Geodesically Complete Universe

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This talk is about solving cosmological equations analytically without approximations, and discovering new phenomena that could not be noticed with approximate solutions. We found all the solutions of the Friedmann equations for a specific model, including all the zero-size-bounce solutions that do not violate the null energy condition, as well as all the finite-size-bounce solutions, and then discovered model independent phenomena. Among them is the notion of geodesic completeness for the geometry of the universe. From this we learned a few new general lessons for cosmology. Among them is that anisotropy provides a model independent attractor mechanism to some specific initial values for cosmological fields, and that there is a period of antigravity in the history of the universe. The results are obtained only at the classical gravity level. Effects of quantum gravity or string theory are unknown, they are not even formulated, so there are new theoretical challenges.

I. INTRODUCTION

This work started with the discovery of new techniques for solving the Friedmann equations analytically [1], and finding the complete set of solutions for the gravitational system that includes a scalar field $\sigma(x^\mu)$ with a special, but still typical, potential $V(\sigma)$ in a cosmological model [1][2]. The special techniques were direct outcomes from 2T-physics [3] and 2T-gravity [4][5]. This led to an emergent formulation of gravity in 1T-physics with an additional local conformal (Weyl) symmetry. The cosmological model that emerged in this approach had a strong overlap with the cosmological colliding branes scenario [6][7] developed from worldbrane notions [8] in M-theory [9]. This connection led to a fruitful collaboration between our group, that includes C.H. Chen, and the group that includes Paul Steinhardt and Neil Turok. Our collaborative work has partially appeared in [8] in M-theory [9]. Here I will summarize results that include some that are about to be published in [11].

Our approach begins with the standard action typically used in cosmological models that describe a scalar field $\sigma(x^\mu)$ minimally coupled to gravity

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} R(g) - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right\} + \text{radiation + matter}$$

Although not necessary, the scalar field $\sigma$ could be identified with the dilaton in string theory, written in the Einstein frame. For the purpose of studying cosmology, the relevant degrees of freedom are homogeneous fields that are only time dependent. The space dependence is ignored in the leading approximation. It can be argued [12] that the space gradients are responsible for non-leading effects as compared to the dominant effects parameterized by only time dependent fields that we study here (including anisotropy).

Our cosmological analysis of this theory involves geometries with or without spacial curvature, as well as anisotropy, and includes radiation and other relativistic matter approximated as relativistic dust described by an energy-momentum tensor consistent with conformal symmetry. The geometries we study have the form

$$ds^2 = -dt^2 + a^2(t) d\tau^2 + ds_3^2 = a^2(\tau) \left( -d\tau^2 + ds_3^2 \right),$$

where $a^2(\tau)$ is the scale factor that tracks the size of the universe. We will use conformal time $\tau$ which is related to cosmic time $t$, by $dt = a(\tau) d\tau$, as in Eq.(2). The 3-dimensional part $ds_3^2$ of the geometries we study include

- the isotropic FRW universe with or without curvature $K$,

$$\left(ds_3^2\right)_{FRW} = \frac{dr^2}{1-Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2); \quad K = \frac{k}{r_0^2},$$

- the anisotropic Kasner metric with zero curvature

$$\left(ds_3^2\right)_{Kasner} = e^{-2\sqrt{2/3}\kappa_1} (dz)^2 + e^{2\sqrt{2/3}\kappa_1} \left( e^{\sqrt{2\kappa_2}} (dx)^2 + e^{-\sqrt{2\kappa_2}} (dy)^2 \right),$$

where $\alpha_1(\tau)$ and $\alpha_2(\tau)$ are two dynamical degrees of freedom in the metric along with $a(\tau)$, and
• the anisotropic Bianchi IX metric with curvature $K$

\[
(ds^2)_{IX} = e^{-2\sqrt{2/3}\kappa\alpha_1}(d\sigma_z)^2 + e^{\sqrt{2/3}\kappa\alpha_1}\left(e^{\sqrt{2/3}\kappa\alpha_2}(d\sigma_x)^2 + e^{-\sqrt{2/3}\kappa\alpha_2}(d\sigma_y)^2\right),
\]

where $d\sigma_i$, which are given in [13], depend on space coordinates (and $K$) but not on $\tau$. The Bianchi IX metric reduces to the FRW metric with curvature $K$ in the isotropic limit $\alpha_i \to 0$, and it reduces to the anisotropic Kasner metric when the curvature vanishes $K \to 0$.

The relevant degrees of freedom are $\sigma (\tau), a_E (\tau), \alpha_1 (\tau), \alpha_2 (\tau)$. Here we have attached the label $E$ on the scale factor $a_E (\tau)$ to emphasize that it is defined in the Einstein frame. We do this because we will also define other frames to solve the equations. The equations of motion (namely the Friedmann-type equations) are obtained by computing $R_{\mu\nu}$ energy term for the anisotropy involving $\frac{\partial}{\partial \tau}$.

Eventually we work in the gauge $\frac{\partial}{\partial \tau}$ of motion for $e$ and $\frac{\partial}{\partial \tau}$.

