Operational quantification of continuous variable correlations

Carles Rodó,1 Gerardo Adesso,1,2 and Anna Sanpera1,3

1Grupo de Física Teórica, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain.  
2Dipartimento di Fisica “E. R. Caianiello”, Università degli Studi di Salerno, Via S. Allende, 84081 Baronissi (SA), Italy.  
3ICREA, Institució Catalana de Recerca i Estudis Avancats, Barcelona, 08021 Spain.  

(Dated: November 12, 2007)

We quantify correlations (quantum and/or classical) between two continuous variable modes in terms of how many correlated bits can be extracted by measuring the sign of two local quadratures. On Gaussian states, such ‘bit quadrature correlations’ majorize entanglement, reducing to an entanglement monotone for pure states. For non-Gaussian states, such as photonic Bell states, ideal and real de-Gaussified photon-subtracted states, and mixtures of pure Gaussian states, the bit correlations are shown to be a **monotonic** function of the negativity. This yields a feasible, operational way to quantitatively measure non-Gaussian entanglement in current experiments by means of direct homodyne detection, without a full tomographical reconstruction of the Wigner function.

Quantum information with continuous variables (CVs), relying on quadrature entanglement as a resource, has witnessed rapid and exciting progresses recently, also thanks to the high degree of experimental control achievable in the context of quantum optics [1]. While Gaussian states (coherent, squeezed, and thermal states) have been originally the preferred resources for both theoretical and practical implementations, a new frontier emerges with non-Gaussian states (Fock states, Schrödinger’s cats, ...). The latter can be highly non-classical, possess in general more entanglement than Gaussian states [2], and are useful to overcome some limitations of the Gaussian framework such as entanglement distillation [3] and universal quantum computation [4]. Therefore, it is of central relevance to provide proper ways to quantify non-Gaussian entanglement in a way which is experimentally accessible.

At a fundamental level, the difficulty in the investigation of entanglement – quantum correlations – can be traced back to the subtle task of distinguishing it from classical correlations [5]. Correlations can be regarded as classical if they can be induced onto the subsystems solely by local operations and classical communication, necessarily resulting in a mixed state. On the other hand, if a pure quantum state displays correlations between the subsystems, they are of genuinely quantum nature (entanglement). We adopt here a pragmatic approach: if two systems are **in toto** correlated, then this correlation has to be retrieved between the outcomes of some local measurements performed on them. We, therefore, investigate **quadrature correlations** in CV states. We are also motivated by the experimental adequacy: field quadratures can be efficiently measured by homodyne detection, without the need for complete state tomography. Specifically, in this paper, we study optimal correlations in bit strings obtained by digitalizing the outcomes of joint quadrature measurements on a two-mode CV system. First, we apply our procedure to Gaussian states (GS), finding that bit quadrature correlations provide a clear-cut quantification of the total correlations between the two modes. They are monotonic with the entanglement on pure states, and can be arbitrarily large on mixed states, the latter possibly containing arbitrarily strong additional classical correlations. We then address non-Gaussian states (NGS), for which the exact detection of entanglement generally involves measurements of high-order moments [6]. The underlying idea is that for NGS obtained by de-Gaussifying an initial pure GS and/or by mixing it with a totally uncorrelated state, our measure based entirely on second moments is still expected to be a (quantitative) witness of the quantum part of correlations only, i.e. entanglement. We show that this is indeed the case for relevant NGS including photon-subtracted states, photonic Bell states, and mixtures of Gaussian states. Notably, the complete entanglement picture in a recently demonstrated coherent single-photon-subtracted state [7] is precisely reproduced here in terms of quadrature correlations only. Our results render non-Gaussian entanglement significantly more accessible in a direct, practical fashion.

