THE NAVIER-SLIP THIN-FILM EQUATION IN THREE DIMENSIONS:
EXISTENCE AND UNIQUENESS

MANUEL V. GNANN AND MIRCEA PETRACHE

Abstract. We consider the thin-film equation $\dot{h} + \nabla \cdot (h^2 \nabla h) = 0$ in physical space dimensions (i.e., one dimension in time $t$ and two lateral dimensions with $h$ denoting the height of the film in the third spatial dimension), which corresponds to the lubrication approximation of the Navier-Stokes equations of a three-dimensional viscous thin fluid film with Navier-slip at the substrate. This equation can have a free boundary (the contact line), moving with finite speed, at which we assume a zero contact angle condition (complete-wetting regime). Previous results have focused on the $1+1$-dimensional version, where it has been found that solutions are not smooth as a function of the distance to the free boundary. In particular, a well-posedness and regularity theory is more intricate than for the second-order counterpart, the porous-medium equation, or the thin-film equation with linear mobility (corresponding to Darcy dynamics in the Hele-Shaw cell). Here, we prove existence and uniqueness of classical solutions that are perturbations of an asymptotically stable traveling-wave profile. This leads to control on the free boundary and in particular its velocity.

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1. The setting

1.1. Formulation of the free-boundary problem. We study the thin-film equation

\[ \partial_t h + \nabla \cdot (h^2 \nabla \Delta h) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad (y, z) \in \{ h > 0 \} \]  
(1.1a)

in 1 + 2 dimensions. Here, \( t \) denotes the time variable and \((y, z) \in \mathbb{R}^2\) the base point of the fluid film with height \( h = h(t, y, z) \) (cf. Figure 1). The differential operators in (1.1a) read

\[ \nabla := \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \text{and} \quad \Delta := \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \]

It is known that (1.1) allows for solutions evolving with finite speed of propagation (cf. Bernis, 1996a; Grün, 2002, 2003; Hulshof and Shishkov, 1998), that is, a free boundary \( \partial \{ h > 0 \} \) (the contact line) will appear. This is a moving line in three-dimensional physical space, which evolves in time and forms the triple junction between the three phases liquid, gas, and solid. Here, we assume a zero contact angle at the contact line (the angle between the interfaces liquid-gas and liquid-solid), that is,

\[ (\nu, \nabla h) = 0 \quad \text{at} \quad \partial \{ h > 0 \}, \]
(1.1b)

where \( \nu = \nu(t, y, z) \in \mathbb{R}^2 \) denotes the (inner) unit normal of the free boundary \( \partial \{ h > 0 \} \). Reformulating equation (1.1a) in divergence form as

\[ \partial_t h + \nabla \cdot (hV) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad (y, z) \in \{ h > 0 \}, \]
(1.1c)

we read off the transport velocity \( V = h\nabla \Delta h \) of the film height, for which the boundary value on \( \partial \{ h > 0 \} \) has to equal the velocity \( V_0 = V|_{z=0} \) of the free boundary. Thus we impose another condition at the contact line, which reads

\[ h\nabla \Delta h = V_0 \quad \text{at} \quad \partial \{ h > 0 \}. \]
(1.1d)

1.2. The thin-film equation with general mobility. In fact, equation (1.1a) is a special case of the general thin-film equation

\[ \partial_t h + \nabla \cdot (h^n \nabla \Delta h) = 0 \quad \text{in} \quad \{ h > 0 \}, \]
(1.2)

where \( n \in (0, 3) \) determines the mobility \( h^n \) entering (1.2). Note that the upper and lower bounds on \( n \) are due to the following observations:

(i) For \( n \leq 0 \), equation (1.2) is not degenerate anymore and therefore has infinite speed of propagation. Secondly, non-negativity of solutions (cf. Bernis and Friedman, 1990) is not ensured anymore. Therefore, in these cases solutions to (1.2) bear no physical interpretation as fluid films.

(ii) For \( n = 3 \), equation (1.2) is the lubrication approximation of the Navier-Stokes equations with no slip at the substrate. Then the variables \( h \) and \( z \) in the spatial part of (1.2) have a critical scaling leading to a singularity of \( h \) at the contact point. Furthermore, the contact point is fixed for all times as a moving contact point would lead to infinite dissipation (cf. Dussan V. and Davis, 1974; Huh and Scriven, 1971; Moffatt, 1964). As the degeneracy increases with \( n \), the free boundary cannot move for all \( n > 3 \), that is, a physical interpretation ceases to be valid.

In fact, the integer cases \( n = 1 \) and \( n = 2 \) carry most physics. The case of linear mobility \( n = 1 \) can be interpreted as the lubrication approximation of the Darcy flow in the Hele-Shaw cell (cf. Giacomelli and Otto, 2003; Knüpfer and Masmoudi, 2013, 2015). In this case the fluid is trapped between two narrow walls, so that the flow field is in good approximation laminar and parallel to the walls and the dependence on the coordinate perpendicular to the walls is a Poiseuille-type parabola. Hence, it appears that the \( 1 + 1 \)-dimensional case is the physically most relevant one for linear mobility \( n = 1 \).
In a physical 1 + 2-dimensional lubrication model, linear mobilities can be reproduced by assuming a nonlinear slip condition in which the slip length diverges like $h^{-1}$ as $h \to 0$. However, in this case it may be argued that a more natural choice (in line with the original work of Navier (1823), where the slip condition together with the Navier-Stokes equations has been proposed first) is to use a quadratic mobility $n = 2$, corresponding to an $h$-independent slip length. This is in fact the lubrication approximation of the Navier-Stokes equations with Navier slip at the substrate. This is the case on which we chose to concentrate in the present work.

Note that the dissipation functional in the Navier-slip case is given by the (scaled) sum of the dissipation functionals for inner friction only (no-slip case) and purely outer friction (Darcy), so that Navier slip merely is a balance of these two contributions. We refer to the reviews by Bonn, Eggers, Indekeu, Meunier, and Rolley (2009); de Gennes (1985); Oron, Davis, and Bankoff (1997) for (non-rigorous) derivations of \(1.1a\) starting from the Navier-Stokes equations with Navier slip at the liquid-solid interface and to Jäger and Mikelić (2001) for a rigorous derivation of the Navier-slip condition due to a rough liquid-solid interface.

1.3. Weak solutions to the thin-film equation. We emphasize that a well-established global existence theory of weak solutions to \(1.2\) has been developed, starting with Bernis and Friedman (1990) and later on upgraded to the stronger entropy-weak solutions by Beretta, Bertsch, and Dal Passo (1995); Bertozzi and Pugh (1996), which also exist in higher dimensions (cf. Dal Passo, Garcke, and Grün, 1998; Grün, 2004b). An alternative gradient-flow approach leading to generalized minimizing-movement solutions (that are weak solutions as well) is due to Loibl, Matthes, and Zinsl (2016); Matthes, McCann, and Savaré (2009); Otto (1998). Qualitative properties of weak solutions have been the subject of for instance the works of Bernis (1996a,b); Grün (2002, 2003); Hulshof and Shishkov (1998), where finite speed of propagation has been proved, Bertsch, Dal Passo, Garcke, and Grün (1998); Dal Passo, Giacomelli, and Grün (2001); Fischer (2013, 2014, 2016); Giacomelli and Grün (2006); Grün (2004a), where waiting-time phenomena have been considered, or Carlen and Ulusoy (2014); Carrillo and Toscani
(2002); Matthes, McCann, and Savaré (2009), where the intermediate asymptotics of (1.2) have been investigated. Partial-wetting boundary conditions have been considered by Bertsch, Giacomelli, and Karali (2005); Esselborn (2016); Mellet (2015); Otto (1998). We refer to Ansini and Giacomelli (2004); Bertozzi (1998); Giacomelli and Shishkov (2005) for detailed reviews.

Nevertheless, unlike in the porous-medium case (1.3), this theory does neither give uniqueness of solutions nor enough control at the free boundary to give an expression like (1.1d) a classical meaning. Furthermore, the regularity of the free boundary as a sub-manifold of $(0, \infty) \times \mathbb{R}^2$ appears to be inaccessible within this theory. This explains the interest in a well-posedness and regularity theory of classical solutions to (1.1).

1.4. Well-posedness and classical solutions. Well-posedness and regularity for zero contact angles in the Hele-Shaw case (equation (1.2) with $n = 1$) have been treated by Bringmann, Giacomelli, Knüpfer, and Otto (2016); Giacomelli and Knüpfer (2010); Giacomelli, Knüpfer, and Otto (2008); Gnann (2015); Gnann, Ibrahim, and Masmoudi (2017) in $1+1$ dimensions and by John (2015); Seis (2017) in any number of spatial dimensions while nonzero contact angles have been the subject of the works of Knüpfer and Masmoudi (2013, 2015) for $1+1$ dimensions only. The remarkable result is that solutions are smooth functions in the distance to the free boundary only for the linear-mobility case, i.e., for the case $n = 1$. The reason for this feature is the strong analogy to the porous-medium equation

$$h_t - \Delta h^m = 0 \quad \text{in} \quad \{h > 0\},$$

where $m > 1$, which is also degenerate-parabolic, but additionally satisfies a comparison principle. In fact, the spatial part of the linearizations in the works of Giacomelli, Knüpfer, and Otto (2008); Gnann (2015); John (2015) is nothing else but the square of the spatial part of the corresponding linearization of (1.3). For (1.3) a well-established well-posedness and regularity theory (giving smooth solutions) is available (cf. Angenent, 1988; Daskalopoulos and Hamilton, 1998; Kienzler, 2016; Koch, 1999; Seis, 2015), which in fact transfers to (1.2) with $n = 1$ and zero contact angle on $\partial \{h > 0\}$. In what follows, we give support for the insight that the strong analogy to the porous-medium equation (1.3) is lost when passing to general mobilities $n \in (0, 3) \setminus \{1\}$, in particular Navier slip with $n = 2$:

Well-posedness and regularity for the $1+1$-dimensional counterpart of (1.1), i.e.,

$$h_t + \partial_z \left( h^2 \partial_z^2 h \right) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad z \in \{h > 0\}, \quad (1.4)$$

subject to complete-wetting boundary conditions, have already been treated by Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016). There, perturbations of traveling waves

$$h = \begin{cases} x^\frac{3}{2} & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0 \end{cases} \quad \text{with} \quad x := z - \frac{3}{8} t, \quad (1.5)$$

have been investigated (the wave velocity is without loss of generality normalized to $\frac{3}{4}$ and the contact point is without loss of generality at $z = 0$ at initial time $t = 0$). The result of Giacomelli, Gnann, Knüpfer, and Otto (2014) establishes well-posedness of solutions for sufficiently regular initial data and a partial regularity result, which has been upgraded by Gnann (2016) to obtain full regularity in form of

$$h = x^\frac{3}{2} \left( 1 + \sum_{\beta \leq j + \beta k < N} a_{jk}(t)x^{j + \beta k} + O(x^N) \right) \quad \text{as} \quad x \searrow 0, \quad \text{where} \quad x := z - Z_0(t), \quad (1.6)$$

the variable $Z_0(t)$ denotes the contact point, and where the number

$$\beta := \frac{\sqrt{13} - 1}{4} \quad (1.7)$$

has been introduced. Here, the $a_{jk}(t)$ are (at least) continuous functions of time $t$ and the order $N \in \mathbb{N}$ of expansion (1.6) can be chosen arbitrarily large. Note that expansion (1.6) is in line with the findings of Giacomelli, Gnann, and Otto (2013), where it has been found that source-type self-similar solutions $h(t, z) = t^{-\frac{1}{2}} H(x)$ with $x := t^{-\frac{1}{4}} z$, have the form

$$H(x) = C \left[ |x| \pm 1 \right]^\frac{3}{2} \left( 1 + v \left( |x| \pm 1, |x| \pm 1 \right)^3 \right) \quad \text{as} \quad (-1, 1) \ni x \rightarrow \mp 1,$$

where $v = v(x_1, x_2)$ is analytic around $(x_1, x_2) = (0, 0)$ with $v(0, 0) = 0$ (cf. Belgacem, Gnann, and Kuehn, 2016, Appendix, for the generalization to higher dimensions). The analysis of Gnann (2016) heavily uses the fact that spatial and temporal regularity can only be treated jointly, so that expansion (1.6) also implies higher regularity in time, in particular of the contact point $Z_0 = Z_0(t)$. This was also used in the works of Knüpfer (2011), where partial-wetting boundary conditions (i.e., a fixed nonzero
contact angle at the triple junction) have been treated. For the case of general mobilities, we refer to Belgacem, Gnann, and Kuehn (2016); Giacomelli, Gnann, and Otto (2013) for source-type self-similar solutions with complete- and partial-wetting boundary conditions (partially also in higher dimensions), to Knüpfer (2015, 2017) for general solutions in the 1 + 1-dimensional case and partial-wetting boundary conditions, and to Degtyarev (2017) for general solutions in higher dimensions with partial-wetting boundary conditions, where similar features can be observed.

This work is concerned with developing a well-posedness and stability analysis for the Navier-slip thin-film equation with complete-wetting boundary conditions in physical dimensions 1 + 2. We are aware of only three works (which have been mentioned afore) establishing well-posedness and regularity of the free boundary in more than 1 + 1 dimensions. The first two, John (2015); Seis (2017), treat the linear mobility case in arbitrary dimensions, showing that level sets (and therefore also the free boundary) are analytic manifolds of time and space. The third paper due to Degtyarev (2017) treats the quadratic mobility case in arbitrary dimensions, under partial wetting conditions, i.e., condition (1.1b) is replaced by the less degenerate condition \((\nu, \nabla h) = g\) for a function satisfying point-wise bounds \(g \geq \varepsilon > 0\). In this case local-in-time existence of a unique smooth solution (implying smoothness of the free boundary as well) have been shown.

1.5. Perturbations of traveling waves. Due to the choice of complete-wetting boundary conditions (cf. (1.1b)), the generic situation is the one in which the thin fluid film will ultimately cover the whole surface. Close to the contact line, assumed here to be almost straight, one may model this situation by a traveling wave solution to (1.1a), which we assume for convenience to move in the \(z\)-direction:

\[
h_{TW}(t, y, z) := H(x), \quad \text{where} \quad x = z - Vt \quad (1.8a)
\]

and \(H = H(x)\) is a fixed profile moving with velocity \(V\). Using (1.8) in (1.1a) gives the fourth-order ordinary differential equation (ODE)

\[
-V \frac{dH}{dx} + \frac{d}{dx} \left( H^2 \frac{d^3 H}{dx^3} \right) = 0 \quad \text{for} \quad x > 0.
\]

Using (1.1b), we obtain \(\frac{d}{dx} H(0) = 0\), and up to changing the variable \(x\) by a translation we have \(\hat{c}(h > 0) = \{z = Vt\} = \{x = 0\}\) in this case, which gives the condition \(H(0) = 0\). Condition (1.1d) gives \(H \frac{d^3 H}{dx^3}(0) = V\) and by an easy integration we find

\[
H \frac{d^3 H}{dx^3} = V \quad \text{for} \quad x > 0.
\]

Hence, the traveling-wave profile \(H\) is given by

\[
H(x) = \begin{cases} 
  x^\frac{2}{3} & \text{for} \quad x > 0, \\
  0 & \text{for} \quad x \leq 0,
\end{cases} \quad (1.8b)
\]

where \(V = -\frac{3}{8}\) is the rescaled velocity of the wave.

Our main result will establish well-posedness and stability of perturbations of the traveling-wave profile (1.8). As a starting point, we transform equation (1.1a) for \(h\) to an equation for the perturbation of \(h_{TW}\). Therefore, we define new coordinates depending on the function \(h\) and denoted by \((t, x, y)\), which are related to the original coordinates \((t, y, z)\) via the property that (cf. Figure 2)

\[
h(t, y, Z(t, x, y)) = x^\frac{2}{3} \quad \text{for} \quad t, x > 0 \quad \text{and} \quad y \in \mathbb{R}. \quad (1.9)
\]

The transformation interchanges dependent and independent variables and we refer to it as the hodograph transform. Since the active coordinate is the variable \(z\), under the assumption of a profile that is strictly monotone in \(z\) the transformation (1.9) is indeed well-defined, and we find that condition (1.1b) is automatically verified. Under this change of coordinates, the boundary \(\hat{c}(h > 0)\) is transformed into the coordinate hyperplane \(\{x = 0\}\). Furthermore, in case of the traveling-wave solution \(h = h_{TW}\) given in (1.8a) for \(V = -\frac{3}{8}\), we directly find by comparison to (1.8b) that (1.9) is satisfied by

\[
Z = Z_{TW}(t, x, y) := x - \frac{3}{8} t. \quad (1.10)
\]
In Appendix A.1 we provide details on how the free-boundary problem (1.1) transforms under the change of coordinates (1.9) into

\[
Z_t + F^{-1} \left( \frac{D_y^2}{2} - D_yG(D_x - \frac{1}{2}) - GD_y(D_x + \frac{3}{2}) + G(D_x + \frac{3}{2})G(D_x - \frac{1}{2}) \right) + F(D_x + \frac{3}{2})F(D_x - \frac{1}{2}) \left( D_yG - G(D_x + \frac{1}{2})G - F(D_x + \frac{1}{2})F \right) = 0
\]

for \((t, x, y) \in (0, \infty)^2 \times \mathbb{R}\), where

\[
F := Z_x^{-1} \quad \text{and} \quad G := Z_x^{-1}Z_y.
\]

1.6. The nonlinear Cauchy problem. As a next step, we linearize equation (1.11) around the traveling-wave solution (1.10) by setting

\[
v := Z - Z_{TW} \quad \text{with} \quad Z_{TW}(t, x, y) = x - \frac{3}{8}t.
\]

Thus \(Z_x = -\frac{3}{8} + v_t\) and by (A.2) we observe that

\[
F^{-1} = 1 + v_x, \quad F = 1 - v_x + \text{h.o.t.}, \quad \text{and} \quad G = v_y + \text{h.o.t.},
\]

where h.o.t. denotes terms of higher order (super-linear in \(\{v_x, v_y\}\) or containing a term of the form \(w\partial\), where \(w \in \{v_x, v_y\}\) and \(\partial \in \{\partial_x, \partial_y\}\)). We use this in (1.11) and obtain

\[
D_yG - G(D_x + \frac{1}{2})G - F(D_x + \frac{1}{2})F = -\frac{1}{2} + x^{-1}(D_x^2 + D_y^2)v + \text{h.o.t.}
\]

Furthermore, we also have the operator identity

\[
D_y^2 - D_yG(D_x - \frac{1}{2}) - GD_y(D_x + \frac{3}{2}) + G(D_x + \frac{3}{2})G(D_x - \frac{1}{2}) + F(D_x + \frac{1}{2})F(D_x - \frac{1}{2})
\]

\[
= D_y^2 + (D_x + \frac{3}{2})(D_x - \frac{1}{2}) + \frac{1}{2}x^{-1}(D_x^2 + D_y^2) + \text{h.o.t.}
\]

Utilizing this in (1.11), we obtain the nonlinear Cauchy problem

\[
x\partial_t v + q(D_x)v + D_x^2r(D_x)v + D_y^4v = N(v) \quad \text{for} \quad (t, x, y) \in (0, \infty)^2 \times \mathbb{R},
\]

\[
v_{|t=0} = \nu^{(0)} \quad \text{for} \quad (x, y) \in (0, \infty) \times \mathbb{R},
\]

for given initial data \(\nu^{(0)} = \nu^{(0)}(x, y) : (0, \infty) \times \mathbb{R} \to \mathbb{R}\), where we have introduced the polynomials

\[
q(\zeta) := (\zeta + \frac{1}{2})\zeta(\zeta^2 - \frac{3}{2}\zeta - \frac{1}{2}) = (\zeta + \frac{1}{2})(\zeta + \beta - \frac{1}{2})\zeta(\zeta - \beta - 1),
\]

\[
r(\zeta) := 2(\zeta + 1)(\zeta + \frac{1}{2}),
\]

with the irrational root

\[
\beta = \frac{\sqrt{13} - 1}{4} \in (\frac{1}{2}, 1).
\]
The nonlinearity \( N(v) \) is given by
\[
N(v) := -x F^{-1} \left( D_y^2 - D_y G (D_x - \frac{1}{2}) - GD_y (D_x + \frac{1}{2}) + G (D_x + \frac{3}{2}) G (D_x - \frac{1}{2}) \right) x F (D_x + \frac{3}{2}) F (D_x - \frac{1}{2}) \right) \left( D_y G - G (D_x + \frac{3}{2}) G (D_x + \frac{1}{2}) F \right) + \frac{3}{8} x + q(D_x) v + D_y^2 r(D_x) v + D_y^4 v
\]
and is super-linear in \( D^\ell v \), where \( \ell \in \mathbb{N}_0^2 \) with \( 1 \leq |\ell| \leq 4 \). We postpone a precise characterization of its algebraic structure to later sections (cf. §4) and first concentrate on the characterization of the linear operator
\[
x \partial_t + L = x \partial_t + \mathcal{L} (x, \partial_x, \partial_y) := x \partial_t + q(D_x) + D_y^2 r(D_x) \Delta_y + D_y^4.
\]

1.7. The linearized evolution and loss of regularity at the free boundary. In this section, we develop a heuristic understanding of the properties of the linear equation \((x \partial_t + L) u = f\), that is (cf. (1.18)), we study the inhomogeneous linear problem

\[
x \partial_t v + q(D_x) v + D_y^2 r(D_x) v + D_y^4 v = f \quad \text{for} \quad (t, x, y) \in (0, \infty)^2 \times \mathbb{R},
\]

for a given right-hand side \( f \). It appears to be convenient to apply the Fourier transform in the variable \( y \), that is, we set
\[
\hat{v}(t, x, \eta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta y} v(t, x, y) \, dy.
\]

Thus, problem (1.19) is transformed into
\[
x \partial_t \hat{v} + \hat{q}(D_x) \hat{v} - x^2 \eta^2 r(D_x) \hat{v} + x^4 \eta^4 \hat{v} = \hat{f} \quad \text{for} \quad (t, x, \eta) \in (0, \infty)^2 \times \mathbb{R},
\]

and \( \eta \neq 0 \) yields the higher-order corrections and the equation is mainly dominated by the linear operator \( \hat{q}(D_x) \). The kernel of this operator is given by
\[
\ker q(D_x) = \text{span}\{x^{-\gamma} : \gamma \text{ is a root of } q(D_x)\} = \text{span}\{x^{-\frac{1}{2}}, x^{-\frac{1}{2} - \beta}, x^\delta, x^{1+\beta}\},
\]
so that one may expect respective powers to also appear in the solution. Yet, note that from (1.15a), since \( \beta = \frac{1}{4} \) \( \eta^2 q_0 = \frac{1}{4} \), the roots \(-\frac{1}{2}\) and \(\frac{1}{2} - \beta\) of \( q(\zeta) \) are negative and therefore the powers \( x^{-\frac{1}{2}}, x^{\frac{1}{2} - \beta}\) cannot appear in the expansion of the solution near \( x = 0 \) (otherwise, because of \( Z = Z_{TW} + v \), the contact line would be undefined). Therefore, a linear combination of \( x^\delta \) and \( x^{1+\beta} \) is necessary and because of the addend \( x^\delta \partial_x \hat{v} \) a fixed-point iteration (and subsequently undoing the Fourier transform in \( y \), cf. (1.20)) will give
\[
D^\ell \hat{v}(t, x, y) = D^\ell \left( v_0(t, y) + v_1(t, y) x + v_1 + \beta(t, y) x^{1+\beta} + v_2(t, y) x^2 \right) + \ell x^{2+\beta}
\]
as \( x \searrow 0 \), where we have defined the boundary values (or traces)
\[
v_0(t, y) := \lim_{x \searrow 0} v(t, x, y),
\]
\[
v_1(t, y) := \lim_{x \searrow 0} x^{-1} (v(t, x, y) - v_0(t, y)),
\]
\[
v_1 + \beta(t, y) := \lim_{x \searrow 0} x^{-1-\beta} (v(t, x, y) - v_0(t, y) - v_1(t, y) x),
\]
\[
v_2(t, y) := \lim_{x \searrow 0} x^{-2} (v(t, x, y) - v_0(t, y) - v_1(t, y) x - v_1 + \beta(t, y) x^{1+\beta}),
\]
and where \( \delta \in (0, 2 \beta - 1) \) will be fixed later, \( \ell = (\ell_x, \ell_y) \in \mathbb{N}_0^2 \) with \( |\ell| := \ell_x + \ell_y \leq L + 4 \) and \( \ell_y \leq L_y + 4 \) for some fixed \( L, L_y \in \mathbb{N}_0 \), and where we use the convention \( D^\ell := D_x^{\ell_x} D_y^{\ell_y} \). The function \( v_0 = v_0(t, y) \) determines the position of the contact line, i.e., \( Z_0(t, y) := Z(t, 0, y) = -\frac{y}{2} t + v_0(t, y) \) is the \( z \)-coordinate of the contact line, so that
\[
\mathbb{R} \ni y \mapsto -\frac{y}{2} t + v_0(t, y) \in \mathbb{R}^2
\]
is the graph of the free boundary. In the analysis for the \( 1+1 \)-dimensional counterpart (1.4) by Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016) a slightly different transformation has been used, that is, perturbations \( u := F - 1 \), where \( F = Z^{-1} \), have been studied. The equation for the derivative in
Despite that we expect in view of (1.22) to apply the derivative $B_x$ to (1.24) for the expansion (1.24) is in line with the findings of Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016). However, again the equation cannot be formulated in $v_x$ only, so that we expect in view of (1.22)

$$D^j u(t, x, y) := D^j \tilde{c}_x v(t, x, y) = D^j (u_0(t, y) + u_\beta(t, y) x^\beta + u_1(t, y) x) + o(\epsilon^{1+\delta}) \quad \text{as} \quad x \searrow 0,$$

with some $\delta \in (0, 2\beta - 1)$, $|\ell| \leq L + 3$ and $\ell_y \leq L_y + 4$, where $u_0 = v_1$, $u_\beta = (1 + \beta)^{-1} v_{1+\beta}$, and $u_1 = \frac{1}{2} v_2$. Expansion (1.24) is in line with the findings of Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016) for the 1+1-dimensional case.

We further notice that by inserting expansion (1.22) into (1.21a), we obtain the following expansion for the right-hand side $f$:

$$D^j f(t, x, y) = D^j (f_1(t, y) x \pm f_2(t, y) x^2) + o(\epsilon^{2+\delta}) \quad \text{as} \quad x \searrow 0,$$

where $\delta \in (0, 2\beta - 1)$, $\ell \in \mathbb{N}_0^6$ with $|\ell| \leq L$ and $\ell_y \leq L_y$. Linear estimates will depend on condition (1.25). Note that it is nontrivial to see that the nonlinearity $N(v)$ (cf. (1.17)) meets this constraint.

In what follows we will not distinguish anymore in the notation between Fourier transformed quantities $\hat{f} = \hat{f}(t, x, \eta)$ and functions $f = f(t, x, y)$ since this will be apparent from the context and from the choice of letters $y$ versus $\eta$.

### 2. Main Results and Outline

#### 2.1. Norms.

Our main results concern existence, uniqueness, and stability of solutions $v$ to the nonlinear Cauchy problem (1.14) and control of the free boundary $\hat{c}[h > 0]$ for initial data $v^{(0)}$ with small norm $\|v^{(0)}\|_{\text{init}}$, where

$$\left\|v^{(0)}\right\|_{\text{init}}^2 := \left\|D_x v^{(0)}\right\|_{k, -\delta}^2 + \left\|q(D_x) D_x v^{(0)}\right\|_{k, \delta}^2 + \left\|\bar{q}(D_x) D_x v^{(0)}\right\|_{k, \delta+1}^2 + \left\|(D_x - 3)(D_x - 2) \tilde{q}(D_x) D_x v^{(0)}\right\|_{k, \delta+1}^2 + \left\|D^{(4)}_x v^{(0)}\right\|_{k - 2, -\delta+2}^2.$$

Here, $\tilde{q}(\zeta) := (\zeta + \frac{1}{2})(\zeta + \beta + \frac{1}{2})(\zeta - 1)(\zeta - \beta - 1)$ is a fourth-order polynomial, the integer parameters $k, \tilde{k}, \tilde{\delta}$ and the positive parameter $\delta$ fulfill $k \geq 0, \tilde{k} \geq 0, \tilde{\delta} \geq 2$, and $\delta \in (0, \frac{1}{10})$, and we have introduced the norm

$$\|v\|_{k, \alpha}^2 := \sum_{0 \leq j + j' \leq k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{-2\alpha} \left(D^j_x D^{j'}_y v\right)^2 x^{-2} dx dy \quad \text{(A.5)}$$

with $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. We will also use the shorthand $\|v\|_{\alpha} := \|v\|_{0, \alpha}$. The norm $\|\cdot\|_{\text{Sol}}$ for the solution $v$ is more involved and given by

$$\left\|v\right\|_{\text{Sol}}^2 := \int_I \left(\left\|\partial_t D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{c}_t \tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{c}_t \tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{c}_t \tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{c}_t \tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 \right) dt$$

$$+ \int_I \left(\left\|\partial_t D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k - 2, -\delta + \frac{1}{2}}^2 \right) dt$$

$$+ \int_I \left(\left\|D_x v\right\|_{k + 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k + 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k + 2, -\delta + \frac{1}{2}}^2 + \left\|\tilde{q}(D_x) D_x v\right\|_{k + 2, -\delta + \frac{1}{2}}^2 \right) dt.
where \( I \subseteq [0, \infty) \) is the closed time interval on which problem (1.14) is solved and we have \( \hat{k} \geq 3, \tilde{k} \geq 4, \hat{k} \geq 4, \) and \( \delta \in (0, \frac{1}{10}) \). The significance of these norms will become clear from the outline in §2.4 and the heuristic discussions in §3.1 and §3.2 while a rigorous justification of the corresponding estimates is the subject of §3.4. Here, we just briefly motivate their choice:

First notice that for locally integrable \( v \) with \( \|v\|_\alpha < \infty \) for some \( \alpha \in \mathbb{R} \) we necessarily have \( v = o \left( x^{\alpha + \frac{1}{2}} \right) \) as \( x \searrow 0 \) almost everywhere in \( y \) for a subsequence in \( x \). By a standard embedding, \( \|v\|_{2,\alpha} < \infty \) implies \( v = o(x^\alpha) \) as \( x \searrow 0 \) classically. In this sense, the larger \( \alpha \), the better the decay of \( v \) as \( x \searrow 0 \), that is, \( \alpha \) is linked to the regularity of \( v \) at the free boundary \( \{x = 0\} \). On the other hand, just increasing \( k \) only leads to more regularity in the bulk \( \{x > 0\} \) but this regularity is lost as \( x \searrow 0 \) because each derivative \( D_x^{(\alpha,5)} x \partial_x \) or \( D_y^{(\alpha,5)} x \partial_x \) carries the factor \( x \). Since we expect \( v \) and therefore also generic initial data \( v^{(0)} \) to have an expansion of the form (1.22), our initial-data norm \( \|v\|_{\text{init}} \) cannot have contributions \( \|v^{(0)}\|_{k,\alpha} \) with \( \alpha \geq 0 \) because \( v^{(0)} = \lim_{x \searrow 0} v^{(0)} \neq 0 \) in general. Nevertheless, we need to use larger weights \( \alpha \) as otherwise scaling-wise a control of the Lipschitz constant of \( v^{(0)} \) would be impossible. This control is necessary because only for \( v^{(0)} \) with small Lipschitz constant the function \( Z = Z(t,x,y) \) (cf. (1.13)) is strictly increasing in \( x \) and thus the transformation (1.9) is well-defined, as reflected by the occurrence of factors \( (1 + v^{(0)})^{-1} \) in the nonlinearity \( N(v) \) (cf. §4.1).

A way to allow for norms \( \|v\|_{k,\alpha} \) with larger \( \alpha \) is to apply operators \( D_x - \gamma \) to \( v \), where \( \gamma \in \{ k + \beta \ell : (k,\ell) \in \mathbb{N} \times \{0,1\} \} \), since \( x^\gamma \) spans the kernel of \( D_x - \gamma \). Indeed, it is immediate that \( D_x v = O(x) \) and \( \hat{q}(D_x)D_x v = O(x^2) \) as \( x \searrow 0 \) given expansion (1.22). Then it can be proved that \( \|v^{(0)}\|_{\text{init}} \) controls the Lipschitz constant of \( v^{(0)} \) and \( \|v\|_{\text{Sol}} \) the supremum in time of the Lipschitz constant of \( v \) (cf. Lemma 3.4 and 3.9).

2.2. Main result. For our main result we need to make the following assumptions on the number of derivatives \( \hat{k}, k, \) and \( \tilde{k} \):

**Assumptions 2.1.** The numbers \( \hat{k}, k, \) and \( \tilde{k} \) entering the definitions of the norms \( \|v\|_{\text{Sol}} \) (cf. (2.3)), \( \|v\|_{\text{init}} \) (cf. (2.1)), and \( \|v\|_{\text{th}} \) (cf. (2.19)) meet the conditions

\[
4 \leq \hat{k} \quad \text{and} \quad \max(\hat{k},18) \leq k \leq \tilde{k}.
\]

(2.4)

We remark that we can satisfy conditions (2.4) for the explicit choice \( \hat{k} = \tilde{k} = 18 \) and \( k = 4 \). Our main result reads as follows:

**Theorem 2.2** (well-posedness and stability). Suppose that \( \delta \in (0, \frac{1}{2} - \frac{1}{2}) \), the numbers \( \hat{k}, k, \tilde{k} \in \mathbb{N} \) satisfy conditions (2.4) of Assumptions 2.1, and \( I = [0, \infty) \). Then there exists \( \varepsilon > 0 \) such that if \( v^{(0)} = v^{(0)}(x,y) : (0,\infty) \times \mathbb{R} \to \mathbb{R} \) is locally integrable and satisfies \( \|v^{(0)}\|_{\text{init}} < \varepsilon \) then problem (1.14) has a unique solution \( v = v(t,x,y) : (0,\infty) \times \mathbb{R} \to \mathbb{R} \) which is locally integrable and satisfies the a-priori estimate

\[
\|v\|_{\text{Sol}} \leq C \|v^{(0)}\|_{\text{init}},
\]

where \( C = C \left( \hat{k}, k, \tilde{k}, \delta \right) > 0 \) only depends on \( \hat{k}, \tilde{k}, k, \) and \( \delta \). Furthermore, we have \( \|v(t,\cdot,\cdot)\|_{\text{init}} \to 0 \) as \( t \to \infty \), i.e., the traveling wave \( v_{\text{TW}} \equiv 0 \) is asymptotically stable.

We recall that (1.14a) is obtained from (1.1a) via the coordinate changes (1.9) and (1.13). Inverting these transformations, the above result states that we can ensure global existence and uniqueness under the condition that in a suitable norm the initial data \( h^{(0)} \equiv 0 \) is close to the traveling wave \( h_{\text{TW}}(t,z) = H(z-Vt) \) (cf. (1.8)). Moreover, as we will find in §2.3, in the original coordinates the solution \( h \) approaches the traveling wave \( h_{\text{TW}} \) as \( t \to \infty \) and in this sense, \( h_{\text{TW}} \) is asymptotically stable.

We note that the principal improvement in the above statement with respect to previous works for Navier slip and complete-wetting boundary conditions by Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016), is that we work in the physical space dimensions. Due to this, we meet the difficulty of designing appropriate norms that account for the additional tangential coordinate \( y \). This leads to the additional contributions \( D^2_{x}v(D_x v) \) and \( D^2_y v \) in the linearized evolution (1.19a) which need to be absorbed in the linear estimates (cf. 3.2), leading to a restriction of the range of weights \( \alpha \) for which the spatial part of the linear operator is coercive. As a consequence, we obtain a hierarchy of estimates that need to be combined appropriately in order to obtain sufficient control on the solution \( v \). As explained already in §1.7, the transformations of Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016) are not directly applicable in our case. In particular, the boundary value \( v_0 = v_0(t,y) \), determining
the position of the contact line \( Z_0(t, y) \) (cf. (1.9)\&(1.13)), cannot be eliminated from our problem. Furthermore, the nonlinearity \( \mathcal{N}(v) \) cannot be expressed as a sum of multilinear forms as in the works of Giacomelli, Gnann, Knüpfer, and Otto (2014); Gnann (2016), but is rather a rational function of \( D^\ell v : \ell \in \mathbb{N}_0^2, 1 \leq |\ell| \leq 4 \). This complicates the symmetry considerations and the algebraic structure of the nonlinear terms discussed in §4.1 and makes the proof of appropriate estimates for \( \mathcal{N}(v) \) (cf. §4.2) more involved.

A rather natural question in the context of parabolic problems is how the regularity of the free boundary propagates. This question has been studied by Kienzler (2016); Koch (1999); Seis (2015) for the porous-medium equation (1.3) and by John (2015); Seis (2017) in the case of the thin-film equation with linear mobility (i.e., for the case of (1.2) with \( n = 1 \)), but it has not been addressed in physical dimensions for the more realistic case \( n = 2 \) with complete-wetting boundary conditions treated here. We do not prove a regularizing effect in the tangential variables and we only prove partial regularity in the normal variables. Note that through the a-priori estimate (2.5) and finiteness of

\[
\int_I \| (D_x - 3)(D_x - 2) \tilde{q}(D_x) D_x v \|_{k+2, \alpha + \frac{3}{2}}^2 \ dt
\]

(cf. (2.3)) in conjunction with Lemma 3.8 and the fact that \( 1, x, x^{1+\beta} \) and \( x^2 \) are in the kernel of \( (D_x - 3)(D_x - 2) \tilde{q}(D_x) D_x \), we infer that the solution obeys the asymptotic expansion

\[
D^\ell v(t, x, y) = D^\ell (v_0(t, y) + v_1(t, y)x + v_{1+\beta}(t, y)x^{1+\beta} + v_2(t, y)x^2) + o(x^{2+\beta}) \tag{2.6}
\]

as \( x \searrow 0 \) almost everywhere, where we have \( \ell \in \mathbb{N}_0^2 \) satisfying the bounds \( |\ell| \leq k + 1 \) and \( \ell_y \leq k + 2 \). Here, the coefficients \( v_0, v_1, v_{1+\beta}, v_2 \) fulfill a-priori estimates in terms of \( \| v(x) \|_{\text{init}} \) in the following normed spaces (cf. Lemma 3.9):

\[
\begin{align*}
v_0 & \in BC^0 \left( [0, \infty) ; BC^\ell (\mathbb{R}) \right) \quad \text{for} \quad \ell = \min \left\{ k - 1, k - 1 \right\} ; \tag{2.7a} \\
v_1 & \in BC^0 \left( [0, \infty) ; BC^\ell (\mathbb{R}) \right) \quad \text{for} \quad \ell = \min \left\{ k - 1, k - 1 \right\} ; \tag{2.7b} \\
v_{1+\beta} & \in L^2 \left( [0, \infty) ; BC^\ell (\mathbb{R}) \right) \quad \text{for} \quad \ell = \min \left\{ k + 1, k + 1 \right\} ; \tag{2.7c} \\
v_2 & \in L^2 \left( [0, \infty) ; BC^{k+1} (\mathbb{R}) \right) . \tag{2.7d}
\end{align*}
\]

An analogous partial regularity result has been found by Giacomelli, Gnann, Knüpfer, and Otto (2014, Eq. (3.1)) and upgraded to (1.6) by Gnann (2016, Eq. (2.4)). Our expectation is that the full regularity study of Gnann (2016) can be adapted for proving the precise power expansion of the solution near the free boundary to arbitrary order in the higher-dimensional case considered here. More precisely, our expectation is that the unique solution \( v \) to the nonlinear Cauchy problem (1.14) fulfills

\[
v(t, x, y) = \sum_{(k, \ell) \in \mathbb{N}_0^2} v_{k+\beta \ell}(t, y)x^{k+\beta \ell} + O \left( x^N \right) \quad \text{as} \quad x \searrow 0 \quad \text{classically}, \tag{2.8}
\]

where \( N \in \mathbb{N} \) is arbitrary and the functions \( v_{k+\beta \ell} = v_{k+\beta \ell}(t, y) \) are (at least) continuous. On the other hand, the main new difficulties due to the introduction of the extra dimensions appear already for the setting present here, and the the extension to (2.8) following Gnann (2016) would add a second layer of technical detail. Therefore, we believe that investigating the regularizing effect in the tangential as well as normal variables is better suited for a separate future work.

We note that we expect our strategy to be applicable also for (nonphysical) higher space dimensions, at the condition of replacing the present framework based on \( L^p \)-norms by a framework using higher-integrability \( L^p \)-norms with \( p > 2 \). This is required in order to obtain the spatial regularity in the extra dimensions, for which in the current framework we use the Sobolev embedding of \( W^{k, 2} \)-spaces into \( BC^\ell \)-spaces in our nonlinear estimates. For the case of \( 1 + 2 \) dimensions, we may already directly extend our bounds to \( L^p \)-bounds via the openness of the range of exponents for which \( L^p \)-estimates are valid, as available for instance due to Kalton and Mitrea (1998, Thm. 2.5). However, for more general dimension \( d + 1 \) where \( d > 2 \), we would require to use more general Sobolev embeddings, and the full strength of Calderón-Zygmund estimates for \( q(D_x) \). Note that while the basic embeddings are available also in \( L^p \), it is not yet clear how to prove the required \( L^p \)-maximal-regularity for the linearized operator for general \( p \in (2, \infty) \) (see the study of Prüss and Simonett (2008), where this question has been addressed but not completely solved). While being mathematically challenging, studying the \( L^p \)-theory and extending our
results to general dimensions would add a further layer of technical difficulties to the current work, while not being directly motivated by a physical model. Therefore, we leave this endeavor to future work.

2.3. Transformation into the original set of variables. We reformulate the statement of Theorem 2.2 in terms of the quantities appearing in the original problem (1.1). First, due to (1.13) we have that

\[ Z(t,x,y) = Z_{TW}(t,x,y) + v(t,x,y) = x - \frac{3}{8} t + v(t,x,y). \]  

Via (1.9) we find

\[ h (t, y - \frac{3}{8} t + v(t,x,y)) = x^2 \]  

for \( t, x > 0 \) and \( y \in \mathbb{R} \), and we have seen that through (A.1), equation (1.1a) for \( h(t,y,z) \) is equivalent to (1.11), which in terms of \( v \) is re-expressed as (1.14a).

The assumed smallness condition \( \|e^{(0)}\|_{\text{init}} < \varepsilon \) at the initial time \( t = 0 \) means that \( Z(0,x,y) = x + e^{(0)}(x,y) \) is a small perturbation of the linear profile \( x \). By (2.10) this in turn means that \( h^{(0)}(y,z) = h(0,y,z) \) is a small perturbation of the travelling-wave profile \( h_{\text{TW}} \) (cf. (1.8)) at time \( t = 0 \). Indeed, using estimate (3.59a) of Lemma 3.9 we find

\[ \left\| D^\ell \nabla e^{(0)} \right\|_{BC^\ell((0,\infty),\mathbb{R})} \leq C \left\| e^{(0)} \right\|_{\text{init}} \]  

for \( 0 \leq |\ell| \leq \min\{\tilde{k} - 2, \tilde{k} - 2\} \), where \( C = C \left( \tilde{k}, \tilde{k}, \delta \right) \) is a constant depending only on \( \tilde{k}, \delta \).

The hodograph transform (1.9) is well defined due to the point-wise estimate

\[ |\nabla v(t,x,y)| < 1 \]  

for \( t, x > 0 \) and \( y \in \mathbb{R} \). Property (2.12) is a consequence of the bound \( \|e^{(0)}\|_{\text{init}} < \varepsilon \), of the a-priori estimate (2.5), and of

\[ \sup_{t \geq 0} \left\| D^\ell v_x \right\|_{BC^\ell((0,\infty),\mathbb{R})} \leq C \left\| v \right\|_{\text{sol}} \]  

for \( 0 \leq |\ell| \leq \min\{\tilde{k} - 2, \tilde{k} - 2\} \) coming from estimate (3.59b) of Lemma 3.9, and where \( C = C \left( \tilde{k}, \tilde{k}, \delta \right) \) is a constant depending only on \( \tilde{k}, \delta \).

Furthermore, because of (2.11), \( \|v(t,\cdot,\cdot)\|_{\text{init}} \leq \|v\|_{\text{sol}} \) (cf. (2.1) and (2.3)), and the a-priori estimate (2.5), we have that \( v(t,x,y) \) stays close to the linear profile \( x \) for all \( t > 0 \) and Theorem 2.2 further implies that \( \|D^\ell v_x(t,\cdot,\cdot)\|_{BC^\ell((0,\infty),\mathbb{R})} \to 0 \) as \( t \to \infty \) for \( 0 \leq |\ell| \leq \min\{\tilde{k} - 2, \tilde{k} - 2\} \) giving stability of the travelling wave because of (2.10).

The explicit computations pertaining to the remainder of this subsection concentrate on finding expansions of \( h \) and the velocity \( V = h \nabla \Delta h \) close to the free boundary and are contained in Appendix A.2. Our results can be re-expressed in terms of the original formulation as follows. If we parametrize

\[ \hat{c}(y,z) \in \mathbb{R} \times [0, \infty) : h(t,y,z) > 0 \]  

and

\[ \mathcal{C}(y,z) \in \mathbb{R} \times [0, \infty) : h(0,y,z) > 0 \]  

then we have almost everywhere

\[ h(t,y,z) = z^2 \left( \frac{1}{1 + v_1} \right)^2 - \frac{3}{2} \frac{v_1 + \beta}{(1 + v_1)^{2+\beta}} z^{\beta} - \frac{3}{2} \frac{v_2}{(1 + v_1)^2} \bar{z} + o \left( z^{1+\delta} \right) \]  

as \( \bar{z} \to 0 \),

where the coordinate \( \bar{z} \) is given in terms of the distance to the free boundary as \( \bar{z} := z - Z_0(t,y) \).

