Stability of vortex lines in liquid $^3$He-$^4$He mixtures at zero temperature

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Abstract

At low temperatures and $^3$He concentrations below $\sim 6.6\%$, there is experimental evidence about the existence in liquid helium mixtures, of stable vortices with $^3$He-rich cores. When the system is either supersaturated or submitted to a tensile strength, vortices lose stability becoming metastable and eventually completely unstable, so that their cores freely expand. Within a density functional approach, we have determined the pressure-$^3$He concentration curve along which this instability appears at zero temperature.

64.60.My, 67.40.Vs, 67.57.Fg, 67.60.-g
The structure, stability and dynamics of quantized vortices in superfluid $^4$He have been extensively studied either experimental or theoretically (for a systematic review see [1], and for recent work, see for example [2–6] and Refs. therein). However, lesser work has been done in the case of $^3$He-$^4$He solutions [1]. An interesting aspect of the vortex structure in these mixtures was disclosed by Williams and Packard [7], who provided experimental evidence that, in diluted $^3$He-$^4$He solutions at low temperature (T), stable vortices present $^3$He condensation onto the core.

In this work we address the problem of the stability of a vortex line as a function of pressure (P) and $^3$He concentration ($x$). In particular, we will discuss the implications that the existence of $^3$He-rich vortices may have on the critical supersaturation of isotopic helium mixtures, and will determine the T=0 vortex spinodal line as a function of $x$.

The interest in studying the stability of vortex lines at negative pressures stems from recent attempts to describe the phenomenon of quantum cavitation in liquid $^4$He [3]. In particular, it has been found [2] that at T=0, vortex lines become unstable at P ∼ -8 bar, whereas the spinodal point is at P ∼ -9.4 bar [8], thus quantitatively showing that vorticity raises the spinodal line (see also [3]).

In $^3$He-$^4$He mixtures the situation is more complex. Indeed, a vortex may become unstable either increasing the $^3$He concentration, or submitting the solution to a tensile strength that may originate a negative pressure. A characteristic of the zero temperature P-$x$ phase diagram that makes more intricate the study of vortex lines in this system, is the existence of a demixing line $P_d(x)$, experimentally determined from saturation up to 20 bar [9]. A recent calculation [10] has found that line to continue down to $x \sim 2.4\%$ and $P\sim -3.1$ bar, which is the T=0 spinodal point of pure $^3$He. The existence in the negative pressure, metastable region, of this equilibrium line between pure $^3$He and the mixture, affects the description of cavitation caused either by bubble growing [10,11] or by vortex destabilization.

Instabilities caused by supersaturation were studied in [12] within the so-called Hollow Core Model (HCM), adapted to helium mixtures replacing the hollow by a $^3$He-rich core. This study was carried out at positive pressures, where the model works better. We have
predicted that at $P=0$, vortices become completely unstable for $x > 8.2\%$.

Although HCM might seem too crude a model, at positive pressures and close to the
demixing line it bears the basic physical ingredients, so it can be used as an useful guide
to understand the appearance of unstable vortices. Let us just recall that within HCM, the
total energy per unit vortex length as a function of the core radius $R$ reads [12]:

$$\Omega_{HCM} = SR + VR^2 + E_0 \ln(R_\infty/R).$$

(1)

It is the sum of an S-”surface term”, plus a V-”volume term”, plus a kinetic energy term.
When the vortex is in the stable phase, all three terms are positive, and only stable vortices
may exist.

On the contrary, in the metastable phase the factor $V$ in the volume term becomes
negative. Depending on whether the system is underpressured or supersaturated, this factor
is proportional either to the difference $\Delta P$ between the pressure in the hollow and in the
bulk, or to the difference $\Delta \mu$ between the chemical potential of $^3$He in the mixture and of
pure $^3$He [11]. As a consequence, the stable vortex becomes metastable and there exists a
critical vortex configuration for which the potential barrier has a maximum located at a core
radius $R_c$. At the saddle point, when the energy difference between critical and metastable
vortices vanishes, the vortex core freely expands.

