Generalized Zeta Function Regularization and the Multiplicative Anomaly

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1 Introduction

It is well known that (Euclidean) Partition function is a very important object in Relativistic Quantum Field Theory: the full propagator and all other n-point correlation functions can be computed by it. The formalism can be extended also in curved space-time [1]. The relativistic nature of quantum fields, namely the fact that an infinite number of degrees of freedom is involved, plays a crucial role. As a result, ultraviolet divergences are present, and regularization and renormalization are necessary.

In the one-loop approximation or in the external field approximation, one may describe quantum (scalar) field by means of path (Euclidean) integral and expressing the Euclidean partition function in terms of functional determinants associated with differential operators. Namely, the partition function is proportional to

\[ Z_{1} = (\det L)^{-1/2}, \]  

(1)

with \( L \) an elliptic self-adjoint non negative differential, the fluctuation operator. Then, the computation of Euclidean one-loop partition function reduces to the computations of functional determinants. The functional determinants are divergent, ultraviolet divergences are present and may be regularized by making use of suitable regularization.

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As a simplest and illustrative example, let us consider \( \lambda \phi^4 \) self-interacting scalar field. Let us split the quantum field as \( \phi = \Phi_0 + \eta \), where \( \Phi_0 \) is a classical background field. Thus the one-loop fluctuation operator is

\[
L = -\Delta + m^2 + \frac{\lambda}{2} \Phi_0^2.
\]

We recall that in gauge theory, \( A \) is singular due to the gauge invariance and a gauge fixing ghost contributions are necessary. The one-loop quantum partition function \( Z[A], S_0 \) being the classical action

\[
Z[A] \approx e^{-S_0} \int d[\eta] e^{-\frac{1}{2} \int d^4x \eta L \eta}
\]

reduces to a Gaussian functional integral, and as well known, it can be computable in terms of the real eigenvalues \( \lambda_n \) of the fluctuation operator, namely \( L \phi_n = \lambda_n \phi_n \). Since \( \phi = \sum_n c_n \phi_n \), the formal functional measure \( d[\phi] \) may be defined as (\( \mu \) arbitrary renormalization parameter)

\[
d[\phi] = \prod_n \frac{dc_n}{\sqrt{\mu}}.
\]

As a consequence, the one-loop quantum "prefactor" is

\[
Z_1[L] = \prod_n \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dc_n e^{-\frac{1}{2} \lambda_n c_n^2} = \left[ \det(\mu^{-2} L) \right]^{-1/2}
\]

and the one-loop Euclidean Effective Action reads

\[
\Gamma_E = -\log Z = S_0 + \frac{1}{2} \log(\det(\mu^{-2} L)).
\]

What about the evaluation of the above functional determinants? Recall the well known Schwinger argument: one starts from the formal relation

\[
\log \det L = \text{Tr} \log L.
\]

Thus

\[
\delta \log \det L = \text{Tr}(L^{-1} \delta L)
\]

and consequently one arrives at the formal expression

\[
\langle \log \det L \rangle = - \left( \int_0^\infty dt t^{-1} \text{Tr} e^{-t L} \right).
\]

With regard to this expression, for large \( t \), there are no problems, since \( L \) is assumed to be non negative, but for small \( t \), the Heat Kernel expansion in regular smooth and without boundary case and \( D = 4 \), reads (see for example [2]).
\[ \text{Tr} e^{-tL} \simeq \sum_{r=0}^{\infty} A_r t^{r-2}. \]  

(10)

It follows that the Schwinger representation of functional determinants is divergent at \( t = 0 \), and one needs for a regularization. One of the simplest and most useful is the dimensional regularization \([3]\), which in our formulation consists in the replacement

\[ t^{-1} \rightarrow \frac{r^{\epsilon-1}}{\Gamma(1+\epsilon)}. \]  

(11)

As a result, the related regularized functional determinant with \( \epsilon \) sufficiently large is

\[ \log \text{det} L(\epsilon) = -\int_0^\infty dt \frac{r^{\epsilon-1}}{\Gamma(1+\epsilon)} \text{Tr} e^{-tL} = -\frac{\zeta(\epsilon, L)}{\epsilon}, \]  

