On the Existence of Periodic Orbits for the Fixed Homogeneous Circle Problem

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Abstract

We prove the existence of some types of periodic orbits for a particle moving in Euclidean three-space under the influence of the gravitational force induced by a fixed homogeneous circle. These types include periodic orbits very far and very near the homogeneous circle, as well as eight and spiral periodic orbits.

In this paper we use geometric arguments to demonstrate the existence of some types of periodic orbits for the movement of a particle in Euclidean three-space $\mathbb{R}^3$ on which the only acting force is the gravitational force induced by a fixed homogeneous circle. The study presented is purely analytical.

Interestingly all we could find in the literature about the fixed homogeneous circle problem were a few different ways of expressing the potential function. These expressions appear in classical potential theory books. Among these expressions are the one expressed in terms of elliptic integrals of the first kind and the one using the arithmetic-geometric mean given by Gauss. Essentially all expressions of the potential known today had already appeared in Poincare’s Théorie du Potentiel Newtonien [3], published first in 1899. Hence little has been done, at least in the past century, in the study of this problem. It is interesting to note that the results proved here use only elementary geometric constructions (but the technical details are sometimes a little involving).

Before we state our main results we fix some notation. We are interested in the study of the movement in $\mathbb{R}^3$ of a particle $P$ under the influence of the gravitational force induced by a fixed homogeneous circle $C$. Denote by $\mathbf{r} = (x, y, z) \in \mathbb{R}^3 - C$ the position of the particle $P$ and by $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$ its velocity. According to Newton’s Law the movement of $P$ obeys the following second order differential equation:

$$\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$$

(0.1)

where $V$ denotes the potential energy induced by $C$. The expression of $V$ is given by $V(\mathbf{r}) = -\int_C \frac{\lambda}{|\mathbf{r} - \mathbf{u}|}$, where $\lambda$ is the constant mass density of the circle $C$. In this introduction we consider the fixed homogeneous circle $C$ contained in the $xy$-plane and centered at the origin. Also, by rescaling we can consider the circle with radius equal to one (see section 1). Then the mass of $C$ is given by $M = 2\pi \lambda$.

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It is intuitively obvious (see section 1.2 of [2] for more details) that the \( z \)-axis, the horizontal plane (i.e. the \( xy \)-plane, which contains the circle) and any vertical plane (i.e. any plane that contains the \( z \)-axis) are invariant subspaces of our problem. By an invariant subspace \( \Lambda \) we mean that any movement that begins tangentially in \( \Lambda \) stays in \( \Lambda \) (in the future as well as in the past). Our problem restricted to the horizontal plane is a central force problem which outside the circle is given by an attractive force, hence it possesses periodic circular orbits. Inside the circle the force is repulsive and contains no periodic orbits; the dynamics in the horizontal plane is studied in [2].

Our first two results show the existence of periodic orbits of the fixed homogeneous circle problem restricted to a vertical plane. The first result gives periodic orbits very far and periodic orbits very close to the circle in any vertical plane. (In the next Theorem \( \text{dist} \) denotes Euclidean distance.)

**Theorem A.** Let \( \epsilon > 0 \) and \( \Lambda \) a vertical plane. Then there exist periodic orbits \( r(t), s(t) \) in \( \Lambda \) of the fixed homogeneous circle problem such that:

i. \( |r(t)| \geq \frac{1}{\epsilon} \), for all \( t \) (i.e. \( r(t) \) is far from the circle). Moreover, the trace of \( r(t) \) is a simple closed curve, symmetric with respect to the horizontal plane and the \( z \)-axis, and encloses the origin.

ii. \( \text{dist}(s(t), C) \leq \epsilon \), for all \( t \) (i.e. \( s(t) \) is close to the circle). Moreover, the trace of \( s(t) \) is a simple closed curve, symmetric with respect to the horizontal plane, and encloses the fixed homogeneous circle.

Theorem A will follow essentially from a very geometric result about perturbations of certain central forces [1]. Our next result shows the existence of infinitely many “figure eight” periodic orbits in any vertical plane. Before we state this result we have to say what we understand for a figure eight orbit. To fix ideas let the vertical plane \( \Lambda \) be the \( xz \)-plane. Generalizations to any \( \Lambda \) are straightforward. We write \((x, z)\) for the coordinates of a point in the \( xz \)-plane. We say that an orbit \( \mathbf{r}(t) = (x(t), z(t)), t \in [0, \tau], \tau > 0, \) of the fixed homogeneous circle problem in \( \Lambda \) is an essential part of a symmetric figure eight orbit if \( \mathbf{r}(t) \) satisfies: (1) \( z(0) = 0, \) \( x(0) > 1, \) (2) \( \mathbf{r}(\tau) = (0, 0), \) (3) \( \dot{x}(0) = 0, \) (4) \( z(t) > 0, \) \( t \in (0, \tau) \). Note that we do not demand any condition about the angle between \( (-1, 0) \) and \( \dot{\mathbf{r}}(\tau) \). This angle can be less than \( \frac{\pi}{2} \) as in the figure to the left below, or larger than \( \frac{\pi}{2} \) as in the figure to the right below.

![Figure 0.3](image-url)
Now, if \( r(t) \) is an essential part of a symmetric figure eight orbit we can use the symmetry of the problem to define a periodic orbit. Explicitly, using \( r(t) \) define \( \bar{r} : \mathbb{R} \rightarrow \Lambda \) in the following way:

\[
\bar{r}(t) = \begin{cases} 
  r(t - 4n\tau), & t \in [4n\tau, (4n + 1)\tau], \\
  \varphi_1 \varphi_2 r((4n + 2)\tau - t), & t \in [(4n + 1)\tau, (4n + 2)\tau], \\
  \varphi_1 r(t - (4n + 2)\tau), & t \in [(4n + 2)\tau, (4n + 3)\tau], \\
  \varphi_2 r(4n\tau - t), & t \in [(4n - 1)\tau, (4n)\tau].
\end{cases}
\]

Here \( n \) denotes an integer and \( \varphi_1, \varphi_2 \) are reflections with respect to the \( z \) and \( x \) axes, respectively. It is straightforward to verify that \( \bar{r}(t) \) is a periodic orbit of the fixed homogeneous circle problem in \( \Lambda \) with period \( 4\tau \). Moreover, the trace of \( \bar{r} \) is symmetric with respect to the \( x \) and \( z \) axes. We say that \( \bar{r}(t) \) is a symmetric figure eight periodic orbit in \( \Lambda \) and \( r(t) \) is the essential part of it.

**Theorem B.** In any vertical plane \( \Lambda \) there are infinitely many geometrically distinct symmetric figure eight periodic orbits of the fixed homogeneous circle problem, such that their essential parts are imbeddings.

![Symmetric figure eight periodic orbit](image)

**Figure 0.4:** Symmetric figure eight periodic orbit.

By geometrically distinct orbits we mean orbits whose traces are different. There is a clear similarity between the the fixed homogeneous circle problem (restricted to a vertical plane) and the planar symmetric Euler problem (i.e. the problem of two fixed centers with equal masses). For the symmetric Euler problem, which is a well-studied integrable problem, we could not find in the literature any reference related to the existence of symmetric figure eight orbits. The techniques developed to prove Theorem B above also work to prove the existence of symmetric figure eight periodic orbits of the planar symmetric Euler problem.

**Corollary.** There are infinitely many geometrically distinct symmetric figure eight periodic orbits of the planar symmetric Euler problem, such that their essential parts are imbeddings.

Our third result shows the existence of spiral periodic orbits close to the circle. Before we state this result we have some comments. Since our problem is invariant by rotations around the \( z \)-axis, the problem can be reduced in a canonical way to a problem with two degrees of freedom. In fact, every orbit \( r(t) = (x(t), y(t), z(t)) \) can be written as \( r(t) = (r(t)\cos \varphi(t), r(t)\sin \varphi(t), z(t)) \), i.e the cylindrical coordinates of \( r(t) \) are \( (r(t), \varphi(t), z(t)) \), and \( (r(t), z(t)) \) satisfy a system of equations (see (5.3) in section 5). Moreover, once we know \( r(t) \) we can obtain \( \varphi(t) \) by integration. We say that \( (r(t), z(t)) \) is the canonical projection orbit of \( r(t) \). Geometrically, \( (r(t), z(t)) \) is obtained from \( r(t) \) by a projection “of a book onto one of its pages”: think of \( \mathbb{R}^3 \) as book with an infinite number of pages, each page being a half plane having the \( z \)-axis as boundary; then identify all pages to one in the obvious way (like closing the book). Note that under this projection the circle \( \mathcal{C} \) projects to a point \( x_C \) with \( rz \) coordinates \((1,0)\). We say that an orbit is circular if its canonical projection orbit
is an equilibrium position of the system formed by the first two equations of (5.3). It is proved in [2] that all circular orbits lie in the horizontal plane. Moreover, it is also proved in [2] that there is a radius \( r_0, 1 < r_0 < 2 \) (that does not depend on the mass) such that a circular orbit in the horizontal plane with radius \( r \) is stable if and only if \( r > r_0 \). We say that an orbit (not contained in a vertical plane) is a spiral orbit if its canonical projection orbit is periodic. Note that not every spiral orbit is periodic. Indeed a spiral orbit is periodic if and only if \( \frac{\varphi(\tau)}{2\pi} \) is a rational number, where \( \tau \) is the period of the canonical projection orbit (see Lemma 5.1).

**Theorem C.** For every \( \epsilon > 0 \) and \( K \neq 0 \) there is a spiral periodic orbit \( r(t) \) with angular momentum \( K \) and such that \( \text{dist}(r(t), C) \leq \epsilon \), for all \( t \). Moreover, its canonical projection orbit is a simple closed curve, symmetric with respect to the horizontal plane and encloses the point \( x_C \).

![Figure 0.5: Spiral periodic orbit in \( \mathbb{R}^3 \).](image)

Note that the orbits mentioned in Theorem C above lie in a surface of revolution homeomorphic to a two-torus. It follows easily from the proof of Theorem C that there are also spiral orbits which are dense in these tori.

In section 1 we present some preliminary facts. Theorems A, B, C are proved in sections 2, 3, 5 respectively. In section 4 we show how use the methods of the proof of Theorem B to prove its Corollary. Many of our results can be generalized to problems induced by other symmetric objects. We give some examples of this in section 6. Finally, the paper has two appendices. In the first one we study “interval pointing forces” in the half plane. The results in this appendix are used to prove the last part of the statement of Theorem B: the essential part of the orbits are imbeddings. These results apply not just to any orbit in a vertical plane (periodic or not) but to any orbit of a system with a force that “points to an interval”. In the second appendix we prove Proposition 3.1.

This problem was proposed by H. Cabral, who also suggested some questions about it. We are grateful to him. We are also grateful to T. Stuchi and J. Koiller who suggested some of the generalizations shown here. Finally, we want to thank M. Sansuke for drawing most of the figures that appear in this paper.

1 Preliminaries.

We will need to consider circles with variable radius and mass. Write \( V(r, \rho, M) \) to denote the potential induced by \( C \) contained in the \( xy \)-plane and centered at the origin, with radius \( \rho \) and mass \( M \), and \( \nabla V(r, \rho, M) \) to denote the gradient (with respect to \( r \)) of \( V(r, \rho, M) \).

**Lemma 1.1** The potential \( V \) of the fixed homogeneous circle problem satisfies the following identities:

- (i) \( V(r, \rho, cM) = cV(r, \rho, M) \), for \( c \in \mathbb{R} \),
- (ii) \( V(c r, c \rho, M) = \frac{1}{c} V(r, \rho, M) \), for \( c > 0 \),
- (iii) \( \nabla V(c r, c \rho, M) = \frac{1}{c^2} \nabla V(r, \rho, M) \), for \( c > 0 \),
Remark 1.3

Lemma 1.4

Proof. It follows directly from the definition of $V(r, \rho, M) = -\frac{M}{2\pi} \int_0^{2\pi} \frac{d\theta}{|r - \rho e^{i\theta}|}$, where $e^{i\theta} = (\cos \theta, \sin \theta, 0)$.

