Research Article

Blow up of Coupled Nonlinear Klein-Gordon System with Distributed Delay, Strong Damping, and Source Terms

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1. Introduction

In the present paper, we consider the following system:

\[\begin{align*}
\frac{du}{dt} + m_1 u_x \tau_1 + \sum_{i=1}^N \int_0^t g(t-s) \omega_i u_x(s) + \mu_1 v_x(s) ds = f_1(u, v), \\
\frac{dv}{dt} + m_2 v_x \tau_2 + \sum_{i=1}^N \int_0^t g(t-s) \omega_i v_x(s) + \mu_2 v_x(s) ds = f_2(u, v), \\
\end{align*}\]

(1)

where

\[\begin{align*}
\begin{cases}
f_1(u, v) = a_1 |u + v|^{2+\mu} - |u|^{\mu} - |v|^{\mu} - (u + v) \cdot f_3(u, v), \\
f_2(u, v) = a_2 |u + v|^{2+\nu} - |u|^{\nu} - |v|^{\nu} - (u + v) \cdot f_3(u, v),
\end{cases}
\end{align*}\]

(2)

and \(m_1, m_2, \omega_1, \omega_2, \mu_1, \mu_2, a_1, a_2, b_1 > 0,\) and \(\tau_1, \tau_2\) are the time delay with \(0 \leq \tau_1 < \tau_2,\) and \(\mu_1, \mu_2\) are a \(L^\infty\) functions, and \(g, h\) are differentiable functions.

Viscous materials are the opposite of flexible materials that have dissipate mechanical energy and the ability to store.

The mechanical properties of viscous materials are so important that we find them in many applications of natural sciences. Many authors have been concerned with this problem in recent decades.

If there is only one equation and if \(\omega_1 = 0\), that is, for absence of \(\Delta u\), and \(\mu_1 = \mu_2 = 0.\) Our problem (1) has been studied by Berrimi and Messaoudi [1]. Using Galerkin’s method they proved the result of local existence. They also made it clear that the local solution is global in time under suitable conditions and at the same rate of decaying (exponential or polynomial) of the kernel \(g.\) In addition, the authors themselves demonstrated that the dissipation can
where $g$ satisfies

$$
\int_0^\infty g(s)ds < (2p - 4)/(2p - 3).
$$

The initial data was backed by negative energy as

$$
\int u_0 u_1 dx > 0.
$$

In [5], Song and Xue considered the following problem:

$$
\begin{aligned}
& u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = |u|^{p-2}u, \\
& u(x, 0) = u_0(x), \
& u_t(x, 0) = u_1(x),
\end{aligned}
$$

where the authors proved the exponential growth result under suitable assumptions. The authors in [8] studied the following problem:

$$
\begin{aligned}
& u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = |u|^{p-2}u, \\
& u(x, t) = 0, \
& x \in \partial \Omega, 
\end{aligned}
$$

where they showed a blow up result if $p > m$ and established the global existence. In the coupled equation case, the authors in [9] studied the following system:

$$
\begin{aligned}
& u_{tt} - \Delta u + |u_t|^{m-2}u_t = f_1(u, v), \\
& v_{tt} - \Delta v + |v_t|^{p-2}v_t = f_2(u, v),
\end{aligned}
$$

with $f_1$ and $f_2$ nonlinear functions satisfying appropriate conditions. According to certain restrictions imposed on the initial data and parameters, they obtained numerous
results on the existence of weak solutions. They obtained many results on the presence of weak solutions. In addition, by using the same techniques similar to that in [10] with negative initial, energy blows up for a finite period of time.

In [11], the authors have proved the solution of the problem:

\[
\begin{align*}
\begin{cases}
  u_{tt} - \Delta u + \left( a|u|^k + b|v|^\ell \right) u_t |u_t|^{m-2} = f_1(u, v), \\
v_{tt} - \Delta v + \left( a|u|^\ell + b|v|^k \right) v_t |v_t|^{n-2} = f_2(u, v),
\end{cases}
\end{align*}
\]

where under some restrictions on positive initial energy for certain conditions on the functions \( f_1 \) and \( f_2 \), the authors proved the blow up in finite time of solution.

