Integral equation of quantum stochastic process

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Abstract

To describe stochastic quantum processes I propose an integral equation of Volterra type which is not generally transformable to any differential one. The process is a composition of ordinary quantum evolution which admits presence of a quantum bath and reductions to pure states. It is proved that generically solutions stabilize asymptotically for $t \to +\infty$ to a universal limit - the projection onto the state with maximal available entropy. A number of typical methods of finding solutions of the equation are proposed.

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I Introduction

In classical evolution of a system-thermal bath it is observed existing of time arrow. In quantum case there are a number of attempts to modify equations to gain non-invertability of the dynamics. A basis for a formal stochastic disturbance is usually a differential equation as the generalized Schrödinger equation\textsuperscript{1} or Lindblat equation, see for example \textsuperscript{2}. The first modification leading to Hugstone equation is a model of single measurement process. Nevertheless, it realizes one reduction event as a long-time limit and the method is not applicable here, where I assume that reductions moments are points on the time axis. In the second approach the basis Lindblat equation is only a constant-coefficients approximation of complete quantum evolution valid for relatively short time regime, compare \textsuperscript{3}. In \textsuperscript{4} it is described a quantum stochastic process which avoids formalizing to differential equations at all. It consists from mixed quantum evolutions and reductions, where it is assumed that the reduction skips arise in time moments treated as a Poisson stochastic process with a characteristic parameter $\nu$.

In section \textbf{II} I propose the new equation governing the evolution of this type, which for $\nu = 0$ is equivalent to the mere quantum equation \textsuperscript{(3)}. In section \textbf{IV} the asymptotic theorem is proved. Solutions of some special versions of the equation are investigated in section \textbf{III}. In the appendix fulfilling of some genericity conditions for the integral equations based on hamiltonian evolution are elaborated.

II Integral equation

In the article I restrict myself to finite dimensional case. Let a quantum evolution of a system in a system-bath pair is defined by

$$A_\alpha : \mathbb{R} \to B(\mathcal{H})$$

$$\sum_\alpha A_\alpha A_\alpha^\dagger = \sum_\alpha A_\alpha^\dagger A_\alpha = 1$$

where $\mathcal{H} = \mathbb{C}^n$, $B(\mathcal{H})$ is the linear, bounded operators set on $\mathcal{H}$, $\alpha = 1, \ldots, N$, compare also with \textsuperscript{4} \textsuperscript{3}. Now, let $0 = t_0 < \ldots < t_i < t_{i+1} < \ldots$ be a realization of a Poisson process with the mean number of events in unit intervals equals to $\nu \geq 0$. Between two neighboring points in which quantum reductions take place $t_i$, $t_{i+1}$, the evolution is defined by the doubly stochastic
matrix \((2\Sigma)\)

\[
M_{ij}(t) := \sum_{\alpha} Tr P_i A_{\alpha}(t) P_j A^\dagger_{\alpha}(t)
\]  

\[
M_{ij}(0) = 1,
\]

where \(t := t_{i+1} - t_i\), an orthonormal basis \(|i⟩ ∈ \mathcal{H}, i = 1, ..., n\) is chosen and \(P_i := |i⟩ ⟨i|\). As one parameter unitary group evolution is determined by a self-adjoint operator in quantum bath presence case it is determined by a set of operators \(B_{ab} = B_{\alpha a}^\dagger\), \(A_{ab}(t), \alpha \equiv ab\) are solutions of

\[
\dot{A}_{ab}(t) = -i \sum_c B_{ac} A_{cb}(t) = -i \sum_c A_{ac}(t) B_{cb}
\]

with the initial condition \(A_{ab}(0) = \delta_{ab} \frac{1}{\sqrt{n_2}}\), where \(n_2\) is the dimension of the bath Hilbert space, \(a, b, c = 1, 2, \ldots, n_2\).