Corollary 1.2

Let $r, \zeta, M, N$ be positive numbers. If $r(t)$ is a solution of $\ddot{r}(t) = -\nabla V(r, \rho, M)$ then $s(t) = \frac{\zeta}{r} \left( \sqrt{\frac{M^2}{N^2}} t \right)$ is a solution of $\ddot{s}(t) = -\nabla V(s, \zeta, N)$.

Proof. It follows from Lemma 1.1 by a direct calculation.

Remark 1.3

Note that if $r(t)$ and $s(t)$ are as above, then they have the same qualitative properties. For instance, if $r(t)$ is periodic (with period $T$) then $s(t)$ is also periodic (with period $\frac{T\sqrt{M^3}}{\sqrt{N^2}}$).

Note that Lemma 1.1 and Corollary 1.2 imply that in the study the fixed homogeneous circle problem we can assume the mass and the radius to be equal to one. We will also need the following Lemma.

Lemma 1.4

Let $q \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ open and $V : \Omega \to \mathbb{R} C^1$ in $\Omega$. If $r(t)$ is solution of $\ddot{r}(t) = -\nabla V(r)$ then $s(t) = r(t) + q$ is a solution of $\ddot{s}(t) = -\nabla W(s)$, where $W : \Omega + q \to \mathbb{R}$, $W(s) = V(s - q)$. Here $\Omega + q = \{ u + q ; u \in \Omega \}$.

Proof. It follows by direct substitution.

If $V$ and $W$ are as in the Lemma above, we say that $W$ is obtained from $V$ by a translation.

2 Proof of Theorem A.

This section has two subsections. In the first one we prove part (i) and in the second one we prove part (ii). Without loss of generality we can assume that $\Lambda$ is the $xz$-plane.

2.1 Periodic solutions far from the fixed homogeneous circle.

First we show that the potential of the fixed homogeneous circle, with radius $\epsilon$ small, can be regarded as a perturbation of the potential of the Kepler problem. Fix $M > 0$. Denote by $C_\epsilon$ the fixed homogeneous circle contained in the $xy$-plane, centered at the origin, with radius $\epsilon$ and mass $M$. The potential $V(r, \epsilon)$ induced by $C_\epsilon$ is given by:

$$V(r, \epsilon) = -\frac{M}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{(x - \epsilon \cos \theta)^2 + (y - \epsilon \sin \theta)^2 + z^2}}$$

where $r = (x, y, z)$. Note that $V(r, \epsilon)$ is defined and analytic in $\{ (r, \epsilon) ; r \notin C_\epsilon \}$. Note also that it makes sense to allow non-positive values for $\epsilon$. In particular $V(r, 0)$ is the potential induced by a point of mass $M$, located at the origin. We have that $V(r, \epsilon)$ is defined in $\{ (x, y, z, \epsilon) ; x^2 + y^2 \neq \epsilon^2 \text{ or } z \neq 0 \}$ and $V$ is even, that is $V(r, \epsilon) = V(r, -\epsilon)$.

Proposition 2.1

The potential $V(r, \epsilon)$ induced by the fixed homogeneous circle $C_\epsilon$ with radius $\epsilon$, is a second order perturbation (with respect to $\epsilon$) of the Kepler potential, that is,

$$V(r, \epsilon) = -\frac{M}{\|r\|^4} + \epsilon^2 f(r, \epsilon)$$
where \( f \) is analytic in \( \{ (r, \epsilon) = (x, y, z, \epsilon) \in \mathbb{R}^4 : 0 \neq x^2 + y^2 \neq \epsilon^2 \text{ or } z \neq 0 \} \).

**Proof.** Since \( V(r, \epsilon) = V(r, -\epsilon) \), a standard analytic continuation argument proves the Proposition. ■

Because \( M \) is fixed we have that if \( \epsilon \to 0 \) then \( \lambda \to +\infty \). Since the problem of the fixed homogeneous circle, with fixed mass \( M \), can be regarded as a perturbation of Kepler problem we have the following result:

**Proposition 2.2** Let \( C \) be a circle in the \( xz \)-plane, centered at the origin, and let \( U \) be an open neighborhood of \( C \) of the form \( C \subset U \subset (\mathbb{R}^2 - \{(0,0)\}) \). Then there exists \( \delta_0 > 0 \) such that for each \( \epsilon, \, 0 < \epsilon < \delta_0 \), there exists a periodic solution of \( \ddot{r} = -\nabla V(r, \epsilon) \) (restricted to the \( xz \)-plane) in \( U \). Moreover, the trace of this solution is a simple closed curve, symmetric with respect to the \( x \) and \( z \) axes, and encloses the origin.

**Proof.** The Proposition follows from Proposition 2.1 and Theorem 0.1 of [1]. ■

**Remark 2.3** Let \( r(t, x, v, \epsilon) \) denote a solution of \( \ddot{r} = -\nabla V(r, \epsilon) \) (restricted to the \( xz \)-plane) with initial conditions \( x, v \). Let \( r_0(t) = r(t, x_0, v_0, 0) \) be (circular) solution with trace \( C \) and \( r_\epsilon(t) = r(t, x_0, v, \epsilon) \) a periodic solution given by Proposition 2.2. It follows from Theorem 0.1 of [1] that we can choose \( v_\epsilon \) as close to \( v_0 \) as we want.

**Proof of part (i) of Theorem A.** Let \( C \) be a circle as in Proposition 2.2, with radius 2 and \( U = \{ p \in \mathbb{R}^2 : 1 < |p| < 3 \} \). By the Proposition above there is a \( \delta_0 > 0 \) such that for each \( \epsilon, \, 0 < \epsilon < \delta_0 \), there exists a solution \( r_\epsilon(t) \) of \( \ddot{r} = -\nabla V(r, \epsilon) \) in \( U \) with \( r_\epsilon(t) \) periodic satisfying \( 1 < |r_\epsilon(t)| < 3 \), and whose trace is a simple closed curve, symmetric with respect to the \( x \) and \( z \) axes, and encloses the origin. By Corollary 1.2 \( r(t) = \frac{1}{\epsilon} r_\epsilon(\epsilon^{3/2} t) \) is a solution of \( \ddot{r} = -\nabla V(r, 1) \). Moreover, \( r(t) \) satisfies \( \frac{1}{\epsilon} < |r(t)| < \frac{2}{3} \), for all \( t \). By the properties of \( r_\epsilon(t) \) we have that \( r(t) \) is also periodic and its trace is a simple closed curve, symmetric with respect to the \( x \) and \( z \) axes, and encloses the origin. ■

**Remark 2.4** We can assume that the periodic solutions given by Theorem A (i) intersect the \( x \)-axis transversally in exactly two points of the form \((x_0, 0), (-x_0, 0)\) (see Remark 2.1 (1) of [1]).

Note that if the period of \( r_\epsilon(t) \) is \( \tau \) then the period of \( s(t) \) is \( \tau/\epsilon^{3/2} \). Hence if \( \epsilon \) is small the period of \( r(t) \) is large. In this way the periodic orbits obtained here have large periods.

### 2.2 Periodic solutions near the fixed homogeneous circle.

In this subsection instead of fixing the mass \( M \) we fix the density \( \lambda > 0 \). Let \( V(x, y, z; \rho) \) be the potential induced by the fixed homogeneous circle, contained in the \( xy \)-plane, centered at the origin, with constant density \( \lambda \) and with radius \( \rho \).

Now, consider the fixed homogeneous circle with constant density \( \lambda \), contained in the \( xy \)-plane and passing through the origin, with radius \( \frac{1}{\epsilon} \) and center \( \left( \frac{1}{\epsilon}, 0, 0 \right) \), as the figure 2.6. Note that the \( xz \)-plane is a invariant subspace of this new problem. Denote the potential induced by this translated fixed

![Figure 2.6: Fixed homogeneous circle centered at \((\frac{1}{\epsilon}, 0, 0)\).](image)
homogeneous circle (and restricted to the \(xz\)-plane) by \(W(x, z; \epsilon)\). We have that \(W(x, z; \epsilon) = V\left((x,0,z) - (\frac{1}{\epsilon},0,0)\right)\). That is, \(W\) is obtained from \(V\) by a translation. Note that when \(\epsilon\) tends to zero, the mass \(M = 2\pi\lambda\frac{1}{\epsilon}\) tends to infinity. Let \(\nabla W\) be the gradient of \(W\) (with respect to \(x, z\)). We extend now \(\nabla W\) to \(\epsilon = 0\). Define \(\nabla W(x, z; 0) := 64\lambda\frac{(x,z)}{r^2 + z^2}, (x, z) \neq (0, 0)\). Let \(A = \{(x, z; \epsilon) ; (0, 0) \neq (x, z) \neq (\frac{1}{\epsilon}, 0)\}\). Then \(\nabla W(x, z; \epsilon)\) is defined on \(A\).

**Proposition 2.5** \(\nabla W\) is continuous on \(A\).

The proof of this Proposition appears in [2]. Note that \(\nabla W(x, z; 0) = 64\lambda\nabla \ln(\sqrt{x^2 + z^2})\) and recall that \(64\lambda\ln(\sqrt{x^2 + z^2}) = 32\lambda\ln(x^2 + z^2)\) is the potential induced by the infinite wire (with constant density \(32\lambda\) and infinite mass) orthogonal to \(xz\)-plane intersecting the \(xz\)-plane in the origin. Hence the problem of the fixed homogeneous circle with large radius, constant density \(\lambda\), can be regarded as a perturbation of the problem of the infinite homogeneous straight wire with density \(32\lambda\).

**Proposition 2.6** Let \(C\) be a circle in the \(xz\)-plane with center at origin and let \(U\) be an open set that contains \(C\) of the form \(C \subset U \subset (\mathbb{R}^2 - \{(0, 0)\})\). Then there is a \(\delta_0 > 0\) such that for each \(\epsilon, 0 < \epsilon < \delta_0\), there exists a periodic solution in \(U\) of \(\dot{r} = -\nabla W(r, \epsilon)\) (restricted to the \(xz\)-plane). Moreover, the trace of this solution is a simple closed curve, symmetric with respect to the \(x\)-axis, and encloses the origin.

**Proof.** It follows from Proposition 2.5 and Theorem 0.2 of [1].

**Proposition 2.7** There is a \(\delta_0 > 0\) such that for each \(\epsilon, 0 < \epsilon < \delta_0\), there exists a periodic solution \(r_\epsilon(t)\) of \(\dot{r} = -\nabla V\left(r, \frac{1}{\epsilon}\right)\) (restricted to the \(xz\)-plane) whose trace is a simple closed curve, symmetric with respect to the \(x\) axis, that contains \(C\) with center at origin and radius \(\frac{1}{\epsilon}\). Moreover, \(r_\epsilon(t)\) satisfies \(\frac{1}{3} < \text{dist}(r_\epsilon(t), C) < 1\).

**Proof.** It follows directly from Lemma 1.4 and from Proposition 2.6 setting \(U = \{\ p \in \mathbb{R}^2; \frac{1}{3} < \|p\| < 1\}\) and \(C\) with radius \(\frac{1}{3}\).

**Remark 2.8** It follows from Theorem 0.2 of [1] that we can choose \(r_\epsilon\) in Proposition 2.7 with initial velocity \(v_\epsilon = (0, v_\epsilon)\) equal to some fixed velocity \(v_0 = (0, v_0)\), \(v_0 > 0\). Also the initial position \(r_\epsilon(0) = x_\epsilon = (x_\epsilon, 0)\) can be such that \(x_\epsilon - (\frac{1}{\epsilon}, 0)\) is equal to some fixed \(x_0 = (x_0, 0)\), \(x_0 > 0\) \((x_0, v_0)\) are the initial conditions of a circular orbit of an unperturbed problem, see [1]).

**Proof of part (ii) of Theorem A.** Let \(\delta_0\) and \(r_\epsilon(t)\) be as in Proposition 2.7, with \(\epsilon < \delta_0\). By Corollary 1.2, we have that \(s(t) = \epsilon r_\epsilon(t)\) is a solution of \(\dot{r} = -\nabla V(r, 1)\). Moreover, \(\frac{\epsilon}{3} < \text{dist}(s(t), C) < \epsilon\), and \(s\) satisfies the required properties.

