Self-induction and magnetic effects in electron transport through a photon cavity

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We explore higher order dynamical effects in the transport through a two-dimensional nanoscale electron system embedded in a three-dimensional far-infrared photon cavity. The nanoscale system is considered to be a short quantum wire with a single circular quantum dot defined in a GaAs heterostructure. The whole system, the external leads and the central system are placed in a constant perpendicular magnetic field. The Coulomb interaction of the electrons, the para- and diamagnetic electron-photon interactions are all treated by a numerically exact diagonalization using step-wise truncations of the appropriate many-body Fock spaces. We focus on the difference in transport properties between a description within an electric dipole approximation and a description including all higher order terms in a single photon model. We find small effects mostly caused by an electrical quadrupole and a magnetic dipole terms that depend strongly on the polarization of the cavity field with respect to the transport direction and the photon energy. When the polarization is aligned along the transport direction we find indications of a weak self-induction that we analyze and compare to the classical counterpart, and the self-energy contribution of high-order interaction terms to the states the electrons cascade through on their way through the system. Like expected the electron-photon interaction is well described in the dipole approximation when it is augmented by the lowest order diamagnetic part for a nanoscale system in a cavity in an external magnetic field.

I. INTRODUCTION

The non-perturbative coupling of two-dimensional (2D) electrons in a magnetic field in a heterostructure with high-quality-factor terahertz or far-infrared photons has been achieved [1]. The coupling of electronic systems in circuit QED planar microwave cavities to external leads gives the hope that this will also be accomplished for the terahertz systems [2–6]. Several groups have modeled various aspects of the transport of electrons through nanoscale systems placed in photon cavities [7–11], just to mention few. Like for most physical phenomena, different modeling approaches have been applied ranging from non-equilibrium Green functions [12–14] to master equations of various types [15–17]. The modeling efforts have not been straight forward, and some authors have emphasized the role of the geometrical shape of the systems [9], while other have explored how appropriate the use of a single cavity mode is [18], with respect to causality, or if relativistic corrections have to be taken into account [19], besides the fundamental issues on the derivation of an appropriate description when using a master equation approach [20, 21].

Various approaches have been used for the matter-photon interactions in different systems of spins, electrons, or atoms placed in a cavity. Naturally, most can be referred back to an electrical dipole interaction, expressed either in terms of a space integral over the inner product of the vector potential A and the charge current density j, or the integral of the inner product of the electrical field and the position operator [22, 23]. Some models have included the much weaker diamagnetic interaction, that is often referred to as the A-square term [24–27].

Analytical or high-order numerical methods lead to results of high accuracy within the electrical dipole interaction approximation, relying or not [28, 29] on the rotating wave approximation [22, 23].

The dipole approximation is known to be valid if the wavelength of the electromagnetic field is usually much larger than the size of the matter system. Magnetic effects in the interaction will only be evident as the variation of the electromagnetic vector potential is taken into account within the matter or the electron system. They are thus bound to be tiny in most cases.

Earlier, we have shown that high order transitions can be important when describing the approach of systems to a steady state in which otherwise low order forbidden transitions play a role [30]. This has lead us to the questions: What about the higher order electric and magnetic terms of the para- and diamagnetic electron-photon interactions? Do they lead to effects, that among others, can be understood as self-induction in time-dependent

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electron transport through a system with a nonzero spatial extension in a photon cavity?

To explain our ideas, let us consider the following set-up: Initially, the external leads are coupled to an empty central system, i.e., neither an electron nor a photon are present, the initial system is in its vacuum state. The coupling to the external leads offers only electrons to the system, no photons. We consider an external photon reservoir coupled to the cavity to be at zero temperature, so no photons enter the system from the photon reservoir. The electron-photon coupling in the central systems turns all many-body states into either pure photon states, or electrons states dressed with cavity photons. Thus, after the coupling to the leads the photon expectation value in the cavity increases and can become large depending on which dressed electron states are in the bias window defined by the external leads. In this set-up the photon content of the central system will reach a maximum at some intermediate time and decrease as the system reaches a steady state.

In a classical system this emergence of an electromagnetic component in transport could lead to the consideration of inductive effects if higher order terms were considered for the electron-photon interaction. For a “few-body” quantum system one expects the identification of such inductive effects probably to be obscured by the quantum details of countably few transport paths or transitions through the system. Comparison with classical circuits is further hindered by the simplicity of the approach to view few-body high frequency quantum circuits using classical terms as “lumped elements”, with separable capacitive and inductive elements.

In quantum electrodynamics (QED) all higher order terms of an electron-photon interaction, describing processes during which an electron interacts with its own electromagnetic field, are included in the self-energy of the electron. Here, in a confined central system the discrete electron states all acquire a higher-order self-energy that depends on the geometry of both the electron states and the photon field in the cavity in addition to other properties the states have.

The paper is organized as follows: Section II gives a short description of the model with the time-independent properties in the Subsection II A, but the time-dependent transport in Subsection II B. Results are presented in Section III, with general quantities outlined in subsection III A, while differences between the models with different electron-photon interactions are displayed in Subsection III B. Conclusions are drawn in Section IV.

