Existence of stationary vortex sheets for the 2D incompressible Euler equation

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Abstract. We construct a new type of planar Euler flows with localized vorticity. Let $\kappa_i \neq 0$, $i = 1, \ldots, m$, be $m$ arbitrarily fixed constants. For any given nondegenerate critical point $x_0 = (x_{0,1}, \ldots, x_{0,m})$ of the Kirchhoff–Routh function defined on $\Omega^m$ corresponding to $(\kappa_1, \ldots, \kappa_m)$, we construct a family of stationary planar flows with vortex sheets that have large vorticity amplitude and concentrate on curves perturbed from small circles centered near $x_{0,i}$, $i = 1, \ldots, m$. The proof is accomplished via the implicit function theorem with suitable choice of function spaces.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded or unbounded domain. We consider the stationary Euler equation

$$\begin{cases}
\mathbf{u} \cdot \nabla \mathbf{u} = - \nabla P, & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \\
\mathbf{u} \cdot \mathbf{n} = 0, & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $\mathbf{u} = (u_1, u_2)$ is the velocity field, $P$ is the scalar pressure, and $\mathbf{n}$ is the outward unit normal of $\partial \Omega$.

In a planar flow, the vorticity is defined as the third component of the curl of the velocity field $(u_1, u_2, 0)$, namely, $\omega = \partial_1 u_2 - \partial_2 u_1$. Taking the curl of the first equation in (1.1), we find that $\omega$ satisfies the following vorticity equation:

$$\mathbf{u} \cdot \nabla \omega = 0 \quad \text{in } \Omega.$$  

(1.2)

The velocity is recovered by the Biot–Savart law

$$\mathbf{u} = \nabla \perp (-\Delta)^{-1} \omega,$$

where $(x_1, x_2) \perp = (x_2, -x_1)$ and the operator $(-\Delta)^{-1}$ is given by

$$(-\Delta)^{-1} \omega(x) = \int_{\Omega} G(x - y) \omega(y) dy.$$
Here, $G(x, y)$ is the Green function of $-\Delta$ in $\Omega$ with zero Dirichlet data. So $G(x, y)$ takes the form 
$$G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} - H(x, y)$$ 
with $H(x, y)$ the harmonic part of $G(x, y)$. We denote $\psi = (-\Delta)^{-1} \omega$ to be the stream function, then the velocity field can be derived by $u = \nabla \perp \psi$.

In the last century, the two-dimensional Euler equation has been intensively studied, and the global well-posedness of the vorticity equation with initial data in $L^1 \cap L^\infty$ was proved by Yudovich in the classical paper [45]. However, many physical phenomena possess strong and irregular fluctuations, such as fluids with small viscosity, where flows tend to separate from rigid walls and sharp corners [5, 37]. To model this phenomenon mathematically, the most natural way is to think of a solution to the Euler equation, in which the velocity changes sign discontinuously across a stream line. This discontinuity induces vorticity concentrated on a curve, which is only a measure rather than a bounded function.

A velocity discontinuity in an inviscid 2D flow is called a vortex sheet, whose vorticity concentrates as a measure along a curve. Suppose that $\omega$ is a weak solution to the Euler equation concentrated on a finite number of closed curves $\Gamma_i$ parameterized by $z_i(\theta)$. Namely, for any test function $\phi \in C^\infty_c(\Omega)$, $\omega$ is a measure such that

$$\int_{\Omega} \phi(x) d\omega(x) = \sum_i \int_{\Gamma_i} \gamma_i(\alpha) \phi(z_i(\alpha))|z_i'(\alpha)| d\alpha,$$

where $\gamma_i(\alpha)$ is the vorticity strength at $z_i(\alpha)$. Then, the equation of the sheet can be derived by the Birkhoff–Rott operator in a domain [6, 23, 34, 37, 41]

$$BR(z, \gamma)(x) := \frac{1}{2\pi} P.V. \int \frac{(x-z(\alpha))^\perp \gamma(\alpha)|z'(\alpha)|}{|x-z(\alpha)|^2} d\alpha + \int \nabla \perp H(x, z(\alpha)) \gamma(\alpha)|z'(\alpha)| d\alpha,$$

where $P.V.$ stands for Cauchy principal value of an integral. Equation (1.3) yields the motion of the sheet

$$u(z_i(\theta)) = -BR(z_i(\theta))$$

with $BR(z_i(\theta)) := -\sum_j BR(z_j, \gamma_j)(z_i(\theta))$.

Significant efforts have been made in mathematical study of the theory of vortex sheet. In the elegant paper [20], Delort proved global existence of weak solutions with an initial $L^2_{\text{loc}}$ velocity and a positive measure vorticity. Later, the proof was simplified by Majda [36]. Duchon and Robert [21] established global existence for a class of initial data concentrated closed to a line. Existence in different setting of vortex sheet with a distinguished sign was also obtained in [22, 42]. For vorticity without a definite sign, only partial results on the existence are known under some additional assumptions [35, 43, 44]. Note that uniqueness for such solutions still remains open.

On the other hand, blow up may occur in the motion of vortex sheet. Indeed, singular formulation was conjectured by Birkhoff [6], and by Birkhoff and Fisher [7]. In [39], Moore showed the possibility that the curvature blows up in finite time even though the initial data are analytic. Moore’s result was also supported by numerical study [30]. Ill-posedness for vortex sheet problem in the space $H^s$ with $s > \frac{3}{2}$ was obtained by Caflisch and Orellana [9]. These results demonstrate that the study of
vortex sheet is extremely delicate, and hence exact solutions, in particular relative equilibria, are of great importance since their structures persist for long time.

Nevertheless, very few relative equilibria are known. For the vortex sheets in \( \mathbb{R}^2 \), except for circles and lines, the only nontrivial examples include: uniformly rotating segment \([4]\), in which the vorticity is supported on a segment of length \( 2a \) with density

\[
y(x) = \Omega_\tau \sqrt{a^2 - x^2}, \quad \text{for} \ x \in [-a, a]
\]

and angular velocity \( \Omega_\tau \). A generalization of the rotating segment is the Protas–Sakajo class \([40]\), which is made out of segments rotating about a common center of rotation with endpoints at the vertices of a regular polygon. Recently, a new class of vortex sheet was obtained in \([24]\) via degenerate bifurcation from rotating circles. Note that the existence of nontrivial steady vortex sheet in \( \mathbb{R}^2 \) is not apparent in view of the rigidity results obtained in \([23]\), where the authors showed for uniformly rotating vortex sheets with angular velocity \( \Omega_\tau \leq 0 \) and strength \( \gamma > 0 \), only trivial solutions exist.

In a domain \( \Omega \subset \mathbb{R}^2 \), as far as we know, there seems no nontrivial stationary vortex sheet is known so far. The purpose of the present paper is to construct a family of stationary vortex sheets for a domain (bounded or not), whenever the Kirchhoff–Routh function possesses nondegenerate critical points.

For any given integer \( m > 0 \), and \( m \) real numbers \( \kappa_1, \kappa_2, \ldots, \kappa_m \), define the Kirchhoff–Routh function on \( \Omega^m = \{ x = (x_1, x_2, \ldots, x_m) \mid x_i \in \Omega, \text{for} \ i = 1, \ldots, m \} \) as follows:

\[
W_m(x_1, x_2, \ldots, x_m) = - \sum_{i \neq j}^{m} \kappa_i \kappa_j G(x_i, x_j) + \sum_{i=1}^{m} \kappa_i^2 H(x_i, x_i).
\]

It is known that the location of \( m \)-point vortices with strength \( \kappa_i \ (i = 1, \ldots, m) \) in \( \Omega \) must be a critical point of \( W_m \) (see, e.g., \([31, 32]\)). Results on the existence and nondegeneracy of critical points for \( W_m \) can be found in \([2, 3]\). In \([25]\), it was proved that if \( \Omega \) is convex, then there is no critical point of \( W_m \) in \( \Omega^m \) with \( m \geq 2 \) and \( \kappa_i > 0 \) for all \( i = 1, \ldots, m \). Let us point out that although the nondegeneracy of critical points for the Kirchhoff–Routh functions in a general domain is not an easy issue, it is true for most of the domains, as proved in \([1, 38]\). On the other hand, in \([8]\), it was shown that in a convex domain, \( W_1 \) has a unique critical point, which is also nondegenerate. In a recent paper \([12]\), the first author, Yan, and Yu obtained some existence and results on the nondegeneracy of critical points of the Kirchhoff–Routh function for unbounded domains.

Giving a nondegenerate critical point \( x_0 = (x_{0,1}, \ldots, x_{0,m}) \in \Omega^m \) of \( W_m \), for \( \varepsilon \) small, we will construct a branch of vortex sheets concentrated on a finite number of closed curves \( \Gamma_i \). Moreover, each \( \Gamma_i \) is the perturbation of a small circle with radius \( \varepsilon \) centered at some point \( x_{\varepsilon, i} \in \Omega \) close to \( x_{0,i} \), and the vorticity \( \omega \mid_{\Gamma_i} \) satisfies

\[
\int_0^{2\pi} y_i(a) |z_i'(a)| \, da = \kappa_i.
\]

This result shows the rich diversity of stationary vortex sheet solutions despite that the well-posedness is not fully understood.

Our main theorem is as follows.

**Theorem 1.1** Let \( \Omega \subset \mathbb{R}^2 \) be a domain (may be unbounded), and let \( \kappa_i \neq 0 \ (i = 1, \ldots, m) \) be \( m \) given numbers. Suppose that \( x_0 = (x_{0,1}, \ldots, x_{0,m}) \in \Omega^m \) with \( x_{0,i} \neq x_{0,j} \) for \( i \neq j \), is an isolated critical point of \( W_m \) defined by \((1.5)\) satisfying the nondegeneracy
Lemma 2.1 in [23], we are able to formulate the conditions that the Birkhoff–Rott integral (1.3) satisfies for a stationary vortex sheet into a system of 2

Remark 1.4 Fixing and hence (i) moreover, it holds that \( \gamma_i = \frac{\kappa_i + O(\varepsilon)}{2\pi\varepsilon} \) and each \( \Gamma_i \) is a perturbation of a small circle with radius \( \varepsilon \) and centered at some point \( x_{\varepsilon, i} \in \Omega \) satisfying \( |x_{\varepsilon, i} - x_{0, i}| = O(\varepsilon) \).

(ii) As \( \varepsilon \to 0^+ \), one has in the sense of measure

\[

\omega_{\varepsilon, \tau} \to \sum_{i=1}^{m} \kappa_i \delta(x - x_{0, i}) \text{ weakly},

\]

where \( \delta(x - x_{0, i}) \) is the Dirac delta function concentrated at the point \( x_{0, i} \).

(iii) For any \( i = 1, \ldots, m \), the interior of \( \Gamma_i \) is convex.

Remark 1.2 Our result does not rely on the sign of \( \kappa_i \), which is essential in the global existence of the initial problem as mentioned above. As we shall see in Section 5, the parameter \( \tau \) stands for the projection on the kernel of the linearized operator, and it slightly affects the shape of \( \Gamma_i \). More precisely, \( \Gamma_i \) in Theorem 1.1 takes the form

\[

\Gamma_i = \{ x_{\varepsilon, i} + \varepsilon(1 + \varepsilon(f_{\varepsilon, i}(\theta) + \tau f_{0, i}(\theta)))(\cos \theta, \sin \theta) | \theta \in [0, 2\pi) \} \text{ for some } f_{\varepsilon, i} \text{ depending on } \varepsilon \text{ and } \tau \text{ and some fixed } f_{0, i} \text{ in the kernel.}

\]

Remark 1.3 For simplicity, all the scales of \( \Gamma_i \) \( (i = 1, \ldots, m) \) are chosen to be of the same order. However, this is not necessary, and one may construct vortex sheet concentrated on \( \Gamma_i \) with the scale of \( \varepsilon_i \) \( (i = 1, \ldots, m) \) different from each other.

Remark 1.4 Fixing \( \tau \in (-\tau_0, \tau_0) \) in Theorem 1.1, say \( \tau = 0 \), we obtain a family of solutions with vortex sheet \( \omega_0 \) parameterized only by \( \varepsilon \), which is of special interest since it is closely related to the classical problem: regularization of point vortices for the Euler equation. This means justifying the weak formulation for point vortex solutions of the incompressible Euler equations by approximating these solutions with more regular solutions. In fact, the vortex sheets obtained in Theorem 1.1 belong to the space \( H^{-1}(\Omega) \), whereas the point vortices solution belongs to \( H^{-1-\sigma}(\Omega) \) for any \( \sigma > 0 \), which is more singular than a vortex sheet. Thus, our result can be regarded as a desingularization of point vortices by vortex sheet in some way. For more literature on the desingularization problem, we refer to [10, 11, 17, 28, 33] and the references therein.

