Can a vector field be responsible for the curvature perturbation in the Universe?

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(Dated: September 15, 2018)

I investigate the possibility that the observed curvature perturbation is due to a massive vector field. To avoid generating a large scale anisotropy the vector field is not taken to be driving inflation. Instead it is assumed to become important after inflation when it may dominate the Universe and imprint its perturbation spectrum before its decay, as in the curvaton scenario. It is found that, to generate a scale invariant spectrum of perturbations, the mass-squared of the vector field has to be negative and comparable to the Hubble scale during inflation. After inflation the mass-squared must become positive so that the vector field engages into oscillations. It is shown that, such an oscillating vector field behaves as pressureless matter and does not lead to large scale anisotropy when it dominates the Universe. The possibility of realising this scenario in supergravity is also outlined.

I. INTRODUCTION

Observations of the curvature perturbation in the Universe strongly suggest that it is generated during inflation by the gravitational production of particles. The field, whose quantum fluctuations are responsible for the particle production is typically considered to be a scalar field; one of the many flat directions that are envisaged in theories beyond the standard model.

Very little has ever been discussed about gravitational production of vector fields during inflation (see Refs. [1, 2]). This is mostly because, to achieve particle production in a de-Sitter background, the field in question must be light enough for its Compton wavelength to extend beyond the horizon. However, a massless vector field is conformally invariant and, therefore, it does not couple to the inflating gravitational background, which means that it does not undergo particle production. Hence, vector field generation during inflation has been ignored.

In this work I investigate the possibility that a vector field with non-zero mass undergoes indeed particle production during inflation. My motivation was originally the possibility that a small, albeit non-zero, mass may lead to something interesting. However, I have found that this is not a promising direction, as it is shown below. Nevertheless, I have discovered that a negative mass-squared comparable to the Hubble scale can indeed result to the desired scale-invariant superhorizon spectrum of perturbations.

In contrast to previous work [1, 2] the vector field considered is not assigned to the task of driving inflation. This is so in order to avoid generating a large scale anisotropy, which is otherwise inevitable (see, however, Ref. [3]). The curvature perturbations are produced in the same spirit as in the curvaton scenario [4]. Thus, it is assumed that the vector field is subdominant during inflation. Consequently, particle production gives rise to isocurvature perturbations, which turn adiabatic at some point after inflation if the vector field manages to dominate the Universe before its decay. What I find is that an oscillating massive vector field does not result in a large scale anisotropy even when it dominates the Universe. Therefore, a vector field can indeed realise the curvaton scenario.

Throughout the paper I use natural units, where \( c = \hbar = 1 \). The signature of the metric is (1,-1,-1,-1).

II. THE EQUATIONS OF MOTION

The Lagrangian density for a massive vector field with mass \( m \) is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_{\mu} A^{\mu},
\]

where, for an Abelian field, the field strength tensor is

\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}.
\]

Employing the above one obtains the field equations for the vector field:

\[
[\partial_{\mu} + (\partial_{\mu} \sqrt{-\det[g_{\mu\nu}]})(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) + m^2 A^{\nu} = 0, \tag{3}
\]

where \( \det[g_{\mu\nu}] \) is the determinant of the metric tensor \( g_{\mu\nu} \).

Since we are interested in particle production during inflation we assume that, to a good approximation, the spacetime is spatially flat homogeneous and isotropic (we consider anisotropic expansion later). Hence we use the flat-FRW metric:

\[
ds^2 = dt^2 - a^2(t) dx^i dx^i,
\]

where \( a = a(t) \) is the scale factor of the Universe, \( x^i \) are Cartesian spatial coordinates with \( i = 1,2,3 \) and Einstein summation is assumed. Employing the above metric into
Eq. (3) we obtain the temporal component ($\nu = 0$) of the field equations:

$$\nabla \cdot \dot{A} - \nabla^2 A_t + (am)^2 A_t = 0,$$

(5)

where the dot denotes derivative with respect to the cosmic time $t$ and $\nabla$ stands for the divergence while $\nabla^2 \equiv \partial_i \partial_i$ is the Laplacian. Similarly, we obtain the spatial component ($\nu = i$):

$$\ddot{A} + H \dot{A} - a^{-2}[\nabla^2 A - \nabla(\nabla \cdot A)] + m^2 A = \nabla(\dot{A}_t + HA_t),$$

(6)

where $H \equiv \dot{a}/a$ is the Hubble parameter and in the right hand side of the above $\nabla$ denotes the gradient.

Now, contracting Eq. (3) with $\partial_i$ we obtain an integrability condition, which reads

$$(am)^2 \dot{A}_t - m^2 \nabla \cdot A + 3H(\nabla^2 A_t - \nabla \cdot \dot{A}) = 0.$$  

(7)

Combining the above with Eq. (5) we find

$$\dot{A}_t + 3HA_t - a^{-2} \nabla \cdot A = 0.$$  

(8)

Plugging this into Eq. (3) we obtain

$$\ddot{A} + H \dot{A} + m^2 A - a^{-2} \nabla^2 A = -2H \nabla A_t.$$  

(9)

We expect inflation to homogenise the vector field and, therefore,

$$\partial_i A_\mu = 0 \quad \forall \quad \mu \in [0,4].$$  

(10)

Enforcing this condition into Eq. (4) we obtain

$$A_t = 0.$$  

(11)

Using Eqs. (10) and (11) into Eq. (3) we find

$$\ddot{A} + H \dot{A} + m^2 A = 0.$$  

(12)

The above is reminiscent to the Klein-Gordon equation of a homogeneous scalar field in an expanding Universe, with the crucial difference that the friction term does not feature a factor of $3$.

