GEOMETRIC CONSTRUCTION OF HOPF CYCLIC CHARACTERISTIC CLASSES

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Abstract. In earlier joint work with A. Connes on transverse index theory on foliations, cyclic cohomology adapted to Hopf algebras has emerged as a decisive tool in deciphering the total index class of the hypoelliptic signature operator. We have found a Hopf algebra $H_n$, playing the role of a ‘quantum structure group’ for the ‘space of leaves’ of a codimension n foliation, whose Hopf cyclic cohomology is canonically isomorphic to the Gelfand-Fuks cohomology of the Lie algebra of formal vector fields. However, with a few low-dimensional exceptions, no explicit construction was known for its Hopf cyclic classes. This paper provides an effective method for constructing the Hopf cyclic cohomology classes of $H_n$ and of $H_n$ relative to $O_n$, in the spirit of the Chern-Weil theory, which completely elucidates their relationship with the characteristic classes of foliations.

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Key words and phrases. Hopf algebras; Hopf cyclic cohomology; characteristic classes of foliations.

Supported in part by the National Science Foundation award DMS-1300548.
In our joint work with A. Connes on the local index formula for transversely elliptic operators on foliations [10], a certain Hopf algebra $H_n$ turned out to play the role of a ‘quantum structure group’ for the ‘space of leaves’ of any codimension $n$ foliation. Moreover, the Hopf-version of cyclic cohomology, which emerged in the same paper, was shown for $H_n$ to be canonically isomorphic via a quasi-isomorphism of van Est type to the Gelfand-Fuks cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^n$. This isomorphism furnished the decisive tool in relating the total index class of the hypoelliptic signature operator [9] to the characteristic classes of foliations. Furthermore, the transplantation of the Gelfand-Fuks classes in the Hopf cyclic cohomological framework broadened the scope of their applicability, as illustrated by the work on modular Hecke algebras [13], which gave a ‘modular’ interpretation to the basic Hopf cyclic cocycles of $H_1$. However, apart from $H_1$ (cf. [10, 22]), explicit cocycle representatives for all Hopf cyclic cohomology classes were known only for $H_2$ (cf. [23]).

The present paper provides a geometric method for representing the Hopf cyclic cohomology classes of $H_n$, and of $H_n$ relative to $O_n$, by concrete cocycles, in the spirit of Chern-Weil theory. Besides giving an effective construction of the Hopf cyclic characteristic classes, this procedure renders their relationship with the characteristic classes of foliations completely transparent.

In addition to ideas and results from [10] as well as their subsequent refinements obtained in collaboration with B. Rangipour ([22, 23, 24]), our approach uses as key additional ingredients the ‘differentiable’ modification, defined à la Haefliger, of the Bott bicomplex [1, 4] for equivariant cohomology and of Dupont’s simplicial de Rham DG-algebra [14]. In their standard version the above complexes compute the $\text{Diff}(M)^\delta$-equivariant cohomology of a manifold. We first show that their differentiable counterparts deliver Haefliger’s differentiable cohomology [18] of the étale groupoid associated to the tautological action of $\text{Diff}(M)^\delta$, and thus the geometric characteristic classes of foliations. We next prove that the quasi-isomorphism of van Est type constructed in [10] transits through the differentiable simplicial de Rham DG-algebra before landing in the differentiable Bott complex. The former being graded commutative, its cohomology classes can be constructed by the usual Chern-Weil procedure [5]. The transition to the Bott complex is effected by integration along the fibers. Although not multiplicative, this operation provides a quasi-isomorphism which offers the advantage of being explicitly computable. Employing then chain maps (from [10]
and \([24]\), we transfer the representative cocycles, constructed in terms of connection and curvature, from the differentiable Bott complex to the original cyclic model \([10]\) for the Hopf cyclic cohomology of \(H_n\) as well as to the quasi-isomorphic model of Chevalley-Eilenberg type constructed in \([24]\).

The upshot is a concrete construction of bases for the Hopf cyclic cohomology of \(H_n\) and also for \(H_n\) relative to \(O_n\), in both cohomological models mentioned above, on a par with the classical geometric construction of characteristic classes of foliations \([2, 3, 20]\). In particular this construction provides “minimal” representative cocycles for all Hopf cyclic cohomology classes, reproducing the known feature of the Gelfand-Fuks cohomology of being representable by cocycles involving jets of order no higher than two of the formal vector fields.

1. Characteristic cocycles in differentiable cohomology

1.1. Differentiable equivariant cohomology. Let \(M\) be a smooth oriented manifold of dimension \(n\), and let \(G = \text{Diff}(M)\) be its group of diffeomorphisms. Regarding \(G\) as a discrete group, the equivariant cohomology \(H^\bullet_G(M, \mathbb{R})\), originally defined by means of the homotopy quotient as \(H^\bullet(EG \times_G M, \mathbb{R})\), can be expressed in terms of de Rham complexes. These are associated to the simplicial manifold

\[
\Delta_G M = \{\Delta_G M[p] := G^p \times M\}_{p \geq 0},
\]

with face maps \(\partial_i : \Delta_G M[p] \to \Delta_G M[p - 1], 1 \leq i \leq p\), given by

\[
\partial_i(\phi_1, \ldots, \phi_p, x) = \begin{cases} 
(\phi_2, \ldots, \phi_p, x), & i = 0, \\
(\phi_1, \ldots, \phi_i, \phi_{i+1}, \ldots, \phi_p, x), & 1 < i < p, \\
(\phi_1, \ldots, \phi_{p-1}, \phi_p(x)), & i = p.
\end{cases}
\]

and degeneracies

\[
\sigma_i(\phi_1, \ldots, \phi_p, x) = (\phi_1, \ldots, \phi_i, e, \phi_{i+1}, \ldots, \phi_p, x), \quad 0 \leq i \leq p.
\]

The first such complex (cf. \([11, 14]\)) is the total complex of the bicomplex \(\{C^\bullet(G, \Omega^\bullet(M)), \delta, d\}\) defined as follows: \(C^p(G, \Omega^q(M))\) is spanned by cochains

\[
c(\phi_1, \ldots, \phi_p) \in \Omega^p(M), \quad \phi_1, \ldots, \phi_p \in G,
\]

d is the de Rham differential, and \(\delta\) is the group cohomology boundary

\[
\delta c(\phi_1, \ldots, \phi_{p+1}) = \sum_{i=0}^{p} (-1)^i c(\partial_i(\phi_1, \ldots, \phi_{p+1})) \\
+ (-1)^{p+1} \phi_{p+1}^e c(\phi_1, \ldots, \phi_p).
\]
Instead of the action groupoid notation implicitly used in the above formulas it will be more convenient to work with the homogeneous bicomplex \( \{ \mathcal{C}^\bullet (G, \Omega^\bullet (M)), \delta, d \} \). Its \((p,q)\)-cochains

\[
\bar{c}(\rho_0, \ldots, \rho_p) \in \Omega^p(M), \quad \rho_0, \ldots, \rho_p \in G,
\]
satisfy the covariance condition

\[
(1.1) \quad \bar{c}(\rho_0 \rho, \ldots, \rho_p \rho) = \rho^* \bar{c}(\rho_0, \ldots, \rho_p), \quad \forall \rho, \rho_i \in G,
\]
and the group cohomology boundary is given by

\[
\tilde{\delta} \bar{c}(\rho_0, \ldots, \rho_p) = \sum_{i=0}^{p} (-1)^i \bar{c}(\rho_0, \ldots, \rho_i, \ldots, \rho_p),
\]
the ‘check’ mark signifying the omission of the element underneath.

The passage between the two isomorphic bicomplexes is via the relations

\[
(1.2) \quad \bar{c}(\phi_1, \ldots, \phi_p) = \bar{c}(\phi_1 \cdots \phi_p, \phi_2 \cdots \phi_p, \ldots, \phi_p, e),
\]
resp. \( \bar{c}(\rho_0, \ldots, \rho_p) = \rho^*_p \bar{c}(\rho_0 \rho_1^{-1}, \rho_1 \rho_2^{-1}, \ldots, \rho_{p-1} \rho_p^{-1}) \).

The second complex computing \( H^*_G(M, \mathbb{R}) \) is Dupont’s de Rham complex (cf. [14]) of compatible forms \( \{ \Omega^\bullet(|\triangle_G M|), d \} \) on the geometric realization \(|\triangle_G M|\).

By definition,

\[
\Omega^\bullet(|\triangle_G M|) \subset \prod_{p=0}^{\infty} \Omega^\bullet(\Delta^p \times \triangle_G M[p])
\]
consists of sequences \( \omega = \{ \omega_p \}_{p \geq 0} \), with \( \omega_p \in \Omega^\bullet(\Delta^p \times \triangle_G M[p]) \), such that for all morphisms \( \mu \in \Delta(p,q) \) in the simplicial category,

\[
(1.3) \quad (\mu_\bullet \times \text{Id})^* \omega_q = (\text{Id} \times \mu^\bullet)^* \omega_p \in \Omega^\bullet(\Delta^p \times \triangle_G M[q]).
\]

Here \( \Delta^p = \{ t = (t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \mid t_i \geq 0, \ t_0 + \ldots + t_p = 1 \} \), \( \mu_\bullet : \Delta^p \to \Delta^q \) (resp. \( \mu^\bullet : \Delta_G M[q] \to \Delta_G M[p] \)), stands for the induced cosimplicial (resp. simplicial) map, and \( \Omega^k(\Delta^p \times \triangle_G M[q]) \) denotes the \( k \)-forms on \( \Delta^p \times \triangle_G M[q] \) which are extendable to smooth forms on \( V^p \times \triangle_G M[q] \), where \( V^p = \{ t = (t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \mid t_0 + \ldots + t_p = 1 \} \).

By [14] Thm 2.3, the operation of integration along the fibers

\[
(1.4) \quad \oint_{\Delta^p} : \Omega^\bullet(\Delta^p \times \triangle_G M[p]) \to \Omega^{*-p}(\triangle_G M[p])
\]
Establishes a quasi-isomorphism between the complexes \( \{ \Omega^\bullet(|\triangle_G M|), d \} \) and \( \{ C^{\text{tot}}(G, \Omega^\bullet(M)), \delta, d \} \).

As in the case of the Bott complex, there is a homogeneous description of the simplicial de Rham complex, \( \{ \Omega^\bullet(|\triangle_G M|), d \} \), consisting of
the $G$-invariant compatible forms on the geometric realization $|\bar{\Delta}_G M|$. The simplicial manifold $\bar{\Delta}_G M$ is defined as follows:

$$\bar{\Delta}_G M = \{ \bar{\Delta}_G M[p] := G^{p+1} \times M \}_{p \geq 0},$$

with face maps $\bar{\partial}_i : \bar{\Delta}_G M[p] \to \bar{\Delta}_G M[p - 1], \; 1 \leq i \leq p,$ given by

$$\bar{\partial}_i (\rho_0, \ldots, \rho_p, x) = (\rho_0, \ldots, \rho_i, \ldots, \rho_p), \quad 0 \leq i \leq p,$$

and degeneracies

$$\bar{\sigma}_i (\rho_0, \ldots, \rho_p, x) = (\rho_0, \ldots, \rho_i, \rho_i, \ldots, \rho_p), \quad 0 \leq i \leq p.$$

The compatible forms $\omega = \{ \omega_p \}_{p \geq 0} \in \Omega^* (|\bar{\Delta}_G M|)$ satisfy the invariance condition

$$(1.5) \quad \omega (\rho_0, \ldots, \rho_p) = (\rho^{-1})^* \omega (\rho_0 \rho, \ldots, \rho_p \rho), \quad \forall \rho, \rho_i \in G.$$

For the purposes of this paper, the relevant cohomology is the differentiable modification of the above constructs, in the sense of Haefliger [18, Ch.4, §4]. In the case of the Bott complex the modification amounts to pass to the subcomplex of differentiable cochains $\{ C^*_{d \text{tot}} (G, \Omega^*(M)), \delta \pm d \},$ resp. $\{ C^*_{d \text{tot}} (G, \Omega^*(M)), \delta \pm d \},$ which is defined as follows: a cochain $\omega \in \bar{C}^p (G, \Omega^*(M))$ is differentiable if for any local chart $U \subset M$ with coordinates $(x^1, \ldots, x^n)$,

$$(1.6) \quad \omega (\rho_0, \ldots, \rho_p, x) = \sum f_I (x, j^{k_1}_x (\rho_0), \ldots, j^{k_p}_x (\rho_p)) \, dx^I,$$

with $f_I$ smooth functions of $x \in U$ and the $k$-jets at $x$ of $\rho_0, \ldots, \rho_p$, for some $k \in \mathbb{N}$, and $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_q}$ with $I = (i_1 < \ldots < i_q)$ running through the set of strictly increasing $q$-indices.