Next, we shall sketch the basic ideas used to prove the main result. Thanks to Lemma 2.1 in [23], we are able to formulate the conditions that the Birkhoff–Rott integral (1.3) satisfies for a stationary vortex sheet into a system of \( 2m \) coupled integrodifferential equations \( \mathcal{F}_{i,1}(\varepsilon, x, f, g) = 0 \) and \( \mathcal{F}_{i,2}(\varepsilon, x, f, g) = 0, \ i = 1, \ldots, m \).

We expect that the case \( (\varepsilon, x, f, g) = (0, x_0, 0, 0) \) corresponds to the point vortices and hence \( (0, x_0, 0, 0) \) is a solution to \( \mathcal{F}_{i,1} = 0 \) and \( \mathcal{F}_{i,2} = 0 \) provided that \( x_0 \) is a critical point of \( \mathcal{W}_m \). Therefore, the first step is to extend \( \mathcal{F}_{i,1} \) and \( \mathcal{F}_{i,2} \) such that \( \varepsilon \leq 0 \) is allowed. Then, one can verify that \( \mathcal{F}_{i,1}(0, x_0, 0, 0) = \mathcal{F}_{i,2}(0, x_0, 0, 0) = 0 \) does hold when \( x_0 \) is a critical point of \( \mathcal{W}_m \), and hence we obtain a trivial solution. To apply the implicit function theorem at the solution \( (0, x_0, 0, 0) \), the Gateaux derivative of \( \mathcal{F} := (\mathcal{F}_{1,1}, \mathcal{F}_{1,2}, \ldots, \mathcal{F}_{m,1}, \mathcal{F}_{m,2}) \) should be an isomorphism, which
unfortunately fails. Detailed calculations show that $D\mathcal{F}$ has a $2m$-dimensional kernel $\prod_{i=1}^{m}\{(a\cos(\theta) + b\sin(\theta)), \kappa_i(a\cos(\theta) + b\sin(\theta))\} | (a, b) \in \mathbb{R}^2\}$. Hence, we have to consider the equations in quotient spaces and impose the conditions $-\kappa_i \int \mathcal{F}_{i,1} \sin(\theta) d\theta = \int \mathcal{F}_{i,2} \cos(\theta) d\theta$ and $\kappa_i \int \mathcal{F}_{i,1} \cos(\theta) d\theta = \int \mathcal{F}_{i,2} \sin(\theta) d\theta$ for all $i = 1, \ldots, m$. Although these conditions seem to be complicated, we successfully convert them into a concise equation $\nabla \mathcal{W}_m(x) = O(\varepsilon)$, which is solvable near $x_0$ due to the nondegeneracy of $\nabla \mathcal{W}_m$ at $x_0$. Finally, we can apply the implicit function theorem to obtain the existence. The convexity of the interior of $\Gamma_i$ follows from calculating the curvature directly. We point out that our procedure of proving Theorem 1.1 borrows the idea of Lyapunov–Schmidt reduction and local bifurcation theory.

The ideas and methods introduced in the present paper may be widely applied to a variety of situations and other models. For example, one may consider an ideal fluid with an irrotational background flow $u_0 = \nabla \psi_0$, where $\psi_0$ is a given harmonic function. In this case, the Kirchhoff–Routh function is given by (see [12])

$$\mathcal{W}_{m,\psi_0}(x_1, x_2, \ldots, x_m) = -\sum_{i \neq j}^{m} \kappa_i \kappa_j G(x_i, x_j) + \sum_{i=1}^{m} \kappa_i^2 H(x_i, x_i) + 2 \sum_{i=1}^{m} \kappa_i \psi_0(x_i).$$

Although $\mathcal{W}_{m,\psi_0}$ is slightly different from $\mathcal{W}_m$ given by (1.5), we believe that our method can be modified to construct vortex sheets near critical points of $\mathcal{W}_{m,\psi_0}$ in this situation. In addition, for domains with some symmetry properties, such as the unit disk or the half-space, one may modify our method by considering function spaces with certain symmetries to construct solutions with vortex sheets near degenerate critical points of the Kirchhoff–Routh function.

We would like to make a brief remark on the approach of constructing vortex patches via the contour dynamics equation, which shares a similar spirit as the construction of vortex sheet we consider in the present paper. Many celebrated contributions have been made with the contour dynamics equation method in recent years (see, e.g., [13–16, 18, 19, 26–29] and the references therein). However, since a vortex patch is actually a bounded function, the contour dynamics equation is more regular than the equations of a vortex sheet. Hence, more effort is needed in the process of our proof.

This paper is organized as follows. In Section 2, we derive the equations that the Birkhoff–Rott integral satisfies for a stationary vortex sheet and define the function spaces which will be used later. In Section 3, we extend the functionals and show their $C^1$ regularity. Section 4 is devoted to study the linearization operators, where we prove that the derivative is an isomorphism in quotient spaces. In Section 5, we choose $x$ properly such that the range of our functional belongs to the quotient spaces and apply the implicit function theorem to prove Theorem 1.1.

## 2 Formulation and functional setting

Since $\omega$ is a stationary sheet, using Lemma 2.1 in [23], we derive the following equations that the BR equation (1.4) and $\gamma_i$ satisfy:

\begin{equation}
\begin{aligned}
BR(z_i(\theta)) \cdot n(z_i(\theta)) &= 0,
\end{aligned}
\end{equation}

where \( n(z_i(\theta)) \) is the unit normal vector of \( \Gamma_i \) at \( z_i(\theta) \), and

\[
(2.2) \quad BR(z_i(\theta)) \cdot s(z_i(\theta)) \frac{y_i(\theta)}{|z_i'(\theta)|} = C,
\]

where \( s(z_i(\theta)) \) is the unit tangential vector. Note that (2.2) can be rewritten as

\[
(2.3) \quad (I - P_0) \left[ BR(z_i(\theta)) \cdot s(z_i(\theta)) \frac{y_i(\theta)}{|z_i'(\theta)|} \right] = 0,
\]

where \( P_0 \) is the projection to the mean value defined by \( P_0 f := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \).

Since \( x_0 \in \Omega^m \), we can take \( r_0 > 0 \) sufficiently small such that \( B_{r_0}(x_0,i) \subset \Omega \) for all \( i = 1, \ldots, m \), where \( B_{r_0}(x_0,i) \) is the ball with radius \( r_0 \) and centered at \( x_0,i \). We aim to construct vortex sheets localized near \( x_0 \). Thus, for \( \varepsilon > 0 \) small, we assume that \( z_i(i = 1, \ldots, m) \) are of the following form:

\[
z_i(\theta) = x_i + \varepsilon R_i(\theta)(\cos \theta, \sin \theta)
\]

with \( R_i(\theta) = 1 + \varepsilon f_i(\theta) \), and \( x_i \in B_{r_0}(x_0,i) \) to be chosen later. We also assume

\[
y_i(\alpha) = \frac{\kappa_i + \varepsilon g_i(\alpha)}{2|z_i'(\alpha)|}.
\]

We end this section by introducing some notations and definitions that will be used in this paper and reformulating equations (2.1) and (2.3). Denote the mean value of integral of \( g \) on the unit circle by

\[
\int g(\tau) d\tau := \frac{1}{2\pi} \int_0^{2\pi} g(\tau) d\tau
\]

and set

\[
A(\theta, \alpha) := 4 \sin^2\left(\frac{\theta - \alpha}{2}\right), \quad A_{ij} = |x_i - x_j|^2,
\]

\[
B(f, \theta, \alpha) := 4(f(\theta) + f(\alpha)) \sin^2\left(\frac{\theta - \alpha}{2}\right) + \varepsilon \left( (f(\theta) - f(\alpha))^2 + 4f(\theta)f(\alpha) \sin^2\left(\frac{\theta - \alpha}{2}\right) \right),
\]

\[
B_{ij}(\theta, \alpha) = 2(x_i - x_j) \cdot ((\cos \theta, \sin \theta) - (\cos \alpha, \sin \alpha)) + 2\varepsilon (x_i - x_j) \cdot (f_i(\theta)(\cos \theta, \sin \theta) - f_j(\theta)(\cos \alpha, \sin \alpha)) + \varepsilon ((1 + \varepsilon f_i(\theta))(\cos \theta, \sin \theta) - (1 + \varepsilon f_j(\alpha))(\cos \alpha, \sin \alpha))^2.
\]

For \( k \geq 3 \), we will also frequently use the function spaces given in the following, whose norms are naturally defined as norms of product spaces.

\[
X^k = \left\{ g \in H^k \mid g(\theta) = \sum_{j=1}^{\infty} a_j \cos(j\theta) + b_j \sin(j\theta) \right\},
\]

\[
X^k_i := \left\{ (f_1, f_2) \in X^{k+1} \times X^k \mid -\kappa \int f_1(\theta) \cos(\theta) d\theta = \int f_2(\theta) \cos(\theta) d\theta, \quad -\kappa \int f_1(\theta) \sin(\theta) d\theta = \int f_2(\theta) \sin(\theta) d\theta \right\},
\]

where \( \kappa \) is a constant.
For given $\mathbf{f} = (f_1, \ldots, f_m)$ and $\mathbf{g} = (g_1, \ldots, g_m)$, denote $\tilde{g}_{i,t}(t) = \kappa_i + \varepsilon g_i(t)$. Then, we can reduce equations (2.1) and (2.3) to

\begin{equation}
0 = \mathcal{F}_{i,1}(\varepsilon, \mathbf{x}, \mathbf{f}, \mathbf{g})
\end{equation}

\begin{equation}
0 = \mathcal{F}_{i,2} := (I - P_0)\tilde{\mathcal{F}}_{i,2},
\end{equation}

where $\tilde{\mathcal{F}}_{i,2}$ is given by

\begin{equation}
\tilde{\mathcal{F}}_{i,2}(\varepsilon, \mathbf{x}, \mathbf{f}, \mathbf{g}) := \frac{\tilde{g}_{i,t}(\theta)}{\varepsilon(R_i(\theta))^2 + (R_i'(\theta))^2} P.V. \int \frac{\varepsilon f_i'(\theta) R_i(\theta) \sin(\theta - \alpha)}{A(\theta, \alpha) + \varepsilon B(f_i(\theta), \alpha)} \tilde{g}_{i,t}(\alpha) d\alpha
\end{equation}

\begin{equation}
+ \frac{\tilde{g}_{i,t}(\theta)}{\varepsilon(R_i(\theta))^2 + (R_i'(\theta))^2} P.V. \int \frac{\varepsilon f_i'(\theta) R_i(\theta) (\cos(\theta - \alpha) - 1)}{A(\theta, \alpha) + \varepsilon B(f_i(\theta), \alpha)} \tilde{g}_{i,t}(\alpha) d\alpha
\end{equation}

\begin{equation}
+ \frac{\tilde{g}_{i,t}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} P.V. \int \frac{R_i(\theta) (f_i(\alpha) - f_i(\theta))}{A(\theta, \alpha) + \varepsilon B(f_i(\theta), \alpha)} \tilde{g}_{i,t}(\alpha) d\alpha
\end{equation}

\begin{equation}
+ \sum_{j \neq i} \frac{\tilde{g}_{i,t}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \int \frac{(x_i - x_j)^2 \cdot \left( R_i(\theta) (-\sin \theta, \cos \theta) + \varepsilon f_i'(\theta) (\cos \theta, \sin \theta) \right)}{A_{ij} + \varepsilon B_{ij}(\theta, \alpha)} \tilde{g}_{j,t}(\alpha) d\alpha
\end{equation}

\begin{equation}
+ \sum_{j \neq i} \frac{\tilde{g}_{i,t}(\theta)}{R_i(\theta)^2 + (R_i'(\theta))^2} \int \frac{-\varepsilon R_i^2(\theta) + \varepsilon^2 f_i'(\theta) R_i(\theta) \sin(\theta - \alpha) + \varepsilon R_i(\theta) R_i(\alpha) \cos(\theta - \alpha)}{A_{ij} + \varepsilon B_{ij}(\theta, \alpha)} \tilde{g}_{j,t}(\alpha) d\alpha
\end{equation}
3 Extension and regularity of functionals

To apply the implicit function theorem at $\varepsilon = 0$, we need to extend the functions $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ defined in Section 2 to $\varepsilon \leq 0$ and check the $C^1$ regularity.

