The Solvability of a Class of Convolution Equations Associated with 2D FRFT

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Abstract: In this paper, the solvability of a class of convolution equations is discussed by using two-dimensional (2D) fractional Fourier transform (FRFT) in polar coordinates. Firstly, we generalize the 2D FRFT to the polar coordinates setting. The relationship between 2D FRFT and fractional Hankel transform (FRHT) is derived. Secondly, the spatial shift and multiplication theorems for 2D FRFT are proposed by using this relationship. Thirdly, in order to analyze the solvability of the convolution equations, a novel convolution operator for 2D FRFT is proposed, and the corresponding convolution theorem is investigated. Finally, based on the proposed theorems, the solvability of the convolution equations is studied.

Keywords: fractional Fourier transform; convolution theorem; solvability; convolution integral equation

1. Introduction

In recent years, convolution-type singular integral equations have received increasing attention from many mathematicians due to the wide applications in the field of engineering mechanics, fracture mechanics, and so on. They have formed a relatively perfect theoretical system [1–8]. Solvability is one of the essential issues of equation theory, and has been studied in-depth by many researchers [9–13]. Praha et al. [11] studied the solvability and the explicit solutions for a class of singular integral equations by using inverse fast Fourier transform in the class of discontinuous coefficients. Applying Fourier transform (FT), Li and Ren [12] studied the solvability for a class of singular integro-differential equations involving convolutional operators. A series of research results [11–13] have fully shown that FT is one of the important tools in the study of convolution integral equation theory. However, with the deepening of the research, the equation becomes more and more complex, and using the traditional Fourier method to study some equations is limited. Therefore, finding a more flexible tool than Fourier transform has become one of the research hotspots.

The fractional Fourier transform (FRFT), a form of fractional powers of the classical FT, was originally introduced in 1980 by Namias [14]. The definition of FRFT for a function \( f(t) \in L^2(\mathbb{R}) \) is given by [15]

\[
\mathcal{F}^\alpha (u) = \mathcal{F}^\alpha [f(t)](u) = \begin{cases} 
\int_{-\infty}^{\infty} f(t) K_{\alpha}(t,u) dt, & \alpha \neq n\pi, \\
f(u), & \alpha = 2n\pi, \\
f(-u), & \alpha = (2n+1)\pi, 
\end{cases}
\] (1)
with the transform kernel

$$K_\alpha(t, u) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}} \exp\left(\frac{t^2 + u^2}{2}\cot\alpha - it\cdot u\csc\alpha\right), \quad (2)$$

where $n \in \mathbb{Z}$. Note that when $\alpha = \frac{\pi}{2}$, FRFT reduces to the classical FT. Due to the free parameter $\alpha$, FRFT is more flexible than the classical FT without incurring any additional computation [16]. Recently, FRFT has received much attention from researchers, and many useful properties of FT have been extended to FRFT [17–21], including the convolution theorem [17,18], uncertainty principle [19,20], sampling theory [21], etc. Anh et al. [18] introduced two new convolutions associated with FRFT and established the complete solvability of the corresponding convolution equations. However, compared with the wide applications of FT in equation theory, few papers have applied FRFT to the equation analysis as far as we know. It is therefore interesting and worthwhile to investigate convolution integral equation by using FRFT.

In this paper, we consider the following type of convolution integral equation

$$\lambda \varphi(t) + \int_0^\infty g(s)f(t-s)\xi(t,s)ds = f(t), \quad (3)$$

where $\lambda \in \mathbb{C}$, $f, g \in L^2(\mathbb{R}^2)$ are given and $\varphi$ is unknown, and the kernel function $\xi(t,s)$ is

$$\xi(t,s) = \exp(\frac{|s|^2}{2}\cot\alpha) \exp(-it\cdot s\cot\alpha). \quad (4)$$

The main objective of this paper is to propose a novel method to study the solvability of convolution Equation (3) by using two-dimensional (2D) FRFT in polar coordinates. The main contributions of this paper can be stated as follows:

First, a new representation of 2D FRFT in polar coordinates is investigated, and two important properties of 2D FRFT in polar coordinates are studied in details. The relationship between 2D FRFT and FRHT is also obtained. Second, a novel convolution operator for 2D FRFT in polar coordinates is investigated, and the corresponding convolution theorem is also proposed. Third, based on the derived convolution theorem, the solvability of the convolution Equation (3) is studied. The results of this paper not only studied some useful properties of FRFT in polar coordinates but also provided a new way to study the solvability of integral equations.

