Composition of Roofs in Derived Category

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Abstract

In that paper, we prove that the composition of two roofs is another roof by using mapping cone of a morphism of cochain complexes.

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1 Introduction

Assume that \( \mathcal{A} \) is an abelian category. P. Aluffi defines a mapping cone \( MC(f) \) of a morphism \( f: \mathcal{A} \rightarrow \mathcal{B} \) in \( C(\mathcal{A}) \) and homotopy between two morphisms in that category in [Al]. Also, in [Kr], H. Krause defines triangulated category and the localizing class. After that, he proves that the homotopic category \( K(\mathcal{A}) \) is triangulated in Section 2.5.

Using this information, we prove that for a given upside down roof in the localization of \( K(\mathcal{A}) \), we can obtain a regular roof in that category. This allows us to compute the composition of two regular roofs in the localization of homotopic category.
The collection $C(A)$ consisting of all cochain complexes in an abelian category $A$ forms an abelian category. It is easy to show that the set of morphisms of that category is an abelian group, finite products and coproducts exist since they exist in $A$.

**Definition 2.0.1.** A morphism $f$ between cochain complexes is quasi isomorphism if it induces an isomorphism in cohomology.

For a given morphism $f : A \to B$ between cochain complexes $A$ and $B$, we define a mapping cone $MC(f)$ as $MC(f)^i = A[i] \oplus B^i = A^{i+1} \oplus B^i$ for all $i$. Here, we get the morphisms

$$d_{MC(f)}^i : MC(f)^i \to MC(f)^{i+1},$$
$$d_{MC(f)}^i(a, b) = (-d_A^{i+1}(a), f^{i+1}(a) + d_B^i(b))$$

between those objects.

$MC(f)$ is a cochain complex since $d_{MC(f)}^{i+1} \circ d_{MC(f)}^i = 0$.

**Definition 2.0.2.** A homotopy $k$ between two morphisms of cochain complexes $f, g : A \to B$ is a collection of morphisms $k^i : A^i \to B^{i-1}$ such that for all $i$,

$$g^i - f^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i.$$

The above morphisms $f$ and $g$ are homotopic if there is a homotopy between them. We use the following diagram to show that homotopy and use the symbol $f \sim g$ to mean there exists a homotopy between the morphisms $f$ and $g$.

**Definition 2.0.3.** A morphism $f : A \to B$ is a homotopy equivalence if there is a morphism $g : B \to A$ such that $f \circ g \sim id_B$ and $g \circ f \sim id_A$. 

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A and $B$ are homotopy equivalent if there is a homotopy equivalence $A \to B$.

**Proposition 2.0.1.** [Al] If $f, g : A \to B$ are homotopic, then $H^\bullet(f) = H^\bullet(g)$.

**Corollary 2.0.1.** If $f : A \to B$ is homotopy equivalence, then $H^\bullet(A) \cong H^\bullet(B)$.

Every homotopy equivalence is a quasi isomorphism, but every quasi isomorphism may not be a homotopy equivalence.

## 3 Triangulated Categories

**Definition 3.0.4.** [Kr] Assume that $\mathcal{A}$ is an additive category with an equivalence $\mathcal{F} : \mathcal{A} \to \mathcal{A}$. A triangle $(f, g, h)$ in $\mathcal{A}$ is a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \mathcal{F}(X)$ for all objects $X, Y$ and $Z$ in $\mathcal{A}$.

A morphism between two triangles $(f_1, g_1, h_1)$ and $(f_2, g_2, h_2)$ is a triple $(k_1, k_2, k_3)$ of morphisms in $\mathcal{A}$ making the following diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y \\
\downarrow{k_1} & & \downarrow{k_2} \\
X' & \xrightarrow{f_2} & Y'
\end{array}
\quad\quad
\begin{array}{ccc}
Y & \xrightarrow{g_1} & Z \\
\downarrow{k_3} & & \downarrow{\mathcal{F}(k_1)} \\
Y' & \xrightarrow{g_2} & Z'
\end{array}
\quad\quad
\begin{array}{ccc}
Z & \xrightarrow{h_1} & \mathcal{F}(X) \\
\downarrow{k_3} & & \downarrow{k_3} \\
Z' & \xrightarrow{h_2} & \mathcal{F}(X')
\end{array}
\]

The category $\mathcal{A}$ is called pre-triangulated if it has a class of exact triangles satisfying the following conditions.

