D-BRANE ACTIONS, INTRINSIC GEOMETRY AND DUALITY

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Abstract. We discuss an alternative form of the supersymmetric D-p-brane action which is quadratic in derivatives of $X$ and linear in $F_{\mu\nu}$. This action involves an auxiliary worldvolume tensor and generalises the simplification of the Nambu-Goto action for $p$-branes using a symmetric metric. When the worldvolume gauge field is abelian, it appears as a Lagrange multiplier, and solving the constraint gives the dual form of the $(p + 1)$-dimensional action with a $p - 2$ form gauge field instead of a vector gauge field. This is illustrated by the example of the dual D-2-brane action, for which the known result is recovered.

1. Supersymmetric D-p-Brane Actions

The effective action for superstring theory in the presence of a D-p-brane is the sum $S = S_D + S_{(p)}$ of the bulk supergravity action $S_D$ and the effective world-volume action $S_{(p)}$. The latter is the sum of a Dirac-Born-Infeld type action [1, 2] and a Wess-Zumino term [3, 4], and it was shown in refs. [5, 6, 7, 8], to be invariant under a local fermionic symmetry which reflects the fact that the D-brane is BPS-satured and breaks half the space-time supersymmetries [9].

We begin with a brief review of the usual form of $S_{(p)}$. The (flat) superspace coordinates are the $D = 10$ space-time coordinates $X^i$ and the Grassmann coordinates $\theta$, which are space-time spinors and world-volume scalars. For the IIA superstring (even $p$), $\theta$ is Majorana but not Weyl while in the IIB superstring there are two Majorana-Weyl spinors $\theta_\alpha$ ($\alpha = 1, 2$) of the same chirality. The superspace (global) supersymmetry transformations are

$$\delta_\epsilon \theta = \epsilon \ , \ \delta_\epsilon X^i = \epsilon \Gamma^i \theta.$$  

(1)
The world-volume theory has global type IIa or type IIb super-Poincaré symmetry and is constructed using the supersymmetric one-forms $\partial_\mu \theta$ and
\[ \Pi^i_\mu = \partial_\mu X^i - \bar{\theta} \Gamma^i \partial_\mu \theta. \] (2)

The induced world-volume metric is
\[ G_{\mu\nu} = G_{ij} \Pi^i_\mu \Pi^j_\nu. \] (3)

The supersymmetric world-volume gauge field-strength $F$ is defined by
\[ F = F - B, \] and a convenient choice for the two-form $B$ is [10]
\[ B = -\theta \Gamma_{11} d\theta (dX^i + \frac{1}{2} \theta \Gamma_i d\theta) \] (4)
when $p$ is even or the same formula with $\Gamma_{11}$ replaced with the Pauli matrix $\tau_3$ when $p$ is odd. With the choice (4), $\delta_\epsilon B$ is an exact two-form and $F$ is supersymmetric for an appropriate choice of $\delta_\epsilon A$ [10, 6].

The leading contribution to the effective supersymmetric D-$p$-brane action with constant dilaton and zero cosmological constant then takes the form
\[ S_{(p)} = -T_p \int d^p \sigma e^{-\phi} \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu})} + T_p \int_{W_{p+1}} C e^F \] (5)

The constant $T_p$ is the $p$-volume tension with mass dimension $p + 1$, and $W_{p+1}$ denotes the $(p + 1)$-dimensional worldvolume of the brane. Here $C$ represents the complex of differential forms $C = \sum_{r=0}^{9} C^{(r)}$ where the $C^{(r)}$ are the pull-backs of superspace forms $C^{(r)} = d\theta T^{(r-2)} d\theta$ for certain $r - 2$ forms $T^{(r-2)}$ given explicitly in [7, 8, 5, 6], and it is understood that the $p + 1$ form part of $C e^F$, which is $C^{(p+1)} + C^{(n-2)} F + \frac{1}{2} C^{(n-4)} F^2 + \ldots$, is selected. The bosonic part of the Wess-Zumino term gives the coupling of the brane to the background Ramond-Ramond $r$-form gauge fields (where $r$ is odd for type IIa and even for type IIb). The case of the 9-form potential is special because its field equation forces the dual of its field strength to be a constant $m$. The backgrounds with $m = 0$ are the familiar type IIa ones, while backgrounds with $m \neq 0$ are those of massive type IIa supergravity [11]. In the $m \neq 0$ case the Wess-Zumino term given in (5) requires $m$-dependent Chern-Simons modifications, and the bosonic part of these were derived in [12]. The constant $m$ will be taken to be zero here, so that the 9-form potential vanishes; the generalisation of the discussion below to the more general situation will be given elsewhere.