We are interested in the generation of superhorizon perturbations of the vector field, which might be responsible for the curvature perturbations in the Universe. Therefore, we perturb the vector field around the homogeneous value $A_\mu (t)$ as follows:

$$A_\mu (t, x) = A_\mu (t) + \delta A_\mu (t, x) \quad \Rightarrow$$

$$A(t, x) = A(t) + \delta A(t, x) \quad \& \quad A_t (t, x) = \delta A_t (t, x),$$

(13)

where we took into account Eq. (11). In the above $A(t)$ satisfies Eq. (12). In view of Eqs. (12) and (13), Eqs. (5) and (6) become

$$\nabla \cdot (\dot{\delta A}) - \nabla^2 \delta A_t + (am)^2 \delta A_t = 0$$

(14)

$$(\delta \ddot{A}) + H(\delta \dot{A}) + m^2 \delta A - a^{-2} \nabla^2 \delta A = -2H \nabla \delta A_t.$$  

(15)

Now, let us switch to momentum space by Fourier expanding the perturbations:

$$\delta A_\mu (t, x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \delta A_\mu (t, k) \exp (ik \cdot x).$$  

(16)

Using the above, Eq. (14) becomes

$$\delta A_t + \frac{i\partial_l (k \cdot \delta A)}{k^2 + (am)^2} = 0.$$  

(17)

where $k^2 \equiv k \cdot k$. Using this and Eq. (10) we can write Eq. (15) as

$$(\delta \ddot{A}) + H(\delta \dot{A}) + m^2 \delta A - \left(\frac{k}{a}\right)^2 \delta A + 2H \frac{k\partial_t (k \cdot \delta A)}{k^2 + (am)^2} = 0.$$  

(18)

We can rewrite the above in terms of the components parallel and perpendicular to $k$, defined as:

$$\delta A^\| \equiv \frac{k (k \cdot \delta A)}{k^2} \quad \& \quad \delta A^\perp \equiv \delta A - \delta A^\|. \quad (19)$$

Thus, we obtain the following equations of motion for the vector field perturbations in momentum space:

$$\left[\partial^2_t + H \partial_t + m^2 + \left(\frac{k}{a}\right)^2\right] \delta A^\| = 0$$

(20)

$$\left[\partial^2_t + \left(1 + \frac{2k^2}{k^2 + (am)^2}\right) H \partial_t + m^2 + \left(\frac{k}{a}\right)^2\right] \delta A^\perp = 0.$$  

(21)

III. PARTICLE PRODUCTION

To investigate particle production during inflation we need to solve the equation of motion for the perturbations of the field. The integration constants are then evaluated by matching the solution to the vacuum at early times (when $k/aH \to +\infty$), i.e. by demanding

$$\lim_{\frac{k}{aH} \to +\infty} \delta A_k = \frac{1}{\sqrt{2k}} \exp (ik/aH),$$

(22)

where $\delta A_k \equiv \delta A (t, k)$ and we note that at early times the perturbation in question is well within the horizon, which means that $a \to 1$ and $k/aH \to kt$.

Afterwards we evaluate the solution at late times, when the perturbation is superhorizon in size (i.e. when $k/aH \to 0^+$). The power spectrum is obtained by

$$P_A = \frac{k^3}{2\pi^2} \left| \lim_{\frac{k}{aH} \to 0^+} \delta A_k \right|^2.$$  

(23)

We assume that, during inflation, $H$ is constant.
A. The transverse component

Solving Eq. (21) and matching to the vacuum in Eq. (22) we obtain the solution
\[
\delta A_k = \frac{a^{-1/2}}{1-i} \sqrt{\frac{2\pi}{H(1-e^{-2\pi\nu})}} \left[ J_\nu(k/aH) - e^{i\pi\nu} J_{-\nu}(k/aH) \right],
\]
where with \( J_\nu \) we denote Bessel functions of the first kind and
\[
\nu \equiv \sqrt{\frac{1}{4} - \left( \frac{m^2}{H^2} \right)}. \tag{24}
\]
The above solution at late times approaches
\[
\lim_{\tau \to 0^+} \delta A_k = \frac{a^{-1/2}}{1-i} \sqrt{\frac{2\pi}{H(1-e^{-2\pi\nu})}} \times
\]
\[
\left[ \frac{1}{\Gamma(1+\nu)} \left( \frac{k}{2aH} \right)^\nu - \frac{e^{i\pi\nu}}{\Gamma(1-\nu)} \left( \frac{k}{2aH} \right)^{-\nu} \right]. \tag{26}
\]
Hence, using Eq. (24) we find that the dominant contribution to the power spectrum is
\[
\mathcal{P}_A \approx 8\pi\Gamma(1-\nu)^{-2} \left( \frac{aH}{2\pi} \right)^2 \left( \frac{k}{2aH} \right)^{3-2\nu}. \tag{27}
\]
Now, considering a light vector field with \( m \ll H \) we see that \( \nu \approx \frac{1}{2} - \left( \frac{m}{H} \right)^2 \). As a result the dominant term in the power spectrum is simply the vacuum value
\[
\mathcal{P}_A^{\text{vac}} = \left( \frac{k}{2\pi} \right)^2, \tag{28}
\]
which agrees with the expectations, since, when \( m \to 0 \) the vector field becomes conformally invariant and, therefore, it is not gravitationally produced because it does not couple to the expanding gravitational background. In fact, for \( m \ll H \), the largest contribution to the power spectrum due to a non-zero mass is
\[
\delta \mathcal{P}_A = 2\pi \left( \nu - \frac{1}{2} \right) \left( \frac{aH}{2\pi} \right)^2 \left( \frac{k}{2aH} \right)^3, \tag{29}
\]
which is subdominant to the vacuum value for superhorizon scales.

