Uniform approximation of fractional derivatives and integrals with application to fractional differential equations

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\textbf{Abstract.} It is well known that for every $f \in C^m$ there exists a polynomial $p_n$ such that $p_n^{(k)} \rightarrow f^{(k)}$, $k = 0, \ldots, m$. Here we prove such a result for fractional (non-integer) derivatives. Moreover, a numerical method is proposed for fractional differential equations. The convergence rate and stability of the proposed method are obtained. Illustrative examples are discussed.

\section{Introduction}

The analysis and design of many physical systems require the solution of a fractional differential equation \cite{1, 16}. Examples include fractional oscillation equations \cite{18, 21}, linear and nonlinear fractional Bagley–Torvik equations \cite{10, 17}, Basset equations \cite{9}, fractional Lorenz systems \cite{3}, and population models with fractional order derivatives \cite{24, 29}.

Several methods have recently been proposed to address both linear and nonlinear \textit{Fractional Ordinary Differential Equations} (FODEs): an analytical method to solve linear FODEs is given in \cite{15}, homotopy methods for the analysis of linear and nonlinear FODEs are used in \cite{4, 14}, Adams multistep methods for nonlinear FODEs are investigated in \cite{13}, and a differential transform method is developed in \cite{5}. Another widely used procedure consists in transforming a differential equation with fractional derivatives into a system of differential equations of integer order \cite{22}. Other numerical schemes to solve FODEs can be found in \cite{12, 26}.

Here we propose the use of Bernstein polynomials to approximate fractional derivatives and integrals. Numerical algorithms based on Grünwald and modified Grünwald approximation formulas...
were proposed and analyzed in [7, 11] for the approximate evaluation of certain Hadamard integrals, where Bernstein polynomials are used to derive an error bound [7, 11]. Recently, Bernstein polynomials have been used to approximate the solution of fractional integro-differential equations [19], fractional heat- and wave-like equations [25], and multi-dimensional fractional optimal control problems [2]. For more on approximation of fractional derivatives and its applications, we refer the reader to [23] and references therein.

Despite numerous results available in the literature, there is still a significant demand for readily usable numerically algorithms to handle mathematical problems involving fractional derivatives and integrals. To date, such algorithms have been developed but only to a rather limited extent. Here we specify, develop and analyze, a general numerical algorithm and present such scheme in a rigorous but accessible way, understandable to an applied scientist.

The paper is organized as follows. Section 2 reviews, briefly, the necessary definitions and results concerning Bernstein polynomials. We also recall the standard definitions of fractional integral and fractional derivative in the sense of Riemann–Liouville and Caputo. In Section 3 we introduce the basic idea of our method. Uniform approximation formulas for fractional derivatives and integrals are proposed. Some useful results, and the convergence of the approximation formulas, are proved. In Section 4 we analyze the results obtained in Section 3 computationally. Finally, in Section 5 we apply our approximation formula for fractional derivatives to the study of a FODE. We begin by proving the convergence and stability of the proposed numerical method. Then, some concrete examples are given. The examples considered show that our method is effective for solving both linear and nonlinear FODEs in a computationally efficient way.

2 Preliminaries

The Weierstrass approximation theorem is a central result of mathematics. It states that every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated by a polynomial function.

Theorem 1 (The Weierstrass approximation theorem). Let $f \in C[a, b]$. For any $\varepsilon > 0$, there exists a polynomial $p_n$ such that $|f(x) - p_n(x)| \leq \varepsilon$ for all $x \in [a, b]$.

There are many proofs to the Weierstrass theorem. One of the most elegant proofs uses Bernstein polynomials.

Definition 1. Let $f$ be continuous on $[0, 1]$. The Bernstein polynomial of degree $n$ with respect to $f$ is defined as
\[
B_n(f; x) = \sum_{i=0}^{n} \binom{n}{i} f\left(\frac{i}{n}\right) x^i (1-x)^{n-i}.
\]

The next theorem is due to Bernstein, and provides a proof to the Weierstrass theorem. Bernstein’s Theorem not only proves the existence of polynomials of uniform approximation, but also provides a simple explicit representation for them.

Theorem 2 (Bernstein’s theorem [6, 20]). Let $f$ be bounded on $[0, 1]$. Then, $\lim_{n \to \infty} B_n(f; x) = f(x)$ at any point $x \in [0, 1]$ at which $f$ is continuous. Moreover, if $f \in C[0, 1]$, then the limit holds uniformly in $[0, 1]$. 
Corollary 1. If \( f \in C[0, 1] \) and \( \varepsilon > 0 \), then one has, for all sufficiently large \( n \), that \( |f(x) - B_n(f;x)| \leq \varepsilon \) for all \( 0 \leq x \leq 1 \).

