Smooth Livšic regularity for piecewise expanding maps

Matthew Nicol     Tomas Persson

July 26, 2010

Abstract

We consider the regularity of measurable solutions $\chi$ to the cohomological equation

$$\phi = \chi \circ T - \chi,$$

where $(T, X, \mu)$ is a dynamical system and $\phi: X \to \mathbb{R}$ is a $C^k$ valued cocycle in the setting in which $T: X \to X$ is a piecewise $C^k$ Gibbs–Markov map, an affine $\beta$-transformation of the unit interval or more generally a piecewise $C^k$ uniformly expanding map of an interval. We show that under mild assumptions, bounded solutions $\chi$ possess $C^k$ versions. In particular we show that if $(T, X, \mu)$ is a $\beta$-transformation then $\chi$ has a $C^k$ version, thus improving a result of Pollicott et al. [23].

Mathematics Subject Classification 2010: 37D50, 37A20, 37A25.

1 Introduction

In this note we consider the regularity of solutions $\chi$ to the cohomological equation

$$\phi = \chi \circ T - \chi,$$ (1)

where $(T, X, \mu)$ is a dynamical system and $\phi: X \to \mathbb{R}$ is a $C^k$ valued cocycle. In particular we are interested in the setting in which $T: X \to X$ is a piecewise $C^k$ Gibbs–Markov map, an affine $\beta$-transformation of the unit interval.
or more generally a piecewise $C^k$ uniformly expanding map of an interval. Rigidity in this context means that a solution $\chi$ with a certain degree of regularity is forced by the dynamics to have a higher degree of regularity. Cohomological equations arise frequently in ergodic theory and dynamics and, for example, determine whether observations $\phi$ have positive variance in the central limit theorem and have implication for other distributional limits (for examples see [20, 2]). Related cohomological equations to Equation (1) decide on stable ergodicity and weak-mixing of compact group extensions of hyperbolic systems [11, 20, 19] and also play a role in determining whether two dynamical systems are (Hölder, smoothly) conjugate to each other.

Livšic [13, 14] gave seminal results on the regularity of measurable solutions to cohomological equations for Abelian group extensions of Anosov systems with an absolutely continuous invariant measure. Theorems which establish that a priori measurable solutions to cohomological equations must have a higher degree of regularity are often called measurable Livšic theorems in honor of his work.

We say that $\chi: X \to \mathbb{R}$ has a $C^k$ version (with respect to $\mu$) if there exists a $C^k$ function $h: X \to \mathbb{R}$ such that $h(x) = \chi(x)$ for $\mu$ a.e. $x \in X$.

Pollicott and Yuri [23] prove Livšic theorems for Hölder $\mathbb{R}$-extensions of $\beta$-transformations ($T: [0,1) \to [0,1)$, $T(x) = \beta x \pmod{1}$ where $\beta > 1$) via transfer operator techniques. They show that any essentially bounded measurable solution $\chi$ to Equation (1) is of bounded variation on $[0,1-\epsilon)$ for any $\epsilon > 0$. In this paper we improve this result to show that measurable coboundaries $\chi$ for $C^k$ $\mathbb{R}$-valued cocycles $\phi$ over $\beta$-transformations have $C^k$ versions (see Theorem 2).

Jenkinson [10] proves that integrable measurable coboundaries $\chi$ for $\mathbb{R}$-valued smooth cocycles $\phi$ (i.e. again solutions to $\phi = \chi \circ T - \chi$) over smooth expanding Markov maps $T$ of $S^1$ have versions which are smooth on each partition element.