The remainder of this paper is structured as follows. Section 2 provides a brief introduction of fractional Hankel transform (FRHT) and 2D FRFT. In Section 3, we first generalize the 2D FRFT to the polar coordinates setting and derive the relationship between 2D FRFT and FRHT. Applying this relationship, then we study the spatial shift and multiplication theorem for 2D FRFT. In Section 4, we first propose a novel convolution operator based on the convolution Equation (3) and investigate the corresponding convolution theorem. Using the convolution theorem and 2D FRFT, then we establish the solvability of convolution Equation (3). In Section 5, we conclude the paper.

2. Preliminary

In this section, we mainly review some basic facts on the 2D FRFT and FRHT, which will be needed throughout the paper.

2.1. Fractional Hankel Transform

The FRHT of a function $f$ for an angle $\alpha$ is defined as follows [22]

$$\mathcal{H}_n^\alpha[f](\rho) = \int_0^\infty f(r)K_n^\alpha(r, \rho)rdr, \quad (5)$$
where the kernel is
\[
K_n^\alpha(r, \rho) = \frac{\exp[i(1+n)\left(\frac{\pi}{2} - \alpha\right)]}{\sin\alpha} \exp\left[-\frac{i}{2}(r^2 + \rho^2) \cot\alpha\right] J_n\left(\frac{r\rho}{\sin\alpha}\right),
\]  
and \(J_n\) is the \(n\)-th-order Bessel function and \(n\) is an integer.

For \(\alpha = \frac{\pi}{2}\), FRHT becomes the conventional \(n\)-th-order Hankel transform
\[
\mathcal{H}_n[f](\rho) = \mathcal{H}_n^\frac{\pi}{2}[f](\rho) = \int_0^\infty f(r) J_n(r\rho) r dr,
\]
the inversion formula is given by
\[
f(r) = \int_0^\infty \mathcal{H}_n[f](\rho) J_n(r\rho) r d\rho,
\]
and the following relation is satisfied
\[
\mathcal{H}_n^\alpha[f](\rho) = D_{n,\alpha}(\rho) \mathcal{H}_n[f_0](\rho),
\]
where \(D_{n,\alpha}(\rho) = \frac{\exp[i(1+n)(\frac{\pi}{2} - \alpha) - \frac{i}{2}r^2 \cot\alpha]}{\sin\alpha}\) and \(f_0(r) = f(r) \exp(-\frac{i}{2}r^2 \cot\alpha)\).

By (9), we can get
\[
\mathcal{H}_n^\alpha[\hat{f}](\rho) = D_{n,\alpha}(\rho) \mathcal{H}_n[\hat{f}](\rho \csc\alpha),
\]
where \(\hat{f}(r) = f(r) \exp(ir^2 \cot\alpha)\) and \(\tilde{f}(r) = f(r) \exp(i\frac{\pi}{2}r^2 \cot\alpha)\).

2.2. 2D Fractional Fourier Transform

The 2D FRFT of a function \(f(x)\) with the angle \(\alpha\) is defined as [23]
\[
\mathcal{F}^\alpha(u) = \mathcal{F}^{\alpha}[f(t)](u) = \begin{cases} 
\int_0^{\infty} \int_0^{\infty} f(t) K_n(t, u) dt, & \alpha \neq n\pi, \\
f(u), & \alpha = 2n\pi, \\
f(-u), & \alpha = (2n+1)\pi,
\end{cases}
\]
where \(n \in \mathbb{Z}, t = (t_1, t_2) \in \mathbb{R}^2, u = (u_1, u_2) \in \mathbb{R}^2, dt = dt_1 dt_2\) and the kernel is
\[
K_n(t, u) = \sqrt{\frac{1 - i \cot\alpha}{2\pi}} \exp\left(i\frac{|t|^2 + |u|^2}{2} \cot\alpha - it \cdot u \csc\alpha\right),
\]
where \(|t|^2 = t_1^2 + t_2^2, |u|^2 = u_1^2 + u_2^2\) and \(t \cdot u = t_1 u_1 + t_2 u_2\).