Let us first show the continuity of these functionals. Letting $V$ be the unit ball centered at origin in $(X^{k+1} \times X^k)^m$ and $B_{r_0}(x_0)$ be the ball centered at $x_0$ in $\Omega^m$ with radius $r_0$, we have the following proposition.

**Proposition 3.1** The functionals $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ can be extended from $(-\varepsilon_0, \varepsilon_0) \times B_{r_0}(x_0) \times V$ to $X^k \times X^k$ as continuous functionals.

**Proof** Throughout the proof, we will frequently use the following Taylor’s formula:

\[
\frac{1}{(A + B)^k} = \frac{1}{A^k} - \frac{1}{A^{k-1}} \int_0^1 B (A + tB)^{k-1} dt.
\]

Let us consider $\mathcal{F}_{i,1}$ first. We need to prove that $\partial^l \mathcal{F}_{i,1} \in L^2$ for $l = 0, 1, \ldots, k$. For the first term

\[
\mathcal{F}_{i,1} := \frac{1}{\varepsilon} \int_{\mathcal{P}} \theta \left( \frac{1}{A + \varepsilon B} \right)^{\frac{1}{2}} \left( \frac{1}{A + \varepsilon B} \right)^{\frac{1}{2}} \sin(\theta - \alpha) \left( \kappa_i + \varepsilon g_i(\alpha) \right) d\alpha,
\]

since $R_i(x) = 1 + \varepsilon f_i(x)$, the possible singularity caused by $\varepsilon = 0$ may occur only when we take zeroth-order derivative of $\mathcal{F}_{i,11}$. Thus, we first show that $\mathcal{F}_{i,11} \in L^2$. We decompose the kernel into two parts

\[
\frac{1}{A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha)} = \frac{1}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} \left( 1 + 2\varepsilon f_i(\theta) + \varepsilon^2 \left( f_i(\theta)^2 + f_i'(\theta)^2 \right) \right) + K_R,
\]

where $K_R := \frac{1}{A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha)} - \frac{1}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} \left( 1 + 2\varepsilon f_i(\theta) + \varepsilon^2 \left( f_i(\theta)^2 + f_i'(\theta)^2 \right) \right)$ is more regular than $\frac{1}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)}$. Indeed, by using (3.1), we calculate

\[
\sin(\theta - \alpha) K_R = \frac{\sin(\theta - \alpha)}{A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha)} - \frac{\sin(\theta - \alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} \left( 1 + 2\varepsilon f_i(\theta) + \varepsilon^2 \left( f_i(\theta)^2 + f_i'(\theta)^2 \right) \right)
\]

\[
= \frac{\sin(\theta - \alpha)}{1 + 2\varepsilon f_i(\theta) + \varepsilon^2 \left( f_i(\theta)^2 + f_i'(\theta)^2 \right)}
\]
\[
\frac{(1 + 2\varepsilon f_i(\theta) + \varepsilon^2 f_i(\theta)^2 + f_i'(\theta)^2))4 \sin^2 \left(\frac{\theta - \alpha}{2}\right) - A(\theta, \alpha) - \varepsilon B(f_i, \theta, \alpha)}{4 \sin^2 \left(\frac{\theta - \alpha}{2}\right) (A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha))}
\]

\[
= \frac{\varepsilon \sin(\theta - \alpha)}{1 + 2\varepsilon f_i(\theta) + \varepsilon^2 (f_i(\theta)^2 + f_i'(\theta)^2))}
\]

\[
= \frac{f_i(\theta) - f_i(\alpha) + \varepsilon f_i(\theta)(f_i(\theta) - f_i(\alpha)) + \varepsilon \frac{4f_i'(\theta)^2 \sin^2 \left(\frac{\theta - \alpha}{2}\right) - (f_i(\theta) - f_i(\alpha))^2}{4 \sin^2 \left(\frac{\theta - \alpha}{2}\right)}}{A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha)}
\]

Then, it is easy to see that

\[
\frac{1}{\varepsilon} \int_{\mathcal{K}_R} \sin(\theta - \alpha)(1 + \varepsilon f_i(\alpha))(\kappa_i + \varepsilon g_i(\alpha)) d\alpha
\]

\[
= \int_{\mathcal{F}_{i, \alpha}} \left(1 + \varepsilon f_i(\alpha)\right)(\kappa_i + \varepsilon g_i(\alpha))(f_i(\theta) - f_i(\alpha)) \sin(\theta - \alpha) + O(\varepsilon)
\]

\[
= \int_{\mathcal{F}_{i, \alpha}} \kappa_i(f_i(\theta) - f_i(\alpha)) \sin(\theta - \alpha) + O(\varepsilon)
\]

where \(\mathcal{R}_{\alpha}\) is regular and bounded. Hence, to prove \(\mathcal{F}_{i, \alpha} \in L^2\), we only need to estimate the rest term \(\int_{\mathcal{F}_{i, \alpha}} \frac{1}{1 + 2\varepsilon f_i(\theta) + \varepsilon^2 (f_i(\theta)^2 + f_i'(\theta)^2)} \left(1 + \varepsilon f_i(\alpha)\right)(\kappa_i + \varepsilon g_i(\alpha)) d\alpha\). By the odd symmetry and (3.1), we have

\[
\frac{1}{1 + 2\varepsilon f_i(\theta) + \varepsilon^2 (f_i(\theta)^2 + f_i'(\theta)^2)} \left(1 + \varepsilon f_i(\alpha)\right)(\kappa_i + \varepsilon g_i(\alpha)) d\alpha
\]

\[
= \frac{1}{1 + 2\varepsilon f_i(\theta) + \varepsilon^2 (f_i(\theta)^2 + f_i'(\theta)^2)} \left(1 + \varepsilon f_i(\alpha)\right)(\kappa_i + \varepsilon g_i(\alpha)) d\alpha
\]

\[
= \int_{\mathcal{F}_{i, \alpha}} \frac{1}{4 \sin^2 \left(\frac{\theta - \alpha}{2}\right)} \left(1 + \varepsilon f_i(\alpha)\right)(\kappa_i + \varepsilon g_i(\alpha)) d\alpha
\]

Using the expansion \(f_i(\alpha) = f_i(\theta) + O(|\sin \left(\frac{\theta - \alpha}{2}\right)|)\) and \(g_i(\alpha) = g_i(\theta) + O(|\sin \left(\frac{\theta - \alpha}{2}\right)|)\), then we find

\[
P.V \int_{\mathcal{F}_{i, \alpha}} \frac{1}{4 \sin^2 \left(\frac{\theta - \alpha}{2}\right)} \left(1 + \varepsilon f_i(\alpha)\right)(\kappa_i + \varepsilon g_i(\alpha)) d\alpha = O(1),
\]
where we have used the fact \( \left| \frac{\sin(\theta - \alpha)}{4 \sin^2(\frac{\theta - \alpha}{2})} O(\sin \left( \frac{\theta - \alpha}{2} \right)) \right| \leq C \). Therefore, it holds that \( P.V \int \frac{(\kappa_i f_i(\alpha) + \varepsilon g_i(\alpha)) \sin(\theta - \alpha)}{4 \sin^2(\frac{\theta - \alpha}{2})} d\alpha \) belongs to \( L^\infty \) and hence belongs to \( L^2 \). Moreover, \( \mathcal{R}_{112} \) is regular and bounded. We conclude that \( \mathcal{F}_{i,11} \in L^\infty \). Furthermore, it holds

\[
(3.3)
\]

\[
\mathcal{F}_{i,11} = \frac{1}{1 + 2 \varepsilon f_1(\theta) + \varepsilon^2 (f_i(\theta)^2 + f_1(\theta)^2)} \frac{1}{\varepsilon} P.V. \int \frac{\sin(\theta - \alpha)}{4 \sin^2(\frac{\theta - \alpha}{2})} (1 + \varepsilon f_i(\alpha))(\kappa_i + \varepsilon g_i(\alpha)) d\alpha 
\]

\[
+ \frac{1}{\varepsilon} \int \mathcal{R}_R \sin(\theta - \alpha)(1 + \varepsilon f_i(\alpha))(\kappa_i + \varepsilon g_i(\alpha)) d\alpha 
\]

\[
= P.V. \int \frac{g_i(\alpha) \sin(\theta - \alpha)}{4 \sin^2(\frac{\theta - \alpha}{2})} d\alpha + \varepsilon \mathcal{R}_{11},
\]

where \( \mathcal{R}_{11} = \mathcal{R}_{111} + \mathcal{R}_{112} \) is regular and bounded.

Next, we prove that \( \partial^k \mathcal{F}_{i,11} \in L^2 \). To simplify notation, we rewrite \( \mathcal{F}_{i,11} \) as follows by changing the variable \( \alpha \) to \( \theta - \alpha \):

\[
\mathcal{F}_{i,11} := \frac{1}{\varepsilon} P.V. \int \frac{(1 + \varepsilon f_i(\theta - \alpha)) \sin(\alpha)}{A(\theta, \theta - \alpha) + \varepsilon B(f_i, \theta, \theta - \alpha)} (\kappa_i + \varepsilon g_i(\theta - \alpha)) d\alpha.
\]

Taking \( k \)th derivatives of \( \mathcal{F}_{i,11} \), we see that the most singular term is

\[
P.V. \int \frac{\partial^k f_i(\theta - \alpha) \sin(\alpha)}{A(\theta, \theta - \alpha) + \varepsilon B(f_i, \theta, \theta - \alpha)} (\kappa_i + \varepsilon g_i(\theta - \alpha)) d\alpha 
\]

\[
+ P.V. \int \frac{(1 + \varepsilon f_i(\theta - \alpha)) \sin(\alpha)}{A(\theta, \theta - \alpha) + \varepsilon B(f_i, \theta, \theta - \alpha)} \partial^k g_i(\theta - \alpha) d\alpha 
\]

\[- P.V. \int \frac{(1 + \varepsilon f_i(\theta - \alpha))(\kappa_i + \varepsilon g_i(\theta - \alpha)) \sin(\alpha)}{A(\theta, \theta - \alpha) + \varepsilon B(f_i, \theta, \theta - \alpha)} \left[ 4(\partial^k f_i(\theta) + \partial^k f_i(\theta - \alpha)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) 
\]

\[
+ 2\varepsilon (f_i(\theta) - f_i(\theta - \alpha))(\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)) 
\]

\[
+ 4\varepsilon (\partial^k f_i(\theta) f_i(\theta - \alpha) + f_i(\theta) \partial^k f_i(\theta - \alpha)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \right] d\alpha 
\]

\[
=: I_1 + I_2 + I_3.
\]

We first deal with \( I_1 \). By the splitting of the kernel \( (3.2) \), we derive

\[
I_1 = P.V. \int \frac{\partial^k f_i(\theta - \alpha) \sin(\alpha)}{A(\theta, \theta - \alpha) + \varepsilon B(f_i, \theta, \theta - \alpha)} (\kappa_i + \varepsilon g_i(\theta - \alpha)) d\alpha 
\]

\[
= P.V. \int \frac{\partial^k f_i(\alpha) \sin(\theta - \alpha)}{A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha)} (\kappa_i + \varepsilon g_i(\alpha)) d\alpha
\]
\[
\frac{1}{1 + 2\epsilon f_1(\theta) + \epsilon^2 (f_1(\theta)^2 + f_1'(\theta)^2)} P.V. \int \frac{\partial^k f_1(\alpha) (\kappa_i + \epsilon g_i(\alpha)) \sin(\theta - \alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha
\]

\[
+ P.V. \int \mathcal{K}_R \sin(\theta - \alpha) \partial^k f_1(\alpha)(\kappa_i + \epsilon g_i(\alpha)) d\alpha.
\]

Noting that \(|\mathcal{K}_R \sin(\theta - \alpha)| \leq C\epsilon\), we have

\[
\|P.V. \int \mathcal{K}_R \sin(\theta - \alpha) \partial^k f_1(\alpha)(\kappa_i + \epsilon g_i(\alpha)) d\alpha\|_{L^2} \leq C\|f_1\|_{H^k}(1 + \|g\|_{L^2})
\]

is bounded. Since \(P.V. \int \frac{\partial^k f_1(\alpha)(\kappa_i + \epsilon g_i(\alpha)) \sin(\theta - \alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha\) is the Hilbert transformation of the function \(\partial^k f_1(\alpha)(\kappa_i + \epsilon g_i(\alpha))\), we have

\[
\left\|P.V. \int \frac{\partial^k f_1(\alpha)(\kappa_i + \epsilon g_i(\alpha)) \sin(\theta - \alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha\right\|_{L^2} \leq \|\partial^k f_1(\alpha)(\kappa_i + \epsilon g_i(\alpha))\|_{L^2} \leq C\|f_1\|_{H^k}(1 + \|g\|_{L^2}).
\]

Similarly, one can check that \(\|I_2\|_{L^2} \leq C(1 + \|f_1\|_{L^2})\|g\|_{H^k}\).

