ABELIAN VARIETIES WITH MANY ENDOMORPHISMS
AND THEIR ABSOLUTELY SIMPLE FACTORS

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Abstract. We characterize the abelian varieties arising as absolutely simple factors of GL₂-type varieties over a number field k. In order to obtain this result, we study a wider class of abelian varieties: the k-varieties A/k satisfying that End₀^k(A) is a maximal subfield of End₀^k(A). We call them Ribet-Pyle varieties over k. We see that every Ribet-Pyle variety over k is isogenous over ¯k to a power of an abelian k-variety and, conversely, that every abelian k-variety occurs as the absolutely simple factor of some Ribet-Pyle variety over k. We deduce from this correspondence a precise description of the absolutely simple factors of the varieties over k of GL₂-type.

1. Introduction

Let k be a number field. An abelian variety A over k is said to be of GL₂-type if its algebra of k-endomorphisms End₀^k(A) = Q ⊗ Z End_k(A) is a number field of degree equal to the dimension of A. The aim of this note is to characterize the abelian varieties over ¯k that arise as absolutely simple factors of GL₂-type varieties over k.

The interest in abelian varieties over Q of GL₂-type arose in connection with the Shimura-Taniyama conjecture on the modularity of elliptic curves over Q, and its generalization to higher dimensional modular abelian varieties over Q. To be more precise, to each A/Q of GL₂-type is attached a compatible system of λ-adic representations ρ_A,λ: G_Q → GL_2(E_λ), where E = End_Q^0(A) and the λ’s are primes of E. As a consequence of Serre’s conjecture on Galois representations these ρ_A,λ are modular; that is, there exists a newform f ∈ S_2(Γ_1(N)) such that ρ_A,λ ≃ ρ_{f,λ} for all primes λ of E, where ρ_{f,λ} is the λ-adic representation attached to f (see [4] for the details).

The study of the Q-simple factors of GL₂-type varieties over Q was initiated by K. Ribet in [4], in which the one-dimensional factors where characterized: they are the elliptic curves C/Q that are isogenous to all their
Galois conjugates, also known as elliptic $\mathbb{Q}$-curves. This result was completed by Ribet’s student E. Pyle in her PhD thesis [3], where she characterized the higher dimensional $\overline{\mathbb{Q}}$-simple factors as a certain type of abelian $\mathbb{Q}$-varieties called building blocks. More concretely, an abelian variety $B/\mathbb{Q}$ is an abelian $\mathbb{Q}$-variety if it is $\text{End}_{\mathbb{Q}}(B)$-equivariantly isogenous to all of its Galois conjugates; this means that for each $\sigma \in G_{\mathbb{Q}}$ there exists an isogeny $\mu_{\sigma} : \sigma B \to B$ such that $\varphi \circ \mu_{\sigma} = \mu_{\sigma} \circ \sigma \varphi$ for all $\varphi \in \text{End}_{\mathbb{Q}}(B)$. A building block is an abelian $\mathbb{Q}$-variety $B$ whose endomorphism algebra is a central division algebra over a totally real field $F$, with Schur index $t \leq 2$ and reduced degree $t[F : \mathbb{Q}] = \dim B$. The following statement is Proposition 1.3 and Proposition 4.5 of [3].

**Theorem 1.1** (Ribet-Pyle). Let $A/\mathbb{Q}$ be an abelian variety of $GL_2$-type such that $A_{\overline{\mathbb{Q}}}$ does not have complex multiplication. Then $A_{\overline{\mathbb{Q}}}$ decomposes up to $\overline{\mathbb{Q}}$-isogeny as $A_{\overline{\mathbb{Q}}} \sim B^n$ for some building block $B/\overline{\mathbb{Q}}$. Conversely, if $B/\overline{\mathbb{Q}}$ is a building block then there exists a $GL_2$-type variety $A/\mathbb{Q}$ such that $A_{\overline{\mathbb{Q}}} \sim B^n$ for some $n$.

Observe that this result establishes a correspondence between abelian varieties of $GL_2$-type over $\mathbb{Q}$ without CM and building blocks. In the last chapter of Pyle’s thesis, a series of questions were posed about whether a similar correspondence holds for $GL_2$-type varieties over other fields $k$. The goal of this note is to establish such correspondence when $k$ is a number field. In this case, the analogous of building blocks are abelian $k$-varieties (that is, varieties $B/\overline{k}$ equivariantly isogenous to $\sigma B$ for all $\sigma \in G_k$) whose endomorphism algebra is a central division algebra over a field $F$ with Schur index $t \leq 2$ and $t[F : \mathbb{Q}] = \dim B$. We call these varieties building $k$-blocks.

We prove in Section 3 that every $GL_2$-type variety $A/k$ such that $A_{\overline{k}}$ does not have CM is $\overline{k}$-isogenous to the power of a building $k$-block. Conversely, every building $k$-block arises as the $\overline{k}$-simple factor of some variety over $k$ of $GL_2$-type. In other words, we construct a correspondence

\[
\begin{array}{c}
\{A/k \text{ of } GL_2\text{-type without CM}\} \\
\text{k-isogeny}
\end{array} \leftrightarrow 
\begin{array}{c}
\{\text{building } k\text{-blocks } B/\overline{k}\} \\
\text{k-isogeny}
\end{array}
\]