The result of [11] has been extended by the authors in [12], where they studied the following system:

\[
\begin{align*}
\begin{cases}
  u_{tt} - \Delta u + \int_0^\infty g(s) \Delta u(t-s) ds + \left( a|u|^k + b|v|^\ell \right) u_t |u_t|^{m-2} = f_1(u, v), \\
v_{tt} - \Delta v + \int_0^\infty h(s) \Delta v(t-s) ds + \left( a|u|^\ell + b|v|^k \right) v_t |v_t|^{n-2} = f_2(u, v),
\end{cases}
\end{align*}
\]

they proved that the solutions of a system of wave equations with degenerate damping, viscoelastic term and strong nonlinear sources acting in both equations at the same time are globally nonexisting provided that the initial data are sufficiently large in a bounded domain of \( \Omega \).

As complement to these works, we are working to prove the blow up result with distributed delay of problem (1), under appropriate assumptions, and we prove these results using the energy method. In the following, let \( c_i, \delta_i > 0, i = 1, \cdots, 12 \).

The present paper is organized as follows. In Section 2, we give some necessarily assumptions for the main result. In Section 3, we prove the blow up result.

2. Assumptions

We consider the following suitable assumptions.

(A1) \( g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \) are differentiable and decreasing functions such that

\[
g(t) \geq 0, \quad 1 - \int_0^\infty g(s) ds = l_i > 0.
\]

h(t) \geq 0, \quad 1 - \int_0^\infty h(s) ds = l_i > 0.
\]

(A2) There exists a constants \( \xi_1, \xi_2 > 0 \) such that

\[
g'(t) \leq -\xi_1 g(t), \quad t \geq 0,
\]

\[
h'(t) \leq -\xi_2 h(t), \quad t \geq 0.
\]

(A3) \( \mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R} \) are \( L^\infty \) functions so that

\[
\left( \frac{2\delta - 1}{2} \right) \int_{\tau_1}^{\tau_2} |\mu_2(q)| dq < \mu_1, \quad \delta > \frac{1}{2},
\]

\[
\left( \frac{2\delta - 1}{2} \right) \int_{\tau_1}^{\tau_2} |\mu_4(q)| dq < \mu_2, \quad \delta > \frac{1}{2}.
\]

3. Blow up

In this section, we obtain the proof of the blow up result of the solution of problem (1). First, of all in [13], we introduce the new variables

\[
y(x, \rho, q, t) = u(x, t - \rho q),
\]

\[
z(x, \rho, q, t) = v(x, t - \rho q),
\]

then, we obtain

\[
\begin{align*}
Qy_t(x, \rho, q, t) + y_p(x, \rho, q, t) = 0, \\
y(x, 0, q, t) = u_0(x, t), \\
Qz_t(x, \rho, q, t) + z_p(x, \rho, q, t) = 0, \\
z(x, 0, q, t) = v_0(x, t).
\end{align*}
\]

Let us denote by

\[
gou = \int_{\Omega} \int_0^\infty \int_0^\infty g(t-s) |u(t) - u(s)|^2 ds dx.
\]

Therefore, problem (1) get the following form:
with initial and boundary conditions
\[
\begin{aligned}
    u(x, t) &= 0, \quad v(x, t) = 0, \quad x \in \partial \Omega, \\
    y(x, \rho, q, 0) &= f_0(x, \rho, q), \quad z(x, \rho, q, 0) = k_0(x, \rho, q), \\
    u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \\
    v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x),
\end{aligned}
\]

(23)

where
\[
(x, \rho, q, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).
\]

(24)

**Theorem 1.** Assume (14), (16), and (17) hold. Let
\[
\begin{align*}
-1 < \rho < &\frac{4 - n}{n - 2}, \quad n \geq 3, \\
\rho \geq &-1, \quad n = 1, 2.
\end{align*}
\]

(25)

For any initial data,
\[
(u_0, u_1, v_0, v_1, f_0, k_0) \in \mathcal{H},
\]

(26)

where
\[
\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)) \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)).
\]

(27)

then, problem (22) has a unique solution
\[
u \in C([0, T]; \mathcal{H}),
\]

(28)

for some \( T > 0 \).

**Lemma 2.** There exists a function \( F(u, v) \) such that
\[
F(u, v) = \frac{1}{2\rho + 2} [\sigma_1 |u|^{\rho + 2} + \sigma_2 |v|^{\rho + 2}]
\]

(29)

where
\[
\frac{\partial F}{\partial u} = f_1(u, v), \quad \frac{\partial F}{\partial v} = f_2(u, v),
\]

(30)

we take \( \sigma_1 = \sigma_2 = 1 \) for convenience.