To estimate \(I_3\), we split the kernel as follows:

\[
\frac{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)}{(A(\theta, \theta - \alpha) + \epsilon B(f_1, \theta, \theta - \alpha))^2} = \frac{1}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} \cdot \frac{1}{(1 + 2\epsilon f_1(\theta) + \epsilon^2 (f_1(\theta)^2 + f_1'(\theta))^2)^2} + \tilde{\mathcal{K}}_R,
\]

where \(\tilde{\mathcal{K}}_R\) satisfies \(|\tilde{\mathcal{K}}_R \sin \alpha| \leq C\). Since convolution with the kernel \(\frac{\sin \alpha}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)}\) defines the Hilbert transformation, we find that

\[
P.V. \int \frac{(1 + \epsilon f_1(\theta - \alpha))(\kappa_i + \epsilon g_i(\theta - \alpha)) \sin(\alpha)}{(A(\theta, \theta - \alpha) + \epsilon B(f_1, \theta, \theta - \alpha))^2} \sin^2 \left( \frac{\theta - \alpha}{2} \right) (\partial^k f_1(\theta)(\partial^k f_1(\alpha) f_1(\theta - \alpha) + f_1(\theta) \partial^k f_1(\theta - \alpha))) d\alpha
\]

belongs to \(L^2\) due to the \(L^2\) boundedness of Hilbert transformation and the regularity of \(\tilde{\mathcal{K}}_R\). For the remaining term in \(I_3\)

\[
2\epsilon P.V. \int \frac{(1 + \epsilon f_1(\theta - \alpha))(\kappa_i + \epsilon g_i(\theta - \alpha)) \sin(\alpha)}{(A(\theta, \theta - \alpha) + \epsilon B(f_1, \theta, \theta - \alpha))^2} (f_1(\theta) - f_1(\theta - \alpha)) (\partial^k f_1(\theta) - \partial^k f_1(\theta - \alpha)) d\alpha,
\]

we decompose the kernel

\[
\frac{(1 + \epsilon f_1(\theta - \alpha))(\kappa_i + \epsilon g_i(\theta - \alpha)) \sin(\alpha)}{(A(\theta, \theta - \alpha) + \epsilon B(f_1, \theta, \theta - \alpha))^2} (f_1(\theta) - f_1(\theta - \alpha)) = \frac{(1 + \epsilon f_1(\theta))(\kappa_i + \epsilon g_i(\theta)) f_1'(\theta)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} + \tilde{\mathcal{K}}_R,
\]

where \(\tilde{\mathcal{K}}_R\) satisfies \(|\tilde{\mathcal{K}}_R \sin \alpha| \leq C\). Then, we deduce

\[
P.V. \int \frac{(1 + \epsilon f_1(\theta - \alpha))(\kappa_i + \epsilon g_i(\theta - \alpha)) \sin(\alpha)}{(A(\theta, \theta - \alpha) + \epsilon B(f_1, \theta, \theta - \alpha))^2} (f_1(\theta) - f_1(\theta - \alpha)) (\partial^k f_1(\theta) - \partial^k f_1(\theta - \alpha)) d\alpha
\]

\[
= (1 + \epsilon f_1(\theta))(\kappa_i + \epsilon g_i(\theta)) f_1'(\theta) P.V. \int \frac{\partial^k f_1(\theta) - \partial^k f_1(\theta - \alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha
\]

\[
+ P.V. \int \tilde{\mathcal{K}}_R (\partial^k f_1(\theta) - \partial^k f_1(\theta - \alpha)) d\alpha
\]

\[
= (1 + \epsilon f_1(\theta))(\kappa_i + \epsilon g_i(\theta)) f_1'(\theta) \left(-\Delta\right)^{\frac{1}{2}} (\partial^k f_1(\theta)) d\alpha + P.V. \int \tilde{\mathcal{K}}_R (\partial^k f_1(\theta) - \partial^k f_1(\theta - \alpha)) d\alpha.
\]
By Fourier transformation and the Hardy inequality, we obtain
\[ \| (-\Delta)^{\frac{1}{2}} (\partial^k f_i) \|_{L^2} \leq \| \nabla \partial^k f_i \|_{L^2} \leq \| f_i \|_{H^{k+1}} \]
and
\[ \| \mathcal{K}_R (\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)) \|_{L^2} \leq \| \nabla \partial^k f_i \|_{L^2} \leq \| f_i \|_{H^{k+1}}. \]
Consequently, we have \( \partial^k \mathcal{F}_{i,11} \in L^2 \) and hence \( \mathcal{F}_{i,11} \in H^k \).

Now, we turn to the second term
\[ \mathcal{F}_{i,12} := \frac{1}{1 + \epsilon f_i(\theta)} P.V. \int \frac{f_i'(\theta)(1 + \epsilon f_i(\alpha))(1 - \cos(\theta - \alpha))}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \left( \kappa_i + \epsilon g_i(\alpha) \right) d\alpha. \]
Since \( |1 - \cos(\theta - \alpha)| = \sin^2 \left( \frac{\theta - \alpha}{2} \right) \), the kernel of this term is actually regular and bounded. Therefore, it is easy to see that \( \mathcal{F}_{i,12} \in H^k \). Moreover, by (3.1), we find
\[ \mathcal{F}_{i,12} = \frac{1}{1 + \epsilon f_i(\theta)} P.V. \int \frac{f_i'(\theta)(1 + \epsilon f_i(\alpha))(1 - \cos(\theta - \alpha))}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \left( \kappa_i + \epsilon g_i(\alpha) \right) d\alpha \]
\[ = P.V. \int \frac{\kappa_i f_i'(\theta)(1 - \cos(\theta - \alpha))}{A(\theta, \alpha)} d\alpha + \epsilon \mathcal{R}_{i,12} \]
(3.4) \[ = \frac{\kappa_i}{2} f_i'(\theta) + \epsilon \mathcal{R}_{i,12}, \]
where \( \mathcal{R}_{i,12} \) is smooth and we have used the identity \( 1 - \cos(\theta - \alpha) = 2 \sin^2 \left( \frac{\theta - \alpha}{2} \right) = \frac{\sin^2(\theta)}{2} \).

For \( \mathcal{F}_{i,13} \), taking kth derivatives of \( \mathcal{F}_{i,13} \), we see that the most singular terms are
\[ \epsilon \partial^{k+1} f_i(\theta) P.V. \int \frac{f_i(\theta) - f_i(\alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \left( \kappa_i + \epsilon g_i(\alpha) \right) d\alpha \]
\[ + \frac{\epsilon f_i'(\theta)}{1 + \epsilon f_i(\theta)} P.V. \int \frac{\nabla f_i(\theta) - \nabla f_i(\theta - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \left( \kappa_i + \epsilon g_i(\theta - \alpha) \right) d\alpha \]
\[ + \frac{\epsilon f_i'(\theta)}{1 + \epsilon f_i(\theta)} P.V. \int \frac{f_i(\theta) - f_i(\theta - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \nabla g_i(\theta - \alpha) d\alpha \]
\[ - \frac{\epsilon f_i'(\theta)}{1 + \epsilon f_i(\theta)} P.V. \int \frac{f_i(\theta) - f_i(\theta - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \nabla g_i(\theta - \alpha) \left( \kappa_i + \epsilon g_i(\theta - \alpha) \right) d\alpha \]
\[ =: J_1 + J_2 + J_3 + J_4. \]
Since \( P.V. \int \frac{f_i(\theta) - f_i(\alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \left( \kappa_i + \epsilon g_i(\alpha) \right) d\alpha \in L^2 \), by Taylor's expansion of \( f_i(\alpha) \) at \( \alpha = \theta \), we know that the first term \( J_1 \) is bounded in \( L^2 \). To deal with \( J_2 \), we split the kernel
\[ \frac{\kappa_i + \epsilon g_i(\theta - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} = \frac{\kappa_i + \epsilon g_i(\theta)}{4 \sin^2 \left( \frac{\theta}{2} \right)} \mathcal{K}_R, \]
where \(|\hat{K}_R \sin \frac{a}{2}| \leq C\). Therefore, we conclude

\[
\frac{\varepsilon f_1''(\theta)}{1 + \varepsilon f_1(\theta)} P.V. \int \frac{\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)}{A(\theta, \theta - \alpha) + \varepsilon B(f_i, \theta, \theta - \alpha)} (\kappa_i + \varepsilon \eta_i(\theta - \alpha)) d\alpha \\
= \frac{\varepsilon f_1''(\theta)(\kappa_i + \varepsilon \eta_i(\theta))}{1 + \varepsilon f_1(\theta)} P.V. \int \frac{\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)}{4 \sin^2 \left( \frac{\alpha}{2} \right)} d\alpha \\
+ \frac{\varepsilon f_1''(\theta)}{1 + \varepsilon f_1(\theta)} P.V. \int \hat{K}_R (\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)) d\alpha \\
= \frac{\varepsilon f_1''(\theta)(\kappa_i + \varepsilon \eta_i(\theta))}{1 + \varepsilon f_1(\theta)} (-\Delta)^{\frac{1}{2}} (\partial^k f_i)(\theta) \\
+ \frac{\varepsilon f_1''(\theta)}{1 + \varepsilon f_1(\theta)} P.V. \int \hat{K}_R (\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)) d\alpha.
\]

By Fourier transformation and Hardy inequality, we obtain

\[\|(\partial^k f_i)\|_{L^2} \leq \|\nabla (\partial^k f_i)\|_{L^2} \leq \|f_i\|_{H^{k+1}}\]

and

\[\left\|\int \hat{K}_R (\partial^k f_i(\theta) - \partial^k f_i(\theta - \alpha)) d\alpha\right\|_{L^2} \leq \|\nabla (\partial^k f_i)\|_{L^2} \leq \|f_i\|_{H^{k+1}}\]

We can show that the remaining terms \(I_3\) and \(I_4\) are bounded in \(L^2\) similarly. Moreover, it can be seen that

\[(3.5)\]

\[F_{i,13} = \frac{1}{1 + \varepsilon f_1(\theta)} P.V. \int \frac{\varepsilon f_1''(\theta)(f_i(\theta) - f_i(\alpha))}{A(\theta, \alpha) + \varepsilon B(f_i, \theta, \alpha)} (\kappa_i + \varepsilon \eta_i(\alpha)) d\alpha = \varepsilon R_{i,13},\]

where \(R_{i,13}\) is regular.

Since \(H(x, y)\) is smooth in \(\Omega\), the terms \(F_{i,14}, F_{i,15},\) and \(F_{i,16}\) are apparently smooth and belong to \(H^k\). Furthermore, we have

\[(3.6)\]

\[F_{i,14} + F_{i,15} + F_{i,16} = \sum_{j \neq i} \frac{\kappa_i(x_i - x_j)(- \sin \theta, \cos \theta)}{|x_i - x_j|^2} d\alpha - \sum_{j=1}^{m} 2\pi f \kappa_j \nabla H(x_i, x_j) \cdot (- \sin \theta, \cos \theta) d\alpha + \varepsilon R_{i,14}
\\
= \sum_{j \neq i} 2\pi f \kappa_i \nabla G(x_i, x_j) \cdot (- \sin \theta, \cos \theta) d\alpha
\\
- 2\pi f \kappa_i \nabla H(x_i, x_i) \cdot (- \sin \theta, \cos \theta) d\alpha + \varepsilon R_{i,14},\]

where \(R_{i,14}\) is bounded and smooth.

