Bipartite theory of irredundant set

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Abstract

The bipartite version of irredundant set, edge-vertex irredundant set and vertex-edge irredundant set are introduced. Using the bipartite theory of graph, \( IR_{ve}(G) + \gamma(G) \leq |V| \) and \( \gamma_{ve}(G) + IR(G) \leq |V| \) are proved.

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1. Introduction

All graphs considered here are simple and undirected. [4,5] suggests that given any problem, say P, on an arbitrary graph \( G \), there is very likely a corresponding problem Q on a bipartite graph \( G' \), such that a solution for Q provides a solution for P. The bipartite theory of graphs was introduced in [4] and a parameter called \( X \)-domination number of a bipartite graph was defined. Let \( G = (X, Y, E) \) be a bipartite graph with \( |X| = p \) and \( |Y| = q \). Two vertices \( u \) and \( v \) in \( X \) are \( X \)-adjacent if they have a common adjacent vertex \( y \in Y \). Let \( y \in X \) and \( \Delta_Y = \max\{|N_Y(u) : y \in X\} \) where the \( X \)-neighbor set \( N_Y(u) \) is defined as \( N_Y(u) = \{v \in X : u \text{ and } v \text{ are } X \text{-adjacent}\} \).

A subset \( X \subseteq X \) is an \( X \)-dominating set [4] if every \( x \in X - D \) is \( X \)-adjacent to some vertex in \( D \). The minimum cardinality of a \( X \)-dominating set is called \( X \)-domination number and is denoted by \( \gamma_X(G) \).

We say a vertex \( x \in X \) hyper \( Y \)-dominates \( y \in Y \) if \( y \in N(x) \) or \( y \in N(N_Y(x)) \). A subset \( S \subseteq X \) is a hyper \( Y \)-dominating set [6] if every \( y \in Y \) is hyper \( Y \)-dominated by a vertex of \( S \). The minimum cardinality of a hyper \( Y \)-dominating set is called hyper \( Y \)-domination number and is denoted as \( \gamma_{hY}(G) \).

Given an arbitrary graph \( G = (V, E) \), a vertex \( u \in V(G) \) ve-dominates an edge \( vw \in E(G) \) if (a) \( u = v \) or \( u = w \) (\( u \) incident to \( vw \)) or (b) \( uv \) or \( uw \) is an edge in \( G \). A subset \( S \subseteq V(G) \) is a vertex-edge dominating set [3] if for all edges \( e \in E(G) \), there exists a vertex \( v \in S \) such that \( v \) dominates \( e \). The minimum cardinality of a ve-dominating set of \( G \) is called the vertex-edge domination number and is denoted as \( \gamma_{ve}(G) \).

An edge \( e = uv \in E(G) \) ev-dominates a vertex \( w \in V(G) \) if (i) \( u = w \) or \( v = w \) (\( w \) is incident to \( e \)) or (ii) \( uw \) or \( vw \) is an edge in \( G \). \( (w \) is adjacent to \( u \) or \( v \)). A set \( S \subseteq E(G) \) is an edge-vertex dominating set [3] if for all vertices \( v \in V(G) \), there exists an edge \( e \in S \) such that \( e \) dominates \( v \). The minimum cardinality of a ev-dominating set of \( G \) is called the edge-vertex domination number and is denoted as \( \gamma_{ev}(G) \).

Observation: 1. Let \( G \) be an arbitrary graph. A vertex \( u \in V(G) \) ve-dominates the edge \( e \in E(G) \) if and only if the edge \( e \) ev-dominates the vertex \( u \in V(G) \).
2. Bipartite Construction

The bipartite graph $VE(G)$ constructed from an arbitrary graph $G = (V, E)$ is defined as in [4]. $VE(G) = (V, E, F)$ is defined by the edges $F = \{(u, e) : e = (u, v) \in E\}$. $VE(G) \cong S(G)$, where $S(G)$ denotes the subdivision graph of $G$.

The bipartite graph $EV(G)$ [4] constructed from an arbitrary graph $G = (V, E)$ is defined as $EV(G) = (E, V, J)$ where $J = \{(e, u)(e, v) : e = (u, v) \in E\}$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set [2] if every $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. The minimum cardinality of a dominating set of a graph $G$ is called the domination number and is denoted by $\gamma(G)$.

