THE MULTIPLE GAMMA FUNCTIONS AND THE MULTIPLE $q$-GAMMA FUNCTIONS

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Abstract. We give an asymptotic expansion (the higher Stirling formula) and an infinite product representation (the Weierstrass product representation) of the Vignéras multiple gamma functions by considering the classical limit of the multiple $q$-gamma functions.

1. Introduction

The multiple gamma function was introduced by Barnes. It is defined to be an infinite product regularized by the multiple Hurwitz zeta functions [2], [3], [4], [5]. After his discovery, many mathematicians have studied this function: Hardy [7], [8] studied this function from his viewpoint of the theory of elliptic functions, and Shintani [20], [21] applied it to the study on the Kronecker limit formula for zeta functions attached to certain algebraic fields.

In the end of 70’s, Vignéras [24] redefined the multiple gamma function to be a function satisfying the generalized Bohr-Morelup theorem. Furthermore, Vignéras [24], Voros [23], Vardi [23] and Kurokawa [12], [13], [14], [15] showed that it plays an essential role to express gamma factors of the Selberg zeta functions of compact Riemann surfaces and the determinants of the Laplacians on some Riemannian manifolds.

As we can see from these studies, the multiple gamma functions are fundamental for the analytic number theory: See also [16], [17]. However we do not think that the theory of the multiple gamma functions has been fully explored.

On the other hand, the second author of this paper introduced a $q$-analogue of the Vignéras multiple gamma functions and showed it to be characterized by a $q$-analogue of the generalized Bohr-Morelup theorem [24].

In this paper, we will establish an asymptotic expansion formula (the higher Stirling formula) and an infinite product representation (the

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Weierstrass product representation) of the Vignéra’s multiple gamma functions by considering the classical limit of the multiple $q$-gamma functions. In order to get these results, we will use the method developed in [22]. Namely, by making use of the Euler-MacLaurin summation formula, we derive the Euler-MacLaurin expansion of the multiple $q$-gamma functions. Taking the classical limit, we are led to the Euler-MacLaurin expansion of the Vignéra’s multiple gamma functions. The higher Stirling formula and the Weierstrass product representation follow from this expansion formula.

This paper is organized as follows. In Section 2, we give a survey of the multiple gamma functions and its $q$-analogue. In Section 3, we derive the Euler-MacLaurin expansion of the multiple $q$-gamma functions by using the Euler-MacLaurin summation formula. In Section 4, we consider the classical limit of the multiple $q$-gamma functions rigorously and give an asymptotic expansion formula of the Vignéra’s multiple gamma functions. In Section 5, the Weierstrass product representation of this function is derived.

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2. A SURVEY OF THE MULTIPLE GAMMA FUNCTION AND THE MULTIPLE $q$-GAMMA FUNCTION

2.1. The gamma function. The following are well-known facts in the classical analysis: The Bohr-Morellup theorem says that the gamma function $\Gamma(z)$ is characterized by the three conditions,

\begin{align*}
(1) & \quad \Gamma(z+1) = z\Gamma(z), \\
(2) & \quad \Gamma(1) = 1, \\
(3) & \quad \frac{d^2}{dz^2} \log \Gamma(z+1) \geq 0 \quad \text{for} \quad z \geq 0.
\end{align*}

The gamma function is meromorphic on $\mathbb{C}$, and has an infinite product representation

$$\Gamma(z+1) = e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}},$$

where $\gamma$ is the Euler constant. This is usually called the Weierstrass product formula.
Another representation of the gamma function is derived from the Hurwitz zeta function:

\[ \zeta(s, z) := \sum_{k=0}^{\infty} \frac{1}{(z + k)^s} \quad \text{for} \quad \Re s > 1. \quad (2.2) \]

It is well-known that

\[ \frac{\Gamma(z)}{\sqrt{2\pi}} = \exp(\zeta'(0, z)), \]

where \( \zeta'(0, z) = \frac{d}{dz}\zeta(s, z)|_{s=0}. \)

The gamma function has an asymptotic expansion formula, i.e. the Stirling formula,

\[
\log \Gamma(z + 1) \sim \left( z + \frac{1}{2} \right) \log(z + 1) - (z + 1) - \zeta'(0) + \sum_{r=1}^{\infty} \frac{B_{2r}}{[2r]_2} \frac{1}{(z + 1)^{2r-1}},
\]

as \( z \to \infty \) in a sector \( \Delta_\delta := \{ z \in \mathbb{C} | \arg z < \pi - \delta \} \) \( (0 < \delta < \pi) \), where

\[ z e^{tz} e^{z} - 1 = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} z^n, \]

\( B_k = B_k(0) \) (the Bernoulli number), \( \zeta(s) \) is the Riemann zeta function, \( \zeta'(s) = \frac{d}{ds}\zeta(s) \) and \( [x]_r = x(x-1)\cdots(x-r+1) \). Note that

\( \zeta'(0) = -\log \sqrt{2\pi} \).

2.2. The Barnes \( G \)-function.\[ \]Barnes \[ \]introduced the function \( G(z) \) which satisfies

(1) \( G(z + 1) = \Gamma(z)G(z), \)

(2) \( G(1) = 1, \)

(3) \( \frac{d^3}{dz^3} \log G(z + 1) \geq 0 \quad \text{for} \quad z \geq 0, \)

and he called this the “\( G \)-function”. He proved that the \( G \)-function has an infinite product representation.

\[ G(z + 1) = e^{-z\zeta'(0)-\frac{z^2}{2} \gamma - \frac{z^2}{2}} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right)^k \exp \left( -z + \frac{z^2}{2k} \right) \]

and an asymptotic expansion

\[ \log G(z + 1) \sim \left( \frac{z^2}{2} - \frac{1}{12} \right) \log(z + 1) - \frac{3}{4} z^2 - \frac{z}{2} + \frac{1}{3} + z\zeta'(0) - \log A + O\left( \frac{1}{z} \right) \]
as $z \to \infty$ in the sector $\Delta_\delta$, where $A$ is called the Kinkelin constant. Voros showed this constant can be written with the first derivative of the Riemann zeta function (cf [25], [23])

$$\log A = -\zeta'(-1) + \frac{1}{12}.$$ 

2.3. The Barnes multiple gamma function. We assume that $\omega_1, \omega_2, \cdots, \omega_n$ lie on the same side of some straight line through the origin on the complex plane. The Barnes zeta function [5] is defined as

$$\zeta_n(s, z; \omega) := \sum_{k_1, k_2, \cdots, k_n=0}^{\infty} \frac{1}{(z + k_1\omega_1 + \cdots + k_n\omega_n)^s}.$$ 

where $\omega := (\omega_1, \omega_2, \cdots, \omega_n)$.

This is a generalization of the Hurwitz zeta function. As a generalization of the formula (2.2), Barnes [5] introduced his multiple gamma functions through

$$\frac{\Gamma_n(z, \omega)}{\rho_n(\omega)} := \exp(\zeta'_n(0, z; \omega)).$$

where

$$\log \rho_n(\omega) := -\lim_{z \to 0} \left[ \zeta'_n(0, z; \omega) + \log z \right].$$

It is easy to see that $\Gamma_n(z, \omega)$ satisfies the functional relation

$$\frac{\Gamma_n(z, \omega)}{\Gamma_n(z + \omega_i, \omega)} = \frac{\rho_n-1(\omega(i))}{\Gamma_n-1(z, \omega(i))},$$

where $\omega(i) := (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_n)$.

2.4. The Vignéras multiple gamma function. As a generalization of the gamma function and the $G$-function, Vignéras [24] introduced a hierarchy of functions which she called “the multiple gamma functions”.

**Theorem 2.1.** There exists a unique hierarchy of functions which satisfy

1. $G_n(z + 1) = G_{n-1}(z)G_n(z)$,
2. $G_n(1) = 1$,
3. $\frac{d^{n+1}}{dz^{n+1}} \log G_n(z + 1) \geq 0$ for $z \geq 0$,
4. $G_0(z) = z$. 

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Applying Dufresnoy and Pisot’s results, she showed that these functions satisfying the above properties are uniquely determined and that $G_n(z + 1)$ has an infinite product representation

$$G_n(z + 1) = \exp \left[ -z E_n(1) + \sum_{h=1}^{n-1} \frac{p_h(z)}{h!} \left( \psi_n^{(h)}(0) - E_n^{(h)}(1) \right) \right] \left( 1 + \frac{z}{s(m)} \right)^{(-1)^n} \exp \left\{ \sum_{l=0}^{n-1} \frac{(-1)^{n-l}}{n-l} \left( \frac{z}{s(m)} \right)^{n-l} \right\} \prod_{m \in \mathbb{N}^{n-1} \times \mathbb{N}^*} \left[ 1 + \frac{z}{s(m)} \right]^{(-1)^n} \exp \left\{ \sum_{l=0}^{n-1} \frac{(-1)^{n-l}}{n-l} \left( \frac{z}{s(m)} \right)^{n-l} \right\},$$

where

$$E_n(z) := \sum_{m \in \mathbb{N}^{n-1} \times \mathbb{N}^*} \left[ \left\{ \sum_{l=0}^{n-1} \frac{(-1)^{n-l}}{n-l} \left( \frac{z}{s(m)} \right)^{n-l} \right\} + (-1)^n \log \left( 1 + \frac{z}{s(m)} \right) \right],$$

$$\psi_{n-1}(z) := \log G_{n-1}(z + 1),$$

$$\frac{e^{tz} - 1}{e^z - 1} = 1 + \sum_{k=0}^{\infty} p_k(t) \frac{z^k}{k!}$$

for $s(m) := m_1 + m_2 + \cdots + m_n$ for $m = (m_1, m_2, \ldots, m_n)$, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

The Vignéras multiple gamma function can be regarded as a special case of the Barnes multiple gamma function. Namely

$$G_n(z) = \Gamma_n(z, (1, 1, \cdots, 1))^{(-1)^{n-1}} \times (\text{the normalization factor}).$$

In this paper, we will use the word “the multiple gamma function” to refer the Vignéras multiple gamma function.

2.5. The $q$-gamma function. Throughout this paper, we suppose $0 < q < 1$. A $q$-analogue of the gamma function was introduced by Jackson [9], [10].

$$\Gamma(z + 1; q) = (1 - q)^{-z} \prod_{k=1}^{\infty} \left( \frac{1 - q^{z+k}}{1 - q^k} \right)^{-1}.$$  

Askey [11] pointed out that this function satisfies a $q$-analogue of the Bohr-Mörellup theorem. Namely, $\Gamma(z; q)$ satisfies

- $\Gamma(z + 1; q) = [z]_q \Gamma(z; q)$,
- $\Gamma(1; q) = 1$,
- $\frac{d^2}{dz^2} \log \Gamma(z + 1; q) \geq 0$ for $z \geq 0$,

where $[z]_q := (1 - q^z)/(1 - q)$.  

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As $q$ tends to $1 - 0$, $\Gamma(z; q)$ converges $\Gamma(z)$ uniformly with respect to $z$. A rigorous proof of this fact was given by Koornwinder [11].

