Maximum Entanglement in Squeezed Boson and Fermion States

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A class of squeezed boson and fermion states is studied with particular emphasis on the nature of entanglement. We first investigate the case of bosons, considering two-mode squeezed states. Then we construct the fermion version to show that such states are maximum entangled, for both bosons and fermions. To achieve these results, we demonstrate some relations involving squeezed boson states. The generalization to the case of fermions is made by using Grassmann variables.

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Boson coherent states have played an important role in the production and detection of Bose-Einstein condensates. The experimental tools of laser cooling, magnetic and magneto-optic traps have advanced tremendously [1, 2]. This has lead to a consideration of producing a degenerate Fermi gas as well as condensates of rare isotopes, a fact that has been achieved experimentally [2, 3, 4, 5, 6]. These results suggest that a careful study of the Fermi coherent state and Fermi density operators is needed. Some of these concepts have been introduced for fermion systems long ago [7, 8, 9, 10], but only recently, Cahill and Glauber [11] have discussed notions such as $P$-function, $Q$-function and Wigner function for fermions; all of them are described as a counterpart of the Boson systems and are made possible through the use of Grassmann variables.

In addition to the idea of achieving a Fermi degenerate gas, there is a great deal of interest in studying entangled states of multipartite systems [12, 13]. The existence of entangled states is directly related to the nature of the quantum mechanics formalism, based on the structure of Hilbert space and the superposition principle. The present interest in entangled states is strongly driven by the understanding that this is the focal point of studies leading to teleporting of quantum states, from one locus to another, which is also the basic ingredient of quantum computers [14, 15]. The necessity to study such entangled states is greatly increased with the suggestion that the conditions for teleporting, however, require specific states characterized by maximum entanglement. Measure of entanglement has been discussed in the literature in different ways [16, 17, 18, 19, 20, 21], and the identification and construction of maximum entangled states have also been addressed [22, 23].

Consider the state $|\psi\rangle_{ab}$ of a bipartite system $(A, B)$. Define the reduced density operator $\rho_a$, by

$$\rho_a = \text{Tr}_B(|\psi\rangle_{ab}\langle\psi|),$$

where $\text{Tr}_B$ stands for the trace over the variables of the subsystem $B$. A measure of the amount of entanglement of the state $|\psi\rangle_{ab}$ is given by the von Neumann entropy associated with the reduced density operator [22],

$$S_a = -\text{Tr}(\rho_a\ln \rho_a).$$

The question addressed in this paper is concerned with the maximum entangled state within the set of all pure states having a given (fixed) reduced energy.

The entropy $S_a$ is a homogeneous function of first degree in its dependency on $E_a$, the energy of the system $A$. Then we wish to maximize $S_a[\rho_a]$ we require $\delta S_a[\rho_a] = 0$ under the constraints

$$E_a = \langle H_a \rangle = \text{Tr}(\rho_a H_a), \quad \text{Tr}\rho_a = 1,$$

where $H_a$ is the energy operator of the $A$-component. Following standard procedures, we derive then a constraint equation for $\rho_a$, that is $\lambda_0 - 1 + \lambda_1 H_a - \ln \rho_a = 0$, where $\lambda_0$ and $\lambda_1$ are Lagrange multipliers. Solving this equation, we get a Gibbs-like density operator, that is

$$\rho_a = \frac{1}{Z} \exp(\lambda_1 H_a),$$

where $Z = \exp(1 - \lambda_1)$. Multiplying the constraint equation by $\rho_a$, taking the trace and using Eqs. (2) and (3), we get

$$\ln Z = \lambda_1 E_a + S_a.$$

For the sake of convenience, let us write $\lambda_1 = -\tau$, then we have $-\tau^{-1} \ln Z = E_a - \tau^{-1} S_a$. The function $F(\tau) = -\tau^{-1} \ln Z$ describes the Legendre transform of $S_a$ since we assume that $\tau = \partial S_a/\partial E_a$. Here $\tau$ is an intensive parameter describing the fact that the energy average $E_a = \langle H_a \rangle$, given by Eq. (2), is constant in the state described by $\rho_a$. Therefore, a state $|\psi\rangle_{ab}$ with reduced energy $E_a$ is a maximum entangled state when the corresponding reduced density matrix, defined in Eq. (1), is written as a canonical Gibbs-ensemble distribution, explicitly given by Eq. (4), such that $Z = [1 - \exp(-\tau)]^{-1}$. Using this approach, examples of maximum entanglement states were explicitly constructed in [23], considering as a guide the thermofield dynamics (TFD) formalism.

