Toroidal Compactification of Heterotic 6D Non-Critical Strings Down to Four Dimensions

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Abstract

The low-energy limit of the 6D non-critical string theory with $N = 1$ SUSY and $E_8$ chiral current algebra compactified on $T^2$ is generically an $N = 2 U(1)$ vector multiplet. We study the analog of the Seiberg-Witten solution for the low-energy effective action as a function of $E_8$ Wilson lines on the compactified torus and the complex modulus of that torus. The moduli space includes regions where the Seiberg-Witten curves for $SU(2)$ QCD are recovered as well as regions where the newly discovered 4D theories with enhanced $E_{6,7,8}$ global symmetries appear.

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1 Introduction

One of the many new challenges that arose in the recent developments in physics is to obtain a microscopic description of the 6D non-critical (and sometimes tensionless) strings.

In 6D there are a few kinds of such theories. The first is the theory with $\mathcal{N} = 2$ which was discovered in [1] and is related to type-IIB on a K3 with an $A_1$ singularity and which we will refer to as type-II TS-theory. The second is the theory with $\mathcal{N} = 1$ and global $E_8$ symmetry which is related to small $E_8$ instantons [2, 3] and which we will refer to as Heterotic TS-theory (or HTS-theory). There are other, even more mysterious theories which are related to type-IIB at special values of the coupling constant or F-theory on a $\mathbb{Z}_3$ orbifold [3, 4, 5].

In six dimensions, the low-energy of the first two theories is known (except for the superconformal tensionless string point) and is a free tensor multiplet of $\mathcal{N} = 2$ [1] or $\mathcal{N} = 1$ according to the theory.

One way to probe a microscopic structure is to compactify the theory down to lower dimensions.

It was argued in [1] that compactification of type-II TS-theory on $\mathbb{T}^2$ is related, at scales much below the compactification scale, to $\mathcal{N} = 4$ SU(2) Yang-Mills theory. In a recent paper [6] we studied some aspects of the low-energy of type-II theory compactified to 2D on some 4-manifolds. The main tool there was the relation between type-II on $\mathbb{T}^2$ and $\mathcal{N} = 4$ Yang-Mills theory which made it possible to read off the partition function from results of Vafa and Witten [7] by a further compactification on $\mathbb{T}^2$ (down to zero dimensions).

In order to implement this technique to the heterotic TS-theory we need to know the low-energy of HTS-theory on $\mathbb{T}^2$. The low-energy will be a function of 9 complex background parameters which are the $E_8$ boundary conditions (Wilson lines) along the $\mathbb{T}^2$ and the complex structure $\sigma$ of $\mathbb{T}^2$.

At a generic point in the moduli space the 4D low energy is given by a $U(1)$ vector multiplet which is the dimensional reduction of the 6D tensor multiplet and so the question arises what is the Seiberg-Witten elliptic curve that describes that low-energy.

The answer encodes some interesting physical information. There are regions of moduli space where by appropriate scaling of the parameters we can reach $\mathcal{N} = 2$ QCD [9] and...
there are other regions where we reach the newly discovered 4D theories with enhanced $E_{6, 7, 8}$ global symmetry [8, 9, 10]. Indeed, the relation between those theories and tensionless strings has been anticipated by Seiberg.

The paper is organized as follows:

1. Section (2): Generalities regarding the compactification on $T^2$.
2. Section (3): We derive the Seiberg-Witten curve as a function of the modulus of the torus and the $E_8$ Wilson lines.
3. Section (4): We discuss appropriate scaling limits which reproduce the QCD results [11] for massive $N_f = 3$ (and hence also $N_f = 0, 1, 2$) as well as the asymptotically non-free $N_f = 8$!
4. Section (5): Discussion.

