Quantum affine algebras at small root of unity

Simon Lentner
Algebra and Number Theory, University Hamburg,
Bundesstraße 55, D-20146 Hamburg
simon.lentner@uni-hamburg.de

Abstract. We study the Frobenius-Lusztig kernel for quantum affine algebras at root of unity of small orders that are usually excluded in literature. These cases are somewhat degenerate and we find that the kernel is in fact mostly related to different affine Lie algebras, some even of larger rank, that exceptionally sit inside the quantum affine algebra. This continues the authors study for quantum groups associated to finite-dimensional Lie algebras in [Len14c].

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1. Introduction

A quantum affine algebra $U_q(g)$ is a Hopf algebra over the field of rational functions $\mathbb{C}(q)$ and can be viewed as a deformation of the universal enveloping of an affine Lie algebra $g$. It is a special case of the Drinfel’d-Jimbo quantum group and the next logical step after quantum groups associated to finite-dimensional Lie algebras $g$. Lusztig has in [Lusz94] studied an integral form in this situation, which is a Hopf algebra $U_q^{\mathbb{Z}[q,q^{-1}]}(g)$ over the ring $\mathbb{Z}[q,q^{-1}]$, and one may again perform a specialization to a specific value $q \in \mathbb{C}^\times$ and thus obtain a complex Hopf algebra $U_q(g)$. For $q$ not a root of unity these algebras behave similar to $U_q(g)$ but for $q$ an $\ell$-th root of unity, the algebra and its representation theory becomes significantly more interesting. The theory of quantum affine algebras is much less developed than the theory for finite Lie algebras $g$ - Lusztig proves several structural results in [Lusz94] with certain restrictions on the order of $q$; among others he defined a “small quantum group” $Ru$ (which is now infinite-dimensional) and a Frobenius-homomorphism to the ordinary universal enveloping algebra of $g$.

Notable subsequent results were a PBW-basis established in [Beck94] and a complete description of the representation theory of the Drinfel’d Jimbo quantum group in [CP95] and for the specialization in [CP97], both for $g$ an untwisted affine Lie algebra. Among others they prove a factorization theorem of representations into representations of the ordinary universal enveloping algebra of $g$ and representations with highest weight “less then $\ell$”. This is analogous to the respective results for finite-dimensional $g$ (and to the Steinberg factorization theorem for Lie groups over finite fields) and closely related to the Frobenius homomorphism.

In [CP98] the study was extended to twisted affine Lie algebras. As an application, the authors mention that e.g. the symmetry of the affine Toda field theories are governed by quantum affine algebras and they voice the hope that their study could help understand affine Toda theories with certain specific values of the coupling constant.

The aim of this article is to clarify the Hopf algebra structure of the restricted specialization $U_q^\mathbb{Z}(g)$ to such values $q$ where Lusztig’s restrictions on $q$ are violated. For these small roots of unity, the quantum group severely degenerates. The author has in [Len14c] performed such a study already in the case of $g$ a finite Lie algebra and found a similar Frobenius homomorphism to a universal enveloping algebra.
In these degenerate cases, neither the kernel nor the image of the Frobenius homomorphism are associated to the initial Lie algebra $\mathfrak{g}$. As an application we noted in [Len14c] that e.g. the case $\mathfrak{g} = B_n, \ell = 4$ seem to be closely related to the vertex algebra of $n$ symplectic fermions; here the kernel of the Frobenius homomorphism is a small quantum group of type $A_1 \times n$ and the image is the universal enveloping of $C_n = sp_{2n}$. It is to be expected that a similar study for affine $\mathfrak{g}$ would explain exceptional behaviour for affine Toda field theories at certain small values of the coupling constants.

In this article we concentrate on the study of the kernel of the (yet-to-be constructed) Frobenius homomorphism for affine $\mathfrak{g}$. More precisely, we define a suitable Hopf subalgebra $u^\ell_q(\mathfrak{g})$ and describe its structure. It turns out to be mostly governed by subsystems of the dual root system, while in several exotic cases the quantum group itself changes into a quantum group associated to a different Lie algebra of larger rank. In one set of cases it even collapses to an infinite tower of quantum groups of finite Lie algebras. Altogether we find:

**Theorem (4.2).** Let $\mathfrak{g}$ be an affine Lie algebra and $q$ and $\ell$-th root of unity. We shall define a Hopf subalgebra $u^\ell_q(\mathfrak{g})^+ \subset U^\ell_q(\mathfrak{g})^+$ with the following properties:

- $u^\ell_q(\mathfrak{g})^+$ consists of all degrees (roots) $\alpha$ with $\ell_\alpha \neq 1$.
- Except for cases marked deaffinized, $u^\ell_q(\mathfrak{g})^+$ is generated by primitives $E_{\alpha_i(0)}$, spanning a braided vector space $M$ (in the deaffinized cases $u^\ell_q(\mathfrak{g})^+$ is an infinite extension tower, see Section 7).
- Lusztig’s $R \subset u^\ell_q(\mathfrak{g})^+$ and equality only holds for trivial and generic cases.
- Except for cases marked deaffinized and (possibly) exotic $u^\ell_q(\mathfrak{g})^+$ is coradically graded and hence maps onto the Nichols algebra $\mathcal{B}(M) = u_q(\mathfrak{g}^{(0)})^+$.

For untwisted affine $\mathfrak{g}$ we prove (and else conjecture) they are isomorphic.

- Except for cases marked deaffinized, the $u^\ell_q(\mathfrak{g})^+$ fulfills $\ell_i \neq 1$ as well as Lusztig’s non-degeneracy condition $\ell_\alpha_i \geq -a_{ij} + 1$. Hence we have determined subalgebras on which Lusztig’s theory in [Lusz94] may be applied.

We explicitly describe the type $\mathfrak{g}^{(0)}, q'$ of $M, \mathcal{B}(M)$ as follows:

| $\mathfrak{g}$ | $\ell$ | $M$ | $q'$ | comment |
|----------------|--------|-----|-----|---------|
| $A_1^{(1)}$    | $\ell = 4$ | $A_2^{2x}$ | $q$ | deaffinized |
| $B_n^{(1)}, D_{n+1}^{(2)}$ | $\ell = 4$ | $A_4^{2n+1}$ | $q$ | short roots, deaffinized |
| $C_n^{(1)}, A_{2(n-1)}^{(2)}$ | $\ell = 4$ | $D_n^{(1)}$ | $q$ | short roots |
| $F_4^{(1)}, E_6^{(2)}$ | $\ell = 4$ | $D_4^{(1)}$ | $q$ | short roots |
| $G_2^{(2)}, D_4^{(3)}$ | $\ell = 3, 6$ | $A_2^{(1)}$ | $q$ | short roots |

(continues on next page)
\begin{tabular}{|c|c|c|}
\hline
$A_{1}^{(2)}$ & $\ell = 4$ & $A_{1}^{2}\times q$ \textit{very short roots, deaffinized} \\
$A_{2}^{(1)}$ & $\ell = 8$ & $A_{1}^{2n}\times q$ \textit{very short roots} \\
$A_{2}^{(2)}$ & $\ell = 4$ & $A_{1}^{2n}\times q$ \textit{very short roots, deaffinized} \\
$A_{2n}^{(2)}$ & $\ell = 8$ & $A_{2n-1}^{(2)} q$ \textit{not very long roots} \\
$A_{1}^{(2)}$ & 3 & $A_{2}^{(1)} q$ \textit{exotic} \\
$A_{2}^{(2)}$ & 6 & $A_{2}^{(1)} -q$ \textit{exotic} \\
$A_{2}^{(2)}$ & 3, 6 & $A_{2}^{(2)} q$ \textit{(pseudo-)exotic} \\
$G_{2}^{(1)}$ & 4 & $A_{3}^{(1)} \bar{q}$ \textit{exotic} \\
$D_{4}^{(3)}$ & 4 & $D_{4}^{(1)} q, \bar{q}, -1$ \textit{exotic} \\
\hline
\end{tabular}

All cases not included in the list are generic cases $g^{(0)} = g$ with $u_{q}^{\ell}(g)^{+} = R^{\ell}u^{+}$.

We shall now discuss the approach and results of this article in more detail:

Lusztig defines in [Lusz94] Chp. 36 the subalgebra $Ru$ to be generated by all $E_{\alpha_{i}}, \ell_{\alpha_{i}} \neq 1$ and works under the additional restriction

$$\ell_{\alpha_{j}} \geq 2 \Rightarrow \ell_{\alpha_{i}} \geq -a_{ij} + 1 \quad (*)$$

(and $g$ not containing odd cycles). Then he establishes a Frobenius homomorphism with kernel $Ru$. Note that in contrast in the finite case he had in [Lusz90b] Thm. 8.3 defined $u$ without restrictions on $q$ as being generated by all root vectors $E_{\alpha}$ with $\ell_{\alpha} \neq 1$, but has refrained from doing so in the affine case by lack of root vectors (it would be interesting to now use the results in [Beck94]). Note that frequently already in the finite case $u$ violating $\ell_{\alpha_{i}} \neq 1$ is not generated by simple root vectors.

The aim of this article is to nevertheless find a suitable subalgebra $u_{q}^{\ell}$ as in [Lusz90b] and more importantly calculate the structure of $u_{q}^{\ell}$ in all cases violating either $\ell_{\alpha_{i}} \neq 1$ or (*). In the first case the type of $u_{q}^{\ell}$ will be determined by a subset of roots (mostly the short roots), which will be characterized by a subsystem of the dual root system. In the second exotic cases the root system severely changes (compare the only finite example $G_{2}, \ell = 4$).

In Section 3 we determine the subsystems of the affine root systems consisting of all roots divisible by a fixed integer. This extends results in e.g. [Car05] Prop. 8.13. in the case of a finite $g$ and is responsible for most of the root system data in the main theorem.
In Section 4 we formulate the main theorem and prove the general statements. The final case-by-case analysis is performed in the remaining sections 5-7.

In Section 5 we consider all degenerate cases where some simple roots fail $\ell_{\alpha_i} \neq 1$; this leaves only a subset of generators $E_{\alpha_i}^{(0)} = E_{\alpha_i}$ for $u_q^\ell(\mathfrak{g})$. We first give a general approach to find more exceptionally primitive elements $E_{\alpha_i}^{(0)}$ for small $q$. Namely, the Lusztig reflection operator is still defined with respect to the root system of $\mathfrak{g}$, whereas the “true” root system generated by simple root vectors is now smaller. The images of such “inappropriate” reflections turn out to be new primitives. We then give a generic argument that shows the dual subsystems found in Section 3 characterizes the quantum affine algebra generated by all $E_{\alpha_i}^{(0)}$.

In Section 6 we turn to the exotic cases where all $\ell_{\alpha_i} \neq 1$, so all $E_{\alpha_i}^{(0)} = E_{\alpha_i}$, but the second condition (*) is violated. These cases are very interesting, because the $E_{\alpha_i}$ only generate a (smaller) different quantum affine algebra than expected, so one again has to add additional primitives and the rank is now larger than $\mathfrak{g}$. A similar phenomenon has been observer by the author already in the case $\mathfrak{g} = G_2, \ell = 4$, where the corresponding $u_q^\ell(\mathfrak{g})^+$ is isomorphic to $u_q^\ell(A_3)^+$. Approach these cases one-by-one and apply the theory of Nichols algebras to determine the subalgebra generated by the $E_{\alpha_i}$. Then we find additional primitives (sometimes using techniques from Section 5, some by guess-and-check), until we finally account for all roots with $\ell_{\alpha_i} \neq 1$.

We finally turn to cases where Section 7 returns one or more copies of $A_1^{(1)}, \ell = 4$. One could accept this result, but it again violates (*), so we study it a bit further. Surprisingly this is the most misbehaved case: The $E_{\alpha_0}, E_{\alpha_1}$ span a braided vector space of type $A_1 \times A_1$. They generate a (quasi-) classical universal enveloping of type $A_2$ (so there is so-called nontrivial liftings). The entire algebra $u_q(A_1^{(1)})$ can be described as an infinite tower of extensions by other $A_2$ algebras. We can only make such observations by using Drinfel’d alternative generating system as determined by [Beck94], which views the algebra as an explicit affinization (not a mere Cartan matrix).

We close in section 8 by stating some open questions that were out of the scope of this paper.
2. Preliminaries

2.1. Affine Lie algebras. Our exposition is largely from [Kac84] Sec. 1. Affine Lie algebras are characterized by the fact that the Cartan matrix is positive semi-definite and there is a unique isotropic root \( \delta \mathbb{Z} \) of length 0, which is clearly not in the Weyl group orbit of any simple roots. Proper parabolic subsystems always correspond to finite dimensional Lie algebras and there is a common choice of such a parabolic subsystem \( \{ \alpha_1, \ldots, \alpha_{\text{rank}(g)} - 1 \} \) corresponding to a finite root system \( \bar{\Delta} \) which is extended by an additional simple root \( \alpha_0 \). Affine Lie algebras are classified: The so-called twisted affine Lie algebras \( X_n^{(1)} = \hat{X}_n \) are obtained by extending a finite Lie algebra of type \( X_n \) by the negative highest root \( \alpha_0 \); they can be realized by centrally extending the loop algebra \( X_n \otimes \mathbb{C}[t, t^{-1}] \). The other so-called twisted affine Lie algebras can be realized similarly by an extension involving an outer automorphism of \( X_n \); this is what Kac’s notation \( X_n^{(2)}, D_4^{(3)} \) refers to, which is well established especially in physics. Other authors such as Carter, Fuchs, etc. denote the twisted affine Lie algebras in a way that emphasizes the similar Weyl group and (equivalently) how a parabolic finite Lie algebra with root system \( \bar{\Delta} \) is extended by a different \( \alpha_0 \) then in the untwisted case. For example, both \( G_2^{(1)}, D_4^{(3)} \) have affine Weyl groups \( G_2 \) and can be obtained from extending \( \bar{\Delta} = G_2 \). The author would prefer the second notation, but sticks with the more common one.

Another less direct construction the author finds convenient (and may not be new) is to obtain affine Lie algebras from simply-laced untwisted Lie algebras by the folding procedure: Let \( g \) be a Lie algebra and \( f \) an automorphism of the Dynkin diagram, then we may consider the Lie subalgebra \( g' \) fixed by \( f \). Note this is not the same as the twisted realizations using an automorphism of the finite root system!

Note also that as in the finite case one may form (in non-simply-laced cases) the dual root system consisting of rescaled coroots \( \alpha_i^\vee = \frac{2}{(\alpha_i, \alpha_i)} \alpha_i \), which switches long and short roots. We summarize all Dynkin diagrams and then all properties:
Two important numbers associated to the affine Lie algebra is the superscript number \( k = (1), (2), (3) \) and the number \( a_0 = 1 \) for all cases except \( A_2^{(2)}, A_2^{(2)} \) have \( a_0 = 2 \). We will frequently distinguish the cases \( a_0^k = 1, 2, 3, 4 \). We denote \( n = \text{rank}(\bar{\Delta}) = \text{rank}(\Delta) − 1 \). We summarize the description of the root system of the affine Lie algebras from [Kac84] Sec. 1.4:

The isotropic roots (i.e. length 0 and hence not in the Weyl orbit of a simple root) are

\[ \Delta^{\text{im}} = \delta \mathbb{Z} \setminus \{0\}, \quad \delta := a_0 \alpha_0 + \theta \]

where \( \theta \) is the highest root of \( \bar{\Delta} \) for \( a_0 k = 1, 4 \) and the highest short root for \( a_0 k = 2, 3 \). Hence it is often convenient to draw \( \bar{\Delta} \) in the root system \( \Delta \) by projecting \( \delta = 0 \) and hence drawing \( \alpha_0 = -a_0^{-1} \theta \).