**Quadrature measurements and bit correlations.**— We consider a bipartite CV system of two bosonic modes, A and B, described by an infinite-dimensional Hilbert space. The probability distribution associated to the measurement of the rotated position quadrature $\hat{X}_A(\theta)$ in mode A with outcome $x_A^0$ and uncertainty $\sigma$ is given by $P_A(x_A^0) = \text{Tr} \left[ \rho_A \hat{R}_A(\theta) \hat{\sigma}(x_A) \hat{R}_A(\theta)^{\dagger} \right] = \text{Tr} \left[ \hat{R}_A(\theta)^{\dagger} \rho_A \hat{R}_A(\theta) \hat{\sigma}(x_A) \right]$, where $\hat{\sigma}(x_A)$ is a single-mode Gaussian (squeezed) state with first moments $\{ x_A, 0 \}$ and covariance matrix $\text{diag}(\sigma^2, 1/\sigma^2)$. Here $\hat{R}_A(\theta)$ is a unitary operator describing a rotation of $\theta$ on mode A, corresponding to a symplectic transformation given by $R_A(\theta) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) [8]$. Thus, one can either measure the rotated quadrature on the state (passive view) or antirotate the state and measure the unrotated quadrature (active view). Similarly we define $P_B(x_B^0)$ for mode B. The probability distribution associated to a joint measurement of the rotated quadratures $\hat{X}_A(\theta)$ and $\hat{X}_B(\varphi)$, is given by $P_{AB}(x_A^0, x_B^0) = \text{Tr} \left[ \rho_{AB} \hat{R}_A(\theta) \otimes \hat{R}_B(\varphi) \hat{\sigma}(x_A) \otimes \hat{\sigma}(x_B) \right] \hat{R}_A(\theta)^{\dagger} \otimes \hat{R}_B(\varphi)^{\dagger}$.  

We digitalize the obtained outputs by assigning the bits $+(-)$ to the positive (negative) values of the measured quadrature. This digitalization transforms each joint quadrature measurement into a pair of classical bits. A string of such correlated bits can be used e.g. to distill a quantum
key 6, 10. Let us adopt a compact notation by denoting (at given angles \( \theta, \phi \)) \( P_{AB}^{\theta, \phi} = P_{AA}(\pm x_A^\theta, \pm x_B^\phi) \), and \( P_{AB}^{\theta, \phi} = P_{AB}(\pm x_A^\theta, \pm x_B^\phi) \). The conditional probability that the bits of the corresponding two modes coincide is given by \( P_{AB}^{\theta, \phi} = (P_{AB}^{\theta, \phi} + P_{AB}^{\theta, \phi}) / \sum_{\{a=\pm, b=\pm\}} P_{AB}^{\theta, \phi} \). Correspondingly, the conditional probability that they differ is \( P_{AB}^\theta = (P_{AB}^{\theta, \phi} + P_{AB}^{\theta, \phi}) / \sum_{\{a=\pm, b=\pm\}} P_{AB}^{\theta, \phi} \). Trivially, \( P_{AB}^\theta + P_{AB}^\phi = 1 \). If \( P_{AB}^\theta > P_{AB}^\phi \) the measurement outcomes display correlations, otherwise they display anticorrelations. Notice that, if the two modes are completely uncorrelated, \( P_{AB}^\theta = P_{AB}^\phi = 1/2 \). For convenience, we normalize the strength of bit correlations as

\[
B(|x_A^\theta|, |x_B^\phi|) = 2|P_{AB} - 1/2| = |P_{AB} - P_{AB}^\phi | \tag{1}
\]

so that for a completely uncorrelated state \( B(|x_A^\theta|, |x_B^\phi|) = 0 \). The interpretation of Eq. (1) in terms of correlations is meaningful if a fairness condition is satisfied: on each single mode, the marginal probabilities associated to the outcomes “+” or “−” must be the same: \( P_{AB}^\theta = P_{AB}^\phi \). For a two-mode CV system, whose state is described by a Wigner function \( W \), we define the ‘bit quadrature correlations’ \( Q \) as the average probability of obtaining a pair of classically correlated bits (in the limit of zero uncertainty) optimized over all possible choices of local quadratures

\[
Q(\bar{\rho}) = \sup_{\theta, \phi} \int dx_A dx_B W(x_A^\theta, x_B^\phi) \lim_{\sigma \to 0} B^\sigma(|x_A^\theta|, |x_B^\phi|), \tag{2}
\]

where \( W(x_A^\theta, x_B^\phi) = \int dp_A dp_B W(\theta_A, \rho_A, x_A^\theta, x_B^\phi) \) is the marginal Wigner distribution of the (rotated) positions, and \( \{ x_A^\theta, x_B^\phi \} = (R(\theta) \otimes R(\phi)) \{ x_A, x_B, p_A, p_B \} \). After some algebra, we can rewrite Eq. (2) as

\[
Q(\bar{\rho}) = \sup_{\theta, \phi} \left| \frac{\rho_{A,B}^{\theta,\phi}(\bar{\rho})}{L_{A,B}^{\theta,\phi}(\bar{\rho})} \right| \tag{2}
\]

\[
= \int dx_A dx_B \text{sgn}(x_A^\theta, x_B^\phi) W(x_A^\theta, x_B^\phi) \text{sgn}(x_A^\theta, x_B^\phi) \tag{2}
\]

is the ‘sign-binned’ quadrature correlation function, which has been employed e.g. in proposed tests of Bell inequalities violation for CV systems [11]. While this form is more suitable for an analytic evaluation on specific examples, the definition Eq. (2) is useful to prove the following general properties of \( Q(\bar{\rho}) \):

