INVERTIBLE DIRAC OPERATORS AND HANDLE ATTACHMENTS ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. For spin manifolds with boundary we consider Riemannian metrics which are product near the boundary and are such that the corresponding Dirac operator is invertible when half-infinite cylinders are attached at the boundary. The main result of this paper is that these properties of a metric can be preserved when the metric is extended over a handle of codimension at least two attached at the boundary. Applications of this result include the construction of non-isotopic metrics with invertible Dirac operator, and a concordance existence and classification theorem.

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1. Introduction

Surgery constructions can be used in differential geometry to show that under certain assumptions a geometric structure on a manifold can be transported to another manifold by applying certain surgeries. The most prominent examples are probably the results concerning positive scalar curvature, for an overview see [28]. In [18, Theorem A] Gromov and Lawson showed that if a closed manifold $M$ possesses a metric of positive scalar curvature, then a manifold obtained from $M$ by a surgery of codimension $\geq 3$ also possesses a metric with positive scalar curvature. This was generalized to handle attachments on manifolds with boundaries by Gajer in [15] and Carr in [12]. Those results not only gave existence results for positive scalar curvature but also allowed conclusions on the topology of the space of all metrics of positive scalar curvature $\mathcal{R}_{psc}(M) \subset \mathcal{R}(M)$. For example in [12], [22, Chapter 4, Theorem 7.7], it is shown that that $\mathcal{R}_{psc}(M)$ has infinitely many connected components if $\dim M = 4m - 1$, $m \geq 2$. This result is sharp in the sense that $\mathcal{R}_{psc}(S^3)$ is connected, see [24]. There are also newer works that examine the higher homotopy groups of the moduli space of positive scalar curvature, see for example [13], [9].

If the manifold is spin, similar results can be obtained for metrics with invertible Dirac operator.

Theorem 1.1. [3, Theorem 1.2] Let $(M, g)$ be a closed Riemannian spin manifold of dimension $n$ and let $\tilde{M}$ be obtained from $M$ by surgery of dimension $k$ where $0 \leq k \leq n - 2$. Then $\tilde{M}$ carries a metric $\tilde{g}$ for which $\dim \ker D^{\tilde{g}} \leq \dim \ker D^{g}$.

If the codimension is $\geq 3$ this result is a special case of [7, Theorem 1.2]. In the spirit of the generalization of the surgery result for positive scalar curvature to manifolds with boundary, the first author showed in [14, Proposition 2.5] that a metric with invertible Dirac operator on a closed spin manifold $M$ can be extended to a metric with invertible Dirac operator over the trace $W$ of a surgery of codimension $\geq 3$ on $M$.

From the Lichnerowicz formula it follows that every metric with positive scalar curvature on a closed spin manifold has invertible Dirac operator, that is $\mathcal{R}_{psc}(M) \subset \mathcal{R}_{inv}(M) \subset \mathcal{R}(M)$. Thus, obstructions to invertibility of the Dirac operator give obstructions to the existence of positive scalar curvature. One of the main obstructions is given by the index of the Dirac operator which is equal to the $\alpha$-genus of the manifold, compare [20].

The aim of this paper is to generalize Theorem [14] to handle attachments for manifolds with boundary. This will also be an extension of [14, Proposition 2.5] to codimension 2. For that we have to fix a notion of invertibility for Dirac operators on complete but non-compact manifolds. We will use the following conventions.

On a compact manifold with boundary we always assume that any Riemannian metric has a product structure near the boundary. When considering spectral properties of the Dirac operator we attach half-infinite cylinders (with the natural product metric) at the boundary and consider the Dirac operator acting on smooth $L^2$-sections on the resulting complete manifold.

Our goal is to show that if a metric $g$ with invertible Dirac operator is given on a manifold with boundary $M$, then there is a metric $g''$ with invertible Dirac operator on the manifold $M''$ obtained by attaching a handle of codimension $\geq 2$.
at the boundary of $M$ (see below). Moreover the metrics $g$ and $g''$ coincide outside an arbitrarily small neighbourhood of the handle attachment sphere.

For a compact spin manifold $M$ we denote by $\mathcal{R}^{\text{inv}}(M)$ the set of metrics with invertible Dirac operator and product structure near the boundary.

We assume as given a compact Riemannian spin manifold with boundary $(M, g)$, where $\dim M = n + 1$, and we assume that $D^g$ is invertible. Let $\partial g := g|_{\partial M}$ be the induced metric on the boundary so that $g = \partial g + dt^2$ in a neighbourhood of the boundary. The handle attachment (and the corresponding surgery on the boundary) is specified by a spin-structure preserving embedding $f : S^k \times B^{n-k} \to \partial M$.

For $k \geq 2$ the spin structure on $S^k \times B^{n-k}$ is uniquely determined. But for $k = 1$ there are two spin structures where we only allow the spin structure on $S^1$ which bounds a disks. Otherwise we cannot assure that the handle attachment will be compatible with the spin structures. From now on, when having a handle attachment for $k = 1$ we always mean implicitly that $S^1$ is equipped with the bounding spin structure.

The image of $S^k \times \{0\}$ is called the handle attachment sphere and is denoted by $S$. In Figure 1 the surgery sphere (a zero dimensional $S^0$) is represented by the two dots in the boundary of $M$.

![Figure 1. Handle attachment.](image)

More precisely (for pictures see the strategy described in Section 3.1), the manifold $M_\infty'$ will be obtained from $M_\infty$ in the following way: $S^k \times B^{n-k+1}$ will be embedded in $M_\infty$, where $B^{n-k+1}$ is the $(n-k+1)$-dimensional half disk $B^{n-k+1} := \{ (x_1, \ldots, x_{n-k+1}) \mid \sum x_i^2 \leq 1, x_{n-k+1} \leq 0 \}$.

The image of this embedding will be removed and replaced by $B^{k+1} \times S^{n-k-1}$, where $S^{n-k-1} := S^{n-k-1} \cap B^{n-k} \cong B^{n-k-1}$ denotes the lower hemisphere. Restricting the handle attachment to the boundary $\partial M$ the embedded $S^k \times B^{n-k}$ is replaced by $B^{k+1} \times S^{n-k-1}$. Thus, on the boundary we also have a $k$-dimensional surgery.

**Theorem 1.2.** Let $M$ be a manifold with boundary and $g \in \mathcal{R}^{\text{inv}}(M)$. Let $M''$ be obtained by a handle attachment as described above where $n-k \geq 2$. Then for any given neighbourhood of the surgery sphere there is a metric $g'' \in \mathcal{R}^{\text{inv}}(M'')$ such that $g'' = g$ outside that neighbourhood.
Note, that the construction described above is exactly the attachment of a $k$-handle at the boundary $\partial M$.

We indicate some applications of this result.

In Section 4 we will show that the space $\mathcal{R}^{\text{inv}}(S^3)$ is—in contrast to $\mathcal{R}^{\text{psc}}(S^3)$—not path-connected. More generally, let $(M, g \in \mathcal{R}^{\text{inv}}(M))$ be a closed 3-dimensional Riemannian spin manifold. Then, in Proposition 4.3 we construct infinitely many non-concordant metrics with invertible Dirac operators that are pairwise bordant but not concordant. This generalizes a result from the first author \cite[Theorem 3.3]{14} to dimension 3.

In Section 5 we discuss the concordance classification of metrics with invertible Dirac operator. These considerations are mainly based on the work of Stolz \cite{29}. The case of invertible Dirac operators is easier than the positive scalar curvature case since we have larger range of handle attachments available, and we do not have to take the fundamental group into account, see Section 5.2. Hence, one motivation to examine the invertible Dirac operator case is that it gives a simplified illustration of this circle of ideas.

In Section 6 we will prove a genericity result. Proposition 6.1 gives that if the subset $\mathcal{R}^{\text{inv}}(M_{\text{rel}} h) \subset \mathcal{R}(M_{\text{rel}} h)$ is non-empty, then it is open with respect to the $C^1$-topology and dense with respect to the $C^\infty$-topology. Here the notation $\text{rel} h$ means that we only consider metrics with a fixed boundary metric $h \in \mathcal{R}^{\text{inv}}(\partial M)$.

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2. Preliminaries

The following subsections contain material in preparation for the proof of Theorem 1.2.

2.1. Notation. The flat metric on $\mathbb{R}^n$ is denoted by $\xi^n$, the round metric of radius 1 on $S^n$ is denoted by $\sigma^n$. Let $B^n(r)$ denote the $n$-dimensional ball of radius $r$, and let $B^n := B^n(1)$. Let $S^n(r)$ denote the $n$-dimensional sphere of radius $r$, and let $S^n := S^n(1)$.

The spinor bundle of a Riemannian spin manifold $(M, g)$ is denoted by $\Sigma M$ (The construction of the spinor bundle and its dependence on the Riemannian metric is discussed in Section 2.3). The spinor bundle is a complex vector bundle which is fiberwise equipped with a hermitian metric denoted by $\langle \cdot, \cdot \rangle_g$. Sections of the spinor bundle are called spinors. The space of all smooth spinors with compact support, denoted by $C_c^\infty(\Sigma M)$, carries a scalar product

$$\langle \varphi_1, \varphi_2 \rangle_g = \int_M \langle \varphi_1, \varphi_2 \rangle_g \, dv^g, \quad \varphi_1, \varphi_2 \in C_c^\infty(\Sigma M).$$

The completion of $C_c^\infty(\Sigma M)$ with respect to this scalar product is denoted by $L^2(\Sigma M, g)$. Let $D^g$ be the corresponding classical Dirac operator. We denote by $H^1(\Sigma M, g)$ those spinors $\varphi \in L^2(\Sigma M, g)$ with $\|\nabla^g \varphi\|_{L^2(\Sigma M, g)} + \|\varphi\|_{L^2(\Sigma M, g)} < \infty$, where $\nabla^g$ is the Levi-Civita connection lifted to the spinor bundle.
In particular, let \( M \) be a compact subset such that \( \text{Spec} \) of the Dirac operators on \( M \) holds:

\[
\lambda \in \text{Spec}, \quad \text{for example if it is an eigenvalue for which the corresponding eigenspace is compact.}
\]



Moreover, the following decomposition principle holds for the essential spectrum.

Next we state properties of the spectrum on manifolds with cylindrical ends. To make things more precise, the manifold \( M \) is assumed to have a neighbourhood \( \partial M \times (-t_0,0] \) of its boundary where the metric has the product form \( \partial g + dt^2 \). Let \( M_\infty \) be the manifold \( M \) with half-infinite cylindrical ends attached,

\[
(M_\infty, g) := (M, g) \cup \partial M \ (\partial M \times [0, \infty), \partial g + dt^2).
\]

By a slight abuse of notation we use the same symbol \( g \) for the metric on \( M \) and on \( M_\infty \). Note that \( (M_\infty, g) \) contains a cylindrical part \( (\partial M \times (-t_0, \infty), \partial g + dt^2) \).

The Dirac operator \( D^g \) has a self-adjoint extension to a bounded operator

\[
D^g : H^1(\Sigma M_\infty) \to L^2(\Sigma M_\infty),
\]

see [3], Section 3.6.2. This operator is invertible with a bounded inverse if and only if it has a spectral gap \( (-\lambda, \lambda) \) around zero, that is if there is \( \lambda > 0 \) such that \( \|D^g \varphi\|_{L^2} \geq \lambda \|\varphi\|_{L^2} \) for all \( \varphi \in H^1(\Sigma M_\infty) \). We call the spectral gap maximal if \( \lambda \) or \(-\lambda\) is in the spectrum of \( D^g \).

The norms \( \|\varphi\|_{H^1(g)} \) and \( \|\varphi\|_{L^2(g)} + \|D^g \varphi\|_{L^2(g)} \) are equivalent on cylindrical manifolds which follows from the corresponding statement on compact manifolds, see for example [2], Corollary 3.2.4.

The boundary manifold \( (\partial M, \partial g) \) is a compact manifold without boundary which carries an induced spin structure [25], Page 200]. Thus, the Dirac operator \( D^g \) is a self-adjoint operator \( H^1(\Sigma \partial M) \to L^2(\Sigma \partial M) \) with discrete spectrum.

**Proposition 2.1.** Assume that \( D^g \) has a maximal spectral gap \( (-\Lambda, \Lambda) \), \( \Lambda > 0 \), around zero. That is, assume \( \|D^g \varphi\|_{L^2} \geq \Lambda \|\varphi\|_{L^2} \) for all \( \varphi \in H^1(\Sigma \partial M) \). Then...
We differentiate Proof of Part (3).

Since \( n \) depends on \( \lambda \), and we obtain \( l \).

Thus, \( \lambda \) decays exponentially with rate \( \sqrt{\Lambda^2 - \lambda^2} \) on the cylindrical end.

In particular, for the exponential decay of an eigenspinor \( \varphi \) corresponding to an eigenvalue \( \lambda \) with \( \Lambda^2 \geq \lambda^2 \) it holds that

\[
\int_{\partial M \times \{t_1\}} |\varphi|^2 \, dv^{\partial g} \leq 2^{-1}(\Lambda^2 - \lambda^2)^{-1/2} e^{-2\sqrt{\Lambda^2 - \lambda^2} t_1} \int_{\partial M \times [0,1]} |\varphi|^2 \, dv^g
\]

for \( 0 \leq t_1 \).

We need (3) as a quantitative version of (2). It shows that the decay rate only depends on \( \lambda, \Lambda \) but not on \( g \).

Proof of Part (3). We differentiate \( l(t)^2 := \int_{\partial M_t} |\varphi|^2 \, dv^{\partial g} \) where \( \partial M_t := \partial M \times \{t\} \) and we obtain \( l'(t)l(t) = \int_{\partial M_t} \langle \nabla^{\partial g}_{\partial_t} \varphi, \varphi \rangle \, dv^{\partial g} \). Here and in the rest of the proof, \( \langle \cdot, \cdot \rangle \) denotes the real part of the hermitian scalar product. Differentiating again and using the Cauchy-Schwarz inequality, we get

\[
l''(t)l(t) + l'(t)^2 = \int_{\partial M_t} \langle \nabla_{\partial_t}^{\partial g} \varphi, \nabla_{\partial_t}^{\partial g} \varphi \rangle \, dv^{\partial g} + \int_{\partial M_t} \langle (\nabla_{\partial_t}^{\partial g})^2 \varphi, \varphi \rangle \, dv^{\partial g}
\]

\[
\geq \left( \int_{\partial M_t} \langle \nabla_{\partial_t}^{\partial g} \varphi, \varphi \rangle \, dv^{\partial g} \right)^2 + \int_{\partial M_t} \langle (\nabla_{\partial_t}^{\partial g})^2 \varphi, \varphi \rangle \, dv^{\partial g},
\]

and, thus,

\[
l''(t)l(t) \geq \int_{\partial M_t} \langle \nabla_{\partial_t}^{\partial g} \varphi, \varphi \rangle \, dv^{\partial g}.
\]

Using the Schrödinger-Lichnerowicz formula and \( \text{scal}^g = \text{scal}^{\partial g} \) we write the square of the Dirac operator \( D^g \) on the cylinder as

\[
(D^g)^2 = (D^{\partial g})^2 + (\nabla_{\partial_t}^{\partial g})^2 = (D^{\partial g})^2 - (\nabla_{\partial_t}^{\partial g})^2.
\]

We obtain

\[
\lambda^2 l(t)^2 = \int_{\partial M_t} \langle (D^{\partial g})^2 \varphi, \varphi \rangle \, dv^{\partial g}
\]

\[
= \int_{\partial M_t} D^{\partial g} \varphi \, dv^{\partial g} - \int_{\partial M_t} \langle (\nabla_{\partial_t}^{\partial g})^2 \varphi, \varphi \rangle \, dv^{\partial g}
\]

\[
\geq \Lambda^2 l(t)^2 - l''(t)l(t).
\]

Thus, \( l''(t) \geq (\Lambda^2 - \lambda^2) l(t) \). Note that \( l(t) > 0 \) for all \( t \), since the zero set of an eigenspinor has zero \( n - 1 \) Hausdorff measure, see [3]. Next we will show that \( l' \leq 0 \).