II. MODEL

We model a short quantum wire of length $L_z = 180$ nm in the $x$-direction with a parabolic confinement with characteristic energy $\hbar \Omega_w = 2.0$ meV in the $y$-direction. We assume a hard wall confinement at the ends in the $x$-direction and a constant external magnetic field $B = Bz$ with strength $B = 1.0$ T perpendicular to the two-dimensional quantum wire assumed to be formed in a GaAs heterostructure with effective mass $m^* = 0.067m_e$, dielectric constant $\epsilon_c = 12.4$, and $g^* = -0.44$. Embedded in the short wire is a quantum dot potential as is shown in Fig. 1 The short quantum wire with the dot is defined by the potential

$$V(x, y) = \theta \left( \frac{L_z}{2} - |x| \right) \left[ \frac{1}{2} m^* \Omega_w^2 y^2 - e V_g + \sum_{i=1}^{2} \exp \left\{ -\beta^2 x^2 - \beta^2 (y - y_{0i})^2 \right\} \right],$$

formed by two strongly overlapping Gaussian dips in order to get an almost circular shape in the parabolic confinement. The parameters are $V_d = -6.0$ meV, $\beta = 0.018$ nm$^{-1}$, defining their depth and extent, $y_{01} = -48$ nm, and $y_{02} = +48$ nm defining their location. $V_g$ is a plunger gate voltage used to raise or lower certain states into the bias window. Here, it will be set to $V_g = 2.47$ meV.

A. Time-independent properties

The Hamiltonian of the central system is

$$H_S = \int d^2r \psi^\dagger \left\{ \frac{\pi^2}{2m^*} + V(r) \right\} \psi(r) + H_{EM} + H_{Coul} + H_Z + \frac{1}{c} \int d^2r \mathbf{j}(r) \cdot \mathbf{A} + \frac{\epsilon^2}{2m^*e^2} \int d^2r \rho(r) A^2_z,$$

FIG. 1. The potential, $V$, defining the shape of the central system, a circular quantum dot embedded in a short quantum wire. The length of the wire is 180 nm, extending from $x/a_w \approx -4.3$ to $x/a_w \approx +4.3$, where $a_w \approx 20.75$ nm is the effective magnetic length for the parabolic confinement in the $y$-direction with characteristic energy $\hbar \Omega_w = 2.0$ meV. The tiny gaps at either end of the short wire indicate the onset of the external leads with same confinement in the $y$-direction.
in terms of the fermionic field operators \( \psi \) and \( \psi^\dagger \) for elec-
trons, the probability density \( \rho = \psi^\dagger \psi \), and the charge
current density \( \mathbf{j} = -e\{\psi^\dagger \mathbf{p} \psi + \mathbf{p}^\dagger \psi^\dagger \psi\}/(2m^* \) \), with \( \mathbf{p} = (p + e\mathbf{A}_{\text{ext}}/c) \), where \( \mathbf{A}_{\text{ext}} = (- By, 0, 0) \) is the vector
potential defining the external constant magnetic field \( \mathbf{B} \). The
external magnetic field together with the parabolic confine-
ment energy \( \hbar \Omega_0 \) define the effective confinement energy \( \hbar \Omega_\mathbf{p} = \hbar (\omega^2 + \Omega_0^2/\omega_0^2)^{1/2} \) and the effective mag-
netic length \( a_w = (\hbar/(m^* \Omega_0))^{1/2} \), where \( \omega_0 = (eB_{\text{ext}})/(mc) \)
is the cyclotron frequency. We will explore properties of the cen-
tral system for \( B = 1.0 \, \text{T} \), such that \( a_w \approx 20.75 \)
nm and \( \hbar \Omega_\mathbf{p} \approx 2.642 \, \text{meV} \). \( H_{\text{EM}} = \hbar \omega a/\alpha \) is the Hamil-
tonian for the single-mode cavity with energy \( \hbar \omega \), \( H_Z \) is the Zeeman term for the electrons, and \( H_{\text{Coul}} \) is the Coulomb interaction of the electrons with a kernel
\[
V_{\text{Coul}}(\mathbf{r} - \mathbf{r}') = \frac{e^2}{\kappa e \sqrt{|\mathbf{r} - \mathbf{r}'|^2 + n^2}} \quad (3)
\]
where a small regularization parameter \( \kappa e/a_w = 3 \times 10^{-7} \)
has been used. The electron-photon interactions terms in the 3rd line of Eq. (2) are the paramagnetic- and the
diamagnetic parts, respectively.

