SISO H-OPTIMAL SYNTHESIS WITH INITIALLY SPECIFIED STRUCTURE OF CONTROL LAW

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Abstract. The paper is devoted to particular cases of H-optimization problems for LTI systems with scalar control and external disturbance. The essence of these problems is to find an output feedback optimal controller having initially given structure to attenuate disturbances action with respect to controlled variable and control. An admissible set of controllers can be additionally restricted by the requirement to assign given poles spectrum of the closed-loop system. Specific features of the posed problems are considered and three simple numerical methods of synthesis are proposed to design correspondent H-optimal controllers. To show the simplicity and effectiveness of the proposed approach and the benefits of developed methods, illustrative examples are enclosed to the paper.

1. Introduction. In recent years, various problems of H-optimal control theory have attracted considerable attention in connection with an evident attempt to suppress external disturbances impact on the controlled output of closed-loop systems. These problems have initiated various investigations in control theory and signal processing since the 1940s. The main objective of the H-optimal synthesis is to find a stabilizing controller minimizing $H_2$ or $H_\infty$ performance indices for the closed-loop system [1], [2], [6], [8].

Two computational approaches are widely used for the practical solution of the aforementioned problems. The first one is based on algebraic matrix Riccati equations ("2-Riccati" approach [1]-[2]), and the second one is connected with linear matrix inequalities ("LMI" technique [4]). Correspondent methods are successfully implemented in MATLAB package.

However, the global structure of obtained optimal controllers may be not so convenient for practical implementation. Because of this drawback, the problems of H-optimization taking into account various controller structure restrictions have become of current importance. Problems in this area have been studied since the late 1980s, and are known to be extremely difficult, often falling into the category of NP-hard problems. To solve them, usually two general ideas are used: the first is

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based on a certain parameterization of admissible controller sets [6], and the other one provides direct optimization on the sets of coefficients, taking into account asymptotic stability requirement [7].

Furthermore, global classical H-optimal controllers have an additional disadvantage: there is no guarantee that the closed-loop system poles are placed on the left-half complex plane in a desirable way. This can worsen transient processes in comparison with no optimal controllers. Such situations are discussed in detail in [9] and in the references therein.

Let us notice that to bypass the aforementioned obstacles, constructive methods of synthesis were proposed in [5], [7], [9]. Being similar to the classical H-optimization, these approaches are based on 2-Riccati and LMI techniques or allow reduction of synthesis to finite-dimensional constrained optimization problems.

Nevertheless, in our opinion, there exists a possibility of essentially raising an efficiency of synthesis for a partial situation of SISO systems with a controlled plant, having scalar control and disturbance inputs. In this case, it is more suitable to use spectral methods in frequency domain if the order of systems is not so large. The theoretical framework for efficient spectral approach is proposed in publications [3], [10]–[12].

It is obvious that the effective numerical implementation of the algorithms for optimal synthesis is a highly crucial issue for control systems with an adoptive tuning in real-time operating regime. If we implement these algorithms in stationary laboratories, the time length of calculations is not so significant, but we cannot say the same with respect to embedded systems or onboard control systems of autonomous moving robots.

In connection with the previously mentioned circumstances, this paper, which partly expands and enhances the result presented in the conference publication [12], is devoted to the problem of SISO LTI H-optimal synthesis with initially specified structure of a controller and with initially given pole placement in the complex plane. The proposed methods are in the range of the general spectral approach ideas discussed in [3], [10], [11]. All the basic positions of the accepted approach are presented by the theorems with the correspondent proofs.

The paper is organized as follows. In Section 2, equations of a controlled plant are presented and the problems of H-optimal synthesis are posed, taking into account initially specified controller structure. Section 3 is devoted to direct search of the optimal controller based on the numerical parameterization of the controller admissible set. In Section 4, we discuss the problem with the additional restriction of an admissible set to assign desirable pole placement for the closed-loop connection. Section 5 includes resulting numerical algorithms of synthesis that reflect the theoretical framework obtained above. Section 6 presents illustrative examples to show the simplicity and effectiveness of the proposed approaches. Finally, Section 7 concludes this paper by discussing the overall results of the investigation.

2. Problem statement. Here we consider a problem of feedback control laws synthesis for LTI plant with mathematical model of the state space form

\[
\dot{x} = Ax + bu + pd(t),
\]

\[
y = cx,
\]

\[
e_1 = y, \quad e_2 = cu,
\]

(1)

where \(x \in \mathbb{R}^n\) is the state vector, \(y, u, \xi, d\) are the scalar values: \(y\) is the measured variable, \(u\) is the control, \(d(t)\) represents external disturbances, acting to the system.
Variables $e_1$ and $e_2$ are the components of the controlled vector $e = \begin{pmatrix} e_1 & e_2 \end{pmatrix}^T$. All components of the matrices $A$, $b$, $p$, $c$, and parameter $c_0$ are given constants.

Let us suppose that the pairs $\{A, b\}$, $\{A, c\}$ are controllable and observable respectively.

We shall treat the external input $d = d(t)$ below as the stationary ergodic random process with given rational spectral power density of the form

$$S_d(\omega) = S_1(s)S_1(-s)|_{s=j\omega}, S_1(s) = N(s)/T(s),$$

where $N$ and $T$ are Hurwitz polynomials with degrees $p_s$ and $q_s$ correspondingly, $p_s \leq q_s$, $T(s) = (s - \tau_1)(s - \tau_2)\ldots(s - \tau_{q_s})$.

It is necessary to design a controller with tf-model of the following form

$$u = W(s)y,$$

where $W(s) \equiv W_1(s)/W_2(s)$; $W_1$, $W_2$ are polynomials. We are going to find a transfer function $W$ of the controller $[3]$ as a solution of the synthesis problem. If this function is determined by any way, we obtain a closed-loop connection $[1]$, $[3]$ with the equation

$$e = H(s, W)d.$$

Here we have $H(s, W) \equiv \begin{pmatrix} H_y(s, W) & c_0H_u(s, W) \end{pmatrix}^T$, where $H_y$ and $H_u$ are transfer functions of the closed-loop system from $d$ to $y$ and to $u$ respectively:

$$H_y(s, W) = W_2C_d/\Delta, \quad H_u(s, W) = W_1C_d/\Delta;$$

$$\Delta(s) = A(s)W_2(s) - B(s)W_1(s),$$

i.e. $\Delta$ is the characteristic polynomial of this system, where

$$A(s) = \text{det}(Es - A), \quad B(s) = A(s)c(ES - A)^{-1}b, \quad C_d(s) = A(s)c(ES - A)^{-1}p.$$  

In accordance with $[4] - [7]$, it is convenient to consider a scalar generalized transfer function $H(s, W)$ such that $H(s, W)H(-s, W) = H^T(-s, W)H(s, W)$ or

$$H(s)H(-s) = H_y(s)H_y(-s) + c_0^2H_u(s)H_u(-s),$$

where direct dependency from $W$ is omitted.

