Offsets of a regular trifolium

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Abstract. The non-uniqueness of a rational parametrization of a rational plane curve may influence the process of computing envelopes of 1-parameter families of plane curves. We study envelopes of family of circles centred on a regular trifolium and its offsets, paying attention to different parametrizations. We use implicitization both to show that two rational parametrizations of a curve are equivalent, and to determine an implicit equation for the envelope under study. The derivation of an implicit equation of an offset follows another path, leading to new developments of the package GeoGebra Discovery. As an immediate symbolic result, we obtain that in the general case the offset curve of a regular trifolium is an algebraic curve of degree 14. We illustrate this fact by providing a GeoGebra applet that computes such curves automatically and visualizes them in a web browser.

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1. Envelopes and offsets

Let $C_t$ be a family of plane curves, parameterized by the real $t$, given by the equation $F(x, y, t) = 0$.

Definition 1 (Impredicative; [13]). A plane curve $E$ is called an envelope of the family $\{C_k\}$ if the following properties hold:

(i) every curve in the family $\{C_t\}$ is tangent to $E$;
(ii) to every point $M$ on $E$ is associated a value $t(M)$ of the parameter $t$, such that $C_t$ is tangent to $E$ at the point $M$;
(iii) The function $t(M)$ is non-constant on every arc of $E$.

Two other definitions exist, also mentioned by [13]. One of them reads as follows:
Definition 2 (Synthetic; [13]). The envelope of the given family of curves is the set of limit points of intersections of nearby curves $C_t$.

This definition is somehow problematic, as it would request the definition of a topological set of curves, in order to have a reasonable definition of a limit. Nevertheless, such a point of view has been used for a simple case by [10].

The most used and usable definition is called analytic by [13]; note that [2], (sections 9.6.7 and 14.6.1) gives this one only. It has been derived form the limit definition as theorem by [10].

Definition 3 (Analytic; [13]). Let $C_t$ be a 1-parameter family of curves given by an equation $F(x,y,t) = 0$. The envelope of the family $\{C_t\}$ is the set of all points $(x,y)$ such that there is a $t$ verifying the system of equations

$$\begin{align*}
F(x,y,t) &= 0, \\
\frac{\partial F(x,y,t)}{\partial t} &= 0.
\end{align*}$$

Following a Wikipedia page1, [3] reports on a 4th definition of an envelope, namely: the envelope is the curve that bounds the planar region described by the points belonging to the curves in the family. In the case of our study, this is the offset of the trifolium at distance 1, as coined by [16], page 10. Actually, the first three definitions mentioned above are equivalent, but the last one is not equivalent to them. There exist examples where envelope and offset are identical, such as in [10] and [16], but other examples enlighten the difference; see [7]. The family studied in this paper is an example with new properties. They are studied in section 4.

Note that the difference between envelope and offset is of the utmost importance for industrial applications, such as the determination of safety zones in industrial plants and entertainment parks. This issue is evoked in [7].

2. A regular trifolium: different presentations

We study various constructions on the basis of a regular trifolium (called also regular trefoil). We recall the basic definitions, referring to [11], where other constructions are also proposed.

Definition 4. A regular trifolium is a plane algebraic curve, given by the implicit equation

$$(x^2 + y^2)^2 = ax(x^2 - 3y^2)$$

where $a$ is a positive real parameter

The parameter $a$ influences the size of the trifolium, not its shape. Two examples are shown in Figure 1, obtained using GeoGebra, a Dynamic Geometry System (freely downloadable from http://geogebra.org). An important reason for using this software is that it enables dynamical mouse-driven

1http://en.wikipedia.org/wiki/Envelope_(mathematics)
experiment. Animation can be obtained with a CAS also, requiring some programming. The differences between the two kinds of animation are discussed in [7]. A plot can be obtained also using an \texttt{implicitplot} command of a Computer Algebra System. WLOG, we will work in the case where \( a = 1 \).

