Moist Shallow Water Response to Localized Tropical Forcing: Initial Value Problems

D.L. Suhas *\(^{1}\) and Jai Sukhatme\(^{1,2}\)

\(^{1}\)Centre for Atmospheric and Oceanic Sciences, Indian Institute of Science, Bangalore, 560012, India
\(^{2}\)Divecha Centre for Climate Change, Indian Institute of Science, Bangalore, 560012, India

Abstract

The response of spherical moist and dry shallow water systems to localized tropical imbalances are examined. The nonlinear dry response consists of a combination of Rossby, Kelvin and mixed Rossby-Gravity waves, depending on the nature of the initial condition. Remarkably, most of the power in the nonlinear solution follows linear dispersion curves quite accurately. In contrast, with a meridionally varying saturation profile, the long time moist solution consists of only westward propagating modes that are of a large scale, low frequency and confined to the subtropics. When the saturation profile is also allowed to vary with longitude, apart from a westward quadrupole, there is a distinct eastward propagating intraseasonal response at long times. In fact, the early eastward response is dominated by a Kelvin wave that changes into slow, predominantly rotational wave packets which arc out to the midlatitudes and return to the tropics. This carries over to a realistic saturation profile (derived from reanalysis based precipitable water) in the boreal summer, but with the eastward response restricted to the northern hemisphere. In the boreal winter, this component consists of a subtropically confined quadrupole in addition to weaker midlatitudinal disturbances. The nature of the eastward propagation is well described to first order by moist potential vorticity considerations. Finally, in all the moist simulations, the tropical equivalent depth closely matches linear rapid condensation estimates.

Key Words : Initial Value Problem, Nonlinear Solutions, Moist Shallow Water Equations

1 Introduction

The response of the atmosphere to localized near-equatorial forcing, usually in terms of a heat source, has been useful in developing an understanding of the waves that can exist at low latitudes (Matsuno [1966]), as well as the non-axisymmetric nature of the tropical circulation (Gill [1980]). Further, within the framework of simplified dynamical models such as the shallow water system or single layer vorticity equations, the inclusion of various base states has highlighted the emergent global teleconnections (Lau and Lim [1984]; Sardeshmukh and Hoskins [1988]), equatorial trapping (Zhang and Webster [1989]) and tropical stationary waves that form with continuous forcing (Kraucunas and Hartmann [2007]). Time dependent solutions in a linear shallow water setting (Kacimi and Khouider [2018]), as well as in more complicated models,

*Corresponding author: D.L. Suhas, suhasdl.mysore@gmail.com
such as the linear and nonlinear primitive equations with episodic forcing restricted to specified timescales (Salby and Garcia, 1987; Garcia and Salby, 1987), steady forcing (Webster, 1972; Jin and Hoskins, 1995) and the initial value problem (Hoskins and Jin, 1991) with spatially varying background flows have also been pursued to examine the character of waves generated in the tropics, their propagation to extratropical regions and the evolution towards a steady state circulation. On a related note, the response to specific tropical phenomena such as the heating associated with Madden-Julian Oscillation (MJO; Matthews et al., 2004; Bao and Hartmann, 2014; Monteiro et al., 2014) and El Niño (Lin et al., 2007; Shaman and Tziperman, 2016) have also been analysed to delineate their global character.

Quite reassuringly, on the steady front, reanalysis data suggests that at low latitudes, stationary waves in the upper troposphere are equatorial Rossby waves and are mainly a response to tropical heating (Dima et al., 2005). Similarly, space-time spectral analysis of tropical fields (such as outgoing longwave radiation) that capture evolving disturbances show a signature of linear equatorial waves (Wheeler and Kiladis, 1999). But, there are two caveats regarding the wavenumber-frequency spectra from data: (i) The wave speeds are significantly lower than predictions from the dry shallow water theory (Matsumo, 1966). This reduction in speed is attributed to the coupling of moist processes with large scale dynamics resulting in the so called convectively coupled equatorial waves (CCEWs; Kiladis et al., 2009). (ii) More importantly, there is significant power in regions of the wavenumber-frequency diagram that do not correspond to traditional linear, dry equatorial waves — specifically, the MJO (Zhang, 2005). Indeed, interactive moisture is believed to play a vital, possibly essential, role in the MJO (Sobel and Maloney, 2013; Adames and Kim, 2016).

