THE RITT PROPERTY OF SUBORDINATED OPERATORS IN THE GROUP CASE

FLORENCE LANCIEN AND CHRISTIAN LE MERDY

Abstract. Let $G$ be a locally compact abelian group, let $\nu$ be a regular probability measure on $G$, let $X$ be a Banach space, let $\pi : G \to B(X)$ be a bounded strongly continuous representation. Consider the average (or subordinated) operator $S(\pi, \nu) = \int_G \pi(t) \, d\nu(t) : X \to X$. We show that if $X$ is a UMD Banach lattice and $\nu$ has bounded angular ratio, then $S(\pi, \nu)$ is a Ritt operator with a bounded $H^\infty$ functional calculus. Next we show that if $\nu$ is the square of a symmetric probability measure and $X$ is $K$-convex, then $S(\pi, \nu)$ is a Ritt operator. We further show that this assertion is false on any non-$K$-convex space $X$.

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1. Introduction

Let $G$ be a locally compact abelian group and let $M(G)$ denote the Banach algebra of all bounded regular Borel measures on $G$. Let $X$ be a complex Banach space and let $B(X)$ denote the Banach algebra of all bounded operators on $X$. Let $\pi : G \to B(X)$ be a representation, that is, $\pi(t+s) = \pi(t)\pi(s)$ for any $t, s \in G$, and $\pi(e) = I_X$ (where $e$ and $I_X$ denote the unit of $G$ and the identity operator on $X$, respectively). Assume further that $\pi$ is bounded, that is $\sup_{t \in G} \| \pi(t) \| < \infty$, and that $\pi$ is strongly continuous, that is, for any $x \in X$, the mapping $t \mapsto \pi(t)x$ is continuous from $G$ into $X$. To any probability measure $\nu \in M(G)$, one can associate the average operator

$$S(\pi, \nu) = \int_G \pi(t) \, d\nu(t) \in B(X),$$

where the integral is defined in the strong sense.

In this paper we are interested in the following two questions.

Q.1 When is $S(\pi, \nu)$ a Ritt operator?

Q.2 When does $S(\pi, \nu)$ admit a bounded $H^\infty$ functional calculus?

Background on Ritt operators and their $H^\infty$ functional calculus (along with references) will be given in Section 2 below.

Average operators appear in various contexts, notably in ergodic theory. When $X$ is a function space, the behaviour of the norm of the powers of $S(\pi, \nu)$, the almost everywhere convergence of these powers and various maximal and oscillation inequalities were widely studied, see \[11, 20, 21, 29\] and the references therein. Recent papers on these topics \[8\]
show the importance of the Ritt property and $H^\infty$ functional calculus in the behaviour of the powers of operators $S(\pi, \nu)$. This is the source of motivation for this paper.

Let $U \in B(X)$ be an invertible operator such that $\sup_{k \in \mathbb{Z}} ||U^k|| < \infty$. Then the mapping $\pi: \mathbb{Z} \to B(X)$ defined by $\pi(k) = U^k$ for any $k \in \mathbb{Z}$ is a representation and any bounded representation of $\mathbb{Z}$ on $X$ has this form. A probability measure on $\mathbb{Z}$ is given by a sequence $\nu = (c_k)_{k \in \mathbb{Z}}$ of nonnegative real numbers such that $\sum_k c_k = 1$. In this case, we have

$$S(\nu, \pi) = \sum_{k=-\infty}^{\infty} c_k U^k.$$ 

Thus $S(\nu, \pi)$ is subordinated to $U$ in the sense of [13]. The special case $G = \mathbb{Z}$ therefore indicates that average operators [11] may be regarded as subordinated operators in the context of group representations. Subordination operators induced by probability measures on the semigroup $\mathbb{N}$ were extensively studied recently [4, 13, 14, 15]. In this context a major question is to determine when $\sum_{n=0}^{\infty} c_k T^k$ is a Ritt operator for a nonnegative sequence $(c_k)_{k \geq 0}$ with $\sum_k c_k = 1$ and a power bounded $T: X \to X$. Question Q.1 in the present paper should be considered as its analogue in the group case.

Our results in Sections 3-5 emphasize the role of Banach space geometry in these issues. In Section 3 we recall the so-called bounded angular ratio (BAR) condition and extend some of the results in [8]. We show that if $\nu$ has BAR and $X$ is a UMD Banach lattice, then $S(\pi, \nu)$ is a Ritt operator and it admits a bounded $H^\infty$ functional calculus for any $\pi$ as above.

Section 4 deals with the case when $X$ is a $K$-convex Banach space and $\nu$ is the square of a symmetric probability measure. In this case we show (see Theorem [11]) that for any bounded strongly continuous representation $\pi: G \to B(X)$, $S(\pi, \nu)$ is a Ritt operator. In the case when $\nu = \lambda_p^X$ is the regular representation on $L^p(G; X)$ (see [8] for the definition), this result is a discrete analogue of Pisier’s Theorem [35] showing the analyticity of the tensor extension of any convolution semigroup associated with a family of symmetric probability measures.

Section 5 provides examples of pairs $(\pi, \nu)$ for which Q.1 (and hence Q.2) has a negative answer. These examples further show that the $K$-convexity assumption is unavoidable in Theorem [11].

We conclude this introduction with a few notations and conventions. Unless otherwise specified, $G$ denotes an arbitrary locally compact abelian group, equipped with a fixed Haar measure $dt$. For any $1 \leq p \leq \infty$, we let $L^p(G)$ denote the $L^p$-space associated to this measure. We let $\hat{G}$ denote the dual group of $G$ and, for any $\nu \in M(G)$, we let $\hat{\nu}: \hat{G} \to \mathbb{C}$ denote the Fourier transform of $\nu$. For any measurable subset $V \subset G$, we let $|V|$ and $\chi_V: G \to \mathbb{R}$ denote the Haar measure of $V$ and the characteristic function of $V$, respectively.

For any $t \in G$, let $\lambda_p(t): L^p(G) \to L^p(G)$ be the translation operator defined by $[\lambda_p(t)f](s) = f(s - t)$ for any $f \in L^p(G)$. We say that an operator $T: L^p(G) \to L^p(G)$ is a Fourier multiplier if $T\lambda_p(t) = \lambda_p(t)T$ for any $t \in G$.

Let $(\Omega, \mu)$ be a measure space. For any $1 \leq p \leq \infty$ and for any Banach space $X$, we let $L^p(\Omega; X)$ be the Bochner space of measurable functions $f: \Omega \to X$ (defined up to almost
everywhere zero functions) such that the norm function \( \| f(\cdot) \| \) belongs to \( L^p(\Omega) \) (see e.g. [12, Chapter II]). For \( p \neq \infty \), the algebraic tensor product \( L^p(\Omega) \otimes X \) is dense in \( L^p(\Omega; X) \).

Let \( 1 \leq p < \infty \), let \( T: L^p(\Omega_1) \to L^p(\Omega_2) \) and let \( S: X \to X \) be bounded operators. If \( T \otimes S: L^p(\Omega_1) \otimes X \to L^p(\Omega_2) \otimes X \) extends to a bounded operator from \( L^p(\Omega_1; X) \) into \( L^p(\Omega_2; X) \), then we let

\[
T \otimes S: L^p(\Omega_1; X) \to L^p(\Omega_2; X)
\]

denote this extension. It is well-known that if \( T \) is positive (i.e. \( T(f) \geq 0 \) for any \( f \geq 0 \)), then \( T \otimes S \) has a bounded extension for any \( S: X \to X \).

