On the use of running $\alpha_s$ in calculations of radiative energy loss of fast partons in a quark-gluon plasma

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The incorporation of running $\alpha_s$ for the gluon emission vertex in calculations of radiative parton energy loss in a quark-gluon plasma is discussed. It is argued that the virtuality scale for running $\alpha_s$ for induced gluon emission is determined by the square of the transverse momentum of the emitted gluon rather than by the square of the invariant mass of the final two-parton state often used in the literature.

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An important feature of quantum chromodynamics (QCD) is the decrease of the effective coupling constant $\alpha_s = g^2/4\pi$ with increase of the particle virtualities. This property plays a significant role in the dynamics of parton showers in jet production in hard processes. As is known, in the leading logarithmic approximation parton cascade allows a probabilistic description in terms of successive $a \to bc$ decays $\llbracket 1 \rrbracket$ with the distribution in Feynman’s variable $x = E_b/E_a$ and the transverse momentum $k_\perp$ of parton $b$ given by the expression

$$dw = \frac{dk_\perp^2}{k_\perp^2} \frac{\alpha_s(k_\perp^2)}{4\pi} P_{bc}(x)dx , \quad (1)$$

where $P_{bc}(x)$ is the splitting function for the $a \to bc$ transition in the DGLAP equation. The use of Eq. $\llbracket 1 \rrbracket$ in Monte Carlo generators, e.g., PYTHIA $\llbracket 2 \rrbracket$ (together with the condition of angular ordering for a soft region $x < 1 \llbracket 3,4 \rrbracket$) makes it possible to describe a huge amount of data on the jet physics. The emission of gluons is the dominant process in evolution of jets in the soft region. It is important that the virtuality in the argument of running $\alpha_s$ is determined only by the transverse momentum of the emitted gluon, and is independent of the longitudinal variable $x \llbracket 3,4 \rrbracket$. For this reason, the emission of soft gluons at $x \ll 1$ is independent of the energy of parton $a$; i.e., the situation in QCD is similar to the emission of soft photons in the Low theorem in QED $\llbracket 5 \rrbracket$. This property would be violated, e.g., when the square of the invariant mass of the $bc$ system, $M_{bc}^2 = k_\perp^2/\Delta(x(1-x))$, is used for the characteristic virtuality in running $\alpha_s$. The expression $\alpha_s(k_\perp^2)$ in Eq. $\llbracket 1 \rrbracket$ for jets in vacuum can be obtained within the diagrammatic technique in the momentum representation after summation over the masses of states to which the emitted gluon can transit $\llbracket 3 \rrbracket$.

The problem of the choice of the running charge argument for the gluon emission becomes more complicated in the case of a parton shower in the medium. Such a situation occurs in the case of jet production in collisions of heavy nuclei at RHIC and LHC energies when a hot quark-gluon plasma (QGP) is produced in the initial stage of an $AA$ collision at proper time $\tau_0 \sim 0.5 - 1$ fm. The development of the parton cascade at times

$$\tau \sim \tau_0 \div L_{QGP} \quad (\text{here, } L_{QGP} \sim (1-2)R_A \text{ is the size of the QGP, where } R_A \text{ is the nucleus radius}) \quad \text{occurs in the QGP. The energy losses of fast partons in the QGP result in jet quenching, which is, in particular, manifested in the strong suppression in } AA \text{ collisions of spectra of particles with high } p_T \text{ (which is characterized by the nuclear modification factor } R_{AA} \text{) observed at the RHIC and LHC. The energy losses in the QGP for RHIC and LHC conditions are due primarily to the induced gluon emission caused by multiple scattering of fast partons in the medium} \llbracket 7,12 \rrbracket$.

In contrast to the shower cascade in vacuum, it is reasonable to analyze radiation loss in the medium in the coordinate representation in the noncovariant perturbation theory because the description in terms of usual Feynman diagrams in the momentum representation is impossible because of a huge number of diagrams. In this case, all fast particles between collisions with constituents of the medium are described by plane waves on the mass shell, and are not characterized by virtualities as in the Feynman diagram formalism. The virtuality of particles can be qualitatively determined using the uncertainty relation $\Delta p \Delta L \sim 1$ from the size of the spatial region $\Delta L$ filled by a plane wave between decays of particles or their rescatterings on particles of the medium (which are usually modeled by the static Debye-screened color centers). At present, jet quenching is usually analyzed using formulas for the single-gluon spectrum under the assumption of independent gluon emission for multigluon processes $\llbracket 13 \rrbracket$. The single-gluon spectrum for massive partons for arbitrary magnitude of the Landau-Pomeranchuk-Migdal effect can be obtained within the light cone path integral (LCPI) approach $\llbracket 8,14,15 \rrbracket$. The calculation of the single-gluon spectrum with running $\alpha_s$ involves the problem of the choice of the running coupling constant in the decay vertex. The effect of the running coupling constant is particularly strong for LHC energies where the jet energy range is much wider than that for RHIC experiments.

