WEIGHTED ALEXANDROV-FENCHEL INEQUALITIES IN HYPERBOLIC SPACE AND A CONJECTURE OF GE, WANG AND WU

FREDERICO GIRÃO, NEILHA M. PINHEIRO, DIEGO PINHEIRO, AND DIEGO RODRIGUES

Abstract. We consider a conjecture made by Ge, Wang and Wu regarding weighted Alexandrov–Fenchel inequalities for horospherically convex hypersurfaces in hyperbolic space (a bound, for some physically motivated weight function, of the weighted integral of the $k$th mean curvature in terms of the area of the hypersurface). We prove an inequality very similar to the conjectured one. Moreover, when $k$ is zero and the ambient space has dimension three, we give a counterexample to the conjectured inequality.

1. Introduction

Let $\Sigma$ be a convex hypersurface in $\mathbb{R}^n$, $n \geq 3$. The Alexandrov–Fenchel inequalities [1, 2] state that

\[ \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} H_k \, d\Sigma \right)^{n-k} \geq \left( \frac{1}{\omega_{n-1}} \int_{\Sigma} H_{k-1} \, d\Sigma \right)^{n-k-1}, \]

for $k = 1, \ldots, n-1$, where $\omega_{n-1}$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and $H_k$ is the normalized $k$th mean curvature of $\Sigma$, that is,

\[ H_k = \frac{1}{C^k_{n-1}} \sigma_k, \]

$k = 0, 1, \ldots, n-1$, with $\sigma_k$ being the $k$th elementary symmetric function of the principal curvature vector $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$. Moreover, the equality holds if and only if $\Sigma$ is a round sphere. In [14], using a certain inverse curvature flow, Guan and Li showed that (1) still holds for any $\Sigma$ which is star-shaped and $k$-convex (which means that $\sigma_i(\lambda) \geq 0$ for $i = 0, 1, \ldots, k$).

The $k = 1$ case of (1), namely,

\[ \int_{\Sigma} H_1 \, d\Sigma \geq \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \]

where $|\Sigma|$ is the area of $\Sigma$, is a key step in the proof of the Penrose inequality for graphs, given by Lam in [16] (see also [5] and [20]). More generally, the cases of (1) for which $k$ is odd were used in a crucial way to establish, for...
graphs, versions of the Penrose inequality in the context of the so called Gauss–Bonnet–Chern mass [10] (see also [17] and [7]).

Let us now consider the hyperbolic $n$-space $\mathbb{H}^n$ to be the ambient space. We will work with two models of $\mathbb{H}^n$: the warped product model and the Poincaré ball model. The former consists of $\mathbb{R}_+ \times S^{n-1}$ endowed with the metric

$$dr^2 + (\sinh^2 r) h,$$

where $h$ is the round metric on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The later consists of the unit ball 

$$B^n = \{ x \in \mathbb{R}^n; |x| \leq 1 \}$$

endowed with the metric

$$\left( \frac{2}{1 - |x|^2} \right)^2 \delta,$$

where $| |$ denotes the Euclidean norm and $\delta$ denotes the Euclidean metric.

A hypersurface $\Sigma$ in $\mathbb{H}^n$ is said to be star-shaped if it can be written as a graph over a geodesic sphere centered at the origin. We say that $\Sigma$ is strictly mean-convex if its mean curvature $H_1$ is positive everywhere. Also, $\Sigma$ is said to be horospherically convex if all of its principal curvatures are greater than or equal to 1.

We consider the function $\rho: \mathbb{H}^n \to \mathbb{R}$ which in the warped product model is given by

$$\rho = \cosh r.$$ 

When working with the Poincaré model, the function $\rho$ has the expression

$$\rho = \frac{1 + |x|^2}{1 - |x|^2}.$$ 

We consider also the support function $p: \Sigma \to \mathbb{R}$, which is defined by

$$p = \langle D\rho, \xi \rangle,$$

where $\xi$ is the outward unit normal vector to $\Sigma$ and where $\langle , \rangle$ denotes the hyperbolic metric and $D$ denotes its Levi-Civita connection.

