THE TORIC FROBENIUS MORPHISM AND A CONJECTURE OF ORLOV

MATTHEW R BALLARD, ALEXANDER DUNCAN, AND PATRICK K. MCFADDIN

Abstract. We combine the Bondal-Uehara method for producing exceptional collections on toric varieties with a result of the first author and Favero to expand the set of varieties satisfying Orlov’s Conjecture on derived dimension.

1. Introduction

To solve a longstanding question originating with work of Auslander, Rouquier used a new invariant of a triangulated category \( T \) [Rou06]. This invariant is a measure of the minimal homological complexity of the category \( T \). He focused particular attention on the case where \( T = D^b(\text{coh} X) \) for a scheme \( X \) of finite type over a field \( k \). In this case, he showed his invariant is always at least the Krull dimension of \( X \) with equality in certain situations, like Grassmannians \( \text{Gr}(r, n) \). Orlov then asked if Rouquier’s invariant, henceforth known as the Rouquier dimension, is actually equal to the Krull dimension in the case \( X \) is smooth and projective. He showed the answer was yes in the case of curves [Orl09].

Despite being a simply stated question, Orlov’s Conjecture seems difficult to address in general. Indeed, supporting evidence comes from individual constructions specialized to particular examples [BF12, BFK17, Rou08, Orl09]. For toric varieties, we give a more robust method based on an idea of Bondal [Bon06] refined by Uehara [Ueh14] utilizing the toric Frobenius morphism. The main result asserts that if the Bondal-Uehara method produces a tilting bundle then Orlov’s Conjecture holds. We close with applications that illustrate the potency of this simple idea.

2. Toric Frobenius and generation time

2.1. Generation time and the Rouquier dimension. Let \( T \) be a triangulated category. Recall that this guarantees that for any map \( f : A \to B \) in \( T \) there is a triangle

\[
A \xrightarrow{f} B \to C(f) \to A[1].
\]

Generally, the assignment \( f \mapsto C(f) \) is only well-defined up to an isomorphism of \( C(f) \). Even so one commonly calls \( C(f) \) the cone over \( f \). In [Rou08], Rouquier, building on work of [BvB03], introduced a notion of dimension of a triangle category. This notion measures the homological complexity of the category by, roughly, counting cones. Let us be a bit more precise.

The first author was partially supported by NSF DMS-1501813. He would also like to thank the Institute for Advanced Study for providing a wonderful research environment. These ideas were developed during his membership.

The second author was partially supported by NSA grant H98230-16-1-0309.
Definition 2.1. If $S$ is a full subcategory of $T$, let $\langle S \rangle$ denote the full subcategory of $T$ containing $S$ and which is closed under finite coproducts, summands, and translations.

Let $S_1$ and $S_2$ be full subcategories of $T$. Let $S_1 \ast S_2$ be the full subcategory of $T$ consisting of objects $A$ such that there exists a triangle

$$S_1 \to A \to S_2 \to S_1[1]$$

with $S_i$ an object of $S_i$. Define $S_1 \diamond S_2 := \langle S_1 \ast S_2 \rangle$.

One inductively defines

$$\langle S \rangle_0 := \langle S \rangle$$

$$\langle S \rangle_{n+1} := \langle S \rangle_n \diamond \langle S \rangle.$$

One says that $S$ generates $T$ if any object of $T$ is isomorphic to an object of $\langle S \rangle_n$ for some $n$, possibly depending on the object. One says that $S$ strongly generates $T$ if there exists an $n$ such that the inclusion $\langle S \rangle_n \to T$ is an equivalence. One says that the generation time of $S$ is the minimal $n$ such that $\langle S \rangle_n \to T$ is an equivalence.

If $S$ consists of a single object $S$, then one also says that $S$ (strongly) generates if $S$ does. The generation time of the object $S$ is the generation time of $S$.

Definition 2.2. The Rouquier dimension of $T$ is

$$\text{rdim } T := \min \{n \mid \exists \text{ an object } S \text{ so that } \langle S \rangle_n \cong T \}$$

with the notation $\infty$ used if the set is empty. In other words, the Rouquier dimension is the minimal generation time among any of the objects. For a $k$-scheme $X$, we denote $\text{rdim } D^b(\text{coh } X)$ by $\text{rdim } X$.

In [Orl09], Orlov made the following conjecture.

