A polynomial formula for the solution of 3D reflection equation

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Abstract
We introduce a family of polynomials in $q^2$ and four variables associated with the quantized algebra of functions $A_{q^2}(C_2)$. A new formula is presented for the recent solution of the three-dimensional reflection equation in terms of these polynomials, specialized to the eigenvalues of the $q$-oscillator operators.

Keywords: 3D reflection equation, tetrahedron equation, quantized algebra of functions, $q$-oscillator algebra

1. Introduction

In three-dimensional (3D) quantum integrable systems, an important role is played by the Zamolodchikov tetrahedron equation [18] and the Isaev–Kulish 3D reflection equation [4]:

$$R_{356}R_{246}R_{145}R_{123} = R_{123}R_{145}R_{246}R_{356},$$

$$R_{456}R_{489}K_{5379}R_{269}R_{258}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}R_{269}K_{3578}R_{489}R_{456}.$$ (1.1) (1.2)

They are equalities among linear operators acting on the tensor product of six and nine vector spaces, respectively. The indices specify the components in the tensor product on which the operators $R$ and $K$ act nontrivially. They serve as a 3D analogue of the Yang–Baxter [1] and the reflection equations, postulating certain factorization conditions on straight strings, which undergo the scattering $R$ and the reflection $K$ by a boundary plane. We call the scattering and the reflection operators 3D $R$ and 3D $K$ for short.

The first nontrivial solution $K$ to the 3D reflection equation (1.2) was constructed in [7] based on the representation theory [16] of the quantized algebra of functions $A_{q^2}(C_2)$ [12]. It is essentially obtained as the intertwiner of the two equivalent irreducible $A_{q^2}(C_2)$ modules $F_{q^2} \otimes F_{q} \otimes F_{q^2} \otimes F_{q^2} \simeq F_{q} \otimes F_{q^2} \otimes F_{q^2} \otimes F_{q^2}$, where $F_{q^2}$, $F_{q^2}$ are the Fock spaces of the

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$q$-oscillators $(a^\pm, k), \langle A^\pm, K \rangle$, (see (2.1)). $K$ is also characterized as the transition matrix of the Poincaré–Birkhoff–Witt (PBW) bases of the positive part of $U_q(C_2)$ [9] (see (2.9)). Matrix elements of $K$ are polynomials in $q$ whose $q \to 0$ limits are still known to yield a decent set-theoretical solution to the 3D reflection equation [7]. So far, a general formula for $K$ is only available in [7, Theorem 3.4], which consists of sums of ratios of many $q$-factorials (see (3.1)).

The status contrasts with the relevant 3D $R$, the companion object in (1.2), for which a number of results have been established. It was originally obtained as the intertwiner of the quantized algebra of functions $A_q(A_2)$ [6] and found later in a quantum geometry consideration [2]. They were shown to be the same object in [7]. Several formulas are available including [2, equation (30)], [3, equation (58)], [7 equation (2.20)] (a correction of the misprint in [6, p194]), [15, equation (104)], and [8, equation (9)].

The purpose of this paper is to provide the relatively intact 3D $K$ with the new explicit formulas (3.2) and (3.7), which are more structural than the previous one in [7]. We introduce a family of polynomials $[Q_{b,c}(x, y, z, w)](b, c) \in \mathbb{Z}_{\geq 0}$ with coefficients in $\mathbb{Z}[q^2]$ that are characterized by a system of $q$-difference equations. The equations are polynomial forms of the intertwining relations for the 3D $K$, where the four variables $x, y, z, w$ correspond to the four positive roots of $C_2$ (see (2.8)). The elements of $K$ are expressed as a specialization of the polynomials to the eigenvalues of the $q$-oscillator operators $k$ and $K$ (see example 3.5).

Our new formula may be viewed as a generalization of the analogous result on the 3D $R$ in [2, 3]. It is still a cumbersome expression, reflecting a significantly more involved nature of $K$ compared with $R$. However, extracting the polynomial structure implies an ‘analytic continuation’ of the eigenvalues of $k, K$ to generic variables, which is an important step toward a possible extension to the modular double-setting. A more detailed account on this will be given in section 5. See also [3, 10] and references therein for the generalization and application of the 3D $R$ associated with the modular double. We hope to report on this issue elsewhere.

In section 2 the origin of the 3D $K$ is recalled based on [7] and [9]. In section 3 the polynomials $Q_{b,c}(x, y, z, w)$ are introduced, and some basic properties are established. The new formulas for $K$ is presented with a proof. In section 4 a review of the analogous result on the 3D $R$ [2, 3] is given for comparison. It also includes supplementary $q$-difference equations, an example of the proof of the integral formula [15, equation (104)], and a derivation of the new formula for the matrix elements of the 3D $R$ [8, equation (9)]. Section 5 contains a summary and an outlook. The appendix lists the intertwining relations of $K$ and the corresponding $q$-difference equations of $Q_{b,c}(x, y, z, w)$.

We assume that $q$ is generic and use the notation

$$(z; q)_n = \prod_{j=0}^{n-1} (1 - q^j z), \quad (q)_n = (q; q)_n, \quad (\eta, \ldots, \eta, s_1, \ldots, s_n)_q = \prod_{i=1}^{m} (q)_{\eta_i} \prod_{i=1}^{n} (q)_{s_i}$$

in terms of $(z; q)_\infty = \prod_{j>0} (1 - q^j z)$. They will only be used for integer indices; therefore the assumption $|q| < 1$ is not necessary, despite the formal appearance of the infinite product. The last symbol will be used without assuming $\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{n} s_i$. An important consequence of these definitions is the support property: $1/(q)_n = 0$ for $n \in \mathbb{Z}_{<0}$. Thus $(\eta, \ldots, \eta, s_1, \ldots, s_n)_q = 0$ if $\{\eta, \ldots, \eta\} \cap \mathbb{Z}_{<0} = \emptyset$ and $\{s_1, \ldots, s_n\} \cap \mathbb{Z}_{<0} \neq \emptyset$. 

\[ \quad \]

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2. Origin of 3D $K$

2.1. $K$ as the intertwiner of $A_q(C_n)$ modules

We recall the definition of $A_q(C_n)$ [12], where it was denoted by $\text{Fun}(\text{Sp}_q(n))$. First, we introduce the structure constants $\delta_{ij}^{kl} = \delta_{ij}^{kl} + \delta_{ij}^{kl}$ by

$$
\sum_{i,j,l} R_{ij,kl} E_{il} \otimes E_{jl} = q \sum_{i,l} E_{il} \otimes E_{il} + \sum_{i,j,l} E_{il} \otimes E_{jl} + q^{-1} \sum_{i,j} E_{il} \otimes E_{jl} + (q - q^{-1}) \sum_{i,j} E_{il} \otimes E_{jl} - (q - q^{-1}) \sum_{i,j} E_{il} \otimes E_{jl},
$$

$C_q = \delta_{ij}^{kl} e_{ij} q^{\epsilon_{ij}}, \quad i' = 2n + 1 - i, \quad \epsilon_i = 1(1 \leq i \leq n), \quad \epsilon_i = -1(n < i \leq 2n)$,

$(q_1, ..., q_{2n}) = (n - 1, n - 2, ..., 1, 0, 0, -1, ..., -n + 1)$.

Here $E_{ij}$ is a matrix unit, and the indices are summed over $\{1, 2, ..., 2n\}$ under the specified conditions. The $R_{ij,kl} E_{il} \otimes E_{jl}$ is a limit of the quantum $R$ matrix for the vector representation of $U_q(C_n)$ given in [5, equation (3.6)].

The quantized algebra of functions $A_q(C_n)$ is a Hopf algebra generated by $T = (t_{ij})_{i,j \in \{1,2\}}$, with the relations symbolically expressed as $R(T \otimes T) = (T \otimes T)R$ and $TCT^{-1} = T^{CT}$. Explicitly they read

$$
\sum_{i,j,l} R_{ij,kl} E_{il} \otimes E_{jl} = \sum_{i,j} C_{ij} C_{il} t_{ij} t_{ik} = \sum_{j,k} C_{ij} C_{jk} t_{ij} t_{ik} = -\delta_{im},
$$

where $\delta_{im} = 1$ if $i = m$ and 0 otherwise. The coproduct is given by $\Delta(t_{ij}) = \sum_{l} t_{il} \otimes t_{lj}$. We will use the symbol $\Delta$ to also mean the multiple coproducts like $\Delta \Delta \Delta = \Delta \otimes \Delta \otimes \Delta$, etc.

