On the low-energy limit of the QED $N -$ photon amplitudes

Louise C. Martin$^a$, Christian Schubert$^{a,b}$, Victor M. Villanueva Sandoval$^a$

$^a$ Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio C-3, Apdo. Postal 2-82
C.P. 58040, Morelia, Michoacán, México

$^b$ California Institute for Physics and Astrophysics
366 Cambridge Ave., Palo Alto, CA 94306, USA

Abstract

We derive an explicit formula for the low energy limits of the one-loop on-shell massive $N -$ photon amplitudes, for arbitrary $N$ and all helicity assignments, in scalar and spinor QED. The two-loop corrections to the same amplitudes are obtained for up to the ten point case. All photon amplitudes with an odd number of ‘$+$’ helicities are shown to vanish in this limit to all loop orders.
1 Introduction: On-shell QED photon amplitudes

In recent years substantial progress has been made in the computation of on-shell one-loop amplitudes. This has been due to the development of new techniques [1, 2] which provide alternatives to the standard Feynman diagrammatic approach, as well as to progress in the calculation of the basic integrals [3, 4, 5, 6, 7, 8, 9, 10]. Much of this work has been concerned with massless amplitudes, which are computationally the most accessible ones. It led to a number of unexpectedly simple results for certain special helicity configurations of photon or gluon amplitudes (see [2] for a review). A particularly striking result is Mahlon’s observation that the massless one-loop QED $N$-photon amplitudes with all helicities equal vanish on-shell for all $N \geq 6$ [11].

For the corresponding amplitudes involving massive loops little is known beyond the four-point case [12, 13, 14, 15]. In the present paper, we will investigate the QED $N$ photon amplitudes in the limit of low photon energies, i.e. with photon momenta such that all kinematic invariants $k_i \cdot k_j$ are small compared to $m^2$ (see [16] for a discussion of this approximation for the photon scattering case). As is well-known (see, e.g., [17]), in this limit the photon amplitudes are directly related to the QED effective Lagrangian $\mathcal{L}(F)$ for a background field with a constant field strength tensor $F_{\mu\nu}$. Namely, to obtain the amplitude with photon momenta $k_1, \ldots, k_N$ and polarisation vectors $\varepsilon_1, \ldots, \varepsilon_N$, define for every leg the field strength tensor

\[ F_{i}^{\mu\nu} \equiv k_i^{\mu} \varepsilon_i^{\nu} - k_i^{\nu} \varepsilon_i^{\mu} \quad (1.1) \]

and

\[ F_{\text{tot}} \equiv \sum_{i=1}^{N} F_i \quad (1.2) \]

The corresponding amplitude is then obtained by inserting $F_{\text{tot}}$ into the effective Lagrangian, and extracting the terms involving each $F_1, \ldots, F_N$ precisely once:

\[ \Gamma^{(EH)}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] = \mathcal{L}(iF_{\text{tot}}) \bigg|_{F_1, \ldots, F_N} \quad (1.3) \]

At one loop, the QED effective Lagrangian for the constant field strength case is just the well-known Euler-Heisenberg Lagrangian [18, 19] whose weak
field expansion is known in closed form. Nevertheless, it appears that (1.3) was previously applied only to the textbook case of photon-photon scattering.

In [20, 21] G.V. Dunne and one of the authors had considered the special case of constant self-dual background fields, and derived closed-form expressions for the corresponding two-loop Euler-Heisenberg Lagrangian and its scalar QED analogue. Due to the well-known correspondence between self-dual fields and helicity eigenstates [22, 23, 24, 25, 26] this Lagrangian still contains the full information on the low energy limit of the all ‘+’ component of the $N$-photon amplitudes in the helicity decomposition. The relation (1.3) could thus be used to derive simple closed-form expressions for these amplitudes not only at one but also at two loops, for arbitrary $N$, in scalar and spinor QED.

In the present paper we extend the same approach to the case of arbitrary helicity assignments. Applying (1.3) to the Euler-Heisenberg Lagrangian and its scalar QED analogue will allow us to obtain closed-form expressions for the low energy limits of the one-loop $N$-photon amplitudes with arbitrary helicity assignments. The standard spinor helicity technique [27, 28, 29] will turn out highly useful in working out the algebra of the field strength tensors $F_i$.

