GYSIN MAPS, DUALITY AND SCHUBERT CLASSES

by

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Abstract. — We establish a Gysin formula for Schubert bundles and a strong version of the duality theorem in Schubert calculus on Grassmann bundles. We then combine them to compute the fundamental classes of Schubert bundles in Grassmann bundles, which yields a new proof of the Giambelli formula for vector bundles.

0. Introduction

The present paper is a continuation of [DP15]. We apply Gysin formulas for flag bundles to study Schubert calculus on Grassmann bundles. We shall work in the framework of intersection theory of [Ful84]. Recall that a proper morphism $F: Y \to X$ of nonsingular algebraic varieties over an algebraically closed field yields an additive map $F_*: A^\bullet Y \to A^\bullet X$ of Chow groups induced by push-forward cycles, called the Gysin map. The theory developed in [Ful84] allows also one to work with singular varieties, or with cohomology. In this paper, $X$ will be always nonsingular. For possibly singular $Y$, and for a vector bundle $R \to Y$, if $a$ is a polynomial in the Chern classes of $R$, by push-forward of $a$ along $F$, we shall mean $F_*(a \cap [Y])$, that we will abusively denote $F_*(a)$.

Intersection theory on Grassmann bundles was studied in [Gro58], [Lak72], [KL74], [Las74] and [Sco81]. We consider Schubert bundles $\Omega_\lambda(E\bullet) \to X$ in the Grassmann bundles of $d$-planes $G_d(E) \to X$, where $E \to X$ is a vector bundle of rank $n \geq d$ on a variety $X$, $E\bullet$ is the reference flag of bundles determining the Schubert bundles and $\lambda \subseteq (n - d)^d$ are partitions (see (1)).

In Sect. 1, for strict partitions $\mu \subseteq (n)^d$ with $d$ parts, we study generalized flag bundles $F_\mu(E\bullet) \to X$ introduced by Kempf and Laksov [KL74] (see (4)), and following ideas of [DP15], we give a Gysin formula along these. We get an expression in the spirit of [Dar15] and [DP15], presenting the push-forward as some specific coefficient of a certain polynomial, depending only on the Segre classes of the reference flag of bundles $E\bullet$ (see Theorem 1.2).

The flag bundles $F_\nu(E\bullet)$ are well-known desingularizations of Schubert bundles $\Omega_\lambda(E\bullet)$ (see (2) for the relation between $\nu$ and $\lambda$), which were used by many authors. We shall also use them, but in a different way: with the help of the desingularization $F_\nu(E\bullet) \to \Omega_\lambda(E\bullet)$, we define the Gysin map along the Schubert bundle $\omega_\lambda: \Omega_\lambda(E\bullet) \to X$. This leads to a compact formula for the push-forward along $\omega_\lambda$ of a Schur polynomial of the universal subbundle on $G_d(E)$ (see Proposition 1.5).

Then, in Sect. 2, we push further the study of the Gysin map along $\pi: G_d(E) \to X$ started in [DP15]. We investigate the combinatorics of partitions with at most $d$ parts and establish several formulas for Littlewood–Richardson coefficients. Then we establish the main new result of the present paper, namely a formula for the push-forward along $\pi$ of the product of two Schur polynomials $s_{\nu_1}s_{\nu_2}$ of the universal subbundle $U \to G_d(E)$. This is done in Theorem 2.6, and should be interpreted as a relativization of the classical duality of (absolute) Schubert
calculus, as well as an extension of it, since we also compute the intersection in positive degree, \(i.e.\) when \(|a| + |b| > \text{rank} \, G_d(E)\). We call it the strong duality theorem in Grassmann bundles.

Note that a related formula was established in [JLP81]: the push-forward of the product of a Schur polynomial of the universal subbundle times another Schur polynomial of the universal quotient bundle. In Sect. 3, we show that the latter formula is a consequence of the former, using a formula for Laplace expansion of Schur functions that can be of independent interest.

In Sect. 4, using the Gysin formulas of Sect. 1 and Sect. 2, we give a new derivation of the Giambelli formula for Schubert bundles. Previous proofs were done by Kempf–Laksov [KL74], Lascoux [Las74] in the framework of classical intersection theory, and Anderson–Fulton [Fu107, And12] in the framework of equivariant cohomology in intersection theory. In [AF15], Anderson and Fulton gave also a variety of Kempf–Laksov formulas for degeneracy loci. Note that like the approaches of [KL74], [Las74], [AF15], and others, our approach uses the desingularization of Schubert varieties (or degeneracy loci) by chains of projective bundles, and various Gysin formulas for them. In our approach however, we use two new Gysin formulas for flag bundles (Proposition 4.4 in [DP15] and Proposition 1.5) and the reasoning is then based on the comparison of these two formulas. To the best of our knowledge, this method is new.

This approach fits in a broader context than this of the present paper, and has the noteworthy feature that, like in [AF15], it should be adaptable to the symplectic and orthogonal settings, with some technical adjustments. The main idea is summarized in the following Figure 1, where we denote by \(F(1, \ldots, d)(E) \rightarrow X\) the bundle of flags of subspaces of dimensions 1, \ldots, \(d\) in the fibers of \(E\).

![Figure 1](image)

The desingularization (a) and how we use it (b).

We also recall the standard approach of Kempf and Laksov (with our notation) for comparison:

\[
\begin{align*}
\Omega_d(E) & \xleftarrow{\text{subvar}} G_d(E) \\
\Omega_d(E) & \xleftarrow{\text{bir}} G_d(E)
\end{align*}
\]

In their approach, one works with \(G_d(E)\) as the base variety. To compute the fundamental class \([\Omega_d(E)]\) in the Chow group of \(G_d(E)\) one first computes the fundamental class \([s(F_\nu(E))]\) in the Chow group of \(G_d(E) \times_X F_\nu(E)\) and then one computes the push-forward along \(p_1\) of this specific class.

In our approach, we work above the base variety \(X\) and we use Gysin formulas for different morphisms satisfied for all classes; we do not need to compute the class \([F_\nu(E)]\) in \(A^*(F(1, \ldots, d)(E))\). There are two ways to push a class from \(\Omega_d(E)\) to \(X\). Either we use the desingularization \(F_\nu(E)\) studied in Sect. 1 and we push-forward along \(\vartheta_d\), or we regard the Schubert bundle \(\Omega_d(E)\) as a subvariety of the Grassmann bundle \(G_d(E)\) and we push-forward along \(\pi\). In the latter case, the push-forward formula involves the fundamental class \([\Omega_d(E)]\) in \(G_d(E)\). It remains to compare the expressions obtained in each way for some generators of the Chow group of \(\Omega_d(E)\). The only serious obstacle to compute \([\Omega_d(E)]\) \(\in A^*(G_d(E))\) may be the difficulty to solve this system of equations.