Nicol and Scott [15] have obtained measurable Livšic theorems for certain discontinuous hyperbolic systems, including $\beta$-transformations, Markov maps, mixing Lasota–Yorke maps, a simple class of toral-linked twist map and Sinai dispersing billiards. They show that a measurable solution $\chi$ to Equation (1) has a Lipschitz version for $\beta$-transformations and a simple class of toral-linked twist map. For mixing Lasota–Yorke maps and Sinai dispersing billiards they show that such a $\chi$ is Lipschitz on an open set. There is an error in [15, Theorem 1] in the setting of $C^2$ Markov maps — they only prove measurable solutions $\chi$ to Equation (1) are Lipschitz on each element $T\alpha$, $\alpha \in \mathcal{P}$, where $\mathcal{P}$ is the defining partition for the Markov map, and not that the solutions are Lipschitz on $X$, as Theorem 1 erroneously states. The
error arose in the following way: if $\chi$ is Lipschitz on $\alpha \in \mathcal{P}$ it is possible to extend $\chi$ as a Lipschitz function to $T\alpha$ by defining $\chi(Tx) = \phi(x) + \chi(x)$, however extending $\chi$ as a Lipschitz function from $\alpha$ to $T^2\alpha$ via the relation $\chi(T^2x) = \phi(Tx) + \chi(Tx)$ may not be possible, as $\phi \circ T$ may have discontinuities on $T\alpha$. In this paper we give an example, (see Section 3), which shows that for Markov maps this result cannot be improved on.

Gouëzel [7] has obtained similar results to Nicol and Scott [15] for cocycles into Abelian groups over one-dimensional Gibbs–Markov systems. In the setting of Gibbs–Markov system with countable partition he proves any measurable solution $\chi$ to Equation (1) is Lipschitz on each element $T\alpha$, $\alpha \in \mathcal{P}$, where $\mathcal{P}$ is the defining partition for the Gibbs–Markov map.

In related work, Aaronson and Denker [1, Corollary 2.3] have shown that if $(T, X, \mu, \mathcal{P})$ is a mixing Gibbs–Markov map with countable Markov partition $\mathcal{P}$ preserving a probability measure $\mu$ and $\phi: X \to \mathbb{R}^d$ is Lipschitz (with respect to a metric $\rho$ on $X$ derived from the symbolic dynamics) then any measurable solution $\chi: X \to \mathbb{R}^d$ to $\phi = \chi \circ T - \chi$ has a version $\tilde{\chi}$ which is Lipschitz continuous, i.e. there exists $C > 0$ such that $d(\tilde{\chi}(x), \tilde{\chi}(y)) \leq C\rho(x, y)$ for all $x, y \in T(\alpha)$ and each $\alpha \in \mathcal{P}$.

Bruin et al. [4] prove measurable Livšic theorems for dynamical systems modelled by Young towers and Hofbauer towers. Their regularity results apply to solutions of cohomological equations posed on Hénon-like mappings and a wide variety of non-uniformly hyperbolic systems. We note that Corollary 1 of [4, Theorem 1] is not correct — the solution is Hölder only on $M_k$ and $TM_k$ rather than $T^jM_k$ for $j > 1$ as stated for reasons similar to those given above for the result in Nicol et al. [15].

2 Main results

We first describe one-dimensional Gibbs–Markov maps. Let $I \subset \mathbb{R}$ be a bounded interval, and $\mathcal{P}$ a countable partition of $I$ into intervals. We let $m$ denote Lebesgue measure. Let $T: I \to I$ be a piecewise $C^k$, $k \geq 2$, expanding map such that $T$ is $C^k$ on the interior of each element of $\mathcal{P}$ with $|T'| > \lambda > 1$, and for each $\alpha \in \mathcal{P}$, $T\alpha$ is a union of elements in $\mathcal{P}$. Let $P_n := \bigvee_{j=0}^n T^{-j}\mathcal{P}$ and $J_T := \frac{d(\nu \circ T)}{dm}$. We assume:

(i) (Big images property) There exists $C_1 > 0$ such that $m(T\alpha) > C_1$ for all $\alpha \in \mathcal{P}$.

(ii) There exists $0 < \gamma_1 < 1$ such that $m(\beta) < \gamma_1^n$ for all $\beta \in P_n$. 