This can be seen as a natural generalization of the results of Ribet and Pyle to a wider class of abelian varieties. Moreover, it is worth noting that varieties over $k$ of $GL_2$-type play a similar role as their counterparts over $\mathbb{Q}$ with respect to modularity: they are conjectured to be modular, at least when $k$ is totally real, in a similar sense as they are known to be modular for $k = \mathbb{Q}$. Indeed, if $A/k$ is of $GL_2$-type and $k$ is a totally real number field, a generalization of the Shimura-Taniyama conjecture predicts the existence of a Hilbert modular form $f$ such that $\rho_{A,\lambda} \simeq \rho_{f,\lambda}$ for all primes $\lambda$ of $E = \text{End}_k^0(A)$. See [1, Conjecture 2.4] for a precise statement.
Observe that in correspondence (1) the objects in the right hand side are $k$-varieties whose endomorphism algebra satisfies certain conditions. Instead of proving (1) directly, what we do is to construct as a previous step a more general correspondence, in which the right hand side is enlarged to all abelian $k$-varieties. As we will see, the varieties that correspond to them in the left hand side are then varieties $A/k$ characterized by the fact that $A_{\bar{k}}$ is a $k$-variety and $\text{End}_{k}^{0}(A)$ is a maximal subfield of $\text{End}_{\bar{k}}^{0}(A)$. We call the varieties satisfying these properties Ribet-Pyle varieties, because they arise naturally in this generalization of the results of Ribet and Pyle. Section 2 is devoted to the study of Ribet-Pyle varieties and their absolutely simple factors, and we obtain the following main result.

**Theorem 1.2.** Let $k$ be a number field and let $A/k$ be a Ribet-Pyle variety. Then $A_{\bar{k}}$ decomposes up to $\bar{k}$-isogeny as $A_{\bar{k}} \sim B^n$ for some abelian $k$-variety $B/\bar{k}$. Conversely, if $B/\bar{k}$ is a $k$-variety then there exists a Ribet-Pyle variety $A/k$ such that $A_{\bar{k}} \sim B^n$ for some $n$.

This result gives some insight into the nature of the correspondences of Theorem 1.1 and its generalization (1). Indeed, what we do in Section 3 is to prove that varieties over $k$ of GL$_2$-type without CM are Ribet-Pyle varieties, and then we obtain (1) by applying Theorem 1.2 to GL$_2$-type varieties.

### 2. Ribet-Pyle varieties

Let $k$ be a number field. In this section we establish and prove the correspondence between abelian $k$-varieties and Ribet-Pyle varieties of Theorem 1.2. We begin by giving the relevant definitions.

**Definition 2.1.** An abelian variety $B/\bar{k}$ is an abelian $k$-variety if for each $\sigma \in G_k$ there exists an isogeny $\mu_{\sigma} : \sigma B \to B$ compatible with the endomorphisms of $B$; i.e., such that for all $\varphi \in \text{End}_{\bar{k}}(B)$ the following diagram is commutative

\[
\begin{array}{ccc}
\sigma B & \xrightarrow{\mu_{\sigma}} & B \\
\downarrow{\sigma \varphi} & & \downarrow{\varphi} \\
\sigma B & \xrightarrow{\mu_{\sigma}} & B \\
\end{array}
\]

**Definition 2.2.** An abelian variety $A$ defined over $k$ is a Ribet-Pyle variety if $A_{\bar{k}}$ is an abelian $k$-variety and $\text{End}_{k}^{0}(A)$ is a maximal subfield of $\text{End}_{\bar{k}}^{0}(A)$.

**Remark 2.3.** We remark that not all abelian varieties $A$ defined over $k$ satisfy that $A_{\bar{k}}$ is a $k$-variety. Indeed, although in this case the identity is an obvious isogeny between $\sigma A$ and $A$, it is not necessarily compatible with $\text{End}_{\bar{k}}(A)$ in general.

One of the directions of the correspondence that we aim to establish follows almost immediately from the definitions.
Proposition 2.4. Let $A/k$ be a Ribet-Pyle variety. Then it decomposes up to $k$-isogeny as $A_k \sim B^n$, for some simple abelian $k$-variety $B$ and some $n$.

Proof. Let $F$ be the center of $\text{End}_k^0(A)$ and let $\varphi$ be an element of $F$. Since $A_k$ is a $k$-variety, for each $\sigma \in G_k$ we have that

$$\sigma \varphi = \mu_{\sigma}^{-1} \cdot \varphi \cdot \mu_{\sigma},$$

for some isogeny $\mu_{\sigma} : \sigma A_k \rightarrow A_k$. Since $A$ is defined over $k$ the isogeny $\mu_{\sigma}$ belongs to $\text{End}_k^0(A)$. Then $\sigma \varphi = \varphi$ because $\varphi$ belongs to the center of $\text{End}_k^0(A)$. This gives the inclusion $F \subseteq \text{End}_k^0(A)$. By hypothesis $\text{End}_k^0(A)$ is a field, so $F$ is a field as well and this implies that $A_k \sim B^n$ for some simple variety $B$ and some $n$. Next, we show that $B$ is a $k$-variety. By fixing an isogeny $A_k \sim B^n$ the center of $\text{End}_k^0(B)$ can be identified with $F$, and each compatible isogeny $\mu_{\sigma} : \sigma A_k \rightarrow A_k$ gives rise to an isogeny $\nu_{\sigma} : \sigma B \rightarrow B$. The relation (3) implies that $\psi = \nu_{\sigma} \circ \sigma \psi \cdot \nu_{\sigma}^{-1}$ for all $\psi \in Z(\text{End}_k^0(B)) \simeq F$, so that the map

$$\text{End}_k^0(B) \longrightarrow \text{End}_k^0(B)$$

$$\psi \longmapsto \nu_{\sigma} \circ \sigma \psi \cdot \nu_{\sigma}^{-1}$$

is a $F$-algebra automorphism. By the Skolem-Noether Theorem it is inner, and there exists an element $\alpha_{\sigma} \in \text{End}_k^0(B)^*$ such that

$$\nu_{\sigma} \circ \sigma \psi \cdot \nu_{\sigma}^{-1} = \alpha_{\sigma}^{-1} \circ \psi \cdot \alpha_{\sigma},$$

for all $\psi \in \text{End}_k^0(B)$. The isogeny $\alpha_{\sigma} \cdot \nu_{\sigma}$ satisfies the compatibility condition (2) and we see that $B$ is a $k$-variety.