## 2 Compactification on $T^2$

To compactify HTS on $T^2$ we need to define a complex structure $\sigma$ for the $T^2$ and $E_8$ flat gauge connections which serve as backgrounds. Let the area of $T^2$ be denoted by $A$. In the low-energy limit of 4D we will find generically a $U(1)$ vector multiplet of $\mathcal{N} = 2$ supersymmetry. This is because generically only the 6D tensor multiplet is massless and it gives rise to the $U(1)$ vector-multiplet while all the other modes are massive. The low-energy is thus given by [12]:

$$\frac{1}{4\pi} \text{Im} \left[ \int d^4\theta a_D(a) \bar{a} + \int d^2\theta \frac{1}{2} \frac{\partial a_D}{\partial a} W_a W^a \right]$$

The gauge coupling constant is given by

$$\tau = \frac{i}{g^2} + \frac{\theta}{2\pi} = \frac{\partial a_D}{\partial a} \quad (2)$$

Seiberg and Witten expressed $a_D$ and $a$ of $\mathcal{N} = 2$ Yang-Mills theory as a function of a third complex variable $u$ which was defined in the microscopic theory as

$$u = \frac{1}{2} \langle \text{tr} \{ \phi^2 \} \rangle \quad (3)$$

In our case, we do not know of any microscopic description of HTS-theory so we do not know whether there exists a “natural” $u$-variable. At present, we can only ask the question
of what is the function $a(\tau)$. Thus, in our case $u$ will be only an ‘auxiliary’ variable which parameterizes the moduli space (but given the simple result that we will find, it will be very tempting to conjecture that $u$ is a true microscopic variable).

The functions $a(u)$ and $a_D(u)$ are give by a Seiberg-Witten curve \cite{12,11}:

$$y^2 = x^3 - f(u)x - g(u)$$

(4)

where the functions $f, g$ depend on $\sigma$ and on the flat $E_8$ gauge connection that we chose.

2.1 Geometrical symmetries

The moduli space of flat $E_8$ gauge connections is given by a pair of commuting $E_8$ group elements (Wilson lines) in the Cartan sub-algebra of $E_8$. We will take a Cartan sub-algebra which is a double cover of

$$SO(2)^8 \subset SO(16) \xleftarrow{\mathbb{Z}_2} Spin(16) \subset E_8.$$  

(5)

Let $T^2$ be given by

$$\mathbb{C}/\{z \sim z + 2, z \sim z + 2\sigma\}$$

(6)

Let $\alpha_i$ be the charge under the $i$-th $SO(2)$ of the Wilson loop around the one cycle of the torus (the one form 0 to 2) and let $\beta_i$ be the charge under the $i$-th $SO(2)$ of the Wilson loop around the other cycle (from 0 to $2\sigma$). Then we define

$$w_i = \alpha_i + \beta_i \sigma$$

(7)

A point in the moduli space is given by

$$(w_1, \ldots, w_8)$$

(8)

with the identifications:

$$(w_1, \ldots, w_8) \sim (w_1+n_1+m_1\sigma, \ldots, w_8+n_8+m_8\sigma), \quad n_i, m_i \in \mathbb{Z}, \quad \sum_1^8 n_i \equiv \sum_1^8 m_i \equiv 0 \pmod{2}$$

(9)

The condition of $0 \pmod{2}$ is because of the $\mathbb{Z}_2$ projection in (3).
On top of that there are further identifications because of the $E_8$ Weyl group which is generated by

$$\begin{align*}
(w_1, \ldots, w_8) & \rightarrow (w_{\psi_1}, \ldots, w_{\psi_8}), \quad \psi \in S_8 \\
(w_1, \ldots, w_8) & \rightarrow ((-1)^{\epsilon_i}w_1, \ldots, (-1)^{\epsilon_8}w_8), \quad \sum_{i=1}^8 \epsilon_i \equiv 0 \pmod{2} \\
(w_1, \ldots, w_8) & \rightarrow (w_1 - \frac{\sum_{i}^8 w_i}{4}, \ldots, w_8 - \frac{\sum_{i}^8 w_i}{4})
\end{align*}$$

The general form of the Seiberg-Witten curve that we are looking for is:

$$x^3 - f(u; \sigma, w_1 \ldots w_8)x - g(u; \sigma, w_1 \ldots w_8)$$

$f$ and $g$ are analytic functions of the variables. The argument for analyticity in the $w_i$-s is that they can be realized as VEVs of scalar components of background fields (as in similar cases in [13]).