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(iii) (Bounded distortion) There exists $0 < \gamma_2 < 1$ and $C_2 > 0$ such that $|1 - \frac{\gamma(x)}{\gamma(y)}| < C_2 \gamma_2^n$ for all $x, y \in \beta$ if $\beta \in P_n$.

Under these assumptions $T$ has an invariant absolutely continuous probability measure $\mu$ and the density of $\mu$, $h = \frac{d\mu}{dm}$ is bounded above and below by a constant $0 < C^{-1} \leq h(x) \leq C$ for $m$ a.e. $x \in I$.

Note that a Markov map satisfies (i), (ii) and (iii) for finite partition $P$.

It is proved in [15] for the Markov case (finite $P$), and in [7] for the Gibbs–Markov case (countable $P$) that if $\phi: I \to \mathbb{R}$ is Hölder continuous or Lipschitz continuous, and $\phi = \chi \circ T - \chi$ for some measurable function $\chi: I \to \mathbb{R}$, then there exists a function $\chi_0: I \to \mathbb{R}$ that is Hölder or Lipschitz on each of the elements of $P$ respectively, and $\chi_0 = \chi$ holds $\mu$ (or $m$) a.e. A related result to [7] is given in [4, Theorem 7] where $T$ is the base map of a Young Tower, which has a Gibbs–Markov structure.

Fried [6] has shown that the transfer operator of a graph directed Markov system with $C^{k,\alpha}$-contractions, acting on a space of $C^{k,\alpha}$-functions, has a spectral gap. If we apply his result to our setting, letting the contractions be the inverse branches of a Gibbs–Markov map we can conclude that the transfer operator of a Gibbs–Markov map acting on $C^k$-functions has a spectral gap. As in Jenkinson’s paper [10] and with the same proof, this gives us immediately the following proposition, which is implied by the results of Fried and Jenkinson:

**Proposition 1.** Let $T: T \to I$ be a mixing Gibbs–Markov map such that $T$ is $C^k$ on each partition element and $T^{-1}: T(\alpha) \to \alpha$ is $C^k$ on each partition element $\alpha \in P$. Let $\phi: I \to \mathbb{R}$ be uniformly $C^k$ on each of the partition elements $\alpha \in P$. Suppose $\chi: I \to \mathbb{R}$ is a measurable function such that $\phi = \chi \circ T - \chi$. Then there exists a function $\chi_0: I \to \mathbb{R}$ such that $\chi_0$ is uniformly $C^k$ on $T\alpha$ for each partition element of $\alpha \in P$, and $\chi_0 = \chi$ almost everywhere.

### 3 A counterexample

We remark that in general, if $\phi = \chi \circ T - \chi$, one cannot expect $\chi$ to be continuous on $I$ if $\phi$ is $C^k$ on $I$. We give an example of a Markov map $T$ with Markov partition $P$, a function $\phi$ that is $C^k$ on $I$, and a function $\chi$ that is $C^k$ on each element $\alpha$ of $P$ such that $\phi = \chi \circ T - \chi$, yet $\chi$ has no version that is continuous on $I$. 


Let $0 < c < \frac{1}{4}$, Put $d = 2 - 4c$. Define $T: [0, 1] \to [0, 1]$ by

$$T(x) = \begin{cases} 
2x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{4} \\
(d(x - \frac{1}{2}) + \frac{1}{2} & \text{if } \frac{1}{4} < x < \frac{3}{4} \\
2x - \frac{3}{2} & \text{if } \frac{3}{4} \leq x \leq 1
\end{cases}.$$