In the present paper, we treat \(A\) and we heavily rely on the strong duality theorem to simplify this last step. For \(d \leq n\), we let \(U_d\) (or simply \(U\) when \(d\) is fixed) denote the universal subbundle on the Grassmann bundle \(G_d(E)\), as well as its pullbacks to the different varieties appearing in Figure 1a. Since no confusion could arise, we will indeed drop the pullback notation.
(with a few exceptions). We express the sought Schubert class as a certain linear combination

$$[\Omega_\lambda(E_x)] = \sum \alpha_\mu \delta_\mu(U),$$

with unknowns $\alpha_\mu$, and we consider the push-forwards of the Schur classes $s_\mu(U)$ along $\omega_1$. Using the duality theorem to express the intersection of two Schur classes leads to a triangular invertible system in the unknowns $\alpha_\mu$. Solving this system, we get an expression for $[\Omega_\lambda(E_x)]$ in a familiar form of a determinant in the Chern classes of differences between the universal quotient bundle and bundles from the reference flag $E_*$. In our computations we use extensively the algebra and combinatorics of skew Schur functions and the algorithm of straightening (see Sect. 1.3).

Before entering the subject, a few words about the notation. The Greek letters $\alpha, \beta, \gamma, \delta$; $\lambda, \mu, \nu, \rho$ always denote partitions. For $\ell \geq 0$, we denote $(\ell)^d$, the rectangular partition with $d$ parts of length $\ell$. We denote by $\subseteq$ the containment relation of (the diagrams of) partitions. The conjugate partition of $\alpha$ is denoted $\alpha^\vee$. For $d \leq n$ fixed, we let $\rho$ denote the triangular partition $(\overline{d}, \overline{d-1}, \ldots, 1)$.

We denote by $\alpha \sqcup \beta$ the concatenation of partitions $\alpha$ and $\beta$; it is in general not a partition. Given two $d$-tuples of integers $K = (k_1, \ldots, k_d)$, $K' = (k'_1, \ldots, k'_d)$, we write $K \pm K' = (k_1 \pm k'_1, \ldots, k_d \pm k'_d)$. Lastly for a $d$-tuple of integers $K = (k_1, \ldots, k_d)$, we denote $K^\circ = (k_d, \ldots, k_1)$.

### 1. Gysin formulas for Kempf–Laksov bundles and for Schubert bundles

#### 1.1. Desingularization of Schubert bundles. —

Let $E \to X$ be a rank $n$ vector bundle on a variety $X$ with a reference flag of bundles $E_1 \subseteq \cdots \subseteq E_n = E$ on it, where rank $E_i = i$. Let $\pi: G_d(E) \to X$ be the Grassmann bundle of subspaces of dimension $d$ in the fibers of $E$. For any partition $\lambda \subseteq (n-d)^d$, one defines the Schubert bundle $\Omega_\lambda: \Omega_\lambda(E_\bullet) \to X$ in $G_d(E)$ over the point $x \in X$ by

$$\Omega_\lambda(E_\bullet)(x) := \{V \in G_d(E)(x) : \dim(V \cap E_{n-d-\lambda+i}(x)) \geq i, \text{ for } i = 1, \ldots, d\}.$$  

In this description, the only non-trivial conditions correspond to indices $i$ for which $\lambda_i > 0$.

We denote by

$$(v_1, \ldots, v_d) := (n-d - \lambda_1 + 1, \ldots, n-d - \lambda_d + d)^\circ = (n-d - \lambda_1 + d, \ldots, n-d - \lambda_1 + 1),$$

the dimensions of the spaces of the reference flag involved in the definition of $\Omega_\lambda(E_\bullet)$—in reverse order—. To a partition $\alpha \subseteq (n-d)^d$, one associates a dual partition $\alpha^\circ \subseteq (n-d)^d$ by setting

$$\alpha^\circ = (n-d)^d - \alpha^\vee,$$

as illustrated below:

![Diagram](attachment:image.png)

With this notation

$$(2) \quad \nu := \lambda^\circ + \rho.$$

So $\nu$ is a strict partition, and furthermore, $\rho_i \leq v_i \leq v_1 = n - \lambda_1 \leq n$, for any $i$.

Note that the above definition of $\Omega_\lambda(E_\bullet)$ can be restated using the strict partition with $d$ parts $\nu \subseteq (n)^d$ with the conditions

$$(3) \quad \dim(V \cap E_{\nu, i}(x)) \geq d + 1 - i, \text{ for } i = 1, \ldots, d.$$  

For a strict partition $\mu \subseteq (n)^d$ with $d$ parts, consider the flag bundle $\delta_\mu: F_\mu(E_\bullet) \to X$ defined over the point $x \in X$ by

$$(4) \quad F_\mu(E_\bullet)(x) := \{0 \subseteq V_1 \subseteq \cdots \subseteq V_d \in F(1, \ldots, d)(E)(x) : V_{d+1-i} \subseteq E_{\nu_i}(x), \text{ for } i = 1, \ldots, d\}.$$  

We will call Kempf–Laksov flag bundles such bundles $\delta_\mu$ introduced notably in [KL74].
These appear naturally as desingularizations of Schubert bundles. For a partition $\lambda \subseteq (n-d)^d$, denoting $\nu = \lambda^c + \rho$ as above, by (3), the forgetful map $F(1, \ldots, d(E)) \to G\nu(U)$ induces a birational map $F_\nu(E) \to \Omega(E)$; on the Schubert cell defined over the point $x \in X$ by

$$\Omega(E)(x) \coloneqq \{V \in G\nu(U)(x) : \dim(V \cap E_\nu(x)) = d + 1 - i, \text{ for } i = 1, \ldots, d\},$$

which is open dense in $\Omega(E)$, the inverse map is $V \mapsto (V \cap E_\nu(x), \ldots, V \cap E_\nu(x))$. This is the map $\varphi$ fitting in Figure 1a. It establishes a desingularization of $\Omega(E)$ (see [KL74]).

Later, to complete the study of Schubert bundles, we will fix a partition $\lambda$ and consider $F_\nu(E)$ for $\nu = \lambda^c + \rho$, but we shall first study Kempf–Laskov bundles $F_\nu(E) \to X$ in themselves.

### 1.2. Gysin formulas for Kempf–Laskov flag bundles

Regarding our goals, a central feature of Kempf–Laskov flag bundles is that these can be regarded as chains of projective bundles of lines defined by the reference flag of bundles $E$ and the universal subbundles $U_1, \ldots, U_d$.

**Lemma 1.1.** Let $\mu \subseteq (n)^d$ be a strict partition with $d$ parts. For $e = 1, \ldots, d - 1$, the forgetful map $F(1, \ldots, d - e + 1)(E) \to F(1, \ldots, d - e)(E)$ induces a map $F(\mu_{i_{e-1}, \mu})(E) \to F(\mu_{i_{e-1}, \mu})(E)$, isomorphic to $P(E_{\mu_i}/U_{e-1})$. Lastly, the bundle $F(\nu)(E) \to X$ is isomorphic to $P(E_{\mu_e})$.