A more dynamical definition is as follows:

\textbf{Definition 5.} A regular trifolium is the trajectory of the second intersection point between a line and a circle turning around one of their common points in the same direction, and the circle turning four times as fast as the line.

We will use this definition in order to derive a rational parametrization of the regular trifolium, whence showing that this curve is a rational curve.

Consider the circle \( C \) through the origin and whose center has coordinates \((1/2, 0)\). Its implicit equation is \((x - 1/2)^2 + y^2 = 1/4\). A general rotation matrix about the origin is The rotated circle \( C_s \) is thus given by the equation

\[
\left( x \cos s + y \sin s - \frac{1}{2} \right)^2 + \left( y \cos s - x \sin s \right)^2 = \frac{1}{4}. \tag{3}
\]

Now consider the line \( L : y = x \). Rotating the line at a speed one fourth of the rotation speed of the circle, we have a line \( L_s \) whose equation is

\[
\cos \frac{s}{4} x + y \sin \frac{s}{4} = 0. \tag{4}
\]

The circle \( C_s \) and the line \( L_s \) intersect at two points, the origin and a second point whose coordinates are given as follows (for the sake of simplicity, we substitute \( t = s/4 \)):

\[
\begin{align*}
x(t) &= 4 \sin^4 t - 3 \sin^2 t, \\
y(t) &= -\sin t \cos t \left( 4 \sin^2 t - 3 \right).
\end{align*} \tag{5}
\]

Using this parametrization, we obtain Figure 2; an experimental checking using a DGS or a CAS shows that this plot is identical to Figure 1a. This
identity may be checked by substitution of the two expressions for $x(t)$ and $y(t)$ into Equation (2). Note that the parametrization (5) is not rational. In order to have benefit of Theorem 4.7 in [16], we need a rational presentation.

[12] proposes another trigonometric parametric presentation (we modify it in order to be coherent with our previous presentations):

$$\begin{align*}
x &= a \cos t + \cos 2t, \\
y &= a \sin t - \sin 2t.
\end{align*}$$

(6)

This parametrization shows that a regular trifolium is a special case of a hypotrochoid. It can be verified, either experimentally with software or using trigonometric manipulations, that for $a = 1/2$, we obtain the same curve studied earlier. A regular trifolium is a special case of trifolium, in which the three leaves are isometric, two of them being obtained from the third one by a rotation of angle $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ (see [6]).

The following result is well-known, we emphasize it because of its importance for the subsequent sections. We will show two proofs.

**Proposition 6.** A regular trifolium is a rational curve.

This proposition means that a regular trifolium has a rational parametrization. It is well-known that such a parametrization, when it exists, is not unique. We derive now two parametrizations of a regular trifolium. The first one will be obtained when using the following Lemma.

**Lemma 7.** A rational parametrization of the unit circle centred at the origin is as follows:

$$\begin{align*}
x &= \frac{1-t^2}{1+t^2}, \\
y &= \frac{2t}{1+t^2}.
\end{align*}$$

(7)

**Proof.** The proof is straightforward; we give it as a promo, because we will use the same method later in a more complicated setting. Consider the unit circle $U$ centred at the origin and a line $L$ through the point $B(-1,0)$ (Figure 3). The tangent to $U$ through $B$ is parallel to the $y$-axis, all other lines through
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Figure 3. Geometric construction for parameterizing the unit circle $U$

$B$ have a slope $t$ and intersect $U$ at a 2nd point $B_t$. The coordinates of $B_t$ are the solutions of the system of equations

\[
\begin{cases}
x^2 + y^2 = 1, \\
y = t(x + 1).
\end{cases}
\]

The result is obtained by substitution. Note that this parametrization is valid for every point but $B$.

Following [16] (p. 90), we recall that a rational parametrization can be stated by means of rational maps. Let $C$ be a rational affine curve over the field $K$ and $P(t) \in \mathbb{R}(t)^2$ be a rational parametrization of $C$. This parametrization induces a rational map

\[ P : \mathbb{A}^1(K) \longrightarrow C \]

\[ t \mapsto P(t) \]

and $P(\mathbb{A}^1(K))$ is a dense subset of $C$ (dense in the sense of Zariski topology).