The functions $f, g$ are invariant under (9,10-12), and in particular (9) implies that $f, g$ are doubly-periodic in each $w_i$ separately, with periods $2$ and $2\sigma$.

Under modular transformations

$$\sigma \rightarrow -\frac{1}{\sigma}, \quad w_i \rightarrow \frac{w_i - 2}{\sigma}$$

The functions $f, g$ are, however, not modular invariant but as we will see, have modular weights 4 and 6.

### 2.2 The central charges

As was explained in [11], when there are global $U(1)$-s present in the low-energy, the $\mathcal{N} = 2$ supersymmetry algebra contains central charges corresponding to those $U(1)$-s. In [11], the global $U(1)$-s where flavor symmetries and the central charges where

$$Z = n_e a + n_m a_D + \sum_i S_i \frac{m_i}{\sqrt{2}}$$

where $S_i$ are the $U(1)$ integer charges. In our case, the global $U(1)$-s are the eight $SO(2)$-s in the subgroup $SO(2)^8 \subset E_8$ that is unbroken by the Wilson lines. The masses $m_i$ should

\[2\text{In what follows, we will be sloppy and write } SO(16) \subset E_8 \text{ instead of } Spin(16).\]
be replaced by the $w_i$-s. This can be inferred by representing the $w_i$-s again as VEVs of background vector-multiplets. We will also see in section (4) that at the scaling limits that reproduce QCD the $w_i$-s become the masses.

Thus:

$$Z = n_e a + n_m a_D + \sum_i S_i \frac{w_i}{2}$$  \hspace{1cm} (16)

This cannot be the full central charge because this formula is not periodic in $w_i$. To correct it, we need to add two more charges corresponding to the momenta around the two compact $T^2$ directions:

$$Z = n_e a + n_m a_D + \sum_i S_i \frac{w_i}{2} + P_5 + P_6 \sigma$$  \hspace{1cm} (17)

(we can set $R_5 = 2\pi$ where $R_5$ is the size of the $T^2$ cycle from 0 to 1). Now it is clear that the transformation $w_i \rightarrow w_i + 2N + 2M \sigma$ must be accompanied by $P_5 \rightarrow P_5 - NS_i$ and $P_6 \rightarrow P_6 - MS_i$ as is also clear physically.

Under modular transformations of $T^2$:

$$\sigma \rightarrow -\frac{1}{\sigma}$$  \hspace{1cm} (18)

we must have

$$a \rightarrow -\frac{1}{\sigma} a_D, \quad a_D \rightarrow -\frac{1}{\sigma} a$$  \hspace{1cm} (19)

The multiplication by $\sigma$ expresses the fact that after changing the axes of the $T^2$ we also change the units from $R_5 = 2\pi$ to $R_6 = 2\pi$ which multiplies masses by $|\sigma|$. The interchange between $a$ and $a_D$ is the physical expectation that monopoles and electrons are strings wrapped on different cycles of the $T^2$.

