Competing quantum paramagnetic ground states of the Heisenberg antiferromagnet on the star lattice

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We investigate various competing paramagnetic ground states of the Heisenberg antiferromagnet on the two dimensional star lattice which exhibits geometric frustration. Using slave particle mean field theory combined with a projective symmetry group analysis, we examine a variety of candidate spin liquid states on this lattice, including chiral spin liquids, spin liquids with Fermi surfaces of spinons, and nematic spin liquids which break lattice rotational symmetry. Motivated by connection to large-N SU(N) theory as well as numerical exact diagonalization studies, we also examine various valence bond solid (VBS) states on this lattice. Based on a study of energetics using Gutzwiller projected states, we find that a fully gapped spin liquid state is the lowest energy spin liquid candidate for this model. We also find, from a study of energetics using Gutzwiller projected wave functions and bond operator approaches, that this spin liquid is unstable towards two different VBS states — a VBS state which respects all the Hamiltonian symmetries and a VBS state which exhibits $\sqrt{3} \times \sqrt{3}$ order — depending on the ratio of the Heisenberg exchange couplings on the two inequivalent bonds of the lattice. We compute the triplon dispersion in both VBS states within the bond operator approach and discuss possible implications of our work for future experiments on candidate materials.

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I. INTRODUCTION

In recent years, several geometrically frustrated spin-1/2 magnets have been synthesized which appear to not order magnetically even at temperatures well below the characteristic exchange couplings. Among these are the quasi-two-dimensional (2D) triangular organic material $\kappa$-BEDT(TCN)$_{3}$,1 the kagome lattice herbertsmithite2 and distorted kagome lattice volborthite3 and the three-dimensional hyperkagome lattice magnet $\text{Na}_4\text{Ir}_3\text{O}_8$. A large class of these magnets appear to exhibit gapless spin liquid behavior down to very low temperatures, leading to the exciting possibility that they may possess exotic ground states with fractionalized excitations.5,6,7,8,9,10,11,12,13 Others among these have been proposed to weakly order into singlet valence bond solid (VBS) states which break lattice symmetries.14,15,16,17 As yet, there is no clear picture of what combination of geometric effects and spin interactions will lead to spin liquid ground states or VBS ground states; this necessitates a theoretical and experimental exploration of various new lattice geometries as well as possible ring-exchange interactions beyond the simplest Heisenberg spin exchange interaction.

In this paper, we focus on understanding several competing ground states of the nearest neighbor $S = 1/2$ Heisenberg model

$$H = \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j,$$

on the 2D star lattice, shown in Fig. 1, as a function of $J_t/J_c$ where $J_t$ and $J_c$ are the exchange couplings on the ‘triangle bonds’ and ‘expanded bonds’. Our motivation for this study is two-fold. First, the recent synthesis of a ‘star lattice’ organic Iron Acetate quantum magnet4 raises the possibility that a $S = 1/2$ variant may possibly be synthesized in the near future and our results should be applicable to such systems. Second, this lattice has a sufficiently different geometry from more commonly studied quantum magnets — it may be viewed either as a variant of the kagome lattice or as a decorated honeycomb lattice — which allows us to explore the effect of this new lattice geometry on possible spin liquid physics and valence bond solid phases in quasi-2D systems.

Motivated by interpolating between this spin-1/2 SU(2) model and an SU(N) generalization at large N which permits a mean field solution with fermionic spinon excitations,19,20,21 we examine a large number of interesting U(1) spin liquids as candidate ground states of the nearest neighbor $S = 1/2$ Heisenberg model on the 2D star lattice. Guided by earlier work on the kagome lattice,22 we focus on spin liquid states denoted by $\text{SL}[\Phi_\Delta, \Phi_\nabla, \Phi_{\text{dodecagon}}]$ where $\Phi_\Delta$, $\Phi_\nabla$, and $\Phi_{\text{dodecagon}}$ denote, respectively, the “fictitious” fluxes seen by the fermionic spinons as they move around an elementary plaquette of the lattice: an up triangular plaquette $\triangle$, a down triangular plaquette $\nabla$, or the 12-site dodecagon plaquette. In terms of the original spin variable, the fluxes on the triangular plaquettes correspond to scalar spin chiralities of the form $\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{S}_3$, while $\Phi_{\text{dodecagon}}$ is related to an operator defined by the twelve spins around the dodecagon loop. Depending on the flux values, these spin liquids represent gapped chiral spin liquids which break time-reversal symmetry or states with gapless Fermi surfaces of spinons, or gapped spin liquids with no broken symmetries.

From a study of energetics of various flux values using Gutzwiller projected wave function numerics for the physical case of $N = 2$, we show that a particular gapped spin liquid, which we denote as $\text{SL}[0, 0, \pi]$, which does not break lattice or time-reversal symmetries emerges as a favorable candidate over a wide range of $J_t/J_c$. This is in striking contrast to earlier work on the kagome lattice from two perspectives. First, as we show, the effect of projection is far more dramatic on
the star lattice when compared with the kagome lattice; a numerical Gutzwiller projection of the mean field states leads to a complete reordering of the energies of the candidate spin liquids. Second, unlike the kagome lattice case where the lowest energy variational state of this form is a spin liquid liquids. Second, unlike the kagome lattice case where the lowest energy variational state of this form is a spin liquid — we therefore know that it is ultimately unstable towards spinon confinement at low energies unlike the Dirac fermion states whose stability depends on the number of fermion flavors N. We find that the SL[0, 0, π] state naturally forms strong dimers on the ‘expanded bonds’ for small values of Jt/Je thus leading to a confined state, a Jt-dimer VBS, which respects all symmetries of the Hamiltonian. For large Jt/Je, numerical exact diagonalization (ED) studies of this model carried out in a restricted nearest-neighbor dimer basis showed signatures of √3 × √3 ordering. We argue that another motivation to study possible dimer orders that break lattice symmetries is that such ordering often appear quite naturally in the large-N fermionic SU(N) theory as recognized in the early work of Affleck and Marston and shown in various other models studied recently. Inspired by these results, we consider various candidate VBS phases from different perspectives — a large-N route, a bond operator formalism, and Gutzwiller projected wave function numerics. All of these point to a transition to a √3 × √3 ordered VBS phase for large enough Jt/Je ≳ 2 − 2.5, leading us to the phase diagram shown in Fig. 1. Our result is in broad agreement with the ED study although the transition point estimated from our work is somewhat larger than the exact diagonalization (ED) result which yields (Jt/Je)crit ≡ 1.3; the ED result may, however, may suffer from significant finite size effects. We then discuss possible routes by which the SL[0, 0, π] state might be unstable towards such √3 × √3 VBS order instead of the Je-dimer VBS. We present results for the triplon dispersion in both VBS states which can be tested in inelastic neutron scattering experiments on candidate materials.

Finally, although the various other interesting spin liquids we study do not appear to be energetically viable ground states for the nearest neighbor Heisenberg model on the star lattice, they have energies which are close to the ground state. They might thus be stabilized as ground states by small changes in the Hamiltonian, such as further neighbor exchange or spin-phonon coupling, or they might be relevant to understanding the intermediate energy scale properties or finite temperature physics of materials which realize this model. We therefore elucidate some of the properties of these spin liquid states.

This paper is organized as follows. We begin, in Section II, by formulating the mean field theory of the Heisenberg model on this lattice in a slave particle description using fermionic spinons which we relate to a large-N SU(N) approach. Based on this, we classify and study the physical properties of a number of candidate spin liquid ansatzes. We next turn, in Section III, to a study of dimerized states on this lattice, and present a group theoretic classification of √3 × √3 orders as well as a large-N justification of specific candidate VBS phases. Section IV contains a discussion of the energetics of various spin liquid states using mean field theory as well as a Gutzwiller projected wave function study and bond operator approaches of candidate VBS phases. Section V discusses the various ways in which the SL[0, 0, π] state, which is the lowest energy spin liquid state, might be unstable towards VBS ordering as a result of spinon interactions and from a Gutzwiller wave function approach. The phase transition between VBS phases is described in Section VI. Section VII contains a discussion of the triplon dispersion in both VBS states which we think are realized in this model. We conclude with a discussion about experimental implications in Section VIII. Details of various calculations are contained in Appendices A-C.

II. SPIN LIQUID PHASES ON THE STAR LATTICE

A. Formulation of the mean field theory

We investigate the ground state of the S=1/2 Heisenberg antiferromagnet on the star lattice, which can be described by

![FIG. 1: (Color online) Top: Structure of the star lattice depicting the six-site unit cell, the chosen lattice basis vectors a1=2x and a2 = x + √3y, and the bonds with Heisenberg exchange couplings Jx (‘expanded bonds’) and Jt (‘triangle bonds’). Bottom: Phase diagram of the antiferromagnetic Heisenberg model on the star lattice. For Jt/Je ≪ 1, the ground state is a valence bond solid (VBS) phase (Jx-dimer VBS) in which every dimer sits on the expanded links connecting neighboring triangles. For Jt/Je ≫ 1, the ground state is a VBS with an 18-site unit cell (the columnar 18-site VBS).](image-url)
the following Hamiltonian,

\[ H = \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \]  

(2)

where \( i \) and \( j \) indicate the positions of nearest neighbor spin pairs. Natural description of this lattice system requires the consideration of two inequivalent links, that is, one link lying on a triangle (a triangular link) and the other link connecting two neighboring triangles (an expanded link). We assign two different exchange couplings \( J_o \) and \( J_t \) on expanded and triangular links, respectively. We can also label a site \( i \) by pairs \((R, n)\) where \( R \) denotes the location of a unit cell and \( n \) labels the six sites inside a single unit cell. (See Fig. 1.)

To construct spin liquid states we introduce the fermionic spinon operators, \( f_\sigma \ (\sigma = \uparrow \text{ or } \downarrow) \) to represent the spin operator:

\[ S_i^\alpha = \frac{1}{2} \sum_{\sigma_1, \sigma_2} f_{i,\sigma_1}^\dagger \sigma_{\sigma_1, \sigma_2} f_{i,\sigma_2}, \quad (\alpha = x, y, z). \]  

(3)

Since this representation alone contains unphysical local configurations enlarging the Hilbert space, we have to impose the following local constraint, \( f_i^\dagger f_i + f_i f_i^\dagger = 1 \), to recover the physical Hilbert space. Using the fermionic spinon representation of the spin operator, the Heisenberg spin Hamiltonian can be rewritten as follows.

\[ H = - \sum_{\sigma_1, \sigma_2 < i,j>} \frac{J_{ij}}{2} f_{i,\sigma_1}^\dagger f_{j,\sigma_1} f_{j,\sigma_2}^\dagger f_{i,\sigma_2}. \]  

(4)

Here we have dropped unimportant constant terms.

To decouple the four fermion interaction term we define spin singlet order parameters, \( \chi_{ij} = \frac{1}{2} \sum_\sigma \langle f_{i,\sigma}^\dagger f_{j,\sigma} \rangle \). After imposing the single occupancy constraint using the Lagrange multipliers \( \mu_i \), the mean field Hamiltonian is given by

\[ H_{MF} = - \sum_{\sigma < i,j>} J_{ij} \langle f_{i,\sigma}^\dagger f_{j,\sigma} \chi_{ij}^* + h.c. \rangle + \sum_{<i,j>} 2J_{ij} |\chi_{ij}|^2 + \sum_{i,\sigma} \mu_i (f_{i,\sigma}^\dagger f_{i,\sigma} - 1). \]  

(5)

To describe the phase fluctuation of the mean field ansatz, we express \( \chi_{ij} \) as \( \chi_{ij} = \chi_{ij} e^{i\theta_{ij}} \), which leads to the following Hamiltonian,

\[ H_{U(1)} = - \sum_{\sigma < i,j>} J_{ij} \langle f_{i,\sigma}^\dagger f_{j,\sigma} \bar{\chi}_{ij} e^{-ia_{ij}} + h.c. \rangle + \sum_{i,\sigma} \mu_i (f_{i,\sigma}^\dagger f_{i,\sigma} - 1). \]  

(6)

In the above Hamiltonian \( H_{U(1)} \), the local \( U(1) \) gauge symmetry of the spin Hamiltonian which comes from the local conservation of the fermion number is manifest via the following gauge transformation,

\[ f_i \rightarrow f_i e^{i\theta_i}, \quad a_{ij} \rightarrow a_{ij} - \theta_i + \theta_j. \]  

(7)

Here \( a_{ij} \) describing the phase fluctuation of \( \chi_{ij} \) plays the role of the spatial components of the \( \text{U}(1) \) gauge field. Namely, we have reformulated the quantum spin model as the problem of the spinons strongly coupled to the \( \text{U}(1) \) gauge field.

