Interactive Combinatorial Bandits: Balancing Competitivity and Complementarity

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Abstract

We study non-modular function maximization in the online interactive bandit setting. We are motivated by applications where there is a natural complementarity between certain elements: e.g., in a movie recommendation system, watching the first movie in a series complements the experience of watching a second (and a third, etc.). This is not expressible using only submodular functions which can represent only competitiveness between elements. We extend the purely submodular approach in two ways. First, we assume that the objective can be decomposed into the sum of monotone submodular and supermodular function, known as a BP objective. Here, complementarity is naturally modeled by the supermodular component. We develop a UCB-style algorithm, where at each round a noisy gain is revealed after an action is taken that balances refining beliefs about the unknown objectives (exploration) and choosing actions that appear promising (exploitation). Defining regret in terms of submodular and supermodular curvature with respect to a full-knowledge greedy baseline, we show that this algorithm achieves at most $O(\sqrt{T})$ regret after $T$ rounds of play. Second, for those functions that do not admit a BP structure, we provide analogous regret guarantees in terms of their submodularity ratio; this is applicable for functions that are almost, but not quite, submodular. We numerically study the tasks of movie recommendation on the MovieLens dataset, and selection of training subsets for classification. Through these examples, we demonstrate the algorithm's performance as well as the shortcomings of viewing these problems as being solely submodular.

1 Introduction

Many problems in machine learning are inherently discrete, such as the tasks of choosing a subset of features for prediction, or a subset of training data for classification. These problems are often naturally framed as the maximization of a function with diminishing returns. For instance, in the training-data selection task, the marginal benefit to a learner of selecting an additional training point is much larger if they have selected only a small number of points so far. This general behavior is well-modeled with submodular functions. In addition to the aforementioned tasks, submodular functions have been used to model recommender systems, summarization, coverage, and other phenomena. In their seminal paper, Nemhauser et al.
showed that the greedy algorithm, which sequentially adds the element that increases the objective value the most, selects a near-optimal solution of \( k \) elements \(^{22}\).

In order to employ the greedy algorithm, the objective function must be known exactly beforehand. However, applications of interest often take place in interactive settings. Here, the optimizer interacts with the environment to update its beliefs about the objective, which it does not know a-priori. This has led to development of modified greedy algorithms for interactive settings, with guarantees \(^{8,12,14}\).

In this paper, we argue that submodular functions alone often do not capture certain nuances in these motivating applications. Consider, for example, the task of recommending an item (news, book, movie etc.) to a user. Traditionally, this was understood as a static prediction task, and was solved using techniques such as collaborative or content-based filtering. However, static approaches are unable to take into account the dependencies between different items being recommended. For instance, someone who has already watched a documentary on a particular topic may not benefit from being recommended another documentary on the same topic—this is an instance of competitiveness between the recommended items since the two documentaries fulfill a similar purpose. To capture this into the model, submodular functions are used \(^{3}\). However, there are other manners of dependency that submodularity does not capture; for instance, a person may receive a greater utility from watching Godfather II after already having watched Godfather I; we say that these items have a complementary relationship.

Complementarity naturally arises in other applications that are traditionally interpreted as submodular. Consider the problem of active learning to select training points for a classification task to be labeled by a human \(^{2,5}\). Here, choosing two points nearby on opposite sides of a decision boundary could be greatly beneficial to understanding the local behavior of the decision boundary. Next, consider the goal of document summarization: choosing representative sentences to create a paragraph-length summary. Often, a single sentence is meaningful only in the context of other sentences, an instance of complementarity between the sentences. In the supplement, we situate our work in the landscape of other approaches adopted for these problems, such as reinforcement learning.

In this paper, we concern ourselves with the intriguing goal of addressing both competitiveness and complementarity simultaneously in the interactive setting. Complementarities are well-modeled by supermodular functions. Hence, we propose to model our dual aims as the sum of a submodular and supermodular function (called BP functions), introduced by Bai et al. in the offline setting \(^{4}\). In Figure 1 and Table 1, we illustrate the modeling power of BP functions in comparison with the purely submodular counterparts from the literature. In both cases, the output of the greedy algorithm on the BP function can be seen to capture the complementary relationship between items while also retaining the original submodular structure.

To address cases where the objective does not admit a BP decomposition, we also study functions with bounded submodular curvature; this characterization is especially useful to provide guarantees for functions that are almost submodular. Hence, through the following contributions, we extend the interactive setting to two function-classes of practical interest.

Contributions.

1. Define the notion of \( \alpha \)-regret appropriate to the interactive maximization of BP-functions - see Equation (1). We show that the contextual-UCB-algorithm developed by \(^{19}\) can be successfully applied to this algorithm to obtain \( O(\sqrt{T}) \) regret.
Figure 1: Comparison of greedy algorithm selection on submodular (second panel) and BP (third panel) objectives for subset selection of 100 points of training data from a ground set of 400 points. The first panel depicts the entire training (ground) set. The BP function results in the selection of points near the decision boundary—i.e. points of opposite class that are proximal. The details are provided in Section 5.

2. Define the notion of $\alpha$-regret appropriate to functions with bounded submodularity ratio in Equation (2), and provide $O(\sqrt{T})$ regret guarantees.

3. To obtain the regret bounds, we redeveloped the proof based on a worst-case linear program construction by Conforti et al. [9]. Additionally, the proof-structure provides a receipe to provide regret guarantees for a wider variety of function-classes for which the greedy algorithm performs well on offline settings.

4. We study the application of the proposed algorithm to movie recommendation on the MovieLens dataset, and corroborate our theory. Additionally, we verify our expectations on an active learning example on a synthetic dataset. Through an investigation of the selections by the recommender in both cases, we illustrate the pitfall of modeling a function as submodular when a BP function is more appropriate.

| SM Objective          | BP Objective                                      |
|-----------------------|---------------------------------------------------|
| 0 Lion King, The      | Godfather, The                                    |
| 1 Speed               | Godfather: Part II, The                           |
| 2 Godfather, The      | Godfather: Part III, The                          |
| 3 Godfather: Part II, The | Star Wars: Episode I - The Phantom Menace         |
| 4 Terminator, The     | Memento                                           |
| 5 Good Will Hunting   | Harry Potter and the Sorcerer’s Stone             |
| 6 Memento             | Star Wars: Episode II - Attack of the Clones      |
| 7 Harry Potter and the Sorcerer’s Stone | Harry Potter and the Chamber of Secrets |
| 8 Dark Knight, The    | Star Wars: Episode III - Revenge of the Sith      |
| 9 Inception           | Dark Knight, The                                  |

Table 1: Comparison of the selections of the greedy algorithm on submodular and BP objectives for movie recommendation, on a toy ground set of 23 movies from the MovieLens dataset. The submodular objective is the facility location objective, chosen from [8]. In the BP objective, there is an additional reward at each step for choosing a movie that is complementary with previously selected movies; this results the desirable joint selection of groups of movies from the same series. The task is formalized mathematically in Section 5 and experimental details are provided in the supplement.
2 Related work

