Convex Hulls of Varieties and Entanglement Measures Based on the Roof Construction

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Abstract

In this paper we study the problem of calculating the convex hull of certain affine algebraic varieties. As we explain, the motivation for considering this problem is that certain pure-state measures of quantum entanglement, which we call polynomial entanglement measures, can be represented as affine algebraic varieties. We consider the evaluation of certain mixed-state extensions of these polynomial entanglement measures, namely convex and concave roofs. We show that the evaluation of a roof-based mixed-state extension is equivalent to calculating a hyperplane which is multiply tangent to the variety in a number of places equal to the number of terms in an optimal decomposition for the measure. In this way we provide an implicit representation of optimal decompositions for mixed-state entanglement measures based on the roof construction.

1 Introduction

The burgeoning field of quantum information science is beginning to provide us with hints that quantum systems are capable of performing spectacularly powerful information processing tasks. An example of this is the existence of
a quantum algorithm, Shor’s algorithm, which can factor in polynomial time [Sho94, Sho97]. A consequence of discoveries such as Shor’s factoring algorithm is the emergence of a widespread belief that, parallel to the physical resource of information, there is a corresponding resource which quantum systems support called quantum information. The study of how quantum information can be manipulated and processed is the central goal of quantum information science.

So what is quantum information? It is folklore within the quantum information community that it may be quantum entanglement, the unique property possessed by quantum states of bipartite systems not expressible as simple tensor products. A classic example of an entangled state is the singlet $|\Psi^\rightarrow\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, which is commonly considered to be the fundamental unit of entanglement. There are a number of fascinating information processing tasks which use entangled states as a resource, such as teleportation [BBC+93] and enhanced classical communication [NC00].

Throughout the progress of the research conducted in quantum information science, in particular in the study of quantum entanglement, a number of fascinating mathematical problems have been encountered. Within the theory of quantum entanglement and quantum communication these problems can be broadly described as the classification of positive operators on tensor products of vector spaces. This classification problem is central to these theories. Many of the stumbling blocks currently encountered by researchers in quantum information science are due to incomplete results in this classification program.

Despite the many problems encountered, a fruitful interplay between mathematics and physics has already resulted from progress in this classification program. An example of this is the recent work on indecomposable positive linear maps. In this case a result obtained previously by Woronowicz [Wor76] was applied by Paweł Horodecki to construct quantum states which have the property that they are bound entangled [Hor97] (for a review of bound entanglement and distillable entanglement see the paper [HH01]). The subsequent work on bound entanglement was then fed back into mathematics by Terhal [Ter01], who used this work in physics to construct new indecomposable positive linear maps.

Bound entanglement is a particular nonlocal phenomena exhibited by certain mixed states which falls within the framework of mixed-state entanglement, or the study of nonlocal quantum features of mixed states. Understanding mixed-state entanglement is an immediate and pressing priority in quantum information science. This is because a number of expected developments in areas such as quantum communication cannot take place until certain questions about mixed-state entanglement are answered.

Quite apart from the ramifications for quantum information, we believe that the study of mixed-state entanglement will lead to new and interesting results in mathematics. In particular, there are suggestive indications (see, for example, the work on concurrence for some striking results [HW97, Woo98, Uhl00, AWDM01, Osb02]) that there are deep results waiting to be found in the field.

1 Mixed states are probabilistic mixtures of quantum states.
of matrix analysis.

The purpose of this paper is to provide a method to evaluate certain functions on the space of positive operators, called mixed-state entanglement measures. To do so, in §2 we begin by establishing some results in convexity theory. These results allow us to state our main result, a necessary condition for constructing convex hulls of certain subsets. In §3 we make the observation that quantum state-space can be represented by the convex hull of a quadratic algebraic variety. Finally, in §4 we combine these two results to provide an implicit representation for a family of mixed-state entanglement measures, namely those based on the roof construction.

2 Preliminaries

In this section we introduce the main object of study for this paper: the convex or concave roof. We also establish some general results concerning the construction and evaluation of roofs, the most important result being Proposition 2.6. We illustrate the definitions and results throughout this section in terms of a simplified example.

2.1 Properties of Convex and Concave Roofs

We begin by establishing some notation. Let $V$ denote the subset of $\mathbb{R}^n$ formed from the intersection of the zero-sets of a collection of polynomials $\ell_0, \ldots, \ell_a \in \mathbb{R}[x_1, \ldots, x_n]$, which we write as

$$V = \mathcal{Z}(\ell_0, \ldots, \ell_a).$$

The set $V$ is known as an affine algebraic variety. We assume that this set is compact in the standard topology on $\mathbb{R}^n$, irreducible, and nonsingular, i.e., $V$ is a nonsingular variety. (For a gentle introduction to varieties and algebraic geometry, including definitions of the terms we use here, see [CLO97] and [Har95]. For a more complete reference see [Har77].) We have assumed that $V$ is irreducible and nonsingular in order to avoid various pathologies associated with the definition of the tangent space for these spaces. It should be noted that it is possible, with a little extra work, to extend the results of this section to include these cases. However, for our applications the assumption of nonsingularity imposes no limitations, so we ignore singular varieties.

Let $f$ be an arbitrary continuous scalar-valued function defined on $V$. (The assumption of continuity is invoked so that subsequent constructions are at least continuous.) It suffices for most of our applications to take this function to be a polynomial, in which case $f$ is an element of the coordinate ring of $V$, $f \in \mathbb{R}[V]$. An example variety $V$ and function $f$ is shown in Figure 1. The objective of this paper is to construct and evaluate a scalar-valued function $g$ which is equal to $f$ on $V$, but which is defined over all of $\text{conv } V$, the convex hull of $V$. Obviously there are many ways to define such functions. To proceed, we single out two
special possibilities for $g$, defined by the formulas:

$$(\text{conv } f)(r) \triangleq \inf \left\{ \sum_{j=1}^{m} p_j f(x_j) \mid x_j \in V, \sum_{j=1}^{m} p_j x_j = r \right\}$$ (2)

and

$$(\text{conc } f)(r) \triangleq \sup \left\{ \sum_{j=1}^{m} p_j f(x_j) \mid x_j \in V, \sum_{j=1}^{m} p_j x_j = r \right\},$$ (3)

where $r$ is an arbitrary point in conv $V$, $p_j \in \mathbb{R}$ is a probability distribution, and $m$ is an arbitrary positive integer. A simple consequence of Carathéodory’s Theorem is that $m$ is bounded above by the dimension of $V$, $m \leq v = \dim V$ [Roc70].

**Definition 2.1** ([Unh98]). The functions conv $f$ and conc $f$ defined by (2) and (3) are called the **convex roof** and **concave roof** of $f$, respectively. We refer to the procedure of constructing conv $f$ or conc $f$ as **roofing** $f$.

One of our main objectives is to evaluate conv $f$ or conc $f$ at an arbitrary point $r \in \text{conv } V$. Obviously this can be done by simply carrying out the optimisation in (2) and (3). Unfortunately, evaluating conv $f$ at a point $r \in \text{conv } V$ according to (2) and (3) is, in general, very difficult. For this reason, in this section, we provide an alternative way to evaluate these functions.
Figure 2: The three dimensional convex hull \( \text{conv}(\text{gr} f) \) of the graph of \( f(x, y) = x^3 \) on \( Z(x^2 + y^2 - 1) \). Note that this is a three-dimensional convex set. The upper boundary is shaded red through green and the lower boundary is shaded blue.