The cohomology of the total complex $\{ \bar{C}^*_{d \text{tot}} (G, \Omega^*(M)), \delta \pm d \}$ will be denoted $H^*_{d\bar{\Delta}_G} (M, \mathbb{R})$.

In the case of Dupont’s complex, the differentiable simplicial de Rham complex is the subcomplex $\{ \Omega^*_{d\bar{\Delta}} (|\bar{\Delta}_G M|), d \} \subset \{ \Omega^* (|\bar{\Delta}_G M|), d \}$ consisting of the $G$-invariant compatible forms $\{ \omega_p \}_{p \geq 0}$ whose components satisfy the analogous condition:

$$(1.7) \quad \omega_p (t; \rho_0, \ldots, \rho_p, x) = \sum f_{I,J} (t; x, j^{k_1}_x (\rho_0), \ldots, j^{k_p}_x (\rho_p)) \, dt^I \wedge dx^J,$$

with $f_{I,J}$ smooth in all variables.

We denote by $H^*_{d\bar{\Delta}} (|\Delta_G M|, \mathbb{R})$ the cohomology of the differentiable simplicial de Rham complex $\{ \Omega^*_{d\bar{\Delta}} (|\Delta_G M|), d \}$.

**Theorem 1.1.** The chain map $\int_{\Delta^*} : \Omega^*_{d\bar{\Delta}} (|\bar{\Delta}_G M|) \to \bar{C}^*_{d\bar{\Delta}} (G, \Omega^*(M))$ induces an isomorphism $H^*_{d\bar{\Delta}} (|\Delta_G M|, \mathbb{R}) \cong H^*_{d\bar{\Delta}_G} (M, \mathbb{R}).$
Proof. Clearly, the integration along the fibers maps $\Omega^\bullet_d(\Delta G M)$ to $C^\bullet_d(G, \Omega^\bullet(M))$. Moreover, the natural chain maps in both directions as well as the chain homotopies relating them in the proof of Theorem 2.3 in [14] preserve the differentiable subcomplexes. □

1.2. Explicit van Est-Haefliger isomorphism. We denote by $F^k M$ the frame bundle of order $k \in \mathbb{N} \cup \infty$, formed of $k$-jets at 0 of local diffeomorphisms $\phi$ from a neighborhood of $0 \in \mathbb{R}^n$ to a neighborhood of $\phi(0) \in M$. Thus, $F^1 M = FM$ is the usual frame bundle, while by definition $F^\infty M := \varprojlim F^k M$. Each $F^k M$ is a principal bundle over $M$ with structure group $G^k$ formed of $k$-jets at 0 of local diffeomorphisms of $\mathbb{R}^n$ preserving 0. The group $G$ operates on the left on each $F^k M$ by:

$$\phi \cdot \tilde{j}_0^\infty(\rho) := \tilde{j}_0^\infty(\phi \circ \rho), \quad \phi \in G, \rho \in F^k M.$$ 

Let now $a_n$ be Lie algebra of formal vector fields on $\mathbb{R}^n$. Any $v \in a_n$ can be represented as $v = \tilde{j}_0^\infty \left(\frac{d}{dt} \big|_{t=0} \rho_t\right)$, with $\{\rho_t\}_{t \in \mathbb{R}}$ a 1-parameter group of local diffeomorphisms of $\mathbb{R}^n$; it thus gives rise to a $G$-invariant vector field on $F^\infty M$, defined at a point $\tilde{j}_0^\infty(\phi) \in F^\infty M$ by

$$\tilde{v} \big|_{\tilde{j}_0^\infty(\phi)} = \tilde{j}_0^\infty \left(\frac{d}{dt} \big|_{t=0} (\phi \circ \rho_t)\right).$$

Dually, any $\omega \in C^m(a_n)$, where $C^*\bullet(a_n)$ denotes the Gelfand-Fuks cohomology complex [15] of $a_n$, gives rise to a $G$-invariant form $\tilde{\omega} \in \Omega^m(F^\infty M)$, characterized by

$$\tilde{\omega}(\tilde{v}_1, \ldots, \tilde{v}_m) = \omega(v_1, \ldots, v_m).$$

Moreover, the assignment

$$(1.8) \quad \omega \in C^\bullet(a_n) \mapsto \tilde{\omega} \in \Omega^\bullet(F^\infty M)^G$$

is a DGA-isomorphism, by means of which we shall tacitly identify the two DG-algebras.

Choosing a torsion-free affine connection $\nabla$ on $M$, one defines a cross-section $\sigma_\nabla : FM \to F^\infty M$ of the natural projection $\pi_1 : F^\infty M \to FM$, by the formula

$$(1.9) \quad \sigma_\nabla(u) = \tilde{j}_0^\infty(\exp_{\nabla} \circ u), \quad u \in F_x M.$$ 

This cross-section is clearly $GL_n$-equivariant

$$(1.10) \quad \sigma_\nabla \circ R_a = R_a \circ \sigma_\nabla, \quad a \in GL_n.$$
as well as Diff-equivariant,

\[(1.11) \quad \sigma_{\nabla^\phi} = \phi^{-1} \circ \sigma_{\nabla} \circ \phi, \quad \forall \phi \in G.\]

Here \(\nabla^\phi = \phi_*^{-1} \circ \nabla \circ \phi_*\), or more precisely the derivative whose connection form is the pull-back \(\phi^*(\omega_{\nabla})\) of the connection form of \(\nabla\).

Let \(\bar{\Delta}_G FM\) be the simplicial manifold associated to the action of \(G\) by prolongation on \(FM\). We define the maps \(\sigma_p : \Delta^p \times \bar{\Delta}_G FM[p] \to F^\infty M, \ p \in \mathbb{N}\), by the formula

\[(1.12) \quad \sigma_p(t; \rho_0, \ldots, \rho_p, u) = \sigma_{\nabla(t; \rho_0, \ldots, \rho_p)}(u),\]

where \(\nabla(t; \rho_0, \ldots, \rho_p) = \sum_{i=0}^{p} t_i \nabla_{\rho_i}\), \(t \in \Delta^p\).

Manifestly, the collection \(\hat{\sigma} = \{\sigma_p\}_{p \geq 0}\) descends to the geometric realization of \(\bar{\Delta}_G FM\), yielding a well-defined map \(\hat{\sigma} : |\bar{\Delta}_G FM| \to F^\infty M\); moreover, this map is \(\text{GL}_n\)-equivariant.

**Theorem 1.2.** (1) Let \(\omega \in C^*(a_n)\) then \(\hat{\sigma}^*(\tilde{\omega}) \in \Omega^*_d(|\bar{\Delta}_G FM|)\), and the map \(C_{\nabla} : C^*(a_n) \to \Omega^*_d(|\bar{\Delta}_G FM|)\), defined by

\[(1.13) \quad C_{\nabla}(\omega) = \hat{\sigma}^*(\tilde{\omega}) \in \Omega^*_d(|\bar{\Delta}_G FM|),\]

is a quasi-isomorphism of \(DG\)-algebras.

(2) The map \(C_{\nabla}\) is \(\text{GL}_n\)-equivariant and, by restriction to the subcomplex of \(O_n\)-basic cochains, it induces a quasi-isomorphism of \(DG\)-algebras \(C_{\nabla}^O : C^*(a_n, O_n) \to \Omega^*_d(|\bar{\Delta}_G (PM, O_n)|)\); here \(PM = FM/ O_n\) and \(O_n\)-basic forms on \(FM\) are identified with forms on \(PM\).

**Proof.** Since \(\tilde{\omega}\) is \(G\)-invariant, \(\hat{\sigma}^*(\tilde{\omega})\) is indeed a compatible form. It is also quite obvious that it belongs to the differentiable subcomplex \(\Omega^*_d(|\bar{\Delta}_G FM|)\). Furthermore, \(\hat{\sigma}^*(d\tilde{\omega}) = d(\hat{\sigma}^*(\tilde{\omega}))\), and so \(C_{\nabla}\) is a well-defined map of complexes.

Observe now that for any connection \(\tilde{\nabla}\),

\[(\pi_1 \circ \sigma_{\tilde{\nabla}})(u) = j_0^1(\exp_{\tilde{\nabla}} \circ u) = u, \quad u \in F_\Sigma M.\]

Upgrading both \(\pi_1\) and \(\hat{\sigma}\) in the obvious way to simplicial maps \(\text{Id} \times \pi_1 : |\bar{\Delta}_G F^\infty M| \to |\bar{\Delta}_G FM|\) and \(\text{Id} \times \hat{\sigma} : |\bar{\Delta}_G FM| \to |\bar{\Delta}_G F^\infty M|\), it follows that

\[(\text{Id} \times \pi_1) \circ (\text{Id} \times \hat{\sigma}) = \text{Id}.\]

Hence \((\text{Id} \times \hat{\sigma})^* : \Omega^*_d(|\bar{\Delta}_G F^\infty M|) \to \Omega^*_d(|\bar{\Delta}_G FM|)\) is a left inverse for \((\text{Id} \times \pi_1)^* : \Omega^*_d(|\bar{\Delta}_G FM|) \to \Omega^*_d(|\bar{\Delta}_G F^\infty M|)\). Because the fibers of \(\pi_1\) are contractible, \((\text{Id} \times \pi_1)^*\) induces an isomorphism in (differentiable) cohomology. Therefore so does its inverse \((\text{Id} \times \hat{\sigma})^*\).
On the other hand, the usual horizontal homotopy (cf. [18, §IV.4]), defined by the formula

\[(H\alpha)_{p-1}(t; \rho_0, \ldots, \rho_{p-1}, j_0^\infty(\rho)) = \alpha_p(t; \rho_0^{-1}, \rho_1, \ldots, \rho_{p-1}, j_0^\infty(\rho)), \quad \alpha \in \Omega_d^*(|\hat{\Delta}_G F^\infty M|),\]

shows that the natural inclusion of $\Omega^*(F^\infty M)^G$ into $\Omega_d^*(|\hat{\Delta}_G F^\infty M|)$ is also quasi-isomorphism. Recalling the identification (1.8), the proof following explicit form of the van Est-Haefliger isomorphism [18, §IV.4], the proof is achieved by noting that when restricted to $\Omega^*(F^\infty M)^G$ the map $(\mathrm{Id} \times \Delta \sigma)^*$ coincides with $C_\nabla$.

The second claim has a similar proof. Identifying the $O_n$-basic forms on $F^\infty M$ with forms on $P^\infty M = F^\infty M/ O_n$, the appropriate homotopy takes the form

\[(H\alpha)_{p-1}(t; \rho_0, \ldots, \rho_{p-1}, j_0^\infty(\rho) O_n) = \int_{O_n} \alpha_p(t; k^{-1} \rho_0^{-1}, \rho_1, \ldots, \rho_{p-1}, j_0^\infty(\rho) O_n) dk.\]

Combining the above theorem with Theorem [1.1] one obtains the following explicit form of the van Est-Haefliger isomorphism [18, §IV.4].

**Theorem 1.3.** The maps $D_\nabla = \int_{\Delta^*} C_\nabla : C^*(a_n) \to \bar{C}_d^{\text{tot}}^* (G, \Omega^*(FM))$, resp. $D_\nabla^{O_n} = \int_{\Delta^*} C_\nabla^{O_n} : C^*(a_n, O_n) \to \bar{C}_d^{\text{tot}}^* (G, \Omega^*(PM))$, are quasi-isomorphisms of complexes.

### 1.3. Characteristic cocycles.

Although explicit, the map $D_\nabla$ is quite intricate and thus not amenable to concrete computations. Instead, we proceed now to describe an alternative construction of the Diff-equivariant geometric characteristic classes, in the spirit of the Chern-Weil theory (cf. [5]), in terms of cocycles manufactured out of the connection and curvature forms.