The corresponding inverse formula is given by \(f(t) = \mathcal{F}^{-\alpha}[\mathcal{F}^\alpha(u)](t)\).

From the above definition, it is noted that, for \(\alpha = 2n\pi\) or \(\alpha = (2n+1)\pi\), the 2D FRFT is not particular interest for our objective in this work. Hence, without loss of generality, we set \(\alpha \neq n\pi\) in the following sections unless stated otherwise.

3. 2D FRFT in Polar Coordinates and Its Properties

In this section, we first generalize the 2D FRFT to the polar coordinates setting and derive the relationship between 2D FRFT and FRHT. Based on this relationship, we then discuss the spatial shift property and multiplication theorems for 2D FRFT in polar coordinates. The technology method used here is based on an extension of a celebrated result concerning the relationship between 2D FT and Hankel transform (HT) [24–26].
3.1. 2D Fractional Fourier Transform in Polar Coordinates

**Definition 1.** Let polar coordinates \( t_1 = r \cos \theta, t_2 = r \sin \theta, u_1 = \rho \cos \phi, u_2 = \rho \sin \phi \), the 2D FRFT in polar coordinates is defined as

\[
F^\alpha(\rho, \phi) = F^\alpha[f(r, \theta)] = \int_0^\infty \int_{-\pi}^{\pi} f(r, \theta) K_\alpha(r, \theta; \rho, \phi) r dr d\theta, \quad \alpha \neq n\pi, \tag{13}
\]

where

\[
K_\alpha(r, \theta; \rho, \phi) = \sqrt{\frac{1 - i \cot \alpha}{2\pi}} \exp \left( \frac{i r^2 + \rho^2}{2} \cot \alpha - i r \rho \cos(\theta - \phi) \csc \alpha \right). \tag{14}
\]

Hence, in terms of polar coordinates, the FRFT operation transforms the spatial position radius and angle \((r, \theta)\) to the frequency radius and angle \((\rho, \phi)\). We use \( t = (t_1, t_2) \) to represent \((r, \theta)\) in physical polar coordinates and \( u = (u_1, u_2) \) to denote the frequency vector \((\rho, \phi)\) in frequency polar coordinates.

Moreover, the following exponent expansion formulas are valid [27]

\[
\exp(-it \cdot u) = \exp[-ipr \cos(\phi - \theta)] = \sum_{m=-\infty}^{\infty} i^{-m} J_m(p r) \exp(-im\theta) \exp(im\phi), \tag{15}
\]

\[
\exp(it \cdot u) = \exp[ipr \cos(\phi - \theta)] = \sum_{m=-\infty}^{\infty} i^m J_m(p r) \exp(im\theta) \exp(-im\phi). \tag{16}
\]

The relation between 2D FRFT and 2D FT satisfies

\[
F^\alpha(\rho, \phi) = \sqrt{\frac{1 - i \cot \alpha}{2\pi}} \exp \left( \frac{i p^2}{2} \cot \alpha \right) F[\tilde{f}](\rho \csc \alpha, \phi). \tag{17}
\]

3.2. Relationship Between the 2D FRFT and the FRHT

It is well known that a HT can be obtained by a 2D FT. In the same case, we can also establish the connection between 2D FRFT and FRHT.

The Fourier series are defined as

\[
f(t) = f(r, \theta) = \sum_{n=-\infty}^{\infty} f_n(r, \theta) \exp(in\theta), \tag{18}
\]

and

\[
F(u) = F(\rho, \phi) = \sum_{n=-\infty}^{\infty} F_n(\rho, \phi) \exp(in\phi), \tag{19}
\]

where \( f_n(r) \) and \( F_n(\rho) \) can be interpreted in terms of a HT as [24,28]

\[
f_n(r) = 2\pi i^n H_n[f_n](r), \tag{20}
\]

\[
F_n(\rho) = 2\pi i^{-n} H_n[f_n](\rho). \tag{21}
\]

