CONFORMALLY-KÄHLER RICCI SOLITONS AND BASE METRICS FOR WARPED PRODUCT RICCI SOLITONS

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ABSTRACT. We investigate Kähler metrics conformal to gradient Ricci solitons, and base metrics of warped product gradient Ricci solitons. The latter we name quasi-solitons. A main assumption that is employed is functional dependence of the soliton potential, with the conformal factor in the first case, and with the warping function in the second. The main result in the first case is a partial classification in dimension $n \geq 4$. In the second case, Kähler quasi-soliton metrics satisfying the above main assumption are shown to be, under an additional genericity hypothesis, necessarily Riemannian products. Another theorem concerns quasi-soliton metrics satisfying the above main assumption, which are also conformally Kähler. With some additional assumptions it is shown that such metrics are necessarily base metrics of Einstein warped products, that is, quasi-Einstein.

1. INTRODUCTION

The study of the Ricci flow [Hami] has inspired the introduction of a metric type generalizing the Einstein condition. A gradient Ricci soliton is a Riemannian metric satisfying

$$\text{Ric} + \nabla df = \lambda g, \quad \lambda \text{ constant}. $$

The function $f$ is called the soliton potential. Such solitons are further referred to as shrinking, steady or expanding, depending on the sign of $\lambda$.

In this paper we consider Ricci solitons in two settings: the case where they are conformal to Kähler metrics, and the case where they are warped products. Conformal classes of Ricci solitons have been studied recently in [JaWy, CMMR]. Kähler metrics in such a conformal class, with nontrivial conformal factor, have been examined in [Mas1, Derd]. Warped product Ricci solitons, on the other hand, have been studied extensively when the base of the warped product is one dimensional (cf. [CCGG]). For example, the cigar soliton and the Bryant soliton belong to this category.

In each case we focus on an auxiliary metric which at least partially determines the soliton. In the first case that would be the associated Kähler metric in the conformal class, and in the second case it is the induced metric on the base of the warped product. We call the latter metric a (gradient Ricci) quasi-soliton, in analogy with how base metrics of Einstein warped products are often called quasi-Einstein metrics. We consider only quasi-soliton metrics which are Kähler, or conformally Kähler.

A common thread for these two cases of auxiliary metrics is the appearance of two Hessians in their defining equation. One of these is the Hessian of the soliton...
potential $f$, while the other Hessian depends on the case: it is that of the conformal factor $\tau$ in the first case, and that of the the warping function $\ell$ in the second.

These equations are, of course, more complex than the original Ricci soliton equation, and handling them in full generality still appears beyond reach. Our strategy is thus to consider mainly the case where functional dependence of the above two functions holds, in either setting. In other words, we require

\begin{equation}
(1.1) \quad d\tau \wedge df = 0 \quad \text{in the first case, and} \quad d\ell \wedge df = 0 \quad \text{in the second.}
\end{equation}

In the latter case we call the metric a special quasi-soliton.

An example where the first of these conditions occurs in the Kähler conformally-soliton case, is when the conformal factor $\tau$ is additionally a potential for a Killing vector field of the Kähler metric (a Killing potential). The latter condition has been studied in [Mas1] and plays a role in Theorem 7.3. It turns out that the first of Conditions (1.1) also implies, generically, the existence of a Killing potential which, however, is of a more general kind, being only functionally dependent on $\tau$, rather than being $\tau$ itself. An instance of this more general setting has first been considered in [Derd].

Another metric type that plays an important role in all our main theorems is the SKR metric, i.e. a metric that admits a so-called special Kähler-Ricci potential. This notion that was first introduced in [DeM1], [DeM2] for the purpose of classifying conformally-Kähler Einstein metrics. In all our main theorems the argument yields a Ricci-Hessian equation of the form

\[ \alpha \nabla d\tau + \text{Ric} = \gamma g, \]

for functions $\alpha$ and $\gamma$. The theory of SKR metrics which is then applied is closely tied to such equations.

The main results in this article are Theorems 6.2, 7.2 and 7.3. The content of the first of these is a partial classification of Kähler metrics conformal to gradient Ricci solitons in dimension $n \geq 4$ with the first of the conditions in (1.1). Theorem 7.2 presents a reducibility result for special quasi-soliton metrics which are Kähler. The conclusion of this theorem, that the metric is a Riemannian product, is analogous to a similar result for quasi-Einstein metrics [CaSW]. Theorem 7.3 mixes the two main themes of this paper, as it involves special quasi-soliton metrics that are conformal to an irreducible Kähler metric. With some additional assumptions, the conclusion of the theorem is that the metric must in fact be quasi-Einstein.

The structure of the paper is as follows. After some preliminaries in §2, we give several forms for the conformally soliton equation in §3. We then determine in §4 in the context of the first metric type considered, certain implications of the assumption that vector fields that occur in the conformally soliton equation are classically distinguished. One such assumption which does not occur in nontrivial cases has, nonetheless, an interesting classification, which we give in an appendix in §8. In §5 we recall the salient features of SKR metric theory. The main theorem in the conformally Kähler case is given in §6 and the two main theorems for special quasi-soliton metrics appear in §7.

The author acknowledges the contribution of Andrzej Derdzinski to this work, most significantly in §6 and the appendix. The paper is dedicated to Vanessa Gunter, whose insightful suggestion led to the results of §7.
2. Preliminaries

Let $(M, g)$ be a Riemannian manifold of dimension $n$, and $\tau : M \to \mathbb{R}$ a $C^\infty$ function. We write metrics conformally related to $g$ in the form $\hat{g} = \tau^{-2}g$, with $\tau$ a smooth function. We recall a few conformal change formulas. The covariant derivative is

\begin{equation}
\hat{\nabla}_wu = \nabla_wu - (d_u \log \tau)u - (d_u \log \tau)w + \langle w, u \rangle \nabla \log \tau,
\end{equation}

where $d_u$ denotes the directional derivative of a vector field $u$ and the angle brackets stand for $g$. It follows that the $\hat{g}$-Hessian and $\hat{g}$-Laplacian of a $C^2$ function $f$ are given by

\begin{equation}
\hat{\nabla} df = \nabla df + \tau^{-1}[2 d\tau \otimes df - g(\nabla \tau, \nabla f)g],
\end{equation}
\begin{equation}
\hat{\Delta} f = \tau^2 \Delta f - (n - 2)\tau g(\nabla \tau, \nabla f),
\end{equation}

where $d\tau \otimes df = (d\tau \otimes df + df \otimes d\tau)/2$. Finally, the well-known formula for the Ricci tensor of $\hat{g}$ is given by

\begin{equation}
\hat{\text{Ric}} = \text{Ric} + (n - 2)\tau^{-1}\nabla d\tau + [\tau^{-1}\Delta \tau - (n - 1)\tau^{-2}|\nabla \tau|^2]g,
\end{equation}

with $\Delta$ denoting the Laplacian and the norm $| \cdot |$ is with respect to $g$.

Recall that a (real) vector field $w$ on a complex manifold $(M, J)$ is holomorphic if the Lie derivative $\mathcal{L}_wJ = 0$.

**Proposition 2.1.** Let $\hat{\nabla}$ be a torsion-free connection on a complex manifold $(M, J)$. For any vector field $w$,

\[ \mathcal{L}_wJ = \hat{\nabla}_wJ + [J, \hat{\nabla} w], \]

where the square brackets denote the commutator.

