Cubic magnets with Dzyaloshinskii-Moriya interaction at low T

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Ground state and spin-wave spectrum of cubic magnets with the Dzyaloshinskii-Moriya interaction (DMI) such as MnSi and FeGe are studied theoretically. Following interactions are taken into account: conventional isotropic exchange, the DMI, anisotropic exchange, magnetic dipole interaction and Zeeman energy. In the classical approximation these interactions determine the helix wave-vector \( k \) and critical field \( H_c \) for the transition to the "ferromagnetic" spin configuration. This field depends on the sample form due to the demagnetization. The linear spin-wave theory is developed. The spin-wave spectrum depends strongly on the magnetic field. At \( H > H_c \) we have quadratic spectrum with the gap linearly increasing with the field. Below \( H_c \) the spectrum is gapless and strongly anisotropic. It is a result of incommensurate magnetic structure when the DMI breaks the total spin conservation law and umklapp processes appear connecting the spin-wave excitations with momenta \( q \) and \( q \pm k \) with different energies. For \( q \parallel k \) the spectrum is linear. For other \( q \) directions it have very complex form determined by solution of infinite set of linear equations connecting the states with \( q \) and \( q \pm n k \) where \( n = 1, 2, \ldots \). Restriction to \( n = 1 \) gives six equations which general solution remains complex. For \( q \perp k \) there are two modes: one has the gap equal to \( AK \sqrt{2} \) where \( A \) is the spin-wave stiffness at \( q \gg k \). The second is gapless and proportional to \( q^2 \). At \( q \gg k \) all branches merge and the anisotropy of the spectrum disappears. These results change insignificantly in \( n = 2 \) approximation.

The classical energy depends on the field component along the helix axis \( k \) only. However it is known experimentally that rather weak perpendicular field rotates the helix and its axis is settled along the field. This quantum phenomenon is a consequence of the umklapp too. The spin-wave spectrum is unstable at infinitesimal perpendicular field. If the gap \( \Delta \) is the introduced the spectrum becomes stable if \( g_{\mu B} H_{\perp} < \Delta \). In this field there are the magnetization along \( H_{\perp} \) and deformation of the helix. The gap appears due to cubic anisotropy and the spin-wave interaction considered in the Hartree-Fock approximation.

Peculiar properties of the ESR and neutron scattering in the helical magnets are considered and possibilities of corresponding experimental studies are discussed.

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I. INTRODUCTION

Itinerant cubic magnets MnSi, FeGe, FeSiCo, etc have attracted a lot of attention last years due to their specific electronic and magnetic properties. The former are characterized by closeness to quantum phase transition, which is achieved at high pressure (see and references therein). Corresponding theory was developed in. The last is related to the \( P2_13 \) symmetry which allows the Dzyaloshinskii-Moriya interaction (DMI) responsible for magnetic helix structure. In cubic crystals the DMI fixes the the sense of the helix (right or left-handed spiral) but can’t determine its direction. It is stabilized by very weak anisotropic exchange interaction (AEI) only and . The helix structure is very sensitive to external magnetic field. Two field-induced transitions were observed and and references therein. In low field it is transition to the state with the helix axis along the field. Then, there is the second transition to the "ferromagnetic" state. These transitions were described by the mean-field theory which contains a lot of phenomenological parameters. It should be noted also that critical properties of MnSi near transition temperature \( T_c \) are very unusual too. Corresponding experimental studies and its theoretical explanation may be found in Refs. and.

In this paper we present theoretical description of the low-temperature properties of the cubic magnets with the DMI. We microscopically evaluate the ground-state energy and spin-wave spectrum. In particular we demonstrate that the critical fields mentioned above are related to the parameters of the spin-wave spectrum in agreement with existing experimental data.

The outline of this paper is follows. In Sec. II the theoretical model is formulated. Along with the exchange interaction , the DMI and AEI considered we include also the magnetic dipole interaction and the Zeeman energy. Following we introduce all three spin components in each lattice point of the helix and demonstrate that in "ferromagnetic" state the perpendicular spin components responsible for the spin-waves remain rotating. Sec. III is devoted to consideration of the ground state energy. The critical field for the transition to the "ferromagnetic" state is determined with the demagnetization corrections. In Sec. IV the spin-wave Hamiltonian is considered. It consists of two parts: direct and umklapp. The first part is a conventional spin-wave Hamiltonian. Its diagonalization determines the spin-wave spectrum. The second one describes transitions between spin-waves with momenta \( q \) and \( q \pm k \) and
different energies. These transitions are a result of the incommensurate helical structure and low symmetry of the DMI. This umklapp interaction is considered in Sec.V. It is shown that the umklapps created by the field perpendicular to the helix wave-vector \( \mathbf{k} \) are a cause of its turn to the direction of the field. At zero field the umklapps bring to strong anisotropy of the spin-wave spectrum: excitations with momentum along and perpendicular to \( \mathbf{k} \) have different energies. Spin configuration in weak perpendicular field is discussed in Sec. VI. Origin of the spin-wave gap \( \Delta \) is studied in Sec. VII. There are two contributions to \( \Delta^2 \): the spin-wave interaction considered in the Hartree-Fock approximation and the cubic anisotropy. If the last contribution is negative and sufficiently strong instead of the helical magnetic order the chiral spin liquid state is realized. Possibilities of the ESR and neutron scattering studies of the dynamical phenomena investigated in the paper is discussed in Sec. VIII. The principal results are summarized and discussed in Sec. IX. Some details of calculations are presented in Appendixes A–D.

II. MODEL

There are following principal interactions in cubic magnets without center of inversion: i) conventional exchange interaction, ii) the DMI, iii) the AEI, iv) magnetic dipole interaction and Zeeman energy. We investigate spin configuration in magnetic field and the spin-wave spectrum using these interactions. Then we discuss role of the cubic anisotropy in formation of the spin-wave gap.

There are four magnetic ions in the unit sell of the magnets under study. We will consider here the total spin of the unit sell as a basic magnetic entity, completely neglecting internal movements of the cell spins and corresponding optical spin-wave branches. Interested in spin dynamics, we can use the magnetic density approximation or the effective spin-lattice model with total spin of the unit cell \( \mathbf{S}(\mathbf{R}) \) and usual commutation relations

\[
[S_\alpha(R), S_\beta(R)] = i\varepsilon_{\alpha\beta\gamma}S_\gamma(R)
\]  

The DMI destroys ferromagnetism and is responsible for the long periodical spin-density wave. For each spin we have right-handed local orthogonal coordinate frame with basic unit vectors \( \hat{\zeta}_R, \hat{\eta}_R \) and \( \hat{\xi}_R \) which for helical structure are given by

\[
\hat{\zeta}_R = \hat{\epsilon} \sin \alpha + (\hat{\alpha} \cos \mathbf{k} \cdot \mathbf{R} + \hat{\beta} \sin \mathbf{k} \cdot \mathbf{R}) \cos \alpha,
\]

\[
\hat{\eta}_R = -\hat{\alpha} \sin \mathbf{k} \cdot \mathbf{R} + \hat{\beta} \cos \mathbf{k} \cdot \mathbf{R},
\]

\[
\hat{\xi}_R = \hat{\epsilon} \cos \alpha - (\hat{\alpha} \cos \mathbf{k} \cdot \mathbf{R} + \hat{\beta} \sin \mathbf{k} \cdot \mathbf{R}) \sin \alpha.
\]

where \( \hat{\alpha} \times \hat{\beta} = \hat{\epsilon}, \hat{\beta} \times \hat{\epsilon} = \hat{\alpha} \) and \( \hat{\epsilon} \times \hat{\alpha} = \hat{\beta} \).

Corresponding projections of the spin operators have well known form

\[
S^\zeta_R = S - (a^+ a)_R,
\]

\[
S^\eta_R = \frac{1}{2 \sqrt{S}} \left[ a_R - a^+_R - \frac{(a^+ a^2)_R}{2S} \right],
\]

\[
S^\xi_R = \frac{1}{2 \sqrt{S}} \left[ a_R + a^+_R - \frac{(a^+ a^2)_R}{2S} \right],
\]

here \( S \) is a parameter connected to the cell magnetization by \( M = g \mu_B S/a^3 \) where \( \mu_B > 0 \) and \( g \approx 2 \). For \( MnSi \) magnetic moment per spin \( 0.4 \mu_B \) is strongly reduced in comparison with the paramagnetic state value \( 1.4 \mu_B \) and the unit-cell spin \( S = 0.2 \times 4 = 0.8 \). For the above-mentioned principal interactions in the \( \mathbf{R} \) space we have

\[
H = H_{\text{EX}} + H_{\text{DM}} + H_{\text{AE}} + H_Z H_D
\]

\[
H_{\text{EX}} = -\frac{1}{2} \sum J_{RR'} S_R \cdot S_{R'},
\]

\[
H_{\text{DM}} = \frac{1}{2} \sum D_{RR'} (\nabla - \nabla') S_R \times S_{R'},
\]

\[
H_{\text{AE}} = \frac{1}{2} \sum F_{RR'} [(\nabla \cdot S_R') (\nabla \cdot S_{R'}') + (\nabla \cdot S_R') (\nabla \cdot S_R)] + (\nabla \cdot S_R') (\nabla \cdot S_{R'}),
\]

\[
H_D = \frac{g \mu_B}{2} \sum \left( S_R \cdot S_R - (S_R' \cdot \hat{R})(S_{R'}' \cdot \hat{R}') \right) ||\mathbf{R} - \mathbf{R}'||^{-3},
\]

\[
H_Z = \mu_B \mathbf{H} \cdot \sum S_R.
\]
It is convenient to represent these expressions in \( \mathbf{q} \) space

\[
H_{EX} = \frac{1}{2} \sum J_{\mathbf{q}} \mathbf{S}_{\mathbf{q}} \cdot \mathbf{S}_{-\mathbf{q}},
\]

\[
H_{DM} = \sum i D_{\mathbf{q}} \mathbf{q} \cdot [\mathbf{S}_{\mathbf{q}} \times \mathbf{S}_{-\mathbf{q}}],
\]

\[
H_{AE} = \frac{1}{2} \sum_{\nu=x,y,z} F_{\mathbf{q} \nu} S_{\mathbf{q} \nu} S_{\mathbf{q} - \mathbf{q} \nu},
\]

\[
H_D = \frac{\omega_0}{2} \sum (\mathbf{S}_{\mathbf{q}} \cdot \hat{q}) (\mathbf{S}_{-\mathbf{q}} \cdot \hat{q}).
\]

\[
H_Z = N^{1/2} \mathbf{H} \cdot \mathbf{S}_0.
\]

where \( N \) is total number of the cells, \( \mathbf{S}_0 = N^{-1/2} \sum \mathbf{S}_\mathbf{R} \exp(i\mathbf{q} \cdot \mathbf{R}) \), \( \hat{q} = \mathbf{q} / q, \omega_0 = 4\pi (g\mu_B)^2 / v_0 \) is the characteristic energy of the dipole interaction and \( v_0 \) is the unit cell volume and we omitted the isotropic part of the dipole interaction which has the same symmetry as the exchange one. The DMI and AEI are of the first and second order of the spin-orbit interaction respectively. So we have \( J \gg D \gg F \). Value of the dipole energy \( \omega_0 \) will be given in the last Sec..