2. Ritt operators and their \( H^\infty \) functional calculus

An operator \( T: X \to X \) is called power bounded if there exists a constant \( C_0 > 0 \) such that

\[
\forall n \geq 0, \quad \| T^n \| \leq C_0.
\]

Then a power bounded \( T \) is called a Ritt operator if there exists a constant \( C_1 > 0 \) such that

\[
\forall n \geq 1, \quad n \| T^n - T^{n-1} \| \leq C_1.
\]

Ritt operators can be characterized by a spectral condition, as follows. Let

\[
\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}
\]

be the open unit disc. For any \( T \in B(X) \), let \( \sigma(T) \) denote the spectrum of \( T \). Then \( T \) is a Ritt operator if and only if \( \sigma(T) \subset \overline{\mathbb{D}} \) and there exists a constant \( K > 0 \) such that

\[
\forall z \in \mathbb{C} \setminus \overline{\mathbb{D}}, \quad \| (z - T)^{-1} \| \leq \frac{K}{|z - 1|}.
\]

This result goes back to [31, 33, 34], see also [3] for complements.

Let us now turn to functional calculus. For any angle \( \gamma \in (0, \frac{\pi}{2}) \), consider the so-called Stolz domain \( B_\gamma \) as sketched in Figure 1. In analytic terms, \( B_\gamma \) is defined as the interior of the convex hull of 1 and the disc \( \{ z \in \mathbb{C} : |z| < \sin \gamma \} \).

![Figure 1. Stolz domain](image-url)
It turns out that if $T$ is a Ritt operator, then $\sigma(T) \subset \overline{B}\gamma$ for some $\gamma \in (0, \frac{\pi}{2})$. In particular we have $\sigma(T) \subset \mathbb{D} \cup \{1\}$.

One important feature of the above solz domains is that for any $\gamma \in (0, \frac{\pi}{2})$ there exists a constant $C_\gamma > 0$ such that
\begin{equation}
\forall z \in B\gamma, \quad |1 - z| \leq C_\gamma (1 - |z|).
\end{equation}

Let $\mathcal{P}$ be the algebra of all complex polynomials. Let $\gamma \in (0, \frac{\pi}{2})$. Following [26] (to which we refer for more information), we say that an operator $T : X \to X$ has a bounded $H^\infty(B\gamma)$ functional calculus if there exists a constant $C \geq 1$ such that
\begin{equation}
\forall \varphi \in \mathcal{P}, \quad \|\varphi(T)\| \leq C \sup\{||\varphi(z)|| : z \in B\gamma\}.
\end{equation}
A routine argument shows that this condition implies
\[\sigma(T) \subset \overline{B}\gamma.\]

Further if $T$ satisfies (4), then $T$ is a Ritt operator. Indeed (5) applied to $z \mapsto z^n$ immediately implies that $\|T^n\| \leq C$ for any $n \geq 0$. Next define $\varphi_n \in \mathcal{P}$ by
\[\varphi_n(z) = n(z^n - z^{n-1})\]
for any $n \geq 1$. Then for any $z \in B\gamma$, we have
\[|\varphi_n(z)| \leq C_\gamma n|z|^{n-1}(1 - |z|)\]
by (4). An elementary computation shows that
\[\forall t \in (0, 1), \quad nt^{n-1}(1 - t) \leq \left(1 - \frac{1}{n}\right)^{n-1}.\]
We deduce that
\[\sup\{|\varphi_n(z)| : z \in B\gamma, \ n \geq 1\} < \infty.\]
Thus (5) implies the boundedness of the sequence $(\varphi_n(T))_{n \geq 1}$, hence an estimate (2). We record this simple fact for further use.

**Lemma 1.** Let $T : X \to X$ be an operator satisfying (2) for some $\gamma \in (0, \frac{\pi}{2})$. Then $T$ is Ritt operator.

Condition (5) is close but different from the notion of polynomial boundedness. Recall that an operator $T : X \to X$ is called polynomially bounded if there exists a constant $C \geq 1$ such that $\|\varphi(T)\| \leq C \sup\{||\varphi(z)|| : z \in \mathbb{D}\}$. Clearly if $T$ admits a bounded $H^\infty(B\gamma)$ functional calculus, then it is polynomially bounded. However there exist polynomially bounded Ritt operators which do not admit any bounded $H^\infty(B\gamma)$ functional calculus [25].

**Definition 2.** We say that a Ritt operator $T \in B(X)$ has a bounded $H^\infty$ functional calculus if it admits a bounded $H^\infty(B\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.

Recent papers show the relevance of this notion for the study of Ritt operators. It is proved in [16, 26] that for a large class of Banach spaces $X$, a Ritt operator $T : X \to X$ has a bounded $H^\infty$ functional calculus in the above sense if and only if $T$ and its adjoint $T^* : X^* \to X^*$ satisfy certain square functions estimates which naturally arise in the harmonic analysis of the discrete semigroup $(T_n)_{n \geq 0}$. We refer the reader to the papers [27, 28] for applications of this characterization of bounded $H^\infty$ functional calculus.
We now briefly discuss sectorial operators, which will be used in Section 3. For any angle \( \omega \in (0, \pi) \), set
\[
\Sigma_\omega = \{ \lambda \in \mathbb{C}^* : |\text{Arg}(\lambda)| < \omega \}.
\]
Recall that a closed operator \( A : D(A) \to X \) with dense domain \( D(A) \subset X \) is called sectorial of type \( \omega \) if its spectrum is included in \( \Sigma_\omega \) and for any \( \alpha \in (\omega, \pi) \) there exists a constant \( K_\alpha > 0 \) such that
\[
\forall \lambda \in \mathbb{C} \setminus \Sigma_\alpha, \quad \| (\lambda - A)^{-1} \| \leq \frac{K_\alpha}{|\lambda|}.
\]
A simple connection between the Ritt condition and sectoriality is that an operator \( T \in B(X) \) is a Ritt operator if and only if \( A = I_X - T \) is sectorial of type \( < \frac{\pi}{2} \) and \( \sigma(T) \subset \mathbb{D} \cup \{1\} \). This follows from comparing (3) and (6), see [33] for details.

Let \( \mathcal{R} \) be the algebra of all rational functions with nonpositive degree and poles in the closed half-line \( \mathbb{R}_- \). For any \( \phi \in \mathcal{R} \) and any sectorial operator \( A \), \( \phi(A) \) is a well-defined bounded operator on \( X \). For any \( \alpha \in (0, \pi) \), we say that a sectorial operator \( A \) has a bounded \( H^\infty(\Sigma_\alpha) \) functional calculus if there exists a constant \( C \geq 1 \) such that
\[
\forall \phi \in \mathcal{R}, \quad \| \phi(A) \| \leq C \sup \| \phi(\lambda) \| : \lambda \in \Sigma_\alpha \}.
\]
This definition is equivalent to other classical ones that the interested reader will find e.g. in [17, 24].

A strong connection between \( H^\infty \) calculus for sectorial and Ritt operators is given by the following statement.

**Theorem 3.** ([26 Proposition 4.1]) Let \( T \in B(X) \) be a Ritt operator and let \( A = I_X - T \). Then \( T \) has a bounded \( H^\infty \) functional calculus (in the sense of Definition 2) if and only if there exists \( \alpha \in (0, \frac{\pi}{2}) \) such that \( A \) has a bounded \( H^\infty(\Sigma_\alpha) \) functional calculus.

### 3. The BAR condition and UMD Banach lattices

Let \( X \) be an arbitrary Banach space. We adopt the notation \([1]\) for any \( \nu \in M(G) \) and any bounded strongly continuous representation \( \pi : G \to B(X) \).