However, if the field is not effectively massless but instead we have
\[
m^2 \approx -2H^2 \Rightarrow \nu \approx 3/2, \tag{30}
\]
then we find that a scale invariant spectrum of perturbations is indeed recovered with
\[
\mathcal{P}_A \approx a^2 \left( \frac{H}{2\pi} \right)^2 \tag{31}
\]
as in the case of a massless scalar field. We will discuss the reason for this result in Sec. V. For the time being we note that the transverse component of a light vector field cannot be gravitationally generated during inflation.

B. The longitudinal component

Turning our attention now to \( \delta A_k^\parallel \), the first thing to point out is that Eq. (21) is impossible to solve analytically. One can only approximate the solutions in some extreme cases.

Consider first that \( k \ll am \). Then it is evident that the equation assumes the same form as the case of \( \delta A_k^\perp \) and, therefore, the results are identical to the previous section. However, in the opposite case, when \( k \gg am \), the equation becomes
\[
\left[ \partial_t^2 + 3H \partial_t + m^2 + \left( \frac{k}{a} \right)^2 \right] \delta A^\parallel \approx 0, \tag{32}
\]
which, in fact, is identical to the equation of motion for a perturbation of a light scalar field in a de-Sitter background. The solution of the above, after matching to the vacuum in Eq. (22), is of identical form to Eq. (24) with the crucial difference that, this time
\[
\nu \equiv \sqrt{\frac{9}{4} - \left( \frac{m^2}{H^2} \right)}. \tag{33}
\]
Thus, the dominant contribution to the power spectrum is again given by Eq. (27).

Considering, therefore, an effectively massless field with \( m \ll H \) we have
\[
\nu \approx \frac{3}{2} - \eta \quad \text{with} \quad \eta = \frac{1}{3} \left( \frac{m}{H} \right)^2. \tag{34}
\]
In view of the above the power spectrum in Eq. (21) becomes
\[
\mathcal{P}_A \approx a^2 \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{2a} \right)^{2\eta}, \tag{35}
\]
which is approximately scale invariant approaching the value of Eq. (31) for extremely light fields. Parameterising the scale dependence of the perturbations in the usual manner
\[
\mathcal{P}_A(k) \propto k^{n_s-1}, \tag{36}
\]
we obtain for the spectral index result
\[
n_s = 1 + 2\eta, \tag{37}
\]
which is the usual finding in the case of a light scalar field.\(^1\)

The similarity between the longitudinal component of a massive vector field to a scalar field can be understood if one considers that the mass of the vector field is due to the Higgs mechanism, in which case the longitudinal component
\[
^1 \text{There is no contribution from } \epsilon \equiv -\dot{H}/H^2 \text{ to the spectral index because we have taken } H = \text{const}.\]
component corresponds to the scalar degree of freedom (the Goldstone boson) which has been “consumed” by the vector field in the Higgs process. However, we should also note here that, in the limit of \(m \rightarrow 0\) the longitudinal component of the vector field becomes unphysical. (The superhorizon spectra of \(\delta A^L_k\) and \(\delta A^\perp_k\) in the case when \(m \ll H\) are shown in Figure 1.) Thus, we see that a light vector field may indeed obtain an almost scale invariant superhorizon perturbation spectrum, through its longitudinal component. However, the price to pay is that the condition \(k \gg am\) has to be satisfied throughout the superhorizon evolution of the perturbations in question. This means that, for the cosmological scales, we must satisfy the constraint

\[
a_* H_* \equiv k_* > m a_{end} \Rightarrow m < \epsilon^{-N_*} H_* ,
\]

where ‘*’ denotes the epoch when the cosmological scales exit the horizon during inflation and ‘end’ denotes the end of inflation, while \(N\) corresponds to the number of remaining e-foldings of inflation. For the cosmological scales we have \(N_* \gtrsim 45\) and also, typically \(H_* \lesssim 10^{13}\)GeV. Hence, we find that the mass of the vector field has to be \(m \lesssim O(100)\) eV. Such an extremely light field cannot decay before nucleosynthesis and, therefore, cannot be responsible for the curvature perturbations imprinted on the thermal bath of the Hot Big Bang. The only possible solution would be to consider a subsequent period of inflation, occurring after the decay of the vector field, which would last long enough to drastically diminish \(N_*\), but not too long to render the original inflationary period irrelevant. This highly contrived scenario, however, suffers from another problem, as will be shown in the next section.

IV. EVOLUTION DURING AND AFTER INFLATION

As we mentioned in the introduction, in order to avoid large scale anisotropy in the Universe, we do not consider the possibility that the vector field in question can act as the inflaton field. Instead we assume that the vector field may imprint its spectrum of curvature perturbations onto the Universe at some time after the end of inflation, when it becomes dominant (or nearly dominant) before its decay. In other words, we assume that our vector field acts as a curvaton field. In this case the amplitude of the curvature perturbations is determined by the dynamics of the field after inflation as well.

