Special moments

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Dedicated to the memory of David Robbins

In this article, we show that a linear combination $\bar{X}$ of $n$ independent, unbiased Bernoulli random variables $\{X_i\}$ can match the first $2n$ moments of a random variable $Y$ which is uniform on an interval. More generally, for each $p \geq 2$, each $X_i$ can be uniform on an arithmetic progression of length $p$. All values of $\bar{X}$ lie in the range of $Y$, and their ordering as real numbers coincides with dictionary order on the vector $(X_1, \ldots, X_n)$.

The construction involves the roots of truncated $q$-exponential series. It applies to a construction in numerical cubature using error-correcting codes [3]. For example, when $n = 2$ and $p = 2$, the values of $\bar{X}$ are the 4-point Chebyshev quadrature formula.

1. INTRODUCTION

One of the standard proofs of the central limit theorem establishes that the moments of the normalized sum

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}$$

of $n$ i.i.d. centered random variables with finite moments converge to the moments of a Gaussian random variable. This fact raises the question of when the first $n$ moments of a random variable $Y$ can be matched by a linear combination of independent copies of another variable $X$. It is an easy exercise with cumulants that this is impossible when $X$ is not Gaussian. In this article we will show that for every $n$, $Y$ can be the uniform distribution on an interval if $X$ is unbiased Bernoulli. More generally $X$ can be uniform on an arithmetic progression of length $p$ for any $p \geq 2$.

We conjecture that the moments of most absolutely continuous distributions cannot be matched by those of a linear combination of Bernoulli random variables. In this sense the uniform distribution on an interval has “special moments”.

**Theorem 1.** Let $p \geq 2$ be an integer, let $X$ be a uniformly random variable on the set

$$\{p-1, p-3, p-5, \ldots, 1-p\},$$

and let $X_1, X_2, \ldots, X_n$ be independent copies of $X$. Then there exist unique constants

$$a_1 > a_2 > \cdots > a_n > 0$$

such that the first $2n$ moments of

$$\bar{X} = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

agree with the first $2n$ moments of a random variable $Y$ which is uniform on $[-1, 1]$. Moreover

$$\sum_{j=1}^{n} |a_j - p^{-j}| < \frac{1}{p^n(p-1)}.$$
Let $Z_{p,n}$ be the range of $\tilde{X}$. Since $\tilde{X}$ has the same first $2n$ moments as $Y$, indeed trivially the same $(2n+1)$st moment as well, the equation
\[
\frac{1}{2} \int_{-1}^{1} P(x) dx = \frac{1}{p^n} \sum_{\xi \in Z_{p,n}} P(\xi)
\]
holds for any polynomial $P$ of degree at most $2n+1$. A weighted set $Z$ with this property up to some degree $t$ is called an \textit{(interpolatory) $t$-quadrature formula}. E.g., Simpson’s rule and Gaussian quadrature are standard quadrature formulas. Our quadrature formula $Z_{p,n}$ is highly inefficient for general $p$ and $n$, but its special structure is useful for the higher-dimensional cubature problem for integration on the $k$-cube $[-1,1]^k$. Elsewhere we combine the product formula $Z_{2,n}$ with binary error-correcting codes, in particular extended BCH codes, to obtain a $(2n+1)$-cubature formula on $[-1,1]^k$ with equal weights and $O(k^p)$ points. (The asymptotic bound is with $n$ fixed and $k \to \infty$.)

2. THE PROOF

We will write $a_{n,j}$ for $a_j$, to make clear that they depend on $n$.

**Lemma 2.** The random variables $\tilde{X}$ and $Y$ have the same first $2n$ moments if and only if
\[
\sum_{j=1}^{n} a_{n,j}^{2k} = \frac{1}{p^{2k} - 1}
\]
for all $1 \leq k \leq n$.

**Proof.** Recall that $Y$ is uniformly random on $[-1,1]$ and that $X$ is uniformly random on $\{p - 1, p - 3, p - 5, \ldots, 1 - p\}$. Thus if they are independent, then $X + Y$ is uniformly random on $[-p,p]$, so
\[
X + Y \overset{d}{=} pY. \tag{2}
\]
Here “$\overset{d}{=}$” means equality of distribution.