Submodular maximization with bounded curvature. Nemhauser et al. [22] studied the performance of the greedy algorithm for maximizing a monotone non-decreasing submodular set function subject to a cardinality constraint and provided a \(1 - \frac{1}{e}\) approximation ratio for this problem. While Nemhauser et al. [21] showed that the \(1 - \frac{1}{e}\) factor cannot be improved under polynomial number of function value queries, the performance of the greedy algorithm is usually closer to the optimum in practice. In order to theoretically quantify this phenomenon, Conforti et al. [9] introduced the notion of curvature \(\kappa \in [0, 1]\) for submodular functions—this is defined in Section 4. The constant \(\kappa\) measures how far the function is from being modular. The case \(\kappa = 0\) corresponds to modular functions and larger \(\kappa\) indicates that the function is more curved. Conforti et al. [9] showed the greedy algorithm applied to monotone non-decreasing submodular maximization subject to a cardinality constraint has a \(\frac{1}{\kappa}(1 - e^{-\kappa})\) approximation ratio. Therefore, for general submodular functions (\(\kappa = 1\)), the same \(1 - \frac{1}{e}\) approximation ratio is obtained. However, if \(\kappa < 1\), \(\frac{1}{\kappa}(1 - e^{-\kappa}) > 1 - \frac{1}{e}\) holds and as \(\kappa \to 0\), the approximation ratio tends to 1. More recently, Sviridenko et al. [28] proposed two approximation algorithms for the more general problem of monotone non-decreasing submodular maximization subject to a matroid constraint and obtained a \(1 - \frac{1}{\kappa}\) approximation ratio for these two algorithms. They also provided matching upper bounds for this problem showing that the \(1 - \frac{\kappa}{e}\) approximation ratio is indeed optimal. Later on, the notion of curvature was extended to continuous submodular functions as well and similar bounds were derived for the maximization problem [23, 24, 25].

BP maximization. Bai et al. first introduced the problem of maximizing a BP function \(h = f + g\) (Definition 4.1) subject to a cardinality constraint or \(p\) matroid constraints [4]. They showed that this problem is NP-hard to approximate to any factor without further assumptions. However, if the supermodular function \(g\) has a bounded curvature (i.e., \(\kappa^g < 1\)), it is possible to obtain approximation ratios for this problem. In particular, for the setting with a cardinality constraint, they analyzed the greedy algorithm along with a new algorithm (SemiGrad) and provided a \(\frac{1}{\kappa_f}(1 - e^{-(1-\kappa^g)\kappa_f})\) approximation ratio. Note that for general supermodular functions (\(\kappa^g = 1\)), the approximation ratio is 0 and as \(\kappa^g \to 0\), the bound tends to that of Conforti et al. for monotone non-decreasing submodular maximization subject to a cardinality constraint. Bai et al. also showed that not all monotone non-decreasing set functions admit a BP decomposition. However, in cases where such a decomposition is available, one can compute the curvature of submodular and supermodular terms in linear time and compute the bound. More recently, Liu et al. proposed a distorted version of the greedy algorithm for this problem and provided an improved \(\min\{1 - \frac{\kappa_f}{e}, 1 - \kappa^g e^{-(1-\kappa^g)}\}\) approximation ratio [20].

Non-submodular maximization. Das et al. [10] introduced the notions of submodularity ratio \(\gamma\) and generalized curvature \(\alpha\) for general monotone non-decreasing set functions (defined in Section 4) and showed that the greedy algorithm obtains the approximation ratio \(\frac{1}{\alpha}(1 - e^{-\alpha\gamma})\) under cardinality constraints [6, 10]. Unlike the BP decomposition, the notions of submodularity ratio and generalized curvature can be defined for any monotone non-decreasing set function.

Adaptive and Interactive Submodularity. First, onsider papers on Gaussian-process bandits, which paves the way for work in interactive submodularity. Srinivas et al. consider the problem of optimizing an unknown function \(f\) that is either sampled from a Gaussian process or has
bounded RKHS norm [27]. They develop an upper-confidence bound (UCB) approach that achieves sublinear regret with respect to the optimal. Krause et al. extend the setting from Srinivas et al. to the contextual setting where the function $f_{z_t}$ being optimized now depends also on a context $z_t$ that varies with time [19]. Chen et al. [8] is the work most related to ours. They employ a similar UCB algorithm as above to optimize an unknown submodular function in an interactive setting. They define regret as the suboptimality gap with respect to a full-knowledge greedy strategy at the final round. They define a different notion of pointwise regret as the difference between the algorithms rewards and that of the greedy strategy at that stage, treating the past choices as fixed. By viewing the submodular problem as a special case of contextual bandits, they observe that this accumulation of pointwise regret is precisely bounded by Krause et al. [19]. Then, they modify Nemhauser et al.’s seminal proof to relate their target notion of regret with the pointwise regrets. Golovin et al. [12] also consider an adaptive submodular problem. They assume more knowledge of the structure of the submodular objective than [8].

3 Notation and Problem Formulation

First, we recall some essential definitions. A set function $f : 2^V \to \mathbb{R}$ is monotone non-decreasing if $f(A \cup \{v\}) \geq f(A)$ for all $A \subseteq V, v \in V$. It is normalized if $f(\emptyset) = 0$. For convenience, we refer to the collection of Monotone Non-decreasing Normalized set functions as MNN functions. A set function $f : 2^V \to \mathbb{R}$ is called submodular if for all $A \subseteq B \subseteq V$ we have

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

An equivalent definition of submodularity that shows better the diminishing returns property is as follows: for all $A \subseteq B \subseteq V$ and any element $v \not\in B$ we have

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B).$$

This states that the benefit of adding $v$ to the smaller set $A$ is larger than that of adding $v$ to the larger superset $B$. A function $g : 2^V \to R$ is said to be supermodular if $-g$ is submodular. It has the property of increasing returns where the presence of an item enhances the utility of selecting another item. Now, we are ready to formulate our problem mathematically.

Setup. The optimization takes place over the course of $T$ timesteps. In each timestep $i \in [T]$:  

1. The optimizer encounters one of $m$ functions from the set $\{h_1 \ldots h_m\}$. 
2. The optimizer does not know the exact form of the function but knows the index $u_i \in [m]$ of the function $h_{u_i}$ presented in round $i$, and a feature-vector $\phi_{u_i}$ associated with that index. 
3. We denote using $S_{u_i,i}$ the set that contains all items that the optimizer has selected for function $u_i$ until the end of round $i$. 
4. The optimizer performs the action $v_i \in V$, where $V$ is a finite ground set. 
5. The optimizer updates record of actions chosen for index $u_i$ as $S_{u_i,i} = S_{u_i,i-1} \cup v_i$. For all other $u_j \neq u_i$, $S_{u_j,i} = S_{u_j,i-1}$. 
6. The environment presents the optimizer with noisy marginal gain $y_i = h_{u_i}(v_i|S_{u_i,i-1}) + \epsilon_i$. 


Above, the gain notation $h_{u_i}(v|A) = h_{u_i}(v + A) - h_{u_i}(A)$ is the marginal gain of including element $v$ to set $A$. All feature-vectors are chosen from set $\Phi$, and we assume that the identity of the utility function $h_k$ is determined uniquely by $\phi_k$; hence, we abuse notation to use $h_k$ and $h_{\phi_k}$ interchangeably. Below, Vignette 1 and Vignette 2 formalize the task from Table 1 and Figure 1 respectively. This illustrates the applicability of our theoretical framework to our motivating applications, which we explore further numerically in Section 5.

Vignette 1 (Movie Recommendation). Each utility function $h_k$ can be thought of as capturing the preferences of a single user, and the index $u_i$ reveals which user has arrived at timestep $i$. The action $v_i$ performed in timestep $i$ is a movie that the optimizer recommends to user $u_i$. The feature-vector $\phi_k$ contains relevant context about the user that would inform their preferences - for instance, their age, their favorite movies and genres etc. The gain notation $h_{u_i}(v|A)$ is the enjoyment user $u_i$ derives from watching movie $v$, having already watched all movies in set $A$.