A. The evolution of the vector field

Assume that the homogenised vector field lies along the \(z\)-direction

\[
A_\mu = (0, 0, 0, A_z(t)) .
\]

Now, \(A_z(t)\) satisfies Eq. (12). Solving this equation during inflation, where \(H \approx \text{const.}\) we obtain

\[
A_z(t) = \left[ -\frac{\dot{A}_z(0)}{H R} - \frac{1 - R}{2R} A_z(0) \right] e^{-\frac{1}{2}H \Delta t(1 + R)} + \left[ \frac{\dot{A}_z(0)}{H R} + \frac{1 + R}{2R} A_z(0) \right] e^{-\frac{1}{2}H \Delta t(1 - R)},
\]

where \(R \equiv \sqrt{1 - 4\left(\frac{m}{H}\right)^2}\), \(\Delta t\) is the elapsed time and ‘(0)’ denotes the initial value. When \(m \ll H\) the above give

\[
A_z(t) \approx \frac{\dot{A}_z(0)}{H} + A_z(0) \exp \left[ -H \Delta t \left(\frac{m}{H}\right)^2 \right] - \frac{\dot{A}_z(0)}{H} + \frac{1}{2} A_z(0) (\frac{m}{H})^2 A_z(0) \exp (-H \Delta t) .
\]

Therefore, for a light vector field during inflation we find \(A_z \rightarrow A_z(0) + A_z(0)/H = \text{const.}\)

After the end of inflation solving Eq. (12) we obtain

\[
A_z(t) = t^x [c_1 J_x(mt) + c_2 Y_x(mt)],
\]

where \(c_1, c_2\) are constants of integration, \(Y_x\) is a Bessel function of the second kind and

\[
x = \frac{1 + 3w}{6(1 + w)}
\]

with \(w\) being the barotropic parameter \(w = 0 \{w = \frac{1}{3}\}\) for matter \{radiation\} domination.

When \(m \ll H\) we find

\[
A_z(t) \approx t^x \left[ \frac{c_1}{\Gamma(x + 1) \left(\frac{mt}{2}\right)^x} - c_2 \frac{\Gamma(x)}{\pi} \left(\frac{mt}{2}\right)^{-x} \right] .
\]
From the above, since \((mt/2) \sim m/H \ll 1\), we see that, in both cases, \(A_z(t) \approx \text{const}\). Thus, we have verified that both during and after inflation, when \(m \ll H\) the vector field is overdamped and remains frozen.

Now, consider the opposite case \(m \gg H\). In this case, from Eq. (11) we have

\[
A_z(t) \approx t^{-\frac{1}{2 + 1 + 2}} \sqrt{\frac{2}{\pi m}} \left\{ c_1 \cos \left[ mt - \frac{\pi}{2} \left( x + \frac{1}{2} \right) \right] + c_2 \sin \left[ mt - \frac{\pi}{2} \left( x + \frac{1}{2} \right) \right] \right\}.
\]

Hence, we see that, when the vector field becomes heavy then it engages in damped harmonic oscillations with envelope decreasing as shown in Eq. (45).

\[
\bar{A}_z(t) \propto a^{-1/2}.
\]  

**B. The energy momentum tensor**

Now that we know how the vector field evolves during and after inflation we can compute if and when it will come to dominate the Universe, in order to imprint its superhorizon perturbation spectrum. To find this we follow the evolution of the energy-momentum tensor of the vector field.

Using Eq. (1), the energy momentum tensor for \(A_\mu\) is

\[
T_{\mu\nu} = \frac{1}{2} g_{\rho\sigma} F_{\rho\sigma} F^{\mu\nu} - F_{\mu\nu} F_{\rho\sigma} \Gamma^\rho_F^\sigma + \frac{m^2}{2} (A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} A^2) .
\]

In order to imprint its perturbation spectrum onto the Universe the vector field must dominate (or nearly dominate) the Universe, in accordance to the curvaton scenario. Since this may result into large scale anisotropy we consider the following metric

\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2) - b^2(t)dz^2,
\]

where \(b(t)\) is the scale factor along the \(z\)-direction. In view of the above, the energy-momentum tensor can be written in the form

\[
T^\nu_\mu = \text{diag}(\rho_A, -p_\perp, -p_\perp, +p_\perp) ,
\]

where

\[
\rho_A \equiv \rho_{\text{kin}} + V, \quad p_\perp \equiv \rho_{\text{kin}} - V
\]

with

\[
\rho_{\text{kin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2b^2} \bar{A}_z^2
\]

\[
V = -\frac{1}{2} m^2 A_\mu A^\mu = \frac{1}{2b^2} m^2 \bar{A}_z^2 .
\]

From Eq. (49) we see that the energy momentum tensor for our vector field resembles the one of a perfect fluid, with the crucial difference that the pressure along the longitudinal direction is of opposite sign to the pressure along the transverse directions. Thus, if the pressure is non-zero and the vector field dominates the Universe, then large scale anisotropy will be generated. This is the reason we did not consider that \(A_z\) can play the role of the inflaton field in the first place.