In the present work we extend that analysis to show that two-mode squeezed states (TMSs) are maximum entangled, within a class of states with fixed energy, for both fermions and bosons. We first investigate the case of bosons and, then, we construct the fermion version, extending a preliminary study by Chaturvedi et al. [10]. For the case of fermions, the situation is more intricate, demanding the notion of coherent fermion state and density operator, which is achieved by using Grassmann variables. In any case we explore the similarities among these squeezed states with those found in the TFD formalism [24, 25, 26].
In order to analyze entanglement, we derive some general properties of the usual Caves-Schumaker (CS) state for bosons [27]. Consider a two-boson system specified by the operators \(a\) and \(b\) obeying the algebra \([a, a^\dagger] = [b, b^\dagger] = 1\), \([a, b] = 0\), with unitary displacement operators \(D_a(\xi) = \exp[\xi a - a^\dagger\xi^*]\) and \(D_b(\eta) = \exp[\eta b - b^\dagger\eta^*]\), and define the two-mode squeezing operator
\[
S_{ab}(\gamma) = \exp[\gamma(a^\dagger b - ab)],
\]
with \(\gamma\) being a real non-negative number for simplicity. Using standard TFD formulas [24, 25], we get the useful relations
\[
a(\gamma) = S_{ab}(\gamma)aS_{ab}^\dagger(\gamma) = u(\gamma)a - v(\gamma)b^\dagger,
b(\gamma) = S_{ab}(\gamma)bS_{ab}^\dagger(\gamma) = u(\gamma)b - v(\gamma)a^\dagger,
\]
and the corresponding relations for \(a^\dagger(\gamma)\) and \(b^\dagger(\gamma)\), where \(u(\gamma) = \cosh \gamma\) and \(v(\gamma) = \sinh \gamma\).

First, consider the TMSS defined by \(|\gamma\rangle_{ab} = S_{ab}(\gamma)|0\rangle_{ab}\), where \(|0\rangle_{ab} = |0\rangle_a \otimes |0\rangle_b \equiv |0\rangle_a |0\rangle_b\) is the two-mode vacuum such that \(a|0\rangle_a = b|0\rangle_b = 0\). For this squeezed state, we have \(a(\gamma)|\gamma\rangle_{ab} = b(\gamma)|\gamma\rangle_{ab} = 0\). Another important result is that \(S_{ab}(\gamma)\) is a canonical transformation, that is, \([a(\gamma), a^\dagger(\gamma)] = \gamma, [b(\gamma), b^\dagger(\gamma)] = 1\) and \([a(\gamma), b(\gamma)] = 0\). Now, using the operator identity \(\exp[\gamma(a + B)] = \exp[(\tanh \gamma)B] \exp[\ln \cosh \gamma C] \exp[(\tanh \gamma)A]\), with \(A = -ab, B = a^\dagger b^\dagger\) and \(C = [A, B] = -a^\dagger b - ab^\dagger\), the TMSS can be written as
\[
|\gamma\rangle_{ab} = \frac{1}{\cosh \gamma} \exp[(\tanh \gamma)a^\dagger b^\dagger]|0\rangle_{ab},
\]
(5)

Changing the parametrization by taking \(\cosh \gamma = [1 - \exp(-\tau)]^{-1/2}\), so that \(\tanh \gamma = \exp(\tau/2), 2\), and defining \(Z(\tau) = [1 - \exp(-\tau)]^{-1} = \text{Tr}[-\tau a^\dagger a]\), we find
\[
|\gamma\rangle_{ab} = \frac{1}{\sqrt{Z(\tau)}} \sum_{n=0}^{\infty} e^{-\tau n/2}|n\rangle_a|n\rangle_b.
\]
(6)

Therefore, the TMSS \(|\gamma\rangle_{ab}\) can be written as
\[
|\gamma\rangle_{ab} = \sqrt{f_a(\tau)} \sum_{n=0}^{\infty} |n\rangle_a|n\rangle_b,
\]
(7)

where \(f_a(\tau) = \exp(-\tau a^\dagger a)/Z(\tau)\).

