Automated Proofs of Unique Normal Forms w.r.t. Conversion for Term Rewriting Systems

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July 4, 2018

Abstract

The notion of normal forms is ubiquitous in various equivalent transformations. Confluence (CR), one of the central properties of term rewriting systems (TRSs), concerns uniqueness of normal forms. Yet another such property, which is weaker than confluence, is the property of unique normal forms w.r.t. conversion (UNC). Famous examples having UNC but not CR include the TRSs consisting of S,K,I-rules for the combinatory logic supplemented with various pairing rules (de Vrijer, 1999). Recently, automated confluence proof of TRSs has caught attentions leading to investigations of automatable methods for (dis)proving CR of TRSs; some powerful confluence tools have been developed as well. In contrast, there have been little efforts on (dis)proving UNC automatically yet. Indeed, there are few tools that are capable of (dis)proving UNC; furthermore, only few UNC criteria have been elaborated in these tools. In this paper, we address automated methods to prove or disprove UNC of given TRSs. We report automation of some criteria known so far, and also present some new criteria and methods for proving or disproving UNC. Presented methods are implemented over the confluence prover ACP (Aoto et al., 2009) and an experimental evaluation is reported.

1 Introduction

The notion of normal forms is ubiquitous in various equivalent transformations—normal forms are objects that can not be transformed further. Two crucial issues arise around the notion of normal forms—one is whether any object has a normal form and the other is whether they are unique, so that normal forms can represent the equivalence classes of objects. The former issue arises various kinds of termination problems. For the latter, the notion of confluence (CR), namely that $s \leftarrow \circ \rightarrow t$ implies $s \rightarrow \circ \leftarrow t$ for any objects $s, t$, is most well-studied. Here, $\rightarrow$ is the reflexive transitive closure of an equivalent transformation $\rightarrow$, and $\circ$ stands for the composition. In fact, in the efforts of proving uniqueness of the normal forms, one encounters the situation of analyzing ‘local’ peaks $s \leftarrow \circ \rightarrow t$, and then, in order to apply the induction, one needs to consider (general) peaks $s \leftarrow \circ \rightarrow t$. This naturally leads to the notion of confluence. In term rewriting, confluence of various systems, as well as general theories of confluence for establishing confluence of systems in various classes of rewriting systems have been investigated (see e.g. [Toy05] for a survey).

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Yet another such a property is the property of unique normal forms w.r.t. conversion (UNC), namely that two convertible normal forms are identical, i.e. \( s \overset{\leftrightarrow}{\rightarrow} t \) with normal forms \( s, t \) implies \( s = t \). Interestingly, CR implies UNC, and this implication is proper, i.e. UNC does not imply CR. Thus, even if the system lacks CR, there still exists a hope that the system retains UNC.

In term rewriting, famous examples having UNC but not CR include term rewriting systems (TRSs) consisting of S,K,I-rules for the combinatory logic supplemented with various pairing rules \cite{Ko80, dV99}, whose non-CR have been shown in \cite{Ko80}. In contrast to CR, the property UNC directly captures the uniqueness of normal forms in equivalence classes of objects, which is one of the motivation for verifying CR. Therefore, it is anticipated that many applications would be considered, once much more powerful techniques to achieve UNC were obtained.

Compared to CR, however, analyzing UNC is not (yet) very straightforward. Indeed, not much has been studied on UNC in the field of term rewriting—below, we present the short list of known results on UNC in term rewriting.

**Proposition 1** \cite{KS16}. Any non-\( \omega \)-overlapping TRS has UNC.

This recent proposition is, in fact, an old open problem known as Chew’s problem \cite{Che81, MO01}, and properly generalizes one of the earliest UNC results that strongly non-overlapping TRSs have UNC \cite{Ko80, dV99}.

**Proposition 2** \cite{Mid90}. UNC is modular for the direct sum.

This is one of the earliest results on the modularity of TRSs, where a property \( \varphi \) of TRSs is modular for the direct sum if \( \varphi(R) \) and \( \varphi(S) \) implies \( \varphi(R \cup S) \) for TRSs \( R \) and \( S \) over disjoint sets of function symbols. Modularity holds for some cases having an overlap between sets of function symbols, namely, for layer-preserving decomposition \cite{AT96} and persistent decomposition \cite{AT97} (we refer to \cite{AYT09} for these terminologies).

It is undecidable whether \( R \) has UNC for a given TRS \( R \) in general. But for some subclasses of TRSs, it is known that it is decidable whether the given TRS in the classes has UNC. Concerning the (un)decidability results, we here only present ones on the positive side, despite some important negative ones are known as well.

**Proposition 3** \cite{DHLT90}. UNC is decidable for left-linear right-ground TRSs.

**Proposition 4** \cite{RMV17}. UNC is decidable for shallow TRSs.

For the former result, we remark that for the class of left-linear right-ground TRSs first-order theory of rewriting is decidable \cite{DHLT90}. For the latter result, we remark that, in contrast to UNC, CR is undecidable for flat TRSs \cite{MOJ06}, which is a subclass of shallow TRSs. Another obvious class for which UNC is decidable is terminating TRSs.

**Proposition 5** \cite{TO01}. Any non-duplicating weight-decreasing joinable TRS has UNC.

This criterion is based on a closure condition of conditional critical pairs, arising from conditional linearization of TRSs. In contrast to various critical pair closure conditions for ensuring confluence (e.g. \cite{Hue80, vO97, Gra96, Oku98}), few such criteria have been known for UNC.

Recently, automated confluence proof of TRSs has caught attentions leading to investigations of automatable methods for (dis)proving CR of TRSs; some powerful confluence tools have been

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1The uniqueness of normal forms w.r.t. conversion is also often abbreviated as UN in the literature; here, we prefer UNC to distinguish it from a similar but different notion of unique normal forms w.r.t. reduction (UNR), following the convention employed in CoCo (Confluence Competition).
developed as well, such as ACP [AYT09, CSI [NFM17], Saigawa [HK12] for TRSs, and also tools for other frameworks such as conditional TRSs and higher-order TRSs. This leads to the emergence of the Confluence Competition (CoCo2) yearly efforts since 2012.

In contrast, there have been little efforts on (dis)proving UNC automatically yet. Indeed, there are few tools that are capable of (dis)proving UNC; furthermore, only few UNC criteria have been elaborated in these tools. In CoCo 2017, the category of UNC runs for the first time[3]. Techniques used by participants are summarized as follows: (1) UNC is decidable for ground TRSs (in polynomial time) [Fel16], (2) UNC is decidable for left-linear right-ground TRSs [DHLT90] and (3) any non-ω-overlapping TRS has UNC [KS16].

In this paper, we address automated methods to prove or disprove UNC. Main contributions of the paper are summarized as follows.

• We report new UNC criteria based on the conditional linearization technique, namely that TRSs have UNC if their conditional linearization is parallel-closed or linear strongly closed (Theorems 10 and 13). We also report on automation of these criteria. Contrast to the earlier result (UNC of strong non-overlapping TRSs) based on the conditional linearization technique, these results are not subsumed by Proposition 1.

• We present a UNC criterion which generalizes Proposition 5 given in [TO01], and show how one can effectively check the criterion. To be more precise on the first item, we present a critical pair criterion ensuring the (abstract) weight-decreasing joinability, which is slightly general than the one given in [TO01].

• We present a novel method, UNC completion, for proving and disproving UNC, and show its correctness (Theorem 26). The method is another application of an abstract UNC principle behind the conditional linearization technique. It turns out that the method is much effective for proving and disproving UNC of our testbed from Cops (Confluence problems) database, compared to the conditional linearization approach.

• We give a transformational method effective for (dis)proving UNC, named rule reversing transformation, and show its correctness (Theorem 28). The transformation experimentally turns out to work effectively when combined with the UNC completion.

• We present a simple UNC criterion, named right-reducibility (Theorem 31).

• We implement UNC criteria except for the decidability results (Propositions 3 and 4), and an experimental evaluation is performed on a testbed from Cops database. Our implementation is built over our confluence prover ACP [AYT09] and is freely available.

The rest of the paper is organized as follows. After introducing necessary notions and notations in Section 2, we first revisit the conditional linearization technique for proving UNC, and obtain new UNC criteria based on this approach in Section 3. In Section 4, we present a slightly generalized version of the critical pair criterion presented in the paper [TO01], and report an automation of the criterion based on Proposition 5. In Section 5, we present our novel methods for proving or
disproving UNC. We show an experiment of the presented methods in Section 6, and report our
confluence prover ACP which newly supports UNC (dis)proving in Section 7. Section 8 concludes.
Most proofs are given in the Appendix.

2 Preliminaries

We now fix notions and notations used in the paper. We assume familiarity with basic notions in
term rewriting (e.g. [BN98]).

We use \( \sqcup \) to denote the multiset union and \( \mathbb{N} \) the set of natural numbers. A sequence of objects
\( a_1, \ldots, a_n \) is written as \( \bar{a} \). Negation of a predicate \( P \) is denoted by \( \neg P \).

The composition of relation \( R \) and \( S \) is denoted by \( R \circ S \). Let \( \rightarrow \) be a relation on a set \( A \). The
reflexive transitive (reflexive, symmetric, equivalent) closure of the relation \( \rightarrow \) is denoted by \( \Rightarrow \)
(resp. \( \Rightarrow^*, \leftrightarrow \)). The set NF of normal forms w.r.t. the relation \( \rightarrow \) is given by \( NF = \{ a \in A \mid a \rightarrow b \)
for no \( b \in \mathbb{N} \} \). The relation \( \rightarrow \) has unique normal forms w.r.t. conversion (denoted by UNC\( (\rightarrow) \))
if \( a \Rightarrow^* b \) and \( a, b \in NF \) imply \( a = b \). The relation \( \rightarrow \) is confluent (denoted by CR\((\rightarrow) \)) if
\( \Leftrightarrow \circ \Rightarrow^* \subseteq \Rightarrow^* \circ \Leftrightarrow^* \). When we consider two relations \( \rightarrow_1 \) and \( \rightarrow_2 \), the respective sets of normal forms w.r.t. \( \rightarrow_1 \) and \( \rightarrow_2 \) are denoted by NF1 and NF2. The following proposition, which is proved easily, is a basis of the conditional linearization technique, which will be used in Sections 3 and 4.

Proposition 6 ([KdV90] [dV99]). Suppose \( (1) \rightarrow_0 \subseteq \rightarrow_1 \), \( (2) \text{CR}(\rightarrow_1) \), and \( (3) \text{NF}_0 \subseteq \text{NF}_1 \). Then, \( \text{UNC}(\neg_0) \).

The set of terms over the set \( \mathcal{F} \) of arity-fixed function symbols and denumerable set \( V \) of
variables is denoted by \( T(\mathcal{F}, V) \). The set of variables (in a term \( t \)) is denoted by \( V(t) \) (resp. \( V(t) \)). A
term \( t \) is ground if \( V(t) = \emptyset \). We abuse the notation \( V(t) \) and denote by \( V(e) \) the set of variables occurring in
any sequence \( e \) of expressions. The subterm of a term \( t \) at a position \( p \) is denoted by \( t|_p \). The root position is denoted by \( \epsilon \). A context is a term containing a special constant \( \square \)
called hole). If \( C \) is a context containing \( n \)-occurrences of the hole, \( C[t_1, \ldots, t_n] \) denotes the term
obtained from \( C \) by replacing holes with \( t_1, \ldots, t_n \) from left to right; we write \( C[t_1, \ldots, t_n]_{p_1, \ldots, p_n} \)
if the occurrences of holes in \( C \) are at the positions \( p_1, \ldots, p_n \). For positions \( p_1, \ldots, p_n \) in a term \( s \), the expression \( s[t_1, \ldots, t_n]_{p_1, \ldots, p_n} \) denotes the term obtained from \( s \) by replacing subterms
at the positions \( p_1, \ldots, p_n \) with terms \( t_1, \ldots, t_n \) respectively. We denote by \( |t|_x \) the number of occurrences of a variable \( x \) in a term \( t \). Again, we abuse the notation \( |t|_x \) and denote by \( |e|_x \)
the number of occurrences of a variable \( x \) in any sequence of expressions \( e \). A term \( t \) is linear if
\( |t|_x \leq 1 \) for any \( x \in V(t) \). A substitution \( \sigma \) is a mapping from \( V \) to \( T(\mathcal{F}, V) \) such that the set
dom(\( \sigma \)) = \( \{ x \in V \mid \sigma(x) \neq x \} \), called the domain of \( \sigma \), is finite. Each substitution is identified
with its homomorphic extension over \( T(\mathcal{F}, V) \). For simplicity, we often write \( ts \) instead of \( \sigma(t) \) for
substitutions \( \sigma \) and terms \( t \). A most general unifier \( \sigma \) of terms \( s \) and \( t \) is denoted by mgu(\( s, t \)).