We model a rectangular photon-cavity with dimensions \( a_x \times b_y \times d_z \), using the Coulomb gauge for the quantized
vector potential \( \mathbf{A}_z \) for the single-mode photon of the
field. The electric field of the cavity photons is aligned to the transport in the \( x \)-direction (with the unit vector \( \mathbf{e}_x \) in the \( T_{011} \) mode, and perpendicular to it (defined by the unit vector \( \mathbf{e}_y \) in the \( T_{101} \) mode). The general configuration of the electric and magnetic com-
ponents of the cavity field are schematically displayed in
Fig. 2 for the two linear polarizations. The quantized

\[
\mathbf{A}_{\gamma i}(\mathbf{r}) = \hat{e}_i \cdot \mathbf{A} \{ a + a^\dagger \} u_i(z) \quad (4)
\]

where \( i = x \) or \( y \) labels the direction of the polarization
and
\[
\begin{align*}
u_x(z) &= \cos \left( \frac{\pi y}{b_c} \right) \cos \left( \frac{\pi z}{a_c} \right) \quad \text{and}, \\
u_y(z) &= \cos \left( \frac{\pi x}{a_c} \right) \cos \left( \frac{\pi z}{d_c} \right) \quad (5)
\end{align*}
\]

\( a \) is the annihilation operator for a cavity photon, and \( a^\dagger \)
is the corresponding creation operator. The magnitude of the vector potential, \( \mathbf{A} \), and the electron-photon coupling
constant are related by \( g_{\text{EM}} = e\mathbf{A} \omega a/\alpha \). For the cavity
with photon energy \( \hbar \omega = 2.63 \, \text{meV} \) the ratio of the size of the cavity to the length of the central system, \( a_c/b_x \), is 26.40, and for the energy \( \hbar \omega = 0.98 \, \text{meV} \) it is 70.7, so as expected the higher-order terms will be small. Simplicitly, one might expect the higher photon energy
leads to increased higher order effects, but this view will be
challenged below.

For an easier comparison of the electron-photon interaction
used in other models we use the vector potential (4) and the field operators for the electrons to rewrite the
Hamiltonian for the electron-photon interactions as [30]
\[
H_{e-\text{EM}} = g_{\text{EM}} \sum_{ij} d_i^\dagger d_j^\dagger \mathbf{g}_{ij} \{ a + a^\dagger \} \\
+ \frac{g_{\text{EM}}^2}{\hbar \Omega_w} \sum_{ij} d_i^\dagger d_j^\dagger \mathbf{g}_{ij} \left[ \left( a^\dagger a + \frac{1}{2} \right) + \frac{1}{2} (a^\dagger a + a a) \right],
\quad (6)
\]

where \( d_i \) is the annihilation operator for an electron in the
single-particle state labeled by \( i \), and \( d_i^\dagger \) is the cor-
responding creation operator. The definition of the coupling
matrices (in the \( z = 0 \) plane) [30] in terms of the wavefunctions of the original single-electron basis \( \{|a\} \)
is
\[
g_{ab}^{di} = \langle a|\{u_i(0)\}^2|b\rangle, \quad (7)
\]

with \( i = x \) or \( y \) for the diamagnetic electron-photon in-
teraction. If the size of the cavity would be assumed to
approach infinity (compared to \( L_x \) \( u_i(0) \) \to 1 and then
\( g_{ab}^{di} \to \delta_{a,b} \). The coupling matrix for the paramagnetic
interaction is written symbolically as
\[
g_{ab}^{pi} = \frac{a_w}{2\hbar} \langle a|\{\hat{e}_i \cdot \mathbf{\pi} \} u_i(0)|b\rangle, \quad (8)
\]

where the basis \( \{|a\} \) consists of the original single-
electron states of the wire-dot system with no interac-
tion to the photons. As the counter rotating terms in the
electron-photon interactions are important [32–34],
especially for the self-energy of the dressed states [30] we do not apply the rotating wave approximation. The eigenstates of the central system described by the Hamiltonian (6) are denoted by \( |\tilde{\mu}\rangle \).

If in Eqs. (7-8) the spatial variation of the vector po-
tential of the photon field, \( \mathbf{A}_z(\mathbf{r}) \) over the electron system
is neglected, we have a kind of a dipole interaction,
but including the diamagnetic electron-photon interac-
tion. Note, that in this case we still use the exact numerical
diagonalization that thus includes higher order terms
of the dipole origin into the self-energies of the dressed electron many-body states. We will notice below that at least the lowest order of the diamagnetic interaction is necessary due to the strong external magnetic field having orbital effects.

B. Transport description

The external leads have the same parabolic confinement as the short quantum wire and are subjects to the same perpendicular magnetic field. Their coupling to the central system at $t = 0$ is described by the Hamiltonian

$$H_T = \theta(t) \sum_i \int dq \left( T_{qi} c_{qi}^{\dagger} d_i + (T_{qi})^* d_i^{\dagger} c_{qi} \right).$$

Electrons in the short wire are created (annihilated) by the operators $d_i^{\dagger}$ ($d_i$), and in the leads by the operators $c_{qi}^{\dagger}$ ($c_{qi}$). The quantum number $q$ stands both for the continuous momenta in the leads and the appropriate subband index. Due to the shape of the central system it is essential to account for the fact that states in the leads and the central system couple differently well or badly depending on their energy and the shape of their probability densities. Thus, the coupling matrix $T_{qi}$ is calculated using the probability density of each single-electron state of the lead $l$ and the central electron system in the contact region that is defined to extend approximately one $a_w$ into each subsystem [35–37]. See also Appendix A in Ref. [30]. The coupling of the external leads to the central system has an overall strength $g_0 g_{lR} a_w^{3/2} = 0.101$ meV, when the average dimensionless coupling constant $g_0 = 1.0$.