The CS states are introduced by the application of \(S_{ab}(\gamma)\) to a two-mode coherent state, that is \(|\xi, \eta\rangle_{ab} = S_{ab}(\gamma)D_a(\xi)D_b(\eta)|0\rangle_{ab}\). We can show that \(|\xi, \eta\rangle_{ab} = |\xi, \eta, \gamma\rangle_{ab}\), with \(|\xi, \eta, \gamma\rangle_{ab} = D_a(\xi)D_b(\eta)S_{ab}(\gamma)|0\rangle_{ab}\), where
\[
\left(\begin{array}{c}
\xi \\
\eta
\end{array}\right) = B_B(\gamma) \left(\begin{array}{c}
\xi \\
\eta
\end{array}\right),
\]
(8)

\(B_B(\gamma)\) being the matrix form associated to \(S_{ab}(\gamma)\) given by
\[
B_B(\gamma) = \left(\begin{array}{cc}
u(\gamma) & -v(\gamma) \\
v(\gamma) & u(\gamma)
\end{array}\right).
\]
(9)

For \(\xi = \eta = 0\), the CS state reduces to the TMSS \(|\gamma\rangle_{ab} = S_{ab}(\gamma)|0\rangle_{ab}\), which has the same structure as the thermal vacuum state used to introduce TFD [24, 25]. We explore this result to show that \(|\xi, \eta, \gamma\rangle\) can be used to define a Gibbs-like density.

Following the prescription discussed before, taking \(\rho_{ab} = |\xi, \eta, \gamma\rangle\langle\xi, \eta, \xi|, \gamma\rangle\), we have to perform the calculation of the reduced density matrix, say \(\rho_a = \text{Tr}_b\rho_{ab}\). Using the notation, \(|r\rangle_b = n^{-1/2}(b^\dagger)^n|0\rangle_b\) (similarly for mode \(a\)), we can write the matrix elements \(|s|\rho_a|t\rangle = \sum_{r, m, n} a(s)[b](r)|p_{ab}(r)b(t)\rangle_a\) as
\[
|s|\rho_a|t\rangle = \sum_{r, m, n} a|s|D_a(\xi)\sqrt{f_a(\tau)}|m|D_b(\eta)|n\rangle_b \times b|m|D_b(\eta)^\dagger|r\rangle_{ba}b|m\sqrt{f_a(\tau)}D_a(\xi)^\dagger|t\rangle_a.
\]

Changing the order of the matrix elements in the \(b\) mode, and using the completeness relation, we obtain \(|s|\rho_a|t\rangle = \langle s|D_a(\xi)f_a(\tau)D_a(\xi)^\dagger|t\rangle\). Thus, we get
\[
\rho_a = D_a(\xi)f_a(\tau)D_a(\xi)^\dagger(\xi) = Z^{-1}(\tau) \exp(-\tau a^\dagger(\xi)a(\xi)),
\]
(10)

where \(a(\xi) = D_a(\xi)aD_a^\dagger(\xi)\) is the displaced annihilation operator; \(\rho_a\) is, therefore, a Gibbs-like density. In particular, for \(\xi = 0\), we find \(\rho_a = f_a(\tau)\) showing that the TMSS \(|\gamma\rangle_{ab}\) also generates a Gibbs-like density.

Using the displaced Fock’s basis, \(\{D_a(\xi)|n\rangle\}_{a}\), we show that the reduced von Neumann entropy for a CS state is
\[
S(\tau) = \frac{\tau}{e^{\tau - 1}} - \ln(1 - e^{-\tau}).
\]
(11)

Thus, all CS states, with the same (fixed) squeezing parameter, have the same amount of entanglement independent of the displacement parameters. Among them, the one having the smallest energy is the TMSS \((\xi = 0)\); its reduced energy \((E_a = \text{Tr}(\rho_a a^\dagger a))\) is given by \(E(\tau) = (e^\tau - 1)^{-1}\). Since both actions of displacing and squeezing the vacuum lead to states with greater energy, the TMSS is the maximum entangled state when the energy is fixed.

Let us now analyze comparatively the amount of entanglement of the TMSS. Consider the state \(|\Psi^{(N)}\rangle_{ab} = N^{-1} \sum_{n=0}^{N-1} |n\rangle_a |n\rangle_b\) which has reduced energy and entropy given by \(E(\tau) = (N - 1)/2\) and \(S(N) = \ln N\). This state has the greatest amount of entanglement among all pure states belonging to the finite \((N^2)\) dimensional subspace spanned by \(|0\rangle_a |0\rangle_b, |0\rangle_a |1\rangle_b, \ldots, |N - 1\rangle_a |N - 1\rangle_b\), corresponding to equal occupation probability. Naturally, as \(N \to \infty\), both energy and amount of entanglement of \(|\Psi^{(N)}\rangle_{ab}\) goes to \(\infty\).

