Implementing the fanout operation with simple pairwise interactions

Stephen Fenner*  Rabins Wosti*
University of South Carolina  University of South Carolina

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Abstract

It has been shown that, for even \( n \), evolving \( n \) qubits according to a Hamiltonian that is the sum of pairwise interactions between the particles, can be used to exactly implement an \(( n + 1)\)-qubit fanout gate using a particular constant-depth circuit [arXiv:quant-ph/0309163]. However, the coupling coefficients in the Hamiltonian considered in that paper are assumed to be all equal. In this paper, we generalize these results and show that for all \( n \), including odd \( n \), one can exactly implement an \(( n + 1)\)-qubit parity gate and hence, equivalently in constant depth an \(( n + 1)\)-qubit fanout gate, using a similar Hamiltonian but with unequal couplings, and we give an exact characterization of which couplings are adequate to implement fanout via the same circuit.

We also investigate pairwise couplings that satisfy an inverse square law and give planar arrangements of four qubits that (together with a target qubit) are adequate to implement 5-qubit fanout.

Keywords: constant-depth quantum circuit; quantum fanout gate; Hamiltonian; pairwise interactions; spin-exchange interaction; Heisenberg interaction

1 Introduction

1.1 Previous work

In the study of classical Boolean circuit complexity, the fanout operation—where a Boolean value on a single wire is copied into any number of wires—is taken for granted as cost-free. The picture is very different, however, with quantum circuits with unitary gates, where the number of wires is fixed throughout the circuit. There, fanout gates are known to be very powerful primitives for making shallow quantum circuits [7][10][13]. It has been shown that in the quantum realm, fanout, parity (see below), and \( \text{Mod}_q \) gates (for any \( q \geq 2 \)) are all equivalent up to constant depth and polynomial size [7][11]. That is, each gate above can be simulated exactly by a constant-depth, polynomial-size quantum circuit using any of the other gates above, together with standard one- and two-qubit gates (e.g., \text{C-NOT}, \text{H}, and \text{T}). This is not true in the classical case, where, for example, parity cannot be computed by constant-depth Boolean circuits with fanout and unbounded AND-, OR-,
and NOT-gates $\text{I, O, S, and C}$, and $\text{Z}$. Furthermore, using fanout gates, in constant depth and polynomial size one can approximate sorting, arithmetical operations, phase estimation, and the quantum Fourier transform $[10, 13]$. Fanout gates can also exactly implement $n$-qubit threshold gates, unbounded AND-gates (generalized Toffoli gates), and OR-gates in constant depth $[14]$. Since long quantum computations may be difficult to maintain due to decoherence, shallow quantum circuits may prove much more realistic, at least in the short term, and finding ways to implement fanout would then lend enormous power to these circuits.

On the negative side, fanout gates so far appear hard to implement by traditional quantum circuits. There is mounting theoretical evidence that fanout gates cannot be simulated in small (sublogarithmic) depth and small width, even if unbounded AND-gates are allowed $[3, 12]$. Therefore, rather than trying to implement fanout with a traditional small-depth quantum circuit, an alternate approach would be to evolve an $n$-qubit system according to one or more (hopefully implementable) Hamiltonians, along with a minimal number of traditional quantum gates. It was shown in $[4, 5]$ that simple Hamiltonians using spin-exchange (Heisenberg) interactions do exactly this. Those papers presented a simple quantum circuit for computing gates. It was shown in $[4, 5]$ that simple Hamiltonians using spin-exchange (Heisenberg) interactions (equivalent to fanout) that included two invocations of the Hamiltonian along with a constant number of one- and two-qubit Clifford gates.

More recently, Guo et al. $[8]$ presented a method for implementing fanout on a mesh of qubits. Their approach involves a series of modulated long-range Hamiltonians applied to the qubits obeying inverse power laws.

1.2 The current work

This paper revisits the spin-exchange Hamiltonians considered in $[3, 5]$. A major weakness of that work is that it assumes all the pairwise couplings between the spins to be equal. This is physically unrealistic since we expect couplings between spins that are spatially far apart to be weaker than those between spins in close proximity.

In this paper, we show that $n$-qubit fanout can still be implemented by the exact same circuit $C_n$ given in $[5]$, even with a wide variety of unequal pairwise couplings. We also give an exact characterization of which couplings are allowed so that $C_n$ implements fanout.