To describe the representations of $A_q(C_2)$, we introduce the Fock space $F_q = \bigoplus_{m \geq 0} C(q) |m\rangle$ equipped with the $q$-oscillator operators $a^+, a^-, k$ acting as

$$
|k, m\rangle = q^m |m\rangle, \quad a^+ |m\rangle = |m + 1\rangle, \quad a^- |m\rangle = (1 - q^2m) |m - 1\rangle. \quad (2.1)
$$

Let $F_q^+, A^+, A^-$ and $K$ denote the corresponding $q$-oscillator operators with $q$ replaced by $q^2$.

Thus $K |m\rangle = q^{2m} |m\rangle, \quad A^+ |m\rangle = |m + 1\rangle, \quad A^- |m\rangle = (1 - q^{4m}) |m - 1\rangle$.

Now we consider the $n = 2$ case. Then for $T = (t_{ij})_{i,j \in \{1,2\}}$ the maps

$$
\pi_1(T) = \begin{pmatrix}
\mu_1 a^- & \alpha_1 k & 0 & 0 \\
\beta_1 k & \nu_1 a^+ & 0 & 0 \\
0 & 0 & \nu_1^{-1} a^- & q \beta_1^{-1} k \\
0 & 0 & q \alpha_1^{-1} k & \mu_1^{-1} a^+
\end{pmatrix} (q^{-1} \alpha_1 \beta_1 = \mu_1 \nu_1 = \epsilon = \pm 1), \quad (2.2)
$$

$$
\pi_2(T) = \begin{pmatrix}
\rho 1 & 0 & 0 & 0 \\
0 & \mu_2 A^- & \alpha_2 K & 0 \\
0 & \beta_2 K & \mu_2^{-1} A^+ & 0 \\
0 & 0 & 0 & \rho^{-1} 1
\end{pmatrix} (\rho = \pm 1, \quad \alpha_2 \beta_2 = -q^2) \quad (2.3)
$$

give the irreducible representations $\pi_i: A_q(C_2) \rightarrow \text{End}(F_q^+)$ [7]. Here $I$ denotes the identity operator on $F_q^+$. The parameters $\alpha_i, \beta_i, \mu_i, \nu_i$ are to obey the constraints in the parentheses. According to [16], irreducible representations of $A_q(C_2)$ are labeled by the elements of the
Weyl group $W(C_2)$ up to a torus degree of freedom. The $W(C_2)$ is a Coxeter system generated by the simple reflections $s_1$ and $s_2$ with the relations $s_1^2 = s_2^2 = 1$, $s_1s_2s_1 = s_2s_1s_2$. (We employ the convention such that the indices 1 and 2 correspond to the short and the long simple root of $C_2$, respectively.) The $\pi$ is the irreducible $A_q(C_2)$ module corresponding to $s_i$. The irreducible representation for $s_is_js_i$ is given by $\pi_i \otimes \pi_j \otimes \pi_i \otimes \pi_j (i \neq j)$. We write such a tensor product representation as $\pi_{ij}$ for short.

The relation $s_1s_2s_1 = s_2s_1s_2$ implies the equivalence $\pi_{1212} \simeq \pi_{2121}$. Therefore, there is a unique map

$$\Psi : F_q \otimes F_{q^2} \otimes F_q \otimes F_{q^2} \longrightarrow F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q$$

characterized by the intertwining relations and the normalization:

$$\pi_{1212}(\Delta(f')) \circ \Psi = \Psi \circ \pi_{1212}(\Delta(f')) \quad (\forall f' \in A_q(C_2)),
\Psi([0] \otimes [0] \otimes [0] \otimes [0]) = [0] \otimes [0] \otimes [0] \otimes [0].$$

(2.4)

We find it convenient to work with $\mathcal{H}$ defined by

$$\mathcal{H} = \mathcal{H}_{234} : F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q \longrightarrow F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q,$$

where $\mathcal{H}_{234} : x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto x_4 \otimes x_3 \otimes x_2 \otimes x_1$ is the linear operator reversing the order of the tensor product. The intertwining relation (2.4) is translated into

$$\pi_{2121}(\Delta(f')) \circ \mathcal{H} = \mathcal{H} \circ \pi_{2121}(\Delta(f')) \quad (\forall f' \in A_q(C_2)),$
$$
(2.5)

where $\Delta(t_{ij}) = \sum t_{ij}^{1/2} t_{ij}^{1/2} t_{ij}^{1/2} t_{ij}^{1/2}$.

We introduce the matrix elements of $\mathcal{H}$ by

$$\mathcal{H}(i') \otimes j' \otimes k' \otimes l' = \sum_{a,b,c,d} \mathcal{H}_{a,b,c,d}^{i',j',k',l'}(a) \otimes (b) \otimes (c) \otimes (d).$$

We set $K_{ijkl}^{a,b,c,d} = e^{b+c} \mu_1^{c-k} (\mu_2)^{-b} \mathcal{H}_{a,b,c,d}^{i,j,k,l}$ by using the parameters in (2.2) and (2.3). It turns out that $K_{ijkl}^{a,b,c,d}$ is a polynomial in $q$ free from all the other parameters [7]. More precisely, $K_{ijkl}^{a,b,c,d} \in q^\eta Z[q^2]$ holds, where $\eta = 0, 1$ is specified by $\eta \equiv bd + jl$ mod 2. It satisfies [7, equation (3.25)], which in the present notation reads

$$K_{ijkl}^{a,b,c,d} = \left( \begin{array}{cccc} j & l \\ b & c & d & q \end{array} \right) K_{ijkl}^{a,b,c,d}.$$ 

(2.6)

We introduce the (parameter-free) 3D reflection operator $K \in \text{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$ by

$$K(i') \otimes j' \otimes k' \otimes l' = \sum_{a,b,c,d} K_{ijkl}^{a,b,c,d}(a) \otimes (b) \otimes (c) \otimes (d).$$

(2.7)

The appendix lists the relation (2.5) for $K$ with $f = t_{i-1,j-1}$ as $\langle ij \rangle$. The 3D reflection equation (1.2) follows from the equivalence $\pi_{21212323} \simeq \pi_{32212321}$ of the irreducible representations for $A_q(C_3)$, reflecting the two reduced expressions $s_1s_2s_3s_2s_1s_2s_3 = s_3s_2s_3s_2s_1s_2s_3$ of the longest element of the Weyl group $W(C_3)$. Further properties of $K$ are available in [7].

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We warn that $K$ and $\mathcal{H}$ here are denoted by $\mathcal{H}$ and $K$, respectively in [7].
2.2. $K$ as the transition coefficient of PBW bases

Let $U_q^+(C_2)$ be the positive part of the quantized universal enveloping algebra of $U_q(C_2)$. It is an associative algebra generated by $e_1$ and $e_2$ obeying the $q$-Serre relations:

$$e_1^3e_2 - [3]_q e_1^2 e_2 e_1 + [3]_q e_2 e_1^2 e_2 - e_2 e_1^3 = 0, \quad e_2^2e_1 - [2]_q e_2 e_1 e_2 + e_1 e_2^2 = 0,$$

where $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$. According to the general theory [11], the $U_q^+(C_2)$ admits two natural PBW bases $\{ \phi_{abcd}^{(1)} \}$ and $\{ \phi_{abcd}^{(2)} \}$ corresponding to the reduced expressions $s_1 s_2 s_1 s_2$ and $s_2 s_1 s_2 s_1$ of the longest element of $W(C_2)$. Explicitly they read

$$\phi_{abcd}^{(1)} = \frac{[a]_q! [b]_q! [c]_q! [d]_q!}{[2]_q! [1]_q! [m]_q! [n]_q!}, \quad \phi_{abcd}^{(2)} = \frac{[b]_q! [c]_q! [d]_q!}{[2]_q! [1]_q! [m]_q! [n]_q!},$$

where $[m]_q! = [m]_q [m-1]_q \cdots [1]_q$. As the special case $g = C_2$ of the result for general simple Lie algebra $g$ [9], one has

$$\phi_{abcd}^{(2)} = \sum_{ijkl} K_{ijkl}^{abcd} \phi_{ijkl}^{(1)}.$$  (2.9)

This relation in turn characterizes the matrix element $K_{ijkl}^{abcd}$ of the intertwiner $K$ also as the transition coefficient of the PBW bases. See [14, 17] for the recent development on this topic.