Thus, while small perturbations to the dry shallow water system produce the classical spectrum of tropical waves (Matsumo, 1966; Vallis, 2019), realizing the importance of moist processes, attempts have been made to include the effect of moisture on dynamics while retaining the simplified nature of the shallow water framework. The intent of these simplified models is to examine if there are any new modes with moisture that do not exist in the dry system — in particular, whether a signature of a MJO like system appears on the inclusion of moisture. Usually linear momentum equations with bulk aerodynamic evaporative protocols and either an explicit (Vallis and Penn, 2019), or implicit (Solodoch et al., 2011; Yang and Ingersoll, 2013) treatment of water vapour and possibly even boundary layer convergence (Wang et al., 2016) have been explored. In fact, linear analyses that include moisture gradients have also been pursued (Sobel et al., 2001; Sukhatme, 2013; Monteiro and Sukhatme, 2015). Indeed, all these systems show the presence of a slow, large-scale, eastward propagating mode, albeit via differing mechanisms, that is absent without moisture. Apart from MJO related investigations, it is worth mentioning that moist dynamics in relatively simple models has also been utilized in understanding the genesis, structure and propagation of CCEWs (Kuang, 2008; Khouider and Majda, 2008; Dias et al., 2013) and tropical cyclones (Schecter and Dunkerton, 2009).

Here, following Gill et al. (1979); Gill (1982), we consider the full nonlinear dynamics of a layer of fluid that is subject to condensation and evaporation by means of mass loss and gain (Bouchut et al., 2009). We focus on the initial value problem in the absence of large scale drag or damping and explore the nonlinear response of a moist shallow water system on a sphere to localized tropical forcing. In particular, we investigate the waves produced with differing saturation profiles that range from being purely a function of latitude to those derived from seasonal mean maps of precipitable water in reanalysis data. In essence, this allows us to systematically compare the adjustment of tropical imbalances in dry and moist settings, and to assess whether the nature of the moist environment affects the atmospheric response. Section 2 contains the basic equations and a description of the initial conditions. Section 3 shows the response of a dry system and Section 4 considers the corresponding moist initial value problem.
with different saturation profiles. The manuscript then concludes with a discussion of results in Section 5.

2 Equations and Numerical Setup

The moist shallow water equations with mass and momentum forcing, radiative damping and momentum drag, precipitation and evaporation take the form (Gill et al., 1979; Bouchut et al., 2009; Sukhatme, 2013; Monteiro and Sukhatme, 2015),

\[
\begin{align*}
\frac{\partial \omega}{\partial t} + \nabla \cdot (u \omega) &= -\frac{\omega}{\tau_m} + F_\omega, \\
\frac{\partial \delta}{\partial t} - k \cdot \nabla \times (u \omega) &= -\nabla^2 \left( \frac{\mathbf{u} \cdot \mathbf{u}}{2} + gh \right) - \frac{\delta}{\tau_m} + F_\delta, \\
\frac{\partial h}{\partial t} + \nabla \cdot (u h) &= -\left( h - H \right) \tau_r + S - \chi(P - \mathcal{E}), \\
\frac{\partial q}{\partial t} + \nabla \cdot (u q) &= -P + E.
\end{align*}
\]

Here, \( u = (u, v) \) is the horizontal flow, \( \omega \) and \( \omega_a \) are the relative and absolute vorticity, and \( h(x, y, t) \) is the depth of the fluid (\( H \) is the mean undisturbed depth, taken to be a constant for most of the simulations). The drag in the vorticity and divergence equations has the same timescale \( \tau_m \), \( \tau_r \) is the radiative damping timescale, \( S \) is the mass forcing and \( F_\omega, F_\delta \) are vorticity and divergence forcing terms. Further, \( q \) is the column water vapour, \( q_s \) is a prescribed moist saturation profile, \( P \) and \( \mathcal{E} \) are the moisture sink and source terms. \( \chi \) is an effective specific heat for this system.

Precipitation is dependent on column water vapour (Muller et al., 2009), and takes a Betts-Miller form (Betts, 1986),

\[
P = (q - q_s) \delta \frac{(q - q_s)}{\tau_c},
\]

where \( \tau_c \) is the condensation timescale. Evaporation takes the similar form with \( \mathcal{E} = (q_s - q) \delta (q_s - q)/\tau_e \), where \( \tau_e \) is the condensation timescale. Therefore, a parcel either experiences condensation (if \( q > q_s \)) or evaporation (if \( q < q_s \)), and we set their timescales to be equal, i.e., \( \tau_c = \tau_e \).

Combining the last two equations in (I), setting \( m = h - \chi q \), we obtain,

\[
\frac{Dm}{Dt} + m \nabla \cdot \mathbf{u} = -\left( h - H \right) \frac{1}{\tau_r} + S.
\]

A moist potential vorticity (PV) equation that results is,

\[
\frac{D}{Dt} \left[ \frac{\omega_a}{m} \right] = \frac{\omega_a (h - H)}{m^2 \tau_r} - \frac{S \omega_a}{m^2} - \frac{\omega}{m \tau_m} + \frac{F_\omega}{m}.
\]

Thus, when the sources are switched off and \( \tau_r, \tau_m \to \infty \), moist PV, i.e., \( (\omega_a/m) \) is materially conserved (Monteiro and Sukhatme, 2015).

