that now because the broken symmetry is continuous, we can start from a fully-fledged Ward identity, e.g. for $U(1) \otimes U(1)$ we have

$$- \langle \sigma \rangle \Gamma_{\bar{\psi} \psi}(p_\pi = 0) = \frac{i}{2} \{ S_F^{-1}, \gamma_5 \}. \quad (2.78)$$

Using relations (2.3) and (2.12) we arrive at a result analogous to (2.62):

$$Z_M^2 \propto Z_\pi g^2. \quad (2.79)$$

To complete the argument we need a further Ward identity:

$$\Gamma_{\bar{\sigma} \bar{\psi} \psi} + \langle \sigma \rangle \Gamma_{\sigma \bar{\psi} \psi}(p_\pi = 0) = \frac{i}{2} \{ \Gamma_{\sigma \bar{\psi} \psi}, \gamma_5 \}. \quad (2.80)$$

The four-point function $\Gamma_{\bar{\sigma} \bar{\psi} \psi}$ is constructed from superficially convergent graphs, so needs no renormalization. Assuming $\gamma_5$ commutes with the scalar vertex, we see that $Z_\sigma = Z_\pi = Z_\phi$ as promised. Equivalent arguments hold for the $SU(2) \otimes SU(2)$ model.

We end this section with a comment about proving renormalizability. We have stressed the physical equivalence of renormalizability and hyperscaling for model field theories of this kind, and, we suspect, for a variety of other models with a non-trivial fixed point in $2 < d < 4$ [6,26]. A crucial point in our argument has been the existence of a chiral symmetry leading to identities such as (2.62, 79). The model will remain renormalizable so long as this symmetry is only broken by soft terms, i.e. a bare fermion mass. (The effects of introducing a symmetry-breaking term $(\bar{\psi}_i \psi_i)^3$ into $\mathcal{L}$ and the consequent non-renormalizability of the model have been discussed in [22].) It has been stated [9,22] that the renormalizability of the model follows from power-counting considerations alone, once the leading order form of the scalar propagator (2.7) has been established. We feel that this ignores the necessity for the consistency relations between the renormalization constants we have presented here – hence the chiral symmetry of the model must play a fundamental role in the full proof. Of course, this is not a new situation: power-counting alone does not suffice to prove the renormalizability of gauge theories. Moreover, the $SU(2) \otimes SU(2)$ model provides a specific example where next-to-leading order corrections render scalar exchange harder than leading order, since $\eta > 4 - d$. Power-counting arguments to analyze the degree of divergence of the graphs would fail in this case unless the momentum dependence of the dressed vertices is taken into account. The demonstration that $O(1/N_f)$ and higher corrections to scaling dimensions cancel in the correct fashion, so as to leave the set of primitively divergent graphs given by the leading order predictions unchanged, depends on consistency relations derived from Ward identities.
3. Lattice Formulation of the Gross-Neveu Model

The Gross-Neveu model in its bosonized form (1.2) may be formulated in three dimensions on a spacetime lattice using the following action:

\[
S = \sum_{i=1}^{N_f/2} \left( \sum_{x,y} \bar{\chi}_i(x) M_{x,y} \chi_i(y) + \frac{1}{8} \sum_{x} \bar{\chi}_i(x) \chi_i(x) \sum_{\langle \tilde{x}, x \rangle} \sigma(\tilde{x}) \right) + \frac{N_f}{4g^2} \sum_{\tilde{x}} \sigma^2(\tilde{x}), \tag{3.1}
\]

where \( \chi_i, \bar{\chi}_i \) are Grassmann-valued staggered fermion fields defined on the lattice sites, the auxiliary scalar field \( \sigma \) is defined on the dual lattice sites, and the symbol \( \langle \tilde{x}, x \rangle \) denotes the set of 8 dual lattice sites \( \tilde{x} \) surrounding the direct lattice site \( x \). The lattice spacing \( a \) has been set to one for convenience. The fermion kinetic operator \( M \) is given by:

\[
M_{x,y} = \frac{1}{2} \sum_{\mu} \eta_{\mu}(x) \left[ \delta_{y,x+\hat{\mu}} - \delta_{y,x-\hat{\mu}} \right], \tag{3.2}
\]

where \( \eta_{\mu}(x) \) are the Kawamoto-Smit phases \((-1)^{1+\cdots+x_{\mu-1}}\).

The Gross-Neveu model in two dimensions was first formulated using auxiliary fields on the dual sites in reference [27]. We can motivate this particular scheme by considering a unitary transformation to fields \( u \) and \( d \) [28]:

\[
u_{i}^{\alpha}(Y) = \frac{1}{4\sqrt{2}} \sum_{A} \Gamma_{A}^{\alpha} \chi_{i}(A; Y),
\]

\[
d_{i}^{\alpha}(Y) = \frac{1}{4\sqrt{2}} \sum_{A} B_{A}^{\alpha} \chi_{i}(A; Y). \tag{3.3}
\]

Here \( Y \) denotes a site on a lattice of twice the spacing of the original, and \( A \) is a lattice vector with entries either 0 or 1, which ranges over the corners of the elementary cube associated with \( Y \), so that each site on the original lattice corresponds to a unique choice of \( A \) and \( Y \). The \( 2 \times 2 \) matrices \( \Gamma_{A} \) and \( B_{A} \) are defined by

\[
\Gamma_{A} = \tau_{1}^{A_{1}} \tau_{2}^{A_{2}} \tau_{3}^{A_{3}},
\]

\[
B_{A} = (-\tau_{1})^{A_{1}} (-\tau_{2})^{A_{2}} (-\tau_{3})^{A_{3}}, \tag{4.1}
\]

where the \( \tau_{\mu} \) are the Pauli matrices. Now, if we write

\[
q_{i}^{\alpha}(Y) = \left( \begin{array}{c}
u_{i}^{\alpha}(Y) \\ d_{i}^{\alpha}(Y) \end{array} \right)^{a}, \tag{3.5}
\]

and interpret \( q \) as a 4-spinor with two flavors counted by the latin index \( a \), then the fermion kinetic term of the action (3.1) may be recast in Fourier space as follows:

\[
S_{\text{kin}} = \int \frac{d^{3}k}{(2\pi)^{3}} \sum_{i} \sum_{\mu} \frac{i}{2} \left\{ \tilde{q}_{i}(k)(\gamma_{\mu} \otimes 1_{2}) q_{i}(k) \sin 2k_{\mu} + \tilde{q}_{i}(k)(\gamma_{4} \otimes \tau_{\mu}^{*}) q_{i}(k) (1 - \cos 2k_{\mu}) \right\}. \tag{3.6}
\]
For the classical continuum limit lattice spacing $\alpha$ spacings in the time direction, and antiperiodic boundary conditions are imposed on the fermion fields. In $k$ extends over the range $k_\mu \in (-\pi/2, \pi/2)$. At non-zero temperature the lattice has finite extent in the temporal direction, and $\int dk_0$ is replaced by a sum over $N_\tau/2$ modes, where $N_\tau$ is the number of lattice spacings in the time direction, and antiperiodic boundary conditions are imposed on the fermion fields. In the classical continuum limit lattice spacing $\alpha \to 0$, the flavor non-diagonal terms vanish as $O(\alpha)$, and we recover the standard Euclidian form $\bar{q}_j \phi q_j$, where the flavor index $j$ now runs from 1 to $N_f$.

The interaction term between the fermions and the auxiliary requires a little care. We label the dual site $(x + \frac{1}{2}, x + \frac{1}{2}, x + \frac{1}{2})$ by $(A; \tilde{Y})$ where $x$ corresponds to $(A; Y)$. In this particular labelling the $\sigma$ fields are not all equivalent. For instance, for $A = (0, 0, 0)$ it is easy to show the interaction term transforms to

$$\sigma(0; \tilde{Y}) \bar{q}_i(Y)(14 \otimes 12)q_i(Y).$$

(3.8)

For $A = (1, 0, 0)$, however, it is more complicated:

$$\sigma(1; \tilde{Y}) \left[ \bar{q}_i(Y)(14 \otimes 12)q_i(Y) + \frac{1}{2} \Delta^+_1(\bar{q}_i(Y)(14 \otimes 12 + i\gamma_1 \gamma_4 \otimes \tau^*_1)q_i(Y)) \right],$$

(3.9)

where $\Delta^+_1 f(Y) \equiv f(Y + \hat{\mu}) - f(Y) \simeq 2a \partial_\mu f(Y)$. We see that there is an extra term containing non-covariant and flavor non-singlet interactions, which is formally $O(\alpha)$. If we used a formulation in which the $\sigma$ fields lived on the direct lattice sites, then such non-covariant terms would contribute at $O(\alpha^0)$ [27]. It is straightforward to extend this analysis to all $A$; we conclude that the interaction term may be written

$$S_{int} = \sum_Y \left( \sum_A \sigma(A; \tilde{Y}) \right) \bar{q}_i(Y)(14 \otimes 12)q_i(Y) + O(\alpha),$$

(3.10)

which is clearly of the same form as the equivalent term in (1.2). In principle the $O(\alpha)$ non-covariant terms could be important once quantum loop corrections are computed, and should be analyzed further. In the two-dimensional lattice Gross-Neveu model their effect is discussed in [29], where it is argued that in a perturbative expansion, even the on-site auxiliary formulation (with non-covariances present at $O(\alpha^0)$) yields the correct physics, that is with no important continuum symmetries violated. This analysis appears to be readily extended to the $1/N_f$ expansion in the three dimensional case.