Note that the equality of the weight on the both sides implies $K_{ijkl}^{abcd} = 0$ unless $(a + b + c, b + 2c + d) = (i + j + k, j + 2k + l)$.

3. Polynomial formula

3.1. $K$ in terms of $Q_{x,y,z,w}$

In [7, theorem 3.4] the matrix element $K_{ijkl}^{abcd}$ was expressed as a finite sum of the form

$$K_{ijkl}^{abcd} = \delta_{ij}^{a+b+c} \delta_{jk}^{b+2c+d} \sum_{\alpha,\beta} (-1)^{\alpha} q^\Phi \left( \frac{r_1}{s_1}, \ldots, \frac{r_6}{s_6}, \frac{r_7}{s_7}, \frac{r_8}{s_8} \right) q^i.$$  (3.1)

Here $r_i, s_i$ are at most linear and $\Phi$ is at most quadratic in $\alpha, \beta, \gamma, \lambda$. The indices $a, b, c, d, i, j, k, l$ enter many places and their dependence is quite involved.

Our aim here is to provide an alternative formula which is more structural. It is expressed in terms of polynomials in the four variables that accommodate the eigenvalues of $K$ and $k$ in the four components of $|i \otimes j \rangle \otimes |k \rangle \otimes |l \rangle \in F_q \otimes F_q \otimes F_q \otimes F_q$.

**Theorem 3.1.** (i) There is a family of polynomials $\{ Q_{x,y,z,w}(x, y, z, w) \}|(b, c) \in Z_{\geq 0}^2 \}$ in variables $x, y, z, w$ characterized by the $q$-difference equations $E22$–$E55$ in the appendix and
the condition $Q_{0,0}(x, y, z, w) = 1$. (ii) The following formulas are valid for the matrix elements of the 3D $K$:

$$K_{i,j,k,l}^{a,b,c,d} = \delta_{i+a+j+k}^{a+b+c+d} \delta_{j+2k+l}^{b+2c+d} \frac{q^{x+y} q^{y+z}}{(q^2)^c} Q_{b,c}^{i,j,k,l} (q^4, q^{2j}, q^{4l}, q^{2l}).$$  

(3.2)

$$\phi_K = (a - k)(d - j) + (b - l)(c - i) - 2(b - j)(c - k),$$  

(3.3)

$$q_{b,c} = 3b(b - 1) + 2c(3c - 2) + 8bc.$$  

(3.4)

**Proof.** The intertwining relations (2.5) for $K$ are listed in (22)–(55) in the appendix. Substituting (3.2) into them, one finds that they are equivalent to E22–E55 as illustrated there along (24). The normalization condition $K_{0,0,0,0}^{0,0,0,0} = 1$ also matches $Q_{0,0}(x, y, z, w) = 1$. Thus there is a unique family of functions $\{Q_{b,c}(x, y, z, w)\}_{(b, c) \in \mathbb{Z}_+^2}$ satisfying E22–E55 due to the unique existence of the intertwiner $K$. Combining E22 and E55 in the appendix, one can derive

$$Q_{b,c}(x, y, z, w) = wy(z - 1)q^{4b+8c-4} Q_{b-1,c}^{x,y,z,w}$$

$$+ wx(y - 1)yzq^{4b+8c-4} Q_{b-1,c}^{x,y-2,z,w}$$

$$+ (w - 1)(y - 1)q^{4b+8c-6} Q_{b-1,c}^{x,y-2,z,w}$$

$$+ w(x - 1)y(z - 1)q^{4b+8c-4} Q_{b-1,c}^{x,y,z-2,w}$$

$$+ (w - 1)(x - 1)q^{6b+8c-6} Q_{b-1,c}^{x,y,z-2,w}.$$  

(3.5)

$$Q_{b,c}(x, y, z, w) = -w^2y(z - 1)q^{4b+8c-8} Q_{b-1,c}^{x,y,z,w}$$

$$+ wx(y - 1)q^{4b+8c-6} (q^{2b+2c} - q^2wyz) Q_{b-1,c}^{x,y,z,w}$$

$$- (w - 1)w(y - 1)q^{4b+8c-8} Q_{b-1,c}^{x,y,z,w}$$

$$+ w(x - 1)(z - 1)q^{4b+8c-10} (q^{2b+2c} - q^2wyz)$$

$$\times Q_{b-1,c}^{x,y,z,w}$$

$$+ (w - 1)(x - 1)q^{6b+8c-10} (q^{2b+2c} - q^2wyz)$$

$$\times Q_{b-1,c}^{x,y,z,w}.$$  

(3.6)

By induction on $b$ and $c$, they tell that $Q_{b,c}(x, y, z, w)$ is a polynomial in $x, y, z, w$.

The power $\phi_K$ in (3.3) is invariant under the exchange $(a, b, c, d) \leftrightarrow (i, j, k, l)$. Therefore (2.6) implies another general formula:

$$K_{i,j,k,l}^{a,b,c,d} = \delta_{i+a+j+k}^{a+b+c+d} \delta_{j+2k+l}^{b+2c+d} q^{x+y} q^{y+z} (q^2)^c Q_{b,c}^{i,j,k,l} (q^4, q^{2j}, q^{4l}, q^{2l}).$$  

(3.7)
If one switches from $q$ to $p = q^{-1}$ and introduces the functions of $p, x, y, z, w$ by
$$
\hat{Q}_{b,c}(x, y, z, w) = p^b x c Q_{b,c}(x, y, z, w) \big|_{q \rightarrow p^{-1}},
$$
the recursion relations (3.5) and (3.6) slightly simplify as
$$
\hat{Q}_{b,c}(x, y, z, w) = w y^2 (z - 1) p^{2b-2} \hat{Q}_{b-1,c}(x, y, p^2 z, w) 
+ w x (y - 1) y z^2 p^{2b+k-2} \hat{Q}_{b-1,c}(x, p^2 y, z, p^2 w) 
+ (w - 1) (y - 1) p^{2b-2} \hat{Q}_{b-1,c}(p^2 x, p^2 y, p^2 z, w) 
+ (w - 1) (x - 1) y \hat{Q}_{b-1,c}(p^2 x, y, z, p^2 w),
$$
(3.8)

$$
\hat{Q}_{b,c}(x, y, z, w) = -w^2 y (z - 1) p^{2b+4k-2} \hat{Q}_{b+1,c}(x, y, p^2 z, w) 
- w x (y - 1) p^{2b+4k-6} \left( w y z p^{2b+2c} - p^2 \right) \hat{Q}_{b+1,c}(x, p^2 y, z, p^2 w) 
- w (x - 1) (z - 1) p^{2b-2} \left( w y z p^{2b+2c} - p^2 \right) \hat{Q}_{b+1,c}(p^2 x, p^2 y, p^2 z, w) 
+ (w - 1) (x - 1) \left( 1 - w y z p^{2b+2c} \right) \hat{Q}_{b+1,c}(p^2 x, y, z, p^2 w).
$$
(3.9)

Here are some examples of $Q_{b,c}(x, y, z, w)$'s with small $b, c$:

$q_{1,0}(x, y, z, w) = w x y^2 - w - x y + 1,
q_{0,1}(x, y, z, w) = q^2 \left( w x y z - w - x + 1 \right) - w z \left( w x y z - w - x y + 1 \right),
q_{2,0}(x, y, z, w) = q^6 (w - 1) (x y - 1) + q^4 \left( -w^2 x y^2 + w^2 + w x y - w + x y^2 - x y \right)
- q^3 x y^2 (w x y z - w - x + 1) + w x y z^2 \left( w x y z - w - x y + 1 \right)
- q^4 \left( w - w^2 + x y - x w y - x y + 2 x y^2 \right),
q_{1,1}(x, y, z, w) = q^{10} (w - 1) (x y - 1) - q^8 (w - 1) w z (x y - 1)
+ q^6 \left( -w^2 x y z + w^2 z + w x y^2 z + 2 w x y z - w z + x^2 y - x y \right)
+ q^4 w z \left( w^2 x y^2 z - w^2 + w x y^2 z - x w y z - w x y + w - x^2 y^2 + x y \right)
+ q^3 w x y^2 z \left( w x y z - w - x + 1 \right) - w^2 x y^2 \left( w x y z - w - x + 1 \right).