The shallow water system is solved in spherical geometry using the SHTns library (Schaeffer, 2013). All results are reported at a resolution of 512 (longitude) \( \times \) 256 (latitude), triangularly truncated corresponding to a maximum resolved wavenumber of 170. We use a 3rd order Adams-Bashforth integrator for time stepping. A \( \Delta^4 \) hyperviscosity is used for small scale dissipation. The mean depth of the fluid is fixed at 300 m so as to have phase speeds near that of the first baroclinic mode in the tropics (Kraucunas and Hartmann, 2007). Planetary radius (\( a \)), rotation rate (\( \Omega \)) and acceleration due to gravity (\( g \)) are set to that of the Earth (Suhas et al., 2017).
The timescales of both evaporation and condensation are relatively rapid, and are fixed to be 0.1 days (we have performed runs with varying $\tau_c, \tau_e$, and as long as these timescales are less than a day, they do not affect the nature of the results presented). In this work, we focus on initial value problems hence there is no large-scale drag or damping in the equations, specifically, $\tau_m, \tau_r = \infty$ in Equation 1. Steady and statistically stationary solutions, that include drag and damping, are explored in a companion paper. With regard to the choice for $\chi$ and $\max(q_s)$, this is based on the speed of moist Kelvin waves in the tropics. Specifically, with $H = 300$ m, the dry Kelvin wave speed is $\approx 54$ m/s. Now, in the limit of rapid condensation, linear analysis of Equation 1 suggests an equivalent depth of $h_e = H - \chi Q$. Setting $Q = \max(q_s) = 0.05$, we choose $\chi = 5000$ which yields a moist Kelvin wave speed of $\approx 22$ m/s. Another way to look up on this choice is to note that moisture couples with $\chi$ to force the height in Equation 1 thus we set these two so that their product is less than, but of the same order as the mean depth of the fluid.

Four kinds of initial conditions used are (i) Height imbalance, here, initially an anomaly is introduced in the height field which is localized on the equator. The anomaly is Gaussian in shape with a peak magnitude of 10m and decays by about $30^\circ$ and $10^\circ$ in longitude and latitude, respectively. A negative anomaly acts as a mass sink and is representative of lower tropospheric convergence. (ii) Vorticity imbalance, here, initially the system is forced by a localized vortex on the equator that has the same size as the aforementioned height anomaly. This forcing creates a setup similar to Suhass et al. (2017) used to study superrotation in the spherical dry SWEs. (iii) Large-scale purely divergent initial conditions that are localized by a Gaussian shape factor. (iv) Random large-scale initial height field also localized to the equatorial region. The spectrum of this random perturbation takes the form, $A e^{i \theta}$, where $\theta$ is a random number in $[0, 2\pi)$ and $A$ is the wavelength-dependent forcing amplitude with forcing restricted to total wavenumbers between 1 and 10. A meridional Gaussian profile (similar to previous cases but is not a function of longitude) is multiplied to this random anomaly, thereby confining it to tropics but unbounded zonally. We mainly discuss the height imbalances (Gaussian and random), though in some places novel features of vorticity and divergent forcing are highlighted.

3 Dry Response

Starting with a height imbalance, the evolution of height and velocity anomalies for the dry problem are shown in Figure 1. Specifically, responses to a Gaussian mass sink and a large-scale random height perturbation are shown in the upper and lower panels, respectively. In both cases, adjacent highs and lows are evident on the equator and these spread out into off-equatorial gyres. On comparing the upper and lower panels of Figure 1, the initial random height forcing appears to generate a spectrum of large to small scale features, whereas the Gaussian forcing remains restricted to fairly large spatial scales. A wavenumber-frequency analysis of the solutions is shown in Figure 2. In each case, this diagram is constructed using a window of 50 days (Days 251-300) of the simulation. Interestingly, late and early time (not shown) diagrams do not show much change, i.e., the response generated lasts through the course of the entire simulation (we have checked till 1000 days). Quite clearly, the system generates a family of Rossby and Kelvin waves with an equivalent depth of about 300 m (which was the mean depth of the fluid used in the simulation). Further, the equatorial deformation radius ($\sqrt{(gh_e)^{1/2}/\beta}$) is about $14^\circ$, and thus the response spreads out to a significantly larger latitudinal extent than this length scale. Noticeably, the eastward Kelvin peaks have more power and smaller time periods than the westward Rossby waves. Specifically, the eastward (westward) components have periods of about 8-9 days (25-28 days), respectively. While the Rossby and Kelvin portions of the diagrams are similar in the two cases, the random height initial condition appears “busier”
Figure 1: Height anomaly with velocity quivers at Days 25 and 250 for the dry case. Upper two panels are the response to a Gaussian mass sink (indicated by a black circle). The lower two panels are for a large-scale random height field imbalance.