In [6], de Lima together with the first named author showed the following Alexandrov–Fenchel-type inequality: if $\Sigma$ is a star-shaped and strictly mean-convex hypersurface in $\mathbb{H}^n$, $n \geq 3$, then

$$(2) \quad \int_\Sigma \rho H_1 \, d\Sigma \geq \omega_{n-1} \left[ \left( \frac{\frac{3}{\omega_{n-1}}} \right)^{\frac{n-2}{n-1}} + \left( \frac{\frac{1}{\omega_{n-1}}} \right)^{\frac{n}{n-1}} \right],$$

with the equality occurring if and only if $\Sigma$ is a geodesic sphere centered at the origin. The proof uses, among other ingredients, two monotone quantities along the inverse mean curvature flow (IMCF) and an inequality due to Brendle, Hung and Wang [3]. Inequality (2) was conjectured by Dahl, Gicquaud and Sakovich in [4], where they found an explicit formula for the mass of an asymptotically hyperbolic graph; (2) was then the only thing left to show in order to proved the Penrose inequality in this context.

In [11], Ge, Wang and Wu defined the Gauss–Bonnet–Chern mass for asymptotically hyperbolic manifolds. In order to establish, in this context, the Penrose inequality for graphs, they showed, for odd $k$, the following
weighted Alexandrov–Fenchel-type inequality: if $\Sigma$ is a horospherically convex hypersurface in $\mathbb{H}^n$, then it holds

$$\int_{\Sigma} \rho H_k \, d\Sigma \geq \omega_{n-1} \left[ \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2n}{(k+1)(n-1)}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2(n-k-1)}{(k+1)(n-1)}} \right]^{\frac{k+1}{2}},$$

with the equality occurring if and only if $\Sigma$ is a geodesic sphere centered at the origin. They accomplished this by an induction argument (from $j$ to $j+2$), with the base case being inequality (2).

Also in [11] it was conjectured that (3) holds for even values of $k$ as well. They remarked that the induction argument (from $j$ to $j+2$) still works in this case. Thus, it would be enough to show the validity of (3) for $k = 0$, that is,

$$\int_{\Sigma} \rho \, d\Sigma \geq \omega_{n-1} \left[ \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2n}{n}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{2} \right]^\frac{1}{2}.$$

Now let’s state the main results of this paper. Our first main result shows the existence of a counterexample to (4) when $n = 3$.

**Theorem 1.1.** There exists a horospherically convex hypersurface $\Gamma$ in $\mathbb{H}^3$ such that

$$\int_{\Gamma} \rho \, d\Gamma < \omega_{n-1} \left[ \left( \frac{|\Gamma|}{\omega_{n-1}} \right)^{\frac{2n}{n}} + \left( \frac{|\Gamma|}{\omega_{n-1}} \right)^{2} \right]^\frac{1}{2}.$$

Our second main result is an inequality very similar to (4). The precise statement is the following:

**Theorem 1.2.** Let $\Sigma$ be a star-shaped hypersurface in $\mathbb{H}^n$ satisfying $H_1 \geq 1$. It holds that

$$\int_{\Sigma} \rho \, d\Sigma > \omega_{n-1} \left[ \left( \frac{n-1}{n} \right)^{2} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2n}{(k+1)(n-1)}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{2} \right]^\frac{1}{2}.$$

We now state our third and final main result, which is an inequality very similar to (3).

**Theorem 1.3.** If $\Sigma$ is a horospherically convex hypersurface in $\mathbb{H}^n$ and $k \in \{0,1,\ldots,n-1\}$ is even, then it holds

$$\int_{\Sigma} \rho H_k \, d\Sigma > \omega_{n-1} \left[ \left( \frac{n-1}{n} \right)^{2} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2n}{(k+1)(n-1)}} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{2} \right]^\frac{k+1}{2}.$$

### 2. Variation Formulae

Let $\psi_0 : \Sigma \to \mathbb{H}^n$ be a closed, isometrically immersed oriented hypersurface. We consider a one-parameter family $\Psi(t,\cdot) : \Sigma \to \mathbb{H}^n$ of isometrically immersed hypersurfaces evolving according to

$$\frac{\partial \psi}{\partial t} = F \xi,$$
with $\Psi(0, \cdot) = \psi_0$, where $\xi$ is the outward unit normal to $\Psi(t, \cdot) : \Sigma \rightarrow \mathbb{H}^n$ and $F$ is a general speed function.

