Convergence Rate of the Symmetrically Normalized Graph Laplacian

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Abstract

This short note aims at (re)proving that the symmetrically normalized graph Laplacian
$L = \text{Id} - D^{-1/2}W D^{-1/2}$ (from a graph defined from a Gaussian weighting kernel on a sampled
smooth manifold) converges towards the continuous Manifold Laplacian when the sampling
become infinitely dense. The convergence rate with respect to the number of samples $N$ is
$O(1/N)$.

There exist discrete operators which are the equivalent of the gradient and the divergence
operators defined on continuous manifold. They share with them some common properties and
they converge also to their continuous counterparts for a sufficiently fine sampling of the underlying
manifold.

The first one relies on the definition of edge derivative $[1, 2]$. For a smooth function $f : V \to \mathbb{R}$,
the edge derivative of $f$ on $u \in V$ along the edge $e = (u, v) \in E$ formed by the connected vertices
$u$ and $v \in V$ reads

$$\nabla_e f(u) = \sqrt{w(u, v)/2d(v)} f(v) - \sqrt{w(u, v)/2d(u)} f(u),$$

where $d(u) = \sum_{v \in V} w(u, v)$ is the degree of the vertex $u$. From this relation, we have obviously
$\nabla_e f(u) = -\nabla_e f(v)$.

The gradient of $f$ is then defined globally as the vector field $\nabla f : E \to \mathbb{R}$ defined on the edge
set $E$ as $\nabla f(e) = \nabla_e f(u)$ for $e = (u, v) \in E$. This gradient can be represented as a linear operator
$\nabla \in \mathbb{R}^{N \times N}$, so that

$$\nabla f(u) = \{ \nabla f(u, v) : v \in V \} \in \mathbb{R}^N,$$

corresponds to the gradient of $f$ on $u \in V$ seen as a vector of $\mathbb{R}^N$. The norm of this object on each
$u \in V$ is defined naturally as

$$\| \nabla f(u) \|^2 = \sum_{v \in V} |\nabla f(u, v)|^2 = \sum_{v \sim u} |\nabla f(u, v)|^2.$$

The scalar products $\langle f, g \rangle = \sum_{u \in V} f(u)g(v)$ and $\langle F, G \rangle = \sum_{e \in E} F(e)G(e)$ between two real
functions $f$ and $g$ in the Hilbert space $\ell^2(V) = \{ h : \sum_{u \in V} |h(u)|^2 < \infty \}$ and two vector fields $F$ and $G$ in $\ell^2(E) = \{ H : \sum_{e \in E} |H(e)|^2 < \infty \}$, defined then the adjoint of the gradient, i.e. the graph
Since \( Y \), we get \( D \) with \( \Delta \)

Using matrix notations, this operator corresponds actually to the common (symmetric) normalized Laplacian defined on graph, i.e.

\[
\Delta = D^{-1/2}WD^{-1/2} - \text{Id},
\]

with \( D \in \mathbb{R}^{N \times N} \) (\( N = \#V \)) a diagonal matrix such that \( D_{uu} = d(u) \) and \( W \in \mathbb{R}^{N \times N} \) the weight matrix with \( W_{uv} = w(u, v) \).

Interestingly, the graph Laplacian converges to the continuous Laplace-Beltrami operator \([4]\) on the manifold underlying the graph definition.

**Lemma 1.** If the vertex set \( V = \{v_1, \cdots, v_N\} \subset \mathcal{M} \) consists of \( N \) points taken uniformly and independently at random on a \( m \)-dimensional compact manifold \( \mathcal{M} \) embedded in \( \mathbb{R}^n \), then, for any smooth function \( f : V \to \mathbb{R} \) and for a weighting function \( w(u, v) = \exp\{-\|u - v\|^2/2\epsilon\} \), the normalized graph Laplacian defined by the graph \( G = (V, E = V \times V, w) \) satisfies

\[
\frac{1}{\epsilon} \lim_{N \to \infty} \Delta f(u) = \frac{1}{2} \Delta_M f(u) + O(\epsilon^{1/2}),
\]

with \( \Delta_M \) the Laplace-Beltrami operator defined on \( \mathcal{M} \).

**Proof.** We follow a similar development to the one given in \([5]\). With the hypothesis of the Lemma, for any function \( h \in \ell^2(V) \), we have

\[
[Wh](u) = \sum_j w(u, v_j) h(v_j) = \sum_{j \neq i} \exp\{-\|u - v_j\|^2/2\epsilon\} h(v_j) + h(u).
\]

Since \( Y_j = w(u, v_j) h(v_j) \) are iid for \( j \neq i \), by the law of large numbers we have

\[
\sum_{j \neq i} w(u, v_j) h(v_j) \simeq (N - 1) \mathbb{E}_M \left[ \exp\{-\|u - \cdot\|^2/2\epsilon\} h(\cdot) \right] = \frac{N-1}{\text{vol}_M} \int_\mathcal{M} \exp\{-\|u - y\|^2/2\epsilon\} h(y) \, d_M y,
\]

where the integral is performed on the manifold with the local infinitesimal volume element \( d_M y \).