The controller $[3]$ is aimed to suppress an external disturbance $d(t)$ with respect to output vector $e$. Using a function $H$, we can introduce the following performance indices for the closed-loop system:

$$J_2(W) = \|H(s, W)S_1(s)\|_2^2 = \|H(s, W)S_1(s)\|_2^2,$$

$$J_\infty(W) = \|H(s, W)S_1(s)\|_\infty^2 = \|H(s, W)S_1(s)\|_\infty^2.$$  

The functionals $J_2$ and $J_\infty$ are given on the admissible sets $RH_2$ and $RH_\infty$ correspondently $[3]$. The less a value of these functionals is then the controller $[3]$ provides the better suppression of an external disturbance.

Therefore, it is quite suitable to set up the following optimization problems:

$$J_2(W) = \|H(s, W)S_1(s)\|_2^2 \rightarrow \inf_{W \in \Omega_{a2} \subset \Omega_2},$$

$$J_\infty(W) = \|H(s, W)S_1(s)\|_\infty^2 \rightarrow \inf_{W \in \Omega_{a\infty} \subset \Omega_\infty},$$

$$\Omega_2 = \{ W : H(s, W) \in RH_2 \}, \quad \Omega_\infty = \{ W : H(s, W) \in RH_\infty \}.$$  

Here $\Omega_{a2}$ and $\Omega_{a\infty}$ are admissible sets of controllers: remark that if $\Omega_{a2} = \Omega_2$ and $\Omega_{a\infty} = \Omega_\infty$, we have the classical situation of $H$-theory with some specialities,
which are considered in [3], [10] in details. For these cases, the infima in (11), (12) can be achieved.

As for the restrictions \( \Omega_a^2 \) and \( \Omega_a^\infty \) of the controllers (3) sets, they can be determined by any additional requirements with respect to closed-loop stability. It is evident that theirs introducing essentially sophisticate optimization problems (11), (12).

Here we shall consider a commonly used situation, when the mentioned restrictions are determined by given structural requirements to controllers.

**Definition 1.** We say that the controller (3) have a given structure \( \mathcal{R}\{\mu, \nu\} \), using notation \( W \in \mathcal{R}\{\mu, \nu\} \), if the following equalities hold

\[
\deg W_1(s) = \mu, \quad \deg W_2(s) = \nu.
\]

**Definition 2.** The sets of the controllers (3) with transfer functions \( W \in \Omega_2 \cap \mathcal{R} \) or \( W \in \Omega_\infty \cap \mathcal{R} \) are called admissible if correspondent intersections are not empty.

In the residual sections, we shall concentrate our discussion on two kinds of the optimization problems (11) and (12). The first of them is determined by immediate vector parameterization of the mentioned admissible sets. The second one assumes additional restrictions of these sets determined by the assignment of desirable poles spectrum for the closed-loop connection.

**3. Direct approach to optimization.** The obvious way to decide the problems posed above is using a numerical parameterization for the admissible controller sets. Because of the controllers structure is initially given, one can easy make such a parameterization considering any vector \( h \in \mathbb{R}^p \) of adjustable parameters for the function \( W = W(s, h) = W_1(s, h)/W_2(s, h) \).

In particular, the vector \( h \) can be composed putting together all coefficients of the polynomials \( W_1 \) and \( W_2 \). For this situation we have \( p = \nu + \mu + 1 \), accepting that the coefficient for \( s^\nu \) is equal to one. Nevertheless, in general case the nature of the vector \( h \) could be arbitrary.

**Definition 3.** We shall say that the parameterization is complete if the structure \( \mathcal{R}\{\mu, \nu\} \) and the composition of adjustable parameters are such that it is possible to assign arbitrary poles spectrum for a closed-loop system by the correspondent choice of the vector \( h \in \mathbb{R}^p \). Otherwise, we shall say that the parameterization is incomplete.

If a parameterization is done by any way, we can transform the mentioned problems to the equivalent finite-dimensional representation for \( i = 2, \infty \):

\[
J_i(h) \rightarrow \inf_{h \in \Omega_{hi}} \Omega_{hi} = \{h \in \mathbb{R}^p : W(s, h) \in \Omega_{ai}\}.
\]

It is quite suitable to distinguish two different situations with respect to a solution of the problems (14) subject to the completeness of admissible sets parameterization.

A) Let us firstly consider a complete variant of the vector parameterization. For this case, the existence of the solution is guarantied. This allows us to simplify the situation transforming it to the equivalent unconditional extremum problem. We shall use below a concept of stability degree for any polynomial as a distance from its roots to the imaginary axis on the complex plane.
Lemma 1. For any vector \( \gamma \in R^{n_d} \) a stability degree of the following polynomial
\[
\Delta^*(s, \gamma) = \begin{cases} 
\tilde{\Delta}^*(s, \gamma), & \text{if } n_d \text{ is even}; \\
(s + a_{n_d+1}(\gamma, \alpha))\tilde{\Delta}^*(s, \gamma), & \text{if } n_d \text{ is odd},
\end{cases}
\] (15)
is not less than the initially given real constant \( \tilde{\alpha} > 0 \), and conversely, if a stability degree of some polynomial \( \Delta(s) \) is not less than the constant \( \tilde{\alpha} > 0 \), then there exists the vector \( \gamma \in R^{n_d} \) such that the identity \( \Delta(s) \equiv \Delta^*(s, \gamma) \) holds, where
\[
\tilde{\Delta}^*(s, \gamma) = \prod_{i=1}^{n_q} (s^2 + a_1^i(\gamma, \tilde{\alpha})s + a_0^i(\gamma, \tilde{\alpha})),
\] (16)
\[a_1^i(\gamma, \tilde{\alpha}) = 2\tilde{\alpha} + \gamma_{11}^2, a_0^i(\gamma, \tilde{\alpha}) = \tilde{\alpha}^2 + \gamma_{12}^2 + \gamma_{22}^2, i = 1, n_q, a_{n_q+1}(\gamma, \tilde{\alpha}) = \gamma_{0}^2 + \tilde{\alpha}, \gamma = \{\gamma_{11}, \gamma_{12}, \gamma_{22}, \ldots\}, n_q = \lfloor n_d/2 \rfloor.\]

Proof. A validation of this claim directly follows from the elementary features of quadratic trinomials in the formulae (15), (16).

Let implement Lemma 1 to construct a computational method of the problems (1) solution. Remark preliminary that using this claim we can additionally restrict admissible sets, introducing the following constraint: a degree of stability for closed-loop system must be not less than the given value \( \tilde{\alpha} \).

To this end, let give any vector \( \gamma \in R^{n_d} \) and compute the auxiliary polynomial \( \Delta^*(s, \gamma) \) by the formulae (15), (16). Next, let determine the vector-parameter \( h \in R^n \) of the controller (3), providing the identity
\[
\Delta_3(s, h) \equiv \Delta^*(s, \gamma),
\] (17)
\[\Delta_3(s, h) = A(s)W_2(s, h) - B(s)W_1(s, h), \text{ (deg } \Delta_3 = n_d) \text{ is the characteristic polynomial of the closed-loop system. Applying the method of indeterminate coefficients to this identity, we obtain the following system of nonlinear equations}
\]
\[Q(h) = \lambda(\gamma)
\] (18)
with respect to the unknown components of the vector \( h \). Remark that this system is consistent for any \( \gamma \in R^{n_d} \) due to the completeness of a parameterization. In general case, if the system (18) has no unique solution, we can represent the vector \( h \) as the aggregate \( h = \{ h, h_c \} \). Here \( h_c \in R^{n_c} \) is a free components vector, which can be given arbitrarily, and \( h \) is a vector, which must be uniquely determined as a solution of the system (18) for a given \( h_c \).