In case of functions that are twice differentiable, \( \alpha \), an asymptotic error term for the Bernstein polynomials is easily obtained.

Theorem 3 (Voronovskaya’s theorem \([6,20]\)). Let \( f \) be bounded on \([0, 1]\). For any \( x \in [0, 1] \) at which \( f''(x) \exists \), limit \( \lim_{n \to \infty} 2n(B_{n}(f;x) - f(x)) = x(1 - x)f''(x) \).

In contrast with other methods of approximation, the Bernstein polynomials yield smooth approximations. If the approximated function is differentiable, not only do we have \( B_{n}(f;x) \to f(x) \) but also \( B'_{n}(f;x) \to f'(x) \). A corresponding statement is true for higher derivatives. Therefore, Bernstein polynomials provide simultaneous approximation of the function and its derivatives.

Theorem 4 (\([6,20]\)). If \( f \in C^{p}[0, 1] \), then \( \lim_{n \to \infty} B^{(p)}_{n}(f;x) = f^{(p)}(x) \) uniformly on \([0, 1]\).

Remark 1. All results formulated above for \([0, 1]\) are easily transformed to \([a, b]\) by means of the linear transformation \( y = \frac{x-a}{b-a} \) that converts \([a, b]\) into \([0, 1]\).

In Section 3 we are going to develop the result presented in Theorem 4 for fractional derivatives. For an analytic function \( f \) over the interval \([a, b]\), the left and right sided fractional integrals in the Riemann–Liouville sense, \( aI_{a}^{\alpha}f(x) \) and \( bI_{b}^{\alpha}f(x) \), respectively, are defined by

\[
aI_{a}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(\tau)}{(x-\tau)^{1-\alpha}} d\tau, \quad x \in [a, b],
\]

and

\[
bI_{b}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{f(\tau)}{(\tau-x)^{1-\alpha}} d\tau, \quad x \in [a, b],
\]

where \( \alpha > 0 \) is a real number. For \( m - 1 \leq \alpha < m \), the left and right sided Riemann–Liouville fractional derivatives are defined by

\[
aD_{a}^{\alpha}f(x) := \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dx^{m}} \int_{a}^{x} \frac{f(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, \quad x \in [a, b],
\]

and

\[
bD_{b}^{\alpha}f(x) := \frac{(-1)^{m}}{\Gamma(m-\alpha)} \frac{d^{m}}{dx^{m}} \int_{x}^{b} \frac{f(\tau)}{(\tau-x)^{\alpha+1-m}} d\tau, \quad x \in [a, b],
\]

respectively. The left and right sided Caputo fractional derivatives are defined by

\[
cD_{a}^{\alpha}f(x) := \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, \quad x \in [a, b],
\]

and

\[
cD_{b}^{\alpha}f(x) := \frac{(-1)^{m}}{\Gamma(m-\alpha)} \int_{x}^{b} \frac{f^{(m)}(\tau)}{(\tau-x)^{\alpha+1-m}} d\tau, \quad x \in [a, b],
\]

respectively, where \( m \) is the integer such that \( m - 1 \leq \alpha < m \). The next theorem gives a relation between Caputo and Riemann–Liouville derivatives.
Theorem 5 ([21]). If \( f \in C^m[0,1] \) and \( \alpha \in [m-1,m) \), then
\[
\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \int_0^x f^{(k)}(t) (x-t)^{k-\alpha} \, dt, \quad k = 0, 1, \ldots, m-1.
\]

\[
C_0^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt.
\]

For more on fractional calculus see, e.g., [16, 21].

3 Fractional derivatives and integrals of the Bernstein polynomials

We begin by computing the left and right sided fractional derivatives and integrals of the Bernstein polynomials with respect to a function \( f \). Then, we prove that Theorem 4 can be formulated for non-integer derivatives and integrals.

3.1 Left and right sided Caputo fractional derivatives of \( B_n(f;x) \)

The Bernstein polynomials (2.1) can be written in the form
\[
B_n(f;x) = \sum_{i=0}^{n} \binom{n}{i} (n-i)^n (-1)^i f \left( \frac{i}{n} \right) x^i (1-x)^{n-i}.
\]

The left sided Riemann–Liouville fractional derivative of \( B_n(f;x) \) on \([0,1]\) is
\[
_0^c D_x^\alpha B_n(f;x) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \sum_{i=0}^{n} \binom{n}{i} (n-i)^n (-1)^i f \left( \frac{i}{n} \right) x^i (1-x)^{n-i}.
\]

Another representation of the Bernstein polynomials with respect to function \( f \) is obtained when we use the binomial expansion of \( x^i = (1-(1-x))^i \):
\[
B_n(f;x) = \sum_{i=0}^{n} \binom{n}{i} (n-i)^n (-1)^i f \left( \frac{i}{n} \right) (1-x)^{n-i}.
\]

Using (3.2), we get the right sided Riemann–Liouville fractional derivative of \( B_n(f;x) \) over the interval \([0,1]\) as
\[
_1^c D_x^\alpha B_n(f;x) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \sum_{i=0}^{n} \binom{n}{i} (n-i)^n (-1)^i f \left( \frac{i}{n} \right) (1-x)^{n-i}.
\]

The left and right sided Caputo fractional derivatives of the Bernstein polynomials with respect to a function \( f \) on \([0,1]\) are easily obtained from Theorem 5.