The parton cascade in medium is not ordered in the parton virtualities, as in the case of the Feynman diagrams corresponding to cascading of partons in vacuum $\llbracket 1 \rrbracket$. Consequently, the reasoning used in $\llbracket 3 \rrbracket$ to determine the virtuality scale for running $\alpha_s$ is inapplicable. For the induced gluon emission in the case of not too strong
the Landau-Pomeranchuk-Migdal effect, squares of the transverse momenta of gluons at $L_f \lesssim L_{QGP}$ (where $L_f \sim 2\omega/m_g^2$ is the coherence length for emission of a gluon with the energy $\omega$) are concentrated in a region up to several units of $m_g^2$, where $m_g$ is the gluon quasiparticle mass in the QGP ($m_g \sim 400$ for RHIC and LHC conditions). This estimate corresponds to the diffusion relation for the typical transverse distance passed by a parton in the $\rho$ plane on the longitudinal length $L$ [8]s:

$$\rho \sim \sqrt{L/\omega}.$$  

(2)

At $L = L_f$, this gives $k_{\perp} \sim 1/\rho \sim m_g$. However, when $L_f$ becomes much larger than the QGP size, Eq. (2) includes $L_{QGP}$ rather than $L_f$. In this regime, $k_{\perp}^2 \sim m_g^2(L_f/L_{QGP})$, which can lead to significant virtualities in running $\alpha_s$ (particularly for a small QGP produced in $pp$ and $pA$ collisions [18, 19]).

Two methods are currently in use for incorporating the running coupling in decay vertices when calculating radiative energy losses. The first method involves $\alpha_s(k_{\perp}^2)$, whereas the second method includes $\alpha_s(k_{\perp}^2/x(1-x))$, which corresponds to the squared invariant mass of the final two-parton state. The first method was used in analyses [18, 20, 22], which were based on the LCPI formalism [8, 14, 15], and in [17, 24], which were based on the generalization of the AMY formalism [12] to the case of a finite-size QGP. The second method was used in the well known CUJET model [25, 28] based on the GLV formalism [10] for a thin medium. The squared invariant mass of the two-parton state (including the parton masses) for $Q^2$ in the running charge within the GLV formalism was also used in [29, 32]. The use of $\alpha_s(k_{\perp}^2/x(1-x))$ gives a steeper increase of the nuclear modification factor with $p_T$, which is due to suppression of the induced gluon emission with increasing energy of the initial parton (because of a decrease in $x$ at a fixed energy of the gluon $\omega$ and, correspondingly, an increase of the squared invariant mass $k_{\perp}^2/x(1-x)$). This receipt gives better agreement with the LHC data on $R_{AA}$, that show a steep increase of $R_{AA}$ with the hadron transverse momentum. In this work, we show that there are simple physical arguments against the use of $\alpha_s(k_{\perp}^2/x(1-x))$ in calculations of the radiative energy losses of fast partons.

For definiteness, we consider the induced gluon emission for the $q \to gq$ process. It is assumed that a fast quark is produced at $z = 0$ (the $z$ axis is chosen along the momentum of the initial quark), and passes through a slab of the medium of thickness $L$, which simulates the interaction in the finite state for a jet produced in $AA$ collisions (where $L \sim L_{QGP}$). The medium is described as a system of color centers. We consider first the case of a quite thin dilute medium, when it is enough to account for only single rescattering of the fast partons on one of the color centers. The $q \to gq$ process with allowance for only single interactions with constituents of the medium is described by the diagrams shown in Fig. 1. In this description, each fast particle is described by a plane wave with a certain transverse momentum, which changes sharply at rescattering on a color center with exchange by one or two $t$-channel gluons [33]. These diagrams for arbitrary $x = \omega/E_q$ were calculated in [34]. The square of the sum of diagrams shown in Fig. 1 after integration over transverse momenta gives the gluon spectrum in the Feynman variable $x$ in the form of the sum

$$\frac{dP}{dx} = \frac{dP_{vac}}{dx} + \frac{dP_{in}}{dx}.$$  

(3)