To study the evolution of the energy density of the vector field we begin by assuming that originally \(\rho_A\) is subdominant, in an isotropic Universe. In this case we can employ Eq. (12), which can be written as

\[
\ddot{A}_z + m^2 A_z \left[ 1 + \left( \frac{H}{m} \right)^2 \frac{\dot{A}_z}{HA_z} \right] = 0 .
\]

When \(m \gg H\) we have shown that \(A_z\) oscillates harmonically with envelope decreasing as shown in Eq. (45). Using this we can approximate \(\dot{A}_z/HA_z \approx A_z/HA_z = -\frac{\dot{A}_z}{HA_z}\). Inserting this into the above and considering that \(m \gg H\) we obtain

\[
\ddot{A}_z + m^2 A_z \approx 0 ,
\]

which verifies that, since the oscillation period is much smaller that the Hubble time, the oscillations are practically harmonic. This means that, on average,

\[
\bar{A}_z^2 = m^2 \bar{A}_z^2
\]

Now, Eq. (12) can also be written as

\[
\frac{d}{dt} \left( \frac{1}{2} A_z^2 + \frac{1}{2} m^2 A_z^2 \right) + HA_z = 0 .
\]

In view of Eq. (54) the above is recast as

\[
\frac{d}{dt} (A_z^2) + HA_z^2 = 0 .
\]

From Eqs. (54), (55) and (56), considering also Eq. (58) we have

\[
\bar{A}_z^2 = a^2 \bar{\rho}_A .
\]

Hence, Eq. (58) suggests that

\[
\bar{\rho}_A \propto a^{-3}.
\]

This means that, when the vector field begins oscillating, its density scales as pressureless matter. This is not surprising because Eq. (54) implies that, on average \(\bar{\rho}_{\text{kin}} = \bar{V}\), which means that \(\bar{p}_{\perp} = 0\). Hence, all the fluid energy is contained in the longitudinal direction.

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2 In the limit \(m \to 0\) it is easy to see that \(T_{\mu\nu}\) becomes traceless as it should. Also, \(\rho_A \to \rho_{\text{kin}}\). In the case of isotropic expansion (\(b = a\)) (i.e. when \(A_z\) is subdominant) the solution of Eq. (12) suggests, \(\bar{A}_z \propto a^{-3}\). Hence, \(\bar{\rho}_A = \frac{1}{2} (\bar{A}_z/a)^2 \propto a^{-4}\), which is the conformal invariant result for radiation.
pressure components in Eq. (18) are equal to zero. The result in Eq. (18) is also obtained by considering that
\[ \rho_A = 2\sqrt{\frac{\dot{a}}{a}} = a^{-2}m^2A_z^2 \propto a^{-3}, \]
where we also used Eq. (16).

The above behaviour does not change if the Universe becomes anisotropic. If this is so, then it is easy to verify that, using the metric in Eq. (17) and Eq. (49), the equivalent to Eq. (18) is
\[ \dot{A}_z + (2H_a - H_b)\dot{A}_z + m^2A_z = 0, \]
where \( H_a \equiv \dot{a}/a \) and \( H_b \equiv \dot{b}/b \). Considering \( m \gg H_a, H_b \) one arrives at Eq. (18), from which all the results follow.

The fact that the average pressure of an oscillating vector field is zero along both the longitudinal and the transverse directions implies that \textit{when the density of the vector field dominates the Universe this does not cause anisotropic expansion}. Hence, we can use the curvaton mechanism to imprint the superhorizon perturbation spectrum onto the Universe without the danger of generating a large scale anisotropy, provided that the vector field is oscillating at domination.

C. Generating the curvature perturbation

The curvaton mechanism transforms the spectrum of perturbations of the vector field into a spectrum of curvature perturbations as follows. As discussed by Lyth and Wands in Ref. [2], on a foliage of spacetime corresponding to spatially flat hypersurfaces, the curvature perturbation attributed to each of the Universe components (labelled by the index \( n \)) is given by
\[ \zeta_n \equiv -H\frac{\delta \rho_n}{\rho_n}, \]
where \( \rho_n \) and \( \delta \rho_n \) are, respectively, the density and its perturbation of the component in question.

The total curvature perturbation \( \zeta(t) \), which is also given by Eq. (18) with \( \rho_n \) and \( \delta \rho_n \) replaced, respectively, by the total density of the Universe \( \rho = \sum_n \rho_n \) and its perturbation \( \delta \rho \), may be calculated as follows. Using the fact that \( \delta \rho = \sum_n \delta \rho_n \) and the continuity equation \( \dot{\rho}_n = -3H(\rho_n + p_n) \), where \( p_n \) is the pressure of the \( n \)-th component of the Universe, it is easy to find that
\[ \zeta = \sum_n \rho_n + p_n \frac{\rho}{\rho + p} \zeta_n, \]
where \( p = \sum_n p_n \) is the total pressure. Now, since in the curvaton scenario, all contributions to the curvature perturbation other than the curvaton’s are negligible, we find that
\[ \zeta = \zeta_A \left( \frac{1}{1 + w} \right) \frac{\rho_A}{\rho} \bigg| \text{dec}, \]
where \( \zeta \approx 2 \times 10^{-5} \) is the observed curvature perturbation and \( \zeta_A \) is the partial curvature perturbation of the vector field, for which we have assumed that the pressure is zero, when it is oscillating as previously discussed. The right hand side of this equation is evaluated at the time when the vector field decays and this is indicated by the subscript ‘dec’. If the vector field decays after it dominates the Universe then \( \rho \rightarrow \rho_A \), which means that \( \zeta \approx \zeta_A \).

To obtain \( \zeta_A \) we employ Eq. (18) and the continuity equation, which suggest
\[ \zeta_A = \left. \frac{\delta \rho_A}{3\rho_A} \right| \text{dec} \approx \frac{2}{3} \left. \frac{\delta A_z}{A_z} \right| \text{dec} \approx \frac{2}{3} \left. \frac{\delta A_z}{A_z} \right| \text{dec}, \]
where the bar denotes the amplitude of the oscillating vector field at the time of decay [c.f. Eq. (18)]. In the above we have considered that, after the end of inflation, the ratio \( \delta A_z/A_z \) remains constant because either the vector field and its perturbation are frozen (when \( m \ll H \)) or they are oscillating (when \( m \gg H \)), with the same equation of motion. This is indeed so since the perturbations in question are superhorizon in size when the vector field decays, so spatial gradients in Eq. (18) are negligible and, hence, Eq. (18) becomes of identical form to Eq. (18).