III. CLASSICAL ENERGY

Replacing \( \mathbf{S}_\mathbf{R} \) by \( \mathbf{S}_0 \) we get the classical ground-state energy per unit cell in the following form

\[
E_{cl} = -\frac{S^2}{2} ( J_0 \sin^2 \alpha + J_0 \cos^2 \alpha ) - S^2 D_0 (k \cdot [\hat{a} \times \hat{b}]) \cos^2 \alpha + \left( \frac{S^2 F}{4} \right) k^2 (\hat{a}_x^2 + \hat{b}_y^2) + k^2 (\hat{a}_y^2 + \hat{b}_y^2) + k^2 (\hat{a}_x^2 + \hat{b}_y^2) \cos^2 \alpha + Sh || \sin \alpha + \frac{S^2}{2} \omega_0 N_{cc} \cos^2 \alpha, \tag{6}
\]

here \( h || = g\mu_B H || \), \( H || \) is the field component along the \( c \)-axis and \( N_{cc} \) is the corresponding component of the demagnetization tensor. It is important to note that the classical energy depends on the field along \( c \) axis only. Meanwhile experiment shows that the system is very sensitive to weak perpendicular field and \( \parallel \mathbf{H} \parallel \). We will explain below the nature of this quantum phenomenon.

We are interested by small \( k \) only \((ka \ll 1)\) when \( J_0 = J_0 - \frac{4k^2}{A} \) and \( D_0 = D_0 \). As we will see below \( A \) is the spin-wave stiffness constant. At \( H || = 0 \) this ground state energy was studied in [4]. From Eq.(6) for components of the vector \( \mathbf{k} \) we get

\[
Ak_{\nu} + \frac{SF}{2} k_{\nu} (\hat{a}_x^2 + \hat{b}_y^2) = SD_0 [\hat{a} \times \hat{b}]_{\nu}. \tag{7}
\]

from this expression we obtain

\[
Ak^2 + \frac{SF}{2} I(\mathbf{k}) = SD_0 (k \cdot [\hat{a} \times \hat{b}]), \tag{8}
\]

where \( I = \sum k^2 (\hat{a}_x^2 + \hat{b}_y^2) \) is a cubic invariant. Inserting \( k_{\nu} \) from Eq.(7) into Eq.(6) we get

\[
E_{cl} = \left\{ -\frac{(S^2 D_0^2}{2A} + \frac{S^2 F^2 I^2}{8} + \frac{S^2 F I}{4} \right\} \cos^2 \alpha + Sh || \sin \alpha + \frac{S^2 N_{cc}}{2} \omega_0 \sin^2 \alpha, \tag{9}
\]

where the \( F^2 \) term may be omitted. There are two possibilities. For negative \( F \) the classical energy (6) is minimal for \( \mathbf{k} \) along one of the cubic diagonals and \( I = 2k^2 / 3 \). If \( F > 0 \) the minimum is for \( \mathbf{k} \) along the cubic edge and \( I = 0 \). In both cases the spins rotate in the plane perpendicular to the vector \( \mathbf{k} \) and

\[
\mathbf{k} = \frac{SD_0 [\hat{a} \times \hat{b}]}{A + SF I / 2}. \tag{10}
\]

For \( D_0 > 0 \) and \( D_0 < 0 \) we have the right and left-handed helix respectively. The classical energy depends on the field projection onto the vector \( \mathbf{k} \) (\( c \)-axis) only and from Eq.(6) for we obtain

\[
\sin \alpha = \begin{cases} -H || / H_c, & H || < H_c, \\ -1, & H || > H_c, \end{cases} \tag{11}
\]
where the critical field is given by

$$g\mu_B H_c = h_c = Ak^2 + \frac{SF}{2} I + S\omega_0 N_{cc},$$

(12)

here the last term is a result of the demagnetization and the intrinsic critical field is determined by $H_c^{int} = H_c - 4\pi N_{cc} M$, where magnetization in the high-field ”ferromagnetic” state $M$ is determined as $4\pi \mu_B M = \omega_0$ and $g\mu H_c^{int} = h_c^{int} = Ak^2 + SF I/2 \approx Ak^2$. For $H > H_c$ we have ferromagnetic spin configuration, but according to Eqs. (2) and (3) rotation persists in the perpendicular, spin-wave components of the spin density.

IV. SPIN-WAVES

We are interested in the long-wave spin-waves with $q \lesssim k$. At $q \gg k$ one can neglect all interactions except the exchange one and the spin-wave spectrum must be the same as in ferromagnets: $\epsilon_q = Aq^2$. In the low-$q$ region the hierarchy of the interactions becomes very important. The ferromagnetic exchange $J$ is the strongest one. The DMI and AEI are result of the weak spin-orbit coupling $\lambda$. The former and the last are of order $\lambda J$ and $\lambda^2 J$ respectively. According to Eq.(10) $k \sim \lambda$.

We are dealing with incommensurate spin structure where umklapp processes are important which mix excitations with momenta $q$ and $q \pm k$ and different energies. It is useful to represent the unit vectors given by Eq.(2) as

$$\hat{c} = \hat{c}\sin \alpha + (A e^{i k_R} + A^* e^{-i k_R}) \cos \alpha,$$
$$\hat{q}_R = i A e^{ik_R} - i A^* e^{-ik_R},$$
$$\hat{q}_R = \hat{c}\cos \alpha - (A e^{i k_R} + A^* e^{-i k_R}) \sin \alpha$$

(13)

where $A = (\hat{a} - i \hat{b})/2$, $A \cdot A = A^* \cdot A^* = 0$, $A \cdot A^* = 1/2$, $A \times \hat{c} = -i A$, $A^* \times \hat{c} = i A^*$, and $A \times A^* = i \hat{c}/2$. Using this representation we obtain $S_q = S_{q-\alpha}^c \hat{c} + S_{q}^A A + S_{q}^{A^*} A^*$ where

$$S_{q-\alpha}^c = S_{q}^c \sin \alpha + S_{q}^\xi \cos \alpha,$$
$$S_{q-\alpha}^A = \sum_{q} S_{q-k}^\xi \cos \alpha - S_{q-k}^\xi \sin \alpha + i S_{q-k}^\eta,$$
$$S_{q+\alpha}^A = S_{q+k}^\xi \cos \alpha - S_{q+k}^\xi \sin \alpha - i S_{q+k}^\eta$$

(14)

In the DMI interference terms appear proportional to $S_{q}^c S_{q-k}^A$ and $S_{q}^c S_{q+k}^{A^*}$. They are responsible for umklapp processes with $q \rightarrow q \pm k$. At the same time the processes with $q \pm 2k$ appear in the $S^A S^A$ and $S^A S^{A^*}$ terms. Due to algebra of the vectors $\hat{c}, A$ and $A^*$ these double umklappes are in $H_{AE}$ and $H_0$ only. Besides as we see below the perpendicular field initiate processes with $q \pm k$ too.

Here we evaluate the direct contribution to the spin-wave energy. The umklappes will be considered in the next section.

The exchange energy does not contain the umklappes and has the form

$$H_{EX} = -\frac{1}{2} \sum_i (J_q \sin^2 \alpha + J_{q,k} \cos^2 \alpha) S_{q-\alpha}^c S_{q}^c + \sum_i (J_q \cos^2 \alpha + J_{q,k} \sin^2 \alpha) S_{q-\alpha}^\xi S_{q-\alpha}^\xi + (J_q \cos^2 \alpha + J_{q,k} \sin^2 \alpha) S_{q-\alpha}^\eta S_{q-\alpha}^\eta + i (S_{q-\alpha}^\xi S_{q-\alpha}^\xi - S_{q-\alpha}^\eta S_{q-\alpha}^\eta) N_{q,k} \cos \alpha + (J_q - J_{q,k})(S_{q-\alpha}^\xi S_{q-\alpha}^\eta - S_{q-\alpha}^\eta S_{q-\alpha}^\xi) \sin \alpha \cos \alpha - i (S_{q-\alpha}^\xi S_{q-\alpha}^\eta - S_{q-\alpha}^\eta S_{q-\alpha}^\xi) N_{q,k} \sin \alpha$$

(15)

where $J_{q,k} = (J_{q+k} + J_{q-k})/2 \approx J_0 - A(q^2 + k^2) / S$ and $N_{q,k} = (J_{q+k} - J_{q-k})/2 \approx -2 Aqk / S$. Direct part of $H_{DM}$ is given by

$$H_{DM}^D = -D_0 (\hat{a} \times \hat{b}) \sum_i [S_{q-\alpha}^c S_{q-\alpha}^c \cos^2 \alpha + S_{q-\alpha}^\xi S_{q-\alpha}^\xi + S_{q-\alpha}^\eta S_{q-\alpha}^\eta \sin^2 \alpha - (S_{q-\alpha}^\xi S_{q-\alpha}^\eta + S_{q-\alpha}^\eta S_{q-\alpha}^\xi) \cos \alpha]$$
$$+ i D_0 \sum_i (\hat{a} \times \hat{b}) [(S_{q-\alpha}^\xi S_{q-\alpha}^\eta - S_{q-\alpha}^\eta S_{q-\alpha}^\xi) \cos \alpha + (S_{q-\alpha}^\xi S_{q-\alpha}^\eta - S_{q-\alpha}^\eta S_{q-\alpha}^\xi) \sin \alpha],$$

(16)

where we neglected $q$-dependence of the DM interaction. For the anisotropic exchange we have

$$H_{AE}^A = \frac{F}{2} \sum_{q,v=x,y,z} \{(\epsilon_v q_v)^2 (S_{q}^c \sin \alpha + S_{q}^\xi \cos \alpha)(S_{q-\alpha}^c \sin \alpha + S_{q-\alpha}^\xi \cos \alpha)$$
$$+ \frac{(q_v^2 + k_v^2)(\hat{a}_v^2 + \hat{b}_v^2)}{2}$$
$$-[S_{q}^\xi \cos \alpha - S_{q-\alpha}^\xi \sin \alpha](S_{q-\alpha}^c \cos \alpha - S_{q-\alpha}^\xi \sin \alpha) + S_{q-\alpha}^\eta S_{q-\alpha}^\eta] - 2i q_v k_v (\hat{a}_v^2 + \hat{b}_v^2)(S_{q}^c \cos \alpha - S_{q}^\xi \sin \alpha) S_{q-\alpha}^\eta,$$

(17)
In the case of the dipole interaction we obtain

$$H_D^{(d)} = \frac{\omega_0}{2} \sum \left\{ (\hat{q} \cdot \hat{c})^2 (S_{q}^c \sin \alpha + S_{q}^\xi \cos \alpha) (S_{-q}^c \sin \alpha + S_{-q}^\xi \cos \alpha) \\
+ \frac{q_q^2}{2|q|} [(S_{q}^\xi \cos \alpha - S_{q}^c \sin \alpha)(S_{-q}^\xi \cos \alpha - S_{-q}^c \sin \alpha) + S_{q}^y S_{-q}^y] \right\},$$

where \( q_q^2 = q_a^2 + q_b^2 \) and \(|q, k|^2 = (|q + k|^2 + |q - k|^2)/2 \). In this expression we have taken into account that the vector \( k \) is perpendicular to the \( ab \) plane.