Similarly, we can define the Fourier expansion of 2D FRFT

\[
F^\alpha(u) = F^\alpha(\rho, \phi) = \sum_{n=-\infty}^{\infty} F^\alpha_n(\rho, \phi) \exp(in\phi). \tag{22}
\]
By (19) and (21), we can obtain

\[
\sqrt{1 - i \cot \alpha \over 2\pi} \exp \left( i \frac{\rho^2}{2} \cot \alpha \right) \mathcal{F}[\tilde{f}](\rho \csc \alpha, \phi) = 2\pi \sqrt{1 - i \cot \alpha \over 2\pi} \exp \left( i \frac{\rho^2}{2} \cot \alpha \right) \cdot \sum_{n=-\infty}^{\infty} i^{-n} \mathcal{H}_n[\tilde{f}_n](\rho \csc \alpha) \exp(i n \phi),
\]

(23)

where \( \tilde{f}_n(r) = f_n(r) \exp (\frac{i}{2} r^2 \cot \alpha) \).

Using (10) and (17), it follows that

\[
\mathcal{F}^{\alpha}(\rho, \phi) = 2\pi \sqrt{1 - i \cot \alpha \over 2\pi} \exp \left( i \frac{\rho^2}{2} \cot \alpha \right) \sum_{n=-\infty}^{\infty} i^{-n} \frac{1}{\mathcal{D}_{n,\alpha}(\rho)} \mathcal{H}_n[\tilde{f}_n](\rho) \exp(i n \phi),
\]

(24)

where \( \tilde{f}_n(r) = f_n(r) \exp(i r^2 \cot \alpha) \).

According to (22) and (24), we have

\[
\mathcal{F}^{\alpha}_n(\rho) = \frac{\sqrt{2\pi(1 - i \cot \alpha)} \sin \alpha \exp \left( i \frac{\rho^2}{2} \cot \alpha \right) i^{-n} \mathcal{H}_n[\tilde{f}_n](\rho)}{\exp[i(1 + n)(\frac{\rho^2}{2} - \alpha)]}.
\]

(25)

From (5) and (25), we can derive the following result

\[
\mathcal{F}^{\alpha}_n(\rho) = \sqrt{2\pi(1 - i \cot \alpha) \exp \left( i \frac{\rho^2}{2} \cot \alpha \right) i^{-n} \int_{0}^{\infty} f_n(r) \exp \left( i \frac{\rho^2}{2} \cot \alpha \right) f_n(r \rho \csc \alpha) r dr}.
\]

(26)

The corresponding 2D inverse FRFT is written as

\[
f(t) = f(r, \theta) = \int_{0}^{\infty} \int_{-\pi}^{\pi} \mathcal{F}^{\alpha}(\rho, \phi) K_{-\alpha}(r, \theta; \rho, \phi) \rho d \rho d \phi.
\]

(27)

Substitute the expansions (16), (17) into (27) and we have

\[
f(t) = f(r, \theta) = \int_{0}^{\infty} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \mathcal{F}^{\alpha}_n(\rho) \exp(i n \phi) \sqrt{1 + i \cot \alpha \over 2\pi} \exp \left( -i \frac{\rho^2}{2} \cot \alpha + i \rho \cos(\theta - \phi) \csc \alpha \right) \rho d \rho d \phi
\]

\[
= \sum_{n=-\infty}^{\infty} \sqrt{2\pi(1 + i \cot \alpha)} \exp(-i \frac{\rho^2}{2} \cot \alpha) \cdot \sum_{m=-\infty}^{\infty} i^m f_n(r \rho \csc \alpha) \exp(i m \phi) \exp(-i m \phi) \rho d \rho d \phi
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i^m f_n(r \rho \csc \alpha) \exp(i m \phi) \exp(-i m \phi) \rho d \rho d \phi
\]

\[
= \sum_{m=0}^{\infty} \int_{0}^{\infty} \exp(-i \frac{\rho^2}{2} \cot \alpha) \mathcal{F}^{\alpha}_n(\rho) f_n(r \rho \csc \alpha) \rho d \rho \exp(i m \phi),
\]

(28)

since

\[
\int_{-\pi}^{\pi} \exp(i m \phi) d \phi = 2\pi \delta_{m0} = \begin{cases} 2\pi, & n = 0, \\ 0, & otherwise, \end{cases}
\]