In fact, write $(\mathcal{L}_wJ)u = \mathcal{L}_w(Ju) - J\mathcal{L}_wu$ and replace each Lie derivative by the Lie brackets, and each of these by the torsion free condition for $\hat{\nabla}$, giving $\hat{\nabla}_wJu - \hat{\nabla}_Jwu - J\hat{\nabla}_wu + J\hat{\nabla}_wu$. The first and third terms together give $(\hat{\nabla}_wJ)(u)$, while the second and fourth terms give $[J, \hat{\nabla} w](u)$.

**Proposition 2.2.** Let $(M, J)$ be a complex manifold with a Hermitian metric $\hat{g}$. Given a $C^2$ function $f$ on $M$, set $w = \hat{\nabla} f$. Then $\hat{\nabla} df$ is $J$-invariant if and only if $[J, \hat{\nabla} w] = 0$.

In fact, $\hat{\nabla} df(Ja, b) = \hat{g}(Ja, \hat{\nabla}_b w) = -\hat{g}(a, J\hat{\nabla}_bw) = -\hat{g}(a, J(\hat{\nabla} w)(b))$ while $-\hat{\nabla} df(a, Jb) = -\hat{g}(a, \hat{\nabla}_Jbw) = -\hat{g}(a, (\hat{\nabla} w)(Jb))$.

In the following well-known proposition, $\iota_v$ denotes interior multiplication by a vector field, while $\delta$ denotes the divergence operator.

**Proposition 2.3.** Let $\sigma$ be a smooth function on a Kähler manifold such that $v = \nabla \sigma$ is a holomorphic gradient vector field. Then $2\iota_v\text{Ric} = -dY$ and $2\delta \nabla \sigma = dY$ for $Y = \Delta \sigma$.

For a proof, see [DeM, (5.4) and (2.9)c)].
3. Various forms of the conformally-soliton equation

Let $g$ be a Riemannian metric and $\tau$ a smooth function on a given manifold, for which $\hat{g} = g/\tau^2$ is a gradient Ricci soliton with soliton potential $f$. The soliton equation for $\hat{g}$, together with its associated scalar equation, are

\begin{align}
\text{(3.1)} & \quad \text{i)} \quad \hat{\text{Ric}} + \hat{\nabla} df = \lambda \hat{g}, \quad \text{with } \lambda \text{ constant.} \\
& \quad \text{ii)} \quad \hat{\Delta} f - \hat{g}(\hat{\nabla} f, \hat{\nabla} f) + 2\lambda f = a, \quad \text{for a constant } a.
\end{align}

To obtain this in terms of $g$, we apply Equation (2.3) and the second equation in (2.2) to Equation (3.1.i). The result is

\begin{align}
\text{(3.2)} & \quad \text{Ric} + (n - 2)\tau^{-1}\nabla d\tau + \nabla df + 2\tau^{-1}d\tau \otimes df = \gamma g.
\end{align}

for

\begin{align}
\gamma & = \tau^{-2}[\lambda + (n - 1)|\nabla \tau|^2] - \tau^{-1}[\Delta \tau - g(\nabla \tau, \nabla f)],
\end{align}

with $|\nabla \tau|^2 = g(\nabla \tau, \nabla \tau)$.

We will now rewrite Equation (3.2) in a different form. Specifically, for the vector fields $v = \nabla \tau$ and $w = \tau^2\nabla f$, Equation (3.2) is equivalent to

\begin{align}
\text{(3.4)} & \quad \text{Ric} + \alpha L_v g + \beta L_w g = \gamma g,
\end{align}

with $\alpha = (n - 2)\tau^{-1}/2$, $\beta = (2\tau^2)^{-1}$, and $L$ denoting the Lie derivative. To show this, recall that for any vector fields $a, b$

\begin{align}
\text{(3.5)} & \quad (L_w g)(a, b) = g(\nabla_a w, b) + g(a, \nabla_b w),
\end{align}

or $L_w g = [\nabla w + (\nabla w)^*]$, where $*$ denotes the adjoint and $\flat$ is the isomorphism associated with lowering of an index. Now clearly $L_v g = L_{\nabla \tau} g = 2\nabla d\tau$. To compute the Lie derivative term for $w$, write $w = h\nabla f$, then

\begin{align}
L_w g & = [\nabla (h \nabla f) + (\nabla (h \nabla f))^*]_b = h\nabla df + dh \otimes df + h\nabla df + df \otimes dh \\
& \quad = 2h\nabla df + 2dh \otimes df.
\end{align}

Setting $h = \tau^2$ and dividing by $2\tau^2$ gives $\nabla df + 2\tau^{-1}d\tau \otimes df = (2\tau^2)^{-1}L_{\tau^2 \nabla f} g = \beta L_w g$.

Another form for Equation (3.2) is obtained as follows. It is natural to combine the two Hessian terms into one. For this, set

\begin{align}
\mu = \log \tau, \quad \theta = f + (n - 2) \log \tau, \quad \psi = 2\theta - (n - 2)\mu.
\end{align}

Then (3.2), (3.3) and (3.1.ii) read

\begin{align}
\text{(3.6)} & \quad \text{i)} \quad \text{Ric} + \nabla d\theta + d\mu \otimes d\psi = \gamma g, \quad \gamma = \lambda e^{-2\mu} - \Delta \mu + g(\nabla \theta, \nabla \mu), \\
& \quad \text{ii)} \quad e^{2\mu}[\Delta f - g(\nabla \theta, \nabla f)] + 2\lambda f = a.
\end{align}

To derive (3.6.ii) one uses the second equation in (2.2), which, in terms of $\mu$, reads

\begin{align}
e^{-2\mu} \hat{\Delta} f = \Delta f - (n - 2)g(\nabla \mu, \nabla f).
\end{align}
Let \( g \) be a metric which is Kähler with respect to a complex structure \( J \) on a manifold \( M \), and conformal to a gradient Ricci soliton. Equation (3.4) then holds, and the \( J \)-invariance of \( g \) and its Ricci curvature implies that

\[
\alpha \mathcal{L}_v g + \beta \mathcal{L}_w g \text{ is } J\text{-invariant.}
\]

Applying (3.5) to the relation \( \mathcal{L}_x g(\cdot, J\cdot) = -\mathcal{L}_x g(J\cdot, \cdot) \), for both \( x = v \) and \( x = w \), and recalling that \( J^* = -J \), we see that (4.1) is equivalent to the vanishing of a commutator:

\[
[\alpha(\nabla v + (\nabla v)^*) + \beta(\nabla w + (\nabla w)^*), J] = 0,
\]

or

\[
[(\mathcal{L}_w g)^\sharp, J] = 0,
\]

where \( \sharp \) denotes the isomorphism acting by the raising of an index.

The most obvious case where (4.2) holds is when both summands vanish separately, so that, \( w \), for example, satisfies

\[
[(\mathcal{L}_w g)^\sharp, J] = 0.
\]

We wish to study relations between these two vanishing conditions for \( v \) and \( w \). We first note that (4.3) includes as special cases the following three classical types of vector fields (the first being, of course, a special case of the second):

- a Killing vector field \( (\mathcal{L}_w g = 0) \),
- a conformal vector field \( ((\mathcal{L}_w g)^\sharp = hI \), for a function \( h \) and \( I \) the identity),
- a holomorphic vector field \( ([\nabla w, J] = 0 \) on a Kähler manifold).

This last type is holomorphic by Proposition 2.1 in the Kähler case, and it is indeed a special case, as \( [\nabla w, J]^* = [(\nabla w)^*, J] \) and (4.3) is equivalent to \( [\nabla w + (\nabla w)^*, J] = 0 \).