**Remark 2.9** (1) By the symmetry of the problem, we have periodic solutions enclosing the point \((-1, 0)\) and periodic solutions enclosing the point \((1, 0)\).
(2) If the period of \(r_\epsilon(t)\) is \(\tau\) then the period of \(s(t)\) is \(\tau\epsilon\). Hence, if \(\epsilon\) is small the period of \(s(t)\) is small. In this way the periodic orbits obtained above have small period.
(3) We can assume that the periodic solutions given by Theorem A (ii) intersect the \(x\)-axis transversally in exactly two points of the form \((x_0, 0), (x'_0, 0)\) with \(x_0 > 1, 0 < x'_0 < 1\). (see Remark 3.2 of [1]).

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3 Proof of Theorem B: Figure Eight Orbits.

Without loss of generality we assume the vertical plane to be the $xz$-plane, which we identify with $\mathbb{R}^2$ with coordinates $(x, z)$. We define the half vertical plane $\mathbb{R}^2_+ = \{(x, z) \in \mathbb{R}^2; z > 0 \}$ and the closed half vertical plane $\mathbb{R}^2_+ = \{(x, z) \in \mathbb{R}^2; z \geq 0 \}$. In this section we consider the potential $V(r)$ of the circle of radius 1 contained in the $xy$-plane and centered at the origin. We will use the following notation: we will write $r(t, x, v)$ for a solution $r(t)$ of $\ddot{r} = -\nabla V(r)$ (restricted to the $xz$-plane) with $r(0) = x$, $\dot{r}(0) = v$.

Note that the potential $V$ has singularities at the points $(-1, 0)$ and $(1, 0)$, which are the points of intersection of the circle $C$. In the proof of Proposition 3.1 we show that we can choose the circle of radius 1 contained in the $r$-notation: we will write $r(t, x, v)$ for a solution $r(t)$ of $\ddot{r} = -\nabla V(r)$ (restricted to the $xz$-plane) with $r(0) = x$, $\dot{r}(0) = v$.

The proof of this Proposition is given in appendix B. The necessity of writing the limit $\lim_{t \to t_0^-} z(t)$ in the statement above instead of just $z(t_0)$ is due to the fact that the solution can approach the circle. In the proof of Proposition 3.1 we show that we can choose $T_0 = 2 \left( 1 + \frac{2}{\min\{A, 1\}} \right)$, where $A = \frac{M}{(1 + R_0^2)}$ and $R_0 < \infty$ is the radius of Hill’s region of energy $\delta$.

**Proposition 3.1** For all $\delta < 0$ there exists $T_0 > 0$ such that if $r(t) = (x(t), z(t))$ is a maximal solution of $\ddot{r} = -\nabla V(r)$ (restricted to the $xz$-plane) with $z(0) = 0$, $\dot{z}(0) > 0$ and energy $E(r) \leq \delta$, then $\lim_{t \to t_0^-} z(t) = 0$ for some $t_0 \in (0, T_0]$.

The proof of this Proposition is given in appendix B. The necessity of writing the limit $\lim_{t \to t_0^-} z(t)$ in the statement above instead of just $z(t_0)$ is due to the fact that the solution can approach the circle. In the proof of Proposition 3.1 we show that we can choose $T_0 = 2 \left( 1 + \frac{2}{\min\{A, 1\}} \right)$, where $A = \frac{M}{(1 + R_0^2)}$ and $R_0 < \infty$ is the radius of Hill’s region of energy $\delta$.

**Proof of Theorem B.** From the comments in the introduction it is enough to prove the existence of infinitely many essential parts of symmetric figure eight orbits. The fact that these orbits are imbeddings follows from the results in appendix A. We will first prove the existence of one essential part of a symmetric figure eight orbit, and at the end of the section we indicate how to obtain infinitely many symmetric figure eight orbits.

We will denote by $s_0(t)$ a solution of $\ddot{r} = -\nabla V(r)$ obtained from Theorem A (i), that passes outside (and far from) the fixed homogeneous circle, and by $s_1(t)$ a solution of $\ddot{r} = -\nabla V(r)$ close to the circle, obtained from Theorem A (ii), that encloses the point $(1, 0)$ (see figure). Write $\dot{x}_i = s_i(0)$, $\dot{v}_i = \dot{s}_i(0)$, $i = 0, 1$. That is, with the notation introduced above $s_i(t) = (x(t), \dot{x}_i)$ and $\dot{v}_i = (\dot{v}_i, 0)$ in the positive $x$-axis. It is clear that we can also assume that $\dot{x}_0 > \dot{x}_1 > 0$ and $\dot{v}_0 > \dot{v}_1 > 0$. By Remark 2.3 we can choose $s_0$ such that it intersects transversally the $x$-axis in exactly two points $(\tilde{x}_0, 0)$ and $(-\tilde{x}_0, 0)$. Also, by Remark 2.3(3), we can choose $s_1$ such that it intersects transversally the $x$-axis in exactly two points $(\tilde{x}_1, 0)$ and $(-\tilde{x}_1, 0)$, with $0 < \tilde{x}_1 < 1$. 

![Diagram of figure eight orbit](image)
Before we continue with the proof of the Theorem consider following claims, which we will prove later. Let \( h_1 = E(s_i(t)) \) denote the energy of \( s_i(t), \ i = 0, 1 \). We denote by \( [x_1, x_0] = \{(x, 0); x_1 \leq x \leq x_0 \} \) and \( \{\bar{v}_0, \bar{v}_1\} = \{(0, v); \bar{v}_0 \leq v \leq \bar{v}_1\} \). Let \( A = ([x_1, x_0] \times \{\bar{v}_0\}) \cup ([x_1] \times \{\bar{v}_0, \bar{v}_1\}) \).

**Claim 3.2** We can choose \( s_0(t) \) and \( s_1(t) \), such that \( \bar{v}_0 < \bar{v}_1 \) and \( h_1 < h_0 < 0 \).

**Claim 3.3** For every \( (t, x, v) \in A \), the energy of \( r(t) = r(t, x, v) \) is less or equal \( h_0 \).

Before we state the last claim, we need some comments. For \( (t, x, v) \in A \), let \( \tilde{t}(x, v) = \inf\{t > 0; \tilde{z}(t) = 0\} \), where \( (x(t), z(t)) = r(t, x, v) \). By Proposition 3.1 and Claim 3.3 \( \tilde{t}(x, v) \) exists and \( \tilde{t}(x, v) \leq T_{h_0} \), for all \( (t, x, v) \in A \). By continuity \( \tilde{z}(\tilde{t}(x, v)) = 0 \). Since \( \tilde{z}(0) = v > 0, v = (0, v) \), we have that \( \tilde{t}(x, v) > 0 \). Hence \( \tilde{z}(t) > 0 \), for all \( t \in (0, \tilde{t}(x, v)) \), and \( z(t) \) is increasing on \( [0, \tilde{t}(x, v)) \). Define \( \tilde{z}(x, v) = z(\tilde{t}(x, v)) \), with \( (t, x, v) \in A \) and \( (x(t), z(t)) = r(t, x, v) \). Since \( \tilde{z}(x, v) > 0 \), the second equation of Claim 3.2 implies that \( \tilde{z}(x, v) \) is a local maximum of \( z(t) \).

The following claim says that there exists a hight \( \xi > 0 \) such that all solutions \( r(t, x, v) \), with \( (x, v) \in A \), “pass” this minimum height.

**Claim 3.4** There is \( \xi > 0 \) such that \( \tilde{z}(x, v) \geq \xi \) for all \( (t, x, v) \in A \).

Assuming the claims and Proposition 3.1, we prove the Theorem. Let \( n \) be such that \( \frac{1}{n} < \xi \) where \( \xi \) is as in Claim 3.4 and let \( E_n \) be the line \( E_n = \{(x, \frac{1}{n}); x \in \mathbb{R}\} \). Since \( z(0) = 0 \) we have that \( 0 = z(0) < \frac{1}{n} < \tilde{z}(x, v) = z(\tilde{t}(x, v)) \). By the intermediate value theorem we have that there exists \( t_1 = t_1(x, v) \) such that (1) \( 0 < t_1(x, v) < \tilde{t}(x, v) \) and (2) \( z(t_1(x, v)) = \frac{1}{n} \), that is \( r(t_1(x, v), x, v) \in E_n \).

Since \( \tilde{z}(t) > 0 \) for \( t \in (0, \tilde{t}(x, v)) \), we have \( \tilde{z}(t_1(x, v)) > 0 \). It follows that \( r(t_1) \) is not horizontal, that is, the intersection of \( r \) with \( E_n \) at \( t_1(x, v) \) is transversal. For \( (x, v) \in A \) let \( t_2 = t_2(x, v) = \inf\{t > \tilde{t}(x, v); r(t, x, v) \in E_n\} \). Note that \( t_2 \) exists by the intermediate value theorem and Proposition 3.1. Moreover, we have \( t_2(x, v) < T_{h_0} \) and \( z(t) > 0, t \in (0, t_2(x, v)) \).

We show now that \( \tilde{z}(t_2(x, v)) = 0 \). If \( \tilde{z}(t_2(x, v)) = 0 \) then \( t_2(x, v) \) is local maximum of \( z \) (because, since \( z(t_2(x, v)) = 1/n > 0 \), \( \tilde{z}(t_2(x, v)) < 0 \) by the second equation of Claim 3.2). It follows that \( z \) is increasing for \( t < t_2(x, v), t \) near \( t_2(x, v) \). Since \( z(\tilde{t}) \geq \xi > 1/n, \tilde{t} = \tilde{t}(x, v) \), we have that there exists \( t', \tilde{t} < t' < t_2(x, v) \) with \( z(t') = 1/n \), which contradicts the definition of \( t_2(x, v) \). This shows that \( \tilde{z}(t_2(x, v)) \neq 0 \). It follows that \( r(t_2(x, v)) \) is not horizontal thus the intersection of \( r \) with \( E_n \) at \( t = t_2(x, v) \) is transversal. Hence, we have functions \( t_1, t_2: A \to \mathbb{R} \) with

(1) \( 0 < t_1(x, v) < \tilde{t}(x, v) < t_2(x, v) < T_{h_0} \),

(2) \( r(t_1(x, v), x, v) \in E_n \) and \( r(t, x, v) \) intersects \( E_n \) transversally at \( t = t_1, t_2 \),

(3) \( t_1, t_2 \) are the first two times where \( r(t, x, v) \) intersects (transversally) \( E_n \), that is, \( r(t, x, v) \notin E_n \), for all \( t \in (0, t_2), t \neq t_1 \).

Note that \( t_1 \) and \( t_2 \) depend on \( n \). We show now that \( t_1 \) and \( t_2 \) are continuous on \( A \), for each fixed \( n \). Let \( (x_k, v_k) \to (x, v) \) be a convergent sequence in \( A \). Choose \( a, b > 0 \) such that \( t_1(x, v) < a < t_2(x, v) < b \). Then \( r(t, x, v), a \leq t \leq b \), intersects \( E_n \) transversally in a single point (certainly this point is \( r(t_2(x, v), x, v) \)). Also \( r(t, x, v), 0 \leq t \leq a \), intersects transversally \( E_n \) in a single point (this point is \( r(t_1(x, v), x, v) \)). Consequently for \( k \) sufficiently large, \( r(t, x_k, v_k), 0 \leq t \leq a \), and \( r(t, x_k, v_k), a \leq t \leq b \), also intersect \( E_n \) transversally in a single point (for this we can use, for example, Proposition 1.4 of [1]). It follows that \( r(t, x_k, v_k), 0 \leq t \leq b \), intersects \( E_n \) transversally in two points.
and these intersections happen at times $t_1(x_k, v_k)$ and $t_2(x_k, v_k)$. By the continuous dependence of solutions of O.D.E. (e.g. Proposition 1.5 of [1]) we have that $\lim_{k \to +\infty} t(x_k, v_k) = t(x, v)$, where $t(x', v')$, $(x', v')$ close to $(x, v)$, is the time in which $r(t, x', v')$, $0 \leq t \leq a$, intersects $E_n$ transversally. By the uniqueness of the intersections, $t(x_k, v_k) = t_1(x_k, v_k)$, which implies that $t_1$ is continuous. In the same way we have that $\lim_{k \to +\infty} t(r_k(a), r_k(a)) = t(r(a), \dot{r}(a))$, where $r_k(a) = r(a, x_k, v_k)$ and $t(r_k(a), r_k(a))$ is the time in which $r(t, x_k, v_k)$, $a \leq t \leq b$, intersects $E_n$ transversally. By the uniqueness of the intersections, $t(r_k(a), \dot{r}(a)) = t_2(x_k, v_k)$ and $t(r(a), \dot{r}(a)) = t_2(x, v)$. Then $t_2(x_k, v_k) \to t_2(x, v)$, which implies that $t_2$ is continuous.