**Proposition 2.1.** Along the flow (6), the following evolution equations hold:

The area element $d\Sigma$ evolves as

$$\frac{\partial}{\partial t} d\Sigma = F \sigma_1 d\Sigma.$$  \hspace{1cm} (7)

In particular, $|\Sigma|$, the area of $\Sigma$, evolves as

$$\frac{d}{dt} |\Sigma| = \int_{\Sigma} F \sigma_1 d\Sigma.$$  \hspace{1cm} (8)

The function $\rho$ evolves as

$$\frac{\partial \rho}{\partial t} = pF.$$  \hspace{1cm} (9)

**Proof.** Formulas (7) and (8) are well known (see, for example, [15]). Equation (9) is proven, for example, in [6] (Proposition 3.2).

Of particular interest to us is the case $F = -p$, so that $\Sigma$ evolves according to

$$\frac{\partial \Psi}{\partial t} = -p\xi.$$  \hspace{1cm} (10)

This flow will be called support function flow (SFF).

From now on we use the Poincaré ball model to represent the hyperbolic space.

Next we consider, for each $t \in [0, \infty)$, the hypersurface $\varphi_t : \Sigma \rightarrow \mathbb{B}^n$ defined by

$$\varphi_t = e^{-t}\psi_0.$$  \hspace{1cm} (11)

Notice that if $\Phi : \mathbb{R} \times \Sigma \rightarrow \mathbb{B}^n$ is defined by

$$\Phi(t, p) = \varphi_t(p),$$

then it satisfies the differential equation

$$\frac{\partial \Phi}{\partial t} = -\Phi.$$  \hspace{1cm} (12)

We have that (11) defines, for any hypersurface $\Sigma_0$ in $\mathbb{H}^n$ a 1-parameter family $\{\Sigma_t\}_{t \geq 0}$ of hypersurfaces in $\mathbb{H}^n$. Whenever no confusion arises, we will write only $\Sigma$ to denote $\Sigma_t$.

**Remark 2.2.** Notice that, from the Euclidean point of view (that is, by endowing $\mathbb{B}^n$ with the Euclidean metric $\delta$), $\Sigma_t$ is just the image of $\Sigma_0$ under the homothety of center in the origin and ratio $e^{-t}$.

**Proposition 2.3.** The flow (10) exists for all time.

**Proof.** By the same argument given in Proposition 1.3.4 of [19], as long as the flow (12) exists, then the flow

$$\frac{\partial \Phi}{\partial t} = -\langle \Phi, \xi \rangle \xi$$  \hspace{1cm} (13)
also exists. Since (12) exists for all time, (13) also exists for all time. However, when working with the ball model, a simple computation shows that

\[ D\rho = X, \]

where \( X \) is the vector field that associates to each \( x \in \mathbb{B}^n \) the vector \( x \). Thus, \( \langle \Phi, \xi \rangle \) is the support function and the flow (13) coincides with the flow (10). □

**Remark 2.4.** The argument given in Proposition 1.3.4 of [19] actually shows that the flows (10) and (12) are, up to reparametrization, the same flow. For this reason, we will abuse notation and also denote by \( \{\Sigma_t\}_{t \geq 0} \) the 1-parameter family of hypersurfaces defined by (10). Again, whenever no confusion arises, we will write only \( \Sigma \) to denote \( \Sigma_t \).

