Four-class Skew-symmetric Association Schemes

Jianmin Ma\textsuperscript{a}, Kaishun Wang\textsuperscript{b,∗}

\textsuperscript{a}Oxford College of Emory University, Oxford, GA 30054, USA
\textsuperscript{b}Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China

Abstract

An association scheme is called skew-symmetric if it has no symmetric adjacency relations other than the diagonal one. In this paper, we investigate 4-class skew-symmetric association schemes. In recent work by the first author it was discovered that their character tables fall into three types. We now determine their intersection matrices. We then determine the character tables for 4-class skew-symmetric pseudocyclic association schemes, the only known examples of which are cyclotomic schemes. As a result, we answer a question raised by S.Y. Song in 1996. We characterize and classify 4-class imprimitive skew-symmetric association schemes. We also prove that none of 2-class Johnson schemes admits a 4-class skew-symmetric fission scheme. Based on three types of character tables above, a short list of feasible parameters is generated.

Keywords: association scheme, fusion scheme, fission scheme, skew-symmetric scheme, cyclotomic scheme, pseudocyclic scheme

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1. Introduction

A d-class association scheme $\mathcal{X}$ is a pair $(X, \{R_i\}_{i=0}^d)$, where $X$ is a finite set, and each $R_i$ is a nonempty subset of $X \times X$ satisfying following axioms:

(i) $R_0 = \{(x, x) \mid x \in X\}$ is the diagonal relation;
(ii) $X \times X = R_0 \cup R_1 \cup \ldots \cup R_d$, $R_i \cap R_j = \emptyset$ ($i \neq j$);
(iii) for each $i$, $R_i^T = R_i$ for some $0 \leq i' \leq d$, where $R_i^T = \{(y, x) \mid (x, y) \in R_i\}$;
(iv) there exists integers $p^k_{ij}$ such that for all $(x, y) \in R_k$ and all $i, j, k$,

$$p^k_{ij} = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|.$$

$\mathcal{X}$ is also called an association scheme with $d$ classes (a $d$-class association scheme, or even simply a scheme). The subsets $R_i$ are called the adjacency relations of $\mathcal{X}$. The integers $p^k_{ij}$ are called the intersection numbers of $\mathcal{X}$, and $k_i (= p^i_{ii})$ is called the valency of $R_i$. Furthermore, $\mathcal{X}$ is called commutative if $p^k_{ij} = p^l_{ji}$ for all $i, j, k$; $\mathcal{X}$ is called symmetric if $R_i^T = R_i$ for all $i$. 

\textsuperscript{∗}Corresponding author

Email addresses: Jianmin.Ma@emory.edu (Jianmin Ma), wangks@bnu.edu.cn (Kaishun Wang)
In the rest of this paper, all association schemes are assumed to be commutative. By a theorem of Higman [10], association schemes with at most four classes are commutative. We refer readers to Bannai and Ito’s book [1] for the general theory of association schemes.

Let $\mathfrak{X} = (X, (R_i)_{i=0}^d)$ be a commutative association scheme with $|X| = n$. The adjacency matrix $A_i$ of $R_i$ is the $n \times n$ matrix whose $(x, y)$-entry is 1 if $(x, y) \in R_i$ and 0 otherwise. By the adjacency or Bose-Mesner algebra $\mathfrak{A}$ of $\mathfrak{X}$ we mean the algebra generated by $A_0, \ldots, A_d$ over the complex field. Axioms (i)-(iv) are equivalent to the following:

$$A_0 = I, \quad \sum_{j=0}^d A_j = J, \quad A_i^T = A_i, \quad A_iA_j = \sum_{k=0}^d p_{ij}^k A_k,$$

where $I$ and $J$ are the identity and all-one matrices of order $n$, respectively.

Since $\mathfrak{A}$ consists of commuting normal matrices, it has a second basis consisting of primitive idempotents $E_0 = J/n, \ldots, E_d$. Let

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^d q_{ij}^k E_k,$$

where $\circ$ is the Hadamard (entry-wise) product. The coefficients $q_{ij}^k$ are called the Krein numbers of $\mathfrak{X}$. The Krein condition asserts that all $q_{ij}^k$ are nonnegative reals [1, Theorem II.3.8]. The integers $m_e = \text{rank} E_e$ are called the multiplicities of $\mathfrak{X}$. In particular, $m_0 = 1$ is said to be trivial. If $m_1 = \cdots = m_d$, $\mathfrak{X}$ is called pseudocyclic. A pseudocyclic scheme is equivalenced, i.e., all non-diagonal relations have the same valency (see [4, p.48] and references there).

For $i = 0, 1, \ldots, d$, let

$$A_i = \sum_{j=0}^d p_{ji} E_j.$$

The following $(d + 1) \times (d + 1)$ matrices is called the character table of $\mathfrak{X}$:

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_{ji} & \cdots & 1 \end{pmatrix}, \quad 1 \leq j, i \leq d.$$

Both intersection numbers $p_{ij}^e$ and Krein numbers $q_{ij}^e$ can be calculated from $P$ [1, Section II.3]:

$$p_{ij}^e = \frac{1}{nk_e} \sum_{h=0}^d m_h p_{ih} p_{jh} \overline{p}_{ih}, \quad q_{ij}^e = \frac{m_j m_i}{n} \sum_{h=0}^d 1 \overline{p}_{ih} p_{jh} \overline{p}_{ih},$$  \hspace{1cm} (1)

where $\overline{p}_{ih}$ is the complex conjugate of $p_{ih}$.

For $i = 0, 1, \ldots, d$, the $i$-th intersection matrix $B_i$ is defined to be the $(d + 1) \times (d + 1)$ matrix whose $(j, k)$ entry is $p_{ij}^{jk}$. The character table $P$ determines matrices $B_i$, and vice versa.