Now, take another parametrization of the TMSS by writing \(\tau = \ln(\chi + 1) - \ln(\chi - 1)\); the limit situations of zero and infinite squeezing correspond to \(\chi = 1\) (\(\gamma = 0\), \(\tau = \infty\)) and \(\chi = \infty\) (\(\gamma = \infty\), \(\tau = 0\)), respectively. The reduced energy and the amount of entanglement are then written as \(E(\chi) = (\chi - 1)/2\) and \(S(\chi) = [(\chi + 1)\ln(\chi + 1) - (\chi - 1)\ln(\chi - 1) - 2\ln 2]/2\). We find that both \(E(\chi)\) and \(S(\chi)\) go to \(\infty\) as \(\chi \to \infty\), with \(S(\chi) \sim \ln \chi\) in this limit. In Fig. 1, we plot the difference between \(S(\chi)\) and \(\ln \chi\), showing explicitly that the TMSS has an amount of entanglement greater than that of the state \(|\Psi^{(N)}\rangle_{ab}\) with the same energy, for all \(N \geq 2\).
We now consider a similar analysis for fermions. A coherent fermion state \[ |\alpha\rangle = \exp(i\alpha a - a^\dagger \alpha)|0\rangle \]
can be defined by introducing a displacement-like operator \( D(\alpha) \), where \( \alpha \) is a Grassmann variable, trying to reproduce formally the basic results of the boson case. This is accomplished in the following way. Consider \( \alpha \) and \( \beta \) two Grassmann numbers, then \( \langle \alpha, \beta \rangle = \alpha \beta + \beta \alpha = 0 \), and moreover, \( \langle \alpha, a \rangle = \langle \alpha, a^\dagger \rangle = 0 \), where the fermion operator \( a \) and \( a^\dagger \) satisfy the anticommutation relation \( \{ a, a^\dagger \} = 1 \). Notice that we maintain the same notation for the creation and annihilation operators as that for bosons. The complex conjugation is an antilinear mapping \( * : \alpha \rightarrow \alpha^* \) such that, for a general expression involving Grassmann numbers and the operators \( a \) and \( a^\dagger \), we have \( (\alpha \alpha^* + c\beta \beta^*)^* = \alpha \alpha^* + c^* \beta \beta^* \); \( c \in \mathbb{C} \) and \( \langle \alpha , \alpha \rangle = \langle \alpha^* , \alpha^* \rangle = 1 \). The Grassmann variable \( \alpha \) is considered independent of \( \alpha^* \).

The fermion displacement operator is defined by
\[
D_a(\alpha) = \exp(\alpha a - a^\dagger \alpha),
\]
such that \( D_a(\alpha) a D_a^\dagger(\alpha) = a - \alpha \), with the coherent state given by \( |\alpha\rangle = D_a(\alpha)|0\rangle \) and \( a|\alpha\rangle = \alpha|\alpha\rangle \). We can then prove that the dual coherent state is given by \( \langle \alpha | = \langle 0 | D_a^\dagger(\alpha) \),
with \( D_a^\dagger(\alpha) = D_a^{-1}(\alpha) \). As a consequence
\[
\langle \alpha | \beta \rangle = \exp(\alpha^* \beta - \frac{1}{2} \alpha^* \alpha - \frac{1}{2} \beta^* \beta)
\]
and \( \langle \alpha | D_a^\dagger(\alpha) = \langle \alpha | a^\dagger \). In terms of the number basis, the state \( |\alpha\rangle \) is written as
\[
|\alpha\rangle = e^{-\alpha^* \alpha/2} \sum_n (-\alpha)^n |n\rangle
\]
and, then, we have \( \langle n | \alpha \rangle = \exp(-\alpha^* \alpha/2) \langle n | \alpha \rangle \).

The integration is defined, as usual, by \( \int d\alpha = 0 \) and \( \int d\alpha^\dagger = 0 \). Note that, in particular, we have \( \int d\alpha^* \alpha^* = 1 \) (resulting in \( (\alpha^* \alpha^*) = -\alpha^* \alpha^* \)), \( \int d\alpha^* \alpha \alpha^* = 1 \) and \( \int d\alpha^* \alpha |\alpha\rangle \langle \alpha | = 1 \).