By (3.3)–(3.6), we conclude

\[(3.7)\]

\[F_{i,1}(\varepsilon, x, f, g) = P.V. \int \frac{\sin(\theta - \alpha) f_i(\alpha)}{4 \sin^2 \left( \frac{\theta}{2} \right)} d\alpha + \frac{\kappa_i f''_i(\theta)}{2} + \sum_{j \neq i} 2\pi f \kappa_j \nabla G(x_i, x_j) \cdot (- \sin \theta, \cos \theta)
\\
- 2\pi f \kappa_i \nabla H(x_i, x_i) \cdot (- \sin \theta, \cos \theta) + \varepsilon R_{i,1},\]
where \( \mathcal{R}_1 := \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{13} + \mathcal{R}_{14} \) is regular. Hence, we can define

\[
\mathcal{F}_{1,1}(0, \mathbf{x}, f, g) := P.V.\int \frac{\sin(\theta - \alpha)g_i(\alpha)}{4 \sin^2(\frac{\pi \alpha}{2})} \, d\alpha + \frac{\epsilon i}{2} f'_i(\theta) + \Sigma_{j \neq i} 2\pi \kappa_j \nabla G(x_i, x_j) \cdot (-\sin \theta, \cos \theta)
\]

\[ -2\pi \kappa_i \nabla H(x_i, x_j) \cdot (-\sin \theta, \cos \theta). \]

Next, we prove the continuity of \( \mathcal{F}_{1,1} \). By (3.7) and the definition of \( \mathcal{F}_{1,1}(0, \mathbf{x}, f, g) \), one can easily check that \( \mathcal{F}_{1,1} \) is continuous with respect to \( \epsilon \) at \( \epsilon = 0 \). Thus, we only need to prove that \( \mathcal{F}_{1,1} \) is continuous with respect to \( \epsilon \) for \( \epsilon \neq 0 \). However, it is easy to see that the continuity of \( \mathcal{F}_{1,1} \) with respect to \( \epsilon \) is a consequence of its continuity with respect to \( f \) and \( g \) when \( \epsilon \neq 0 \), on which we will focus below.

We only prove the continuity of \( \mathcal{F}_{1,11} \) with respect to \( f_i \) and \( g_i \), and the continuity of other terms in \( \mathcal{F}_{1,1} \) can be proved by a similar or even easier way. We will use the following notations: for a general function \( h \), we denote \( \Delta h = h(\theta) - h(\alpha), \ h = h(\theta), \ h = \hat{h}(\alpha), \) and

\[
D(h) = \epsilon^2 (\Delta h)^2 + 4(1 + \epsilon h)(1 + \epsilon \hat{h}) \sin^2 \left( \frac{\theta - \alpha}{2} \right).
\]

To show the continuity of \( \mathcal{F}_{1,11} \) with respect to \( f_i \), let \( (f_1, g), (f_2, g) \in X^k \). Then, we can calculate the difference

\[
\mathcal{F}_{1,11}(\epsilon, f_2, g) - \mathcal{F}_{1,11}(\epsilon, f_1, g) = P.V.\int \frac{(f_2(\alpha) - f_1(\alpha)) \sin(\theta - \alpha)}{D(f_1)} (\kappa + \epsilon g(\alpha)) \, d\alpha
\]

\[ + \frac{1}{\epsilon} P.V.\int (1 + \epsilon f_1(\alpha))(\kappa + \epsilon g(\alpha)) \sin(\theta - \alpha) \left( \frac{1}{D(f_1)} - \frac{1}{D(f_2)} \right) \, d\alpha =: K_1 + K_2. \]

For the first term \( K_1 \), since \( \frac{1}{D(f_1)} \) has the same singularity as \( \frac{1}{\sin^2(\frac{\pi \alpha}{2})} \), it is easy to prove

\[ \|K_1\|_{\mathcal{H}^k} \leq C \|f_1 - f_2\|_{\mathcal{H}^{k+1}} \]

by the technique we have used before. For the second term \( K_2 \), since

\[
\frac{1}{D(f_2)} - \frac{1}{D(f_1)} = \frac{\epsilon^2 ((\Delta f_1)^2 - (\Delta f_2)^2) + 4\epsilon ((f_1 - f_2)(1 + \epsilon \tilde{f}_2) + (\tilde{f}_1 - \tilde{f}_2)(1 + \epsilon f_1)) \sin^2 \left( \frac{\theta - \alpha}{2} \right)}{D(f_1)D(f_2)}
\]

\[ = \epsilon ((\Delta f_1 + \Delta f_2)(\Delta f_1 - \Delta f_2) + 4((f_1 - f_2)(1 + \epsilon \tilde{f}_2) + (\tilde{f}_1 - \tilde{f}_2)(1 + \epsilon f_1)) \sin^2 \left( \frac{\theta - \alpha}{2} \right)}{D(f_1)D(f_2)} \]

it holds that the singularity of \( \frac{1}{D(f_2)} - \frac{1}{D(f_1)} \) is also of the order \( O \left( \frac{1}{4 \sin^2(\frac{\pi \alpha}{2})} \right) \), the same as the kernel in \( \mathcal{F}_{1,11} \) itself. Therefore, using argument similar to the above, we can prove that \( \|K_2\|_{\mathcal{H}^k} \leq C \|f_2 - f_1\|_{\mathcal{H}^{k+1}} \), which shows the continuity of \( \mathcal{F}_{1,11} \) with respect to \( f_i \). Notice that \( \mathcal{F}_{1,11} \) is linear with respect to \( g_i \). Then, the continuity of \( \mathcal{F}_{1,11} \) with respect to \( g \) can be obtained by argument similar to the proof of boundedness of \( \mathcal{F}_{1,11} \) in \( \mathcal{H}^k \).

We have shown that the conclusion of Proposition 3.1 holds true for \( \mathcal{F}_{1,1} \). The fact that \( \mathcal{F}_{1,2} \) is well defined and continuous can be verified in a similar way. Attention should be paid to the fact that the projection operator \( I - P_0 \) eliminates all constant
terms in \( \tilde{\mathcal{F}}_{i,2} \), which also removes singularity in \( \mathcal{F}_{i,1} \). By (3.1), we obtain
\[
\tilde{\mathcal{F}}_{i,2}(\epsilon, x, f, g) = \kappa_i^2 f_i(\theta) - \kappa_i^2 P.V. \int \frac{f_i(\theta) - f_i(\alpha)}{4 \sin^2 \frac{\theta + \epsilon}{2}} d\alpha - \kappa_i g_i(\theta)
\]
(3.9)
\[
- \sum_{j \neq i} 2\pi \kappa_j \nabla^\perp G(x_i, x_j) : (- \sin \theta, \cos \theta)
+ 2\pi \kappa_i^2 \nabla^\perp H(x_i, x_i) : (- \sin \theta, \cos \theta).
\]
(3.10)
where \( \mathcal{R}_2 \) is smooth. Thus, we define
\[
\tilde{\mathcal{F}}_{i,2}(0, x, f, g)(\theta) = \kappa_i^2 f_i(\theta) - \kappa_i^2 P.V. \int \frac{f_i(\theta) - f_i(\alpha)}{4 \sin^2 \frac{\theta + \epsilon}{2}} d\alpha - \kappa_i g_i(\theta)
\]
(3.11)
\[
- \sum_{j \neq i} 2\pi \kappa_j \nabla^\perp G(x_i, x_j) : (- \sin \theta, \cos \theta)
+ 2\pi \kappa_i^2 \nabla^\perp H(x_i, x_i) : (- \sin \theta, \cos \theta).
\]

Our next proposition concerns the \( C^1 \) regularity.

**Proposition 3.2** The Gateaux derivatives \( \partial_{(f, g)} \mathcal{F}_{i,1} \) and \( \partial_{(f, g)} \tilde{\mathcal{F}}_{i,2} \) exist and are continuous.

**Proof** We first prove that the derivative of \( \mathcal{F}_{i,1} \) with respect to \( f_i \) exists and is as follows:
\[
\partial f_i \mathcal{F}_{i,1} h = Fh, \quad \forall h \in X^{k+1},
\]
where \( Fh \) is given by
\[
Fh := P.V. \int \frac{h(\alpha) \sin(\theta - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)}(\kappa_i + \epsilon g_i(\alpha)) d\alpha
\]
(3.12)
\[
- P.V. \int \frac{(1 + \epsilon f_i(\alpha))(\kappa_i + \epsilon g_i(\alpha)) \sin(\theta - \alpha)}{(A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha))^2} \left[ 4(h(\theta) + h(\alpha)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \right] d\alpha.
\]
To prove (3.11), one needs to verify
\[
\lim_{t \to 0} \left\| \frac{\mathcal{F}_{i,1}(\epsilon, f_i + th, g_i) - \mathcal{F}_{i,1}(\epsilon, f_i, g_i) - Fh}{t} \right\|_{H^s} = 0.
\]
(3.13)
Using the notations given in the proof of Proposition 3.1, we deduce
\[
\mathcal{F}_{i,1}(\epsilon, f_i + th, g_i) - \mathcal{F}_{i,1}(\epsilon, f_i, g_i) = Fh
\]
(3.14)
\[
= \frac{1}{t} \int (1 + \epsilon f_i(\alpha))(\kappa_i + \epsilon g_i(\alpha)) \sin(\theta - \alpha)
\times \left( \frac{1}{D(f_i + th)} - \frac{1}{D(f_i)} + \frac{2\epsilon^2 \Delta f_i \Delta h + 4\epsilon((1 + \epsilon f_i) h + (1 + \epsilon f_i(\alpha)) \hat{h}) \sin^2 \left( \frac{\theta - \alpha}{2} \right)}{D(f_i)^2} \right)
\]
\[
+ \int h(\alpha)(\kappa_i + \epsilon g_i(\alpha)) \sin(\theta - \alpha) \left( \frac{1}{D(f_i + th)} - \frac{1}{D(f_i)} \right) dy
\]
(3.15)
\[
=: F_1 + F_2.
\]
By the mean value theorem, we find

\[
\frac{1}{D(f_i + th)} - \frac{1}{D(f_i)} = O \left( \frac{t \epsilon}{4 \sin^2 \left( \frac{\pi y_2}{2} \right)} \right),
\]

\[
\frac{1}{D(f_i + th)} - \frac{1}{D(f_i)} + t \frac{2 \epsilon^2 \Delta f_i \Delta h + 4 \left( \epsilon \tilde{R} h + \epsilon \tilde{h} R \right) \sin^2 \left( \frac{\pi y_2}{2} \right)}{D(f_i)^2} = O \left( \frac{t^2 \epsilon^2}{4 \sin^2 \left( \frac{\pi y_2}{2} \right)} \right),
\]

which means that the kernels in \( F_1 \) and \( F_2 \) are of the same order as the kernel in \( F_{i,11} \).

Therefore, by argument similar to Proposition 3.1, we have

\[
\| F_i \|_{H^k} + \| F_2 \|_{H^k} \leq C t \| h \|_{X^{k+1}}.
\]

Letting \( t \to 0 \), we obtain (3.13) and hence obtain the existence of Gateaux derivative of \( F_{i,11} \). To prove the continuity of \( \partial_{f_i} F_{i,11}(\epsilon, f_i, g_i) h \), one just needs to verify by definition. Since there is no other new idea than the proof of continuity for \( F_{i,11} \), we omit it therefore. The existence and continuity of Gateaux derivatives of other terms in \( F_{i,1} \) and \( F_{i,2} \) can be obtained via similar argument, which we leave out here. Noting that \( F_{i,1} \) is linearly dependent on \( g \) and \( F_{i,2} \) is quadratically dependent on \( g \), it is much easier to compute their Gateaux derivatives with respect to \( g \), so we leave them to our reader. For readers' convenience, we also write down the derivatives of \( F_{i,1} \) and \( F_{i,2} \) in the following form directly without proof here.