with activity on the mixed Rossby-Gravity branch as well as isolated peaks scattered through the wavenumber-frequency domain. But by and large, quite remarkably, even though this is a fully nonlinear solution, the wavenumber-frequency characteristics of the solution lie along the linear dispersion curves. This is reminiscent of particular equatorially trapped solutions in the nonlinear non-divergent barotropic system \cite{Constantin2012}.

Based on the wavenumber-frequency diagrams, a decomposition of the solution into eastward and westward moving structures is shown in Figure 3. Here, we have used a window of 100 days to reconstruct the eastward and westward signals and we show these components for the Gaussian initial condition. Further a Hanning window is used to reduce the spectral leakage. The decomposition is robust with respect to changes in the window length. As we would expect, this yields off-equatorial Rossby gyres (westward) and Kelvin waves (eastward) consisting of high-low pairs with maximum amplitude on the equator that travel in the eastward direction. Further, Rossby waves of different scales are visible in the westward propagating component of the solution. The contribution of the eastward and westward components to the kinetic and potential energy as a function of time is shown in the first panel of Figure 4. As expected from the wavenumber-frequency diagram, these contributions fluctuate with time and moreover, the eastward component has greater energy than the westward portion. This is especially clear for the Gaussian case, while the ratio of the east to west energy hovers around unity for the random initial condition.
Figure 2: Wavenumber-frequency plot of height field for the dry case. The first and second panels are for Gaussian and random initial forcing, respectively. A 50 day period (Day 251-300) was used to construct the diagram. The latitudinal band used is from 15°N to 15°S. The power is normalized and plotted in log scale. Dispersion curves are plotted for an equivalent height of 300 m.

A spatial power spectrum of the kinetic energy (KE), shown in the second panel of Figure 4 suggests the energy spreading out from the initially large scale localized source. For the Gaussian initial condition, there are some signs of power-law scaling (with a -5 exponent), but the range is quite limited. The spectrum for the random height imbalance suggests a much more robust transfer of energy to small scales and a power-law scaling appears more clear (with a -3 exponent). The contribution of rotational portion to the KE is comparable to the divergent part at the largest scales, but it is quite small over the smaller scales. A map of the rotational and divergent components of the flow (not shown) aligns quite well with the westward and eastward response, respectively.

4 Moist Response

4.1 Moist solutions with a purely latitudinal saturation profile

The corresponding solutions for the moist simulation are now examined. We first consider the case where the saturation profile is purely a function of latitude. In particular, $q_s$ has a maximum at the equator and falls off with increasing latitude. Figure 5 shows the height and velocity anomalies, where the upper two panels are for a Gaussian sink and the lower two for a random height imbalance. Figure 6 shows the early and late time wavenumber-frequency diagrams for the Gaussian case (the random height forcing produces a similar diagram and is not shown). Early in the evolution (before Day 50), we see Rossby and Kelvin waves, though with a much smaller equivalent depth (about 50 m). The smaller equivalent depth can be estimated by a linear analysis of Equation 1 in the rapid evaporation-condensation limit, this yields $h_e = H - \chi Q$, where $H$ is the mean depth of the fluid and $Q$ is the scale factor for the precipitable water. As the wavenumber-frequency diagrams are constructed from data within 15°N/S, the domain averaged $q_s \approx \max(q_s)$, i.e., we are probably justified in taking $Q = 0.05$. With $\chi = 5000$, this yields $h_e = 50$ m.

Noticeably, even at early times, smaller time period disturbances are absent in the moist case when compared to the dry simulation. In fact, given a smaller deformation radius (approxi-
Figure 3: Eastward and westward component of height field at Day 50 for the dry simulation. These are for the Gaussian height imbalance. First ten zonal wavenumbers are retained and a 100 day window is used.