For a hypersurface \( \Sigma \) in \( \mathbb{H}^n \) we define the quantity \( I(\Sigma) \) by

\[ \mathcal{I} = \int_{\Sigma} \rho \, d\Sigma. \]

**Proposition 2.5.** Along the flow (10) the following evolution equations hold:

- The area \( \vert \Sigma \vert \) evolves as
  \[ \frac{d}{dt} \vert \Sigma \vert = -(n - 1) \mathcal{I}. \]  

- The quantity \( \mathcal{I} \) evolves as
  \[ \frac{d\mathcal{I}}{dt} = \vert \Sigma \vert - n \int_{\Sigma} \rho^2 \, d\Sigma. \]

**Proof.** We have

\[ \Delta_{\Sigma} \rho = (n - 1) \rho - p\sigma_1 \] and

\[ \rho^2 = 1 + p^2 + \langle \nabla \rho, \nabla \rho \rangle. \]

Identities (16) and (17) are proven, for example, in [11] (Lemma 7.1). Integrating (16) we get

\[ (n - 1) \mathcal{I} = \int_{\Sigma} p\sigma_1 \, d\Sigma. \]

Equation (14) follows from (8) and (15). Multiplying (16) by \( \rho \) and integrating yields

\[ - \int_{\Sigma} \langle \nabla \rho, \nabla \rho \rangle \, d\Sigma = (n - 1) \int_{\Sigma} \rho^2 \, d\Sigma - \int_{\Sigma} \rho \sigma_1 \, d\Sigma. \]

Using (9), (7), (19) and (17) we find

\[ \frac{d\mathcal{I}}{dt} = \int_{\Sigma} \frac{\partial}{\partial t} \rho \, d\Sigma + \int_{\Sigma} \rho \frac{\partial}{\partial t} \sigma_1 \, d\Sigma \]

\[ = - \int_{\Sigma} p^2 \, d\Sigma - \int_{\Sigma} \rho \sigma_1 \, d\Sigma \]

\[ = - \int_{\Sigma} p^2 \, d\Sigma - \int_{\Sigma} \langle \nabla \rho, \nabla \rho \rangle \, d\Sigma - (n - 1) \int_{\Sigma} \rho^2 \, d\Sigma \]

\[ = \vert \Sigma \vert - n \int_{\Sigma} \rho^2 \, d\Sigma, \]
as wished.

For a hypersurface $\Sigma$ in $\mathbb{H}^n$, define the quantity $P(\Sigma)$ by

$$P = \left[ \frac{n-1}{|\Sigma|} \frac{|\Sigma|}{\omega_{n-1}} \right]^{-2} \left[ I^2 - |\Sigma|^2 \right].$$

**Proposition 2.6.** Along the flow (11) it holds

$$\frac{dP}{dt} \leq 0.$$ 

Moreover, the equality holds at $t$ if and only if $\Sigma_t$ is a geodesic sphere centered at the origin.

**Proof.** First, note that Hölder’s inequality applied to (15) gives

$$\frac{dI}{dt} \leq \left| \frac{\Sigma}{|\Sigma|} \right| - n \left| \frac{\Sigma}{|\Sigma|} \right|^2,$$  

with the equality holding if and only if $\rho$ is constant on $\Sigma$, that is, if and only if $\Sigma$ is a geodesic sphere centered at the origin.

Now, a straightforward computation together with (21), (14) and (15) yields

$$\frac{dP}{dt} \leq \left[ \frac{2I \frac{dI}{dt} - 2|\Sigma| \frac{d|\Sigma|}{dt}}{|\Sigma|} - \frac{2n}{n-1} \left( I^2 - |\Sigma|^2 \right) \frac{d|\Sigma|}{dt} \right]$$

$$\leq \left[ \omega \left( \frac{|\Sigma|}{\omega} \right)^{-\frac{n-1}{n}} \right]^2$$

$$= 0.$$ 

The equality holds in (22) if and only if it also holds in (21), which occurs if and only if $\Sigma$ is a geodesic sphere centered at the origin. \(\Box\)

**Proposition 2.7.** Along the flow (12) the quantity $P$ defined by (20) satisfies

$$\frac{dP}{dt} \leq 0.$$ 

Furthermore, the equality holds at $t$ if and only if $\Sigma_t$ is a geodesic sphere centered at the origin.

**Proof.** In order to compute the variation of $P$ along (12), we can disregard tangential motions, that is, instead of the flow (12), we can consider the flow (13) which, as argued in the proof of Proposition 2.3, coincides with the flow (10). Hence, the proposition follows from Proposition 2.6. \(\Box\)

A hypersurface $\Sigma$ in $\mathbb{H}^n$ can also be seen as an Euclidean hypersurface (just endow $\mathbb{H}^n$ with the Euclidean metric $\delta$).