Since \( \int_0^\infty l(t)^2 \, dt < \infty \) we have

\[
\lim_{t \to \infty} \int_{\partial M_t} \langle \nabla_{\partial_t}^{\partial g} \varphi, \varphi \rangle \, dv^{\partial g} = \lim_{t \to \infty} l'(t)l(t) = \lim_{t \to \infty} \left( \frac{1}{2} l(t)^2 \right)' = 0,
\]

(1) [8, Lemma 3.20] [26, Section 4] in the interval \((-\Lambda, \Lambda)\) the spectrum of \( D^g : H^1(\Sigma_{M,\infty}) \to L^2(\Sigma_{M,\infty}) \) consists of finitely many eigenvalues of finite multiplicity. Further, the essential spectrum of \( D^g \) is equal to \((-\infty, -\Lambda] \cup [\Lambda, \infty)\).

(2) [8, Lemma 3.21] [26, Section 4] any eigenspinor of \( D^g \) on \( M_{\infty} \) to the eigenvalue \( \lambda \) decays exponentially with rate \( \sqrt{\Lambda^2 - \lambda^2} \) on the cylindrical end.
which is used in the third step of the following computation.

\[
\lambda^2 \int_T^\infty l(t)^2 \, dt = \int_T^\infty \int_{\partial M_t} \langle (D^g)^2 \varphi, \varphi \rangle \, dv^g \, dt \\
= \int_T^\infty \int_{\partial M_t} \left( \langle (D^g)^2 \varphi, \varphi \rangle - \langle \nabla^g_{\partial_t} \varphi, \varphi \rangle \right) \, dv^g \, dt \\
= \int_T^\infty \int_{\partial M_t} \langle D^g \varphi, \varphi \rangle + |\nabla^g_{\partial_t} \varphi|^2 \rangle \, dv^g \, dt + \int_{\partial M_T} \langle \nabla^g_{\partial_t} \varphi, \varphi \rangle \, dv^g, \\
\geq \Lambda^2 \int_T^\infty l(t)^2 \, dt + l'(T)l(T).
\]

Since \( \Lambda^2 \geq \lambda^2 \) we conclude that \( l'(T) \leq 0 \). Together with \( l''(t) \geq (\Lambda^2 - \lambda^2)l(t) \) we have

\[
l(t) \geq e^{\sqrt{\Lambda^2 - \lambda^2}(t_1 - t)}l(t_1)
\]

for \( t \geq 0 \). Integrating and applying the Cauchy-Schwarz inequality we get

\[
l(t) \leq \int_0^1 l(t)e^{-\sqrt{\Lambda^2 - \lambda^2}(t_1 - t)} \, dt \\
\leq \left( \int_0^1 l(t)^2 \, dt \right)^{1/2} \left( \int_0^1 e^{-2\sqrt{\Lambda^2 - \lambda^2}(t_1 - t)} \, dt \right)^{1/2} \\
\leq 2^{-1/2}(\Lambda^2 - \lambda^2)^{-1/4}e^{-\sqrt{\Lambda^2 - \lambda^2}(t_1 - t)} \left( \int_0^1 l(t)^2 \, dt \right)^{1/2}.
\]

\[\square\]

**Lemma 2.2.** Let \((M, g)\) be a Riemannian manifold, let \( K \subset M \) be a compact subset, and let \( \Lambda > 0 \). Then there is a constant \( C = C(K, M, g, \Lambda) \) such that

\[\|\varphi\|_{C^2(K, g)} \leq C\|\varphi\|_{L^2(M, g)}\]

for any spinor \( \varphi \) on \((M, g)\) satisfying \( D^g \varphi = \lambda \varphi \) where \(|\lambda| < \Lambda\).

Note that \( M \) is not assumed to be compact. The proof of Lemma 2.2 is similar to Lemma 2.2 in [3].

**Lemma 2.3** (Ascoli’s Theorem, [1] Theorem 1.34]). Let \((M, g)\) be a Riemannian manifold and let \( K \subset M \) be a compact subset. Suppose that \( \varphi_i \) is a bounded sequence in \( C^2(K) \), then a subsequence of \( \varphi_i \) converges in \( C^1(K) \).

### 2.3. Comparing spinors for different metrics.

Let \( M \) be an \( n \)-dimensional spin manifold with Riemannian metrics \( g \) and \( g' \). In this subsection we review the method for comparing spinors for \( g \) and \( g' \) following Bourguignon and Gauduchon [10].

There is a unique endomorphism \( b^g_{g'} \) of \( TM \) which is positive, symmetric with respect to \( g \) and satisfies \( g(X, Y) = g'(b^g_{g'} X, b^g_{g'} Y) \) for all \( X, Y \in TM \). Since \( b^g_{g'} \) maps \( g \)-orthonormal frames to \( g' \)-orthonormal frames, this gives an \( SO(n) \)-principal bundle map \( b^g_{g'} : SO(M, g) \to SO(M, g') \). If the spin structures \( \text{Spin}(M, g) \) and \( \text{Spin}(M, g') \) are equivalent then the map \( b^g_{g'} \) lifts to a \( \text{Spin}(n) \)-principal bundle map \( \beta^g_{g'} : \text{Spin}(M, g) \to \text{Spin}(M, g') \). From this we get a map between the spinor bundles
\( \Sigma^g M \) and \( \Sigma^{g'} M \) which we will denote with the same symbol,

\[
\beta^g_{g'} : \Sigma^g M = \text{Spin}(M,g) \times_\sigma \Sigma_n \to \text{Spin}(M,g') \times_\sigma \Sigma_n = \Sigma^{g'} M
\]

\[
\psi = [s, \varphi] \mapsto \beta^g_{g'} \psi = [\beta^g_{g'} s, \varphi]
\]

where \((\sigma, \Sigma_n)\) is the complex spinor representation. The map \( \beta^g_{g'} \) preserves fiberwise the length of the spinors.

Let the Dirac operator \( D^{g'} \) act on sections of \( \Sigma^{g'} M \) as the operator

\[
^g D^{g'} := (\beta^g_{g'})^{-1} \circ D^{g'} \circ \beta^g_{g'}.
\]

Compared with the Dirac operator \( D^g \) on \( \Sigma^g M \) there is the following relation, see [10, Théorème 20],

\[
^g D^{g'} = D^g \psi + A^g_{g'} (\nabla^g \psi) + B^g_{g'} (\psi)
\]

(1)

where \( A^g_{g'} \) and \( B^g_{g'} \) are pointwise vector bundle maps whose pointwise norms are bounded by

\[
|A^g_{g'}| \leq C |g - g'|_g, \quad |B^g_{g'}| \leq C (|g - g'|_g + |\nabla^g (g - g')|_g).
\]

(2)

When \( g' \) and \( g \) are conformal with \( g' = F^2 g \) for a positive smooth function \( F \) we have

\[
^g D^{g'} (F^{-\frac{n-1}{2}} \psi) = F^{-\frac{n+1}{2}} D^g \psi.
\]

(3)

2.4. Removal of singularities. The next Lemma tells us that a spinor in \( L^2 \) which is harmonic outside a subset of codimension two can be extended to a harmonic spinor everywhere.

**Lemma 2.4.** Let \( (M, g) \) be a compact \((n+1)\)-dimensional manifold with boundary \( \partial M \), and let \( g \) be product on \( \partial M \times [-t_0, 0] \). Moreover, let \( S \subset \partial M \) be a compact submanifold of dimension \( k \leq n - 2 \). Let the manifold \( M_\infty \) be obtained from \( M \) as described above. Let \( B \subset M_\infty \) be a submanifold (possibly with boundary) of dimension \( k+1 \) with \( S \times [-t_0, \infty) \subset B \) and such that \( B \backslash (S \times (-t_0, \infty)) \) is a compact submanifold with boundary. Assume that \( \varphi \) is a spinor with \( \|\varphi\|_{L^2(M_\infty)} < \infty \) and \( D^g \varphi = 0 \) weakly on \( M_\infty \backslash B \). Then \( D^g \varphi = 0 \) holds weakly also on \( M_\infty \).

Note that the Lemma includes in the case \( B = S \times [-t_0, \infty) \).

**Proof.** The proof follows the method of [3, Lemma 2.4]. Let \( \psi \) be a compactly supported spinor. We will show that \( \int_{M_\infty} \langle \varphi, D^g \psi \rangle \, dv^g = 0 \).

Let \( U_B(\delta) \) consist of the points in \( M_\infty \) with distance to \( B \) less than \( \delta \). Let \( \eta : M_\infty \to [0,1] \) be a smooth cut-off function with \( \eta = 1 \) on \( U_B(\delta) \), \( \eta = 0 \) on
\[ M_\infty \setminus U_B(2\delta) \text{ and } |\text{grad}^g \eta| \leq 2/\delta. \] We compute
\[
\left| \int_{M_\infty} \langle \varphi, D^g \psi \rangle \; dv^g \right| = \left| \int_{M_\infty} \langle \varphi, D^g((1-\eta)\psi + \eta \psi) \rangle \; dv^g \right| \\
\leq \left| \int_{M_\infty} \langle \varphi, D^g((1-\eta)\psi) \rangle \; dv^g \right| + \left| \int_{M_\infty} \langle \varphi, \eta D^g \psi \rangle \; dv^g \right| \\
+ \left| \int_{M_\infty} \langle \varphi, \text{grad}^g \eta \cdot \psi \rangle \; dv^g \right| \\
\leq \left| \int_{M_\infty \setminus U_B(\delta)} \langle \varphi, D^g((1-\eta)\psi) \rangle \; dv^g \right| + \| \varphi \|_{L^2(U_B(2\delta))} \| D^g \psi \|_{L^2} \\
+ \frac{2}{\delta} \| \varphi \|_{L^2(U_B(2\delta))} \| \psi \|_{L^2(U_B(2\delta))}.
\]
The first term vanishes since \( D^g \varphi = 0 \) weakly on \( M_\infty \setminus B \) and \((1-\eta)\psi\) is compactly supported on \( M_\infty \setminus B \). The second summand goes to 0 as \( \delta \to 0 \). To estimate the third term note that
\[
\| \psi \|_{L^2(U_B(2\delta))}^2 \leq \max |\psi|^2 \text{vol}(U_B(2\delta) \cap \text{supp} \psi) \\
\leq \max |\psi|^2 C(\psi) \text{vol}_{k+1}(B \text{supp} \psi)(2\delta)^{n-k}
\]
where \( \text{vol}_k \) measures the \( k \)-dimensional volume, \( C(\psi) > 0 \) and \( B \text{supp} \psi \) denotes a compact subset of \( B \) such that \((U_B(2\delta) \cap \text{supp} \psi) \subset U_{\text{Bsupp} \psi}(2\delta) \). Then,
\[
\frac{2}{\delta} \| \varphi \|_{L^2(U_B(2\delta))} \| \psi \|_{L^2(U_B(2\delta))} \leq C\delta^{\frac{n-k}{2} - 1} \| \varphi \|_{L^2(U_B(2\delta))}
\]
where \( C \) only depends on \( \psi \) and, thus, with \( n-k \geq 2 \) this term also tends to 0 as \( \delta \to 0 \).

\[ \square \]

3. Handle attachment

In this section the proof of Theorem 1.2 is given in a sequence of steps. We begin by giving an overview and explaining the strategy of the proof.

3.1. Overview of the proof. We will use a similar construction as Carr in [12] where it is proved that the existence of positive scalar curvature metrics on manifolds with boundary is preserved under handle attachment of codimension \( \geq 3 \). For this the manifold is doubled in order to obtain a closed manifold and the handle attachment construction is split into two steps to make the construction of the new metric easier.

We will also split the surgery into two steps, but we work with the original manifold with attached cylindrical ends since we are interested in the invertibility of the Dirac operator.

We now describe the topological construction, and then explain how the metric will be obtained.

3.1.1. Topological strategy. Let \((M, g)\) be the initial manifold with product structure near the boundary on \((-\delta, 0) \times \partial M\). Moreover let \((M_\infty, g)\) be \(M\) with cylindrical ends attached, and let \(S \subset \partial M\) be the handle attachment sphere, where \(S\) is diffeomorphic to \(S^k\).
First we construct a surgery along $S^k \times (B^{n-k} \times (-\epsilon, \epsilon) \hookrightarrow \partial M \times (-t_0, \infty) \subset M_\infty$ where $S^k \times \{0\}$ is mapped to $S$, see the first picture in Figure 2, where $S$ is indicated as the dots inside the circles. By replacing the image of $S^k \times (B^{n-k} \times (-\epsilon, \epsilon) \cong S^k \times B^{n-k+1}$ by $B^{k+1} \times S^{n-k}$ we obtain $M'_\infty$.

Second we embed $S^k \times (B^{n-k} \times (c, \infty))$ into the part of $M'_\infty$ which lies “above” the first surgery, that is in $\partial M \times (c, \infty)$ for certain $c$. Moreover, we embed $B^{k+1} \times B^{n-k}$ into the attached handle $B^{k+1} \times S^{n-k}$ of the first surgery. Gluing both along its part of the boundary lying in $\partial M$, that is $S^k \times B^{n-k} \subset \partial M$, we obtain an embedding

$$S^k \times B^{n-k} \times (c, \infty) \sqcup B^{k+1} \times B^{n-k} \cong B^{k+1} \times B^{n-k} \hookrightarrow M'_\infty$$

The second surgery will replace the embedded $B^{k+1} \times B^{n-k}$ by $B^{k+2} \times S^{n-k-1}$.

---

Figure 2. Surgery divided in two steps

Note that after cutting $M'_\infty$ along the former boundary of $M$ (which is $\partial M \times \{0\} \subset M_\infty$) we already get the desired surgery, $S^k \times B^{n-k+1}$ is replaced by $B^{k+1} \times S^{n-k+1}$. Thus, topologically this would suffice. But in order to obtain a metric which has product structure near the boundary we have the second surgery which produces a cylinder above the boundary and which does not change the topology below the boundary. Thus, after both steps we still have the desired surgery on the manifold with boundary and additionally we already got the corresponding manifold with attached cylinders.

3.1.2. Metric strategy. One of the main tasks in the proof is to construct approximations of the metric such that the handles can be glued into the manifold and such that the metrics are easily extended to the handles. Moreover, this has to be done in such a way that the new metrics can be chosen to be arbitrarily close to the old one but still have an invertible Dirac operator on the manifold before and after surgery.

We now explain the steps in the proof.

- In Step 1, before starting with the first surgery, we approximate $g$ by metrics $\tilde{g}_\delta$. The new metrics will have product structure on a small tubular neighbourhood of $S \times (-t_0 + \delta, \infty)$. This product structure is not only a product in the direction tangential to the boundary as before but also product of $S$ and the normal directions inside the boundary. Moreover, the new metric will coincide with $g$ outside a larger tubular neighbourhood
A compact spin manifold with boundary. The manifold \( M \) can be smoothly approximated in steps by surgery using the embedding as in the beginning of Section 3 in [3]. Let \((M, g)\) be an embedding and set \( S := i(S^k) \). Let \( \pi^\nu : \nu \to S \) be the normal bundle of \( S \) in \((\partial M, \partial g)\). We assume that a trivialization of \( \nu \) is given through a vector bundle map \( \iota : S^k \times \mathbb{R}^{n-k} \to \nu \) such that \( (\pi^\nu \circ \iota)(p, 0) = i(p) \) for \( p \in S^k \). Further we assume that \( \iota \) is fiberwise an isometry when the fibers \( \mathbb{R}^{n-k} \) of \( S^k \times \mathbb{R}^{n-k} \) are given the standard metric, and the fibers of \( \nu \) have the metric induced by \( \partial g \). We get the embedding \( f \) by setting \( f := \exp^\nu \circ \iota : S^k \times B^{n-k}(R) \to \partial M \) for sufficiently small \( R \). We define open neighborhoods \( U_S(R) \) of \( S \) in \( \partial M \) by

\[
U_S(R) := (\exp^\nu \circ \iota)(S^k \times B^{n-k}(R))
\]

for \( R \) small enough. For a point \( x \in \partial M \) set \( r(x) := d(x, S) \) to be the distance from \( x \) to \( S \). Again, let \( h \) denote the pullback by \( i \) to \( S^k \) of the restriction of \( g \) to the tangent bundle of \( S \),

\[
h := i^*(g|_{TS \times TS}).
\]
Our goal is to perturb the metric $g$ slightly so that the map $f$ becomes an isometry if its domain is equipped with the product metric $h + \xi^{n-k}$. The next lemma gives an estimate of how much this fails for the metric $g$.