Conjecture 2.3. Let $X$ be a smooth quasiprojective variety. Then the Rouquier dimension and the Krull dimension coincide, i.e.,

$$\text{rdim } X = \dim X.$$

Rouquier had already given the lower bound in [Rou08, Proposition 7.16].

Proposition 2.4. For a reduced separated scheme $X$ of finite type over a field, we have $\text{rdim } X \geq \dim X$.

Thanks to this result, verifying Orlov’s Conjecture for a given $X$ amounts to finding a particular nice generator whose generation time is $\dim X$. We will also need the following basic property of Rouquier dimension, which follows from [Rou08, Lemma 3.4]. Recall that a functor is dense if every object is isomorphic to a summand of an object in the image.

Lemma 2.5. Let $F : S \to T$ be a dense exact functor of triangulated categories. Then

$$\text{rdim } S \geq \text{rdim } T.$$

2.2. A result on generation time for tilting objects. Now, we restrict ourselves to a simpler class of generators.

Definition 2.6. Let $T$ be a triangulated category. An object $T$ of $T$ is called a tilting object if the following two conditions hold:

1. $\text{Hom}_T(T[T[i]]) = 0$ for all $i \neq 0$;
(2) $T$ is a generator for $T$.

For these, one can give a more easily computable upper bound on the generation time. The following is a consequence of [BF12, Theorem 3.2].

**Theorem 2.7.** Let $X$ be a smooth and projective variety. Suppose that $T$ is a tilting object in $D^b(\text{coh } X)$ and let

$$m_0(T) := \max \{ m \mid \text{Hom}(T, T \otimes \omega_X^{-1}[m]) \neq 0 \}.$$  

The generation time of $T$ is bounded above by $\dim X + m_0(T)$. In particular, if there exists a $T$ with $m_0(T) = 0$, then Orlov’s Conjecture holds.

2.3. Toric Frobenius and the anti-nef cone. To search for tilting objects, we follow Bondal [Bon06] and turn to the toric Frobenius morphism. Let $X$ be a (split) smooth projective toric variety of dimension $n$ with fixed torus embedding $T \hookrightarrow X$ and take $\ell \in \mathbb{N}$. Define the $\ell$th Frobenius map on $T = \mathbb{G}_m^n$ to be

$$(x_1, \ldots, x_n) \mapsto (x_1^\ell, \ldots, x_n^\ell).$$

This uniquely extends to an endomorphism of $X$ which will be denoted $F_\ell$ and called the $\ell$th Frobenius morphism. Each sheaf $(F_\ell)_*(\mathcal{O}_X)$ splits into line bundles.

**Definition 2.8.** Let $\text{frob}(X)$ denote the union of all line bundles arising as direct summands of $(F_\ell)_*(\mathcal{O}_X)$ as $\ell$ varies over $\mathbb{Z}^+$. Thomsen provides an explicit description for $(F_\ell)_*(\mathcal{O}_X)$ and shows that the set $\text{frob}(X)$ is finite [Tho00, Proposition 6.1]. Taking a sufficiently divisible $\ell$, we recover the following:

**Proposition 2.9.** There exists an $\ell$ such that $(F_\ell)_*(\mathcal{O}_X)$ contains every line bundle in $\text{frob}(X)$.

Recall that for a normal variety $X$, a Cartier $D$ divisor on $X$ is nef if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$. Let $N^1(X)$ be the quotient group of Cartier divisors by the subgroup of numerically trivial divisors. The nef cone $\text{nef}(X)$ is the cone in $N^1(X) \otimes \mathbb{R}$ given by positive span of the nef divisors, and the anti-nef cone is the cone $\text{fen}(X) := -\text{nef}(X) \subset N^1(X) \otimes \mathbb{R}$. For smooth projective toric varieties, $\text{Pic}(X) = N^1(X)$.

**Definition 2.10.** We denote the intersection by $\text{bu}(X) := \text{frob}(X) \cap \text{fen}(X) \subset \text{Pic}(X)$.

2.4. Main result. We can now state the main result. Let

$$T_{\text{bu}}(X) := \bigoplus_{L \in \text{bu}(X)} L.$$ 

**Theorem 2.11.** Let $X$ be a smooth projective toric variety. If $T_{\text{bu}}(X)$ is a tilting object and $-K_X$ is nef, then the generation time of $T_{\text{bu}}$ is $\dim X$. In particular, Orlov’s Conjecture holds for $X$.