Let introduce the following notation for the common solution of the nonlinear system (18):
\[h = h^* = \{ h^*(h_c, \gamma), h_c \} = h^*(\gamma, h_c) = h^*(\varepsilon),\] (19)
where by \( \varepsilon = \{ \gamma, h_c \} \) we denote an arbitrary vector of independent parameters with the dimension
\[\lambda = \dim \varepsilon = \dim \gamma + \dim h_c = n_d + n_c.\] (20)

Using controller (3) with the obtained transfer function \( W(s, h^*) \), we can compute the value of the functional \( J_1(h^*) = J_1(h^*(\varepsilon)) = I(\varepsilon) \) to be minimized. In this connection, the following claim is valid:

Theorem 1. If the optimization problems (14) have a solution \( h_0 \in \Omega_{h_i} \), providing desirable stability degree \( \tilde{\alpha} \), then there exists a point \( \varepsilon_0 \in R^\lambda \) such that
\[h_0 = h^*(\varepsilon_0), \quad \varepsilon_0 = \arg \inf_{\varepsilon \in R^\lambda} I(\varepsilon),\] (21)
Inversely, if a point \( \varepsilon_0 \in \mathbb{R}^r \) exists, satisfying (21), then the vector \( \mathbf{h}_0 = \mathbf{h}^*(\varepsilon_0) \) is a solution of the correspondent problem (14). By other words, the problems (14) are equivalent in specified sense to the following unconditional extremum problem

\[
I(\varepsilon) \rightarrow \inf_{\varepsilon \in \mathbb{R}^r} .
\]  

(22)

**Proof.** Let suppose that the following equalities hold:

\[
\mathbf{h}_0 = \arg \inf_{\mathbf{h} \in \Omega_{h_i}} J_i(\mathbf{h}), \quad J_{i0} = J_i(\mathbf{h}_0).
\]  

(23)

Correspondent closed-loop system has the characteristic polynomial \( \Delta_3(s, \mathbf{h}_0) \) with a stability degree not less than \( \bar{\alpha} \). Hence, in accordance with Lemma 1, there exists a point \( \gamma = \gamma_0 \in \mathbb{R}^{nd} \) such that \( \Delta_3(s, \mathbf{h}_0) = \Delta^*(s, \gamma_0) \), where \( \Delta^* \) is polynomial, determined by the formulae (15), (16). This means that there exists a point \( \varepsilon_0 = \{\gamma_0, \mathbf{h}_0\} \in \mathbb{R}^r \) such that \( \mathbf{h}_0 = \mathbf{h}^*(\varepsilon_0), I(\varepsilon_0) = J_{i0} \), where \( \mathbf{h}_0 \) is correspondent free component of the vector \( \mathbf{h}_0 \).

It remains to show that there no exists a point \( \varepsilon_{01} \in \mathbb{R}^r \) such that \( I(\varepsilon_{01}) < J_{i0} \). Really, let us suppose reciprocal, i.e. \( J_i(\mathbf{h}^*(\varepsilon_{01})) = I(\varepsilon_{01}) = J_{i0} = J_i(\mathbf{h}_0) \) for the point \( \mathbf{h}^*(\varepsilon_{01}) \). However, this is impossible because of (23). The inverse claim can be proven similarly. \( \square \)

B) Next, let us suppose that parameterization is incomplete. Note that despite of the admissible sets are nonempty, here we cannot directly apply a preceding approach because of the system (18) does not have a solution for any \( \gamma \in \mathbb{R}^{nd} \), i.e. for any poles spectrum. Nevertheless, let us consider one particular situation, when vector \( \mathbf{h} \) consists of the coefficients of the transfer function \( W(s, \mathbf{h}) \). It is a matter of simple calculation to verify that the system (18) now is linear of the form

\[
\mathbf{Gh} = \chi(\gamma),
\]  

(24)

where the matrix \( \mathbf{G} \) with dimension \( r \times p \) has constant components, \( \text{rank} \ \mathbf{G} = r \).

Using pseudo inversion \( \mathbf{G}^+ = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \), obtain the best approximation of the solution in the form

\[
\mathbf{h}^* = \mathbf{h}^*(\gamma) = \mathbf{G}^+ \chi(\gamma).
\]  

(25)

For the controller (3) with the obtained transfer function \( W(s, \mathbf{h}^*) \) we can compute the value of the functional \( J_i(\mathbf{h}^*) = J_i(\mathbf{h}^*(\gamma)) = I(\gamma) \) to be minimized, and the value \( f(\gamma) = \|\chi(\gamma) - \mathbf{G}^+ \chi(\gamma)\| \) of the misalignment for the system (24). It is obvious that the following statement is next to Theorem 1:

**Corollary 1.** If the optimization problems (14) have a solution \( \mathbf{h}_0 \in \Omega_{h_i} \), providing desirable stability degree, then there exists a point \( \gamma_0 \in \mathbb{R}^{nd} \) such that

\[
\mathbf{h}_0 = \mathbf{h}^*(\gamma_0), \quad \gamma_0 = \arg \inf_{\gamma \in \mathbb{R}^{nd}: f(\gamma)=0} I(\gamma).
\]  

(26)

Inversely: if a point \( \gamma_0 \in \mathbb{E}^{nd} \) exists, satisfying (26), then the vector \( \mathbf{h}_0 = \mathbf{h}^*(\gamma_0) \) is a solution of the correspondent problem (14).

**Proof.** This statement directly follows from Theorem 1. \( \square \)

By other words, the problems (14) are equivalent to (26) which, in turn, can be reduced to the unconditional extremum problem of a form

\[
J(\gamma) = I(\gamma) + \rho^2 f(\gamma) \rightarrow \inf_{\gamma \in \mathbb{R}^{nd}},
\]  

(27)

where \( \rho \) is a real weight multiplier.
4. **Synthesis with poles assignment.** Let us consider a particular case of the optimization problem (11) with the functional $J_2(W)$, introducing an admissible set $\Omega_{a2}$ of the following form:

$$\Omega_{a2} = \{W \in \mathbb{R}\{\mu, \nu\} : A(s)W_2(s) - B(s)W_1(s) \equiv Q(s)\}. \quad (28)$$

Here $Q(s) = (s - \lambda_1) (s - \lambda_2) \ldots (s - \lambda_k)$ is given desirable characteristic polynomial of the closed-loop system with the roots $\lambda_i, i \in \{1, k\}$ which belong to the initially given spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $k = \max\{n + \nu, m + \mu\}$.