Let \(\overline{M}(T)\) be the average doubly matrix obtaining by summing up all realizations of the Poisson process. The result is:

\[
\overline{M}(T) = \sum_{n=0}^{\infty} \int_0^T dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 M(T - t_n) \circ \ldots \circ M(t_2 - t_1) \circ M(t_1) \nu^n e^{-\nu T}.
\]

Following a structural similitude to Weinberg-Van Winter equation\(^5\) one obtains from \((6)\) the integral equation:

\[
\overline{M}(t) = e^{-\nu T}(M(T) + \nu \int_0^T M(T - t) \overline{M}(t) e^{\nu t} dt).
\]

In the way I have built a dynamical system entirely defined by \(B_{ab} ∈ B(\mathcal{H})\) and \(\nu ≥ 0\) by equations \((3), (7)\). The functions \(A_{\alpha}(t)\) given by \((3)\) are analytical, so \(M(t)\) as well. Then performing \(k\) differentiations of \((7)\) I gain:

\[
\overline{M}^{(k)}(T) = e^{-\nu T}[M^{(k)}(T) + L_k(T) + \nu \int_0^T M(T - t) \overline{M}(t) e^{\nu t} dt]
\]

where \(L_k\) are defined by the recurrence formula:

\[
L_{k+1}(T) = \nu M(T) \overline{M}^{(k)}(0) + (1 - \nu)L_k(T) - \nu M^{(k)}(T).
\]

Generally, the solution of \((7)\) is always uniquely defined by \(M(t)\), an integrable map from \(\mathbb{R}\) to the set of general stochastic matrices and given by the convergent series \((3)\).
Remark 1. It is appeared that if one takes a different distribution of reductions events and still wants to keep an analog of equation (7) it needs to be again a Poissonian distribution. Nevertheless, the integral equation generalizes itself to the following one:

\[
\bar{M}(T) = a(T)M(T) + \int_0^T M(T - t)\bar{M}(t)b(t, T)dt,
\]

where \(a, b\) are nonnegative, continuous functions such that

\[
a(T) + \int_0^T a(t_1)b(t_1, T)dt_1 + \int_0^T \int_0^{t_1} a(t_2)b(t_2, t_1)b(t_1, T)dt_2dt_1 + \ldots = 1
\]

or equivalently

\[
\int_0^T b(t, T)dt = -a(T) + 1.
\]

At the moment still if \(M(t)\) are stochastic matrices (or \(2\Sigma\)) then \(\bar{M}(t)\) are of the same type.

III Solving the equation

The following property of doubly stochastic matrices will be of interest.

Definition 1. For a doubly stochastic matrix \(M : \mathbb{R}^n \to \mathbb{R}^n\) its compression \(c(M)\) is defined by \(c(M) := ||M|\triangle||\), where \(M|\triangle\) is the restriction of \(M\) to the subspace of vectors \(v^i\) fulfilling \(\sum_{i=1}^n v^i = 0\).

Clearly \(0 \leq c(M) \leq 1\), see 4. Now, I may consider some special situations delivering more information about evident solutions constructing.

Example 2. The most simple one is \(M(t) = M = const.\). Then equation (7) is the following:

\[
\dot{M} = \nu(M - 1)\bar{M}.
\]

Then for \(M = M^T\) the solution \(\exp(\nu(M - 1)t)\) has the limit for \(t \to \infty\) of the form

\[
\begin{pmatrix}
1 & 0 \\
0 & \Theta
\end{pmatrix}.
\]

In the case \(c(M) < 1\) the limit is \(\Theta\), where \(c(\Theta) = 0\).
Example 3. Let $M(t) = \alpha(t) \cdot 1 + (1 - \alpha(t)) \cdot \Theta$, where $\alpha, \beta$ are an integrable on finite intervals functions of values in the interval $[0, 1]$. Then equation (7) is reduced to

$$\beta(T) = e^{-\nu T} \alpha(T) + \nu e^{-\nu T} \int_0^T \alpha(T - t) \beta(t) e^{\nu t} dt,$$

(15)

where $\beta$ is uniquely defined through $\bar{M}(t) = \beta(t) \cdot 1 + (1 - \beta(t)) \cdot \Theta$. In the similar way for each $M_1, M_2 \in 2\Sigma$ such that $\{\alpha M_1 + (1 - \alpha)M_2, \alpha \in [0, 1]\}$ is closed for matrices multiplication linear integral equations arise of the form $\beta = \Omega_\alpha \beta$, where $\Omega_\alpha$ transforms any integrable input function $\beta$ of values in $[0, 1]$ into an output $\Omega_\alpha \beta(t) \in [0, 1]$. The solutions of the equations (generically) stabilize in infinity to a number from $[0, 1]$ as one may conclude from the asymptotic theorem in section [15].

In the case of locally constant functions $\alpha(t)$ except finite number of discontinuities in bounded intervals equation (15) may be viewed as a sequences of differential equations defining and solving step by step. I consider a simple 0, 1 input function.