If $c = \frac{1}{8}$, then the partition

$$\mathcal{P} = \left\{ \left[0, \frac{1}{8}\right], \left[\frac{1}{8}, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2} - \frac{1}{4d}\right], \left[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}\right], \left[\frac{1}{2} + \frac{1}{4d}, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{7}{8}\right], \left[\frac{7}{8}, 1\right] \right\}$$

is a Markov partition for $T$. Define $\chi$ such that $\chi$ is $0$ on $\left[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}\right]$ and $1$ on $\left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}\right]$. On $[0, \frac{1}{4})$ we define $\chi$ so that $\chi(0) = 1$ and $\lim_{x \to \frac{1}{4}^+} \chi(x) = 0$, and on $(\frac{3}{4}, 1]$ we define $\chi$ so that $\chi(1) = 0$ and $\lim_{x \to \frac{3}{4}^-} \chi(x) = 1$. For any natural number $k$, this can be done so that $\chi$ is $C^k$ except at the point $\frac{1}{2}$ where it has a jump. One easily check that $\phi$ defined by $\phi = \chi \circ T - \chi$ is $C^k$. This is illustrated in Figures 1–4.

4 Livšic theorems for piecewise expanding maps of an interval

Let $I = [0, 1)$ and let $m$ denote Lebesgue measure on $I$. We consider piecewise expanding maps $T: I \to I$, satisfying the following assumptions:

(i) There is a number $\lambda > 1$, and a finite partition $\mathcal{P}$ of $I$ into intervals, such that the restriction of $T$ to any interval in $\mathcal{P}$ can be extended to a $C^2$-function on the closure, and $|T'| > \lambda$ on this interval.

(ii) $T$ has an absolutely continuous invariant measure $\mu$ with respect to which $T$ is mixing.

(iii) $T$ has the property of being weakly covering, as defined by Liverani in [12], namely that there exists an $n_0$ such that for any element $\alpha \in \mathcal{P}$

$$\bigcup_{j=0}^{n_0} T^j(\alpha) = I.$$

For any $n \geq 0$ we define the partition $\mathcal{P}_n = \mathcal{P} \vee \cdots \vee T^{n+1} \mathcal{P}$. The partition elements of $\mathcal{P}_n$ are called $n$-cylinders, and $\mathcal{P}_n$ is called the partition of $I$ into $n$-cylinders.

We prove the following two theorems.
Figure 1: The graph of $T$.

Figure 2: The graph of $\chi$.

Figure 3: The graph of $\chi \circ T$.

Figure 4: The graph of $\phi = \chi \circ T - \chi$. 
Theorem 1. Let \((T, I, \mu)\) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). Let \(\phi : I \to \mathbb{R}\) be a Hölder continuous function, such that \(\phi = \chi \circ T - \chi\) for some measurable function \(\chi\), with \(e^{-\chi} \in L_1(m)\). Then there exists a function \(\chi_0\) such that \(\chi_0\) has bounded variation and \(\chi_0 = \chi\) almost everywhere.

For the next theorem we need some more definitions. Let \(A\) be a set, and denote by \(\text{int} A\) the interior of the set \(A\). We assume that the open sets \(T(\text{int} \alpha), \alpha \in \mathcal{P}\), cover \(\text{int} I\).

We will now define a new partition \(Q\). For a point \(x\) in the interior of some element of \(\mathcal{P}\), we let \(Q(x)\) be the largest open set such that for any \(x_2 \in Q(x)\), and any \(m\)-cylinder \(C_m\), there are points \((y_{1,k})_{k=1}^n\) and \((y_{2,k})_{k=1}^n\), such that \(y_{1,k}\) and \(y_{2,k}\) are in the same element of \(\mathcal{P}\), \(T(y_{i,k+1}) = y_{i,k}\), \(T(y_{1,1}) = x\), \(T(y_{2,1}) = x_2\), and \(y_{1,n}, y_{2,n} \in C_m\). (This forces \(n \geq m\).)

Note that if \(Q(x) \cap Q(y) \neq \emptyset\), then for \(z \in Q(x) \cap Q(y)\) we have \(Q(z) = Q(x) \cup Q(y)\). We let \(Q\) be the coarsest collection of connected sets, such that any element of \(Q\) can be represented as a union of sets \(Q(x)\).

