Open Gromov-Witten Theory of $K_{P^2}, K_{P^1 \times P^1}, K_{WP[1,1,2]}, K_{F_1}$ and Jacobi Forms

Bohan Fang, Yongbin Ruan, Yingchun Zhang and Jie Zhou

May 15, 2018

Abstract

It was known through the efforts of many works that the generating functions in the closed Gromov-Witten theory of $K_{P^2}$ are meromorphic quasi-modular forms [CI14, LP17, CI18] basing on the B-model predictions [BCOV94, ABK08, ASYZ14]. In this article, we extend the modularity phenomenon to $K_{P^1 \times P^1}, K_{WP[1,1,2]}, K_{F_1}$. More importantly, we generalize it to the generating functions in the open Gromov-Witten theory using the theory of Jacobi forms where the open Gromov-Witten parameters are transformed into elliptic variables.

Contents

1 Introduction 2

2 A brief review of the solution of Remodeling Conjecture 6
  2.1 Toric Calabi-Yau 3-orbifolds and mirror curves . . . . . . . . . . . . . . . . . . 6
  2.2 Mirror symmetry from remodeling conjecture . . . . . . . . . . . . . . . . . . 8

3 Geometry of genus one mirror curves 12
  3.1 Basic definitions on modular forms and Jacobi forms . . . . . . . . . . . . . 13
  3.2 Uniformizations of genus one algebraic curves . . . . . . . . . . . . . . . . . . 16
  3.3 Ramification points for hyperelliptic curves of genus one . . . . . . . . . . . 17
  3.4 One-parameter subfamilies of genus one mirror curve families . . . . . . . . 18
  3.5 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19

4 Proof of main theorems 26
  4.1 Expansions of basic ingredients in topological recursion . . . . . . . . . . . . 26
  4.2 Modular properties of $\{\omega_{g,n}\}_{g,n}$ and ring structure . . . . . . . . 29
  4.3 Holomorphic anomaly equations . . . . . . . . . . . . . . . . . . . . . . . . . . 36

Appendix A Some explicit formulae 41
  A.1 $K_{P^2}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42
  A.2 $K_{F_1}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42
  A.3 $K_{P^1 \times P^1}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 43
  A.4 $K_{WP[1,1,2]}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 43
1 Introduction

Toric Calabi-Yau (CY) manifolds/orbifolds have always occupied a special place in geometry and physics. The combinatorial nature of these objects make them a fertile ground to test new ideas and techniques. In principle, the Gromov-Witten (GW) theory of such a space has been computed in the 90’s by the localization technique. However, the solution is in terms of a complicated graph sum formula which makes it not so useful for actual computations. Then, the topological vertex formalism was developed in early 2000 [AKMV05]. In this past decade, a new formulation of its B-model in terms of mirror curve leads to the Remodeling Conjecture [BKMnP09] via topological recursion [EO07]. This conjecture for toric CY’s has been proved by the first author and his collaborators [FLZ16]. However, the topological recursion formalism computes the open GW invariants via a recursion algorithm. From the mathematical point of view, the ultimate goal is to compute its generating functions by closed formula in some sense. These generating functions are quite complicated and it is rare that they can be expressed as elementary functions. The next attractive classes of functions are modular forms from number theory. Indeed, a great deal of efforts were spent to show that the generating functions of closed GW invariants of $K_{P^2}$ are meromorphic quasi-modular forms [CI14, LP17, CI18] basing on the earlier results in [BCOV93, BCOV94, KZ99, CKYZ99, YY04, ABK08, GKMW07, ALM10, HKR08, ASYZ14].

The main purpose of this article is to push further the interaction between GW theory and modular forms to the cases $K_{\mathbb{P}^1 \times \mathbb{P}^1}, K_{\mathbb{P}^{[1,1,2]}}, K_{F_1}$. More importantly, we generalize it to open GW theory. Open GW theory has an additional open parameter keeping track of the number of boundaries. Our key idea is to replace meromorphic quasi-modular forms by certain "quasi-meromorphic Jacobi forms" (see Section 3.1 for the precise definition), while the open parameter is translated into a certain function of the elliptic variable $z$ of the latter.

Let’s briefly recall the definition of Jacobi forms here. A meromorphic function $\Phi$ on $\mathbb{C} \times \mathcal{H}$ is a meromorphic Jacobi form of weight $k \in \mathbb{Z}$, index $\ell \in \mathbb{Z}$ for the modular group $\Gamma(1) = SL_2(\mathbb{Z})$ if it satisfies

- $\Phi\left( \frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k e^{2\pi i \ell \frac{z^2}{c\tau + d}} \Phi(z, \tau), \quad \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(1),$

- $\Phi(z + m\tau + n, \tau) = e^{-2\pi i (m^2\tau + 2nz)} \Phi(z, \tau), \quad \forall m, n \in \mathbb{Z}.$

together with some regularity condition. It is “modular” in $\tau$ and “elliptic” in $z$. See Section 3.1 (e.g., Definition 3.4) for detailed definitions on holomorphic and meromorphic Jacobi forms for a modular subgroup $\Gamma < SL_2(\mathbb{Z})$. The set of all such meromorphic Jacobi forms gives a graded ring $\mathcal{F}(\Gamma)$. Adjoining the quasi-modular Eisenstein series $E_2$, we get the graded ring of quasi-meromorphic Jacobi forms.

Example 1.1. The Weierstrass $\wp$-function

$$\wp(u, \tau) = \frac{1}{u^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{(u + m\tau + n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

is a meromorphic Jacobi form of weight 2, index 0 for the modular group $SL_2(\mathbb{Z})$. 

The main result of this article is about the generating series $F_{g,n}$ of open GW invariants (called open-closed GW potential) – for its definition see (2.12) below. It collects open-closed GW invariants of genus $g$ with $n$ boundary components into a generating series. When $n = 0$ this is the usual closed GW potential and is denoted by $F_g$. The quantity $F_{g,n}$ is a formal power series of the closed parameters $Q = (Q_1, \ldots)$ and open parameters $X = (X_1, \ldots, X_n)$. In our four examples, $Q = Q_1$ or $Q = (Q_1, Q_2)$, corresponding to the Kähler parameters $T_1$ or $(T_1, T_2)$ by $Q_k = e^{T_k}, k = 1, 2$.

Recall that the mirror map, which can be derived within GW theory by using Givental’s $I$-functions, is a bi-holomorphic map $\mathbf{m} : \Delta_Q \to \Delta_q$ from a certain neighborhood $\Delta_Q$ of the large radius limit $Q = 0$ in $\mathbb{A}^2$ to such an one $\Delta_q$ of the large complex structure limit $q = (q_1, q_2) = 0$ in $\mathbb{A}^2$. The parameters $q$ appear naturally as complex parameters in the defining equations of the mirror curve family $\chi : C \to \mathcal{U}_C$ of the toric CY, see [HV00]. Induced by the mirror map $\mathbf{m}$, the parameter $X_k$ gets identified with a function of a rational function $x_k$ on the mirror curve. They are explicit hypergeometric-like series (see for example (4.59)) with nice leading order (2.14).

As will be discussed in Section 3.4 when there are two Kähler parameters we need to restrict to a non-trivial one-parameter family so that we can employ the theory of modular forms. Namely, we choose a rational affine curve $\mathcal{U}_{\text{res}}$ in the base $\mathcal{U}_C$ of the mirror curve family whose Zariski closure includes $q = 0$, then we take the fiber product to get the restriction of the family $\chi_{\text{res}} : C_{\text{res}} \to \mathcal{U}_{\text{res}}$. From the perspective of the A-model, this corresponds to the restriction to the preimage under $\mathbf{m}$ of the (analytification) of the subvariety $\mathcal{U}_{\text{res}} \cap \Delta_q$.

After the restriction to the one-parameter family, $q_1$ and $q_2$ are modular functions (for a certain modular subgroup $\Gamma$ depending on $\chi_{\text{res}}$) in the complex structure parameter $\tau$ for the mirror curve lying on the upper-half plane $\mathcal{H}$. In this way $Q$ also becomes a function of $\tau$, although it is not modular (see e.g. Example 4.9). In fact $\tau$ has a purely A-model expression $\tau = \partial_t^2 F_{0,t}$, where $t$ is a certain $\mathbb{Z}$-linear combination of $T_1$ and $T_2$ (or simply $T_1$ for the one Kähler parameter case). The parameter $t$ is called the flat coordinate for the one-parameter subfamily. A different choice of such combination amounts to an $SL_2(\mathbb{Z})$-transformation on $\tau$ which still represents the same complex structure of the mirror curve. See Section 3.4 for details on this.

We then make use of the uniformization to express (see Lemma 3.16) the rational function $x$ on the mirror curve as $x(u, \tau)$, in terms of meromorphic modular and Jacobi forms. Here $u \in \mathbb{C}$ is the universal cover of the mirror curve, which is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$ with $\tau \in \mathcal{H}$. Thus one may regard the formal series $F_{g,n}(Q, X_1, \ldots X_n)$ as one in $(\tau, u_1, \ldots, u_n)$.

One of the main results of this article is to show that above $F_{g,n}(Q, X_1, \ldots X_n)$ for the examples $K_{\mathbb{P}^2}, K_{\mathbb{P}^1 \times \mathbb{P}^1}, K_{\mathbb{WP}[1,1,2]}, K_{F_1}$ are quasi-meromorphic Jacobi forms under the mirror map $\mathbf{m}$.

**Theorem** (Theorem 4.5). The following statements hold for $d_{X_1} \ldots d_{X_n} F_{g,n}$ with $2g - 2 + n > 0$.

1. The differential $d_{X_1} \ldots d_{X_n} F_{g,n}$ is a quasi-meromorphic Jacobi form of total weight $n$.

2. The closed GW potential $F_g$ is a meromorphic quasi-modular form of weight zero.

Here we say the differential $d_{X_1} \ldots d_{X_n} F_{g,n}$ is Jacobi if its coefficient with respect to the basis $du_1 \otimes \cdots \otimes du_n$, which is a meromorphic function in any $u_k$, is Jacobi in $(u_k, \tau)$.
Part 2 of the above theorem applied to $K_{12}$ recovers the results in \cite{LP17} recently proved basing on the earlier results in e.g. \cite{BCOV94, ASYZ14}. Restricting to the subfamily obtained by setting $T_1 = T_2$ for the $K_{P_1 \times P_1}$ case, it recovers the results in \cite{Lho18} on this particular subfamily.

Applying some elementary properties of modular forms and Jacobi forms, we obtain the following corollary of the above theorem.

**Corollary** (Corollary 4.6). The Taylor coefficients of $F_{g,n}$ in a certain $X$-expansion are meromorphic quasi-modular forms.

**Example 1.2** (Proposition 4.1). The disk potential $\partial_x W := (\log y)/x$ involves the logarithm of a meromorphic Jacobi form. For the $K_{P_1}$ case, this is

$$\partial_x W = \frac{\log (\kappa^2 \varphi(u) + \frac{3}{2}(-4)^{\frac{1}{2}} \kappa^2 \varphi(u) + \frac{9}{8} \varphi^3 - \frac{1}{2})}{-3(-4)^{\frac{1}{2}} \kappa^2 \varphi(u) - \frac{9}{8} \varphi^3}.$$ 

**Example 1.3** (Proposition 4.2). The annulus potential is

$$\omega_{0,2}(u_1, u_2) = (\varphi(u_1 - u_2) + \frac{\pi^2}{3} E_2) du_1 \boxtimes du_2.$$ 

It is a quasi-meromorphic Jacobi form of weight 2, index 0.

**Example 1.4** (Theorem 4.5). For the $(g, n) = (0, 3)$ case, one has

$$d_x d_x d_x F_{0,3} = \omega_{0,3}(u_1, u_2, u_3) = \sum_{r \in R^c} \left( 2\left[ \frac{1}{\Lambda} \right]_{-2} \cdot \prod_{k=1}^3 (\varphi(u_k - u_r) + \frac{\pi^2}{3} E_2) \right) du_1 \boxtimes du_2 \boxtimes du_3.$$ 

It is a quasi-meromorphic Jacobi form of total weight 3, index 0 for a certain modular group. For the $K_{P_2}$ case, one has

$$\left[ \frac{1}{\Lambda} \right]_{-2} = \frac{1}{\varphi^{\eta^2}(u_r)} \frac{1}{\kappa^3} \left( 1 - 3(-4)^{\frac{1}{2}} \kappa^2 \varphi(u_r) - \frac{9}{8} \varphi^3 \right) (\varphi(u_r) + \frac{3}{4} \varphi^2 - 2(-4)^{\frac{1}{2}} \kappa^2))$$

In Example 1.2 and Example 1.4 above,

$$\phi(\tau) = \Theta_{A_2}(2\tau) \frac{\eta(3\tau)}{\eta(\tau)^3}, \quad \kappa = \zeta_6 2^{\frac{3}{2}} 3^\frac{1}{2} \pi^{-1} \frac{\eta(3\tau)}{\eta(\tau)^3},$$

with $\Theta_{A_2}(2\tau)$, $\eta(3\tau)\eta(\tau)^{-3}$ modular forms for the modular group $\Gamma_0(3)$, see \cite{Zag08, Mai09, Mai11} for details. Also

$$R^c = \{ u_r = \frac{1}{2}, \frac{1+\tau}{2}, \frac{1}{2} \},$$

and

$$\varphi(\frac{1}{2}) = \frac{\pi^2}{3} (\theta_3^4 + \theta_4^4), \quad \varphi(\frac{\tau}{2}) = \frac{\pi^2}{3} (-\theta_2^4 - \theta_3^4), \quad \varphi(\frac{1+\tau}{2}) = \frac{\pi^2}{3} (\theta_2^4 - \theta_4^4),$$

4
with \( \varphi''(u_r) = 6\varphi^2(u_r) - \frac{2}{3}\pi^4E_4 \). More formulae for the \( K_{\mathbb{P}^1 \times \mathbb{P}^1}, K_{\mathbb{WP}[1,1,2]} \) and \( K_{F_1} \) cases can be found in Appendix A.

With a structure theorem in Theorem 4.4 in hand, we can combine the holomorphic anomaly equation of [EOM07] with the above results and arrive at the Yamaguchi-Yau type equation. We refer to Theorem 4.8 for its full form, but just state the result for the closed sector here.

**Theorem (Theorem 4.8).** The closed GW potentials \( F_g \) \((g \geq 2)\) satisfy

\[
\frac{\partial}{\partial \eta_1} F_g = \frac{\partial Y}{\partial \eta_1} \cdot \frac{1}{2} \left( \partial_t \partial_j F_{g-1} + \sum_{g_1 + g_2 = g} \partial_t F_{g_1} \cdot \partial_t F_{g_2} \right).
\]

Here the prime \(^{'}\) in the summation on the right hand side means the range for the summation is such that the equations are strictly recursive, and \( \eta_1 = (\pi^2/3)E_2 \).

We refer to Section 4.3 for the definition of \( S'' \) and \( Y \), but only remark that in our cases \( \partial Y S'' / \partial \eta_1 \) is a constant number. For example for \( \mathcal{X} = K_{\mathbb{P}^2}, \) this is \( 3/(2\pi^2) \), see Example 4.9. Our theorem recovers the Yamaguchi-Yau equation for \( K_{\mathbb{P}^2} \) as shown in [LP17] recently proved basing on the earlier results in e.g. [BCOV94, ASYZ14]. Restricting to the subfamily obtained by setting \( T_1 = T_2 \) for the \( K_{\mathbb{P}^1 \times \mathbb{P}^1} \) case, this theorem recovers the Yamaguchi-Yau equation proved in [Lho18] on this particular subfamily.

**Outline of the proof**

We review toric geometry, mirror symmetry and the remodeling conjecture for our four examples in Section 2. The four examples in the article are chosen for the fact that their mirror curves are genus one algebraic curves equipped with hyperelliptic structures determined by the structure of the branes. The genus one and hyperelliptic structure allow us to apply some arithmetic geometry of elliptic curves in the study of topological recursion on the mirror curve. In Section 3 we show that the hyperelliptic structure implies the ramification points are identified with the group of 2-torsion points on the elliptic curve. We also explicitly give uniformization results for the mirror curve families in Section 3. In Section 4 an examination following the procedure in topological recursion then shows that the differentials \( \{\omega_{g,n}\}_{g,n} \), produced by residue calculus near the ramification points, are quasi-meromorphic Jacobi forms lying in a ring with very simple generators. A structure theorem of \( \{\omega_{g,n}\}_{g,n} \), relating the weights, poles of the quasi-meromorphic Jacobi forms \( \{\omega_{g,n}\}_{g,n} \) to the genus \( g \) and number of boundary components \( n \), follows by induction. This then offers a rigorous proof of the Yamaguchi-Yau type holomorphic anomaly equations for \( d_{X_1} \ldots d_{X_n} F_{g,n} \), basing on the equations [EOM07] satisfied by \( \{\omega_{g,n}\}_{g,n} \) and the proof in [FLZ16] stating that \( d_{X_1} \ldots d_{X_n} F_{g,n} = \omega_{g,n} \). Furthermore, on the mirror curve there is a distinguished point around which the expansions of GW potentials give rise to open GW invariants. The results on uniformization imply that the Taylor coefficients at this point of the GW potentials which are now regarded as quasi-meromorphic Jacobi forms, are meromorphic quasi-modular forms.

We remark that the hyperelliptic structure is crucial in our discussion. There are 12 other local toric CY 3-folds whose mirror curves are in hyperelliptic forms. In principle,
our technique applies to these examples as well. At this point, we are unsure about the compatibility of their hyperelliptic structures and the remodeling conjecture—we hope to come back in another time (see Remark 2.8 for more technical discussion). A more exciting future direction is the case of genus two mirror curve, in which the ramification data can be also made intrinsic from the hyperelliptic structure. We will leave it to another paper.

**Acknowledgement**

B. F. would like to thank Chiu-Chu Melissa Liu and Zhengyu Zong for enlightening discussion, and is partially supported by the Recruit Program of Global Experts in China. Y. R. is partially supported by NSF grant DMS 1405245 and NSF FRG grant DMS 1159265. Y. Z. is supported by China Scholarship Council grant No. 201706010026. J. Z. would like to thank Murad Alim, Florian Beck, Kathrin Bringmann, Xiaoheng Jerry Wang and Baosen Wu for useful discussions. J. Z. is partially supported by German Research Foundation Grant CRC/TRR 191.

**2 A brief review of the solution of Remodeling Conjecture**

**2.1 Toric Calabi-Yau 3-orbifolds and mirror curves**

The toric Calabi-Yau 3-orbifolds $\chi = K_{P^2}, K_{P^1 \times P^1}, K_{WP[1,1,2]}, K_{F_1}$ are defined via following triangulated polytopes in $N'_R \cong \mathbb{R}^2$ (see Figure 1). Here $N' = \mathbb{Z}^2$ and $N'_R \cong N' \otimes \mathbb{Z} \mathbb{R}$.

![Figure 1: The defining polytopes of $K_{P^2}, K_{P^1 \times P^1}, K_{WP[1,1,2]}$ and $K_{F_1}$ respectively. The polytopes are in gray and their triangulations are in solid lines. They are fitted into a triangle in dashed lines, which ensures that the mirror curves are in hyperelliptic forms.](image)

We denote the defining polytope by $\Delta$. We notice that all of these polytopes are contained in a triangle with vertices $(0, -1), (0, 1), (4, -1)$ – the goal is to ensure that the mirror curves are in hyperelliptic form (see [CLS11] for more detailed definition of toric varieties and orbifolds).