These arguments can be made quantitative within the density functional approach. To
this end, we resort to the density functional of Refs. [10,11] which describes the basic ther-
modynamical properties of liquid helium mixtures at zero temperature. For the sake of
simplicity, we address the problem of a vortex line. Using cylindrical coordinates and taking
the vortex line as $z$-axis, the density profiles depend on the $r$-distance to the $z$-axis and are
obtained solving the Euler-Lagrange equations

$$\frac{\delta \omega(\rho_3, \rho_4)}{\delta \rho_q} = 0, \ q = 3, 4,$$

(2)

where $\omega(\rho_3, \rho_4)$ is the grand potential density functional [10,11] to which we have added
a centrifugal term $\hbar^2 \rho_4 n^2/(2m_4 r^2)$ [2] associated with the superfluid flow. We choose the
quantum circulation number \( n=1 \) because it corresponds to the most stable vortex \[13\], \( m_4 \) is the \( ^4\text{He} \) atomic mass, and \( \rho_q \) are the particle densities of each helium isotope.

For given \( P \) and \( x \), Eqs. (2) are solved imposing that at long distances from the \( z \)-axis, \( \rho_q \) equals that of the metastable, homogeneous liquid \( \rho^h_q \) (\( x \) is simply \( \rho^h_3/(\rho^h_3+\rho^h_4) \)), and that \( \rho_4 \) and the \( r \)-derivative of \( \rho_3 \) are zero on the \( z \)-axis. Notice that the metastable and critical configurations are solutions of these equations for the same \( P \) and \( x \) conditions. The barrier height per unit vortex length is:

\[
\Delta \Omega = 2\pi \int r dr [\omega(\rho^c_3, \rho^c_4) - \omega(\rho^m_3, \rho^m_4)] ,
\]

where \( \rho^c_q \) and \( \rho^m_q \) are the particle densities of the critical and metastable vortices, respectively. It is worth to note that \( \Delta \Omega \) is a finite quantity: there is no need to introduce any \( r \)-cutoff as it would have been unavoidable if we had described either configuration separately.

Depending on the situation of the metastable vortex in the \( P-x \) plane, there may exist two different kinds of critical configurations. To illustrate it, we show in Fig. 1 the \( P-x \) phase diagram at \( T=0 \) \[14\] and three selected metastable configurations labeled 1 to 3. The grey zone represents the stable region, and the dashed line is the demixing line. Configuration 1 is underpressed, configuration 2 is supersaturated and configuration 3 is both. In all three cases, the metastable configuration corresponds to a rather compact vortex filled with \( ^3\text{He} \) whose radius increases with increasing \( x \). This is not the case for the critical vortex. Indeed, as configuration 2 is in the supersaturated region, the critical vortex may have a large \( R_c \) radius if that point is close enough to the demixing line, and its core is filled with almost pure \( ^3\text{He} \). \( R_c \) diverges at the demixing line, and also \( \Delta \Omega \). Since configuration 1 is underpressed and undersaturated, the critical vortex also has a large radius provided point 1 is close to the \( P=0 \) line, but its core is almost empty, with the surface covered by \( ^3\text{He} \) (Andreev states). \( R_c \) diverges at the \( P=0 \) line, and also \( \Delta \Omega \). As configuration 3 is underpressed and supersaturated, it has two possible critical configurations, one bearing the characteristics of configuration 1, and another bearing those of configuration 2.

These possibilities are displayed in Fig. 2. Fig. 2 (a) corresponds to a type 1 configu-
ration with \( P = -1.66 \) bar, \( x = 1\% \), whereas Fig. 2 (b) corresponds to a type 2 configuration with \( P = 0.91 \) bar, \( x = 8\% \).