**Lemma 1 (Normalization):** \( 0 \leq Q(\bar{\rho}) \leq 1 \).

**Proof.** It follows from the definition of \( Q(\bar{\rho}) \), as both \( B \) and the marginal Wigner distribution range between 0 and 1. □

**Lemma 2 (Zero on product states):** \( Q(\bar{\rho}_A \otimes \bar{\rho}_B) = 0 \).

**Proof.** For a product state the probabilities factorize i.e. \( P_{AB}^{\theta,\phi} = P_A^{\theta} P_B^{\phi} \) and so \( P_{AB}^\theta = P_{AB}^\phi \), where we have used the fairness condition. Namely \( B = 0 \), hence the integral in Eq. (2) trivially vanishes. □

**Lemma 3 (Local symplectic invariance):** Let \( U_{A,B} \) be a unitary operator amounting to a single-mode symplectic operation \( S_{A,B} \) on the local phase space of mode \( A,B \) [2]. Then \( Q(U_{A,B} \otimes U_{B,A}) = Q(\bar{\rho}) \).

**Proof.** Any single-mode symplectic operation \( S \) can be decomposed in terms of local rotations and local squeezings (Euler decomposition). By definition Eq. (2) is invariant under local rotations, so we need to show that local squeezings, described by symplectic matrices of the form \( Z_x = \text{diag}(1/r, r) \), also leave \( Q(\bar{\rho}) \) invariant. Adopting the passive view, the action of local squeezings on the covariance matrix of each Gaussian state \( \sigma(x_A, x_B) \) is irrelevant, as we take eventually the limit \( \sigma \to 0 \). The first moments are transformed as \( d_{A,B} \mapsto Z_{x_A}^{-1} d_{A,B} \), so that \( B_{A,B}^\sigma(\{x_A, x_B\}) \mapsto B_{A,B}^\sigma(\{sx_A, tx_B\}) \). On the other hand, the Wigner distribution is transformed as \( W(\xi) \mapsto W((Z_x^s \otimes Z_x^t) \xi) \). Summing up, local squeezings transform \( \xi = \{ x_A, p_A, x_B, p_B \} \) into \( \xi_{st} = \{ sx_A, p_A/s, tx_B, p_B/t \} \). As Eq. (2) involves integration over the four phase space variables \( d^4\xi \), we change variables noting that \( d^4\xi = d^4\xi_{st} \), to conclude the proof. □

It follows from Lemmas (1–2) that if \( Q > 0 \), then the state necessarily possesses correlations between the two modes. Lemma 3, moreover, suggests that \( Q \) embodies not only a qualitative criterion, but might be interpreted as \( \text{bona fide} \) operational quantifier of CV correlations. We will now show that this is the case for various important classes of states.

**Gaussian states.**—Even though entanglement of GS is already efficiently accessible via their covariance matrix [1], we use such states as ‘test-beds’ for understanding the role of \( Q \) in discriminating CV correlations. The covariance matrix \( \gamma \) of any two-mode GS \( \rho \) can be written in standard form as:

\[
\gamma = \left( \begin{array}{cc}
\alpha & \delta \\
\delta & \beta
\end{array} \right), \quad \alpha = \lambda_{a} \beta_{2}, \quad \beta = \lambda_{b} \beta_{2}, \quad \delta = \text{diag}(c_{x}, -c_{p}),
\]

where, without loss of generality, we adopt the convention \( c_{x} \geq |c_{p}| \). The covariance matrix \( \gamma \) describes a physical state if \( \lambda_{a,b} \geq 1 \) and \( \Delta < 1 + \Delta \gamma \), with \( \Delta = \Delta \alpha + \Delta \beta + 2 \Delta \gamma \). The **negativity** [13], quantifying entanglement between the two modes, reads \( N(\bar{\rho}) = \max \{ 0, (1 - \nu^2)/(2\nu) \} \), where \( \nu^2 = [\Delta - (\Delta^2 - 4 \Delta \gamma)^{1/2}]/2 \) with \( \Delta = \Delta - 4 \Delta \delta \). For two-mode GS, Eq. (2) evaluates to:

\[
Q(\bar{\rho}) = (2/\pi) \arctan(c_{x}/\sqrt{\lambda_{a} \lambda_{b} - c_{x}^2}), \tag{3}
\]