We shall spend a lot of time discussing properties of the graph of \( f \) on \( V \), denoted \( \text{gr} f \), which is the subset of \( \mathbb{R}^{n+1} \) defined by

\[
\text{gr} f \triangleq \{ (r, f(r)) | r \in V \}. \tag{4}
\]

Associated with \( \text{gr} f \) is the convex hull of the graph of \( f \), which is the subset of \( \mathbb{R}^{n+1} \) given by

\[
\text{conv}(\text{gr} f) \triangleq \left\{ (x, \mu) \left| \sum_{j=1}^{v'+1} p_j(x_j, f(x_j)) \right. \right\}, \tag{5}
\]

where \( x_j \in \mathbb{R}^n \), \( p_j \) is a probability distribution, and we have invoked Carathéodory’s Theorem to bound the number of terms in the sum by \( v' + 1 = \dim(\text{conv} V) + 1 \).

See Figure 2 for an example of what \( \text{conv}(\text{gr} f) \) looks like for the variety \( V \) and function \( f \) introduced in Figure 1.

The connection between \( \text{conv}(\text{gr} f) \) and \( \text{conv} f \) and \( \text{conc} f \) is provided by the following lemma.

**Lemma 2.2.** The convex and concave roofs of \( f \) are given by

\[
(\text{conv} f)(r) = \inf \{ \mu | (r, \mu) \in \text{conv}(\text{gr} f) \} \tag{6}
\]

and

\[
(\text{conc} f)(r) = \sup \{ \mu | (r, \mu) \in \text{conv}(\text{gr} f) \}, \tag{7}
\]

respectively.
Proof. The equality of (6) and (2) (respectively, (7) and (2)) follows immediately from the definition of (5).

Remark 2.3. Lemma 2.2 shows us that the convex roof of \( f \) is the “lower boundary” of the convex hull of the graph of \( f \). Similarly, the concave roof \( \text{conc} f \) is the “upper boundary” of \( \text{conv} \text{gr} f \). This is illustrated for \( \text{conv} f \) in Figure 3.

The convex and concave roofs of \( f \) are distinguished from all other functions \( g \) on \( \text{conv} V \) equal to \( f \) on \( V \) by the following result.

Lemma 2.4 (Lemma A-3, [Uhl98; Uhl03]). The convex roof (respectively, concave roof) of \( f \) is, pointwise, the largest convex (respectively, smallest concave) function equal to \( f \) on \( V \).

We now turn to the discussion of the main result of this section: an alternative way to evaluate \( \text{conv} f \) and \( \text{conc} f \). This result takes the form of a necessary condition which limits the space over which the infimum and supremum in (2) and (3) needs to be taken.

To begin, we make use of the fundamental duality between convex sets and intersections of halfspaces to write \( \text{conv} M \), where \( M \subset \mathbb{R}^n \) is some bounded set, as an intersection in \( \mathbb{R}^n \) of a collection of halfspaces,

\[
\text{conv} M = \bigcap_{H_{(m, \omega)} \in \Omega} H_{(m, \omega)},
\]

Figure 3: Convex roof for the function \( f(x, y) = x^3 \) on \( Z(x^2 + y^2 - 1) \). Note that the graph of \( \text{conv} f \) is the lower boundary of the convex set \( \text{conv} \text{gr} f \).

\[\text{conv f} \]

\[\text{conc f} \]

\[\text{conv(M)} \]

\[\bigcap_{H_{(m, \omega)} \in \Omega} H_{(m, \omega)} \]

\[\text{conv}(M) = \bigcap_{H_{(m, \omega)} \in \Omega} H_{(m, \omega)}, \quad (8)\]
where \( H_{(m,\omega)} = \{ x \in \mathbb{R}^n \mid m \cdot x \leq \omega \} \) is a halfspace, and \( \Omega \) is the set of all halfspaces containing \( M \). Where obvious we abuse notation and write \( H_{(m,\omega)} \in \Omega \) to refer to the hyperplane \( H_{(m,\omega)} = \{ x \in \mathbb{R}^n \mid m \cdot x = \omega \} \) associated with a halfspace \( H_{(m,\omega)} \in \Omega \).

**Definition 2.5.** A hyperplane \( H \) in \( \mathbb{R}^n \) is said to be \( m \)-tangent to a nonsingular variety (or indeed any sufficiently well-behaved manifold) \( M \subset \mathbb{R}^n \) at points \( x_1, \ldots, x_m \in M \) if \( H \) contains both the points \( x_1, \ldots, x_m \in M \) and their associated tangent spaces \( T_{x_j}M \). If \( m = 2 \) or \( 3 \) then \( H \) is said to be bitangent or tritangent to \( M \), respectively.

The set \( \Omega \) contains three types of halfspace. Firstly, there are those halfspaces which completely contain \( M \) and whose associated hyperplane does not contain any point of \( M \). Secondly, there are those halfspaces whose associated hyperplane contains one or more points of \( M \). These halfspaces are said to support \( M \). Finally, of those halfspaces which support \( M \) at some collection of points \( x_1, \ldots, x_m \) there are those whose associated hyperplane is \( p \)-tangent to \( M \) at some subset of \( p \) points of \( \{ x_1, \ldots, x_m \} \) (because of the assumption of nonsingularity, \( M \) has a well-defined tangent space at each point).

One of our main results, for this subsection, is a necessary condition for the hyperplane \( H \) associated with some halfspace \( H \in \Omega \) to support \( M \):

**Proposition 2.6.** Let \( H \in \Omega \) be a hyperplane that supports a nonsingular variety (or manifold) \( M \subset \mathbb{R}^n \) at \( m \) points \( x_1, \ldots, x_m \in M \). The (translate of an) \((n-1)\)-dimensional vector space defined by \( H \) must contain, as subspaces, the vector spaces \( T_{x_j}M \), \( j = 1, \ldots, m \) and the vector space formed from the vectors \( x_j - x_k \), \( \forall j \neq k = 1, \ldots, m \). In particular, if \( H \in \Omega \) supports \( M \) at \( x_1, \ldots, x_m \), it must be \( m \)-tangent to \( M \) at \( x_1, \ldots, x_m \).

**Proof.** A hyperplane \( H \in \Omega \) which supports \( M \) at \( m \) points \( x_1, \ldots, x_m \in M \) contains these points so, in particular, it must contain all the vectors \( x_j - x_k \), \( \forall j \neq k = 1, \ldots, m \). In addition, if \( H \) did not contain \( T_{x_j}M \) as a subspace, for all \( j \), then some part of \( H \) would lie inside the convex hull of \( M \) near some point \( x_j \), contradicting \( H \in \Omega \). The second statement is simply a restatement of the first.

**Remark 2.7.** Proposition 2.6 provides a necessary condition for a hyperplane \( H \in \Omega \) to support \( M \). In general this condition is not sufficient as there exist hyperplanes \( H' \notin \Omega \) which satisfy the condition of Proposition 2.6. The effect of this proposition is to identify a subset of \( \Phi \subset \Omega \) such that the convex hull of \( M \) can still be found by intersecting members of \( \Phi \).