One way to construct new association schemes is by merging or splitting relations in an existing scheme. More precisely, a partition $\Lambda_0, \Lambda_1, \ldots, \Lambda_e$ of the index set $\{0, 1, \ldots, d\}$ is said to be admissible [12] if $\Lambda_0 = \{0\}, \Lambda_1 \neq \emptyset$ and $\Lambda_j' = \Lambda_j$ for some $j$ $(1 \leq i, j \leq e)$, where $\Lambda_j' = \{d' | \alpha \in \Lambda_j\}$. Let $R_{\Lambda} = \bigcup_{\alpha \in \Lambda_i} R_\alpha$. If $\mathfrak{X} = (X, (R_\alpha)_{\alpha \in \Lambda})$ becomes an association scheme,
it is called a fusion scheme of \( X \), while \( X \) is called a fission scheme of \( Y \). If every admissible partition gives rise to a fusion scheme, \( X \) is called amorphous \([12]\).

In \([2]\), Bannai and Song raised a question regarding the existence of 4-class amorphous association schemes with the diagonal relation being the only symmetric adjacency relation. We showed the nonexistence of such schemes in \([17]\), and this implies the nonexistence of amorphous association schemes with at least 4 classes with this property. These results bring our attention to what we call skew-symmetric association schemes, i.e., association schemes with no symmetric adjacency relations other than the diagonal one. In \([17]\), 4-class skew-symmetric schemes are classified by their character tables, which fall into three types. In this paper, we investigate 4-class skew-symmetric association schemes.

The balance of this paper is structured as follows. We first determine the intersection matrices of 4-class skew-symmetric association schemes (see Section 2). We then determine the character tables for 4-class skew-symmetric pseudocyclic association schemes. As a result, we answer a question raised by Song \([22]\) about cyclotomic schemes (see Section 3). In Section 4, we classify 4-class imprimitive skew-symmetric association schemes. In Section 5, we generate a short list of feasible parameters. We conclude with some remarks in Section 6.

We conclude this section by briefly mentioning the status of association schemes with up to 4 classes. Two-class skew-symmetric association schemes are regular tournaments. Two-class symmetric association schemes (or equivalently, strong regular graphs) have been widely studied \([4, 6]\). There are a few papers about 3-class symmetric association schemes \([18, 7]\). Three-class nonsymmetric schemes have been investigated by Song \([21, 22]\), Goldbach and Claasen \([8, 9]\) and Jørgensen \([13]\). Though it is relatively easy to characterize the parameter sets of 3-class nonsymmetric schemes, it is much hard to construct primitive examples \([13]\). Nonetheless, some families of 3-class non-symmetric schemes were constructed on Galois rings in characteristic 4 \([15, 16]\). In \([22]\), Song initiated the study of 4-class skew-symmetric association schemes.

2. Character tables and Intersection matrices

Let \( X = (X, \{R_0, R_1, R_2, R_1^T, R_2^T\}) \) be a skew-symmetric association scheme. So the symmetrization \( \tilde{X} \) of \( X \) is a 2-class symmetric association scheme, where \( \tilde{X} = (X, \{R_0, R_1 \cup R_1^T, R_2 \cup R_2^T\}) \).

The concept of a strong regular graph is essentially the same as that of a 2-class symmetric association scheme. A regular graph \((X, R)\), with \( n \) vertices and valency \( k \), is called strongly regular if any pair of adjacent vertices have \( \lambda \) common neighbors, and any two distinct non-adjacent vertices have \( \mu \) common neighbors. We say this is an \((n, k, \lambda, \mu)\)-strongly regular graph. It is easy to verify that \( \tilde{X} = (X, \{R_0, R, \tilde{R}\}) \) is a symmetric association scheme, where \((X, \tilde{R})\) is the complement (graph) of \((X, R)\). The parameters \( n, k, \lambda, \mu \) of \((X, R)\) determine those of \( \tilde{X} \).

The following result has been proved in \([17]\).

**Theorem 2.1.** Let \( X = (X, \{R_0, R_1, R_2, R_1^T, R_2^T\}) \) be a skew-symmetric association scheme and let \( \tilde{X} = (X, \{R_0, R_1 \cup R_1^T, R_2 \cup R_2^T\}) \) be the symmetrization of \( X \). Let

\[
\tilde{P} = \begin{pmatrix}
1 & k_1 & k_2 \\
1 & r & t \\
1 & s & u
\end{pmatrix}
\]

be the character table of \( \tilde{X} \), where 1, \( m_1 \) and \( m_2 \) are the multiplicities of the corresponding row entries (eigenvalues). The entries of \( \tilde{P} \) and the multiplicities \( m_i \) can be calculated from the
parameters \((n, k, \lambda, \mu)\) of the strongly regular graph \((X, R_1 \cup R_2^t)\), where \(k = k_1\). With a possible rearrangement of rows and columns, the character table of \(X\) has the following form:

\[
P = \begin{bmatrix}
1 & k_1/2 & k_2/2 & k_3/2 & k_4/2 & 1 \\
1 & \rho & \tau & \bar{\tau} & \bar{\rho} & m_1/2 \\
1 & \sigma & \omega & \bar{\omega} & \bar{\sigma} & m_2/2 \\
1 & \bar{\sigma} & \bar{\omega} & \omega & \sigma & m_2/2 \\
1 & \bar{\rho} & \bar{\tau} & \tau & \rho & m_1/2 \\
\end{bmatrix}
\]

The entries \(\rho, \omega, \tau, \text{ and } \sigma\) are one of the three cases:

(i) \(\rho = r/2, \ \sigma = (s + \sqrt{b})/2, \ \tau = (t + \sqrt{z})/2, \ \omega = u/2, \) where \(b = nk_1/m_2, z = nk_2/m_1\).