Although various integral representations have been derived for the corresponding two-loop effective Lagrangians [30, 31, 32, 33, 34, 35, 36], for the case of a general constant field none of them is sufficiently explicit to obtain corresponding all-$N$ formulas at the two-loop level. Nevertheless, we will use the formulas given in [30, 31, 32] to obtain the weak-field expansions of these two-loop effective Lagrangians up to the order ($F^{10}$), which will allow us to compute the corresponding photon amplitudes up to the ten point case.

## 2 One loop spinor QED

Let us begin with spinor QED at the one-loop level. We will use the standard integral representation of the Euler-Heisenberg Lagrangian [18],

$$L_{\text{spin}}^{(1)} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left[ \frac{e^2 ab}{\tanh(eaT) \tan(ebT)} - \frac{e^2}{3} (a^2 - b^2) - \frac{1}{T^2} \right]$$

(2.1)

Here $T$ denotes the propertime of the loop fermion, and $a, b$ are related to the two invariants of the Maxwell field by $a^2 - b^2 = B^2 - E^2, ab = E \cdot B$. 

3
The charge $e$ will often be set to unity in the following. The subtraction of the terms of zeroeth and second order in $a, b$ corresponds to on-shell renormalization. These terms are not relevant for our purposes and will be omitted in the following. The invariants $a, b$ can be related to the field strength tensor $F_{\mu\nu}$ and its dual \[ \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \]

\[
\begin{align*}
a^2 &= \frac{1}{4} \sqrt{(F_{\mu\nu} F^{\mu\nu})^2 + (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}} \\
b^2 &= \frac{1}{4} \sqrt{(F_{\mu\nu} F^{\mu\nu})^2 + (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu})^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}
\end{align*}
\tag{2.2}
\]

We wish to use this Lagrangian to obtain the low energy limit of the on-shell $N$-photon amplitude for arbitrary $N \geq 4$ and with an arbitrary helicity assignment. Due to Furry’s theorem we can, of course, assume that $N$ is even. Since in the abelian case the ordering of the legs does not matter we shall further assume that legs $1, \ldots, K$ carry the helicity ‘+’ and the remaining ones helicity ‘-’. Also, by CP invariance flipping all helicities is equivalent to changing all momenta from outgoing to ingoing. It is therefore sufficient to consider the case $K \geq N - K$. To construct suitable polarisation vectors we use the standard spinor helicity formalism. In this formalism, a polarisation vector with circular polarisation ‘$\pm$’ for a photon with momentum $k$ is written as

\[ \varepsilon_{\mu}^{\pm} = \pm \frac{\langle q^\mp | \gamma_\mu | k^\pm \rangle}{\sqrt{2 \langle q^\mp | q^\pm \rangle}} \]

(2.3)

Here $\langle q^\pm | k^\mp \rangle = u_{\pm}(q) u_{\mp}(k)$ etc. are basic spinor products, and $q$ is a reference momentum (see [37] for details and conventions). Changes of the reference momentum amount to gauge transformations. As usual we will use the notation

\[ \langle ij \rangle \equiv \langle k_i^- | k_j^+ \rangle \]

\[ [ij] \equiv \langle k_i^+ | k_j^- \rangle \]

(2.4)

So, let us use (2.3) with some arbitrary choice of reference momenta $q_i$ to define polarisation vectors $\varepsilon_i^+, \ldots, \varepsilon_K^+, \varepsilon_{K+1}^-, \ldots, \varepsilon_N^-$. In the corresponding field strength tensor for leg $i$ we work in Minkowski space with $\eta = \text{diag}(1, -1, -1, -1)$ and $\varepsilon_{0123} = 1$. 

\[ \text{We work in Minkowski space with } \eta = \text{diag}(1, -1, -1, -1) \text{ and } \varepsilon_{0123} = 1. \]
\[ F_{\pm \mu \nu} \equiv k_{ij} e_{\pm}^{\mu \nu} - \varepsilon_{\pm}^{\mu \nu} k_{i\nu} \]  