Formally, the $n$-qubit fanout gate is the $(n + 1)$-qubit unitary operator $F_n$ is defined such that $F_n |x_1, \ldots, x_n, c⟩ = |x_1 \oplus c, \ldots, x_n \oplus c, c⟩$ for all $x_1, \ldots, x_n, c \in \{0, 1\}$. The $n$-qubit parity gate is the $(n+1)$-qubit unitary operator $P_n$ such that $P_n |x_1, \ldots, x_n, t⟩ = |x_1, \ldots, x_n, t \oplus x_1 \oplus \cdots \oplus x_n⟩$ for all $x_1, \ldots, x_n, t \in \{0, 1\}$. It was shown in $[11]$ that $F_n = H^{(n+1)} P_n H^{(n+1)}$, where $H$ is the 1-qubit Hadamard gate. Thus $F_n$ and $P_n$ are equivalent in constant depth, and any circuit implementing $P_n$ can be converted to one implementing $F_n$ by conjugating with a bank of Hadamard gates.

The circuit $C_n$ implements $P_n$ and is shown in Figure 1. Here, the 1-qubit Clifford gate $G_b$ is either $S$, $I$, $S^\dagger$, or $Z$, depending on $b$ mod 4, where $I$ is the identity, $S$ satisfies $S |b⟩ = i^b |b⟩$ for $b \in \{0, 1\}$, and $Z$ is the Pauli $z$-gate. The unitary operator $U_n$ is defined as follows: for all $x = x_1 \cdots x_n \in \{0, 1\}^n$, letting $w = x_1 + \cdots + x_n$,

$$U_n |x⟩ = i^{w(n-w)} |x⟩.$$  \hfill (1)

\footnote{Fanout on $n$ qubits can be implemented by a $O(\log n)$-depth circuit with $O(n)$ many C-NOT gates.}
and a similar calculation shows the same is true for odd \( n \). For the full result and its proof, see Appendix A.

We consider Hamiltonians of the form \( H_n = \sum_{1 \leq i < j \leq n} J_{i,j} Z_i Z_j \), where \( Z_i \) and \( Z_j \) are Pauli Z-gates acting on the \( i \)th and \( j \)th qubits, respectively, and the \( J_{i,j} \) are real coupling constants (in units of energy). \( H_n \) is a simplified version of the spin-exchange interaction where only the \( z \)-components of the spins are coupled. It bears some resemblance to a quantum version of the Ising model, as described in [15], but with no transverse field and allowing long-range as well as nearest-neighbor couplings. In [5] it was shown that \( U_n = e^{-iH_n t} \) for a certain time \( t \), provided all the coupling constants \( J_{i,j} \) are equal.

In this paper, we exactly characterize when \( H_n \) can be run to implement \( U_n \) by proving the following result in Section 3:

**Theorem 1.1.** \( U_n \propto e^{-iH_n t} \) for some \( t > 0 \) if and only if there exists a constant \( J > 0 \) such that

1. all \( J_{i,j} \) are odd integer multiples of \( J \), and
2. the graph \( G \) on vertices \( 1, \ldots, n \) with edge set \( \{(i, j) : i < j \text{ and } J_{i,j}/J \equiv 3 \pmod{4}\} \) is Eulerian\(^2\), that is, all its vertices have even degree.

Furthermore, if \( t \) exists, we can set \( t := \pi \hbar/4J \).

Our result gives more flexibility in the coupling constants, allowing stronger and weaker couplings for spins placed nearer and farther apart, respectively. For example, suppose we have four identical spins arranged in the corners of a square. The spins diagonally opposite each other may have coupling constant \( J \) whereas neighboring spins can have coupling constant \( 3J \). The corresponding \( g_{i,j} = 1 \) is thus odd for neighboring spins, but this arrangement can be used to implement \( U_4 \), because the edges connecting neighboring spins form a square, which is Eulerian. For the spins arranged in the corners of a regular cube, neighboring spins may have coupling constant \( 7J \), spins on the diagonal ends of each face may have coupling constant \( 3J \) and the antipodal spins may have coupling constant \( J \). Thus, the corresponding \( g_{i,j} \) for antipodal spins is even while it is odd for the neighboring and diagonal spins and therefore, this arrangement can be used to implement \( U_8 \) because the edges connecting the neighboring spins and the spins on the diagonal ends of each face of a regular cube form an Eulerian graph. Similarly, for spins arranged on the corners of a regular octahedron, the graph of neighboring spins is Eulerian, so neighboring spins can have coupling \( 3J \) and antipodal spins \( J \).

We also investigate how spatial arrangements of qubits whose couplings obey an inverse square law can be used to implement \( U_n \) exactly. We find severe limitations on such arrangements. In

\(^2\)We use this term in the looser sense that the graph need not be connected.
particular, we show that no three qubits can lie on the same line, and this generally rules out any kind of mesh arrangement. Such arrangements therefore cannot implement $U_n$, assuming an inverse square law, unless extra physical barriers are used to moderate the couplings between certain pairs of qubits.