Inspired by Moak’s works [18], the authors [22] derived a representation of the $q$-gamma function

$$
\log \Gamma(z; q) = (z - \frac{1}{2}) \log \left( \frac{1 - q^z}{1 - q} \right) + \log q \int_1^z \frac{\xi - q^\xi}{1 - q^\xi} d\xi + C_1(q) + \frac{1}{12} \log q + \sum_{k=1}^m \frac{B_{2k}}{(2k)!} \left( \frac{\log q}{q^z - 1} \right)^{2k-1} \tilde{M}_{2k-1}(q^z)
$$

+ $R_{2m}(z; q)$,

where

$$C_1(q) = -\frac{1}{12} \log q - \frac{1}{12} \log q - \frac{1}{12} q - 1 + \int_0^\infty B_2(t) \left( \frac{\log q}{q^{t+1} - 1} \right)^2 q^{t+1} dt,$$

$$R_{2m}(z; q) = \int_0^\infty \frac{\overline{B}_{2m}(t)}{(2m)!} \left( \frac{\log q}{q^{t+z} - 1} \right)^{2m} \tilde{M}_{2m}(q^{t+z}) dz,$$

and the polynomial $\tilde{M}_n(x)$ is defined by the recurrence relation

$$\tilde{M}_1(x) = 1, \quad (x^2 - x) \frac{d}{dx} \tilde{M}_n(x) + nx \tilde{M}_n(x) = \tilde{M}_{n+1}(x).$$

Each term of the formula (2.5) converges uniformly as $q \to 1 - 0$. So we get another proof of the uniformity of the classical limit of $\log \Gamma(z; q)$.

2.6. The multiple $q$-gamma function. Recently, one of the authors [19] constructed the function $G_n(z; q)$ which satisfies a $q$-analogue of the generalized Bohr-Morellup theorem:

**Theorem 2.2.** There exists a unique hierarchy of functions which satisfy

1. $G_n(z + 1; q) = G_{n-1}(z; q) G_n(z; q),$
2. $G_n(1; q) = 1,$
3. $\frac{d^{n+1}}{dz^{n+1}} \log G_{n+1}(z + 1; q) \geq 0 \quad \text{for} \quad z \geq 0,$
4. $G_0(z; q) = [z]_q.$
We call it “the multiple $q$-gamma function”. It is given by the following infinite product representation \[19\]

\[G_n(z + 1; q) := (1 - q)^{-\binom{z}{n}} \prod_{k=1}^{\infty} \left\{ \left( \frac{1 - q^{z+k}}{1 - q^k} \right)^{\binom{n}{n-k}} (1 - q^k)g_n(z,k) \right\}\]  

for $n \geq 1$, where

\[g_n(z, u) = \left( z - u \right) - \left( \frac{z - u}{n-1} \right).\]

In the next section, we will derive a representation of the multiple $q$-gamma function like (2.5) and consider its classical limit. This limit formula gives some important properties of the multiple gamma functions.

3. The Euler-MacLaurin expansion of $G_n(z + 1; q)$

By means of the Euler-MacLaurin summation formula

\[\sum_{r=M}^{N-1} f(r) = \int_M^N f(t) dt + \sum_{k=1}^{n} \frac{B_k}{k!} \left\{ f^{(k-1)}(N) - f^{(k-1)}(M) \right\}\]

\[+ (-1)^{n-1} \int_M^{N-1} \frac{B_n(t)}{n!} f^{(n)}(t) dt \quad \text{for} \quad f \in C^n[M, N],\]

we give an expansion formula of the multiple $q$-gamma functions which we call the Euler-MacLaurin expansion \[22\]. This formula plays an important role in the following sections.

**Proposition 3.1.** Suppose $\Re z > -1$ and $m > n$, then

\[
\log G_n(z + 1; q) = \left\{ \binom{z + 1}{n} + \sum_{r=1}^{n} \frac{B_r}{r!} \left( -\frac{d}{dz} \right)^{r-1} \left( \binom{z}{n-1} \right) \right\} \log \left( \frac{1 - q^{z+1}}{1 - q} \right)
\]

\[+ \sum_{r=1}^{n} \left\{ \left( -\frac{d}{dz} \right)^{r-1} \left( \binom{z}{n-1} \right) \right\} \times \int_1^{z+1} \frac{\xi^r q^\xi \log q}{r!} \frac{d\xi}{1 - q^\xi}\]

\[+ \sum_{j=0}^{n-1} G_{n,j}(z) C_j(q) + \sum_{r=1}^{m} \frac{B_r}{r!} F_{n,r-1}(z; q) - R_{n,m}(z; q),\]

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where

\[ F_{n,r-1}(z; q) := \left[ \frac{d^{r-1}}{dt^{r-1}} \left\{ \left( -t \right)^{n-1} \log \left( \frac{1 - q^{z+t}}{1 - q^{z+1}} \right) \right\} \right]_{t=1}, \]

\[ C_j(q) := -\sum_{r=1}^{n+1} \frac{B_r}{r!} f_{j+1,r-1}(q) \]

\[ + \frac{(-1)^n}{(n+1)!} \int_1^\infty B_{n+1}(t) \left\{ \frac{d^{n+1}}{dt^{n+1}} \left\{ t^j \log \left( \frac{1 - q^t}{1 - q} \right) \right\} \right\} dt, \]

\[ f_{j+1,r-1}(q) := \left[ \frac{d^{r-1}}{dt^{r-1}} \left\{ t^j \log \left( \frac{1 - q^t}{1 - q} \right) \right\} \right]_{t=1}, \]

\[ R_{n,m}(z; q) := \frac{(-1)^{m-1}}{m!} \int_1^\infty \left[ B_m(t) \left\{ \frac{d^n}{dt^n} \left\{ \left( -t \right)^{n-1} \log \left( \frac{1 - q^{z+t}}{1 - q^{z+1}} \right) \right\} \right\} \right] dt. \]

The polynomial \( G_{n,j}(z) \) is introduced through

\[ \left( \frac{z - u}{n - 1} \right) = \sum_{j=0}^{n-1} G_{n,j}(z) u^j. \]

**Proof.** From (2.7) and the definition of \( G_{n,j}(z) \), we obtain

\[ \log G_n(z + 1; q) = -\left( \frac{z}{n} \right) \log(1 - q) - \sum_{k=1}^\infty \left( \frac{-k}{n-1} \right) \log(1 - q^{z+k}) \]

\[ + \sum_{j=0}^{n-1} G_{n,j}(z) \left\{ \sum_{k=1}^\infty k^j \log(1 - q^k) \right\}. \]

Applying the Euler-MacLaurin summation formula, we have

\[ \sum_{k=1}^\infty \left( \frac{-k}{n-1} \right) \log(1 - q^{z+k}) \]

\[ = \int_1^\infty \left( \frac{-t}{n - 1} \right) \log(1 - q^{z+t}) dt \]

\[ - \sum_{r=1}^m B_r \left\{ \left( \frac{d}{dt} \right)^{r-1} \left( \frac{-t}{n - 1} \right) \right\} \left|_{t=1} \right. \log(1 - q^{z+1}) \]
\[- \sum_{r=1}^{m} \frac{B_r}{r!} F_{n,r-1}(z; q) + R_{n,m}(z; q).\]

Let $L_r(z)$ and $L_r$ be
\[L_r(z) := \frac{Li_r(q^z)}{\log^{-1} q}, \quad L_1(z) := -\log(1 - q^z), \quad L_r := L_r(q),\]
where $Li_r(z)$ is Euler’s polylogarithm
\[Li_r(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^r}.\]

From
\[\log(1 - q^{z+t}) = -\sum_{l=1}^{\infty} \frac{q^{(t+z)l}}{l},\]
it follows that
\[
\int_{1}^{\infty} \left( \frac{-t}{n-1} \right) \log(1 - q^{t+z}) dt \tag{3.3}
\]
\[= \sum_{j=0}^{n-1} (-1)^j \frac{n_{-1} S_j}{(n-1)!} \sum_{r=0}^{j} \frac{(-1)^r j!}{(j-r)!} L_{r+2}(z+1),\]
where $n_{-1} S_j$ is the Stirling number of the first kind defined by
\[\sum_{j=0}^{n} n_{-1} S_j u^j = [u]_n,\]
where $[u]_n = u(u-1) \cdots (u-n+1)$. Substituting (3.3) to (3.2), we have
\[
\sum_{k=1}^{\infty} \left( \frac{-k}{n-1} \right) \log(1 - q^{z+k}) \tag{3.4}
\]
\[= \sum_{j=0}^{n-1} (-1)^j \frac{n_{-1} S_j}{(n-1)!} \sum_{r=0}^{j} \frac{(-1)^r j!}{(j-r)!} L_{r+2}(z+1)
\]
\[+ \sum_{r=1}^{m} \frac{B_r}{r!} \left\{ \left( \frac{d}{dt} \right)^{r-1} \left( \frac{-t}{n-1} \right) \right\} \bigg|_{t=1} L_1
\]
\[+ \sum_{r=1}^{m} \frac{B_r}{r!} F_{n,r-1}(z; q) + R_{n,m}(z; q).\]
Similarly,
\[ \sum_{k=1}^{\infty} k^j \log(1 - q^k) \] (3.5)
\[ = \sum_{r=0}^{j} \frac{(-1)^r j!}{(j - r)!} L_{r+2} + \sum_{r=1}^{j} \frac{B_r}{r!} \frac{j!}{(j + 1 - r)!} L_1 + C_j(q). \]

$L_r(z)$ and $L_r$ cause divergence as $q \to 1 - 0$, but we can prove that these divergent terms vanish. In order to simplify such terms, let us express $L_r(z)$ as the sum of $L_r$ and convergent terms.

**Lemma 3.2.**

\[ L_{l+1}(z) = z^l l! \log \left( \frac{1 - q^z}{1 - q} \right) \]
\[ + \sum_{r=0}^{l} \frac{(z - 1)^{l-r}}{(l-r)!} L_{r+1} + \sum_{r=1}^{l} \frac{(-1)^r z^{l-r}}{(l-r)!} T_r(z), \]

where
\[ T_r(z) = \int_1^z \frac{\xi^r q^\xi \log q}{1 - q^\xi} d\xi. \]

**Proof.** By partial integration, we have
\[
\int_1^z \xi^n q^\xi \log q d\xi = \sum_{r=0}^{n} \frac{n!(-1)^r}{(n - r)!} \log^r q \left\{ z^{n-r} Li_{r+1}(q^z) - Li_{r+1}(q) \right\}. \]

Let $L_k(z)$ be
\[ L_{k+1}(z) = \frac{Li_{k+1}(q^z) - Li_{k+1}(q)}{\log^k q}. \]

Then (3.6) reads
\[
L_{n+1}(z) = - \sum_{k=0}^{n-1} \frac{(-1)^{n-k} z^{n-k}}{(n-k)!} L_{k+1}(z) \]
\[ - \sum_{k=0}^{n-1} \frac{(-1)^{n-k}(z^{n-k} - 1)}{(n-k)!} L_{k+1} + (-1)^n T_n(z). \]

In order to write $L_{n+1}(z)$ with $L_{n+1}$ and $T_k(z)$, we use induction on $n$. It can be seen that
\[ L_{n+1}(z) = \frac{z^n L_1(z)}{n!} + \sum_{k=0}^{n} \frac{(z - 1)^{n-k}}{(n-k)!} L_{k+1} + \sum_{k=1}^{n} \frac{(-1)^k z^{n-k}}{(n-k)!} T_k(z), \]
which is equivalent to the claim of Lemma 3.2.

