On Pre-$\gamma$-$I$-Open Sets In Ideal Topological Spaces

HARIWAN ZIKRI IBRAHIM
Department of Mathematics, Faculty of Science, University of Zakho, Kurdistan Region-Iraq

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ABSTRACT
In this paper, the author introduce and study the notion of pre-$\gamma$-$I$-open sets in ideal topological space.

Keywords: $\gamma$-open, pre-$\gamma$-$I$-open sets.

1. INTRODUCTION
In 1992, Jankovic and Hamlett introduced the notion of $I$-open sets in topological spaces via ideals. Dontchevich 1999 introduced pre-$I$-open sets, Kasaharain 1979 defined an operation $\alpha$ on a topological space to introduce $\alpha$-closed graphs. Following the same technique, Ogata in 1991 defined an operation $\gamma$ on a topological space and introduced $\gamma$-open sets. In this paper, some relationships of pre-$\gamma$-$I$-open, pre-$I$-open, preopen, pre-$\gamma$-open, $\gamma$-p-open, $\gamma$-preopen, $I$-open, $\delta$-open, $R$-$I$-open, $\alpha$-$I$-open, semi-$I$-open, b-$I$-open and weakly $I$-local closed sets in ideal topological spaces are discussed.

2. PRELIMINARIES
Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset $A$ of $X$, the closure of $A$ and the interior of $A$ will be denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. A subset $A$ of a space $(X, \tau)$ is said to be regular open [N. V. Velicko, 1968] if $A$ is the union of all regular open sets containing $A$. The closure of $A$ and the interior of $A$ are called $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. Let $A$ be the value of $\text{Int}(A)$ and defined to be the set of all $\gamma$-closed sets containing $A$. A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-regular [H. Ogata, 1991] if for each $x \in X$ and for each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $\gamma(U)$ contained in $V$. It is also to be noted that $\tau = \tau_\gamma$ if and only if $X$ is a $\gamma$-regular space [H. Ogata, 1991].

An ideal is defined as a nonempty collection $I$ of subsets $X$ satisfying the following two conditions:
1. If $A \in I$ and $B \subseteq A$, then $B \in I$.
2. If $A \in I$ and $B \in I$, then $A \cup B \in I$.

For an ideal $I$ on $(X, \tau)$, $(X, \tau, I)$ is called an ideal topological space or simply an ideal space. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $P(X)$ is the set of all subsets of $X$, a set operator $(.)^*:P(X) \rightarrow P(X)$ called a local function [E. Hayashi, 1964], [K. Kuratowski, 1966] of $A$ with respect to $\tau$ and $I$ is defined as follows for a subset $A$ of $X$, $A^*(I, \tau) = \{x \in X: U \cap A \in I \text{ for each neighborhood } U \text{ of } x\}$. A Kuratowski closure operator $\text{Cl}(\gamma)$ for a topology $\tau_\gamma(I, \tau)$, called the $\gamma$-topology, finer than $\tau$, is defined by $\text{Cl}(\gamma) = AU A^*(I, \tau)$ [D. Jankovic and T. R. Hamlett, 1990]. We will simply write $A^*$ for $A^*(I, \tau)$ and $\tau^*$ for $\tau_\gamma(I, \tau)$.

Recall that $A \subseteq (X, \tau, I)$ is called $\gamma$-dense-in-itself [E. Hayashi, 1964] (resp. $\tau^*$-closed [D. Jankovic and T. R. Hamlett, 1990] and $\gamma$-perfect [E. Hayashi, 1964]) if $A \subseteq A^*$ (resp. $A \subseteq A^*$ and $A = A^*$).