A systematic way of studying the coupled spinon and gauge field system is to consider the large-N reformulation of the problem extending the spin \( \text{SU}(2) \) symmetry to \( \text{SU}(N) \) (with \( N \) even).\[20\] We let the flavor index \( \alpha \) run from 1 to \( N \) and modify the single occupancy constraint as,

\[ \sum_{\alpha} f_{i,\alpha}^\dagger f_{i,\alpha} = \frac{N}{2}. \]  

(8)

In addition, we scale the interaction strength \( J_{ij}/2 \) to be \( J_{ij}/N \) to make each term of the Hamiltonian to be of order \( N \). The resulting large-N Hamiltonian is given by

\[ H = - \sum_{\alpha, \beta = 1}^{N} \sum_{<i,j>} \frac{J_{ij}}{N} f_{i,\alpha}^\dagger f_{j,\alpha} f_{j,\beta}^\dagger f_{i,\beta} + \sum_{i,\alpha} \mu_i (f_{i,\alpha}^\dagger f_{i,\alpha} - \frac{N}{2}). \]  

(9)

To treat the quartic interactions we perform a mean field decoupling by introducing \( \text{SU}(N) \) singlet valence bond \( , \chi_{ij} \equiv \frac{1}{N} \sum_\alpha \langle f_{i,\alpha}^\dagger f_{j,\alpha} \rangle \). Assuming the valence bond amplitude is a complex number, we obtain the mean field Hamiltonian given by

\[ H_{MF} = - \sum_{\alpha, \beta = 1}^{N} \sum_{<i,j>} J_{ij} \langle f_{i,\alpha}^\dagger f_{j,\alpha} f_{j,\beta} f_{i,\beta} \chi_{ij}^* + h.c. \rangle + N \sum_{<i,j>} J_{ij} |\chi_{ij}|^2 + \sum_{i,\alpha} \mu_i (f_{i,\alpha}^\dagger f_{i,\alpha} - \frac{N}{2}). \]  

(10)

Since the fluctuations of \( \chi_{ij} \) and the average local density \( \frac{1}{\sqrt{N}} \sum_\alpha \langle f_{i,\alpha} \rangle \) scale as \( 1/\sqrt{N} \), we can safely neglect those fluctuations in the large-N limit justifying the mean field approximation.

Here we consider the following mean field ansatz \( \bar{\chi}_{ij} = |\chi_{ij}| e^{i\phi_{ij}} \) where \( |\chi_{ij}| \equiv \chi_{e} \) on expanded links and \( |\chi_{ij}| \equiv \chi_{t} \) on triangular links, respectively. We specify the various flux patterns inside the elementary plaquettes, i.e., the triangles and the dodecagons. The flux inside a triangle \( \Phi_\triangle \), for example, is defined in the following way, \( e^{i\phi_{\triangle}} \equiv e^{i(\phi_{ij} + \phi_{jk} + \phi_{ki})} \) where \( (ijk) \) indicates the three corners of a triangle traversed along the counterclockwise direction. The flux inside a dodecagon is also defined in the same manner. Since the flux inside a closed loop is a gauge invariant object, different spin liquid ansatz can be distinguished based on the flux values inside the elementary plaquettes. In particular, we use the term \( \text{SL}[\Phi_\triangle, \Phi_\psi, \Phi_{dodecagon}] \) to represent the ansatz which has the fluxes \( \Phi_\triangle \) inside an up-pointing triangle, \( \Phi_\psi \) inside a down-pointing triangle and \( \Phi_{dodecagon} \) inside a dodecagon.
problem, the flat band is lying exactly at the Fermi level on the star lattice.

The flatness of the band at the Fermi energy is not the generic property of the uniform spin liquid. There are perturbations which do not break any lattice symmetry but spoil the flatness by generating curvature. The third nearest neighbor hopping is such an example. However, the quadratic degeneracy at the zone center is protected by the point group symmetry of the underlying unit cell.

To understand the stability of the spin liquid we derive the low energy effective Hamiltonian, which describes the states near the zone center, expanding the Hamiltonian up to the quadratic order of the momentum \( \mathbf{k} \). The procedure for deriving the effective Hamiltonian is outlined in the Appendix A.3. The resulting Hamiltonian is given by,

\[
H_{\text{eff}} = \frac{1}{m_{\text{eff}}} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \psi^\dagger (\mathbf{k}) h_{\text{eff}}(\mathbf{k}) \psi(\mathbf{k}),
\]

in which

\[
h_{\text{eff}}(\mathbf{k}) = (k_x^2 + k_y^2) \tau_0 - (k_x^2 - k_y^2) \tau_x - 2k_x k_y \tau_x,
\]

where the Pauli matrix \( \tau_i \) is acting on the two-component space of the continuum field \( \psi(\mathbf{k}) = (\psi_1, \psi_2) \) which describes the two low energy states near the \( \Gamma \) point.

The SL[0,0,0] state respects all the space group symmetry of the lattice. In particular, if we choose the gauge in which \( \chi_{ij} = \chi_\ell \) on every triangular link and \( \chi_{ij} = \chi_e \) on every expanded link with \( \chi_\ell \) and \( \chi_e \) being real constants, the action of the symmetry generators on the spin operator, \( S_i \), is the same as that on the spinon operator, \( f_{i,\sigma} \). Since we consider the low energy excitations near the zone center, we focus on the action of the point group symmetry on the continuum fields. The \( D_6 \) point group of the star lattice consists of twelve symmetry generators and is generated by the two elements, \( C_\pi \) and \( R_y \). Here \( C_\pi \) means the \( \pi \) rotation with respect to the center of a decagon and \( R_y \) indicates the reflection about the \( x \)-axis. The details on the elements of the \( D_6 \) point group are discussed in the Sec. IIIIB.

Under the \( C_\pi \) and \( R_y \), the continuum field \( \psi \) transforms in the following way,

\[
C_\pi : \psi \rightarrow e^{-i \frac{\pi}{2} \tau_x} \psi,
R_y : \psi \rightarrow \tau_x \psi,
\]

meaning all the fermion bilinears \( \psi^\dagger \tau_\alpha \psi \) (\( \alpha = x, y, z \)) are forbidden by the point group symmetry. Note that the \( \psi^\dagger \tau_y \psi \) breaks time-reversal symmetry as well since it shows sign change under complex conjugation.

Next we consider the fermion bilinear terms that contain spatial derivatives. Because the dynamical critical exponent is two, the terms with a single spatial derivative are relevant and those with two spatial derivatives are marginal perturbations. Investigating the transformation rule under the \( D_6 \) point group symmetry, it can be easily checked that \( \psi^\dagger \tau_y \psi \) transforms as the one dimensional \( A_2 \) irreducible representation and \( \psi^\dagger \tau_x \psi, \psi^\dagger \tau_\alpha \psi \) forms a basis for the two dimensional \( E_2 \) irreducible representation. Similarly, the transformation properties of derivative terms can be determined.

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**B. Properties of competing spin liquid phases**

In this section we discuss the characteristic properties of various spin liquid phases and their instabilities. In particular, we focus on translationally invariant mean field states which have nonzero \( |\chi_{ij}| \) on every link of the lattice. Extensive discussion on possible dimerized phases is given later in Sec. III. As shown in the previous studies about the spin liquid phases on the square\(^{21}\) and kagome\(^{22}\) lattices, the inclusion of additional spin interactions can change the relative energetics of different spin liquid phases. Therefore it is useful to understand the nature of various competing spin liquid states which are the potential ground states of spin Hamiltonians beyond the nearest neighbor Heisenberg model.

(a) SL[0,0,0] : Uniform spin liquid state

To describe this state we introduce two real mean field order parameters, \( \chi_e \) and \( \chi_\ell \), which lie on the expanded and triangular links, respectively. Since the unit cell contains six sites we obtain six different bands inside the first Brillouin zone. Among the six bands, the third and fourth bands near the Fermi level show an interesting structure displayed in Fig. 2. There is a flat band lying on the Fermi energy which is touching another dispersive band at the zone center, \( \Gamma (\mathbf{k} = 0) \). The flat band comes from the existence of the localized eigenstates, which occur due to the destructive interference of hopping amplitudes between the localized units.\(^{23}\) The flat band of the uniform spin liquid on the kagome lattice emerges owing to the same reason. However, in contrast to the kagome lattice...
At first, the linear derivative term, \((\partial_x, \partial_y)\) transforms as a two dimensional \(E_1\) irreducible representation. To have terms with linear derivatives in the Hamiltonian, the product of the fermion bilinear and the derivative must be invariant under the point group symmetry operations. Using the decompositions of \(E_1 \otimes A_2 = E_1\) and \(E_1 \otimes E_2 = B_1 \oplus B_2 \oplus E_1\), we see that every product of fermion bilinears and the linear derivative is not invariant under the point group symmetry. Therefore linear derivative terms are not allowed. On the other hand, we have a second derivative term \((\partial_x^2 - \partial_y^2, \partial_x \partial_y)\) making a two dimensional \(E_2\) irreducible representation. Using \(E_2 \otimes A_2 = E_2\) and \(E_2 \otimes E_2 = A_1 \oplus A_2 \oplus E_2\), we see that there is a term following \(A_1\) irreducible representation, which is nothing but \(\psi^\dagger [2(\partial_x \partial_y) \tau_x + (\partial_x^2 - \partial_y^2) \tau_z] \psi\). Therefore in addition to the isotropic \(\psi^\dagger (\partial_x^2 + \partial_y^2) \psi\) term, \(\psi^\dagger [(2 \partial_x \partial_y) \tau_x + (\partial_x^2 - \partial_y^2) \tau_z] \psi\) is the only term allowed by symmetry. Since these terms are already present in the Hamiltonian, the low energy properties of the SL\([0,0,0]\) are not spoiled by these marginal perturbations. However, these perturbations add curvature to the flat band.

Finally, we discuss the effect of the four fermion interaction terms on the stability of the SL\([0,0,0]\) state. Though a simple power counting shows that the four fermion interactions are marginal, they are actually marginally relevant. Recently, the effects of the four fermion interaction on the quadratic perturbations add curvature to the flat band. However, these perturbations curtail the flat band.

### (b) SL\([0,0,\pi]\) state

The SL\([0,0,\pi]\) state supports \(\pi\) flux piercing the decagons. Even though this state does not break the translational symmetry, the mean field description requires doubling of the unit cell. Here we consider the doubling of the unit cell along the \(a_1\) direction. For the lattice vectors \(2a_1\) and \(a_2\), the reciprocal lattice vectors are given by

\[
G_1 = \left(\frac{\pi}{2}, -\frac{\pi}{2\sqrt{3}}\right), \quad G_2 = \left(0, \frac{2\pi}{\sqrt{3}}\right).
\]

The reduced Brillouin zone corresponding to the above reciprocal lattice vectors is depicted in Fig. 3(a).

For the mean field description of the SL\([0,0,\pi]\) state, we have chosen the flux configuration as described in Fig. 3(b). Since there are twelve sites inside the unit cell, we have twelve bands within the Brillouin zone. The mean field spinon dispersion of the low energy bands near the Fermi level is described in Fig. 4. This state does not have a spinon Fermi surface and shows a gapped spectrum. The lower flat band (valence band) is doubly degenerate and the upper band (conduction band) is dispersive.

According to the projected wave function study that is discussed in detail later in Sec. [IVB], the SL\([0,0,\pi]\) state has the lowest ground state energy among the various spin liquid ansatz over a wide parameter range. Unfortunately, however, the SL\([0,0,\pi]\) state is unstable once gauge fluctuation is allowed. Since the spinon spectrum has a finite gap, the low energy excitations are described by the compact U(1) gauge theory. In 2+1 dimension, the compact U(1) gauge theory is confining, which means that free spinons with unit gauge charge can only make charge neutral bound states. In addition, the interaction between spinons can also induce various kinds of broken symmetry states. Extensive discussion on the instability of the SL\([0,0,\pi]\) state and its relation with candidate valence bond solid phases are given in Sec. [V].

### (c) SL\([\frac{\pi}{2}, \frac{\pi}{2}, \pi]\) : A chiral spin liquid state

FIG. 4: (Color online) The low energy spinon bands of the SL\([0,0,\pi]\) state near the Fermi level. The lower band (valence band) is flat and doubly degenerate. The upper band (conduction band) has small dispersion. The locations of the conduction band minimum (maximum) are described by \(m_1\) (\(M_1\)). The dispersion is plotted along the \(k_y = \sqrt{3} k_x + \frac{\pi}{2\sqrt{3}}\) line passing the \(m_1\) and \(m_2\).
Counterclockwise motion along an triangular link results in the flux angles. On the other hand, the other reflection (symmetry properties of the where \( \hat{C} \) connecting the center of a dodecagon with the mid-point of on the star lattice. The reflection (to recognize that there are two different reflection symmetries to the mid-point of the expanded link connecting two triangles. In both cases, the combination of the time-reversal, parity transformation, and inversion (\( T \cdot P_1 \cdot I \)) is equivalent to the identity operation under which both the \( \hat{C}_{\text{uniform}} \) and \( \hat{C}_{\text{staggered}} \) are invariant.