Now, in Eq. (18) \( \rho_V(t) \) is the homogeneous zero-mode, which has no \( k \)-dependence. This means that the \( \zeta_A(k) \) has the same \( k \)-dependence as the perturbation \( \zeta_A(k) \).

Hence, a scale-invariant superhorizon spectrum of vector field perturbations can give rise to a scale-invariant superhorizon curvature perturbation spectrum provided the vector field is heavy and engages into oscillations before it dominates the Universe.

The above are in stark contrast to the case when the vector field dominates while it is still light. As we have shown, in this case \( A_z \) is frozen to a constant value, which suggests that \( \rho_{\text{kin}} \approx 0 \) and \( p_L = -\rho_A = -V \). This implies that the Universe inflates along the transverse direction but not along the longitudinal direction because the longitudinal pressure is positive (c.f. Eq. (48)). Hence, when \( m \ll H \) we have to demand that \( \rho_A \) is subdominant in order to retain isotropy.

We should note here that, for a light vector field, \( \rho_A \) is not constant. Indeed, while \( A_z \approx \text{const.} \), we see from Eq. (18) that \( \rho_A = V \propto a^{-2} \) (with \( a = b \)). Hence, despite the fact that the vector field is frozen, its density decreases as \( a^{-2} \). This undermines even further the possibility of using a light vector field to generate the curvature perturbation, because its density during inflation is exponentially suppressed. Bearing in mind that the field needs to dominate (or nearly dominate) after inflation and decay before nucleosynthesis while having a tiny mass due to the bound in Eq. (18), it is easy to see that a successful scenario is rather unlikely (in fact it is inviable). However, if we abandon the light field assumption we might be able to attain the desired result as we discuss in the next section.
V. THE "PHYSICAL" VECTOR FIELD

The careful reader might have been alarmed by the fact that the results in Eqs. (31) or (35) appear to be proportional to powers of the scale factor. Similarly, the components of the energy momentum tensor as shown in Eqs. (30) and (31) also appear to bear an explicit dependence to the scale factor $b$. Both these findings are not expected because quantities such as the power spectrum or the density and pressure are observables and should not depend on the normalisation of the scale factor. The problem is overcome if one realises that $A_t$ is more like a 'comoving' quantity, which has the Universe expansion factored out. As implied by the form of $\delta V_k$ in Eqs. (31) and (35) and also on the form of $V$ in Eq. (31), one can assume that, in an isotropic Universe, the spatial components of the "physical" vector field may be defined as

$$V_i \equiv A_i/a(t).$$

(65)

From Eqs. (12) and (65) we find that the spatial components $V_i(t)$ during inflation satisfy the equation

$$\ddot{V} + 3H\dot{V} + (2H^2 + m^2)V = 0,$$

(66)

where we considered $H \approx \text{const}$. After perturbing $V_\mu$ around the homogeneous value and following the same procedure as in Sec. III, we obtain the equations of motion for the transverse component of the perturbations of the vector field:

$$\left[\ddot{\delta V} + 3H\dot{\delta V} + 2H^2 + m^2 + \left(\frac{k}{a}\right)^2\right] = 0,$$

(67)

where

$$\delta V_i(t, x) = \int \frac{d^3k}{(2\pi)^3/2} \frac{\delta V_i(t, k)}{k} \exp(ik \cdot x),$$

(68)

\[\text{Eq. (68) is also motivated as follows. Suppose that } m = 0. \text{ Then the non-zero components of the stress tensor can be expressed in terms of an electric and a magnetic field } E_i \text{ and } B_i \text{ respectively as: } F_{0i} = E_i \text{ and } F_{ij} = -\epsilon_{ijk}B_k, \text{ where } \epsilon_{ijk} \text{ is the totally anti-symmetric Levi-Civita tensor. In this case the energy momentum tensor in Eq. (67) suggests that the energy density of the field is:}

$$\rho_A \equiv T^{00} = \frac{1}{a^2} \left(\frac{E_iE_i}{a^2} + B_iB_i\right).$$

Since, the energy density is a physical quantity we realise that the physical electric field is $E_i/a$ and the physical magnetic field is $B_i/a^2$. Now, $B_i$ is defined as $B_i \equiv \epsilon_{ijk}\partial_jA_k$. This means

$$B \equiv \nabla \times A \Rightarrow B/a^2 = \hat{n} \times V,$$

where the hat denotes derivatives with respect to physical coordinates $x^i \equiv ax^\mu$ in contrast to comoving coordinates $x^i$ [c.f. Eq. (4)], such that $\partial_i = \hat{n}/a$. Hence we see that, since $B/a^2$ is a physical quantity, so is $V$.

\[\text{i.e. } \delta V_k = \delta A_k/a(t). \text{ From Eqs. (66) and (67) it is evident that the equation of motion for the transverse component of the "physical" vector field is remarkably similar to a scalar field (notice the factor of } 3H \text{ in the friction term of Eq. (66)) of mass-squared}

$$m^2 = 2H^2 + m^2.$$  

(69)

Hence, we expect that particle production will take place when $m \ll H$, i.e. in the case when

$$m^2 \approx -2H^2.$$  

(70)

Indeed, we have already seen that, when the condition in Eq. (70) is valid, the transverse component of the vector field obtains a scale invariant superhorizon spectrum of perturbations. In view of Eq. (31) we see that

$$\mathcal{P}_V \approx \left(\frac{H}{2\pi}\right)^2.$$  

(71)

exactly as in the case of a scalar field, where we used that $\mathcal{P}_V \equiv a^{-2}P_A$ as is evident by Eq. (28) and the fact that $\delta V_k = \delta A_k/a$.