One notices that these $Q_{b,c}(x, y, z, w)$ are polynomials that are also in $q^2$. To show it in general we introduce
$$
\mathcal{S}_{b,c} = \left\{ (r, s, t, u) \in \mathbb{Z}_{\geq 0}^4 \mid \min (u-t, 2r-s, b-s+2t-u, c-r+s-t) \geq 0 \right\},
$$
(3.10)
which is a finite subset of $\{(r, s, t, u) \in \mathbb{Z}_c^4 \mid s \geq 0, b \leq c, t \leq u \leq b + 2c \}$.

Proposition 3.2. (i) Let $Q_{b,c}(x, y, z, w) = \sum_{r,s,t,u} D_{b,c,r,s,t,u}^{b,c} x^r y^s z^t w^u$ with $D_{b,c,r,s,t,u}^{b,c}$ independent of $x, y, z, w$. Then $D_{b,c,r,s,t,u}^{b,c} = 0$ unless $(r, s, t, u) \in \mathcal{S}_{b,c}$.
(ii) $Q_{b,c}(x, y, z, w) \in \mathbb{Z}[q^2, x, y, z, w].$

(iii) $q^{-m_b}Q_{b,c}(x, y, z, w) \in \mathbb{Z}[q^{-2}, x, y, z, w].$

Proof. (i) By induction on $b$ and $c$, it suffices to show that all the monomials $x^iy^jz^kw^u$, which survive possible cancellations in the right-hand sides of (3.5) and (3.6), satisfy

$$S \in \mathbb{Z}[x, y, z, w].$$

This assumes $D_{r,s,t,u} = 0$ unless $(r, s, t, u) \in \mathcal{S}_{b-1,c}$, and $D_{r,s,t,u} = 0$ unless $(r, s, t, u) \in \mathcal{S}_{b,c-1}$, respectively. We illustrate the procedure for (3.6). The treatment of (3.5) is completely similar. First, consider the condition $-s = u'$. The right-hand side of (3.6) contains no prefactor $w^u$ such that $-s < u'$; therefore, this condition is trivially satisfied. Second, consider the condition $-t = u'$. The terms in the right-hand side of (3.6) that apparently break it are

$$w^2yzq^{4b+8c-8}Q_{b,c-1}(x, y, z, w) \big|_{2w=2}$$

where $[2w=2]$ means the contribution of the monomials $x^iy^jz^kw^u$ such that $2r = s$. This vanishes due to $Q_{b,c-1}(x, y, z, w) \big|_{2w=2} = Q_{b,c-1}(q^{-4}x, q^{-2}y, q^{-2}z, w) \big|_{2w=2}$ and $Q_{b,c-1}(q^{-4}x, y, z, q^{-2}w) \big|_{2w=2}$. Similarly, the proof of the third and the fourth conditions $-t = u'$ and $-s = u'$ in (3.10) reduce to checking

$$w^2yzq^{4b+8c-8}Q_{b,c-1}(x, y, z, w) \big|_{b=x-s-2+u}$$

where $[b=x-s-2+u]$ means the contribution of the monomials $x^iy^jz^kw^u$. As in the previous case, it is easy to see that these terms pairwise cancel.

(ii) By induction on $b$ and $c$, it suffices to show $Q_{b,c}(x, y, z, w) \in \mathbb{Z}[q^2, x, y, z, w]$ from (3.5) and (3.6) by assuming

$$Q_{b-1,c}(x, y, z, w) \in \mathbb{Z}[q^2, x, y, z, w] \quad \text{and} \quad Q_{b,c-1}(x, y, z, w) \in \mathbb{Z}[q^2, x, y, z, w],$$

respectively. This can easily be checked term by term in the right-hand sides. For instance, consider the contribution in (3.5) whose $q$-dependent part is $q^{4b+8c-8}Q_{b-1,c}(q^{-4}x, q^{-2}y, q^{-2}z, w)$. It consists of the monomials of the form $q^{4b+8c-4-2s-4t}D_{r,s,t,u}^{b-1,c}x^iy^jz^kw^u$ with $D_{r,s,t,u}^{b-1,c} \in \mathbb{Z}[q^2]$ by the assumption. From (i) we know $D_{r,s,t,u}^{b-1,c} = 0$ unless $(r, s, t, u) \in \mathcal{S}_{b-1,c}$. This ensures $4b + 8c - 4 - 4r + 2s - 4t \in 2\mathbb{Z}_{\geq 0}$ if $D_{r,s,t,u}^{b-1,c} \neq 0$. 

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(iii) As in (ii), one can show \( \hat{Q}_{b,c}(x, y, z, w) \in \mathbb{Z}[p^2, x, y, z, w] \) using (3.8) and (3.9).

By a similar inductive argument it is easy to show

**Proposition 3.3.**

\[
\lim_{q \to 0} Q_{b,c}(x, y, z, w) = (-1)^r \left( x y z w \right)^{b+c-1} Q_{1,0}(x, y, z, w),
\]

\[
\lim_{q \to \infty} q^{-2} Q_{b,c}(x, y, z, w) = 1 - x y z w - w x y z + x w y z + y z w x + z w x y,
\]

\[
Q_{b,c}(x, 1, 1, w) = (-1)^r x^{b+c} y^{b+2r} (x^{-1}; q^2)_{b+c} (w^{-1}; q^2)_{b+2c},
\]

\[
Q_{b,c}(x, y, 1, 1) = (-1)^r x^{b+c} y^{b+r} (y^{-1}; q^2)_{b+2c},
\]

\[
Q_{b,c}(1, 1, z, w) = (-1)^r (2 w z^{-1}; q^4)_{b+c},
\]

where \( (b, c) \neq (0, 0) \) in the first two relations.

Thus the power \( q^{b,c} \) (3.4) gives the exact degree of \( Q_{b,c}(x, y, z, w) \) in \( q \). The third result, for instance, reflects \( K_{0,0,0,0} = 0 \) unless \( (a, d) = (i - b - c, l - b - 2c) \in \mathbb{Z}_{\geq 0}^4 \).

Now we present an explicit form of \( Q_{b,c}(x, y, z, w) \).

**Theorem 3.4.** The following formula is valid:

\[
Q_{b,c}(x, y, z, w) = q^{b,c} \sum_{(r, t, u) \in \mathbb{Z}_{\geq 0}} (-1)^r q^{\phi_Q} \psi^\nu \phi_{Qx}^{b,c} C_{r,s,t,u}^{b,c} x^r y^s z^t w^u,
\]

(3.11)

\[
C_{r,s,t,u}^{b,c} = \frac{b, u - t}{b + 2t - s - u, 2r - s} \left( q^3 \right) \left( q^4 \right)^{a-r-s-t} \sum_{(a, b, t) \in \mathbb{Z}_{\geq 0}^3} (-1)^a q^{\phi_T} \psi^\nu \phi_{Qx}^{b,c} \Xi_{a, b, t, u},
\]

(3.12)

\[
\Xi_{a, b, t, u} = \left( \begin{array}{c} b - s + t - \alpha, 2r - s + \beta \\ \alpha, \beta, \gamma, u - t - \alpha, t - \beta, b - s - \alpha + \beta, s - \beta - \gamma \end{array} \right)^t,
\]

(3.13)

\[
\phi_Q = (s - 2t + u)^2 + 2r (r + 2r + 1) - (2b - 1) (s + u) - 4c (r + t),
\]