Vignette 2 (Active Learning). We are choosing training points to be labeled for $m$ related tasks on the same dataset - for instance classification, object detection and captioning. The function $h_k(A)$ is the test accuracy of a classifier $f(A)$, trained on set $A$, on the $k_{th}$ task. Choosing an action $v_i$ is tantamount to choosing a training point to be labeled for task $u_i \in [m]$.

Our primary goal of the paper is to propose an algorithm that achieves sublinear regret for different classes of functions. We specify the function classes of interest and appropriate regret formulations for these classes in Section 4.

4 Algorithm and analysis

In this section, we first propose our algorithm for the online problem (Section 4.1) and then, we analyze its performance in Section 4.2 and Section 4.3.

4.1 MNN-UCB algorithm

In Algorithm 1 we propose the MNN-UCB algorithm for the interactive online problem. The algorithm is inspired by [27], which was utilized by [8, 19] as well. Since the function to be optimized is not known beforehand, the algorithm first estimates this from data non-parametrically. At timestep $i \in [T]$, the optimizer has the following useful historical data for each timestep $j < i$: (i) item-selections $v_j$ (ii) the context of these item selections, which includes $\phi_j$ and the collection of previously chosen items $S_{u_j,i-1}$ (iii) the reward $y_j$ that was achieved. The algorithm computes a Bayesian posterior distribution to aggregate this information into a mean and covariance estimate for the rewards at time $i$ of each arm (lines 1-9). For our results, we assume that the marginal gain function $h_k(\cdot|\cdot): V \times 2^V \times \Phi \to \mathbb{R}$ has a bounded RKHS norm with respect to some kernel $k$; this is the same kernel $k$ we refer to in the algorithm.

The mean estimate $\mu_i$ takes the form of a weighted average of all past rewards; the weight for $y_j$ is proportional to the similarity in contexts of timesteps $j$ and $i$, captured in the vector $k_{i-1}(v)$. Then, the algorithm then chooses the item $v$ that achieves the highest Upper Confidence Bound (UCB) $\mu_{i-1}(v) + \sqrt{\beta_i \sigma_{i-1}(v)}$. The parameter $\beta_i$ controls the algorithm’s bias towards either exploration or exploitation. The functions $k_{i-1}(v), \mu_{i-1}(v), \sigma_{i-1}(v)$ are defined for all $v \in V \setminus S_{u_i,i-1}$.
Algorithm 1 MNN-UCB

**Input** set of items $\Omega$, mean $\mu_0 = 0$, variance $\sigma_0$

1: Initialize $S_i \leftarrow \emptyset$ for all $i \in [m]$; $V_i \leftarrow V$
2: for $i = 1, 2, 3 \ldots T$ do  // The below lines use $\forall v, v' \in V_i$
3:     Let $k_{i-1}(v) \in \mathbb{R}^{-1}$ have $j$-th entry $k((v_j, S_{u_i,j-1}, \phi_{u_i}))(v, S_{u_i,i-1}, \phi_{u_i}))$
4:     Let $K_{i-1} \in \mathbb{R}^{-1} \times \mathbb{R}^{-1}$ with $(j, t)$-th entry $k((v_j, S_{u_i,j-1}, \phi_{u_i}))(v_t, S_{u_t,i-1}, \phi_{u_t}))$
5:     Update $y_{i-1} \leftarrow [y_1, y_2, \ldots, y_{i-1}]^T$
6:     Define a kernel function $k_{i-1}(v, v') = k(v, v') - k_{i-1}(v)^T (K_{i-1} + \sigma^2 I)^{-1} k_{i-1}(v')$
7:     Estimate $\mu_{i-1}(v) \leftarrow k_{i-1}(v)^T (K_{i-1} + \sigma^2 I)^{-1} y_{i-1}$
8:     Estimate $\sigma_{i-1}(v) \leftarrow \sqrt{k_{i-1}(v, v)}$
9:     Select an item $v_i \leftarrow \arg\max_{v \in V_i} \mu_{i-1}(v) + \sqrt{\beta_i} \sigma_{i-1}(v)$
10:    Update $S_{u_i,i} \leftarrow S_{u_i,i-1} \cup \{v_i\}$ and $S_{u_j,i} \leftarrow S_{u_j,i-1}, \forall u_j \neq u_i$
11:    Obtain feedback $y_i = h_{u_i}(v_i|S_{u_i,0} - 1) + \epsilon_i$; $V_i \leftarrow V_i \setminus v_i$
12: end for

For readability, we assume that the ground set $V_i = V$ is the same for all functions, which holds for our motivating applications; however, the results extend seamlessly to the distinct $V_i$ case as well. Define $z_j \in 2^V \times \Phi$ as $z_j = (S_{u_j,0} - 1, \phi_{u_j})$ to be the context for prediction in timestep $j$, containing the feature-vector for user $u_j$ and all recommendations provided to this user in the past. Then the kernel function $k : (V, Z) \times (V, Z)$ measures the similarity between two (action, context) pairs, where $Z = 2^V \times \Phi$.

4.2 Performance guarantee: Submodular + Supermodular (BP) functions

In this section, we study utility functions $h_k$ that admit the form below.

**Definition 4.1** (BP Function). A utility function $h$ is said to be BP if it admits the decomposition

$$h = f + g,$$

where $f$ is submodular, $g$ is supermodular, and both functions are also MNN.

Bai et al. [4] studied the offline BP maximization problem subject to a cardinality constraint and showed that the problem is NP-hard to approximate even in the offline setting. However, it is possible to obtain approximation bounds under further assumptions. To better understand these assumptions, we first introduce the notion of curvature for submodular and supermodular functions below.

**Definition 4.2** (Submodular curvature). Denote the submodular curvature for $f_k$ as

$$\kappa_{f,k} = 1 - \min_{v \in V} \frac{f_k(v|V \setminus \{v\})}{f_k(v)}.$$

**Definition 4.3** (Supermodular curvature). Denote the supermodular curvature for $g_k$ as:

$$\kappa_{g,k} = 1 - \min_{v \in V} \frac{g_k(v)}{g_k(v|V \setminus \{v\})}.$$
These quantities are measures of how far the functions are from being modular. If the curvature is zero, the function is modular and larger curvature corresponds to the function being more curved. Importantly, this can be calculated in linear time. Bai et al. analyzed the greedy algorithm for the cardinality-constrained BP maximization problem and provided a \( \frac{1}{e/2} \left( 1 - e^{-(1-\kappa)\epsilon} \right) \) approximation ratio for this problem. If the supermodular function is fully curved (i.e., \( \kappa^g = 1 \)), the bound reduces to 0. However, \( \kappa^g < 1 \) leads to a strictly positive approximation ratio. Inspired by the approximation ratio for the offline problem, we define the regret metric below.

Let \( T_k \) represent the number of items selected for function \( h_k \) by the final round, \( T \); we let \( S_k = S_k,T_k \) refer to the final set selected for \( h_k \) and drop the second index \( T_k \) for simplicity of notation. Let

\[
S_k^* = \arg \max_{|S| \leq T_k} h_k(S).
\]

be the set that maximizes the payoff of function \( h_k \) with at most \( T_k \) elements. Choosing the full-knowledge greedy solver as our baseline to compare against, we define the following regret metric:

\[
R_T := \sum_{k=1}^m \frac{1}{\kappa_{k,f}} \left( 1 - e^{-(1-\kappa_k)\epsilon_{k,f}} \right) h_k(S_k^*) - h_k(S_k).
\] (1)

Below, we refer to \( h(.|.) : V \times 2^V \times \Phi \rightarrow \mathbb{R} \), with the context included in the function's domain.