The multiplicity of each root \( m\delta \) is \( n = \text{rank}(\bar{\Delta}) \) with the following exceptions:

\[ \dagger \text{A very remarkable fact is that the simply-laced } A_1^{(1)} \text{ of rank } n+1 \text{ can be folded from } A_k^{(1)}(n+1)−1 \text{ of rank } k(n+1) \text{ for any } k \text{ via a rotation of order } k. \text{ As a consequence, there is also an infinite series of folding for } A_1^{(1)} \text{ as well as an infinite series of folding } C_1^{(1)} \text{ of rank } n+1 \text{ from } C_2^{(1)} \text{ of rank } 2(n+1)−1. \]
| g       | $A_{2n-1}^{(2)}$ | $D_{n+1}^{(2)}$ | $E_6^{(2)}$ | $D_4^{(3)}$ |
|---------|-----------------|-----------------|-------------|-------------|
| m       | 2 ↗ m           | 2 ↗ m           | 2 ↗ m       | 3 ↗ m       |
| mult$(m\delta)$ | $n-1$        | 1               | 2           | 1           |

The real roots (i.e. in the Weyl orbit) $\Delta_{re}$ are as follows

- $a_0k = 1$, $\Delta_{re} = \bar{\Delta} + \delta \mathbb{Z}$
- $a_0k = 2, 3$, $\Delta_{re} = (\bar{\Delta}_{\text{short}} + \delta \mathbb{Z}) \cup (\bar{\Delta}_{\text{long}} + \delta k \mathbb{Z})$
- $a_0k = 4$, $\Delta_{re} = (\bar{\Delta}_{\text{short}} + \delta \mathbb{Z}) \cup (\bar{\Delta}_{\text{long}} + \delta k \mathbb{Z}) \cup \left(\frac{1}{2}(\bar{\Delta}_{\text{long}} + \delta) + \delta \mathbb{Z}\right)$

and all real roots have multiplicity 1.

2.2. Affine quantum groups. In [Lusz94] Sec. 1.2 Lusztig defines a Hopf algebra $f$ over $\mathbb{Q}(q)$ associated to a Cartan datum. In modern terminology, consider the category $\mathcal{C}_Q^{(q)}$ of $\mathbb{N}$-graded $\mathbb{Q}(q)$-vector spaces $V$ with braiding on homogeneous elements $x_\alpha \otimes y_\beta \mapsto q^{(\alpha, \beta)}$. Let $V \in \mathcal{C}_Q^{(q)}$ be the vector space spanned by symbols $E_\alpha$.

Let $f' = TV$ be the tensor algebra which becomes a Hopf algebra in the braided category $\mathcal{C}_Q^{(q)}$ by defining $\Delta(E_\alpha) = 1 \otimes E_\alpha + E_\alpha \otimes 1$. Then, there is a unique symmetric Hopf pairing defined by

$$(E_\alpha, E_\beta) = \delta_{\alpha, \beta} (1 - q^{-(\alpha, \alpha)})^{-1}$$

Now let $I$ be the radical of the Hopf pairing, then define the Hopf algebra

$$f := f'/I$$

In Sec. 36 an integral form, i.e. a Hopf algebra $\mathcal{A}f$ over the ring $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$ is defined and then the specialization $Rf$ restricted to a specific value $q \in \mathbb{C}^\times$ via $\otimes_\mathcal{A} R$ where we may take $R = \mathbb{C}_q$ the field $\mathbb{C}$ with $q$ acting by the specified value. Note this is only possible for good values $q^{(\alpha, \alpha)} \neq 1$ and leads to what we today call the Nichols algebra $\mathcal{B}(V)$. These restrictions are always in place throughout the reminder of [Lusz94].

On the other hand in Sec. 3 Lusztig proceeds, without any restrictions on $q$, as in [Lusz90a, Lusz90b] for finite Cartan datum. He takes the Drinfel’d-Jimbo quantum group $U$ over $\mathbb{Q}(q)$, defines an integral form (restricted form) $\mathcal{A}U$ over the ring $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$ and performs again specialization $RU$. We again take $R = \mathbb{C}_q$ and denote this Hopf algebra over $\mathbb{C}$ by $U_q^{\mathfrak{g}}(\mathfrak{g})$. Moreover, the Borel part $U_q^{\mathfrak{g}}(\mathfrak{g})^+$ is again a Hopf algebra in the Yetter-Drinfel’d modules over the root lattice $\Lambda_R$ of...
g. Compare the authors introduction in [Len14c].

From this point the case of affine Lie algebras start to exhibit an incomplete picture. Lusztig does not prove that reflection provides a PBW-basis, see Lusztig’s respective question in [Lusz94] Sec. 40.2.3, although he gives a sketch how such a fact might be proven, using an infinite sequence of Weyl group elements of ascending length. He establishes a Frobenius homomorphism if the root of unity fulfills the restrictions, namely \( q^{(\alpha_i, \alpha_i)} \neq 1 \) and the Lie algebra Dynkin diagram does not contain odd cycles. He shows that under these restrictions on \( q \) the kernel of the Frobenius homomorphism, which we call today Frobenius-Lusztig kernel, is generated by all \( E_{\alpha_i} \) with \( q^{(\alpha_i, \alpha_i)} \).

Note that in the completely worked out case in [Lusz90a] [Lusz90b] for finite root systems the Frobenius-Lusztig kernel is shown to be generated by all root vectors \( E_{\alpha}, q^{(\alpha, \alpha)} \neq 1 \). As the author has worked out in [Len14c], this Hopf algebra is for arbitrary \( q \) associated to a different Lie algebra and in one exotic case \( G_2, q = \pm i \) not even generated just by the simple root vectors \( E_{\alpha_i} \). The author also provides a different proof for the Frobenius homomorphism that includes arbitrary \( q \).

The aim of this article is to consider all the affine Lie algebras \( g \) and the roots of unity \( q \) that violate the condition \( q^{(\alpha_i, \alpha_i)} \neq 1 \) and realize the affine Frobenius-Lusztig kernel (of usually different Lie type) inside \( U_L^q(\mathfrak{g})^+ \) that corresponds precisely to the set of all roots \( q^{(\alpha, \alpha)} \neq 1 \).

3. THE SUBSYSTEM OF LONG ROOTS

A natural question for any root system \( \Delta \) is the subsystem of roots divisible by a fixed number \( t \in \mathbb{N} \), though for affine Lie algebras the author has not found this question addressed in literature explicitly. The aim of this section is to determine this subsystem. We note already at this point, that the structure of the Frobenius-Lusztig kernel will often (but not always) be described by the set of short roots, hence the set of long roots in the dual root system. In the finite case treated by the author in [Len14c] these were in fact all but a single exotic case \( G_2, q = \pm i \).
Lemma 3.1. Let $\Delta$ be a root system and $t \in \mathbb{N}$, then the subset
$$\Delta^t := \{ \alpha \in \Delta \mid t(\alpha, \alpha) \}$$
is a root system. Note that $\Delta^2 = \Delta$ and arbitrary $\Delta^t$ may be empty. By convention, isotropic roots are contained in any $\Delta^t$.

Proof. We only have to prove that for $\alpha, \beta \in \Delta^t$ and $\alpha + \beta \in \Delta$ we have $\alpha + \beta \in \Delta^t$:
$$(\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + (\beta, \beta) + 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)}(\alpha, \alpha) \in t\mathbb{Z}$$since the Cartan matrix $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

We consider the cases $\Delta^t = \Delta$ generic and all cases with $\Delta^t$ only isotropic roots trivial. Hence for simply-laced root systems there are only generic and trivial cases.

3.1. Finite root systems. For completeness and later use we first we give the table with all nontrivial/nongeneric cases for finite root systems $\Delta$, which has been implicitly already used in [Len14c] and has can be found in [Car05] Prop. 8.13. , including the new set $\Pi^t$ of positive simple roots $\alpha^t_i$ for $\Delta^t$:

Lemma 3.2. Let $\Delta$ be a connected finite root system, then all $\Delta^t$ are equal to $\Delta$ or empty except the following cases:

| $\Delta$ | $t$  | $\Delta^t$                                  | $\Pi^t$            |
|----------|------|---------------------------------------------|--------------------|
| $B_n$    | 4    | $D_n$ $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n$ | $\alpha^t_i$       |
| $C_n$    | 4    | $A_1^{\infty}$ $\alpha_n, \alpha_n + 2\alpha_{n-1}, \alpha_n + 2\alpha_{n-1} + 2\alpha_{n-2}, \ldots$ |                      |
| $F_4$    | 4    | $D_4$ $\alpha_1, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4$ |                      |
| $G_2$    | 3, 6 | $A_2$ $\alpha_1, \alpha_1 + 3\alpha_2$     | $\Pi^t$            |

Note that our choice of $\Pi^t$ is minimal in the sense that any $\alpha^t_i \in \Pi^t$ contains precisely one simple root $\Delta^t \cap \Pi$ and only with multiplicity one.

Proof. Since all root lengths are 2, 4 resp. 2, 6 we only have to consider the cases in the statement. We proceed case-by-case and first show that the simple roots $\alpha^t_i$ indeed have the claimed new Cartan matrix; then we show by counting that we have indeed found all new roots.

- Let $\Delta = B_n$ and $t = 4$. The simple roots $\alpha^t_i := \alpha_i, \ 1 \leq i \leq n - 1$ are indeed long and have a Cartan matrix of type $A_{n-1}$. For the root $\alpha^t_n := \alpha_{n-1} + 2\alpha_n$
we check
\[(\alpha_n^t, \alpha_n^t) = (\alpha_{n-1}, \alpha_{n-1}) + 4(\alpha_n, \alpha_{n-1}) + 4(\alpha_n, \alpha_n) = 4 - 8 + 8 = 4\]
\[(\alpha_n^t, \alpha_{n-1}^t) = (\alpha_{n-1}, \alpha_{n-1}) + 2(\alpha_n, \alpha_{n-1}) = 4 - 4 = 0\]
\[(\alpha_n^t, \alpha_{n-2}^t) = (\alpha_{n-1}, \alpha_{n-2}) + 2(\alpha_n, \alpha_{n-2}) = -2\]
\[(\alpha_n^t, \alpha_{i<n-2}^t) = 0\]

This is the Cartan matrix of \(D_n\) with center node \(\alpha_{n-2}\) as claimed. It is known that \(B_n\) has \(2n^2\) roots and \(2n(n - 1)\) long roots. Since \(D_n\) has also \(2n(n - 1)\) roots we see that \(\Delta^t = D_n\).

• Let \(\Delta = C_n\) and \(t = 4\). We check that all roots \(\alpha_i^t = \alpha_n + 2\alpha_n \cdots + 2\alpha_{n-i+1}\) by induction:

\[(\alpha_1^t, \alpha_1^t) = (\alpha_n, \alpha_n) = 4\]
\[(\alpha_{i+1}^t, \alpha_{i+1}^t) = (\alpha_i^t, \alpha_i^t) + 4(\alpha_{n-i}, \alpha_i^t) + 4(\alpha_i, \alpha_i)\]
\[= (\alpha_i^t, \alpha_i^t) + 4(\alpha_{n-i}, \alpha_n + \cdots + 2\alpha_{n-i+1}) + 4(\alpha_i, \alpha_i)\]
\[= 4 - 8 + 8 = 4\]

Next we convince ourselves that all \(\alpha_i^t\) have a \(A_1^{\times n}\) Cartan matrix i.e. are orthogonal (this does not mean \(\alpha_i^t + \alpha_j^t \notin \Delta\), let \(i > j\):

\[(\alpha_i^t, \alpha_j^t) = (\alpha_j^t, \alpha_j^t) + (2\alpha_{n-j} + \cdots + 2\alpha_{n-i} + 1, \alpha_i^t) = 4 - 4 = 0\]

Hence the subsystem generated by the \(\alpha_i^t\) is indeed of type \(A_1^{\times n}\). It is known that \(C_n\) has \(2n^2\) roots and \(2n\) long roots. Since \(A_1^{\times n}\) has also \(2n\) roots we have \(\Delta^t = A_1^{\times n}\).

• Let \(\Delta = F_4\) and \(t = 4\). We again calculate the Cartan matrix of \(\alpha_i^t := \alpha_1, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\) as follows:

\[(\alpha_1^t, \alpha_1^t) = (\alpha_1, \alpha_1) = 4\]
\[(\alpha_1^t, \alpha_2^t) = (\alpha_1, \alpha_2) = -2\]
\[(\alpha_1^t, \alpha_3^t) = (\alpha_1, \alpha_2 + 2\alpha_3) = -2\]
\[(\alpha_1^t, \alpha_4^t) = (\alpha_1, \alpha_2 + 2\alpha_3 + 2\alpha_4) = -2\]
\[(\alpha_2^t, \alpha_2^t) = (\alpha_2, \alpha_2) = 4\]
\[(\alpha_2^t, \alpha_3^t) = (\alpha_2, \alpha_2 + 2\alpha_3) = 4 - 4 = 0\]
\[(\alpha_2^t, \alpha_4^t) = (\alpha_2, \alpha_2 + 2\alpha_3 + 2\alpha_4) = 0\]
\[(\alpha_t^3, \alpha_t^3) = (\alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3)\]
\[= (\alpha_2, \alpha_2) + 4(\alpha_2, \alpha_3) + 4(\alpha_3, \alpha_3) = 4 - 8 + 8 = 4\]
\[(\alpha_t^3, \alpha_t^1) = (\alpha_t^3, \alpha_t^3) + (\alpha_2 + 2\alpha_3, 2\alpha_4) = 4 - 4 = 0\]
\[(\alpha_t^4, \alpha_t^1) = (\alpha_t^3, \alpha_t^3) + 2(\alpha_2 + 2\alpha_3, 2\alpha_4) + 4(\alpha_4, \alpha_4) = 4 - 8 + 8 = 4\]

Hence the set of roots \(\alpha_t^i\) generate a (rescaled) subsystem of type \(D_4\) with center node \(\alpha_t^1\) as claimed. It is known that \(F_4\) has 48 roots and 24 long roots. Since \(D_4\) has also \(2n(n - 1) = 24\) roots we have \(\Delta^t = D_4\).