(12)

where the generalized zeta function associated with \( L \), defined for \( \text{Re} s > 2 \)

\[ \zeta(s, L) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} e^{-tL}, \]  

(13)

has been introduced. In order to be able to handle the cutoff, one makes use of the celebrated Seeley Theorem: If \( L \) is elliptic and differential operator, defined on a smooth and compact manifold, the analytic continuation of \( \zeta(s, L) \) in the whole complex space \( s \) is regular at \( s = 0 \). Making use of this dimensional regularization, and making a Taylor expansion at \( \epsilon = 0 \), one arrives at

\[ \log \text{det} L(\epsilon) = -\frac{1}{\epsilon} \zeta(0, L) - \zeta'(0, L) + O(\epsilon). \]  

(14)

Thus, one obtains a justification of the zeta-function regularized functional determinant \([4, 5, 6]\), namely

\[ \log \text{det} L = -\zeta'(0, L). \]  

(15)

In four dimension, the computable Seeley-de Witt coefficient \( A_2 = \zeta(0, L) \) controls the ultraviolet divergence, while \( \zeta'(0, L) \) gives the finite contribution, and this, in general, is difficult to evaluate (see for example \([2]\) and references therein).

# 2 Multiplicative anomaly in the regular case

In some cases, if, for example, one is dealing with a vector valued fields (charged scalar field), the \( L \) becomes a matrix valued differential operator. In the evaluation of the determinant, one first computes the algebraic one. As a consequence, one is dealing with products of operators. A crucial point arises: the zeta-function regularized determinants do not satisfy the relation \( \det(AB) = \det A \det B \), or equivalently

\[ \text{Indet}(AB) = \text{Indet} A + \text{Indet} B. \]  

(16)
In fact, in general, there exists the so-called multiplicative anomaly, which may be defined as:

\[ a(A, B) = \ln \det(AB) - \ln \det(A) - \ln \det(B). \]  

(17)

Here it is left understood that the determinants of the two elliptic operators, A and B are regularized by means of the zeta-function regularization. This multiplicative anomaly has been discovered by Wodzicki (see for example [7, 8, 9] and references therein).

In the simple but important case in which A and B are two commuting invertible self-adjoint elliptic operators of second order, the multiplicative anomaly can be evaluated by the Wodzicki formula (a discussion can be found in [10] and references therein).

\[ a(A, B) = \frac{1}{8} \text{res} \left[ (\ln(AB^{-1}))^2 \right], \]  

(18)

where the non-commutative residue, denoted by res, related to a classical pseudo-differential operator Q of order zero may defined by the logarithmic term in t of the following generalized heat-kernel expansion

\[ \text{Tr}(Qe^{-tH}) = \sum_j c_j t^{(j-D)/2} - \frac{\text{res} Q}{2} \ln t + O(t \ln t), \]  

(19)

where H is an elliptic non negative operator of second order, irrelevant for the evaluation of resQ.

However, from a practical point of view, the non-commutative residue can also be evaluated by means of the local formula found by Wodzicki, namely

\[ \text{res} Q = (2\pi)^{-D} \int_M dx \int_{|k|=1} Q_{-D}(x,k) dk. \]  

(20)

Here the homogeneity component of order \(-D\) of the complete symbol appears. Recall that a classical pseudo-differential operator Q of order zero has a complete symbol \(e^{ikx}Qe^{-ikx}\), admitting the following asymptotics expansion, valid for large \(|k|\)

\[ Q(x,k) \simeq \sum_{j=0}^{\infty} Q_{-j}(x,k). \]  

(21)

In this expansion, the related coefficients satisfy the homogeneity property \(Q_j(x, \lambda k) = \lambda^{-j} Q_{-j}(x,k)\).