(3.14)

\[
\phi_C = \alpha (a + 1 + 2t) + \beta (\beta - 1 - 2a + 2b - 4r) + \gamma (\gamma - 1 - 4r),
\]

(3.15)

\[
\psi_{r,s} = s (4r - s + 1).
\]

(3.16)

In view of the support property of the symbols in (1.3), the sum (3.12) is limited to those \( \alpha, \beta, \gamma \) such that all the lower entries in (3.13) are nonnegative, which also ensures that all the upper entries are nonnegative. Thus it ranges over a finite subset of \( \{(a, \beta, \gamma) \in \mathbb{Z}_{\geq 0}^3 | \alpha \leq u - t, \beta \leq t, \gamma \leq s \} \). The dependence of \( \Xi_{a, b, t, u} \) on \( (b, c, r, s, t, u) \) has been suppressed in the notation for simplicity.
From proposition 3.2 (ii), we know \( q^{\psi_0 - \psi_0} c_{r,s,t,u}^{b,c} \in \mathbb{Z}[q^2] \). Actually, computer experiments suggest the following conjecture

\[
C_{r,s,t,u}^{b,c} \in \mathbb{Z}[q^2], \quad \lim_{q \to 0} C_{r,s,t,u}^{b,c} = 1. \tag{3.17}
\]

In general \( \Xi_{\alpha, \beta, \gamma} \) is not necessarily a polynomial but a rational function of \( q^2 \). In the special case \( b = 0 \) or \( c = 0 \), \( C_{r,s,t,u}^{b,c} \) admits a simple formula:

\[
C_{r,s,t,u}^{b,0} = \left( u, b + 2t - s - u, 2r - s \right) q^r \quad \text{for} \quad r \leq \min(2, u, b + 2t - s - u, 2r - s), \tag{3.18}
\]

\[
C_{r,s,t,u}^{0,c} = \left( s, r - 2t - s + u, 2r - s \right) q^r \quad \text{for} \quad r \leq \min(2, s, r - 2t - s + u, 2r - s). \tag{3.19}
\]

The latter is equivalent to \( A_{r,s,t,u}^{0,c} = (q^2)(q^2)^2(q^4)^r \) in (3.20), which can be verified by following Steps (i)–(v) in section 3.2.

**Example 3.5.** The following is the list of the nonzero \( K_{i,j,k,l}^{3,1,0,2} \):

\[
\begin{align*}
K_{1,3,0,0}^{3,1,0,2} &= -q^6(1 - q + q^2)(1 + q + q^2), \\
K_{2,1,1,0}^{3,1,0,2} &= -q^{10}(1 - q + q^2)(1 + q + q^2), \\
K_{2,2,0,1}^{3,1,0,2} &= (1 + q^2)(1 - q^2 + q^4 - q^6 + q^8 - q^{10} - q^{14}), \\
K_{3,0,1,1}^{3,1,0,2} &= q^6(1 + q^2)(1 - q^2 + q^4 - q^6 + q^8 - q^{10} - q^{14}), \\
K_{3,1,0,2}^{3,1,0,2} &= q^6(1 + q^2 - q^{14} - q^{16} - q^{18}), \\
K_{4,0,0,3}^{3,1,0,2} &= q^{14}(1 - q + q^2)(1 + q + q^2)(1 - q^{16}).
\end{align*}
\]

According to (3.2) they are expressed by various special values of \( Q_{i,0}(x, y, z, w) = wxy^2z - w - xy + 1 \) as

\[
\begin{align*}
K_{1,3,0,0}^{3,1,0,2} &= \frac{Q_{1,0}(q^4, q^6, 1, 1)}{q^3(1 - q^2)}, \\
K_{2,1,1,0}^{3,1,0,2} &= \frac{Q_{1,0}(q^6, q^2, q^4, 1)}{1 - q^2}, \\
K_{2,2,0,1}^{3,1,0,2} &= \frac{Q_{1,0}(q^8, q^4, 1, q^2)}{1 - q^2}, \\
K_{3,0,1,1}^{3,1,0,2} &= \frac{q^6Q_{1,0}(q^{12}, 1, q^4, q^2)}{1 - q^2}, \\
K_{3,1,0,2}^{3,1,0,2} &= \frac{q^6Q_{1,0}(q^{12}, q^2, 1, q^4)}{1 - q^2}, \\
K_{4,0,0,3}^{3,1,0,2} &= \frac{q^{14}Q_{1,0}(q^{16}, 1, 1, q^6)}{1 - q^2}.
\end{align*}
\]

On the other hand, according to (3.7) they are also expressed in terms of \( Q_{b,c}(q^{12}, q^2, 1, q^4) \) with various \( b, c \). For instance, one has
3.2. Proof of theorem 3.4

We find it convenient to introduce \( A_{r,s,t,u}^{b,c} \) by

\[
C_{r,s,t,u}^{b,c} = \left( s, t, b - s + 2t - u, 2r - s \right) q^2 \left( u - t, c - r + s - t \right) q^t A_{r,s,t,u}^{b,c}.
\]

Then substitution of (3.11) into the difference equations E22–E55 in the appendix leads to the recursion relations for \( A_{r,s,t,u}^{b,c} \). In what follows, we outline how they can be solved in steps (i)–(v) to yield (3.11)–(3.16). An important feature in the derivation is the boundary condition \( A_{r,s,t,u}^{b,c} = 0 \) unless \( (r, s, t, u) \in \mathcal{S}_{b,c} \) which stems from proposition 3.2 (i).

(i) Combining E35 and E45, one gets

\[
A_{r,s,t,u}^{b,c} = A_{r,s,t,u-1}^{b,c} + q^{2a} A_{r,s,t,u-1}^{b-1,c} \quad (t < u),
\]

which leads to

\[
A_{r,s,t,u}^{b,c} = \sum_{a=0}^{u-t} q^{a(\alpha+2t+1)} \left( \alpha, u - t - \alpha \right) q^t A_{r,s,t,u}^{b-\alpha,c}.
\]

Henceforth we concentrate on \( A_{r,s,t,u}^{b,c} \) with \( (b - s + t, c + s - r - t, 2r - s) \geq 0 \).

(ii) From E24, one gets

\[
A_{r,s,t,u}^{b,c} = q^{2(b-s+t)} \left( 1 - q^{2t} \right) A_{r,s-1,t-1,u}^{b,c} + \left( 1 - q^{2(b-s+t)} \right) A_{r,s-1,t-1,u}^{b,c} \quad (b - s + t \geq 0),
\]

which leads to

\[
A_{r,s,t,u}^{b,c} = \sum_{\beta=(t-s),s}^{t-s} q^{(t-s)-\beta(b-s+t-\beta)} \left( \beta, t - \beta, s - t + \beta, b - s + t - \beta \right) q^t A_{r,s-\beta+\beta,0,0}^{b,c}.
\]

where \( (x)_s = \max(x, 0) \). Under the assumption \( (b - s + t, c + s - r - t, 2r - s) \geq 0 \), all the summands \( A_{r,s,0,0}^{b,c} \) appearing here satisfy the condition \( (r, m, 0, 0) \in \mathcal{S}_{b,c} \) in (3.10).

Henceforth we concentrate on \( A_{r,s,0,0}^{b,c} \) with \( (b - s + c + s - r - t, 2r - s) \geq 0 \).

(iii) By setting \( t = u = 0 \) in the recursion relation of \( A_{r,s,t,u}^{b,c} \) derived from E34, one gets

\[
A_{r,s,0,0}^{b,c} = A_{r,s,0,0}^{b-1,c} \quad (b > s).
\]

(iv) By setting \( t = u = 0 \) in the recursion relation of \( A_{r,s,t,u}^{b,c} \) derived from E32, one gets

\[
(1 - q^{4+4c}) A_{r,s-1,0,0}^{b-1,c+1} - q^{2+4r-2s} \left( 1 - q^{4(c-r+s)} \right) A_{r,s-1,0,0}^{b,c} - \left( 1 - q^{2(2r+s+1)} \right) A_{r,s,0,0}^{b,c} = 0.
\]

Setting \( s = b \) and applying (3.21) further, one finds

\[
(1 - q^{4+4c}) A_{r,b-1,0,0}^{b-1,c+1} - q^{2+4r-2b} \left( 1 - q^{4(b+c-r)} \right) A_{r,b-1,0,0}^{b-1,c} - \left( 1 - q^{2(2r-b+1)} \right) A_{r,b,0,0}^{b,c} = 0
\]

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for $2r \geq b$, $b + c \geq r$. This allows one to decrease $b$, leading to

$$A_{r,b,0,0}^{b,c} = \sum_{r=0}^{b-(r-c)} (-1)^r q^{(r+1+4r-2b)} \left( b, 2r - b \right) \left( b + c - r, b + c - r \right) q^{c, b + c - r - \gamma} A_{r,0,0,0}^{0,b+c-r}.$$  

Henceforth we concentrate on $A_{r,b,0,0}^{b,c}$ with $c \geq r$.