Thus we see that the lattice action (3.1) reproduces the bosonised Gross-Neveu model, at least in the classical continuum limit. Most importantly, (3.1) has a discrete $Z_2$ global invariance under

$$\chi_i(x) \mapsto (-1)^{x_1 + x_2 + x_3} \chi_i(x); \quad \bar{\chi}_i(x) \mapsto -(-1)^{x_1 + x_2 + x_3} \bar{\chi}_i(x); \quad \sigma(\tilde{x}) \mapsto -\sigma(\tilde{x}),$$

(3.11a)
\[ q_i(Y) \mapsto (\gamma_5 \otimes 1)q_i(Y); \quad \bar{q}_i(Y) \mapsto -\bar{q}_i(Y)(\gamma_5 \otimes 1); \quad \sigma(\bar{x}) \mapsto -\sigma(\bar{x}). \quad (3.11b) \]

It is this symmetry, corresponding to the continuum form \( \psi \mapsto \gamma_5 \psi, \bar{\psi} \mapsto -\bar{\psi}\gamma_5 \), which is spontaneously broken at strong coupling, signalled by the appearance of a non-vanishing condensate \( <\bar{\chi}\chi> \) or equivalently \( <\bar{q}(1_4 \otimes 1_2)q> \). As discussed in Sec. 2, an equivalent order parameter is the scalar field expectation \( \Sigma_0 \).

To leading order in \( 1/N_f \), we can compute the fermion tadpole explicitly to yield the lattice gap equation, relating \( \Sigma_0 \) to \( 1/g^2 \) (Cf. Eq.(2.4)). Due to the simplicity of the loop integral, the contributions from the \( O(a) \) terms in (3.10) vanish, and we find

\[
\frac{\Sigma_0}{g^2} = \frac{1}{V} \text{tr} S_F = \int_{\pi/2}^{\pi/2} \frac{d^3k}{(2\pi)^3} \frac{1}{4} \sum_\mu [\sin 2k_\mu (\gamma_4 \otimes \tau^\mu_2) + (1 - \cos 2k_\mu)(\gamma_4 \otimes \tau^\mu_1)] + \Sigma_0 (1_4 \otimes 1_2) \sum_\mu \sin^2 k_\mu + \Sigma_2^2. \quad (3.12)
\]

Using standard techniques, we arrive at a form suitable for numerical quadrature:

\[
\frac{1}{g^2} = \int_0^\infty d\alpha e^{-\alpha^2 + \frac{1}{2}\Sigma_0^2} I_0^2(\frac{\alpha}{2}), \quad (3.13)
\]

where \( I_0 \) is the modified Bessel function. A plot of \( \Sigma_0 \) vs. \( 1/g^2 \) is shown in Fig. 5 – we see that in this regularisation scheme, to leading order in \( 1/N_f \), the bulk critical point \( 1/g^c \simeq 1.011a^{-1} \).

The action (3.1) was numerically simulated using the hybrid Monte Carlo algorithm [30], in which the Grassmann fields are replaced by real bosonic pseudofermion fields \( \phi(x) \) governed by the action

\[
S = \sum_{x,y} \sum_{i,j=1}^{N_f/2} \frac{1}{2} \phi_i(x)(M^T M)^{-1}_{x y i j} \phi_j(y) + \frac{N_f}{4g^2} \sum_{\bar{x}} \sigma^2(\bar{x}), \quad (3.14)
\]

where

\[
M_{x y i j} = M_{x y} \delta_{i j} + \delta_{x y} \delta_{i j} \frac{1}{8} \sum_{\bar{x}} \sigma(\bar{x}). \quad (3.15)
\]

Note \( M \) is strictly real, and that in this model we are able to work directly in the chiral limit bare fermion mass \( m = 0 \), since the matrix \( M \) has non-vanishing diagonal entries, and can always be inverted. Integration over \( \phi \) yields the functional measure \( \sqrt{\text{det}(M^T M)} \equiv \text{det}M \) if the determinant of \( M \) is positive semi-definite. This condition is fulfilled if \( N_f/2 \) is an even number.

The hybrid Monte Carlo algorithm works by evolving the fields through a fictitious “simulation time” \( \tau \) using a Hamiltonian \( H = \pi^2/2 + S[\sigma] \), where \( \pi(\bar{x}) \) is a momentum field conjugate to \( \sigma \). The fields are sampled after evolving through a period of fixed or variable \( \tau \) known as a trajectory. Detailed balance is maintained by accepting or rejecting the entire trajectory according to a Metropolis step calculated using the probability weight \( \exp(-\Delta H) \), and ergodicity by regenerating the \( \{\pi\} \) after each trajectory using gaussian random numbers. The Hamiltonian dynamics is simulated by finite difference equations parameterized by
some fundamental interval \( d\tau \): in the limit \( d\tau \to 0 \) we expect the dynamics to conserve energy, and hence the Metropolis acceptance rate to approach one. The art of hybrid Monte Carlo lies in tuning the parameters of the simulation so that \( d\tau \) can be made as large as possible, to reduce the amount of computer time per trajectory, whilst maintaining a reasonably high acceptance rate. One method, as originally discussed in [30], is to alter the couplings of the “guidance Hamiltonian” used to evolve the system through the trajectory; only the “acceptance Hamiltonian” used in the Metropolis step need be exact. The idea is that couplings, masses, etc. may receive \( O(d\tau) \) corrections, so that the algorithm is in effect producing exact Hamiltonian dynamics at a different point in coupling space. In the work presented here we have found that the acceptance rate of the algorithm may be optimized by tuning the parameter \( N_f \) in the action (3.14), so that the guidance Hamiltonian uses \( N_f' > N_f \). Interestingly, because quantum fluctuations are \( O(1/N_f) \), the amount of computer time needed to achieve comparable statistical accuracy is roughly independent of \( N_f \): the only requirement that increases with \( N_f \) is the storage of the pseudofermion fields.

The hybrid Monte Carlo algorithm proved to run sufficiently quickly and efficiently that unusually quantitative results could be obtained for large enough \( N_f \). Considerable details about the runs will be given below as different computer experiments are described, but a few words about the parameters used in the algorithm should be recorded here. In order to maintain a high acceptance rate so the algorithm would produce configurations that explored phase space rapidly, we tuned both \( N_f' \) and \( d\tau \). Typically as the lattice size was increased \( d\tau \) had to be taken smaller and \( N_f' \) approached \( N_f \). For example, on a \( 6^3 \) lattice when simulating the \( N_f = 12 \) theory, the choices \( N_f' = 13.35 \) and \( d\tau = 0.25 \) gave acceptance rates greater than 93% for all couplings of interest. To maintain this acceptance rate on a \( 12^3 \) lattice we picked \( N_f' = 12.38 \) and \( d\tau = 0.125 \), and on a \( 24^3 \) lattice \( N_f' = 12.17 \) and \( d\tau = 0.050 \). Measurements were taken only after a reasonable time interval had passed – it is pointless to take measurements after each sweep because sequential measurements are highly correlated, but measurements should not be taken so infrequently that information is lost. For most of our runs we chose a trajectory length \( \tau \) equal to half a time unit – in other words \( d\tau \) multiplied by the number of sweeps between refreshment was chosen to be 0.50. Note that on the \( 24^3 \) lattice a trajectory rises to 10 sweeps and the runs are proportionally more compute intensive than on small lattices. Of course, we never assumed that sequential trajectories were statistically independent. We used standard binning methods to calculate the variances in our measurements for each observable. In a typical run of 5,000 trajectories, say, a list of 5,000 measurements of the order parameter, etc. would be made and those lists would be analyzed to find the number of truly statistically independent measurements. Near the critical point we found the usual symptoms of critical slowing down – tens of trajectories were needed to decorrelate measurements of observables such as the order parameter. Variances could be calculated after each run and
the run extended if greater accuracy were needed. In our tables of results to be discussed below, we typically list the number of trajectories used and the statistical error bars for our raw data sets. Runs as long as 10,000 trajectories were performed near the system’s critical point to extract scaling laws and critical indices with good accuracy and confidence. Runs of this length are ten times as long as those currently practised in lattice QCD. We have such good statistics here because of the lower dimensionality of our model and the fact that each sweep has much fewer floating point operations. In addition, the number of sweeps of the conjugate gradient routines needed to effectively invert the lattice Dirac operator during each sweep is much smaller than in lattice QCD. Typically, only 10 - 100 conjugate gradient sweeps are needed for each inversion, while in lattice QCD ten times as many are needed in the present generation of simulations. The reason for the difference presumably lies in the fact that Eq. (1.1) has only a discrete rather than a continuous chiral symmetry. We very carefully monitored the convergence of the various conjugate gradient routines we used. There was the conjugate gradient used in the guidance Hamiltonian, that used in the acceptance Hamiltonian, and that used in measurements of the chiral condensate. The stopping residuals were typically chosen per lattice site to be $10^{-4}$, $10^{-6}$, and $10^{-4}$ respectively. Since conjugate gradient routines are only approximate, they could introduce an unwanted systematic error into the algorithm. Therefore, we carefully checked that our observables were insensitive to the size of the stopping residuals. Since the conjugate gradient algorithm converges monotonically at an exponential rate, it is relatively cheap (in computer time) to choose the stopping residual very safely and conservatively. Lengthy test runs were made to assure ourselves that all was in order.

We have not chosen to explore different values of $N_f$ systematically in this study, but rather to thoroughly explore the system’s critical behavior with the choice $N_f = 12$. However, in pilot studies on small lattices we did comparative runs at $N_f = 6$, 12, and 24. The results from a $12^4$ lattice for the expectation value $\Sigma_0 = <\sigma>$ vs, $1/g^2$ are plotted in Fig. 5. We see that chiral symmetry is indeed spontaneously broken, and for $1/g^2$ between 0.5 and 0.8 the measurements are in fair agreement with the leading order prediction (3.13): the points lie systematically below the line, which we interpret as being due to $O(1/N_f)$ corrections as predicted by Eq.(2.19). In Table II we give values for $\Sigma_0(N_f = \infty)$ given by (3.13) together with the measured normalized deviations $N_f(\Sigma_0(\infty) - \Sigma_0(N_f))$. The numbers for $N_f = 24$ and 12 are consistent within errors, implying that the deviation is $O(1/N_f)$. For $N_f = 6$ the deviation is consistently larger, possibly because for this small value $O(1/N^2_f)$ effects are becoming significant. A similar trend was found in a high-statistics study of the two-dimensional Gross-Neveu model [31]; here $N_f$ was varied between 4 and 120, and the $O(1/N_f)$ correction calculated exactly using a lattice regularization. It was found that an $O(1/N^2_f)$ term was required to accurately fit data for $N_f \leq 12$. More work is needed in three dimensions.
before reliable conclusions can be drawn; for one thing, at \( N_f = 6 \) the model does not have a positive definite determinant, and runs at adjacent values of \( N_f \) will be needed before the results of Table II can be taken seriously.

