In validating the Robustness Conjecture for Geometric Quantum Gates

R. K. L. Colmenar,1,∗ Utkan Güngördü,2,3,1 and J. P. Kestner1

1Department of Physics, University of Maryland Baltimore County, Baltimore, MD 21250, USA
2Laboratory for Physical Sciences, College Park, Maryland 20740, USA
3Department of Physics, University of Maryland, College Park, Maryland 20742, USA

Geometric quantum gates are conjectured to be more resilient than dynamical gates against certain types of error, which makes them ideal for robust quantum computing. However, there are conflicting claims within the literature about the validity of that robustness conjecture. Here we use dynamical invariant theory in conjunction with filter functions in order to analytically characterize the noise sensitivity of an arbitrary quantum gate. Under certain conditions, we find that there exists a transformation of the Hamiltonian that leaves invariant the final gate and noise sensitivity (as characterized by the filter function) while changing the phase from geometric to dynamical. Our result holds for a Hilbert space of arbitrary dimensions, but we illustrate our result by examining experimentally relevant single-qubit scenarios and providing explicit constructions of such a transformation.

One of the biggest roadblocks in quantum computing is developing techniques that enable control of quantum information under a certain error threshold [1]. Among the plethora of potential candidates for robust quantum control, geometric quantum computation (GQC) [2] stands out owing to its elegant formulation in terms of concepts from differential geometry and topology. Put simply, a geometric quantum gate is a type of quantum gate for which it is possible to attribute a geometric interpretation to the accumulated phase. The usual paradigm is to generate a desired quantum gate in a basis of cyclic states. After an adiabatic [3] or nonadiabatic [4] cyclic evolution, these states accumulate a phase that depends on the qubit’s spectrum. If the computational basis is encoded in an energetically non-degenerate (degenerate) subspace of the total Hilbert space, the computational basis accumulates an Abelian (non-Abelian) phase [5, 6]. This phase can be decomposed into a dynamical and a geometric component. A geometric gate is naturally produced when the dynamical component of the total phase vanishes, though that condition is not necessary [7]. Further extension to noncyclic evolution has also been made [8]. Experimentally, geometric gates have been realized in nuclear magnetic resonance [9–11], trapped-ion [12, 13], solid-state [14–18], and superconducting qubits [19–22].

The primary motivation for using GQC is the robustness conjecture which claims that geometric gates are intrinsically more robust than dynamical gates [23]. This is typically supported by the reasoning that since geometric phase is a global feature of quantum evolution, then it must be intrinsically resilient to noise that only generates local perturbations in the system’s evolution path. Thus, a majority of the effort on GQC focuses on finding experimentally feasible ways of eliminating dynamical phase contributions in a gate. Numerous studies on geometric gates, both theoretical [2, 24–33] and experimental [21, 34, 35], have shown evidence to support the robustness conjecture. However, there are also studies that report control situations in which geometric gates are not intrinsically more robust than dynamical gates and, in some cases, can even increase sensitivity to noise [23, 36–43].

In this letter, we analyze the robustness of geometric and dynamical quantum gates to coherent noise. We use dynamical invariant theory [44] in conjunction with filter functions [45, 46] in order to analytically characterize how noise sensitivity changes with the type of accumulated phase. We show that it is possible to find, under certain conditions, a transformation of the Hamiltonian that leaves the gate and filter function invariant but continuously varies the nature of the phase from purely geometric to purely dynamical. The existence of such a transformation consequently invalidates the most general form of the robustness conjecture for geometric gates, and our analysis applies equally to adiabatic and nonadiabatic geometric gates. We explicitly illustrate our result in experimentally relevant single-qubit cases, including both Abelian and non-Abelian geometric gates. Our result may reconcile decades of seemingly contradictory claims on geometric gate robustness within the literature. Furthermore, our result calls into question the primary motivation for using GQC.

We begin by briefly describing how a geometric gate is generated. We temporarily restrict our attention to the Abelian case. A natural framework for considering geometric phase is through the theory of dynamical invariants [44]. Consider a qubit system whose evolution is governed by some Hamiltonian $H(t)$. A dynamical invariant $I(t)$ is a solution to the Liouville-von Neumann equation

$$i \frac{\partial I(t)}{\partial t} - [H(t), I(t)] = 0,$$

(1)

where we use units such that $\hbar = 1$. The eigenvectors $|\phi_n(t)\rangle$ of $I(t)$ are related to the solutions of the Schrödinger equation by a global phase factor: $|\psi_n(t)\rangle = e^{i\alpha_n(t)}|\phi_n(t)\rangle$, where $\alpha_n(t)$ are the Lewis-
Riesenfeld phases given by [47]

\[
\alpha_n(t) = \alpha_{n,g}(t) + \alpha_{n,d}(t),
\]

\[
\alpha_{n,g}(t) = \int_0^t \langle \phi_n(t')|i\partial_{t'}|\phi_n(t') \rangle \, dt',
\]

\[
\alpha_{n,d}(t) = -\int_0^t \langle \phi_n(t')|H(t')|\phi_n(t') \rangle \, dt',
\]

and the subscripts \(g\) and \(d\) denote the geometric and dynamical components, respectively. Within this framework, the evolution operator \(U(t)\) can be expressed as

\[
U(t) = \sum_n e^{i\alpha_n(t)} |\phi_n(t)\rangle \langle \phi_n(0)|.
\]

A geometric gate is produced if the final accumulated dynamical phase is zero, which can be ensured by, for example, carefully choosing the Hamiltonian so that the integral in Eq. (4) vanishes or by using composite pulses [48].