The universal connection and curvature forms $\vartheta = (\vartheta^i_j)$ and $R = (R^i_j)$, defined as in [1 §2], generate a DG-subalgebra $CW^*(a_n)$ of $C^*(a_n)$. By the Gelfand-Fuks theorem (cf. [15, 16]), the inclusion $CW^*(a_n) \hookrightarrow C^*(a_n)$ is a quasi-isomorphism. Actually, there is a faithful embedding of the truncated Weil complex $\hat{W}(\mathfrak{gl}_n)$ which identifies it with the subcomplex $CW^*(a_n)$ of $C^*(a_n)$. We recall that $\hat{W}(\mathfrak{gl}_n) = W(\mathfrak{gl}_n)/\mathcal{I}_{2n}$, where $W(\mathfrak{gl}_n) = \wedge^* \mathfrak{gl}_n^* \otimes S(\mathfrak{gl}_n)$ is the Weil algebra of $\mathfrak{gl}_n$, and $\mathcal{I}_{2n}$ is the ideal generated by the elements of $S(\mathfrak{gl}_n)$ of degree $> 2n$. These DG-algebras are $\mathrm{GL}_n$ algebras as well. Let $CW^*(a_n, O_n)$, resp. $\hat{W}(\mathfrak{gl}_n, O_n)$,
denote their subalgebras consisting of $O_n$-basic elements. The above identification $\tilde{W}(\mathfrak{g}_n) \equiv CW^\bullet(\mathfrak{a}_n)$ then restricts to an identification $\tilde{W}(\mathfrak{g}_n, O_n) \equiv CW^\bullet(\mathfrak{a}_n, O_n)$.

With $\nabla$ being as before a fixed torsion-free connection and $\omega_\nabla = (\omega^i_j)$, resp. $\Omega_\nabla = (\Omega^i_j)$, denoting its matrix-valued connection, resp. curvature form on $FM$, one has the naturality relation:

**Lemma 1.4.** $\sigma^*_\nabla(\tilde{\vartheta}^i_j) = \omega^i_j$ and $\sigma^*_\nabla(\tilde{R}^i_j) = \Omega^i_j$.

**Proof.** (Cf. [12, Lemma 18].) Since \eqref{1.14} $R^i_j = d\tilde{\vartheta}^i_j + \partial^i_k \wedge \partial^j_k$, the second identity is a consequence of the first. To prove the first, we note that by \eqref{1.11} the operator $\omega_\nabla \mapsto \sigma^*_\nabla(\tilde{\vartheta}^i_j)$, acting on the (affine) space of torsion-free connections on $FM$, is natural, i.e. $G$-equivariant. The uniqueness of such operators on torsion-free connections (cf. [21, §25.3]) ensures that the only such operator is the identity. \[\square\]

In homogeneous group coordinates (see \eqref{1.12}), the simplicial connection form-valued matrix $\tilde{\omega}_\nabla = \{\tilde{\omega}^i_j\}_{p \in \mathbb{N}}$ associated to $\nabla$ has components

\[
\tilde{\omega}_p(t; \rho_0, \ldots, \rho_p) := \sum_{i=0}^p t_i \rho_i^*(\omega_\nabla),
\]

and the simplicial matrix-valued curvature form $\tilde{\Omega}_\nabla := d\tilde{\omega}_\nabla + \tilde{\omega}_\nabla \wedge \tilde{\omega}_\nabla$ has components $\tilde{\Omega}_p = \tilde{\Omega}_p^{(1,1)} + \tilde{\Omega}_p^{(0,2)}$, given by

\[
\tilde{\Omega}_p(t; \rho_0, \ldots, \rho_p) = \sum_{i=0}^p dt_i \wedge \rho_i^*(\omega_\nabla) + \sum_{i,j=0}^p t_i t_j \rho_i^*(\omega_\nabla) \wedge \rho_j^*(\omega_\nabla).
\]

The forms $\tilde{\omega}^i_j$ and $\tilde{\Omega}^i_j$ clearly belong to the differentiable de Rham complex $\Omega^\bullet(\tilde{\Delta}_G FM)$.

In view of the above discussion, Theorem\ref{1.2} together with Lemma\ref{1.4} have the following consequence.

**Corollary 1.5.** (1a) The forms $\tilde{\omega}^i_j$ and $\tilde{\Omega}^i_j$ generate a DG-subalgebra $CW^\bullet_d(\tilde{\Delta}_G FM)$, and $C_\nabla$ restricts to an isomorphism of $CW^\bullet(\mathfrak{a}_n) \equiv \tilde{W}(\mathfrak{g}_n)$ onto $CW^\bullet_d(\tilde{\Delta}_G FM)$.

(2a) By restriction to the respective DG-subalgebras of $O_n$-basic elements, $C_\nabla$ induces an isomorphism of $CW^\bullet(\mathfrak{a}_n, O_n) \equiv \tilde{W}(\mathfrak{g}_n, O_n)$ onto $CW^\bullet_d(\tilde{\Delta}_G PM)$. 
The operation of integration along the fibers does not preserve the cup product (which is graded commutative at the source but not in the target). Nevertheless, by Theorem 1.3, it still induces isomorphism in cohomology.

**Corollary 1.6. (1b)** The restriction of \( \mathcal{D}_\nabla \) to \( CW^*(a_n, O_n) \) is a quasi-isomorphism to \( \{ C^{\text{tot}}_d (G, \Omega^*(FM)), \delta \pm d \} \).

**Corollary 1.6. (2b)** The restriction of \( \mathcal{D}_0 \) to \( CW^*(a_n, O_n) \) is a quasi-isomorphism to \( \{ C^{\text{tot}}_d (G, \Omega^*(PM)), \delta \pm d \} \).

To describe concrete bases of cohomology classes constructed in the Chern-Weil manner, we let \( I(\mathfrak{gl}_n) = S(\mathfrak{gl}_n)^{GL_n} \) be the algebra of invariant polynomials, or equivalently, the subalgebra of \( GL_n \)-basic elements of \( W(\mathfrak{gl}_n) \). Once the torsion-free connection \( \nabla \) is chosen, to any polynomial \( P \in I(\mathfrak{gl}_n) \) one associates a closed simplicial differential form \( P(\hat{\Omega}) \in \Omega^d(\Delta_G FM) \). On the total space of the frame bundle this form is exact, and can be expressed as a boundary by a standard transgression formula (cf. [6]):

\[
P(\hat{\Omega}) = d(TP(\hat{\omega})), \quad \text{with}
\]

\[
(1.17) \quad TP(\hat{\omega}) = k \int_0^1 P(\hat{\omega}, \hat{\Omega}_t, \ldots, \hat{\Omega}_t) dt, \quad k = \deg(P),
\]

where \( \hat{\Omega}_t = t\hat{\Omega} + (t^2 - t)\hat{\omega} \wedge \hat{\omega} \).

Corollary 1.6 allows now to transfer a Vey basis [10] of \( H^*(a_n) \) to a basis of \( H_d(\Delta_G FM, \mathbb{R}) \) as follows. Let \( \{ c_k \}_{1 \leq k \leq n} \) be a system of generators of the algebra \( I(\mathfrak{gl}_n) \), for example the coefficients of the powers of \( t \) in the expansion

\[
(1.18) \quad \det \left( \text{Id} - \frac{t}{2\pi i} A \right) = \sum_{k=1}^n t^k c_k(A), \quad A \in \mathfrak{gl}_n(\mathbb{C}).
\]

These give the classical Chern forms \( c_k(\hat{\Omega}) \in \Omega^{2k}_d(\Delta_G FM) \), and by transgression the Chern-Simon forms \( TC_k(\hat{\omega}) \in \Omega^{2k-1}_d(\Delta_G FM) \).

The image \( C_k(\hat{\Omega}) = \int_{\Delta^*} c_k(\hat{\Omega}) \) is the cocycle \( C_k(\hat{\Omega}) = \{ C_k^{(p)}(\hat{\Omega}) \}_{p \geq 0} \) whose components in homogeneous group coordinates are

\[
(1.19) \quad C_k^{(p)}(\hat{\Omega})(\phi_0, \ldots, \phi_p) = (-1)^p \int_{\Delta^p} c_k \left( \hat{\Omega}(t; \phi_0, \ldots, \phi_p) \right).
\]

Similarly, the image \( TC_k(\hat{\Omega}) = \int_{\Delta^*} TC_k(\hat{\Omega}) \) is the transgressed cochain \( TC_k(\hat{\Omega}) = \{ TC_k^{(r)}(\hat{\Omega}) \} \) with homogeneous components given...
by

\[ TC^{(r)}(\hat{\omega}_V)(\phi_0, \ldots, \phi_r) = (-1)^p \int_{\Delta^p} TC_k(\hat{\omega}_V(t; \phi_0, \ldots, \phi_r)). \]

The Vey basis can now be transferred as follows. Consider the collection \( V_n \) of all pairs \((I, J)\) of subsets of \( \{1, \ldots, n\} \) of the form \( I = \{i_1 < \ldots < i_p\} \) and \( J = \{j_1 \leq \ldots \leq j_q\} \), such that \( |J| = j_1 + \ldots + j_q \leq n \), \( i_1 \leq j_1 \) and \( i_1 + |J| > n \).

**Corollary 1.7.** With \((I, J)\) running over the set \( V_n \), the forms

\[ TC_I(\hat{\omega}_V) \wedge c_J(\hat{\Omega}_V) := TC_{i_1}(\hat{\omega}_V) \wedge \ldots \wedge TC_{i_p}(\hat{\omega}_V) \wedge c_{j_1}(\hat{\Omega}_V) \wedge \ldots \wedge c_{j_q}(\hat{\Omega}_V) \]

are closed and their cohomology classes form a basis of \( H^*_d(|\triangle GFM|, \mathbb{R}) \). The cocycles obtained by their integration along fibers,

\[ C_{I, J}(\nabla) := \int_{\Delta^*} TC_I(\hat{\omega}_V) \wedge c_J(\hat{\Omega}_V), \quad (I, J) \in V_n, \]

provide a complete set of representatives for a basis of \( H^*_d(GFM, \mathbb{R}) \).

The same procedure applies to the relative case. The representatives of the even Chern classes are still given by the formula \( \text{(1.19)} \) with \( k = 2i \), while the odd Chern forms can be transgressed as follows (cf. \textbf{[17, Prop. 5]}). Denote by \( \mathfrak{gl}_n = \mathfrak{s}_n \oplus \mathfrak{o}_n \) the standard decomposition into symmetric and skew-symmetric parts, and let \( \mathfrak{s} : \mathfrak{gl}_n \to \mathfrak{s}_n \), resp. \( \mathfrak{o} : \mathfrak{gl}_n \to \mathfrak{o}_n \), be the corresponding projections. Then

\[ c_{2k-1}(\hat{\Omega}_V) = d(Tc_{2k-1}(\hat{\omega}_V)), \quad \text{with} \]

\[ TC_{2k-1}(\hat{\omega}_V) = (2k - 1) \int_0^1 c_{2k-1} \left( \mathfrak{s}(\hat{\omega}_V), \hat{\Omega}_t, \ldots, \hat{\Omega}_t \right) dt, \]

where \( \hat{\Omega}_t = t\mathfrak{s}(\hat{\Omega}_V) + \mathfrak{o}(\hat{\Omega}_V) + (t^2 - 1)\mathfrak{s}(\hat{\omega}_V) \wedge \mathfrak{s}(\hat{\omega}_V) \).

To construct a Vey basis, one now takes the collection \( VO_n \) of all pairs \((I, J)\) of (possibly empty) subsets of \( \{1, \ldots, n\} \), with \( I = \{i_1 < \ldots < i_p\} \) containing only odd integers and \( J = \{j_1 \leq \ldots \leq j_q\} \), with \( |J| \leq n \), such that \( i_0 \leq j_0 \) and \( i_0 + |J| > n \). Here \( i_0 = i_1 \) if \( I \neq \emptyset \) or \( i_0 = \infty \) otherwise, and \( j_0 \) stands for the smallest odd integer in \( J \) or \( j_0 = \infty \) if there is none.

**Corollary 1.8.** With \((I, J)\) running over the set \( VO_n \), the forms

\[ TC_I(\hat{\omega}_V) \wedge c_J(\hat{\Omega}_V) := TC_{i_1}(\hat{\omega}_V) \wedge \ldots \wedge TC_{i_p}(\hat{\omega}_V) \wedge c_{j_1}(\hat{\Omega}_V) \wedge \ldots \wedge c_{j_q}(\hat{\Omega}_V) \]

are closed and their classes form a basis of \( H^*_d(|\triangle GPM|, \mathbb{R}) \).
The cocycles obtained by their integration along fibers,

\[ C_{I,J}(\nabla) := \oint_{\Delta^*} \Delta \cdot Tc_I(\hat{\omega}_{\nabla}) \wedge c_J(\hat{\Omega}_{\nabla}), \quad (I,J) \in VO_n, \]

form a complete set of representatives for a basis of \( H_{d,G}(PM,\mathbb{R}) \).

In particular, the representatives of the Chern classes are the cocycles \( C_{\emptyset,k}(\nabla) \equiv C_k(\hat{\Omega}_{\nabla}) \), with \( k \) even.

2. Cyclic cohomological models for the Hopf algebra \( \mathcal{H}_n \)

For the convenience of the reader, we collect here a modicum of salient facts from [10, 11, 23, 24] about the Hopf algebra \( \mathcal{H}_n \) and the Hopf cyclic cohomological models associated to it.