(v) By setting $b = s = t = u = 0$ in the recursion relation of $A_{r,s,t,u}^{b,c}$ derived from E53, one gets $A_{r,s,t,u}^{b,c} = (1 - q^{4r-2}) A_{r-1,0,0,0}^{b,c}$. Thus $A_{r,0,0,0}^{b,c} = \frac{(q^{3b})_{b}}{(q_{b})_{b}} A_{0,0,0,0}^{b,c}$. By setting $b = r = s = t = u = 0$ in the recursion relation of $A_{r,s,t,u}^{b,c}$ derived from E23 or E54, one gets $A_{r,0,0,0}^{b,c} = A_{0,0,0,0}^{b,c}$. From $Q_{0,0}(x, y, w, z) = 1$ it follows that $A_{0,0,0,0}^{b,c} = 1$. Therefore $A_{r,0,0,0}^{b,c} = \frac{(q^{3b})_{b}}{(q_{b})_{b}} (c \geq r)$.

Synthesizing the results in steps (i)–(v), one obtains a formula for $A_{r,s,t,u}^{b,c}$ as a triple sum with respect to $(a, \beta, \gamma)$ over a finite subset of $\mathbb{Z}_{\geq 0}^3$. We express the ratio of the $q$-factorials contained in the summand by using the symbol (1.3). Then a little inspection shows that the sum can actually be relaxed to $(a, \beta, \gamma) \in \mathbb{Z}_{\geq 0}^3$ due to its support property mentioned after (1.3). Together with (3.20), one arrives at (3.12)–(3.16). This completes the proof of theorem 3.4.

4. Analogous result on 3D $R$

Our result may be viewed as a generalization of the analogous fact for the 3D $R$ [2, 3]. We review it here with a few supplements for comparison, as it is a closely related object associated with the quantized algebra of functions $A_q(A_2)$ [6, 7] and constitutes the companion in the 3D reflection equation (1.2). As an application, the expression of the 3D $R$ (4.20) announced in [8, equation (9)] is derived here for the first time. See [2, 3, 7] for more aspects.

The 3D $R$ is the linear operator on $F_q \otimes F_q \otimes F_q$, whose parameter (except $q$) free part is characterized, up to a normalization, by

$$R(a^z \otimes k \otimes 1) = (a^z \otimes 1 \otimes k + k \otimes a^z \otimes a^z) R,$$

$$R(1 \otimes k \otimes a^z) = (k \otimes 1 \otimes a^z + a^z \otimes a^z \otimes k) R,$$

$$R(1 \otimes a^z \otimes 1) = (a^z \otimes 1 \otimes a^z - q k \otimes a^z \otimes k) R,$$

$$[R, k \otimes k \otimes k] = 0.$$  

They follow either as the intertwining relations for the irreducible $A_q(A_2)$ modules [6] or from a quantum geometry consideration [2, 3]. Actually there should be nine intertwining relations in total, reflecting the $3 \times 3$ matrix nature of $A_q(A_2)$. The last one is given by

$$R(a^z \otimes a^- \otimes a^+ - q k \otimes 1 \otimes k) = (a^- \otimes a^+ \otimes a^- - q k \otimes 1 \otimes k) R.$$  

Define the matrix elements of the $R$ by $R(\{i\} \otimes \{j\} \otimes \{k\}) = \sum_{a,b,c} R_{a,b,c}^{a,b,c} \{a\} \otimes \{b\} \otimes \{c\}$. We adopt the normalization $R_{0,0,0}^{0,0,0} = 1$. Setting [2]

$$R_{a,b,c}^{a,b,c} = \delta_{a+b}^{a+b} \delta_{a+c}^{a+c} \delta_{b+c}^{b+c} \left( q^{(a-\gamma)(c-\gamma)} P_1 \left( q^{2a}, q^{2b}, q^{2c} \right) / (q^2) \right),$$  

where
the relations (4.1) and (4.2) are translated into the \(q\)-difference equations:

\[
\begin{align*}
\rho(x, y, z) - \rho(x, y, z) - q^{2-2h}x \left(1 - q^{2h}\right) (1 - q^{2-2h}yz) \rho_{b-1}(x, y, z) &= 0, \\
(1 - x)\rho(x, y, z) - q^{-2h}z \left(1 - q^{-2hx}\right) \rho_b(x, y, z) - \rho_{b+1}(x, y, z) &= 0, \\
(1 - z)\rho(x, y, -q^{-2h}z) - q^{-2hx} \left(1 - q^{-2h}yz\right) \rho_b(x, y, z) - \rho_{b+1}(x, y, z) &= 0, \\
\rho(x, q^2y, z) - \rho(x, y, z) + q^{4-4hx}yz \left(1 - q^{2h}\right) \rho_{b-1}(x, y, z) &= 0, \\
y\rho_{b+1}(x, y, z) + (1 - y)\rho(x, q^{-2}y, z) - \left(1 - q^{-2hx}\right) \left(1 - q^{-2h}yz\right) \rho_b(x, y, z) &= 0,
\end{align*}
\]

and the condition \(\rho(x, y, z) = 1\). It is known that \(R = R^{-1}\) [7, proposition 2.4]. Applying this to (4.1) and (4.2), one can extract another set of \(q\)-difference equations:

\[
\begin{align*}
\rho(x, y, z) - q^{2-2h}z\rho(x, y, z) - (1 - z)\rho(x, q^2y, -q^{-2}z) &= 0, \\
\rho(x, y, z) - q^{-2hx}\rho(x, y, q^{-2}z) - (1 - x)\rho(x, q^2y, z) &= 0, \\
q^{-2hx}(1 - y)\rho(x, q^{-2}y, q^{-2}z) + (1 - x)\rho(x, q^2y, z) - \left(1 - q^{-2hx}\right) \rho_b(x, y, z) &= 0, \\
q^{-2hx}(1 - y)\rho(x, q^2y, q^{-2}z) - \left(1 - q^{-2hx}\right) \rho_b(x, y, z) &= 0, \\
q^{-2hx}(1 - y)\rho(x, q^{-2}y, q^{-2}z) - (1 - z)\rho(x, y, q^{-2}z) - \left(1 - q^{-2hx}\right) \rho_b(x, y, z) &= 0, \\
q^{-2hx}(1 - y)\rho(x, q^{-2}y, q^{-2}z) - \left(1 - q^{-2hx}\right) \rho_b(x, y, z) &= 0, \\
q^{-2hx}(1 - y)\rho(x, q^{-2}y, q^{-2}z) - (1 - x)(1 - z)\rho(x, q^{2h}x, q^{-2}z) + \rho_{b+1}(x, y, z) &= 0, \\
q^{-2hx}(1 - y)\rho(x, q^{-2}y, q^{-2}z) - \left(1 - x\right)(1 - z)\rho(x, q^{2h}x, q^{-2}z) + \rho_{b+1}(x, y, z) &= 0,
\end{align*}
\]

The recursion (4.15), which was adopted as the defining relation of \(\rho_b(x, y, z)\) in [2], is a member of these compatible system of difference equations. Any one of the recursions (4.5), (4.6), or (4.16) also determines \(\rho_b(x, y, z)\) uniquely as a polynomial in \(x, y, z\). (More precisely, \(\rho_b(x, y, z) \in q^{-2h(b+1)}Z[q^2, x, y, z]\) holds.) The original problem (4.1) is symmetric with respect to the interchange of the first and third components. Accordingly \(\rho_b(x, y, z) = \rho_b(z, y, x)\) holds, and there are four pairs of relations in (4.3)–(4.16) connected by this symmetry. The solution admits an explicit formula [3]
in terms of the \( q \)-hypergeometric series 
\[
\phi_q(a, b; c, q, z) = \sum_{n \geq 0} \frac{(a, b; q)_n (c, q; q)_n}{(c, q; q)_n (b, q; q)_n} z^n, 
\]
which is actually terminating in (4.17) due to the entry \( q^{-2b} \). It is most easily established from (4.6) by gathering the terms in powers of \( x \).