The following statement gives the other direction of the correspondence between $k$-varieties and Ribet-Pyle varieties in the number field case.

Theorem 2.5. Let $k$ be a number field, and let $B/k$ be a simple abelian $k$-variety. Then there exists a Ribet-Pyle variety $A/k$ such that $A_k \sim B^n$ for some $n$.

Before giving the proof of Theorem 2.5 we shall need some preliminary results.

Cohomology classes and splitting fields. Let $k$ be a number field and let $B/k$ be a simple abelian $k$-variety. Let $B$ be its endomorphism algebra and let $F$ be the center of $B$. Since $B$ has a model over a finite extension of $k$, we can choose for each $\sigma \in G_k$ a compatible isogeny $\mu_{\sigma} : \sigma B \rightarrow B$ in such a way that the set $\{\mu_{\sigma}\} \subset \text{End}_k(G_k)$ is locally constant; more precisely, such that $\mu_{\sigma} = \mu_{\tau}$ if $\sigma B = \tau B$. Then we can define a map $c_B : G_k \times G_k \rightarrow F^*$ by means of $c_B(\sigma, \tau) = \mu_{\sigma} \cdot \mu_{\tau} \cdot \mu_{\sigma\tau}^{-1}$. It is easy to check that $c_B$ is a continuous 2-cocycle of $G_k$ with values in $F^*$ (considering the trivial action of $G_k$ in $F^*$). Its cohomology class $[c_B] \in H^2(G_k, F^*)$ is an invariant of the isogeny
We say that a map $\beta$ satisfies (4) if, as we vary $\chi$, the class of $\epsilon$ exists continuous maps $\beta$ such that $H^2(G_k, F^*)$ is trivial, which means that there exist continuous maps $\beta : G_k \to \overline{F^*}$ such that

$$c_B(\sigma, \tau) = \beta(\sigma)\beta(\tau)\beta(\sigma\tau)^{-1}.$$  

We say that a map $\beta$ satisfying (4) is a splitting map for the cocycle $c_B$. If $\chi : G_k \to \overline{F^*}$ is a character then $\beta' = \beta\chi$ is another splitting map for $c_B$. In fact, as we vary $\chi$ through all the characters from $G_k$ to $\overline{F^*}$ we obtain all the splitting maps for $c_B$. For a splitting map $\beta$, we will denote by $E_\beta$ the field $F(\{\beta(\sigma)\}_{\sigma \in G_k}) \subseteq \overline{F}$. The extension $E_\beta / F$ is finite because $\beta$ is continuous.

Let $m$ be the order of $[c_B]$ in $H^2(G_k, F^*)$, and let $d$ be a continuous map $d : G_k \to F^*$ expressing $c_B$ as a coboundary:

$$c_B(\sigma, \tau)^m = d(\sigma)d(\tau)d(\sigma\tau)^{-1}.$$  

We define a map

$$\epsilon_\beta : G_k \to \overline{F^*}$$

$$\sigma \mapsto \beta(\sigma)^m/d(\sigma).$$

By (4) and (5) we see that $\epsilon_\beta : G_k \to \overline{F^*}$ is a continuous character.

Lemma 2.6. For each nonnegative integer $n$ there exists a splitting map $\beta$ such that $F(\zeta_n) \subseteq E_\beta$, where $\zeta_n$ is a primitive $n$-th root of unity in $\overline{F}$.

Proof. Let $\beta'$ be a splitting map for $c_B$, and let $r$ be the order of $\epsilon_\beta$. Let $e = \gcd(n, r)$ and let $\chi : G_k \to \overline{F^*}$ be a character of order $mn/e$, where $m$ is the order of $[c_B]$ in $H^2(G_k, F^*)$. Then the character $\chi^m \epsilon_\beta$ is the character that corresponds to the splitting map $\beta = \chi\beta'$ and its order is $nr/e$, which is a multiple of $n$. Therefore $E_\beta$ contains a primitive $n$-th root of unity $\zeta_n$. \qed

Cyclic splitting fields of simple algebras. Let $\mathcal{A}$ be a central simple algebra over a number field $F$. A well-known result of central simple algebras over number fields guarantees the existence of fields $L$ cyclic over $F$ that split $\mathcal{A}$ (i.e. with $\mathcal{A} \otimes_F L \simeq M_n(L)$ for some $n$). In order to prove Theorem 2.5 we use a similar result, but with the extension $L$ being cyclic over $\mathbb{Q}$ and such that $LF$ splits $\mathcal{A}$. Although this is probably also well-known, for lack of reference we include a proof based on the Grunwald-Wang Theorem.

Theorem 2.7 (Grunwald-Wang Theorem). Let $M$ be a number field, and let $\{(v_1, n_1), \ldots, (v_r, n_r)\}$ be a finite set of pairs, where each $v_i$ is a place of $M$ and each $n_i$ is a positive integer such that $n_i \leq 2$ if $v_i$ is a real place, and $n_i = 1$ if $v_i$ is a complex place. Let $m$ be the least common multiple of...
the $n_i$’s, and let $n$ be a positive integer divisible by $m$. Then there exists a cyclic extension $L/M$ of degree $n$ such that for each $i$ the degree $[L_{v_i} : M_{v_i}]$ is divisible by $n_i$.