A straightforward application of Fubini’s Theorem shows that for any \( \nu_1, \nu_2 \in M(G) \), we have
\[
S(\pi, \nu_2)S(\pi, \nu_1) = S(\pi, \nu_2 * \nu_1).
\]
This implies that for any \( \varphi \in \mathcal{P} \) and for any \( \nu \in M(G) \),
\[
\varphi(S(\pi, \nu)) = S(\pi, \nu_\varphi),
\]
where \( \nu_\varphi := \varphi(\nu) \in M(G) \) is obtained by applying polynomial functional calculus in the Banach algebra \( M(G) \).

For any \( 1 \leq p \leq \infty \), let
\[
C_{\nu,p}^X : L^p(G; X) \longrightarrow L^p(G; X)
\]
be the convolution operator defined by setting \( C_{\nu,p}^X(f) = \nu * f = \int_G f(\cdot - t) \, d\nu(t) \) for any \( f \) in \( L^p(G; X) \). Note that if \( p \neq \infty \) and \( \lambda_p^X : G \to B(L^p(G; X)) \) denotes the regular representation defined by
\[
[\lambda_p^X(t)f](s) = f(s - t), \quad f \in L^p(G; X), \quad s, t \in G,
\]

[33]
then we have
\[
C_{\nu,p}^X = S(\lambda_{\nu,p}^X, \nu).
\]
For convenience we will write $C_{\nu,p}$ instead of $C_{\nu,p}^X$ when $X = \mathbb{C}$. Observe that with the notation introduced at the end of Section 1, we have
\[
C_{\nu,p}^X = C_{\nu,p} \otimes I_X
\]
for any $1 \leq p < \infty$.

We will use the following well-known transference principle, which is a variant of [9, Theorem 2.4]. We include a proof for the sake of completeness.

**Proposition 4.** Let $\pi: G \to B(X)$ be a bounded strongly continuous representation and set
\[
\|\pi\| = \sup_{t \in G} \|\pi(t)\|.
\]
Then for any $1 < p < \infty$, we have
\[
\|S(\pi, \nu): X \to X\| \leq \|\pi\|^2 \|C_{\nu,p}^X \cdot L^p(G; X) \to L^p(G; X)\|.
\]

**Proof.** We first assume that $\nu$ has support in a compact subset $K \subset G$. Let $V$ be an arbitrary open set, with $0 < |V| < \infty$. Let $x \in X$ and define $f: G \to X$ by setting
\[
f(t) = \chi_{V-K}(t)\pi(-t)x, \quad t \in G.
\]
The assumptions imply that $V - K$ has a finite Haar measure, hence $f$ belongs to $L^p(G; X)$ with
\[
\|f\|_p^2 \leq |V - K|\|\pi\|_p^p|x|_p^p.
\]
For any $s \in V$, we may write
\[
S(\pi, \nu)x = \pi(s)\pi(-s)S(\pi, \nu)x = \pi(s) \int_G \pi(t-s)x \, d\nu(t),
\]
which yields
\[
\|S(\pi, \nu)x\| \leq \|\pi\| \left\| \int_G \pi(t-s)x \, d\nu(t) \right\|.
\]
Integrating over $V$, we deduce
\[
|V|\|S(\pi, \nu)x\|_p^p \leq \|\pi\|_p^p \int_G \chi_V(s) \left\| \int_G \pi(t-s)x \, d\nu(t) \right\|_p^p \, ds.
\]
Since $\nu$ has support in $K$, $\int_G \pi(t-s)x \, d\nu(t)$ is equal to $\int_G \chi_{V-K}(s-t)\pi(t-s)x \, d\nu(t)$ for any $s \in V$. Consequently,
\[
|V|\|S(\pi, \nu)x\|_p^p \leq \|\pi\|_p^p \int_G \left\| \int_G \chi_{V-K}(s-t)\pi(t-s)x \, d\nu(t) \right\|_p^p \, ds.
\]
By definition,
\[
[C_{\nu,p}^X(f)](s) = \int_G \chi_{V-K}(s-t)\pi(t-s)x \, d\nu(t)
\]
for a.e. \( s \in G \). Hence we obtain
\[
|V||S(\pi, \nu)x|^p \leq \|\pi\|^p\|C^X_{\nu,p}\|\|f\|^p
\leq \|C^X_{\nu,p}\|^p|V - K||\pi\|^{2p}\|x\|^p.
\]
We deduce that
\[
\|S(\pi, \nu)x\| \leq \left(\frac{|V - K|}{|V|}\right)^{\frac{1}{p}}\|\pi\|^2\|C_{\nu,p}\||x|.
\]

The group \( G \) is abelian, hence amenable. Thus according to Folner’s condition (see e.g. [9, Chapter 2]), we can choose \( V \) such that \( \frac{|V - K|}{|V|} \) is arbitrary close to 1. Hence the above inequality shows that \( \nu \) satisfies (11), in the case when \( \nu \) has compact support.

Now consider an arbitrary \( \nu \in M(G) \) (without any assumption on its support). Since this measure is regular, there is a sequence \((K_n)_n \geq 1\) of compact subsets of \( G \) such that
\[
|\nu|(G) = \lim_{n \to \infty} |\nu|(K_n).
\]
Define \( \nu_n = \nu|_{K_n} \) for any \( n \geq 1 \). Then \( \|\nu_n - \nu\|_{M(G)} \to 0 \) hence \( \|C^X_{\nu_n,p} - C^X_{\nu,p}\| \to 0 \) when \( n \to \infty \). Further for any \( x \in X \), \( S(\pi, \nu_n)x \to S(\pi, \nu)x \) when \( n \to \infty \), by Lebesgue’s Theorem. By the first part of the proof, each \( \nu_n \) satisfies (11), hence \( \nu \) satisfies (11) as well.

Following [8] we say that a probability measure \( \nu \in M(G) \) has bounded angular ratio (BAR in short) if there exists a constant \( K \geq 1 \) such that
\[
|1 - \tilde{\nu}(\xi)| \leq K(1 - |\tilde{\nu}(\xi)|), \quad \xi \in \hat{G}.
\]
It is well-known that this holds true if and only if there exists an angle \( \gamma \in (0, \frac{\pi}{2}) \) such that \( \tilde{\nu}(\xi) \in B_\gamma \) for any \( \xi \in \hat{G} \). The relevance of BAR for the study of Q.1 and Q.2 is shown by the following elementary result.

**Lemma 5.** Let \( \nu \in M(G) \) be a probability measure and let \( H \) be a Hilbert space. Then \( C^H_{\nu,2} \) is a Ritt operator if and only if \( \nu \) has BAR. In this case, the operator \( C^H_{\nu,2} \) admits a bounded \( H^\infty \) functional calculus.

**Proof.** For any \( T \in B(L^2(G)) \), \( T \otimes I_H \) extends to a bounded operator on \( L^2(G; H) \) with \( \|T \otimes I_H\| = \|T\| \). This immediately implies that \( \sigma(C^H_{\nu,2}) = \sigma(C_{\nu,2}) \). Further \( C^H_{\nu,2} \) is a normal operator on \( L^2(G; H) \), hence
\[
\forall \varphi \in \mathcal{P}, \quad \|\varphi(C^H_{\nu,2})\| = \sup\{|\varphi(z)| : z \in \sigma(C_{\nu,2})\}.
\]
Applying Fourier transform, we have
\[
\sigma(C_{\nu,2}) = \{\tilde{\nu}(\xi) : \xi \in \hat{G}\},
\]
hence \( \nu \) has BAR if and only if there exists \( \gamma \in (0, \frac{\pi}{2}) \) such that \( \sigma(C_{\nu,2}) \subset B_\gamma \). Combining this equivalence with (12) yields the result. \( \square \)
It follows from [19] and Lemma [5] that if a probability measure \( \nu \in M(G) \) is such that \( S(\pi, \nu) \) is a Ritt operator for all bounded strongly continuous representations \( \pi \) acting on Hilbert space, then \( \nu \) necessarily has BAR. The next theorem shows that if we consider representations acting on UMD Banach lattices, this necessary condition is also sufficient.