**Lemma 3.** (see [12]). There exist two positive constants \( c_0 \) and \( c_1 \) such that
\[
\frac{c_0}{2\rho + 2} \left( |u|^{2(\rho + 2)} + |v|^{2(\rho + 2)} \right) \leq F(u, v) \leq \frac{c_1}{2\rho + 2} \left( |u|^{2(\rho + 2)} + |v|^{2(\rho + 2)} \right).
\]

(31)

We define the energy functional (see, e.g., [14–16] and reference therein).

**Lemma 4.** Assume (14), (16), (17), and (25) hold, let \( (u, v, y, z) \) be a solution of (22), then \( E(t) \) is nonincreasing, that is,
\[
E(t) = \frac{1}{2\rho + 2} \left( |u_t|^2 + |v_t|^2 + m_1 |u|^2 + m_2 |v|^2 \right) + \frac{1}{2} l_2 |\nabla u|^2 + \frac{1}{2} l_2 |\nabla v|^2 + \frac{1}{2} (g_0 u) + \frac{1}{2} (h_0 v) + \frac{1}{2} K(y, z) - \int_{\Omega} F(u, v) dx
\]

(32)

satisfies
\[
E'(t) \leq -c_3 \left( |u_t|^2 + |v_t|^2 + |u|^2 + |v|^2 \right)
\]

(33)

where
\[
K(y, z) = \int_{\Omega} \int_{\tau_1}^{\tau_2} \left\{ \mu_2(y) |y|^2(x, \rho, q, t) \right. + \left. |\mu_2(y)|^2(x, \rho, q, t) \right\} dx dt
\]

(34)

**Proof.** By multiplying (3.4)\(_1\), (3.4)\(_2\) by \( u_t, v_t \) and integrating over \( \Omega \), we get
\[
\frac{d}{dt} \left\{ \frac{1}{2\rho + 2} \left( |u_t|^2 + |v_t|^2 + m_1 |u|^2 + m_2 |v|^2 \right) + \frac{1}{2} l_2 |\nabla u|^2 + \frac{1}{2} l_2 |\nabla v|^2 \right\}
\]

(35)
and, from (3.4)_3, (3.4)_4, we have
\[
\frac{d}{dt} \frac{1}{2} \int_{\Omega \cup \partial \Omega} \int_{\Gamma_1} \| \mu_2(q) \|_{\mathcal{P}}(x, \rho, q, t) d\eta d\rho + \int_{\Omega} \int_{\Gamma_1} |\mu_2(q)| |y^2(x, \rho, q, t)| d\eta d\rho dx \]
\[
= -\frac{1}{2} \int_{\Omega} \int_{\Gamma_1} r_{\Omega}^2 |\mu_2(q)| |y_\rho^2 d\eta d\rho dx,
\]
\[
+ \frac{1}{2} \int_{\Omega} \int_{\Gamma_1} |\mu_2(q)| |y^2(x, 0, q, t)| d\eta d\rho dx,
\]
\[
- \frac{1}{2} \int_{\Omega} \int_{\Gamma_1} |\mu_2(q)| |y^2(x, 1, q, t)| d\eta d\rho dx,
\]
\[
= \frac{1}{2} \left( \int_{\Gamma_1} |\mu_2(q)| |y^2(x, 1, q, t)| d\eta d\rho \right) \| \mu_2 \|_2^2
\]
\[
- \frac{1}{2} \int_{\Omega} \int_{\Gamma_1} |\mu_2(q)| |y^2(x, 1, q, t)| d\eta d\rho dx,
\]
\[
(36)
\]
Now, we define the functional
\[
\mathcal{H}(t) = -E(t) = -\frac{1}{2} \| u_1 \|_2^2 - \frac{1}{2} \| v_1 \|_2^2 - \frac{m_1}{2} \| u_2 - m_2 \|_2^2 \| v_2 \|_2^2
\]
\[
- \frac{1}{2} (\| \nabla u_1 \|_2^2 - \| \nabla v_1 \|_2^2 - \frac{1}{2} (\| \nabla u_2 \|_2^2 - \| \nabla v_2 \|_2^2 - \frac{1}{2} (\| \nabla u_2 \|_2^2 - \| \nabla v_2 \|_2^2) )
\]
\[
- \frac{1}{2} K(y, z) + \frac{1}{2(p+2)} \left[ \| u + v \|_{2(p+2)}^2 + 2 \| u \|_{2(p+2)}^2 \right].
\]
\[
(39)
\]
**Theorem 5.** Assume (14)–(17) and (25) hold. Assume further that \( E(0) < 0 \), then the solution of problem (22) blow up in finite time.