Figure 4: Evolution of the ratio of eastward to westward component of Energy (KE+PE) with time for the dry case is shown in panel 1. Panel 2 shows the KE spectra averaged over Day 200-300. Black dashed lines in Panel 2 have slopes of -3 and -5. In both panels solid lines: Gaussian, dashed lines: Random initial forcing.
mately 9°) these solutions are confined to the subtropics with almost no signature extending out into the midlatitudes. Further, at late times, power is exclusively concentrated around large-scale Rossby waves (westward) with almost no eastward propagating features. Thus, the initial height imbalance adjusts to a large scale, westward propagating Rossby quadrupole. The time period of the westward moving quadrupole is approximately 75 days. Note that, as our WK plots use a 50 day window, we use Hovmöller plots (not shown) to estimate the time period. The evaporation and condensation fields that go along with this emergent structure, in the Gaussian and random cases, are shown in Figure [7]. In both cases, condensation (evaporation) occurs behind (ahead of) the anticyclonic gyres. This is consistent with moist (dry) air experiencing condensation (evaporation) as it moves anti-cyclonically away from (towards) the equator. Further, moist interactions, i.e., condensation and evaporation, align well with regions of convergence and divergence, respectively. Thus, even though the initially imposed convergence is on the equator, at late times this field attains a maximum off the equator as dictated by the flanks of the adjoining Rossby gyres.

A decomposition of the solution into eastward and westward portions for the Gaussian case at early times is shown in Figure [8]. A clear large-scale Kelvin wave is seen in the eastward direction while Rossby gyres appear in the westward component of the solution. As time goes on, the eastward component dies out while the Rossby gyres consolidate into a well formed quadrupole structure (Day 250 in Figure [5]). The loss of the eastward features is seen via the decay of the eastward to westward energy in Figure [9]. Further, the spatial KE spectra suggest almost no transfer of energy out of the initial large length scales. Interestingly, in contrast to the dry run, almost all the energy in the moist simulation is contained in the rotational portion of the flow.

4.2 Moist solutions with latitudinal and longitudinal saturation profile

In reality, precipitable water in the troposphere has marked variations in the latitudinal and longitudinal directions (Sukhatme, 2012; Monteiro and Sukhatme, 2015). Taking this into account, we now examine the response to an initial height imbalance with \( q_s \) decaying with latitude and from 180° longitude. Now, the position of the source can be important, specifically, we have three cases where the initial source is to the west, aligned with or to the east of the peak in \( q_s \). It turns out that the position of the source affects the response, but only in the initial stages. Indeed, depending on where the source is located, the Rossby wave response is slow, muted and fast, when the source is to the east of, aligned with and to the west of the peak in \( q_s \). Here, we present results for the case when the source is to the west of the peak in \( q_s \). The height and velocity anomalies for a Gaussian imbalance are shown in Figure [10] (the random forcing produces a similar response and is not shown). As a guide to the eye, a map is included beneath the height anomalies. A striking feature of the solution is the presence, at later times (Day 250 in Figure [10]), of a wavetrain that expands out to the midlatitudes. A tropical wavenumber-frequency diagram (not shown) indicates the presence of Rossby and Kelvin waves at early times and follows linear dispersion curves with an equivalent depth of 160 m. This matches quite well with \( h_e = H - \chi Q \), where \( Q \) is the tropical (15° band straddling the equator) average of \( q_s \).

The westward component of height anomalies is again dominated by the large-scale Rossby waves (not shown). Its quadrupole structure is confined to sub-tropics, but has a greater meridional extent as compared to the case where \( q_s \) was purely a function of latitude. This is in line with a larger deformation scale (approximately 12°) in the present situation. Given that the response moves out to the midlatitudes, we construct a Hovmöller plot in the latitude band of 30° to 60°. As seen in Figure [11] this immediately suggests cohesive eastward movement.
Figure 5: Height anomaly with velocity quivers at Days 25 and 250 when $q_s$ is a function of latitude with a maximum at the equator (shown in brown contours). Upper two panels are the response to a Gaussian mass sink (indicated by a black circle). The lower two panels are for a large-scale random height field imbalance.
Figure 6: Wavenumber frequency plot of height field when $q_s$ is a function of latitude with a maximum at the equator. The system is initially forced by a Gaussian mass sink. Early (Day 1 - 50) and late (Day 251-300) periods are shown in the first and second panels, respectively. The latitudinal band used is from 15°N to 15°S. The power is normalized and plotted in log scale. Dispersion curves are plotted for an equivalent depth of 50 m.

Figure 7: Moisture anomaly (deviation from the saturation state) at Day 250. First panel is for the Gaussian sink and the second panel corresponds to a random height imbalance. Positive (negative) values indicate the region of precipitation (evaporation). Height anomalies are displayed in black contours. Here, $q_s$ is a function of latitude with a maximum at the equator (shown in brown contours).
Figure 8: Eastward and westward component of height field for the moist case at Day 50 when $q_s$ is a function of latitude with a maximum at the equator (shown in brown contours). First ten zonal wavenumbers are retained and a 100 day window centred at Day 50 is used. The system is initially forced by a Gaussian mass sink as shown in Figure 5.

