Two-dimensional self-consistent quantum-corrected geometries

with a constant dilaton field

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It is argued that the existence of constant dilaton field solutions is a
generic feature of string-inspired dilaton gravity. Such solutions arise in the
extreme limit of black hole metrics. It is shown that in a strong coupling
region quantum effects give rise to two horizons in thermal equilibrium that
has no classical counterpart.

The existence of black hole solutions in two-dimensional (2D) dilaton gravity gives
us hope for better understanding some key issues of black hole physics (singularities, information loss, etc.) which are difficult to analyze in the more realistic 4D case. One of such issues is the nature of the extreme state with a zero surface gravity which is believed to represent an object qualitatively different from nonextreme black holes either in topological or thermodynamic respects. Quantum-corrected geometry of 2D extreme black holes was analyzed in [3]. Meanwhile, one important class of solutions with a zero surface gravity was overlooked in that study - solutions with a constant dilaton field. The existence of such kind of solutions was pointed out in [4] but for the classical case only. In the quantum case the similar solution was found [5] for the RST model [6].

The aim of the present paper is to show that the existence of solutions with a constant
dilaton field is a generic feature of 2D dilaton gravity. (Although I deal with the standard
form of the action generalization is straightforward.) Solutions under discussion are of
special interest since they describe the metric near a horizon of extreme 2D black holes
(as indicated in [4] with the reference to R. B. Mann’s observation). In this sense the
relationship between extreme 2D black holes and solutions in question resemble that between
the Bertotti-Robinson spacetime and extreme Reissner-Nordström 4D black holes (see, for example, [7] and references therein) that may promote a better understanding of what happens to a 4D black hole dressed by quantum fields near the extreme state. However, in contrast to the 4D case where the computation of the stress-energy tensor is a very complicated problem, in the 2D spacetimes it is known exactly. Moreover, it turns out that, as we will see below, field equations with backreaction taken into account are also solved for a constant dilaton field exactly to give a very simple result. It is worth stressing that, whereas the RST model for which solutions at hand were found in [5] is exactly solvable itself, we need not in what follows imposing such a restriction: in general field equations with inhomogeneous dilaton are not solvable but, nevertheless, ”degenerate” solutions under discussion can be found exactly.

In classical 2D dilaton gravity they may have one of the following forms [4]:

\[ ds^2 = -dt^2 \exp(-2\alpha t) + dl^2 \]

\[ ds^2 = -dt^2 \sinh^2 \alpha t + dl^2 \]

More exactly, there exist also solutions with \( \cosh^2 \alpha t \) at \( dt^2 \) but, as there is no horizon in that case we will not discuss them further. Here \( \alpha = \sqrt{\lambda} \) where \( \lambda > 0 \) is a cosmological constant. The metric 1 with the zero surface gravity corresponds to extreme black holes in the sense explained above. It is interesting that the metric 2 is intimately connected with the issue of extreme black holes too: as follows from results of [8], such a form of the metric arises in the extreme limit of nonextreme black holes. Thus, both metrics are related to the issue of the extreme state describing situation in both topological sectors. Below we will see that quantum effects (which are not supposed to be small) leave the qualitative character of 1, 2 unchanged but give rise to one more type of solutions which was absent in a classical theory.

Let a system be described by the action \( I = I_0 + I_q \) where \( I_0 \) is the standard action of 2D dilaton gravity and \( I_q \) is a Polyakov-Liouville action [9]. According to [1]
\[ I_0 = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^\Phi (R + L) \]

\[ L = (\nabla \Phi)^2 + \lambda - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \]

Here \( F_{\mu\nu} \) is electromagnetic field. We do not write down explicitly the boundary terms which are necessary for the variation procedure to be self-consistent. In two dimensions \( F_{\mu\nu} = Fe_{\mu\nu} \) where \( e_{\mu\nu} = e_{[\mu\nu]} \), \( e_{01} = \sqrt{-g} \). The electromagnetic-field equation reads

\[ (e^\Phi F)_{,\alpha} = 0 \]

(4)

The field equation which is obtained by varying \( \Phi \) has the form

\[ 2\Box \Phi + (\nabla \Phi)^2 - R - \lambda - F^2 = 0 \]

(5)