Definition 2.1. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be
1. preopen [A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, 1982] if $A \subseteq \text{Int}(\tau_\gamma(I, \tau))$.
2. pre-$\gamma$-open [H. Z. Ibrahim, 2012] if $A \subseteq \tau_\gamma(I, \tau) \subseteq \text{Int}(\tau_\gamma(I, \tau))$.
3. $\gamma$-preopen [G. S. S. Krishnan and K. Balachandran, 2006] if $A \subseteq \tau_\gamma(I, \tau) \subseteq \text{Int}(\gamma(I, \tau) \subseteq \text{Cl}(I))$.
4. $\gamma$-p-open [A. B. Khalaf and H. Z. Ibrahim, 2011] if $A \subseteq \text{Int}(\tau_\gamma(I, \tau) \subseteq \text{Cl}(I))$.
5. $I$-open [D. Jankovic and T. R. Hamlett, 1992] if $A \subseteq \text{Int}(A^*)$. 

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6. R-I-open [S. Yuksel, A. Acikgoz and T. Noiri, 2005] if $A = \text{Int}(\text{Cl}^*(A))$.
7. pre-I-open [J. Dontchev, 1999] if $A \subseteq \text{Int}(\text{Cl}^*(A))$.
8. semi-I-open [E. Hatir and T. Noiri, 2002] if $A \subseteq \text{Cl}^*(\text{Int}(A))$.
9. α-I-open [E. Hatir and T. Noiri, 2002] if $A \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A)))$.
10. b-I-open [A. C. Guler and G. Aslim, 2005] if $A \subseteq \text{Int}(\text{Cl}^*(A)) \cup \text{Cl}^*(\text{Int}(A))$.
11. Weakly I-local closed [A. Keskin, T. Noiri and S. Yuksel, 2004] if $A = U \cap K$, where $U$ is an open set and $K$ is a $*$-closed set in $X$.
12. Locally closed [N. Bourbaki, 1966] if $A = U \cap K$, where $U$ is an open set and $K$ is a closed set in $X$.

**Definition 2.2.** [S. Yuksel, A. Acikgoz and T. Noiri, 2005] A point $x$ in an ideal space $(X, \tau, I)$ is called a $\delta_I$-cluster point of $A$ if $\text{Int}(\text{Cl}^*(U)) \cap A \neq \emptyset$ for each neighborhood $U$ of $x$. The set of all $\delta_I$-cluster points of $A$ is called the $\delta_I$-closure of $A$ and will be denoted by $\text{Cl}^I(A)$. A set is said to be $\delta_I$-closed if $\text{Cl}^I(A) = A$. The complement of a $\delta_I$-closed set is called a $\delta_I$-open set.

**Lemma 2.3.** [E. G. Yang, 2008] A subset $V$ of an ideal space $(X, \tau, I)$ is a weakly I-local closed set if and only if there exists $K \in \tau$ such that $V = K \cap \text{Cl}^*(V)$.

**Definition 2.4.** [E. Ekici and T. Noiri, 2009] An ideal topological space $(X, \tau, I)$ is said to be $\ast$-extremally disconnected if the $\ast$-closure of every open subset $V$ of $X$ is open.

**Theorem 2.5.** [E. Ekici and T. Noiri, 2009] For an ideal topological space $(X, \tau, I)$, the following properties are equivalent:
1. $X$ is $\ast$-extremally disconnected.
2. $\text{Cl}^I(\text{Int}(V)) \subseteq \text{Cl}^I(\text{Cl}^*(V))$ for every subset $V$ of $X$.

**Lemma 2.6.** [D. Jankovic and T. R. Hamlett, 1990] Let $(X, \tau, I)$ be an ideal topological space and $A, B$ subsets of $X$. Then
1. If $A \subseteq B$, then $A^* \subseteq B^*$.
2. If $U \in \tau$, then $U \cap A^* \subseteq (U \cap A)^*$.
3. $A^*$ is closed in $(X, \tau)$.

Recall that $(X, \tau)$ is called submaximal if every dense subset of $X$ is open.

**Lemma 2.7.** [R. A. Mahmoud and D. A. Rose, 1993] If $(X, \tau)$ is submaximal, then $P(O)(X, \tau) = \tau$.

**Corollary 2.8.** [J. Dontchev, 1999] If $(X, \tau)$ is submaximal, then for any ideal $I$ on $X$, $P(O)(X) = \tau$.

Where $P(O)(X)$ is the family of all pre-I-open subsets of $(X, \tau, I)$.