The \( \text{SL}[\frac{\pi}{2}, \frac{\pi}{2}, \pi] \) is characterized by nonzero \( \hat{C}_{\text{uniform}} \) but with vanishing \( \hat{C}_{\text{staggered}} \). Therefore it breaks time-reversal and parity transformation while it is invariant under the combined operation. It is a chiral spin liquid state which has a finite energy gap. In Fig. 6(a) we plot the spinon dispersion near the Fermi level corresponding to the valence and conduction bands. The energy gap is minimum at the momentum \( Q = (\pi/3, 0) \) and \(-Q\). If we expand the mean field Hamiltonian near the dispersion minimum, \( Q \) and \(-Q\) in the limit of large \( J_1/J_e \), we can get the following effective low energy Hamiltonian,

\[
H_{\text{eff}} = \int \frac{d^2 q}{(2\pi)^2} \Psi^\dagger(q) \left\{ v_F(q_x \tau_x + q_y \tau_y) - m \tau_z \right\} \Psi(q),
\]

where the Fermi velocity \( v_F = J_e \chi_c / \sqrt{3} \) and the mass \( m = \langle J_2 \chi_c^2 \rangle / (\sqrt{3} J_1 \chi_t) \). In the above we define the eight compo-
nant Dirac fermion field, $\Psi^\dagger = (\varphi^\dagger_{1,\alpha,\sigma},\varphi^\dagger_{2,\alpha,\sigma})$ in which 1 and 2 are the two-component Dirac indices, $\alpha$ and $\sigma$ are indices for the nodes ($\pm \mathbf{Q}$) and spins. The Pauli matrix $\tau_x$ acts on the two-component Dirac space. For later convenience we define two additional Pauli matrices, $\vec{\mu}$ and $\vec{\sigma}$ acting on the nodal and spin spaces, respectively.

Since the mass term has the same sign in the two nodal positions, integrating out fermions leads to the Chern-Simons gauge field action. As a consequence, the charge neutral spinon Hall conductivity should be finite. The Chern-Simons term stabilizes the spin liquid ground state by providing a finite mass to the U(1) gauge field. Therefore the U(1) gauge field can only mediate a short range interaction between the spinons, which makes the fractionalized particles (the spinons) to be the elementary excitations of the spin liquid ground state.\(^{22}\)

(d) SL$[-\frac{\pi}{2}, -\frac{\pi}{2}, 0]$: A nematic spin liquid state

The SL$[-\frac{\pi}{2}, -\frac{\pi}{2}, 0]$ is characterized by nonzero $C_{\text{staggered}}$ but with vanishing $C_{\text{uniform}}$. Therefore it breaks both the time-reversal and inversion operation but is invariant under the parity transformation. Because the fluxes of the two triangles within the unit cell have opposite sign, the six-fold rotational symmetry is broken down to the three-fold symmetry. (See Fig. 3(b).) Thus it is a nematic spin liquid.

The mean field spinon dispersion corresponding to the two bands near the Fermi energy is plotted in Fig. 6(b). The spin liquid ansatz has a spinon Fermi surface which consists of an electron pocket at the $\mathbf{K} = (2\pi/3, 0)$ point and a hole pocket at the $-\mathbf{K}$ point.

Expanding the mean field Hamiltonian using $J_\chi e/ J_\chi t$ as an expansion parameter, the following effective low energy Hamiltonian can be obtained,

$$H_{\text{eff}} = \int \frac{d^2 q}{(2\pi)^2} \Psi^\dagger(q) \begin{pmatrix} v_F [q_x \tau_x + q_y \tau_y] - M \mu_z \end{pmatrix} \Psi(q),$$

(16)

where the fermi velocity $v_F = J_\chi e/\sqrt{3}$ and the “staggered” field $M = (J^2_\chi^2)/(\sqrt{3} J_\chi t)$. Since the effective chemical potentials coming from the “staggered” field $M$ have the opposite signs at the two nodal points, we have both an electron pocket (at the $\mathbf{K}$ point) and a hole pocket (at the $-\mathbf{K}$ point) on the fermi surface.

In contrast to the SL$[\frac{\pi}{2}, \frac{\pi}{2}, \pi]$ which has a gapped spinon spectrum, the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state has gapless low energy excitations. To confirm that the low energy description based the above effective Hamiltonian in Eq. (16) is valid after including the fluctuation beyond the mean field description, we have to check whether there are relevant perturbations which are allowed by symmetry. Especially, some of the fermion bilinears, which are made of $\Psi$, can potentially generate various mass terms which spoil the low energy description of Eq. (16).

To judge the stability of this spin liquid state, we have to understand how the symmetries of the microscopic Hamiltonian are realized in the effective continuum theory. Even though the original spin Hamiltonian is invariant under the full space group transformations, after the gauge theory formulation of the problem, the symmetry of the mean field Hamiltonian is realized projectively. That is, under the symmetry transposition $S$ with the mapping $i \rightarrow S(i)$, the spinon operator $f_{i,\sigma}$ transforms in the following way,

$$S : f_{i,\sigma} \rightarrow G_S(i) f_{S(i),\sigma},$$

where $G_S(i)$ is a phase factor which depends on the symmetry operation $S$, and a local coordinate $i$. The group of the symmetry operations which make the mean field Hamiltonian invariant is called the projective symmetry group (PSG)\(^{38,39}\).

To perform the PSG analysis we have to specify the symmetry group of the spin Hamiltonian. The star lattice has the $D_6$ point group symmetry generated by the six-fold rotation symmetry with respect to the center of a dodecagon and the reflections. However, due to the finite fluxes inside triangles, the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state breaks some $D_6$ point group symmetries. Especially, the six-fold rotational symmetry is broken down to three-fold rotational symmetry. The point group symmetry of the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state is generated by the $2\pi/3$ rotation ($C_{2\pi/3}$) around the center of the dodecagon and the reflection ($R_y$) which maps $y \rightarrow -y$. The symmetry operations which generate the point group of the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state are depicted in Fig. 7. The SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state is also invariant under the translations ($T_{a1}$ and $T_{a2}$) by the lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. In combination with the above point group symmetry, the translational symmetry defines the space group of the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state. In addition, the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$ state is invariant under the combination $(T \cdot J)$ of the time-reversal ($T$) and inversion ($I$) as well as the spin rotation. Finally, it has the charge conjugation symmetry ($C^*$) via the mapping $f_{i,\alpha} \rightarrow \epsilon_i f_{i,\alpha}^\dagger$ where $\epsilon_i = 1$ for $i=1,2,3$ and -1 for $i=4,5,6$. Under these symmetry operations the continuum field $\Psi$ transforms as follows.

FIG. 7: (Color online) The generators of the point group symmetry of the SL$[-\frac{\pi}{2}, \frac{\pi}{2}, 0]$. $R_y$ maps $y \rightarrow -y$ while $C_{2\pi/3}$ induce the rotation by $2\pi/3$ with respect to the center of the dodecagon.
The following two fermion bilinears, \( \tilde{C} \) and \( \tilde{T} \), are defined in Sec. III B.

The effective Hamiltonian in Eq. (16) leads to the mass gap of 2√2 for the pair \( \tilde{C} \) and \( \tilde{T} \). Since these mass terms are anticommuting with the effective Hamiltonian in Eq. (16), the pair \( \tilde{C} \) and \( \tilde{T} \) transform nontrivially under the space group operations. Its symmetry property is consistent with the terminology defined in Sec. III B, \( \tilde{C} \) and \( \tilde{T} \) transform as the \( E_3 \) irreducible representation under the enlarged point group \( G_{ivb} \). The detailed discussion on the group theory for the star lattice is given in Sec. III B.

(e) SL[\( \frac{\pi}{2}, \frac{\pi}{2}, 0 \)] and SL[\( -\frac{\pi}{2}, \frac{\pi}{2}, \pi \)]

We obtain SL[\( \frac{\pi}{2}, \frac{\pi}{2}, 0 \)] (SL[\( \frac{\pi}{2}, \frac{\pi}{2}, \pi \)] (SL[\( -\frac{\pi}{2}, \frac{\pi}{2}, 0 \)] states. Due to the introduction of the additional \( \pi \) flux, the mean field description requires unit cell doubling although the actual physical wave function maintains the translational invariance.

The spinon dispersion of the SL[\( \frac{\pi}{2}, \frac{\pi}{2}, 0 \)] state is described in Fig. 8. The structure of the low energy spectrum of SL[\( \frac{\pi}{2}, \frac{\pi}{2}, 0 \)] is similar to that of SL[\( \frac{\pi}{2}, \frac{\pi}{2}, \pi \)], except that the number of the momentum points which support low energy excitations is doubled. Both of them are characterized by finite \( \tilde{C}_{unif} \) indicating the time-reversal and parity symmetry breaking. Therefore the SL[\( \frac{\pi}{2}, \frac{\pi}{2}, 0 \)] state is also a chiral spin liquid state. The low energy excitations can be described by the effective Hamiltonian similar to Eq. (15), which can be obtained following the same procedure we used to derive Eq. (15) for SL[\( \frac{\pi}{2}, \frac{\pi}{2}, \pi \)].

In Fig. 8(b) we have drawn the low energy spinon excitation spectrum of the SL[\( \frac{\pi}{2}, \frac{\pi}{2}, \pi \)]. There are two electron pockets (around \( m_1 \) and \( m_3 \)) and two hole pockets (around \( m_2 \) and \( m_4 \)). It is characterized by finite \( \tilde{C}_{staggered} \) showing broken time reversal and inversion symmetry. Since the fluxes inside up-triangles and down-triangles have opposite signs, the six-fold rotational symmetry is broken down to three-fold rotational symmetry. Therefore it is another nematic spin liquid state.

III. DIMER PHASES

A. Large-N approach

According to the pioneering work by D.S. Rokhsar when the lattice system is dimerizable, the best mean field ansatz is one of dimerized states in the large-N limit of the SU(N)-generalized Heisenberg model. Here we call a lattice to be dimerizable when it is possible to make every site belong to a dimer and a lattice site be paired with one and only one of its neighboring site. In particular, when every dimer is lying on the link which has the maximum exchange coupling \( J_{max} \), the dimer state belongs to the ground state manifold of the mean field Hamiltonian. In terms of the variable \( \chi_{ij} \), we have finite \( \chi_{ij} \) only on the dimers lying on the link which has the maximum spin coupling \( J_{max} \).

The star lattice is dimerizable with respect to \( J_\phi \). Therefore when \( J_\phi \) is larger than \( J_f \), it has a unique dimerized ground state (we call it the \( J_\phi \)-dimer VBS) in which every dimer is lying on an expanded link connecting neighboring triangles.
we include the fluctuations investigated the $1/N$ correction systematically for a similar treatments to the ground state energy. In Ref. 19, Read and Sachdev phases is lifted by fluctuations, we consider the $1/N$ correction energy of dimerized states when $t_i > J_e$. This is because the star lattice is not dimerizable with respect to the $J_t$ links and every dimer configuration defined on the star lattice contains a finite number of dimers lying on the $J_e$ links. Therefore it is possible that the translationally invariant mean field ansatz can be the ground state even in the large-$N$ limit.

When $J_i > J_e$, we have to maximize the number of the dimers lying on triangular links to minimize the ground state energy of dimerized states. Since every triangle can support a single dimer at most (we call the triangle with a dimer lying on it a filled triangle), the remaining unpaired lattice point of the filled triangle has to be a part of the dimer lying on an expanded link. In other words, every dimer lying on an expanded link is connecting two filled triangles and this describes a representative local dimer configuration of the lowest energy dimerized states when $J_i > J_e$. (See Fig. 10(a).) Using this local dimer configuration as a building block we can construct infinite number of degenerate dimerized ground states.

In Fig. 9 we describe the geometric arrangement of singlet dimers of the $J_e$-dimer VBS phase.

On the other hand, the Rokhsar’s general theorem cannot be applied when $J_t$ is larger than $J_e$. This is because the star lattice is not dimerizable with respect to the $J_t$ links and every dimer configuration defined on the star lattice contains a finite number of dimers lying on the $J_e$ links. Therefore it is possible that the translationally invariant mean field ansatz can be the ground state even in the large-$N$ limit.

In the star lattice problem with $J_t > J_e$, the dimerized ground states are constructed by repeating the representative local dimer configuration displayed in Fig. 10(a). In this ground state manifold, the smallest flippable loop contains 18 sites with a dodecagon at the center, which is shown in Fig. 10(b). Here when the bright (blue) thick link is occupied by a dimer, the neighboring dark (black) thick link is empty and vice versa. By interchanging the roles played by the bright (blue) links and the dark (black) links, two degenerate dimerized phases can be connected.