In order to compute \( Z_0(t,y) \), we may express \( \hat{c}[h > 0] \) as the solution of the following system of ODEs

\[ \dot{Y},Z(t,y) = V_0(t,Y(t,y)) \]  

for \( (t,y) \in (0,\infty) \times \mathbb{R} \),

\[ (Y,Z)(0,y) = (y,Z_0^{(0)}(y)) \]  

for \( y \in \mathbb{R} \).

where \( V_0(t,y) \), appearing also in (1.1d), can be characterized as the first term of the asymptotic expansion of the advection velocity \( V \) from (1.1c)

\[ V(t,y,Z(t,x,y)) = V_0(t,y) + V_1(t,y)x + o \left( x^{1+\delta} \right) \]  

as \( x \to 0 \)

and can be expressed in terms of the function \( v \) from Theorem 2.2 as

\[ V_0 = \begin{pmatrix} v_0^{(y)} \vspace{1em} \\ v_0^{(z)} \end{pmatrix} = \begin{pmatrix} 3 + (v_0)_y^2 \\ 8(1 + v_1)^2 \end{pmatrix} \begin{pmatrix} (v_0)_y \vspace{1em} \\ 1 \end{pmatrix}. \]
2.4. Outline. We start our study in §3 by studying the linearization (1.19a) of (1.11), in order to determine suitable norms which allow for a maximal-regularity estimate of the linear problem (1.19) that reads
\[ \|v\|_{\text{Sol}} \leq C \left( \|v^{(0)}\|_{\text{init}} + \|f\|_{\text{rhs}} \right), \tag{2.18} \]
where \( C = C \left( \bar{k}, \bar{k}, \bar{k}, \delta \right) \) is a constant depending only on \( \bar{k}, \bar{k}, \bar{k} \) and \( \delta \), and the norm \( \|\cdot\|_{\text{rhs}} \) is defined through
\[ \|f\|_{\text{rhs}}^2 := \int_I \| (D_x - 1) f \|_{k-2,-\delta+\frac{1}{2}}^2 \ dt + \int_I \left( \| \tilde{q}(D_x - 1)(D_x - 1) f \|_{k-2,-\delta+\frac{1}{2}}^2 + \| \tilde{q}(D_x - 1)(D_x - 1) f \|_{k-2,-\delta+\frac{1}{2}}^2 \right) dt + \int_I \left( \| (D_x - 4)(D_x - 3) \tilde{q}(D_x - 1)(D_x - 1) f \|_{k-2,-\delta+\frac{1}{2}}^2 + \| D_y \|^2_{k-4,-\delta+\frac{1}{2}} \right) dt. \tag{2.19} \]
Note that for estimate (2.18) to yield maximal regularity, the solution norm \( \|\cdot\|_{\text{Sol}} \) has to control 4 spatial and 1 temporal derivative more than the norm \( \|\cdot\|_{\text{rhs}} \) for the right-hand side \( f \). It is known from the theory of linear (higher-order) parabolic equations that \( L^2 \) maximal regularity follows from coercivity of the spatial part of the linear operator (cf. Mielke (1987)). For equation (1.19a) we cannot obtain such a coercivity bound for unweighted Sobolev norms, but only for weighted ones. Essentially, our elliptic estimates are based on the study of the one-dimensional operator \( q(D_x) \) in (1.19a) for which a quantitative result is given by Lemma 3.1 in §3.1 giving coercivity with respect to \( \|\cdot\|_{\alpha} \) (cf. (2.2)), for \( \alpha \in \left( -\frac{N+2}{2}, 0 \right) \), that is, \( q(D_x)v \|_{\alpha} \geq C \|v\|_{2,\alpha} \), with \( C = C(\alpha) \) depending only on \( \alpha \). Since we furthermore need to absorb the terms coming from the operators \( D_y^2(D_x) \) and \( D_y^4 \), the coercivity range is further restricted to \( \left( -\frac{1}{3}, 0 \right) \) (cf. §3.2.1) and we arrive at a maximal-regularity estimate of the form
\[ \sup_{\delta \in J} \| D_y^j v \|^2_{k,-\delta+1+j} + \int_I \left( \| \partial_t D_y^j v \|^2_{k-2,-\delta+\frac{1}{2}+j} + \| D_y^j v \|^2_{k+2,-\delta-\frac{1}{2}+j} \right) dt \leq C \left( \| D_y^j v \|^2_{k,-\delta+1+j} + \int_I \| D_y^j f \|^2_{k-2,-\delta-\frac{1}{2}+j} \right) \]
for \( \delta \in \left( 0, \frac{1}{3} \right) \), \( j \geq 0 \), \( k \geq 2 \), and where \( C = C(\delta) \) depends only on \( \delta \), but not at the free boundary \( \{ x = 0 \} \). In particular, due to the negative weight \( -\delta \in \left( -\frac{1}{3}, 0 \right) \), control of both the Lipschitz constant of \( v \) and even the spatial free boundary determined by \( v|_{|\xi| = 0} = v_0 = v_0(y) \) fails due to the scaling in \( x \) of the norms that appear above. For control of \( v_0 \) we would have to allow for weights with \( \alpha > 0 \) while for control of the Lipschitz constant \( v \), we would require \( \alpha < 1 \). Yet, these weights are excluded as a nonzero boundary value \( v_0 \) would lead to blow-up of terms appearing on the left-hand side of estimate (3.19). As a first step, we apply the operator \( D_x \) to equation (1.19a), canceling \( v_0 \) in expansion (1.22) and leading to \( D^2 v = O(x) \) as \( x \to 0 \), in which \( \ell = (\ell_x, \ell_y) \in \mathbb{N}_0^2 \), \( \ell \leq L + 4 \), and \( L_y \leq L_y + 4 \), where \( L \) and \( L_y \) are the total number of \( D \)- and \( D_y \)-derivatives, respectively, controlled for \( f \). The resulting equation is (3.20), i.e.,
\[ (x\partial_t + \tilde{q}(D_x))D_x v + D_y^2 \tilde{q}(D_x)v + D_y^4 \tilde{q}(D_x + 3)v = (D_x - 1)f, \]
where \( \tilde{q}(\zeta) \) is a fourth-order real polynomial in \( \zeta \) and \( \tilde{q}(D_x) \) has coercivity range \( (0, 1) \). Unlike in the 1+1-dimensional case, the equation does not have a closed form in \( D_x v \) and \( (D_x - 1)f \), so that additional remnant terms appear that need to be absorbed with (3.19). We consider weights \( \delta \) and \( 1 - \delta \) in this range. For weight \( 1 - \delta \) the resulting estimate is (3.37) presented at the end of §3.2.3. In order to allow for control of the norm \( \|v_0\|_{B^0(\mathbb{R}_+ \times (0,x))} \) by Sobolev embedding as in Lemma 3.9, we require to use weights \( 2 \pm \delta \). Therefore, firstly in §3.2.4 we apply the operator \( \tilde{q}(D_x - 1) \) to (3.20) leading to equation (3.39), i.e.,
\[ (x\partial_t + \tilde{q}(D_x - 1)\tilde{q}(D_x) D_x v - \eta^2 x^2\tilde{q}(D_x + 1)\tilde{r}(D_x)v + \eta^4 x^4\tilde{q}(D_x + 3)(D_x + 3)v = \tilde{q}(D_x - 1)(D_x - 1)f, \]
Now the coercivity range of \( \tilde{q}(D_x - 1) \) contains the interval \( (1, 2) \), which allows for the weight \( 2 - \delta \), and the operator \( \tilde{q}(D_x) = (D_x + \frac{1}{2}) (D_x + \beta - \frac{1}{2}) (D_x - 1) (D_x - \beta - 1) D_x \) cancels expansion (1.22) up to order \( O(x^2) \). Since equation (3.39) does not have a closed form in \( \tilde{q}(D_x) D_x v \) and \( \tilde{q}(D_x - 1)(D_x - 1)f \) as in the 1+1-dimensional case, we again need to restrict the range of admissible weights and absorb...
the remnant contributions coming from the additional terms in (3.39) by those coming from (3.37). This leads to estimate (3.44) from §3.2.4.

A third step is needed in order to reach a coercivity range including weights $2 + \delta$, which -- together with $2 - \delta$ -- are required for estimating the norm $\|v_\text{sol}\|_{BC^0(I \times (0,\infty) \times \mathbb{R})}$ as in Lemma 3.9. For this reason, we apply $(D_x - 4)(D_x - 3)$ to (3.39), and obtain equation (3.46), i.e.
\[
(x \tilde{c}_1 + \tilde{q}(D_x))(D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v - \eta^2 x^2 \tilde{r}_1(D_x)D_x v - \eta^4 x^4 \tilde{r}_2(D_x)D_x v = (D_x - 4)(D_x - 3)\tilde{q}(D_x) - 1(D_x - 1)f.
\]
In order to reach the final estimate with weight $2 + \delta$ via this equation we also make use of versions of (3.37) and (3.44) with weights $\delta$ and $1 + \delta$, respectively. These bounds will appear in (3.38), (3.45), respectively. The resulting bound is (3.51) from §3.2.5. Finally, by suitably combining the previous estimates, we obtain a maximal-regularity estimate of the form (2.18) with slightly more complicated norms. These results are summarized in §3.2.6. Then in §3.3.1 we reabsorb some of the terms in the norms, in order to simplify their form. This allows to reach (2.18) itself with norms $\|\cdot\|_{\text{ini}}$ as introduced in (2.1), $\|\cdot\|_{\text{Sol}}$ as given by (2.3), and $\|\cdot\|_{\text{rhs}}$ as defined in (2.19).

In §3.3.2 several properties of the norms and in particular embeddings and control of the coefficients $v_0$, $v_1$, and $v_{1 + \beta}$ are discussed. The reasoning mainly relies on elementary estimates, combined with elliptic-regularity estimates based on Hardy’s inequality. The embeddings are necessary to rigorously define appropriate function spaces, but also for the treatment of the nonlinear Cauchy problem (1.14). In order to lighten the presentation, many of the proofs of this section have been outsourced to Appendix B. In §3.4 the treatment of the linear problem is concluded by discussing all arguments to make estimate (2.18) rigorous and to prove existence and uniqueness for the linear problem (1.19) (cf. Proposition 3.20). This is achieved through a time-discretization procedure, relying on a thorough understanding of the resolvent equation
\[
xv + q(D_x) v - \eta^2 x^2 r(D_x) v + \eta^4 x^4 v = f \quad \text{for} \quad x > 0,
\]
which is (3.61), discussed in §3.4.1. This is done through a matching argument of solution manifolds with convenient asymptotic properties as $x \searrow 0$ and $x \to \infty$ (cf. Proposition 3.16). In particular the construction of the solution manifold with convenient asymptotics as $x \to \infty$ is quite different from the arguments of Giacomelli, Gnann, Knüpfer, and Otto (2014, §6), because of the additional terms coming from the tangential direction $\tilde{c}_v$. Note that the resolvent equation (3.61) is in essence nothing but the time-discretized linear equation (1.19a). A solution for the linear problem (1.19) is obtained by compactness in the limit in which the time step tends to zero. The bounds for the discrete case are the same as in the equations coming from the computations of, and leading to, §3.2.6, with the only exception that the continuous time derivative has to be replaced by a discrete difference quotient. This allows to prove the bounds for (1.19) with the usual time derivative in the limit. Note that our approach does not essentially rely on the time-discretization argument and a semi-group approach appears to be applicable, too (see monographs by Lunardi (1995); Pazy (1983) and the work of Mielke (1987) for this approach). Yet, firstly the mathematical ingredients of both approaches are essentially the same, namely a solid understanding of the resolvent equation (3.61) has to be obtained, and secondly also the resulting estimates will be identical.

In order to apply the maximal-regularity estimate (2.18) valid for the linearized evolution (1.19) to the actual nonlinear problem (1.14) to prove well-posedness and the a-priori estimate (2.5), we dedicate §4 to studying the nonlinearity $\mathcal{N}(v)$ defined in (1.17). We note that $\mathcal{N}(v)$ is in fact a local rational function in $x$ and $\mathcal{D} := \{D^\ell v : \ell \in \mathbb{N}_0^3, 1 \leq \ell \leq 4\}$ and super-linear in $\mathcal{D}$, too. In §4.1 we deal the algebraic structure of $\mathcal{N}(v)$ and derive suitable decompositions that in particular make the nontrivial expansion (1.25) apparent also on the level of the nonlinear problem (1.14). In §4.2 we then prove our main estimate for the nonlinearity (cf. (4.16) of Proposition 4.1), i.e.,
\[
\|\mathcal{N}(v^{(1)}) - \mathcal{N}(v^{(2)})\|_{\text{rhs}} \leq C \left(\|v^{(1)}\|_{\text{sol}} + \|v^{(2)}\|_{\text{sol}}\right) \|v^{(1)} - v^{(2)}\|_{\text{sol}},
\]
provided that $\|v^{(j)}\|_{\text{sol}} \leq c$, for constants $C = C(\tilde{k}, \hat{k}, \tilde{k}, \delta)$ and $c = c(\tilde{k}, \hat{k}, \tilde{k}, \delta)$ depending only on $\tilde{k}, \hat{k}, \tilde{k}, \delta$. The above estimate allows to establish Lipschitz continuity of $\mathcal{N}(v)$ in the norm $\|\cdot\|_{\text{rhs}}$ (appearing on the right-hand side of the maximal-regularity estimate (2.18)) with a small Lipschitz constant, if $v$ belongs to a small $\|\cdot\|_{\text{sol}}$-ball centered at 0. This allows to use (2.18) for the linearized evolution (1.19) in order to solve (1.11) and to prove the a-priori estimate (2.5) of Theorem 2.2 using the contraction-mapping theorem with the underlying norm $\|\cdot\|_{\text{sol}}$ (cf. §4.3). The uniqueness will follow by a classical argument based on the continuity of $t \mapsto \|v(t, \cdot)\|_{\text{ini}}$, as shown in Corollary 3.15 from §3.3.3.
2.5. **Notation.** If $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to [0, \infty)$ we write

(i) $f(x) = O(g(x))$ as $x \to a \in \mathbb{R} \cup \{ \pm \infty \}$ if there exist constants $C > 0$ and a neighborhood $U$ of $a$ such that for all $x \in U$ there holds $|f(x)| \leq C g(x)$.

(ii) $f(x) = o(g(x))$ as $x \to a \in \mathbb{R} \cup \{ \pm \infty \}$ if $\lim_{x \to a} |g(x)/f(x)| = 0$.

If $A, B \in \mathbb{R}$ and $\mathcal{P}$ is a set of parameters, then

(i) we will write $A \lesssim B$ if there exists a constant $C > 0$, with $C$ only depending on $\mathcal{P}$, such that $A \leq C B$. We write $S \sim B$ if $A \lesssim B$ and $B \lesssim A$.

(ii) we say that property $(X)$ holds if $A \gg B$ in case there exists a constant $C > 1$, with $C$ only depending on $\mathcal{P}$, such that $(X)$ holds whenever $A \geq CB$.

We write $\mathbb{N} := \{1, 2, 3, \ldots \}$ and we denote by $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots \}$ the set of natural numbers including 0. If $a \in \mathbb{R}$ then we denote by $[a]$ the integer part of $a$, i.e., the largest integer smaller than $a$.

For a multi-index $\alpha \in \mathbb{N}_0^d$ of the form $\alpha = (\alpha_1, \ldots, \alpha_k)$ we write $|\alpha| := \alpha_1 + \cdots + \alpha_k$. We will indicate by $\ell \in \mathbb{N}_0$ the multi-index of derivatives $\ell = (\ell_x, \ell_y)$ and in that case we write $D^\ell := D_y^{\ell_y} D_x^{\ell_x}$, as already indicated in (A.5).

For a complex number $z \in \mathbb{C}$ we write $z^*$ for its complex conjugate.

If $E_1, E_2, \ldots, E_N$ are a finite number of expressions of the form $E_i = \prod_{j=1}^N D_i^{j,j} f_i$, for $i = 1, \ldots, N$, then we write $E_1 \times E_2 \times \ldots \times E_N$ to indicate that operators $D$ within the expression $E_i$ act on everything to their right within $E_i$. Otherwise, we write $f_1, f_2$, or $f_3$ if the derivative shall act on the function $f$ only.

For derivatives we usually use the notation $\partial_1, \partial_2, \partial_3$ whenever several variables play a role in our computations, while total derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are used in order to emphasize that we deal with (in-)equalities depending of only one variable, i.e., ordinary differential equations (ODE) theory is used.

Our function space norms will be denoted by $|\cdot|, ||\cdot||, \|\cdot\|, \|\cdot\|$ if they are norms on functions depending respectively on 1, 2, or 3 of the variables $t, x, y$ appearing in our original problem.

If $\Omega \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ and $X$ is a normed vector space, we will denote by $BC^0(\Omega; X)$ the space of $f : \Omega \to X$ which are continuous and bounded, and we endow this space with the norm $\|f\|_{BC^0(\Omega; X)} := \sup_{x \in \Omega} |f(x)|$. For the case $X = \mathbb{R}$ we will denote this space simply by $BC^0(\Omega)$ and the norm notation will be either one of $\|f\|_{BC^0(\Omega)}$, $\|\|_{BC^0(\Omega)}$, $\|f\|_{BC^0(\Omega)}$ in case $\Omega$ has dimension $d = 1, 2$ or 3, respectively. For more general $l \in \mathbb{N}_0$ we denote by $BC^l(\Omega; X)$ the space of bounded continuous functions having all partial derivatives up to order $l$ bounded and continuous, with norm equal to the $BC^0$-norms of all such partial derivatives.

We will denote by $I = [0, T] \subset [0, \infty)$ a time interval starting at zero, which will appear in our norms.

If $A, B \subset \mathbb{R}^d$ we write $A \subset B$ in case there exists a compact set $K$ and an open set $U$ such that $A \subset K \subset U \subset B$.

3. **Linear theory**

3.1. **Coercivity.** Our aim is to derive suitable maximal-regularity estimates for problem (1.19) that are strong enough to control the nonlinearities. We consider a general fourth-order operator $Q(D_x)$, where $Q(\zeta)$ is a polynomial in $\zeta$ of degree $4$ with real roots $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4$. A key concept is the coercivity of $Q(D_x)$ with respect to the inner product

$$(f, g)_\alpha := \int_0^\infty x^{-2\alpha} f(x) g^*(x) \, \frac{dx}{x}.$$ 

More precisely our aim is to find $\alpha \in \mathbb{R}$ such that

$$(v, Q(D_x)v)_\alpha \geq |v|_{2,\alpha}^2 \quad \text{for} \quad v : (0, \infty) \to \mathbb{R} \text{ smooth with } |v|_{2,\alpha} < \infty,$$ 

where we have introduced the norm

$$(v)_{k,\alpha}^2 := k \int_0^\infty x^{-2\alpha} |D^k x^0 \alpha| \, \frac{dx}{x}, \quad \text{with} \quad k \in \mathbb{N}_0, \alpha \in \mathbb{R}.$$ 

In what follows, we will also write $|v|_\alpha := |v|_{0,\alpha}$. If (3.1) holds, then we say that $Q(D_x)$ is coercive with respect to $(\cdot, \cdot)_\alpha$. A sufficient criterion for this property to hold is due to Giacomelli, Gnann, Knüpfer, and Otto (2014, Prop. 5.3). We need a quantitative version of this result, stated as follows:
Lemma 3.1 (cf. Giacomelli, Gnann, Knüpfer, and Otto (2014, Prop. 5.3)). Suppose that $\alpha \in \mathbb{R}$ and $v : (0, \infty) \to \mathbb{R}$ is smooth with $|v|_{2,\alpha}^2 < \infty$, then

\[ (Q(D_x)v, v)_\alpha = (D_x - \alpha)^2 |v|_{2,\alpha}^2 + \omega(\alpha) (D_x - \alpha)|v|_{2,\alpha}^2 + Q(\alpha) |v|_{2,\alpha}^2, \]  
\[(3.3a) \]

where

\[ \omega(\alpha) := - \sum_{1 \leq j < k \leq 4} (\gamma_j - \alpha)(\gamma_k - \alpha) = 2(\sigma^2 - 3(\alpha - m)^2) \]  
\[(3.3b) \]

and

\[ m = \frac{1}{4} \sum_{j=1}^{4} \gamma_j, \quad \sigma = \sqrt{\frac{1}{4} \sum_{j=1}^{4} (\gamma_j - m)^2}. \]  
\[(3.3c) \]

In particular, the operator $Q(D_x)$ is coercive with respect to $(\cdot, \cdot)_\alpha$ if the weight $\alpha$ meets the necessary condition

\[ \alpha \in (-\infty, \gamma_1) \cup (\gamma_2, \gamma_3) \cup (\gamma_4, +\infty) \]  
\[(3.4a) \]

and additionally

\[ \alpha \in \left[ m - \frac{\sigma}{\sqrt{3}}, m + \frac{\sigma}{\sqrt{3}} \right]. \]  
\[(3.4b) \]

Since the quantitative version has not been stated by Giacomelli, Gnann, Knüpfer, and Otto (2014, Prop. 5.3), we briefly outline the reasoning once more:

Proof of Lemma 3.1. Passing to the variable $s := \ln x$ and employing the Fourier transform in $s$ (which as in our usual convention is not highlighted in the notation), we obtain

\[
(Q(D_x)v, v)_\alpha = \int_{0}^{\infty} x^{-2\alpha} v(x)Q(D_x)v^*(x) \frac{dx}{x} = \int_{-\infty}^{\infty} x^{-\alpha} v(x)Q(D_x + \alpha) x^{-\alpha} v^*(x) \frac{dx}{x} = \int_{-\infty}^{\infty} w(s)Q(\hat{\xi} + \alpha) w^*(s) ds = \int_{-\infty}^{\infty} Q(\xi + \alpha) |w(\xi)|^2 d\xi
\]

\[
= \int_{-\infty}^{\infty} \left( \xi^4 - \sum_{1 \leq j < k \leq 4} (\gamma_j - \alpha)(\gamma_k - \alpha)\xi^2 + 4 \left( \prod_{j=1}^{4} (\gamma_j - \alpha) \right)|w(s)|^2 \right) ds = \left| (D_x - \alpha)^2 v^2 \right|_{2,\alpha} - \sum_{1 \leq j < k \leq 4} (\gamma_j - \alpha)(\gamma_k - \alpha) |D_x - \alpha)|v|_{2,\alpha}^2 + \sum_{j=1}^{4} (\gamma_j - \alpha) |v|_{2,\alpha}^2.
\]

Then indeed condition (3.4a) ensures positivity of $Q(\alpha)$ and condition (3.4b) ensures non-negativity of $\omega(\alpha)$. In order to understand how the second condition (3.4b) comes up, observe

\[
\omega(\alpha) = - \sum_{1 \leq j < k \leq 4} \gamma_j \gamma_k + 12 m \alpha - 6 \alpha^2 = - \sum_{1 \leq j < k \leq 4} \gamma_j \gamma_k + 6 m^2 - 6(\alpha - m)^2
\]

\[
= \frac{1}{2} \sum_{j=1}^{4} \gamma_j^2 - \frac{1}{2} \sum_{j,k=1}^{4} \gamma_j \gamma_k + 6 m^2 - 6(\alpha - m)^2 = 2 \left( \frac{1}{4} \sum_{j=1}^{4} \gamma_j^2 - m^2 - 3(\alpha - m)^2 \right)
\]

\[
= 2(\sigma^2 - 3(\alpha - m)^2).
\]

\[ \square \]

In the specific situation of the operator $q(D_x)$ defined through (1.15a) (the left-hand side of (1.21a) neglecting the $D_y$-terms), the roots of $q(\zeta)$ are in strictly increasing order $\gamma_1 := -\frac{1}{2}, \gamma_2 := \frac{1}{2} - \beta, \gamma_3 := 0, \gamma_4 := 1 + \beta$. This gives the mean of the zeros as $m = \frac{1}{4}$ and their variance as $\sigma^2 = \frac{13}{16}$. In our case, we have from (3.4)

\[
m - \frac{\sigma}{\sqrt{3}} \approx -0.2287 < \gamma_2 (1.15a) \frac{1}{2} - \beta = \frac{3 - \sqrt{13}}{4} \approx -0.1514.
\]

On the other hand

\[
m + \frac{\sigma}{\sqrt{3}} \approx 0.7287 > \gamma_3 = 0.
\]
Therefore, we have
\[\gamma_1 < m - \frac{\sigma}{\sqrt{3}} < \gamma_2 < \gamma_3 < m + \frac{\sigma}{\sqrt{3}} < \gamma_4.\]

This implies:

Lemma 3.2. The operator \(q(D_x)\) given through (1.15a) is coercive with respect to \((\cdot, \cdot)_\alpha\) if
\[\alpha \in \left(\frac{1}{2} - \beta, 0\right) = \left(-\frac{\sqrt{3} - 3}{4}, 0\right) \cup \left(-\frac{1}{10}, 0\right).\] (3.5)

Observe that in view of (1.22), the boundary value \(v_0\) of the solution \(v\) to (1.14) is in general non-vanishing, which makes the condition \(\alpha < 0\) in (3.5) intuitive.

3.2. Heuristic treatment of the linear equation. In this part, we derive estimates for the linear equation that are sufficient in order to treat the nonlinear problem (1.14). The following arguments are based on the assumption that sufficiently regular solutions already exist on a time interval \(I = [0, T] \subseteq [0, \infty)\). A rigorous justification is based on treating the resolvent problem associated to (1.14) in conjunction with a time-discretization argument. We postpone these arguments to §3.4.

3.2.1. A basic weak estimate. We assume that \(-\delta\) belongs to the range (3.5) and test (1.21a) with \(v\) in the inner product \((\cdot, \cdot)_\delta\)
\[\langle x\partial_t v, v \rangle_{\delta} + (q(D_x)v, v)_{\delta} - \eta^2 \langle x^3 r(D_x)v, v \rangle_{\delta} + \eta^4 \langle x^4 v, v \rangle_{\delta} = (f, v)_{\delta}.\] (3.6)

We treat each term in (3.6) separately: For the first term we obtain
\[\langle x\partial_t v, v \rangle_{\delta} = \frac{1}{2} \frac{d}{dt} \|v\|_{2,\delta}^2.\] (3.7)

For the second term in (3.6) we use the coercivity of \(q(D_x)\) (cf. (3.1)) in the quantitative version (3.3) of Lemma 3.1 under the assumptions (3.5) of Lemma 3.2 as
\[\langle q(D_x)v, v \rangle_{\delta} = \left[(D_x + \delta)^2 \|v\|^2_{\delta} + \omega(-\delta) [(D_x + \delta)v]^2_{\delta} + q(-\delta) \|v\|^2_{\delta},\right.\] (3.8)
where \(\omega(-\delta)\) is defined in (3.3b) and the polynomial \(q(-\delta)\) was introduced in (1.15a). For the term in (3.6) proportional to \(\eta^2\), we have
\[-\eta^2 \langle x^3 r(D_x)v, v \rangle_{\delta} = \left[-2\eta^2 ((D_x + 1)(D_x + \frac{1}{2}) v, x^2 v + q(-\delta) v)_{\delta}^2 \right.\] (1.15b)
\[= 2\eta^2 \left((D_x + \frac{1}{2}) v, (D_x - 1) x^2 v + q(-\delta) v \right)_{\delta} \right.\]
\[= 2\eta^2 \left((D_x + \frac{1}{2}) v, (D_x + 1 + 2\delta) v \right)_{\delta} \right.\]
\[= \eta^2 \left(2 \left[(D_x + 1 + \delta) \|v\|^2_{\delta - 1} - \delta(2\delta + 1) \|v\|^2_{\delta - 1} \right.\right.\] (3.9)
where we have used the skew-symmetry of \(D_x\) with respect to \((\cdot, \cdot)_0\) and \(v, (D_x) v \rangle_{\delta - 1} = -\delta \|v\|^2_{\delta - 1}\)
(which both immediately follow through integration by parts). Finally, for the term in \(\eta^4\) we have
\[\eta^4 \langle x^4 v, v \rangle_{\delta} = \eta^4 \|v\|^2_{\delta - 2}.\] (3.10)

For the right-hand side of (3.6), we use Young’s inequality and obtain
\[((f, v)_{\delta} \leq \frac{1}{2\varepsilon} \|f\|^2_{\delta} + \frac{\varepsilon}{2} \|v\|^2_{\delta}\] for any \(\varepsilon > 0.\) (3.11)

The contribution from the last term in (3.9) will be absorbed by the last term of (3.8) and (3.10) as follows:
\[\eta^2 \delta(2\delta + 1) \|v\|^2_{\delta - 1} \leq 2 \eta^2 \left(\|v\|^2_{\delta - 2}\right) \left(\delta \|v\|^2_{\delta} + \frac{1}{2}\right) \|v\|^2_{\delta - 1}\]
\[\leq K(\delta) \left(q(-\delta) \|v\|^2_{\delta} + \eta^4 \|v\|^2_{\delta - 2}\right) \text{ for } \delta \in (0, \frac{1}{10}),\] (3.12)
where \(K(\delta) \in (0, 1)\). Estimate (3.12) follows by Young’s inequality, that is,
\[2 \eta^2 \|v\|_{\delta - 2} \left(\delta \|v\|^2_{\delta - 2}\right) \leq C \delta^2 (\delta + 1)^2 \|v\|^2_{\delta} + C^{-1} \eta^4 \|v\|^2_{\delta - 2}\] with \(C > 1\).

Then we have
\[\eta^2 \delta - C \delta^2 (\delta + 1)^2 \eta^2 \leq \left((1 - C)\delta^3 + (1 - C)\delta^2 - \left(\frac{C}{4} + 1\right) \frac{1}{8}\right)\]
and in the limit \(C \to 1\) the expression is positive for \(\delta \in (0, \frac{1}{10})\), which is more restrictive than (3.5).
We combine (3.7), (3.8), (3.9), (3.10), (3.12) and then absorb the second sum in (3.11) with $\varepsilon$ small enough depending on $\delta$. This gives

$$\frac{d}{dt} |v|_{-\frac{3}{2}}^2 + \sum_{\ell=0}^{2} \eta^{2\ell} |v|_{-\delta-\frac{1}{2}}^{2} \lesssim |f|^2$$

for $\delta \in (0, \frac{1}{10})$. (3.13)

This is a weak estimate, as the regularity gain in space is only 2 (i.e., up to two derivatives $D_x^2, D_y^2$ more compared to those acting on the right-hand side $f$), while equation (1.21a) is of order 4. Upgrading it to a strong estimate will be addressed in what follows.

3.2.2. A basic strong estimate. For upgrading (3.13) to a strong estimate, we test (1.21a) against

$$\eta^{2\ell} \langle \cdot, (-D_x - 1 - 2\ell - 2\delta)^{k-\ell} D_x^{k-\ell} v \rangle_{-\delta-\ell} =: \langle \cdot, S \rangle_{-\delta-\ell},$$

where $k \geq 2$ and $0 \leq \ell \leq k$. We obtain the following terms:

$$\langle \bar{x}\partial_x v, S \rangle_{-\delta-\ell} = \frac{1}{2} \int \eta^{2\ell} \left| D_x^{k-\ell} v \right|_{-\delta-\frac{1}{2}}^2,$$

(3.15a)

$$\langle q(D_x) v, S \rangle_{-\delta-\ell} = \eta^{2\ell} \langle q(D_x)(D_x - 1)^{k-\ell-2} v, (D_x + 1 + 2\ell + 2\delta)^2 D_x^{k-\ell} v \rangle_{-\delta-\ell} \geq \eta^{2\ell} \left( \frac{1}{2} |v|_{k+\ell+2-\delta}^2 - C |v|_{-\delta-\ell}^2 \right),$$

(3.15b)

$$-\eta^{2\ell} \langle r(D_x) v, S \rangle_{-\delta-\ell} = \eta^{2\ell+1} \langle r(D_x)(D_x + 1)^{k-\ell-1} v, (D_x + 1 + 2\ell + 2\delta)^2 D_x^{k-\ell} v \rangle_{-\delta-\ell-1} \geq \eta^{2\ell+1} \left( |v|_{k+1+\ell-\delta-1}^2 - C |v|_{-\delta-\ell-1}^2 \right),$$

(3.15c)

$$\eta^{4\ell} \langle x^4 v, S \rangle_{-\delta-\ell} = \eta^{2\ell+2} \langle (D_x + 3)^{k-\ell} v, D_x^{k-\ell} v \rangle_{-\delta-\ell-2} \geq \eta^{2\ell+2} \left( \frac{1}{2} |v|_{k+\ell-\delta-\ell-2}^2 - C |v|_{-\delta-\ell-2}^2 \right),$$

(3.15d)

where $\eta^{2\ell} (f, S)_{-\delta-\ell} \lesssim k^{1/2} \eta^{2\ell} (f, |v|_{-\delta-\ell})$ comes from interpolation estimates for intermediate terms. We do not have the coefficient $\frac{1}{2}$ in (3.15c) because $r(\cdot)$ has leading coefficient 2 (cf. (1.15b)). For the right-hand side we obtain through integration by parts and Young’s inequality

$$\eta^{2\ell} (f, S)_{-\delta-\ell} \lesssim k^{1/2} \eta^{2\ell} (f, |v|_{-\delta-\ell})$$

for $\ell = 0$, $\eta^{2\ell} (f, |v|_{-\delta-\ell})$ for $\ell = 1$,

$$\eta^{2\ell} (f, |v|_{-\delta-\ell})$$

for $\ell \in \{0, 1\}$, $\eta^{2\ell} (f, |v|_{-\delta-\ell})$ for $2 \leq \ell \leq k$,

(3.16)

where $\varepsilon > 0$ is arbitrary. By absorbing the $v$-terms of (3.16) on the left-hand side, we obtain:

$$\frac{d}{dt} \left( \eta^{2\ell} \left| D_x^{k-\ell} v \right|_{-\delta-\ell-\frac{1}{2}}^2 \right) + \sum_{\ell'=0}^{\ell} \eta^{2\ell+\ell'} \left| v \right|_{k-\ell-\ell'}^2 \lesssim \delta \eta^{2\ell} \left| D_x^{k-\ell} v \right|_{-\delta-\ell-\ell'}^2 - C |v|_{-\delta-\ell-\ell'}^2$$

for $\ell \in \{0, 1\}$, $\eta^{2\ell} \left| f \right|_{k-\ell-\delta-2}^2$ for $2 \leq \ell \leq k$,

(3.17)

where $C \gg k, \delta$ and $k \geq 2$. An induction argument, starting with the base (3.13), leads to

$$\frac{d}{dt} \left( \sum_{\ell=0}^{k-2} C \eta^{2\ell} \left| D_x^{k-\ell} v \right|_{-\delta-\ell-\frac{1}{2}}^2 \right) + \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| v \right|_{k+2-\ell-\delta-\ell}^2 \lesssim k^{1/2} \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| f \right|_{k-2-\ell-\delta-\ell}^2,$$

(3.18)

Next we return to physical space $(t, x, y)$ and define the norms $\|v\|_{k, \alpha}$ through

$$\|v\|_{k, \alpha}^2 := \sum_{0 \leq j, j' \leq k} \int_0^\infty \eta^{2j} \left| D_y^{j'} v \right|_{\alpha + \frac{j}{2}}^2 \, dy = \sum_{0 \leq j, j' \leq k} \int_0^\infty \left| D_y^{j'} D_x^{j'} v \right|_{\alpha + \frac{j}{2}}^2 \, dy = \sum_{0 \leq j, j' \leq k} \int_0^\infty \int_{(0,x) \times \mathbb{R}} x^{-2\alpha} \left( D^{j'} D_x^{j'} v \right)^2 \, dx \, dy = \sum_{0 \leq j \leq k} \int_0^\infty \int_{(0,x) \times \mathbb{R}} x^{-2\alpha} \left( D_x^j v \right)^2 \, dx \, dy \, dx \, dy,$$

where $\ell = (\ell_x, \ell_y) \in \mathbb{N}_0^2$ with $|\ell| := \ell_x + \ell_y$, $D^\ell := D_x^{\ell_x} D_y^{\ell_y}$, and $d\mu(x, y) = x^{-2} \, dx \, dy$. Again, we use the short-hand notation $\|v\|_{\alpha} := \|v\|_{0, \alpha}$. Integration in time of (3.17) multiplied with $\eta^{2\ell}$, where $j \in \mathbb{N}_0$ is
arbitrary, Fourier back transform, and a standard interpolation estimate for the norm (3.18) give
\[ \sup_{t \in I} \| D_y^j v \|_{k, -\delta - 1 + j}^2 + \int_I \| D_y^j v \|_{k, -\delta - 1 + j}^2 \, dt \lesssim_k \| D_y^j v_{t=0} \|_{k, -\delta - 1 + j}^2 + \int_I \| D_y^j f \|_{k, -2, -\delta - \frac{3}{2} + j}^2 \, dt. \]

With help of the linear equation (1.19a), we can also obtain control on the time derivative \( \partial_t v \) according to
\[ \int_I \| \partial_t D_y^j v \|_{k, -2, -\delta - \frac{3}{2} + j}^2 \, dt \lesssim_k \int_I \left( \| f \|_{k, -2, -\delta - \frac{1}{2} + j}^2 + \| q(D_x - j) D_y^j v \|_{k, -2, -\delta - \frac{1}{2} + j}^2 \right) \, dt \\
+ \int_I \left( \| D_y^j r(D_x - j) D_y^j v \|_{k, -2, -\delta - \frac{1}{2} + j}^2 + \| D_y^j D_y^j v \|_{k, -2, -\delta - \frac{1}{2} + j}^2 \right) \, dt \]
and thus obtain
\[ \sup_{t \in I} \| D_y^j v \|_{k, -\delta - 1 + j}^2 + \int_I \left( \| \partial_t D_y^j v \|_{k, -\delta - \frac{1}{2} + j}^2 + \| D_y^j v \|_{k, -\delta - \frac{1}{2} + j}^2 \right) \, dt \]
\[ \lesssim_{j,k} \sup_{t \in I} \| D_y^j v_{t=0} \|_{k, -\delta - 1 + j}^2 + \int_I \| D_y^j f \|_{k, -2, -\delta - \frac{3}{2} + j}^2 \, dt \quad \text{for} \quad \delta \in (0, \frac{1}{M}), \tag{3.19} \]

where \( j \in \mathbb{N}_0 \). While estimate (3.19) is maximal in the sense that one derivative of the first order or subsequent derivatives more are controlled for the solution \( v \) minus the right-hand side \( f \) (the maximal gain in regularity possible in view of equation (1.19a)), the estimate is too weak in order to treat the nonlinearity \( N(v) \) (cf. (1.17)). The reason for this is a loss of regularity of the employed norms on approaching the boundary \( \{ x = 0 \} \) since increasing the number of derivatives \( D_x^{(A,4)} x \partial_x \) does not change the scaling in \( x \), while taking more derivatives \( D_y^{(A,4)} x \partial_y \) deteriorates the scaling in \( x \). Hence, no matter how large we take \( k \), the norms \( \sup_{t \in I} \| D_y^j v \|_{k, -\delta - 1 + j} \) cannot control the norm \( \| \nabla v \|_{BC^0(I \times (0,\infty) \times \mathbb{R})} \) as \(-\delta - 1 < 1\) on the other hand, control of the norm \( \| \nabla v \|_{BC^0(I \times (0,\infty) \times \mathbb{R})} \) appears to be necessary in order to estimate the nonlinearity as from (1.17) we recognize that factors \( F^{(1,12),(1,13)}(1 + v_x)^{-1} \) need to be controlled. This problem can be overcome by combining parabolic estimates of the form (3.19) with elliptic regularity estimates which will be dealt with in §3.2.3, §3.2.4, and §3.2.5.

3.2.3. Higher regularity I. In view of expansion (1.22) we can only hope for estimates with negative weights for \( v \) since \( v_0 = v_x = 0 \) is not necessarily zero (unless the free boundary is a straight line) and norms \( \| v \|_\alpha \) can be infinite otherwise. Applying \( D_x - 1 \) to (1.19), we get an equation in terms of \( D_x v \) and \( v \), which is more conveniently formulated with \( \eta \), i.e., Fourier-transformed in the \( y \)-variable:
\[ (x \partial_x + \tilde{q}(D_x)) D_x v - \eta^2 x^2 \tilde{r}(D_x) v + \eta^4 x^4 (D_x + 3) v = (D_x - 1) f, \tag{3.20} \]
where \( \tilde{q}(\zeta) \) and \( \tilde{r}(\zeta) \) are monic polynomials of degrees \( 4 \) and \( 3 \), respectively:
\[ \tilde{q}(\zeta) := (\zeta + \frac{1}{2})(\zeta + \beta - \frac{1}{2})(\zeta - 1)(\zeta - \beta - 1), \tag{3.21a} \]
\[ \tilde{r}(\zeta) := 2(\zeta + 1)^2(\zeta + \frac{1}{2}). \tag{3.21b} \]

For \( D_x v \) instead of \( v \) we have indeed \( D_x v \) \( \sqrt{1} \) \( O(x) \) as \( x \to 0 \) and thus we expect to obtain maximal-regularity estimates with increased weight exponents. The subsequent reasoning is similar to the one leading from (1.21a) to (3.13), only that this time we test against \( D_x v \) instead of \( v \) in the inner product \( (\cdot, \cdot)_{\alpha} \):
\[ (x \partial_x, D_x v)_{\tilde{\alpha}} + (\tilde{q}(D_x) D_x v, D_x v)_{\tilde{\alpha}} - \eta^2 (x^2 \tilde{r}(D_x) v, D_x v)_{\tilde{\alpha}} + \eta^4 (x^4 (D_x + 3) v, D_x v)_{\tilde{\alpha}} = ((D_x - 1) f, D_x v)_{\tilde{\alpha}}. \tag{3.22} \]

This gives
\[ (x \partial_x D_x v, D_x v)_{\tilde{\alpha}} = \frac{1}{2} \frac{d}{dt} ||D_x v||_{\tilde{\alpha} - \frac{3}{2}}. \tag{3.23} \]
as in (3.7) for the first term of (3.22). The new mean \( \bar{m} \) and variance \( \bar{\sigma} \) of the zeros of \( \tilde{q}(\zeta) \), characterizing the coercivity range of \( \tilde{q}(\zeta) \) as in Lemma 3.1, are given by
\[ \bar{m} = \frac{1}{2} \quad \text{and} \quad \bar{\sigma}^2 (1.15a)(1.16) \frac{1}{2} \left( \beta^2 + \frac{1}{2} \beta + \frac{3}{4} \right) = \frac{3}{4}, \]
Then indeed for  we meeting (3.24) we have for the second term in (3.22)

\[ (q(D_x)D_x v, D_x v) \overset{(3.30)}{\leq} |D_x v|_{\tilde{\alpha}}^2. \]  

For the third term in (3.22) we obtain through integration by parts and interpolation

\[ x(D_x + 3) v, D_x v) \overset{(3.21b)}{\geq} \frac{1}{2} |D_x v|_{\tilde{\alpha} - 2}^2 - C |v|_{\tilde{\alpha} - 2}^2, \]  

where  is sufficiently large. By interpolation furthermore

\[ (x^4(D_x + 3) v, D_x v) \overset{(3.31d)}{\geq} \frac{1}{2} |D_x v|_{\tilde{\alpha} - 2}^2 - C |v|_{\tilde{\alpha} - 2}^2. \]  

We let  for  with criterion (3.4) we infer that  is coercive with respect to if

\[ \tilde{\alpha} \in \left( \frac{1}{2} - \beta, 1 \right) \cap \left[ \tilde{m} - \frac{\tilde{\sigma}}{\sqrt{3}} \tilde{m} + \frac{\tilde{\sigma}}{\sqrt{3}} \right] = [0, 1). \]  

Then we use the right-hand side of (3.22) we may use the elementary

\[ (D_x - 1)f, D_x v) \leq \frac{1}{2\epsilon} |(D_x - 1)f|_{\tilde{\alpha}}^2 + \epsilon |D_x v|_{\tilde{\alpha}}^2, \]  

for  > 0. Now we combine (3.23), (3.25), (3.26), (3.27), and (3.28) in (3.20) tested against  in the inner product  and get by absorption of the terms by  in (3.28) and interpolation

\[ \frac{d}{dt} |D_x v|_{\tilde{\alpha} - \frac{1}{2}}^2 + |D_x v|_{\tilde{\alpha} - \frac{1}{2}}^2 + \eta^2 |D_x v|_{\tilde{\alpha} - 1}^2 + \eta^4 |D_x v|_{\tilde{\alpha} - 2}^2 \lesssim |(D_x - 1)f|_{\tilde{\alpha}}^2 + \epsilon |D_x v|_{\tilde{\alpha}}^2. \]  

We let  \( k \geq 2 \),  \( 0 \leq \ell \leq k \), and test equation (3.20) against

\[ T := \eta^{2\ell}(-D_x - 1 - 2\tilde{\alpha})^{k-\ell}D_x^{k+1-\ell}v \]  

in the  inner product, giving

\[ \eta^{2\ell}((D_x^\ell) v, T)_{\tilde{\alpha} + \frac{1}{2}} + \eta^2 ((D_x^\ell) v, T)_{\tilde{\alpha} - 1} + 2\eta^{2(\ell+1)}((D_x^\ell) v, T)_{\tilde{\alpha} - 2} + \eta^{2(\ell+2)}((D_x^{\ell+1}) v, T)_{\tilde{\alpha} - 2} \]

Then the terms appearing in (3.30) can be simplified according to

\[ \eta^{2\ell}((D_x^\ell) v, T)_{\tilde{\alpha} - 1} \geq \eta^{2\ell}((D_x^{\ell+1}) v, T)_{\tilde{\alpha} + 1}, \]  

for  \( \ell \geq 2 \). Now we insert (3.31a) and (3.32a), with  > 0 sufficiently small, into (3.30) and arrive at

\[ \frac{d}{dt} \left( \eta^{2\ell} |D_x^{k+1-\ell}v|_{\tilde{\alpha} - \frac{1}{2}}^2 \right) + \sum_{\ell = 0}^2 \eta^{2(\ell+\ell')} \left( |D_x v|_{\tilde{\alpha} - \ell-\ell'}^2 \right) \lesssim \eta^2 |f|_{\tilde{\alpha} - \ell, \tilde{\alpha} - \ell}^2, \]  

for  \( \ell \in \{0, 1\} \).

\[ \frac{d}{dt} \left( \eta^{2\ell} |D_x^{k+2-\ell}v|_{\tilde{\alpha} - \frac{1}{2}}^2 \right) + \sum_{\ell = 0}^2 \eta^{2(\ell+\ell')} \left( |D_x v|_{\tilde{\alpha} - \ell-\ell'}^2 \right) \lesssim \eta^2 |f|_{\tilde{\alpha} - \ell, \tilde{\alpha} - \ell}^2, \]  

for  \( 0 \leq \ell \leq k \).
Then we infer inductively from (3.29) and (3.33) that
\[
\frac{d}{dt} \left( \sum_{\ell=0}^{k} \hat{C}_\ell \eta^{2\ell} \left| \tilde{D}_x^{k+1-\ell} v \right|^2 \right) + \sum_{\ell=0}^{k+2} \eta^{2\ell} \left| D_x v \right|^2 \lesssim \frac{1}{\alpha - \tilde{\alpha} - \ell} \left( \sum_{\ell=0}^{k} \hat{C}_\ell \eta^{2\ell} \left| \tilde{D}_x^{k+1-\ell} v \right|^2 \right) + \sum_{\ell=0}^{k+2} \eta^{2\ell} \left| D_x v \right|^2 \lesssim \frac{1}{\alpha - \tilde{\alpha} - \ell} \left( \sum_{\ell=0}^{k} \hat{C}_\ell \eta^{2\ell} \left| \tilde{D}_x^{k+1-\ell} v \right|^2 \right) + \sum_{\ell=0}^{k+2} \eta^{2\ell} \left| D_x v \right|^2 ,
\]
with constants \( \hat{C}_\ell > 0 \) and \( \tilde{\alpha} \in [0, 1] \). Passing to physical space \((t, x, y)\) by Fourier back transforming in \( \eta \), we get with the same steps as those leading from (3.17) to (3.19)
\[
\sup_{t \in I} \left( \| \tilde{D}_y^j D_x v \|_{k,\tilde{\alpha} - 1 + j}^2 + \int_I \left( \| \hat{\eta}_i \tilde{D}_y^j D_x v \|_{k,\tilde{\alpha} - \frac{3}{2} + j}^2 + \| \tilde{D}_y^j D_x v \|_{k,\tilde{\alpha} - \frac{1}{2} + j}^2 \right) dt \right) \lesssim \tilde{k}, \tilde{\alpha} \left( \| \tilde{D}_y^j D_x v \|_{k,\tilde{\alpha} - 1 + j}^2 + \int_I \left( \| \hat{\eta}_i \tilde{D}_y^j (D_x - 1) f \|_{k,\tilde{\alpha} - \frac{3}{2} + j}^2 + \| \tilde{D}_y^j (D_x - 1) f \|_{k,\tilde{\alpha} - \frac{3}{2} + j}^2 \right) dt \right) \quad (3.35)
\]
where \( \tilde{j} \in \mathbb{N}_0 \) is arbitrary. Observe that we can absorb the remnant terms, i.e.,
\[
\sum_{\ell=1}^{2} \int_I \left( \| D_y^j \bar{v} \|_{\bar{\alpha} + \frac{1}{2} + j}^2 dt \right) \quad (3.36)
\]
in the second line of (3.35), by those appearing on the left-hand side of (3.19) if \( \bar{\alpha} = 1 - \delta \), \( \alpha \) is in the intersection of \((-1, 0)\) (cf. (3.19)) and the range (3.24) (which simply leads to \( \delta \in (0, \frac{1}{4}) \) again), and if \( j = j + 1 \). Then adding a large multiple of (3.19) to (3.35) to eliminate (3.36) (where for convenience we make the choice \( k := \tilde{k} - 1 \), we obtain a higher-order maximal-regularity estimate of the form
\[
\sup_{t \in I} \left( \| D_y^j \bar{v} \|_{k, - \delta - 1 + j}^2 + \| D_y^{j-1} D_x \bar{v} \|_{k, - \delta - 1 + j}^2 \right) + \int_I \left( \| \hat{\eta}_i D_y^j \bar{v} \|_{k, - 3, - \delta - 1 + j}^2 + \| \tilde{D}_y^j (D_x - 1) f \|_{k, - 3, - \delta - 1 + j}^2 \right) dt \lesssim \tilde{k}, \tilde{\alpha} \left( \| D_y^j \bar{v} \|_{k, - \delta - \frac{1}{2} + j}^2 + \| D_y^{j-1} (D_x - 1) f \|_{k, - \delta - \frac{1}{2} + j}^2 \right) dt \quad (3.37)
\]
for \( \delta \in (0, \frac{1}{4}) \), where \( \tilde{k} \geq 3 \) and \( j \geq 1 \). Note that throughout the paper we use without mention the equivalence \( \sup a + \sup b = \sup(a + b) \) valid if \( a, b \geq 0 \), which leads to a correction of our estimates by universal constants. Estimate (3.37) gives stronger control of the solution \( \bar{v} \) in terms of the initial data \( \bar{v}_{t=0} \) and the right-hand side \( f \), as the weight \( \alpha \) has increased by \( +1 \). Yet, estimate (3.37) is still insufficient to treat the nonlinear problem (1.14). The reason is that the scaling of neither \( \sup_{t \in I} \| D_y^j \bar{v} \|_{k, - \delta - 1 + j} \) nor \( \sup_{t \in I} \| D_y^{j-1} D_x \bar{v} \|_{k, - \delta - 1 + j} \) is sufficient in order to get control of \( \| \nabla \bar{v} \|_{BCO(I \times (0, \mathbb{R}^3))} \) (for details we refer to the discussion of such scalings at the end of §2.1). This requires to apply another polynomial in \( D_x \) to (3.20), which will be the subject of §3.2.4.