The time evolution of the central system under the influence of the reservoirs, the external leads and the photon reservoir, is calculated using a Markovian master equation for the reduced density operator $\rho_S$ in the Liouville space of transitions [38–40], that has been derived from a generalized master equation [41, 42] in a many-body Fock space by applying a Markovian approximation and a vectorization of the matrices [30, 43, 44]. The coupling of the cavity to the external photon reservoir is $\kappa = 1.0 \times 10^{-5}$ meV, and the average photon number in the reservoir is set by the parameter $n_R$ [45].

From the generalized master equation we derive the mean current [35–37, 45] into the central system, $I_L$, from the left lead (L), and the mean current out of the central system, $I_R$, into the right lead (R). For comparison with simple classical circuit it would be convenient to consider the mean net current through the central system. Due to the capacitance and charging of the central system this quantity is not well defined, and we use the mean current, $I = I_L + I_R$, as a measure of it. Convenitely, in the regime when $I_L = I_R$, $I_L$ is double this net current through the system, but the net current into the central system, $I_i = I_L - I_R$, vanishes.

In addition to calculating the mean current, $I$, we evaluate the mean number of electrons in the central system, $N_e$, the mean number of photons $N_g$, the mean value of the $z$-component of the total spin $S_z$, and the Rényi-2 entropy of the central system [46–48]

$$S = -k_B \ln [Tr(\rho_S^2)].$$

The mean entropy (10) serves as a sensitive measure of changes in the central system, and we only use it for that purpose here.

Information about details in the numerical computations is found in Appendix A.

III. RESULTS

We first concentrate on properties of the system with the full electron-photon interactions. Then we analyze the differences of these quantities between a system with the full electron-photon interactions and a system where the dipole approximation is considered.

A. Transport results for numerically exact one-mode electron-photon interaction

The external homogeneous magnetic field $B = 1.0$ T. We select the chemical potentials of the left and right lead to be $\mu_L = 1.7$ meV and $\mu_R = 1.4$ meV defining a bias window $\Delta \mu$ of 0.3 meV. The plunger gate voltage $V_g$ is set such that $-eV_g = 2.47$ meV. Further more we will explore the transport properties of the system for two values of the cavity photon energy, $\hbar \omega = 0.98$, and 2.63 meV. The one-electron ground state of the quantum dot (and the central system) is a typical circular symmetric state with vanishing angular momentum, while the next one-electron states in energy are typical circular ring states with unit angular momenta quantum numbers. In the Fock-Darwin energy spectrum for a circular symmetric parabolically confined quantum dot in a magnetic field the radial wavefunctions are the same for the $M = \pm 1$ states [49]. In our case they are very similar, but the higher one (in the bias window) look slightly elliptical due to the opening of the dot into the short wire in the $x$-direction. In our quantum dot the energy difference between these two states is close to the lower photon frequency, leading to a very special Rabi-splitting as will be seen below. We select these circular states of a quantum dot in an external magnetic field to enhance possible self-induction in the system. At the same time the one-electron ground state is in resonance with the first excited one-electron state.

As Figure 3 shows the one electron ground state $| \bar{1} \rangle$ and its partner with opposite electron spin $| \bar{2} \rangle$ (corresponding to the $M = 0$ states of the Fock-Darwin spectrum) are well below the bias window. For the case of $\hbar \omega = 2.63$ meV only the two spin partners of the second excited one-particle states $| \bar{6} \rangle$ and $| \bar{7} \rangle$ (corresponding to the higher
\[ M = \pm 1 \text{ Fock Darwin states, } |4\rangle \text{ and } |5\rangle \text{ correspond to the lower ones) are within the bias window, but there is a Rabi-resonance between two spin partners of the ground state and states slightly above the bias window, such that the lower Rabi branch is either of the spin partners } |9\rangle \text{ or } |10\rangle, \text{ and the upper branch is } |12\rangle \text{ or } |13\rangle. \text{ This can best be seen by the mean photon content of these dressed one-electron states which is close to half a photon.}

In this case the photon content of the dressed one-electron states in the bias window is small. When the photon energy is \( \hbar \omega = 0.98 \text{ meV} \) there is, on the other hand, a Rabi-resonance between the second and the third excited one-electron states in the system on top of a resonance between the ground state and the first excited state. We thus end up with six one-electron states in the bias window, all with a noninteger photon content. For this case one would expect a faster charging of the system and it should likewise approach the steady state as most transitions needed would be photon aided. We also notice from Fig. 3 that in the case of \( \hbar \omega = 0.98 \text{ meV} \) (left subfigures) the photon content of the dressed states is more dependent on the polarization of the photon field than for \( \hbar \omega = 2.63 \text{ meV} \) (right subfigures).