By other words, this section is devoted to $H_\infty$-optimization problem (11), where the set $\Omega_{a2}$ of the controllers (3) with specified structure $\mathcal{R}$ is restricted by the requirement to provide the poles spectrum $\Lambda$ for the closed-loop connection (1), (3). One should bear in mind, however, that a correspondent restriction may be found empty or consisting from the only element. To avoid this trivial situation, let turn to the following statement:

**Lemma 2.** For the existence of infinitely many controllers (3) in the range of the structure $\mathcal{R}\{\mu, \nu\}$, providing arbitrary desirable spectrum $\Lambda$ of the closed-loop system poles, it is necessary and sufficient that the following inequalities hold

$$\mu > n - 1, \ \nu > m - 1. \quad (29)$$

*Proof.* Consider the polynomial equation $A(s)W_2(s) - B(s)W_1(s) \equiv Q(s)$ with respect to the unknown polynomials $W_1$ and $W_2$. This one can be transformed to the equivalent linear system, which consists of $k$ linear algebraic equations with respect to $P(s) = \nu + \mu + 1$ coefficients of these polynomials. A solution of this system exists for any $Q$ if and only if $p \geq k$ or $\nu + \mu + 1 \geq \max\{n + \nu, m + \mu\}$, from where it directly follows that $\mu \geq n - 1, \ \nu \geq m - 1$. However, the equalities here must be excluded, because they lead to the unique solution. \hfill \Box

Let us suppose that for the given structure $\mathcal{R}\{\mu, \nu\}$ the condition (29) is fulfilled, i.e. the problem (11) has nontrivial solution. To obtain the method of its constructive search, it is convenient to implement the polynomial parameterization of the admissible set $\Omega_{a2}$ on the base of the following claim:

**Theorem 2.** For the controller (3) with the structure $\mathcal{R}\{\mu, \nu\}$ provides given desirable spectrum $\Lambda$ of closed-loop system poles, it is necessary and sufficient that the polynomials $W_1$ and $W_2$ satisfy the identities

$$W_1(s) \equiv \alpha(s) - A(s)P(s), \quad W_2(s) \equiv \beta(s) - B(s)P(s). \quad (30)$$

Here $W_\alpha(s) = \alpha(s)/\beta(s) \in \Omega_{a2}$ is the transfer function of any controller (3) with the mentioned properties, $P(s)$ is any polynomial such that

$$r = \deg P(s) = \min\{\mu - n, \nu - m\}. \quad (31)$$

*Proof.* Sufficiency: In accordance with hypothesis of a theorem, we have the identity

$$A(s)\beta(s) - B(s)\alpha(s) \equiv Q(s). \quad (32)$$

Given an arbitrary polynomial $P(s)$ with the degree (31), we can compute polynomials $W_1$ and $W_2$ by the formulae (30). One can directly see that for this case $W = W_1/W_2 \in \mathcal{R}$ and, furthermore, $W \in \Omega_{a2}$ because of we have characteristic polynomial

$$\Delta(s) = A(\beta - BP) - B(\alpha - AP) \equiv Q.$$
Necessity: Let there exists any controller with a transfer function satisfying the identity
\[ \Delta(s) = A(s)W_2(s) - B(s)W_1(s) \equiv Q(s). \]

After deducting (32) we obtain \( AP_2 - BP_1 \equiv 0 \), where \( P_1 \equiv W_1 - \alpha, P_2 \equiv W_2 - \beta \), therefore \( P_2 \equiv BP, P_1 \equiv AP \). Here \( P \) is any polynomial with the degree (31), i.e. the representation (30) is valid that proves this theorem.

Thus, Theorem 2 determines one-to-one correspondence between the set \( \Omega_{a2} \) of the admissible transfer functions \( W(s) \) and the set \( \Omega_P \) of polynomials \( P(s) \) with the degree (31). Such a correspondence allows claiming that the problem (11) for this case is equivalent to the following essentially simpler optimization problem
\[ J_2(P) = J_2(W(P)) = \| H(s, W(P))S_1(s) \|_2^2 \rightarrow \min_{P \in \Omega_P}. \] (33)

To decide this problem, let us preliminary present the functional \( J_2 \) in explicit dependency from the polynomial-parameter \( P(s) \). On the base of the formulae (5) we obtain
\[ J_2(P) = \| HS_1 \|_2^2 = \frac{1}{2\pi j} \int_{-\infty}^{\infty} (H_yH_y + c_0^2H_yH_u) S_ds = \]
\[ \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{1}{Q} \left[ (\beta\beta + c_0^2\alpha\bar{\alpha} + (BB + c_0^2AA) PP - \\
- (B\beta + c_0^2A\bar{\alpha}) P - (B\beta + c_0^2\bar{\bar{A}}\alpha) \right] \frac{C_0C_{\varepsilon}C_{\varepsilon}}{QTT} ds, \] (34)
where a bar under letter denotes changing a sign of the argument “s”.

Let us introduce additional notations
\[ Q_T(s) = Q(s)T(s) = (s - e_1)(s - e_2) \ldots (s - e_\varepsilon), \]
\[ \text{deg}(Q_T) = \varepsilon, e_i = \lambda_i, i = 1, ..., e_i = \tau_i, i = k + 1, \varepsilon. \] (35)
Remark that in accordance with the finiteness \( J(P) \) for any \( P \in \Omega_P \) the improper integral
\[ \frac{1}{2\pi j} \int_{-\infty}^{\infty} (BB + c_0^2A\bar{\bar{A}}) \frac{PC_dC_dN\bar{N}}{QTT} ds \]
converges that is the relationship
\[ \text{deg}[(BB + c_0^2A\bar{\bar{A}}) P\bar{C}_dC_dN\bar{N}] \leq \text{deg} QTT - 2 \]
holds or, equivalently,
\[ r + 1 < k + q_s = \varepsilon. \] (36)

The main statement, which determines a method of synthesis with poles assignment, is as follows:

**Theorem 3.** The following polynomial
\[ P_0(s) = h_r s^r + h_{r-1} s^{r-1} + \ldots + h_1 s + h_0 \]
is the solution of the problem (33) if the vector \( h \) of its coefficients is determined by the equality
\[ h = \left( h_r \ h_{r-1} \ldots h_1 \ h_0 \right)^T = M^{-1}m. \] (37)
Here the matrix $\mathbf{M}$ consists of the first $r + 1$ rows of the $\varepsilon \times (r + 1)$ matrix $\mathbf{M}^*$ and vector $\mathbf{m}$ consists of the first $r + 1$ components of the vector $\mathbf{m}^* \in \mathbb{R}^\varepsilon$, where

$$\mathbf{M}^* = \begin{pmatrix} \mathbf{M} \\ \mathbf{M}_1 \end{pmatrix} = \sum_{i=1}^\varepsilon \mathbf{q}_i \mathbf{g}_i, \quad \mathbf{m}^* = \begin{pmatrix} \mathbf{m} \\ \mathbf{m}_1 \end{pmatrix} = \sum_{i=1}^\varepsilon \mathbf{q}_i \xi_i. \quad (38)$$

Formulae (32) include the vectors

$$\mathbf{q}_i = \begin{pmatrix} 1 & q_{i-2} & \ldots & q_i & q_0^i \end{pmatrix}^T \in \mathbb{R}^\varepsilon (i = \overline{1, \varepsilon}),$$

presenting the coefficients of the polynomials

$$Q_i(s) = \frac{Q_T(s)}{s - e_i} = s^{\varepsilon - 1} + q_{i-2}^i s^{\varepsilon - 2} + \ldots q_1^i s + q_0^i. \quad (39)$$

Formulae (32) include also the additional vectors

$$\mathbf{g}_i = \gamma_i \begin{pmatrix} e_i^r & e_i^{r-1} & \ldots & e_i & 1 \end{pmatrix} \in \mathbb{R}^{r+1} (i = \overline{1, \varepsilon}),$$

where $G_d = B\bar{B} + c_0^2 \bar{A}\bar{A}$, $F_d = \bar{B}\beta + c_0^2 \bar{A}\alpha$;

$$\gamma_i = \frac{G_a(e_i)N(-e_i)C(e_i)C_d(-e_i)}{Q_T(-e_i)Q_T'(e_i)} \quad (41),$$

$$\xi_i = \frac{F_d(e_i)N(-e_i)C(e_i)C_d(-e_i)}{Q_T(-e_i)Q_T'(e_i)} \quad (42),$$

are additional complex-value coefficients.