We refer the reader to [19] for background and information on the UMD property and to [38] for a more specific study for Banach lattices. We merely recall that any UMD Banach space is reflexive, that Hilbert spaces and \( L^p \)-spaces for \( 1 < p < \infty \) are UMD and that the UMD property is stable under taking subspaces and quotients.

**Theorem 6.** Let \( \nu \in M(G) \) be a probability measure with BAR. Let \( X \) be a UMD Banach lattice. For any bounded strongly continuous representation \( \pi: G \to B(X) \), \( S(\pi, \nu) \) is a Ritt operator with a bounded \( H^\infty \) functional calculus.

This theorem is close in spirit to [13]. Furthermore it extends some of the results of [8]. Indeed it is shown in [8, Proposition 5.2] that a probability measure \( \nu \) has BAR if and only if \( C_{\nu,p} \) is a Ritt operator for any \( 1 < p < \infty \). Further [8, Theorem 5.6] says that if \( X \) is an \( L^p \)-space for some \( 1 < p < \infty \) and \( \nu \) has BAR, then \( S(\pi, \nu) \) is a Ritt operator for any bounded strongly continuous representation \( \pi: G \to B(X) \). \( H^\infty \) functional calculus is not discussed in [8].

For the proof of Theorem 6 will use complex interpolation, for which we refer to [5, 23]. Given any compatible couple \( (X_0, X_1) \) of Banach spaces and any \( \theta \in [0, 1] \), we let \( [X_0, X_1]_\theta \) denote the interpolation space defined by [23, Section 4]. The interpolation theorem ensures that if \( T: X_0 + X_1 \to X_0 + X_1 \) is a linear operator such that \( T: X_0 \to X_0 \) and \( T: X_1 \to X_1 \) boundedly, then \( T: [X_0, X_1]_\theta \to [X_0, X_1]_\theta \) boundedly for any \( \theta \in [0, 1] \). In the context of Ritt operators, we have the following.

**Proposition 7.** ([6]) Let \( (X_0, X_1) \) be a compatible couple of Banach spaces and let \( T: X_0 + X_1 \to X_0 + X_1 \) be a linear operator such that \( T: X_0 \to X_0 \) is power bounded and \( T: X_1 \to X_1 \) is a Ritt operator. Then for any \( \theta \in (0, 1) \), \( T: [X_0, X_1]_\theta \to [X_0, X_1]_\theta \) is a Ritt operator.

This result was established by Blunck [6] in the case when \( X_0, X_1 \) are \( L^p \)-spaces. However the proof works as well in the more general setting of interpolation couples so we omit it.

**Proof of Theorem 6.** Let \( \nu \in M(G) \) be a probability measure with BAR. According to Definition [2] and Lemma [1], it suffices to show the existence of \( \gamma \in (0, \frac{\pi}{2}) \) and \( C \geq 1 \) such that

\[
\forall \varphi \in \mathcal{P}, \quad \|\varphi(S(\pi, \nu))\| \leq C \sup\{\|\varphi(z)\| : z \in B_\gamma\}.
\]

By (7) and Proposition 4, we have

\[
\|\varphi(S(\pi, \nu))\| \leq \|\pi\|^2\|\varphi(C_{\nu,2}^X)\|
\]

for any \( \varphi \in \mathcal{P} \). Hence it suffices to show that \( T = C_{\nu,2}^X \) satisfies (5) for some \( \gamma \in (0, \frac{\pi}{2}) \).

Since \( X \) is a UMD Banach lattice, it follows from [38] that there exist a compatible couple \( (Y, H) \) and some \( \theta \in (0, 1) \) such that \( H \) is a Hilbert space, \( Y \) is a UMD Banach space and \( X = [Y, H]_\theta \) isometrically. By [3, Theorem 5.1.2], this implies that

\[
L^2(G; X) = [L^2(G; Y), L^2(G; H)]_\theta \quad \text{isometrically.}
\]
By Lemma 5, $C^H_{\nu,2}$ is a Ritt operator. Further $C^Y_{\nu,2}$ is a contraction hence it follows from Lemma 7 that $C^X_{\nu,2}$ is a Ritt operator.

Now we set

$$A^X = I_{L^2(G;X)} - C^X_{\nu,2}$$

and we similarly define $A^Y$ on $L^2(G;Y)$ and $A^H$ on $L^2(G;H)$. Then we consider

$$T_t = e^{-t}e^{tC_{\nu,2}}, \quad t \geq 0.$$ 

It is plain that $(T_t)_{t \geq 0}$ is a $c_0$-semigroup of contractions on $L^2(G)$. Its negative generator is equal to $I_{L^2} - C_{\nu,2}$. Moreover $T_t$ is a positive operator (in the lattice sense) for any $t \geq 0$. Since $Y$ is UMD, Theorem 6 asserts that the negative generator of $(T_t \otimes I_{\mathcal{Y}})_{t \geq 0}$ has a bounded $H^\infty(\Sigma_\alpha)$ functional calculus for any $\alpha \in (\frac{\pi}{2}, \pi)$. By construction, this negative generator is $(I_{L^2} - C_{\nu,2}) \otimes I_{\mathcal{Y}}$, which is equal to $A^Y$.

On the other hand, $C^H_{\nu,2}$ has a bounded $H^\infty$ functional calculus by Lemma 5 hence by Theorem 3, there exists $\beta < \frac{\pi}{2}$ such that $A^H$ admits a bounded $H^\infty(\Sigma_\beta)$ functional calculus. By Theorem 8, we finally obtain that $C^X_{\nu,2}$ has a bounded $H^\infty$ functional calculus.

□

In the next corollary we focus on the simple case $G = \mathbb{Z}$.

**Corollary 8.** Let $(c_k)_{k \in \mathbb{Z}}$ be a sequence on nonnegative real numbers such that $\sum_{k=-\infty}^{\infty} c_k = 1$ and there exists a constant $K \geq 1$ such that

$$\left| 1 - \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \right| \leq K \left( 1 - \left| \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \right| \right), \quad \theta \in \mathbb{R}.$$ 

(1) Let $X$ be a UMD Banach lattice and let $U: X \to X$ be an invertible operator such that

$$\sup_{k \in \mathbb{Z}} \| U^k \| < \infty.$$ 

Then

$$V = \sum_{k=-\infty}^{\infty} c_k U^k$$

is a Ritt operator with a bounded $H^\infty$ functional calculus.

(2) Assume that $c_k = 0$ for any $k \leq -1$. Let $(\Omega, \mu)$ be a measure space, let $1 < p < \infty$ and let $T: L^p(\Omega) \to L^p(\Omega)$ be a positive operator with $\| T \| \leq 1$. Then

$$S = \sum_{k=0}^{\infty} c_k T^k$$

is a Ritt operator with a bounded $H^\infty$ functional calculus.
Proof. Part (1) is the translation of Theorem 6 in the case $G = \mathbb{Z}$.