Now, define $f : A \to E_n$ by $f(x, v) = r(t_2(x, v), x, v)$. Since $r$ and $t_2$ are continuous, $f$ is continuous. Recall that $s_0(t)$ and $s_1(t)$ intersect transversally the $x$-axis exactly in two points. Then, for $n$ sufficiently large, $s_0(t)$ and $s_1(t)$ also intersect $E_n$ transversally in exactly two points, and we have that $f(x_0, \tilde{v}_0) = (x', \frac{1}{n})$, with $x' < 0$, $f(x_1, \tilde{v}_1) = (x'', \frac{1}{n})$, with $x'' > 0$. Since $A$ is connected and $f$ is continuous, by the Intermediate value Theorem there exists $(x_0, v_0) \in A$ such that $f(x_0, v_0) = (0, \frac{1}{n})$. Hence we have a sequence $\{ (x_n, v_n) \} \in A$ such that $r(t_n, x_n, v_n) = (0, \frac{1}{n})$, for some $t_n \in (0, T_{h_0})$ (in fact $t_n = t_2(x_n, v_n)$). Since $A$ is compact and $t_n \in (0, T_{h_0})$, there exists a subsequence $(t_m, x_m, v_m)$ of $(t_n, x_n, v_n)$ such that $(t_m, x_m, v_m) \to (\tau, \bar{x}, \bar{v}) \in (0, T_{h_0}) \times A$. Consequently $\lim_{m \to +\infty} r'(t_m, x_m, v_m) = r'(\tau, \bar{x}, \bar{v})$. On the other hand, $r(t_m, x_m, v_m) = (0, \frac{1}{n}) \to (0, 0)$, which implies that $r(\tau, \bar{x}, \bar{v}, 1) = (0, 0)$. Therefore $r : [0, \tau] \to \mathbb{R}^2$, $r(t) = r(t, \bar{x}, \bar{v})$ satisfies (1), (2), (3) of the definition of an essential part of a symmetric figure eight orbit given in the introduction.

![Figure 3.8: Obtaining the solution $r(t)$.](image)

To finish the proof of the Theorem we show that $r(t) = r(t, \bar{x}, \bar{v})$ satisfies also property (4). Let $(x_m(t), z_m(t)) = r(t, x_m, v_m)$. Note that $z_m(t) \geq 0$, for all $t \in [0, t_m]$ (because $t_m = t_2(x_m, v_m)$). Then $\ddot{z}(t) \geq 0$, for all $t \in [0, \tau]$, where $(\bar{x}(t), \dot{z}(t)) = r(t, \bar{x}, \bar{v})$. If $\ddot{z}(t') = 0$ for some $t' \in (0, \tau)$, we have $\ddot{z}(t') = 0$ ($t'$ is a point of local minimum), that is, $(\bar{x}, \bar{z})$ is tangent to the $x$-axis at $t = t'$. It follows that $(\bar{x}(t), \dot{z}(t))$ is contained in the $x$-axis, for all $t \in (0, \tau)$ (because the $x$-axis is an invariant subspace). This is a contradiction. Then $\ddot{z}(t) > 0$, $t \in (0, \tau)$. ■

**Proof of Claim 3.2** First we prove that we can choose $s_0(t)$ with small negative energy and small velocity. Let $r_0(t)$ and $r_1(t)$ be as in Remark 2.8. Recall that $s_0(t) = \frac{1}{\epsilon} r_1(\epsilon^{3/2} t)$. Let $h, \tilde{h}$ be the energy of $r_0$ and $r_1$ respectively. We have $h < 0$. If $v_\epsilon$ is close to $v_0$ we have that $\tilde{h}$ is close to $h$. In particular, we can choose $v_\epsilon$ with energy $\tilde{h} = E(r_\epsilon(t)) < 0$. Note that $\tilde{v}_0 = s_0(0) = \epsilon \dot{r}_1(0) = \epsilon v_\epsilon$. Hence we can assume $||\tilde{v}_0|| = ||\tilde{v}_0|| \leq \epsilon (2 ||v_0||)$. Also, $E(s_0(t)) = \frac{1}{\epsilon^2} ||\tilde{v}_0(0)||^2 + V(\tilde{x}_0, 0) = \frac{1}{\epsilon} ||v_\epsilon||^2 + cV(x_\epsilon, \epsilon) = \epsilon E(r_\epsilon(t)) = \epsilon \tilde{h} < 0$, where $V(x, \epsilon)$ is as in section 2.1 (for the second equality see Lemma 1.1). This shows that we can choose $s_0$ with small velocity $\tilde{v}_0$ and small negative energy.

We now deal with $s_1(t)$. Let $r_\epsilon, v_\epsilon = (0, v_\epsilon)$, $v_0 = (0, v_0)$ as in Remark 2.8. Recall that $s_1(t) = \epsilon r_\epsilon(t)$ (since we can choose $v_\epsilon$ close to $v_0$, we can assume that $0 < \epsilon < ||v_0||$). Since $0 < a < ||v_0|| < b$, for some constants $a, b$ independent of $\epsilon$. A simple calculation shows that $\dot{s}_1(0) = \dot{r}_\epsilon(0) = v_\epsilon$. We have
$E(s_1(t)) = \frac{1}{2}||\dot{s}_1(0)||^2 + V(s_1(0)) < \frac{1}{2}b^2 + V(s_1(0))$ and that $dist(s_1(0), C) = e||x_0||$, where $x_0$ (fixed) is as in Remark 2.8. Since $V(x) \to -\infty$, when $dist(x, C) \to 0$ (see [2], Lemma 2.4), we have that for any $n > 0$ we can choose $\epsilon$ sufficiently small such that $E(s_1(t)) < -n$. In particular, we can choose $s_0(t)$ and $s_1(t)$ such that $E(s_1(t)) < E(s_0(t))$. Also, since $||v_1|| = ||v_r|| > a$, and $||v_0||$ can be chosen small, it follows that we can choose $||\dot{v}_0|| < ||\dot{v}_1||$. This proves the claim. ■

Proof of Claim 3.3: It follows from claim 3.2 and from the fact that $V(\overline{x}_1) \leq V(x) \leq V(\overline{x}_0)$, for all $x \in [\overline{x}_1, \overline{x}_0]$. ■

Proof of Claim 3.4: Suppose that there is no such $\xi > 0$. Then there exists a sequence $(x_n, v_n)$ such that $\overline{z}(x_n, v_n) \leq \frac{1}{n}$ and $(x_n, v_n)$ converges to some $(\overline{x}, \overline{v}) \in A$. Hence $r(t, x_n, v_n) \to r(t, \overline{x}, \overline{v})$ and $\tilde{r}(t, x_n, v_n) \to \tilde{r}(t, \overline{x}, \overline{v})$, which implies that $z_n(t) \to \overline{z}(t)$ and $\dot{z}_n(t) \to \dot{z}(t)$ where $(x_n(t), z_n(t)) = r(t, x_n, v_n)$ and $(\tilde{z}(t), \dot{z}(t)) = r(t, \overline{x}, \overline{v})$.

Since $\tilde{z}(t) > 0$, for $t \in [0, \tilde{t}(\overline{x}, \overline{v}))$, we have that there exists $\chi > 0$ such that $\tilde{z}(t) \geq \chi$, for all $t \in [0, \frac{1}{2}\tilde{t}(\overline{x}, \overline{v})]$. Hence for $n$ sufficiently large $\tilde{z}_n(t) \geq \frac{\chi}{2} > 0$, for all $t \in [0, \frac{1}{2}\tilde{t}(\overline{x}, \overline{v})]$. Also, since $z_n \to \tilde{z}$ we have for $n$ sufficiently large, $z_n(\frac{1}{2}\tilde{t}(\overline{x}, \overline{v})) \geq \frac{\chi}{2}(\frac{1}{2}\tilde{t}(\overline{x}, \overline{v})) > 0$. Note that $r(t, x_n, v_n)$ is defined in $[0, \frac{1}{2}\tilde{t}(\overline{x}, \overline{v})]$, for $n$ sufficiently large. By the definition of $\tilde{t}$, we have that $\tilde{t}(x_n, v_n) > \frac{1}{2}\tilde{t}(\overline{x}, \overline{v})$ and, since $z_n(t)$ is increasing in $[0, \tilde{t}(\overline{x}, \overline{v})]$ we get $z(x_n, v_n) = z_n(\tilde{t}(x_n, v_n)) > z_n(\frac{1}{2}\tilde{t}(\overline{x}, \overline{v})) \geq \frac{1}{2}(\frac{1}{2}\tilde{t}(\overline{x}, \overline{v})) = \eta > 0$ for all $n$ sufficiently large, a contradiction. ■

Up to now have proved only the existence of one symmetric figure eight orbit. We show now how to obtain infinitely many symmetric figure eight orbits. First, note that the symmetric figure eight orbit constructed above has its initial values $(x, v)$ in the set $A$. This set depends only on the choice of the initial values of the solutions $s_0$ and $s_1$. These initial values are $\overline{x}_0 = (\overline{x}_0, 0)$, $\overline{v}_0 = (0, \overline{v}_0)$ for $s_0$ and $\overline{x}_1 = (\overline{x}_1, 0)$, $\overline{v}_1 = (0, \overline{v}_1)$ for $s_1$. Its clear from the proof of claim 3.2 that we can choose $\overline{v}_0 > 0$ as small as we want and $\overline{x}_1 > 1$ as close to 1 as we want. Hence are can choose a sequence $\{A_n\}$ of disjoint sets, all satisfying the statements of claims 3.2 and 3.3 (see figure below). Since for each $A_n$ we have a symmetric figure eight orbit with initial values on $A_n$, we obtain in this way infinitely many geometrically distinct figure eight orbits.

4 Proof of the Corollary to Theorem B: Symmetric figure eight orbits in the symmetric Euler problem.

Consider the $xy$-plane and two fixed centers with mass $M$ located at $(\rho, 0) = \rho e_1$, $(-\rho, 0) = -\rho e_1$, $\rho > 0$. Consider a particle $P$ moving in the $xy$-plane under the influence of the gravitational attraction induced by the two fixed centers. The potential of this problem is given by
\[ U(r) = -\frac{M}{\|r + pe_1\|} - \frac{M}{\|r - pe_1\|}. \]

Write \( U(r, \rho, M) = -\frac{M}{\|r + pe_1\|} - \frac{M}{\|r - pe_1\|} \) to express the fact that \( U \) depends on the mass \( M \) and the distance \( \rho \). \( U \) satisfies the same properties as \( V \) (section 1). For \( \rho \) small \( U(r, \rho, M) \) is a symmetric perturbation of the Kepler potential \( U(r, 0, M) = -\frac{2M}{r} \). We can then obtain (using the results in [2]), as in the case of the fixed homogeneous circle problem, periodic symmetric orbits \( s_0(t) \) far from the two fixed centers. Moreover, a similar argument as the one used in the proof of claim 3.2 shows that we can choose \( s_0(t) \) with small negative energy, and also such that \( \|s_0(t)\| \) is small.