We let $\chi$ be one of these orbifolds, and $\mathbb{T} \cong (\mathbb{C}^*)^3$ be the dense algebraic torus inside $\chi$. The Calabi-Yau torus $\mathbb{T}' \subset \mathbb{T}$ preserves the Calabi-Yau forms. By the construction of the toric orbifold $\mathbb{T}' = N'_R \otimes \mathbb{Z} \mathbb{C}^*$. Let $\pi'_R : \chi \rightarrow \mathbb{R}^2$ be the moment map of is maximal compact subgroup $T'_R$. Let $\chi^1$ be the union of the $\mathbb{T}'$-invariant 1-suborbifolds of $\chi$, which are either weighted projective lines or gerbes over $\mathbb{C}$. The toric graph of $\chi$ is the image $\pi'_R(\chi^1)$. It is a trivalent graph, and the images of gerby $\mathbb{C}$ are rays while the images of weighted projective lines are segments. Each vertex is the image of a $\mathbb{T}'$-fixed stacky point.
In this article, we will only consider so called outer Aganagic-Vafa Lagrangian brane $L$. It is in the inverse image $\pi^{-1}_R(pt)$, where the point $pt$ is on an outer leg of the toric diagram, as shown in Figure 2. With an extra condition that when presenting $X$ as a GIT quotient $C^{p+3}/(C^*)^p$ the sum of $p + 3$ complex coordinates on $L$ is a constant, $L$ is a Lagrangian submanifold in $X$ diffeomorphic to $R^2 \times S^1$.

The framing of an Aganagic-Vafa brane is simply a choice of $f \in \mathbb{Z}$, which determines a one-dimensional subtorus $T'_f$ of the two-dimensional torus $T'$: let $w_1 = (1,0)$, $w_2 = (0,1)$ be lattice points in $M' = \text{Hom}(N',\mathbb{Z})$, and thus characters of $T'$, then define $T'_f = \ker(w_2 - fw_1)$. Together $(L,f)$ is a framed Aganagic-Vafa brane.

Given a toric CY 3-fold with a framed Aganagic-Vafa brane, there is a standard procedure to write down its mirror curves. Let $(m_1,n_1), \ldots, (m_{p+3},n_{p+3})$ be integral points in the defining polytope. For our examples, all defining polytopes have a triangle inside with vertices $(1,0), (0,1)$ and $(0,0)$. By a permutation, we require $(m_i,n_i) = (1,0), (0,1), (0,0)$ for $i = 1,2,3$ respectively. The affine mirror curve equation is then

$$H(x,y) = x + y + 1 + \sum_{i=1}^{p} a_i(q)x^{m_i+3}y^{n_i+3} = 0. \quad (2.1)$$

This is an affine curve in $(\mathbb{C}^*)^2$, denoted by $C^\circ$. It has a natural compactification into a compactified mirror curve $C$ in the toric orbifold $\mathbb{P}_\Delta$ given by the defining polytope $\Delta$. The standard toric construction of $\mathbb{P}_\Delta$ is defined as the Zariski closure of

$$\mathbb{P}_\Delta = \{[x^iy^{-5} : y^m : 1 : x^{m_4}y^{m_4} : \ldots : x^{m_{p+3}y^{n_{p+3}}}]\} \subseteq \mathbb{P}^{p+2}, \quad (2.2)$$

while $H$ could be regarded as a section of certain line bundle over $\mathbb{P}_\Delta$. Then the zero set $C$ is the compactification of $C^\circ$. We denote the Zariski open subset $U_C \subseteq (\mathbb{C}^*)^p$ on which $C^\circ$ and $C$ are smooth. When $q \in U_C$, the topological quantities of the mirror curve are recapitulated from the toric data, where $3 + p$ is the number of integer points in $\Delta$, while $g$ is the number of interior integer points in $\Delta$:

$$h^1(C^\circ) = g + p + 2, \quad h^1(C) = 2g. \quad (2.3)$$
Meanwhile, $h^2_{CR}(\mathcal{X}) = p$ and $h^1_{CR}(\mathcal{X}) = g$. In particular, the genus of $C$ is $g$. Therefore we have a family of smooth compactified mirror curves $C$ over $\mathcal{U}_C$ with fiber $C$ and $C^0 \subset C$ is the family of smooth affine mirror curves with fiber $C^0$.

We list our four examples in details below.

**Example 2.1.** $\mathcal{X} = K_{\mathbb{P}^2}$ (the canonical bundle over $\mathbb{P}^2$). The affine mirror curve $C^0$ is
\[ x + y + 1 + q_1x^3/y = 0. \tag{2.4} \]

The compactified mirror curve $C$ sits in $\mathbb{P}_\Delta = \mathbb{P}^2/\mu_5$.

**Example 2.2.** $\mathcal{X} = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ (the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$). The affine mirror curve $C^0$ is
\[ x + y + 1 + q_1x^2 + q_2x^2/y = 0. \tag{2.5} \]

The compactified mirror curve $C$ sits in $\mathbb{P}_\Delta = (\mathbb{P}^1 \times \mathbb{P}^1)/\mu_2$.

**Example 2.3.** $\mathcal{X} = K_{W[1,1,2]}$ (the canonical bundle over $W[1,1,2]$). The affine mirror curve $C^0$ is
\[ x + y + 1 + q_1x^2 + q_2x^4/y = 0. \tag{2.6} \]

The compactified mirror curve $C$ sits in $\mathbb{P}_\Delta = W[1,1,2]/\mu_2$.

**Example 2.4.** $\mathcal{X} = K_{F_4}$. The affine mirror curve $C^0$ is
\[ x + y + 1 + q_2x^2y^{-1} + q_1x^3y^{-1} = 0. \tag{2.7} \]

For a generic choice of parameters $q = (q_1, q_2, \ldots, q_p) \in \mathcal{U}_C$, the affine mirror curve $C^0$ of $\mathcal{X}$ is holomorphic Morse with respect to the covering $x : C^0 \to \mathbb{C}^*$. For our examples $\mathcal{X} = K_{\Sigma}$ for $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, W[1,1,2], F_4$, the number of ramification points is 3 for $S = \mathbb{P}^2$ and 4 for others – which is the same as the $p + g + 1 = \dim H^1_{CR}(\mathcal{X})$. We denote by $R$ the divisor of ramification points of $x : C \to \mathbb{P}^1$ on the mirror curve $C$ (those on $C^0$ are called finite ramification points).

### 2.2 Mirror symmetry from remodeling conjecture

Let’s briefly review the definition of open GW theory Stable maps to orbifolds with Lagrangian boundary conditions and their moduli spaces have been introduced in [CP12, Section 2]. Let $(\Sigma, x_1, \ldots, x_n)$ be a prestable bordered orbifold Riemann surface with $n$ interior marked points in the sense of [CP12, Section 2]. Then the coarse moduli space $(\tilde{\Sigma}, \tilde{x}_1, \ldots, \tilde{x}_n)$ is a prestable bordered Riemann surface with $n$ interior marked points, defined in [KL01, Section 3.6] and [Liu02, Section 3.2]. We define the topological type $(g, h)$ of $\Sigma$ to be the topological type of $\tilde{\Sigma}$ (see [Liu02, Section 3.2]).

Let $(\Sigma, \partial \Sigma)$ be a prestable bordered orbifold Riemann surface of type $(g, h)$, and let $\partial \Sigma = R_1 \cup \cdots \cup R_h$ be union of connected component. Each connected component is a circle which contains no orbifold points. Let $\varphi : (\Sigma, \partial \Sigma) \to (\mathcal{X}, \mathcal{L})$ be a (bordered) stable map in the sense of [CP12, Section 2]. The topological type of $\varphi$ is given by the degree $b' = \varphi_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$ and $\mu = \varphi_*[R_i] \in H_1(\mathcal{L}; \mathbb{Z})$. Given $b' \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$ and $\mu = (\mu_1, \ldots, \mu_h) \in H_1(\mathcal{L}; \mathbb{Z})^h$.

\[ \bar{b} = (\bar{b}_1, \ldots, \bar{b}_h) \in H_2(\mathcal{L}; \mathbb{Z})^h. \tag{2.8} \]
Let $\overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \mu)$ be the moduli space of stable maps of type $(g,h)$, degree $\beta'$, winding numbers and twisting $\mu$, with $n$ interior marked points.

There are evaluation maps (at interior marked points)

$$\text{ev}_j : \overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \mu) \to \mathcal{I}\mathcal{X}, \quad j = 1, \ldots, n,$$

(2.9)

where $\mathcal{I}\mathcal{X}$ is the inertia stack of $\mathcal{X}$.

Let $T'_R \cong U(1)^2$ be the maximal compact subgroup of $T' \cong (\mathbb{C}^*)^2$. Then the $T'_R$-action on $\mathcal{X}$ is holomorphic and preserves $\mathcal{L}$, so it acts on the moduli spaces $\overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \mu)$.

Given $\gamma_1, \ldots, \gamma_n \in H^*_T;C_R(X;Q) = H^*_T;C_R(X;Q)$, we define

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g, \beta', \mu} \colon := \int_{[\overline{M}_{g,\beta',\mu}]} \prod_{i=1}^{n} \left( \text{ev}_i^* \gamma_i \right)_{|F} \in Q(w_1, w_2)$$

(2.10)

where $F \subset \overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \mu)$ is the $T'_R$-fixed points set of the $T'_R$-action on $\overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} \mid \beta', \mu)$ and $Q(w_1, w_2)$ is the fractional field of $H^*_{T';R}(\mathcal{X};Q) \cong Q[w_1, w_2]$. As shown in [FLT13], it turns out that the open GW invariant $\langle \gamma_1, \ldots, \gamma_n \rangle_{g, \beta', \mu} \in Q(w_2/w_1)$, and specifying a framing $f$ amounts to setting $w_2/f = w_1$. Then the following open GW invariant

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g, \beta', \mu} (\mathcal{X}, f) := \iota_{T'_R}^* \gamma_1 \ldots \gamma_n \overline{\gamma} \mathcal{X}, \mathcal{L}, T'_R \rangle \in Q$$

(2.11)

for the rest of this paper we only consider the framing zero $f = 0$ case, namely setting the equivariant parameters $w_2 = 0$ and $w_1 = 1$, and simply write $\langle \ldots \mathcal{X}, \mathcal{L} \rangle$ for $\langle \ldots \mathcal{X}, \mathcal{L}, 0 \rangle$.

Define the open-closed GW potential

$$F^X_{g,n}(T; X_1, \ldots, X_n)$$

(2.12)

where $T = T_1H_1 + \ldots + T_pH_p$, where $\{H_a\}$ the integral basis of $H^2(\mathcal{X})$ in the extended Kähler cone $\overline{\mathcal{C}}^X$. When $n = 0$ this becomes the closed GW potential and we denote it by $F^X_g(T)$. For simplicity we use $F^X_{g,n}$ and $F^X_g$ for $F^X_{g,n}(T; X_1, \ldots, X_n)$ and $F^X_g(T)$ respectively.

The remodeling conjecture of [BKMnP09, BKMnP10] relates open-closed GW potential $F^X_{g,n}(T; \mathcal{X}, \mathcal{L})$ to Eynard-Orantin’s topological recursion invariants $\omega_{g,n}$. The full version of this conjecture, including the orbifold cases, is proved in [FLZ16].

Let $K_1(C; \mathbb{C}) = \ker(H_1(C; \mathbb{C}) \to H_1((\mathbb{C}^*)^2; \mathbb{C})) \cong \mathbb{Z}^2$ where the map $H_1(C^0; \mathbb{C}) \to H_1((\mathbb{C}^*)^2; \mathbb{C})$ is induced from

$$\mathbb{C}^0 \xrightarrow{(x,y)} (\mathbb{C}^*)^2.$$  

(2.13)
The enumerative mirror symmetry is corrected by the mirror map. We refer the reader to \cite{FLT13} Section 4.1 and 4.2] for the explicit form of the mirror map. We only list its asymptotic behavior here:

\[ T_a = \log q_a + o(q), a = 1, \ldots, p', \]
\[ T_a = q_a + o(q), a = p' + 1, \ldots, p, \]
\[ X_i = x_i(1 + O(q)) . \]  

Notice that since we have fixed \( q_a \) as in Examples 2.1, 2.2, 2.3 and 2.4, this particular asymptotic behavior determines each \( H_a \).

By the explicit construction in \cite{FLZ16} Section 5.5], the closed mirror maps are given by period integrals. Over a neighborhood of \( q = 0 \) in \( U_C \) (although 0 is taken away from \( U_C \) since the mirror curve is not smooth there), there are (families of) cycles \( \tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_p \in K_1(C^0; C) \) such that

\[ T_a = \int_{A_a} \log y \frac{dx}{x}, \quad a = 1, \ldots, p. \]  

They are called \textit{closed mirror maps}. The integrals are well-defined up to constants since the cycles are in \( K_1 \).

We define the bifundamental, a.k.a. Bergmann kernel, \( \omega_{0,2} \) as follows.

- \( \omega_{0,2} \) is a symmetric meromorphic form on \( C^2 \), with the only pole at the diagonal, i.e. for any local coordinate \( z \)
  \[ \omega_{0,2}(z_1, z_2) = \frac{dz_1 \boxtimes d\bar{z}_2}{(z_1 - \bar{z}_2)^2} + \text{holomorphic part.} \]  

- We require
  \[ \int_{z \in \tilde{\mathcal{A}}_a} \omega_{0,2}(z, w) = 0, \quad a = 1, \ldots, p. \]

That is, \( \omega_{0,2} \) vanishes on the \( A \)-cycles of a Torelli’s marking. Here each \( \tilde{\mathcal{A}}_a \) is the image of \( \tilde{\mathcal{A}}_a \) when passing to \( H_1(C; \mathbb{C}) \). Notice that \( \{\tilde{\mathcal{A}}_a\}_{a=1}^g \) span a Lagrangian subspace in \( H_1(C; \mathbb{C}) \).

\textbf{Remark 2.5} (Anti-holomorphic completion of the fundamental bidifferential). The fundamental bidifferential \( \omega_{0,2} \) constructed in the previous subsection depends on the choice of a Lagrangian subspace of \( H_1(C; \mathbb{C}) \) on curves over a neighborhood of 0 in \( U_C \), which cannot be extended over the entire \( U_C \). Let \( A_1, \ldots, A_g, B_1, \ldots, B_g \in H_1(C; \mathbb{Z}) \) be a Torelli’s marking, i.e. \( A_i \cap A_j = B_i \cap B_j = 0 \) and \( A_i \cap B_j = \delta_{ij} \). Define the coordinates \( \tau_{ij} \).

\[ \int_{A_i} \omega_i = \delta_{ij}, \quad \int_{B_j} \omega_i = \tau_{ij}, \]  

where \( \{\omega_i\} \) form a basis in \( \Omega^1(C) \). One can then define the modified cycles following \cite{EO07}

\[ A_i = A_i - \sum_{k_{ij}} (B_j - \sum_{l=1}^g \tau_{lj} A_l), \quad B_j = B_i - \sum_{j=1}^g A_j. \]

They also form a Lagrangian subspace in \( H_1(C; \mathbb{C}) \). We use the cycles \( A_1, \ldots, A_g \) to define the bidifferential \( \hat{\omega}_{0,2} \), this is what is called the Schiffer kernel, see \cite{Tyu78}. It turns out that
\( \tilde{\omega}_{0,2} \) does not depend on the choice of \( A_1, \ldots, A_g \), and is defined for curves over the entire \( \mathcal{U}_C \). From the definition, by regarding \( \text{Im} \tau_{ij} \) as formal variables, then the Schiffer kernel has the "holomorphic limit"

\[
\lim_{\text{Im} \tau_{ij} \to \infty} \tilde{\omega}_{0,2} = \omega_{0,2}.
\] (2.20)

See [Fay77, Tyu78, Tak01] for details.

Define \( R^\circ \) to be the ramification locus of \( x : \mathbb{C}^o \to \mathbb{P}^1 \). We assume \( R^\circ \) has only simple ramifications which is the case of our examples for generic \( q \in \mathcal{U}_C \). The Eynard-Orantin topological recursion defines \( \omega_{g,n} \) \((2g - 2 + n > 0)\) recursively as follows

\[
\omega_{g,n}(p_1, \ldots, p_n) = \sum_{p_0 \in R^\circ} \text{Res}_{p \to p_0} \frac{\int_{\xi = p}^p B(p_n, \xi)}{2(\lambda(p) - \lambda(p^*))} \left( \omega_{g-1,n+1}(p, p^*, p_1, \ldots, p_{n-1}) \right) + \sum_{g_1 + g_2 = g} \sum_{J \cup K = \{1, \ldots, n-1\}} \sum_{J \cap K = \emptyset} \omega_{g_1,|J|+1}(p, p_J) \omega_{g_2,|K|+1}(p^*, p_K),
\] (2.21)

where \( \lambda = \log y \frac{dx}{x} \), \( \omega_{0,1} = 0 \), and for any \( p \in \mathbb{C}^o \) around a simple ramification point \( p_0 \), \( p^* \neq p \) is the unique point that has the same \( x \)-coordinate. Eynard-Orantin has shown that \( \omega_{g,n} \) is symmetric, and at most has poles at ramification points.

The mirror \( \mathbb{C} \) has a distinguished point \( s_0 = (x, y) = (0, -1) \).

We call this the open large radius limit point or open GW point.

**Theorem 2.6** (Open-sector remodeling conjecture and disk mirror theorem [FLT13, FLZ16] restricted to our cases). **Under the open-closed mirror map** (2.14), for \( X = K_S \) where \( S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{W}[1, 1, 2], \mathcal{F}_1 \), we have the following statements.

- **When** \( 2g - 2 + n > 0 \),
  \[
  d_{x_1} \ldots d_{x_n} F_{g,n} = \omega_{g,n}.
  \] (2.23)

- **When** \( (g, n) = (0, 2) \),
  \[
  d_{x_1} d_{x_2} F_{0,2} = \omega_{0,2} - \frac{d x_1 \boxtimes d x_2}{(x_1 - x_2)^2}.
  \] (2.24)

- **When** \( (g, n) = (0, 1) \),
  \[
  (x \frac{\partial}{\partial x})^2 F_{0,1} = x \frac{\partial}{\partial x} \log y.
  \] (2.25)

We understand \( \omega_{g,n} \) or \( \log y \) as expansions by power series in \( x_1, \ldots, x_n \) (or \( x \)) at the open large radius limit point \( s_0 = (0, -1) \). Notice that \( \omega_{0,2} \) is singular at the diagonal so we need to subtract its singular part first.
Theorem 2.7 (Closed-sector remodeling conjecture for $g > 1$, [FLZ16], restricted to our cases). Under the same assumption as the previous theorem,

$$
F_g = \frac{1}{2 - 2^g} \sum_{p_0 \in R} \text{Res}_{p \to p_0} \left( d^{-1} \lambda(p) \cdot \omega_{g,1}(p) \right),
$$

(2.26)

where $d^{-1} \lambda = \int \lambda$, which can be locally defined near each ramification point, and the constant ambiguity does not affect the result.

Remark 2.8. The remodeling conjecture is true for any toric CY 3-orbifold under generic framing as shown in [FLZ16]. Since we restrict to framing zero, whether the remodeling conjecture holds for $X = K_S$ for $S$ being other toric Fano orbifolds in framing zero is unknown to us. In such cases, the affine mirror curves have 3 or 4 $x$-ramification points and which is less than the dimension of $H_{\text{CR}}^*(\mathcal{X})$. The number of the ramification points of $xy^{-f}$ and the dimension of $H_{\text{CR}}^*(\mathcal{X})$ are equal for a generic framing $f$ and this fact is essential in the proof of the remodeling conjecture. We hope by taking limit to $f = 0$ one may still recover the remodeling conjecture, and then our argument in this paper automatically extends to all 12 cases when the mirror curves are of genus one and in hyperelliptic forms.