Now, we are ready to state our main result. The assumptions below are inherited from Krause et al. [19], and ensure that the challenge of function-estimation can be tackled.

**Theorem 1.** Suppose that the true marginal gain function \( h(.|.) \) has a small RKHS norm according to some kernel \( k \), i.e., \( ||h(.|.)||_k \leq B \), and each function \( h_k \) is BP according to Definition 4.1. The noise variables \( \epsilon_t \) satisfy \( \mathbb{E}[\epsilon_t | \epsilon_1, \epsilon_2, \ldots, \epsilon_{t-1}] = 0 \) for all \( t \in \mathbb{N} \) and are uniformly bounded by \( \sigma \). Let \( \delta \in (0,1), \beta_t = 2B^2 + 300\gamma_1 \ln^3(t/\delta) \) and \( C_1 = 8/\log(1 + \sigma^{-2}) \). Then, the accumulated regret of MNN-UCB over \( T \) rounds, as defined in [1], is bounded as follows:

\[
\Pr \left\{ R_T \leq \sqrt{C_1 T \beta_T} \gamma_T + 2, \forall T \geq 1 \right\} \geq 1 - \delta.
\]

The high-level proof framework for Theorem 1 is based on [4] where they analyzed the greedy algorithm for offline BP maximization subject to a cardinality constraint. Let \( h_k \in \{h_1, \ldots, h_m\} \) and consider the ordering of \( S_k \) based on the order of selection of the items by the algorithm. The proof follows from (1) Showing that the greedy algorithm has a nontrivial overlap with the optimal solution (2) Providing a lower-bound on the performance of any selection with this level of overlap (3) Showing that the pointwise “errors” of Algorithm 1 with respect to the greedy strategy at that stage is bounded. To address step (2), we partition all different instances of the problem \( \max_{S:|S| \leq T_k} h_k(S) \) according to the set of indices \( S_k^* \cap S_k \) where the decision of the algorithm is consistent with that of the optimal solution. Then, we can analyze the worst-case approximation ratio of each of these groups of problems instances by constructing and solving Linear Programs (LPs) based on the properties of the problem instances [9].

Adapting the proof from Bai et al. [4] to our online setting requires several changes and new techniques in step (3). Define the instantaneous regret at round \( i \in [T] \) as the difference between the maximum possible utility that is achievable in the round and the actually received utility \( r_i = \sup_{v \in V} h_u(v|S_{u,i-1}) - h_u(v_i|S_{u,i-1}) \). The main building block to formulate the LPs (Lemma 2 in the Appendix) needs to incorporate \( r_i \)'s into the inequalities and is different from [4]. Given
that the LPs are constructed differently, analyzing the structure of these LPs and investigating the worst-case instance for each group of problem instances (Lemma 1 in the Appendix) is also different. Finally, in order to use the result of Lemma 1 to obtain regret bounds, we need to relate the \( \sum_{i=1}^{T} r_i \) term appearing in the approximation guarantee of Lemma 1 to the regret metric \( R_T \).

## 4.3 Performance guarantee: General MNN functions

Before extending our results to arbitrary MNN functions, we first recap some important definitions.

**Definition 4.4 (Submodularity ratio, \([6, 10]\)).** The submodularity ratio of a non-negative set function \( h(\cdot) \) is the largest scalar \( \gamma \) such that

\[
\sum_{v \in S \setminus A} h(v|A) \geq \gamma h(S|A), \forall S,A \subseteq V.
\]

The submodularity ratio measures to what extent \( h(\cdot) \) has submodular properties. For a nondecreasing function \( h(\cdot) \), it holds that \( \gamma \in [0,1] \) always, and \( h(\cdot) \) is submodular if and only if \( \gamma = 1 \).

**Definition 4.5 (Generalized curvature, \([6]\)).** The curvature of a non-negative function \( h(\cdot) \) is the smallest scalar \( \alpha \) such that

\[
h(v|A \cup \{v\} \setminus S)) \geq (1 - \alpha)h(v|A \setminus \{v\}), \quad \forall S,A \subseteq V, v \in A \setminus S.
\]

Note that unlike the notions of submodular and supermodular curvature, the submodularity ratio and generalized curvature parameters are information theoretically hard to compute in general. Bian et al. analyzed the greedy algorithm for maximizing a general non-negative and monotone non-decreasing set function with submodularity ratio \( \gamma \) and generalized curvature \( \alpha \) subject to a cardinality constraint and obtained a \( \frac{1}{\alpha} (1 - e^{-\alpha \gamma}) \) approximation ratio for this problem \([6]\). In this section, we assume that each utility function \( h_k \) is non-submodular with submodularity ratio \( \gamma_k \) and generalized curvature \( \alpha_k \).

Let \( T_k \) and \( S_k^* \) be as defined previously. Hence, choosing the full-knowledge greedy solver as our baseline to compare against, we define the following regret metric:

\[
\mathcal{R}_T := \sum_{k=1}^{m} \frac{1}{\alpha_k} [1 - e^{-\alpha_k \gamma_k}] h_k(S_k^*) - h_k(S_k).
\]  

(2)

Below, we provide a comparable guarantee for this setting as in Section 4.2.

**Theorem 2.** Suppose that each function \( h_k \) has submodularity ratio \( \gamma_k \) (Definition 4.4) and generalized curvature \( \alpha_k \) (Definition 4.5). Additionally, assume the conditions on the gain function and noise variables from Theorem 1 hold, and the constants \( \beta_t, B, C_1 \) are defined as in Theorem 1. Then, the accumulated regret of MNN-UCB over \( T \) rounds, as defined in (2) is bounded as follows:

\[
\Pr\{\mathcal{R}_T \leq \sqrt{C_1 T \beta_T \gamma_T} + 2, \forall T \geq 1\} \geq 1 - \delta.
\]

The proof involves adapting the proof from Bian et al. to the interactive setting, which is also based on the LP argument from Conforti et al. The extension of our argument from Theorem 1 to this case illustrates the generality of our approach to providing guarantees in the interactive setting, and suggests that it could apply to other cases of interest where the greedy algorithm is known to perform well in the offline setting.
5 Numerical Experiments

In this section, we study the performance of Algorithm 1 for the tasks of movie recommendation and active learning.

5.1 Movie Recommendation

In this section, we use the MovieLens dataset. This contains a movie rating matrix $M$ where $M_{i,j}$ is the rating of the $i$th user for the $j$th movie. In addition, the dataset contains a list of genres that are applicable to each movie. In order to obtain a reliable and dense matrix, we consider the 900 most active users and 1600 most popular movies amongst these users, denoted as $V = \{v_1 \ldots v_{1600}\}$. We use the singular-value thresholding approach from Cai et al. [7] in order to fill in the missing entries of this matrix.