The next theorem shows that \( _0^c D_x^\alpha B_n(f;x) \to \frac{1}{\Gamma(1-\alpha)} \int_0^x f(t) \, dt \) as \( n \to \infty \) uniformly on \([0,1]\). The same result holds for the right derivative: \( _1^c D_x^\alpha B_n(f;x) \to \frac{1}{\Gamma(1-\alpha)} \int_0^x f(t) \, dt \) as \( n \to \infty \) uniformly on \([0,1]\). Along the text, we use \( \| \cdot \| \) to denote the uniform norm over the interval \([0,1]\), that is, \( \|g\| = \max_{0 \leq x \leq 1} |g(x)| \) for \( g \in C[0,1] \).
**Theorem 6.** Let $\alpha$ be a nonnegative real number and $m \in \mathbb{N}$ be such that $m - 1 \leq \alpha < m$. If $f \in C^m[0, 1]$ and $\varepsilon > 0$, then $\|C_0^\alpha f - C_0^\alpha B_n(f)\| < \varepsilon$.

**Proof.** We wish to show that, given $\varepsilon > 0$ and $f \in C^m[0, 1]$, there exists an integer $n > m$ such that $\|C_0^\alpha f - C_0^\alpha B_n(f)\| < \varepsilon$, where $B_n(f)$ is the Bernstein polynomial of degree $n$ with respect to function $f$. Using Theorem 4, we have $\|f^{(m)} - B_n^{(m)}(f)\| < \varepsilon$. Thus,

$$\begin{align*}
\|C_0^\alpha f(x) - C_0^\alpha B_n(f;x)\| &= \frac{1}{\Gamma(m - \alpha)} \left( \int_0^x (x-t)^{m-\alpha-1} \frac{d^m}{dt^m} (f(t) - B_n(f;t)) \, dt \right) \\
&\leq \frac{1}{\Gamma(m - \alpha)} \left( \int_0^x (x-t)^{m-\alpha-1} \|f^{(m)}(t) - B_n^{(m)}(f)(t)\| \, dt \right) \\
&\leq \frac{1}{\Gamma(m - \alpha)} \left( \int_0^x (x-t)^{m-\alpha-1} \|f^{(m)} - B_n^{(m)}(f)\| \, dt \right) \\
&= \varepsilon \frac{x^{m-\alpha}}{\Gamma(m - \alpha + 1)} < \varepsilon.
\end{align*}$$

**Remark 2.** Theorem 6 is a generalization of Theorem 4 if $\alpha \in \mathbb{N}$, then Theorem 6 reduces to Theorem 4.

The next theorem gives an asymptotic error term for the Caputo fractional derivatives of the Bernstein polynomials with respect to functions $f \in AC^{m+2}[0, 1]$, where $AC^{m+2}[0, 1]$ denotes the space of real functions $f$ that have continuous derivatives up to order $m + 1$ with $f^{(m+1)}$ absolutely continuous on $[0, 1]$. The theorem shows that if $\alpha \in [m - 1, m)$, then $C_0^\alpha B_n(f;x) = C_0^\alpha B_n(f;x)$.

**Theorem 7.** Let $m - 1 \leq \alpha < m$ and $f \in AC^{m+2}[0, 1]$. Then, for any $x \in [0, 1]$,

$$\lim_{n \to \infty} \left[ 2n(C_0^\alpha f - C_0^\alpha B_n(f)) \right] = C_0^\alpha f(x)(1-x)f''(x). \quad (3.4)$$

**Proof.** The result follows from Theorem 5 the properties of the left sided Caputo fractional derivative, and the fact that under assumptions on $f$ the right-hand side of (3.4) exists almost everywhere on $[0, 1]$ (see, e.g., [15]).

Similar results to those of Theorem 8 and Theorem 7 hold for the right sided Caputo fractional derivatives:

**Theorem 8.** If $m - 1 \leq \alpha < m$, $f \in C^m[0, 1]$, and $\varepsilon > 0$, then $\|C_0^\alpha f - C_0^\alpha B_n(f)\| < \varepsilon$.