(2.5)

the dependence on \( q_i \) already drops out, as is easily verified. Using standard manipulations (see, e.g., [37]) the following identities are found to hold

\[ \{ F_i^+, F_j^+ \}^{\mu \nu} = -\frac{1}{2} [ij]^2 \eta^{\mu \nu} \]  

(2.6)

\[ \{ F_i^-, F_j^- \}^{\mu \nu} = -\frac{1}{2} (ij)^2 \eta^{\mu \nu} \]  

(2.7)

\[ [F_i^+, F_j^-] = 0 \]  

(2.8)

\[ \text{tr} (F_i^+ F_j^-) = 0 \]  

(2.9)

Moreover, as expected on general grounds [22, 23, 24, 25, 26] one finds the self-duality properties

\[ \tilde{F}^\pm_i = \mp i F_i^\pm \]  

(2.10)

With the help of these relations it is easy to compute the two Maxwell invariants for the case of \( F = F_{\text{tot}} \):

\[ \frac{1}{4} F_{\text{tot}, \mu \nu} F_{\text{tot}}^{\mu \nu} = \chi_+ + \chi_- \]  

(2.11)

\[ \frac{1}{4} \tilde{F}_{\text{tot}, \mu \nu} \tilde{F}_{\text{tot}}^{\mu \nu} = -i (\chi_+ - \chi_-) \]

where we have introduced

\[ \chi_+ \equiv \frac{1}{2} \sum_{1 \leq i < j \leq N} [ij]^2 \]  

\[ \chi_- \equiv \frac{1}{2} \sum_{1 \leq i < j \leq N} (ij)^2 \]  

(2.12)

Using (2.11) in (2.2) yields
\[ a = \sqrt{\chi^+} + \sqrt{\chi^-} \]
\[ b = -i(\sqrt{\chi^+} - \sqrt{\chi^-}) \]

(2.13)

The choice of sign for \( a, b \) does not matter since \( a \) and \( b \) appear only squared in the Lagrangian (2.1). Similarly, there is no need to introduce a sign convention for \( \sqrt{\chi^\pm} \).

Using (2.13) in (2.1) we get (omitting the subtraction terms)

\[ L^{(1)}_{\text{spin}}(iF_{\text{tot}}) = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \frac{(\sqrt{\chi^+} + \sqrt{\chi^-})(\sqrt{\chi^+} - \sqrt{\chi^-})}{\tan((\sqrt{\chi^+} + \sqrt{\chi^-})T) \tan((\sqrt{\chi^+} - \sqrt{\chi^-})T)} \]

(2.14)

As we explained in the introduction, the right hand side constitutes a generating functional for the one-loop (on-shell) photon amplitudes. The \( N \)-photon amplitude will be obtained by a double truncation of this formal expression: First, it must be expanded in powers of \( \chi^+, \chi^- \), and only the part of order \( F_{\text{tot}}^N \) kept from this series. Then, from the result those terms should be extracted involving each individual \( F_i \) just once.

Using the Taylor series,

\[ \frac{x}{\tan x} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n} \]

(2.15)

(the \( B_{2n} \) are Bernoulli numbers) the first step yields

\[ L^{(1)}_{\text{spin}}(iF_{\text{tot}}) = -\frac{m^4}{8\pi^2} \sum_{N=4}^{\infty} \left( \frac{2e}{m^2} \right)^N \sum_{K=0}^{N} c^{(1)}_{\text{spin}} \left( \frac{K}{2}, \frac{N-K}{2} \right) \chi^+ \chi^- \]

(2.16)

where

\[ c^{(1)}_{\text{spin}} \left( \frac{K}{2}, \frac{N-K}{2} \right) = (-1)^{\frac{N}{2}}(N-3)! \sum_{k=0}^{K} \sum_{l=0}^{N-K} (-1)^{N-K-l} \frac{B_{k+l} B_{N-k-l}}{k!(K-k)!(N-K-l)!} \]

