Some Algebraic Symmetries
of (2,2)-Supersymmetric Systems

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ABSTRACT
The Hilbert spaces of supersymmetric systems admit symmetries which are often related to the topology and geometry of the (target) field-space. Here, we study certain (2,2)-supersymmetric systems in 2-dimensional spacetime which are closely related to superstring models. They all turn out to possess some hitherto unexploited and geometrically and topologically unobstructed symmetries, providing new tools for studying the topology and geometry of superstring target spacetimes, and so the dynamics of the effective field theory in these.

TANSTAAFL?

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1. Introduction

It has been known by now for quite some time [1] that there exists a formal but rather precise analogy between supersymmetry and exterior calculus. This analogy derives from the fact that the generators of supersymmetry are anticommuting and so nilpotent differential operators just as exterior derivatives are.

We now turn to our case of interest: the 2-dimensional (2,2)-supersymmetric gauged linear \( \sigma \)-models [2], exemplified here by a simple Landau-Ginzburg/Calabi-Yau model. For simplicity, we consider a simple hypersurface in a projective space; generalizations to intersections of hypersurfaces in toric varieties and the corresponding more general gauged linear \( \sigma \)-models are notationally tedious but straightforward. Indeed, the same analysis will apply to gauged \( \sigma \)-models with non-Abelian gauge symmetries, and so geometries of complete intersections of hypersurfaces within non-abelian orbit spaces, not just toric varieties. In the present case, the commuting canonical coordinates are \( \phi^\mu, \bar{\phi}^{\overline{\mu}} \), with \( \mu = 0, \ldots, n \), the map immersing the world sheet (Riemann surface) into the target space \( \mathbb{P}^n \) for which the \( \phi^\mu \) serve as homogeneous coordinates. The supersymmetric ground states turn out to be further constrained to \( X \), the hypersurface \( W(\phi)=0 \) in \( \mathbb{P}^n \).

The anticommuting variables, \( \psi^\mu_+, \psi^\mu_- \) are local sections of \( K^{\pm1/2} \otimes \phi^* (T_X) \) and \( \bar{K}^{\pm1/2} \otimes \bar{\phi}^* (\bar{T}_X) \), where \( K \) is the canonical bundle of the world-sheet. They satisfy the equal-time anticommutation relations

\[
\{ \psi^\mu_\pm, \psi^{\mu'}_{\pm} \} = g^{\mu\mu'} ,
\]

where \( g^{\mu\mu'} \) is a metric on the target space \( X \). Owing to (1), half of the \( \psi_+ \)'s and half of the \( \psi_- \)'s can be interpreted as creation operators, the other half then being annihilation operators. That is, there are two possible choices of Clifford-Dirac vacua (and their conjugates):

\[
|\eta; b, q\rangle_{cc} = \eta^\mu_{\overline{\nu}_1 \cdots \overline{\nu}_q} (\phi, \bar{\phi}) \psi^\mu_1 \cdots \psi^\mu_b + \psi^{\overline{\nu}_1} \cdots \psi^{\overline{\nu}_q} |0\rangle_{cc} ,
\]

and

\[
|\omega; p, q\rangle_{ac} = \omega^\mu_{\nu_1 \cdots \nu_p} (\phi, \bar{\phi}) \psi^\mu_1 \cdots \psi^\mu_p \psi^{\nu_1} \cdots \psi^{\nu_q} |0\rangle_{ac} .
\]

Using these two distinct vacua, two distinct types of states (and their conjugates) can be defined:

\[
|\eta; b, q\rangle_{cc} = \eta^\mu_{\overline{\nu}_1 \cdots \overline{\nu}_q} (\phi, \bar{\phi}) \psi^\mu_1 \cdots \psi^\mu_b + \psi^{\overline{\nu}_1} \cdots \psi^{\overline{\nu}_q} |0\rangle_{cc} ,
\]

where \( \psi^\mu_+ \equiv g_{\mu\nu} \psi^{\overline{\nu}+} \), and

\[
|\omega; p, q\rangle_{ac} = \omega^\mu_{\nu_1 \cdots \nu_p} (\phi, \bar{\phi}) \psi^\mu_1 \cdots \psi^\mu_p \psi^{\nu_1} \cdots \psi^{\nu_q} |0\rangle_{ac} .
\]