We now specialise the results we have just obtained to describe the structure of \( \text{conv(gr} f) \). As demonstrated by Corollary 2.2, the convex and concave roofs of a function \( f \) are the top and bottom halves of the boundary of \( \text{conv(gr} f) \). It is useful to obtain a characterisation of this boundary. To do so, we make use of the observations above and a lemma of Uhlmann [Uhl98] to describe the nature of these two hypersurfaces.
It turns out that the surfaces in $\mathbb{R}^{n+1}$ defined by either $\text{conv} f$ or $\text{conc} f$ are comprised of a union of a family of polytopes (convex hulls of finite sets of points) which, pairwise, have zero intersection. To see this\(^3\), note that any point $(r, \mu)$ of $\text{conv} f$ may be written as a finite sum $(r, \mu) = \sum_{j=1}^{v+1} \lambda_j(x_j, f(x_j))$. We call this sum an optimal $(v + 1)$-decomposition of $r$ for $\text{conv} f$. (Where the convexity or concavity is obvious we simply refer to this sum as an optimal decomposition.) A lemma of Uhlmann (lemma 3, \cite{Uhl98}) shows us that something stronger actually applies, namely that the polytope defined by the convex hull of the points $(x_1, f(x_1)), \ldots, (x_{v+1}, f(x_{v+1}))$ also lies entirely within the surface defined by $\text{conv} f$. The consequence of this observation is that the surface defined by $\text{conv} f$ (which is one half of the boundary of $\text{conv}(\text{gr} f)$) is made completely from a (pairwise disjoint) union of polytopes. Further, each of these polytopes $P$ is a piece of the hyperplane which supports $\text{gr} f$ at the extreme points $(x_1, f(x_1)), \ldots, (x_{v+1}, f(x_{v+1}))$ (because the polytope lives on the boundary of the convex set $\text{conv}(\text{gr} f)$). The structure of $\text{conv} f$ is illustrated for the example of $f = x^3$ in Figure 3, where, in this case, it may be seen that $\text{conv} f$ is comprised of line segments and one triangle.

For a given $r \in \text{conv} V$, we are interested in determining the polytope $P$ on the boundary of $\text{conv}(\text{gr} f)$ which contains $(r, f(r))$. Equivalently, given $r \in \text{conv} V$ we are interested in determining an optimal $m$-decomposition for $\text{conv} f$ at $r$. One way of approaching this problem is to simply search the entire space of all sequences of points $(x_1, f(x_1)), \ldots, (x_{m+1}, f(x_{m+1}))$ which achieve the infimum in (2) and check to see if their convex hull contains $(r, f(r))$. Proposition 2.6 indicates that this is not necessary because we need only search through sequences of points $(x_1, f(x_1)), \ldots, (x_{m+1}, f(x_{m+1}))$ for which there exists an $m$-tangent hyperplane passing through them. This statement is the content of the following corollary.

**Corollary 2.8.** Let $r \in \text{conv}(V)$. If $\{\lambda_j, x_j\}$ is an optimal $m$-decomposition for $\text{conv} f$ at $r$ then there is a hyperplane $H$ passing through $x_j, \forall j$ which is $m$-tangent at $x_j, \forall j$.

### 2.2 Calculating Optimal Decompositions for Polynomial Roofs

In this subsection we utilise Corollary 2.8 to provide an implicit representation for an optimal decomposition of a point $r \in \text{conv} V$ for $\text{conv} f$. We show that when $f$ is a polynomial an optimal decomposition for $\text{conv} f$ at a point $r \in \text{conv} V$ may be found by simultaneously solving a set of polynomial equations. In principle this enables us to invoke the machinery of algebraic geometry, in particular Gröbner bases methods, to solve the problem \cite{CLO97, CLO98, Har95, Har77}.

\(^3\)From now on we will phrase most of our results in terms of the convex roof of $f$. This does not entail any loss of generality as all statements about $\text{conv} f$ extend in an obvious way to $\text{conc} f$. 
Corollary 2.8 shows that to find an optimal decomposition for \( \text{conv} f \) at a point \( r \in \text{conv} V \) we need only search over points \( x_1, \ldots, x_m \) for which there exists an \( m \)-tangent hyperplane \( H \) passing through these points. If we were able to obtain an expression for the basis for the tangent space \( T_{x_j} V \) at \( x_j \), then we could reduce this problem to establishing the linear dependence of a collection of vectors in the following way.

Consider the tangent space \( T_{x_j} V \) at a point \( x_j \in V \). Suppose we had a formula, in terms of the components of \( x_j \), for each the basis vectors \( \alpha_1^{(j)}, \ldots, \alpha_t^{(j)} \) of \( T_{x_j} V \), where \( t = \dim T_{x_j} V \). Using this formula we could rephrase Proposition 2.6 as the statement that for an \( m \)-tangent hyperplane \( H \) supporting \( V \) at \( x_1, \ldots, x_m \), the matrix \( R \) whose rows are formed from the vectors \( \alpha_1^{(j)}, \ldots, \alpha_t^{(j)} \) and \( x_j - x_k \), \( \forall j \neq 1, \ldots, m \) must have rank less than or equal to \( n-1 \). This condition can be imposed by requiring \( [HJ90] \) that all the \( (n \times n) \)-submatrices of \( R \) have zero determinant. If all the entries of \( R \) were polynomials in \( x_1, \ldots, x_m \), then this condition becomes the requirement that a set of polynomials vanish.

Using the ingredients discussed in the previous paragraph we are now able to express an optimal decomposition for a point \( r \in \text{conv} V \) as the simultaneous solution to a set of polynomials. The key observation is that it is possible to take the components of the basis vectors \( \alpha_1, \ldots, \alpha_t \) for the tangent space \( TV \) of an algebraic variety \( V \) (embedded in affine space) to be polynomials in the coordinate vector \( x \in V \). To see this recall [Har95] that the tangent space to \( V \), or the Zariski tangent space, is defined to be the null space of the matrix \( J_{jk} \equiv \frac{\partial f}{\partial x_k} \), where \( V = Z(\ell_1, \ldots, \ell_a) \). Because the entries of \( J \) are polynomials in the components of \( x \), the components of the basis vectors \( \alpha_j \) for the null space of \( J \) can be taken to be polynomials in \( x \) as well. (To see this, find the basis for the null space of \( J \) via row reduction. The entries will be rational functions of \( x_j \). To get polynomial entries just clear denominators.)