Varying a metric we obtain field equations \( T_{\mu\nu} = 0 \) where \( T_{\mu\nu} \equiv \frac{\delta I}{\delta g^{\mu\nu}} \). Explicit calculation gives for the classical part of \( T_{\mu\nu} \):

\[ T_{11}^{\text{cl}} = e^\Phi \frac{\pi}{\pi} (F^2 + U - \Phi_{,1}^1) \]

(6)

\[ T_{00}^{\text{cl}} = e^\Phi \frac{\pi}{\pi} (F^2 + U - \Phi_{,0}^0) \]

(7)

Here

\[ U = \Box \Phi + \frac{(\nabla \Phi)^2}{2} - \frac{\lambda}{2} - \frac{F^2}{2} \]

(8)

semicolon denotes covariant derivative. It follows from (3) that \( U = R/2 \).

We will look for static solution, so the metric can be represented in the form

\[ ds^2 = -dt^2 \mu^2 + dl^2 \]

(9)

where \( \mu = \mu(l) \). Then the Riemann curvature \( R = -2\mu''/\mu \). For solutions with a constant \( \Phi \)

\[ T_{11}^{\text{cl}} = \pi^{-1} e^\Phi (F^2 - \mu''/\mu) = T_{00}^{\text{cl}}. \]

(10)

Now let us make use the explicit expressions for the quantum part of \( T_{\mu\nu} \) in a static geometry which can be obtained from the Polyakov-Liouville action [9]:

\[ \]
\begin{align*}
T_{1q}^1 &= \frac{\kappa}{\pi} \left( \frac{\mu^2}{\mu^2} - \frac{4\pi^2 T_{H}^2}{\mu^2} \right), \\
T_{0q}^0 &= -T_{1q}^1 - \frac{\kappa R}{\pi}
\end{align*}

(11)

where the quantum parameter \( \kappa = N/24 \), \( N \) is a number of scalar fields in a multiplet.

Consider first solutions with a zero surface gravity (prototype of extreme black holes). Then, making substitution \( \mu = \exp(\sigma) \) one can infer form eqs. 10, 11 and \( T_1^1 - T_0^0 = 0 \) the equation \( \sigma'' = 0 \) whence \( \sigma = c l + d \), where \( c \) is a constant to be determined, without loss of generality one can put \( d = 0 \) by a proper rescaling time. Then according to 10, 11

\( T_{1cl}^1 = \pi^{-1} e^\Phi (F^2 - c^2), T_{1q}^1 = \pi^{-1} \kappa c^2 \). Taking into account eq. \( T_1^1 = 0 \) and eq. 5 we obtain

\begin{align*}
2c^2 - \lambda - F^2 &= 0 \\
F^2 - c^2 + \tilde{\kappa}c^2 &= 0
\end{align*}

(12)

where \( \tilde{\kappa} = \kappa e^{-\Phi} \). Solving 12 one easily obtains

\begin{align*}
c^2 &= \lambda (1 + \tilde{\kappa})^{-1} \\
F^2 &= \lambda (1 - \tilde{\kappa})(1 + \tilde{\kappa})^{-1} \\
R &= -2\lambda (1 + \tilde{\kappa})^{-1}
\end{align*}

(13)

It is seen from 13 that solutions under considerations exist only for \( \lambda > 0 \) as it takes place in the classical case 4. Meanwhile, a qualitatively new phenomenon occurs in a quantum domain when \( \tilde{\kappa} = 1 \): \( F^2 = 0 \) whereas for a classical system cancellation of electromagnetic field with a constant dilaton one implies that \( \lambda = 0 \), so spacetime is flat. Thus, in a strong coupling regime effects of back reaction make it possible the existence of the anti-de Sitter like solutions without electromagnetic field. If \( \tilde{\kappa} > 1 \) solutions under consideration are absent.

Let now the temperature be finite. It is convenient to normalize time in such a way that \( T_H = (2\pi)^{-1} \). Repeating calculations step by step, we obtain that formulae 13 for curvature and electromagnetic field hold true, the function \( \mu = c^{-1} \sinh c l \). Thus, in both considered cases quantum corrected geometry reads
\[ ds^2 = -e^{-2\lambda}dt^2 + dl^2 \]  
(quantum analog of 14) or

\[ ds^2 = -\frac{\sinh^2 cl}{c^2}dt^2 + dl^2 \]  
(quantum analog of 15). It is seen that quantum corrections do not change qualitatively the character of spacetime. In particular, the limiting transition of a black hole to the extreme state with a temperature finite in any point outside a horizon is performed in the same manner as for a bare hole. It is worth stressing that, while 14 or its classical counterpart 1 represent the metric of an extreme black hole only in the vicinity of a horizon [4] eq. 15 describes the metric of a nearly extreme black hole in the whole manifold as it follows from [8].