Upon the formal identification \( \psi^{\overline{\nu}+} \sim d\bar{\nu} \), \( \psi_{\mu+} \sim \partial_\mu \) and \( \psi^\mu_+ \sim dz^\mu \), we have that \( |\eta; b, q\rangle \) correspond to \( \wedge^b T_X \)-valued Dolbeault \( q \)-forms, whereas \( |\omega; p, q\rangle \) correspond to \( (p, q) \)-forms. Furthermore, those \( |\eta; b, q\rangle \) and \( |\omega; p, q\rangle \) which are annihilated by all supercharges represent zero-energy states, correspond to \textit{harmonic} forms, and so encode information about global geometry and topology of the hypersurface \( W(\phi)=0 \) within \( \mathbb{P}^n \).
2. Two Ubiquitous $SL(2, \mathbb{C})$ Actions

The case at hand falls in a very well studied category: the hypersurface $W(\phi) = 0$ in $\mathbb{P}^n$ is Kähler. Now, the $p,q$-forms on all Kähler manifolds admit a so-called Lefschetz $SL(2)$ action, depending, besides the canonical wedge product and Hodge star operator, only on the choice of the Kähler metric \[3\]. The fermionic analogue of this $SL(2)$ action has been known for some time (see Refs. [4,5], and references therein).

2.1. The standard Lefschetz $SL(2, \mathbb{C})$ algebra

Acting on the wave-functions (4), (5) and their conjugates, we define two “ladder” operators:

\[
L_+ \overset{\text{def}}{=} g_{\mu \nu} \psi^\mu_+ \psi^-_\nu, \quad L_- \overset{\text{def}}{=} g_{\mu \nu} \psi^\mu_- \psi^\nu_+ ,
\]

and calculate their commutator:

\[
h \overset{\text{def}}{=} [L_+, L_-] = g_{\mu \nu} g_{\rho \sigma} \left[ \psi^\mu_+ \psi^-_\nu, \psi^\rho_- \psi^\sigma_+ \right] ,
\]

\[
= g_{\mu \nu} g_{\rho \sigma} \left( \psi^\mu_+ \left\{ \psi^-_\nu, \psi^\rho_- \right\} \psi^\sigma_+ - \psi^\rho_- \left\{ \psi^\mu_+, \psi^-_\nu \right\} \psi^\sigma_+ \right) ,
\]

\[
= g_{\mu \nu} \left( \psi^\mu_+ \psi^-_\nu - \psi^\rho_- \psi^\sigma_+ \right) .
\]

Using the anticommutation relations (1), this expression can be brought into the more standard form:

\[
h = g_{\mu \nu} \left( \psi^\mu_+ \psi^-_\nu + \psi^\rho_- \psi^\sigma_+ \right) - (n+1) ,
\]

where now the creation operators (when acting on $\oplus_{p,q} |\omega; p, q\rangle_{ac}$) are to the left of the annihilation operators. The monomials $\psi^\mu_+ \psi^-_\nu$ and $\psi^\rho_- \psi^\sigma_+$ are simply (fermion) number operators, and $h$ then simply stands for the total (fermion) number operator, shifted so that its eigenvalues on $|\omega; p, q\rangle$ range from $(p+q) = -(n+1)$ to $(p+q) = (n+1)$.

A similar calculation verifies that

\[
[ h, L_\pm ] = \pm 2L_\pm ,
\]

whence $\{L_\pm, h\}$ form an $SL(2)$ algebra. It is similarly easy to verify that:

\[
L_\pm |\omega; p, q\rangle_{ac} \mapsto |\omega'; p\pm 1, q\pm 1\rangle_{ac} ,
\]

so that this $SL(2)$ action coincides with the standard Lefschetz $SL(2)$ action. In the usual layout of the Hodge diamond, where the $(p, q) = (0, 0)$- and $(n, n)$-forms are at the bottom and top corners, respectively, this $SL(2)$ acts vertically. In fact, $\psi^\mu_+, \psi^-_\nu$ play the rôles of Griffiths and Harris’s formal basis elements $e_k, \tilde{e}_k$, while $\psi^\mu_-, \psi^\nu_+$ play the roles of their duals, $i_k, \tilde{i}_k$. Finally, we complexify $g_{\mu \nu} \rightarrow (g_{\mu \nu} + iB_{\mu \nu})$, using the antihermitian 2-form $B$ familiar from 2-dimensional $\sigma$-models related to string theory. The $SL(2)$ transformation parameters thus become complex, producing the complexified Lefschetz $SL(2, \mathbb{C})$ action.
2.2. The mirror $SL(2, \mathbb{C})$ algebra