Our work differs from the recent work of Guo et al. \[8\] in a number of respects. They adapt a state transfer protocol of Eldredge et al. \[2\] that, given an arbitrary 1-qubit state $\alpha |0\rangle + \beta |1\rangle$, produces the GHZ-like state $\alpha |0\cdots0\rangle + \beta |1\cdots1\rangle$ on $n$ qubits. Their protocol uses long-range interactions on a mesh of qubits by sequentially turning on and off various Hamiltonians to implement a cascade of C-NOT gates, where different Hamiltonians must be applied at different times. Our scheme runs a simple, swap-invariant Hamiltonian twice, together with a constant number of 1-qubit gates and a C-NOT gate connecting to the target. Unlike in \[8\], our scheme needs no ancilla qubits. If the pairwise couplings must satisfy an inverse-square law, however, then our scheme has the disadvantages described above.

2 Preliminaries

We let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ the set of nonnegative integers. We choose physical units so that $\hbar = 1$. For $n \in \mathbb{N}$ and bit vector $x \in \{0,1\}^n$, we let $w(x)$ denote the Hamming weight of $x$, and we let $x_i$ denote the $i^{th}$ bit of $x$, for $1 \leq i \leq n$. We use $[n]$ to denote the set $\{1,\ldots,n\}$. For $x,y,\alpha \in \mathbb{R}$ with $\alpha > 0$, we write $x \equiv_{\alpha} y$ to mean that $(x-y)/\alpha$ is an integer, and we let $x \mod \alpha$ denote the unique $y \in [0,\alpha)$ such that $x \equiv_{\alpha} y$. For bits $a,b \in \{0,1\}$ we write $a \oplus b$ to mean $(a+b) \mod 2$. For vectors or operators $U$ and $V$ of the same type, we write $U \propto V$ to mean there exists $\theta \in \mathbb{R}$ such that $U = e^{i\theta}V$, i.e., $U$ and $V$ differ by a global phase factor.

We use the symbol ‘:=’ to mean “equals by definition.”

3 Main Results

We consider a particular type of Hamiltonian $H_n$, acting on a system of $n \in \mathbb{N}$ qubits, as the weighted sum of pairwise $Z$-interactions among the qubits in analogy to spin-exchange (Heisenberg) interactions:

$$ H_n := \sum_{1 \leq i < j \leq n} J_{i,j} Z_i Z_j , $$

where $Z_k$ is the Pauli $Z$-gate acting on the $k^{th}$ qubit for $k \in [n]$, and for $1 \leq i < j \leq n$, $J_{i,j} \in \mathbb{R}$ is the coupling coefficient between the $i^{th}$ and $j^{th}$ qubits. It will be convenient in the sequel to define $J_{j,i} := J_{i,j}$ for all $1 \leq i < j \leq n$. $H_n$ differs from the usual (isotropic) Heisenberg interactions in that only the $z$-components of the spins are coupled.

Let $x = x_1 \cdots x_n \in \{0,1\}^n$ be a vector of $n$ bits, where each $x_i$ denotes the $i^{th}$ bit of $x$. Notice that $Z_i Z_j |x\rangle = (-1)^{x_i + x_j} |x\rangle$ for $1 \leq i < j \leq n$, that is, $Z_i Z_j$ flips the sign of $|x\rangle$ iff $x_i \neq x_j$. Further, for $t, \theta \in \mathbb{R}$, let

$$ V_n := V_n(t, \theta) := e^{-i\theta} e^{-iH_n t} $$

be the unitary operator realized by evolving the Hamiltonian $H_n$ of Equation (2) for time period $t$, where $\theta$ represents a global phase factor that may be introduced into the system. It has been explicitly shown in \[5\] that for $n \equiv 4 \mod 2$, if $V_n \propto U_n$ (see Equation (1)), one can realize the parity gate $P_n$ (and thus the fanout gate $F_n$) in constant additional depth for $n$ qubits via the quantum circuit.
This fact indeed holds for all \( n \), and we give a unified proof in Appendix A that the circuit of Figure 1 works for all \( n \). Further, it was shown in the same paper that \( V_n \propto U_n \) if all the \( J_{i,j} \) are equal, and we give an updated proof of this in Appendix B where we prove the following:

**Lemma 3.1.** For \( n \geq 1 \), let \( H_n := J \sum_{1 \leq i < j \leq n} Z_i Z_j \) for some \( J > 0 \) (in units of energy). Then \( U_n = V_n(t, \theta) \) for some \( \theta \in \mathbb{R} \), where \( t := \pi/(4J) \) and \( V_n(t, \theta) \) is as in Equation (3).