**Proposition 2.8.** Let $F$ be a number field and let $\mathcal{D}$ be a central division algebra over $F$. There exists a cyclic extension $L/\mathbb{Q}$ such that $LF$ is a splitting field for $\mathcal{D}$.

**Proof.** Let $F'$ be the Galois closure of $F$. Let $n = [F' : \mathbb{Q}]$ and let $t$ be the Schur index of $\mathcal{D}$. Let $\{p_1, \ldots, p_s\}$ be the set of primes of $F$ where $\mathcal{D}$ ramifies, and let $\{p_1, \ldots, p_t\}$ be the set of primes of $\mathbb{Q}$ below $\{p_1, \ldots, p_s\}$. The Grunwald-Wang Theorem, when applied to the primes $p_i$ with $n_i = tn$, and to the infinite place of $\mathbb{Q}$ with $n_\infty = 2$, guarantees the existence of a cyclic extension $L/\mathbb{Q}$ of degree $2tn$ such that $[L_p : \mathbb{Q}_p] = tn$ for all $p$ belonging to $\{p_1, \ldots, p_t\}$ and $L_v = \mathbb{C}$ for all archimedean places $v$ of $L$. Let $K = LF$.

If $p$ is a prime of $F$ dividing $p$, and $\mathfrak{P}$ is a prime of $K$ dividing $p$, the fields $L_p$ and $F_p$ can be seen as subfields of $K_{\mathfrak{P}}$. Then the degree $g = [L_p \cap F_p : \mathbb{Q}_p]$ divides $n$, so $[L_p : L_p \cap F_p] = \frac{tn}{g} = [F_p L_p : F_p]$ and we see that $t$ divides $[K_{\mathfrak{P}} : F_p]$. Therefore, $K$ is a totally imaginary extension of $F$ such that, for every prime $p$ of $F$ ramifying in $\mathcal{D}$ and for every prime $\mathfrak{P}$ of $K$ dividing $p$, the index $[K_{\mathfrak{P}} : F_p]$ is a multiple of the Schur index of $\mathcal{D}$. This implies that $K$ is a splitting field for $\mathcal{D}$ (see [2] Corollary 18.4 b and Corollary 17.10 a)).

**Corollary 2.9.** Every central division $F$-algebra is split by an extension of the form $F(\zeta_m)$ for some $m$.

**Proof.** By the previous proposition there exists a cyclic extension $L/\mathbb{Q}$ such that $LF$ splits $\mathcal{D}$. The field $L$ is contained in a field of the form $\mathbb{Q}(\zeta_m)$ by the Kronecker-Weber Theorem, and then $F(\zeta_m)$ splits $\mathcal{D}$.

**Construction of Ribet-Pyle varieties.** In this paragraph we perform the construction of Ribet-Pyle varieties having a $k$-variety $B$ as simple factor. Recall that $\mathcal{B}$ denotes $\text{End}_k^0(B)$, $F$ is the center of $\mathcal{B}$ and $t$ denotes the Schur index of $\mathcal{B}$. Fix also a locally constant set of isogenies $\{\mu_\sigma : \sigma B \to B\}_{\sigma \in G_k}$, let $c_B$ be the cocycle constructed with these isogenies and let $\beta$ be a splitting map for $c_B$.

Let $n$ be the degree $[E_\beta : F]$, and fix an injective $F$-algebra homomorphism

$$\phi : E_\beta \longrightarrow M_n(F) \subseteq M_n(\mathcal{B}) \simeq \text{End}_k^0(B^n).$$

The elements of $E_\beta$ act as endomorphisms of $B^n$ up to isogeny by means of $\phi$. Let $\hat{\mu}_\sigma$ be the diagonal isogeny $\hat{\mu}_\sigma : \sigma B^n \to B^n$ consisting in $\mu_\sigma$ in each factor. 
Proposition 2.10. There exists an abelian variety $X_\beta$ over $k$ and a $\bar k$-isogeny $\kappa: B^n \to X_\beta$ such that $\kappa^{-1} \circ \kappa = \phi(\beta(\sigma))^{-1} \circ \hat \mu_\sigma$ for all $\sigma \in G_k$. Moreover, the $k$-isogeny class of $X_\beta$ is independent of the chosen injection $\phi$.

Proof. Let $\nu_\sigma$ be the isogeny defined as $\nu_\sigma = \phi(\beta(\sigma))^{-1} \circ \hat \mu_\sigma$. In order to prove the existence of $X_\beta$, by [4, Theorem 8.1] we need to check that $\nu_\sigma \circ \nu_\tau \circ \nu_\sigma^{-1} = 1$. By the compatibility of $\mu_\sigma$ we have that:

$$\nu_\sigma \circ \nu_\tau \circ \nu_\sigma^{-1} = \phi(\beta(\sigma))^{-1} \circ \hat \mu_\sigma \circ \phi(\beta(\tau))^{-1} \circ \hat \mu_\tau \circ \phi(\beta(\sigma)) = \phi(\beta(\sigma))^{-1} \circ \phi(\beta(\tau))^{-1} \circ \mu_\sigma \circ \mu_\tau \circ \phi(\beta(\sigma)) \circ \phi(\beta(\tau)) = \phi(\beta(\sigma))^{-1} \circ \beta(\tau)^{-1} \circ \beta(\sigma)^{-1} \circ \beta(\tau) = \phi(c_B(\sigma, \tau)^{-1} \circ c_B(\sigma, \tau) = c_B(\sigma, \tau)^{-1} \circ c_B(\sigma, \tau) = 1.$$ 

Now suppose that $\phi$ and $\psi$ are $F$-algebra homomorphisms $E_\beta \to M_n(F)$, and let $X_{\beta, \phi}$ and $X_{\beta, \psi}$ denote the varieties constructed by the above procedure using $\phi$ and $\psi$ respectively to define the action of $E_\beta$ on $B^n$. We aim to see that $X_{\beta, \phi}$ and $X_{\beta, \psi}$ are $k$-isogenous.