The existence of the mirror map among (families of) Calabi-Yau 3-folds implies that there exists a $Y$, the mirror model of the manifold $X$, such that $H^q(X, \wedge^p T^*_X) \approx H^q(Y, \wedge^p T_Y)$ and $H^q(X, \wedge^p T_X) \approx H^q(Y, \wedge^p T^*_Y)$; note that $\wedge^p T_X = \wedge^{n-p} T^*_X$ on Calabi-Yau $n$-folds. Therefore, the mirror map identifications may also be stated as $H^{p,q}(X) \approx H^{n-p,q}(Y)$. The standard Lefschetz $SL(2)$ action on $H^{p,q}(X)$ is then mapped to an action on $H^{n-p,q}(Y)$, where it now acts horizontally! Similarly, the standard Lefschetz $SL(2)$ action on $H^{p,q}(Y)$ has a pre-image on $H^{n-p,q}(X)$, where it acts horizontally.

In the Landau-Ginzburg model for $X$, this horizontal Lefschetz-like $SL(2)$ action is easy to identify. Recall that $X$ is defined as the $W=0$ hypersurface within $\mathbb{P}^n$; let then $W_{\mu\nu} = \partial_{\mu} \partial_{\nu} W$. We now define another two ‘ladder’ operators:

$$\Gamma_+ \overset{\text{def}}{=} W_{\mu\nu} \psi_+^\mu \psi^-_\nu, \quad \Gamma_- \overset{\text{def}}{=} W_{\mu\nu} \psi^-_\mu \psi^\nu_+, \quad (9a, b)$$

where $W_{\mu\nu} \overset{\text{def}}{=} g_{\kappa\pi} W^{\kappa\sigma} g_{\sigma\pi}$, with $W^{\kappa\sigma}$ being the matrix-inverse of $W_{\kappa\sigma}$: $W_{\mu\nu} W^{\mu\sigma} = \delta^\sigma_\nu$. Furthermore,

$$\mu \overset{\text{def}}{=} [\Gamma_+, \Gamma_-] = g_{\mu\nu}[\psi_+^\mu \psi^-_\nu - \psi^-_\mu \psi^\nu_+] \quad (9c)$$

Again, we may rewrite this as in the more standard way as

$$\mu = g_{\mu\nu}[\psi_+^\mu \psi^-_\nu + \psi^-_\mu \psi^\nu_+] - (n+1) \quad (9d)$$

Again, the creation operators (now when acting on $\oplus_{b,q} |h; b, q\rangle_{cc}$) are to the left of the annihilation operators. The monomials $\psi_+^\mu \psi_+^\nu$ and $\psi^-_\mu \psi^-_\nu$ are again the (fermion) number operators, and $\mu$ then simply stands for the total (fermion) number operator, shifted so that its eigenvalues on $|h; b, q\rangle$ range from $(b+q) = -(n+1)$ to $(b+q) = (n+1)$.

A quick calculation verifies that

$$[\mu, \Gamma_\pm] = \pm 2\Gamma_\pm \quad (10)$$

whence $\{\Gamma_\pm, \mu\}$ form another $SL(2)$ algebra. It is similarly easy to verify that:

$$\Gamma_\pm |\omega; p, q\rangle_{ac} \mapsto |\omega'; p\mp 1, q\pm 1\rangle_{ac} \quad (11)$$

so that this second $SL(2)$ action coincides with the mirror map pre-image of the standard Lefschetz $SL(2)$ action on the mirror, $Y$; it acts horizontally on the Hodge diamond of $X$.