**Proof.** We will apply Theorem 2.7 to $T := T_{\text{bu}}(X)$. Let’s compute $\text{Hom}(T, T \otimes \omega_X^{-1}[m])$ to show that $m_0(T) = 0$. Note that $T$ is a direct summand of $F_\ell*\mathcal{O}$ for some $\ell$. Thus, to get the desired vanishing, we first observe that

$$\text{Hom}(L, F_\ell*\mathcal{O} \otimes \omega_X^{-1}[m]) \cong \text{Hom}(L, F_\ell*\omega_X^{-\ell}[m])$$
using the projection formula. Using adjunction, we have
\[ \text{Hom}(L, F^\ell)(\omega_X^{-\ell}[m]) \cong \text{Hom}(F^\ell, L, \omega_X^{-\ell}[m]) \cong \text{Hom}(L^\ell, \omega_X^{-\ell}[m]) \cong H^m(X, (L \otimes \omega_X)^{-\ell}). \]
Since \((L \otimes \omega_X)^{-\ell}\) is nef, its higher cohomology vanishes. \qed

**Remark 2.12.** One sees immediately from [BF12, Theorem 2.1] that this also computes the global dimension of the finite dimensional algebra \(A = \text{End}_X(T)\) as \(\text{dim} X\).

The next result shows that Orlov’s Conjecture can propagate, in \(K\)-negative ways, to other birational models, or chambers in the secondary fan. Following the argument of [BFK17, Proposition 5.2.5, Corollary 5.2.6], we see that:

**Proposition 2.13.** If Orlov’s Conjecture holds for a smooth projective nef-Fano toric DM stack \(X\) that is isomorphic in codimension \(\geq 1\) to a smooth projective toric DM stack \(Y\), then Orlov’s Conjecture holds for \(Y\).

Thus we have the following:

**Corollary 2.14.** Let \(X\) be a smooth projective variety isomorphic in codimension \(\geq 1\) to a smooth projective toric nef-Fano \(Y\) with \(T_{\text{bu}}(Y)\) a tilting object. Then Orlov’s Conjecture holds for \(X\).

2.5. **Examples.** Despite being fairly innocuous, we can leverage the results of Section 2.4 into a healthy increase of positive examples of Orlov’s Conjecture.

**Proposition 2.15.** Orlov’s Conjecture holds for all smooth Fano toric threefolds.

*Proof.* One can now apply Theorem 2.11 thanks to [Ueh14] which guarantees that \(T_{\text{bu}}(Y)\) is tilting for \(Y\) a smooth toric Fano variety of dimension at most 3. \qed

In [PN17], the Bondal-Uehara method is used to exhibit exceptional collections for a subset of smooth toric Fano fourfolds but, in fact, \(m_0(T) = 0\) for all \(T\) produced by Prabhu-Naik.

**Proposition 2.16.** Orlov’s Conjecture holds for all smooth toric Fano fourfolds.

*Proof.* This follows immediately from [PN17, Theorem 7.8] since being a pullback exceptional collection, in the language of [BF12, Section 3.2], includes the vanishing of \(m_0(T)\). \qed

**Remark 2.17.** We can use Corollary 2.14 to expand the list of toric varieties satisfying Orlov’s Conjecture to all those coming from \(K\)-negative birational maps starting from Propositions 2.15 and Proposition 2.16.

These results hold for all the corresponding arithmetic toric varieties as well.

**Lemma 2.18.** Let \(X\) be a smooth, quasi-projective variety over \(k\) and \(L/k\) a Galois extension. Then Orlov’s Conjecture holds for \(X\) if and only if it holds for \(X_L\).

*Proof.* If it holds for \(X\), then [Sos14, Proposition 5.4] says it holds for \(X_L\). Conversely, the projection \(\pi : X_L \to X\) is a dense functor as \(\pi_* \pi^* E \cong E^{\oplus |G|}\) for \(G = \text{Gal}(L/k)\). So this follows from Lemma 2.5 and Proposition 2.4. \qed
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Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: ballard@math.sc.edu
URL: http://people.math.sc.edu/ballard/

Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: duncan@math.sc.edu
URL: http://people.math.sc.edu/duncan/

Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: pkmcfaddin@gmail.com
URL: http://mcfaddin.github.io/