The validity of the robustness conjecture can be tested using filter functions [45, 46], which provide a convenient method of quantifying the gate fidelity’s susceptibility to noise of a given spectral composition. A noisy \(su(N)\) Hamiltonian can be decomposed as

\[
H(t) = H_c(t) + H_e(t),
\]

where \(H_c(t)\) is the ideal deterministic control Hamiltonian and \(H_e(t)\) is the stochastic error Hamiltonian,

\[
H_c(t) = \mathbf{h}_c(t) \cdot \mathbf{\sigma}, \quad H_e(t) = \sum_q \delta_q(t) \mathbf{\chi}_q [\mathbf{h}_c(t)] \cdot \mathbf{\sigma},
\]

where \(q\) indexes a set of uncorrelated stochastic variables \(\delta_q(t)\), \(\mathbf{\chi}_q\) is the vector describing the first-order sensitivity of the control Hamiltonian to \(\delta_q(t)\), and \(\mathbf{\sigma}\) is a vector comprising the \(N^2 - 1\) traceless Hermitian generators of \(su(N)\). Most commonly the sensitivity vector \(\mathbf{\chi}_q\) is of the general linear form

\[
\mathbf{\chi}_q [\mathbf{h}_c(t)] = \mathbf{a}_q + M_q \mathbf{h}_c(t),
\]

where \(\mathbf{a}_q\) is independent of the control (i.e., additive noise) and \(M_q\) is likewise a constant real matrix accounting for sensitivity linearly proportional to some subset of the control (e.g., multiplicative noise).

For sufficiently weak noise, we can compactly express the ensemble averaged gate infidelity as

\[
\langle I \rangle \approx \frac{1}{2\pi} \sum_q \int_{-\infty}^{\infty} d\omega \, S_q(\omega) F_q(\omega),
\]

where \(S_q(\omega)\) denotes the power spectral density for the stochastic variable \(\delta_q(t)\) and \(F_q(\omega)\) is the corresponding filter function.

Denote the \(N \times N\) unitary evolution operator generated by the control Hamiltonian, \(H_c\), in the absence of noise as the time-ordered exponential

\[
U_c(t) = T e^{-i \int_0^t dt' H_c(t')} \equiv e^{-i\theta(t)\cdot \mathbf{\sigma}/2}.
\]

\(U_c\) can also be represented via its adjoint representation, \(R\), defined through

\[
U_c(x \cdot \mathbf{\sigma}) U_c^\dagger = (R x) \cdot \mathbf{\sigma} \implies R_{ij} = \text{tr} (\sigma_i U_c \sigma_j U_c^\dagger)/N.
\]

For example, in the case of \(N = 2\), \(R(t) = \exp (\theta(t)\cdot \mathbf{L})\), where \(\mathbf{L}\) is the vector of generators of \(su(2)\) isomorphic to \(\mathbf{\sigma}\). In general, the filter function can be interpreted geometrically as the magnitude of a complex vector

\[
F_q(\omega) = \mathbf{R}(\omega) \cdot \mathbf{R}(\omega)^*,
\]

\[
\mathbf{R}(\omega) = \int_0^T R^\dagger(t) \mathbf{\chi}_q (\mathbf{h}_c(t)) e^{-i\omega t} dt,
\]

where \(T\) is the gate time.

We calculate the filter function of two different control Hamiltonians, \(H_c(t)\) and \(\tilde{H}_c(t)\), which, without loss of generality, can be related by a time-dependent unitary transformation: \(\tilde{H}_c = V H_c V^\dagger - i V \dot{V} V^\dagger\). A unitary transformation generally results in a geometric and dynamical phase shift that preserves the overall Lewis-Riesenfeld phase [49, 50]:

\[
\tilde{\alpha}_{n,g}(t) = \alpha_{n,g}(t) + \int_0^t \langle \phi_n(t')|iV^\dagger \dot{V}|\phi_n(t') \rangle \, dt',
\]

\[
\tilde{\alpha}_{n,d}(t) = \alpha_{n,d}(t) - \int_0^t \langle \phi_n(t')|iV^\dagger \dot{V}|\phi_n(t') \rangle \, dt'.
\]

The resulting transformation of the control evolution operator is \(\tilde{U}_c(t) = V(t)U_c(t)V^\dagger(t)\). Likewise the transformation of the adjoint representation of the evolution, \(R(t)\), denoting the adjoint representation of \(V\) as \(Q\), is

\[
\tilde{R}(t) = Q(t)R(t)Q^\dagger(t).
\]

We wish to consider a transformation such that the new Hamiltonian implements the same final gate as the original, \(\tilde{U}_c(T) = U_c(T)\), which imposes the constraint \(V(0) = V(T) = 1\) and likewise for \(Q\).