(ii) \(\rho = (r + \sqrt{b})/2, \ \sigma = s/2, \ \tau = t/2, \ \omega = (u + \sqrt{c})/2, \) where \(y = nk_1/m_1, c = nk_2/m_2\).

(iii) \(\rho = (r + \sqrt{g})/2, \ \tau = (t + \sqrt{z})/2, \ \sigma = (s + \sqrt{b})/2, \ \omega = (u - \sqrt{c})/2, \) where all of \(b, c, y, z\) are positive and satisfy the following equations:

\[
m_1y + m_2b = nk_1, \quad m_1z + m_2c = nk_2, \quad m_1\sqrt{yz} - m_2\sqrt{bc} = 0.
\]

If any of \(y, z, b, c\) is taken to be a free variable, the remaining variables can be solved. Let \(z\) be the free variable. We have

\[
b = \frac{m_1k_1}{k_2m_2}, \quad y = \frac{k_1(nk_2 - m_1z)}{k_2m_1}, \quad c = \frac{nk_2 - m_1z}{m_2}.
\]

We denote by \(P_1\), \(P_{II}\) and \(P_{III}\) the three character tables in the above theorem, and refer them as type I, II, and III, respectively. For a skew-symmetric association scheme \(\mathcal{X} = (X, \{R_0, R_1, R_2, R_1^t, R_2^t\})\), the intersection matrices \(B_1\) and \(B_2\) determine \(B_3\) and \(B_4\). For typographic convenience, we display only the principal part \(B_1^{(0)}\) of an intersection matrix \(B_i\), i.e., the lower-right 4 by 4 submatrix.

**Theorem 2.2.** With the notation in Theorem 2.1, Suppose that the strongly regular graph \((X, R_1 \cup R_2^t)\) has parameters \(n, k, \lambda, \mu\). Let \(k = k_1\) and \(k_2 = n - k - 1\).

(i) For Theorem 2.1(i), we have

\[
P_1^{(0)} = \begin{bmatrix}
\frac{j+s}{4} & \frac{k(k-1-u)}{4k_2} & \frac{k(k-1-u)}{4k_2} & \frac{k-3}{4} \\
\frac{k-1-u}{4} & \frac{k-\mu}{4} & \frac{k-\mu}{4} & \frac{k-1-u}{4} \\
\frac{k-1-u}{4} & \frac{k-\mu}{4} & \frac{k-\mu}{4} & \frac{k-1-u}{4} \\
\frac{j+s}{4} & \frac{k(k-1-u)}{4k_2} & \frac{k(k-1-u)}{4k_2} & \frac{k-3}{4} \\
\end{bmatrix}
\]

\[
P_2^{(0)} = \begin{bmatrix}
\frac{k-1-u}{4} & \frac{k-\mu}{4} & \frac{k-\mu}{4} & \frac{k-1-u}{4} \\
\frac{k-1-u}{4} & \frac{(a-2k+\mu-2)x}{4} & \frac{(a-2k+\mu-2)x}{4} & \frac{k(k-\mu)}{4} \\
\frac{k-1-u}{4} & \frac{(a-2k+\mu-2)x}{4} & \frac{(a-2k+\mu-2)x}{4} & \frac{k(k-\mu)}{4} \\
\frac{k-1-u}{4} & \frac{(a-2k+\mu-2)x}{4} & \frac{(a-2k+\mu-2)x}{4} & \frac{k(k-\mu)}{4} \\
\end{bmatrix}
\]
(ii) For Theorem \ref{thm:example} \( (ii) \),

\[
B_1^{(0)} = \begin{bmatrix}
\frac{k_{-4}}{4} & \frac{(k-1-1)L}{4k_2} & \frac{(k-1-1)L}{4k_2} & \frac{-3L}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\end{bmatrix}.
\]

\[
B_2^{(0)} = \begin{bmatrix}
\frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\frac{-L}{4} & \frac{k_{-4}}{4} & \frac{k_{-s}}{4} & \frac{k_{-s}}{4} \\
\end{bmatrix}.
\]

(iii) For Theorem \ref{thm:example} \( (iii) \), we have

\[
B_1^{(0)} = \begin{bmatrix}
\frac{n_k+1}{4k} & \frac{n_{k-r}+n_{k+2}r+2\Phi}{4k} & \frac{n_{k-r}+n_{k-2}r+2\Phi}{4k} & \frac{n_k-3}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\end{bmatrix},
\]

\[
B_2^{(0)} = \begin{bmatrix}
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\frac{n_{k-r}+n_{k-2}r-\Pi}{4k} & \frac{n_k(n_{k-r}+\Gamma+\Pi)}{4k^2} & \frac{n_k(n_{k-r}+\Gamma-2\Phi)}{4k^2} & \frac{n_k+n_{k-r}+2\Phi+\Pi}{4k} \\
\end{bmatrix},
\]

where

\[\Gamma = m_1 r z + m_2 s c, \quad \Phi = m_1 r \sqrt{bc} - m_2 s \sqrt{bc}, \quad \Pi = m_1 r y + m_2 b s.\]

Proof. We now calculate the intersection number \( p_{11} \) in \( (i) \) as an example.