Figure 9: Evolution of eastward to westward ratio of energy (KE+PE) with time is shown in panel 1. Panel 2 shows KE spectra. The spectra is averaged over Day 200-300. Here, $q_s$ is a function of latitude with a maximum at the equator. Black dashed lines in panel 2 have slopes of -3 and -5. In both panels, solid lines are for the Gaussian sink and dashed lines are for the random height forcing.
Figure 10: Height anomaly with velocity quivers at Days 25 and 250 when $q_s$ is a function of latitude and longitude with a maximum at the equator and 180° longitude (shown in brown contours). The system is initially forced by a Gaussian mass sink (indicated by a black circle) to the west of the peak in $q_s$.

between 120° to 240° longitudes. Thus, in contrast to the case when $q_s$ was purely a function of latitude, a feature that stands out is the presence of an eastward moving low-frequency mode. We isolate this eastward component and its evolution with time is shown in Figure 12. Quite clearly, the eastward signal begins as a Kelvin wave restricted to the tropics (as in Day 25) and then arcs out to the midlatitudes (by Day 150). In fact, by Day 250, this signal has curved back towards the tropics from the midlatitudes. Indeed, the midlatitude extension seen in Figure 10 is principally due to the eastward component of the solution. Similar features are seen when the source position is aligned with, and to the east of, the peak in $q_s$. Judging from Figure 12, in the Northern hemisphere this signal starts propagating in a North-East direction (to the midlatitudes) and then changes to a South-East direction (to the tropics). Further, the velocity anomaly quivers show this off-equatorial response to be primarily rotational in character moving eastward at a speed of about 0.5°/day.

The movement of this eastward component can be explained in terms of moist PV. In the absence of mass and momentum sources (or sinks), Equation 3 implies the conservation of moist PV. Similar to the dry Rossby waves which rely on the conservation of PV, here we have an equivalent moist rotational wave (i.e., "moist Rossby" wave) that relies on the conservation of moist PV. Further its direction of propagation can be well described by the moist PV gradients. Near the equator moist PV increases polewards as shown in Figure 12 and as per their dry counterparts, the moist Rossby waves propagate in a westward direction. But away from the equator in the Northern hemisphere, the moist PV gradient points North-East to the west of dateline, equatorward at the dateline and North-West to the east of dateline. In essence, moist PV gradient has a clockwise (anti-clockwise) rotation in the Northern (Southern) hemisphere. Moist Rossby waves propagate to the "west" of the PV gradient and experience a similar rotation but with a 90° phase lag. This naturally results in the eastward propagation of our moist Rossby waves at 180° longitude where the moist PV gradient is inverted (equatorward).

The ratio of eastward to westward energy hovers around 0.1 to 0.2 at later times (not shown). This indicates that the westward mode has most of the energy, but also suggests that the eastward piece does not die away (mainly in the midlatitudes) as was the situation when $q_s$ was only dependent on latitude. KE spectra (not shown) indicate the dominance of rotational portion of the flow and the inhibition of energy transfer to smaller scales. But this inhibition is not as strong as in the purely meridionally varying saturation profile. Further, maps of evaporation/condensation fields are consistent with Figure 7.
Figure 11: Hoømoller plot of height anomalies for the moist case where $q_s$ is a function of latitude and longitude with a maximum at the equator and 180° longitude. The system is initially forced by a Gaussian mass sink to the west of the peak in $q_s$ as shown in Figure 10. The height anomalies are averaged over the latitudes 30°N to 60°N.

Figure 12: Eastward component of height field when $q_s$ is a function of latitude and longitude with a maximum at the equator and 180° longitude. First ten zonal wavenumbers are retained and a 50 day window is used. The system is initially forced by a Gaussian mass sink as shown in Figure 10. The moist PV is shown in green contours.
4.3 Moist solutions with a realistic saturation profile

The responses studied so far have been with idealized saturation profiles. To see if the intuition developed carries over to more realistic scenarios we now use climatological mean precipitable water vapor for the months of January and July obtained from NCEP-NCAR reanalysis data as saturation profiles. Note that, to have a smooth $q_s$ field only the first five spherical harmonics are considered.

