A Proposed Casimir-Like Effect Between Contaminants in Ideal Bose-Einstein Condensates

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Abstract

It is hypothesized that, within Bose-Einstein condensates, contaminants will form a potential that effects the energy state of a condensate. While assuming a system governed by the Gross-Pitaevskii equation, contaminants are modelled as boundary conditions for the wave function of the condensate. It is then found that the energy of the system depends directly upon the distance between contaminants. Energy is minimized as two particles either come together or move apart depending on the nature of the condensate. This is due to the presence of induced standing waves in the condensate between two contaminants, similar to the attractive effect caused by standing electromagnetic waves in a vacuum, the Casimir effect. Quantum calculations are also done to determine the expected strength of the “contaminant in condensate” effect.

1 Introduction

Much attention has previously been given to the behavior of atoms on a macroscopic level within Bose-Einstein condensates [1], especially since their first observation [2]. However, relatively little work has been done on the behavior of
contaminants within Bose-Einstein condensates, such as He\(^3\) within a He\(^4\) condensate with the exception of how to eliminate them, for example as mentioned in [3] and [4]. Here the existence of a “Casimir-like” effect is proposed within a Bose-Einstein condensate where the wave function of the condensate functions as the background electromagnetic field of the classical Casimir effect [5]. The proposed effect will be referred to as the “contaminant in condensate” (CIC) effect.

2 Predictions

The idea of a Casimir-like force between contaminants in Bose-Einstein condensate is conceptually simple. For the sake of similar simplicity in our first calculation, the following assumptions were made.

1. Any given contaminant particle can be taken to occupy a cubic volume element or a point.

2. Tunnelling does not occur through a given contaminant, i.e. the potential in a contaminant is infinite and can be represented by an infinite square well.

3. The Bose-Einstein Condensate is ideal.

4. There are no other external potentials.

5. In the absence of any potential, the energy of the wave function of a Bose-Einstein condensate goes to 0. In general, the wave function of a Bose-Einstein condensate will always be in the ground state.

The Gross-Pitaevskii equation representing the state of the condensate between two contaminants should thus take the form of a one-dimensional Schrödinger wave equation:
\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r)\psi(r) = \mu \psi(r) \]  
(1)

where \(\psi(r)\) is the wave function of the condensed state, \(m\) is the mass of a given component boson, and \(\mu\) is the chemical potential as usual. \(V(r)\) is given in this case by an infinite square well located at the position of each of the contaminants. This gives a potential for the condensate between two contaminants of:

\[ \mu = \frac{\hbar^2 \pi^2}{8ma^2} \]  
(2)

Note the reason a one-dimensional Schrödinger wave equation is used even in the case of cube-shaped contaminants is that in directions parallel to the faces of contaminants which are facing one another, the energy of the wave function of the condensate is equal to 0, as per assumption 5 above.

The energy of the condensate superstate is inversely proportional to the square of the distance between the two contaminants. Taking the negative of the gradient of (2) with respect to distance \(a\) results in a force:

\[ F = \frac{\hbar^2 \pi^2}{4ma^3} \]  
(3)

between two contaminants.

To be slightly more sophisticated, relativistic effects can be taken into account by using the Klein-Gordon equation [6]:

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi = 0 \]  
(4)

rather than the Schrödinger equation. This has both positive and negative energy solutions given by:

\[ E_n = \pm \sqrt{\frac{m^2 c^4}{\hbar^2} + \frac{c^2 \pi^2 n^2}{a^2}} \]  
(5)

where \(n = 1, 2, 3 \ldots\). This begs the question of which energy level a condensate superstate will naturally settle to. To resolve this issue, it is assumed that
a superstate will obtain the lowest energy value not otherwise occupied. By assuming some analogy of the Dirac “electron sea” to exist for this model, it is speculated that this occurs at either the positive or negative case of \( n = 1 \).

### 2.1 Gross-Pitaevskii Approach

Let us now drop assumptions 3 and 4 from above and include the entire Gross-Pitaevskii (GP) equation [7]:

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + V(r, t)\psi(r, t) + U_0|\psi(r, t)|^2\psi(r, t) = \mu\psi(r, t) \tag{6}
\]

to allow calculations in a more physically realistic model where \( U_0 = \frac{4\pi\hbar^2}{m} \alpha \) gives the interaction, attractive or repulsive, between two atoms with scattering radius \( \alpha \). In this case \( V(r) \) is given by:

\[
V(x, y, z) = \frac{1}{2}m(\omega^2x^2 + \omega^2y^2 + \omega^2z^2) + \sum_j B(x, y, z, x_1, y_1, z_1), \tag{7}
\]

where the function \( B \) is given for each contaminant \( j \) by:

\[
B(x, y, z, x_1, y_1, z_1) = \begin{cases} 
\infty & \text{if } x = x_1, y = y_1, z = z_1 \\
0 & \text{if otherwise}
\end{cases} \tag{8}
\]

and \( x_1, y_1, \) and \( z_1 \) (or alternatively \( r_1 \)) is the position of each contaminant \( j \).