By Eq. \( (1) \) and Theorem \ref{thm:example} \( (ii) \), we can obtain the \((1,1)\)-entry \( p_{11}^{(0)} \) of \( B_i^{(0)} \):

\[p_{11}^{(0)} = \frac{k_1^3 + m_1 r^3 + s^3 m_2 + n k_1}{4 n k_1}.
\]

Let \( \tilde{A}_0, \tilde{A}_1, \tilde{A}_2 \) and \( \tilde{E}_0, \tilde{E}_1, \tilde{E}_2 \) be the adjacency matrices and primitive idempotents of \( \tilde{X} \) in Theorem \ref{thm:example} Then \( \tilde{A}_1 = k_1 \tilde{E}_0 + r \tilde{E}_1 + s \tilde{E}_2 \) and thus \( \tilde{A}_1 = k_1^3 \tilde{E}_0 + r^3 \tilde{E}_1 + s^3 \tilde{E}_2 \) under the usual matrix multiplication. Take the trace of both sides of this identity:

\[
\text{trace}(k_1^3 \tilde{E}_0 + r^3 \tilde{E}_1 + s^3 \tilde{E}_2) = k_1^3 + m_1 r^3 + m_2 s^3,
\]

\[
\text{trace}(\tilde{A}_1) = \text{sum of all entries of} (\tilde{A}_1^2 \circ \tilde{A}_1).
\]

Since \( \tilde{A}_1^2 = k_1 \tilde{A}_0 + m_1 \tilde{A}_1 + m_2 \tilde{A}_2 \), \( \tilde{A}_1^2 \circ \tilde{A}_1 = \tilde{A} \tilde{A}_1 \). Hence, \( \text{trace}(\tilde{A}_1) = n k_1 \lambda \) and thus \( k_1^3 + m_1 r^3 + m_2 s^3 = n k_1 \lambda \). So \( p_{11}^{(0)} = (\lambda + s)/4 \). The other intersection numbers can be obtained in a similar way. \( \square \)
Since the character tables $P_I$ and $P_{II}$ are completely determined by $\hat{P}$, we can check the parameter set $(n,k,\lambda,\mu)$ of a strongly regular graph for possible fissions. Furthermore, that intersection numbers are nonnegative integers put some arithmetic restrictions on the parameters $n,k,\lambda,\mu$. Note the intersection matrices in Theorem 2.2 (ii) can be obtained from those in (i) by switching $r$ and $s$, and $t$ and $u$. Therefore, we will state these restrictions for the case (i).

**Lemma 3.1.** Let $q$ be a finite field with $q$ elements, and $\alpha$ be a primitive element of $GF(q)$. For a fixed divisor $d$ of $q - 1$, define
\[
(x,y) \in R_i \text{ if } x-y \in \alpha^i(\alpha^d), \quad 1 \leq i \leq d.
\]
These relations $R_i$ define a pseudocyclic association scheme. We denote it by Cyc($q,d$), called the $d$-class cyclotomic scheme over $GF(q)$. The scheme Cyc($q,d$) is symmetric if and only if $(q - 1)/d$ is even or $q$ is a power of 2.

The intersection numbers of Cyc($q,d$) are given by cyclotomic numbers of order $d$ (for example, see [3, Chapter 2] for cyclotomy). The intersection number $p_{ij}^d$ is the number of nonzero elements $z \in GF(q)$ such $(x,z) \in R_i$ and $(z,y) \in R_j$ for any $(x,y) \in R_k$, which can be reduced to the number of solutions $s$ in $\alpha^{i+j}(\alpha^d)$ such that $1 + s$ is in $\alpha^{k+i}(\alpha^d)$. The latter is the cyclotomic number $(j - i, k - i)$. It is not difficult to prove the follow result.

**Lemma 3.2.** Suppose that $P$ is the character table of a 4-class skew-symmetric scheme whose symmetrization is Cyc($q,2$) for $q \equiv 5 \mod 8$. Then $P$ has the following form:
\[
P = \begin{bmatrix} 1 & f & f & f & f \\ 1 & \rho & \rho & \tau & \bar{\tau} \\ 1 & \bar{\rho} & \rho & \bar{\tau} & \tau \\ 1 & \tau & \bar{\tau} & \rho & \bar{\rho} \\ 1 & \bar{\tau} & \bar{\rho} & \rho & \tau \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}, \quad \text{where } (*) \quad \begin{bmatrix} f = \frac{1}{4}(q - 1) \\ \rho + \bar{\rho} = \frac{1}{2}(-1 + \sqrt{q}) \\ \tau + \bar{\tau} = \frac{1}{2}(-1 - \sqrt{q}) \\ \rho\bar{\rho} + \tau\bar{\tau} = \frac{1}{2}(3q + 1) \end{bmatrix}.
\]
Song [22] gave two solutions for the system $(*)$ and he also raised the following questions:

**Question 1.** Is there any other solution for the system $(*)$ that yields a feasible fission table $P$ for given $\hat{P}$ (from the symmetrization of $P$), for a large prime power $q \equiv 5 \mod 8$?
that are not Paley graphs. A conference graph on Paley graphs are conference graphs. But the converse is not true; there are conference graphs

Theorem 3.3. tables of its 4-class skew-symmetric fission schemes.

1 mod 4, then there is no danger of confusion, we also use \( z \) for some integer \( m \). Since \( z \) is an integer. Since \( z \) can obtain that \( z = \frac{1}{2}(q-1) \) for some integer \( h \), where

\[
p_{11} = \frac{\lambda + r}{4} = \frac{q + 2\sqrt{q - 7}}{16}, \quad p_{11}^2 = \frac{k(k - \lambda - 1 - t)}{4k_2} = \frac{q + 2\sqrt{q + 1}}{16}.
\]

Now \( p_{11}^2 - p_{11} = 1/2 \), a contradiction.