We will see in the next theorem that the Killing case does not lead to important Kähler conformally soliton metrics. However, Kähler metrics with a Killing field of the form \( w = \tau^2 \nabla f \) can be classified, as we show in the appendix.

To state the next result, we continue to assume \( g \) is Kähler and conformal to a gradient Ricci soliton \( \hat{g} \), but now on a manifold of dimension \( n > 2 \). With notations as above for \( \tau, f, v \) and \( w \) we have

**Theorem 4.1.** The following conclusions follow for the vector fields \( v \) and \( w \):

1. If \( w \) is a Killing or, more generally, a conformal vector field for \( g \), then \( \hat{g} \) is Einstein.
2. If \( w \) is a holomorphic vector field and either \( v \) is holomorphic as well, or \( \hat{\nabla} df \) is \( J \)-invariant, then \( \text{span}_C\{v\} = \text{span}_C\{w\} \) away from the zero sets of \( v \) and \( w \).

**Proof.** The key to both parts is that \( w = \tau^2 \nabla f \) is also the \( \hat{g} \)-gradient of \( f \), i.e. \( w = \hat{\nabla} f \). Therefore \( \mathcal{L}_w g = \mathcal{L}_{\hat{\nabla} f} g = 2\hat{\nabla} df \). It follows that in the case where \( w \) is Killing, or, more generally, conformal, the Ricci soliton equation in (3.1) reduces, using Schur’s lemma, to the Einstein equation. This proves (1).

To prove (2), note first that the combination of Propositions 2.1 and 2.2 for a Kähler metric yields the result that the vector field \( v = \nabla \tau \) is holomorphic exactly when \( \nabla d\tau \) is \( J \)-invariant. This in turn is equivalent, by (2.3) and the fact that the metric \( g \) and its Ricci curvature are \( J \)-invariant, to \( \hat{\text{Ric}} \) being \( J \)-invariant. Finally, the latter condition is equivalent to \( \hat{\nabla} df \) being \( J \)-invariant, by the soliton equation.
in (3.1). The combination, again, of Propositions 2.1 and 2.2 but this time for a hermitian metric, yields equivalence of the latter condition with $L_{\hat{\nabla} f} J = \hat{\nabla} f J$, or

$$L_w J = \hat{\nabla} w J.$$  

Now from (2.1), for any vector field $u$

$$(\hat{\nabla} w J) u = \hat{\nabla} w (Ju) - \tau^{-1} (d_w \tau) Ju - \tau^{-1} (d_J u \tau) w + \langle w, Ju \rangle \tau^{-1} v$$

$$= [J \nabla_w (u) - \tau^{-1} (d_w \tau) Ju - \tau^{-1} (d_J u \tau) Jw + \langle w, u \rangle \tau^{-1} Jv]$$

$$= \tau^{-1} (-\langle v, Ju \rangle w + \langle w, Ju \rangle v + \langle v, u \rangle Jw - \langle w, u \rangle Jv)$$

$$= \tau^{-1} (\langle Jv, u \rangle w - \langle Jw, u \rangle v + \langle v, u \rangle Jw - \langle w, u \rangle Jv)$$

where we used the fact that $\nabla_w J = 0$, and the angle brackets denote $g$. Combining this with (4.4) we see that as $w$ is holomorphic, the last expression vanishes for every vector field $u$. Substituting first $u = v$ and then $u = Jv$ shows that away from the zeros of $v$, the vector fields $w$ and $Jw$ are in $\text{span} \{ v, Jv \}$. As this reasoning is symmetric for $v$ and $w$, the result follows.

In known examples the manifolds on which $g$ and $\hat{g}$ live are locally total spaces of holomorphic line bundles over manifolds admitting a Kähler-Einstein metric, and there in fact $\text{span}_\mathbb{R} v = \text{span}_\mathbb{R} w$.

## 5. SKR Metrics

We recall here some facts from [DeM1] and [Mas1] on the notion of an SKR metric, i.e. a Kähler metric $g$ admitting a special Kähler-Ricci potential $\sigma$. For the definition, recall that a smooth function $\sigma$ on a Kähler manifold $(M, J, g)$ is called a Killing potential if $J \nabla \tau$ is a Killing vector field. The definition of a special Kähler-Ricci potential consists then of the requirement that $\sigma$ is a Killing potential and, at each noncritical point of it, all nonzero tangent vectors orthogonal to the complex span of $\nabla \sigma$ are eigenvectors of both the Ricci tensor and the Hessian of $\sigma$, considered as operators. This rather technical definition implies that a Ricci-Hessian equation holds for $\sigma$ on a suitable open set (see [DeM1, Remark 7.4]), namely

$$\text{Ric} + \alpha \nabla d\sigma = \gamma g,$$

for some functions $\alpha, \gamma$ which are functionally dependent on $\sigma$.

We say that Equation (5.5) is a standard Ricci-Hessian equation if $\alpha d\alpha \neq 0$ whenever $d\sigma \neq 0$. This condition will appear in all our main theorems. However, even if it does not hold over the entire set where $d\sigma \neq 0$, these theorems will hold, with the same proofs, on any open subset of $\{ d\sigma \neq 0 \}$ where $\alpha d\alpha \neq 0$. We have

**Proposition 5.1.** A Kähler metric on a manifold of dimension at least four is an SKR metric, provided it satisfies a standard Ricci-Hessian Equation of the form (5.5) with $d\alpha \wedge d\sigma = d\gamma \wedge d\sigma = 0$.

This result appears in [Mas1, Proposition 3.5] with proof referenced from [DeM1], a proof that has to be interpreted with the aid of [Mas1, Remark 3.6]. Note also that in dimension greater than four, if the Ricci-Hessian equation of a Kähler metric
satisfies \( d\alpha \wedge d\sigma = 0 \) then it automatically also satisfies \( d\gamma \wedge d\sigma = 0 \) (see [Mas1, Proposition 3.3]).

If an SKR metric is locally irreducible, the theory of such metrics (see §4 of [Mas1]) implies that a pair of equations holds on the open set where the Ricci-Hessian equation (5.5) holds:

\[
\begin{align*}
\sigma - c & \quad \text{or} \quad (\sigma - c)^2 \phi'' + (\sigma - c)[m - (\sigma - c)\alpha]\phi' - m\phi = K, \\
- (\sigma - c)\phi'' + (\alpha(\sigma - c) - (m + 1))\phi' + \alpha\phi &= \gamma
\end{align*}
\]

Here \( \phi \) is defined pointwise as the eigenvalue of the Hessian of \( \sigma \), considered as an operator, corresponding to the eigendistribution \([\text{span}_C \nabla \sigma] \), and \( c \) is a constant. This eigenvalue and \( \sigma \) are functionally dependent, so that the primes represent differentiations with respect to \( \sigma \). Furthermore, \( K \) is a constant whose exact expression in terms of SKR data will not concern us, while \( m = \dim(M)/2 \). We further have the following relations between \( \phi \), \( \Delta \sigma \) and \( Q := g(\nabla \sigma, \nabla \sigma): \)

\[
\Delta \sigma = 2m\phi + 2(\tau - c)\phi', \quad Q = 2(\tau - c)\phi.
\]

In analyzing equations such as (5.6) we will repeatedly use in §7 the following elementary lemma, taken from [Mas1].