Example 4. Let $\alpha(t) = \alpha_{2k} = 1$ for $t \in [2k\tau, (2k + 1)\tau)$ and $\alpha(t) = \alpha_{2k+1} = 0$ for $t \in [(2k + 1)\tau, (2k + 2)\tau)$, where $k \in \mathbb{N}$ and $\tau > 0$. Then (15) is transformable to

$$\dot{\beta}(T) = \nu \sum_{k=1}^i \beta(T - k\tau)e^{-\nu k\tau}(\alpha_k - \alpha_{k-1})$$

(16)

with $T \in [i\tau, (i + 1)\tau)$ or adopting periodicity

$$\dot{\beta}(T + 2\tau) = e^{-2\nu \tau} \dot{\beta}(T) - \nu \beta(T + \tau)e^{-\nu \tau} + \nu \beta(T)e^{-2\nu \tau}$$

(17)

The initial conditions for each intervals are $\beta(0) = \alpha(0)$ and the discontinuity in $t = k\tau$ are given by $\beta(k\tau^+) - \beta(k\tau^-) = (-1)^k \exp(-\nu k\tau)$ for $k \geq 1$. In the way solving the integral equation in an interval one needs to possess already solutions of last two. First two intervals need to be solved independently. Here $\beta(T) = 1$ for $T \in (0, \tau)$ and $\beta(T) = 1 + (\nu \tau - \nu T - 1) \exp(-\nu \tau)$ in $[\tau, 2\tau)$.

For effective finding of solutions I return to an analytical input.

Example 5. Let

$$\alpha(t) = \frac{1}{2} + \frac{\cos(t)}{2} = \frac{1}{2} + \frac{e^{it}}{4} + \frac{e^{-it}}{4}$$

(18)
and \( \nu = 1 \). I assume that the solution of (15) has a form
\[
e^t \beta(t) = a(t) + b(t)e^{it} + \bar{b}(t)e^{-it}
\]
where \( a, b \) smooth functions, \( a(t) \in \mathbb{R}, b(t) \in \mathbb{C} \). Then the linear, constant coefficient equations follow
\[
a^{(3)} - \ddot{a} + \dot{a} - \frac{1}{2}a = 0,
\]
\[
b^{(3)} + (3i - 1)\ddot{b} - 2(i + 1)\dot{b} - \frac{1}{4}b = 0
\]
with the initial conditions:
\[
a(0) = \dot{a}(0) = \ddot{a}(0) = \frac{1}{2},
\]
\[
b(0) = \frac{1}{4},
\]
\[
\dot{b}(0) = -\frac{1}{4},
\]
\[
\ddot{b}(0) = \frac{1}{4}(i - 1).
\]
Similarly, the equivalent, linear, constant coefficients differential system of equations of order \( 2N+1 \) may be constructed for \( M(t) = \sum_{n=-N}^{N} A_n \exp(int) \), where \( A_n \) are constant matrices.

**IV Asymptotic theorem**

I define a restriction on \( 2\Sigma_{\mathbb{R}^+} \) maps

**Definition 2.** Let \( M : \mathbb{R}_+ \to \mathcal{M}(n \times n, \mathbb{R}) \) be a measurable map into doubly stochastic matrices. It is said to be *generic* if \( \exists (\delta < 1) \mu(I_\delta) > 0 \), where \( I_\delta := \{ t \in \mathbb{R}_+: c(M(t)) \leq \delta \} \) and \( \mu \) the Lebesgue measure of the real, non-negative numbers space \( \mathbb{R}_+ \).

The above genericity condition naturally appears for \( M(t) \) generated by physical quantum systems defined in section II, for details see appendix A. Now, the main theorem may be proved.

**Theorem 1.** Let \( M(t) \) be a generic map. Then \( \lim_{t \to \infty} M(t) = \Theta \).
Proof. One has the following estimation

\[
c(M(t)) \leq \sum_{n=0}^{\infty} \int_{0}^{T} dt_{n} \int_{0}^{t_{n}} dt_{n-1} \ldots \int_{0}^{t_{2}} dt_{1} c(M(T - t_{n})) \circ \ldots \circ c(M(t_{2} - t_{1})) \circ c(M(t_{1})))\nu^{n} e^{-\nu T}. \tag{26}
\]

so the problem is reduced to considering equation (15) with \(\alpha\) differing from 1 on a positive measure set. Changing variables in the integral of the equation and making transformation \(T \to T/\nu\) lead to elimination of \(\nu\) (\(\nu = 1\)). Let

\[
\delta = \int_{0}^{\infty} \alpha(t)e^{-t}dt. \tag{27}
\]