Let
\[ u(r, \epsilon) = -\frac{M}{\|r\|} - \frac{M}{\|r + 2\epsilon e_1\|} = -\frac{M}{\|r\|} - \epsilon M. \]

Note that \( U(r, \frac{1}{\epsilon}, M) \) is obtained by a translation from \( U(r, \epsilon) \). Note also that for \( \epsilon \) small we can consider \( U(r, \epsilon) \) as a perturbation of Kepler potential \( U(r, 0) = -\frac{M}{\|r\|} \). Let \( \tilde{v} = (0, \tilde{v}) \), \( \tilde{v} > 0 \), be a velocity such that \( \tilde{r}(t) \) is a circular solution of the Kepler problem \( \ddot{r} = -\nabla U(r, 0) \), with \( \dot{r}(0) = (1, 0) = e_1 \) and \( \tilde{r}(0) = \tilde{v} \). Note that \( \tilde{v} = \sqrt{M} \). Let also \( \tilde{t} > 0 \) be such that \( \tilde{r}(\tilde{t}) = (0, -1) \) is the first (positive) time such that \( \tilde{r}(\tilde{t}) = (0, -1) \). Note that \( \tilde{r}(t) \), \( 0 \leq t \leq \tilde{t} \), intersects transversally the negative \( x \)-axis in a single point. Let \( r_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t)) \) be a solution of \( \ddot{r} = \nabla U(r, \epsilon) \), with \( r_\epsilon(0) = e_1 \), \( \dot{r}_\epsilon(0) = \tilde{v} \).

By Proposition 1.4 of [2], for \( \epsilon \) sufficiently small, \( r_\epsilon(t) \), \( 0 \leq t \leq \tilde{t} \), intersects transversally the negative \( x \)-axis in exactly one point \( r_\epsilon(t_\epsilon) \) and \( y_\epsilon(t_\epsilon) > 0 \), \( 0 < t < t_\epsilon \). Since \( r_\epsilon \) is close to \( \tilde{r} \) we can assume that \( -2 < x_\epsilon(t_\epsilon) \). Let \( \tilde{r}_\epsilon := r_\epsilon(t_\epsilon) + \frac{1}{\epsilon} e_1 \).

Then \( \tilde{r}_\epsilon(t) \) is a solution of \( \tilde{r} = \nabla U(r, \frac{\epsilon}{\epsilon}, M) \). Let \( s_\epsilon(t) := \tilde{r}_\epsilon(\epsilon^{-3/2} t) \). It follows from Corollary 1.2 that \( s_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t)) \) is a solution of \( \ddot{r} = \nabla U(r, 1, M) \), and it is easy to verify that \( s_\epsilon(0) = (1 + \epsilon, 0) \), \( s_\epsilon(\epsilon^{3/2} t_\epsilon) = (x_\epsilon, 0), 0 < x_\epsilon < 1, y_\epsilon(t) > 0, 0 < t < \epsilon^{3/2} t_\epsilon \), and \( \dot{s}_\epsilon(0) = \frac{1}{\epsilon} \tilde{v} \).

Then
\[ E(s_\epsilon(t)) = E(s_\epsilon(0)) = \frac{1}{2\epsilon} \|	ilde{v}\|^2 + U(s_\epsilon(0), 1, M) = M \frac{1}{2\epsilon} - M + \frac{M}{2 + \epsilon} = -M \frac{1}{2\epsilon} \left(1 + \frac{2\epsilon}{2 + \epsilon}\right). \]

Hence for \( \epsilon \) sufficiently small we can choose \( s_1(t) = s_\epsilon(t) \) with large and negative energy. Note also that \( \|s_1(0)\| = \frac{1}{\sqrt{\epsilon}} \|\tilde{v}\| = \frac{\sqrt{M}}{\sqrt{\epsilon}} \) is large. The solutions \( s_0(t) \) and \( s_1(t) \) constructed above satisfy the same properties as the solutions \( s_0(t) \) and \( s_1(t) \) constructed at the beginning of the proof of Theorem B.

For the case of the symmetric Euler problem, Claim 3.2 follows from the choice of \( s_0 \) and \( s_1 \) above; the proofs of Claims 3.3 and 3.4 are identical. The proof of Proposition 3.3 is similar, just estimate \( \frac{\partial}{\partial y} U(x, y) = \frac{M}{\|r - e_1\|^3} + \frac{M}{\|r + e_1\|^3} \), where \( U(x, y) = U((x, y), 1, M) \). But a simple calculation shows that if \( r(t) = (x(t), y(t)) \) has energy less or equal \( \delta < 0 \), then
\[ \left(\frac{M}{\|r - e_1\|^3} + \frac{M}{\|r + e_1\|^3}\right) \geq \frac{2M}{(1 + R_\delta)^3}, \]

where \( R_\delta < +\infty \) is the radius of Hill’s region with energy \( \delta < 0 \), that is, \( R_\delta = \sup \{ \|r\| : U(r) \leq \delta \} \) (it can be verified directly that \( R_\delta \leq \frac{2}{\delta} + 1 \)). Then \( y(t) \) satisfies \( \|\dot{y}\| \geq A |y| \), with \( A = \frac{2M}{(1 + R_\delta)^3} \). The rest of the proofs of Proposition 3.4 and of Theorem B for this case are similar.

As in the case of the fixed homogeneous circle, we can prove that for the symmetric Euler problem we also have infinitely many symmetric figure eight orbits.
5 Proof of Theorem C: Periodic Spiral Orbits.

In this section we fix the density \( \lambda \) of the circle and denote by \( V(\mathbf{r}, \frac{1}{\varepsilon}) = V(x, y, z; \frac{1}{\varepsilon}) \) the potential at the point \( \mathbf{r} = (x, y, z) \in \mathbb{R}^3 \) induced by the fixed homogeneous circle with density \( \lambda \), contained in the \( xy \)-plane, centered at the origin and with radius \( \frac{1}{\varepsilon} \). Since \( V \) is invariant by rotations around the \( z \)-axis we can reduce our problem in a canonical way to a problem with two degrees of freedom. The Lagrangian in cylindrical coordinates can be written as:

\[
L(\mathbf{r}, \varphi, z, \dot{\mathbf{r}}, \dot{\varphi}, \dot{z}) = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2) - V_1(r, z),
\]

where \( V_1(r, z) = V(r \cos \varphi, r \sin \varphi, z; \frac{1}{\varepsilon}) \). It is then straightforward to verify that the system \( \dot{\mathbf{r}} = -\nabla V(\mathbf{r}, \frac{1}{\varepsilon}) \) in these coordinates is given by:

\[
\begin{align*}
\dot{r} &= \frac{K^2}{r^3} - \frac{\partial V_1}{\partial r}(r, z) \\
\dot{z} &= -\frac{\partial V_1}{\partial z}(r, z) \\
\dot{\varphi}(t) &= \frac{K/\varepsilon}{r^2}
\end{align*}
\]

(5.3)

where \( \frac{K}{\varepsilon} \) is the (constant) angular momentum.

Remarks.

(1) \( K = 0 \) implies that \( \varphi \) is constant. Then the particle moves on the vertical plane determined by the \( z \)-axis and by the vector \((\cos \varphi, \sin \varphi, 0)\).

(2) The first two equations of system (5.3) can be rewritten as \( \dot{\mathbf{r}} = -\nabla V(\mathbf{r}, \frac{1}{\varepsilon}) \), with \( \mathbf{r} = (r, z) \) and

\[
\nabla V(r, z; \frac{1}{\varepsilon}) = -K^2 \frac{1}{r^3} + V(r, 0, z; \frac{1}{\varepsilon})
\]

Note that, if \( (r(t), z(t)) \) is a solution of the first two equations of (5.3), defining \( \varphi(t) = \int_0^t \frac{K/\varepsilon}{r^2(s)} ds \), we have that \( (r(t), \varphi(t), z(t)) \) is a solution of (5.3). Then \( (r(t) \cos \varphi(t), r(t) \sin \varphi(t), z(t)) \) is a solution of \( \dot{\mathbf{r}} = -\nabla V(\mathbf{r}; \frac{1}{\varepsilon}) \).

Lemma 5.1 Let \( (r(t), z(t)) \) be a periodic solution of the first two equations of (5.3) with period \( \tau \) and \( \varphi(t) = \int_0^t \frac{K/\varepsilon}{r^2(s)} ds \). Then \( \mathbf{r}(t) = (r(t) \cos \varphi(t), r(t) \sin \varphi(t), z(t)) \) is a periodic solution of \( \dot{\mathbf{r}} = -\nabla V(\mathbf{r}; \frac{1}{\varepsilon}) \) if and only if \( \frac{\varphi(t)}{2\pi} = \frac{p}{q} \) is rational.

Proof. Suppose \( \frac{\varphi(t)}{2\pi} = \frac{p}{q}, p, q \in \mathbb{Z} \). Let \( R_\theta \) denote the rotation about the \( z \)-axis by an angle \( \theta \). Note that \( (R_\theta)^l = R_\theta \) for every \( l \in \mathbb{Z} \). Since \( (r(t), z(t)) \) is \( \tau \)-periodic and \( \dot{\varphi} = \frac{K/\varepsilon}{r^2} \) a direct calculation shows that:

\[
\mathbf{r}(\tau) = R_{\varphi(\tau)} \mathbf{r}(0) \quad \text{and} \quad \dot{\mathbf{r}}(\tau) = R_{\varphi(\tau)} \dot{\mathbf{r}}(0).
\]

This implies \( \mathbf{r}(q \tau) = R_{q \varphi(\tau)} \mathbf{r}(0) \) and \( \dot{\mathbf{r}}(q \tau) = R_{q \varphi(\tau)} \dot{\mathbf{r}}(0) \). Hence \( \dot{\mathbf{r}}(q \tau) = \dot{\mathbf{r}}(0) \). Conversely, suppose that \( (r(t) \cos \varphi(t), r(t) \sin \varphi(t), z(t)) \) is \( \tau_0 \)-periodic, for some \( \tau_0 \). Then, \( z(t) \) and \( r(t) = \sqrt{r(t)^2 \cos^2 \varphi(t) + r(t)^2 \sin^2 \varphi(t)} \) are \( \tau_0 \)-periodic. Let \( \tau \) be the minimal period of \( (r(t), z(t)) \). From the definition of \( \varphi \) it follows that \( \varphi(k \tau) = k \varphi(\tau) \), for all \( k \in \mathbb{Z} \). It also follows that \( \tau_0 = n \tau \). Hence, since \( r(t) > 0 \), for all \( t \), and \( r(t) \) is \( \tau_0 \)-periodic, we have \( \cos \varphi(t + n \tau) = \cos \varphi(t + \tau_0) = \cos \varphi(t) \).

In the same way \( \sin \varphi(t + n \tau) = \sin \varphi(t + \tau_0) = \sin \varphi(t) \), which implies that \( \varphi(t + n \tau) - \varphi(t) = 2p \tau \), for some \( p \in \mathbb{Z} \). Evaluating in \( t = 0 \), we have \( n \varphi(\tau) = \varphi(n \tau) = 2p \tau \), and therefore \( \frac{\varphi(t)}{2\pi} = \frac{p}{n} \). □
Recall that in section 2.2 we denoted by $W(x, z; \epsilon)$ the potential at the point $(x, z)$ (in the $xz$-plane) induced by a circle in the $xy$-plane of radius $\frac{1}{\epsilon}$ centered at $\left(\frac{1}{\epsilon}, 0, 0\right)$. Now, for $K \in \mathbb{R}$ consider the system:
\[
\begin{aligned}
\dot{r} &= \frac{k^2}{(r - \frac{1}{\epsilon})^2} - \frac{\partial W}{\partial r}(r, z; \epsilon) \\
\dot{z} &= \frac{\partial W}{\partial z}(r, z; \epsilon)
\end{aligned}
\tag{5.4}
\]

Remarks. (1) System (5.4) can be rewritten as $\dot{r} = -\nabla W(r, \epsilon)$, where
\[
W(r, \epsilon) = W(r, z; \epsilon) = \frac{k^2}{2(r - \frac{1}{\epsilon})^2} + W(r, z; \epsilon).
\]

(2) Note that $\nabla$ is obtained from $W$ by a translation. Indeed $\nabla\left(x - \frac{1}{\epsilon}, z, \frac{1}{\epsilon}\right) = W(x, z; \epsilon)$.

Recall that a key ingredient in the proof of Theorem A, part (ii), was the fact that for $K = 0$ system (5.4) is a perturbation of the infinite homogeneous straight wire problem. The next Lemma says that the same is true for any $K$.

**Lemma 5.2** For every $K$ the system given by (5.4) behaves as a perturbation of the infinite homogeneous straight wire problem.