**Proposition 2.9.** [H. Ogata, 1991] Let $\gamma : \tau \rightarrow \rho(X)$ be a regular operation on $\tau$. If $A$ and $B$ are $\gamma$-open, then $A \cap B$ is $\gamma$-open.

**3. Pre-$\gamma$-I-Open Sets**

**Definition 3.1.** A subset $A$ of an ideal topological space $(X, \tau, I)$ with an operation $\gamma$ on $\tau$ is called pre-$\gamma$-I-open if $A \subseteq \tau(I, \text{Int}(\text{Cl}^*(A)))$.

We denote by $P(I)(X, \tau, I)$ the family of all pre-$\gamma$-I-open subsets of $(X, \tau, I)$ or simply write $P(I)(X, \tau)$ or $P(I)(X)$ when there is no chance for confusion with the ideal.

**Theorem 3.2.** Every $\gamma$-open set is pre-$\gamma$-I-open.

**Proof.** Let $(X, \tau, I)$ be an ideal topological space and $A$ a $\gamma$-open set of $X$. Then $A = \tau(I, \text{Int}(\text{Int}(\tau A))) = \tau(I, \text{Int}(\text{Cl}^*(A)))$.

The converse of the above theorem is not true in general as shown in the following example.

**Example 3.3.** Consider $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, c\}\}$ and $I = \{\{a\}, \{b\}\}$. Define an operation $\gamma$ on $\tau$ by $\gamma(A) = X$ for all $A \in \tau$. Then $A = \{a, b\}$ is a pre-$\gamma$-I-open set which is not $\gamma$-open.

**Theorem 3.4.** Every pre-$\gamma$-I-open set is pre-$\gamma$-open.

**Proof.** Let $(X, \tau, I)$ be an ideal topological space and $A$ a pre-$\gamma$-I-open set of $X$. Then, $A \subseteq \tau(I, \text{Int}(\text{Cl}^*(A))) \subseteq \tau(I, \text{Int}(\text{Cl}^*(A)))$.

The converse of the above theorem is not true in general as shown in the following example.

**Example 3.5.** Consider $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, c\}\}$ and $I = \{\{a\}, \{c\}\}$. Define an operation $\gamma$ on $\tau$ by $\gamma(A) = X$ for all $A \in \tau$. Set $A = \{c\}$, since $A^* = \emptyset$ and $\text{Cl}^*(A) = A$, then $A$ is a pre-$\gamma$-I-open set which is not pre-$\gamma$-I-open.

**Theorem 3.6.** Every pre-$\gamma$-I-open set is pre-$\gamma$-open.

**Proof.** Let $(X, \tau, I)$ be an ideal topological space and $A$ a pre-$\gamma$-I-open set of $X$. Then, $A \subseteq \tau(I, \text{Int}(\text{Cl}^*(A))) \subseteq \tau(I, \text{Int}(\text{Cl}^*(A)))$.

The converse of the above theorem is not true in general as shown in the following example.

**Example 3.7.** Consider $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, c\}\}$ and $I = \{\{a\}, \{c\}\}$. Define an operation $\gamma$ on $\tau$ by $\gamma(A) = X$ for all $A \in \tau$. Then $A = \{a\}$ is a pre-$\gamma$-I-open set which is not pre-$\gamma$-I-open.
Example 3.9. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}, \{a\}\}$ and $I = \{\phi, \{b\}\}$. Define an operation $\gamma$ on $\tau$ by $\gamma(A) = X$ for all $A \in \tau$. Then $A = \{b, c\}$ is a $\gamma$-preopen set which is not pre-$\gamma$-I-open.

Theorem 3.10. Every pre-$\gamma$-I-open set is $\gamma$-p-open.

Proof. Let $(X, \tau, I)$ be an ideal topological space and $A$ a pre-$\gamma$-I-open set of $X$. Then, $A \subseteq \tau_\gamma$-Int$(Cl^*(A)) \subseteq \tau_\gamma$-Int$(Cl(A)) \subseteq Int(\tau_\gamma$-Int$(Cl(A)))$.