From the assumption about \(\alpha\) is that \(\delta < 1\). Now, let \(\beta(T) \leq b_{0}\) for \(T \in [t_{0}, \infty)\). From (15) I have

\[
\beta(T) \leq e^{-T}\alpha(T) + e^{-T}\int_{0}^{t_{0}} \alpha(T - t)\beta(t)e^{t}dt + b_{0}\delta. \tag{28}
\]

I define \(\epsilon = (1 - \delta)/2\). Let \(t_{0}', t_{0}' \geq t_{0}\), be such that two first terms of (28) are less or equal to \(b_{0}\epsilon\) for \(t \geq t_{0}'\). Then \(\beta(T) \leq b_{0}(\delta + \epsilon)\) for \(T \geq t_{0}'\). Continuing the procedure one reaches \(\lim_{T \to \infty} \beta(T) = 0\).

Nevertheless, even for \(c(M(t)) = 1\) an asymptotic stabilization can exist as the example is showing.

Example 6. Let \(M(t) = P\), where \(P\) is a cyclic group generator such that \(P^{i} \neq 1\) for \(i = 1, \ldots, k - 1\) and \(P^{k} = 1\). I also assume that \(c(\sum_{i=1}^{k} \alpha_{i}P^{i}) = 1\) for \(\alpha_{i} \geq 0\) and \(\sum_{i=1}^{k} \alpha_{i} = 1\). It may be easy realized by the matrix

\[
\begin{pmatrix}
P_{\sigma} & 0 \\
0 & P_{\sigma}
\end{pmatrix}. \tag{29}
\]

\(P_{\sigma}\) is a permutation. Then the solution (3) has the form

\[
\bar{M}(t) = \exp(-\nu t)[P^{k}f_{1}(\nu t) + Pf_{2}(\nu t) + \ldots + P^{k-1}f_{k}(\nu t)]P, \tag{30}
\]

where the analytical functions, \(f_{i}(z)\), are formally defined by \(\exp(\alpha_{i}z) = (\alpha_{i})^{k}f_{1}(z) + \ldots + (\alpha_{i})^{k-1}f_{k}(z)\) with \(\alpha_{k}^{k} = 1\). Then \(\lim_{t \to \infty} \bar{M}(t) = \frac{1}{k} \sum_{i=1}^{k} P^{k}\).

Therefore, a generalization may be proposed.
Theorem 2. Let $M \in 2\Sigma^\mathbb{R}_+$ be a right-side continuous map and $M(0) = 1$. If $\bar{M}(t)$ fulfills the integral equation then

$$\lim_{t \to \infty} \bar{M}(t) = \begin{pmatrix} \Theta_1 & 0 & 0 & 0 \\ 0 & \Theta_2 & 0 & 0 \\ & & \ddots & \vdots \\ 0 & 0 & 0 & \Theta_N \end{pmatrix}. \quad (31)$$

Proof. I put $S_M := \{M(t); t \in \mathbb{R}_+\}$. Let $\bar{S}_M$ be the closure of $S_M$ with respect to matrices multiplication and $\mathcal{I} := \{P_\sigma; \sigma \in I\}$ be a minimal permutations’ subset spanning $\bar{S}_M$. Then $\mathbb{R}^n = \bigoplus_i V_i \oplus V_{id}$, where $V_i \neq \{0\}$ are minimal invariant subspaces of $P \circ A_3$, $P^{-1} \in \mathcal{I}$, spanned by the canonical basis vectors and such that $\dim V_i > 1$, where $A_3$ denotes all doubly stochastic matrices obtained from $\mathcal{I}$ as baricentric points, see proposition \[.\] On $V_{id}$, $V_{id} \perp V_i$, elements of $A_3$ acts as identity. $1 \in A_3$, so $P = 1$ may be chosen.