To prove part (2), we apply the Akcoglu-Sucheston dilation theorem for positive contractions $[11]$: there exist a measure space $(\Omega', \mu')$, a surjective isometry $U : L^p(\Omega') \to L^p(\Omega')$ and two contractions $J : L^p(\Omega) \to L^p(\Omega')$ and $Q : L^p(\Omega) \to L^p(\Omega)$ such that $T^k = QU^kJ$ for any $k \geq 0$.

Let $S = \sum_{k \geq 0} c_k T^k$ and $V = \sum_{k \geq 0} c_k U^k$. For any integer $n \geq 1$, we have

$$S^n = \sum_{k \geq 0} c(n)_k T^k \quad \text{and} \quad V^n = \sum_{k \geq 0} c(n)_k U^k,$$

where $c(n) \in \ell^1$ is defined by $c(1) = (c_k)_{k \geq 0}$ and $c(n) = c(1) \cdots c(1)$ ($n$ times). Then the above dilation property implies that $S^n = QV^n J$ for any $n \geq 0$. Thus $\varphi(S) = Q \varphi(V) J$ for any $\varphi \in \mathcal{P}$ and hence

$$\forall \varphi \in \mathcal{P}, \quad \|\varphi(S)\| \leq \|\varphi(V)\|.$$ 

The operator $V$ has a bounded $H^\infty$ functional calculus by part (1). By the above inequality, $S$ also has a bounded $H^\infty$ functional calculus. \qed

Remark 9. Let $\nu \in M(G)$ be a probability measure with BAR and let $1 < p < \infty$. According to Theorem 6, $\nu_p$ is a Ritt operator with a bounded $H^\infty$ functional calculus.

Let $X$ be an $S\Omega_p$ space, that is, a quotient of a subspace of an $L^p$-space. For any $T$ in $B(L^p(G))$, $T \otimes I_X$ extends to a bounded operator on $L^p(G; X)$ with $\|T \otimes I_X\| = \|T\|$. This implies that

$$\forall \varphi \in \mathcal{P}, \quad \|\varphi(\nu_p^X)\| = \|\varphi(\nu_p)\|.$$ 

The proof of Theorem 6 therefore shows that for any bounded strongly continuous representation $\pi : G \to B(X)$, $\pi(\nu, \nu)$ is a Ritt operator with a bounded $H^\infty$ functional calculus.

It would be interesting to characterize the class of all Banach spaces $X$ with this property.

## 4. Subordination on K-convex spaces

Let $\Omega_0$ denote the compact group $\{-1, 1\}^\mathbb{N}$ equipped with its normalized Haar measure and for any $n \geq 1$, let $\varepsilon_n : \Omega_0 \to \mathbb{R}$ denote the Rademacher function defined by setting $\varepsilon_n(\Theta) = \theta_n$ for any $\Theta = (\theta_i)_{i \geq 1} \in \Omega_0$. Let $\text{Rad}_2 \subset L^2(\Omega_0)$ denote the closed linear span of the $\varepsilon_n$ and let $Q : L^2(\Omega_0) \to L^2(\Omega_0)$ be the orthogonal projection with range equal to $\text{Rad}_2$.

A Banach space $X$ is called $K$-convex if $Q \otimes I_X$ extends to a bounded operator on $L^2(\Omega_0; X)$. In this case, we let

$$K_X = \|Q \otimes I_X : L^2(\Omega_0; X) \to L^2(\Omega_0; X)\|.$$ 

This number is called the $K$-convexity constant of $X$. This property plays a fundamental role in Banach space theory, in relation with the notions of type and cotype. In his fundamental paper [35], Pisier showed that $X$ is $K$-convex if and only if it admits a non trivial Rademacher type. We refer to [32] for more information on these topics.

$L^p$-spaces are $K$-convex whenever $1 < p < \infty$. More generally we have the following well-known fact.

**Lemma 10.** Let $1 < p < \infty$ and let $X$ be a $K$-convex Banach space. There exists a constant $C > 0$ such that whenever $(\Omega, \mu)$ is a measure space, the space $L^p(\Omega; X)$ is $K$-convex and $K_{L^p(\Omega; X)} \leq C$. 


Proof. It follows from the Khintchine-Kahane inequalities (see e.g. [30, Theorem 1.e.13]) that $Q$ extends to a bounded projection $Q_p: L^p(\Omega_0) \to L^p(\Omega_0)$, that $X$ is $K$-convex if and only if $Q_p \otimes I_X$ extends to a bounded operator on $L^p(\Omega_0; X)$, and that in this case,
\[
A_pK_X \leq \|Q_p \otimes I_X: L^p(\Omega_0; X) \to L^p(\Omega_0; X)\| \leq B_pK_X
\]
for some universal constants $0 < A_p < B_p$. By Fubini’s Theorem, the boundedness of $Q_p \otimes I_X$ on $L^p(\Omega_0; X)$ implies that $Q_p \otimes I_{L^p(\Omega; X)}$ is bounded on $L^p(\Omega_0; L^p(\Omega; X))$, with
\[
\|Q_p \otimes I_{L^p(\Omega; X)}: L^p(\Omega_0; L^p(\Omega; X)) \to L^p(\Omega_0; L^p(\Omega; X))\| = \|Q_p \otimes I_X: L^p(\Omega_0; X) \to L^p(\Omega_0; X)\|.
\]
The result follows at once.

Let $\nu \in M(G)$ be a probability measure. We say that $\nu$ is symmetric provided that $\nu(-V) = \nu(V)$ for any measurable $V \subseteq G$. It is clear that $\nu$ is symmetric if and only if $\hat{\nu}$ is real valued. In this case, $\hat{\nu}$ is actually valued in $[-1, 1]$.

We will say that such a measure $\nu$ is a square if there exists another symmetric probability measure $\eta$ such that
\[
\nu = \eta \ast \eta.
\]
In this case, $\hat{\nu} = \hat{\eta}^2$ is valued in $[0, 1]$, hence $\nu$ has BAR.

Our main result is the following.

**Theorem 11.** Let $\nu \in M(G)$ be a symmetric probability measure. Assume that $\nu$ is a square and let $X$ be a $K$-convex Banach space.

1. For any $1 < p < \infty$, the convolution operator $C^X_{\nu,p} = \nu \ast \cdot : L^p(G; X) \to L^p(G; X)$ is a Ritt operator.

2. For any bounded strongly continuous representation $\pi: G \to B(X)$, $S(\pi, \nu)$ is a Ritt operator.

If we think of Ritt operators as the discrete analogues of bounded analytic semigroups, part (1) of the above theorem should be regarded as a discrete version of [35, Theorem 1.2]. Its proof will make crucial use of the following result of Pisier.

**Theorem 12.** ([35, Theorem 3.1]) Let $Z$ be a $K$-convex Banach space. There exists a constant $C > 0$ only depending on the $K$-convexity constant $K_Z$ such that for any integer $n \geq 1$ and for any $n$-tuple $(P_1, \ldots, P_n)$ of mutually commuting contractive projections on $Z$,
\[
\left\| \sum_{k=1}^n (I_Z - P_k) \prod_{1 \leq j \neq k \leq n} P_j \right\| \leq C.
\]

For any integer $n \geq 2$, the Haar measure on $G^n$ can be defined as the $n$-fold product of the Haar measure on $G$. Then for any $1 \leq p < \infty$, we regard the $n$-fold tensor product $L^p(G) \otimes \cdots \otimes L^p(G)$ as a subspace of $L^p(G^n)$ is the usual way. We recall that
\[
L^p(G) \otimes \cdots \otimes L^p(G) \subset L^p(G^n)
\]
is a dense subspace.
Lemma 13. Let \( 1 < p < \infty \). Let \( n \geq 2 \) and \( m \geq 1 \) be two integers and let \( \{\nu_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m} \) be a family in \( M(G) \). For any \( i, j \) as above, let

\[
S_{ij} = C_{\nu_{ij}, p} = \nu_{ij}^* : L^p(G) \longrightarrow L^p(G)
\]

be the corresponding convolution operator. Then

\[
\left\| \sum_{j=1}^{m} S_{nj} \cdots S_{1j} \Xi X \right\|_{B(L^p(G; X))} \leq \left\| \sum_{j=1}^{m} S_{nj} \cdots \Xi S_{1j} \Xi X \right\|_{B(L^p(G^n; X))}.
\]

Proof. In the first part of the proof, we assume that all the measures \( \nu_{ij} \) have a compact support. Taking their union, we obtain a compact set \( K \subset G \) such that \( \nu_{ij} \) has support included in \( K \) for any \( 1 \leq i \leq n, 1 \leq j \leq m \). Let \( V \subset G \) be an arbitrary open subset with \( 0 < |V| < \infty \). We set \( W = V - K \) for convenience.