In contrast with the robustness conjecture, we now show that, in many experimentally relevant scenarios, for any control scheme \(\mathbf{h}_c(t)\) that generates a gate geometrically there exists a different control \(\tilde{\mathbf{h}}_c(t)\) that generates the same gate dynamically with an identical filter function. To state it more operationally, we must find some choice of transformation \(Q(t)\) such that a) \(Q(T) = Q(0) = 1\) so as to produce the same gate, b) a geometric phase is traded for a dynamical one via Eqs. (14)-(15), and c) the integrands in Eq. (13) are equal for a given noise \(\delta_q\) so as to have the same filter function. For the latter, it suffices if

\[
Q^\dagger(t) \mathbf{\chi}_q [\tilde{\mathbf{h}}_c(t)] = \mathbf{\chi}_q [\mathbf{h}_c(t)].
\]
It is important to note that the sensitivity vector $\chi_q$ is not itself directly transformed since the underlying physical noise mechanism is unchanged; it is simply evaluated as a function of the new control input

$$\tilde{h}_c(t) = Q(t)h_c(t) + h_{QQ}(t), \quad h_{QQ}(t) \equiv \operatorname{tr}(-iQQ^T\Lambda_i)/N,$$

with $\Lambda$ the $(N^2 - 1)$-dimensional vector of generators of $\mathfrak{su}(N)$ isomorphic to $\sigma$, so Eq. (17) becomes

$$Q^T(t)a_q + Q(t)M_qQ(t)h_c(t) + h_{QQ}(t) = a_q + M_qh_c(t).$$

If we can find a $Q(t)$ that satisfies Eq. (19), we will have two Hamiltonians that result in identical filter functions. This simple fact is the crux of this paper.

While the existence of a solution to the nonlinear Eq. (19) is not obvious, we simplify by taking the more restrictive condition that the first (second) term on the lhs must separately equal the first (second) term on the rhs. Thus, one should choose $Q(t)$ such that i) $a_q$ is an eigenvector of $Q(t)$, ii) $[Q(t), M_q] = 0$, and iii) $h_{QQ}(t)$ is in the null space of $M_q$. Parameterizing as $Q(t) = T\exp\{\int \omega(t) \cdot d\Lambda dt\}$, these conditions become i) $\omega_1(t) = 0$ if $a_q \in \text{Col} \Lambda$, ii) $\omega(t) \cdot [\Lambda, M_q] = 0$, and iii) $\omega(t) \in \text{null} M_q$. In practice it is typically easy to satisfy these conditions.

To illustrate, consider a single qubit under a generic $\mathfrak{su}(2)$ control Hamiltonian

$$h_c(t) = \frac{1}{2} \begin{pmatrix} \Omega(t) \cos(\varphi(t)) \\ \Omega(t) \sin(\varphi(t)) \end{pmatrix} \Delta(t)$$

This form is pertinent to a variety of qubit implementations such as superconducting qubits [51], quantum dot spin qubits [52], and NMR qubits [53] to name a few, corresponding to the rotating wave approximation for a two-level system driven by an oscillating field with amplitude $\Omega$ at a carrier frequency detuned from resonance by $\Delta$, and with phase $\varphi$. Suppose that this qubit is subject to independent additive fluctuations in the resonance frequency, $\Delta \rightarrow \Delta + \delta\Delta$, and in the phase, $\varphi \rightarrow \varphi + \delta\varphi$, as well as multiplicative amplitude noise, $\Omega \rightarrow \Omega(1 + \delta\Omega)$, i.e., in terms of Eq. (8),

$$a_\Delta = \frac{1}{2} \hat{z}, \quad M_\Delta = 0,$$

$$a_\varphi = 0, \quad M_\varphi = \frac{1}{2} (\hat{y}\hat{x}^T - \hat{x}\hat{y}^T),$$

$$a_\Omega = 0, \quad M_\Omega = \frac{1}{2} (\hat{x}\hat{x}^T + \hat{y}\hat{y}^T).$$

Note that this noise model encompasses most, if not all, scenarios treated in the literature of nonadiabatic geometric gates.

The three flavors of Eq. (19) corresponding to $q = \Delta, \varphi, \Omega$ are all satisfied by choosing $Q(t) = e^{\nu(t)\Lambda_\omega}$ (and hence $h_Q = \nu(t)\hat{z}/2$) where $\Lambda_\omega$ is the $z$-rotation generator in $\mathfrak{so}(3)$ and $\nu(t)$ is any function satisfying $\nu(0) = \nu(T) = 0$. Thus, for any geometric gate produced by a particular choice of $Q(t)$, $\varphi(t)$, and $\Delta(t)$ in Eq. (20), one can implement the same gate with identical robustness by using the modified control

$$\tilde{h}_c(t) = \frac{1}{2} \begin{pmatrix} \Omega(t) \cos(\varphi(t) + \nu(t)) \\ \Omega(t) \sin(\varphi(t) + \nu(t)) \end{pmatrix} \Delta(t) + \nu(t)$$

and the free parameter $\nu(t)$ allows a way to tune the nature of the Lewis-Riesenfeld phase as indicated in Eqs. (14) and (15). An explicit example of converting the nonadiabatic geometric gate of Ref. [30] to an equally robust dynamical gate in this way is provided in the Supplemental Material.

This result could explain the contradictory claims in the literature regarding the robustness of geometric gates, in that a study would support the conjecture if it were considering a geometric gate pulse that happened to be relatively robust compared to generic choices of dynamical gate pulses, whereas a study would not support the conjecture if it were considering a generic geometric gate pulse. Note that if a particular control Hamiltonian constraint is enforced (e.g., only two-axis control) such that the transformation defined by the solution to Eq. (19) results in a Hamiltonian that is inaccessible, a geometric gate could naturally emerge as preferable (as is indeed the case for a strictly two-axis Hamiltonian with static multiplicative amplitude error [48]). Likewise, there are unusual error models (e.g., $a_q = 0$, $M_q = 1$) for which only trivial, time-independent solutions to Eq. (19) exist, breaking the equivalence of geometric and dynamical phases and again opening the door to an optimal choice for that particular model. Generically, though, we have shown that there is no inherent difference in the robustness of geometric and dynamical phases.