I define a compression in $V_i$ by $c_i(M) := c(M|_{V_i})$ for $M \in A_3$. If $\forall(M \in A_3) c_i(M) = 1$ then $P' \circ M|_{V_i}$ is decomposable for $P'$ being an admitted permutation of the basis vectors from $V_i$. Therefore, for all interior $M \in A_3^i$ the compression $c_i(M) < 1$, also the next consequence is that $\Theta_i \in A_3^i$, where $i$ have denoted $A_3^i := A_3|_{V_i}$. Now, I restrict considerations to one subspace $V_i$ and for simplicity omit the index. Let $\mathcal{G} \subset \bar{S}_M$ be a minimal set that does not belong to any $A_3$, $\mathcal{I} \not\subset \mathcal{J}$. $\mathcal{G}$ is finite and $c(\sum_k \alpha_k s_k) < 1$ for $\alpha_k > 0$, $\sum_k \alpha_k = 1$ and $\{s_k\} = \mathcal{G}$. Each $s_k$ has a form $M(t_{N_k}) \circ \ldots \circ M(t_1)$. Let $N := \max\{N_k\}$. At the moment I return to equation \[ with $\nu = 1$.

Equivalently, it may be written in the form:

$$\bar{M}(T) = e^{-T} M(T) + e^{-T} \int_0^T M(T - t_1) M(t_1) dt_1 + \ldots \quad (32)$$

$$+ e^{-T} \int_0^T \int_0^{t_N} \ldots \int_0^{t_{N-1}} M(T - t_2) \circ \ldots \circ M(t_2 - t_1) \bar{M}(t_1) e^{t_1} dt_1 \ldots dt_N.$$ 

I will check the limit of $W(T) := \sum_t \gamma_t \bar{M}(T + T_t)$, where $\gamma_t > 0$, $\sum_t \gamma_t = 1$ and

$$T_k := \sum_{n_k = 1}^{N_k} t_{n_k}. \quad (33)$$

$k$ are indices of $s_k$.

The final step is like as in theorem \[.\] Having an upper bound $b_0$ of $c(W(t))$ in $[t_0, \infty)$ one improves it, here using \[, to $(1 + \delta)b_0/2$ on $[t_0', \infty)$.
in the next step, where $t'_0 \geq t_0 + 1$ may be found. Now, I will show it. The matrix $W(T)$ has the form:

$$W(T) = \sum_l \gamma_l \left( e^{-(T+T_l)} M(T + T_l) + \int_0^{T+T_l} M(T + T_l - t_1) M(t_1) dt_1 + \ldots \right) + \sum_l \gamma_l e^{-(T+T_l)} \int_0^{T+T_l} \int_0^{t_N} \ldots \int_0^{t_0} M(T + T_l - t_N) \circ \ldots \circ M(t_2 - t_1) M(t_1) e^{t_1} dt_1 \ldots dt_N + \sum_l \gamma_l e^{-(T+T_l)} \int_0^{T+T_l} \int_0^{t_N} \ldots \int_0^{t_0} M(T + T_l - t_N) \circ \ldots (34) \right.$$  

One may verify that

$$c \left( \sum_l \gamma_l \int_0^{t_N} M(t_N - \epsilon_N) \circ \ldots \circ M(t_1 + \epsilon - \epsilon_1) \underline{M}(T + \epsilon_1) e^{T_1 + \epsilon_1} d^N \epsilon \right)$$

$$< \sum_l \gamma_l e^{-T_l} \mu(\Delta) \sup_\Delta c(\underline{M}(T + \epsilon_1)), \quad (35)$$

where $\Delta$ is a measurable set of $\mathbb{R}^N_+$, $\mu(\Delta) > 0$ and $\underline{s}_l = M(t_N) \circ \ldots \circ M(t_1)$ with additional $M(0) = 1$ if it is necessary for $N = N$. In the way I obtain an estimation for the last term of $(34)$ by $\delta b_0$, $\delta < 1$. The first terms of $(34)$ have vanishing compression for $T \to \infty$. The proof is completed.

In other words the limit doubly stochastic matrix projects initial state $p$ onto the maximal entropy state available for $p$. Theorem 2 covers all analytical situations arising from quantum physics. Nevertheless, for keeping the result $(31)$ it is also enough to assume that $M(t)$ differs from maps in theorem 2 on a set of measure zero.