2.1. Canonical representation and the standard cyclic model.

The Hopf algebra \( \mathcal{H}_n \) serves as a “quantum” analogue of the structure group of the universal “space of leaves” for codimension \( n \) foliations. As such, it arises naturally as the symmetry structure of the convolution algebra \( C_c^\infty(\bar{\Gamma}_n) \) of the \( \acute{e}tale \) groupoid \( \bar{\Gamma}_n \) of germs of local diffeomorphisms of \( \mathbb{R}^n \) acting by prolongation on the frame bundle \( F\mathbb{R}^n \). For the clarity of the exposition it is convenient to replace \( C_c^\infty(\bar{\Gamma}_n) \) by the crossed product algebra \( \mathcal{A} = C_c^\infty(F\mathbb{R}^n) \rtimes G \), where \( G = \text{Diff} \mathbb{R}^n \) is treated as a discrete group.

In order to implement the operational construction of \( \mathcal{H}_n \), one identifies \( F\mathbb{R}^n \) with the affine group \( G = \mathbb{R}^n \times \text{GL}_n \) in the obvious way, and one endows it with the canonical form \( \theta = (\theta^k) = (y^{-1} dx)^k \) and with the flat connection \( \omega = (\omega^i_j) = (y^{-1} dy)^i_j \). With the usual summation convention, the basic horizontal vector fields are \( X_k = y^\mu_i \partial_\mu \), and the fundamental vertical vector fields are \( Y^j_i = y^\mu_i \partial^j_\mu \), \( i,j,k = 1,\ldots,n \), where \( \partial_\mu = \frac{\partial}{\partial x^\mu} \) and \( \partial^j_\mu = \frac{\partial}{\partial y_j^\mu} \). The collection \( \{X_k,Y^j_i\} \) forms the standard basis of the Lie algebra \( \mathfrak{g} \) of left-invariant vector fields on \( G \).

The group \( G \) acts on \( F\mathbb{R}^n \) by prolongation,

\[ \phi(x,y) := (\phi(x),\phi'(x) \cdot y), \quad \text{where} \quad \phi'(x)^i_j := \partial^j_\mu \phi^i(x). \]

The algebra \( \mathcal{A} \) can be regarded as the subalgebra of the endomorphism algebra \( \text{End}_C(C_c^\infty(F\mathbb{R}^n)) \) generated by the multiplication and the translation operators

\[ M_f(\xi) = f \xi, \quad U^*_\varphi(\xi) = \xi \circ \varphi, \quad f, \xi \in C_c^\infty(F\mathbb{R}^n), \varphi \in G. \]

Letting the vector fields \( Z \in \mathfrak{g} \) act on \( \mathcal{A} \) by

\[ Z(f U^*_\varphi) = Z(f) U^*_\varphi, \quad f U^*_\varphi \in \mathcal{A}, \]
the resulting operators in $\mathcal{L}(\mathcal{A})$ satisfy generalized Leibnitz rules, which in the Sweedler notation take the form

$$Z(ab) = \sum_{(Z)} Z_{(1)}(a) Z_{(2)}(b), \quad a, b \in \mathcal{A}. \tag{2.2}$$

In particular, $X_k \in \mathcal{L}(\mathcal{A})$ satisfy

$$X_k(ab) = X_k(a)b + a X_k(b) + \delta^i_i(a) Y^j_i(b),$$

where $\delta^i_j(f U^*_0) = \gamma^i_j(\phi) f U^*_0$, with

$$\gamma^i_j(\phi)(x, y) = (y^{-1} \cdot \phi'(x)^{-1} \cdot \partial_x \phi'(x) \cdot y)^i_j y^\mu; \tag{2.3}$$

The operators $\delta^i_j \in \mathcal{L}(\mathcal{A})$ are derivations, but their successive commutators with the $X_i$’s yield operators $\delta^i_{jk \ell_1 \ldots \ell_r} = [X_{\ell_r}, \ldots, X_{\ell_1}, \delta^i_{jk}] \ldots]$, which involve multiplication by higher order jets of diffeomorphisms

$$\delta^i_{jk \ell_1 \ldots \ell_r}(f U^*_0) = \gamma^i_{jk \ell_1 \ldots \ell_r}(\phi) f U^*_0 \quad \text{where}$$

$$\gamma^i_{jk \ell_1 \ldots \ell_r}(\phi) = X_{\ell_r} \cdots X_{\ell_1} (\gamma^i_{jk}(\phi)), \quad \phi \in \mathcal{G}, \tag{2.4}$$

and satisfy progressively more elaborated Leibnitz rules. The subspace $\mathfrak{h}_n$ of $\mathcal{L}(\mathcal{A})$ generated by the operators $X_k$, $Y^i_j$, and $\delta^i_{jk \ell_1 \ldots \ell_r}$ forms a Lie algebra $\mathfrak{h}_n$. By definition, $\mathcal{H}_n$ is the algebra of operators in $\mathcal{L}(\mathcal{A})$ generated by $\mathfrak{h}_n$ and the scalars. For $n > 1$ the operators $\delta^i_{jk \ell_1 \ldots \ell_r}$ are not algebraically independent. They are subject to the “structure identities”

$$\delta^i_{jk} - \delta^i_{j k} = \delta^s_{j k} \delta^i_{s \ell} - \delta^s_{j \ell} \delta^i_{s k}, \tag{2.5}$$

reflecting the flatness of the standard connection. The algebra $\mathcal{H}_n$ is isomorphic to the quotient $\mathcal{A}(\mathfrak{h}_n)/\mathcal{I}$ of the universal enveloping algebra $\mathcal{A}(\mathfrak{h}_n)$ by the ideal $\mathcal{I}$ generated by the above identities. It possesses a distinguished character $\delta : \mathcal{H}_n \to \mathbb{C}$, which extends the modular character of $\mathfrak{g}l_n(\mathbb{R})$, and is induced from the character of $\mathfrak{h}_n$ defined by

$$\delta(Y^i_i) = \delta^i_i, \quad \delta(X_k) = 0, \quad \delta(\delta^i_{jk \ell_1 \ldots \ell_r}) = 0. \tag{2.6}$$

As coalgebras, $\mathcal{H}_n$ and $\mathcal{A}(\mathfrak{h}_n)$ differ drastically however. The coproduct of $\mathcal{H}_n$ stems from the interplay between the action of $\mathcal{H}_n$ and the product in $\mathcal{A}$. More precisely, it confers to $\mathcal{H}_n$ the only Hopf algebra structure for which $\mathcal{A}$ is a left $\mathcal{H}_n$-module algebra. Concretely, the formula (2.2) extends to all $h \in \mathcal{H}_n$,

$$h(ab) = \sum_{(h)} h_{(1)}(a) h_{(2)}(b), \quad h_{(1)}, h_{(2)} \in \mathcal{H}_n, \quad a, b \in \mathcal{A}, \tag{2.7}$$
and this uniquely determines a coproduct $\Delta : \mathcal{H} \to \mathcal{H}_n \otimes \mathcal{H}_n$, by
\[
\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}.
\]

The counit is $\varepsilon(h) = h(1)$, and there is a canonical twisted antipode determined by the canonical trace of the crossed product algebra $\mathcal{A}$, namely
\[
\tau(f U^*_\varphi) = \begin{cases} 
\int_{FIR^n} f \varpi, & \text{if } \varphi = Id, \\
0, & \text{otherwise};
\end{cases}
\]

here $\varpi$ is the volume form attached to the canonical framing given by the flat connection $\varpi = \bigwedge_{k=1}^n \theta^k \wedge \bigwedge_{(i,j)} \omega^j_i$ (ordered lexicographically).

This trace is $\delta$-invariant with respect to the action of $\mathcal{H}_n$, that is
\[
\tau(h(a)) = \delta(h) \tau(a), \quad h \in \mathcal{H}_n, \, a \in \mathcal{A}.
\]

The Leibnitz rule (2.7) together with the fact that the pairing $(a, b)$ is non-degenerate also ensure the existence and uniqueness of an anti-automorphism $S_\delta : \mathcal{H}_n \to \mathcal{H}_n$, $S_\delta^2 = \text{Id}$, satisfying
\[
\tau(h(a) b) = \tau(a S_\delta(h)(b)), \quad h \in \mathcal{H}_n, \, a, b \in \mathcal{A},
\]

as well as the involutive property
\[
S_\delta^2 = \text{Id}.
\]

Finally, the antipode of $\mathcal{H}_n$ is $S = \tilde{\delta} * S_\delta$, where $\tilde{\delta}$ is the convolution inverse of $\delta$.

Using (2.10), the standard Hopf cyclic model for $\mathcal{H}_n$ is “imported” from the standard cyclic model of the algebra $\mathcal{A}$, via the characteristic map
\[
(2.12) \qquad h^1 \otimes \ldots \otimes h^q \in \mathcal{H}_n^\otimes_q \longmapsto \chi_\tau(h^1 \otimes \ldots \otimes h^q) \in C^q(\mathcal{A}),
\]

where
\[
\chi_\tau(h^1 \otimes \ldots \otimes h^q)(a^0, \ldots, a^q) = \tau(a^0 h^1(a^1) \ldots h^q(a^q)), \quad a^i \in \mathcal{A}.
\]

This map is faithful and gives rise to a cyclic structure on $\{C^q(\mathcal{H}_n; \delta) : \mathcal{H}_n^\otimes_q\}_{q \geq 0}$, with faces, degeneracies and cyclic operator given by
\[
\begin{align*}
\delta_0(h^1 \otimes \ldots \otimes h^{q-1}) &= 1 \otimes h^1 \otimes \ldots \otimes h^{q-1}, \\
\delta_j(h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{q-1}, \quad 1 \leq j \leq q - 1, \\
\delta_n(h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes h^{q-1} \otimes 1; \\
\sigma_i(h^1 \otimes \ldots \otimes h^{q+1}) &= h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{q+1}, \quad 0 \leq i \leq q; \\
\tau_q(h^1 \otimes \ldots \otimes h^q) &= S_\delta(h^1) \cdot (h^2 \otimes \ldots \otimes h^q \otimes 1).
\end{align*}
\]
The cyclicity condition $\tau_q^{q+1} = \Id$ is satisfied precisely because of the involutive property (2.11), to which is actually equivalent.

The periodic Hopf cyclic cohomology $HP^\bullet(\mathcal{H}_n; \mathbb{C}_\delta)$ of $\mathcal{H}_n$ with coefficients in the modular pair $(\delta, 1)$ is, by definition (cf. [10, 11]), the $\mathbb{Z}_2$-graded cohomology of the total complex $CC^{tot\bullet}(\mathcal{H}_n; \mathbb{C}_\delta)$ associated to the bicomplex $\{CC^{*\bullet}(\mathcal{H}_n; \mathbb{C}_\delta), b, B\}$, where

$$CC^{p,q}(\mathcal{H}_n; \mathbb{C}_\delta) = \left\{ \begin{array}{ll}
C^{q-p}(\mathcal{H}_n; \mathbb{C}_\delta), & q \geq p, \\
0, & q < p, \end{array} \right.$$ 

$$b = \sum_{k=0}^{q+1} (-1)^k \delta_k, \quad B = \sum_{k=0}^q (-1)^k \tau_q^k \sigma_{q-1} \tau_q.$$

To define the periodic Hopf cyclic cohomology of $\mathcal{H}_n$ relative to $O_n$, one considers the quotient $Q_n = \mathcal{H}_n \otimes \mathcal{U}(a_n) \mathbb{C} \equiv \mathcal{H}_n / \mathcal{H}_n U^+(a_n)$, which is an $\mathcal{H}_n$-module coalgebra with respect to the coproduct and counit inherited from $\mathcal{H}_n$. Then $\{C^q(\mathcal{H}_n, O_n; \mathbb{C}_\delta) := (Q_n^{\otimes q})^{O_n}\}_{q \geq 0}$, is endowed with a cyclic structure given by restricting to $O_n$-invariants the operators

$$\delta_0(c^1 \otimes \ldots \otimes c^{q-1}) = \hat{1} \otimes c^1 \otimes \ldots \otimes c^{q-1},$$
$$\delta_i(c^1 \otimes \ldots \otimes c^{q-1}) = c^1 \otimes \ldots \otimes \Delta c^i \otimes \ldots \otimes c^{q-1}, \quad 1 \leq i \leq q - 1;$$
$$\delta_n(c^1 \otimes \ldots \otimes c^{q-1}) = c^1 \otimes \ldots \otimes c^{q-1} \otimes \hat{1};$$
$$\sigma_i(c^1 \otimes \ldots \otimes c^{q+1}) = c^1 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{q+1}, \quad 0 \leq i \leq q;$$
$$\tau_q(\hat{h}^1 \otimes c^2 \otimes \ldots \otimes c^q) = S_\delta(h^1) \cdot (c^2 \otimes \ldots \otimes c^q \otimes \hat{1}).$$

The resulting periodic cyclic cohomology is denoted $HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$.