Cahill and Glauber \cite{Cahill} introduced the following coherent fermion state representation for a density operator, i.e. \( \rho = \int d^2 \alpha P(\alpha) | -\alpha \rangle \langle -\alpha | \), where \( P(\alpha) \) is the corresponding \( P \)-function. Notice that the density operator
\[
\rho_\alpha = | -\alpha \rangle \langle \alpha |
\]
possesses the expected properties to be taken as representing the coherent state \( |\alpha\rangle \). First, it is normalized, \( \text{Tr} \rho_\alpha = 1 \). This property can be proved from the matrix representation of \( \rho_\alpha \), by calculating \( \langle m| \rho_\alpha |n\rangle = \exp(\alpha^* \alpha)(-\alpha)^m(\alpha^*)^n \), giving rise to
\[
\rho_\alpha = \left( 1 - \alpha^* \alpha \right) \left( \frac{\alpha^*}{\alpha \alpha^*} \right).
\]
Secondly, \( \text{Tr}(\rho_\alpha a^\dagger a) = \alpha^* \alpha \), which is similar to the boson case. Observe that \( \rho_\alpha \), although not being hermitian, is introduced in such way that \( \rho^\dagger = \rho \).

Using the properties described above, we can prove that the displaced fermion number state is given by
\[
D_a(\alpha)|n\rangle = (a^\dagger - \alpha^* a)^n|n\rangle,
\]
with \( n = 0, 1 \). Another property useful for calculations but reflecting also the nature of \( \rho_\alpha \) is given by
\[
\langle m| \rho_\alpha |n\rangle = (1 - 1 \rangle \langle 0 |\rangle \langle 0 | \langle 0 | \langle m | = (-1)^m(|n\rangle \langle m | \langle m | \langle n | \langle m | ,
\]
where \( \langle m| \rho_\alpha |n\rangle = (m - \alpha^* | \langle n | \langle n | \langle m | \langle m | \langle n | \langle m | \langle n | \langle m | \langle m | \langle m | . \)

Let us now consider a two-fermion system, specified by the operators \( a \) and \( b \) satisfying the algebra \( \{ a, a^\dagger \} = \{ b, b^\dagger \} = 1 \), with all the other anticommutation relations being zero. A fermionic two-mode squeezed vacuum state is defined by \( |\gamma\rangle_{ab} = S_{ab}(\gamma)|0\rangle_{ab} \), where \( \gamma \) is still a real number and \( S_{ab}(\gamma) = \exp(\gamma a^\dagger b^\dagger - ab) \). Some useful formulas can be derived using \( S_{ab}(\gamma) \), that is,
\[
\begin{align*}
\langle a | \gamma \rangle &= S_{ab}(\gamma)\langle a | \gamma \rangle = u(\gamma) a - v(\gamma) b, \\
\langle b | \gamma \rangle &= S_{ab}(\gamma)\langle b | \gamma \rangle = u(\gamma) b + v(\gamma) a,
\end{align*}
\]
where now \( u(\gamma) = \cos(\gamma) \) and \( v(\gamma) = \sin(\gamma) \). Thus, for the two-mode squeezed vacuum state \( |\gamma\rangle_{ab} \) we have \( a(\gamma) |\gamma\rangle_{ab} = b(\gamma) |\gamma\rangle_{ab} = 0 \), since \( a|0\rangle = a|0\rangle \). The squeezing operator \( S_{ab}(\gamma) \) is a canonical transformation, in the sense that, as in the case of bosons, \( \{ a(\gamma), a(\gamma) \} = \{ b(\gamma), b(\gamma) \} = 1 \) and \( a(\gamma), b(\gamma) \rangle = 0 \). The matrix form \( B_F(\gamma) \) associated to \( S_{ab}(\gamma) \) is
\[
B_F(\gamma) = \left( \begin{array}{cc}
u(\gamma) & u(\gamma) \\
v(\gamma) & -u(\gamma)
\end{array} \right).
\]