**Proof.** Fix $x \in X$. Over a point $(V_1, \ldots, V_{d-e}) \in F(\mu_{i_{e-1}, \mu})(E)(x)$, one has $V_{d-e} \subseteq E_{\mu_{i_{e-1}}}(x) \subseteq E_{\mu_e}(x)$ and the fiber of $F(\mu_{i_{e-1}, \mu})(E) \to F(\mu_{i_{e-1}, \mu})(E)$ over $(V_1, \ldots, V_{d-e})$ consists of subspaces $V_{d-e+1} \subseteq V_{d-e+1}' \subseteq E_{\mu_e}(x)$. Since $V_i = i$, the result follows.

To sum up, we obtain a chain of projective bundles of lines

$$F(\mu_{i_{e-1}, \mu})(E) \to F(\mu_{i_{e-1}, \mu})(E) \to \cdots \to F(\mu_{i_{e-1}, \mu})(E) \to F(\nu)(E) \to X,$$

which is the same as

$$(5) \quad P(E_{\mu_{i_1}}/U_{d-1}) \to P(E_{\mu_{i_1}}/U_{d-2}) \to \cdots \to P(E_{\mu_{i_e}}/U_1) \to P(E_{\mu_e}) \to X.$$

As in [DP15], one can deduce a Gysin formula for $F_\nu(E) \to X$ from the described structure of chain of projective bundles (5). For the sake of completeness, and also because we make different notational conventions, let us recall the main lines of the argument.

For a Laurent polynomial $P$ in $d$ variables $t_1, \ldots, t_d$, and a monomial $m$, we denote by $[m](P)$ the coefficient of $m$ in the expansion of $P$. Clearly, for any second monomial $m'$, one has $[mm'](Pm') = [m](P)$, a property that we will use repeatedly. For a vector bundle $E \to X$ or rank $r$, recall the Gysin formula for the projective bundle of lines $p: P(E) \to X$

$$(6) \quad p_*(\xi^\ell) = s_{i_{e-1}}(E) = \left[t^{r-1}\right](\ell s_{1/\ell}(E)),
$$

where $\xi = c_1(O_{P(E)}(1))$, $s_i(E)$ is the $i$th Segre class, and $s_{1/\ell}(E)$ the Segre polynomial evaluated in $1/\ell$ (this yields a Laurent polynomial). The idea is simply to iterate this formula.

In our push-forward formulas, for $d$ fixed and for a symmetric polynomial $f$ in $d$ variables, by $f(\ell)$ we shall mean $f$ specialized with the Chern roots of $U^\nu$.

**Theorem 1.2.** For a rank $n$ vector bundle $E \to X$ on a variety $X$ with a reference flag $E_\bullet$ on it, and for a strict partition $\mu \subseteq (n)^d$ with $d \leq n$ parts, the push-forward along $\delta_\mu$: $F_\nu(E) \to X$ of a symmetric polynomial $f$ in $d$ variables is

$$(\delta_\mu)_*(f(\ell)) = \prod_{i=1}^d \left(f(t_1, \ldots, t_d) \prod_{1 \leq j \leq d} (t_i - t_j) \prod_{1 \leq i \leq d} s_{1/\ell}(E_{\mu_i})\right).$$

**Proof.** We numerate the Chern roots $\xi_1, \ldots, \xi_d$ of $U^\nu$ by taking

$$\xi_i := -c_1(U_{d+1-i}/U_{d-i}).$$

For notational convenience, we will denote $\xi^\nu := (\xi_1, \ldots, \xi_d, \xi_{e+1}, \ldots, \xi_d)$ the $d$-tuple obtained from $(\xi_1, \ldots, \xi_d)$ after replacement of the first $e$ roots $\xi_i$ by formal variables $t_i$.

For $\ell_1 \leq \ell_2 \leq d - 1$, let us denote by $f(\ell_1^\nu)$ the Gysin map along $F(\mu_{i_{2e-1}, \mu})(E) \to F(\mu_{i_{2e-1}, \mu})(E)$ and for $\ell_2 = d$, let us denote by $f(\ell_1^d)$ the Gysin map along $F(\mu_{i_{2e+1}, \mu})(E) \to X$.

We will prove by induction on $\ell = 0, \ldots, d$ that

$$(*) \quad \int_0^\ell f(\ell) = \prod_{i=1}^d \left(f(t_1^\nu) \prod_{1 \leq i \leq \ell} (t_i - t_j) \prod_{1 \leq i \leq \ell} s_{1/\ell}(E_{\mu_i} - U_{d-\ell})\right).$$
For $e = 0$, this is the definition of $f(U)$ and for $e = d$, since $\int_0^d = (d_\mu)$, this is the announced formula. Assume that the formula holds for $e < d$.

By Lemma 1.1, $F_{(\mu_1+1,\ldots,\mu_d)}(E_t) \rightarrow F_{(\mu_1+1,\ldots,\mu_d)}(E_t)$ is isomorphic to $P(E_{\mu_{i+1}}/U_{d-\mu_{i+1}})$, with the notation of (6), this projective bundle has rank

$$r - 1 = \mu_{i+1} - (d - (e + 1)) - 1,$$

so by (6) one has

$$\int_0^{e+1} \gamma^e_{i+1} = [t^{\mu_i - (d+1-c+1)}_e(\gamma^e_{i+1} t^1 t^{1/\gamma^e_{i+1}} (E_{\mu_{i+1}} - U_{d-\mu_{i+1}}))].$$

Now by the induction hypothesis (*),

$$\int_0^{e+1} f(U) = \int_0^{e+1} \int_0^\nu f(U) = \left[ \prod_{i=1}^e \frac{1}{t^{\mu_i - (d+1-c)}} \right] \left( \int_0^{e+1} P(t^e_{\nu}) \right),$$

where $P = P(E_{\mu_1}, \ldots, E_{\mu_e})$ is the Laurent polynomial in $d$ variables such that

$$P(t^e_{\nu}) = f(t^e_{\nu}) \prod_{1 \leq i \leq e} (t_i^1 - t_i) \prod_{1 \leq i \leq e} s_1^{1/\gamma^e_{i+1}}(E_{\mu_i} - U_{d-\mu_i}).$$

Note that the Segre classes of the universal bundle $U_{d-\mu}$ are polynomials in $\xi_{e+1}, \ldots, \xi_d$. To apply (*), we prefer regard $P(t^e_{\nu})$ as a polynomial in $\xi_{e+1}$, according to the formula

$$P(t^e_{\nu}) = f(t^e_{\nu}) \prod_{1 \leq i < j \leq e} (t_i^1 - t_j) \prod_{1 \leq i \leq e} s_1^{1/\gamma^e_{i+1}}(E_{\mu_i} - U_{d-\mu_i})(1 - \xi_{e+1}/t_i).$$