(29)

where \( \delta_{nm} \) denotes the Kronecker delta function.
3.3. Spatial Shift Theorem

**Theorem 1.** Let \( t_0 = (t_3, t_4) \in \mathbb{R}^2, t_3 = r_0 \cos \theta_0, t_4 = r_0 \sin \theta_0, \) using polar coordinates. Then, we have

\[
f(t - t_0) = 2\pi \csc \alpha \exp \left( -i \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{2} \cot \alpha \right) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp(-i(m-n)\theta_0) \exp(im\theta)
\]

\[
= \int_0^\infty f_n(v) \exp(i \frac{v^2}{2} \cot \alpha) R_m^n(v, r, r_0) \, dv,
\]

where \( R_m^n(v, r, r_0) = \int_0^\infty f_n(\frac{vp \csc \alpha}{m}) f_m(r \rho \csc \alpha) f_{m-n}(r_0 \rho \csc \alpha) \, d\rho. \)

**Proof.** According to (11), it follows that

\[
f(t - t_0) = \int_0^\infty \int_{-\pi}^{\pi} F_n^m(\rho, \phi) \sqrt{1 + i \cot \alpha} \, \exp \left( -i \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{2} \cot \alpha \right) 
\]

\[
\cdot \exp \left( i\rho \cos(\theta - \phi) \csc \alpha - iv_0 \rho \cos(\theta - \phi) \csc \alpha \right) \, d\rho \, d\phi
\]

\[
= \sqrt{1 + i \cot \alpha} \, \exp \left( -i \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{2} \cot \alpha \right) 
\]

\[
\cdot \int_0^\infty \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} F_n^m(\rho) \exp(im\phi) \, \exp\left( -i \frac{\rho^2}{2} \cot \alpha \right) 
\]

\[
\cdot \sum_{m=-\infty}^{\infty} i^m f_m(\frac{\rho \csc \alpha}{m}) \exp(-im\phi) 
\]

\[
\cdot \sum_{k=-\infty}^{\infty} i^{-k} f_k(\frac{\rho \csc \alpha}{m}) \exp(-ik\theta_0) \exp(ik\phi) \, d\phi \, d\rho.
\]

For \( k = m - n, \) performing the integration over \( \phi, \) the formula (32) becomes

\[
f(t - t_0) = \sqrt{2\pi(1 + i \cot \alpha)} \, \exp \left( -i \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{2} \cot \alpha \right) 
\]

\[
\cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} i^m \exp(-i(m-n)\theta_0) \exp(im\theta)
\]

\[
\cdot \int_0^\infty F_n^m(\rho) \exp\left(-i \frac{\rho^2}{2} \cot \alpha \right) f_m(\frac{\rho \csc \alpha}{m}) f_{m-n}(r_0 \rho \csc \alpha) \, d\rho.
\]

By (26), we obtain

\[
f(t - t_0) = 2\pi \csc \alpha \exp \left( -i \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{2} \cot \alpha \right) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp(-i(m-n)\theta_0)
\]

\[
\cdot \exp(im\theta) \int_0^\infty f_n(v) \exp(i \frac{v^2}{2} \cot \alpha) f_n(\frac{vp \csc \alpha}{m}) f_m(r \rho \csc \alpha) f_{m-n}(r_0 \rho \csc \alpha) \, d\rho.
\]
3.4. Multiplication Theorem

**Definition 2.** The convolution operation $\ast$ of 2D FRFT in polar coordinates for two series $f_k(r)$ and $g_k(r)$ is defined by

$$ (f_k \ast g_k)(r) = \sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r). \quad (35) $$