Using this dataset, we setup an interactive BP maximization problem, as formulated in Vignette 1. We group the 900 users into $m = 10$ groups using the $k$-means algorithm, and design a BP objective for each user-group. Let the set $L$ refer to the collection of all genres in the ground set. The objective for the $k$th group is decomposed as $h_k(A) = \sum_{v \in A} m_k(v) + \lambda_1 f_k(A) + \lambda_2 g_k(A)$, where the modular part $m_k(v)$ is the average rating for movie $v$ amongst all users in group $k$. The submodular part is the sum of $|L|$ concave over modular functions, one for each genre; it encourages the recommender to maintain a balance across genres in chosen suggestions:

$$f_k(A) = \sum_{g \in L} \log \left(1 + q_{k,g}(A)\right).$$

Above, $q_{k,g}(\cdot)$ is a modular function which measures how well set $A$ does at including movies from genre $g$ that are highly rated by user group $k$. The matrix $1600 \times 1600$ matrix $C$ measures the complementarity between movies in terms of their genre-overlap. Then, the supermodular part is the sum-sum-dispersion function:

$$g_k(A) = \sum_{v_i \in A} \sum_{v_j \in A: v_j \neq v_i} C_{i,j}.$$

The constants $\lambda_1, \lambda_2$ were chosen such that the submodular part slightly dominates the supermodular part, in order to ensure that the recommender sufficiently searches over different movie genres before committing to a smaller number of genres that it believes the user is disposed towards. The definition of matrix $C$ and function $q_{k,g}(\cdot)$, and choices of all constants are specified in the supplement; the code is contained in notebook “Figure 2.”

For Algorithm 1 we choose the RBF kernel for movies, the linear kernel for users and the jaccard kernel for a history of recommendations. The composite kernel $k((u,v,A),(u',v',A')) = \kappa_1 k_{user}(u,u') + \kappa_2 k_{movie}(v,v') + \kappa_3 k_{history}(A,A')$ for $\kappa_1, \kappa_2, \kappa_3 > 0$. In Figure 2 (left), we compare the rewards of the MNN-UCB algorithm on the BP objective (magenta, labeled BP-UCB) to baselines over $T = 100$ timesteps. It performs much more comparably to the greedy algorithm than the random baseline, which selects an unseen movie at each timestep uniformly at random. If we give MNN-UCB feedback only about the submodular part and not the supermodular part, then it performs notably worse (green, labeled SM-UCB), even though the supermodular part is only a small part of the overall BP objective. SM-UCB overfits to the submodular part of

1https://grouplens.org/datasets/movielens/
the function and performs worse than random selection on the supermodular part; hence, it only does slightly better overall. This underscores the pitfall of viewing a BP problem as solely submodular.

Define a proxy for regret $\bar{R}_i = \sum_{k\in[m]} h_k(G_{k,i}) - h_k(S_{k,i})$, where $G_{k,i}$ is the selection of the greedy algorithm for the $k$th objective until timestep $i$, and $S_{k,i}$ is as defined in Section 3. This is an upper bound on our regret $R_i$ since the greedy algorithm is guaranteed to do no worse than the baseline in $\bar{R}_i$. Nevertheless, we show in Figure 2 (right) that $\bar{R}_i/i$ decays for BP-UCB, in contrast to SM-UCB. Note that in practice, the greedy algorithm is known to perform much better than its theoretical guarantee, so the decay of $\bar{R}_i$ is expected to be faster than the plotted curve.

5.2 Active Learning

This corresponds to Vignette 2 with $m = 1$ task. The synthetic data for Figure 1 is drawn from two Gaussian distributions, and separating its quantiles into the two classes. In order to apply the Naive-Bayes formulation of active learning in Equation (5) of \cite{29}, we discretize the space into 56 boxes of equal size, and set $f(A) = f_{NB}(A)$. The supermodular part is the sum-sum-dispersion function as above $g(A) = \sum_{v_j \in A} \sum_{v_i \in A \setminus v_j} B_{i,j}$. Here $B_{i,j} = 0$ if $(v_i, v_j)$ are from the same class, and $B_{i,j} = 1/\text{dist}(v_i, v_j)$ if $(v_i, v_j)$ are from the opposite class; this encourages the selection of proximal points from different classes. In the BP function $h(A) = \lambda_1 f(A) + \lambda_2 g(A)$, the submodular part strongly dominates the supermodular part. This ensures that the learner samples evenly across the entire space as in Wei et al., but also has a preference for complementary points. The code is contained in the notebook entitled “Figure 1,” and details are contained in the supplement.

6 Discussion

Future Directions Since computing the Bayesian posterior distribution at each timestep in Algorithm 1 is time-intensive, it is of interest to consider faster nonparametric variants for the posterior update to improve practical feasibility. In addition, it might be interesting to design variants of parametric-UCB algorithms. Additionally, it is of interest to design notions of regret that compare against baselines in the literature other than greedy. This would require
a modification of the Algorithm 1 and associated regret analysis from [27, 19]. Further, it would be informative to conduct an empirical comparison of the performance and robustness of submodularity-based approaches for the motivating problems of recommendation systems, active learning with commonly employed approaches for these problems.

**Social impact.** Consider our motivating applications. The employment of poor summarization techniques could exacerbate the spread of ambiguous or incorrect information; hence, studying it mathematically has positive social impact. Active learning has tremendous ability to broadly enhance all types of learning-based systems; thus, the net-impact of this technique is difficult to predict and largely depends on the overall impact of machine learning in the future. Recommendation systems are commonly highlighted as having negative social impact on society, largely due to flexible modern approaches such as reinforcement learning (RL). Short of legislative action on such systems, strengthening mathematically grounded approaches such as ours to compete with RL is the best course to alleviate this problem; we elaborate in the supplement.
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A Applications and Role of MNN functions

Comparison with RL for recommender systems For recommender systems, the dependencies between past and future recommendations may be modeled through a changing “state variable,” leading to adopting reinforcement learning (RL) solutions [1]. These have been tremendously effective at maximizing engagement; however, Kleinberg et al. [17] highlight that a key oversight of these approaches is that the click and scroll-time data that platforms observe is not representative of the users’ actual utilities: "research has demonstrated that we often make choices in the moment that are inconsistent with what we actually want." Hence, Kleinberg et al. advocate to encode diminishing returns of addictive but superficial content into the model, in the manner that we do with submodular functions. Further, RL systems are incentivized to manipulate users’ behavior [30, 15], mood [18] and preferences [11]; this inspires the use of principled mathematical techniques, as in the present work, to design systems to behave as we want rather than simply following the trail of the unreliable observed data.

Comparison with common approaches for Active Learning In their survey paper, Settles [26] compare submodularity-based approaches with other approaches for active learning. The main benefit of framing the active learning problem as submodular is that the greedy algorithm can be employed, which is much less computationally expensive than other common active learning approaches. While submodularity has been shown to be relevant to active learning [13, 29, 16], Settles remark that in general, the active learning problem cannot be framed as submodular.

In our paper, by extending the classes of functions that can be optimized online, we take a step towards addressing this limitation of submodularity. Further, an open question outlined in [26] is that of multi-task active learning, which has not been explored extensively in previous work. However, our formulation in Vignette 2 naturally extends to this multi-task setting.

B Details on Numerical Simulations

Details for Table 1 The chosen toy ground set of 23 elements is detailed in Table 2. The submodular function is the facility location function; we chose this function because it is used in prior work [8] for the task of movie recommendation. The supermodular part is the sum-sum-dispersion function, and the weights that capture the complementarity between movies are specified in the python notebook code/table-1.ipynb in the attached code.

From Table 1 we notice that with the submodular objective, the greedy algorithm chooses the first two movies in the Godfather series but does not choose the third. Similarly, it chooses the first Harry Potter but not the subsequent ones. Contrasting, with the BP function, the greedy algorithm chooses all elements from the series in both cases. This behavior cannot be encoded using solely a submodular function, but it is very easy to do so with a BP function.