The two main goals of the current work are (1) to show that equality of the \( J_{i,j} \) is not necessary and (2) to determine exactly for which values of \( J_{i,j} \) this is possible. We will use Lemma 3.1 to establish Theorem 1.1, the proof of which is the goal of this section.

Let \( H_n \) be as in Equation (3) for arbitrary \( J_{i,j} \). For \( x \in \{0, 1\}^n \) and \( t, \theta_1 \in \mathbb{R} \), we have

\[
V_n(t, \theta_1)|x\rangle = \exp \left( -i \theta_1 - i \sum_{1 \leq i < j \leq n} J_{i,j} t (-1)^{x_i + x_j} \right) |x\rangle \\
= \exp \left( -i \theta_1 - i \sum_{1 \leq i < j \leq n} k_{i,j} (-1)^{x_i + x_j} \right) |x\rangle 
\]

(setting \( k_{i,j} := J_{i,j} t \) for convenience)

Using the fact that \( i = e^{\pi i/2} \) and equating exponents, the condition that \( V_n(t, \theta_1) = U_n \) is seen to be equivalent to

\[
\theta_1 + \sum_{1 \leq i < j \leq n} k_{i,j} (-1)^{x_i + x_j} \equiv 2\pi - \left( \frac{\pi}{2} \right) w(x)(n - w(x)) \tag{5}
\]

holding for all \( x = x_1 \cdots x_n \in \{0, 1\}^n \). Lemma 3.1 yields a similar phase congruence in the case where \( k_{i,j} = J t = \pi/4 \) for all \( i < j \): there exists \( \theta_2 \in \mathbb{R} \) such that for all \( x \in \{0, 1\}^n \),

\[
\theta_2 + \frac{\pi}{4} \sum_{1 \leq i < j \leq n} (-1)^{x_i + x_j} \equiv 2\pi - \left( \frac{\pi}{2} \right) w(x)(n - w(x)). \tag{6}
\]

Subtracting Equation (5) from Equation (6) and rearranging, we get that \( V_n(t, \theta) = U_n \) is equivalent to

\[
\sum_{1 \leq i < j \leq n} \left( k_{i,j} - \frac{\pi}{4} \right) (-1)^{x_i + x_j} \equiv 2\pi \theta_2 - \theta_1 \quad \forall x \in \{0, 1\}^n,
\]

or equivalently, setting \( f_{i,j} := k_{i,j} - \pi/4 \) for all \( 1 \leq i < j \leq n \),

\[
\sum_{1 \leq i < j \leq n} f_{i,j} (-1)^{x_i + x_j} \equiv 2\pi \theta_2 - \theta_1 \quad \forall x \in \{0, 1\}^n. \tag{7}
\]

Substituting the zero vector for \( x \) in Equation (7) implies \( \theta_2 - \theta_1 \equiv 2\pi \sum_{i<j} f_{i,j} \), so Equation (7) can be rewritten as

\[
\sum_{i<j} f_{i,j} (-1)^{x_i + x_j} \equiv 2\pi \sum_{i<j} f_{i,j}
\]

\[
\sum_{i<j} f_{i,j} ((-1)^{x_i + x_j} - 1) \equiv 2\pi 0
\]

\[
\sum_{i<j : x_i \neq x_j} f_{i,j} \equiv 0 \quad \forall x \in \{0, 1\}^n. \tag{8}
\]

We have thus established the following lemma:
Lemma 3.2. Let $H_n$ be as in (2) and let $t \in \mathbb{R}$ be arbitrary. There exists $\theta \in \mathbb{R}$ such that $V_n(t, \theta) = U_n$, if and only if Equation (8) holds, where $f_{i,j} := f_{i,j}t - \pi/4$ for all $1 \leq i < j \leq n$.

Lemma 3.3. Let $\{f_{i,j}\}_{1 \leq i < j \leq n}$ be real numbers such that Equation (8) holds. Then $f_{i,j} \equiv \pi/2 \ 0$ for all $1 \leq i < j \leq n$.