For our model we chose the following potential energy

\[
V (\sigma) = \left(\frac{6}{\kappa^2}\right)^2 \left(c \sinh^4 \left(\frac{\kappa^2}{6} \sigma\right) + b \cosh^4 \left(\frac{\kappa^2}{6} \sigma\right)\right),
\]

After insuring stability by requiring $(b + c) > 0$, the plot of this potential, which looks like a well or double well for various values and signs of $b, c$, shows that it has features that are similar to other potentials used in cosmological applications. The reason for choosing this specific potential is that we can use some special tricks to solve the Friedmann equations exactly in the FRW universe and find all of the solutions analytically [1-11]. We can similarly solve the equations analytically for a few other potentials, as presented in a future publication.

II. BCST TRANSFORMATION AND WEYL SYMMETRY

The tricks for solving the equations are actually deeply related to the gauge symmetries and shadow phenomena in 2T-Physics [3-11]. But, in the current setting with the potential $V (\sigma)$ in (8), it is possible to introduce them directly as the Bars-Chen-Steinhardt-Turok (BCST) transformation of the fields $(a_{E}^2, \sigma)$ to a new basis $(\phi_s, s_s)$ as follows

\[
a_{E}^2 = |z|, \quad z = \frac{\kappa^2}{6}(\phi_s^2 - s_s^2), \quad \sigma = \frac{\sqrt{6}}{4\kappa} \ln \left(\frac{(\phi_s + s_s)^2}{(\phi_s - s_s)^2}\right).
\]
Then the action $S^E_{\text{eff}}$ in Eq. (9) is rewritten in the new basis as follows

$$
S^E_{\text{eff}} = \int d\tau \left\{ \frac{1}{\alpha} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} b^2 + \frac{3}{2} \left( \phi^2 - s^2_\gamma \right) \left( \dot{\phi}^2 + \dot{s}^2_\gamma \right) \right] - e \cdot \left( \phi^2 f(s_\gamma/\phi_\gamma) + \rho_0 - \frac{3}{2} \left( \phi^2 - s^2_\gamma \right) v(\alpha_1, \alpha_2) \right) \right\},
$$

(10)

where $\phi^2 f(s_\gamma/\phi_\gamma) = b\phi^2 + cx^2$ when $V(\sigma)$ is given as in Eq. (8).

In the absence of anisotropy, $\alpha_i \to 0$, $v(\alpha_1, \alpha_2) \to 1$, this form of the action, with the special potential, shows that the fields $(\phi_\gamma, s_\gamma)$ are decoupled from each other, except for the zero energy condition that results from the $e$-equation of motion. The decoupled second order field equations for $\phi_\gamma(\tau)$ and $s_\gamma(\tau)$ are easily solved because they are analogous to a particle in a potential. The solutions are given in terms of Jacobi elliptic functions. The zero energy condition merely relates the integration variables $E_{\phi}, E_s$ (energy levels for $\phi_\gamma, s_\gamma$) in the form $E_{\phi} = E + \rho_0$ and $E_s = E$, so that there is only one integration variable, $E$, and two initial values $\phi_\gamma(\tau_0), s_\gamma(\tau_0)$. Due to time translation symmetry one of the initial values has no physical significance; therefore $E, \phi(\tau_0)$ are the only two integration parameters. Together with the 4 parameters of the model $(b, c, K, \rho_0)$, these 6 parameters determine the properties of all the solutions in the absence of anisotropy.

This shows that the BCST transformation makes the Friedmann equations solvable analytically and leads to the complete set of solutions given in [1][10][11] as outlined below.

Fig.1- Gravity $\phi^2 > s^2$ in left/right quadrants. Antigravity $\phi^2 < s^2$ in top/bottom quadrants.

Eq. (9) is only half of the BCST transformation between the $(a^s_E, \sigma)$ and $(\phi_\gamma, s_\gamma)$ bases. The form in Eq. (9) is initially defined only for the region of the $(\phi_\gamma, s_\gamma)$ plane that satisfies $(\phi^2_\gamma - s^2_\gamma) > 0$, which corresponds to the left and right quadrants as depicted in Fig.1. The cosmological singularity at $a^s_E = 0$ corresponds to the 45° lines, which may be considered a “lightcone” in the $(\phi_\gamma, s_\gamma)$ space. The study of geodesic completeness at the $a^s_E = 0$ singularity amounts to the question of how geodesics cross the lightcone in $(\phi_\gamma, s_\gamma)$ field space. Through our explicit analytic solutions [1][10][11] we learned that generic trajectories in the absence of anisotropy cross this lightcone and continue to develop in the entire $(\phi_\gamma, s_\gamma)$ plane. Therefore, geodesic completeness leads us to expand the domain of the Einstein frame beyond the $(\phi^2_\gamma - s^2_\gamma) > 0$ region by including the full $(\phi_\gamma, s_\gamma)$ plane. This extension of the field domain, including the top and bottom quadrants, reminds us of a similar extension of spacetime (rather than field space) via the Kruskal-Szekeres coordinates of the Schwarzschild blackhole.