Indeed, in Fig. 4 we see this photon enhancement of all transitions leading to a faster charge for \( \hbar \omega = 0.98 \text{ meV} \) (upper) than for \( \hbar \omega = 2.63 \text{ meV} \) (lower). This can be verified in Fig. 4 by the difference in the mean photon content for either case with different photon energy. We notice that the steady states for these two cases are different as can be seen by their difference in the mean spin \( z \)-component. We come back to this below.

Fig. 5 displays the mean currents for either photon energy, the current from the left lead into the central system, \( I_L \) labeled (L), the current from the central system into the right lead, \( I_R \) labeled (R), and the current \( I = I_L + I_R \) labeled (T). As \( n_R = 0 \) and there are several one-electron states below the bias window and the lowest two-electron states are above it, the system is in a Coulomb blockade in the steady state. If \( n_R = 1 \), or higher, the current will not vanish in the steady state.

Importantly, there is a time interval just when the system is getting fully charged where the left and right currents are almost the same. During this time interval the
currents $I_{lR}(R)$, and the current $I = I_{lR}(R)$, for photon energy $h\omega = 0.98$ meV (upper), and $h\omega = 2.63$ meV (lower). $g_{EM} = 0.1$ meV, $B = 1.0$ T, $\bar{n}_{R} = 0$, $\kappa = 1.0 \times 10^{-3}$ meV, $-\epsilon V_{k} = 2.47$ meV, $L_{x} = 180$ nm, and $g_{0}g_{L}R_{\omega}a_{\omega}^{1/2} = 0.101$ meV.

The results shown in Figs. 4 and 5 allow us to estimate two time scales relevant for classical circuits. First, using the size of the bias window, $\Delta \mu = 0.3$ meV, and the average current outside the steady state, $I = 0.1$ nA, we arrive at the contact resistance $R \approx 3$ M$\Omega$. Secondly, from the geometry of the electron system we estimate its coefficient of inductance to be $L \approx 36$ Hz. Together these give the time scale of the inductance to be $\tau_{L} \approx 1.2 \times 10^{-20}$ s = $1.2 \times 10^{-8}$ ps. This shows plainly that we cannot expect to observe this $\tau_{L}$ in the present system.

From the charging time $\tau_{C} \approx 10^{6}$ ps (see Fig. 6) and the contact resistance we estimate the capacity of the electron system to be approximately $C \approx 1$ pf. If our system could be compared to an RCL circuit this allows us to extract the resonance frequency of oscillations to be $\omega_{0} = 1/\sqrt{LC} \approx 5.3$ (ps)$^{-1}$, or their energy $h\omega_{0} \approx 3.5$ meV. Interestingly, this resonance energy is inside the part of the energy spectrum we are exploring, but as we are dealing with few particles in a system with discrete energy levels we can not expect to find this classical resonance energy in our quantum system.

Looking at Fig. 4 and 6 one has to wonder why there are no Rabi-oscillations seen in the density or the occupation of states like has been seen for different dot systems [30, 31]. We would expect them, at least for the case of the photon energy $h\omega = 0.98$ meV, when there are Rabi-split states in the bias window. The answer lies in the combination of high external magnetic
field ($B = 1.0$ T), and the simple circular dot potential in the central system. Like, before we draw an analogy with the Fock-Darwin energy spectrum [49], mentioned above, the lowest Rabi-resonance is between states with opposite angular momenta, but having the same radial wavefunctions. The resonance thus leads to circular oscillations in the local current density in the central system. These divergence-free oscillations can not lead to oscillations in charge density. We have thus “transverse” Rabi-oscillations that will be clear in the current, but not in the density. This behavior will even be clear on a higher resolution scale introduced below.

This resonance between the lowest lying ring-formed states corresponding to the lowest $M = \pm 1$ Fock-Darwin states would not appear in the lowest order dipole approximation, and at the same time there is a resonance between the one-electron ground state and the lower of the $M = \pm 1$ states that would be enabled in a dipole approximation. The Rabi-resonance caused by the diamagnetic part of the electron-photon interactions occurs even for constant vector potential inside the electron system, as it is facilitated by the external constant magnetic field, that leads to divergent-free rotating currents in the system [25].

