bosonization theory of fermions interacting via a confining potential

Tai-Kai Ng

Department of Physics, Hong Kong University of Science and Technology,
Clear Water Bay Road, Kowloon, Hong Kong
(March 24, 2022)

Abstract

We study in this paper the properties of a gas of fermions interacting via a scalar potential $v(q) = 4\pi e^2/q^2$ for $q < \Lambda << k_F$ at dimensions larger than one, where $\Lambda$ is a high momentum cutoff and $k_F$ is the fermi wave vector. In particular, we shall consider the $e^2 \to \infty$ limit where the potential becomes confining. Within a bosonization approximation, effective Hamiltonians describing the low energy physics of the system are constructed, where we show that the system can be described as a fermi liquid formed by chargeless quasiparticles which has vanishing wavefunction overlap with the bare fermions in the system.

PACS Numbers: 71.27.+a, 74.25.-q, 11.15.-q
I. INTRODUCTION

In the last few years, there has been enormous interests in the study of $U(1)$ gauge theories of fermionic systems in dimensions higher than one, as a result of appearance of effective gauge theories in the $t - J$ type models \[^{[1]}\] and also in the studies of the $\nu = \frac{1}{2}$ Fractional Quantum Hall state \[^{[2,3]}\]. In particular, the behaviour of the systems in the confinement regime where charge excitations are "confined" are of special interests. In this paper we shall apply a bosonization procedure to study a system of charge $e$ spinless fermions interacting with scalar gauge field $\phi$ in dimensions higher than one, with effective long distance gauge field action $L_\phi = (\nabla^2 \phi)^2$. The problem is equivalent to fermions interacting via scalar potential $v(q) = 4\pi e^2 / q^2$ for $q < \Lambda$, where $\Lambda << k_F$ is a high momentum cutoff, $k_F$ is the fermi wave vector. In particular we are interested at the $e^2 \to \infty$ limit where the potential becomes confining. In this limit any non-uniform charge density fluctuations $\langle \rho(\vec{q}) \rangle \neq 0$ with any wave vector $q < \Lambda$ cost infinite electric field energy. As a result any physical state $|\psi\rangle$ which survives in this limit satisfies the constraint $\rho(\vec{q})|\psi\rangle = 0$, where $\rho(\vec{q})$ is the charge density operator.

In this paper we shall show that within a bosonization approximation the projection to the physical states $|\psi\rangle$ where density fluctuations are forbidden can be carried out in the $e^2 \to \infty$ limit, the bosonization approximation being a generalization of usual bosonization method in one dimension to dimensions higher than one \[^{[4-7]}\]. Within the approximation effective low energy Hamiltonians which describes the particle-hole as well as single-particle excitations in the system can be constructed. In particular, the single particle excitations are described by operators which commutes with charge operator $\rho(\vec{q})$, and has zero-overlap with the bare fermion operators in the system. The ground state of the system can be described as a fermi liquid of these new single particle operators and corresponding to a kind of ‘marginal’ fermi liquid in the original fermion description. The organization of our paper is as follows: in section two we shall outline our mathematical formulation of the bosonization procedure where the nature of the bosonization approximation will be explained. In section III we
shall study in detail the eigenstates and eigenvalues of the particle-hole excitation spectrum in the bosonization approximation for arbitrary values of coupling constant $e^2$. In particular we shall show how the projection to the physical Hilbert space where charged excitations are forbidden is achieved in the $e^2 \to \infty$ limit. In section IV we shall consider the single particle excitations where ‘physical’ single particle operators which commute with density operators are constructed and the equations of motion for these single-particle operators are derived. We shall show that the system can be described as a marginal fermi liquid of the original fermions. Our results will be summarized in section V where some further comments will be given.

II. MATHEMATICAL FORMULATION

We consider a gas of spinless fermions interacting via scalar potential $v(q) = 4\pi e^2/q^2$ in momentum space. The Hamiltonian of the system is,

$$H = \sum_k \epsilon_k f_k^+ f_k + \frac{1}{2L^d} \sum_{q \neq 0, q < \Lambda} v(q) \rho(q) \rho(-q),$$

where $\epsilon_k = (\tilde{k})^2 \Phi/2m$ and $f^+(f)$’s are fermion annihilation(creation) operators. We have set $\hbar = 1$ to simplify notation. $\rho(q) = \sum_k f_{k+q/2}^+ f_{k-q/2}$ is the density operator for the fermions. $L^d$ is the volume of the system. $\Lambda << k_F$ is a high momentum cutoff. Notice that the Hartree ($\tilde{q} = 0$) interaction energy does not appear in the Hamiltonian as in usual Coulomb gas problem. The difference between our model and usual Coulomb gas problem has to be emphasized here. In usual Coulomb gas problem the high momentum cutoff $\Lambda$ is taken to be infinity, or satisfies $\Lambda >> k_F$. In this limit a Wigner crystal is expected to be formed (in 3D) when $e^2$ becomes large because the potential energy term becomes dominating at length scale $\sim k_F^{-1}$. In our problem where $\Lambda << k_F$, the interaction is effective only at length scale $>>$ inter-particle spacing and formation of Wigner crystal is not possible since the fermions are essentially non-interacting at distance scale of inter-particle spacing. Instead the system should remain in a liquid state and a new treatment
to the problem is necessary. We note that bosonization method is a natural tool to attack the problem in this limit [3, 8]. The bosonization procedure can be formulate most easily by introducing the Wigner function operators $\rho_{\vec{k}}(\vec{q}) = f_{\vec{k}+\vec{q}/2}^+ f_{\vec{k}-\vec{q}/2}^-$. We shall work in the path-integral formulation where the Wigner operators are introduced in the action of the system at imaginary time through Langrange multiplier fields, 