**Theorem 2.** Let $h(t) = f(t)g(t)$, where $f(t) = f(r, \theta) = \sum_{k=-\infty}^{\infty} f_k(r) \exp(ik\theta)$, $g(t) = g(r, \theta) = \sum_{k=-\infty}^{\infty} g_k(r) \exp(ik\theta)$ and $h(t) = h(r, \theta) = \sum_{k=-\infty}^{\infty} h_k(r) \exp(ik\theta)$. Then,

$$ h_k = f_k \ast g_k. \quad (36) $$

**Proof.** According to (13) and (14), we get the following result

$$ F^\alpha[h(r, \theta)] = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} f(r, \theta) g(r, \theta) K_\alpha(r, \theta; \rho, \phi) r dr d\theta $$

$$ = \int_{0}^{\infty} \int_{0}^{\pi} \sum_{n=-\infty}^{\infty} f_n(r) \exp(in\theta) \sum_{m=-\infty}^{\infty} g_m(r) \exp(im\theta) $$

$$ \cdot \frac{1}{\sqrt{1-i\cot \alpha}} \exp\left(\frac{i^2 + \rho^2}{2} \cot \alpha - ir\rho \cos(\theta - \phi) \csc \alpha\right) r dr d\theta. \quad (37) $$

Using (15), (32), we have

$$ F^\alpha[h(r, \theta)] = \int_{0}^{\infty} \int_{0}^{\pi} \sum_{n=-\infty}^{\infty} f_n(r) \exp(in\theta) \sum_{m=-\infty}^{\infty} g_m(r) \exp(im\theta) $$

$$ \cdot \frac{1}{\sqrt{1-i\cot \alpha}} \exp\left(\frac{i^2 + \rho^2}{2} \cot \alpha - ir\rho \cos(\theta - \phi) \csc \alpha\right) r dr d\theta. \quad (38) $$

For $n = k - m$, performing the integration over $\theta$, we obtain

$$ F^\alpha[h(r, \theta)] = \sum_{k=-\infty}^{\infty} \frac{\sqrt{2\pi(1-i\cot \alpha) \exp(i^2/2 \cot \alpha)i^{-k}}}{\sqrt{1-i\cot \alpha}} $$

$$ \cdot \int_{0}^{\infty} \sum_{m=-\infty}^{\infty} f_{k-m}(r) \exp(i^2/2 \cot \alpha) g_m(r) h_k(r) \exp(ik\phi) r dr. \quad (39) $$

Using (22) and (26), we obtain

$$ F^\alpha_k(\rho) = \sqrt{2\pi(1-i\cot \alpha) \exp(i^2/2 \cot \alpha)i^{-k}} \int_{0}^{\infty} \sum_{m=-\infty}^{\infty} f_{k-m}(r) \exp(i^2/2 \cot \alpha) g_m(r) h_k(r) \exp(ik\phi) r dr $$

$$ = \sqrt{2\pi(1-i\cot \alpha) \exp(i^2/2 \cot \alpha)i^{-k}} \int_{0}^{\infty} h_k(r) \exp\left(\frac{i^2}{2} \cot \alpha\right) l_v(\rho \csc \alpha) r dr. \quad (40) $$

Hence

$$ h_k(r) = \sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r). \quad (41) $$

We have

$$ (fg)_k = f_k \ast g_k. \quad (42) $$
4. Solvability for One Class of Convolution Equations

In this section, we first rewrite the convolution integral Equation (3) in polar coordinates. Based on the convolution integral equation, a novel convolution operator is proposed, and the corresponding convolution theorem is investigated. Then the solvability of the convolution equation is established using the 2D FRFT and convolution theorem.

Using polar coordinates \( t = (t_1, t_2) = (r \cos \theta, r \sin \theta), \ s = (s_1, s_2) = (s \cos \theta_s, s \sin \theta_s) \), the convolution integral Equation (3) can be rewritten as

\[
\lambda \varphi(t) + \int_0^\infty g(s) f(t - s) \xi(t, s) ds = f(t), \quad (43)
\]

where \( \lambda \in \mathbb{C}, f, g \in L^2(\mathbb{R}^2) \) are given and \( \varphi \) is unknown, and the kernel function \( \xi(t, s) \) is

\[
\xi(t, s) = \exp(ir^2 \cot \alpha) \exp(-irr_s \cos(\theta - \theta_s) \cot \alpha). \quad (44)
\]

Based on (43) and (44), a novel convolution operator is proposed and the corresponding convolution theorem is investigated in the following subsection.

4.1. Convolution Theorem

**Definition 3.** The convolution operation \( \star^2 \) of 2D FRFT for two functions \( f(t) \) and \( g(t) \) is defined by

\[
f(t) \star^2 g(t) = \int_0^\infty g(s) f(t - s) \xi(t, s) ds, \quad (45)
\]

where \( \xi(t, t_0) \) is given by (44).