Applying (3.4), (3.5) and Lemma 3.2 to (3.1), we obtain

\[
\log G_n(z + 1; q) = - \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1)!} S_j \sum_{l=1}^{j} \frac{(-1)^l j!}{(j-l)!} \sum_{r=0}^{l} \frac{z^{l-r}}{(l-r)!} L_{r+2}
\]

\[
+ \sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \frac{(-1)^r j!}{(j-r)!} L_{r+2}
\]

\[
+ \left\{ \frac{z}{n} - \sum_{r=1}^{m} B_r r! \left( \frac{d}{dt} \right)^{r-1} \left( \frac{-t}{n-1} \right) \right|_{t=1} \right. 
\]

\[
- \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1)!} S_j \sum_{l=0}^{j} \frac{(-1)^l j!}{(j-l)!} \frac{z^{l+1}}{(l+1)!} 
\]

\[
+ \sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \frac{B_r}{r!} \frac{j!}{(j+1-r)!} \right\} L_1
\]

\[
+ \left\{ - \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1)!} S_j \sum_{l=0}^{j} \frac{(-1)^l j!}{(j-l)!} \frac{(z+1)^{l+1}}{(l+1)} 
\]

\[
+ \sum_{j=0}^{n-1} B_{j+1} \left( \frac{d}{dt} \right)^{j} \left( \frac{-t}{n-1} \right) \left|_{t=1} \right. \right\} \times \log \left( \frac{1-q^{z+1}}{1-q} \right)
\]

\[
- \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1)!} S_j \sum_{l=0}^{j} \frac{(-1)^l j!}{(j-l)!} \sum_{r=0}^{l} \frac{(-1)^r (z+1)^{l-r}}{(l-r)!} T_{r+1}(z + 1)
\]

\[
+ \sum_{r=0}^{m} B_r F_{n,r-1}(z; q) + \sum_{j=1}^{n-1} G_{n,j}(z) C_j(q)
\]

\[- R_{n,m}(z; q).\]
We prove that the coefficients of the divergent terms in (3.8) vanish. By the following lemma, we can see that the first term and the second term in (3.8) are canceled out.

**Lemma 3.3.**

\[
\sum_{j=0}^{n-1} \frac{(-1)^j n_{j-1} S_j}{(n-1)!} \sum_{l=1}^{j} \frac{(-1)^l j!}{(j-l)!} \sum_{r=0}^{l} \frac{z^{l-r}}{(l-r)!} L_{r+2} = \sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \frac{(-1)^r j!}{(j-r)!} \frac{u^r}{r!} L_{r+2}.
\]

**Proof.** We consider the exponential generating functions of the coefficients of \( L_{r+2} \) in both sides. The generating function of the left hand side is

\[
\sum_{j=0}^{n-1} \frac{(-1)^j n_{j-1} S_j}{(n-1)!} \sum_{l=0}^{j} \frac{(-1)^l j!}{(j-l)!} \sum_{r=0}^{l} \frac{z^{l-r}}{(l-r)!} u^r L_{r+2} = \sum_{j=0}^{n-1} \frac{(-1)^j n_{j-1} S_j}{(n-1)!} (1 - z - u)^j = \binom{z + u - 1}{n - 1}.
\]

On the other hand, the generating function of the right hand side is

\[
\sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \frac{(-1)^r j!}{(j-r)!} \frac{u^r}{r!} = \sum_{j=1}^{n-1} G_{n,j}(z)(1 - u)^j = \binom{z + u - 1}{n - 1}.
\]

Therefore, the coefficients of \( L_{r+2} \) in both sides coincide. Thus, the claim follows. \( \square \)
Next, we prove the coefficient of $L_1$ vanishes. Since

$$
\sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1)!} S_j \sum_{l=0}^{j} \frac{(-1)^l j!}{(j-l)! (l+1)!} = \int_0^{z} \left( \frac{t-1}{n-1} \right) dt
$$

and

$$
\sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \frac{B_r}{r!} \frac{j!}{(j+1-r)!}
$$

we have

$$
\left( \text{the coefficient of } L_1 \right)
$$

$$
= \left( \frac{z}{n} \right) - \int_0^{z} \left( \frac{t-1}{n-1} \right) dt + \sum_{r=0}^{m} B_r \left( \frac{d}{dz} \right)^{r-1} \left( \frac{z-1}{n-1} \right) - \sum_{r=0}^{m} \frac{B_r}{r!} \left( \frac{d}{dz} \right)^{r-1} \left( \frac{z-1}{n-1} \right)_{z=0}.
$$

We prove that the right hand side of the above formula is equal to zero. In a formal sense,

$$
\left( e^{-\frac{1}{n}} - 1 \right)^{-1} + \left( \frac{d}{dz} \right)^{-1} = \sum_{r=1}^{\infty} \frac{B_r}{r!} \left( \frac{d}{dz} \right)^{r-1}.
$$
Imposing the boundary condition at \( z = 0 \), we make the both sides above act on \( \binom{z-1}{n-1} \). Then,

\[
\left( e^{-n\frac{d}{dz}} - 1 \right)^{-1} \binom{z - 1}{n - 1} + \int_0^z \binom{t - 1}{n - 1} \, dt + \sum_{r=1}^n \frac{B_r}{r!} \left[ -\left( -\frac{d}{dt} \right)^{r-1} \binom{t - 1}{n - 1} \right]_{t=0}^{t=z} = 0
\]

because \( \binom{t-1}{n-1} \) is a polynomial of \((n-1)\)-degree. Since \( F(z) := -(\frac{z}{n})^n \) satisfies

\[
F(0) = 0, \quad \left( \frac{z - 1}{n - 1} \right) = F(z - 1) - F(z),
\]

it can be seen that

\[
\left( e^{-n\frac{d}{dz}} - 1 \right)^{-1} \binom{z - 1}{n - 1} = -\binom{z}{n}.
\]

Thus we have the formula

\[
-\binom{z}{n} + \int_0^z \binom{t - 1}{n - 1} \, dt + \sum_{r=1}^n \frac{B_r}{r!} \left[ -\left( -\frac{d}{dt} \right)^{r-1} \binom{t - 1}{n - 1} \right]_{t=0}^{t=z} = 0, \quad (3.9)
\]

which shows that the coefficient of \( L_1 \) is equal to zero. Hence we have proved that the all coefficients of \( L_r \) vanish in (3.8).

Finally, we calculate the coefficients of \( \log \left( \frac{1-q^{z+1}}{1-q} \right) \) and \( T_r(z+1) \). Using the formula (3.9), we have

\[
\left( \text{the coefficient of } \log \left( \frac{1-q^{z+1}}{1-q} \right) \text{ in (3.1)} \right) = \binom{z + 1}{n} + \sum_{r=1}^n \frac{B_r}{r!} \left( -\frac{d}{dt} \right)^{r-1} \binom{z}{n - 1}.
\]

In order to calculate the coefficients of \( T_r(z+1) \) in (3.8), we note that

\[
\binom{z - u}{n - 1} = \sum_{j=0}^{n-1} \frac{n-1S_j}{(n-1)!} (z-u)^j
\]

\[
= \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1)!} \sum_{l=0}^j \frac{(-1)^l j!}{(j-l)!} \sum_{r=0}^l \frac{(-1)^r (z+1)^{l-r} u^r}{(l-r)! r!}.
\]
Using this identity, we have

\[
\text{(the coefficient of } T_r(z+1) \text{ in (3.8))}
\]

\[
= \left( \frac{d}{dt} \right)^{r-1} \left( z - u \right) \bigg|_{u=0} = \left( -\frac{d}{dz} \right)^{r-1} \binom{z}{n-1}.
\]

Substituting (3.10) and (3.11) in (3.8), we obtain Proposition 1.1.

4. The classical limit of $G_n(z+1;q)$ and the asymptotic expansion of $G_n(z+1)$

In this section, we study the classical limit of $G_n(z+1;q)$ using the Euler-MacLaurin expansion. We will see that this limit formula gives an asymptotic expansion for the multiple gamma functions, which is a generalization of the Stirling formula for the gamma function.

4.1. The classical limit of $G_n(z+1;q)$. First we consider the classical limit of the Euler-MacLaurin expansion in the domain \{ $z \in \mathbb{C} | \Re z > -1$ \}.

**Proposition 4.1.** Suppose $\Re z > -1$ and $m > n$.

\[
\lim_{q \to 1-0} \log G_n(z+1;q) = \left\{ \binom{z+1}{n} + \sum_{r=1}^{n} \frac{B_r}{r!} \left( -\frac{d}{dz} \right)^{r-1} \binom{z}{n-1} \right\} \log(z+1)
\]

\[
- \sum_{r=1}^{n} \left\{ \frac{(-d/dz)^{r-1}}{r!(n-1)} \binom{z}{n-1} \right\} \frac{1}{r!r} (z+1)^r - 1
\]

\[
- \sum_{j=0}^{n-1} G_{n,j}(z) C_j + \sum_{r=1}^{m} \frac{B_r}{r!} F_{n,r-1}(z)
\]

\[
- R_{n,m}(z),
\]

where

\[
C_j := - \sum_{r=1}^{n+1} \frac{B_r}{r!} \left( \frac{d}{dt} \right)^{r-1} \{ t^j \log t \} \bigg|_{t=1} + \frac{(-1)^n}{(n+1)!} \int_{1}^{\infty} B_{n+1}(t) \left( \frac{d}{dt} \right)^{n+1} \{ t^j \log t \} \, dt
\]
\[ F_{n,r-1} := \left( \frac{d}{dt} \right)^{r-1} \left\{ \left( -t \right) \log \left( \frac{z + t}{z + 1} \right) \right\} \bigg|_{t=1}, \]

\[ R_{n,m}(z) := \frac{(-1)^{m-1}}{m!} \int_1^\infty \overline{B}_m(t) \left( \frac{d}{dt} \right)^m \left\{ \left( -t \right) \log \left( \frac{z + t}{z + 1} \right) \right\} dt. \]

Furthermore, this convergence is uniform on any compact set in \( \{ z \in \mathbb{C} | \Re z > -1 \} \).

Proof. Taking Proposition 3.1 into account, we must show that
\[ \lim_{q \to 1-0} \log \left( \frac{1 - q^{z+1}}{1 - q} \right) = \log (z + 1), \]  
(4.1)
\[ \lim_{q \to 1-0} \int_1^{z+1} \frac{\xi^r q^\xi \log q}{r!} d\xi = -\int_1^\infty \frac{\xi^{r-1}}{r^r} d\xi = -\frac{1}{r!r} \{ (z + 1)^r - 1 \}, \]  
(4.2)
\[ \lim_{q \to 1-0} F_{n,r-1}(z; q) = F_{n,r-1}(z), \]  
(4.3)
\[ \lim_{q \to 1-0} R_{n,m}(z; q) = R_{n,m}(z), \]  
(4.4)
\[ \lim_{q \to 1-0} C_j(q) = C_j, \]  
(4.5)
and further have to show that this convergence is uniform. Here we prove only (4.4). The other formulas can be verified in a similar way.