$$S = \int_0^\beta d\tau \left[ \sum_{\vec{k}} f_{\vec{k}}^+(\tau) \left( \frac{\partial}{\partial \tau} + \epsilon_k - \mu \right) f_{\vec{k}}(\tau) - \sum_{\vec{k},\vec{q} \neq 0} i\lambda_{\vec{k}}(\vec{q},\tau)(\rho_{\vec{k}}(\vec{q},\tau) - f_{\vec{k}+\vec{q}/2}^+(\tau)f_{\vec{k}-\vec{q}/2}(\tau)) \right]$$

$$+ \frac{1}{2L^d} \sum_{\vec{q} \neq 0, q < L, \vec{k}, \vec{k}'} v(q) \rho_{\vec{k}}(\vec{q},\tau) \rho_{\vec{k}'}(-\vec{q},\tau),$$

where $\mu$ is the chemical potential. $\lambda_{\vec{k}}(\vec{q})$ are Lagrange multiplier fields introduced to enforce the constraint that the Wigner operators are given by $\rho_{\vec{k}}(\vec{q}) = f_{\vec{k}+\vec{q}/2}^+ f_{\vec{k}-\vec{q}/2}^-$. In particular, the original Hamiltonian (1) is recovered once the $\lambda_{\vec{k}}(\vec{q})$ field is integrated out.

Integrating out first the fermion fields $f(f^+)$'s we obtain an action in terms of $\rho_{\vec{k}}(\vec{q})$ and $\lambda_{\vec{k}}(\vec{q})$ fields,

$$S = \frac{1}{\beta} F_0 - Tr ln \left[ \hat{1} - \hat{G}_0 \hat{\lambda} \right] - \sum_{\vec{k},\vec{q} \neq 0, q < L} i\lambda_{\vec{k}}(\vec{q}) \rho_{\vec{k}}(\vec{q}) + \frac{1}{2L^d} \sum_{\vec{q} \neq 0, q < L, \vec{k}, \vec{k}'} v(q) \rho_{\vec{k}}(\vec{q},\tau) \rho_{\vec{k}'}(-\vec{q},\tau),$$

where $F_0$ is the free energy for an non-interacting Fermi gas. $\hat{G}_0$ and $\hat{\lambda}$ are infinite matrices in wave vector and frequency space, with matrix elements given by

$$\left[ \hat{G}_0 \right]_{k,k'} = \delta_{k,k'} g_0(k) \quad g_0(k) = \frac{1}{i\omega_n - \epsilon_k},$$

and

$$\left[ \hat{\lambda} \right]_{k,k'} = i \sqrt{\beta} \frac{\lambda_{\vec{k}+\vec{q}}(\vec{k'} - \vec{k}, i\omega_n - i\omega_{n'})}{\sqrt{2}} = \frac{i \sqrt{\beta}}{2} \lambda_{k+k'}(k - k'),$$

where $k = (\vec{k}, i\omega_n)$ and $\epsilon_k = \epsilon_k - \mu$. The $Tr ln [1 - G_0 \lambda]$ term can be expanded in a power series of $i\lambda_{\vec{k}}(\vec{q})$ field,

$$Tr ln \left[ \hat{1} - \hat{G}_0 \hat{\lambda} \right] = -Tr \left[ \hat{G}_0 \hat{\lambda} \right] - \frac{1}{2} Tr \left[ \hat{G}_0 \hat{\lambda} \right]^2 - \frac{1}{3} Tr \left[ \hat{G}_0 \hat{\lambda} \right]^3 + O(\hat{\lambda}^4).$$

Keeping terms to second order in $\hat{\lambda}$ (Gaussian approximation), we obtain
\[ Trln \left[ 1 - \hat{G}_0 \lambda \right] \sim \frac{1}{2 \beta} \sum_{k,q} g_0(k + \frac{q}{2}, g_0(k - \frac{q}{2}) \lambda_k(q) \lambda_k(-q). \] (6)

Notice that the first order term in \( \hat{\lambda} \) gives the usual Hartree self-energy and is excluded in the problem. The \( i\lambda_k(q) \) fields in Action (3) can be integrated out in Gaussian approximation, resulting in a quadratic action in terms of \( \rho_k(q) \) fields only. We obtain

\[ S_\rho = \frac{1}{2L^d} \sum_{k,k',\vec{q},i\omega_n} \left[ -\frac{1}{\chi_{0k}(\vec{q}, i\omega_n)} (L^d \delta_{k,k'}) + v(q) \right] \rho_k(\vec{q}, i\omega_n) \rho_{k'}(-\vec{q}, -i\omega_n), \] (7)

where

\[ \chi_{0k}(\vec{q}, i\omega_n) = \frac{1}{\beta} \sum_{\Delta \Omega_n} g_0(\vec{k} + \vec{q}/2, i\omega_n + i\Omega_n) g_0(\vec{k} - \vec{q}/2, i\Omega_n) = \frac{n_{\vec{k} - \vec{q}/2} - n_{\vec{k} + \vec{q}/2}}{i\omega_n - \frac{\vec{k} \cdot \vec{q}}{m}}, \] (8)