Estimate (3.37) is \( \delta \)-sub-critical with respect to the addend \( v_x \) in expansion (1.22) for \( v \) while estimate (3.19) is \( \delta \)-sub-critical with respect to the addend \( v_0 \) of \( v \) in (1.22). For convenience for the subsequent reasoning, we will also use an estimate which is \( \delta \)-super-critical with respect to \( v_0 \). Therefore, we use, in addition to the previous choices, also the values \( \tilde{\alpha} = \delta, \tilde{k} = k, \) and \( \tilde{j} = j - 1 \) where \( j \geq 1 \) in (3.35), so that in view of (3.36) the remnant contributions are \( \int_I \| D_y^j \bar{v} \|_{\bar{\alpha} - \frac{1}{2}}^2 dt \) with \( \ell = 1, 2 \). By the interpolation of the norm (3.18) we have the interpolation estimate
\[
\| D_y^{j+1-\ell} v \|_{\bar{\alpha} - \frac{1}{2} + j} \lesssim \| D_y^{j+\ell-1} v \|_{\bar{\alpha} - \frac{3}{2} + j} + \| D_y^{j+\ell-1} v \|_{\bar{\alpha} - \frac{1}{2} + j} .
\]
Then we notice that \( \int_I \| D_y^j \bar{v} \|_{\bar{\alpha} - \frac{1}{2}}^2 dt \) can be absorbed by \( \int_I \| D_y^{j-1} \bar{v} \|_{k+2, - \delta - \frac{1}{2} + j}^2 dt \) appearing on the left-hand side of (3.19) with \( j \) replaced by \( j - 1 \) while \( \int_I \| D_y^j \bar{v} \|_{\bar{\alpha} - \frac{1}{2} + j}^2 dt \) can be absorbed by \( \int_I \| D_y^j \bar{v} \|_{k+2, - \delta - \frac{1}{2} + j}^2 dt \).
appearing in (3.19) as well where \( j \) is the same and \( k \) is replaced by \( \bar{k} - 1 \). By combining these estimates, we find an inequality of the same form as (3.37) for the exponent \( \bar{\alpha} = \delta \):

\[
\sup_{t \in I} \left( \left\| D_{y}^{-1} v \right\|_{k, \bar{k} - 2 + j}^2 + \left\| D_{y}^{-1} D_{x} v \right\|_{k, \bar{k} - 2 - \frac{j}{2}}^2 + \left\| D_{y} D_{x} v \right\|_{k, \bar{k} - 2 - 1 - j}^2 \right) \\
+ \int_{I} \left( \left\| \partial_{1} D_{y}^{-1} v \right\|_{k, \bar{k} - 2, \bar{k} - \frac{j}{2}}^2 + \left\| \partial_{1} D_{y}^{-1} D_{x} v \right\|_{k, \bar{k} - 2, \bar{k} - \frac{j}{2} - \frac{1}{2}}^2 + \left\| \partial_{1} D_{y} D_{x} v \right\|_{k, \bar{k} - 2, \bar{k} - \frac{j}{2} - \frac{1}{2} + j}^2 \right) dt \\
+ \int_{I} \left( \left\| D_{y}^{-1} v \right\|_{k+1, \bar{k} - 2, \bar{k} - \frac{j}{2} - \frac{1}{2}}^2 + \left\| D_{y}^{-1} D_{x} v \right\|_{k+1, \bar{k} - 2, \bar{k} - \frac{j}{2} - \frac{1}{2} + j}^2 + \left\| D_{y} D_{x} v \right\|_{k+1, \bar{k} - 2, \bar{k} - \frac{j}{2} - \frac{1}{2} + 2j}^2 \right) dt \leq \\
\bar{k}, \bar{k} \cdot \delta \left\| D_{y}^{-1} v \right\|_{(t-0)^2, k, \bar{k} - 2 + j}^2 + \left\| D_{y}^{-1} D_{x} v \right\|_{(t-0)^2, k, \bar{k} - 2 - \frac{j}{2}}^2 + \left\| D_{y} D_{x} v \right\|_{(t-0)^2, k, \bar{k} - 2 - 1 - j}^2 \\
+ \int_{I} \left( \left\| D_{y}^{-1} f \right\|_{k, \bar{k} - 2, \bar{k} - \frac{j}{2} + j}^2 + \left\| D_{y}^{-1} (D_{x} - 1) f \right\|_{k, \bar{k} - 2, \bar{k} - \frac{j}{2} + j}^2 + \left\| D_{y} f \right\|_{k, \bar{k} - 2, \bar{k} - \frac{j}{2} + j}^2 \right) dt,
\]

(3.38)

where \( \delta \in (0, \frac{1}{3}) \), \( k \geq 2 \), \( \bar{k} \geq 3 \), and \( j \geq 1 \).

3.2.4. Higher regularity II. Next, we apply \( \bar{q}(D_{x} - 1) \) to (3.20):

\[
(x \partial_{t} + \bar{q}(D_{x} - 1)) \bar{q}(D_{x}) D_{x} v - \eta^{2} x^{2} \bar{r}_{1}(D_{x}) D_{x} v + \eta^{4} x^{4} \bar{r}_{2}(D_{x}) v = \bar{q}(D_{x} - 1)(D_{x} - 1) f,
\]

(3.39)

where in view of (3.21) the sixth-order operator \( \bar{r}_{2}(D_{x}) \) is defined through the polynomial

\[
\bar{r}_{1}(\zeta) = \zeta^{-1} \bar{q}(\zeta + 1) \bar{r}(\zeta) = 2 \left( \zeta + \frac{3}{2} \right) \left( \zeta + 1 \right)^{2} \left( \zeta + \frac{1}{2} \right) \left( \zeta + \beta + \frac{1}{2} \right) \left( \zeta - \beta \right)
\]

(3.40a)

and the fifth-order operator \( \bar{r}_{2}(D_{x}) \) is defined through

\[
\bar{r}_{2}(\zeta) = \bar{q}(\zeta + 3)(\zeta + 3) \left( 3 \bar{r}_{1}(\zeta) \right) \left( \zeta + \frac{5}{2} \right) \left( \zeta + \beta + \frac{3}{2} \right) \left( \zeta + 3 \beta + 2 \right).
\]

(3.40b)

The motivation for this step is that in view of expansion (1.22) we expect \( \bar{q}(D_{x}) D_{x} v(x) = O(x^{2}) \) as \( x \to 0 \), as the powers \( x \) and \( x^{1+\beta} \) are in the kernel of \( \bar{q}(D_{x}) \) (cf. (3.21a)). This enables us to obtain estimates in norms with larger weights and therefore stronger control at the free boundary \( \{ x = 0 \} \).

We test (3.39) with \( \bar{q}(D_{x}) D_{x} v \) in the inner product \( (\cdot, \cdot)_{\bar{\alpha}} \), where \( \bar{\alpha} \) is in the coercivity range of \( \bar{q}(D_{x} - 1) \). The latter is the coercivity range of \( \bar{q}(D_{x}) \) shifted by +1, that is, \( \bar{\alpha} \in (1, 2) \). This gives similarly as before in (3.29)

\[
\frac{d}{dt} \left| \bar{q}(D_{x}) D_{x} v \right|_{\bar{\alpha} - \frac{1}{2}}^{2} + \left| \bar{q}(D_{x}) D_{x} v \right|_{\bar{\alpha} - \frac{3}{2}}^{2} + \eta^{2} \left| D_{x} v \right|_{\bar{\alpha} - 1}^{2} + \eta^{4} \left| v \right|_{\bar{\alpha} - 2}^{2}
\]

(3.41)

The only difference in the argumentation leading to (3.41) compared to the reasoning before (3.29) is the second but last term \( \left| D_{x} v \right|_{\bar{\alpha} - 1}^{2} \) in the second line, which can be obtained as in (3.26) noting that \( \bar{q}(\zeta + 1) \) has a zero in \( \zeta = 0 \). With the same reasoning as before, we can upgrade the weak estimate (3.41) to

\[
\frac{d}{dt} \left( \sum_{\ell = 0}^{k} C_{\ell} \eta^{2\ell} \left| D_{x}^{2-\ell} \bar{q}(D_{x}) D_{x} v \right|_{\bar{\alpha} - \frac{1}{2}, \bar{\alpha} - \ell}^{2} + \left| \bar{q}(D_{x}) D_{x} v \right|_{\bar{\alpha} - \frac{3}{2}, \bar{\alpha} - \ell}^{2} \right) + \sum_{\ell = 0}^{k+2} \eta^{2\ell} \left| \bar{q}(D_{x}) D_{x} v \right|_{\bar{\alpha} - \frac{1}{2}, \bar{\alpha} - \ell}^{2}
\]

(3.42)

with constants \( C_{\ell} > 0 \) and \( \bar{\alpha} \in (1, 2) \). Then from the above we obtain a strong estimate of the form:

\[
\sup_{t \in I} \left| \bar{q}(D_{x}) D_{x} v \right|_{\bar{\alpha} - 1 + j}^{2} + \int_{I} \left( \left| \partial_{1} \bar{q}(D_{x}) D_{x} v \right|_{k - 2, \bar{\alpha} - 2 - \frac{j}{2}}^{2} + \left| \bar{q}(D_{x} - 1) D_{x} v \right|_{k - 2, \bar{\alpha} - 1 + j}^{2} \right) dt
\]

\[
\leq \left| \bar{q}(D_{x}) D_{x} v \right|_{(t-0)^{2}, k, \bar{k} - 2 + j}^{2} + \int_{I} \left| \bar{q}(D_{x} - 1) (D_{x} - 1) f \right|_{k - 2, \bar{\alpha} - 1 + j}^{2} dt
\]

(3.43)

The last line in (3.43) contains the renmant terms, which can be absorbed by adding a large multiple of (3.37) provided \( j = j + 2 \) and \( \bar{\alpha} = -\delta + 2 \) with \( \delta \in (0, \frac{1}{3}) \) (where for convenience we choose \( \bar{k} := \bar{k} - 1 \)).
Thus, estimate (3.43) upgrades to

$$\sup_{t \in I} \left( \left\| D_y^j v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2 \right)$$

$$+ \int_I \left( \left\| \partial_t D_y^j v \right\|_{k-1,\delta-1+j}^2 + \left\| \partial_t D_y^{j-1} D_x v \right\|_{k-1,\delta-1+j}^2 + \left\| \partial_t D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2 \right) dt$$

$$+ \int_I \left( \left\| D_y^j D_x v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2 \right) dt$$

$$\lesssim_{k,\delta} \left\| D_y^{j-1} v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2$$

for $\delta \in (0, \frac{1}{10})$, where $k \geq 4$ and $j \geq 2$.

Estimate (3.44) is $\delta$-sub-critical with respect to the addend $v_2 x^2$ in expansion (1.22). For later purposes we also choose the weight $\tilde{\alpha} = \delta + 1$ in (3.43) being $\delta$-super-critical with respect to the addend $v_1 x$ in (1.22). Choosing $\bar{k} := k$ and $\bar{j} := j - 2$, where $j \geq 2$, it then suffices to bound the remnant terms $\int_I \left\| D_y^{j+\epsilon-2} D_x v \right\|_{k-\frac{3}{2}+\epsilon}^2 dt$, where $\epsilon = 1, 2$. We now use that $\int_I \left\| D_y^{j+\epsilon-2} D_x v \right\|_{k-\frac{3}{2}+\epsilon}^2 dt$ can be absorbed by $\int_I \left\| D_y^{j-1} v \right\|_{k-1,\delta-\frac{3}{2}+j}^2 dt$ appearing on the left-hand side of (3.38), where $k$ is replaced by $\bar{k} - 1$ and $\bar{j}$ is replaced by $\bar{j} - 1$. By combining these estimates, we find the following bounds for the case $\alpha = \delta + 1$

$$\sup_{t \in I} \left( \left\| D_y^j v \right\|_{k-1,\delta-2+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-1,\delta-2+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-2+j}^2 \right)$$

$$+ \sup_{t \in I} \left( \left\| D_y^j v \right\|_{k-2,\delta-1+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-2,\delta-1+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2 \right) dt$$

$$+ \int_I \left( \left\| \partial_t D_y^j v \right\|_{k-2,\delta-1+j}^2 + \left\| \partial_t D_y^{j-1} D_x v \right\|_{k-2,\delta-1+j}^2 + \left\| \partial_t D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2 \right) dt$$

$$+ \int_I \left( \left\| D_y^j D_x v \right\|_{k-2,\delta-1+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-2,\delta-1+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2 \right) dt$$

$$\lesssim_{\bar{k},\bar{j},\delta} \left\| D_y^{j-1} v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-1} D_x v \right\|_{k-1,\delta-1+j}^2 + \left\| D_y^{j-2} \tilde{q}(D_x) D_x v \right\|_{k,\delta-1+j}^2$$

for $\delta \in (0, \frac{1}{10})$, $\bar{k} \geq 3$, $\bar{j} \geq 4$, and $\bar{j} \geq 2$.

3.2.5. Higher regularity III. Estimate (3.45) is still insufficient as the scaling of the norms in the first three lines is sub-critical for controlling $\left\| \nabla v \right\|_{BC^0(I \times (0,\infty) \times \mathbb{R})}$. Hence, we apply $(D_x - 4)(D_x - 3)$ to equation (3.39) leading us to

$$(x \partial_t + \tilde{q}(D_x))(D_x - 3)(D_x - 2)\tilde{q}(D_x) D_x v - \eta^2 x^2 \tilde{r}_1(D_x)(D_x - 1) D_x v + \eta^4 x^4 \tilde{r}_2(D_x) D_x v$$

$$= (D_x - 4)(D_x - 3)\tilde{q}(D_x - 1)(D_x - 1)f,$$

where in view of (3.21b)

$\tilde{r}_1(\zeta) = (\zeta - 2)\tilde{r}_1(\zeta)$

(3.40a)

$$= \frac{2}{\zeta + \frac{3}{2}} (\zeta + 1)^2 \left( \zeta + \frac{1}{2} \right) (\zeta + \beta + \frac{1}{2}) (\zeta - \beta) (\zeta - 2),$$

$$\tilde{r}_2(\zeta) = (\zeta + 1)\tilde{r}_2(\zeta)$$

(3.21a)(3.40b)

$$= \left( \zeta + \frac{7}{2} \right) (\zeta + \beta + \frac{5}{2}) (\zeta + 3) (\zeta + 2) (\zeta - \beta + 2) (\zeta + 1)$$

(3.47b)
are polynomials of order 7 and 6, respectively, and where we have introduced the operator \( \hat{q}(D_x) \) fulfilling
\[(D_x - 4)(D_x - 3)\hat{q}(D_x - 1) = \hat{q}(D_x)(D_x - 3)(D_x - 2), \]
i.e., the root 2 of \( \hat{q}(\zeta - 1) \) (cf. (3.21a)) is replaced by 4 and therefore \( \hat{q}(\zeta) \) is given by
\[
\hat{q}(\zeta) = \left( \zeta - 1 \right) \left( \zeta + \frac{3}{2} \right) \left( \zeta - 2 \right) \left( \zeta - 4 \right).
\] (3.47c)

Our motivation for applying the operator \( (D_x - 4)(D_x - 3) \) is that we want the resulting operator \( \hat{q}(D_x) \) to be coercive with respect to a weight which is \( \delta \)-supercritical with respect to the term \( v_2x^2 \)
from expansion (1.22). We are led to prefer the weight \( \delta + 2 \) since remnant contributions can be easily absorbed using estimates from the previous steps (see below).

Note that applying operators \( D_x - \gamma \), where \( \gamma - 1 \) is a root of \( \hat{q}(\zeta) \), preserves the structure of the \( \eta^0 \)-terms on the left-hand side of (3.39). Since the root 2 is the third-largest root of \( \hat{q}(\zeta - 1) \) and by criterion (3.4a) of Lemma 3.1 limits the coercivity range, we are lead to apply \( D_x - 3 \) to (3.39), resulting in the linear operator
\[(D_x - \frac{1}{2}) (D_x + \frac{3}{2}) (D_x - 2) (D_x - 3) \]
acting on \( v \) in the \( \eta^0 \)-term. This operator has coercivity range containing a neighborhood around 2. It also produces a factor \( D_x - 1 \) acting on \( D_xv \) in the \( \eta^2 \)-term, so that the fist two terms \( v_0 \) and \( v_1x \) in expansion (1.22) are canceled and in a weak estimate with weight \( \delta + 2 \), where \( \delta > 0 \), the resulting term remains finite. However, in the resulting equation the \( \eta^4 \)-term has no \( D_x \) acting on \( v \), which because of the boundary value \( v_0 \) (cf. (1.22)) would lead to a blow-up in a weak estimate with weight \( \delta + 2 \) where \( \delta > 0 \). For this reason, we need the operator to be applied to (3.39) to have a factor \( (D_x - 4) \), so that after commuting with \( x^4 \), the operator \( D_x \) acts on \( v \) in the \( \eta^4 \)-term.

Further note that the operator \( (D_x - 3)(D_x - 2)\hat{q}(D_x)D_x \) acting on \( v \) (cf. (3.21a)) does not cancel the addend \( O(x^{1+2\beta}) \) from (1.22), which as it turns out later, also cannot be excluded via the nonlinear theory (cf. §4.1). However, the weight \( \delta + 2 \) which we introduce for \( \delta \in (0, \frac{1}{10}) \) as before, verifies \( \delta + 2 < 1 + 2\beta \) in view of (1.16), thus also avoids the aforementioned \( O(x^{1+2\beta}) \)-terms.

It remains to verify coercivity of \( \hat{q}(D_x) \) employing criterion (3.4) of Lemma 3.1. We find that the mean of the zeros of \( \hat{q}(\zeta) \) is given by \( \bar{m} = 2 \) and the root of the variance fulfills \( \bar{\sigma} > \frac{3}{4} > \sqrt{\frac{3}{10}} \), so that indeed weights \( 2 + \delta \) with \( \delta \in (0, \frac{1}{10}) \) are admissible.

Testing (3.46) with \( (D_x - 3)(D_x - 2)\hat{q}(D_x)D_x \) in the inner product \( \langle \cdot, \cdot \rangle_{\delta + 2} \), we obtain analogous to the computations in §3.2.1 and §3.2.3
\[
\frac{d}{dt} \left| (D_x - 2)\hat{q}(D_x)D_xv \right|_{\delta + \frac{3}{2}}^2 + \left| (D_x - 2)\hat{q}(D_x)D_xv \right|_{\delta + 2}^2 + \eta^2 \left| (D_x - 1)D_xv \right|_{\delta + 1}^2 + \eta^4 \left| D_xv \right|_{\delta}^2 
\leq_{\delta} \left| (D_x - 4)(D_x - 3)\hat{q}(D_x - 1)(D_x - 1)f \right|_{\delta + 2}^2 + \eta^2 \left| \hat{q}(D_x)D_xv \right|_{\delta + 1}^2 + \eta^4 \left| D_xv \right|_{\delta}^2.
\] (3.48)

Upgrading the weak estimate (3.48) follows the lines of the respective discussion when passing from the weak estimates (3.13), (3.29) and (3.41) to the strong estimates (3.17), (3.34), and (3.42), respectively. Therefore, we skip all details and arrive at
\[
\frac{d}{dt} \left( \sum_{\ell=0}^{k} \tilde{C}_\ell \eta^{2\ell} \left| D_x^{k-\ell}(D_x - 3)(D_x - 2)\hat{q}(D_x)D_xv \right|_{\delta + \frac{3}{2}}^2 + \left| (D_x - 3)(D_x - 2)\hat{q}(D_x)D_xv \right|_{\delta + 2}^2 \right) + \sum_{\ell=0}^{k+2} \eta^{2\ell} \left| (D_x - 3)(D_x - 2)\hat{q}(D_x)D_xv \right|_{\delta + 2}^2 + \eta^2 \left| \hat{q}(D_x)D_xv \right|_{\delta + 1}^2 + \eta^4 \left| D_xv \right|_{\delta}^2
\leq_{k,\delta} \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| (D_x - 4)(D_x - 3)\hat{q}(D_x - 1)(D_x - 1)f \right|_{\delta + 2}^2 + \eta^2 \left| \hat{q}(D_x)D_xv \right|_{\delta + 1}^2 + \eta^4 \left| D_xv \right|_{\delta}^2.
\] (3.49)

\footnote{In the framework of the more general thin-film equation (1.2), where \( \beta \) is a function of the mobility exponent \( n \), this restricts our analysis to those \( n \) for which \( \beta > \frac{1}{2} \).}
with constants \( \hat{k} \geq 2 \), \( \hat{C}_\ell > 0 \), and \( \delta \in (0, \frac{1}{10}) \). In integrated form and after employing equation (3.46) to obtain control on the time derivative \( \hat{c}_t v \), we find

\[
\begin{align*}
\sup_{t \in I} \| D_y^j (D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v \|_{k,\delta+1+j}^2 \\
+ \int_I \left( \| \hat{c}_t D_y^j (D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v \|_{k-2,\delta+\frac{1}{2}+j}^2 + \| D_y^j \hat{q}(D_x)D_x v \|_{k+2,\delta+\frac{3}{2}+j}^2 \right) dt \\
\lesssim_{k,\delta} \| D_y^j (D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v_{t=0} \|_{k,\delta+1+j}^2 \\
+ \int_I \left( \| D_y^j (D_x - 4)(D_x - 3)\hat{q}(D_x - 1)(D_x - 1)f \|_{k-2,\delta+\frac{1}{2}+j}^2 dt \\
+ \int_I \| D_y^{j+1}\hat{q}(D_x)D_x v \|_{\delta+\frac{3}{2}+j}^2 dt + \int_I \| D_y^{j+2}D_x v \|_{\delta+\frac{3}{2}+j}^2 dt. \tag{3.50} \right)
\end{align*}
\]

for \( j \geq 0 \).

In order to absorb the remnant terms

\[
\int_I \left( \| D_y^{j-1}D_x v \|_{k,\delta-\frac{3}{2}+j}^2 + \| D_y^{j-2}\hat{q}(D_x)D_x v \|_{k+1,\delta-\frac{3}{2}+j}^2 \right) dt,
\]

we choose \( j := j - 3 \) and assume \( j \geq 3 \), so that the remnant terms can be bounded by

\[
\int_I \left( \| D_y^{j-1}D_x v \|_{k,\delta-\frac{3}{2}+j}^2 + \| D_y^{j-2}\hat{q}(D_x)D_x v \|_{k+1,\delta-\frac{3}{2}+j}^2 \right) dt,
\]

appearing on the left-hand side of (3.45) with \( \hat{k} \) replaced by \( \hat{k} - 1 \) and \( \hat{k} \) replaced by \( \hat{k} - 1 \), where \( \hat{k} \geq 4 \).

This results in the estimate

\[
\begin{align*}
\sup_{t \in I} \left( \| D_y^{j-1}v \|_{k-2,\delta-2+j}^2 + \| D_y^{j-1}D_x v \|_{k-2,\delta-2+j}^2 + \| D_y^{j-2}\hat{q}(D_x)D_x v \|_{k-1,\delta-2+j}^2 \right) \\
+ \sup_{t \in I} \left( \| D_y^{j-3}(D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v \|_{k,\delta-2+j}^2 + \| D_y^j v \|_{k-2,\delta-1+j}^2 \right) \\
+ \int_I \left( \| \hat{c}_t D_y^{j-1}v \|_{k-4,\delta-\frac{3}{2}+j}^2 + \| \hat{c}_t D_y^{j-1}D_x v \|_{k-4,\delta-\frac{3}{2}+j}^2 + \| \hat{c}_t D_y^{j-2}\hat{q}(D_x)D_x v \|_{k-3,\delta-\frac{3}{2}+j}^2 \right) dt \\
+ \int_I \left( \| \hat{c}_t D_y^{j-3}(D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v \|_{k,\delta-\frac{3}{2}+j}^2 + \| \hat{c}_t D_y^{j-1}v \|_{k-4,\delta-\frac{3}{2}+j}^2 + \| \hat{c}_t D_y^{j-1}D_x v \|_{k,\delta-\frac{3}{2}+j}^2 \right) dt \\
+ \int_I \left( \| D_y^{j-3}(D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v \|_{k,\delta-\frac{3}{2}+j}^2 + \| D_y^{j-2}\hat{q}(D_x)D_x v \|_{k+1,\delta-\frac{3}{2}+j}^2 \right) dt \\
\lesssim_{k,\delta} \| D_y^{j-1}v_{t=0} \|_{k-2,\delta-2+j}^2 + \| D_y^{j-1}D_x v_{t=0} \|_{k-2,\delta-2+j}^2 + \| D_y^{j-2}\hat{q}(D_x)D_x v_{t=0} \|_{k-1,\delta-2+j}^2 \\
+ \| D_y^{j-3}(D_x - 3)(D_x - 2)\hat{q}(D_x)D_x v_{t=0} \|_{k,\delta-2+j}^2 + \| D_y^j v_{t=0} \|_{k-2,\delta-1+j}^2 \right) \\
+ \int_I \left( \| D_y^{j-1}f \|_{k-4,\delta-\frac{3}{2}+j}^2 + \| D_y^{j-1}(D_x - 1)f \|_{k-4,\delta-\frac{3}{2}+j}^2 \right) dt \\
+ \int_I \| D_y^{j-2}\hat{q}(D_x - 1)(D_x - 1)f \|_{k-3,\delta-\frac{3}{2}+j}^2 \right) dt \\
+ \int_I \left( \| D_y^{j-3}(D_x - 4)(D_x - 3)\hat{q}(D_x - 1)(D_x - 1)f \|_{k-2,\delta-\frac{3}{2}+j}^2 + \| D_y^j f \|_{k-4,\delta-\frac{3}{2}+j}^2 \right) dt \tag{3.51}
\end{align*}
\]

for \( \delta \in (0, \frac{1}{10}) \), where \( \hat{k} \geq 4 \), \( \hat{k} \geq 4 \), and \( j \geq 3 \).

3.2.6. Maximal regularity for the linear degenerate-parabolic equation. In this section, we combine estimates (3.37) with \( j = 1 \), (3.44) with \( j = 2 \), (3.45) with \( j = 2 \), and (3.51) with \( j = 3 \) to obtain maximal regularity for the linear degenerate-parabolic problem (1.19) in form of

\[
\| v \|_{sol} \lesssim_{\hat{k},\hat{k},\delta} \| v^{(0)} \|_{\text{init}} + \| f \|_{\text{rhs}}, \tag{3.52}
\]

where we have introduced the following norms:
The norm $\| \cdot \|_{\text{sol}}$ for the solution $v$ is defined through
\[
\| v \|_{\text{sol}}^2 := 
\sup_{t \in I} \left( \| D_y v \|_{k-1,-\delta}^2 + \| D_x v \|_{k,-\delta}^2 + \| D_y D_x v \|_{k-1,-\delta}^2 + \| \tilde{q}(D_x) D_x v \|_{k,\delta}^2 \right) 
+ \sup_{t \in I} \left( \| D_y^2 v \|_{k-2,-\delta+1}^2 + \| D_y D_x v \|_{k-1,-\delta+1}^2 + \| \tilde{q}(D_x) D_x v \|_{k,-\delta+1}^2 + \| D_y^2 D_x v \|_{k-2,\delta+1}^2 \right) 
+ \sup_{t \in I} \left( \| D_y \tilde{q}(D_x) D_x v \|_{k-1,-\delta+1}^2 + \| (D_x - 3)(D_x - 2) \tilde{q}(D_x) D_x v \|_{k,-\delta+1}^2 + \| D_y^2 \tilde{q}(D_x) D_x v \|_{k-2,\delta+1}^2 \right) 
+ \int_I \left( \| \tilde{c}_1 D_y v \|_{k-3,-\delta+\frac{1}{2}}^2 + \| \tilde{c}_1 D_x v \|_{k-2,-\delta+\frac{1}{2}}^2 + \| \tilde{c}_1 D_y D_x v \|_{k-3,-\delta+\frac{1}{2}}^2 + \| \tilde{c}_1 \tilde{q}(D_x) D_x v \|_{k-2,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \left( \| \tilde{c}_1 D_y^2 v \|_{k-4,-\delta+\frac{1}{2}}^2 + \| \tilde{c}_1 \tilde{q}(D_x) D_x v \|_{k-3,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \left( \| \tilde{c}_1 D_y^2 \tilde{q}(D_x) D_x v \|_{k-4,\delta+\frac{1}{2}}^2 + \| \tilde{c}_1 \tilde{q}(D_x) D_x v \|_{k-3,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \left( \| \tilde{c}_1 \tilde{q}(D_x) D_x v \|_{k-1,\delta+\frac{1}{2}}^2 + \| (D_x - 3)(D_x - 2) \tilde{q}(D_x) D_x v \|_{k+2,\delta+\frac{1}{2}}^2 + \| D_y^3 \tilde{q}(D_x) D_x v \|_{k,3,\delta+\frac{1}{2}}^2 \right) dt. \tag{3.53}
\]

Its equivalence to the norm $\| \cdot \|_{\text{sol}}$ defined in (2.3) is given through Lemma 3.6.

The norm $\| \cdot \|_{\text{init}}$ for the initial data $v^{(0)}$ is given through
\[
\left( \| v^{(0)} \|_{\text{init}}^2 \right)^{\frac{1}{2}} := 
\| D_y v^{(0)} \|_{k-1,-\delta}^2 + \| D_x v^{(0)} \|_{k,-\delta}^2 + \| D_y D_x v^{(0)} \|_{k-1,-\delta}^2 + \| \tilde{q}(D_x) D_x v^{(0)} \|_{k,\delta}^2 
+ \| D_y^2 v^{(0)} \|_{k-2,-\delta+1}^2 + \| D_y D_x v^{(0)} \|_{k-1,-\delta+1}^2 + \| \tilde{q}(D_x) D_x v^{(0)} \|_{k,-\delta+1}^2 + \| D_y^2 D_x v^{(0)} \|_{k-2,\delta+1}^2 
+ \| D_y \tilde{q}(D_x) D_x v^{(0)} \|_{k-1,-\delta+1}^2 + \| (D_x - 3)(D_x - 2) \tilde{q}(D_x) D_x v^{(0)} \|_{k+2,\delta+1}^2 + \| D_y^3 v^{(0)} \|_{k-2,\delta+2}^2. \tag{3.54}
\]

The norm $\| \cdot \|_{\text{rhs}}$ for the right-hand side $f$ is determined by
\[
\left( \| f \|_{\text{rhs}}^2 \right)^{\frac{1}{2}} := 
\int_I \left( \| D_y f \|_{k-3,-\delta+\frac{1}{2}}^2 + \| (D_x - 1)f \|_{k-2,-\delta+\frac{1}{2}}^2 + \| D_y (D_x - 1)f \|_{k-3,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \left( \| \tilde{q}(D_x - 1)(D_x - 1)f \|_{k-4,-\delta+\frac{1}{2}}^2 + \| D_y^2 f \|_{k-4,\delta+\frac{1}{2}}^2 + \| D_y (D_x - 1)f \|_{k-3,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \left( \| \tilde{q}(D_x - 1)(D_x - 1)f \|_{k-4,\delta+\frac{1}{2}}^2 + \| D_y^2 (D_x - 1)f \|_{k-4,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \left( \| D_y \tilde{q}(D_x - 1)(D_x - 1)f \|_{k-3,\delta+\frac{1}{2}}^2 + \| (D_x - 4)(D_x - 3)\tilde{q}(D_x - 1)(D_x - 1)f \|_{k-2,\delta+\frac{1}{2}}^2 \right) dt 
+ \int_I \| D_y^3 f \|_{k-4,\delta+\frac{1}{2}}^2 dt. \tag{3.55}
\]

Here, $k \geq 3$, $\tilde{k} \geq 4$, and $\delta \in (0, \frac{1}{10})$. Further conditions on the number of derivatives $k$, $\tilde{k}$, and $\delta$ will be given when treating the nonlinearity $\mathcal{N}(v)$ (cf. (1.17)) in §4.2. A simplification of the norms $\| \cdot \|_{\text{init}}$ and $\| \cdot \|_{\text{rhs}}$ is provided in Lemmata 3.4 and 3.5 of §3.3.2. A rigorous justification of the maximal-regularity estimate (3.52) will be the subject of §3.4.

3.3. Norms and their properties.
3.3.1. Simplifications of our norms. The following weighted trace estimate gives control on the time trace, which follows directly by using the fundamental theorem of calculus and Young’s inequality. The precise version included below is convenient because it can be used to estimate the supremum terms from the definition (3.53) of $\|v\|_{\text{sol}}$ by the integral terms appearing in the same norm.

**Lemma 3.3** (time trace). Assume that $T > 0$, $I = [0, T] \subseteq [0, \infty)$, and let $w \in L^1_{\text{loc}}(I \times (0, \infty) \times \mathbb{R})$. Then there holds

$$\sup_{t \in I} |w|_{t, \gamma}^2 \lesssim_{t, \gamma} |w|_{t = 0, \gamma}^2 + \int_I |\tilde{c_t}w|_{t-2, \gamma-\frac{1}{2}}^2 dt + \int_I |w|_{t, \gamma+\frac{1}{2}}^2 dt,$$

and

$$\sup_{t \in I} |w|_{t, \gamma}^2 \lesssim_{t, \gamma} \frac{1}{T} \left( \int_I |\tilde{c_t}w|_{t-2, \gamma-\frac{1}{2}}^2 dt + \int_I |w|_{t, \gamma+\frac{1}{2}}^2 dt \right),$$

for $\gamma \in \mathbb{R}$ and $\ell \geq 2$.

Next, we give the following simplification of the norms $\|\cdot\|_{\text{init}}$, $\|\cdot\|_{\text{rhs}}$, and $\|\cdot\|_{\text{sol}}$, whose proofs can be found in §B.1.

**Lemma 3.4.** Suppose $\delta \in (0, \frac{1}{M})$, $\tilde{k} \geq 1$, $\tilde{k} \geq 2$, $\tilde{k} \geq 2$, and define the norm $\|v^{(0)}\|_{\text{init}}$ for a locally integrable function $v^{(0)} : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ through (2.1), i.e.,

$$\|v^{(0)}\|_{\text{init}}^2 := \left\|D_x v^{(0)}\right\|_{k,-\delta}^2 + \left\|\tilde{q}(D_x)D_x v^{(0)}\right\|_{k,\delta}^2 + \left\|\tilde{q}(D_x)D_x v^{(0)}\right\|_{k,-\delta+1}^2 + \left\|(D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v^{(0)}\right\|_{k,\delta+1}^2 + \left\|D_y^3 v^{(0)}\right\|_{k-2,-\delta+2}^2.$$

Then the norms $\|\cdot\|_{\text{init}}^p$ (cf. (3.54)) and $\|\cdot\|_{\text{init}}$ are equivalent on the space of all locally integrable $v^{(0)} : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $\|v^{(0)}\|_{\text{init}} < \infty$.

**Lemma 3.5.** Suppose $\delta \in (0, \frac{1}{M})$, $\tilde{k} \geq 3$, $\tilde{k} \geq 4$, $\tilde{k} \geq 4$, and define the norm $\|f\|_{\text{rhs}}$ for a locally integrable function $f : I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$, where $I = [0, T] \subseteq [0, \infty)$, through (2.19), i.e.,

$$\|f\|_{\text{rhs}}^2 := \int_I \|D_x - 1\|f\|_{k,-2,-\delta+\frac{1}{2}}^2 dt + \int_I \left( \left\|\tilde{q}(D_x - 1)D_x f\right\|_{k-2,\delta+\frac{1}{2}}^2 + \left\|\tilde{q}(D_x - 1)(D_x - 1)f\right\|_{k-2,-\delta+\frac{1}{2}}^2 \right) dt + \int_I \left( \left\|D_x - 4\right\|\tilde{q}(D_x - 1)(D_x - 1)f\|_{k-2,\delta+\frac{1}{2}}^2 + \left\|D_y^3 f\right\|_{k-4,-\delta+\frac{1}{2}}^2 \right) dt.$$

Then the norms $\|\cdot\|_{\text{rhs}}^p$ (cf. (3.55)) and $\|\cdot\|_{\text{rhs}}$ are equivalent on the space of all locally integrable $f : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R}$ with $\|f\|_{\text{rhs}} < \infty$.

Thanks to Lemma 3.3 we can also prove the following analogue of Lemmata 3.4 and 3.5.

**Lemma 3.6.** Suppose that $\delta \in (0, \frac{1}{M})$, $\tilde{k} \geq 3$, $\tilde{k} \geq 4$, $\tilde{k} \geq 4$, and consider the norm $\|v\|_{\text{Sol}}$ for a locally integrable function $v : I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ for $I = [0, T]$ with $T \in (0, \infty]$ as

$$\|v\|_{\text{Sol}}^2 := \left( \int_I \left( \left\|\tilde{c_t}D_x v\right\|_{k,\delta-\frac{1}{2}}^2 + \left\|\tilde{c_t}\tilde{q}(D_x)D_x v\right\|_{k-2,\delta-\frac{1}{2}}^2 + \left\|\tilde{c_t}D_y v\right\|_{k,\delta-\frac{1}{2}}^2 \right) dt + \int_I \left( \left\|\tilde{c_t}(D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v\right\|_{k,-\delta+\frac{1}{2}}^2 + \left\|\tilde{c_t}D_y^3 v\right\|_{k-4,-\delta+\frac{1}{2}}^2 \right) dt + \int_I \left( \left\|D_x v\right\|_{k-2,\delta+\frac{1}{2}}^2 + \left\|\tilde{q}(D_x)D_x v\right\|_{k,\delta+\frac{1}{2}}^2 + \left\|\tilde{q}(D_x)D_x v\right\|_{k-2,\delta+\frac{1}{2}}^2 \right) dt + \int_I \left( \left\|D_x - 3\right\|D_x - 2\|\tilde{q}(D_x)D_x v\|_{k-2,\delta+\frac{1}{2}}^2 + \left\|D_y^3 v\right\|_{k-4,-\delta+\frac{1}{2}}^2 \right) dt.$$

Then the norms $\|\cdot\|_{\text{Sol}}$ (cf. (3.53)) and $\|\cdot\|_{\text{Sol}}$ are equivalent on the space of all locally integrable $v : I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $\|v\|_{\text{Sol}} < \infty$.

**Remark 3.7.** Note that the constant coming into the equivalence of norms of Lemma 3.6 blows up as $T \downarrow 0$, unlike the one in Lemma 3.5.
3.3.2. Elliptic regularity and embeddings. Note that the norms $\|\cdot\|_{\text{sol}}, \|\cdot\|_{\text{init}},$ and $\|\cdot\|_{\text{rhs}}$ (cf. (3.53), (2.1), and (2.19)) contain terms in $\tilde{q}(D_\gamma D_\lambda w, D_j^\beta D_\lambda w$ $(j \in \{0,1\}),$ or $D_j^\beta w$ $(j \in \{0,1,2\},$ where $w \in \{v,v^{(0)},f\},$ that are not explicit enough in terms of $w$ in the sense that they do not change on changing boundary values such as $w_0,$ $w_1,$ or $w_{1+\beta}.$ On the other hand, we are precisely interested in control on these terms, which require to study the elliptic regularity of the operators $\tilde{q}(D_\gamma D_\lambda)$ and $D_\lambda.$ Therefore, we consider the following direct consequence of Hardy’s inequality (cf. Giacomelli, Knüpf, and Otto (2008, Lemma A.1), Giacomelli, Gnann, Knüpf, and Otto (2014, Lemma 7.4), Gnann (2016, Proposition 3.1 and Lemma 3.1)).

**Lemma 3.8.** Let $\ell, \bar{\gamma} : (0, \infty) \to \mathbb{R}$ be locally integrable functions and $\rho, \gamma, \bar{\gamma} \in \mathbb{R}$ be constants such that $\gamma < \rho < \bar{\gamma}$ and
\[
\lim_{x \to \infty} x^{-\rho} f(x) = \lim_{x \to 0} x^{-\rho} \bar{\gamma} = 0 \quad \text{for a subsequence.}
\]
Then
\[
\left| (D - \gamma) f_\rho \right|_{L^2} \geq 2 \gamma \rho \left| f \right|_{L^2}, \quad \text{and} \quad \left| (D - \bar{\gamma}) \bar{\gamma} \right|_{L^2} \geq \bar{\gamma} \rho \left| \bar{\gamma} \right|_{L^2},
\]
provided the norms in (3.58) yield a finite value.

The following Lemma yields $\text{BC}^0$-bounds on the solution $v$ to (1.14), and its gradient. Its proof is contained §B.2.

**Lemma 3.9.** Suppose that $\delta \in (0, \frac{1}{10}), \bar{k} \geq 3, k \geq 4$ and $\bar{k} \geq 4.$ Then for functions $v^{(0)} : (0, \infty) \times \mathbb{R} \to \mathbb{R},$ $v : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R},$ and $f : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R}$ that are locally integrable with $\|v^{(0)}\|_{\text{init}} < \infty,$ $\|v\|_{\text{sol}} < \infty,$ and $\|f\|_{\text{rhs}} < \infty,$ the expansion (1.22) in powers of $x$ holds, where the terms satisfy the following estimates, with constants only depending on $k,$ $\bar{k},$ and $\delta$:
\[
\left\| \frac{D^\alpha v_0^{(0)}}{\text{BC}^0((0,\infty)_x \times \mathbb{R})} + \left| \frac{D^\alpha v_1^{(0)}}{\text{BC}^0(\mathbb{R}^n)} \right| + \left| \frac{D^\alpha v_2^{(0)}}{\text{BC}^0((0,\infty)_x \times \mathbb{R})} \right| + \left| \frac{D^\alpha v_3^{(0)}}{\text{BC}^0(\mathbb{R}^n)} \right| \right\|_{\text{init}} \lesssim \left\| v^{(0)} \right\|_{\text{init}} \quad (3.59a)
\]
and
\[
\sup_{t \in I} \left( \left\| \frac{D^\alpha v_0^{(0)}}{\text{BC}^0((0,\infty)_x \times \mathbb{R})} + \left| \frac{D^\alpha v_1^{(0)}}{\text{BC}^0(\mathbb{R}^n)} \right| + \left| \frac{D^\alpha v_2^{(0)}}{\text{BC}^0((0,\infty)_x \times \mathbb{R})} \right| + \left| \frac{D^\alpha v_3^{(0)}}{\text{BC}^0(\mathbb{R}^n)} \right| \right\|_{\text{sol}} \lesssim \left\| v^{(0)} \right\|_{\text{sol}} \quad (3.59b)
\]
for $0 \leq \left| \epsilon \right| \leq \min \left\{ \bar{k} - 2, k - 2 \right\},$ $0 \leq \ell' \leq \min \left\{ \bar{k} - 1, k - 1 \right\},$ and $0 \leq \ell'' \leq \min \left\{ \bar{k} - 2, k - 2 \right\},$
\[
\int_I \left( \left| \frac{D^\alpha v_{1+\beta}}{\text{BC}^0(\mathbb{R}^n)} \right|^2 + \left| \frac{D^\alpha v_{2+\beta}}{\text{BC}^0(\mathbb{R}^n)} \right|^2 \right) dt \lesssim \left\| f \right\|_{\text{sol}}^2 \quad (3.59c)
\]
for $0 \leq \ell \leq \min \left\{ \bar{k} + 1, k + 1 \right\}$ and $0 \leq \ell' \leq \bar{k} + 1,$ and
\[
\int_I \left( \left| \frac{D^\alpha f_1}{\text{BC}^0(\mathbb{R}^n)} \right|^2 + \left| \frac{D^\alpha f_2}{\text{BC}^0(\mathbb{R}^n)} \right|^2 \right) dt \lesssim \left\| f \right\|_{\text{sol}}^2 \quad (3.59d)
\]
for $0 \leq \ell \leq \bar{k} - 3$ and $0 \leq \ell' \leq \bar{k} - 3.$

3.3.3. Approximation results. Our aim is to approximate the solution $v,$ the initial data $v^{(0)}$ and the right-hand side $f$ in the norms $\|\cdot\|_{\text{sol}}, \|\cdot\|_{\text{init}},$ and $\|\cdot\|_{\text{rhs}},$ where the approximating sequences have well-controlled behavior as $x \to 0$ and $x \to \infty.$ The following two definitions specify the necessary asymptotic behavior precisely:

**Definition 3.10.** A function $f : (0, \infty) \times \mathbb{R} \to \mathbb{C}$ such that $f(\cdot, \eta) \in C^{\infty}((0, \infty))$ for every $\eta \in \mathbb{R},$ satisfies (G1) if for every compact interval $J \subset \mathbb{R},$ there exists $\varepsilon > 0$ and a function $\hat{f} : [0, \varepsilon] \times [0, \varepsilon^{1+\beta}] \times J \to \mathbb{C}$ such that
\begin{itemize}
  \item[(i)] $f(x, \eta) = \hat{f}(x, x^{1+\beta}, \eta)$ for $(x, \eta) \in (0, \varepsilon^{1+\beta}] \times J$ and for each $\eta \in J$ the function $\hat{f}(\cdot, \cdot, \eta)$ is analytic;
  \item[(ii)] the map $J \ni \eta \mapsto \hat{f} \in \Omega,$ where
\end{itemize}
\[
\Omega := \left\{ \tilde{g} : \|\tilde{g}\|_\Omega := \sum_{(k,\ell) \in \mathbb{N}_0^2} \frac{\varepsilon^{k+1+\beta}\ell}{k!\ell!} \|\tilde{g}_{x_1} \tilde{g}_{x_2} \tilde{g}(0,0)\| < \infty \right\},
\]
is well-defined and continuous.
Note that the power $1 + \beta$ above is different from the power $\beta$ appearing in Giacomelli, Gnann, Knüpfer, and Otto (2014) (compare to §1.7 and in particular the discussion leading to (1.24)) but the ensuing properties and the use we make of this condition is in part analogous. As already outlined in §1.7 (see the discussion after (1.21)) the terms $-\eta^2x^2 r(D_x) v$ and $\eta^4 x^3 v$ in (1.19a) are, for fixed $\eta \in \mathbb{R}$, merely higher-order corrections.

We make an analogous definition for the asymptotic behavior at $x = \infty$.

**Definition 3.11.** Suppose that $\eta \in \mathbb{R}$ is fixed. A function $f : (0, \infty) \times \mathbb{R} \to \mathbb{C}$ such that $f(\cdot, \eta) \in C^\infty((0, \infty))$ for every $\eta \in \mathbb{R}$, meets $(G_x)$ if

(i) for $\eta = 0$, for all $j \in \mathbb{N}_0$, and all $\nu \in [0, 2^\frac{3}{2})$ there holds

$$\limsup_{x \to \infty} e^{\nu x^{1/4}} \frac{d^j f}{dx^j}(x) < \infty;$$

(ii) for $\eta \neq 0$, for all $\nu \in [0, |\eta|]$, and all $j \in \mathbb{N}_0$ there holds

$$\limsup_{x \to \infty} e^{\nu x^{1/4}} \frac{d^j f}{dx^j}(x) < \infty;$$

(iii) for all $x_0 > 0$, $k \geq 0$, and $\nu \in [0, |\eta|]$ the mapping $(\mathbb{R} \setminus \{0\}) \ni \eta \mapsto f(\cdot, \eta) \in \mathcal{V}$ is well-defined and continuous, where

$$\mathcal{V} := \left\{ g : |g|_{k,x_0,\nu} := \max_{j=0,\ldots,k} \sup_{x \geq x_0} e^{\nu x^{1/4}} \left| \frac{d^j g}{dx^j}(x) \right| \right\}.$$

Note that item (i) coincides with the choice in Giacomelli, Gnann, Knüpfer, and Otto (2014, Def. 6.2, $(G_x)$).

For the proof of the following approximation results, see §B.3.

**Lemma 3.12.** Suppose $\tilde{k} \geq 3, \tilde{k} \geq 4, \tilde{k} \geq 4$, and $\delta \in (0, \frac{1}{10})$. Then for each integrable $v^{(0)} : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $\|v^{(0)}\|_{\text{init}} < \infty$ there exists a sequence $(v^{(0,n)})_n$ in $C^\infty((0, \infty) \times \mathbb{R})$ such that $\|v^{(0,n)} - v^{(0)}\|_{\text{init}} \to 0$ as $n \to \infty$ and

(i) $v^{(0,n)}$ is smooth in $\{(x, \eta) : x > 0, \eta \neq 0\}$,

(ii) $v^{(0,n)}(x, \eta) = v_0^{(0,n)}(\eta) + v_1^{(0,n)}(\eta)x$ for $x \ll_n 1$, where $v_0^{(0,n)}, v_1^{(0,n)}$ are smooth, so that in particular $v^{(0,n)}$ satisfies $(G_0)$ from Definition 3.10.

(iii) $v^{(0,n)}(x, \eta) = 0$ for $x \gg_n 1$, so that in particular $v^{(0,n)}$ satisfies $(G_x)$ from Definition 3.11.

**Lemma 3.13.** Suppose $\tilde{k} \geq 3, \tilde{k} \geq 4, \tilde{k} \geq 4$, and $I \subseteq (0, \infty)$ is an interval. Then for each locally integrable $f : I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $\|f\|_{\text{hs}} < \infty$ there exists a sequence $(f^{(n)})_n$ in $C^\infty(I \times (0, \infty) \times \mathbb{R})$ such that $\|f^{(n)} - f\|_{\text{hs}} \to 0$ as $n \to \infty$ and

(i) $f^{(n)}$ is smooth in $\{(t, x, \eta) : t \in I, x > 0, \eta \neq 0\}$,

(ii) for every $t \in I$ we have $f^{(n)}(t, x, \eta) = f_1^{(n)}(t, \eta)x + f_2^{(n)}(t, x)x^2$ as $x \searrow 0$ for $x \ll_n 1$, where $f_1^{(n)}$ and $f_2^{(n)}$ are smooth on $I \times \mathbb{R}$, so that in particular $f^{(n)}(t, \cdot, \cdot)$ satisfies $(G_0)$ from Definition 3.10.