**Lemma 3.1** ([3 Lemma 3.1]). For sufficiently small $R > 0$ there is a constant $C > 0$ so that

$$G := \partial g - (f^{-1})^* (h + \xi^{n-k})$$

satisfies

$$|G| \leq Cr, \quad |\nabla G| \leq C$$
on $U_S(R)$.

We are now ready to go through the steps of the proof.

3.2. **Step 1: Approximating by product metrics.** We show that the metric on $(M, g)$ can be perturbed to have product form near the surgery sphere, the argument follows [3 Proposition 3.2]. We recall that the metric $g$ has by assumption a cylindrical structure $g = \partial g + dt^2$ in a neighbourhood $\partial M \times (-t_0, 0]$ of the boundary.

**Proposition 3.2.** The metric $g \in \mathcal{R}^{\text{inv}}(M)$ can be arbitrarily closely approximated by metrics $\overline{g}_\delta \in \mathcal{R}^{\text{inv}}(M)$ which have

$$\overline{g}_\delta = \partial \overline{g}_\delta + dt^2 = h + \xi^{n-k} + dt^2$$
on $U_S(\delta) \times (-t_0 + 2\delta, \infty)$ and

$$\overline{g}_\delta = g$$
on outside $U_S(2\delta) \times (-t_0 + \delta, \infty)$.

Before discussing the proof of this Proposition we define the metrics $\overline{g}_\delta$.

Let $\chi: \mathbb{R} \to [0, 1]$ be a smooth decreasing function with $\chi = 1$ on $(-\infty, 1]$, $\chi = 0$ on $[2, \infty)$, and $-2 \leq \chi' \leq 0$. On the part of $(M, g)$ which is isometric to $(\partial M \times (-t_0, \infty), \partial g + dt^2)$ we define a cut-off function

$$\eta(x, t) := \chi((x + t_0)/\delta),$$

where $\delta > 0$ is a small parameter. This has the property that $\eta(x, t) = 1$ if $x \in U_S(\delta)$ and $t \geq -t_0 + 2\delta$, and $\eta(x, t) = 0$ if $x \in \partial M \setminus U_S(2\delta)$ or if $t \leq -t_0 + \delta$. We define the metrics

$$\overline{g}_\delta := \eta(f^{-1})^* (h + \xi^{n-k}) + (1 - \eta) \partial g + dt^2$$
on $\partial M \times (-t_0, \infty)$ and we extend them by setting $\overline{g}_\delta := g$ on the rest of $M$. The metric $\overline{g}_\delta$ has the required product structure where $\eta = 1$, that is on $U_S(\delta) \times (-t_0 + 2\delta, \infty)$. Further, we have $\overline{g}_\delta = g$ outside $U_S(2\delta) \times (-t_0 + \delta, \infty)$. From

$$\overline{g}_\delta - g = \eta((f^{-1})^* (h + \xi^{n-k}) - \partial g)$$

(together with (2) and Lemma 3.1) we get that

$$|A^{\partial g}_{\overline{g}_\delta}| \leq C \eta r, \quad |B^{\partial g}_{\overline{g}_\delta}| \leq C n \eta \langle \text{grad} \eta \rangle r$$

for some $C > 0$. The metric $\overline{g}_\delta$ restricted to the boundary $\partial M$ gives the boundary metric

$$\partial \overline{g}_\delta = \eta(f^{-1})^* (h + \xi^{n-k}) + (1 - \eta) \partial g$$

$$= \chi(r/\delta)(f^{-1})^* (h + \xi^{n-k}) + (1 - \chi(r/\delta)) \partial g.$$
Figure 3. Approximating with a product metric.

Figure 3 shows \((M_\infty, g)\) to the left and \((M_\infty, \overline{g}_\delta)\) with the product region shaded to the right.

We begin by proving that the boundary metrics have uniform spectral gaps around zero. The proof is very similar to [3, Proposition 3.2].

**Lemma 3.3.** There are constants \(\Lambda, \delta_0 > 0\) such that the Dirac operator of the closed manifold \((\partial M, \partial g_\delta)\) has a spectral gap \((-\Lambda, \Lambda)\) for all \(\delta < \delta_0\).

**Proof.** We argue by contradiction and assume that there is a sequence \(\delta_i \to 0\) such that

\[
D^{\partial g_{\delta_i}} \varphi_i = \lambda_i \varphi_i
\]

where \(\lambda_i \to 0\) and \(\varphi_i\) are spinors on \((\partial M, \partial g_{\delta_i})\) with \(\int_{\partial M} |\varphi_i|^2 \, dv_{\partial g_{\delta_i}} = 1\). The proof continues exactly as in [3] and uses [1] and [4]. \(\square\)

We are now ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** The metrics \(\overline{g}_\delta\) have the required product structure, we need to show that \(\overline{g}_\delta \in R^{\text{inv}}(M)\) when \(\delta\) small enough. We proceed by assuming the contrary: there exists a sequence \(\delta_i \to 0\) such that the operators \(D^{\overline{g}_{\delta_i}}\) are not invertible.

From Lemma 3.3 we know that there are constants \(\Lambda, \delta_0 > 0\) such that the restriction \(\partial g_{\delta} := g_{\delta}|_{\partial M \times \{0\}}\) has a spectral gap \((-\Lambda, \Lambda)\) for all \(\delta < \delta_0\). From Proposition 2.1 it then follows that the essential spectrum of \(D^{\overline{g}_{\delta_i}}\) has the same gap, and, thus, the non-invertibility comes from a zero eigenvalue. Hence, there is a sequence of \(L^2\)-spinors \(\varphi_i\) on \((M_\infty, \overline{g}_{\delta_i})\) with \(D^{\overline{g}_{\delta_i}} \varphi_i = 0\) and \(\int_{M_\infty} |\varphi_i|^2 \, dv_{\overline{g}_{\delta_i}} = 1\).

First, we note that \(\overline{g}_{\delta_i} = g\) on \(M_\infty \setminus (U_S(2\delta_i) \times [-t_0 + \delta_i, \infty))\). Set \(U(\delta) := U_S(\delta) \times (-t_0, \infty)\).

Fix \(\gamma > 0\). Then for all \(i\) with \(2\delta_i < \gamma\) and all compact subsets \(K\) of \(M_\infty \setminus U(\gamma) \subset M_\infty \setminus (U_S(2\delta_i) \times [-t_0 + \delta_i, \infty))\) we have from Lemma 2.2 that there is a constant
$C = C(K, M_\infty \setminus U(\gamma), g)$ with

$$\|\varphi_i\|_{C^2(K)} \leq C\|\varphi_i\|_{L^2(M_\infty \setminus U(\gamma), g)} \leq C.$$  

From the Theorem of Ascoli, Lemma 2.3, we obtain that $\varphi_i \to \varphi$ strongly in $C^1(K)$ and $D^g \varphi = 0$ weakly on each $K$. Moreover, $\varphi_i \to \varphi$ weakly in $L^2(M_\infty \setminus U(\gamma), g)$ and $\|\varphi\|_{L^2(M_\infty \setminus U(\gamma), g)} \leq 1$. Thus, if $\gamma \to 0$ we obtain that $D^g \varphi = 0$ weakly on $M_\infty \setminus (S \times [-t_0, \infty))$ and $\varphi \in L^2(M_\infty, g)$. From Lemma 2.4 we then have $D^g \varphi = 0$ weakly on $M_\infty$.

It remains to show that $\varphi$ is not identically zero. We prove this by contradiction and assume that $\varphi = 0$. Thus, due to the Rellich-Kondrakov Theorem $\varphi_i \to 0$ in $L^2(g)$ on compact subsets. In particular, $\int_K |\varphi_i|^2 \, d\gamma_i \to 0$ as $i \to \infty$ for each compact $K \subset M_\infty$, since $|\gamma_i - g| \to 0$ on compact $K$.

To study $\varphi_i$ on $\partial M \times (0, \infty)$ we set $l_i(t)^2 := \int_{\partial M \times \{t\}} |\varphi_i|^2 \, d\gamma_i$. From part (3) of Proposition 2.1 we have $l_i(t)^2 \leq (2\Lambda)^{-1} e^{-2\Lambda(s-1)} \int_0^1 l_i(t)^2 \, dt$. Integrating this gives us

$$\int_1^\infty l_i(s)^2 \, ds \leq \frac{1}{2\Lambda} \int_1^\infty e^{-2\Lambda(s-1)} \, ds \int_0^1 l_i(t)^2 \, dt = \frac{1}{4\Lambda^2} \int_0^1 l_i(t)^2 \, dt$$

and, thus,

$$1 = \int_{M_\infty} |\varphi_i|^2 \, d\gamma_i = \int_M |\varphi_i|^2 \, d\gamma_i + \int_0^\infty l_i(t)^2 \, dt \leq \int_M |\varphi_i|^2 \, d\gamma_i + \left(1 + \frac{1}{4\Lambda^2}\right) \int_0^1 l_i(t)^2 \, dt \leq \left(1 + \frac{1}{4\Lambda^2}\right) \int_{M \cup (\partial M \times [0,1])} |\varphi_i|^2 \, d\gamma_i,$$

which gives a contradiction since $\varphi_i$ is supposed to tend to zero in $L^2(g)$ on the compact set $M \cup (\partial M \times [0,1])$. Thus, we obtained a nontrivial $L^2(g)$-harmonic spinor $\varphi$ on $(M_\infty, g)$ which contradicts the assumption that $g \in \mathcal{R}^{\text{inv}}(M)$. \hfill $\square$

After the first step we replace $g$ by $\gamma_{\delta_0}$ for some $\delta_0$ sufficiently small, we also set $-t_1 := -t_0 + 2\delta_0$ and $R_{\text{max}} := \delta_0$ (and perhaps we make the spectral gap of $D^g$ a bit smaller). The conclusion of this first step is then that we may assume that

$$\gamma_{\delta_0} = \partial g + dt^2 = h + \xi^{n-k} + dt^2$$

on $U_2(R_{\text{max}}) \times (-t_1, \infty)$ and

$$\gamma_{\delta_0} = g$$

outside $U_2(2R_{\text{max}}) \times (-t_1 - R_{\text{max}}, \infty)$, while the spectral gap is the same.

From now on we assume that the metric $g$ has already the form $\gamma_{\delta_0}$.

3.3. **Step 2: First surgery.** We now perform a standard surgery of codimension $n-k+1$ on $(M_\infty, g)$ along the embedding

$$S^k \times B^{n-k+1} = \mathcal{S}_k \times B^{n-k} \times (-\varepsilon, \varepsilon) \to \partial M \times (-t_1, \infty) \subset M_\infty$$

with $f^p : (x, y, t) \mapsto (f(x, y), -2p + t)$ where
where \(-t_1 < -2\rho - \varepsilon < -2\rho + \varepsilon < -\rho\). Here \(\rho\) is a parameter which will be specified later, and the first equality in the embedding comes from the choice of a diffeomorphism \(B^{n-k+1} \simeq B^n \times (-\varepsilon, \varepsilon)\).

Denote by \(M'_\infty\) the resulting manifold after surgery and by \(M'\) the same manifold without the cylindrical end. We will construct a family of metrics \(g'_\rho\) on \(M'_\infty\) which coincide with \(g\) outside the distance \(\rho\) of the surgery sphere.

On \(U_S(R_{\text{max}}) \times (-t_1, \infty)\) the metric \(g\) has the product form

\[
g = \partial g + dt^2 = h + \xi^{n-k} + dt^2 = h + \xi^{n-k+1}.
\]

The surgery in this step is centered around the surgery sphere \(S_\rho := S \times \{\rho\} \subset \partial M \times (-t_1, 0]\). We write the flat metric \(\xi^{n-k+1}\) in polar coordinates around 

\[(0, -2\rho) \in B^{n-k}(R_{\text{max}}) \times (-t_1, 0],\]

and we get

\[
g = h + d\hat{r}^2 + \hat{r}^2 \sigma^{n-k}
\]

where \(\hat{r} = \sqrt{r^2 + (t + 2\rho)^2}\) is the distance to the point \((0, -2\rho)\) and \(r\) is the distance to \(S \times (-t_0, \infty)\). Set \(U_{S_\rho}(R) := \{\hat{r} \leq R\} \subset M_\infty\). Figure 4 shows the placement of \(S_\rho\) and \(U_{S_\rho}(R)\).

We divide \(M\) into three pieces:

\begin{align*}
\{A\} & \quad M \setminus U_{S_\rho}(R_{\text{max}}/2) \\
\{B\} & \quad U_{S_\rho}(R_{\text{max}}/2) \setminus U_{S_\rho}(\rho/2) \simeq S^k \times (\rho/2, R_{\text{max}}/2) \times S^{n-k} \\
\{C\} & \quad U_{S_\rho}(\rho/2) \simeq S^k \times B^{n-k+1}(\rho/2)
\end{align*}

The manifold \(M'\) after surgery is obtained by replacing \(\{C\}\) by \(\{C'\} \quad B^{k+1} \times S^{n-k}\),

see Figure 5.

We define metrics \(g'_\rho\) on \(M'\) by

\begin{align*}
\{A\} & \quad g'_\rho := g \\
\{B\} & \quad g'_\rho := h + d\hat{r}^2 + \alpha_\rho(\hat{r})^2 \sigma^{n-k}
\end{align*}
\(\{C'\} \ g'_\rho := H + (2\rho/3)^2 \sigma^{n-k}\)

where the function \(\alpha_\rho\) is as in Figure 6 and \(H\) is a metric on \(B^{k+1}\) which is equal to \(d\tilde{r}^2 + h\) near the boundary which is possible since near the boundary \(B^{k+1}\) is diffeomorphic to \(S^k \times [0, \varepsilon]\). Figure 5 shows \((M_\infty, g)\) to the left and \((M'_\infty, g'_\rho)\) after surgery to the right.

Define the subset \(U'(R) \subset M'_\infty\) by \(M'_\infty \setminus U'(R) = M_\infty \setminus U_{S_\rho}(R)\) for \(R > \rho/2\). Note that \(\alpha_\rho(\tilde{r}) = \tilde{r}\) on \([\rho, \rho/2 R_{\max}]\) and, thus, \(g'_\rho = g\) on \(M'_\infty \setminus U'(\rho)\). Note also that the definition of \(g'_\rho\) does not involve \(\alpha_\rho(\tilde{r})\) for \(\tilde{r} > R_{\max}/2\). This part is defined so that we easily can extend the function \(\alpha_\rho\) to all of \(M'_\infty\). We set \(\alpha_\rho = 1\) on \(M'_\infty \setminus U'(R_{\max})\) and \(\alpha_\rho = 2\rho/3\) on \(\{C'\}\).
We define a conformally related metric on $M'$ by
\[
\tilde{g}_\rho := \alpha_\rho^{-2} g'_\rho.
\]
On \{B\} + \{C'\} we have that $\tilde{g}_\rho$ is a product metric,
\[
\tilde{g}_\rho = \alpha_\rho^{-2} H + \sigma^{n-k},
\]
where $H$ is defined as $dt^2 + h$ on \{B\}. For the proof of Proposition 3.5 we need the following Lemma, similar to \cite{3} Proposition 3.5.