**Proof.** It is a matter of simple calculation to verify that the functional (34) is strongly differentiable, i.e. for any polynomial $P \in \Omega_P$ this one has Frechet differential equal to its first variation

$$\delta J(P) = \delta_1 J(P) + \delta_2 J(P), \quad (43)$$

$$\delta_1 J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left( \frac{B\bar{B} + c_0^2 \bar{A}\bar{A}}{QQ} - \frac{\bar{B}\beta + c_0^2 \bar{A}\alpha}{QQ} \right) C_d \bar{C}_d S_d \delta \bar{P} ds,$$

$$\delta_2 J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left( \frac{B\bar{B} + c_0^2 \bar{A}\bar{A}}{QQ} - \frac{\bar{B}\beta + c_0^2 \bar{A}\alpha}{QQ} \right) C_d \bar{C}_d S_d \delta \bar{P} ds.$$

In this case the polynomial $P_0(s)$ is optimal if and only if $\delta J(P_0) = 0 \quad \forall \delta P \in \Omega_P$ that for $P = P_0$ is equivalent to

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left( \ldots \right) C_d \bar{C}_d S_d \delta \bar{P} ds = 0 \quad \forall \delta \bar{P} \in \Omega_P. \quad (44)$$

The obtained equality (44) we can treat as an equation with respect to polynomial $P(s)$. To decide this equation, let us transform its integrand making the separations of the following strictly proper fractions:

$$\frac{C_d \bar{C}_d N\bar{N} \left( B\bar{B} + c_0^2 \bar{A}\bar{A} \right) P}{QQT} = \frac{G_1(s, P)}{Q_T(s)} + \frac{G_2(s, P)}{Q_T(-s)}, \quad (45)$$

$$\frac{C_d \bar{C}_d N\bar{N} \left( \bar{B}\beta + c_0^2 \bar{A}\alpha \right)}{QQT} = \frac{F_1(s)}{Q_T(s)} + \frac{F_2(s)}{Q_T(-s)}. \quad (46)$$
The polynomials $G_1$ and $F_1$ here can be presented with the help of Lagrange interpolation formulae as

$$G_1(s, P) = \sum_{i=1}^{\varepsilon} \frac{Q_T(s)}{s - e_i} \gamma_i P(e_i), \quad F_1(s) = \sum_{i=1}^{\varepsilon} \frac{Q_T(s)}{s - e_i} \xi_i,$$  \hspace{1cm} (47)

with the coefficients $\gamma_i$ [41] and $\xi_i$ [42]. Then we can rewrite the equation (44) as follows:

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{K(P)}{Q_T} \delta Pds + \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{L(P)}{Q_T} \delta Pds = 0,$$  \hspace{1cm} (48)

where

$$K(s, P) = G_1 - F_1 = \sum_{i=1}^{\varepsilon} \frac{Q_T(s)}{s - e_i} [\gamma_i P(e_i) - \xi_i].$$  \hspace{1cm} (49)

It is easy to see that if the first term in (48) is equal to zero then the second one also is and vice versa. This allows presenting the equation with respect to polynomial $P(s)$ in the following form:

$$I_1 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{K(s, P)}{Q_T(s)} \delta P(-s)ds = 0.$$  \hspace{1cm} (50)

Here the expression $\rho(s) = K(P)\delta P/Q_T$ is a rational fraction with Hurwitz denominator. Then the equality (50) holds if and only if a degree of its numerator is less or equal to a degree of denominator minus two: $\deg K + r \leq \varepsilon - 2 \Leftrightarrow \deg K \leq \varepsilon - 1 - (r + 1)$, i.e.

$$\deg K < \varepsilon - (r + 1).$$  \hspace{1cm} (51)

Nevertheless, in general case $\deg K = \varepsilon - 1$, and from (51) we have $r + 1 < \varepsilon$. Then from (51) it follows that the solution of (50) is a polynomial $P = P_0(s)$ such that $r + 1$ high-order coefficients of the polynomial $K(s, P)$ are equal to zero. Let realize this requirement for the polynomial $P(s) = (s^r - s^{r-1} \ldots s 1)^T \in R^{r+1}$ is the vector of its coefficients. Then, using (38) – (42), we obtain from (49)

$$K(s, P) = \sum_{i=1}^{\varepsilon} \left( \begin{array}{cccc} s^{r-1} & s^{r-2} & \ldots & s 1 \end{array} \right) q_i(g, h - \xi_i) \equiv \left( \begin{array}{cccc} s^{r-1} & s^{r-2} & \ldots & s 1 \end{array} \right) (M^*h - m^*).$$

It follows from this presentation that the equality of $r + 1$ high-order coefficients to zero is equivalent to (37), and theorem is proven.

**Corollary 2.** Controller (3) with the transfer function $W = W_0(s) \equiv W_{01}(s)/W_{02}(s)$ is a solution of the problem (11) for the admissible set $\Omega_{02}$ if the polynomials $W_{01}$, $W_{02}$ satisfy the identities of the form

$$W_{01}(s) \equiv \alpha(s) - A(s)P_0(s),$$  
$$W_{02}(s) \equiv \beta(s) - B(s)P_0(s)$$  \hspace{1cm} (52)

for the optimal polynomial $P_0(s)$, presented by the formulae (37) – (42).

**Proof.** Validity of the claim immediately follows from the equivalency of the problems (11) and (33).
5. **Numerical algorithms.** Theoretical background presented above allows us to summarize the results of consideration as the following algorithms for numerical calculations.

**Algorithm 1.** To decide the problems (14) with complete parameterization taking into account desirable stability degree $\tilde{\alpha}$ we need to execute the following steps:

1. Give a point $\gamma \in R^{n_{\delta}}$ and construct desirable polynomial $\Delta^*(s, \gamma)$ via the formulae (15), (16).

2. In accordance with the identity $\Delta_3(s, h) \equiv \Delta^*(s, \gamma)$ form the system (17) of nonlinear equations, which is always consistent, and give the vector $h_c \in R^{n_{\delta}}$ of free components if any.

3. Decide the system (17) for the given vector $\varepsilon = \{\gamma, h_c\} \in R^\lambda$, obtaining the point $h^*(\varepsilon)$.

4. Calculate the value of the function $I(\varepsilon) = J_1(h^*(\varepsilon))$ for the closed-loop system (11), (12) with the transfer function $W = W(s, h^*(\varepsilon))$ of a controller.

