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Knit Products of Graded Lie Algebras and Groups

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Abstract. If a graded Lie algebra is the direct sum of two graded sub Lie algebras, its bracket can be written in a form that mimics a "double sided semidirect product". It is called the knit product of the two subalgebras then. The integrated version of this is called a knit product of groups — it coincides with the Zappa-Szép product. The behavior of homomorphisms with respect to knit products is investigated.

Introduction

If a Lie algebra is the direct sum of two sub Lie algebras one can write the bracket in a way that mimics semidirect products on both sides. The two representations do not take values in the respective spaces of derivations; they satisfy equations (see 1.1) which look "derivatively knitted" — so we call them a derivatively knitted pair of representations. These equations are familiar for the Frölicher-Nijenhuis bracket of differential geometry, see [1] or [2, 1.10]. This paper is the outcome of my investigation of what formulas 1.1 mean algebraically. It was a surprise for me that they describe the general situation (Theorem 1.3). Also the behavior of homomorphisms with respect to knit products is investigated (Theorem 1.4).

The integrated version of a knit product of Lie algebras will be called a knit product of groups — but it is well known to algebraists under the name Zappa-Szép product, see [3] and the references therein. I present it here with different notation in order to describe afterwards again the behavior of homomorphisms with respect to this product. This gives a kind of generalization of the method of induced representations.

1. Knit products of graded Lie algebras

1.1. Definition. Let $A$ and $B$ be graded Lie algebras, whose grading is in $\mathbb{Z}$ or $\mathbb{Z}_2$, but only one of them. A derivatively knitted pair of representations $(\alpha, \beta)$ for $(A, B)$ are graded Lie algebra homomorphisms $\alpha : A \to \text{End}(B)$ and $\beta : B \to \text{End}(A)$ such that:

$$\alpha(a)[b_1, b_2] = [\alpha(a)b_1, b_2] + (-1)^{|a||b_1|}[b_1, \alpha(a)b_2] - \left((-1)^{|a||b_1|} \alpha(\beta(b_1)a)b_2 - (-1)^{|a|+|b_1|}|b_2| \alpha(b_2)a) b_1 \right)$$
Here \(|a|\) is the degree of \(a\). For (non-graded) Lie algebras just assume that all degrees are zero.

1.2. Theorem. Let \((\alpha, \beta)\) be a derivatively knitted pair of representations for graded Lie algebras \(A = \bigoplus A_k\) and \(B = \bigoplus B_k\). Then \(A \oplus B := \bigoplus_{k,l} (A_k \oplus B_l)\) becomes a graded Lie algebra \(A \oplus (\alpha, \beta) B\) with the following bracket:

\[
[(a_1, b_1), (a_2, b_2)] := \left( [a_1, a_2] + \beta(b_1)a_2 - (-1)^{|b_1||a_1|}[a_2, \beta(b_1)]a_1, 
[b_1, b_2] + \alpha(a_1)b_2 - (-1)^{|a_2||b_1|}[b_2, \alpha(a_1)]b_1 \right)
\]

The grading is \((A \oplus B)_k := A_k \oplus B_k\).

Proof: Obviously this bracket is graded anticommutative. The graded Jacobi identity is checked by computation.

We call \(A \oplus (\alpha, \beta) B\) the knit product of \(A\) and \(B\). If \(\beta = 0\) then \(\alpha\) has values in the space of (graded) derivations of \(A\) and \(A \oplus 0\) is an ideal in \(A \oplus (\alpha, 0) B\) and we get a semidirect product of graded Lie algebras. Note also that \([[a, 0], (0, b)] = \((-1)^{|b||a|}(\alpha(a)b)\). This is the key to the following theorem.

1.3. Theorem. Let \(A\) and \(B\) be graded Lie subalgebras of a graded Lie algebra \(C\) such that \(A + B = C\) and \(A \cap B = 0\). Then \(C\) as graded Lie algebra is isomorphic to a knit product of \(A\) and \(B\).

Proof: For \(a \in A\) and \(b \in B\) we write

\[
[a, b] := \alpha(a)b - (-1)^{|a||b|}\beta(b)a
\]

for the decomposition of \([a, b]\) into components in \(C = B + A\). Then \(\beta : B \to \text{End}(A)\) and \(\alpha : A \to \text{End}(B)\) are linear. Now decompose both sides of the graded Jacobi identity

\[
[a, [b_1, b_2]] = [[a, b_1], b_2] + (-1)^{|a||b_1|}[b_1, [a, b_2]]
\]

and compare the \(A\)- and \(B\)-components respectively. This gives equation 1.1 for \(\alpha\) and that \(\beta\) is a graded Lie algebra homomorphism. The rest follows by interchanging \(A\) and \(B\). Now we decompose \([a_1 + b_1, a_2 + b_2]\) and see that \(C = A \oplus (\alpha, \beta) B\).