Now suppose that \( C(q) \) has a 4-class skew-symmetric fission scheme with character table of type III:

\[
P_{III} = \begin{bmatrix}
1 & f & f & f & f \\
1 & \rho & \tau & \bar{\rho} & \bar{\tau} \\
1 & \tau & \bar{\rho} & \rho & \bar{\tau} \\
1 & \bar{\tau} & \rho & \bar{\rho} & \tau \\
1 & \bar{\rho} & \tau & \rho & \bar{\tau}
\end{bmatrix}, \quad \begin{cases} f = \frac{1}{4}(q - 1) \\
\rho = \frac{1}{2}(-1 + \sqrt{q + 2\sqrt{q - z}}) \\
\tau = \frac{1}{2}(-1 - \sqrt{q + 2\sqrt{q - z}})
\end{cases}
\]

We can calculate the first intersection matrices \( B_1 \) of the scheme from Theorem 2.2 (iii):

\[
B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & A & B & D & C \\
0 & E & E & B & D \\
f & A & E & E & A
\end{bmatrix}, \quad \begin{cases} 16qA = q^2 - 7q + 2\sqrt{q(q - 2z)} \\
16qB = q^2 + q + 2\sqrt{q(q - 2z)} + 8\sqrt{q}\sqrt{3(q - z)} \\
16qC = q^2 + q + 6\sqrt{q(q - 2z)} \\
16qD = q^2 + q + 2\sqrt{q(q - 2z)} - 8\sqrt{q}\sqrt{3(q - z)} \\
16qE = q^2 - 3q - 2\sqrt{q(q - 2z)}
\end{cases}
\]

Since \( A + E = (q - 5)/8 \), \( q \equiv 5 \) mod 8. So \( q \) can not be a perfect square. Otherwise, \( \sqrt{q} \) is an odd integer and thus \( \sqrt{q} \equiv 1 \) or 3 mod 4. It follows that \( q \equiv 1 \) mod 8, a contradiction.

Now \( q \) is a nonsquare integer. So \( q - 2z = g \sqrt{q} \) for some integer \( g \) in order for \( A \) to be an integer. Since \( A, C, E \) are integers, it follows that \( g \equiv 1 \) mod 4. In order for \( B \) to be an integer, \( \sqrt{z(q - z)} = h \sqrt{q} \) for some integer \( h \). Substituting \( z = (q - g \sqrt{q})/2 \) into \( \sqrt{z(q - z)} = h \sqrt{q} \), we can obtain that \( q = g^2 + 4h^2 \).

Conversely, if \( q \equiv 5 \) mod 8 and if there exist integer \( s,t \) such that \( q = g^2 + 4h^2 \) with \( g \equiv 1 \) mod 4, then \( P_{III} \) is the character table of a putative scheme, where \( z = (q - g \sqrt{q})/2 \). \( \square \)
We now determine the character table of the cyclotomic scheme \( \text{Cyc}(q,4) \) for \( q \equiv 5 \mod 8 \).

**Theorem 3.4.** Let \( q = 4f + 1 \) be a prime power with \( f \) odd. So \( \text{Cyc}(q,4) \) is a 4-class skew-symmetric association scheme. Let \( g \) and \( h \) be defined by \( q = g^2 + 4h^2, g \equiv 1 \mod 4 \) and \( (g,q) = 1 \); these conditions determine \( g \) uniquely, and \( h \) up to a sign. Then the following hold:

(i) \( \text{Cyc}(q,4) \) has the following character table \( P \):

\[
\begin{align*}
1 & f & f & f & f & f \\
1 & \rho & \tau & \tilde{\rho} & \tilde{\tau} & \tilde{\tau} \\
1 & \tilde{\rho} & \rho & \tilde{\rho} & \tau & \tau \\
1 & \tilde{\tau} & \rho & \tilde{\rho} & \tau & \tau \\
\end{align*}
\]

\( f = \frac{1}{4}(q - 1) \), where \( \rho = \frac{1}{4}(-1 + \sqrt{1 + \sqrt{1 - 2q - 2g \sqrt{q}}} \right) \) and \( \tau = \frac{1}{4}(-1 - \sqrt{1 - 2q + 2g \sqrt{q}} \right) .

(ii) \( \text{Cyc}(q,4) \) has the following intersection matrices \( B_1, B_2 \):

\[
\begin{align*}
0 & 1 & 0 & 0 & 0 \\
0 & A & B & D & C \\
0 & E & E & B & D \\
f & A & E & E & A \\
\end{align*}
\]

\[
\begin{align*}
0 & 1 & 0 & 0 \\
0 & E & E & B & D \\
0 & D & A & C & B \\
f & E & A & E & A \\
\end{align*}
\]

\[
\begin{align*}
0 & B & E & D & E \\
\end{align*}
\]

\( 16A = q - 7 + 2g \)
\( 16B = q + 1 + 2g + 8h \)
\( 16C = q - 1 + 6g \)
\( 16D = q + 1 + 2g - 8h \)
\( 16E = q - 3 - 2g \)

\( \text{Proof.} \) Since the intersection numbers \( p^i_j \) of \( \text{Cyc}(q,4) \) are given by the cyclotomic numbers \( (j - i, k - i) \), which are given in \([20, \text{Proposition 11}]\). From the intersection matrices, we can calculate the character table \( P \). \( \Box \)

We note that a putative 4-class skew-symmetric fission scheme of \( C(q) \) has intersection matrices of the form in Theorem 3.4 (ii), where \( q = g^2 + 4h^2 \) with \( g \equiv 1 \mod 4 \). In particular, Theorem 3.3 answers Question 1, in which \( q \) is a prime power with \( q \equiv 5 \mod 8 \). Question 2 remains open. From number theory (e.g., [11, §17.6]), \( q = g^2 + 4h^2 \) has a solution if and only if in the prime factorization of \( q \), every prime factor \( \equiv 3 \mod 4 \) has an even exponent. In particular, if \( q \equiv 5 \mod 8 \) is a prime, there is essentially one way to express \( q \) as a sum of two squares. So \( \text{Cyc}(q,2) \) has a unique fission table, which is realized by \( \text{Cyc}(q,4) \).