A set $F \subseteq E(G)$ of edges in a graph $G = (V, E)$ is called an edge dominating set [2] if every $e \in E(G)$ is either an element of $F$ or is adjacent to an element of $E - F$. The minimum cardinality of an edge dominating set of a graph $G$ is called the edge domination number and is denoted by $\gamma_1(G)$.

**Theorem 2.1** [4] For any graph $G$,
(a) $\gamma_X(VE(G)) = \gamma(G)$
(b) $\gamma_X(EV(G)) = \gamma_1(G)$.

**Theorem 2.2** [6] For any graph $G$,
(a) $\gamma_{hY}(VE(G)) = \gamma_{ve}(G)$
(b) $\gamma_{hY}(EV(G)) = \gamma_{ev}(G)$.

3. Irredundant sets

3.1. Vertex-edge irredundant set

A vertex $v \in S \subseteq V(G)$ has a private edge $e = uw \in E(G)$ (with respect to a set $S$), if: 1. $v$ is incident to $e$ or $v$ is adjacent to either $u$ or $w$, and 2. for every vertices $x \in S - \{v\}$, $x$ is not incident to $e$ and $x$ is not adjacent to either $u$ or $w$.

A set $S$ is a vertex-edge irredundant set [3] (simply a ve-irredundant set) if every vertex $v \in S$ has a private edge. The vertex-edge irredundance of a graph $G$ is the cardinality of a maximal ve-irredundant set with minimum number of vertices and is denoted by $ir_{ve}(G)$. The upper vertex-edge irredundance number of a graph $G$ is the cardinality of a maximum
A ve-irredundant set of vertices and is denoted by $IR_{ve}(G)$.

**Theorem: 3.1.1** [3] Every minimal ve-dominating set is a maximal ve-irredundant set.

### 3.2. Edge-vertex irredundant set

An edge $e = uv \in F \subseteq E(G)$ has a private vertex $w \in V(G)$ (with respect to a set $F$), if: 1. $e$ is incident to $w$, and 2. for all edges $f = xy \in F - \{e\}$, $f$ is not incident to $w$ and neither $x$ nor $y$ is adjacent to $w$.

A set $F$ is an edge-vertex irredundant set [3] (simply a ev-irredundant set) if every edge $e \in F$ has a private vertex. The edge-vertex irredundance of a graph $G$ is the cardinality of a maximal ev-irredundant set with minimum number of vertices and is denoted by $ir_{ev}(G)$. The upper edge-vertex irredundance number of a graph $G$ is the cardinality of a maximum ev-irredundant set of vertices and is denoted by $IR_{ev}(G)$.

**Theorem 3.2.1:**[3] Every minimal ev-dominating set of $G$ is a maximal ev-irredundant set.

### 3.3. Hyper $Y$− Irredundant set

Let $G = (X,Y,E)$ be a bipartite graph. Let $S \subseteq X$. A vertex $x \in S$ has a private hyper $Y$−neighbor $y \in Y$ if 1. $x$ is adjacent to $y$ or $y \in N(N_Y(x))$ and 2. for all vertices $x_1 \in S - \{x\}$, $x_1$ is not adjacent to $y$ and $y \notin N(N_Y(x_1))$.

A set $S$ is hyper $Y$−irredundant set if every $v \in S$ has a private hyper $Y$−neighbor. The hyper $Y$−irredundance number of a graph $G$ is the minimum cardinality of a maximal hyper $Y$−irredundant set of vertices and is denoted by $ir_{HY}(G)$. The upper hyper $Y$−irredundance number of a graph $G$ is the maximum cardinality of a maximal hyper $Y$−irredundant set of vertices and is denoted by $IR_{HY}(G)$.

**Theorem: 3.3.1** A hyper $Y$−dominating set $S$ is a minimal hyper $Y$−dominating set if and only if it is hyper $Y$−dominating set and hyper $Y$−irredundant set.
Proof: Let \( S \) be a hyper \( Y \)-dominating set. Then \( S \) is a minimal hyper \( Y \)-dominating set if and only if \( \forall u \in S, \exists y \in Y \) which is not hyper \( Y \)-dominated by \( S - \{u\} \). Equivalently, \( S \) is a minimal hyper \( Y \)-dominating set if and only if \( \forall u \in S, u \) has at least one private hyper \( Y \)-neighbour. Thus \( S \) is minimal hyper \( Y \)-dominating set if and only if it is hyper \( Y \)-irredundant set.

Conversely, let \( S \) be both hyper \( Y \)-dominating and hyper \( Y \)-irredundant.