1.4. Now let \(\Phi : A \oplus (\alpha, \beta) B \to A' \oplus (\alpha', \beta') B'\) be a linear mapping between knit products. Then \(\Phi\) can be decomposed into \(\Phi(a, b) := (f(a) + \psi(b), g(b) + \varphi(a))\) for linear mappings \(\varphi : A \to B', \psi : B \to A', f : A \to A',\) and \(g : B \to B'\).
Theorem. In this situation $\Phi$ is a graded Lie algebra homomorphism if and only if the following conditions hold:

\[
\varphi([a_1, a_2]) = [\varphi(a_1), \varphi(a_2)] + \alpha'(f(a_1))\varphi(a_2) - (-1)^{|a_1||a_2|}\alpha'(f(a_2))\varphi(a_1)
\]
\[
\psi([b_1, b_2]) = [\psi(b_1), \psi(b_2)] + \beta'(g(b_1))\psi(b_2) - (-1)^{|b_1||b_2|}\beta'(g(b_2))\psi(b_1)
\]
\[
[f(a), g(b)] = f(\beta(b)a) - \beta'(g(b))f(a) - (-1)^{|a||b|}(\beta(\varphi(a))\varphi(b))
\]
\[
[g(a), \varphi(a)] = \varphi(\beta(b)a) - \alpha'(\psi(b))\varphi(a) - (-1)^{|a||b|}((\varphi(a)b) - \alpha'(f(a))g(b))
\]
\[
f([a_1, a_2]) = [f(a_1), f(a_2)] + \beta'(\varphi(a_1))f(a_2) - (-1)^{|a_1||a_2|}\beta'(\varphi(a_2))f(a_1)
\]
\[
g([b_1, b_2]) = [g(b_1), g(b_2)] + \alpha'(\psi(b_1))g(b_2) - (-1)^{|b_1||b_2|}\alpha'(\psi(b_2))g(b_1)
\]

If $f$ and $g$ are graded Lie algebra homomorphisms the last pair of equations obviously simplifies.

Proof: A long but straightforward computation. 

This theorem can be used to build representations of $C$ out of representations of $A$ and $B$.

2. Knit products of groups

2.1. Definition. Let $A$ and $B$ be groups. An automorphically knitted pair of actions $(\alpha, \beta)$ for $(A, B)$ are mappings $\alpha : B \times A \rightarrow A$ and $\beta : B \times A \rightarrow B$ such that:

1. $\tilde{\alpha} : B \rightarrow \{\text{bijections of } A\}$ is a group homomorphism, so $\alpha_b \circ \alpha_{b_2} = \alpha_{b_1 b_2}$ and $\alpha_e = Id_A$, where $\alpha_b(a) := \alpha(b, a)$.
2. $\tilde{\beta} : A \rightarrow \{\text{bijections of } B\}$ is a group anti homomorphism, i.e., $\beta^{a_1} \circ \beta^{a_2} = \beta^{a_2 a_1}$ and $\beta^e = Id_B$, where $\beta^a(b) = \beta(b, a)$.
3. $\alpha_b(a_1, a_2) = \alpha_b(a_1) \cdot \alpha_{\beta_e(b)}(a_2)$.
4. $\beta^a(b_1, b_2) = \beta^{a_2}(a)(b_1) \cdot \beta^a(b_2)$.

2.2. Theorem. Let $(\alpha, \beta)$ be an automorphically knitted pair of actions for $(A, B)$. Then $A \times B$ is a group $A \times (\alpha, \beta) B$ with the following operations:

\[
(a_1, b_1) \cdot (a_2, b_2) := (a_1 \cdot \alpha_{b_1}(a_2), \beta^{a_2}(b_1), b_2)
\]
\[
(a, b)^{-1} := (\alpha_{b^{-1}}(a^{-1}), \beta^{a^{-1}}(b^{-1})).
\]

Unit is $(e, e)$. $A \times \{e\}$ and $\{e\} \times B$ are subgroups of $A \times (\alpha, \beta) B$ which are
isomorphic to $A$ and $B$, respectively. If $\alpha = \text{Id}_A$ then $\{e\} \times B$ is a normal subgroup of $A \times (\alpha, \beta) B$ and we have a semidirect product; similarly if $\beta = \text{Id}_B$.

If $A$ and $B$ are topological groups or Lie groups and $\alpha$, $\beta$ are continuous or smooth, then $A \times (\alpha, \beta) B$ is also a topological group or Lie group, respectively.

The proof is routine.

We will call $A \times (\alpha, \beta) B$ the \textit{knit product} of $A$ and $B$ in analogy with section 1. In algebra, with different notation, this product is well known under the name \textit{Zappa-Szép product}. I owe this remark to G. Kowol.

2.3. \textbf{Theorem}. Let $G$ be a group, let $A$ and $B$ be subgroups such that $G = A.B$ and $A \cap B = \{e\}$. Then $G$ is isomorphic to a knit product of $A$ and $B$.