For an Euclidean hypersurface $\Sigma$ we define the quantity $Q(\Sigma)$ by
\[ Q(\Sigma) = \left[ \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n+1}{n-1}} \right]^{-1} \int_\Sigma |x|^2 (d\Sigma)_\delta, \]

where \( |\Sigma|_\delta \) and \((d\Sigma)_\delta\) are the area and the area element of \( \Sigma \) with respect to the metric induced by the Euclidean metric.

The next proposition relates the quantities \( P \) and \( Q \).

**Proposition 2.8.** It holds

\[ (23) \quad \lim_{t \to \infty} P(\Sigma_t) = Q(\Sigma_0). \]

**Proof.** First, note that since \( P(\Sigma_t) \) is decreasing and bounded below (by 0), the limit on the left hand side of \((23)\), in fact, exists.

Also, since the quantities \( I_2 - |\Sigma|^2 \) and \( \omega_n^{n-1} (|\Sigma|/\omega_n^{n-1})^{n+1} \) converge to 0, l’Hôpital’s rule together with \((14), (15)\) and a straightforward computation give

\[ \lim_{t \to \infty} P(\Sigma) = \lim_{t \to 0} \left[ \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n+1}{n-1}} \right]^{-1} \int_\Sigma (\rho^2 - 1) d\Sigma. \]

Let \( \epsilon > 0 \) be given. Take \( \eta > 0 \) such that \( |x| < \eta \) implies

\[ \left| \left( \frac{1}{1 - |x|^2} \right)^{n+1} - 1 \right| < \epsilon \quad \text{and} \quad \left| \left( \frac{1}{1 - |x|^2} \right)^{n-1} - 1 \right| < \epsilon. \]

Using that

\[ \rho = \frac{1 + |x|^2}{1 - |x|^2} \]

and that

\[ d\Sigma = \left( \frac{2}{1 - |x|^2} \right)^{n-1} (d\Sigma)_\delta \]

we have, for each \( \Sigma \) contained in \( \{ x \in \mathbb{B}^n; |x| < \eta \} \), that

\[ \left| \frac{\int_\Sigma (\rho^2 - 1) d\Sigma}{2^{n+1} \int_\Sigma |x|^2 (d\Sigma)_\delta} - 1 \right| \leq \frac{\int_\Sigma |x|^2 \left( \frac{1}{1 - |x|^2} \right)^{n+1} - 1}{\int_\Sigma |x|^2 (d\Sigma)_\delta} < \epsilon \]

and

\[ \left| \frac{\left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{n-1} - 1}{2^{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)} \right| \leq \frac{\int_\Sigma \left( \frac{1}{1 - |x|^2} \right)^{n-1} - 1}{|\Sigma|_\delta} < \epsilon. \]
Hence,

\[
\lim_{t \to \infty} \frac{\int_{\Sigma} (\rho^2 - 1) \, d\Sigma}{2^{n+1} \int_{\Sigma} |x|^2 \, (d\Sigma)_\delta} = 1 \quad \text{and} \quad \lim_{t \to \infty} \frac{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)}{2^{n-1} \left(\frac{|\Sigma|}{\omega_{n-1}}\right)} = 1.
\]

Using (24) and the scale invariance of the quantity \( Q \) we have

\[
\lim_{t \to \infty} \frac{\mathcal{P}(\Sigma_t)}{\mathcal{Q}(\Sigma_t)} = \lim_{t \to \infty} \frac{\mathcal{P}(\Sigma_t)}{\mathcal{Q}(\Sigma_t)}
\]

\[
= \lim_{t \to \infty} \left[ \left( \frac{\int_{\Sigma} \rho^2 - 1 \, d\Sigma}{\omega_{n-1} \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n+1}{n-1}}} \right) \left( \frac{\int_{\Sigma} |x|^2 \, (d\Sigma)_\delta}{\omega_{n-1} \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n+1}{n-1}}} \right)^{-1} \right]
\]

\[
= \left( \lim_{t \to \infty} \frac{\int_{\Sigma} (\rho^2 - 1) \, d\Sigma}{2^{n+1} \int_{\Sigma} |x|^2 \, (d\Sigma)_\delta} \right) \left( \lim_{t \to \infty} \frac{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)}{2^{n-1} \left(\frac{|\Sigma|}{\omega_{n-1}}\right)} \right)^{-\frac{n+1}{n-1}}
\]

\[
= 1.
\]

\[\square\]