\[ n_{\vec{k}} = \theta(-\xi_{\vec{k}}) \] at zero temperature is the free fermion occupation number. \( S_\rho \) can be expressed in terms of canonical boson fields by introducing

\[ \rho_k(\vec{q}, i\omega_n) = \sqrt{|\Delta_k(\vec{q})|} \left( \theta(\Delta_k(\vec{q})) a_k^+(\vec{q}, i\omega_n) + \theta(-\Delta_k(\vec{q})) a_k(-\vec{q}, -i\omega_n) \right), \] (9a)

where \( \Delta_k(\vec{q}) = n_{\vec{k} - \vec{q}/2} - n_{\vec{k} + \vec{q}/2}. \) Correspondingly, we also have

\[ \rho_{k'}(-\vec{q}, -i\omega_n) = \sqrt{|\Delta_k(\vec{q})|} \left( \theta(\Delta_k(\vec{q})) a_k(\vec{q}, i\omega_n) + \theta(-\Delta_k(\vec{q})) a_k^+(-\vec{q}, -i\omega_n) \right). \] (9b)

Putting eqs. (9) back into \( S_\rho, \) we obtain after some straightforward manipulations,

\[ S_\rho = \frac{1}{2} \sum_{\vec{k},\vec{q},i\omega_n} (-i\omega_n + \frac{|\vec{k} \cdot \vec{q}|}{m}) a_k^+(\vec{q}, i\omega_n) a_k(\vec{q}, i\omega_n) + \frac{1}{2L^d} \sum_{\vec{k},\vec{k}',\vec{q},i\omega_n} v(q) \sqrt{|\Delta_k(\vec{q})\Delta_{k'}(\vec{q})|} \theta(\Delta_k(\vec{q})) \theta(\Delta_{k'}(\vec{q})) \times \left( a_k^+(\vec{q}, i\omega_n) a_k(\vec{q}, i\omega_n) + a_k(\vec{q}, i\omega_n) a_k^+(\vec{q}, i\omega_n) \right) \]

\[ + a_{-\vec{k}}(-\vec{q}, -i\omega_n) a_{\vec{k}}^+(\vec{q}, i\omega_n) + a_{-\vec{k}}(-\vec{q}, -i\omega_n) a_{\vec{k}}^+(-\vec{q}, -i\omega_n). \]

Notice that \( S_\rho \) is an action for interacting bosons described by boson fields \( a(a^+)\lambda(\vec{q}) \) satisfying usual boson commutation relations \( [a_k(\vec{q}), a_k^+(\vec{q}')] = \delta_{kk'} \delta_{\vec{q} \vec{q}'} \) and \( [a_a, a_\beta] = [a_a^+, a_\beta^+] = 0 \) and with kinetic energies \( |\vec{k} \cdot \vec{q}|/m \) and interaction term of form \( \sim v(q)(a^+ a + a^+ a^+ + a a + a a^+). \)

In this form the dynamics of the original fermion system is described completely in terms of boson fields(bosonized). Notice that we have so far restricted ourselves to the Gaussian approximation. Higher order interaction terms between bosons will appear in a cumulant...
expansion of the $\lambda_{\vec{k}}(\vec{q})$ fields [8]. We shall assume in the following that these high-order terms do not modify qualitatively the physics described by the Gaussian theory. Notice also that in the $\vec{q} \to 0$ limit, $\Delta_{\vec{k}}(\vec{q}) \to -\delta(\epsilon_{\vec{k}} - \mu)(\frac{\vec{k} \cdot \vec{q}}{m})$ and the usual ”tomographic” bosonization procedure based on subdivision of Fermi surface into disjoint patches at small $\vec{q}$ is recovered [3-7]. Our bosonization procedure can be viewed as a generalization of the tomographic bosonization method for small wave vector $\vec{q}$ to arbitrary values of $\vec{q} < \Lambda$.

To understand the nature of bosonization theory and Gaussian approximation, we first evaluate the free energy associated with $S_{\rho}$. Integrating out the $\rho_{\vec{k}}(\vec{q})$ or $a(\vec{a}^+)_{\vec{k}}(\vec{q})$ fields in $S_{\rho}$ and using the fact that $\text{Tr} [v(q)\hat{\chi}_0(\vec{q},i\omega)]_{\vec{k},\vec{k}'} = \chi_{0\vec{k}}(\vec{q},i\omega_n)$ and $\chi_{0}(\vec{q},i\omega_n)$ is the usual Lindhard function, we obtain

$$F_{\rho} = \frac{1}{2\beta} \sum_{\vec{k},\vec{q},i\omega_n} \ln(\chi_{0\vec{k}}(\vec{q},i\omega_n)) + \frac{1}{2\beta} \sum_{\vec{q},i\omega_n} \ln (1 - v(q)\chi_{0}(\vec{q},i\omega_n)), \quad (11)$$