- Let \(\Delta = G_2\) and \(t = 3, 6\). We again calculate the Cartan matrix of the short roots \(\alpha_t^i = \alpha_1, \alpha_1 + 3\alpha_2\) as follows:
\[(\alpha_1^t, \alpha_1^t) = (\alpha_1, \alpha_1) = 6\]
\[(\alpha_1^t, \alpha_2^t) = (\alpha_1, \alpha_1) + 3(\alpha_1, \alpha_2) = 6 - 9 = -3\]
\[(\alpha_2^t, \alpha_2^t) = (\alpha_1, \alpha_1) + 6(\alpha_1, \alpha_2) + 9(\alpha_2, \alpha_2) = 6 - 18 + 18 = 6\]

Hence the set of roots \(\alpha_t^i\) generate a (rescaled) subsystem of type \(A_2\) as claimed. It is known that \(G_2\) has 6 roots and 3 long roots. Since \(A_2\) has also 3 roots we have \(\Delta^t = A_2\).
3.2. Affine root systems. We now determine the nontrivial/nongeneric cases for the affine Lie algebras without considering the isotropic roots.

Theorem 3.3. Let $\Delta$ be a connected affine root system, then all $\Delta^t$ are equal to $\Delta$ or only consist of isotropic roots except the following cases:

| $\Delta$      | $t$ | $\Delta^t$                                                                 | $\Pi^t$                      |
|--------------|-----|---------------------------------------------------------------------------|-------------------------------|
| $B_n^{(1)}$  | 4   | $D_n^{(1)}$ $\times_n \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n$ | $\Pi^t$                      |
| $C_n^{(1)}$  | 4   | $\left( A_1^{(1)} \right)^n \alpha_n, \alpha_n + 2\alpha_{n-1}, \alpha_n + 2\alpha_{n-1} + 2\alpha_{n-2}, \cdots$ | $\alpha_0, \alpha_0 + 2\alpha_1, \alpha_0 + 2\alpha_1 + 2\alpha_2, \cdots$ |
| $F_4^{(1)}$  | 4   | $D_4^{(1)}$ $\alpha_0, \alpha_1, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4$ | $\alpha_0, \alpha_1, \alpha_1 + 3\alpha_2$ |
| $G_2^{(1)}$  | 3, 6| $A_2^{(1)}$ $\alpha_0, \alpha_1, \alpha_1 + 3\alpha_2$ | $\alpha_0, \alpha_1, \alpha_1 + 3\alpha_2$ |
| $D_n^{(2)}$  | 4   | $D_n^{(1)}$ $\alpha_0', \alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n$ | $\alpha_0 := 2\alpha_0 + \alpha_1$ |
| $A_2^{(2)}$  | 4   | $\left( A_1^{(1)} \right)^n \alpha_n, \alpha_n + 2\alpha_{n-1}, \alpha_n + 2\alpha_{n-1} + 2\alpha_{n-2}, \cdots$ | $\alpha_0', \alpha_0' + 2\alpha_1, \alpha_0' + 2\alpha_1 + 2\alpha_2, \cdots$ |
| $E_6^{(2)}$  | 4   | $D_4^{(1)}$ $\alpha_0', \alpha_4, \alpha_3, \alpha_3 + 2\alpha_2, \alpha_3 + 2\alpha_2 + 2\alpha_1$ | $\alpha_0 := 2\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3$ |
| $D_4^{(3)}$  | 3, 6| $A_2^{(1)}$ $\alpha_0', \alpha_2, \alpha_2 + 3\alpha_1$ | $\alpha_0 := 3\alpha_0 + 3\alpha_1 + \alpha_2$ |
| $A_2^{(2)}$  | 4, 8| $A_1^{(1)}$ $\alpha_0 + \alpha_1, \alpha_1$ | $\alpha_0 := 2\alpha_0 + \alpha_1$ |
| $A_2^{(2)}$  | 4   | $A_2^{(2)}$ $\alpha_0', \alpha_1, \cdots, \alpha_1$ | $\alpha_0 := 2\alpha_0 + \alpha_1$ |
| $A_2^{(2)}$  | 8   | $\left( A_1^{(1)} \right)^n \alpha_n, \alpha_n + 2\alpha_{n-1}, \alpha_n + 2\alpha_{n-1} + 2\alpha_{n-2}, \cdots$ | $\alpha_0', \alpha_0' + 2\alpha_1, \alpha_0' + 2\alpha_1 + 2\alpha_2, \cdots$ |

Note that our choice of $\Pi^t$ is minimal in the sense that any $\alpha_i \in \Pi^t$ contains precisely one simple root $\Delta^t \cap \Pi$ and only with multiplicity one.

Remark 3.4. With the exception of $A_2^{(2)}$ the subsystem $\Delta^t$ turns out to be the affinization of the respective subsystem of the finite root system $\bar{\Delta}^t$ (this is more serious for the disconnected cases in the second and last row, where $\Delta^t$ has higher rank $2n$). Also it coincides whenever $\bar{\Delta}$ coincides. It would be nice to find a more
systematic reason for this. Our proof uses a nice inclusion between the respective root systems, which may be known.

Proof. All root lengths are 2, 4 resp. 2, 6 except 2, 8 for \( A_2^{(2)} \) and 2, 4, 8 for \( A_{2n}^{(2)} \), hence we only have to consider the cases in the statement. We proceed similarly as (and using heavily) Lemma 3.2. Namely, we “guess” a set of positive simple roots \( \Pi^t \subset \Delta^t \), calculate the root system generated by \( \Pi^t \) and compare to the set \( \Delta^t \) to show we indeed have found all. The set \( \Delta^t \) can be easily read off the explicit form of \( \Delta^t \) in Subsection 2.1, so we distinguish the three cases for \( a_0 k \). Note from the Dynkin diagrams, that roots in \( \bar{\Delta}^t \) have the same length as in \( \Delta^t \) (no rescaling) except for \( A_{2}^{(2)} \) and \( A_{2n}^{(2)} \), which is the case \( a_0 k = 4 \).

Let \( a_0 k = 1 \), i.e. \( \Delta \) an untwisted affine Lie algebra. Then \( \Delta^{re} = \bar{\Delta} + \delta \mathbb{Z} \) and we only have to check \( t = 4 \). Obviously \( \alpha + \delta \mathbb{Z} \) with \( \alpha \in \bar{\Delta} \) is long (i.e. length 4 resp 6) iff \( \alpha \) is. We hence finds

\[
\Delta^{t, re} = \bar{\Delta}^{long} + \delta \mathbb{Z}
\]

We want to prove that \( \Delta^t \) is indeed the root system of the untwisted affine Lie algebra associated to \( \bar{\Delta}^t \) (note \( D_3^{(1)} := A_3^{(1)} \)); for the latter we have determined the root system and a set of simple roots \( \alpha^t_i, 1 \leq i \leq n \) in Lemma 3.2. We further notice from the Dynkin diagram of the untwisted affine Lie algebras that \( \alpha^t_0 := \alpha_0 \) is always a long root. We start with the cases where \( \bar{\Delta}^t \) is connected, in these cases \( \Pi^t := \{ \alpha^t_0 \} \cup \bar{\Pi}^t \) will already be a set of simple roots for the affinization:

- Let \( \Delta = B_n^{(1)}, n \geq 3 \) and \( t = 4 \), then \( \bar{\Delta}^t = D_n \) (note \( D_3 = A_3 \)) with simple roots \( \alpha^t_i := \alpha_i, 1 \leq i \leq n - 1 \) and \( \alpha^t_n := \alpha_{n-1} + 2 \alpha_n \). We convince ourselves that \( \Pi^t := \{ \alpha^t_0 \} \cup \bar{\Pi}^t \) with \( \alpha^t_0 := \alpha_0 \) is of type \( D_n^{(1)} \) resp. \( A_3^{(1)} \) (rescaled) and \( \delta_{B_n^{(1)}} = \delta_{D_n^{(1)}} \) under this correspondence:

\[
\begin{align*}
(\alpha^t_0, \alpha^t_1) &= (\alpha_0, \alpha_1) = 0 \\
(\alpha^t_0, \alpha^t_2) &= (\alpha_0, \alpha_2) = -2 \\
(\alpha^t_0, \alpha^t_3) &= \begin{cases} (\alpha_0, \alpha_2 + 2 \alpha_3) = -2, & n = 3 \\
(\alpha_0, \alpha_3) = 0, & n \geq 4 \end{cases} \\
(\alpha^t_0, \alpha^t_i) &= 0, & i \geq 4 \\
\delta_{D_n^{(1)}} &= \alpha^t_0 + \theta_d \\
&= \alpha^t_0 + \alpha^t_1 + 2 \alpha^t_2 + 2 \alpha^t_3 + \cdots + \alpha^t_{n-1} + \alpha_n
\end{align*}
\]
\[= \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + \alpha_{n-1} + (\alpha_{n-1} + \alpha_n)\]
\[= \alpha_0 + \theta_{B_n} = \delta_{B_n}^{(1)}\]

- Let \(\Delta = F_4^{(1)}\) and \(t = 4\), then \(\bar{\Delta}' = D_4\) with simple roots \(\alpha_i' := \alpha_1, \alpha_2, \alpha_2 + 2\alpha_3, \alpha_3 + 2\alpha_3 + 2\alpha_4\) and \(\alpha_4'\) the center node. We convince ourselves that \(\Pi' := \{\alpha_0'\} \cup \bar{\Pi}'\) with \(\alpha_0' := \alpha_0\) is of type \(D_4^{(1)}\) (rescaled) and \(\delta_{F_4^{(1)}} = \delta_{D_4^{(1)}}\) under this correspondence:

\[(\alpha_0', \alpha_1') = (\alpha_0, \alpha_1) = -2\]
\[(\alpha_0', \alpha_2') = (\alpha_0, \alpha_2) = 0\]
\[(\alpha_0', \alpha_3') = (\alpha_0, \alpha_2 + 2\alpha_3) = 0\]
\[(\alpha_0', \alpha_4') = (\alpha_0, \alpha_2 + 2\alpha_3 + 2\alpha_4) = 0\]
\[\delta_{D_4^{(1)}} = \alpha_0' + \theta_{D_4}\]
\[= \alpha_0' + 2\alpha_1' + \alpha_2' + \alpha_3' + \alpha_4'\]
\[= \alpha_0 + 2\alpha_1 + \alpha_2 + (\alpha_2 + 2\alpha_3) + (\alpha_2 + 2\alpha_3 + 2\alpha_4)\]
\[= \alpha_0 + 2\alpha_2 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\]
\[= \alpha_0 + \theta_{D_4} = \delta_{D_4}^{(1)}\]

- Let \(\Delta = G_2^{(1)}\) and \(t = 3, 6\), then \(\bar{\Delta}' = A_2\) with simple roots \(\alpha_i' := \alpha_1, \alpha_1 + 3\alpha_2\). We convince ourselves that \(\Pi' := \{\alpha_0'\} \cup \bar{\Pi}'\) with \(\alpha_0' := \alpha_0\) is of type \(A_2^{(1)}\) (rescaled) and \(\delta_{G_2^{(1)}} = \delta_{A_2^{(1)}}\) under this correspondence:

\[(\alpha_0', \alpha_1') = (\alpha_0, \alpha_1) = -3\]
\[(\alpha_0', \alpha_2') = (\alpha_0, \alpha_1 + 3\alpha_2) = -3\]
\[\delta_{A_2^{(1)}} = \alpha_0' + \theta_{A_2}\]
\[= \alpha_0' + \alpha_1' + \alpha_2'\]
\[= \alpha_0 + \alpha_1 + (\alpha_1 + 3\alpha_2)\]
\[= \alpha_0 + \theta_{A_2} = \delta_{A_2}^{(1)}\]

We now turn to the case \(C_n^{(1)}, n \geq 2\) and \(t = 4\), where \(\bar{\Delta}' = A_4^{(1)n}\) is disconnected. Here we wish to prove \(\Delta' = (A_4^{(1)})^n\). This is slightly more complicated than the previous cases, because we want multiple affinizations resp. the set of simple roots \(\Pi' := \{\alpha_0\} \cup \bar{\Pi}'\) will not suffice to generate \(\Delta'\). We proceed ad-hoc: Consider the diagram automorphism \(f\) on \(C_n^{(1)}\) switching \(\alpha_i \leftrightarrow \alpha_{n-i}\). Define \(\Pi' := \bar{\Pi}' \cup f(\Pi')\)
(which are long roots). We calculate the Cartan matrix of \( A_1^{(1)} \times n \) with each \( \alpha_i^t \) a simple root and \( \alpha_{0,i}^t := f(\alpha_{n-i+1}^t) \) the respective affinization node:

\[
(\alpha_i^t, \alpha_{0,j}^t) = (\alpha_i^t, f(\alpha_{n-j}^t)) = (\alpha_n + 2\alpha_{n-1} + \cdots + 2\alpha_{n-i+1}, \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-j}) = \]

We distinguish three cases:

- If \( i < j \) then clearly \( (\alpha_i^t, \alpha_{0,j}^t) = 0 \)
- If \( i > j \) we have \( 0 < n - i + 1 \leq n - j < n \) and consider the nonempty subsum \( 2\beta := 2\alpha_{n-i+1} + \cdots + 2\alpha_j \). Since \( \beta \) is in the in the parabolic \( A_{i-j} \) subsystem of \( C_n^{(1)} \) it is a short root. Hence:

\[
(\alpha_i^t, \alpha_{0,j}^t) = (\alpha_n + 2\alpha_{n-1} + \cdots + 2\alpha_{n-j+1} + 2\beta, \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-i} + 2\beta) = 2(\alpha_n + 2\alpha_{n-1} + \cdots + 2\alpha_{n-j+1}, \beta) + 2(\beta, \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-i}) + 4(\beta, \beta) = -4 - 4 + 8 = 0
\]

- If \( i = j \) we calculate:

\[
(\alpha_i^t, \alpha_{0,i}^t) = (\alpha_n + 2\alpha_{n-1} + \cdots + 2\alpha_{n-i+1}, \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-i})
\]

\[
= \begin{cases} 
  (\alpha_n, 2\alpha_{n-1}) = -4, & i = 1 \\
  (2\alpha_{n-i+1}, 2\alpha_{n-1}) = -4, & 1 < i < n \\
  (2\alpha_1, \alpha_0) = -4, & i = 0 
\end{cases}
\]

 Altogether we have shown that \( \Pi^t \) defined above generates a (rescaled) root system of type \( (A_1^{(1)})^\times n \). We now check explicitly that this already accounts for all roots in \( \Delta^{t, re} = \pm \alpha_i^t + \delta \mathbb{Z} \). This is again, because in every copy of \( A_1^{(1)} \) we have the same \( \delta \) under the correspondence:

\[
\delta_{(A_1^{(1)})^i} = \theta_{(A_1^{(1)})_i} = \alpha_{0,i}^t + \alpha_i^t \\
= (\alpha_n + 2\alpha_{n-1} + \cdots + 2\alpha_{n-i+1}) + (\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-i}) \\
= \alpha_0 + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n \\
= \alpha_0 + \theta_{C_n} = \delta_{C_n^{(1)}}
\]

