Abstract

We examine T-duality transformations for supersymmetric strings with target space geometry with compact abelian isometries. We consider the partition function of these models and we show that although T-duality is not a symmetry, due to an anomaly, it relates type IIA to type IIB strings. In this way we extend the corresponding result for toroidal compactification to the general case of non-trivial backgrounds with abelian isometries and for world sheets of any genera.

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In a previous work [1] we have examined the transformation properties of the \( \sigma \)-model partition function under T-duality [2]–[9]. In particular, we have considered a N=1 supersymmetric 2D \( \sigma \)-model with target space a manifold \( M \) with a compact abelian isometry. For this model we have written down the corresponding partition function \( Z[g, b, s] \) which in general depends on the metric \( g_{\mu\nu} \) and the antisymmetric field \( b_{\mu\nu} \) of \( M \), as well as on the spin structure \( s \) of the world sheet. By performing the standard duality transformation for targets with abelian isometries [3], we found that, besides the well-known transformations of the target metric, the antisymmetric field and the dilaton shift [6],[7], there also exists a transformation of the fermion fields [1],[3],[9]. In particular, left and right-handed fermions are transformed differently under duality giving rise to a corresponding non-trivial transformation of the fermionic measure. We have calculated the Jacobian of this transformation using Fujikawa’s method [10] and we have found an “anomaly” which depends on the parity of the spin structure of the world sheet [1]. As a result, T-duality, due to this anomaly, is not strictly speaking a symmetry of the N=1 supersymmetric \( \sigma \)-model. Here we will show that for the string case, this anomaly is what one needs to establish the type IIA/IIB string duality in the general case.

That the type IIA theory is perturbatively T-dual to type IIB has long been known for the 10-dimensional flat Minkowski space compactified on an odd dimensional torus [11]. The crucial observation here was that besides the \( R \to 1/R \) transformation, one has to transform the fermionic left movers as well in order to preserve supersymmetry. For compactification on a circle in the ninth direction for example, the fermionic transformation is \( \psi^0_L \to -\psi^0_L \). This transformation flips the sign of \( \Gamma^{11}_L \), alters the GSO projection and change the type IIA theory to type IIB and vice versa. We will extend here this result for non-trivial backgrounds with compact abelian isometries and for world sheets of any genera.

Let us consider a \( N = 1 \) supersymmetric \( \sigma \)-model defined on a 2-dim space-time \( \Sigma \) with metric of \((-,+)\) signature. The target space \( M \) is parametrized by the scalars \((X^I, I, J = 1, \cdots, D)\), its metric is \( g_{IJ} \) and the antisymmetric field \( b_{IJ} \). The action for this model, in the conformal gauge and in the conventions of [12] is

\[
S = \frac{1}{2} \int d^2 \sigma \left( (g_{IJ} + B_{IJ}) \partial_+ X^I \partial_- X^J - ig_{IJK} \psi^J_+ D_+ \psi^K_+ - ig_{IJK} \psi^J_- D_- \psi^K_- \right),
\]

\( (1) \)
where
\[ D_{\pm} \psi^I_{\pm} = \partial_{\pm} \psi^I_{\pm} + \left( \Gamma^I_{KL} \pm \frac{1}{2} H^I_{KL} \right) \partial_{\pm} X^K \psi^L_{\pm}, \]

with \( H_{IJK} = \partial_I b_{JK} + \text{cycl. perm.} \) the torsion and the Riemann tensor is evaluated with the torsionful connection \( \Gamma^I_{JK} + \frac{1}{2} H^I_{JK} \). The action (1) is invariant under the supersymmetry transformations
\[ \delta_{\pm} X^I = \mp i \epsilon_{\pm} \psi^I_{\mp}, \]
\[ \delta_{\pm} \psi^I_{\mp} = \mp \partial_{\mp} X^I \epsilon_{\pm}, \]
\[ \delta_{\pm} \psi^I_{\pm} = \pm i \psi^I_{\pm} \epsilon_{\mp} \left( \Gamma^I_{JK} \pm \frac{1}{2} H^I_{JK} \right) \psi^K_{\mp}, \]
as well as under reparametrizations (diffeomorphisms) of \( M \)
\[ X''^I = X'^I(X^J), \quad \psi''^I = \frac{\partial X'^I}{\partial X^J} \psi^J, \]
which, moreover, commute with the supersymmetry transformations of eq. (3). The partition function of the theory is obtained by integrating over all \((X^I, \psi^I)\) fields and it is given by
\[ Z[g, b, s] = \int d\mu.bd\mu.ue^{-iS} = \int \left[ \sqrt{g} \mathcal{D}X^I \frac{1}{\sqrt{g}} \mathcal{D}\psi^I \right] e^{-iS}, \]
where \( g = \det(g_{IJ}) \) is the determinant of the target-space metric. It depends, in general, on the background metric and antisymmetric field and moreover on the spin structure \( s \) of the world sheet. Analytic continuation to Euclidean time is necessary for a proper definition of \( Z[g, b, s] \). However, such a continuation can be performed after the duality transformation. We have not considered ghosts contributions because the ghost sector is not affected by the presence of the background metric and thus, it is irrelevant for our purposes.