Setup for movie recommendation in Figure 2 In this section, we fill in details of the construction of functions $f_k, g_k$ that we omit from the main paper due to space and readability. Recall that set $L$ contains all genres for movies in the ground set. For each element $v \in V$, define a vector $r(v) \in \{0, 1\}^{|L|}$. Here, each entry corresponds to a genre and is 1 if the genre is associated with the movie $v$. Then let $N_v = r(v)^\top 1$ denote the number of genres for movie $v$. 16
Lion King, The  
Speed  
True Lies  
Aladdin  
Dances with Wolves  
Batman  
Godfather, The  
Godfather: Part II, The  
 Terminator, The  
Indiana Jones and the Last Crusade  
Men in Black  
Good Will Hunting  
Godfather: Part III, The  
Star Wars: Episode I - The Phantom Menace  
Gladiator  
Memento  
Shrek  
Harry Potter and the Sorcerer’s Stone  
Star Wars: Episode II - Attack of the Clones  
Harry Potter and the Chamber of Secrets  
Star Wars: Episode III - Revenge of the Sith  
Dark Knight, The  
Inception  

Table 2: Ground set for Table I

Specification of $q_{k,g}(A)$ Recall that $m_k(v)$ is the average rating for movie $v$ amongst users in group $k$.

$$q_{k,g}(A) = \sum_{v \in A} \mathbb{1}(m_k(v) > \tau) \mathbb{1}(v \text{ has genre } g) \frac{m_k(v)}{N_v}$$

Above, $\mathbb{1}$ is the indicator function.

Construction of matrix $C$: The matrix $C$ is specified as follows:

$$C_{i,j} = \cos \theta_{i,j},$$

where $\theta_{i,j}$ is the angle between vector $r(v_i)$ and $r(v_j)$. Hence, the supermodular part $g_k(A)$ is the same for all $k$, and the function definition does not change for each user group $k$.

Active Learning. Here, we elaborate on the choice of submodular function. Assume our features are discrete - each point $v \in V$ has features $x_v \in \mathcal{X}$ (where $\mathcal{X}$ is some finite set) and binary label $y_v \in \{0,1\}$, denoted by the orange and blue colors in Figure I. Then, for any $(x \in \mathcal{X}, y \in \{0,1\})$ and for any subset of training points $S \subseteq V$, we can define

$$m_{x,y}(S) = \sum_{v \in S} \mathbb{1}(x_v = x \wedge y_v = y)$$

as the empirical count of the joint occurrence of $(x,y)$ in $S$. Then, inspired by the construction in Wei et al., we define the submodular part $f$ as

$$f(A) = \sum_{x \in \mathcal{X}} \sum_{y \in \{0,1\}} \sqrt{m_{x,y}(V)} \log(m_{x,y}(S))$$

To obtain the finite set $\mathcal{X}$, we discretize our 2-dimensional features into 56 boxes. The square-root in the expression above does not occur in the original paper, and was introduced by us due to better empirical performance. The intuition for constructing $f(\cdot)$ in this way is that the feature $x$ should appear alongside label $y$ in the chosen subset with roughly the same frequency as in the ground training set.

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C Proof of Theorem 1

The proof follows from:

1. Showing that the MNN-UCB algorithm has a nontrivial overlap with the optimal solution. This is tackled in Lemma 2.

2. Providing a lower-bound on the performance of any selection with this level of overlap. This is done in Lemma 1.

3. The two steps above enable us to express regret $R_T$ in terms of the pointwise “errors” of Algorithm 1 with respect to the greedy strategy at each stage. We show, in the main proof below, that the accumulation of these quantities is bounded.

These three steps together complete our argument.

Proof of Theorem 1 Define the instantaneous regret at round $i$ as the difference between the maximum possible utility that is achievable in the round and the actually received utility

$$r_i = \sup_{v \in V} h_{u_i}(v|S_{u_i,i-1}) - h_{u_i}(v_i|S_{u_i,i-1})$$

Define the accumulated instantaneous regret until round $i$ as

$$R_i = \sum_{j=1}^{i} r_j$$

Recognize that $R_i$ is different than $R_j$. From Lemma 1 it follows that

$$R_T = \sum_{k=1}^{m} \sum_{i=1}^{T} \mathbb{I}(u_i = k) r_i = R_T$$

Now, we can model the problem of the present work as a contextual bandit problem in the vein of [19]. Here, the context in the $i_{th}$ round is $z_i = (\phi_{u_i}, S_{u_i,i-1})$. Now we invoke Theorem 1 in [19], using the third assumption on the RKHS norm of the gain function, which follows from our assumption on $\|h(\cdot, \cdot)\|_k$. Thus, we have that

$$\Pr\{R_T \leq \sqrt{C_1 T \beta_T \gamma_T} + 2, \forall T \geq 1\} \geq 1 - \delta$$

Combining this with Equation (3), our argument is complete.

Notation for Lemma 1, Lemma 2 Recall that we use $S_{k,i}$ to refer to the ordered set of elements chosen for function $h_k$ until round $i$, and $S_k$ to refer to the ordered final set of items chosen for function $h_k$ until round $T$. For the following two lemmas, we restrict attention to the $k_{th}$ function $h_k$. Let the ordered set $M = \{m_1, m_2, ... m_T \}$ denote the set of timesteps $\{i \in [T]: u_i = k\}$ where the $k_{th}$ user arrived to the optimizer. Hence, $S_{k,m_j}$ refers to the first $i$ elements chosen for $h_k$.

In order to be consistent with the notation of [4], let $s_j = v_m$ be the $j_{th}$ element of $S_k$. Then, we define $a_j = h_k(s_j|{s_1, s_{j-1}})$ be the gain of the $j_{th}$ element chosen for function $k$. In the
we wish to show that this is subtle implicit. 

Since the second and third inequality are just statements about the linear program, they follow directly from the arguments in [4] when we substitute $\alpha$.

Next, recognize that $T_k$ is an ordered set. We let $C \subseteq [T_k]$ denote the indices (in increasing order) of elements in $S_k$ that are also in $S_k^*$. For instance, for $S_k = \{s_1 \ldots s_3\}$ and $S_k \cap S_k^* = \{s_1, s_2, s_3\}$, we have $C = \{1, 2, 3\}$. Hence, $j \in C \iff s_j \in S_k \cap S_k^*$. Further, define filtered sets $C_t = \{c \in C|c \leq t\}$.

Now, we are ready to state our main lemma. The proof of the lemma follows using the same outline as the approximation-guarantee of the greedy algorithm on BP functions. However, the MNN-UCB algorithm is not greedy, so we keep record of the deviation of MNN-UCB from the choice made by greedy at each stage. This results in the new second term in the equation below.

Lemma 1. For any $h_k \in \{h_1 \ldots h_m\}$ with curvatures $\kappa_{f,k}$ and $\kappa_{g,k}$, MNN-UCB is guaranteed to choose elements $S_k$ such that

\[
h_k(S_k) \geq \frac{1}{\kappa_{f,k}} \left[ 1 - \left( 1 - \frac{(1 - \kappa_{g,k})\kappa_{f,k}}{T_k} \right)^{T_k} \right] h_k(S_k^*) - \sum_{j=1}^{T_k} r_m j.
\]

Proof of Lemma 1. From Lemma 2, we have that MNN-UCB obeys $T_k$ different inequalities, and we wish to show that this is sufficient to obey the inequality above. In order to complete the argument, we consider the worst-case overall gain if these $T_k$ inequalities are satisfied; and show that this worst case sequence satisfies the desired, and hence MNN-UCB must satisfy the desired as well.