Replacing in Eqs.(15)-(18) \( S_q^c \) by \( N^{1/2} S_0 q \) we obtain a sum of terms proportional to \( S_0^c \) which cancels corresponding contribution from the Zeeman energy. In the sum \( H_{EX} + H_D^{(d)} + H_{AE}^d \) the terms proportional to \( \sin \alpha \) and \( \cos \alpha \) cancel too due to Eq.(8), if one neglects small terms of order of \( F k \cdot q \).

We consider now linear spin-wave theory. From Eqs. (3), and (15)-(18) follows

$$H_{SW}^{(d)} = \sum [E_q^a q_a^+ a_q + \frac{B_q}{2} (a_{q-a} + a_{q-a}^+)],$$

where

$$E_q = A q^2 + \left( \frac{A k^2}{2} + \frac{S F I}{2} \right) \cos^2 \alpha + \frac{S \omega_0}{2} (q^2 - \frac{q^2}{|q, k|^2}) \cos^2 \alpha,$$

$$B_q = \left( \frac{A k^2}{2} + \frac{S F I}{2} \right) \cos^2 \alpha + \frac{S \omega_0}{2} (q^2 - \frac{q^2}{|q, k|^2}) \cos^2 \alpha,$$

The spin-wave energy is determined as \( \epsilon_q = (E_q^2 - B_q^2)^{1/2} \). For \( H_{||} > H_c \) we have

$$\epsilon_q = A q^2 + h_{||} - h_c + \frac{S \omega_0 q_2^2}{|q, k|^2},$$

At \( H_{||} < H_c \) the energy has the form

$$\epsilon_q = \left\{ \left( A q^2 + \frac{S \omega_0 q_2^2}{|q, k|^2} \right) \left[ A q^2 + (A k^2 + S \omega_0 q^2) \cos^2 \alpha + \frac{S \omega_0 q_2^2}{|q, k|^2} \sin^2 \alpha \right] \right\}^{1/2}$$

and we have the gapless excitation with linear dispersion at \( q \ll k \). In these equations the factor \( q_2^2/|q, k|^2 \) is singular if \( q \rightarrow \pm k \) and should be replaced by \( q_2^2/2 \) or \( N_{||}/2 = (N_{aa} + N_{bb})/2 \) if \( q_{||} = 0 \). This replacement becomes evident if we put \( q = k + q_{||} \).

V. UMKLAPP INTERACTION

There are following umklapp contributions to the Hamiltonian: i) Interaction of the perpendicular magnetic field with \( S^c \) spin component. ii) Interference of the longitudinal (\( c \)) and transverse (\( A \) and \( A^* \)) spin components in the DM and dipole interactions. All these contributions mix excitations with \( q \) and \( q \pm k \). The dipole interaction contains also the (\( AA \)) and (\( A^* A^* \)) terms which mix waves with \( q \) and \( q \pm 2k \). We will show below that this contribution may be neglected. The umklapps in the anisotropic exchange is neglected too.

The umklapp contribution to the Hamiltonian consists from terms odd and even in operators \( a \) and \( a^+ \). It is convenient to write out general expression for it and then extract the bilinear part which contribute to the spin-wave spectrum and linear one considered in the next section. So we have

$$H_{U}^{(U)} = - \hbar \cdot A (S_{q}^c k \sin \alpha - i S_{-q}^c k \sin \alpha) + \hbar \cdot A^* (S_{q}^c \sin \alpha + i S_{-q}^c \sin \alpha) + \omega_0 (q \cdot \hat{q})(q \cdot \hat{A}) S_q^y S_{-q}^y \sum_{q} \{ [-2 D_0(q \cdot \hat{A}) + \omega_0 (\hat{c} \cdot \hat{q})(\hat{q} \cdot \hat{A})] S_q^y S_{-q}^y \} + [2 D_0(q \cdot \hat{A}^*) + \omega_0 (\hat{c} \cdot \hat{q})(\hat{q} \cdot \hat{A}^*)] S_q^y S_{-q}^y \sum_{q} \{ [-2 D_0(q \cdot \hat{A}^*) + \omega_0 (\hat{c} \cdot \hat{q})(\hat{q} \cdot \hat{A}^*)] S_q^y S_{-q}^y \}.$$
where \( H_U = H_D^{(U)} + H_U^{(U)} \), and \( S^c, S^A \) and \( S^{A^r} \) are given by Eq. (14). In this expression we included also terms linear in \( S_k^c \). They are not contribute to the spin-wave energy and will be considered in next section.

The term \( S^c \) in the expression for \( S^c_q \) leads to demagnetization of the perpendicular field if \( H_\parallel \neq 0 \) which may appear due to low symmetry of the sample. In this case \( -\mathbf{h} \cdot \mathbf{A} \) is replaced by

\[
P = -(\mathbf{h} \cdot \mathbf{A} + S \omega_0 N_{cA} \sin \alpha),
\]

where \( N_{cA} = (N_{cA} - i N_{dA})/2 \) is corresponding off-diagonal component of the tensor \( N_{\alpha\beta} \). It is an unusual feature which has to be clarified. The tensor \( N_{\alpha\beta} \) is diagonal if the coordinates are along the principal axes or the ellipsoidal sample. If \( \mathbf{k} \) is not along one of these axes the off diagonal components of \( N_{\alpha\beta} \) appear. The \( \delta_{q\pm k} \) terms in \( S_{qA}^{A^r} \) do not contribute due to the condition \( \mathbf{q} \cdot \mathbf{A} = 0 \). As a result for the bilinear spin-wave contribution we obtain

\[
H_{SW}^{(U)} = \sum_{\mathbf{q}} \frac{1}{2} \left( \left| P \cos \alpha + R_{-\mathbf{q} - \mathbf{k}}(1 - \sin \alpha) \right| \right.
\]

\[
\left. - R_{\mathbf{q}}(1 + \sin \alpha)a_{q+k}^+a_{\mathbf{q}} + \left| P^\ast \cos \alpha \right| \right)
\]

\[
+ R_{-\mathbf{q}}(1 - \sin \alpha) - R_{-\mathbf{q} - \mathbf{k}}(1 + \sin \alpha)\right) a_{q-k}^+a_{\mathbf{q}}
\]

\[
+ R_{\mathbf{q}}[a_{\mathbf{q}}a_{-\mathbf{q} - \mathbf{k}}(1 - \sin \alpha) - a_{\mathbf{q}}^+a_{-\mathbf{q} + \mathbf{k}}(1 + \sin \alpha)]
\]

\[
+ R_{-\mathbf{q}}[a_{-\mathbf{q} - \mathbf{k}}^+a_{-\mathbf{q}}(1 - \sin \alpha) - a_{\mathbf{q}}a_{-\mathbf{q} + \mathbf{k}}(1 + \sin \alpha)]\right)
\]

where

\[
R_{\mathbf{q}} = \frac{S \cos \alpha}{2} \left[ -2D_0(\mathbf{q} \cdot \mathbf{A}) + \omega_0(\hat{\mathbf{q}} \cdot \hat{\mathbf{q}})(\hat{\mathbf{q}} \cdot \mathbf{A}) \right].
\]

Here again \( R_{\mathbf{q}} \) contains off diagonal component of \( N_{\alpha\beta} \). From these equations we see that the umklapp interaction contributes to the spin-wave energy at \( H_\parallel < H_c \) only and do not affect excitations with \( \mathbf{q} \) along \( \mathbf{k} \). At the same time from Eqs. (15)-(18) for the direct interaction follows that at \( H_\parallel > H_c \) the zero-point vibrations disappear.

For consideration of the umklapp interaction we will use equations of motion

\[
\omega G_{A,B}(\omega) + G_{[H,A],B}(\omega) = \langle [A, B] \rangle,
\]

where \( H = H_{SW}^{(d)} + H_{SW}^{(U)} \) and the Green functions are determined by

\[
G_{A,B}(\omega) = -i \int_0^\infty e^{i\omega t} \langle [A(t), B(0)] \rangle = -\langle A, B \rangle > \omega
\]

where the expression in the right hand side coincides with the conventional determination of the generalized susceptibility.

We will use below the following notations

\[
G_{a_{\mathbf{q}}a_{\mathbf{q}}} = G_{\mathbf{q}}, \quad G_{a_{\mathbf{q} + \mathbf{k}}a_{\mathbf{q}}} = G_{\pm 1},
\]

\[
F_{a_{\mathbf{q}}a_{\mathbf{q}}} = F_{\mathbf{q}}, \quad F_{a_{-\mathbf{q} + \mathbf{k}}a_{\mathbf{q}}} = F_{\pm 1}.
\]

Neglecting the umklapp interaction we have

\[
G_{\mathbf{q}}(\omega) = \frac{E_{\mathbf{q}} + \omega}{\omega^2 - c_{\mathbf{q}}^2}; \quad F_{\mathbf{q}}(\omega) = -\frac{B_{\mathbf{q}}}{\omega^2 - c_{\mathbf{q}}^2},
\]

where \( E_{\mathbf{q}}, B_{\mathbf{q}} \) and \( c_{\mathbf{q}} \) are given by Eqs. (20)–(23). If \( H_\parallel > H_c \) we have \( F = 0 \)

Using Eq.(28) we obtain infinite set of equations which contain along with conventional Green functions \( G \) and \( F \) the functions \( G_{\pm 1} \) and \( F_{\pm 1} \). Below we will restrict mainly by \( n = 1 \) approximation. Its validity will be discussed later. As a result we obtain six linear equations for functions (30) (see Appendix A). In general form their solution is very complex. So we will consider two limiting cases: \( q \to 0 \) and \( q \gtrsim k \).

The \( q \to 0 \) case. Corresponding equations are analyzed in Appendix B. Neglecting the dipole interaction we obtain the final result for weak perpendicular field in the form

\[
G_{\mathbf{q}}(\omega) = \frac{(Ak^2/2)\cos^2 \alpha + \omega}{\omega^2 - c_{\mathbf{q}}^2 + (h_+^2/2)\cos^4 \alpha};
\]

\[
F_{\mathbf{q}}(\omega) = -\frac{(Ak^2/2)\cos^2 \alpha}{\omega^2 - c_{\mathbf{q}}^2 + (h_+^2/2)\cos^4 \alpha},
\]
where $h_2^2 = (g\mu_B)^2(H_2^2 + H_3^2)$. We neglected here the small terms proportional to $h_1^2$ in numerators as well as the contribution of the slightly split modes with the energies $\epsilon \simeq \omega_k$ (See Appendix B). The functions $G_\pm$ and $F_\pm$ are of order of $h_1$ and are evaluated in Appendix B. In these expressions the square of the spin-wave energy in the transverse field at $q = 0$ is given by Eq.(B5)

$$\epsilon_q^2(H_\perp) = \Delta^2 - (h_2^2/2)\cos^4 \alpha.$$  

(33)

For the gapless spin-waves $\epsilon_0^2(H_\perp)$ is negative for any $H_\perp$ and magnetic subsystem is unstable. It means that the helix axis $k$ must turn and stands along the field. For finite gap $\Delta$, the spin-wave spectrum remains stable up to $h_\perp \sim \Delta$ and then there is the first order transition to the parallel state. In the intermediate case, when $H_\parallel \neq 0$ there is complex behavior governed by the equation of state derived in the next Sec. Similar situation takes place in conventional antiferromagnets: If the field is along the sublattice magnetization the first order spin-flop transition occurs at $H = \Delta$ where $\Delta$ is the spin-wave gap. For the inclined field there is the rotation of the sublattices along with the first order transition. We consider origin of the gap below.