where the first term in $F_{\rho}$ is coming from the kinetic energy of the $a(\vec{a}^+)$ bosons and the second term is the usual random-phase approximation (RPA) correction to free energy for interacting fermions. The presence of the RPA term in $F_{\rho}$ suggests that our present Gaussian approximation is essentially the same as RPA theory for interacting fermions, except that the extra 'bosonization' assumption in our theory which gives rise to the extra kinetic term in $F_{\rho}$. The 'bosonized' action $S_{\rho}$ assumes that the excitations in a fermion system can be fully represented by particle-hole pairs which are treated as independent bosons in the Gaussian approximation. Note that particle-hole pairs are not all independent in a fermion system because of Pauli exclusion principle. For example, two particle hole pairs $f_{\vec{k}}^+ f_{\vec{p}}$ and $f_{\vec{k}}^+ f_{\vec{p}'}$ are not independent excitations because they both involve creation of fermions in state $\vec{k}$. However, they are treated as independent bosons here as long as $\vec{p} \neq \vec{p}'$. Notice, however that in one dimension the situation becomes different when the fermion spectrum is linearized near the Fermi surface. In this limit the entire particle-hole excitation spectrum can be represented rigorously by bosons [9] and the Gaussian approximation becomes "exact". At
higher dimensions, it is also believed that the bosonization approximation is good as long as the interaction cutoff satisfies $\Lambda < k_F$, and as long as interaction with transverse gauge field is excluded.

Despite these approximations in the Gaussian theory, the bosonized form of the action has the advantage that within the approximation the full excitation spectrum and the ground and excited states wavefunctions of the system can be obtained easily. This allows us to study the properties of the system in great detail, as we shall see in the following.

III. PARTICLE-HOLE EXCITATION SPECTRUM

The eigenstates and eigenvalue spectrum described by the action $S_\rho$ can be obtained by diagonalizing the bosonized action using a generalized Bogoliubov transformation. We introduce for each wave vector $\vec{q}$ the Bogoliubov transformation

$$a_{\vec{k}}(\vec{q}) = \sum_{\vec{k}'} \left[ \alpha_{\vec{k}\vec{k}'} \gamma_{\vec{k}}(\vec{q}) + \beta_{\vec{k}\vec{k}'} \gamma^+_{\vec{k}}(-\vec{q}) \right],$$

$$a_{-\vec{k}}(-\vec{q}) = \sum_{\vec{k}'} \left[ \alpha_{\vec{k}\vec{k}'} \gamma_{-\vec{k}}(-\vec{q}) + \beta_{\vec{k}\vec{k}'} \gamma^+_{-\vec{k}}(\vec{q}) \right],$$

and with correspondingly,

$$\gamma_{\vec{k}}(\vec{q}) = \sum_{\vec{k}'} \left[ \alpha^*_{\vec{k}\vec{k}'} a_{\vec{k}'}(\vec{q}) - \beta_{\vec{k}\vec{k}'} a^+_{\vec{k}'}(-\vec{q}) \right],$$

$$\gamma_{-\vec{k}}(-\vec{q}) = \sum_{\vec{k}'} \left[ \alpha^*_{\vec{k}\vec{k}'} a_{-\vec{k}'}(-\vec{q}) - \beta_{\vec{k}\vec{k}'} a^+_{-\vec{k}'}(\vec{q}) \right],$$

where we assume that the $\gamma(\gamma^+)$ operators diagonalized the Hamiltonian, i.e.

$$H_\rho = \sum_{\vec{k}} E_{\vec{k}}(\vec{q}) \gamma^+_{\vec{k}}(\vec{q}) \gamma_{\vec{k}}(\vec{q}) + E_G,$$

where $E_{\vec{k}}(\vec{q})$ are the eigen-energies and $E_G$ is the ground-state energy of the system. Notice that a collective mode may appear in the system and is also included in the sum $\sum_{\vec{k}}$. The matrix elements $\alpha$ and $\beta$ satisfies the orthonormality condition

$$\sum_{\vec{k}'} \left[ \alpha_{\vec{k}\vec{k}'} \alpha^*_{\vec{k}\vec{k}'} - \beta_{\vec{k}\vec{k}'} \beta^*_{\vec{k}\vec{k}'} \right] = \delta_{\vec{k}\vec{k}'},$$

$$\sum_{\vec{k}'} \left[ \alpha_{\vec{k}\vec{k}'} \beta^*_{\vec{k}\vec{k}'} - \beta_{\vec{k}\vec{k}'} \alpha^*_{\vec{k}\vec{k}'} \right] = 0.$$
Writing down the equation of motions for $a_{\vec{k}}(\vec{q})$ in terms of $\gamma(\gamma^+_{\vec{k}}(\vec{q})$ in Ref. [14], we obtain the Bogoliubov equations

\[
\begin{align*}
(E_{\vec{k}'}(\vec{q}) - \frac{[\vec{k},\vec{q}]}{m})\alpha_{\vec{k}\vec{k}'} &= \frac{v(q)}{L^d} \sum_{\vec{k}''} \theta(\Delta_{\vec{k}''}(\vec{q})) \theta(\Delta_{\vec{k}''}(\vec{q}')) \sqrt{|\Delta_{\vec{k}''}(\vec{q})\Delta_{\vec{k}''}(\vec{q}')|} (\alpha_{\vec{k}''\vec{k}''} + \beta_{\vec{k}''\vec{k}''}^*), \\
(E_{\vec{k}'}(\vec{q}) + \frac{[\vec{k},\vec{q}]}{m})\beta_{\vec{k}\vec{k}'} &= -\frac{v(q)}{L^d} \sum_{\vec{k}''} \theta(\Delta_{\vec{k}''}(\vec{q})) \theta(\Delta_{\vec{k}''}(\vec{q}')) \sqrt{|\Delta_{\vec{k}''}(\vec{q})\Delta_{\vec{k}''}(\vec{q}')|} (\alpha_{\vec{k}''\vec{k}''} + \beta_{\vec{k}''\vec{k}''}^*).
\end{align*}
\]