Let \( a_0k = 2,3 \), i.e. \( \Delta = D_{n+1}^{(2)}, E_6^{(2)}, A_{2n-1}^{(2)}, D_4^{(3)} \). Note that we still have \( a_0 = 1 \), but now \( \theta \) is the highest short root, \( \alpha_0 \) is always a short root (hence not in \( \Delta^t \)) and the set of roots is

\[
\Delta^{re} = (\bar{\Delta}_{short} + \delta \mathbb{Z}) \cup (\bar{\Delta}_{long} + \delta \mathbb{Z})
\]
We wish to prove uniformly that for these twisted affine Lie algebras $\Delta'$ is the same as for the respective untwisted affine Lie algebra $\Delta'$ with same $\bar{\Delta}$. We will do so by choosing a long root $\alpha'_0$ such that the subsystem generated by $\{\alpha'_0\} \cup \Pi$ is precisely $\Delta'$ and $k\delta_\Delta = \delta_{\Delta'}$ under this correspondence. This then reduces the problem completely to $\Delta'$, since the long roots of $\Delta$ are precisely $\Delta_{long} + \delta k \mathbb{Z}$:

- Let $\Delta = D^{(2)}_{n+1}$ and $t = 4$, then we choose $\alpha'_0 := 2\alpha_0 + \alpha_1$. It is clear (from the parabolic subsystem $B_n$ excluding $\alpha_n$) that $\alpha'_0$ is long and that $\{\alpha'_0\} \cup \Pi$ is of type $\Delta' = B^{(1)}_n$ and we check

$$\delta_{B^{(1)}_n} = \alpha'_0 + \theta_{B_n}$$

$$= (2\alpha_0 + \alpha_1) + (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$$

$$= 2 (\alpha_0 + (\alpha_1 + \alpha_2 + \cdots + \alpha_n))$$

$$= 2 (\alpha_0 + \theta_{short}) = 2\delta_{D^{(2)}_{n+1}}$$

- Let $\Delta = A^{(2)}_{2n-1}$, $n \geq 3$ and $t = 4$, then we choose $\alpha'_0 := 2\alpha_0 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n$ (the highest root in the parabolic subsystem $C_n$ excluding $\alpha_1$), then we have to check $\{\alpha'_0\} \cup \Pi$ is indeed $\Delta' = C^{(1)}_n$ and $\delta_{C^{(1)}_n} = 2\delta_{A^{(2)}_{2n-1}}$. It is clear from the subsystem that $\alpha'_0$ is orthogonal on all but $\alpha_1$ and we calculate that now:

$$\delta_{C^{(1)}_n} = \alpha'_0 + \theta_{C_n}$$

$$= (2\alpha_0 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n)$$

$$+ (2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n)$$

$$= 2 (\alpha_0 + \alpha_1 + \alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n)$$

$$= 2 \left( \alpha_0 + \theta_{short} \right) = 2\delta_{A^{(2)}_{2n-1}}$$

- Let $\Delta = E^{(2)}_6$ and $t = 4$, then we choose $\alpha'_0 := 2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3$ (the highest long root of the parabolic subsystem $C_4$), then we have to check $\{\alpha'_0\} \cup \Pi$ is indeed $\Delta' = F^{(4)}_4$ (note that $\alpha_0$ is at the other end of the
diagram!) and $\delta_{F_4^{(1)}} = 2\delta_{E_6^{(2)}}$:

\[(\alpha'_0, \alpha_1) = (2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1) = -2 + 4 - 2 = 0\]
\[(\alpha'_0, \alpha_2) = (2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_2) = -2 + 4 - 2 = 0\]
\[(\alpha'_0, \alpha_3) = (2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_3) = -4 + 4 = 0\]
\[(\alpha'_0, \alpha_4) = (2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4) = -2\]
\[\delta_{F_4^{(1)}} = \alpha' + \theta_{F_4}\]
\[= (2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3) + (2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4)\]
\[= 2(\alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4)\]
\[= 2(\alpha_0 + \theta_{F_4}^{short}) = 2\delta_{E_6^{(2)}}\]

Let $\Delta = D_4^{(3)}$ and $t = 3, 6$, then we choose $\alpha'_0 := 3\alpha_0 + 3\alpha_1 + \alpha_2$ (a long root in the $G_2$ subsystem generated by $\alpha_0 + \alpha_1, \alpha_2$), then we have to check \{$\alpha'_0$\} $\cup \Pi$ is indeed $\Delta' = G_2^{(1)}$ (note that $\alpha_0$ is at the other end of the diagram!) and $\delta_{G_2^{(1)}} = 3\delta_{D_4^{(3)}}$:

\[(\alpha'_0, \alpha_1) = (3\alpha_0 + 3\alpha_1 + \alpha_2, \alpha_1) = -3 + 6 - 3 = 0\]
\[(\alpha'_0, \alpha_2) = (3\alpha_0 + 3\alpha_1 + \alpha_2, \alpha_2) = -9 + 6 = -3\]
\[\delta_{G_2^{(1)}} = \alpha' + \theta_{G_2}\]
\[= (3\alpha_0 + 3\alpha_1 + 2\alpha_2) + (3\alpha_1 + \alpha_2)\]
\[= 3(\alpha_0 + 2\alpha_1 + \alpha_2)\]
\[= 3(\alpha_0 + \theta_{G_2}^{short}) = 3\delta_{D_4^{(3)}}\]

Let finally $a_0, k = 4$, then we have the more exceptional cases $A_2^{(2)}, A_2^{(2)}$ with $a_0 = 2, k = 2$ and $\theta$ again the highest root and where the finite root system $\tilde{\Delta} = A_1, C_n$ is rescaled by $2$ resp. $\sqrt{2}$. The explicit set of all roots is in both cases

$$\Delta^{re} = \left(\tilde{\Delta}^{short} + \delta \mathbb{Z}\right)_{(\alpha, \alpha) = 4} \cup \left(\tilde{\Delta}^{long} + \delta k \mathbb{Z}\right)_{(\alpha, \alpha) = 8} \cup \frac{1}{2}\left(\tilde{\Delta}^{long} + \delta\right)_{(\alpha, \alpha) = 2} + \delta \mathbb{Z}$$
Let first $A_2^{(2)}$ and $t = 8$ (or equivalently $t = 4$ since all roots have length 2, 8), then the set of roots of length 8 is explicitly
\[ \Delta_{t, \text{re}} = \bar{\Delta} + \delta k \mathbb{Z} = \{ \pm \alpha_1 \} + (4\alpha_0 + 2\alpha_1) \mathbb{Z} \]
We choose the long roots $\alpha_t^1 := \alpha_1$ and $\alpha_t^0 := 4\alpha_0 + \alpha_1 = -\alpha_1 + 2\delta$ (the reflection of $\alpha_1$ on $\alpha_0$). We check that $\Pi' = \{ \alpha_0', \alpha_1' \}$ generates a subsystem of type $A_1^{(1)}$
\[ (\alpha_0', \alpha_1') = (2\delta - \alpha_1, \alpha_1) = -8 \]
and since under this correspondence $\delta_{A_1^{(1)}} = \alpha_0' + \alpha_1' = 2\delta_{A_2^{(2)}}$ we have in fact equality $\Delta_{t, \text{re}} = A_1^{(1)}$.

Let now $A_2^{(2)}$, $n \geq 2$ and $t = 4$ with $\bar{\Delta} = C_n$, then the set of roots of length 8 is explicitly
\[ \Delta_{\text{re}} = (C_n^{\text{short}} + \delta \mathbb{Z}) \cup (C_n^{\text{long}} + 2\delta \mathbb{Z}) \]
This reminds strongly on $A_2^{(2)}_{2n-1}$ (after rescaling the generators by $\sqrt{2}$). Indeed, choosing $\alpha'_0 := 2\alpha_0 + \alpha_1$ (a long root in the parabolic subsystem $C_2$ generated by $\alpha_0, \alpha_1$) we calculate the Cartan matrix and check that also $\delta$ corresponds:
\[ (\alpha_0', \alpha_1) = (2\alpha_0 + \alpha_1, \alpha_1) = -4 + 4 = 0 \]
\[ (\alpha_0', \alpha_2) = (2\alpha_0 + \alpha_1, \alpha_2) = -4 \]
\[ \delta_{A_2^{(2)}_{2n-1}} = \alpha_0' + \theta_{C_n}^{\text{short}} \]
\[ = (2\alpha_0 + \alpha_1) + (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n) \]
\[ = 2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n = a_0\alpha_0 + \theta_{C_n} = \delta_{A_2^{(2)}}. \]
This shows $\Delta_{t, \text{re}} = A_2^{(2)}_{2n-1}$ for $t = 4$. For $t = 8$ we may hence equally look at roots of length 4 in $A_2^{(2)}_{2n-1}$, which we have already seen is $(A_1^{(1)})^n$. This concludes the proof of Theorem 3.3.

\[ \square \]

4. MAIN THEOREM

Lemma 4.1 ([Lusz94] Lm. 35.2.2). Assume (35.1.2) that
a) For any $i \neq j$ with $\ell_{\alpha_j} \geq 2$ we have $\ell_{\alpha_i} \geq -a_{ij} + 1$ (with $a_{ij}$ the Cartan matrix).
b) The root system is without odd cycles, i.e. $\mathfrak{g} \neq A_1^{(1)}, 2|n$.
Then $\mathfrak{Rf}$ is generated by all $E_{\ell_{\alpha_i}}, \ell_{\alpha_i} \geq 2$ and $E_{\ell\alpha_i}^{(\ell_{\alpha_i})}$.
Lusztig defines in [Lusz94] Chp. 36 the subalgebra $R_u$ to be generated by all $E_\alpha$, $\ell_\alpha \neq 1$ and establishes under the restriction on $g, \ell$ above a Frobenius homomorphism with kernel $R_u$. Note that in contrast in [Lusz90b] Thm. 8.3 he had defined $u$ without restrictions on $q$ as being generated by all root vectors $E_\alpha$ with $\ell_\alpha \neq 1$, but has refrained from doing so in the affine case by lack of root vectors. Note that frequently already in the finite case $u$ for $g, \ell$ violating a) is not generated by simple root vectors.

The aim of this article is the following theorem, that describes in all cases violating a) a subalgebra $u^{\zeta}_q(g)$ that has properties similar to $u$ defined in the finite case and could serve as a kernel of a Frobenius homomorphism without restrictions on $\ell$.

**Theorem 4.2.** Let $g$ be an affine Lie algebra and $q$ and $\ell$-th root of unity. We shall define a Hopf subalgebra $u^{\zeta}_q(g)^+ \subset U^{\zeta}_q(g)^+$ with the following properties:

- $u^{\zeta}_q(g)^+$ consists of all degrees (roots) $\alpha$ with $\ell_\alpha \neq 1$.
- Except for cases marked deaffinized, $u^{\zeta}_q(g)^+$ is generated by primitives $E_{\alpha^{(0)}}$, spanning a braided vector space $M$ (in the deaffinized cases $u^{\zeta}_q(g)^+$ is an infinite extension tower, see Section 7).
- Lusztig’s $R_u \subset u^{\zeta}_q(g)^+$ and equality only holds for trivial and generic cases.
- Except for cases marked deaffinized and (possibly) exotic $u^{\zeta}_q(g)^+$ is coradically graded and hence maps onto the Nichols algebra $B(M) = u^{\zeta}_q(g^{(0)})^+$. For untwisted affine $g$ we prove (and else conjecture) they are isomorphic.
- Except for cases marked deaffinized, the $u^{\zeta}_q(g)^+$ fulfills $\ell_i \neq 1$ as well as Lusztig’s non-degeneracy condition $\ell_{\alpha_i} \geq -a_{ij} + 1$. Hence we have determined subalgebras on which Lusztig’s theory in [Lusz94] may be applied.
We explicitly describe the type $\mathfrak{g}^{(0)}, q'$ of $M, \mathcal{B}(M)$ as follows:

| $\mathfrak{g}$ | $\ell$ | $M$ | $q'$ | comment |
|----------------|--------|-----|------|---------|
| $A_1^{(1)}$    | $\ell = 4$ | $A_1^{2x}$ | $q$ | deaffinized |
| $B_n^{(1)}, D_{n+1}^{(2)}$ | $\ell = 4$ | $A_1^{2n\times}$ | $q$ | short roots, deaffinized |
| $C_n^{(1)}, A_{2n-1}^{(2)}$ | $\ell = 4$ | $D_n^{(1)}$ | $q$ | short roots |
| $F_4^{(1)}, E_6^{(2)}$ | $\ell = 4$ | $D_4^{(1)}$ | $q$ | short roots |
| $G_2^{(2)}, D_4^{(3)}$ | $\ell = 3, 6$ | $A_2^{(1)}$ | $q$ | short roots |
| $A_2^{(2)}$    | $\ell = 4$ | $A_2^{2x}$ | $q$ | very short roots, deaffinized |
| $A_2^{(1)}$    | $\ell = 8$ | $A_1^{2n\times}$ | $q$ | very short roots |
| $A_{2n}^{(2)}$ | $\ell = 4$ | $A_2^{2n\times}$ | $q$ | very short roots, deaffinized |
| $A_{2n}^{(2)}$ | $\ell = 8$ | $A_{2n-1}^{(2)}$ | $q$ | not very long roots |
| $A_{2n}^{(2)}$ | $3$ | $A_2^{(1)}$ | $q$ | exotic |
| $A_2^{(2)}$    | $6$ | $A_2^{(1)}$ | $-q$ | exotic |
| $A_{2n}^{(2)}$ | $3, 6$ | $A_2^{(2)}$ | $q$ | (pseudo-)exotic |
| $G_2^{(1)}$    | $4$ | $A_3^{(1)}$ | $\bar{q}$ | exotic |
| $D_4^{(3)}$    | $4$ | $D_4^{(1)}$ | $q, \bar{q}, -1$ | exotic |

All cases not included in the list are generic cases $\mathfrak{g}^{(0)} = \mathfrak{g}$ with $u^\mathfrak{g}_q(\mathfrak{g})^+ = Ru^+$. 

**Proof.** The proof of this theorem will occupy the reminder of the article and consists of a precise description of the respective subalgebras. The proof proceeds as follows:

We first determine all cases $\mathfrak{g}, q$ where $U^\mathfrak{g}_q(\mathfrak{g})$ fails Lusztig’s condition. This is easily done in the following Lemma 4.3. We define two subcases:

- **Degenerate Cases:** When there are simple roots with $\ell_\alpha = 1$, then these simple root vectors $E_\alpha$ are not contained in $Ru$. We give an explicit set of primitive elements $E_\alpha^{(0)}$ with $\ell_\alpha \neq 1$, determine their root system $\mathfrak{g}^{(0)}$ and show it contains precisely the real roots $\alpha$ of $\mathfrak{g}$ with $\ell_\alpha \neq 1$. This is done in Section 3 by linking $\mathfrak{g}^{(0)}$ to subsystems of the dual root system $\mathfrak{g}^\vee$ of $\mathfrak{g}$.