Claim: \( S \) is a minimal hyper \( Y \)-dominating set.

If \( S \) is not minimal hyper \( Y \)-dominating set, there exists \( v \in S \) for which \( S - \{v\} \) is hyper \( Y \)-dominating. Since \( S \) is hyper \( Y \)-irredundant, \( v \) has a private hyper \( Y \)-neighbor of \( u \). By definition \( u \) is not hyper \( Y \)-adjacent to any vertex in \( S - \{v\} \). That is, \( S - \{v\} \) is not hyper \( Y \)-dominating set, a contradiction. Hence, \( S \) is a minimal hyper \( Y \)-dominating set.

Theorem: 3.3.2 Every minimal hyper \( Y \)-dominating set is a maximal hyper \( Y \)-irredundant set.

Proof: Every minimal hyper \( Y \)-dominating set \( S \) is hyper \( Y \)-irredundant set.

Claim: \( S \) is a maximal hyper \( Y \)-irredundant set.

Suppose \( S \) is not maximal hyper \( Y \)-irredundant set. Then there exists a vertex \( u \in X - S \) for which \( S \cup \{u\} \) is hyper \( Y \)-irredundant. There exists at least one vertex \( y \in Y \) which is a private hyper \( Y \)-neighbor of \( u \) with respect to \( S \cup \{u\} \). That is no vertex in \( S \) is hyper \( Y \)-adjacent to \( y \). Hence, \( S \) is not a hyper \( Y \)-dominating set, a contradiction. Hence, \( S \) is a maximal hyper \( Y \)-irredundant set.

Theorem: 3.3.3 For any graph \( G \),

\[
\text{ir}_{hY}(VE(G)) = \text{ir}_{ev}(G)
\]

\[
\text{ir}_{hY}(EV(G)) = \text{ir}_{ev}(G).
\]

Proof: Let \( S \) be a \( \text{ir}_{hY} \)-set of \( VE(G) = (X,Y,E) \). Every \( x \in S \) has a private hyper \( Y \)-neighbor \( y \in Y \). \( x \) is adjacent to \( y \) or \( y \in N(N_Y(x)) \) and for all vertices \( x_1 \in S - \{x\}, x_1 \) is not adjacent to \( y \) and \( y \notin N(N_Y(x_1)) \). In graph \( G \), \( x \in S \subseteq V \) is incident with \( y \in E \) or \( x \) is adjacent to either \( u \) or \( v \) where \( y = uv \) and for every \( x_1 \in S - \{x\}, y \in E \) is not incident with
$x_1$ and $x_1$ is not adjacent to either $u$ or $v$. $S$ is a vertex edge irredundant set.

$$ir_{ve}(G) \leq |S| = ir_{hY}(VE(G)),$$

Let $U$ be a $ir_{ve}$-set of $G$. Every vertex $v \in S$ has a private edge $e = uw$ with respect to $U$. Equivalently, $v$ is incident with $e$ or $v$ is adjacent to either $u$ or $w$ and for every $x \in U - \{v\}$, $x$ is not incident with $e$ and $x$ is not adjacent to either $u$ or $w$. In $VE(G)$, every $v \in S$ has private hyper $Y-$neighbor $e$. Therefore, $U \subseteq X$ is a hyper $Y-$irredundant set of $VE(G)$. Hence, $ir_{hY}(VE(G)) \leq |U| = ir_{ve}(G)$.

Similarly (b) can be proved.

### 3.4. X-Irredundant set

Let $G = (X,Y,E)$ be a bipartite graph. Let $S \subseteq X$. Let $u \in S$. A vertex $v$ is a private $X$-neighbor of $u$ with respect to $S$ if $u$ is the only point of $S$, $X$-adjacent to $v$.

A set $S$ is $X$-irredundant set if every $u \in S$ has a private $X$-neighbor. The $X$-irredundance number of a graph $G$ is the cardinality of a maximal $X$-irredundant set of vertices with minimum cardinality and is denoted by $ir_X(G)$. The upper $X$-irredundance number of a graph $G$ is the cardinality of a $X$-irredundant set of vertices with maximum cardinality and is denoted by $IR_X(G)$.

**Theorem 3.4.1** A $X$-dominating set $S$ is a minimal $X$-dominating set if and only if it is $X$-dominating and $X$-irredundant.