Theorem 2. Let \((T, I, \mu)\) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). If \(\phi : I \to \mathbb{R}\) is a continuously differentiable function, such that \(\phi = \chi \circ T - \chi\) for some function \(\chi\) with \(e^{-\chi} \in L_1(m)\), then there exists a function \(\chi_0\) such that \(\chi_0\) is continuously differentiable on each element of \(Q\) and \(\chi_0 = \chi\) almost everywhere. If \(T'\) is constant on the elements of \(\mathcal{P}\), then \(\chi_0\) is piecewise \(C^k\) on \(Q\) if \(\phi\) is in \(C^k\). If for each \(r\), \(\frac{1}{T'(r)}\) is in \(C^k\) with derivatives up to order \(k\) uniformly bounded, then \(\chi_0\) is piecewise \(C^k\) on \(Q\) if \(\phi\) is in \(C^k\).

It is not always clear how big the elements in the partition \(Q\) are. The following lemma gives a lower bound on the diameter of the elements in \(Q\).

Lemma 1. Assume that the sets \(\{T(\text{int} \alpha) : \alpha \in \mathcal{P}\}\) cover \((0, 1)\). Let \(\delta\) be the Lebesgue number of the cover. Then the diameter of \(Q(x)\) is at least \(\delta/2\) for all \(x\).

Proof. Let \(C_m\) be a cylinder of generation \(m\). We need to show that for some \(n \geq m\) there are sequences \((y_{1,k})_{k=1}^n\) and \((y_{2,k})_{k=1}^n\) as in the definition of \(Q\) above.

Take \(n_0\) such that \(\mu(T^{n_0}(C_m)) = 1\). Write \(C_m\) as a finite union of cylinders of generation \(n_0\), \(C_m = \bigcup D_i\). Then \(R := [0, 1] \setminus T^{n_0}(\bigcup \text{int} D_i)\) consists of finitely many points. Let \(\varepsilon\) be the smallest distance between two of these points.

Let \(I_\delta\) be an open interval of diameter \(\delta\). Let \(n_1\) be such that \(\delta \lambda^{-n_1} < \varepsilon\). Consider the full pre-images of \(I_\delta\) under \(T^{n_1}\). By the definition of \(\delta\), there is
at least one such pre-image, and any such pre-image is of diameter less than \( \varepsilon \). Hence any pre-image contains at most one point from \( R \).

If the pre-image does not contain any point of \( R \), then \( I_\delta \) is contained in some element of \( Q \) and we are done. Assume that there is a point \( z \) in \( I_\delta \) corresponding to the point of \( R \) in the pre-image of \( I_\delta \). Assume that \( z \) is in the right half of \( I_\delta \). The case when \( z \) is in the left part is treated in a similar way. Take a new open interval \( J_\delta \) of length \( \delta \), such that the left half of \( J_\delta \) coincides with the right half of \( I_\delta \).

Arguing in the same way as for \( I_\delta \), we find that a pre-image of \( J_\delta \) contains at most one point of \( R \). If there is no such point, or the corresponding point \( z_{J_\delta} \) is not equal to \( z \), then \( I_\delta \cup J_\delta \) is contained in an element in \( Q \) and we are done.

It remains to consider the case \( z = z_{J_\delta} \). Let \( I_\delta = (a, b) \) and \( J_\delta = (c, d) \). Then the intervals \((a, z)\) and \((z, d)\) are both of length at least \( \delta/2 \), and both are contained in some element of \( Q \). This finishes the proof.

**Corollary 1.** If \( \beta > 1 \) and \( T: x \mapsto \beta x \pmod{1} \) is a \( \beta \)-transformation then clearly \( T \) is weakly covering and \( Q = \{(0, 1)\} \), so in this case Theorem 2 and Theorem 1 of [15] imply that \( \chi_0 \) is in \( C^k \) if \( \phi \) is in \( C^k \).

**Remark 1.** If \( T: x \mapsto \beta x + \alpha \pmod{1} \) is an affine \( \beta \)-transformation, then \( Q = \{(0, 1)\} \), and hence if \( e^{-x} \) is in \( L_1(m) \) then \( \chi \) has a \( C^k \) version.

