Hadronic $\tau$ Decay Based Determinations of $|V_{us}|$

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I review sum rule determinations of $|V_{us}|$ employing hadronic $\tau$ decay data, taking into account recent HFAG updates of exclusive $\tau$ branching fractions and paying special attention to the impact of the slow convergence of the relevant integrated $D = 2$ OPE series and the potential role of contributions of as-yet-unmeasured higher multiplicity modes to the strange inclusive spectral distribution. In addition to conventional flavor-breaking sum rule determinations, information obtainable from mixed $\tau$-electroproduction sum rules having much reduced OPE uncertainties, and from sum rules based on the inclusive strange decay distribution alone, is also considered. Earlier discrepancies with the expectations of 3-family unitarity are found to be reduced, both the switch to $D = 2$ OPE treatments favored by self-consistency tests and the increase in the strange branching fractions playing a role in this reduction.

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1 Introduction

Recent determinations of $|V_{us}|$ using flavor-breaking (FB) hadronic $\tau$ decay sum rules [1, 2, 3, 4] yield results $\sim 3\sigma$ low compared to both 3-family unitarity expectations, and those from $K_{\mu3}$ and $K_{\mu2}$ analyses [5, 6]. The $\tau$ determinations employ finite energy sum rules (FESRs) which, for a kinematic-singularity-free correlator, $\Pi$, with spectral function, $\rho$, take the form (valid for arbitrary $s_0$ and analytic $w(s)$)

$$\int_{s_0}^{s_0} w(s)\rho(s)\,ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s)\Pi(s)\,ds . \quad (1)$$

$|V_{us}|$ is obtained by setting $\Pi = \Delta \Pi_{\tau} \equiv \left[ \Pi_{V+A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)} \right]$, with $\Pi_{V/A;ij}^{(J)}(s)$ the spin $J = 0,1$ components of the flavor $ij$, vector (V) or axial vector (A) current two-point functions. For large enough $s_0$, the OPE can be used on the RHS, while for $s_0 \lesssim m_\tau^2$, the $\rho_{ij}^{(J)}$ needed on the LHS are related to the inclusive differential distributions, $dR_{V/A;ij}/ds$, with $R_{V/A;ij} \equiv \Gamma[\tau^{-}\rightarrow \nu_\tau$ hadrons$V/A;ij(\gamma)]/\Gamma[\tau^{-}\rightarrow \nu_\tau e^-\nu_e(\gamma)]$, by [7]

$$\frac{dR_{V/A;ij}}{ds} = \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2} \left[ w_\tau(y_\tau)\rho_{V/A;ij}^{(0+1)}(s) - w_L(y_\tau)\rho_{V/A;ij}^{(0)}(s) \right] \quad (2)$$

with $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1 - y)(2 + y)$, $w_L(y) = 2y(1 - y)^2$, $V_{ij}$ the flavor $ij$ CKM matrix element, and $S_{EW}$ a short-distance electroweak correction.

The $J = 0+1$ combination, $\Delta \Pi_{\tau}$, is employed due to the extremely bad behavior of the integrated $J = 0$, $D = 2$ OPE series [8]. Fortunately, $J = 0$ spectral contributions are dominated by the accurately known $K$ and $\pi$ pole terms, with residual continuum contributions numerically negligible for $ij = ud$, and determinable phenomenologically via dispersive [9] and sum rule [10] analyses for $ij = us$. Subtracting the $J = 0$ contributions from $dR_{V+A;ij}/ds$, one can evaluate the re-weighted $J = 0+1$ integrals $R_{V+A;ij}^w(s_0) \equiv 12\pi^2 S_{EW} |V_{ij}|^2 \int_{s_0}^{s_0} \frac{ds}{m_\tau^2} w(s) \rho_{V+A;ij}^{(0+1)}(s)$ and FB differences

$$\delta R_{V+A}^w(s_0) = \frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \frac{R_{V+A;us}^w(s_0)}{|V_{us}|^2} = 12\pi^2 S_{EW} \int_{s_0}^{s_0} \frac{ds}{m_\tau^2} w(s) \Delta \rho_\tau(s) \quad (3)$$

Taking $|V_{ud}|$ and any OPE parameters from other sources, Eq. (11) then yields [11]

$$|V_{us}| = \sqrt{\frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2}} \left[ \frac{R_{V+A;us}^w(s_0)}{|V_{us}|^2} - \delta R_{V+A}^w,_{OPE}(s_0) \right] . \quad (4)$$

The OPE contribution in Eq. (11) is at the few-to-several-% level of the $ud$ spectral integral term for weights used previously in the literature [11, 2, 3], making modest accuracy for $\delta R_{V+A}^w,_{OPE}(s_0)$ sufficient for a high accuracy determination of $|V_{us}|$ [11].