**Lemma 3.4.** Let $s$ be such that $\rho < s < 2s < R_{\text{max}}/2$ and assume that $D^{g'_\rho} \psi'_\rho = 0$. Then
\[
\int_{U'(2s)\setminus U'(s)} |\psi'_\rho|^2 \, du^{g'_\rho} \geq \frac{1}{8} \int_{U'(s)} |\psi'_\rho|^2 \, du^{g'_\rho}.
\]

**Proof.** We make the conformal change $\tilde{g}_\rho := \alpha_\rho^{-2} g'_\rho$ and set
\[
\tilde{\psi}_\rho := \alpha_\rho^2 g'_\rho \psi'_\rho,
\]
observe here that we are working on the manifold $M$ which has dimension $n + 1$. From \cite{3} we then have
\[
D^{\tilde{g}_\rho} \tilde{\psi}_\rho = 0.
\]
Choose a cut-off function $\eta$ on $M'$ with $\eta = 1$ on $U'(s)$, $\eta = 0$ on $U'(2s)$. Since $d\eta$ is supported in $M' \setminus U'(\rho)$ where $g'_\rho = g$ we may assume
\[
|d\eta|_{g'_\rho} \leq 2/s
\]
which implies
\[
|d\eta|_{g'_\rho}^2 = \alpha_\rho^2 |d\eta|_{g'_\rho}^2 \leq 4\alpha_\rho^2 / s^2.
\]
We have
\[
D^{\tilde{g}_\rho} (\eta \tilde{\psi}_\rho) = \text{grad}^{\tilde{g}_\rho} \eta \cdot \tilde{\psi}_\rho
\]
which is supported in $U'(2s) \setminus U'(s)$ and can be estimated by
\[
|D^{\tilde{g}_\rho} (\eta \tilde{\psi}_\rho)|^2 = |\text{grad}^{\tilde{g}_\rho} \eta|_{\tilde{g}_\rho}^2 |\tilde{\psi}_\rho|^2 \leq \frac{4\alpha_\rho^2}{s^2} |\tilde{\psi}_\rho|^2.
\]
Since $\tilde{g}_\rho = \alpha_\rho^{-2} H + \sigma^{n-k}$ on $U'(2s)$ we have a lower spectral bound, see \cite{3} Lemma 2.5
\[
\int_{U'(2s)} |D^{\tilde{g}_\rho} (\eta \tilde{\psi}_\rho)|^2 \, dv^{\tilde{g}_\rho} \geq \frac{(n - k)^2}{4} \int_{U'(2s)} |\eta \tilde{\psi}_\rho|^2 \, dv^{\tilde{g}_\rho}
\]
\[
\geq \int_{U'(2s)} |\eta \tilde{\psi}_\rho|^2 \, dv^{\tilde{g}_\rho}.
\]
Using \cite{5} we get for the left-hand side,
\[
\int_{U'(2s)} |D^{\tilde{g}_\rho} (\eta \tilde{\psi}_\rho)|^2 \, dv^{\tilde{g}_\rho} \leq \frac{4}{s^2} \int_{U'(2s)\setminus U'(s)} \alpha_\rho^2 |\tilde{\psi}_\rho|^2 \, dv^{\tilde{g}_\rho}
\]
\[
= \frac{4}{s^2} \int_{U'(2s)\setminus U'(s)} \alpha_\rho |\psi'_\rho|^2 \, dv^{g'_\rho}
\]
\[
\leq \frac{8}{s} \int_{U'(2s)\setminus U'(s)} |\psi'_\rho|^2 \, dv^{g'_\rho}
\]
where we used that $\alpha_\rho \leq 2s$ in the final step, recall here that $\rho < s$ and, thus, $U'(2s) \setminus U'(s) = U_{S_\rho}(2s) \setminus U_{S_\rho}(s)$. Inserted in (6) we get
\begin{equation}
\frac{8}{s} \int_{U'(2s) \setminus U'(s)} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho} \geq \int_{U'(2s)} |\eta \psi_\rho'|^2 \ dv^{\tilde{g}_\rho}.
\end{equation}
Here we have for the right-hand side,
\begin{align*}
\int_{U'(2s)} |\eta \psi_\rho'|^2 \ dv^{\tilde{g}_\rho} &\geq \int_{U'(s)} |\tilde{\psi}_\rho|^2 \ dv^{\hat{g}_\rho} \\
&= \int_{U'(s)} \alpha_\rho^{-1} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho} \\
&\geq \frac{1}{s} \int_{U'(s)} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho},
\end{align*}
where we used that $\alpha_\rho \leq s$ in the final step. Inserted in (7) we get
\begin{equation}
\frac{8}{s} \int_{U'(2s) \setminus U'(s)} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho} \geq \frac{1}{s} \int_{U'(s)} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho},
\end{equation}
or
\begin{equation}
\int_{U'(2s) \setminus U'(s)} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho} \geq \frac{1}{8} \int_{U'(s)} |\psi_\rho'|^2 \ dv^{\tilde{g}_\rho}
\end{equation}
which is the claim of the Lemma. \qed

**Proposition 3.5.** $g_\rho' \in \mathcal{R}^{\text{inv}}(M')$ for all sufficiently small $\rho$.

**Proof.** To prove this Proposition we first observe that since the boundary metric $\partial g'_\rho = \partial g$ is independent of $\rho$ it follows that the essential spectrum of $D^{\tilde{g}_\rho}$ has a gap around zero which is independent of $\rho$, see Proposition 2.1. We can then proceed as in the proof of Theorem 1.2 of [3] and assume that there is a sequence $\rho_i \to 0$ so that $D^{\tilde{g}_{\rho_i}}$ has a harmonic spinor $\varphi_i \in L^2(M'_\infty, g'_{\rho_i})$. We normalize $\int_{M'_\infty} |\varphi_i|^2 \ dv^{\tilde{g}_{\rho_i}} = 1$.

Now let $\delta > 0$. For all $\rho_i < \delta$ we have $M'_\infty \setminus U'(\rho_i) = M'_\infty \setminus U_{S_\rho}(\rho_i)$ and on this part $g_{\rho_i} = g$. Note that $Z_\delta := M'_\infty \setminus U_{S_\rho}(3\delta) \subset M'_\infty \setminus U'(\rho_i)$. Then, by Lemma 2.2 we know that for each compact subset $K \subset Z_\delta$ there is a constant $C > 0$ with
\begin{equation}
\|\varphi_i\|_{C^2(K)} \leq C\|\varphi_i\|_{L^2(Z_{\delta}, g)}.
\end{equation}
Thus, $\|\varphi_i\|_{C^2(K)} \leq C$. By Ascoli’s Theorem, Lemma 2.3 we know that $\varphi_i$ then converges strongly in $C^1(K)$ to a spinor $\varphi$. Since $Z_\delta$ tends to $M'_\infty \setminus (S \times \{0\})$ as $\delta \to 0$ a diagonal subsequence argument tells us that $\varphi \in C^1_{\text{loc}}(M'_\infty \setminus (S \times \{0\}))$ and $D^g \varphi = 0$ on $M'_\infty \setminus (S \times \{0\})$. From $\|\varphi_i\|_{L^2(Z_{\delta}, g)} \leq 1$ the spinors $\varphi_i$ converge weakly in $L^2$, the limit has to be the same spinor $\varphi$. Thus $\|\varphi\|_{L^2(Z_{\delta}, g)} \leq \lim \inf \|\varphi_i\|_{L^2(Z_{\delta}, g)} \leq 1$ and $\|\varphi\|_{L^2(M'_\infty, g)} \leq 1$. Now Lemma 2.4 on removal of singularities tells us that $D^g \varphi = 0$ weakly on $(M'_\infty, g)$.

It remains to show that $\varphi$ is not identically zero. In the same way as in the proof of Proposition 3.2 one shows that $\varphi_i \to \varphi$ on compact subsets of $(M'_\infty, g)$ and
\begin{equation}
\int_{M'_\infty \setminus (\partial M \times [0,1])} |\varphi_i|^2 \ dv^{\tilde{g}_{\rho_i}} \geq c
\end{equation}
for a positive constant $c$. This means that the $\varphi_i$ cannot escape to infinity. Assuming that $\varphi = 0$ we get

$$c \leq \int_{M' \cup (\partial M \times [0,1])} |\varphi_i|^2 \, dv_{g_{\rho_i}}' = \int_{(M \setminus U') \cup (\partial M \times [0,1])} |\varphi_i|^2 \, dv_{g_{\rho_i}}' + \int_{U'(\rho_i)} |\varphi_i|^2 \, dv_{g_{\rho_i}}' = \left( \int_{(M \setminus U_{\rho_i}(\rho_i)) \cup (\partial M \times [0,1])} |\varphi_i|^2 \, dv_{g_{\rho_i}}' + \int_{U'(\rho_i)} |\varphi_i|^2 \, dv_{g_{\rho_i}}' \right) \to 0$$

Hence, we still have to rule out that $\varphi_i$ concentrates in the limit only in the attached neck. This follows immediately from Lemma 3.4. For $\rho_i < s < 2s < R_{\max}/2$ we have $U'(2s) \setminus U'(s) = U_{S_{\rho_i}}(2s) \setminus U_{S_{\rho_i}}(s) \subset M$ and with Lemma 3.4 we get that

$$\int_{U_{S_{\rho_i}}(2s) \setminus U_{S_{\rho_i}}(s)} |\varphi_i|^2 \, dv_{g_{\rho_i}}' \geq \frac{1}{8} \int_{U'(s) \setminus U'(\rho_i)} |\varphi_i|^2 \, dv_{g_{\rho_i}}' \geq c$$

which contradicts that $\varphi_i \to 0$ on compact subsets of $M_\infty$.

Thus, the harmonic spinors $\varphi_i$ converge to a non-zero harmonic $L^2$-spinor on $(M_\infty, g)$ as $i \to \infty$ which gives a contradiction since there are no such spinors for the metric $g$.

3.4. Step 3: Approximating with a product metric again. We have now performed the first surgery, and we fix a metric $g' := g'_{\rho_0}$ on $M'$ with the properties we need. That is $g' \in R^{inv}(M')$ and the metric is unchanged except near the surgery sphere so the product structure from Step 1 in a neighbourhood of $S \times [0,\infty]$ is preserved. The radius of this neighbourhood will be again denoted by $R_{\max}$.

The surgery in the previous step consisted of removing a neighbourhood $S^k \times B^{n-k+1}$ (this was $\{C\}$) of the surgery surface, where the radius of the ball $B^{n-k+1}$ is small. The boundary of the resulting manifold is diffeomorphic to $S^k \times S^{n-k}$, and the surgery is completed by attaching $B^{k+1} \times S^{n-k}$ (which we called $\{C'\}$).

We now define a submanifold $B \simeq B^{k+1}$ of $M'$, in $M'_\infty$ we have $B \simeq \mathbb{R}^{k+1} \simeq B^{k+1} \cup S^k \times [0,\infty)$. In $\{A\}$ and $\{B\}$ introduced in the previous subsection we set $B := S \times [-\rho/2,\infty)$ (with respect to the cylindrical structure), in part $\{C'\}$ we set $B := B^{k+1} \times \{p\}$ where $p \in S^{n-k}$ is chosen so that $B$ is a smooth connected submanifold. The position of $B$ in $(M'_\infty, g')$ is illustrated in the left of Figure 7.

Let $i' : B^{k+1} \to M'$ be the corresponding embedding with $i'(B^{k+1}) = B$. The submanifold $B$ has a natural trivialization of its normal bundle.

In this section we will show that the metric $g'$ can be deformed to have a product structure in an arbitrarily small neighborhood of $B$. In the subset $U_S(R_{\max}) \times (0,\infty)$ of the cylindrical end we already have

$$g' = \partial g + dt^2 = h + \xi^2 + dt^2$$

We will extend this product structure to a neighborhood of all of $B$.

Let $\pi' : \nu' \to B$ be the normal bundle of $B$ in $(M', g')$ and assume that a trivialization of $\nu'$ is given through a vector bundle map $i' : B^{k+1} \times \mathbb{R}^{n-k} \to \nu'$ such that $(\pi' \circ i')(p, 0) = i'(p)$ for $p \in B^{k+1}$. Further we assume that $i'$ is fiberwise an isometry when the fibers $\mathbb{R}^{n-k}$ of $B^{k+1} \times \mathbb{R}^{n-k}$ are given the standard metric, and the fibers of $\nu'$ have the metric induced by $g'$. For sufficiently small $R$ we
get an embedding \( f' := \exp' \circ i' : B^{k+1} \times B^{n-k}(R) \to M' \). We define an open neighborhood of \( B \) by

\[
U_B(R) := (\exp' \circ i')(B^{k+1} \times B^{n-k}(R))
\]

for \( R \) small enough. Let \( h' \) denote the pullback by \( i' \) to \( B^{k+1} \) of the restriction of \( g' \) to the tangent bundle of \( B \), and let \( r(x) \) be the distance from the point \( x \in M' \) to \( B \). Note that in the cylindrical end \( \partial M' \times [-t_1, \infty) \) we have \( h' = h + dt^2 \) and \( r \) coincides with the previous definition.

**Proposition 3.6.** The metric \( g' \in \mathcal{R}^{\text{inv}}(M') \) can be arbitrarily closely approximated by metrics \( \overline{g}'_\delta \in \mathcal{R}^{\text{inv}}(M') \) which have the form

\[
\overline{g}'_\delta = h' + \xi^{n-k}
\]

on \( U_B(\delta) \) and

\[
\overline{g}'_\delta = g'
\]

outside \( U_B(2\delta) \) and on the cylindrical end \( \partial M' \times [0, \infty) \).

We now define the metrics \( \overline{g}'_\delta \) and then prove that they have the required properties. Let \( \chi \) be the cut-off function introduced in Subsection 3.2 and set \( \eta(x) := \chi(r(x)/\delta) \) where \( \delta > 0 \) is a small parameter. We define

\[
\overline{g}'_\delta := \eta(f'^{-1})^*(h' + \xi^{n-k}) + (1 - \eta)g'
\]

To the right in Figure 7 we have \((M'_\infty, \overline{g}'_\delta)\) with the product region shaded.

![Figure 7. Second approximation with product metric.](image)

**Proof.** We need to prove \( \overline{g}'_\delta \in \mathcal{R}^{\text{inv}}(M') \) for small \( \delta \), the other properties are clear. Again we argue by contradiction and assume that there is a sequence \( \delta_i \to 0 \) such that \( \overline{g}'_{\delta_i} \notin \mathcal{R}^{\text{inv}}(M') \).

Since \( \partial \overline{g}'_\delta = \partial g' = \partial g \) is independent of \( \delta \) we have uniform gaps around zero in the essential spectrum of \( D\overline{g}'_\delta \), therefore the assumption that \( \overline{g}'_{\delta_i} \notin \mathcal{R}^{\text{inv}}(M') \)
implies the existence of harmonic spinors \( \varphi'_i \) on \((M'_\infty, \mathcal{g}'_i)\) with \( D_{\mathcal{g}'_i} \varphi'_i = 0 \) and \( \int_{M'_\infty} |\varphi'_i|^2 \, dv_{\mathcal{g}'_i} = 1 \).