**Lemma 5.2.** For a system

\[
\begin{align*}
A\phi'' + B\phi' + C\phi &= D \\
\phi' + p\phi &= q
\end{align*}
\]

with rational coefficients, either \( A(p^2 - p') - Bp + C = 0 \) holds identically, or else the solution is given by \( \phi = (D - A(q' - pq) - Bq)/(A(p^2 - p') - Bp + C) \).

Metrics with a special Kähler-Ricci potential have been completely classified [DeM1, DeM2]. One result that will be used below is that for an irreducible SKR metric, the function \( \phi \) is nowhere zero on the open dense set where \( d\sigma \neq 0 \).

### 6. Functional Dependence

Recall Equation (3.6.i):

\[
\text{Ric} + \nabla d\theta + d\mu \otimes d\psi = \gamma g, \quad \gamma = \lambda e^{-2\mu} - \Delta \mu + g(\nabla \theta, \nabla \mu),
\]

with \( \mu = \log \tau, \theta = f + (n-2) \log \tau \) and \( \psi = 2\theta - (n-2)\mu \). This was one of the forms of Equation (3.2) characterizing a metric \( g \) conformal to a gradient Ricci soliton. If \( g \) is also Kähler on a manifold \((M, J)\) of real dimension at least four, constancy of \( \theta \) implies that \( g \) is in fact Kähler-Einstein. This follows since, in this case, the above relation defining \( \psi \) shows that the term \( d\mu \otimes d\psi \) is just a constant multiple of \( d\mu \otimes d\mu \), and the latter vanishes as it is the only term in (6.1) that is not \( J \)-invariant.

Note that \( f \) cannot be constant on a nonempty open subset of \( M \) without being constant everywhere in \( M \), by a real-analyticity argument stemming from [Ive2]. Hence the same holds for \( \theta \), because we see from the previous paragraph that constancy of \( \theta \) on a nonempty open set implies the same for \( f \).

**Proposition 6.1.** Assume \( g \) is Kähler and conformal to a gradient Ricci soliton in dimension \( n \geq 4 \) with \( \theta \) nonconstant. If

\[
df \wedge d\tau = 0
\]
(equivalently, \( d\mu \wedge d\theta = 0 \)) then \( g \) satisfies on an open dense set a Ricci-Hessian equation of the form

\[
\alpha \nabla d\sigma + \text{Ric} = \gamma g,
\]

for appropriate functionally dependent functions \( \alpha, \sigma \).

In fact, in the set where \( d\theta \neq 0 \), choose any function \( t \) of \( \theta \) with \( dt \neq 0 \), so that \( \theta \) and \( \mu \) become functions of \( t \), on some interval of the variable \( t \). For the moment, \( t \) is not further specified. Denoting \( \dot{t} = \frac{dt}{d\theta} \), we have

\[
\nabla d\theta + d\mu \odot d\psi = \dot{\theta} \nabla dt + [\ddot{\theta} + 2\dot{\mu} \dot{\theta} - (n - 2)\dot{\mu}^2] dt \odot dt.
\]

Next, we choose a function \( \sigma \) of \( t \) such that \( \dot{\sigma} > 0 \) and

\[
\dot{\sigma}/\dot{\theta} = [\ddot{\theta} + 2\dot{\mu} \dot{\theta} - (n - 2)\dot{\mu}^2]/\dot{\theta}
\]
on the open dense set where \( \dot{\theta} \neq 0 \). The right hand side of this equation is given, so that this stipulation amounts to the requirement that an (easily solvable) ODE holds for \( \sigma \), with an essentially unique solution.

We now fix \( t = \sigma \). For this choice, \( (6.4) \) becomes

\[
\ddot{\theta} + 2\dot{\mu} \dot{\theta} - (n - 2)\dot{\mu}^2 = 0,
\]

which holds on the image under \( \sigma \) of an open dense set, namely the intersection of the noncritical set of \( \sigma \), with points where \( \dot{\theta} \neq 0 \). It follows from \( (6.5) \) and \( (6.3) \) that

\[
\nabla d\theta + d\mu \odot d\psi = \dot{\theta} \nabla ds, \text{ with } \alpha = \dot{\theta}.
\]

This translates the first of Equations \( (6.1) \) into a Ricci-Hessian equation.

We now record some relations that will be used in the next theorem, with assumptions as in Proposition \( 6.1 \). Let \( Q = g(\nabla \sigma, \nabla \sigma) \), \( Y = \Delta \sigma \) and \( s \) the scalar curvature of \( g \). First, from \( (6.1) \),

\[
\gamma = \lambda e^{-2\mu} - \dot{\mu} Y + (\alpha \dot{\mu} - \ddot{\mu}) Q
\]
as \( \Delta \mu = \ddot{\mu} Y + \dot{\mu} Q \) and \( g(\nabla \theta, \nabla \mu) = \alpha \dot{\mu} Q \). Next, we have

\[
(6.7)
\]

\begin{itemize}
  \item[i)] \( \alpha Y + s = v \gamma \),
  \item[ii)] \( \alpha dY + Y \dot{\alpha} d\sigma + ds = nd \gamma \),
  \item[iii)] \( \alpha dY + \ddot{\alpha} dQ + ds = 2 d \gamma \),
  \item[iv)] \( \alpha dQ - dY = 2 \gamma d \sigma \).
\end{itemize}

These equations are obtained in succession by taking the \( g \)-trace of \( (6.2) \); forming the \( d \)-image of \( (6.7) \); finally, applying twice the divergence operator \( 2\delta \) and, separately, interior multiplication by \( \nabla \sigma \), i.e. \( 2\nabla^2 \sigma \), to \( (6.2) \) and using Proposition \( 2.3 \) and the Bianchi relation \( 2\delta \text{Ric} = ds \).

Further relations are obtained by subtracting \( (6.7) \) from \( (6.7) \), then applying \( \ldots \wedge d\sigma \) to \( (6.8) \), \( d \) to \( (6.7) \), and \( d \) followed by \( \ldots \wedge d\sigma \) to \( (6.6) \), which yield

\[
(6.8)
\]

\begin{itemize}
  \item[a)] \( Y \dot{\alpha} d\sigma - \ddot{\alpha} dQ = (n - 2) d \gamma \),
  \item[b)] \( \dot{\alpha} d\sigma \wedge dQ = (n - 2) d \gamma \wedge d\sigma \),
  \item[c)] \( \dot{\alpha} d\sigma \wedge dQ = 2 d \gamma \wedge d\sigma \),
  \item[d)] \( d \gamma \wedge d\sigma = (\alpha \dot{\mu} - \ddot{\mu}) dQ \wedge d\sigma - \dot{\mu} dY \wedge d\sigma \).
\end{itemize}

We can now state the following partial classification theorem.
Theorem 6.2. Let $g$ be a Kähler metric conformal to a gradient Ricci soliton $\hat{g}$ on a manifold $M$ of dimension $n \geq 4$, so that Equations (3.2) and (6.1) hold. If $df \wedge d\tau = 0$ (equivalently, $d\mu \wedge d\theta = 0$), then one of the following must occur:

$$
\begin{align*}
(i) & \quad \text{ } g \text{ is a Kähler-Ricci soliton,} \\
(ii) & \quad \text{ } g \text{ satisfies a Ricci-Hessian equation, and if it is standard, } g \text{ is an SKR metric,} \\
(iii) & \quad n = 4 \text{ and } \hat{g} \text{ is an Einstein metric,} \\
(iv) & \quad n = 4 \text{ and } \hat{g} \text{ is a non-Einstein steady gradient Ricci soliton (} \lambda = 0). \\
\end{align*}
$$

The Ricci-Hessian equation in (ii) holds on an open dense set.