The action (5) is supersymmetric and invariant under local kappa symmetry [7, 8, 5, 6]. However, just as in the case of the usual Nambu-Goto
action for a $p$-brane, the non-linearity of (5) is inconvenient for many purposes. In particular, dualising the action has proved difficult in this approach, and has only been achieved for $p < 5$ \[10, 13, 14, 15, 16, 17\]. It is therefore useful to generalise the simplification of the Nambu-Goto action using an auxiliary world-volume metric $g_{\mu\nu}$ to the case of actions involving world-volume gauge fields in addition to the induced metric $G_{\mu\nu}$. This was achieved in ref. \[18\] by introducing an auxiliary tensor field

$$k_{\mu\nu} \equiv g_{\mu\nu} + b_{\mu\nu}$$

with both a symmetric part $g_{\mu\nu}$ and an antisymmetric part $b_{\mu\nu}$. The supersymmetric D-brane action which is classically equivalent to (5) is

$$S = -\frac{1}{2}T_p \int d^n\sigma e^{-\phi} \sqrt{-k} \left[ k^{\mu\nu} (G_{\mu\nu} + F_{\mu\nu}) - (p - 1)\Lambda \right] + T_p \int_{W_n} C e^F,$$

(7)

where $k \equiv \det k_{\mu\nu}$ and $\Lambda$ is a constant. The inverse tensor $k^{\mu\nu}$ satisfies

$$k^{\mu\nu} k_{\nu\rho} = \delta^{\mu}_{\rho}.$$ 

(8)

For $p \neq 1$, the $k_{\mu\nu}$ field equation implies

$$G_{\mu\nu} + F_{\mu\nu} = \Lambda k_{\nu\mu},$$

(9)

and substituting back into (7) yields the supersymmetric D-brane action in the DBI form (5), with the constants $T_p, T'_p$ related by

$$T'_p = \Lambda^{\frac{1}{2}(p-1)} T_p.$$ 

(10)

For $p = 1$, the action (7) is invariant under the generalised Weyl transformation

$$k_{\mu\nu} \rightarrow \omega(\sigma) k_{\mu\nu},$$

(11)

and the $k_{\mu\nu}$ field equation implies

$$G_{\mu\nu} + F_{\mu\nu} = \frac{1}{2} k^{\rho\sigma} (G_{\rho\sigma} + F_{\rho\sigma}) k_{\mu\nu}.$$ 

(12)

The first term in the action (7) is linear in $(G_{\mu\nu} + F_{\mu\nu})$ and quadratic in $\partial X$, so it is easier to analyse than (5). Moreover, being classically equivalent to (5), the action (7) is supersymmetric and invariant under local kappa symmetry; this can also be checked explicitly.
2. Dual Super D-\(p\)-brane Actions

The dualisation of the action (7) can be achieved by adding a Lagrange term \(\frac{1}{2} \tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_{[\mu}A_{\nu]}))\) imposing the constraint \(F = dA\). Here the antisymmetric tensor density \(\tilde{H}^{\mu\nu}\) is a Lagrange multiplier which can be integrated out to regain the original action (7). Alternatively, integrating out over \(A_{\mu}\) imposes the constraint

\[\partial_{\mu} \tilde{H}^{\mu\nu} = 0,\]  

which can be solved in terms of a \(p - 2\) form \(\tilde{A}\):

\[\tilde{H}^{\mu\nu} = \frac{1}{(p - 1)!} \epsilon^{\mu\nu\rho\gamma_1...\gamma_{p-2}} \partial_{[\rho} \tilde{A}_{\gamma_1...\gamma_{p-2}]},\]  

where \(\epsilon^{\mu\nu\rho...}\) is the alternating tensor density. Now \(F\) is an auxiliary 2-form occurring algebraically; we emphasize this by rewriting \(F \rightarrow L\), so that the action is

\[-\frac{1}{2} T_p' \int d^{p+1}\sigma \left\{ \sqrt{-k} [k^{\mu\nu}(N_{\mu\nu} + L_{\mu\nu}) - (p - 1)\Lambda] + \frac{1}{2} \tilde{H}^{\mu\nu} L_{\mu\nu} \right\} + T_p \int_{W_{p+1}} C e^{L-B}.\]  