The proof goes on exactly as the proof of Proposition 3.2. We note that \( \mathcal{g}'_i = g' \) on \( M'_\infty \setminus U_B(2\delta) \). We fix \( \gamma \) small enough. Then for all \( i \) with \( 2\delta_i < \gamma \) and all compact subsets \( K \subset M'_\infty \setminus U_B(\gamma) \subset M'_\infty \setminus U_B(2\delta_i) \) we get with Lemma 2.2 that

\[
\|\varphi'_i\|_{C^2(K)} \leq C \|\varphi'_i\|_{L^2(M'_\infty \setminus U_B(\gamma), \mathcal{g}')} \leq C
\]

where \( C \) is a constant only depending on \((K, M'_\infty \setminus U_B(\gamma), \mathcal{g}')\). From the Theorem of Ascoli, Lemma 2.3, we obtain that \( \varphi'_i \to \varphi' \) strongly in \( C^1(K) \) and \( D\mathcal{g}' \varphi' = 0 \) weakly on each \( K \). Moreover, \( \varphi'_i \to \varphi' \) weakly in \( L^2(M'_\infty \setminus U_B(\gamma), \mathcal{g}') \) and \( \|\varphi'\|_{L^2(M'_\infty \setminus U_B(\gamma), \mathcal{g}')}, \leq 1 \). Thus, if \( \gamma \to 0 \) we obtain that \( D\mathcal{g}' \varphi' = 0 \) weakly on \( M'_\infty \setminus B \) and \( \varphi' \in L^2(M'_\infty, \mathcal{g}') \). From Lemma 2.4 we then have \( D\mathcal{g}' \varphi' = 0 \) weakly on \( M'_\infty \). And again it remains to show that \( \varphi' \) does not vanish identically. This is done exactly as in Proposition 3.2 using part (3) of Proposition 2.1.

After this step we replace \( g' \) by \( \mathcal{g}'_{\delta_0} \) for some \( \delta_0 \) sufficiently small and define \( R'_{\max} := \delta_0 \).

3.5. Step 4: Second surgery. In this section, we perform surgery (or “half-surgery”) on \( B \) in \((M', g')\) to produce \((M'', g'')\). Here \( \rho > 0 \) is again a parameter which will be adjusted later. The aim is to replace a neighbourhood of \( B \) which is diffeomorphic to \( B^{k+1} \times B^{n-k} \) (see \( F \) below) by \( B^{k+2} \times S^{n-k-1} \) (see \( F' \) below).

On \( U_B(R'_{\max}) \) the metric \( g' \) has the product form

\[
g' = h' + \xi^{n-k} = h' + dr^2 + r^2 \sigma^{n-k-1},
\]

and in the cylindrical end where \( h' = h + dt^2 \) we have

\[
g' = h + dt^2 + \xi^{n-k} = h + dr^2 + r^2 \sigma^{n-k-1} + dt^2.
\]

We divide \( M' \) into three pieces, see Figure 9.

\[
\{D\} = M' \setminus U_B(R'_{\max}/2),
\{E\} = U_B(R'_{\max}/2) \setminus U_B(\rho/2) \simeq B^{k+1} \times (\rho/2, R'_{\max}/2) \times S^{n-k-1},
\{F\} = U_B(\rho/2) \simeq B^{k+1} \times B^{n-k}(\rho/2),
\]

and we divide the cylindrical end of \( M'_{\infty} \) in corresponding pieces,

\[
\partial M' \times [0, \infty) = \{\overline{D}\} + \{\overline{E}\} + \{\overline{F}\},
\]

where

\[
\{\overline{D}\} = \{\partial D\} \times [0, \infty), \quad \{\overline{E}\} = \{\partial E\} \times [0, \infty), \quad \{\overline{F}\} = \{\partial F\} \times [0, \infty)
\]

come from a decomposition of the boundary \( \partial M' = \partial M \) into three pieces

\[
\{\partial D\} = \partial M' \setminus U_S(R'_{\max}/2),
\{\partial E\} = U_S(R'_{\max}/2) \setminus U_S(\rho/2) \simeq S^k \times (\rho/2, R'_{\max}/2) \times S^{n-k-1},
\{\partial F\} = U_S(\rho/2) \simeq S^k \times B^{n-k}(\rho/2).
\]

Finally, we set

\[
\{D_\infty\} = \{D\} + \{\overline{D}\}, \quad \{E_\infty\} = \{E\} + \{\overline{E}\}, \quad \{F_\infty\} = \{F\} + \{\overline{F}\},
\]

so that \( M'_{\infty} = \{D_\infty\} + \{E_\infty\} + \{F_\infty\} \).

Let \( B^{k+2} \) denote the lower half of the \( (k + 2) \)-dimensional disk. Let \( H' \) be a metric on \( B^{k+2} \) which is equal to \( H + dt^2 \) near the horizontal part of the boundary (For the definition of \( H \) see Step 2.) and equal to \( h' + dr^2 \) near the hemisphere
part of the boundary, see Figure 8. Near the corners these metrics coincide as $h + dr^2 + dt^2$.

The manifold $M''$ after surgery is obtained by replacing $\{F\}$ by

$$\{F'\} = B^{k+2} \times S^{n-k-1}$$

and $\{\overline{F}\}$ by $\{\overline{F'}\} = \{\partial F'\} \times [0, \infty)$, where

$$\{\overline{F'}\} = B^{k+1} \times S^{n-k-1}.$$

We define metrics $g''_\rho$ on $M''$ by

$$\{D\} g''_\rho := g'$$

$$\{E\} g''_\rho := h + dr^2 + \alpha_\rho(r)^2 \sigma^{n-k}$$

$$\{F'\} g''_\rho := H' + (2\rho/3)^2 \sigma^{n-k-1}$$

($\alpha_\rho$ is as defined in Figure 6 when replacing $R_{\text{max}}$ by $R'_{\text{max}}$) and on the cylindrical end $g''_\rho := \partial g''_\rho + dt^2$ where

$$\{D\} \partial g''_\rho := \partial g'$$

$$\{E\} \partial g''_\rho := h + dr^2 + \alpha_\rho(r)^2 \sigma^{n-k-1}$$

$$\{F'\} \partial g''_\rho := H + (2\rho/3)^2 \sigma^{n-k-1}$$

In Figure 9 we have $(M''_{\infty}, g'_\rho)$ before surgery to the left and $(M''_{\infty}, g''_\rho)$ after surgery to the right. Note that the boundary manifold $(\partial M'', \partial g''_\rho)$ is the result of surgery on $\partial M$ along the embedding $f: S^k \times B^{n-k} \to \partial M$.

For $R > \frac{\rho}{2}$ we define $U''(R) \subset M''_{\infty}$ by $M''_{\infty} \setminus U''(R) = M''_{\infty} \setminus U_B(R)$. Note that on $M''_{\infty} \setminus U''(R)$ we have $g''_\rho = g'$. Further, we define the subset $\partial U''(R) \subset \partial M''$ by $\partial M'' \setminus \partial U''(R) = \partial M \setminus U_2(R)$ for $R > \frac{\rho}{2}$. Note that $\partial g''_\rho = \partial g$ on $\partial M'' \setminus \partial U''(R)$.

**Proposition 3.7.** $g''_\rho \in R^{\text{inv}}(M'')$ for all sufficiently small $\rho$.

Before proving the Proposition we need to show that the boundary metrics have a uniform spectral gap. For this we need the following Lemma, similar to [3, Proposition 3.5] and Lemma 3.4.

**Lemma 3.8.** Choose $\rho$ and $s$ so that $\rho < s < 2s < R'_{\text{max}}/2$ and assume that $\psi_\rho$ are spinors on $(\partial M'', \partial g''_\rho)$ satisfying

$$D^{\partial g''_\rho} \psi_\rho = \lambda_\rho \psi_\rho$$
where $32\lambda^2 \rho^2 s^2 < 1/2$. Then
\[
\frac{1}{128} \int_{\partial U''(s)} |\psi_p|^2 \, dv^{g''} \leq \int_{\partial U''(2s) \setminus \partial U''(s)} |\psi_p|^2 \, dv^{g''}.
\]

Proof. We make the conformal change $\partial \tilde{g}_p = \alpha_p^{-2} \partial g''_p$ (the function $\alpha_p$ was defined in Step 2) and set
\[
\tilde{\psi}_p = \alpha_p^{n-1} \beta \partial \tilde{g}_p \psi_p.
\]
We then have
\[
D^{\partial g_p} \tilde{\psi}_p = \lambda_p \alpha_p \tilde{\psi}_p.
\]
Choose $s$ so that
\[
\rho < s < 2s < R_{\text{max}}'/2
\]
and choose a cut-off function $\eta$ on $\partial M''$ with $\eta = 1$ on $\partial U''(s)$ and $\eta = 0$ on $\partial M'' \setminus \partial U''(2s)$. Since $d\eta$ is supported in $\partial U''(2s) \setminus \partial U''(s) \subseteq \{\partial E\}$ we may assume that
\[
|d\eta|_{\partial g_p'} \leq 2/s
\]
which implies
\[
|d\eta|_{\partial g_p'}^2 = \alpha^2_p |d\eta|_{\partial g'_p}^2 \leq 4\alpha^2_p / s^2.
\]
We have
\[
D^{\partial g_p} (\eta \tilde{\psi}_p) = \text{grad}^{\partial g_p} \eta \cdot \tilde{\psi}_p + \eta \lambda_p \alpha_p \tilde{\psi}_p,
\]
so
\[
|D^{\partial g_p} (\eta \tilde{\psi}_p)|^2 \leq 2|\text{grad}^{\partial g_p} \eta | \tilde{\psi}_p|^2 + 2\lambda_p^2 \alpha_p^2 |\eta \tilde{\psi}_p|^2
\]
\[
\leq \frac{8\alpha^2_p}{s^2} |\tilde{\psi}_p|^2 + 2\lambda_p^2 \alpha_p^2 |\eta \tilde{\psi}_p|^2
\]
(8)
where the first term is supported in \(\partial U''(2s) \setminus \partial U''(s)\). Since \(\partial \bar{g}_\rho = \sigma^{n-k-1} + \alpha_\rho^{-2} H\) on \(\partial U''(2s)\) we have a lower spectral bound, see [8] Lemma 2.5,

\[
\int_{\partial U''(2s)} |\mathcal{D}^{\bar{g}_\rho}(\eta\bar{\psi}_\rho)|^2 \, dv^{\bar{g}_\rho} \geq \frac{(n - k - 1)^2}{4} \int_{\partial U''(2s)} |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho}
\geq \frac{1}{4} \int_{\partial U''(2s)} |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho}. \tag{9}
\]

Using [8] we get for the left-hand side,

\[
\int_{\partial U''(2s)\setminus\partial U''(s)} |\mathcal{D}^{\bar{g}_\rho}(\eta\bar{\psi}_\rho)|^2 \, dv^{\bar{g}_\rho}
\leq \frac{8}{s^2} \int_{\partial U''(2s)\setminus\partial U''(s)} \alpha_\rho^2 |\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho} + 2\lambda_\rho^2 \int_{\partial U''(2s)} \alpha_\rho^2 |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho}
= \frac{8}{s^2} \int_{\partial U''(2s)\setminus\partial U''(s)} \alpha_\rho |\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho} + 2\lambda_\rho^2 \int_{\partial U''(2s)} \alpha_\rho^2 |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho}
\leq \frac{16}{s} \int_{\partial U''(2s)\setminus\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''} + 8\lambda_\rho^2 s^2 \int_{\partial U''(2s)} |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho},
\]

where we used that \(\alpha_\rho \leq 2s\) in the final step. Inserted in (9) we get

\[
\frac{16}{s} \int_{\partial U''(2s)\setminus\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''} \geq \left(\frac{1}{4} - 8\lambda_\rho^2 s^2\right) \int_{\partial U''(2s)} |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho}. \tag{10}
\]

Here we have for the right-hand side,

\[
\int_{\partial U''(2s)} |\eta\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho} \geq \int_{\partial U''(s)} |\bar{\psi}_\rho|^2 \, dv^{\bar{g}_\rho}
= \int_{\partial U''(s)} \alpha_\rho^{-1} |\psi_\rho|^2 \, dv^{g_\rho''}
\geq \frac{1}{s} \int_{\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''},
\]

where we in the final step used that \(\alpha_\rho \leq s\). Inserted in (10) we get

\[
\frac{16}{s} \int_{\partial U''(2s)\setminus\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''} \geq \left(\frac{1}{4} - 8\lambda_\rho^2 s^2\right) \frac{1}{s} \int_{\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''},
\]

or

\[
\int_{\partial U''(2s)\setminus\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''} \geq \frac{1 - 32\lambda_\rho^2 s^2}{64} \int_{\partial U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''},
\]

from which the lemma follows.

We also need one more version of this estimate.

**Lemma 3.9.** Let \(\rho < s < 2s < R_{\text{max}}/2\) and assume that \(\psi_\rho\) are harmonic \(L^2\)-spinors on \((M''_\rho, g''_\rho)\), that is \(\mathcal{D}^{g''_\rho}\psi_\rho = 0\) and \(\int_{M''_\rho} |\psi_\rho|^2 \, dv^{g''_\rho} < \infty\). Then

\[
\frac{1}{8} \int_{U''(s)} |\psi_\rho|^2 \, dv^{g_\rho''} \leq \int_{U''(2s)\setminus U''(s)} |\psi_\rho|^2 \, dv^{g''_\rho}.
\]

**Proof.** The proof is similar to the ones for Lemmas 3.4 and 3.8.

We can now show that the boundary metrics \(\partial g''_\rho\) have a uniform spectral gap.
Lemma 3.10. There is a $\Lambda > 0$ such that $\text{Spec} D^{g''_\rho} \cap [-\Lambda, \Lambda] = \emptyset$ for all sufficiently small $\rho$.

**Proof.** For a contradiction assume that there is a sequence $\rho_i \to 0$ such that there are eigenspinors $\varphi_i \in L^2(\partial M'', \partial g''_\rho)$ with $D^{g''_\rho} \varphi_i = \lambda_i \varphi_i$ and $\lambda_i \to 0$. We normalize the eigenspinors by $\int_{\partial M''} |\varphi_i|^2 \, dv^{g''_\rho} = 1$.

Fix $\delta > 0$. Then with $\partial g''_\rho \equiv \partial g$ on $\partial M'' \setminus \partial U''(\delta)$ for all $i$ with $\rho_i < \delta$ and from Lemma 2.2 we get that for those $i$ $\varphi_i$ is uniformly bounded in $C^2(\partial M'' \setminus \partial U''(\delta), g)$. Due to Ascoli’s theorem, Lemma 2.3, we get that $\varphi_i \to \varphi$ strongly in $C^1(\partial M'' \setminus \partial U''(\delta))$ and $D^{g''} \varphi = 0$ weakly on $\partial M \setminus U_S(\delta)$. Letting $\delta$ tend to zero and taking a diagonal sequence we find that $D^{g''} \varphi = 0$ weakly on $\partial M \setminus S$. Since $\|\varphi_i\|_{L^2(\partial M \setminus U_S(\rho_i))} \leq 1$, we get $\varphi \in L^2(\partial M \setminus S)$. Using the result on removal of singularities on closed manifolds in [3] Lemma 2.4 we see that $D^{g''} \varphi = 0$ holds weakly on $\partial M$.

It remains to show that $\varphi$ does not vanish identically. For a fixed $s < R'_{\max}/4$ Lemma 3.8 gives

$$
\frac{1}{128} \int_{\partial U''(s)} |\varphi_i|^2 \, dv^{g''_\rho} \leq \int_{\partial U''(2s) \setminus \partial U''(s)} |\varphi_i|^2 \, dv^{g''_\rho},
$$

$$
\leq \int_{\partial M'' \setminus \partial U''(s)} |\varphi_i|^2 \, dv^{g''_\rho},
$$

for all $i$ with $\rho_i < s$ and $\lambda_{g_i} < \frac{4}{\sqrt{\pi}}$. Therefore, we get

$$
\frac{1}{128} \int_{\partial M''} |\varphi_i|^2 \, dv^{g''_\rho} \leq (1 + \frac{1}{128}) \int_{\partial M'' \setminus \partial U''(s)} |\varphi_i|^2 \, dv^{g''_\rho},
$$

and

$$
1 \leq 129 \int_{\partial M \setminus U_S(s)} |\varphi_i|^2 \, dv^{g'}.
$$

Since $\partial M \setminus U_S(s)$ is compact, $\varphi$ cannot vanish identically. Thus, $\varphi$ is a harmonic spinor on $(\partial M, g')$ which gives the required contradiction. \qed

Finally, we are ready to prove Proposition 3.7.