**Proof.** The Lemma follows from Proposition 2.5 and the fact that $\lim_{\epsilon \to 0} \frac{k^2}{(x - \frac{1}{\epsilon})^2} = 0$ uniformly on compacts. 

We will use the following notation. Let $U \subset \{(x, 0, z); x > 0\} \subset \mathbb{R}^2$. Define the rotation of $U$ about the $z$-axis: $\text{rot } U = \{(x \cos \varphi, x \sin \varphi, z); \varphi \in \mathbb{R}, (x, z) \in U\}$. In cylindrical coordinates, we have $\text{rot } U = \{(r, \varphi, z); (r, z) \in U\}$. The next result, together with some rescaling, will imply Theorem C.

**Proposition 5.3** Let $C$ be a circle in the $xz$-plane, with center at the origin and let $U$ be an open bounded set that contains $C$, of the form $C \subset U \subset (\mathbb{R}^2 - \{(0, 0)\})$. Let $K \neq 0$. Then for each $\epsilon_0 > 0$ there exist $\epsilon$, $0 < \epsilon < \epsilon_0$, and a periodic solution $r_\epsilon(t) = (r_\epsilon(t) \cos \varphi_\epsilon(t), r_\epsilon(t) \sin \varphi_\epsilon(t), z_\epsilon(t))$ of $\dot{r} = -\nabla W \left( r, \frac{1}{\epsilon} \right)$ in $\text{rot} \left( U + \left\{(\frac{r}{\epsilon}, 0)\right\} \right)$ with angular momentum $K/\epsilon$. Moreover, the trace of $(r_\epsilon(t), z_\epsilon(t))$ is a simple closed curve, symmetric with respect to the $x$-axis, and encloses the fixed homogeneous circle.

**Proof.** Let $C$ be a circle in the $rz$-plane with center at the origin and $U$ be an open bounded set that contains $C$ of the form $C \subset U \subset \mathbb{R}^2 - \{(0, 0)\}$. Without loss of generality we can assume $K > 0$. Let $x_0$ be the unique point in $C \cap \text{positive } r - \text{axis}$. We will denote by $\tilde{r}_{\nu, \epsilon}(t) = (\tilde{r}_{\nu, \epsilon}(t), \tilde{z}_{\nu, \epsilon}(t))$ a solution of $\dot{r} = -\nabla W(r; \epsilon)$, with $\tilde{r}(0) = x_0$ and $\tilde{r}(0) = \nu$.

Let $\tilde{r}_{\nu, 0}(t)$ be a circular solution of $\dot{r} = -\nabla W(r; 0)$ whose trace is $C$. Consider $[0, \tilde{t}]$, a time interval in which $\tilde{r}_{\nu, 0}(t)$ intersects transversally the closed segment $E = \{(r, 0); -\infty < r \leq 0\}$ in a single point. By the continuous dependence of the solutions (e.g. see Proposition 1.5 of [1]) we have that there is a $\delta > 0$ such that, if $\|\nu - \nu_0\| < \delta$, $|\epsilon| < \delta$, then $\tilde{r}_{\nu, \epsilon}(t) = (\tilde{r}_{\nu, \epsilon}(t), \tilde{z}_{\nu, \epsilon}(t))$, $t \in [0, \tilde{t}]$, ...
intersects $E$ in a single point, and $t(v, \epsilon)$ is continuous, where $t(v, \epsilon)$ is such that $\bar{r}_{V, \epsilon}(t(v, \epsilon)) \in E$. Define $\tau(v, \epsilon) := 2t(v, \epsilon)$ and $V_0 = \{ (1 + s)v_0; |s| < \frac{\delta}{|V_0|} \}$. We can assume

1. $\tau(v, \epsilon)$ is bounded,
2. $|\bar{r}_{V, \epsilon}(t)|$ is bounded,
3. $0 < \gamma < \tau(v, \epsilon)$ for some constant $\gamma$.

Let $r_{V, \epsilon}(t) = (r_{V, \epsilon}(t), z_{V, \epsilon}(t)) = (\bar{r}_{V, \epsilon}(t) + \frac{\epsilon}{r_{V, \epsilon}(t)}, z_{V, \epsilon}(t))$. Note that $r_{V, \epsilon}(t)$ is a solution of $\ddot{r} = -\nabla V(r, \epsilon)$). We can also choose $\delta$ small such that $r_{V, \epsilon}(t) > 0$, for all $\epsilon \in (-\delta, \delta)$. Hence we have that $(r_{V, \epsilon}, \varphi, z_{V, \epsilon})$ is solution of \([5,3]\), where $\varphi(t) := \varphi_{V, \epsilon}(t) = \int_0^t \frac{K/\epsilon ds}{r_{V, \epsilon}(s)}$. Define $\Theta : V_0 \times [0, \delta] \to \mathbb{R}$ by

$$\Theta(v, \epsilon) = \begin{cases} \frac{1}{2\pi} \int_0^{\tau(v, \epsilon)} \frac{K/\epsilon ds}{r_{V, \epsilon}(s)} = \frac{1}{2\pi} \int_0^{\tau(v, \epsilon)} \frac{K/\epsilon ds}{(r_{V, \epsilon}(s) + \frac{\epsilon}{r_{V, \epsilon}(s)})^2}, & \text{for } \epsilon \neq 0 \\ 0, & \text{for } \epsilon = 0. \end{cases}$$

From (1) and (2) above and from the fact that $\tau$ is continuous it follows that $\Theta$ is continuous in $V_0 \times [0, \delta]$. Note that $\Theta(v, \epsilon) = \frac{2V_{\epsilon}(\tau(v, \epsilon))}{2\pi}$. Since $\tau(v, \epsilon) \geq \gamma > 0$ (see (3) above) and we are assuming $K > 0$, we have that $\Theta(v, \epsilon) > 0$ for $\epsilon > 0$. Also $\Theta(v, 0) = 0$, for all $v$.

Let $\epsilon_0 > 0$. By the Addendum to Theorem 0.2 of \([1]\) (taking $\epsilon_0 > 0$ even smaller, if necessary) there exists a compact connected set $V \subset V_0 \times [0, \epsilon_0]$ with $V_\epsilon = V \cap V_0 \times \{ \epsilon \} \neq \emptyset$, for all $\epsilon \in [0, \epsilon_0]$ such that for $(v, \epsilon) \in V$, $r_{V, \epsilon}(t)$ is a periodic solution of $\ddot{r} = -\nabla V(r, \epsilon)$ whose trace is a simple closed curve symmetric with respect to the $r$-axis, and encloses the origin.

Since $V_{\epsilon_0} \neq \emptyset$ and $V_0 \neq \emptyset$, there exist $(v_{\epsilon_0}, \epsilon_0), (v_0, 0) \in V$. Moreover, $\Theta(v_{\epsilon_0}, \epsilon_0) > 0$ and $\Theta(v_0, 0) = 0$. Since $V$ is connected we have that $[0, \Theta(v_{\epsilon_0}, \epsilon_0)] \subset \Theta(V)$. Then there exists a rational number $\frac{p}{q} \in [0, \Theta(v_{\epsilon_0}, \epsilon_0)]$ and hence there exists $(v, \epsilon) \in V$, $0 < \epsilon \leq \epsilon_0$, such that $\Theta(v, \epsilon) = \frac{p}{q}$. From Lemma \([7,1]\) it follows that $(r_{V, \epsilon}(t) \cos \varphi_{V, \epsilon}(t), r_{V, \epsilon}(t) \sin \varphi_{V, \epsilon}(t), z_{V, \epsilon}(t))$ is a periodic solution of $\ddot{r} = -\nabla V(r, \epsilon)$, that satisfies the properties of the statement of the Proposition \([5,3]\). □

**Proof of Theorem C.** In Proposition \([5,3]\) take $C$ with radius $\frac{1}{2} \mathbb{R}$ and $U = \{ p \in \mathbb{R}^2; \frac{1}{3} < p < 1 \}$. Then there exist $\epsilon$, $0 < \epsilon < \epsilon_0$, and a periodic solution $r(t) = (r(t) \cos \varphi(t), r(t) \sin \varphi(t), z(t))$ of $\ddot{r} = -\nabla V(r, \epsilon)$, with angular momentum $K/\epsilon$. Moreover, the trace of $(r(t), z(t))$ is a simple closed curve symmetric with respect to the $r$-axis, and encloses the fixed homogeneous circle. By the choice of $U$ we have $\frac{1}{3} < dist(r(t), C) < 1$.

Let $r(t) = cr_\epsilon(\frac{1}{\epsilon} t)$, $z(t) = \epsilon z_\epsilon(\frac{1}{\epsilon} t)$, $\varphi(t) = \varphi_\epsilon(\frac{1}{\epsilon} t)$, and $r(t) = (r(t) \cos \varphi(t), r(t) \sin \varphi(t), z(t))$, i.e. $r(t) = cr_\epsilon(\frac{1}{\epsilon} t)$. By Corollary \([12]\) we have that $r(t)$ is a solution of our original problem $\ddot{r} = -\nabla V(r, 1) = -\nabla V(r)$. Also, by the properties of $r_\epsilon$ we have that $r(t)$ satisfies: (1) $\frac{\epsilon}{\pi} < dist(r(t), C) < \epsilon$, (2) writing $r_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t), z_\epsilon(t))$ we see that $r(t) = (\epsilon x_\epsilon(\frac{1}{\epsilon} t), \epsilon y_\epsilon(\frac{1}{\epsilon} t), \epsilon z_\epsilon(\frac{1}{\epsilon} t))$ has angular momentum $\epsilon (\dot{y_\epsilon} x_\epsilon - \dot{x_\epsilon} y_\epsilon) = \epsilon \lambda_\epsilon = K$. □

6 Some Generalizations.

In this section we indicate how the existence of some periodic orbits remains true if we add some other body $B$ to the fixed homogeneous circle $C$. For a Lebesgue measurable $B \subset \mathbb{R}^3$ and $\rho \in \mathbb{R}$, $B_\rho$ denotes the set $\{ pb; b \in B \}$. Let $\lambda$ be a positive finite measure on $B$. The (total) mass of $B$ is $M_B = \int_B \lambda = \lambda(B) > 0$. For $\rho > 0$, $\lambda_\rho$ denotes the measure on $B_\rho$ induced by $\lambda$, i.e. $\lambda_\rho(A_\rho) = \lambda(A)$, for all Lebesgue measurable $A \subset B$. For $r \notin B$, define $V(r, \rho, M) = M_B \int_{B_\rho} \frac{\lambda_\rho(u)}{\|r-u\|} du$. It is straightforward
to check that $V(r, \rho, M)$ satisfies the statements of Lemma 1.1 and Corollary 1.2. By a change of variable we obtain

$$V(r, \rho, M) = \frac{M}{M_B} \int_B \frac{\lambda(u)}{\|r - \rho u\|} du.$$  

The results in section 2.2 can be generalized as follows. Let $C_\rho$ be the fixed homogeneous circle on the $xy$-plane, centered at the origin, with radius $\rho$. For $\rho = 1$, write $C_1 = C$ and we consider $C$ with fixed mass $M_0$. Let $B$ be such that:

1. $\text{dist}(C, B) = d > 0$,
2. $(B, \lambda)$ is symmetric with respect to the $xz$-plane and the $xy$-plane,
3. $0 < M_B < \infty$, where $M_B$ is the mass of $B$.