**Theorem 9.** Let $m - 1 \leq \alpha < m$ and $f \in AC^{m+2}[0, 1]$. Then, for any $x \in [0, 1]$,

$$\lim_{n \to \infty} \left[ 2n(C_0^\alpha f - C_0^\alpha B_n(f)) \right] = C_0^\alpha f(x)(1-x)f''(x).$$
3.2 Left and right sided Riemann–Liouville fractional integrals of $B_n(f; x)$

In order to derive the left and right sided Riemann–Liouville fractional integrals of the Bernstein polynomials, we replace $\alpha$ by $-\alpha$ in (3.1) and (3.3):

$$0 I_1 B_n(f; x) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} f \left( \frac{i}{n} \right) \frac{(-1)^j \Gamma(i+j+1)}{\Gamma(i+j+1+\alpha)} x^{i+j+\alpha}$$

and

$$x I_1 B_n(f; x) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} f \left( \frac{i}{n} \right) \frac{(-1)^j \Gamma(n-i+j+1)}{\Gamma(n-i+j+1+\alpha)} (1-x)^{n-i+j+\alpha}.$$ 

**Theorem 10.** If $m - 1 \leq \alpha < m$, $f \in C[0, 1]$, and $\varepsilon > 0$, then

$$\|0 I_1^\alpha f - 0 I_1^\alpha B_n(f)\| < \varepsilon$$

uniformly on the interval $[0, 1]$.

**Proof.** Using Theorem 2, we have $\|f - B_n(f)\| < \varepsilon$. Therefore,

$$\left|0 I_1^\alpha f(x) - 0 I_1^\alpha B_n(f; x)\right| = \left| \frac{1}{\Gamma(\alpha)} \left( \int_0^x (x-t)^{\alpha-1} (f(t) - B_n(f; t)) dt \right) \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left( \int_0^x (x-t)^{\alpha-1} |f(t) - B_n(f; t)| dt \right)$$

$$\leq \frac{1}{\Gamma(\alpha)} \left( \int_0^x (x-t)^{\alpha-1} \|f - B_n(f)\| dt \right)$$

$$= \varepsilon \frac{x^\alpha}{\Gamma(\alpha+1)} < \varepsilon.$$

**Theorem 11.** If $m - 1 \leq \alpha < m$, $f \in C[0, 1]$, and $\varepsilon > 0$, then

$$\|x I_1^\alpha f - x I_1^\alpha B_n(f)\| < \varepsilon$$

uniformly on the interval $[0, 1]$.

**Proof.** The proof is similar to the proof of Theorem 10.

**Theorem 12.** Let $m - 1 \leq \alpha < m$ and $f \in AC^{m+2}[0, 1]$. Then, for any $x \in [0, 1]$,

$$\lim_{n \to \infty} [2n(0 I_1^\alpha B_n(f; x) - 0 I_1^\alpha f(x))] = 0 I_1^\alpha (x(1-x)f''(x)).$$

(3.5)

**Proof.** The expression on the right-hand side of (3.5) exists almost everywhere on the interval $[0, 1]$ by 16. Equality (3.5) follows from Theorem 3.

Analogous result holds for the right sided Riemann–Liouville fractional integral:

**Theorem 13.** Let $m - 1 \leq \alpha < m$ and $f \in AC^{m+2}[0, 1]$. Then, for any $x \in [0, 1]$,

$$\lim_{n \to \infty} [2n(x I_1^\alpha B_n(f; x) - x I_1^\alpha f(x))] = x I_1^\alpha (x(1-x)f''(x)).$$
4 Numerical experiments

We present two examples in order to illustrate the numerical usefulness of Theorems 6–13. To explore the dependence of the error with the parameter $n$, we use the following definitions:

$$E(n, \alpha) = \max_{1 \leq j \leq N} \left| \sum_{k=0}^{\infty} x^k \frac{x^k}{k^{2-\alpha}} \right|, \quad \alpha \in (0, 1],$$

$$E(n, -\alpha) = \max_{1 \leq j \leq N} \left| \sum_{k=0}^{\infty} x^k \frac{x^k}{k^{2+\alpha}} \right|, \quad \alpha \geq 0.$$

In our simulations we choose $N = 100$. The experimentally order of convergence (EOC) is considered as in [8]: for $0 D_n^\alpha y(x)$ $EOC = \log_2 \frac{E(2n, \alpha)}{E(n, \alpha)}$, while for $0 D_n^\alpha y(x)$ $EOC = \log_2 \frac{E(2n, -\alpha)}{E(n, -\alpha)}$. The following notations are used throughout: $y^{(\alpha)}(x) := 0 D_n^\alpha y(x), y^{(-\alpha)}(x) := 0 D_n^\alpha y(x), B_n^{(\alpha)}(y, x) := 0 D_n^\alpha B_n(y, x), B_n^{(\alpha)}(y, x) := \sum_{k=0}^{\infty} x^k \frac{x^k}{k^{2-\alpha}}; \quad \alpha \in (0, 1],$