(iii) $f^{(n)}(t, x, \eta) = 0$ for $x \gg_n 1$, so that in particular $f^{(n)}(t, \cdot, \cdot)$ satisfies $(G_x)$ from Definition 3.11 for all $t \in I$.

**Lemma 3.14.** Suppose $\tilde{k} \geq 3, \tilde{k} \geq 4, \tilde{k} \geq 4$, and $\delta \in (0, \frac{1}{10})$, and $I \subseteq (0, \infty)$ is an interval. Then for each locally integrable $v : I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $\|v\|_{\text{sol}} < \infty$ there exists a sequence $(v^{(n)})_n$ in $C^\infty(I \times (0, \infty) \times \mathbb{R})$ such that $\|v^{(n)} - v\|_{\text{sol}} \to 0$ as $n \to \infty$ and

(i) if $I = (0, \infty)$ then $v^{(n)}(t, x, \eta) = 0$ whenever $t \geq 2n$,

(ii) $v^{(n)}$ is smooth in $\{(t, x, \eta) : t \in I, x > 0, \eta \neq 0\}$,

(iii) for every $t \in I$ there holds

$$v^{(n)}(t, x, \eta) = v_0^{(n)}(t, \eta) + v_1^{(n)}(t, \eta)x + v_1^{(n)}(t, \eta)x^{1+\beta} + v_2^{(n)}(t, \eta)x^2$$

for $x \ll_n 1$,

where $v_0^{(n)}, v_1^{(n)}, v_1^{(n)}$ are smooth, so that in particular $v^{(n)}(t, \cdot, \cdot)$ satisfies $(G_0)$ from Definition 3.10 for all $t \in I$.

(iv) $v^{(n)}(t, x, \eta) = 0$ for $x \gg_n 1$, so that in particular $v^{(n)}(t, \cdot, \cdot)$ satisfies $(G_x)$ from Definition 3.11 for all $t \in I$.

We then note the following corollary of Lemma 3.14:
Corollary 3.15. Suppose that \( k \geq 3, \delta \geq 4, k \geq 4, \delta \in (0, \frac{1}{10}) \), and \( I \subseteq (0, \infty) \) is an interval. Then for each locally integrable \( v : \mathbb{R} \rightarrow \mathbb{R} \) with \( \|v\|_{\text{sol}} < \infty \), using the notation of Lemma 3.14, the following hold:

(i) If \( I = (0, \infty) \) then there holds \( \lim_{t \to \infty} v_0(t, y) = 0 \) for all \( y \in \mathbb{R} \), as well as \( \lim_{t \to \infty} \|v(t, \cdot)\|_{\text{init}} = 0 \).

(ii) The function \( v_0(t, y) := \lim_{t \to \infty} v(t, x, y) \) is continuous and bounded.

(iii) If for \( \tau \in (0, 1) \) we define \( I_\tau := [0, \tau) \) and we denote by \( \|v\|_{\text{sol}, \tau} \) the norm \( \|v\|_{\text{sol}} \) as defined in (3.53) with the choice \( I = I_\tau \), then the functions \( (0, 1) \ni \tau \mapsto \|v\|_{\text{sol}, \tau} \) and \( (0, 1) \ni \tau \mapsto \|v(t, \cdot)\|_{\text{init}} \) are continuous and bounded and \( \|v\|_{\text{sol}, \tau} \to \|v(0)\|_{\text{init}} \) as \( \tau \to 0 \).

3.4. Rigorous treatment of the linear equation. The construction of solutions to the linear problem (1.19) with appropriate estimates is based on a time-discretization argument (cf. §3.4.2) which in turn is based on a solid understanding of the corresponding resolvent equation (cf. §3.4.1). Note that the reasoning of §3.4.2 partially follows Giacomelli, Gnann, Knüpfer, and Otto (2014, §7) while the treatment of the resolvent problem is similar to Giacomelli, Gnann, Knüpfer, and Otto (2014, §6). We will thus emphasize differences and introduce some simplifications to the approach of Giacomelli, Gnann, Knüpfer, and Otto (2014).

3.4.1. The resolvent equation. Suppose we are given \( v(t) = v(t)(x) \) for a given time \( t > 0 \). Then an approximate solution of (1.19a) at time \( t + \delta t \), where \( \delta t > 0 \) is small, can be found by solving

\[
\frac{v(t + \delta t) - v(t)}{\delta t} + q(D_x) v(t + \delta t) - \eta^2 x^2 r(D_x) v(t + \delta t) + \eta^4 x^4 v(t + \delta t) = \frac{1}{\delta t} \int_t^{t+\delta t} f(t') \, dt'
\]

for \( x > 0 \). Replacing \( \frac{1}{\delta t} \int_t^{t+\delta t} f(t') \, dt' \) by \( f = f(x) \), setting \( \lambda := \frac{1}{\delta t} \), and writing \( v \) instead of \( v(t + \delta t) \), we end up with the resolvent equation

\[
\lambda x v + q(D_x) v - \eta^2 x^2 r(D_x) v + \eta^4 x^4 v = f \quad \text{for} \quad x > 0.
\]

By scaling \( x \) and \( y \), we can assume without loss of generality \( \lambda = 1 \), so that the resolvent equation simplifies to

\[
\lambda x v + q(D_x) v - \eta^2 x^2 r(D_x) v + \eta^4 x^4 v = f \quad \text{for} \quad x > 0.
\]

Equation (3.61) is for every fixed \( \eta \in \mathbb{R} \) an ODE in \( x \), i.e., we consider \( \eta \) merely as a parameter in the problem. In what follows by a slight abuse of notation, we will be indicating solutions that are well-controlled as \( \eta \to 0 \). However, our aim is not to construct just some solution to (3.61) but solutions that are well-controlled as \( x \to 0 \) and \( x \to \infty \). This is detailed in conditions \((G_0)\) and \((G_\infty)\) of Definitions 3.10 and 3.11.

The main result of §3.4.1 reads as follows.

Proposition 3.16. Suppose that \( f : (0, \infty) \times \mathbb{R} \to \mathbb{C} \) such that \( f(\cdot, \eta) \in C^{2}\left((0, \infty)\right) \) for every \( \eta \in \mathbb{R} \), satisfies \((G_0)\) with \( (f, \partial_x f)(0, 0, \eta) = (0, 0) \), \((G_\infty)\), and is such that all derivatives \( \frac{\partial^j f}{\partial x^j} \) for \( j \geq 0 \) are continuous in \( \{(x, \eta) : x > 0, \eta \neq 0\} \). Then there exists exactly one solution \( v : (0, \infty) \times \mathbb{R} \to \mathbb{C} \), such that for every \( \eta \in \mathbb{R} \) there holds \( v(\cdot, \eta) \in C^{2}\left((0, \infty)\right) \), to the resolvent equation (3.61) such that for \( k = 0, 1, 2 \) there holds \( |v|_{k, -\delta-k} < \infty \) for some \( \delta \in (0, \frac{1}{10}) \). Moreover \( v \) satisfies conditions \((G_0)\) and \((G_\infty)\), and \( \frac{\partial^j v}{\partial x^j} \) for \( j \geq 0 \) is continuous in \( \{(x, \eta) : x > 0, \eta \neq 0\} \).

Note that for later purpose the continuous dependence on \( \eta \) will not be necessary but the weaker measurability is sufficient.

The strategy for constructing a solution to (3.61) proceeds in three steps:

(i) We construct a 2-dimensional manifold of solutions to (3.61) for \( x \ll 1 \) meeting \((G_0)\) (cf. Definition 3.10).

(ii) We construct a 2-dimensional manifold of solutions to (3.61) for \( x \gg 1 \) fulfilling \((G_\infty)\) (cf. Definition 3.11).

(iii) We find exactly one solution to (3.61) satisfying contemporarily the above two conditions, by intersecting above the two 2-dimensional solution manifolds in 4-dimensional phase space spanned by coordinates \( (v, \frac{dw}{dx}, \frac{dx}{dx}, \frac{dx^3}{dx^3}) \).
Then define Lemma 3.18. For all compact intervals $[\eta, \xi, \zeta \in (0, \infty))$ solves (3.61) with $f(x) = 0$ in $(0, \infty)$ and for $k = 0, 1, 2$ there holds $|v|_{k-\delta-\xi k < \infty}$, then $v(x) = 0$.

For the convenience of the reader, we will give a proof of Proposition 3.16 and Lemma 3.17 at the end of §3.4.1. In order for all the ingredients completing the above sketch of proof to be in order, we will first pass to constructing solutions to (3.61) as described above, separately near $x = 0$ and near $x = \infty$.

Construction of solutions for $x \ll 1$. The treatment for $x \ll 1$ is essentially the same as the one by Giacomelli, Gnann, Knüpfer, and Otto (2014) since – as mentioned twice before – the extra terms coming from the additional dimension are of order $O(x^2)$ and $O(x^4)$, respectively, i.e., they are perturbative terms in the fixed-point problem for $x \ll 1$. Note that we first concentrate on constructing solutions to (3.61) for $x \ll 1$ and $\eta \in \mathbb{R}$ fixed, so that we initially suppress the dependence on $\eta$ in the notation. The continuous dependence on $\eta$ will be discussed at the end of the paragraph.

Here the leading-order operator in (3.61) is $q(D_x)$ and the corresponding homogeneous equation has two linearly independent bounded solutions $x^\alpha, x^{1+\beta}$ corresponding to the positive roots $0, 1 + \beta$ of the polynomial $q(\zeta)$. As a consequence, we expect that a generic bounded solution of $q(D_x)v = f$ behaves like $v(x) \approx a_1 x^{1+\beta} + o(x^{1+\beta})$ as $x \ll 0$, where $a_1, a_2 \in \mathbb{R}$ provided $f(x) = o(x^{1+\beta})$ as $x \ll 0$. Then we unfold the singular behavior for $0 \ll x \ll 1$ by using two variables $x_1, x_2$ rather than the single variable $x$, and employing the following substitutions:

$$v(x) \mapsto \hat{v}(x_1, x_2), \quad f(x) \mapsto \hat{f}(x_1, x_2), \quad D = x^2 \frac{\partial}{\partial x} \mapsto \tilde{D} := x_1 \frac{\partial}{\partial x_1} + (1 + \beta) x_2 \frac{\partial}{\partial x_2},$$

where on the characteristic $(x_1, x_2) = (x, x^{1+\beta})$ we can identify

$$D^j v(x) = \tilde{D}^j \tilde{v}(x, x^{1+\beta}) \quad \text{and} \quad D^j f(x) = \tilde{D}^j \tilde{f}(x, x^{1+\beta}) \quad \text{for} \quad j \in \mathbb{N}_0.$$ 

For fixed $\eta \in \mathbb{R}$, we then consider the unfolded problem

$$x_1 \tilde{v} + q(\tilde{D}) \tilde{v} - \eta^2 x_1^3 \tilde{f}(\tilde{D}) \tilde{v} + \eta^2 x_1^2 \tilde{v} = \tilde{f} \quad \text{in} \quad Q := [0, \varepsilon] \times [0, L],$$

subject to

$$(\tilde{v}, \tilde{c}_{x_2}) (0, 0) = (a_1, a_2),$$

where $a_1, a_2 \in \mathbb{R}$ are parameters and by compatibility necessarily $(\bar{f}, \tilde{c}_{x_2} \bar{f}) (0, 0) = (0, 0)$ holds true.

Problem (3.62) can be solved with a fixed-point argument, for which we define the norms (compare to item (ii) in Definition 3.10)

$$|\tilde{f}|_\omega := \sum_{(k, \ell) \in \mathbb{N}_0^2} \frac{\varepsilon^k L^\ell}{k! \ell!} |\tilde{c}_{x_1}^{\varepsilon} \tilde{c}_{x_2}^{\varepsilon} \tilde{f}(0, 0)| \quad \text{and} \quad |\tilde{v}|_{4, \omega} := \sum_{m=0}^4 |\tilde{D}^m \tilde{v}|_\omega,$$

with parameters $\varepsilon, L > 0$. Note that finiteness of $|\tilde{f}|_\omega$ is equivalent to the property that the series

$$\sum_{(k, \ell) \in \mathbb{N}_0^2} \frac{\varepsilon^k L^\ell}{k! \ell!} |\tilde{c}_{x_1}^{\varepsilon} \tilde{c}_{x_2}^{\varepsilon} \tilde{f}(0, 0)| x_1^k x_2^\ell$$

absolutely converges in $Q$. We obtain the following result being the analogue of Giacomelli, Gnann, Knüpfer, and Otto (2014, Lem. 6.5), though the norms (3.63) are chosen slightly differently and are more in line with the reasoning of Belgacem, Gnann, and Kuehn (2016, §4.4):

Lemma 3.18. For all compact intervals $J \subset \mathbb{R}$, there exists $\varepsilon > 0$ such that for all $\eta \in J$, any $L > 0$, and any $a_1, a_2 \in \mathbb{R}$, for any $\tilde{f} = \tilde{f}(x_1, x_2)$ analytic in $Q = [0, \varepsilon] \times [0, L]$ with $(\tilde{f}, \tilde{c}_{x_2} \tilde{f}) (0, 0) = (0, 0)$ and $|\tilde{f}|_\omega < \infty$, problem (3.62) has an analytic solution $\tilde{v}$, which additionally satisfies

$$|\tilde{v}|_{4, \omega} \lesssim |a_1| + |a_2| + |\tilde{f}|_\omega.$$ (3.64)

Now define $v^{(j)}(x) := \tilde{v}(x, x^{1+\beta})$ solving (3.61) for $x \ll \varepsilon$ with

0) $(a_1, a_2) = (0, 0)$ and given $f = \tilde{f}(x, x^{1+\beta})$ for $j = 0$,

1) $(a_1, a_2) = (1, 0)$ and $f(x) = 0$ for $j = 1$,

2) $(a_1, a_2) = (0, 1)$ and $f(x) = 0$ for $j = 2$.

Then

$$v^{(0)}(x) = -\frac{8}{9} \frac{d f}{d x}(0) x + O\left(x^2\right) \quad \text{as} \quad x \ll 0,$$

(3.65a)

$$v^{(1)}(x) = 1 + O\left(x^2\right) \quad \text{as} \quad x \ll 0,$$

(3.65b)

$$v^{(2)}(x) = x^{1+\beta} + O\left(x^2\right) \quad \text{as} \quad x \ll 0,$$

(3.65c)
and \( v := v^{(0)} + a_1 v^{(1)} + a_2 v^{(2)} \) is a solution to (3.61) on the interval \((0, \varepsilon]\) that can be extended to a solution for all \( x > 0 \) by standard ODE theory. Furthermore, if \( \mathcal{J} \ni \eta \rightarrow f(\cdot, \eta) \) is continuous in \( \{ f : \| f \|_\omega < \infty \} \), then the solution map \( \mathcal{J} \ni \eta \rightarrow v \) is continuous in \( \{ \bar{w} : \| \bar{w} \|_{1, \omega} < \infty \} \).

**Proof of Lemma 3.18.** Step 1: Linear solution operator. We first observe that for \( \tilde{f} \) analytic on \( Q \) satisfying
\[
(f, \partial_{x_x} \tilde{f})(0, 0) = (0, 0) \quad \text{and} \quad \| \tilde{f} \|_\omega < \infty,
\]
there exists an analytic solution \( T \tilde{f} \) with \( \sum_{m=0}^{4} \| D^m T \tilde{f} \|_\omega < \infty \) of
\[
q(D) T \tilde{f} = \tilde{f} \quad \text{in} \quad Q, \quad \text{with} \quad (T \tilde{f}, \partial_{x_x} T \tilde{f})(0, 0) = (0, 0),
\]
where
\[
T = T_{\alpha} T_{\beta + \epsilon} T_{\gamma} T_{\delta}
\]
and \( T_{\gamma} \) is determined through
\[
(D - \gamma) T \tilde{g} = \tilde{g} \quad \text{in} \quad Q, \quad \text{subject to} \quad (T \tilde{g}, \partial_{x_x} T \tilde{g})(0, 0) = (0, 0),
\]
and where by compatibility we need to assume \( (\tilde{g}, \partial_{x_x} \tilde{g})(0, 0) = (0, 0) \). Further assuming that \( \tilde{g} \) is analytic with \( \| \tilde{g} \|_{1, \omega} \), we may define \( T \tilde{g} \) through
\[
\frac{\partial^k \partial^\ell}{x_{x_x}^k \gamma_x^\ell} T \tilde{g}(0, 0) := \frac{\partial^k \partial^\ell \tilde{g}(0, 0)}{k + (1 + \varepsilon) \ell - \varepsilon}, \quad \text{where} \quad (k, \ell) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\},
\]
and
\[
(T \tilde{g}, \partial_{x_x} T \tilde{g})(0, 0) = (0, 0),
\]
so that \( T \tilde{g}(x_1, x_2) = \sum_{(k, \ell) \in \mathbb{N}^2} \frac{1}{k! \ell!} \frac{\partial^k \partial^\ell \tilde{g}(0, 0)}{k + (1 + \varepsilon) \ell - \varepsilon} x_{x_x}^k \gamma_x^\ell \) is indeed an analytic solution to (3.67). We find moreover that
\[
\sum_{m=0}^{4} \| D^m T \tilde{g} \|_\omega \lesssim \| \tilde{g} \|_\omega,
\]
which, by the product decomposition (3.67) and due to the commutation relation \( DT_{\gamma} = T_D \), immediately upgrades to the maximal-regularity estimate
\[
\| T \tilde{f} \|_{1, \omega} \lesssim \| \tilde{f} \|_\omega.
\]

**Step 2: Fixed-point argument.** We may reformulate problem (3.62) in terms of the fixed-point problem
\[
\bar{v} = \mathcal{T}[\bar{v}] \quad \text{for} \quad \mathcal{T}[\bar{v}] := a_1 + a_2 x_2 + T[\tilde{f}] + T[-x_1 \bar{v} + \eta^2 x_1^2 r(D) \bar{v} - \eta^4 x_1^4 \bar{v}],
\]
where the operator \( \mathcal{T} \) acts on the space of analytic functions \( \bar{v} \) with \( \| \bar{v} \|_{1, \omega} < \infty \). Now notice that with help of (3.63) and (3.69), by the sub-multiplicativity of \( \| \cdot \|_\omega \) (cf. Giacomelli, Gnann, and Otto (2013, Lem. 3(b)) for an analogous case) we have
\[
|\mathcal{T}[\bar{v}] - \mathcal{T}[\bar{w}]|_{1, \omega} \leq C_2 (\varepsilon + \eta^2 \varepsilon^2 + \eta_1^4 \varepsilon^4) |\bar{v} - \bar{w}|_{1, \omega},
\]
where \( C_1, C_2 > 0 \), and for \( \bar{v}, \bar{w} \) such that \( \| \bar{v} \|_{1, \omega} < \infty \) and \( \| \bar{w} \|_{1, \omega} < \infty \) we have
\[
|\mathcal{T}[\bar{v}] - \mathcal{T}[\bar{w}]|_{1, \omega} \leq C_2 (\varepsilon + \eta^2 \varepsilon^2 + \eta^4 \varepsilon^4) |\bar{v} - \bar{w}|_{1, \omega}. \tag{3.72}
\]
Estimates (3.71) and (3.72) imply that \( \mathcal{T} \) is a contraction, provided \( C_2 (\varepsilon + \eta^2 \varepsilon^2 + \eta^4 \varepsilon^4) < 1 \), which can be uniformly fulfilled for \( 0 < \varepsilon \ll 1 \) if \( \eta \in \mathcal{J} \). The contraction-mapping theorem yields existence of a unique solution to (3.70) and therefore also to (3.62). The bound (3.64) follows immediately from (3.71) for \( 0 < \varepsilon \ll \min \{1, 1/\eta \} \) . Inserting the power series \( \bar{v}(x_1, x_2) = \sum_{(k, \ell) \in \mathbb{N}^2} \frac{\partial^k \partial^\ell f(0, 0)}{k! \ell!} x_{x_x}^k \gamma_x^\ell \) in (3.62a), we find
\[
\partial_{x_x} \bar{v}(0, 0) = \frac{\partial_{x_x} \tilde{f}(0, 0) - a_1}{q(1)} = \left(1.15a \right) \frac{\partial_{x_x} \tilde{f}(0, 0)}{q(1)} - a_1, \tag{3.73}
\]
from which we conclude that (3.65) holds true.

**Step 3: Continuous dependence on \( \eta \).** From the fixed-point equation (3.70) it follows that for a given fixed point \( \bar{v} \) for given right-hand side \( f \) with \( \| f \|_{1, \omega} < \infty \), we have
\[
|\bar{v}(\cdot, \eta_1) - \bar{v}(\cdot, \eta_2)|_{1, \omega} \lesssim C_1 \| \tilde{f}(\cdot, \eta_1) - \tilde{g}(\cdot, \eta_2) \|_{1, \omega} + C_2 \| \varepsilon \|_{1, \omega} + \eta^2 \varepsilon^2 + \eta_1^4 \varepsilon^4 \| \varepsilon \|_{1, \omega} + \eta^4 \varepsilon^4 \| \varepsilon \|_{1, \omega}.$
where \(C_1, C_2 > 0\) are as in (3.71) and (3.72). Under the same smallness assumption on \(\varepsilon\) giving \(C_2 (\varepsilon + \eta_1^2 \varepsilon^2 + \eta_1^4 + \varepsilon^4) < 1\), we get from (3.71) that \(|\hat{v}(\cdot, \eta_2)|_{L^\infty}\) is uniformly bounded and hence from (3.73) that
\[
|\hat{v}(\cdot, \eta_1) - \hat{v}(\cdot, \eta_2)|_{L^\infty} \to 0 \quad \text{as} \quad \eta_1 \to \eta_2 \quad \text{in} \quad \mathcal{F}.
\]
\[\square\]

**Construction of solutions for** \(x \gg 1\). Like in the previous paragraph, by abuse of notation we suppress the dependence of \(v\) and \(f\) on \(\eta\) except for those parts, in which the continuous dependence on \(\eta\) is considered.

Our aim is to prove the following result which is analogous to Giacomelli, Gnann, Knüpfer, and Otto (2014, Lem. 6.6):

**Lemma 3.19** (resolvent equation for \(x \gg 1\)). Assume that \(f \in C((0, \infty) \times \mathbb{R})\) with \(f(\cdot, \eta) \in C^\infty((0, \infty))\) for every \(\eta \in \mathbb{R}\) satisfies \((G_\eta)\). Then there exists a two-parameter family of smooth solutions of (3.61) of the form
\[
v(x) = v(x^\epsilon) + a_3 v^{(3)}(x) + a_4 v^{(4)}(x) \quad \text{for} \quad x > 0, \quad \text{where} \quad a_1, a_2 \in \mathbb{R},
\]
\(v(x^\epsilon)\) is a smooth solution to (3.61), and \(v^{(3)} = v^{(3)}(x), v^{(4)} = v^{(4)}(x)\) are two linearly independent solutions to (3.61) with \(f(x) \equiv 0\), such that \(v(x^\epsilon), v^{(3)}, \) and \(v^{(4)}\) satisfy \((G_\eta)\).

**Proof of Lemma 3.19.** We first study the homogeneous version of the resolvent equation, i.e., equation (3.61) with \(f(x) \equiv 0\). Afterwards we find a particular solution to (3.61) in the general case. Notice that in the case \(\eta = 0\), equation (3.61) takes the form
\[
xv + q(D_x)v = f \quad \text{for} \quad x > 0,
\]
which is the same form as Giacomelli, Gnann, Knüpfer, and Otto (2014, Eq. (6.1)) except for the fact that \(q(\zeta)\) is a different fourth-order polynomial. However, the leading coefficient of \(q(\zeta)\) here is the same as that of the polynomial \(p(\zeta)\) in Giacomelli, Gnann, Knüpfer, and Otto (2014, Eq. (6.1)), which allows to apply precisely the same strategy as in the proof of Giacomelli, Gnann, Knüpfer, and Otto (2014, Lem. 6.6), leading to the statement of Lemma 3.19 for \(\eta = 0\). Hence, in what follows we concentrate on the case \(\eta \neq 0\).

**Step 1: Finding two linearly independent solutions to the homogeneous resolvent equation.**

Collecting only the leading-order terms in the homogeneous version of the resolvent equation (3.61), we find that the principal part in \(\frac{d}{dx}\) and \(\eta\) is given by
\[
x^4 \left(4 \frac{d^4}{dx^4} + 2 \eta^2 \frac{d^2}{dx^2} + 4 \eta^4\right) = x^4 \left(4 \frac{d^4}{dx^4} + 2 \eta^2 \frac{d^2}{dx^2} + 4 \eta^4\right)^2 = x^4 \left(\frac{d}{dx} - \eta\right)^2 \left(\frac{d}{dx} + \eta\right)^2.
\]
The kernel of this operator is spanned by \(\{e^{\pm \eta |x|}, x e^{\pm \eta |x|}\}\). In what follows, we will construct two linearly independent solutions that are stable as \(x \to \infty\), so that we neglect algebraic factors for the time being - we are lead to factor out \(e^{-|\eta|x}\) from \(v\), i.e., we define
\[
v = e^{-|\eta|x} \phi
\] (3.74)
and hence the homogeneous version of the resolvent equation (3.61) turns into
\[
x\phi + q(D_x) \phi + \eta |x| ( -4D_x^4 - 3D_x^2 + D_x + \frac{9}{8}) \phi + \eta^2 x^2 (4D_x^2 + 6D_x + 2) \phi = 0.
\]
Next, it turns out to be convenient to apply the scaling \(\eta |x| \to x\), leading to the simplification
\[
|\eta|^{-1} x\phi + q(D_x) \phi + x ( -4D_x^4 - 3D_x^2 + D_x + \frac{9}{8}) \phi + x^2 (4D_x^2 + 6D_x + 2) \phi = 0.
\] (3.75)
We will now consider the leading-order operator \(x^2 \mathcal{A}\), where
\[
\mathcal{A} := \left(4 \frac{d^4}{dx^4} - 4 \frac{d^2}{dx^2} + 4\right) \left( \frac{d^4}{dx^4} + \frac{1}{2} D_x + \frac{1}{4}\right) = \left(4 \frac{d^4}{dx^4} - 2\right) \left( D_x + 1\right) \left( D_x + \frac{1}{2}\right).
\] (3.76a)
The criteria for selecting \(\mathcal{A}\) as above are that \(x^2 \mathcal{A}\) contains all terms from (3.75) having the maximal number of 4 derivatives while at the same time collecting the terms \(x^2 \left(4D_x^2 + 6D_x + 2\right)\) with the largest pre-factor \(x^2\) (which scaling-wise give the leading order in \(x\) for (3.75) as \(x \to \infty\)). Furthermore, it has the useful factorization (3.76a), which makes it easier to find an inverse with appropriate estimates. Utilizing (3.76a) in (3.75), we arrive at
\[
\mathcal{A} \phi = -|\eta|^{-1} x^{-1} \phi + \frac{3}{2} x^{-1} \frac{d}{dx} \left( D_x^2 - \frac{1}{4}\right) \phi - 3x^{-1} \left( D_x^2 + D_x + \frac{3}{8}\right) \phi.
\] (3.76b)
Now we notice that the powers $x^{-\frac{1}{2}}$ and $x^{-1}$ are in the kernel of the operator $A$, so that we may set
\[ \phi = x^{\alpha}(1 + \psi), \quad \text{where} \quad \alpha \in \{-\frac{1}{2}, -1\}. \] (3.77)
Our goal is to construct solutions $\psi$ with $\psi \to 0$ as $x \to \infty$. Indeed, undoing the transformations (3.74), (3.77), and the scaling of $x$, this will result in two linearly independent solutions $v^{(1)}$ and $v^{(2)}$ to the homogeneous version of the resolvent equation (3.61) with the asymptotic behavior
\[ v^{(j)} = x^{\alpha_j}e^{-|q|\sqrt{x}(1 + o(1))} \quad \text{as} \quad x \to \infty, \quad \text{where} \quad \alpha_1 = -\frac{1}{2}, \ \alpha_2 = -1. \] (3.78)
The decay as given in Definition 3.11 will also immediately follow from the fixed-point argument.

In the two situations formulated in (3.77) we consider
\[ A_{-\frac{1}{2}} := \left( \frac{d}{dx} - 2 \right)^2 (D_x + \frac{1}{2}) D_x \quad \text{and} \quad A_{-1} := \left( \frac{d}{dx} - 2 \right)^2 (D_x - \frac{1}{2}) D_x, \] (3.79a)
and with this notation (3.76) becomes
\[ A_{\alpha} \psi = -|\eta|^{-1} x^{-1} + x^{-1} P_{\alpha}(0) + x^{-2} R_{\alpha}(0) \]
\[ -|\eta|^{-1} x^{-1} \psi + x^{-1} \frac{d}{dx} Q_{\alpha}(D_x) \psi + x^{-2} R_{\alpha}(D_x) \psi. \] (3.79b)
where $Q_{\alpha}(\zeta)$, $P_{\alpha}(\zeta)$, and $R_{\alpha}(\zeta)$ are real-valued polynomials of degree less or equal to 2.

Next we invert the operator $A_{\alpha}$ and prove estimates in suitable norms. We first note the product structure of $A_{\alpha}$ induces a product structure of the solution operator $T = T_{-2}\mathbb{S}_{\frac{1}{2}}S_0$, where $T_{-2}$ is a solution of $(\frac{d}{dx} - 2)T_{-2} = \varrho$ and $S_\mu \varrho$ solves $(D_x - \mu)S_\mu \varrho = \varrho$. We use the explicit definitions (which, abstractly speaking, have the role of fixing the behavior of $T_{-2} \varrho$ and $S_\mu \varrho$ as $x \to \infty$):
\[ T_{-2} \varrho(x) := -\int_{-\infty}^{x} e^{2(x-x')} \varrho(x') \, dx' = -\int_{0}^{x} e^{-2x'} \varrho(x+x') \, dx' \] (3.80)
\[ S_{\mu} \varrho(x) := -x^\mu \int_{-\infty}^{x} \varrho(x') \, dx' = -\int_{1}^{x} x^{-\mu} \varrho(x) \, dx. \] (3.81)
For proving estimates for $T_{-2}$, observe that for $x \geq x_0 > 0$ we have
\[ x |T_{-2} \varrho(x)| \leq \left( \int_{x_0}^{\infty} e^{-2x'} \varrho(x+x') \, dx' \right) \sup_{x \geq x_0} x |\varrho(x)| \leq \frac{1}{2} \sup_{x \geq x_0} x |\varrho(x)|, \]
that is,
\[ \sup_{x \geq x_0} x |T_{-2} \varrho(x)| \leq \frac{1}{2} \sup_{x \geq x_0} x |\varrho(x)|, \]
and because of $\frac{d}{dx} T_{-2} \varrho = 2T_{-2} \varrho + \varrho$ and therefore also $\frac{d}{dx} T_{-2} \varrho = 2 \frac{d}{dx} T_{-2} \varrho + \frac{d}{dx} \varrho$, we have
\[ \max_{j=0,\ldots,J+1} x \left( \frac{d}{dx} T_{-2} \varrho(x) \right) \leq J \max_{j=0,\ldots,J} x \left( \frac{d}{dx} \varrho(x) \right) \text{ for every } J \in \mathbb{N}_0. \] (3.82)
Considering $S_{\mu}$, where $\mu \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$, we estimate
\[ x |S_{\mu} \varrho(x)| \leq x \left( \int_{1}^{\infty} r^{-\mu} x^{-1} x^{-1} \frac{dr}{r} \right) \sup_{x \geq x_0} x |\varrho(x)| = \frac{1}{1 + \mu} \sup_{x \geq x_0} x |\varrho(x)|, \]
that is,
\[ \sup_{x \geq x_0} x |S_{\mu} \varrho(x)| \leq \frac{1}{1 + \mu} \sup_{x \geq x_0} x |\varrho(x)|, \]
and the relation $D_x^{k+1} S_{\mu} \varrho = \mu D_x^k S_{\mu} \varrho + D_x^k \varrho$ leads to the upgraded version
\[ \max_{k=0,\ldots,K+1} x \left| D_x^k S_{\mu} \varrho(x) \right| \leq K \max_{k=0,\ldots,K} x \left| D_x^k \varrho(x) \right| \text{ for every } K \in \mathbb{N}_0. \] (3.83)
Now we use the factorization $T = T_{-2}\mathbb{S}_{\frac{1}{2}}S_0$ together with the commutation relation $\frac{d}{dx} D_x = (D_x + 1) \frac{d}{dx}$, so that (3.82) and (3.83) result in the maximal-regularity estimate
\[ |T \varrho|_{J+2,K+2,x_0} \lesssim_{J,K} |\varrho|_{J,K,x_0} \text{ for all } J, K \in \mathbb{N}_0, \] (3.84a)
where
\[ |\varrho|_{J,K,x_0} := \max_{k=0,\ldots,K} x \sup_{x \geq x_0} \left| \frac{d}{dx}^k D_x^j \varrho(x) \right|. \] (3.84b)
Notice that estimate (3.84a) reflects a regularity gain of two $\frac{d}{dx}$ derivatives and two $D_x$-derivatives, which is the maximal gain in regularity to be expected from (3.79a).

Next, we discuss the fixed-point argument for the full equation (3.79b) on the space

$$\Psi_{J,K,x_0} := C^\infty([x_0, \infty))^{(J+2,K+2,x_0)}.$$ 

First apply the solution operator $T$ to both sides of (3.79b) and obtain $\psi = T[\psi]$ for

$$T[\psi] := T\left[ -|\eta|^{-1} x^{-1} + x^{-1} P_\alpha(0) + x^{-2} R_\alpha(0) \right]$$
$$+ T\left[ -|\eta|^{-1} x^{-1} \psi + x^{-1} \frac{d}{dx} Q_\alpha(D_x) \psi + x^{-1} P_\alpha(D_x) \psi + x^{-2} R_\alpha(D_x) \psi \right].$$

(3.85)

For establishing the self-map and contraction property of $T$ in $\Psi_{J,K,x_0}$, observe that

$$\left| T\left[ -|\eta|^{-1} x^{-1} + x^{-1} P_\alpha(0) + x^{-2} R_\alpha(0) \right] J+2,K+2,x_0 \right| \lt C_1 \left( |\eta|^{-1} + |P_\alpha(0)| + |R_\alpha(0)| x_0^{-1} \right)$$

(3.86a)

for $C_1 = C_1(J,K) > 0$ and secondly

$$\left| T\left[ -|\eta|^{-1} x^{-1} \psi + x^{-1} \frac{d}{dx} Q_\alpha(D_x) \psi + x^{-1} P_\alpha(D_x) \psi + x^{-2} R_\alpha(D_x) \psi \right] J+2,K+2,x_0 \right| \lt C_2 \left( |\eta|^{-1} x_0^{-1} + x_0^{-1} + x_0^{-2} \right) \psi_{J+2,K+2,x_0}$$

(3.86b)

for $C_2 = C_2(J,K) > 0$. From (3.86) it is immediate that $T$ is a self-map in $\Psi_{J,K,x_0}$ and estimate (3.86b) also yields the contraction property of $T$ provided

$$C_2 \left( |\eta|^{-1} x_0^{-1} + x_0^{-1} + x_0^{-2} \right) < 1$$

(3.87)

which is true for $x_0 \gg J,K,|\eta| 1$ uniformly in $\eta \in J \subseteq \mathbb{R}\{0\}$. Hence, for all $J,K \in \mathbb{N}_0$ there exists an $x_0 = x_0(J,K) > 0$ such that (3.86) and therefore also (3.79b) has a solution $\psi = \psi_{J,K}$ with finite norm $\|\psi\|_{J+2,K+2,x_0}$. Next, we verify the continuity in $\eta$.

For a fixed point $\psi$ to (3.85) we have with the same constants $C_1, C_2 > 0$ of (3.86)

$$|\psi(\cdot, \eta_1) - \psi(\cdot, \eta_2)|_{J+2,K+2,x_0} \lt C_1 \frac{|\eta_1 - \eta_2|}{|\eta_1| |\eta_2|} + C_2 \left( |\eta_1|^{-1} x_0^{-1} + x_0^{-1} + x_0^{-2} \right) |\psi(\cdot, \eta_1) - \psi(\cdot, \eta_2)|_{J+2,K+2,x_0}$$

(3.88)

$$+ C_2 \frac{|\eta_1 - \eta_2|}{x_0 |\eta_1| |\eta_2|} |\psi(\cdot, \eta_2)|_{J+2,K+2,x_0}.$$ 

(3.89)

Under the same smallness assumption (3.87) as for the contraction property, we obtain from (3.85) and (3.86) uniform boundedness of $|\psi(\cdot, \eta)|_{J+2,K+2,x_0}$ for $\eta \in J \subseteq \mathbb{R}\{0\}$ and from (3.89) also

$$|\psi(\cdot, \eta_1) - \psi(\cdot, \eta_2)|_{J+2,K+2,x_0} \to 0 \quad \text{as} \quad \eta_1 \to \eta_2 \quad \text{in} \quad J.$$

By standard theory, the solution $\psi = \psi_{J,K}$ can be uniquely extended to a solution of (3.79b) for all $x > 0$ that is also continuous for $\eta \in J \subseteq \mathbb{R}\{0\}$. Now for pairs of indices $(J,K), (J',K') \in \mathbb{N}_0^2$ such that $J \leq J'$ and $K \leq K'$ we may choose $x_0 > 0$ large enough such that the fixed points $\psi_{J,K}$ and $\psi_{J',K'}$ exist. Because $\Psi_{j',K',x_0} \subseteq \Psi_{J,K,x_0}$, the uniqueness part of the contraction-mapping theorem gives $\psi_{J,K}(x) = \psi_{J',K'}(x)$ on $[x_0, \infty)$ and the unique extension property yields the same identity on the whole interval $(0, \infty)$. Undoing the transformations (3.74), (3.77), and the scaling of $x$, the two values $\alpha = -\frac{1}{2}$ and $\alpha = -1$ yield two solutions $v^{(3)}$ and $v^{(4)}$ with asymptotics as in (3.78) that are therefore linearly independent. Note that $v^{(3)}$ and $v^{(4)}$ also meet the decay and continuity properties of condition $(G_{x_0})$ of Definition 3.11, because $\|\psi\|_{J+2,K+2,x_0} < \infty$, where $J \in \mathbb{N}_0$ (or $K \in \mathbb{N}_0$) can be chosen arbitrarily large if $x_0 > 0$ is sufficiently large.

**Step 2: Finding a particular solution for the full equation.** Recall that we only consider the case $\eta \neq 0$, as justified at the beginning of the proof. Although the reasoning below for $\eta \neq 0$ is similar to the construction of a particular solution for $\eta = 0$ (cf. Giacomelli, Gnann, Knüpfer, and Otto (2014, Lem. 6.6(a)) for details), we nevertheless provide all details in the situation at hand for the sake of clarity. We apply again the scaling $x |\eta| \mapsto x$ to equation (3.61) and obtain

$$|\eta|^{-1} x v + q(D_x)v - x^2 r(D_x)v + x^4 v = f \quad \text{for} \quad x > 0.$$ 

(3.90)
By expressing equation (3.90) in terms of $\frac{d}{dx}$-derivatives rather than $D_x$-derivatives, after which we divide by $x^4$, we can rewrite it in the following form

$$\left(\frac{d^2}{dx^2} - 1\right)^2 v = x^{-4} f - x^{-1} P \left(x^{-1}, \frac{d}{dx}, \eta\right) v,$$  

(3.91)

where $P(a, \xi, \eta)$ is a third-order polynomial in $(a, \xi)$ given by

$$P(a, \xi, \eta) = 5\xi^3 + 3a\xi^2 - \frac{9}{8}a^2\xi - 5\xi + |\eta|^{-1} a^2 - a.$$  

(3.92)

We now use the truncation function $\chi_n(x) = \chi_1 \left(\frac{x}{n}\right)$, where $\chi_1$ is a smooth positive function $\chi_1(x) = 1$ on $[2, \infty)$ and $\chi_1(x) = 0$ on $(-\infty, 1]$. Then consider the equation

$$\left(\frac{d^2}{dx^2} - 1\right)^2 v = \chi_n \left(x^{-4} f - x^{-1} P \left(x^{-1}, \frac{d}{dx}, \eta\right) v\right).$$  

(3.93)

We will find a solution for truncated data for $n \gg 1$ by a contraction principle, and then we use unique continuation to extend it to a solution of (3.61) on the whole interval $(0, \infty)$. Note that the operator on the right-hand side of (3.93) is translation-invariant in $x$, and thus in order to find a particular solution to (3.93), we will apply a fixed-point argument, where we invert the operator $\left(\frac{d^2}{dx^2} - 1\right)^2$ by convolving with the associated fundamental solution $g$, defined as the unique solution to the equation

$$\left(\frac{d^2}{dx^2} - 1\right)^2 g = \delta_0$$  

subject to $\lim_{x \to \pm \infty} g(x) = 0$.

By classical ODE methods (e.g. Fourier transforming the above equation), we find

$$g(x) = \begin{cases} \frac{1}{4} e^{-x} + \frac{1}{4} xe^{-x} & \text{for } x > 0, \\ \frac{1}{4} e^{-x} - \frac{1}{4} xe^{-x} & \text{for } x < 0. \end{cases}$$

Using the convolution with $g$, we desire to solve the following equation (which is equivalent to (3.93)):

$$\mathcal{T}_n[v] = v \quad \text{with} \quad \mathcal{T}_n[v] := g * (\chi_n x^{-4} f) - g * (\chi_n x^{-1} P \left(x^{-1}, \frac{d}{dx}, \eta\right) v).$$  

(3.94)

In order to obtain a contraction, note that for $\nu \in (0, 1)$ and for $j = 1, \ldots, 4$, $e^{\nu |x|} |\frac{d}{dx}|^j g(x)$ is exponentially decaying as $x \to \pm \infty$ and absolutely integrable. Based on this, for $\nu \in (0, 1)$ and for $N \in \mathbb{N}$ we define the norms $||v||_{N, \nu, \infty}$ and the space $X_N$ as follows:

$$||v||_{N, \nu, \infty} := \sup_{x \in \mathbb{R}} \max_{0 \leq j < N} \left|e^{\nu x} |\frac{d}{dx}|^j v(x)\right| \quad \text{and} \quad X_N := \left\{v \in C^4(\mathbb{R}) : ||v||_{N, \nu, \infty} < \infty\right\}.$$  

Then using the decay of $e^{\nu |x|} |\frac{d}{dx}|^j g(x)$ and the property that $e^{\nu f} (f * g) = (e^{\nu f}) * (e^{\nu g})$, we find

$$\sup_{x \in \mathbb{R}} \left|e^{\nu x} |\frac{d}{dx}|^j \left(\frac{d}{dx}\right)^{\min(4,j)} \left(g * (\chi_n x^{-4} f) - g * (\chi_n x^{-1} P \left(x^{-1}, \frac{d}{dx}, \eta\right) v)\right)\right|$$  

$$\leq \sup_{x \in \mathbb{R}} \left|e^{\nu x} |\frac{d}{dx}|^j \left(\chi_n x^{-1} P \left(x^{-1}, \frac{d}{dx}, \eta\right) v\right)\right|$$

giving the first bound

$$\sup_{x \in \mathbb{R}} \left|e^{\nu x} |\frac{d}{dx}|^j \left(g * (\chi_n x^{-4} f)\right)\right| \leq C_2 \frac{1 + |\eta|^{-1}}{n} \sup_{x \geq n, 1 \leq k \leq \max(j - 3)} \max_{0 \leq j < N} \left|e^{\nu x} \frac{d^k}{dx^k} v\right|$$  

(3.95)

for some $C_2 = C_2(j) > 0$. Along the same lines, we also get the following estimate for the first term in (3.93):

$$\sup_{x \in \mathbb{R}} \left|e^{\nu x} |\frac{d}{dx}|^j \left(g * (\chi_n x^{-1} P \left(x^{-1}, \frac{d}{dx}, \eta\right) v)\right)\right| \leq C_1 \sup_{x \in \mathbb{R}} \left|e^{\nu x} \left(\frac{d}{dx}\right)^{\max(0,j-4)} (\chi_n f)\right|$$  

(3.96)

for some $C_1 = C_1(j) > 0$. The above inequalities (3.95) and (3.96) imply

$$|\mathcal{T}_n[v]|_{j, \nu, \infty} \leq C_1 |\chi_n f|_{j-4, \nu, \infty} + C_2 \frac{1 + |\eta|^{-1}}{n} |v|_{j, \nu, \infty} \quad \text{for } j \geq 4$$  

(3.97a)

and

$$|\mathcal{T}_n[v] - \mathcal{T}_n[w]|_{4, \nu, \infty} \leq C_2 \frac{1 + |\eta|^{-1}}{n} |v - w|_{4, \nu, \infty}.$$

(3.97b)

As the bounds (3.97) imply that for $C_2 \frac{1 + |\eta|^{-1}}{n} < 1$, which is true for large enough $n$ provided $\eta \in \mathcal{F} \subseteq \mathbb{R} \setminus \{0\}$, the mapping $\mathcal{T}_n$ is a contraction on the space $X_\nu$, we have existence of a unique fixed point to
For the fixed point properties (i) and (ii) of Definition 3.11. Hence, it remains to prove continuity in $f$. Now taking (3.93) for any $v \in (0,1)$, which by undoing the scaling $x \mapsto |\eta| x$ in particular implies the decay bounds (i) and (ii) of Definition 3.11. In order to obtain bounds on more than 4 derivatives, we note that

$$
|v|_{N,\nu,\infty} \leq |g \ast (\chi_n f)|_{N,\nu,\infty} + |g \ast (\chi_n x^{-1} P_0 (x^{-1}, \frac{d}{dx}) v)|_{N,\nu,\infty} \\
\lesssim_N |\chi_n f|_{N-4,\nu,\infty} + \frac{1}{n} |v|_{N-1,\nu,\infty},
$$

which can be iterated and yields estimates for any order of derivatives. As the above fixed-point argument also yields uniqueness of solutions, we find that our so-described solution to (3.93) is the same for all $n$, can be proved to exist provided that $f$ satisfies ($G_\infty$), can be extended to all $x > 0$, and satisfies properties (i) and (ii) of Definition 3.11. Hence, it remains to prove continuity in $\eta$.

For the fixed point $v$ we have from (3.94)

$$
|\mathcal{T}_n[v(\cdot, \eta_1)] - \mathcal{T}_n[v(\cdot, \eta_2)]|_{j,\nu} \leq C_1 |f(\cdot, \eta_1) - f(\cdot, \eta_2)|_{j-4,\nu} + C_2 \frac{1 + |\eta_1 - \eta_2|}{n} |v(\cdot, \eta_1) - v(\cdot, \eta_2)|_{j,\nu} + C_2 \frac{|\eta_1 - \eta_2|}{n |\eta_1|} |v(\cdot, \eta_2)|_{j,\nu},
$$

(3.98)

where $j \geq 4$. Therefore, under the assumption $C_2 \frac{1 + |\eta_1 - \eta_2|}{n} < 1$ for $\eta \in \mathcal{F} \subseteq \mathbb{R} \setminus \{0\}$, we have uniform boundedness of $|v(\cdot, \eta_2)|_{j,\nu}$ from (3.97a) and thus continuity in $\mathcal{F}$ because of (3.98). The continuity statement of Definition 3.11(iii) follows by standard ODE theory, since the obtained continuity in $\eta$ propagates from $x \geq 2n$ to all $x > 0$. 

**Proof of existence and uniqueness of solutions to the resolvent equation.** As already noted, the uniqueness result (Lemma 3.17) is not difficult to obtain, so we only sketch the arguments.

**Proof sketch of Lemma 3.17 (uniqueness).** It suffices to choose $\delta \in (0, \frac{1}{10})$ and to test the homogeneous version of the resolvent equation (3.61) with a cut-off $\chi_n \ast v$ in the inner product $\langle \cdot, \cdot \rangle_{-\delta,4}$, where $\chi_n(x) := \chi \left( \frac{\log x}{n} \right)$ with $\chi \in C^\infty((-\infty, \infty))$, $\chi(s) = 1$ on $B_1(0)$, and $\chi(s) = 0$ on $(B_2(0))^c$. This is analogous to the treatment of the time evolution equation (3.6) in §3.2.1 except for the fact that now the cut-off $\chi_n$ has to be computed with the differential operators $q(D_x)$ and $r(D_x)$. Note that the commutators $[q(D_x), \chi_n]$ and $[r(D_x), \chi_n]$ exclusively consist of addends in which at least one $D_x = \delta_x$ (where $s := \log x$) acts on $\chi_n$, giving a pre-factor $O \left( \frac{1}{\delta^4} \right)$. Otherwise following the computations going from (3.6) to (3.13), we obtain

$$
|\chi_n v|_{-\delta,4}^2 + \sum_{\ell=0}^2 \eta^{2\ell} |\chi_n v|^\ell_{-\delta-\ell} \leq O \left( \frac{1}{n} \right).
$$

Now taking $n \to \infty$, we conclude that $v(x) = 0$ as desired. 