## 3 The Seiberg-Witten curve

In this section we will derive the functions $f$ and $g$ in:

$$y^2 = x^3 - f(u; \sigma, w_1 \ldots w_8) x - g(u; \sigma, w_1 \ldots w_8)$$  \hspace{1cm} (20)

The plan is to first obtain the degree of $f$ and $g$ as polynomials in $u$ from what we physically expect from the region of moduli space where the tension of the strings is large. This region will also determine the leading coefficients in $f$ and $g$ as a function of $\sigma$. Then we will find
the relations between the coefficients of the \( u^k \)-s in \( f \) and \( g \) and the external variables \( \sigma \) and \( w_1 \ldots w_8 \) by using (17) and the identification [14] of the central charges and the poles of the 1-form \( \lambda \) which determines \( a \) and \( a_D \) according to [12]:

\[
\frac{da}{du} = \frac{d}{du} \oint d\lambda = \oint \frac{dx}{y}.
\]  

(21)

### 3.1 Large \( u \)

In 6D, large tension means that the scalar component of the tensor multiplet is large compared to the inverse of the area of the compactified torus. We will assume that in 4D large string tension corresponds to large \( u \). Thus we are interested in the region of large \( u \) and fixed \( q \) and \( \vec{w} \) (but not necessarily small).

To first order, we can forget about the strings altogether and find the dimensional reduction of \( B_{\mu
u}^{(-)} \) to 4D. \( B_{\mu
u}^{(-)} \) is quantized to have integral periods on \( T^2 \) and this sets the unit of electric charge in 4D. We find that to first order the 4D coupling constant is

\[
\frac{i}{g^2} + \frac{\theta}{2\pi} = \sigma
\]  

(22)

In fact, this result is independent of the Wilson lines \( \vec{w} \).

Now let \( \Phi \) be the tension of the string (which is the super-partner of \( B_{\mu
u}^{(-)} \) in 6D) and let

\[
B = \langle \int_{T^2} B_{\mu
u}^{(-)} \rangle
\]  

(23)

For large tension the HTS-instanton expansion is an expansion in powers of

\[
e^{-\Phi R_5 R_6 - iB}
\]  

(24)

From the curve’s point of view, the expansion parameter is \( \frac{1}{u} \) (for simplicity we take a right-angled \( T^2 \) with sides \( R_5 \) and \( R_6 \)). So, we identify

\[
u = e^{\Phi R_5 R_6 + iB}.
\]  

(25)

The mass of a string that is wrapped around \( R_6 \) is approximately (in units where \( R_5 = 2\pi \)):

\[
M^2 \approx (\Phi R_6)^2 + \left( \frac{B}{R_5} \right)^2 = |\log u|^2
\]  

(26)
The term with $B$ appears because when $B$ is turned on it shifts the boundary conditions around $R_5$ and is like “fractional momentum”. We deduce that for large $u$

$$a \approx \log u, \quad \frac{da}{du} \approx \frac{1}{u}$$

As in [11] we write down the general curve which at large $u$ has modular parameter $\sigma$:

$$y^2 = x^3 - \frac{1}{4}g_2(\sigma)\beta_2 x u^{2\alpha} - \frac{1}{4}\beta_3 g_3(\sigma) u^{3\alpha}$$

(28)

here $u^{2\alpha}$ is the leading (highest power in $u$) coefficient of $x$ and $u^{3\alpha}$ is the leading free term. $\alpha$ and $\beta$ will be determined shortly.

$$\frac{1}{4}g_2(\sigma) = 15\pi^{-4} \sum_{m,n \in \mathbb{Z}, \neq 0} \frac{1}{(m\sigma + n)^4}$$

$$\frac{1}{4}g_3(\sigma) = 35\pi^{-6} \sum_{m,n \in \mathbb{Z}, \neq 0} \frac{1}{(m\sigma + n)^6}$$

This curve has

$$\int \frac{dx}{y} = \frac{1}{2\sqrt{2}\beta u^\alpha}$$

(29)

comparing to (27) we find

$$\alpha = 2, \quad \beta = \frac{1}{8}$$

(30)

so for large $u$:

$$y^2 = x^3 - \left(\frac{1}{256}g_2(\sigma)u^4 + \mathcal{O}(u^3)\right)x - \frac{1}{2048}g_3(\sigma)u^6 + \mathcal{O}(u^5)$$

(31)

We conclude that $f$ is a polynomial of degree 4 and $g$ is of degree 6 and we have also determined the leading coefficients as a function of $\sigma$.