We now return to the matrix \( R \) which we introduced in the previous two paragraphs. This matrix can be taken to have \( m \) rows consisting of the vectors\(^ 5 \) \( x_j - x_{j+1}, j = 1, \ldots, m \), where we identify \( j = 1 \) with \( j = m + 1 \), as well as \( nt \) rows consisting of the basis vectors \( \alpha_1^{(j)}, \ldots, \alpha_t^{(j)}, j = 1, \ldots, m \). There are \( \binom{m+t+1}{n} \binom{n}{m} \) \( n \times n \)-submatrices of \( R \). We denote the determinants of each of these submatrices by \( \Delta_\alpha(x_1, \ldots, x_m) \), where \( \alpha \) runs over all the \( \binom{m+t+1}{n} \binom{n}{m} \) possible submatrices. In order for a hyperplane to be \( m \)-tangent to \( \text{gr}(f) \) we require that all of the polynomials \( \Delta_\alpha(x_1, \ldots, x_m) \) simultaneously vanish. By adjoining the equations \( \ell_j(x_k), j = 1, \ldots, a, k = 1, \ldots, m \) (which ensure that \( x_j \) lie in \( V \)) and the equations \( 0 = \sum_{j=1}^m p_j x_j - r, 0 = \sum_{j=1}^m p_j - 1 \) (which ensure that \( r \) lies in the convex hull of \( x_j \)) we obtain a set of conditions for \( \{p_j, x_j\} \) to be an optimal \( m \)-decomposition for \( r \in \text{conv} V \). This discussion constitutes a proof of the following proposition.

**Proposition 2.9.** An optimal \( m \)-decomposition for \( r \in \text{conv} V \) consists of \( m \)

\(^4\)By which we mean polynomials in the individual components of the vector \( x_j \).

\(^5\)The other vectors \( x_j - x_{k}, j \neq (k + 1) \) are linearly dependent with this set.
points $x_j \in \mathbb{R}^{n+1}$ and $m$ variables $p_j$ which satisfy the following equations:

$$0 = \ell_j(x_k), \quad \forall j = 1, \ldots, a, \forall k = 1, \ldots, m,$$

$$0 = \Delta_\alpha(x_1, \ldots, x_m), \quad \forall \alpha = 1, \ldots, \binom{m+1}{m},$$

$$0 = \sum_{j=1}^{m} p_j x_j - r,$$

$$0 = \sum_{j=1}^{m} p_j - 1.$$ 

Proposition 2.9 has two important consequences for the evaluation of convex roofs. The first is that it provides an implicit representation for the optimal decomposition of a point $r \in \text{conv} V$ for $\text{conv} f$. This is in contrast to standard procedure to calculate an optimal decomposition, i.e., via direct minimisation of (2). In general this minimisation must be performed numerically and, as a consequence, there is no guarantee that a calculated decomposition is optimal (It could just be a local minima). The polynomials of Proposition 2.9 therefore provide a certificate of optimality.

The second consequence of Proposition 2.9 is that the space required to search through to calculate an optimal decomposition is dramatically reduced. Often, in practice, an infinite search space is replaced with a finite search space.

To illustrate Proposition 2.9 we write out the equations that must be solved to find a bitangent plane for the example introduced in Figure 1. These equations read

$$0 = x_1^2 + y_1^2 - 1,$$

$$0 = x_2^2 + y_2^2 - 1,$$

$$0 = \begin{vmatrix} y_1 & -x_1 & x_1^2y_1 \\ y_2 & -x_2 & x_2^2y_2 \\ x_1 - x_2 & y_1 - y_2 & x_1^3 - x_2^3 \end{vmatrix},$$

$$0 = px_1 + (1-p)x_2 - r_x,$$

$$0 = py_1 + (1-p)y_2 - r_y.$$ 

By solving these equations we were able to calculate the convex hull and convex roof illustrated in Figure 2 and Figure 3.

3 The Geometry of State Space: the Poincaré Sphere

In this paper we are interested in calculating roofs for certain scalar-valued functions defined on the state-space of quantum systems. Initially, it seems that this problem is unrelated to the results of the previous section because state-space (a Hilbert space) has no obvious geometric character. In this section we
show that in fact pure state-space for quantum systems of dimension \( D \) may be represented by the common zero-loci of a collection of polynomial equations. This provides the first piece of information — the domain \( V \) for the function \( f \) we wish to roof — needed to apply the results of \( \mathbb{R}^n \) to our problem. (We discuss the second piece of information, the construction of the function \( f \) we wish to roof, in the next section.) We note that there are many papers dealing with generalisations of the Bloch sphere construction for higher-dimensional quantum systems. The generalisation we recall here seems to date from 1997, where it was considered in detail in the papers of [KMSM97, AMM97]. The observation that this embedding of state-space into \( \mathbb{R}^n \) is a quadratic variety, to the best of our knowledge, original.

### 3.1 Poincaré Spheres Based on \( \mathfrak{su}(D) \)

The Bloch sphere representation for a rank-2 density operator\(^6\) is a fundamental conceptual tool in quantum mechanics (see, for example, [NC00] and [Pre98] for a discussion of the Bloch sphere). The fact that the space of all rank-2 density operators corresponds to a geometrical object as simple as the sphere \( S^2 \) and its interior in \( \mathbb{R}^3 \) has led to many discoveries. This is due, in part, to the fact that it is possible to employ geometric intuition when dealing with systems whose states or dynamics can be represented on the Bloch sphere. It is therefore very desirable to consider generalisations of the Bloch sphere to see if there is a simple geometrical object that represents the state-space for a \( D \)-level system. Unfortunately there is no object which possesses quite the simplicity of the Bloch sphere. However, as we argue, there is one natural way to generalise the Bloch sphere construction which is useful when discussing certain aspects of rank-\( D \) density operators.

In this section we discuss a generalisation of the Bloch sphere representation for density operators of rank-\( D \), which we call the Poincaré sphere (we choose this terminology in order to avoid confusion with the Bloch sphere which specifically refers to \( S^2 \)). The first step in the Poincaré sphere construction is to choose an operator basis with respect to which an arbitrary rank-\( D \) density operator can be expanded. We choose the set of traceless hermitian generators\(^7\) 

\[
\lambda^j, \ j = 1, \ldots, D^2 - 1
\]

for the group \( SU(D) \). (As we will show, this choice is entirely arbitrary. Any other basis for the operator space may be obtained via a linear transformation from this one.) We represent the generators in an orthonormal basis \( \{ |v_a\rangle \} \). The generators are given in terms of the set of operators 

\[
\Gamma \triangleq \{ \Gamma_a, \Gamma_{ab}^{(+)}, \Gamma_{ab}^{(-)} \}
\]

defined by

\[
\Gamma_a \triangleq \frac{1}{\sqrt{a(a-1)}} \left( \sum_{b=1}^{a-1} |v_b\rangle\langle v_b| - (a-1)|v_a\rangle\langle v_a| \right), \quad 2 \leq a \leq D,
\]

\[\tag{18}\]

\(^6\)Any rank-2 density operator \( \rho = p|v_1\rangle\langle v_1| + (1-p)|v_2\rangle\langle v_2| \), where \( |v_j\rangle \) are the eigenvectors of \( \rho \), may be thought of as the state of an effective two-level system (qubit) whose states are “logical zero” \( |0\rangle \triangleq |v_1\rangle \) and “logical one” \( |1\rangle \triangleq |v_2\rangle \).

\(^7\)These matrices are also known as the Gellmann matrices.
\[ \Gamma_{ab}^{(+)} \triangleq \frac{1}{\sqrt{2}}(|v_a\rangle\langle v_b| + |v_b\rangle\langle v_a|), \quad 1 \leq a < b \leq D, \] (19)

\[ \Gamma_{ab}^{(-)} \triangleq -\frac{i}{\sqrt{2}}(|v_a\rangle\langle v_b| - |v_b\rangle\langle v_a|), \quad 1 \leq a < b \leq D. \] (20)

The set \( \Gamma \) consists of \( D^2 - 1 \) hermitian operators. We assume that the set \( \Gamma \) is ordered, and we set \( \lambda^j \) to be equal to the \( j \)th element of \( \Gamma \). Note that, in this paper, we suppose that roman indices \( j \) run from 1 to \( D^2 - 1 \).