An equation is a pair \( \langle l, r \rangle \) of terms, which is denoted by \( l \approx r \). When we indistinguish lhs and rhs of
the equation, we write \( l \approx r \). We identify equations modulo renaming of variables. For a set
or sequence \( \Gamma \) of equations, we denote by \( \Gamma \sigma \) the set or the sequence obtained by replacing each
equation \( l \approx r \) by \( l \sigma \approx r \sigma \). An equation \( l \approx r \) satisfying \( l \notin V \) and \( V(r) \subseteq V(l) \) is a rewrite rule
and written as \( l \rightarrow r \). A rewrite rule \( l \rightarrow r \) is linear if \( l \) and \( r \) are linear terms; it is left-linear
(right-linear) if \( l \) (resp. \( r \)) is a linear term. A rewrite rule \( l \rightarrow r \) is non-duplicating if \( |l|_x \geq |r|_x \)
for any \( x \in V(l) \). A term rewriting system (TRS, for short) is a finite set of rewrite rules. A
TRS is linear (left-linear, right-linear, non-duplicating) if so are all rewrite rules. A rewrite step of a
TRS \( \mathcal{R} \) (a set \( \Gamma \) of equations) is a relation \( \rightarrow_{\mathcal{R}} \) (resp. \( \leftrightarrow_{\mathcal{R}} \)) over \( T(\mathcal{F}, V) \) defined by \( s \rightarrow_{\mathcal{R}} t \) iff
s = C[la] and s = C[rs] for some l → r ∈ \( R \) (resp. \( l \cong r \in \Gamma \)) and context C and substitution σ. The position p such that \( C[p] = \square \) is called the redex position of the rewrite step, and we sometimes write \( s \rightarrow_{p,R} t \) to indicate the redex position of this rewrite step explicitly. A rewrite sequence is (finite or infinite) consecutive applications of rewrite steps. A rewrite sequence of the form \( t_1 R \leftarrow t_0 \rightarrow_R t_2 \) is called a local peak.

Let \( l_1 \rightarrow r_1 \) and \( l_2 \rightarrow r_2 \) be rewrite rules such that \( \forall(l_1) \cap \forall(l_2) = \emptyset \). Suppose that there exists a position \( p \) in \( l_2 \) such that \( l_2[p] \) and \( l_1 \) are unifiable and \( \sigma = \text{mgu}(l_1, l_2[p]) \). A local peak \( l_2[p] \rightarrow_{R} l_2 \rightarrow_{R} r_2 \sigma \) is called a critical peak of the rewrite rule \( l_1 \rightarrow r_1 \) over the rewrite rule \( l_2 \rightarrow r_2 \), provided that it is not the case that \( p = \epsilon \) and \( l_1 \rightarrow r_1 \) and \( l_2 \rightarrow r_2 \) are identical. The term pair \( (l_2[p], r_2) \) is called a critical pair in \( R \). It is called an overlay critical pair if \( p = \epsilon \); it is called an inner-outer critical pair if \( p \neq \epsilon \). The set of (overlay, inner-outer) critical pairs from rules in a TRS \( R \) is denoted by \( \text{CP}(R) \) (resp. \( \text{CP}_{\text{out}}(R), \text{CP}_{\text{in}}(R) \)).

Let \( l \approx r \) be an equation and let \( \Gamma \) be a sequence \( s_1 \approx t_1, \ldots, s_k \approx t_k \) of equations. An expression of the form \( \Gamma \approx l \approx r \) is called a conditional equation. A conditional equation \( \Gamma \approx l \approx r \) is a conditional rewrite rule if \( l \notin V \); in this case \( \Gamma \approx l \approx r \) is written as \( l \rightarrow r \approx \Gamma \). The sequence \( \Gamma \) is called the condition part of the conditional rewrite rule. A finite set of conditional rewrite rules is called a conditional term rewriting system (CTRS, for short). A CTRS is left-linear if so are all rewrite rules. CTRS \( R \) is said to be of type \( \beta \) (type 1) if \( \forall(r) \subseteq \forall(l) \cup \forall(\epsilon) \) (resp. \( \forall(r) \subseteq \forall(l) \)) for all \( l \rightarrow r \approx \epsilon \in \Gamma \).

The notion of critical pairs of TRSs is naturally generalized to the notion of conditional critical pairs of CTRSs. Let \( l_1 \rightarrow r_1 \leftarrow \Gamma_1 \) and \( l_2 \rightarrow r_2 \leftarrow \Gamma_2 \) be conditional rewrite rules such that \( \forall(l_1, r_1, \Gamma_1) \cap \forall(l_2, r_2, \Gamma_2) = \emptyset \). Suppose that \( l_2[p] \) and \( l_1 \) are unifiable and \( \sigma = \text{mgu}(l_1, l_2[p]) \). Then the ternary relation of a sequence of equations and two terms \( \Gamma_1 \sigma, \Gamma_2 \sigma \Rightarrow (l_2[p] \sigma, r_2 \sigma) \) is called a conditional critical pair, provided that it is not the case that \( p = \epsilon \) and \( l_1 \rightarrow r_1 \leftarrow \Gamma_1 \) and \( l_2 \rightarrow r_2 \leftarrow \Gamma_2 \) are identical. Here, \( \Gamma_1 \sigma, \Gamma_2 \sigma \) is a sequence of equations obtained by the juxtaposition of sequences \( \Gamma_1 \sigma \) and \( \Gamma_2 \sigma \). It is called overlay if \( p = \epsilon \); it is called inner-outer if \( p \neq \epsilon \). The set of conditional critical pairs from conditional rewrite rules in a CTRS \( R \) is denoted by \( \text{CP}(R) \) (resp. \( \text{CP}_{\text{out}}(R), \text{CP}_{\text{in}}(R) \)). A CTRS \( R \) is orthogonal if it is left-linear and \( \text{CP}(R) = \emptyset \).

Several types of CTRSs are distinguished according to how the condition part of the conditional rewrite rules is interpreted to define the rewrite steps. In this paper, we are interested in semi-equational CTRSs where the equations in condition parts are interpreted by convertibility \( \leftrightarrow \).

Formally, the conditional rewrite step \( \rightarrow_{\pi} \) of a semi-equational CTRS \( R \) is defined, using auxiliary relations \( \rightarrow_{\pi}^{(n)} \) \((n \geq 0)\), like this:

\[
\begin{align*}
\rightarrow_{\pi}^{(0)} & = \emptyset \\
\rightarrow_{\pi}^{(n+1)} & = \{(C[\sigma]), C[rs]) \mid l \rightarrow r \leftarrow s_1 \approx t_1, \ldots, s_k \approx t_k \in R, \\
& \quad \forall i \ (1 \leq i \leq k), s_i \sigma \leftrightarrow_{\pi}^{(n)} t_i \sigma \} \\
\rightarrow_{\pi} & = \bigcup_{n \in \mathbb{N}} \rightarrow_{\pi}^{(n)}
\end{align*}
\]

The rank of conditional rewrite step \( s \rightarrow_{\pi} t \) is the least \( n \) such that \( s \rightarrow_{\pi}^{(n)} t \).

Let \( R \) be a TRS or CTRS. The set of normal forms w.r.t. \( \rightarrow \) is written as \( \text{NF}(R) \). A (C)TRS \( R \) has UNCs (CR) if UNCs(\( \rightarrow_R \)) (resp. CR(\( \rightarrow_R \))) on the set \( T(F, V) \). Let \( E \) be a set or sequence of equations or rewrite rules. We denote \( \approx_E \) the congruence closure of \( E \). We write \( \vdash_E l \approx r \) if \( l \approx_E r \). For sets or sequences \( \Gamma \) and \( \Sigma \) of equations, we write \( \vdash_E \Sigma \) if \( \vdash_E l \approx r \) for all \( l \approx r \in \Sigma \), and \( \Gamma \vdash_E \Sigma \) if \( \vdash_E \Gamma \sigma \) implies \( \vdash_E \Sigma \sigma \) for any substitution \( \sigma \).
3 Conditional linearization revisited

The plan of this section is as follows: We first revisit the conditional linearization technique for proving UNC in Section 3.1. Then, we present two new UNC criteria based on this approach in Section 3.2. We remark on automation of check of the criteria in Section 3.3.

3.1 Conditional linearization

A conditional linearization is a translation from TRSs to CTRSs which eliminates non-left-linear rewrite rules, say \( f(x, x) \rightarrow r \), by replacing them with a corresponding conditional rewrite rules, such as \( f(x, y) \rightarrow r \iff x \approx y \). Formally, let \( l = C[x_1, \ldots, x_n] \) with all variable occurrences in \( l \) displayed (i.e. \( V(C) = \emptyset \)). Note here \( l \) may be a non-linear term and some variables in \( x_1, \ldots, x_n \) may be identical. Let \( l' = C[x'_1, \ldots, x'_n] \) where \( x'_1, \ldots, x'_n \) are mutually distinct fresh variables and \( \delta \) be a substitution such that \( \delta(x'_i) = x_i \) (1 \( \leq \) i \( \leq \) n) and \( \text{dom}(\sigma) = \{x'_1, \ldots, x'_n\} \). A conditional rewrite rule \( l' \rightarrow r' \Leftarrow \Gamma \) is a conditional linearization of a rewrite rule \( l \rightarrow r \) if \( r' = r \) and \( \Gamma \) is a sequence of equations of the form \( x_i \approx x_j \) (1 \( \leq \) i \( \leq \) j \( \leq \) n) such that \( x'_i \approx \Gamma \) \( x'_j \) iff \( x'_i \delta = x'_j \delta \) holds for any \( 1 \leq i, j \leq n \). A conditional linearization of a TRS \( \mathcal{R} \) is a semi-equational CTRS (denoted by \( \mathcal{R}^L \)) obtained by replacing each rewrite rule with its conditional linearization. Note that the results of conditional linearizations are not unique, and any results of conditional linearization is a left-linear CTRS of type 1.

Conditional linearization is useful for showing UNC of non-left-linear TRSs. The key observation is \( \text{CR}(\mathcal{R}^L) \) implies UNC(\( \mathcal{R} \)). For this, we use Proposition 6 for \( \rightarrow_0 := \rightarrow \mathcal{R} \) and \( \rightarrow_1 := \rightarrow \mathcal{R}^L \). Clearly, \( \rightarrow \mathcal{R} \subseteq \rightarrow \mathcal{R}^L \), and thus the condition (1) of Proposition 6 holds. Suppose \( \text{CR}(\mathcal{R}^L) \). Then, one can easily show that \( \text{NF}(\mathcal{R}) \subseteq \text{NF}(\mathcal{R}^L) \) by induction on the rank of conditional rewrite steps. Thus, the condition (2) of Proposition 6 implies its condition (3). Hence, \( \text{CR}(\mathcal{R}^L) \) implies UNC(\( \mathcal{R} \)).

Now, for semi-equational CTRSs, the following confluence criterion is known.

Proposition 7 ([BK86, O'D77]). Orthogonal semi-equational CTRSs are confluent.

A TRS \( \mathcal{R} \) is strongly non-overlapping if CCP(\( \mathcal{R}^L \)) = \( \emptyset \). Hence, it follows:

Proposition 8 ([KoV90, V99]). Strongly non-overlapping TRSs have UNC.

As we mentioned in the introduction, this proposition is subsumed by Proposition 1.

3.2 UNC by conditional linearization

We now give some simple extensions of Proposition 8 which are easily incorporated from [Hue80], but are not subsumed by Proposition 1. For this, let us recall the notion of parallel rewrite steps. A parallel rewrite step \( s \rightarrow_R t \) is defined like this: \( s \rightarrow_R t \) iff \( s = C[l_1\sigma_1, \ldots, l_n\sigma_n] \) and \( t = C[r_1\sigma_1, \ldots, r_m\sigma_m] \) for some rewrite rules \( l_1 \rightarrow r_1, \ldots, l_n \rightarrow r_n \in \mathcal{R} \) and context \( C \) and substitutions \( \sigma_1, \ldots, \sigma_m \) (\( \sigma \geq 0 \)). Let us write \( \Gamma \vdash_R u \rightarrow v \) if \( \Gamma \vdash_R \sigma \rightarrow_R v \) for any substitution \( \sigma \). We define \( \Gamma \vdash_R u \rightarrow_R v \) etc. analogously.

The following notion is a straightforward extension of the corresponding notion of [Hue80, Toy88].