Let $C$ denote the image of $\phi$. The map $\phi(x) \mapsto \psi(x): C \to M_n(F)$ is a $F$-algebra homomorphism. Since $C$ is simple and $M_n(F)$ is central simple over $F$, by the Skolem-Noether Theorem there exists an element $b$ in $M_n(F)$ such that $\phi(x) = b \psi(x) b^{-1}$ for all $x$ in $E_\beta$. By the defining property of $X_{\beta, \phi}$ and $X_{\beta, \psi}$, there exist $k$-isogenies $\kappa: B^n \to X_{\beta, \phi}$ and $\lambda: B^n \to X_{\beta, \psi}$ such that

$$\kappa^{-1} \circ \kappa = \phi(\beta(\sigma))^{-1} \circ \hat \mu_\sigma = b \circ \psi(\beta(\sigma))^{-1} \circ b^{-1} \circ \hat \mu_\sigma,$$

$$\lambda^{-1} \circ \lambda = \psi(\beta(\sigma))^{-1} \circ \hat \mu_\sigma.$$ 

The $\bar k$-isogeny $\nu = \kappa \circ b \circ \lambda^{-1}: X_{\beta, \psi} \to X_{\beta, \phi}$ is in fact defined over $k$, since for each $\sigma$ of $G_k$ we have that

$$\nu^{-1} \circ \nu = \lambda \circ b^{-1} \circ \kappa^{-1} \circ \kappa \circ b \circ \lambda^{-1} = \lambda \circ b^{-1} \circ b \circ \psi(\beta(\sigma))^{-1} \circ b^{-1} \circ \hat \mu_\sigma \circ b \circ \lambda^{-1} = \lambda \circ \psi(\beta(\sigma))^{-1} \circ \hat \mu_\sigma \circ b \circ \lambda^{-1} \circ b \circ \lambda^{-1} = \lambda \circ \lambda^{-1} \circ \lambda \circ \lambda^{-1} = 1,$$

where we used the compatibility of $\hat \mu_\sigma$ with the endomorphisms of $B^n$ in the third equality, and the expressions (6) and (7) in the second and fourth equality respectively.

Proposition 2.11. The algebra $\text{End}^0_k(X_\beta)$ is isomorphic to the centralizer of $E_\beta$ in $M_n(B)$. 

Proof. \( \text{End}_k^0(\mathcal{X}_\beta) \) is isomorphic to \( M_n(\mathcal{B}) \) and every endomorphism of \( \mathcal{X}_\beta \) up to \( \bar{k} \)-isogeny is of the form \( \kappa \ast \psi \ast \kappa^{-1} \), for some \( \psi \in \text{End}_k^0(B^n) \). For \( \sigma \) in \( G_k \) we have:

\[
\sigma((\kappa \ast \psi \ast \kappa^{-1})) = \kappa \ast \psi \ast \kappa^{-1} \iff \sigma \kappa \ast \sigma \psi \ast \sigma \kappa^{-1} = \kappa \ast \psi \ast \kappa^{-1} \iff \kappa^{-1} \ast \sigma \kappa \ast \sigma \psi \ast (\kappa^{-1} \ast \sigma \kappa)^{-1} = \psi \iff \beta(\sigma) \ast \mu_\sigma \ast \sigma \psi \ast \mu_\sigma^{-1} \ast \beta(\sigma)^{-1} = \psi \iff \beta(\sigma) \ast \psi \ast \beta(\sigma)^{-1} = \psi.
\]

Thus the endomorphisms of \( \mathcal{X}_\beta \) defined over \( k \) are exactly the ones coming from endomorphisms \( \psi \) that commute with \( \beta(\sigma) \), for all \( \sigma \) in \( G_k \). Now the proposition is clear, since the \( \beta(\sigma) \)'s generate \( E_\beta \).

\[\square\]

Corollary 2.12. The algebra \( \text{End}_k^0(\mathcal{X}_\beta) \) is isomorphic to \( E_\beta \otimes_F \mathcal{B} \).

Proof. Let \( C \) be the centralizer of \( E_\beta \) in \( M_n(\mathcal{B}) \). In view of Proposition 2.11 we have to prove that \( C \simeq E_\beta \otimes_F \mathcal{B} \). It is clear that \( E_\beta \) is contained in \( C \). Moreover, \( \mathcal{B} \) is contained in \( C \) because the elements of \( E_\beta \) can be seen as \( n \times n \) matrices with entries in \( F \), and these matrices commute with \( \mathcal{B} \) (which is identified with the diagonal matrices in \( M_n(\mathcal{B}) \)). Since \( E_\beta \) and \( \mathcal{B} \) commute there exists a subalgebra of \( C \) isomorphic to \( E_\beta \otimes_F \mathcal{B} \), which has dimension \( nt^2 \) over \( F \). By the Double Centralizer Theorem we know that

\[ [C : F][E_\beta : F] = [M_n(\mathcal{B}) : F] = nt^2, \]

and from this we obtain that \( [C : F] = nt^2 \), hence \( C \) is isomorphic to \( E_\beta \otimes_F \mathcal{B} \).

\[\square\]

At this point we have at our disposal all the tools needed to prove Theorem 2.5.