Therefore the fluctuation corrections pick the patterns that maximize the number of the 18-site dimer units participating in the 18-site flippable loops. In Fig. 11(a) we show the valence bond solid order which has the maximum number of the 18-site flippable loops. Among the six neighboring dodecagons around an 18-site dimer unit, three can be the centers of the 18-site dimer units. This is in contrast to the case of the kagome lattice problem. There, none of the six neighboring hexagons around a perfect hexagon can be perfect hexagons. In fact, the valence bond solid ground state of the kagome lattice, which contains 36-site within the unit cell, results from the condition of maximizing the number of the perfect hexagons. The similar idea was also applied to the square-kagome antiferromagnet. In the star lattice problem with $J_t > J_e$, the dimerized ground states are constructed by repeating the representative local dimer configuration displayed in Fig. 10(a). In this ground state manifold, the smallest flippable loop contains 18 sites with a dodecagon at the center, which is shown in Fig. 10(b). Here when the bright (blue) thick link is occupied by a dimer, the neighboring dark (black) thick link is empty and vice versa. By interchanging the roles played by the bright (blue) links and the dark (black) links, two degenerate dimerized phases can be connected.

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where $C$ perform the detailed symmetry analysis on the bond order unit have the same finite value of the resonance energy gain from the smallest flippable loop. In addition we also consider another low energy valence bond order which is dis-ordered states. (See Fig. 12(b)).

The star lattice has the three-fold degeneracy of the columnar 18-site VBS comes except the central one support the 18-site dimer unit. The 18-site valence bond solid emerges. By maximizing the number of the square plaquette half of them (two square plaquettes) can support two parallel dimers. Based on the similarity with the square lattice problem, we can call the valence bond solid in Fig. 11(a) as a columnar 18-site valence bond solid. In this figure all the dodecagons in the three-fold degeneracy in this phase as well. In the recent numerical study by G. Misguich et al., this phase was sug-gested as a possible valence bond solid ground state when compared to those on the square and kagome lattice problem, however, since the length of the flippable loop is quite large compared to those on the square and kagome lattice problem, the energy difference of the two candidate valence bond solids in Fig. 11 could be very small.

**B. Group theoretical approach to $\sqrt{3} \times \sqrt{3}$ bond orders**

The columnar and box 18-site VBS phases discussed in the above section break the lattice translational symmetry and are described by the enlarged $\sqrt{3} \times \sqrt{3}$ unit cell. Here we perform the detailed symmetry analysis on the bond order that are compatible with the $\sqrt{3} \times \sqrt{3}$ enlarged unit cell.

(a) Group theory for the star lattice

The star lattice has the $D_6$ point group symmetry. The twelve elements of the $D_6$ group are as follows.

$$D_6 = \{E, C_6, C_6^2, C_6^3, C_6^4, C_6^5, R_a, R_b, R_c, R_A, R_B, R_C\},$$

where $C_6$ means the rotation by $2\pi/6$ around the center of a dodecagon and $E$ is the identity operator. $R_a$ indicates the reflection with respect to a plane orthogonal to the lattice plane. The reflection planes (denoted by $\alpha$) are described in the Fig. 12(a).

The space group, $G_S$, of the star lattice is generated by the translation group, $G_T$, and the $D_6$ point group. An element of the space group can be written using the Seitz operator $\{gD_6|\}$, where $D_6$ is an element of the $D_6$ group and $t= n_1a_1 + n_2a_2, (n_1$ and $n_2$ are integers.) The action of a Seitz operator on a lattice point $r$ is defined as $\{gD_6|\} r = gD_6 r + t$.

Note that the translation group, $G_T$, is an invariant subgroup of the space group $G_S$ and the point group $D_6$ is the corre-sponding factor group, i.e., $D_6 = G_S/G_T$.

To understand the symmetry of the enlarged unit cell, we define another translation group, $G_{T, b}$, whose elements can be written as,

$$G_{T, b} = \{E|n_1b_1 + n_2b_2\}; n_1, n_2 \in \mathbb{Z}$$

where $b_1 = a_1 + a_2$ and $b_2 = 2a_1 - a_2$ are the lattice vectors corresponding to the $\sqrt{3} \times \sqrt{3}$ ordered state. (See Fig. 12(b)) Since $G_{T, b}$ is an invariant subgroup of the space group $G_S$, the enlarged point group $G_{P, b}$ can be defined as the factor group $G_{P, b} = G_S/G_{T, b}$. Therefore the elements of the space group can be written as $\{gG_{P, b}|\} n_1b_1 + n_2b_2\}$ in which $gG_{P, b}$ is an element of the enlarged point group, $G_{P, b}$. (Note that similar approach was used by Hermele et al. to investigate the sym-metry properties of the object invariant under the translations by $2a_1$ and $2a_2$ on the kagome lattice.)

The construction of the $G_{P, b}$ group is straightforward, whose elements can be written using the Seitz operator $\{gD_6|\}$ with $t=0, a_1, a_2$. The 36 elements of the $\{gD_6|\}$ can be grouped into the nine conjugate classes,
TABLE I: The Character table of the enlarged point group $G_{P,h}$

| $\Gamma$ | $E_C$ | $T_C$ | $E_T$ | $T_T$ | $E_{CT}$ | $T_{CT}$ | $R_1$ | $R_{1T}$ | $R_{2T}$ | $R_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_2$ | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $B_1$ | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 |
| $B_2$ | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $E_1$ | 2 | 2 | -1 | 1 | -2 | 0 | 0 | 0 | 0 | 0 |
| $E_2$ | 2 | 2 | -1 | 1 | -2 | 0 | 0 | 0 | 0 | 0 |
| $E_3$ | 2 | 2 | -1 | 1 | -2 | 0 | 0 | 0 | 0 | 0 |
| $E_4$ | 2 | 2 | -1 | 1 | -2 | 0 | 0 | 0 | 0 | 0 |
| $Q$ | 4 | -2 | -2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

(a)

The enlarged point group $G_{P,h}$ whose decomposition into the irreducible representations is given by

$\Gamma_{\text{bond}} = 2A_1 + B_2 + E_1 + 2E_2 + 2E_3 + E_4 + 3Q$.

Notice that $\Gamma_{\text{bond}}$ supports two independent $E_3$ irreducible representations (we call them as $E_3^A$ and $E_3^B$, respectively) and one $E_4$ irreducible representation. The bond ordering patterns which constitute a basis of each irreducible representation are displayed in Fig. 13 and Fig. 14 describing $E_3^A$, $E_3^B$, and $E_4$ irreducible representations, respectively.

A bond order transforming as an $E_3$ irreducible representation can be represented by a linear combination of states like

$$|E_3\text{bond order}\rangle = \alpha_1|E_3^A(a)\rangle + \alpha_2|E_3^B(b)\rangle + \alpha_3|E_3^C(a)\rangle + \alpha_4|E_3^D(b)\rangle,$$

in which $|E_3^A(a)\rangle = \sum c_l|l\rangle$, with $c_l$ specified in Fig. 13(a). The other three basis states $|E_3^B(a)\rangle$, $|E_3^C(b)\rangle$, and $|E_3^D(c)\rangle$ are defined following the same way.

Interestingly the bond ordering patterns of the two valence bond solid states, the columnar and box 18-site VBS, are given by the following superposition of states,
FIG. 15: (Color online) Description of the pair of the $E_4$ bond ordered states which form a basis of the $E_4$ irreducible representation.  
(a) $c_1=\sqrt{3}$ for the thick light (red) links, $-\sqrt{3}$ for the thick dark (black) links, and zero for the thin solid links.  
(b) In each triangle the link belonging to the central dodecagon has the $c_1=2$ (-2) if it has red (black) color.  
The other two links of the triangle have $c_1=1$ (-1) if they have red (black) colors.

$$|\text{Columnar VBS}\rangle \propto |\text{uniform}\rangle - |E_3^A(b)\rangle - |E_3^B(b)\rangle,$$

$$|\text{Box VBS}\rangle \propto |\text{uniform}\rangle + \frac{1}{2}|E_3^A(b)\rangle + \frac{1}{2}|E_3^B(b)\rangle,$$

where $|\text{uniform}\rangle \equiv \sum_l |l\rangle$.  
Since both the columnar and box 18-site VBS are invariant under the reflections $R_a$, $R_b$, and $R_c$, $|E_3^A(a)\rangle$ and $|E_3^A(a)\rangle$ have no contribution.  
Superpositions of the $|E_3^A(b)\rangle$ and $|E_3^B(b)\rangle$ can induce more general bond ordering patterns other than those described in Fig. 11.  
Finally, since $E_4$ irreducible representation always breaks the reflections $R_a$, $R_b$, and $R_c$.  
(See Fig. 15), we neglect bond orders transforming as $E_4$ irreducible representation.

IV. GROUND STATE ENERGY

In this section we compare the ground state energies of various spin liquid states.

A. Mean field theory

The ground state energies of various spin liquid states for $J_t=2J_e$ are shown in Table II.  
In addition to the translationally invariant spin liquid states with finite $J_e$ and $J_t$, we have also considered a decoupled dimer phase for comparison.  
A decoupled dimer phase, which has the lowest mean field ground state energy for $J_t > J_e$, can be built based on the local dimer configuration described in Fig. 10(a).  
The columnar 18-site valence bond solid (VBS) displayed in Fig. 11(a) is an example.  
According to the mean field calculation, the dimer state has lower ground state energy than any other translationally invariant spin liquids.  
Among the spin liquid phases with translational invariance, the four ansatz having the $\pi/2$ flux inside triangles have lower energies than those having zero flux inside triangles, i.e., SL[0, 0, 0] and SL[0, 0, $\pi$].

| Dimer            | $E_{MF}$ (unprojected) | $E_{MF}$ (projected) |
|------------------|------------------------|----------------------|
| Dimer            | -0.625                 | -0.625               |
| SL[0, 0, 0]      | -0.498                 | -0.647               |
| SL[$\frac{\pi}{2}$, $\frac{\pi}{2}$, $\pi$] | -0.553                 | -0.624               |
| SL[$-\frac{\pi}{2}$, $\frac{\pi}{2}$, 0] | -0.553                 | -0.616               |
| SL[0, 0, $\pi$] | -0.498                 | -0.654               |
| SL[$\frac{\pi}{2}$, $\frac{\pi}{2}$, 0] | -0.552                 | -0.617               |
| SL[$-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\pi$] | -0.552                 | -0.614               |

TABLE II: The ground state energies of the various mean field ansatz when $J_t=2J_e$.  
The energies are measured in unit of $J_e$.

B. Projected wave function study

In the above mean field calculation, the single occupancy constraint is imposed only on average.  
Therefore the mean field wave functions contain unphysical states with zero or two fermions at a point.  
To obtain physical spin wave functions we therefore perform a numerical Gutzwiller projection on the mean field wave functions.  
The ground state energies of the projected states are computed numerically using the variational Monte Carlo (VMC) method.  
The resulting energies are displayed in Fig. 16 for a range of $J_t/J_e$ where we have optimized the state with respect to $\chi_t/\chi_e$ for each value of $J_t/J_e$.  
Table II shows the numerical energy values for $J_t/J_e=2.0$ to facilitate a comparison with the mean field numerics.  
We find that Gutzwiller projection dramatically changes the relative ordering of the various states, and that the state SL[0, 0, $\pi$] appears, upon projection, to be the lowest energy spin liquid over the entire parameter range.

C. Bond operator approach

According to the projected wave function study, SL[0, 0, $\pi$] state is the ground state over a wide parameter space.  
However, the SL[0, 0, $\pi$] state is unstable due to the confinement in the 2+1 dimensional pure gauge theory.  
It is also inconsistent with our expectation for the $J_e \gg J_t$ limit.  
When $J_e \gg J_t$, the $J_e$-dimer VBS phase (See Fig. 9) is the exact ground state.  
In addition the recent exact diagonalization study shows that the $J_e$-dimer VBS phase remains as the ground state up to the isotropic limit of $J_e = J_t/2$.  
Therefore SL[0, 0, $\pi$] state should have higher energy than $J_e$-dimer VBS phase at least in some finite range of $0 \leq J_t/J_e \leq 1$.  
This discrepancy comes from the lack of the interdimer interaction in the decoupled dimer limit.  
For the description of dimerized phases beyond the decoupled dimer limit, we undertake the self-consistent bond operator approach.  
If the correction coming from the inter-dimer interaction is significant, we also have to check the possibility that the true ground state is a valence bond solid even when $J_t > J_e$.

In the bond operator formulation, the dimer singlet degrees of freedom are used as natural building blocks and the quantum corrections coming from the triplet fluctuations can systematically be investigated.  
Here we present a brief explanation of the bond operator formulation.  
Let us consider the
Reduced such that each of the above states can be created from a singlet state, $|s\rangle$, and three triplet states, $|t_x\rangle$, $|t_y\rangle$ and $|t_z\rangle$. Then, the singlet and triplet boson operators are introduced such that each of the above states can be created from the vacuum $|0\rangle$ as follows:

$$|s\rangle = s^d |0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$,

$$|t_x\rangle = t_x^d |0\rangle = -\frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$,

$$|t_y\rangle = t_y^d |0\rangle = \frac{i}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$,

$$|t_z\rangle = t_z^d |0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$.