Once we replace the Bergman kernel $\omega_{0,2}$ by the Schiffer kernel $\hat{\omega}_{0,2} = S$, we denote the recursion result by $\hat{\omega}_{g,n}$. Furthermore we use $\hat{F}_g$ to denote the right hand side of (2.26) after replacing $\omega_{0,2}$ by $\hat{\omega}_{0,2}$. We have $\lim_{\text{int} \to \infty} \hat{\omega}_{g,n} = \omega_{g,n}$ and $\lim_{\text{int} \to \infty} \hat{F}_g = F_g$.

3 Geometry of genus one mirror curves

Hereafter by a curve $C$ we mean a smooth projective variety over $\mathbb{C}$ of pure dimension one. Our technique requires the affine mirror curve (2.1) to be in hyperelliptic form. That is, the equation of the curve (2.1) can be transformed by a bi-regular morphism into the form

$$
\hat{y}^2 = g(x)
$$

(3.1)

after the simple change of variables

$$
\hat{y} = y + h(x)
$$

(3.2)

where $h(x)$ is a quadratic polynomial in $x$. In particular one writes

$$
y^* = -y - 2h(x),
$$

(3.3)

then the action $*: (x, y) \mapsto (x, y^*)$ gives the hyperelliptic involution.

The remodeling conjecture relates $\omega_{g,n}^f$ to $F_{g,n}^{X_L}$. The main tool of our investigation is the hyperelliptic form of the mirror curve $\hat{y}^2 = g(x)$, for which the ramification points of $x$ have very nice properties.
3.1 Basic definitions on modular forms and Jacobi forms

In topological recursion, we need to represent various ingredients in terms of modular forms and Jacobi forms.

We now give very quick definitions, without explaining many of the subtitles. See e.g., [Zag08] for a quick introduction to modular forms. Readers who are familiar with these concepts can skip this subsection.

**Definition 3.1 (Modular forms).** A holomorphic function $\phi: \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the upper-half plane, is called a (holomorphic) modular form of weight $k \in \mathbb{Z}$ for the modular group $\Gamma < SL_2(\mathbb{Z})$ if it satisfies

1. $\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \phi(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,
2. $\phi$ has sub-exponential growth at infinity $\tau \to i\infty$.

The factor $(c\tau+d)^k$ is called the automorphy factor. In this paper, we shall need to work with modular forms with non-trivial multiplier systems. The definition is as follows. Fix a modular group $\Gamma < SL_2(\mathbb{Z})$. A function $v: \Gamma \to U(1)$ is called a multiplier system of weight $k$ for $\Gamma$ if it satisfies $v(-1) = (-1)^k$ and

$$v(\gamma_1\gamma_2) = v(\gamma_1)v(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma$$

for some function $w$ valued in $\{\pm 1\}$ making $v(\gamma)(c\tau+d)^k$ into an automorphy factor. Replacing the automorphy factor $(c\tau+d)^k$ in Definition 3.1 by $v(\gamma)(c\tau+d)^k$ one defines modular forms of weight $k$ with respect to the multiplier system $v$. See [Ran77, Sch12] for further details.

One can also define the variants quasi-modular forms and almost-holomorphic modular forms [KZ95].

**Definition 3.2 (Quasi-modular forms).** A holomorphic function $\phi: \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the upper-half plane, is called a quasi-modular form of weight $k \in \mathbb{Z}$ for the modular group $\Gamma < SL_2(\mathbb{Z})$ if it satisfies

1. There exist holomorphic functions $f_j: \mathcal{H} \to \mathbb{C}, j = 1, 2, \cdots, k$, such that
   $$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \phi(\tau) + \sum_{j=1}^{k} c^j (c\tau+d)^{k-j} f_j, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$
2. $\phi$ has subexponential growth at infinity $\tau \to i\infty$.

**Definition 3.3 (Almost-holomorphic modular forms).** A real analytic function $\phi: \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the upper-half plane, is called an almost-holomorphic modular form of weight $k \in \mathbb{Z}$ for the modular group $\Gamma < SL_2(\mathbb{Z})$ if it satisfies

1. $\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{b\tau+a}{d\tau+c}\right) = (c\tau+d)^k \phi(\tau, \bar{\tau}), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

2 Throughout this work, we always assume that $\Gamma$ contains $-1$. 

13
2. $\phi$ has polynomial growth in $1/\text{Im}\tau$ as $\tau \to i\infty$.

A typical example of a quasi-modular form is the Eisenstein series $E_2$ of weight 2 and of an almost-holomorphic modular form is

$$\hat{E}_2 = E_2 + \frac{3}{\pi} \frac{1}{\text{Im}\tau}.$$ (3.5)

In fact, we have the following structure theorem due to [KZ95]. The set of modular forms, quasi-modular forms, almost-holomorphic modular forms for the modular group $\Gamma$ form graded rings. Denote them by $M(\Gamma), \tilde{M}(\Gamma), \hat{M}(\Gamma)$, respectively. Then

$$\tilde{M}(\Gamma) = M(\Gamma)[E_2], \quad \hat{M}(\Gamma) = M(\Gamma)[\hat{E}_2].$$ (3.6)

For the full modular group $\Gamma(1) = \text{SL}_2(\mathbb{Z})$ one has $M(\Gamma(1)) = \mathbb{C}[E_4, E_6]$, where $E_4, E_6$ are the Eisenstein series of weight 4, 6 respectively.

The above definitions generalize to the corresponding objects with multiplier systems. For the examples later studied in this paper, all of the modular forms with non-trivial multiplier systems arise from uniformization of some elliptic curve families (see Section 3.5) and are usually explicit functions\(^3\) of $\theta$-constants or $\eta$-functions whose multiplier systems are very explicit. By passing to a smaller modular subgroup if necessary, we can assume that their multiplier systems are integer powers of the same quadratic multiplier system. In these cases, the sets of modular forms, quasi-modular forms, and almost-holomorphic modular forms form graded rings respectively. The structure in (3.6) still holds. For this reason, we will usually ignore this subtlety on multiplier system in this work. Further in what follows by a modular form (also its variants) we mean one with a possibly non-trivial multiplier system.

We shall also encounter the notion of Jacobi forms [EZ84] whose definition is given as follows.

**Definition 3.4** (Jacobi forms). A holomorphic function $\Phi : \mathbb{C} \times \mathcal{H} \to \mathbb{C}$ is a (holomorphic) Jacobi form of weight $k \in \mathbb{Z}$, index $\ell \in \mathbb{Z}_{>0}$ for the modular group $\Gamma < \text{SL}_2(\mathbb{Z})$ if it is “modular in $\tau$ and elliptic in $z$” in the sense that

- $\Phi\left(\frac{z}{c\tau+a}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i \frac{a^2 + b^2}{c^2}} \Phi(z, \tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$

- $\Phi(z + m\tau + n, \tau) = e^{-2\pi i (m^2 \tau + 2mn)} \Phi(z, \tau), \quad \forall m, n \in \mathbb{Z}.$

together with some regularity condition

- in the Fourier expansion

$$\Phi(z, \tau) = \sum_{n,r} c(n, r) e^{2\pi i n \tau} e^{2\pi i rz},$$

one has $c(n, r) = 0$ unless $4\ell n \geq r^2$.

\(^3\)In this paper we follow the convention in [Zag08] for the $\theta$-constants.
where \( J \) denotes the ring of weak Jacobi forms and the fractional field (respecting the bigrading) \( \mathcal{M}(\Gamma) \) of (holomorphic) modular forms. We also denote the fractional field (respecting the bigrading) of the ring \( J \) of weak Jacobi forms for the full modular group \( SL_2(\mathbb{Z}) \) by \( \mathcal{J} \). We call it the ring of meromorphic Jacobi forms. It includes in particular the Weierstrass elliptic functions \( \wp \) and \( \wp' \) which are proportional to \( B/A, C/A^2 \) respectively according to e.g., [DMZ12].

In this work, we shall need to work with "meromorphic modular forms", "meromorphic Jacobi forms" which are defined with the requirement of holomorphy replaced by meromorphicity. To be more precise, fix a modular group \( \Gamma \), we denote the ring of meromorphic modular forms by \( \mathcal{M}(\Gamma) \). This is the fractional field (respecting the grading) of ring \( M(\Gamma) \) of (holomorphic) modular forms. We also denote the factional field (respecting the bigrading) of the ring \( J \) of weak Jacobi forms for the full modular group \( SL_2(\mathbb{Z}) \) by \( \mathcal{J} \). We call it the ring of meromorphic Jacobi forms. It includes in particular the Weierstrass elliptic functions \( \wp \) and \( \wp' \) which are proportional to \( B/A, C/A^2 \) respectively according to e.g., [DMZ12].

We finally introduce the following definitions, borrowing the terminologies "quasi" and "almost" from the \( \tau \)-part of the corresponding functions.

**Definition 3.5** (Quasi-meromorphic Jacobi forms and almost-meromorphic Jacobi forms). Fix a modular group \( \Gamma < SL_2(\mathbb{Z}) \). We define the ring of meromorphic quasi-modular forms, and almost-meromorphic modular forms to be

\[
\mathcal{M}(\Gamma) = M(\Gamma)[E_2], \quad \hat{\mathcal{M}}(\Gamma) = M(\Gamma)[\hat{E}_2],
\]

where \( M(\Gamma) \) is the fractional field of the ring \( M(\Gamma) \) of (holomorphic) modular forms for \( \Gamma \).

We define the ring of quasi-weak Jacobi forms and almost-weak Jacobi forms for the full modular group \( SL_2(\mathbb{Z}) \) to be

\[
\mathcal{J} = J[E_2], \quad \hat{\mathcal{J}} = J[\hat{E}_2],
\]

where \( J \) is the rings of weak Jacobi forms for \( SL_2(\mathbb{Z}) \) in (3.7). We define the ring of quasi-meromorphic Jacobi forms and almost-meromorphic Jacobi forms for the full modular group \( SL_2(\mathbb{Z}) \) to be

\[
\mathcal{J} = J[E_2], \quad \hat{\mathcal{J}} = J[\hat{E}_2],
\]

where \( \mathcal{J} \) is the fractional field of the ring \( J \) of weak Jacobi forms for \( SL_2(\mathbb{Z}) \).

For the modular group \( \Gamma < SL_2(\mathbb{Z}) \), we define the (multi-)graded rings of quasi-meromorphic Jacobi forms and almost-meromorphic Jacobi forms for the modular group \( \Gamma < SL_2(\mathbb{Z}) \) to be

\[
\mathcal{J}(\Gamma) = J \otimes \hat{\mathcal{M}}(\Gamma) = \hat{\mathcal{J}} \otimes \mathcal{M}(\Gamma), \quad \hat{\mathcal{J}}(\Gamma) = J \otimes \hat{\mathcal{M}}(\Gamma) = \hat{\mathcal{J}} \otimes \mathcal{M}(\Gamma).
\]
Following [KZ95], one can define the so-called "constant term map". It is defined by regarding $\text{Im} \tau$ as a formal variable and then sending it to infinity, hence the same as the "holomorphic limit". It has the effect of replacing $\hat{E}_2$ by $E_2$ and of mapping the "almost" objects to "quasi" objects. We also denote this operation by $\lim_{\text{Im} \tau \to \infty}$.

### 3.2 Uniformizations of genus one algebraic curves

In topological recursion, we shall need to explicitly express many quantities, including rational functions on a curve $C$ of genus one, in terms of modular and Jacobi forms. This is possible with the help of the classical uniformization theorem of Klein-Poincaré which says that any plane curve $C$ admits a parametrization in terms of automorphic functions.

First let $C$ be a Riemann surface of genus one. It is a classic fact that $C$ can be uniformized by the complex plane $\mathbb{C}$. That is to say, there exists a lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \tau$, $\tau \in \mathcal{H}$ depending on $C$ such that $C$ is biholomorphic to $\mathbb{C}/\Lambda$. The uniformization is then provided by the universal covering map

$$\pi : C \rightarrow \mathbb{C}/\Lambda \cong C.$$ (3.12)

Explicitly the covering map is constructed by the Weierstrass elliptic functions, as the inverse of the Abel-Jacobi map in terms of Abelian integrals. In particular, the Abel-Jacobi map provides a uniformizing parameter $u$ on $C$.

Let now $C$ be a curve of genus one. In this work, by a uniformization of the algebraic curve $C$ we mean an explicit map $\pi$, expressing elements in the rational functional field $k(C)$ of $C$ in terms of automorphic functions in the local uniformizing parameter $u$.

Consider the case where $C$ is already in the Weierstrass normal form, which in the affine patch $Z = 1$ of the plane $\mathbb{P}^2$ with homogeneous coordinates $[X, Y, Z]$ is given by

$$Y^2 = 4X^3 - aX - b,$$ (3.13)

for some $(a, b)$ such that the curve $C$ is smooth. A uniformization of $C$ is provided by the Weierstrass elliptic functions

$$X = \wp(u, \tau), \quad Y = \partial_u \wp(u, \tau).$$ (3.14)

Here to obtain $\tau, u$ from $C$, we first choose a Torelli marking $\{A, B\}$ on the curve $C$ and a reference point $O$ for the Abel-Jacobi map $u$. Then we take

$$\tau = \frac{\int_B \frac{dX}{Y}}{\int_A \frac{dX}{Y}}, \quad u(p) = \int_O^p \frac{dX}{Y}, \quad \forall p \in C.$$ (3.15)

A uniformization is determined up to translation (inducing shift of origin in the group law on $C$) and scaling (inducing homothety on $C$) on $C$. The translation ambiguity can be fixed by requiring that the origin $O$ in the group law to be $[0, 1, 0]$ for example, while the homothety can be uniquely determined by requiring $a, b$ to be the modular forms $g_2 = (4/3)\pi^4 E_4, g_3 = (8/27)\pi^6 E_6$.

A general curve $C$ of genus one is bi-regular to a plane curve in Weierstrass normal form. A uniformization for $C$ can then be obtained by transforming the curve into the Weierstrass
normal form, and then applying the results for the latter.

The notion of uniformization makes sense for the relative version. Let $\chi : C \to U_C$ be a family of curves, that is, a flat proper morphism $\chi : C \to U_C$ between algebraic varieties such that for any geometric point $b \in U_C$, the fiber $C_b$ is a curve. By a uniformization of the curve family $C$ we mean an explicit holomorphic map $\pi : C \times H \to C$, which restricts to uniformizations fiberwisely.

A particularly interesting family\(^5\) is the Weierstrass normal form

$$W : Y^2Z = 4X^3 - aXZ^2 - bZ^3$$

(3.16)

defined over $U_W := \mathbb{A}^2 - \Delta, \Delta = \{(a,b)|a^3 - 27b^2 = 0\}$. This serves as the reference family for the construction of uniformization for families of curves of genus one.

**Lemma 3.6.** Any family $C \to U_C$ of curves of genus one admits a uniformization via the Weierstrass normal form. That is, there exists a morphism $U_C \to U_W$ such that $C = U_C \times_{U_W} W$, while a uniformization is obtained by pulling back a uniformization of $W$.

**Proof.** The existence of uniformization is well known: one simply reduces the defining equations of the curve family $C$ in $U_C \times \mathbb{P}^N$, for some ambient projective space $\mathbb{P}^N$, into the Weierstrass normal form.

For the cases that we are interested in, the family $C$ is usually defined by a complete intersection with small $N$. Practically, reducing the defining equations to the Weierstrass normal form can be done following the algorithms in e.g., [Con96].

### 3.3 Ramification points for hyperelliptic curves of genus one

We will need to identify the ramification points for a hyperelliptic cover $p : C \to \mathbb{P}^1$ of a genus one curve $C$ as the 2-torsion points on its Jacobian. The statement is as follows.

**Lemma 3.7.** Suppose the genus one curve $C$ is equipped with a hyperelliptic structure $p : C \to \mathbb{P}^1$.

1. The set of ramification points $R$ are identified with the group of 2-torsion points of the group law, with the origin of the group law chosen to be any of the ramification points.

2. Under the Abel-Jacobi map with the reference point chosen to be any of the ramification points, the involution on $C$ exchanging the two sheets of the hyperelliptic cover $p : C \to \mathbb{P}^1$ is induced by the map $u \mapsto -u$ on the Jacobian variety of $C$.

**Proof.**

1. Taking any two of the branch points $b_1, b_2$, denote the corresponding ramification points by $r_1, r_2$. Then we have for the divisor class

$$p^*([b_1] - [b_2]) = p^*([b_1]) - p^*([b_2]) = 2[r_1] - 2[r_2] = 2([r_1] - [r_2]).$$

(3.17)

Since the left hand side is principal, so is the right hand side $2([r_1] - [r_2])$. Then $[r_1] - [r_2]$ is a 2-torsion on the Jacobian of $C$.

---

\(^5\)This is a universal family with the base having a moduli stack interpretation. See e.g. [Kat76, Dub94] for a nice account on this.
Picking once and for all any of the ramification points makes the genus one curve $C$ an elliptic curve whose origin $O$ in the group law is the chosen point. By the property of the Abel-Jacobi map (with reference $O$) as an isomorphism, we see that the corresponding difference of $r_1, r_2$ in the group law of the elliptic curve $C$ is a 2-torsion point in the group law.

2. Recall that the uniformization of the algebraic curve (3.13) and the Abel-Jacobi map $u$ are related through the Weierstrass elliptic functions in (3.14), with which the origin $O$ of the group law of the elliptic curve $C$ is mapped to $[0, 1, 0]$ in the homogenized coordinates of $[\wp, \wp', 1]$. It is a classical fact that rational function field $k(C)$ of a genus 1 curve $C$ is generated by $\wp, \wp'$ with the algebraic relation given by the Weierstrass equation

$$k(C) \cong C(\wp, \wp')/\langle (\wp')^2 - (4\wp^3 - g_2\wp - g_3) \rangle.$$  

(3.18)

The Galois group for the Galois extension $k(C)$ of the field $C(\wp)$ is generated by $*: \wp \mapsto \wp, \wp' \mapsto -\wp'$. It is induced by the reflection $u \mapsto -u$ in the $u$-plane which is the universal cover of the elliptic curve $C$.

We claim that the local involution around any ramification point of any hyperelliptic cover $p: C \to \mathbb{P}^1$ of the genus one curve $C$ must be the above one. To see this, we simply observe that by analytic continuation this local involution determines an index 2 rational subfield over $C$. The fixed locus of this involution includes at least the ramification point. Up to isomorphism there is only one such index 2 subfield, namely, $C(\wp)$. This shows that the desired statement is true.

3.4 One-parameter subfamilies of genus one mirror curve families

In later discussions in topological recursion, we only consider the cases when $C$ is one of mirror curve families in Examples 2.1, 2.2, 2.3 and 2.4. These are families of genus one curves to which Lemma 3.6 applies. We also take the hyperelliptic structure $p: C \to \mathbb{P}^1$ on any fiber $C$ in the family $C$ to be the hyperelliptic structure $x$ determined by the brane structure. We can then apply Lemma 3.7.

Although the techniques below apply to more general families with one-dimensional bases, we are mainly interested in the so-called one-parameter families of curves, namely those whose bases are Zariski open subsets of $\mathbb{P}^1$. This implies that the bases are rational curves and hence admit rational parametrizations. As a consequence of Lemma 3.6, we have the following result.

**Lemma 3.8.** Consider a non-trivial one-parameter family $C \to U_C$ of curves of genus one. Any rational function in $k(C/U_C)$ is a rational function of $\wp, \wp'$, with coefficients lying in the fractional field $M(\Gamma)$ of the ring $M(\Gamma)$ of modular forms whose modular group $\Gamma$ depends on $C$.