We fix some \( f \in L^p(G; X) \) and introduce \( F : G^n \rightarrow X \) defined by

\[
F(s_1, s_2, \ldots, s_n) = \chi_W(s_2) \cdots \chi_W(s_n) f(s_1 + \cdots + s_n).
\]

It is plain that \( F \) belongs to \( L^p(G^n; X) \), with

\[
\|F\|_p^p = |W|^{n-1}\|f\|_p^p.
\]

We claim that for any \( j \), for a.e. \( s_1 \in G \) and for a.e. \( s_2, \ldots, s_n \) in \( V \), we have

\[
[S_{nj} \cdots S_{1j} \Xi X(f)](s_1 + s_2 + \cdots + s_n) = [(S_{nj} \cdots \Xi S_{1j} \Xi X)(F)](s_1, \ldots, s_n).
\]

Indeed applying \( S_{2j} \Xi X, \ldots, S_{nj} \Xi X \) successively to \( S_{1j} \Xi X(f) \), we have

\[
[S_{nj} \cdots S_{2j} S_{1j} \Xi X(f)](s_1 + s_2 + \cdots + s_n) = \int_{G^{n-1}} [S_{1j}(f)](s_1 + s_2 + \cdots + s_n - t_2 - \cdots - t_n) \, d\nu_{2j}(t_2) \cdots d\nu_{nj}(t_n)
\]

for a.e. \( (s_1, s_2, \ldots, s_n) \) in \( G^n \). It follows from the fact that \( \nu_{2j}, \ldots, \nu_{nj} \) have support in \( K \) that whenever \( s_2, \ldots, s_n \) belong to \( V \), this is equal to

\[
\int_{G^{n-1}} \chi_W(s_2 - t_2) \cdots \chi_W(s_n - t_n) [S_{1j}(f)](s_1 + s_2 + \cdots + s_n - t_2 - \cdots - t_n) \, d\nu_{2j}(t_2) \cdots d\nu_{nj}(t_n),
\]

and hence to

\[
\int_{G^{n-1}} \chi_W(s_2 - t_2) \cdots \chi_W(s_n - t_n) \left( \int_{G} f \left( \sum_{i=1}^{n} (s_i - t_i) \right) \, d\nu_{ij}(t_1) \right) \, d\nu_{2j}(t_2) \cdots d\nu_{nj}(t_n)
\]

\[
= \int_{G^n} \chi_W(s_2 - t_2) \cdots \chi_W(s_n - t_n) f \left( \sum_{i=1}^{n} (s_i - t_i) \right) \, d\nu_{1j}(t_1) d\nu_{2j}(t_2) \cdots d\nu_{nj}(t_n)
\]

\[
= \int_{G^n} F(s_1 - t_1, s_2 - t_2, \ldots, s_n - t_n) \, d\nu_{1j}(t_1) d\nu_{2j}(t_2) \cdots d\nu_{nj}(t_n)
\]

\[
= [(S_{nj} \cdots \Xi S_{1j} \Xi X)(F)](s_1, \ldots, s_n)
\]

as claimed.
To derive the estimate (14) from this identity, we note that by translation invariance,
\[ \left\| \sum_j S_{nj} \cdots S_{1j} \otimes I_X(f) \right\|_p^p = \int_G \left\| \sum_j \left[ S_{nj} \cdots S_{1j} \otimes I_X(f) \right] (s_1 + z) \right\|_p^p ds_1. \]
for any \( z \in G \). Hence integrating over \( V^{n-1} \),
\[ |V|^{n-1} \left\| \sum_j S_{nj} \cdots S_{1j} \otimes I_X(f) \right\|_p^p = \int_{V^{n-1}} \int_G \left\| \sum_j \left[ S_{nj} \cdots S_{1j} \otimes I_X(f) \right] (s_1 + s_2 + \cdots + s_n) \right\|_p^p ds_1 \cdot ds_2 \cdots ds_n. \]
According to (15), this implies that
\[ |V|^{n-1} \left\| \sum_j S_{nj} \cdots S_{1j} \otimes I_X(f) \right\|_p^p \leq \left\| \sum_j S_{nj} \otimes \cdots \otimes S_{1j} \otimes I_X \right\|_p^p \| F \|_p^p \leq \left\| \sum_j S_{nj} \otimes \cdots \otimes S_{1j} \otimes I_X \right\|_p^p |V - K|^{n-1} \| f \|_p^p. \]
This shows that
\[ \left\| \sum_j S_{nj} \cdots S_{1j} \otimes I_X \right\| \leq \left( \frac{|V - K|}{|V|} \right)^{\frac{n-1}{p}} \left\| \sum_j S_{nj} \otimes \cdots \otimes S_{1j} \otimes I_X \right\|. \]
As indicated in the proof of Proposition 4 we can choose \( V \) such that \( \frac{|V - K|}{|V|} \) is arbitrary close to 1. Consequently the above estimate implies (14) under the assumption that all \( \nu_{ij} \) have compact support.

The argument at the end of Proposition 4 can be easily adapted to deduce the general result from this special case. Details are left to the reader. \( \square \)

**Proof of Theorem 14.** Let \( \pi \) be as in part (2) and let \( S = S(\pi, \nu) \in B(X) \). It follows from (7) and Proposition 4 that for any integer \( n \geq 1 \),
\[ \| S^n - S^{n-1} \| \leq \| \pi \|^2 \| (C^X_{\nu,2})^n - (C^X_{\nu,2})^{n-1} \|. \]
Hence part (2) is a consequence of part (1). We now aim at proving part (1).

It is plain that \( C^X_{\nu,p} \) is a contraction, hence a power bounded operator (this does not require the \( K \)-convexity assumption).

We fix \( 1 < p < \infty \) and set \( T = C^X_{\nu,p} : L^p(G) \rightarrow L^p(G) \). By assumption, there exists a symmetric probability measure \( \eta \in M(G) \) such that \( T \) is the square of \( C^X_{\eta,p} \).

For any \( 1 \leq q \leq \infty \), \( C_{\eta,q} = \nu^{\ast} \) is a positive contraction on \( L^q(G) \). Since \( \eta \) is symmetric, the operator \( C_{\eta,2} \) is selfadjoint on the Hilbert space \( L^2(G) \). Further \( C_{\eta,\infty}(1) = 1 \), because \( \eta \) is a probability measure. It therefore follows from Rota’s dilation Theorem (see e.g. [11] p.
that there exist a measure space \((\Omega, \mu)\), two positive contractions \(J: L^p(G) \to L^p(\Omega)\) and \(Q: L^p(\Omega) \to L^p(G)\), as well as a conditional expectation \(E: L^p(\Omega) \to L^p(\Omega)\) such that
\[
T = QEJ \quad \text{and} \quad I_{L^p(G)} = QJ.
\]
In the sequel we will make essential use of the fact that \(E\) is a positive contraction and \(E^2 = E\).