**Proof:** Let $S$ be a $X$-dominating set. Then $S$ is a minimal $X$-dominating set if and only if for every $u \in S$ there exists $v \in X - (S - \{u\})$ which is not $X$-dominated by $S - \{u\}$. Equivalently, $S$ is a minimal $X$-dominating set if and only if $\forall u \in S$, $u$ has at least one private $X$-neighbor with respect to $S$. Thus $S$ is minimal $X$-dominating set if and only if it is $X$-irredundant.

Conversely, Let $S$ is both $X$-dominating and $X$-irredundant.

**Claim:** $S$ is a minimal $X$-dominating set.

If $S$ is not a minimal $X$-dominating set, then there exists $v \in S$ for which $S - \{v\}$ is $X$-dominating. Since $S$ is $X$-irredundant, $v$ has a private $X$-neighbor of with respect to $S$ say $u$ ($u$ may be equal to $v$). By definition, $u$ is not $X$-adjacent to any vertex in $S - \{v\}$. Therefore, $S - \{v\}$ is not a $X$-dominating set, a contradiction. Hence, $S$ is a minimal $X$-dominating set.
Theorem: 3.4.2 Every minimal $X$-dominating set is a maximal $X$-irredundant set.

Proof: Every minimal $X$-dominating set $S$ is $X$-irredundant.

Claim: $S$ is a maximal $X$-irredundant set.

Suppose $S$ is not a maximal $X$-irredundant set. Then there exists a vertex $u \in X - S$ for which $S \cup \{u\}$ is $X$-irredundant. Therefore, there exists at least one vertex $x$ which is a private $X$-neighbor of $u$ with respect to $S$. Hence, no vertex in $S$ is $X$-adjacent to $x$. Thus $S$ is not $X$-dominating set, a contradiction. Hence, $S$ is maximal $X$-irredundant set.

A vertex $v$ is a private neighbor of a vertex $u$ in a set $S \subseteq V(G)$ with respect to $S$ if $N[v] \cap S = \{u\}$. The private neighbor set of $u$ is defined as $pn[u,S] = \{v : N[v] \cap S = \{u\}\}$. A set $S$ is called irredundant set [2] if for every vertex $u \in S$, $pn[u,S] \neq \emptyset$. The irredundance number of a graph $G$ is the cardinality of a maximal irredundant set with minimum number of vertices and is denoted by $ir(G)$. The upper irredundance number of a graph $G$ is the cardinality of a maximum irredundant set of vertices and is denoted by $IR(G)$.

Theorem: 3.4.3 For any graph $G$,
(a) $ir_X(VE(G)) = ir(G)$
(b) $ir_X(EV(G)) = ir^1(G)$

Proof: Let $S$ be a $ir_X$ set of $VE(G) = (X,Y,E^1)$. Every $v$ has a private $X$-neighbor $u$. Equivalently, $v$ is $X$-adjacent to $u$ and no other vertex in $S$ is $X$-adjacent to $u$. In $G$, $v \in S$ is the only vertex adjacent to $u$ and no other vertex in $S$ is adjacent to $u$. Therefore, $S$ is an irredundant set of $G$.

$ir(G) \leq |S| = ir_X(VE(G))$.

Let $U$ be an $ir-$ set of $G$. For every vertex $v \in U$, $pn[v,U] \neq \emptyset$. Every vertex $v \in U$ has at least one private neighbor with respect to $u$. In $VE(G)$, that is every vertex $v \in U$ has at least one private $X$-neighbor. Therefore, $U$ is an $X$-irredundant set. Hence, $ir_X(VE(G)) \leq |U| = ir(G)$. Hence, $ir_X(VE(G)) = ir(G)$.
(b) Let $S$ be an $ir_X$ set of $EV(G) = (X,Y,E^1)$. Every $e$ has a private $X$-neighbor $f$. Equivalently, $e$ is $X$-adjacent to $f$ and no other vertex in $S$ is $X$-adjacent to $f$. In $G$, $e \in S$ is the only edge adjacent to $f$ and no other edge in $S$ is adjacent to $f$. Therefore, $S$ is an edge irredundant set of $G$. Hence, $ir^1(G) \leq |S| = ir_X(EV(G))$.

Let $U$ be a $ir^1-$ set of $G$. For every edge $e \in U$, $pn[e,U] \neq \phi$. Hence, every edge $e \in U$ has at least one private neighbor. That is, in $EV(G)$, every vertex $e \in U$ has at least one private $X$-neighbor. Therefore, $U$ is an $X$-irredundant set in $EV(G)$. Thus, $ir_X(EV(G)) \leq |U| = ir^1(G)$. Hence, $ir_X(EV(G)) = ir^1(G)$.