**Proof.** From (32), we have
\[
E(t) \leq E(0) \leq 0.
\]

Therefore,
\[
\mathcal{H}'(t) = -E'(t)
\]
\[
\geq c_3 \left( \| u_1 \|_2^2 + \| u_2 \|_2^2 + \int_{\Omega} \int_{\Gamma_1} |\mu_2(q)| y^2(x, 1, q, t) d\eta d\rho dx
\]
\[
+ \| v_1 \|_2^2 + \| v_2 \|_2^2 + \int_{\Omega} \int_{\Gamma_1} |\mu_4(q)| z^2(x, 1, q, t) d\eta d\rho dx \right)
\]
\[
(41)
\]
\[
\text{hence,}
\]
\[
\mathcal{H}'(t) \geq c_3 \int_{\Gamma_1} \int_{\Gamma_1} |\mu_2(q)| y^2(x, 1, q, t) d\eta d\rho dx \geq 0,
\]
\[
(42)
\]
\[
\mathcal{H}'(t) \geq c_3 \int_{\Gamma_1} \int_{\Gamma_1} |\mu_4(q)| z^2(x, 1, q, t) d\eta d\rho dx \geq 0,
\]
\[
(43)
\]
\[
0 \leq \mathcal{H}(0) \mathcal{H}'(t) \leq \frac{1}{2(p+2)} \left[ \| u + v \|_{2(p+2)}^2 + 2 \| u \|_{2(p+2)}^2 \right]
\]
\[
\leq \frac{c_1}{2(p+2)} \left[ \| u \|_{2(p+2)}^2 + \| v \|_{2(p+2)}^2 \right].
\]
\[
(44)
\]
\[
\mathfrak{H}'(t) = \mathcal{H}'(t) + \frac{1}{2} \int_{\Omega} \int_{\Gamma_1} (u u_1 + v v_1) dx + \frac{3}{2} \int_{\Omega} \int_{\Gamma_1} (\mu_2 u^2 + \mu_3 v^2) dx
\]
\[
+ \frac{1}{2} \int_{\Omega} \int_{\Gamma_1} (w_1 (\nabla u_1)^2 + w_2 (\nabla v_1)^2) dx,
\]
\[
(45)
\]
where \( \epsilon > 0 \) to be assigned later and
\[
0 < \alpha < \frac{2p+2}{4(p+2)} < 1.
\]
\[
(46)
\]
By multiplying (3.4)₁, (3.4)₂ by \( u, v \) and with a derivative of (45), we get

\[
\mathcal{K}′(t) = (1 - \alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon \left(\|u\|_{2}^{2} + \|v\|_{2}^{2} + \|u\|_{2}^{2} + \|v\|_{2}^{2}\right)
\]

We obtain, from (47),

\[
\mathcal{K}′(t) \geq (1 - \alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon \left(\|u\|_{2}^{2} + \|u\|_{2}^{2} + \|v\|_{2}^{2} + \|v\|_{2}^{2}\right)
\]

Using Young’s inequality, we get

\[
\varepsilon \int_{\Omega} \int_{\tau_{1}}^{\tau_{2}} |\mu_{2}(Q)|u(y(x, 1, q, t))d\tilde{q}d\tilde{q}
\]

and we have

\[
\varepsilon \int_{0}^{t} g(t-s)ds \int_{\Omega} \nabla u \cdot \nabla u(s)dxds
\]

We therefore, using (43) and by setting \( \delta_{1}, \delta_{2} \) so that, 1/4\( \delta_{1} \), \( c_{3} = \kappa \mathbb{H}^{-\alpha}(t)/2 \) and 1/4\( \delta_{2} c_{3} = \kappa \mathbb{H}^{-\alpha}(t)/2 \), substituting in (50), we get

\[
\mathcal{K}′(t) \geq (1 - \alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon \left(\|u\|_{2}^{2} + \|u\|_{2}^{2} + \|v\|_{2}^{2} + \|v\|_{2}^{2}\right)
\]