The field equation for \(L\) is then

\[\sqrt{-k} k^{\mu\nu} + \frac{1}{2} \tilde{H}^{\mu\nu} + \frac{\delta f(L)}{\delta L_{\mu\nu}} = 0,\]  

where the potential \(f(L) = Ce^{L-B}\) is a polynomial of order \([\frac{1}{2}(p + 1)]\) in \(L\) (i.e. the integer part of \(\frac{1}{2}(p + 1)\)), so that the field equation (16) is of order \([\frac{1}{2}(p + 1) - 1]\), which is at most quartic for \(p < 9\). In particular, it is quadratic for \(p < 5\), so that the dual supersymmetric actions for the corresponding D-branes should be obtainable straightforwardly; this will be illustrated in an example below, and will be discussed further elsewhere. Here we will consider only the cases in which the action (7) is linear in \(F\).

3. The D-Membrane

The dualisation of (the bosonic part of) the D-string action starting from the form (7) was discussed in ref. [18]; here we consider the case of the D-membrane of type IIa superstring theory. For \(p = 2\), (7) reduces to the following form of the D-2-brane action in the string metric,

\[S = -\frac{1}{2} T_2' \int d^3\sigma e^{-\phi} \sqrt{-k} [k^{\mu\nu}(G_{\mu\nu} + F_{\mu\nu}) - \Lambda] + T_2 \int W_3 C^{(3)} + C^{(1)}F.\]  

\[\hat{H}^{\mu\nu} = \frac{1}{(p - 1)!} \epsilon^{\mu\nu\rho\gamma_1...\gamma_{p-2}} \partial_{[\rho} \tilde{A}_{\gamma_1...\gamma_{p-2}]},\]  

where \(\epsilon^{\mu\nu\rho...}\) is the alternating tensor density. Now \(F\) is an auxiliary 2-form occurring algebraically; we emphasize this by rewriting \(F \rightarrow L\), so that the action is

\[-\frac{1}{2} T_p' \int d^{p+1}\sigma \left\{ \sqrt{-k} [k^{\mu\nu}(N_{\mu\nu} + L_{\mu\nu}) - (p - 1)\Lambda] + \frac{1}{2} \tilde{H}^{\mu\nu} L_{\mu\nu} \right\} + T_p \int_{W_{p+1}} C e^{L-B}.\]  

The field equation for \(L\) is then

\[\sqrt{-k} k^{\mu\nu} + \frac{1}{2} \tilde{H}^{\mu\nu} + \frac{\delta f(L)}{\delta L_{\mu\nu}} = 0,\]  

where the potential \(f(L) = Ce^{L-B}\) is a polynomial of order \([\frac{1}{2}(p + 1)]\) in \(L\) (i.e. the integer part of \(\frac{1}{2}(p + 1)\)), so that the field equation (16) is of order \([\frac{1}{2}(p + 1) - 1]\), which is at most quartic for \(p < 9\). In particular, it is quadratic for \(p < 5\), so that the dual supersymmetric actions for the corresponding D-branes should be obtainable straightforwardly; this will be illustrated in an example below, and will be discussed further elsewhere. Here we will consider only the cases in which the action (7) is linear in \(F\).
with $T'_2 = \sqrt{\Lambda} T_2$. Adding a Lagrange multiplier term $\frac{1}{2} \tilde{H}^{\mu \nu}(F_{\mu \nu} - 2 \partial_{[\mu} A_{\nu]})$ to this action and integrating out the gauge field $A$ gives the constraint (13), which is solved in $p + 1 = 3$ dimensions by

$$\tilde{H}^{\mu \nu} = \epsilon^{\mu \nu \rho} \partial_\rho S$$

(18)

