NOTE ON SIMULATION PRICING OF \( \pi \)-OPTIONS

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ABSTRACT. In this work, we adapt a Monte Carlo algorithm introduced by Broadie and Glasserman [5] to price a \( \pi \)-option. This method is based on the simulated price tree that comes from discretization and replication of possible trajectories of the underlying asset’s price. As a result this algorithm produces lower and upper bounds that converge to the true price with the increasing depth of the tree. Under specific parametrization, this \( \pi \)-option is related to relative maximum drawdown and can be used in the real-market environment to protect a portfolio against volatile and unexpected price drops. We also provide some numerical analysis.

KEYWORDS. \( \pi \)-option \* American-type option \* optimal stopping \* Monte Carlo simulation

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1. Introduction

In this paper we analyze $\pi$-options that depends on so-called relative drawdown and can be used in hedging against volatile and unexpected price drops or by speculators betting on falling prices. These options are the contracts with a payoff function:

$$g(S_T) = (M_T^a S_T^b - K)^+$$

in case of the call option and:

$$g(S_T) = (K - M_T^a S_T^b)^+$$

in the case of put option where

$$S_t = S_0 \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma B_t \right)$$

is an asset price in the Black-Scholes model under martingale measure, that is, $r$ is a risk-free interest rate, $\sigma$ is an asset’s volatility and $B_t$ is a Brownian motion. Moreover,

$$M_t = \sup_{w \leq t} S_w$$

is a running maximum of the asset price and $T$ is its maturity. Finally, $a$ and $b$ are some chosen parameters.

Few very well known options are particular cases of $\pi$-option. In particular, taking $a = 0$ and $b = 1$ produces American option and by choosing $a = 1$ and $b = 0$ we derive Lookback option. Another interesting case is when $-a = b = 1$ and $K = 1$. Then the pay-out function $(K - M_T^a S_T^b)^+ = 1 - \frac{S_T^b}{M_T^a} = \frac{M_T - S_T}{M_T} = D_T^R$ equals the relative drawdown $D_T^R$ defined as a quotient of the difference between maximum price and the present value of the asset and the past maximum price. In other words, $D_T^R$ corresponds to the percentage drop in price from its maximum; see Figure 6.

Monte Carlo simulations are widely used in pricing in financial markets they have proved to be valuable and flexible computational tools to calculate the value of various options as witnessed by the contributions of [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 15, 14, 16, 17, 18, 19, 20, 21, 22].

In this paper we adapt a Monte Carlo algorithm proposed in 1997 by Broadie and Glasserman [5] to price $\pi$-options. This numerical method replicates possible trajectories of the underlying asset’s price by a simulated price-tree. Then, the values of two estimators, based on the price-tree, are obtained. They create an upper and a lower bound for the true price of the option and, under some additional conditions, converge to that price. The first estimator compares the early exercise payoff of the contract to its expected continuation value (based on the successor nodes) and decides if it is optimal to hold or exercise the option. This estimation technique is one of the most popular ones used for pricing American-type derivatives. However, as shown by Broadie and Glasserman [5], it overestimates the true price of the option. The second estimator also compares the expected continuation value and early exercise payoff, but in slightly different way, which results in underestimation of the true price. Both Broadie-Glasserman Algorithms (BGAs) are explained and described precisely in Section 2. The price-tree that we need to generate is parameterized by the number of nodes and also by the number of branches in each node. Naturally, the bigger the numbers of nodes and branches, the more accurate price estimates we get. The obvious drawback of taking a bigger price-tree is that the computation time increases significantly with the size of the tree. However, in this paper we show that one can take a relatively small price-tree and still the results are satisfactory.

In this paper we use BGA to price the $\pi$-option on relative drawdown for the Microsoft Corporation’s (MSFT) stock. Input parameters for the algorithm are based on real market data, that is, daily closing prices of MSFT stock from 6.11.2017 to 9.11.2018. Moreover, we provide an exemplary situation in which we explain the possible application of $\pi$-option on relative drawdown to the protection against volatile price movements. We also compare this type of option to an American put and outline the difference between these two contracts.