To complete the BCST transformation, we also need the inverse transformation which involves the four quadrants in the $(\phi_\gamma, s_\gamma)$ space. This is given by

$$
\phi_\gamma = \pm \begin{cases} 
\frac{\sqrt{3}}{\alpha} \sqrt{|z|} \cosh \left( \frac{\alpha z}{\sqrt{6}} \right), & \text{if } z > 0 \\
\frac{\sqrt{3}}{\alpha} \sqrt{|z|} \sinh \left( \frac{\alpha z}{\sqrt{6}} \right), & \text{if } z < 0
\end{cases},
$$

$\gamma$

$$
s_\gamma = \pm \begin{cases} 
\frac{\sqrt{3}}{\alpha} \sqrt{|z|} \cosh \left( \frac{\alpha z}{\sqrt{6}} \right), & \text{if } z > 0 \\
\frac{\sqrt{3}}{\alpha} \sqrt{|z|} \sinh \left( \frac{\alpha z}{\sqrt{6}} \right), & \text{if } z < 0
\end{cases},
$$

(11)

Actually the $\phi^2_\gamma > s^2_\gamma$ and $\phi^2_\gamma < s^2_\gamma$ field domains are separated from each other by the spacetime singularity at $a^s_E = 0$; so $\phi^2_\gamma > s^2_\gamma$ is satisfied in a patch of spacetime, while $\phi^2_\gamma < s^2_\gamma$ is satisfied in a different patch of spacetime. These patches communicate with each other through the cosmological singularity, or equivalently through the whole lightcone in the $(\phi_\gamma, s_\gamma)$ plane.

Allowing $(\phi_\gamma, s_\gamma)$ to be extended to the full $(\phi_\gamma, s_\gamma)$ plane, including the top and bottom quadrants (as guided by the solutions), defines a continuation of the Einstein frame to a larger domain of field space. We found that
the full \((\phi, s, \gamma)\) provides a geodesically complete geometry, while the domain \(\phi^2 > s^2\) by itself is geodesically incomplete. We find that in the extended domain the physics described by \(S_{\text{eff}}^\gamma\) includes patches of spacetime in which there is gravity when \(\phi^2 > s^2\), or antigravity when \(\phi^2 < s^2\). The actions \(S_{\text{eff}}^\gamma, S_{\text{eff}}^E\) are equivalent to each other via the BCST transformation only in the sector in which \((\phi^2 - s^2)\) is positive. In the antigravity patches the inverse BCST transformation produces again an Einstein frame, and an action similar to \(S_{\text{eff}}\), but with a negative Newton constant.

To better understand the meaning of the different patches we need the complete theory, not only the effective theory \(S_{\text{eff}}\) for the cosmological degrees of freedom. This is given from first principles by a slight extension of the Einstein action by including only gauge degrees of freedom associated with a local scaling symmetry, called Weyl symmetry. The local scaling transformation, given by

\[
(\phi(x), s(x)) \to (\phi(x), s(x)) e^{\lambda(x)} \quad \text{and} \quad g_{\mu\nu}(x) \to g_{\mu\nu}(x) e^{-2\lambda(x)},
\]

(12)
does not allow the usual Einstein-Hilbert term, but allows conformally coupled scalars. The following action contains two conformally coupled scalars \(\phi, s\), interacting with the curvature term with the coefficient \(\frac{1}{12}\) dictated by the gauge symmetry

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g^{\mu\nu} \partial_{\mu} s \partial_{\nu} s + \frac{1}{12} (\phi^2 - s^2) R(g) - \phi^2 f \left( \frac{s}{\phi} \right) \right).
\]

(13)

Here \(f(z)\) is an arbitrary function of the scale invariant ratio \(z \equiv \frac{s}{\phi}\). Fermionic and gauge fields, as well as more conformally coupled scalars, can be added, as part of a complete model, without breaking the conformal gauge symmetry \(E\). Here \((\phi^2 - s^2)/12\) appears as a gravitational parameter that replaces the Newton constant \(1/2\kappa^2\).

To have the possibility for it to be positive, we must have one scalar, namely \(\phi\), to have the wrong sign kinetic term. If \(\phi\), like \(s\), has the correct sign, then we would end up with a purely negative gravitational parameter. Hence the relative minus sign in \((\phi^2 - s^2)\) is required. The wrong sign kinetic term makes \(\phi\) potentially a ghost. However, the local gauge symmetry compensates for the ghost, thus insuring unitarity of the theory.

The usual Einstein gravity in Eq. (1) can be obtained by choosing a gauge for the local scale symmetry. In particular, in the Einstein gauge we denote the fields by \(\phi_E(x), s_E(x), g_{\mu\nu}^{E}(x)\) with a subscript \(E\), and we choose \(\frac{1}{12} (\phi_E^2(x) - s_E^2(x)) = \frac{\kappa^2}{12}\) for all spacetime \(x^\mu\), thus generating the gravitational constant. Then writing

\[
\phi_E(x) = \pm \sqrt{\kappa^2 \left( \frac{\kappa \sigma(x)}{\sqrt{6}} \right)}, \quad s_E(x) = \pm \sqrt{\kappa^2 \sinh \left( \frac{\kappa \sigma(x)}{\sqrt{6}} \right)},
\]

we find that the conformally gauge invariant action \(S_{\text{eff}}^\gamma\) yields the familiar action in Eq. (1) with an Einstein-Hilbert term \(\frac{1}{2\kappa^2} R(g_E)\) and a minimally coupled scalar field \(\sigma\), and any potential energy \(V(\sigma)\).