To eliminate unphysical states from the enlarged Hilbert space, the following constraint needs to be imposed on the bond-particle Hilbert space:

$$s^d s + t^d_\mu t^{\dagger}_\mu = 1,$$

where $\alpha = x, y,$ and $z$, and we adopt the summation convention for the repeated indices hereafter unless mentioned otherwise.

Constrained by this equation, the exact expressions for the spin operators can be written in terms of the bond operators:

$$S_{R\alpha} = \frac{1}{2} (s^d t_\alpha + t^{\dagger}_\alpha s - i \varepsilon_{\alpha\beta\gamma} t^{\dagger}_\beta t_\gamma)$$,

$$S_{L\alpha} = \frac{1}{2} (-s^d t_\alpha - t^{\dagger}_\alpha s - i \varepsilon_{\alpha\beta\gamma} t^{\dagger}_\beta t_\gamma),$$

where $\varepsilon_{\alpha\beta\gamma}$ is the third-rank totally antisymmetric tensor with $\varepsilon_{xyz} = 1$.

Utilizing the bond operator representation of spin operators, the Heisenberg spin Hamiltonian in Eq. (2) can be rewritten solely in terms of bond particle operators. Since all dimers of $J_e$-dimer VBS phase are symmetry equivalent, the singlet condensate density $\langle s_1 \rangle$ and the chemical potential $\mu$ can be set to be $\langle s_1 \rangle = s$ and $\mu = \mu$ in our mean field theory. Here $i$ denotes the location of dimers. The hard-core constraint on the bond-particle operators is imposed by adding the following Lagrange multiplier term, $H_i = -\sum_i (s^2 + t^{\dagger}_i t_i - 1)$. The resulting Hamiltonian can be written as follows:

$$H = N \epsilon_0 + H_{\text{Quad}} + H_{\text{Quartic}},$$

where

$$H_{\text{Quad}} = \left( \frac{J_e}{4} - \mu \right) \sum_i t^{\dagger}_i t_i$$

$$+ \frac{J_x s^2}{4} \sum_{(i,j)} \{ t^{\dagger}_i t_j + t_i t_j + \text{H. c.} \},$$

and

$$H_{\text{Quartic}} = \frac{J_e}{4} \sum_{(i,j)} \varepsilon_{\alpha\beta\gamma} \varepsilon_{\alpha\mu\nu} t^{\dagger}_{i\beta} t^{\dagger}_{j\gamma} t^{\dagger}_{i\mu} t^{\dagger}_{j\nu}.$$  

In the above, $N$ is the number of unit cells and

$$\epsilon_0 = 3 \left[ \mu (1 - s^2) - \frac{3}{4} J_e s^2 \right].$$
The quartic interactions between triplet particles are decoupled using the mean field order parameters $P$ and $Q$, where $P \equiv \langle t_{j_1}^\dagger t_{j_2} \rangle$ and $Q \equiv \langle t_{j_1}^\dagger t_{j_2}^\dagger \rangle$. Here $P$ and $Q$ denote the diagonal and off-diagonal triplet correlations between neighboring dimers. These two order parameters $P$ and $Q$ together with $\bar{s}$ and $\mu$ are determined self-consistently by solving the coupled saddle point equations.\textsuperscript{44,45}

The ground state energy of $J_c$-dimer VBS phase obtained from the self-consistent bond operator calculation is displayed and compared to the energy of SL$[0,0,\pi]$ state in Fig. 17. Here we have obtained the energy of $J_c$-dimer VBS phase in three different ways. If we neglect the inter-dimer couplings (the decoupled dimer limit) completely, the energy is independent of $J_f/J_c$. The inclusion of the inter-dimer couplings (the decoupled dimer limit) completely, the energy is independent of $J_f/J_c$. The inclusion of the inter-dimer interaction lowers the ground state energy significantly. In the end, $J_c$-dimer VBS phase has the lower ground energy than the SL$[0,0,\pi]$ state over the entire parameter range of $0 \leq J_f/J_c \leq 1$ when we include the quartic interactions. The inter-dimer interactions generate huge correction to the ground state energy of dimerized phases.

Now we concentrate on the other limit where $J_f > J_c$. In contrast to the $J_c > J_f$ limit, it is nontrivial to identify the ground state even when we restrict our attention to valence bond solid phases. Taking into account the information from the exact diagonalization study and 1/N fluctuation from the large-N limit, we suggest the columnar 18-site VBS phase as a promising candidate for the ground state as explained below.

We apply the bond operator approach to the columnar 18-site VBS phase. The nine dimers within the unit cell can be divided into two groups. One group is made of the six dimers lying on the triangular links. Note that all these six dimers are lying on a dodecagon. (See the central dodecagon in Fig. 11a.) We call such a dodecagon surrounded by six dimers as a “perfect” dodecagon. The remaining three dimers lying on the expanded links make the other group. Every dimer belonging to the same group is symmetry equivalent as one can easily notice from the patterns around the central “perfect” dodecagon in Fig. 11a. To apply the bond operator approach we have to introduce two independent sets of order parameters to distinguish the two different groups of dimers. We use $\bar{s}_c$ and $\mu_c$ ($\bar{s}_t$ and $\mu_t$) to indicate the singlet condensate density and the chemical potential corresponding to the expanded (triangular) link. To decouple the quartic triplet interactions we introduce two sets of the order parameters, that is, $\{P_{pp}, Q_{pp}\}$ and $\{P_{ep}, Q_{ep}\}$. $P_{pp} (Q_{pp})$ describes the diagonal (off-diagonal) correlation between the neighboring dimers lying on a “perfect” dodecagon. On the other hand $P_{ep} (Q_{ep})$ describes the diagonal (off-diagonal) correlation between a dimer lying on an expanded link and its neighboring dimer lying on a “perfect” dodecagon. We have determined the eight parameters $\bar{s}_c, \mu_c$ ($\bar{s}_t, \mu_t$) self-consistently by solving the coupled saddle point equations.

The self-consistent solution shows that $P_{ep}=Q_{ep}=0$, $\bar{s}_c^2=1$ and $\mu_c=3/4J_c$. Since $P_{pp}$ and $Q_{pp}$ describe the coupling between the dimers lying on expanded links and the dimers lying on “perfect” dodecagons, these two groups of dimers are completely decoupled when $P_{pp}=Q_{pp}=0$. In this situation, every dimer lying on expanded links is decoupled from the surrounding, leading to $\bar{s}_c^2=1$ and $\mu_c=3/4J_c$. The triplet fluctuations are confined inside every isolated perfect dodecagon, which is reflected in the finite $P_{pp}$ and $Q_{pp}$ values. This interesting structure would result in the highly localized triplet excitation spectrum.
The ground state energies of the columnar 18-site VBS and SL[0,0,π] are compared in Fig. 18. The energies of the columnar 18-site VBS are obtained in three different ways again. That is, for the decoupled dimer limit, including the inter-dimer coupling neglecting quartic interactions, and finally including the inter-dimer quartic interactions. For clarity we also calculated the energy difference relative to the decoupled dimer energy as shown in Fig. 18(b). Interestingly, there is a critical ratio \((J_t/J_c)_c \approx 2.4\) beyond which the columnar 18-site VBS becomes the ground state when we include the quartic triplet interactions. Even though the critical ratio \((J_t/J_c)_c\) is a bit larger than the suggested phase boundary from the numerical study\(^{26}\), the existence of the critical values of \((J_t/J_c)_c\) is quite encouraging. In particular, because the slopes of the lines in Fig. 18(a) are almost parallel, small additional energy correction could induce a large shift of the crossing point as shown in Fig. 18(b). Since the simple Hartree-Fock approximation does not take into account the fluctuations coming from the cooperative interaction between the dimers on the expanded links and those on the perfect dodecagons, we expect that the quantum correction beyond the Hartree-Fock limit could shift the energy level crossing point down to \((J_t/J_c)_c \approx 1.3\) as suggested by the numerical study\(^{26}\).

V. INSTABILITY OF SL[0,0,π] SPIN LIQUID AND VALENCE BOND SOLIDS

Summarizing the previous discussions, \(J_c\)-dimer VBS phase is the ground state for \(J_t/J_c < (J_t/J_c)_c\) while the columnar 18-site VBS is the ground state in the opposite limit of \(J_t/J_c > (J_t/J_c)_c\). (Here \((J_t/J_c)_c \leq (J_t/J_c)_c\).) Although this result is obtained based on the energy comparison with SL[0,0,π] state, one may still expect that the two valence bond solid states are intimately related to the SL[0,0,π] state. In particular, the \(J_c\)-dimer VBS and the columnar 18-site VBS may arise as a consequence of the confinement in the SL[0,0,π] spin liquid state. In this section we describe the possible relation between these two valence bond solid phases and the SL[0,0,π] state.

A. Spinon confinement and uniform bond orders

Due to the finite spinon gap, the U(1) gauge field is the only low energy excitation in the SL[0,0,π] state in the long wavelength limit. Since the compact U(1) gauge theory without matter field is confining in 2+1 dimension, we expect that the monopole proliferation would lead the SL[0,0,π] ansatz to some confined phases. To understand the properties of the confined phases resulting from the monopole condensation, we have to determine the symmetry properties of the monopole operators.

Here we discuss the possibility that the monopole operators are invariant under all possible symmetry transformations. The \(J_c\)-dimer VBS phase, which is the ground state for \(J_t/J_c < 1\), is invariant under space group operations. The bond ordering pattern of the \(J_c\)-dimer VBS phase belongs to the trivial \(A_1\) irreducible representation of the \(D_6\) point group. Therefore if we interpret the \(J_c\)-dimer VBS phase to be induced by the confinement transition, which is reasonable in the limit of \(J_t/J_c \ll 1\), this reflects the fact that monopole operators are invariant under symmetry transformations.

Extending the group theory analysis we performed in Sec. III B, we investigate all possible bond orders invariant under the space group operations. These are displayed in Fig. 19. Here we have finite singlet correlation \((c_l \neq 0)\) only on the thick solid (red) links. The bond order in Fig. 19(a) is nothing but the \(J_c\)-dimer VBS phase. On the other hand, the bond order in Fig. 19(b) has finite \(c_l\) only on the triangular links (we call it as a \(J_t\)-bond ordered phase). We expect the \(J_t\)-bond ordered phase is the natural low energy bond ordering pattern when \(J_t \gg J_c\). Since the arbitrary superposition of these two orders follows the same \(A_1\) irreducible representation, we expect the actual ground states would have finite \(c_l\) values over all the links on the lattice. However, it is natural to expect that the \(c_l\) on the expanded (triangular) link would be larger than that on the triangular (expanded) link when \(J_t > J_c\). Therefore the bond ordered phase corresponding to the \(A_1\) irreducible representation successfully describes the low energy manifold over the whole parameter range of \(J_t/J_c\). Interestingly, the recent work by Choy and Kim\(^{38}\) has suggested that the same bond ordered states are the ground states of the same model Hamiltonian in the strong quantum limit based on the bosonic \(Sp(N)\) approach.

Since the spinon bandstructure of the SL[0,0,π] state does not change qualitatively by varying \(J_t/J_c\), we expect that the change of \(J_t/J_c\) ratio would not affect the trivial monopole quantum number. Therefore if the ordered phase is coming from the confinement transition, it will transform trivially under the symmetries. However, it is also possible that the instability of the SL[0,0,π] state is caused by the interactions between spinons leading to some broken symmetry phases. We discuss about this possibility in the following section.
B. Instability induced by interactions between spinons

Here we investigate the instability of the SL[0,0,π] state coming from the interactions between spinons and the symmetry properties of the resulting ordered phase. Especially we focus on the instability towards the states with the $\sqrt{3} \times \sqrt{3}$-type translational symmetry breaking. As shown in Fig. 4, the valence band is completely flat without any preferred momentum. However, the conduction band supports several dispersion minima. The four minimum points of the conduction band are given by

$$ m_3 = -m_2 = \left( \frac{\pi}{6}, 0 \right) \quad \text{and} \quad m_4 = -m_1 = \left( \frac{\pi}{3}, \frac{\pi}{2\sqrt{3}} \right). $$

Interestingly if we double the vectors connecting neighboring minimum points, they sit on the Brillouin zone corners which are nothing but the momentum corresponding to the $\sqrt{3} \times \sqrt{3}$ ordering. Motivated by this observation we study the symmetry properties of the bound states made of low energy fermions near the conduction band minima.

We introduce the fermion fields $\Psi_i$ which describe the low energy excitations near the four conduction band minima $m_i$ ($i=1,2,3$ and 4),

$$ \Psi_i(x) \sim \sum_{n=1}^{12} e^{-i m_i \cdot x} (\nu_i)_n^* f_n(x), $$

where $\nu_i$ is the eigenvector of the mean field Hamiltonian at the momentum $m_i$, and $f_n$ is a slowly varying fermion field near the conduction band minimum, with $n$ labelling the twelve sites within the unit cell. To determine the symmetry of bound states made of the above low energy fermions, we have to understand how the symmetries of the microscopic Hamiltonian are realized in the effective continuum fields, $\Psi_i$.