The technical Lemmata 3.18 and 3.19 provide all ingredients for proving existence to the resolvent equation (3.61) as stated in Proposition 3.16. This is essentially the same as Giacomelli, Gnann, Knüpfer, and Otto (2014, Prop. 6.3) except for the proof of continuity in $\eta$.

**Proof of Proposition 3.16.** We use the notation of Lemmata 3.18 and 3.19, i.e., $v^{(0)} + a_1 v^{(1)} + a_2 v^{(2)}$ with $a_1, a_2 \in \mathbb{R}$ and $v^{(2)} + a_3 v^{(3)} + a_4 v^{(4)}$ with $a_3, a_4 \in \mathbb{R}$ define two-parameter solution families to (3.61) satisfying the decay conditions ($G_0$) (cf. Definition 3.10) and ($G_\infty$) (cf. Definition 3.11), respectively. We first show that the family $(v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)})$ is linearly independent. Afterwards we show that we can construct a $v \in C^\infty((0, \infty))$ solving the resolvent equation (3.61) and meeting ($G_0$) and ($G_\infty$). Note that $v$ is unique, since uniqueness already holds for a larger class of functions (cf. Lemma 3.17).

**Step 1: linear independence.** Suppose that $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ exist with

$$
a_1 v^{(1)}(x) + a_2 v^{(2)}(x) - a_3 v^{(3)}(x) - a_4 v^{(4)}(x) \equiv 0.
$$

Then we see that $v := a_1 v^{(1)} + a_2 v^{(2)} = a_3 v^{(3)} + a_4 v^{(4)}$ satisfies ($G_0$) and ($G_\infty$) and is a solution to the resolvent equation (3.61) with $f(x) = 0$. Lemma 3.17 gives $v(x) \equiv 0$, i.e., $a_1 v^{(1)}(x) + a_2 v^{(2)}(x) = 0$ and $a_3 v^{(3)}(x) + a_4 v^{(4)}(x) = 0$ and linear independence of the tuples $(v^{(1)}, v^{(2)})$ and $(v^{(3)}, v^{(4)})$ implies $a_1 = a_2 = a_3 = a_4 = 0$.

**Step 2: existence.** For given right-hand side $f \in C^\infty((0, \infty))$ satisfying ($G_0$) with $(f, \frac{\partial f}{\partial \eta}) (0,0) = (0,0)$ and ($G_\infty$), the function $v^{(0)} - v^{(0)}$ defines a smooth solution to the homogeneous version of the resolvent equation (3.61). Since $(v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)})$ is a fundamental system to the
left-hand side of (3.61), there exist \((a_1, a_2, a_3, a_4) \in \mathbb{R}^4\) such that
\[
v^{(x)} - v^{(0)} = a_1 v^{(1)} + a_2 v^{(2)} - a_3 v^{(3)} - a_4 v^{(4)}.
\] (3.99)

Now define \(v := v^{(0)} + a_1 v^{(1)} + a_2 v^{(2)} = v^{(x)} + a_3 v^{(3)} + a_4 v^{(4)}\). Then by construction \(v\) satisfies \((G_0)\) and \((G_{x, \eta})\) is and the solution to the inhomogeneous resolvent equation (3.61).

**Step 3: Continuity in \(\eta\).** We know from Lemma 3.18 that the analytic extensions of \(v^{(0)}\), \(v^{(1)}\), and \(v^{(2)}\) depend continuously on \(\eta \in \mathcal{F} \subset \mathbb{R}\) in a small \(\mathcal{F}\)-dependent neighborhood of the origin. In particular \(\frac{d^j v^{(j)}}{dx^j}\) for \(j = 0, 1, 2\) and \(k \geq 0\) depend continuously on \(\eta\) at a given sufficiently small \(x > 0\). By standard ODE theory this implies that the extensions of \(\frac{d^j v^{(j)}}{dx^j}\) for \(j = 0, 1, 2\) and \(k \geq 0\) to all \(x > 0\) depend continuously on \(\eta\). Lemma 3.19 further yields that \(\frac{d^j v^{(j)}}{dx^j}\), \(\frac{d^j v^{(3)}}{dx^j}\), and \(\frac{d^j v^{(4)}}{dx^j}\) for \(k \geq 0\) depend continuously on \(\eta \in \mathbb{R} \setminus \{0\}\). Hence, it suffices to show that the coefficients \(a_1, a_2, a_3,\) and \(a_4\) of (3.99) are continuous functions of \(\eta \in \mathbb{R} \setminus \{0\}\).

Note that differentiation of (3.99) yields
\[
\frac{d^k v^{(k)}}{dx^k} = a_1 \frac{d^k v^{(1)}}{dx^k} + a_2 \frac{d^k v^{(2)}}{dx^k} - a_3 \frac{d^k v^{(3)}}{dx^k} - a_4 \frac{d^k v^{(4)}}{dx^k}
\]
for \(k = 0, 1, 2, 3\).

Then linear independence of \((v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)})\) implies by Liouville’s formula that the Wronskian
\[
\det \begin{pmatrix} v^{(1)} & v^{(2)} & v^{(3)} & v^{(4)} \\ \frac{d v^{(1)}}{dx} & \frac{d v^{(2)}}{dx} & \frac{d v^{(3)}}{dx} & \frac{d v^{(4)}}{dx} \\ \frac{d^2 v^{(1)}}{dx^2} & \frac{d^2 v^{(2)}}{dx^2} & \frac{d^2 v^{(3)}}{dx^2} & \frac{d^2 v^{(4)}}{dx^2} \\ \frac{d^3 v^{(1)}}{dx^3} & \frac{d^3 v^{(2)}}{dx^3} & \frac{d^3 v^{(3)}}{dx^3} & \frac{d^3 v^{(4)}}{dx^3} \end{pmatrix}
\]
is non-vanishing for all \(x > 0\), so that the continuous dependence of \(a_1, a_2, a_3,\) and \(a_4\) on \(\eta \in \mathbb{R} \setminus \{0\}\) follows by Cramer’s rule.

3.4.2. **Time discretization for the linear equation.** In this section, we prove existence, uniqueness, and maximal regularity for the linear equation (1.19) (cf. Giacomelli, Gnann, Knüpfer, and Otto (2014, Prop. 7.6) for the equivalent result for the 1 + 1-dimensional counterpart of the free boundary problem (1.1)):

**Proposition 3.20.** Suppose \(I = [0, T] \subseteq [0, \infty), \hat{k}, \tilde{k}, \hat{\tilde{k}} \in \mathbb{N}\) with \(\hat{k} \geq 3, \tilde{k} \geq 4, \hat{\tilde{k}} \geq 4,\) and \(\delta \in (0, \frac{1}{10})\).

Then for every locally integrable \(f: I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}\) such that \(\|f\|_{\text{rhs}} < \infty\), there exists exactly one solution \(v: I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}\) of the linear degenerate-parabolic problem (1.19a) that is locally integrable with \(\|v\|_{\text{sol}} < \infty\) and \(v|_{t=0} = v^{(0)}\). This solution obeys the maximal-regularity estimate
\[
\|v\|_{\text{sol}} \lesssim_{\hat{k}, \tilde{k}, \delta, a} \left\|v^{(0)}\right\|_{\text{init}} + \|f\|_{\text{rhs}}.
\] (3.100)

Moreover, \(v\) is continuous, the function \(q(D_x)v + D^2_x r(D_x)v + D^4_y v\) is continuous and defined classically, and the initial condition constraint \(v|_{t=0} = v^{(0)}\) holds in the classical sense.

The last statement of the proposition follows from \(\|v\|_{\text{sol}} < \infty\) via Lemmata 3.9 and 3.3, and the approximation results from Corollary 3.15, and is rather independent from the rest of the proof.

The proof of the main statement of Proposition 3.20 will proceed as follows:

(i) On each discretization interval \([m-1]\delta t, m\delta t\), where \(\delta t := \frac{T}{M}\) with \(M \in \mathbb{N}\) and \(m = 1, \ldots, M\), we solve the resolvent problem (3.61) via Proposition 3.16.

(ii) We derive estimates dealing with one discrete time step \(t \to t + \delta t\) for \(\delta t = \frac{T}{M}\), where \(M \in \mathbb{N}\) is arbitrary, which are discretized versions of the estimates that have been heuristically derived in \$3.2\$ (assuming that sufficiently regular solutions to (1.19) exist, see Lemma 3.21). We also use the simple trace estimate (3.57) of Lemma 3.3 to show that the operation of fixing \(v\) at \(t = 0\) is robust with respect to our approximation.

(iii) We perform a piecewise constant in time approximation of the data \(f^{(m\delta t)}\) and a piecewise affine in time interpolation of the solutions \(v^{(m\delta t)}\) in order to have objects defined in continuous time, while retaining the estimates (ii).

(iv) We treat the discretized problem on a non-vanishing interval \(I = [0, T]\) by summing the estimates obtained in (iii).

(v) The solution in the continuum is obtained from (iv) in the limit \(\delta t \to 0\), using (ii) coupled with a compactness argument. By (i), (ii), this limit also fulfills the linear equation (1.19a) and it recovers the desired initial condition \(v|_{t=0}\). Via (ii) and (iii) we finally obtain the bounds stated in Proposition 3.20.
Consider the discretization (3.60) of the linear equation (1.19a) for a time step $0 \mapsto \delta t$ from §3.4.1, i.e.,

$$\frac{x u^{(\delta t)} - u^{(0)}}{\delta t} + q(D_x)u^{(\delta t)} - |\eta|^2 x^2 r(D_x)v^{(\delta t)} + |\eta|^4 x^4 v^{(\delta t)} = f^{(\delta t)},$$

(3.101)

where $f^{(\delta t)} := \frac{1}{\delta t} \int_0^{\delta t} f(t') \, dt'$. Up to approximating $v^{(0)}, f^{(\delta t)}$ as in Lemmata 3.12 and 3.13, we can assume that $v^{(0)}$ and $f^{(\delta t)}$ satisfy the following assumptions:

\begin{enumerate}[(D1)]  
  
  \item For all $\eta \in \mathbb{R}$ we have $v^{(0)}(\cdot, \eta) \in C^\infty((0, \infty))$ such that for all $j \geq 0$ the function $\partial^j_x v^{(0)}$ is continuous in $\{(x, \eta) : x > 0, \eta \neq 0\}$, and $v^{(0)}$ satisfies $(G_0)$ and $(G_\infty)$ (see Lemma 3.12).
  
  \item For all $\eta \in \mathbb{R}$ we have $f^{(\delta t)}(\cdot, \eta) \in C^\infty((0, \infty))$ such that for all $j \geq 0$ the function $\partial^j_x f^{(\delta t)}$ is continuous in $\{(x, \eta) : x > 0, \eta \neq 0\}$, and $f^{(\delta t)}$ satisfies $(G_0)$ with $(\tilde{f}^{(\delta t)}, \partial_x \tilde{f}^{(\delta t)})(0, 0) = (0, 0)$ and $(G_\infty)$ (see Lemma 3.13).
\end{enumerate}

The following estimates are the discrete analogues of estimates (3.17), (3.34), (3.42), and (3.49):

**Lemma 3.21.** Suppose we are given $v^{(0)}$ and $f^{(\delta t)}$ satisfying (D1) and (D2), respectively. Then there exists a solution $u^{(\delta t)}$ to (3.101) satisfying $(G_0)$ and $(G_\infty)$, and for $\tilde{k} \geq 3$, $k \geq 4$, $\tilde{k} \geq 4$, and $\delta \in (0, \frac{1}{10})$ fulfilling the estimates

\begin{align}
\frac{1}{\delta t} \left( \sum_{\ell=0}^{k} \tilde{C}_\ell \eta^{2\ell} \left| D^{\tilde{k}+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} + \left| D^{k+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} - \sum_{\ell=0}^{\tilde{k}} \tilde{C}_\ell \eta^{2\ell} \left| D^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} - \left| D^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} \right) & \\
+ \sum_{\ell=0}^{\tilde{k}+2} \eta^{2\ell} \left| \partial_t v^{(\delta t)} \right|^{2}_{k+2-\ell, -\delta - \ell} & \lesssim_{k, \alpha} \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| \tilde{D}^{k-\ell}_{x} v^{(\delta t)} \right|^{2}_{k-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}-\ell}_{x} v^{(\delta t)} \right|^{2}_{\tilde{k}-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell} - \left| \tilde{D}^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{k+1-\ell, \alpha - \ell} \left| \tilde{D}^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{k+1-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell}
\end{align}

(3.102a)

with constants $C_\ell > 0$ and $\delta \in (0, \frac{1}{10})$;

\begin{align}
\frac{1}{\delta t} \left( \sum_{\ell=0}^{\tilde{k}} \tilde{C}_\ell \eta^{2\ell} \left| D^{\tilde{k}+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} + \left| D^{k+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} - \sum_{\ell=0}^{k} \tilde{C}_\ell \eta^{2\ell} \left| D^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} - \left| D^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} \right) & \\
+ \sum_{\ell=0}^{\tilde{k}+2} \eta^{2\ell} \left| D^{k+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\tilde{k}+2-\ell, \alpha - \ell} & \lesssim_{k, \alpha} \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| \tilde{D}^{\tilde{k}-\ell}_{x} v^{(\delta t)} \right|^{2}_{\tilde{k}-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell} - \left| \tilde{D}^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{k+1-\ell, \alpha - \ell} \left| \tilde{D}^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{k+1-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell}
\end{align}

(3.102b)

with constants $\tilde{C}_\ell > 0$ and $\tilde{\alpha} \in (0, 1)$;

\begin{align}
\frac{1}{\delta t} \left( \sum_{\ell=0}^{k} \tilde{C}_\ell \eta^{2\ell} \left| D^{\tilde{k}-\ell}_{x} \tilde{q}(D_x)D_x v^{(\delta t)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} + \left| \tilde{q}(D_x)D_x v^{(\delta t)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} \right) & \\
- \sum_{\ell=0}^{k} \tilde{C}_\ell \eta^{2\ell} \left| D^{\tilde{k}-\ell}_{x} \tilde{q}(D_x)D_x v^{(0)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} - \left| \tilde{q}(D_x)D_x v^{(0)} \right|^{2}_{\alpha - \frac{1}{2} - \ell} & \lesssim_{k, \alpha} \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| \tilde{D}^{\tilde{k}-\ell}_{x} v^{(\delta t)} \right|^{2}_{\tilde{k}-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(\delta t)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell} - \left| \tilde{D}^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{k+1-\ell, \alpha - \ell} \left| \tilde{D}^{k+1-\ell}_{x} v^{(0)} \right|^{2}_{k+1-\ell, \alpha - \ell} \left| \tilde{D}^{\tilde{k}+1-\ell}_{x} v^{(0)} \right|^{2}_{\tilde{k}+1-\ell, \alpha - \ell}
\end{align}

(3.102c)
with constants $\tilde{C}_\ell > 0$ and $\delta \in (1, 2)$;

\[
\frac{1}{\delta t} \left( \sum_{\ell = 0}^{k+2} \tilde{C}_\ell \eta^{2\ell} \left| D_x^{k-\ell} (D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v^{(\delta t)} \right|^2 \right)_{\delta + \frac{\ell}{2}} 
- \frac{1}{\delta t} \sum_{\ell = 0}^{k} \tilde{C}_\ell \eta^{2\ell} \left| D_x^{k-\ell} (D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v^{(0)} \right|^2 \right)_{\delta + \frac{\ell}{2}} 
+ \frac{1}{\delta t} \sum_{\ell = 0}^{k+2} \tilde{C}_\ell \eta^{2\ell} \left| D_x^{k-\ell} (D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v^{(0)} \right|^2 \right)_{\delta + \frac{\ell}{2}} 
\leq \frac{1}{2} \sum_{\ell = 0}^{k+2} \eta^{2\ell} \left| D_x^{k-\ell} (D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v^{(\delta t)} \right|^2 \right)_{\delta + \frac{\ell}{2}} 
+ \eta^2 \left| \tilde{q}(D_x)D_x v^{(\delta t)} \right|^2 \right)_{\delta + \frac{\ell}{2}} 
+ \eta^4 \left| D_x v^{(\delta t)} \right|^2 \right)_{\delta + \frac{\ell}{2}} 
(3.102d)
\]

with constants $\tilde{C}_\ell > 0$ and $\delta \in (0, \frac{1}{10})$.

**Proof of Lemma 3.21.** By scaling according to $x \mapsto \frac{x}{\eta}$ and $\eta \mapsto \delta \eta$, we need to solve the resolvent equation (cf. (3.61))

\[
x v^{(\delta t)} + q(D_x) v^{(\delta t)} - |e|^{4} x^2 r(D_x) v^{(\delta t)} + |\theta|^{4} x^4 v^{(\delta t)} = g, \text{ where } g := f^{(\delta t)} + x v^{(0)}. \quad (3.103)
\]

Then $g$ also satisfies $(G_0)$ with $(\hat{g}, \hat{\varepsilon}_x, \hat{g})(0,0) = (0,0)$ and $(G_x)$. Therefore, we are in the setting of Proposition 3.16, i.e., there exists a solution $v^{(\delta t)} \in C^{\infty}((0, \infty))$ to (3.103) satisfying $(G_0)$ and $(G_x)$ such that all derivatives $\partial^j v^{(\delta t)}$ for $j \geq 0$ are continuous in $\{(x, \eta) : x > 0, \eta \neq 0\}$.

The reasonings for the four cases in (3.102) are alike and follow the derivations of (3.17), (3.34), (3.42) and (3.49), respectively. Therefore, we describe the procedure only for the first case (3.102a) and leave the details in the other cases (3.102b) and (3.102c) to the reader. Consider the un-rescaled equation (3.101) and test it with $v^{(\delta t)}$ with respect to the inner product $(\cdot, \cdot)_{\delta}$ for $\delta \in (0, \frac{1}{10})$:

\[
\frac{1}{\delta t} \left( x (v^{(\delta t)} - v^{(0)})^2 \right)_{\delta - \frac{1}{2}} + \frac{1}{\delta t} \left( q(D_x) v^{(\delta t)} - q(D_x) v^{(0)} \right)_{\delta - \frac{1}{2}} 
+ \frac{1}{\delta t} \left( q(D_x) v^{(\delta t)} - q(D_x) v^{(0)} \right)_{\delta - \frac{1}{2}} 
(3.104)
\]

This equation is the discrete version of (3.6), and is treated in the same way, except for the first term, for which rather than (3.7) we obtain

\[
\left( x (v^{(\delta t)} - v^{(0)})^2 \right)_{\delta - \frac{1}{2}} \geq \left( v^{(\delta t)} \right)_{\delta - \frac{1}{2}}^2 - \frac{1}{2} \left( v^{(\delta t)} \right)_{\delta - \frac{1}{2}}^2 - \frac{1}{2} \left( v^{(0)} \right)_{\delta - \frac{1}{2}}^2. \quad (3.105)
\]

The bound (3.105), together with estimates (3.8)-(3.12), allows to obtain the following analogue of (3.13):

\[
\frac{1}{\delta t} \left( \left| v^{(\delta t)} \right|_{\delta - \frac{1}{2}}^2 \right)_{\delta - \frac{1}{2}} + \sum_{\ell = 0}^{k+2} \eta^{2\ell} \left| \tilde{q}(D_x) v^{(\delta t)} \right|^2 \right)_{\delta + \frac{\ell}{2}} \leq 2 \left( f^{(\delta t)} \right)_{\delta - \frac{1}{2}}^2 \quad \text{for } \delta \in (0, \frac{1}{10}). \quad (3.106)
\]

This can be upgraded to a strong estimate like in §3.2.2, by testing (3.101) against (3.14), i.e., against $\eta^{2\ell} (\cdot, -D_x - 1 - 2\varepsilon + 2 - \delta) D_x^{k-\ell} v^{(\delta t)}_{\delta - \frac{1}{2}} =: (\cdot, S)_{\delta - \frac{1}{2}}$. We proceed as for estimates (3.15a)-(3.16), with the only difference appearing for the treatment of the time derivative term (3.15a), which is discretized here. Similar to the discussion leading to (3.105), we find the time-discrete analogue of (3.15a):

\[
\frac{1}{\delta t} \left( x (v - v^{(0)}) \right)_{\delta - \frac{1}{2}} = \frac{1}{\delta t} \left( x (vS)_{\delta - \frac{1}{2}} - (xv^{(0)} - S)_{\delta - \frac{1}{2}} \right) 
(3.107)
\]

Adding the bound (3.107) to the bounds (3.15b)-(3.16), afterwards summing over $\ell$ and reabsorbing by using (3.106) (precisely like in the steps in §3.2.2 leading up to (3.17)), we obtain the time-discrete estimate (3.102a).
We are now ready to prove our main result for the linear equation:

**Proof of Proposition 3.20:** Throughout the proof, estimates may depend on \( \hat{k}, \tilde{k}, \hat{\delta}, \) and \( \delta. \) By writing \( [0, \infty) = \bigcup_{T \in \mathbb{N}} [0, T) \) and using the uniqueness part of Proposition 3.20 for \( T < \infty, \) we may without loss of generality assume that \( T \) is finite. As already mentioned in the discussion preceding the proof, up to approximating as in Lemmata 3.12 and 3.13, we may assume that (D1) and (D2) hold for \( v^{(0)} \) and \( f^{(dt)} \), respectively. We subdivide the proof into several parts.

**Step 1:** Bounds for a single time interval. We first deduce the integral version of (3.102a) which is the discrete analogue of estimate (3.19). To this aim, for equal time steps \( \delta t \) we iteratively solve for

\[ m = 1, 2, \ldots, M = \frac{T}{\delta t} \]

\[ x \frac{v^{(m\delta t)} - v^{((m-1)\delta t)}}{\delta t} + q(D_x)v^{(m\delta t)} - \eta^2 x^2 r(D_x)v^{(m\delta t)} + \eta^4 x^4 v^{(m\delta t)} = f^{(m\delta t)} \]  
(3.108a)

on the time interval \([(m-1)\delta t, m\delta t)\), where

\[ f^{(m\delta t)} := \frac{1}{\delta t} \int_{(m-1)\delta t}^{m\delta t} f(t') \, dt'. \]  
(3.108b)

We then apply estimate (3.102a) with the substitutions

\[ v^{(\delta t)} \mapsto v^{((m-1)\delta t)}, \quad v^{(0)} \mapsto v^{((m-1)\delta t)}, \quad f^{(0)} \mapsto f^{(m\delta t)}, \]  
(3.109)

and we sum the resulting estimates over \( m = 1, \ldots, M. \) The sum of the discretized time derivatives, coming from the first term in (3.102a), form a telescoping sum, so that after multiplying by \( \delta t \) we find

\[ \sum_{\ell=0}^{k} C_{\ell} \eta^{2\ell} \left| D_x^{k-\ell} v(T) \right|^2_{[-\delta-\frac{1}{2}, -\delta]} + \left| v(T) \right|^2_{[-\delta-\frac{1}{2}, -\delta]} + \delta t \sum_{m=1}^{M} \sum_{\ell=0}^{k} \eta^{2\ell} \left| v^{(m\delta t)} \right|^2_{[-\delta-\delta, -\delta]} \leq \sum_{\ell=0}^{k} C_{\ell} \eta^{2\ell} \left| D_x^{k-\ell} v^{(0)} \right|^2_{[-\delta-\frac{1}{2}, -\delta]} + \left| v^{(0)} \right|^2_{[-\delta-\delta, -\delta]} + \delta t \sum_{m=1}^{M} \sum_{\ell=0}^{k-2} \eta^{2\ell} \left| f^{(m\delta t)} \right|^2_{[-\delta, -\delta]} \]  
(3.110)

The constant in estimate (3.110) does not depend on \( \delta t \) and \( M, \) and upon increasing its value, we can replace the first two terms in (3.110) by

\[ \max_{0 \leq m \leq M} \left( \sum_{\ell=0}^{k} C_{\ell} \eta^{2\ell} \left| D_x^{k-\ell} v^{(m\delta t)} \right|^2_{[-\delta-\frac{1}{2}, -\delta]} + \left| v^{(m\delta t)} \right|^2_{[-\delta-\delta, -\delta]} \right). \]

Multiplying with \( \eta^{2\ell} \) and applying the inverse Fourier transform in \( \eta, \) we obtain from (3.110)

\[ \max_{0 \leq m \leq M} \left\| D_y^j v^{(m\delta t)} \right\|^2_{k,-\delta-1+j} + \delta t \sum_{m=1}^{M} \left\| D_y^j v^{(m\delta t)} \right\|^2_{k+2,-\delta-\frac{1}{2}+j} \leq \left\| D_y^j v^{(0)} \right\|^2_{k,-\delta-1+j} + \delta t \sum_{m=1}^{M} \left\| D_y^j f^{(m\delta t)} \right\|^2_{k-2,-\delta-\frac{1}{2}+j} \]  
(3.111)

In order to justify convergence of the integrals underlying inverse Fourier transform, we note that we are working under the hypothesis that the terms on the right-hand side of (3.111) are bounded. Due to the local continuity in \( \eta \) (cf. Proposition 3.16), which implies (global) measurability in \( \eta, \) we find due to (3.110) also the integrals required for defining the left-hand side of (3.111) to converge.

**Step 2:** Interpolation in time. We define the continuum version of the left-hand side \( f \) by a piecewise constant extension

\[ \Phi_M(t, x) := \sum_{m=1}^{M} f^{(m\delta t)}(x) \mathbb{1}_{[(m-1)\delta t, m\delta t)}(t), \]  
(3.112)

and for approximating the solution \( v \) itself for time steps \( T = 0, \delta t, \ldots, M \delta t, \) define the piecewise affine interpolant

\[ \Psi_M(t, x) := \sum_{m=1}^{M} \left( \frac{t - (m-1)\delta t}{\delta t} v^{(m\delta t)}(x) + \frac{m\delta t - t}{\delta t} v^{((m-1)\delta t)}(x) \right) \mathbb{1}_{[(m-1)\delta t, m\delta t)}(t). \]  
(3.113)

We note that for \( 1 \leq m \leq M \) and for \( t \in [(m-1)\delta t, m\delta t) \), based on (3.108a), (3.112), and (3.113) there holds

\[ x \partial_t \Psi_M + q(D_x)v^{(m\delta t)} - \eta^2 x^2 r(D_x)v^{(m\delta t)} + \eta^4 x^4 v^{(m\delta t)} = \Phi_M, \]  
(3.114)
Then we use the same reasoning as the one preceding (3.19) in order to bound $\partial_t D_y \Psi_M$ via equation (3.114). On the time interval $[(m-1)\delta t, m\delta t]$ we have:

$$\|\partial_t D_y \Psi_M\|_{k-2, -\delta - \frac{1}{2} + j}^2 \lesssim \|D_y D_y^{(m\delta t)}\|_{k+2, -\delta - \frac{1}{2} + j}^2 + \|D_y^2 \Phi_M\|_{k-2, -\delta - \frac{1}{2} + j}^2. \tag{3.115}$$

On each time interval $[(m-1)\delta t, m\delta t]$ we may use the triangle inequality for (3.113) in order to bound the norm of $D_y^2 \Psi_M$ by the sum of the norms of $D_y^2 e^{i(m-1)\delta t}$ and $D_y^2 v^{(m\delta t)}$ and finally integrate (3.115) on $t \in [(m-1)\delta t, m\delta t]$. Summing these estimates over $m$ allows to pass from (3.111) to the following bound for $\Psi_M$:

$$\sup_{t \in I} \|D_y^2 \Psi_M\|_{k-1, -\delta - 1 + j}^2 + \|D_y^{j-1} D_x \Psi_M\|_{k, -\delta - 1 + j}^2 + \|\partial_t D_y \Psi_M\|_{k-3, -\delta - \frac{1}{2} + j}^2 + \|\partial_t D_y^2 \Phi_M\|_{k+2, -\delta - \frac{1}{2} + j}^2 \lesssim \|D_y^{(0)}\|_{k, -\delta - 1 + j}^2 + \int_I \|D_y^2 \Phi_M\|_{k-2, -\delta - \frac{1}{2} + j}^2 \, dt. \tag{3.116}$$

By the same reasoning applied to (3.102b), (3.102c), and (3.102d), we obtain discrete integral analogues thereof, which are also discrete versions of (3.37), (3.44), (3.45) and (3.51), respectively. These are again obtained directly from (3.37), (3.44), (3.45) and (3.51) by replacing $(v, v|_{t=0}, f)$ by $(\Psi_M, v^{(0)}, \Phi_M)$. For example the bound corresponding to (3.37) reads

$$\sup_{t \in I} \|D_y^2 \Psi_M\|_{k-1, -\delta - 1 + j}^2 + \|D_y^{j-1} D_x \Psi_M\|_{k, -\delta - 1 + j}^2 + \|\partial_t D_y \Psi_M\|_{k-3, -\delta - \frac{1}{2} + j}^2 + \|\partial_t D_y^2 \Phi_M\|_{k+2, -\delta - \frac{1}{2} + j}^2 \lesssim \|D_y^{(0)}\|_{k, -\delta - 1 + j}^2 + \int_I \|D_y^2 \Phi_M\|_{k-2, -\delta - \frac{1}{2} + j}^2 \, dt. \tag{3.117}$$

and precisely in the same way we can write down the analogues to (3.44), (3.45) and (3.51), which we omit for the sake of keeping the presentation concise.

**Step 3: Passage to the continuum limit and existence of a solution.** Let $T$ be fixed and consider a sequence of finer and finer subdivisions of $[0, T]$ in equal disjoint small intervals, i.e., consider the limit $M \to \infty$ and $\delta t = \frac{T}{M}$.

In order to obtain that the right-hand sides in (3.117) and of the higher-order discretized versions of estimates (3.44), (3.45) and (3.51) are bounded, we will use the bounds on $v^{(0)}$ which follow from the assumption $\|v^{(0)}\|_{\text{init}} < \infty$, and for the remaining terms we use bounds following from $\|f\|_{\text{rhs}} < \infty$ and the fact that for any $j, j' \in \mathbb{N}_0$ there holds

$$\int_I \|D_y^j D_x^{j'} \Phi_M\|_{k, \varrho}^2 \, dt \lesssim \int_I \|D_y^j D_x^{j'} f\|_{k, \varrho}^2 \, dt. \tag{3.118}$$

In order to check (3.118) we just have to recall the definition (3.112) of $\Phi_M$ and the definition (3.108b) of $f^{(m\delta t)}$. Indeed, using Jensen’s inequality in order to interchange the $\delta t$-interval average in (3.108b) with the norm squared, we find

$$\int_I \|D_y^j D_x^{j'} f\|_{k, \varrho}^2 \, dt = \int_I \left(\sum_{m=1}^M \|f^{(m\delta t)}(t, \cdot, \cdot)\|_{k, \varrho}^2\right)^{\frac{1}{2}} \, dt \geq \sum_{m=1}^M \int_{[(m-1)\delta t, m\delta t]} \|D_y^j D_x^{j'} f\|_{k, \varrho}^2 \, dt \tag{3.119}$$

As a consequence of (3.118), we have

$$\|\Phi_M\|_{\text{rhs}} \leq \|f\|_{\text{rhs}} \tag{3.120}$$

and thus the inequalities (3.117), and similarly the discretized analogues of (3.44), (3.45) and (3.51) continue to hold if we use $f$ rather than $\Phi_M$ on the right-hand side. These three inequalities also combine
by precisely the same mechanism that led from (3.37), (3.44), (3.45), and (3.51) to the definition of the norms $\|\cdot\|_{\text{sol}}, \|\cdot\|_{\text{init}}$, and $\|\cdot\|_{\text{rhs}}$, and allow us to prove for $I = [0, T)$

$$\|\Psi_M\|_{\text{sol}} \leq \|\vartheta(0)\|_{\text{init}} + \|f\|_{\text{rhs}},$$

where the constant is independent of $M$. By using Lemmata 3.4 and 3.5, the norms $\|\cdot\|_{\text{init}}$ and $\|\cdot\|_{\text{rhs}}$ can be replaced by the simpler $\|\cdot\|_{\text{init}}$ and $\|\cdot\|_{\text{rhs}}$ (cf. (2.1)&(2.19)), respectively, so that we end up with

$$\|\Psi_M\|_{\text{sol}} \leq \|\vartheta(0)\|_{\text{init}} + \|f\|_{\text{rhs}}. \tag{3.121}$$

Due to the bounds (3.120) and (3.121) we find that a subsequence of $(\Phi_M, \Psi_M)_M$ (which we denote by $(\Phi_M, \Psi_M)$ again) weak-∗-converges to $(f,v)$, where $v : I \times (0,x) \times \mathbb{R} \to \mathbb{R}$ is locally integrable. By weak lower-semicontinuity of the norms, we find that the maximal-regularity estimate (3.100) holds in the limit as well.

In order to ensure the validity of the boundary condition $v|_{t=0} = \vartheta(0)$ in the limit, we note that the function $\vartheta(0)$ is the initial value for every $\Psi_M$ regardless of the choice of $M$. Then we use the time-trace bounds (3.57) of Lemma 3.3 for the norms $\|\cdot\|_{\text{sol}}$ with indices as appearing in the definition (3.53) of $\|\cdot\|_{\text{sol}}$ in order to be able to define the evaluation at a specific $t$ in the limit.

In order to pass to the limit $M \to \infty$ in equation (3.114), we first rewrite it in terms of $\Psi_M$ only:

$$x \partial_t \Psi_M + q(D_x)\Psi_M - \eta^2 x^2 r(D_x)\Psi_M + \eta^4 x^4 \Psi_M = \Phi_M + R_M, \tag{3.122a}$$

with

$$R_M := q(D_x)\tilde{R}_M - \eta^2 x^2 r(D_x)\tilde{R}_M + \eta^4 x^4 \tilde{R}_M, \tag{3.122b}$$

$$\tilde{R}_M := \Psi_M - \sum_{m=1}^{M} v^{(m\delta t)} \mathbb{I}_{\{(m-1)\delta t, m\delta t\}}. \tag{3.122c}$$

In order to prove distributional convergence to zero of the remainder $R_M$, it suffices to prove the distributional convergence to zero of $\tilde{R}_M$. For this we first use (3.108a) leading to

$$v^{(m\delta t)} - v^{(m\delta t)} = \frac{\delta t}{x} \left( -\Phi_M + q(D_x) v^{(m\delta t)} - \eta^2 x^2 r(D_x) v^{(m\delta t)} + \eta^4 x^4 v^{(m\delta t)} \right). \tag{3.123}$$

Note that on one side, the $\Phi_M$-terms in (3.123) already weak-∗-converges to $f$, so that when multiplied by $\delta t = \frac{x}{T}$ it converges to zero. For the remaining terms, we note that the linear operator $q(D_x) - \eta^2 x^2 r(D_x) + \eta^4 x^4$ acting on $v^{(m\delta t)}$ is independent of $t$ and contains no $t$-derivatives. Therefore, we may consider directly the distributional convergence the term

$$\frac{\delta t}{x} \sum_{m=1}^{M} \left( q(D_x) v^{(m\delta t)} - \eta^2 x^2 r(D_x)v^{(m\delta t)} + \eta^4 x^4 v^{(m\delta t)} \right) \mathbb{I}_{\{(m-1)\delta t, m\delta t\}} = \frac{\delta t}{x} \left( q(D_x) - \eta^2 x^2 r(D_x) + \eta^4 x^4 \right) \sum_{m=1}^{M} v^{(m\delta t)} \mathbb{I}_{\{(m-1)\delta t, m\delta t\}}, \tag{3.124}$$

and so we consider the function

$$\sum_{m=1}^{M} v^{(m\delta t)} \mathbb{I}_{\{(m-1)\delta t, m\delta t\}}. \tag{3.125}$$

In view of (3.113) and by using (3.121) together with (3.59b) of Lemma 3.9, we find the bound

$$\max_{1 \leq m \leq M} \left( \left\| v^{(m\delta t)} \right\|_{BC^0((0,x) \times \mathbb{R})} + \left\| v^{(m\delta t)} \right\|_{BC^0((0,x) \times \mathbb{R})} \right) \lesssim \sup_{t \in I} \left( \left\| \partial_x \Psi_M(t, \cdot, \cdot) \right\|_{BC^0((0,x) \times \mathbb{R})} + \left\| \partial_x \Psi_M(t, \cdot, \cdot) \right\|_{BC^0((0,x) \times \mathbb{R})} \right) \lesssim \|\Psi_M\|_{\text{sol}}, \tag{3.126}$$

which shows that the gradient of (3.125) is bounded in the $\|\cdot\|_{BC^0((0,x) \times \mathbb{R})}$-norm. We now note that the operator which in (3.124) is applied to (3.125) is, after inverse Fourier transform, equal to $x^{-1} \delta t \left( q(D_x) - D_y^2 r(D_x) + D_y^4 \right)$. The latter is in divergence form and tends to zero distributionally as $M \to \infty$, due to the factor $\delta t$. Therefore, the boundedness of (3.126) implies that distributionally in the
limit $M \to \infty$ the term (3.124) tends to zero, and therefore (3.122) gives as a distributional limit along any subsequence $M \to \infty$ the desired equation (1.19a).

This completes the proof of existence of a solution $v$ meeting (3.100).

Step 4: Uniqueness of solutions. For proving uniqueness of the solution $v$, we use the following elementary arguments, which are very similar to the proof of Lemma 3.17 about uniqueness of the resolvent equation: Without loss of generality, we may assume $f = 0$, $u^{(0)} = 0$ by linearity. Since $\|v\|_{\text{sol}} < \infty$, we have in particular

$$\sup_{t \in [0, T]} \|v\|_{k, \beta}^2 + \int_I \|v\|_{k+2, \beta-1}^2 \, dt < \infty. \quad (3.127)$$

Testing (1.19a) with $\chi_n^2 v$ where $\chi_n$ is a cut-off $\chi_n(x) = \left(\frac{\log x}{n}\right)$ such that $\chi$ is a smooth non-negative function $\chi(x) = 1$ on $[-1, 1]$ and $\chi(x) = 0$ on $\mathbb{R} \setminus [-2, 2]$, we can proceed along the steps leading from (3.6) to (3.13) and integrate in $t \in I$. Note, however, that remnant terms appear when commuting the multiplication by $\chi_n$ with the operators $q(D_x)$ and $r(D_x)$. Due to the scaling of $\chi_n$ in $n$, we find that $R_n = O \left(\frac{1}{n}\right)$ as $n \to \infty$. The bounds we have are

$$\sup_{t \in [0, T]} \frac{1}{2} \|\chi_n v(t)\|_{k, \beta-1}^2 + \int_I \|\chi_n v\|_{k+2, \beta-1}^2 \, dt \leq \frac{1}{2} \|\chi_n v_{|t=0}\|_{k, \beta-1}^2 + \int_I R_n \, dt. \quad (3.128)$$

Since $v_{|t=0}$ we have that the first term on the right-hand side of (3.128) vanishes, and as $n \to \infty$ so does the second term. Then by dominated convergence we find $v = 0$. \hfill \Box

4. Nonlinear theory

4.1. The structure of the nonlinearity. We begin by making some observations on the structure of the nonlinearity $N(v)$ given through (1.17), i.e.,

$$N(v) := -xF^{-1} \left( D_y^2 D_y G(D_x - \frac{1}{2}) - G D_y (D_x + \frac{3}{2}) G(D_x - \frac{1}{2}) + F(D_x + \frac{3}{2}) F(D_x - \frac{1}{2}) \right) \left( D_y G - G(D_x + \frac{1}{2}) G - F(D_x + \frac{1}{2}) F \right) + \frac{3}{8} x + q(D_x)v + D_y^2 r(D_x)v + D_y^3 v.$$

Recall that due to (A.2) and (1.13) we have

$$F^{-1} = Z_x = v_x + 1 \quad \text{and} \quad G = Z_x^{-1} Z_y = \frac{v_y}{v_x + 1}. \quad (4.1)$$

Our main objective is to re-write $N(v)$ in a form that reflects expansion (1.25), i.e., almost everywhere

$$D^\ell N(v) = D^\ell \left( (N(v))_1 (t, y)x + (N(v))_2 (t, y)x^2 \right) + o \left( x^{2+\beta} \right) \quad \text{as} \quad x \searrow 0,$$

where $\ell \notin \mathbb{N}_0$ with $|\ell| \leq k + 5$ and $\ell_y \leq k - 2$, given that $v$ meets expansion (2.6), i.e., almost everywhere

$$D^\ell v(t, x, y) = D^\ell \left( v_0(t, y) + v_1(t, y)x + v_1(t, y)x^{1+\beta} + v_2(t, y)x^2 + o \left( x^{2+\beta} \right) \right) \quad \text{as} \quad x \searrow 0, \quad (4.2)$$

where $\ell \notin \mathbb{N}_0$ with $|\ell| \leq k + 9$ and $\ell_y \leq k + 2$. Because of $q(0) \overset{(1.15a)}{=} 0$, we have $D^\ell q(D_x)v = O(x)$ as $x \searrow 0$, where $|\ell| \leq L$, and the asymptotics $D^\ell D_y^2 r(D_x)v = O(x^2)$ and $D^\ell D_y^3 v = O(x^4)$ as $x \searrow 0$ for $|\ell| \leq L$ are trivial, so that indeed $D^\ell N(v) = O(x)$ as $x \searrow 0$ for $|\ell| \leq L$ holds true. In order to see that the contribution $O \left( x^{1+\beta} \right)$ is canceled due to the structure of the nonlinearity $N(v)$, we separate the terms appearing in (1.17) by defining the operators

$$A_1 := D_y^2 - D_y G(D_x - \frac{1}{2}) - G D_y (D_x + \frac{3}{2}) G(D_x - \frac{1}{2}), \quad (4.3a)$$

$$A_2 := G(D_x + \frac{3}{2}) G(D_x - \frac{1}{2}) + F(D_x + \frac{3}{2}) F(D_x - \frac{1}{2}), \quad (4.3b)$$

and expressions

$$B_1 := D_y G, \quad B_2 := G(D_y + \frac{1}{2}) G + F(D_x + \frac{1}{2}) F. \quad (4.3c)$$

Then we write

$$N(v) =: N^{(1)}(v) + N^{(2)}(v), \quad (4.4a)$$

where

$$N^{(1)}(v) := -xF^{-1} (A_1 B_1 + A_2 B_1 - A_1 B_2) + D_y^2 r(D_x)v + D_y^3 v, \quad (4.4b)$$

$$N^{(2)}(v) := x F^{-1} A_2 B_2 + \frac{3}{8} x + q(D_x)v. \quad (4.4c)$$
Thus $\mathcal{N}^{(1)}(v)$ is the combination of terms from (1.17) which contain factors of the form $D_x^2, D_y G$ or $GD_y$, and $\mathcal{N}^{(2)}(v)$ contains only products of $x, D_x, F, G$.

4.1.1. The structure of $\mathcal{N}^{(1)}(v)$. By series expansion of the factors $F = (1 + v_x)^{-1}$ for $|v_x| < 1$, we can write $\mathcal{N}^{(1)}(v)$ as a convergent series of terms of the form

$$T \left( a^{(1)}, \ldots, a^{(m)}, v \right) := c \left( a^{(1)}, \ldots, a^{(m)} \right) x^{1-m} \prod_{j=1}^{m} D^{a^{(j)}},$$

where coefficients $c(a^{(1)}, \ldots, a^{(m)})$ and multi-indices $a^{(j)} = (a_x^{(j)}, a_y^{(j)})$ satisfy (with notations and conventions like in (A.5))

$$m \geq 2, \quad a^{(1)}, \ldots, a^{(m)} \in \mathbb{N} \setminus \{(0,0)\}, \quad \left| a^{(1)} \right| \geq 2, \quad a_y^{(1)} \geq 1,$$

$$m \leq \sum_{j=1}^{m} \left| a^{(j)} \right| \leq m + 3, \quad c \left( a^{(1)}, \ldots, a^{(m)} \right) \in \mathbb{R}. \quad (4.5b)$$

Since $D^j D^{a^{(j)}}, v = O(x)$ for $j \in \{0, \ldots, m\}$ and $D^j D^{a^{(j)}}, v = O(x^2)$ as $x \to 0$, each term in (4.5a) behaves as $D^j T (a^{(1)}, \ldots, a^{(m)}, v) = O(x^2)$ as $x \to 0$. As a result, we have $\mathcal{N}^{(1)}(v) = O(x^2)$ as well.

4.1.2. The structure of $\mathcal{N}^{(2)}(v)$. Introducing the 4-linear form

$$\mathcal{M} \left( H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)} \right) := xH^{(1)} \left( D_x + \frac{3}{2} \right) H^{(2)} \left( D_x - \frac{1}{2} \right) H^{(3)} \left( D_x + \frac{3}{2} \right) H^{(4)},$$

we may decompose the term $\mathcal{N}^{(2)}(v)$ from (4.4a) as follows:

$$\mathcal{N}^{(2)}(v) := \mathcal{N}^{(2,1)}(v) + \mathcal{N}^{(2,2)}(v) + \mathcal{N}^{(2,3)}(v) + \mathcal{N}^{(2,4)}(v). \quad (4.7a)$$

with

$$\mathcal{N}^{(2,1)}(v) := \mathcal{M}(1, F, F, F) + q(D_x)v + \frac{3}{8} x,$$

$$\mathcal{N}^{(2,2)}(v) := \mathcal{M} \left( F^{-1}G, G, G, G \right), \quad (4.7b)$$

$$\mathcal{N}^{(2,3)}(v) := \mathcal{M}(1, F, G, G), \quad (4.7c)$$

$$\mathcal{N}^{(2,4)}(v) := \mathcal{M} \left( F^{-1}G, G, F, F \right). \quad (4.7d)$$

For treating the terms appearing in (4.7), we keep expansion (4.2) in mind in order to keep track of the $x$-power series contributions appearing in this regime. We introduce the notation

$$\phi := (1 + v_1)^{-1} (v - v_0 - v_1 x) \quad \text{and} \quad \psi := v - v_0 \quad (4.8)$$

leading to

$$F^{(1.12)} = \left( 1 + v_1 \right)^{-1} (1 + \phi x)^{-1}, \quad (4.9a)$$

$$F^{-1}G^{(1.12)} = (v_0)_y + \psi_y, \quad (4.9b)$$

$$G^{(1.12)} = \left( 1 + v_1 \right)^{-1} (1 + \phi x)^{-1} ((v_0)_y + \psi_y). \quad (4.9c)$$

Employing (4.8) and (4.9) for (4.7b) gives through power series expansion in $\phi$ (using $q(0)$

$$\mathcal{N}^{(2,1)}(v) = \left( -\frac{3}{8} (1 + v_1)^{-3} + q(1)v_1 + \frac{3}{4} \right) x$$

$$- \left( 1 + v_1 \right)^{-3} x \left( D_x + \frac{3}{2} \right) \left( -\frac{1}{2} \phi + \frac{1}{2} \left( D_x - \frac{1}{2} \right) \phi \right)$$

$$+ (1 + v_1) q(D_x) \phi$$

$$+ (1 + v_1)^{-3} \sum_{\tau_2 + \tau_3 + \tau_4 \geq 2} (-1)^{\tau_1} \mathcal{M}(1, \phi x^{\tau_2}, \phi x^{\tau_3}, \phi x^{\tau_4}).$$

Hence, with help of (1.15a)

$$\mathcal{N}^{(2,1)}(v) = -\frac{3}{8} (1 + v_1)^{-3} \left( 6v_1^2 + 8v_1^3 + 3v_1^4 \right) x$$

$$+ (1 + v_1)^{-3} \left( 4v_1 + 6v_1^2 + 4v_1^3 + v_1^4 \right) q(D_x) \phi$$

$$+ (1 + v_1)^{-3} \sum_{\tau_2 + \tau_3 + \tau_4 \geq 2} (-1)^{\tau_1} \mathcal{M}(1, \phi x^{\tau_2}, \phi x^{\tau_3}, \phi x^{\tau_4}). \quad (4.10)$$
With an analogous reasoning, using (4.15a), (4.16), and (4.9) in (4.7c), we also obtain
\[
\mathcal{N}^{(2.2)}(v) = -\frac{3}{8} (1 + v_1)^{-3} (v_0)^4 x \\
- (1 + v_1)^{-3} (v_0)^3 q (D_x) \phi \\
+ (1 + v_1)^{-3} (v_0)^3 (D_x + \frac{1}{2}) (D_x - \frac{3}{2} D_x - \frac{5}{8}) D_y \psi \\
+ (1 + v_1)^{-3} \sum_{|\mu| + |\tau| \geq 2 \atop \mu_j + \nu_j = 1 for j \in \{1,2\}} (-1)^{|\tau|} \mathcal{M} \left( (\psi_{y j}^{\mu j} (v_0)^\nu \phi_x^{\tau j})^4 \right).
\]  
(4.11)

For dealing with \(\mathcal{N}^{(2.3)}(v)\) (cf. (4.7d)), we may use (4.15a) and (4.9) once more and arrive at
\[
\mathcal{N}^{(2.3)}(v) = -\frac{3}{8} (1 + v_1)^{-2} (v_0)^2 x \\
- (1 + v_1)^{-3} (v_0)^2 q (D_x) \phi \\
- \frac{3}{8} (1 + v_1)^{-3} (v_0)^2 (\frac{2}{3} D_x + \frac{4}{3}) D_y \psi \\
+ (1 + v_1)^{-3} \sum_{|\mu| + |\tau| \geq 2 \atop \mu_j + \nu_j = 1 for j \in \{1,2\}} (-1)^{|\tau|} \mathcal{M} \left( (\psi_{y j}^{\mu j} (v_0)^\nu \phi_x^{\tau j})^4 \right).
\]  
(4.12)

For \(\mathcal{N}^{(2.4)}(v)\) (cf. (4.7e)), employing (4.15a) and (4.9) gives
\[
\mathcal{N}^{(2.4)}(v) = -\frac{3}{8} (1 + v_1)^{-3} (v_0)^2 x \\
- (1 + v_1)^{-3} (v_0)^2 q (D_x) \phi \\
- \frac{3}{8} (1 + v_1)^{-3} (v_0)^2 (\frac{2}{3} D_x + \frac{4}{3}) D_y \psi \\
+ (1 + v_1)^{-3} \sum_{|\mu| + |\tau| \geq 2 \atop \mu_j + \nu_j = 1 for j \in \{1,2\}} (-1)^{|\tau|} \mathcal{M} \left( (\psi_{y j}^{\mu j} (v_0)^\nu \phi_x^{\tau j})^4 \right).
\]  
(4.13)

By combining (4.7a), (4.10), (4.11), (4.12), and (4.13), we find
\[
\mathcal{N}^{(2)}(v) = -\frac{3}{8} (1 + v_1)^{-3} (6v_1^2 + 8v_1^3 + 3v_1^4 + 2(v_0)^2 + (v_0)^3) x \\
+ (1 + v_1)^{-3} (4v_1^2 + 6v_1^3 + 4v_1^4 + v_1^4 - 2(v_0)^2 - (v_0)^3) q (D_x) \phi \\
+ (1 + v_1)^{-3} (v_0)^2 (\frac{2}{3} D_x + \frac{4}{3}) (D_x^2 - \frac{2}{3} D_x - \frac{5}{8}) D_y \psi \\
+ (1 + v_1)^{-3} \sum_{|\mu| + |\tau| \geq 2 \atop \mu_j + \nu_j = 1 for j \in \{1,2\}} (-1)^{|\tau|} \mathcal{M} \left( (\psi_{y j}^{\mu j} (v_0)^\nu \phi_x^{\tau j})^4 \right).
\]  
(4.14a)

where \((\mu, \nu, \tau) \in (\mathbb{N}^d_0)^3\) and
\[
\mathcal{I} := \{|\mu| + |\tau| \geq 2, \tau_1 = 0, \mu = \nu = 0, 0 \}
\cup \{ |\mu| + |\tau| \geq 2, \tau_1 = 0, \mu_j + \nu_j = 1 for j \in \{1,2,3,4\} \}
\cup \{ |\mu| + |\tau| \geq 2, \tau_1 = 0, \mu_j + \nu_j = 0 for j \in \{1,2\}, \mu_j + \nu_j = 1 for j \in \{3,4\} \}
\cup \{ |\mu| + |\tau| \geq 2, \tau_1 = 0, \mu_j + \nu_j = 1 for j \in \{1,2\}, \mu_j + \nu_j = 0 for j \in \{3,4\} \}
\]  
(4.14b)

Note that the precise definition (4.14b) of \(\mathcal{I}\) is not essential for the subsequent arguments and that we will simply use \(\mathcal{I} \subseteq \{|\mu| + |\tau| \geq 2\}\). Regarding expression (4.14a), we remark that due to (4.2) and (4.8) we have almost everywhere
\[
D^f \phi(t, x, y) = D^f \left( (1 + v_1(t, y))^{-1} (v_1 + \beta(t, y) x^{1+\beta} + v_2(t, y) x^2) \right) + o(x^{2+\beta}) \quad as \quad x \searrow 0,
\]  
(4.15)

where \(\ell \in \mathbb{N}_0^d\) with \(|\ell| \leq \ell_k + 9\) and \(\ell_y \leq \ell_k + 2\).