The following two propositions relate the geometry of \( \Sigma \) as a hypersurface in \( \mathbb{H}^n \) with the geometry of \( \Sigma \) as an Euclidean hypersurface.

**Proposition 2.9.** Let \( \psi : \Sigma \to \mathbb{B}^n \) be so that, as a hypersurface in \( \mathbb{H}^n \), its mean curvature satisfies \( H_1 \geq 1 \). Then, as an Euclidean hypersurface, \( \Sigma \) is mean-convex.

**Proof.** In Poincaré’s model for \( \mathbb{H}^n \), the hyperbolic metric is given by

\[
\langle \xi, \xi \rangle = \phi^2 \delta,
\]

where

\[
\phi = \frac{2}{1 - |x|^2}.
\]

In particular, since \( \langle \xi, \xi \rangle = 1 \), it follows that

\[
\delta(\phi \xi, \phi \xi) = 1,
\]

that is,

\[
|\phi \xi| = 1.
\]

The well known formula for the mean curvature under a conformal change of metric gives

\[
H_1 = \phi^{-1} H_1^\delta + \phi^{-1} \xi(\phi),
\]

where \( H_1^\delta \) denotes the mean curvature of \( \Sigma \) as an Euclidean hypersurface. Using that

\[
\xi(\phi) = \phi^2 \delta(\xi, \psi),
\]

we find

\[
H_1^\delta = \phi \left( H_1 - \phi \delta(\xi, \psi) \right) > 0
\]

since \( H_1 \geq 1 \) and, by Cauchy’s inequality together with (26),

\[
\phi \delta(\xi, \psi) \leq |\phi | |\xi| \cdot |\psi| = |\phi \xi| \cdot |\psi| = |\psi| < 1.
\]
Proposition 2.10. Let $\psi_0 : \Sigma \rightarrow \mathbb{B}^n$ be such that, as an Euclidean hypersurface, $\Sigma$ is strictly convex. Then, there exists $T \in [0, \infty)$ for which $\varphi_t : \Sigma \rightarrow \mathbb{B}^n$ given by $\varphi_t = e^{-t} \psi_0$ is horospherically convex for each $t \geq T$.

**Proof.** Let $b$ and $b^\delta$ be the second fundamental forms of $\Sigma$ and $\Sigma^\delta$, respectively. A well known formula in conformal geometry gives

$$b = \phi b^\delta + \phi \xi(\phi) \delta,$$

where $\phi$ is defined by (25). Together with (27), this gives

$$b = \phi b^\delta + \phi^3 \delta(\xi, \psi) \delta.$$

Thus, for any vector $v$ we have

$$b(v, v) = \phi b^\delta(v, v) + \phi^3 \delta(\xi, \psi) \delta(v, v).$$

Hence, using the convexity of $\Sigma$, we find

$$b(v, v) \geq \phi b^\delta(v, v).$$

Dividing both sides by $\phi^2 \delta(v, v)$ we get

$$\frac{b(v, v)}{\phi^2 \delta(v, v)} \geq \phi^{-1} \frac{b^\delta(v, v)}{\delta(v, v)}.$$ 

Now, let $b_t$ and $b^\delta_t$ be the second fundamental forms of $\Sigma_t$ and $\Sigma^\delta_t$, respectively. The previous inequality gives

$$\frac{b_t(v, v)}{\phi_t^2 \delta(v, v)} \geq \phi_t^{-1} \frac{b^\delta_t(v, v)}{\delta(v, v)}.$$ 

Also, since $b^\delta_t = e^t b^\delta$, we have

$$\frac{b_t(v, v)}{\phi_t^2 \delta(v, v)} \geq \phi_t^{-1} e^t \frac{b^\delta(v, v)}{\delta(v, v)}.$$ 

Therefore, since $\phi_t$ converges uniformly to 2 as $t$ goes to infinity and $\Sigma_t$ is strictly convex as an Euclidean hypersurface, we can choose $T \in [0, \infty)$ so that all of the principal curvatures of $\Sigma_t$ are no less than 1, for each $t \geq T$. 