The converse of the above theorem is not true in general as shown in the following example.

Example 3.11. Consider $X = \{a, b, c, d\}$ with $\tau = P(X)$ and $I = \{\phi\}$. Define an operation $\gamma$ on $\tau$ by $\gamma(A) = X$ for all $A \in \tau$. Then $A = \{c, d\}$ is a $\gamma$-p-open set which is not pre-$\gamma$-I-open.

Remark 3.12. We have the following implications but none of these implications are reversible.

The intersection of two pre-$\gamma$-I-open sets need not be pre-$\gamma$-I-open as shown in the following example.

Example 3.13. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, c\}\}$ and $I = \{\phi, \{b\}\}$. Define an operation $\gamma$ on $\tau$ by $\gamma(A) = X$ for all $A \in \tau$. Set $A = \{a, b\}$ and $B = \{b, c\}$. Since $A^* = B^* = X$, then both $A$ and $B$ are pre-$\gamma$-I-open. But on the other hand $A \cap B = \{b\} \not\subseteq \gamma P_\gamma IO(X, \tau)$.

Theorem 3.14. Let $(X, \tau, I)$ be an ideal topological space and $\{A_\alpha : \alpha \in \Delta\}$ a family of subsets of $X$, where $\Delta$ is an arbitrary index set. Then,

1. If $A_\alpha \in \gamma P_\gamma IO(X, \tau)$ for all $\alpha \in \Delta$, then $U_{\alpha \in \Delta} A_\alpha \in \gamma P_\gamma IO(X, \tau)$.
2. If $A \in \gamma P_\gamma IO(X, \tau)$ and $U \in \tau_\gamma$, then $A \cap U \in \gamma P_\gamma IO(X, \tau)$. Where $\gamma$ is a regular operation on $\tau$.

Proof. 1. Since $\{A_\alpha : \alpha \in \Delta\} \subseteq \gamma P_\gamma IO(X, \tau)$, then $A_\alpha \subseteq \tau_\gamma$-Int$(Cl^*(A_\alpha))$ for each $\alpha \in \Delta$. Then we have $U_{\alpha \in \Delta} A_\alpha \subseteq U_{\alpha \in \Delta} \tau_\gamma$-Int$(Cl^*(A_\alpha)) \subseteq \tau_\gamma$-Int$(U_{\alpha \in \Delta} Cl^*(A_\alpha)) \subseteq \tau_\gamma$-Int$(Cl^*(U_{\alpha \in \Delta} A_\alpha))$. This shows that $U_{\alpha \in \Delta} A_\alpha \in \gamma P_\gamma IO(X, \tau)$.

2. By the assumption, $A \subseteq \tau_\gamma$-Int$(Cl^*(A))$ and $U = \tau_\gamma$-Int$(U)$. Thus using Lemma 2.6, we have $A \cap U \subseteq \tau_\gamma$-Int$(Cl^*(A)) \cap \tau_\gamma$-$Int(U) = \tau_\gamma$-$Int(Cl^*(U)) \cap (A \cap U)$ $= \tau_\gamma$-$Int((A \cup (A \cap U)) \subseteq \tau_\gamma$-$Int(Cl^*(A) \cup (A \cap U)) \subseteq \tau_\gamma$-$Int(Cl^*(A \cap U))$. This shows that $A \cap U \in \gamma P_\gamma IO(X, \tau)$.

Proposition 3.15. For an ideal topological space $(X, \tau, I)$ with an operation $\gamma$ on $\tau$ and $A \subseteq X$ we have:

1. If $I = \{\phi\}$, then $A$ is pre-$\gamma$-I-open if and only if $A$ is pre-$\gamma$-open.
2. If $I = P(X)$, then $P_\gamma IO(X) = \tau_\gamma$.