Let \(n \geq 1\) be an integer. We may write
\[
-n(T^n - T^{n-1}) = nT^{n-1}(I_{L^p} - T) = \sum_{j=1}^{n} T^{n-j}(I_{L^p} - T)T^{j-1}.
\]
We apply Lemma 13 with \(m = n\) and the family \(\{S_{ij}\}_{1 \leq i, j \leq n}\) of convolution operators on \(L^p(G)\) defined by
\[
S_{jj} = I_{L^p} - T \quad \text{and} \quad S_{ij} = T \quad \text{if} \quad i \neq j.
\]
This yields
\[
\|n(T^n - T^{n-1}) \times I_X\|_{B(L^p(G^n; X))} \leq \]
\[
\left\|\sum_{j=1}^{n} T \times \cdots \times T \times (I_{L^p} - T) \times T \times \cdots \times T \times I_X\right\|_{B(L^p(G^n; X))}.
\]
For convenience we let
\[
Y = L^p(\Omega) \quad \text{and} \quad Z_n = L^p(\Omega^n; X).
\]
According to (10), we have
\[
\underbrace{T \times \cdots \times T}_{n-j} \times (I_{L^p} - T) \times \underbrace{T \times \cdots \times T}_{j-1} = \]
\[
\left(\underbrace{Q \times \cdots \times Q}_{n}\right) \left(\underbrace{E \times \cdots \times E \times (I_Y - E) \times E \times \cdots \times E}_{n-j}\right) \left(\underbrace{J \times \cdots \times J}_{n}\right)
\]
for any \(j = 1, \ldots, n\). Thus the operator in (17) is the composition of the following three operators:

(i) the operator \(J \times \cdots \times J \times I_X\), which is a well-defined contraction from \(L^p(G^n; X)\) into \(L^p(\Omega^n; X)\), because \(J\) is a positive contraction;

(ii) the operator
\[
\sum_{j=1}^{n} E \times \cdots \times E \times (I_Y - E) \times \underbrace{E \times \cdots \times E}_{j-1} \times I_X: Z_n \to Z_n,
\]
which is well-defined because \(E\) is positive.

(iii) the operator \(Q \times \cdots \times Q \times I_X\), which is a well-defined contraction from \(L^p(\Omega^n; X)\) into \(L^p(G^n; X)\), because \(Q\) is a positive contraction.
We deduce that
\[ n \| (T^n - T^{n-1}) \otimes I_X \|_{B(L^p(G;X))} \leq \left\| \sum_{j=1}^{n} E \otimes \cdots \otimes E (I_Y - E) \otimes \sum_{j=1}^{n} E \otimes \cdots \otimes E I_X \right\|_{B(Z_n)}. \]

Now for any \( j = 1, \ldots, n \), define \( P_j : Z_n \to Z_n \) by
\[ P_j = \underbrace{I_Y \otimes \cdots \otimes I_Y} \otimes E \underbrace{I_Y \otimes \cdots \otimes I_Y} \otimes I_X. \]

Then the operator in the right-hand side of the above inequality is equal to
\[ \sum_{k=1}^{n} (I_{Z_n} - P_k) \prod_{1 \leq j \neq k \leq n} P_j. \]

Since \( E \) is a positive contraction, each \( P_j \) is a contraction. Moreover the \( P_j \) mutually commute. Further \( Z_n \) is \( K \)-convex and \( \sup_{n \geq 1} K_{Z_n} < \infty \) by Lemma 10. It therefore follows from Theorem 12 that the operators in (18) are uniformly bounded. Consequently,
\[ \sup_{n \geq 1} n \| (T^n - T^{n-1}) \otimes I_X \|_{B(L^p(G;X))} < \infty. \]

According to (10), this proves that \( C_{\nu,p}^X = T \otimes I_X \) is a Ritt operator.

We will show in Proposition 14 below that if a symmetric probability \( \nu \) has a density we do not need to assume that \( \nu \) is a square in the statement of Theorem 11.

We identify any \( h \in L^1(G) \) with the measure \( \nu \) with density \( h \) and use the notations \( C_{h,p}, C_{h,p}^X, S(\pi,h) \) instead of \( C_{\nu,p}, C_{\nu,p}^X, S(\pi,\nu) \). With this convention, \( h \) is a probability when \( h \in L^1(h)_+ \) and \( \|h\|_1 = 1 \) and \( h \) is symmetric when \( h(t) = h(-t) \) for a.e. \( t \in G \). Note that a symmetric probability \( h \in L^1(G) \) has BAR if and only if if there exists \( a \in (-1,1] \) such that
\[ \forall \xi \in \widehat{G}, \quad a \leq \hat{h}(\xi) \leq 1. \]

A classical result asserts that for any \( h \in L^1(G) \), we have \( \sigma(C_{h,p}) = \sigma(C_{h,2}) \) for any \( 1 \leq p < \infty \) (see e.g. [14]). This implies that
\[ \sigma(C_{h,p}^X) = \sigma(C_{h,2}) \]
for any \( 1 \leq p < \infty \) and any Banach space \( X \). Indeed let \( \lambda \in \mathbb{C} \setminus \sigma(C_{h,1}) \). The operator \( \lambda I_{L^1} - C_{h,1} \) is a Fourier multiplier hence its inverse is a Fourier multiplier on \( L^1(G) \). Consequently there exists \( \rho \in M(G) \) such that \( (\lambda I_{L^1} - C_{h,1})^{-1} = C_{\rho,1} = \rho^{\star} : L^1(G) \to L^1(G) \). This implies that \( \lambda I_{L^p(X)} - C_{h,p}^X \) is invertible, with inverse equal to \( C_{\rho,p}^X \). Hence \( \sigma(C_{h,p}^X) \) is included in \( \sigma(C_{h,1}) = \sigma(C_{h,2}) \). The reverse inclusion is clear.

Proposition 14. Let \( h \in L^1(G)_+ \) with \( \|h\|_1 = 1 \). Assume that \( h \) is symmetric. If \( h \) has BAR and \( X \) is a \( K \)-convex Banach space, then \( C_{h,p}^X \) is a Ritt operator for any \( 1 < p < \infty \). Furthermore for any bounded strongly continuous representation \( \pi : G \to B(X) \), \( S(\pi,h) \) is a Ritt operator.

Proof. As in the proof of Theorem 11 it suffices to prove the first assertion.

We let \( T = C_{h,p}^X \). By the BAR assumption and Lemma 5, \( C_{h,2} \) is a Ritt operator hence \(-1 \notin \sigma(C_{h,2})\). Applying (19) we deduce that \( I + T \) is invertible.
By Theorem [11] $T^2$ is a Ritt operator. Hence the sequence $(n(T^{2n} - T^{2(n-1)}))_{n \geq 1}$ is bounded. Writing

$$T^{2n} - T^{2(n-1)} = T^{2(n-1)}(T^2 - I) = T^{2(n-1)}((T - I)(T + I)),$$

we deduce that the sequence $(nT^{2(n-1)}(T - I))_{n \geq 1}$ is bounded. This immediately implies that the sequence $(nT^n(T - I))_{n \geq 1}$ itself is bounded, that is, $T$ is a Ritt operator.

Applying the above result in the case $G = \mathbb{Z}$, we derive the following.