The vector \( |\gamma\rangle_{ab} \) can be cast in a TFD state. To see that we write
\[
|\gamma\rangle_{ab} = [1 - \gamma (ba - a^\dagger b^\dagger) + \frac{\gamma^2}{2!} (ba - a^\dagger b^\dagger)^2 + ...]|0\rangle_{ab}.
\]
Using the relations \( (ba - a^\dagger b^\dagger)^2n|0\rangle_{ab} = (-1)^n|0\rangle_{ab} \) and introducing the reparametrization \( u(\gamma) = \cos \gamma = 1 + e^{-\tau} \), \( v(\gamma) = \sin \gamma = (1 + e^{-\tau})^{-1/2} \), we obtain
\[
|\gamma\rangle_{ab} = \frac{1}{\sqrt{1 + e^{-\tau}}} |1 + e^{-\tau/2} a^\dagger b^\dagger |0\rangle_{ab}.
\]
Defining \( Z(\tau) = 1 + e^{-\tau} \), Eq. (18) reads
\[
|\gamma\rangle_{ab} = \frac{1}{\sqrt{Z(\tau)}} e^{-\tau N/2} (|0\rangle_\alpha|0\rangle_\beta + |1\rangle_\alpha|1\rangle_\beta),
\]
where $N = a^† a$, the fermion number operator for the mode $a$, is such that $N|n⟩_a = n|n⟩_a$. Therefore, we obtain

$$|γ⟩ = \sqrt{f_a(τ)} \sum_{n=0}^{1} |n⟩_α |n⟩_β,$$

with $f_a(τ) = Z^{−1}(τ) \exp(−τ a^† a)$.

With these results, we can prove the following statement. Given the two fermion displacement operators, $D_a(α) = \exp(a^† α - α^∗ a)$ and $D_b(β) = \exp(b^† β - β^∗ b)$, where $α$ and $β$ are Grassmann numbers, then

$$S_{ab}(γ)D_a(α)D_b(β) = D_a(\bar{α})D_b(\bar{β})S_{ab}(γ)$$

(20)

where

$$\begin{pmatrix} \bar{α} \\ \bar{β}^* \end{pmatrix} = B_f(γ) \begin{pmatrix} α \\ β^* \end{pmatrix}.$$ (21)

Thus, the fermion version of the CS state, defined by

$$|α, β, γ⟩ = S_{ab}(γ)D_a(α)D_b(β)|0⟩_{ab},$$

is related to the state $|α, β, γ⟩ = D_a(α)D_b(β)S_{ab}(γ)|0⟩_{ab}$ by the transformation given in Eqs. (20) and (21). As in the bosonic case, when $α = β = 0$ we have the two-fermion squeezed vacuum state $|γ⟩_{ab} = S_{ab}(γ)|0⟩_{ab}$.

Now we turn our attention to the nature of the entanglement in squeezed fermion states. Considering the states $|α, β, γ⟩$ and inspired by the definition of the density operator given in Eq. (14), we introduce the following density matrix

$$ρ_{ab} = |α, β, γ⟩⟨γ, β, α|.$$ (14)

Performing the trace in the mode $b$ and using the properties derived before, we can prove that $ρ_a = D_a(α)f_a(τ)D_a(α)†$, similar to the boson case. Thus we find that the state $|α, β, γ⟩$ has reduced density operator in the form of a Gibbs-like density. The reduced entropy is thus maximal. However, in the fermionic case, the CS states are not in general physical states since they involve Grassmann variables. The two-fermion squeezed vacuum state $|γ⟩_{ab}$ is nevertheless physical and maximally entangled.

It is worth mention that, in the case of fermions, there is another class of physical states having maximum entanglement for a given value of the reduced energy. In fact, one can show that the state

$$|γ⟩'_{ab} = \left( |0⟩_a |1⟩_b + e^{-τ/2} |1⟩_a |0⟩_b \right) / \sqrt{Z(τ)}$$ (22)

has reduced density operator $ρ_a'$ identical to the reduced density operator $ρ_a$ associated with the state $|γ⟩_{ab}$; therefore, these states have identical reduced energy and entropy.

Summarizing, in this paper we have analyzed a class of two-mode squeezed boson and fermion states, looking for explicit realization of maximum entangled states with fixed energy. We investigate the case of bosons, and then, construct the fermion version, to show that such states, in both cases, are maximum entangled. For achieving these results we have demonstrated some relations involving the squeezed boson states, which are then extended to the case of fermions. The calculations for fermions are performed with a generalization of the density fermion operator introduced by Cahill and Glauber [11].

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