Note that the less expected possibility here is (iv). However, the theorem shows it cannot occur when $M$ is compact, as it is well-known that compact manifolds do not admit non-Einstein steady gradient Ricci solitons (see [Ive1]).

Proof. If $\theta$ is constant, we have seen $g$ is Kähler-Einstein, a special case of (i). Assume from now on that $\theta$ is nonconstant. Then by Proposition 6.1, $g$ satisfies the Ricci-Hessian equation (6.2) on an open dense set.

When $\alpha$ is constant, so is $\gamma$, by (6.8.a) and thus (6.2) gives (i). Next, we assume in the rest of this proof that $\alpha$ is nonconstant.

If $n > 4$ (or, $dQ \wedge d\sigma = 0$ everywhere), then $d\gamma \wedge d\sigma = 0$, as verified by subtracting (6.8.c) from (6.8.b) (or, using (6.8.c)). If the Ricci-Hessian equation is standard, taking into consideration that $d\alpha \wedge d\sigma = 0$ because $\alpha = \dot{\theta}$, Proposition 5.1 implies (ii).

So assume $n = 4$ and $dQ \wedge d\sigma \neq 0$ somewhere in $M$ (and, consequently, almost everywhere, by an argument involving real-analyticity, valid in dimension four). By (6.7.iv), (6.8.c) and (6.8.d), $(\dot{\alpha} + 2\alpha \dot{\mu} - 2\ddot{\mu})dQ - 2\dot{\mu}dY$ and $2\dot{\mu}(dY - \alpha dQ)$ are both functional multiples of $d\sigma$. Adding these two relations, we obtain $(\dot{\alpha} - 2\ddot{\mu})dQ \wedge d\sigma = 0$, so that (6.5) with $n = 4$ gives $\dot{\alpha} = 2\ddot{\mu}$ and

$$
(6.10) \quad a) \ \alpha = 2(\ddot{\mu} + p), \quad b) \ 2\dot{\alpha} + \alpha^2 = 4p^2, \quad c) \ 4(\alpha \dot{\mu} - \ddot{\mu}) = (3\alpha + 2p)(\alpha - 2p),
$$

for a constant $p$, where a) is obtained by integration, b) using a) and (6.5) with $n = 4$, while c) follows from a) and b) by algebraic manipulations that use again $\dot{\alpha} = 2\ddot{\mu}$.

Also, as $\dot{\theta} = \alpha$,

$$
(6.11) \quad i) \ \dot{f} = 2p, \quad ii) \ p [e^{2\mu}(Y - \alpha Q) + 2\lambda \sigma] \text{ is a constant.}
$$

In fact, differentiating the relation $\theta = f + (n - 2)\mu$ with $n = 4$ and (6.10a) give (6.11i). Thus, $f$ equals $2p \sigma$ plus a constant. Hence $\Delta f = 2p Y$, and (6.11ii) follows from (3.6ii) and (6.10a). If $p = 0$ then $f$ is constant, and this, by the soliton equation (first equation in (3.1)), implies (iii).

Suppose, finally, that $p \neq 0$ while $n = 4$ and $dQ \wedge d\sigma \neq 0$ somewhere. As a consequence of (6.8a) and (6.10b)

$$
(6.12) \quad 4d\gamma = (4p^2 - \alpha^2)(Yd\sigma - dQ),
$$

On the other hand, (6.9), (6.10a) and (6.10c) give

$$
(6.13) \quad 4\gamma = 4\lambda e^{-2\mu} + (\alpha - 2p)[(\alpha + 2p)Q + 2(\alpha Q - Y)].
$$
Since \( p \neq 0 \), (6.11.ii) yields \( \alpha Q - Y = e^{-2\mu}(2\lambda\sigma - b) \) for some constant \( b \), so that (6.13) and (6.12) become

\[
\begin{align*}
\text{(6.14)} \quad & \quad \text{a) } 4\gamma = e^{-2\mu}[4\lambda + (2\lambda\sigma - b)(2\alpha - 4p)] + (\alpha^2 - 4p^2)Q. \\
& \text{b) } 4d\gamma = (4p^2 - \alpha^2)[\alpha Q d\sigma - e^{-2\mu}(2\lambda\sigma - b) d\sigma - dQ].
\end{align*}
\]

Thus \( (4p^2 - \alpha^2)[\alpha Q d\sigma - e^{-2\mu}(2\lambda\sigma - b) d\sigma] \) equals the sum of \( Q d(\alpha^2 - 4p^2) \) and \( d[e^{-2\mu}(4\lambda + (2\lambda\sigma - b)(2\alpha - 4p))] \), since both expressions coincide with \( 4d\gamma + (4p^2 - \alpha^2) dQ \), which for the former is clear from (6.14.b), and for the latter follows if one applies \( d \) to (6.14.a). This equation yields \( 4e^{-2\mu}(2\lambda\sigma - b)(2p - \alpha)\alpha = 0 \), as seen by evaluating these expressions via the first two parts of (6.10), and subtracting the former expression from the latter. As we are assuming \( \alpha \) is not constant, it follows necessarily that \( \lambda \) (and \( b \)) must be zero. This gives (iv), completing the proof. \( \square \)

7. Quasi-solitons

Many of the original examples of gradient Ricci solitons arise as warped products over a one dimensional base (cf. [CCCG]). We consider here the case of an arbitrary base.

Let \( \bar{g} \) be a warped product (gradient Ricci) soliton metric on a manifold \( M = B \times F \), so that

\[
\bar{g} = g_B + \ell^2 g_F := g + \ell^2 g_F, \quad \text{Ric} + \nabla df = \lambda g,
\]

where \( \ell \) is the (pullback of) a function on the base \( B \) and \( \lambda \) is constant. When \( \bar{g} \) is Einstein, the base metric \( g = g_B \) is sometimes called quasi-Einstein. Similarly, in our case we will call \( g_B \) a quasi-soliton metric and drop the subscript \( B \) in the notation for \( g_B \)-dependent quantities.

**Proposition 7.1.** With notations as above, the soliton equation for \( \bar{g} \) (see (7.1)) is equivalent to the system

\[
\begin{align*}
\text{(7.2)} \quad & \quad \text{Ric} - \frac{k}{\ell} \nabla d\ell + \nabla df = \lambda g, \quad k = \dim(F), \\
& \text{Ric}_F = \nu g_F, \quad \text{where } \nu \text{ is given by} \\
& \nu + \ell d\nabla f \ell - \ell^2 \ell \# = \lambda \ell^2, \quad \text{for } \ell \# = \ell^{-1} \Delta \ell + (k - 1)\ell^{-2} |\nabla \ell|^2.
\end{align*}
\]

In particular the fiber metric is Einstein if \( \dim(F) > 2 \), and \( f \) turns out to be a function with vanishing fiber covariant derivative (see below), so that we regard it as a function on \( B \). Unlike the quasi-Einstein case, the third scalar equation in (7.2) does not appear to be a consequence of the first.