The hermitian operators \( \lambda^j \) form a representation for the Lie algebra \( su(D) \), which allows us to write

\[ \lambda^j \lambda^k = \frac{1}{D} \delta^{jk} + d^{jkl} \lambda^l + i f^{jkl} \lambda^l, \] (21)

where \( f^{jkl} \) are the completely antisymmetric structure constants of \( su(D) \), and the completely symmetric coefficients \( d^{jkl} \) are given by

\[ \{ \lambda^j, \lambda^k \} = \frac{2}{D} \delta^{jk} + 2 d^{jkl} \lambda^l. \] (22)

By adjoining the operator

\[ \lambda^0 = \frac{1}{\sqrt{D}} I \] (23)

to the \( D^2 - 1 \) hermitian generators we obtain a complete basis for the space of all \( D \times D \) hermitian operators. Further, this basis is orthonormal under the Hilbert-Schmidt inner product:

\[ \text{tr}(\lambda^\alpha \lambda^\beta) = \delta^{\alpha\beta}. \] (24)

(In the following greek indices are assumed to run from 0 to \( D^2 - 1 \).)

Using the operator basis (19), (20) we can obtain a geometric representation for the space of all pure states of a \( D \)-dimensional quantum system. Recall that a hermitian operator \( \rho \) represents a state (either pure or mixed) of a quantum system if it satisfies the following conditions:

\[ \text{tr}(\rho) = 1, \quad \rho^\dagger = \rho, \quad \text{and} \quad \rho^2 \leq \rho, \] (25)

where by \( \rho^2 \leq \rho \) we mean that \( \rho - \rho^2 \) is positive semidefinite and we require equality when \( \rho \) is a pure state. (These conditions assert that \( \rho \) is positive semidefinite with trace unity.) Utilising the fact that any density operator may be expanded uniquely in terms of the operator basis as

\[ \rho = \frac{1}{D} c_\alpha \lambda^\alpha, \] (26)

\[ ^8 \text{We are employing the Einstein summation convention over repeated indices, so that } d^{jkl} \lambda^l \text{ means } \sum_{l=1}^{D^2-1} d^{jkl} \lambda^l. \]
where the coefficients $c_\alpha \in \mathbb{R}$ are given by

$$c_\alpha = D \text{tr}(\rho \lambda^\alpha),$$

we can translate the pure-state conditions into geometric conditions on $c_\alpha$.

Firstly, normalisation implies that

$$\rho = I + c_j \lambda^j.$$  

In this way the components $c$ can be regarded as a vector in a $(D^2 - 1)$-dimensional real vector space (this is where we impose the hermiticity requirement), which we can identify with $\mathbb{R}^{D^2-1}$. If the density operator is a pure state $\rho = |\psi\rangle\langle\psi|$, then $\text{tr}(\rho^2) = 1$ implies that

$$|c|^2 = D(D - 1).$$

Before we complete our description of the geometric conditions we need to introduce three vector operations on the vectors $c$ defining pure states. The first such operation is simply the euclidean inner product,

$$a \cdot b = a_j b^j.$$  

The second operation is a generalisation of the cross-product formula in 3 dimensions, the wedge or outer product, which is constructed using the antisymmetric structure constants

$$(a \wedge b)_l = a_j b_k f^{jk}_l.$$  

The third operation is a completely symmetric vector product, the star product

$$(a \star b)_l = a_j b_k d^{jk}_l.$$  

The vector operations satisfy a number of useful relations. To see this, let $U \in SU(D)$ be a group element. Let $O(U)$ be the operator transforming in the adjoint representation of $SU(D)$ that represents $U$ whose matrix elements are given by the formula

$$[O(U)]_{\alpha\beta} = \text{tr}(\lambda^\alpha U \lambda^\beta U^\dagger).$$

The vector operations defined above transform in the following way under the adjoint representation of $SU(D)$,

$$O(U) a \cdot O(U) b = a \cdot b,$$

$$O(U) a \wedge O(U) b = O(U)(a \wedge b),$$

$$O(U) a \star O(U) b = O(U)(a \star b).$$

By definition the inner product is an $SO(D^2-1)$-scalar. Both the star and wedge product of two vectors transforming in the adjoint representation of $SU(D)$ give rise to another vector transforming in the adjoint representation.
We can now state a geometric version of the final pure-state condition. Equation (29) is not a sufficient condition for the vector $c$ to represent a pure state. This is a consequence of the fact that the identity $\text{tr}(\rho^2) = 1$ does not imply $\rho^2 = \rho$ when $D > 2$. The further constraints due to the anticommutation relations (22) must be taken into account. If these constraints are taken into account then the following condition, along with (29), is necessary and sufficient for the state represented by $c$ to be pure. The additional constraint on $c$ is known as the star-product formula [KMSM97, AMM97].

$$c \star c = (D - 2)c.$$  
(37)

With these constraints, the space of all pure states is the subvariety $\mathcal{P}^* \subset S^{D^2 - 1}$ which is the common locus of zeros of the following set of quadratic polynomials,

$$\ell_0(c) = |c|^2 - D(D - 1),$$  
(38)

$$\ell_l(c) = c^T D^l c - e^T c,$$  
(39)

where $D^l$ are real symmetric matrices with entries given by $[D^l]_{jk} = d^{ljk}t$, and $e_j$ is a row-vector whose $j$th entry is one and the rest zero.

The space $\mathcal{P}$ of all pure states may be envisioned as a section $\mathcal{P}^*$ of the sphere $S^{D^2 - 1}$. The space of all mixed states $\mathcal{D}^*$ in this representation (the Poincaré sphere) is simply given by the convex hull of this quadratic variety, $\mathcal{D}^* = \text{conv } \mathcal{P}^*$.

To get an idea of what $\mathcal{P}^*$ looks like as a subspace of $S^{D^2 - 1}$ consider the expression

$$\text{tr}(\rho\sigma) = \frac{1}{D} \frac{r \cdot s}{D^2},$$  
(40)

where $\rho = \frac{I + r \cdot \lambda}{D}$ and $\sigma = \frac{I + s \cdot \lambda}{D}$ are any two pure states. Using the formula $r \cdot s = |r||s| \cos \theta$ we find that the angle $\theta$ between the vectors representing $\rho$ and $\sigma$ is given by

$$\theta = \cos^{-1} \left( \frac{D \text{tr}(\rho\sigma) - 1}{D - 1} \right).$$  
(41)

This angle is largest when $\text{tr}(\rho\sigma)$ is zero. In this way we see that widest angle $\theta_{\max}$ between any two points in $\mathcal{P}^*$ is given by $\cos^{-1} \left( \frac{1}{1 - D} \right)$, so that if $r$ represents a pure quantum state, $-r$ cannot, unless $D = 2$.