Here, the first term is the contribution from the usual vacuum decay $q \to gq$ (diagram a in Fig. 1), and the second term (square of the sum of the sum of the diagrams $b, c, d$ in Fig. 1 and the interference terms of the diagram $a$ and the sum of the diagrams $e, f, g, h$ in Fig. 1) corresponds to the $q \to gq$ transition induced by the interaction with a scattering center. The induced gluon spectrum corresponding to the diagrams in Fig. 1 can be represented in a compact form within the LCPI approach [8]. It corresponds to the leading in the medium density contribution to the total spectrum for arbitrary number of rescatter-
ings in the LCPI method, and can be written in the form

\[
\frac{dP_{in}}{dx} = \int_{0}^{L} dz \, n(z) \frac{d\sigma_{eff}^{BH}(x, z)}{dx},
\]

(4)

where \( n(z) \) is the particle number density, and \( d\sigma_{eff}^{BH}/dx \) is the effective Bethe-Heitler cross section including the effect of finite size of the medium. In the LCPI formalism, this cross section can be represented by the single diagram shown Fig. 2, where ellipsis is the cross section for the interaction of the color-singlet \( q\bar{q}g \) system with a color center, \( \sigma_3 \), and sets of three lines on the right and left correspond to the Green’s function for the Hamiltonian

\[
H = \frac{q^{2} + \epsilon^{2}}{2M},
\]

(5)

where \( M = E_{x}(1 - x) \), and \( \epsilon^{2} = m_{q}^{2}x^{2} + m_{g}^{2}(1 - x) \) (generally, for the \( a \rightarrow be \) transition, \( \epsilon^{2} = m_{q}^{2}(1 - x) + m_{x}^{2}x - m_{g}^{2}x(1 - x) \)). The Hamiltonian (5) describes evolution in \( z \) coordinate of the wavefunction of the \( q\bar{q}g \) pair in the \( \rho \) plane (in this case, the antiquark in the \( q\bar{q}g \) system is located at the center of mass of the \( q\bar{q}g \) pair [8]). Representing integrals with respect to \( z_{1,2} \) in Fig. 2 in terms of the light cone wavefunctions in the \( \rho \) representation, one can represent \( \sigma_{eff}^{BH}(x, z) \) in the form

\[
\frac{d\sigma_{eff}^{BH}}{dx} = \frac{1}{2} \sum_{\{\lambda\}} \text{Re} \int d\rho \, \Psi_{\{\lambda\}}^{*}(\rho, x) \sigma_{3}(\rho, x) \Psi_{\{\lambda\}}^{\mu_{3}}(\rho, x, z),
\]

(6)

where \( \{\lambda\} \) is the set of parton helicities, \( \Psi_{\{\lambda\}}(\rho, x) \) is the usual light cone wavefunction for the \( q \rightarrow q\bar{g} \) transition, and \( \Psi_{\{\lambda\}}^{\mu_{3}}(\rho, x, z) \) is the light cone wavefunction modified by the effect of finite size of the region of the longitudinal coordinate \( z_{1} \) for the gluon emission vertices \( 0 < z_{1} < z \).

The three-body cross section \( \sigma_{3} \) can be expressed in terms of the well known dipole cross section for a color-singlet \( q\bar{q} \) pair [8], which is given by the formula

\[
\sigma_{q\bar{q}}(\rho, z) = C_{F} C_{F} \int dq \, q(5g_{s}^{2}) \frac{[1 - \exp(iq\rho)]}{[q^{2} + \mu_{D}^{2}(z)]^{2}},
\]

(7)

where \( C_{F} \) are the color quadratic Casimir operator for the quark and thermal parton (quark or gluon), and \( \mu_{D}(z) \) is the local Debye mass 1. The three-body cross section is represented in terms of the dipole cross section (7) as

\[
\sigma_{3}(\rho, x, z) = \frac{9}{8} [\sigma_{q\bar{q}}(\rho, z) + \sigma_{q\bar{q}}(1 - x) \rho, z] - \frac{1}{8} \sigma_{q\bar{q}}(x, \rho, z),
\]

(8)

1 Expression (7) is written in the form allowing for the use of running \( \alpha_{s} \). The square of the momentum through the gluon is a natural virtuality scale in the case of the \( t \)-channel gluon exchanges.