### 3.2 The $E_8$ Wilson lines

The curve looks like:

$$y^2 = x^3 - \sum_{k=0}^{4} f_k(\sigma, w_1 \ldots w_8) u^k x - \sum_{l=0}^{6} g_l(\sigma, w_1 \ldots w_8) u^l$$

(32)

We have determined $f_4$ and $g_6$ above as a function $\sigma$ so there are $4 + 6 = 10$ remaining coefficients to determine. Out of those, one coefficient can be set by a shift:

$$u \rightarrow u + \text{const}$$

(33)
Since we do not know what \( u \) is microscopically anyway this shift does not matter. It thus seems that we are left with 9 unknown coefficients but there are only 8 variables \( w_1 \ldots w_8 \) (since we have already fixed \( \sigma \) to determine \( f_4 \) and \( g_6 \)) so it seems that we have a puzzle. However, precisely when the degrees are 4 and 6 we can use a second degree of freedom:

\[
\begin{align*}
u &\rightarrow \beta u, \\
x &\rightarrow \beta^2 x, \\
y &\rightarrow \beta^3 y
\end{align*}
\]

which leaves the holomorphic 2-form

\[
du \wedge \frac{dx}{y}
\]

invariant and thus will not change the physical low-energy.

It remains to determine the coefficients as a function of the \( w_i \)-s. In fact, we will read off the inverse function from the results of [11]. They showed that the central charges \( w_i \) in (17) are integrals of

\[
\Omega = du \wedge \frac{dx}{y}
\]

over 2-cycles of the total space of the elliptic curve fibered over the \( u \)-plane. In our case, the total space is precisely the almost Del Pezzo surface which appears in the F-theory description of HTS-theory [3, 16, 15]. This is of course no coincidence, as we have learned in the past year how abstract Seiberg-Witten curves “come to life” in string theory [14].

This almost Del Pezzo surface has the \( E_8 \) lattice as a sub-lattice of its \( H^2(\mathbb{Z}) \) cohomology [3]. If we denote the 2-cycles which correspond to this sub-lattice as \( e_1 \ldots e_8 \) then they can be chosen so as not to intersect the fiber at \( u = \infty \) where \( \Omega \) has a pole (see appendix (A) of [16]).

Thus, from the curve (32) we can get two points on the \( E_8 \) root lattice

\[
\int_{e_i} \text{Im}[du \wedge \frac{dx}{y}], \quad \int_{e_i} \text{Re}[du \wedge \frac{dx}{y}]
\]

This corresponds in a natural way to the two Wilson lines on the \( T^2 \) thus completing the map between \( w_1 \ldots w_8, \sigma \) and the coefficients \( f_k \) and \( g_l \).

4 QCD at scaling limits

In this section we will tune the \( w_i, u \) and \( \sigma \) in such a way as to reproduce \( SU(2) \) Yang-Mills with matter in appropriate scaling limits, and in particular see that we can reproduce the
curves of $[1]$. 

To begin with, let us take the Wilson line along the horizontal direction of the compact $T^2$ (i.e. the real axis) to be (in the adjoint representation 248 of $E_8$):

$$ W = \begin{pmatrix} I_{120 \times 120} & 0 \\ 0 & -I_{128 \times 128} \end{pmatrix} \quad (38) $$

This is the special Wilson loop that was used in [2] and has the property that the $E_8$ heterotic string on an $S^1$ of radius $R_5$ is T-dual to the $SO(32)$ heterotic string on $S^1$ of radius $\frac{1}{R_5}$ [17]. The $SO(32)$ small instanton has been described by Witten in the low-energy as a field theory [18]. It is an $SU(2)$ gauge theory with $\frac{1}{2}$-hypermultiplets in the $\left( 2, \bar{32} \right)$ of $Sp(1) \times SO(32)$. In our case, $SO(32)$ is broken by the T-dual of $W$ and we end up with 16 $\frac{1}{2}$-hypermultiplets in $E_8$.