We will assume now, that \( M \) has a compact abelian isometry, orthogonal, for simplicity, to the surfaces of transitivity, so that the target space metric may be written as
\[ g_{IJ}(X^K) = (g_{ij}(X^k), g_{00}(X^k)), \]
where \( X^K = (X^k, X^0 = X^D) \), \((i, j, k = 1, \cdots, D - 1)\) are coordinates adapted to the congruence \( \partial/\partial X^0 \). The latter generates an isometry which corresponds to a \( X^0 \rightarrow X^0 + \text{const.} \) symmetry. The duality transformation is performed by gauging this symmetry and adding a Lagrange multiplier which constraints the field strength to vanish. The integration of the
multiplier gives back the original model while by integrating out the gauge field, the dual model is obtained. Applying this procedure to the present case, we get the dual action

\[ \tilde{S} = \frac{1}{2\pi} \int d^2\sigma \left( (\tilde{g}_{IJ} + \tilde{b}_{JK}) \partial_+ X^I \partial_- X^J - i\tilde{g}_{IJ} \tilde{\psi}_+^I D_+ \tilde{\psi}_+^J - i\tilde{g}_{IJ} \tilde{\psi}_-^I D_- \tilde{\psi}_-^J + \frac{1}{2} \tilde{\psi}_+^I \tilde{\psi}_+^J \tilde{\psi}_-^K \tilde{\psi}_-^L \tilde{R}_{IJKL}(\tilde{\Gamma}_+) \right). \] (7)

The dual metric \( \tilde{g}_{IJ} \) and antisymmetric field \( \tilde{b}_{IJ} \) are given by the Buscher’s transformation [3] and the dilaton shift has been omitted. In addition, the fermions \( \tilde{\psi}^I \) in the dual theory are related to the original ones by

\[ \tilde{\psi}^0 = -\bar{\gamma} g_{0I} \psi^I + b_{0I} \psi^I, \] (8)

\[ \tilde{\psi}^i = \psi^i, \] (9)

where \( \bar{\gamma} = \sigma^3 \) is the corresponding \( \gamma^5 \) matrix in 2 dimensions. This transformation is necessary in order the dual action, as the original one, to have manifest \( N=1 \) supersymmetry [1],[3] and it can also be derived from the supersymmetry transformations in eq.(3) [9].

The transformation of the fermion fields in eqs.(8,9) now, effects the fermionic measure \( d\mu_f \) in the partition function. In particular, we found in [1] that the Jacobian of the transformation \( \psi^I \to \tilde{\psi}^I \) is the parity of the spin structure \((-1)^n\) where \( n \) is the number of (positive chirality) zero modes of the Dirac operator. As a result, the original and the dual theories are not the same but rather satisfy

\[ Z[g,b,s] = (-1)^n Z[\tilde{g},\tilde{b},s]. \] (10)

Thus, for odd world-sheet spin structures, the \( N=1 \) supersymmetric \( \sigma \)-model is not invariant under T-duality. On the other hand, since in string theory modular invariance requires a sum over all spin structures, the string partition function cannot be invariant under T-duality as well. However, as we will see in a moment, T-duality maps type IIA to type IIB strings and vice versa and in this sense it is a symmetry of type II theory. To establish this, let us first consider world sheet topology of a 2-torus which is the first non-trivial case where the anomaly shows up (since \( n = 0 \) for the 2-sphere). In this case there exist four spin structures labeled as \(((+,+),(+,-),(-,+),(-,-))\) which correspond to the possible boundary conditions for the left- right-handed fermions on the torus [14],[15]. The partition function may then be written as a sum over all these structures as

\[ Z[g,b] = \eta_{(+,+)} Z[g,b,(+,+)] + \eta_{(+,-)} Z[g,b,(+,-)] + \eta_{(-,+)} Z[g,b,(-,+)] + \eta_{(-,-)} Z[g,b,(-,-)], \] (11)
where the coefficients $\eta$ have to be specified by modular invariance \[13\]. It follows then from eqs.(10,11) that the partition function of the dual theory is

$$Z[\tilde{g}, \tilde{b}] = -\eta(\cdot,+)Z[g,b,(\cdot,+)] + \eta(-,+)Z[g,b,(-,+)] + \eta(-,-)Z[g,b,(-,-)],$$  \hspace{1cm} (12)$$

since only the $(+,+)$ spin structure has a zero mode. This means that $\tilde{\eta}(\cdot,+) = -\eta(\cdot,+)$. Thus for example, if $\eta(\cdot,+) = +1$ so that left- and right-handed fermions are of the same chirality corresponding to type IIB string, $\tilde{\eta}(\cdot,+) = -1$ in the dual theory and left- and right-handed fermions are of the opposite chirality corresponding to type IIA theory and vice versa \[15\].