\textbf{Proof}: Let $b.a = \alpha(b,a)\beta(b,a)$ be the unique decomposition of $b.a$ in $G = A.B$. Then

$$a_1 b_1 a_2 b_2 = a_1 \alpha(b_1, a_2) \beta(b_1, a_2) b_2 = (a_1 \alpha_b(a_2)).(\beta^a_2(b_1)b_2).$$

So it remains to show that $(\alpha, \beta)$ satisfies the conditions of 2.1. Obviously we have $\alpha(e, a) = a$, $\beta(e, a) = e$, $\alpha(b, e) = e$, $\beta(b, e) = b$. Comparing coefficients in the law of associativity of $G$ gives two equations. Setting suitable elements in these equations to $e$ gives all conditions of 2.1. \hfill \blacksquare

2.4. Let $\Phi = (\Phi_1, \Phi_2) : A \times (\alpha, \beta) B \rightarrow A' \times (\alpha', \beta') B'$ be a mapping between knit products of groups. We put

(1) \hspace{1cm} f(a) := \Phi_1(a, e), \quad g(b) := \Phi_2(e, b)

(2) \hspace{1cm} \varphi(b) := \Phi_1(e, b), \quad \psi(a) := \Phi_2(a, e)

Then we have $f : A \rightarrow A'$, $g : B \rightarrow B'$, $\varphi : B \rightarrow A'$, $\psi : A \rightarrow B'$. $\Phi$ is a group homomorphism if and only if

(3) \hspace{1cm} \begin{align*}
\Phi_1(a_1 \alpha_b(a_2), \beta^a_2(b_1)b_2) &= \Phi_1(a_1, b_1).\alpha'_{\Phi_2(a_1, b_1)}(\Phi_1(a_2, b_2)) \\
\Phi_2(a_1 \alpha_b(a_2), \beta^a_2(b_1)b_2) &= \beta'^{\Phi_1(a_2, b_2)}(\Phi_2(a_1, b_1))\Phi_2(a_2, b_2).
\end{align*}

Now we set in (3) suitable elements to $e$, use (1) and (2) and get in turn

(4) \hspace{1cm} \begin{align*}
\Phi_1(a_1, b_2) &= f(a_1).\alpha'_{\psi(a_1)}(\varphi(b_2)) \\
\Phi_2(a_1, b_2) &= \beta'^{\varphi(b_2)}(\psi(a_1))g(b_2)
\end{align*}

(5) \hspace{1cm} \begin{align*}
\varphi(b_1 b_2) &= \varphi(b_1).\alpha'_{\psi(a_2)}(\varphi(b_2)) \\
\psi(a_1 a_2) &= \beta'^{f(a_2)}(\psi(a_1)).\psi(a_2)
\end{align*}

(6) \hspace{1cm} \begin{align*}
\Phi_1(\alpha_b(a_2), \beta^a_2(b_1)) &= \varphi(b_1).\alpha'_{\varphi(b_1)}(f(a_2)) \\
\Phi_2(\alpha_b(a_2), \beta^a_2(b_1)) &= \beta'^{f(a_2)}(g(b_1)).\psi(a_2)
\end{align*}
Knit Products of Graded Lie Algebras and Groups

\[ \begin{align*}
(f) & \quad f(a_1 a_2) = f(a_1).\alpha'_{\varphi(a_1)}(f(a_2)) \\
(g) & \quad g(b_1 b_2) = \beta'_{\varphi(b_2)}(g(b_1)).g(b_2)
\end{align*} \]

If \( f \) and \( g \) are homomorphisms of groups then (g) implies:

\[ \begin{align*}
(f') & \quad f(a_2) = \alpha'_{\varphi(a_1)}(f(a_2)) \\
(g') & \quad g(b_1) = \beta'_{\varphi(b_2)}(g(b_1))
\end{align*} \]

Now we decompose the left hand sides of (4) with the help of (e) and get:

\[ \begin{align*}
(h) & \quad f(\alpha_b(a_2)).\alpha'_{\varphi(\alpha_a(a_2))}(\varphi(\beta^{a_2}(b_1))) = \varphi(b_1).\alpha'_{g(b_1)}(f(a_2)) \\
& \quad \beta'_{\varphi(\beta^{a_2}(b_1))}(\psi(\alpha_b(a_2))).g(\beta^{a_2}(b_1)) = \beta'_{f(a_2)}(g(b_1)).\psi(a_2)
\end{align*} \]

2.5. Theorem. Let \( A \times_{(\alpha, \beta)} B \) and \( A' \times_{(\alpha', \beta')} B' \) be knit products of groups and let \( f : A \to A' \), \( g : B \to B' \), \( \varphi : B \to A' \), \( \psi : A \to B' \) be mappings such that (f), (g), and (h) from 2.4 hold. We define \( \Phi = (\Phi_1, \Phi_2) : A \times_{(\alpha, \beta)} B \to A' \times_{(\alpha', \beta')} B' \) by 2.4(e), then \( \Phi \) is a homomorphism of groups. If \( f \) and \( g \) are homomorphisms, then we may use (g') instead of (g).

Proof: It suffices to check (3) of 2.5. This is a difficult computation using 2.4 (a)-(h).

For topological groups and Lie groups all the expected assertions about continuity and smoothness are true.

This theorem may be used to construct representations of \( A \times_{(\alpha, \beta)} B \) out of representations of \( A \) and \( B \) — a sort of generalized induced representation procedure.

Starting from the equations 2.1 for a knit product of Lie groups and deriving the equations of 1.1 for their Lie algebras is a very interesting exercise in calculus on Lie groups.

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