4. Imprimitivity

In this section, we investigate 4-class skew-symmetric imprimitive association schemes. An association scheme \( (X, \{R_i\}_{i=0}^3) \) is imprimitive if the union of some relations is an equivalence relation that is not \( R_0 \) or \( X \times X \), and primitive otherwise.

Let \( \mathcal{X} = (X, \{R_0, R_1, R_2, R_3\}) \) be a skew-symmetric association scheme. If \( \mathcal{X} \) is imprimitive, then \( R_0 \cup R_1 \cup R_2 \) or \( R_0 \cup R_2 \cup R_3 \) is an equivalence relation on \( X \). Without loss of generality, we assume that \( R_0 \cup R_1 \cup R_2 \) is an equivalence relation. So \( (X, R_1 \cup R_2) \) is the union of certain copies of a complete graph, \( gK_f \), where \( n = |X| = gf \). Thus, \( (X, R_2 \cup R_3) \) is the complete multipartite graph \( gK_f \). The symmetrization \( \overline{\mathcal{X}} \) has the following character table:

\[
\overline{P} = \begin{pmatrix}
1 & f - 1 & f(g - 1) \\
1 & f - 1 & -f \\
1 & -1 & 0 \\
\end{pmatrix}
\]

\( g - 1 \) .
By Theorem 4.1, $\mathcal{X}$ has one of the following character tables:

$$P_1 = \begin{bmatrix}
1 & f^{-1} & f(g-1) & (f-1) \\
1 & f^{-1} & f(g-1) & (f-1) \\
1 & \sigma & 0 & 0 \\
1 & \sigma & 0 & 0 \\
1 & \rho & 0 & 0 \\
1 & \rho & 0 & 0 \\
\end{bmatrix}, \quad \text{where} \quad \begin{cases}
\sigma = \frac{1}{2}(-1 + \sqrt{-f}) \\
\tau = \frac{1}{2}(-f + \sqrt{-gf^2})
\end{cases}.
$$

$$P_{II} = \begin{bmatrix}
1 & f^{-1} & f(g-1) & (f-1) \\
1 & \rho & -\frac{f}{2} & -\frac{f}{2} \\
1 & -\frac{1}{2} & \omega & \bar{\omega} & -1/2 \\
1 & -\frac{1}{2} & \bar{\omega} & \omega & -1/2 \\
1 & \bar{\rho} & -\frac{f}{2} & -\frac{f}{2} & \rho
\end{bmatrix}, \quad \text{where} \quad \begin{cases}
\rho = \frac{1}{2} \sqrt{-f(g-1)g^{-1}} \\
\omega = \frac{1}{2} \sqrt{-f(g-1)/g^{-1}}
\end{cases}.
$$

$$P_{III} = \begin{bmatrix}
1 & f^{-1} & f(g-1) & (f-1) \\
1 & \rho & \tau & \bar{\rho} \\
1 & \sigma & \omega & \bar{\omega} & \bar{\sigma} \\
1 & \bar{\sigma} & \bar{\omega} & \omega & \sigma \\
1 & \bar{\rho} & \bar{\tau} & \rho & \rho
\end{bmatrix}, \quad \text{where} \quad \begin{cases}
\rho = \frac{1}{2} \sqrt{-f(g-1)(g-1)} \\
\sigma = \frac{1}{2} \sqrt{-1 + \bar{f}/\sigma} \\
\tau = \frac{1}{2} \sqrt{-f + \sqrt{z}} \\
\omega = \frac{1}{2} \sqrt{-(f-\bar{f})g^{-1}}
\end{cases}.
$$

By Theorem 2.1, the intersection matrices $B_1, B_2$ for $P_1$ has the following principal parts:

$$\begin{bmatrix}
1 & f - 3 & 0 & 0 & f + 1 \\
0 & 2(f - 1) & 0 & 0 & f \\
0 & 0 & 2(f - 1) & 0 & f \\
f - 3 & 0 & 0 & f - 3 & f
\end{bmatrix} \cdot \begin{bmatrix}
0 & 2(f - 1) & 0 & 0 \\
0 & (g - 3)f & (g + 1)f & 0 \\
2f & (g - 3)f & (g - 3)f & 2f \\
2f & (g - 1)f & (g - 1)f & 2f & (g - 1)
\end{bmatrix}.$$

It follows that $f, g \equiv 3 \mod 4$.

The intersection matrix $B_1$ for $P_{II}$ has negative entries and thus $P_{II}$ is not realizable.

For $P_{III}$, the parameter $y$ can be calculated from Theorem 2.1 (iii): $y = \frac{1}{f^{-1}/(f-1)}(gf^2 - z)$. So $z < gf^2$. By Theorem 2.2 (iii), we can obtain the intersection numbers. In particular, $P_{12}^1 = \frac{z - gf^2}{2f^2}$ and hence $z \geq gf^2$, a contradiction. So $P_{III}$ is not realizable either. Thus, we have proved following result, and the "if part" was mentioned in 22 Lemma 2.3.

**Theorem 4.1.** Let $\mathcal{X}$ be an imprimitive 2-class association scheme with character table given in $\mathcal{X}$. $\mathcal{X}$ has a putative 4-class skew-symmetric fission scheme if and only if $f, g \equiv 3 \mod 4$. In this case, the character table of this fission scheme is realized as $P_1$.