**Proof.** To derive the equations, we need the well-known Ricci curvature formulas for warped products (see [ONe]), and additionally, similar equations for the Hessian of \( f \). For the latter we use the covariant derivative formulas for warped products, together with the known fact that for a \( C^1 \) function defined on the base, the gradient of its pull-back equals the pull-back of its base gradient.
Let, $x$, $y$ denote lifts of vector fields on $B$, and $u$, $v$ lifts of vector fields on $F$. Then we have
\begin{align}
\nabla_x y \text{ is the lift of } \nabla_x y \text{ on } B, \\
\nabla_x v = \nabla_v x = d_x \log(\ell) v, \\
[\nabla_v w]^F = \text{the lift of } \nabla_v^F w \text{ on } F, \\
[\nabla_v w]_B = -\bar{g}(v, w) \nabla \log(\ell).
\end{align}

Hence,
\begin{equation}
(7.4)
\nabla df(x, y) = \bar{g}(\nabla_x \nabla f, y) = \bar{g}(\nabla_x (\nabla f)_B, y) + \bar{g}(\nabla_x (\nabla f)^F, y) = \\
g(\nabla_x (\nabla f)_B, y) + \bar{g}(d_x \log(\ell) (\nabla f)^F, y) = g(\nabla_x (\nabla f)_B, y), \\
\nabla df(x, v) = \bar{g}(\nabla_x \nabla f, v) = \bar{g}(\nabla_x (\nabla f)_B, v) + \bar{g}(\nabla_x (\nabla f)^F, v) = \\
d_x \log(\ell) \bar{g}(\nabla f)^F, v) = \ell d_x g_F((\nabla f)^F, v), \\
\nabla df(v, w) = \bar{g}(\nabla_v \nabla f, w) = \bar{g}(\nabla_v (\nabla f)_B, w) + \bar{g}(\nabla_v (\nabla f)^F, w) = \\
d_v (\nabla f)_B(\log(\ell)) \bar{g}(v, w) + \bar{g}(\nabla_v (\nabla f)^F, w) - \bar{g}(v, (\nabla f)^F) \bar{g}(\nabla \log(\ell), w) = \\
\ell d_v (\nabla f)_B(\log(\ell)) g_F(v, w) + \ell^2 g_F(\nabla_v (\nabla f)^F, w).
\end{equation}

We combine these with the Ricci curvature formulas
\begin{align}
Ric(x, y) &= Ric_B(x, y) - (k/\ell) \nabla d\ell(x, y), \\
Ric(x, v) &= 0, \\
Ric(v, w) &= Ric_F(v, w) - \ell^g(v, w).
\end{align}

We now notice that the soliton equation applied to $x$ and $v$ implies that $(\nabla f)^F = 0$ so that $f$ can be regarded as the pull-back of a function on $B$. This readily gives Equations (7.2).

In analogy with the previous section, we will be considering quasi-soliton metrics for which $f$ and $\ell$ are functionally dependent, that is
\begin{equation}
df \wedge d\ell = 0.
\end{equation}

We call such metrics special quasi-soliton metrics.

It is known that Kähler quasi-Einstein metrics do not exist on a compact manifold, and in general must be certain Riemannian product metrics \cite{CaSW}. Similarly we show

**Theorem 7.2.** Let $g$ be a Kähler special quasi-Einstein metric on a manifold $M$ of dimension at least four. Then $g$ satisfies a Ricci-Hessian equation on an open set. If this equation is standard, then $g$ is a Riemannian product there. If the dimension is greater than four, then one of the factors in this product is a Kähler-Einstein manifold of codimension two.

**Proof.** As the quasi-soliton metric is special, we have $\nabla df = f' \nabla d\ell + f'' d\ell \otimes d\ell$, with the prime denotes differentiation with respect to $\ell$. The first of Equations (7.2) then becomes
\begin{equation}
(7.6) \quad Ric + (f' - k/\ell) \nabla d\ell + f'' d\ell \otimes d\ell = \lambda g.
\end{equation}
In analogy with Proposition 6.1, we introduce a function $\sigma$ with $d\ell \wedge d\sigma = 0$ and rewrite the special quasi-Einstein Equation (7.6) as
\[(7.7) \quad \text{Ric} + \tilde{\alpha}\ell' \nabla d\sigma + (\tilde{\alpha}\ell'' + f''\ell^2)d\sigma \otimes d\sigma = \lambda g,\]
for $\tilde{\alpha} = f'(\ell) - k/\ell$, with the convention that primes on $\ell$ represent differentiations with respect to $\sigma$, while primes on $f$ still represent differentiations with respect to $\ell$. The restriction on the open set where an ODE analogous to (6.4) holds is $\tilde{\alpha} := \tilde{\alpha}\ell' \neq 0$ (corresponding to $\dot{\theta} \neq 0$ in Proposition 6.1). On that set, Equation (7.7) becomes a Ricci-Hessian equation of the form
\[\text{Ric} + \alpha \nabla d\sigma = \lambda g, \quad \alpha = \tilde{\alpha}\ell',\]
provided we choose $\sigma$ so that the differential equation
\[(7.8) \quad \tilde{\alpha}\ell'' + f''\ell^2 = 0\]
also holds.

Assuming the Ricci-Hessian equation is standard, Proposition 5.1 now shows that $g$ is an SKR metric on the open set described above. If $g$ is irreducible, the theory of SKR metrics gives the two equations (5.6), which now take the form
\[(7.9) \quad (\sigma - c)^2\phi'' + (\sigma - c)[m - (\sigma - c)\alpha]\phi' - m\phi = K,\]
\[\quad - (\sigma - c)\phi'' + [\alpha(\sigma - c) - (m + 1)]\phi' + \alpha\phi = \lambda,\]
where $\phi$ is defined pointwise as the eigenvalue of the Hessian of $\sigma$, mentioned in §5.

Adding the first of Equations (7.9) to $(\sigma - c)$ times the second replaces the latter with the first order equation
\[\quad -(\sigma - c)\phi' + [(\sigma - c)\alpha - m]\phi = K + (\sigma - c)\lambda.\]
Denote the ratio of the second coefficient of this equation to the first by $p$, the ratio of the third to the first by $q$ and the coefficients of the first of Equations (7.9) by $A, B, C, D$. We wish to invoke Lemma 5.2. An easy computation gives the two relations
\[(7.10) \quad A(p^2 - p') - Bp + C = \alpha'(\sigma - c)^2,\]
\[\quad D - A(q' - pq) - Bq = 0.\]
According to this lemma, the solution $\phi$ is the ratio of the second term to the first, if the latter is nonzero. However, as mentioned at the end of §5, the function $\phi$ is nowhere zero on the set where $d\sigma \neq 0$ when $g$ is irreducible. Hence the only possibility is that the first term in (7.10) vanishes identically, i.e. $\alpha$ is constant, so that $g$ is additionally a gradient Ricci soliton. Writing this condition explicitly we get, with primes now denoting solely differentiations with respect to $\sigma$,
\[\quad (f \circ \ell)' - k\ell'/\ell = b,\]
where $b$ is constant. But Equation (7.8) can also be written as
\[\quad (f \circ \ell)'' - k\ell''/\ell = 0.\]
Differentiating the first of these two equations and combining it with the second shows that $\ell$ is constant, hence $g$ is Einstein. But this means $\alpha \equiv 0$, contradicting that the Ricci-Hessian equation for $g$ is standard. Hence $g$ must be reducible. The structure of the Riemannian product constituting $g$ follows from SKR theory. □
Next we consider the problem of whether quasi-soliton metrics can be conformally Kähler. This is certainly possible for quasi-Einstein metrics (see [Mas2, BHJM]). We have the following result, analogous in form and in proof to the previous one, though it requires more assumptions and is computationally more difficult.