Hence, using (*),

$$\int_0^{e+1} P(t^e_{\nu}) = \left[ t^{\mu_i - (d+1-c+1)}_e \right] \left( f(t^e_{\nu}) \prod_{1 \leq i < j \leq e} (t_i^1 - t_j) \prod_{1 \leq i \leq e} s_1^{1/\gamma^e_{i+1}}(E_{\mu_i} - U_{d-\mu_i}) (1 - t_i^1/t_i) \right)
= \left[ t^{\mu_i - (d+1-c+1)}_e \right] \left( \frac{1}{t_1 \cdots t_e} f(t^e_{\nu}) \prod_{1 \leq i < j \leq e} (t_i^1 - t_j) \prod_{1 \leq i \leq e} s_1^{1/\gamma^e_{i+1}}(E_{\mu_i} - U_{d-\mu_i}) \right).$$

It follows that

$$\int_0^\nu f(U) = \left[ \prod_{i=1}^e \frac{t^{\mu_i - (d+1-c) \mu_i - (d+1-c+1)}_e}{t^{\mu_i - (d+1-c)}_e} \right] \left( \frac{1}{t_1 \cdots t_e} f(t^e_{\nu}) \prod_{1 \leq i < j \leq e} (t_i^1 - t_j) \prod_{1 \leq i \leq e} s_1^{1/\gamma^e_{i+1}}(E_{\mu_i} - U_{d-\mu_i}) \right).$$

Multiplying the extracted monomial and the polynomial by $t_1 \cdots t_e$ one obtains

$$\int_0^\nu f(U) = \left[ \prod_{i=1}^e \frac{t^{\mu_i - (d+1-c) \mu_i - (d+1-c+1)}_e}{t^{\mu_i - (d+1-c)}_e} \right] \left( f(t^e_{\nu}) \prod_{1 \leq i < j \leq e} (t_i^1 - t_j) \prod_{1 \leq i \leq e} s_1^{1/\gamma^e_{i+1}}(E_{\mu_i} - U_{d-\mu_i}) \right).$$

This is (*) for $e + 1$, whence by induction the formula (*) holds for any $e = 0, \ldots, d$, and this finishes the proof.

### 1.3. Push-forward of Schur classes.

We can now specialize Theorem 1.2, to get a formula for the push-forward of Schur polynomials. This is done in Proposition 1.3.

For any partition $\alpha = (a_1, \ldots, a_d)$, recall that the Schur polynomial $s_\alpha \in \mathbb{Z}[t_1, \ldots, t_d]$ can be defined by the formula

$$s_\alpha(t_1, \ldots, t_d) = \frac{\det(t^{a_i + d}_{i+1})}{\prod_{1 \leq i \leq d} (t_i^1 - t_i)}.$$

Note that in particular for $\alpha = (0)$, one has $s_1$ the complete symmetric function of degree $i$. It is convenient to set $s_i = 0$ for $i < 0$. Then the Jacobi–Trudi identity states

$$s_\alpha(t_1, \ldots, t_d) = \det(s_{\alpha - i}((t_1, \ldots, t_d)))_{1 \leq i \leq d}.$$

This identity allows us to generalize the Segre classes of $E$ for any $K = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ in a natural way by setting

$$s_K(E) = \det(s_{\alpha - i}((E)))_{1 \leq t \leq d}.$$
For a \( d \)-tuple \( K = (k_1, \ldots, k_d) \) that is not necessarily a partition, \( s_K(E) \) is either 0 or \( \pm s_\alpha(E) \) for some partition \( \alpha \). Indeed, for any permutation \( w \) of \( \{1, \ldots, d\} \)
\[
\det(s_{k-i+1}(E))_{1 \leq i \leq d} = (-1)^w \det(s_{k-w^{-1}+i}(E))_{1 \leq i \leq d'}
\]
where \((-1)^w\) denotes the sign of \( w \). Let us thus define the action of \( w \) on \( d \)-tuples by
\[
(w \cdot K)_i = k_{w(i)} - w(i) + i.
\]
This yields the formula
\[
s_K(E) = (-1)^w s_{w \cdot K}(E).
\]
Now, for a fixed \( K \) two situations can occur: either two rows of the matrix \((s_{k-i+1}(E))_{1 \leq i \leq d} \) are equal or, permuting the rows, columns can be ordered, in which case there exists a unique permutation \( w \) of \( \{1, \ldots, d\} \) such that \((k_{w(1)} - w(1) + j)_{i=1, \ldots, d} \) is a strict partition (this does not depend on the number \( j \) of the column). In the first case, the determinant is 0, and we will say that \( K \) cannot be straightened. In the second case \( \alpha := w \cdot K \) is a partition, that we will call the straightening of \( K \). We will write \( K \leadsto_{w} \alpha \) when \( \alpha \) is the straightening of \( K \) by the permutation \( w \).

One can generalize further the Segre classes by considering skew Schur functions, defined for two partitions \( \alpha \) and \( \beta \) by
\[
s_{\alpha/\beta}(E) := \det(s_{\alpha_i-i+1}(E))_{1 \leq i \leq d'}.
\]
This definition can in turn be applied to \( d \)-tuples \( K \) and \( L \) not necessarily partitions. Again, if either \( K \) or \( L \) has no straightening, then \( s_{K/L}(E) = 0 \), since two rows or two columns in the determinant are identical, and if \( K \leadsto_{w} \alpha \), \( L \leadsto_{w} \beta \), reasoning separately on rows and then on columns, one obtains
\[
s_{K/L}(E) = (-1)^{(w \cdot \alpha) - (w \cdot \beta)} s_{\alpha/\beta}(E).
\]

**Proposition 1.3.** — For a strict partition \( \mu \subseteq (n)^d \) with \( d \) parts, and any partition \( \alpha \), the push-forward along \( \vartheta_\mu: f_\mu(E_\bullet) \to X \) of the Schur class \( s_\alpha(U) \) is
\[
(\vartheta_\mu)_*(s_\alpha(U)) = \det(s_{\alpha_i-i+1+1}(E_\mu_i))_{1 \leq i \leq d'}.
\]

**Proof.** — Applying the result of Theorem 1.2 to \( f = s_\alpha \) one gets
\[
(\vartheta_\mu)_*(s_\alpha(U)) = [t_{d}^{\mu_{d}-1} \cdots t_{1}^{\mu_{1}-1}] \left( \prod_{1 \leq i < j \leq d} (t_i - t_j) \prod_{1 \leq i \leq d} s_{1/t_i}(E_\mu_i) \right).
\]
Now, by definition
\[
s_\alpha(t_1, \ldots, t_d) \prod_{1 \leq i \leq d} (t_i - t_j) = \det(t_i^{\alpha_i-i+1})_{1 \leq i \leq d'}.
\]
Hence, dividing the \( j \)th column of the determinant and also the extracted monomial by \( t_j^{\mu_j-1} \), for \( j = 1, \ldots, d \), one gets
\[
(\vartheta_\mu)_*(s_\alpha(U)) = [1] \left( \prod_{1 \leq i \leq d} s_{1/t_i}(E_\mu_i) \det(t_i^{\alpha_i-i+1})_{1 \leq i \leq d'} \right).
\]
Then using again the linearity of the determinant with respect to columns, one obtains:
\[
(\vartheta_\mu)_*(s_\alpha(U)) = \det([1] \left( t_i^{\alpha_i-i+1} \prod_{1 \leq i \leq d} s_{1/t_i}(E_\mu_i) \right))_{1 \leq i \leq d'} = \det(s_{\alpha_i-i+1+1}(E_\mu_i))_{1 \leq i \leq d'}. \]

This is the announced formula.