**Proof.** Consider the map, given by the $j$-invariant, from $U_C$ to the modular curve $SL_2(\mathbb{Z}) \backslash \mathcal{H}^*$. This map can again be obtained by reducing the equation for the curve family to the Weierstrass normal form for which the $j$-invariant is the standard one $j = 1728a^3/(a^3 - 27b^2)$. The map induces an orbifold structure on $U_C$ which extends across the preimages of the cusp on the modular curve. The coarse moduli is the compactification of $U_C$ which by our
assumption of an one-parameter family is \( \mathbb{P}^1 \). By deleting the orbifold points and cusps on the compactified orbifold and looking at the monodromy representation, we obtain the usual monodromy group which is of finite index in \( SL_2(\mathbb{Z}) \). We then take the modular group \( \Gamma \) to be the monodromy group with \(-1 \in SL_2(\mathbb{Z})\) adjoint.

The one-parameter families studied in this work are obtained by specializing a possibly multi-parameter mirror curve family \( \chi : C \to \mathcal{U} \) to non-trivial one-parameter sub-families. For \( \mathcal{X} = K_{\mathbb{P}^2} \), \( C \) is an one-parameter family, for which the base \( \mathcal{U} \) is actually the thrice punctured \( \mathbb{P}^1 \). For the other cases \( K_S, S = \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{W}[1,1,2], \mathbb{F}_1 \), the base \( \mathcal{U} \) is two-dimensional. We take a rational affine curve \( \mathcal{U}_{\text{res}} \) in \( \mathcal{U} \), such that the restriction of the family \( C \) to \( \mathcal{U}_{\text{res}} \), denoted by \( C_{\text{res}} \), has non-constant complex structures. Moreover, in the partial compactification of \( \mathcal{U} \) where the point \((q_1, \ldots, q_p) = 0\) is included, we require 0 is also in the closure of \( \mathcal{U}_{\text{res}} \). Then we denote the one-parameter compactified mirror curve family by \( \chi_{\text{res}} : C_{\text{res}} \to \mathcal{U}_{\text{res}} \), and the affine mirror curve family by \( \chi_{\text{aff}} : C_{\text{aff}} \to \mathcal{U}_{\text{aff}} \).

Let \( A, B \) be cycles in \( K_1(C^0; \mathbb{Z}) \) on a fiber \( C \) such that passing to \( H_1(C; \mathbb{Z}) \), their images \( \bar{A}, \bar{B} \) constitute a Torelli marking. We can recover the complex structure parameter \( \tau \) of \( C \) from
\[
\int_A \lambda = t, \quad \int_B \lambda = tB, \quad \frac{\partial B}{\partial t} = \tau.
\]
This definition is compatible with (3.15) and Remark 2.5. The parameter \( t \) is called the flat coordinate. Under the mirror map, such a coordinate \( t \) is the Kähler parameter \( T_1 \) for \( K_{\mathbb{P}^2} \), or a linear combination of \( T_1, T_2 \) for the other cases when such a linear combination of \( \bar{A}_1 \) and \( \bar{A}_2 \) is primitive in \( H_1(C; \mathbb{Z}) \). After restricting to \( \mathcal{U}_{\text{res}} \), all of \( t, T_1, T_2 \) are functions of \( \tau \). By [FLZ16, Theorem 7.10], we know that \( \tau \) has an \( A \)-model description
\[
\frac{\partial^2}{\partial t^2} F_0 = \tau.
\]

### 3.5 Examples

In this section, we give the uniformizations for the mirror curve families of \( K_{\mathbb{P}^2}, K_{\mathbb{P}^1 \times \mathbb{P}^1}, K_{\mathbb{W}[1,1,2]} \) and \( K_{\mathbb{F}_1} \), displayed in Example 2.1, 2.2, 2.3, 2.4 respectively. For each of these examples, the \( \wp \)-uniformization in Lemma 3.6 is derived by transforming the curve family \( C \) to the Weierstrass normal form (3.13), with the coordinates carefully so that the coefficients in the degree 1 and 0 terms in the resulting Weierstrass normal form become exactly \(-g_2, -g_3\) respectively. The derivations are straightforward and we omit the details here.

In all of our examples, the curve in the chosen affine patch is defined by the equation \((y + h(x))^2 = g(x)\) as shown in (3.1) and (3.2). For the \( K_{\mathbb{P}^2} \) and \( K_{\mathbb{F}_1} \) cases, the degree of \( g(x) \) is 3. Taking the origin \( O \) for the group law to be the ramification point \( \infty = [0, 1, 0] \) fixes the ambiguity in the shift \( \epsilon \) of the argument in \( \wp(u + \epsilon), \wp'(u + \epsilon) \) for the uniformization to be zero. For the other cases, we choose once and for all a ramification point \( O \) to be the origin. Then in the rational functions \( x(\wp, \wp'), y(\wp, \wp') \) in terms of \( \wp(u + \epsilon), \wp'(u + \epsilon) \), we have that \([x_0, y_0, 1] := [x, y, 1]|_{u=0}\) is the coordinate for the chosen ramification point \( O \). With these choices, the hyperelliptic involution is induced by \( u \to -u \) as shown in Lemma 3.7.

We shall also discuss the subtlety on multiplier systems mentioned in Section 3.1. One of the main results, proved by a case by case analysis below, is the following
Lemma 3.9. Consider the local toric Calabi-Yau 3-folds $X = K_S, S = \mathbb{P}^2, \mathbb{WP}[1,1,2], \mathbb{P}^1 \times \mathbb{P}^1, F_1$. Consider non-trivial one-parameter subfamilies of the mirror curves with hyperelliptic structure determined by the corresponding brane. Then the values of the rational functions $x, y$ at the ramification points are meromorphic modular forms in $\mathcal{M}(\Gamma(2) \cap \Gamma)$ for some modular group $\Gamma$ depending on the one-parameter subfamily.

3.5.1 $K_{\mathbb{P}^2}$

The affine part of the mirror curve given in Example 2.1 is equivalent to

$$x^3 + y^2 + y - 3\phi xy = 0. \quad (3.21)$$

The parameter $\phi$ is related to the parameter $q_1$ in Example 2.1 by $q_1 = (-3\phi)^{-3}$. It is uniformized by

$$x = (-4)^{-1/2} \kappa^2 \varphi(u) + \frac{3}{4} \phi^2, \quad y = \kappa^3 \varphi'(u) - \left(\frac{1 - 3\phi x}{2}\right), \quad (3.22)$$

with

$$\phi(\tau) = \Theta_{A_2}(2\tau) \frac{\eta(3\tau)}{\eta(\tau)^3}, \quad \kappa = \zeta_6 2^{-1/2} 3^{1/2} \eta(3\tau) \eta(\tau)^{-3}, \quad (3.23)$$

where $\Theta_{A_2}$ is the $\theta$-function for the $A_2$-lattice and $\eta$ is the $\eta$-function as a modular form. The quantities $\phi, \kappa$ are modular forms for $\Gamma_0(3)$ with non-Dirichlet multiplier systems. By passing to the smaller modular subgroup $\Gamma_0(9)$, we see that both $\Theta_{A_2}(2\tau)$ and $\kappa$, and hence $\phi$, are modular forms with the same quadratic multiplier system, which is given by the Dirichlet character $\chi_{-3}$ taking the values 1, $-1$ on 1, $-1$ modulo 3 respectively. See [BB91, BBG94, BBG95, Mai09, Mai11] for details. This confirms the discussion on multiplier systems in Section 3.1.

Under the uniformization, the point $\infty = [0,1,0]$ corresponds to the origin $O$ of the group law, which is given by $u = 0$ on the Jacobian. The values of $x, y$ at the ramification points $u = 1/2, \tau/2, 1 + \tau/2$ are meromorphic modular forms for $\Gamma(2) \cap \Gamma_0(9)$, by the standard fact that the values of $\varphi, \varphi'$ at these points are weight-two modular forms with trivial multiplier systems for $\Gamma(2)$. See Section 4.1.3 for more details on this.

This family admits furthermore a uniformization via Jacobi $\theta$-functions compatible with the above Weierstrass $\wp$-uniformization in the sense that the origins for the group law are the same. See [Dol97] for details. It turns out that the open GW point $[0, -1, 1]$ in (2.22) is a 3-torsion point.

3.5.2 $K_{\mathbb{WP}[1,1,2]}$

The affine part of the mirror curve given in Example 2.3 is equivalent to

$$y^2 + x^4 + y + b_4 x^2 y + b_0 xy = 0. \quad (3.24)$$

The parameters $b_0, b_4$ are related to those in Example 2.3 by $q_1 = b_4 b_0^{-2}, q_2 = b_0^{-4}$. The rational function $x$ induces a hyperelliptic structure on the mirror curve with generic $b_4, b_0$. Another different hyperelliptic structure for the mirror curve is induced from the equation

$$y^2 + 1 + x^2 y + b_4 y + b_0 xy = 0. \quad \text{The discussion below applies similarly to this case.}$$
The \( \wp \)-uniformization can be obtained from the algorithm in [Con96]. It is accomplished by the following sequence of change of coordinates which induce bi-regular maps on the curves. First we make the change of coordinates

\[
\alpha = 2^{\frac{3}{2}} \kappa^2 X - \frac{1}{12} (b_0^2 + 2b_4), \quad \beta = \kappa^3 Y - \frac{1}{2} b_0 (\alpha + \frac{1}{2} b_4), \tag{3.25}
\]

where \( \kappa \) is some constant arising from homothety. Then we set

\[
x = \beta^{-1} \left( 2^{\frac{3}{2}} \kappa^2 X + \frac{1}{3} (b_4 - \frac{1}{4} b_0^2) \right), \quad y = - \frac{1}{2} + x (\alpha x - \frac{1}{2} b_0) - \frac{1}{2} (1 + b_0 x + b_4 x^2). \tag{3.26}
\]

Then the equation for the curve becomes the Weierstrass normal form

\[
Y^2 = 4X^3 - aX - b, \tag{3.27}
\]

with

\[
a = \kappa^{-4} \frac{(b_0^4 - 8b_0^2 b_4 + 16b_4^2 - 48)}{2^{1} \cdot 24}, \quad b = -\kappa^{-6} \frac{(b_0^2 - 4b_4)(b_0^4 - 8b_0^2 b_4 + 16b_4^2 - 72)}{864}. \tag{3.28}
\]

The \( j \)-invariant is

\[
j = \frac{(b_0^4 - 8b_0^2 b_4 + 16b_4^2 - 48)^3}{(b_0^2 - 4b_4 + 8)(b_0^2 - 4b_4 - 8)}. \tag{3.29}
\]

From these computations it is easy to see that the parameters \( b_0, b_4 \) enter the discriminant and the \( j \)-invariant through the combination

\[
s = (b_0^2 - 4b_4)^2 = \frac{(1 - 4q_1)^2}{q_2}, \quad j = \frac{(s - 48)^3}{s - 64}. \tag{3.30}
\]

We recognize (see for instance [Mai09]) that \( s \) is a Hauptmodul \( t_2 + 64 \) for \( \Gamma_0(2) \). Up to an \( SL_2(\mathbb{Z}) \) transform, one has

\[
t_2 = 64 \frac{(\theta_2^4(2\tau) + \theta_3^4(2\tau))^2}{\theta_4^8(2\tau)} - 64. \tag{3.31}
\]

Solving \( a = g_2, b = g_3 \), we obtain

\[
\kappa = 2^{-\frac{3}{2}} \pi^{-1} \theta_4^{-2}(2\tau). \tag{3.32}
\]

This is a modular form for \( \Gamma_0(4) \) with a non-Dirichlet multiplier systems (see for instance [Mai09]). By passing to the smaller modular subgroup \( \Gamma_0(8) \), with the Dirichlet character \( \chi^{-4} \).

We now consider the shift \( \epsilon \) in \( X = \wp(u + \epsilon), Y = \wp'(u + \epsilon) \). It is such that the point \([x_0, y_0, 1]\) is a ramification point for \( (3.24) \). By completing square, we see that \( (3.24) \) is transformed into

\[
(y + h(x))^2 = g(x), \quad h(x) = \frac{1}{2} (1 + b_0 x + b_4 x^2), \quad g(x) = -x^4 + h^2(x). \tag{3.33}
\]
In particular, the coordinate for the branch point $x_0$ satisfy the equation $g(x_0) = 0$ which for generic parameters $(b_0, b_4)$ has four distinct finite solutions. These four solutions are given by

$$x = -b_0 \pm \sqrt{\frac{b_0^2 - 4(b_4 + 2)}{2(b_4 + 2)}}, \quad -b_0 \pm \sqrt{\frac{b_0^2 - 4(b_4 - 2)}{2(b_4 - 2)}}. \tag{3.34}$$

Recall that $s = (b_0^2 - 4b_4)^2$ is a modular function $t_2 + 64$ for $\Gamma_0(2)$, we claim that the square roots $(b_0^2 - 4b_4 - 8)^{1/2}, (b_0^2 - 4b_4 + 8)^{1/2}$ are also modular functions, by passing to a smaller modular subgroup $\Gamma < \Gamma_0(2)$. Indeed, from the formulae in [Mai09], we see that

$$(b_0^2 - 4b_4 - 8) = t_4 \tag{3.35}$$

for a Hauptmodul $t_4$ for $\Gamma_0(4)$. Up to a $SL_2(\mathbb{Z})$ transform on $\tau$, it is given by

$$t_4(\tau) = 2^8 \frac{\eta^8(4\tau)}{\eta^8(\tau)}. \tag{3.36}$$

Hence $(b_0^2 - 4b_4 - 8)^{1/2}$ is a modular form for $\Gamma_0(4)$ with a quadratic multiplier system. By passing to $\Gamma_0(8)$ it turns out to be a modular function with trivial multiplier system. We also have $(b_0^2 - 4b_4 + 8) = (t_8 + 4)^2$ for a certain Hauptmodul $t_8$ for the modular group $\Gamma_0(8)$.

Therefore by passing to the smaller modular subgroup $\Gamma_0(8)$, the roots do not create trouble in discussing modularity. Furthermore, by making use of a $\theta$-uniformization similar to the $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ case, we see that the open GW point (2.22) is a $4$-torsion point.

One can obtain interesting non-trivial one-parameter families by restricting to one-dimensional subspaces in the $(b_0, b_4)$-space with non-constant $b_0^2 - 4b_4$ which determines the complex structure through the $j$-invariant above. For example, by restricting to $b_4 = 0$, we see that we get one-parameter family parametrized by $b_0$ such that $b_0^4$ is the Hauptmodul $t_2 + 64$ for $\Gamma_0(2)$. This corresponds to the one-parameter family

$$(q_1, q_2) = (0, s), \quad s = (t_2 + 64)^{-1}. \tag{3.37}$$

According to the discussions in Section 3.4, after the restriction both $b_0, b_4$ become modular functions for a certain modular group $\Gamma$ depending on the subfamily. By passing to the intersection with $\Gamma_0(8)$ which incorporates the multiplier system for $\kappa$ in the uniformization and the issue on roots of modular forms above, we see that the values of $x$ and hence of $y = -h(x)$ at any ramification point are modular functions.

### 3.5.3 $K_{\mathbb{P}^1 \times \mathbb{P}^1}$

Then affine part of the mirror curve given in Example 2.2 is equivalent to

$$y^2 + (1 + x + q_1 x^2) y + q_2 x^2 = 0. \tag{3.38}$$

We follow the algorithm in [Con96] to reduce it to the Weierstrass normal form. This is accomplished by the following sequence of change of coordinates which induce bi-regular maps on the curves. First we set

$$\alpha = 2^3 \kappa^2 X + \frac{1}{12} (-1 - 2q_1 + 4q_2), \quad \beta = \kappa^3 Y - \frac{1}{2} (\alpha + \frac{1}{2} q_1), \tag{3.39}$$

22
where again $\kappa$ is some constant arising from homothety. Then we make a change of coordinates
\[
x = \beta^{-1} \left( 2^2 \kappa^2 X + \frac{1}{6} (1 + 2q_1 - 4q_2) \right), \quad y = -\frac{1}{2} + x(ax - \frac{1}{2}) - \frac{1}{2} (1 + x + q_1 x^2). \tag{3.40}
\]
Then the equation for the curve becomes the Weierstrass normal form
\[
Y^2 = 4X^3 - aX - b \tag{3.41}
\]
with
\[
a = 2^{-\frac{1}{3}} 24^{-\frac{1}{6}} \kappa^{-4} (16q_1^2 - 16q_1q_2 + 16q_2^2 - 8q_1 - 8q_2 + 1), \quad b = 864^{-\frac{1}{6}} \kappa^{-6} (4q_1 + 4q_2 - 1)(16q_1^2 - 40q_1q_2 + 16q_2^2 - 8q_1 - 8q_2 + 1). \tag{3.42}
\]
The $j$-invariant is
\[
j = \frac{(16q_1^2 - 16q_1q_2 + 16q_2^2 - 8q_1 - 8q_2 + 1)^3}{q_1^2 q_2^2 (16q_1^2 - 32q_1q_2 + 16q_2^2 - 8q_1 - 8q_2 + 1)}. \tag{3.43}
\]
From this it is easy to see that the parameters $q_1, q_2$ determine the complex structure of the curve through
\[
s = 16q_1^2 + q_2^2 - 8q_1q_2 + \frac{1}{q_1q_2}, \quad j = \frac{(s - 16)^3}{s - 32}. \tag{3.44}
\]
We recognize that $s$ is a Hauptmodul for $\Gamma_0(2)$. Similar discussions in Section 3.5.2 on the shift $\epsilon$ and on values of $x, y$ at ramification points apply.

One can obtain one-parameter subfamilies by restrictions to one-dimensional spaces with non-constant $j$.

- Taking $q_1 = q_2 = s$, we have
  \[
j(s) = \frac{(1 - 16s + 16s^2)^3}{s^4(1 - 16s)}. \tag{3.45}\]
  We recognize that $s$ is the Hauptmodul $-1/t_4$ for $\Gamma_0(4)$, see e.g. [Mai09] for details.

  One can then solve for $\kappa$ to be
  \[
  \kappa = 2^{-\frac{5}{8}} \pi^{-1} \theta_2^{-2}(2\tau). \tag{3.46}\]
  A similar computation as in the previous cases by using $\theta$-uniformization shows that the open GW point (2.22) is an 8-torsion.

- Taking $q_1 = \frac{1}{4}, q_2 = s$ or $q_1 = s, q_2 = \frac{1}{4}$ leads to
  \[
j(s) = 64 \frac{(-3 + 4s)^3}{-1 + s}. \tag{3.47}\]
  We recognize that $s$ is the Hauptmodul $1 + t_2/64$ for $\Gamma_0(2)$. One can solve for $\kappa$ to be
  \[
  \kappa = \zeta_4 2^{-\frac{5}{8}} \pi^{-1} \theta_{D_4}^{-1}(2\tau) \theta_{D_4}^{-4}(2\tau). \tag{3.48}\]
Similar to earlier discussions, the subtlety on the multiplier systems arising from taking roots of modular forms can be resolved by passing to a smaller modular subgroup if needed. Unlike the previous cases, it is not completely trivial to derive an explicit \( \theta \)-uniformization and to determine the \( u \)-coordinate of the open GW point in these cases.

According to the discussions in Section 3.4, after the restriction, both \( q_1, q_2 \) become modular functions for a certain modular group \( \Gamma \) depending on the subfamily in Lemma 3.8. Similar to Section 3.5.2, by passing to a smaller modular subgroup of \( \Gamma \) if needed, we see that the values of \( x \) and hence \( y = -h(x) \) at any ramification point are modular functions.