where the optimal quadratures are the standard unrotated positions (\( \theta = \phi = 0 \)). First, we notice that \( Q = 0 \Leftrightarrow \rho \) describes a product state: for GS, \( Q > 0 \) is then necessary and sufficient for the presence of correlations. Second, we...
observe that for pure GS [reducible, up to local unitary operations, to the two-mode squeezed states $\hat{\rho}_s \equiv |\phi_s^\pm\rangle\langle\phi_s^\pm|$ characterized by $\lambda_{ab} = \cosh(2r)$ and $c_r = cp = \sinh(2r)$], Eq. (3) yields a monotonic function of the negativity (see Fig. 1): $Q$ is thus, as expected, an operational entanglement measure for pure two-mode GS. Third, we find that for mixed states $Q$ majorizes entanglement. Given a mixed GS $\hat{\rho}_N$ with negativity $N$, it is straightforward to see that $Q(\hat{\rho}_N)$, [Eq. (3)], is always greater than $Q(|\psi_N^\pm\rangle\langle\psi_N^\pm|) = (2/\pi) \arctan[(\nu^2 - 1)/(2\nu)]$, with $|\psi_N^\pm\rangle$ being a pure two-mode squeezed state with the same negativity $N$. Hence $Q$ quantifies total correlations, and the difference $Q(\hat{\rho}_N) - Q(|\psi_N^\pm\rangle\langle\psi_N^\pm|)$ (where the first term is due to total correlations and the second to quantum ones) can be naturally regarded as an operational measure of classical correlations [15]. We have evaluated $Q$ on random two-mode GS as a function of their entanglement, conveniently scaled to $2N/(1 + 2N)$, as shown in Fig. 1. Note that for any entanglement content there exist maximally correlated GS with $Q = 1$, and also that separable mixed GS can achieve an arbitrary $Q$ from 0 to 1, their correlations being only classical.

**Non-Gaussian states.—** Let us now turn our attention towards NGS, whose entanglement and correlations are, in general, encoded in higher moments as well. We focus on the most relevant NGS recently discussed in the literature and some-
squeezing $r$ which are kept free. We then evaluate Eq. (2) as a function of $r$ for different values of $R$. Unlike the previous cases, optimal correlations in the state $\hat{\rho}_{\text{exp}}$ occur between momentum operators ($\theta = \varphi = \pi/2$). Also for this realistic mixed case, the correlation measure $Q$ reproduces precisely the behaviour of the negativity, as obtained in [7] after full Wigner tomography of the produced state $\hat{\rho}_{\text{exp}}$. In particular, the negativity ($q$) increases with the squeezing $r$, and decreases with $R$. Below a threshold squeezing which ranges around $\sim 3$ dB, the NGS exhibits more entanglement (larger $Q$) than the original two-mode squeezed state. Our results depicted in Fig. 4 compare extremely well to the experimental results (Fig. 6 of [2]) where the negativity is plotted as a function of $r$ for different $R$'s.

Mixtures of Gaussian states etc. Recent papers [19 20] dealt with mixed NGS of the form $\hat{\rho}_m = p|\phi_+\rangle\langle\phi_+| + (1-p)|00\rangle\langle00|$, with $0 \leq p \leq 1$. They have a positive Wigner function yet they are NGS (but for the trivial instances $p = 0,1$). Clearly, the de-Gaussification here reduces entanglement and correlations in general. The negativity of such states reads

$$
N_m(\rho_m) = p N(\phi_+) = p(e^{2r} - 1)/2
$$

is increasing both with $r$ and with $p$. The same dependency holds for the bit correlations, $Q(\rho_m) = (2p/\pi) \arctan\left\{ N_m \left[ 1 - 1/\rho_{max} + 1/p \right] \right\}$, which again is a monotonic function of $N_m$ for any $p$. We further studied other NGS including photon-added and squeezed Bell-like states [18], and their mixtures with the vacuum: for all we found a direct match between entanglement and $Q$ [21].

Discussion.--- By analyzing the maximal number of correlated bits ($Q$) that can be extracted from a CV state via quadrature measurements, we have provided an operational quantification of the entanglement content of several relevant NGS (including the useful photon-subtracted states). Crucially, one can experimentally measure $Q$ by direct homodyne detections (of the quadratures displaying optimal correlations only), in contrast to the much more demanding full tomographical state reconstruction. One can then easily invert the (analytic or numeric) monotonic relation between $Q$ and the negativity to achieve a direct entanglement quantification from the measured data. Our analysis demonstrates the rather surprising feature that entanglement in the considered NGS can thus be detected and experimentally quantified with the same complexity as if dealing with GS.