To understand the implications of this relation between $X$ and $Y$, we review the relations among cumulants, moments, and their generating functions $\tilde{M}(t)$. For a general random variable $X$, the $k$th moment is denoted $\mu_k(X)$, the exponential generating function of all of the moments is the moment function $M_X(t)$, the cumulant function $K_X(t)$ is its logarithm, and the cumulants $\kappa_k(X)$ are defined by $K_X(t)$ as their exponential generating function. In formulas,
\[
\mu_k(X) \overset{def}{=} E[X^k], \quad M_X(t) \overset{def}{=} E[e^{tX}] = \sum_{k=0}^{\infty} \frac{\mu_k(X) t^k}{k!}, \quad K_X(t) \overset{def}{=} \ln M_X(t), \quad \kappa_k(X) \overset{def}{=} \frac{K_X(t)^k}{k!} = K_X(t)
\]
This framework is designed so that first, cumulants carry the same information as moments, and second, cumulants are additive, i.e.,
\[
K_X(t) + K_Y(t) = K_{X+Y}(t),
\]
for independent random variables $X$ and $Y$.

Equation (2) yields the cumulant generating function equation
\[
K_X(t) + K_Y(t) = K_Y(pt),
\]
which we can write as a relation between individual cumulants:
\[
\kappa_k(X) + \kappa_k(Y) = p^k \kappa_k(Y).
\]
The odd cumulants of $X$ and $Y$ vanish since they are even random variables. The even cumulants thus satisfy the relation
\[
\kappa_{2k}(X) = (p^{2k} - 1) \kappa_{2k}(Y).
\]
Since
\[
\kappa_{2k}(\tilde{X}) = \sum_j b_{n,j}^{2k} \kappa_{2k}(X),
\]
it suffices for the $a_{n,j}$’s to satisfy the stated power sum relation.

This condition is also necessary provided that each $\kappa_{2k}(Y) \neq 0$. To check this, we will establish that $\kappa_{2k}(X) \neq 0$ when $p = 2$. The moment function with imaginary argument, $M_X(\imath t)$, is also called the characteristic function of $X$ (meaning the Fourier transform of the distribution of $X$). In this case, its logarithm is:
\[
K_X(\imath t) = \log \cos t, \quad K_X(\imath t)' = -\tan t.
\]
The relation
\[
(\tan t)' = (\tan t)^2 + 1
\]
implies that the tangent function has strictly positive odd derivatives, so
\[
(-1)^{k+1} \kappa_{2k}(X) > 0
\]
for all $k$.

For convenience let $q = p^2$, and let:
\[
b_{n,j} = a_{n,j}^2, \quad r_{n,j} = \frac{1}{b_{n,j}}.
\]
Lemma can then be restated as
\[
\sum_{j=1}^{n} b_{k,j}^k = \frac{1}{q^k - 1}
\]
for all $1 \leq k \leq n$. This implies a unique solution for the $a_{n,j}$’s provided that each $b_{n,j}$ is real and positive. For convenience we will study a polynomial whose roots are $r_{n,j}$ for $1 \leq j \leq n$. 

Lemma 3. If
\[ \sum_{j=1}^{n} b_{n,j}^k = \frac{1}{q^k - 1} \]
for all \(1 \leq k \leq n\), then
\[ F_{q,n}(x) = \prod_{j=1}^{n} (1 - b_{n,j}x) = \sum_{k=0}^{n} (1 - q)(1 - q^2) \cdots (1 - q^k). \]
Proof. Let
\[ p_k = \sum_{j=1}^{n} b_{n,j}^k \]
be the \(k\)th power sum of the \(b_{n,j}\)'s, and let \(e_k\) be the corresponding elementary symmetric function, so that
\[ \prod_{j=1}^{n} (1 + b_{n,j}x) = 1 + \sum_{k=1}^{n} e_k x^k. \]
Since the first \(n\) elementary symmetric functions determine the first \(n\) power sums and vice versa, and since our desired value
\[ p_k = \frac{1}{q^k - 1} \]
does not depend on \(n\), we can derive each \(e_k\) by taking the limit \(n \to \infty\) and finding \(b_{n,j}\)'s to match all \(p_k\)'s. Let
\[ b_{\infty,j} = q^{-j}. \]
Then
\[ p_k = \sum_{j=1}^{\infty} b_{\infty,j}^k = \frac{1}{q^k - 1} \]
since the left side is a geometric series. Moreover
\[ e_k = \sum_{1 \leq j_1 < j_2 < \cdots < j_k} b_{\infty,j_1} b_{\infty,j_2} \cdots b_{\infty,j_k} = \frac{1}{(q - 1)(q^2 - 1) \cdots (q^k - 1)} \]
by a routine combinatorial exercise. Another way to recognize these values of \(p_k\) and \(e_k\) is that they are the principal specialization of the ring \(\Lambda\) of symmetric functions [4, §7.8], transported by the fundamental involution \(\omega\) and a sign involution \(\sigma\):
\[ \omega(e_k) = h_k, \quad \sigma(e_k) = (-1)^k e_k. \]
To conclude, our explicit choice for the \(b_{\infty,j}\)'s establishes that the given \(p_k\)'s are consistent with the claimed \(e_k\)'s.