The relation (2.9) in \([5]\) (or Eq. (10) in \([6]\)) explains that

\[
\frac{1}{(2\pi\epsilon)^{m/2}} \int_\mathcal{M} \exp\{-\|u - y\|^2/2\epsilon\} h(y) \, d_M y = h(u) + \frac{\epsilon}{2} [E(u) h(u) + \Delta_M h(u)] + O(\epsilon^{3/2}),
\]

where \( E(u) = \frac{1}{\mathcal{M}} S(u) \) and \( S \) is the scalar curvature of \( \mathcal{M} \) \([4]\). Therefore,

\[
[Wh](u) = \frac{(N-1)(2\pi\epsilon)^{m/2}}{\text{vol}_M} \left[ h(u) + \frac{\epsilon}{2} [E(u) h(u) + \Delta_M h(u)] + O(\epsilon^{3/2}) \right] + h(u) = \frac{(N-1)(2\pi\epsilon)^{m/2}}{\text{vol}_M} \left[ h(u) + \frac{\epsilon}{2} [E(u) h(u) + \Delta_M h(u)] + O(\epsilon^{3/2}, 1/(N\epsilon^{m/2})) \right],
\]

(1)
where the notation $O(\mu, \nu)$ means $|O(\mu, \nu)| < C\mu + D\nu$ for two positive values $\mu, \nu \ll 1$. Taking $h = 1$, we get then

$$d(v_j) = \frac{(N-1)(2\pi)^{m/2}}{\text{vol} M}[1 + \frac{1}{2} E(v_j) + O(\epsilon^{3/2}, 1/(N\epsilon^{m/2}))].$$

(2)

Therefore,

$$\Delta f(u) = \sum_j \frac{w(u,v_j)}{\sqrt{d(u)d(v_j)}} f(v_j) - f(u)$$

$$= \frac{1}{\sqrt{1 + \frac{1}{2} E(u)}} \frac{\text{vol} M}{(N-1)(2\pi)^{m/2}} \sum_{j \neq i} w(u,v_j) \frac{f(v_j)}{\sqrt{1 + \frac{1}{2} E(v_j)}} - f(u) + O(\epsilon^{3/2}, 1/(N\epsilon^{m/2})).$$

Taking now $h(t) = f(t)g(t)$ for $t \in M$ with $g(t) = \frac{1}{\sqrt{1 + \frac{1}{2} E(t)}}$ in (1), we get

$$\frac{\text{vol} M}{(N-1)(2\pi)^{m/2}} \sum_{j \neq i} w(u,v_j) f(v_j) g(v_j)$$

$$= [g(u)f(u)(1 + \frac{1}{2} E(u)) + \frac{1}{2} \Delta_M h(u) + O(\epsilon^{3/2}, 1/(N\epsilon^{m/2}))].$$

However,

$$\Delta_M h(u) = f(u)\Delta_M g(u) + 2 \langle \nabla_M f(u), \nabla_M g(u) \rangle_{T_u M} + g(u) \Delta_M f(u),$$

where the scalar product $\langle \cdot, \cdot \rangle_{T_u M}$ occurs in the tangent plane $T_u M \simeq \mathbb{R}^d$ of $M$ on $u$ with the metric of the manifold $[\mathcal{M}]$. For any function $s$ on $M$, the gradient $\nabla_M s(u) \in T_u M$ is composed of the derivative of $s$ according to a local system of coordinates (or chart) isomorphic to $\mathbb{R}^d$.

From the definition of $g$, it is clear that any differential operator $D_M$ of $g$ with respect to this local system in $T_u M$ is of order $D_M g(u) = O(\epsilon)$. Consequently,

$$\frac{1}{2} \Delta_M h(u) = \frac{1}{2} g(u) \Delta_M f(u) + O(\epsilon^2),$$

that provides the final result,

$$\Delta f(u) = g^2(u)f(u)(1 + \frac{1}{2} E(u)) + \frac{1}{2} \Delta_M f(u) - f(u) + O(\epsilon^{3/2}, 1/(N\epsilon^{m/2}))$$

$$= \frac{1}{2} \Delta_M f(u) + O(\epsilon^{3/2}, 1/(N\epsilon^{m/2})).$$

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