We now extend this treatment to the non-Abelian case. Unlike the Abelian case, where ensuring the dynamical phase is zero at the final time is a constraint on the geometric pulse design, non-Abelian geometric quantum computing typically encodes the computational basis in an energetically degenerate subspace of the full Hilbert space such that any dynamical phase is either automatically zero at all times or can be treated as a global phase factor. We generalize our previous framework and denote the eigenvectors of $I(t)$ by $|\phi_{n;\alpha}(t)\rangle$ where $\alpha \in \{1, 2, \ldots, d_n\}$ labels the orthonormal basis vectors of a $d_n$-fold degenerate subspace corresponding to the $n^{th}$ eigenvalue. The propagator of Eq. (5) generalizes to [50]

$$U(t) = \sum_n \sum_{a,b=1}^{d_n} u_{n;ab}(t) |\phi_{n;a}(t)\rangle \langle \phi_{n;b}(0)|,$$

where the eigenstates accumulate a non-Abelian phase
\[ u_n(t) = T e^i \int_0^t A_n(t') + E_n(t') dt', \]  
\[ A_{n,\alpha}(t) = \langle \phi_{n,\alpha}(t) | i \partial_t | \phi_{n,\alpha}(t) \rangle, \]  
\[ E_{n,\alpha}(t) = -\langle \phi_{n,\alpha}(t) | H(t) | \phi_{n,\alpha}(t) \rangle. \]

Thus, if we consider the effect of a time-dependent unitary transformation \( V \) on the Hamiltonian, we get an expression for the change in the geometric and dynamical components of the Lewis-Riesenfeld phase similar to Eqs. (14) and (15):

\[ \dot{A}_{n,\alpha}(t) = A_{n,\alpha}(t) + \left[ V | \phi_{n,\alpha}(t) \rangle, \right. \]  
\[ \dot{E}_{n,\alpha}(t) = E_{n,\alpha}(t) - \left. \left. \langle \phi_{n,\alpha}(t) | V^\dagger V | \phi_{n,\alpha}(t) \rangle \right\} \right]. \]

To illustrate our result, we consider a three-level system in a \( \Lambda \) configuration where the states \(|0\rangle \) and \(|1\rangle \) are coupled to an excited state \(|e\rangle \) [54]. The \( k \leftrightarrow e \) transition \((k = 0, 1)\) is separately driven by a laser pulse with fixed polarization and frequency. Following our notation in Eq. (7), the system-laser interaction is described by the following rotating-frame Hamiltonian belonging to an \( su(3) \) algebra:

\[
    h_{e}(t) = \begin{pmatrix}
    0 & \Delta_0 - \Delta_1 + \nu(t) \\
    0 & 0 \\
    \Delta_0 - \Delta_1 & 0 \\
    \Omega(t) \cos \left( \frac{\nu(t)}{2} \right) & \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) \\
    \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) & \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) \\
    -\Omega(t) \cos \left( \frac{\nu(t)}{2} \right) & -\Omega(t) \cos \left( \frac{\nu(t)}{2} \right) \\
    \Delta_0 - \Delta_1 & 0 \\
    \Delta_0 + \Delta_1 & 0 \\
    \end{pmatrix}
\]

where \( \theta \) and \( \varphi \) are fixed angles determined by the laser polarization, \( \Delta_k \) are detunings which can be independently varied, \( \Omega(t) \) describes the pulse amplitude envelope, and \( \sigma \) is chosen to comprise the Gell-Mann matrices. If we impose the constraint that \( \int_0^T \Omega(t) dt = \pi \) and drive the qubit at resonance \((\Delta_k = 0)\), the evolution produces a purely geometric gate which, when projected in the computational space spanned by \(|0\rangle, |1\rangle \), yields [54]

\[
    \text{proj}_{\{|0\rangle, |1\rangle \}} [U(T)] = \begin{pmatrix}
    \cos \theta & e^{-i \varphi} \sin \theta \\
    e^{i \varphi} \sin \theta & -\cos \theta
    \end{pmatrix}.
\]

It is possible to generate any single-qubit operation by applying Eq. (32) with different values of \( \theta \) and \( \varphi \).

Suppose that this qubit is subject to independent additive fluctuations in the laser detunings, \( \Delta_k \to \Delta_k + \delta \Delta_k \), in their relative strength, \( \theta \to \theta + \delta \theta \), and in their relative phase, \( \varphi \to \varphi + \delta \varphi \), as well as multiplicative amplitude noise, \( \Omega \to \Omega(1 + \delta \Omega) \). Then, in terms of Eq. (8), we have

\[
    (a_{\Delta_k})_i = (-1)^k \pi \delta_{i,3} + \frac{\pi}{\sqrt{3}} \delta_{i,8}, \quad M_{\Delta_k} = 0,
\]

\[
    a_{\Omega} = 0, \quad M_{\Omega} = E_{4,4} + E_{5,5} + E_{6,6},
\]