### 3.5.4 \( K_{F_1} \)

The affine part of the mirror curve given in Example 2.4 is equivalent to

\[
y^2 + (1 + x)y + q_2x^2 + q_1x^3 = 0. \tag{3.49}
\]

For this hyperelliptic structure, we apply the linear transformation

\[
x = (-q_1)^{-1}4^13_1^1k^2X + \frac{1 - 4q_2}{12q_1}, \quad y = k^3Y - \frac{1 + x}{2}, \tag{3.50}
\]

where \( k \) is an undetermined constant arising from homothety. Then the equation (3.49) is transformed to the Weierstrass normal form

\[
y^2 = 4X^3 - aX - b \tag{3.51}
\]

with

\[
a = \frac{(1 - 4q_2)^2 + 24q_1}{24 \cdot 2^13_1^3}, \quad b = \frac{(1 - 4q_2)^3 + 36(1 - 4q_2)q_1 + 216q_1^2}{-864k^6q_1^2}. \tag{3.52}
\]

The \( j \)-invariant is given by

\[
j = \frac{(1 - 8q_2 + 16q_1^2 + 24q_1)^3}{q_1^2(q_2 - 8q_1^2 + 16q_1^3 - q_2 + 36q_1q_2 - 27q_1^2)}. \tag{3.53}
\]

We can obtain interesting subfamilies by restricting the above two-parameter family to one-dimensional ones.

- Taking \( q_2 = 0, q_1 = s \), we obtain

\[
j = \frac{(1 + 24s)^3}{s^3(1 + 27s)}. \tag{3.54}
\]

We recognize that \( s \) is a Hauptmodul for \( \Gamma_0(3) \), see e.g. [Mai09] for details. This is consistent with the observation that setting \( q_1 = 0 \) in (3.49) reduces the mirror curve of \( K_{F_1} \) to the mirror curve of \( K_{P_2} \). In particular, the open GW point (2.22) is a 3-torsion.
Taking \( q_2 = 1/4, q_1 = s \), we obtain

\[
j = \frac{8 \cdot 1728}{8 - 27s}.
\]

(3.55)

In particular \( s \) is a Hauptmodul for \( \Gamma(1) = SL_2(\mathbb{Z}) \). One can solve for \( \kappa \) to be

\[
\kappa = \zeta_4 3^{\frac{1}{4}} 2^{-\frac{5}{8}} \pi^{-\frac{1}{2}} E_6^{-\frac{1}{6}}.
\]

(3.56)

Similar to earlier discussions, the subtlety on the multiplier systems arising from taking roots of modular forms can be resolved by passing to a smaller modular subgroup if needed. Determining the \( u \)-coordinate of the open GW point (2.22) reduces to computing the zero of \( \wp \), which is an interesting question on its own [EZ82] (see also [DI08]).

Taking \( q_1 = q_2 = s \), we obtain

\[
j = \frac{(16s^2 + 16s + 1)^3}{s^4(16s + 1)}.
\]

(3.57)

We recognize that \( s \) is the Hauptmodul \( 1/t_4 \) for \( \Gamma_0(4) \). One can solve for \( \kappa \) to be

\[
\kappa = \zeta_6^5 \pi^{-\frac{1}{2}} \theta_2^{-\frac{3}{2}} (2\tau) \theta_4^{-\frac{3}{2}} (2\tau).
\]

(3.58)

Similar to earlier discussions, the subtlety on the multiplier systems arising from taking roots of modular forms can be resolved by passing to a smaller modular subgroup if needed. Deriving the \( u \)-coordinate for the open GW point is more complicated in this case.

**Remark 3.10.** Another hyperelliptic structure is

\[
y^2 + (1 + x + q_1 x^2) y + q_2 x^3 = 0.
\]

(3.59)

The underlying algebraic curves are bi-regular, with the bi-regular map easily identified from the relations to the toric characters in (2.2). The \( \wp \)-uniformization is again derived from the algorithm in [Con96]. The details are as follows. We first make the change of variables

\[
\alpha = 2^\frac{5}{2} \kappa^2 X - \frac{1}{12} (2q_1 + 1), \quad \beta = \kappa^3 Y - \frac{1}{2} (\alpha + \frac{1}{2} q_1 - q_2).
\]

(3.60)

Then we set

\[
x = \beta^{-1} \left( \alpha + \frac{1}{2} q_1 \right), \quad y = -\frac{1}{2} + x(ax - \frac{1}{2}) + \frac{1}{2} (1 + x + q_1 x^2).
\]

(3.61)

The Weierstrass normal form is the same as the one for the first hyperelliptic structure as it should be. The different hyperelliptic structures have different ramification data and open GW points. One can consider the special one-parameter sub-families as above. The discussion in Section 3.5.2 on the values of \( x, y \) at the ramification points also applies here.
Remark 3.11. Invoking the correspondence between the linear relations in the homogeneous quotient construction of toric variety and the Mori cone of curves in the toric variety, we see that the above specializations correspond to different walls in the second fan, which models the moduli space of Kähler structures of the A-model. Hence topological recursion, when combined with the modularity studied in this work, provides a promising tool in studying the phase transition and wall crossing phenomena, along the lines in e.g. [Wit93, CKYZ99, ALM10]. We hope to return to this in a future work.

4 Proof of main theorems

In this section we prove the main theorems for the examples $\mathcal{X} = K_S$ for $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{W}P[1,1,2], \mathbb{F}_1$. We will start from a general discussion on the modularity of the differentials $\{\omega_{g,n}\}_{g,n}$ produced from applying topological recursion to a genus one mirror curve $C$ whose affine part is given by the equation (2.1) with hyperelliptic structure given by $x$. The proof of modularity is mainly based on the results in Lemma 3.6, Lemma 3.7, and Lemma 3.8 which reveal some arithmetic properties of the ramification points.

We shall only focus on one-parameter subfamilies. However, many of the results for the one-parameter subfamilies, such as the structure for the ring in Theorem 4.4 and the holomorphic anomaly equations in Theorem 4.8 can be easily generalized to topological recursion for the full multi-parameter families. The only difference is the lack of a better understanding on the moduli space interpretation of the rest of the parameters (other than the complex structure modulus) from the view point of the mirror curve.

4.1 Expansions of basic ingredients in topological recursion

4.1.1 Local coordinates for expansions

We use $[x_1, x_2, x_3]$ to denote a point on the (compactified) mirror curve $C$, which are the first three homogeneous coordinates of $\mathbb{P}_\Delta = \mathbb{P}^{p+2}$ in (2.2) – namely $x = x_1/x_3$ and $y = x_2/x_3$. For a generic mirror curve, the set $R^\circ$ of finite (i.e., in the $x_3 = 1$ patch) ramification points is a subset of the affine mirror curve $C^\circ$.

In Section 3.5, we have made the choice of origin for the group law for the mirror curve. For $\mathcal{X} = K_{\mathbb{P}^2}, \mathbb{F}_1$ the shift $\epsilon$ in uniformization formula has chosen to be zero. Accordingly, we have $R^\circ = \{u = \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \}$. For the other two cases $\mathcal{X} = K_{\mathbb{P}_1 \times \mathbb{P}_1}, K_{\mathbb{W}P[1,1,2]}$, we have $R^\circ = \{u = 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \}$. According to Part 2 of Lemma 3.7, the hyperelliptic involution $*$ on the mirror curve is induced by the involution $u \mapsto -u$ on the Jacobian. We also use $*$ to denote the induced actions on functions and differentials.

We need the notion of local uniformizer for the calculus on the mirror curve $C$. In what follows, we always use the local uniformizer near a point $p$ corresponding to $u(p)$ under uniformization. We shall also identify a point $p \in C$ with its $u$-coordinate which is defined modulo translation by elements in the lattice.

---

6 Only the affine part of the curve is relevant in topological recursion.

7 The should not be confused with the Kähler parameter discussed earlier.
4.1.2 The log-differential and Bergmann/Schiffer kernel

The basic ingredients in Eynard-Orantin topological recursion are the log-differential \( \lambda = \log y \cdot dx/x \) and the Bergman kernel \( B \). The differential \( \lambda \) depends on the choice of the local coordinates \( x, y \) as displayed in (2.1).

Instead of the Bergmann kernel \( B \) in [EO07] (which produces differentials \( \{ \omega_{g,n} \}_{g,n} \)) we usually work with the Schiffer kernel \( S \) (which produces differentials \( \{ \hat{\omega}_{g,n} \}_{g,n} \)). The Schiffer kernel is independent of the Torelli marking, as defined in Section 2.2.

In the genus one case, the Schiffer kernel is given by

\[
S(u_1, u_2) = (\wp(u_1 - u_2) + \hat{\eta}_1) du_1 \boxtimes du_2, \quad \hat{\eta}_1 = 2\zeta(2)E_2 = \frac{\pi^2}{3} (E_2 + \frac{-3}{\pi \text{Im} \tau}).
\]  

(4.2)

Here although the quantity \( \tau \) depends on the Torelli marking, the Schiffer kernel \( S \) does not. An advantage, besides being modular, is that it keeps track of part of the combinatorics in topological recursion through the non-holomorphic dependence in \( \tau \). This will be used later in the discussion of holomorphic anomaly equations in Section 4.3.

Through this work, we are only interested in the coefficient part of the differential \( \omega_{g,n} \) with respect to the trivialization \( du_1 \boxtimes du_2 \cdots \boxtimes du_n \), constructed from topological recursion. By abuse of terminology, we say \( \omega_{g,n} \) has modular properties (like being Jacobi forms) if its coefficient has so. Hence the Schiffer kernel \( S \) is regarded as an almost-meromorphic Jacobi form according to Definition 3.5. Similarly, the Bergmann kernel \( B \) is quasi-meromorphic Jacobi form.

4.1.3 Modularity of Taylor coefficients of Jacobi forms at torsion points

The following result proves to be useful in discussing modularity of Taylor coefficients of meromorphic Jacobi forms [EZ84]. Suppose \( \Phi \) is a meromorphic Jacobi form of weight \( m \), then its \( k \)th Taylor coefficient at \( x_0 + y_0 \tau \) is a meromorphic modular form of weight \( m + k \) for the modular group consisting of matrices \( \gamma \in SL_2(\mathbb{Z}) \) such that \( \gamma(x_0 + y_0 \tau) = x_0 + y_0 \tau \mod Z \oplus \tau Z \). See [Dol97] for a nice exposition of these facts.

Consider the case \( \Phi = \wp \) which is a meromorphic Jacobi form of weight 2 with level \( SL_2(\mathbb{Z}) \). At the 2-torsion points, the modular group can be taken to be \( \Gamma(2) \). The same statement is true for the meromorphic Jacobi form \( \wp' \), and higher derivatives of \( \wp \). In the higher derivative cases, we can alternatively use the algebraic relation \( (\wp')^2 = 4\wp^3 - 2g_2 \wp - g_3 \) satisfied by \( \wp \) and \( \wp' \) in (3.18) and then apply induction. This when combined with Lemma 3.6, Lemma 3.7, Lemma 3.8 and Lemma 3.9 would imply that the differentials produced by

\[8\] The differential \( \lambda \), which involves logarithm, is derived as the dimension reduction of the Calabi-Yau form of the non-compact CY 3-fold [KY99, AV00, AKV02] and relates to mirror symmetry. Its rigorous definition uses mixed Hodge structure [Bar93, Shi97, KM10]. In the current genus one case, we understand the logarithm via the formal group of the elliptic curve [Sil09]. In the literature, sometimes another version \( \lambda = ydx \) is used. While much easier to deal with, \( ydx \) is not directly related to toric CY 3-folds by mirror symmetry.
For later use, we recall the values of $\wp$

\[
e_1 := \wp\left(\frac{1}{2}\right) = 2\zeta(2)(\theta_2^2 + \theta_3^2),
\]

\[
e_2 := \wp\left(\frac{\tau}{2}\right) = 2\zeta(2)(-\theta_2^4 - \theta_3^4),
\]

\[
e_3 := \wp\left(\frac{1+\tau}{2}\right) = 2\zeta(2)(\theta_2^4 - \theta_3^4).
\] (4.3)

See [Zag08] for the convention of the $\theta$-constants above. As explained earlier in Section 4.1.3 these are modular forms for $\Gamma(2)$ with trivial multiplier systems. We also denote

\[
\hat{e}_k := e_k + \hat{\eta}_1, \quad k = 1, 2, 3, \quad \hat{e}_0 := \hat{\eta}_1. \quad (4.4)
\]

The following Laurent expansion of $\wp$ at $u = 0$ is also useful

\[
\wp(u) = \frac{1}{u^2} + \sum_{k=1}^{\infty} (2k + 1)2\zeta_{2k+2}E_{2k+2}u^{2k},
\] (4.5)

where $\zeta_{2k+2}$ is the $\zeta$-value and $E_{2k+2}$ is the Eisenstein series of weight $2k + 2$ with normalized leading term in the Fourier expansion to be 1.

### 4.1.4 Local expansions near the ramification points

In topological recursion one needs to study residues of quantities around ramification points of $x : C \to \mathbb{P}^1$ which gets identified with the group of 2-torsion points, according to Lemma 3.7.

For later use, we now study $\lambda - \lambda^*$ around the ramification points in $R^\circ$. Note that vanishing locus of $y$ is away from $R^\circ$, hence $\log y$ is single-valued if we fix a branch of logarithm once and for all. We shall choose the principal branch which takes the value 0 when $y = 1$.

We simplify $\lambda - \lambda^*$ by making use of the results on uniformization as follows. From Lemma 3.8 we know for an one-parameter subfamily, $x, y$ are rational functions in $\wp(u + \epsilon), \wp'(u + \epsilon)$ for some shift $\epsilon$, with coefficients lying in the fractional field $\mathcal{M}(\Gamma)$ of the ring $M(\Gamma)$ modular forms for some modular group $\Gamma$ depending on the curve family $C$. Under the involution $*$ the rational function $x$ is fixed while for $y$ we have

\[
y = \frac{y + y^*}{2} + \frac{y - y^*}{2}, \quad y^* = \frac{y + y^*}{2} - \frac{y - y^*}{2}, \quad (4.6)
\]

Furthermore since $y \neq 0$ at a ramification point in $R^\circ$ where $y - y^* = 0$, we know $y + y^*$ is not vanishing at a ramification point in $R^\circ$. We then have

\[
\lambda - \lambda^* = \log y \frac{dx}{x} - \log y^* \frac{dx^*}{x^*} = \log \left(\frac{y + y^*}{2} + \frac{y - y^*}{2} \right) \frac{dx}{x}. \quad (4.7)
\]
At a finite ramification point we also have \( x \neq 0, dx = 0 \), we then define

\[
\Lambda := 2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \left( \frac{y - y^*}{y + y^*} \right)^{2k+1} \partial_u x \frac{du}{x},
\]

which is an expression for \((\lambda - \lambda^*)\) near each ramification point. The vanishing order of \( y - y^* \) at the ramification point is 1 since the curve \( C \) is smooth. According to the results on uniformization \( \Lambda \) is a meromorphic Jacobi form, its weight is 1 coming from the \( dx/x \) part: the coefficient part has weight zero.

We can further expand the above expression \((4.8)\) in terms of the local uniformizing parameter \( T = u - u_r \), where \( u_r \) is the \( u \)-coordinate of the ramification point \( r \in \mathbb{R}^\circ \). Then we have

\[
\varphi(u + \epsilon) = \varphi(T + u_r + \epsilon) .
\]

When \( u_r + \epsilon = 0 \) modulo \( \mathbb{Z} \oplus \tau \mathbb{Z} \), the Laurent expansion of \( \varphi(u + \epsilon), \varphi'(u + \epsilon) \) in the local uniformizer \( T \) follow from \((4.5)\). Otherwise we have the Taylor expansion

\[
\varphi(u + \epsilon) = \left( \sum_{k=0}^{\infty} \frac{T^k}{k!} \varphi^{(k)}(u_r + \epsilon) \right) .
\]

We can also expand the Schiffer kernel \((4.2)\) around a ramification point \( r \in \mathbb{R}^\circ \) with respect to one of its arguments. The expansion in \( T = u - u_r \) is

\[
S(u, v) = (\varphi(T + u_r - v) + \tilde{\eta}_1) dT \otimes dv = \left( \sum_{k=0}^{\infty} \frac{T^k}{k!} (\varphi^{(k)}(u_r - v) + \tilde{\eta}_1^{(k)}) \right) dT \otimes dv .
\]

One has \( \tilde{\eta}_1^{(k)} = 0 \) unless \( k = 0 \) in which case \( \tilde{\eta}_1^{(k)} = \tilde{\eta}_1 \).

### 4.2 Modular properties of \( \{ \omega_{g,n} \}_{g,n} \) and ring structure

The differentials \( \hat{\omega}_{g,L+2}^{\ast + 2 + I + 1} \) are constructed recursively in \([EO07]\) through

\[
\hat{\omega}_{g,L+1}(u_0, u_1) = \sum_{r \in \mathbb{R}^\circ} \text{Res}_{v=r} K(u_0, v) \cdot \left[ \hat{\omega}_{g-1,L+2}(v, v^*, u_1) + \sum_{s_1, s_2, I, K} \sum_{s = s_1 + s_2, I' = I' \cup K'} \hat{\omega}_{g_1, I+1}(v, u_I) \cdot \hat{\omega}_{g_2, K+1}(v^*, u_K) \right] .
\]

Here the notation \( \Sigma' \) means that the range in the sum is such that the construction is strictly recursive. We have also used the notations \( I, J, K \) to denote the sets of indices and the corresponding cardinalities. The quantity \( \hat{\omega}_g = \hat{\omega}_{g,0} \) is called genus \( g \) free energy, is defined in \([EO07]\) through

\[
\hat{\omega}_g := \frac{1}{(2 - 2g)} \sum_{r \in \mathbb{R}^\circ} \text{Res}_{v=r} (d^{-1} \lambda \cdot \hat{\omega}_{g,1}) .
\]
In the above constructions (4.12) and (4.13), the quantity $K$ is the recursion kernel [EO07] defined by

$$K(u, v) = \frac{d^{-1}S}{\lambda(v) - \lambda(v^*)} = \frac{d^{-1}S}{\lambda(v) - \lambda^*(v)}.$$  \hspace{1cm} (4.14)

where

$$d^{-1}S := \frac{1}{2} \int_{2u_r + v^*}^v S(u, \bullet).$$ \hspace{1cm} (4.15)

Again we understand the logarithm in the denominator of $K$ from the formal group point of view [Sil09] as before. This means that both (4.14) and (4.15) are expressed in terms of Laurent series in the local uniformization $T = v - u_r$ near a ramification point $u_r \in R^o$. The shift $2u_r$ in the lower bound $2u_r + v^* = 2u_r - v$ in (4.15) is needed such that $d^{-1}S$ vanishes at the ramification point $v = u_r$, i.e., $T = 0$. The quantity $d^{-1}\lambda$ in (4.13) is defined in a similar way such that $2(d^{-1}\lambda)'(v) = \lambda(v) - \lambda^*(v)$.

The differentials $\omega_{g,n}, 2g - 2 + n \leq 0$, that is $(g, n) = (0,1), (0,2), (1,0)$, are dealt with separately below. For the $(g, n) = (0,1)$ case, the differential $\omega_{0,1}$ is defined in [EO07] to be zero.