Here we follow the same procedure which we use to determine the transformation properties of the continuum field for the SL[-$\pi/3$, $\pi/3$, 0] state in Sec. II B. To extract the necessary information on the transformations of the continuum field, we consider a finite system of a 6×12 unit cell. By solving the mean field Hamiltonian on this finite system, we determine the properties of the eigenvectors at the four momentum points, $m_i$. The detailed explanation of the procedure as to how to determine the symmetry of the continuum fields is discussed in the Appendix III.

Through the projective symmetry group analysis, we determine the following transformation properties of the continuum fields,

$$ T_{a_1} : \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & e^{-i \frac{\pi}{2}} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & e^{i \frac{\pi}{2}} & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}, $$

$$ T_{a_2} : \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i \frac{\pi}{2}} & 0 & 0 & 0 \\ 0 & e^{i \frac{\pi}{2}} & 0 & 0 \\ 0 & 0 & 0 & e^{i \frac{\pi}{2}} \\ 0 & 0 & 0 & e^{-i \frac{\pi}{2}} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}, $$

We first investigate the symmetry of all possible fermion bilinears which can be written as $\Psi_i^\dagger M_{ij} \Psi_j$. For the description of the 4 by 4 unitary matrix $M_{ij}$ we introduce two sets of the pauli matrices $\tau_i$ and $\mu_j$. Here $\tau_i$ is acting on the space spanned by either $(\Psi_1, \Psi_3)$ or $(\Psi_2, \Psi_4)$. On the other hand $\mu_j$ is defined in the space spanned by $(\Psi_1, \Psi_2)$ or $(\Psi_3, \Psi_4)$. Among the sixteen possible bilinears, there are only two terms which can form a basis of the enlarged point group $G_{PB}$. These are $\Psi_1^\dagger T_3 \mu_0 \Psi_2$ and $\Psi_1^\dagger T_2 \tau_0 \mu_0 \Psi_2$ which transform as the $A_2$ and $A_1$ irreducible representation of the enlarged point group $G_{PB}$, respectively. However, since these two bilinears have zero total momentum, they cannot describe the $\sqrt{3} \times \sqrt{3}$ type symmetry breaking. The net momentums carried by the other fourteen bilinears are neither zero nor $K=\frac{2\pi}{3}(0,0)$. Therefore the transformation properties of them are not compatible with the symmetry of the $\sqrt{3} \times \sqrt{3}$ type enlarged unit cell.

Next we consider the instability in the particle-particle channels. Defining the pairing amplitude as $\Delta_{ij} \equiv \Psi_i \Psi_j$, we have sixteen different $\Delta_{ij}$. We first omit the indices for the spin degrees of freedom and study how they transform under the space group symmetry operations. It can easily be checked that the sixteen pairing amplitudes are divided into the four different sets $\Pi_1$, $\Pi_2$, $\Omega_1$ and $\Omega_2$ which transform independently. The six-component vectors $\Pi_i$ and the two-component vectors $\Omega_i$ are given by

$$ \Pi_1 = \begin{pmatrix} e^{-i \frac{\pi}{2}} \Delta_{11} + e^{i \frac{\pi}{2}} \Delta_{33} \\ e^{-i \frac{\pi}{2}} \Delta_{11} - e^{i \frac{\pi}{2}} \Delta_{33} \\ \Delta_{13} + \Delta_{31} \\ e^{-i \frac{\pi}{2}} \Delta_{22} + e^{i \frac{\pi}{2}} \Delta_{44} \\ e^{-i \frac{\pi}{2}} \Delta_{22} - e^{i \frac{\pi}{2}} \Delta_{44} \\ \Delta_{24} + \Delta_{42} \end{pmatrix}, $$

$$ \Pi_2 = \begin{pmatrix} e^{-i \frac{\pi}{2}} \Delta_{12} + e^{i \frac{\pi}{2}} \Delta_{34} \\ e^{-i \frac{\pi}{2}} \Delta_{12} - e^{i \frac{\pi}{2}} \Delta_{34} \\ \Delta_{14} - \Delta_{32} \\ e^{-i \frac{\pi}{2}} \Delta_{21} + e^{i \frac{\pi}{2}} \Delta_{43} \\ e^{-i \frac{\pi}{2}} \Delta_{21} - e^{i \frac{\pi}{2}} \Delta_{43} \\ \Delta_{41} - \Delta_{23} \end{pmatrix}, $$

$$ \Omega_1 = \begin{pmatrix} \Delta_{13} - \Delta_{31} \\ \Delta_{24} - \Delta_{42} \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} \Delta_{14} + \Delta_{32} \\ \Delta_{41} + \Delta_{23} \end{pmatrix}. $$
Under a space group symmetry operation $S$, they transform in the following way,

$$\Pi_i \to A_S(\Pi_i)\Pi_i \quad \text{and} \quad \Omega_i \to B_S(\Omega_i)\Omega_i,$$

where $A_S(\Pi_i)$ ($B_S(\Omega_i)$) is the 6 by 6 (2 by 2) matrix representing the symmetry operation $S$, whose detailed expressions are displayed in the Appendix.  

Since the pairing amplitude $\Delta_{ij}$ is not a gauge invariant object, we have to look into the symmetry of the bilinears such as $D_{ijk,kl}\equiv\Delta^*_{ij}\Delta_{kl}$. Even though the number of all possible tensors $D_{ijk,kl}$ is very large, we can reduce the complexity of the symmetry analysis by focusing on the objects carrying the momentum compatible with the $\sqrt{3} \times \sqrt{3}$ ordering. This idea leads us to exclude $D_{ijk,kl}$ made of the basis $\Pi_2$ and $\Omega_2$. In addition, $\Pi_1$ and $\Omega_1$ have opposite spin parities, that is, $\Pi_1$ is spin singlet while $\Omega_1$ is spin triplet. Therefore all we have to consider are the terms like $\Pi_1^3 M_{01}^3 \Pi_1$ and $\Omega_1^3 M_{00}^3 \Omega_1$.

First, we define a set of pauli matrices $\tau_i$ acting on the space spanned by $\Omega_1$. Using the transformation properties of $\Omega_1$ under the space group, we obtain a pair $(\Omega_1^1 \tau_x \Omega_1, \Omega_1^1 \tau_y \Omega_1)$ which has the momentum $k=(\frac{2\sqrt{3}}{3},0)$ and transforms as the $E_3$ irreducible representation of the enlarged point group $G_{Pb}$.

To understand the symmetry of $\Pi_1^3 M_{01}^3 \Pi_1$, we introduce a set of the Gell-Mann matrices $\lambda_{ij}$ ($i=1,...,8$) as well as the Pauli matrices $\tau_i$. For convenience, we also define the matrix, $\lambda_9=\sqrt{\frac{2}{3}}I_3$ where $I_3$ is the $3 \times 3$ identity matrix. The Gell-Mann matrices are acting on the space spanned by either $(\Pi_{1,1}, \Pi_{1,2}, \Pi_{1,3})$ or $(\Pi_{1,4}, \Pi_{1,5}, \Pi_{1,6})$ and $\tau_i$ connects these two three-component vectors. Here $\Pi_{1,n}$ indicates the $n$th component of $\Pi_1$.

We have examined the symmetry of all possible bilinears $\Pi_1^3 M_{01}^3 \Pi_1$ and found out that there are only two sets of bilinears which have the momentum $K$ compatible with the $\sqrt{3} \times \sqrt{3}$ ordering. These are given by

$$X_{E_3} = \left( \begin{array}{c} \frac{2\sqrt{3}}{3} \Pi_1^3 \tau_1 \lambda_0 \Pi_1 + \frac{1}{4} \Pi_1^3 \tau_1 \lambda_9 \Pi_1 \\ \frac{2\sqrt{3}}{3} \Pi_1^3 \tau_2 \lambda_0 \Pi_1 + \frac{1}{4} \Pi_1^3 \tau_2 \lambda_9 \Pi_1 \end{array} \right),$$

$$X_Q = \left( \begin{array}{rr} \Pi_1^3 \tau_1 \lambda_3 \Pi_1 \\ \frac{1}{4} \Pi_1^3 \tau_1 \lambda_8 \Pi_1 - \frac{2\sqrt{3}}{3} \Pi_1^3 \tau_1 \lambda_9 \Pi_1 \\ \Pi_1^3 \tau_2 \lambda_3 \Pi_1 \\ \frac{1}{4} \Pi_1^3 \tau_2 \lambda_8 \Pi_1 - \frac{2\sqrt{3}}{3} \Pi_1^3 \tau_2 \lambda_9 \Pi_1 \end{array} \right).$$

In the above $X_{E_3}$ transforms as the two dimensional $E_3$ irreducible representation of the enlarged point group $G_{Pb}$. On the other hand $X_Q$ constitutes a basis of the four dimensional $Q$ irreducible representation. Therefore the instability given by $X_{E_3}$ has the symmetry consistent with the $\sqrt{3} \times \sqrt{3}$ orders which we have discussed in detail in previous sections.

Since we are considering an instability from a gapped phase, to stabilize the resulting ordered state, the condensation energy should be larger than the excitation gap. However, because the magnitude of the energy gap reduces as $J_1/J_e$ increases, the instability can occur beyond the critical value of $J_1/J_e$. Finally, since the condition of $\langle X_{E_3} \rangle \neq 0$ does not constrain the magnitude of $\langle \Pi_1 \rangle$ or $\langle \Pi_1 \rangle$ can have both zero and nonzero values. If $\langle \Pi_1 \rangle \neq 0$ while $\langle X_{E_3} \rangle = 0$, we have $Z_2$ spin liquid supporting fractionalized quasi-particles and breaking the translational symmetry at the same time. This state is similar to the Amperean paired state, which is recently suggested as a possible ground state of the organic compound $\kappa$(BEDT-TTF)$_2$Cu$_2$(CN)$_3$. On the other hand, if $\langle \Pi_1 \rangle = 0$ while $\langle X_{E_3} \rangle \neq 0$, we have more conventional phase transformation as an $E_3$ irreducible representation.

In conclusion, the symmetry analysis of the low energy fermions near the conduction band minima shows that the instability of the SL[0,0,π] state from the particle-particle channel supports valence bond solid phases, which have the $\sqrt{3} \times \sqrt{3}$ unit cell transforming as $E_3$ irreducible representations of the enlarged point group $G_{Pb}$. Remember that the $J_e$-dimer VBS phase is induced via the monopole condensation from the SL[0,0,π] state for small $J_1/J_e$. Therefore we can obtain the valence bond solid ground states both for $J_1/J_e < 1$ and $J_1/J_e \gg 1$ limits from the instability of the SL[0,0,π] state.

### C. Projected wave function approach

We next assess the stability of the SL[0,0,π] spin liquid towards columnar dimer order in the Guzwiller projected state. In order to do this, we include a parameter δ in the mean field Hamiltonian which corresponds to strengthening the fermion hopping on those bonds which dimerize in the ‘classical’ columnar dimer state, shown in Fig. 11(a), and Guzwiller project the resulting state. Clearly, if $\delta t \gg 1$, the resulting wave function will be precisely the ‘classical’ columnar dimer pattern. The energy change of the weakly distorted state as a function of the distortion parameter $\delta t$ serves as a measure of the inverse susceptibility of the SL[0,0,π] state towards columnar dimer order. As seen from Fig. 20 the SL[0,0,π] state is stable, with a positive inverse susceptibility, for $J_1/J_e \gtrsim 2$, but is unstable, with a negative inverse susceptibility for $J_1/J_e \lesssim 2$. Further, the optimal $\delta t$ appears to increase continuously for $J_1/J_e \gtrsim 2$. This suggests that the SL[0,0,π] state possibly undergoes a continuous transition into a state with $\sqrt{3} \times \sqrt{3}$ columnar dimer order at $J_1/J_e \approx 2.0$. We discuss the phase transition more carefully in the next section.