Finally we note that for \( N_f = 6, 12 \), chiral symmetry is restored for \( 1/g^2 \geq 0.9 \), and for \( N_f = 24 \) it is restored for \( 1/g^2 \geq 1.0 \). Finite volume effects to be discussed in the next section make the precise determination of the critical coupling \( 1/g_c^2 \) difficult. In the rest of the paper we will restrict our attention to the case \( N_f = 12 \), and concentrate on high precision studies in the region \( 1/g^2 \geq 0.7 \), using a much finer grid of \( 1/g^2 \) values. This will enable a quantitative description of the critical scaling properties of the model.

### 4. Symmetric Lattice Simulations and the Bulk Critical Point

Our emphasis in this section is the discovery of the critical point predicted by the large-\( N_f \) expansion and a numerical calculation of its critical indices, \( \beta \) (magnetic), \( \gamma \) (susceptibility) and \( \nu \) (correlation length). This program will be very successful for \( \beta \) and \( \gamma \) while an accurate determination of \( \nu \) will only occur once the behavior of the model at non-zero fermion density has been investigated [32]. To find the critical point we can simply calculate \( \Sigma_0 \), the vacuum expectation value of the auxiliary field. Since \( \Sigma_0 \) is proportional to the chiral condensate \( \langle \bar{\psi}\psi \rangle \) and since its vacuum expectation value spontaneously breaks chiral symmetry, it is a particularly convenient order parameter. On small lattices, however, vacuum tunneling processes which take \( \Sigma_0 \) to \(-\Sigma_0\) can obscure the critical point. The same problem affects computer simulation calculations of the magnetization in the Ising model, for example, and can be controlled by adding a small symmetry breaking field to the action. We chose not to do that here, although later we shall use this trick to measure the critical index \( \delta \). Instead, we monitored each computer simulation run for vacuum tunneling events. Away from the critical point in the symmetry-broken phase such events were so rare that good measurements of \( \Sigma_0 \) and its susceptibility \( \chi \), given by the variance of \( \Sigma_0 \), were possible. There are presumably two reasons for this. First, we chose \( N_f = 12 \) which is sufficiently large that fluctuations and tunneling processes were unlikely. And, second, our lattices were sufficiently large \( (8^3 - 20^3) \) that the probability of tunneling events was highly suppressed. However, in the immediate vicinity of the critical point tunneling events were sufficiently common that our computer data was not useful. Luckily we shall see that the scaling laws (critical indices) we seek were apparent in the data over regions of coupling where tunneling was not a problem.

In Figs. 6 – 9 we show the data for \( \Sigma_0 \) vs. \( 1/g^2 \) and the reciprocal of the susceptibility \( 1/\chi \) vs. \( 1/g^2 \) on both the ordered and disordered sides of the transition. Our raw computer data, statistical errors and
number of trajectories of the hybrid Monte Carlo algorithm used for each measurement can be found in Tables III – VI. For $1/g^2 < 1/g_c^2$, the critical point, we expect a non-analytic vanishing of $\Sigma_0$,

$$\Sigma_0 = C \left( \frac{1}{g_c^2} - \frac{1}{g^2} \right)^\beta,$$

where $\beta$ is the magnetic critical index. This simple power behavior is only expected for couplings $1/g^2$ sufficiently near the critical point $1/g_c^2$. In leading order of the large $N_f$ expansion $\beta = 1$. Similarly the susceptibility should diverge at $1/g_c^2$:

$$\chi = A \left( \frac{1}{g_c^2} - \frac{1}{g^2} \right)^{-\gamma} \quad g^2 > g_c^2$$

$$\chi = B \left( \frac{1}{g^2} - \frac{1}{g_c^2} \right)^{-\gamma} \quad g_c^2 > g^2$$

with the same index $\gamma$ above and below the transition. In leading order of the large $N_f$ expansion $\gamma = 1$.

The data plotted in Figs. 6 – 9 are beautifully fitted by the large $N_f$ predictions $\beta = 1$ and $\gamma = 1$. We are unable to extract the small finite $N_f$ correction (for $N_f = 12$, $\gamma = 1 + 8/N_f \pi^2 + O(N_f^{-2}) \simeq 1.068$, $\beta = 1 + O(N_f^{-2})$) expected in this simulation. It may be necessary to work very close to the critical point to find evidence for $\gamma = 1.068$ rather than $\gamma = 1$, and that would require larger lattices to evade vacuum tunnelling apparent in the figures and the Tables III – VI which accompany them. Note that the error bars on the measured $\Sigma_0$ values are smaller than the circles themselves of the figures. The error bars on the susceptibility $\chi$ are considerably larger ($5 – 10\%$) because it measures the width of the fluctuations in the order parameter. In Fig. 10 we plot $\ln \Sigma_0$ vs. $\ln(1/g_{c20}^2 - 1/g^2)$ where $1/g_{c20}^2 \simeq 0.955$ is the best estimate for the critical coupling of a $20^3$ lattice inferred from Fig. 9. The straight-line fit in Fig. 10 gives $\beta = 1.00$ and is in almost perfect agreement with the measurements at couplings $1/g^2 = 0.70 – 0.825$. Comparing Tables V and VI we see that over these couplings the $\Sigma_0$ measurements are in agreement to better than $1\%$ on the $16^3$ and $20^3$ lattices. Closer to the critical point we do not have comparable control over finite size effects. It would be interesting to redo Fig. 10 on a larger lattice and see if the $\Sigma_0$ values at $1/g^2 > 0.825$ approach the scaling curve. Similar curves for the susceptibility $\chi$ can be made (ln $\chi$ vs. ln$(1/g_{c20}^2 - 1/g^2)$) and the critical index $\gamma$ found to be $1.0(1)$. Much greater statistics would be needed to reduce the error on the determination of $\gamma$ to one percent where the finite $N_f$ corrections in Eq. (1.3) could be probed. A major barrier to our obtaining a higher precision determination of $\gamma$ is the uncertainty in the exact critical coupling and the fact that our estimates of it are hindered by vacuum tunnelling.

One notices from Figs. 6 – 9 that the peak of the system’s susceptibility shifts with the linear lattice size $L$. Finite size scaling arguments relate the size dependence to the correlation length exponent $\nu$;

$$\frac{1}{g_c^2(L)} - \frac{1}{g_c^2} = a L^{-\frac{1}{\nu}},$$

$$31$$
where $1/g_c^2(L)$ is the coupling where $\chi$ peaks on a $L^3$ lattice and $1/g_c^2$ is the infinite volume limit. Unfortunately the results shown in Figs. 6 – 9 do not determine $1/g_c^2(L)$ accurately enough to determine $\nu$ well. In Table VII we give the $1/g_c^2(L)$ values with error bars and in Fig. 11 we plot Eq. (4.3) for $\nu = 1$ and infer $1/g_c^2 \simeq 1.00$. In the next section of this article we shall study the four-fermi model on asymmetric lattices (nonzero temperature) and will achieve quantitative results for $\nu$.

5. Asymmetric Lattice Simulations, the Critical Temperature, and the Index $\nu$

We simulated the four-fermi model on asymmetric lattices, $N_r \times N^2$, to determine its critical temperature $T_c$ where chiral symmetry is restored and to determine the order of this transition. Lengthy runs, several tens of thousands of trajectories of the hybrid Monte Carlo algorithm, in the vicinity of the phase transition on lattices with $N_r$ ranging from 2 to 12 and $N$ set to $3N_r$ showed no convincing evidence for metastability or tunnelling between a symmetric and an asymmetric vacuum. So, we concluded that the transition was second order in agreement with the large $N_f$ analysis of this transition [17,32]. In Fig. 12 we show the histograms of measurements of $\Sigma$ (we reserve the subscript zero for the zero temperature value) in the ground state at couplings $1/g^2 = 0.86, 0.865$ and $0.870$ on a $10 \times 30^2$ lattice. At $1/g^2 = 0.860$ the system is in a chirally broken phase with $\Sigma$ very small, $0.039(1)$, but definitely nonzero. At $1/g^2 = 0.865$ vacuum tunnelling occurs between two states with $\Sigma = \pm 0.030$, and finally, at $1/g^2 = 0.870$ the distribution of $\Sigma$ measurements is centered around the origin indicating symmetry restoration. For $1/g^2$ values decreasing below $1/g_{\beta c}^2(N_r = 10) = 0.865(5)$, the mean values of $\Sigma$ increase smoothly, which is indicative of a second order phase transition. Since $T = 1/N_r = 0.10$ in lattice units, we see that the critical temperature measured in units of $\Sigma_0$ is $T_c/\Sigma_0 = 0.64(4)$ in this case. (Note from Fig. 9 that $\Sigma_0 = 0.157(1)$ at coupling $1/g^2 = 0.865$ at zero temperature.) For values of $1/g^2$ greater than $1/g_{\beta c}^2(N_r = 10)$ the mean values of $\Sigma$ on the $10 \times 30^2$ lattice were always consistent with zero and histograms of particular runs were well fitted with normal distributions centered at zero. Similar investigations on lattices with $N_r$ ranging from 2 to 12 led to the estimates of the critical couplings $1/g_{\beta c}^2(N_r)$ recorded in Table VIII. Using Fig. 9 to read off zero temperature vacuum expectation values of $\Sigma$ at these couplings produces the estimates of $T_c/\Sigma_0$ shown in Table IX and plotted in Fig. 13.