Now, if we employ the condition in Eq. (70) into Eq. (69) we find that, during inflation, $R = 3$ and so

$$A_z(t) \approx \frac{1}{3H} \left\{2HA_z(0) + \dot{A}_z(0)e^{H\Delta t} + \left[HA_z(0) - \dot{A}_z(0)e^{-2H\Delta t}\right]\right\}.$$  

(72)

Hence, after a Hubble time we see that $A_z \propto e^{H\Delta t} \propto a$, which means that $V_z \equiv A_z/a \approx \text{const}$. [c.f. Eq. (65)] and, therefore the "physical" vector field remains frozen during inflation.

Expressing the components of $T_{\mu\nu}$ in terms of $V_z$ we obtain

$$\rho_A = \frac{1}{2} [V_z^2 + 2HV_z\dot{V}_z + (H^2 + m^2)V_z^2].$$  

(73)

$$p_\perp = \frac{1}{2} [V_z^2 + 2HV_z\dot{V}_z + (H^2 - m^2)V_z^2].$$  

(74)

During inflation, when we require Eq. (70) to hold, we find

$$\rho_A = -\frac{1}{2} H^2V_z^2 \text{ and } p_\perp = \frac{3}{2} H^2V_z^2,$$

(75)

where we have used that $V_z \approx \text{const.}$ during inflation. Since $V_z$ is frozen during inflation, both the energy density and the pressure are constant. The fact that the energy density appears negative is not surprising: according to Eq. (70) the mass of the vector field is tachyonic, which is a similar situation to the case when a scalar field is placed on top of a potential hill.

VI. VECTOR CURVATON IN SUPERGRavity

From the above we see that we may have a chance to generate a scale invariant superhorizon spectrum of
perturbations through the use of a vector field, provided the condition in Eq. (70) holds during inflation, at least when the cosmological scales exit the horizon. How can we achieve this?

In particle physics, vector fields obtain masses through the Higgs mechanism. The masses are due to the kinetic term of the Higgs field $\phi$

$$\mathcal{L}_{D\phi} = D_\mu \phi (D^\mu \phi)^* ,$$  \hspace{1cm} (76)

where $^*$ here denotes charge conjugation and

$$D_\mu \equiv \partial_\mu + ig A_\mu$$  \hspace{1cm} (77)

is the covariant derivative with $g$ being the gauge coupling. This means that the Higgs kinetic term in Eq. (76) generates a mass term for the vector field $A_\mu$:

$$\mathcal{L}_m = g^2 |\phi|^2 A_\mu A^\mu .$$  \hspace{1cm} (78)

From this term it is in principle possible to obtain a mass of order $H$ during inflation for the vector field. Indeed, if the Higgs field is light during inflation then particle production would generate a condensate of magnitude

$$\langle \phi^2 \rangle \sim H^2 \min \{ H \Delta t, H^2/m^2_\phi \} ,$$  \hspace{1cm} (79)

where $m_\phi \ll H$ is the mass of the Higgs field and $\Delta N = H \Delta t$ is the number of elapsed e-foldings of inflation. The second term in the brackets above corresponds to the Bunch-Davis result \cite{5}, whereas the first term corresponds to the case of a flat potential \cite{6}. Hence, we see that, provided $m_\phi < H/\sqrt{\Delta N}$ the vector field obtains a mass

$$m \sim g \sqrt{\Delta N} H$$  \hspace{1cm} (80)

where, if inflation is not very long, one might have $g \sqrt{\Delta N} \sim \mathcal{O}(1)$. However, this mass is not tachyonic and cannot satisfy the condition in Eq. (70).

One may envisage a possibility to obtain a tachyonic mass in the context of supergravity, where the kinetic term of the scalar fields is multiplied by the Kähler metric:

$$\mathcal{L}_{D\phi} = K_{\phi\phi^*} D_\mu \phi (D^\mu \phi)^* ,$$  \hspace{1cm} (81)

Indeed, if the Kähler potential includes a term of the form

$$\Delta K = -|\phi|^2$$  \hspace{1cm} (82)

then we may indeed obtain a tachyonic mass for the vector field. If this mass is close to satisfying Eq. (70) it is possible to generate an approximately scale invariant spectrum of perturbations. Slight deviations from the condition in Eq. (70) generate a tilt on the spectrum, with spectral index $n_s = 1 + 2\tilde{\eta}$, where

$$\tilde{\eta} = \frac{1}{3} \frac{\dot{m}^2}{H^2} = \frac{1}{3} \left( 2 + \frac{m^2}{H^2} \right) ,$$  \hspace{1cm} (83)

where we used Eq. (89).

However, in view of Eqs. (51) and (52), we see that a negative mass-squared for the vector field requires a negative kinetic term for the scalar field $\phi$. Hence, $\phi$ is rendered a ghost field, with all the unpleasant side effects this might imply (e.g. breaking of Lorentz invariance). Still, this scenario may find use in the context of the so-called ghost inflation model, which uses exactly such a field \cite{5}. However, it is not clear whether a ghost condensate would still satisfy Eq. (79).