For both January and July cases we obtain robust eastward and westward moving components. The westward response in both seasons is a circumnavigating Rossby quadrupole. Here, we focus on the eastward response which is shown in Figures 13 and 14 for the months of January and July, respectively. With the January saturation profile, the eastward component has a clear Kelvin wave signature early on (Day 25 and 50) which transforms to a quadrupole structure (Day 150 and 250) in the tropics during later stages. Weaker mid-latitude eastward moving anomalies also form near the Southern Africa and Australia in the Southern Hemisphere and over Northern Asia and the Pacific in the Northern Hemisphere. At long times (around Day 250), as seen in Figure 13, the midlatitude response is overshadowed by the quadrupole in the tropics. This consists of negative anomalies over Africa (on either side of the equator) that extend westward over the Atlantic to South and Central America. This is complemented by positive anomalies in the northern equatorial west Pacific and Australia. The velocity anomaly quivers suggests that this quadrupole is rotational in character. The changing nature of this eastward response can also be seen via the Hovmöller diagrams in Figure 15. Specifically, near the equator (0° to 10° S), we observe Kelvin waves that decay with time. Subsequently, the quadrupole structure emerges in the subtropics (15° to 25° S), its propagation is slower than the Kelvin wave and is dependent on longitude. In particular, this structure is stronger and moves slower in the eastern hemisphere (especially from 60° to 120° longitude). In the midlatitudes (30° to 60° S), an even slower moving signal is apparent, though it is largely in the eastern hemisphere.

In contrast, the July saturation profile yields a response, shown in Figure 14 that begins as a Kelvin wave — much like in January — and then appears as a predominant eastward moving wave train originating in the subtropics, moving over the Tibetan plateau region into the midlatitudes up to North America. In fact, during the boreal summer the longer time features are almost exclusively restricted to the Northern Hemisphere. This propagation is guided by the moist PV, as is evident from the contours of precipitable water and the position of height anomalies. Also, it is worth noting that the strength of eastward anomalies remains relatively steady with time. Once again, the velocity anomalies along with the height fields suggest that this eastward signal is predominantly rotational in character. In all, Figure 14 suggests a teleconnection from a heat source in the equatorial western Indian Ocean to North America. The condensation and evaporation fields for both January and July, are shown in Figure 16. For July, a strong North-East oriented wave train is seen that spans the Indian sub-continent to the western Pacific. For an observer located in India this appears as an oscillation (wet and dry) with an approximate time period of 40 to 50 days. Note that as the wave train progresses into the midlatitudes the condensation and evaporation anomalies are absent (or weak) and reappear as the signal curves back towards the equator over the southern tip of North America. For January, consistent with the background precipitable water field, moisture anomalies are mainly seen in the eastern hemisphere, and as noted in Figure 15 the signal propagates slowly in this region. In both January and July, even though the initially imposed convergence was on the equator, at later times both the westward and eastward rotational response conspire to create off-equatorial maxima in this field (not shown), and are aligned with regions of condensation.
Figure 13: Eastward component of height field at Days 25, 50, 150 and 250 for the moist case during the boreal winter. First ten zonal wavenumbers are retained and a 50 day window is used. The system is initially forced by a Gaussian mass sink (black dashed contours). Here, $q_s$ is derived from the January climatology of precipitable water vapor obtained from NCEP-NCAR reanalysis data (first 5 harmonics shown in brown contours).
Figure 14: Eastward component of height field at Days 25, 50, 150 and 250 for the moist case during the boreal summer. First ten zonal wavenumbers are retained and a 50 day window is used. The system is initially forced by a Gaussian mass sink (black dashed contours). Here, $q_s$ is derived from the July climatology of precipitable water vapor obtained from NCEP-NCAR reanalysis data (first 5 harmonics shown in brown contours).

Figure 15: Hovmöller plot of eastward component of height anomalies for the moist case during the boreal winter. The system is initially forced by a Gaussian mass sink. Here, $q_s$ is derived from the January climatology of precipitable water vapor obtained from NCEP-NCAR reanalysis data. The three panels are for the tropics ($0^\circ$S - $10^\circ$S), sub-tropics ($15^\circ$S - $25^\circ$S) and mid-latitudes ($30^\circ$S - $60^\circ$S).
5 Discussion and Conclusions

Along the lines of initial height imbalances, experiments with vorticity and divergence initial conditions were also conducted. While many of the features of the dry and moist responses were similar, we point out some of the differences that stood out. For a vorticity imbalance (Gaussian vortex at the equator), the principal difference is that the westward moving component is a quadrupole which is anti-symmetric about the equator. Further, in the dry case, Kelvin waves are absent and instead there is a significant presence of mixed Rossby-Gravity waves along with the Rossby waves. In contrast, an initial anomaly that is purely divergent in nature results in a predominant Kelvin wave with a weaker Rossby component. However, the introduction of moisture in both these case results in a slow moving, large scale Rossby waves at later stages. Similar to the initial height imbalance experiments, here too, the eastward component decays over time. Finally, latitudinal and longitudinally varying as well as realistic moist saturation profiles yield similar responses as in the initial height anomaly simulations.