These results should be compared to the large $N_f$ prediction [17,32],

$$\frac{T_c}{\Sigma_0} = \frac{1}{2 \ln 2} \simeq 0.72,$$

which is plotted in Fig. 13. Our $N_r = 12$ result is within a standard deviation of the exact result and the
trend of the simulation results, $T_c/\Sigma_0$ increasing slowly with $N_\tau$, is quite satisfactory.

Next we can use the data in Table VIII to obtain a better estimate of the correlation length index $\nu$. The shifts of the critical couplings $1/g^2_{\beta c}(N_\tau)$ with lattice extent in the temporal direction should scale as

$$\frac{1}{g^2_{\beta c}(N_\tau)} - \frac{1}{g^2_c} = bN_\tau^{1/\nu}, \quad (5.2)$$

where $1/g^2_c$ is the large volume limit of the bulk critical coupling. From Fig. 10 we know that $1/g^2_c$ is between 0.95 and 1.00, but unfortunately cannot be specified with any greater precision. In Fig. 14 we plot Eq. (5.2) with the data of Table VIII for the choices of $1/g^2_c = 0.950, 0.976$ and 0.995. In each case linear fits of $\ln(1/g^2_c - 1/g^2_{\beta c}(N_\tau))$ vs. $\ln N_\tau$ are possible for $N_\tau = 6, 8, 10$ and 12. The slopes determine the critical index $\nu$, and we have from the fits $\nu = 0.81$ if $1/g^2_c = 0.950, \nu = 0.94$ if $1/g^2_c = 0.976$ and $\nu = 1.07$ if $1/g^2_c = 0.995$. Clearly our precision for a good determination of $\nu$ is limited by our uncertainty in the critical coupling. Nonetheless, our best estimate for $\nu$,

$$\nu = 0.94(13), \quad (5.3)$$

is consistent with and close to the large $N_\tau$ prediction of unity. In the second paper of this series [32] where we study the four-fermi model at nonzero chemical potential a more accurate and independent determination of $\nu$ will be given which lies within the error bars of Eq. (5.3).

6. The Critical Index $\delta$ for the Bulk and the Finite Temperature Transition

The critical index $\delta$ controls the response of a magnetic system’s magnetization $M$ at criticality to a small external magnetic field $h$:

$$M|_{T=T_c} \propto h^{1/\delta}. \quad (6.1)$$

For the four-fermi model $M$ maps into the chiral order parameter $\Sigma_0$ (or $<\bar{\psi}\psi>$) and $h$ maps into a bare fermion mass (which explicitly breaks the chiral symmetry), so

$$\Sigma_0|_{g^2=g^2_c} \propto m^{1/\delta}. \quad (6.2)$$

For large $N_f$ the index $\delta$ is predicted to be 2, Eq.(1.3). We attempted to measure $\delta$ by introducing a bare fermion mass into the theory’s action, and simulating the system on $8^3$, $16^3$ and $24^3$ lattices at criticality but variable $m$. Since we do not know $1/g^2_c$ exactly we did these simulations at $1/g^2 = 1.00$ and 0.975. In addition, since we do not know beforehand how small $m$ must be taken to see the scaling law Eq. (6.2), we simulated $m$ values ranging from 0.00375 to 0.25 in lattice units and searched numerically for a simple
scaling form, Eq. (6.2). We found that very accurate measurements of $\Sigma_0$ were possible. Vacuum tunnelling was always suppressed in these measurements because of the explicit symmetry breaking in the action. Our computer results are given in Tables X and XI.

In Fig. 15 we plot the data for a sequence of lattices $8^3, 16^3$ and $24^3$. In fact, $\ln \Sigma_0$ is plotted against $\ln m$ for $1/g^2 = 1.00$. We see finite size effects at the smallest $m$ values as expected – as $m$ decreases larger and larger lattices are needed to measure $\Sigma_0$’s nonzero value. For $m$ ranging from 0.0625 to 0.01625 we see the $8^3$ and $16^3$ data approaching the $24^3$ data from above. A linear fit over this range of $m$ gives $\delta = 2.0$ in almost perfect agreement with the large $N_f$ predictions. Note that fermion masses less than 0.01625 were not used in the fit because the $16^3$ and $24^3$ lattices do not give consistent values for $\Sigma_0$, indicating that finite size effects are still important here. Presumably, if one simulated $32^3$ lattices, a wider “scaling window” would be seen. Note that error bars on the $\Sigma_0$ measurements are smaller than the symbols in the figure.

In Fig. 16 we show identical plots for simulation runs at $1/g^2 = 0.975$ which could also be the bulk critical point. The value obtained for $\delta$ for the fit here is slightly larger, 2.21, than that discussed above. We again see that without better knowledge of the bulk critical coupling our uncertainty in the critical index $\delta$ will be about 10%. Uncertainty in the position of critical points usually limits the precision of finite size scaling studies in other statistical mechanics models as well.

Finally, we measured $\delta$ for the finite temperature rather than the bulk critical point. This is particularly interesting because the large $N_f$ prediction for $\delta$ associated with the finite temperature transition is 3 [17], indicative of a Gaussian model rather than the nontrivial scaling theory at the bulk critical point. To determine this critical index we used the largest lattice studied in Sec. 4, $12 \times 36^2$ and added a bare mass term to the action. In this case the critical coupling for the finite temperature transition on the $12 \times 36^2$ lattice was determined accurately in Sec. 4, $1/g^2_{c}(N_T = 12) = 0.880(5)$, so a relatively high precision measurement of its $\delta$ is possible. The computer data are recorded in Table XII. In Fig. 17 we show $\ln \Sigma_0$ vs. $\ln m$ for $m$ values ranging from 0.01 to 0.000625. Linear fits to the data produce the result $\delta = 3.0(1)$ in excellent agreement with our analytic expectations.

7. The Scalar Correlation Function

In this section we report on measurements of the scalar two-point function $<\sigma(0)\sigma(x)>$, and our attempts to understand them using the predictions of the $1/N_f$ expansion described in Sec. 2. Since we have been interested for the most part in the bulk properties of the model, we did not perform any special
simulations on lattices which are longer in one direction, which is normal practice in numerical QCD, where the requirement is to examine the asymptotic behavior of hadronic correlators in order to extract hadron masses. Further studies may benefit from this approach; however, as we shall see, the relation between the asymptotic decay of the correlator and the particle mass is by no means so straightforward in this model, and indeed, in the symmetric phase we are only able to extract a width by fitting to the sub-asymptotic behavior! Instead, we performed measurements on $20^3$ lattices at the values of the coupling $1/g^2$ used elsewhere.

As usual, in order to simplify the fit and maximize statistics we projected onto the zero spatial momentum sector by summing over timeslices, and define the correlator:

$$C(x) = \frac{1}{N_T} \sum_{\tau=1}^{N_T} \frac{1}{V_s^2} \left\langle \left( \sum_{\vec{y}} \sigma(\vec{y}; \tau) \right) \left( \sum_{\vec{y}} \sigma(\vec{y}; \tau + x) \right) \right\rangle,$$  \hspace{1cm} (7.1)

where $\vec{y}$ labels the $V_s$ sites existing on each of $N_T$ timeslices. Of course, the quantity of interest is really the connected correlator

$$C_c(x) = C(x) - \frac{1}{V_s^2} \langle \sum_{\vec{y}} \sigma(\vec{y}; 0) \rangle \langle \sum_{\vec{y}} \sigma(\vec{y}; x) \rangle.$$  \hspace{1cm} (7.2)

In our work we assume the expectation value $\langle \sigma \rangle$ is translationally invariant, and extract $\langle \sigma \rangle^2$ as a fitted parameter in fits of $C(x)$. This is a much more stable procedure than subtracting the measured value $\Sigma_0^2$ from the data before fitting, because the vacuum and the other quantities entering the fit are highly correlated. Of course, we expect agreement between the two quantities.

In order to parameterize the decay of $C_c(x)$, we use the leading order expressions for $D_\sigma$ given in Sec. 2. First we discuss fits in the broken phase, where the scalar is expected to be a genuine massive bound state. From (2.7) we have

$$C_c(x) \propto \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{(k^2 + M^2)F(1, \frac{1}{2}; \frac{3}{2}; -\frac{k^2}{M^2})},$$  \hspace{1cm} (7.3)

where here $M$ denotes the scalar mass. Immediately we see the difference between this model and QCD: the presence of the hypergeometric function in the denominator introduces branch cuts in the complex $k$-plane along the imaginary $k$-axis in the ranges $[iM, i\infty)$ and $(-i\infty, -iM]$. Instead of a discrete series of poles, corresponding to a set of bound states in the channel in question, which is the case in a confining theory such as QCD, we have a complicated spatial dependence due to a strongly-interacting fermion – anti-fermion continuum for $k > M_\sigma = 2\Sigma_0$ (to leading order in $1/N_f$) [9]. Performing the contour integral around the branch cut in the upper half-plane, we find

$$C_c(x) \propto P(x; M) = \int_1^\infty dv \frac{v e^{-Mv}}{(u^2 - 1)[\pi^2 + 4(\coth^{-1}v)^2]},$$

$$= \int_0^1 dt \frac{(1 + e^{-1/t})}{t^2(2 + e^{-1/t})} \frac{\exp(-Mx(1 + e^{-1/t}))}{\pi^2 + \ln^2(1 + 2e^{1/t})} + \int_0^{e/e+1} \frac{1}{u(1 - u)} \frac{e^{-Mx/u}}{\pi^2 + 4(\tanh^{-1}u)^2]}.$$

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where the second form is suitable for numerical quadrature. Due to the contribution of the fermion–antifermion continuum, \( P(x; M) \) decays faster than \( e^{-Mx} \), and the corrections to the latter form cannot be expressed as a discrete sum of exponentials over higher states. This means that we no longer have methods such as a search for a plateau in the effective mass plot to tell us when the form (7.4) might become valid: we must simply trust its validity over the whole range of \( x \), or at least that range exceeding the correlation length, where we might expect the continuum approximation to be accurate. Note that in ref. [31] the scalar correlator in the two-dimensional Gross-Neveu model, which is asymptotically free, was analyzed in this way, and it was found that the inclusion of a free fermion continuum in the fit did improve the results. In our case, however, the fermions are strongly interacting, so we are forced to try (7.4).