The comparison of $y^{(\alpha)}(x)$ with $B_n^{(\alpha)}(y, x)$ and $y^{(-\alpha)}(x)$ with $B_n^{(\alpha)}(y, x)$, for $n = 5, 10, 15, 20$ and $\alpha = \frac{5}{4}$, is shown in Fig. 1 and Fig. 2 respectively. In Tables 1 and 2 the values of $EC(n)$ and $EI(n)$, for $n = 40, 60, 80, 100$ and $\alpha = \frac{5}{4}$, are reported for $x \in [0, 1]$. The experimentally order of convergence for $\alpha = \frac{1}{4}, \frac{3}{4}, \frac{1}{2}$ and $\frac{3}{4}$ is illustrated in Tables 3 and 4 with $n = 20, 40, 80, 160, 320$.  

![Fig. 1](image_url) Example 1 $y^{(\alpha)}(x)$ versus $B_n^{(\alpha)}(y, x)$ (left) and plot of $EC(n)$ (right) with $\alpha = \frac{5}{4}$ and $n = 5, 10, 15, 20$. 

![Fig. 2](image_url) 

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Fig. 2  Example 1: $y^{(-\alpha)}(x)$ versus $B^{(-\alpha)}(y;x)$ (left) and plot of $EI(n)$ (right) with $\alpha = \frac{1}{2}$ and $n = 5, 10, 15, 20$.

| $x$ | $EC(40)$ | $EC(60)$ | $EC(80)$ | $EC(100)$ |
|-----|----------|----------|----------|-----------|
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.2 | 0.0061036942 | 0.0040621962 | 0.0030440502 | 0.0024339892 |
| 0.4 | 0.0073719719 | 0.0049132237 | 0.0036843732 | 0.0029472355 |
| 0.6 | 0.0085767960 | 0.0039249810 | 0.0029463970 | 0.0023583970 |
| 0.8 | 0.0094944290 | 0.0033920100 | 0.0025800000 | 0.0020813500 |
| 1.0 | 0.0106309420 | 0.0070992270 | 0.0053288670 | 0.0042652620 |

Table 1  Example 1: values of $EC(n)$ for $n = 40, 60, 80, 100$, $\alpha = \frac{1}{2}$, and some values of $x \in [0, 1]$.

| $x$ | $EI(40)$ | $EI(60)$ | $EI(80)$ | $EI(100)$ |
|-----|----------|----------|----------|-----------|
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.2 | 0.0008344262 | 0.0005544267 | 0.0004153698 | 0.0003320800 |
| 0.4 | 0.0022223574 | 0.0014794349 | 0.0011087740 | 0.0008866339 |
| 0.6 | 0.0036002970 | 0.0023986370 | 0.0017983900 | 0.0014382600 |
| 0.8 | 0.0043154540 | 0.0028772970 | 0.0021580890 | 0.0017265270 |
| 1.0 | 0.0053416963 | 0.0022786810 | 0.0017029750 | 0.0013675460 |

Table 2  Example 1: values of $EI(n)$ for $n = 40, 60, 80, 100$, $\alpha = \frac{1}{2}$, and some values of $x \in [0, 1]$.

| $n$ | $EOC$, $\alpha = \frac{1}{2}$ | $EOC$, $\alpha = \frac{1}{2}$ | $EOC$, $\alpha = \frac{3}{4}$ |
|-----|-------------------------------|-------------------------------|-------------------------------|
| 20  | 0.9997991137                   | 0.9997991137                   | 0.9997991137                   |
| 40  | 0.9997834530                   | 0.9997834530                   | 0.9997834530                   |
| 80  | 0.9996406642                   | 0.9996406642                   | 0.9996406642                   |
| 160 | 0.9993064563                   | 0.9993064563                   | 0.9993064563                   |
| 320 | 0.9996330493                   | 0.9996330493                   | 0.9996330493                   |

Table 3  Example 1: experimentally determined order of convergence for $D_x^\alpha e^x$, $\alpha = \frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{4}$, and different values of $n$. 