### 3.2 Poincaré Spheres from General Operator Bases

In the previous subsection we introduced a geometric representation for the convex set of all density operators for a $D$ dimensional quantum system. In order to do so we expanded an arbitrary density operator $\rho$ in terms of a specific operator basis, the generalised Gellmann matrices. In this subsection we briefly summarise the analogous results for when we expand $\rho$ in terms of an arbitrary (hermitian) operator basis.
Our main motivation for investigating Poincaré sphere representations for $\rho$ with respect to different operator bases is to better study states $\rho$ of a multipartite system. Because we are interested in situations when a pure-state density operator $\rho$ is a product state $\rho = \sigma \otimes \omega$, it is desirable, in some cases, to expand $\rho$ in terms of an operator basis which makes this property manifest, such as the basis $\{\lambda^\alpha \otimes \lambda^\beta\}$ formed from the tensor product of two Gellmann bases.

If we expand a density operator $\rho$ in terms of an operator basis different from the Gellmann matrices we find that the generalised Poincaré sphere so derived is different. It turns out however, that the new Poincaré sphere is related to the original sphere via an overall isometry. To see this we need to derive a more general geometric form of the pure state conditions (25). For clarity we limit ourselves to complete hermitian orthonormal operator bases $\mu^\alpha, \alpha = 1, \ldots, D^2$.

The geometric pure state conditions analogous to (38) and (39) follow from expanding the (hermitian) product $\mu^\alpha \mu^\beta$ in terms of the complete basis,

$$
\mu^\alpha \mu^\beta = \chi^{\alpha\beta \gamma} \mu^\gamma, \tag{42}
$$

where $\chi^{\alpha\beta \gamma}$ are the expansion coefficients. Following the analysis detailed in the previous subsection we can write out the pure state conditions in terms of $c_\alpha$:

$$
c_\alpha \tau^\alpha = 1, \tag{43}
$$

$$
c_\alpha c^\alpha = 1, \tag{44}
$$

$$
c_\alpha c_\beta \chi^{\alpha\beta \gamma} = c_\gamma, \tag{45}
$$

where $\tau^\alpha = \text{tr}(\mu^\alpha)$. The pure state conditions (43), (44), and (45) say that $c_\alpha$ must be a vector in $\mathbb{R}^{D^2}$ which lies in the intersection of the hypersphere (44) with a hyperplane (43) and $D^2$ quadric surfaces (45).

The algebraic variety $V$ defined by (43), (44), and (45) representing the space of all pure states of a $D$-dimensional quantum system is related to the $\mathfrak{su}(D)$ Poincaré sphere in a simple way. Any complete orthonormal hermitian operator basis $\mu^\alpha$ can be found as an orthogonal rotation of the $\mathfrak{su}(D)$ basis,

$$
\mu^\alpha = O^\alpha_\beta \lambda^\beta, \tag{46}
$$

where $O^\alpha_\beta \in SO(D^2)$. Say we wish to represent $\rho$ with respect to some other operator basis, i.e. write $\rho = c_\alpha \mu^\alpha = d_\alpha \lambda^\alpha$, and we want to know how the geometric pure state conditions for $c_\alpha$ and $d_\alpha$ are related. This relationship can be elucidated by making use of the formula $c_\alpha = d_\alpha O^\alpha_\beta$. By substituting this formula into the geometric pure state conditions (43), (44), and (45) we find that $d_\alpha$ must satisfy

$$
d_\alpha \tau'^\alpha = 1, \tag{47}
$$

$$
d_\alpha d^\alpha = 1, \tag{48}
$$

$$
d_\alpha d_\beta \chi^{\alpha\beta \gamma} = d_\gamma, \tag{49}
$$

15
where \( \tau'_{\alpha} = O^\alpha_\beta \tau^\beta \) and \( \chi'^{\alpha\beta}_{\gamma} = O^\alpha_{\alpha'} O^\beta_{\beta'} \chi'^{\alpha'}_{\beta'} O^{\gamma'}_{\gamma} \). (When \( \lambda^\alpha \) is the Gellmann basis these conditions reduce to (38) and (39).) In this way we see that the generalised Poincaré representation is unique up to an isomorphism of varieties. The geometric representation of the space of all rank-\( D \) mixed states is simply given by the convex hull of the variety \( V \) defined by (47), (48), and (49). We note, in passing, that the variety \( V' \) defined by (49) is a geometric representation of the extreme points of the positive cone of nonnegative \( D \times D \) matrices. The space of all positive linear combinations of points in \( V' \) is a geometric representation for the positive cone of nonnegative \( D \times D \) matrices.

4 Entanglement Measures

In this section we briefly review the definitions and some results that have been obtained in the theory of entanglement. In particular we introduce a set of axioms for a function to be considered an entanglement measure. We then introduce a family of polynomial functions which satisfy these axioms. Finally, we demonstrate that the method of §2 can be applied to evaluate these functions.

Suppose we have a bipartite quantum system \( AB \) composed of two subsystems \( A \) and \( B \) whose respective Hilbert spaces, \( \mathcal{H}_A \) and \( \mathcal{H}_B \), have dimensions \( \dim \mathcal{H}_A = d_A \) and \( \dim \mathcal{H}_B = d_B \). Our problem, at its most elementary level, is to answer questions such as: given a pure state \( \rho = |\psi\rangle\langle\psi| \) of \( AB \), when is this state expressible as a tensor product \( \rho = \sigma \otimes \omega \)? More precisely, one of the general problems we are interested in is the following. Given an arbitrary mixed state \( \rho \) of a multipartite quantum system \( A_1 A_2 \cdots A_n \), when is \( \rho \) expressible as a convex combination of tensor-product states:

\[
\rho = \sum_{j=1}^{s \geq \text{rank} \rho} p_j \sigma_{A_1}^{(j)} \otimes \sigma_{A_2}^{(j)} \otimes \cdots \otimes \sigma_{A_n}^{(j)},
\]

where \( \sigma_{A_k}^{(k)} \) are pure states? This problem is fundamental in the theory of entanglement and has received a great deal of attention in recent times. This work has culminated in the development of several criteria to determine when a state can be written in the form (50).

Deciding if a mixed state \( \rho \) is expressible as in (50) is significantly complicated by the fact that there are many ways to write a mixed state as a convex combination of pure states. Fortunately there is a recipe which can be utilised to relate all the different decompositions of \( \rho \) in terms of convex mixtures of pure states [Sch36, Jay57, HJW93]. To describe this recipe we first write \( \rho \) in terms of the decomposition

\[
\rho = \sum_{j=1}^{r} p_j |u_j\rangle\langle u_j| = \sum_{j=1}^{r} |\pi_j\rangle\langle \pi_j|,
\]

where \( r \) is the rank of \( \rho \), \( \{ |u_j\rangle \} \) are the eigenvectors of \( \rho \) and \( |\pi_j\rangle = \sqrt{p_j} |u_j\rangle \) are the subnormalised eigenvectors of \( \rho \). It turns out that any other decomposition
of \( \rho \) can be found via the equation

\[
\rho = \sum_{j=1}^{s \geq r} |v_j\rangle\langle v_j|,
\]

(52)

where \( |v_j\rangle = \sum_{k=1}^{r} U_{jk}^* |u_k\rangle \) and \( U \) is any right-unitary matrix. (The number \( s \) in Eq. (52) may take any integer value greater than or equal to the rank of \( \rho \). This is because \( \rho \) may be realised from an ensemble consisting of arbitrarily many pure states.)