4. Main Result

For any graph $G$, $IR_{rev}(G) + \gamma(G) \leq |V|$ and $\gamma_{rev}(G) + IR(G) \leq |V|$ are proved using bipartite theory of graphs, which are open problem in [3].

**Theorem 4.1** Let $G = (X,Y,E)$ be a bipartite graph with $N_Y(x) \neq \phi$ for every $x \in X$. Then $IR_{hY}(G) + \gamma_X(G) \leq |X|$.

**Proof:** Let $S$ be a $IR_{hY}$ set of $G$. Then, $S$ is a maximal hyper $Y$-irredundant set. Therefore, $S$ is a hyper $Y$-irredundant set. That is every $x \in S$ has a private hyper $Y$-neighbor $y \in Y$. Then $x$ is adjacent to $y$ or $y \in N(N_Y(x))$ and for all vertices $x_1 \in S - \{x\}$, $x_1$ is not adjacent to $y$ and $y \notin N(N_Y(x))$.

**Case (i):** $x$ is adjacent with $y$.

Since $N_Y(v) \neq \phi$, $x$ has $X$-neighbours. Let $z$ be any $X$-neighbour of $x$. Suppose $z \in S$. Then $z$ is not adjacent to $y$ and $y \notin N(N_Y(z))$. But $y \in N(Y(x))$, since $x$ is a $X$-neighbour of $z$, a contradiction. Therefore, any $X$-neighbour of $x$ is in $X - S$.

**Case (ii):** $y \in N(N_Y(x))$.

Vertices in $N(y)$ are in $X - S$. Then $N(y) \subseteq X - S$. Other wise, we get a contradiction to $y \in Y$ is a private hyper $Y$-neighbour of $x \in S$. Hence, for every $x \in S$ there exists $x_1 \in X - S$ such that $x$ and $x_1$ are $X$-adjacent. That is, $X - S$ is a $X$-dominating set. Therefore, $\gamma_X(G) \leq |X - S| = |X| - IR_{hY}(G)$. Hence, $IR_{hY}(G) + \gamma_X(G) \leq |X|$.
Corollary: 4.2 For any graph $G$, 
(a) $IR_{ve}(G) + \gamma(G) \leq |V|$

(b) $IR_{ev}(G) + \gamma_1(G) \leq |E|.$

Theorem: 4.3 Let $G = (X, Y, E)$ be a bipartite graph with $N_Y(x) \neq \phi$ for every $x \in X$ then $IR_X(G) + \gamma_{hY}(G) \leq |X|.$

Proof: Let $S$ be a $IR_X$ set of $G$. Every element $x \in S$ has a private $X$-neighbor. Consider the set $X - S$. Since $X - S$ is a $X$-dominating set elements of $Y$ are either adjacent to $X - S$ or adjacent to vertices which are $X$-adjacent to elements of $X - S$. Therefore, $X - S$ is a hyper $Y$-dominating set. Therefore, $\gamma_{hY} \leq |X - S| = |X| - IR_X$. Hence, $IR_X + \gamma_{hY} \leq |X|.$

Corollary: 4.4 For any graph $G$, 
(a) $\gamma_{ve}(G) + IR(G) \leq |V|$

(b) $\gamma_{ev}(G) + IR^1(G) \leq |E|.$

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References

[1] Bondy J. A., Murthy U. S. R., Graph theory with applications, London Macmillan (1976).
[2] Haynes T. W., Hedetniemi. S. T. and Slater P. J., Fundamentals of Domination in graphs, Marcel Dekker, New York, (1998).
[3] Jason Robert Lewis, Vertex-edge and edge-vertex parameters in graphs, (Ph. D Thesis), Clemson University, August 2007.
[4] Stephen Hedetniemi, Renu Laskar, A Bipartite theory of graphs I, Congressus Numerantium, Volume 55; pp. 5–14, December 1986.
[5] Stephen Hedetniemi, Renu Laskar, A Bipartite theory of graphs II, Congressus Numerantium, Volume 64; pp. 137-146, November 1988.
[6] Swaminathan V. and Venkatakrishnan Y. B., *Hyper \( Y \)-domination in Bipartite graphs*, International Mathematical Forum, Volume 4, No. 20, pp. 953-958, (2009).

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