Proof of Theorem 2.5. By Corollary 2.9 there exists an integer \( m \) such that \( F(\zeta_m) \) splits \( \mathcal{B} \). Let \( \beta \) be a splitting map for \( c_B \) with \( E_\beta \) containing \( F(\zeta_m) \); the existence of such a \( \beta \) is guaranteed by Lemma 2.6. Consider the variety \( \mathcal{X}_\beta \) defined as in Proposition 2.10. By Corollary 2.12 we have that \( \text{End}_k^0(\mathcal{X}_\beta) \simeq E_\beta \otimes_F \mathcal{B} \), and this later algebra is in turn isomorphic to \( M_t(E_\beta) \) because \( E_\beta \) is a splitting field for \( \mathcal{B} \). Therefore, there exists an abelian variety \( A_\beta \) defined over \( k \) such that \( \mathcal{X}_\beta \sim_k A_\beta^t \) and \( \text{End}_k^0(A_\beta) \simeq E_\beta \). Clearly \( A_\beta \) is \( \bar{k} \)-isogenous to \( B^{n/t} \), where \( n = [E_\beta : F] \), and we claim that it is a Ribet-Pyle variety. First of all, it is easily seen that the power of a \( k \)-variety is also a \( k \)-variety. This implies that \((A_\beta)_k \) is a \( k \)-variety. Moreover, we have that \( [\text{End}_k^0(A_\beta) : F] = [E_\beta : F] = n \), and the dimension of the ambient algebra is \( [\text{End}_k^0(A_\beta) : F] = (\frac{nt}{t})^2[B : F] = n^2 \). This implies (cf. Proposition 13.1) that \( \text{End}_k^0(A) \) is a maximal subfield of \( \text{End}_k^0(A) \).

\[\square\]
Proposition 2.13. Let $B$ be a $k$-variety and let $A/k$ be a Ribet-Pyle variety having $B$ as $\bar{k}$-simple factor. Then $A$ is $k$-isogenous to the variety $A_\beta$ obtained by applying the above procedure to some cocycle $c_B$ attached to $B$ and some splitting map $\beta$ for $c_B$.

Proof. Let $B = \text{End}_k^0(B)$, let $F$ be the center of $B$ and let $t$ be the Schur index of $B$. Let $E$ be the maximal subfield $\text{End}_k^0(A)$ of $\text{End}_k^0(A)$, and fix an embedding of $E$ into $\overline{F}$. Let $\kappa$ be an isogeny $\kappa: B^\text{nt} \to A_{\bar{k}}$. We have the relation $[E : F] = nt$. Let $\{\mu_\sigma: \sigma B \to B\}_{\sigma \in G_k}$ be a locally constant set of compatible isogenies and denote by $\tilde{\mu}_\sigma: \text{End}_k^0(B) \to \text{End}_k^0(B)$ the diagonal of $\mu_\sigma$. Define $\beta(\sigma) = \kappa \circ \tilde{\mu}_\sigma \circ \kappa^{-1}$, which is a compatible isogeny $\beta(\sigma): A_{\bar{k}} \to A_{\bar{k}}$. The fact that $\beta(\sigma)$ is compatible implies that

$$
(8) \quad \beta(\sigma) \circ \varphi = \sigma \varphi \circ \beta(\sigma)
$$

for all $\sigma$ in $G_k$ and for all $\varphi \in \text{End}_k^0(A)$. In particular, when applied to elements $\varphi$ of $E$ this property says that $\beta(\sigma)$ lies in $C(E)$, the centralizer of $E$. But $C(E)$ is equal to $E$, because $E$ is a maximal subfield. Thus $\beta(\sigma)$ belongs to $E$ and it is an isogeny defined over $k$. Now we have that

$$
c_B(\sigma, \tau) = \mu_\sigma \circ \mu_\tau \circ \mu_{\sigma \tau}^{-1} = \tilde{\mu}_\sigma \circ \tilde{\mu}_\tau \circ \tilde{\mu}_{\sigma \tau}^{-1}
$$

$$
= \beta(\sigma) \circ \sigma \beta(\tau) \circ \beta(\sigma \tau)^{-1} = \beta(\sigma) \circ \beta(\tau) \circ \beta(\sigma \tau)^{-1},
$$

and we see that the map $\sigma \mapsto \beta(\sigma)$ is a splitting map for $c_B$. We have already seen the inclusion $E_\beta \subseteq E$. From (8) it is clear that $C(E_\beta) \subseteq E$, and taking centralizers and applying the Double Centralizer Theorem we have that $E = C(E) \subseteq C(C(E_\beta)) = E_\beta$. Thus $E = E_\beta$ and, in particular, $[E_\beta : F] = nt$.

Now we define a $\bar{k}$-isogeny $\hat{\kappa}: (B^{\text{nt}})^t \to A_{\bar{k}}^t$ as the diagonal isogeny associated to $\kappa$, and we make $E_\beta$ act on $B^{nt}$ by means of $\hat{\kappa}$. It is easy to check that $\hat{\kappa}^{-1} \circ \hat{\kappa} = \tilde{\kappa}^{-1} \circ \tilde{\kappa} \circ \mu_\sigma$, so $A^t$ satisfies the property defining $X_\beta$. By the uniqueness property of $X_\beta$ we have that $A^t \sim_k X_\beta$, and so $A_\beta \sim_k A$. □

Remark 2.14. The hypothesis that $k$ is a number field has been used only in order to guarantee the existence of splitting maps for $c_B$, by means of Tate’s theorem on the triviality of $H^2(G_k, \overline{F}^\ast)$. Since Tate’s theorem is valid for any global or local field $k$, Theorem [1.2] is valid for any global or local field $k$ as well.