Example 2. Let $y(x) = \sin(x), x \in [0, 1]$. The Caputo fractional derivative and the Riemann–Liouville fractional integral of $y$ are given in [28]:
\[
\begin{align*}
\mathcal{C}_0 D^\alpha x y(x) &= y^{(\alpha)}(x) = x^{1-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(2k + 2 - \alpha)}, \quad \alpha \in (0, 1],
0^\alpha D_x y(x) &= y^{(-\alpha)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1+\alpha}}{\Gamma(2k + 2 + \alpha)}, \quad \alpha \geq 0.
\end{align*}
\]

The comparison of $y^{(\alpha)}(x)$ with $B_n^{(\alpha)}(y; x)$ and $y^{(-\alpha)}(x)$ with $B_n^{(-\alpha)}(y; x)$, for $n = 5, 10, 15, 20$ and $\alpha = \frac{3}{4}$, is shown in Fig. 3 and Fig. 4 respectively. In Tables 5 and 6, EC$(n)$ and EI$(n)$ for $n = 40, 60, 80, 100$, $\alpha = \frac{3}{4}$, and some values of $x \in [0, 1]$, are reported. The experimentally order of convergence for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ is shown in Table 7 for $\mathcal{C}_0 D^\alpha x \sin(x)$ and in Table 8 for $0^\alpha D_x^\alpha \sin(x)$.

Fig. 3 Example 2: $y^{(\alpha)}(x)$ versus $B_n^{(\alpha)}(y; x)$ (left) and EC$(n)$ (right) for $\alpha = \frac{3}{4}$ and $n = 5, 10, 15, 20$.

5 Application to fractional differential equations

Consider the nonlinear fractional differential equation
\[
\mathcal{C}_0 D^\alpha x(t) = f(t,x(t)) \tag{5.1}
\]
subject to $x^{(k)}(0) = 0, k = 0, \ldots, m - 1$, where $m - 1 \leq \alpha < m$ and $t \in [0, 1]$. Note that a fractional differential equation (5.1) with nonzero initial conditions $x^{(k)}(0) = x_k$ can be easily transformed to a
Fig. 4 Example 2: \( y^{(-\alpha)}(x) \) versus \( B_n^{(-\alpha)}(y;x) \) (left) and \( EI(n) \) (right) for \( \alpha = \frac{3}{4} \) and \( n = 5, 10, 15, 20 \).

| \( n \) | \( EOC, \alpha = \frac{1}{4} \) | \( EOC, \alpha = \frac{1}{2} \) | \( EOC, \alpha = \frac{3}{4} \) |
|--------|----------------|----------------|----------------|
| 20     | 0.9997752559   | 0.9992561676   | 0.9923368788   |
| 40     | 0.99969076505  | 0.9965126525   | 0.9962296261   |
| 80     | 0.9984573422   | 0.9982238029   | 0.9981122106   |
| 160    | 0.9992402809   | 0.9991070234   | 0.999044572    |
| 320    | 0.9994769560   | 0.9996091945   | 0.998994139    |

Table 6 Example 2: experimentally determined order of convergence for \( \frac{\Gamma}{n!} D_x^\alpha \sin(x) \), \( \alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \), and different \( n \).
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| $n$  | $EOC, \alpha = \frac{1}{4}$ | $EOC, \alpha = \frac{1}{3}$ | $EOC, \alpha = \frac{1}{2}$ |
|------|-----------------|-----------------|-----------------|
| 20   | 0.9958096867    | 0.9969600807    | 0.9981760620    |
| 40   | 0.9979038936    | 0.9984739450    | 0.9990790991    |
| 80   | 0.9989581171    | 0.9992369945    | 0.9995370214    |
| 160  | 0.9994912812    | 0.9996141316    | 0.999773614     |
| 320  | 0.9997385874    | 0.9998309959    | 0.9999065568    |

Table 8 Example (2) experimentally determined order of convergence for $\alpha D_t^\alpha \sin(x), \alpha = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$, and different $n$.

problem (5.1) with vanishing initial conditions. The advantage of considering zero initial values of $x$ and its derivatives up to order $m-1$ is that under such conditions the Riemann–Liouville and the Caputo derivatives coincide. Let $x(t) \approx x_n(t) = \sum_{i=m}^{n} c_i (1-t)^{n-i}$, where $c_i = x(i/n), i = m, \ldots, n$, are unknown coefficients to be determined. Using (3.1),

$$C_0 D_t^\alpha x(t) \approx \sum_{i=m}^{n} \sum_{j=0}^{n-i} c_i \binom{n-i}{j} \frac{(-1)^j}{\Gamma(i+j+1)} t^{i+j-\alpha}.$$ 

We approximate (5.1) by $C_0 D_t^\alpha x_n(t) = f(t, x_n(t))$ and, considering equidistant nodes $t_j = jh$ in the interval $[0, 1]$, $j = 0, \ldots, n-m$, $h = \frac{1}{n}$, we transform this system into $C_0 D_t^\alpha x_n(t_j) = f(t_j, x_n(t_j))$, $j = 0, \ldots, n-m$. The $n-m+1$ unknown coefficients $c_i, i = m, \ldots, n$, are found by solving this algebraic system of $n-m+1$ equations.