The problem described in the previous paragraphs is known as the separability problem — a state \( \rho \) being separable if it is expressible as in (50), and entangled otherwise. By itself this is an interesting mathematical problem. However, there are deeper reasons for why we are interested in answering it. These stem from the fact that entanglement, the property possessed by entangled states, seems to be a fundamentally new kind of resource like free energy or information. Motivated by this, researchers are interested in working out when a state contains some of this resource (i.e. deciding when \( \rho \) is not separable). Ultimately, however, researchers are more interested in working out how much of this resource is present in a given state.

In this area of research, the theory of entanglement, a unified approach to answering questions such as the separability of a state \( \rho \) and the measurement of the entanglement of \( \rho \) has been developed. This approach is based on studying the properties of certain functions, called entanglement measures, on state-space which are zero for all separable states, and which are nonzero for all other states.

At the current time there are only hints that entanglement measures actually measure the resource character of entanglement\(^9\). We ignore this problem for the time being, concentrating on the more basic problem of evaluating certain entanglement measures.

### 4.1 Pure-State Entanglement Measures

What properties must a scalar-valued function \( F : \mathcal{H} \to \mathbb{R} \) possess in order for it to be described as an entanglement measure? This problem has been widely studied in the physics literature and a partial consensus has been reached\(^{10}\).

For bipartite quantum systems with state-space isomorphic to \( \mathbb{C}^M \otimes \mathbb{C}^N \), with \( N \leq M \), it has been shown \(^{\text{Vid00}}\) that entanglement measures are in bijective correspondence with unitarily invariant, concave (and hence continuous) real-valued functions on the space of density operators over \( \mathbb{C}^M \). More precisely, consider any (continuous) real-valued function \( f : \mathcal{D}(\mathbb{C}^M) \to \mathbb{R} \) on the space \( \mathcal{D}(\mathbb{C}^M) \) of density operators \( \rho \) on \( \mathbb{C}^M \) which satisfies:

\(^{9}\)This has only been conclusively established for the entanglement for pure-states of a bipartite system \(^{\text{BDSW96, PR97, Vid00, Nie00}}\).

\(^{10}\)To date, the physics behind the measurement of entanglement for multipartite quantum systems has not been satisfactorily elucidated. Only in the bipartite case has agreement been reached. The conditions we use for a function to be a multipartite entanglement measure follow \(^{\text{Vid00}}\).
1. invariance under the transformation of \( \rho \) by any unitary matrix \( U : \mathbb{C}^M \rightarrow \mathbb{C}^M \), i.e.
\[
f(U\rho U^\dagger) = f(\rho), \quad \text{and}
\]
\[f(\rho) \geq p f(\omega_1) + (1 - p)f(\omega_2),\]
where \( \rho = p\omega_1 + (1 - p)\omega_2 \), \( p \in [0, 1] \), and \( \omega_1, \omega_2 \in \mathcal{D}(\mathbb{C}^M) \).

Using \( f \) a (pure-state) entanglement measure \( F \) on \( \mathcal{H} \) can be constructed by defining
\[
F(|\phi\rangle) \triangleq f(\text{tr}_B(|\phi\rangle\langle\phi|)),
\]
where \( \text{tr}_B \) denotes the partial trace over subsystem \( B \).

Using the correspondence between pure-state bipartite entanglement measures and unitarily invariant concave functions we propose a family of entanglement measures using for \( f \) the functions
\[
f_a(\rho) \triangleq 2(1 - \text{tr}(\rho^a)),
\]
where \( a \in \mathbb{N}, a > 1 \). The functions \( f_a \) are easily seen to be unitarily invariant and concave. The family \( f_a \) has one further property, namely that they are zero for pure states \( \rho \). This means that the entanglement measures \( f_a \) that they give rise to under the construction discussed above are zero if and only if \( |\phi\rangle \) is a product state.

If we use the Poincaré sphere representation of \( \mathcal{H} \) we see that \( f_a \) are polynomial functions of the coordinate vector \( c_{\alpha} \) for a state \( |\phi\rangle \). This can be established as follows. First expand the state \( |\phi\rangle \) in terms of the complete orthonormal operator basis given by \( \{ \lambda^\alpha \otimes \lambda^\beta | \alpha = 1, \ldots, M^2, \beta = 1, \ldots, N^2 \} \), which is the tensor product of the Gellmann matrices of sides \( M \) and \( N \) respectively:
\[
\rho = c_{\alpha\beta}\lambda^\alpha \otimes \lambda^\beta.
\]
Taking the partial trace over \( B \) yields
\[
\text{tr}_B(\rho) = c_{\alpha 0}\lambda^\alpha,
\]
because \( \lambda^\alpha \) is traceless for \( \alpha \neq 0 \). If we now calculate \( f_a \) we find that
\[
f_a(|\phi\rangle) = 2(1 - \text{tr}((c_{\alpha 0}\lambda^\alpha)^a)),
\]
which is a polynomial in the coefficients \( c_{\alpha\beta} \). This function is defined on the Poincaré sphere for the tensor-product operator basis \( \{ \lambda^\alpha \otimes \lambda^\beta \} \). The entanglement measure \( f_a \) can also be written as a polynomial in terms of the position vector \( d_\alpha \) defining \( \rho \) in any other operator basis by replacing \( c_{\alpha 0} \) with \( O_{\alpha\beta}d_\beta \), where \( O \) is the \( SO(M^2N^2) \) operator connecting the two operator bases.
4.2 Mixed-State Entanglement Measures from the Roof Construction

In the previous subsection we discussed a set of conditions for a function on state-space to be considered an entanglement measure. We introduced a family of entanglement measures which are polynomial functions of the vector representing a state in the Poincaré sphere representation. In this subsection we discuss a technique to construct entanglement measures defined for all states, both pure and mixed, from pure-state entanglement measures.

Any mixed state $\rho$ can be written as a convex combination $\{p_j, |\phi_j\rangle\}$ of pure states,

$$\rho = \sum_{j=1}^{r} p_j |\phi_j\rangle \langle \phi_j|.$$  

(60)

As we discussed at the beginning of this section this representation is not unique. In order to measure the entanglement of a mixed state $\rho$ it is necessary to take account of this nonuniqueness of the pure state decomposition. Each decomposition of $\rho$ in terms of pure states represents a way of manufacturing the mixed state $\rho$, because $\rho = \sum_{j=1}^{r} p_j |\phi_j\rangle \langle \phi_j|$ can be thought of as being produced from a quantum source emitting the pure state $|\phi_j\rangle$ with probability $p_j$. If we now imagine that entanglement is an expensive quantity that “costs” $F(\langle \phi_j\rangle) = f(\text{tr}_{B}(|\phi_j\rangle \langle \phi_j|))$, then it is natural to ask what is the minimal cost of producing $\rho$. This leads us to define a quantity which is called the (generalised) entanglement of formation (see [BDSW96] for the original definition and further discussions):

$$F(\rho) \triangleq \inf_{\{q_j, |\psi_j\rangle\}} \sum_{j=1}^{r} q_j f(\text{tr}_{B}(|\psi_j\rangle \langle \psi_j|)),$$  

(61)

where $\rho = \sum_{j=1}^{s} q_j |\psi_j\rangle \langle \psi_j|$, which quantifies the minimum expected cost\(^{11}\) of preparing $\rho$.