**Theorem 7.3.** Let $M$ be a manifold of dimension $n = 2m > 4$ and $g$ an irreducible Kähler metric on $M$ conformal to a special quasi-soliton $\hat{g} = g/\tau^2$ having warping function $\ell$, potential $f$ and appropriate constants $k$ and $\lambda$. Assume $\tau$ is a Killing potential for $g$ and $d\ell \wedge d\tau = 0$. Then $g$ satisfies a Ricci-Hessian equation. If the latter is standard, then $\hat{g}$ is quasi-Einstein.

**Proof.** Being a special quasi-soliton, $\hat{g}$ satisfies Equation (7.6), i.e.

\[
\hat{\text{Ric}} + \mu \hat{\nabla} d\ell + \chi d\ell \otimes d\ell = \lambda \hat{g},
\]

for $\mu = f'(-\ell) - k/\ell$ and $\chi = f''(-\ell)$.

Using (2.3) and the first equation in (2.2), we see that $g$ satisfies

\[
\text{Ric} + (n - 2)\tau^{-1}\nabla d\tau + (\tau^{-1}\Delta \tau - (n - 1)\tau^{-2}Q)g + \\
\mu(\nabla d\ell + 2\tau^{-1}d\tau \otimes d\ell - \tau^{-1}g(\nabla \tau, \nabla \ell)g) + \chi d\ell \otimes d\ell = \lambda \tau^{-2}g,
\]

with $Q = g(\nabla \tau, \nabla \tau)$. Since $d\ell \wedge d\tau = 0$, writing $d\ell = \ell'(\tau) d\tau$ and rearranging terms, we rewrite this equation as

\[
\text{Ric} + \alpha \nabla d\tau + (\mu(\ell'' + 2\tau^{-1}\ell') + (\ell')^2 \chi) d\tau \otimes d\tau \\
= (\lambda \tau^{-2} - \tau^{-1}\Delta \tau + (\alpha + \tau^{-1})\tau^{-1}Q)g, \quad \text{for } \alpha = (n - 2)\tau^{-1} + \mu \ell'.
\]

As $g$ is Kähler and $\tau$ is a Killing potential, the term with $d\tau \otimes d\tau$ is the only one which is not $J$-invariant. Hence its coefficient must vanish:

\[
\mu(\ell'' + 2\tau^{-1}\ell') + (\ell')^2 \chi = 0.
\]

As a result, Equation (7.13) is Ricci-Hessian:

\[
\text{Ric} + \alpha \nabla d\tau = \gamma g, \quad \text{where } \gamma = \lambda \tau^{-2} - \tau^{-1}\Delta \tau + (\alpha + \tau^{-1})\tau^{-1}Q.
\]

Since clearly $d\alpha \wedge d\tau = 0$, and $n > 4$, as mentioned in §5, we also have $d\gamma \wedge d\tau = 0$. Under the assumption that the Ricci-Hessian equation is standard, we conclude from Proposition 5.1 that $(g, \tau)$ is an SKR metric with $\tau$ the special Kähler-Ricci potential. As in the previous theorem, irreducibility of $g$ again implies that two ODE’s hold for the horizontal Hessian eigenvalue function $\phi$. They are

\[
(\tau - c)^2 \phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K \\
- (\tau - c)\phi'' + (\alpha(\tau - c) - (m + 1))\phi' + \alpha\phi = \gamma = \\
\lambda \tau^{-2} - \tau^{-1}(2m\phi + 2(\tau - c)\phi') + (\alpha + \tau^{-1})\tau^{-1}2(\tau - c)\phi
\]

where $K, c$ are constants, and we have used formulas (5.7) giving $\Delta \tau$ and $Q$ in terms of $\phi$. 

\[
(\tau - c)^2 \phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K \\
- (\tau - c)\phi'' + (\alpha(\tau - c) - (m + 1))\phi' + \alpha\phi = \gamma = \\
\lambda \tau^{-2} - \tau^{-1}(2m\phi + 2(\tau - c)\phi') + (\alpha + \tau^{-1})\tau^{-1}2(\tau - c)\phi
\]

where $K, c$ are constants, and we have used formulas (5.7) giving $\Delta \tau$ and $Q$ in terms of $\phi$. 

\[
(\tau - c)^2 \phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K \\
- (\tau - c)\phi'' + (\alpha(\tau - c) - (m + 1))\phi' + \alpha\phi = \gamma = \\
\lambda \tau^{-2} - \tau^{-1}(2m\phi + 2(\tau - c)\phi') + (\alpha + \tau^{-1})\tau^{-1}2(\tau - c)\phi
\]
Simplifying the second equation, we then replace it by a first order equation as in
the previous theorem, to obtain the equivalent system
\begin{equation}
(\tau - c)^2 \phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K,
\end{equation}
\begin{equation}
\frac{(\tau - c)(\tau - 2c)}{\tau}\phi' - \left(\frac{(\tau - c)(\tau - 2c)}{\tau}2(\tau - c)^2 - m(\tau - 2c)\right)\phi
\end{equation}
\begin{equation}
= \frac{K\tau^2 + \lambda\tau - \lambda c}{\tau^2}.
\end{equation}

Naming the coefficients $A$, $B$, $C$, $D$, $p$, $q$ as before, we now apply Lemma 5.2 to
the system (7.17). This time the computation of the two quantities used in the
lemma is quite laborious, though still elementary. A symbolic computational program
simplifies the result to the following.
\begin{equation}
A(p^2 - p') - Bp + C = \frac{(\tau - c)^2((\tau - 2c)\tau\alpha' + 2(\tau - c)\alpha + 2 - 2m)}{\tau(\tau - 2c)},
\end{equation}
\begin{equation}
D - A(q' - pq) - Bq = 0.
\end{equation}

By the lemma and the fact $\phi$ is nowhere zero, solutions are only possible if the first
expression vanishes identically, so that $\alpha$ solves
\begin{equation}
(\tau - 2c)\tau\alpha' + 2(\tau - c)\alpha + 2 - 2m = 0.
\end{equation}

The solutions of this take the form
\begin{equation}
\alpha = \frac{n - 2}{\tau} + \frac{C}{\tau(\tau - 2c)},
\end{equation}
where $C$ is a constant. As (7.13) and the second of Equations (7.16) imply that the
form of $\alpha$ determines that of $\gamma$, we have the following outcome. If $c = 0$, the metric
$g$ is conformal to a gradient Ricci soliton [Mas1, Proposition 2.4], while if $c \neq 0$ then
$g$ is conformal to a quasi-Einstein metric [Mas2]. (The case $C = 0$ is a special case
of both these types, where $g$ is conformally Einstein [DeM1].)

To rule out the case that $\hat{g}$ is a nontrivial gradient Ricci soliton, we note first that
the expression defining $\alpha$ in (7.13), when compared to that in (7), results in
\begin{equation}
(f \circ \ell)' - k\ell'/\ell = \frac{C}{\tau(\tau - 2c)}.
\end{equation}

Additionally, Equation (7.14) can also be written as
\begin{equation}
(f \circ \ell)'' - k\ell''/\ell + 2((f \circ \ell)' - k\ell'/\ell)\tau^{-1} = 0.
\end{equation}

Substituting the first of these equations in the last term of the second, and combining
the result with the derivative of the first equation gives, after eliminating
$(f \circ \ell)'' - k\ell''/\ell$ and rearranging terms
\begin{equation}
k\ell^2/\ell^2 = \frac{2C}{\tau^2(\tau - 2c)} + \left(\frac{C}{\tau(\tau - 2c)}\right)' = -\frac{2cC}{\tau^2(\tau - 2c)^2}.
\end{equation}

Hence the Ricci soliton case $c = 0$ implies that $\ell$ is constant, so that comparing
the two expressions for $\alpha$ again yields $C = 0$, i.e. that $\hat{g}$ is Einstein, which is, of course,
a special case of the quasi-Einstein condition.