Fig. 3 shows the barrier height per unit length as a function of \( P \) for \( x = 1 \) to 9\%. For the sake of illustration, we display for \( x = 4\% \), the barriers corresponding to the two kinds of critical vortices already discussed: the dashed (solid) line is \( \Delta \Omega \) for empty- (filled-) core configurations. Notice that these curves have different slopes because they diverge at different pressures, the former at \( P = 0 \), and the later at \( P = P_d \). For a given \( x \), the \( P \)-value at which \( \Delta \Omega \) is negligible defines a point along the vortex spinodal curve. That curve is the solid line in Fig. 1.

Fig. 4 shows the core radius of the saddle configurations as a function of \( x \) (solid line). Following [6], we have defined that radius as the \( r \)-value at which the superfluid circulation current \( \rho_s^4 (r) / r \) has a maximum. We have found that metastable vortices in the mixture have a core radius larger than in pure \( ^4\text{He} \) [2]. This is in agreement with the experimental findings for stable vortices [1]. Also shown in that figure is the radius of the stable vortex at \( P = 0 \) (dashed line), which is actually metastable above \( x = 6.6\% \).

The above results have implications on the critical supersaturation degree \( \Delta x_{cr} \) of isotopic helium solutions at low temperatures. Recent experiments [14-15] have found \( \Delta x_{cr} \) below \( \sim 1\% \), whereas classical nucleation theory yields \( \sim 10\% \) [12,16]. The microscopically calculated spinodal line [17] is about the same value. Within the HCM, we have argued [12] that the rather small degree of supersaturation experimentally found could be due to destabilization of \( ^3\text{He} \)-rich vortices. The density functional approach yields \( \Delta x_{cr} \) values around 2\% (we recall that the maximum solubility at \( P = 0 \) is \( \sim 6.6\% \) [18]). This is shown in Fig. 5 for \( P = 0, 0.5 \) and 1 bar. A discrepancy with experiment still exists. It is unclear whether considering more realistic vortex geometries, like vortex rings, could bring theory closer to experiment. Other possibilities to improve on the agreement, such as vortex destabilization due to quantum tunneling through, or to thermal activation over the barrier, seem to be ruled out. The former because of the extremelly small quantum-to-thermal crossover temperature [19], and the later because of the large mass of the critical vortex [12].
Vortex destabilization at negative pressures may also have implications on the phenomenon of cavitation in isotopic helium solutions [10,11]. Realistic calculations [20] indicate that for a $^3$He concentration as small as $x \sim 1\%$, cavitation driven either by quantum or by thermal fluctuations triggers phase separation if the system is submitted to a tensile strength of 8.2 bar. If there are metastable vortices in the mixture, the present calculations show that for much smaller tensile strengths, of about 5.2 bar, the solution undergoes phase separation. A delicate question is whether $^3$He atoms have enough time to diffuse into the vortex core on the time scale of current cavitation experiments [21] when their concentration is too small.

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FIGURES

FIG. 1. $P-x$ phase diagram at $T=0$. The grey zone represents the stable region. The dashed curve is the demixing line, which ends at the $P=-3.12$ bar, $x = 2.43\%$ cross, and the solid curve is the vortex spinodal line.

FIG. 2. Panel (a): vortex profiles for $x =1\%$ and $P=-1.66$ bar. Panel (b): vortex profiles for $x = 8\%$ and $P=0.91$ bar. The solid lines represent the total particle density, and the dash-dotted (dashed) lines, the $\rho_4$ ($\rho_3$) densities. Critical (metastable) configurations are denoted as $\rho^c$ ($\rho^m$).

FIG. 3. Barrier height per unit vortex length as a function of $P$ for the indicated $^3$He concentrations. At $x = 4\%$, the dashed line corresponds to empty-core configurations, whereas the solid line corresponds to filled-core ones.

FIG. 4. Solid line, radius of the saddle vortex core. Dashed line, radius of the stable vortex at $P=0$.

FIG. 5. Barrier height per unit vortex length as a function of $x$ for $P=0$, 0.5 and 1 bar. The corresponding critical $x$-values are 8.46, 8.74 and 8.97%, respectively.
(a) 

![Graph showing density distributions](graph(a).pdf)

(b) 

![Graph showing density distributions](graph(b).pdf)