This paper is organized as follows. In next section we describe we present Broadie-Glasserman Algorithm. Section 3 uses this algorithm to numerically study $\pi$-options for the Microsoft Corporation’s stock. Finally,
in last section, we state our conclusions and recommendations for further research into this new and interesting topic.

2. Monte Carlo algorithm

There are no known explicit formulas for the general price of π-option. Therefore we propose a Monte Carlo method of pricing of this financial instrument. In this section we present a detailed description of used algorithm. In particular, we give formulas for two estimators, one biased low and one biased high, that under certain conditions converge to the theoretical price of the option.

2.1. Preliminary notations. We adapt the Monte Carlo method introduced by Broadie and Glasserman in [5] for pricing American options. In this algorithm, values of two estimators are calculated on the so-called price-tree that represents the underlying’s behavior over time. This tree is parameterized by the number of nodes \( n \) and the number of branches in each node - denoted by \( l \). For example the tree with parameters \( n = 2, l = 3 \) is depicted in Figure [1]

In order to apply the numerical algorithm we have to discretize the price process given in (3) by considering the time sequence \( t_0 = 0 < t_1 < \ldots < t_n = T \) with \( t_i = i T/n \) for \( i = 0, \ldots, n \). By \( S_{t_i}^{l_1, \ldots, l_i} \) we denote the asset’s price at the time \( t_i = i T/n \). The upper index \( l_1, \ldots, l_i \) associated with \( t_i \) describes the branch selection (see Figure [2]) in each of the tree nodes and allows us to uniquely determine the path of the underlying’s price process up to time \( t_i \). Similarly we define \( M_{t_i}^{l_1, \ldots, l_i} = \max_{k \leq i} S_{t_k}^{l_1, \ldots, l_k} \).

We introduce the state variable \( \tilde{S}_{t_i}^{l_1, \ldots, l_i} = (S_{t_i}^{l_1, \ldots, l_i}, M_{t_i}^{l_1, \ldots, l_i}) \) as well.

We relate with it the payoff of immediate exercise (for π put) at time \( t_i \) in the state \( \tilde{S}_{t_i}^{l_1, \ldots, l_i} \)

\[
 h_{t_i}(\tilde{S}_{t_i}^{l_1, \ldots, l_i}) = (K - S_{t_i}^{a_{l_1}, \ldots, l_i} M_{t_i}^{b_{l_1}, \ldots, l_i})^+
\]

and the expected value of holding the option from \( t_i \) to \( t_{i+1} \), given asset’s value \( \tilde{S}_{t_i}^{l_1, \ldots, l_i} \) at time \( t_i \)

\[
 g_{t_i}(\tilde{S}_{t_i}^{l_1, \ldots, l_i}) = E \left[ e^{-r} f_{t_{i+1}}(\tilde{S}_{t_{i+1}}^{l_1, \ldots, l_{i+1}}) \big| \tilde{S}_{t_i}^{l_1, \ldots, l_i} \right]
\]

where

\[
 f_{t_i}(\tilde{S}_{t_i}^{l_1, \ldots, l_i}) = \max\{h_{t_i}(\tilde{S}_{t_i}^{l_1, \ldots, l_i}), g_{t_i}(\tilde{S}_{t_i}^{l_1, \ldots, l_i})\}
\]

is the option value at time \( t_i \) in state \( \tilde{S}_{t_i}^{l_1, \ldots, l_i} \). Note that

\[
 f_{t_n}(\tilde{S}_{t_n}^{l_1, \ldots, l_n}) = f_T(\tilde{S}_T^{l_1, \ldots, l_n}) = h_T(\tilde{S}_T^{l_1, \ldots, l_n}) = (K - S_{T}^{a_{l_1}, \ldots, l_n} M_{T}^{b_{l_1}, \ldots, l_n})^+.
\]