We have fitted \( C(x) \) for \( 1/g^2 \) ranging from 0.7 to 0.975, using the three-parameter form \( a[P(x; M) + P(L-x; M)] + \Sigma_0^2 \), where \( L = 20 \) is the length of the lattice. The results are presented in Table XIII and plotted in Fig. 18 (error bars are not shown for clarity). We used a least squares fit with a correlated \( \chi^2 \) function [33] to allow for correlation between timeslices. Errors were estimated using standard binning procedures. For comparison we also tried a standard single pole fit of the form \( a[e^{-Mx} + e^{-M(L-x)}] + \Sigma_0^2 \). In both cases we fitted for \( x \) ranging between 2 and 9: this produced \( \chi^2 \) values per degree of freedom \( \leq 1 \) in most cases (the maximum \( \chi^2 \) per d.o.f. found was 1.64).

We can make the following observations. The values of \( M \) obtained by the branch cut form \( P(x; M) \) lie systematically below those obtained from the single pole form, as expected: however the fluctuations in the two are clearly correlated. We see that \( M \) decreases as \( g^2 \to g_{c+}^2 \approx 0.96 \) on the \( 20^3 \) lattice, but does not reach zero as required by the hypothesis that it represents an inverse correlation length. The points at \( 1/g^2 = 0.95, 0.975 \) clearly lie outside this trend, probably because they are in or very close to the symmetric phase. The scatter of the \( M \) values about the downward trend is accounted for by the errors given in Table XIII. We also plot the fitted values of \( 2\Sigma_0 \) (which coincide within error bars for both fits), which at leading order in \( 1/N_f \) corresponds to the two fermion threshold (note that even for a correlation length of 10, the \( O(1/N_f) \) corrections predicted by Eq.(2.15) amount to just 5\% for \( N_f = 12 \). The line drawn through these points is that obtained from a fit to the data of Table VI, so we see the fitted values of \( \Sigma_0 \) coincide with good accuracy with the bulk averages, which is a consistency check of our fitting method.

If the scalar field interpolates a true bound state, then \( M \leq 2\Sigma_0 \). We expect the difference, representing the binding energy, to be \( O(1/N_f) \). We see that our fits for \( M \) using \( P(x; M) \) obey this criterion only for \( 1/g^2 < 0.875 \); moreover there is no clear region of the plot where the ratio \( M : 2\Sigma_0 \) is constant, so no estimate of \( M \) in physical units can be made. Given the systematic uncertainties in the fitting procedure highlighted
by the difference between the two fitted forms, we interpret this as an indication that the leading order form (7.3.4) is still not adequate to fit the measured correlator. Two further avenues suggest themselves. First, one could calculate the leading order scalar propagator in an explicit lattice regularization: this would probably need to be done numerically, but might, for instance, modify parameters such as the \( \pi^2 \) in the denominator of (7.4) which at present are put into the fits “by hand”. Secondly, one could bite the bullet and attempt to calculate \( D_\sigma(k^2) \) to \( O(1/N_f) \) in the broken phase, in order to be able to parameterize the separation between the bound state pole and the two-fermion threshold. Our attempts to do this in a naive way did not yield useful results.

Next we consider the scalar correlator in the symmetric phase. As discussed in Sec. 2, in this case the scalar field does not interpolate a stable massive particle state, but rather an unstable resonance. The resonance width \( \mu \) serves as an inverse correlation length, and can be extracted from the data as follows.

We start from the definition (7.1) of \( C(x) \) and the leading order form for \( D_\sigma(k^2) \) given in (2.26):

\[
C(x) \propto \int_0^\infty \frac{\cos kx}{k + \mu} = \text{Re} \left( e^{-i\mu x} E_1(-i\mu x) \right) \equiv g(\mu x).
\]

Here \( E_1(z) \) is the exponential integral function defined by [34]:

\[
E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt.
\]

For \( \mu x \to \infty \) we have

\[
g(\mu x) = \frac{1}{(\mu x)^2} \left( 1 + O\left( \frac{1}{(\mu x)^2} \right) \right).
\]

In this phase the scalar correlator decays asymptotically with a power law: the corrections to this power law at small \( \mu x \) enable us to extract the physical scale \( \mu \). Because of the power-law form, it is not sufficient to sum over just forward and backward propagating signals to get an accurate fit: one must sum over the signals from all “images” corresponding to propagation an arbitrary number of times around the periodic lattice. Using the asymptotic form (7.7) and the Euler-McLaurin formula, we can approximate this sum:

\[
C(x) \propto Q(x; \mu) = \sum_{n=-\infty}^{\infty} g(\mu (x + nL))
\]

\[
= \sum_{n=-N}^{N} g(\mu (x + nL)) + \frac{1}{\mu^2 L} \left( \frac{1}{NL + x} + \frac{1}{NL - x} \right)
\]

\[
- \frac{1}{2\mu^2} \left( \frac{1}{(NL + x)^2} + \frac{1}{(NL - x)^2} \right) + \frac{L}{6\mu^2} \left( \frac{1}{(NL + x)^3} + \frac{1}{(NL - x)^3} \right) + O(N^{-4}).
\]

In Table XIV and Fig. 18 we show the results of the two-parameter fit \( C(x) = aQ(x; \mu) \) to results obtained for \( 1/g^2 \) between 1.1 and 0.925. Once again, we fitted to \( x \) values 2 – 9, and the \( \chi^2 \) per d.o.f. was
in most cases less than one, with a maximum value 1.58. Fits of the standard simple pole form in this region were of worse quality and yielded higher values of $\mu$: moreover if we constrained the vacuum $\Sigma_0 = 0$, then these fits became disasterously poor. In most cases we found the series truncation in (7.8) was adequate for $N = 5$. Only for the very smallest $\mu$ at $1/g^2 = 0.925$ did we find some weak dependence of the fit on $N$ – here we quote the $N = 40$ result.

From Fig. 18 we see that in the symmetric phase the fitted parameter $\mu$ decreases fairly smoothly to zero as $g^2 \to g^2_{c-}$, and has every appearance of a genuine inverse correlation length. The data are not yet of sufficient quality to make a meaningful extraction of the exponent $\nu$, however. It is also not clear how reliable the points at $1/g^2 = 0.925$ and 0.95 are – perhaps a study on different sized lattices might be useful here. Overall, however, given the limitations of our statistics and lattice sizes, we find the results of the fit quite satisfactory. This is probably a sign that the leading order form of the propagator in the symmetric phase (2.26) is not grossly changed by $1/N_f$ corrections: this is of course verified by (2.35), where we find the only change to $D_\sigma(k^2)$ is an $O(1/N_f)$ correction to the power of $k$.

To sum up, we have identified a correlation length associated with the auxiliary field $\sigma$ in both phases of the model, which increases as $1/g^2 \to 1/g^2_{c\pm}$. Leading order $1/N_f$ calculations predict that the inverse correlation length should decrease linearly in this limit in either phase, ie. $\nu = 1 + O(1/N_f)$. In the symmetric phase our accuracy appears limited principally by statistics and lattice volume; the leading order prediction is qualitatively verified, but we are unable to fit directly for the exponent $\nu$. In the broken phase, the correlation length we extract from our fits does not diverge, and we suspect an improved understanding of the $1/N_f$ or $O(\alpha)$ corrections in this phase will be required before more progress can be made.

8. Summary

In this article we have attempted to reconcile two alternative views of the strongly-coupled critical point which is generic to the model four-fermi Lagrangian (1.1). In the language of statistical mechanics, we have characterized the fixed point by calculating its critical exponents, which in turn define its universality class. The indices satisfy consistency conditions known as hyperscaling relations, which supports the idea of a single divergent correlation length in the critical region. The critical correlation functions display non-canonical scaling, that is, the operators corresponding to physical observables acquire anomalous dimensions.

In particle physics language, we have seen that for some sufficiently large coupling, the most stable vacuum is one in which chiral symmetry is spontaneously broken, leading to a dynamically-generated fermion
mass. An expansion in powers of \(1/N_f\) about the critical point is exactly renormalizable for a continuum of dimensionality \(2 < d < 4\) — this behavior is unlike standard weak coupling perturbation expansions (WCPE) about the Gaussian fixed point, which are exactly renormalizable only for a single critical value of \(d\). Interactions in the model occur via exchange of scalar degrees of freedom. The composite nature of these particles ensures that the physical interaction strength remains non-vanishing as the UV cutoff is sent to infinity — the theory is non-trivial.