5. Using any convenient numerical descend method, give new point $\varepsilon = \{\gamma, h_c\} \in R^\lambda$ and repeat calculations of the steps 1 – 4, minimizing the function $I(\varepsilon) = J_1(h^*(\varepsilon))$.

6. If the point $\varepsilon_0 = \arg \min_{\varepsilon \in R^\lambda} I(\varepsilon)$ is found, take the vector $h_0 = h^*(\varepsilon_0)$ as a solution of the problem (14), and determine optimal transfer function $W = W_0(s) = W(s, h_0)$ as a solution of correspondent problem (11) or (12).

7. If the value $I_0 = \inf_{\varepsilon \in R^\lambda} I(\varepsilon)$ cannot be achieved in the point $\varepsilon_0 \in R^\lambda$ with finite norm, finish computational process in any pertinent point $\bar{\varepsilon}_0$ from the minimizing sequence $\{\varepsilon_i\}$, which is generated by the descend method, and accept $\varepsilon_0 = \bar{\varepsilon}_0$ as a required solution, consequently compute $h_0 = h^*(\varepsilon_0)$ and construct transfer function $W = W_0(s) = W(s, h_0)$.

**Algorithm 2.** The following numerical scheme allows to solve the problems (14) with incomplete parameterization for given stability degree $\tilde{\alpha}$ of the closed-loop connection:

1. Give a point $\gamma \in R^{n_{\delta}}$ and construct desirable polynomial $\Delta^*(s, \gamma)$ via the formulae (15), (16).

2. In accordance with the identity $\Delta_3(s, h) \equiv \Delta^*(s, \gamma)$ form the system (24) of linear algebraic equations with respect to the vector $h \in R^{p}$ of the adjustable parameters.

3. For the system (24) find the best approximation $h^* = h^*(\gamma) = (G^T\Gamma)^{-1}G^T\chi(\gamma)$ of the solution.

4. Calculate the value $J_1(h^*) = J_1(h^*(\gamma)) = I(\gamma)$ of the functional, and the value $f(\gamma) = \|\chi(\gamma) - \Gamma^T\chi(\gamma)\|$ of the misalignment for the system (24). For the given weight multiplier $\rho$ calculate the function $J(\gamma)$ (27).

5. Using any convenient numerical descend method, give new point $\gamma \in R^p$ and repeat calculations of the steps 1 – 4, minimizing the function $J(\gamma)$.

6. If desirable stability degree is achieved, decrease the multiplier $\rho$, otherwise increase this one, and repeat the steps 1–5.

7. The point $\gamma_0 = \arg \min_{\gamma \in R^{n_{\delta}}} J(\gamma)$ obtained for the minimal $\rho$ accept as a solution of the problem (27). Correspondingly, take the vector $h_0 = (G^T\Gamma)^{-1}G^T\chi(\gamma_0)$ as a solution of the problem (14) and determine correspond transfer function $W = W_0(s) = W(s, h_0)$ as a solution of correspondent problem (11) or (12).
Algorithm 3. To decide the problems \cite{11} on the admissible set \cite{28} with given structure \( \Re\{\mu, \nu\} \) and desirable poles spectrum \( \Lambda \) we need to carry out the following steps:

1. Check the validity of the conditions \( \mu > n - 1, \nu > m - 1 \) and determine the number \( k = \max\{n + \nu, m + \mu\} \) of the poles to be assigned and the degree \( r = \min\{\mu - n, \nu - m\} \) of the polynomial parameter \( P(s) \).

2. Give desirable spectrum \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) of the roots for characteristic polynomial, which must be distinct from the roots \( \tau_i (i = 1, q_s) \) of the polynomial \( T(s) \). Form the auxiliary polynomial \( Q_T(s) \) of degree \( \varepsilon = k + q_s \) by the formulae \cite{35}.

3. Design the reference controller \cite{3} with the transfer function

\[
W_0(s) = \alpha(s)/\beta(s) \in \Re\{\mu, \nu\},
\]

satisfying \( \eqref{32} \).

4. Calculate the vectors \( q_i = \left( 1 \ q_i^{r-2} \ldots \ q_i q_i^0 \right)^T \in \Re^c \ (i = 1, \varepsilon) \) such that the identities \( \eqref{33} \) hold; the vectors \( g_i \) with the help of the formulae \( \eqref{40}, \eqref{41} \), and the complex numbers \( \xi_i \) \cite{42}.

5. Calculate matrix \( M^* \) and vector \( m^* \) \cite{38} and extract from them the upper blocks \( M \) and \( m \) with dimensions \( (r + 1) \times (r + 1), (r + 1) \times 1 \) correspondently.

6. Find the vector \( h_0 = M^{-1}m \) of coefficients for the optimal parameter \( P_0(s) = \left( s^r \ s^{r-1} \ldots \ s \ 1 \right) h_0 \) and form the transfer function \( W_0 = W_{01}/W_{02} \) of the optimal controller

\[
W_{01}(s) = \alpha(s) - A(s)P_0(s), \quad W_{02}(s) = \beta(s) - B(s)P_0(s).
\]

Remark that the above algorithm is in sequential order, while no ‘going back’ procedures are involved.

6. Examples of synthesis. Example 1. Let us implement Algorithm 1 to design stabilizing controller \cite{3} with the given structure \( \Re\{\mu, \nu\} \), where \( \mu = 2, \nu = 1 \), for the controlled plant presented by the model \cite{1}, having the following matrices:

\[
A = \begin{pmatrix}
0 & 0 & 0.200 \\
0.250 & 0 & 0.290 \\
0 & 0.500 & -1.15
\end{pmatrix}, \quad b = \begin{pmatrix}
-2.00 \\
0.500 \\
0
\end{pmatrix},
\]

\[
p = \begin{pmatrix}
2 & 0 & 0
\end{pmatrix}^T, \quad c = \begin{pmatrix}
0 & 0 & 4
\end{pmatrix}.
\]

We accept here that the external disturbance \( d(t) \) is Gaussian white noise with the unit spectral density: \( N(s) = T(s) \equiv 1, p_s = q_s = 0 \). Let also suppose that \( c_0 = 1 \).

The aim of synthesis is to find the transfer function

\[
W = W_0(s) = W_{01}(s)/W_{02}(s) \in \Re\{\mu, \nu\}
\]

of the controller \cite{3}, providing stability degree \( \alpha = 0.7 \) and minimizing the functional \( J_\infty(W) = \|H(s, W)\|_\infty^2 = \|H(s, W)\|_\infty^2 \).

Accepting higher-order coefficient of a numerator \( W_{01} \) as one, let us compose all remaining coefficients to the vector \( h \in \Re^4 \) of adjustable parameters. It is a matter of simple calculation to verify that such parameterization is complete for this system. Moreover, with the help of vector \( h \) we can uniquely assign desirable poles of the closed-loop system, i.e. here we have \( \varepsilon = \gamma \) and \( \lambda = p \) for the unconditional extremum problem \cite{22}.
To estimate the result of synthesis, we can use the solution of the classical global optimization problem \((11)\) for admissible set \(\Omega_{\infty} = \Omega_{\infty}\). With the help of the methods presented in the papers [3], [10] we obtain

\[ J_0 = \arg \min_{W \in \Omega_{\infty}} J_{\infty}(W) = 4.38. \]

It is obvious that for a controller \((3)\) to be designed the inequality \(J_{\infty_0} = J_{\infty}(W_0) > J_0\) holds, i.e. the value \(J_{\infty_0}\) is the lower estimation for the optimum \(J_{\infty_0}\).