To continue the example mentioned first in Section 1
\[ F_{4,\leq n}(x) = 1 - \frac{x}{3} \cdot \frac{x^2}{45} - \frac{x^3}{2835}. \]
Its roots are
\[ \left( \frac{1}{a_{3,1}}, \frac{1}{a_{3,2}}, \frac{1}{a_{3,3}} \right) \approx (3.997956, 16.80465, 42.19739), \]
so that
\[ (a_{3,1}, a_{3,2}, a_{3,3}) \approx (.500128, .243941, .153942). \]
Therefore
\[ \bar{x} \approx \pm .500128 \pm .243941 \pm .153942 \]
when \(p = 2\) and \(n = 3\).

We will need the \(q\)-Pochhammer symbol [2]:
\[ (a;q)_k = \begin{cases} \prod_{j=0}^{k-1} (1 - aq^j) & k > 0 \\ \prod_{j=0}^{k-1} (1 - aq^{-j})^{-1} & k < 0 \\ 1 & k = 0 \end{cases} \]
We also define
\[ F_{q,n}(x) = \frac{x^n}{(q;q)_n}, \quad F_{q}(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} \]
\[ F_{q,\leq n}(x) = \sum_{k=0}^{n} \frac{x^k}{(q;q)_k}, \quad F_{q,> n}(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{(q;q)_k}. \]
(The polynomial \(F_{q,\leq n}(x)\) was already used in Lemma 3.) We will often use the relation
\[ \frac{F_{q,n}(x)}{F_{q,n-1}(x)} = \frac{x}{1 - q^u}. \]
(4)

The function \(F_q(x)\) is related to the standard Jackson \(q\)-exponential function \(e_q(x)\) [2] by
\[ F_q(x) = e_q \left( \frac{x}{1 - q} \right). \]
(In some works, \(F_q(x)\) itself is called a \(q\)-exponential.) By the proof of Lemma 3, \(F_q(x)\) is even periodic with period \(1\), so
\[ F_q(x) = \prod_{j=1}^{\infty} (1 - q^{-j}x) = \frac{1}{(x;q)_\infty}. \]
(5)
This identity holds for all \(q\) as an equality of formal power series. We will need the stronger fact that it is an equality of entire analytic functions when \(q > 1\).

To establish Theorem 1 we would like to understand the effect of truncation on the first \(n\) zeroes of \(F_q(x)\) when \(q \geq 4\).

Lemma 4. If \(q \geq 4\), then \(F_{q,\leq n}(x)\) has \(n\) distinct, positive roots.
Proof. It suffices to show that the value
\[ f_k,\leq n = F_{q,\leq n}(q^{k+\frac{1}{2}}) \]
alters in sign as \(k\) ranges from 0 to \(n\). The terms of \(f_k,\leq n\) are
\[ f_k,j = \frac{q^{j(k+\frac{1}{2})}}{(q;q)_j}. \]
These alternate in sign in \( j \) and we claim that \( f_{k,\leq n} \) has the same sign as \( f_{k,k} \). This claim will imply the lemma.

By equation (4), the sequence \( \{f_{k,j}\} \) is unimodal in \( j \) and achieves its maximum at \( j=k \). This already implies that \( f_{k,\leq n} \) has the same sign as \( f_{k,k} \) if \( k=0 \) or \( k=n \). If \( 1<k<n \), then

\[
\left| \sum_{j=0}^{k-1} f_{k,j} \right| < \left| f_{k,k-1} \right| \quad \text{and} \quad \left| \sum_{j=k+1}^{n} f_{k,j} \right| < \left| f_{k,k+1} \right|.
\]

Finally

\[
\frac{\left| f_{k,k-1} \right| + \left| f_{k,k+1} \right|}{\left| f_{k,k} \right|} = \frac{q^k - 1}{q^{k+1} - 1} < \frac{2}{q^2} \leq 1.
\]

The first equality once again comes from equation (4), while the last inequality is the only step that requires \( q \geq 4 \) instead of merely \( q > 1 \). Thus in each case \( f_{k,k} \) is the dominant term in \( f_{k,\leq n} \).