The rotation of the helix axis was observed in $FeGe_2$ and $MnSi$ and $\beta$. In both cases $H_\perp$ is much less than $H_\parallel$. For $MnSi$ the transition to parallel state is at $H_\perp \sim 0.1T$ and $H_\parallel \approx 0.6T$. It is the reason of the approximation with $n = 1$. Any additional $n$ produces small correction of order of $H_\perp^2/H_\parallel^2 \ll 1$.

The $q \gtrsim k$ case. As above we consider the first-order umklapps connecting $q$ and $q \pm k$. We consider $q \perp k$ case only where the umklapp interaction the dipole interaction is absent (see Eq.(27)). We put also $H = 0$. Then we discussed briefly the role of small $q_\parallel$. We demonstrate also that $n = 2$ approximation changes does not change qualitatively the $n = 1$ results.

If $q \perp k$ we have two interacting modes with $q$ and $q \pm k$ which energies according to Eq. (23) are given by

$$\epsilon_q = Aq_\perp(q_\perp^2 + k^2)^{1/2},$$

$$\epsilon_{q \pm k} = A[(q_\perp^2 + k^2)(q_\perp^2 + 2k^2)]^{1/2}.$$  

(34)

The excitation $\epsilon_q$ is the gapless and linear in $q$ at $q \ll k$. The gap of the double degenerated mode is equal to $Ak^2\sqrt{2}$. The umklapp interaction lifts this degeneracy (see Appendix C) and we have three different modes. One of them remains non-renormalized and coincides with the gapped mode in Eq.(34) and two other are given by

$$\epsilon_{\pm} = A[k^2 + 4q_\perp^2 + k^2(2 + 8q_\perp^2 + 17q_\perp^4/k^2 + 8q_\perp^6/k^4)]^{1/2},$$  

(35)

where the plus mode has the same gap as in Eq.(34) but other dependence on $q_\perp$.

The non-renormalized mode is irrelevant as corresponding expressions for the Green functions given by Eqs.(C5) do not have poles at this energy. It is evident that for other directions of $q$ the Green functions must have all three poles. The gapless mode is strongly renormalized and at small $q_\perp$ we have quadratic dispersion with $\epsilon_- = Aq_\perp^2/2$. Asymptotically at $q \gg k$ all three modes tends to $Aq_\perp^2$ and the umklapps become unimportant. The umklapp renormalization is seen clearly in Fig. 1 where the full and dashed lines correspond to minus and plus modes in Eq(35) and the dash-dotted line to gapless mode with $q \parallel k$ respectively.

If $q \parallel k$ the umklapp interaction disappears (see Eq. (27)). General expression for arbitrary directed $q$ is very complex. But it becomes very simple for small $q_\perp \ll k$. In this case from Eq. (C3) we get

$$\epsilon_q = A[q_\perp^4/2 + k^2q_\perp^2]^{1/2}.$$  

(36)

Cancellation of the $q_\perp^2$ term is the most striking feature of this expression. So we have to discus validity of our $n = 1$ approximation. Corresponding equations of motion are given by Eq. (A1) where at $H = 0$ and $q = q_\perp$ and the amplitude of the umklapps $R_q = -SD_0(q \cdot A)$. So at small $q_\perp \ll k$ and $\omega \ll Ak^2$ we have an expansion in powers of $q_\perp^2$ as $G_\pm$ and $F_\pm$ are proportional to $1/\epsilon_k$. So the $n = 1$ approximation gives correct result for $q_\perp^2$ term at small $\omega$ and above cancellation of $q_\perp^2$ survives in higher approximations. It shown in Appendix C for $n = 2$. However the $q_\perp^4$ remains with slightly changed coefficient and instead Eq.(36) we obtain

$$\epsilon_q = A[3q_\perp^4/8 + k^2q_\perp^2]^{1/2}$$  

(37)

It is follow from the above argumentation that further approximations can not change coefficients in this expression. For $q_\perp \gg k$ we have $1/q_\perp$ expansion and the umklapp renormalization disappears. Unfortunately at $q_\perp \sim k$ and $\omega \sim Ak^2$ the results of $n = 1$ approximation are qualitatively correct only. It is demonstrated in the end of Appendix C.
VI. PERPENDICULAR SUSCEPTIBILITY AND HELIX DEFORMATION.

We have demonstrated that the terms linear in operators \( a \) and \( a^+ \) cancel in the diagonal part of the interaction given by Eqs. (15)-(18) due to equilibrium conditions for the classical energy. However, there are not similar conditions for the terms with \( a_{\pm k} \) and \( a^+_{\pm k} \) in the interaction (24) with the perpendicular field and we have

\[
H_Z^U = P \left( \frac{S}{2} \right)^{\frac{3}{2}} \left[ -a_{-k}(1 - \sin \alpha) + a^+_k(1 + \sin \alpha) \right] + h.c. \tag{38}
\]

This expression may be considered as perturbation which gives rise addition contribution in the energy as well as some non-zero parts of the average values of the transverse spin components \( < S_R^R > \) proportional to \( h_\perp \). But we will use more general method, which leads in the first approximation to the same results. We consider operators \( a_{\pm k} \) and \( a^+_{\pm k} \) as c-numbers. Corresponding c-number terms appear in Eqs.(19) and (26) and we obtain additional contribution to the ground-state energy

\[
E_1 = P\sqrt{S/2}(a_k^+(1 + s) - a_{-k}(1 - s)) + [Pc - R_0(1 + s)]a_k^+a_0
\]

\[
+ [Pc + R_0(1 - s)]a_0^+a_{-k} + R_0[a_0a_{-k}(1 - s) - a_0^+a^+_k(1 + s)] + h.c. \tag{39}
\]

\[
+ E_k([a_k^2 + (a_{-k}^2)] + B_k(a_k^+a_{-k} + a_{-k}^+a_k) + E_0|a_0|^2 + B_0(a^2 + a^2)/2,
\]

where \( s = \sin \alpha, c = \cos \alpha, R_0 = (S/2)\omega_0N_CL\cos \alpha \) and in the classical limit \( a^+ = a^* \).

The energy \( E_1 \) is minimal at the following obvious equilibrium conditions \( \partial E_1/\partial a_{\pm k} = 0 \) which lead again in \( n = 1 \) approximation to six linear equations. Corresponding matrix coincides with \( M \) given by Eq.(B2) at \( \omega = 0 \) and we get equation

\[
M(0)Z = V, \tag{40}
\]

where column \( Z = (a_0, a_0^+, a_{-k}, a_k^+, a_k, a_{-k}^+) \) and

\[
V = (0, 0, -P^*\sqrt{S/2}(1 - s), -P^*\sqrt{S/2}(1 + s), P\sqrt{S/2}(1 + s), P\sqrt{S/2}(1 - s)). \tag{41}
\]

In the case when dipole contribution \( R_0 = 0 \) solution of these equations is presented in Appendix B. Using above-mentioned equilibrium conditions one can show

\[
E_1 = \frac{P}{2} \left( \frac{S}{2} \right) (a_k + a_k^+)(1 + s) - (a_{-k} + a_{-k}^+)(1 - s) + c.c., \tag{42}
\]

and for the field depended part of the ground-state energy we have

\[
E = -\frac{Sh^2}{Ak^2} - \frac{Sh^2}{2Ak^2(1 + \cos^2 \alpha)}[\Delta^2 - (h_\perp^2/2)\cos^2 \alpha], \tag{43}
\]

where the first term is the Zeeman energy for \( H < H_C \).

The helix axis has to rotate at \( g\mu_BH_\perp \sim \Delta \). According to experimental data the rotation was observed at \( H_\perp \ll H_c \). So we can put \( \cos^2 \alpha = 1 \) and obtain

\[
E = \frac{Sh^2}{Ak^2} \left[ -\cos^2 \Psi - \sin^2 \Psi \right], \tag{44}
\]

where \( \Psi \) is the angle between the helix axis \( k \) and the field. For small \( h_\perp \) this energy is minimal at \( \Psi = 0 \) i.e the field along \( k \). The real direction of the helix axis is determined by competition between the magnetic energy and anisotropic exchange given by Eq. (9). Eq.(44) is not valid at \( h \sim \Delta \) as it was derived in the linear approximation when the amplitudes \( a \) and \( a^+ \) given by (B7) should be small.

The perpendicular field deforms the helix structure producing the higher harmonics. In our \( n = 1 \) approximation there is the second harmonic only. The static contribution to the transverse spin components has now the form

\[
S_R = iAe^{iR_k}(S_0^k e^{iR} + S_0^k e^{-iR}) + c.c., \tag{45}
\]
where \( i S^\parallel_{a,k} = \sqrt{S/2}(a_{+,k} - a_{+,k}^+) \) and we put \( \alpha = 0 \). From these expressions using Eqs. (B7) we obtain the transverse uniform magnetization\(^{18}\)

\[
S^\parallel_U = -\frac{Sh_\perp \Delta^2}{2AK^2(\Delta^2 - h^2_\perp /2)}
\]  

The second harmonic of the helix structure induced by the perpendicular field is given by

\[
S^\perp_R = -\frac{2\Delta^2}{2AK^2(\Delta^2 - h^2_\perp /2)} [A(A \cdot h)e^{2iKR} + A^*(A^* \cdot h)e^{-2iKR}]
\]  

As we have mentioned above corresponding second-order Bragg reflections were observed\(^{18}\) and\(^{20}\).

### VII. CUBIC ANISOTROPY AND THE GAP PROBLEM

We postulated above existence of the spin-wave gap. Now we demonstrate that there are at least two contributions to the gap: cubic anisotropy and interaction between spin-waves considered in the Hartree-Fock (HF) approximation. The former may have arbitrary sign and the last is positive. So different contributions to the gap may compete. Changing the sign and strength of the cubic anisotropy by pressure one can get the quantum phase transition from the ordered to the spin-liquid state. This phenomenon was observed in \( MnS \)\(^{19,20}\) and\(^{21}\).