2.2. Estimators. We will give now the formulas for the estimators \( \Theta \) and \( \Phi \) which overestimate and underestimate the true price of the option, respectively. Then we will state main theorem showing that both estimators are asymptotically unbiased and that they converge to the theoretical price of the π-option.
2.2.1. The $\Theta$ estimator. The formula for the estimator is recursive and given by:

$$\Theta_t_i = \max \left\{ h_{t_i}(\tilde{S}_{l_1}, \ldots, l_i), e^{-r \frac{1}{T} \sum_{j=1}^{l} \Theta_{t_{i+1}}(l_{i+1}, \ldots, l_{i+j})} \right\}, \quad i = 0, \ldots, n - 1.$$  

At the option’s maturity, $T$, the value of the estimator is given by

$$\Theta_T = f_T(\tilde{S}_T).$$

The $\Theta$ estimator, at each node of the price tree, chooses the maximum of the payoff of the option’s early exercise at time $t_i$, $h_{t_i}(\tilde{S}_{l_1}, \ldots, l_i)$, and the expected continuation value, i.e. the discounted average payoff of successor nodes. Figure 3 below shows how the value of $\Theta$ estimator is obtained given the certain realization of a price-tree. All calculations are also shown here:
Figure 2. Branch selecting.

- **a**
  
  Holding value: \( \frac{0 + \frac{10}{3}}{3} e^{-0.05} \approx 0.028 \)
  
  Early exercise: 0

- **b**
  
  Holding value: \( \frac{5}{3} + \frac{10}{3} + \frac{13}{3} e^{-0.05} \approx 0.052 \)
  
  Early exercise: \( \frac{10}{110} \approx 0.091 \)

- **c**
  
  Holding value: \( \frac{10}{3} + \frac{15}{3} + \frac{7}{3} e^{-0.05} \approx 0.086 \)
  
  Early exercise: \( \frac{20}{110} \approx 0.182 \)

- **d**
  
  Holding value: \( \frac{0.028 + 0.091 + 0.182}{3} e^{-0.05} \approx 0.095 \)
  
  Early exercise: \( \frac{10}{110} \approx 0.091 \)
2.2.2. The Φ estimator. The Φ estimator is also defined recursively. Before we give the formula we need to introduce an auxiliary function ξ by

\[
\xi^j_{i_1, \ldots, i_l} = \begin{cases} 
    h_{t_i}(\tilde{S}^1_{i_1^{t_i}, \ldots, i_l}), & \text{if } h_{t_i}(\tilde{S}^1_{i_1^{t_i}, \ldots, i_l}) \geq \alpha - \frac{\pi}{l - 1} \sum_{k=1}^{l} \frac{1}{l - 1} \sum_{k \neq j} \Phi_{i_1^{t_i}, \ldots, i_l, k} \\
    \exp \left( -\frac{\pi}{l - 1} \sum_{k=1}^{l} \frac{1}{l - 1} \sum_{k \neq j} \Phi_{i_1^{t_i}, \ldots, i_l, k} \right), & \text{if } h_{t_i}(\tilde{S}^1_{i_1^{t_i}, \ldots, i_l}) < \alpha - \frac{\pi}{l - 1} \sum_{k=1}^{l} \frac{1}{l - 1} \sum_{k \neq j} \Phi_{i_1^{t_i}, \ldots, i_l, k}
\end{cases}
\]  

for \( j = 1, \ldots, l \). Now we can define the Φ estimator in the following way:
\begin{align}
\Phi_{t_1^{i_1}, \ldots, t_i^{i_i}} &= \frac{1}{T} \sum_{j=1}^{T} \xi_{t_1^{i_1}, \ldots, t_i^{i_i}}^j \\
\Phi_T &= f_T(S_T).
\end{align}