B. Exploring dynamical differences

All the results described above have been calculated using a numerically exact diagonalization for the single-mode electron-photon interactions. We now look for dynamic effects of the higher order terms in the results. To this end we assume the spatial variation of the vector potential $\mathbf{A}$, to vanish inside the electron system itself. One might call that a dipole approximation, but within it we also keep the diamagnetic interaction, which is usually not done in a dipole approximation, and we use the exact numerical diagonalization for both cases. Within this approximation we calculate the same quantities as in subsection III A and then compare the two sets of results by defining $\Delta N_e = N_e^{\text{exact}} - N_e^{\text{dipole}}$, $\Delta N_\gamma = N_\gamma^{\text{exact}} - N_\gamma^{\text{dipole}}$, and $\Delta I = I^{\text{exact}} - I^{\text{dipole}}$. Fig. 7 displays $\Delta N_e$ and $\Delta N_\gamma$ in the central system for the whole time range from the transient regime to the steady state. The figure shows results for both photon energies and polarizations, and for the photon energy $\hbar \omega = 2.63$ meV it shows results for three different values of the overall system-lead coupling coefficient $g_0$. We notice that especially in the intermediate time range (ITR) the difference in the electron and photon number becomes larger for the $x$-polarization than the $y$-polarization.

Three of the subfigures of Fig. 7 show irregular oscillations looking like noise. This is not the case here. When a higher resolution is used for a narrower time interval (as will be done below) the irregular oscillations are replaced by very regular Rabi oscillations.

In the lower panels of Fig. 7 for photon frequency $\hbar \gamma = 2.63$ meV, when no photon replicas or other electron states with a high expectation value of photons is in the bias window we see that in the ITR both the expectation values for electrons and photons is reduced for the $x$-polarized cavity field. The time variation of both $\Delta N_e$ and $\Delta N_\gamma$ can be correlated with changes in the occupation of the states of the central system seen in the lower panel of Fig. 6 if one considers at the same time the change in the self-energies of the corresponding states due to the higher-order electron-photon interaction shown in lower panel of Fig. 15 in Appendix B, and the changes in the occupation of the states induced by the same terms seen in the lower panel of Fig. 14 in the same Appendix. The photon energy in the lower panels in Fig. 7 $\hbar \omega = 2.63$ meV is just below the effective confinement energy of the short quantum wire, $\hbar \Omega_w = 2.642$ meV favoring polarization of the electron charge density in the $x$-direction, but at the same time allowing for a small polarization in the $y$-direction. In the upper panels the photon energy is $\hbar \gamma = 0.98$ meV, so polarization of the charge density in the $y$-direction can only be very tiny, but as there is now a multiple Rabi resonance with the
states in the bias window, the system can effectively be polarized in the \( x \)-direction. Correspondingly, the time variation of both \( \Delta N_e \) and \( \Delta N_\gamma \) in the upper panels of Fig. 7 is tiny for the \( y \)-polarized cavity field, but for the \( x \)-polarized field the effects are larger and \( \Delta N_\gamma \) shows clear signs of Rabi-oscillations.

For the lower photon energy, \( \hbar \omega = 0.98 \text{ meV} \), we see in Appendix B in the upper panel of Fig. 15 that, indeed, the self-energies of the relevant states due to the higher-order terms in the electron-photon interactions are vanishingly small for the \( y \)-polarized cavity field, but for the \( x \)-polarized field the states in the bias window acquire no or relatively large positive or negative self-energy contribution. The early in time photon active transitions seen in the differences in the occupation in the upper panel of Fig. 14 in Appendix B lead both to the strong response in \( \Delta N_e \) in the upper left panel of Fig. 7 for the early and the intermediate time.

All the results in Fig. 7 are for the case when no photons flow from the photon reservoir to the cavity, i.e. the mean value of photons in the cavity is \( \bar{n}_R = 0 \), and as the there are several one-electron states below the bias window the system ends up in a steady state of a Coulomb-blockade. Figure 8 shows how \( \Delta N_e \) looks for a system with \( \bar{n}_R = 1 \), that does not end up in a Coulomb-blockade, but the current through the system is maintained by the photons the reservoir supplies. Initially, the \( \Delta N_e \) shows similar behavior as in the upper left panel of Fig. 7, but in the late intermediate time there are changes due to the nonvanishing current through the system.

Commonly, in classical circuits the current is analyzed in order to find traits of self-induction. In Fig. 9 we display the change in the current \( I = I_L + I_R \) for the higher photon energy, \( \hbar \omega = 2.63 \text{ meV} \), for both polarizations of the cavity field, and four values of the overall coupling to the external leads \( g_0 \). For the \( x \)-polarized cavity field (upper panel) the difference in the current caused by the higher-order electron-photon interactions terms increases with growing overall coupling and is generally positive. The higher-order terms lead to more current through the central system. On the contrary, for a \( y \)-polarized cavity field (lower panel) the current is weakly suppressed by the higher-order terms, increasing with the overall cou-


FIG. 7. The differences in the mean electron \( \Delta N_e \) (left) and photon \( \Delta N_\gamma \) (right) numbers for a system with an exact, and a system with a dipole approximation for the electron-photon interactions for a single cavity mode for \( \hbar = 0.98 \text{ meV} \) (upper), and for \( \hbar = 2.63 \text{ meV} \) (lower) with \( g_0 = 1.0 \), \( \bar{n}_R = 0 \), \( g_{\text{EM}} = 0.1 \text{ meV} \), \( B = 1.0 \text{ T} \), \( \kappa = 1.0 \times 10^{-5} \text{ meV} \), \( -eV_g = 2.47 \text{ meV} \), and \( L_x = 180 \text{ nm} \).
The change in the current $\Delta I$ for the lower photon energy $\hbar \omega = 0.98$ meV is displayed in Fig. 11. On the scale appropriate for $\Delta I$ in the case of $x$-polarization of the cavity field $\Delta I$ for the $y$-polarization is vanishingly small. The mean change $\Delta I$ for the $x$-polarized field is positive as before, but a complex pattern of Rabi-oscillations emerges at the ITR, that will be analyzed below.