Since
\[ \lim_{q \to 1-0} \left( \frac{d}{dt} \right)^{r-1} \left\{ \left( -t \right) \log \left( \frac{1 - q^{z+t}}{1 - q^{z+1}} \right) \right\} \left( \frac{d}{dt} \right)^{r-1} \left\{ \left( -t \right) \log \left( \frac{z + t}{z + 1} \right) \right\}, \]
in order to show (4.4), it is sufficient to prove that the procedure of taking the classical limit commutes with the integration. Let us introduce polynomials \( M_r(x) \) through
\[ \frac{d^r}{dz^r} \log(1 - q^{z+t}) = - \left( \frac{\log q}{1 - q^{z+t}} \right)^r q^{z+t} M_r(q^{z+t}) \]
(cf.[18]). They satisfy the recurrence relation.

\[ M_1(x) = 1, \quad (x^2 - x) \frac{d}{dx} M_n(x) + \{(r-1)x + 1\} M_r(x) = M_{r+1}(x) \]
and \( M_r(1) = (r-1)! \). Using this we have
\[ \int_1^\infty \overline{B}_m(t) \left( \frac{d}{dt} \right)^m \left( \frac{-t}{n-1} \right) \log \left( \frac{1 - q^{z+t}}{1 - q^{z+1}} \right) dt \]
\begin{align*}
&= \frac{1}{(n-1)!} \sum_{j=1}^{n-1} \left\{ (-1)^j \sum_{l=0}^j \binom{m}{l} [j]l \right. \\
&\quad \times \int_1^\infty B_m(t) t^{j-l} \left( \frac{\log q}{1 - q^{t+z}} \right)^{m-l} q^{t+z} M_{m-l}(q^{t+z}) dt \}.
\end{align*}

Therefore we have to show
\begin{align*}
&\lim_{q \to 1-0} \int_1^\infty B_m(t) \left\{ t^{j-l} \left( \frac{\log q}{1 - q^{t+z}} \right)^{m-l} q^{t+z} M_{m-l}(q^{t+z}) \right\} dt \\
&= \int_1^\infty \lim_{q \to 1-0} B_m(t) \left\{ t^{j-l} \left( \frac{\log q}{1 - q^{t+z}} \right)^{m-l} q^{t+z} M_{m-l}(q^{t+z}) \right\} dt.
\end{align*}

**Lemma 4.2.** Suppose that \( \alpha \in \mathbb{N} \) be fixed and that \( y_0 \) and \( y_1 \) are fixed constants such that \(-1 < y_0 < y_1\). Then there exists a constant \( C \) depending on \( y_0 \) and \( y_1 \) such that
\[
0 < \left( \frac{\log q}{q^{t+y} - 1} \right)^\alpha < \frac{C}{(t+y)^\alpha} \quad \text{for} \quad y \in [y_0, y_1], \quad 1 < t < \infty, \quad 0 < q < 1.
\]

**Proof.** Letting \( q = e^{-\delta} \), we define
\[
\varphi(t, y, \delta) := \begin{case}
\left( \frac{e^{-(t+y)\delta}}{e^{-(t+y)\delta} - 1} \right)^\alpha e^{-\delta(t+y)} & \delta > 0 \\
1 & \delta = 0.
\end{case}
\]

Then, \( \varphi(t, y, \delta) \) is positive, continuous and bounded on
\[
D := \left\{ (t, y, \delta) \mid 1 \leq t \leq +\infty, \quad \delta \geq 0, \quad y_0 \leq y \leq y_1 \right\}.
\]

So we can put \( C := \sup_D \varphi(t, y, \delta). \)

Suppose \(-1 < y_0 \leq \Re z \leq y_1\). By Lemma 4.2, there exists a constant \( C \) depending only on \( y_0 \) and \( y_1 \) such that
\[
\left| t^{j-l} \left( \frac{\log q}{1 - q^{t+z}} \right)^{m-l} q^{t+z} M_{m-l}(q^{t+z}) \right| \leq t^{j-l} \frac{C}{(t+\Re z)^{m-l}}
\]
for \( 1 \leq t < +\infty, \quad 0 \leq q < 1. \)

Since \( m - j \geq 2 \), the right hand side above is integrable over \( 1 \leq t < +\infty \). Therefore, Lebesgue’s convergence theorem ensures (4.6). We have thus proved the limit formula (4.4). Next we show that this
convergence is uniform on any compact set in the domain \( \{ z \in \mathbb{C} | \Re z > -1 \} \).

Put

\[
\Phi(z, q) := \int_1^\infty B_m(t) t^{j-l} \left( \frac{\log q}{1 - q^z t} \right)^{m-l} q^{l+z} M_{m-l}(q^z t) dt.
\]

From the consideration above, we see that there exists a constant \( C \) depending only on \( y_0 \) and \( y_1 \) such that

\[
|\Phi(z, q)| \leq C \int_1^\infty B_m(t) \frac{dt}{t^2} \text{ for } \Re z \in [y_0, y_1] \text{ and } 0 < q \leq 1.
\]

Hence \( \{ \Phi(z, q) | 0 < q \leq 1 \} \) is a uniformly bounded family of functions and

\[
\Phi(z, q) \to \int_1^\infty B_m(t) \frac{t^{j-l} (-1)^{m-l} (m-l-1)!}{(t+z)^{m-l}} dt \text{ as } q \to 1 - 0.
\]

By Vitali’s convergence theorem, this convergence is uniform on any compact set in the domain \( \{ z \in \mathbb{C} | \Re z > -1 \} \).

The constant \( C_j \) in Proposition 4.1 can be expressed in terms of the special value of the Riemann zeta function.

**Lemma 4.3.**

\[
C_j = -\zeta'(-j) - \frac{1}{(j+1)^2}.
\]

**Proof.** From the definition of \( \zeta(s) \),

\[
\zeta'(s) = -\sum_{k=1}^{\infty} \frac{\log k}{k^s} \text{ for } \Re s > 1.
\]

By the Euler-MacLaurin summation formula, we obtain

\[
\zeta'(s) = -\frac{1}{(s-1)^2} + \sum_{r=1}^{n} \frac{B_r}{r!} \left( \frac{d}{dt} \right)^{r-1} \{ t^{-s} \log t \} \bigg|_{t=1} + \frac{1}{n!} \int_1^\infty B_n(t) \left( \frac{d}{dt} \right)^n \{ t^{-s} \log t \} dt.
\]

Since

\[
\left( \frac{d}{dt} \right)^n \{ t^{-s} \log t \} = -\frac{\partial}{\partial s} \left[ t^{-s-n} \right]_n + \left[ t^{-s-n} \log t \right]_n,
\]

we have

\[
\Phi(z, q) \to \int_1^\infty B_m(t) \frac{t^{j-l} (-1)^{m-l} (m-l-1)!}{(t+z)^{m-l}} dt \text{ as } q \to 1 - 0.
\]
(4.7) can be analytically continued to \(\{ z \in C | \Re z > -n + 1 \}\). So if we put \( s = -j, n = j + 2 \), then we obtain
\[
\zeta'(-j) = -\frac{1}{(j+1)^2} + \sum_{r=1}^{j+2} \frac{B_r}{r!} \left( \frac{d}{dt} \right)^{r-1} \left\{ t^r \log t \right\} \bigg|_{t=1} \\
- \frac{(-1)^{j+1}}{(j+2)^2} \int_1^\infty \frac{B_{j+2}(t)}{B_{j+2}(t)} \left( \frac{d}{dt} \right)^{j+2} \left\{ t^j \log t \right\} dt \\
= -\frac{1}{(j+1)^2} - C_j.
\]

\[\square\]

Next we prove that the limit function in Proposition 4.1 coincides with the multiple gamma function.

**Theorem 4.4.** Suppose \( m > n \). Then, as \( q \to 1 - 0 \), \( G_n(z+1; q) \) converges to \( G_n(z+1) \) uniformly on any compact set in the domain \( C \setminus \mathbb{Z}_{\leq 0} \) and
\[
\log G_n(z+1) = \left\{ \binom{z+1}{n} + \sum_{r=1}^{n} \frac{B_r}{r!} \left( -\frac{d}{dz} \right)^{r-1} \left( \frac{z}{n-1} \right) \right\} \log(z+1) \\
- \sum_{r=1}^{n} \left\{ \left( -\frac{d}{dz} \right)^{r-1} \left( \frac{z}{n-1} \right) \right\} \times \frac{1}{r!} \{(z+1)^r - 1\} \\
- \sum_{j=0}^{n-1} G_{n,j}(z) \left\{ \zeta'(-j) + \frac{1}{(j+1)^2} \right\} + \sum_{r=1}^{m} \frac{B_r}{r!} F_{n,r-1}(z) \\
- R_{n,m}(z).
\]

**Proof.** In the domain \( \{ z \in C | \Re z > -1 \} \), we have already proved the existence of the limit function and uniformity of the convergence. Let us put
\[
\tilde{G}_n(z+1) := \lim_{q \to 1-0} G_n(z+1; q).
\]

Because of the uniformity of the convergence, we have particularly
\[
\lim_{q \to 1-0} \left[ \left( \frac{d}{dz} \right)^{n+1} \{ \log G_n(z+1; q) \} \right] = \left( \frac{d}{dz} \right)^{n+1} \{ \log \tilde{G}_n(z+1) \}.
\]
so that, from Theorem 2.2, $\tilde{G}_n(z+1)$ satisfies the conditions in Theorem 2.1. Namely

\begin{align*}
1) \quad & \tilde{G}_n(z + 1) = \tilde{G}_{n-1}(z)\tilde{G}_n(z) \\
2) \quad & \left( \frac{d}{dz} \right)^{n+1} \log \tilde{G}_n(z + 1) \geq 0 \quad \text{for} \quad z \geq 0 \\
3) \quad & \tilde{G}_n(1) = 1 \\
4) \quad & \tilde{G}_0(z + 1) = z + 1.
\end{align*}

Since a hierarchy of such functions is uniquely determined, so $G_{n+1}(z+1) = \tilde{G}_n(z+1)$ in \{z $\in$ $\mathbb{C}$|Re$z$ $>$ $-1$\}. Thus the claim of the theorem in the case that Re$z$ $>$ $-1$ has been proved for \{z $\in$ $\mathbb{C}$|Re$z$ $>$ $-1$\}.

Next, we show that in \{z $\in$ $\mathbb{C}$|Re$z$ $\leq$ $-1$, z $\neq$ $-1$\},

\[ G_n(z + 1; q) \rightarrow G_n(z + 1) \quad \text{as} \quad q \rightarrow 1 - 0 \]

and that the convergence is uniform on any compact set in this domain. For the proof, we use induction on $n$.

The case that $n = 1$ was considered by Koornwinder [11]. Let $K$ be a compact set in \{z $\in$ $\mathbb{C}$|2 $<$ Re$z$ $\leq$ $-1$, z $\neq$ $-1$\}. If $q$ is sufficiently close to 1 then $[z + 1]_q \neq 0$ on $K$. Therefore as $q \rightarrow 1 - 0$,

\[ \Gamma(z + 1; q) = \frac{\Gamma(z + 2; q)}{[z + 1]_q} \]

uniformly converges to

\[ \frac{\Gamma(z + 2)}{z + 1} = \Gamma(z + 1) \]

on $K$.

We assume that

\[ G_{n-1}(z + 1; q) \rightarrow G_{n-1}(z + 1) \quad \text{as} \quad q \rightarrow 1 - 0 \]

and that the convergence is uniform on $K$.

From (2.7), we see that, if $q$ is sufficiently close to 1, $G_{n-1}(z + 1; q)$ has no poles and no zeros on $K$, neither has $G_{n-1}(z + 1)$ from (2.5).

Therefore, as $q \rightarrow 1 - 0$,

\[ G_n(z + 1; q) = \frac{G_n(z + 2; q)}{G_{n-1}(z + 1; q)} \]

uniformly converges to

\[ \frac{G_n(z + 2)}{G_{n-1}(z + 1)} = G_n(z + 1) \]

on $K$. 20
Repeating this procedure, we can verify, for any $n$, $G_n(z+1)$ converge to $G_n(z+1)$ in a compact set in the domain $\{-3 < \Re z \leq -2, z \neq -2\}$, $\{-4 < \Re z \leq -3, z \neq -3\}$, · · · . Thus the claim of the theorem is proved.