Complete expression for the gap at arbitrary \( H \) is very complex. So we present here results for \( H = 0 \) and \( H > H_c \). We begin with the cubic anisotropy. As is well known the single ion cubic anisotropy has the form

\[
V = K \sum_{R} (S^4_{x,R} + S^4_{y,R} + S^4_{z,R}).
\]  

Using Eqs. (3), (13) and (14) for zero magnetic field (\( \alpha = 0 \)) we obtain

\[
V = 6S^4K \sum_{\nu=x,y,z} |A_\nu|^4 + 6S^4K \sum_{k,\nu=x,y,z} \{2|A_\nu|^2c_\nu a_k^* a_k + [|A_\nu|^2c_\nu^2 - 2|A_\nu|^4](a_k a_{-k} + a_k^* a_{-k}^+)\}.
\]  

Neglecting in Eq. (9) the \( F^2 \) term for the classical energy we have

\[
E_{cl} = \frac{S^2F}{4} \sum_{\nu=x,y,z} k^2_\nu(\hat{a}_\nu^2 + \hat{b}_\nu^2) + \frac{3S^4K}{8} \sum_{\nu=x,y,z} (a_\nu^2 + b_\nu^2)^2.
\]  

The spin-wave terms in Eq.(49) give additional contribution to \( E_{q} \) and \( B_{q} \) in Eq.(19) which are given by

\[
\delta E_{q} = 12S^4K \sum_\nu c_\nu^2 |A_\nu|^2; \quad \delta B_{q} = 12S^4K \sum_\nu (c_\nu^2 |A_\nu|^2 - 2|A_\nu|^4),
\]  

and for the anisotropic contribution to the spin-wave gap we obtain

\[
\Delta^2_{c} = \frac{3}{2} S^3 K_h c \sum_\nu (\hat{a}_\nu^2 + \hat{b}_\nu^2)^2,
\]  

where \( h_c = E_0 + B_0, h_c, E_0 \) and \( B_0 \) are given by Eqs. (12), (20) and (21). This equation for the gap holds at \( \nu = 0 \) when \( q_\nu \rightarrow N_c \). For small \( \nu \) we must replace \( h_c \) by \( h_c^{int} + S\omega_0 q_\nu^2 \). So at \( \nu = 0 \) and \( \nu \neq 0 \) the gap depends on the sample form and angle between \( \nu \) and the helix axis respectively. We see that \( \Delta^2_c \) is positive for \( K > 0 \) only. In ferromagnets this sign of \( K \) corresponds to the easy directions along the cubic diagonals.

Let us consider now the classical energy and the gap for orientations of the helix vector \( k \) along \([1,1,1], [1,0,0] \) and \([1,1,0] \) directions labeled as 1, 2, and 3 respectively. We can choose \( \hat{c}_1 = (1,1,1)/3^{1/2} \), \( \hat{a}_1 = (1,-1,0)/2^{1/2} \) and \( \hat{b}_1 = (1,1,-2)/6^{1/2} \), \( \hat{c}_2 = (1,0,0), \hat{a}_2 = (0,1,0) \) and \( \hat{b}_2 = (0,0,1) \), \( \hat{c}_3 = (1,1,0)/2^{1/2}, \hat{a}_3 = (1,-1,0)/2^{1/2} \) and \( \hat{b}_3 = (0,0,1) \). For the classical energy we have

\[
E_{cl1} = \frac{S^2Fk^2}{6} + \frac{S^4K}{2}; \quad E_{cl2} = \frac{3S^4K}{4}; \quad E_{cl3} = \frac{S^2Fk^2}{8} + \frac{9S^4K}{16}.
\]  

From these expressions we see that structures \( (1,1,1) \) is realized if \( S^2Fk^2 < 3S^4K/2 \). Otherwise we have \( (1,0,0) \) structure. These expressions are derived at \( T = 0 \). However interplay between two anisotropy constants \( F \) and \( K \)
must determine the helix direction at all \( T \) and explain the transition from \((1, 0, 0)\) to \((1, 1, 1)\) structure in \( FeGe \) with decreasing \( T_c \).

Corresponding expressions for the gap are given by

\[
\Delta^2_{c1} = 4S^3K h_c; \quad \Delta^2_{c2} = 6S^3K h_c; \quad \Delta^2_{c3} = (9/2)KS^3 h_c. \tag{54}
\]

We consider now contribution to \( \Delta^2 \) arising from the spin-wave interaction considered in the HF approximation. It is well-known that in the Heisenberg ferromagnets there is not a gap due to the total spin conservation law. The DM interaction brakes this law. From Eqs. (15) and (16) we have

\[
H_{EX} + H_{DM} = -\frac{1}{2} \sum (\{J_{q,k} - J_0 + 2D_q(q \cdot [\hat{a} \times \hat{b}])\}(S^\alpha q^\alpha - a^\alpha q + S^\beta q^\beta) + (J_q - J_0)S^\gamma q^\gamma \{S^\xi q^\xi\}) \tag{55}
\]

where we have taken into account that \( \sum S_q S_{-q} = NS(S+1) \) and the odd terms with \( S^\xi S^\xi \) surviving at arbitrary \( q \) are zero in the HF approximation. We replace also \( \textbf{D} \) by \( D_q \) as the HF contribution is saturated by large \( q \). Using now Eq.(3) for the spin-wave interaction energy we obtain

\[
V_I = -\frac{1}{2} \sum \{V_I[a^+_{2+2} a^+_{2-3} a_3 + \frac{1}{2}(a_1 - a_1^+)a^+_{1+2+3}a^+_{2}a_3] - \frac{1}{2}(J_1 - J_0)(a_1 + a_1^+)a^+_{1+2+3}a^+_{2}a_3\}, \tag{56}
\]

where \( 1, 2, 3 = q_{1,2,3} \) and \( V_I = J_{1,k} - J_0 + 2D_q(k \cdot [\hat{a} \times \hat{b}]) \).

The HF contribution to \( \Delta^2 \) is calculated in Appendix D by the method used in \(^22\) for evaluation HF corrections to the spin-wave energy in 2D \( CuO_2 \) planes where the gap appearing due to pseudopolar interaction between neighboring \( Cu^{2+} \) ions. From Eqs. (52) and (D7) we obtain the final result

\[
\Delta^2 = \frac{3S^3K h_c}{2} \sum (\hat{a}^2_\nu + \hat{b}^2_\nu) + \frac{Ak^2 h_c}{4S} \sum \frac{D_q}{D_0}, \tag{57}
\]

where the last sum has to be less than unity.

In strong field \( H > H_c \) we have \( \cos \alpha = 0 \) and zero-point fluctuation disappear \((B_q = 0 \text{ in Eq. (21)}\). As a result one can show that there is not the HF contribution to the gap and the classical energy and the gap are given by

\[
E_{el} = \frac{S\omega_0 N_{ec}}{2} - Sh + S^4K \sum \hat{c}^4_\nu \tag{58}
\]

\[
\Delta = h - h_c + 4S^3K \sum [-\hat{c}^4_\nu + \frac{3}{2} \hat{c}^2_\nu (\hat{a}^2_\nu + \hat{b}^2_\nu)].
\]

For the cases listed above cubic contributions to the classical energy and the gap are given by

\[
E_{el1} = \frac{KS^4}{3}; \quad E_{el2} = KS^4; \quad E_{el3} = \frac{KS^4}{2}; \tag{59}
\]

\[
\Delta_1 = \frac{8KS^3}{3}; \quad \Delta_2 = -4KS^3; \quad \Delta_3 = KS^3.
\]

VIII. EPR AND NEUTRON SCATTERING

Both phenomena the EPR and the inelastic magnetic scattering are described by the spin susceptibility. We outline here some of its features. For simplicity we use the \( n = 1 \) approximation.

From Eq.(13) and definition (29) we obtain following general expression for the susceptibility

\[
\chi_{\alpha\beta}(q, \omega) = < S^\alpha_q, S^\alpha_{-q} > \omega \hat{c}_\alpha \hat{c}_\beta + < S^A_q, S^A_{-q} > \omega A_\alpha A_\beta^* + < S^A_q, S_A^q_{-q} > \omega A_\alpha^* A_\beta
\]

\[
+ < S^\alpha_q, S^\alpha_{-q} > \omega \hat{c}_\alpha A_\beta + < S^A_q, S^A_{-q} > \omega A_\alpha \hat{c}_\beta + < S^A_q, S^A_{-q} > \omega \hat{c}_\alpha A_\beta + < S^A_q, S^A_{-q} > \omega A_\alpha^* \hat{c}_\beta \tag{60}
\]

where \( - < A, B > \omega \) is the generalized susceptibility \( \text{chi}_{AB}(\omega) \). In the first three terms both spin components have opposite momenta and describe the direct processes . All other are the umklapp terms. We will use below the linear spin-wave theory and consider two limiting cases \( H = 0 \) and \( H > H_c \). The main attention will be paid to direct part of the susceptibility. The umklapp part will be commented briefly in the end of this Sec.
Zero-field case. 
There are now two contributions to the susceptibility along the k and in ab plane. For the first using Eqs. (14) and (31) we have

\[ \chi_{cc}(q, \omega) = -\frac{S\Delta^2}{h_c(\omega^2 - \Delta^2)}, \tag{61} \]

where due to non-Hermiticity of the interaction (56) we must to distinguish between F and F\(^+\). In the EPR case when \( q = 0 \) we obtain

\[ \chi_{cc}(\omega) = -\frac{S\Delta^2}{h_c(\omega^2 - \Delta^2)}. \tag{62} \]

This susceptibility describes respond to the external ac field. Connection between external and intrinsic susceptibilities is given by well known equation (see for example\(^{23} \))

\[ \chi_{\alpha\beta} = \chi_{\alpha\beta}^{\text{Int}} - 4\pi\omega_0\chi_{\alpha\beta}^{\text{Int}}N_{\mu\nu}\chi_{\mu\nu}, \tag{63} \]

and for \( \chi_{cc}^{\text{int}} \) we obtain the same equation as for \( \chi_{cc} \) with replacement \( h_c \) and \( \Delta^2 \) by corresponding intrinsic quantities. It should mention that there has to be very small contribution of the \( \epsilon_1 \) mode (more precisely \( \epsilon_+ \) mode). It appears if in Eqs. (B9-10) we retain terms proportional to \( (E_0 - B_0)^2 \) \( R^2 = \Delta^2 |R|^2 / (E_0 + B_0) \). Moreover in \( n = 2 \) approximation the \( \epsilon_2 \) mode has to appear too. But we neglect both these contributions.

For \( q \gg k \) the susceptibility is strongly anisotropic in q space. For \( q \parallel k \) the umklapp interaction is zero and from we obtain

\[ \chi_{cc}(q\parallel, \omega) = -\frac{SAq^2_\parallel}{\omega^2 - \epsilon^2_{q\parallel}}, \tag{64} \]

where \( \epsilon^2_{q\parallel} = Aq^2_\parallel[A(q^2_\parallel + k^2) + S\omega_0q^2_\parallel] \).