A trigonometric parametrization of the unit circle centred at the origin is given by $P(u) = (\cos u, \sin u)$, $u \in \mathbb{R}$. On the one hand, this parametrization is a bijection between the unit circle and every interval $[2k\pi, 2(k + 1)\pi)$. On the other hand, Equations (7) define a bijection from $\mathbb{R}$ onto the unit circle but the point $B$. Therefore, for almost every real $u$, there exists a real $t$ such that $\cos u = \frac{1-t^2}{1+t^2}$ and $\sin u = \frac{2t}{1+t^2}$. By substitution into the trigonometric parametrization of the curve given above, and after simplification, we obtain the following result:

Proposition 8 (First rational parametrization). A rational parametrization of the regular trifolium is given by

\[
\begin{align*}
x(t) &= -\frac{4t^2(3t^2-1)(t^2-3)}{(t^2+1)^4}, \\
y(t) &= -\frac{2t(t-1)(t+1)(3t^2-1)(t^2-3)}{(t^2+1)^4}.
\end{align*}
\]
Another rational parametric presentation for the regular trifolium can be obtained, not applying directly Lemma 7, but using the method of its proof.

**Proposition 9 (Second rational parametrization).** A rational parametrization of the regular trifolium is given by

\[
\begin{align*}
    x(t) &= -\frac{3t^2-1}{(t^2+1)^2}, \\
    y(t) &= -\frac{t(3t^2-1)}{(t^2+1)^2}.
\end{align*}
\]

(9)

**Proof.** The regular trifolium is a quartic with a triple point at the origin. The \(y\)-axis is tangent to the curve at the origin; by substitution of \(x = 0\) into Equation 2, we prove that there is no other point of intersection. Any other line through the origin (with equation \(y = tx\), the parameter \(t\) being the slope of the line) through the origin intersects the trifolium at one extra point, as we see from the solution of system of equations 10; see Figure 4. The coordinates of this point are the solutions of the following system of

\[
\begin{align*}
    (x^2 + y^2)^2 &= x(x^2 - 3y^2), \\
    y &= tx.
\end{align*}
\]

(10)

The solution is obtained by substitution and simplification\(^2\). \(\square\)

**Remark 10.** Other rational parametrizations of the unit circle can be obtained, using other families of lines, such as the families given by equations \(y = t(x-1)\) (lines through \(A(1,0)\)), or \(y-1 = tx\) (lines through \(B(0,1)\), etc.). Following the same process as above, we obtain different rational presentations for the regular trifolium. Nevertheless, it can be proven that all of them provide the same implicit equation for the envelope that we will study.

\(^2\)Of course, these computations may be performed using a CAS with the `solve` command. The output may show both the origin and the extra (variable) point.
3. The 1-parameter family of unit circles centred on the regular trifolium

We consider now the family of unit circles $C_t$ centred on the regular trifolium described in Section 2. The index $t$ is actually the parameter that we use for the parametric presentation of the trifolium. Here we will use the formulas derived in Prop. 9, namely:

\[
\begin{align*}
    x(t) &= -\frac{3t^2 - 1}{(t^2 + 1)^2}, \\
    y(t) &= -\frac{t(3t^2 - 1)}{(t^2 + 1)^2}.
\end{align*}
\]

A general equation for the family of unit circles centred on the trifolium is of the form $F(x, y, t) = 0$, where

\[
F(x, y, t) = \left(x + \frac{3t^2 - 1}{(t^2 + 1)^2}\right)^2 + \left(y + \frac{t(3t^2 - 1)}{(t^2 + 1)^2}\right)^2 - 1.
\]