Recall the definitions \( \hat{g}_{i,\epsilon}(t) = \kappa_i + \epsilon g_i(t), R_i(t) = 1 + \epsilon f_i(t) \). For any \( h_1 \in X^{k+1} \) and \( h_2 \in X^k \), we have

(3.14)

\[
\partial_{f_i} F_{i,1}(\epsilon, x, f_i, g_i) h_1 = P.V. \int g_{i,\epsilon}(\alpha) d\alpha
\]

\[
- P.V. \int \frac{h_i(\alpha) \sin(\theta - \alpha)}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) \left[ 4(h_1(\theta) + h_1(\alpha)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) + 2 \epsilon (f_i(\alpha) - f_i(\alpha)) \left( h_1(\theta) - h_1(\alpha) \right) + 4 \epsilon (h_1(\theta) f_i(\alpha) + h_1(\alpha) f_i(\theta)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \right] d\alpha
\]

\[
- \epsilon h_i(\theta) \frac{P.V. \int h_i(\alpha) (1 - \cos(\theta - \alpha))}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) d\alpha
\]

\[
+ \epsilon \frac{1}{R_i(\theta)} P.V. \int \frac{h_i(\alpha) (1 - \cos(\theta - \alpha))}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) d\alpha
\]

\[
- \frac{1}{R_i(\theta)} P.V. \int \frac{h_i(\alpha) (1 - \cos(\theta - \alpha))}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) d\alpha
\]

\[
+ \frac{1}{R_i(\theta)} P.V. \int \frac{h_i(\alpha) (1 - \cos(\theta - \alpha))}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) d\alpha
\]

\[
- \frac{1}{R_i(\theta)} P.V. \int \frac{h_i(\alpha) (1 - \cos(\theta - \alpha))}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) d\alpha
\]

\[
+ \frac{1}{R_i(\theta)} P.V. \int \frac{h_i(\alpha) (1 - \cos(\theta - \alpha))}{A(x, \alpha) + \epsilon B(f_i, \alpha)} \hat{g}_{i,\epsilon}(\alpha) d\alpha
\]

\[
+ O(\epsilon),
\]

(3.15)

\[
\partial_{f_i} F_{i,1}(\epsilon, x, f_i, g_i) h_1 = O(\epsilon),
\]
\[
(3.16) \quad \partial_{e_i}F_{i,1}(\epsilon, x, f, g)h_2 = \text{P.V.} \int \frac{R_i(\alpha)A(\theta, \alpha)\sin(\alpha - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \frac{h_2(\alpha)}{h_2(\alpha) d\alpha} \\
+ \frac{\epsilon}{R_i(\theta)} \text{P.V.} \int \frac{f'_i(\theta)R_i(\alpha)(1 - \cos(\alpha - \alpha))}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} h_2(\alpha) d\alpha + \frac{\epsilon}{R_i(\theta)} \text{P.V.} \int \frac{f'_i(\theta)(f_i(\theta) - f_i(\alpha))}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} h_2(\alpha) d\alpha \\
- \frac{2\pi \epsilon}{R_i(\theta)} \int \nabla H(z_i(\theta), z_i(\alpha)) \cdot (R_i(\theta)(-\sin \theta, \cos \theta) + \epsilon f'_i(\theta)(\cos \theta, \sin \theta)) h_2(\alpha) d\alpha,
\]

\[
(3.17) \quad \partial_{e_i}F_{i,1}(\epsilon, x, f, g)h_2 = \frac{\epsilon}{R_i(\theta)} \int \frac{(x_i - x_j \cdot [R_i(\theta)(-\sin \theta, \cos \theta) + \epsilon f'_i(\theta)(\cos \theta, \sin \theta)]}{A_{ij} + \epsilon B_{ij}(\theta, \alpha)} h_2(\alpha) d\alpha \\
+ \frac{\epsilon}{R_i(\theta)} \int \frac{\epsilon^2 f'_i(\theta)R_i(\alpha)\cos(\alpha - \alpha) + \epsilon R_i(\theta)R_i(\alpha)\sin(\alpha - \alpha)}{A_{ij} + \epsilon B_{ij}(\theta, \alpha)} h_2(\alpha) d\alpha \\
- \frac{2\pi \epsilon}{R_i(\theta)} \int \nabla H(z_i(\theta), z_i(\alpha)) \cdot [R_i(\theta)(-\sin \theta, \cos \theta) + \epsilon f'_i(\theta)(\cos \theta, \sin \theta)] h_2(\alpha) d\alpha
= O(\epsilon),
\]

\[
(3.18) \quad \partial_{f_i}F_{i,2}(\epsilon, x, f, g)h_1 = -\frac{2\tilde{g}_{i,\epsilon}(\theta)(R_i(\theta)h_1(\theta) + f'_i(\theta)h'_1(\theta))}{(R_i(\theta)^2 + (R'_i(\theta))^2)^2} \text{P.V.} \int \frac{f'_i(\theta)R_i(\alpha)\sin(\alpha - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \tilde{g}_{i,\epsilon}(\alpha) d\alpha \\
+ \frac{\tilde{g}_{i,\epsilon}(\theta)}{R_i(\theta)^2 + (R'_i(\theta))^2} \text{P.V.} \int \frac{h'_i(\theta)R_i(\alpha)\sin(\alpha - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \tilde{g}_{i,\epsilon}(\alpha) d\alpha \\
- \frac{\tilde{g}_{i,\epsilon}(\theta)}{R_i(\theta)^2 + (R'_i(\theta))^2} \text{P.V.} \int \frac{f'_i(\theta)R_i(\alpha)\sin(\alpha - \alpha)}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \left[ 4(h_1(\theta) + h_1(\alpha)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \right] d\alpha \\
+ 2\epsilon(f_i(\theta) - f_i(\alpha))(h_1(\theta) - h_1(\alpha)) + 4\epsilon(h_1(\theta)f_i(\alpha) + h_1(\alpha)f_i(\theta)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \right] d\alpha \\
- \frac{2\tilde{g}_{i,\epsilon}(\theta)(R_i(\theta)h_1(\theta) + f'_i(\theta)h'_1(\theta))}{(R_i(\theta)^2 + (R'_i(\theta))^2)^2} \text{P.V.} \int \frac{R_i(\theta)R_i(\alpha)\cos(\alpha - \alpha) - 1}{A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha)} \tilde{g}_{i,\epsilon}(\alpha) d\alpha \\
+ \frac{\tilde{g}_{i,\epsilon}(\theta)}{R_i(\theta)^2 + (R'_i(\theta))^2} \text{P.V.} \int \frac{h_i(\theta)R_i(\alpha) + R_i(\theta)h_1(\alpha)}{(A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha))} \tilde{g}_{i,\epsilon}(\alpha) d\alpha \\
- \frac{\tilde{g}_{i,\epsilon}(\theta)}{\epsilon(R_i(\theta)^2 + (R'_i(\theta))^2)} \text{P.V.} \int \frac{R_i(\theta)R_i(\alpha)\cos(\alpha - \alpha) - 1}{(A(\theta, \alpha) + \epsilon B(f_i, \theta, \alpha))} \tilde{g}_{i,\epsilon}(\alpha) d\alpha \\
\times \left[ 4(h_1(\theta) + h_1(\alpha)) \sin \left( \frac{\theta - \alpha}{2} \right) + 2\epsilon(f_i(\theta) - f_i(\alpha))(h_1(\theta) - h_1(\alpha)) \\
+ 4\epsilon(h_1(\theta)f_i(\alpha) + h_1(\alpha)f_i(\theta)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \right] d\alpha
\[ - \frac{2\tilde{g}_{i,e}(\theta)(R_i(\theta)h_1(\theta) + \varepsilon f_i' \theta)h_1'(\theta))}{(R_i(\theta)^2 + (R_i')^2)^2} \int_{P.V.} R_i(\theta)(f_i(a) - f_i(\theta)) \frac{A(\theta,a) + \varepsilon B(f_i,\theta,a)}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} \tilde{g}_{i,e}(a)da \\
+ \frac{\tilde{g}_{i,e}(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{\varepsilon h_1(\theta)(f_i(a) - f_i(\theta)) + R_i(\theta)(h_1(a) - h_1(\theta))}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} \tilde{g}_{i,e}(a)da \\
- \frac{\tilde{g}_{i,e}(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{\varepsilon h_1(\theta)(f_i(a) - f_i(\theta)) + R_i(\theta)(h_1(a) - h_1(\theta))}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} \tilde{g}_{i,e}(a)da \\
+ 2\varepsilon(f_i(\theta) - f_i(a))(h_1(\theta) - h_1(a)) + 4\varepsilon(h_1(\theta)f_i(a) + h_1(a)f_i(\theta)) \sin^2 \left( \frac{\theta - \alpha}{2} \right) \] \\
+ O(\varepsilon),

\[(3.19) \quad \partial_{f_i} \mathcal{U}_{i,2}(\varepsilon, x, f, g) h_1 = O(\varepsilon),\]

\[(3.20) \quad \partial_{f_i} \mathcal{U}_{i,2}(\varepsilon, x, f, g) h_2 \]

\[= \frac{h_2(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{\varepsilon f_i'(\theta)R_i(a) \sin(\theta - a)}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} \tilde{g}_{i,e}(a)da \\
+ \frac{\tilde{g}_{i,e}(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{\varepsilon f_i'(\theta)R_i(a) \sin(\theta - a)}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} h_2(a)da \\
+ \frac{h_2(\theta)}{R(\theta)^2 + (R')^2} P.V. \int \frac{R_i(\theta)R_i(a)(\cos(\theta - a) - 1)}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} \tilde{g}_{i,e}(a)da \\
+ \frac{\tilde{g}_{i,e}(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{R_i(\theta)R_i(a)(\cos(\theta - a) - 1)}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} h_2(a)da \\
+ \frac{\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{R_i(\theta)(f_i(a) - f_i(\theta))}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} h_2(a)da \\
+ \frac{\varepsilon \tilde{g}_{i,e}(\theta)}{R_i(\theta)^2 + (R_i')^2} P.V. \int \frac{R_i(\theta)(f_i(a) - f_i(\theta))}{A(\theta,a) + \varepsilon B(f_i,\theta,a)} h_2(a)da \\
+ \frac{\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i')^2} \int \frac{(x_{i,a} - x_{j,a})^2 \cdot [R_i(\theta)(- \sin \theta, \cos \theta) + \varepsilon f_i'(\theta)(\cos \theta, \sin \theta)]}{A_{ij} + \varepsilon B_{ij}(\theta, a)} \tilde{g}_{i,e}(a)da \\
+ \frac{\varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i')^2} \int \frac{-\varepsilon R_i^2(\theta) + \varepsilon^2 f_i'(\theta)R_i(a) \sin(\theta - a) + \varepsilon R_i(\theta)R_i(a) \cos(\theta - a)}{A_{ij} + \varepsilon B_{ij}(\theta, a)} \tilde{g}_{i,e}(a)da \\
- \sum_{i,j} \frac{2\pi \varepsilon h_2(\theta)}{R_i(\theta)^2 + (R_i')^2} \int \nabla^\perp H(z_i(\theta), z_j(a)) \cdot (R_i(\theta)(- \sin \theta, \cos \theta) + \varepsilon f_i'(\theta)(\cos \theta, \sin \theta)) \times \tilde{g}_{i,e}(a)da \\
- \frac{2\pi \varepsilon \tilde{g}_{i,e}(\theta)}{R_i(\theta)^2 + (R_i')^2} \int \nabla^\perp H(z_i(\theta), z_j(a)) \cdot (R_i(\theta)(- \sin \theta, \cos \theta) + \varepsilon f_i'(\theta)(\cos \theta, \sin \theta)) h_2(a)da,\]
and

\begin{equation}
\partial_{g_{ij}}\mathcal{F}_{i,2}(\epsilon, x, f, g)h_2 = \frac{x_{g_{ij}}(\theta)}{R_j(\theta)^2 + (\kappa_j)^2} \left\{ \frac{f}{\kappa_j} \left[ R_i(\theta)(-\sin \theta \cos \theta) + \epsilon f_i'(\theta)(\cos \theta, \sin \theta) \right] - \kappa_j^2 \frac{f(\theta) - f(a)}{4 \sin^2 \left( \frac{\theta - a}{2} \right)} d\alpha \right\} h_2(\alpha) d\alpha
\end{equation}

\[ + \epsilon \kappa_j^2 f_i(\theta) \sin \theta \left( R_i(\theta) \right) \sin(\theta - a) \right\} h_2(\alpha) d\alpha
\]

\[ - 2\pi F \nabla^1 H(z_i(\theta), z_j(\alpha)) \cdot [R_i(\theta)(-\sin \theta, \cos \theta) + \epsilon f_i'(\theta)(\cos \theta, \sin \theta)] h_2(\alpha) d\alpha \}
\]

\[ = O(\epsilon).
\]

4 Linearization and isomorphism

In this section, we study the linearization of the functionals defined in Section 2. Denote \( \mathcal{F}_i := (\mathcal{F}_{i,1}, \mathcal{F}_{i,2}) \) and \( \mathcal{F} := (\mathcal{F}_1, \ldots, \mathcal{F}_m) \).

By (3.8) and (3.10), one can check that \((0, x, 0, 0)\) is a solution to \( \mathcal{F} = 0 \) if and only if \( x \) is a critical point of \( \mathcal{W}_m \). Now, we take \( x_0 \) to be a critical point of \( \mathcal{W}_m \), and hence \((0, x_0, 0, 0)\) is a solution to \( \mathcal{F} = 0 \). We study the linearization of \( \mathcal{F} \) at \((0, x_0, 0, 0)\).