There is no analytic solution to the Gross-Pitaevskii equation. For an approximately solution, the technique of Edwards and Burnett [8] will be used. It will be assumed that the kinetic energy of the condensate is significantly less than the external potential and contribution of interatomic forces. The GP equation can thus be changed to:

\[
\left(\frac{1}{2}m\omega^2r^2 + \sum_j B(\mathbf{r}, \mathbf{r}_j)\right)\psi(\mathbf{r}) + NU_0|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = \mu\psi(\mathbf{r}), \tag{9}
\]

the solution of which, using the factor \( \sum_j B(\mathbf{r}, \mathbf{r}_j) \) as boundary, conditions is:
This has the advantage of going to zero as $r$ goes to a critical value $r_c$, thus fitting well to the wavefunction for a condensate between two infinite point potentials. The potential associated with this wavefunction is:

$$\mu = \frac{m \omega^2 r_c^2}{2}.$$  \hspace{1cm} (11)

By replacing $r_c$ with $a/2$, we get a force associated with the presence of a condensate superstate between two contaminants of:

$$F(a) = -\frac{m \omega^2 a}{4}$$  \hspace{1cm} (12)

Note the rest of the condensate, disregarding the effect of atom density, is unaffected by the distance between the two contaminants and thus has no effect on the CIC effect.

### 2.2 Quantum Approach

Finally, by taking a quantum approach, as used in the Bogoliubov approach [9], rather than a classical approach, we can find a way to eliminate assumption 5 and take into account an entire excitation spectrum of an ideal condensate. Given the nature of a Bose-Einstein condensate, assumption 5 is likely to be physically valid. However, in the interest of thoroughness, we will show how it can be disregarded here. The usual derivation of the Casimir effect shall be used and will be recounted here [5]. We shall assume that a Bose-Einstein condensate is governed by the massive scalar field equation:

$$\frac{1}{c^2} \frac{\partial^2 \phi(t, x)}{\partial t^2} - \frac{\partial^2 \phi(t, x)}{\partial x^2} + \frac{m^2 c^2}{\hbar^2} \phi(t, x) = 0$$  \hspace{1cm} (13)

Solutions of this equation are given by:
\[ \varphi_n^{(\pm)}(t, x) = \sqrt{\frac{c}{\omega_n}} e^{\pm i \omega_n t} \sin k_n x \quad (14) \]

\[ \omega_n = \sqrt{\frac{m^2 c^4}{\hbar^2} + c^2 k_n^2} \quad (15) \]

\[ k_n = \frac{\pi n}{a}, n = 1, 2, \ldots \quad (16) \]

We now quantize this field by using the expansion:

\[ \varphi(t, x) = \sum_n [\varphi_n^{(-)}(t, x)a_n + \varphi_n^{(+)}(t, x)a_n^+] \quad (17) \]

with commutation relations for the annihilation and creation operators

\[ [a_n, a_{n'}^+] = \delta_{n,n'}, [a_n, a_{n'}] = [a_{n'}^+, a_n^+] = 0 \quad (18) \]

and vacuum state defined by:

\[ a_n |0\rangle = 0. \quad (19) \]

The desired quantity is the energy density:

\[ T_{00}(x) = \frac{\hbar c}{2} (c^{-2} [\partial_t \varphi(x)]^2 + [\partial_x \varphi(x)]^2) \quad (20) \]

which from equations (12)-(18) is:

\[ \langle 0 | T_{00}(x) | 0 \rangle = \frac{\hbar}{2a} \sum_{n=1}^{\infty} \omega_n - \frac{m^2 c^4}{2a \hbar} \sum_{n=1}^{\infty} \cos 2k_n x \omega_n. \quad (21) \]

Total energy across an interval length \( a \) is:

\[ E(a) \int_0^a \langle 0 | T_{00}(x) | 0 \rangle dx = \frac{\hbar}{2} \sum_{n=1}^{\infty} \omega_n. \quad (22) \]

This value is evaluated by introducing a damping function, with the result:
\[ E(a) = -\frac{mc^2}{4} - \frac{\hbar c}{4\pi a} \int_{2\mu}^{\infty} \frac{\sqrt{y^2 - 4\mu^2}}{ey - 1} dy \] (23)

where \( \mu \equiv \frac{mc}{\hbar} \). For \( \mu \gg 1 \) as is the case in our calculations:

\[ E(a) \approx -\frac{mc^2}{2} - \frac{\sqrt{\mu} \hbar c}{4\sqrt{\pi a}} e^{-2\mu}. \] (24)

It should also be noted that the quantum approach gives another rather convenient method for calculating the force of attraction between two contaminants due to a single condensate superstate. To do so we associate with the superstate a scalar propagator for a spin 0 particle:

\[ \frac{1}{k^2 - m^2 + i\epsilon}. \] (25)

If we then let the superstate “propagate” from one distance to another rather than from one position to another, one can use Feynman’s path integral approach to calculate the energy and force associated with the creation and propagation of a superstate. This turns out to be, using the usual methods [10]:

\[ E(a) = -\frac{1}{4\pi a} e^{-ma}. \] (26)

### 3 Discussion

It is interesting to note the similarities of the above effect to the Casimir effect in a vacuum. In the Casimir effect, standing waves due to an electromagnetic field in a vacuum result in an attractive force between any two objects in what would otherwise be a vacuum. However, the Casimir effect requires summation over all possible excitation modes of the standing waves. As we assume only one excitation mode for a Bose-Einstein condensate, we remove that step in the computation.
The pressing issue that occurs is whether the above hypothesis actually could be observed to occur. It is clear that this would be difficult, as contaminants obviously tend to prevent a Bose-Einstein condensate from forming and destabilize the condensate around them. In addition, the strength of the proposed interaction is so small that it would be very difficult to directly observe for systems as small as a few dozen atoms large.

Rather than direct observation, an alternative approach is to note that the potential energy that forms a condensate supersate must come from the kinetic energy of its components. The introduction of contaminants into a Bose-Einstein condensate which do not couple to it and that are cooled to the level of the surrounding condensate should thus paradoxically lower the temperature of the surrounding condensate. The reduction in temperature should proportional to the energy of the induced condensate superstates.

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