## 5 Proof of Theorem 1

We continue to assume that \((T, I, \mu)\) is a piecewise expanding map satisfying assumptions (i), (ii) and (iii). For a function \( \psi: I \to \mathbb{R} \) we define the weighted transfer operator \( \mathcal{L}_\psi \) by

\[
\mathcal{L}_\psi f(x) = \sum_{T(y) = x} e^{\psi(y)} \frac{1}{|d_y T|} f(y).
\]

The proof is based on the following two facts, that can be found in Hofbauer and Keller’s papers [8, 9]. The first fact is

There is a function \( h \geq 0 \) of bounded variation such that if \( f \in L^1 \) with \( f \geq 0 \) and \( f \neq 0 \), then \( \mathcal{L}_0 f \) converges to \( h \int f \, dm \) in \( L^1 \).
The second fact is

Let $f \in L^1$ with $f \geq 0$ and $f \neq 0$ be fixed. There is a function $w \geq 0$ with bounded variation, a measure $\nu$, and a number $a > 0$, depending on $\phi$, such that

$$a^n \mathcal{L}_\phi^n f \to w \int f \, d\nu,$$

in $L^1$.

For $f$ of bounded variation, these facts are proved as follows. Theorem 1 of [8] gives us the desired spectral decomposition for the transfer operator acting of functions of bounded variation. Proposition 3.6 of Baladi’s book [3] gives us that there is a unique maximal eigenvalue. This proves the two facts for $f$ of bounded variation. The case of a general $f$ in $L^1$ follows since such an $f$ can be approximated by functions of bounded variation.

Using that $T$ is weakly covering, we can conclude by Lemma 4.2 in [12], that $h > \gamma > 0$. The proof of this fact in [12] goes through also for $w$, and so we may also conclude that $w > \gamma > 0$.

Let us now see how Theorem 1 follows from these facts. The following argument is analogous to the argument used by Pollicott and Yuri in [23] for $\beta$-expansions. We first observe that $\phi = \chi \circ f - \chi$ implies that

$$\mathcal{L}_\phi^n 1(x) = \sum_{T^n(y) = x} e^{S_n \phi(y)} \frac{1}{|d_y T^n|} = \sum_{T^n(y) = x} e^{\chi(T^n y) - \chi(y)} \frac{1}{|d_y T^n|} = e^{\chi(x)} \sum_{T^n(y) = x} e^{-\chi(y)} \frac{1}{|d_y T^n|} = e^{\chi(x)} \mathcal{L}_0^n e^{-\chi(x)}.$$

Since $a^n \mathcal{L}_\phi^n 1 \to w$ and $e^{-\chi} \mathcal{L}_\phi^n 1 = \mathcal{L}_\phi^n e^{-\chi} \to h \int e^{-\chi} \, dm$ we have that $a^n \mathcal{L}_\phi^n 1$ converges to $w$ in $L^1$ and $\mathcal{L}_\phi^n 1$ converges to $he^{\chi} \int e^{-\chi} \, dm$ in $L^1$. By taking a subsequence, we can achieve that the convergences are a.e. Therefore, we must have $a = 1$ and

$$w(x) = e^{\chi(x)} h(x) \int e^{-\chi} \, dm, \quad \text{a.e.}$$

It follows that

$$\chi(x) = \log w(x) - \log h(x),$$

almost everywhere. Since $h$ and $w$ are bounded away from zero, their logarithms are of bounded variation. This proves the theorem.
6 Proof of Theorem 2

We first note that it is sufficient to prove that $\chi_0$ is continuously differentiable on elements of the form $Q(x)$.

Let $x$ and $y$ satisfy $T(y) = x$. Then by $\phi = \chi \circ T - \chi$ we have $\chi(x) = \phi(y) + \chi(y)$.