Solving these equations we find that in general there exists two kinds of solutions: (i) particle-hole continuum, with $E_{\vec{k}}(\vec{q}) = |\vec{k},\vec{q}|/m$, and (ii) collective modes, with energy $E_0(\vec{q})$ outside the particle-hole continuum satisfying the RPA eigenvalue equation $1 - v(q)\chi_0(\vec{q}, E_0(\vec{q})) = 0$. Most of the detailed mathematics can be found in Ref [10]. We obtain finally,

\[
\begin{align*}
\alpha_{\vec{k}\vec{k}'} &= \delta_{\vec{k}\vec{k}'} + \frac{\theta(\Delta_{\vec{k}''}(\vec{q}))\theta(\Delta_{\vec{k}''}(\vec{q}')) \sqrt{|\Delta_{\vec{k}''}(\vec{q})\Delta_{\vec{k}''}(\vec{q}')|} v_{eff}(q, |\vec{k}',\vec{q}|/m)}{L^d(|\vec{k}',\vec{q}| - |\vec{k},\vec{q}|)}, \\
\beta_{\vec{k}\vec{k}'} &= \frac{-\theta(\Delta_{\vec{k}''}(\vec{q}))\theta(\Delta_{\vec{k}''}(\vec{q}')) \sqrt{|\Delta_{\vec{k}''}(\vec{q})\Delta_{\vec{k}''}(\vec{q}')|} v_{eff}(q, -|\vec{k}',\vec{q}|/m)}{L^d(|\vec{k}',\vec{q}| + |\vec{k},\vec{q}|)}
\end{align*}
\]

for the particle-hole continuum spectrum $\vec{k}'$, where $v_{eff}(q, \omega) = v(q)/(1 - v(q)\chi_0(q, \omega))$ is the RPA screened interaction, and

\[
\begin{align*}
\alpha_{\vec{k}0} &= \frac{1}{L^{d/2}} \frac{\theta(\Delta_{\vec{k}''}(\vec{q})) \sqrt{|\Delta_{\vec{k}''}(\vec{q})|}}{(E_0(\vec{q}) - |\vec{k},\vec{q}|) \left[- \frac{\partial\chi_0(q, \omega)}{\partial\omega}\right]_{\omega = E_0(\vec{q})}}, \\
\beta_{\vec{k}0} &= -\frac{1}{L^{d/2}} \frac{\theta(\Delta_{\vec{k}''}(\vec{q})) \sqrt{|\Delta_{\vec{k}''}(\vec{q})|}}{(E_0(\vec{q}) + |\vec{k},\vec{q}|) \left[- \frac{\partial\chi_0(q, \omega)}{\partial\omega}\right]_{\omega = E_0(\vec{q})}},
\end{align*}
\]

for the collective mode $E_0(\vec{q})$. Notice that in the boson representation, the collective mode exists as an non-perturbative effect due to interaction and cannot be obtained from analytic continuation of the non-interacting boson modes.

Next we examine the solutions of the bosonized Hamiltonian in the $e^2 \to \infty$ limit. First we consider the collective mode. Using the result that $\chi_0(\vec{q}, \omega) \to \frac{n_0 e^2}{m \omega^2}$ in the limit $\omega >> k_F q/m$, where $n_0$ is the fermion density [14], it is easy to see that in the limit $e^2 \to \infty$, the collective mode frequencies are given by $E_0(q) = \omega_P$, where $\omega_P = (\frac{4\pi n_0 e^2}{m})^{\frac{1}{2}}$ is the plasma
frequency. Notice that $\omega_p \to \infty$ as $e^2 \to \infty$, indicating that plasma oscillations are outside the physical spectrum in this limit.

Despite the vanishing of collective excitation in the physical spectrum, the particle-hole excitation spectrum with excitation energies $|k.q|/m$ survives in bosonization theory in the limit $e^2 \to \infty$. In this limit $v_{\text{eff}}(q, |k.q|/m) \to -1/\chi_0(q, |k.q|/m)$ and the coefficients $\alpha_{kk'}$’s and $\beta_{kk'}$’s remain regular, indicating that the particle-hole excitation spectrum is not qualitatively modified by effect of confinement. It is instructive to examine the charge fluctuations carried by the particle-hole excitations by examining the commutator $[\rho(q), \gamma^+_{k}(q')]$. In particular, the commutator will be zero if creating a particle-hole eigen-excitation does not introduce any charge fluctuations in the system, which is what we expect in the $e^2 \to \infty$ limit. Using Eqs. (9), (13) and (16a), and the usual boson commutation rules, it is straightforward to show that

$$
[r(q), \gamma^+_{k}(-q')] = \delta_{qq'} \theta(\Delta_k(q)) \sqrt{|\Delta_k(q)|} \times \frac{1}{1 - v(q)\chi_0(q, |k.q|/m)},
$$

(17)

and vanishes in the limit $e^2 \to \infty$.