1. A triangle is exact if it is isomorphic to an exact triangle.
2. For all objects $X$ in $\mathcal{A}$, the triangle $0 \xrightarrow{} X \xrightarrow{id} X \xrightarrow{} 0$ is exact.
3. Each morphism $f : X \to Y$ can be completed to an exact triangle $(f, g, h)$.
4. A triangle $(f, g, h)$ is exact if and only if the triangle $(g, h, -\mathcal{F}(f))$ is exact.
5. Given two exact triangles $(f_1, g_1, h_1)$ and $(f_2, g_2, h_2)$, each pair of maps $k_1$ and $k_2$ satisfying $k_2 \circ f_1 = f_2 \circ k_1$ can be completed to a morphism:

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y \\
\downarrow{k_1} & & \downarrow{k_2} \\
X' & \xrightarrow{f_2} & Y'
\end{array}
\quad\quad
\begin{array}{ccc}
Y & \xrightarrow{g_1} & Z \\
\downarrow{k_3} & & \downarrow{\mathcal{F}(k_1)} \\
Y' & \xrightarrow{g_2} & Z'
\end{array}
\quad\quad
\begin{array}{ccc}
Z & \xrightarrow{h_1} & \mathcal{F}(X) \\
\downarrow{k_3} & & \downarrow{k_3} \\
Z' & \xrightarrow{h_2} & \mathcal{F}(X')
\end{array}
\]

$\mathcal{A}$ is a triangulated category if in addition it satisfies the following axiom.
6. **The Octahedral Axiom**: Given exact triangles \((f_1, f_2, f_3), (g_1, g_2, g_3)\) and \((h_1, h_2, h_3)\) with \(h_1 = g_1 \circ f_1\), there exists an exact triangle \((k_1, k_2, k_3)\) making the following diagram commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y \\
\downarrow h_1 & & \downarrow g_1 \\
X & \xrightarrow{f_2} & U \\
\downarrow g_2 & & \downarrow h_2 \\
Y & \xrightarrow{f_3} & \mathcal{F}(X) \\
\downarrow h_3 & & \downarrow g_3 \\
V & \xrightarrow{k_1} & \mathcal{F}(X)
\end{array}
\]

\[\mathcal{F}(Y) \xrightarrow{\mathcal{F}(f_2)} \mathcal{F}(U)\]

**Remark 3.0.1.** If \(A\) is a pretriangulated category, then \(A^{op}\) is a pretriangulated category, too.

### 4 The Localization of A Category

**Definition 4.0.5.** [Kr] Assume that \(A\) is a category and \(F\) is a class of maps in \(A\). \(F\) is a localizing class if the following conditions are satisfied.

1. If \(f, g\) are composable maps in \(F\), then \(g \circ f\) is in \(F\).

2. The identity map \(id_A\) is in \(F\) for all \(A \in A\).

3. If \(f : A \rightarrow B\) is in \(F\), then every pair of maps \(B' \rightarrow B\) and \(A \rightarrow A''\) in \(A\) can be completed to a pair of commutative diagrams;

\[
\begin{align*}
A' & \longrightarrow A \quad A \longrightarrow A'' \\
\downarrow f' & \quad \downarrow f \\
B' & \longrightarrow B \quad B \longrightarrow B''
\end{align*}
\]

such that \(f'\) and \(f''\) are in \(F\).

4. If \(f, g : A \rightarrow B\) are maps in \(A\), then there is some \(h : A' \rightarrow A\) in \(F\) with \(f \circ h = g \circ h\) if and only if there is some \(k : B \rightarrow B''\) in \(F\) with \(k \circ f = k \circ g\).

**Definition 4.0.6.** [Kr] Assume that \(A\) is a category and \(F\) is a class of maps in \(A\). The localization of \(A\) with respect to \(F\) is a category \(A[F^{-1}]\) together with a functor \(\mathcal{F} : A \rightarrow A[F^{-1}]\) such that \(\mathcal{F}(f)\) is an isomorphism for all \(f\) in \(F\) and any functor \(G : A \rightarrow B\) such that \(G(f)\) is an isomorphism for all \(f\) in \(F\) factors uniquely through \(\mathcal{F}\).