\[
    a_{\theta} = 0, \quad M_{\theta} = \frac{1}{2} (E_{5,7} - E_{7,5} + E_{6,4} - E_{4,6}),
\]

\[
    a_{\varphi} = 0, \quad M_{\varphi} = \frac{1}{2} (E_{5,4} - E_{6,5} + E_{6,7} - E_{7,6}),
\]

where \( E_{i,j} \) is a square matrix which shows the value 1 at the position \((i, j)\) and zeros elsewhere [55]. It is straightforward to verify that the transformation \( Q(t) = e^{i \nu(t) \Lambda} \), where \( \Lambda_i \) are the adjoint representations of the Gell-Mann matrices [56], uniquely satisfies all the previously specified criteria. (If any one of these noise sources is irrelevant, there is more freedom in the transformation.) Thus, for any holonomic gate produced by a particular choice of \( \theta, \varphi, \) and \( \Omega(t) \) in Eq. (32), one can implement the same gate with identical robustness using the modified control

\[
    h_c(t) = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 & \Delta_0 - \Delta_1 + \nu(t) \\
    0 & \Omega(t) \cos \left( \frac{\nu(t)}{2} \right) & \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) & 0 & 0 & 0 & 0 \\
    \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) & \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) & 0 & 0 & 0 & 0 & 0 \\
    -\Omega(t) \cos \left( \frac{\nu(t)}{2} \right) & -\Omega(t) \cos \left( \frac{\nu(t)}{2} \right) & 0 & 0 & 0 & 0 & 0 \\
    \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) & \Omega(t) \sin \left( \frac{\nu(t)}{2} \right) & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & \Delta_0 - \Delta_1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \Delta_0 + \Delta_1 & 0 & 0 \\
    \end{pmatrix}
\]

where the free parameter \( \nu(t) \) breaks the degeneracy of an equal-detuning setting, and similar to the Abelian case, provides a way to tune the nature of the Lewis-Riesenfeld phase as indicated in Eqs. (29) and (30).

Note that the difference compared to the Abelian case where the dynamical and geometric contributions to the phase are easily separable, as in Eqs. (14) and (15). In the non-Abelian case, the gate accumulates matrix-valued dynamical and geometric phase components at each time step, as seen in Eq. (26), which generally do not commute. Thus, the inseparability of the phase’s time-ordered integral can lead to nontrivial dynamical contributions even if the integral of \( \mathcal{E}(t) \) in Eq. (28) is zero. This is why non-Abelian purely geometric gates are typically defined to have \( \mathcal{E}(t) = 0 \) within the computational basis [54, 57]. Therefore, to illustrate that a non-Abelian gate is no longer purely geometric, it is sufficient to show that the transformation \( V \) results in a nontrivial \( \mathcal{E} \) in Eq. (30). We present an explicit example in the Supplemental Information of converting the nonadiabatic holonomic gate of Ref. [54] into an equally robust non-holonomic gate.

On the other hand, it is possible to simultaneously diagonalize both matrices in special cases where
\[A(t), E(t) = 0\] which allows the decoupling of the geometric and dynamical phase contribution (for example, in adiabatic non-Abelian geometric gates [6]). In such cases, it is again straightforward to separate and tune the two types of phase.

In summary, we examine the broadband noise-resilience of geometric and dynamical gates using filter functions and show that there exists no intrinsic difference. We illustrate this explicitly in a one-qubit scenario for both the Abelian and non-Abelian case. Our argument applies to both adiabatic and nonadiabatic gates and does not impose any speed restriction on the control. Our result reconciles the apparent contradictory functions and show that there exists no intrinsic difference of geometric gates. Since geometric gates are not inherently more robust than dynamical gates, then the use of GQC becomes a question of experimental convenience.

RKLC and JPK acknowledge support from the National Science Foundation under Grant No. 1915064, and UG from the Army Research Office under Grant No. W911NF-17-1-0287.