### 2.1 Non interacting charged boson field

Let us consider a physical example: a free charged boson field at finite temperature \(\beta = 1/T\) and chemical potential \(\mu\). The related grand canonical partition function is standard and reads
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\[ Z_{\beta,\mu} = \int_{\phi(\tau)=\phi(\tau+\beta)} D\phi_i e^{-\frac{1}{2} \int_0^\beta d\tau \int d^3x \phi A_{ij} \phi_j}, \]  
(22)

where

\[ A_{ij} = (L_\tau + L_3 - \mu^2) \delta_{ij} + 2\mu \epsilon_{ij} \sqrt{L_3}, \quad L_3 = -\Delta_3 + m^2, \]  
(23)

\( \Delta_3 \) being the Laplace operator on \( R^3 \), continuous spectrum \( k^2 \) and \( L_\tau = -\partial_\tau^2 \), discrete spectrum over the Matsubara frequencies \( \omega_n^2 = \frac{4\pi^2}{\beta^2} \). Thus, the grand canonical partition function may be written as (see, for example, [11] and references therein)

\[ \ln Z_{\beta,\mu} = -\ln \det |A_{ik}|. \]  
(24)

Now the algebraic determinant, denoted by \( |A| \), can be evaluated and gives

\[ |A_{ik}| = (K_+,K_-), \]  
(25)

with

\[ K_\pm = L_3 + (\sqrt{L_3} \pm i\mu)^2. \]  
(26)

However, it is easy to show that another factorization exists [11], i.e.

\[ |A_{ik}| = (L_+,L_-), \]  
(27)

where

\[ L_\pm = L_\tau + (\sqrt{L_3} \pm \mu)^2. \]  
(28)

Now a simple calculation gives

\[ |A_{ik}| = L_+ L_- = K_+ K_-, \]  
(29)

and in both cases one is dealing with the product of two pseudo-differential operators (ΨDOs), the couple \( L_+ \) and \( L_- \) being also formally self-adjoint. Thus, the partition function may be written as

\[ \ln Z_{\beta,\mu} = -\ln \det K_+ - \ln \det K_- + a(K_+,K_-), \]  
(30)

or as

\[ \ln Z_{\beta,\mu} = -\ln \det L_+ - \ln \det L_- + a(L_+,L_-). \]  
(31)

The evaluation of the multiplicative anomalies which appear in the above expressions can be done making use of the Wodzicki formula and a complete agreement is found between the two expressions of the partition function. Thus, if one neglects the multiplicative anomaly, one arrives at a mathematical inconsistency [11].
3 Multiplicative anomaly in the singular case

However there exist cases in which the analytic continuation of the zeta function is not regular at \( z = 0 \). Recall that the usual Seeley Theorem, is based on standard heat kernel expansion (here \( D = 4 \), and boundaryless case)

\[
\text{Tr} e^{-tL} \simeq \sum_{j=0}^{\infty} A_j t^{j-2}.
\]

As a consequence the standard meromorphic continuation admits only simple poles

\[
\zeta(s|L) = \frac{1}{\Gamma(s)} \left[ \sum_{j=0}^{\infty} \frac{A_j(L)}{s+j-2} + J(s) \right],
\]

the function \( J(s) \) being analytic. It follows that \( \zeta(s|L) \) is regular at \( s = 0 \) and \( \zeta(0|L) = A_2(L) \), and \( \zeta'(0|L) \) is well-defined and gives the regularized expression for \( \det \ln L \).

If we have a non standard Heat-Kernel expansion

\[
\text{Tr} e^{-tL} \simeq \sum_{j=0}^{\infty} A_j t^{j-2} + \sum_{j=0}^{\infty} P_j \ln t t^{j-2}
\]

namely, additional \( \ln t \) terms are present, one has a generalization of Seeley result:

\[
\zeta(s|L) = \frac{1}{\Gamma(s)} \left[ \sum_{j=0}^{\infty} \frac{A_j(L)}{s+j-2} - \sum_{j=0}^{\infty} \frac{P_j(L)}{(s+j-2)^2} + J(s) \right].
\]

As a consequence, double poles are present and, in general, \( \zeta(s|L) \) may have a simple pole at \( s = 0 \). This may happen with pseudodifferential operators or differential operators defined on non compact manifolds.

Within this new contest, two issues have to be discussed. The first one is: how \( \det \ln L \) may be defined and the second one: how the Multiplicative Anomaly may be computed in these singular cases?