Definition 9. A semi-equational CTRS \( \mathcal{R} \) is parallel-closed if (i) \( \Gamma \vdash_R u \rightarrow_R v \) for any inner-outer conditional critical pair \( \Gamma \Rightarrow (u, v) \) of \( \mathcal{R} \), and (ii) \( \Gamma \vdash_R u \rightarrow_R \sigma \uparrow \uparrow \downarrow \rightarrow_R v \) for any overlay conditional critical pair \( \Gamma \Rightarrow (u, v) \) of \( \mathcal{R} \).
We now come to our first extension of Proposition 8, the proof, which is very similar to the one for TRSs, is given in Appendix A.

**Theorem 10.** Parallel-closed semi-equational CTRSs are confluent.

**Corollary 11.** A TRS \( \mathcal{R} \) has UNC if \( \mathcal{R}^L \) is parallel-closed.

Next, we incorporate the strong confluence criterion of TRSs [Hue80] to semi-equational CTRSs in the similar way.

**Definition 12.** A semi-equational CTRS \( \mathcal{R} \) is strongly closed if \( \Gamma \vdash u \rightarrow^* \circ \ oversim{\rightarrow} \oversim{\rightarrow} v \) and \( \Gamma \vdash u \rightarrow^* \circ \ oversim{\rightarrow} v \) for any critical pair \( \Gamma \Rightarrow \langle u, v \rangle \) of \( \mathcal{R} \).

Similar to the proof of Theorem 10, the following theorem is obtained in the same way as in the proof for TRSs.

**Theorem 13.** Linear strongly closed semi-equational CTRSs are confluent.

**Corollary 14.** A right-linear TRS \( \mathcal{R} \) has UNC if \( \mathcal{R}^L \) is strongly closed.

**Example 15.** Let
\[
\mathcal{R} = \{ f(x,x,g(y)) \rightarrow h(y,x), g(a) \rightarrow f(a,b), h(x,y) \rightarrow h(a,y), f(x,x,y) \rightarrow h(a,x) \}
\]

Since \( \mathcal{R} \) is overlapping, not shallow and not right-ground, neither Propositions 1, 2 and 3 apply. By conditional linearization, we obtain
\[
\mathcal{R}^L = \{ f(x_1,x_2,g(y)) \rightarrow h(y,x_1) \leftarrow x_1 \approx x_2 (a), g(a) \rightarrow f(a,b), (b) h(x,y) \rightarrow h(a,y) (c), f(x_1,x_2,y) \rightarrow h(a,x_1) \leftarrow x_1 \approx x_2 (d) \}
\]

We have
\[
\text{CCP}_{in}(\mathcal{R}^L) = \{ x_1 \approx x_2 \Rightarrow \langle f(x_1,x_2,f(a,b,b)), h(a,x_1) \rangle \}
\]

and
\[
\text{CCP}_{out}(\mathcal{R}^L) = \{ x_1 \approx x_2 \Rightarrow \langle h(a,x_1), h(y,x_1) \rangle \}.
\]

By \( f(x_1,x_2,f(a,b,b)) \rightarrow_{(a)} h(a,x_1) \) and \( h(a,x_1) \leftarrow_{(c)} h(y,x_1) \), \( \mathcal{R}^L \) is parallel-closed (or linear strongly closed). Thus, from Corollary 11 (or Corollary 14), it follows that \( \mathcal{R} \) has UNC.

### 3.3 Automation

Even though proofs are rather straightforward, it is not at all obvious how the conditions of Theorems 10 and 13 can be effectively checked.

Let \( \mathcal{R} \) be a semi-equational CTRS. Let \( \Gamma \Rightarrow \langle u, v \rangle \) be an inner-outer conditional critical pair of \( \mathcal{R} \), and consider to check \( \Gamma \vdash u \rightarrow^* v \). For this, we construct the set \( \text{Red} = \{ v' \mid \Gamma \vdash u \rightarrow^* v' \} \) and check whether \( v \in \text{Red} \).

To construct the set \( \text{Red} \), we seek the possible redex positions in \( u \). Suppose we found conditional rewrite rules \( l_1 \rightarrow r_1 \leftarrow \Gamma_1, l_2 \rightarrow r_2 \leftarrow \Gamma_2 \in \mathcal{R} \) and substitutions \( \theta_1, \theta_2 \) such that
u = C[l_1 \theta_1, l_2 \theta_2]. Then we obtain u \rightarrow_R C[r_1 \theta_1, r_2 \theta_2] if \vdash \Gamma \Gamma_1 \theta_1 and \vdash \Gamma \Gamma_2 \theta_2, i.e. s \sim_R t for any equations s \approx t in \Gamma_1 \theta_1 \cup \Gamma_2 \theta_2. Now, for checking \vdash \Gamma u \rightarrow v, it suffices to consider the case \vdash \Gamma \Gamma. Thus, we may assume s' \sim_R t' for any s' \approx t' in \Gamma. Therefore, the problem is to check whether s' \sim_R t' for s' \approx t' in \Gamma implies s \sim_R t for any equations s \approx t in \Gamma_1 \theta_1 \cup \Gamma_2 \theta_2.

To check this, we use the following sufficient condition: s \approx \Gamma t for all s \approx t \in \Gamma_1 \theta_1 \cup \Gamma_2 \theta_2. Note there \approx_{\Gamma} is the congruence closure of \Gamma. Since congruence closure of a finite set of equations is decidable \cite{BN98}, this approximation is indeed automatable.

Example 16. Let

\[
\mathcal{R} = \left\{ \begin{array}{ll}
P(Q(x)) & \rightarrow P(R(x)) \quad \approx x \approx A \quad (a) \\
Q(H(x)) & \rightarrow R(x) \quad \approx S(x) \approx H(x) \quad (b) \\
R(x) & \rightarrow R(H(x)) \quad \approx S(x) \approx A \quad (c) 
\end{array} \right.
\]

Then we have CCP(\mathcal{R}) = CCP_{\approx}(\mathcal{R}) =

\{S(x) \approx H(x), H(x) \approx A \Rightarrow (P(R(x)), P(R(H(x))))\}

Now, in order to apply rule (c) to have P(R(x)) \rightarrow_R P(R(H(x))), we have to check the condition S(x) \sim_{\Gamma} R. This holds, since we can suppose S(x) \sim_{\Gamma} H(x) and H(x) \sim_{\Gamma} A. This is checked by S(x) \approx_{\Sigma} A, where \Sigma = \{S(x) \approx H(x), H(x) \approx A\}.

4 Automating UNC proof of non-duplicating TRSs

In this section, we show a slight generalization of the UNC criterion based on Proposition \cite{TO01}, and show how the criterion can be decided. First, we briefly capture necessary notions and notations from the paper \cite{TO01}.

A left-right separated (LR-separated) conditional rewrite rule is l \rightarrow r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n such that (i) l \notin \mathcal{V} is linear, (ii) \mathcal{V}(l) = \{x_i\}, and \mathcal{V}(r) \subseteq \{y_i\} (iii) \{x_i\} \cap \{y_i\} = \emptyset, and (iv) x_i \neq x_j for i \neq j. Here, note that some variables in y_1, \ldots, y_n can be identical. A finite set of LR-separated conditional rewrite rules is called an LR-separated conditional term rewriting system (LR-separated CTRS, for short). An LR-separated conditional rewrite rule l \rightarrow r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n is non-duplicating if |r|_y \leq |y_1, \ldots, y_n|_y for all y \in \mathcal{V}(r).

The LR-separated conditional linearization translated TRSs to LR-separated CTRSs. This is given as follows: Let C[y_1, \ldots, y_n] \rightarrow r be a rewrite rule, where \mathcal{V}(C) = \emptyset. Here, some variables in y_1, \ldots, y_n may be identical. Then, we take fresh distinct n variables x_1, \ldots, x_n, and put
\[ C[x_1, \ldots, x_n] \rightarrow r \leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n \] as the result of the translation. It is easily seen that the result is indeed an LR-separated conditional rewrite rule. It is also easily checked that if the rewrite rule is non-duplicating then so is the result of the translation (as an LR-separated conditional rewrite rule). The LR-separated conditional linearization \( \mathcal{R}^{LRS} \) of a TRS \( \mathcal{R} \) is obtained by applying the translation to each rule.

It is shown in [TO01] that semi-equational non-duplicating LR-separated CTRSs are confluent if their conditional critical pairs satisfy some closure condition, which makes the rewrite steps ‘weight-decreasing joinable’. By applying the criterion to LR-separated conditional linearization of TRSs, they obtained a criterion of UNC for non-duplicating TRSs. Note that rewriting in LR-separated CTRSs is (highly) non-deterministic; even reducts of rewrite steps at the same position by the same rule is generally not unique, not only reflecting semi-equational evaluation of the conditional part but also by the \( \mathcal{V}(l) \cap \mathcal{V}(r) = \emptyset \) for LR-separated conditional rewrite rule \( l \rightarrow r \leftarrow c \). Thus, how to effectively check the sufficient condition of weight-decreasing joinability is not very clear, albeit it is mentioned in [TO01] that the decidability is clear.

For obtaining an algorithm for computing the criterion, we introduce ternary relations parameterized by an LR-separated CTRS \( \mathcal{R} \) and \( n \in \mathbb{N} \) as follows.

**Definition 17.** The derivation rules for \( \Gamma \vdash^R u \sim_n v \) and \( \Gamma \vdash^R u \rightarrow_n v \) are given in Figure 1. Here, \( n \in \mathbb{N} \) and \( \Gamma \) is a multiset of equations.

Intuitively, \( \Gamma \vdash^R u \sim_n v \) means that \( u \leftrightarrow^R v \) using the assumption \( \Gamma \) where the number of rewrite steps is \( n \) in total (i.e. including those used in checking conditions). Main differences to the relation \( \sim \) in [TO01] are twofold:

1. Instead of considering a special constant \( \bullet \), we use an index of natural number. The number of \( \bullet \) corresponds to the index number.

2. Auxiliary equations in \( \Gamma \) are allowed in our notation of \( \Gamma \vdash^R u \sim_n v \). On the contrary, \( \Gamma \) in \( \sim \) in [TO01] does not allow auxiliary equations in \( \Gamma \).

The former is rather a notational convenience; however, this is useful to designing the effective procedure to check the UNC criteria presented below. The latter is convenient to prove the satisfiability of constraints on such expressions. We refer to Appendix B for more precise comparison with [TO01].

The following is a slight generalization of the main result of [TO01]. A proof is given in Appendix B.

**Theorem 18.** A semi-equational non-duplicating LR-separated CTRS \( \mathcal{R} \) is weight-decreasing joinable if for any critical pair \( \Gamma \Rightarrow (s, t) \) of \( \mathcal{R} \), either (i) \( \Gamma \vdash^R s \sim_{\leq 1} t \), (ii) \( \Gamma \vdash^R s \leftrightarrow_{\leq 2} t \), or (iii) \( \Gamma \vdash^R s \rightarrow_i \circ_{j} t \) with \( i + j \leq 2 \) and \( \Gamma \vdash^R t \rightarrow_{i'} \circ_{j'} s \) with \( i' + j' \leq 2 \).

Thus, any non-duplicating TRS \( \mathcal{R} \) has UNC if all CCPs of \( \mathcal{R}^{LRS} \) satisfy some of conditions (i)–(iii).

Thanks to our new formalization of sufficient condition, decidability of the condition follows.

**Theorem 19.** The condition of Theorem 18 is decidable.

**Proof.** We show that each condition (i)–(iii) is decidable. Let \( \Gamma \) be a (finite) multiset of equations, \( s, t \) terms, and \( \bar{s}, \bar{t} \) sequences of terms. The claim follows by showing the following series of sets are finite and effectively constructed one by one: (a) \( \text{SIM}_0(\Gamma, s) = \{ (\Sigma, t) \mid \Gamma \setminus \Sigma \vdash^R s \sim_0 t \} , \)
(b) $\text{SIM}_0(\Gamma, s') = \{\langle \Sigma, \vec{t} \rangle \mid \Gamma \vdash_{R} s \sim_0 \vec{t} \}$, (c) $\text{RED}_1(\Gamma, s, t) = \{\Sigma \mid \Gamma \vdash_{R} s \rightarrow_1 t\}$, (d) $\text{SRS}_{0\mathbf{10}}(\Gamma, s, t) = \{\Sigma \mid \Gamma \vdash_{R} s \sim_0 \circ \rightarrow_1 t\}$, (e) $\text{SIM}_1(\Gamma, s, t) = \{\Sigma \mid \Gamma \vdash_{R} s \sim_1 t\}$, (f) $\text{SIM}_1(\Gamma, s', \vec{t}') = \{\Sigma \mid \Gamma \vdash_{R} s \sim_1 \vec{t}'\}$, and (g) $\text{RED}_2(\Gamma, s, t) = \{\Sigma \mid \Gamma \vdash_{R} s \rightarrow_2 t\}$.