- **Exotic Cases:** Now assume all $\ell_\alpha \neq 1$ but the condition $\ell_\alpha \geq -a_{ij} + 1$ fails. In these cases all $E_\alpha^{(0)} := E_\alpha \in Ru$, but usually they generate a Nichols algebra of different type than $\mathfrak{g}$ and do not include all roots with $\ell_\alpha \neq 1$. We determine case-by-case additional primitive elements $E_\alpha^{(0)}$ in
\( U_q^\xi(g)^+ \), calculate the Nichols algebra generated by them and verify that it contains all roots with \( \ell_\alpha \neq 1 \). This is done in Section \[6\]

- **Deaffinized Cases:** A specific case with \( \ell_\alpha \neq 1 \) but violating \( \ell_\alpha \geq -a_{ij} + 1 \) is \( g = A_1^{(1)} \), \( \ell = 4 \). It is the such only exotic case where \( Ru \) is finite-dimensional and an infinite tower of copies is needed to cover all roots with \( \ell_\alpha \neq 1 \). This case is dealt with in Section \[7\]

Moreover there are quite a few degenerate cases where \( g^{(0)} \) still fails \( \ell_\alpha \geq -a_{ij} + 1 \), namely precisely those with \( (g^{(0)} = A_1^{(1)}) \times n \).

Having determined a suitable set of primitive elements \( E_{\alpha_i}^{(0)} \), the remaining assertions of the theorem follow by standard arguments:

- For degenerate cases we prove now that \( u_q^\xi(g) \) is coradically graded: In most cases \( g^{(0)} \) has the same rank as \( g \) i.e. is generated by \( E_{\alpha_0}^{(0)}, \ldots E_{\alpha_n}^{(0)} \) with \( \alpha_i^{(0)} \) a basis of \( \mathbb{R}^n \). Then there is a linear function \( f : \mathbb{N}^n \to \mathbb{R} \) fulfilling \( f(\alpha_i^{(0)}) = 1 \) and since \( U_q^\xi(g) \) is \( \mathbb{N}^n \)-graded, the assertion follows. For the exceptional cases with \( g^{(0)} = (A_1^{(1)}) \times n \), i.e. rank of \( g^{(0)} \) = 2n = 2rank(g) - 2 we convince ourselves from the specific \( \alpha_i^{(0)} \) given in Lemma \[5.2\]

\[
\alpha_1^{(0)} , \ldots \alpha_n^{(0)} = \alpha_n + \alpha_{n-1} + \ldots \\
\alpha_{n+1}^{(0)} , \ldots \alpha_{2n}^{(0)} = \alpha_0 + \alpha_1 + \ldots
\]

that nevertheless the linear function \( f(\alpha_0) = f(\alpha_n) = 1 \) and \( f(\alpha_i) = 0 \) else fulfills \( f(\alpha_i^{(0)}) = 1 \). This again shows the assumption.

- By the universal property of the Nichols algebra, every Hopf algebra generated by a braided vector space \( M \) of primitive elements maps onto the Nichols algebra \( B(M) \). For the exotic cases the weaker statement is that \( \text{gr}(u_q^\xi(g)) \) maps onto \( B(M) \).

- By construction the root system of \( B(M) \) precisely coincides with the roots contained in \( u_q^\xi(g) \). For untwisted Lie algebras \( g \) we have a PBW-basis of \( U_q^\xi(g) \) by [Beck94] Prop. 6.1, which implies the map to \( B(M) \) is an isomorphism. We conjecture this to be true also for the twisted affine \( g \).

For the deaffinized cases it is clear we do not get an isomorphism to the (finite-dimensional) Nichols algebra, but we nevertheless expect an isomorphism to the explicit extension tower algebra \( u_q((A_1^{(1)}) \times n) \).
We easily determine all cases in question, note that we are in the following only interested in cases violating Lusztig’s first condition:

**Lemma 4.3.** The cases of affine quantum groups \( g, \ell \) where the set \( E_{\alpha_i}^{(0)} \) fails Lusztig’s condition are the following:

a) Cases with \( \Delta^{\vee, t} = \Delta^\vee \) i.e. \( \alpha_i^{(0)} = \alpha_i \) which have not yet appeared as being exceptional, but violate either condition of Lemma 4.1:

\[
\begin{array}{c|c}
 g, \Delta & \ell \\
\hline
 A_1^{(1)} & 4 \\
 A_2^{(2)}, A_2^{(1)} & 3, 6 \\
 G_2^{(1)}, D_4^{(3)} & 4 \\
 A_n^{(1)}, 2|n & \neq 1, 2 \\
\end{array}
\]

Note that also the only finite exotic example 6.1 \( g = G_2, \ell = 4 \) would fall into this case.

b) Cases with \( \emptyset \neq \Delta^{\vee, t} \neq \Delta^\vee \) where the root subsystem \((\Delta^{\vee, t})^\vee\) is in case a) and hence violates either condition in Lemma 4.1:

\[
\begin{array}{c|c|c}
 g, \Delta & \ell & ((\Delta^\vee)^t)^\vee \\
\hline
 B_n^{(1)}, D_{n+1}^{(2)}, A_2^{(2)}, A_2^{(1)} & 4 & (A_1^{(1)})^{\times n} \\
 G_2^{(1)}, D_4^{(3)} & 3, 6 & A_2^{(1)} \\
\end{array}
\]

**Proof.** a) We assume \( \Delta^{\vee, t} = \Delta^\vee \). Besides clarifying the root system in question, this assumption will exclude several values for \( \ell \), for which we have to consider the appropriate subsystem in b). For the first condition in Lemma 4.1 we reformulate

\[
\ell_{\alpha_i} > -a_{ij}
\]

\[
\Leftrightarrow \frac{\ell}{\gcd(\ell, (\alpha_i, \alpha_i))} > -\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}
\]

\[
\Leftrightarrow \lcm(\ell, (\alpha_i, \alpha_i)) > -2(\alpha_i, \alpha_j)
\]

Hence we have to check all cases with \( \lcm \leq -2(\alpha_i, \alpha_j) = 0, 2, 4, 6, 8 \). For \( \ell = 1, 2 \) we have \( \ell | (\alpha, \alpha) \) hence \( \Delta^{\vee, t} = \emptyset \) and this is not included in this case, also \( (\alpha_i, \alpha_i) = 2, 4, 6, 8 \), this leaves

\[
\ell = 3, 6, \ (\alpha_i, \alpha_i) = 2, 6, \ -2(\alpha_i, \alpha_j) = 6, 8
\]

\[
\ell = 4, \ (\alpha_i, \alpha_i) = 2, 4, \ -2(\alpha_i, \alpha_j) = 4, 6, 8
\]
Notice now that case-by-case several of these cases are degenerate in the sense that $q^{(\alpha_i, \alpha_k)} = 1$ and hence do not belong in this case a): For $\ell = 3, 6$ cases with $(\alpha_i, \alpha_i) = 6$ (namely $G^{(1)}_2, D^{(3)}_4$) are degenerate, which leaves only $A^{(1)}_2, A^{(2)}_2$. For $\ell = 4$ cases with $(\alpha_i, \alpha_i) = 4, 8$ are degenerate while simply-laced $g$ have $-2(\alpha_i, \alpha_j) = -2$, which only leaves the cases $A^{(1)}_2, G^{(1)}_2, D^{(3)}_4$.

The second condition in Lemma 4.1 is violated for $A^{(1)}_n$ for $2 | n$ and arbitrary $\ell \neq 1, 2$ (again for $\ell = 1, 2$ it is degenerate).

b) If $\Delta^{\mathcal{L}, \mathcal{V}} \neq \Delta^{\mathcal{V}}$ violates the conditions in Lemma 4.1 then it has to appear in a) above. We simply check the table in Lemma 5.2 and find that the only cases are for one $B^{(1)}_n, D^{(2)}_{n+1}, A^{(2)}_2, A^{(2)}_2$ at $\ell = 4$ which yield $(A^{(1)}_1)^n$. On the other hand we have $G^{(1)}_2, D^{(3)}_4$ for $\ell = 3, 6$ which yield $A^{(1)}_2$ having an odd cycle.

\[\square\]

5. Degenerate cases

5.1. Preliminaries on primitives. We start by the following observation in our context, that appears e.g. throughout [Heck09]. We shall use it in what follows to construct exceptionally primitive elements for small $\ell$. Note that Lusztig’s reflection operator is defined in terms of the Cartan matrix of $g$, not intrinsically with respect to the braiding matrix (which may be of different type) in [Heck09] and also may not always be expressed as iterated braided commutators.

Lemma 5.1. Let $\alpha, \beta$ such that $\ell | 2(\alpha, \beta)$, or equivalently $q^{(\alpha, \beta)} = \pm 1$ i.e. the braiding is symmetric. Then

a) The braided commutator $[x, y]$ of primitive elements in degree $\alpha, \beta$ is again primitive.

b) Let $\alpha = \alpha_i$ and $x$ a primitive element in degree $\beta$ with braided commutator $[E_{\alpha_i}, x] = [F_{\alpha_i}, x] = 0$, then Lusztig’s reflection $T^{\prime\prime}_{i, 1}(x)$ is again primitive.

c) Let $\ell | (\alpha_i, \alpha_i)$, i.e. trivial self-braiding $q^{(\alpha_i, \alpha_i)} = 1$, then $T^{\prime\prime}_{i, 1}$ maps primitive elements to primitive elements without further assumptions.

Proof. a) This is a standard argument: Let the braiding of some elements $x_1, x_2$ be given by $x_1 \otimes x_2 \mapsto q_{12} x_2 \otimes x_1$ and $x_2 \otimes x_1 \mapsto q_{21} x_1 \otimes x_2$. Then the assumption
\( q_{12}q_{21} = 1 \) implies for primitive elements \( x_1, x_2 \):

\[
\Delta([x_1, x_2]) := \Delta(x_1x_2 - q_{12}x_2x_1) = x_1x_2 \otimes 1 + x_1 \otimes x_2 + q_{12}x_2 \otimes x_1 + 1 \otimes x_1x_2 - q_{12}(x_2x_1 \otimes 1 + x_2 \otimes x_1 + q_{21}x_1 \otimes x_2 + 1 \otimes x_2x_1) = [x_1, x_2] \otimes 1 + 1 \otimes [x_1, x_2]
\]

Note this is the only case where \([x_1, x_2] = \pm [x_2, x_2]\) are linearly dependent.

b) In [Lusz94] Sec. 37.3 a relation between reflection and comultiplication in \( U_q^\mathcal{L} \) is given as follows:

\[
(T'_{i,-1} \otimes T'_{i,-1})\Delta(T'_{i,1}x) = \left( \sum_{n} q_{i}^{n(n-1)/2} \{ n \} \cdot F_{i}^{(n)} \otimes E_{i}^{(n)} \right) \Delta(x) \cdot \left( \sum_{n} (-1)^{n} q_{i}^{-n(n-1)/2} \{ n \} \cdot F_{i}^{(n)} \otimes E_{i}^{(n)} \right)
\]

where \( \{ n \}_{\alpha} = \prod_{a=1}^{n} (q_{i}^{a} - q_{i}^{-a}) = (q_{i} - q_{i}^{-1})^{n} \cdot [n]_{q_{i}}! = (q_{i} - q_{i}^{-1})^{n} \cdot [n]_{q_{i}}! \)

\[
T'_{i,-1}T'_{i,1} = \text{id}
\]

With the assumption \( E_{i}x - q^{n(\alpha_{i},\beta)}x E_{i} = 0 \) and \( F_{i}x - q^{-n(\alpha_{i},\beta)}xF_{i} = 0 \) we calculate

\[
\begin{align*}
n_{q}E_{i}^{(n)}x n'q_{i}E_{i}^{(n')} &= (q_{i} - q_{i}^{-1})^{n + n'} E_{i}^{n}x E_{i}^{n'} = (q_{i} - q_{i}^{-1})^{n + n'} q^{n(\alpha_{i},\beta)} \cdot x E_{i}^{n + n'} \\
n_{q}F_{i}^{(n)}x n'q_{i}F_{i}^{(n')} &= (q_{i} - q_{i}^{-1})^{n + n'} F_{i}^{n}x F_{i}^{n'} = (q_{i} - q_{i}^{-1})^{n + n'} q^{-n(\alpha_{i},\beta)} \cdot x F_{i}^{n + n'}
\end{align*}
\]

With \( \Delta(x) = x \otimes 1 + 1 \otimes x \) we calculate

\[
(T'_{i,-1} \otimes T'_{i,-1})\Delta(T'_{i,1}x) = \left( \sum_{n} q_{i}^{n(n-1)/2} \{ n \} \cdot F_{i}^{(n)} \otimes E_{i}^{(n)} \right) (x \otimes 1 + 1 \otimes x) \cdot \left( \sum_{n} (-1)^{n} q_{i}^{-n(n-1)/2} \{ n \} \cdot F_{i}^{(n)} \otimes E_{i}^{(n)} \right)
\]

\[
= \sum_{n,n'=0}^{\ell_{i}} q_{i}^{n(n-1)/2 - n'(n'-1)/2} (-1)^{n'(n'-1)/2} q^{n(\alpha_{i},\beta)} \cdot F_{i}^{(n)}x F_{i}^{(n')} \otimes E_{i}^{(n)}E_{i}^{(n')}
\]

\[
+ \sum_{n,n'} q_{i}^{n(n-1)/2 - n'(n'-1)/2} (-1)^{n'(n'-1)/2} q^{n(\alpha_{i},\beta)} \cdot F_{i}^{(n)}F_{i}^{(n')} \otimes E_{i}^{(n)}E_{i}^{(n')}
\]
$$= \sum_{n,n'} q_i^{n(n-1)/2-n'(n'-1)/2} (-1)^n (q_i - q_i^{-1})^{n+n'} q^{2n(\alpha, \beta)} \cdot x F_i^{n+n'} \otimes E_i^{(n')}$$

$$+ \sum_{n,n'} q_i^{n(n-1)/2-n'(n'-1)/2} (-1)^n (q_i - q_i^{-1})^{n+n'} \cdot F_i^{(n)} F_i^{(n')} \otimes x E_i^{n+n'}$$

$$= \sum_{m=0}^m q_i^{-m^2} (q_i - q_i^{-1})^m \left( \sum_{n=0}^m (-1)^n \binom{m}{n} q^{2n(\alpha, \beta)} q_n q^{n(m-1)} \right) \cdot x F_i^m \otimes E_i^m$$

$$+ \sum_{m=0}^m q_i^{-m^2} (q_i - q_i^{-1})^m (-1)^m \left( \sum_{n=0}^m (-1)^n \binom{m}{n} q^{n(m-1)} \right) \cdot F_i^m \otimes x E_i^m$$

where we substituted $m = n + n'$. For $m > 0$ the second sum vanishes by [Lusz94] Sec. 1.3.4:

$$\sum_{n=0}^m (-1)^n \binom{m}{n} q^{n(m-1)} = 0$$

The first sum vanishes for $m > 0$ the same reason because of the additional assumption $q^{2(\alpha, \beta)} = 1$. Hence only the term $m = 0$ remains and by the inverse property $T_{i,-1} T_{i,1}' = \text{id}$ shows the assertion

$$(T_{i,-1}' \otimes T_{i,-1}') \Delta(T_{i,1}'(x)) = x \otimes 1 + 1 \otimes x$$

$$\Delta(T_{i,1}' x) = T_{i,1}'(x) \otimes 1 + 1 \otimes T_{i,1}'(x)$$

c) The assumption $q^{(\alpha_i, \alpha_i)} = 1$ amounts to $q_i = q^{(\alpha_i, \alpha_i)/2} = \pm 1$ hence $\{n\}_{\alpha_i} = 0$ except $\{0\}_{\alpha_i} = 1$. Hence only the terms $n, n' = 0$ in b) remain which shows the assertion without using the assumption on the vanishing braided commutator.

$$\square$$

5.2. Primitives from the dual root system. Let $\mathfrak{g}$ be an affine Lie algebra with root system $\Delta$ of rank $n+1$ and consider the dual root system $\Delta^\vee$. In Theorem 3.3 we have determined the type and a set $\Pi'$ of simple roots $\alpha_0', \ldots, \alpha_N'$ (or empty) for the subsystem $(\Delta^\vee)^t$ of roots with length divisible by $t$.