With regard to the first issue, the starting point is to observe that the functional determinant of self-adjoint operator \( L \) is formally a “divergent infinite product”

\[
\prod_n \lambda_n,
\]

where \( \lambda_n \) are the eigenvalues of \( L \). The divergence is present because \( L \) is a self-adjoint unbounded operator. To deals with it, Mathematicians introduce a canonical regularization by considering the analytic continuation of associated zeta function \( \zeta(s|L) = \sum \lambda^{-s} \) and then by definition.
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\[
\prod_{k=1}^{\infty} \lambda_k \equiv e^{-\left(\text{Res}\left(\frac{\zeta(s)}{s}\right)\right)_{s=0}},
\]

where \(\text{Res}\) is the usual Cauchy residue.

In the regular case, one Taylor expands \(\zeta(s)\) at \(s = 0\) and one obtains

\[
\prod_{k=1}^{\infty} \lambda_k \equiv e^{-\zeta'(0)},
\]

in agreement with Ray-Singer-Hawking prescription. However, if one has a simple pole, namely \(\zeta(s|L) = \frac{\omega(s)}{s}\), then

\[
\prod_{k=1}^{\infty} \lambda_k \equiv e^{-\frac{\omega''(0)}{2}},
\]

in agreement with [12, 13]. This prescription is quite general and is valid for generic singular behaviour of \(\zeta\) at \(s = 0\).

What about the second issue? To our knowledge, Wodzicki approach and associated formula are valid only in the regular case. With regard to this issue, we note that, in general, one may proceed defining the regularized functional determinant of the operator \(L\) as regularization of a divergent product, namely

\[
\ln \text{det} L = -\text{Res}\left(\frac{\zeta(s|L)}{s^2}\right)_{s=0}.
\]

3.1 A multiplicative anomaly formula for shift operators

Consider elliptic differential self-adjoint operators: \(H = H_0 + V_1\) and \(H_V = H + V = H_0 + V_2\) with \(V = V_2 - V_1\) constant shifts. The main idea is to express all quantities as a function of \(\zeta(s|H)\). Now, the spectral theorem gives

\[
\zeta(s|H_V) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-V)^n}{n!} \Gamma(s+n)\zeta(s+n|H).
\]

\[
\zeta(s|H H_V) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-V)^n}{n!} \Gamma(s+n)\zeta(2s+n|H).
\]

Note that here only the meromorphic continuation of \(\zeta(H|s)\) appears.

Recalling that the Multiplicative Anomaly may be defined as

\[
\mathcal{A} = \ln \text{det}(H(H + V)) - \ln \text{det}(H + V) - \ln \text{det} H,
\]

one has

\[
\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_V,
\]
with
\[ \mathcal{A}_0 = - \operatorname{Res} \left( \frac{\zeta(2s|\mathcal{H}) - 2\zeta(s|\mathcal{H})}{s^2} \right)_{s=0}, \tag{45} \]

\[ \mathcal{A}_V = - \operatorname{Res} \left( \frac{1}{s^2 \Gamma(s)} \sum_{n=1}^{\infty} \frac{(-V)^n}{n!} \Gamma(s+n) \left[ \zeta(2s+n|\mathcal{H}) - \zeta(s+n|\mathcal{H}) \right] \right)_{s=0}. \tag{46} \]

A direct computation shows that the first term does not give any contribution to the Cauchy residue, and in the second term, only a finite number of terms survive, the ones corresponding to the poles for \( \operatorname{Re} s > 0 \), and these are, say \( n_0 \). Thus, one has
\[ \mathcal{A} = - \operatorname{Res} \left( \frac{1}{s^2 \Gamma(s)} \sum_{n=1}^{n_0} \frac{(-V)^n}{n!} \Gamma(s+n) \left[ \zeta(2s+n|\mathcal{H}) - \zeta(s+n|\mathcal{H}) \right] \right)_{s=0}. \tag{47} \]

This is a new result and gives a general expression for the Multiplicative Anomaly in the case considered. It should be noted that the above expression involves only the meromorphic continuation of \( \zeta(s|\mathcal{H}) \), and we remind that this follows from the Heat-Kernel expansion of heat trace \( \operatorname{Tr} e^{-t\mathcal{H}} \).