Proof. 1. By Theorem 3.4, we need to show only sufficiency. Let $I = \{\phi\}$, then $A^* = Cl(A)$ for every subset $A$ of $X$. Let $A$ be pre-$\gamma$-open, then $A \subseteq \tau_\gamma$-$Int(Cl(A)) = \tau_\gamma$-$Int(A^*) \subseteq \tau_\gamma$-$Int(A \cup A^*) = \tau_\gamma$-$Int(Cl^*(A))$ and hence $A$ is pre-$\gamma$-I-open.2. Let $I = P(X)$, then $A^* = \phi$ for every subset $A$ of $X$. Let $A$ be any pre-$\gamma$-I-open set, then $A \subseteq \tau_\gamma$-$Int(Cl^*(A)) = \tau_\gamma$-$Int(A \cup A^*) = \tau_\gamma$-$Int(A \cup \phi) = \tau_\gamma$-$Int(A)$ and hence $A$ is $\gamma$-open. By Theorem 3.2, we obtain $P_\gamma IO(X) = \tau_\gamma$.

Remark 3.16. 1. If a subset $A$ of a $\gamma$-regular space $(X, I, \tau)$ is open then $A$ is pre-$\gamma$-I-open.
2. If a subset $A$ of a submaximal space $(X, I, \tau)$ is pre-$\gamma$-I-open then $A$ is open.
3. If $(X, I, \tau)$ is $\gamma$-regular space and $I = P(X)$, then $A$ is pre-$\gamma$-I-open if and only if $A$ is open.

Remark 3.17. Let $(X, I, \tau)$ be a $\gamma$-regular space and $I = P(X)$. Then

1. If $A$ is $R$-I-open then $A$ is pre-$\gamma$-I-open.
2. If \( A \) is \( \delta \)-open then \( A \) is pre-\( \gamma \)-I-open.
3. If \( A \) is regular open then \( A \) is pre-\( \gamma \)-I-open.
4. If \( A \) is \( \delta \)-open then \( A \) is pre-\( \gamma \)-I-open.

**Remark 3.18.** For an ideal topological space \((X, \tau, I)\) with an operation \( \gamma \) on \( \tau \) and \( I = P(X) \) we have:
1. If \( A \) is pre-\( \gamma \)-I-open then \( A \) is open.
2. If \( A \) is pre-\( \gamma \)-I-open then \( A \) is \( \alpha \)-I-open.
3. If \( A \) is pre-\( \gamma \)-I-open then \( A \) is semi-I-open.

**Proposition 3.19.** Let \((X, \tau, I)\) be an ideal topological space and \(A\) a subset of \(X\). If \( A \) is closed and pre-\( \gamma \)-I-open, then \( A \) is R-I-open.

**Proof.** Let \( A \) be pre-\( \gamma \)-I-open, then we have \( A \subseteq \tau_{\gamma} (\mathit{Cl}(A)) \subseteq \mathit{Cl}(A) \subseteq \mathit{Int}(A) \), hence \( A \) is I-open.

**Remark 3.20.** Let \((X, \tau, I)\) be a \( \gamma \)-regular space. If \( A \subseteq (X, \tau, I) \) is R-I-open, then \( A \) is pre-\( \gamma \)-I-open.

**Remark 3.21.** If \((X, \tau, I)\) is \( \gamma \)-regular space and \( I = \{ \phi \} \), then
1. \( A \) is pre-\( \gamma \)-I-open if and only if \( A \) is pre-open.
2. \( A \) is pre-\( \gamma \)-I-open if and only if \( A \) is \( \gamma \)-pre-open.
3. \( A \) is \( \gamma \)-pre-open if and only if \( A \) is p-pre-open.

**Proposition 3.22.** Let \((X, \tau, I)\) be an ideal topological space and \(A\) a subset of \(X\). If \( I = \{ \phi \} \) and \( A \) is pre-\( \gamma \)-I-open, then \( A \) is I-open.

**Proof.** Let \( A \) be pre-\( \gamma \)-I-open, then we have \( A \subseteq \tau_{\gamma} (\mathit{Cl}(A)) \subseteq \mathit{Cl}(A) \subseteq \mathit{Int}(A) \), hence \( A \) is I-open.

**Remark 3.23.** If \((X, \tau, I)\) is a \( \gamma \)-regular space and \( I = \delta \), then \( A \) is \( \delta \)-open if \( A \) is pre-\( \gamma \)-I-open.