Another interesting result is the integral formula [15, equation (104)]:
\[
\oint \frac{\pi}{2} = \sum_{b \geq 0} \left( -q^{2b+1} \right) P_b(x, y, z) = q^{-b(b-1)}(q^2)_b \oint \frac{du}{2\pi i u^{b+1}} \frac{(-q^{2b}xyzu; q^2)_\infty (-u; q^2)_\infty}{(-xu; q^2)_\infty (-zu; q^2)_\infty}, 
\]
where the integral (4.18) encircles \( u = 0 \) anti-clockwise, picking up the residue. Equivalently, the generating series is factorized as
\[
\sum_{b \geq 0} q^{b(b-1)} u^b \left( q^2 \right)_b P_b(x, y, z) = \frac{(-xu; q^2)_\infty (-u; q^2)_\infty}{(-xu; q^2)_\infty (-zu; q^2)_\infty}. 
\]

Substitution of (4.18) into (4.3)–(4.16) gives rise to two situations. In the simple case the integrands just sum up to zero under an appropriate rescaling of \( u \). The other case requires a slight maneuver. Let us illustrate it along (4.15) as an example. After the substitution of (4.18) and replacement of \( u \) by \( q^2u \) in the second term, one is left to show
\[
\oint \frac{du}{u^{b+2}} \frac{(-q^{2b}xyzu; q^2)_\infty (-q^2u; q^2)_\infty}{(-xu; q^2)_\infty (-zu; q^2)_\infty} (1 - y)u(1 + u) - (1 - x)(1 - z)u + (1 - q^{2b+2})(1 + u) = 0. 
\]
By setting \( f(u) = (-q^{2b}xyzu; q^2)_\infty (-u; q^2)_\infty / ((-xu; q^2)_\infty (-zu; q^2)_\infty) \), this is identified with the identity \( 0 = \oint \frac{du}{u^{b+2}} f(q^2u) - q^{2b+2}f(u) \). All the relations (4.3)–(4.16) can be verified in a similar manner.

Due to (4.18), matrix elements of the 3D \( R \) are expressed as [3]
\[
R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j+k} \oint \frac{du}{2\pi i u^{b+1}} \frac{(-q^{2+a+c+i}u; q^2)_\infty (-q^{-i-k}u; q^2)_\infty}{(-q^{a+c}u; q^2)_\infty (-q^{i-k}u; q^2)_\infty}. 
\]
Note that the ratio of the infinite products equals \( (-q^{i-k}u; q^2)_\infty / (-q^{-i-k}u; q^2)_\infty \) because of \( a - c = i - k \). Applying the standard expansion to it and collecting the coefficients of \( u^b \), one gets
\[
R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j+k} \sum_{\lambda, \mu \geq 0} (-1)^{\lambda+b+c-\lambda i} (\lambda + a) \left( \begin{array}{c} \lambda \newline \mu \end{array} \right) \left( \begin{array}{c} i \newline \mu - \lambda \end{array} \right) q^2, 
\]
summed over \( \lambda, \mu \in \mathbb{Z}_{\geq 0} \) under the constraint \( \lambda + \mu = b \). This was announced in [8, equation (9)].
5. Concluding remarks

In this paper we have introduced a family of polynomials \( Q_{b,c}(x, y, z, w) \in \mathbb{Z}[q^2, x, y, z, w] \) characterized by the recursion relations (3.5) and (3.6), or more generally E22–E55 in the appendix. They form a compatible set of \( q \)-difference equations associated with the ‘maximal’ representation \( \pi_{1212} \approx \pi_{1212} \) of the quantized algebra of functions \( A_q(C_2) \). The variables \( x, y, z, w \) correspond to the four positive roots of \( C_2 \), as indicated in (2.8) and (2.9). We have shown some basic properties of the polynomials in proposition 3.2 and proposition 3.3. The \( q \)-difference equations are solved in section 3.2, and a new formula of the 3D K, the solution [7] to the 3D reflection equation (1.2), is obtained in theorems 3.1 and 3.4. We have also included an expanded review on the closely related result on the 3D R in section 4. It is an interesting question if the family of polynomials \( Q_{b,c}(x, y, z, w) \) admits a factorizable generating series analogous to (4.19). Another challenge will be to establish a similar polynomial formula for the intertwiner of \( A_q(G_2) \) [9, section 4.4] for which no general expression has been constructed.

We remark that the system of intertwining relations \( \langle \rangle \langle \rangle \langle \rangle \langle \rangle \) in the appendix is autonomous in the sense that the apparent \( q \) can completely be removed by replacing \( k, K \) with \( q^{-1/k}, q^{-1/K} \), respectively. The new ones act on the Fock space by \( k|m\rangle = q^{m+1/2}|m\rangle, K|m\rangle = q^{2m+1}|m\rangle \). The resulting relations \( k a^2 = q^2 a^2k, a^2a^2 = 1 - q^{2k^2} \) and \( K a^2 = q^{22} A^2 K, A^2 A^2 = 1 - q^{22} K^2 \) can be realized in terms of the Weyl pairs \( (k, w) \) and \( (K, W) \) satisfying \( kw = qkw \) and \( kW = q^2KW \) by

\[
a^+ = \left(1 - q^{-1}k^2\right)^{1/2}w, \quad a^- = \left(1 - qk^2\right)^{1/2}w^{-1},
A^+ = \left(1 - q^{-2}K^2\right)^{1/2}W, \quad A^- = \left(1 - q^2K^2\right)^{1/2}W^{-1}.
\]

It is an interesting problem to seek a solution \( K \) to the intertwining relations \( \langle \rangle \langle \rangle \langle \rangle \langle \rangle \) in the appendix for the canonical representation of the Weyl pairs (called a non-compact representation of the \( q \)-oscillator algebra [13]). Especially, the autonomous feature mentioned above indicates a possible extension to the modular double setting, where not only \( q \) but also its modular dual \( \tilde{q} \) ((log \( q \))(log \( \tilde{q} \)) = const) enters everywhere compatibly via the Faddeev non-compact quantum dilogarithm [3, 10]. Such an analysis effectively poses an analytic continuation of the eigenvalues of \( k \) and \( K \) away from \( q^{2+2z}\) and \( q^{1+2z} \). The result in this paper may be viewed as a first step in this direction.

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Appendix. Difference equations for \( Q_{b,c}(x, y, z, w) \)

Let \( (ij) \) be the intertwining relation for (2.5) \( K \) (rather than \( X \)) with the choice \( f = t_{i-1,j-1} \). They all become independent of the parameters in (2.2) and (2.3), other than \( q \). Explicitly they read as follows (see [7, appendix A]):
The relations (25) and (52) imply the factor \(\delta_{a^{+}b^{+}c^{+}d^{+}}\delta_{a^{+}b^{+}c^{+}d^{+}}\) in (3.2). The other ones are translated into difference equations of \(Q_{\alpha,\beta}(x, y, z, w)\). For instance, consider equation (24).

In terms of the matrix elements defined by (2.7), it reads

\[
q^{b^{+}2c^{+}}(1 - q^{2d^{+}2})K_{i,j,k,l}^{a^{+}b^{+}c^{+}d^{+}} + q^{2b^{+}c^{+}}(1 - q^{2d^{+}2})K_{i,j,k,l}^{a^{+}b^{+}c^{+}d^{+}} + q^{2b^{+}c^{+}}(1 - q^{2d^{+}2})K_{i,j,k,l}^{a^{+}b^{+}c^{+}d^{+}}.
\]
Substituting (3.2) into this and setting \((x, y, z, w) = (q^4, q^2, q^4, q^2)\), one gets the difference equation E24 given below. Similarly, the equation \((ij)\) is cast into Eij below.