According to Definition 3, an envelope of this family of unit circles, if it exists, is determined by the solutions of the following system of equations (the second one is displayed after simplification):

\[
\begin{align*}
    \left(x + \frac{3t^2 - 1}{(t^2 + 1)^2}\right)^2 + \left(y + \frac{t(3t^2 - 1)}{(t^2 + 1)^2}\right)^2 - 1 &= 0, \\
    -6t^6y + (-12x - 18)t^5 + 18t^4y + (8x + 60)t^3 + 22t^2y + (20x - 18)t - 2y &= 0.
\end{align*}
\]

Denote

\[
D(t) = (9t^6 - 45t^4 + 75t^2 + 1)\left(t^2 + 1\right)^2.
\]

Then solutions of the System (11) are given by the two following parametrizations, each one defining an envelope of the family of circles, and their union being also an envelope of this family:

\[
\begin{align*}
    x(t) &= \frac{27\left(-3t^4 + 12t^2 - 1\right)\sqrt{D(t)(3t^2 - 5)^2(t^2 + 1) - 81t^{10} + 567t^8 - 1530t^6 + 1566t^4 - 357t^2 - 5}}{D(t)(3t^2 - 5)}, \\
    y(t) &= \frac{t\left[-27t^8 + 144t^6 - 270t^4 + 72t^2 + 2\sqrt{D(t)(3t^2 - 5)^2(t^2 + 1) + 1}\right]}{D(t)},
\end{align*}
\]

and

\[
\begin{align*}
    x(t) &= \frac{27\left(3t^4 - 12t^2 + 1\right)\sqrt{D(t)(3t^2 - 5)^2(t^2 + 1) - 81t^{10} + 567t^8 - 1530t^6 + 1566t^4 - 357t^2 - 5}}{D(t)(3t^2 - 5)}, \\
    y(t) &= \frac{-t\left(27t^8 - 144t^6 + 270t^4 - 72t^2 + 2\sqrt{D(t)(3t^2 - 5)^2(t^2 + 1) - 1}\right)}{D(t)}.
\end{align*}
\]

Figures 5(a) and 5(b) show the two separate components, which will be denoted by $E_1$ and $E_2$ respectively. In Figure 5(c), both the trifolium and the “full” envelope $E = E_1 \cup E_2$ are displayed. A first analysis of the display reveals the following properties:

- The roles of the components $E_1$ and $E_2$ are not symmetric.
- It seems that $E_1$ is twice tangent to $C$ and $E_2$ is tangent once.
- The tree points of tangency are the vertices of an equilateral triangle.
Of course, these remarks have to be proven. This can be done by algebraic manipulations and CAS aided solution of systems of equations.

![Figure 5. The envelopes of the family of unit circles](image)

**Remark 11.** The first and second envelopes are themselves disjoint unions of subcurves, with obvious discontinuities. These plots have been obtained with Maple. A first attempt with the `plot` command yielded within one second a picture with superfluous straight segments connecting the separate subcurves. Addition of the option `discont=true` eliminated these superfluous, but each computation required about one minute.

The full envelope is also the disjoint union of two components: an external one (with three bulbs) and an internal one (looking like 3 fishes connected by their mouth). We will now use algebraic manipulations in order to determine an implicit polynomial equation for the envelope. The properties of the polynomial will indicate whether each of these components is an algebraic curve, or of they are impossible to distinguish by that way.

We transform Equations (12) into polynomial equations, using squaring and transferring form side to side. We obtain the two following polynomials in three variables $x, y, t$:

$$P_1(x, y, t) = 9t^{14}x^2 - 9t^{14} - 9t^{12}x^2 + 54t^{12}x + 45t^{12} - 51t^{10}x^2 - 180t^{10}x + 120t^{10} + 67t^8x^2 + 18t^8x - 678t^8 + 283t^6x^2 + 648t^6x + 647t^6 + 261t^4x^2 + 250t^4x - 567t^4 + 79t^2x^2 - 148t^2x + 90t^2 + x^2 - 2x,$$

$$P_2(x, y, t) = 9t^{14}y^2 + 54t^{12}y - 9t^{12}y^2 + 45t^{12} - 180t^{11}y - 51t^{10}y^2 - 447t^{10} + 18t^9y + 67t^8y^2 + 1106t^8 + 648t^7y + 283t^6y^2 - 462t^6 + 250t^5y + 261t^4y^2 - 111t^4 - 148t^3y + 79t^2y^2 - 99t^2 - 2ty + y^2.$$