2.2. Bicrossed product and Chevalley-Eilenberg cyclic model.

The Hopf algebra $\mathcal{H}_n$ can be reconstructed as bicrossed product of a matched pair of Hopf algebras of classical type (cf. [10, 23]). This structure arises naturally from the canonical splitting of the group $G$ as a set-theoretical product $G = G \cdot N$ of the group $G$ of affine motions of $\mathbb{R}^n$ and the group

$$N = \{ \psi \in G; \ \psi(0) = 0, \ \psi'(0) = \Id \}.$$ 

If $\phi \in G$ and $\phi_0 := \phi - \phi(0)$, then its canonical decomposition is

$$\phi = \varphi \circ \psi, \quad \varphi \in G, \ \psi \in N,$$

where

$$\varphi(x) = \phi'_0(0) \cdot x + \phi(0), \quad x \in \mathbb{R}^n \quad (2.14)$$
$$\psi(x) = \phi'_0(0)^{-1}(\phi(x) - \phi(0)).$$
Reversing the order in the above decomposition one simultaneously obtains a pair of well-defined operations, one of $N$ on $G$ and the other of $G$ on $N$:

$$\psi \circ \varphi = (\psi \triangleright \varphi) \circ (\psi \triangleleft \varphi), \quad \text{for } \varphi \in G \text{ and } \psi \in N.$$  

The operation $\triangleright$ is a left action of $N$ on $G$, and $\triangleleft$ is a right action of $G$ on $N$. Via the identification $G \simeq F\mathbb{R}^n$, one recognizes $\triangleright$ as being exactly the action by prolongation (2.1).

To reconstruct $H_n$ one actually uses the pronilpotent group of jets

$$\mathfrak{N} := \{ j^\infty_0(\psi) \mid \psi \in N \},$$

on which the jet components are regarded as affine coordinates. Thus, the algebra $F$ of regular functions on $\mathfrak{N}$ consists of polynomial expressions in the coordinates

$$\alpha^i_{jj_1j_2\ldots j_r}(\psi) = \partial_{j_r} \cdots \partial_{j_1} \partial_j \psi^i(x) \mid_{x=0}, \ 1 \leq i, j, j_1, j_2, \ldots, j_r \leq n.$$ 

Since $\alpha^i_j(\psi) = \delta^i_j$, and for $r \geq 1$ the coefficients $\alpha^i_{jj_1j_2\ldots j_r}(\psi)$ are symmetric in the lower indices but otherwise arbitrary, $F$ can be viewed as the free commutative algebra over $\mathbb{C}$ generated by the indeterminates $\{ \alpha^i_{jj_1j_2\ldots j_r}; 1 \leq j < j_1 < j_2 < \cdots < j_r \leq n \}$.

The algebra $F$ inherits from the group $\mathfrak{N}$ a canonical Hopf algebra structure, in the standard fashion, with the coproduct $\Delta : F \rightarrow F \otimes F$, the antipode $S : F \rightarrow F$, and the counit $\varepsilon : F \rightarrow \mathbb{C}$ determined by

$$\Delta(f)(\psi_1, \psi_2) = f(\psi_1 \circ \psi_2), \quad \psi_1, \psi_2 \in N,$$

$$S(f)(\psi) = f(\psi^{-1}), \quad \psi \in N, \quad f \in F,$$

$$\varepsilon(f) = f(e).$$

The coefficients of the Taylor expansion at $e \in G$ of the prolongation of $\psi \in N$,

$$\eta^i_{j\ell_1\ldots \ell_r}(\psi) = \gamma^i_{j\ell_1\ldots \ell_r}(\psi)(e),$$

are easily seen to be regular functions on $\mathfrak{N}$, which also generate $F$ as an algebra. Letting $H_{ab}$ denote the (commutative) Hopf subalgebra of $H_n$ generated by the operators $\{ \delta^i_{j\ell_1\ldots \ell_r}; 1 \leq i, j, \ell_1, \ldots, \ell_r \leq n \}$, one proves using the structure identities (2.5) that the assignment $\eta : H_{ab} \rightarrow F$ defined as

$$\eta(\delta^i_{j\ell_1\ldots \ell_r}) = \eta^i_{j\ell_1\ldots \ell_r}, \quad \forall 1 \leq i, j, \ell_1, \ldots, \ell_r \leq n .$$

defines an isomorphism of Hopf algebras. With $\mathfrak{g}$ denoting the Lie algebra of $G$, let $U := U(\mathfrak{g})$ be its universal enveloping algebra. The right action $\triangleleft$ of $G$ on $N$ induces an action of
\( g \) on \( \mathcal{F} \),

\[
(2.18) \quad (X \triangleright f)(\psi) = \frac{d}{dt} \bigg|_{t=0} f(\psi \circ \exp tX), \quad f \in \mathcal{F}, \ X \in g.
\]

and hence a left action \( \triangleright \) of \( U \) on \( \mathcal{F} \). Explicitly, for any \( u \in U \),

\[
(2.19) \quad (u \triangleright \eta^i_{jk\ell_1...\ell_r})(\psi) = u(\gamma^i_{jk\ell_1...\ell_r}(\psi))(e), \quad \psi \in N.
\]

The right hand side of \((2.19)\), before evaluation at \( e \in G \), describes the effect of the action of \( u \in U \) on \( \delta^i_{jk\ell_1...\ell_r} \in \mathcal{H}_{ab} \). From the very definition \((2.17)\), it follows that \( \eta : \mathcal{H}_{ab} \to \mathcal{F} \) identifies the \( U \)-module \( \mathcal{H}_{ab} \) with the \( U \)-module \( \mathcal{F} \). In particular \( \mathcal{F} \) is \( U \)-module algebra.

On the other hand, \( U \) carries a natural right \( \mathcal{F} \)-comodule structure \( \nabla : U \to U \otimes \mathcal{F} \), which can be suggestively described by assigning to each element \( u \in U \) a function from \( N \) to \( U \) defined by

\[
(2.20) \quad (\nabla u)(\psi) = \tilde{u}(\psi)(e), \quad \text{where} \quad \tilde{u}(\psi) = U_{\psi} u U_{\psi}^*.
\]

This coaction actually endows \( U(g) \) with the structure of a right \( \mathcal{F} \)-comodule coalgebra.

Thus equipped, \( U \) and \( \mathcal{F} \) form a matched pair of Hopf algebras, i.e. (with the usual conventions of notation) satisfy the compatibility conditions

\[
\epsilon(u \triangleright f) = \epsilon(u) \epsilon(f), \quad u \in U, \ f \in \mathcal{F}
\]

\[
\Delta(u \triangleright f) = u_{(1)} \triangleright (u_{(2)} \triangleright f) = u_{(1)} \triangleright f, \quad u \in U, \ f \in \mathcal{F}
\]

\[
\nabla(1) = 1 \otimes 1,
\]

\[
\nabla(uv) = u_{(1)} \triangleright v_{(0)} \otimes u_{(1)} \triangleright (u_{(2)} \triangleright v_{(1)}),
\]

\[
u_{(1)}(u_{(1)} \triangleright v_{(1)} \triangleright (u_{(2)} \triangleright f)) = u_{(1)} \triangleright u_{(2)} \triangleright f.
\]

One can then form the bicrossed product Hopf algebra \( \mathcal{F} \triangleright \triangleleft U \), which has the crossed coproduct \( \mathcal{F} \triangleright \triangleleft \mathcal{F} \) as underlying coalgebra, the crossed product \( \mathcal{F} \triangleright \triangleleft U \) as underlying algebra, and whose the antipode is given by

\[
S(f \triangleright \triangleleft u) = (1 \triangleright \triangleleft S(u_{(0)}))(S(f_{(1)}) \triangleright \triangleleft 1).
\]

The reconstruction of \( \mathcal{H}_n \) is now made precise by the statement that the Hopf algebras \( \mathcal{F} \triangleright \triangleleft U \) and \( \mathcal{H}_n^{\text{cop}} \) are canonically isomorphic [23, Thm. 2.15], via the identification which at the level of vector spaces can be described as \( \eta^{-1} \otimes \text{Id}_{U} : \mathcal{F} \triangleright \triangleleft U \to \mathcal{H}_n^{\text{cop}} \).

Exploiting the bicrossed product structure, and taking advantage of the extended framework for Hopf cyclic cohomology with coefficients [19], the complex \( CC^{\text{tot}}(\mathcal{H}_n; \mathbb{C}_g) \) can be replaced (cf. [23, 24]) by quasi-isomorphic bi-cyclic complexes. The latter amalgamate two classical
types of cohomological constructs, Lie algebra cohomology with coefficients and coalgebra cohomology with coefficients, with the essential distinction though that the coefficients are not only acted upon but also 'act back'.

The first such bicomplex \(C^{\bullet, \bullet}(\wedge g^*, \otimes F)\) is described by the diagram

\[
\begin{array}{c}
\cdots \\
\vdots \\
\vdots \\
\cdots \\
\end{array}
\]

\[
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array}
\]

\[
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{array}
\]

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
\cdots \\
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\]

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\begin{array}{c}
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\[
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{array}
\]

the coboundary \(\partial_q\) involves the action of \(g\) on the coefficients \(C_\delta \otimes F^q\), while \(b_F\) and \(B_F\) involve the coaction \(\nabla_g\). More precisely,

\[
b_F(1 \otimes \alpha \otimes f^1 \otimes \ldots \otimes f^q) = 1 \otimes \alpha \otimes 1 \otimes f^1 \otimes \ldots \otimes f^q + \sum_{1 \leq i \leq q} (-1)^i 1 \otimes \alpha \otimes f^1 \otimes \ldots \otimes \Delta(f^i) \otimes \ldots \otimes f^q + (-1)^{q+1} 1 \otimes \alpha_{<\ldots>} \otimes f^1 \otimes \ldots \otimes f^q \otimes S(\alpha_{<\ldots>});
\]

\[
B_F = \left( \sum_{i=0}^{q-1} (-1)^{(q-1)i} \tau_F^i \right) \sigma \tau_F (\text{Id} - (-1)^q \tau_F), \quad \text{with}
\]

\[
\tau_F(1 \otimes \alpha \otimes f^1 \otimes \ldots \otimes f^q) = 1 \otimes \alpha_{<\ldots>} \otimes S(f^1) \cdot (f^2 \otimes \ldots \otimes f^q \otimes S(\alpha_{<\ldots>}))
\]

and \(\sigma(1 \otimes \alpha \otimes f^1 \otimes \ldots \otimes f^q) = \varepsilon(f^q) \otimes \alpha \otimes f^1 \otimes \ldots \otimes f^{q-1}\).