We have argued, in general terms in section 1 and in more detail in section 2, that these descriptions are equivalent, and that renormalizability, hyperscaling and non-triviality are inextricably linked in this model. The composite nature of the \(\sigma\) field is signalled by its large anomalous dimension. The main results of the explicit calculations of \(O(1/N_f)\) corrections in section 2 are as follows: firstly, at this order and beyond, logarithmic divergences appear, which necessitate vertex, wavefunction, and mass renormalizations. The set of primitively divergent diagrams is such that the renormalization constants are overdetermined, which means that renormalizability depends on a non-trivial cancellation between different one-particle irreducible (1PI) vertex functions. This cancellation can be viewed as a constraint on the critical indices equivalent to hyperscaling, but ultimately results from the existence of Ward identities arising from the chiral symmetry of the model. Secondly, we have seen that the critical exponents get explicit \(1/N_f\) corrections, implying that theories with different \(N_f\) belong to different universality classes. As we calculate to higher order in \(1/N_f\), we improve our understanding of the fixed point theory — this is in contrast to WCPE, where the fixed point is known to be Gaussian from the start, and perturbative corrections serve to parameterize trajectories in or near the critical hypersurface, that is, to relate massless theories at differing momentum scales. In the latter case we say the WCPE interaction is marginal, whereas in the four-fermi model the interaction \((\bar{\psi}\psi)^2\) is relevant, since any departure from the critical coupling strength \(g^2 \neq g_c^2\) results in the theory becoming massive, with a finite correlation length, either in the conventional sense \((t > 0)\), or by generating a dimensionful width for the \(\sigma\) resonance \((t < 0)\). So far we have no reason to suspect the critical hypersurface has a dimension greater than zero [22]. In simple terms, the \(1/N_f\) expansion can never yield a “running coupling constant”, since \(N_f\) is a basic parameter of the theory and is not itself renormalized. Thirdly, as a result of the consistency between the different 1PI amplitudes, or if preferred the chiral symmetry of the model, we see that in the deep Euclidean limit \(k^2 \to \infty\) the four-fermi scattering amplitude has no dependence on \(1/N_f\), apart from a trivial overall factor, and assumes a universal form \(A_d/N_f k^{d-2}\). This is the universal interaction which characterizes the UV fixed point. It is interesting that despite the fact that we have used the \(1/N_f\) expansion to calculate radiative corrections, the \(1/N_f\) corrections manifest themselves in the IR properties of the theory, such as the bulk critical exponents, and presumably the bound
state spectrum, about which we have little knowledge at present. Finally, we have shown that the picture we have developed is readily extended to models with a continuous chiral symmetry, so that there appear to be no problems incorporating massless Goldstone excitations.

On the numerical side, our main result is that the leading order predictions of the $1/N_f$ expansion have been verified in a non-perturbative simulation, suggesting that the $1/N_f$ expansion is accurate at least for $N_f \geq 12$. The critical exponents $\beta, \gamma, \delta$ and $\nu$ have all been directly estimated and found to be distinctly non-Gaussian – this is strong evidence for the presence of anomalous dimensions. Moreover, simulations at non-zero temperature support a crossover from the leading order result $\delta = d - 1$ to the Gaussian mean field value $\delta = 3$. We have not been able to measure explicit $1/N_f$ corrections in this study, but are able to estimate the extra computational effort required. We have also laid the groundwork for simulations at smaller $N_f$, where quantum fluctuations will be larger, and the $1/N_f$ corrections easier to detect. It is, of course, possible that for some sufficiently small $N_f$ the picture we have developed breaks down and the $1/N_f$ expansion loses its applicability – one suggestion has been that the transition becomes first order [35]. Another direction, which we have explored in a separate paper [32], is to simulate the model at non-vanishing chemical potential. Unlike the case for lattice gauge theories, the action (3.1) remains real under this modification, which means that we are able to study a relativistic theory of interacting fermions at non-zero density on far larger systems than are otherwise possible. As reported in [32], these simulations have enabled an independent estimate of the exponent $\nu$.

Progress on understanding the scalar two-point function has been slower, highlighting the difficulties in understanding the spectrum of a theory which is both strongly coupled and non-confining. The main novelty and success here was in the symmetric phase, where a “mass” (inverse correlation length) was extracted from a fit to the pre-asymptotic behavior of the correlator. Further progress in this direction probably depends upon a more complete understanding of $O(1/N_f)$ corrections in the broken phase, which is technically much tougher than the calculation presented here. Of course, in the long run one would wish for a complete understanding of the bound-state spectrum.

We hope that many of the aspects we have learned about this model will be of relevance to strongly coupled quantum electrodynamics in four dimensions, which has been the focus of much recent work by ourselves and others. Both models exhibit a continuous phase transition from a “massless” phase to one in which mass is dynamically generated – and both models appear to scale with non-Gaussian critical exponents [36]. The great hope is, of course, that QED$_4$ will turn out to be the first known example of a non-trivial non-asymptotically free theory in four dimensions. As yet, we have only analytic arguments based
on self-consistent approximations, and numerical simulations, which, however powerful, are always in need of some kind of interpretation. In the models discussed here we have the additional benefit of a systematic approximation, which has enabled us to make precise calculations and gain physical insight. Perhaps the ultimate result of this work will be the development of a concise vocabulary to describe strongly-coupled UV fixed points.

Appendix

Here we give some technical details of how to perform various momentum integrals in $d$-space. First, we define our Dirac matrices for all values of $d$ to be $4 \times 4$, traceless, and hermitian, so that the trace always yields a factor of 4. Alternative definitions are possible [21], but the results are qualitatively unchanged, at least to $O(1/N_f)$.

We define the integral over Euclidean $d$-momentum $p$ as follows:

\[
\int p f(p^2) = \frac{2}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^{\Lambda} p^{d-1} f(p^2) dp;
\]

\[
\int p_\mu f(p^2) = 0; \quad \int p_\mu p_\nu f(p^2) = \frac{\delta_{\mu\nu}}{d} \int p^2 f(p^2).
\] (A.1)

Although the formulæ are superficially similar, this is not dimensional regularisation; the cutoff $\Lambda$ appears explicitly in (A.1) and is used to regulate divergent integrals. The following formula, which holds for $d \in (2, 4)$, serves to evaluate loop integrals involving only fermi propagators:

\[
\int_p \frac{1}{(p^2 + M^2)^n} = \delta_{n1} \frac{2\Lambda^{d-2}}{(4\pi)^{\frac{d}{2}}(d-2)\Gamma(\frac{d}{2})} + \frac{M^{d-2n} \Gamma(n - \frac{d}{2})}{\Gamma(n)(4\pi)^{\frac{d}{2}}}. \] (A.2)

Finally, we give a brief outline of how to evaluate the two-loop integral (2.29). With the momentum routing specified the integral over $p$ is fairly straightforward: after taking the trace it reduces to several components which can each be brought to the form (A.2) using a suitable Feynman parameterization and momentum shift. If only one Feynman parameter is required in a particular sub-integral, it is then simple to extract divergent terms of the power-law form (2.30). If there are two or more Feynman parameters, then there will be an intermediate integral with a denominator, which is a function of $q$ and external momentum,
raised to a non-integer power $s$. The following formula is needed:

\[
\int A + Bx + Ca^2 \over (a + bx + cx^2)^s \ dx = \left( A - \frac{bB}{2c} + \frac{C(b^2 - 2ac)}{2c^2} \right) \left( \frac{4c}{\Delta} \right)^s \left( x + \frac{b}{2c} \right) F \left( s, \frac{1}{2}; \frac{3}{2}; -\frac{4c^2}{\Delta} (x + \frac{b}{2c})^2 \right) + \left( B - \frac{bC}{c} \right) \frac{2(1-s)c}{\Delta + 4c^2 (x + \frac{b}{2c})^2}](A.3)
\]

where $\Delta \equiv 4ac - b^2$ and $F$ is the hypergeometric function. This result is exact. However, it turns out that the momentum-dependent logarithmic divergences (2.31) naturally arise from the lower limit of the Feynman parameter integration, i.e. from setting $x = 0$ above. Also, in practice, the coefficients $b, c$ are $O(q^2)$, where $q$ is the remaining loop momentum, but $a$ is $O(q^0)$. These two considerations make the following simplifications possible:

\[
\int_0^1 \frac{dx}{(a + bx + cx^2)^s} = -\frac{b}{\Delta(s-1)a^{s-1}} + O\left( \frac{1}{q^1} \right); \quad \quad (A.4a)
\]

\[
\int_0^1 \frac{x \ dx}{(a + bx + cx^2)^s} = \frac{2}{\Delta(s-1)a^{s-2}} + \frac{(3-2s)}{\Delta(s-2)(s-1)a^{s-2}} + O\left( \frac{1}{q^s} \right); \quad \quad (A.4b)
\]

\[
\int_0^1 \frac{x^2 \ dx}{(a + bx + cx^2)^s} = \frac{2(3-2s)b}{\Delta^2(s-2)(s-1)a^{s-3}} + \frac{2(5-2s)b}{\Delta^2(s-3)(s-2)a^{s-3}} + O\left( \frac{1}{q^s} \right). \quad \quad (A.4c)
\]

With judicious choice of Feynman parameterization these formulae suffice to evaluate (2.29). For example, suppose after taking the trace, introducing Feynman parameters $x$ and $y$, and shifting the momentum $p$, we are left with the following $p$ sub-integral:

\[
I(q,k) = 8 \int_0^1 dx \int_0^{1-x} dy \int_p \frac{xk^2 + y k.q}{[p^2 + x(1-x)k^2 + y(1-y)q^2 - 2xyk.q]^3}. \quad (A.5)
\]

We can do the integration over $p$ using (A.2):

\[
I(q,k) = 4 \frac{\Gamma(3 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \int_0^{1-x} dy \frac{xk^2 + y k.q}{[x(1-x)k^2 + (q^2 - 2xk.q)y - q^2y^2]^{3 - \frac{d}{2}}}. \quad (A.6)
\]

Now use (A.4) to do the integral over $y$; only the lower limit is relevant:

\[
I(q,k) = 4 \frac{\Gamma(3 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx x^{\frac{d}{2} - 1}(1-x)^{\frac{d}{2}-2}(k^2)^{\frac{d}{2}-1} \left( \frac{1}{(q^2 + 4x(1-x)k^2)^{\frac{d}{2}} + O\left( \frac{1}{q^{\frac{d}{2}}} \right)} \right). \quad (A.7)
\]