4.2. Asymptotic expansion of $G_n(z+1)$. Let us call the expression (4.8) the Euler-MacLaurin expansion of $G_n(z+1)$. We should note that (4.8) is valid for $z \in \mathbb{C} \setminus \{\Re z \leq -1\}$. We show that it gives an asymptotic expansion of $G_n(z+1)$ as $|z| \to \infty$, i.e. the higher Stirling formula.

**Theorem 4.5.** Let $0 < \delta < \pi$, then

$$
\log G_n(z+1)
\sim \left\{ \left( \frac{z+1}{n} \right) + \sum_{r=1}^{n} \frac{B_r}{r!} \left( -\frac{d}{dz} \right)^{r-1} \left( \frac{z}{n-1} \right) \right\} \log(z+1)
- \sum_{r=1}^{n} \left\{ \left( -\frac{d}{dz} \right)^{r-1} \left( \frac{z}{n-1} \right) \right\} \times \frac{1}{r!r} \{(z+1)^r - 1\}
- \sum_{j=0}^{n-1} G_{n,j}(z) \left\{ \zeta'(-j) + \frac{1}{(j+1)^2} \right\} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} F_{n,2r-1}(z)
$$

as $|z| \to \infty$ in the sector $\{z \in \mathbb{C}||\arg z| < \pi - \delta\}$.

**Proof.** Straightforward calculation shows that

$$
F_{n,r-1}(z) = \sum_{l=1}^{r-1} \binom{r-1}{l} \left\{ \left( \frac{d}{dt} \right)^{r-1-l} \left( -t \right) \right\} \left\{ \left( \frac{d}{dt} \right)^l \log(z+t) \right\} \bigg|_{t=1}
= \sum_{l=1}^{r-1} \left( \sum_{k=0}^{n-1} S_k (-1)^k [k]_{r-1-l} \right) \frac{(-1)^{l-1}(l-1)!}{(z+1)^l}.
$$

Thus, if $r > n$, then $F_{n,r-1}(z) = O(z^{-r+n})$ as $|z| \to \infty$. So we can see that

$$
\frac{F_{n,2r-1}(z)}{F_{n,2r-3}(z)} = O(z^{-2}) \quad \text{as} \quad |z| \to \infty. \quad (4.9)
$$

Furthermore we can see that
\[ R_{n,2m}(z) = \frac{-1}{(2m)!} \int_1^\infty B_{2m}(t) \left( \frac{d}{dt} \right)^{2m} \left\{ \binom{-t}{n-1} \log \left( \frac{z+t}{z+1} \right) \right\} dt \]

\[ = \frac{-1}{(2m)!} \sum_{j=1}^{n-1} (-1)^j n_{-1} S_j \sum_{l=0}^{j} [j]_{l} (m - l - 1)! \int_1^\infty \frac{B_{2m}(t) t^j}{(t+z)^{2m-j}} dt. \]

Noting that

\[ \frac{1}{|z+t|} < \frac{1}{|t||\sin \delta|} \]

in the sector \( \{ z \in \mathbb{C} : |\arg z| < \pi - \delta \} \) and that \( |B_{2m}(t)| \leq |B_{2m}| \) for \( 0 \leq t \leq 1 \), we have

\[ \left| \int_1^\infty B_{2m}(t) \left( \frac{d}{dt} \right)^{2m} \left\{ \binom{-t}{n-1} \log \left( \frac{z+t}{z+1} \right) \right\} dt \right| \]

\[ \leq \frac{|B_{2m}|}{(2m)!} \sum_{j=1}^{n-1} \sum_{l=0}^{j} n_{-1} S_j \binom{n}{l} [j]_{l} (2m - l - 1)! \int_1^\infty \frac{dt}{|\sin \delta|^{2m-j} t^{2m-j}}. \]

Hence,

\[ |R_{n,2m}(z)| = O(z^{-2m-1+n}) = o(F_{n,2m-1}(z)) \]

as \( |z| \to \infty \) in the sector. \( \square \)

Let us exhibit some examples of the higher Stirling formula. In the case that \( n = 1 \), we obtain

\[ \log G_1(z+1) = \log \Gamma(z+1) \]

\[ \sim \left( z + \frac{1}{2} \right) \log(z+1) - (z+1) - \zeta'(0) \]

\[ + \sum_{r=1}^{\infty} \frac{B_{2r}}{[2r]_2} \frac{1}{(z+1)^{2r-1}}. \]

This is the Stirling formula since \( \zeta'(0) = -\frac{1}{2} \log(2\pi) \).
Furthermore, in the case that \( n = 2 \), we obtain

\[
\log G_2(z + 1) \\
\sim \left( \frac{z^2}{2} - \frac{1}{12} \right) \log(z + 1) - \frac{3}{4} z^2 - \frac{z}{2} + \frac{1}{4} \\
- z\zeta'(0) + \zeta'(-1) \\
- \frac{1}{12} \frac{1}{z + 1} + \sum_{r=2}^{\infty} \frac{B_{2r}}{[2r]_3} \frac{1}{(z + 1)^{2r-1}} (z - 2r + 1),
\]

which coincides with the formula (2.3). In the case that \( n = 3, 4, \) and 5, we have the following results.

**Proposition 4.6.** The higher Stirling formula for \( n = 3, 4 \) and 5 are as follows:

\[
\log G_3(z + 1) \\
\sim \left( \frac{z^3}{6} - \frac{z^2}{4} + \frac{1}{24} \right) \log(z + 1) - \frac{11}{36} z^3 + \frac{5}{24} z^2 + \frac{z}{3} - \frac{13}{72} \\
- \frac{z^2 - z}{2} \zeta'(0) + \frac{2z - 1}{2} \zeta'(-1) - \frac{1}{2} \zeta'(-2) \\
+ \frac{1}{12} \frac{1}{z + 1} + \sum_{r=2}^{\infty} \frac{B_{2r}}{[2r]_4} \frac{1}{(z + 1)^{2r-1}} \left( z^2 - (6r - 11)z + (4r^2 - 16r + 16) \right).
\]

\[
\log G_4(z + 1) \\
\sim \left( \frac{z^4}{24} - \frac{z^3}{6} + \frac{z^2}{6} - \frac{19}{720} \right) \log(z + 1) \\
- \frac{4}{72} z^4 + \frac{2}{9} z^3 + \frac{z^2}{8} - \frac{11}{36} z + \frac{31}{144} \\
- \frac{z^3 - 3z^2 + 2z}{6} \zeta'(0) + \frac{3z^2 - 6z + 2}{6} \zeta'(-1) - \frac{z - 1}{2} \zeta'(-2) + \frac{1}{6} \zeta'(-3) \\
- \frac{1}{12} \frac{1}{z + 1} + \frac{1}{720} \frac{1}{(z + 1)^3} \left( 6z^2 + \frac{13}{2} z + \frac{5}{2} \right)
\]
\[
+ \sum_{r=3}^{\infty} \frac{B_{2r}}{[2r]_6} \frac{1}{(z + 1)^{2r-1}} \left\{ z^3 - (12r - 27)z^2 + (20r^2 - 94r + 111)z
- (8r^3 - 56r^2 + 134r - 109) \right\}.
\]

\[\log G_5(z + 1) \sim \left( \frac{z^5}{120} - \frac{z^4}{16} + \frac{11}{72} z^3 - \frac{z^3}{8} + \frac{3}{160} \right) \log(z + 1) \]

\[- \frac{137}{7200} z^5 + \frac{39}{320} z^4 - \frac{461}{2160} z^3 + \frac{z^2}{1440} - \frac{323}{1440} + \frac{5639}{43200} \]

\[- \frac{z^4 - 6z^3 + 11z^2 - 6z}{24} \zeta'(0) + \frac{4z^3 - 18z^2 + 22z - 6}{24} \zeta'(-1) \]

\[- \frac{6z^2 - 18z + 11}{24} \zeta'(-2) + \frac{2z - 3}{12} \zeta'(-3) - \frac{1}{24} \zeta'(-4) \]

\[\frac{1}{12} \frac{1}{z + 1} - \frac{1}{720} \frac{1}{(z + 1)^3} \left( \frac{35}{4} z^2 + \frac{45}{4} z + \frac{9}{2} \right) \]

\[+ \sum_{r=3}^{\infty} \frac{B_{2r}}{[2r]_6} \frac{1}{(z + 1)^{2r-1}} \left\{ z^4 - (20r - 54) z^3 + (70r^2 - 375r + 506) z^2
- \left( \frac{200}{3} r^3 - 540r^2 + \frac{4420}{3} r - 1354 \right) z
+ 16r^4 - \frac{536}{3} r^3 + \frac{754r^2}{3} - \frac{4279}{3} r + 1021 \right\}.\]

5. The Weierstrass Product Representation for the \( G_n(z + 1) \)

By calculating the formula (4.8) in the case of \( m = n + 1 \), we derive the Weierstrass product representation for the multiple gamma functions. Main theorem of this section is the following:

**Theorem 5.1.** For \( n \in \mathbb{N} \), we have

\[ G_n(z + 1) = \exp \left( F_n(z) \right) \prod_{k=1}^{\infty} \left\{ \left( 1 + \frac{z}{k} \right)^{-\frac{k}{n+1}} \exp \left( \Phi_n(z,k) \right) \right\}, \]
where

\[
F_n(z) := \sum_{j=0}^{n-1} G_{n,j}(z)Q_j(z) + \sum_{r=0}^{n-2} \frac{1}{r!} \left( \frac{\partial}{\partial u} \right)^r \left( \frac{z-u}{n-1} \right)_{u=z} \times \zeta'(-r) \\
- \int_0^z \left( \frac{z-u}{n-1} \right) du \times \gamma,
\]

\[
\Phi_n(z, k) := \frac{1}{(n-1)!} \sum_{\mu=-1}^{n-1} \left\{ \sum_{r=\mu+1}^{n-1} \frac{S_r z^r}{r-\mu} \right\} (1)^{\mu+1} k^\mu,
\]

\[
Q_j(z) := P_j(z+1) - \sum_{r=0}^{j} \binom{j}{r} z^r P_{j-r}(1)
\]
\[
+ \frac{1}{j+1} \sum_{r=1}^{j+1} \binom{j+1}{r} B_{j+1-r}(z) \sum_{l=1}^{r} \frac{(-1)^{l-1} z^l}{l},
\]

\[
P_j(x) := \sum_{r=0}^{j+1} \frac{B_r}{r!} \varphi_{j,r} x^{j-r+1},
\]

\[
\varphi_{j,r} := \left( \frac{d}{dt} \right)^r \left\{ \frac{t^{j+1}}{j+1} \log t - \frac{t^{j+1}}{(j+1)^2} \right\} \bigg|_{t=1}.
\]

Proof of this theorem will be carried out through the sections 5.1 ~ 5.4, and some examples of this representation will be discussed in the section 5.5.

5.1. **Rewriting the Euler-MacLaurin expansion of** \(G_n(z+1)\). In this section we prove the following proposition.

**Proposition 5.2.**

\[
\log G_n(z+1) = \sum_{j=0}^{n-1} G_{n,j}(z)K_j(z)
\]

where

\[
K_j(z) := \frac{B_{j+1}(z+1)}{j+1} \log(z+1) - \zeta'(-j) + P_j(z+1)
\]
\[
+ \sum_{k=1}^{\infty} \left[ P_j(z+k+1) - P_j(z+k) + \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{z+k} \right) \right].
\]
Furthermore the infinite sum of the last term is absolutely convergent.