These polynomials generate an ideal $J = \langle P_1, P_2 \rangle$ in the polynomial ring $\mathbb{R}[x, y, t]$. Using Maple’s command `EliminationIdeal`, we eliminate the variable $t$ (i.e. we compute a projection onto $\mathbb{R}[x, y]$). The obtained ideal is generated by a polynomial of degree 28 in two variables $x$ and $y$. This polynomial has exactly two irreducible components, each of degree 14, namely:
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\[ F_1(x,y) = 256x^{14} + 1792x^{12}y^2 + 5376x^{10}y^4 + 8960x^8y^6 + 8960x^6y^8 + 5376x^4y^{10} + 1792x^2y^{12} + 256y^{14} - 512x^{13} - 1024x^{11}y^2 + 2560x^9y^4 + 10240x^7y^6 + 12800x^5y^8 + 7168x^3y^{10} + 1536xy^{12} + 3840x^{12} - 12032x^{10}y^2 - 78848x^8y^4 - 122368x^6y^6 - 71936x^4y^8 - 8960x^2y^{10} + 3584y^{12} - 5728x^{11} + 14752x^9y^2 + 50752x^7y^4 + 6464x^5y^6 - 51680x^3y^8 - 27872xy^{10} + 6080x^{10} + 120064x^8y^2 + 336000x^6y^4 + 333056x^4y^6 + 107968x^2y^8 - 3072y^{10} - 5952x^9 - 22688x^7y^2 - 435936x^5y^4 - 288480x^3y^6 + 75424xy^8 - 56823x^8 - 376540x^6y^2 + 232758x^4y^4 - 358620x^2y^6 - 26359y^8 + 35664x^7 + 360802x^5y^2 + 636938x^3y^4 + 201190xy^6 + 56109x^6 - 176542x^4y^2 - 256867x^2y^4 - 19328y^6 - 38090x^5 - 387676x^3y^2 - 68626xy^4 - 15173x^4 + 339668x^2y^2 + 25677y^4 + 14394x^3 - 67846xy^2 - 3289x^2 - 90y^2 + 242x, \]

\[ F_2(x,y) = 1024x^{14} + 7168x^{12}y^2 + 21504x^{10}y^4 + 35840x^8y^6 + 35840x^6y^8 + 21504x^4y^{10} + 7168x^2y^{12} + 1024y^{14} - 2048x^{13} - 4096x^{11}y^2 + 10240x^9y^4 + 40960x^7y^6 + 51200x^5y^8 + 28672x^3y^{10} + 6144xy^{12} - 3072x^{12} - 27648x^{10}y^2 - 67584x^8y^4 - 71680x^6y^6 - 39936x^4y^8 - 15360x^2y^{10} - 4096y^{12} + 13632x^{11} + 13632x^9y^2 - 81792x^7y^4 - 190848x^5y^6 - 149952x^3y^8 - 40896x^{10} - 4992x^{10} - 28416x^8y^2 - 54528x^6y^4 - 49152x^4y^6 - 23424x^2y^8 - 5376y^{10} - 34944x^9 + 15552x^7y^2 + 152640x^5y^4 + 312384x^3y^6 + 99648xy^8 + 28793x^8 + 258596x^6y^2 + 220566x^4y^4 + 35492x^2y^6 + 44729y^8 + 35078x^7 - 35078x^5y^2 - 175390x^3y^4 - 105234xy^6 - 60633x^6 - 297828x^4y^2 - 104613x^2y^4 - 73514y^6 - 35046x^5 + 70092x^3y^2 + 105138xy^4 + 38880x^4 + 77760x^2y^2 + 38880y^4 + 23328x^3 - 69984xy^2. \]

An implicit plot of \( F_2 \) is identical to the envelope displayed in Figure 5(c). Plotting \( F_1 \) yields a totally different figure, which is irrelevant to our purpose. See [3] for a discussion of irrelevant components in such situations.