We continue our example case with \( q=4 \) and \( n=3 \) to illustrate Lemma (4) and its proof:

\[
\begin{align*}
F_{k,\leq 3}(2) &= 1 - \frac{2}{3^2} + \frac{4}{45} - \frac{8}{2835} = 1189/2835, \\
F_{k,\leq 3}(8) &= 1 - \frac{8}{3^3} + \frac{64}{45} - \frac{512}{2835} = 1205/2835, \\
F_{k,\leq 3}(32) &= 1 - \frac{32}{3^4} + \frac{1024}{45} - \frac{32768}{2835} = 4339/2835, \\
F_{k,\leq 3}(128) &= 1 - \frac{128}{3^5} + \frac{16384}{45} - \frac{2097152}{2835} = -1183085/2835.
\end{align*}
\]

In this example we can see that the \( k \)th sum is dominated by its \( k \)th term.

Since \( F_{k,\leq 3}(x) \) is a cubic polynomial with four values that alternate in sign, it therefore has distinct, positive roots, previously noted to be

\[
(r_{3,1}, r_{3,2}, r_{3,3}) \approx (3.997956, 16.80465, 42.19739).
\]

Lemma (4) is also implied by the final lemma, Lemma (5) but the proof of Lemma (5) is much more complicated. It may unfortunately be as taxing for the reader as it was for the author.

**Lemma 5.** Let \( q \geq 4 \). Let \( r_{n,j} \)

\[
r_{n,1} < r_{n,2} < \cdots < r_{n,n}
\]

be the roots of \( F_{q,\leq n}(x) \) and let \( k = n+1-j \). Then \( r_{n,j}q^{-j} \) lies between 1 and

\[
c_k = \begin{cases} 
1 - 2q^{-1} & k = 1 \\
1 + (-1)^k q^{-\left(k+\frac{1}{2}\right)} & k > 1
\end{cases}
\]

for every \( 1 \leq j \leq n \).

**Proof.** Let

\[
\hat{F}_{q,m}(x) = (-1)^{n+1} F_{q,m}(x)
\]

with \( m \) independent of \( n \), and extend this notation to all of the definitions in (4).

The proof consists of three steps. In the first step, we will show that the lemma follows from the inequality

\[
\hat{F}_{q}(c_k q^l) > \hat{F}_{q,>n}(c_k q^l)
\]

and we will show that both sides are positive. In the second step, we will show that this inequality follows from the inequality

\[
h_{q,k} \overset{\text{def}}{=} \lim_{n \to \infty} \frac{\hat{F}_{q}(c_k q^l)}{F_{q,>n}(c_k q^l)} \geq 1,
\]

where \( k \) is fixed in taking the limit. Finally in the third step, we will show that

\[
h_{q,k} > 1.
\]

**Step 1.** We claim that equation (7) implies that

\[
\hat{F}_{q,\leq n}(x) = \hat{F}_{q}(x) - \hat{F}_{q,>n}(x)
\]

changes sign as \( x \) passes from \( q^l \) to \( c_k q^l \). In the estimates for this step we will assume that

\[
q^l \leq x \leq c_k q^l \quad \text{or} \quad c_k q^l \leq x \leq q^j,
\]

except when we explicitly state more general conditions.

By equation (3), \( \hat{F}_{q}(q^j) \) vanishes when \( j \geq 1 \). When \( x > 0 \),

\[
\hat{F}_{q,>n+1}(x) > 0
\]

by the definition of \( \hat{F}_{q,>n+1}(x) \). Moreover

\[
\hat{F}_{q,>n}(x) > 0
\]

because, by equation (4), the series for \( \hat{F}_{q,>n}(x) \) is alternating and decreasing when \( 0 < x < q^{n+2} - 1 \). Thus

\[
\hat{F}_{q,>n}(q^j) > 0
\]

since \( j \leq n \) and the argument \( x = q^j \) is thus in the required range. Equation (5) now tells us both that

\[
\hat{F}_{q,\leq n}(q^j) < 0
\]

willy-nilly, and that

\[
\hat{F}_{q,\leq n}(c_k q^l) > 0
\]

is equivalent to equation (7). This establishes the first claim of this step.