3. Varieties over $k$ of GL$_2$-type and $k$-varieties

Let $k$ be a number field. In this section we characterize the absolutely simple factors of the varieties over $k$ of GL$_2$-type, in the case where they do not have complex multiplication.

Proposition 3.1. Let $A/k$ be an abelian variety of GL$_2$-type such that $A_{\bar{k}}$ does not have complex multiplication. Then $A$ is a Ribet-Pyle variety.
Proof. By [6] Proposition 1.5] we can suppose that \( A \) does not have any simple factor with CM. Let \( A \simeq B_1^{n_1} \times \cdots \times B_r^{n_r} \) be the decomposition of \( A \) into simple abelian varieties up to isogeny. Since \( E = \text{End}_{A}(A) \) is a field it acts on each factor \( B_i^{n_i} \), and so it acts on the homology with rational coefficients \( H_1((B_i^{n_i})_C, \mathbb{Q}) \), which is a vector space of dimension \( 2 \dim B_i^{n_i} \) over \( \mathbb{Q} \). Thus \( 2 \dim B_i^{n_i} \) is divisible by \([E : \mathbb{Q}] = \dim A\). But \( \dim A \geq \dim B_i^{n_i} \), so either \([E : \mathbb{Q}] = \dim B_i^{n_i}\) or \(2[E : \mathbb{Q}] = \dim B_i^{n_i}\).

The later is not possible, because it would mean that \( B_i^{n_i} \) has CM by \( E \). Thus \( \dim A = \dim B_i^{n_i} \) and \( A \) has only one simple factor up to isogeny; say \( A \simeq B \).

Next, we see that \( E \) is a maximal subfield of \( \text{End}_{A}(A) \). Let \( C \) be the centralizer of \( E \) in \( \text{End}_{A}(A) \), and let \( \varphi \) be an element in \( C \). A priori \( \varphi(A) \) is isogenous to \( B^r \) for some \( r \leq n \). Since \( \varphi \in C \), the field \( E \) acts on \( \varphi(A) \); as before this implies that \([E : \mathbb{Q}] \) divides \( 2 \dim B^r \). But \([E : \mathbb{Q}] = \dim A = \dim B^r \), therefore \( r = n \) or \( r = n/2 \). Again \( r = n/2 \) is not possible, because then \( B^r \) would be a factor of \( A \) with CM by \( E \). Thus \( r = n \) and \( \varphi \) is invertible in \( \text{End}_{A}^0(A) \). This implies that \( C \) is a field, and then \( E \) is a maximal subfield of \( \text{End}_{A}^0(B) \).

Finally, we see that \( A \) is an abelian \( k \)-variety. For each \( \sigma \in G_k \) the map

\[
\text{End}_{A}^0(A) \, \longrightarrow \, \text{End}_{A}^0(A) \\
\varphi \, \longmapsto \, \sigma \varphi
\]

is the identity when restricted to \( E \). Since \( E \) is a maximal subfield, it contains the center \( F \) of \( \text{End}_{A}^0(A) \), so (9) is a \( F \)-algebra isomorphism. By the Skolem-Noether Theorem there exists an element \( \mu_\sigma \in \text{End}_{A}^0(A)^* \) such that \( \sigma \varphi = \mu_\sigma^{-1} \circ \varphi \circ \mu_\sigma \), and we see that \( \mu_\sigma \) is a compatible isogeny in the sense of Definition 2.1.

\[
\begin{align*}
\text{Definition 3.2.} \quad & \text{A building } k \text{-block is an abelian } k \text{-variety } B/k \text{ such that } \\
& \text{End}_{A}^0(B) \text{ is a central division algebra over a field } F, \text{ with Schur index } t \leq 2 \\
& \text{and reduced degree } t[F : \mathbb{Q}] = \dim B.
\end{align*}
\]

\[
\begin{align*}
\text{Theorem 3.3.} \quad & \text{Let } k \text{ be a number field and let } A/k \text{ be an abelian variety of } \\
& \text{GL}_2\text{-type such that } A \text{ does not have CM. Then } A \simeq B^n \text{ for some building } \\
& k \text{-block } B. \text{ Conversely, if } B \text{ is a building } k \text{-block then there exists a variety } \\
& A/k \text{ of GL}_2\text{-type such that } A \simeq B^n \text{ for some } n.
\end{align*}
\]

Proof. By Proposition [3.1] \( A \) is a Ribet-Pyle variety, and by Proposition 2.4 we have that \( A \simeq B^n \) for some \( k \)-variety \( B \). Let \( B = \text{End}_{A}^0(B) \), let \( F \) be the center of \( B \) and let \( t \) be its Schur index. Then \( E = \text{End}_{A}^0(A) \) is a maximal subfield of \( \text{End}_{A}^0(A) \simeq M_n(B) \), which has dimension \( n^2 t^2 \) over \( F \). Therefore \([E : F] = nt\), and multiplying both sides of this equality by \([F : \mathbb{Q}]\) we see that \([E : \mathbb{Q}] = \dim A = nt[F : \mathbb{Q}]\). The equality \( t[F : \mathbb{Q}] = \dim B \) follows. Since \( B \) is a division algebra of \( \mathbb{Q} \)-dimension \( t^2[F : \mathbb{Q}] \) that acts
on $H_1(B_C, \mathbb{Q})$, which has $\mathbb{Q}$-dimension $2 \dim B = 2t[F : \mathbb{Q}]$, we see that necessarily $t \leq 2$ and $B$ is a building $k$-block.