Even if we manage to account for all the above we still need to ensure that the vector field dominates the Universe after the end of inflation, without causing a large-scale anisotropy. However, having a tachyonic mass as described above does not lead to the oscillating behaviour which we would prefer after the end of inflation. To remedy this, we can add another term in the Kähler potential, coupling the Higgs field to some other scalar $\psi$:

$$\Delta K = -|\phi|^2 + \frac{|\phi|^2 |\psi|^2}{M^2} ,$$  \hspace{1cm} (84)

where $M$ is an appropriate cutoff scale. We can then assume that the scalar field $\psi$ is heavy during inflation and, therefore, $\psi \approx 0$. However, at or after the end of inflation a phase transition gives non-zero vacuum expectation values (VEVs) to both scalar fields. In this scenario the vacuum mass of the vector field would be

$$m^2 = g^2 \left( \frac{M^2_\psi}{M^2} - 1 \right) M^2_\phi ,$$  \hspace{1cm} (85)

where $M_\phi$ and $M_\psi$ are the VEVs of $\phi$ and $\psi$ respectively. From the above we see that there is a chance that, after inflation, the mass of the vector field ceases to be tachyonic provided $M_\psi \gtrsim M$. This also means that, in the vacuum, $\phi$ is not a ghost.

A similar way to attain the desired results but without casting doubt in the validity of Eq. (84) is as follows. One can consider that instead of the mass term changing signs after inflation, the kinetic term of the vector field does so. Indeed, in supergravity the kinetic term of a vector field is multiplied by the gauge kinetic function $f$:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} f F_{\mu\nu} F^{\mu\nu} .$$  \hspace{1cm} (86)

The gauge kinetic function is a holomorphic function of the fields of the theory. Suppose that it is of the form

$$f(\psi) = -1 + \frac{\psi^2}{M^2} ,$$  \hspace{1cm} (87)

where $M$ is an appropriate cutoff scale. Then, similarly as before, we may assume that $\psi$ is heavy during inflation and, therefore, driven to zero. This introduces a change of sign of the kinetic term of the vector field. Suppose, now, that the mass of the vector field is $m \approx \sqrt{2} H_*$, where $H_*$ is the Hubble scale during inflation. Then, for
our considerations, the reversal of the sign of the kinetic term is entirely equivalent to the reversal of the sign of the mass-term because changing the overall sign of the vector Lagrangian density in Eq. (11) does not affect the equations of motion in Eq. (33). Therefore, particle production of the vector field will take place during inflation. However, the physics will be affected because one has to consider that the sign for the gravitational Lagrangian density is not reversed and we may still run into trouble in the same manner as with the ghost condensate above. Now, after inflation, a phase transition may send $\psi$ to a non-zero value $M_\psi$. This changes the sign of the kinetic term (rendering the vacuum stable) provided $M_\psi \gtrsim M$. In this case, in the post-inflation era our vector field undergoes the desired oscillations and can become a successful curvaton.

Another way to comply with the requirement in Eq. (70) is to assign a “potential” $U(\xi)$ to the vector field, in the manner considered in Refs. [1, 2], where $\xi \equiv A_\mu A^\mu$. For example, that way we may assume a negative mass-squared with the vacuum stabilised through a self-coupling of the vector field of the form:

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 A_\mu A^\mu + \frac{1}{4} \lambda (A_\mu A^\mu)^2, \quad (88)$$

i.e. we assume $U = -\frac{1}{2} m^2 \xi + \frac{1}{4} \lambda \xi^2$, with $m \approx \sqrt{2} H_*$. Such self-couplings might arise when considering non-Abelian vector fields. Note that, the above assumption generates a positive mass-squared in the vacuum for our vector field. This means that, while the vector field may lie on top of the $U$-potential hill during inflation, after inflation $H(t) < m$ and the vector field begins oscillating in a potential $U \propto \xi$, corresponding to isotropic pressure-less matter as we have shown.\footnote{4}

Finally, note that the vector field may not need to oscillate and dominate the Universe after inflation, in order to imprint its curvature perturbation spectrum. One may consider a modulated-reheating scenario \cite{7}, where the vector field controls the decay rate of the inflaton through a coupling such as the one in Eq. (78).

It is evident that the above scenarios are quite contrived and involve a number of tunings.

\section{VII. CONCLUSIONS}

In summary we have shown that, in principle, a vector field can indeed be responsible for the observed curvature perturbation in the Universe. In order to be so the mass-squared of the vector field has to be $m^2 \approx -2H^2$ during inflation. The vector field must be subdominant during inflation to avoid generating a large-scale anisotropy. Hence, particle production generates an originally isocurvature perturbation, which can become adiabatic if, after inflation, the vector field dominates the Universe before its decay, in accordance to the curvaton scenario. Indeed, we have shown that, if after inflation the mass-squared of the vector field becomes positive, then the field begins oscillating. We have demonstrated that an oscillating vector field scales as pressureless matter with the Universe expansion and does not cause any large scale anisotropy even when it dominates the Universe. Hence, provided the mass of the vector field complies to the above requirements, the ‘vector curvaton’ scenario can account successfully for the observations.

Admittedly, the condition for the mass of the vector field during inflation is hard to achieve and, at best, amounts to a certain level of tuning. Still, using a ‘vector curvaton’ may be additionally motivated by the fact that no scalar fields have been observed as yet in nature.

\section{Acknowledgments}

I would like to thank D. H. Lyth and J. McDonald for discussions and the referee for insightful comments.

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