The formula for this estimator is more complicated. Thus we provide a detailed explanation of the mechanism behind the algorithm in the following part of this section. In our explanation we refer to Figure 4.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\begin{tikzpicture}
\node at (0,0) {100};
\node at (1,1) {110};
\node at (2,2) {115};
\node at (3,3) {120};
\node at (4,4) {120};
\node at (5,5) {0};
\draw (0,0) -- (1,1) -- (2,2) -- (3,3) -- (4,4);
\end{tikzpicture}
\caption{Price-tree}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\centering
\begin{tikzpicture}
\node at (0,0) {0};
\node at (1,1) {0.091};
\node at (2,2) {0.182};
\node at (3,3) {0.061};
\node at (4,4) {0.087};
\node at (5,5) {0.045};
\node at (6,6) {0.118};
\node at (7,7) {0.091};
\node at (8,8) {0.136};
\node at (9,9) {0.045};
\draw (0,0) -- (1,1) -- (2,2) -- (3,3) -- (4,4) -- (5,5) -- (6,6) -- (7,7) -- (8,8) -- (9,9);
\end{tikzpicture}
\caption{Evaluation of $\Phi$ estimator}
\end{subfigure}
\caption{Explanation of $\Phi$ estimator}
\end{figure}
Theorem 1. Both $\Theta$ and $\Phi$ are consistent and asymptotically unbiased estimators of the option value. They both converge to the true price of the option as the number of price-tree branches, $l$, increases to infinity. For finite $l$:

- The bias of the $\Theta$ estimator is always positive, i.e.,
  $$\mathbb{E}[\Theta_0(l)] \geq f_0(\tilde{S}_0).$$

- The bias of the $\Phi$ estimator is always negative, i.e.,
  $$\mathbb{E}[\Phi_0(l)] \leq f_0(\tilde{S}_0).$$

On every realization of price-tree, the low estimator $\Phi$ is always less than or equal to the high estimator $\Theta$, i.e.,
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$$P(\Phi_{t_1,\ldots,t_i} \leq \Theta_{t_1,\ldots,t_i}) = 1.$$  

3. Numerical analysis

In this section we will present results of the numerical analysis. First we use above described algorithm to price the American option. This will allow us to confirm that our Monte Carlo algorithm produces precise estimates of options’ prices. Next we price $\pi$-options for a number of combinations of parameters. We also consider $\pi$-option on the drawdown using the real-market Microsoft data and we compare it with American put, which is one of the most popular tool for protecting our portfolio against price drops.

To compare prices of American and $\pi$-option we implement above described algorithm for standard American option pricing. We decided to analyze the underlying Microsoft Corporation assets. The data is taken from www.finance.yahoo.com. In Table 1 we present all results.

| Strike | Low Est. | High Est. | Estimated Price | Real Price | Abs. Perc. Err. |
|--------|----------|-----------|-----------------|------------|-----------------|
| $80$   | $20.16$  | $20.55$   | $20.36$         | $20.33$    | $0.14\%$       |
| $85$   | $15.10$  | $15.54$   | $15.32$         | $15.35$    | $0.19\%$       |
| $90$   | $10.28$  | $10.62$   | $10.45$         | $10.43$    | $0.19\%$       |
| $95$   | $5.84$   | $6.02$    | $5.93$          | $5.89$     | $0.68\%$       |
| $100$  | $2.54$   | $2.60$    | $2.57$          | $2.51$     | $2.36\%$       |
| $105$  | $0.76$   | $0.77$    | $0.77$          | $0.73$     | $5.33\%$       |
| $110$  | $0.16$   | $0.16$    | $0.16$          | $0.14$     | $13.35\%$      |

Table 1. Comparison of the estimated and ‘real’ American option prices with different strikes. Absolute percentage errors are also included.