**Proof of Proposition 3.7.** From Lemma 3.10 and Proposition 2.1 we know that $D^{g''}$ has a uniform gap in the essential spectrum for all small $\rho$. We argue by contradiction and assume that there are $L^2$-harmonic spinors $\varphi_i$ for $g''_\rho$ as $\rho_i \to 0$. We normalize the harmonic spinors by $\int_{M'_\infty} |\varphi_i|^2 \, dv^{g''_\rho} = 1$. The goal is to prove that these converge to an $L^2$-harmonic spinor on $(M'_\infty \setminus B, g')$, which then gives an $L^2$-harmonic spinor on $(M'_\infty, g')$ and, thus, a contradiction.

The next step is similar to the proof of 3.2. Fix $\delta > 0$ small enough. Note that for all $i$ with $\rho_i < \delta$ we have $(M'_{\infty} \setminus U''(\delta), g''_\rho) = (M'_{\infty} \setminus U_B(\delta), g')$. By Lemma 2.2 we obtain that $\varphi_i$ is uniformly bounded in $C^2(K)$ for any compact subset $K \subset M'_{\infty} \setminus U_B(\delta)$. From Ascoli’s Theorem, Lemma 2.3, we get $\varphi_i \to \varphi$ strongly in $C^1(K)$ and $D^{g'} \varphi = 0$ weakly on each $K$. Thus, $\varphi \in C^1(M'_{\infty} \setminus U_B(\delta))$. Hence, if $\delta \to 0$, we get a spinor $\varphi \in C^1_{loc}(M'_{\infty} \setminus B)$ with $D^{g'} \varphi = 0$ weakly on $M'_{\infty} \setminus B$. Using Lemma 2.4 we see that $D^{g'} \varphi = 0$ on $M'_{\infty}$. \qed
It remains again to show that $\varphi$ does not vanish identically. For a fixed $\delta \in (0, R'_{\text{max}})$ we get from Lemma 3.9 that
\[
\frac{1}{8} \int_{U''(\delta)} |\varphi_i|^2 \, dv^{g''_{\rho_i}} \leq \int_{U''(2\delta) \setminus U''(\delta)} |\varphi_i|^2 \, dv^{g''_{\rho_i}} \leq \int_{M'_{\text{max}} \setminus U''(\delta)} |\varphi_i|^2 \, dv^{g''_{\rho_i}},
\]
for all $i$ with $\rho_i < \delta$. It follows that
\[
\frac{1}{8} = \frac{1}{8} \int_{M'_{\text{max}}} |\varphi_i|^2 \, dv^{g''_{\rho_i}} \leq \frac{9}{8} \int_{M'_{\text{max}} \setminus U''(\delta)} |\varphi_i|^2 \, dv^{g''_{\rho_i}},
\]
or
\[
\frac{1}{9} \leq \int_{M'_{\text{max}} \setminus U''(\delta)} |\varphi_i|^2 \, dv^g.
\]
Since we know from Proposition 2.1 that each harmonic spinor $\varphi_i$ decays exponentially, this implies as in the proof of Proposition 3.2 that $\varphi$ cannot converge to zero on compact subsets. Hence, $\varphi$ cannot be identically zero. Thus, $\varphi$ is a nontrivial $L^2$-harmonic spinor on $(M'_{\text{max}}, g')$ which is a contradiction. \(\square\)

Proof of Theorem 1.2. The theorem now follows by choosing $g'' = g''_{\rho_0}$ with $\rho_0$ sufficiently small. \(\square\)

4. Non-isotopic Metrics with Invertible Dirac Operator

In this section we show that $\mathcal{R}^{\text{inv}}(M)$ has infinitely many components if $\dim M = 3$. This extends previous results using surgery techniques for positive scalar curvature [12, Theorem 4], [22, Chapter 4, Theorem 7.7], and for invertible Dirac operator [14, Theorem 3.3], where $\dim M = 4m - 1$, $m \geq 2$. For the case of metrics with invertible Dirac operator the fact that $\mathcal{R}^{\text{inv}}(M)$ has infinitely many components follows more easily from the explicit examples of spectral flow constructed in [20], [4] using families of Berger metrics. The motivation for the argument given here is primarily to illustrate the use of surgery techniques to prove spectral results for the Dirac operator.

Definition 4.1. Let $M$ and $N$ be closed Riemannian spin manifolds with metrics $g^0, g^1 \in \mathcal{R}^{\text{inv}}(M)$ and $h \in \mathcal{R}^{\text{inv}}(N)$.

1. $g^0$ and $g^1$ are called concordant if there exists a metric $g \in \mathcal{R}^{\text{inv}}([0, 1] \times M)$ with $g|_{\{i\} \times M} = g^i$ for $i = 0, 1$.

2. $g^0$ and $g^1$ are called isotopic if there exists a smooth path of metrics $g_t$ in $\mathcal{R}^{\text{inv}}(M)$ ($t \in \mathbb{R}$) with $g_t = g^0$ for $t \leq 0$ and $g_t = g^1$ for $t \geq 1$.

3. $g^0$ and $h$ are called bordant if there is a manifold $W$ with a metric $g^W \in \mathcal{R}^{\text{inv}}(W)$ and $\partial(W, g^W) = (M, g^0) \sqcup (N^-, h)$ where $N^-$ denotes the manifold $N$ equipped with the reverse orientation.

Both isotopy and concordance are equivalence relations [14, Corollary 2.2]. The corresponding sets of equivalence classes are denoted by $\tilde{\pi}_0 \mathcal{R}^{\text{inv}}(M)$ for the concordance classes and by $\pi_0 \mathcal{R}^{\text{inv}}(M)$ for the isotopy classes. Isotopic metrics are concordant [14, Corollary 2.1], this is the reason why non-concordant metrics can be used to detect path components in $\mathcal{R}^{\text{inv}}(M)$.
We will use the handle attachment result to construct non-concordant metrics in $R^{\text{inv}}(S^3)$—and the same for other 3-manifolds—from a handle decomposition of a 4-manifold with non-zero index.

**Lemma 4.2.** There are 4-dimensional spin manifolds $Y^i$, $i \in \mathbb{Z}$, with boundary $\partial Y^i = S^3$, and metrics $g^{Y^i} \in R^{\text{inv}}(Y^i)$ for which $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i - j)$ where $c \neq 0$.

**Proof.** Let $Y^0$ be the 4-dimensional ball $B^4$ with a “torpedo” metric $g^{Y^0} \in R^{\text{inv}}(Y^0)$ of positive scalar curvature, such that $\partial g^{Y^0}$ is the standard round metric on $S^3$, see for example [31] Section 1.3]. For positive $i$ we define the manifolds $Y^i$ as the connected sum of $i$ copies of the K3 surface with an open disc removed. For negative $i$ we set $Y^i := (Y^{-i})^{-}$. Using the spin bordism invariance of $\alpha$ we have

$$\alpha(Y^i \cup_{S^3} (Y^j)^-) = (i - j)\alpha(K3) = c(i - j)$$

where $c := \alpha(K3) \neq 0$. It remains to find metrics $g^{Y^i} \in R^{\text{inv}}(Y^i)$ for $i > 0$.

From [17] Corollary 6.3.19 we know that there exists a handle decomposition of the K3 surface which starts from the 4-dimensional ball $B^4$, then attaches a number of 2-handles $B^2 \times B^2$, before finishing by attaching a $B^4$. This means that $Y^i$ can be obtained by attaching a number of 2-handles to an initial $B^4$. Starting with the metric $g^{Y^0}$ on $B^4$ we apply Theorem 1.2 to extend it over the 2-handles to a metric $g^{Y^i} \in R^{\text{inv}}(Y^i)$. \hfill $\Box$

Let $h^i \in R^{\text{inv}}(S^3)$ be defined by $h^i := g^{Y^i}|_{S^3}$.

**Proposition 4.3.** Suppose $M$ is a closed 3-dimensional Riemannian spin manifold and $g \in R^{\text{inv}}(M)$. Then there are metrics $g^i \in R^{\text{inv}}(M)$, $i \in \mathbb{Z}$, such that $g^i$ is bordant to $g$ but $g^i$ is not concordant to $g^j$ for $i \neq j$.

**Proof.** By Theorem 1.2 there is for $i \in \mathbb{Z}$ a metric $g^i$ on $M \# S^3 = M$ which is bordant to $g \cup h^i$ on $M \cup S^3$. The metric $h^i \in R^{\text{inv}}(S^3)$ is bordant to zero through the bordism $(Y^i, g^{Y^i})$, using [14] Proposition 2.1 we can attach this bordism to the handle attachment bordism and conclude that $g^i$ is bordant to $g$. Denote by $(W^i, g^{W^i})$ the bordism between $(M, g^i)$ and $(M, g)$ we have now constructed. The manifold $W^i$ is diffeomorphic to the boundary connected sum of $[0, 1] \times M$ and $Y^i$.

For $i, j \in \mathbb{Z}$ assume that $g^i, g^j \in R^{\text{inv}}(M)$ are concordant. We then find a metric with invertible Dirac operator on the closed manifold $W^i \cup (W^j)^-$ obtained by attaching the identical (but oppositely oriented) boundary components $(M, g)$ to each other, and by attaching $(M, g^i)$ to $(M, g^j)$ using a concordance of the metrics. Then $\alpha(W^i \cup (W^j)^-) = 0$. Further, $W^i \cup (W^j)^-$ is diffeomorphic to the connected sum $(S^3 \times M) \# (Y^i \cup_{S^3} (Y^j)^-)$, so

$$0 = \alpha(W^i \cup (W^j)^-) = \alpha(S^1 \times M) + \alpha(Y^i \cup_{S^3} (Y^j)^-) = \alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i - j)$$

and we conclude that $i = j$. \hfill $\Box$

5. Concordance theory

In this section we study the concordance classes of metrics with invertible Dirac operator on a manifold with boundary. Following closely the work by Stolz for positive scalar curvature we prove an existence and classification theorem, see [30], [29]. For previous work in the positive scalar curvature case see [19], [16].
Let \( M \) be a manifold with boundary, and let \( h \in \mathcal{R}^{\text{inv}}(\partial M) \). We define \( \mathcal{R}(M \text{ rel } h) \) as the set of Riemannian metrics \( g \) on \( M \) for which \( \partial g = h \). Further we set \( \mathcal{R}^{\text{inv}}(M \text{ rel } h) := \mathcal{R}(M \text{ rel } h) \cap \mathcal{R}^{\text{inv}}(M) \).

By \( \Omega_n^{\text{pin}} \) we denote the ordinary spin bordism group of dimension \( n \). We also define

\[
\Omega_n^{\text{inv}} := \{(M, g) \mid M \text{ is a closed spin } n\text{-manifold, } g \in \mathcal{R}^{\text{inv}}(M)\}/\sim,
\]

where the equivalence relation \( \sim \) is defined by \((M_0, g_0) \sim (M_1, g_1)\) if there is a spin manifold \( W \) with \( \partial W = M_0 \cup M_1 \) and a metric \( H \in \mathcal{R}^{\text{inv}}(W) \) such that \( H|_{\partial W} = g_0 \cup g_1 \) and all involved orientations and spin structures are compatible.

**5.1. Manifolds with corners.** To study concordances of metrics on manifolds with boundary it is necessary to extend most of the theory and results obtained so far to manifolds with corners.

A manifold \( M \) of dimension \( n \) with corners of codimension 2 is a smooth manifold with charts modelled on open sets in \( \mathbb{R}^{n-2} \times (\mathbb{R}^2) \). Points with a neighbourhood diffeomorphic to a neighbourhood of the boundary of \( \mathbb{R}^{n-2} \times (\mathbb{R}^2) \) constitute the boundary \( \partial M \) of \( M \). Points with a neighbourhood diffeomorphic to a neighbourhood of the corner of \( \mathbb{R}^{n-2} \times (\mathbb{R}^2) \) constitute the corner \( \partial^2 M \) of \( M \). We assume that the boundary itself constitutes an embedded submanifold with boundary in \( M \). We consider only Riemannian metrics \( g \) on \( M \) which have a product structure \( g = \partial g + dt^2 \) near \( \partial M \) and a double product structure \( g = \partial^2 g + dt^1_\partial + dt^2_\partial \) near \( \partial^2 M \).

As for manifolds with boundary we let \( M_\infty \) be the manifold \( M \) with half-infinite cylindrical ends attached,

\[
(M_\infty, g) := (M, g) \cup (\partial M \times [0, \infty), \partial g + dt^2) \cup (\partial^2 M \times [0, \infty)^2, \partial^2 g + dt^1_\partial + dt^2_\partial).
\]

Now, \((M_\infty, g)\) is a complete Riemannian spin manifold. Thus, as in the case of manifolds with boundaries we can define the notion of invertibility of the Dirac operator and we have corresponding results for its spectrum.

We say that \((M, g)\) has invertible Dirac operator if the Dirac operator of \((M_\infty, g)\) is invertible when it acts on \(L^2\)-sections of the spinor bundle.

The next proposition gives information about the spectral theory on those manifolds and is a version of Proposition 2.1 for manifolds with corners.

**Proposition 5.1.** Let \((M, g)\) be a Riemannian spin manifold with corners \( X_i \). Let the boundary \( \partial M \) be decomposed into finitely many manifolds with boundaries \( N_i \) such that each boundary \( \partial N_i \) is a corner \( X_{j(i)} \). Assume that the Dirac operator on \((X_i, \partial^2 g)\) and the Dirac operator on \((N_i, \partial g)\) are invertible. Moreover, let \( M_T := M \cup (\bigcup_i N_i \times [0, T]) \cup (\bigcup_i X_i \times [0, T]^2) \) with the obvious identifications of the boundaries. Then the following holds.

1. \((M_\infty, g)\) is invertible of Proposition 6.1]{27]}. The Dirac operator on \((M_\infty, g)\) is invertible.

2. \((M_\infty, g)\) is invertible of Proposition 2.19]{27]. There are constants \( c, C > 0 \) such that for all harmonic spinors \( \varphi \) on \( M_\infty \)

\[
\int_{M \setminus M_T} |\varphi|^2 dv^g \leq Ce^{-cT} \|\varphi\|^2_{L^2(M_\infty)} \tag{11}
\]

for all \(L^2\)-harmonic spinors \( \varphi \) on \( M_\infty \).
(3) [27] from the proof of Prop. 2.19 Let \( \Lambda > 0 \) be such that the Dirac operators on \((N_i, \partial g)\) have a spectral gap on \((-\Lambda, \Lambda)\). Then in [11] the constants can be chosen as \( c = \Lambda \) and \( C = 2 \).

Bordisms of manifolds with boundary are naturally manifolds with corners. Such a bordism gives rise to a boundary bordism between the boundaries. For manifolds with boundary there are obvious extensions of the definitions of concordance and isotopy to \( \mathcal{R}^\text{inv}(M_{rel} h) \). Note that the concordance relation for manifolds with boundary then assumes an invertible Dirac operator on a manifold with corners.

Elementary constructions can be performed for metrics with invertible Dirac operator. A product \( M \times N \) with corners has invertible Dirac operator if at least one of the factors has. Attaching isometric boundary components by a sufficiently long attaching cylinder preserves invertibility of the Dirac operator, compare [14, Proposition 2.1]. Stretching an isotopy of metrics with invertible Dirac operator produces a concordance, compare [14, Proposition 2.3]

For a smooth manifold \( M \) with corner there is a procedure to round the corner, producing a smooth manifold \( \tilde{M} \) with boundary. Next we show that corners can be rounded while preserving invertibility of the Dirac operator. Let \( \tau \) be a metric on a two-dimensional triangular domain \( T \) which is a product near the boundary lines and a double product near the corners.