Another convenient gauge choice is the \(\gamma\)-gauge in which we denote the fields by \(\phi_{\gamma}(x), s_{\gamma}(x)\) with a subscript \(\gamma\). In this gauge we choose the conformal factor of the metric \(a_E^2(x^\mu) = 1\) for all spacetime \(\{x^\mu\}\).

In particular, in the FRW, Kasner or Bianchi IX spacetimes of Eqs. (25) taken in the gamma gauge \(a_E^2(\tau) = 1\), we obtain simple expressions for the curvature, such as \(R(g_{\gamma}^{\text{FRW}}) = 6K\), etc. Inserting these expressions in the full action in Eq. (13) we derive the gauge fixed action \(S_{\text{eff}}^\gamma\) for cosmological applications, precisely as given in Eq. (10).

It is useful to point out that the following quantity is gauge invariant in all spacetime

\[
z(x^\mu) = \frac{\kappa^2}{6} (-g(x))^\frac{1}{2} (\phi^2(x) - s^2(x)).
\]

In the Einstein gauge in the gravity sector given in Eq. (14) it is equal to \(z(x^\mu) = (-g(x))^\frac{1}{2} = a_E^2(x)\), while in the Einstein gauge in the antigravity sector \((\text{interchanging cosh and sinh in Eq. (14)})\) it is equal to \(z(x^\mu) = -(-g(x))^\frac{1}{2} = -a_E^2(x)\). Furthermore, in the \(\gamma\)-gauge it is equal to \(z(x^\mu) = \frac{\kappa^2}{6} (\phi_{\gamma}^2 - s_{\gamma}^2)\). Equating these quantities to each other by gauge invariance immediately gives part of the BCST transformation in Eq. (9). This shows that the BCST transformation in Eqs. (9, 11) is actually the gauge transformation from the Einstein gauge (both gravity and antigravity regions) to the \(\gamma\)-gauge and vice-versa.

Now it is clear that the \(\gamma\)-gauge of the action \(S_{\text{eff}}^\gamma\) is valid in the entire \((\phi, s)\) plane in all spacetime, for all values of the gauge invariant \(z(x)\). But the usual Einstein theory as given in Eq. (1) is possible only in the left and right quadrants of Fig.1 where \((\phi_{\gamma}^2 - s_{\gamma}^2) > 0\), or equivalently when the gauge invariant \(z(x)\) is positive. To give an Einstein frame description of the top and bottom quadrants of Fig.1, where \((\phi^2 - s^2) < 0\)
and the gauge invariant $z(x)$ is negative, we need an Einstein gauge just like Eq. (14) by interchanging cosh with sinh. The result of gauge fixing in the region $(\phi^2 - s^2) < 0$ is an Einstein action just like (1), but with the gravitational constant $1/2\kappa^2$ replaced by $-1/2\kappa^2$, and the wrong sign kinetic term for the sigma field. Evidently, in this system there is antigravity rather than gravity since the Newton constant has the opposite sign.

Comparing the gravity/antigravity regions further in their respective Einstein frames, note that the potential energy $V(\sigma)$ does not change sign, but is replaced by a new one $\tilde{V}(\sigma)$ which is related to the old one in Eq. (6) by interchanging cosh with sinh. Any additional terms in a complete action do not change sign either, but any terms that initially contain $\phi, s$ would look different in the gravity/antigravity sectors by the interchange of cosh with sinh. The $\gamma$-gauge interpolates smoothly between the gravity and antigravity regions and this is why, among other things, it is helpful with the geodesic completeness of the geometry.

After discussing the classical solutions of the cosmological equations, we will return to the antigravity region to discuss possible instabilities due to the sign changes in the full action (14), and what they may mean.

### III. PROPERTIES OF THE SOLUTIONS

First, recall the geodesic incompleteness of the geometry in the usual Einstein frame, which is an old problem that has been set aside. This is typical for any FRW cosmological solution with a big bang singularity, because the geometry is incomplete at the singularity as $a^2_{E}(\tau) \to 0$. A massive particle in the flat FRW universe satisfies the geodesic equation $\frac{d^2 x}{d\tau^2} = \bar{p}^2 / \sqrt{\bar{E}^2 + m^2 a^2_{E}(\tau)}$ where $\bar{p}$ is the conserved momentum of the particle. From this it is straightforward to compute that it takes a finite amount of conformal time, cosmic time, or proper time, to reach the big bang singularity from anywhere in the universe. Therefore the particle trajectory is artificially stopped at the singularity in a finite amount of proper time. This is geodesic incompleteness. It begs the question: what was there just before the big bang? What would we find if we could allow our watch to run beyond the singularity where it was artificially stopped? A geodesically complete geometry would provide at least a partial answer to this question.

Of course, modifications of the geometry or even a completely different description may be expected from quantum gravity or string theory. Unfortunately, the question remains even more obscure in these formalisms because of the lack of proper understanding of how they should be applied to the cosmological setting. It is therefore not unreasonable to at least, at first, try to answer such a question in the setting of classical gravity. Having the complete set of solutions analytically allowed us to answer such questions, and understand how to complete the geometry in the classical physics setting, thus leading to some surprising behavior, as follows.