E22: \[ q^{-2i\delta_{-4}b + 4 + 1} Q_{b,c+1}(x, y, z, w) + q^{-2i\delta_{-4}b + 6 + 1} (wyz - q^{2b + 4b + 2}) Q_{b+1,c}(x, y, z, w) \]
\[ + (w - 1)(y - 1)Q_{b,c}(x, q^{-2}y, z, q^{-2}w) + wy(z - 1)q^{2b} Q_{b,c}(x, y, q^{-2}z, w) = 0, \]

E23: \[ - q^{-2i\delta_{-4}b + 4 + 1} Q_{b,c+1}(x, y, z, w) - wq^{-2i\delta_{-4}b + 6 + 1} Q_{b+1,c}(x, y, z, w) \]
\[ + wix(y - 1)q^{-2i\delta_{-4}b + 2} Q_{b,c}(x, q^{-2}y, z, w) + (w - 1)(x - 1)Q_{b,c}(q^{-2}x, y, z, q^{-2}w) \]
\[ + w(x - 1)(z - 1)q^{-2b} Q_{b,c}(q^{-2}x, q^{-2}y, q^{-2}z, w) = 0, \]

E24: \[ (wyzq^{-2i\delta_{-4}b + 2} - 1) Q_{b,c}(x, y, z, w) + (1 - w)Q_{b,c}(x, y, z, q^{-2}w) \]
\[ - w(z - 1)q^{-2b} Q_{b,c}(x, q^{-2}y, q^{-2}z, w) - w(y - 1)q^{-2i\delta_{-4}b + 2} Q_{b,c}(q^{-2}x, q^{-2}y, q^{-2}z, w) = 0, \]

E32: \[ q^{-i\delta_{-4}b - 4b + 4c} (wyz - x y z^2) Q_{b,c}(x, y, z, w) \]
\[ - y (q^{2b} - 1)q^{-i\delta_{-4}b - 4c} (q^{4b + c} - x y z^2) Q_{b-1,c+1}(x, y, z, w) \]
\[ - yzq^{-i\delta_{-4}b - 4c} Q_{b-1,c}(x, y, z, w) + (y - 1)Q_{b,c}(x, q^{-2}y, z, w) \]
\[ + y(z - 1)q^{-2b} Q_{b,c}(x, y, q^{-2}z, q^{-2}w) = 0, \]

E33: \[ wq^{-i\delta_{-4}b - 8} (q^{4b + c} - x y z^2) Q_{b,c}(x, y, z, w) \]
\[ + (q^{2b} - 1)q^{-i\delta_{-4}b - 4c} (q^{4b + c} - x y z^2) Q_{b-1,c+1}(x, y, z, w) \]
\[ + q^{-i\delta_{-4}b - 4c} Q_{b+1,c}(x, y, z, w) \]
\[ + (x - 1)Q_{b,c}(q^{-2}x, y, z, w) + x(y - 1)q^{-2i\delta_{-4}b + 2} Q_{b,c}(x, q^{-2}y, z, q^{-2}w) \]
\[ + (x - 1)(z - 1)q^{-2b} Q_{b,c}(q^{-2}x, q^{-2}y, q^{-2}z, q^{-2}w) = 0, \]

E34: \[ (wyzq^{-2i\delta_{-4}b + 2} - 1) Q_{b,c}(x, y, z, w) \]
\[ + z (q^{4c} - 1)q^{-2i\delta_{-4}b + 2} (q^{4b + 2c} - q^4 w y z) Q_{b+1,c-1}(x, y, z, w) \]
\[ + (q^{2b} - 1)q^{-2i\delta_{-4}b + 2} (q^{2b + 2c} - w y z) Q_{b-1,c}(x, y, z, w) \]
\[ - (z - 1)q^{2b} Q_{b,c}(x, q^{-2}y, q^{-2}z, q^{-2}w) - (y - 1)q^{-2i\delta_{-4}b + 2} Q_{b,c}(q^{-2b} x, q^{-2}y, z, q^{-2}w) = 0, \]

E35: \[ - wq^{-i\delta_{-4}b - 4b + 4c} Q_{b+1,c-1}(x, y, z, w) \]
\[ - Q_{b,c}(x, y, z, q^{-2}w) \]
\[ - w(q^{2b} - 1)y q^{-i\delta_{-4}b - 4c} Q_{b-1,c}(x, y, z, w) + Q_{b,c}(x, y, z, w) = 0, \]

E36: \[ wq^{-i\delta_{-4}b - 4b + 4c} Q_{b-1,c+1}(x, y, z, w) \]
\[ + wq^{2b} (w y z - q^{2b+2c}) Q_{b,c}(x, y, z, w) \]
\[ - (w - 1)q^{2b} Q_{b,c}(x, y, z, q^{-2}w) = 0, \]

E37: \[ q^{4c} - 1)q^{-2i\delta_{-4}b + 2} (q^{4b + 2c} - q^4 w y z) Q_{b+1,c-1}(x, y, z, w) \]
\[ + wq^{2b} (w y z - q^{2b+2c}) - yz Q_{b,c}(x, y, z, w) \]
\[ - (w - 1)x (y - 1)q^{4b} Q_{b,c}(x, q^{-2}y, q^{-2}z, q^{-2}w) \]
\[ - (w - 1)(x - 1)q^{4b+4c} Q_{b,c}(q^{-4}x, q^{-2}y, z, q^{-2}w) + Q_{b+1,c}(x, y, z, w) = 0, \]

E38: \[ q^{-i\delta_{-4}b - 8} (q^{4b + c} - x y z^2) Q_{b,c}(x, y, z, w) \]
\[ + q^{-i\delta_{-4}b - 4c} Q_{b+1,c}(x, y, z, w) \]
\[ + (x - 1)Q_{b,c}(q^{-2}x, y, z, w) + x(y - 1)q^{-2i\delta_{-4}b + 2} Q_{b,c}(x, q^{-2}y, z, q^{-2}w) \]
\[ + (x - 1)(z - 1)q^{-2b} Q_{b,c}(q^{-2}x, q^{-2}y, q^{-2}z, q^{-2}w) + y Q_{b,c}(x, y, z, w) = 0, \]
E45: \[ wxyz^2 (q^{2b} - 1) q^{4c} Q_{b-1,c}(x, y, z, w) - wz (q^{4c} - 1) Q_{b+1,c-1}(x, y, z, w) \]
\[ - wq^{-2b} Q_{b,c}(x, y^2, z, w) + (w - 1) Q_{b,c}(x, y, q^4z, q^{-2}w) + Q_{b,c}(x, y, z, w) = 0, \]

E53: \[ 3yq^{-2(b+2c)} Q_{b,c}(x, y, z, q^2w) + (x - 1) Q_{b,c}(q^{-x}q^2y, q^2z, w) \]
\[ + x(y - 1)q^{-c} Q_{b,c}(x, q^{-2}y, q^4z, w) + Q_{b,c}(x, y, z, w) = 0, \]

E54: \[ y(q^{2b} - 1) Q_{b-1,c}(x, y, z, w) + q^{2b}(q^{4c} - 1) \]
\[ \times (q^{2b+2c-2} - wyz) Q_{b,c-1}(x, y, z, w) \]
\[ = q^{-4b-4c+6} Q_{b,c}(x, y^2, z, w) + yq^{-6b-8c+6} Q_{b,c}(q^4x, y, q^2z, w) \]
\[ - (y - 1)q^{-4b-8c+6} Q_{b,c}(q^2x, q^{-2}y, q^4z, w) = 0, \]

E55: \[ - wzq^{2b}(q^{4c} - 1) Q_{b,c-1}(x, y, z, w) + (q^{2b} - 1) Q_{b-1,c}(x, y, z, w) \]
\[ - q^{-4b-8c+6} Q_{b,c}(x, y, q^2, z, w) + q^{-6b-8c+6} Q_{b,c}(x, q^2y, q^2z, w) = 0. \]

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