For 0 < \( a < 1 \), from (39),

\[
\varepsilon \int_{0}^{t} g(t-s)ds \int_{\Omega} \nabla u \cdot \nabla u(s)dxds
\]
substituting in (51), we get

\[ \mathcal{K}'(t) \geq \left[(1 - \alpha) - \epsilon \kappa \mathcal{H}'^{\alpha} \mathcal{H}'(t)\right. \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) + 1 \right]\left[\left\| u_{t}^{2} \right\|_{2(p+2)}^{2} + \left\| v_{t}^{2} \right\|_{2(p+2)}^{2} + \left\| v_{1}^{2} \right\|_{2(p+2)}^{2} + \left\| v_{2}^{2} \right\|_{2(p+2)}^{2}\right] \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) \int_{0}^{t} g(s)ds \right. \]
\[ \left. - \left(1 + \frac{1}{2} \int_{0}^{t} g(s)ds \right) \right\| \nabla u \|^2_{2} \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) \int_{0}^{t} h(s)ds \right. \]
\[ \left. - \left(1 + \frac{1}{2} \int_{0}^{t} h(s)ds \right) \right\| \nabla v \|^2_{2} \]
\[ - \epsilon \frac{\mathcal{H}^{\alpha}(t)}{2c_{5}K} \int_{t_{1}}^{t} |\mu_{2}(\xi)|d\xi \left\| u \|^2_{2} \right. \]
\[ + \epsilon \frac{\mathcal{H}^{\alpha}(t)}{2c_{5}K} \int_{t_{1}}^{t} |\mu_{2}(\xi)|d\xi \left\| v \|^2_{2} \right. \]
\[ + \epsilon (p + 2)(1 - \alpha)K(y, z) \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) - 1 \right] \left( go\nabla u + ho\nabla v \right) \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) \right] \left( \| u \|^2 + \| v \|^2 + \| v_{1} \|^2 + \| v_{2} \|^2 \right) \]
\[ + \epsilon 2(p + 2)(1 - \alpha)\mathcal{H}(t). \]

(53)

Since (25) hold, we obtain by using (44) and (46)

\[ \mathcal{H}^{\alpha}(t)\| u \|^2_{2} \leq c_{4}\left(\| u \|^2_{2} \right) + 2\| \nabla u \|^2_{2}, \]

(54)

\[ \mathcal{H}^{\alpha}(t)\| v \|^2_{2} \leq c_{5}\left(\| v \|^2_{2} \right) + 2\| \nabla v \|^2_{2}, \]

(55)

for some positive constants \(c_{4}, c_{5}\). By using (46) and the algebraic inequality,

\[ B^{\theta} \leq (B + 1) \leq \left(1 + \frac{1}{B} \right) (B + b), \quad \forall B > 0, \quad 0 < \theta < 1, \quad b > 0, \]

(56)

we have, \( \forall t > 0 \)

\[ \| u \|^2_{2(p+2)} \leq d \left(\| u \|^2_{2(p+2)} + \mathcal{H}(0) + \mathcal{H}(t) \right), \]

(57)

\[ \| v \|^2_{2(p+2)} \leq d \left(\| v \|^2_{2(p+2)} + \mathcal{H}(t) \right), \]

(58)

where \( d = 1 + (1/\mathcal{H}(0)) \). Also, since

\[ (x + y)^{\theta} \leq C(x^{\theta} + y^{\theta}), \quad \forall x, y > 0, \quad \gamma > 0, \]

(59)

we conclude

\[ \| v \|^2_{2(p+2)} \| u \|^2_{2} \leq c_{6}\left(\| v \|^2_{2(p+2)} + \| u \|^2_{2} \right) \]
\[ \leq c_{7}\left(\| v \|^2_{2(p+2)} + \| u \|^2_{2} \right), \]

(60)

\[ \| u \|^2_{2(p+2)} \| v \|^2_{2} \leq c_{8}\left(\| u \|^2_{2(p+2)} + \| v \|^2_{2} \right) \]
\[ \leq c_{9}\left(\| u \|^2_{2(p+2)} + \| v \|^2_{2} \right), \]