Thus, let us implement Algorithm 1 for the solution of the problem \((11)\) \((i = \infty)\), transforming it to the equivalent unconditional extremum problem \((22)\), \(\varepsilon = \gamma\), \(\lambda = p\). Starting computation process at the point \(\gamma = 0 \in \mathbb{R}^4\), where \(I(\gamma) = 181\), we consequently obtain the following values of the functional \(N\) is the number of iteration for Nelder-Mead method):

\[ \begin{align*}
N = 20 & \Rightarrow I = 180, \\
N = 30 & \Rightarrow I = 119, \\
N = 35 & \Rightarrow I = 35.3, \\
N = 40 & \Rightarrow I = 8.90, \\
N = 107 & \Rightarrow I = 7.70.
\end{align*} \]

For the 114-th iteration we finally obtain:

\[ \gamma_0 = \begin{pmatrix} 3.49 & 2.94 & -4.13 & 0.413 \end{pmatrix}^T, I_0 = J_{\infty_0} = I(\gamma_0) = 6.59, \]

\[ W_0(s) = \frac{s^2 + 1.65s + 0.481}{0.00205s + 1.06}, \]

that is quite consistent with the mentioned lower estimation. Obtained controller provides the following poles of the closed-loop system: \(\lambda_1 = -3390\), \(\lambda_2 = -455\), \(\lambda_3 = -1.24\), \(\lambda_4 = -0.751\), i.e. we achieve desirable stability degree.

**Example 2.** Let us consider numerical example of synthesis via Algorithm 3 for the controlled plant \((1)\) with the matrices given as follows:

\[ A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0.5 \\ -1 \\ 0.5 \end{pmatrix}, \]

\[ p = \begin{pmatrix} 0.5 & 0 & 0 \end{pmatrix}^T, \quad c = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix}. \]

In accordance with \((7)\) we obtain

\[ A(s) = s^3 - s^2 + 2, \quad B(s) = s^2 - 2s + 1, \quad C_d(s) = 1, \]

i.e. \(n = \deg A = 3\), \(m = \deg B = 2\).

We accept here that the disturbance \(d(t)\) and the weight multiplier \(c_0\) are the same as for preceding case.

First, in accordance with Algorithm 3, let fixed the controller’s structure \(\Re\{\mu, \nu\}\), assigning \(\mu = \nu = 4\). In conformity with \((29)\), this allows us to assign \(k = \max\{n + \nu, m + \mu\} = 7\) poles by no unique way.

Obtain also \(r = \min\{\mu - n, \nu - m\} = 1 = \deg(P)\).

By the second step, let us specify desirable spectrum of the poles as \(\Lambda = \{-1, -1.5, -2, -2.5, -3, -3.5, -4\}\), defining polynomials

\[ Q_T = Q = s^7 + 17.50s^6 + 127.8s^5 + 503.1s^4 + 1152s^3 + 1527s^2 + 1082s + 315.0 \]

i.e. here we have \(\kappa = k = 7\).

The third step consists of a reference controller choice with the transfer function \(W_a(s) = \alpha(s)/\beta(s)\), which provides desirable spectrum \(\Lambda\). Solving the polynomial
equation \( A(s)\beta(s) - B(s)\alpha(s) = Q(s) \) with respect to polynomials \( \alpha(s) \) and \( \beta(s) \) of the 4-th degree, obtain
\[
\alpha(s) = s^4 + 588s^3 + 2407s^2 - 210s - 2030, \\
\beta(s) = s^4 + 19.5s^3 + 733.6s^2 + 2466s - 857.5.
\]

The forth step allows us to arrive at the following auxiliary values and vectors:
\[
\gamma_1 = 0, \quad \gamma_2 = 4.01\cdot10^{-4}, \quad \gamma_3 = -2.18\cdot10^{-3}, \quad \gamma_4 = 5.22\cdot10^{-3}, \quad \gamma_5 = -6.08\cdot10^{-3}, \\
\gamma_6 = 3.38\cdot10^{-3}, \quad \gamma_7 = -7.25\cdot10^{-4}; \\
\xi_1 = 0, \quad \xi_2 = -0.1902, \quad \xi_3 = 0.7256, \quad \xi_4 = -1.152, \quad \xi_5 = 0.7970, \\
\xi_6 = -0.1983, \quad \xi_7 = -0.0007252; \\
g_1 = (0 0), g_2 = (-601 401), g_3 = (4.36 - 2.18)\cdot10^{-3}, \\
g_4 = (-13.1 5.22)\cdot10^{-3}, \quad g_5 = (18.2 - 6.08)\cdot10^{-3}, \\
g_6 = (-11.8 3.38)\cdot10^{-3}, \quad g_7 = (2.90 - 0.725)\cdot10^{-3}.
\]

The fifth step results to the matrices
\[
M^* = 10^{-3} \begin{pmatrix}
0 & 0.0222 \\
-0.05242 & 0.3885 \\
-0.9173 & 2.784 \\
-6.003 & 10.25 \\
-14.23 & 19.56 \\
-16.09 & 19.66 \\
-6.993 & 7.937
\end{pmatrix},
\quad m^* = \begin{pmatrix}
-0.01906 \\
-0.3125 \\
-2.062 \\
-6.951 \\
-12.79 \\
-12.70 \\
-5.091
\end{pmatrix},
\]
and we can extract the 2 \( \times \) 2 block \( M \) and 2 \( \times \) 1 block \( m \) as follows:
\[
M = 10^{-3} \begin{pmatrix}
0 & 0.0222 \\
-0.05242 & 0.3885
\end{pmatrix},
\quad m = \begin{pmatrix}
-0.01906 \\
-0.3125
\end{pmatrix}.
\]

The sixth step gives the optimal polynomial
\[
P_0(s) = -401.6s - 858.4,
\]
and, at last, we can form the transfer function \( W_0(s) = W_{01}(s)/W_{02}(s) \) of the optimal controller, where
\[
W_{01}(s) = 402.3s^4 + 1045s^3 + 1548s^2 + 592.7s - 313.1, \\
W_{02}(s) = s^4 + 421.1s^3 + 788.9s^2 + 1151s + 0.9306.
\]

Obtained solution provides the following minimum value of the functional:
\[
J(P_0) = J(W(P_0)) = \|H(s, W_0)S_1(s)\|_2^2 = 8.383,
\]
that is essentially better then for the reference controller:
\[
J(0) = J(W(0)) = \|H(s, W_0)S_1(s)\|_2^2 = 33.19.
\]