*As an example, removing entirely the OPE corrections from the recent HFAG $s_0 = m_\tau^2$, $w = w_\tau$ determination, $|V_{us}|$ is shifted by only $\sim 3\%$, from 0.2174(23) [4] to 0.2108(19).
Estimating the error on $\delta R_{V+A}^{\text{OPE}}(s_0)$ is complicated by the slow convergence of the leading dimension $D = 2$ OPE series, $[\Delta \Pi^\tau]_{D=2}^{\text{OPE}}$. To four loops \[ \left[ \Delta \Pi^\tau(Q^2) \right]_{D=2}^{\text{OPE}} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[ 1 + \frac{7}{3} \alpha_s + 19.93\alpha_s^2 + 208.75\alpha_s^3 + d_4\alpha_s^4 + \cdots \right] \] (5) with $\alpha = \alpha_s(Q^2)/\pi$, and $\alpha_s(Q^2)$ and $m_s(Q^2)$ the running coupling and strange quark mass in the \(\overline{MS}\) scheme. Since $\alpha(m^2_\tau) \simeq 0.1$, convergence at the spacelike point on $|s| = s_0$ is marginal at best and conventional error estimates may significantly underestimate the truncation uncertainty. Consistency checks are, however, possible. Assuming both the data and OPE error estimates are reliable, $|V_{us}|$ should be independent of $s_0$ and $w(s)$. On the OPE side, results obtained using $D = 2$ truncation schemes differing only at orders beyond the truncation order should agree to within the truncation uncertainty estimate. We consider three commonly used truncation schemes: the contour improved (CIPT) prescription, used with either the truncated expression for $[\Delta \Pi^\tau]_{D=2}^{\text{OPE}}$, or, after partial integration, the correspondingly truncated Adler function series, and the truncated fixed-order (FOPT) prescription.

2 $|V_{us}|$ from various FESRs employing $\tau$ decay data

Results below are based on updated 2010 HFAG hadronic and lepton-universality-constrained leptonic $\tau$ BFs [4], supplemented by SM $K_{\mu2}$ and $\pi_{\mu2}$ expectations for $B_K$ and $B_\pi$. The publicly available ALEPH $ud$ distribution [12], rescaled to reflect the resulting normalizations $R_{V+A;us} = 0.1623(28)$, $R_{V+A;ud} = 3.467(9)$, is used for $\rho_{V+A;ud}(s)$. Though improved exclusive $us$ BFs are available from BaBar and Belle, a completed inclusive $us$ distribution is not. The ALEPH inclusive $us$ distribution [13], however, corresponds to exclusive BFs with significantly larger errors, and, sometimes, significantly different central values [4]. Following Ref. [14], we “partially update” $\rho_{V+A;us}(s)$, rescaling the ALEPH distribution mode by mode with the ratio of new to old BFs. This procedure works well when tested using BaBar $\tau \to K^-\pi^+\pi^-\nu_\tau$ data [15], but is likely less reliable for modes ($K3\pi$, $K4\pi$, \ldots) estimated using Monte Carlo rather than measured by ALEPH. OPE input is specified in Ref. [16].

For $s_0 = m^2_\tau$, $w = w_\tau$, the $ud$ and $us$ spectral integrals needed in the FB $\Delta \Pi^\tau$ FESR are determined by the corresponding inclusive BFs. Conventional last-term-retained\oplus residual-scale-dependence $D = 2$ OPE truncation error estimates yield a combined theoretical uncertainty of 0.0005 on $|V_{us}|$ in this case [3].

The left panel of Fig. [1] shows $|V_{us}|$ versus $s_0$ for each of the three prescriptions for the $w_\tau$-weighted $D = 2$ OPE series. The two CIPT treatments give similar results, but show poor $s_0$-stability. The FOPT prescription yields significantly improved, \[d_4 \sim 2378\] [11] for the as-yet-undetermined 5-loop coefficient $d_4$. \[\footnote{We use the estimate $d_4 \sim 2378$ [11] for the as-yet-undetermined 5-loop coefficient $d_4$.}
though not perfect, \( s_0 \)-stability. For all \( s_0 \), the FOPT-CIPT difference is significantly greater than the nominally estimated 0.0005 theoretical error. The integrated \( D = 2 \) series is also better behaved for FOPT. The FOPT version of \( \delta R_{V+A}^{OPE}(m^2) \) is a factor of \( \sim 2 \) larger than either of the two CIPT versions, suggesting that the integrated \( D = 2 \) convergence is indeed slow, and the resulting truncation uncertainty large. The \( s_0 = m^2 \) version of the better behaved FOPT prescription yields

\[
|V_{us}| = 0.2193(3)_{\text{ud}}(19)_{us}(19)_{th} , \tag{6}
\]

\( \sim 2.3\sigma \) below 3-family unitarity expectations, the theory error reflecting the sizeable \( D = 2 \) FOPT-CIPT difference. The right panel of Fig. 1 compares the results from FB FESRs corresponding to three additional weights, \( w_{10}, \hat{w}_{10}, \) and \( w_{20} \), constructed in Ref. [17] to improve convergence of the integrated CIPT \( D = 2 \) series, with those of the \( w_r \) case. Improved \( s_0 \)-stability is observed, together with a reduced weight-choice dependence. For \( \hat{w}_{10} \) (which shows the best \( s_0 \)-stability), \( |V_{us}| = 0.2188 \) at \( s_0 = m^2 \).