4.2.1 Disk potential

The mirror counterpart of the superpotential $W$ is a primitive [AV00, AKV02] of the differential $\lambda$, integrated along a certain chain on the curve $C$. By definition, its derivative $\partial_x W$, called the disk potential, satisfies

$$\frac{\partial W}{\partial x} = \lambda = \log y \cdot \frac{1}{x}.$$ \hspace{1cm} (4.16)

From Lemma 3.8 we arrive at the following result.

**Proposition 4.1.** The disk potential $\partial_x W$ is the logarithm of a meromorphic Jacobi form whose modular group $\Gamma$ is determined by the one-parameter subfamily of the mirror curve family $C$.

4.2.2 Annulus potential

The differential $\omega_{0,2}$ is mirror to the annulus amplitude. It is defined to be the Bergmann kernel $B$ and is the holomorphic limit of the Schiffer kernel $\omega_{0,2} := S$. It is a quasimeromorphic Jacobi form.

The quantity $d^{-1}S$ is a "formal" almost-meromorphic Jacobi form of "formal" weight 1 in the sense that its derivative (in $v$) is an almost-meromorphic Jacobi form of weight 2. The recursion kernel $K$, as the quotient of $d^{-1}S$ by the Jacobi form in (4.8) is also regarded as a "formal" almost-meromorphic Jacobi form.

**Proposition 4.2.** The annulus amplitude $\omega_{0,2} = B$ is a weight 2, index 0, level $\Gamma(1)$, quasi-meromorphic Jacobi form. It is symmetric in its arguments. The recursion kernel $K = d^{-1}\omega_{0,2}/(\lambda - \lambda^*)$ is a formal almost-meromorphic Jacobi form of formal weight 0.
4.2.3 Higher genus modularity

We will use topological recursion to prove the modularity of \( \{ \hat{\omega}_{g,n} \}_{g,n} \) for higher \((g, n)\).

Genus one closed case

The quantity \( \hat{\omega}_{1,0} = \hat{F}_1 \), called genus one free energy, involves the Bergmann \( \tau \)-function \( \tau_\mathcal{B} \) \cite{EO07}. In the current genus one case, the Bergman \( \tau \)-function, as an analytic invariant, is given by \cite{KK,KK04a,KK04b}

\[
\tau_\mathcal{B} = \eta^2(\tau). \tag{4.17}
\]

The genus one free energy \( \hat{F}_1 \) is then defined to be

\[
\hat{F}_1 = -\frac{1}{2} \ln \tau_\mathcal{B} - \frac{1}{24} \ln \prod_{r \in \mathcal{R}} \left| \frac{dy}{d(x - x(r))} \right|^2 |_r \ln \det Y, \quad Y = -\pi / \text{Im} \tau. \tag{4.18}
\]

The second term can be computed to be the logarithm of a modular function using Lemma 3.6, Lemma 3.7, Lemma 3.8 and Lemma 3.9. Taking the holomorphic limit (setting \( \text{Im} \tau \to \infty \)), we define \( dF_1 := \lim_{\text{Im} \tau \to \infty} d\hat{F}_1 \). It is shown that in [FLZ16, Theorem 7.9] that \( dF_1^\mathcal{X} = dF_1 \). We therefore arrive at the following result.

**Theorem 4.3.** Up to addition by a constant, the genus one closed GW potential \( F_1^\mathcal{X}(\tau) \) is the logarithm of a meromorphic modular form whose modular group \( \Gamma \) is determined by the one-parameter subfamily of the mirror curve family \( \mathcal{C} \).

Higher genera

Note that in higher genus recursion for \( \{ \hat{\omega}_{g,n} \}_{g,n} \), the disk potential \( W \) and genus one free energy \( \hat{F}_1 \) do not enter, hence no logarithms of almost-meromorphic Jacobi forms will appear.

We define the total weight of the coefficient of \( \omega_{g,n}(u_1, \ldots, u_n) \) with respect to the trivialization \( du_1 \otimes \cdots \otimes du_n \) to be the integer \( k \) in Definition 3.4 under the transformation

\[
(u_1, u_2, \ldots, u_n, \tau) \mapsto \left( \frac{u_1}{c \tau + d}, \frac{u_2}{c \tau + d}, \ldots, \frac{u_n}{c \tau + d}, \frac{a \tau + b}{c \tau + d} \right). \tag{4.19}
\]

We also consider the corresponding weight with respect to the argument \( u_k \), with the other arguments among \( (u_1, u_2, \ldots, u_n) \) fixed.

**Theorem 4.4.** The following statements hold for \( \hat{\omega}_{g,n} \) with \( 2g - 2 + n > 0 \).

1. The differential \( \omega_{g,n}(u_1, \ldots, u_n), n \neq 0 \) is symmetric in its arguments. In each argument, it only has poles at the ramification points in \( \mathbb{R}^n \). At any of the ramification point, the order of pole in any argument is at most \( 6g + 2n - 4 \). Furthermore, the sum of orders of poles over all arguments in each term in \( \hat{\omega}_{g,n}(u_1, \ldots, u_n) \) is at most \( 6g + 4n - 6 \).

2. The differential \( \omega_{g,n}(u_1, \ldots, u_n), n \neq 0 \) is a differential polynomial in \( S(u_k - u_r), k = 1, 2, \ldots, n, r \in \mathbb{R} \). The coefficients of \( \hat{\omega}_{g,n} \) regarded as a differential polynomial in \( S(u_k - u_r), k = 1, 2, \ldots, n, r \in \mathbb{R} \) are elements in the ring

\[
\mathcal{K} := \mathcal{M}(\Gamma(2) \cap \Gamma) \otimes \mathbb{C}[\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{\eta}_1]. \tag{4.20}
\]
In particular, $\hat{\omega}_{g,n}(u_1, \ldots, u_n), n \neq 0$ is an almost-meromorphic Jacobi form with level $\Gamma(2) \cap \Gamma$, where the modular group $\Gamma$ is determined by the one-parameter subfamily of the mirror curve family $C$. Its total weight is $n$.

3. The quantity $\hat{F}, g \geq 2$ is an almost-meromorphic modular form of weight zero, lying in the ring $\hat{K}$ given in (4.20).

Proof. 1. The proofs of the first two statements follow by induction basing on the recursion formula (4.12), as in [EO07].

For the third statement, denote by $N_{g, I}$ the maximum of the order of pole among all arguments and all ramification points in $\hat{\omega}_{g, I+1}$ for any $g, I$ not necessarily satisfying the condition $2g - 2 + (I + 1) > 0$. By induction it is easy to show that

$$N_{g, I} + 2 \leq \max_{g_1, g_2, I, K} \{(N_{g_1, I} + 2) + (N_{g_2, K} + 2)\},$$

where the maximum is taken over all possible partitions of $g$ and $I$. Direct computations for the first few $(g, n)$’s show that $N_{0, 1} = 0, N_{0, 2} = 2, N_{1, 0} = 4$. The estimate (4.21) and the initial values imply that $N_{g, I} \leq 6g + 2I - 2$ when $2g - 2 + (I + 1) > 0$.

For the last statement, denote similarly by $\tilde{N}_{g, I}$ the maximum of the sum of orders of pole over all arguments in $\hat{\omega}_{g, I+1}$, for any $g, I$ not necessarily satisfying the condition $2g - 2 + (I + 1) > 0$. Again by induction we see that

$$\tilde{N}_{g, I} + 2 \leq \max_{g_1, g_2, I, K} \{(\tilde{N}_{g_1, I} + 2) + (\tilde{N}_{g_2, K} + 2)\},$$

Direct computation shows that $\tilde{N}_{0, 1} = 2, \tilde{N}_{0, 2} = 6, \tilde{N}_{1, 0} = 4$. The estimate (4.22) and the initial values imply that $\tilde{N}_{g, I} \leq 6g + 4I - 2$ when $2g - 2 + (I + 1) > 0$.

2. We again prove by induction. Near the ramification point $u_r$, we choose the local parameter $T = v - u_r$ in order to evaluate the residues.

We first consider the genus zero case. The initial few cases can be computed directly for which the statement holds. Assume the statement is true for $\omega_{0, n}$ with $n \leq |I|$. For $\omega_{0, I+1}$, we divide the terms in the recursive construction (4.12) of $\omega_{0, I+1}$ into two cases: those with $|J|, |K| > 1$, and those with one of them equal to 1. For the first case, from the recursion, the $v$-dependent terms in the term

$$\omega_{0, I+1}(v, u_I)\omega_{0, K+1}(v^*, u_K)$$

with $|J|, |K| > 1$ (and hence $|I| > 3$), are differential polynomials in $S(T + \delta_i)$ where $\delta_i \in R^0 \cup \{0\}$, with coefficients lying in $\hat{K}$. Pick any term among all possible ramification points and all partitions in the sum for the recursion. From (4.5) and (4.11) we see that $\omega_{0, I+1}(v, u_I)\omega_{0, K+1}(v^*, u_K)$ is an element in

$$\hat{K}[E_{2k+1}, k \geq 1, S^{(m \geq 0)}(\delta), \delta \neq 0](T) \otimes C[S^{(m \geq 0)}(u_i - u_r), i \in I = J \cup K].$$

We introduce the notation $[\cdot]_n$ for the degree $n$ Laurent coefficient at the corresponding point. We also denote the $m$th derivative by the superscript $(m)$. Then the ring above is

$$\hat{K}[S]_{m \in \mathbb{Z}}(\delta), \delta \in R^0 \cup \{0\} ((T)) \otimes C[S^{(m \geq 0)}(u_i - u_r), i \in I = J \cup K].$$
For the second case where one of the cardinalities $|J|, |K|$, say $|J|$, is 1, the ring is changed to
\[ \hat{K} \left[ [S]_{m \in \mathbb{Z}}(\delta), \delta \in R^\circ \cup \{0\}, S^{(m \geq 0)}(u_r - u_I) \right] (\mathbb{T}) \otimes \mathbb{C}[S^{(m \geq 0)}(u_k - u_r), k \in K]. \] (4.25)

We also have from (4.15) that
\[ d^{-1} S \in \mathbb{C} \left[ S^{(m \geq 0)}(u_r - u_0) \right] [T]. \] (4.26)

Applying chain rule to (4.8), we obtain
\[ \Lambda = \sum_{m \geq 2} [\Lambda]_m T^m \]
\[ \in \mathbb{C} \left[ \frac{1}{x} |_{u_r} \frac{1}{y + y^*} |_{u_r}, x^{(m \geq 1)} |_{u_r}, (y - y^*)^{(m \geq 1)} |_{u_r}, (y + y^*)^{(m \geq 0)} |_{u_r} \right] T^2 [T]. \] (4.27)

Lemma 3.8 for uniformization shows that (recall the expression for $y^*$ from (3.3)),
\[ x, y, y^* = -y - 2h(x) \in \mathcal{M}(\Gamma) \otimes \mathbb{C}(\varphi(u + \epsilon), \varphi'(u + \epsilon)). \] (4.28)

Lemma 3.9 shows that $x|_{u_r}, y|_{u_r}$ and hence
\[ \varphi(u_r + \epsilon), \varphi'(u_r + \epsilon) \in \mathcal{M}(\Gamma(2) \cap \Gamma), \] (4.29)

as the map from $(x, y)$ to $(\varphi(u + \epsilon), \varphi'(u + \epsilon))$ is a bi-regular map with coefficients being elements in $\mathcal{M}(\Gamma)$ from uniformization. From the algebraic relation (3.18) between $\varphi, \varphi'$, we see that
\[ \varphi^{(m \geq 0)}(u_r + \epsilon) \in \mathcal{M}(\Gamma(2) \cap \Gamma). \] (4.30)

Combing the above results we obtain $\Lambda \in \mathcal{M}(\Gamma(2) \cap \Gamma) T^2 [T]$ and hence
\[ \frac{1}{\Lambda} \in \mathcal{M}(\Gamma(2) \cap \Gamma) T^{-2} [T]. \] (4.31)

Due to the order of pole behavior in Part 1, all of the formal Laurent and power series above can be replaced by their finite truncations depending on $g, n$. Multiplying the expansions of the above ingredients and collecting the degree $-1$ coefficients, we see that $\omega_{0, t+1}$ is a differential polynomial in $S(u_i - u_r), i \in I \cup \{0\}, r \in R^\circ$, and the coefficients are elements in the ring
\[ \hat{K} \left[ [S]_{m \in \mathbb{Z}}(\delta), \delta \in R^\circ \cup \{0\} \right] \otimes \mathcal{M}(\Gamma(2) \cap \Gamma). \] (4.32)

The results in Section 4.1.3 tells that $[S]_{m \in \mathbb{Z}, \delta}(\delta), \delta \in R^\circ \cup \{0\}$ are weight-two holomorphic modular forms for $\Gamma(2)$ with trivial multiplier systems, while we have
\[ \{ [S]_0(\delta), \delta \in R^\circ \cup \{0\} \} = \{ \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4 \}. \] (4.33)

Since $\mathcal{M}(\Gamma) \otimes M(\Gamma(2)) \subseteq \mathcal{M}(\Gamma(2) \cap \Gamma)$, the statement on the ring then follows.
The higher genus differentials are constructed from the genus zero ones. Since all ingredients are differential polynomials with coefficients in the ring \( \hat{\mathcal{C}} \), the conclusion follows automatically.

Observe that taking the \( u \)-derivative of an almost-meromorphic Jacobi form of index 0 increases the weight by one. As long as its Laurent coefficients are concerned, the recursion kernel \( K \) can be regarded as an almost-meromorphic Jacobi form of weight 0. By tracing the degrees in the recursion formula \( (4.12) \), and the weight 2 of \( \hat{\omega}_{0,2} \) computed before, we then immediately see the total weight of \( \hat{\omega}_{g,n} \) as an almost-meromorphic Jacobi form is \( n \).

3. This follows from the proof of Part 2 and the definition of \( \hat{F}_g \) in \( (4.13) \).

According to the proof of Remodeling Conjecture \cite{BKMnP09, FLZ16}, the GW potentials \( d_{X_1} \cdots d_{X_n} \hat{F}_{g,n} \) and \( F_g \) for the toric CY 3-fold \( \mathcal{X} \) coincide with the differentials \( \omega_{g,n} \) and \( F_g \) produced by topological recursion for the mirror curve, using the Bergmann kernel \( B \). Observe that the non-holomorphic dependences in \( \tau \) of \( \hat{\omega}_{g,n}, \hat{F}_g, 2g - 2 + n > 0 \) are polynomial in \( 1/\text{Im}\tau \). Taking the holomorphic limit, we arrive at the following easy consequence of Theorem 4.4.

**Theorem 4.5.** Consider the local toric Calabi-Yau 3-folds \( \mathcal{X} = K_S, S = \mathbb{P}^2, WP[1, 1, 2], \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \). Consider non-trivial one-parameter subfamilies of the mirror curves with hyperelliptic structure determined by the corresponding brane. The following statements hold.

The GW potentials \( d_{X_1} \cdots d_{X_n} \hat{F}_{g,n} = \omega_{g,n}, 2g - 2 + n > 0, n > 0 \), as the holomorphic limits of the differentials \( \hat{\omega}_{g,n}(u_1, \ldots, u_n) \) which are almost-meromorphic Jacobi forms, are quasi-meromorphic Jacobi forms. The structure as quasi-meromorphic Jacobi forms is as exhibited in Theorem 4.4 with the Schiffer kernel \( S \) replaced by the Bergmann kernel \( B \).

The GW potentials \( F_g = \omega_{g,0}, g \geq 2 \), as the holomorphic limits of the differentials \( \hat{\omega}_{g,0} \) which are almost-meromorphic modular forms, are meromorphic quasi-modular forms lying in the ring

\[
\mathcal{K} := \mathcal{M}(\Gamma(2) \cap \Gamma) \otimes \mathbb{C}[e_1, e_2, e_3][\eta_1].
\]

(4.34)

Recall that in all our cases, the open GW point \( s_0 \) in \( (2.22) \) given by \( [x, y, 1] = [0, -1, 1] \) exists on the mirror curve \( C \) independent of the generic complex parameters \( (q_1, \cdots) \). The expansion of \( F_{g,n} \) in terms of \( X \) enumerates open GW invariants \( \{n_{g,d,\mu}\}_{d,\mu} \)

\[
F_{g,n} = \sum_{\mu \geq 1} X^\mu \sum_{d \geq 0} n_{g,d,\mu} Q^d,
\]

(4.35)

where \( \mu = (\mu_1, \cdots, \mu_n), X^\mu := X_1^{\mu_1} \cdots X_n^{\mu_n} \). See \( (2.12) \) for the more detailed expression of this. In our examples, after restriction to an one-parameter subfamily, we have \cite{AKV02, FLT13} (for the \( K_{g^2} \) case there is no \( q_2^2 \) term)

\[
X_k = x_k \cdot c_3 Q^c q_1^{\ell_1} q_2^{\ell_2},
\]

(4.36)

for some \( c, c_1, c_2 \in \mathbb{Q}, c_3 \in \mathbb{C} \). Rewrite the generating series \( (4.35) \) as

\[
F_{g,n} = \sum_{\mu \geq 1} (Q^{-c} X)^\mu \sum_{d \geq 0} n_{g,d,\mu} Q^d c \sum k \mu_k.
\]

(4.37)
The ring structure \[ (4.34) \] in Theorem \[ 4.5 \] above exhibits nice structure of the Taylor coefficients in this expansion.

**Corollary 4.6.** With the same assumptions as Theorem \[ 4.3 \] above. The degree-\( \mu \) Taylor coefficients \[ \sum_{d \geq 0} n_{g,d,\mu} Q^{d+c} \sum \mu_k \] in the expansion \[ (4.37) \] of \( F_{g,n} \) are meromorphic quasi-modular forms in the ring \( K \) in \[ (4.34) \].

**Proof.** Part 1 of Theorem \[ 4.4 \] tells that generically the differential \( \omega_{g,n} \) does not have singularity at the open GW point \( (2.22) \), which avoids the ramification points. Hence developing Taylor expansion makes sense and we have, recall that \( \mu_k \geq 1 \),

\[
\sum_d n_{g,d,\mu} Q^{d+c} \sum \mu_k = \prod_{k=1}^{\infty} \mu_k^{n-1} \partial (Q^{-\epsilon} X_k) \mu_k^{n} \Big|_{x=0} F_{g,n} = \prod_{k=1}^{n} \partial (Q^{-\epsilon} X_k) \mu_k^{n} \Big|_{x=0} F_{g,n}.
\]

Theorem \[ 4.5 \] shows that

\[
\frac{\partial \omega_{g,n}}{\partial u_1 \cdots \partial u_n} \in K[\varphi^{(m \geq 0)}(u_k - u_r), k \in \{1,2,3 \ldots n\}, r \in R^\circ].
\]

By using the algebraic relation \( (3.18) \) between \( \varphi \) and \( \varphi' \), the above ring can be reduced to

\[
K[\varphi(u_k - u_r), \varphi'(u_k - u_r), k \in \{1,2,3 \ldots n\}, r \in R^\circ].
\]

The chain rule says

\[
\partial_x (Q^{-\epsilon} X) = \partial_x (Q^{-\epsilon} X) \cdot \partial_u x, \quad \frac{\partial}{\partial (Q^{-\epsilon} X)} = \frac{1}{\partial_x (Q^{-\epsilon} X)} \cdot \frac{1}{\partial_u x} \cdot \partial u.
\]

From \( (4.36) \) we obtain

\[
\partial_{x_k} (Q^{-\epsilon} X_k) = c_3 q_1^{c_1} q_2^{c_2}.
\]

According to the discussion in Section \[ 3.4 \] and Section \[ 3.5 \], after the restriction to an one-parameter subfamily, both \( q_1, q_2 \) become modular functions for a certain modular group \( \Gamma \) depending on the subfamily. Hence so is \( \partial_{x_k} (Q^{-\epsilon} X_k) \), where the subtlety of taking roots of modular functions can be addressed similarly as in Section \[ 3.5 \]. The same argument in establishing \( (4.31) \) in the proof of Part 2 of Theorem \[ 4.4 \] shows that

\[
\varphi(u + \epsilon)|_{u=u_0}, \quad \varphi'(u + \epsilon)|_{u=u_0}, \quad \partial_{u}^{m \geq 0} x|_{u=u_0} \in \mathcal{M}(\Gamma).
\]

This implies that the values at the open GW point of terms arising from differentials of the term \( \partial_{x_k} x_k \) also lie in \( \mathcal{M}(\Gamma) \).