### 5.1 Convergence and stability

We start to prove an important result about the error committed when solving the fractional differential equation (5.1) with our approximate method (Theorem 14). The second result asserts that the approximate solutions of (5.1) are stable with respect to the right-hand side of the fractional differential equation (Theorem 15). The proof of both results make use of the following Gronwall-type result for fractional integral equations:

**Lemma 1** ([8]). Let $\alpha, T, \varepsilon_1, \varepsilon_2 \in \mathbb{R}^+$. Moreover, assume that $\delta : [0, T] \rightarrow \mathbb{R}$ is a continuous function satisfying the inequality

$$|\delta(t)| \leq \varepsilon_1 + \frac{\varepsilon_2}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} |\delta(x)| \, dx$$

for all $t \in [0, T]$. Then, $|\delta(t)| \leq \varepsilon_1 E_{\alpha,1}(\varepsilon_2 t^\alpha)$ for $t \in [0, T]$.

**Theorem 14.** Let $x$ be the solution of

$$C_0 D_t^\alpha x(t) = f(t, x(t)), \quad x^{(k)}(0) = 0, \quad k = 0, \ldots, [\alpha] - 1,$$

(5.2)

where $f$ satisfies a Lipschitz condition in its second argument on $[0, 1]$. If $x_n(t) = B_n(x, t)$ is the approximate solution of (5.2), then $x_n(t)$ is convergent to $x(t)$ as $n \to \infty$, and $|x_n(t) - x(t)| \leq O(h)$.

**Proof.** As in the proof of Theorem 7, if $x^{(k)}(0) = 0, k = 0, \ldots, [\alpha] - 1$, then

$$C_0 D_t^\alpha x_n(t) = \alpha D_t^\alpha x_n(t) = f(t, x_n(t)) + O(h).$$

If we write $x$ and $x_n$ in the form of the equivalent Volterra integral equation

$$x(t) = x(0) + \int_0^t f(s, x(s)) \, ds,$$

and

$$x_n(t) = x_n(0) + \int_0^t f(s, x_n(s)) \, ds,$$

we can apply using the above Lemma 1 and obtain

$$|x(t) - x_n(t)| \leq \varepsilon_1 E_{\alpha,1}(\varepsilon_2 t^\alpha).$$

This completes the proof of Theorem 14.
\[ x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} f(z, x(z)) \, dz \]

and

\[ x_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (f(z, x_n(z)) + O(h)) \, dz, \]

subtracting we obtain the relation

\[ x(t) - x_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (f(z, x(z)) - f(z, x_n(z))) \, dz + O(h) \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \, dz \right). \]

Using the Lipschitz condition on \( f \) in the first term on the right-hand side,

\[ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (f(z, x(z)) - f(z, x_n(z))) \, dz \right| < \Lambda \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \, dz \right). \]

If we put \( Y(t) = |x(t) - x_n(t)| \), then we have

\[ |Y(t)| \leq O(h) + \Lambda \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} Y(z) \, dz \right). \quad (5.3) \]

In view of Lemma 1, (5.3) allow us to conclude that \( |Y(t)| \leq O(h)E_{1,\alpha}(\Lambda t^\alpha) \) for \( t \in [0, 1] \), and the proof is complete.

**Theorem 15.** Let \( x_n(t) \) and \( x_n'(t) \) be approximate solutions of (5.2) with the right-hand side of the fractional differential equation given by \( f \) and \( f' \), respectively. If \( f \) and \( f' \) satisfy a Lipschitz condition in its second argument on \([0, 1]\), then \( |x_n(t) - x_n'(t)| \leq C \| f - f' \| \) for any \( n \).

**Proof.** As in the proof of Theorem 7 if \( x_n^{(k)}(0) = 0 \), \( k = 0, 1, \ldots \), \( [\alpha] - 1 \), then \( 0D_0^\alpha x_n(t) = f(t, x_n(t)) + O(h) \) and \( 0D_0^\alpha x_n'(t) = f'(t, x_n'(t)) + O(h) \). Now, we write the approximate solutions \( x_n \) and \( x_n' \) in the Volterra integral equation form:

\[ x_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (f(z, x_n(z)) + O(h)) \, dz \quad (5.4) \]

and

\[ x_n'(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (f'(z, x_n'(z)) + O(h)) \, dz. \quad (5.5) \]

Subtracting (5.5) from (5.4), we obtain the relation

\[ x_n(t) - x_n'(t) = \int_0^t \frac{f(z, x_n(z)) - f(z, x_n'(z))}{\Gamma(\alpha)(t-z)^{1-\alpha}} \, dz + O(h) \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \, dz \right) \]

\[ = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(z, x_n(z)) - f(z, x_n'(z))}{(t-z)^{1-\alpha}} \, dz \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(z, x_n'(z)) - f(z, x_n'(z))}{(t-z)^{1-\alpha}} \, dz + O(h) \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad t \in [0, 1]. \]