Finally, to illustrate the optimal features of the obtained result, let us consider the function
\[
J(h_0, h_1) = J(P) = \|H(s, W(h_0, h_1))S_1(s)\|_2^2,
\]
depending on two coefficients of the polynomial \( P(s) = h_1s + h_0 \). It is evident that the point of its global minimum is \( h_0 = (-401.6 - 858.4)^T \in R^2 \). Figure 1 illustrates this feature, where a fragment of the surface \( J = J(h_0, h_1) \) is shown. Besides, Figure 2 presents cross-sections of this surface for fixed values of the parameter \( h_1 \). These figures confirm the optimality of the obtained result.
Example 3. Finally, let us present numerical example of synthesis, using Algorithm 2 for the solution of the problem (14) with incomplete parameterization, providing desirable stability degree $\bar{\alpha}$ of the closed-loop connection. Let accept the following matrices for the controlled plant (1):

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
\[ \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. \]

In accordance with (7), we have
\[ A(s) = s^3 + s^2 + s - 1, \quad B(s) = 1, \quad C_d(s) = 0.5, \]
i.e. \( n = \deg A = 3, \quad m = \deg B = \deg C_d = 0. \)

We shall treat the external disturbance \( d = d(t) \) as the stationary ergodic random process with given rational spectral power density (2), where
\[ N(s) = 2, \quad T(s) = s + 1. \]

Let also suppose that \( c_0 = 1 \), and that the desirable stability degree is \( \hat{\alpha} = 0.3. \)

The aim of synthesis for this case is to find the transfer function \( W = W_0(s) = W_{01}(s)/W_{02}(s) \in \mathbb{R}\{\mu, \nu\} \) of the controller (3), providing the mentioned stability degree and minimizing the functional \( J_2(W) = \| \mathbf{H}(s, W) \|_2^2 = \| H(s, W) \|_2^2. \)

Let us accept the structure of the controller as follows:
\[ u = k_1 x_1 + k_2 x_2 \Leftrightarrow u = k_1 y + k_2 \dot{y} \Leftrightarrow u = (k_2 s + k_1) y. \] (53)

In accordance with (53), a controller’s structure \( \mathbb{R}\{\mu, \nu\} \) is determined by the equalities \( \mu = 1, \nu = 0 \) and by the vector \( \mathbf{h} = ( k_1 \quad k_2 )^T \in \mathbb{R}^2 \) of adjustable parameters. One could easy check that such parameterization is incomplete for the closed-loop system: with the help of vector \( \mathbf{h} \), we cannot assign arbitrary desirable roots of its characteristic polynomial. This makes actual the second situation for the problem (14), which can be resolved by the implementation of Algorithm 2.

1. Given an initial point \( \gamma = (0 \quad 0 \quad 0)^T \), we construct desirable polynomial \( \Delta^*(s, \gamma) = s^3 + 0.9 s^2 + 0.27 s + 0.027 \) via the formulae (15), (16).

2. In accordance with the identity \( \Delta_3(s, \mathbf{h}) = \Delta^*(s, \gamma) \) let form the system (24) of linear algebraic equations with respect to the vector \( \mathbf{h} \in \mathbb{R}^2 \) of the adjustable parameters, obtaining the matrix \( \mathbf{\Gamma} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}^T \) and the vector \( \chi(\gamma) = (−0.100 \quad 0.730 \quad 1.03)^T \). Let us also find the matrix \( \mathbf{\Gamma}^+ = (\mathbf{\Gamma}^T \mathbf{\Gamma})^{-1} \mathbf{\Gamma}^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \).

3. Let find the best approximation of the solution for the system (24):
\[ \mathbf{h}^* = \mathbf{h}^*(\gamma) = \mathbf{\Gamma}^+ \chi(\gamma) = \begin{pmatrix} 1.03 \\ 1.03 \end{pmatrix}, \quad \text{i.e.} \quad k_1 = -1.03, \quad k_2 = 0.730. \]

4. Let us calculate the value \( J_2(h^*) = J_2(\mathbf{h}^*(\gamma)) = I(\gamma) = 17.5 \) of the functional, and the value \( f(\gamma) = \| \chi(\gamma) - \mathbf{\Gamma}^+ \chi(\gamma) \| = 0.100 \) of the misalignment for the system (24). Given the weight multiplier \( \rho = 10 \), let calculate the function (27)
\[ J(\gamma) = 18.5. \]

Remark that a stability degree of the obtained closed-loop connection is \( \alpha = 0.176 \) that is not satisfactorily for this case.

5. Using MATLAB function \texttt{fminsearch}, let us implement a numerical descend method for the function \( J(\gamma) \) minimization, giving new points \( \gamma \in \mathbb{R}^3 \) and repeating correspondent calculations for steps 1–4. As a result, we obtain the following results after 150 iterations:
\[ \gamma = (0.222 \quad 3.40 \quad -0.223)^T, \quad J_2(h^*) = J_2(\mathbf{h}^*(\gamma)) = I(\gamma) = 1.13, \quad f(\gamma) = \| \chi(\gamma) - \mathbf{\Gamma}^+ \chi(\gamma) \| = 7.87 \cdot 10^{-7}, \quad \alpha = 0.325. \]
We can observe that a desirable stability degree is achieved, however the value of $J_2$ functional is larger than its global minimum $J_{20} = 0.762$.

To decrease the mentioned value, let us decrease the weight multiplier, accepting $\rho = 0.024$, and repeating the steps 1–5. We obtain almost the same results for the values of the functionals $I(\gamma)$ and $f(\gamma)$, but stability degree is $\alpha = 0.301$. A subsequent decreasing of $\rho$ leads to the worst values of $\alpha$.

In such a way, we accept the obtained results from the step 5: the point $\gamma_0 = (0.222, 3.40, -0.223)^T$ is a solution of the problem [27]. Correspondingly, the vector $h_0 = \Gamma^+ \chi(\gamma_0) = \begin{pmatrix} -5.07 \\ -10.9 \end{pmatrix}$ (i.e. $k_1 = -5.07$, $k_2 = -10.9$) is a solution of the problem [14], and correspondent transfer function $W = W_0(s) = W(s, h_0) = -10.9s - 5.07$ is a solution of the problem [11].

**Conclusions.** The purpose of this paper is the development of constructive methods for the $H$-optimal SISO LTI control law synthesis, taking into account initially given structural and modal restrictions. The urgency of the problem is determined by both the presence of the numerous dynamical requirements for the closed-loop system and the need to make a computational procedure the most effective for adoptive changeover in real-time regime of operating. In particular, these circumstances play a crucial role with respect to embedded systems or for onboard control systems of autonomous moving robots.

Central attention here is paid to the issue of the control laws synthesis in the range of given structure of theirs transfer functions. Two possible situations are considered: with complete and incomplete manner of controller set numerical parameterization. In the first case, the basic method is proposed to solve the problem by transforming it into the equivalent unconditional optimization. The other option uses the method mentioned above, and an additional penalty function to address incompleteness of parameterization.

In addition, special attention is paid to the optimal synthesis with initially given allocation of characteristic polynomial roots. As for the structure, this is a particular case of complete parameterization; but in order to meet an additional modal requirement, special method is proposed using specific polynomial parameterization of admissible set of the controllers.

The developed theoretical framework is summarized in the form of three numerical algorithms. Illustrative examples show the effectiveness of the proposed approach and the benefits of developed methods.

The research results can be expanded to other kinds of performance indices and can be developed with transport delays taken into account.

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