**Theorem 3.** Let \( y(t) = f(t) \star^2 g(t) \). Then,

\[
F^\alpha[y](u) = \Psi(u)F^\alpha[g](u)F^\alpha[f](u), \quad (46)
\]

where \( \Psi(u) = \sqrt{2\pi \csc \alpha} \exp(-i\frac{u^2}{2} \cot \alpha) \).

**Proof.** According to (18) and (30), we have

\[
y(t) = f(t) \star^2 g(t) = \int_0^\infty \int_0^\pi \int_{-\infty}^{\infty} g_1(r_1) \exp(ik \theta_s) \exp(ir^2 \cot \alpha) \exp(-irr_s \cos(\theta - \theta_s) \cot \alpha)
\]

\[
\cdot 2\pi \csc \alpha \exp\left(-i \frac{r^2}{2} + \frac{2rr_s \cos(\theta - \theta_s)}{\cot \alpha}\right) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp(-im \theta_s \exp(im \phi))
\]

\[
\cdot \int_0^\infty \int_0^\infty f_n(v) \exp\left(i \frac{v^2}{2} \cot \alpha\right) J_n(rp \csc \alpha) J_m(rp \csc \alpha) J_{m-n}(rs \csc \alpha) vd\theta v dp dr \theta_s. \quad (47)
\]
So it follows from (26) and (29) that
\[
y(t) = 4\pi^2 \csc \alpha \exp\left(-\frac{r^2}{2} \cot \alpha\right) \sum_{m=-\infty}^{\infty} \exp(i m \theta) \\
\cdot \int_0^\infty \int_0^\infty \int_0^\infty g_{m-n}(r) \exp\left(i \frac{r^2}{2} \cot \alpha\right) I_{m-n} (r \rho \csc \alpha) r \delta \, dr \\
\cdot \int_0^\infty f_n(v) \exp\left(i \frac{v^2}{2} \cot \alpha\right) I_m (r \rho \csc \alpha) \rho \, d\rho
\]
\[= \sum_{m=-\infty}^{\infty} \frac{2\pi}{1-i \cot \alpha} \csc \alpha \exp\left(-\frac{r^2}{2} \cot \alpha\right) i^m \\
\cdot \int_0^\infty \exp(-i \rho^2 \cot \alpha) \sum_{n=-\infty}^{\infty} F_{m-n}^\alpha [g](\rho) F_{n}^\alpha [f](\rho) I_m (r \rho \csc \alpha) \rho \, d\rho \exp(i m \theta).\] (48)

According to (28), we have
\[
F_{m}^\alpha [y](\rho) = \sqrt{\frac{2\pi}{1-i \cot \alpha}} \exp(-\frac{\rho^2}{2} \cot \alpha) \sum_{n=-\infty}^{\infty} F_{m-n}^\alpha [g](\rho) F_{n}^\alpha [f](\rho).\] (49)

In Subsection 3.4, it was shown that the convolution of two sets of FRFT coefficients is equivalent to multiplication of the functions so that \(F_{m}^\alpha [g] \ast F_{m}^\alpha [f] = (F\alpha [g] F\alpha [f])_m\), hence it follows from (49) that
\[
F\alpha [y](u) = \Psi(u) F\alpha [g](u) F\alpha [f](u).\] (50)
\[\square\]

**Remark 1.** By putting \(\alpha = \frac{\pi}{2}\) in the above theorem, we obtain a convolution theorem for the 2D FT in polar coordinates \[24\].

### 4.2. Solvability Analysis

Using the proposed convolution operation \(\ast^2\), the convolution Equation (43) can be rewritten as
\[
\lambda \varphi(t) + (g \ast^2 \varphi)(t) = f(t),\] (51)
where \(\lambda \in \mathbb{C}\), \(f, g \in L^2(\mathbb{R}^2)\) are given and \(\varphi\) is unknown.

In order to obtain main result, we give the following lemma.