Before ending this section let us examine our results obtained so far from bosonization theory. Within the Gaussian approximation, we find a RPA-like excitation spectrum with both collective modes and particle-hole excitations. As the coupling constant $e^2$ increases, the energy of the collective mode rises continuously to infinity whereas the particle-hole excitation spectrum remains unaltered. The charge fluctuations carried by the particle-hole excitations are projected out gradually as $e^2$ increases, resulting in chargeless particle-hole excitations at $q < \Lambda$ in the confinement limit $e^2 \to \infty$. Notice that within the Gaussian approximation, the confinement state analytically continues to the usual Fermi liquid state and there is no phase transition in between. The theory thus suggests that the confinement state of a gas of fermions is a Fermi liquid state, with however chargeless-quasi-particles constituting the Fermi liquid. It also suggests that this is a rather unusual Fermi liquid state, since bare fermions in the system carries charge $e$, and the quasi-particles must have vanishing overlap with bare fermions if they carry zero charge.
IV. SINGLE-PARTICLE PROPERTIES

In usual bosonization theory for one-dimensional systems, the single particle properties of the system can be determined once a rigorous representation of the single-particle operator in terms of density operators \( \rho_L(q) \) and \( \rho_R(q) \) are obtained [5]. In higher dimensions, this procedure becomes inadequate for two reasons: (1) The corresponding procedure requires that the bosonized representation of single-particle operator \( \psi_{\bar{b}}(\vec{r}) \) satisfies the commutation relations

\[
[\psi_{\bar{b}}(\vec{r}), \rho_{\vec{k}}(\vec{q})] = e^{-i(\vec{k} + \vec{q}/2).\vec{r}} \int d^{d'} r' e^{i(\vec{k} - \vec{q}/2).\vec{r}'} \psi_{\bar{b}}(\vec{r}')
\]

for all possible momenta \( \vec{k} \) and \( \vec{q} \). We have not been able to find a representation which satisfies this criteria [6–8], and even if we can find such a representation, the theory is still approximate because in dimensions higher than one, the boson representation using Wigner operators is not exact and violates the Pauli exclusion principle, at least in Gaussian approximation. (2) More importantly, unlike in one dimension where the elementary excitations are collective density waves, we have seen in last section that in dimensions higher than one the particle-hole excitation spectrum is fermi-liquid like, implying that fermionic quasi-particles exist in dimensions higher than one. It is thus important to construct directly the quasi-particle operators in this case.

To construct the quasi-particle operators we first consider the equation of motion of the bare fermion operator \( \psi(\vec{r}) = \frac{1}{L^d} \sum_{\vec{k}} e^{-i\vec{k}.\vec{r}} f_{\vec{k}} \) at imaginary time,

\[
\frac{\partial \psi(\vec{r})}{\partial \tau} = \frac{1}{2m} \nabla^2 \psi(\vec{r}) - \frac{1}{L^d} \sum_{\vec{q}} v(q) \rho_c(\vec{q}) e^{i\vec{q}.\vec{r}} \psi(\vec{r}) = \frac{1}{2m} \nabla^2 \psi(\vec{r}) - \frac{1}{L^d} \sum_{\vec{q}} [v(q) \rho_{ph}(\vec{q}) + v(q) \rho_c(\vec{q})] e^{i\vec{q}.\vec{r}} \psi(\vec{r}),
\]

where

\[
v(q) \rho_{ph}(q) = \sum_{\vec{k}} \sqrt{\Delta_{\vec{k}}(\vec{q})} \theta(\Delta_{\vec{k}}(\vec{q})) \left[ v_{eff}(q, -|\vec{k}.\vec{q}|/m) \gamma^+_{\vec{k}}(\vec{q}) + v_{eff}(q, |\vec{k}.\vec{q}|/m) \gamma^-_{\vec{k}}(-\vec{q}) \right],
\]

\[
v(q) \rho_c(\vec{q}) = L^{d/2} \left( -\frac{\partial \chi_0(q, \omega)}{\partial \omega} \right)^{-\frac{1}{2}}_{\omega = E_0(\vec{q})} \left[ \gamma^+_{\vec{q}} + \gamma^-_{\vec{q}} \right],
\]

10
where \( v(q)\rho_{ph}(\vec{q}) \sim \gamma_{k}^{\pm}(\vec{q}) + \gamma_{-k}(\vec{q}) \) describes the coupling of the particle-hole excitations to the fermion operator, and \( v(q)\rho_{c}(\vec{q}) \sim \gamma_{0}^{\pm}(\vec{q}) + \gamma_{0}(\vec{q}) \) describes coupling of the collective excitations (plasmons) to the fermion operator. In the \( e^2 \to \infty \) limit, the interaction between fermions and particle-hole excitations are regular and finite, and the strong confinement effect shows up only in the interaction between fermions and collective mode excitations (plasmons). The effect of plasmons on the one-particle properties can be estimated perturbatively by evaluating the plasmon contribution to fermion self-energy \( \Sigma \) and renormalization factor \( z \) to second order using Eqs. (18) and (19). We find that \( \Sigma \sim \int_{L^{-1}}^{A} d^{d}qv(q) \), and \( z \sim \frac{\partial \Sigma}{\partial \omega} \sim \Sigma/\omega_{p} \). Notice that both \( \Sigma \) and \( z \) goes to infinity as \( e^2 \to \infty \). Furthermore, the integrals carry also infrared divergence at dimensions \( d \leq 2 \) for any finite \( e^2 \). The divergence of \( z \) at \( d = 2 \) for finite \( e^2 \) was interpreted as signature of a marginal fermi liquid [12].