It was explained in [2] how those states can be derived from the $E_8$ point of view as winding states of the string. The special value of $W$ was important because for all other values of $W$, the T-dual radius does not go to $\infty$ as $R_5 \to 0$.

Now let us compactify on a second direction of radius $R_6 = R_5 \text{Im} \sigma$ and turn on $SO(16) \subset E_8$ Wilson loops as well as an expectation value for the scalar $\phi$ of the 6D tensor multiplet and for $f_{T^2, B_{\mu\nu}^{(-)}}$ and see what happens after T-duality.

The tensor multiplet expectation values become $SU(2) = Sp(1)$ Wilson loops and we also get $SO(16)$ Wilson loops.

The $SO(16)$ and $Sp(1)$ Wilson loops will generate masses $m_i$ to the $\left( 2, 16 \right)$ $\frac{1}{2}$-hypermultiplets.

The $SU(2)$ gauge coupling is 1 at the 6D string scale [18] and in 4D it will be of order

$$ \frac{1}{g^2} = R_2 \frac{1}{R_1} = \text{Im} \sigma \quad (39) $$

at scale $\text{Im} \sigma$ (the area of $T^2$ on the $SO(32)$ side) and it will continue according to the RGE to the low-energy.

Note that we need to start with both $R_6$ and $\frac{1}{R_5}$ large so that the $T^2$ on the $SO(32)$ side will be large and we could use the low-energy description of the small instanton.

In the discussion that follows we will “forget” about the previous section and check what are the restrictions that arise just from the requirement of compatibility with the QCD curves.
4.1 Perturbing to small masses

Let’s move a little bit away from the point with the special Wilson lines (38). This will correspond to giving the 16 hyper-multiplets 8 bare masses

\[ m_1 \ldots m_8 \]  

according to

\[ \delta w_i = \frac{R_5}{2\pi} m_i \]  

In the future we will set \( R_5 = 2\pi \).

4.2 Small \( q, u \) and \( m_1 \ldots m_8 \)

To begin with, we take small \( q = e^{2\pi i \sigma} \) and \( u \) and set \( \delta w_i = m_i \) with all the 8 \( m_i \) being small. In this region, as long as \( u^4 \gg q \), the low-energy coupling constant is given by:

\[ e^{2\pi i \tau} = \prod \left( u - m_i^2 \right) q \]  

and the mass of the \( W \) boson is approximately

\[ m_W = \frac{1}{2} \sqrt{2u} \]  

The corresponding curve has to be of the form:

\[ y^2 = (x - u + q^{1/2} \psi(u, m, q^{1/2}))(x^2 - \frac{64q}{u^2} \prod (u - m_i^2) + qx\chi(u, m, q^{1/2}) + q^{3/2} \xi(u, m, q^{1/2})) \]  

where \( \psi, \chi, \xi \) are as yet unknown functions.

By shifting \( x \) and requiring the RHS to be a polynomial in \( u \) we can determine some of the singular (in \( u \)) parts in \( \psi, \chi \) and \( \xi \). After some algebra we find that the curve is of the form:

\[ y^2 = x^3 - \frac{1}{3} u^2 x - \frac{2}{27} u^3 - 16(\prod m_i)(x + \frac{1}{3} u)q^{1/2} \]  

\[ + (A(x + \frac{1}{3} u) + 64T_7(u))q + \mathcal{O}(q^{3/2}) \]  

(45)
where
\[ T_7(u) \equiv \frac{1}{u} \left( \prod (u - m_i^2) - \prod m_i^2 \right) \]
\[ (46) \]
A is an unknown coefficient which depends on \( m_1 \ldots m_8 \), and all the curves that are polynomials in \( u \) and of the form (44) can be reached from (45) by the changes:
\[
\begin{align*}
x & \rightarrow x + \chi_0(u) + \chi_1(u)q^{1/2} + \chi_2(u)q \\
u & \rightarrow u + \psi_1(u)q^{1/2} + \psi_2(u)q
\end{align*}
\]
\[ (47) \]
where the coefficients \( \chi_0, \chi_1, \psi_1, \psi_2 \) are polynomials in \( u \) with an implicit dependence on the masses \( m_1 \ldots m_8 \).