We can always find a localization like that.
Definition 4.0.7. Assume that $\mathcal{A}$ is a category and $F$ is a localizing class. The objects of $\mathcal{A}[F^{-1}]$ are the objects of $\mathcal{A}$. The morphisms $A \to B$ in $\mathcal{A}[F^{-1}]$ are equivalence classes of diagrams $A \xrightarrow{f} B' \xrightarrow{g} B$ with the morphism $f$ in $F$ for all objects $A$ and $B$ in the category $\mathcal{A}[F^{-1}]$. We will call those morphisms as regular roofs.

A pair $(f, g)$ is also called a fraction because it is written as $g \circ f^{-1}$ in $\mathcal{A}[F^{-1}]$.

Remark 4.0.2. The functor $\mathcal{F} : A \to \mathcal{A}[F^{-1}]$ sends a map $f : A \to B$ to the pair $(id_A, f)$.

Definition 4.0.8. $(f, g)$ and $(f', g')$ are equivalent if there exists a commutative diagram with $f''$ in $F$;

5 Composition of Two Roofs

Definition 5.0.9. [Al] Assume that $\mathcal{A}$ is an abelian category. $K(\mathcal{A})$ is a category whose objects are the objects in $C(\mathcal{A})$ and the set of morphisms is

$$\text{Hom}_{K(\mathcal{A})}(A, B) = \text{Hom}_{C(\mathcal{A})}(A, B) / \sim$$

where $\sim$ is homotopy relation.

If $f \circ g \sim id$ in $C(\mathcal{A})$, then $f \circ g = id$ in $K(\mathcal{A})$. As a result, homotopy equivalences in $C(\mathcal{A})$ become isomorphisms in $K(\mathcal{A})$ and we say that $K(\mathcal{A})$ is obtained by inverting all homotopy equivalences in $C(\mathcal{A})$. It is an additive category, but not abelian in general since homotopic maps don’t have same kernels and cokernels.

In [Kr], H. Krause proves that $K(\mathcal{A})$ is a triangulated category.

Remark 5.0.3. The set of quasi isomorphisms in $K(\mathcal{A})$ for a given abelian category $\mathcal{A}$ forms a localizing class.

Theorem 5.0.2. [Al] Assume that $\mathcal{A}$ is an abelian category and we have two morphisms $L \xrightarrow{\alpha} K$ and $M \xrightarrow{\beta} K$ with $\beta$ is a quasi isomorphism for objects $L$ and $M$ in $K(\mathcal{A})$. Then, there exists a cochain complex $K$, morphisms $K \to L$ which is quasi isomorphism
and $K \to M$ in $K(A)$ such that the following diagram commutes.

$$
\begin{array}{c}
K \\
\gamma_2 \\
\downarrow \\
L \\
\alpha \\
\downarrow \\
K
\end{array}
\quad
\begin{array}{c}
\gamma_1 \\
\downarrow \\
M \\
\beta \\
\downarrow \\
K
\end{array}
$$

(1)

**Proof.** Assume that $\gamma$ is the composition $L \longrightarrow K \longrightarrow MC(\beta)$, $K = MC(\gamma)[-1]$ and $K^i = L^i \oplus M^i \oplus K^{i-1}$. We define morphisms $K^i \to L^i$, $(l, m, k) \to l$ and $K^i \to M^i$, $(l, m, k) \to -m$ as in [Al].

We want to prove that $L$ and $M$ are connected by a regular roof as well.

For the rest of the proof, we need to show that the Diagram 1 commutes.

$H^\ast(M) \cong H^\ast(K)$ since $\beta$ is a quasi isomorphism. This implies that $MC(\beta)$ is exact, so $H^\ast(MC(\beta)) = 0$.