### VI. PHASE TRANSITION BETWEEN VBS PHASES

The results in previous sections show that there is a phase transition between the $J_e$-dimer VBS phase and the columnar 18-site VBS phase with increasing $J_1/J_e$. To describe the phase transition between these two VBS phases, we construct a Landau-Ginzburg free energy introducing a two-component vector $(\Phi_1, \Phi_2)$, which transforms as an $E_3$ irreducible representation of the enlarged point group $G_{Pb}$. Since the VBS phases are time-reversal invariant, we can use two real numbers, $\Phi_1$ and $\Phi_2$. The Landau-Ginzburg free energy can be
written using all possible invariants made of \( \Phi_1 \) and \( \Phi_2 \). In particular, it is important to note that there is a third order invariant, which belongs to the \( A_1 \) irreducible representation of the following decomposition,

\[
E_3 \otimes E_3 \otimes E_3 = A_1 \oplus B_1 \oplus 3E_3.
\]

In terms of \( \Phi_1 \) and \( \Phi_2 \), the third order invariant is given by

\[
\Phi_2 (3\Phi_1^2 - \Phi_2^2).
\]

Straightforward extension of the same group theoretical analysis to quartic order shows that there is only one quartic invariant of \((\Phi_1^2 + \Phi_2^2)^2\). Collecting all invariants up to quartic order, the Landau-Ginzburg free energy density is written as

\[
f = \alpha(\Phi_1^2 + \Phi_2^2) + \lambda \Phi_2 (3\Phi_1^2 - \Phi_2^2) + u(\Phi_1^2 + \Phi_2^2)^2.
\]

For convenience we define a complex variable \( \Phi \) as follows,

\[
\Phi \equiv \Phi_2 - i\Phi_1 \equiv |\Phi|e^{i\theta}.
\]

Then the free energy density is given by

\[
f = \alpha|\Phi|^2 + u|\Phi|^4 - \lambda|\Phi|^3 \cos(3\theta).
\]

Given \( u > 0 \) and \( \lambda > 0 \), the above mean field free energy predicts two different phases separated by a first order transition point at \( \alpha = \alpha_c = \lambda^2/(4u) \). For \( \alpha > \alpha_c \), we have a disordered phase with \( |\Phi| = 0 \). On the other hand, ordered phases with \( |\Phi| \neq 0 \) and \( \theta = \frac{2\pi m}{3} \) (\( m=0,1,2 \)) appear when \( \alpha < \alpha_c \). The three ordered states describe the three-fold degenerate VBS phases with \( \sqrt{3} \times \sqrt{3} \) pattern.

FIG. 20: Change in energy of the Gutzwiller projected SL[0, 0, π] spin liquid state upon including a distortion \( \delta t \) corresponding to increased fermion hopping amplitude on the dimerized bonds of the \( \sqrt{3} \times \sqrt{3} \) columnar dimer state shown in Fig. 11(a). For \( J_1/J_c = 1.9, 2.0 \), the SL[0, 0, π] state is found to be stable against this distortion, while it is seen to be unstable for \( J_1/J_c = 2.1, 2.2 \). (This calculation was carried out on a system with 12 \( \times \) 12 unit cells, i.e., with 864 spins. The statistical errors on the computed energy are of the order of the symbol size.)

FIG. 21: Triplet dispersions of valence bond solid ground states along high symmetry directions of the Brillouin zone. The dispersions are obtained from the self-consistent bond operator mean field theory. (a) Triplet dispersion of the \( J_e \)-dimer VBS phase for \( J_1/J_c = 1 \). (b) Triplet dispersion of the columnar 18-site VBS phase for \( J_1/J_c = 3 \).

The above effective action is nothing but the action for the three dimensional \( Z_3 \)-clock model. Previous Monte Carlo simulations suggest that the phase transition associated with the \( Z_3 \) symmetry breaking is weakly first order.\cite{51,52,53} The wave function numerics in the previous section, however, are not inconsistent with this scenario considering the small system size. In addition, the nature of the transition may not have been completely settled.\cite{24}

VII. TRIPLON DISPERSION IN THE VBS STATES

In Fig. 21, we plot the triplet dispersions of the two VBS phases, which are obtained from the bond operator mean field theory. The triplet dispersion of the \( J_e \)-dimer VBS state is shown in Fig. 21(a). Since the unit cell of the \( J_e \)-dimer VBS state is composed of three dimers, we have three triplet modes in the spectrum. Here we neglect the fact that each triplet particle has three components (\( x, y, \) and \( z \)) when we count the number of bands. Interestingly, the lowest band is flat and touches another dispersive band at the Brillouin zone center. The emergence of this band touching has a topological origin.\cite{33,35} The flat band reflects the existence of localized eigenstates. In fact, there are two different types of localized eigenstates. The first set is given by states which are confined within dodecagons. Each dodecagon supports a single localized eigenstate. In addition, there are “non-contractible” loop states constituting the second group of localized eigenstates. In contrast to the states confined within dodecagons, these loop states extend over the whole lattice system one-dimensionally. The topological characteristics of these loop
states can be easily understood using the periodic boundary condition, under which a two-dimensional system has a torus geometry. In this situation, there are two independent loop states winding the torus once. Since these states cannot be shrunk to points, they are “non-contractible”. Counting the number of independent localized states carefully, we see that the number of degenerate localized eigenstates is larger than the number of dimers on the lattice. It means that a single flat band is not enough to support all independent localized eigenstates. Therefore additional degrees of freedom must be provided by another band, leading to the band touching.

The triplet dispersion of the columnar 18-site VBS state is displayed in Fig. 21 (b). We have nine triplet bands which are all flat within our mean field approach. The degeneracies of the flat bands are given by 3, 1, 2, 2, and 1 counting from the bottom to the top bands. In contrast to the case of the $J_t$-dimer VBS phase, the flat structure emerges simply because the dimers on expanded links are completely decoupled from those on neighboring “perfect” dodecagons. The lowest flat band is triply degenerate, which comes from the three dimers on expanded links. The remaining six dimers of the unit cell lying on “perfect” dodecagons constitute the other six bands with higher energies. Therefore we expect that the nature of the valence bond solid ground states can be demonstrated via neutron scattering experiments measuring triplet dispersion spectra.

VIII. DISCUSSION

In summary, we have shown that the ground state of the nearest neighbor Heisenberg model on the star lattice undergoes a phase transition from “the $J_t$-dimer VBS phase” which respects all lattice symmetries to “the columnar 18-site VBS phase” which exhibits $\sqrt{3} \times \sqrt{3}$ order with increasing $J_t/J_c$. From the Landau-Ginzburg analysis, this appears to be a conventional quantum phase transition which is described as the thermal transition of the 2 + 1 dimensional $Z_3$-clock model.

If $S = 1/2$ variants of the organic Iron-Acetate magnet can be synthesized, they would be particularly good candidates to study the phase diagram discussed in this paper since it may be possible to pressure tune the ratio $J_t/J_c$ significantly in such systems. Both VBS states obtained here would exhibit a spin gap in uniform susceptibility measurements. We expect a direct signature of the 18-site VBS order to appear in X-ray scattering or neutron diffraction studies which would see a change in the crystal periodicity. Ignoring coupling to phonons, the 18-site VBS state should exhibit a thermal transition in the universality class of the $Z_3$-clock model in $D = 2$ dimensions. In addition, the two VBS phases exhibit quite distinct behaviors in their triplet excitation spectra as discussed above which could be tested using inelastic neutron scattering experiments.

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APPENDIX A: DERIVATION OF THE LOW ENERGY EFFECTIVE HAMILTONIANS

Here we present the details of how we have derived the low energy effective Hamiltonians of the spin liquid states discussed in Sec. II B.

1. Effective Hamiltonian for $SL[\frac{\pi}{2}, \frac{\pi}{2}, \pi]$ 

For the gauge choice depicted in Fig. 5(a), the mean field Hamiltonian corresponding to the $SL[\frac{\pi}{2}, \frac{\pi}{2}, \pi]$ can be written in momentum space as,

$$H_{MF} = -J_t \chi_i \sum_k \sum_{m,n} f^\dagger_k f_{k,m} H(k)_{m,n} f_{k,n}, \quad (A1)$$

in which

$$H(k) = \begin{pmatrix} 0 & i & \lambda z_2 & 0 & 0 \\ -i & 0 & -i & 0 & \lambda \\ -i & i & 0 & 0 & \lambda \lambda z_1^* \\ \lambda z_2 & 0 & 0 & 0 & -i \\ 0 & \lambda & 0 & i & 0 \\ 0 & 0 & \lambda z_1 & i & 0 \end{pmatrix}.$$ 

where $z_1 = e^{i k a_1}$ and $z_2 = e^{i k a_2}$. $m, n$ are indices for the six sites inside a unit cell and $\lambda = J_t \chi_i / J_c \chi_i$. The indices for the spin quantum number are dropped for simplicity. Here we define the Fourier transformation via $f_{k,n} = \frac{1}{\sqrt{N}} \sum_k e^{i k R} f_{k,n}$.

As described in Fig. 6(b), the conduction (valence) band shows the dispersion minimum (maximum) at the momentum $\pm Q$. The energy eigenvalues of $H(k)$ at $k = \pm Q$ are given by

$$E_1^\pm = \sqrt{3/2 + \lambda^2 - 1/2 \sqrt{9 + 12\lambda^2}},$$

$$E_2^\pm = -E_1^\pm . \quad (A2)$$

where $\pm$ indicates the two momentum position $\pm Q$ and 1 and 2 represent the conduction (1) and valence (2) bands, respectively. To make the analytic treatment of the problem possible we focus on the small $\lambda$ limit. Note that the overall spinon band structure does not change upon varying $\lambda$. Expanding the energy eigenvalues in powers of $\lambda$, we get

$$E_1^\pm = -E_2^\pm \simeq \frac{1}{\sqrt{3}} \lambda^2 + O(\lambda^4).$$
The corresponding eigenvectors are

\[
\begin{align*}
(\nu_1^+)^T &= \frac{e^{-i\pi/6}}{\sqrt{3(1+b^2)}} \left( -1 - \frac{1}{3} \lambda^2, -1 - \frac{1}{3} \lambda^2, 1 + \frac{1}{3} \lambda^2, \right. \\
&\quad \quad - \frac{1}{\sqrt{3}} \lambda e^{i\pi/3}, - \frac{1}{\sqrt{3}} \lambda e^{i2\pi/3} \\
(\nu_2^+)^T &= \frac{e^{i\pi/6}}{\sqrt{3(1+b^2)}} \left( -1 \lambda^2, -1 - \frac{1}{3} \lambda^2, 1 + \frac{1}{3} \lambda^2, \right. \\
&\quad \quad 1 + \frac{1}{3} \lambda^2, -1 + \frac{1}{3} \lambda^2, 1 - \frac{1}{3} \lambda^2, \\
(\nu_1^-)^T &= \frac{e^{-i2\pi/3}}{\sqrt{3(1+b^2)}} \left( -1 - \frac{1}{3} \lambda^2, 1 + \frac{1}{3} \lambda^2, -1 - \frac{1}{3} \lambda^2, \
&\quad \quad 1 + \frac{1}{3} \lambda^2, -1 + \frac{1}{3} \lambda^2, 1 - \frac{1}{3} \lambda^2, \\
(\nu_2^-)^T &= \frac{e^{i2\pi/3}}{\sqrt{3(1+b^2)}} \left( -1 - \frac{1}{3} \lambda^2, 1 + \frac{1}{3} \lambda^2, -1 - \frac{1}{3} \lambda^2, \
&\quad \quad 1 + \frac{1}{3} \lambda^2, -1 + \frac{1}{3} \lambda^2, 1 - \frac{1}{3} \lambda^2,
\end{align*}
\]

where the superscript \(T\) means taking transposition. The above eigenvalues and eigenvectors satisfy \(H(\pm Q)\nu^\pm_\alpha = E^\pm_\alpha \nu^\pm_\alpha \) (\(\alpha = 1, 2\)) correctly up to the third order in \(\lambda\).

Now we want to construct the effective Hamiltonian describing the states which have small momentum deviation from \(\pm Q\), i.e., states with \(k = \pm Q + q\). We first define \(\Delta H(q) = H(\pm Q + q) - H(\pm Q)\). Keeping terms which are first order in \(q\) and projecting them into the low energy space spanned by the eigenvectors \(\nu^\pm_\alpha \) (\(\alpha = 1, 2\)), we obtain the following effective Hamiltonian,

\[
H^\pm(q) \equiv H(\pm Q + q) = -\frac{\lambda}{\sqrt{3}} (q_x \tau_x + q_y \tau_y) + \frac{\lambda^2}{\sqrt{3}} \tau_z
\]

Finally, defining the continuum fermion fields using the spinon variables as

\[
\begin{align*}
(\psi^\pm(q))^T &\sim \sum_{n=1}^6 (\nu^\pm_1)^*_n f_{\pm q+n, q+n}, \\
(\Psi(q))^T &\equiv ((\psi^+(q))^T, (\psi^-(q))^T),
\end{align*}
\]

we arrive at the low energy effective Hamiltonian written in Eq. (15) which is nothing but the Hamiltonian for the massive Dirac particles.

2. Effective Hamiltonian for SL\([-\pi/2, \pi/2, 0]\]

The low energy Hamiltonian corresponding to the SL\([-\pi/2, \pi/2, 0]\] can be obtained following the similar steps used to construct the massive Dirac Hamiltonian of the SL\([\pi/2, \pi/2, \pi]\]. Adopting the flux configuration depicted in Fig. [5]b, we have an electron pocket centered at the momentum \(\mathbf{K}\) and an hole pocket centered at the momentum \(-\mathbf{K}\). At each momentum \(\pm \mathbf{K}\) there is a linear band touching as is shown in Fig. [5]b.