We will analyze put $\pi$-option for various combinations of parameters $a$ and $b$. Parameter $a$ is varying from $-1.1$ to $-0.9$ and $b$ parameter between $0.9$ and $1.1$. All input parameters for options pricing, $S_0$, $M_0$, volatility and interest rate are taken from the real-market data and are given in Table 2. The numerical results are presented in Figure 5.

| Parameter | Value |
|-----------|-------|
| $S_0$     | 106.08|
| $M_0$     | 110.83|
| $\sigma$ | 17.03 \%|
| $r$       | 1.5\% |
| $l$       | 65    |
| $K$       | 1     |

Table 2. Input parameters for pricing $\pi$ put option on the Microsoft Corporation stock.

We will now show the impact of $M_t$ and $K$ on the price of $\pi$-option. The maximum price $M_t$ is between 100 and 120 and $K$ ranges between 0.8 and 1. All other parameters for the estimation are given in Table 3 as well. The results are shown in Figure 7.

Recall that $a = -1$ and $b = 1$ the payoff of the $\pi$-option equals

$$(6) \quad \left( K - \frac{S_t}{M_t} \right)^+, $$

where $S_t/M_t$ is the current value of the relative drawdown of the underlying asset. We believe that such contracts could be very efficiently used for hedging and managing portfolio risk against the volatile drops in underlying’s price. One can adjust the payoff function (6) by the appropriate choice of the strike $K$. 
This allows to set the minimal size of drawdown we would like to protect against. For example by setting $K = \frac{9}{10}$ the payoff of our option becomes greater than zero only if the drop in the price of the underlying from its maximum exceeds 10%. We now compare the the prices of American put and $\pi$-option on relative drawdown for Microsoft Corporation stock. As an exemplary environment for the options comparison we choose a time series containing daily closing prices of the Microsoft Corporation’s stock (see Figure 6).
The data spans approximately one year, from 6.11.2017 to 9.11.2018. We use first 9 months to calibrate the historical volatility, which is one of the input parameters in our pricing algorithm. Then, using this historical volatility, we compute prices of our options, both expiring 3 months after the end of calibration period. Input parameters for calculation and estimated options prices are given in Table 4.

Since the payoff of $\pi$-option on relative drawdown with $K = 1$ is always less than 1, in order to compensate against the drop in underlying’s price, we need a certain number of these contracts per each unit of stock in our portfolio. This number is chosen to equal $M_0$. In Table 4 the real price of the single $\pi$-option on relative drawdown contract is 0.0735. To compare it to the American put we initially need to make these instruments pay the same amount in case of a price drop. Therefore we choose to multiply the price of single $\pi$-option on drawdown by 110 (the value of $M_0$). That is why 'the new price of $\pi$ option' in Table 4 equals 0.0735 · 110 = 8.09.

It turns out that $\pi$ option is more expensive than vanilla put which is not a surprise as it initially pays the amount equivalent to present maximum drawdown. However, since the difference in price between these instruments is rather significant, a question emerges whether there exists a situation in which purchasing $\pi$-option on relative drawdown is more profitable than buying vanilla put. In order to answer this question, let us focus on the Microsoft Corporation data from the beginning of this section. On Figure 9 we show...
Figure 8. Daily closing prices of the Microsoft Corporation’s stock. Data spans from 6.11.2017 to 9.11.2018. Vertical dashed line indicates the end of volatility calibration period.

| American put | Parameter | Value | π on drawdown | Parameter | Value |
|--------------|-----------|-------|--------------|-----------|-------|
|              | $S_0$     | 108.13|              | $S_0$     | 108.13|
|              | $K$       | 108.13|              | $M_0$     | 110.83|
|              | $\sigma$ | 24.06%|              | $\sigma$ | 24.06%|
|              | $r$       | 2.25% |              | $r$       | 2.25% |
|              | $l$       | 100   |              | $l$       | 100   |
|              | $T$       | 3     |              | $T$       | 3     |
|              | Option price | $5.47$ | Option price | $8.09$ |

Table 4. Input parameters for computation and estimated options’ prices.