Proof. For convenience, define $f_{j,i} := f_{i,j}$ for all $i < j$. For $a \in [n]$, let $x^{(a)} \in \{0,1\}^n$ be the $n$-bit vector whose $a^{th}$ bit is 1 and whose other bits are all 0. Consider two different bit vectors $x^{(a)}$ and $x^{(b)} \in \{0,1\}^n$ for $a < b$. Also, consider a third bit vector $y \in \{0,1\}^n$ with $w(y) = 2$ such that its bits are set to 1 in exactly the $a$ and $b$ positions, i.e., $y = x^{(a)} \oplus x^{(b)}$. Plugging in $x^{(a)}$, $x^{(b)}$, and $y$, respectively into Equation (8), we have

$$\sum_{j \in [n] : j \neq a} f_{a,j} \equiv_\pi 0$$ (9)

$$\sum_{i \in [n] : i \neq b} f_{i,b} \equiv_\pi 0$$ (10)

$$\sum_{k \in [n] : k \notin \{a,b\}} (f_{a,k} + f_{k,b}) \equiv_\pi 0$$ (11)

Equation (9)+(10)-(11) gives

$$\left( \sum_{j \in [n] : j \neq a} f_{a,j} - \sum_{k \in [n] : k \notin \{a,b\}} f_{a,k} \right) + \left( \sum_{i \in [n] : i \neq b} f_{i,b} - \sum_{k \in [n] : k \notin \{a,b\}} f_{k,b} \right) = 2f_{a,b} \equiv_\pi 0.$$ (12)

Therefore, $f_{a,b} \equiv \pi/2 \ 0$. Since, $a$ and $b$ are chosen arbitrarily, the conclusion follows. \hfill \Box

Definition 3.4. For $n \geq 2$, let $M_n$ be the $2^n \times \binom{n}{2}$ matrix over the 2-element field $\mathbb{F}_2$ with rows $m_x$ indexed by bit vectors $x$ of length $n$ and columns indexed by pairs $(i,j)$ for $1 \leq i < j \leq n$, whose $(x, \{i,j\})^{th}$ entry is $m_{x,\{i,j\}} = x_i \oplus x_j$.

Lemma 3.5. Every matrix $M_n$ defined by Definition 3.4 has rank $n - 1$, and its rows are spanned by any set of $n - 1$ rows $m_x$ for $x$ with Hamming weight 1.

Proof. All scalar and vector addition below is over $\mathbb{F}_2$. Let $S := \{x \in \{0,1\}^n : w(x) = 1\}$ be the set of $n$-bit vectors of Hamming weight 1. For $n$-bit vectors $r$ and $s$, we can write the $(i,j)^{th}$ component of the sum $m_r + m_s$ as

$$(m_r + m_s)_{(i,j)} = m_{r,\{i,j\}} + m_{s,\{i,j\}} = (r_i + r_j) + (s_i + s_j) = (r_i + s_i) + (r_j + s_j) = m_{r+s,\{i,j\}}.$$ (13)

Equation (13) shows that the sum of two rows equals the row in the $M_n$ matrix corresponding to the sum (bitwise XOR) of the index vectors of the two summand rows. With this observation, we can infer that every row in the matrix $M_n$ can be expressed as the sum of the rows indexed by $n$-bit vectors in $S$. In particular, we have

$$\sum_{x \in S} m_x = m_{11\ldots1} = \vec{0}.$$ (14)

This causes a linear dependence among the $n$ vectors in the set $S$. The sum of rows indexed by any nonempty proper subset of $S$, however, results in a row indexed by an $n$-bit vector containing
arbitrarily, this applies to all the vertices of \( G \). The right-hand side of Equation (15) is the degree of the vertex \( x \) is then equivalent to the degree or \( r \) being even. Since, \( r \in [n] \) (and hence \( x \in S \)) was chosen arbitrarily, this applies to all the vertices of \( G \). Finally, from Equation (14) we have for all \( i < j \) that \( J_{i,j}/J \equiv_4 3 \) if and only if \( g_{i,j} \) is odd, and so the theorem follows.

Here is an easy restatement of Theorem 1.1 that avoids graph concepts. (Recall that we set \( J_{i,i} := J_{i,j} \) for all \( i < j \).)
Corollary 3.7. \( U_n \propto e^{-iH_n t} \) for some \( t > 0 \) if and only if there exists a constant \( J > 0 \) such that
1. all \( J_{i,j} \) are odd integer multiples of \( J \), and
2. for every \( i \in [n] \),
\[
\prod_{j : j \neq i} \frac{J_{i,j}}{J} \equiv 1 \, .
\]
Furthermore, if \( t \) exists, we can set \( t := \pi \hbar/4J \).

Proof. Fix \( i \in [n] \). Given that for all \( j \neq i \), either \( J_{i,j}/J \equiv 1 \) or \( J_{i,j}/J \equiv 3 \), the product over all such \( j \) is congruent to 1 (mod 4) if and only if the latter congruence holds for an even number of such \( j \). This is the stated condition on the graph in Theorem 1.1.

4 Couplings Obeying the Inverse Square Law

Here we consider identical qubits as points in Euclidean space, where the inter-qubit couplings satisfy an inverse square law, i.e., \( J_{i,j} \) is proportional to \( d_{i,j}^{-2} \), where \( d_{i,j} \) is the distance between qubits \( i \) and \( j \). For different types of arrangements, we determine whether the resulting couplings can satisfy the conditions of Theorem 1.1 implementing \( U_n \) exactly (up to a phase factor).