Example 20. Let

$$\mathcal{R} = \begin{cases} f(x, x) & \rightarrow h(x, f(x, b)) \\ f(g(y), y) & \rightarrow h(y, f(g(y), c(b))) \\ h(c(x), b) & \rightarrow h(b, b) \\ c(b) & \rightarrow b \end{cases}$$

Since $\mathcal{R}$ is overlapping, not right-ground, and not shallow, Propositions 4, 5, 6 do not apply. Proposition 7 and Theorems 10, 11 do not apply either. By conditional linearization, we obtain $\mathcal{R}^{\mathbf{LRS}} = \begin{cases} f(x_1, x_2) & \rightarrow h(x, f(x, b)) \iff x_1 \approx x, x_2 \approx x \\ f(g(y_1), y_2) & \rightarrow h(y, f(g(y), c(b))) \iff y_1 \approx y, y_2 \approx y \\ h(c(x), b) & \rightarrow h(b, b) \end{cases}$

We have an overlay critical pair

$$\begin{cases} y_1 \approx y \ (a) \\ y_2 \approx y \ (b) \\ g(y_1) \approx x \ (c) \\ y_2 \approx x \ (d) \end{cases} \Rightarrow \langle h(x, f(x, b)), h(y, f(g(y), c(b))) \rangle$$

(Another one is its symmetric version.) Let $\Gamma = \{(a), (b), (c), (d)\}, s = h(y, f(g(y), c(b)))$ and $t = h(x, f(x, b))$. To check the criteria of Theorem 13, we start computing $\text{SIM}_0(\Gamma, s)$ and $\text{SIM}_0(\Gamma, t)$. For example, the former equals to

$$\begin{cases} \langle (a), (b), (c), (d) \rangle, h(y, f(g(y), c(b))) \\ \langle (b), (c), (d) \rangle, h(y_1, f(g(y), c(b))) \\ \langle (b), (d) \rangle, h(y, f(x, c(b))) \\ \langle (a), (c), (d) \rangle, h(y_2, f(g(y), c(b))) \\ \langle (a), (c), (d) \rangle, h(y, f(g(y_2), c(b))) \\ \langle (a), (c) \rangle, h(x, f(g(y), c(b))) \\ \langle (a), (c) \rangle, h(y, f(g(x), c(b))) \\ \langle (c), (d) \rangle, h(y_1, f(g(y_2), c(b))) \\ \langle (c), (d) \rangle, h(y_2, f(g(y_1), c(b))) \\ \langle (c), (d) \rangle, h(y_1, f(g(x), c(b))) \\ \langle (c), (d) \rangle, h(y_2, f(g(x), c(b))) \\ \langle (d) \rangle, h(y_2, f(g(y_1), c(b))) \\ \langle (d) \rangle, h(y_2, f(x, c(b))) \end{cases}$$

We now can check $s \sim_0 t$ does not hold by $\langle \Gamma', t \rangle \in \text{SIM}_0(\Gamma, s)$ for no $\Gamma'$. To check $\Gamma \vdash s \rightarrow_1 t$, we compute $\text{RED}_1(\Gamma, s, t)$. For this, we check there exist a context $C$ and substitution $\theta$ and rule $l \rightarrow r \in \mathcal{R}^{\mathbf{LRS}}$ such that $s = C[l[\theta]]$ and $t = C[r[\theta]]$. In our case, it is easy to see $\text{RED}_1(\Gamma, s, t) = \emptyset$. Next to check $\Gamma \vdash s \sim_1 t$, we compute $\text{SRS}_{0\mathbf{10}}(\Gamma, s, t)$. This is done by, for each $\langle \Gamma', s' \rangle \in \text{SIM}_0(\Gamma, s)$, computing $\langle \Sigma, t' \rangle \in \text{SIM}_0(\Gamma', t)$ and check there exists $\Sigma \in \text{RED}_1(\Sigma', s', t')$. In our case, for $\langle \emptyset, h(x, f(x, c(b))) \rangle \in \text{SIM}_0(\Gamma, s)$ we have $\langle \emptyset, t \rangle \in \text{SIM}_0(\emptyset, t)$,
Input: TRS $\mathcal{R}$, predicates $\varphi, \Phi$
Output: UNC or NotUNC or Failure (or may diverge)

Step 1. Compute the set $\mathcal{CP}(\mathcal{R})$ of critical pairs of $\mathcal{R}$.

Step 2. If $\Phi(u, v)$ for all $\langle u, v \rangle \in \mathcal{CP}(\mathcal{R})$ and $\varphi(\mathcal{R})$ then return UNC.

Step 3. Let $\mathcal{S} := \emptyset$. For each $\langle u, v \rangle \in \mathcal{CP}(\mathcal{R})$ with $u \neq v$ for which $\Phi(u, v)$ does not hold, do:
   (a) If $u, v \in \text{NF}(\mathcal{R})$, then exit with NotUNC.
   (b) If $u \notin \text{NF}(\mathcal{R})$ and $v \in \text{NF}(\mathcal{R})$, then if $\mathcal{V}(v) \subseteq \mathcal{V}(u)$ then exit with NotUNC, otherwise update $\mathcal{S} := \mathcal{S} \cup \{u \rightarrow v\}$.
   (c) If $v \notin \text{NF}(\mathcal{R})$ and $u \in \text{NF}(\mathcal{R})$, then if $\mathcal{V}(u) \subseteq \mathcal{V}(v)$ then exit with NotUNC, otherwise update $\mathcal{S} := \mathcal{S} \cup \{v \rightarrow u\}$.
   (d) If $u, v \notin \text{NF}(\mathcal{R})$ then find $w$ such that $u \overset{\mathcal{R}}{\rightarrow} w (v \overset{\mathcal{R}}{\rightarrow} w)$, and $\mathcal{V}(w) \subseteq \mathcal{V}(v)$ (resp. $\mathcal{V}(w) \subseteq \mathcal{V}(v)$). If it succeeds then update $\mathcal{S} := \mathcal{S} \cup \{v \rightarrow w\}$.

Step 4. If $\mathcal{S} = \emptyset$ then return Failure; otherwise update $\mathcal{R} := \mathcal{R} \cup \mathcal{S}$ and go back to Step 1.

Figure 2: UNC completion procedure parameterized by predicates $\varphi, \Phi$

and $\emptyset \in \text{RED}_1(\emptyset, h(x, f(x, c(b))), t)$. Thus, we know $h(x, f(x, c(b))) \rightarrow_1 h(x, f(x, b))$. Hence, for these overlay critical pairs, we have $y_1 \approx y, y_2 \approx y, g(y_1) \approx x, y_2 \approx x \vdash_{\mathcal{R}} h(y, f(g(y), c(b))) \sim_1 h(x, f(x, b))$. We also have $\mathcal{CP}_m(\mathcal{R}^{\text{LRS}}) = \{ \emptyset \Rightarrow \langle h(b, b), h(b, b) \rangle \}$. For this inner-outer critical pair, it follows that $\vdash_{\mathcal{R}} h(b, b) \sim_0 h(b, b)$ using $\langle \emptyset, h(b, b) \rangle \in \text{SIM}_0(\emptyset, h(b, b))$. Thus, from Theorem 18, $\mathcal{R}^{\text{LRS}}$ is weight-decreasing. Hence, it follows $\mathcal{R}$ has UNC. We remark that, in order to derive $\vdash_{\mathcal{R}} h(b, b) \sim_0 h(b, b)$, we need the reflexivity rule. However, since the corresponding Definition of $\sim$ in the paper [TO01] lacks the reflexivity rule, the condition of weight-decreasing in [TO01] (Definition 9) does not hold for $\mathcal{R}^{\text{LRS}}$. A part of situations where the reflexivity rule is required is, however, covered by the congruence rule; thus the reflexivity rule becomes necessary when there exists a trivial critical pair such as above.

5 UNC completion and other methods

In this section, we present some new approaches for proving and disproving UNC.

Firstly, observe that the conditional linearization does not change the input TRSs if they are left-linear. Thus, the technique has no effects on left-linear rewrite rules. But, as one can easily see, however, it is not at all guaranteed that left-linear TRSs have UNC.

Now, observe that a key idea in the conditional linearization technique is that CR of an approximation of a TRS implies UNC of the original TRS. The first method presented in this section is based on the observation that one can also use the approximation other than conditional linearization. To fit our usage, we now slightly modify Proposition 6.

Lemma 21. Suppose (1) $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{S}} \subseteq \overset{\sim}{\rightarrow}_{\mathcal{R}}$ and (2) $\text{NF}(\mathcal{R}) \subseteq \text{NF}(\mathcal{S})$. Then, (i) If $\text{CR}(\mathcal{S})$ then $\text{UNC}(\mathcal{R})$. (ii) If there exists distinct $s, t \in \text{NF}(\mathcal{S})$ such that $s \overset{\sim}{\rightarrow}_{\mathcal{S}} t$, then $\neg \text{UNC}(\mathcal{R})$. 

Our approximation \( \mathcal{S} \) of a TRS \( \mathcal{R} \) is given by adding auxiliary rules aiming to obtain CR of the TRS \( \mathcal{S} \), in such a way that conditions (1) and (2) of the lemma are guaranteed.

**Definition 22.** A UNC completion procedure is given as Figure 2. Its input are a TRS and two predicates \( \varphi, \Phi \) such that for any TRS \( \mathcal{S} \) satisfying \( \varphi(\mathcal{S}) \) if \( \Phi(u, v) \) for all critical pairs \( \langle u, v \rangle \) of \( \mathcal{S} \), then CR(\( \mathcal{S} \)).

**Example 23** (Cops §254). Let

\[
\mathcal{R} = \begin{cases}
  a & \rightarrow f(c) \\
  a & \rightarrow f(h(c)) \\
  f(x) & \rightarrow h(f(x))
\end{cases}
\]

Since \( \mathcal{R} \) is overlapping, not right-ground, and not shallow, Propositions 2, 3, 4 do not apply. Proposition 2 does not apply either. Now, let us apply the UNC completion procedure to \( \mathcal{R} \) using linear strongly closed criteria for confluence. For this, take \( \varphi(\mathcal{R}) \) as \( \mathcal{R} \) is linear, and \( \Phi(u, v) \) as \( (u \xrightarrow{\sim} v) \land (u \xrightarrow{\sim} v) \). In Step 3, we find an overlay critical pair \( \langle f(h(c)), f(c) \rangle \), for which \( \Phi \) is not satisfied. Since \( f(h(c)) \) and \( f(c) \) are not normal, we go to Step 3(b). Take \( w := f(c) \) and add a rewrite rule \( f(h(c)) \rightarrow f(c) \) to obtain \( \mathcal{R} := \mathcal{R} \cup \{ f(h(c)) \rightarrow f(c) \} \). Now, the updated \( \mathcal{R} \) is linear and strongly closed (and thus, \( \mathcal{R} \) is confluent). Hence, the procedure returns UNC at Step 2.

We now prove the correctness of the procedure. We first present two simple lemmas for this.

**Lemma 24.** Suppose \( l \xrightarrow{\mathcal{R}} r, l \notin \text{NF}(\mathcal{R}) \), and \( l \rightarrow r \) is a rewrite rule. Then, Unc(\( \mathcal{R} \)) iff Unc(\( \mathcal{R} \cup \{ l \rightarrow r \} \)).

**Lemma 25.** Suppose \( s \xrightarrow{\mathcal{R}} t, t \in \text{NF}(\mathcal{R}) \) and \( \mathcal{V}(t) \notin \mathcal{V}(s) \). Then \( \neg \text{Unc}(\mathcal{R}) \).

**Theorem 26.** The UNC completion procedure is correct, i.e. if the procedure returns UNC then Unc(\( \mathcal{R} \)), and if the procedure returns NotUnc then \( \neg \text{Unc}(\mathcal{R}) \).

We now present two simple results, which turn out effective for some examples.

**Definition 27.** Let \( \mathcal{R} \) be a TRS. We write \( \mathcal{R} \sim \mathcal{R}' \) if \( \mathcal{R}' = (\mathcal{R} \setminus \{ l \rightarrow r \}) \cup \{ l \rightarrow l, r \rightarrow l \} \) for some \( l \rightarrow r \in \mathcal{R} \) such that \( r \notin \text{NF}(\mathcal{R}) \) and \( r \rightarrow l \) is a rewrite rule, or \( \mathcal{R}' = \mathcal{R} \setminus \{ l \rightarrow r \} \) for some \( l \rightarrow r \in \mathcal{R} \) such that \( l = r \) and \( l \notin \text{NF}(\mathcal{R} \setminus \{ l \rightarrow r \}) \). Any transformation \( \mathcal{R} \sim \mathcal{R}' \) is called a rule reversing transformation.