Suppose $q$ an $\ell$-th root of unity and let $s \in \{2, 4, 6, 8\}$ denote the longest root length in $\mathfrak{g}$. We find below for each $\mathfrak{g}, \ell$ a (not unique) value $t$, such that $\ell \nmid a$ is equivalent to $t \mid \frac{2s}{a}$ for all root lengths $a \neq 0$ (mostly $t = \ell$). This will allow us to characterize roots with $q^{(\alpha, \alpha)} \neq 1$, i.e. $\ell \mid (\alpha, \alpha)$, as being dual to roots $\alpha^\vee \in (\Delta^\vee)^t$, i.e. $t \mid (\alpha^\vee, \alpha^\vee)$.

More precisely we ultimately wish to prove the following:
**Lemma 5.2.** Via reflection we define the following root vectors $E_{\alpha_i^{(0)}}$ for short roots $\alpha_i^{(0)} \in ((\Pi^\vee)^t)^\vee$. These are primitive elements in $u^L_\lambda(g)$ resp. skew-primitives in $u^L_\lambda(g)$:

| $g, \Delta$ | $\ell$ | $\Delta^\vee$ | $t$ | $(\Delta^\vee)^t$ | $((\Delta^\vee)^t)^\vee$ | $((\Pi^\vee)^t)^\vee$ |
|------------|------|----------------|-----|----------------|----------------|----------------|
| $B_n^{(1)}$ | 4    | $A_{2n-1}^{(2)}$ | 4   | $(A_1^{(1)})^{\times n}$ | $(A_1^{(1)})^{\times n}$ | $\alpha_n, \alpha_n + \alpha_{n-1}, \alpha_n + \alpha_{n-1} + \alpha_{n-2}, \ldots$ |
| $C_n^{(1)}$ | 4    | $D_n^{(2)}$ | 4   | $D_n^{(1)}$ | $D_n^{(1)}$ | $\alpha_0', \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n + \alpha_0$ |
| $F_4^{(1)}$ | 4    | $E_6^{(2)}$ | 4   | $D_4^{(1)}$ | $D_4^{(1)}$ | $\alpha_0', \alpha_1, \alpha_2, \alpha_3 + \alpha_2, \alpha_3 + \alpha_2 + \alpha_1$ |
| $G_2^{(1)}$ | 3, 6 | $D_4^{(3)}$ | 3, 6 | $A_2^{(1)}$ | $A_2^{(1)}$ | $\alpha_0', \alpha_0, \alpha_2, \alpha_1 + \alpha_2$ |
| $D_{n+1}^{(2)}$ | 4    | $C_n^{(1)}$ | 4   | $(A_1^{(1)})^{\times n}$ | $(A_1^{(1)})^{\times n}$ | $\alpha_n, \alpha_n + \alpha_{n-1}, \alpha_n + \alpha_{n-1} + \alpha_{n-2}, \ldots$ |
| $A_{2n-1}^{(2)}$ | 4    | $B_n^{(1)}$ | 4   | $D_n^{(1)}$ | $D_n^{(1)}$ | $\alpha_0, \alpha_0 + \alpha_1, \alpha_0 + \alpha_1 + \alpha_2, \ldots$ |
| $E_6^{(2)}$ | 4    | $F_4^{(1)}$ | 4   | $D_4^{(1)}$ | $D_4^{(1)}$ | $\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_2, \alpha_3 + \alpha_2 + \alpha_4$ |
| $D_4^{(3)}$ | 3, 6 | $G_2^{(1)}$ | 3, 6 | $A_2^{(1)}$ | $A_2^{(1)}$ | $\alpha_0, \alpha_1, \alpha_1 + \alpha_2$ |
| $A_2^{(2)}$ | 4, 8 | $A_2^{(2)}$ | 4, 8 | $A_1^{(1)}$ | $A_1^{(1)}$ | $\alpha_1 + \alpha_0, \alpha_0$ |
| $A_2^{(2)}$ | 4    | $A_2^{(2)}$ | 8   | $(A_1^{(1)})^{\times n}$ | $(A_1^{(1)})^{\times n}$ | $\alpha_0, \alpha_0 + \alpha_1, \alpha_0 + \alpha_1 + \alpha_2, \ldots$ |
| $A_2^{(2)}$ | 8    | $A_2^{(2)}$ | 4   | $A_2^{(2)}$ | $B_n^{(1)}$ | $\alpha_n, \alpha_n - 1, \ldots, \alpha_0$ |

Note in the last block we have a nontrivial self-duality $\alpha_k \leftrightarrow \alpha_{n-k}$ as well as $t \neq \ell$.

**Remark 5.3.**

- For finite root systems the lemma has been proven case-by-case as part of the proof of Thm 5.4 in [Len14c]. The proof strategy via reflections is new.
- There is no coherent definition of affine root vectors available. Here we a-priori take some reflection to yield some $E_\alpha$, but it will become clear in the
proof that up to a sign our specific $E_\alpha$ are independent of the choice of the reflection.

- We do not claim these are all primitives. This will depend on the Nichols algebra structure and fail precisely in the so-called exotic cases.

- The theorem gives only information about real roots. Imaginary roots are by definition included in any $(\Delta^\vee)^t$, but never fulfill $q^{(\delta,\delta)} \neq 1$. Whether they are included in $u_q(g)$ will depend on the Nichols algebra structure and will be unexpected for the exotic cases.

Proof. First we check that the conditions on the roots $\alpha^\vee \in (\Delta^\vee)^t$ defined in Lemma 3.1 for the dual root system $\Delta^\vee$ matches precisely $q^{(\alpha,\alpha)} \neq 1$, i.e. $E_\alpha$ should be in $u_q^\ell$ for $q$ a primitive $\ell$-th root of unity:

Lemma 5.4. Let $s \in \{2, 4, 6, 8\}$ denote the longest root length in $g$. Consider the map

$$f : \Delta^\vee \to \Delta$$

$$\alpha' \mapsto \sqrt{s^2(\alpha')^\vee}, \quad (\alpha')^\vee = \frac{(\alpha',\alpha')}{2} \alpha'$$

Then $f$ maps $(\Delta^\vee)^t$ bijectively to the set of roots $\alpha \in \Delta$ fulfilling $q^{(\alpha,\alpha)} \neq 1$ for the pairs $\ell, t$ in Lemma 5.2. Moreover, for $\ell = 1, 2$ no roots fulfill $q^{(\alpha,\alpha)} \neq 1$, hence $t = \infty$ would be appropriate and for other $\ell$ all roots fulfill $q^{(\alpha,\alpha)} \neq 1$, hence $t = 1$ would be appropriate.

Proof. By definition the condition $q^{(\alpha,\alpha)} = 1$ amounts to $\ell \mid (\alpha, \alpha)$. Let $(\alpha', \alpha') = a$, then we calculate

$$(f(\alpha'), f(\alpha')) = \frac{s}{2} \cdot \left(\frac{2}{(\alpha', \alpha')}\right)^2 \cdot (\alpha', \alpha') = \frac{2s}{(\alpha', \alpha')}$$

Hence $\ell \mid (\alpha, \alpha)$ is equivalent to $\ell \mid \frac{2s}{(\alpha', \alpha')}$. We check all cases:

- For $\ell = 1, 2$ all $\ell \mid (\alpha, \alpha)$ and we will excluding this trivial cases in the following.

- For $s = 4$ and $\ell = 4$ we have $\ell \mid (\alpha, \alpha)$ for long roots $\alpha$ and thus dually $\ell \mid \frac{2s}{(\alpha', \alpha')} = \frac{8}{(\alpha', \alpha')}$ for short roots $\alpha'$. On the other hands, short roots are characterized by being not divisible by $t = 4$.

- For $s = 6$ and $\ell = 3, 6$ we have $\ell \mid (\alpha, \alpha)$ for long roots $\alpha$ and thus dually $\ell \mid \frac{2s}{(\alpha', \alpha')} = \frac{12}{(\alpha', \alpha')}$ for short roots $\alpha'$. On the other hands, short roots are characterized by being not divisible by $t = 3, 6$. 
• For \( s = 8 \), i.e. \((\alpha, \alpha) = 2, 4, 8\) we have two cases \( \ell = 4 \) and \( \ell = 8 \). For \( \ell = 4 \) we have \( \ell \mid (\alpha, \alpha) \) for all but very short roots \( \alpha \) and thus dually \( \ell \mid \frac{2s}{(\alpha', \alpha')} = \frac{16}{(\alpha', \alpha')} \) for all but very long roots \( \alpha' \). On the other hands, such roots are characterized by being not divisible by \( t = 8 \). For \( \ell = 8 \) we have \( \ell \mid (\alpha, \alpha) \) for very long roots \( \alpha \) and thus dually \( \ell \mid \frac{2s}{(\alpha', \alpha')} = \frac{16}{(\alpha', \alpha')} \) for very short roots \( \alpha' \). On the other hands, such roots are characterized by being not divisible by \( t = 4 \).

• This exhausts all \( \ell \) dividing root lengths for affine Lie algebras. For all other values of \( \ell \) we have the generic case \( \ell \nmid (\alpha, \alpha) \) for all real roots \( \alpha \).

We now conclude the proof of Lemma 5.2:
The table in the statement is taken by dualizing the table in Theorem 3.3 hence by Lemma 5.4 the roots \( \alpha^{(0)}_i \in (\Pi^\vee)^+_t \subset ((\Delta^\vee)'\bigwedge)_{(\Delta^\vee)'\bigwedge}^\vee \) in the statement fulfill \( q^{(\alpha^{(0)}_i, \alpha^{(0)}_i)} = 1 \). We wish to show that (in contrast to other roots \( \alpha \in ((\Delta^\vee)'\bigwedge)_{(\Delta^\vee)'\bigwedge}^\vee \)) the \( \alpha^{(0)}_i \) give rise to primitive elements Now Lemma 5.1 that any reflection \( T_{\alpha_j}'(x) \) of a primitive element \( x \) (especially \( x = E_{\alpha_i} \)) on simple roots \( \alpha_j \) with \( q^{(\alpha_j, \alpha_j)} = 1 \). The last condition characterizes the \( \alpha_j \notin ((\Delta^\vee)'\bigwedge)_{(\Delta^\vee)'\bigwedge}^\vee \) or dually \( \alpha_{\vee j} \notin (\Delta^\vee)'\bigwedge \). We now check that the choices \( \alpha^{(0)}_i, \alpha^{(0)}_{\vee i} \) in the statement have all the property that they contain only a unique simple root \( \alpha_{\vee i} \in (\Delta^\vee)'\bigwedge \) and only with multiplicity one (this has already been noticed in Theorem 3.3). The \( \alpha^{(0)}_i, \alpha^{(0)}_{\vee i} \) can hence be obtained by iterated reflection of \( E_{\alpha_k} \) only on simple roots \( \alpha_j \notin ((\Delta^\vee)'\bigwedge)_{(\Delta^\vee)'\bigwedge}^\vee \) and hence \( E_{\alpha^{(0)}_i} \) is primitive as asserted. □

6. EXOTIC CASES

We have established in Lemma 5.2 a set of primitive elements \( E_{\alpha^{(0)}_i} \in U_q^\vee (g) \) associated to the dual of the simple roots in a subsystem \((\Delta^\vee)'\bigwedge \subset \Delta \). However, it is neither clear that these are all primitive elements nor that they indeed generate an affine quantum group of type \(((\Delta^\vee)'\bigwedge)_{(\Delta^\vee)'\bigwedge}^\vee \).

Example 6.1. In [Len14c] the author has already determined for finite root systems the exotic case \( u_{\sqrt{-1}}^\vee (G_2) \) which contains all root vectors but is not generated by \( E_{\alpha_1}, E_{\alpha_2} \). Rather, these primitive elements only generate a quantum group of type \( A_2 \) and \( E_{\alpha_{112}} \) is a new primitive generator. Altogether we found \( u_{\sqrt{-1}}^\vee (G_2)^+ \cong u_{\sqrt{-1}}^\vee (A_3)^+ \).
In this case we were explicitly checking the braiding matrix for \(E_{\alpha_1}, E_{\alpha_2}\) against Heckenberger’s classification of finite-dimensional Nichols algebras \([Heck09]\).

We now turn to the potentially exotic cases in Lemma 4.3 which do not involve odd cycles:

| \(g, \Delta\) | \(\ell\) |
|-------------|--------|
| \(A_2^{(2)}, A_{2n}^{(2)}\) | \(3, 6\) |
| \(G_2^{(1)}, D_4^{(3)}\) | \(4\) |

These were all cases which were not degenerate in the sense that \(\alpha_i^{(0)} = \alpha_i\). We compute in each case the braiding matrix of the braided vector space \(M\) spanned by the \(E_{\alpha_i}\) as we did for the finite root systems in \([Len14c]\) Thm 5.4 and determine the Cartan matrix for the Nichols algebra \(B(M)\) using \([Heck06]\) Sec. 3. Mostly we find that \(B(M)\) is significantly smaller than \(u_\ell^L\) (in terms of real roots) and we thus successively add new primitive elements (some using Lemma 5.1, some by guess-and-check) until we account for all roots \(\alpha\) with \(\ell_\alpha \neq 1\). We treat the cases in order of increasing difficulty:

### 6.1. Case \(G_2^{(1)}\) at \(\ell = 4\).

For \(g = G_2^{(1)}, \ell = 4\) the braiding matrix is

\[
\begin{pmatrix}
q^6 & q^{-3} & 1 \\
q^{-3} & q^6 & q^{-3} \\
1 & q^{-3} & q^2
\end{pmatrix}
\begin{pmatrix}
q^2 & q^{-1} & 1 \\
q^{-1} & q^2 & q^{-1} \\
1 & q^{-1} & q^2
\end{pmatrix}
\]

which is the standard braiding matrix \(q_{ij} = q^{(\alpha_i, \alpha_j)}\) for \(A_3, q' = \bar{q}\). Especially (as in all exotic cases) the \(E_{\alpha_i}\) do not generate the expected root system \(G_2^{(1)}\).