Interestingly, this is not true for all CV states. By definition, $Q$ quantifies correlations encoded in the second canonical moments only. We have realized that there exist also states [e.g. the photonic qutrit state $|\psi_n\rangle = |00\rangle/\sqrt{2} + (|10\rangle + |20\rangle)/2$] which, though being totally uncorrelated up to the second moments ($Q = 0$), are strongly entangled, with correlations embedded only in higher moments. The characterization of such states is an intriguing topic for further study [21]. In this respect, it is even more striking that the measure considered in this paper, based on (and accessible in terms of) second moments and homodyne detections only, provides such an exact quantification of entanglement in a broad class of pure and mixed NGS, whose quantum correlations are encoded nontrivially in higher moments too, and currently represent the preferred resources in CV quantum information. We focused on optical realizations of CV systems, but our framework equally applies to collective spin components of atomic ensembles [22], and radial modes of trapped ions [23].

We thank Ph. Grangier, A. Ourjoumtsev, and M. Piani for discussions. We were supported by EU IP Programme SCALA, ESF PESC QUEDEDIS, MEC (Spanish government) contract FIS2005-01369, CIRIT (Catalan government) contract SGR-00185, and Consolidador-Ingienio 2010 CSD2006-0019 QOIT.

[1] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005); G. Adesso and F. Illuminati, J. Phys. A 40, 7821 (2007).
[2] M. M. Wolf et al., Phys. Rev. Lett. 96, 080502 (2006).
[3] J. Eisert et al., Phys. Rev. Lett. 89, 137903 (2002); J. Fiurášek, ibid. 89, 137904 (2002); G. Giedke and J. I. Cirac, Phys. Rev. A 66, 032316 (2002).
[4] N. C. Menicucci et al., Phys. Rev. Lett. 97, 110501 (2006).
[5] See, e.g., B. Groisman et al., Phys. Rev. A 72, 032317 (2005).
[6] E. Shchukin and W. Vogel, Phys. Rev. Lett. 95, 230502 (2005).
[7] A. Ourjoumtsev et al., Phys. Rev. Lett. 98, 030502 (2007).
[8] Unitary operations $U$ at most quadratic in the creation / annihilation operators amount to symplectic transformations $S: SO(4)^T = \Omega$ in phase space, where the symplectic matrix $\Omega$ encodes the canonical commutation relations. Then: $\hat{\rho} \mapsto U \hat{\rho} U^\dagger$ (on the density matrix) corresponds to $W(\xi) \mapsto W(S^{-1} \xi)$ (on the Wigner function, where $\xi = (x_1, p_1, \ldots, x_n, p_n)$).
[9] M. Navascues et al., Phys. Rev. Lett. 94, 010502 (2005).
[10] C. Rodó et al., Open Sys. Inf. Dyn 14, 69 (2007).
[11] Fair states have necessarily zero first moments, which can be computed without loss of generality. In general, one could take in Eq. (2) the difference between $B$ computed on $\hat{\rho}$ and $\hat{\rho}$ computed on $\hat{\rho}_A \otimes \hat{\rho}_B$. The latter is zero on fair states.
[12] A homodyne Bell test requires measuring two different rotated quadratures per mode, to achieve violation of the bound $|E_A^\phi,\phi + E_A^\phi,-\phi + E_B^\phi,\phi - E_B^\phi,-\phi| \leq 2$. Here, we propose to measure a single quadrature per mode, which displays one-shot optimal correlations, unveiling a powerful quantitative connection with Gaussian and non-Gaussian entanglement.
[13] W. J. Munro et al., Phys. Rev. A 59, 4197 (1999); H. Nha and H. J. Carmichael, Phys. Rev. Lett. 93, 020401 (2004); R. Garcia-Patron et al., Phys. Rev. Lett. 93, 130409 (2004); R. Garcia-Patron et al., Phys. Rev. A 71, 022105 (2005).
[14] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[15] The emerging measure of classical correlations is special to Gaussian states, where the correlated degrees of freedom are the field quadratures. Different approaches to the quantification of classical vs quantum correlations were proposed [3].
[16] J. S. Neergaard-Nielsen et al., Phys. Rev. Lett. 97, 083604 (2006); K. Waki et al., Optics Express 15, 3568 (2007).
[17] A. Kitagawa et al., Phys. Rev. A 73, 042310 (2006).
[18] F. Dell’Anno et al., Phys. Rev. A 76, 022301 (2007).
[19] A. P. Lund et al., arXiv:quant-ph/0605247.
[20] L. M. Mita et al., Phys. Rev. A 65, 062315 (2002).
[21] C. Rodó, G. Adesso, and A. Sanspera, in preparation.
[22] B. Julsgaard et al., Nature 413, 400 (2001).
[23] A. Serafini et al., arXiv:0708.0851.