Equation (5) also tells us that

\[
\hat{F}_{q,>n}(c_k q^l) > 0,
\]

since \( x = c_k q^l \) is also in the required range \( 0 < x < q^{n+2} - 1 \). Finally we confirm that

\[
\hat{F}_{q}(c_k q^l) > 0
\]
Step 2. The goal of this messy step is to reduce equation (7) to its asymptotic limit as $n, j \to \infty$ with $k$ fixed. We will use some preliminary relations for the Pochhammer symbol. The product relation
\[(a; q)_m = (a; q)\ell (aq^{\ell}; q)_{m-\ell}\] holds when $\ell$ is finite (but $m$ need not be). The inversion relation
\[(aq; q)_\ell = \frac{(-a)^\ell q^{(\ell+1)/2}}{(a-1; q)_{-\ell}}\] holds for all finite $\ell$; it follows from the trivial identity
\[1 - a = -a(1 - a^{-1}).\]

The inequality
\[(a; q)_{-\ell} \geq 1\] holds when $0 \leq \ell \leq \infty$ and $0 < a < q$ (with equality only when $\ell = 0$). Finally
\[ (a; q)_{\ell+1} - (a; q)_{\ell} = -aq^\ell (a; q)_\ell\] for all finite $\ell$. We will also use the elementary binomial identity
\[ \binom{\ell + m}{2} = \binom{\ell}{2} + \ell m + \binom{m}{2}.\]

The left side of equation (7) limits to a product of manageable factors:
\[
\tilde{F}_q(c_k q^j) = \frac{(-1)^{n+1}}{(c_k q^j; q)_{-\infty}} = \frac{(-1)^{n+1}(1 - c_k)(c_k q^j; q)_{j-1}}{(c_k q^j; q)_{-m}} \quad \text{(by eq. (10))}
\]
\[
= \frac{(-1)^{k+1}(1 - c_k)q^{(k+1)/2}c_k^{-j}}{(c_k q^j; q)_{-m}} \quad \text{(by eq. (11))}
\]
\[
\geq \frac{(-1)^{k+1}(1 - c_k)q^{(k+1)/2}c_k^{-j}}{(c_k q^j; q)_{-m}} \quad \text{(by eqs. (10), (12)).}
\]

Meanwhile, the right side of equation (7) essentially stabilizes as a power series in $q^{n-2}$:
\[
\tilde{F}_{q,n}(x) = (-1)^{n+1}\sum_{\ell = m}^{\infty} \frac{x^\ell}{(q^\ell; q)_\ell} = (-1)^{n+1}\sum_{\ell = m+1}^{\infty} \frac{x^\ell}{(q^\ell; q)_{\ell+n+1}} = x^{n+1}\sum_{\ell = 0}^{\infty} \frac{(-x)^\ell(1; q)_{-\ell-n-1}}{q^\ell q^{(n-2)/2}} \quad \text{(by eq. (11))}
\]
\[
= x^{n+1}\sum_{\ell = 0}^{\infty} \frac{(-xq^{n-2})^\ell(1; q)_{-\ell-n-1}}{q^\ell}. \quad \text{(by eq. (14))}
\]

To isolate the power series, let
\[ G_{q,n}(t) = \sum_{\ell=0}^{\infty} \frac{(-t)^\ell(1; q)_{-\ell-n-1}}{q^{\ell}}. \]

Then
\[ \tilde{F}_{q,n}(x) = \frac{x^{n+1}G_{q,n}(xq^{-n-2})}{q^{(n-2)/2}}. \]

To complete step 2, we will show that $G_{q,n}(x)$ is monotonic in $n$ and consolidate inequalities. Observe that
\[ G_{q,m+1}(t) - G_{q,m}(t) = \sum_{\ell=0}^{\infty} \frac{(-t)^\ell(1; q)_{-\ell-m-2}}{q^{\ell+\ell+m+2}} \]
by equation (13). This series is alternating decreasing when $0 < t < q - q^{-m-2}$ and $m > 0$, whence
\[ G_{q,m+1}(t) > G_{q,m}(t). \]