(2.17)
Here we have omitted the irrelevant terms of order $\chi_0^0$, $\chi_1^1$. According to the above, the amplitude with $K$ `+' and $N - K$ `-' helicities is obtained from the corresponding term in the sum (2.16) by picking out the terms multilinear in the $F_i$'s. It is immediately seen that such terms exist only if $K$ is an even number. Thus all amplitudes with an odd number of `+' helicities do, in fact, vanish in the low energy limit. For $K$ even, let us define

$$
\chi^K_+ \equiv (\chi^+)^K_{\text{all \ different}} = \frac{(K^2)!}{2^{K^2}} \left\{ [12]^2 [34]^2 \cdots [(K-1)K]^2 + \text{all permutations} \right\}
$$

$$
\chi^{N-K}_- \equiv (\chi^-)^{N-K}_{\text{all \ different}} = \frac{(N-K)^2!}{2^{N-K^2}} \left\{ (K+1)(K+2)^2(K+3)(K+4)^2 \cdots (N-1)(N)^2 + \text{all \ perm.} \right\}
$$

The final result for the amplitude can then be written as

$$
\Gamma_{\text{spin}}^{(1)(EH)}[\varepsilon_1^+; \ldots; \varepsilon_K^+; \bar{\varepsilon}_{K+1}^+; \ldots; \bar{\varepsilon}_N^-] = -\frac{m^4}{8\pi^2} \left( \frac{2e}{m^2} \right)^N e^{(1)_{\text{spin}}(\frac{K}{2}, \frac{N-K}{2})} \chi^K_+ \chi^{N-K}_-
$$

(2.18)

(here and in the following we omit the momenta $k_1, \ldots, k_N$ in the argument of amplitudes).

We remark that the introduction of the variables $\chi_{\pm}$ is not essential in this calculation. An alternative, though less elegant, way of arriving at the same result would be to expand $L(iF_{\text{tot}})$ directly in powers of $F_{\text{tot}}$, perform the truncation to the multilinear part of the order $F_{\text{tot}}^N$ terms, and only after this use the spinor helicity identities (2.6), (2.7).

3 One-loop scalar QED

The scalar QED case is completely analogous, and we will write down only the main formulas. The analogue of the Euler-Heisenberg Lagrangian (2.1) for the scalar QED case was given by Schwinger [38]:

$$
L_{\text{scal}}^{(1)}(1) = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2T} \left[ \frac{e^2ab}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2 - b^2) - \frac{1}{T^2} \right]
$$

(3.1)
Using the Taylor expansion
\[
\frac{x}{\sin x} = -\sum_{n=0}^{\infty} (-1)^n \frac{(2n - 2)B_{2n}}{(2n)!} x^{2n} \tag{3.2}
\]
we can use the same procedure as in the spinor QED case. Again the result vanishes for odd \(K\), and for even \(K\) one obtains a formula analogous to (2.19):
\[
\Gamma_{\text{scal}}^{(1)(EH)} [\varepsilon_1^+; \ldots; \varepsilon_K^+; \varepsilon_{K+1}^-; \ldots; \varepsilon_N^-] = \frac{m^4}{16\pi^2} \left( \frac{2e}{m^2} \right)^N c_{\text{scal}}^{(1)} \left( \frac{K}{2}, \frac{N-K}{2} \right) \chi_K \chi_{N-K} \tag{3.3}
\]
where now
\[
c_{\text{scal}}^{(1)} \left( \frac{K}{2}, \frac{N-K}{2} \right) = (-1)^{\frac{N}{2}} (N-3)! \sum_{k=0}^{N-K} (-1)^{N-K-l} \sum_{l=0}^{K} \frac{(1-2^{1-k-l})(1-2^{1-N+k+l})B_{k+l}B_{N-k-l}}{k!(N-K)!}, \tag{3.4}
\]