To discuss the running coupling constant at the \( q \rightarrow gq \) vertex, it is convenient to represent Eq. (6) in the momentum representation:

\[
\frac{d\sigma_{eff}^{BH}}{dx} = \frac{1}{2(2\pi)^{2}} \sum_{\{\lambda\}} \text{Re} \int d\mathbf{k}_{1} d\mathbf{k}_{2} \Psi_{\{\lambda\}}^{*}(\mathbf{k}_{2}, x) \times \sigma_{3}(\mathbf{q}, x) \Psi_{\{\lambda\}}(\mathbf{k}_{2}, x, z),
\]

(9)

where \( \mathbf{q} = \mathbf{k}_{1} - \mathbf{k}_{2} \). The wavefunctions for fixed \( \alpha_{s} \) are given by the formulas

\[
\Psi_{\{\lambda\}}(\mathbf{k}, x) = \sqrt{\alpha_{s}} \frac{(k_{x} - i \lambda_{3} k_{y})[2 - x + 2ix\lambda_{3} \lambda_{y}]}{\sqrt{2x(k^{2} + \epsilon^{2})}},
\]

(10)

\[
\Psi_{\{\lambda\}}^{\mu_{3}}(\mathbf{k}, x, z) = F(\mathbf{k}, x, z) \Psi_{\{\lambda\}}(\mathbf{k}, x),
\]

(11)

where

\[
F(\mathbf{k}, x, z) = 1 - \exp \left[ i \left( \frac{2}{2M} \right) \frac{z^{2}}{z^{2}} \right].
\]

(12)

The running charge is introduced by changing in Eq. (10) the fixed \( \alpha_{s} \) to the running one. This exactly corresponds to the use of running \( \alpha_{s} \) for the \( q \rightarrow gq \) decay vertices in the diagrams shown in Fig. 1. It is important that Eqs. (6) and (9) include not only the square of diagrams with one-gluon exchanges in Fig. 1 but also the interference between the vacuum diagram \( a \) in Fig. 1 and the diagrams with rescattering of fast partons on the color center because of two-gluon exchanges (diagrams \( e, f, g, h \) in Fig. 1). Only with this interference terms can all \( t \)-channel exchanges be summed in the form of the cross section \( \sigma_{3} \), and the formula for the effective Bethe-Heitler cross section can be obtained in the simple factorized form (6). To keep the form of Eqs. (6) and (9) at their generalization to the case of the running charge, running \( \alpha_{s} \), coinciding with \( \alpha_{s} \) in the vacuum diagram in Fig. 1, i.e., \( \alpha_{s}(k_{g}^{2}) \) (if it is accepted that the vacuum cascade includes this form of running \( \alpha_{s} \) [3]), should be used for diagrams with rescatterings in Fig. 1. The requirement for keeping the factorized form (6) is important because vanishing of \( \sigma_{3}(\rho, x, z) \) at \( \rho \rightarrow 0 \) (related to the property of the color transparency of the dipole cross section (7), i.e., vanishing of \( \sigma_{q\bar{q}}(\rho, z) \) at \( \rho \rightarrow 0 \)) ensures the convergence of the \( \rho \)-integral in Eq. (10). Form (6) implies the absence of the interaction with the medium for a point color-singlet \( q\bar{q}g \) state. The induced transition physically appears because \( t \)-channel gluons can distinguish, for a nonzero \( \rho \), the two-body virtual state \( gq \) and the initial quark because of the color dipole moment of the \( q\bar{q}g \) pair. For this reason, at phenomenological generalization of the formulas for the induced spectrum to the case of the running charge, it is reasonable to require keeping of Eq. (9) having the property of color transparency.

The above reasons in favor of the charge \( \alpha_{s}(k_{g}^{2}) \) in the decay vertex are applicable when \( L_{f} \) is not too small as
compared to the size of the medium, and the contribution of interference with the vacuum amplitude is significant. When $L_{QGP}$ is large, the contribution of color centers with $z > L_f$ dominates, and the contribution to the induced spectrum from the interference of the vacuum diagram with diagrams $e, f, g, h$ in Fig. 1 should obviously be small. In this regime, the induced gluon emission occurs in much the same way as for the case of a quark incident on the scattering center from infinity, and the effective Bethe-Heitler cross section becomes close to the usual QCD Bethe-Heitler cross section. However, the use of $\alpha_s(k^2_f)$ for this regime also seems reasonable. Indeed, the situation with $z > L_f$ is similar to the emission of gluons in the dipole form $[33, 40]$ of the BFKL equation $[41, 42]$. The analysis of radiative corrections to the BFKL equation shows that the virtuality scale for the running charge is determined only by the transverse momentum $[43]$ (in agreement with the direct calculations in the coordinate representation $[44]$ where $k_\perp \sim 1/\rho$). In earlier works $[39, 40, 45]$, the running coupling constant was phenomenologically incorporated in the BFKL equation also by using $\alpha_s(k^2_f)$ (with $k^2_f \sim 1/\rho^2$ for the dipole approach $[40]$). With such a running charge, a good description of the HERA data on the proton structure function $F_2(x, Q^2)$ at small $x$ was obtained in $[39]$ within the dipole BFKL equation. The replacement of $\alpha_s(k^2_f)$ by $\alpha_s(k^2_f/x(1-x))$ would obviously lead to a significant change in the $x$ dependence of the proton structure function. These facts indicate that $\alpha_s(k^2_f)$ rather than $\alpha_s(k^2_f/x(1-x))$ should be used for the regime $z > L_f$. It is noteworthy that the contribution of gluons with $L_f \ll L_{QGP}$ at moderate energies is very significant. In the oscillatory approximation $\sigma_{qg}(\rho) \propto \rho^2$, it gives the dominant contribution to the radiative parton energy loss $[2, 31, 37]$.