3. Proofs of the theorems

We begin with the proof of Theorem 1.2. Let $\Sigma$ be a star-shaped hypersurface in $\mathbb{H}^n$ whose mean curvature satisfies $H_1 \geq 1$. Then $\Sigma^\delta$ is a star-shaped hypersurface in $\mathbb{R}^n$. Moreover, by Proposition 2.4, $\Sigma^\delta$ is strictly mean-convex. By a result proved in [12] it follows that

$$Q(\Sigma^\delta) > \left(\frac{n-1}{n}\right)^2.$$ 

Let $\Sigma_t$, with $\Sigma_0 = \Sigma$, be the one-parameter family of hypersurfaces defined by (11). By Proposition 2.3 and (28) we have

$$\lim_{t \to \infty} P(\Sigma_t) > \left(\frac{n-1}{n}\right)^2.$$
Since, by Proposition 2.7, \( P(\Sigma_t) \) is nonincreasing, we conclude that
\[
P(\Sigma_0) > \left( \frac{n-1}{n} \right)^2,
\]
which is just a rewriting of (5).

**Remark 3.1.** The quantities \( P(\Sigma) \) and \( Q(\Sigma) \) also make sense when \( n = 2 \). Moreover, it is known that if \( \Sigma \subset \mathbb{R}^2 \) is convex, then
\[
Q(\Sigma) > \frac{(2\pi)^2}{54}
\]
(see [21, 22, 14]). Thus, by proceeding as above, one can show that if \( \Sigma \) is a hypersurface in \( \mathbb{H}^2 \) satisfying
\[
\kappa \geq 1,
\]
where \( \kappa \) denotes the geodesic curvature of \( \Sigma \), then it holds that
\[
\int_{\Sigma} |x|^2 \, d\Sigma > 2\pi \left[ \frac{(2\pi)^2}{54} \left( \frac{|\Sigma|}{2\pi} \right)^4 + \left( \frac{|\Sigma|}{2\pi} \right)^2 \right]^{\frac{1}{2}},
\]
that is,
\[
\int_{\Sigma} |x|^2 \, d\Sigma > |\Sigma| \left( \frac{1}{54} |\Sigma|^2 + 1 \right)^{\frac{1}{2}}.
\]
Also, by considering a sequence \( \{\Lambda_n\} \) of convex curves in \( \mathbb{R}^2 \) that converges, in the \( C^0 \) topology, to an equilateral triangle centered at the origin, one can show, by suitably rescaling the terms of \( \{\Lambda_n\} \), that \( 1/54 \) is the largest constant \( \Theta \) for which the inequality
\[
\int_{\Sigma} |x|^2 \, d\Sigma > |\Sigma| \left( \Theta |\Sigma|^2 + 1 \right)^{\frac{1}{2}}
\]
holds for every hypersurface in \( \mathbb{H}^2 \) satisfying (29). We leave the details to the interested reader.

Now let us prove Theorem 1.1. It is proved in [12] that there exists a strictly convex surface \( \Gamma^d \) in \( \mathbb{R}^3 \) such that
\[
Q(\Gamma^d) < 1.
\]
By the scale invariance of \( Q \), we can assume that \( \Gamma^d \subset \mathbb{B}^3 \). Denote by \( \Gamma \) the surface \( \Gamma^d \) when seen as a hypersurface in \( \mathbb{H}^3 \). Let \( \Gamma_t \), with \( \Gamma_0 = \Gamma \), be defined as in (11). Inequality (30) together with Proposition 2.8 give
\[
\lim_{t \to \infty} P(\Gamma_t) < 1.
\]
Thus, there exists \( t_0 \in [0, \infty) \) for which \( P(\Gamma_t) < 1 \), for all \( t \geq t_0 \). To finish the proof, notice that Proposition 2.10 guarantees that \( t_0 \) can be chosen so that \( \Gamma_t \) is horospherically convex, for each \( t \geq t_0 \).