The change in \( u \) is to be expected since we do not know the precise relation between the field theoretic \( u \) and the HTS-theoretic \( u \).

The degree of \( g \) in (45) is 7 and not 6, but by using (47) with
\[
\begin{align*}
\psi_1(u) &= \pm 24u^3 + \mathcal{O}(u^2) \\
\psi_2(u) &= -288u^5 + \mathcal{O}(u^4)
\end{align*}
\]
we can decrease the degree of \( g \) to 6 at the (good) price of increasing the degree of \( f \) to 4.

4.3 The \( N_f = 3 \) scaling limit

Taking \( m_4 \ldots m_8 \) small but fixed and
\[
u = \lambda^2 v, \quad q = \lambda^2 \frac{\Lambda^3}{m_4^2 \ldots m_8^2} \equiv \lambda^2 \Lambda, \quad m_i = \lambda \mu_i, \quad i = 1, 2, 3
\]
\[ (48) \]
We reach the limit of \( N_f = 3 \) \( SU(2) \) gauge theory with three masses \( \mu_1, \mu_2, \mu_3 \), scale \( \Lambda \) and moduli parameter \( v \).

After rescaling
\[
x \rightarrow \lambda^2 x, \quad y \rightarrow \lambda^3 y
\]
\[ (49) \]
We need to recover the curve of [11]:
\[
y^2 = x^2(x - u) - \frac{1}{64} \Lambda_3^2(x - u)^2 - \frac{1}{64}(\mu_1^2 + \mu_2^2 + 2 \mu_3^2) \Lambda_3^2(x - u) \\
+ \frac{1}{4} \mu_1 \mu_2 \mu_3 \Lambda_3 x - \frac{1}{64}(\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2) \Lambda_3^2.
\]
\[ (50) \]
To compare (53) to (50) it will be easier to shift

\[
\begin{align*}
  u & \rightarrow u - 128C_5q \\
  x & \rightarrow x - \frac{1}{3}(u + 128C_5q)
\end{align*}
\]

where we define

\[
C_k \equiv \sum_{i_1 < i_2 < \cdots < i_k} m_{i_1}^2 m_{i_2}^2 \cdots m_{i_k}^2
\]

for \( k = 1 \ldots 8 \).

After the shifts the curve is in the form:

\[
y^2 = x^2(x - u) - 16q^{1/2}x \prod m_i - 64C_5(x - u)^2q
\]

\[
+ (64u^7 - 64C_1u^6 + 64C_2u^5 - 64C_3u^4 + 64C_4u^3 + 64C_6u - 64C_7 + Ax)q + q^{3/2}\psi(u, x, q, m_i)
\]

substituting (48-49) in (45) we find:

\[
y^2 = x^2(x - u) - 16\mu_1\mu_2\mu_3M^5\Lambda x \prod m_i - 64M^{10}(x - u)^2\Lambda^2
\]

\[
+ 64M^{10}(\mu_1^2 + \mu_2^2 + \mu_3^2)\Lambda^2u - 64M^{10}(\mu_1\mu_2^2 + \mu_2\mu_3^2 + \mu_1\mu_3^2)\Lambda^2
\]

\[
+ \frac{1}{\lambda^2}Ax\Lambda^2 + \frac{1}{\lambda^3}\Lambda^3\psi(\lambda^2u, \lambda^2x, \lambda\Lambda, \lambda\mu_i, m_j)
\]

Comparing with (50) we find

\[
\Lambda_3 = -64\Lambda \prod_{j=5}^8 m_j
\]

and

\[
A = -64C_6 + \mathcal{O}(\lambda^3)
\]

where \( \mathcal{O}(\lambda^3) \) can be made of \( \prod m_i \) and \( C_7 \) and so on.