The energy eigenvalues of the degenerate states at \(k = \pm \mathbf{K}\) are given by

\[
\begin{align*}
E^+ &= \sqrt{3 + \lambda^2 - \sqrt{3}} \approx \frac{1}{\sqrt{3}} \lambda^2 + O(\lambda^4), \\
E^- &= -E^+ \approx -\frac{1}{\sqrt{3}} \lambda^2 + O(\lambda^4),
\end{align*}
\]

where \(E^+ (E^-)\) represents the degenerate energy eigenvalue at the momentum \(\mathbf{K} (-\mathbf{K})\). We choose the corresponding eigenvectors in the following way,

\[
\begin{align*}
(\nu_1^+)^T &= \frac{e^{i\pi/3}}{\sqrt{3(1+b^2)}} \left( -1 - \frac{1}{3} \lambda^2, -1 - \frac{1}{3} \lambda^2, 1 + \frac{1}{3} \lambda^2, \
&\quad \quad 1 + \frac{1}{3} \lambda^2, -1 + \frac{1}{3} \lambda^2, 1 - \frac{1}{3} \lambda^2, \\
(\nu_2^+)^T &= \frac{e^{i2\pi/3}}{\sqrt{3(1+b^2)}} \left( -1 - \frac{1}{3} \lambda^2, 1 + \frac{1}{3} \lambda^2, -1 - \frac{1}{3} \lambda^2, \
&\quad \quad 1 + \frac{1}{3} \lambda^2, -1 + \frac{1}{3} \lambda^2, 1 - \frac{1}{3} \lambda^2,
\end{align*}
\]

satisfying \(H(\pm \mathbf{K})\nu^\pm_\alpha = E^\pm_\alpha \nu^\pm_\alpha \) (\(\alpha = 1, 2\)) correctly up to the third order in \(\lambda\). The first order perturbation theory combined with the projection into the low energy space spanned by \(\nu^\pm_\alpha \) (\(\alpha = 1, 2\)) leads to the following Hamiltonian,

\[
H^\pm(q) \equiv H(\pm \mathbf{K} + q) = -\frac{\lambda}{\sqrt{3}} (q_x \tau_x + q_y \tau_y) + \frac{\lambda^2}{\sqrt{3}} \mu_z
\]

Using the continuum fermion field defined in Eq. (14) we obtain the low energy effective Hamiltonian displayed in Eq. (16).

3. Effective Hamiltonian for SL\([0, 0, 0]\]

The mean field band structure of the SL\([0, 0, 0]\) has a flat band lying at the fermi energy, which is touching a dispersive band at the zone center, i.e., at the momentum \(\mathbf{\Gamma} = (0, 0)\). To describe the low energy states near the \(\mathbf{\Gamma}\) point, we use the degenerate perturbation theory again. However, since the two bands touch quadratically, we keep the perturbation expansion...
up to the quadratic order in momentum. For the SL\,[0,0,0],
the mean field Hamiltonian is given by
\[ H_{MF} = -J_t \chi_t \sum_k \sum_{m,n} f_k^\dagger f_{m,n} H(k)_{m,n} f_{k,n}, \]  
(A6)
in which
\[
H(k) = \begin{pmatrix}
0 & 1 & 1 & \lambda z_2^* & 0 & 0 \\
1 & 0 & 0 & \lambda z_1^* & 0 & 0 \\
1 & 1 & 0 & 0 & \lambda z_1^* & 0 \\
\lambda z_1 & 0 & 0 & 1 & 0 & 1 \\
0 & \lambda z_1 & 0 & 1 & 0 & 1 \\
0 & 0 & \lambda z_1 & 1 & 1 & 0
\end{pmatrix}.
\]  

We divide \( H(k) \) into two pieces such that \( H(k) = H_0 + V \) in which \( H_0 \equiv H(k = 0) \). Diagonalization of \( H_0 \) gives the eigenvalues \( E_0 = \{-1, -1, -1, 1, 1, 2, 2, 3, 3, 4\} \). Note that there are two pairs of doubly degenerate eigenvalues. Here we focus on one of the degenerate eigenvalues \( w_0 = -1 + \lambda \) which is lying at the fermi level. We choose the following two degenerate eigenvectors corresponding to \( w_0 \),
\[ (\nu_1)^T = \frac{1}{\sqrt{12}} \{2, -1, -1, 1, -1\}, \]
\[ (\nu_2)^T = \frac{1}{\sqrt{4}} \{0, 1, -1, 0, 1\}, \]

Now we introduce the projection operator \( \hat{P}_0 \) (\( \hat{P}_1 \)) which projects states into (out of) the low energy space spanned by \( \nu_1 \) and \( \nu_2 \). That is, \( \hat{P}_0 \nu_1 = \nu_1 \), \( \nu_2 \), and \( \hat{P}_1 - \hat{I} - \hat{P}_0 \). We also define \( \hat{P}_0 \equiv \hat{P}_0 V \hat{P}_0 \) and \( \hat{V}_1 \equiv \hat{V} - \hat{V}_0 \). Then \( H(k) = H_0 + V_0 + V_1 \). The projected Hamiltonian is given by
\[
H_{P0}(k) = \hat{P}_0 H(k) \hat{P}_0 \\
\approx \hat{P}_0 [H_0 + V_0] \hat{P}_0 + \hat{P}_0 V_1 \hat{P}_1 \frac{1}{w_0 - H_0} \hat{P}_1 V_1 \hat{P}_0,
\]  
(A7)
which is valid up to the quadratic order in \( k \). The resulting Hamiltonian can be written as,
\[
H_{P0}(k) = \frac{\lambda}{2 \lambda - 3} \begin{pmatrix}
k_y^2 & -k_x k_y \\
-k_x k_y & k_x^2
\end{pmatrix},
\]
We define the continuum fields as
\[
(\psi(k))^T \sim \{\sum_{n=1}^{6} (\nu_1^*)^\dagger f_{k,n} + \sum_{n=1}^{6} (\nu_2^*)^\dagger f_{k,n}\}.
\]  
(A8)
Finally, combining the above results we obtain the following low energy effective Hamiltonian,
\[
H_{\text{eff}} = \frac{1}{m_{\text{eff}}} \int \frac{d^2k}{(2\pi)^2} \psi^\dagger(k) h_{\text{eff}}(k) \psi(k),
\]
in which
\[
h_{\text{eff}}(k) = (k_x^2 + k_y^2) \tau_0 - (k_x^2 - k_y^2) \tau_2 - 2k_x k_y \tau_3.
\]

**APPENDIX B: SYMMETRY AND CONTINUUM FIELD OF THE SL\,[\pi/2, \pi/2, 0] STATE**

In this section we show how the continuum fields of the SL\,[\pi/2, \pi/2, 0] state transform under the microscopic symmetries of the lattice. Here we follow the procedures suggested by Hermele et al. in Ref. 10. To obtain the necessary information we consider a finite system with periodic boundary conditions in both \( a_1 \) and \( a_2 \) directions. In particular, to determine how the wave functions at the momentum \( \pm \mathbf{K} \) (we call it as the nodal wave functions) transform under the space group symmetries of the ansatz, we consider the \( 3 \times 3 \) lattice system, that is, the system is periodic under the translation by \( 3a_1 \) and \( 3a_2 \). Since a unit cell (indexed by a vector \( \mathbf{R} \)) contains six sites labeled by \( n \), the finite system consists of 54 sites. The nine points within the Brillouin zone of the finite system contain the the nodal points \( \pm \mathbf{K} \) and respect all the point group symmetries of the ansatz.

Using the eigenvectors \( \nu^2 \) (\( a = +, - \) and \( \alpha = 1, 2 \)) in Eq. (A5), the nodal wave function is given by
\[
\Phi_{a,\alpha}(\mathbf{R}, n) = \frac{e^{i \mathbf{K} \cdot \mathbf{R} (\nu_a^\dagger)^n}}{3}.
\]

Then the continuum field is written as
\[
\Psi_{a,\alpha}(\mathbf{q} = 0) = \sum_{\mathbf{R},n} \Phi_{a,\alpha}^*(\mathbf{R}, n) f_{\mathbf{R},n}.
\]

Now we consider a symmetry operation \( \mathcal{S} \) under which the spinors transform as
\[
\mathcal{S} : f_{\mathbf{R},\sigma} \rightarrow \mathcal{G}_S(i) f_{\mathcal{G}_S(\mathbf{i},\sigma)}.
\]
In particular we consider the following symmetry operations,

(i) The \( a_1 \) translation (\( T_1 \));
\[
T_1 : f_{\mathbf{R},n,\sigma} \rightarrow G_{T_1}(\mathbf{R}, n) f_{\mathbf{R}+\mathbf{a}_1,n,\sigma},
\]

(ii) The \( a_2 \) translation (\( T_2 \));
\[
T_2 : f_{\mathbf{R},n,\sigma} \rightarrow G_{T_2}(\mathbf{R}, n) f_{\mathbf{R}+\mathbf{a}_2,n,\sigma},
\]

(iii) The \( \sigma^k \) rotation (\( C_{2\pi}/3 \));
\[
C_{2\pi}/3 : f_{\mathbf{R},n,\sigma} \rightarrow G_{C_{2\pi}/3}(\mathbf{R}, n) f_{C_{2\pi}/3(\mathbf{R},n),\sigma},
\]

(iv) The \( y \) reflection (\( R_y \));
\[
R_y : f_{(x,y),n,\sigma} \rightarrow G_{R_y}(\mathbf{R}, n) f_{(x,-y),m,\sigma},
\]
where \( \mathbf{R} = (x,y) \) and \((n,m) \in \{(1,4), (2,6), (3,5), (4,1), (5,3), (6,2)\}\).

(v) Time-reversal and inversion (\( T \cdot I \));
\[
T \cdot I : f_{\mathbf{R},n,\sigma} \rightarrow (i\sigma_2)_{\alpha,\sigma'} f_{-\mathbf{R},(n+3),\sigma'},
\]
in which \((n+3)\) represents the remainder when it is divided by six.
(vi) Charge conjugation ($C^*$):

$$C^* : \mathcal{f}_{R, n, \sigma} \rightarrow \epsilon_n \mathcal{f}_{R, n, \sigma},$$

in which $\epsilon_i = 1$ for $i = 1, 2, 3$ and -1 otherwise. For each of the symmetry operation $S$, the matrix representation of the symmetry $S$ is defined as

$$(S)_{S(i),i} \equiv G_S(i).$$

The action of the symmetry operation $S$ on the nodal wave function is given by $S \Phi_{a} = c_a \Phi_{b}$. Now $a$ and $b$ denote the nodal and the two-component Dirac indices collectively. Finally, the transformation of the nodal wave function reflects the action of $S$ on the continuum field such as

$$S : \Psi_a \rightarrow e^{a_{ab}} \Psi_b.$$  

The transformation rule of the continuum field under all the above symmetry operations is summarized in Eq. (17).

**APPENDIX C: EXPRESSIONS OF THE MATRICES $A_S(\Pi_1)$ AND $B_S(\Omega_1)$**

Here we present the expressions of the matrices $A_S(\Pi_1)$ and $B_S(\Omega_1)$ ($S=T_{a_1}, T_{a_2}, R_y$, and $C_{22}$), which are defined in Sec. [VIII]. At first $A_S(\Pi_1)$ are given by

$$A_{T_{a_1}}(\Pi_1) : \begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i\frac{2\pi}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\frac{2\pi}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\frac{2\pi}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\frac{2\pi}{3}} \\ \end{pmatrix},$$

$$A_{T_{a_2}}(\Pi_1) : \begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\frac{2\pi}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{2\pi}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\frac{2\pi}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\frac{2\pi}{3}} \\ \end{pmatrix},$$

$$A_{R_y}(\Pi_1) : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \end{pmatrix},$$

$$A_{C_{22}}(\Pi_1) : \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ \end{pmatrix}.$$

Similarly for $A_S(\Pi_2)$.

$$A_{T_{a_1}}(\Pi_2) : \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \end{pmatrix},$$

$$A_{T_{a_2}}(\Pi_2) : \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \end{pmatrix},$$

$$A_{R_y}(\Pi_2) : \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \end{pmatrix},$$

$$A_{C_{22}}(\Pi_2) : \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix}.$$

In the case of $B_S(\Omega_1)$,

$$B_{T_{a_1}}(\Omega_1) : \begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{-i\frac{2\pi}{3}} \\ \end{pmatrix}, \quad B_{T_{a_2}}(\Omega_1) : \begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{-i\frac{2\pi}{3}} \\ \end{pmatrix},$$

$$B_{R_y}(\Omega_1) : \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ \end{pmatrix}, \quad B_{C_{22}}(\Omega_1) : \begin{pmatrix} 0 & -i \\ i & 0 \\ \end{pmatrix}.$$

Similarly for $B_S(\Omega_2)$.

$$B_{T_{a_1}}(\Omega_2) : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \end{pmatrix}, \quad B_{T_{a_2}}(\Omega_2) : \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \end{pmatrix},$$

$$B_{R_y}(\Omega_2) : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \end{pmatrix}, \quad B_{C_{22}}(\Omega_2) : \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \end{pmatrix}.$$