To characterize the worst-case gains, we define a set of linear programming problems parameterized by $B$.

\[
T(B) = \min_b \sum_{j=1}^{T_k} b_j \quad \text{s.t.} \quad h_k(S_k^*) \leq \alpha \sum_{j \in [t-1]} b_j + \sum_{j \in B_{t-1}} b_j + \frac{T_k - |B_{t-1}|}{1 - \beta} b_t, \quad \forall t \in [T_k]
\]

In the above, $b = [b_1 \ldots b_{T_k}]$ is a vector in $\mathbb{R}^{T_k}$. In the linear program, $b \geq 0$ is the decision variable and $T_k, \alpha, \beta$ are fixed values. The parameter of the LP $B \subseteq [T_k]$, and $B_t = \{j \in B|j \leq t\}$ is the filtered set. Note that the constraints are linear in $b$ with non-negative coefficients.

To show the result, we hope to show the following chain of inequalities:

\[
h_k(S_k) + \sum_{j=1}^{T_k} r_m j \geq T(C) \geq T(\phi) \geq \xi_k h_k(S_k^*)
\]

In the above,

\[
\xi_k = \frac{1}{\kappa_{f,k}} \left[ 1 - \left( 1 - \frac{(1 - \kappa_{g,k})\kappa_{f,k}}{T_k} \right)^{T_k} \right]
\]

First, recognize that $T(\cdot)$ is exactly the LP considered in [4], modulo notation differences. Since the second and third inequality are just statements about the linear program, they follow directly from the arguments in [4] when we substitute $\alpha = \kappa_{f,k}$ and $\beta = \kappa_{g,k}$.
For the first inequality, we have from Lemma 2 that \( b_j = a_j + r_{m_j} \) is a feasible solution for the linear program \( T(C) \), if we choose \( \alpha = \kappa_{f,kt} \beta = \kappa_k \). Hence,

\[
T(C) \leq \sum_{j=1}^{T_k} b_j = \sum_{j=1}^{T_k} a_j + \sum_{j=1}^{T_k} r_{m_j} = h_k(S_k) + \sum_{j=1}^{T_k} r_{m_j}
\]


The lemma below is a modified version of Equation (19) in [4], with all \( a_i \)'s replaced with \( (a_i + r_i) \).

**Lemma 2.** Using the notation above and for \( S_k \) as chosen by MNN-UCB, it follows that \( \forall t \in [T_k] \),

\[
h_k(S_k^*) \leq \kappa_{f,k} \sum_{j \in [t-1], C_{t-1}} (a_j + r_{m_j}) + \sum_{j \in C_{t-1}} (a_j + r_{m_j}) + \frac{T_k - |C_{t-1}|}{1 - \kappa_k^g} (a_t + r_{m_t})
\]

**Proof of Lemma 2** By the properties of BP functions from Lemma C.2 in [4], it follows for all \( t \in [T_k] \) that

\[
h_k(S_k^*) \leq \kappa_f \sum_{j \in [t-1], C} a_j + \sum_{j \in C_{t-1}} a_j + h_k(S_k^* \setminus S_{k,m_{t-1}}) \leq \kappa_f \sum_{j \in [t-1], C} (a_j + r_{m_j}) + \sum_{j \in C} (a_j + r_{m_j}) + h_k(S_k^* \setminus S_{k,m_{t-1}})
\]

The inequality above follows because the coefficients on the first two summations are positive and \( r_{m_j} \geq 0 \). Now, we must simplify the third term to obtain the desired. For any feasible \( v \),

\[
h_k(v | S_{k,m_{t-1}}) \leq \sup_v h_k(v | S_{k,m_{t-1}}) \leq h_k(s_t | S_{k,m_{t-1}}) + r_{m_t}
\]

The first inequality follows from the definition of \( \sup \) and the second follows from the definition of \( r_j \) in the proof of Theorem 1 above. Now, apply inequality (iv) from Lemma C.1 in [4]:

\[
h_k(S_k^* \setminus S_{k,m_{t-1}} | S_{k,m_{t-1}}) \leq \frac{1}{1 - \kappa_k^g} \sum_{v \in S_k^* \setminus S_{k,m_{t-1}}} h(v | S_{k,m_{t-1}}) \leq \frac{1}{1 - \kappa_k^g} \sum_{v \in S_k^* \setminus S_{k,m_{t-1}}} h_k(s_t | S_{k,m_{t-1}}) + r_{m_t}
\]

The second line follows from Equation (8).

We have that

\[
|S_k^* \setminus S_{k,m_{t-1}}| = |S_k^*| - |S_k^* \cap S_{k,m_{t-1}}| = T_k - |S_k^* \cap S_{k,m_{t-1}}|
\]

Hence,

\[
h_k(S_k^* \setminus S_{k,m_{t-1}} | S_{k,m_{t-1}}) \leq \frac{T_k - |S_k^* \cap S_{k,m_{t-1}}|}{1 - \kappa_k^g} \left[ h_k(s_t | S_{k,m_{t-1}}) + r_{m_t} \right]
\]

Recognizing that \( |S_k^* \cap S_{k,m_{t-1}}| = |C_{t-1}| \) completes the argument.
Proof of Theorem 2

The proof follows the same LP construction of Conforti et al. [9] as Theorem 1. The main contribution lies showing Lemma 4; this shows that the offline counterpart, Lemma 1 from Bian et al. [6], holds in the online setting as well.

**Proof of Theorem 2.** Define $r_i$ and $R_i$ as in the proof of Theorem 1. From Lemma 3, it follows that

$$ R_T \leq \sum_{k=1}^{m} \sum_{i=1}^{T} \mathbb{I}(u_i = k)r_i = R_T $$

(9)

As in the proof of Theorem 1, combining Theorem 1 of [19] with Equation (9), our argument is complete.

Define $S_k, a_j, C$ as in the proof for BP functions. Below, we present the counterparts of the lemmas in the proof of the BP functions for the present case. The proof for Lemma 4 is different than Lemma 2 due to the change in the class of functions being considered. The similarity of the two proofs suggests the generality of our proof technique and indicates that it may be analogously applied to other classes of functions as well.

**Lemma 3.** For any $h = h_k \in \{h_1 \ldots h_m\}$ with curvatures $\kappa_f$ and $\kappa_g$, MNN-UCB is guaranteed to choose elements $S_k$ such that

$$ h_k(S_k) \geq \frac{1}{\kappa_f}[1 - e^{-\alpha_k \gamma_k}]h_k(S_k^*) - \sum_{j=1}^{T_k} r_{m_j} $$

**Proof of Lemma 3.** We consider again the parameterized LP $T(B)$, but this time with the constants set as $\alpha = \alpha_k, \beta = 1 - \gamma_k$. To show the result, we hope to show the following chain of inequalities:

$$ h_k(S_k) + \sum_{j=1}^{T_k} r_{m_j} \geq T(C) \geq T(\phi) \geq \xi_k h_k(S_k^*) $$

(10)

In the above,

$$ \xi_k = \frac{1}{\alpha_k} \left[ 1 - \left( 1 - \frac{\gamma_k \alpha_k}{T_k} \right)^{T_k} \right] $$

Similarly to the argument in Lemma 1, the first two inequalities follow directly from the arguments in [4] when we substitute $\alpha = \alpha_k$ and $\beta = 1 - \gamma_k$. Under the same choice of constants, we have from Lemma 4 that $b_j = a_j + r_{m_j}$ is a feasible solution for the linear program $T(C)$. Hence,

$$ T(C) \leq \sum_{j=1}^{T_k} b_j = \sum_{j=1}^{T_k} a_j + \sum_{j=1}^{T_k} r_{m_j} = h_k(S_k) + \sum_{j=1}^{T_k} r_{m_j} $$

Recognizing that

$$ \frac{1}{\alpha_k} \left[ 1 - \left( 1 - \frac{\gamma_k \alpha_k}{T_k} \right)^{T_k} \right] \geq 1 - e^{-\alpha_k \gamma_k} $$

completes the argument.