(61)

substituting (58) and (61) in (55), we get

\[ \mathcal{H}^{\alpha}(t)\| u \|^2_{2} \leq c_{10}\left(\| v \|^2_{2(p+2)} + \| u \|^2_{2(p+2)} \right) + c_{10}\mathcal{H}(t), \]

(62)

\[ \mathcal{H}^{\alpha}(t)\| v \|^2_{2} \leq c_{11}\left(\| u \|^2_{2(p+2)} + \| v \|^2_{2(p+2)} \right) + c_{11}\mathcal{H}(t), \]

(63)

Combining (53) and (63), using (31), we get

\[ \mathcal{K}'(t) \geq \left[(1 - \alpha) - \epsilon \kappa \mathcal{H}'^{\alpha} \mathcal{H}'(t)\right. \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) + 1 \right]\left[\left\| u_{t}^{2} \right\|_{2(p+2)} + \left\| v_{t}^{2} \right\|_{2(p+2)} + \left\| v_{1}^{2} \right\|_{2(p+2)} + \left\| v_{2}^{2} \right\|_{2(p+2)}\right] \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) \int_{0}^{t} g(s)ds \right. \]
\[ \left. - \left(1 + \frac{1}{2} \int_{0}^{t} g(s)ds \right) \right\| \nabla u \|^2_{2} \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) \int_{0}^{t} h(s)ds \right. \]
\[ \left. - \left(1 + \frac{1}{2} \int_{0}^{t} h(s)ds \right) \right\| \nabla v \|^2_{2} \]
\[ - \epsilon \frac{\mathcal{H}^{\alpha}(t)}{2c_{5}K} \int_{t_{1}}^{t} |\mu_{2}(\xi)|d\xi \left\| u \|^2_{2} \right. \]
\[ + \epsilon \frac{\mathcal{H}^{\alpha}(t)}{2c_{5}K} \int_{t_{1}}^{t} |\mu_{2}(\xi)|d\xi \left\| v \|^2_{2} \right. \]
\[ + \epsilon (p + 2)(1 - \alpha)K(y, z) \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) - 1 \right] \left( go\nabla u + ho\nabla v \right) \]
\[ + \epsilon \left[ (p + 2)(1 - \alpha) \right] \left( \| u \|^2 + \| v \|^2 + \| v_{1} \|^2 + \| v_{2} \|^2 \right) \]
\[ + \epsilon 2(p + 2)(1 - \alpha)\mathcal{H}(t) \]
\[ + \epsilon \left( \frac{\lambda_{1} + \lambda_{2}}{2c_{5}K} \right) \left( \| u \|^2 + \| v \|^2 \right) \]
\[ + \epsilon \left( \frac{\lambda_{1} + \lambda_{2}}{2c_{5}K} \right) \mathcal{H}(t) \]
\[ \left. \right) \mathcal{H}(t), \]

(64)

where \( \lambda_{1} = c_{10} \int_{t_{1}}^{t} |\mu_{2}(\xi)|d\xi, \lambda_{2} = c_{11} \int_{t_{1}}^{t} |\mu_{4}(\xi)|d\xi. \)

In this case, we take \( a > 0 \) small enough, then

\[ \alpha_{1} = (p + 2)(1 - \alpha) - 1 > 0, \]

(65)

assuming

\[ \max \left\{ \int_{0}^{t} g(s)ds, \int_{0}^{t} h(s)ds \right\} \leq \frac{(p + 2)(1 - \alpha) - 1}{(p + 2)(1 - \alpha) - (1/2)} \]
\[ = \frac{2\alpha_{1}}{2\alpha_{1} + 1}, \]

(66)
we have
\[ a_2 = \left\{ (p + 2)(1 - a) - 1 - \int_0^1 g(s)ds \left( (p + 2)(1 - a) - \frac{1}{2} \right) \right\} > 0, \]
\[ a_3 = \left\{ (p + 2)(1 - a) - 1 - \int_0^1 h(s)ds \left( (p + 2)(1 - a) - \frac{1}{2} \right) \right\} > 0, \]
(67)

choose \( \kappa \) so large that
\[ a_4 = a_0 - \frac{\lambda_1 + \lambda_2}{2c_2 \kappa} > 0, \]
\[ a_5 = 2(p + 2)(1 - a) - \frac{\lambda_1 + \lambda_2}{2c_2 \kappa} > 0, \]
(68)

fix \( \kappa \) and \( a \), we appoint \( \epsilon \) small enough so that
\[ a_6 = (1 - a) - \epsilon \kappa > 0. \]
(69)