Let $V_C(r, \rho, M)$ be the gravitational potential induced by $C_\rho$ with total mass $M$ and $V_B(r, \rho, M) = \frac{M}{M_B} \int_B \frac{\lambda(u)}{\|r - u\|} du = \frac{M}{M_B} \int_B \frac{\lambda(u)}{\|r - \rho u\|} du$, the potential induced by $B_\rho$ with mass $M$. Note that the $xz$-plane is an invariant subspace of both $V_C(r, \rho, M)$ and $V_B(r, \rho, M)$. Let $V(r) = V_C(r, 1, M_0) + V_B(r, 1, M_B)$ be the gravitational potential induced by $C \cup B$. Let also $V(r, \epsilon) = V_C(r, \frac{1}{\epsilon}, M_0) + V_B(r, \frac{1}{\epsilon}, M_B)$ and $W(r, \epsilon) = W_C(r, \epsilon) + W_B(r, \epsilon)$, which is obtained from $V(r, \epsilon)$ by translating the origin to $(\frac{1}{\epsilon}, 0, 0)$. We have $\nabla W(r, \epsilon) = \nabla W_C(r, \epsilon) + \nabla W_B(r, \epsilon)$. Note that

$$\nabla W_B(r, \epsilon) = \epsilon \int_B \frac{\epsilon r - u - e_1}{\|\epsilon r - u - e_1\|^3} \lambda du.$$  

Since $\text{dist}(C, B) = d > 0$, and $e_1 = (1, 0, 0) \in C$, we have that $\|u - e_1\| \geq d$, for all $u \in B$. Hence $\lim_{\epsilon \to 0} \nabla W_B(r, \epsilon) = 0$ and it can be shown that this limit is uniform on compacts. Therefore we can apply the methods used in section 2.2 to prove that there are periodic orbits in the $xz$-plane, close to $C$. In the pictures below (from left to right), (1) $B$ is a ball, (2) $B$ is a three dimensional set with rotational symmetry, (3) $B$ can be chosen to be any two of the three homogeneous circles, (4) $B$ is an annulus in the $xy$-plane.

Analogously, it is not difficult to show that the results in section 3 can also be generalized. For example we can consider $C$ as above and (1) $B$ is the union of two inner circles and $C$ is the outer circle, (2) $B$ is annulus.

Proceeding as in section 3 we can prove the existence of infinitely many figure eight periodic solutions for these two cases. Finally, repeating the process used in section 5 we can prove the existence of spiral solutions for bodies $C \cup B$, where $B$ has rotational symmetry around the $z$-axis and satisfies $\text{dist}(C, B) = d > 0$. Here are some examples.
A Interval Pointing Forces.

We identify \( \mathbb{R} \) with \( \mathbb{R} \times \{0\} \subset \{(x_1,x_2), x_2 \geq 0\} \subset \mathbb{R}^2 \). We will consider open sets \( \Omega \subset \mathbb{R}^2 \) satisfying \( \mathbb{R}^2_+ \subset \Omega \). Let \( I \subset \mathbb{R} \subset \mathbb{R}^2 \) be an interval and \( x \in \mathbb{R}^2, v \in \mathbb{R}^2 \). We say that \( v \) points to \( I \) at \( x \) if either \( v = 0 \) or the infinite ray \( R(x,v) \) that begins at \( x \) and has direction \( v \), intersects \( I \), i.e. \( R(x,v) \cap I \neq \emptyset \).

Remarks A.1 (1) Note that if \( x \in \mathbb{R} - I \) and \( v \) points to \( I \) at \( x \) then \( v \) is horizontal.
(2) The following special case will be used later. Suppose \( I = (-\infty, 0] \). If \( v = (v_1,v_2) \neq 0 \) and \( x = (x_1,x_2) \in \mathbb{R}^2_+ \) then \( v \) points to \( I \) at \( x \) if and only if (i) \( \langle v, x^- \rangle = x_1 v_2 - x_2 v_1 \geq 0 \), (ii) \( v_2 < 0 \). Here \( (a,b)^- = (-b,a) \).

Statement (i) says that the (oriented) angle from \( v \) to \(-x\) is non-negative, and statement (ii) says that \( v \) points downward. Let \( \alpha(t) \) be a curve in \( \mathbb{R}^2_+ \). We say that \( \alpha \) points to \( I \) at \( t = t_0 \) if \( \dot{\alpha}(t) \) points to \( I \) at \( \alpha(t) \). We say that \( \alpha \) points to \( I \) if \( \alpha \) points to \( I \) at \( t \), for all \( t \) in the domain of \( \alpha \).

![Figure A.12: \( \nu \) points to \( I \) at \( x \).](image)

![Figure A.13: \( \alpha \) points to \( I \).](image)

![Figure A.14: \( F \) points to \( I \).](image)

Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) satisfying \( \mathbb{R}^2_+ \subset \Omega \). Let \( F : \Omega \to \mathbb{R}^2 \). We say that \( F \) points to \( I \) if \( F(x) \neq 0 \) and \( F(x) \) points to \( I \) at \( x \), for every \( x \in \mathbb{R}^2_+ \).

Remarks A.2 (1) If \( F \) is continuous and points to \( I \) and \( x \in \mathbb{R} \cap \Omega \) then \( F(x) \) is horizontal (but it may happen that \( F(x) = 0 \)). Hence \( \mathbb{R} \cap \Omega \) is an invariant subspace of \( F \).
(2) In all definitions above if an object points to \( I \) and \( I \subset I' \) then it also points to \( I' \).
(3) Let \( C \) be a homogeneous fixed circle in the \( xy \)-plane, centered at the origin with radius \( \rho \). Let \( V \) be the potential function induced by \( C \). It is not difficult to see that \( \nabla V \) (restricted to the \( xz \)-plane) points to \([-\rho, \rho]\).

Proposition A.3 Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) with \( \mathbb{R}^2_+ \subset \Omega \). Let \( I \subset \mathbb{R} \subset \mathbb{R}^2 \) be a closed interval and let \( F : \Omega \to \mathbb{R}^2 \) be a continuous map which points to \( I \). Let also \( r : [0,b) \to \Omega \cap \mathbb{R}^2_+ \) be a solution of \( \dot{r} = F(r) \). Suppose that \( r(t) = (x(t),z(t)) \) points to \( I \) at \( t = 0 \). Then \( r \) points to \( I \). Moreover, if \( r \) extends to a solution \( r : [0,b] \to \Omega \cap \mathbb{R}^2_+ \), with \( r(b) \in \Omega \cap \mathbb{R} \), then \( r(b) \in I \).

Proof. First we prove the Proposition for \( I = (-\infty, 0] \). Since \( \dot{r} = F(r) \) and \( F \) points to \( I \) by Remark
for all \( t \in [0, b) \). Let \( h(t) = \langle \dot{r}(t), \dot{r}^+(t) \rangle = x(t) \dot{z}(t) - z(t) \dot{x}(t) \). Since \( r(t) = (x(t), z(t)) \) points to \( I \) at \( t = 0 \), we have either \( \dot{r}(0) = 0 \) or \( h(0) \geq 0 \), \( \dot{z}(0) < 0 \). In any case \( h(0) \geq 0 \). Therefore, differentiating \( h \) and using (A.5) we obtain that \( h(t) \geq 0 \). Hence \( h(t) \geq 0 \), for all \( t \in [0, b) \). This proves that \( \langle \dot{r}(t), \dot{r}^+(t) \rangle = x(t) \dot{z}(t) - z(t) \dot{x}(t) \geq 0 \), for all \( t \in [0, b) \). Also, since (by A.5) \( \dot{z}(t) < 0 \) and \( \dot{z}(0) < 0 \) we have that \( \dot{z}(t) < 0 \), for all \( t \in (0, b) \). Then, by remark A.1 (2), it follows that \( r \) points to \( I \). This proves the first part of the Proposition (for \( I = (-\infty, 0] \)). Suppose now that \( r \) extends to a solution \( r : [0, b] \to \Omega \cap \mathbb{R}^2_+ \), with \( r(b) \in \Omega \cap \mathbb{R} \). Let us assume that \( r(b) \notin I \). Since \( r \) is continuous \( \dot{r}(b) \) points to \( I \) at \( r(b) \). Then, by remark A.1 (1), \( \dot{r}(b) \) is horizontal. Since \( \Omega \cap \mathbb{R} \) is an invariant subspace (see remark A.2 (1)) it follows that \( r(t) \in \Omega \cap \mathbb{R} \), for all \( t \in [0, b) \). A contradiction since \( r(t) \in \mathbb{R}^2_+ \), for \( t \in [0, b) \). This proves the Proposition for the case \( I = (-\infty, 0] \). Using translations and reflections, we can prove that the Proposition also holds for intervals \( I = (-\infty, a], I = [a, \infty) \). For the case \( I = [a, b] \) apply the Proposition to \( (-\infty, b] \) and \( [a, \infty) \). This proves the Proposition.

**Proposition A.4** Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) with \( \mathbb{R}^2_+ \subset \Omega \). Let \( I \subset \mathbb{R} \subset \mathbb{R}^2 \) be a closed interval and let \( F : \Omega \to \mathbb{R}^2 \) be a continuous map which points to \( I \). Let also \( r : [0, b) \to \Omega \cap \mathbb{R}^2_+ \) with \( r(0) \in \mathbb{R} \setminus I \) and \( r(0, b) \subset \mathbb{R}^2_+ \) such that \( r|_{(0, b)} \) is a solution of \( \ddot{r} = F(r) \). Then \( r \) is one-to-one.

**Proof.** Note that the hypothesis: \( r(0) \in \mathbb{R} \setminus I \) implies that we can assume that \( \mathbb{R} \neq I \). Since \( F \) points to \( I \) we can assume that \( F \) points to a semi-infinite closed interval and without loss of generality we can also assume that \( I = (-\infty, 0] \).

Write \( r = (x, z) \). Since \( F \) is continuous we have that \( r \) is a \( C^1 \) map. Note that \( z(t) > 0 \) for \( t \in (0, b) \). It follows from this and the fact that \( F \) points to \( I \) that (see remark A.1 (2)) \( \ddot{r}(t) = (x(t), z(t)) \) satisfy (A.5) for all \( t \in (0, b) \).

Now, if \( \ddot{z}(t) \neq 0 \) for all \( t \in [0, b) \) then \( z \) is an increasing function, thus one-to-one, and we have nothing to prove. We suppose then that there is a \( t_0 \in (0, b) \) such that \( \ddot{z}(t_0) = 0 \). Since \( \ddot{z}(t) < 0 \), \( z(t_0) \) is a maximum value of \( z \) and \( t_0 \) is unique, that is, it is the only \( t \in [0, b) \) where \( \ddot{z} \) vanishes. We also have that \( \ddot{z}(t) > 0 \), for \( t \in [0, t_0) \) and \( \ddot{z}(t) < 0 \), for \( t \in [t_0, b) \).

**Claim 1.** \( \dot{z}(t_0) < 0 \).

**Proof of Claim 1.** Define \( s : [0, t_0] \to \mathbb{R}^2_+ \) by \( s(t) = r(t_0 - t) \). Write \( s = (\dot{x}, \dot{z}) \). If \( \dot{z}(t_0) = 0 \) then \( s(0) = 0 \). Hence \( s \) points to \( I \), at \( t = 0 \). By Proposition A.3 \( r(0) = s(t_0) \in I \), a contradiction. Therefore \( \dot{z}(t_0) \neq 0 \). Suppose that \( \dot{z}(t_0) > 0 \) (see figure A.15). Then \( \dot{z}(0) < 0 \). Hence, for \( t \) close to 0, \( \dot{z}(t) > 0 \) for \( t < t_0 \), we have that \( \dot{z}(t) < 0 \) for \( t > 0 \). Therefore, since \( s \) is \( C^1 \), \( s \) points to \( I = (-\infty, 0] \) for \( t \) close to 0. By the Proposition above \( s(t_0) \in I \), which is a contradiction since \( s(t_0) = r(0) \notin I \). This proves the claim.

By the claim above, for \( t \) close to \( t_0 \), \( \dot{z}(t) < 0 \). Since we have that \( \dot{z}(t) < 0 \) for \( t > t_0 \) and \( r \) is \( C^1 \), it follows that \( r \) points to \( I = (-\infty, 0] \) for \( t \) close to \( t_0 \) and \( t > t_0 \). Then, by the Proposition above, \( r \) points to \( I \) at \( t \in (t_0, b) \).

![Figure A.15](image-url)
Let $s(t) = r(t_0 - t)$ as in proof of the claim 1. Suppose $r$ is not one-to-one. Let $\tilde{t} = \min \{ t \in (t_0, b) : r(t) = r(t') \text{ for some } t' \in [0, t_0] \}$. Note that $\tilde{t}$ exists and $\tilde{t} > t_0$, since $\dot{x}(t_0) < 0$. Let $t'$ be such that $r(\tilde{t}) = r(t')$. Note that $t'$ is uniquely defined and $t' \in (0, t_0)$ because $z$ is increasing on $[0, t_0]$. Write $d = \tilde{t} - t_0$ and $c = t_0 - t'$ and define $\alpha = s|_{[0, c]}$ and $\beta : [0, d] \to \Omega \cap \mathbb{R}^2_+$, $\beta(t) = r(t - t_0)$.