However, note that the action of the ladder operators $L_\pm$ and $\Gamma_\pm$ is swapped when acting on the $|\eta; b, q\rangle_{cc}$:

$$L_\pm |\eta; b, q\rangle_{cc} \mapsto |\eta'; b\mp 1, q\pm 1\rangle_{cc}, \quad \text{horizontal action;} \quad (12a)$$

$$\Gamma_\pm |\eta; b, q\rangle_{cc} \mapsto |\eta'; b\pm 1, q\pm 1\rangle_{cc}, \quad \text{vertical action.} \quad (12b)$$

Moreover, straightforward calculations show that these two $SL(2)$ actions commute, whence $\{L_\pm, h\}$ and $\{\Gamma_\pm, \mu\}$ generate an $SL(2)_L \times SL(2)_\Gamma$. On $\oplus_{p,q} |\omega; p, q\rangle_{ac}$, the first factor acts vertically and the second one horizontally, while on $\oplus_{b,q} |\eta; b, q\rangle_{cc}$ their actions are swapped. Therefore, $SL(2)_\Gamma$ generated by $\{\Gamma_\pm, \mu\}$ is the (mirror map pre-image of the) mirror of $SL(2)_L$ generated by $\{L_\pm, h\}$.
Finally, it is obvious that the Bogoliubov transformation $\psi_+^\mu \leftrightarrow \psi_-^\mu$ becomes exactly the ‘relative sign change’ in the action of the $U(1)_L \times U(1)_R$ of the corresponding super-conformal field theory, and so is the mirror map $\{L_+^\mu, L_-^\mu, h\} \leftrightarrow \{\Gamma_+^\mu, \Gamma_-^\mu, \mu\}$. It is equally clear that the same field redefinition also swaps the two $SL(2, \mathbb{C})$ actions, $\{L_+^\mu, L_-^\mu, h\} \leftrightarrow \{\Gamma_+^\mu, \Gamma_-^\mu, \mu\}$, proving that these are indeed the mirror (pre)images of each other; see also Ref. [9].

### 3. Discussion

The main result proven, we now address some additional issues in turn.

#### 3.1. The mirror map and marginal operators

The definition of $\{\Gamma_\pm, \mu\}$ uses, most crucially, the Hessian of the defining polynomial, $W$. Notice that $g_{\mu\nu}$ is in fact a Kähler metric, and so also a Hessian: $g_{\mu\nu} = \partial_{\mu}\partial_{\nu}K$. Since the two $SL(2, \mathbb{C})$ algebras are mirror (pre)images of each other, the Kähler potential $K$ for the metric $g_{\mu\nu}$ and the defining polynomial (superpotential) $W$ must be mirrors of each other. In 2-dimensional (2,2)-superspace, the ‘Kähler potential’ function $K$ is defined only up to the addition of terms each of which is annihilated at least by one of the four superderivatives [9]. This ‘undefinedness’ is far larger than in spacetimes of more than 2 dimensions! Also unlike its familiar 4-dimensional counterpart, the superpotential in 2-dimensions is similarly ‘undefined’, although in more restrictive way [9].

Furthermore, the definition of $\Gamma_-\bar{}$ involves the matrix inverse of the Hessian of $W$. This exists provided the determinant of the Hessian is non-zero, and which allows $W$ to be mildly singular: $dW$ may vanish, as long as the locus of $dW=0$ are only nodes (double points). Mirror symmetry then implies that $K$ may be ‘singular’ in the sense that $dK$ may vanish, as long as the (Hermitian) matrix of second derivatives, $g_{\mu\nu}$, remains invertible. But this, and nothing more is precisely the ‘standard’ requirement of the Kähler potential! So, since one never expects anything more of $K$, mirror symmetry suggests that:

- Superpotentials should also be allowed to singularize, as long as their Hessians are invertible [10,11].

In the (2,2)-supersymmetric field theory, the mirror relation between the Kähler potential and the superpotential may come as a surprise. While the latter is a purely chiral function, the former is a neither chiral nor anti-chiral, but real. Whereas the latter enters the Lagrangian as an $F$-term and does not renormalize \(^1\), the former figures in a $D$-term, 

\(^1\) See, e.g., p.358 of Ref. [12] for an important caveat to this ‘theorem’.

\[\text{Table} \]

| Generators | on $|\omega; p, q\rangle_{a,c}$ | on $|\omega; b, q\rangle_{c,c}$ |
|------------|---------------------------------|---------------------------------|
| $\{L_+, L_-, h\}$ | Lefschetz $SL(2, \mathbb{C})$ | ‘mirror’ $SL(2, \mathbb{C})$ |
| $\{\Gamma_+, \Gamma_-, \mu\}$ | ‘mirror’ $SL(2, \mathbb{C})$ | Lefschetz $SL(2, \mathbb{C})$ |
not protected by the usual non-renormalization theorems. However, this real function does
give rise to a collection of twisted-chiral marginal operators (one for each \((a, c)\)-modulus),
just as the superpotential produces a collection of chiral marginal operators (one for each
\((c, c)\)-modulus) \([13]\). Of course, the Bogoliubov transformation \(\psi^+ \leftrightarrow \psi^\Gamma\) (a.k.a. mirror
map) also swaps the chiral and the twisted-chiral fields, again verifying that:

- \(D\)-terms can yield twisted-chiral marginal operators,
the mirror map (pre)images of the \(F\)-term chiral marginal operators.