In the two remaining figures we show the Fourier power spectrum for the oscillations in the difference of the mean photon number, $\Delta N_\gamma$ for the two different photon energies, starting with $\hbar \omega = 0.98$ meV in Fig. 12. The Fourier transform is taken using 40000 points equispaced in the time interval from 65.82 – 8065.45 ps. In the upper panel of Fig. 12 the whole energy range from 0 – 6.0 meV is seen and clearly there is an order of magnitude between the two spectra for the $x$- and the $y$-polarization of the cavity field. Numerical noise is visible, specially for the highest energies in the $y$-polarization. The lower panel of Fig. 12 presents the low energy range of the same spectra with three main peaks caused by the Rabi-splitting of the states in the bias window.

In the upper panel of Fig. 12 there are a small groups of peaks just below 2.0 meV, and again just below 4.0 meV. These peaks can all be correlated to active transitions between one-electron states in the system. It is interesting to note that from the energy spectra shown in the left panels of Fig. 3 for $\hbar \omega = 0.98$ meV there are clear differences in energy, and specially in the mean photon number of many states for the two different polarizations of the cavity field. This difference is minimal for the spectra for $\hbar \omega = 2.63$ meV seen in the right panels of Fig. 3, and for one state in the lower panel of Fig. 12.
but the Fourier power spectra for ∆Nγ seen in Fig. 13 again shows the order of magnitude between the spectra for the two polarizations.

Here are no Rabi-split states in the bias window and thus no low-energy Rabi-resonances are seen, instead we notice pairs of resonances emerging corresponding to the Rabi-split states above the bias window, indicating that these states do participate, even though very weakly, in the electron transport through the central system. Here, we have analyzed the difference in the mean photon number ∆Nγ, similar analysis for the difference in the current ∆I reveals corresponding information. The resonances in ∆I are weaker, but in addition we then see peaks caused by transitions between dressed electron states with only a minimal photon content.

### IV. CONCLUSIONS

As expected, the dipole approximation for the electron-photon interaction together with the lowest order diamagnetic terms, is a good approximation to describe the interactions of electrons and photons in nanoscale systems embedded in a cavity.

In our central system with discrete energy spectrum and few particles far away from the possibility of a semi-classical interpretation all correspondence to classical self-induction in simple circuits is lost.

Another difficulty comes from the fact that we consider a photon cavity with a single photon mode, one fundamental frequency. Anybody using a Laplace transformation to analyze a classical circuit driven by a step potential has noticed that the results always imply a spread of frequencies producing the inductance effects. Nevertheless, the question about the self-induction is interesting in order to acquire a more complete understanding of the quantum transport of electrons through photon cavities, and a cavity leads to a dominance of their fundamental frequency in the process, as only processes with the fundamental frequency of the cavity or integer multiples thereof will form a standing wave (a cavity mode) with high intensity in the central system.

Our modeling of the electron transport through the central system, a nanoscale electron system embedded in a three-dimensional photon cavity with a single FIR photon mode relies on a linear many-body space of photons dressed electron states \{\hat{\mu}\}. These states have been constructed using a step-wise numerically exact diagonalization and truncations for all the interactions present in the isolated central system [50]. By keeping this space large enough we have used it to describe the dynamical evolution of the system as it is coupled to the external leads with different chemical potentials. This evolution
FIG. 12. The Fourier power spectrum for the difference in the mean photon number for a system with an exact, and a system with a dipole approximation for the electron-photon interactions for a single cavity mode. The two panels show different energy intervals. $\hbar \omega = 0.98 \text{ meV}$, $g_{EM} = 0.1 \text{ meV}$, $g_0 = 1.0$, and $g_0 g_{LR} a_0^2 = 0.101 \text{ meV}$.

FIG. 13. The Fourier power spectrum for the difference in the mean photon number for a system with an exact, and a system with a dipole approximation for the electron-photon interactions for a single cavity mode. The two panels show different energy intervals. $\hbar \omega = 2.63 \text{ meV}$, $g_{EM} = 0.1 \text{ meV}$, $g_0 = 1.0$, and $g_0 g_{LR} a_0^2 = 0.101 \text{ meV}$.

The differences in the transport properties are caused by high-order terms in the electron-photon interactions. The strongest terms contributing the most are terms with electrical quadrupole and magnetic dipole momenta, but we cannot separate the two contributions.