Assume that \( M \) is a manifold with corners and \( g \in \mathcal{R}^\text{inv}(M) \). We replace the corner piece \( \partial^2 M \times [0, \infty)^2 \) of \((M_\infty, g)\) by a part of \(((\partial^2 M \times T)_\infty, \partial^2 g + \tau)\), see Figure 10. If \( M \) and \( \partial^2 M \times T \) are sufficiently far apart we can use a cut-off function with sufficiently small gradient to conclude that the resulting manifold with boundary \((\tilde{M}, \tilde{g})\) has invertible Dirac operator, compare [14, Proposition 2.1].

Next we extend Theorem 1.2 to manifolds with corners.

**Theorem 5.2.** Let \((M, g)\) be a manifold with corners and \( g \in \mathcal{R}^\text{inv}(M) \). Let \( M'' \) be obtained by a handle attachment outside a neighbourhood of the corners and of codimension at least two. Then for any given neighbourhood of the surgery sphere there is a metric \( g'' \in \mathcal{R}^\text{inv}(M'') \) such that \( g'' = g \) outside this neighbourhood.

**Proof.** In principle the proof follows the proof of Theorem 1.2 since the handle attachment is done outside a neighbourhood of the corners. The steps explained in
the strategy \([3.1]\) remain the same. But one has to make sure that all the auxiliary lemmas can be adapted to the new situation. Next, we will describe the required changes in those lemmas and in the proof.

Step 1: In Lemma \([3.3]\) the boundary \((\partial M, \partial g)\) will now be itself a manifold with boundary. Thus, the statement is then just Proposition \([2.1]\). The proof of Proposition \([3.2]\) is done for corners analogously as before. But we now use Proposition \([5.1]\) instead of Proposition \([2.1]\). Moreover, the formulation of the Lemma \([2.4]\) for the removal of singularities has to be adapted to manifolds with corners. But its proof is exactly the same provided that \(S\) is placed outside a neighbourhood of the corners.

Step 2 and 3 can be done in the same way using Proposition \([5.1]\).

Step 4: The auxiliary Lemma \([3.10]\) is now needed for manifolds with boundary which is exactly the result of Theorem \([1.2]\). The rest of this step is done analogously to the adaptations discussed before. \(\square\)

5.2. The \(R_n^{\text{inv}}\) groups and statement of the Theorem. Following \([29]\) Definition 4.1] we define \(R_n^{\text{inv}} := \{(M, h) \mid M \text{ is a spin } n\text{-manifold}, h \in R^{\text{inv}}(\partial M)\}/\sim,\) where \(\partial M\) and \(M\) are allowed to be empty, and \(M\) is not required to be connected. The equivalence relation \(\sim\) is defined by \((M_0, h_0) \sim (M_1, h_1)\) if

- there is a spin manifold \(V\) with \(\partial V = \partial M_0 \sqcup \partial M_1\) and a metric \(H \in R^{\text{inv}}(V)\) such that \(H|_{\partial V} = h_0 \sqcup h_1,\)
- there is a spin manifold \(W\) with boundary \(M_0 \cup_{\partial M_0} V \cup_{\partial M_1} M_1,\)
- the orientations and spin structures on all manifolds involved are compatible in the obvious ways.

This is illustrated in Figure 11. The equivalence class of \((M, h)\) is denoted by \([M, h]\).

![Figure 11. The equivalence relation in \(R_n^{\text{inv}}\).](image)

Note that in contrast to the concordance theory of positive scalar curvature metrics the definition of the \(R_n^{\text{inv}}\) groups does not involve a fixed fundamental group. The reason is that the Surgery Theorem \([1.2]\) also allows handle attachment of codimension 2. Thus, any manifold and the manifold obtained by killing the fundamental group via codimension 2 surgeries represent the same element in \(R_n^{\text{inv}}\).

The set \(R_n^{\text{inv}}\) is an abelian group when addition is defined as disjoint union and the zero element is given by the equivalence class of the empty manifold. The
groups $R_n^{\text{inv}}$ are defined to fit in the sequence of abelian groups

$$\ldots \to R_{n+1}^{\text{inv}} \xrightarrow{\partial} \Omega_n^{\text{inv}} \xrightarrow{i} \Omega_n^{\text{spin}} \xrightarrow{j} R_n^{\text{inv}} \to \ldots$$

where the maps are defined by $\partial([M,g]) := [\partial M, g]$, $i([M,g]) := [M]$, and $j([M]) := [M,-]$. It is not complicated to see that this sequence is exact at $\Omega_n^{\text{inv}}$ and at $R_n^{\text{inv}}$.

Our main theorem follows [29, Theorem 5.4].

**Theorem 5.3.** Let $M$ be a connected spin manifold of dimension $n \geq 4$.

1. $R_n^{\text{inv}}(M)$ is nonempty if and only if $[M,h] \in R_n^{\text{inv}}$ vanishes.
2. If $R_n^{\text{inv}}(M)$ is nonempty then $R_{n+1}^{\text{inv}}$ acts freely and transitively on $\tilde{\pi}_0 R_n^{\text{inv}}(M)$.

For closed manifolds we get the following Corollary as a special case.

**Corollary 5.4.** Let $M$ be a closed connected spin manifold of dimension $n \geq 4$.

1. $R_n^{\text{inv}}(M)$ is nonempty if and only if $[M,-]$ is zero in $R_n^{\text{inv}}$.
2. If $R_n^{\text{inv}}(M)$ is nonempty then $R_{n+1}^{\text{inv}}$ acts freely and transitively on $\tilde{\pi}_0 R_n^{\text{inv}}(M)$.

**5.3. Proof of the Theorem.** The next Lemma is similar to [29, Lemma 4.3].

**Lemma 5.5.** Let $M$ be a manifold of dimension $n$, and let $h \in \mathcal{R}_n^{\text{inv}}(\partial M)$. Suppose $C$ is obtained from $M$ by removing the interior of a compact codimension 0 submanifold $N \subset M$. Assume $h$ can be extended to a metric $H$ on $C$ with invertible Dirac operator (and product near the boundary as usual). Then $(M,h) \sim (N,H|_{\partial N})$, so they define the same element in $R_n^{\text{inv}}$.

**Proof.** For the equivalence of $(M,h)$ and $(N,H|_{\partial N})$ in $R_n^{\text{inv}}$ the connecting part $V$ consists of $(C,H)$ with the cylinder $(\partial M \times I, h + dt^2)$ attached at $\partial M$, which has invertible Dirac operator by assumption. The role of $W$ is played by $\tilde{M} \times I$ which is the product manifold $M \times I$ with corners rounded, see Figure 12.

![Figure 12](image)

The following Corollary is immediate.

**Corollary 5.6.** If $h$ extends to a metric with invertible Dirac operator on all of $M$, then $[M,h] = 0$ in $R_n^{\text{inv}}$. 
Theorem 5.7. Let $M, V$ be spin manifolds of dimension $n \geq 4$ with boundary. Assume that $\partial M = \partial V$ and that there is a spin bordism from $V$ to $M$ for which the boundary bordism is a product $\partial M \times I$. If the inclusion $M \hookrightarrow W$ is a 1-equivalence, then a metric $g \in R^{\text{inv}}(V \cup h)$ can be extended to a metric $G$ on $W$ with invertible Dirac operator such that $G = h + dt^2$ on the boundary bordism.

Proof. The bordism $W$ can be built from $V \times I$ by attaching handles of codimension $\geq 2$ outside a neighborhood of $\partial V \times I$. For closed manifolds this is proved in [21, Chapter 8, Proposition 3.1], the argument works also in our setting. By Theorem 5.2 a metric in $R^{\text{inv}}(V \cup h)$ can be extended over $W$ as required.

We prove that every element of $R^{\text{inv}}_n$ is represented by the ball $B^n$ and a metric on its boundary, this is parallel to [29, Proposition 5.8]. We define addition on $\tilde{\pi}_0 R^{\text{inv}}(S^{n-1})$ by taking connected sum of metrics with invertible Dirac operator. This makes $\tilde{\pi}_0 R^{\text{inv}}(S^{n-1})$ into an abelian group. Recall that $\tilde{\pi}_0 R^{\text{inv}}(M)$ denotes the set of concordance classes as given in Definition 4.1.

Proposition 5.8. Let $n \geq 5$. For $[M, h] \in R^{\text{inv}}_n$ there is a $q \in R^{\text{inv}}(S^{n-1})$ so that $[M, h] = [B^n, q]$. The inclusion of elements of the form $[B^n, q]$ into $R^{\text{inv}}_n$ induces a group isomorphism between $R^{\text{inv}}_n$ and $\tilde{\pi}_0 R^{\text{inv}}(S^{n-1})$.

Proof. Let $[M, h] \in R^{\text{inv}}_n$. By making surgeries in the interior we may assume that $M$ is connected and simply connected. Take an embedding of $B^n$ in the interior of $M$ and apply the Extension Theorem 5.7 to the 1-equivalence $S^n \hookrightarrow W = M \setminus \text{int} B^n$. This gives a metric which extends $h$ on $\partial M$ to a metric $G$ on $W$. From Lemma 5.5 the first statement then follows with $q$ taken as the restriction of $G$ to $S^{n-1}$.

If $[M_0, h_0] = [M_1, h_1]$ in $R^{\text{inv}}_n$, then there is a bordism $(V, H)$ from $\partial M_0$ to $\partial M_1$ with $H \in R^{\text{inv}}(V)$ such that $H|_{\partial V} = h_0 \sqcup h_1$. From this it is not complicated to use Theorem 5.7 to find a concordance between the corresponding $q_0, q_1 \in R^{\text{inv}}(S^{n-1})$. Further, it is easy to see that the disjoint union $[M_0 \sqcup M_1, h_0 \sqcup h_1]$ corresponds to $[B^n \sqcup B^n, q_0 \sqcup q_1]$, which in turn is equivalent to the pair consisting of $B^n$ and the connected sum metric $q_0 \# q_1$ on $S^{n-1}$.

We are now ready to prove the first part of Theorem 5.3.

Proof of Theorem 5.3 (1). From Corollary 5.6 we know that $[M, h] = 0$ if $h$ extends to a metric with invertible Dirac operator on all of $M$, which is one direction of the claim.

For the other direction, suppose $[M, h] = 0$. This means that $(M, h)$ is equivalent to the empty manifold. By definition of the equivalence relation we then know that

- there is a manifold $V$ with $\partial V = \partial M$, and a metric $H \in R^{\text{inv}}(V)$ with $H|_{\partial V} = h$,
- there is a manifold $W$ with boundary $M \cup_{\partial M} V$,
- all manifolds have compatible spin structures,

see the left of Figure 13. By performing surgeries in the interior we may change $W$ to be connected and simply connected. Then, we introduce corners so that $W$ becomes a bordism from $V$ to $M$ which is a product vertical bordism of the boundaries, see the right of Figure 13. Since $M$ is connected the inclusion $M \hookrightarrow W$ is a 1-equivalence, and from the Extension Theorem 5.7 we conclude that the metric
$H$ extends to a metric on $W$ with invertible Dirac operator. In particular this metric, when restricted to $M$, gives an invertible extension of $h$ to $M$. \hfill \Box

Next we prove the second part of Theorem 5.3. For this we follow [29] and construct a pairing

$$i : \pi_0 R^{\text{inv}}(M \text{ rel } h) \times \pi_0 R^{\text{inv}}(M \text{ rel } h) \rightarrow R^{\text{inv}}_{n+1}$$

with the properties

\begin{itemize}
  \item $i([g_0], [g_1]) + i([g_1], [g_2]) = i([g_0], [g_2])$ for $[g_0], [g_1], [g_2] \in \pi_0 R^{\text{inv}}(M \text{ rel } h)$,
  \item For every $[g_0] \in \pi_0 R^{\text{inv}}(M \text{ rel } h)$ the map $i_{[g_0]} : \pi_0 R^{\text{inv}}(M \text{ rel } h) \rightarrow R^{\text{inv}}_{n+1}$ is a bijection, where $i_{[g_0]}([g]) = i([g_0], [g])$.
\end{itemize}

Using this pairing we define an action of $R^{\text{inv}}_{n+1}$ on $\pi_0 R^{\text{inv}}(M \text{ rel } h)$ by $x \cdot [g] = i^{-1}(x)$ for $x \in R^{\text{inv}}_{n+1}$ and $[g] \in \pi_0 R^{\text{inv}}(M \text{ rel } h)$. From the first property of $i$ it follows that this defines an action, and from the second property it follows that the action is free and transitive.

As a first step we define the pairing on metrics,

$$i : R^{\text{inv}}(M \text{ rel } h) \times R^{\text{inv}}(M \text{ rel } h) \rightarrow R^{\text{inv}}_{n+1}.$$ 

Let $\tilde{M} \times I$ be $M \times I$ with the corners rounded. Then $\partial(\tilde{M} \times I) = (-M) \cup \partial M \times I \cup M$. Take $g_0, g_1 \in R^{\text{inv}}(M \text{ rel } h)$. By stretching the interval $I$ we may assume that the metric $g_0 \cup h + dt^2 \cup g_1$ has invertible Dirac operator on the closed manifold $\tilde{M} \times I$, see Section 5.1. We define

$$i(g_0, g_1) := [\tilde{M} \times I, g_0 \cup h + dt^2 \cup g_1] \in R^{\text{inv}}_{n+1}.$$ 

**Lemma 5.9.** If $g_0$ and $g_1$ are concordant then $i(g_0, g_1) = 0$.

**Proof.** The fact that $g_0$ and $g_1$ are concordant means that the metrics extend to a metric with invertible Dirac operator on $M \times I$. By the discussion in Section 5.1 we get a metric with invertible Dirac operator on $M \times I$ which has $g_0 \cup h + dt^2 \cup g_1$ as boundary. From Corollary 5.6 we get that $i(g_0, g_1) = [\tilde{M} \times I, g_0 \cup h + dt^2 \cup g_1] = 0$. \hfill \Box
Lemma 5.10. For $g_0, g_1, g_2 \in \mathcal{R}_{n+1}^{\text{inv}}(M \times \sigma)$ we have

$$i(g_0, g_1) + i(g_1, g_2) + i(g_2, g_0) = 0.$$ 

Proof. Since $h \in \mathcal{R}_{n+1}^{\text{inv}}(\partial M)$ we have that $h + \sigma \in \mathcal{R}_{n+1}^{\text{inv}}(\partial M \times S)$. Attach $(M \times I, g_0 + dt^2)$, $i = 0, 1, 2$, to $(\partial M \times S, h + \sigma)$ as in Figure 14. This gives $(V, H)$ in the equivalence relation for $R_{n+1}^{\text{inv}}$. If we glue this manifold with three copies of $M \times I$ we get a closed manifold diffeomorphic to $\partial M \times D^2 \cup M \times S^1$, which is the boundary of $M \times D^2$ with corner rounded. We set $W$ in the equivalence relation for $R_{n+1}^{\text{inv}}$ to be $M \times D^2$ with corner rounded. With $(V, H)$ and $W$ chosen like this we conclude that $i(g_0, g_1) + i(g_1, g_2) + i(g_2, g_0) = 0$ in $R_{n+1}^{\text{inv}}$. Setting $g_1 = g_0$ we see that $i(g_2, g_1) = -i(g_1, g_2)$, and the claim of the Lemma follows.

We now prove that $i_{[g_0]}$ is injective. Suppose that $i_{[g_0]}([g_1]) = i_{[g_0]}([g_2])$, then

$$0 = -i_{[g_0]}([g_1]) + i_{[g_0]}([g_2]) = -i([g_0], [g_1]) + i([g_0], g_2]) = i([g_1], [g_0]) + i([g_0], [g_2]) = i([g_1], [g_2]).$$

If the interval $I$ is long enough the metric $g_1 + h + dt^2 \cup g_2$ has invertible Dirac operator on $(-M) \cup \partial M \times I \cup M$. Since $i([g_1], [g_2]) = [M \times I, g_1 + h + dt^2 \cup g_2] = 0$ it follows from part (1) of Theorem 5.3 that the metric $g_1 \cup h + dt^2 \cup g_2$ extends to a metric $G \in \mathcal{R}_{n+1}^{\text{inv}}(M \times I \text{ rel } (g_1 \cup h + dt^2 \cup g_2))$, again for all sufficiently long intervals $I$.