The complete set of solutions of the cosmological equations have been analyzed in detail in [1] [10] [11]. These solutions are obtained with no restrictions on either the parameters of the model or the initial conditions on the fields. Before the important role of anisotropy is taken into account near the singularity at $a^2_{E} = 0$, there are 6 unrestricted parameters in our solutions as indicated following Eq.(10). All 6 parameters are available to try to fit cosmological data far away from the singularity.

In the absence of anisotropy, the generic solution for $(\phi_\gamma(\tau), s_\gamma(\tau))$ behaves in various detailed ways in various regions of the 6 parameter space. This is given in the Appendix of [11] where we identified 25 different regions of the 6 parameter space in which the analytic expression is different for each case separately. The trajectory of the generic solution in the absence of anisotropy can be plotted parameterically in the $(\phi_\gamma, s_\gamma)$ plane of Fig.1. This shows that the generic trajectory crosses the lightcone in Fig.1 at any point (as determined by the 6 parameters) so that $\phi^2_\gamma(\tau) - s^2_\gamma(\tau)$ keeps changing sign as the universe moves back and forth from the gravity patch to the antigravity patch. Of course, to do so, the scale factor $a^2_{E}(\tau)$ vanishes at each crossing, so that the gravity/antigravity patches are connected to each other only through the cosmological singularity.

The behavior of the generic solution near the singularity changes dramatically in the presence of anisotropy. As will be outlined below, an important effect of anisotropy is to focus the trajectory of the generic solution to pass through the origin of the $(\phi_\gamma, s_\gamma)$ plane, such that the generic trajectory can cross the lightcone in Fig.1 only at the origin where $(\phi_\gamma, s_\gamma)$ vanish simultaneously.

#### A. Zero-size and Finite-size Bounces in the Gravity Patch

By restricting the parameter space or initial conditions, we find that there is a subset of special solutions that are geodesically complete in the Einstein frame purely in the gravity patch. These are the zero-size and finite-size bounces shown in Figs.(2,3,4,5). As seen in Figs.(4,5), their trajectories never reach into the antigravity sector and they never cross the lightcone in Fig.1 except at the origin.
The zero-size-bounce (see Figs.(2,3)) describes a cyclic universe that contracts to zero size and then bounces back smoothly from zero size. It expands up to either a finite size (when \( b < 0 \)) or infinite size (when \( b > 0 \)) and then turns around to repeat the cycle. In these solutions, as described in detail in [10][11], the behavior of \( a_E^2(\tau) \) near the singularity at \( \tau \sim 0 \) is smooth (if \( \rho_0 \) or \( K \) take generic values, then \( a_E^2(\tau) \sim \tau^2 \to 0 \); if both \( \rho_0 \), \( K \) vanish or take some special values, then \( a_E^2(\tau) \sim \tau^6 \to 0 \)). Also, the behavior of the potential and kinetic energy terms for the scalar field \( \sigma(\tau) \) near the singularity are surprising. Namely, contrary to the generic solution, the potential energy dominates over the kinetic energy so that the equation of state \( w(\tau) \) is negative near the singularity. According to common lore, it was thought that such zero-size-bounce solutions would not exist because they would violate the null energy condition (NEC). However, this is not the case. The NEC is satisfied because there is a singularity at zero size, and this is the exception allowed according to the NEC theorems. We found all such solutions and classified them in [10][11], thus providing a rich class of examples of zero-size-bounce cyclic universes, characterized by arbitrary values of the parameters \( \rho_0 \), \( K \), \( b \), \( c \) plus one additional quantized parameter.

To make the zero-size-bounce happen, a synchronization of initial conditions and a quantization condition among the 6 available parameters must be imposed. These properties are illustrated in Fig.3, where it is seen \( s(0) = \phi(0) = 0 \) is imposed as an initial condition, and the periods of oscillations of \( \phi(\tau) \) and \( s(\tau) \) are quantized relative to each other. Hence such solutions are characterized by 4 continuous and 1 quantized parameter, rather than the 6 continuous parameters of the generic solutions. In that sense the zero-size-bounce solutions are a set of measure zero. So, statistically, it does not seem likely that the universe would choose such a solution over the generic solution.

However, anisotropy plays a very important role in providing an attractor mechanism such that, for typical initial conditions away from the singularity, all trajectories are attracted to the origin of the \((\phi, s)\) plane and can cross the lightcone in Fig.1 only at the origin. In that sense the type of solution depicted in Figs.(2-5) is not too far from being generic once anisotropy is taken into account. In particular the behavior away from the origin is a good approximation. Nevertheless, what goes on in the singularity region is quite different as seen in Figs.(8,9) and discussed below.
we use units with $\kappa$ parameters, so this is not a set of measure zero, but it is a restricted region of parameter space or initial values.

where the potentials, the momenta ($p$, $\sigma$, $\alpha$) all the Friedmann equations to obtain the behavior of the general solution near the singularity. In the absence of such cycles the minimum size is not necessarily the same, as this depends on the parameters. Such solutions occur when the parameters satisfy, $\rho_0 < K^2/16b$ and $\phi_{\text{min}}^2 (\tau_0) > K/4b > s_{\text{max}}^2 (\tau_0)$. Note that there are still 6 parameters, so this is not a set of measure zero, but it is a restricted region of parameter space or initial values. The analytic solution is given explicitly in [10][11].

\[ \phi (\tau) \text{ and } s (\tau) \text{ for the finite-size-bounce.} \]

\[ |\phi| > |s| \text{ for all } \tau. \]