The above consideration concerns the case of a low density of the medium when the Landau-Pomeranchuk-Migdal effect is small. In the case of a dense medium, the modified wavefunction $\Psi^m_{\lambda}(x, \rho, z)$ in Eq. (6) should be calculated including any number of rescatterings, which can be taken into account by adding the imaginary potential $v = -i\hbar(z)\sigma_3(\rho, z)/2$ to the Hamiltonian $[34]$. However, this does not change the situation for the first decay vertex $g \rightarrow gg$, where it is necessary to use $\alpha_s(k^2_f)$. In this case, in calculations in the coordinate representation with the use of the diffusion relation $[2]$, the typical value of $k^2_f$ in $\alpha_s$ can be related to the typical longitudinal distance dominating in the representation of $\Psi^m_{\lambda}(\rho, x, z)$ in terms of the integral of the Green’s function over $z$ (details can be found in $[46]$). It is worth noting that thermal effects in the dense QGP can result in the modification of $\alpha_s$ at virtualities of about several temperatures of the QGP (this phenomenologically corresponds to dependence of $\alpha_s$ on $z$). A parameterization of $\alpha_s(Q^2)$ with freezing in the region of small $Q^2$ at a certain value $\alpha_s^F$ is widely used in current works on jet quenching. For processes in vacuum, $\alpha_s^F \approx 0.7 - 0.8$ $[29, 47]$. The analyses of the RHIC and LHC data on the nuclear modification factor $R_{AA}$ $[18, 20, 22]$ show that the QGP can reduce $\alpha_s^F$ to $\sim 0.4 - 0.6$. However, this does not change the behavior of the running charge in the region of large $k^2_f$, where the running charges for the two virtuality scales $k^2_f$ and $k^2_f/x(1-x)$ are still significantly different.

The quantitative difference in the induced gluon spectrum between two virtuality scales for $\alpha_s$ at the decay vertex appears quite noticeable. In order to quantitatively estimate the difference between these two variants, we have calculated the gluon spectrum for the QGP with the initial temperature $T = 400$ MeV at the proper time $\tau_0 = 0.5$ fm (which approximately corresponds to central Pb + Pb collisions at $\sqrt{s} = 2.76$ TeV at the LHC) for the purely longitudinal expansion of the QGP within the 1+1D Bjorken model. Our calculations within the LCPI approach (including all possible rescatterings) show that, in the gluon energy range $\omega \sim 1 - 3$ GeV (dominating in jet quenching) at an increase in the energy of the initial parton from 20 to 150 GeV, the spectrum in the variants with $\alpha_s(k^2_f)$ and $\alpha_s(k^2_f/x(1-x))$ increases by $3 - 10\%$ and decreases by $23 - 30\%$, respectively. Because of this difference in the behavior of the induced spectrum with increase of the initial parton energy, the nuclear modification factor $R_{AA}(p_T)$ for the variant with $\alpha_s(k^2_f/x(1-x))$ grows with $p_T$ more rapidly, which gives a better agreement with the LHC data on $R_{AA}$ . However, our analysis shows that the variant with $\alpha_s(k^2_f/x(1-x))$ contradicts to well established facts, and, hence, is theoretically unsatisfactory.

The method of incorporation of the running charge by changing the fixed $\alpha_s$ in Eqs. (10), (11) to running $\alpha_s(k^2_f)$ is also not absolutely strict. The point is that the possibility of separating the contribution associated with evolution of $\alpha_s$ in higher radiative corrections to Eqs. (6) and (9) keeping the form of these formulas is not obvious. Indeed, the inclusion of rescattering of virtual s-channel gluons can result in the appearance of interaction with the medium of the four-body parton states, which lead to a modification of the evolution of $\alpha_s$ by the medium effects. Nevertheless, under the assumption that such effects are small, and Eqs. (6), (9) provide a good approximation for the running charge as well, the above facts indicate that $\alpha_s(k^2_f)$ is preferable.

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References