4 Two-loop scalar and spinor QED

The following integral representation was obtained in [30] for the two-loop generalization of the Euler-Heisenberg Lagrangian (2.1):
\[
\mathcal{L}_{\text{spin}}^{(2)} = \frac{\alpha}{16\pi^3} \int_0^\infty dT \left\{ \int_0^T dT' \left[ K(T, T') - \frac{K_0(T)}{T'} \right] + K_0(T) \left[ \ln(m^2 T) + \gamma - \frac{5}{6} \right] \right\} \tag{4.1}
\]
where
\[
K(T, T') = e^{-m^2(T+T')} \left\{ \frac{a^2 b^2}{PP'} \left[ 4m^2 (S S' + P P') I_0 + I \right] - \frac{1}{TT'(T+T')} \left[ 4m^2 + \frac{2}{T+T'} + \frac{a^2 - b^2}{3} \left( 2m^2 (2T^2 + 2T'^2 - TT') - \frac{5TT'}{T+T'} \right) \right] \right\}
\]
\[ K_0(T) = e^{-m^2T} \left( 4m^2 - \frac{\partial}{\partial T} \right) \left( \frac{ab}{\tanh(aT) \tan(bT)} - \frac{1}{T^2} - \frac{a^2 - b^2}{3} \right) \]  

\[(4.2)\]

\[ I_0 = \frac{1}{B - A} \ln \left( \frac{B}{A} \right) \]

\[ I = \frac{q - p}{B - A} I_0 - \frac{q}{B - A} \]

\[ p = 2 \frac{a^2 \cos(b(T' - T))}{\sinh(aT) \sinh(aT')} \]

\[ q = 2 \frac{b^2 \cosh(a(T' - T))}{\sin(bT) \sin(bT')} \]

\[ P = \sinh(aT) \sin(bT), \quad S = \cosh(aT) \cos(bT) \]

\[ A = a(\coth(aT) + \coth(aT')), \quad B = b(\cot(bT) + \cot(bT')) \]  

\[(4.3)\]

Here \( \gamma \) is the Euler-Mascheroni constant. The charge \( e \) has been set to unity.

In contrast to the one-loop formula \((2.1)\) it is not known how to obtain from this integral representation a closed-form expression for the coefficients of the weak field expansion. Therefore, at two loops we contend ourselves with a calculation of this expansion to a certain order. Using MATHEMATICA we have found it straightforward to compute this expansion up to the order \((F^{10})\). As in the one-loop case, from the resulting polynomial in \( a, b \) we can directly read off the helicity amplitudes for \( N = 4, 6, 8, 10 \). To obtain a nonvanishing result, again we have to assume that not only \( N \) but also \( K \) are even. Its form is analogous to \((2.19)\):

\[ \Gamma^{(2)(EH)}_{\text{spin}}[\varepsilon_1^+; \ldots; \varepsilon_K^+; \varepsilon_{K+1}^-; \ldots; \varepsilon_N^-] = -\frac{\alpha \pi m^4}{8\pi^2} \left( \frac{2e}{m^2} \right)^N c^{(2)}_{\text{spin}}(\frac{K}{2}, N - \frac{K}{2}) \chi_+^+ \chi_-^- \]  

\[(4.4)\]

where the coefficients \( c^{(2)}_{\text{spin}}(\frac{K}{2}, N - \frac{K}{2}) \) are given in the appendix.

For the scalar QED case, we use the similar representation \([31]\)

\[ \mathcal{L}^{(2)}_{\text{scal}} = -\frac{\alpha}{32\pi^3} \int_0^\infty dT \left\{ \int_0^T dT' \left[ \tilde{K}(T, T') - \frac{\tilde{K}_0(T)}{T'} \right] + \tilde{K}_0(T) \left( \ln(m^2T) + \gamma - \frac{7}{6} \right) \right\} \]  

\[(4.5)\]

where now
\[ K(T, T') = e^{-m^2(T + T')} \left\{ \frac{a^2 b^2}{PP'} \left[ m^2 \frac{I_0 - \tilde{I}}{2} \right] \right. \]
\[ \left. - \frac{1}{TT'(T + T')} \left[ m^2 - \frac{1}{T + T'} - \frac{a^2 - b^2}{6} \left( m^2 (T + T')^2 - m^2 TT' - \frac{11TT'}{(T + T')} \right) \right] \right\} \]
\[ \tilde{K}_0(T) = e^{-m^2T} \left( m^2 + \frac{1}{2} \frac{\partial}{\partial T} \left( \frac{ab}{\sinh(aT) \sin(bT)} - \frac{1}{T^2} + \frac{a^2 - b^2}{6} \right) \right) \]