To prove the desired statement, it remains to show

\[
\varphi(u_k - u_r)|_{u=u_0}, \quad \varphi'(u_k - u_r)|_{u=u_0} \in \mathcal{M}(\Gamma(2) \cap \Gamma).
\]
This is automatically true for those cases in which \( u_{s_0} \) is identified with a torsion point according to the discussion in Section 4.1.3. In general, we use (4.29), (4.43) and the additional formula for \( \wp \) which tells that

\[
\wp(u_0 - u_r) = \frac{1}{4} \left( \wp'(u_0 + \epsilon) + \wp'(u_r + \epsilon) \right)^2 - \wp(u_0 + \epsilon) - \wp(u_r + \epsilon). \tag{4.45}
\]

\[\square\]

**Remark 4.7.** For each \( g, n, \omega_{g,n} \) is an \( n \)-variable differential polynomials in \( \wp \). By carefully keeping track of the degrees in the generators including the derivatives of the Weierstrass-\( \wp \) functions and the meromorphic quasi-modular forms basing on the structure of the coefficient ring in (4.27), we can see that for each fixed \( n \), there are only finitely many possible terms (see e.g. Example 4.9) with numbers being coefficients. Again using the algebraic relation (3.18) between \( \wp \) and \( \wp' \), we can further reduce the number of generators since differential polynomials in \( \wp \) are polynomials in \( \wp, \wp' \). This structure tells that determining \( \omega_{g,n} \) can be reduced to a finite computation. In particular, knowing the first few terms (depending on \( g, n \)) in the expansion of \( \omega_{g,n} \), which can in principle be computed from the A-model of the mirror symmetry side, would then be enough to fix \( \omega_{g,n} \) completely.

### 4.3 Holomorphic anomaly equations

In [BCOV93] [BCOV94], it is argued from physics that the closed string free energies \( F_g \) satisfy a system of recursive equations called holomorphic anomaly equations. To be more precise, choose a set of coordinates \( s = \{ s^i \} \) on the moduli space of complex structures of the mirror CY 3-fold \( X \) of the CY 3-fold \( \mathcal{X} \). Then the equations are

\[
\hat{\partial}_k \hat{F}_g = \frac{1}{2} C_{ij}^k \left( D_i D_j \hat{F}_{g-1} + \sum'_{g_1 + g_2 = g} D_i \hat{F}_{g_1} \cdot D_j \hat{F}_{g_2} \right), \tag{4.46}
\]

where the summation \( \sum' \) means that the range is such that the equations are strictly recursive.

In the equation one has \( C_{ij}^k := e^{2K} G^{ij} C^{ij} \overline{C}_{jk} \) where \( K \) is the Kähler potential for the Weil-Petersson metric on the moduli space, \( G_{ij} \) is the metric tensor, \( C_{ijk} \) is the Yukawa coupling. And \( D_i \) is the \( t^* \)-connection [BCOV94]. All of these quantities can be determined mathematically by the theory of variation of Hodge structures for the mirror CY 3-fold \( X \).

Note that the equation is tensorial and hence does not depend on the choice of the local coordinate system \( s \) on the moduli space. It is conjectured in general that the holomorphic limit of \( \hat{F}_g \) coincides with the genus \( g \) GW potential \( F_g \) of \( \mathcal{X} \) by mirror symmetry.

It is shown in [EOM07] that \( \hat{F}_g \), \( g \geq 2 \) and \( \omega_{g,1+1} \) produced by topological recursion from any spectral curve (and in particular for our mirror curves) satisfy such equations: \( \hat{F}_g \) satisfies (4.46) while \( \omega_{g,1+1} \) satisfies a similar set of equations:

\[
\hat{\partial}_k \omega_{g,1+1} = -\frac{1}{2} C_{ij}^k \left( D_i D_j \omega_{g-1,1} + \sum'_{g_1 + g_2 = g} \sum'_{l+j+k} D_i \omega_{g_1,1} \cdot D_j \omega_{g_2,k+1} \right). \tag{4.47}
\]

\[\text{The affine mirror curve } C^0 \text{ is reduced from the mirror CY 3-fold } \hat{X}, \text{ which we did not discuss.}\]
While the mirror curve is reduced from the CY 3-fold $\tilde{X}$ and its parameters have Calabi-Yau geometry interpretation, the coefficients in the equation are defined purely in terms of data on the mirror curve.

In our examples we restrict ourselves to an one-parameter subfamily, and we use the flat coordinate $t$ as defined in Section 3.4. We also use $t$ as the label for taking derivatives. It follows from [EOM07] that

$$C_i^{tt} = \frac{1}{(2 \text{Im} \tau)^2} \left( \frac{\partial^3 F_0}{\partial t^3} \right).$$

(4.48)

While $F_0$ agrees with the A-model genus zero GW potential, here we may just regard it given purely in terms of mirror curve information by (3.20) (the ambiguity in the integration constants does not play a role here).

Now we translate the above differential equations (4.47) for the non-holomorphic (in $t$) differentials $\tilde{\omega}_{g,n}$, which are defined by using the Schiffer kernel $S$, into equations for the corresponding holomorphic differentials $\omega_{g,n}$ defined using the Bergman kernel $B$. From Theorem 4.4, we know that the $\tilde{\omega}_{g,n}$’s are polynomials of almost-meromorphic Jacobi forms and almost-meromorphic modular forms, with the only nontrivial non-holomorphic dependence in $t$ entering through the Schiffer kernel $S$ and the non-holomorphic (in $\tau$) generators $\tilde{e}_a, a = 0, 1, 2, 3$ in (4.4). Therefore, by the chain rule, the anti-holomorphic derivative on the left hand side of (4.47) is nothing but

$$\frac{\partial}{\partial \bar{t}} \sum_{g=0}^{3} \frac{\partial \tilde{e}_a}{\partial \bar{t}} + \sum_{k,r} \frac{\partial S_{kr}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial \bar{t}},$$

(4.49)

where $S_{kr} = S(u_k - u_r)$ stands for the Schiffer kernel with argument $u_k, u_r, k = 1, 2, \cdots n, r \in R$. From the explicit formulae for the generators $\tilde{e}_a$ in (4.4) and for the Schiffer kernel $S$ in (4.2), this can be simplified into

$$\frac{\partial \tilde{\eta}_1}{\partial \bar{t}} \sum_{a=0}^{3} \frac{\partial \tilde{e}_a}{\partial \bar{t}} + \frac{\partial \tilde{\eta}_1}{\partial \bar{t}} \sum_{k,r} \frac{\partial \bar{t}}{\partial \bar{t}}.$$

(4.50)

Hence (4.47) becomes

$$\left( \sum_{a=0}^{3} \frac{\partial}{\partial \bar{e}_a} + \sum_{k,r} \frac{\partial}{\partial S_{kr}} \right) \tilde{\omega}_{g,1+1} = \frac{1}{2} \frac{C_i^{tt}}{\partial \tilde{\eta}_1} \cdot \left( D_t D_t \omega_{g-1,1+1} + \sum_{g_1+g_2=g} \sum_{l=1..K} D_t \omega_{g_1,l+1} \cdot D_t \omega_{g_2,k+1} \right).$$

(4.51)

The term $C_i^{tt}$ is usually rewritten with the help of results computed from the Weil-Petersson metric on the moduli space, by introducing the so-called propagator [BCOV94] $S^{tt}$ defined to be a solution to

$$\partial_t S^{tt} = C_i^{tt}.$$

(4.52)

The flat coordinate $t$ is the Kähler normal coordinate. The derivatives $D_t$ in the above equations get simplified into ordinary derivatives due to the properties of the Kähler normal coordinate and the non-compactness of the CY 3-fold (which implies that the regular
period near the large complex structure limit is a constant function). The computations for \( S^{tt} \) in [ASYZ14] for the cases in our study yield explicit results for them in terms of almost-holomorphic modular forms (actually we can take any solution to (4.52) whose non-holomorphic dependence has no ambiguity). The structure theorem for almost-holomorphic modular forms [KZ95] tells that their nontrivial anti-holomorphic dependences are in polynomials in \( Y := -\pi / \text{Im}\tau \). For the current cases the quantities \( S^{tt} \) are in fact linear in \( Y \).

This then leads to

\[
\frac{\partial}{\partial t} S^{tt} = \frac{\partial}{\partial \tau} S^{tt} = \frac{\partial}{\partial \eta_1} S^{tt}.
\]  

(4.53)

The BCOV type holomorphic anomaly equation (4.47) for \( \tilde{\omega}_{g,n} \) is finally translated into the Yamaguchi-Yau type [YY04] functional equation

\[
\left( \frac{3}{\sigma_a \cdot \sigma_k \cdot \sigma_r} \right) \tilde{\omega}_{g,I+1} = \frac{\partial}{\partial \eta_1} S^{tt} \frac{1}{2} \left( \partial_t \partial_t \tilde{\omega}_{g-1,I+1} + \sum_{g_1 + g_2 = g} \sum_{I = J \cup K} \partial_t \partial_t \tilde{\omega}_{g_1,I+1} \cdot \partial_t \partial_t \tilde{\omega}_{g_2,K+1} \right).
\]  

(4.54)

Due to the structure for \( \tilde{\omega}_{g,n} \) in Theorem 4.4, this identity is an identity for polynomials in \( Y \) (with coefficients being holomorphic quantities). Therefore, we can take the degree zero term in \( Y \) (called the holomorphic limit). Observe that the holomorphic limit of the holomorphic derivatives of \( Y \) vanish in the holomorphic limit. This then yields a functional equation for the differentials \( \omega_{g,n} \) produced by using the Bergmann kernel \( B \) (in what follows \( B_{kr} = B(u_k - u_r) \))

\[
\left( \frac{\partial}{\partial \eta_1} + \sum_{k,r} \frac{\partial}{\partial B_{kr}} \right) \omega_{g,I+1} = \frac{\partial}{\partial \eta_1} S^{tt} \frac{1}{2} \left( \partial_t \partial_t \omega_{g-1,I+1} + \sum_{g_1 + g_2 = g} \sum_{I = J \cup K} \partial_t \omega_{g_1,I+1} \cdot \partial_t \omega_{g_2,K+1} \right).
\]  

(4.55)

Note that the other generators discussed in Theorem 4.4 are considered to be independent of \( B \). The reason is that they are so before the holomorphic limit: \( S \) includes the transcendental quantity \( Y \) while the others do not. Plainly, that \( B \) is not modular permits us to distinguish it from the rest of the generators which are all modular. This is what makes \( B \) algebraically independent of the rest. This algebraic independence is a property that one can not easily argue if one had used \( \wp(u, v) du \otimes dv \), for example, as a replacement of the Bergmann kernel.

Combing the proof of Remodeling Conjecture for toric CY’s in [BKMnP09, FLZ16], it follows then that the GW potentials satisfy the above Yamaguchi-Yau type functional equations.

We now summarize the above discussions as follows. As before, the only interesting cases are \( 2g - 2 + n > 0 \) when discussing modularity, with the rest isolated cases already easily computed.
Theorem 4.8. Consider the local toric Calabi-Yau 3-folds $X = K_S, S = \mathbb{P}^2, \mathbb{WP}[1,1,2], \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$. Consider non-trivial one-parameter subfamilies of the mirror curves with hyperelliptic structure determined by the corresponding brane. The following statements hold.

The GW potentials $\omega_{g,n}, 2g - 2 + n > 0, n > 0$ satisfy the holomorphic anomaly equations

$$
\left( \frac{\partial}{\partial \eta_1} + \sum_{k,r} \frac{\partial}{\partial B_{kr}} \right) \omega_{g,l+1} = \frac{\partial Y S^{|t|}}{\partial Y \eta_1} \cdot \frac{1}{2} \left( \partial_t \partial_t \omega_{g-1,l+1} + \sum_{g_1+g_2=g} \sum_{I=J}^{'} \partial_I \omega_{g_1,l+1} \cdot \partial_J \omega_{g_2,K+1} \right),
$$

(4.56)

where the prime $'$ in the summation on the right hand side means the range for the summation is such that the equations are strictly recursive. The quantity $S^{|t|}$ is defined to be a solution to (4.52), which can alternatively be computed from the Weil-Petersson geometry of the moduli space of complex structures of the mirror CY 3-fold $\hat{X}$.

The closed GW potentials $F_{g_0, g} \geq 2$ satisfy

$$
\frac{\partial}{\partial \eta_1} F_g = \frac{\partial Y S^{|t|}}{\partial Y \eta_1} \cdot \frac{1}{2} \left( \partial_t \partial_t F_{g-1} + \sum_{g_1+g_2=g} \partial_I F_{g_1} \cdot \partial_J F_{g_2} \right).
$$

(4.57)

Here again the prime $'$ in the summation on the right hand side means the range for the summation is such that the equations are strictly recursive, and the quantity $S^{|t|}$ is as above.

Proof. The $n > 0$ case has been proved in the above discussions, we only need to prove the statements for the $n = 0$ case. For this part, we first observe that in its definition (4.13), the non-holomorphicity of $\hat{F}_g$ in $\tau$ only comes from the $\hat{\omega}_{g,1}$ part. Theorem 4.4 for $\hat{F}_g$ tells that $\hat{F}_g$ is a polynomial of finitely many generators which are almost-holomorphic modular forms, and the only non-holomorphic generator is $\hat{\eta}_1$. This allows to translate the non-holomorphic derivative on $\hat{F}_g$ into the derivative with respect to the generator $\hat{\eta}_1$. We then apply the holomorphic anomaly equation for $\hat{F}_g$ in [EOM07], and take the degree zero term in $Y$ of both sides of the corresponding holomorphic anomaly equation. The result then follows. $\square$

Example 4.9 ($K_{P^2}$ continued). The natural parameters in the generating series of open GW invariants are the closed modulus $T$ and the open modulus $X$.

The closed modulus $T$ is the Kähler normal coordinate with respect to Weil-Petersson metric on the moduli space of Kähler structures of the CY 3-fold $K_{P^2}$, near the large volume limit. In the B-model this is the flat coordinate, defined as a period integral in Section 3.4. Explicitly it is, see [CKYZ99, AV00, AKV02, Bat93, Sti97, Hos04, KM10],

$$
T = \log(-1) + \log \frac{-q_1}{27} + \sum_{k \geq 1} \frac{(3k)!}{(k!)^3} \frac{1}{k} \left( \frac{-q_1}{3^3} \right)^k, \quad q_1 = -3^3 \frac{\eta(3\tau)^9}{\Theta_A^3(2\tau) \eta(\tau)^3}.
$$

(4.58)

Its derivative in the variable $\log(-q_1)$ is related to the $\theta$-function of the $A_2$-lattice and is a modular form for $\Gamma_0(3)$. The quantity $Q = e^T$ is related to modular variable $e^{2\pi i \tau}$ of the mirror curve by an infinite product [Moh02, Sti06, Zho14].
The open modulus $X$ is described by an integral along a carefully chosen chain in $C$. According to [AV00, AKV02], one has

$$X = \exp \left( \log x + \log(-1) + \frac{1}{3} \sum_{k \geq 1} \frac{(3k)!}{(k!)^3} \frac{1}{k} \left( \frac{-q^1}{3^3} \right)^k \right).$$

(4.59)

With this choice, the open modulus and the affine coordinate $u$ on the Jacobian is related by using the uniformization in Section 3.5.1

$$X = \left( (-4)^{\frac{1}{2}} \kappa^2 \phi + \frac{3}{4} \phi^2 \right) \cdot \exp \left( \frac{1}{3} \left( T + 2 \log(-1) - \log \frac{-q^1}{27} \right) \right).$$

(4.60)

Thanks to the identification in Section 3.5.1 that the open GW point (2.22) is a 3-torsion point and the results in Section 4.1.3, the coefficients in the expansion in $X$ of the GW potentials $\{\omega_{g,n}\}_{g,n}$ are meromorphic quasi-modular forms in $\tau$.

The ring (4.34) in Theorem 4.5 is a subring of the following

$$\mathbb{C}[e_1, e_2, e_3, \eta_1] \left[ \frac{1}{1 - 3\phi x(u_r)}, \frac{1}{x(u_r)}, \kappa, \kappa^{-1}, \phi, \phi''(u_r), \phi^{(m \geq 2)}(u_r), u_r \in \{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \} \right].$$

(4.61)

Regarded as a polynomial in $\eta_1$, the coefficient of any element in this ring is a meromorphic modular form of level $\Gamma(2) \cap \Gamma_0(9)$ as shown in Section 3.5.1. Using Theorem 4.4 and the algebraic relation (3.18) between $\wp, \wp'$, we see that $\omega_{g,n}$ lies in a ring with only finitely many generators.

In the computation of genus one free energy, using the uniformization in Section 3.5.1 it is straightforward to compute

$$\frac{dy}{d(x - x(r))^\frac{1}{2}} |_{r} = \frac{\partial_y y}{(2 - \partial_y x)^\frac{1}{2}} |_{r} = \kappa^2 \left( \frac{2\phi''(u_r)}{(-4)^\frac{1}{2}} \right)^\frac{1}{2}. $$

(4.62)

Using the results in (4.3), we obtain

$$\prod_r \wp''(u_r) = -\frac{1}{2} \Delta = -\frac{1}{2} (2\pi)^{12} \eta^{24}. $$

(4.63)

where $\Delta$ is the Dedekind $\Delta$-function and $\eta$ is the $\eta$-function. Hence we get, up to addition by constant,

$$- \frac{1}{24} \ln \prod_r \frac{dy}{d(x - x(r))^\frac{1}{2}} |_{r} = - \frac{1}{24} \ln (\kappa^6 \eta^{12}). $$

(4.64)

Combining the above formula for the Bergmann $\tau$-function, we therefore get

$$\hat{F}_1 = \frac{1}{2} \ln \tau_B - \frac{1}{12} \ln \prod_r \frac{dy}{d(x - x(r))^\frac{1}{2}} |_{r} + \frac{1}{2} \ln \det Y $$

$$= -\frac{1}{2} \log \left( \eta(\tau) \eta(3\tau) \sqrt{\text{Im} \tau} \sqrt{\text{Im} 3\tau} \right). $$

(4.65)
This agrees with the results in [ABK08, HKR08, ASYZ14] obtained by other means.

For the CY 3-fold $K_{P^2}$, the mirror curve family is an one-parameter family. Under the flat coordinate $t$, we have from [ASYZ14] (see also [Zho14]) that

$$S^{tt} = \frac{1}{2} \frac{13E_2(3\tau) + E_2(\tau)}{4} + \frac{1}{2} \frac{-3}{\pi \text{Im} \tau} = \frac{1}{2} \frac{13E_2(3\tau) + E_2(\tau)}{4} + \frac{3}{2\pi^2} \gamma$$  (4.66)

Hence in Theorem 4.8 we have

$$\frac{\partial \bar{t} S^{tt}}{\partial \bar{t} \bar{t}} = \frac{\partial \gamma S^{tt}}{\partial \gamma \bar{t}} = \frac{3}{2\pi^2}.$$  (4.67)

### A Some explicit formulae

Some explicit formulae for the disk potential, annulus potential, $\omega_{0,3}$, and $\omega_{1,1}$ for certain special one-parameter families of our four examples are collected in this appendix. The general expressions are displayed below.