for some scalar field $S$. Substituting this back in the action and using a density $\tilde{k}^{\mu \nu} = \sqrt{-\tilde{k}} k^{\mu \nu}$ with $\tilde{k} \equiv \det \tilde{k}^{\mu \nu} = -\sqrt{-k}$ gives

$$S = -\frac{\sqrt{\Lambda}}{2} T_2 \int d^3 \sigma \left\{ e^{-\phi} \left[ \tilde{k}^{\mu \nu} (N_{\mu \nu} + L_{\mu \nu}) + \Lambda \tilde{k} \right] + \epsilon^{\mu \nu \rho} K_\rho L_{\mu \nu} \right\}$$

$$+ T_2 \int_{W_3} C^{(3)} - C^{(1)} B,$$

(19)

where $L$ is an auxiliary two-form and we have defined

$$K_\rho \equiv \frac{1}{2} \partial_\rho S - \frac{1}{\sqrt{\Lambda}} C^{(1)}_\rho.$$

(20)

Integrating out $L$ yields the constraint

$$e^{-\phi} \tilde{k}^{[\mu [\nu]} + \epsilon^{\mu \nu \rho} K_\rho = 0,$$

(21)

so that

$$\tilde{k}^{\mu \nu} = \tilde{g}^{\mu \nu} + \tilde{H}^{\mu \nu}$$

(22)

where $\tilde{H}^{\mu \nu} = -\epsilon^{\mu \nu \rho} e^\phi K_\rho$, and $\tilde{g}^{\mu \nu}$ is the symmetric tensor density defined by $\tilde{g}^{\mu \nu} = \tilde{k}^{(\mu \nu)}$. Then $\det(\tilde{k}^{\mu \nu}) = \det(\tilde{g}^{\mu \nu}) \Omega$, where $\Omega = \det(\delta^\nu_\rho + \tilde{H}_\nu^{\mu \rho})$. Using the identity $\det(1 + X) = 1 - \frac{1}{2} tr X^2$ valid for $n = 3$ and $X$ antisymmetric, we find

$$\Omega = 1 - \frac{1}{2} \tilde{H}_\mu^{\nu} \tilde{H}^\nu_\mu - \frac{1}{2} e^{2\phi} g_{\mu \rho} g_{\nu \sigma} \epsilon^{\rho \sigma \kappa \delta} K_\kappa K_\delta$$

$$= 1 + \frac{1}{2} e^{2\phi} g^{\mu \nu} K_\mu K_\nu,$$

(23)

where $g^{\mu \nu} = \frac{1}{\sqrt{-g}} \tilde{g}^{\mu \nu}$ is a symmetric tensor with inverse $g_{\mu \nu}$, and $g \equiv \det(g_{\mu \nu})$. Using these results, the first term in (19) becomes

$$S = -\frac{\sqrt{\Lambda}}{2} T_2 \int d^3 \sigma e^{-\phi} \sqrt{-g} [g^{\mu \nu} G_{\mu \nu} - \Lambda \Omega] + \frac{\sqrt{\Lambda}}{2} T_2 \int_{W_3} B \wedge K.$$

(24)

The field equation for the metric $g_{\mu \nu}$ implies

$$g_{\mu \nu} = \frac{1}{\Lambda} \left( G_{\mu \nu} - \frac{\Lambda}{2} e^{2\phi} K_\mu K_\nu \right).$$

(25)
Substituting this back in (24) and adding the second term in (19) yields the dual action

\[
S = -T_2 \int d^3 \sigma e^{-\phi} \sqrt{-\det \left[ G_{\mu \nu} - \frac{\Lambda}{8} \epsilon^{2\phi} (\partial_\mu S - \frac{2}{\sqrt{\Lambda}} C^{(1)}_\mu)(\partial_\nu S - \frac{2}{\sqrt{\Lambda}} C^{(1)}_\nu) \right]}
+ T_2 \int W_3 \left( -C^{(3)} + \frac{\sqrt{\Lambda}}{2} B dS \right)
\]

which agrees with that of [10, 13, 15, 16, 17] if we set \( \Lambda = 4 \). The dilaton dependence can be removed by a rescaling of the superspace coordinates, and this yields the standard M theory membrane action with a circular dimension, as expected on the basis of the relationship between type IIa superstring theory and M theory.

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