Proof. From the definition of $G_{n,r}(z)$, we have

$$
\left( -\frac{d}{dt} \right)^{r-1} \left( \frac{z}{n-1} \right) = \left( \frac{d}{du} \right)^{r-1} \left( \frac{z-u}{n-1} \right) \bigg|_{u=0} = (r-1)!G_{n,r-1}(z),
$$

and hence

$$
\sum_{r=1}^{n} \left\{ \left( -\frac{d}{dz} \right)^{r-1} \left( \frac{z}{n-1} \right) \right\} \bigg|_{z=0} \times \frac{1}{r!} \{ (z+1)^r - 1 \}
$$

$$
= \sum_{j=0}^{n-1} G_{n,j}(z) \frac{(z+1)^{j+1} - 1}{(j+1)^2}.
$$

Since

$$
\left( \frac{-t}{n-1} \right) = \left( \frac{z-(z+t)}{n-1} \right) = \sum_{j=0}^{n-1} G_{n,j}(z)(z+t)^j,
$$

we have, for $1 \leq r \leq n-1$

$$
F_{n,r-1}(z) = \sum_{j=0}^{n-1} G_{n,j}(z)(z+1)^{j+1-r} \varphi_{j,r}.
$$

In order to calculate the coefficient of $\log(z+1)$, we use the following lemma.

**Lemma 5.3.**

$$
\left( \frac{z+1}{n} \right) + \sum_{j=0}^{n-1} G_{n,j}(z) \frac{B_{j+1}}{j+1} = \sum_{j=0}^{n-1} G_{n,j}(z) \frac{B_{j+1}(z+1)}{j+1}.
$$

**Proof.** Since the both sides are polynomials in $z$, it is sufficient to prove that the formula holds for a sufficiently large, arbitrary integer $N$. Noting that

$$
\frac{B_{j+1}(N+1) - B_{j+1}}{j+1} = \begin{cases} 
\sum_{l=1}^{N} \frac{N^{l-j}}{(l+1)!} & (j > 0) \\
\frac{N^j}{j+1} & (j = 0),
\end{cases}
$$

we have

$$
\sum_{j=0}^{n-1} G_{n,j}(N) \frac{B_{j+1}(N+1)}{j+1} - \sum_{j=0}^{n-1} G_{n,j}(N) \frac{B_{j+1}}{j+1}
$$
\[
= \sum_{l=1}^{N} \sum_{j=0}^{n-1} G_{n,j}(N)^j l^j + G_{n,0}(N)
\]
\[
= \sum_{l=1}^{N} \binom{N-l}{n-1} + \binom{N}{n-1}
\]
\[
= \binom{N}{n} + \binom{N}{n-1}
\]
\[
= \binom{N+1}{n}.
\]

\[\square\]

From (5.2) and Lemma 5.3, it follows that
\[
\left( z + 1 \right) + \sum_{r=1}^{n} B_r \left( - \frac{d}{dt} \right)^{r-1} \left( \begin{array}{c} z \\ (n-1) \end{array} \right) \quad (5.5)
\]
\[
= \left( z + 1 \right) + \sum_{j=0}^{n-1} G_{n,j}(z) \frac{B_{j+1}}{j+1}
\]
\[
= \sum_{j=0}^{n-1} G_{n,j}(z) \frac{B_{j+1}(z+1)}{j+1}.
\]

Next we calculate \( R_{n,n+1}(z) \). From the definition of \( G_{n,r}(z) \), we have
\[
R_{n,n+1}(z) \quad (5.6)
\]
\[
= \frac{(-1)^n}{(n+1)!} \int_{1}^{\infty} B_{n+1}(t) \left( \frac{d}{dt} \right)^{n+1} \left\{ \left( z + (z+t) \right) \log \left( \frac{z + t}{z+1} \right) \right\} dt
\]
\[
+ \sum_{j=0}^{n-1} G_{n,j}(z) \left[ \frac{(-1)^n}{(n+1)!} \int_{1}^{\infty} B_{n+1}(t) \left( \frac{d}{dt} \right)^{n+1} \left\{ (z+t)^j \log \left( \frac{z + t}{z+1} \right) \right\} \right] dt
\]
\[
= \sum_{j=0}^{n-1} G_{n,j}(z) \left[ \sum_{k=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int_{0}^{1} B_{n+1}(t) \left( \frac{d}{dt} \right)^{n+1} \left\{ (z+t+k)^j \log \left( \frac{z + t + k}{z+1} \right) \right\} \right] dt.
\]
Here we have, for \( j \leq n - 1 \),
\[
\int_0^1 B_{n+1}(t) \left( \frac{d}{dt} \right)^{n+1} \left\{ (z + t + k)^j \log \left( \frac{z + t + k}{z + 1} \right) \right\} dt = O(k^{j-n-1}) \quad \text{as} \quad k \to \infty,
\]
so that the infinite sum in (5.6) converges absolutely because \( j-n-1 \leq -2 \).

By means of the Euler-MacLaurin summation formula, we have
\[
\frac{(-1)^n}{(n+1)!} \int_0^1 B_{n+1}(t) \left( \frac{d}{dt} \right)^{n+1} \left\{ (z + t + k)^j \log \left( \frac{z + t + k}{z + 1} \right) \right\} dt = \sum_{r=0}^{j+1} \left( \frac{B_r}{r!} \right) [j]_{r-1} \{ (z + k)^j \cdot (z + k+1)^{j+1-r} \} \left( \frac{1}{r+1} \right) \log(z + 1)
\]
\[
- \sum_{r=1}^{j+1} \frac{B_r}{r!} [j]_{r-1} \{ (z + k)^j \cdot (z + k+1)^{j+1-r} \} \left( \frac{1}{r+1} \right) \log(z + 1)
\]
\[
- \sum_{r=0}^{n+1} \frac{B_r}{r!} \varphi_{r,j} \left\{ (z + k)^j \cdot (z + k+1)^{j+1-r} \right\}.
\]

The coefficient of \( \log(z + 1) \) of the above formula vanishes because of the following lemma.

**Lemma 5.4.**

\[
\frac{B_{j+1}(z + k + 1)}{j+1} = \frac{(z + k + 1)^{j+1}}{j+1} + \sum_{r=1}^{j+1} \frac{B_r}{r!} [j]_{r-1} (z + k + 1)^{j+1-r}
\]
\[
= (z + k)^j + \frac{(z + k)^{j+1}}{j+1} + \sum_{r=1}^{j+1} \frac{B_r}{r!} [j]_{r-1} (z + k)^{j+1-r}.
\]
Proof. The first equality follows from the identity

\[ B_{j+1}(z+k+1) = \sum_{r=0}^{j+1} \binom{j+1}{r} (z+k+1)^{j+1-r} B_r. \]

The second equality holds because of the Euler-MacLaurin summation formula for \((z+t+k)^j\) on \([0,1]\).

Applying Lemma 5.4 to (5.8), we obtain

\[
R_{n,n+1}(z) = -\sum_{j=0}^{n-1} G_{n,j}(z) \sum_{k=1}^{\infty} \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{z+k} \right) \\
+ \sum_{r=0}^{n+1} \frac{B_r}{r!} \varphi_{j,r} \left\{ (z+k+1)^{j+1-r} - (z+k)^{j+1-r} \right\}.
\]

If \(r \geq j+2\), then \((z+k+1)^{j+1-r} - (z+k)^{j+1-r}\) decreases more rapidly than \(k^{-2}\) as \(k \to \infty\). Thus we have

\[
R_{n,n+1}(z) = -\sum_{j=0}^{n-1} G_{n,j}(z) \left[ -\sum_{r=j+2}^{n+1} \frac{B_r}{r!} (z+1)^{j+1-r} \varphi_{j,r} \\
+ \sum_{k=1}^{\infty} \left\{ \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{z+k} \right) + P_j(z+k+1) - P_j(z+k) \right\} \right],
\]

and the infinite sum is absolutely convergent.

Substituting (5.3), (5.4), (5.5), (5.9) to (4.8), and noting that

\[
- \frac{1}{(j+1)^2} + \sum_{r=1}^{n+1} \frac{B_r}{r!} (z+1)^{j+1-r} - \sum_{r=j+2}^{n+1} \frac{B_r}{r!} (z+1)^{j+1-r} = P_j(z+1),
\]

we obtain (5.1).

The next lemma which will be useful in the section 5.3 is deduced from the above proof.

Lemma 5.5.

\[
\frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{z+k} \right) + P_j(z+k+1) - P_j(z+k) = O(k^{-2}) \quad \text{as} \quad k \to \infty.
\]

In other words, the polynomial \(P_j(z+k+1) - P_j(z+k)\) is a convergent factor such that the infinite sum in (5.9) converges absolutely.
5.2. An infinite product representation for the $\zeta'(-j)$. We derive an infinite product representation for $\zeta'(-j)$ in the same way as in the section 3.1.

**Proposition 5.6.**

\[
\exp(\zeta'(-j)) = \exp(P_j(1)) \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{1}{k}\right)^{\frac{B_{j+1}(k+1)}{j+1}} \exp \left(P_j(k+1) - P_j(k) \right) \right\}
\]

and the infinite product converges absolutely.

**Proof.** From the proof of Lemma [4.3], we have

\[
\zeta'(-j) = -\frac{1}{(j+1)^2} + \sum_{r=1}^{j+2} \frac{B_r}{r!} \varphi_{j,r} 
\]

\[
- \frac{(-1)^{j+1}}{(j+2)!} \int_{1}^{\infty} B_{j+2}(t) \left( \frac{d}{dt} \right)^{j+2} \{t^j \log t\} dt 
\]

\[
= P_j(1) + \frac{B_{j+2}}{(j+2)!} \varphi_{j,j+2} 
\]

\[
- \frac{(-1)^{j+1}}{(j+2)!} \int_{1}^{\infty} B_{j+2}(t) \left( \frac{d}{dt} \right)^{j+2} \{t^j \log t\} dt. 
\]

Furthermore we can see by the same way as (5.7) that

\[
\int_{1}^{\infty} B_{j+2}(t) \left( \frac{d}{dt} \right)^{j+2} \{t^j \log t\} dt = O(k^{-2}) \quad \text{as} \quad k \to \infty, 
\]

(5.10)

Hence we have

\[
\frac{(-1)^{j+1}}{(j+2)!} \int_{1}^{\infty} B_{j+2}(t) \left( \frac{d}{dt} \right)^{j+2} \{t^j \log t\} dt 
\]

\[
= \frac{(-1)^{j+1}}{(j+2)!} \sum_{k=1}^{\infty} \int_{0}^{1} B_{j+2}(t) \left( \frac{d}{dt} \right)^{j+2} \{(t+k)^j \log(t+k)\} dt, 
\]

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and the infinite sum of this formula is absolutely convergent. In the same fashion as in the previous section, we have

\[
(5.11) \quad \int_0^1 B_{j+2}(t) \left( \frac{d}{dt} \right)^{j+2} \{(t+k)^j \log(t+k)\} dt
\]

\[
= - \frac{B_{j+1}(k+1)}{j+1} \log \left( 1 + \frac{1}{k} \right) + P_j(k) - P_j(k+1)
\]

\[
+ \frac{B_{j+2}}{(j+2)!} \left( \frac{1}{k} - \frac{1}{k+1} \right) \varphi_{j,j+2}.
\]