Using the Lipschitz conditions on \( f \) in the first term on the right-hand side, we get

\[ \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(z, x_n(z)) - f(z, x_n'(z))}{(t-z)^{1-\alpha}} \, dz \right| < \frac{\Lambda}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |x_n(z) - x_n'(z)| \, dz. \]
by evaluating the integral representations of the fractional derivatives of $x_n(t)$ and $x'_n(t)$. For the second term on the right-hand side, we have

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^t f(z, x'_n(z)) - f'(z, x'_n(z)) \frac{dz}{(t-z)^{1-\alpha}} \right| < L\|f - f'\| \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} dz \right)$$

$$= L \frac{t^\alpha}{\Gamma(1+\alpha)} \|f - f'\|$$

for $t \in [0, 1]$. If we put $Y(t) = |x_n(t) - x'_n(t)|$, then

$$|Y(t)| \leq O(h) + L'\|f - f'\| + \Lambda \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} Y(z) dz \right). \quad (5.6)$$

In view of Lemma 5.6 allow us to conclude that

$$|Y(t)| \leq (O(h) + L'\|f - f'\|)E_{1,\alpha}(\Lambda t^\alpha) \leq C\|f - f'\|, \quad t \in [0, 1].$$

### 5.2 Illustrative examples

We give four examples of application of our method.

**Example 3.** The following problem is studied in [27]: $c(\frac{C}{}^\alpha D_t \frac{}x)x(t) + kx(t) = f(t), 0 \leq t \leq 1$, subject to $x(0) = 0$ and where $\alpha = \frac{1}{2}, c = 100, k = 10,$ and $f(t) \equiv 1$. The exact solution is shown to be $x_{\text{exact}}(t) = \frac{1}{t} \left( 1 - E_{1,\alpha} \left( -\frac{k}{\alpha} \right) \right) [27]$. Fig. 5 compares the exact solution with the numerical approximations obtained by our method.

**Fig. 5** Example 3: $x_{\text{exact}}$ versus $x_n(t) = B_n(x; t)$ (left) and $E(n) = |x_{\text{exact}}(t) - x_n(t)|$ (right), $n = 5, 10, 15, 20$.

**Example 4.** As a second example, consider the nonlinear ordinary differential equation

$$\left( \frac{C}{}^\alpha D_t x(t) \right) + k(x(t))^2 = f(t)$$

subject to $x(0) = 0$ and $x^{(1)}(0) = 0$, where
Example 4. Consider the fractional oscillation equation

\[ C_0 \mathcal{D}^\alpha t x(t) + x(t) = t e^{-t} \]

subject to \( x(0) = 0 \) and \( x^{(1)}(0) = 0 \). The exact solution is given by

\[ x_{\text{exact}}(t) = t^5 - 3 t^4 + 2 t^3. \]

Following [18], we take \( k = 1 \) and \( \alpha = \frac{3}{2} \). The exact solution is given by \( x_{\text{exact}}(t) = t^5 - 3 t^4 + 2 t^3 \) [18]. Fig. 6 plots the results obtained. It is worthwhile to note that the best numerical solution is obtained for \( n = 5 \) because the exact solution is a polynomial of degree five. Fig. 6 (right) shows that \( E(n) \) grows for \( n > 5 \).

Example 5. Consider the fractional oscillation equation

\[ C_0 \mathcal{D}^\alpha t x(t) + x(t) = t e^{-t} \]

with \( \alpha = \frac{3}{2} \). In Fig. 7 we compare, for several values of \( \alpha \), the exact solution with the approximation obtained by our method with \( n = 15 \) (left), and the case \( \alpha = \frac{3}{2} \) for various values of \( n \) (right).
Example 6. As our last example, we consider the following fractional nonlinear differential equation borrowed from [8]:
\[
\left( C_0 D^1_{0.28} x \right) (t) + \left( t - 0.5 \right) \sin(x(t)) = 0.8 t^3
\]
subject to a nonzero initial condition \( x(0) = x_0 \). The exact solution to this problem is unknown. Let \( z(t) = x(t) - x_0 \). Then the problem is transformed into
\[
\left( C_0 D^1_{0.28} z \right) (t) + \left( t - 0.5 \right) \sin(z(t) + x_0) = 0.8 t^3
\]
subject to the zero initial condition \( z(0) = 0 \). In Fig. 8 we show the numerical solutions \((n = 10 \text{ and } n = 20)\) for several values of the initial condition: \( x_0 = 1.2, 1.3, 1.4, 1.5, \text{ and } 1.6 \).

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