The remaining integral over $x$, and the subsequent integral over $q$ are now straightforward, the final contribution to (2.29) being

\[
J(k) = -\frac{Z \sigma g^2}{N_f} \int_q \frac{A_d}{(q^2)^{\frac{d}{2} - 1} + \mu^{d-2}} I(q,k)
\]

\[
= -8 \frac{Z \sigma g^2}{N_f} \frac{\Gamma(2 - \frac{d}{2})B(2, \frac{d}{2} - 1)A_d}{(4\pi)^d \Gamma(\frac{d}{2})} (k^2)^{\frac{d}{2}-1} \ln \frac{A}{\mu k}. \quad (A.8)
\]
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Table I

Table of critical exponents and critical couplings $1/G_c^2 = 1/g_c^2 \times (4\pi)^{d/2}\Gamma(\frac{d}{2})(d-2)/8\Lambda^{d-2}$ for the three variants of the Gross-Neveu model considered in section 2. The constant $C_d$ is defined in Eq.(2.13).

|       | $Z_2$ | $U(1)_L \otimes U(1)_R$ | $SU(2)_L \otimes SU(2)_R$ |
|-------|-------|-------------------------|---------------------------|
| $\beta$ | $\frac{1}{(d-2)}$ | $\frac{1}{(d-2)}\left[1 + \frac{C_d}{N_f}\right]$ | $\frac{1}{(d-2)}\left[1 + \frac{3 C_d}{2 N_f}\right]$ |
| $\delta$ | $(d-1)\left[1 + \frac{C_d}{N_f}\right]$ | $(d-1)\left[1 + \frac{(d-2) C_d}{(d-1) N_f}\right]$ | $(d-1)\left[1 + \frac{(d-4) C_d}{2(d-1) N_f}\right]$ |
| $\gamma$ | $1 + \frac{(d-1) C_d}{(d-2) N_f}$ | $1 + \frac{2 C_d}{N_f}$ | $1 + \frac{(2d-5) C_d}{(d-2) N_f}$ |
| $\nu$ | $\frac{1}{(d-2)}\left[1 + \frac{(d-1) C_d}{d N_f}\right]$ | $\frac{1}{(d-2)}\left[1 + \frac{2(d-1) C_d}{d N_f}\right]$ | $\frac{1}{(d-2)}\left[1 + \frac{2(d-1) C_d}{d N_f}\right]$ |
| $\eta$ | $4 - d - \frac{2(d-1) C_d}{d N_f}$ | $4 - d - \frac{2(d-2) C_d}{d N_f}$ | $4 - d + \frac{(4-d) C_d}{d N_f}$ |
| $\frac{1}{G_c^2}$ | $1 - \frac{(d-1)}{2 N_f}$ | $1 - \frac{(d-2)}{N_f}$ | $2\left[1 - \frac{(2d-5)}{2 N_f}\right]$ |
Table II

Vacuum expectation value $\Sigma_0(N_f = \infty)$ as predicted by the leading order lattice gap equation, together with measured deviations $N_f(\Sigma_0(\infty) - \Sigma_0(N_f))$ vs. coupling $1/g^2$, for $N_f = 6, 12, 24.$

| $1/g^2$ | $\Sigma_0(\infty)$ | $24(\Sigma_0(\infty) - \Sigma_0(24))$ | $12(\Sigma_0(\infty) - \Sigma_0(12))$ | $6(\Sigma_0(\infty) - \Sigma_0(6))$ |
|---------|---------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| .5      | .8351               | .199(14)                             | .227(12)                             | .267(10)                             |
| .6      | .6401               | .310(19)                             | .319(12)                             | .382(11)                             |
| .7      | .4711               | .408(22)                             | .402(13)                             | .517(15)                             |
| .8      | .3165               | .542(34)                             | .550(24)                             | .634(30)                             |
Table III

Vacuum expectation value $\Sigma_0$ and its susceptibility $\chi$ vs. coupling $1/g^2$ on a $8^3$ lattice. The number of trajectories of the hybrid Monte Carlo algorithm at each value of $1/g^2$ are listed in column 4.

| $1/g^2$ | $\Sigma_0$     | $\chi$     | Trajectories |
|---------|----------------|------------|--------------|
| .70     | .4397(12)      | .265(8)    | 5,000        |
| .725    | .3974(14)      | .308(12)   | 5,000        |
| .75     | .3544(16)      | .400(14)   | 5,000        |
| .775    | .3114(20)      | .534(25)   | 5,000        |
| .80     | .2693(20)      | .721(31)   | 7,500        |
| .825    | .2136(21)      | 1.485(35)  | 7,500        |
| .85     | .1579(19)      | 2.07(14)   | 10,000       |
| .875    | .1152(19)      | 1.75(18)   | 10,000       |
| .90     | –              | 1.38(18)   | 10,000       |
| .925    | –              | .916(44)   | 10,000       |
| .95     | –              | .633(41)   | 7,500        |
| .975    | –              | .489(32)   | 7,500        |
| 1.00    | –              | .363(21)   | 5,000        |
| 1.025   | –              | .270(9)    | 5,000        |
| 1.05    | –              | .215(7)    | 5,000        |
Table IV

Vacuum expectation value $\Sigma_0$ and its susceptibility $\chi$ vs. coupling $1/g^2$ on a $12^3$ lattice.

| $1/g^2$ | $\Sigma_0$     | $\chi$     | Trajectories |
|---------|----------------|------------|--------------|
| .70     | .4319(10)      | .511(11)   | 5,000        |
| .725    | .3905(12)      | .621(13)   | 5,000        |
| .75     | .3482(12)      | .695(14)   | 5,000        |
| .775    | .3091(13)      | .876(25)   | 5,000        |
| .80     | .2655(15)      | .974(33)   | 5,000        |
| .825    | .2255(15)      | 1.06(7)    | 7,000        |
| .85     | .1786(17)      | 2.34(17)   | 7,000        |
| .875    | .1262(21)      | 3.28(75)   | 10,000       |
| .90     | .0853(25)      | 3.38(81)   | 10,000       |
| .925    | –              | 6.38(2.00) | 10,000       |
| .95     | –              | 3.75(1.00) | 10,000       |
| .975    | –              | 2.18(22)   | 5,000        |
| 1.00    | –              | 1.87(21)   | 5,000        |
| 1.025   | –              | 1.53(16)   | 5,000        |
| 1.05    | –              | 1.22(4)    | 5,000        |
Table V

Vacuum expectation value $\Sigma_0$ and its susceptibility $\chi$ vs. coupling $1/g^2$ on a $16^3$ lattice.

| $1/g^2$ | $\Sigma_0$   | $\chi$   | Trajectories |
|---------|--------------|----------|--------------|
| .70     | .4323(15)    | .546(24) | 1,000        |
| .725    | .3888(15)    | .556(25) | 1,000        |
| .75     | .3462(17)    | .741(27) | 1,000        |
| .775    | .3065(18)    | .826(45) | 1,000        |
| .80     | .2625(21)    | .988(60) | 1,000        |
| .825    | .2219(22)    | 1.19(8)  | 1,000        |
| .85     | .1747(23)    | 1.85(30) | 2,000        |
| .875    | .1306(25)    | 3.05(35) | 2,000        |
| .90     | .0885(35)    | 4.12(1.12) | 2,000    |
| .925    | –            | 10.6(4.0) | 2,000        |
| .95     | –            | 4.97(1.60) | 2,000        |
| .975    | –            | 2.94(50)  | 1,000        |
| 1.00    | –            | 3.22(50)  | 1,000        |
| 1.025   | –            | 1.22(14)  | 1,000        |
| 1.05    | –            | .97(8)    | 1,000        |
Table VI

Vacuum expectation value $\Sigma_0$ and its susceptibility $\chi$ vs. coupling $1/g^2$ on a $20^3$ lattice.

| $1/g^2$ | $\Sigma_0$         | $\chi$         | Trajectories |
|---------|-------------------|----------------|--------------|
| .70     | .4317(2)          | .611(15)       | 5,000        |
| .725    | .3883(2)          | .633(15)       | 5,000        |
| .75     | .3460(2)          | .746(25)       | 5,000        |
| .775    | .3045(3)          | .837(51)       | 5,000        |
| .80     | .2616(3)          | .948(55)       | 7,500        |
| .825    | .2189(3)          | 1.21(8)        | 7,500        |
| .85     | .1800(4)          | 1.40(9)        | 7,500        |
| .875    | .1376(5)          | 2.24(15)       | 10,000       |
| .90     | .0873(15)         | 4.07(1.15)     | 10,000       |
| .925    | .0406(20)         | 3.86(1.16)     | 10,000       |
| .95     | –                 | 7.49(2.50)     | 7,500        |
| .975    | –                 | 4.19(1.20)     | 7,500        |
| 1.00    | –                 | 2.28(32)       | 7,500        |
| 1.025   | –                 | 1.66(15)       | 5,000        |
| 1.05    | –                 | 1.35(10)       | 5,000        |
| 1.075   | –                 | 1.05(7)        | 5,000        |
| 1.10    | –                 | .770(2)        | 5,000        |
Table VII

Estimates of the bulk critical point on $L^3$ lattices with $L = 8, 12, 16, 20$.

| $L$ | $1/g_c^2(L)$   |
|-----|----------------|
| 8   | .867(8)        |
| 12  | .925(20)       |
| 16  | .930(20)       |
| 20  | .950(20)       |

Table VIII

Estimates of the finite temperature critical points on $N_{\tau} \times N^2$ lattices for $N_{\tau} = 2, 4, 6, 8, 10$ and 12.

| $N_{\tau}$ | $1/g_{\beta c}^2$ |
|------------|--------------------|
| 2          | .47(1)             |
| 4          | .695(5)            |
| 6          | .785(5)            |
| 8          | .833(5)            |
| 10         | .865(5)            |
| 12         | .880(20)           |
Table IX