Conversely, let $B$ be a building $k$-block. In particular it is a $k$-variety, and by Theorem 2.5 there exists a Ribet-Pyle variety $A/k$ such that $A_\mathbb{C} \sim B^n$ for some $n$. The field $E = \text{End}_k^0(A)$ is a maximal subfield of $\text{End}_k^0(A) \simeq M_n(B)$, which means that $[E : F] = nt$. Multiplying both sides of this equality by $[F : \mathbb{Q}]$ we see that $[E : \mathbb{Q}] = nt[F : \mathbb{Q}] = n \dim B = \dim A$, and so $A$ is a variety of $\text{GL}_2$-type.

In the case $k = \mathbb{Q}$ the center of the endomorphism algebra of a building $k$-block is necessarily totally real, but for arbitrary number fields $k$ a priori it can be either totally real or CM. That is why in Definition 3.2 the field $F$ is not required to be totally real. However, if $k$ admits a real embedding then exactly the same argument of [3, Theorem 1.2] shows that $F$ is necessarily totally real. In addition, there are some extra restrictions on the endomorphism algebra.

**Proposition 3.4.** Let $k$ be a number field that admits a real embedding. Let $B$ be a building $k$-block, let $\mathcal{B} = \text{End}_k^0(B)$ and let $F = Z(\mathcal{B})$. Then $F$ is totally real and $\mathcal{B}$ is either isomorphic to $F$ or to a totally indefinite division quaternion algebra over $F$.

**Proof.** We view $k$ as a subfield of $\mathbb{C}$ by means of a real embedding $k \hookrightarrow \mathbb{R}$. Let $A/k$ be a $\text{GL}_2$-type variety such that $A_\mathbb{C} \sim B^n$. Let $E$ be the maximal subfield $\text{End}_k^0(A)$ of $\text{End}_k^0(A)$, and identify $F$ with $Z(\text{End}_k^0(A))$; under this identification $F$ is contained in $E$. Let $t$ be the Schur index of $B$ and let $m = 2 \dim B/[\mathcal{B} : \mathbb{Q}]$, for which we have that $mt = 2$.

The division algebra $\mathcal{B}$ belongs a priori to one of the four types of algebras with a positive involution, according to Albert’s classification (see for instance [5, Proposition 1]). However, type III is not possible; indeed by [5, Proposition 15] the variety $B$ would then be isogenous to the square of a CM abelian variety.

To see that type IV is also not possible, suppose that $F$ is a CM extension of a totally real field $F_0$. Let $\Phi$ denote the complex representation of $\mathcal{B}$ on the space of differential forms $H^0(B_C, \Omega^1)$. For every real embedding $\nu$ of $F_0$ let $\chi_\nu, \overline{\chi}_\nu$ be the two complex-conjugate irreducible representations of $\mathcal{B}$ extending $\nu$. Let $r_\nu$ and $s_\nu$ be the multiplicities of $\chi_\nu$ and $\overline{\chi}_\nu$ in $\Phi$. For each $\nu$ we have that $r_\nu + s_\nu = 2$; moreover, the equality $r_\nu = s_\nu = 1$ is not possible for all $\nu$ (cf. [3, Propositions 18 and 19]). This implies that $\text{Tr}(\Phi)|_F = \sum r_\nu \chi_\nu|_F + s_\nu \overline{\chi}_\nu|_F$ takes non-real values. On the other hand, if we denote by $\Psi$ the complex representation of $\text{End}_k^0(A)$ on $H^0(A_C, \Omega^1)$, then $\text{Tr}(\Psi) = n \text{Tr}(\Phi)$. Since $A$ is defined over $k$ we can take a basis of the differentials defined over $k$, and with respect to this basis the elements of $E$ are represented by matrices with coefficients in $k$. Since $F \subseteq E$, the trace
of $\Psi$ restricted to $F$ takes values in $k \subseteq \mathbb{R}$, giving a contradiction with the fact that $\text{Tr}(\Phi)|_F$ takes non-real values. \hfill \Box

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References

[1] H. Darmon. *Rigid local systems, Hilbert modular forms, and Fermat’s last theorem.* Duke Math. J. **102** (2000), no. 3, 413–449.

[2] R. S. Pierce. *Associative Algebras.* Graduate Texts in Mathematics, 88. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982.

[3] E. Pyle. *Abelian varieties over $\mathbb{Q}$ with large endomorphism algebras and their simple components over $\mathbb{Q}$.* Modular curves and abelian varieties, Progress in Math., vol. 224 (2002), pp. 189–239.

[4] K. A. Ribet. *Abelian varieties over $\mathbb{Q}$ and modular forms.* Algebra and topology 1992 (Taejŏn), 53–79, Korea Adv. Inst. Sci. Tech., Taejŏn, 1992. Reprinted on *Modular curves and abelian varieties*, 241–261, Progr. Math. 224, Birkhäuser, Basel, 2004.

[5] G. Shimura. *On analytic families of polarized abelian varieties and automorphic functions.* Ann. of Math. (2) **78** (1963) 149–192.

[6] G. Shimura. *Class fields over real quadratic fields and Hecke operators.* Ann. of Math. (2) **95** (1972), 130–190.

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