- **Disk potential**
  $$\partial_x W = \log y \cdot \frac{1}{x}.$$  (A.1)

- **Annulus potential**
  $$\omega_{0,2}(u_1, u_2) = B(u_1, u_2) = (\varphi(u_1 - u_2) + \eta_1) du_1 \boxtimes du_2.$$  (A.2)

- **Recursion kernel $K = d^{-1}S/\Lambda$,**
  $$S(u_1, u_2) = (\varphi(u_1 - u_2) + \bar{\eta}_1) du_1 \boxtimes du_2,$$
  $$\Lambda = 2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \left( \frac{y - y^*}{y + y^*} \right)^{2k+1} \partial_u x^\frac{1}{x} du.$$  (A.3)

Here $d^{-1}S$ is as defined in (4.15), and the expression $y^* = -y - 2h(x)$ in (3.3) is determined from the mirror curve equation as in (3.1) and (3.2).

- **$\omega_{0,3}$**
  $$\omega_{0,3}(u_1, u_2, u_3) = \sum_{r \in \mathbb{Z}^3} \left( 2 \left[ \frac{1}{\Lambda} \right]_{-2} \prod_{k=1}^{2} (\varphi(u_k - u_r) + \eta_1) \right) du_1 \boxtimes du_2 \boxtimes du_3.$$  (A.4)

- **$\omega_{1,1}$**
  $$\omega_{1,1}(u_1) = \sum_{r \in \mathbb{Z}^3} \left( \frac{1}{24} \left[ \frac{1}{\Lambda} \right]_{-2} \varphi(2)(u_1 - u_r) + \eta_1 \left[ \frac{1}{\Lambda} \right]_{-2} \varphi(u_1 - u_r) + \frac{1}{4} \left[ \frac{1}{\Lambda} \right] \eta \varphi(u_1 - u_r) \right) du_1.$$  (A.5)
In the above we have used the notation \([-\cdot\]_n\) to denote the degree \(n\) Laurent coefficient at the corresponding point in consideration. Direct computations show that

\[
\frac{1}{A_2} - 2 = \frac{1}{|A_2|} = \frac{a_0}{a_0}, \quad \frac{1}{A_1} = -\frac{a_2}{a_0} + \frac{a_1^2}{a_0^2},
\]

where

\[
a_0 = 2[x^*_1][y - y^*_1] = \frac{[x][y + y^*_0]}{[x]}_0\]
\[
a_1 = 2[x^*_1][y - y^*_1] \cdot \frac{[x][y + y^*_1] + [x][y + y^*_0]}{[x]}_0\]
\[
+ 2[x^*_1][y - y^*_1]^2 + [x^*_2][y - y^*_1],
\]
\[
a_2 = \frac{2[x^*_1][y - y^*_1]^3}{3[x][y + y^*_0]^3}_0\]
\[
+ \frac{2[x^*_1][y - y^*_1] + 2[x^*_2][y - y^*_1]^2 + 2[x^*_3][y - y^*_1]}{[x][y + y^*_0]}\]
\[
- \frac{2([x^*_1][y - y^*_1]^2 + [x^*_2][y - y^*_1]) ([x][y + y^*_1] + [x][y + y^*_0])}{[x]^2_0[(y + y^*_1)^3]_0}\]
\[
+ \frac{2([x^*_1][y - y^*_1]^2 + [x^*_2][y + y^*_1] + [x][y + y^*_1]^2 + [x][y + y^*_0]^2)}{[x]^2_0[(y + y^*_0)^3]_0}\]
\[
- \frac{2([x^*_1][y - y^*_1]^2 + [x][y + y^*_0]^2) + [x][y + y^*_0]^2)}{[x]^3_0[(y + y^*_0)^3]_0}.
\]

**A.1 \(K_{P^2}\)**

The affine part of the mirror curve given in Example 2.1 is equivalent to

\[
y^2 + (x + 1)y + q_1x^3 = 0, \quad q_1 = (-3\phi)^{-3}.
\]

The set of finite ramification points is \(R^\circ = \{\frac{1}{4}, \frac{1}{2}, \frac{1 + \pi}{2}\}\). Uniformization gives

\[
x = -3(4\frac{1}{2})\kappa^2\phi(u) - \frac{9}{4}\phi^3, \quad y = \kappa^3\phi'(u) - \frac{1 + x}{2}.
\]

with

\[
\phi(\tau) = \Theta_{A_2}(2\tau) - \frac{\eta(3\tau)}{\eta(\tau)^3}, \quad \kappa = \zeta_6 2^{-\frac{1}{2}} 3^{\frac{1}{2}} \pi^{-1} \frac{\eta(3\tau)}{\eta(\tau)^3}.
\]

**A.2 \(K_{F_1}\)**

The affine part of the mirror curve given in Example 2.4 is

\[
y^2 + (1 + x)y + q_2x^3 + q_1x^3 = 0.
\]
The set of finite ramification points is \( R^\circ = \{ \frac{1}{2}, \frac{7}{2}, \frac{1+7}{2} \} \). The uniformization is given by
\[
x = (-q_1)^{-\frac{1}{3}43\kappa^2\wp(u) + \frac{1 - 4q_2}{12q_1}, \quad y = \kappa^3\wp'(u) - \frac{1 + x}{2},
\]
(A.19)

Taking the special one-parameter family \( q_1 = q_2 = s \), we have
\[
s = 2 - 8\frac{\eta^8(\tau)}{\eta^8(4\tau)}, \quad \kappa = \frac{\pi^3}{6} s^{-1} \theta_2^{-\frac{3}{2}}(2\tau) \theta_4^{-\frac{1}{2}}(2\tau).
\]
(A.20)

**A.3** \( K_{P^1 \times P^1} \)

Then affine part of the mirror curve given in Example 2.2 is equivalent to
\[
y^2 + (1 + x + q_1x^2)y + q_2x^2 = 0.
\]
(A.21)

The set of finite ramification points is \( R^\circ = \{ 0, \frac{1}{2}, \frac{7}{2}, \frac{1+7}{2} \} \). The uniformization is given by the iteration of the following changes of coordinates (for some \( \epsilon \) and \( \kappa \))
\[
\alpha = 2\frac{3}{2}\kappa^2\wp(u + \epsilon) + \frac{1}{12}(-1 - 2q_1 + 4q_2), \quad \beta = \kappa^3\wp'(u + \epsilon) - \frac{1}{2}(\alpha + \frac{1}{2}q_1),
\]
(A.22)
\[
x = \beta^{-1} \left( 2\frac{3}{2}\kappa^2\wp(u + \epsilon) + \frac{1}{6}(1 + 2q_1 - 4q_2) \right), \quad y = -\frac{1}{2} + x(ax - \frac{1}{2}) - \frac{1}{2}(1 + x + q_1x^2).
\]
(A.23)

Taking the special one-parameter subfamily \( q_1 = q_2 = s \), we have
\[
s = 2 - 8\frac{\eta^8(\tau)}{\eta^8(4\tau)}, \quad \kappa = 2^{-\frac{7}{2}} \pi^{-1} \theta_2^{-2}(2\tau).
\]
(A.24)

**A.4** \( K_{WP[1,1,2]} \)

The affine part of the mirror curve given in Example 2.3 is equivalent to
\[
y^2 + x^4 + y + b_4x^2y + b_0xy = 0, \quad q_1 = b_4b_0^{-4}, q_2 = b_0^{-2}.
\]
(A.25)

The set of finite ramification points is \( R^\circ = \{ 0, \frac{1}{2}, \frac{7}{2}, \frac{1+7}{2} \} \). The following combination is independent of the specialization to an one-parameter subfamily
\[
(b_0^2 - 4b_4)^2 = 64 \frac{(\theta_2^4(2\tau) + \theta_3^4(2\tau))^2}{\theta_4^8(2\tau)},
\]
(A.26)

up to an \( SL_2(\mathbb{Z}) \)-transform on \( \tau \).

The uniformization is given by the iteration of the following changes of coordinates (for some \( \epsilon \) and \( \kappa \))
\[
\alpha = 2\frac{3}{2}\kappa^2\wp(u + \epsilon) - \frac{1}{12}(b_0^2 + 2b_4), \quad \beta = \kappa^3\wp'(u + \epsilon) - \frac{1}{2}b_0(\alpha + \frac{1}{2}b_4),
\]
(A.27)
\[
x = \beta^{-1} \left( 2\frac{3}{2}\kappa^2\wp(u + \epsilon) + \frac{1}{5}(b_4 - \frac{1}{4}b_0^2) \right), \quad y = -\frac{1}{2} + x(ax - \frac{1}{2}b_0) - \frac{1}{2}(1 + b_0x + b_4x^2).
\]
(A.28)

Taking the special one-parameter subfamily \( (q_1, q_2) = (0, s) \) that is \( b_4 = 0 \), we have
\[
s = 64^{-1} \frac{\theta_4^8(2\tau)}{(\theta_2^4(2\tau) + \theta_3^4(2\tau))^2}, \quad \kappa = 2^{-\frac{7}{2}} \pi^{-1} \theta_2^{-2}(2\tau).
\]
(A.29)
References

[ABK08] Mina Aganagic, Vincent Bouchard, and Albrecht Klemm, Topological Strings and (Almost) Modular Forms, Commun.Math.Phys. 277 (2008), 771–819.

[AKMV05] Mina Aganagic, Albrecht Klemm, Marcos Marino, and Cumrun Vafa, The topological vertex, Commun.Math.Phys. 254 (2005), 425–478.

[AKV02] Mina Aganagic, Albrecht Klemm, and Cumrun Vafa, Disk instantons, mirror symmetry and the duality web, Z. Naturforsch. A 57 (2002), no. 1-2, 1–28. MR 1906661 (2003f:81183)

[ALM10] Murad Alim, Jean Dominique Länge, and Peter Mayr, Global Properties of Topological String Amplitudes and Orbifold Invariants, JHEP 1003 (2010), 113.

[ASYZ14] Murad Alim, Emanuel Scheidegger, Shing-Tung Yau, and Jie Zhou, Special polynomial rings, quasi modular forms and duality of topological strings, Adv. Theor. Math. Phys. 18 (2014), no. 2, 401–467. MR 3273318

[AV00] Mina Aganagic and Cumrun Vafa, Mirror symmetry, D-branes and counting holomorphic discs, arXiv:hep-th/0012041

[Bat93] Victor V Batyrev, Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori, Duke Mathematical Journal 69 (1993), no. 2, 349–409.

[BB91] J. M. Borwein and P. B. Borwein, A cubic counterpart of Jacobi’s identity and the AGM, Trans. Amer. Math. Soc. 323 (1991), no. 2, 691–701. MR 1010408 (91e:33012)

[BBG94] J. M. Borwein, P. B. Borwein, and F. G. Garvan, Some cubic modular identities of Ramanujan, Transactions of the American Mathematical Society 343 (1994), no. 1, 35–47. MR 1243610 (94j:11019)

[BBG95] Bruce C. Berndt, S. Bhargava, and Frank G. Garvan, Ramanujan’s theories of elliptic functions to alternative bases, Transactions of the American Mathematical Society 347 (1995), no. 11, 4163–4244. MR 1311903 (97h:33034)

[BCOV93] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Holomorphic anomalies in topological field theories, Nucl.Phys. B405 (1993), 279–304.

[BCOV94] , Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun.Math.Phys. 165 (1994), 311–428.

[BKMnP09] Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti, Remodeling the B-model, Comm. Math. Phys. 287 (2009), no. 1, 117–178. MR 2480744

[BKMnP10] , Topological open strings on orbifolds, Comm. Math. Phys. 296 (2010), no. 3, 589–623. MR 2628817

[CI14] T. Coates and H. Iritani, A Fock sheaf for Givental quantization, arXiv:1411.7039 [math.AG].
[CI18] T. Coates and H. Iritani, *Gromov-Witten Invariants of Local $\mathbb{P}^2$ and Modular Forms*, arXiv:1804.03292 [math.AG].

[CKYZ99] T.M. Chiang, A. Klemm, Shing-Tung Yau, and E. Zaslow, *Local mirror symmetry: Calculations and interpretations*, Adv.Theor.Math.Phys. 3 (1999), 495–565.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094)

[Con96] Ian Connell, *Elliptic curve handbook*.

[CP12] Cheol-Hyun Cho and Mainak Poddar, *Holomorphic orbidiscs and Lagrangian Floer cohomology of symplectic toric orbifolds*, Journal of Differential Geometry 98.1 (2014): 21-116.

[DI08] W Duke and ÖImamoğlu, *The zeros of the Weierstrass $\wp$-function and hypergeometric series*, Mathematische Annalen 340 (2008), no. 4, 897–905.

[DMZ12] A. Dabholkar, S. Murthy, and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms*, arXiv:1208.4074 [hep-th].

[Dol97] Igor V Dolgachev, *Lectures on modular forms*. Fall 1997/98.

[Dub94] B. Dubrovin, *Geometry of 2D topological field theories*. Integrable systems and quantum groups. Springer, Berlin, Heidelberg, 1996. 120-348.

[EO07] Bertrand Eynard and Nicolas Orantin, *Invariants of algebraic curves and topological expansion*, Communications in Number Theory and Physics 1.2 (2007): 347-452.

[EOM07] Bertrand Eynard, Nicolas Orantin, and Marcos Marino, *Holomorphic anomaly and matrix models*, Journal of High Energy Physics 2007 (2007), no. 06, 058.

[EZ82] M Eichler and D Zagier, *On the zeros of the Weierstrass $\wp$-function*, Mathematische Annalen 258 (1982), no. 4, 399–407.

[EZ84] M. Eicher and D. Zagier, *On the theory of Jacobi forms*, Progress in Math. 255 (1984).

[Fay77] John D Fay, *Fourier coefficients of the resolvent for a Fuchsian group*, Journal für die reine und angewandte Mathematik 293 (1977), 143–203.

[FLT13] B. Fang, C.-C. Liu, and H.-H. Tseng, *Open-closed Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks*, arXiv:1212.6073 [math.AG].

[FLZ16] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *On the remodeling conjecture for toric Calabi-Yau 3-orbifolds*, arXiv:1604.07123 [math.AG].

[GKMW07] Thomas W. Grimm, Albrecht Klemm, Marcos Marino, and Marlene Weiss, *Direct Integration of the Topological String*, JHEP 0708 (2007), 058.
[HKR08] Babak Haghighat, Albrecht Klemm, and Marco Rauch, Integrability of the holomorphic anomaly equations, JHEP 0810 (2008), 097.

[Hos04] S. Hosono, Central charges, symplectic forms, and hypergeometric series in local mirror symmetry, Mirror symmetry. V, 405–439, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc., Providence, RI, 2006.

[HV00] Kentaro Hori and Cumrun Vafa, Mirror symmetry, ArXiv e-prints (2000).

[Iri09] H. Iritani, An integral structure in quantum cohomology and mirror symmetry for toric orbifolds, Adv. Math. 222 (2009), no. 3, 1016–1079. MR 2553377 (2010j:53182)

[Kat76] Nicholas M Katz, p-adic interpolation of real analytic Eisenstein series, Annals of Mathematics (1976), 459–571.

[KK] A Kokotov and D Korotkin, Bergmann tau-function on Hurwitz spaces and its applications, arXiv preprint math-ph/0310008, 03–101.

[KK04a] A Kokotov and D Korotkin, Tau-functions on Hurwitz spaces, Mathematical Physics, Analysis and Geometry 7 (2004), no. 1, 47–96.

[KK04b] Aleksey Kokotov and Dmitry Korotkin, Tau-functions on spaces of Abelian differentials and higher genus generalizations of Ray-Singer formula, Tau-functions on spaces of Abelian differentials and higher genus generalizations of Ray-Singer formula. J. Differential Geom. 82 (2009), no. 1, 35–100.

[KL01] Sheldon Katz and Chiu-Chu Melissa Liu, Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc, Adv. Theor. Math. Phys. 5 (2001), no. 1, 1–49. MR 1894336 (2003e:14047)

[KM10] Yukiko Konishi and Satoshi Minabe, Local B-model and mixed Hodge structure, Advances in Theoretical and Mathematical Physics 14 (2010), no. 4, 1089–1145.

[KZ95] Masanobu Kaneko and Don Zagier, A generalized Jacobi theta function and quasimodular forms, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 165–172. MR 1363056 (96m:11030)

[KZ99] A. Klemm and E. Zaslow E, Local mirror symmetry at higher genus. AMS IP STUDIES IN ADVANCED MATHEMATICS. 2001;23:183-208.

[Lho18] H. Lho, Gromov-Witten invariants of Calabi-Yau manifolds with two Kähler parameters. arXiv:1804.04399 [math.AG].

[Liu02] Melissa Chiu-Chu Liu, Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an S¹-equivariant pair. arXiv:math/0210257 [math.SG].

[LP17] H. Lho and R. Pandharipande, Stable quotients and the holomorphic anomaly equation. arXiv:1702.06096 [math.AG].
Robert S. Maier, On rationally parametrized modular equations. J. Ramanujan Math. Soc. 24 (2009), no. 1, 1–73. MR 2514149 (2010f:11060)

Robert S. Maier, Nonlinear differential equations satisfied by certain classical modular forms, Manuscripta Math. 134 (2011), no. 1-2, 1–42. MR 2745252 (2012d:11095)

Kenji Mohri, Exceptional string: Instanton expansions and Seiberg-Witten curve, Rev.Math.Phys. 14 (2002), 913–975.

Robert A. Rankin, Modular forms and functions, Cambridge University Press, Cambridge, 1977.

Bruno Schoeneberg, Elliptic modular functions: an introduction, vol. 203, Springer Science & Business Media, 2012.

Joseph H Silverman, The arithmetic of elliptic curves, vol. 106, Springer Science & Business Media, 2009.

Jan Stienstra, Resonant Hypergeometric Systems and Mirror Symmetry. Proceedings of the Taniguchi Symposium 1997. Integrable Systems and Algebraic Geometry. World Scientific, 1998.

Jan Stienstra, Mahler measure variations, Eisenstein series and instanton expansions, Mirror symmetry. V, AMS/IP Stud. Adv. Math., vol. 38, Amer. Math. Soc., Providence, RI, 2006, pp. 139–150. MR 2282958 (2008d:11095)

Leon A Takhtajan, Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves. current correlation functions, Letters in Mathematical Physics 56 (2001), no. 3, 181–228.

Andrei Nikolaevich Tyurin, On periods of quadratic differentials, Russian mathematical surveys 33 (1978), no. 6, 169–221.

E. Witten, Phases of $N = 2$ theories in two dimensions, Nuclear Physics B 403 (1993), 159–222.

Satoshi Yamaguchi and Shing-Tung Yau, Topological string partition functions as polynomials, JHEP 0407 (2004), 047.

Don Zagier, Elliptic modular forms and their applications, The 1-2-3 of modular forms, Universitext, Springer, Berlin, 2008, pp. 1–103. MR 2409678 (2010b:11047)

Jie Zhou, Arithmetic Properties of Moduli Spaces and Topological String Partition Functions of Some Calabi-Yau Threefolds, Harvard Ph. D. Thesis (2014).

Beijing International Center for Mathematical Research, Peking University, 5 Yiheyuan Road, Beijing 100871, China
Email: bohanfang@gmail.com

Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109, USA
E-mail: ruan@umich.edu

Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109, USA
E-mail: yc.zhang15@pku.edu.cn

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
Email: zhouj@math.uni-koeln.de