The most interesting result consists in the possibility to have \( \mu = c^{-1} \sin cl \). In this case

\[ ds^2 = -\frac{\sin^2 cl}{c^2}dt^2 + dl^2 \]  
(16)

\[ c^2 = -\lambda(1 + \kappa) \]

\[ R = -2\lambda(1 + \kappa)^{-1} \]

\[ F^2 = -\lambda(\kappa - 1)(\kappa + 1)^{-1} \]

These solutions exist only if \( \lambda < 0 \), \( \kappa \geq 1 \) and have no classical analog. From the physical viewpoint they represent two horizons (at \( l = 0 \) and \( l = \pi/c \)) in thermal equilibrium. It was shown in [10] that such a metric always arises when radii of a black hole and cosmological horizons coalesce (but the proper distance between them remains finite as seen from 16) that generalizes a series of observations made in some particular cases [11]. Now the curvature \( R > 0 \) (de Sitter type case). Thus, quantum effects change the structure of spacetime drastically. Although the solution at hand can exists only in a strong coupling regime and in this sense is pure quantum, the curvature of spacetime is classical in that it is proportional to the cosmological constant similarly to what takes place with constant dilaton field solutions.
in the RST model \([5]\). The quantum coupling parameter in our problem can take arbitrary values, the electromagnetic field strength being adjusted to it according to \([13]\) or \([16]\) whereas in the limit \(F = 0\) (as was assumed in \([5]\)), solutions under discussion exist only for one preferable value \(\tilde{\kappa} = 1\) in agreement with \([5]\) as it follows from \([13], [16]\). While for solutions \([14], [15]\) the energy density of quantum fields \(-\pi^{-1}\kappa c^2\) is negative for the spacetime \([16]\) it is positive and is equal to \(\pi^{-1}\kappa c^2\).

It is worth dwelling upon the following subtlety. When the dilaton is constant, the curvature term in Lagrangian becomes dynamically irrelevant because of topological invariance of the Ricci scalar in two dimensions, so one can wonder where not-flat solutions originate from. It is instructive in this respect to compare the classical and quantum solutions. In the first case the nontrivial contribution to field equations comes from the \(\lambda\) - term and electromagnetic field, so it is the electromagnetic force only which curves the geometry. As a result, \(R = -2F^2\) \([1]\) as seen from \([13]\), so in the limit \(F = 0\) the curvature goes to zero as well and the solution is necessarily flat. In the quantum case, however, account for the Polyakov - Liouville action brings about the new contribution to field equations: back reaction of quantum fields. As a result, the curvature is due to either the electromagnetic force or back reaction.

It is worthwhile to note that for all values of the coupling parameter \(\tilde{\kappa} \neq 1\) the regions with \(\tilde{\kappa} < 1\) and \(\tilde{\kappa} > 1\) are strongly separated: the metric under discussion can be only of the anti-de Sitter type in the first case and only of the de Sitter one in the second case. However, the point \(\tilde{\kappa} = 1\) is exceptional in the sense that either the de Sitter or anti-de Sitter solution may exist depending on the sign of \(\lambda\). As in that case \(F = 0\), the nonzero curvature arises due to the back reaction of quantum fields only.

All metrics considered above arise in a self-consistent way, with one-loop effects of back reaction of quantum fields taken into account exactly. They possess the following attractive feature: whereas on a horizon of a generic two-dimensional black hole weak divergencies of stress-energy tensors inevitably occur \([3], [12]\) such divergencies are absent for spacetimes at
hand. Indeed, the relevant quantity $\mu^{-2}(T_{1q}^1 - T_{0q}^0)$ vanishes for all three solutions \[12\]. It gives some hint that such degenerate solutions could be suitable candidates to the final state of a black hole after evaporation of it. The considered examples can also serve as a toy model for a more complicated question about properties of four-dimensional extreme and nearly extreme black holes dressed by one-loop effects of quantum backreaction.

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