The divergence in single-particle self-energy indicates that the bare fermions is not a good starting point for constructing quasi-particle operators. To find a better starting point we first look for a canonical transformation for the single-particle operator which diagonalize the interaction term between fermions and plasmons. The kinetic energy of fermions and interaction with particle-hole excitations will be treated afterward. We obtain [13]

\[
\psi_{Q}(\vec{r}) = e^{\phi(\vec{r})}\psi(\vec{r}),
\]

(20a)

where

\[
\phi(\vec{r}) = \frac{1}{L^{d/2}} \sum_{\vec{q}} \frac{e^{i\vec{q}.\vec{r}}}{E_{0}(\vec{q})} \left[ -\frac{\partial \chi_{0}(q,\omega)}{\partial \omega} \right]^{1/2} \left( \gamma_{0}^{+}(\vec{q}) - \gamma_{0}(-\vec{q}) \right),
\]

(20b)

which represents a fermion operator "dressed" by plasmon modes. It is straightforward to obtain the equation of motion of the "dressed" fermion \( \psi_{Q} \),

\[
\frac{\partial \psi_{Q}(\vec{r})}{\partial \tau} = \frac{1}{2m} \left( \nabla^{2}\psi_{Q}(\vec{r}) - [\nabla^{2}\phi(\vec{r}) - (\nabla\phi(\vec{r}))^{2}]\psi_{Q}(\vec{r}) - 2\nabla\phi(\vec{r}).\nabla\psi_{Q}(\vec{r}) \right) - \frac{1}{L^{d}} \sum_{\vec{r}} v(q)\rho_{ph}(\vec{q})e^{i\vec{q}.\vec{r}}\psi_{Q}(\vec{r}),
\]

(21)

where we have used the results \([H, \gamma_{0}(\vec{q})] = -E_{0}(\vec{q})\gamma_{0}(\vec{q}) \) and \([\gamma_{k}(\vec{q}), \gamma_{0}^{+}(\vec{q})] = 0 \), etc. for \( \vec{k} \neq 0 \) and the usual commutator between bare fermions and density operators to derive
the above equation. We have also neglected constant energy terms coming from normal ordering of operators in Eq. (21). It is clear from Eq. (21) that the direct coupling between fermions and plasmons is eliminated in the equation of motion of $\psi_Q(\vec{r})$. However, interaction between fermions and particle-hole excitations remains in the equation of motion. Moreover, an indirect coupling to plasmon is also generated from the "dressed" fermion kinetic energy term as is in the similar "small polaron" problem [13]. It is easy to see by direct power counting of $e^2$ in the $\phi(\vec{r})$ field that the coupling of the "dressed" fermion to plasmons through kinetic energy term is much weaker than the original fermion-plasmon coupling.

In particular, we find that the self-energy correction of "dressed" fermions from $\phi(\vec{r})$ fields remains finite in the limit $e^2 \to \infty$. It is also straightforward to show that the infra-red divergence in fermion self-energy at two dimensions is removed for the dressed fermions because of the much weaker coupling to plasmons.

These results suggest that the dressed fermion operators $\psi_Q(\vec{r})$'s constitute a valid starting point to construct chargeless quasi-particles in the $e^2 \to \infty$ limit. The other interaction effects can be treated perturbatively. To show that this is indeed the case we first consider the commutation relation between charge and dressed fermion operators. It is straightforward to show that

$$[\rho(\vec{q}), \psi_Q(\vec{r})] = \left( \frac{2\chi_0(q, E_0(\vec{q}))}{E_0(\vec{q}) - \left. \frac{\partial \chi_0(q, \omega)}{\partial \omega} \right|_{\omega = E_0(\vec{q})}} - 1 \right) e^{-i\vec{q}.\vec{r}} \psi_Q(\vec{r}), \quad (22)$$

which vanishes in the limit $e^2 \to \infty$, when $E_0(\vec{q}) \to \omega_P \to \infty$, indicating that the dressed single particle operators $\psi_Q(\vec{r})$'s are indeed 'chargeless' in the $e^2 \to \infty$ limit. To show that $\psi_Q(\vec{r})$ and $\psi_Q(\vec{r'})$ defines independent quasi-particles when $\vec{r} \neq \vec{r'}$ we check the commutation relation between the dressed fermion operators themselves. We obtain

$$[\psi_Q(\vec{r}), \psi_Q^+(\vec{r'})] \sim \frac{1}{n_0(\pi |\vec{r} - \vec{r'}|)^{d-1}} \hat{O}(\vec{r}, \vec{r'}),$$

in the limits $e^2 \to \infty$ and $|\vec{r} - \vec{r'}| \to \infty$, where

$$\hat{O} = \left( \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|} \right) \left[ \psi^+(\vec{r'}) e^{\phi(\vec{r})} (\nabla \psi(\vec{r})) e^{\phi^+(\vec{r'})} - (\nabla \psi^+(\vec{r'})) e^{\phi(\vec{r})} \psi(\vec{r}) e^{\phi^+(\vec{r'})} \right].$$
The vanishing of the commutator between different $\psi_Q$ operators separated by large distances at dimensions larger than one indicates that they can be used to construct independent quasi-particle operators when describing the dynamics of these systems at long distance. Notice that in one dimension, such a construction is not possible because of the long-rangeness of the commutation relation. In fact, the only fermionic operators which commute with density operators $\rho(\vec{q})$’s are the ladder operators which raise or lower the number of particles in the system by one. There are only two independent ladder operators in the system, which change the number of left-going and right-going fermions and cannot be used to construct local quasi-particle excitations.