**Theorem 28.** Let \( \mathcal{R}' \) be a TRS obtained by a rule reversing transformation from \( \mathcal{R} \). Then, Unc(\( \mathcal{R} \)) iff Unc(\( \mathcal{R}' \)).

**Example 29.** Let

\[
\mathcal{R} = \begin{cases}
  a & \rightarrow f(a) \\
  h(c, a) & \rightarrow b \\
  h(a, x) & \rightarrow h(x, f(x))
\end{cases}
\]

Since \( \mathcal{R} \) is overlapping and not shallow, Propositions 1, 4 do not apply. Proposition 2 does not apply either. Since it is left-linear, conditional linearization technique does not apply. Note here that \( f(a) \notin \text{NF}(\mathcal{R}) \) because of the rule \( a \rightarrow f(a) \in \mathcal{R} \). Thus, one can apply the rule reversing transformation to obtain

\[
\mathcal{R}' = \begin{cases}
  a & \rightarrow a \\
  f(a) & \rightarrow a \\
  h(c, a) & \rightarrow b \\
  h(a, x) & \rightarrow h(x, f(x))
\end{cases}
\]
Now, it is easy to check $R'$ is left-linear and development closed, and thus $R'$ is confluent. Thus, from Theorem 28, we conclude $R$ has UNC.

**Definition 30.** A TRS $R$ is said to be right-reducible if $r \not\in \text{NF}(R)$ for all $l \rightarrow r \in R$.

**Theorem 31.** Any right-reducible TRS has UNC.

**Example 32** (Cops §126).

$$R = \{ f(f(x,y),z) \rightarrow f(f(x,z),f(y,z)) \}$$

The state of the art confluence tools fail to prove confluence of this example. However, it is easy to see $R$ is right-reducible, and thus, UNC is easily obtained automatically.

### 6 Experiment

We have tested various methods presented so far. The methods used in our experiment are summarized as follows.

- **(sno)** UNC($R$) if $R$ is strongly non-overlapping.
- **(ω)** UNC($R$) if $R$ is non-$\omega$-overlapping.
- **(pcl)** UNC($R$) if $R^L$ is parallel-closed.
- **(scl)** UNC($R$) if UNC($R$) is right-linear and $R^L$ is strongly closed.
- **(wd)** UNC($R$) if $R$ is non-duplicating and weight-decreasing joinable by the condition of Theorem 18.
- **(sc)** UNC completion using strongly-closed critical pairs criterion for linear TRSs.
- **(dc)** UNC completion using development-closed critical pairs criterion for left-linear TRSs.

| without (rev) | (sno) | (ω) | (pcl) | (scl) | (wd) | (sc) 1/2/3 | (dc) 1/2/3 | (rr) | (cp) | all |
|---------------|-------|-----|-------|-------|-----|-----------|-----------|-----|-----|-----|
| YES           | 7     | 7   | 7     | 0     | 2   | 0/6/8     | 0/6/9     | 35  | 0   | 47  |
| NO            | 0     | 0   | 0     | 0     | 0   | 14/33/41  | 14/34/41  | 0   | 42  | 58  |
| YES+NO        | 7     | 7   | 7     | 0     | 2   | 14/39/49  | 14/40/50  | 35  | 42  | 105 |
| timeout (60s) | 0     | 0   | 0     | 0     | 0   | 2/7/17    | 4/10/19   | 0   | 0   | –   |

| with (rev) | (sno) | (ω) | (pcl) | (scl) | (wd) | (sc) 1/2/3 | (dc) 1/2/3 | (rr) | (cp) | all |
|------------|-------|-----|-------|-------|-----|-----------|-----------|-----|-----|-----|
| YES        | 3     | 3   | 3     | 0     | 0   | 24/42/45  | 24/35/39  | 35  | 0   | 62  |
| NO         | 0     | 0   | 0     | 0     | 0   | 15/39/44  | 15/40/44  | 0   | 35  | 56  |
| YES+NO     | 3     | 3   | 3     | 0     | 0   | 39/81/89  | 39/75/83  | 35  | 35  | 118 |
| timeout (60s) | 0     | 0   | 0     | 0     | 0   | 3/4/8     | 3/5/9     | 0   | 0   | –   |

| both | (sno) | (ω) | (pcl) | (scl) | (wd) | (sc) 1/2/3 | (dc) 1/2/3 | (rr) | (cp) | all |
|------|-------|-----|-------|-------|-----|-----------|-----------|-----|-----|-----|
| YES+NO | 7     | 7   | 7     | 0     | 2   | 39/82/90  | 39/78/85  | 35  | 42  | 127 |

Table 1: Test on basic criteria
(rr) UNC(\(\mathcal{R}\)) if \(\mathcal{R}\) is right-reducible.

(cp) \(\neg\)UNC(\(\mathcal{R}\)) by adhoc search of a counterexample for UNC(\(\mathcal{R}\)).

(rev) Rule reversing transformation, combined with other criteria above.

Here, we remark that (sno) is subsumed by (\(\omega\)) and just included for the reference. For the implementation of non-\(\omega\)-overlapping condition, we need unification over infinite terms; our implementation is based on the algorithm in [Jal84]. The last one (rev) is used combined with the other methods. For (sc) and (dc), we employed an approximation of \(\rightarrow\) by \(\rightarrow\circ\) in Step 3(d). We employed a heuristics for (rev) the first kind of transformation is tried only when the term length of \(l\) is less than that of \(r\). For (cp), we use an adhoc search based on rule reversing, critical pairs computation, and rewriting.

We test on the 144 TRSs from the Cops (Confluence Problems) database\(^4\) of which no confluence tool has proven confluence nor terminating. The motivation of using such testbed is as follows: If a confluent tool can prove CR, then UNC is obtained by confluent tools. If \(\mathcal{R}\) is terminating then CR(\(\mathcal{R}\)) iff UNC(\(\mathcal{R}\)), and thus the result follows also from the result of confluence tools. Assuming dedicated termination or confluence tools are used at first, we haven’t elaborated on sophisticated combination with confluence proofs in ACP.

In Table 1 we summarize the results. Out test is performed on a PC with 2.60GHz cpu with 4G of memory. The column headings show the technique used. The number of examples for which UNC is proved (disproved) successfully is shown in the row titled ‘YES’ (resp. ‘NO’). In the columns below (sc) and (dc), we put \(l/n/m\) where each \(l, n, m\) denotes the scores for the 1-round (2-rounds, 3-rounds) UNC completion. The columns below ‘all’ show the numbers of examples succeeded in any of the methods.

The columns below the row headed ‘with (rev)’ are the results for which methods are applied after the rule reversing transformation. The columns below the row headed ‘both’ show the numbers of examples succeeded by each technique, where the techniques are applied to both of the original TRSs and the TRSs obtained by the rule reversing transformation.

3 rounds UNC completions (sc), (dc) with rule reversing are most effective, but they also record most timeouts. Simple methods (rr), (cp) are also effective for not few examples. There is only a small number of examples in the testbed for which weight-decreasing criterion or critical pairs criteria for conditional linearization work. Rule reversing (rev) is only worth incorporated for UNC completions. For other methods, the rule reversing make the methods less effective; for methods (sno), (\(\omega\)), (pcl), (scl) and (wd), this is because the rule reversing transformation generally increases the number of lhs of the rules. In total, UNC of the 127 problems out of 144 problems have been solved by combining our techniques. All the details of the experiment are found in http://www.nue.ie.niigata-u.ac.jp/tools/acp/experiments/ppdp18-sbm/.

7 Tool

The experiment in the previous section reveals how presented methods for UNC (dis)proving should be combined—we have incorporated UNC (dis)proof methods (\(\omega\)), (pcl), (scl), (wd), (rr), (cp), (rev+sc)/3 and (rev+dc)/3 into our confluence tool ACP [AYT09].

ACP originally intends to (dis)prove confluence of TRSs; we have extended it to also deal with (dis)proving UNC of TRSs. Since ACP facilitates CR proof methods, it is easy to use confluence

\(^4\)Cops can be accessed from http://cops.uibk.ac.at/ which currently includes 438 TRSs.
criteria other than strong-closedness and development-closedness; thus, we add yet another UNC completion procedure in which confluence check is performed only to the final result of completion.

We have also incorporated modularity results: we have incorporated extensions of Proposition 2, namely persistent decomposition [AT97], and layer-preserving decomposition [AT96]. Using these decomposition methods, our tool first try to decompose the problem into smaller components if possible.

ACP is written in SML/NJ and provided as the heap image of SML. The new version (ver. 0.62) is downloadable from [http://www.nue.ie.niigata-u.ac.jp/tools/acp/](http://www.nue.ie.niigata-u.ac.jp/tools/acp/).

To run the UNC (dis)proving, it should be invoked like this:

```bash
$ sml @SMLload=acp.x86-linux -p unc filename
```

Other tools that support UNC (dis)proving include CSI [NFM17], which is a powerful confluence prover supporting UNC proof for non-ω-overlapping TRSs and a decision procedure of UNC for ground TRSs, and FORT [RM16], which implements decision procedure for first-order theory of left-linear right-ground TRSs based on tree automata. Our new methods are also effective for TRSs outside the class of non-ω-overlapping TRSs and that of left-linear right-ground TRSs.

A comparison of our tool and these tools (CSI ver. 1.1 and FORT ver. 1.0) is given in Table 2(a). The diagram on the center (right) in Table 2 shows the distribution of problems for which some tool can show UNC (resp. non-UNC). There are 22 problems for which UNC has been newly proved automatically, and 11 problems for which UNC has been newly disproved automatically. Since most of success of CSI (FORT) is due to the decision procedure for ground TRSs (left-linear right-ground TRSs), the size of the problem sets of ground TRS vs. non-ground TRSs (left-linear right-ground TRSs vs. non-left-linear or non-right-ground TRSs) highly affect the result. In Table 3(b), we present a comparison of ACP and CSI distinguishing case of ground TRSs and non-ground TRSs, and that of ACP and FORT distinguishing case of left-linear right-ground TRSs and other TRSs. Our methods work for many of left-linear right-ground TRSs, but takes much longer time.

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**Table 2: Comparison of UNC proofs (1)**

|                | ACP 66 | CSI 41 | FORT 38 |
|----------------|--------|--------|---------|
| YES            | 22     | 4      | 4       |
| NO             | 65     | 43     | 34      |
| YES+NO         | 126    | 84     | 72      |
| time           | 13m    | 25m    | 55s     |

**Table 3: Comparison of UNC proofs (2)**

|                | 58 ground | 86 non-ground |
|----------------|-----------|---------------|
|                | ACP       | CSI           | ACP       | CSI           |
| YES            | 33        | 34            | 33        | 7             |
| NO             | 23        | 24            | 37        | 19            |
| YES+NO         | 56        | 58            | 70        | 26            |
| time           | 2.5m      | 70s           | 11.5m     | 25m           |

|                | 72 LL-RG | 72 non-LL-RG |
|----------------|----------|--------------|
|                | ACP      | FORT         | ACP      | FORT         |
| YES            | 37       | 38           | 29       | –             |
| NO             | 33       | 34           | 27       | –             |
| YES+NO         | 70       | 72           | 56       | –             |
| time           | 2.8m     | 40s          | 11.4m    | –             |

---
than decision procedures in CSI or FORT.

8 Conclusion

In this paper, we have studied automated methods for (dis)proving UNC of TRSs. We have presented some new methods for (dis)proving UNC of TRSs. Presented methods, except for the decidability results (Propositions $3$ and $4$), have been implemented over our confluence tool ACP. Our tool is capable of UNC (dis)proofs for TRSs outside the class of non-$\omega$-overlapping TRSs and that of left-linear right-ground TRSs, for which class UNC dis(proof) had been already implemented by tools CSI and FORT, respectively.

We have not yet incorporated the decidability results (Propositions $3$ and $4$). Currently, our tool lacks a sophisticated infrastructure for implementing efficient decision procedures. Incorporating these methods to our tool remains as our future work. It is shown in [dV99] that $CL_{sp}$, the S.K.I-rules for the combinatory logic supplemented with surjective pairing, has UNC. Our tool, however, can not handle this example; this is theoretically so, even with the help of Propositions $3$ and $4$.

The argument used in [dV99] for showing UNC of $CL_{sp}$ seems hardly automatable. Thus, more powerful methods to prove UNC automatically should be investigated. Lastly, another future plan is to extend our tools to deal with NFP and UNR, and conditional rewriting as well.