**Corollary 15.** Let $(c_k)_{k \in \mathbb{Z}}$ be a sequence on nonnegative real numbers such that

$$\forall \theta \in \mathbb{R}, \quad 0 \leq \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \leq 1.$$

Let $X$ be a $K$-convex Banach space and let $U : X \to X$ be an invertible operator such that $\sup_{k \in \mathbb{Z}} \|U^k\| < \infty$. Then

$$S = \sum_{k=-\infty}^{\infty} c_k U^k$$

is a Ritt operator.

5. A FAMILY OF COUNTEREXAMPLES

In Theorem [17] below we provide examples which show two facts. First BAR does not imply that Q.1. has an affirmative answer on general Banach spaces, although Theorem 6 says that this is the case on UMD Banach lattices. Second, the $K$-convexity assumption in Theorem [11] is optimal.

In the sequel we let $\ell^1_n$ (resp. $\ell^\infty_n$) denote the $n$-dimensional $L^1$-space (resp. $L^\infty$-space).

We give some background on the so-called regular operators, which will be used in the proof of Theorem [17]. Let $(\Omega, \mu)$ be a measure space, let $1 < p < \infty$ and let $T : L^p(\Omega) \to L^p(\Omega)$ be a bounded operator. We say that $T$ is regular if there exists a constant $C \geq 0$ such that

$$\left\| \sup_{1 \leq k \leq n} |T(f_k)| \right\|_{L^p} \leq C \left\| \sup_{1 \leq k \leq n} |f_k| \right\|_{L^p}$$

for any integer $n \geq 1$ and any $f_1, \ldots, f_n$ in $L^p(\Omega)$. In this case, we let $\|T\|_r$ denote the smallest $C$ for which this property holds; this is called the regular norm of $T$. We mention (although we will not use it) that an operator $T$ is regular if and only if it is a linear combination of positive bounded operators $L^p(\Omega) \to L^p(\Omega)$.

This definition can be reformulated as follows: a bounded operator $T$ on $L^p(\Omega)$ is regular if and only if the tensor extensions $T \otimes I_{\ell^p_n} : L^p(\Omega; \ell^\infty_n) \to L^p(\Omega; \ell^\infty_n)$ are uniformly bounded. It is not hard to deduce that $T$ is regular if and only if $T \otimes I_X$ extends to a bounded operator on $L^p(\Omega; X)$ for any Banach space $X$ and that $\|T \otimes I_X : L^p(\Omega; X) \to L^p(\Omega; X)\| \leq \|T\|_r$ in this case. This immediately implies that a bounded operator $T : L^p(\Omega) \to L^p(\Omega)$ is regular if and only if its adjoint $T^* : L^p'(\Omega) \to L^p'(\Omega)$ is regular, and that

$$\|T^*\|_r = \|T\|_r$$
in this case. (Here $p' = p/(p - 1)$ denotes the conjugate of $p$.) We refer to [37, 40] for details and complements.

We say that a Banach space $X$ contains the $\ell_1^n$ uniformly if there exists a constant $C \geq 1$ such that for any $n \geq 1$, there exist $x_1, x_2, \ldots, x_n$ in $X$ such that

$$\sum_{i=1}^{n} |\alpha_i| \leq \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq C \sum_{i=1}^{n} |\alpha_i|$$

for any complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. This is equivalent to the existence, for any $n \geq 1$, of an $n$-dimensional subspace of $X$ which is $C$-isomorphic to $\ell_1^n$. It is shown in [35] that a Banach space contains the $\ell_1^n$ uniformly if and only if it is not $K$-convex. This leads to the following.

**Lemma 16.** Let $X$ be a Banach space and assume that $X$ is not $K$-convex. Let $T : L^p(\Omega) \to L^p(\Omega)$ be a bounded operator such that $T \otimes I_X$ extends to a bounded operator $T \otimes I_X : L^p(\Omega; X) \longrightarrow L^p(\Omega; X)$.

Then $T$ is regular.

**Proof.** Since $X$ is not $K$-convex, there exist a constant $C \geq 1$ such that for any $n \geq 1$, $X$ contains an $n$-dimensional subspace $C$-isomorphic to $\ell_1^n$. As a consequence of the tensor extension assumption, we obtain that for any $n \geq 1$,

$$\left\| T \otimes I_{\ell_1^n} : L^p(\Omega; \ell_1^n) \longrightarrow L^p(\Omega; \ell_1^n) \right\| \leq C \| T \otimes X \|.$$

Since $L^p(\Omega; \ell_1^n)^* = L^{p'}(\Omega; \ell_\infty^n)$ isometrically, this implies that for any $n \geq 1$,

$$\left\| T^* \otimes I_{\ell_\infty^n} : L^{p'}(\Omega; \ell_\infty^n) \longrightarrow L^{p'}(\Omega; \ell_\infty^n) \right\| \leq C \| T \otimes X \|.$$

This shows that $T^*$ is regular. By (20), the operator $T$ is regular as well. \hfill \Box

**Theorem 17.** Assume that $G$ is a non discrete locally compact abelian group and that $X$ is a non $K$-convex Banach space. Then there exists a symmetric probability measure $\eta$ on $G$ such that for any $1 < p < \infty$, $C_{\eta^2, p} : L^p(G; X) \to L^p(G; X)$ is not a Ritt operator.

**Proof.** Let $\delta_e$ denote the Dirac measure at point $e$ (the unit of $G$). Since $G$ is non discrete, there exists a symmetric probability measure $\eta$ on $G$ such that

$$\tau = \delta_e + \eta^2$$

is not invertible in the Banach algebra $M(G)$. We refer to the proof of [39, Theorem 5.3.4] for this fact, which is a generalization of the Wiener-Pitt Theorem (see in particular the last three lines of p. 107 in the latter reference).

We fix some $1 < p < \infty$. By [3, Proposition 5.2] (or by Theorem [3], $C_{\eta^2, p} : L^p(G) \to L^p(G)$ is a Ritt operator. In particular, $-1 \in \sigma(C_{\eta^2, p})$. Thus the operator

$$S = I_{L^p} + C_{\eta^2, p}$$

is invertible.
Assume now that $C_{\eta^2,p}^X : L^p(G;X) \to L^p(G;X)$ is a Ritt operator. Then similarly, the operator $I_{L^p(X)} + C_{\eta^2,p}^X$ is invertible. By (10), $I_{L^p(X)} + C_{\eta^2,p}^X = S \otimes I_X$ hence the invertibility of this operator implies that $S^{-1} \otimes X$ extends to a bounded operator $S^{-1} \otimes I_X : L^p(G;X) \longrightarrow L^p(G;X)$.

Since $X$ is not $K$-convex, it follows from Lemma 16 that $S^{-1}$ is regular.

The operator $S^{-1}$ is a Fourier multiplier. Hence according to Arendt’s description of regular Fourier multipliers [2, Proposition 3.3], there exists a measure $\rho \in M(G)$ such that $S^{-1}(f) = \rho \ast f$ for any $f \in L^p(G)$. By construction, $S(f) = \tau \ast f$ for any $f \in L^p(G)$. Hence we have $\rho \ast \tau \ast f = f$ for any $f \in L^p(G)$. This yields $\rho \ast \tau = \delta_e$ and contradicts the fact that $\tau$ is not invertible in the Banach algebra $M(G)$. □

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E-mail address: florence.lancien@univ-fcomte.fr

E-mail address: clemerdy@univ-fcomte.fr

Laboratoire de Mathématiques de Besançon, UMR 6623, CNRS, Université Bourgogne Franche-Comté, 25030 Besançon Cedex, FRANCE