In particular,
\[ G_{q,\infty}(t) \defeq \sum_{\ell=0}^{\infty} \frac{(-t)^\ell(1; q)_{-\ell}}{q^{\ell}} > G_{q,n}(t) \]
when $0 < t < q - q^{-n-2}$, by induction on $m \geq n$. In particular
\[ \tilde{F}_{q,n}(c_k q^j) < \frac{(c_k q^j)^{n+1}}{q^{(n+2)/2}}G_{q,\infty}(c_k q^{-k-1}) \]

because $t = c_k q^{-k-1}$ is well within range given that $k \geq 1$, $c_k < q$, and $q \geq 4$. Thus we have estimates for both sides of equation (7). Upon close examination, they sacrifice less and less as $n \to \infty$. Combining the estimates,
\[ \frac{\tilde{F}_q(c_k q^j)}{\tilde{F}_{q,n}(c_k q^j)} \geq \frac{(-1)^{k+1}(1 - c_k)q^{(k+1)/2}c_k^{-j}}{(c_k q^j; q)_{-m}} \quad \text{(by eqs. (10), (12)).} \]

The right side is the limit $h_{k,n}$ defined previously. We have shown that the lemma follows from the inequality $h_{k,n} \geq 1$. It is also necessary if our use of the intermediate value theorem in step 1 is to work for all $n$.

Step 3. We will need the Pochhammer symbol estimate
\[ (a; q)_{-\ell} > 1 - \frac{a}{q - 1}. \]

We claim that it holds when $0 < a < q$ and $q > 1$. By equation (12)
\[ (a; q)_{-\ell} = (1 - aq^{-\ell})(a; q)_{-\ell} \geq (a; q)_{-\ell} - aq^{-\ell} \]
for any $\ell \geq 1$, with equality only when $\ell = 1$. Thus by induction,
\[ (a; q)^{-1}_{-\ell} \geq 1 - \sum_{m=1}^{\ell} aq^{-m}, \]
again with equality only when $\ell = 1$. Now we sum the geometric series in the limit $\ell \to \infty$.

We will also need the estimate

$$G_{q,n}(t) < (1; q)_\infty,$$  \hspace{1cm} (17)

which holds when $0 < \tau < 1$ because the power series for $G_{q,n}(t)$ is then alternating decreasing. We apply equations (16) and (17) to the right side of equation (15) to obtain

$$h_{q,k} > (1 - k^{t+1}) (1 - c_k) q^{(k+1)} c_k^{-k}$$

$$\cdot (1 - \frac{c_k}{q-1}) (1 - \frac{c_k^{-1}}{q-1}) (1 - \frac{1}{q-1})$$

defining $\tilde{h}_{q,k}$.

when $k \neq 2$. The inequality holds when $k = 2$ as well, but it is not adequate because our proof is a close shave in this case. So we will define $\tilde{h}_{q,2}$ differently. We refine equation (16),

$$(a; q)_\infty = (1 - \frac{a}{q}) (aq^{-1}; q)_\infty > (1 - \frac{a}{q}) (1 - \frac{a}{q(q-1)}),$$

to obtain

$$h_{q,2} > (a - 1) q^2 a^{-2} (1 - \frac{a}{q}) (1 - \frac{a}{q(q-1)})$$

$$\cdot (1 - \frac{1}{q}) (1 - \frac{1}{q(q-1)})$$

defining $\tilde{h}_{q,2}$.

If we apply equation (16) to the first occurrence of $c_k$ here, we learn that

$$\tilde{h}_{q,k} = 4 c_k^{1-k} (1 - \frac{c_k}{q-1}) (1 - \frac{c_k^{-1}}{q-1}) (1 - \frac{1}{q-1})$$

when $k > 2$, while

$$\tilde{h}_{q,1} = 2 c_1^{-2} (1 - \frac{c_1}{q-1}) (1 - \frac{c_1^{-1}}{q-1}) (1 - \frac{1}{q-1})$$

$$\tilde{h}_{q,2} = 4 c_2^{3} (1 - \frac{c_2}{q-1}) (1 - \frac{c_2^{-1}}{q-1}) (1 - \frac{1}{q}) (1 - \frac{1}{q(q-1)})$$