Zero temperature vacuum expectation values of $\Sigma_0$ (from Fig. 9) at the couplings of the finite temperature critical points recorded in Table VII, and $T_c$ estimates for $N_\tau$ ranging from 2 through 12 in units of $\Sigma_0$.

| $N_\tau$ | $\Sigma_0$  | $T_c/\Sigma_0$ |
|---------|-------------|----------------|
| 2       | –           | –              |
| 4       | .44(1)      | .57(1)         |
| 6       | .29(1)      | .57(2)         |
| 8       | .21(1)      | .60(3)         |
| 10      | .157(10)    | .64(4)         |
| 12      | .125(10)    | .67(5)         |
Table X

Vacuum expectation value $\Sigma_0$ vs. fermion bare mass $m$ at coupling $1/g^2 = 1.00$ on $8^3$, $16^3$ and $24^3$ lattices. The number of trajectories used for each measurement are given in the last column.

| $m$  | $\Sigma_0(8^3)$ | $\Sigma_0(16^3)$ | $\Sigma_0(24^3)$ | Trajectories |
|------|-----------------|------------------|------------------|--------------|
| .005 | .0496(12)       | .0614(10)        | .0594(10)        | 10,000       |
| .008 | .0703(13)       | .0783(9)         | .0814(10)        | 10,000       |
| .0125| .0931(5)        | .106(1)          | .106(1)          | 10,000       |
| .01625| .118(2)        | .122(1)          | .123(1)          | 10,000       |
| .020 | .138(1)         | .138(1)          | .137(1)          | 10,000       |
| .025 | .155(1)         | .154(1)          | .153(1)          | 10,000       |
| .0375| .190(1)         | .187(1)          | .187(1)          | 10,000       |
| .050 | .218(1)         | .213(1)          | .213(1)          | 5,000        |
| .0625| .239(1)         | .234(1)          | .235(1)          | 5,000        |
| .075 | .257(1)         | .253(1)          | .252(1)          | 5,000        |
| .10  | .287(1)         | .282(1)          | .282(1)          | 5,000        |
| .15  | .327(1)         | –                | –                | 5,000        |
| .20  | .355(1)         | –                | –                | 5,000        |
| .25  | .377(1)         | –                | –                | 5,000        |
Table XI

Vacuum expectation value $\Sigma_0$ vs. fermion bare mass $m$ at coupling $1/g^2 = 0.975$ on $8^3$, $16^3$ and $24^3$ lattices.

| $m$  | $\Sigma_0(8^3)$ | $\Sigma_0(16^3)$ | $\Sigma_0(24^3)$ | Trajectories |
|------|-----------------|------------------|------------------|--------------|
| .005 | .0647(10)       | .0705(5)         | .0767(3)         | 10,000       |
| .008 | .0943(8)        | .0968(3)         | .0980(3)         | 10,000       |
| .0125| .118(2)         | .123(1)          | .122(1)          | 10,000       |
| .01625| .134(1)        | .135(1)          | .129(1)          | 10,000       |
| .020 | .150(1)         | .152(1)          | .152(1)          | 10,000       |
| .025 | .170(1)         | .170(1)          | .169(1)          | 10,000       |
| .0375| .207(1)         | .203(1)          | .202(1)          | 10,000       |
| .050 | .233(1)         | .229(1)          | .228(1)          | 5,000        |
| .0625| .254(1)         | .250(1)          | .249(1)          | 5,000        |
| .075 | .272(1)         | .267(1)          | .267(1)          | 5,000        |
| .10  | .300(1)         | .297(1)          | .297(1)          | 5,000        |
Table XII

Vacuum expectation value $\Sigma$ vs. fermion bare mass $m$ on a $12 \times 36^2$ lattice at coupling $1/g^2 = 0.8775$. The number of trajectories used for each measurement are given in the last column.

| $m$    | $\Sigma$    | Trajectories |
|--------|-------------|--------------|
| .01    | .1826(8)    | 1,000        |
| .005   | .1471(4)    | 1,000        |
| .0025  | .1151(2)    | 1,000        |
| .0018  | .1084(2)    | 1,000        |
| .00125 | .0903(10)   | 1,200        |
| .0009  | .0805(10)   | 1,200        |
Table XIII

Results of fits to the scalar correlator in the broken phase, using both the branch cut form $P(x; M)$ and the usual simple pole form.

| $1/g^2$ | $M$     | $\Sigma_0^2$ | $\chi^2$ | $M$     | $\Sigma_0^2$ | $\chi^2$ |
|---------|---------|--------------|----------|---------|--------------|----------|
| .7      | .708(98)| .18606(1)    | 3.3      | .812(94)| .18606(1)    | 2.9      |
| .725    | .750(95)| .15069(1)    | 4.2      | .836(87)| .15069(1)    | 4.3      |
| .75     | .494(69)| .11958(1)    | 3.4      | .607(64)| .11958(1)    | 3.7      |
| .775    | .388(58)| .092382(18)  | 3.3      | .513(51)| .092391(15)  | 5.1      |
| .8      | .433(58)| .068532(17)  | 6.2      | .553(51)| .068539(15)  | 5.9      |
| .825    | .348(48)| .047656(23)  | 8.2      | .478(40)| .047667(17)  | 6.7      |
| .85     | .320(57)| .032503(30)  | 2.1      | .450(50)| .032519(22)  | 1.9      |
| .875    | .290(36)| .018956(25)  | 3.1      | .433(27)| .018982(16)  | 5.1      |
| .9      | .281(42)| .007620(32)  | 4.7      | .428(29)| .007653(19)  | 2.5      |
| .925    | .247(55)| .001906(46)  | 3.1      | .387(39)| .001941(23)  | 2.4      |
| .95     | .354(57)| .000812(24)  | 2.0      | .475(51)| .000821(19)  | 2.9      |
| .975    | .312(61)| .000383(29)  | 3.1      | .459(51)| .000407(19)  | 2.5      |
Table XIV

Results of fitting the scalar correlator in the symmetric phase to the form $Q(x; \mu)$.

| $1/g^2$ | $\mu$       | $\chi^2$ |
|---------|-------------|----------|
| 1.1     | 0.437(55)   | 5.8      |
| 1.075   | 0.283(33)   | 3.1      |
| 1.05    | 0.159(14)   | 9.5      |
| 1.025   | 0.156(13)   | 4.6      |
| 1.0     | 0.146(12)   | 4.0      |
| 0.975   | 0.075(5)    | 4.7      |
| 0.95    | 0.041(2)    | 8.3      |
| 0.925   | 0.019(1)    | 4.8      |
Figure Captions

**Figure 1**: $O(1/N_f)$ corrections to the fermion self-energy and the fermion-scalar vertex.

**Figure 2**: $O(1/N_f)$ correction to the gap equation.

**Figure 3**: $O(1/N_f)$ corrections to the scalar propagator.

**Figure 4**: Schematic diagram showing $1/N_f$ corrections to the four-fermion scattering amplitude.

**Figure 5**: Plot of $\Sigma_0$ vs. $1/g^2$ on a $12^3$ lattice, for $N_f = 24$ (squares), $N_f = 12$ (circles), and $N_f = 6$ (triangles). The solid line is the leading order solution to the lattice gap equation (3.13). Errors are smaller than the size of the symbols.

**Figure 6**: Plot of the data recorded in Table III. $\chi^{-1}$, the reciprocal of the susceptibility, is plotted multiplied by $10^{-1}$.

**Figure 7**: Plot of the data recorded in Table IV. $\chi^{-1}$ is plotted multiplied by $10^{-1}$.

**Figure 8**: Plot of the data recorded in Table V. $\chi^{-1}$ is plotted multiplied by $10^{-1}$.

**Figure 9**: Plot of the data recorded in Table VI. $\chi^{-1}$ is plotted multiplied by $10^{-1}$.

**Figure 10**: Plot of $\ln \Sigma_0$ vs. $\ln(1/g_c^2 - 1/g^2)$ from Table VI, $1/g_c^2 = 0.955$.

**Figure 11**: Plot of the bulk critical point’s dependence on lattice size, $1/g_c^2(L)$ vs. $1/L$.

**Figure 12**: Histograms of $\Sigma$ measurements on a $10 \times 30^2$ lattice for couplings $1/g^2 = 0.870, 0.865$ and $0.860$.

**Figure 13**: Plot of the finite temperature critical point’s dependence on the temporal extent $N_\tau$ of the asymmetric $N_\tau \times N^2$ lattice.

**Figure 14**: Plot of $\ln(1/g_c^2 - 1/g_{\beta c}^2(N_\tau))$ vs. $\ln N_\tau$ for three estimates (0.995, 0.976, 0.950) of the infinite volume critical point.

**Figure 15**: Plot of $\ln \Sigma_0$ vs. $\ln m$ for $8^3$, $16^3$ and $24^3$ lattices at $1/g^2 = 1.00$, an estimate of the bulk critical point.

**Figure 16**: Plot of $\ln \Sigma_0$ vs. $\ln m$ for $8^3$, $16^3$ and $24^3$ lattices at $1/g^2 = 0.975$, an estimate of the bulk critical point.

**Figure 17**: Plot of $\ln \Sigma$ vs. $\ln m$ on a $12 \times 36^2$ lattice at $1/g^2 = 0.8775$, the finite temperature critical point for this lattice size.

**Figure 18**: Plot of fits of the inverse correlation length vs. $1/g^2$ for a $20^3$ lattice. The pluses show the scalar mass $M$ in the broken phase obtained using a fit to a branch cut (7.4), the stars show $M$ values obtained
using a simple pole fit, and the circles show the 2 fermion threshold $2\Sigma_0$ obtained using the branch cut fit. In the symmetric phase the crosses show the scalar width $\mu$ fitted using the form (7.8).