The above bicomplex has a homogeneous version \(C^{\bullet, \bullet}_{F^q}(\wedge g^*, \otimes F)\), with

\[
C^{\bullet, \bullet}_{F^q}(\wedge g^*, \otimes F) := (\wedge^p g^* \otimes F^q+1)F,
\]

defined as follows. An element \(\sum \alpha \otimes \tilde{f} \in (\wedge^p g^* \otimes F^q+1)F\) if it satisfies the \(F\)-coinvariance condition:

\[
\sum \alpha_{<\ldots>} \otimes \tilde{f} \otimes S(\alpha_{<\ldots>}) = \sum \alpha \otimes \tilde{f}_{<\ldots>} \otimes \tilde{f}_{<\ldots>};
\]

here for \(\tilde{f} = f^0 \otimes \ldots \otimes f^q\), we have denoted

\[
\tilde{f}_{<\ldots>} \otimes \tilde{f}_{<\ldots>} = f^0_{(1)} \otimes \ldots \otimes f^q_{(1)} \otimes f^0_{(2)} \otimes \ldots \otimes f^q_{(2)}.
\]
The identification between the two complexes is made by the isomorphism

\[ I : \wedge^p g^* \otimes \mathcal{F}^{\otimes q} \xrightarrow{\sim} (\wedge^p g^* \otimes \mathcal{F}^{\otimes q+1})^F, \quad I(\alpha \otimes \tilde{f}) = \alpha_{<0>} \otimes f_1 \otimes S(f_1^{(2)}) f_2^{(1)} \otimes \cdots \otimes S(f_q^{(1)}) f_{q+1} \otimes S(\alpha_{<1>} f_q^{(2)}). \]

A closely related bicomplex replaces the tensor powers of the algebra \( \mathcal{F} \) with homogeneous cochains on the group of jets \( \mathfrak{N} \) with values in \( \wedge g^* \), namely \( \tilde{C}^{\ast, \ast}(\mathfrak{N}, \wedge g^*) \), defined as follows:

\[ \tilde{C}^q(\mathfrak{N}, \wedge g^*) = \{ c : \mathfrak{N}^{q+1} \to \wedge^p g^* \mid c(\psi_0, \ldots, \psi_q) = \psi^1 \circ c(\psi_0, \ldots, \psi_q), \forall \psi \in \mathfrak{N} \}, \]

with boundary operators \( \tilde{\partial} \) and \( \tilde{B} \),

\[ \begin{array}{ccc}
\tilde{C}^0(\mathfrak{N}, \wedge^2 g^*) & \xrightarrow{\tilde{b}} & \tilde{C}^1(\mathfrak{N}, \wedge^2 g^*) \\
\tilde{b} & \xrightarrow{\tilde{B}} & \tilde{b} \\
\tilde{C}^0(\mathfrak{N}, g) & \xrightarrow{\tilde{b}} & \tilde{C}^1(\mathfrak{N}, g^*) \\
\tilde{b} & \xrightarrow{\tilde{B}} & \tilde{b} \\
\tilde{C}^0(\mathfrak{N}, C) & \xrightarrow{\tilde{b}} & \tilde{C}^1(\mathfrak{N}, C) \\
\tilde{b} & \xrightarrow{\tilde{B}} & \tilde{b} \\
\end{array} \]

defined as follows:

\[ (\tilde{\partial} c)(\psi_0, \ldots, \psi_q) = \partial c(\psi_0, \ldots, \psi_q) - \sum_k \alpha^k \wedge (Z_k \circ c)(\psi_0, \ldots, \psi_q) \]

where \( \{ Z_k \} \) and \( \{ \alpha_k \} \) are dual bases of \( g \) and \( g^* \), and

\[ (Z \circ c)(\psi_0, \ldots, \psi_q) = \sum_i \frac{d}{dt} \bigg|_{t=0} c(\psi_0, \ldots, \psi_i \circ \exp(tZ), \ldots, \psi_q); \]

\[ \tilde{b} c(\psi_0, \ldots, \psi_{q+1}) = \sum_{i=0}^{q+1} (-1)^i c(\psi_0, \ldots, \hat{\psi}_i, \ldots, \psi_{q+1}); \]

\[ \tilde{B} = \left( \sum_{i=0}^{q-1} (-1)^i (q-1)^{q-1} i \tilde{\tau} \right) \tilde{\sigma} \tilde{\tau}, \quad \tilde{\tau}(c)(\psi_0, \ldots, \psi_q) = c(\psi_1, \ldots, \psi_q, 0), \]

and \( \tilde{\sigma}(c)(\psi_0, \ldots, \psi_{q-1}) = c(\psi_0, \ldots, \psi_{q-1}, \psi_q-1) \).
It is isomorphic to the bicomplex \((2.22)\) via the chain map

\[
\kappa : C^\bullet_\mathcal{F}(\wedge g^*, \bigotimes \mathcal{F}) \to \bar{C}^\bullet(\mathcal{N}, \wedge^p g^*), \]

\[
\kappa(\sum \alpha \otimes \tilde{f})(\psi_0, \ldots, \psi_q) = \sum f^0(\psi_0) \ldots f^q(\psi_q)\alpha.
\]

Finally, since \(\mathcal{F}\) is commutative one can restrict both sides to the quasi-isomorphic subcomplexes of totally antisymmetric cochains

\[
(2.24) \quad \kappa_\wedge := \kappa \circ \alpha_\mathcal{F} : C^\bullet_\mathcal{F}(\wedge g^*, \wedge \mathcal{F}) \to \bar{C}^\bullet_\wedge(\mathcal{N}, \wedge g^*),
\]

where \(\alpha_\mathcal{F} : C^\bullet_\mathcal{F}(\wedge g^*, \wedge \mathcal{F}) \to C^\bullet_\wedge(\wedge g^*, \bigotimes \mathcal{F})\) is the antisymmetrization map. Diagrammatically,

\[
\begin{array}{c}
\Lambda^2 g^* \xrightarrow{b_\wedge} (\Lambda^2 g^* \otimes \Lambda^2 \mathcal{F})^\mathcal{F} \xrightarrow{b_\wedge} (\Lambda^2 g^* \otimes \Lambda^3 \mathcal{F})^\mathcal{F} \xrightarrow{b_\wedge} \ldots \\
\downarrow{\partial} \downarrow{\partial} \downarrow{\partial} \\
g^* \xrightarrow{b_\wedge} (g^* \otimes \Lambda^2 \mathcal{F})^\mathcal{F} \xrightarrow{b_\wedge} (g^* \otimes \Lambda^3 \mathcal{F})^\mathcal{F} \xrightarrow{b_\wedge} \ldots \\
\downarrow{\partial} \downarrow{\partial} \downarrow{\partial} \\
\mathbb{C} \xrightarrow{b_\wedge} (\mathbb{C} \otimes \Lambda^2 \mathcal{F})^\mathcal{F} \xrightarrow{b_\wedge} (\mathbb{C} \otimes \Lambda^3 \mathcal{F})^\mathcal{F} \xrightarrow{b_\wedge} \ldots
\end{array}
\]

The boundary operators of the bicomplex \(C^\bullet_\mathcal{F}(\wedge g^*, \bigotimes \mathcal{F})\) acquire a simpler form when restricted to \(C^\bullet_\wedge(\wedge g^*, \wedge \mathcal{F})\). Thus, \(B_\mathcal{F} = 0\) and the others are given by

\[
b_\wedge(\alpha \otimes f^0 \wedge \cdots \wedge f^q) = \alpha \otimes 1 \wedge f^0 \wedge \cdots \wedge f^q;
\]

\[
\partial_\wedge(\alpha \otimes f^0 \wedge \cdots \wedge f^q) = \partial \alpha \otimes f^0 \wedge \cdots \wedge f^q = \sum_k \alpha^k \wedge \alpha \otimes \wedge Z_k \triangleright (f^0 \wedge \cdots \wedge f^q).
\]

The total cohomology of the above bicomplex is canonically isomorphic to \(HP^\bullet(\mathcal{H}_n; \mathbb{C}_d)\) and will be denoted by \(HP^\bullet(\mathcal{H}_n)\). The relative (to \(O_n\)) version of the above cohomology will be denoted \(HP^\bullet(\mathcal{H}_n, O_n)\). It is canonically isomorphic to \(HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_d)\), via the quasi-isomorphism of relative complexes obtained by restricting to \(O_n\)-basic cochains on both sides. This amounts to replacing \(g = \mathbb{R}^n \rtimes gl_n(\mathbb{R})\) by \(g / o_n\), and then restricting to \(O_n\)-invariant cochains.

The group \(O_n\) acts on \(\mathcal{N}\) by the restriction of the right action of \(G\).
The chain map $\kappa_\Lambda$ induces a quasi-isomorphism
\begin{equation}
\kappa_\Lambda^{O_n} : C^{\mathrm{tot}\,\cdot}(\wedge(g/o_n)^*, \wedge F)^{O_n} \to \bar{C}_\Lambda^{\mathrm{tot}\,\cdot}(\Omega, \wedge (g/o_n)^*)^{O_n}.
\end{equation}

## 3. Hopf cyclic characteristic cocycles

For the transfer of the characteristic cocycles to Hopf cyclic cohomology we shall use two analogues of the classical van Est isomorphism. The first one, recalled below, was established in [10] and provided the means to identify the Hopf cyclic cohomology of $H_n$ with the Gelfand-Fuks cohomology of the Lie algebra $a_n$. The second one, derived in §3.2, identifies the Hopf cyclic cohomology to the differentiable cohomology. The transferral proper of the characteristic cocycles is then achieved in §3.3.

### 3.1. From Lie algebra to Hopf cyclic cohomology

We begin by recalling the first quasi-isomorphism, in the form refined in [24]. The setting is very similar to that described in §1.1, only here it is specialized to $M = \mathbb{R}^n$, endowed with the standard flat connection, still denoted by $\nabla$. As in §2.2, we identify $F\mathbb{R}^n$ and the affine group $G$.

Instead of the map $\hat{\sigma} : |\Delta_G FM| \to F^\infty M$ of (1.12), we now consider the map $\hat{\varsigma} : |\hat{\Delta_N} F\mathbb{R}^n| \to F^\infty \mathbb{R}^n$, whose simplicial components are defined, in homogeneous group coordinates, by
\begin{equation}
\varsigma_p(t; \psi_0, \ldots, \psi_p, \varphi) = \varphi \cdot (s_{(\psi_0, \ldots, \psi_p)}(t) \triangleleft \varphi)^{-1}, \quad \text{where}
\end{equation}
\begin{equation}
s_{(\psi_0, \ldots, \psi_p)}(t) = \sum_{i=0}^{p} t_i j^\infty_0(\psi_i), \quad \varphi \in G, \; \psi_0, \ldots, \psi_p \in \mathbb{N}.
\end{equation}

The composition $D = \int_{\Delta^*} \circ \hat{\varsigma}^*$ defines a new map of complexes $D : C^{\cdot}(a_n) \to \bar{C}_d^{\mathrm{tot}\,\cdot}(N, \Omega^*(\hat{F}M))$, which satisfies the enhanced covariance property
\begin{equation}
D(\omega)(\psi_0 \triangleleft \phi, \ldots, \psi_p \triangleleft \phi) = \phi^* (D(\omega)(\psi_0, \ldots, \psi_p)), \; \forall \phi \in G.
\end{equation}

Taking $\phi \in G$, this relation shows that $D(\omega)$ is completely determined by its values at the identity $e \in G$. One is led then to define
\begin{equation}
\mathcal{E}(\omega)(\psi_0, \ldots, \psi_p) := D(\omega)(\psi_0, \ldots, \psi_p) \big|_{\varphi=e} \in \wedge^\cdot g^*.
\end{equation}

A more explicit expression for $\mathcal{E}(\omega)$ is obtained as follows. Fix a basis $\{\alpha_k\}$ of $g^*$, and denote by $\{\tilde{\alpha}_k\}$ the corresponding left invariant forms on $G$, and by $\{Z_k^i\}$ be the dual basis of left invariant vector fields.
Define \( \nu(\varphi, \psi) := \varphi \circ (\psi \circ \varphi)^{-1} \), and let \( \iota : G \to G \) be the inversion map \( \iota(\rho) = \rho^{-1} \). Then

\[
\mathcal{E}(\omega) = \int_{\Delta^r} \xi^r(\mu(\omega)), \quad \text{where} \quad \mu(\omega) = \sum_{|I|=r} \iota^*(i_{Z_I} \nu^*(\tilde{\omega})) \otimes \alpha_I,
\]

with \( I = (i_1 < \ldots < i_r) \) and \( \alpha_I = \alpha_{i_1} \wedge \ldots \wedge \alpha_{i_r} \).

or in a more suggestive notation,

\[
(3.4) \quad \mathcal{E}(\omega)(\psi_0, \ldots, \psi_p) = \int_{\Delta(\psi_0, \ldots, \psi_p)} \mu(\omega).
\]

The way in which \( \mathcal{D}(\omega) \) can be recovered from \( \mathcal{E}(\omega) \) is made precise by the following identity:

\[
\mathcal{D}(\omega)(\psi_0, \ldots, \psi_p) \mid_\varphi = \int_{\Delta(\psi_0 \circ \varphi, \ldots, \psi_p \circ \varphi)} \tilde{\mu}(\omega) \mid_\varphi,
\]

where \( \tilde{\mu}(\omega) = \sum_{|I|=r} \iota^*(i_{Z_I} \nu^*(\tilde{\omega})) \otimes \tilde{\alpha}_I \).

Note that the only difference between \( \mu(\omega) \) and \( \tilde{\mu}(\omega) \) is the replacement of the \( \alpha_I \in \wedge^* g^* \) by the associated left invariant forms \( \tilde{\alpha}_I \in \Omega^*(G) \).

Thus, denoting by \( L_\varphi \) the left translation by \( \varphi \in G \), the above identity can be stated in the equivalent form

\[
(3.5) \quad \mathcal{D}(\omega)(\psi_0, \ldots, \psi_p) \mid_\varphi = L_\varphi^* (\mathcal{E}(\omega)(\psi_0 \circ \varphi, \ldots, \psi_p \circ \varphi)).
\]

The first part of the van Est theorem, proved in [10] and in the form stated below in [21], can be formulated as follows.