$$
MC(\beta)^i = M[1]^i \oplus K^i = M^{i+1} \oplus K^i,
$$

$$
d^i_{MC(\beta)}: MC(\beta)^i \to MC(\beta)^{i+1},
$$

$$
d^i_{MC(\beta)}(m, k) = (-d^{i+1}_M, \beta^{i+1}(m) + d^i_K(k)).
$$

We define $\gamma^i(l) = (0, \alpha^i(l))$ and

$$
MC(\gamma)^i = L[1]^i \oplus M^{i+1} \oplus K^i = L^{i+1} \oplus M^{i+1} \oplus K^i,
$$

where $d^i_{MC(\gamma)} : MC(\gamma)^i \to MC(\gamma)^{i+1}$ with

$$
d^i_{MC(\gamma)}(l, m, k) = (-d^{i+1}_L(l), \gamma^{i+1}(l) + d^i_{MC(\beta)}(m, k)) =
$$

$$
(-d^{i+1}_L(l), -d^{i+1}_M(m), \alpha^{i+1}(l) + \beta^{i+1}(m) + d^i_K(k)).
$$

$K = MC(\gamma)[-1],

$$
MC(\gamma)[-1]^i = MC(\gamma)^{i-1} = L^i \oplus M^i \oplus K^{i-1} = K^i
$$

and $d^i_K = -d^{i-1}_{MC(\gamma)}$ with

$$
d^i_K = (d^i_L(l), d^i_M(m), -\alpha^i(l) - \beta^i(m) - d^{i-1}K(k)).
$$

Assume that $h^i : K^i \to K^{i-1}$ takes $(l, m, k)$ to $-k$. We need to show that

$$
\alpha^i \circ \gamma_2^i - \beta^i \circ \gamma_1^i = d^{i-1}_K \circ h^i + h^{i+1} \circ d^i_K.
$$
for all $i \in \mathbb{Z}$ which shows that $\alpha \circ \gamma_2$ and $\beta \circ \gamma_1$ are homotopic maps in $K(A)$. This will show that they are same maps.

For all $(l, m, k) \in K^i$,

$$(\alpha^i \circ \gamma^i_2 - \beta^i \circ \gamma^i_1)(l, m, k) = \alpha^i(\gamma^i_2(l, m, k)) - \beta^i(\gamma^i_1(l, m, k)) = \alpha^i(l) - \beta^i(-m) = \alpha^i(l) + \beta^i(m)$$

since $A$ is additive. On the other hand,

$$(d^{i-1}_K \circ h^i + h^{i+1} \circ d^i_K)(l, m, k) = d^{i-1}_K(h^i(l, m, k)) + h^{i+1}(d^i_N(l, m, k)) = d^{i-1}_K(-k) + \alpha^i(l) + \beta^i(m) + d^{i-1}_K(k) = \alpha^i(l) + \beta^i(m).$$

This shows the maps are homotopic and the diagram is commutative.

We need to show that $\gamma_2$ is a quasi isomorphism. We have an exact triangle;

$$\begin{array}{ccc}
L & \rightarrow & L[1] + MC(\beta) \\
\downarrow & & \downarrow \\
MC(\beta) & \rightarrow & MC(\beta)
\end{array}$$

This triangle is isomorphic to an exact triangle;

$$\begin{array}{ccc}
MC(\beta) & \rightarrow & MC(\beta) \\
\downarrow & & \downarrow \\
L[1] & \rightarrow & L[1] + MC(\beta)
\end{array}$$

Then, we take its cohomology and the triangle still will be exact.

$$\begin{array}{ccc}
H^*(MC(\beta)) & \rightarrow & H^*(MC(\beta)) \\
\downarrow & & \downarrow \\
H^*(L[1]) & \rightarrow & H^*(L[1] + MC(\beta))
\end{array}$$

$H^*(MC(\beta)) = 0$, so $H^*(L[1]) \cong H^*(L[1] + MC(\beta))$ and

$$L[1] + MC(\beta) = L[1] + M[1] + K = K[1].$$

Consequently, $H^*(L[1]) \cong H^*(K[1])$. This means $H^*(L) \cong H^*(K)$, hence $\gamma_2$ is a quasi isomorphism. □
The pair \((f \circ f'', \ g' \circ g'')\) is the composition of two pairs \((f, g)\) and \((f', g')\) as in the following commutative diagram;

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B' & \xrightarrow{g} & C' & \xrightarrow{g'} & C \\
\downarrow{f''} & & \downarrow{g''} & & \downarrow{f''} & & \downarrow{g''} \\
B' & \xrightarrow{f'} & C'' & \xleftarrow{g'} & C \\
\end{array}
\]

References

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[Kr] H. Krause, Derived Categories, Resolutions, and Brown Representability, arxiv: math/0511047v3, 2006.