We wish to determine more primitives: We have already seen explicitly in \([Len14c]\) Thm. 5.4 that for \(g = G_2, \ell = 4\) (with indices 1, 2 switched) the element

\[
E_{221} := -q^2(E_1E_2^{(2)} - q^{-6}E_2^{(2)}E_1) - qE_{12}E_2
\]

is a primitive and not in the subalgebra generated by \(E_{\alpha_1}, E_{\alpha_2}\). The new braided vector space spanned by \(E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+2\alpha_2}\) then the new extended braiding...
matrix is easily calculated:

\[
q^{(\alpha_0, \alpha_1+2\alpha_2)} = q^{-3} = q \\
q^{(\alpha_1, \alpha_1+2\alpha_2)} = q^{6-6} = +1 \\
q^{(\alpha_2, \alpha_1+2\alpha_2)} = q^{-3+4} = q \\
q^{(\alpha_1+2\alpha_2, \alpha_1+2\alpha_2)} = q^{6-24+8} = q^2
\]

\[
\begin{pmatrix}
\bar{q}^2 & \bar{q}^{-1} & 1 & \bar{q}^{-1} \\
\bar{q}^{-1} & \bar{q}^2 & \bar{q}^{-1} & 1 \\
1 & \bar{q}^{-1} & \bar{q}^2 & \bar{q}^{-1} \\
\bar{q}^{-1} & 1 & \bar{q}^{-1} & \bar{q}^2
\end{pmatrix}
\]

This is a standard braiding matrix \( q_{ij} = q^{(\alpha_i, \alpha_j)} \) for the affine Lie algebra \( A_3^{(1)} \), \( q' \) with \( q' := \bar{q} \). Especially the 6 roots generated by \( E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+2\alpha_2} \) account for all roots in \( G_2 \). We furthermore check that indeed under this correspondence

\[
\delta_{A_3^{(1)}} = \alpha_0 + \theta_{A_3} \\
= \alpha_0 + \alpha_1 + \alpha_2 + (\alpha_1 + 2\alpha_2) \\
= \alpha_0 + 2\alpha_1 + 3\alpha_2 \\
= \alpha_0 + \theta_{G_2} = \delta_{G_2^{(1)}}
\]

This also shows that the isotropic roots in \( G_2^{(1)}, A_3^{(1)} \) coincide. So the primitive elements \( E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+2\alpha_2} \) generate an affine quantum group of type \( A_3^{(1)} \) which contains all roots of \( G_2^{(1)} \).

6.2. Case \( A_2^{(2)} \) at \( \ell = 3, 6 \).

For \( g = A_2^{(2)} \) the braiding matrix is for \( \ell = 3 \)

\[
\begin{pmatrix}
q^2 & q^{-4} \\
q^{-4} & q^8
\end{pmatrix}
= 
\begin{pmatrix}
q^2 & q^{-1} \\
q^{-1} & q^2
\end{pmatrix}
\]

respectively for \( \ell = 6 \)

\[
\begin{pmatrix}
q^2 & q^{-4} \\
q^{-4} & q^8
\end{pmatrix}
= 
\begin{pmatrix}
(-q)^2 & (-q)^{-1} \\
(-q)^{-1} & (-q)^2
\end{pmatrix}
\]

The braiding matrix for \( \ell = 3 \) is the standard braiding matrix \( q_{ij} = q^{(\alpha_i, \alpha_j)} \) for \( A_2, q \). For \( \ell = 6 \) the braiding matrix is the standard braiding matrix \( q_{ij} = q'^{(\alpha_i, \alpha_j)} \).
for $A_2, q'$ with $q' := -q$ again of order 3.

In both cases this does not generated the root system $A_2^{(2)}$.

We wish to determine more primitives and treat both cases simultaneous using $\epsilon := q^3 = \pm 1$. Similar to the $G_2^{(1)}$-case we find a primitive element $E_{0001} := E_0^{(3)} E_1 - E_1 E_0^{(3)} + \cdots$ in degree $3\alpha_0 + \alpha_1$, which cannot be obtained by commutators of $E_0, E_1$.

The new braided vector spaces spanned by $E_{\alpha_0}, E_{\alpha_1}, E_{3\alpha_0 + \alpha_1}$, then the new extended braiding matrices for $\ell = 3$ resp. $\ell = 6$ are easily calculated:

$$q^{(\alpha_0, 3\alpha_0 + \alpha_1)} = q^{6-4} = q^2 = q^{-1} \text{ resp. } (-q)^{-1}$$

$$q^{(\alpha_1, 3\alpha_0 + \alpha_1)} = q^{-12+8} = q^{-4} = q^{-1} \text{ resp. } (-q)^{-1}$$

$$q^{(3\alpha_0 + \alpha_1, 3\alpha_0 + \alpha_1)} = q^{18-24+8} = q^2$$

These are the standard braiding matrices $q_{ij} = q^{(\alpha_i, \alpha_j)}$ of type $A_2^{(2)}$ with $q' := q$ for $\ell = 3$ resp. $q' = -q$ for $\ell = 6$.

We now want to convince ourselves that the affine quantum group of type $A_2^{(1)}$ generated by the three primitives $E_{\alpha_0}, E_{\alpha_1}, E_{3\alpha_0 + \alpha_1}$ in $A_2^{(2)}$ does indeed generate the full root system of $\Delta = A_2^{(2)}$, which is by Section 2.1

$$\Delta^\text{re} = \left( \{ \pm \alpha_1 \} + 2\delta A_2^{(2)} \mathbb{Z} \right) \cup \left( \frac{1}{2} \{ \pm \alpha_1 \} + \delta A_2^{(2)} + \delta A_2^{(2)} \mathbb{Z} \right)$$

We first compare $\delta A_2^{(2)}, \delta A_2^{(1)}$ under this correspondence:

$$\delta A_2^{(1)} = \alpha_0 + \theta A_2$$

$$= \alpha_0 + \alpha_1 + (3\alpha_0 + \alpha_1)$$

$$= 2(2\alpha_0 + \alpha_1) = 2\delta A_2^{(2)}$$

The verify that the real roots are in bijection it hence suffices to find matching fundamental domains of the action $+2\delta A_2^{(2)}$ on both root systems. The following
works:

\[
\{\pm \alpha_1\} \cup \left(\frac{1}{2}(\{\pm \alpha_1\} + \delta_{A_2^{(2)}}) + \delta_{A_2^{(2)}}\right) \cup \left(\frac{1}{2}(\{\pm \alpha_1\} + \delta_{A_2^{(2)}}) - 2\delta_{A_2^{(2)}}\right)
\]

\[= \{\pm \alpha_1\} \cup \{\alpha_{00011}, \alpha_{00001}\} \cup \{-\alpha_{0001}, -\alpha_{00011}\} \]

\[= \{\pm \alpha_1, \pm \alpha_{0001}, \pm \alpha_{00011}\} \]

As a side remark, note this gets significantly nicer if one rotates $A_2^{(1)}$ such that $\alpha'_0 = \alpha_{0001}$. Hence the real roots of $A_2^{(1)}$ and $A_2^{(2)}$ coincide under our correspondence.

Note however this fails for the isotropic roots: In $A_2^{(2)}$ we have isotropic roots $\delta_{A_2^{(2)}}$ with multiplicities 1, while in $A_2^{(1)}$ we have the isotropic roots $\delta_{A_2^{(1)}} = 2m\delta_{A_2^{(2)}}$ with multiplicity 2. The author does not have an explanation for this.

6.3. **Case $D_4^{(3)}$ at $\ell = 4$.**

For $g = D_4^{(3)}$, $\ell = 4$ the braiding matrix is

\[
\begin{pmatrix}
q^2 & q^{-1} & 1 \\
q^{-1} & q^2 & q^{-3} \\
1 & q^{-3} & q^6
\end{pmatrix}
= \begin{pmatrix}
q^2 & q^{-1} & 1 \\
q^{-1} & q^2 & q^{-3} \\
1 & q^{-3} & q^6
\end{pmatrix}
\]

This is not a braiding matrix of the form $q_{ij} = q^{\alpha_i + \alpha_j}$, but it is nevertheless of type $A_3$ by [Heck06] with $q_{ii} = q^2 = -1 = (q_{ij}q_{ji})^{-1}$ for $i, j$ adjacent. It could be rewritten as a Doi twist of $A_3, q$ or $A_3, \bar{q}$. Especially (as in all exotic cases) the $E_{\alpha_i}$ do not generate the expected root system $D_4^{(3)}$.

We wish to determine more primitives: We have already seen explicitly in [Len14c] Thm. 5.4 that for $g = G_2, \ell = 4$ the element

\[
E_{112} := -q^2(E_2E_1^{(2)} - q^{-6}E_1^{(2)}E_2) - qE_{12}E_1
\]

is a primitive and not in the subalgebra generated by $E_{\alpha_1}, E_{\alpha_2}$. The new braided vector space spanned by $E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_2}, E_{2\alpha_1 + \alpha_2}$ then the new extended braiding
matrix is easily calculated:

\[ q^{(\alpha_0, 2\alpha_1 + \alpha_2)} = q^{-2} = -1 \]
\[ q^{(\alpha_1, 2\alpha_1 + \alpha_2)} = q^{4-3} = q \]
\[ q^{(\alpha_2, 2\alpha_1 + \alpha_2)} = q^{-6+6} = 1 \]
\[ q^{(2\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2)} = q^{8-12+6} = q^2 \]

\[
\begin{pmatrix}
q^2 & q^{-1} & 1 & -1 \\
q^{-1} & q^2 & \bar{q}^{-1} & \bar{q}^{-1} \\
1 & \bar{q}^{-1} & q^2 & 1 \\
-1 & \bar{q}^{-1} & 1 & q^2 \\
\end{pmatrix}
\]

This is not a braiding matrix of the form \( q_{ij} = q^{(\alpha_i, \alpha_j)} \), but it is nevertheless of type \( D_4 \) with center node \( \alpha_1 \) by [Heck06] with \( q_{ii} = q^2 = -1 = (q_{ij} q_{ji})^{-1} \) for \( i, j \) adjacent (note especially now two non-adjacent nodes \( \alpha_0, 2\alpha_1 + \alpha_2 \) anticommute). It could be rewritten as a Doi twist of \( D_4, q \) or \( D_4, \bar{q} \). Especially still the \( E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_2}, E_{2\alpha_1 + \alpha_2} \) do not generate the expected root system \( D_4^{(3)} \).

We wish to determine more primitives: We observe that \( E_0, E_{112} \) anticommute in the \( D_4 \) subalgebra established above even though \( (\alpha_0, \alpha_{112}) = -2 \) in the root system \( D_4^{(3)} \) (this is a typical effect in exotic cases). We get hence from lemma 5.1 that the reflection of \( E_{112} \) on \( \alpha_0 \) is a primitive element in degree \( 2\alpha_0 + 2\alpha_1 + \alpha_2 \). The new braided vector space spanned by \( E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_2}, E_{2\alpha_1 + \alpha_2}, E_{2\alpha_0 + 2\alpha_1 + \alpha_2} \), then the new extended braiding matrix is easily calculated:

\[ q^{(\alpha_0, 2\alpha_0 + 2\alpha_1 + \alpha_2)} = q^{4-2} = q^2 = -1 \]
\[ q^{(\alpha_1, 2\alpha_0 + 2\alpha_1 + \alpha_2)} = q^{-2+4-3} = q^{-1} \]
\[ q^{(\alpha_2, 2\alpha_0 + 2\alpha_1 + \alpha_2)} = q^{-6+6} = 1 \]
\[ q^{(2\alpha_1 + \alpha_2, 2\alpha_0 + 2\alpha_1 + \alpha_2)} = q^{-4+4} = 1 \]
\[ q^{(2\alpha_0 + 2\alpha_1 + \alpha_2, 2\alpha_0 + 2\alpha_1 + \alpha_2)} = q^{8-8+8-12+6} = q^2 \]

\[
\begin{pmatrix}
q^2 & q^{-1} & 1 & -1 & -1 \\
q^{-1} & q^2 & \bar{q}^{-1} & \bar{q}^{-1} & q^{-1} \\
1 & \bar{q}^{-1} & q^2 & 1 & 1 \\
-1 & \bar{q}^{-1} & 1 & q^2 & 1 \\
-1 & q^{-1} & 1 & 1 & q^2 \\
\end{pmatrix}
\]
This is not a braiding matrix of the form \( q_{ij} = q^\ell(\alpha_i, \alpha_j) \), but it is nevertheless of type \( D_4^{(1)} \) with center node \( \alpha_1 \) by [Heck06] with \( q_{ii} = q^2 = -1 = (q_{ij}q_{ji})^{-1} \) for \( i, j \) adjacent.

We now want to convince ourselves that the affine quantum group of type \( D_{4}^{(1)} \) generated by the four primitives \( E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_2}, E_{2\alpha_0 + \alpha_2} \) in \( D_{4}^{(3)} \) does indeed generate the full root system of \( \Delta = D_{4}^{(3)} \), which is by Section 2.1

\[
\Delta^{re} = \left( \tilde{\Delta}^{short} + \delta \mathbb{Z} \right) \cup \left( \tilde{\Delta}^{long} + \delta k \mathbb{Z} \right)
\]

With \( \tilde{\Delta} = G_2 \) and \( \delta_{D_{4}^{(3)}} = \alpha_0 + \theta_{G_2}^{short} = 3 \alpha_{0112} \), this is explicitly

\[
\Delta^{re} = \left( \{ \pm \alpha_1, \pm \alpha_2, \pm \alpha_{112} \} + \alpha_{0112} \mathbb{Z} \right) \cup \left( \{ \alpha_2, \pm \alpha_{1112}, \pm \alpha_{11122} \} + 3 \alpha_{0112} \mathbb{Z} \right)
\]

On the other hand we have for the root system \( \Delta' = D_{4}^{(1)} \) that \( \tilde{\Delta}' = D_4 \) and under the correspondence

\[
\delta_{D_{4}^{(1)}} = \alpha_0' + \theta_{D_4}^{long} = \alpha_0' + (2\alpha_1' + \alpha_2' + \alpha_3' + \alpha_4') \\
= \alpha_0 + (2\alpha_1 + \alpha_2 + \alpha_{112} + \alpha_{00112}) \\
= 3\alpha_0 + 6\alpha_1 + 3\alpha_2 = 3\delta_{D_{4}^{(3)}}
\]

We hence have to convince ourselves whether some fundamental domain for the action \( +3\delta_{D_{4}^{(3)}} \) coincides, say

\[
\tilde{\Delta}^{short} \cup \left( \tilde{\Delta}^{short} + \delta_{D_{4}^{(3)}} \right) \cup \left( \tilde{\Delta}^{short} + 2\delta_{D_{4}^{(3)}} \right) \cup \tilde{\Delta}^{long} \overset{?}{=} \tilde{\Delta}'
\]

This is not completely true, but a slightly more complicated fundamental domain on the right-hand side suffices: As in the finite \( G_2 \) example 6.1, the roots in \( \tilde{\Delta} = G_2 \) are in bijection with the \( A_3 \) subsystem in \( \tilde{\Delta}' = D_4 \) generated by \( \alpha_1, \alpha_2, \alpha_{112} \). The three roots of \( \Delta^{short} + 2\delta_{D_{4}^{(3)}} \) are the three larger roots in \( D_4 \) containing \( \alpha_4' \), namely

\[
\alpha_1 + 2\alpha_{0112} = \alpha_1 + \alpha_{112} + \alpha_{00112} = \alpha_1' + \alpha_3' + \alpha_4' \\
\alpha_{12} + 2\alpha_{0112} = \alpha_1 + \alpha_2 + \alpha_{112} + \alpha_{00112} = \alpha_1' + \alpha_2' + \alpha_3' + \alpha_4' \\
\alpha_{112} + 2\alpha_{0112} = 2\alpha_1 + \alpha_2 + \alpha_{112} + \alpha_{00112} = 2\alpha_1' + \alpha_2' + \alpha_3' + \alpha_4'
\]
The three roots of $\Delta^{\text{short}} + 2\delta_{D_4^{(3)}}$ however correspond to shifted, negative versions of the remaining roots $\alpha_4, \alpha'_4 + \alpha_1 + \alpha'_2 + \alpha'_4$ in $D_4$, namely:

$$\begin{align*}
\alpha_1 + \alpha_{0112} &= -\alpha_1 - \alpha_2 - 3\alpha_{0112} - (\alpha'_1 + \alpha'_2 + \alpha'_4) + \delta_{D_4^{(1)}} \\
\alpha_{12} + \alpha_{0112} &= -\alpha_1 - 3\alpha_{0112} - (\alpha'_1 + \alpha'_4) + \delta_{D_4^{(1)}} \\
\alpha_{112} + \alpha_{0112} &= -3\alpha_{0112} - \alpha'_4 + \delta_{D_4^{(1)}}
\end{align*}$$

Hence the real roots of $D_4^{(1)}$ and $D_4^{(3)}$ coincide under our correspondence.