**Theorem 3.1.** For any \( \omega \in C^*(a_n) \), \( \mathcal{E}(\omega) \in C^*_\mathcal{F}(\wedge g^*, \wedge \mathcal{F}) \) and the resulting map \( \mathcal{E} : C^*(a_n) \to C^*_\mathcal{F}(\wedge g^*, \wedge \mathcal{F}) \) is a quasi-isomorphism. The induced map \( \mathcal{E}^\mathcal{O}_n : C^*(a_n, O_n) \to C^*_\mathcal{F}(\wedge (g/o_n)^*, \wedge \mathcal{F})^\mathcal{O}_n \) is also a quasi-isomorphism.

The full version of the van Est theorem (cf. [10]) involves the map \( \Phi \) of Connes [8 III.2.6], so we recall its definition specialized to our context.

Consider the DG-algebra, \( \mathcal{B}_G(G) = \Omega^*_c(G) \otimes \mathbb{C}[G'] \), where \( G' = G \setminus \{e\} \), with the differential \( d \otimes \text{Id} \). One labels the generators of \( \mathbb{C}[G'] \) as \( \gamma_\phi, \phi \in G \), with \( \gamma_e = 0 \), and one forms the crossed product \( \mathcal{C}_G(G) = \mathcal{B}_G(G) \times G \), with the commutation rules

\[
U_{\phi_2}^* \omega U_{\phi_1} = \phi^* \omega, \quad \omega \in \Omega^*_c(G),
\]

\[
U_{\phi_2} \gamma_{\phi_2} U_{\phi_1} = \gamma_{\phi_2 \circ \phi_1} - \gamma_{\phi_1}, \quad \phi_1, \phi_2 \in G.
\]
\( \mathcal{C}_G(G) \) is also a DG-algebra, equipped with the differential

\[
\partial(bU^*_\phi) = \partial bU^*_\phi - (-1)^{\partial b} b \gamma_\phi U^*_\phi, \quad b \in \mathcal{B}_G(G), \quad \phi \in \mathcal{G},
\]

A cochain \( \lambda \in \bar{\mathcal{C}}^q(\mathcal{G}, \Omega^p(G)) \) determines a linear form \( \tilde{\lambda} \) on \( \mathcal{C}_G(G) \) as follows:

\[
\tilde{\lambda}(bU^*_\phi) = 0 \quad \text{for} \quad \phi \neq 1;
\]

if \( \phi = 1 \) and \( b = \omega \otimes \gamma_{\rho_1} \cdots \gamma_{\rho_q} \) then

\[
\tilde{\lambda}(\omega \otimes \gamma_{\rho_1} \cdots \gamma_{\rho_q}) = \int_G \lambda(1, \rho_1, \ldots, \rho_q) \wedge \omega.
\]

The map \( \Phi \) from \( \bar{\mathcal{C}}^\bullet(\mathcal{G}, \Omega^\bullet(G)) \) to the \((b,B)\)-complex of the algebra \( \mathcal{A} = C^\infty_c(G) \rtimes \mathcal{G} \) is now defined for \( \lambda \in \bar{\mathcal{C}}^q(\mathcal{G}, \Omega^p(G)) \) by

\[
\Phi(\lambda)(a^0, \ldots, a^m) = \frac{p!}{(m+1)!} \sum_{j=0}^m (-1)^{j(m-j)} \tilde{\lambda}(da^{i+1} \cdots da^m a^0 da^1 \cdots da^j)
\]

where \( m = \dim G - p + q, \quad a^0, \ldots, a^m \in \mathcal{A} \).

By [8, III.2.δ, Thm. 14], \( \Phi \) is a chain map to the total \((b,B)\)-complex of the algebra \( \mathcal{A} \).

It is shown in [10, pp. 233-234], that if \( \lambda \in \bar{\mathcal{C}}^q(\mathcal{G}, \Omega^p(G)) \) is of the form \( \lambda = D(\omega) \) with \( \omega \in C(\mathcal{a}_n) \) then \( \Phi(\lambda) \) has the expression

\[
\Phi(\lambda)(a^0, \ldots, a^q) = \sum_{\alpha} \tau(a^0 h^1_\alpha(a^1) \cdots h^q_\alpha(a^q)), \quad h^i_\alpha \in \mathcal{H}_n;
\]

the tensor \( \sum_{\alpha} h^1_\alpha \otimes \cdots \otimes h^q_\alpha \in \mathcal{H}_n^{\otimes q} \) is uniquely determined, because the characteristic map [2.12] is faithful. Via the corresponding identification, \( \Phi(\lambda) \) becomes a chain in the \((b,B)\)-complex which defines the Hopf cyclic cohomology of \( \mathcal{H}_n \). By restricting \( \Phi \) to the subcomplex

\[
\bar{\mathcal{C}}^{\text{tot}}_D(\mathcal{G}, \Omega^\bullet(G)) := D(C(\mathcal{a}_n)) \subset \bar{\mathcal{C}}^{\text{tot}}_D(\mathcal{G}, \Omega^\bullet(G)),
\]

one thus obtains a map

\[
\Phi_D : \bar{\mathcal{C}}^{\text{tot}}_D(\mathcal{G}, \Omega^\bullet(G)) \to \text{CC}^{\text{tot}}(\mathcal{H}_n, \mathbb{C}_\delta)
\]

(3.11)

\[
\Phi_D(\lambda) = \sum_{\alpha} h^1_\alpha \otimes \cdots \otimes h^q_\alpha \in \mathcal{H}_n^{\otimes q}.
\]

\( \Phi_D \) is tautologically a chain map, because the Hopf cyclic structure of \( \mathcal{H}_n \) was imported from that of the cyclic complex of \( \mathcal{A} \).

Furthermore, by restriction to \( \mathcal{O}_n \)-basic forms on \( G \) one obtains the relative version of the above chain map, which lands in the relative version of the above Hopf cyclic complex \( \mathcal{H}_n^2(\mathcal{O}_n; \delta) \).
Theorem 3.2. The composition $\Phi_d \circ \mathcal{D} : C^\bullet(a_n) \to CC_{\text{tot}}^\bullet(H_n; \mathbb{C}_\delta)$, together with its restriction $C^\bullet(a_n, O_n) \to CC_{\text{tot}}^\bullet(H_n, O_n; \mathbb{C}_\delta)$, are quasi-isomorphisms.

3.2. From Hopf cyclic to differentiable cohomology. In [24, §3.2] we have constructed another map of bicomplexes, $\Theta : C^\bullet_2(\wedge g^*, \wedge F) \to \tilde{C}^\bullet_2(G, \Omega^*(G))$, and we are now in a position to prove that it too is a quasi-isomorphism.

In order to define it, we recall the isomorphism $\eta : \mathcal{H}_{\text{ab}} \to F$ of (2.17), and denote its inverse $\delta = \eta^{-1}$. Given $f \in F$, one defines the function $\gamma(f) : G \to C^\infty(G)$ by

$$\delta(S(f))(U_\phi) = \gamma(f)(\phi) U^*_\phi, \quad \forall \phi \in G. \tag{3.12}$$

The left hand side uses the natural action of $\mathcal{H}_n$ on the crossed product algebra $\hat{A} = C^\infty(G) \rtimes G$. The function $\gamma(f)(\phi) \in C^\infty(G)$, depends smoothly (in fact algebraically) on the components of the $k$-jet of $\phi$, for some $k \in \mathbb{N}$. For example, one can easily see that

$$\gamma(S(\eta^i_{jk}))(\phi^{-1}) = \gamma^i_{jk}(\phi), \quad \phi \in G. \tag{3.13}$$

With this notation, $\Theta : C^\bullet_2(\wedge g^*, \wedge F) \to \tilde{C}^\bullet_2(G, \Omega^*(G))$ is given by the formula

$$\Theta\left( \sum_{|I| = q} \alpha_I \otimes \mathit{I}f_0 \wedge \cdots \wedge \mathit{I}f_p \right)(\phi_0, \ldots, \phi_p) = \sum_I \sum_{\sigma \in S_{n+1}} (-1)^\sigma \gamma(S(\mathit{I}f^{\sigma(0)}))(\phi_0^{-1}) \cdots \gamma(S(\mathit{I}f^{\sigma(p)}))(\phi_p^{-1}) \tilde{\alpha}_I; \tag{3.14}$$

here, as in §3.1, $\tilde{\alpha}_I$ stands for the left-invariant form on $G$ corresponding to $\alpha_I \in \wedge g^*$.

We shall first show that the map $\Theta$ satisfies a property completely similar to that described by formula (3.5). Given the form of the expression in the right hand side of (3.14), to justify this it suffices to prove the following lemma.

Lemma 3.3. For any $f \in F$, $\psi \in \mathbb{N}$ and $\varphi \in G$, one has

$$\gamma(S(f))(\psi^{-1})(\varphi) = \gamma(S(f))(\psi \circ \varphi)^{-1})(e). \tag{3.15}$$

Proof. Using the cocycle property of $\gamma^i_{jk}$ and the fact that $\varphi \in G$ is affine, one has for any $\psi \in \mathbb{N}$,

$$\gamma^i_{jk}(\psi \varphi) = \gamma^i_{jk}(\psi) \circ \varphi + \gamma^i_{jk}(\varphi) = \gamma^i_{jk}(\psi) \circ \varphi.$$
By successive differentiation with respect to left invariant vector fields $X_k$, one obtains
\[ \gamma^i_{jkl_1\ldots l_r} (\psi \varphi) = \gamma^i_{jkl_1\ldots l_r} (\psi) \circ \varphi. \]

Letting $e = (0, 1)$ be the base frame we note that, by our identification of $G$ with $F\mathbb{R}^n$, $\varphi(e) \equiv \varphi$. Thus, the above identity evaluated at $e$ gives
\[ (3.16) \quad \gamma^i_{jkl_1\ldots l_r} (\psi \varphi)(e) = \gamma^i_{jkl_1\ldots l_r} (\psi)(\varphi(e)). \]

Again by the cocycle property, if $\rho \in G$ then
\[ \gamma^i_{jk}(\rho \phi) = \gamma^i_{jk}(\rho) \circ \phi + \gamma^i_{jk}(\phi) = \gamma^i_{jk}(\phi). \]

Therefore, by differentiation,
\[ (3.17) \quad \gamma^i_{jkl_1\ldots l_r} (\rho \phi) = \gamma^i_{jkl_1\ldots l_r} (\phi), \quad \rho \in G, \quad \phi \in G. \]

Writing now $\psi \varphi = (\psi \triangleright \phi)(\psi \triangleleft \varphi)$, for $\varphi \in G$, $\psi \in N$, on applying (3.17) one obtains
\[ \gamma^i_{jkl_1\ldots l_r} (\psi \varphi) = \gamma^i_{jkl_1\ldots l_r} (\psi \triangleleft \varphi), \]

and so by (3.16),
\[ (3.18) \quad \gamma^i_{jkl_1\ldots l_r} (\psi)(\varphi) = \gamma^i_{jkl_1\ldots l_r} (\psi \triangleleft \varphi)(e). \]

On the other hand, the identity (3.13) is valid for higher order jets. Indeed, applying the definition (3.12) to $f = \eta^i_{jkl}$, one has
\[ \gamma(S(\eta^i_{jkl}))(\phi^{-1}) U^*_\phi = \delta^i_{jkl} (U^*_\phi) = [X_\ell, \delta^i_{jkl}](U^*_\phi) = X_\ell (\delta^i_{jkl}(U^*_\phi)) = X_\ell (\gamma^i_{jkl}(\phi)) U^*_\phi. \]

Repeated applications give the general identity
\[ (3.19) \quad \gamma(S(\eta^i_{jkl_1\ldots l_r}))(\phi^{-1}) = \gamma^i_{jkl_1\ldots l_r}(\phi). \]

The relations (3.16) and (3.19) taken together imply
\[ \gamma(S(\eta^i_{jkl_1\ldots l_r}))(\psi^{-1})(\varphi) = \gamma(S(\eta^i_{jkl_1\ldots l_r}))(\psi \triangleleft \varphi)^{-1}(e), \]

which proves the statement for a set of generators of the algebra $F$.

To complete the proof it remains to notice that
\[ \gamma(f_1 f_2) = \gamma(f_1) \gamma(f_2), \quad f_1 f_2 \in F, \]

and therefore both sides of the desired relation behave multiplicatively. \[ \Box \]

Relying on this lemma, we can now prove a key relation between the maps of complexes constructed before.