Write $\alpha = (u_1, u_2)$ and $\beta = (v_1, v_2)$. Then $\alpha$ and $\beta$ have the following properties: (property (d) follows from the minimality of $\tilde{t}$ and claim 1)

(a) $\alpha$ does not point to $I$ at $t$, for all $t \in [0, c]$ and $\beta(t)$ points to $I$ at $t \in (0, d)$,
(b) $\alpha(0) = \beta(0)$ and $\alpha(c) = \beta(d)$,
(c) $u_2(t) < 0$ for $t \in (0, c]$ and $v_2(t) < 0$ for $t \in (0, d]$,
(d) If $u_2(t) = v_2(t^*)$ then $v_1(t) < u_1(t^*)$, for $t \neq 0, c$.

Claim 2. The (oriented) angle from $\dot{\alpha}(c)$ to $\dot{\beta}(d)$ is non-negative and less than $\pi$.

Proof of Claim 2. First, since both vectors point downward, we have that this angle is less that $\pi$. Write $z_0 = u_2(0) = v_2(0) = z(t_0)$, and $z_1 = u_2(c) = v_2(d) = z(t') = z(\tilde{t})$. By property (c) above we can write $u_1$ and $v_1$ in terms of $z \in [z_1, z_0]$. That is, there is $t = t(z)$, $z \in [z_1, z_0]$ such that $u_2(t(z)) = z$. Then we write $u_1(z) = u_1(t(z))$. Similarly we can write $v_1(z)$. Note that, by properties (b) and (d), $u_1(z_1) - v_1(z_1) = 0$, and $u_1(z) - v_1(z) \geq 0$, for $z \in (z_1, z_0)$. Hence, $\frac{d}{dz}(u_1(z) - v_1(z)) \mid_{z_1} \geq 0$. But $\frac{d}{dz}(u_1(z)) \mid_{z_1} = \frac{u_1(c)}{u_2(c)}$ and $\frac{d}{dz}(v_1(z)) \mid_{z_1} = \frac{v_1(d)}{v_2(d)}$. Therefore $\frac{u_1(c)}{u_2(c)} \geq \frac{v_1(d)}{v_2(d)}$ and it follows that $\langle \dot{\beta}(d), \dot{\alpha}(c) \rangle \geq 0$. This proves claim 2.

By the claim above and property (a), $\alpha$ does not point to $I$ at $c$. It follows that $\beta$ does not point to $I$ at $d$. This contradicts property (a) and proves the Proposition. ■

Remark. Let us assume that $I = [a, b]$ is bounded. If we shoot a particle $P$ upward from the $x$-axis, by the Proposition above (the connected piece in $\mathbb{R}^2_+$ of) the solution behaves, with respect to self-intersections and landing, in the following way. If we shoot from outside $I$, say, to the right of $I$, we have no self-intersections.

![Figure A.16](image_url) This is what happens if $r$ is not one-to-one.

Figure A.17: The figure to the left can happen. The two figures to the right cannot happen.

Also, the particle can land on $I$, or to the left of $I$, but not to the right of $I$. (To see that the particle $P$ cannot land to the right of $I$ see the proof of claim 1 in the proof of the Proposition above, and use also the fact that the orbit has no self-intersections.)

![Figure A.18](image_url) The figure in the middle cannot happen. The figure to the right cannot happen.

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If we shoot from inside \( I \) then we may have self-intersections but then the particle should land inside \( I \), otherwise, inverting time, we obtain one of the previous cases. Also, in this case, if the orbit of \( P \) has no self-intersections then it may land anywhere. Of course, the particle may not land on the \( x \)-axis at all, and escape to infinity. The same analysis can be made for closed semi-infinite intervals \( I \).

## B Proof of Proposition 3.1

To prove Proposition 3.1 we will use the following Lemmas and remarks. Write \( \Lambda_A = \frac{2}{\min\{1, A\}} + 1 \), for \( A > 0 \). Note that \( \Lambda_A > 1 \). In the following two Lemmas we assume \( z(t) \) to be twice differentiable.

**Lemma B.1** \( \text{Let } z(t) \text{ be defined in } [0, a] \text{ and such that (i) } \dot{z} \leq -Az, \ A > 0, \ (ii) \ z(0) = z_0 > 0, \ (iii) \ z(0) = 0. \) If \( a > \Lambda_A \) then there is \( t_0 \in [0, \Lambda_A] \) such that \( z(t_0) = 0 \).

**Proof.** Suppose that \( z(t) \neq 0 \) for all \( t \in [0, \Lambda_A] \). Then \( z(t) > 0 \) and \( \dot{z}(t) \) is decreasing in \([0, \Lambda_A]\). Since \( \dot{z}(0) = 0 \), we have \( \dot{z}(t) < 0, \ t \in (0, \Lambda_A] \). Then \( z \) is decreasing in \([0, \Lambda_A]\). Note also that \( z \) is defined in \( 1 \) because \( 1 < \Lambda_A < a \). We claim that \( z(1) \leq (\min\{1, A\})z_0^{\frac{\alpha}{2}} \). We have two cases: \( z(1) \geq \frac{z_0}{2} \) or \( z(1) \leq \frac{z_0}{2} \). Suppose first that \( z(1) \geq \frac{z_0}{2} \). This implies that \( z(t) \geq \frac{z_0}{2} \) for all \( t \in [0, 1] \). Then

\[
\dot{z}(1) = \int_0^1 \dot{z}(t) dt \leq \int_0^1 -Az(t) dt \leq -A \int_0^1 z(t) dt \leq -Az_0^\alpha.
\]

Suppose now that \( z(1) \leq \frac{z_0}{2} \). By the intermediate value theorem there exists \( t' \in (0, 1) \) such that \( \dot{z}(1) < \dot{z}(t') = \frac{(z(1) - z(0))}{t} \leq -\frac{z_0}{t} \). This proves our claim.

Since \( \dot{z} \) is decreasing in \([0, \Lambda_A]\), for \( 1 \leq t \leq \Lambda_A \) we have that \( \dot{z}(t) \leq \dot{z}(1) \leq -\alpha \), with \( \alpha = (\min\{A, 1\})\frac{z_0}{2} \). Hence, \( z(t) - z(1) = \int_1^t \dot{z}(s) ds \leq -\alpha(t - 1) \). It follows that \( z(t) \leq z(1) - \alpha(t - 1) \leq z_0 - \alpha(t - 1) \), for \( 1 \leq t \leq \Lambda_A \). Therefore \( z(\Lambda_A) = z\left(\frac{z_0}{\alpha} + 1\right) \leq z_0 - \alpha(\Lambda_A - 1) = 0 \), which is a contradiction. It follows that there exists \( t_0 \in [0, \Lambda_A] \) such that \( z(t_0) = 0 \). \( \blacksquare \)

**Lemma B.2** \( \text{Let } z(t) \text{ be defined in } [0, +\infty) \text{ and such that (i) } \dot{z} \leq -Az, \ A > 0, \ (ii) \ z(0) = 0, \ (iii) \ z(0) > 0. \) Then there exists \( t_0 \in [0, a] \) such that \( z(t_0) = 0 \).

**Proof.** Suppose that \( \dot{z}(t) \neq 0 \), for all \( t \). Then \( \dot{z}(t) > 0 \) for all \( t \). Hence \( z \) is an increasing function. Then for all \( t \geq 1 \), \( 0 < z(t) \leq z(t) \) and \( \dot{z}(t) = \dot{z}(t) + z(t) ds \leq \dot{z}(t) - A \int_1^t z(s) ds \leq \dot{z}(1) - Az(1)(t - 1) \).

Evaluating at \( t = 1 + \frac{\dot{z}(1)}{Az(1)} \) we have \( \dot{z}(1 + \frac{\dot{z}(1)}{Az(1)}) \leq 0 \), which is a contradiction. \( \blacksquare \)

Before proving Proposition 3.1 we have the following remark.

**Remarks B.3** Let \( \delta < 0 \). By Lemma 2.3 of \([2]\), there exists \( R_\delta, 0 < R_\delta < +\infty, \) such that \( \{ r; V(r) \leq \delta \} \subset B(R_\delta) \) where \( B(R_\delta) \) is the ball centered at the origin of radius \( R_\delta \). Therefore, if \( r(t) \) is a solution of \( \ddot{r} = -\nabla V(r) \) (restricted to the \( xz \)-plane) with \( E(r(t)) \leq \delta < 0 \), then \( \|r(t)\| \leq R_\delta \). Moreover, if \( C \) is the fixed homogeneous circle of radius one centered at the origin and \( u \in C \), we have: \( \|r - u\| \leq \|r\| + \|u\| \leq R_\delta + 1 \). Hence, \( \frac{1}{\|r - u\|^2} \geq \frac{1}{(R_\delta + 1)^2} \), and follows that \( \lambda \frac{du}{\|r - u\|^2} \geq \lambda \frac{du}{(R_\delta + 1)^2} = \frac{M}{(R_\delta + 1)^2} \).

**Proof of Proposition 3.1** From \([8, 2]\) we have \( \ddot{z} = -\frac{\partial V}{\partial z}(x, z) = -\lambda z \int_C \frac{du}{\|r - u\|^2} \). Note that \( \ddot{z} < 0 \), if \( z > 0 \) and \( \ddot{z} > 0 \), if \( z < 0 \). Let \( r(t) = (x(t), z(t)) \) be a solution of \( \ddot{r} = -\nabla V(r) \) (restricted to the \( xz \)-plane), with \( E(r(t)) \leq \delta < 0, \ z(0) = 0 \) and \( \dot{z}(0) > 0 \). Then \( |\dot{z}| = \lambda |z| \int_C \frac{du}{\|r - u\|^2} \geq \frac{M}{(R_\delta + 1)^2} |z| \) (see
Remark B.3). Set \( A = \frac{M}{(R_\delta+1)^2} \) and \( \Lambda_A = \frac{2}{\min\{1, \Lambda\}} + 1 \). We have \( |\ddot{z}| \geq A|\dot{z}| \). If \( z > 0 \), then \( \ddot{z} < 0 \), and the inequality becomes \( -\ddot{z} \geq Az \), for \( z \geq 0 \). Let \((a, b), a < 0 < b\), be the maximal interval on which \( r(t) \) is defined. If \( b < \infty \), \( r(t) \) collides (see [2], Theorem A), hence \( \lim_{t \to b^-} z(t) = 0 \), and we have nothing to prove. Suppose then that \( b = \infty \). By Lemma B.2 there exists \( \bar{t} \in (0, \infty) \), such that \( \dot{z}(\bar{t}) = 0 \). Replacing \( \bar{t} \) by \( \min\{t > 0 ; \dot{z}(t) = 0\} \), we can assume that \( \dot{z}(t) > 0, t \in [0, \bar{t}] \). Then \( z(t) \) is increasing in \([0, \bar{t}]\) and \( z(t) > 0, t \in (0, \bar{t}) \). Applying Lemma B.1 to the function \( t \mapsto z(\bar{t} - t), t \in [0, \bar{t}] \), we obtain that \( \bar{t} \leq \Lambda_A \). We will show that there exists \( t_0 \in [0, 2\Lambda_A] \), such that \( z(t_0) = 0 \). Suppose that \( z(t) \neq 0, t \in (0, 2\Lambda_A] \). Since \( \bar{t} \in (0, 2\Lambda_A) \) and \( z(\bar{t}) > 0 \) we have \( z(t) > 0, t \in (0, 2\Lambda_A] \). Hence there exists \( c \) such that \( z(t) > 0, t \in (0, c) \), with \( 2\Lambda_A < c < \infty \). Note that \( c - \bar{t} > \Lambda_A \). Applying Lemma B.1 to the function \( t \mapsto z(t + \bar{t}), t \in [0, c - \bar{t}] \) we obtain a contradiction. This proves the Proposition.

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