From (5.10) and (5.11), it follows that

\[
\frac{B_{j+1}(k+1)}{j+1} \log \left( 1 + \frac{1}{k} \right) + P_j(k+1) - P_j(k) = O(k^{-2}) \text{ as } k \to \infty.
\]

Hence we have

\[
\zeta'(-j) = P_j(1) + \sum_{k=1}^{\infty} \log \left[ \left( 1 + \frac{1}{k} \right)^{\frac{B_{j+1}(k+1)}{j+1}} \exp \{P_j(k+1) - P_j(k)\} \right],
\]

and the infinite sum is absolutely convergent. \qed

**Remark 5.7.** In a similar fashion to Proposition 5.6, an infinite product representation of \( \zeta'(-j, z) \) \((:= \frac{d}{ds}\zeta(s, z)|_{s=-j})\) can be given as follows:

\[
\exp(\zeta'(-j, z)) = \exp \left( - \frac{z^{j+1}}{j+1} \log z + P_j(z) \right)
\]

\[
\times \prod_{k=0}^{\infty} \left[ \left( \frac{z+k+1}{z+k} \right)^{\frac{B_{j+1}(z+k+1)}{j+1}} \exp \{P_j(z+k+1) - P_j(z+k)\} \right].
\]

**5.3. Good representation for \( K_j(z) \).** In this section, we give a "good" representation for \( K_j(z) \). The following two lemmas are useful in subsequent arguments.
Lemma 5.8. Let $k$ be a positive integer and $k \to \infty$, then we have

(1) \[ k^r \log \left( 1 + \frac{z}{k} \right) + \sum_{l=1}^{r+1} \frac{(-1)^l z^l}{l} k^{r-l} = O(k^{-2}) \]

(2) \[ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( 1 + \frac{z}{k} \right) + \frac{1}{j + 1} \sum_{r=0}^{j+1} B_{j+1-r}(1) \sum_{l=1}^{r+1} \frac{(-1)^l}{l} (z + k)^{l-r} \]
\[ = O(k^{-2}). \]

Proof. (1) is clear. (2) follows from the identity

\[ B_{j+1}(z + k + 1) = \sum_{r=0}^{j+1} B_{j+1-r}(1)(z + k)^r. \]

\[ \square \]

Lemma 5.9. There exists a unique polynomial $A(k, z)$ such that

\[ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( 1 + \frac{z}{k} \right) + A(k, z) = O(k^{-2}). \]

$A(k, z)$ is equal to $P_j(z + k + 1) - P_j(z + k)$.

Proof. By integrating the both side of

\[ B_{j+1}(z + 1) = \sum_{r=0}^{j+1} B_{j+1-r}(1)z^r \]
from $-1$ to $0$, we get the formula

\[ \frac{1}{j + 1} \sum_{r=0}^{j+1} \binom{j + 1}{r} B_{j+1-r}(1) \frac{(-1)^r}{r+1} = 0. \] (5.13)

From Lemma 5.8 (2) and (5.13), a polynomial $A(k, z)$ satisfying

\[ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( 1 + \frac{z}{k} \right) + A(k, z) = O(k^{-2}) \quad \text{as} \quad k \to \infty, \]

is uniquely determined and the polynomial $P_j(z + k + 1) - P_j(z + k)$ satisfies (5.14) by Lemma 5.4. Therefore, $A(k, z) = P_j(z + k + 1) - P_j(z + k)$.

\[ \square \]

Using these lemmas, we prove the following proposition.
Proposition 5.10. Let $k$ be a positive integer and define

$$(I)_k := \frac{1}{j+1} \left\{ \sum_{r=0}^{j+1} \binom{j+1}{r} B_{j+1-r}(z) \sum_{l=1}^{r+1} \frac{(-1)^{l-1}z^l}{l} k^{l-r} \right\},$$

$$(II)_k := -(I)_{k+1},$$

$$(III)_k := \sum_{r=0}^{j} z^{j-r} \{ P_r(k+1) - P_r(k) \} - \frac{1}{k} \frac{z^{j+1}}{j+1},$$

and

$$(IV)_k := \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1}z^l}{l} k^{r-l}.$$ 

Then we have

$$\frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{z+k} \right) + P_j(z+k+1) - P_j(z+k)$$

$$= \{ \frac{B_{j+1}(z+k)}{j+1} \log \left( \frac{k}{z+k} \right) + (I)_k \}$$

$$+ \{ \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{k+1} \right) + (II)_k \}$$

$$+ \{ \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{k+1}{k} \right) + (III)_k \}$$

$$+ \{ -(z+k)^j \log \left( 1 + \frac{z}{k} \right) + (IV)_k \}$$

$$- \frac{1}{j+1} \left\{ \sum_{r=0}^{j+1} \frac{(-1)^r z^{r+1}}{r+1} B_{j+1-r}(z) \right\} \left( \frac{1}{k} - \frac{1}{k+1} \right).$$

Furthermore each term in the right hand side decreases like $O(k^{-2})$ as $k \to \infty$.

Proof. Making use of the identity,

$$\frac{B_{j+1}(z+k) - B_{j+1}(z+k+1)}{j+1} = -(z+k)^j,$$
we have
\[
\frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{z+k} \right)
\]
\[
= \frac{B_{j+1}(z+k)}{j+1} \log \left( \frac{k}{z+k} \right) + \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{z+k+1}{k+1} \right)
\]
\[
+ \frac{B_{j+1}(z+k+1)}{j+1} \log \left( \frac{k+1}{k} \right) - (z+k)^j \log \left( \frac{z+k}{k} \right).
\]

By Lemma 5.8 (1), (5.12) and the identity
\[
\frac{B_{j+1}(z+k+1)}{j+1} = \frac{1}{j+1} \sum_{r=0}^{j+1} \binom{j+1}{r} B_{j+1-r}(z) k^r,
\]
we can see that each term in the right hand side of (5.15) is \(O(k^{-2})\).

On the other hand, it can be seen that \((I)_k \sim (IV)_k\) are polynomials of \(z\) and that
\[
(I)_k = \frac{1}{j+1} \left\{ \sum_{r=0}^{j+1} \binom{j+1}{r} B_{j+1-r}(z) \frac{(-1)^r z^{r+1}}{r+1} \right\} \frac{1}{k}
\]
\[
+ \text{(a polynomial of } k),
\]
\[
(II)_k = -\frac{1}{j+1} \left\{ \sum_{r=0}^{j+1} \binom{j+1}{r} B_{j+1-r}(z) \frac{(-1)^r z^{r+1}}{r+1} \right\} \frac{1}{k+1}
\]
\[
+ \text{(a polynomial of } k),
\]
\[
(III)_k = -\frac{z^{j+1}}{j+1} \frac{1}{k} + \text{(a polynomial of } k),
\]
\[
(IV)_k = \left\{ \sum_{r=0}^{j} \frac{(-1)^r}{r+1} \binom{j}{r} \right\} \frac{z^{j+1}}{k} + \text{(polynomial of } k)
\]
\[
= \frac{z^{j+1}}{j+1} \frac{1}{k} + \text{(a polynomial of } k).
\]

Hence, if we put
\[
B(k,z) := (I)_k + (II)_k + (III)_k + (IV)_k
\]
\[
- \frac{1}{j+1} \left\{ \sum_{r=0}^{j+1} \binom{j+1}{r} \frac{(-1)^r z^{r+1}}{r+1} B_{j+1-r}(z) \right\} \left( \frac{1}{k} - \frac{1}{k+1} \right),
\]
then, $B(k, z)$ is a polynomial of $k$, $z$ and it satisfies
\[ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( 1 + \frac{z}{k} \right) + B(k, z) = O(k^{-2}) \quad \text{as} \quad k \to \infty. \]

By Lemma 5.5, we have
\[ B(k, z) = P_j(z + k + 1) - P_j(z + k). \quad (5.17) \]

By (5.16) and (5.17), we can deduce (5.15). \hfill \Box

**Proposition 5.11.**

\[ K_j(z) = Q_j(z) + \sum_{r=1}^{j} \binom{j}{r} z^{j-r} \zeta'(-r) - \frac{z^{j+1}}{j+1} \gamma \]
\[ + \sum_{k=1}^{\infty} \left\{ -(z + k)^k \log \left( 1 + \frac{z}{k} \right) + \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l} k^{r-l} \right\} \]
and the infinite sum of this formula converges absolutely.

**Proof.** The summation of each term in (5.15) from $k = 1$ to $k = \infty$, is absolutely convergent because of Proposition 5.9, and the following calculation is possible. Namely,
\[ \sum_{k=1}^{\infty} \left\{ \frac{B_{j+1}(z + k)}{j + 1} \log \left( \frac{k}{z + k} \right) + (I)_k \right\} \quad (5.18) \]
\[ + \sum_{k=1}^{\infty} \left\{ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( \frac{z + k + 1}{k + 1} \right) + (II)_k \right\} \]
\[ = -\frac{B_{j+1}(z + 1)}{j + 1} \log(z + 1) + \frac{1}{j + 1} \sum_{r=0}^{j+1} \binom{j + 1}{r} B_{j+1-r}(z) \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l}. \]

Noting Theorem 5.6 and
\[ \gamma = \sum_{k=1}^{\infty} \left\{ \log \left( 1 + \frac{1}{k} \right) - \frac{1}{k} \right\}, \]
we have
\[ \sum_{k=1}^{\infty} \left\{ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( 1 + \frac{1}{k} \right) + (III)_k \right\} \quad (5.19) \]
\[ = \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \left\{ \zeta'(-r) - P_r(1) \right\} - \frac{z^{j+1}}{j + 1} \gamma. \]
By (5.18) and (5.19), we obtain

\[
\sum_{k=1}^{\infty} \left\{ \frac{B_{j+1}(z + k + 1)}{j + 1} \log \left( \frac{z + k + 1}{z + k} \right) + P_j(z + k + 1) - P_j(z + k) \right\} = -\frac{B_{j+1}(z + 1)}{j + 1} \log(z + 1)
\]

\[
+ \frac{1}{j + 1} \sum_{r=1}^{j+1} \binom{j + 1}{r} B_{j+1-r}(z) \sum_{l=1}^{r} \frac{(-1)^{l-1}z^l}{l} 
\]

\[
+ \sum_{r=0}^{j} \binom{j}{r} \frac{z}{r} \left\{ \zeta'(-r) - P_r(1) \right\} - \frac{z^{j+1}}{j + 1} \gamma 
\]

\[
+ \sum_{r=0}^{\infty} \left\{ -z^j \log \left( 1 + \frac{z}{k} \right) + \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1}z^l}{l} k^{r-l} \right\}.
\]

The proof is completed by substituting (5.20) to the definition of \( K_j(z) \) in Proposition 5.2.