V Summary and interpretation

I have modified the ordinary quantum evolution of a system-bath through considering an associated stochastic process describing probability transformation, not Hilbert space vectors. The arising formalism offers an integral equation which transgresses the former methods of stochastic modification of differential equation based on adding a linear stochastic term. The closer analysis of the integral equation direct solving shows the difference with the
ordinary differential one, its non-Markovian character. The equation appears in a natural way from physical considerations and reproduces the time arrow - the missing feature of unitary evolution. The relation to the physical origin is stressed by a simple observation. Let $M(t)$ be of the period $2\pi$. If I change the period by a general $\tau > 0$ then $M'(t)$ corresponding to $M'(t) = M(\frac{2\pi}{\tau}t)$ is equal to $\bar{M}(\frac{2\pi}{\tau}t)$, but, now, the integral equation is governed by the new $\nu'$ such that
\[
\tau \nu' = 2\pi \nu.
\] (36)

This is a theoretical suggestion for a universal relation between the period of quantum wave and the associated coefficient of stochastic reduction. The interpretation may be found even for cases excluded from the asymptotic theorem. Namely, if $M(t)$ contains an identity sector arising from eigenvectors of hamiltonian, then in the sector the decay effect does not appear at all. In the way $\nu > 0$ influences only states which during evolution cease to be pure. Appearance of invariant sectors is related to observables commuting with the hamiltonian, which are not also disturbed by the stochastic modification.

A Genericity conditions

The aim of the appendix is to study the maps $M(t)$ from $\mathbb{R}_+$ into $2\Sigma$ arising from quantum evolution. Firstly, I start from a characterization of doubly stochastic matrices with unit compression.

**Proposition 3.** Let $M$ be a doubly stochastic matrix. Then $c(M) = 1$ iff $P \circ M$ is a decomposable matrix for $P$ being a permutation.

**Proof.** The implication ($\Leftarrow$) is obvious. If $\mathbb{R}^n = V_1 \oplus V_2$ is a decomposition then it is enough to put $p = p_1^c + q p_2^c$, $q \neq 1$ to have $p \neq p^c$ and $||M(p)|| = ||P \circ M(p)|| = ||p||$, where $(p^c)^k = 1/\dim V$ and $(p^c)^k = 1/\dim V_i$, $i = 1, 2$.

To show ($\Rightarrow$) I take $M = \sum_i \alpha_i P_i$, where $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$ and $P_i$ are permutations enumerated by $i$. I may assume that $\alpha_1 \neq 0$. Then $P_i^{-1} \circ M = \alpha_1 1 + P_i^{-1} \circ \sum_{i \neq 1} \alpha_i P_i$. Still $c(P_i^{-1} \circ M) = 1$, so $p, p \in \triangle$, exists such that $P_i^{-1} \circ P_ip = p$ and $||M(p)|| = ||p||$. Then the canonical basis vectors for which corresponding components $p^k$ are equal to themselves constitute the invariant subspaces and a decomposition is done. \qed

From the above proposition one states that $c(M) < 1$ is open and dense in $2\Sigma$. 

10
Doubly stochastic matrices can be also built via Kraus representation\(^4\). Then one begins from \(\{A_\alpha(t)\}_{\alpha=1}^N\), defined by equation (5). Generic maps \(M(t)\) defined by (3) appear in the following way.

**Proposition 4.** If at least one \(B'_{ab} \in \{B_{ab}\}\) is not proportional to the identity then a dense and open set of orthonormal bases of \(\mathbb{C}^n\) exists such that \(M(A(t))\) is generic.

**Proof.** Now, \(M(t) = M(A(t))\) defined by an orthonormal basis \(\{|i>\}\) appears to be analytical, so of the form

\[
M(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} M_k,
\]

where \(M_k \in M(n \times n, \mathbb{R})\). First terms of the expansion are \(M_0 = 1, M_1 = 0\) and

\[
(M_2)_{jl} = \frac{2}{n^2} \left( \sum_{a,b,c} |<j|B_{ab}|l>|^2 - |<j|B_{ac}B_{ca}|l>| \delta_{jl} \right).
\]

Basises \(\{|i>\}\) constituting the open and dense set of \(U(n)\) are such that \(j \neq l \Rightarrow |<j|B'_{ab}|l>| \neq 0\), where \(B'_{ab}\) is indicated in the assumption. One finds that

\[
||M(t)p|| = 1 + \sum_{i,j} p_i (M_2)_{ij} p_j t^2 / 2 + o(t^2),
\]

where \(||p|| = 1\). Further

\[
\sum_{i,j} p_i (M_2)_{ij} p_j = \sum_{i,j} \sum_{a,b} |<i|B_{ab}|j>|^2 \left( p_i p_j - \frac{p_i^2 + p_j^2}{2} \right) \leq 0.
\]

Therefore, for a generic basis \(\{|i>\}\) from (38) one obtains \(c(M(t)) < 1\) for a \(t > 0\). The genericity of \(M(t)\) is proved. \(\square\)
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