Example: If one has poles of third order,
\[ \zeta(s|\mathcal{H}) = \frac{1}{\Gamma(s)} \sum_{j=0}^{\infty} \left[ A_j s^j - \frac{P_j}{(s+j-2)^2} + \frac{C_j}{(s+j-2)^3} + J(s) \right]. \tag{48} \]

In this case \( \zeta(s|\mathcal{H}) \) has a pole of second order at \( s = 0 \), and we get
\[ \mathcal{A} = \frac{V^2}{4} \left[ A_0 + (1 - \gamma) P_0 \right] + \frac{1}{24} \left[ 10 - 2\pi^2 - 24\gamma + 12\gamma^2 - G \right] C_0, \tag{49} \]

\( \gamma = -\psi(1) \) being the Euler-Mascheroni constant and \( G \) is a computable constant. In the standard regular case, all \( P_j \) and \( C_j \) are vanishing and one gets the Wodzicki formula result \([7]\).

\[ \mathcal{A} = \frac{1}{4} V^2 A_0 = \frac{1}{96\pi} \int d^4x (V_1 - V_2)^2. \tag{50} \]

Let us conclude this Section with a simple example: a vector valued massive scalar field \( \phi \) defined in \( R \times H_3/\Gamma \), ultrastatic space-time with a non compact hyperbolic manifold with finite volume \([14]\).

\[ I = \int \left[ -\frac{1}{2} \phi \Delta \phi + \frac{m^2 \phi^2}{2} \right] \sqrt{g} d^4x \tag{51} \]

The Heat-Kernel expansion reads
\[ \operatorname{Tr} e^{-tL} \sim \sum_{j=0}^{\infty} [A_j(L) + P_j(L) \ln t] t^{j-2}. \tag{52} \]
A_0 = \frac{\text{Vol}}{16\pi^2}, \quad P_0(L) = 0, \quad P_1(L) = -\frac{\text{Vol}}{16\pi^2} \frac{\pi R}{6v_F}, \quad (53)

P_2(L) = \frac{\text{Vol}}{16\pi^2} \frac{\pi R \delta^2}{6v_F}. \quad (54)

with \( v_F \) finite volume of fundamental domain of hyperbolic non compact manifold and \( \delta^2 = m^2 + \frac{\bar{g}}{G} \). Note that \( P_0 = 0 \), thus the Multiplicative Anomaly is equal to the regular case.

4 Concluding Remarks

The Multiplicative Anomaly is present in dealing with functional determinants of products of differential operators. In the regular case, it is a local functional of the fields and can be computed. In the singular case, where the zeta functions are not analytic at \( s = 0 \), we have shown that it is still a local functional and we have provided a formula for its evaluation.

Within one-loop physics, apparently no new physics seems to be associated with Multiplicative Anomaly, also in the presence of generalized zeta-function regularization. However, its inclusion is necessary for mathematical consistency: charged scalar field at finite temperature is an example.

Furthermore, dealing with spinor fields, one has the Euclidean massive Dirac operator.

\[ K = p_\mu T^{\mu} + iM = A + iM, \quad A^+ = A = p_\mu T^{\mu}. \quad (55) \]

Problem: How to evaluate \( \ln \det A \) ? In \( D = 4 \), with \( L = A^2 \) being the spinorial Laplace operator in curved space, one has [15]

\[ \ln \det K = \frac{1}{2} \ln \det(L + M^2) + \frac{\pi}{2} \zeta(0) L + M^2) + c_1 M^2 A_1(L) + c_2 M^4 A_0(L). \quad (56) \]

where \( A_0 \) and \( A_1 \) are the associated Seeley-de Witt coefficients. Two remarks: first the last term contains \( A_1(L) \equiv \bar{R} \), Ricci scalar, this is Sakarov induced gravity idea [8]. Second, this last term may be interpreted as multiplicative anomaly contribution [15]. With regard to this last issue, recently, I. Shapiro and others have reported a non-trivial non-local M. A. in Quantum ED in curved space-time [16].

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