According to (3.14)–(3.21) at the end of the proof of Proposition 3.2, when \( \epsilon = 0 \) and \( f, g \equiv 0 \), for all \( i = 1, \ldots, m \), the Gateaux derivatives are

\begin{equation}
\begin{cases}
\partial_{f_i} \mathcal{F}_{i,1}(0, x, 0, 0) f = \frac{\epsilon}{2} f_i'(\theta), \\
\partial_{f_j} \mathcal{F}_{i,1}(0, x, 0, 0) f = 0, \quad j \neq i, \\
\partial_{g_i} \mathcal{F}_{i,1}(0, x, 0, 0) g = f \frac{g(a) \sin(\theta - a)}{4 \sin^2 \left( \frac{\theta - a}{2} \right)} d\alpha, \\
\partial_{g_j} \mathcal{F}_{i,1}(0, x, 0, 0) g = 0, \quad j \neq i, \\
\partial_{f_i} \mathcal{F}_{i,2}(0, x, 0, 0) f = \kappa_i^2 f_i(\theta) - \kappa_i^2 \frac{f(\theta) - f(a)}{4 \sin^2 \left( \frac{\theta - a}{2} \right)} d\alpha, \\
\partial_{f_j} \mathcal{F}_{i,2}(0, x, 0, 0) f = 0, \quad j \neq i, \\
\partial_{g_i} \mathcal{F}_{i,2}(0, x, 0, 0) g = -\frac{\alpha_i}{2} g(\theta), \\
\partial_{g_j} \mathcal{F}_{i,2}(0, x, 0, 0) g = 0 \quad j \neq i.
\end{cases}
\end{equation}

Taking \((h_1, h_2) \in X^{k+1} \times X^k\), where

\begin{equation}
\begin{aligned}
 h_1(\theta) &= \sum_{j=1}^{\infty} (a_j \cos(j\theta) + b_j \sin(j\theta)), \\
 h_2(\theta) &= \sum_{j=1}^{\infty} (c_j \cos(j\theta) + d_j \sin(j\theta)),
\end{aligned}
\end{equation}

we will prove that the linearization of \( \mathcal{F}_i \) at \((0, x_0, 0, 0)\) has the following Fourier series form:

\begin{equation}
D_{(f_i, g_i)}\mathcal{F}_i(0, x_0, 0, 0)(h_1, h_2) := \begin{pmatrix}
\partial_{f_i} \mathcal{F}_{i,1}(0, x_0, 0, 0) h_1 + \partial_{g_i} \mathcal{F}_{i,1}(0, x_0, 0, 0) h_2 \\
\partial_{f_i} \mathcal{F}_{i,2}(0, x_0, 0, 0) h_1 + \partial_{g_i} \mathcal{F}_{i,2}(0, x_0, 0, 0) h_2
\end{pmatrix}
\end{equation}
\[(4.3) \quad \sum_{j=1}^{\infty} \left( \hat{a}_j \sin(j\theta) + \hat{b}_j \cos(j\theta) \right), \]

where
\[
\begin{pmatrix} \hat{a}_j \\ \hat{c}_j \end{pmatrix} = M_j \begin{pmatrix} a_j \\ c_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{b}_j \\ \hat{d}_j \end{pmatrix} = N_j \begin{pmatrix} b_j \\ d_j \end{pmatrix},
\]

with \(M_j\) and \(N_j\) two \(2 \times 2\) matrices given in Lemma 4.2.

To compute \(M_j\) and \(N_j\), we need the following identities.

**Lemma 4.1** For all \(j \geq 1\) and \(j \in \mathbb{N}^*\), there hold

\[(4.4) \quad \int \frac{\cos(j\alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha = \frac{1}{2} \sin(j\theta),\]

\[(4.5) \quad \int \frac{\sin(j\alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha = -\frac{1}{2} \cos(j\theta),\]

\[(4.6) \quad \int \frac{\cos(j\theta) - \cos(j\alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha = \frac{j}{2} \cos(j\theta),\]

\[(4.7) \quad \int \frac{\sin(j\theta) - \sin(j\alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha = \frac{j}{2} \sin(j\theta).\]

**Proof** Identities (4.4) and (4.6) were proved in Lemma A.8 [24]. Indeed, (4.4) can be deduced from the identity
\[
\int \frac{\cos(j\alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha = \frac{1}{2} \int \cos(j\alpha) \cot \left( \frac{\theta - \alpha}{2} \right) d\alpha = \frac{1}{2} \mathcal{H}(\cos(j\theta))(\theta),
\]

where \(\mathcal{H}(\cdot)\) is the Hilbert transform on torus and hence \(H(\cos(j\theta)) = \sin(j\theta)\). Identity (4.6) can be obtained by computing the fractional Laplacians
\[
\int \frac{\cos(j\theta) - \cos(j\alpha)}{4 \sin^2 \left( \frac{\theta - \alpha}{2} \right)} d\alpha = \frac{1}{2} (-\Delta)^{\frac{1}{2}} \cos(j\theta) = \frac{j}{2} \cos(j\theta).
\]

Finally, we point out that the identities (4.5) and (4.7) can be derived by calculating derivatives of (4.4) and (4.6), respectively. \(\square\)

Now, we can prove (4.3) and find the explicit formula for \(M_j\) and \(N_j\).

**Lemma 4.2** The derivative of \(\mathcal{F}_i\) at \((0, x_0, 0, 0)\) is given by (4.3) with

\[(4.8) \quad M_j = \begin{pmatrix} -\frac{\kappa_{ij}}{2} & 1/2 \\ (2-j)\kappa_i^2 & -\frac{\kappa_i}{2} \end{pmatrix}, \quad N_j = \begin{pmatrix} \frac{\kappa_{ij}}{2} & 1/2 \\ (2-j)\kappa_i^2 & -\frac{\kappa_i}{2} \end{pmatrix},\]

for any \(j \geq 1\).

Moreover, \(D_{(f,s)}\mathcal{F}_i(0, x_0, 0, 0)\) is an isomorphism from \(X_i^k\) to \(Y_i^k\) and \(D_{(f,g)}\mathcal{F}(0, x_0, 0, 0)\) is an isomorphism from \(X^k\) to \(Y^k\).\]
Proof Using (4.1), (4.2), and Lemma 4.1, we obtain by direct calculations

\[
\partial_{f_i} \mathcal{F}_{i,1}(0, x_0, 0, 0) h_1 = \sum_{j=1}^{\infty} \left( -\frac{\kappa_i j}{2} a_j \sin(j \theta) + \frac{\kappa_i j}{2} b_j \cos(j \theta) \right),
\]

\[
\partial_{g_i} \mathcal{F}_{i,1}(0, x_0, 0, 0) h_2 = \frac{1}{2} \sum_{j=1}^{\infty} (c_j \sin(j \theta) - d_j \cos(j \theta)),
\]

\[
\partial_{f_i} \mathcal{F}_{i,2}(0, x_0, 0, 0) h_1 = \kappa_i^2 \sum_{j=1}^{\infty} (a_j \cos(j \theta) + b_j \sin(j \theta)) - \kappa_i^2 \sum_{j=1}^{\infty} \left( \frac{\kappa_i j}{2} a_j \cos(j \theta) + \frac{\kappa_i j}{2} b_j \sin(j \theta) \right),
\]

and

\[
\partial_{g_i} \mathcal{F}_{i,2}(0, x_0, 0, 0) h_2 = -\frac{\kappa_i^2}{2} \sum_{j=1}^{\infty} (c_j \cos(j \theta) + d_j \sin(j \theta)).
\]

Then, one can easily check that the derivative of \( \mathcal{F}_i \) at \( (0, x, 0, 0) \) is given by (4.3) with

\[
M_j = \begin{pmatrix} -\frac{\kappa_i j}{2} & \frac{1}{2} \\ \frac{\kappa_i^2}{2} - \frac{\kappa_i j}{2} & \frac{\kappa_i^2}{2} \end{pmatrix}, \quad N_j = \begin{pmatrix} \frac{\kappa_i j}{2} & -\frac{1}{2} \\ \frac{\kappa_i^2}{2} + \frac{\kappa_i j}{2} & \frac{\kappa_i^2}{2} \end{pmatrix}.
\]

Now we are going to prove that \( D_{(f_i, g_i)} \mathcal{F}_i(0, x_0, 0, 0) \) is an isomorphism from \( X_k^i \) to \( Y_k^i \). Recall the definition of \( X_k^i \) and \( Y_k^i \) given at the end of Section 2. From the above calculations, one has \( M_1 = \begin{pmatrix} -\kappa_i/2 & 1/2 \\ \kappa_i^2/2 - \kappa_i/2 & \kappa_i^2/2 - \kappa_i/2 \end{pmatrix} \) and \( N_1 = \begin{pmatrix} \kappa_i/2 & -1/2 \\ \kappa_i^2/2 + \kappa_i/2 & \kappa_i^2/2 - \kappa_i/2 \end{pmatrix} \), then it is obvious that \( D_{(f_i, g_i)} \mathcal{F}_i(0, x_0, 0, 0) \) maps \( X_k^i \) to \( Y_k^i \). Hence, only the invertibility needs to be considered.

For \( j \geq 2 \), \( \det(M_j) = -\det(N_j) = \frac{\kappa_i^2(j-1)}{2} > 0 \) which implies that \( M_j \) and \( N_j \) are invertible, and their inverse are given by

\[
M_j^{-1} = \begin{pmatrix} \frac{\kappa_i^2(j-1)}{2j-2} & -1 \\ \frac{\kappa_i(j-1)}{2j-2} & \frac{\kappa_i^2(j-1)}{2j-2} \end{pmatrix}, \quad \forall \ j \geq 2,
\]

and

\[
N_j^{-1} = \begin{pmatrix} \frac{\kappa_i^2(j-1)}{2j-2} & -1 \\ \frac{\kappa_i(j-1)}{2j-2} & \frac{\kappa_i^2(j-1)}{2j-2} \end{pmatrix}, \quad \forall \ j \geq 2.
\]

Thus, for any \( (u, v) \in Y_k^i \) with

\[
u = -\kappa_i p_1 \cos(\theta) + \kappa_i q_1 \sin(\theta) + \sum_{j=2}^{\infty} r_j \cos(j \theta) + s_j \sin(j \theta),
\]

\[
u = -\kappa_i p_1 \cos(\theta) + \kappa_i q_1 \sin(\theta) + \sum_{j=2}^{\infty} r_j \cos(j \theta) + s_j \sin(j \theta),
\]

and

\[
u = -\kappa_i p_1 \cos(\theta) + \kappa_i q_1 \sin(\theta) + \sum_{j=2}^{\infty} r_j \cos(j \theta) + s_j \sin(j \theta),
\]

and
we can write $D_{(f_0, g_0)} \tilde{\mathcal{F}}_j(0, x_0, 0, 0)^{-1}(u, v)$ as

$$D_{(f_0, g_0)} \tilde{\mathcal{F}}_i(0, x_0, 0, 0)^{-1}(u, v) = \left( \begin{array}{c}
-\frac{p_1}{\kappa_i} \cos(\theta) + \frac{q_1}{\kappa_i} \sin(\theta) \\
p_1 \cos(\theta) - q_1 \sin(\theta)
\end{array} \right) + \sum_{j=2}^{\infty} M_j^{-1} \left( \begin{array}{c}
p_j \\
r_j
\end{array} \right) \cos(j\theta) + N_j^{-1} \left( \begin{array}{c}
q_j \\
s_j
\end{array} \right) \sin(j\theta).
$$

Denote

$$\left( \begin{array}{c}
\hat{p}_j \\
\hat{r}_j
\end{array} \right) = M_j^{-1} \left( \begin{array}{c}
p_j \\
r_j
\end{array} \right), \quad \left( \begin{array}{c}
\hat{q}_j \\
\hat{s}_j
\end{array} \right) = N_j^{-1} \left( \begin{array}{c}
q_j \\
s_j
\end{array} \right), \quad \forall \ j \geq 2.
$$

From (4.9) and (4.10), we have the asymptotic behavior: $\hat{p}_j = O(j^{-1}(|p_j| + |r_j|))$, $\hat{r}_j = O(|p_j| + |r_j|)$, $\hat{q}_j = O(j^{-1}(|q_j| + |s_j|))$ and $\hat{s}_j = O(|q_j| + |s_j|)$ as $j \to +\infty$, which implies that $D_{(f_0, g_0)} \tilde{\mathcal{F}}_i(0, x, 0, 0)^{-1}(u, v)$ does belong to $X^k$.