As pointed out in 22, when $f, g \equiv 3 \mod 4$ are prime powers, the fission scheme in Theorem 4.1 is realized as a wreath product of Cyc(2) and Cyc(g, 2). For example, there are two 4-class skew-symmetric association schemes on 21 vertices from the wreath products Cyc(3, 2) wr Cyc(7, 2) and Cyc(7, 2) wr Cyc(3, 2).
5. Lists of small feasible parameters

In order to generate feasible parameters for 4-class skew-symmetric association schemes we shall classify them into three sets according to their symmetrizations:

1. The symmetrization is imprimitive;
2. The symmetrization is pseudocyclic thus a conference type, which is primitive.
3. The symmetrization is non-conference type and primitive.

These three sets cover all possibilities. The tables to follow list information about existence and enumeration of primitive skew-symmetric schemes with 4 classes. In the heading #, a plus sign “+” means that there is at least one such scheme; a question mark “?” that there is no such scheme known; and a number \( m \) that there are \( m \) such schemes.

Case 1 is determined by Theorem 4.1 so each feasible parameter is determined by a pair \((f, g)\). For distinct primes \( f \) and \( g \) with \( f, g \equiv 3 \mod 4 \), there are at least two 4-class skew-symmetric imprimitive association schemes on \( fg \) vertices, realized by the wreath product.

Case 2 is determined by Theorem 3.3. In this case, \( q = g^2 + 4h^2 \) with \( g \equiv 1 \mod 4 \), and the pairs \((g, h)\) can be easily generated (for example, the command \texttt{sum2sqr} in Maple). Each parameter set in this case is determined by the pair \((g, h)\). In particular, if \( q \) is a prime power with \((g, p) = 1\), then Cyc\((q, 4)\) realizes this parameter set. We list the parameter sets up to 325 vertices in Table 1.

Table 1: Feasible parameters of pseudocyclic skew-symmetric schemes

| \( n \) | 5  | 13 | 29 | 37 | 45 | 53 | 61 | 85 | 85 | 101 | 109 | 117 | 125 |
|--------|----|----|----|----|----|----|----|----|----|-----|-----|-----|-----|
| \( g \) | 1  | -3 | 5  | 1  | -3 | -7 | 5  | 9  | -7 | 1   | -3  | 9   | -11 |
| #  | 1  | 1  | 1  | +  | ?  | +  | +  | ?  | +  | +   | ?   | +   | +   |
| \( n \) | 125 | 149| 157| 173| 181| 197| 205| 205| 221| 221| 229| 245| 261 |
| \( g \) | 5  | -7 | -11| 13 | 9  | 1  | -3 | 13 | 5  | -11| -15 | 7   | -15 |
| #  | ?  | +  | +  | +  | +  | +  | ?  | ?  | +  | ?   | ?   | +   | ?   |
| \( n \) | 269 | 277| 293| 317| 325| 325| 325| 325| 325| 325| 325| 325| 325 |
| \( g \) | 13 | 9  | 17 | -11| 1  | 17 | -15|    |    |    |    |    |    |
| #  | +  | +  | +  | +  | ?  | ?  | ?  |    |    |    |    |    |    |

An association scheme in Case 3 has character table given by one of the three types in Theorem 2.1. Character tables of type I or II are uniquely determined by the underlying symmetrizations, while those of type III need an additional parameter \( z \). Here is how we determine \( z \). Starting with the parameter set of a strongly regular graph, we generate the intersection matrices of all skew-symmetric fission schemes with four classes. Then we determine the value of \( z \) using Theorem 2.2 (iii). Based on the table of strongly regular graphs on A. E. Brouwer’s webpage [5], a list is generated for up to 1300 vertices in Table 2. We further rule out some parameter sets with the Krein condition.

We note that the Krein condition rules out several parameter sets in Table 2. It turns out that these parameter sets are from the family of 2-class Johnson schemes. The Johnson scheme \( J(v, 2) \) has as vertices all 2-subsets of a \( v \)-set and two 2-subsets in relation \( R_i \) if their intersection has size \( 2 - i \) (\( 0 \leq i \leq 2 \)). The relation \( R_i \) defines a strongly regular graph, with parameters

\[ n = \frac{v(v - 1)}{2}, \quad k = 2(v - 2), \quad \lambda = v - 2, \quad \mu = 4. \]
Since intersection numbers are nonnegative integer and \( n(n-2k+\lambda) > 0 \), then by Theorem 2.2 (ii), we have the following: if its character table is of type I, then the scheme can not have character table of type I by Corollary 2.3 (ii). If its character table is of type III, we can compute its intersection numbers using Theorem 2.2 (iii). In particular, we have the following:

\[
\begin{align*}
\text{Table 2: Feasible parameters of non-conference type skew-symmetric schemes} \\
\begin{array}{cccccccc}
 n & k & \lambda & \mu & \rho_{\mu} & \nu_{\mu} & \text{type} & z & \# \\
57 & 14 & 1 & 4 & 2^{18} & -518 & \text{III} & 27 & 0 [5] \\
105 & 26 & 13 & 4 & 11^{14} & -2^{90} & \text{III} & 540 & 0 \text{ Krein} \\
253 & 42 & 21 & 4 & 19^{22} & -2^{230} & \text{III} & 2300 & 0 \text{ Krein} \\
273 & 102 & 41 & 36 & 11^{90} & -6^{182} & \text{III} & 364 & ? \\
381 & 114 & 29 & 36 & 6^{254} & -13^{126} & \text{III} & 147 & ? \\
441 & 110 & 19 & 30 & 5^{330} & -16^{110} & \text{II} & ? \\
441 & 110 & 19 & 30 & 5^{330} & -16^{110} & \text{III} & 252 & ? \\
465 & 58 & 29 & 4 & 27^{30} & -2^{434} & \text{III} & 6076 & 0 \text{ Krein} \\
497 & 186 & 55 & 78 & 4^{426} & -2^{709} & \text{III} & 175 & ? \\
729 & 182 & 55 & 42 & 20^{182} & -7^{546} & \text{I} & ? \\
741 & 74 & 37 & 4 & 3^{538} & -2^{702} & \text{III} & 12636 & 0 \text{ Krein} \\
813 & 290 & 109 & 100 & 19^{270} & -10^{532} & \text{III} & 1084 & ? \\
889 & 222 & 35 & 62 & 5^{762} & -3^{122} & \text{II} & ? \\
889 & 222 & 35 & 62 & 5^{762} & -3^{122} & \text{III} & 252 & ? \\
945 & 354 & 153 & 120 & 39^{118} & -6^{826} & \text{II} & ? \\
993 & 310 & 89 & 100 & 10^{662} & -21^{1330} & \text{III} & 363 & ? \\
1065 & 266 & 103 & 54 & 5^{310} & -4^{994} & \text{III} & 10224 & ? \\
1081 & 90 & 45 & 4 & 4^{346} & -2^{1034} & \text{III} & 22748 & 0 \text{ Krein} \\
1225 & 306 & 89 & 72 & 26^{306} & -9^{1918} & \text{III} & 1575 & ? \\
1225 & 510 & 215 & 210 & 20^{1010} & -15^{714} & \text{I} & ? \\
1241 & 310 & 81 & 76 & 18^{510} & -13^{730} & \text{I} & ? \\
\end{array}
\end{align*}
\]

We ask the following:

**Question 3.** Does any Johnson scheme \( J(v, 2) \) admit a 4-class skew-symmetric fission scheme?

We now settle this question in the rest of this section. \( J(v, 2) \) has the following character table:

\[
P = \begin{bmatrix}
1 & 2(v-3) & \frac{1}{2}(v-2)(v-3) \\
1 & v - 4 & -v + 3 \\
1 & -2 & 1 \\
\end{bmatrix}
\]

Since both of the nontrivial multiplicities \( v - 1 \) and \( v(v-3)/2 \) are even, it is necessary that \( v \equiv 3 \mod 4 \) for \( J(v, 2) \) to admit a desired fission scheme. Since \( \lambda + s = v - 4 \equiv 3 \mod 4 \), this fission scheme can not have character table of type I by Corollary 2.3 (ii). If its character table is of type II, then by Theorem 2.2 (ii), we have \( \lambda = \frac{v}{2} = \frac{v}{2} \) and \( s = \frac{v}{2} = \frac{v}{2} \). So \( v = 5 \), a contradiction. If this fission scheme has character table of type III, we can compute its intersection numbers using Theorem 2.2 (iii). In particular, we have the following:

\[
\frac{nk\lambda + \Pi}{4nk} = \frac{v(v-3)^2 - 2z}{2v(v-3)}, \quad \frac{nk(n - 2k + \lambda) + \Gamma}{4nk_2} = \frac{4z}{v(v-3)^2}.
\]

Since intersection numbers are nonnegative integer and \( z > 0 \), either \( z = \frac{1}{4}v(v-3)^2 \) or \( z = \frac{1}{2}v(v-3)^2 \). The Krein numbers can be calculated from Eq. (2). For \( z = \frac{1}{4}v(v-3)^2 \), \( q_{11}^3 = \)

\[
\frac{nk\lambda + \Pi}{4nk} = \frac{v(v-3)^2 - 2z}{2v(v-3)}, \quad \frac{nk(n - 2k + \lambda) + \Gamma}{4nk_2} = \frac{4z}{v(v-3)^2}.
\]
\[
\frac{(v-1)(v-1-\sqrt{2(v-1)})}{4(v-2)^2} < 0, \text{ a contradiction. For } z = \frac{1}{2}(v - 3)^2, \text{ the value } c \text{ in Theorem 2.1 (ii) is not valid, and hence } \frac{z}{v} = \frac{1}{2}(v - 1)(6 - v). \text{ So } v = 3 \text{ and hence } z = 0, \text{ a contradiction. Therefore, we have proved:}
\]

**Proposition 5.1.** No Johnson scheme \(J(v, 2)\) admits a 4-class skew-symmetric fission scheme.

6. Concluding Remarks

**Remark 1.** We do not know any association scheme with character table of type II or III. In view of Theorems 3.3 and 3.4, it is interesting to know which 4-class fission tables of \(C(q)\) or \(\text{Cyc}(q, 2)\) can be realized. Using the small association schemes data of Hanaki and Miyamoto [19], we find that \(\text{Cyc}(q, 4)\) are the only 4-class pseudocyclic skew-symmetric schemes for \(q \leq 30\).

**Remark 2.** The association schemes discussed in the paper are related to some combinatorial designs. Each pseudocyclic scheme give rise to a 2-\((n, f, f - 1)\) design and 2-\((n, f + 1, f + 1)\) design (see [3, p. 48]). A 2-class skew-symmetric association scheme is equivalent to a Hadamard tournament and an imprimitive skew-symmetric association scheme gives rise to a multipartite tournament (see [14]). It seems interesting to pursue further study along this line.

**Remark 3.** Question 3 should be asked in more general form. Which 2-class association scheme admits a (putative) skew-symmetric fission scheme with 4 classes? For example, study the possibility of 4-class skew-symmetric fission scheme of known series of strongly regular graphs. We shall investigate this question in another paper.

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