With increasing of \( q_\perp \) tree different modes appear. However at \( q_\parallel = 0 \) there are two modes (see Appendix C) given by Eqs. (35) and using Eqs. (C5) we get

\[ \chi_{cc}(q_\perp, \omega) = -\frac{SAq^2_\perp}{\epsilon^2_{q_\perp}} \left[ \frac{\epsilon^2_\perp - \epsilon^2_\perp + A(2q^2_\perp + k^2)}{\omega^2 - \epsilon^2_\perp} \right], \tag{65} \]

where \( \epsilon_1 \) is given by the second line in Eq. (34). In the \( n = 2 \) approximation additional modes appear and so on. However amplitudes of these modes decrease with \( n \).

Consider now the second two terms in Eq. (60). By the same way as above we get

\[ \chi^{AA^*}_{\alpha\beta}(q, \omega) = (\delta_{\alpha\beta} - \hat{c}_\alpha\hat{c}_\beta)\chi_{\perp q}(\omega) - \frac{i}{2}\epsilon_{\alpha\beta\gamma}\hat{c}_\gamma C_q(\omega), \tag{66} \]

where we put \( \hat{a}_\alpha\hat{a}_\beta + \hat{b}_\alpha\hat{b}_\beta = \delta_{\alpha\beta} - \hat{c}_\alpha\hat{c}_\beta \) and \( \hat{a}_\alpha\hat{b}_\beta - \hat{b}_\beta\hat{a}_\alpha = \epsilon_{\alpha\beta\gamma}\hat{c}_\gamma \). Perpendicular and chiral susceptibilities are determined as

\[ \chi_{\perp q} = \chi_{q+k}(\omega) + \chi_{q-k}(\omega); \quad C_q(\omega) = \chi_{q+k}(\omega) - \chi_{q-k}(\omega), \tag{67} \]

where

\[ \chi_{q}(\omega) = \frac{S}{8} [F_{q}(\omega) + F_{q}^{\dagger}(\omega) - G_{q}(\omega) - G_{q}(-\omega)]. \tag{68} \]

Functions \( Im\chi_{\perp} \) and \( ImC \) determine parts of the neutron scattering cross section independent on the neutron polarization \( P_0 \) and proportion to it respectively (See for example\(^{23} \)). The chiral contribution to the cross section appears due to the DM interaction and is \( q \) odd in agreement with general theory\(^{23} \).

In the EPR case when \( q \to 0 \) the singular term \( q^2_\perp / q_\parallel k^2 \) in Eqs. (20) and (21) is equal to \( N_{\perp} / 2 = N_{aa} + N_{bb} \) and \( q^2_c = N_{cc} \) respectively. At the same time the umklapp interaction \( |R| = S\omega_0(N_{ac}^2 + N_{bc}^2)^{1/2} / 4 \) is small and we can neglect it in the dispersion Eq.(B4). As a result using Eqs.(20), (21) and (31) we obtain

\[ \chi^{AA^*}_{\alpha\beta}(\omega) = -\frac{1}{2}(\delta_{\alpha\beta} - \hat{c}_\alpha\hat{c}_\beta)\frac{S(2A^2 + S\omega_0N_{cc})}{\omega^2 - \epsilon^2_1}, \tag{69} \]
where
\[ \epsilon_1 = [2(Ak^2)^2 + S\omega_0Ak^2 + (S\omega_0)^2N_{cc}N_{\perp}/2]^{1/2}. \] (70)
and \( C = 0 \). If we take into account the umklapp interaction the resonances appear at \( \omega = \Delta \) and \( \epsilon_{2k} \) and so on.

If \( q \neq 0 \) as above we consider two cases: \( q \parallel k \) and \( q \perp k \). In the first case we have
\[ \chi_\perp = -\frac{S}{4} \left( \frac{Z_-}{\omega^2 - \epsilon_{-}^2} + \frac{Z_+}{\omega^2 - \epsilon_{+}^2} \right); \quad C = -\frac{S}{4} \left( \frac{Z_-}{\omega^2 - \epsilon_{-}^2} - \frac{Z_+}{\omega^2 - \epsilon_{+}^2} \right), \] (71)
where \( Z_{\pm} = A(k \pm q) ^2 + S\omega_0 \) and \( \epsilon_{\pm} = A(k \pm q) ^2[A(k \pm q) ^2 + Ak^2 + S\omega_0] \).

For \( q \perp k \) using (C5) we obtain
\[ \chi_\perp = -\frac{S}{4epsilon_{+} - \epsilon_{-}^2} \left[ \frac{Z_+}{\omega^2 - \epsilon_{-}^2} + \frac{Z_-}{\omega^2 - \epsilon_{+}^2} \right], \] (72)
where \( \epsilon_{\pm} \) are given by Eq. (35), \( Z_{\pm} = \pm2Aq_\perp(2k^2 + q_\perp^2) \pm (\epsilon_{\pm}^2 - \epsilon_{+}^2) \) and \( C = 0 \).

We discuss now briefly off-diagonal terms in Eq.(60). They are a result of the umklapp interaction and has to be less than that considered above. At \( q = 0 \) the \( \hat{c}A \) and \( \hat{A}A \) terms are proportional to off-diagonal components \( N_{ca}(b) \) of the demagnetization tensor in the first and the second degree respectively. Possibility of their experimental study is a special problem which is beyond scope of this paper. At \( q \neq 0 \) these terms are very complex and we do not analyse them here too.

"Ferromagnetic" state \((H > H_c)\) In the longitudinal \( cc \) channel there are the two-spin excitations only which are beyond our consideration. By the same way as above for the transverse susceptibility we obtain
\[ \chi_{\perp\alpha\beta} = -\frac{S}{2} (\hat{c}_\alpha \hat{c}_\beta - \hat{c}_\alpha \hat{c}_\beta) \left( \frac{1}{\omega - \epsilon_{q+k}} - \frac{1}{\omega + \epsilon_{q-k}} \right) - iS \left( \frac{\epsilon_{\alpha\beta}}{2} \right) \hat{c}_\gamma \left( \frac{1}{\omega - \epsilon_{q+k}} + \frac{1}{\omega + \epsilon_{q-k}} \right), \] (73)
where the second term represents the chiral contribution to the susceptibility and the spin-wave energy is given by Eq.(22). At \( q = 0 \) the last term in (22) contains the demagnetization and \( \epsilon_k = Ak^2 + S\omega_0(N_{aa} + N_{bb})/2 + g\mu_B(H - H_c) \). In this case the imaginary part of the chirality is \( \omega \)-even and the chiral contribution to the static susceptibility is equal to zero in agreement with the general theory. Experimentally this contribution could be measured using circularly polarized AC field. For \( q \neq 0 \) the neutron scattering is maximal at \( q = \pm k \) in contrast to conventional ferromagnets.

**IX. SUMMARY AND DISCUSSION**

We begin with a short survey of the main results presented above. We have used the following interactions: conventional isotropic exchange, DMI, anisotropic exchange, magnetic dipole interaction and Zeeman energy. In the classical approximation these interactions determine the helix form and the critical field \( H_c \) of the transition to the "ferromagnetic" spin configuration. This field depends on the sample form due the demagnetization.

The linear spin-wave theory was developed. It was shown that the spin-wave spectrum depends strongly on the magnetic field. At \( H > H_c \) we have quadratic spectrum with the gap linearly increasing with the field [see Eq.(22)]. Below \( H_c \) the spectrum is gapless. It is strongly anisotropic due to incommensurate helical structure and low symmetry of the DMI. As a result umklapp processes appear which connect the spin-waves with momenta \( q \) at small \( Ak \) equal to \( \epsilon_k \) and so on.

The second harmonic with the wave-vector \( 2k \) was observed in and.
We considered two ways of the gap origin: cubic anisotropy and interaction between spin waves. The first contribution is proportional to the strength $K$ of the cubic anisotropy. The second appears is a result of breaking of the total spin conservation law by the DMI. It is positive and disappears in the "ferromagnetic state" at $H > H_c$ due to absence of the zero-point vibrations. But in this region there is the field-induced gap determined by Eq.(22). In ferromagnets the sign of $K$ determines the direction of the easy axis. In our case the helical structure is stable if $\Delta^2$ given by Eq.(57) is positive. Otherwise we have the chiral spin liquid. So there is a question: If change sign of $\Delta^2$ is a reason of the transition to the disordered state in $MnSi$ at high pressure? This problem demands further experimental and theoretical study.

The helix structure leads to peculiar features of the ESR and neutron scattering. In conventional magnets the ESR frequency is equal to the spin-wave gap. In the helical systems due to the umklapp along with this frequency there are more higher resonant excitations corresponding to the spin-waves with $q = n\mathbf{k}$ where $n = 1, 2, \ldots$. At $\mathbf{H} \neq 0$ the chiral channel appears. It may be observed using circular radiation. In neutron scattering below $H_c$ three modes can be studied. The chiral channel exists at $H = 0$ and $q \neq 0$ due to axial-vector nature of the DMI.

Detailed experimental work was done in the case of $MnSi$ compound only. We now compare some of known experimental results obtained at ambient pressure with the predictions of our theory and discuss possibilities of the further experimental studies. The principal parameters are: lattice spacing $a = 4.558\AA$, $T_c \simeq 29K$, $k \simeq 0.035\AA^{-1}$, saturated magnetization $M = 0.4\mu_B/a^3 \simeq 0.016T$, $4\pi M = 0.20T$, critical field $H_c = 0.5 \div 0.6T$ and spin-wave stiffness $A \simeq 52meV\AA^2$. From these data and Eq.(12) we obtain $H_c \simeq Ak^2/(g\mu_B) \simeq 0.55T$. This value coincides with experimentally observed critical field. For more precise comparison one must measure all parameters including the demagnetization $N_{rc}$ using single sample.

To the best of my knowledge the ESR in $MnSi$ was studied only in\textsuperscript{26}. Several resonances were observed but only one was studied qualitatively as a function of the magnetic field. Its frequency in zero field is equal to 0.93T. Using Eq.(70) and taking into account that $S_{c0} = 4\pi g\mu_B M$ we obtain 0.85T. The agreement is within the error bars. According to Eq.(22) at $H = H_c$ the frequency is close to 0.6T $\approx H_c$ and then increases linearly with $H$. This behavior coincides with results of Ref\textsuperscript{26}. Further experimental studies are necessary. First of all it is essential to observe the gap $\Delta$ and its dependence on the perpendicular field, which accordingly with Eq.(33) should be $\Delta = [\Delta^2 - (1/2)(g\mu_B H_\perp)^2] cos^4 \alpha^{1/2}$. The observation of the umklapp resonances $\epsilon_{2k}$ and $\epsilon_{3k}$ would be important too.

The theory developed in this paper explains partly some experimental findings. It was developed in the linear spin-wave approximation. The spin-wave interaction was used for evaluation contribution to the gap only. We also did not evaluate some experimentally observed quantities such as specific heat and the site magnetization. Corresponding results will be published elsewhere.