* ralphkc1@umbc.edu

1. E. Knill, Nature 434, 39 (2005).
2. A. Ekert, M. Ericsson, P. Hayden, H. Inamori, J. A. Jones, D. K. L.Oi, and V. Vedral, Journal of Modern Optics 47, 2501 (2000).
3. M. V. Berry, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 392, 45 (1984).
4. Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
5. J. Anandan, Physics Letters A 133, 171 (1988).
6. P. Zanardi and M. Rasetti, Physics Letters A 264, 94 (1999).
7. S.-L. Zhu and Z. D. Wang, Phys. Rev. Lett. 91, 187902 (2003).
8. D. Kult, J. Áberg, and E. Sjöqvist, Phys. Rev. A 74, 022106 (2006).
9. J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature 403, 869 (2000).
10. J. Du, P. Zou, and Z. D. Wang, Phys. Rev. A 74, 020302 (2006).
11. Y. Ota, Y. Goto, Y. Kondo, and M. Nakahara, Phys. Rev. A 80, 052311 (2009).
12. D. Leibfried, B. DeMarco, V. Meyer, D. Lucas, M. Barrett, J. Britton, W. M. Itano, B. Jelenković, C. Langer, T. Rosenband, and D. J. Wineland, Nature 422, 412 (2003).
13. C.-Y. Guo, L.-L. Yan, S. Zhang, S.-L. Su, and W. Li, Phys. Rev. A 102, 042607 (2020).
14. C. Zu, W.-B. Wang, L. He, W.-G. Zhang, C.-Y. Dai, F. Wang, and L.-M. Duan, Nature 514, 72 (2014).
15. L. Wang, T. Tu, B. Gong, C. Zhou, and G.-C. Guo, Scientific Reports 6, 19048 (2016).
16. Y. Sekiguchi, N. Niikura, R. Kuroiwa, H. Kano, and H. Kosaka, Nature Photonics 11, 309 (2017).
17. N. Ishida, T. Nakamura, T. Tanaka, S. Mishima, H. Kano, R. Kuroiwa, Y. Sekiguchi, and H. Kosaka, Opt. Lett. 43, 2380 (2018).
18. K. Nagata, K. Kuramitani, Y. Sekiguchi, and H. Kosaka, Nature Communications 9, 3227 (2018).
19. X. Tan, D.-W. Zhang, Z. Zhang, Y. Yu, S. Han, and S.-L. Zhu, Phys. Rev. Lett. 112, 027001 (2014).
20. C. Song, S.-B. Zheng, P. Zhang, K. Xu, L. Zhang, Q. Guo, W. Liu, D. Xu, H. Deng, K. Huang, D. Zheng, X. Zhu, and H. Wang, Nature Communications 8, 1061 (2017).
21. Y. Xu, Z. Hua, T. Chen, X. Pan, X. Li, J. Han, W. Cai, Y. Ma, H. Wang, Y. P. Song, Z.-Y. Xue, and L. Sun, Phys. Rev. Lett. 124, 230503 (2020).
22. A. A. Abdumalikov Jr, J. M. Fink, K. Julisson, M. Pechal, S. Berger, A. Wallraff, and S. Filipp, Nature 496, 482 (2013).
23. S.-L. Zhu and P. Zanardi, Phys. Rev. A 72, 020301 (2005).
24. A. Carollo, I. Fuentes-Guridi, M. F. m. c. Santos, and V. Vedral, Phys. Rev. Lett. 90, 160402 (2003).
25. G. De Chiara and G. M. Palma, Phys. Rev. Lett. 91, 090404 (2003).
26. G. De Chiara, A. Lozinski, and G. M. Palma, The European Physical Journal D 41, 179 (2007).
27. Z. S. Wang, C. Wu, X.-L. Feng, L. C. Kwek, C. H. Lai, C. H. Oh, and V. Vedral, Phys. Rev. A 76, 044303 (2007).
28. J. T. Thomas, M. Lababidi, and M. Tian, Phys. Rev. A 84, 042335 (2011).
29. Z.-T. Liang, X. Yue, Q. Lv, Y.-X. Du, W. Huang, H. Yan, and S.-L. Zhu, Phys. Rev. A 93, 040305 (2016).
30. T. Chen and Z.-Y. Xue, Phys. Rev. Applied 10, 054051 (2018).
31. B.-J. Liu, X.-K. Song, Z.-Y. Xue, X. Wang, and M.-H. Yung, Phys. Rev. Lett. 123, 100501 (2019).
32. T. Chen and Z.-Y. Xue, Phys. Rev. Applied 14, 064009 (2020).
33. J. Pachos and P. Zanardi, International Journal of Modern Physics B 15, 1257 (2001).
34. S. Berger, M. Pechal, A. A. Abdumalikov, C. Eichler, L. Steffen, A. Fedorov, A. Wallraff, and S. Filipp, Phys. Rev. A 87, 060303 (2013).
35. F. Kleißler, A. Lazariev, and S. Arroyo-Camejo, npj Quantum Information 4, 49 (2018).
36. A. Nazir, T. P. Spiller, and W. J. Munro, Phys. Rev. A 65, 042303 (2002).
37. A. Blais and A.-M. S. Tremblay, Phys. Rev. A 67, 012308 (2003).
38. P. Solinas, P. Zanardi, and N. Zanghi, Phys. Rev. A 70, 042316 (2004).
39. A. Carollo, I. Fuentes-Guridi, M. F. m. c. Santos, and V. Vedral, Phys. Rev. Lett. 92, 20402 (2004).
40. J. Dajka, M. Mierzejewski, and J. /suppress Luczka, Journal of Physics A: Mathematical and Theoretical 41, 012001 (2007).
41. Y. Ota and Y. Kondo, Phys. Rev. A 80, 024302 (2009).
42. M. Johansson, E. Sjöqvist, L. M. Andersson, M. Ericsson, B. Hessmo, K. Singh, and D. M. Tong, Phys. Rev. A 86, 062322 (2012).
43. W. Dong, F. Zhuang, S. E. Economou, and E. Barnes, arXiv:2103.08029.
44. H. R. Lewis, Phys. Rev. Lett. 18, 510 (1967).
45. T. J. Green, J. Sastrawan, H. Uys, and M. J. Biercuk, New Journal of Physics 15, 095004 (2013).
46. H. Ball and M. J. Biercuk, EPJ Quantum Technology 2,
We distinguish the adjoint representation of a group which is defined in Eq. (11) from the adjoint representation of a Lie algebra which can be calculated using the structure constants of the algebra $f_{ijk}$ obeying $[\sigma_i, \sigma_j] = \sum_k f_{ijk} \sigma_k$ as $[\text{ad}(\sigma_i)]_{jk} = f_{ikj}$.
In this supplement, we provide an explicit example of converting a geometric gate to an equally robust dynamical gate for both the Abelian and non-Abelian case.