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A Omitted Proofs

We first prepare two lemmas to present a proof of Theorem [10]

Lemma 33. Let \( \mathcal{R} \) be a semi-equational CTRS and \( l \to r \Leftarrow \Gamma \in \mathcal{R} \) be left-linear. Suppose \( s \overset{r_1}{\to} l_1 \to r_1 \rightarrow \Gamma_r \theta \), and any redex occurrence of \( l_1 \to r_1 \) is contained in a subterm occurrence of \( \theta(x) \) in \( l_0 \) for some \( x \in V(l) \). Then there exists \( t \) such that \( s \overset{r_1}{\to} l \to r \rightarrow \Gamma \). Proof. Let \( P = \{ p_1, \ldots, p_k \} \) and, for each \( 1 \leq i \leq k \), let \( \alpha_i \) be the subterm occurrence in \( l_0 \) at \( p_i \) and \( \beta_i \) be the subterm occurrence in \( s \) at \( p_i \). For each \( x \in V(l) \), let \( \theta(x) = C[x][\alpha_1, \ldots, \alpha_m] \) with all \( \alpha_1, \ldots, \alpha_m \) in \( \theta(x) \) displayed. Take a substitution \( \theta' \) such that \( \theta'(x) = C[x][\beta_1, \ldots, \beta_m] \). Then, we have \( s = \theta' \) by linearity of \( l \), and moreover, \( \theta'(y) \Rightarrow \theta(y) \) for all \( y \in V \) by definition. From the latter and \( \vdash_\mathcal{R} \Gamma \theta \), we obtain \( \vdash_\mathcal{R} \Gamma \theta' \). Thus, \( s = \theta' \Rightarrow \Gamma \rightarrow \Gamma \theta' \). Let \( r = C'[x_1, \ldots, x_n] \) with all variable occurrences in \( r \) displayed. Then \( \theta' \Rightarrow C'[x_1, \ldots, x_n] \Rightarrow C'[\theta'(x_1), \ldots, x_n] = \theta' \). Thus, \( s \overset{r_1}{\to} l \to r \rightarrow \Gamma \) and the claim is obtained.

Lemma 34. Let \( \mathcal{R} \) be a semi-equational CTRS and \( l_1 \to r_1 \Leftarrow \Gamma_1 \in \mathcal{R} \). Suppose \( s \overset{r_1}{\to} l_1 \to r_1 \rightarrow \Gamma_r \theta \), and the redex occurrence of \( l_1 \to r_1 \) is not contained in any subterm occurrence of \( \theta(x) \) (\( x \in V(l_1) \)) in \( l_1 \). Then \( s \overset{r_1}{\to} l \to r \rightarrow \Gamma \). Proof. Let \( s \overset{r_2}{\to} l \sim l_1 \theta \Leftarrow \Gamma_r \theta \). W.l.o.g. assume \( V(l_1) \cap V(l_2) = \emptyset \). Then we can let \( l_1 \theta \sim l_1'[l_2]' \theta \) and \( s \sim l_1'[l_2]' \theta \). By the condition \( p \in Pos_\mathcal{R}(l_1) \), and hence \( l_1' \theta \theta = l_1' \theta \theta = l_1' \theta \), and thus \( l_1' \theta = l_2 \theta \) and \( l_2 \theta \) is unifiable. Hence, there exists a conditional critical pair \( \Gamma l_1' \theta, \ldots, l_1' \theta \) of \( \mathcal{R} \), where \( \rho \) is an mgu of \( l_1' \theta \) and \( l_2 \theta \). Furthermore, by the definition of mgu, there exists a substitution \( \sigma \) such that \( \sigma \circ \rho = \theta \). Then we have \( s = l_1'[l_2]' \theta = l_1'[l_2]' \theta \rho = \theta \rho = \psi \theta \rho = w \theta \sigma \), and \( \Gamma \theta \sigma = \theta \). Thus, from \( \vdash_\mathcal{R} \Gamma l_1 \theta, \Gamma_2 \theta \), it follows \( \vdash_\mathcal{R} \Sigma \theta \).

Proof of Theorem [10] We show the claim \( t \Rightarrow t_1 \) and \( t \Rightarrow t_2 \) imply \( t_1 \Rightarrow t_3 \) and \( t_2 \Rightarrow t_3 \) for some \( t_3 \). In fact, the proof is almost same as that of the criteria for TRSs. The only essential difference is captured by Lemmas [33] and [34]. For such parallel peak, let \( t \Rightarrow t_1 \) with \( P_1 = \{ p_1, \ldots, p_m \} \) and \( t \Rightarrow t_2 \) with \( P_2 = \{ p_2, \ldots, p_n \} \). We set subterm occurrences \( \alpha_i = t_{p_1} t_{p_1} \) for \( i = 1, \ldots, m \) and \( \beta_j = t_{p_j} t_{p_j} \) for \( j = 1, \ldots, n \). Let \( \Gamma = C_1[\alpha_1, \ldots, \alpha_m] _{p_1, \ldots, p_n} \). Let \( t_{p_k} = l_k \sigma \) with \( l_k \to r_k \Leftarrow \Gamma_k \in \mathcal{R} \). Then, we have \( t_1 = C_1[\gamma_1 \sigma_1, \ldots, \gamma_m \sigma_m] \) for \( \gamma_k \to \gamma_k \in \mathcal{R} \). Let \( \Gamma \sigma_k = \Gamma_k \sigma_k \).
Let us denote by $|t|$ the size of a term $t$. Let $|M| = \sum_{i \in M} |t_i|$. The proof of the claim is by induction on $|Red_{in}(t_1 \leftrightarrow t \rightarrow t_2)|$.

- Case $|I| = 0$. Then for any $p_{k_1}, p_{k_2} \in P_1 \cup P_2$, $k_1 \neq k_2$ implies $p_{k_1} \parallel p_{k_2}$. For notational simplicity, we only consider the case $t = C[\alpha_1, \alpha_0, \beta_2, \beta_3]$, $t = C[\alpha'_1, \alpha'_0, \beta'_2, \beta'_3]$, with $\alpha_i \rightarrow \alpha'_i$ (1 \leq i \leq m) and $\beta_j \rightarrow \beta'_j$ (1 \leq j \leq n). Let $t_3 = C[\alpha'_1, \alpha'_0, \beta'_2, \beta'_3]$, Then $t_1 \rightarrow t_2$ and $t_2 \rightarrow t_3$.

- Case $|I| > 0$.

Let $\gamma_1, \ldots, \gamma_h$ be subterm occurrences of the term $t$ contained in $Red_{out}(t_1 \leftrightarrow t \rightarrow t_2)$.

Then we can write $t = C[\gamma_1, \ldots, \gamma_h]$, $t = C[\gamma_1, \ldots, \gamma_h]$, $t = C[\gamma_1, \ldots, \gamma_h]$, where, for each $1 \leq k \leq h$, $\gamma_k \rightarrow \gamma_k$ with one of them being a root step. It is sufficient to show there are $\gamma_1, \ldots, \gamma_h$ such that $\gamma_k \rightarrow^* \gamma'_k$ and $\gamma_k \rightarrow \gamma_k$ for each $1 \leq k \leq h$.

Suppose $1 \leq k \leq h$.

- Let us consider the case $\gamma_k \rightarrow^* \gamma_k$ and $\gamma_k \rightarrow^* \gamma_k$. Then there exist $l \rightarrow r \rightarrow \Gamma \in \mathcal{R}$ and $\sigma$ such that $\gamma_k = \sigma \Gamma$ and $\gamma_k = \tau \Gamma \sigma$ and $\Gamma \sigma$. Let $\gamma_k = \tilde{C}[\gamma_1, \ldots, \gamma_g]$ where the subterm occurrences $\gamma_1, \ldots, \gamma_g$ are at the respective positions in $P$. Then we can let $\gamma_k = \tilde{C}[\gamma_1, \ldots, \gamma_g]$ with $\gamma_i \rightarrow \gamma_i$ for each $1 \leq i \leq g$.

First, consider the case that that for each $\gamma_i$, there exists $x \in \mathcal{V}(l)$ such that $\gamma_i$ is contained in some $x(\sigma(x))$. Then, by Lemma 43, $\gamma_k \rightarrow^* \gamma_k$.

Otherwise, there exists some $1 \leq i \leq g$ such that $\gamma_i$ is contained in $\sigma(x)$ for no $x \in \mathcal{V}(l)$. Let $p$ be the position of $\gamma_i$ in $\gamma_k$. Then we have $\gamma_k \rightarrow^* \gamma_k \rightarrow^* \gamma_k$. Then, by Lemma 44, $\langle \gamma_k \gamma'_k, \gamma_k \gamma_k \rangle$ is an instance of some CCP $\Gamma \Rightarrow \langle u, v \rangle$, i.e. there exists some $\theta$ such that $\gamma_k \gamma'_k = u \theta$, $\gamma_k = v \theta$ and $\Gamma \theta$. We distinguish two cases.

* Case $p = \epsilon$. Then we have $P = \{\epsilon\}$ and $\gamma_k[\gamma'_k] = \gamma_k$. Furthermore, $\Gamma \Rightarrow \langle u, v \rangle$ is an overlay critical pair, and hence, we have $\Gamma \Rightarrow \epsilon \Rightarrow v \theta$ by the parallel-closed assumption. Thus, $u \theta \rightarrow v \theta$ follows from $\Gamma \theta$ by Definition. Hence we have $\gamma_k = v \theta \rightarrow^* \epsilon \Rightarrow v \theta = \gamma_k[\gamma'_k] = \gamma_k$.

* Case $p \neq \epsilon$. Then, $\Gamma \Rightarrow \langle u, v \rangle$ is an inner-outer critical pair. Hence, we have $\Gamma \Rightarrow u \rightarrow v \theta$ by the parallel-closed assumption. Thus, $u \theta \rightarrow v \theta$ follows from $\Gamma \theta$ by Definition. Hence we have $\gamma_k = u \theta \rightarrow^* \epsilon \Rightarrow u \theta = \gamma_k[\gamma'_k] \rightarrow p \eta \gamma_k$. Now, $Red_{in}(t_1 \leftrightarrow o \rightarrow t_2)$ contains $\gamma_1, \ldots, \gamma_d$. On the other hand, $Red_{in}(\gamma_k \epsilon \rightarrow \gamma_k[\gamma'_k] \rightarrow^* \gamma_k)$ contains only subterm occurrences of $\gamma_1, \ldots, \gamma_p, \gamma_{p+1}, \ldots, \gamma_d$. Thus, we have $|Red_{in}(\gamma_k \epsilon \rightarrow \gamma_k[\gamma'_k] \rightarrow \gamma_k)| < |Red_{in}(t_1 \leftrightarrow o \rightarrow t_2)|$. Thus, by induction hypothesis, $\gamma_k \rightarrow^* \epsilon \Rightarrow^* \gamma_k$.

The case $\gamma_k \rightarrow^* \gamma_k$ and $\gamma_k \rightarrow^* \gamma_k$ this case is proved analogously to the previous case.

Proof of Theorem 132 We here supplement the proof of Theorem 131. For (c), take $S_{I \leftrightarrow R \leftrightarrow C}(s, t) = \{s \mid \text{C}[\sigma] = s, C[\sigma] = t\}$ and then $\text{RED}_1(\Gamma, s, t) = \bigcup_{t \in \text{RED}_1(\Gamma, s, t)} \{\sigma \mid \text{C}[\sigma] = s, C[\sigma] = t\}$, where $\text{hsf}(u_1 \sigma \approx v_1 \sigma, \ldots, u_n \sigma \approx v_n \sigma) = \{u_1 \sigma, \ldots, u_n \sigma\}$ and $\text{hsf}(u_1 \sigma \approx v_1 \sigma, \ldots, u_n \sigma \approx v_n \sigma) = \{u_1 \sigma, \ldots, v_n \sigma\}$. For (d), take $A = \bigcup_{(\varphi, s, t) \in \text{SIM}_0(\psi, t)} \{\varphi, s, t\} \mid (\varphi, s, t) \in \text{SIM}_0(\psi, t)\}$ and $\bigcup \{\text{RED}_1(\Gamma, s, t) \mid (\Gamma, s, t) \in A\}$. For (g), as $\gamma_1 = \gamma_0 \rightarrow \gamma_1 \rightarrow \gamma_0$, take $S_{\text{SRS}_{10}}(\Gamma, s, t) \cup S_{\text{SRS}_{010}}(\Gamma, t, s)$. 19
Now, the condition (i) is equivalent to \((\Sigma, t) \in \text{SIM}_0(\Gamma, s)\) for some \(\Sigma\) or \(\text{SIM}_1(\Gamma, s, t) \neq \emptyset\). The condition (ii) is equivalent to \(\text{RED}_2(\Gamma, s, t) \cup \text{RED}_2(\Gamma, t, s) \neq \emptyset\). The first part of condition (iii) is equivalent to (a) \(\Gamma \vdash_R s \rightarrow_2 \circ \sim_0 t\) or (b) \(\Gamma \vdash_R s \rightarrow_1 \circ \sim_1 t\) or (c) \(\Gamma \vdash_R s \rightarrow_1 \circ \sim_0 t\). (a,c) is equivalent to \(\text{RED}_2(\Sigma, s, t') \cup \text{RED}_2(\Sigma, s, s') \neq \emptyset\) for some \((\Sigma, t') \in \text{SIM}_0(\Gamma, t)\). (b) is equivalent to \(\text{SIM}_1(\Sigma, s', t') \neq \emptyset\) for some \((\Sigma, s') \in \text{RED}_2(\Gamma, s)\). The second part is similar. □

**Proof of Lemma 22.** (i) Suppose \(\vec{s} \vdash_{\mathcal{R}} t\) and \(s, t \in \text{NF}(\mathcal{R})\). Then \(s \vdash_{\mathcal{R}} w \vdash_{\mathcal{C}} t\) for some \(w\) by \(\text{CR}(\mathcal{R})\). But by \(s, t \in \text{NF}(\mathcal{S})\), we obtain \(s = w = t\). (ii) From \(\vdash_{\mathcal{R}} \subseteq \vdash_{\mathcal{S}}\), we have \(\text{NF}(\mathcal{S}) \subseteq \text{NF}(\mathcal{R})\), and thus \(s, t \in \text{NF}(\mathcal{R})\). From \(\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{R}}, s \vdash_{\mathcal{R}} t\).