We claim that $\tilde{h}_{q,k} > 1$. It can be checked directly with symbolic algebra that

$$\tilde{h}_{q,1} = 2 q(q^2 - 4q + 2)(q^2 - 2q + 2) (q-1)^2 (q-2)^2$$

when $q \geq 4$ (indeed when $q \geq 3.718$), and that

$$\tilde{h}_{q,2} = 4 q^4 (q^2 - 4q - 1)(q^2 - 2q + 2)(q^2 - 2)(q^4 - 2q^3 - 4) (q-1)^2 (q^3 + 4q^3)$$

when $q \geq 4$ (indeed when $q \geq 3.974$). When $k > 2$, we claim that

$$c_k^{1-k} > \frac{99}{100}$$

$$1 - \frac{c_k}{q-1} \geq \frac{2}{3}$$

$$1 - \frac{c_k^{-1}}{q-1} > \frac{5}{8}$$

To check the first of these inequalities, we apply equation (16) and take the logarithm of both sides. We want to show that

$$1 + k \log (1 + (-k)^k q^{-(k+1)}) < \log \frac{100}{99},$$

We can assume that $k$ is even, so that $k \geq 4$. We can simplify using the elementary inequalities

$$\log (1 + x) < x \quad \log \frac{1}{1-x} > x.$$ 

Thus it suffices to show that

$$(1 + k) q^{-(k+1)} < \frac{1}{100}.$$ 

This holds easily assuming that $q \geq 4$ and $k \geq 4$. The other three inequalities also hold easily given that $q \geq 4$.

Thus when $k > 2$,

$$\tilde{h}_{q,k} \geq 4 \cdot \frac{99}{100} \cdot 5 \cdot \frac{5}{8} \cdot \frac{2}{3} = \frac{33}{32} > 1.$$ 

This completes step 3 of the proof.

We return to our continuing example with $q = 4$ and $n = 3$.

Recall that

$$F_{4,\leq 3}(x) = 1 - \frac{x}{3} + \frac{x^2}{45} - \frac{x^3}{2835},$$

and that its roots are

$$(r_{3,1}, r_{3,2}, r_{3,3}) \approx (3.997956, 16.80465, 42.19739).$$

The roots are alternately below and above 4, 16, and 64; the first root is very close to 4, the last one not so close. We can expect this pattern because, first,

$$F_4(4^n) = 0$$

for any $n \geq 1$, and second, the difference between $F_{4,\leq 3}(x)$ and $F_4(x)$,

$$F_{4,> 3}(x) = \frac{x^4}{722925} - \frac{x^5}{379552275} + \ldots,$$

is very small when $x$ is small. The series $F_{4,> 3}(x)$ is also dominated by its first term even when $x = 64$. Since $F_{4,> 3}(x)$ is positive when $x \leq 64$, the direction in which it displaces the first three roots of $F_4(x)$ depends only on the sign of the derivative $F'_4(x)$. (The sign of the derivative $F'(x)$ of any differentiable $f(x)$ must alternate between consecutive simple roots.)

Lemma 5 is a careful estimate of the displacement (indeed correct to within a universal constant factor).

To complete the proof of Theorem 1, we return to the definition of $c_k$ in the statement of Lemma 5 and the convention $k = n + 1 - j$. Recall that

$$a_{n,j} = \frac{1}{\sqrt{T_{n,j}}} \quad q = p^2.$$
We claim that by Lemma 5
\[ |p^{-j} - a_{n,j}| < p^{-j} \left| 1 - \frac{1}{\sqrt[2]{c_k}} \right| < \frac{2}{p^{n+3k-1}} = \frac{2}{p^{j+4k-2}}. \] (18)

The first inequality is equivalent to Lemma 5. The second inequality is far from sharp (equation (1) is closer to the truth), but it is convenient to prove Theorem 1. To establish it, we need the elementary inequalities
\[ 1 - \frac{1}{\sqrt[2]{1+x}} < \frac{x}{2} \quad \frac{x}{2} < 1 - \frac{3x^2}{4} \]
for \( x > 0 \). (They follow from the Taylor remainder theorem.) When \( k = 1 \), we want to show that
\[ \frac{1}{\sqrt[2]{1-2p^{-2}}} - 1 < 2p^{-2}. \]
This can be established by symbolic algebra for \( p \geq 2 \) (indeed \( p \geq 1.799 \)). When \( k > 1 \) is even,
\[ 1 - \frac{1}{\sqrt[2]{1+4p^{-k(k+1)}}} < 2p^{-k(k+1)} \leq 2p^{-2k}, \]
using that \( k \geq 2 \). When \( k > 1 \) is odd,
\[ \frac{1}{\sqrt[2]{1-4p^{-k(k+1)}}} - 1 < 2p^{-k(k+1)} + 12p^{-2k(k+1)} < 2p^{-1k(k+1)} < 2p^{-2k}, \]
using that \( p \geq 2 \) and \( k \geq 3 \).
Finally the theorem follows from equation (18) by a geometric sum:
\[(p - 1)p^n \sum_{j=1}^{n} |a_{n,j} - p^{-j}| < \sum_{k=1}^{n} \frac{2(p - 1)}{p^{2k + 1}} < \frac{2p}{p^2 + p + 1} < 1, \]
as desired.