The same process, starting from the parametric equations of the second component leads to the same two polynomials \( F_1(x,y) \) and \( F_2(x,y) \).

The role of the two components can be explored using an animation with Maple. Two snapshots are displayed in Figure 6. Note that for every value of the parameter \( t \), the corresponding circle touches both components. This confirms, as if this was necessary, an interesting detail in the definition of an envelope: its non-uniqueness. Here, each component is a separate envelope of the given family of circles, and their union too.
4. The offset at distance 1 of the regular trifolium

An interactive study with GeoGebra of the given family of unit circles centred on the regular trifolium shows that the offset of the family is different from the envelope. Recall that the circles of the family have the following general equation:

\[
\left( x + \frac{3t^2 - 1}{(t^2 + 1)^2}\right)^2 + \left( y + \frac{t(3t^2 - 1)}{(t^2 + 1)^2}\right)^2 = 1. \tag{14}
\]

Using a slider bar with the Trace On feature to modify the values of the parameter \( t \), a certain number of circles can be plotted. Note that no preview of the envelope is visible here, but some preview of the offset is obtained. See Figure 7, which shows the regular trifolium \( C \) and a certain number of circles of the family.
5. Symbolic experiments with GeoGebra

Recent improvements in GeoGebra, in particular in its experimental version GeoGebra Discovery, allow us to directly study the offsets of the regular trifolium at various distances. Latest versions since December 2020 (freely available at https://github.com/kovzol/geogebra/releases) make it possible to define the algebraic equation (2) with parameter $a = 1$, and then use it as a path for a moving circle. First the user enters this equation in the Input Bar and obtains the object $eq1$, then a distance $f$ will be defined by a segment $AB$. As a next step, a point $C$ is attached to the trifolium curve, and a circle $c$ with center $C$ and radius $f$ will be added. At last, the command $\text{Envelope}(c, C)$ will be used to compute the offset of the trifolium at distance $f$. (The same result can be obtained by using the $\text{Envelope tool}$ and clicking on $c$ and $C$. In this alternative way, the user does not even have to type anything but the equation, and just use the mouse.) Fig. 8 shows how the object $eq2$ can be obtained finally when $f = 0.5$.

![Figure 8](image)

**Figure 8.** The distance 0.5 offset of the regular trifolium made with GeoGebra Discovery

The obtained figure shows some similarities with the previously obtained figures, most visibly the external envelope has a similar geometry. On the other hand, there is an inmost envelope that seems to be a circle. We can verify this fact by opening the CAS View and issue the command

$$\text{Factors(LeftSide(eq2) - RightSide(eq2))}$$

which informs us that the obtained result consists of two factors: one defines a circle with center $(0, 0)$ and radius $1/2$, reported as $4x^2 + 4y^2 - 1$; the other
factor is