Our results are mostly negative in that we rule out several arrangements of qubits. Our results suggest that, for many well-studied geometric arrangements of qubits, e.g., meshes, an unmodulated inverse square law is not useful for an exact implementation of \( U_n \); either the couplings must be modified in some way or one must make do with an approximation of \( U_n \), or both. See Section 5 for further discussion.

On the positive side, we display an arrangement of four qubits in the plane and give necessary and sufficient conditions for any such planar arrangement of four qubits.

Definition 4.1. For \( n > 0 \), we say that a set of positive real numbers \( \{J_{i,j}\}_{1 \leq i < j \leq n} \) is adequate for \( U_n \) if it satisfies the conditions of Theorem 1.1 i.e., there exists \( J > 0 \) such that all the \( J_{i,j} \) are odd integer multiples of \( J \), and for each \( i \in [n] \), there are an even number of \( j \neq i \) such that \( J_{i,j}/J \equiv 3 \).

Theorem 4.2. In any physical arrangement of \( n \geq 3 \) qubits where the coupling constant \( J_{i,j} \) between each pair of distinct qubits \( i, j \in [n] \) is inversely proportional to the square of the distance \( d_{i,j} \) between them, if any three qubits are collinear or form the vertices of a right triangle, then the set of couplings is not adequate for \( U_n \).

Proof. Without loss of generality, suppose qubits 1, 2, 3 are the qubits in question, with either \( d_{1,3} = d_{1,2} + d_{2,3} \) or \( (d_{1,3})^2 = (d_{1,2})^2 + (d_{2,3})^2 \). Equivalently, \( (d_{1,3})^2 = (d_{1,2})^2 + (d_{2,3})^2 + 2bd_{1,2}d_{2,3} \), where \( b \) is either 0 or 1. If such an arrangement is adequate for \( U_n \), then by Theorem 1.1 for each \( 1 \leq i < j \leq 3 \) there exists an integer \( k_{i,j} \geq 0 \) such that \( J_{i,j} = (2k_{i,j} + 1)J \) for some \( J > 0 \). For simplicity, let \( m := 2k_{1,2} + 1 \), \( n := 2k_{2,3} + 1 \), and \( p := 2k_{1,3} + 1 \), noting that \( m, n, \) and \( p \) are odd integers. Since the coupling between qubits \( i \) and \( j \) is proportional to \( d_{i,j}^{-2} \), there exists \( c > 0 \) such that
\[
J_{1,2} = mJ = \frac{c}{(d_{1,2})^2} \, ,
J_{2,3} = nJ = \frac{c}{(d_{2,3})^2} \, ,
J_{1,3} = pJ = \frac{c}{(d_{1,3})^2} \, .
\]
Figure 2: Two possible four-qubit arrangements in the plane whose couplings are adequate for $U_4$ and satisfy an inverse square law. Edges are labeled with the corresponding couplings. Left: an equilateral triangle with a point in the center. Right: points $q_1, \ldots, q_4$ whose cartesian coordinates can be arranged as: $q_1 = (0,0), q_2 = (1,0), q_3 = (-5/2, \sqrt{3}/2), q_4 = (-5/2, -\sqrt{3}/2)$.

From the above equations, we can write

$$c/mJ + c/nJ - c/pJ = (d_{1,2})^2 + (d_{2,3})^2 - (d_{1,3})^2 = -2bd_{1,2}d_{2,3} = -\frac{2bc}{J\sqrt{mn}}$$

$$\frac{2bc}{J\sqrt{mn}} = \frac{c}{pJ} - \frac{c}{mJ} - \frac{c}{nJ}$$

$$\frac{2b}{\sqrt{mn}} = \frac{mn - np - mp}{p\sqrt{mn}}$$

$$2bp\sqrt{mn} = mn - np - mp$$ (16)

The right-hand side of (16) is an odd integer, which requires $b = 1$ and $mn$ to be a perfect square, but then the left-hand side is an even integer. This contradicts the adequacy for $U_n$ of any such arrangement.