**1.4. Push-forward formula for Schubert bundles.** — As corollaries, we get push-forward formulas for Schubert bundles.

**Proposition 1.4.** — The push-forward along \( \varrho_\lambda: \Omega_\lambda(E_\bullet) \to X \) of a symmetric polynomial \( f \) in \( d \) variables is
\[
(\varrho_\lambda)_*(f(U)) = \prod_{i=1}^{d} t_i^{\mu_{i}-1} \left( f(t_1, \ldots, t_d) \prod_{1 \leq i < j \leq d} (t_i - t_j) \prod_{1 \leq i \leq d} s_{1/t_i}(E_\nu_i) \right).
\]
Proof. — Using the notation of Figure 1b:
\[(\delta_\nu)_*(f(U|\Omega_\lambda(E_\nu))) = (\omega_\lambda)_*(\varphi_*(f(U|\Omega_\lambda, E_\nu)\cap F_\nu(E_\nu))) = (\omega_\lambda)_*(f(U|\Omega_\lambda, E_\nu)\cap \Omega_\lambda(E_\nu))].\]
The first equality holds by commutativity, the second by the projection formula, and the third because \(\varphi\) is a desingularization. Thus, dropping the pullback notation
\[(\delta_\nu)_*(f(U)) = (\omega_\lambda)_*(f(U) \cap [\Omega_\lambda(E_\nu)]).\]
For the left hand side, the formula
\[(\delta_\nu)_*(f(U)) = \left[\prod_{i=1}^d t_i^{\nu_i-1}\right] \left[\prod_{1 \leq i \leq d} (t_i - t_j) \prod_{1 \leq i \leq d} s_{1/1}(E_\nu)\right]
\]
holds by Theorem 1.2 for \(\mu = \nu\).

Proposition 1.5. — The push-forward along \(\omega_\lambda\): \(\Omega_\lambda(E_\nu) \to X\) of a Schur class \(s_\lambda(U)\), for a partition \(\alpha\) is
\[(\omega_\lambda)_*(s_\lambda(U)) = \det(s_{\lambda_i-\gamma_i-j}(E_\nu))_{1 \leq i \leq d}^d.\]
Proof. — This follows from the previous considerations, since \(\nu_j = \lambda_j + d + 1 - j\). \(\square\)

2. Combinatorics of partitions with at most \(d\) parts

With the applications of Sections 3 and 4 in mind, we investigate the combinatorics of partitions with at most \(d\) parts, and deduce a strong version of the duality theorem in Grassmann bundles. We use Young tableaux. For all terminology and standard results used in this section, we refer the reader to [Ful97].

2.1. Littlewood–Richardson tableaux. — The Littlewood–Richardson number \(c^\alpha_{\beta\gamma}\) is the number of semi-standard skew tableaux with shape \(\alpha/\beta\) and type \(\gamma\) such that, in the word obtained by concatenation of the rows, from top to bottom, reading from right to left, for any initial part of the word, the value \(i\) occurs not less than the value \(i + 1\), for all \(i = 1, \ldots, d - 1\) ([Ful97, Sect. 5.2],[Mac95, Sect. I.9]). Such tableaux are called Littlewood–Richardson skew tableaux (L-R tableaux).

Lemma 2.1. — If \(T\) is a Littlewood–Richardson skew tableau with at most \(d\) rows and with type \(\gamma\), its \(i\)th row finishes with at least \(\gamma_d\) entries with value \(i\), for \(i = 1, \ldots, d\).

Proof. — The conclusions of the lemma follows from the following observations
- in the \(i\)th row, the values of all entries are at most \(i\);
- the number of entries with value \(i\) in the \(i\)th row (necessarily in the end of the row by the first property) is at least the number of entries with value \(i + 1\) in the \(i + 1\)st row;
- the number of entries with value \(d\) in the last row is \(\gamma_d\).
The first point is proven by induction. The value of the most right top entry is necessarily a 1, since 1 should occur not less than all other values in the word made of the most right top entry. Since \(T\) is semi-standard, all the first row is filled with the value 1. Then assuming the property up to \(i\)th row. It is not possible to have an entry with value larger than \(i + 1\) in the end of the \(i + 1\)st row, since \(i + 1\) shall occur not less than all values \(j > i + 1\) in the word made of all \(i + 1\)st rows plus the last entry of the \(i + 1\)st row. The second point is a consequence of the first point taking the word made of the \(i\) first rows plus all entries with value \(i + 1\) in the end of the \(i + 1\)st row. The entries with value \(i\) are then necessarily in the \(i\)th row, and these with value \(i + 1\) in the \(i + 1\)st row, and \(i\) occurs not less than \(i + 1\). For the last point, again by the first point, all entries \(d\) are in the end of the last row.

Corollary 2.2. — Let \(\alpha, \beta\) be partitions with at most \(d\) parts. For any partition \(\gamma\) and any rectangular partition with \(d\) parts \(\Box \subseteq \gamma\), one has:
\[c^\alpha_{\beta\gamma} = c^\alpha_{\beta(\gamma-\Box)}.\]
Proof. — Assume $\square = (\ell)^d \subseteq \gamma \subseteq \alpha$ (otherwise both coefficients in the sought formula are zero). To prove the lemma, it is thus sufficient to make a bijection between the L-R tableaux with shape $\alpha/\beta$ and type $\gamma$ and the L-R tableaux with shape $\alpha - (\ell)^d/\beta$ and type $\gamma - (\ell)^d$.

We know that $\ell \leq \gamma_d$ and that the $i$th row of the L-R tableaux with shape $\alpha/\beta$ and shape $\gamma$ finishes with at least $\gamma_d$ entries with value $i$.

Now, we claim that removing/adding $\ell$ boxes filled with $i$ in the end of the $i$th row for $i = 1, \ldots, d$ is the sought bijection. This is an easy check. $\square$

2.2. A product formula. — We shall now give a modern treatment of a theorem a Naegelbasc of d determinant of order $d$ in $d$ variables can be expressed as a determinant of order $d$ in $s$.