Then, for \( \beta > 0 \), we estimate (64) and it becomes
\[ \mathcal{R}'(t) \geq \beta \left\{ \mathcal{H}(t) + \| u_t \|^2 + \| v_t \|^2 + \| u \|^2 + \| v \|^2 + \| \nabla u \|^2 \\
+ \| \nabla v \|^2 + (go \nabla u) + (ho \nabla v) + K(y, z) \\
+ \left\{ \| u \|_{L^2(\Omega)} + \| v \|_{L^2(\Omega)} \right\} \right\}, \]
(70)

By (31), for \( \beta_1 > 0 \), we get
\[ \mathcal{R}'(t) \geq \beta_1 \left\{ \mathcal{H}(t) + \| u_t \|^2 + \| v_t \|^2 + \| u \|^2 + \| v \|^2 + \| \nabla u \|^2 \\
+ \| \nabla v \|^2 + (go \nabla u) + (ho \nabla v) + K(y, z) \\
+ \left\{ \| u \|_{L^2(\Omega)} + \| v \|_{L^2(\Omega)} \right\} \right\}, \]
(71)

Using Holder’s and Young’s inequalities, we have
\[ \int_{\Omega} (u u_t + v v_t) \, dx \geq \frac{1}{\theta} \left\{ \| u \|_{L^{1/\theta} \cap L^{\theta}}^{(1/\theta)} + \| u_t \|_{L^{1/(1-\theta)} \cap L^{\theta}}^{(1-\theta)} \right\}, \]
(72)

where \( (1/\mu) + (1/\theta) = 1 \) put \( \theta = 2(1 - \alpha) \), to get
\[ \frac{\mu}{1 - \alpha} = \frac{2}{1 - 2\alpha} \leq 2(p + 2). \]
(73)

Subsequently, for \( s = 2(1 - 2\alpha) \) and by using (39), we get
\[ \| u \|_{L^{2/(p+2)}(\Omega)}^2 \leq d \left( \| u \|_{L^{2/(p+2)}(\Omega)}^2 + \mathcal{H}(t) \right), \]
(74)

Therefore,
\[ \int_{\Omega} (u u_t + v v_t) \, dx \leq c_{12} \left[ \| u \|_{L^{2/(p+2)}(\Omega)}^2 + \| v \|_{L^{2/(p+2)}(\Omega)}^2 + \| u_t \|^2 + \| v_t \|^2 + \mathcal{H}(t) \right]. \]
(75)

Subsequently,
\[ \mathcal{E}^{1/(1-\alpha)}(t) = \left( \mathcal{H}^{1-\alpha} + \epsilon \int_{\Omega} (u u_t + v v_t) \, dx \right) \\
+ c \left( \| u_t \|^2 + \| v_t \|^2 + \| u \|^2 + \| v \|^2 + \| \nabla u \|^2 + \| \nabla v \|^2 + \mathcal{H}(t) \right)^{1/(1-\alpha)} \]
(76)

From (70) and (76), gives
\[ \mathcal{R}'(t) \geq \lambda \mathcal{R}^{1/\alpha}(t), \]
(77)

with \( \lambda > 0 \), this quantity depends on \( \beta \) and \( c \). By simple integration of (77), we obtain
\[ \mathcal{E}^{1/(1-\alpha)}(t) \geq \frac{1}{\mathcal{R}^{-\alpha(1-\alpha)}(0) - \lambda (\alpha/(1 - \alpha))t}, \]
(78)

Hence, \( \mathcal{E}(t) \) in a situation of blow up in time, when
\[ T \leq T^* = \frac{1 - \alpha}{\lambda \alpha \mathcal{R}^{\alpha(1-\alpha)}(0)}, \]
(79)

Then, this completes the proof of the theorem.
4. Conclusion

In this work, we have studied the blow up of the coupled Klein-Gordon system with strong damping, distributed delay, and source terms, under suitable conditions which are so important that we find them in many applications of natural sciences. Many authors have been concerned with this problem in recent decades (see, for example, [17–19]). In the next work, we will try to apply the same technique with a new class of Boussinesq equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see, for example, [20, 21]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Authors’ Contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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