Let $x_1$ be a point in an element of $Q$, and take $x_2 \in Q(x_1)$. We choose pre-images $y_{1,j}$ and $y_{2,j}$ of $x_1$ and $x_2$ such that $T(y_{i,1}) = x_i$ and $T(y_{i,j}) = y_{i,j-1}$. We then have

$$\chi(x_1) - \chi(x_2) = \sum_{j=1}^{n} (\phi(y_{1,j}) - \phi(y_{2,j})) + \chi(y_{1,n}) - \chi(y_{2,n}).$$

We would like to let $n \to \infty$ and conclude that $\chi(y_{1,n}) - \chi(y_{2,n}) \to 0$. By Theorem 1 we know that $\chi$ has bounded variation. Assume for contradiction that no matter how we choose $y_{1,j}$ and $y_{2,j}$ we cannot make $|\chi(y_{1,n}) - \chi(y_{2,n})|$ smaller than some $\varepsilon > 0$. Let $m$ be large and consider the cylinders of generation $m$. For any such cylinder $C_m$, we can choose $y_{1,j}$ and $y_{2,j}$ such that $y_{1,n}$ and $y_{2,n}$ both are in $C_m$. Since $|\chi(y_{1,n}) - \chi(y_{2,n})| \geq \varepsilon$, the variation of $\chi$ on $C_m$ is at least $\varepsilon$. Summing over all cylinders of generation $m$, we conclude that the variation of $\chi$ on $I$ is at least $N(m)\varepsilon$. Since $m$ is arbitrary and $N(m) \to \infty$ as $m \to \infty$, we get a contradiction to the fact that $\chi$ is of bounded variation.

Hence we can make $|\chi(y_{1,n}) - \chi(y_{2,n})|$ smaller that any $\varepsilon > 0$ by choosing $y_{1,j}$ and $y_{2,j}$ in an appropriate way. We conclude that

$$\chi(x_1) - \chi(x_2) = \sum_{j=1}^{\infty} (\phi(y_{1,j}) - \phi(y_{2,j})).$$

If $x_1 \neq x_2$ and $y_{1,j} \neq y_{2,j}$ for all $j$, and we have

$$\frac{\chi(x_1) - \chi(x_2)}{x_1 - x_2} = \sum_{j=1}^{\infty} \frac{\phi(y_{1,j}) - \phi(y_{2,j})}{y_{1,j} - y_{2,j}} \frac{y_{1,j} - y_{2,j}}{x_1 - x_2}.$$

Clearly, the limit of the right hand side exists as $x_2 \to x_1$, and is

$$\sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}.$$

The series converges since $|(T^j)'| > \lambda^j$. This shows that $\chi'(x_1)$ exists and satisfies

$$\chi'(x_1) = \sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}. \quad (4)$$
If $T'$ is constant on the elements of $\mathcal{P}$, then (4) implies that $\chi$ is in $C^k$ provided that $\phi$ is in $C^k$.

Let us now assume that $\frac{1}{(T')}^r$ is in $C^k$ with derivatives up to order $k$ uniformly bounded in $r$. We proceed by induction. Let $g_n = \frac{1}{(T')}^r$. Assume that

$$\chi^{(m)}(x) = \sum_{n=1}^{\infty} \psi_{n,m}(y_n) g_n(y_n),$$

where $$(\psi_{n,m})_{n=1}^{\infty}$$ is in $C^{n-m}$ with derivatives up to order $n - m$ uniformly bounded. Then

$$\chi^{(m+1)}(x) = \sum_{n=1}^{\infty} \left( \psi'_{n,m}(y_n) g_n(y_n) + \psi_{n,m}(y_n) g'_n(y_n) \right) g_n(y_n) = \sum_{n=1}^{\infty} \psi_{n,m+1} g_n(y_n).$$

This proves that there are uniformly bounded functions $\psi_{n,m}$ such that (5) holds for $1 \leq m \leq k$. The series in (5) converges uniformly since $g_n$ decays with exponential speed. This proves that $\chi$ is in $C^k$. \hfill \Box
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