Lemma 2.3. — Let $\alpha$ and $\beta$ be partitions with at most $d$ parts. For any rectangular partition with $d$ parts $\square \geq \beta$, one has

$$s_\alpha(t_1, \ldots, t_d)s_\beta(t_1, \ldots, t_d) = s_{\square + \alpha/\square - \beta^-}(t_1, \ldots, t_d).$$

Proof. — For any skew Schur diagram $\delta$

$$s_\delta(t_1, \ldots, t_d) = \sum_{|T| = \delta} t^T_\ell$$

where the sum is over the semi-standard tableaux $T$ with shape $\delta$ and values in $\{1, \ldots, d\}$ and where for such a tableau $t^T_\ell$ denotes the monomial $t_1^{m_1} \cdots t_d^{m_d}$, with $m_i$ the number of entries of $T$ with value $i$ for $i = 1, \ldots, d$.

As a consequence, it is sufficient to show that there is a bijection between the pairs of tableaux with respective shapes $\alpha, \beta$ and the tableaux with shape $\square + \alpha/\square - \beta^-$. Given a filling $T_\alpha$ of $\alpha$ together with a filling $T_\beta$ of $\beta$, we apply the following process to $T_\beta$: replace each entry $v$ by $d + 1 - v$ and rotate the diagram by half a turn. This yields a skew tableau $T'_\beta$ with shape $[T'_\beta] = \square/\square - \beta^-$. Then the concatenation of $T'_\beta$ and $T_\alpha$ is a tableau with shape $\square + \alpha/\square - \beta^-$ (see Figure 2).

![Figure 2. The sought bijection](image)

Indeed, in the last column of $T'_\beta$, the entry of the $i$th row has value $\leq (d + 1) - (d + 1 - i) = i$ and in the first column of $T_\alpha$ the entry of the $i$th row has value $\geq i$.

The inverse map is the obvious one. Take the filling of the intersection between $\square + \alpha/\square - \beta^-$ and $\square$, rotate it, and apply the involution $v \leftrightarrow d + 1 - v$ to get $T'_\beta$, then take the filling of the remaining part in $\square + \alpha/\square - \beta^-$ to get $T_\alpha$. $\square$

We can deduce a rule for Littlewood–Richardson coefficients.
Corollary 2.4. — With the same notation and hypotheses, for any partition $\gamma$ with at most $d$ parts, the following Littlewood–Richardson numbers coincide:

$$c^\gamma_{\alpha\beta} = c^{\alpha + (\beta - \gamma)}_{\beta}.$$ 

Proof. — Indeed, the coefficient $c^\gamma_{\alpha\beta}$ is also the coefficient $\langle s_\alpha s_\beta, s_\gamma \rangle$ of $s_\gamma$ in the decomposition of the symmetric function $s_\alpha s_\beta$ over the Schur basis, and the coefficient $c^\alpha_{\beta \gamma}$ is also the coefficient $\langle s_\alpha, s_\beta \rangle$ of $s_\gamma$ in the decomposition of the symmetric function $s_\alpha$ over the Schur basis. Now, one has $\langle s_\alpha s_\beta, s_\gamma \rangle = \langle s_\alpha, s_\beta \gamma \rangle.$

\[\square\]

2.3. A strong version of the duality theorem. — The combination of Corollary 2.2 and Corollary 2.4 yields the following proposition.

Proposition 2.5. — For $\alpha, \beta, \gamma$ partitions with at most $d$ parts, if $\Box_\beta \supseteq \beta$ and $\Box' \subseteq \gamma$ are two rectangular partitions with $d$ parts, then

$$c^\gamma_{\alpha\beta} = c^{\alpha + (\beta - \gamma)}_{\beta}.$$ 

We derive the following statement.

Theorem 2.6 (Strong duality theorem). — Let $E \to X$ be a rank $n$ vector bundle over a variety, and let $\alpha, \beta$ be two partitions with at most $d$ parts. For any rectangle partition $(\ell)^d \supseteq \beta$, one has the following pushforward formula along $\pi: G_\alpha(E) \to X$:

$$\pi_* (s_\alpha(U) s_\beta(U)) = s_{(\ell - n + d)\beta + \alpha \ell} (E).$$

In particular, if $\beta \subseteq (n - d)^d$:

$$\pi_* (s_\alpha(U) s_\beta(U)) = s_{n/\beta} (E).$$

We recover the duality theorem in Schubert calculus [Ful97, Sec. 9.4]

Corollary 2.7 (Duality theorem). — For a partition $\alpha$ and a partition $\beta \subseteq (n - d)^d$ one has

$$\pi_* (s_\alpha(U) s_\beta(U)) = \begin{cases} 1 & \text{if } \alpha = \beta^c \\ 0 & \text{if } \alpha \not\supseteq \beta^c \end{cases}.$$ 

Proof of the theorem. — One has

$$s_\alpha(U) s_\beta(U) = \sum c^\gamma_{\alpha\beta} s_\gamma(U),$$

and since $\text{rank}(U) = d$, one can restrict to the partitions $\gamma$ with at most $d$ parts. Using [DP15, Sect. 4] to compute the push-forward, one obtains

$$\pi_* (s_\alpha(U) s_\beta(U)) = \sum c^\gamma_{\alpha\beta} s_{\gamma^c - (n - d)^d} (E).$$

But now, one can assume that $\gamma \supseteq (n - d)^d$, since otherwise the summand indexed by $\mu$ does not contribute, whence

$$\pi_* (s_\alpha(U) s_\beta(U)) = \sum c^{\gamma^c - (n - d)^d}_{\alpha\beta} s_{\gamma^c} (E).$$

Applying Proposition 2.5 with $\Box_\beta = (\ell)^d$ and $\Box' = (n - d)^d$, one obtains

$$\pi_* (s_\alpha(U) s_\beta(E)) = \sum c^{\gamma^c - (n - d)^d}_{\alpha\beta} s_{\gamma^c} (E) = s_{\alpha^c + (\ell - n + d)^d} (E),$$

and this finishes the proof.

\[\square\]
3. A push-forward formula from [JLP81]

Recall that for a partition $\alpha$, the partition $\alpha^-$ is the conjugate to $\alpha$ and that for a partition $\alpha \subseteq (n-d)^d$ the dual partition $\alpha^d$ is by definition $\alpha^d = (n-d)^d - \alpha^-$. Let us denote by $Q$ the universal quotient bundle on $G_d(E) \to X$:

$$0 \to U \to \pi^* E \to Q \to 0.$$ 

Following [JLP81], we are now interested in the push-forward of the product of a Schur function of $U$ and a Schur function of $Q$.