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APPENDIX A: EQUATIONS OF MOTION

Using Eqs.(25), (27) and (28) in the $n = 1$ approximation for functions (29) we obtain

\begin{equation}
(\omega - E)G - BF - [Pc + R_+q(1 - s) - R_-q(1 + s)]G_- + (R_-q + R_+q)(1 + s)F_- \\
- [P^*c + R^*_+q^{-1}(1 - s) - R^*_q(1 + s)]G_+ - (R^*_q + R^*_+q^{-1})(1 - s)F_+ = 1,
\end{equation}

\begin{equation}
BG + (\omega + E)F + (R_-q + R_+q)(1 - s)G_- + [Pc + R_+q^{-1}(1 - s) - R_-q(1 + s)]F_- \\
- (R_-q + R^*_+q^{-1})(1 + s)G_+ + [P^*c + R^*_q(1 - s) - R^*_+q^{-1}(1 + s)]F_+ = 0,
\end{equation}

\begin{equation}
- [P^*c + R^*_+q^{-1}(1 - s) - R^*_q(1 + s)]G_- + (R^*_q + R^*_+q^{-1})(1 - s)F_+ + (\omega - E_-)G_- - B_-F_- = 0,
\end{equation}

\begin{equation}
- (R_-q + R^*_q)(1 + s)G_+ + [P^*c + R^*_+q^{-1}(1 - s) - R^*_q(1 + s)]F_- + B_-G_+ + (\omega + E_-)F_- = 0,
\end{equation}

\begin{equation}
- [Pc + R_-q^{-1}(1 - s) - R_+(1 + s)]G_+ + (R_+(1 - s) - R_-q^{-1}(1 + s)]F_+ + B_+G_+ + (\omega + E_+)F_+ = 0,
\end{equation}

where $s = \sin \alpha$, $c = \cos \alpha$, $E(B) = E(B)_q$, $E(B)_{\pm} = E(B)_{q\pm k}$ and functions $E$ and $B$ are determined by Eqs. (18) and (19).
In matrix form these equations are given by

$$M(q, \omega) = I$$  \hspace{1cm} (A2)

where \( I = (1, 0, 0, 0, 0, 0) \). Determinant of the matrix \( M \) is even function of \( \omega \) and has the following general form

$$\text{Det}[M] = (\omega^2 - \epsilon_{q0}^2)(\omega^2 - \epsilon_{q+}^2)(\omega^2 - \epsilon_{q-}^2)$$  \hspace{1cm} (A3)

We obtain three renormalized energies as roots of this equation. As \( \text{Det}[M] \) is the denominator of the Green functions they have three poles. However at \( q \perp k \) two initial spin-wave energies are equal and only two renormalized branches are physically relevant. Hence the Green functions must have two poles. That it is the case is demonstrated below in Appendixes B and C. There are five renormalized branches in the \( n = 2 \) approximation (see Appendix C) and so on.

General expressions for \( \epsilon_{q0}^2 \) and \( \epsilon_{q\pm}^2 \) are very complex. We consider below two main cases: i. Small \( q \ll k \) and ii. \( H = 0, q \sim k \).

**APPENDIX B: THE \( q \to 0 \) CASE**

From Eqs. (20) and (21) we have

$$E_q = Aq^2 + (Ak^2 \cos^2 \alpha)/2 + (S\omega_0/2)q^2 \cos^2 \alpha, \quad B_q = (1/2)(Ak^2 + \omega_0q_0^2)\cos^2 \alpha,$$

$$E_1 = Ak^2(2 + \cos^2 \alpha)/2 + (S\omega_0/2)(q_{1c}^2 \cos^2 \alpha + (\tilde{q}_q^2/2)(1 + \sin^2 \alpha)], \quad B_1 = (1/2)[Ak^2 + (S\omega_0/2)(\tilde{q}_q^2 - \tilde{q}_q^2/2)]\cos^2 \alpha,$$

where \( E_1 = E_{q \pm k} \) and \( B_1 = B_{q \pm k} \). From Eq. (27) we get \( R_q = (S\omega_0/2)(\hat{c} \cdot \hat{q})(\hat{q} \cdot \mathbf{A}) \cos \alpha \) and \( R_{q \pm k} = 0 \). If \( q = 0 \) we must replace \( \tilde{q}_q^c, \tilde{q}_q^\perp \) and \( (\hat{c} \cdot \hat{q})(\hat{q} \cdot \mathbf{A}) \) by \( N_{cc}, N_{sa} + N_{bb} \) and \( (N_{cc} - iN_{cb})/2 \) respectively. The matrix \( M \) has now the form

$$\begin{pmatrix}
\omega - E_q & -B_q & -Pc - R(1 - s) & R(1 + s) & -P^*c + R^*(1 + s) & -R^*(1 - s) \\
-B_q & \omega + E_q & -P^*c - R^*(1 - s) & R^*(1 - s) & P^*c - R^*(1 + s) & R^*(1 + s) \\
-Pc - R(1 + s) & R(1 - s) & \omega - E_1 & -B_1 & 0 & 0 \\
-R(1 + s) & P^*c - R^*(1 + s) & B_1 & \omega + E_1 & 0 & 0 \\
-Pc + R(1 - s) & R(1 + s) & 0 & 0 & \omega - E_1 & -B_1 \\
R(1 - s) & Pc + R(1 - s) & B_1 & B_1 & \omega + E_1 & 0
\end{pmatrix}$$  \hspace{1cm} (B2)

For the determinant of this matrix using Program Mathematica 5 we obtain

$$\text{Det}[M] = (\omega^2 - \epsilon_0^2)\{(\omega^2 - \epsilon_0^2)(\omega^2 - \epsilon_1^2) - 4|P^2|^2(E_0E_1 + B_0B_1 - |P|^2 \cos^2 \alpha + \omega_0^2 \cos^2 \alpha - 8|R|^2(E_0 - B_0)(E_1 + B_1) + 4(PR^* + P^*R)(\omega^2 - (E_0 - B_0)(E_1 - B_1) - 2|P|^2 \cos^2 \alpha) \sin \alpha \cos \alpha - 8|R|^2(E_0 - B_0)(E_1 + B_1) \sin^2 \alpha + 4[(PR^* + P^*R)^2 - 4|P|^2 R^2]\epsilon_0^2[\sin \alpha \cos \alpha \cos \alpha],$$

$$\epsilon_0^2 = E_{00}^2 - B_0^2 \to \Delta^2$$. Here the last term may be represented as \([|P|^2|R|^2(\omega^2 - \epsilon_1^2) + ((N_{cc} \cdot P_\perp)^2 - N_{cc}^2 P_\perp^2)\omega_0^2 \sin \alpha \cos \alpha \] The \( \omega^2 \) term breaks rotational invariance in the \( ab \) plane and \( \epsilon_1^2 \) is not a root of this equation. But this term is very small and we neglect it.

We assume that \( P, R \) and \( \Delta \) are small in comparison with \( Ak^2 \) which is main parameter of the theory. In this approximation we have

$$\text{Det}[M] = (\omega^2 - \epsilon_0^2)[(\omega^2 - \epsilon_1^2)(\omega^2 - \Delta^2) - 4|P|^2(E_0E_1 + B_0B_1 + \omega^2) \cos^2 \alpha - 8|R|^2 E_1(1 + \sin^2 \alpha) + B_1 \cos^2 \alpha)]$$  \hspace{1cm} (B4)

As complete expressions for resonant modes and the Green functions in this approximation remain very complex we consider two limiting cases: \( R = 0 \) and \( H = 0 \).

**The \( R = 0 \) case.** As we see below the Green functions do not contain pole at \( \omega^2 = \epsilon_1^2 \). For two other energies we have

$$\epsilon_0^2 = \Delta^2 - \frac{h_1^2(Alk^2 + 2S\omega_0N_{cc}) \cos^4 \alpha}{2(Ak^2 + \omega_0N_{1c})/2 \cos^2 \alpha} \cong \Delta^2 - \frac{h_1^2 \cos^4 \alpha}{2},$$

$$\epsilon_1^2 = \epsilon_1^2 + h_1^2 \left[ 1 + \frac{(Alk^2 + 2S\omega_0N_{cc}) \cos^2 \alpha}{2Ak^2 + \omega_0N_{1c} \cos^2 \alpha} \right] \cos^2 \alpha \cong \epsilon_1^2 + \frac{h_1^2(2 + \cos^2 \alpha)}{2} \cos^2 \alpha,$$  \hspace{1cm} (B5)
where $\epsilon_1$ is given by Eq. (67) and in right hand side we neglected the dipole contribution. Corresponding expressions for the Green functions are given by

\begin{align}
G(\omega) &= Z^{-1}[(\omega + E_0)(\omega^2 - \epsilon_1^2) + (h_\perp^2/2)(E_1 - \omega)], \\
F(\omega) &= -Z^{-1}B_0[\omega^2 - \epsilon_1^2 - (h_\perp^2/2)B_1 \cos^2 \alpha], \\
G_-(\omega) &= Z^{-1}(h_+ / 2)[\omega^2 - \epsilon_1^2 + E_0E_1 + B_1B_0 - (h_\perp^2/2) \cos^2 \alpha] \cos \alpha,
F_-(\omega) &= -Z^{-1}(h_+ / 2)[B_0E_1 + B_1E_0 + \omega(B_1 - B_0)] \cos \alpha,
\end{align}

where $Z = (\omega^2 - \epsilon_1^2)(\omega^2 - \epsilon_0^2)$, $h_\perp = (h_\alpha \pm ih_\beta)/2$ and one gets $G_+$ and $F_+$ replacing $h_+$ by $h_-$ in the expressions for $G_-$ and $F_-$. Neglecting the dipole interaction we obtain (32).

Solution of Eq. (39) for the spin deviations frozen in the perpendicular field is given by

\begin{align}
G_k &= (a_k^+)^* = -\sqrt{S} \frac{(A \cdot h)(1 + \cos^2 \alpha + \sin \alpha)\Delta^2}{2 Ak^2(1 + \cos^2 \alpha)[\Delta^2 - (h_\perp^2/2) \cos^2 \alpha]}, \\
A - k &= (a_{-k}^+)^* = \sqrt{S} \frac{(A^* \cdot h)(1 + \cos^2 \alpha - \sin \alpha)\Delta^2}{2 Ak^2(1 + \cos^2 \alpha)[\Delta^2 - (h_\perp^2/2) \cos^2 \alpha]}, \\
A_0 &= A_0^+ = \sqrt{S} \frac{h_\perp^2(E_0 - B_0) \sin \alpha \cos \alpha}{2Ak^2[\Delta^2 - (h_\perp^2/2) \cos^2 \alpha]}.
\end{align}

The H = 0 case. From Eq. (B4) we have now

\begin{equation}
Det[M] = (\omega^2 - \epsilon_1^2)(\omega^2 - \epsilon_0^2)(\omega^2 - \epsilon_1^2),
\end{equation}

where

\begin{align}
\epsilon_0^2 &= \Delta^2 \left[ 1 - \frac{(S\omega_0)^2(N_{ca}^2 + N_{cb}^2)}{(Ak^2 + S\omega_0N_{cc})(2Ak^2 + S\omega_0N_{zz})} \right], \\
\epsilon_1^2 &= \epsilon_1^2 + \frac{\Delta^2(S\omega_0)^2(N_{ca}^2 + N_{cb}^2)}{(Ak^2 + S\omega_0N_{cc})(2Ak^2 + S\omega_0N_{zz})}.
\end{align}

Corresponding expressions for the Green functions are given by

\begin{align}
G(\omega) &= Z^{-1}[(\omega + E_0)(\omega^2 - \epsilon_1^2) + 4|R|^2(E_1 + B_1)], \\
F(\omega) &= -Z^{-1}[B_0(\omega^2 - \epsilon_1^2) + 4|R|^2(E_1 + B_1)], \\
G_-(\omega) &= R^*(\omega + E_0 - B_0)(\omega + E_1 + B_1), \\
F_-(\omega) &= -R^*(\omega + E_0 - B_0)(E_1 + B_1 - \omega),
\end{align}

where $R = (S\omega_0/4)(N_{ca} - iN_{cb})$, $G_+ = -R/R^*G_-$ and $F_+ = -R/R^*F_-$. Note that now it is addition sign minus in expressions for $G_+$ and $F_+$ in comparison with the $R = 0$ case.