**Proof of Lemma 23.** (⇒) Suppose \(\vec{s} \vdash_{\mathcal{R} \cup \{l \rightarrow r\}} t\) with \(s, t \in \text{NF}(\mathcal{R} \cup \{l \rightarrow r\})\). Then from \(l \vdash_{\mathcal{R}} r\), we have \(s \vdash_{\mathcal{R}} t\). Furthermore, by \(l \notin \text{NF}(\mathcal{R})\), \(\text{NF}(\mathcal{R}) = \text{NF}(\mathcal{R} \cup \{l \rightarrow r\})\). Thus, \(s \vdash_{\mathcal{R}} t\) and \(s, t \in \text{NF}(\mathcal{R})\). Hence \(s = t\) by \(\text{UNC}(\mathcal{R})\). (⇐) Suppose \(\vec{s} \vdash_{\mathcal{R}} t\) with \(s, t \in \text{NF}(\mathcal{R})\). Then, by \(\mathcal{R} \subseteq \mathcal{R} \cup \{l \rightarrow r\}\), we have \(s \vdash_{\mathcal{R} \cup \{l \rightarrow r\}} t\). Furthermore, by \(l \notin \text{NF}(\mathcal{R})\), \(\text{NF}(\mathcal{R}) = \text{NF}(\mathcal{R} \cup \{l \rightarrow r\})\). Thus, Suppose \(\vec{s} \vdash_{\mathcal{R} \cup \{l \rightarrow r\}} t\) with \(s, t \in \text{NF}(\mathcal{R} \cup \{l \rightarrow r\})\). Hence, \(s = t\) by \(\text{UNC}(\mathcal{R} \cup \{l \rightarrow r\})\).

**Proof of Lemma 24.** Suppose \(\vec{s} \vdash_{\mathcal{R}} t\) in \(\text{NF}(\mathcal{R})\) and \(x \in \mathcal{V}(t) \setminus \mathcal{V}(s)\). Take a fresh variable \(y\) and let \(t' = \{x := y\}\). Clearly, from \(t \in \text{NF}(\mathcal{R})\) we have \(t' \in \text{NF}(\mathcal{R})\). By \(t' \vdash_{\mathcal{R}} s\) \(\vdash_{\mathcal{R}} t\), we obtain the claim.

**Proof of Theorem 20.** By Lemma 24, each round Step 1–4 keeps whether \(\text{UNC}(\mathcal{R})\) or not. By the assumption on \(\varphi, \Psi\), if \(\text{UNC}\) is returned in Step 2, then \(\text{UNC}(\mathcal{R})\) holds. In Step 3a, \(u, v\) are convertible distinct normal forms and hence \(\neg \text{UNC}(\mathcal{R})\) holds. In Step 3b/c, \(\neg \text{UNC}(\mathcal{R})\) holds by Lemma 25.

**Proof of Theorem 20.** It suffices to show \(\mathcal{R} \sim \mathcal{R}'\) implies \(\text{UNC}(\mathcal{R})\) iff \(\text{UNC}(\mathcal{R}')\). It is easy to see that both conditions ensure that \(\text{NF}(\mathcal{R}) = \text{NF}(\mathcal{R}')\) and \(\vdash_{\mathcal{R}} = \vdash_{\mathcal{R}'}\). From the latter, \(\vdash_{\mathcal{R}} = \vdash_{\mathcal{R}'}\). Thus the claim follows.

**Proof of Theorem 27.** Suppose \(\vec{s} \vdash_{\mathcal{R}} t, s, t \in \text{NF}(\mathcal{R})\) and \(s \neq t\). Then from \(s \neq t\), we have \(s \vdash_{\mathcal{R}} t\), and thus \(s \vdash_{\mathcal{R}} s' \vdash_{\mathcal{R}} t\) for some \(s'\). If \(s \rightarrow_{\mathcal{R}} s'\) then this contradicts \(s \in \text{NF}(\mathcal{R})\). If \(s' \rightarrow_{\mathcal{R}} s\) then \(s' = C[l]\theta\) and \(s = C[r]\theta\) for some \(l \rightarrow r \in \mathcal{R}\), and hence from \(r \notin \text{NF}(\mathcal{R})\) we know \(s \notin \text{NF}(\mathcal{R})\). This is again a contradiction.

## B Comparison to our Definition 17 and Definition 9 of [TO01] and a proof of Theorem 18

The following definition is obtained by adding the rule (refl) to the Definition 9 of [TO01].

**Definition 35.** Let \(\mathcal{R}\) be a non-duplicating LR-separated CTRS. Let \(\Gamma\) be a multiset of equations \(t' \approx s'\) and a fresh constant \(\bullet\). Then relations \(t \sim s\) and \(t \sim s\) on terms are inductively defined as follows:

1. (asp) \(t \sim_{\{t = s\}} s\).
2. (refl) \(t \sim_{\{\}\} t\).
(sym) If \( t \sim s \) then \( s \sim t \).

(trans) If \( t \sim r \) and \( r \sim s \) then \( t \sim s \).

(cntxt) If \( t \sim s \) then \( C[t] \sim C[s] \).

(rule) If \( l \rightarrow r \) \( \equiv x_1 \approx y_1, \ldots, x_n \approx y_n \in R \) and \( x_1 \theta \sim y_1 \theta \) \( \forall i, 1 \leq i \leq n \) then \( C[l\theta] \overset{\sim}{\rightarrow} C[r\theta] \) where \( \Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n \).

(bullet) If \( t \sim s \) then \( t \sim \ast \).

Note \( t \sim s \) in the sense of Definition 9 of \([TO01]\) implies \( t \sim s \) in the sense of Definition \([35]\). On the other hand, \( t \sim s \) in the sense of Definition \([35]\) uses (refl) rule in the derivation, then \( t \sim s \) in the sense of Definition 9 of \([TO01]\) does not hold.

Now, Lemma 3 of \([TO01]\) also follows for our Definition of \( \sim \) and \( \sim\ast \), since the claim holds for the (refl) case trivially.

**Lemma 36** (Lemma 3 of \([TO01]\), generalized). Let \( \Gamma = \{ p_1 \approx q_1, \ldots, p_m \approx q_m, \bullet, \ldots, \bullet \} \) be a multiset in which \( \bullet \) occurs \( k \) times \((k0)\), and let \( \mathcal{P}_1 : p_i \theta \leftrightarrow q_i \theta \) \( \forall i, 1 \leq i \leq m \). (1) If \( t \sim s \) then there exists a proof \( \mathcal{Q} : t\theta \overset{\sim}{\rightarrow} s\theta \) with \( w(\mathcal{Q}) \leq \sum_{i=1}^m k + k \).

Thus, Theorem 1 of \([TO01]\) follows for our Definition of \( \sim \) and \( \sim\ast \).

**Theorem 37** (Theorem 1 of \([TO01]\), generalized). Let \( R \) be a semi-equational non-duplicating LR-separated CTRS. Then \( R \) is weight decreasing joinable if for any critical pair \( \Gamma \vdash \langle s, t \rangle \) of \( R \), either (i) \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \), (ii) \( s \sim \ast \) or \( t \sim \ast \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \), or (iii) \( s \sim \ast \) \( \sim \ast \) then \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \) and \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \). Allow \( \sim \ast \) \( \sim \ast \) and \( k \)-times \((k0)\).

Below, we abbreviate \( \{ \bullet, \ldots, \bullet \} \) as \( \{ \bullet \}^k \).

**Lemma 38.** Let \( \Lambda \) be a multiset of equations. (i) If \( \Lambda \vdash (u \sim_k v) \) then \( u \sim v \) for some \( \Delta = \Lambda' \cup \{ \bullet \}^k \) such that \( \Lambda' \subseteq \Lambda \). (ii) If \( \Lambda \vdash (u \sim_k v) \) then \( u \sim v \) for some \( \Delta = \Lambda' \cup \{ \bullet \}^{k-1} \) such that \( \Lambda' \subseteq \Lambda \). (iii) If \( \Lambda \vdash (u_1, \ldots, u_n) \sim_k (v_1, \ldots, v_n) \) then \( u_j \sim v_j \) \( \forall j, 1 \leq j \leq n \) for some \( \Delta_1, \ldots, \Delta_n \) such that \( \bigcup_i \Delta_i = \Lambda' \cup \{ \bullet \}^k \) for some \( \Lambda' \subseteq \Lambda \).

**Proof.** The proofs of (i)–(iii) proceed by induction on the derivation simultaneously.

For any multiset \( \Delta \) of equations and \( \bullet \), let \( \Delta^* \) be the multiset of \( \bullet \) obtained from \( \Delta \) by removing all equations, and \( \Delta^{eq} \) be the multiset of equations obtained from \( \Delta \) by removing all \( \bullet \). Furthermore, we denote \( |\Delta| \) the length of \( \Delta \).

**Lemma 39.** Let \( \Delta \) be a multiset of equations and \( \bullet \). (i) If \( u \sim v \) then \( \Delta \vdash (u \sim_k v) \) for any \( \Delta \supseteq \Delta^{eq} \) where \( k = |\Delta^*| \). (ii) If \( u \sim v \) then \( \Delta \vdash (u \sim_k v) \) for any \( \Delta \supseteq \Delta^{eq} \) where \( k = |\Delta^*| + 1 \).

(iii) If \( u_j \sim v_j \) \( \forall j, 1 \leq j \leq n \), then \( \Delta \vdash (u_1, \ldots, u_n) \sim_k (v_1, \ldots, v_n) \) for any \( \Delta \supseteq \bigcup_j \Delta_j^{eq} \) where \( k = |\bigcup_j \Delta_j^*| \).
Proof. The proofs of (i)–(iii) proceed by induction on the derivation simultaneously. □

Lemma 40. Let \( \Gamma \) be a multiset of equations. (i) \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \) iff \( \Gamma \vdash s \sim_1 t \).
(ii) \( s \sim \circ t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \) iff \( \Gamma \vdash s \to_1 t \) or \( \Gamma \vdash s \to_2 t \).
(iii) \( s \sim_0 \circ t \) for some \( \Sigma_1, \Sigma_2 \) such that \( \Sigma_1 \cup \Sigma_2 \subseteq \Gamma \cup \{ \bullet \} \) iff \( \Gamma \vdash s \to_1 \circ \sim_0 t \) with \( i + j \leq 2 \).

Proof. (i) (\( \Rightarrow \)) Suppose \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \). Then by Lemma 39 \( \Lambda \vdash s \sim_k t \) for any \( \Lambda \supseteq \Sigma_{eq} \), where \( k = |\Sigma_*| \). If \( \Sigma \subseteq \Gamma \) then \( \bullet \notin \Sigma \), and hence, \( \Lambda \vdash s \sim_0 t \) for any \( \Lambda \supseteq \Sigma_{eq} = \Sigma \), as \( k = |\Sigma_*| = 0 \). Thus, \( \Lambda \vdash s \sim_0 t \) for any \( \Lambda \supseteq \Sigma \). Hence \( \Gamma \vdash s \sim_0 t \). Otherwise, we have \( \bullet \in \Sigma \), and hence, \( \Sigma = \Sigma' \cup \{ \bullet \} \) for some \( \Sigma' \subseteq \Gamma \). Then, \( \Lambda \vdash s \sim_1 t \) for any \( \Lambda \supseteq \Sigma_{eq} = \Sigma' \), as \( k = |\Sigma_*| = 1 \). Thus, \( \Gamma \vdash s \sim_1 t \). Therefore, \( \Gamma \vdash t \sim_1 s \) holds. (\( \Leftarrow \)) Firstly, suppose \( \Gamma \vdash s \sim_0 t \). Then, by Lemma 38 \( s \sim t \) for some \( \Sigma = \Gamma' \cup \{ \bullet \} \) such that \( \Gamma' \subseteq \Gamma \), i.e. \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \). Thus, the claim holds.