In the absence of a new version of the inclusive \( us \) distribution, the experimental error has to be based on the 1999 ALEPH \( us \) covariances, and is 0.0033.

![Graph](image)

**Figure 1:** \( |V_{us}| \) vs. \( s_0 \) for (i) Left panel: the FB \( w_r \) FESR, using the three prescriptions for the \( D = 2 \) OPE series and (ii) Right panel: the FB \( w_{10}, \hat{w}_{10} \) and \( w_{20} \) FESRs, using the CIPT+correlator prescription, with FB \( w_r \) results shown for comparison.

Slow convergence of the integrated \( D = 2 \) OPE series and possible missing higher multiplicity \( us \) spectral strength could both account for the \( s_0 \)-instability of the FB \( w_r \) FESR results. The latter possibility can be tested using FESRs for \( \Pi^{(0+1)}_{V+A;us} \). For \( w(s) \geq 0 \) and \( s_0 \) large enough that the region of missing strength overlaps the range of the \( us \) spectral integral, \( |V_{us}| \) should come out low, while for \( s_0 \) low enough to exclude such overlap, \( |V_{us}| \) should rise back to its true value. Two new OPE terms enter these FESRs: the \( D = 0 \) contribution (known to 5-loops [18]) and a \( D = 4 \) gluon condensate contribution. Excellent agreement between the world average \( \alpha_s \)
value and that obtained from \(ud, J = 0 + 1\) V, A and V+A FESRs \([19]\) shows these ingredients can be reliably evaluated. Results for \(|V_{us}|\) versus \(s_0\), for \(w = w_τ\), are shown in the left panel of Fig. 2. Results for the three \(D = 2\) prescriptions agree with those of the corresponding FB \(w_τ\) FESR treatment. The \(s_0\)-dependence of \(|V_{us}|\) for the two CIPT prescriptions, however, is clearly incompatible with the assumption that the \(D = 2\) OPE representation is reliable and the FB \(w_τ\) instability is due to missing higher multiplicity \(us\) spectral strength. As for the FB \(w_τ\) FESR, the FOPT \(D = 2\) treatment produces improved, though not perfect, \(s_0\)-stability.

The larger-than-expected \(D = 2\) OPE uncertainties of the FB \(τ\) FESRs can be reduced by considering FESRs for \(ΔΠ_M = 9Π_{EM} - 6Π_{V;ud}^{(0+1)} + ΔΠ_τ\) \([20]\). \(Π_{EM}\) is the electromagnetic (EM) correlator, whose spectral function is determined by the bare \(e^+e^- → hadrons\) cross-sections. \(ΔΠ_M\) is the unique FB EM-\(τ\) combination with the same \(Π_{V;A:us}^{(0+1)}\) normalization as \(ΔΠ_τ\) and zero \(O(α_s^0)\) \(D = 2\) coefficient. The \(O(α_s^0)\) \(D = 4\) coefficient is also 0 and the remaining \(D = 2\) coefficients suppressed by factors of \(\sim 5 - 7\) relative to those of \(ΔΠ_τ\). Integrated \(D > 4\) contributions, which are not suppressed \([20]\), can be fitted to data due to their stronger \(s_0\)-dependence. The strong suppression of \(D = 2\) and \(D = 4\) contributions at the correlator level greatly reduces OPE-induced uncertainties \([20]\). At present, use of these FESRs is complicated by inconsistencies (within isospin breaking corrections) of the EM and \(τ\) \(2\pi\) and \(4\pi\) spectral data \([21]\). We illustrate the improved \(s_0\)-stability of the \(ΔΠ_M\) FESRs in the right panel of Fig. 2 for \(w = w_τ\), \(w_2(y) = (1 - y)^2\) and \(w_3(y) = 1 - \frac{3}{2}y + \frac{1}{2}y^3\), assuming the \(τ\) data to be correct for both \(2\pi\) and \(4\pi\). The \(s_0 = m_τ^2, w_τ\) result for \(|V_{us}|\) is 0.2222(20)\(_τ\)(28)\(_EM\), with only experimental errors shown. \(ΔΠ_M\) FESRs, while promising for the future, require resolution of the \(τ\) vs. EM \(2\pi\) and \(4\pi\) discrepancies.

![Figure 2: \(|V_{us}|\) vs. \(s_0\) for (i) Left panel: the \(w_τ\) us V+A FESR, using the three \(D = 2\) OPE prescriptions, and (ii) Right panel: a selection of EM-\(τ\) FESRs.](image-url)
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