**Lemma 3.4.** One has $\Theta \circ \mathcal{E} = \mathcal{D}$. 
Proof. Let $\omega \in C^\bullet(a_n)$ and denote $\varpi = \mathcal{E}(\omega) \in C^\bullet(\wedge g^*, \wedge \mathcal{F})$. By the very definition (3.3),

$$D(\omega)(\psi_0, \ldots, \psi_p)(e) = \varpi(\psi_0, \ldots, \psi_p)(e) \in \wedge g^*,$$

while by (3.5) on the one hand and Lemma 3.3 on the other, for any $\varphi \in G$ one has

$$D(\omega)(\psi_0, \ldots, \psi_p)(\varphi) = L^*_\varphi(\varpi(\psi_0, \ldots, \psi_p)(e)) = \Theta(\varpi)(\psi_0, \ldots, \psi_p)(\varphi).$$

The next important step is to reconcile the two maps denoted by $D$, that defined in Theorem 1.3 and the one defined in §3.1.

Lemma 3.5. With $\nabla$ denoting the standard flat linear connection on $\mathbb{R}^n$, one has $D_\nabla = D$.

Proof. The construction of the two maps starts with two different cross-sections, $\sigma_\nabla : F\mathbb{R}^n \to F^\infty \mathbb{R}^n$ defined by (1.9) and $\varsigma : F\mathbb{R}^n \to F^\infty \mathbb{R}^n$ of (3.1). We need to show that they both lead to the same map from $\Delta_G F\mathbb{R}^n$ to $F^\infty \mathbb{R}^n$.

Let $u \in F\mathbb{R}^n$ be represented as $j_0^1(\varphi) \equiv \rho$, with $\rho \in G$. We claim that for any $\psi \in N$,

$$\sigma_{\nabla \psi}(u) = \rho \cdot j_0^\infty(\psi \circ \rho)^{-1}. \quad (3.20)$$

Indeed with the usual identification $G \cong F\mathbb{R}^n$,

$$\exp_{\rho(0)}^\nabla(u(\xi)) = \exp_{\rho(0)}^\nabla(\rho'(0)\xi) = \rho(0) + \rho'(0)\xi = \rho(\xi), \quad \xi \in \mathbb{R}^n.$$ 

Since the left action of $N$ on $G$ coincides with the natural action on $F\mathbb{R}^n$, $\psi(u) \equiv \psi \circ \rho$. Thus, using the naturality property (1.11) one obtains

$$\sigma_{\nabla \psi}(u) = j_0^\infty(\exp_{\rho(0)}^\nabla \circ u) = j_0^\infty(\psi^{-1} \circ \exp_{\rho(0)}^\nabla \circ \psi(u))$$

$$= j_0^\infty(\psi^{-1} \circ (\psi \circ \rho)) = j_0^\infty(\psi^{-1} \circ (\psi \circ (\psi \circ \rho)) \circ (\psi \circ (\psi \circ \rho)^{-1})$$

$$= j_0^\infty(\psi^{-1} \circ \rho \circ (\psi \circ (\psi \circ (\rho \circ (\psi \circ (\rho)^{-1}) = \rho \cdot j_0^\infty(\psi \circ (\rho)^{-1}. $$

This shows that

$$\sigma_p(t; \psi_0, \ldots, \psi_p, \rho) = \varsigma_p(t; \psi_0, \ldots, \psi_p, \rho).$$

Now let $\phi \in G$, factorized as the product $\varphi \circ \psi$, with $\varphi \in G$ and $\psi \in N$. Then $\sigma_{\nabla \phi} = \sigma_{\nabla \varphi \circ \psi} = \sigma_{\nabla \psi}$, because $\nabla^\varphi = \nabla$.

Therefore, if $\phi_i = \varphi_i \psi_i$, with $\varphi_i \in G$ and $\psi_i \in N$, then for any $\rho \in G$,

$$\sigma_p(t; \phi_0, \ldots, \phi_p, \rho) = \varsigma_p(t; \psi_0, \ldots, \psi_p, \rho), \quad (3.21)$$

which completes the proof. □
Lemmas 3.4 and 3.5 taken together ensure that $\Theta \circ E = D$. Since both $D$ and $E$ are quasi-isomorphisms, so must be $\Theta$. This achieves the proof of the second explicit analogue of the van Est isomorphism:

**Theorem 3.6.** The map $\Theta : C_{\mathcal{F}}^{\text{tot}}(\wedge g^*, \wedge \mathcal{F}) \to C_{d}^{\text{tot}}(G, \Omega^*(G))$ and the induced map $\Theta_{O_n} : C_{\mathcal{F}}^{\text{tot}}(\wedge (g/\phi_n)^*, \wedge \mathcal{F})_{O_n} \to C_{d}^{\text{tot}}(G, \Omega^*(G/\phi_n))$ are quasi-isomorphisms.

### 3.3. Hopf cyclic Vey bases.

As above, $\nabla$ stands for the flat connection on the frame bundle $G \equiv F \mathbb{R}^n \to \mathbb{R}^n$, with connection form $\omega_{\nabla} = (\omega_j^i)$, where

$$\omega_j^i := (y^{-1})^i_\mu dy^\mu_j = (y^{-1} dy)^i_j, \quad i, j = 1, \ldots, n.$$  

Its pull-back under the action (2.1) of $\phi \in G$, is the connection

$$\phi^*(\omega_j^i) = \omega_j^i + \gamma^i_{jk}(\phi) \theta^k. \quad (3.22)$$

Thus, the simplicial connection is

$$\hat{\omega}_{\nabla}(t; \phi_0, \ldots, \phi_p)_{ij} = \sum_{r=0}^{p} t_r \phi_r^*(\omega_j^i) = \omega_j^i + \sum_{r=0}^{p} t_r \gamma^i_{jk}(\phi_r) \theta^k \quad (3.23)$$

and, taking into account that $\Omega_{\nabla} = 0$, the formula (1.16) for the simplicial curvature becomes

$$\hat{\Omega}_{\nabla}(t; \phi_0, \ldots, \phi_p) = \sum_{r=0}^{p} dt_r \wedge \phi_r^*(\omega_{\nabla}) - \sum_{r=0}^{p} t_r \phi_r^*(\omega_{\nabla}) \wedge \phi_r^*(\omega_{\nabla})$$

$$+ \sum_{r,s=0}^{p} t_r t_s \phi_r^*(\omega_{\nabla}) \wedge \phi_s^*(\omega_{\nabla}). \quad (3.24)$$

Both $\hat{\omega}_{\nabla}$ and $\hat{\Omega}_{\nabla}$ are polynomial forms on $\Delta_p$ tensored by left $G$-invariant forms $\{\omega^i_j, \theta^k\}$ multiplied by components of the first-order jet of the prolongation,

$$\gamma^i_{jk}(\phi)(x, y) = (y^{-1} \cdot \phi'(x)^{-1} \cdot \partial_\mu \phi'(x) \cdot y)^i_j y^\mu_k. \quad (3.25)$$

Recall now that in §1.3 (see Corollaries 1.7 and 1.8) we have constructed characteristic cocycles $C_{I,J}(\nabla)$ in the differentiable Bott bi-complex, and also that Theorem 3.6 provides a quasi-isomorphism $\Theta : C_{\mathcal{F}}^{\text{tot}}(\wedge g^*, \wedge \mathcal{F}) \to C_{d}^{\text{tot}}(G, \Omega^*(G))$.

**Theorem 3.7.** The characteristic cocycles $C_{I,J}(\nabla)$, where $(I, J)$ are running over the set $\mathcal{V}_n$, resp. $\mathcal{VO}_n$, are of the form

$$C_{I,J}(\nabla) = \Theta(\kappa_{I,J}); \quad (3.26)$$
\(\kappa_{I,J}\) are explicitly defined cocycles in the bicomplex \(C^\text{tot} \mathcal{F}^\ast(\wedge \mathfrak{g}^\ast, \wedge \mathcal{F})\), and their cohomology classes form a basis of \(\text{HP}^\ast(\mathcal{H}_n)\), resp. \(\text{HP}^\ast(\mathcal{H}_n, \Omega_n)\).

**Proof.** The cocycles \(C_{I,J}(\nabla)\) are homogeneous and totally antisymmetric form-valued group cochains in the differentiable Bott bicomplex. Moreover, their values are combinations of invariant forms on \(G \equiv FR^n\) with coefficients polynomial expressions in \(\gamma_{i_{jk}}'(\phi_r)\)'s. Also, in view of the equality (3.21), one can restrict the simplicial construction to the group \(N\), and thus assume \(\phi_r \in N\).

It then follows from the very definition of the map (3.14) together with the identity (3.13) that the preimage of these cochains in the bicomplex \(C^\text{tot} \mathcal{F}(\wedge \mathfrak{g}^\ast, \wedge \mathcal{F})\) is obtained by simply replacing the \(\gamma_{i_{jk}}\)'s with \(\eta_{i_{jk}}\)'s and the \(G\)-invariant forms \(\tilde{\alpha}_I \in \Omega^\ast(G)\) by their values at the identity, \(\alpha_I \in \wedge^\ast \mathfrak{g}^\ast\).

In view of Theorem 3.6, one obtains this way a basis of Hopf cyclic characteristic classes. \(\square\)

**Remark 3.8.** From the formulas (3.22), (3.23), (3.24) and the very definition of the cocycles \(\kappa_{I,J}\), it is clear that their tensor components are “economically” manufactured solely out of elements from \(\wedge \mathfrak{g}^\ast\) tensored by exterior powers of the algebra generated by \(\{\eta_{i_{jk}}; 1 \leq i, j, k \leq n\}\).

This feature constitutes the analogue of the well-known fact that the Gelfand-Fuks cohomology classes are representable in terms of 2-jets. Returning now to the standard Hopf cyclic cohomological model (see §2.1), we recall that the cocycles \(C_{I,J}(\nabla)\) were obtained by transferring Vey bases of \(H^\ast(a_n)\), resp. \(H^\ast(a_n, \Omega_n)\), via the quasi-isomorphism \(D_{\nabla}\).

Thus, by construction they belong to the subcomplex \(C^\text{tot} \mathcal{F}(G, \Omega^\ast(G))\) and therefore can be further transported via the quasi-isomorphism \(\Phi_d : C^\text{tot} \mathcal{F}(G, \Omega^\ast(G)) \to CC^\text{tot} \mathcal{F}(\mathcal{H}_n; \mathbb{C}_3)\) of (3.11). Since by Theorem 3.2 the composition \(\Phi_d \circ D_{\nabla}\) is a quasi-isomorphism, we conclude that:

**Theorem 3.9.** (1) The cocycles \(c_{I,J}(\nabla) = \Phi_d(C_{I,J}(\nabla))\), with \((I, J) \in \mathcal{V}_n\), form a complete set of representatives for the periodic Hopf cyclic cohomology \(H^P\ast(\mathcal{H}_n; \mathbb{C}_3)\).

(2) The cocycles \(c_{I,J}(\nabla) = \Phi^O_d(C_{I,J}(\nabla))\), with \((I, J) \in \mathcal{V}_0\), form a complete set of representatives for the relative periodic Hopf cyclic cohomology \(H^P\ast(\mathcal{H}_n, \Omega_0; \mathbb{C}_3)\).

**Remark 3.10.** As a final remark, paralleling Remark 3.8 we note that the cocycles \(\Phi_d(C_{I,J}(\nabla))\) are also “economically” constructed. Indeed, their action on tensor products of monomials of the form \(a = f U^a_\phi \in \mathcal{A}\) only involves the vector fields \(X_k, Y^i_j\) applied to the function \(f \in C^\infty(F\mathbb{R}^n)\) and the operators \(\delta^i_{jk}\) applied to the diffeomorphism \(\phi \in G\).
The vector fields appear because
\[ df = \sum_{k=1}^{n} X_k(f) \theta^k + \sum_{i,j=1}^{n} Y^i_j(f) \omega^j_i, \]
while the multiplication operators \( \{ \delta^i_{jk} \} \) show up because of the conjugation relation
\[ U_{\phi} df U_{\phi}^* = \sum_{k=1}^{n} (X_k(f) \circ \phi) \theta^k + \sum_{i,j=1}^{n} (Y^i_j(f) \circ \phi) (\omega^j_i + \gamma^i_{jk}(\phi) \theta^k). \]

Since the characteristic map \( \text{(2.12)} \) is known to be faithful, it follows that the Hopf cyclic cocycles \( c_{I,J}(\nabla) \in \sum_{q \geq 0} H_n^{\otimes q} \) (resp. \( \sum_{q \geq 0} (Q_n^{\otimes q})^{O_n} \)) have their tensorial components made out of the basic generators \( X_k, Y^i_j \) and \( \delta^i_{jk} \), and do not involve any \( \delta^i_{jk} \ldots \) operators of higher order.

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