The main considerations to be kept in mind concerning (4.14) are:
(i) The second line of (4.14a) is \((\phi, \psi)\)-independent but super-linear in \(v\) and forms the \(O(x)\)-contribution of \(N^{(2)}(v)\).

(ii) As a consequence of the fact that \(q(1 + \beta) \frac{1}{1+\alpha} \equiv 0\) and using (4.15), the third line of (4.14a) is of order \(O(x^2)\) as \(x \searrow 0\) and is again nonlinear and at least quadratic in \(v\) (but linear in \(\phi\)).

(iii) Line four of (4.14a) contains one \(D_q\)-derivative acting on \(\psi\), therefore this line is of order \(O(x^2)\) as \(x \searrow 0\) and super-linear in \(v\) but linear in \(\psi\).

(iv) Finally, the last line of (4.14a) is the remainder, being super-linear in \((\phi, \psi)\) (cf. (4.14b)) where \(|\mu| + |\tau| \geq 2\), and thus also in \(v\). Concerning the behavior near 0, we have \(D^j \phi_x = O(x^\beta)\) as \(x \searrow 0\) (with \(\beta \in \left(\frac{1}{2}, 1\right)\) by (1.16)) and \(D^j \psi_y = O(x)\) as \(x \searrow 0\). Thus, recalling that \(M\) also features an extra factor of \(x\), we find that this term is of order \(O(x^{1+\beta})\) as \(x \searrow 0\).

(v) The above considerations also imply that, using the notation (1.22) and (4.2) for the power series coefficient expansion near zero, we have

\[
D^\ell N^{(2)}(v) = D^\ell \left(\left(N^{(2)}(v)\right)_1 x + \left(N^{(2)}(v)\right)_2 x^2\right) + o(x^{2+\delta}) \quad \text{as} \quad x \searrow 0
\]

for \(\ell \in \mathbb{N}_0^d\) with \(|\ell| \leq \bar{k} + 5\) and \(\ell_y \leq \bar{k} - 2\).

4.2. **Nonlinear estimates.** In this section, we derive our main estimate for the nonlinearity \(N(v)\) (cf. (1.17)).

**Proposition 4.1.** Suppose that \(I := (0, T) \subseteq (0, \infty)\) is an interval, \(\delta \in (0, \frac{1}{2} (\beta - \frac{1}{2}))\), and \(\bar{k}, \bar{k}, \text{ and } \bar{k}\) satisfy the bounds (2.4) of Assumptions 2.1. Then

\[
\left\| N\left(v^{(1)}\right) - N\left(v^{(2)}\right) \right\|_{\text{rhs}} \lesssim_{\bar{k}, \bar{k}, \bar{k}, \delta} \left( \left\| v^{(1)} \right\|_{\text{sol}} + \left\| v^{(2)} \right\|_{\text{sol}} \right) \left\| v^{(1)} - v^{(2)} \right\|_{\text{sol}},
\]

where

\[
v^{(j)} = v^{(j)}(t, x, y) : (0, \infty) \times \mathbb{R} \to \mathbb{R}
\]

are locally integrable with \(\left\| v^{(j)} \right\|_{\text{sol}} \lesssim_{\bar{k}, \bar{k}, \bar{k}, \delta} 1\) for \(j = 1, 2\) (cf. (3.53) and (2.19) for the definition of the norms \(\left\| \cdot \right\|_{\text{sol}}\) and \(\left\| \cdot \right\|_{\text{rhs}}\)).

The detailed proof of Theorem 2.2 employing the maximal-regularity estimate (3.100) of Proposition 3.20 and the nonlinear estimate (4.16) of Proposition 4.1 is the subject of §4.3.

In order to prove Proposition 4.1, we show the following auxiliary interpolation result:

**Lemma 4.2.** Assume that for \(i = 1, 2, 3\) there holds \(w^{(i)} \in C^x((0, \infty) \times \mathbb{R}_y) \cap C^0((0, \infty) \times \mathbb{R}_y)\) and suppose that \(\left\| D^\delta w^{(i)} \right\|_{-\delta + \frac{3}{2}} < \infty\). Then we have the following bounds, in which the implicit constants are independent of \(\delta\):

\[
\left\| D^\delta w^{(1)} \times D^\delta w^{(2)} \right\|_{-\delta + \frac{3}{2}} \lesssim \left\| D^\delta w^{(1)} \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w^{(1)} \right\|_{BC^0((0, \infty) \times \mathbb{R}_y)} \left\| D^\delta w^{(2)} \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w^{(2)} \right\|_{BC^0((0, \infty) \times \mathbb{R}_y)}
\]

\[
+ \left\| D^\delta w^{(1)} \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w^{(1)} \right\|_{BC^0((0, \infty) \times \mathbb{R}_y)} \left\| D^\delta w^{(2)} \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w^{(2)} \right\|_{BC^0((0, \infty) \times \mathbb{R}_y)}
\]

and

\[
\left\| D^\delta w^{(1)} \times D^\delta w^{(2)} \times D^\delta w^{(3)} \right\|_{-\delta + \frac{3}{2}} \lesssim \prod_{i=1}^3 \left\| D^\delta w^{(i)} \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w^{(i)} \right\|_{BC^0((0, \infty) \times \mathbb{R}_y)}.
\]

**Proof of Lemma 4.2.** We first prove (4.17a). In this case we can write

\[
\left\| D^\delta w^{(1)} \times D^\delta w^{(2)} \right\|_{-\delta + \frac{3}{2}}^2 = \int_\mathbb{R}_y \int_\mathbb{R} x^{2(\delta - 1)} \left(w_y^{(1)} \right)^2 \left(w_y^{(2)} \right)^2 \, dy \, dx
\]

\[
= - \int_\mathbb{R}_y \int_\mathbb{R} x^{2(\delta - 1)} w_{x}^{(1)} w_{yy}^{(1)} w_y^{(2)} dy \, dx - 2 \int_\mathbb{R}_y \int_\mathbb{R} x^{2(\delta - 1)} \left(w_{y}^{(1)} w_{yy}^{(1)} w_y^{(2)} \right)^2 \, dy \, dx
\]

\[
:= -A - 2B.
\]
Using the Hölder inequality, we can bound the above terms as follows:

\[
-A \leq \left( \int_{-\infty}^{\infty} x^{2(\delta-2)} \left( w_1^{(1)} \right)^6 \, dy \, dx \right)^{\frac{1}{6}} \cdot \left( \int_{-\infty}^{\infty} x^{2(\delta-2)} \left( w_2^{(1)} \right)^6 \, dy \, dx \right)^{\frac{5}{6}},
\]

\[\text{(4.19a)}\]

\[-2B \leq 2 \left\| D_y^2 w^{(1)} \times D_y w^{(1)} \right\|_{-\delta + \frac{3}{2}} \cdot \left\| D_y^2 w^{(2)} \times D_y w^{(2)} \right\|_{-\delta + \frac{3}{2}}.
\]

\[\text{(4.19b)}\]

Now note that writing (4.18) with the choices \( w^{(1)} = w^{(2)} = w \) we may reabsorb \( 2B \) to the left and obtain via (4.19a) the simplified bound

\[
\left\| D_y^2 w \times D_y w \right\|_{-\delta + \frac{3}{2}} \leq \frac{1}{3} \left( \int_{-\infty}^{\infty} x^{2(\delta-2)} w^6 \, dy \, dx \right)^{\frac{1}{6}} \left\| D_y^3 w \right\|_{-\delta + \frac{3}{2}}.
\]

\[\text{(4.20)}\]

Then we can write

\[
\int_{-\infty}^{\infty} x^{2(\delta-2)} w_y^6 \, dy \, dx = -5 \int_{-\infty}^{\infty} x^{2(\delta-2)} w_y w_y^4 \, dy \, dx
\]

\[\leq 5 \left\| D_y^2 w \times D_y w \right\|_{-\delta + \frac{3}{2}} \left( \int_{-\infty}^{\infty} x^{2(\delta-2)} w_y^6 \, dy \, dx \right)^{\frac{1}{2}} \left\| x^{-1} w \right\|_{BC^0((0,\infty),\mathbb{R}_+)}.
\]

and by simplifying and squaring the above,

\[
\int_{-\infty}^{\infty} x^{2(\delta-2)} w_y^6 \, dy \, dx \leq 25 \left\| D_y^2 w \times D_y w \right\|_{-\delta + \frac{3}{2}}^2 \left\| x^{-1} w \right\|_{BC^0((0,\infty),\mathbb{R}_+)}^2.
\]

Applying to (4.20) this gives, again after simplifying terms and squaring,

\[
\left\| D_y^2 w \times D_y w \right\|_{-\delta + \frac{3}{2}} \leq \frac{5}{9} \left\| D_y^3 w \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w \right\|_{BC^0((0,\infty),\mathbb{R}_+)}.
\]

\[\text{(4.21)}\]

and

\[
\left( \int_{-\infty}^{\infty} x^{2(\delta-2)} w_y^6 \, dy \, dx \right)^{\frac{1}{2}} \leq \frac{5}{3} \left\| D_y^3 w \right\|_{-\delta + \frac{3}{2}} \left\| x^{-1} w \right\|_{BC^0((0,\infty),\mathbb{R}_+)}.
\]

\[\text{(4.22)}\]

Now inserting (4.21) and (4.22) into (4.18) and (4.19), we find (4.17a). In order to prove (4.17b) we write:

\[
\left\| D_y w^{(1)} \times D_y w^{(2)} \times D_y w^{(3)} \right\|_{-\delta + \frac{3}{2}} = \left( \int_{-\infty}^{\infty} x^{2(\delta-2)} \left( w^{(1)}_y w^{(2)}_y w^{(3)}_y \right)^2 \, dy \, dx \right)^{\frac{1}{2}}
\]

\[\leq \prod_{i=1}^{3} \left( \int_{-\infty}^{\infty} x^{2(\delta-2)} \left( w^{(i)}_y \right)^6 \, dy \, dx \right)^{\frac{1}{2}}.
\]

Then (4.17b) follows via (4.22).

\[\square\]

Proof of Proposition 4.1. Throughout the proof, estimates will depend on \( \bar{k}, \bar{k}, \bar{k}, \) and \( \delta, \) and for lightness of notation we will not indicate this dependence explicitly. We first prove estimate (4.16) for \( v^{(1)} = v \) and \( v^{(2)} = 0. \) The general case will be dealt with at the end of the proof. We use the decomposition (4.4) and treat the norms \( \left\| \mathcal{N}^{(1)}(v) \right\|_{\text{rhs}} \) and \( \left\| \mathcal{N}^{(2)}(v) \right\|_{\text{rhs}} \) separately.

Estimate of \( \left\| \mathcal{N}^{(1)}(v) \right\|_{\text{rhs}}. \) As we have already noted in (4.5), we can write \( \mathcal{N}^{(1)}(v) \) as a convergent power series of terms of the form (4.5a), i.e.,

\[
T \left( a^{(1)}, \ldots, a^{(m)}, v \right) := c \left( a^{(1)}, \ldots, a^{(m)} \right) x^{1-m} \bigotimes_{j=1}^{m} D^{(j)} v,
\]

where

\[
m \geq 2, \quad a^{(1)}, \ldots, a^{(m)} \in \mathbb{N}_0 \setminus \{(0,0)\}, \quad \left| a^{(1)} \right| \geq 2, \quad a^{(y)} \geq 1,
\]

\[
m \leq \sum_{j=1}^{m} \left| a^{(j)} \right| \leq m + 3, \quad c \left( a^{(1)}, \ldots, a^{(m)} \right) \in \mathbb{R}.
\]

Our goal is to show

\[
\left\| x^{1-m} \bigotimes_{j=1}^{m} D^{(j)} v \right\|_{\text{rhs}} \leq (C \| v \|_{\text{sol}})^m,
\]

\[\text{(4.23)}\]
where \( C = C \left( \bar{k}, \bar{k}, \bar{k}, \delta \right) \) is independent of \( m \). This implies summability of the right-hand side of (4.24) for sufficiently small \( \|v\|_{\text{sol}} \) and since \( m \geq 2 \), the desired

\[
\left\| \mathcal{A}^{(1)}(v) \right\|_{\text{rhs}} \lesssim \|v\|_{\text{sol}}^2 \quad \text{for} \quad \|v\|_{\text{sol}} \ll 1
\]  

(4.24)
is proved.

We take \( f := x^{1-m} \times_{j=1}^m D^{(j)} v \), where \( m \) and \( a^{(j)} \) meet (4.5b), in each of the addends in (2.19), i.e.,

\[
\|f\|_{\text{rhs}}^2 = \int_I \left( \|D_x - 1\| f \right)_{k-2-\delta+\frac{1}{2}}^2 \, dt
\]

\[
+ \int_I \left( \|\tilde{q}(D_x - 1)(D_x - 1)\| f \right)_{k-2-\delta+\frac{1}{2}}^2 \, dt
\]

\[
+ \int_I \left( \|D_x - 4\| f \right)_{k-4-\delta+\frac{1}{2}}^2 \, dt,
\]

individually. Distributing \( D \)-derivatives on the factors \( D^{(j)} v \) demonstrates that it suffices to estimate a term of the form

\[
\int_I \left\| x^{1-m} \times_{j=1}^m D^{(j)} v \right\|_{\rho}^2 \, dt.
\]  

(4.25a)

Without loss of generality \( |b^{(1)}| = \max_j |b^{(j)}| \) and further note that \( 1 \leq |a^{(j)}| \leq 2 \) for all \( 1 \leq j \leq m \). Thus we have

\[
2 \leq |b^{(1)}| \leq \kappa + 4, \quad 1 \leq |b^{(j)}| \leq \left\lfloor \frac{\kappa + 1}{2} \right\rfloor + 2 \quad \text{for} \quad j \in \{2, \ldots, m\},
\]  

(4.25b)

with \( (\kappa, \rho) \) as in the norms (2.19), i.e.,

\[
(\kappa, \rho) \in \left\{ \begin{array}{c}
\left( \bar{k} - 1, -\delta + \frac{5}{2} \right), \left( \bar{k} + 3, \delta + \frac{3}{2} \right), \left( \bar{k} + 3, -\delta + \frac{3}{2} \right), \left( \bar{k} + 5, \delta + \frac{3}{2} \right), \left( \bar{k} - 1, -\delta + \frac{5}{2} \right), \left( \bar{k} + 5, -\delta + \frac{3}{2} \right)
\end{array} \right\}
\]  

(4.25c)

and for \( j \in \{1, \ldots, m\} \) there holds

\[
b^{(j)}_{\rho} \leq \kappa_{\rho}, \quad \text{where} \quad \kappa_{\rho} := \begin{cases} 
\bar{k} - 2 & \text{for} \quad \rho = \mp \delta + \frac{5}{2}, \\
\bar{k} - 2 & \text{for} \quad \rho = \mp \delta + \frac{3}{2}, \\
\bar{k} - 1 & \text{for} \quad \rho = -\delta + \frac{5}{2}.
\end{cases}
\]  

(4.25d)

Note that for the weight \( \rho = -\delta + \frac{5}{2} \) three \( D_{\rho} \)-derivatives appear in the respective term of the norm \( \|\|_{\text{rhs}} \) (cf. (2.19)), so that in that case we have in (4.25) the further restrictions

\[
\sum_{j=1}^m |b^{(j)}| \geq m + 3 \quad \text{and} \quad \sum_{j=1}^m b^{(j)}_{\rho} \geq 3.
\]  

(4.26)

Estimates for the weights \( \rho \in \{ \pm \delta + \frac{1}{2}, -\delta + \frac{3}{2} \} \). We start by considering the terms of in the norm (2.19) corresponding to weights \( \rho \in \{ \pm \delta + \frac{1}{2}, -\delta + \frac{3}{2} \} \).

In view of (4.5b) we can assume without loss of generality that one of the following two cases applies:

(i) \( b^{(1)} \geq 1 \) and \( |b^{(1)}| \geq 2 \),

(ii) \( b^{(2)} = 0 \), \( b^{(2)} \geq 1 \), and \( \|v\|_{\text{sol}} \geq 2 \).

Suppose case (i) is valid. For \( \rho \in \{ \pm \delta + \frac{1}{2}, -\delta + \frac{3}{2} \} \), we can estimate in (4.25) according to

\[
\int_I \left\| x^{1-m} \times_{j=1}^m D^{(j)} v \right\|_{\rho}^2 \leq \int_I \left\| D^{(1)} v \right\|_{\rho}^2 \sup_{t \in I} \left\| x^{-1} D^{(1)} v \right\|_{BC((0,1) \times \mathbb{R}^N)}^2.
\]  

(4.27)

Due to (4.25b)–(4.25d) and (i) we have \( D^{(1)} v = O(x^2) \) as \( x \to 0 \) for \( \rho \in \{ \pm \delta + \frac{1}{2}, -\delta + \frac{3}{2} \} \), and the elliptic-regularity result of Lemma 3.8 yields

\[
\left\| D^{(1)} v \right\|_{\rho} \lesssim \begin{cases} 
\|D_v v\|_{k+1,-\delta+\frac{1}{2}} & \text{for} \quad \rho = -\delta + \frac{1}{2}, \\
\|D_v x v\|_{k+1,\delta+\frac{1}{2}} & \text{for} \quad \rho = \delta + \frac{1}{2}, \\
\|D_v x v\|_{k+1,-\delta+\frac{1}{2}} & \text{for} \quad \rho = -\delta + \frac{3}{2},
\end{cases}
\]
so that in view of the definition of the norm $\|\cdot\|_{\text{sol}}$ we have
\[
\int_I \left\| D^{(1)} v \right\|_p^2 \, dt \leq \| v \|_{\text{sol}}^2.
\]

Furthermore, for $j \geq 2$ we can estimate by Lemma 3.9
\[
\sup_{t \in I} \left\| x^{-1} D^{(j)} v \right\|_{BC^0((0,\infty) \times \mathbb{R}))} \lesssim \sup_{t \in I} \max_{0 \leq l \leq |t| \leq 1 \frac{1}{2}} \left( \left\| D^l v \right\|_{BC^0((0,\infty) \times \mathbb{R}))} + \left\| D^l v_y \right\|_{BC^0((0,\infty) \times \mathbb{R}))} \right) \lesssim \| v \|_{\text{sol}}, \tag{4.28}
\]
because
\[
\left\| \frac{\kappa + 1}{2} \right\| + 2 = \begin{cases}
\frac{3}{4} + 2 & \text{for } \rho = -\delta + \frac{1}{2}, \\
\frac{3}{4} + 4 & \text{for } \rho = \delta + \frac{1}{2}, \\
\frac{3}{4} + 4 & \text{for } \rho = -\delta + \frac{3}{2},
\end{cases} \tag{4.29}
\]
hold true and all the above quantities are less or equal to $\min \{ \frac{1}{2}k, \frac{1}{2}k + \delta \}$ under Assumption 2.1. This implies
\[
\int_I \left\| x^{1-m} \sum_{j=1}^m D^{(j)} v \right\|_p^2 \leq (C \| v \|_{\text{sol}})^{2m}, \tag{4.30}
\]
for a constant $C$ only depending on $\frac{1}{2}k, \frac{1}{2}k + \delta$.

Now suppose that we are in case (ii). Consider first the case $\rho = -\delta + \frac{1}{2}$. Because $b^{(1)}_k \geq 1$ we have $D^{(1)} v = O(x)$ as $x \to 0$. Estimating as in (4.27) and using Lemma 3.8 to absorb terms, we find, similarly to case (i),
\[
\left\| D^{(1)} v \right\|_p \lesssim \| D_x v \|_{k+2, -\delta + \frac{1}{2}} \quad \text{for } \rho = -\delta + \frac{1}{2},
\]
so that we can argue as in the previous case. Next, we treat the case (ii) for weights $\rho \in \{ \delta + \frac{1}{2}, -\delta + \frac{3}{2} \}$.

In case $\rho = \delta + \frac{1}{2}$ we can write
\[
D^{(1)} v = D^{(1)}(v - v_0) = D^{(1)}(v - v_0 - v_1 x) + v_1 x, \tag{4.31a}
\]
and in case $\rho = -\delta + \frac{3}{2}$ we can write, using the same notation $w$,
\[
D^{(1)} v = D^{(1)}(v - v_0) = D^{(1)}(v - v_0 - v_1 x - v_1 x + 1 + \beta) + v_1 x + (1 + \beta)^{\delta} v_1 x + 1 + \beta, \tag{4.31b}
\]
Since $v_1 x$ and $v_1 x + 1 + \beta$ are in the kernel of $\tilde{q}(D_x)D_x$ (cf. (3.53) for the definition of the solution norm in which these operators appear in the respective terms), we can apply Lemma 3.8, and have, with the above notations,
\[
\|w\|_p \lesssim \left\| \tilde{q}(D_x)D_x v \right\|_{k+2, -\delta + \frac{1}{2}} \quad \text{for } \rho = \delta + \frac{1}{2}, \quad \|w\|_p \lesssim \left\| \tilde{q}(D_x)D_x v \right\|_{k+2, -\delta + \frac{3}{2}} \quad \text{for } \rho = -\delta + \frac{3}{2}.
\]

Then, as in case (i) together with (3.59b) of Lemma 3.9, this implies
\[
\int_I \left\| x^{1-m} w \sum_{j=1}^m D^{(j)} v \right\|_p^2 \, dt \leq \int_I \| w \|_p^2 \, dt \times \prod_{j=2}^m \sup_{t \in I} \left\| x^{-1} D^{(j)} v \right\|_{BC^0((0,\infty) \times \mathbb{R}))} \lesssim \| v \|_{\text{sol}}^{2m}.
\]
Additionally, we have
\[
\int_I \left\| x^{1-m} v_1 x \sum_{j=2}^m D^{(j)} v \right\|_p^2 \, dt \leq \sup_{t \in I} \| v_1^{2 BC^0(\mathbb{R}^+)} \times \int_I \left\| D^{(2)} v \right\|_p^2 \, dt \times \prod_{j=3}^m \sup_{t \in I} \left\| x^{-1} D^{(j)} v \right\|_{BC^0((0,\infty) \times \mathbb{R}))}^2 \lesssim \| v \|_{\text{sol}}^{2m}.
\]
because of estimate (3.59b) of Lemma 3.9 and the fact that \( D^{(2)} v = O(x^2) \) as \( x \searrow 0 \), so that this term can be treated as \( D^{(1)} v \) in case (i). This completes the treatment of the weight \( \rho = \delta + \frac{1}{2} \).

Hence, we can restrict ourselves to the weight \( \rho = -\delta + \frac{3}{4} \) for the rest of the proof step and have

\[
\int_I \left\| x^{-m} v_{1+\beta} x^{1+\beta} \sum_{j=1}^{m} D^{(j)} v \right\|^2_{-\delta + \frac{3}{4}} \ dt \\
\leq \int_I \left\| v_{1+\beta} \right\|_{BC^0(R_x)} dt \times \sup_{t \in I} \left\| D^{(2)} v \right\|^2_{-\delta + \frac{3}{4} - \beta} \times \sum_{j=1}^{m} \sup_{t \in I} \left\| x^{-1} D^{(j)} v \right\|^2_{BC^0((0,x) \times R_y)} \\
\lesssim \int_I \left\| v_{1+\beta} \right\|^2_{BC^0(R_x)} dt \times \| v \|^2_{\text{sol}} \tag{4.32}
\]

where Lemma 3.9 has been used in the last line. In order to see (4.32), we note that \( \beta \in \left( \frac{1}{2}, 1 \right) \) implying \( -\delta + \frac{3}{4} - \beta \in (-\delta, -\delta + 1) \) and by using Lemma 3.8, we obtain

\[
\left\| D^{(j)} v \right\|^2_{-\delta + \frac{3}{4} - \beta} \lesssim \| D_v v \|^2_{k-1, -\delta} + \| D_y D_x v \|^2_{k-1, -\delta + 1} + \| D^2_y v \|^2_{k-2, -\delta + 1},
\]

provided

\[
\left\lfloor \frac{k}{2} \right\rfloor + 4 \leq \min\{k - 2, k - 4\}, \tag{4.33}
\]

and the right-hand side of the inequality is bounded in \( BC^0 \) by \( \| v \|^2_{\text{sol}} \). This completes the treatment of case (ii) for \( \rho = -\delta + \frac{3}{4} \).

Estimates for the weight \( \rho = \delta + \frac{3}{4} \). Again, due to (4.5), we can focus on estimating the term

\[
\int_I \left\| (D_x - 4)(D_x - 3)\tilde{q}(D_x - 1)(D_x - 1)x^{-m} \sum_{j=1}^{m} D^{(j)} v \right\|^2_{k-2, \delta + \frac{3}{4}} \ dt, \tag{4.34}
\]

where the \( a^{(j)} \) meet conditions (4.5b). In this case, we make the observation that the operator \( \tilde{q}(D_x - 1) \) (cf. (3.21a) for the definition of \( \tilde{q} \)) figures the factor \( D_x - 2 \). This operator cancels contributions \( O(x^2) \) which we could not exclude by using only the structure (4.5). Distributing \( D \)-derivatives coming from the norm \( \| \cdot \|_{k-2, \delta + \frac{3}{4}} \) shifts the operator \( D_x - 2 \) to \( D_x - 2 - b_y \), where \( b_y \) is the number of \( D_y \)-derivatives applied. Note, however, that due to the presence of an \( x \)-factor in the operator \( D_x = x^\beta D_y \), this also leads to a term of order \( O(x^{2+ b_y}) \) and therefore the \( O(x^2) \) coefficient vanishes if \( b_y > 0 \). If \( b_y = 0 \), we may apply Lemma 3.8 and therefore it suffices to estimate instead of (4.34) a term of the form

\[
\int_I \left\| x^{-m} \sum_{j=1}^{m} D^{(j)} v - x^2 \left( x^{-m} \sum_{j=1}^{m} D^{(j)} v \right) \right\|^2_{\delta + \frac{3}{4}} \ dt, \tag{4.35a}
\]

where the \( b^{(j)} \) fulfill the conditions

\[
2 \leq \left| b^{(1)} \right| \leq \left| k \right| + 9, \quad 1 \leq \left| b^{(j)} \right| \leq \left| \frac{k}{2} \right| + 5, \quad \text{for} \quad j \in \{2, \ldots, m\},
\]

\[
m \leq \sum_{j=1}^{m} \left| b^{(j)} \right| \leq k + m + 8, \quad \text{and} \quad b_y^{(j)} \leq k - 2, \quad \text{for} \quad j \in \{1, \ldots, m\}. \tag{4.35b}
\]

Because of conditions (4.5b), we can assume that one of the following two conditions is fulfilled:

(i) \( b_y^{(1)} \geq 1 \) and \( \left| b^{(1)} \right| \geq 2 \),

(ii) \( b_y^{(1)} = 0 \), \( b_y^{(2)} \geq 1 \), and \( b_y^{(1)} = \left| b^{(1)} \right| = \left| b^{(2)} \right| \geq 2 \).

If (i) holds, then we infer that \( D^{(1)} v = O(x^2) \) as \( x \searrow 0 \) and \( D^{(2)} v = O(x) \) as \( x \searrow 0 \) for \( j \in \{2, \ldots, m\} \). This implies

\[
\left( x^{-m} \sum_{j=1}^{m} D^{(j)} v \right)^2 = \left( D^{(1)} v \right)^2 \times \left( D^{(2)} v \right)^2.
\]
Then we note that, due to (4.35b), if
\[ \int \left| x^{1-m} \times D^{(i)} v - x^2 \left( x^{1-m} \times D^{(i)} v \right) \right|^2 dt \]
\[ \lesssim \int \left| D^{(i)} v - \left( D^{(i)} v \right) x^2 \right|^2 \times x^{1-m} \times D^{(i)} v \right|^{2} dt \]
\[ + \int \left| D^{(i)} v \right|^{2} \times \left( x^{2-m} \times D^{(i)} v - x \times \left( D^{(i)} v \right) \right) \right|^{2} dt. \]  
(4.36)

Then we observe that for the second line in (4.36) we have
\[ \frac{1}{2} \int \left| D^{(i)} v - \left( D^{(i)} v \right) x^2 \right|^2 \times x^{1-m} \times D^{(i)} v \right|^{2} dt \]
\[ \lesssim \int \left| D^{(i)} v - \left( D^{(i)} v \right) x^2 \right|^2 \times \prod_{j=2}^{m} \left| x^{-1} D^{(i)} v \right|^{2} B C^0(I \times (0, \alpha) \times \mathbb{R}) \]
\[ \lesssim \int \left| (D_x - 3)(D_x - 2)\hat{q}(D_x)D_v \right|^2 \frac{1}{k+2, \delta+\frac{7}{2}} dt \times \|v\|_{BC^0(0, \alpha) \times \mathbb{R}}^{2m} \]
where Lemma 3.8 was used in the second-but-last step. The last line of (4.36) can be estimated according to
\[ \int \left| D^{(i)} v \right|^{2} \times \left( x^{2-m} \times D^{(i)} v - x \times \left( D^{(i)} v \right) \right) \right|^{2} dt \]
\[ \lesssim \left| D^{(i)} v \right|^{2} \times \int \left| x^{2-m} \times D^{(i)} v - x \times \left( D^{(i)} v \right) \right|^{2} dt. \]

For the first factor we get
\[ \left| D^{(i)} v \right|^{2} \times \left( x^{2-m} \times D^{(i)} v - x \times \left( D^{(i)} v \right) \right) \right|^{2} \]
so that this term is bounded by \( \|v\|_{\text{sol}} \) in view of (3.59b) of Lemma 3.9. For the second factor we get
\[ \int \left| x^{2-m} \times D^{(i)} v - x \times \left( D^{(i)} v \right) \right|^{2} dt \]
\[ \lesssim \int \left| D^{(i)} v - \left( D^{(i)} v \right) x^2 \right|^2 \times \prod_{j=2}^{m} \left| x^{-1} D^{(i)} v \right|^{2} B C^0(I \times (0, \alpha) \times \mathbb{R}) \]
\[ + \left| D^{(i)} v \right|^{2} \times \int \left| x^{2-m} \times D^{(i)} v - x \times \left( D^{(i)} v \right) \right|^{2} dt. \]

Then we note that, due to (4.35b), if \( \left[ \frac{k}{2} \right] + 5 \leq \tilde{k} + 2 \) (a condition which, in fact, is already implied by (4.33)), then
\[ \int \left| D^{(i)} v - \left( D^{(i)} v \right) x^2 \right|^2 dt \lesssim \int \left| \hat{q}(D_x)D_x \right|^2 \frac{1}{k+2, \delta+\frac{7}{2}} dt \]
\[ \lesssim \|v\|_{\text{sol}}^{2} \]
where Lemma 3.8 has been employed. Due to Lemma 3.8 and using (4.28), the product
\[ \prod_{j=3}^{m} \left| x^{-1} D^{(i)} v \right|^{2} B C^0(I \times (0, \alpha) \times \mathbb{R}) \]
is bounded by \( \|v\|_{\text{sol}}^{2(m-2)} \). Next, by estimate (3.59b) of Lemma 3.9, we find
\[
\left\| \left( D^{(2)} v \right)_1 \right\|_{B^0(I \times \mathbb{R})} \lesssim \|v_0\|_{B^0(I \times \mathbb{R})} + \|v_1\|_{B^0(I \times \mathbb{R})} \lesssim \|v\|_{\text{sol}}.
\]
Iterating the argument on the term
\[
\int_I \left\| x^{3-m} \sum_{j=3}^m D^{(j)} v - x^{3-m} \sum_{j=2}^m \left( D^{(j)} v \right)_1 \right\|^2 dt
\]
upgrades (4.36) (upon enlarging \( C \)) to
\[
\int_I \left\| x^{1-m} \sum_{j=1}^m D^{(j)} v - \left( x^{1-m} \sum_{j=1}^m D^{(j)} v \right)_1 \right\|^2 dt \lesssim (C \|v\|_{\text{sol}})^{2m} \tag{4.37}
\]
and concludes the proof in case (i).

Now suppose that we are in case (ii). Then we have
\[
\left( x^{1-m} \sum_{j=1}^m D^{(j)} v \right)_2 = v_1 \times \left( D^{(2)} v \right)_2 \times \sum_{j=3}^m \left( D^{(j)} v \right)_1
\]
and we may estimate the term (4.35a) according to
\[
\int_I \left\| x^{1-m} \sum_{j=1}^m D^{(j)} v - x^2 \left( x^{1-m} \sum_{j=1}^m D^{(j)} v \right)_2 \right\|^2 dt 
\lesssim \int_I \left\| D^{(1)} (v - v_0 - v_1 x) \times x^{1-m} \sum_{j=2}^m D^{(j)} v \right\|^2 dt 
+ \int_I \left\| v_1 \left( x^{2-m} \sum_{j=2}^m D^{(j)} v - x^2 \left( D^{(2)} v \right)_2 \times \sum_{j=3}^m \left( D^{(j)} v \right)_1 \right) \right\|^2 dt. \tag{4.38}
\]
For the second line in (4.38) we use the decomposition
\[
D^{(1)} (v - v_0 - v_1 x) = D^{(1)} (u - v_0 - v_1 x - v_1 + x^{1+\beta} - v_2 x^2) + (1 + \beta) \alpha_2 v_1 + x^{1+\beta} + 2 \beta (v_1 x)^2,
\]
so that
\[
\int_I \left\| D^{(1)} (v - v_0 - v_1 x) \times x^{1-m} \sum_{j=2}^m D^{(j)} v \right\|^2 dt 
\lesssim \int_I \left\| D^{(1)} v \right\|^2 \|v\|_{\text{sol}}^{2(m-1)} + \int_I \left\| v_1 + \beta D^{(2)} v \right\|^2 \|v\|_{\text{sol}}^{2(m-2)} + \int_I \left\| v_2 \|_{\text{sol}}^{2} \right\| \left\| D^{(2)} v \right\|^2 \|v\|_{\text{sol}}^{2(m-2)}
\]
Then we note that the products
\[
\prod_{j=2}^m \left\| x^{-1} D^{(j)} v \right\|_{B^0(I \times (0,x) \times \mathbb{R})} \quad \text{and} \quad \prod_{j=3}^m \left\| x^{-1} D^{(j)} v \right\|_{B^0(I \times (0,x) \times \mathbb{R})}
\]
are bounded by \( \|v\|_{\text{sol}}^{2(m-1)} \) and \( \|v\|_{\text{sol}}^{2(m-2)} \), respectively, because of (4.28). As in the context of (4.32), we may estimate
\[
\int_I \left\| v_1 + \beta D^{(2)} v \right\|^2 dt \lesssim \|v\|_{\text{sol}}^{4}
\]
under the additional constraints \( \delta \in (0, \frac{1}{2} (\beta - \frac{1}{2})) \) and
\[
\left\lfloor \frac{k}{2} \right\rfloor + 5 \leq \min \left\{ \tilde{k} - 2, \tilde{k} - 4 \right\}
\] (4.39)
being more restrictive than (4.33). Finally, we have \( \int_I \|v_2\|_{BC^0(\mathbb{R})}^2 \, dt \lesssim \|v\|_{\sol}^2 \) by estimate (3.59c) of Lemma 3.9, so that
\[
\int_I \left\| D^{(i)}(v - v_0 - v_1 x) \times x^{1-m} \sum_{j=2}^m D^{(i)} v \right\|_{\delta + \frac{1}{2}}^2 \, dt \lesssim \|v\|_{\sol}^2.
\]
The last line in (4.38) can be estimated through
\[
\int_I \left\| v_1 \left( x^{2-m} \sum_{j=2}^m D^{(i)} v - x^2 \left( D^{(i)} v \right)_2 \right) \sum_{j=3}^m \left( D^{(i)} v \right)_1 \right\|_{\delta + \frac{1}{2}}^2 \, dt \lesssim \|v_1\|_{BC^0(I \times \mathbb{R})} \times \int_I \left\| x^{2-m} \sum_{j=2}^m D^{(i)} v - x^2 \left( D^{(i)} v \right)_2 \sum_{j=3}^m \left( D^{(i)} v \right)_1 \right\|_{\delta + \frac{1}{2}}^2 \, dt,
\]
where \( \|v\|_{\sol} \) bounds \( \|v_1\|_{BC^0(I \times \mathbb{R})} \) by estimate (3.59b) of Lemma 3.9 and the term
\[
\int_I \left\| x^{2-m} \sum_{j=2}^m D^{(i)} v - x^2 \left( D^{(i)} v \right)_2 \sum_{j=3}^m \left( D^{(i)} v \right)_1 \right\|_{\delta + \frac{1}{2}}^2 \, dt
\]
can be treated as in case (i).

Using these bounds in (4.38) concludes the proof of (4.37) in case (ii).

Estimates for the weight \( \rho = -\delta + \frac{3}{2} \). In order to complete the proof of (4.24) it only remains to treat the case \( \rho = -\delta + \frac{3}{2} \). In this case we start from equation (4.5), then apply \( D_{\theta}^\rho \) and the \( D^\rho \)-operator coming from the term \( \|N^{(i)}(v)\|_{k-\delta + \frac{3}{2}} \) which we need to estimate. After commuting \( D_z \) and \( D_y \) derivatives with the \( x^{-1} \)-factors, we find terms of the form
\[
x^{1-m} \sum_{j=1}^m D^{(i)} v,
\]
for which in addition to conditions (4.25b)–(4.25d) and (4.26) there exist indices \( \ell^{(j)} \geq 0 \) for \( j = 1, \ldots, m \) such that
\[
b^{(j)} = c^{(j)} + (0, \ell^{(j)}), \quad \sum_{j=1}^m \ell^{(j)} = 3, \quad 1 \leq |c^{(j)}| \leq \tilde{k}.
\] (4.40)

Thus, up to reordering the terms, we are in one of the following three cases:

(i) \( \ell^{(1)} = 3 \) and \( \ell^{(j)} = 0 \) for \( j = 2, \ldots, m \),
(ii) \( \ell^{(1)} = 2, \ell^{(2)} = 1, \ell^{(j)} = 0 \) for \( j = 3, \ldots, m \),
(iii) \( \ell^{(1)} = \ell^{(2)} = \ell^{(3)} = 1 \) and \( \ell^{(j)} = 0 \) for \( j = 4, \ldots, m \).

In case (i) we can directly estimate
\[
\int_I \left\| x^{1-m} \sum_{j=1}^m D^{(i)} v \right\|_{\delta + \frac{1}{2}}^2 \, dt \lesssim \int_I \left\| D^{(i)} v \right\|_{\delta + \frac{1}{2}}^2 \, dt \times \prod_{j=2}^m \left\| x^{-1} D^{(i)} v \right\|_{BC^0(I \times (0,\infty) \times \mathbb{R})}^2
\]
\[
\lesssim \int_I \left\| D_{\theta}^3 v \right\|_{k, \delta + \frac{3}{2}}^2 \, dt \times \|v\|_{\sol}^{2(m-1)} \lesssim \|v\|_{\sol}^{2m},
\]
provided
\[
\tilde{k} \leq \min\{\tilde{k} - 2, \tilde{k} - 2\}.
\] (4.41)

For case (ii) we have
\[
\int_I \left\| x^{1-m} \sum_{j=1}^m D^{(i)} v \right\|_{\delta + \frac{1}{2}}^2 \, dt \lesssim \int_I \left\| D^{(i)} v \times D^{(j)} v \right\|_{\delta + \frac{1}{2}}^2 \, dt \times \prod_{j=3}^m \left\| x^{-1} D^{(i)} v \right\|_{BC^0(I \times (0,\infty) \times \mathbb{R})}^2
\]
\[
\lesssim \int_I \left\| D^{(i)} v \times D^{(j)} v \right\|_{\delta + \frac{1}{2}}^2 \, dt \times \|v\|_{\sol}^{2(m-2)}.
\] (4.42)
Then we can use estimate (4.17a) of Lemma 4.2, where (with the notation (4.40)) for \( j = 1, 2 \) we define \( w^{(j)} := D^{(j)} v \). By Young's inequality, (4.40) and condition (4.41) allow to bound \( \| x^{-1} w^{(j)} \| _{BC^0(I \times (0, \infty) \times R)} \lesssim \| v \| _{sol} \) via (3.59b) of Lemma 3.9. Thus we find

\[
\int_J \left\| D^{(1)} v \times x^{-1} D^{(2)} v \right\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \lesssim \int_J \left( \left\| D^3_w (1) \right\| ^2 _{-\delta + \frac{2}{3}} + \left\| D^3_w (2) \right\| ^2 _{-\delta + \frac{2}{3}} \right) \frac{dt}{t} \\
\times \sup \left( \left\| x^{-1} w^{(1)} \right\| _{BC^0((0, \infty) \times R)} + \left\| x^{-1} w^{(2)} \right\| _{BC^0((0, \infty) \times R)} \right)
\]

provided (4.41) holds, and thus

\[
\int_J \left\| x^{-1} \right\| _m j=1 D^{(j)} v \left\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \lesssim \| v \| ^{2m} _{sol} \]

in this case as well.

For case (iii), we proceed similarly to the above and write

\[
\int_J \left\| x^{-1} \right\| _m j=1 D^{(j)} v \left\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \lesssim \int_J \left\| D^{(1)} v \times D^{(2)} v \times D^{(3)} v \right\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \times \prod _{j=1} ^m \left\| x^{-1} D^{(j)} v \right\| ^2 _{BC^0(I \times (0, \infty) \times R)}
\]

Defining \( w^{(j)} := D^{(j)} v \) for \( j = 1, 2, 3 \), by using estimate (4.17b) of Lemma 4.2 together with Young's inequality, and by using (4.40) together with (3.59b) of Lemma 3.9 as in case (ii), we get

\[
\int_J \left\| D^{(1)} v \times D^{(2)} v \times D^{(3)} v \right\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \lesssim \int_J \left( \left\| D^3_w (1) \right\| ^2 _{-\delta + \frac{2}{3}} + \left\| D^3_w (2) \right\| ^2 _{-\delta + \frac{2}{3}} + \left\| D^3_w (3) \right\| ^2 _{-\delta + \frac{2}{3}} \right) \frac{dt}{t} \\
\times \left( \left\| x^{-1} w^{(1)} \right\| ^4 _{BC^0(I \times (0, \infty) \times R)} + \left\| x^{-1} w^{(2)} \right\| ^4 _{BC^0(I \times (0, \infty) \times R)} + \left\| x^{-1} w^{(3)} \right\| ^4 _{BC^0(I \times (0, \infty) \times R)} \right)
\]

provided (4.41) holds, which allows also in case (iii) to write

\[
\int_J \left\| x^{-1} \right\| _m j=1 D^{(j)} v \left\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \lesssim \| v \| ^{2m} _{sol} .
\]

Altogether, we obtain the same bound as in (4.27), namely

\[
\int_J \left\| x^{-1} \right\| _m j=1 D^{(j)} v \left\| ^2 _{-\delta + \frac{2}{3}} \frac{dt}{t} \lesssim (C \| v \| _{sol} ) ^{2m} .
\]  

(4.42)

**Estimate of** \( \| \Lambda^{(2)} (v) \| _{rhs} \)**. We now desire to prove the analogue of (4.24), namely that

\[
\| \Lambda^{(2)} (v) \| _{rhs} \lesssim \| v \| ^2 _{sol} \text{ for } \| v \| _{sol} \ll 1 .
\]

(4.43)

In order to prove the above, for each of the four lines appearing in (4.14a), we obtain terms with the basic structure (4.5a), (4.5b), and we distribute derivatives and weights as in each of the three terms from (2.19).

**Estimate of** \( \| \Lambda^{(2)} (v) \| _{rhs} \) **for** \( \rho = -\delta + \frac{1}{2} \)**. For the first term

\[
\left\| (D_x - 1) \Lambda^{(2)} (v) \right\| _{k-2,-\delta + \frac{2}{3}} .
\]
appearing in (2.19), we may perform the same discussion as the one used in treating the analogous terms for \( \mathcal{N}^{(1)}(v) \) in those two cases. Note that the terms appearing in \( \|\mathcal{N}^{(1)}(v)\|_{\text{rhs}} \) with weights \( \rho = -\delta + \frac{1}{2} \) were treated in the same way in both cases (i) and (ii) from that discussion. By splitting as in (4.27), the estimates thereafter leading to the bound (4.30) apply, and give the desired bound

\[
\int_I \left\| (D_x - 1)\mathcal{N}^{(2)}(v) \right\|_{k-2,-\delta+rac{1}{2}}^2 \, dt \lesssim \|v\|^{2m}_{\text{sol}}. \tag{4.44}
\]

**Estimate of** \( \|\mathcal{N}^{(2)}(v)\|_{\text{rhs}} \) **for** \( \rho = -\delta + \frac{3}{2} \). We next control the terms that come from distributing the derivatives appearing in the expression \( \|D^3_{\nu} \mathcal{N}^{(2)}(v)^2\|_k^{-4,-\delta+\frac{3}{2}} \). In this case it suffices to use expansion (4.5) via (4.42) once more, which follows by the same reasoning as done in the case of \( \mathcal{N}^{(1)}(v) \), leading to the bound

\[
\int_I \left\| D^3_{\nu} \mathcal{N}^{(2)}(v) \right\|_{k-4,-\delta+rac{3}{2}}^2 \, dt \lesssim \|v\|^{2m}_{\text{sol}}. \tag{4.45}
\]

**Estimate of** \( \|\mathcal{N}^{(2)}(v)\|_{\text{rhs}} \) **for** \( \rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\} \). We pass to bounding the terms with weight \( \rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\} \), namely

\[
\int_I \left\| \hat{q}(D_x - 1)(D_x - 1)\mathcal{N}^{(2)}(v)^2 \right\|_{k-2,\delta+rac{1}{2}}^2 \, dt, \tag{4.46a}
\]

\[
\int_I \left\| \hat{q}(D_x - 1)(D_x - 1)^2\mathcal{N}^{(2)}(v)^2 \right\|_{k-2,-\delta+rac{3}{2}}^2 \, dt, \tag{4.46b}
\]

and

\[
\int_I \left\| (D_x - 4)(D_x - 3)\hat{q}(D_x - 1)(D_x - 1)\mathcal{N}^{(2)}(v)^2 \right\|_{k-2,\delta+rac{3}{2}}^2 \, dt. \tag{4.46c}
\]

The first line of (4.14a) near \( x = 0 \) for \( \rho \in \{\frac{\delta}{2}, \pm \delta + \frac{3}{2}\} \). We will utilize the decomposition (4.14a) for treating the term (4.46a) for \( x \leq 2 \). To this aim we introduce a cut-off function \( \chi : [0,\infty) \to [0,1] \) that is smooth and meets \( \chi(x) = 1 \) for \( x \in [0,\frac{1}{2}] \) and \( \chi(x) = 0 \) for \( x \in [1,\infty) \). We perform our estimates for \( \chi \mathcal{N}^{(2)}(v) \) first, starting from the structure (4.14).