5.4. A proof of main theorem. Using the results in the sections 5.1 ~ 5.4, we prove Theorem 5.1. By Proposition 5.11, we have

\[
\log G_n(z + 1) = \sum_{j=1}^{n-1} G_{n,j}(z) \left[ Q_j(z) + \sum_{r=0}^{j-1} \binom{j}{r} z^{j-r} \zeta'(-r) - \frac{z^{j+1}}{j + 1} \gamma 
\]

\[
+ \sum_{k=1}^{\infty} \left\{ -z^j \log \left( 1 + \frac{z}{k} \right) + \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=0}^{r+1} \frac{(-1)^{l-1}z^l}{l} k^{r-l} \right\}.
\]

It is easily seen that

\[
\sum_{j=1}^{n-1} G_{n,j}(z) \sum_{r=0}^{j-1} \binom{j}{r} z^{j-r} \zeta'(-r),
\]

\[
= \sum_{r=0}^{n-2} \left\{ \sum_{j=0}^{n-1} G_{n,j}(z) z^{j-r} \right\} \zeta'(-r),
\]

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\[
\begin{align*}
&= \sum_{r=0}^{n-2} \left[ \frac{1}{r!} \left( \frac{\partial}{\partial u} \right)^r \frac{z-u}{n-1} \right]^{u=z}_{u=0} \times \zeta'(-r), \\
\sum_{j=0}^{n-1} G_{n,j}(z) \frac{z^{j+1}}{j+1} = \int_0^z \left( \frac{z-u}{n-1} \right) du, \\
\sum_{j=0}^{n-1} G_{n,j}(z)(z+k)^j = \left( \frac{-k}{n-1} \right). 
\end{align*}
\]

Substituting (5.22), (5.23), (5.24) to (5.21), we have

\[
\log G_n(z+1) \quad (5.25)
\]

\[
= \sum_{j=0}^{n-1} G_{n,j}(z)Q_j(z) + \sum_{r=0}^{n-2} \left[ \frac{1}{r!} \left( \frac{\partial}{\partial u} \right)^r \frac{z-u}{n-1} \right]^{u=z}_{u=0} \times \zeta'(-r) \\
- \int_0^z \left( \frac{z-u}{n-1} \right) du \times \gamma \\
- \sum_{k=1}^{\infty} \left[ \left( \frac{-k}{n-1} \right) \log \left( 1 + \frac{z}{k} \right) + \sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l} k^{r-l} \right].
\]

Thus, in order to prove Theorem 5.1 it is sufficient to show

\[
\sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l} k^{r-l} 
\]

\[
= \frac{1}{(n-1)!} \sum_{\mu=-1}^{n-2} \left\{ \sum_{r=\mu+1}^{n-1} \frac{n^{-1} S_{r-z^{-\mu}}}{r-\mu} \right\} (-1)^{\mu+1} k^\mu. 
\]

Since (5.24) and

\[
(z+k)^j \log \left( 1 + \frac{z}{k} \right) + \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l} k^{r-l} 
\]

\[
= O(k^{-2}) \quad \text{as} \quad k \to \infty,
\]

Thus, in order to prove Theorem 5.1 it is sufficient to show
we have

\[- \left( -\frac{k}{n-1} \right) \log \left( 1 + \frac{z}{k} \right) + \sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l} k^{r-l} \]

\[= O(k^{-2}) \quad \text{as} \quad k \to \infty, \tag{5.27} \]

while we obtain

\[- \left( -\frac{k}{n-1} \right) \log \left( 1 + \frac{z}{k} \right) + \sum_{j=0}^{n-1} G_{n,j}(z) \sum_{r=0}^{j} \binom{j}{r} z^{j-r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1} z^l}{l} \frac{k^{r-l}}{(n-1)!} \]

\[= - \left( -\frac{k}{n-1} \right) \log \left( 1 + \frac{z}{k} \right) + \frac{1}{(n-1)!} \sum_{r=0}^{n-1} (-1)^{r} n_{r-1} S_{r} k^{r} \sum_{l=1}^{r+1} \frac{(-1)^{l-1}}{l} \left( \frac{z}{k} \right)^{l} \]

\[= O(k^{-2}) \quad \text{as} \quad k \to \infty, \tag{5.28} \]

by Lemma 5.8 (1). We can deduce that (5.27) and (5.28) imply (5.26) by the same arguments as in Lemma 5.8. Hence the proof is completed. 

\[\square\]

5.5. Examples of the Weierstrass product representation for $G_{n}(z + 1)$. We give some examples of the Weierstrass product representation for the multiple gamma functions.

In the case that $n = 1$, we have

\[G_{1}(z + 1) = \Gamma(z + 1) = e^{-\gamma z} \prod_{k=1}^{\infty} \left\{ \left( 1 + \frac{z}{k} \right)^{-1} e^{-\frac{z}{k}} \right\}. \]

This is the Weierstrass product representation for the gamma function.

In the case that $n = 2$, we have

\[G_{2}(z + 1) = G(z + 1) = e^{-z\zeta'(0) - \frac{z^2}{2}\gamma - \frac{z^2}{4}} \prod_{k=1}^{\infty} \left\{ \left( 1 + \frac{z}{k} \right)^{k} \exp \left( -z + \frac{z^2}{2k} \right) \right\}. \]

Since $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, this is the Weierstrass product representation for the Barnes $G$-function [2].

In the case that $n = 3, 4$ and 5, we obtain the following results.

**Proposition 5.12.** The Weierstrass product representations in the case that $n = 3, 4$ and 5 are as follows:

\[G_{3}(z + 1) \]

\[= \exp \left\{ -\frac{z^3}{4} + \frac{z^2}{8} + \frac{7}{24} z + \zeta'(-1) - \frac{z(z - 1)}{2} \zeta'(0) - \left( \frac{z^3}{6} - \frac{z^2}{4} \gamma \right) \right\} \]
\[
\times \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-\frac{k(k+1)}{2}} \exp \left\{ \left(\frac{z^3}{6} - \frac{z^2}{4}\right) \frac{1}{k} - \left(\frac{z^2}{4} - \frac{z}{2}\right) + \frac{z}{2^k}\right\},
\]

\[G_4(z + 1)\]

\[= \exp \left\{ \frac{61}{144} z^4 + \frac{13}{18} z^3 + \frac{19}{144} z^2 - \frac{5}{24} z\right\}
- \frac{z^2}{2} \zeta'(-2) + \frac{z^2}{3} \zeta'(-1) - \frac{z^3 - 3 z^2 + 2 z \zeta'(0) - \frac{z^4 - 4 z^3 + 4 z^2}{24}}{6} \gamma\}
\]

\[\times \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{\frac{k(k+1)(k+2)}{6}} \exp \left\{ \left(\frac{z^4}{24} - \frac{z^3}{6} + \frac{z^2}{6}\right) \frac{1}{k}
- \left(\frac{z^3}{18} - \frac{z^2}{4} - \frac{z}{3}\right) + \left(\frac{z^2}{12} - \frac{z}{2}\right) k - \frac{z}{6} k^2\right\},
\]

\[G_5(z + 1)\]

\[= \exp \left\{ -\frac{5}{288} z^5 + \frac{7}{64} z^4 - \frac{173}{864} z^3 - \frac{z^2}{36} + \frac{2827}{17280} z\right\}
+ \frac{z}{6} \zeta'(-3) - \frac{z^2}{4} \zeta'(-2) + \frac{2 z^3 - 9 z^2 + 11 z}{12} \zeta'(-1)
- \frac{z^4 - 6 z^3 + 11 z^2 - 6 z}{24} \zeta'(0) - \frac{6 z^5 - 45 z^4 + 110 z^3 - 90 z^2}{720} \gamma\}
\]

\[\times \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-\frac{k(k+1)(k+2)(k+3)}{24}} \exp \left\{ \left(\frac{z^5}{120} - \frac{z^4}{16} + \frac{11}{72} z^3 - \frac{z^2}{8}\right) \frac{1}{k}
- \left(\frac{z^4}{96} - \frac{z^3}{12} + \frac{11}{48} z^2 - \frac{z}{4}\right) + \left(\frac{z^3}{72} - \frac{z^2}{8} + \frac{11}{24}\right) k
- \left(\frac{z^2}{24} - \frac{z}{4}\right) k^2 + \frac{z}{24} k^3\right\}.
\]

**References**

[1] R. Askey, *The q-Gamma and q-Beta functions* Appl. Anal 8 (1978), pp. 125–141.

[2] E.W. Barnes, *The theory of G-function*, Quat. J. Math 31 (1899), pp. 264–314.
[3] ______, *Genesis of the double gamma function* Proc. London. Math. Soc 31 (1900), pp. 358–381.
[4] ______, *The theory of the double gamma function* Phil. Trans. Royal. Soc (A) 196 (1900), pp. 265–388.
[5] ______, *On the theory of the multiple gamma functions*, Trans. Cambridge. Phil. Soc. 19 (1904), pp. 374–425.
[6] J.Dufresnoy et C.Pisot, *Sur la relation functionalle* \( f(x + 1) - f(x) = \phi(x) \), Bull. Soc. Math. Belgique. 15 (1963), pp. 259–270.
[7] G.H.Hardy, *On the expression of the double zeta-function and double gamma function in terms of elliptic functions*, Trans. Cambridge. Phil. Soc. 20 (1905), pp. 395–427.
[8] G.H.Hardy, *On double Fourier series and especially these which represent the double zeta-function and incommensurable parameters*, Quart. J. Math. 37, (1906), pp. 53–79
[9] F.H.Jackson, *A generalization of the functions* \( \Gamma(n) \) and \( x^n \), Proc. Roy. Soc. London. 74 (1904), pp. 64–72
[10] ______, *The basic gamma function and the elliptic functions*, Proc. Roy. Soc. London. A 76 (1905), pp. 127–144
[11] T.Koornwinder, *Jacobi function as limit cases of q-ultraspherical polynomial*, J. Math. Anal. and Appl 148 (1990), pp. 44–54
[12] N.Kurokawa, *Multiple sine functions and Selberg zeta functions*, Proc. Japan. Acad. 67 A (1991), pp. 61–64
[13] ______, *Multiple zeta functions; an example* Adv. Studies. Pure. Math. 21 (1992), pp. 219–226
[14] ______, *Gamma factors and Plancherel measures* Proc. Japan. Acad. 68 A (1992), pp. 256–260
[15] ______, *On a q-analogues of multiple sine functions*, RIMS. kokyuuroku 843 (1992), pp. 1–10
[16] ______, *Lectures delivered at Tokyo Institute of Technology*, 1993.
[17] Yu. Manin, *Lectures on Zeta Functions and Motives*, Asterisque. 228 (1995), pp. 121–163
[18] D.S.Moak, *The q-analogue of Stirling Formula*, Rocky Mountain J. Math, 14 (1984), pp. 403–413
[19] M.Nishizawa, *On a q-analogue of the multiple gamma functions*, to appear in Lett. Math. Phys. 4 [preprint: hep-th/9408143]
[20] T.Shintani, *On a Kronecker limit formula for real quadratic fields* J. Fac. Sci. Univ. Tokyo Sect. 1A. Vol 24 (1977), pp 167–199
[21] T.Shintani, *A proof of Classical Kronecker limit formula* Tokyo J. Math. Vol.3 (1980), pp 191–199
[22] K.Ueno and M.Nishizawa. *Quantum groups and zeta-functions* in : J.Lukierski, Z.Popowicz and J.Sobczyk (eds.) “Quantum Groups : Formalism and Applications” Proceedings of the XXX-th Karpacz Winter School. pp. 115–126 Polish Scientific Publishers PWN. [preprint: hep-th/9408142]
[23] I.Vardi, *Determinants of Laplacians and multiple gamma functions*, SIAM. J. Math. Anal 19 (1988), pp. 493–507.
[24] M.F.Vignéras, *L’équation fonctionale de la fonction zeta de Selberg de groupe modulaire PSL(2, Z)*, Asterisque. 61 (1979), pp. 235–249.
[25] A.Voros, *Spectral functions, Special functions and the Selberg zeta functions*, Comm. Math. Phys. 110 (1987), pp. 431–465
[26] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Fourth edition, Cambridge Univ. Press

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