**Lemma 3.1.** — For a partition $\alpha$ with at most $d$ parts and a partition $\beta$ with at most $(n-d)$ parts, one has

$$\pi_*(s_\alpha(U)s_{\beta}(Q)) = \sum_{\mu \subseteq (n-d)^d} (-1)^{|\mu|} s_{\delta^\mu/\mu^d}(E)s_{\delta^\mu/\mu^d}(E).$$

**Proof.** — Following [Mac95]

$$s_\beta(Q) = s_\beta(E - U) = \sum_{\mu} s_{\mu}(-U) s_{\delta^\mu/\mu^d}(E) = \sum_{\mu} (-1)^{|\mu|} s_{\delta^\mu/\mu^d}(U) s_{\delta^\mu/\mu^d}(E).$$

Notice that the summand can be nonzero only if $\mu \subseteq \beta$ has at most $n-d$ parts, and $\mu^d$ has at most $d$ parts, thus one can consider only the partitions $\mu \subseteq (d)^{n-d}$. For technical reasons, we will rather make the change of variable $\mu \leftrightarrow \mu^d$ and consider $\mu \subseteq (n-d)^d$, thus

$$s_\beta(Q) = \sum_{\mu \subseteq (n-d)^d} (-1)^{|\mu|} s_{\mu}(U) s_{\delta^\mu/\mu^d}(E).$$

Then, by the projection formula

$$\pi_*(s_\alpha(U)s_{\beta}(Q)) = \sum_{\mu \subseteq (n-d)^d} (-1)^{|\mu|} \pi_*(s_\alpha(U)s_{\mu}(U)) s_{\delta^\mu/\mu^d}(E).$$

According to Theorem 2.6, this last expression yields

$$\pi_*(s_\alpha(U)s_{\beta}(Q)) = \sum_{\mu \subseteq (n-d)^d} (-1)^{|\mu|} s_{\delta^\mu/\mu^d}(E)s_{\delta^\mu/\mu^d}(E),$$

which is the announced formula. \hfill \Box

One gets a new proof of the following result, originally stated in [JLP81].

**Theorem 3.2.** — For two partitions $\alpha, \beta$, one has

$$\pi_*(s_\alpha(U)s_{\beta}(Q)) = s_{(\alpha-(n-d)\beta,\beta)}(E).$$

**Proof.** — Combine Lemma 3.1 and Proposition 3.3 below. \hfill \Box

Remark that by permuting the rows, the right hand side of the formula can be written

$$(-1)^{d(n-d)} s_{(\delta^\beta-(d)^{n-d},\delta^\alpha)}(E),$$

which is closer to the expression of [JLP81] (our convention of signs is different).

**Proposition 3.3 (Laplace expansion for Schur functions).** — For a $d$-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ and a $(n-d)$-tuple $\beta = (\beta_1, \ldots, \beta_{n-d})$, one has

$$s_{\alpha,\beta}(E) = \sum_{\mu \subseteq (n-d)^d} (-1)^{|\mu|} s_{\alpha+(n-d)^d/\mu^d}(E)s_{\delta^\mu/\mu^d}(E).$$

**Proof.** — This is an application of the Laplace expansion for $n \times n$ determinants by complementary minors. Indeed

$$s_{\alpha,\beta}(E) = \det\left(\begin{array}{c|ccc} s_{\alpha,-i+j}(E) & i \in 1, \ldots, d & j \in 1, \ldots, n \\ \hline s_{\beta,-i+j}(E) & i \in 1, \ldots, n & \end{array}\right),$$

thus

$$s_{\alpha,\beta}(E) = \sum_{\gamma_1, \ldots, \gamma_d} (-1)^{i+1+d \gamma_1 + \ldots + \gamma_d} \det(s_{\alpha,-i+j+\gamma}(E))_{i \leq j \leq d} \det(s_{\beta,-i+d+j+\gamma}(E))_{i \leq j \leq n-d},$$

for $\gamma_1, \ldots, \gamma_d$. \hfill \Box
where $\gamma \cup \delta = (n, \ldots, 1)$. Now, adding/subtracting the triangular partition $\rho$ yields a bijection between strict partitions $\gamma \subset (n)^d$ and partitions $\mu \subseteq (n - d)^d$ defined by

$$
\mu := \gamma - \rho \subseteq (n - d)^d.
$$

Furthermore, we can also express $\delta$ in terms of $\mu$. Indeed

$$
\delta_{n-d+1-j} - j = \# \{ \gamma_{d+1-i} - i \leq j \} = \# \{ \mu_{d+1-i} \leq j \} = d - \mu_j.
$$

See Figure 3 for an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{example.png}
\caption{An example for $n = 11$}
\end{figure}

Replacing $\gamma$ and $\delta$ by their expressions in terms of $\mu$ and $\mu^\ast$ in the above formula:

$$
{s_\alpha s_\beta}(E) = \sum_{\mu \subseteq (n-d)^d} (-1)^{\mu + \mu^\ast} \det(s_{\alpha-i+j+\mu+1,1}(E))_{1 \leq i,j \leq d} \det(s_{\beta-i+j-\mu}(E))_{1 \leq i,j \leq n-d}
$$

$$
= \sum_{\mu \subseteq (n-d)^d} (-1)^{\mu} \det(s_{\alpha+(n-d)-i+j-\mu}(E))_{1 \leq i,j \leq d} \det(s_{\beta-i+j-\mu^\ast}(E))_{1 \leq i,j \leq n-d}
$$

$$
= \sum_{\mu \subseteq (n-d)^d} (-1)^{\mu} s_{\alpha/\mu^\ast}(E)s_{\beta/\mu}(E).
$$

This finishes the proof of the proposition. \qed

4. The Giambelli formula for type $A$

We now give a new derivation of the Giambelli formula for vector bundles.

**Theorem 4.1.** — With the notation of Section 1, for a partition $\lambda \subseteq (n-d)^d$, the fundamental class of the Schubert bundle $\Omega_\lambda(E_\ast) \subseteq G_d(E)$ is

$$
[\Omega_\lambda(E_\ast)] = \det(c_{\lambda-i+j}(Q - E_{\nu_1, \ldots}))_{1 \leq i,j \leq d}.
$$

**Proof.** — The Schubert bundle $\iota : \Omega_\lambda(E_\ast) \hookrightarrow G_d(E)$ is a subvariety of $G_d(E)$, thus

- its fundamental class $[\Omega_\lambda(E_\ast)] \in A^\ast(G_d(E))$ admits an expression under the form

$$
[\Omega_\lambda(E_\ast)] = \sum_{\beta \subseteq (n-d)^d} \alpha_{\beta} s_\beta(\iota(U));
$$

- The push-forward of $s_\alpha(U)$ along $\iota$ is

$$
\iota_* s_\alpha(U) = s_\alpha(U)[\Omega_\lambda(E_\ast)] = \sum_{\beta \subseteq (n-d)^d} \alpha_{\beta}s_\beta(U)s_\lambda(\iota(U)).
$$