Note however this fails for the isotropic roots: In $D_4^{(3)}$ we have isotropic roots $m\delta_{D_4^{(3)}}$ with multiplicities $1, 1, 2$ depending on $m \mod 3$, while in $D_4^{(3)}$ we have isotropic roots $m\delta_{D_4^{(1)}} = 3m\delta_{D_4^{(3)}}$ with multiplicity 4. The author does not have an explanation for this.

### 6.4. Case $A_{2n}^{(2)}$ at $\ell = 3, 6$

For $g = A_{2n}^{(2)}$ with $n \geq 2$ the braiding matrix is

$$\begin{pmatrix}
q^2 & q^{-2} & 1 & \ldots & 1 & 1 \\
q^{-2} & q^{4} & q^{-2} & \ldots & \ldots & 1 \\
1 & q^{-2} & q^{4} & q^{-2} & \ldots & \ldots \\
\ldots & \ldots & q^{-2} & q^{4} & q^{-2} & \ldots & \ldots \\
1 & \ldots & \ldots & q^{4} & q^{-4} & \ldots \\
1 & 1 & \ldots & \ldots & q^{-4} & q^{8}
\end{pmatrix}$$

We will verify that the root system and Weyl group in these cases remain $A_{2n}^{(2)}$. This is slightly inconvenient, because we cannot present $u_q^L(g)$ as isomorphic to a different $u_q^L(g')$ with $g', q'$ fulfilling Lusztig’s non-degeneracy assumptions in Lemma 4.1. On the other hand it just means this case is not really degenerate (in contrast to $n = 1$ above).

By [Heck06] Sec. 3 the Cartan matrix of the Nichols algebra of a braided vector space is given by $a_{ii} = 2$ and $a_{ij} = -m_{ij}$ with

$$m_{ij} = \min \{m \mid (m + 1)_{q_{ii}}(q_{ii}^m q_{ij} q_{ji} - 1) = 0\}$$

We have for all $i$ that $q_{ii} = q^2, q^4, q^8$ has order 3 for $\ell = 3, 6$, hence $(m + 1)_{q_{ii}} = 0$ for $m \geq 2$. The term $q_{ij}q_{ji}$ is $= 1$ for non-adjacent $i, j$ and $q^{-4}, q^{-8}$ with again
order 3 for adjacent $i, j$. Hence $q_m^n q_{ij} q_{ji} - 1 = 0$ for $m = 0$ and hence $m_{ij} = 0$ holds precisely for non-adjacent $i, j$. It remains to determine when $q_m^n q_{ij} q_{ji} - 1 = 0$ for $m = 1$ i.e. $q_{ii}(q_{ij} q_{ji}) = 1$. Checking our $2 \times 3$ cases we see this is only possible for $q_{ii} = q^4$, $q_{ij} q_{ji} = q^{-4}$ and $q_{ii} = q^8$, $q_{ij} q_{ji} = q^{-8}$ and $q_{ii} = q^2$, $q_{ij} q_{ji} = q^{-2}$, $\ell = 6$. The last case does not appear for $n \geq 2$, the first case is quite frequent and the second case appears at the last node. Altogether we find the Cartan matrix for $\ell = 3, 6$, and hence the Weyl group, is unchanged:

$$a_{ij} = \begin{pmatrix} 2 & -2 & 1 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ 1 & -1 & 2 & -1 & \cdots & \cdots \\ \cdots & \cdots & -1 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 2 & -2 \\ 0 & 0 & \cdots & \cdots & -1 & 2 \end{pmatrix}$$

7. **Deaffinized cases involving $A_1^{(1)}$ at $\ell = 4$**

Curiously, the case $\mathfrak{g} = A_1^{(1)} = \hat{sl}_2, \ell = 4$ is in some sense the most exceptional case, namely the only one not leading to a (maybe different) affine Lie algebra. This is due to the fact, that all roots are short (hence $u_q^L$ is generated by all $E_{\alpha_i}$), but the Cartan matrix has entries $\pm 2$ (hence the braiding matrix degenerates to $A_1 \times A_1$), but the Hopf algebra is not coradically graded (hence there are nontrivial lifting relations). We shall analyze this case in the following section; not that this in contained in several cases of larger rank in Lemma 5.2.

We first describe the braiding matrix of $\mathfrak{g} = A_1^{(1)}$ for $\ell = 4$:

$$\begin{pmatrix} q^2 & q^{-2} \\ q^{-2} & q^2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

This is not a braiding matrix of the form $q_{ij} = q^{\ell(\alpha_i, \alpha_j)}$, but it is nevertheless of type $A_1 \times A_1$ by [Heck06] with $q_{ii} = -1$ and $q_{ij} q_{ji} = -1$. This implies easily (e.g. Lemma 5.2) that the element in the isotropic degree $\delta = \alpha_0 + \alpha_0$

$$E_{\alpha_0 + \alpha_1} := E_{\alpha_0} E_{\alpha_1} + E_{\alpha_1} E_{\alpha_0}$$

is a new primitive element. For a coradically graded Hopf algebra (e.g. the Nichols algebra) this would imply the commutator is zero as expected for $A_1 \times A_1$; the same holds for all other commutators between higher root vectors and one might
expect our $A_1^{(1)}$ to degenerate to $A_1^{\infty}$. We will see that this is not true: While indeed $\text{gr}(u_q^\infty(g))$ will be of type $A_1^{\times\infty}$ (one copy for each root), some nontrivial commutator relations will survive in $u_q^\infty(g)$.

To pursue our study, we need to calculate in an explicit PBW-basis by Beck for quantum groups of untwisted affine Lie algebras. The given alternative set of generators and relations for $U_q^L(g)$ is essentially due to Drinfel'd and emphasizes the construction of $g$ from a loop algebra (see [Beck94] Thm 4.7):

Let $\bar{I}$ again denote the index set of the finite Lie algebra associated to $g$, then we have generators ("root vectors") $x_{\pm}^{\pm,\alpha i,j}$ in degree $\pm \alpha_i$ and $h_{\alpha_i,k}, k \neq 0$ in degree $k\delta$ and $K_i^{\pm,1}, C^{\pm,1}, D$ in degree 0. We do not give all relations here. Now [Beck94] Prop. 6.1 states that these elements form a PBW-basis for $U_q^L(g)$ and hence the specialization $U_q^L(g)$.

Relation (5) states that $[x_{\alpha_i,0}^{+}, x_{\alpha_i,1}^{-}] \neq 0$ in any specialization $\ell \neq 1, 2$; especially in our case

$$[x_{\alpha_i,0}^{+}, x_{\alpha_i,1}^{-}] = C^{-\frac{1}{2}} K_{\alpha_1} \cdot h_{\alpha_1,1} + C^{\frac{1}{2}} K_{\alpha_1}^{-1} \cdot h_{\alpha_1,1}$$

On the other hand relations (2), (3) states that the commutators $[h_{\alpha_i,k}, h_{\alpha_i,-k}]$ and $[h_{\alpha_i,k}, x_{\pm,\alpha_i,l}]$ contain a term $[ka_{ij}]_q$. This implies in our case $a_{ij} = -2$ and $\text{ord}(q_i) = \text{ord}(q) = \ell = 4$ that all $h_{\alpha_i,k}$ are central in the specialization. Hence the algebra generated by $x_{\alpha_i,0}^{+}, x_{\alpha_i,1}^{-} = E_1, E_0$ is finite-dimensional and has a root system of type $A_2$. It is not coradically graded, rather the respective graded Nichols algebra is of type $A_1^{\times 3}$.

By definition, the reflections of $x_{\alpha_i,0}^{+}, x_{\alpha_i,1}^{-}$ are the higher root vectors $x_{\alpha_i,1}^{+}, x_{\alpha_i,1}^{-}$. The formula in Lemma 5.2 easily shows that they are not primitive. Hence $u_q^\infty A_1^1$ is not generated in degree 1. We have

$$u_q^\infty(A_1^1)\langle E_1, E_0 \rangle \cong u_q^\infty(A_1^1)^+$$

and hence $u_q^\infty(A_1^1)^+$ is an infinite tower extension of $u_q^\infty(A_2)$'s; the higher order lifting relations can be read off directly from the cited relations.
8. Open Questions

We finally give some open questions that the author would find interesting:

Regarding the subsystem of long roots:

**Problem 8.1.** Is there an easier systematic reason why the subsystem of long roots always returns the affinized subsystem of long roots of the corresponding finite Lie algebra - both for twisted and untwisted type?

Regarding the limitations of this article:

**Problem 8.2.** Explicitly determine the algebras $u_L^\infty(g)$ in terms of some $u_q'(g(0))$.

- Is the surjection $u_L^\infty(g)^+ \to B(M) = u_q'(g(0))^+$ also for twisted affine $g$ an isomorphism? By construction this is true on the level of roots, but there seems at present to be no PBW-basis available in this case.
- Are there nontrivial liftings in the exotic cases? It seems this would require explicit calculations.
- Present $u_L^\infty(g)$ as a quotient of the Drinfel’d double of $u_q^\infty(g)^+$.

More generally, describe the affine quantum groups in terms of coradical filtration, Nichols algebra and a Drinfel’d double construction. Can the representations be related to the Yetter-Drinfel’d modules of the Borel part? Can one clarify the impact of the second condition (odd cycles) in [Lusz94]?

Regarding the construction of a Frobenius homomorphism:

**Problem 8.3.** The Hopf subalgebras $u_L^\infty(g)$ constructed in this article are a good candidate for the kernel of a Frobenius homomorphism. In cases where we have a Frobenius homomorphism, namely [Lusz94] for sufficiently large $q$ and [Len14c] for the remaining small $q$ and finite root system, it coincides. Is it possible to extend the approach in [Len14c], namely showing normality of this $u_L^\infty(g)$ and then analyzing the quotient, to construct a Frobenius homomorphism for affine quantum groups for arbitrary $q$? It is to be expected that the quotient is not simply the universal enveloping of the same Lie algebra.

Regarding an observation that puzzled the author during the work on this article:

**Problem 8.4.** While the subalgebras $u_L^\infty(g)^+$ exhaust by construction precisely the real roots, the multiplicities of the isotropic roots sometimes do not coincide:
• We have constructed a map $u_L^q(D_n^{(1)})^+ \to u_L^q(A_{2n-1}^{(2)})^+$ with coinciding $\delta$. However, for $A_{2n-1}^{(2)}$ the multiplicities are $1, 1, 2, 1, 1, 2, \ldots$ while for $D_n^{(1)}$ they are as usual $2, 1, 2, 2, 2, 2, \ldots$. A similar effect occurs for $E_6^{(2)}, D_4^{(3)}$. This could mean two both unlikely things: Either the map is not injective (but this is absurd for a map from a Nichols algebra) or the isotropic root multiplicities in $U_L^q(A_{2n-1}^{(2)})$ are not as for $A_{2n-1}^{(2)}$. This would be surprising, but note that the root system itself does not determine the multiplicities, they are rather calculate from the twisted presentation.

• In the exotic cases there seems a systematic behaviour: Take as example $D_4^{(3)}$ at $\ell = 4$ which has a $D_4^{(1)}$ root system with $3\delta_{D_4^{(3)}} = \delta_{D_4^{(1)}}$. The isotropic root multiplicities are $1, 1, 2, 1, 3, \ldots$ respectively $0, 0, 4, 0, 4, \ldots$ - they seem just to be shifted, the same hold for the other exotic cases. Now: One can easily obtain primitive elements (with trivial self-braiding) for the first isotropic roots in $D_4^{(3)}$, especially they are not in $D_4^{(1)}$, and they turn out to still reside “inside” the first set of commutator relations. Intuitively, do they extend $D_4^{(1)}$ from below? Are their powers or other commutators than the elements in degree $3\delta$ we observe in $D_4^{(1)}$?

Regarding parabolic subalgebras:

**Problem 8.5.** Let $J \subset I$ a subset of simple roots, then we have a parabolic subalgebra $g_J \subset g$ of finite type and $U_L^q(g_J) \subset U_L^q(g)$ and $u_L^q(g_J) \subset u_L^q(g)$. By the Radford projection theorem we can view $u_L^q(g)^+$ as a Hopf algebra in the category of $u_L^q(g_J)^+$-Yetter-Drinfel’d modules (hence morally a module over $u_L^q(g_J)$, which is finite-dimensional). It would be extremely interesting to study this source of infinite-dimensional representations of finite quantum groups. The representation should decompose into infinitely many finite-dimensional modules associated to the finite root strings in $g$.

Regarding applications:

**Problem 8.6.** Are there certain small values for the coupling constants in affine Toda theory such that the symmetry is described by the quantum affine algebras constructed here? Is this degeneracy physically visible? (as in the finite case $B_n, \ell = 4$ for $n$ symplectic Fermions in [Len14c])

**Problem 8.7.** It would be extremely interesting to study quantum groups for hyperbolic Lie algebras, at least in examples, by using techniques from Nichols algebras.
Are there again “deaffinized” cases which decompose into an infinite tower of finite Lie algebras (or something related)?

**Problem 8.8.** It is a curious observation, that the degenerate cases \( g, q \) in \([\text{Len}14c]\) for \( q \) finite (having \( h \) a root system different than \( g \)) seem to match the list of Nichols algebras with root system \( g \) and self-braiding \( q \) over nonabelian groups constructed by the author in \([\text{Len}14a]\) as diagram-folding of different \( u_q(g') \); as if they would be somehow replaced.

One should use a presentation of affine Lie algebras by folding of other affine Lie algebras as indicated in Section 2.1 (this is not the same as the construction of twisted Lie algebras) and use this to construct quantum affine algebras with nonabelian Cartan part resp. affine Nichols algebras over nonabelian groups. Of special interest should be the case \( A_n^{(1)} \) and \( C_n^{(1)} \) which can be folded successively many times.

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