We use the metric $\sigma$ to reintroduce the corners in $\widetilde{M \times I}$, see Figure 15. After suitable stretching of the product structures normal to the attaching boundaries

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{Figure 14.}
\end{figure}
this will give a metric with invertible Dirac operator, that is a concordance from \( g_1 \) to \( g_2 \). We conclude that \([g_1] = [g_2]\), and \( i_{[g_0]} \) is injective.

\[
\begin{align*}
(M \times I, g_1 + dt^2) & \quad (M \times I, g_2 + dt^2) \\
(M \times I, G) & \quad (\partial M \times S, h + \sigma)
\end{align*}
\]

**Figure 15.**

Next we prove surjectivity of \( i_{[g_0]} \). By Proposition 5.8 we know that any element of \( R_{n+1}^{\text{inv}} \) can be represented as \([B^{n+1}, q]\) for some \( q \in R_{\text{inv}}(S^n)\). We must find \( g_1 \in R_{\text{inv}}(M \text{ rel } h) \) such that \( i_{[g_0]}([g_1]) = i([g_0], [g_1]) = [B^{n+1}, q] \).

Remove an open ball from the interior of \( M \times I \) and denote the remaining manifold by \( C \). We then have \( M \times I = B^{n+1} \cup S^n C \). Removing the open ball does not change \( M \times \{1\} \hookrightarrow M \times I \) being a 1-equivalence, so also \( M \times \{1\} \hookrightarrow C \) is a 1-equivalence. By the Extension Theorem 5.7 we can extend the metric \( g_0 \cup q \) on \( M \times \{0\} \cup S^n \) to a metric \( G \) with invertible Dirac operator on \( C \). Set \( g_1 = G|_{M \times \{1\}} \), then

\( i([g_0], [g_1]) = [\widetilde{M} \times I, g_0 \cup h + dt^2 \cup g_1] = [B^{n+1}, q] \)

by Lemma 5.5.

\[ \square \]

### 5.4. The \( R_{n}^{\text{inv}} \) groups and the index.

Using the index of the Dirac operator we can conclude that the group \( R_{n}^{\text{inv}} \) is non-trivial in certain dimensions. Following Bunke [11] and Stolz [29] we sketch the definition of the index map

\[ \theta : R_{n}^{\text{inv}} \to KO_n. \]

For \([M, h] \in R_{n}^{\text{inv}}\) we extend the metric \( h \) to a metric \( g \) on all of \( M \). We view the Dirac operator \( D^g \) as a \( Cl_n \)-linear operator on \( L^2(\Sigma M_\infty) \). Let \( \chi : \mathbb{R} \to [-1, 1] \) be an increasing, odd, smooth function which is constant \( \pm 1 \) outside a bounded interval the size of which is related to the spectral gap of \( D^h \) on \( \partial M \).

The pair \((L^2(\Sigma M_\infty), \chi(D))\) is then a Kasparov module representing \( \theta([M, h]) \in KK(\mathbb{R}, Cl_n) = KO_n \). For details, see Section 9 of [29]. From Theorem 1.2 of [11] it follows that \( \theta : R_{n}^{\text{inv}} \to KO_n \) is well-defined.

For a compact manifold \( M \) without boundary the index map coincides with the ordinary index, \( \theta([M, -]) = \alpha(M) \). Since \( \alpha \) is surjective we conclude that \( \theta \) is also surjective. Further, if \( KO_n \) is non-trivial then \( R_{n}^{\text{inv}} \) is also non-trivial.

From this observation we get a result on existence of metrics with harmonic spinors, see Hitchin [20] and Bär [4] for the case of closed manifolds.
Theorem 5.11. Let $M$ be a spin manifold with boundary, $\dim M = n$ and $h \in \mathcal{R}^{\text{inv}}(\partial M)$. Assume $n$ is such that $\mathcal{R}^{\text{inv}}(M)_{n+1}$ is non-trivial, for example $n \equiv 0, 1, 3, 7 \pmod 8$. Then there is a metric on $M$ which extends $h$ and has non-trivial harmonic $L^2$-spinors.

Proof. If $\mathcal{R}^{\text{inv}}(M \text{ rel } h)$ is empty then all metrics in $\mathcal{R}(M \text{ rel } h)$ have non-trivial harmonic spinors. If $\mathcal{R}^{\text{inv}}(M \text{ rel } h)$ is non-empty it must have several components by Theorem 5.3 (2), so $\mathcal{R}^{\text{inv}}(M \text{ rel } h) \neq \mathcal{R}(M \text{ rel } h)$. □

Inspired by a similar conjecture for the case of positive scalar curvature metrics, [28, Conjecture 5.7], we make the following conjecture.

Conjecture 5.12. The index map $\theta : \mathcal{R}^{\text{inv}}_{n} \to KO_n$ is injective.

Injectivity of the index map means that $h \in \mathcal{R}^{\text{inv}}(\partial M)$ extends to a metric in $\mathcal{R}^{\text{inv}}(\partial M)$ if and only if the index $\theta([M, h])$ vanishes.

6. Genericity of metrics with invertible Dirac operator

From the surgery theorem for the Dirac operator on closed manifolds, Theorem 1.1, it follows that generic metrics on a closed manifold have the minimal dimension allowed by the index theorem, see [3, Theorem 1.1]. In particular, if the index vanishes then a generic metric has invertible Dirac operator. The proof uses the fact that if there is one minimal metric then generic metrics are minimal, surgery can then be used to produce one such metric on a given manifold.

Here the term generic means that the subset of minimal metrics is open in the $C^1$-topology and dense in the $C^\infty$-topology on the set of all Riemannian metrics.

Our goal is to obtain a similar statement for manifolds with boundary. We begin by proving that if there is one metric with invertible Dirac operator then generic metrics with the same boundary have invertible Dirac operator.

Proposition 6.1. Assume that $\mathcal{R}^{\text{inv}}(M \text{ rel } h)$ is nonempty. Then $\mathcal{R}^{\text{inv}}(M \text{ rel } h)$ is open with respect to the $C^1$-topology and dense with respect to the $C^\infty$-topology in $\mathcal{R}(M \text{ rel } h)$.

To prove Proposition 6.1 we need the following lemma.

Lemma 6.2. Let $g, g' \in \mathcal{R}(M_\infty)$ with the boundary metrics $\partial g = \partial g' = h$ on $\partial M$. Then the maps $g' \mapsto \|\beta^g_{g'} \phi\|^2_{L^2(g')} \ (g' \mapsto \|\beta^g_{g'} \phi\|^2_{H^1(g')})$ are uniformly continuous in $\phi \in \Sigma^9 M$ with respect to the $C^0$-topology ($C^1$-topology) on $\mathcal{R}(M \text{ rel } h)$.

Proof. We start with the case of a closed manifold. In local coordinates one sees immediately that the volume element depends continuously on $g$ in the $C^0$-topology and that the Christoffel symbols depend continuously on $g$ in the $C^1$-topology. Since $\beta^g_{g'}$ is fiberwise an isometry, the $L^2$-norm ($H^1$-norm) on a single chart depends continuously on the $C^0$-topology ($C^1$-topology) on $\mathcal{R}(M)$. Hence, the statement is true for closed manifolds.

The Lemma in general is proven by decomposing $M_\infty$ into $M \cup \partial M \times (0, \infty)$. Since $M$ and $\partial M$ are compact and the metrics are constant in the $(0, \infty)$-direction the lemma follows. □

From that lemma we get immediately the following corollary.
Corollary 6.3. Let \( g, g' \in \mathcal{R}(M_{\infty}) \) with the boundary metrics \( \partial g = \partial g' = h \) on \( \partial M \). Then, the norms \( \|D^g(\beta_M'\cdot)\|_{L^2(g')} \) and \( \|D^{g'}\|_{L^2(g)} \) are equivalent. In particular, \( D^g : L^2(\Sigma^g M_\infty) \rightarrow L^2(\Sigma^g M_\infty) \) is invertible if and only if \( D^{g'} : L^2(\Sigma^{g'} M_\infty) \rightarrow L^2(\Sigma^{g'} M_\infty) \) is.

**Proof of Proposition 6.4.** Metrics in \( \mathcal{R}(M \text{ rel } h) \) are the same on the cylindrical end, so the essential spectrum is also the same for such metrics. Since we assume \( \mathcal{R}^{\text{inv}}(M \text{ rel } h) \) to be nonempty, the essential spectrum for each metric in \( \mathcal{R}(M \text{ rel } h) \) is \((-\infty, -\Lambda) \cup [\Lambda, \infty)\) where \( \Lambda > 0 \) is the absolute value of the lowest eigenvalue of \( D^h \), see Proposition 2.1. This means that on \((-\Lambda, \Lambda)\) the spectrum of any metric in \( \mathcal{R}(M \text{ rel } h) \) is discrete and the dimension of the kernel is finite, which allows to carry over the proof from the case of closed manifolds, see [23, Proposition 3.1].

Due to the corollary above it is enough to examine invertibility of the operator \( \mathcal{D}^g \) for a fixed background metric \( \mathcal{g} \in \mathcal{R}(M \text{ rel } h) \).

First, one shows that the map \( g \mapsto \mathcal{D}^g \) from \( \mathcal{R}(M \text{ rel } h) \) to \( \mathcal{B}(H^1(\mathcal{g}), L^2(\mathcal{g})) \) is continuous in the \( C^1 \)-topology on \( \mathcal{R}(M \text{ rel } h) \). Here \( \mathcal{B}(H^1(\mathcal{g}), L^2(\mathcal{g})) \) denotes the space of bounded linear operators from \( H^1(\mathcal{g}) \) to \( L^2(\mathcal{g}) \). That \( \mathcal{D}^g \in \mathcal{B}(H^1(\mathcal{g}), L^2(\mathcal{g})) \) follows immediately from the estimate

\[
\|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})} \leq a\|D^g(\beta_M \varphi)\|_{L^2(\mathcal{g})} \leq an\|\nabla^g(\beta_M \varphi)\|_{L^2(\mathcal{g})}
\]

where \( a, b \) are constants coming from the equivalence of the norms with respect to different metrics, see Lemma 6.2.

Moreover, if \( g \in \mathcal{R}^{\text{inv}}(M \text{ rel } h) \) there is a neighbourhood of \( \mathcal{g} \) with respect to the norm topology on \( \mathcal{B}(H^1(\mathcal{g}), L^2(\mathcal{g})) \) such that all operators in this neighbourhood are also invertible. This is deduced from the following estimate. If \( \varepsilon \) is small enough and \( A \in \mathcal{B}(H^1(\mathcal{g}), L^2(\mathcal{g})) \) lies in the \( \varepsilon \)-neighbourhood of \( \mathcal{D}^g \), we have

\[
\|A \varphi\|_{L^2(\mathcal{g})} \geq \|\mathcal{D}^g \varphi - (\mathcal{D}^g - A) \varphi\|_{L^2(\mathcal{g})}
\]

\[
\geq \|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})} - \|\mathcal{D}^g - A\|_{H^1(\mathcal{g})} \|\varphi\|_{L^2(\mathcal{g})}
\]

\[
\geq \|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})} - b(\|\varphi\|_{L^2(\mathcal{g})} + \|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})})
\]

\[
\geq \|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})} - b(\|\varphi\|_{L^2(\mathcal{g})} + a\|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})})
\]

\[
\geq (1 - ab\varepsilon)\|\mathcal{D}^g \varphi\|_{L^2(\mathcal{g})} - b\varepsilon\|\varphi\|_{L^2(\mathcal{g})}
\]

\[
\geq (C(1 - ab\varepsilon) - b\varepsilon)\|\varphi\|_{L^2(\mathcal{g})},
\]

where \( b \) is the constant describing the equivalence of the norms \( \| \cdot \|_{H^1(\mathcal{g})} \) and \( \| \cdot \|_{L^2(\mathcal{g})} + \|D^g(\cdot)\|_{L^2(\mathcal{g})} \), see for example [27, Prop. 2.7], \( a \) is the constant describing the equivalence of the \( L^2 \)-norms of \( D^g(\beta_M \varphi) \) and \( \mathcal{D}^g \varphi \), and \( C > 0 \) is the infimum of the \( L^2 \)-spectrum of \( D^g \). Together with the \( C^1 \)-continuity of \( g \mapsto D^g \) this shows that \( \mathcal{R}^{\text{inv}}(M \text{ rel } h) \) is open in \( \mathcal{R}(M \text{ rel } h) \) with respect to the \( C^1 \)-topology.

Now let \( g_0 \in \mathcal{R}^{\text{inv}}(M \text{ rel } h) \) and \( g_1 \in \mathcal{R}(M \text{ rel } h) \). Then \( g_t = (1 - t)g_0 + tg_1 \), \( t \in [0, 1] \), is a path in \( \mathcal{R}(M \text{ rel } h) \). The corresponding family of Dirac operators \( D_t := D^{g_t} \) is analytic in \( t \), see [23, Section 11].

We follow the proof of [23, Proposition 11.4] and show that the set \( T := \{ t \in (0, 1) \mid \text{dim ker } D_t > 0 \} \) is discrete from which it follows that \( \mathcal{R}^{\text{inv}}(M \text{ rel } h) \) is \( C^\infty \)-dense in \( \mathcal{R}(M \text{ rel } h) \). Assume that \( s \not\in T \), that is \( D_s \) is invertible. Then \( t \not\in T \) for
all $t$ in a neighbourhood of $s$, so $T$ is closed. Let now $s \in \partial T \cap (0,1)$. We have the orthogonal splittings $H^1 = K \oplus H$ and $L^2 = C \oplus D$ where $K = \ker D_s$ and $C = D_s(H^1)$. Recall that $K$ is finite-dimensional. This induces the decomposition

$$D_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}.$$ 

Note that $d_t : H \to D$ at is invertible at $s = t$, and thus also for $t$ near $s$. If $(x_1, x_2) \in \ker D_t$ for $t$ with invertible $d_t$, then $a_t(x_1) = -b_t(x_2)$ and $c_t(x_1) = -d_t(x_2)$. Thus, $x_2 = -d_t^{-1} \circ c_t(x_1)$ and $R_t(x_1) := (b_t \circ d_t^{-1} \circ c_t - a_t)(x_1) = 0$ where $R_t : K \to C$. Hence we always have that $\dim \ker D_s \geq \dim \ker D_t$ and in particular $\dim \ker D_s = \dim \ker D_t$ if and only if $R_t \equiv 0$. Assume that there is a half-closed interval $I \subset T$ starting or ending at $s$. For $t \in I$ we have $\ker D_t \neq \{0\}$ and thus $\det R_t = 0$. But $R_t$ depends analytically on $t$ which then implies that $R_t = 0$ in the entire neighbourhood of $s$ where $d_t$ is invertible. This contradicts $s \in \partial T$ since it implies that there is a sequence $t_i \to s$ with $\dim \ker D_{t_i} = 0$ and hence $\det R_{t_i} \neq 0$.

From Theorem [5.3] we conclude the following.

**Theorem 6.4.** Let $M$ be an $n$-dimensional spin manifold with boundary, and let $h \in \mathcal{R}^\text{inv}(\partial M)$. Then $\mathcal{R}^\text{inv}(M \text{ rel } h)$ is generic in $\mathcal{R}(M \text{ rel } h)$ if and only if $[M, h] = 0$ in $R_n^\text{inv}$.

If Conjecture [5.12] holds, then the metrics with invertible Dirac operator are generic if and only if the index $\theta([M, h])$ vanishes.

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