If we translate the definition of the mixed state entanglement measure $F$ into the Poincaré sphere representation we find that it coincides precisely with taking a roof of the scalar valued function $f$. In this way the results of §42 can be applied to calculate $F = \text{conv } f$.

5 Conclusions

In this paper we have considered the problem of calculating the convex hull of certain class of affine algebraic varieties. We have shown that this problem is intimately related to a problem studied in quantum information science, namely that of calculating mixed-state entanglement measures. Our principle result is that an optimal decomposition for a point lying in the roof of a function $f$ must

\(^{11}\)It should be noted that the interpretation of $F$ as quantifying the cost of producing the state $\rho$ is subject to the condition that the entanglement of formation is additive on tensor products.
satisfy a set of polynomial equations. This provides an implicit representation for such decompositions.

It is tempting to apply the well-developed techniques in computational commutative algebra to calculate the polynomial entanglement measures we introduced in §4 for mixed states. Unfortunately, apart from the example we considered in Figures 1, 2, and 3 (which pertains to a cubic entanglement measure for real rank-2 density operators) the polynomials that appear in Proposition 2.9 are far too complex for current computational approaches.

One principle future direction suggests itself at this stage. Determining the additivity of convex functionals (particularly entanglement of formation-like quantities) on tensor products of quantum states is a major unsolved problem in quantum information science. It is plausible that the implicit representation afforded by Proposition 2.9 may provide one route to considering this problem.

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References

[AMM97] Arvind, K. S. Mallesh, and N. Mukunda, A generalized Pancharatnam geometric phase formula for three-level quantum systems, J. Phys. A 30 (1997), no. 7, 2417–2431, quant-ph/9605042 MR 98e:81042

[AVDM01] Koenraad Audenaert, Frank Verstraete, and Bart De Moor, Variational characterizations of separability and entanglement of formation, Phys. Rev. A 64 (2001), no. 5, 052304, quant-ph/0006128

[BBC+93] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters, Teleporting an unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. 70 (1993), no. 13, 1895–1899; MR 94a:81004

[BBPS96] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A 53 (1996), no. 4, 2046–2052

[BDSW96] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters, Mixed-state entanglement and quan-
tum error correction, Phys. Rev. A 54 (1996), no. 5, 3824–3851, quant-ph/9604024 MR 97k:81028

[CLO97] David Cox, John Little, and Donal O’Shea, Ideals, varieties, and algorithms, 2nd ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1997; MR 97h:13024

[CLO98] David Cox, John Little, and Donal O’Shea, Using algebraic geometry, Graduate Texts in Mathematics, vol. 185, Springer-Verlag, New York, 1998; MR 99h:13033

[Har77] Robin Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977; MR 57 #3116

[Har95] Joe Harris, Algebraic geometry, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995; MR 97e:14001

[HH01] Paweł Horodecki and Ryszard Horodecki, Distillation and bound entanglement, Quantum Inf. Comput. 1 (2001), no. 1, 45–75; MR 1 910 010

[HJ90] Roger A. Horn and Charles R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1990; MR 91i:15001

[HJW93] Lane P. Hughston, Richard Jozsa, and William K. Wootters, A complete classification of quantum ensembles having a given density matrix, Phys. Lett. A 183 (1993), no. 1, 14–18; MR 94k:81076

[Hor97] Paweł Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, Phys. Lett. A 232 (1997), no. 5, 333–339, quant-ph/9703004 MR 98g:81018

[HW97] Scott Hill and William K. Wootters, Entanglement of a Pair of Quantum Bits, Phys. Rev. Lett. 78 (1997), no. 26, 5022–5025, quant-ph/9703041

[Jay57] E. T. Jaynes, Information theory and statistical mechanics. II, Phys. Rev. 108 (1957), no. 2, 171–190; MR 20 #2898

[KMSM97] G. Khanna, S. Mukhopadhyay, R. Simon, and N. Mukunda, Geometric Phases for SU(3) Representations and Three Level Quantum Systems, Ann. Physics 253 (1997), no. 1, 55–82; MR 98c:81077

[NC00] Michael A. Nielsen and Isaac L. Chuang, Quantum computation and quantum information, Cambridge University Press, Cambridge, 2000; MR 1 796 805

[Nie00] M. A. Nielsen, Continuity bounds for entanglement, Phys. Rev. A 61 (2000), no. 6, 064301, quant-ph/9908086 MR 2001c:81019
[Osb02] Tobias J. Osborne, *Entanglement for rank-2 mixed states*, 2002, quant-ph/0203087

[Per93] Asher Peres, *Quantum theory: concepts and methods*, Kluwer Academic Publishers Group, Dordrecht, 1993; MR 95e:81001

[PR97] Sandu Popescu and Daniel Rohrlich, *Thermodynamics and the measure of entanglement*, Phys. Rev. A 56 (1997), no. 5, R3319–R3321, quant-ph/9610044; MR 98k:81014

[Pre98] John Preskill, Physics 229: Advanced Mathematical Methods of Physics — Quantum Computation and Information. California Institute of Technology, http://www.theory.caltech.edu/people/preskill/ph229/, 1998

[Roc70] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, N.J., 1970; MR 97m:49001

[Sch36] E. Schrödinger, *Probability relations between separated systems*, Proc. Camb. Philos. Soc. 32 (1936), 446–452

[Sho94] Peter W. Shor, *Algorithms for quantum computation: discrete logarithms and factoring*, 35th Annual Symposium on Foundations of Computer Science (Santa Fe, NM, 1994), IEEE Comput. Soc. Press, Los Alamitos, CA, 1994, pp. 124–134, quant-ph/9508027; MR 1 489 242

[Sho97] Peter W. Shor, *Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer*, SIAM J. Comput. 26 (1997), no. 5, 1484–1509; MR 98i:11108

[Ter01] Barbara M. Terhal, *A family of indecomposable positive linear maps based on entangled quantum states*, Linear Algebra Appl. 323 (2001), no. 1–3, 61–73, quant-ph/9810091; MR 2002f:81007

[Uhl98] Armin Uhlmann, *Entropy and optimal decompositions of states relative to a maximal commutative subalgebra*, Open Syst. Inf. Dyn. 5 (1998), no. 3, 209–228, quant-ph/9704017

[Uhl00] Armin Uhlmann, *Fidelity and concurrence of conjugated states*, Phys. Rev. A 62 (2000), no. 3, 032307, quant-ph/9909060; MR 2001h:81044

[Uhl03] Armin Uhlmann, *Concurrence and foliations induced by some 1-qubit channels*, 2003, quant-ph/0301088

[Vid00] Guifrè Vidal, *Entanglement monotones*, J. Mod. Opt. 47 (2000), no. 2–3, 355–376, quant-ph/9807077; MR 2001f:81031

22
[Woo98] William K. Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*, Phys. Rev. Lett. 80 (1998), no. 10, 2245–2248, quant-ph/9709029

[Wor76] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys. 10 (1976), no. 2, 165–183; MR 81m:15014