APPENDIX C: THE $q \gg k$ CASE

It is convenient now to use dimensionless variables $X = \omega/(Ak^2)$, $Y = |q_\perp|/k$, $V = q_\parallel/k$ and $U = -(q_a - iq_b)/k$. In these variables we have $M = (Ak^2)^6m$ and for $m$ we get

\begin{equation}
\begin{pmatrix}
X - W^2 - \frac{1}{2} & -\frac{1}{2} & U & 0 & 0 & U^* & 0 \\
\frac{1}{2} & X + W^2 + \frac{1}{2} & 0 & U & 0 & U^* & 0 \\
U^* & 0 & X - W^2 + 2V - \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
0 & U^* & \frac{1}{2} & X + W^2 - 2V + \frac{3}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X - W^2 - 2V - \frac{3}{2} & -\frac{1}{2} & 0 \\
0 & U & 0 & 0 & \frac{1}{2} & X + W^2 + 2V + \frac{3}{2} & 0 
\end{pmatrix}
\end{equation}

where $W^2 = Y^2 + V^2$.

General expression for $Det[m]$ is very complex. For $V \ll 1$ we have

\begin{equation}
Det[m] = (X^2 - \epsilon_1^2)(X^2 - \epsilon_0^2)(X^2 - \epsilon_1^2)(X^2 - \epsilon_1^2) \\
\epsilon_1^2 = 1 + 4Y^2 + Y^4 \pm (1 + 8Y^2 + 17Y^4 + 8Y^6)^{1/2} + 4V^2
\end{equation}
where $\epsilon_1^2 = (Y^2 + 1)(Y^2 + 2)$ is the energy at $\mathbf{q}_\perp \pm \mathbf{k}$ in the dimensionless units. As above at $Z = 0$ the factor $X^2 - \epsilon_1^2$ cancels in the expressions for Green Functions.

For small $Y$ and $V$ we have

$$X^2 = \frac{Y^4}{2} + V^2; \quad X^2_\perp = 2 + 4Y^2.$$ (C3)

Asymptotic expressions for $\epsilon_\pm$ at $Y \gg 1$ are given by

$$\epsilon_\pm = Y^2 \pm \sqrt{2}Y.$$ (C4)

The Green functions in dimensionless units are given by

$$G(X) = Z^{-1}[(X + Y^2 + 1/2)(X^2 - \epsilon_1^2) + 2Y^2(Y^2 + 3/2 - X)];$$
$$F(X) = -Z^{-1}(X^2 - \epsilon_1^2 + 2Y^2)/2;$$
$$G_-(X) = -Z^{-1}(Y_a + iY_b)[-(X + Y^2 + 1/2)(X + Y^2 + 3/2) + 2Y^2 + 1/4];$$
$$F_-() = Z^{-1}(Y_a + iY_b)(2X - 1)/2,$$

where $\epsilon_1^2 = (Y^2 + 1)(Y^2 + 2)$, $Z = (X^2 - \epsilon_1^2)(X^2 - \epsilon_2^2)$ and $G_+(F_+) = [G_-(F_-)](Y_a - iY_b)/(Y_a + iY_b)$.

To illustrate that $n = 1$ results are at least qualitatively correct we consider now the $n = 2$ approximation. In this case the matrix $M = (Ak^2)^{10}m$ and for $m$ we have

$$U = \begin{pmatrix}
X_+ - \frac{1}{2} & \frac{1}{2} & 0 & 0 & U^* & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & X_+ + \frac{1}{2} & 0 & U & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & U^* & 0 & X_+ - \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & U & 0 & 0 & X_+ - \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & U^* & 0 & 0 & 0 & X_+ - \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & U^* & 0 & 0 & 0 & X_+ - \frac{3}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & U & 0 & 0 & 0 & X_+ - \frac{3}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & U & 0 & 0 & 0 & X_+ - \frac{3}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & U & 0 & 0 & X_+ - \frac{3}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & U & 0 & X_+ - \frac{3}{2}
\end{pmatrix}.$$ (C6)

where $X_+ = X \mp Y^2$. The determinant of $m$ is given by

$$\text{Det}[m] = 1600X^2 - 1760X^4 + 564X^6 - 44X^8 + X^{10} + (7760X^2 - 4868X^4 + 524X^6 - 15X^8)Y^2$$
$$+ (-600 + 9279X^2 - 2462X^4 + 167X^6 - 3X^8)Y^4 + (-585 + 3686X^2 - 597X^4 + 22X^6)Y^6$$
$$+ (-519 + 866X^2 - 107X^4 + 3X^6)Y^8 + (-528 + 27X^2 - 7X^4)Y^{10} + (-144 + 24X^2 - X^4)Y^{12}.$$ (C7)

We see that according to argumentation in Sec.5 we have cancellation of the $Y^2$ term. There is the $Y^2X^2$ term only.

The equation $\text{Det}[m] = 0$ determines now five spin-wave energies generated by $\epsilon_2^2 = Y^2(Y^2 + 1), \epsilon_2^2 = (Y^2 + 1)(Y^2 + 2)$ and $\epsilon_{2k}^2 = (Y^2 + 4)(Y^2 + 5)$. We consider below the three lower branches only.

**Small $Y$ case.** From Eq.(C6) for the gapless branch we have $X^2 = 3Y^4/8$ and the factor 1/2 in (C3) has to be replaced by 3/8. Higher approximations with $n > 2$ can not change this result. Consider now the branches connected to $\epsilon_k$. For $X^2 \approx 2$ the first three terms in Eq.(C7) give $648\chi^2 - 5904\chi Y^2 + 9398Y^4 = 0$, where $\chi = X^2 - 2$. Solutions of this equation are given by

$$X^2 = 2 + \frac{37Y^2}{18} = 2 + 2.1Y^2; \quad X^2 = \frac{127Y^2}{18} = 2 + 7.1Y^2.$$ (C8)

We again obtain small changes of the numerical coefficients.

**Large $Y$.** For large $Y$ we can write $X = WY^2$ and asymptotically $W \to 1$. Retaining only three main terms in powers of $Y$ we obtain the dispersion equation

$$(W - 1)^3 - \frac{8(W - 1) + 15(W - 1)^2}{Y^2} + \frac{40}{Y^4W^2} = 0.$$ (C9)

Its solutions are given by

$$X_\pm = Y^2 \pm \sqrt{2}; \quad X_1 = Y^2 + \frac{5}{2}.$$ (C10)
APPENDIX D: THE HARTREE-FOCK GAP

As the interaction (55) is non-Hermitian the spin-wave Hamiltonian have the form\(^2\)

\[
H_{SW} = \sum |E_q a_q^+ a_q + \frac{1}{2} (B_q^+ a_q a_{-q} + B_q a_{-q}^+ a_q^+)|, \tag{D1}
\]

where \(B_q^* \neq B_q^+\). Corresponding corrections to \(E_q\) and \(B_q\) in Eq. (19) are determined as

\[
\delta E_q = \left( \frac{\delta^2 V_f}{\delta a_q^+ \delta a_q} \right) ; \quad \delta B_q = \left( \frac{\delta^2 V_f}{\delta a_q^+ \delta a_{-q}} \right) ; \quad \delta B_q^+ = \left( \frac{\delta^2 V_f}{\delta a_{-q}^+ \delta a_q} \right). \tag{D2}
\]

At \(q = 0\) we obtain

\[
\delta E_0 = -\frac{1}{2} \sum [(V_0 + V_q - J_q + J_0)n_q + (\frac{1}{2} V_0 + V_q - J_q + J_0)f_q];
\]

\[
\delta B_0 = -\frac{1}{2} \sum V_q f_q; \quad \delta B_0^+ = \delta B_0 - \frac{1}{2} \sum [(V_0 + V_q - 2J_q + 2J_0)n_q - (J_q - J_0)f_q],
\tag{D3}
\]

where expression for \(V_q\) is given below Eq.(55) and

\[
n_q = \frac{E_q - \epsilon_q}{2\epsilon_q}; \quad f_q = \langle a_q a_{-q} \rangle = \langle a_{-q}^+ a_q^+ \rangle = \frac{B_q}{2\epsilon_q}.
\tag{D4}\]

Using Eqs. (20) and (21) and neglecting dipolar terms we get

\[
\Delta^2 = -\frac{Ak^2}{4} \sum [(V_0 + V_q)n_q + (V_0 - J_q + J_0)f_q]. \tag{D5}
\]

Large \(q \sim 1/\epsilon\) give principal contribution to the expression (D3). However Eqs. (20) and (21) determine \(E_q\) and \(B_q\) at \(q \ll 1/\epsilon\) only. Using Eq. (55) we obtain these functions for all \(q\)

\[
E_q = S \left[ J_0 - \frac{J_q k + J_q}{2} + (D_0 - D_q) \frac{Ak^2}{2} \right]; \quad B_q = \frac{S}{2} \left[ J_{q,k} - J_q + 2 \frac{D_q}{SD_0} Ak^2 \right], \tag{D6}
\]

where using Eq.(8) we replaced \(k \cdot [\hat{a} \times \hat{b}]\) by \(Ak^2/(SD_0)\). The function \(B_q \propto k^2\). As \(\epsilon_q = (E_q^2 - B_q^2)^{1/2}\) we have that \(n_q\) is proportional to \((Ak^2)^3\) and may be neglected. In the \(f_q\) term at large \(q\) we have \(V_0 + J_0 - J_q \approx \epsilon_q / S\) and

\[
\Delta^2 = \frac{Ak^2}{8} \sum \left[ J_{q,k} - J_q + 2 \frac{D_q Ak^2}{SD_0} \right], \tag{D7}
\]

where the last term contribute to the gap only as \(\sum J_{q,k} = 0\).
FIG. 1: Fig.1 Spin-wave dispersion for different directions of the wave-vector $q$. (a) $q$ is along the helix axis $k$ (dashed line).
(b) Gapless and gapped branches for $q \perp k$. Solid and dot-dashed lines respectively.

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