**Fig. 6 - Finite-size-bounce,**

\[ |\phi_{\text{min}} (\tau)| > K/4b > |s_{\text{max}} (\tau)|. \]

\[ \text{Finite-size bounce the scale factor never vanishes} \]

\[ \text{the minimum size varies over time} \]

\[ (\phi^2 - s^2) (\tau) = a^2 (\tau) \]

\[ \text{Finite-size-bounce,} \]

\[ a_{E}^2 (\tau) \text{ never vanishes.} \]

The finite-size-bounce (see Figs.(6,7)) describes a universe that contracts up to a minimum non-zero size and then bounces back into an expansion phase up to infinite size. As the universe turns around to repeat such cycles the minimum size is not necessarily the same, as this depends on the parameters. Such solutions occur when the parameters satisfy, $\rho_0 < K^2/16b$ and $\phi_{\text{min}}^2 (\tau_0) > K/4b > s_{\text{max}}^2 (\tau_0)$. Note that there are still 6 parameters, so this is not a set of measure zero, but it is a restricted region of parameter space or initial values. The analytic solution is given explicitly in [10][11].

\[ \frac{z^2}{4z^3} = \frac{\kappa^2}{3} \left[ \frac{p_\sigma^2 + p_\alpha^2}{2z^3} + V (\sigma) + \frac{\rho_0}{z^2} \right] - \frac{6Kv (\alpha_1, \alpha_2)}{\kappa^2 z}, \quad (15) \]

where $p_\alpha^2 \equiv p_1^2 + p_2^2$. For the generic solution (i.e. unlike the zero-size-bounce solution, assuming typically ($p_\sigma, p_1, p_1$) are non-vanishing at the singularity), the kinetic term $\frac{p_\sigma^2 + p_\alpha^2}{2z^3}$ is the dominant term in the energy equation as we approach the singularity $z \to 0$. The next to the leading term is $\frac{\rho_0}{z^2}$ while the potential terms $V (\sigma), \frac{Kv (\alpha_1, \alpha_2)}{\kappa^2 z}$ are subdominant. Keeping the leading and next to the leading terms, it is possible to integrate all the Friedmann equations to obtain the behavior of the general solution near the singularity. In the absence of the potentials, the momenta ($p_\sigma, p_1, p_2$) are all conserved; then the solution is uniquely given as follows (where we use units with $\kappa = \sqrt{6}$)

\[ z (\tau) = 2\tau \left( \sqrt{p_\sigma^2 + p_\alpha^2} + \rho_0 \tau \right), \quad a_E^2 (\tau) = |z (\tau)|, \quad \sigma (\tau) = \frac{p_\sigma}{4\sqrt{p_\sigma^2 + p_\alpha^2}} \ln \left( \frac{(\tau/T)^2}{(\sqrt{p_\sigma^2 + p_\alpha^2} + \rho_0 \tau)^2} \right), \quad (16) \]

as well as

\[ \phi_\gamma (\tau) + s_\gamma (\tau) = \sqrt{T} \left( \sqrt{p_\sigma^2 + p_\alpha^2} + \rho_0 \tau \right) \left( \frac{(\tau/T)^2}{(\sqrt{p_\sigma^2 + p_\alpha^2} + \rho_0 \tau)^2} \right)^{\frac{1}{4}} \left( 1 + \frac{p_\alpha}{\sqrt{p_\sigma^2 + p_\gamma^2}} \right), \quad (17) \]

\[ \phi_\gamma (\tau) - s_\gamma (\tau) = \frac{2\tau}{\sqrt{T}} \left( \frac{(\tau/T)^2}{(\sqrt{p_\sigma^2 + p_\alpha^2} + \rho_0 \tau)^2} \right)^{-\frac{1}{4}} \left( 1 + \frac{p_\alpha}{\sqrt{p_\sigma^2 + p_\gamma^2}} \right). \quad (18) \]

**IV. ANISOTROPY AND THE ATTRACTOR**

It is useful to write the equations of motion for the fields $\sigma, \alpha_1, \alpha_2$ in terms of the canonical momenta instead of their velocities. These are given by $z \dot{\sigma} = p_\sigma$, $z \dot{\alpha}_1 = p_1$, and $z \dot{\alpha}_2 = p_2$. Then the energy constraint that follows from the action $S^\gamma_{\text{eff}}$ or $S^E_{\text{eff}}$ takes the following form when written in terms of $z (\tau) = \phi_\gamma (\tau) - s_\gamma (\tau)$

\[ 2z^2 = \kappa^2 \frac{p_\sigma^2 + p_\alpha^2}{2z^3} + V (\sigma) + \frac{\rho_0}{z^2} - \frac{6Kv (\alpha_1, \alpha_2)}{\kappa^2 z}. \]

\[ \text{Finite-size bounce the scale factor never vanishes} \]

\[ \text{the minimum size varies over time} \]

\[ (\phi^2 - s^2) (\tau) = a^2 (\tau) \]

\[ \text{Finite-size-bounce,} \]

\[ a_E^2 (\tau) \text{ never vanishes.} \]
where $T$ is an integration constant that amounts to an initial value for $\sigma$. Similarly, the solutions for $\alpha_1(\tau), \alpha_2(\tau)$ near the singularity are

$$
\alpha_i(\tau) = \frac{p_i}{4\sqrt{p_i^2 + p_0^2}} \ln \left( \frac{\left(\frac{\tau}{T_i}\right)^2}{\left(\frac{\sqrt{p_i^2 + p_0^2}}{p_0}\right)^2} \right), \ i = 1, 2.
$$