In essence we have explored the response of a nonlinear dry and moist spherical shallow water systems to imbalances in the tropics. With an initial height imbalance, the dry solution consists of an eastward moving Kelvin wave as well as the westward moving Rossby waves. Remarkably the solutions of this fully non-linear problem lie along the linear dispersion curves, with an equivalent depth equal to the mean height of the fluid. These solutions extend to the mid-latitudes and persist for the length of the simulation. Also the eastward component of energy either dominates the westward component (as in Gaussian imbalance) or the two are of similar magnitude (random imbalance). Further the kinetic energy spectra show a transfer of energy to the small scales and the divergent component of KE is comparable to the rotational component at large scales, where as at small scales the spectra is dominated by the divergent component.

Next, the moist solution with a purely meridional variation in the saturation profile was considered. Early on we saw the Rossby and Kelvin waves but these were confined to the subtropics. Also, these waves move slower than their dry counterparts, and their speed is consistent with a reduced equivalent depth that matches well with the predictions from linear rapid condensation estimates. Quite strikingly, the eastward component decays and at long times only the west...

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1 Other explorations of the dry system, especially the relative roles of divergent and rotational components can be found in the Masters project report of A. Baksi [Baksi 2018].
ward moving Rossby gyres remains, i.e., the initial imbalance adjusts to a westward propagating Rossby quadrupole. Further, the off-equatorial position of condensation maxima between adjoining gyres is in accord with outgoing longwave radiation anomalies observed in convectively coupled equatorial Rossby waves (Wheeler et al., 2000). The KE spectra shows no significant transfer of energy to the smaller scales, and in contrast to the dry run, KE is dominated by the rotational portion of the flow.

A distinct low frequency eastward moving component is observed when the saturation profile is allowed to vary both in the latitudinal and longitudinal directions. The eastward component initially comprises of a Kelvin wave confined to tropics. But, with time, this component arcs out to the midlatitudes (to the west of saturation maxima), propagates eastward in the mid-latitudes and then curves back to the tropics (east of the saturation maxima). This slow moving eastward component is rotational in character and relies on the conservation of moist PV. Its propagation can be understood in terms of the moist PV gradient, which in turn is affected by the moist saturation profile. The westward component is again dominated by large scale Rossby quadrupoles. Here too, the speed of these moist waves in the tropics correspond well with a reduced equivalent depth estimated from the rapid condensation limit. The eastward component instead of dying out completely as in purely meridional variation of saturation profile, has a magnitude which is an order less than the westward component. Once again, KE spectra bring out the dominance of the rotational component of the flow.

Finally, more realistic scenarios with saturation profiles following climatological mean precipitable water vapor estimated from reanalysis were considered. The westward response in both seasons consists of a circumnavigating Rossby quadrupole. The eastward propagating signal on the other hand is markedly different, specifically, in the boreal winter, this component transforms from a tropically confined Kelvin wave to a quadrupole structure in the subtropics. Along with this rotational quadrupole, the response also extends out as a weaker signal into the mid-latitudes. Further, given the variation in the moist background, the speed of propagation of the eastward quadrupole depends on longitude. In contrast, during the boreal summer, the moist PV gradient has an inversion near the Tibetan plateau leading to a predominant eastward moving signal originating in the subtropics. Associated with this, the precipitation field shows a strong North-East oriented wave train from the Indian sub-continent to the west Pacific. For an observer fixed in this region, this results in wet and dry oscillations that have a time period of about 40 to 50 days. Also, notably, in the boreal summer, the long time response is confined to the Northern Hemisphere and brings out a teleconnection from the equatorial Indian Ocean to the North American region.

Taken together, the simple moist shallow water model we have considered brings out the changing atmospheric response to localized tropical forcing under different saturation profiles. In all experiments, condensation depletes and evaporation replenishes water vapour in the system, and the saturation profile follows the precipitable water content in a column. Arguably the two most striking results obtained are (a) The change in the solution when the saturation profile goes from being only dependent on latitude to being allowed to vary in both the latitudinal and longitudinal directions, specifically, the long time solution was restricted to a westward Rossby quadrupole in the former, while the latter shows an additional eastward component — a “moist Rossby” wave — that extends out into the midlatitudes. (b) The differences in the eastward component during the boreal winter and summer where we observe a subtropical quadrupole (with accompanying, though weaker, midlatitude extensions) and a northern hemispheric wavetrain from the tropics to the midlatitudes, respectively. Indeed, these perhaps remind one of the eastward moving MJO (Adames and Wallace, 2014) with moist convection and a slower speed of propagation in the eastern hemisphere (Hendon and Salby, 1994; Sobel and Kim, 2012), and the northward propagating summer intraseasonal oscillation (Jiang et al., 2012).
in the winter and summer seasons. While these are fairly idealized experiments, the results point to the likely influence of moisture gradients in shaping the large scale atmospheric response to localized tropical heat sources in each season.

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