**Lemma 1.** Let \(Y(u) := \lambda + \Psi(u) F\alpha [g](u)\). Then we have

1. If \(\lambda \neq 0\), then there exists a constant \(M > 0\), such that \(Y(u) \neq 0\) for every \(|u| > M\).
2. If for all \(u \in \mathbb{R}^2\), \(Y(u) \neq 0\), then \(\frac{1}{\Psi(u)}\) is bounded and continuous on \(\mathbb{R}^2\).

The proof of Lemma 1 is similar to those of [18], and we omit the proof in this paper.

**Theorem 4.** Let \(Y(u) \neq 0\) for all \(u \in \mathbb{R}^2\). Suppose that one of the following two conditions holds:

1. \(\lambda \neq 0\) and \(F\alpha [f](u) \in L^2(\mathbb{R}^2)\);
2. \(\lambda = 0\) and \(F\alpha [f](\parallel u \parallel) \in L^2(\mathbb{R}^2)\).

Then Equation (51) has a solution in \(L^2(\mathbb{R}^2)\) if and only if \(F^{-\alpha} \left(\frac{F\alpha [f]}{\Psi}\right) \in L^2(\mathbb{R}^2)\).

Furthermore, the solution has the form of \(\varphi = F^{-\alpha} \left(\frac{F\alpha [f]}{\Psi}\right)\), where \(F^{-\alpha}\) is the inversion formula of 2D FRFT.
Proof. First, we study the case (1).

Necessity: Suppose that Equation (51) has a solution \( \varphi \in L^2(\mathbb{R}^2) \). Multiplying \( F^\alpha \) to both sides of (51), we get

\[
\lambda F^\alpha[\varphi](u) + F^\alpha(g \ast^2 \varphi)(u) = F^\alpha[f](u). \tag{52}
\]

Using (46), we obtain

\[
\lambda F^\alpha[\varphi](u) + \Psi(u) F^\alpha[g](u) F^\alpha[\varphi](u) = F^\alpha[f](u). \tag{53}
\]

Hence

\[
(\lambda + \Psi(u) F^\alpha[g](u)) F^\alpha[\varphi](u) = F^\alpha[f](u). \tag{54}
\]

Since \( \lambda \neq 0 \), then \( \Upsilon(u) = \lambda + \Psi(u) F^\alpha[g](u) \neq 0 \). The Equation (54) becomes

\[
F^\alpha[\varphi] = \frac{F^\alpha[f]}{\Upsilon}. \tag{55}
\]

According Lemma 1, \( \frac{1}{\Upsilon(u)} \) is bounded and continuous on \( \mathbb{R}^2 \) and \( F^\alpha[f] \in L^2(\mathbb{R}^2) \), we have \( F^\alpha[\varphi] \in L^2(\mathbb{R}^2) \). Applying the inverse transform of \( F^\alpha \) to (55), we obtain the result.

Sufficiency: Let

\[
\varphi = F^{-\alpha}\left(\frac{F^\alpha[f]}{\Upsilon}\right). \tag{56}
\]

We have \( \varphi \in L^2(\mathbb{R}^2) \). Hence, we get \( F^\alpha[\varphi] = \frac{F^\alpha[f]}{\Upsilon} \). That is, \( \Upsilon(F^\alpha[\varphi]) = F^\alpha[f] \). Using (46), we obtain

\[
F^\alpha\{\lambda \varphi(t) + (g \ast^2 \varphi)(t)\}(u) = F^\alpha[f](u). \tag{57}
\]

By the uniqueness of \( F^\alpha \), \( \varphi \) satisfies the Equation (51) for almost every \( t \in \mathbb{R}^2 \), hence (1) is proved. The case of (2) may be proved similarly to that of case (1). \( \square \)

5. Conclusions

In this paper, we have investigated the solvability of a class of convolution equations using 2D FRFT in polar coordinates. Firstly, 2D FRFT is generalized to the polar coordinates setting, and the relationship between 2D FRFT and FRHT is derived. Then applying this relationship, the spatial shift and multiplication theorems for 2D FRFT are obtained. In order to analyze the solvability of the convolution equations, a novel convolution operator for 2D FRFT is proposed, and the corresponding convolution theorem is investigated. Finally, based on the proposed theorems, the solvability of the convolution equations is studied.

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