We also learn that \( \psi = \mathcal{O}(\lambda^4) \) and the curve is:

\[
y^2 = x^2(x - u) - 16q^{1/2}x \prod m_i - 64C_5(x - u)^2q
\]

\[
+ 64(u^7 - C_1u^6 + C_2u^5 - C_3u^4 + C_4u^3 - C_6(x - u) - C_7)q + \chi(m_i)xq + q^{3/2}\psi(u, x, q, m_i)
\]

where \( \chi \) and \( \psi \) are unknown but restricted as above.
When we shift \(u\) and \(x\), we can bring it to the form (32) where:

\[
\begin{align*}
f &= \frac{1}{3} u^2 + 16(u^4 + \prod m_i)q^{1/2} + (64C_6 - \frac{256}{3}C_5u)q + O(q^{3/2}) \\
g &= -\frac{2}{27} u^3 - \frac{16}{3}(u^4 + \prod m_i)uq^{1/2} \\
&\quad + \left\{ \frac{128}{3}C_6 u - 64C_1 u^6 + 64C_2 u^5 - 64C_3 u^4 + 64C_4 u^3 \\
&\quad - \frac{320}{9}C_5 u^2 + \frac{128}{3}C_6 u - 64C_7 - 128(\prod m_i)u^3 \right\}q + O(q^{3/2})
\end{align*}
\]

Here there is still the freedom (47) as long as \(f, g\) stay with degrees 4, 6. Thus, \(u\) in the equation above differs from \(u\) in (32) by a change of variables.

5 Discussion

We have seen that the Coulomb branch of HTS-theory in 4D is described by a Seiberg-Witten curve with a discriminant of degree 12. This low-energy is the same as the one discovered in \([8, 9]\) for a 3-brane near an \(E_8\) singularity of a K3 in F-theory and indeed the structure of the elliptic fibration of (32) is the same as that of F-theory near an \(E_8\) singularity \([19]\).

In fact the relation with tensionless strings was pointed out in \([9]\). HTS-theory also has a Higgs branch which in 6D describes a large \(E_8\) instanton. In 4D this Higgs branch emanates from singularities on the \(u\) plane provided that the Wilson lines are chosen so as to leave a commutant of the \(E_8\) which contains \(SU(2)\) (so that a large instanton can be embedded in it). It is interesting to study the theory compactified to 3D as well and the relation the dual field theory description of \([20]\).

What can we infer from the 4D low-energy curve about the microscopic structure of the 6D theory?

The variable \(u\) might be related to an expectation value of a microscopic HTS operator. For this to be so, it is not necessary for HTS-theory to have local operators. If HTS-theory has only operators that are labeled by surfaces then the 4D operator could be local at \(x\) but to correspond be labeled by the \(T^2\) at a 4D point \(x\). It might be worthwhile to study the instanton terms from strings wrapped on the \(T^2\) (as in \([21]\)) and compare them with the \(\frac{1}{u}\) expansion.

Other related questions are the compactifications of other tensionless string theories.
Like $SU(N)$ Yang-Mills theory the HTS-theories have a natural $N$ which is one plus the instanton number. We studied the $SU(2)$ theory but the $SU(N)$, i.e. small $N + 1$ instantons that coincide, might be related to generalizations of the hyper-elliptic curves of [22, 23].

There are other $\mathcal{N} = 2$ tensionless string theories in 4D [4, 5, 24] which can be realized as part of the spectrum of type-IIB on a Calabi-Yau (or F-theory on a 4-fold CY) and thus the corresponding Seiberg-Witten curves should also be calculable. It might be interesting to study further compactifications of those tensionless string theories to 2D.

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