(19)

where $T_i$ are integration constants that amount to initial values for the $\alpha_i$. The solutions for $(z(\tau), \sigma(\tau))$ are plotted in Fig.8, where we see that $z(\tau) = 0$, $\sigma(\tau) = 0$ changes sign. This shows that the trajectory of the universe inevitably passes through antigravity. Furthermore, the parametric plot in Fig.9 shows that there is an attractor mechanism that forces the trajectory to pass through the origin of the $(\phi, s, s)$ plane.

From these plots it is seen that the universe contracts to zero size with a big crunch, passing through the singularity at $z(-\tau_c) = 0$, it then expands in a region of antigravity while $z(\tau) < 0$, then contracts again to zero size at $z(0) = 0$, and then emerges with a big bang in a region of gravity where $z(\tau) > 0$. The duration of the antigravity phase is

$$
\tau_c = \sqrt{\frac{p^2_p + p^2_\sigma}{p_0^2}}
$$

(20)

The essential parameters that control the properties of this unique solution in the vicinity of the singularity are $\rho_0, \sqrt{p^2_1 + p^2_2}, p_0$. They are related to the kinetic energy of the sigma field, anisotropy and “radiation” (i.e. all relativistic particles in the early universe) respectively. They are all typically non-zero and not negligible in the vicinity of the singularity. Note that even if anisotropy $\sqrt{p^2_1 + p^2_2}$ is infinitesimal, the attraction to the origin $\phi = s = 0$ is inevitable.

The parameter space for which the potentials can be neglected is estimated by inserting the solution back into the the equations and comparing the magnitude of the potential energy terms to the kinetic energy and radiation terms. From this we determined that the radiation term $p_0/z^2(\tau)$ in Eq. (15) beats the potential $V(\sigma)$ by more than two powers of $\tau$ as $\tau \to 0$, and similarly for $\tau \to -\tau_c$, so $V(\sigma(\tau))$ is negligible near the singularity. Similarly $K\nu(\alpha_1, \alpha_2)/z$ is negligible compared to radiation as a function of time near the singularity (even when $K \neq 0$) provided $p_2^2 > 15 (p_1^2 + p_2^2)$. This condition is also compatible with a weaker condition ($p_2^2 > 3 (p_1^2 + p_2^2)$), see (11) that avoids the mixmaster behavior discussed in (12) and is consistent with the discussions in (18, 12).

The attractor mechanism is an almost model independent universal behavior since the potential energy $V(\sigma)$ is negligible close to the singularity. Anisotropy in even tiny amounts forces the universe to behave as described above. The issue of initial conditions for all degrees of freedom in our model becomes partially resolved because at the singularity $z = 0$ we must have the following automatically enforced initial values, independent of any other arbitrarily chosen initial conditions at some other time,

$$
\phi_s = s_s = 0,
$$

(21)

rather than only $\phi_s^2 - s_s^2 = z = 0$. Hence, we must take $\phi_s(0) = s_s(0) = 0$ as the dynamically enforced boundary values at all cosmological singularities. This reduces the number of arbitrary initial values.
Now we see that the zero-size-bounce solutions described in the previous section (Figs.(2-5)) are not really of measure zero. They do take into account effectively the dynamically enforced initial values of Eq. (21). Although these solutions are not capable of describing the detailed (antigravity loop) behavior near the singularity, they do describe the correct analytic behavior away from the singularity where anisotropy becomes negligibly small. A more accurate geodesically complete solution is the combination of the near-singularity and far-from-singularity solutions which can be constructed by matching them at some time sufficiently away from the singularity.

V. ANTIGRAVITY REGIME

The physics during the antigravity period remains cloudy in our investigation so far. We discussed the trajectory of the universe on the average, including the antigravity period. However, once in this regime, we may expect some violent behavior and backreaction because of antigravity, and these may affect the trajectory during the antigravity period. First, because of the opposite sign of the gravitational constant we expect that chunks of similar matter will repel each other rather than attract. This is an invitation to investigate the nature of interactions, including fluctuations on top of our background solution, which so far we have not included in the analysis. Furthermore, there are additional phenomena we should expect due to interactions as seen by investigating the form of the action in the antigravity regime, as follows.

Because of the sign change of the effective Newton constant one may argue that fluctuations of the metric, namely the spin-2 gravitons, will now come with the wrong sign kinetic terms. There are no gauge symmetries to remove them, hence, they will behave like negative energy fluctuations, which could destabilize the vacuum by the emission of negative energy gravitons together with lots of positive energy matter particles. There is however a counter effect. Once particle production begins, it automatically implies a back reaction that changes the energy density $\rho$ for all relativistic matter that enters in the cosmological equations. This effectively amounts to a huge increase of the parameter $\rho_0$ once we enter the antigravity regime. So, rather than solving the cosmological equations with a constant $\rho_0$ in the antigravity region, we should allow $\rho_0$ to increase dramatically by large amounts. Then according to our rough formula in Eq. (20) for the duration of the antigravity regime, the duration gets shortened as $\rho_0$ increases. In other words, particle production by antigravity shortens its duration. The particle production and the corresponding jump of $\rho_0$ need to be better understood. These are effects that oppose each other. It is difficult to estimate the balance of these effects with our current understanding of the corresponding physics.