The approach using exact numerical diagonalizations, or configuration interactions, is essential here, as this view is not easily extensible to formalisms built on mean-field theories, in which one has to guarantee self-consistency at each time step in the calculation. This view underlies the difference between the description of a classical circuit with an inductance and the quantum mechanical one. If the first reaction is that the quantum mechanical view may seem more boring, then one has to appreciate what the quantum mechanical view is really offering. Our calculations indicate that if the full geometry of the system is taken into count, we may not be able to guess what small changes the higher-order effects may lead to, without a full calculation. Here, we have already seen that even the simple question if the higher order
terms will increase or decrease the current through the system is not simple in the quantum mechanical sense. In a direct continuation we are not able to predict what may happen in systems with higher electron-photon coupling and different geometries, even though the effects seen here are by no means large.

Our calculation have been performed for a relatively small system in order to conserve the RAM-memory size needed, but the inductance of the electronic system can be made larger by increasing its size, as there is space enough in the cavity. Increased system size would lead to the need to include more electrons in the system, and the evolution of the results with the system size should point out the path to the properties of the classical system.

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Appendix A: Information about the numerical computations

We use a scheme of step-wise introduction of model complexities and a step-wise truncation of the ensuing many-body spaces to guarantee the accuracy of the calculations and to contain the RAM memory needed [50]. Initially, a large single-electron basis \( \{|\alpha\}\) with 2688 elements is made for the central system, out of which the lowest 53 in energy are used to build a Fock space with 1 0-electron, 52 1-electron, 1326 2-electron, and 16 3-electron states. All together 1395 Fock many-body eigenstates for the noninteracting central system in an external magnetic field. This eigenbasis is used to diagonalize the many-body Hamiltonian of the electrons with their mutual Coulomb interaction. The lowest in energy 512 eigenstates in this basis are then tensor multiplied by the 17 lowest eigenstates of the photon number operator to form a basis with 8704 elements that is used to diagonalize the Hamilton matrix including the electron-photon interactions, Eq. 6. Of the resulting eigenstates \( |\bar{\mu}\rangle\) for the fully interacting central system, the lowest in energy 128 states are used to build the 16384-dimensional Liouville basis of transitions in which the Markovian version of the master equation is solved. The rotating wave approximation is not used for the electron-photon interaction in the central system, but it is used when deriving the dissipation terms for the coupling of the cavity to the external photon reservoir. When constructing these terms special care has been taken as they have to be transformed from the basis of the non-interacting photons to the basis of interacting electrons and photons [20, 21, 51–53]. In the Schrödinger picture used here this is implemented by dismissing all creation terms in the transformed annihilation operators and all annihilation terms in the transformed creation operators. This guarantees that an open system will evolve into the correct physical steady state with respect to the photon decay [31]. The processing time is reduced by a general use of parallelism and a heavy use of off-loading large linear algebra tasks to fast GPU cards [44].

Appendix B: Changes in the self-energy of states and their occupation

![Graph showing changes in self-energy of states over time](image)

FIG. 14. The difference in the time-dependent occupation of selected many-body states \( |\bar{\mu}\rangle\) for a system with an exact, and a system with a dipole approximation for the electron-photon interactions for a single cavity mode, for \( \hbar \omega = 0.98 \) meV (upper), and \( \hbar \omega = 2.63 \) meV (lower). \( g_{EM} = 0.1 \) meV, \( B = 1.0 \) T, \( \bar{n}_R = 0 \), \( -\epsilon V_s = 2.47 \) meV, and \( L_x = 180 \) nm, and \( g_0 g_{LR} \omega^{-3/2} = 0.101 \) meV.

The electron transport through the central system is modeled using a quantum many-body formalism, for the electrons in the central system, the single-photon mode of the cavity electromagnetic field, and the electrons in
the external leads. The search for self-induction and similar higher-order effects where an electron interacts with its own electromagnetic field is carried out in a model based on linear many-body spaces, instead of grids, we thus always have to consider the self-energy of the cavity photon dressed electron states. To aid this exploration we present here Fig. 14 showing the changes in the dynamical occupation of the lowest many-body states of the central system.

In addition, we find it necessary to display in Fig. 15 the underlying changes in the self-energy of each many-body state \( \tilde{\mu} \) caused by all higher-order terms of the electron-photon interactions. In Fig. 15 we show which states are within the bias window for both energies of the cavity photons we have chosen. Note the differences in the energy scales in Fig. 15.

![Figure 15](image)

**FIG. 15.** The difference in the self-energy of the 32 lowest many-body states \( \tilde{\mu} \) for a system with an exact, and a system with a dipole approximation for the electron-photon interactions for a single cavity mode, for \( \hbar\omega = 0.98 \) meV (upper), and \( \hbar\omega = 2.63 \) meV (lower). The states within the two vertical black lines are the states in the bias window. \( g_{EM} = 0.1 \) meV, \( B = 1.0 \) T, \( \tilde{n}_R = 0 \), \( -eV_g = 2.47 \) meV, and \( L_x = 180 \) nm, and \( g_0 g_{LR} a_\omega^{1/2} = 0.101 \) meV.

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