\footnote{See (2.3) in that paper, where the quasi-Einstein case is given by $\alpha = (n - 2)/\tau + a/(\tau(1 + k\tau))$, where $a$ is a constant and $k = -1/2c$. This corresponds to formula (7) with $C = -2ac$.}
8. Appendix: Killing vector fields of the form $w = \tau^2 \nabla f$

We consider here the classification problem for Killing fields of the form of $w = \tau^2 \nabla f$, a form that played an important role in §4. In the following $\tau$ and $f$ will denote smooth functions on a given manifold.

**Proposition 8.1.** On a compact manifold, a Killing field of the form $w = \tau^2 \nabla f$ must be trivial.

**Proof.** First, on a compact manifold $\nabla f$ has zeros, hence so does $w$. Let $p$ be a zero of $w = \tau^2 \nabla f$. Since $\nabla w = 2\tau d\tau \otimes \nabla f + \tau^2 \nabla df$, and at a zero either $\tau = 0$ or $\nabla f = 0$, we see that at a zero $\nabla w$ either equals either zero or $\tau^2 \nabla df$. But in the latter case $\nabla w$ is symmetric, yet it is also skew-symmetric as $w$ is a Killing field, hence $\nabla w$ must be zero in this case as well. However, a Killing field $w$ is uniquely determined by the values of $w$ and $\nabla w$ at one point. As those values are zero at $p$, we see that $w$ must be the zero vector field. □

Without compactness, we have the following classification for such vector fields.

**Theorem 8.2.** A Riemannian metric $g$ with a Killing vector field of the form $w = \tau^2 \nabla f$ is, near generic points, a warped product with a one dimensional fiber. If $g$ is also Kähler, it is, near such points, a Riemannian product of a Kähler metric with a surface metric admitting a nontrivial Killing vector field.

We note here that a surface with a nontrivial Killing vector field can be presented as a warped product with a one dimensional fiber and base.

**Proof.** First, the orthogonal complement $\mathcal{H}$ to $\text{span}(w)$ is generically $[\nabla f]^\perp$, which is obviously integrable. Next, $\mathcal{H}$ is totally geodesic. This follows immediately since $g(x, w) = \text{constant}$ for any geodesic $x(t)$ and Killing field $w$. Alternatively, it can also be shown directly. With $x, y$ denoting vector fields (taking values) in $\mathcal{H}$, we compute that $g(\nabla_x y, w) = -g(y, \nabla_x w) = -g((\nabla w)^*(y), x) = g(\nabla_y w, x) = -g(w, \nabla_y x)$, where in the penultimate step we used the Killing property $(\nabla w)^* = -\nabla w$. One concludes that the sum $\nabla_{xy} + \nabla_{yx}$ is in $\mathcal{H}$, and since the same holds for $\nabla_{xy} - \nabla_{yx}$ by integrability, we see that $\nabla_{xy}$ is in $\mathcal{H}$.

By a result originating in works of Hiepko [Hiep] along with Ponge and Reckziegel [PoRe] (see especially Theorem 3.1 in the survey of Zeghib [Zegh]) a metric is a warped product if and only if it admits two orthogonal foliations, one totally geodesic and the other spherical. In our case we have just shown the foliation orthogonal to $w$ is totally geodesic. The fibers tangent to $\text{span}(w)$, on the other hand, are certainly totally umbilic, as they are one dimensional. This is part of the definition of spherical. The other part is that the mean curvature vector is parallel with respect to the normal connection. We now check this.

Let $w' = w/|w|$ be a unit vector parallel to $w$, defined away from its zeros. The mean curvature vector to the fibers is then, by definition, $n = \nabla_{w'} w'$, which takes values in $\mathcal{H}$. The requirement that $\text{span}(w)$ be spherical amounts to showing that for any $x \in \mathcal{H}$, we have $g(\nabla_{w'} n, x) = 0$. The flow of $w$ certainly preserves itself (as
Suppose the manifold is given by $M = T \times B$, with $B$ the fiber (an interval). Since the base foliation corresponding to $B$ is totally geodesic, parallel transport along one of its leaves with respect to $g$ is the same as parallel transport with respect to the induced metric on this leaf, and therefore it preserves the tangent spaces to these leaves. It is well-known that it also preserves the normal spaces to the leaves; for completeness, we show explicitly that the unit vector field $w'$ perpendicular to the leaves is preserved. If $x$ and $y$ are, as usual, vector fields tangent to the leaves, then $g(w', y) = 0$, so $0 = d_x g(w', y) = g(\nabla_x w', y) + g(w', \nabla_x y) = g(\nabla_x w', y)$ because the leaves are totally geodesic, and similarly $0 = d_x g(w', w') = 2g(\nabla_x w', w')$. So $\nabla_x w'$, being orthogonal to a basis, is zero, i.e. $w'$ is parallel in directions tangent to the leaves.

As $g$ is Kähler, the complex structure $J$ commutes with any $\nabla_x$, so that $Jw'$ is also parallel in leaf directions. But $Jw'$ is itself tangent to leaves of the base foliation. Therefore, by the local de Rham Theorem, the induced metric on any leaf splits locally into a Riemannian product so that $B = N \times I$, where the one dimensional factor $I$ is tangent to $Jw'$, and $N$ is $J$-invariant, hence has holomorphic (and totally geodesic) leaves in $M$.

Armed with this information it remains to show that, near generic points, $g$ is a product of a Kähler metric on $N$ and a local metric of revolution on $I \times F$.

For this we turn to a computation that is based on the formulas (cf. [ONeil]) for the connection of the warped product metric $g = g_B + t^2 g_F$, where the function $l$ is a (lift of) a function on $B$. Let $t$ be a nontrivial vector field tangent to $F$ which is projectable onto $F$. Let $s = Jt$, a vector field tangent to $I$. Then standard formulas for warped products give

\begin{equation}
\nabla_t s = (\nabla_t)^B + (\nabla_t)^F = -|t|^2 \nabla (\log (l)) + ct,
\end{equation}

with $c$ some function, and the last term takes that form because the fiber is one dimensional. Next, as $s$ is tangent to $I$, there is some function $h$ on $M$ such that the vector field $hs$ is projectable onto $I$. Therefore, again by warped product formulas,

\begin{equation}
\nabla_t (hs) = hs (\log (l)) t.
\end{equation}

But $\nabla_t (hs) = (d_t h)s + h \nabla_t s = (d_t h)s + hJ\nabla_t t = (d_t h)s - h |t|^2 J \nabla (\log (l)) + hcs$, by (8.1). Equating this expression with the right hand side of (8.2) and taking components tangent to $N$ gives $h |t|^2 [J \nabla (\log (l))]^N = 0$, so that, away from the zeros of $h$ and $t$, $[J \nabla (\log (l))]^N = 0$. Now each tangent space $T_p N$ is $J$-invariant, so $J$ commutes with the projection to $N$. Hence $\nabla (\log (l))^N = 0$ and so $\nabla (\log (l))$ is parallel to $s$, which means that the warping function $l$ is constant on the leaves of $N$, and only changes along the fibers associated with $I$. Thus $g$ is a Riemannian product of the type claimed above. □
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