**Remark 3.24.** If \((X, \tau, I)\) is \( \gamma \)-regular then \( A \) is pre-\( \gamma \)-I-open if and only if \( A \) is \( \gamma \)-pre-open.

**Proposition 3.25.** If \( A \subseteq (X, \tau, I) \) is \( * \)-perfect and pre-\( \gamma \)-I-open, then \( A \) is \( \gamma \)-open.

**Proof.** Let \( A \) be \( * \)-perfect, then \( A = A^* \) and \( \mathit{Cl}(A) = \tau_{\gamma} (\mathit{Int}(A)) \subseteq \mathit{Int}(A) \), hence \( A \) is \( \gamma \)-open.

**Remark 3.26.** If \( A \subseteq (X, \tau, I) \) is \( * \)-perfect and pre-\( \gamma \)-I-open, then \( A \) is open.

**Proposition 3.27.** If \( A \) is \( * \)-closed in \((X, \tau, I)\) and pre-\( \gamma \)-I-open, then \( A \) is \( \gamma \)-open.

**Proof.** Let \( A \) be pre-\( \gamma \)-I-open, then \( A \subseteq \mathit{Cl}(A) \subseteq \tau_{\gamma} (\mathit{Int}(A)) \subseteq \mathit{Int}(A) \), hence \( A \) is \( \gamma \)-open.

**Remark 3.28.** If \( A \) is \( * \)-closed in \((X, \tau, I)\) and pre-\( \gamma \)-I-open, then \( A \) is open.

**Proposition 3.29.** If \( A \) is \( * \)-perfect in \((X, \tau, I)\) and pre-\( \gamma \)-I-open, then \( A \) is I-open.

**Proof.** Let \( A \) be pre-\( \gamma \)-I-open, then \( A \subseteq \mathit{Cl}(A) \subseteq \tau_{\gamma} (\mathit{Int}(A)) \subseteq \mathit{Int}(A) \), hence \( A \) is I-open.

**Proposition 3.30.** If \( A \) is \( * \)-dense-in-itself in \((X, \tau, I)\) and pre-\( \gamma \)-I-open, then \( A \) is I-open.

**Proof.** Let \( A \) be pre-\( \gamma \)-I-open, then \( A \subseteq \mathit{Cl}(A) \subseteq \tau_{\gamma} (\mathit{Int}(A)) \subseteq \mathit{Int}(A) \) and hence \( A \) is I-open.

**Proposition 3.33.** If \( A \) is \( * \)-extremally disconnected \( \gamma \)-regular space \((X, \tau, I)\) is semi-I-open then \( A \) is pre-\( \gamma \)-I-open.

**Proof.** Let \( A \) be semi-I-open, then \( A \subseteq (\mathit{Cl}(A)) = \tau_{\gamma} (\mathit{Int}(A)) \subseteq \mathit{Int}(A) \), hence \( A \) is pre-\( \gamma \)-I-open.

**Proposition 3.34.** If \( A \) is \( * \)-extremally disconnected \( \gamma \)-regular space \((X, \tau, I)\) is b-I-open and \( I = P(X) \), then \( A \) is pre-\( \gamma \)-I-open.

**Proof.** Let \( A \) be b-I-open, then \( A \subseteq (\mathit{Cl}(A)) = \tau_{\gamma} (\mathit{Int}(A)) \subseteq \mathit{Int}(A) \subseteq \mathit{Cl}(A) \subseteq (\mathit{Cl}(A)) \subseteq \mathit{Int}(A) \), hence \( A \) is pre-\( \gamma \)-I-open.

**Theorem 3.35.** Let \((X, \tau, I)\) be an \( * \)-extremally disconnected \( \gamma \)-regular ideal space and \( V \subseteq X \), the following properties are equivalent:
1. \( V \) is a \( \gamma \)-open set.
2. \( V \) is \( \alpha \)-I-open and weakly \( I \)-local closed.
3. \( V \) is pre-\( \gamma \)-I-open and weakly \( I \)-local closed.
4. \( V \) is pre-\( \gamma \)-I-open and weakly \( I \)-local closed.
5. \( V \) is semi-I-open and weakly \( I \)-local closed.
6. \( V \) is b-I-open and weakly \( I \)-local closed.