(4.6)

\[ \tilde{I} = \frac{\tilde{q} - \tilde{p}}{B - A} I_0 - \frac{\tilde{q} - \tilde{p}}{B - A} \]
\[ \tilde{p} = 2a^2 \left( \coth(aT) \coth(aT') - 3 \right), \quad \tilde{q} = 2b^2 \left( \cot(bT) \cot(bT') + 3 \right) \]

(4.7)

\[ I_0, P, A, B \text{ are as in (4.3)}. \text{ Computation of the weak field expansion to the same order } (F^{10}) \text{ yields} \]

\[ \Gamma^{(2)(EH)}_{\text{scal}} \left[ \varepsilon_1^+; \ldots; \varepsilon_K^+; \varepsilon_{K+1}^-; \ldots; \varepsilon_N^- \right] = \frac{\alpha \pi m^4}{16 \pi^2} \left( \frac{2e}{m^2} \right)^N \left( \frac{K}{2}; \frac{N - K}{2} \right) \chi_K^+ \chi_{N-K}^- \]

(4.8)

with coefficients also given in the appendix.

5 Conclusions

To summarize, we have shown here that the use of the effective action, when combined with spinor helicity techniques, provides a simple and elegant way to obtain information on the low energy limit of the QED N photon amplitudes. This has allowed us to derive an explicit formula for the one-loop N point amplitudes, as well as for the two-loop amplitudes up to the ten-point case. In particular, it has turned out that all amplitudes with an odd number of ‘+’ helicities vanish in the low energy limit. From the approach presented here it is clear that this property follows directly from the fact that the constant field effective action can be written as a function of the two Maxwell invariants. We therefore conclude that this vanishing must persist to all loop orders. Since these amplitudes are not forbidden by any
known symmetries, and indeed, the one-loop four-point 
\((+++-)\) amplitude

is known to be non-vanishing with full momentum [13],
this comes rather unexpected (for the special case of the
amplitudes with all helicities but one positive this vanishing
had been noted already in [21]).

Obviously, the self-duality relations fulfilled by field strength tensors
with definite helicities, eqs. (2.10) play an important part in these
simplifications.

We expect that these relations, as well as the variables \(\chi_{\pm}\), will also
have a useful role to play for the photon amplitudes at full momentum.
One indication for this is the appearance of factors of traces of
products of field strength tensors in the parameter integrals for the
\(N\) - photon amplitudes generated by the Bern-Kosower formalism
[11, 39, 40, 41, 42, 43]. Work in this direction is in progress.

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6 Appendix: Two-loop coefficients

| $c^{(2)}(K/2, N-K)$ | Scalar QED | Spinor QED |
|----------------------|------------|------------|
| $c^{(2)}(5, 0)$      | $\frac{611}{80640\pi^2}$ | $\frac{317}{40320\pi^2}$ |
| $c^{(2)}(4, 1)$      | $\frac{349609}{3628800\pi^2}$ | $-\frac{8707}{1814400\pi^2}$ |
| $c^{(2)}(3, 2)$      | $\frac{688637}{2332800\pi^2}$ | $-\frac{3190547}{8164800\pi^2}$ |
| $c^{(2)}(4, 0)$      | $\frac{67}{12800\pi^2}$ | $\frac{2221}{403200\pi^2}$ |
| $c^{(2)}(3, 1)$      | $\frac{273619}{6350400\pi^2}$ | $-\frac{151379}{6350400\pi^2}$ |
| $c^{(2)}(2, 2)$      | $\frac{2055163}{25401600\pi^2}$ | $-\frac{37763}{282240\pi^2}$ |
| $c^{(2)}(3, 0)$      | $\frac{13}{1920\pi^2}$ | $\frac{7}{960\pi^2}$ |
| $c^{(2)}(2, 1)$      | $\frac{8563}{259200\pi^2}$ | $-\frac{5821}{129600\pi^2}$ |
| $c^{(2)}(2, 0)$      | $\frac{3}{128\pi^2}$ | $\frac{5}{192\pi^2}$ |
| $c^{(2)}(1, 1)$      | $\frac{307}{5184\pi^2}$ | $-\frac{391}{2592\pi^2}$ |
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