3. FINAL REMARKS

The proof of Lemma 5 obtains somewhat more information about the roots \( \{r_{n,j}\} \) of \( F_q \leq n(x) \) than its statement. The proof shows that the sequence
\[ c_{n,k} = r_{n,j} q^{-j} = \frac{r_{n,n+1-k}}{q^{p^n+1-k}} \]
is monotonic in \( n \) for every fixed \( k \). We can also change the bound \( c_k \) to be the solution to the equation
\[ h_{q,k} = \frac{(-1)^{k+1} (1 - c_k) q^{k+2} c_k^{-1} - 1}{(c_k q^{-1} - q^{1/k})^2} = 1 \]
in the range \( q^{-1/2} < c_k < q^{1/2} \). (The equation is taken from equation (15).) Then for this new value of \( c_k \),
\[ \lim_{n \to \infty} c_{n,k} = c_k \]
and
\[ \lim_{k \to \infty} (-1)^{k+1} (1 - c_k) q^{k+1} = (1; q)^{3/2}. \]
The value of this limit has an interesting interpretation when \( q \) is a prime power that may or may not be related to the present work. Its reciprocal is the limiting probability that 3 independent, random \( n \times n \) matrices over the field \( F_q \) are non-singular.

That \( c_{n,k} \) is monotonic in \( n \) follows more directly from the interesting recurrence
\[ P_{q \leq n}(x) = (1 - \frac{x}{q}) P_{q \leq n-1}(x) + \frac{(-x)^n}{q^n (q; q)_n}. \]
This recurrence also shows that \( c_{n,k} \) is near \( c_{n-1,k} \). This was the basis of the author’s first attempted proof of a lemma like Lemma 5. Such an attempt might yet have merit.

Finally we conjecture that Theorem 1 together with its geometric interpretation has a broad generalization to the mixed-base case:

**Conjecture 6.** Let \( p_1, p_2, \ldots, p_n \geq 2 \) be a sequence of integers. Let \( X_1, X_2, \ldots, X_n \) be independent random variables such that \( X_k \) is uniformly distributed on the set
\[ \{ p_k - 1, p_k - 3, p_k - 5, \ldots, 1 - p_k \}. \]
Then there are unique constants
\[ a_1 > a_2 > \cdots > a_n > 0 \]
such that the first 2n moments of
\[ \bar{X} = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \]
agree with the first 2n moments of a random variable \( Y \) which is uniform on \([-1, 1]\). Moreover
\[ \sum_{j=1}^{n} (p_j - 1) |a_j - \prod_{k=1}^{j} p_k^{-1}| < \prod_{k=1}^{n} p_k^{-1}. \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The 6 values of \( \bar{X} \) marked on a ruler when \( n = 2 \) and \( (p_1, p_2) = (2, 3) \) or \( (p_1, p_2) = (3, 2) \).}
\end{figure}

For example, we can confirm Conjecture 6 when \( n = 2 \) and either \( (p_1, p_2) = (2, 3) \) or \( (p_1, p_2) = (3, 2) \). If we again let \( r_j = a_j^{-2} \), then in the first case,
\[ r_1 = 15 - 2 \sqrt{30} \quad r_2 = 20 + 2 \sqrt{30} \]
and
\[ \bar{X} \in \{ \sim \pm .497177 \pm .179737, \sim \pm .497177 \}. \]

In the second case,
\[ r_1 = 20 - 2\sqrt{30} \quad r_2 = 15 + 2\sqrt{30} \]
and
\[ \bar{X} \in \{ \sim \pm .332493 \pm .196288, \sim \pm .196288 \}. \]

The 6 values of \( \bar{X} \) in these two cases are shown in Figure 2.

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