Quite importantly, the definition of the \(SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_\Gamma\) algebra is purely alge-
braic. Thus, it ‘comes for free’, in any \((2,2)\)-supersymmetric model that features a metric
\(g_{\mu\nu}\) and a superpotential \(W\). Geometrically, the \(SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_\Gamma\) is unobstructed
since it acts on the contractible fibres of the bundle \(\wedge(T \times T^*)_X\). In gauged models, \(g_{\mu\nu}\)
is defined upon passing to a ‘gauge slice’: e.g., in Witten’s gauged linear \(\sigma\)-model, the
gauging of the various \(U(1)\) symmetries induces the generalization of the Fubini-Study
metric on the gauge quotient toric variety within which the hypersurface \(W(\phi) = 0\) lies.

Also, note that the \(SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_\Gamma\) algebra is a (small) part of what are gen-
erally known as ‘dynamical symmetry’/‘spectrum-generating’ algebras. That is,

- Given a (judiciously chosen) quarter of the supersymmetric \(|\omega; p, q\rangle_{ac}\)’s, the
  others are obtained by applying the \(SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_\Gamma\) ladder operators.

### 3.2. Extensions

The \(SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_\Gamma\) symmetry found above may be extended in several ways.

#### More fermions

Clearly, \(N\)-fold extended \((2,2)\)-supersymmetry will give rise to \(N\) ‘species’ of fermions, each of which having \(n\) fermions. A more complicated situation occurs in the more general
models of Ref. \([9]\), where the different species of fermions stem from different superfields,
and are therefore not required to be equal in number. Returning to the simple case of
\(N\)-extended supersymmetry, operators of the type (6) and (9) now become \(N \times N\) matrix
operators in the ‘species space’:

\[
L_{ij}^+ = g_{\mu\nu} \psi^{\mu}_+ \psi^\nu_j^+, \quad L_{ij}^- = g_{\mu\nu} \psi^{\mu}_- \psi^\nu_j^-, \quad (13a, b)
\]

\[
h_i^j = g_{\mu\nu} [\psi_i^{\mu} \psi^\nu_j^+ - \psi_i^{\mu} \psi^\nu_j^-], \quad (13c)
\]

and Eqs. (9) now become

\[
\Gamma^{ij}_+ = W_{\mu\nu} \psi^{\mu}_+ \psi^\nu_j^+, \quad \Gamma^{ij}_- = W_{\mu\nu} \psi^{\mu}_- \psi^\nu_j^-, \quad (13d, e)
\]

\[
\mu_j^i = g_{\mu\nu} [\psi^\nu_j^+ \psi^{\mu}_+ - \psi^\nu_j^- \psi^{\mu}_-], \quad (13f)
\]

We will also need:

\[
H^{ij} = W_{\mu\rho} \psi^{\mu}_+ \psi^\rho_j^+, \quad H_{ij} = W_{\nu\sigma} \psi^{\nu}_- \psi^\sigma_j^- , \quad (13g, h)
\]

\[
I^{ij} = W_{\mu\rho} \psi^{\mu}_- \psi^\rho_j^-, \quad I_{ij} = W_{\nu\sigma} \psi^{\nu}_+ \psi^\sigma_j^- , \quad (13i, j)
\]
which are antisymmetric in \(i, j\) and so vanish when there is a single species of fermions. Also, it will be convenient to use

\[
J^i_j \overset{\text{def}}{=} g_{\mu \nu} \psi^i_+ \psi^\mu_j, \quad \text{and} \quad J^i_j \overset{\text{def}}{=} g_{\mu \nu} \psi^i_- \psi^\mu_j,
\]

so that

\[
h^i_j = J^i_j - K^i_j, \quad \text{and} \quad \mu^i = (n+1)\delta^i_j - J^i_j - K^i_j.
\]