Now, by Proposition 1.5 the push-forward to $X$ of $s_\alpha(U)$ is

$$
(\omega_\lambda)_* s_\alpha(U) = \det(s_{\lambda-i+j-\lambda}(E_\nu))_{1 \leq i,j \leq d}.
$$
and by Theorem 2.6 it is

\[(\omega_\lambda)s_\alpha(U) = (\pi \circ \iota)_* s_\alpha(U) = \sum_{\beta \subseteq (n-d)^d} \omega_\beta \pi_* s_\alpha(U)s_\beta(U) = \sum_{\beta \subseteq (n-d)^d} \omega_\beta s_{\alpha//\beta}(E)\]

So, for each partition \(\alpha\), we get the equation in the unknowns \(\omega_\beta\)

\[(\alpha) \sum_{\beta \subseteq (n-d)^d} \omega_\beta s_{\alpha//\beta}(E) = \det\left(s_{\alpha-j-j'(\lambda_\nu)}(E_{\nu})\right)_{1 \leq i, j \leq d}\]

Notice that a subsystem of the set of these equations \((\alpha)\) is triangular and invertible. Indeed, taking a \(\alpha \subseteq (n-d)^d\), if \(\beta^c = \alpha\), then \(s_{\alpha//\beta}(E) = 1\) and if \(\beta^c \not\subseteq \alpha\), e.g. if \(\beta^c < \alpha\) in lexicographic order, then \(s_{\alpha//\beta}(E) = 0\). Note that duality reverse the lexicographic order. Thus, the subsystem made of equations \((\alpha)\) for \(\alpha \subseteq (n-d)^d\) ordered by lexicographic order is a lower triangular system, with 1’s on the diagonal. It is thus invertible and has a unique solution.

We will now modify the right hand side of \((\alpha)\) to make it look alike the left hand side. Firstly with \(j = 1, \ldots, d\), and for any \(i\) we have the expansion

\[s_{\alpha-,i-j-j'(\lambda_\nu)}(E_{\nu}) = s_{\alpha-,i-j-j'(\lambda_\nu)}(E - (E_{\nu})) = \sum_{k_i} c_{k_i}(E - E_{\nu})s_{\alpha-,i-j-j'-k_i}(E),\]

the coefficients of which depend only on \(j\). By linearity with respect to columns

\[\det\left(s_{\alpha-,i-j-j'(\lambda_\nu)}(E_{\nu})\right)_{1 \leq i, j \leq d} = \sum_{\beta \subseteq (n-d)^d} \prod_{k_i} c_{k_i}(E - E_{\nu}) \det\left(s_{\alpha-,i-j-j'(\lambda_\nu)+k_i}(E_{\nu})\right)_{1 \leq i, j \leq d}\]

where \(K\) denotes the \(d\)-tuple \((k_1, \ldots, k_d)\). Next, we use the straightening defined in Sect. 1.3. If \(\lambda^c + K\) has no straightening, then the summand indexed by \(K\) is 0. So we can consider only the \(K\)'s such that \(\lambda^c + K\) has a straightening. Furthermore, the summand indexed by \(K\) is maybe not 0 only if the straightening is contained in \(\alpha \subseteq (n-d)^d\), so we can restrict to \(K\)'s having straightenings contained in \((n-d)^d\). By unicity of the straightening, this set is in bijection with the set of pairs \((\nu, \beta)\) were \(\nu\) is a permutation of \(1, \ldots, d\) and \(\beta\) a partition contained in \((n-d)^d\). Furthermore it is harmless to use \(\beta^c\) instead of \(\beta\). Putting these observations together:

\[\det\left(s_{\alpha-,i-j-j'(\lambda_\nu)}(E_{\nu})\right)_{1 \leq i, j \leq d} = \sum_{\beta \subseteq (n-d)^d} \prod_{w} (-1)^w \prod_{1 \leq j \leq d} c_{k_j}(E - E_{\nu}) s_{\beta,\alpha//\beta}(E),\]

where \(K = K_{w,\beta} = w^{-1} \cdot \beta^c - \lambda^c\). The equation \((\alpha)\) becomes

\[\sum_{\beta \subseteq (n-d)^d} \omega_\beta s_{\beta,\alpha//\beta}(E) = \sum_{\beta \subseteq (n-d)^d} \left(\sum_{w} (-1)^w \prod_{1 \leq j \leq d} c_{k_j}(E - E_{\nu})\right)s_{\beta,\alpha//\beta}(E),\]

Thus we obtain a solution (the unique solution by our above observations) by setting for any \(\beta \subseteq (n-d)^d\)

\[\omega_\beta = \sum_{w} (-1)^w \prod_{1 \leq j \leq d} c_{k_j}(E - E_{\nu}),\]

and the fundamental class of \(\Omega_\lambda(E_\bullet)\) is

\[\left[\Omega_\lambda(E_\bullet)\right] = \sum_{\beta \subseteq (n-d)^d} \left(\sum_{w} (-1)^w \prod_{1 \leq j \leq d} c_{k_j}(E - E_{\nu})\right)s_\beta(U),\]

where \(K = K_{w,\beta} = w^{-1} \cdot \beta^c - \lambda^c\).

We will now improve this formula; we need to use straightening backwards. To any permutation \(w\), one can associate another permutation \(w^c\), with same signature, by setting \(w^c(i) = d + 1 - w(d + 1 - i)\), and a quick computation shows that

\[w^c \cdot (\lambda^c + K) = \beta \iff \beta = w \cdot (\lambda - K^c).\]

We re-index the sum using the involution \(w \leftrightarrow w^c\) (and we use the involution \(K \leftrightarrow K^c\) in the summand) in order to get:

\[\left[\Omega_\lambda(E_\bullet)\right] = \sum_{\beta \subseteq (n-d)^d} \sum_{w} (-1)^w \prod_{1 \leq j \leq d} c_{k_{w(j-1)}}(E - E_{\nu})s_\beta(U),\]
where now

\[ K_{w, \beta} = (K_{w, \beta})^{-1} = w^{-1} \cdot \beta - \lambda. \]

By unicity of the straightening, this in turn leads to

\[ [\Omega_\lambda(E_v)] = \sum_k \prod_{1 \leq j \leq d} c_{k+1-j}(E - E_v) s_{\lambda-k}(U) = \sum_k \prod_{1 \leq j \leq d} c_k(E - E_{v_{d+1-j}}) s_{\lambda-k}(U), \]

since if \( w \cdot (\lambda - K) = \beta \) is the straightening of \( \lambda - K \), then \( s_{\lambda-k}(U) \) is the straightening of \( \lambda - K \), then \( s_{\lambda-k}(U) \) does not contribute. Finally, by linearity with respect to rows

\[ [\Omega_\lambda(E_v)] = \det(c_{\lambda-i+j}(E - U - E_{v_{d+1-j}}))_{1 \leq i, j \leq d}. \]

This yields the announced determinantal formula, replacing \( E - U \) by \( Q \).

\[ \square \]

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