**Proof.** (1) \( \Rightarrow \) (2): It follows from the fact that every \( \gamma \)-open set is open and every open set is \( \alpha \)-I-open and weakly \( I \)-local closed.

(2) \( \Rightarrow \) (3): It follows from Proposition 3.31.

(3) \( \Rightarrow \) (4), (4) \( \Rightarrow \) (5) and (5) \( \Rightarrow \) (6): Obvious.

(6) \( \Rightarrow \) (1): Suppose that \( V \) is a b-I-open set and a weakly \( I \)-local closed set in \( X \). It follows that \( V \subseteq (\mathit{Cl}(V)) \subseteq \mathit{Int}(V) \), since \( V \) is a weakly \( I \)-local closed set, then there exists an open set \( G \) such that \( V = G \cap \mathit{Cl}(V) \). It follows from Theorem 2.5 that \( V \subseteq (\mathit{Cl}(V)) \subseteq (\mathit{Int}(V)) \subseteq (G \cap \mathit{Cl}(V)) \subseteq (\mathit{Int}(V)) \subseteq (G \cap \mathit{Cl}(V)) \subseteq (\mathit{Int}(V)) \subseteq (V) \).

Thus, \( V \subseteq (\mathit{Int}(V)) \) and hence \( V \) is a \( \gamma \)-open set in \( X \).
**Theorem 3.35.** Let \((X, I, \tau)\) be a \(*\)-extremely disconnected \(\gamma\)-regular ideal space and \(V \subseteq X\), the following properties are equivalent:

1. \(V\) is a \(\gamma\)-open set.
2. \(V\) is \(\alpha-I\)-open and a locally closed set.
3. \(V\) is \(\pre\gamma-I\)-open and a locally closed set.
4. \(V\) is \(\pre\gamma\)-I-open and a locally closed set.
5. \(V\) is semi-\(I\)-open and a locally closed set.
6. \(V\) is \(b-I\)-open and a locally closed set.

**Proof.** By Theorem 3.34, it follows from the fact that every open set is locally closed and every locally closed set is weakly \(I\)-local closed.

**Definition 3.36.** A subset \(F\) of a space \((X, \tau, I)\) is said to be \(\pre\gamma-I\)-closed if its complement is \(\pre\gamma-I\)-open.

**Theorem 3.37.** A subset \(A\) of a space \((X, \tau, I)\) is \(\pre\gamma-I\)-closed if and only if \(\tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq A\).

**Proof.** Let \(A\) be a \(\pre\gamma-I\)-closed set of \((X, \tau, I)\). Then \(X-A\) is \(\pre\gamma-I\)-open and hence \(X-A \subseteq \tau_\gamma Int(Cl(\tau_\gamma Int(A))) = X-\tau_\gamma Cl(\tau_\gamma Int(A))\). Therefore, we have \(\tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq A\).

Conversely, let \(\tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq A\). Then \(X-A \subseteq \tau_\gamma Int(Cl(\tau_\gamma Int(A))) \subseteq X\) is \(\pre\gamma-I\)-open. Therefore, \(A\) is \(\pre\gamma-I\)-closed.

**Theorem 3.38.** If a subset \(A\) of a space \((X, \tau, I)\) is \(\pre\gamma-I\)-closed, then \(Cl(\tau_\gamma Int(A)) \subseteq A\).

**Proof.** Let \(A\) be any \(\pre\gamma-I\)-closed set of \((X, \tau, I)\). Since \(\tau_\gamma(I)\) is finer than \(\tau\) and \(\tau\) is finer than \(\tau_\gamma\), we have \(Cl(\tau_\gamma Int(A)) \subseteq \tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq \tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq \tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq A\). Therefore, by Theorem 3.37, we obtain \(Cl(\tau_\gamma Int(A)) \subseteq A\).

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