We now find that the algebra spanned by \(\{1, L_\pm, \Gamma_\pm, J, K, H, e, H, I_\pm, I_\pm, L_\pm, \Gamma_\pm\}\) is:

\[
\begin{align*}
[L^i_i, L^k_k] &= \delta^k_i J^i_i - \delta^i_k K^k_k, & [L^i_i, \Gamma_{kl}] &= \delta^i_k \Gamma_{kl}, & (15a) \\
[L^i_i, \Gamma_{kl}] &= \delta^i_k H^{il}, & [L^i_i, \Gamma_{kl}] &= H_{kl} \delta^i_i, & (15b) \\
[L^i_i, \Gamma_{kl}] &= -I^{ik} \delta^i_j, & [\Gamma_{ij}, \Gamma_{kl}] &= \delta^i_j ((n+1)\delta^i_i - J^i_i) - \delta^i_k K^k_i, & (15c) \\
[H^{ij}, H_{kl}] &= -\delta^{ij}_{[k} J^{j]}_{l]} + (n+1)\delta^{ijkl} \mathbf{1}, & [I^{ij}, I_{kl}] &= -\delta^{ijkl} [k|l] + (n+1)\delta^{ijkl} \mathbf{1}, & (15d) \\
[H_{ij}, L^k_k] &= \delta^k_i \Gamma_{ij} - \delta^k_i \Gamma_{jl}, & [L^i_i, L^j_j] &= \delta^i_j \Gamma^{jk} - \delta^j_i \Gamma^{ik}, & (15e) \\
[H^{ij}, L^k_k] &= \delta^j_i \Gamma_{kl} - \delta^k_l \Gamma^i_j, & [L^j_j, L^k_k] &= \delta^j_k \Gamma_{ij} - \delta^i_k \Gamma_{jl}, & (15f) \\
[H^{ij}, \Gamma_{kl}] &= \delta^j_l \Gamma^i_k - \delta^k_l \Gamma^{ij}, & [I^{ij}, L^k_k] &= \delta^i_l \Gamma^j_k - \delta^j_l \Gamma^i_k, & (15g) \\
[H_{ij}, \Gamma_{kl}] &= \delta^j_l \Gamma_{ij} - \delta^k_l \Gamma_{ij}, & [I_{ij}, L^k_k] &= \delta^i_l \Gamma^j_k - \delta^j_l \Gamma^i_k, & (15h) \\
[J^i_i, L^k_k] &= \delta^j_k L^j_i - \delta^j_k L^i_j, & [J^i_i, L^k_k] &= \delta^j_k L^{ij} - \delta^j_k L_{ij}, & (15i) \\
[J^i_i, \Gamma_{kl}] &= \delta^k_i \Gamma_{ij} - \delta^k_i \Gamma_{jl}, & [J^i_i, \Gamma_{kl}] &= \delta^k_i \Gamma^j_k - \delta^j_k \Gamma^i_k, & (15j) \\
[K^i_i, L^k_k] &= -\delta^j_k L^j_i + \delta^j_k L^i_j, & [K^i_i, L^k_k] &= \delta^j_k \Gamma^i_k, & (15k) \\
[K^i_i, \Gamma_{kl}] &= -\delta^j_k \Gamma_{ij} + \delta^j_k \Gamma_{jl}, & [K^i_i, \Gamma_{kl}] &= \delta^j_k \Gamma^i_k, & (15l) \\
[J^i_i, H^{kl}] &= H^{il}[\delta^j_k], & [J^i_i, H_{kl}] &= \delta^i_k H^l_{ij}, & (15m) \\
[K^i_i, I^{kl}] &= I^{i[l} [\delta^j_k], & [K^i_i, I_{kl}] &= \delta^i_l I^j_k, & (15n) \\
[J^i_i, J^k_k] &= \delta^j_k J^i_i - \delta^i_k J^j_j, & [J^i_i, J^k_k] &= \delta^j_k J^i_i - \delta^i_k J^j_j, & (15o)
\end{align*}
\]

all other commutators being zero. Note that the identity, \(\mathbf{1}\), appears on the right-hand sides of both Eqs. (15d), and so must be included as a generator of the algebra; it of course commutes with all other generators.