For the first line of (4.14a), we have an expression of the form

\[
-\frac{3}{8}(1 + v_1)^{-3}(6v_1^2 + 8v_1^3 + 3v_1^4 + 2v_0^2 + (v_0^2)^2 + (v_0)^4) \chi(x) =: \sigma(t,y)\chi(x)x. \quad \tag{4.47}
\]

By applying \((D_x - 1)\) to (4.47) and noting that \((D_x - 1)x = 0\) we find

\[
(D_x - 1) \left( \sigma(t,y)\chi(x)x \right) = \sigma(t,y)x D_x \chi(x) \quad \tag{4.48}
\]

and due to the definition of \( \chi \) the above expression vanishes for \( x \not\in [\frac{1}{2},1]\). Distributing the remaining \( D_x \)-derivatives appearing in the definition of the norms appearing in (4.46) for \( f = \sigma(t,y)(D_x - 1)\chi(x)x \) yields via (4.48) bounds in the same spirit, that only have extra factors depending on the first \( \kappa \) derivatives of \( \chi \), where \( \kappa \) is as in (4.25c). Here we note that we may always insert an arbitrary multiple of a smooth cut-off \( \vartheta(x) \) which vanishes on \([2,\infty)\) and is equal to 1 on \([0,1]\), for which we then automatically have \( D^\chi \vartheta \chi = \vartheta D^\chi \chi = D^\chi (\vartheta \chi) \). Distributing the \( D_\nu \)-derivatives from the same norms, we have to estimate a series of terms of the form

\[
\int_I \left\| g^{(1)} \right\|_\rho^2 \, dt, \tag{4.49a}
\]

where \( \rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\} \) and

\[
g^{(1)} = \vartheta(1 + v_1)^{-4-m} \times P_{\kappa_y}^\chi \left( \vartheta v_1, D_\nu \vartheta v_1, \ldots, D_\nu^n \vartheta v_1, \vartheta(v_0)_y, D_\nu \vartheta(v_0)_y, \ldots, D_\nu^n \vartheta(v_0)_y \right), \tag{4.49b}
\]

where \( \kappa_y \) is as in (4.25d), \( m \geq 2 \), \( \vartheta(x) \) is a smooth cut-off as above and the polynomial \( P_{\kappa_y}^\chi \) uses multiplication with \( \times \) and has the properties that the total number of \( D_\nu \)-derivatives which appear is not more than \( \kappa_y \) and the degree of a monomial is between 2 and \( m \).

To estimate the terms (4.49), we use the support properties of \( \vartheta \) together with the bound (3.59b) of Lemma 3.9 as follows. First, via (3.59b) of Lemma 3.9, in case \( c\|v\|_{\text{sol}} < 1 \) for a constant \( c \) depending only on the implicit constant in (3.59b), we have

\[
\left\| (1 + v_1)^{-4-m} \right\|_{BC^0(I \times (0,x) \times \mathbb{R})} \lesssim (1 - c\|v\|_{\text{sol}})^{-4-m}. \tag{4.50}
\]
Next, due to the presence of the cut-off \( \hat{\psi} \) we have via (3.59b) of Lemma 3.9, the bound
\[
\| \partial P_{n,m}' \|_{BC^0((x,0) \times R)} \lesssim \| v \|_{\text{sol}}^2 \left( 1 + (C\| v \|_{\text{sol}})^{m-1} \right),
\]
for a constant \( C \) depending only on \( \delta, \tilde{k}, \) and \( k \), provided
\[
\kappa_y \leq \min \{ \tilde{k} - 2, \tilde{k} - 2 \}. \tag{4.52}
\]
These bounds directly lead to
\[
\int_0^T \| g^{(1)} \|_p^2 \, dt \lesssim \| v \|_{\text{sol}}^4 \left( 1 + C\| v \|_{\text{sol}}^2 (m-2) \right) \left( 1 - c\| v \|_{\text{sol}}^{-2(m+4)} \right)
\]
for \( \rho \in \{ \delta + \frac{1}{2}, \pm \delta + \frac{3}{2} \} \) and for a constant \( C > 0 \) depending only on \( \delta, \tilde{k}, \) and \( \tilde{k} \). This is the estimate needed for the first line of (4.14).

The second line of (4.14a) near \( x = 0 \) for \( \rho \in \{ \delta + \frac{1}{2}, \pm \delta + \frac{3}{2} \} \). We pass to treating contributions of the terms for \( f = \chi \mathcal{N}^{(2)}(v) \) coming as a cut-off of the second line of (4.14a), i.e., we look at the terms
\[
f = \chi(1 + v_1)^{-3} (4v_1 + 6v_1^2 + 4v_1 + v_1^2 - 2(v_0)^2 - (v_0)^2) q(D_x) \phi \tag{4.53}
\]
and
\[
\delta \chi(1 + v_1)^{-4} (4v_1 + 6v_1^2 + 4v_1 + v_1^2 - 2(v_0)^2 - (v_0)^2) q(D_x)(v - v_0 - v_1 x). \tag{4.54}
\]
We distribute the remaining \( D \)-derivatives, so that we need to estimate terms of the form
\[
\int_0^T \| g^{(2)} \|_p^2 \, dt, \tag{4.55a}
\]
where \( \rho \in \{ \delta + \frac{1}{2}, \pm \delta + \frac{3}{2} \} \) and
\[
g^{(2)} = (1 + v_1)^{-4-m} \chi \partial P_{n',m'}^x (v_1, D_y v_1, \ldots, D_y^x v_1, (v_0)_y, D_y (v_0)_y, \ldots, D_y^x (v_0)_y)
\times \partial D^b q(D_x)/(D_x - 2)(v - v_0 - v_1 x), \tag{4.55b}
\]
with the same notation for \( P_{n',m'}^x \) and \( \partial \) as for (4.49), and where the monomials contributing to \( P_{n',m'}^x \) have degrees between \( 1 \) and \( m' \). The indices and multi-indices which appear above have to satisfy the following constraints:
\[
|b| \leq \begin{cases} 
\tilde{k} + 2 & \text{for } \rho = \delta + \frac{1}{2}, \\
\tilde{k} + 2 & \text{for } \rho = -\delta + \frac{3}{2}, \\
\tilde{k} + 4 & \text{for } \rho = \delta + \frac{3}{2}, \\
\tilde{k} + 4 & \text{for } \rho = -\delta + \frac{3}{2}, \\
\end{cases} \tag{4.56c}
\]
If expansion (4.2) is valid and because \( x^{1+\beta} \) and \( x^2 \) are in the kernel of \( q(D_x)/(D_x - 2) \), we expect to have \( D^b q(D_x)/(D_x - 2)(v - v_0 - v_1 x) = O(x^{1+2\beta}) \) as \( x \to 0 \). We derive bounds for the individual factors in (4.54) separately. For the first terms we argue precisely as in the previous step, obtaining
\[
\left\| (1 + v_1)^{-4-m} \right\|_{BC^0((x,0) \times R)} \lesssim \left( 1 - c\| v \|_{\text{sol}}^{-4-m} \right), \tag{4.56d}
\]
and
\[
\left\| \partial P_{n',m'}^x \right\|_{BC^0((x,0) \times R)} \lesssim \| v \|_{\text{sol}} \left( 1 + (C\| v \|_{\text{sol}})^{m'-1} \right), \tag{4.57d}
\]
for a constant \( C \) depending only on \( \delta, \tilde{k}, \) and \( \tilde{k} \). For the last term we use the fact that \( 0, 1, \) and \( 1 + \beta \) are zeros of \( q \) (cf. (1.15a)) and thus \( q(D_x) \) vanishes on \( 1, x, x^{1+\beta}, \) and that \( (D_x - 2) \) vanishes on \( x^2 \). This leads to the bounds
\[
\int_0^T \| \partial (x) \partial D^b q(D_x)/(D_x - 2)(v - v_0 - v_1 x) \|_p^2 \, dt \lesssim \left\| \frac{\partial D^b q(D_x)/(D_x - 2)}{D_x^3(D_x - 2)(D_x - 2)q(D_x)D_x v} \right\|_{k+2,\delta+\frac{3}{2}}^2 \, dt \tag{4.58a}
\]
which uses the bound on $|b|$ in (4.55) as well as Lemma 3.8. Estimates (4.56), (4.57), and (4.58) allow to obtain in for $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$

$$
\left\| \int \frac{\|g^{(2)}\|^2}{\rho} \, dt \right\|_{1} \lesssim \|v\|^4_{\text{sol}} \left( 1 + C^{m'-1} \|v\|^2_{\text{sol}}^{2(m'-1)} \right) (1 - c\|v\|^2_{\text{sol}})^{-2(m+4)}
$$

for the terms (4.55).

The third line of (4.14a) near $x = 0$ for $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$. We use $\psi = v - v_0$ in the third line of (4.14a) as well as the commutation relation $(D_x - 2)D_y = D_y(D_x - 1)$. Inserting the cut-off function $\vartheta$ like for the previous proof step, we end up considering terms of the form

$$
\vartheta(1 + v_1)^{-3}(v_0 \tilde{v}^3(D_x + \frac{1}{2})(D_x^2 - \frac{1}{2}D_x - \frac{5}{2}) + (v_0 \varphi \tilde{v}^3(D_x^2 - \frac{5}{2}D_x - \frac{7}{2})) D_y(D_x - 1)(v - v_0).
$$

Note that based on (4.2) there holds $D_y(D_x - 1)(v - v_0) = O(x^{2+\beta})$ as $x \rightarrow 0$ because $D_x - 1$ cancels the contribution $v_1x$ in the expansion (1.22) of $v$ near $x = 0$. The above term requires roughly the same treatment as the one from the second line of (4.2) which we just treated, with the difference that $g^{(3)}$, which appears in the analogue of (4.55), has now the form

$$
g^{(3)} = (1 + v_1)^{-3-m} \times \vartheta P_{\kappa,m}^\infty (v_1, D_y v_1, \ldots, D_y^m v_1, (v_0)_{x,y}, D_y (v_0)_{x,y}, \ldots, D_y^m (v_0)_{y})
$$

with $P_{\kappa,m}^\infty$ consisting of monomials of degrees between 1 and $m'$, and where the constraints on indices are the same as in (4.55) except for the ones on $b$ and $m'$ which now are replaced by the following:

$$
|b| \leq \begin{cases} 
\hat{k} + 5 & \text{for } \rho = \delta + \frac{1}{2}, \\
\hat{k} + 5 & \text{for } \rho = -\delta + \frac{3}{2}, \\
\hat{k} + 7 & \text{for } \rho = \delta + \frac{3}{2},
\end{cases} \quad 1 \leq m' \leq m + 3.
$$

The only other difference to the study of (4.55) is in the fact that the last factor contains $\vartheta(x)D_y(D_x - 1)(v - v_0)$ rather than $\vartheta(x)q(D_x)(D_x - 2)(v - v_0 - v_1x)$. Therefore, we have to replace the bounds (4.58) by

$$
\int_t \|\vartheta(x)D_y(D_x - 1)(v - v_0)\|_{1}^2 \, dt 
$$

$$
\lesssim \begin{cases} 
\int \|\tilde{q}(D_x)D_x v\|^2_{k+2,\delta+\frac{1}{2}} \\
\int \|\tilde{q}(D_x)D_x v\|^2_{k+2,-\delta+\frac{1}{2}} \\
\int \|D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v\|^2_{k+2,\delta+\frac{3}{2}}
\end{cases} \quad \text{for } \rho = \delta + \frac{1}{2}, \quad \text{for } \rho = -\delta + \frac{3}{2}, \quad \text{for } \rho = \delta + \frac{3}{2},
$$

$$
\lesssim \|v\|^2_{\text{sol}}.
$$

By combining estimate (4.63) with precisely the same bounds as (4.56) and (4.57) for the remaining factors from (4.61) we find the bound

$$
\int \|g^{(3)}\|^2_{\rho} \lesssim \|v\|^4_{\text{sol}} \left( 1 + C^{m'-1} \|v\|^2_{\text{sol}}^{2(m'-1)} \right) (1 - c\|v\|^2_{\text{sol}})^{-2(m+3)},
$$

valid for $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$ if $c\|v\|^2_{\text{sol}} < 1$.

The last line in (4.14) near $x = 0$ for $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$. Here we have to treat

$$
\chi(1 + v_1)^{-3} \sum_{(\mu,\nu,\tau) \in \mathbb{Z}} M \left( \psi_{\mu}(v_0)_{\nu} \varphi_{\tau} \right)_{j=1}^4.
$$

Note that $(v_0)_{\nu}^\psi$ can be factored out, since the precise expression (4.6) of $M$ includes only derivatives $D_x$ and products, all of which commute with $(v_0)_{\nu}^\psi$. By distributing $D$-derivatives and inserting the cut-off $\vartheta$ like in the previous cases we find that we have to estimate $\int \|g^{(4)}\|^2_{\rho} \, dt$ for

$$
g^{(4)} = \vartheta(1 + v_1)^{-3-m} \times P_{\kappa,m}^\infty (v_1, D_y v_1, \ldots, D_y^m v_1, (v_0)_{x,y}, D_y (v_0)_{x,y}, \ldots, D_y^m (v_0)_{y})
$$

$$
\times x \times \vartheta D^{(j)} w^{(j)},
$$

with

$$
w^{(j)} \in \{(v - v_0 - v_1x)_x, (v - v_0)_y\}.
$$
Here we have used the definitions of $\psi$ and $\phi$ in (4.8) and again we were allowed to insert as many smooth cut-off factors $\theta$ which vanish on $[1, \infty)$ as required, by the same argument as in the explanation following (4.55). The indices appearing in (4.66) are constrained as follows

$$
\begin{align*}
|b^{(1)}| &\lesssim \begin{cases} 
\frac{k+5}{2} & \text{for } \rho = \delta + \frac{1}{2}, \\
\frac{k+5}{2} & \text{for } \rho = -\delta + \frac{3}{2}, \\
\frac{k+7}{2} & \text{for } \rho = \delta + \frac{3}{2}.
\end{cases}
\end{align*}
$$

$$
\begin{align*}
|b^{(j)}| &\lesssim \begin{cases} 
\frac{k+1}{2} + 2 & \text{for } \rho = \delta + \frac{1}{2}, \\
\frac{k+1}{2} + 2 & \text{for } \rho = -\delta + \frac{3}{2}, \\
\frac{k+1}{2} + 3 & \text{for } \rho = \delta + \frac{3}{2},
\end{cases} \quad j \geq 2.
\end{align*}
$$

For the different terms from (4.66) we estimate the first term as in (4.56), under the condition that $c\|v\|_{\text{sol}} < 1$:

$$
\left\|\frac{1}{(1 + v_1)^{3-m}}\right\|_{BC^0(I \times (0, x) \times \mathbb{R}^p)} \lesssim (1 - c\|v\|_{\text{sol}})^{3-m}.
$$

The estimate of the second term is also similar to the one above, and we obtain, for some $C$ depending only on $\delta$, $\tilde{k}$, and $\tilde{k}$,

$$
\left\|\partial \rho \nu_{n,m'}\right\|_{BC^0(I \times (0, x) \times \mathbb{R}^p)} \lesssim 1 + (C\|v\|_{\text{sol}})^{m'}.
$$

For the remaining product term in (4.66), we first estimate all but the first two terms in the $BC^0(I \times (0, x) \times \mathbb{R}^p)$-norm as follows, for $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$:

$$
\int_I \left\|x \times \partial D^{(o)} w^{(j)}\right\|_{\rho}^2 dt \lesssim \int_I \left\|x \partial D^{(o)} w^{(1)} \times \partial D^{(o)} w^{(2)}\right\|_{\rho}^2 dt \times \prod_{j=3}^n \left\|\partial D^{(o)} w^{(j)}\right\|_{BC^0(I \times (0, x) \times \mathbb{R}^p)}^2.
$$

To justify the above estimate, we observe that $\psi_y = v_y - (v_0)_y$ and $(v - v_0 - v_1)_y = v_y - (v_0)_y - (v_1)_y x$, after which we note that the number of $D$-derivatives acting on such terms is bounded in (4.66c). Therefore, we may use the bound (3.59b) of Lemma 3.9 provided

$$
\max \left\{ \left[ \frac{k+1}{2} + 2 \right], \left[ \frac{k+1}{2} + 3 \right] \right\} \leq \min \left\{ \tilde{k} - 2, \tilde{k} - 2 \right\},
$$

which is implied by (2.4) of Assumption 2.1. We next bound for $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$ as follows:

$$
\int_I \left\|x \partial D^{(o)} w^{(1)} \times \partial D^{(o)} w^{(2)}\right\|_{\rho}^2 dt \lesssim \|v\|_{\text{sol}}^4.
$$

Once (4.71) is proved, combining it with (4.67), (4.68) and (4.69), we find the bound

$$
\int_I \left\|g^{(4)}\right\|_{\rho}^2 dt \lesssim \|v\|_{\text{sol}}^4 \left(1 + Cm^{-1} - \|v\|_{\text{sol}}^{2(m^{-1})} \right) (1 - c\|v\|_{\text{sol}})^{2(m^{-1})},
$$

valid for the weights $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$, for $C$ and $c$ depending only on, $\tilde{k}$, and $\tilde{k}$, and for $v$ such that $c\|v\|_{\text{sol}} < 1$. To perform the bound of the term (4.71) and complete the proof of (4.72), we will separately consider the three values of $\rho$.

The bound (4.71) for $\rho = \delta + \frac{1}{2}$. We study at the same time the two possible values of $u^{(1)}$ given in (4.66b). Note that, in view of (4.2), the first factor entering the norm in (4.66b) is either $x \partial D^{(o)}(v - v_0 - v_1)_x = O(x^{2+\beta})$ or $x \partial D^{(o)}(v - v_0)_y = O(x^2)$ as $x \searrow 0$. Since $2 > 1 + \beta > 1 + \delta$ we may bound according to

$$
\int_I \left\|x \partial D^{(o)} w^{(1)} \times \partial D^{(o)} w^{(2)}\right\|_{\rho}^2 dt \lesssim \int_I \left\|x \partial D^{(o)} w^{(1)}\right\|_{\delta + \frac{1}{2}}^2 dt \times \left\|\partial D^{(o)} w^{(2)}\right\|_{BC^0(I \times (0, x) \times \mathbb{R}^p)}^2.
$$

$$
\lesssim \sum_{\delta < a_1 < \frac{b_1}{2}} \int_I \left\|\partial D^{(o)} D_x(v - v_0 - v_1)_x\right\|_{\delta + \frac{1}{2}}^2 dt \times \|v\|_{\text{sol}}^2
$$

$$
\sum_{\delta < a_1 < \frac{b_1}{2}} \int_I \left\|\partial D^{(o)} D_y(v - v_0)_y\right\|_{\delta + \frac{1}{2}}^2 dt \times \|v\|_{\text{sol}}^2
$$

for $w^{(1)} = (v - v_0 - v_1)_x$

$$
\lesssim \int_I \left\|\partial D^{(o)} D_x v\right\|_{\delta + 2, \delta + \frac{1}{2}}^2 dt \times \|v\|_{\text{sol}}^2 \lesssim \|v\|_{\text{sol}}^4.
$$

(4.73)
where we have commuted the $x$-factor with the $D^b(y)$-derivatives and then used Lemma 3.8 for the final estimate. This concludes the bound of the fourth line of (4.14) near $x=0$ for $\rho = \delta + \frac{3}{2}$.

The bound (4.71) for $\rho = -\delta + \frac{3}{2}$. We first consider the case $w^{(1)} = (v - v_0 - v_1 x)_x$ and write

$$v - v_0 - v_1 x = (v - v_0 - v_1 x - v_{1+\beta} x^{1+\beta}) + v_{1+\beta} x^{1+\beta}. \quad (4.74)$$

Using the triangle inequality and bounds similar to the previous proof step, we may bound

$$\int_I \| x \partial D^b(v^{(1)}_x) \partial D^b(v^{(2)}) \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \int_I \| x \partial D^b(v^{(2)}) \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \sum_{0 \leq a_x \leq b(x_0)} \int_I \| x \partial D^b(v^{(1)}_x) \partial D^b(v^{(2)}) \|_{-\delta + \frac{3}{2}}^2 dt. \quad (4.75)$$

The first term above can be estimated by using Lemma 3.8, which applies because $w = O(x^2)$ as $x \to 0$, leading to a finite $|||\cdot|||_{-\delta + \frac{3}{2}}$-norm in the definition of $|||\cdot|||_{-\delta + \frac{3}{2}}$. We also note that the polynomial $q(\zeta)$ has roots 0, 1, and $1 + \beta$ (cf. (3.21a)) and thus $q(D_x)D_x w = \hat{q}(D_x) D_x v$. As a result we have

$$\int_I \| x \partial D^a D_x w \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \int_I \| \hat{q}(D_x) D_x v \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \int_I \| \hat{q}(D_x) D_x v \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \| v \|_{sol}^2. \quad (4.76)$$

For the second term in (4.75) we estimate according to

$$\int_I \| x^{1+\beta} \partial D^b(v^{(1)}_x) \partial D^b(v^{(2)}) \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \int_I \| \hat{q}(D_x) D_x v \|_{-\delta + \frac{3}{2}}^2 dt \lesssim \| v \|_{sol}^2 \quad (4.77)$$

where we have used the fact that $\partial$ has bounded support in $x$, and we have distinguished which of the two possibilities from (4.66b) is achieved by $w^{(2)}$. Furthermore, we have used the fact that $-\delta + \frac{3}{2} - \beta \in (\delta, -\delta + 1)$ and the elliptic estimates from Lemma 3.8 and once more the fact that 0 and 1 are roots of $q(\zeta)$. Note that again we interpolated $|||\cdot|||_{-\delta + \frac{3}{2}}$ above by $|||\cdot|||_\delta$ and $|||\cdot|||_{-\delta + 1}$. The resulting terms are finite because $D_x(v - v_0 - v_1 x) = O(x^{1+\beta})$ and $D_y(v - v_0) = O(x^2)$ as $x \to 0$, and $2 > 1 + \beta > -\delta + \frac{3}{2}$ since $\beta > \frac{3}{2}$. Further notice that the conditions on $k$ and $\tilde{k}$ that (4.77) imposes, are weaker than (4.70) imposed before. This concludes our discussion of the case $w^{(1)} = (v - v_0 - v_1 x)_x$.

If instead we have $w^{(1)} = (v - v_0)_y$, then we proceed precisely like in the case $\rho = \delta + \frac{1}{2}$, the only difference being the different contribution to $\|v\|_{sol}$ used in the final estimate. We may write

$$\int_I \| x \partial D^b(v^{(1)}_y) \partial D^b(v^{(2)}) \|_\rho^2 dt \lesssim \sum_{0 \leq a_x \leq b(x_0)} \int_I \| x \partial D^a D_y(v - v_0) \|_{-\delta + \frac{3}{2}}^2 dt \| v \|_{sol}^2 \lesssim \int_I \| \hat{q}(D_x) D_x v \|_{k+2, -\delta + \frac{3}{2}}^2 dt \| v \|_{sol}^2 \lesssim \| v \|_{sol}^4, \quad (4.78)$$

which concludes the proof of (4.71) for the case $\rho = -\delta + \frac{3}{2}$.
The bound (4.71) for $\rho = \delta + 2$. We first consider the case $w^{(1)} = (v_0 - v_1 x)$. In this case we write
\[
v - v_0 - v_1 x = (v - v_0 - v_1 x - v_1 x^{1+\beta} - v_2 x^2),
\]
leading to
\[
\int_I \left\| x \partial D^{(1)} w^{(1)}(v - v_0 - v_1 x) x \right\|_{\delta + \frac{1}{2}}^2 dt \\
\lesssim \int_I \left\| x \partial D^{(2)} w_x \right\|_{\delta + \frac{1}{2}}^2 dt \times \left\| D^{(2)} w \right\|_{BC^0(I \times (0, \infty), \mathbb{R}^n)}^2 + \int_I \left\| x \partial D^{(1)} v_0 x \times \partial D^{(2)} w \right\|_{\delta + \frac{1}{2}}^2 dt.
\]
For the second term, we proceed precisely like in the bound (4.77). This is possible because the weight $\delta + \frac{1}{2} - \beta$ lies in the same interval $(\delta, -\delta + 1)$ as in case of the weight $-\delta + \frac{3}{2} - \beta$ appearing in (4.77), and the same discussion as in that case applies here as well.

For the third term, the discussion is again similar to (4.77): by commuting derivatives we find
\[
\int_I \left\| x \partial D^{(1)} w_x \right\|_{\delta + \frac{1}{2}}^2 dt \lesssim \sum_{0 \leq a < b(1)} \left\| \partial D^a D_x w \right\|_{\delta + \frac{1}{2}}^2 dt
\]
Here, we have used the fact that $w = O(x^{1+\beta})$ as $x \searrow 0$, and the fact that $1 + 2\beta > \delta + 2$ (implied by $\delta < \frac{1}{2} (\beta - \frac{1}{2})$), in order to be able to use Lemma 3.8. Moreover we use the fact that $\zeta = (\zeta - 3)(\zeta - 2)\bar{q}(\zeta)\zeta$ has roots $0, 1, 1 + \beta, 2$ (cf. (3.21a)).

For the second term in (4.79), we proceed precisely like in the bound (4.77). This is possible because the weight $\delta + \frac{3}{2} - \beta$ lies in the same interval $(\delta, -\delta + 1)$ as in case of the weight $-\delta + \frac{3}{2} - \beta$ appearing in (4.77), and the same discussion as in that case applies here as well.

For the third term, the discussion is again similar to (4.77): by commuting derivatives we find
\[
\int_I \left\| x^2 \partial D^{(1)} v_2 \times \partial D^{(2)} w \right\|_{\delta + \frac{1}{2}}^2 dt \lesssim \int_I \left\| \partial D_y w \right\|_{BC^0(\mathbb{R}^n)}^2 dt \times \sup_{t \in I} \left\| x^2 \partial D^{(2)} w \right\|_{\delta + \frac{1}{2}}^2
\]
Here, we note once more that $\delta + \frac{3}{2} \in (\delta, -\delta + 1)$ when we interpolate the norms, and we note that $D_x (v - v_0 - v_1 x) = O(x^{1+\beta})$ and $D_y (v - v_0) = O(x^2)$ as $x \searrow 0$. This concludes the discussion for the case $w^{(1)} = (v_0 - v_1 x)$. For the case $w^{(1)} = (v_0 - v_1)_y$ we write
\[
v - v_0 = (v_0 - v_1 x) + v_1 x,
\]
and by the triangle inequality
\[
\int_I \left\| x \partial D^{(1)} w_x \right\|_{\delta + \frac{1}{2}}^2 dt \lesssim \int_I \left\| x \partial D^{(2)} w_x \right\|_{\delta + \frac{1}{2}}^2 dt \times \left\| \partial D^{(2)} w \right\|_{BC^0(I \times (0, \infty), \mathbb{R}^n)}^2 + \int_I \left\| x \partial D^{(1)} D_y v_1 \times \partial D^{(2)} w \right\|_{\delta + \frac{1}{2}}^2 dt
\]
We separately bound the three terms in (4.79).
Now we can estimate
\[
\int \left\| \partial^\alpha D_p q \right\|_{\delta + \frac{3}{2}}^2 \, dt \lesssim \int \left\| D_y \overline{q}(D_x) D_x v \right\|_{k+1, \delta + \frac{3}{2}}^2 \, dt \lesssim \|v\|_{\text{sol}}^2,
\]
where we have used Lemma 3.8, the fact that \( D_y w = O(x^{2+\beta}) \) as \( x \to 0 \), as well as the fact that \( \overline{q}(D_x) D_x w = \overline{q}(D_x) D_x v \) in this case.

For the second term in (4.82) we can estimate
\[
\int \left\| x^2 \partial^\alpha D_y^{(1)}(v_1) y \times \partial D_x^{(2)} w^{(2)} \right\|_{\delta + \frac{3}{2}}^2 \, dt \lesssim \sup_{t \in I} \left| e^{b + 1} v_1 \right|_{\text{BC}^0 (R_k)}^2 \sum_{0 \leq u < \delta} \int \left\| \partial D^a x w^{(2)} \right\|_{\delta + \frac{3}{2}}^2 \, dt
\]
\[
\lesssim \|v\|_{\text{sol}}^2 \times \int \left\| \overline{q}(D_x) D_x v \right\|_{k + 2, \delta + \frac{3}{2}}^2 \, dt \lesssim \|v\|_{\text{sol}}^4,
\]
provided \( |\delta + \frac{3}{2}| + 3 \leq k + 6 \), which, however, is already implied by (4.39), and using the already-imposed assumption (4.52) on \( \kappa_g \) to ensure the applicability of (3.59b) of Lemma 3.9. This concludes the proof of (4.71) for the case \( \rho = \delta + \frac{3}{2} \).

**Conclusion of the estimates for \( N^{(2)}(v) \) near \( x = 0 \) for \( \rho \in \{ \delta + \frac{1}{2}, \pm \delta + \frac{3}{2} \} \).** By summing estimates (4.53), (4.59), (4.64) and (4.72), for the terms (4.46), we find
\[
\int \left\| \overline{q}(D_x - 1)(D_x - 1) \chi N^{(2)}(v) \right\|_{k-2, \delta + \frac{3}{2}}^2 \, dt + \int \left\| \overline{q}(D_x - 1)(D_x - 1) \chi N^{(2)}(v) \right\|_{k-2, \delta + \frac{3}{2}}^2 \, dt
\]
\[
\lesssim_{k, \delta, m} \left( \|v\|_{\text{sol}}^4 + \sum_{n \geq 2} c^{(1)}(n) C_n^2 \|v\|_{\text{sol}}^{2n} \right) \times \sum_{0 \leq m' \leq m} c^{(2)}(m', m) \left( 1 + C_n^{m'} \|v\|_{\text{sol}}^{2m'} \right) (1 - c\|v\|_{\text{sol}})^{-2(m+3)}
\]
\[
\lesssim_{k, \delta, m} \|v\|_{\text{sol}}^4,
\]
provided \( \|v\|_{\text{sol}} < c \) for some constant \( c \) depending only on \( \delta, k, \) and \( k \). The number \( c^{(1)}(n) \) comes from the discussion of terms of the form \( g^{(j)} \) as in (4.72), and bounds the number of possible terms obtainable by distributing derivatives on the terms that have a total of \( n \) factors and appear in the sum in (4.65). The number \( c^{(2)}(m, m') \) is a combinatorial coefficient which bounds the number of terms of the type \( g^{(j)} \) with \( j = 1, \ldots, 4 \), expressed in (4.53), (4.59), (4.64), and (4.72) that appear in the expansion of \( \chi N^{(2)}(v) \).

In order to to justify the last inequality in (4.86), we will prove that \( c^{(1)}(n) \) has polynomial growth in \( n \), so that the first factor in (4.85) is for small enough \( \|v\|_{\text{sol}} \) a geometric series, and that \( c^{(2)}(m, m') \) is bounded depending only on \( k \) and \( k \).

First note that the coefficient \( c^{(2)}(m, m') \) comes directly from distributing \( D_y \)-derivatives across factors coming from the \( (1 + v_1)^{-j} \) term for \( j = 3, 4 \), from at most 4 terms \( v_1 \) and from at most 4 terms \( (n)_y \), as appearing in the formulas for the terms (4.47), (4.54), (4.60), and (4.65), which were the precursors to \( g^{(1)}, g^{(2)}, g^{(3)}, \) and \( g^{(4)} \), respectively. Therefore, \( c^{(2)}(m, m') \) can be bounded with a constant depending on \( k \) and \( k \) only.

The coefficient \( c^{(1)}(n) \) bounds the number of possible terms obtainable by distributing derivatives on the terms that have a total of \( n \) factors and appear in the sum in (4.65). This term thus controls the remaining contribution to \( g^{(4)} \) from (4.66). To bound \( c^{(1)}(n) \), note that in the index set \( I \) in (4.14), only the indices \( \tau_2, \tau_3, \) and \( \tau_4 \) are unbounded. Therefore, the number of terms with a total of \( n \) factors is growing like \( C' n^2 \), with \( C' \) a universal constant. Hence, we can bound the possible number of ways of distributing \( k \) derivatives amongst these terms by
\[
c^{(1)}(n) \leq \left( C' n^2 + k' \right) \lesssim C' n^{2(\max(k,k)+m')},
\]
(4.87)
where $C^m$ and $C^m$ are universal constants. Thus for fixed $\tilde{k}$ and $\tilde{k}$, we can infer that $c^{(1)}(n)$ can be absorbed into $C^m$ upon enlarging $C > 0$ from equation 4.85. Thus, the first factor in (4.85) is bounded by $(1 + C^2) \|v\|^4_{sol}$ provided $\|v\|_{sol} \leq C^{\frac{1}{2}}$. Additionally, provided that $\|v\|_{sol} < \min\{c^{-1}, C^{-\frac{1}{2}}\}$, we find that the terms in the second sum from (4.85) are bounded and the first sum gives a converging geometric series with a pre-factor $\|v\|^4_{sol}$, proving (4.86).

Estimates for $\mathcal{N}^{(2)}(v)$ away from $x = 0$. We now treat the $\|\|_{\kappa, \rho}$-norm contribution of the term

$$(1 - \chi)\mathcal{N}^{(2)}(v)$$

away from $x = 0$ for the cases $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$. In this case we do not use the special structure of $\mathcal{N}^{(2)}(v)$ detailed in (4.14), but rather observe that $\mathcal{N}^{(2)}(v)$ generates factors that can be expressed as a sum of terms of the form (4.5a), with the same bounds on indices as in (4.5b) except for the bound $a^{(1)}_y \geq 1$, which now cannot be ensured. We may use the usual argument concerning nested cut-offs as in the paragraph preceding (4.49), but for $(1 - \chi)$ rather than for $\chi$. Here, we have $D^j(1 - \chi) = D^j(\tilde{\vartheta}(1 - \chi)) = \tilde{\vartheta} D^j(1 - \chi)$ whenever $\tilde{\vartheta}$ is a smooth function having support in $[\frac{1}{2}, \infty)$ and fulfilling $\tilde{\vartheta}(x) = 1$ on $[1, \infty)$. Furthermore, we may introduce any number of factors $\tilde{\vartheta}$ into our terms replacing $(1 - \chi)$, without generating any change in the bounds.

Justified by the structure (4.5a) which holds also for $\mathcal{N}^{(2)}(v)$, and by subdividing the factors $x^{1-m}$ among the terms, we may separately consider the terms of the form

$$\int_I \left\| \sum_{j=1}^{m} D^j(\tilde{\vartheta} w(j)) \right\|^2 dt, \quad \text{where} \quad \begin{cases} w^{(1)}(j) \in \{D_x v, D_y v\} & \text{for } j = 1, \\ w^{(j)}(j) \in \{v_x, v_y\} & \text{for } j \geq 2. \end{cases}$$

(4.88)

Due to the definition of $\mathcal{N}^{(2)}(v)$ (cf. (4.7a)), the multi-indices $b^{(j)}$ appearing in (4.88) satisfy the bounds

$$0 \leq \sum_{j=1}^{m} |b^{(j)}| \leq \kappa + 3,$$

(4.89a)

valid for the choices $(\kappa, \rho) \in \left\{\left(\tilde{k} + 3, \delta + \frac{1}{2}\right), \left(\tilde{k} + 3, -\delta + \frac{3}{2}\right), \left(\tilde{k} + 5, \delta + \frac{1}{2}\right)\right\}$. We further have for $\kappa_y$ as in (4.25d),

$$1 \leq |b^{(j)}| \leq \kappa + 3,$$

$$0 \leq |b^{(j)}| \leq \left\lceil \frac{\kappa + 1}{2} \right\rceil + 1 \quad \text{for } j \in \{2, \ldots, m\},$$

$$0 \leq b^{(j)}_y \leq \kappa_y \quad \text{for } j \in \{1, \ldots, m\}.$$  

(4.89b)

For $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$ we estimate the terms (4.88) as follows:

$$\int_I \left\| \tilde{\vartheta} \times \sum_{j=1}^{m} D^j(\tilde{\vartheta} w(j)) \right\|^2 dt \lesssim \int_I \left\| D^j(\tilde{\vartheta} w(j)) \right\|^2 dt \times \prod_{j=2}^{m} \left\| D^j(\tilde{\vartheta} w(j)) \right\|^2_{BC^0(I \times (0, \infty) \times \mathbb{R})}.$$  

(4.90)

We then treat the terms above arguing as in the previous steps, obtaining the bounds

$$\int_I \left\| D^j(\tilde{\vartheta} w(j)) \right\|^2 dt \lesssim \left\| \tilde{\vartheta} v \right\|^2_{sol} \lesssim \left\| v \right\|^2_{sol},$$

$$\left\| D^j(\tilde{\vartheta} w(j)) \right\|^2_{BC^0(I \times (0, \infty) \times \mathbb{R})} \lesssim \left\| \tilde{\vartheta} v \right\|^2_{sol} \lesssim \left\| v \right\|^2_{sol},$$

(4.91a)

(4.91b)

in the cases $j = 1$ and $j \geq 2$, respectively. Here, we have employed the uniform bounds on the first $|b^{(j)}|$ derivatives of $\tilde{\vartheta}$ as well as the bounds (4.89) on $b^{(j)}$ together with estimate (3.59b) of Lemma 3.9, in order to distribute the derivatives and to include the $\tilde{\vartheta}$-contributions in the implicit constant which depends only on $\tilde{k}$, $\tilde{k}$, and $\delta$.

Combining (4.90) and (4.91), we find

$$\left\| (1 - \chi)\mathcal{N}^{(2)}(v) \right\|_{\text{rhs}}^2 \lesssim \left\| v \right\|^2_{sol}.$$

(4.92)

The bounds (4.44), (4.45), (4.86), and (4.92) conclude our proof of (4.43). Furthermore, (4.24) together with (4.43) give the bound

$$\left\| \mathcal{N}(v) \right\|_{\text{rhs}} \lesssim \left\| v \right\|^2_{sol} \quad \text{for } \left\| v \right\|_{sol} < 1.$$  

(4.93)
Conclusion of the proof. In order to pass from (4.93) to the general bound (4.16), we consider the multilinear terms into which we have decomposed the nonlinearity in the above proof. We note that for an $\ell$-linear form $M(w^{(1)}, \ldots, w^{(\ell)})$ we have the decomposition

$$M(w^{(1,1)} - w^{(1,2)}, \ldots, w^{(j,1)} - w^{(j,2)}, w^{(j+1,2)}, \ldots, w^{(\ell,2)}) = \sum_{j=1}^{\ell} M(w^{(1,j)}, \ldots, w^{(j-1,1)}, w^{(j,1)} - w^{(j,2)}, w^{(j+1,2)}, \ldots, w^{(\ell,2)}).$$

(4.94)

This decomposition can be applied

(i) to the terms (4.5a) that contribute to the expression of $\|N^{(1)}(v)\|_{\text{sol}}$ as well as to the one of the terms form the expression of $\|N^{(2)}(v)\|_{\text{sol}}$ with weights $\rho \in \{-\delta + \frac{1}{2}, -\delta + \frac{3}{2}\}$,

(ii) to the terms (4.49), (4.55), (4.61), (4.66) which give the contributions $g^{(1)}, \ldots, g^{(4)}$ to the contributions of $\chi N^{(2)}(v)$ with weights $\rho \in \{\delta + \frac{1}{2}, \pm \delta + \frac{3}{2}\}$,

(iii) to the terms analogous to (4.5a) which appear in the decomposition of $(1 - \chi)N^{(2)}(v)$.

For $v = v^{(n)}$ with $n \in \{1, 2\}$, all of the above terms have the following form:

$$M(w^{(n,1)}, \ldots, w^{(n,\ell)}) \in \left\{ \prod_{j=1}^{\ell} D^{(j)} v^{(n,j)} \right\},$$

(4.95a)

with

$$w^{(n,j)} \in \left\{ \frac{(1 + v^{(n)})^{1/2}}{\sqrt{v^{(n,j)}}}, \frac{D^{(j)} v^{(n,j)}}{\sqrt{v^{(n,j)}}}, \frac{D^{(j)}(v^{(n)} - v^{(n,j)})}{\sqrt{v^{(n,j)}}} \right\}.$$  

(4.95b)

We can apply formula (4.94) to expressions (4.95), and thereby replace each estimate on $v$ from the previous proof by an estimate on a product of terms of the same form but with $v$ replaced by either one of $v^{(1)}, v^{(2)}$ or $v^{(1)} - v^{(2)}$, with the extra constraint that precisely one of the terms is replaced by the difference $v^{(1)} - v^{(2)}$. This increases the combinatorial coefficient $c^{(1)}(\ell)$ appearing in (4.85) by a factor of $\ell$. Thus for the ensuing estimate in (4.87) $\ell^2$ has to be replaced by $\ell^3$. However, the subsequent discussion shows that any power of $\ell$ is controllable. Applying at each instance the same estimates as those leading to (4.93) and noticing that each one of the products has at least two terms (and therefore there will also be at least one instance of either $v^{(1)}$ or $v^{(2)}$ in each of the factors), we conclude that the thesis holds.

Finally, we simplify conditions (4.33), (4.39), (4.41), (4.52), (4.70), and the conditions $\tilde{k} \geq 3, \hat{k} \geq 4$, and $\hat{k} \geq 4$ required for Lemmata 3.4 and 3.5. First, note that (4.52) translates due to the definition of $\kappa_y$ in (4.25d) into $\max(\hat{k}, \tilde{k}) - 2 = \min(\hat{k}, \tilde{k}) - 2$, so that we find

$$\tilde{k} = \hat{k}.$$  

(4.96)

Next, note that (4.39) is stronger than (4.33) and now gives $\left\lceil \frac{\ell}{2} \right\rceil + 5 \leq \hat{k} - 4$, which, given (4.96), is stronger than (4.70) and equivalent to

$$\hat{k} \geq 18.$$  

Finally, (4.41) gives

$$\tilde{k} \leq \hat{k} - 2.$$  

From the above we obtain the global conditions (2.4) of Assumptions 2.1 on our parameters as appearing in the statement of Proposition 4.1.

\[ \square \]

4.3. Well-posedness, a-priori estimates, and stability for the nonlinear equation.

Proof of Theorem 2.2. Throughout the proof, estimates depend only on $k, \tilde{k}, \hat{k}$, and $\alpha$.

Existence. We fix $\varepsilon > 0$ to be determined later, and consider a locally integrable $v^{(0)} : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with $\|v^{(0)}\|_{\text{init}} < \varepsilon$, like in the assumption of the theorem. Then we will work in the spaces

$$S_{\varepsilon', \varepsilon} := \left\{ v : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R} \text{ locally integrable: } \|v\|_{\text{sol}} < \varepsilon', v|_{t=0} = v^{(0)} \right\},$$

(4.97a)

$$R_{\varepsilon'} := \left\{ f : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R} \text{ locally integrable: } \|f\|_{\text{rhs}} < \varepsilon' \right\},$$

(4.97b)
where \( \varepsilon' > 0 \) will be fixed later depending only on \( k, \tilde{k}, \hat{k}, \) and \( \alpha \). Note that due to the last point of Corollary 3.15, the linear trace operator \( \nu \mapsto \nu^{(0)} \) is continuous and bounded with respect to the norms \( \| \cdot \|_{\text{sol}} \) and \( \| \cdot \|_{\text{init}} \). Thus for \( \varepsilon \) small enough the space \( S_{\varepsilon', \nu^{(0)}} \) is nonempty. We then consider the maps

\[
\mathcal{N} : S_{\varepsilon', \nu^{(0)}} \to R_{\varepsilon'}, \quad T_{\nu^{(0)}} : R_{\varepsilon'} \to S_{\varepsilon', \nu^{(0)}},
\]

where for every \( \nu \in S_{\varepsilon', \nu^{(0)}} \) the nonlinear operator \( \mathcal{N} \), explicitly defined in (1.17), associates the nonlinear right-hand side of our equation (1.14a), and if \( t \in R_{\varepsilon'} \) the linear operator \( T_{\nu^{(0)}} \) associates the solution to the linearized equation (1.19a), as given by Proposition 3.20.

By Proposition 4.1 for the case \( \nu^{(1)} = \nu, \nu^{(2)} = 0 \), we find that if \( \varepsilon' \) is chosen sufficiently small, we have the estimate

\[
\| \mathcal{N}(\nu) \|_{\text{rhs}} \lesssim \| \nu \|^2_{\text{sol}}, \quad (4.98)
\]

which shows that \( \mathcal{N} \) maps \( S_{\varepsilon', \nu^{(0)}} \) to \( R_{\varepsilon'} \) if \( 0 < \varepsilon' < 1 \). Similarly, by the estimate from Proposition 3.20, we find that for fixed \( \nu^{(0)} \), such that \( \| \nu^{(0)} \|_{\text{init}} < \infty \), if \( \varepsilon' < 1 \) then for any \( t \in R_{\varepsilon'} \), we have the bound

\[
\| T_{\nu^{(0)}}(f) \|_{\text{sol}} \lesssim \| \nu^{(0)} \|_{\text{init}} + \| f \|_{\text{rhs}}, \quad (4.99)
\]

and thus \( T_{\nu^{(0)}} \) also maps \( R_{\varepsilon', \nu^{(0)}} \) to \( S_{\varepsilon'} \) if \( \varepsilon' < 1 \) and if \( \| \nu^{(0)} \|_{\text{init}} \lesssim \varepsilon' \). Under this last condition, we may thus define the self-map

\[
T_{\nu^{(0)}} := T_{\nu^{(0)}} \circ \mathcal{N} : S_{\varepsilon', \nu^{(0)}} \to S_{\varepsilon', \nu^{(0)}}, \quad (4.100)
\]

and in order to prove the existence part of our theorem it will suffice to prove that for \( \varepsilon' > 0 \) and \( \varepsilon > 0 \) small enough, the operator \( T_{\nu^{(0)}} \) has a fixed point in \( S_{\varepsilon', \nu^{(0)}} \) for each \( \nu^{(0)} \) such that \( \| \nu^{(0)} \|_{\text{init}} < \varepsilon \). To this aim we note that, again by Propositions 3.20 and 4.1, we may write, for any given \( \nu^{(1)}, \nu^{(2)} \in S_{\varepsilon', \nu^{(0)}}, \)

\[
\left\| T_{\nu^{(0)}}(\nu^{(1)}) - T_{\nu^{(0)}}(\nu^{(2)}) \right\|_{\text{sol}} \lesssim \left( \left\| \nu^{(1)} \right\|_{\text{sol}} + \left\| \nu^{(2)} \right\|_{\text{sol}} \right) \left\| \nu^{(1)} - \nu^{(2)} \right\|_{\text{sol}} \lesssim \left\| \mathcal{N}(\nu^{(1)}) - \mathcal{N}(\nu^{(2)}) \right\|_{\text{rhs}}.
\]

Thus for \( \varepsilon' > 0 \) small enough we find that \( T_{\nu^{(0)}} \) is indeed a contraction, and therefore it has a fixed point. The property that

\[
T_{\nu^{(0)}}(\nu) = \nu.
\]

directly establishes the existence part of the theorem, in view of the definition (4.100).