Despite the constraints given by Theorem 4.2, there are some nontrivial qubit arrangements that are adequate for $U_n$. A trivial arrangement is three qubits forming an equilateral triangle, with all couplings equal (to $J$). Planar arrangements with four qubits are harder to come by, but there are some—ininitely many, in fact. Figure 2 gives two planar arrangements that are adequate for $U_4$. In each of these, the couplings $kJ$ for $k \equiv 1 \pmod{3}$ form a 3-cycle. One can show that any arrangement of points $q_1, \ldots, q_4$ in $\mathbb{R}^2$ with coordinates $q_1 = (0,0), q_2 = (1,0), q_3 = (a,b)$, and $q_4 = (c,d)$ satisfying an inverse square law is adequate for $U_4$ if and only if there exists $k \equiv 1 \pmod{4}$ such that one of the following three sets of congruences holds (up to permutation of the points):

$$k(a^2 + b^2) \equiv 3 \quad k(a^2 + b^2) \equiv 3 \quad k(a^2 + b^2) \equiv 1$$
$$k(c^2 + d^2) \equiv 3 \quad k(c^2 + d^2) \equiv 3 \quad k(c^2 + d^2) \equiv 1$$
$$k((a-1)^2 + b^2) \equiv 1 \quad k((a-1)^2 + b^2) \equiv 3 \quad k((a-1)^2 + b^2) \equiv 1$$
$$k((c-1)^2 + d^2) \equiv 1 \quad k((c-1)^2 + d^2) \equiv 3 \quad k((c-1)^2 + d^2) \equiv 1$$
$$k((a-c)^2 + (b-d)^2) \equiv 3 \quad k((a-c)^2 + (b-d)^2) \equiv 1 \quad k((a-c)^2 + (b-d)^2) \equiv 1$$
5 Conclusions and Further Work

We have exactly characterized which sets of couplings are adequate for \( U_n \) and have given examples of spatial arrangements of qubits sufficient for \( U_n \) where nearer qubits are more strongly coupled than those farther away. We have shown planar arrangements of four identical qubits adequate for \( U_4 \) satisfying the inverse square law, but it remains to be seen if planar arrangements of more than four qubits can be found. This appears difficult at present in light of Theorem 4.2. We have not investigated arrangements in \( \mathbb{R}^3 \), nor have we investigated couplings satisfying other inverse power laws. We have also not considered arrangements of non-identical qubits satisfying the inverse square law. In this latter case, there would be a constant \( c_i \) associated with the \( i^{th} \) qubit such that \( J_{i,j} = c_i c_j / r^2 \), where \( r \) is the distance separating qubits \( i \) and \( j \).

Theorem 4.2 only rules out exact implementations of \( U_n \) and not approximate implementations. It would be interesting to see how useful and feasible these would be.

We have concentrated on implementing the operator \( U_n \), which is constant-depth equivalent to fanout. Studying \( U_n \) instead of \( F_n \) has two theoretical advantages over \( F_n \): \( U_n \) is represented in the computational basis by a diagonal matrix; unlike \( F_n \), which has a definite control and targets, \( U_n \) is invariant under any permutation of its qubits, or equivalently, it commutes with the SWAP operator applied to any pair of its qubits. One can ask if there are other operators that are both constant-depth equivalent to fanout and implementable by a simple Hamiltonian.

The Hamiltonian \( H_n \) only considers the \( z \)-components of the spins. In Heisenberg interactions, the \( x \)- and \( y \)-components should also be included in the Hamiltonian, so that a pairwise coupling between spins \( i \) and \( j \) would be \( J_{i,j}(X_i X_j + Y_i Y_j + Z_i Z_j) \). In [4] it was shown that these Hamiltonians can also simulate fanout provided all the pairwise couplings are equal. We believe we can relax this latter restriction for these Hamiltonians as well.

Finally, the time needed to run our Hamiltonian is inversely proportional to the fundamental coupling constant \( J \). If \( J \) is small relative to the actual couplings in the system, then this gives a poor time-energy trade-off and will likely be more difficult to implement quickly with precision.

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A The Quantum Circuit for Parity

In this section, we show by direct calculation that the circuit $C_n$ shown in Figure 1 implements the parity gate $P_n$, for all $n \geq 1$. The special case for $n \equiv 2 \pmod{4}$ was shown in [5]. Here, $U_n$ is defined by Equation (1), and

$$G_n := \begin{cases} S & \text{if } n \equiv 0 \pmod{4}, \\ I & \text{if } n \equiv 1 \pmod{4}, \\ S^\dagger & \text{if } n \equiv 2 \pmod{4}, \\ Z & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $S$ is the gate satisfying $S|b\rangle = i^b|b\rangle$ for $b \in \{0, 1\}$, $I$ is the identity, and $Z$ is the Pauli $z$-gate. ($G_n$ is chosen so that $G_n|b\rangle = i^{b(1-n)}|b\rangle$.)