**Superalgebras**

Besides the bosonic operators bilinear in the fermions (6) and (9), we can introduce fermionic operators, linear in the \(\psi, \bar{\psi}\):

\[
A_+ = A_{\mu} \psi^\mu_+ , \quad \bar{A}_+ = A_{\mu} \psi^\mu_+ , \\
A_- = A_{\mu} \psi^\mu_- , \quad \bar{A}_- = A_{\mu} \psi^\mu_- ,
\]

(16)
Their anticommutators must be expressible as a linear combination of \(1, L_\pm, h, \Gamma_\pm, \mu\), and the vector space \(A, \bar{A}, B, \bar{B}\) must form a representation of \(SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_R\). The latter requirement forces the commutators of the operators (6) with the \([10]\) to be expanded over the \([16]\). The first requirement produces

\[
\{A_+, \bar{A}_+\} = A_\mu A_\nu g^{\mu \nu} 1 \equiv ||A||^2 1 = \{A_-, \bar{A}_-\}. \tag{17}
\]

Next, we find

\[
[L_+, A_+] = 0, \quad [L_-, A_+] = +A_-, \quad [h, A_+] = +A_+ ,
\]

\[
[L_+, A_-] = +A_+, \quad [L_-, A_-] = 0, \quad [h, A_-] = -A_-, \tag{18}
\]

\[
[L_+, \bar{A}_+] = -\bar{A}_-, \quad [L_-, \bar{A}_+] = 0, \quad [h, \bar{A}_+] = -\bar{A}_+, \quad [L_+, \bar{A}_-] = 0, \quad [L_-, \bar{A}_-] = -\bar{A}_+ , \quad [h, \bar{A}_-] = +\bar{A}_- ;
\]

and

\[
[\Gamma_+, A_+] = -\bar{A}_', \quad [\Gamma_-, A_+] = 0, \quad [\mu, A_+] = -A_+ ,
\]

\[
[\Gamma_+, A_-] = +\bar{A}_', \quad [\Gamma_-, A_-] = 0, \quad [\mu, A_-] = -A_-, \tag{19}
\]

\[
[\Gamma_+, \bar{A}_+] = 0, \quad [\Gamma_-, \bar{A}_+] = +\bar{A}_', \quad [\mu, \bar{A}_+] = +\bar{A}_+ ,
\]

\[
[\Gamma_+, \bar{A}_-] = 0, \quad [\Gamma_-, \bar{A}_-] = -\bar{A}_' , \quad [\mu, \bar{A}_-] = +\bar{A}_- ,
\]

with

\[
A_\pm' \overset{\text{def}}{=} (W_\mu^A A_\nu)^{\psi_{\pm}^\mu} , \quad \text{and} \quad \bar{A}_\pm' \overset{\text{def}}{=} (W_\nu^{A_\mu} A_\mu)^{\psi_{\pm}^\nu} . \tag{20}
\]

The matrices \(W_\mu^A \overset{\text{def}}{=} W_{\mu \rho} g^{\rho \sigma}\) and \(W_\nu^{A_\mu} \overset{\text{def}}{=} W_{\nu \sigma}^{\rho} g^{\mu \sigma}\) act as a (conjugating) linear transformation on the coefficient functions \(A_\mu, A_\nu, \bar{A}_\mu, \bar{A}_\nu\); they are well-defined since the Hessians \(W_{\mu \rho}\) and \(g_{\mu \nu}\) are invertible. The relations \([18]\) assure that \(A_\pm\) and \(\bar{A}_\pm\) transform as two \(SL(2, \mathbb{C})_L\) spin-\(\frac{1}{2}\) doublets, and \([19]\) show that \((A_+, \bar{A}_-)\) and \((A_-, \bar{A}_+)\) are \(SL(2, \mathbb{C})_R\) doublets, twisted by the \(\tilde{W}_-\)-transformation. This guarantees that \(\{A_\pm, \bar{A}_\pm; 1, L_\pm, h, \Gamma_\pm, \mu\}\) generate a supergroup.

Of course, it is also possible to expand the right-hand side of the anticommutators \([17]\) over non-trivial (differential) operators over the bosonic degrees of freedom. One such possibility leads to the well-studied field-space *supersymmetry* algebra; see for example Ref. [3]. If the target manifold \(X\) admits null-vectors, the \(A_\mu, A_\nu\) may be chosen to have zero norm, whereupon they generate a BRST-like subalgebra of the superalgebra discussed in Eqs. \([17]\)–\([20]\). Another possibility is to let the \(A_\mu, A_\nu\) take values in a non-Abelian Lie algebra. These and other such extensions are left for another occasion.

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