A-priori bounds. We now use the above definitions and estimates to find

\[
\| \nu \|_{\text{sol}} \lesssim \| \mathcal{T}_{\nu^{(0)}}(\nu) \|_{\text{sol}} \lesssim \| \mathcal{N}(\nu) \|_{\text{sol}} \lesssim C_1 \left( \| \nu^{(0)} \|_{\text{init}} + \| \mathcal{N}(\nu) \|^2_{\text{sol}} \right) \lesssim C_2 \left( \| \nu^{(0)} \|_{\text{init}} + \| \nu \|^2_{\text{sol}} \right),
\]

with constants \( C_j > 0 \) only depending on \( k, \tilde{k}, \hat{k}, \) and \( \alpha \). Fixing \( \varepsilon' < \frac{1}{C_2} \), we can absorb the last term above to the left obtaining

\[
\| \nu \|_{\text{sol}} \lesssim \frac{1}{1 - C_2 \varepsilon' \| \nu^{(0)} \|_{\text{init}}},
\]

which gives the a-priori bound (2.5).

Uniqueness. Assume that for a given \( \nu^{(0)} \) as in the statement of the theorem, there exist two finite-norm solutions \( \nu^{(1)}, \nu^{(2)} \) to our equation. Then due to the continuity in time as proved in Corollary 3.15 we find that there exists a maximal time \( t^* \) such that \( \nu^{(1)}(t, x, y) = \nu^{(2)}(t, x, y) \) on \( (0, t^*) \times (0, \infty) \times \mathbb{R} \) and \( \nu^{(1)}(t^* + t, \cdot, \cdot) \neq \nu^{(2)}(t^* + t, \cdot, \cdot) \) for all \( t > 0 \) small enough. Then using the traces \( \nu^{(1)}|_{t=t^*} = \nu^{(2)}|_{t=t^*} \) as initial data for our equation on \( I_{\tau} := (t^*, t^* + \tau) \) and using the norms \( \| \cdot \|_{\text{sol}, \tau} \) defined like in (3.53), but where the interval in (3.53) is chosen to be \( I := I_{\tau} \), we can write

\[
\left\| \nu^{(1)} - \nu^{(2)} \right\|_{\text{sol}, \tau} \approx \left\| T_{\nu^{(0)}}(\nu^{(1)}) - T_{\nu^{(0)}}(\nu^{(2)}) \right\|_{\text{sol}, \tau} \lesssim \left( \left\| \nu^{(1)} \right\|_{\text{sol}, \tau} + \left\| \nu^{(2)} \right\|_{\text{sol}, \tau} \right) \left\| \nu^{(1)} - \nu^{(2)} \right\|_{\text{sol}, \tau},
\]

However, due to the last item of Corollary 3.15 we have for \( j = 1, 2 \) that

\[
\left\| \nu^{(j)} \right\|_{\text{sol}, \tau} \rightarrow \| \nu^{(j)} \|_{\text{init}} = \| \nu^{(0)} \|_{\text{init}} < \varepsilon,
\]

which for \( \varepsilon \ll 1 \) contradicts (4.102), concluding the proof of uniqueness.
Stability. We note that as a consequence of the bound \( \|v\|_{\text{sol}} < \infty \) for the solution, by using the first item of Corollary 3.15 we find \( n_0(t, y) \to 0 \) as \( t \to \infty \). Due to the continuity of \( t \mapsto \|v(t, \cdot, \cdot)\|_{\text{ini}} \) as stated in the last item of Corollary 3.15, the stability property \( \lim_{t \to +\infty} \|v(t, \cdot, \cdot)\|_{\text{ini}} = 0 \) directly follows.

This concludes the proof of Theorem 2.2.

\[ \square \]

Appendix A. Coordinate transformations

A.1. Transformations onto a fixed domain. In this appendix, we transform problem (1.1) via the change of variables (1.9). First note that for a function \( f = f(t, y, z) \), the gradient of the transformed function \( \tilde{f}(t, x, y) := f(t, y, Z(t, x, y)) \) is given by:

\[
\begin{pmatrix}
\tilde{f}_t \\
\tilde{f}_x \\
\tilde{f}_y
\end{pmatrix} = \begin{pmatrix}
f_t + f_x Z_t \\
f_x Z_x \\
f_y + f_x Z_y
\end{pmatrix} = \begin{pmatrix} 1 & 0 & Z_t \\
0 & 0 & Z_x \\
0 & 1 & Z_y
\end{pmatrix} \cdot \begin{pmatrix}
f_t \\
f_x \\
f_y
\end{pmatrix}. \tag{A.1}
\]

By inverting the matrix, we can read off the transformation of derivatives from those of \( f \) to those of \( \tilde{f} \) as

\[
\begin{align*}
\partial_t & \mapsto \partial_t - Z_t F \partial_x, \quad \text{where } F := Z_x^{-1}, \\
\partial_y & \mapsto \partial_y - G \partial_z, \quad \text{where } G := Z_x^{-1} Z_y, \\
\partial_z & \mapsto F \partial_x.
\end{align*} \tag{A.2a, A.2b, A.2c}
\]

Combining (1.9) and (A.2) we find for (1.1a)

\[
(\partial_t - F Z_t \partial_x) x^\frac{3}{2} + (\partial_y - G \partial_z) x^3 (\partial_y - G \partial_z) + F \partial_z x^3 F \partial_x \left( (\partial_y - G \partial_z)^2 + (F \partial_x)^2 \right) x^\frac{3}{2} = 0 \tag{A.3}
\]

for \( t, x > 0 \) and \( y \in \mathbb{R} \). Now we may use that

\[
(\partial_y - G \partial_z)^2 + (F \partial_x)^2 x^\frac{3}{2} = 3 \frac{3}{2x^2} (-D_y G + G (D_x + \frac{1}{2}) G + F (D_x + \frac{1}{2}) F),
\]

where we have introduced the operators

\[
D_x := x \partial_x = \partial_s \quad \text{with} \quad s := \ln x \quad \text{and} \quad D_y := x \partial_y. \tag{A.4}
\]

Note that \( D_x \) and \( D_y \) do not commute. For later purpose, we also define

\[
D := (D_x, D_y) \quad \text{(A.4)} \quad \text{and} \quad D^\ell := D_y^\ell D_x^\ell, \quad \text{with} \quad \ell := (\ell_x, \ell_y) \in \mathbb{N}_0^2, \tag{A.5}
\]

where the order of operators is crucial. Next, we observe that

\[
(\partial_y - G \partial_z) x^3 (\partial_y - G \partial_z) + F \partial_z x^3 F \partial_x \left( (\partial_y - G \partial_z)^2 + (F \partial_x)^2 \right) x^\frac{3}{2} = x^\frac{3}{2} \left( D_y^2 - D_y G (D_x - \frac{1}{2}) - GD_y (D_x + \frac{3}{2}) + G (D_x + \frac{1}{2}) G (D_x - \frac{1}{2}) + F (D_x + \frac{3}{2}) F (D_x - \frac{1}{2}) \right)
\]

and as a result equation (A.3) turns into (1.11).

A.2. Advection velocity and expansions near the contact line. We recall that, with the notations introduced in (A.2), and given the definition (1.1c) of the velocity \( V \) under which the height \( h^{-\frac{1}{6}} x^\frac{3}{2} \) of our fluid is advected via (1.1a), we obtain

\[
V := h \nabla h = h \left( \partial_y \right) \left( \partial_y^2 + \partial_z^2 \right) h = x^\frac{3}{2} \left( \partial_y - G \partial_x \right) \left( (\partial_y - G \partial_x)^2 + (F \partial_x)^2 \right) x^\frac{3}{2}
\]

\[
= 3 \frac{3}{2x^2} \left( D_y - GD_x \right) \left( -D_y G + G (D_x + \frac{1}{2}) G + F (D_x + \frac{1}{2}) F \right)
\]

\[
= 3 \frac{3}{2} \left( D_y - G (D_x - \frac{3}{2}) \right) \left( -D_y G + G (D_x + \frac{3}{2}) G + F (D_x + \frac{1}{2}) F \right). \tag{A.6}
\]

We note that the traveling-wave profile \( Z_{TW} = x - \frac{3}{2} t \) meets \( (Z_{TW})_z = 1 \) and \( (Z_{TW})_y = 0 \). Using the power series (2.6) valid as \( x \searrow 0 \), we expand the expression of \( V \) as \( x \searrow 0 \) into powers of \( x \) (cf. (2.16)), i.e., almost everywhere

\[
V(t, y, Z(t, x, y)) = V_0(t, y) + V_3(t, y) x^3 + V_1(t, y) x + o(x^{1+\delta}) \quad \text{as} \quad x \searrow 0. \tag{A.7}
\]
Now we separate the terms based on formulas (A.2) and (1.13), i.e., \( F = (1 + v_x)^{-1} \) and \( G = v_y(1 + v_z)^{-1} \). With the notation as in (1.22) or (2.6), we obtain almost everywhere

\[
D^t F = D^t \left( \frac{1}{1 + v_1} - \frac{(1 + \beta)v_1 + \beta}{(1 + v_1)^2} x^\beta - \frac{2v_2}{(1 + v_1)^2} x \right) + o(x^{1+\delta}),
\]

\[
D^t G = D^t \left( \frac{(v_0)_y}{1 + v_1} - \frac{(1 + \beta)(v_0)_y v_1 + \beta}{(1 + v_1)^2} x^\beta + \frac{(v_1)_y}{1 + v_1} - \frac{2(v_0)_y v_2}{(1 + v_1)^2} x \right) + o(x^{1+\delta}),
\]

where we need to allow for \(|\ell| \leq 2\). With \( P \) defined as in (A.6) and \( D^t P = D^t (P_0 + P_\beta x^\beta + P_1 x) + o(x^{1+\delta}) \) for \(|\ell| \leq 1\), a straightforward computation gives

\[
P_0 = \frac{1}{2} \frac{(v_0)_y}{1 + v_1} + 1 \frac{(v_0)_y}{1 + v_1} + 1,
\]

\[
P_\beta = -\frac{(1 + \beta)^2 v_1 + \beta}{(1 + v_1)^3},
\]

\[
P_1 = -\frac{(v_0)_y}{1 + v_1} + 3 \frac{(v_0)_y (v_1)_y}{1 + v_1} + 4 \left( \frac{(v_0)^2}{1 + v_1} + 1 \right) v_2.
\]

We then find by a direct computation that (A.7) holds with \( V_\beta = 0 \), thus confirming (2.16). More precisely, we find the following expressions for \( V_0 \), \( V_\beta \), and \( V_1 \):

\[
V_0 = \frac{3}{8} \frac{1 + (v_0)_y}{(1 + v_1)^3} \left( \frac{(v_0)_y}{(1 + v_1)^3} - 1 \right), \quad V_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad V_1 = \frac{3}{8} \left( \frac{6(v_0)_y (v_1)_y}{(1 + v_1)^3} - \frac{6(v_0)_y (v_1)_y}{(1 + v_1)^3} + \frac{6(v_0)_y^2 + 2(v_0)_y (v_1)_y}{(1 + v_1)^3} \right).
\]

Due to the advection equation (1.1c), if \( \partial [h > 0] \) is at time \( t = 0 \) the graph

\[
\Gamma_0 := \left\{ (y, Z_0^{(0)}(y)) : y \in \mathbb{R} \right\},
\]

then the advected contact line at time \( t \), which we denote \( \Gamma_t := \partial [h > 0] \), can be parameterized by

\[
\Gamma_t = \{(Y(t, y), Z(t, y)) : y \in \mathbb{R} \},
\]

and the above parameterization evolves under the advection equation (2.15).

We claim that we may express \( \Gamma_t \) as the graph of a regular function, i.e., that there exists a regular function \( Z_0 : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) which satisfies \( Z_0(0, y) = Z_0^{(0)}(y) \) and

\[
\Gamma_t = \{(y, Z_0(t, y)) : y \in \mathbb{R} \} \text{ for all } t \geq 0.
\]

To obtain (A.12), note that under the conditions that

\[
\sup_{t \in I} |(v_0)_y|_{BC^0(\mathbb{R}_y)} < \infty,
\]

the velocity \( V_0 \) is never perpendicular to the \( z \)-axis during our time-evolution. As a consequence of the advection equation \( \partial_t h + \nabla_{y,z} \cdot (hV) = 0 \) we also find that \( V_0 \) remains always orthogonal to \( \partial [h > 0] \), and thus this set remains a graph at all times. As a result of the regularity of \( V_0 \) we also find that \( \partial [h > 0] \) remains regular at all times. We may ensure arbitrarily high regularity depending on the parameters \( \tilde{k}, \bar{k}, \text{ and } \bar{k} \) in our norms according to the bounds (2.7), proved in Lemma 3.9, though the question of smoothing of the free boundary is not investigated here. The above sufficient condition (A.13) is obtained from \( \|v_0^{(0)}\|_{W^{1,1}} < \varepsilon \) which we assumed in Theorem 2.2 due to our main estimate (2.5), coupled with the bound (3.59a) of Lemma 3.9.

In order to invert the hodograph transform and to pass from the understanding of the function \( v \) to that of \( h \), we use the formulas

\[
\tilde{v}(t, x, y) := Z(t, x, y) - Z_0(t, y) = \int_0^y Z_x(t, x', y) dx', \quad Z_x = (Z tw)_x + v_x = 1 + v_x.
\]

From the expansion in powers of \( x \) in (2.6) we find almost everywhere

\[
\tilde{v} = (1 + v_1)x + v_1 x^{1+\beta} + v_2 x^2 + o(x^{2+\delta}) \quad \text{as } x \searrow 0,
\]
from which we infer that $\tilde{z}(t,x,y)$ is monotone in $x$ as $x \searrow 0$ and satisfies $\tilde{z}(t,x,y) \to 0$ as $x \searrow 0$. Furthermore, we can invert (A.15) for $x$ in a neighborhood of 0 and find almost everywhere
\[
x = \frac{1}{1 + v_1} \tilde{z} - \frac{v_1 + \beta}{(1 + v_1)^{2 + \beta}} \tilde{z}^{1 + \beta} - \frac{v_2}{(1 + v_1)^2} \tilde{z}^2 + o(\tilde{z}^{1 + \delta})
\]
(A.16a)
\[
= \frac{\tilde{z}}{1 + v_1} \left( 1 - \frac{v_1 + \beta}{(1 + v_1)^{1 + \beta}} \tilde{z}^{1 + \beta} - \frac{v_2}{(1 + v_1)^2} \tilde{z}^2 + o(\tilde{z}^{1 + \delta}) \right)
\]
(A.16b)
as $\tilde{z} \searrow 0$ and using this in the hodograph transform (1.9), we find via the Taylor expansion of $x_{\tilde{z}}$ near $x = 1$ that almost everywhere
\[
h(t,y,Z(t,x,y)) = x_{\tilde{z}} = \frac{\tilde{z}^{2}}{(1 + v_1)^2} \left( 1 - \frac{3}{2} \frac{v_1 + \beta}{(1 + v_1)^{1 + \beta}} \tilde{z}^{1 + \beta} - \frac{3}{2} \frac{v_2}{(1 + v_1)^2} \tilde{z}^2 + o(\tilde{z}^{1 + \delta}) \right)
\]
(A.17)
as $\tilde{z} \searrow 0$ holds true. Now (A.12), (A.14), and (A.17) allow to re-write the equations expressing the evolution of the moving interface $Z_0(t,y)$ starting at $Z_0^{(0)}(y)$ in the original coordinates, leading to (2.14). We thus observe that up to factoring the traveling-wave profile $\tilde{z}_{\frac{2}{2}}$ the function $h$ has the same form as $v$ itself, and the supremum norm bounds on $v_1$, $v_1 + \beta$, and $v_2$ give bounds on the coefficients of $h$ as well.

**APPENDIX B. PROOFS OF AUXILIARY RESULTS**

**B.1. Proofs of the norm-equivalence lemmata.**

*Proof of Lemma 3.4.* Throughout the proof, estimates depend on $\tilde{k}$, $\tilde{k}$, $\tilde{k}$, and $\delta$. Then the condition $\|v(0)\|_{\text{init}} < \infty$ allows to verify the conditions of Lemma 3.8 and to use estimate (3.58) of that lemma with $\rho = -\delta, \gamma = 0$ or $\rho = \delta$ and $\gamma$ equal to a root of $\tilde{q}(\zeta)$ in
\[
\left\|D_y v(0)\right\|_{\tilde{k} - 1, -\delta} \approx \left\|\partial_y v(0)\right\|_{\tilde{k} - 1, -\delta - 1} \quad \left(3.58\right) \lesssim \left\| \partial_y D_x v(0) \right\|_{\tilde{k} - 1, -\delta - 1} \lesssim \left\|D_y D_x v(0)\right\|_{\tilde{k} - 1, -\delta}
\]

A.1
\[
\left\|D_y D_x v(0)\right\|_{\tilde{k} - 1, \delta} \approx \left\|\partial_y D_x v(0)\right\|_{\tilde{k} - 1, \delta - 1} \quad \left(3.58\right) \lesssim \left\| \partial_y \tilde{q}(D_x) D_x v(0) \right\|_{\tilde{k} - 1, \delta - 1} \lesssim \left\| \tilde{q}(D_x) D_x v(0) \right\|_{\tilde{k}, \delta},
\]
where we also use the fact that $\partial_y$ commutes with $D_x$. A similar reasoning also allows to prove the bounds
\[
\left\|D_y D_x v(0)\right\|_{\tilde{k} - 2, \delta + 1} \lesssim \left\| D_y D_x v(0) \right\|_{\tilde{k} - 1, -\delta + 1} \lesssim \tilde{q}(D_x) D_x v(0) \right\|_{\tilde{k} - \delta + 1};
\]
\[
\left\|D_y D_x v(0)\right\|_{\tilde{k} - 2, \delta + 1} \lesssim \left\| D_y \tilde{q}(D_x) D_x v(0) \right\|_{\tilde{k} - 1, \delta + 1} \lesssim \left\| (D_x - 3) (D_x - 2) \tilde{q}(D_x) D_x v(0) \right\|_{\tilde{k}, \delta + 1}.
\]

The above inequalities and the definition (3.54) directly allow to conclude. \□

*Proof of Lemma 3.5.* Throughout the proof, estimates depend on $\tilde{k}$, $\tilde{k}$, $\tilde{k}$, and $\delta$. The proof follows the same lines as the one of Lemma 3.4. By using the hypothesis $\|f\|_{\text{hyp}} < \infty$, we find that almost everywhere in time $t \in I$ the norms of the form $\|v\|_{\tilde{k}, \alpha}$ appearing in the time intervals defining $\|\cdot\|_{\text{hyp}}$ are all finite. This allows to verify, at such fixed $t$, the hypotheses of Lemma 3.8 and to prove like in Lemma 3.4
\[
\left\| D_y f \right\|_{\tilde{k} - 3, -\delta + \frac{1}{2}} \lesssim \left\| (D_x - 1) f \right\|_{\tilde{k} - 2, -\delta + \frac{1}{2}};
\]
\[
\left\| D_y (D_x - 1) f \right\|_{\tilde{k} - 3, -\delta + \frac{1}{2}} \lesssim \left\| \tilde{q}(D_x - 1) (D_x - 1) f \right\|_{\tilde{k} - 2, -\delta + \frac{1}{2}};
\]
\[
\left\| D_y^2 f \right\|_{\tilde{k} - 4, -\delta + \frac{3}{2}} \lesssim \left\| D_y (D_x - 1) f \right\|_{\tilde{k} - 3, -\delta + \frac{3}{2}} \lesssim \left\| \tilde{q}(D_x - 1) (D_x - 1) f \right\|_{\tilde{k} - 2, -\delta + \frac{3}{2}};
\]
\[
\left\| D_y^2 (D_x - 1) f \right\|_{\tilde{k} - 4, -\delta + \frac{3}{2}} \lesssim \left\| D_y \tilde{q}(D_x - 1) (D_x - 1) f \right\|_{\tilde{k} - 3, -\delta + \frac{3}{2}} \lesssim \left\| (D_x - 3) (D_x - 2) \tilde{q}(D_x - 1) (D_x - 1) f \right\|_{\tilde{k} - 2, -\delta + \frac{3}{2}}.
\]

By comparing the above estimates to the definition (3.55), the thesis follows. \□
Proof of Lemma 3.6. Throughout the proof, estimates depend on \( \tilde{k} \), \( \tilde{k} \), \( \tilde{k} \), and \( \delta \). First, note that the norm \( \|u\|_{\text{sol}} \) can be reexpressed as a sum of terms of the form
\[
\sup_{t \in I} \|w\|_{L^2}^2 + \int_I \|w_t \|_{L^2}^2 \, dt + \int_I \|w_{t^2} \|_{L^2}^2 \, dt.
\]
Thus, by applying (3.57) of Lemma 3.3 we can absorb the supremum terms from the formula (3.53) into the integral terms. By using Lemma 3.8 like in the proof of Lemma 3.5, we can prove the following results, appearing on the right-hand side are finite:
\[
\begin{align*}
\|D_k w\|_{k-3+\delta, \frac{1}{2} + \alpha} & \lesssim \|D_k D_j w\|_{k-2+\delta, \frac{1}{2} + \alpha}, \\
\|D_{\bar{k}} D_j w\|_{k-3+\delta, \frac{1}{2} + \alpha} & \lesssim \|\tilde{q}(D_{\bar{j}})D_k w\|_{k-2+\delta, \frac{1}{2} + \alpha}, \\
\|D_{\bar{k}^2} w\|_{k-4+\delta, \frac{1}{2} + \alpha} & \lesssim \|D_{\bar{k}} D_{\bar{j}} w\|_{k-3+\delta, \frac{1}{2} + \alpha} \lesssim \|\tilde{q}(D_{\bar{j}})D_k w\|_{k-2+\delta, \frac{1}{2} + \alpha}, \\
\|D_{\bar{k}^2} D_{\bar{j}} w\|_{k-4+\delta, \frac{1}{2} + \alpha} & \lesssim \|D_{\bar{k}} q(D_{\bar{j}})D_k w\|_{k-3+\delta, \frac{1}{2} + \alpha} \lesssim \|(D-3)(D-2)\tilde{q}(D_{\bar{j}})D_k w\|_{k-2+\delta, \frac{1}{2} + \alpha}.
\end{align*}
\]
In particular due to the finiteness hypothesis \( \|u\|_{\text{sol}} < \infty \), all the above bounds hold almost everywhere in time \( t \in I \) for the choices
\[
(w, \kappa, \alpha) \in \{(\tilde{c}_L v, 0, 0), (v, 4, 1)\},
\]
where for the case \( v = \tilde{c}_L v \) we also use the fact that \( \tilde{c}_L \) derivatives commute with \( D_x \) and \( D_y \) operators. By inspecting the norm (3.53), we see that this allows to absorb all the integral terms into the ones appearing in the norm \( \|u\|_{\text{sol}} \), as desired. \( \square \)

B.2. Proof of the embedding lemma. We start with two auxiliary lemmata.

Lemma B.1. For \( u \in C^{2}((0, \infty) \times \mathbb{R}) \) and \( \tilde{w} \in C_{c}^{2}(\mathbb{R}) \) and \( \alpha_1 < \gamma \frac{1}{2} < \alpha_2 \). Then there holds
\[
|\tilde{c}_L \tilde{w}|_{BC^{0}(\mathbb{R})} \lesssim \|w\|_{H^{1, \alpha_1}} + \|w + x^\gamma \tilde{w}\|_{H^{1, \alpha_2}}.
\]

Proof. By using the Sobolev embedding in \( \mathbb{R} \), the inequality \( |x^\gamma \tilde{w}| \lesssim 2 \left( |w + x^\gamma \tilde{w}|^2 + |w|^2 \right) \), and the fact that any power of \( x \) is bounded on \([1, 2]\), we find
\[
|\tilde{c}_L \tilde{w}|^2_{BC^{0}(\mathbb{R})} \lesssim \sum_{0 \leq \ell \leq 1} \sum_{0 \leq \ell \leq 1} \int_{\mathbb{R}} |\tilde{c}_L \tilde{w}|^2 \, dx \lesssim \sum_{0 \leq \ell \leq 1} \int_{\mathbb{R}} |\tilde{c}_L \tilde{w}|^2 \, dx \lesssim \|w\|_{H^{1, \alpha_1}} + \|w + x^\gamma \tilde{w}\|_{H^{1, \alpha_2}}.
\]

Lemma B.2. For \( u \in C_{c}^{0}((0, \infty) \times \mathbb{R}) \) and \( \delta > 0 \) there holds
\[
\|u\|^2_{BC^{0}((0, \infty) \times \mathbb{R})} + \|w - w_0\|^2_{BC^{0}((0, \infty) \times \mathbb{R})} + |w_0|^2_{BC^{0}(\mathbb{R})} \lesssim \|D_x w\|^2_{1, \delta} + \|D_x w\|^2_{1, \delta}.
\]

Proof. Reasoning in a similar way as in (Giacomelli et al., 2014, Estimate (8.4)), we use the Sobolev embedding in \( s = \log x \) and Lemma 3.8 to reduce to norms with \( D_x \) only. We choose a smooth cut-off function \( \chi \) such that \( \chi(\log x) = 1 \) for \( x \leq 0 \) and \( \chi(\log x) = 0 \) for \( x \geq 2 \) and obtain
\[
|w_0|^2 + |w|^2_{BC^{0}((0, \infty) \times \mathbb{R})} + |w - w_0|^2_{BC^{0}((0, \infty) \times \mathbb{R})} \lesssim |w|^2_{1, \delta} + |w - w_0|^2_{1, \delta} + |\chi(\log x)|_{1, \delta}.
\]
Here, we first used the triangle inequality, then we used the standard Sobolev embedding on the real line, then for the third estimate we introduced weights which on the support of the functions that appear in
the norms are in both cases $\gtrsim 1$, and the fourth estimate follows from Lemma B.1. Finally in the last estimate we used Lemma 3.8.

By taking the supremum over $y \in \mathbb{R}$ and using standard embeddings, we find

$$\sup_{y \in \mathbb{R}} \left( |D_x w|^2 + |D_y w|^2 \right) \lesssim \int_0^\infty x^{2\delta} \sup_{y \in \mathbb{R}} (D_x w)^2 \frac{dx}{x} + \int_0^\infty x^{-2\delta} \sup_{y \in \mathbb{R}} (D_y w)^2 \frac{dy}{y}$$

$$\lesssim \left( \int_0^\infty x^{2\delta-1} \int (D_x w)^2 dy \right)^{\frac{1}{2}} \left( \int_0^\infty x^{2\delta-1} \int (D_y D_x w)^2 dy \right)^{\frac{1}{2}} \frac{dx}{x}$$

$$+ \frac{1}{2} \int_0^\infty x^{2\delta-1} \int (D_x w)^2 \frac{dx}{x} dy + \frac{1}{2} \int_0^\infty x^{2\delta-1} (D_y D_x w)^2 \frac{dx}{x} dy$$

$$\lesssim \frac{1}{2} \left\| D_x w \right\|^2_{k,\delta} + \frac{1}{2} \left\| D_y D_x w \right\|^2_{j,\delta} + \frac{1}{2} \left\| D_y w \right\|^2_{k,\delta} + \frac{1}{2} \left\| D_x w \right\|^2_{j,\delta}.$$ 

By definition of the norms $\|\|_{k,\delta}$ in (2.2) trivially $\|D_y D_x w\|_{1,\delta} \lesssim \|D_x w\|_{1,\delta}$ and $\|D_y D_x w\|_{\delta} \lesssim \|D_x w\|_{1,\delta}$ concluding the proof. 

**Proof of Lemma 3.9.** Throughout the proof, estimates depend on $\tilde{k}, \tilde{k}, \tilde{q}$, and $\delta$. Up to using a standard approximation and mollification argument by introducing smooth cut-offs in $x, y$, we may assume that all functions belong to $C^\infty([0, \infty) \times \mathbb{R}) \cap C^2((0, \infty) \times \mathbb{R})$.

**Proof of estimates** (3.59a) and (3.59b). Note that (3.59b) follows from (3.59a) by taking a supremum on $t \in I$ and therefore we only prove the latter.

We write here $v_x (0) = -1 D_x v(0)$, after which we use Lemma B.2 for $w = D^\ell_x v \circ \cdot$. Then we find

$$\left\| D^\ell_x v \right\|_{BC^\alpha((0, \infty) \times \mathbb{R})} \leq C \left( \|D_x D_x v\|_{1,\delta} + \|D_y D_x v\|_{1,\delta} \right)$$

$$\lesssim \left\| x^{-1} D_y (D_x + \ell_y - 1) (D_x - 1) D_x v \right\|_{1,\delta} + \left\| x^{-1} D_y (D_x + \ell_y - 1) (D_x - 1) D_x v \right\|_{1,\delta}$$

Now we consider two cases. In case of $\ell_x \geq 1$ or $\ell_y = 0$, we have a term

$$(D_x - 1) D_x v = (D_x - 1) D_x \left( v - v_0 - v_1 \right)$$

appearing in the last two lines of (B.4). Using the triangle inequality, we may bound these terms by

$$(D_x - 1) D_x v \lesssim \sum_{0 \leq |\ell| \leq \ell_x} \left\| D^\ell (D_x - 1) D_x \left( v - v_0 - v_1 \right) \right\|^2_{1,\delta+1}$$

appearing in the last two lines of (B.4). Using the triangle inequality, we may bound these terms by

$$\sum_{0 \leq |\ell| \leq |\ell|} \left\| D^\ell (D_x - 1) D_x \left( v - v_0 - v_1 \right) \right\|^2_{1,\delta+1}$$

for $|\ell| \leq |\ell| \leq k - 2$, and where in the last inequality we have applied Lemma 3.8 together with the fact that $\tilde{q}(D_x)$ figures the factor $D_x - 1$ (cf. (3.21a)). Due to the definition of the initial data norm $\|\|_{init}$ (cf. (2.1)), this allows to bound the first term in (B.4).

If instead $\ell_y \geq 1$ and $\ell_x = 0$, we find in both terms of the last two lines of (B.4) the quantity

$$D_y \left( D_x + \ell_y - 1 \right) D_x v = D_y \left( D_x + \ell_y - 1 \right) D_x \left( v - v_0 \right)$$

and we find that this quantity is $O(x^2)$ as $x \searrow 0$, due to one $x$-factor coming from the $D_y$-operators and the other coming from $v(0) - v_0$. This allows to proceed as above by again applying Lemma 3.8.
Therefore, we find that in either case
\[
\left\| D^\ell_y v_0^{(0)} \right\|_{BC^0((0,x),x)}^2 \lesssim \left\| \partial_x D_x v_0^{(0)} \right\|_{L_{\infty,\ell}}^2 \quad \text{and} \quad \left\| \frac{D_x v_0^{(0)}}{x} \right\|_{L_{\infty,\ell}}^2.
\]
This verifies (3.50a) for the first term on the left-hand side provided \(|\ell| \leq k - 2\). For verifying the estimate for the third term on the left-hand side of (3.50a), we write analogously to (B.4)
\[
\left\| D^\ell_y v_0^{(0)} \right\|_{BC^0((0,x),x)}^2 \lesssim \left\| D^\ell_y v_0^{(0)} + (D_x - 3)(D_x - 2)\tilde{q}(D_x)D_x v_0^{(0)} \right\|_{L_{\infty,\ell}}^2.
\]
Now if \(\ell_x \geq 1\) or \(\ell_y = 0\), we have that at least one \(D_x\) derivative acts on \(v_0^{(0)}\), canceling the \(v_0^{(0)}\) contribution. This, together with the \(x\)-factor coming from the \(D_x\)-operator leads to the expression \(O(x^2)\) as \(x \rightarrow 0\). In the other case when \(\ell_x = 0\) and \(\ell_y \geq 1\) we have at least two factors \(x\) coming from the \(D_y\)-operators, and again the terms are \(O(x^2)\) as \(x \rightarrow 0\). Therefore, we can proceed as in (B.5) and find the bound (B.6) for this term as well.

For the second and fourth terms in (3.50a) we proceed by using (B.2) of Lemma B.1 and then Lemma 3.8, to find
\[
\left\| \tilde{c}_y v_0^{(0)} \right\|_{BC^0(R_{\ell})} \lesssim \left\| v^{(0)} - v_0^{(0)} \right\|_{L_{\infty,\ell}} + \left\| v^{(0)} - v_0^{(0)} \right\|_{L_{\infty,\ell}} \quad \text{and} \quad \left\| \tilde{c}_y v_0^{(0)} \right\|_{BC^0(R_{\ell})} \lesssim \left\| v^{(0)} - v_0^{(0)} \right\|_{L_{\infty,\ell}} + \left\| v^{(0)} - v_0^{(0)} \right\|_{L_{\infty,\ell}}.
\]
This completes the proof of (3.50a), by bounding the second and fourth terms on the left-hand side, provided \(\ell \leq \min \{k - 1, k - 1\}\) and \(\ell \leq \min \{k - 2, k - 2\}\) (cf. (2.1)).

**Proof of estimate** (3.50c). By (B.2) of Lemma B.1 we have
\[
\left\| \tilde{c}_y v_1^{(0)} \right\|_{BC^0(R_{\ell})} \lesssim \left\| v - v_0 - v_1 x \right\|_{L_{\infty,\ell}} + \left\| v - v_0 - v_1 x \right\|_{L_{\infty,\ell}} \quad \text{and} \quad \left\| \tilde{c}_y v_0^{(0)} \right\|_{BC^0(R_{\ell})} \lesssim \left\| \tilde{q}(D_x)D_x v \right\|_{L_{\infty,\ell}} + \left\| \tilde{q}(D_x)D_x v \right\|_{L_{\infty,\ell}}.
\]
We then take the integral of the above bound in \(t\) over \(I\) and note that the resulting term is bounded by \(\|v\|_{sol}^2\) provided \(\ell \leq \min \{k + 1, k + 1\}\) (cf. (3.53)).

For the second term on the left-hand side of (3.50c), we proceed as above:
\[
\left\| \tilde{c}_y f_1 \right\|_{BC^0(R_{\ell})} \lesssim \left\| f - f_1 x \right\|_{L_{\infty,\ell}} + \left\| f - f_1 x \right\|_{L_{\infty,\ell}} \quad \text{and} \quad \left\| \tilde{c}_y f_2 \right\|_{BC^0(R_{\ell})} \lesssim \left\| f - f_1 x \right\|_{L_{\infty,\ell}} + \left\| f - f_1 x \right\|_{L_{\infty,\ell}}.
\]
and along the same lines we obtain
\[
\left\| \tilde{c}_y f_2 \right\|_{BC^0(R_{\ell})} \lesssim \left\| f - f_1 x \right\|_{L_{\infty,\ell}} + \left\| f - f_1 x \right\|_{L_{\infty,\ell}} \quad \text{and} \quad \left\| \tilde{c}_y f_2 \right\|_{BC^0(R_{\ell})} \lesssim \left\| f - f_1 x \right\|_{L_{\infty,\ell}} + \left\| f - f_1 x \right\|_{L_{\infty,\ell}}.
\]
We may then integrate the two above bounds over \( t \in I \) and bound the right-hand sides by \( \|f\|_{\text{rhs}}^2 \) under the conditions \( \ell \leq \bar{k} - 3 \) and \( \ell' \leq \bar{k} - 3 \) (cf. (2.19)).

### B.3. Proofs of the approximation lemmata.

**Proof of Lemma 3.12:** Throughout the proof, estimates depend on \( \bar{k}, \bar{k}, \delta, \).

**Introducing a cut-off in the coordinates \( s \) and \( \eta \).** We have to approximate \( v^{(0)} \) contemporarily in all the addends appearing in \( \|v^{(0)}\|_{\text{init}}^2 \) (cf. (2.1)). In order to adapt to the standard theory of Sobolev spaces, we introduce the Fourier-transformed coordinate \( \eta \in \mathbb{R} \) in the \( \eta \)-direction and the variable \( \varphi = \log x \) in the \( x \)-direction. Through the use of (3.18), we can rewrite the norms by effectively replacing \( D_\varphi \) by \( e^{*} \eta \) and \( D_x \) by \( \hat{\mathbb{c}}_s \). We write for example

\[
\left\| \tilde{q}(D_x) \tilde{q}(D_x) v^{(0)} \right\|_{k,\delta}^2 = \sum_{0 \leq j + j' \leq k} \int_{\mathbb{R}} \eta^{2j} \left| x^{j - \frac{j}{2}} D_x^j \tilde{q}(D_x) v^{(0)} \right| \frac{d\eta}{\delta} = \sum_{0 \leq j + j' \leq k} \left\| \eta^{j} e^{(-\delta - \frac{j}{2}) x} \tilde{q}(\hat{\mathbb{c}}_s)^{j + 1} v^{(0)} \right\|_{L^2(\mathbb{R} \times \mathbb{R}_s)}^2,
\]

and a similar estimate holds for the other norms contributing to \( \|v^{(0)}\|_{\text{init}} \).

We now introduce a cut-off with a smooth function \( \chi_n \) defined as \( \chi_n(s, \eta) := \chi \left( \frac{s}{n}, \frac{\eta}{n} \right) \) with compact support in \( (x, \eta) \) such that

\[
\begin{align*}
\chi(s, \eta) &= 1 \quad \text{for} \quad (s, \eta) \in (-\infty, 1] \times [-1, 1] \\
\chi(s, \eta) &= 0 \quad \text{for} \quad (s, \eta) \notin (-\infty, 2] \times [-2, 2].
\end{align*}
\]

Note that in the norms contributing to \( \|v^{(0)}\|_{\text{init}} \), the difference \( v^{(0)} - \chi_n v^{(0)} \) produces contributions which tend to zero as \( n \to \infty \). For each term in the norm \( \|v^{(0)}\|_{\text{init}} \) we may distribute derivatives. If some of the factors from \( \tilde{q}(\hat{\mathbb{c}}_s) \tilde{q}_s \) fall on the cut-off, then we lose the structure \( \tilde{q}(\hat{\mathbb{c}}_s) \tilde{q}_s \). However, the resulting term can be controlled by Lemma 3.8 again, due to the fact that we may always introduce a second smooth cut-off \( \chi_n \) which equals 1 on the support of \( (1 - \chi_n) \) and zero on \( (-\infty, n - 1] \times [-n + 1, n - 1] \). For example for the term (B.7) we obtain

\[
\sum_{0 \leq j + j' \leq k} \left\| \eta^{j} e^{(-\delta - \frac{j}{2}) x} \tilde{q}(\hat{\mathbb{c}}_s)^{j + 1} (1 - \chi_n) \right\|_{L^2(\mathbb{R} \times \mathbb{R}_s)}^2 \lesssim \sum_{0 \leq j + j' \leq k} \left\| \eta^{j} e^{(-\delta - \frac{j}{2}) x} \chi_n \tilde{q}_s^{j + 1} v^{(0)} \right\|_{L^2(\mathbb{R} \times \mathbb{R}_s)}^2,
\]

where the last term converges to zero as \( n \to \infty \) by dominated convergence due to the fact that \( (\infty, n - 1] \times [-n + 1, n - 1] \to \mathbb{R}^2 \). A similar reasoning for the remaining norms contributing to \( \|v^{(0)}\|_{\text{init}} \) shows that \( \|v^{(0)} - \chi_n v^{(0)}\|_{\text{init}} \to 0 \) as \( n \to \infty \).

The approximants \( \chi_n v^{(0)} \) already satisfy the property \( (G_{\infty}) \) and the smoothness in \( y \) since smoothness in \( y \) is equivalent to decay in \( \eta \).

**Mollification in \( s = \log x \) and \( r = \log \eta \).** The previous step proved that we may without loss of generality restrict ourselves to functions \( v^{(0)} \) such that \( v^{(0)}(x, \eta) = 0 \) for \( (x, \eta) \) with \( \eta > 1 \) or \( |\eta| > 1 \). As a next step, we prove that we can also approximate by smooth functions in \( x \) and \( \eta \). To this aim, we mollify the function \( v^{(0)} \) in the variables \( s = \log x \) and \( \varphi := \log \eta \) (the latter of which gives in the new variables \( s^{2j} = e^{2jr} \) and \( d\varphi = e^{\varphi} d\varphi \)) with a mollifier \( \chi_{\varepsilon} \), where \( \varepsilon > 0 \), \( \chi_{\varepsilon}(s, r) = \chi_1 \left( \frac{s}{\varepsilon}, \frac{r}{\varepsilon} \right) \), and \( \chi_1 \in C^0_0(\mathbb{R}_x^2) \) with \( \int \chi_1(s, r) ds dr = 1 \). Then we know that the mollification commutes with derivatives \( \hat{\mathbb{c}}_s \). Furthermore, we have that

\[
e^{-\rho s + \nu r} \chi_{\varepsilon} * v^{(0)}(s, r) = \int_{\mathbb{R}} e^{-\rho(s-s') + \nu(r-r')} \chi_{\varepsilon} \left( s - s', r - r' \right) e^{-\rho s' + \nu r'} v^{(0)}(s', r') ds' dr' = \chi_{\varepsilon} * v^{(0)},
\]

where \( \chi_{\varepsilon}(s) := e^{-\rho s + \nu r} \chi_{\varepsilon}(s, r) \) and \( v^{(0)}(s, r) := e^{-\rho s + \nu r} v^{(0)}(s, r) \). Hence, upon multiplying the mollifier with an appropriate weight, it can be pulled out of all expressions of the form (B.7) for the norms contributing to \( \|v^{(0)}\|_{\text{init}} \). Then \( \chi_{\varepsilon} \to \delta_0 \) in \( D' \) regardless of the value of \( \rho \) and \( \nu \) and the proof step is concluded.
Decay conditions as \( x \searrow 0 \). By the previous step, we can assume without loss of generality that \( v^{(0)} = v^{(0)}(x, \eta) \in C^\infty((0, \infty) \times \mathbb{R}) \) with \( v^{(0)}(x, \eta) = 0 \) for \( (x, \eta) \) such that \( x \gg 1 \) and \( |\eta| \gg 1 \). Hence, also \( v^{(0)} = v^{(0)}(x, y) \) is smooth. It therefore remains to discuss the asymptotics of \( v^{(0)} \) as \( x \searrow 0 \).

To verify the decay conditions on \( v^{(0)} \), we iteratively apply the following basic reasoning for \( \gamma \in (0, 1) \).

If \( w_1, w_2 \) satisfy \((D_x - \gamma)w^{(1)} = w^{(2)}\) and if \( |w^{(2)}|_{1, \rho} < \infty \) for some \( \rho \in \mathbb{R}\setminus\{\gamma\} \) then we can write

\[
  w^{(1)}(x) = \tilde{w}(x) + w_\gamma x^\gamma,
\]

for some \( w_\gamma \in \mathbb{R} \), where the following explicit formulas for \( \tilde{w} \) hold:

\[
  \tilde{w}(x) = \begin{cases} 
    x^\gamma \int_0^x z^{-\gamma}w^{(2)}(z)\frac{dz}{z} & \text{if } \gamma > \rho, \\
    -x^\gamma \int_0^\infty z^{-\gamma}w^{(2)}(z)\frac{dz}{z} & \text{if } \gamma < \rho.
  \end{cases}
\]

Now note that for the part \( \tilde{w} \) the asymptotics required for applying Lemma 3.8 follow directly from the assumption \( |w^{(2)}|_{1, \rho} < \infty \) and from the explicit formulas (B.10).

By iteratively using expressions (B.9) and (B.10), we obtain that the function \( v^{(0)} \) has the form

\[
  v^{(0)}(x, \eta) = v^{(0)}_0(\eta) + v^{(0)}_1(\eta)x + R(x, \eta) \quad \text{as } x \searrow 0,
\]

with a remainder term \( R \). More precisely, \( R \) can be characterized by noting that by Lemma 3.8 and due to the fact that the operator \( \tilde{q}(D_x)D_x \) cancels the terms \( v^{(0)}_0 \) and \( v^{(0)}_1 \) (cf. (3.21a)), we have

\[
  \|R\|_{k, \delta+1} \lesssim \|(D_x - 3)(D_x - 2)\tilde{q}(D_x)D_xR\|_{k, \delta+1} = \|(D_x - 3)(D_x - 2)\tilde{q}(D_x)D_xv^{(0)}\|_{k, \delta+1} \lesssim \|v^{(0)}\|_{\text{init}} < \infty,
\]

from which it follows that we have \( R(x, \eta) = o(x^{1/2+\delta}) \) as \( x \searrow 0 \) almost everywhere in \( \eta \in \mathbb{R} \). Because of estimate (3.59a) of Lemma 3.9 the coefficients \( \langle v^{(0)}_0 \rangle \) and \( \langle v^{(0)}_1 \rangle \) are continuous in \( \eta \). Now we define

\[
  v^{(0,n)} := v^{(0)}_0 + v^{(0)}_1 x + \chi_n R,
\]

where \( \chi_n(s) := \chi_1 \left( \frac{s}{n} \right) \) and \( \chi_1 \) is a smooth cut-off function such that \( \chi_1(s) = 1 \) on \([-1, \infty)\) and \( \chi_1(s) = 0 \) on \((-\infty, -2]\). Then precisely like in the first proof step, \( v^{(0,n)} = v^{(0)} \) for \( n \gg -n \), and by distributing derivatives, by using Lemma 3.8, and by dominated convergence due to the finiteness of \( \|v^{(0)}\|_{\text{init}} \), we deduce \( \|v^{(0)} - v^{(0,n)}\|_{\text{init}} \to 0 \) as \( n \to \infty \). On the other hand, for \( n \leq -2n \) we have \( v^{(0,n)} := v^{(0)}_0 + v^{(0)}_1 x \), concluding the proof.

Proof of Lemma 3.13. We proceed along the same lines as in the proof of Lemma 3.12, to which we will refer in some parts of the proof.

Truncation in \( s = \log x, \eta \) and \( t \). Almost everywhere in \( t \in I \), the norms in the time integrals of \( \|f\|_{\text{rhs}}^2 \) (cf. (2.19)) are finite, and therefore we can proceed along the lines of the proof of Lemma 3.12. We work in the variables \( t, s := \log x, \eta \), and we introduce again a smooth cut-off function \( \chi_n : I \times [0, \infty) \times \mathbb{R} \to \mathbb{R} \) with values in the interval \([0, 1]\) where \( \chi_n(t, s, \eta) = \chi \left( t, \frac{s}{n}, \frac{\eta}{n} \right) \) and for all \( t \in I \) the function \( \chi(t, \cdot, \cdot) \) satisfies (B.8), and moreover \( \chi_n(t, s, \eta) = 0 \) for \( t \notin I_n \), where \( I_n \subset I \) is a sequence of compact intervals which exhaust \( I \). Then by the same reasoning as in the first step of the proof of Lemma 3.12, after integration over \( I \) as well, we find \( \|f - \chi_n f\|_{\text{rhs}} \to 0 \) as \( n \to 0 \).

Mollification in \( s = \log x, r = \log \eta \) and \( t \). By the previous step we may consider without loss of generality that \( f(t, x, \eta) \) is zero if \( x \gg 1 \) or \( |\eta| \gg 1 \) or \( t \notin I' \) for some compact interval \( I' \subset I \). We then perform a mollification in the variables \( s = \log x \) and \( r = \log \eta \) as in the proof of Lemma 3.12, which allows to approximate \( f \) by functions that are smooth in \( x \) and \( y \) at fixed \( t \). Then we perform a further mollification in \( t \) and obtain approximants that are smooth in \( t, x, \) and \( y \).

Decay conditions as \( x \searrow 0 \). For \( \gamma \in \{1, 2\} \) we apply the reasoning of the corresponding step of Lemma 3.12 for the solution of \((D_x - \gamma)w^{(1)} = w^{(2)}\) via formulas (B.9) and (B.10). This time we find that \( f \) has the form

\[
  f(t, x, \eta) = f_1(t, \eta)x + f_2(t, \eta)x^2 + R(t, x, \eta) \quad \text{as } x \searrow 0,
\]
Lemma 3.9 the coefficients prove continuity in remainder term $r$ and $\sigma$ where $R$ \( p \) in time, i.e.,

\[ \text{Proof of Corollary 3.15: precisely like in Lemma 3.13.} \]

the proof, including the construction of the above-mentioned support properties of convergence in the supremum norm due to (3.59b) of Lemma 3.9. Moreover, since for estimating $\chi(t,s,\eta)$ on $n$, again by using the bound (3.59b) of Lemma 3.9.

The second item follows due to the fact that the required property holds for the approximants constructed in Lemma 3.14, again by using the bound (3.59b) of Lemma 3.9.

Finally, note that the supremum part of the norm (3.53) can be rewritten as $\sup_{t \in I} \| v(t, \cdot, \cdot) \|_{\text{init}}$. From the smoothness properties of the approximants $v^{(n)}$ from Lemma 3.14, we find that $t \rightarrow \| v^{(n)}(t, \cdot, \cdot) \|_{\text{init}}$ are continuous, and from the approximation property $\| v^{(n)} - v \|_{\text{sol}} \rightarrow 0$ we find that these functions are also uniformly convergent and bounded, so that the function $t \rightarrow \| v(t, \cdot, \cdot) \|_{\text{init}}$ is also continuous. Now note that the integral terms in the definition of $\| v \|_{\text{sol}, \tau}$ vanish as $\tau \searrow 0$ by dominated convergence. Thus

\[ \lim_{\tau \searrow 0} \| v \|_{\text{sol}, \tau} = \lim_{\tau \searrow 0} \sup_{t \in I} \| v(t, \cdot, \cdot) \|_{\text{init}} = \| v^{(0)} \|_{\text{init}}, \]

which concludes the proof of the last item as well as of the corollary. \( \square \)
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