The lemma below is a modified version of Lemma 1 in [6].
Lemma 4. Using the notation above and for $S_k$ as chosen by MNN-UCB, it follows that $\forall t \in \{0 \ldots T_k - 1\}$,

$$h_k(S^*_k) \leq \alpha_k \sum_{j \in [r] \cap C_i} (a_j + r_{m_j}) + \sum_{j \in C_i} (a_j + r_{m_j}) + \frac{1}{\gamma_k} (T_k - |C_i|)(a_{t+1} + r_{m_{t+1}})$$

Proof of Lemma 4. The proof follows from the definitions of generalized curvature, submodularity ratio, and instantaneous regret $r_t$.

$$h_k(S^*_k \cup S_{k,m_i}) = h_k(S^*_k) + \sum_{j \in [r]} h_k(s_j|S^*_k \cup S_{k,m_{j-1}})$$

We can split the summation above to separately consider the elements from $S_{k,m_i}$ that do and do not overlap with $S^*_k$.

$$h_k(S^*_k \cup S_{k,m_i}) = h_k(S^*_k) + \sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} h_k(s_j|S^*_k \cup S_{k,m_{j-1}}) + \sum_{j:s_j \in S_{k,m_i} \cap S^*_k} h_k(s_j|S^*_k \cup S_{k,m_{j-1}}) = 0$$

(11)

From the definition of submodularity ratio,

$$h_k(S^*_k \cup S_{k,m_i}) \leq h_k(S^*_k) + \frac{1}{\gamma} \sum_{\omega \in S^*_k \setminus S_{k,m_i}} h_k(\omega|S_{k,m_i})$$

(12)

From the definition of generalized curvature, it follows that

$$\sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} h_k(s_j|S^*_k \cup S_{k,m_{j-1}}) \geq (1 - \alpha_k) \sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} h_k(s_j|S_{k,m_{j-1}}) = (1 - \alpha_k) \sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} a_{j+1}$$

(13)

Then, plugging the inequalities (12) and (13) into (11),

$$h_k(S^*_k) = h_k(S^*_k \cup S_{k,m_i}) - \sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} h_k(s_j|S^*_k \cup S_{k,m_{j-1}})$$

$$\leq \left[ h_k(S^*_k) + \frac{1}{\gamma} \sum_{\omega \in S^*_k \setminus S_k} h_k(\omega|S_k) \right] + \left[ \alpha_k \sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} a_{j+1} - \sum_{j:s_j \in S_{k,m_i} \cap S^*_k} a_{j+1} \right]$$

(14)

Now, we can rearrange and write

$$h_k(S^*_k) - \sum_{j:s_j \in S_{k,m_i} \setminus S^*_k} a_{j+1} = \sum_{j:s_j \in S_{k,m_i} \cap S^*_k} a_{j+1}$$
to simplify Equation (14) as

\[ h_k(S_k^*) = a_k \sum_{j:s_j \in S_{k,m} \setminus S_k^*} a_{j+1} + \frac{1}{\gamma} \sum_{\omega \in S_k \setminus S_{k,m}} h_k(\omega | S_k) + \sum_{j:s_j \in S_{k,m} \cap S_k^*} a_{j+1} \]

\[ \leq a_k \sum_{j:s_j \in S_{k,m} \setminus S_k^*} a_{j+1} + \frac{1}{\gamma} \sum_{\omega \in S_k \setminus S_{k,m}} (a_{t+1} + r_{m_t}) + \sum_{j:s_j \in S_{k,m} \cap S_k^*} a_{j+1} \]

\[ \leq a_k \sum_{j:s_j \in S_{k,m} \setminus S_k^*} (a_{j+1} + r_{m_t}) + \sum_{j:s_j \in S_{k,m} \cap S_k^*} (a_{j+1} + r_{m_t}) + \frac{1}{\gamma} (T_k - |C_t|)(a_{t+1} + r_{m_t}) \]

Equation (15) follows by using the definitions of \( r_t \) and supremum, and Equation (16) follows since \( r_t \geq 0 \).

E A simple approach to guarantee low regret: Why it is too weak

In this section, we provide an alternate proof for the approximation ratio that the greedy algorithm obtains on a BP function in the offline setting. This proof can be very simply extended to the online setting in a manner similar to Chen et al. [8]. However, the approximation ratio obtained is worse than that of Bai et al. [4]; hence, the regret guarantee in the online setting would be provided against a weak baseline. This motivates why we revisit the proof from Bai et al. [4].

**Proposition 1.** For a BP maximization problem subject to a cardinality constraint, \( \max_{S:|S| \leq k} h(S) \) where \( h(S) = f(S) + g(S) \), the greedy algorithm obtains the following guarantee:

\[ h(S_k) \geq (1 - e^{-1 - \kappa^g})h(S^*) \]

where \( S^* = \{v_1^*, \ldots, v_k^*\} = \arg \max_{S:|S| \leq k} h(S) \) and \( \kappa^g \) is the curvature of the supermodular function \( g \).

**Proof.** For \( i < k \), let \( S_i = \{v_1, \ldots, v_i\} \) be the items chosen by the greedy algorithm. We can write:

\[ h(S^*) \leq h(S^* \cup S_i) \]

\[ = h(S_i) + \sum_{j=1}^{k} h(v_j^* | S_i \cup \{v_1^*, \ldots, v_{j-1}^*\}) \]

\[ \leq h(S_i) + \frac{1}{1 - \kappa^g} \sum_{j=1}^{k} h(v_j^* | S_i) \]

\[ \leq h(S_i) + \frac{1}{1 - \kappa^g} \sum_{j=1}^{k} h(v_{i+1} | S_i) \]

\[ = h(S_i) + \frac{k}{1 - \kappa^g} (h(S_{i+1}) - h(S_i)) \]

where the first inequality uses Lemma C.1.(ii) of [4] and the second inequality is due to the update rule of the greedy algorithm. Rearranging the terms, we can write:

\[ h(S^*) - h(S_i) \leq \frac{k}{1 - \kappa^g} \left( [h(S^*) - h(S_i)] - [h(S^*) - h(S_{i+1})] \right) \]

\[ h(S^*) - h(S_{i+1}) \leq (1 - \frac{1 - \kappa^g}{k})(h(S^*) - h(S_i)) \]
Applying the above inequality recursively for \( i = 0, \ldots, k - 1 \), we have:

\[
h(S^*) - h(S_k) \leq (1 - \frac{1 - \kappa g}{k})^k (h(S^*) - h(\emptyset)) = 0
\]

Using the inequality \( 1 - x \leq e^{-x} \) and rearranging the terms, we have:

\[
h(S_k) \geq (1 - e^{-(1 - \kappa g)}) h(S^*)
\]

If \( \kappa_f = 1 \), this approximation ratio matches the obtained approximation ratio for the greedy algorithm in Theorem 3.7 of [4] without the need to change the original proof of the greedy algorithm. \( \Box \)