Of course, quantum gravity effects need to be also included. Therefore, it would be very interesting to study similar circumstances in the framework of string theory, although the requisite technical tools may not be available at this time. To formulate the antigravity aspects in string theory we could use the BCST field transformations given in Eqs. (9,11), but even better would be the inclusion of the analog of the Weyl symmetry in the framework of string theory.

We should also mention that the phenomenon of the sign change of the Newton constant transcends the specific simple model in Eq. (13). The phenomena we have found should also be expected generically in supergravity theories coupled to matter whose formulation include a similar factor that multiplies $R(g)$. In supergravity, that factor is related to what is called the Kahler potential, and this factor, combined with the usual Einstein-Hilbert term, is not generally positive definite [13]. In the past, typically it was assumed that the overall factor is positive and investigations of supergravity proceeded only in the positive regime. A discussion of the field space in the positive sector for general $\mathcal{N}=2$ supergravity can be found in [16]. However, our results suggest that generically the overall factor can and will change sign dynamically, and therefore antigravity sectors similar to our discussion in this paper should be expected in typical supergravity theories. Furthermore, we can also draw an analogy to similar factors that occur in front of the kinetic terms for gauge bosons and scalar fields [13]. These too can potentially change sign. In the case of gauge bosons, since the kinetic term takes the form $(E^2 - B^2)$ where $E, B$ represent electric and magnetic fields, a sign change could be reinterpreted by an electric-magnetic duality transformation. Indeed this type of duality was explored in depth at the quantum level for $\mathcal{N}=2$ super Yang-Mills theory in [17]. It is therefore interesting to point out that something similar may be part of the phenomena in the case of the gravity/antigravity transitions we have uncovered.

Having mentioned a few of the possible physical effects in the antigravity regime, we will proceed by assuming that the destabilizing effects of antigravity are not a disaster, but rather they are the signals of interesting physics that remains to be understood better. We expect that the general nature of our solution survives and still describes the interesting effects we discovered, namely the attractor mechanism and the presence of antigravity for some period of time between a crunch and the following bang.
VI. SUMMARY

In this talk I emphasized the role of analytic solutions of cosmological equations to discuss geodesic completeness through the big bang singularity and the discovery of some new phenomena by focusing on questions that could not be answered with only approximate solutions. Until better understood in the context of quantum gravity, or string theory, our results should be considered to be a first pass for the types of new questions they raise and the answers they provide. Here is a brief summary of the main points we have learned.

- We found new techniques to solve cosmological equations analytically. This led to all the solutions for several special potentials $V(\sigma)$ (only one of those was discussed in this talk). Several model independent general results that followed from this include geodesic completeness, and an attractor mechanism to the origin of the $(\phi, s)$ plane, which resolves partially questions on initial values.

- Antigravity is very hard to avoid. Anisotropy + radiation + kinetic energy of $\sigma$, require the antigravity epoch. For close to a year we tried to find models and mechanisms to avoid antigravity (i.e. when all solutions of a geodesically complete model are included). The failure to find such mechanisms finally led us to take antigravity seriously.

- We studied the Wheeler-deWitt equation to take into account some quantum effects for the same system that we analyzed classically. We could solve some cases exactly, others semi-classically. This will be published at a later time. But let me mention that we arrived substantially to the same conclusions that we learned by studying the purely classical system.

- Will the new insights we obtained survive the effects of a full quantum theory? My expectation is affirmative. Perhaps the most satisfactory approach is to try to study the same issues in the context of string theory. However, this is actually quite challenging since string theory has not been developed sufficiently in the cosmological context.

- I emphasize that the structure of the theory in Eq. (13), the methods of solutions, and the phenomena that ensued, are historically direct predictions of 2T-physics in 4+2 dimensions $[4][5]$. The model corresponds to the conformal shadow of 2T-physics which takes the form of a familiar 1T-field theory setting. Other shadows $[2]$ of the same 2T theory which would be in the form of dual 1T-field theories, could also play an important role to understand the system and its physical interpretation.

- Open questions include: are there observational effects today of a past antigravity period? This is an important ongoing project for our group. This involves the study of small fluctuations and the fitting to current observations of the CMB.

Much remains to be understood, including quantum gravity and string theory effects, but it is clear that previously unsuspected phenomena, including an attractor mechanism and antigravity, are at work close to the cosmological singularity. The technical tools to study such issues in the context of a full quantum theory of gravity are yet to be developed. So this is an important challenge to the theory community, including string theorists, since here are some deep physics issues where string theory, if it is the correct approach at those scales, should play its most important role as a quantum theory of gravity. Perhaps duality transformations of the type that occur in electric-magnetic duality $[17]$ could play a role as mentioned earlier. Our understanding of cosmology and of the beginning of the universe could not be clarified until an understanding of the antigravity effects are obtained in a full quantum theory of gravity, and the associated phenomena are taken into account in comparing to cosmological observations.

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