ABELIAN GEOMETRIC GATING

We begin by considering the Abelian geometric gate proposed in Ref. [1]. The control induces an evolution on the cyclic states that traces out an orange-slice path along the Bloch sphere (see Fig. 1). We present in Eqs. (S1) to (S3) the corresponding control constraints in terms of the Hamiltonian of Eq. (20) with \( \Delta(t) = 0 \):

\[
\begin{align*}
    t & \in [0, T_1] & \int_0^{T_1} \Omega dt = \theta & \quad \varphi = \eta - \frac{\pi}{2}, \\
    t & \in [T_1, T_2] & \int_{T_1}^{T_2} \Omega dt = \pi & \quad \varphi = \eta + \gamma + \frac{\pi}{2}, \\
    t & \in [T_2, T] & \int_{T_2}^{T} \Omega dt = \pi - \theta & \quad \varphi = \eta - \frac{\pi}{2}.
\end{align*}
\]

We denote the generated evolution operator by \( U_0(t) = e^{i n \cdot \sigma} \), where \( n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \). We can produce a two-part composite gate that suppresses additive dephasing \( (\Delta \rightarrow \Delta + \delta \Delta) \) and multiplicative amplitude noise \( (\Omega \rightarrow \Omega + \delta \Omega) \) by applying the same evolution twice [2]: \( U(2T) = U_0^2(T) \). We target a geometric \( X_{\pi/2} \) gate which can be achieved by setting \( \gamma = -\frac{\pi}{8}, \theta = \frac{\pi}{2}, \) and \( \eta = 0 \). To generate its purely dynamical equivalent, we use the transformed Hamiltonian of Eq. (24) with an arbitrary choice of \( \nu(t) = c \sin^2(\pi t/T) \), numerically tuning \( c \) until the geometric phase is zero, which occurs at \( c \approx -0.46186 \). The effect of this transformation on the the evolution path is shown in Fig. 1. We present in Fig. 2 a plot of the control parameters for both geometric and dynamical \( X_{\pi/2} \) gate. We use the computation scheme for cyclic evolutions presented in Ref. [3] to show that the transformed Hamiltonian produces a purely dynamical gate. The geometric and dynamical phases are

\[
\begin{align*}
    \alpha_{+,g} &= \arg \langle \phi_+(0)|\phi_+(T) \rangle + \int_0^T \langle \phi_+(t)|H(t)|\phi_+(t) \rangle \, dt, \\
    \alpha_{+,d} &= -\int_0^T \langle \phi_+(t)|H(t)|\phi_+(t) \rangle \, dt,
\end{align*}
\]

where \( |\phi_+(t)\rangle = U(t)(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T \) and \( U(t) \) is the evolution operator generated by \( H(t) \). The state \( |\phi_+(t)\rangle \) is cyclic since \( |\phi_+(0)\rangle \) is an eigenstate of \( U(T) = X_{\pi/2} \). A comparison of the geometric and dynamical phases for both gates is shown in Fig. 3 and their corresponding filter functions for dephasing and amplitude noise in Fig. 4.
FIG. 1. An illustration of a cyclic state’s evolution along the Bloch sphere using the geometric $X_{\pi/2}$ gate (LEFT) and the dynamical $X_{\pi/2}$ gate (RIGHT). We use the state $|\phi_+(t)\rangle$ which is cyclic since $|\phi_+(0)\rangle$ is an eigenvector of $X_{\pi/2}$. This path is traversed twice and its orientation is determined by the color gradient which begins with red and ends with blue.

FIG. 2. A plot of the control parameters that generate an $X_{\pi/2}$ gate. The subscript “g” (“d”) denote the control parameters that generate a geometric (dynamical) gate. The values are normalized by $\Omega_{\text{max}}$ which denote the maximum value of $\Omega_g(t)$. Note that $\Omega_g(t) = \Omega_d(t)$ and that $\Delta_d(t)$ is non-trivial.
FIG. 3. A comparison of the geometric and dynamical phases generated by the state $|\phi_+(t)\rangle$. The variables with (without) tilde correspond to the dynamical (geometric) $X_\pi$ gate.

FIG. 4. A comparison of the geometric and dynamical $X_\pi$ gate filter functions for additive dephasing and multiplicative amplitude noise when $\Omega_{\text{max}} = 1$. We verify that the control transformation preserves the filter functions.
NON-ABELIAN GEOMETRIC GATING

We now consider the non-Abelian case. Specifically, we consider the non-Abelian geometric gate proposed in Ref. [4]. The control Hamiltonian in this case is given by

\[
H(t) = \Omega(t) \begin{pmatrix}
0 & 0 & e^{-\frac{i\lambda}{2}} \sin \left(\frac{\theta}{2}\right) \\
0 & 0 & -e^{-\frac{i\lambda}{2}} \cos \left(\frac{\theta}{2}\right) \\
e^{\frac{i\lambda}{2}} \sin \left(\frac{\theta}{2}\right) & -e^{-\frac{i\lambda}{2}} \cos \left(\frac{\theta}{2}\right) & 0
\end{pmatrix},
\]

(S6)

where the computational subspace is located in the upper 2 x 2 block. This Hamiltonian results in a π rotation about an axis determined by the angles \(\phi\) and \(\theta\). Thus, one can target any one-qubit gate by producing non-commuting gates through different angle pairs. The generated gate is purely geometric since \(\langle i|H(t)|j \rangle = 0\) where \(i, j \in \{0, 1\}\).