Noticing that by (4.1), we have $\partial_{f_j} \tilde{\mathcal{F}}_{i,1}(0, x_0, 0, 0) h_1$, $\partial_{g_j} \tilde{\mathcal{F}}_{i,1}(0, x_0, 0, 0) h_2$, $\partial_{f_j} \tilde{\mathcal{F}}_{i,2}(0, x_0, 0, 0) h_1$, and $\partial_{g_j} \tilde{\mathcal{F}}_{i,2}(0, x_0, 0, 0) h_2 = 0$, $j \neq i$. Therefore, we find

$$D_{(f, g)} \tilde{\mathcal{F}}(0, x_0, 0, 0) = \text{diag} \left( D_{(f_0, g_0)} \tilde{\mathcal{F}}_1(0, x_0, 0, 0), \ldots, D_{(f_m, g_m)} \tilde{\mathcal{F}}_m(0, x_0, 0, 0) \right),$$

and hence $D_{(f, g)} \tilde{\mathcal{F}}(0, x_0, 0, 0)$ is an isomorphism from $X^k$ to $Y^k$.

The proof of is thus completed.

5 Existence of vortex sheets

In this section, inspired by the classical Crandall–Rabinowitz theorem on bifurcation theory, we use the implicit function theorem to obtain a branch of solutions for arbitrarily fixed small $\varepsilon$.

From the previous sections, we know that $(0, x_0, 0, 0)$ is a solution to $\mathcal{F} = 0$ if and only if $x_0$ is a critical point of $\mathcal{W}_m$. Moreover, $D_{(f, g)} \tilde{\mathcal{F}}(0, x_0, 0, 0)$ is an isomorphism from $X^k$ to $Y^k$. It can be seen from Lemma 4.2 that the kernel of $D_{(f, g)} \tilde{\mathcal{F}}(0, x_0, 0, 0)$ in $(X^{k+1} \times X^k)^m$ is

$$\prod_{i=1}^{m} \{ (a \cos(\theta) + b \sin(\theta), \kappa_i (a \cos(\theta) + b \sin(\theta))) \mid (a, b) \in \mathbb{R}^2 \}.$$

We take arbitrary nontrivial $(f_0, g_0) \in X^k_0$ and define the following new functional:

$$\mathcal{F}(\varepsilon, \tau, x, f, g) := \mathcal{F}(\varepsilon, x, f + \tau f_0, g + \tau g_0).$$

To apply the implicit function theorem, we need to make sure that $\mathcal{F}$ maps a suitable subset of $X^k$ into $Y^k$. This aim will be achieved by choosing $x$ properly. Indeed, letting $V_1 := \{ (f, g) \in X^k \mid \sum_{i=1}^{m} (\|f_i\|_{H^{k+1}} + \|g_i\|_{H^k}) < 1 \} \subset X^k$ be the unit ball, we have the following key proposition.

**Proposition 5.1** The condition that $\mathcal{F}$ maps $(-\varepsilon_0, \varepsilon_0) \times (-\tau_1, \tau_1) \times B_{\varepsilon_0}(x_0) \times V_1$ into $Y^k$ is equivalent to a system of $2m$ equations of the form

$$\nabla \mathcal{W}_m(x) = O_{\tau_1}(\varepsilon),$$

where $O_{\tau_1}(\varepsilon)$ denotes a term that is negligible in the limit $\varepsilon \to 0$.\]
where \( \tau_1 \) is any fixed small positive number and \( O_{\tau_1}(\varepsilon) \) means a vector that is of the order \( \varepsilon \) up to a constant depending on \( \tau_1 \).

**Proof** For arbitrary \( i = 1, \ldots, m \), we take \((f, g) \in V_i \) with

\[
f_i(\theta) = \sum_{j=1}^{\infty} (a_j \cos(j\theta) + b_j \sin(j\theta)),
\]

\[
g_i(\theta) = -\kappa_i a_1 \cos(\theta) - \kappa_i b_1 \sin(\theta) + \sum_{j=2}^{\infty} (\epsilon_j \cos(j\theta) + d_j \sin(j\theta)).
\]

By the definition of \( \mathcal{Y}^k \), in order to make \( S(e_1, \tau, x, f, g) = \mathcal{F}(e_1, f, + \tau f_0, g + \tau g_0) \in \mathcal{Y}^k \), we need to ensure that the following equations hold true:

\[
-\kappa_i \int S_{i,1}(e, x, f + \tau f_0, g + \tau g_0) \sin(\theta) d\theta = \int S_{i,2}(e, x, f + \tau f_0, g + \tau g_0) \cos(\theta) d\theta,
\]

\[
\kappa_i \int S_{i,1}(e, x, f + \tau f_0, g + \tau g_0) \cos(\theta) d\theta = \int S_{i,2}(e, x, f + \tau f_0, g + \tau g_0) \sin(\theta) d\theta,
\]

where \( i = 1, \ldots, m \). By (3.7), (3.9), and calculations in Lemmas 4.1 and 4.2, we obtain

\[
\int S_{i,1}(e, x, f + \tau f_0, g + \tau g_0) \sin(\theta) d\theta = -\kappa_i a_1 - \sum_{j \neq i} 2\pi \kappa_j \partial_{x_{i,j}} G(x_i, x_j) + 2\pi \kappa_i \partial_{x_{i,i}} H(x_i, x_i) + \varepsilon f R_1 \sin(\theta) d\theta,
\]

\[
\int S_{i,1}(e, x, f + \tau f_0, g + \tau g_0) \cos(\theta) d\theta = \kappa_i b_1 + \sum_{j \neq i} 2\pi \kappa_j \partial_{x_{i,j}} G(x_i, x_j) - 2\pi \kappa_i \partial_{x_{i,i}} H(x_i, x_i) + \varepsilon f R_1 \cos(\theta) d\theta,
\]

\[
\int S_{i,2}(e, x, f + \tau f_0, g + \tau g_0) \sin(\theta) d\theta = \kappa_i^2 b_1 - \sum_{j \neq i} 2\pi \kappa_j \partial_{x_{i,j}} G(x_i, x_j) + 2\pi \kappa_i^2 \partial_{x_{i,i}} H(x_i, x_i) + \varepsilon f R_2 \sin(\theta) d\theta,
\]

\[
\int S_{i,2}(e, x, f + \tau f_0, g + \tau g_0) \cos(\theta) d\theta = \kappa_i^2 a_1 - \sum_{j \neq i} 2\pi \kappa_j \partial_{x_{i,j}} G(x_i, x_j) + 2\pi \kappa_i^2 \partial_{x_{i,i}} H(x_i, x_i) + \varepsilon f R_2 \cos(\theta) d\theta.
\]

Then, by the above equations (5.4)–(5.7), we conclude that (5.3) is equivalent to the following equations:

\[
\sum_{j \neq i} \kappa_i \kappa_j \partial_{x_{i,j}} G(x_i, x_j) - \kappa_i^2 \partial_{x_{i,i}} H(x_i, x_i) = \frac{\varepsilon}{4\pi} \left( \int R_2 \cos(\theta) d\theta + \kappa_i \int R_1 \sin(\theta) d\theta \right)
\]
and
\[
\sum_{j \neq i} \kappa_i \kappa_j \partial_{x_{i,j}} G(x_i, x_j) - \kappa_i^2 \partial_{x_{i,i}} H(x_i, x_i) = \frac{\varepsilon}{4\pi} \left( \int R_2 \sin(\theta) d\theta - \kappa_i \int R_1 \cos(\theta) d\theta \right).
\]

Since (5.8) and (5.9) hold for all \( i = 1, \ldots, m \), we arrive at (5.2) and complete the proof of Proposition 5.1. 

Now, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** Since we have the nondegeneracy condition \( \text{deg} \left( \nabla W_m, x_0 \right) \neq 0 \), equation (5.2) is solvable near \( x_0 \) whenever \( \varepsilon \) is small. We solve (5.2) and write the solution \( x_{\varepsilon, \tau} \) in the form \( x_{\varepsilon, \tau} = x_0 + \varepsilon \mathcal{R}_x(\varepsilon, \tau, f, g) \). Then, we know that \( \mathcal{R}_x \) defined on \( (-\varepsilon_0, \varepsilon_0) \times (-\tau_1, \tau_1) \times V_1 \) is at least of \( C^1 \) due to the regularity of \( \mathcal{F} \).

Now, set
\[
\mathcal{F}^*(\varepsilon, \tau, f, g) := \mathcal{F}(\varepsilon, \tau, x_0 + \varepsilon \mathcal{R}_x(\varepsilon, \tau, f, g), f, g).
\]

Then, we conclude from Proposition 5.1 that \( \mathcal{F}^* \) maps \( (-\varepsilon_0, \varepsilon_0) \times (-\tau_1, \tau_1) \times V_1 \) into \( \mathfrak{y}^k \). Moreover, \( \mathcal{F}^* \) is \( C^1 \) continuous with respect to \( f \) and \( g \). Next, we need to verify that \( D_{(f, g)} \mathcal{F}^*(0, 0, 0, 0) \) is an isomorphism from \( \mathcal{X}^k \) to \( \mathfrak{y}^k \). In fact, by the chain rule, we get
\[
D_{(f, g)} \mathcal{F}^* = D_{(f, g)} \mathcal{F} + D_x \mathcal{F} \cdot D_{(f, g)} \left( x_0 + \varepsilon \mathcal{R}_x(\varepsilon, \tau, f, g) \right),
\]
which implies
\[
D_{(f, g)} \mathcal{F}^*(0, 0, 0, 0) = D_{(f, g)} \mathcal{F}(0, x_0, 0, 0).
\]

Therefore, \( D_{(f, g)} \mathcal{F}^*(0, 0, 0, 0) \) is an isomorphism from \( \mathcal{X}^k \) to \( \mathfrak{y}^k \) by Lemma 4.2.

Now, applying implicit function theorem to \( \mathcal{F}^* \) at the point \( (0, 0, 0, 0) \), we obtain that there exist \( \varepsilon_0 > 0 \) and \( 0 < \tau_0 \leq \tau_1 \) such that the solutions set
\[
\left\{ (\varepsilon, \tau, f, g) \in (-\varepsilon_0, \varepsilon_0) \times (-\tau_0, \tau_0) \times V_1 : \mathcal{F}^*(\varepsilon, \tau, f, g) = 0 \right\}
\]
is not empty and can be parameterized by a two-dimensional surface \( (\varepsilon, \tau) \in (-\varepsilon_0, \varepsilon_0) \times (-\tau_0, \tau_0) \rightarrow (\varepsilon, \tau, f_{\varepsilon, \tau, i}(\theta), g_{\varepsilon, \tau, i}) \). So we obtain a family of nontrivial vortex sheet solutions and finish the proof of (i) in Theorem 1.1.

Since (ii) of Theorem 1.1 is obvious, to end our proof, we only need to show the convexity of the interior of \( \Gamma_i \) for \( i = 1, \ldots, m \). This can be done by computing the sign of the curvature. Recall that \( z_i(\theta) = x_{\varepsilon, \tau, i} + \varepsilon R_i(\theta)(\cos \theta, \sin \theta) \) with \( R_i(\theta) = 1 + \varepsilon (f_{\varepsilon, \tau, i}(\theta) + \tau f_{0, i}) \). Given \( \theta \in [0, 2\pi] \), the signed curvature of \( \Gamma_i \) at \( z_i(\theta) \) is
\[
\varepsilon \kappa(\theta) = \frac{R_i(\theta)^2 + 2R_i'(\theta)^2 - R_i(\theta)R_i''(\theta)}{(R_i(\theta)^2 + R_i'(\theta)^2)^{\frac{3}{2}}} = \frac{1 + O(\varepsilon)}{1 + O(\varepsilon)} > 0,
\]
for \( \varepsilon \) and \( \tau \) small, which implies the convexity and thus completes the proof of Theorem 1.1. 

\[\Box\]
We point out that for fixed \( \epsilon \), if \( \tau_1 \neq \tau_2 \) with \( 0 < \tau_1 < \tau_2 < \tau_0 \), then obviously one has \( \omega_{\epsilon, \tau_1} \neq \omega_{\epsilon, \tau_2} \). Thus, we have obtained a large family of stationary solutions with vortex sheet for every \( \epsilon > 0 \) small.

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