\[
O_{1/2}(x, y) = 262144x^{14} - 524288x^{13} + 1835008x^{12}y^2 \\
- 1048576x^{11}y^2 + 540672x^{11} + 5505024x^{10}y^4 \\
- 2359296x^{10}y^2 + 49152x^{10} + 2621440x^9y^4 \\
+ 540672x^9y^2 - 614400x^9 + 9175040x^8y^6 \\
- 5505024x^8y^4 - 5947392x^8y^2 - 352000x^8 \\
+ 10485760x^7y^6 - 3244032x^7y^4 + 3981312x^7y^2 \\
+ 570368x^7 + 9175040x^6y^8 - 2621440x^6y^6 \\
- 7766016x^6y^4 + 9098240x^6y^2 + 213504x^6 \\
+ 13107200x^5y^8 - 7569408x^5y^6 - 10911744x^5y^4 \\
- 570368x^5y^2 - 359424x^5 + 5505024x^4y^{10} \\
+ 1572864x^4y^8 + 1867776x^4y^6 + 1390080x^4y^4 \\
- 2059776x^4y^2 - 277344x^4 + 7340032x^3y^{10} \\
- 5947392x^3y^8 + 13320192x^3y^6 - 2851840x^3y^4 \\
+ 718848x^3y^2 + 62208x^3 + 1835008x^2y^{12} \\
+ 786432x^2y^{10} + 2998272x^2y^8 - 7244800x^2y^6 \\
+ 2440704x^2y^4 - 554688x^2y^2 + 139968x^2 \\
+ 1572864xy^{12} - 1622016xy^{10} + 516096xy^8 \\
- 1711104xy^6 + 1078272xy^4 - 186624xy^2 \\
+ 262144y^{14} - 262144y^{12} - 638976y^{10} + 815360y^8 \\
- 86528y^6 - 277344y^4 + 139968y^2 - 19683,
\]

defining again a curve of degree 14.

We emphasize here that, in the background, the analytic method was used by GeoGebra, that is, computing a Jacobi determinant and then using elimination to obtain an equation in two variables, defining a plane curve. (See [14] and [15] for more details.) In other words: GeoGebra performed an automated proof on the validity of the formula that describes \(O_{1/2}(x, y)\). What is more, GeoGebra’s symbolic capability ensured that the conjectured formula was also obtained in an automated way. (In fact, these two steps: conjecture and proof were performed in one operation.) The computation took less than half of a second on a modern computer.

When choosing \(f = 1\) or other values, similar results can be obtained. For example, when changing \(f\) to 1 the result \((x^2 + y^2 - 1) \cdot F_2(x, y)\) will be delivered, that is, again a product of a circle and the offset. In this case is the computation, however, somewhat slower.

As a consequence, dynamic investigation of the set of offsets for various distances is a bit inconvenient in GeoGebra. When fixing point \(A\) and dragging point \(B\) we reach computation speed between and 1.34 and 1.35 frame
per second (FPS)\(^3\), even if the same distance is defined but with different input points—this difference comes from the very different computational character of the slightly different algebraic translations. In fact, for an enjoyable animation the user could expect at least 5 FPS, so further speedup of the computation could be addressed in future work.

On the other hand, slow movement of point \(B\) can show how the set of offsets look like. Fig. 9 presents how the curves change while \(B\) is dragged from \((1.5, 1.5)\) to \((1, 0.5)\). (Coloring of point \(B\) and the curve \(eq2\) depends on the length of \(f\): we used the RGB-components \((f, 1 - f, 1)\).) An online version of this experiment can be found at [9].

![Figure 9. Various offsets of the trifolium for distances \(f\) between 0.5 and 1](image)

6. Conclusion

We derived the offset curve of a regular trifolium by using different approaches. Automated reasoning based on elimination played an important role in our results. In our last experiment we highlighted that a point-and-click approach by using a Dynamic Geometry System can merge conjecturing and proving to a single step, and dragging some input points a large set of theorems can be proven in a novel way.

\(^3\)Testing was performed on Ubuntu Linux 18.04, Intel(R) Core(TM) i7-4770 CPU @ 3.40GHz. See https://prover-test.geogebra.org/job/GeoGebra_Discovery-art-plotterart/72/artifact/fork/geogebra/test/scripts/benchmark/art-plotter/html/all.html for a detailed output of the benchmarking suite, test cases trifolium-offset1.ggb, ..., trifolium-offset4.ggb. These results are valid for the native version of GeoGebra, that is, GeoGebra Classic 5. The web version, that is, GeoGebra Classic 6, underperforms this speed between 0.23 and 0.24 FPS.
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