Fix any $x_1, \ldots, x_n, t \in \{0, 1\}$. For convenience, we separate the first $n - 1$ qubits, which only participate in $U_n$ and $U_n^\dagger$, letting $\vec{x} := x_1 \cdots x_{n-1}$. We set $w := w(\vec{x}) = x_1 + \cdots + x_{n-1}$ and $W := w + x_n$, the Hamming weight of $x_1 \cdots x_n$. We set $p := W \pmod{2}$, the parity of $x_1 \cdots x_n$, which will be XORed with $t$ in the target qubit. Running the circuit starting with initial state $|\vec{x}\rangle |x_n\rangle |t\rangle$, we have

$$|\vec{x}\rangle |x_n\rangle |t\rangle \xrightarrow{H} 2^{-1/2} |\vec{x}\rangle (|0\rangle + (-1)^{x_n} |1\rangle) |t\rangle$$

$$= 2^{-1/2} (|\vec{x}, 0\rangle + (-1)^{x_n} |\vec{x}, 1\rangle) |t\rangle$$

$$\xrightarrow{U_n} 2^{-1/2} i^{w(n-w)} |\vec{x}, 0\rangle + (-1)^{x_n} i^{(w+1)(n-w-1)} |\vec{x}, 1\rangle |t\rangle$$

$$= 2^{-1/2} e^{i w(n-w)} |\vec{x}, 0\rangle + e^{i n-1-2(w+x_n)} |1\rangle |t\rangle$$

$$\xrightarrow{G_n} 2^{-1/2} e^{i w(n-w)} |\vec{x}, 0\rangle + (-1)^{W} e^{i n-1} |1\rangle |t\rangle$$

$$= 2^{-1/2} e^{i w(n-w)} |\vec{x}, 0\rangle + (-1)^{p} |1\rangle |t\rangle$$

$$\xrightarrow{H} e^{i w(n-w)} |\vec{x}, p\rangle |t\rangle .$$

At this point, the C-NOT gate is applied, resulting in the state $e^{i w(n-w)} |\vec{x}, p\rangle |t \oplus p\rangle$. The remaining gates undo the above action on the first $n$ qubits, resulting in the state $|\vec{x}\rangle |x_n\rangle |t \oplus p\rangle$, which is the same as $P_n$ applied to the initial state.
Finally, we note that $C_n$ only depends on $U_n$ up to an overall phase factor: any gate $V_n \propto U_n$ can be substituted for $U_n$ in the circuit, because any phase factor introduced by applying $V_n$ on the left will be cancelled when $V_n^\dagger$ is applied on the right. This fact is, of course, unnecessary for physical implementation.

B Implementing $U_n$ with Equal Couplings: Proof of Lemma 3.1

In this section give an updated proof of Lemma 3.1 which we restate here:

**Lemma B.1.** For $n \geq 1$, let $H_n := J \sum_{1 \leq i < j \leq n} Z_i Z_j$ for some $J > 0$ (in units of energy). Then $U_n = V_n(t, \theta)$ for some $\theta \in \mathbb{R}$, where $t := \pi/(4J)$ and $V_n(t, \theta)$ is as in Equation (3).

**Proof.** Looking at Equations (1) and (3), we see that for $t, \theta \in \mathbb{R}$, the condition $V(t, \theta) = U_n$ is equivalent to

$$\exp\left(-i\theta - i \sum_{1 \leq i < j \leq n} Jt(-1)^{x_i + x_j}\right) = \exp(i w(x)(n - w(x)))$$

holding for all $x \in \{0, 1\}^n$. Noting that $i = e^{i \pi/2}$ and $Jt = \pi/4$ and equating exponents, this condition becomes

$$\theta + \frac{\pi}{4} \sum_{1 \leq i < j \leq n} (-1)^{x_i + x_j} \equiv 2\pi - \left(\frac{\pi}{2}\right) w(x)(n - w(x))$$

for all $x \in \{0, 1\}^n$ (cf. Equations (4) and (6)). The sum on the left-hand side becomes

$$\sum_{i< j} (-1)^{x_i + x_j} = \frac{1}{2} \sum_{i \neq j} (-1)^{x_i + x_j} = -\frac{n}{2} + \frac{1}{2} \sum_{i} \sum_{j} (-1)^{x_i + x_j} = -\frac{n}{2} + \frac{1}{2} \left( \sum_{i=1}^{n} (-1)^{x_i} \right)^2$$

$$= -\frac{n}{2} + \frac{1}{2} \left( \sum_{i} (1 - 2x_i) \right)^2 = -\frac{n}{2} + \frac{1}{2} (n - 2w(x))^2 = \frac{n^2 - n}{2} - 2w(x)(n - w(x)) .$$

Substituting this back into Equation (17) satisfies it, provided we set $\theta := \pi(n^2 - n)/8$. □