Finally, it is also easy to show that

$$\langle \psi_Q^+(\vec{r})\psi_Q(\vec{r}) \rangle = \langle \psi^+(\vec{r})\psi(\vec{r}) \rangle$$

The fact that the densities of bare fermions and dressed fermions are the same implies that the fermi surface volume of the dressed fermions is exactly the same as that of the bare fermions. Assuming that the dressed fermions form a free fermi gas, we also obtain

$$\langle \psi_Q^+(\vec{r})\psi(\vec{r}) \rangle \approx e^{-\left(\frac{ze^2}{\Lambda} \right)^{1/2} (\int_{k_F-1}^{\Lambda} \frac{d^d q}{q^d})} \langle \psi^+(\vec{r})\psi(\vec{r}) \rangle,$$

to leading order in $e^2$, which vanishes in the $e \to \infty$ limit, indicating that the bare fermions and dressed fermions have zero wavefunction overlap, as is expected on physical ground. A similar calculation shows that the bare-fermion occupation number $n(\vec{k})$ has no discontinuity across fermi surface in the $e^2 \to \infty$ limit, in agreement with marginal fermi liquid picture.

V. SUMMARY

Using a bosonization approximation we studied in this paper a gas of fermions interacting via scalar potential $v(q) = 4\pi e^2/q^2$ for $q < \Lambda << k_F$. In particular we consider the $e^2 \to \infty$ limit where the potential becomes confining. We note that because of the low momentum cutoff $\Lambda << k_F$ in our model, a crystal state is not expected to be formed and a new
treatment of the problem is necessary. Within a Gaussian approximation we find that the particle-hole excitation spectrum of the system is always fermi-liquid like, with the charge carried by the particle-hole excitation vanishing continuously in the $e^2 \to \infty$ limit. Based on this result chargeless fermionic operators are construct which, we believe can be used as the starting point for constructing quasi-particles in the system. The system can be considered as a 'marginal' fermi liquid where the chargeless quasi-particles have vanishing wavefunction overlap with bare fermions in the system.

A major assumption we have made in our theory is that the Gaussian approximation for the Wigner bosons describes at least qualitatively correct the particle-hole excitation spectrum of the system. The Gaussian approximation is believed to be valid as long as the interaction potential is restricted to region of small momentum transfer $q < \Lambda \ll k_F$ \cite{6–8}. Notice that the approximation can be improved systematically via a cummulant expansion \cite{8}. We find that to lowest order of the cummulant expansion there is no extra divergence introduced into the particle-hole excitation spectrum \cite{14}. Notice also that we have considered here only the effect of longitudinal confining gauge fields in gas of fermions. Naively we expect that similar physics will be found in the presence of transverse gauge field. In the $e^2 \to \infty$ limit, the chargeless quasi-particles we have constructed will decouple from the gauge fields and form a (marginal) fermi liquid. However, it is easy to show that the $e^{i(r)}$ operator we have constructed is not able to "screen out" the transverse gauge field because it is constructed from density fluctuations which couples to longitudinal gauge field only. The transverse gauge fields requires rather different treatment and we shall discuss it in a separate paper.

The author thanks Prof. Zhao-bin Su for many helpful questions and comments. This work is supported by HKUGC through RGC grant HKUST6124/98P.
REFERENCES

[1] N. Nagaosa and P.A. Lee, Phys. Rev. Lett. 60, 2450(1990); X.G. Wen and P.A. Lee, Phys. Rev. Lett. 76, 503(1996).

[2] B.I. Halperin, P.A. Lee and N. Read, Phys. Rev. B 47, 7312 (1993); Y.B. Kim, P.A. Lee and X.G. Wen, Phys. Rev. B 52, 17275 (1995).

[3] , see A. Stern, B.I. Halperin, F. von Oppen and S.H. Simon, cond-matt/9812135 for a review.

[4] S. Tomonaga, Prog. Theor. Phys. 5, 544(1950); D.C. Mattis and E.H. Lieb, J. Math. Phys. 6, 304(1965).

[5] see for example, F.D.M. Haldane, J. Phys. C 14, 2585(1981).

[6] F.D.M. Haldane, Helv. Phys. Acta 65, 152(1992).

[7] A. Houghton and J.B. Marston, Phys. Rev. B 48, 7790(1993); A.H. Castro Neto and E. Fradkin, Phys. Rev. Lett. 72, 1393(1994).

[8] P. Kopietz, J. Hermisson and K. Schönhammer, Phys. Rev. B 52, 10877(1995).

[9] D.V. Khveshchenko and P.C.E. Stamp, Phys. Rev. Lett. 71, 2118 (1993); see also B.L. Altshuler, L.B. Ioffe and A.J. Millis, Phys. Rev. B 50, 14048 (1994).

[10] A.H. Castro Neto and E. Fradkin, Phys. Rev. B 49, 10877(1994).

[11] see for example, G.D. Mahan, in Many-Particle Physics, (Plenum Press, New York and London (1990)).

[12] P.A. Bare and X.G. Wen, Phys. Rev. B 48, 8636(1993).

[13] notice that a similar result is also obtained in the case of small polaron problem; see ref.[8] for example.

[14] T.K. Ng, unpublished.