(ii) (\( \Rightarrow \)) Suppose \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \). Then by Lemma 39 \( \Lambda \vdash s \to_1 t \) for any \( \Lambda \supseteq \Sigma_{eq} \), where \( k = |\Sigma_*| + 1 \). If \( \Sigma \subseteq \Gamma \) then \( \bullet \notin \Sigma \), and hence, \( \Lambda \vdash s \to_1 t \) for any \( \Lambda \supseteq \Sigma_{eq} = \Sigma \), as \( k = |\Sigma_*| + 1 = 1 \). Thus, \( \Lambda \vdash s \to_1 t \) for any \( \Lambda \supseteq \Sigma \). Hence \( \Gamma \vdash s \to_1 t \). Otherwise, we have \( \bullet \in \Sigma \), and hence, \( \Sigma = \Sigma' \cup \{ \bullet \} \) for some \( \Sigma' \subseteq \Gamma \). Then, \( \Lambda \vdash s \to_2 t \) for any \( \Lambda \supseteq \Sigma_{eq} = \Sigma' \), as \( k = |\Sigma_*| + 1 = 2 \). Thus, \( \Gamma \vdash s \to_2 t \). Therefore, \( \Gamma \vdash s \to_1 t \) or \( \Gamma \vdash s \to_2 t \) holds. (\( \Leftarrow \)) Firstly, suppose \( \Gamma \vdash s \to_1 t \). Then, by Lemma 38 \( s \sim t \) for some \( \Sigma = \Gamma' \cup \{ \bullet \} \) such that \( \Gamma' \subseteq \Gamma \), i.e. \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \). Next, suppose \( \Gamma \vdash s \to_2 t \). Then, by Lemma 38 \( s \sim t \) for some \( \Sigma = \Gamma' \cup \{ \bullet \} \) such that \( \Gamma' \subseteq \Gamma \), i.e. \( s \sim t \) for some \( \Sigma \subseteq \Gamma \cup \{ \bullet \} \). Thus, the claim holds.

(iii) (\( \Rightarrow \)) Suppose \( s \sim_0 \circ t \) for some \( \Sigma_1, \Sigma_2 \) such that \( \Sigma_1 \cup \Sigma_2 \subseteq \Gamma \cup \{ \bullet \} \). Firstly, if \( \Sigma_1 \cup \Sigma_2 \subseteq \Gamma \), then, as in the proof of (i) and (ii), it follows \( \Gamma \vdash s \to_1 \circ_0 t \). Secondly, if \( \bullet \in \Sigma_1 \), then as in the proof of (i) and (ii), it follows \( \Gamma \vdash s \to_2 \circ_0 t \). Finally, if \( \bullet \in \Sigma_2 \), then as in the proof of (i) and (ii), it follows \( \Gamma \vdash s \to_1 \circ_0 t \) with \( i + j \leq 2 \). (\( \Leftarrow \)) Suppose \( \Gamma \vdash s \to_1 \circ_0 t \) with \( i + j \leq 2 \). Then we have cases (a) \( \Gamma \vdash s \to_1 u \sim_0 t \), (b) \( \Gamma \vdash s \to_2 \sim_0 t \), and (c) \( \Gamma \vdash s \to_1 u \sim_0 t \). In case (a), there exist \( \Gamma_1, \Gamma_2 \) such that \( \Gamma = \Gamma_1 \cup \Gamma_2 \), \( \Gamma_1 \vdash s \to_1 u \) and \( \Gamma_2 \vdash t \sim_0 u \). Then, as in the proof of (i) and (ii), \( s \sim_0 u \) for some for some \( \Sigma_1 \subseteq \Gamma_1 \) and \( \Sigma_2 \subseteq \Gamma_2 \). In case (b), similarly, we have \( s \sim_0 u \) for some for some \( \Sigma_1 \subseteq \Gamma_1 \) and \( u \sim t \) for some for some \( \Sigma_2 \subseteq \Gamma_2 \cup \{ \bullet \} \). In case (c), similarly, we have \( s \sim_0 u \) for some for some \( \Sigma_1 \subseteq \Gamma_1 \cup \{ \bullet \} \) and \( u \sim t \) for some for some \( \Sigma_2 \subseteq \Gamma_2 \). Thus, the claim holds. □

Proof of Theorem 13 It follows immediately from Lemma 40 by noting \( \Gamma \vdash s \to_1 t \) implies \( \Gamma \vdash s \sim_1 t \). □
C Some detailed proofs

Proof of Lemma 38: We prove (i)–(iii) simultaneously by induction on the derivation.

1. Case $\Gamma \vdash \{ u \approx v \} \vDash_R u \sim_0 v$. The claim holds since $u \sim_0 v$ by $\text{asp}$.

2. Case $\Gamma \vDash_R t \sim_0 0$. The claim holds since $t \sim_0 t$ by $\text{refl}$.

3. Case $\Gamma \vDash_R s \sim_i t$ is derived from $\Gamma \vDash_R t \sim_s s$. By induction hypothesis, $t \sim s$ for some $\Delta = \Gamma' \cup \{ \bullet \}$ such that $\Gamma' \subseteq \Gamma$. Then $s \sim t$ by $\text{sym}$, and the claim holds.

4. Case $\Gamma \vdash \Sigma \vdash_R s \sim_{i+j} u$ is derived from $\Gamma \vdash \Sigma \vdash_R s \sim_i t$ and $\Sigma \vdash_R t \sim_j u$ By induction hypothesis, $s \sim t$ for some $\Delta_1 = \Gamma' \cup \{ \bullet \}$ such that $\Gamma' \subseteq \Gamma$, and $t \sim u$ for some $\Delta_2 = \Sigma' \cup \{ \bullet \}$ such that $\Sigma' \subseteq \Sigma$. Take $\Delta = \Delta_1 \cup \Delta_2$. Then $s \sim u$ by $\text{trans}$. Furthermore, $\Delta = \Delta_1 \cup \Delta_2 = \Gamma' \cup \{ \bullet \} \cup \Sigma' \cup \{ \bullet \} = \Gamma' \cup \Sigma' \cup \{ \bullet^i \cdot j \},$ and $\Gamma' \cup \Sigma' \subseteq \Gamma \cup \Sigma$. Hence the claim holds.

5. Case $\Gamma \vDash_R C[s] \sim_i C[t]$ is derived from $\Gamma \vDash_R s \sim_i t$ By induction hypothesis, $s \sim t$ for some $\Delta = \Gamma' \cup \{ \bullet \}$ such that $\Gamma' \subseteq \Gamma$. Then $C[s] \sim C[t]$ by $\text{ctxt}$, and the claim holds.

6. Case $\bigcup_j \Gamma_j \vdash_R \langle u_1, \ldots, u_n \rangle \sim_{k} \langle v_1, \ldots, v_n \rangle$ is derived from $\Gamma_1 \vdash_R \langle u_1 \sim_{i_1} v_1, \ldots, \Gamma_n \vdash_R u_n \sim_{i_n} v_n \rangle$ where $k = \sum_j i_j$. By induction hypothesis, for each $j = 1, \ldots, n$, $u_j \sim_{i_j} v_j$ for some $\Delta_j = \Gamma'_j \cup \{ \bullet \}$ such that $\Gamma'_j \subseteq \Gamma_j$. Since $\bigcup_j \Delta_j = \bigcup_j \Gamma'_j \cup \{ \bullet \}$ and $\bigcup_j \Gamma'_j \subseteq \bigcup_j \Gamma_j$, the claim holds.

7. Case $\Gamma \vDash_R s \sim_i t$ is derived from $\Gamma \vDash_R s \sim_i t$. By induction hypothesis, $s \sim_i t$ for some $\Delta = \Gamma' \cup \{ \bullet^i \}$ such that $\Gamma' \subseteq \Gamma$. Then $s \sim t$ by $\text{bullet}$ and $\Delta \cup \{ \bullet \} = \Gamma' \cup \{ \bullet \}$. Thus, the claim holds.

8. Case $\Gamma \vdash_R C[l\sigma] \sim_{i+1} C[r\sigma]$ is derived from $\Gamma \vdash_R \langle x_1 \sigma, \ldots, x_n \sigma \rangle \sim_i \langle y_1 \sigma, \ldots, y_n \sigma \rangle$ where $l \rightarrow r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n \in \mathcal{R}$. By induction hypothesis, $x_j \sigma \sim \Delta_j(y_j \sigma)$ for $j = 1, \ldots, n$ for some $\Delta_1, \ldots, \Delta_n$ such that $\bigcup_j \Delta_j = \Gamma' \cup \{ \bullet \}$ for some $\Gamma' \subseteq \Gamma$. Then, by $\text{rule}$, we have $C[l\theta] \sim\Delta C[r\theta]$ where $\Delta = \bigcup_j \Delta_j$. Thus, the claim holds.

Proof of Lemma 39: We prove (i)–(iii) simultaneously by induction on the derivation.

1. Case $\text{(asp)}$. We have $t \sim_{\{ \text{t=s} \}} s$. Then $\Lambda \vDash_R t \sim_0 s$ for any $\Lambda \equiv \{ t \approx s \}$ by definition.

2. Case $\text{(refl)}$. We have $t \sim t$. Then $\Lambda \vDash_R t \sim_0 t$ for any $\Lambda$ by definition.

3. Case $\text{(sym)}$. Suppose $s \sim t$ is derived from $t \sim s$. Let $\Lambda \equiv \Gamma^{eq}$. Then by induction hypothesis, $\Lambda \vDash_R t \sim_k s$, where $k = |\Gamma_\bullet|$. Then, it follows $\Lambda \vDash_R t \sim_k s$ by definition.

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4. Case (trans). Suppose $t \sim s$ is derived from $t \sim r$ and $r \sim s$. Let $\Lambda \supseteq (\Gamma \sqcup \Sigma)^{eq}$. Then, there exist $\Lambda_1, \Lambda_2$ such that $\Lambda = \Lambda_1 \sqcup \Lambda_2$, $\Lambda_1 \supseteq \Gamma^{eq}$ and $\Lambda_2 \supseteq \Sigma^{eq}$. Then, by induction hypothesis, $\Lambda_1 \Vdash \Gamma$ where $k_1 = |\Gamma^*|$, and $\Lambda_2 \Vdash \Sigma$ where $k_2 = |\Sigma^*|$. Then, it follows $\Lambda \Vdash \Gamma \sqcup \Sigma$ by definition. As $k_1 + k_2 = |\Gamma^*| + |\Sigma^*| = |(\Gamma \sqcup \Sigma)^*|$, the claim follows.

5. Case (cntxt). Suppose $C[t] \sim C[s]$ is derived from $t \sim s$. Let $\Lambda \supseteq \Gamma^{eq}$. Then by induction hypothesis, $\Lambda \Vdash \Gamma$ where $k = |\Gamma^*|$. Then, it follows $\Lambda \Vdash C[t] \sim_k C[s]$ by definition.

6. Case (rule). Suppose $C|\theta| \sim C[r\theta]$ is derived from $x_1\theta \sim y_i\theta (i = 1, \ldots, n)$, where $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n$ and $l \rightarrow r \leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n \in \mathcal{R}$. Let $\Lambda \supseteq \Gamma^{eq}$. Then, there exist $\Lambda_1, \ldots, \Lambda_n$ such that $\Lambda = \bigcup_j \Lambda_j$ and $\Lambda_j \supseteq \Gamma_j^{eq}$ for each $1 \leq j \leq n$. Hence, by induction hypothesis, $\Lambda_j \Vdash \Gamma_j$ where $k_j = |\Gamma_j^*|$ for each $1 \leq j \leq n$. Then, by definition, $\Lambda \Vdash \langle x_1\theta, \ldots, x_n\theta \rangle \sim_k \langle y_1\theta, \ldots, y_n\theta \rangle$ where $k' = \sum_j k_j = \sum_j |\Gamma_j^*| = |\bigcup_j \Gamma_j^*| = |\Gamma^*|$. Then, by definition, $\Lambda \Vdash C|\theta| \sim_{k' + 1} C[r\theta]$.

7. Case (bullet). Suppose $s \sim t$ is derived from $t \sim s$. Let $\Lambda \supseteq \Gamma^{eq}$. Then by induction hypothesis, $\Lambda \Vdash \Gamma$ where $k = |\Gamma^*| + 1$. Then, it follows $\Lambda \Vdash t \sim_k s$ by definition.

$\square$