Next, let us prove Theorem 1.3. The proof consists of an induction argument very similar to the one given in [11], but with (5) as the base case.

The \( k = 0 \) case follows from Theorem 1.2.

Let \( j \) be an integer such that \( 2j \in \{0, 1, \ldots, n-3\} \) and suppose that the inequality holds for \( k = 2j \), that is, suppose
\[ \int_\Sigma \rho H_{2j} \, d\Sigma > \omega_{n-1} \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \left[ \left( \frac{n-1}{n} \right)^2 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{\frac{2j+1}{2}}. \]

It was proved in [9] (see also [8] and [18]) that

\[ (31) \quad \int_\Sigma H_{2j+2} \, d\Sigma \geq \left| \Sigma \right| \left[ 1 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{j+1}. \]

Hölder’s inequality and (31) give

\[ \left( \int_\Sigma \rho H_{2j+2} \, d\Sigma \right) \left( \int_\Sigma \frac{H_{2j+2}}{\rho} \, d\Sigma \right) \geq \left( \int_\Sigma H_{2k+2} \, d\Sigma \right)^2 \]

\[ \geq \left| \Sigma \right|^2 \left[ 1 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{2(j+1)} \]

\[ > \left| \Sigma \right|^2 \left( \frac{n-1}{n} \right)^2 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{2(j+1)}. \]

Thus, if we set

\[ \alpha = \left| \Sigma \right|^2 \left[ \left( \frac{n-1}{n} \right)^2 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{2(j+1)}, \]

we find that

\[ (32) \quad \int_\Sigma \rho H_{2j+2} \, d\Sigma - \int_\Sigma \frac{H_{2j+2}}{\rho} \, d\Sigma < \int_\Sigma \rho H_{2j+2} \, d\Sigma - \frac{\alpha}{\int_\Sigma \rho H_{2j+2} \, d\Sigma}. \]

It is also known (see [11], Theorem 8.1) that

\[ (33) \quad \int_\Sigma \rho H_{2j+2} \, d\Sigma - \int_\Sigma \frac{H_{2j+2}}{\rho} \, d\Sigma \geq \int_\Sigma \rho H_{2j} \, d\Sigma. \]

Hence, from (33) and the induction hypothesis, we find

\[ \int_\Sigma \rho H_{2j+2} \, d\Sigma - \int_\Sigma \frac{H_{2j+2}}{\rho} \, d\Sigma \]

\[ > \omega_{n-1} \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \left[ \left( \frac{n-1}{n} \right)^2 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{\frac{2j+1}{2}}. \]

Consider the function \( f(t) = t - \alpha t^{-1} \). From (32) and (33) we have

\[ f \left( \int_\Sigma \rho H_{2j+2} \, d\Sigma \right) \]

\[ > \omega_{n-1} \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \left[ \left( \frac{n-1}{n} \right)^2 + \left( \frac{\left| \Sigma \right|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{\frac{2j+1}{2}}. \]
We also have
\[
 f \left( \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \left( \frac{n-1}{n} \right)^2 + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \right) = \left( \frac{n-1}{n} \right)^2 \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \left( \frac{n-1}{n} \right)^2 + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{-\frac{n-2}{n-1}} \right) \right)^{\frac{2n+3}{n}}.
\]

where the last inequality follows from (35). Since \( f \) is increasing on \([0, \infty)\), we find that

\[
\int_{\Sigma} \rho H_{2j+2} d\Sigma > \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \left( \frac{n-1}{n} \right)^2 + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{-\frac{n-2}{n-1}} \right) \right)^{\frac{2n+3}{n}},
\]

which completes the induction.

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E-mail address: fred@mat.ufc.br
E-mail address: neilhamat@gmail.com
E-mail address: diegodsp01@gmail.com
E-mail address: diego.sousa.ismart@gmail.com