the amount each instrument would pay (on each day) throughout the whole 3-month period until options’ maturity. In order to display the difference more clearly, we construct two portfolios $V_{\text{American}}$ and $V_{\pi}$, both consisting of a Microsoft Corporation stock and an option (American put and $\pi$-option on relative drawdown, respectively). We observe them at the end of volatility calibration period. Stock price equals 108.13 at that moment and options prices are taken from Table 4. In Table 5 we show the initial net values of both portfolios. Then we calculate the net value of each portfolio for each day until options’ maturity; see Figure 10. Based on Figure 10 we can observe that the maximum value of portfolio $V_{\text{American}}$ is greater than the one for $V_{\pi}$. Thus when focusing purely at the possible maximum profit over some period of time, then the portfolio containing American option performs better. However, we can notice that $V_{\text{American}}$’s value over time is much more volatile compared to $V_{\pi}$ and it directly follows the behavior of underlying asset (it increases when stock price rises and decreases in the opposite situation). The value $V_{\pi}$ of $\pi$ option portfolio is most of the time non-decreasing. Moreover, $V_{\pi}$ increases its value
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Figure 9. Microsoft Corporation stock closing prices (top) and the corresponding payoffs of $\pi$-option on relative drawdown and American put (bottom) with the parameters from Table 4.

| Portfolio             | $V_{American}$ | $V_\pi$   |
|-----------------------|----------------|-----------|
| Initial stock value   | 108.13         | 108.13    |
| Option premium        | -$5.47         | -$8.09    |
| Option initial payoff | $0             | $2.68     |
| **Portfolio’s net value** | **$102.66** | **$102.72** |

Table 5. Portfolios and their initial net values.

every time stock price reaches a new maximum and essentially does not decrease in case of any price drop. In other words, combining stock and $\pi$-option on drawdown allow us to lock in our profit whenever stock price reaches its new maximum.

This brings us to the conclusion that the purpose of using $\pi$-option on relative drawdown and an American put is completely different. Vanilla American option protects us from stock price drops and ensures us that the current worth of our portfolio will not be less that its initial value. Unfortunately, in this case our portfolio’s value is more volatile and reflects the volatility of the underlying asset. This may result in bigger gains when comparing to the use of $\pi$-option on relative drawdown if the price of the underlying rises significantly and stays on that level until option’s maturity. However, in case of a drop in stock price after the upswing, we do not benefit from the fact that the new maximum has been reached and thus...
the value of our portfolio decreases together with the price of the underlying asset. When looking at the value of $V_\pi$ over time one can notice that combining stock and $\pi$-option on relative drawdown protects us against price drops as well but the volatility of our portfolio is reduced significantly. Additionally, the contract allows us to benefit from the underlying’s price upswings and locks in the profit every time new maximum is reached.

4. Conclusions

In this paper we focus on the numerical pricing of the new derivative instrument - $\pi$ -option. We adapted the Monte Carlo algorithm proposed by Broadie and Glasserman [5] to price this new option. We concentrated on a specific parametrization of this option which we call $\pi$-option on drawdown. We observed that this specific financial instrument is related with so-called relative maximum drawdown. We obtained prices of $\pi$-option on relative drawdown for the Microsoft Corporation stock with different parameters in order to examine the influence of those parameters on option’s premium. Our next step was analysis of two portfolios: first one based on $\pi$-option on relative drawdown and the second one based on American put. We used Microsoft Corporation data as well. It turned out that they behave in a completely different manner. The value of the portfolio containing American put was highly correlated with the underlying’s price movements and thus had an unpredictable and volatile behavior. Combining $\pi$-option on relative drawdown with the underlying not only ensures that the worth of the portfolio will not drop below the initial level, but it also allows us to take advantage of price upswings and to reduce the

Figure 10. Microsoft Corporation stock closing prices (top) and payoffs of portfolios with parameters from Table 5 (bottom).
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portfolio’s volatility at the same time. Similar analysis could be carried out for a geometric Lévy process of asset price. One can also consider regime-switching market.

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