For higher genus Riemann surfaces now, the corresponding to eq.(11) sum will looks like

$$Z[g,b] = \sum_{s_+=1}^{2^g-1(2^g+1)} \eta_{s_+} Z[g,b,s_+] + \sum_{s_-=1}^{2^g-1(2^g-1)} \eta_{s_-} Z[g,b,s_-],$$  \hspace{1cm} (13)$$

since there exist $2^g-1(2^g+1)$ even and $2^g-1(2^g-1)$ odd spin structures on a genus $g$ Riemann surface \[16\]. Under T-duality, only the second sum will change sign since the parity for odd (even) spin structures is $-1(+1)$. This means that the projections in the Ramond sector of left-handed and right-handed fermions in the dual theory will be opposite to the original theory turning a type IIA theory to type IIB and vice versa.

It should be noted that one has to consider global aspects of the procedure as well \[5\]. In particular, the holonomies $\int A$ of the gauge field along non-trivial cycles of the world sheet have to be integer numbers. This specifies the range of the dual field $\tilde{X}^0$ which is restricted to lie on the dual lattice of $X^0$. In this way the dual and the original theory are also exactly equivalent as CFT. Finally, for an even number of abelian isometries, T-duality is an exact symmetry, since each isometry will contribute a $(-1)^n$ factor to the partition function leaving the latter invariant. Thus, type IIA theory is T-dual to type IIB only for targets with odd number of abelian isometries.

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References

[1] A.A. Kehagias, “On the invariance of the string partition function under duality”, preprint TUM-HEP-232/95, hep-th/951212.

[2] K. Kikkawa and M. Yamasaki, Phys. Lett. B 149 (1984) 357;
N. Sakai and I. Senda, Prog. Theor. Phys. 75 (1986) 692.

[3] T.H. Buscher, Phys. Lett. B 159 (1985) 127; Phys. Lett. B 194 (1987) 59; Phys. Lett. B 201 (1988) 466.

[4] S. Cecotti, S. Ferrara and L. Girardello, Nucl. Phys. B 308 (1988) 436;
A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B 322 (1989) 167.

[5] M. Rocek and E. Verlinde, Nucl. Phys. B 373 (1992) 630;
A. Giveon and M. Rocek, Nucl. Phys. B 380 (1992) 128;
A. Giveon and E. Kiritsis, Nucl. Phys. B 411 (1994) 487;
E. Kiritsis, C. Kounnas and D. Lüst, Int. J. Mod. Phys. A9 (1994) 1361;
E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, Phys. Lett. B 336 (1994) 183;
E. Alvarez, L. Alvarez-Gaumé, J.L.F. Barbon and Y. Lozano, Nucl. Phys. B 415 (1994) 71.

[6] P. Ginsparg and C. Vafa, Nucl. Phys. B 289 (1987) 414;
T. Banks, M. Dine, H. Dijkstra and W. Fischler, Phys. Lett. B 212 (1988) 45.

[7] A.A. Tseytlin, Mod. Phys. Lett. A 6 (1991) 1721;
A. Giveon, Mod. Phys. Lett. A 6 (1991) 2843;
E. Kiritsis, Mod. Phys. Lett. A 6 (1991) 2871; Nucl. Phys. B 405 (1993) 109.

[8] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77;
E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, Nucl. Phys. (Proc. Supp.) 41 (1995) 1.

[9] S.F. Hassan, “T-duality and non-local supersymmetries”, preprint CERN-TH-95-98, hep-th/9504148.

[10] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195; Phys. Rev. D 21 (1980) 2848; D 22 (1980) 1499 (E); D 23 (1981) 2262.
[11] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B 322 (1989)310;
    J. Dai, R.G. Leigh and J. Polchinski, Mod. Phys. Lett. A 4 (1989)2073.

[12] Ph. Spindel, A. Servin, W. Troost and A. Van Proyen, Nucl. Phys. B 308 (1988)622;
    Nucl. Phys. B311(1988)465.

[13] K. Pilch, A.N. Schellekens and N.P. Warner, Nucl. Phys. B 287 (1987)362.

[14] L. Alvarez-Gaumé, G. Moore and C. Vafa, Commun. Math. Phys. 106 (1986)1;
    E. D’Hoker and D.H. Phong, Rev. Mod. Phys. 60 (1988)917.

[15] N. Seiberg and E. Witten, Nucl. Phys. B 276 (1986)272.

[16] M.F. Atiyah, Ann. Scient. Ec. Norm. Sup. 4 se 4 (1971)47.