As mentioned in the main text, this is a necessary condition for purely geometric evolution since any non-trivial dynamical components in Eq. (27) will be inseparably mixed with the geometric ones by a time-ordered integral since they generally do not commute. This is in contrast with the Abelian case where the dynamical and geometric components of the phase always commute and can be manipulated separately although the Lewis-Riesenfeld phase remains invariant. Therefore, to show that the generated non-adiabatic gate is no longer holonomic, it is sufficient to show that our transformation yields \(\langle i|H(t)|j \rangle \neq 0\) within the computational subspace.

Since \(H(t)\) commutes with itself at all times, we can calculate the resulting evolution operator \(U(t)\) analytically

\[
U(t) = \exp \left[-i\Omega(t) \begin{pmatrix}
0 & 0 & e^{-\frac{i\lambda}{2}} \sin \left(\frac{\theta}{2}\right) \\
0 & 0 & -e^{-\frac{i\lambda}{2}} \cos \left(\frac{\theta}{2}\right) \\
e^{\frac{i\lambda}{2}} \sin \left(\frac{\theta}{2}\right) & -e^{-\frac{i\lambda}{2}} \cos \left(\frac{\theta}{2}\right) & 0
\end{pmatrix}\right],
\]

(S7)

where we denote \(\Omega(t) \equiv \int_0^t \Omega(s)ds\). We can construct the cyclic states through the eigenvectors and eigenvalues of \(U(t)\)

\[
\lambda_1(t) = 1 \quad |\lambda_1(t)\rangle = \begin{pmatrix} e^{-i\frac{\lambda}{2}} \cos \left(\frac{\theta}{2}\right) \\ e^{i\frac{\lambda}{2}} \sin \left(\frac{\theta}{2}\right) \\ 0 \end{pmatrix}^T,
\]

(S8)

\[
\lambda_2(t) = e^{-i\Omega(t)} \quad |\lambda_2(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\lambda}{2}} \sin \left(\frac{\theta}{2}\right) \\ -e^{i\frac{\lambda}{2}} \cos \left(\frac{\theta}{2}\right) \\ 1 \end{pmatrix}^T,
\]

(S9)

\[
\lambda_3(t) = e^{i\Omega(t)} \quad |\lambda_3(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-i\frac{\lambda}{2}} \sin \left(\frac{\theta}{2}\right) \\ e^{i\frac{\lambda}{2}} \cos \left(\frac{\theta}{2}\right) \\ 1 \end{pmatrix}^T.
\]

(S10)

The cyclic states \(|\phi_i(t)\rangle\) of \(U(T)\) are linear combinations of \(|\lambda_i\rangle\) up to global time-dependent phase which we choose so that \(|\phi_i(0)\rangle = |\phi_i(T)\rangle\):

\[
|\phi_1(t)\rangle = |\lambda_1(t)\rangle, \quad |\phi_2(t)\rangle = \frac{e^{i\Omega(t)} |\lambda_2(t)\rangle - |\lambda_3(t)\rangle}{2}, \quad |\phi_2(t)\rangle = \frac{e^{i\Omega(t)} |\lambda_2(t)\rangle + |\lambda_3(t)\rangle}{2}.
\]

(S11)

We can calculate the change in the dynamical phase contribution under the transformation \(V = e^{-i\lambda\lambda_3(T)}\) (or, equivalently, by \(Q = e^{i\lambda\lambda_3(T)}\) in the adjoint representation) using Eq. (31):

\[
\tilde{\mathcal{E}}_{n,ab}(t) = \frac{1}{4} \begin{pmatrix}
2 \cos(\theta)\nu'(t) & e^{\frac{i\lambda}{2}} \sin(\theta)\nu'(t) \left(1 + e^{2i\Omega(t)}\right) & e^{\frac{i\lambda}{2}} \sin(\theta)\nu'(t) \left(1 - e^{2i\Omega(t)}\right) \\
-\cos(\theta)\nu'(t) \left(1 + e^{2i\Omega(t)}\right) & -2 \cos(\theta) \cos^2(\Omega(t))\nu'(t) & 4\Omega(t) + i \cos(\theta) \sin(2\Omega(t))\nu'(t) \\
e^{-\frac{i\lambda}{2}} \sin(\theta)\nu'(t) \left(1 - e^{-2i\Omega(t)}\right) & 4\Omega(t) - i \cos(\theta) \sin(2\Omega(t))\nu'(t) & -2 \cos(\theta) \sin^2(\Omega(t))\nu'(t)
\end{pmatrix}.
\]

(S12)

Since \(\tilde{\mathcal{E}}(t)\) is nontrivial in the computational subspace, then the transformed gate \(\tilde{U}(T)\) must be non-holonomic by definition, though its filter function is the same as \(U(T)\).

* ralphkc1@umbc.edu

[1] P. Z. Zhao, X.-D. Cui, G. F. Xu, E. Sjöqvist, and D. M. Tong, Phys. Rev. A 96, 052316 (2017).
[2] T. Chen and Z.-Y. Xue, Phys. Rev. Applied 10, 054051 (2018).
[3] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
[4] E. Sjöqvist, D. M. Tong, L. M. Andersson, B. Hessmo, M. Johansson, and K. Singh, New Journal of Physics 14, 103035 (2012).