Turns and special relativity transformations

A A Ketsaris
19-1-83, ul. Krasniy Kazanetz, Moscow 111395, Russian Federation

Abstract.

We advance an universal approach to the construction of kinematics in non-inertial and, in particular, rotating reference frames. On its basis, a 10-dimensional space including three projections of velocity vector and three turn angles in geometric space as additional coordinates to time and geometric coordinates is introduced. In a specific case, the turns in 10-space describe the uniform rotation of reference frame as well as its accelerated motion. Transformations for coordinates and angular velocities are derived. Definitions of dynamic quantities contain a fundamental constant $\Omega$ with dimensionality of angular velocity and the maximal acceleration in addition to the velocity of light. The special relativity relation between energy, impulse, and mass gets changed for particles with moment of inertia. A wave equation obtained describes the rotary and accelerated motions of light wave. A relation between particle moment and the Planck constant representing Bohr’s postulate is added to de Broglie’s relations. A generalized space-time which allows to consider kinematics for derivations of arbitrary order is studied. Differential relations between velocity of arbitrary order and turn angles in the generalized space-time are obtained.

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1. Introduction

It is well known that the non-relativistic view of global disk rotation with constant angular velocity $\omega$ is in contradiction with the relativistic rule of composition of velocities. It is believed that the rotating reference frame can not be realized at distances larger than $c/\omega$ as the rotation velocity should not exceed the velocity of light. Despite it the global rotation model is widely used in the description of relativistic effects in rotating frames such as the Sagnac effect, the length contraction, the time dilation, and others. In doing so it is overlooked that the global disk rotation is accelerated and rotary local motions of disk parts related to each other.

Geometrical turns and Lorentz busts, i.e. turns in the pseudoplane $(x, t)$, make full Lorentz group. The derivations of turn parameters by time are angular velocity $d\theta/dt$ and acceleration $dv/dt$. It is a serious reason for considering both angular velocity and acceleration in the unified way.

‡ E-mail address: ketsaris@sai.msu.su
The progress has been possible in understanding relativistic laws for accelerated motion due to the Caianiello’s idea of the expansion of space-time through components of four-velocity of particle \([6]\). The more important consequence of this model is the existence of the maximal acceleration for any physical processes \([7, 8]\). In our recent work \([9]\) it was shown that a similar increase of space-time dimension (through three components of usual velocity) is required from an uniform approach to differentiation of kinematic quantities. Namely, a derivation of one kinematic variable by other should correspond to a turn angle in plane of these variables. This is possible only if both the function-variable and the argument-variable are involved in the construction of vector space. So in the special relativity theory, the velocity is in correspondence with a turn angle, \(\psi\), in the pseudoplane \((x,t)\):

\[
\frac{dx}{dt} = c \tanh \psi .
\]

By analogy with constant velocity motions, accelerated motions can be considered through correspondence between the acceleration and the turn angle in the pseudoplane \((v,t)\) like \((1)\). Therefore the velocity should be added as a new freedom degree of an expanded space-time.

We shall now present the aforesaid in more rigorous form. Let us write an interval square used in the special relativity as

\[
(ds)^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 (dt)^2 .
\]

Turns in the pseudoplanes \((dx^a, dt)\), where \(a = 1, 2, 3\), preserve the interval square. The parameters of these turns are the velocity coordinates \(v^a = dx^a/dt\). In our previous work \([9]\) the following logical scheme was proposed

(i) to supplement the differentials \(dx^a\) and \(dt\) by the differentials of turn parameters \(dv^a\);

(ii) to construct a vector space on these differentials;

(iii) to introduce the interval square in the specified vector space;

(iv) to consider turns in the pseudoplanes \((dv^a, dt)\).

As a result, it is possible to generalize the special relativity to motions with variable velocity.

The geometric rotations in the planes \((dx^1, dx^2)\), \((dx^3, dx^1)\), \((dx^2, dx^3)\) preserve also the interval square. The parameters of these turns are angles \(\theta_1^1, \theta_3^1, \theta_2^2\), respectively. According to the above approach, a kinematic description of geometric rotations with angular velocity must result from a relation between turns angles in the planes \((d\theta_1^1, dt)\), \((d\theta_3^1, dt)\), \((d\theta_2^2, dt)\) and angular velocity components. Thus the geometrical angles \(\theta_1^1, \theta_3^1, \theta_2^2\) should be included in a list of freedom degrees of space.

The present paper is aimed at applying these ideas to geometric rotations and at constructing the kinematics for derivations of arbitrary order.

The rotation kinematics, including rules of composition of angular velocities and transformations of kinematic variables, is considered in Section 2. A generalization
of dynamic variables (energy, impulse, force, moment) is produced in Section 3 where a wave equation is also modified to describe rotary motion of light wave. Section 4 contains a special relativity generalization to the kinematics of arbitrary order. The conclusions are presented in Section 5.

2. Transformations for rotating reference frames

We supplement the differentials \( dx^a, dt \) by the angle differentials \( d\theta^2_1, d\theta^3_1, d\theta^3_2 \) to construct a vector space on them. Let us introduce an interval square

\[
(ds)^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dt)^2 + R^2(d\theta^2_1)^2 + R^2(d\theta^3_1)^2 + R^2(d\theta^3_2)^2
\]

in this vector space. Here the constant \( R \) adjusts the dimensionality of angles \( d\theta^b_a \) to the dimensionality of interval and has the dimensionality of length. We invoke the fundamental time \( T \) and the fundamental length \( L = cT \) introduced in [9]. Let the fundamental length be related to \( R \) by

\[
L = 2\pi RN,
\]

where \( N \) is the dimensionless parameter. If we divide the interval square by \( L^2 \), we obtain

\[
(d\sigma)^2 \equiv \frac{(ds)^2}{L^2} = -\frac{1}{L^2} \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + \frac{(dt)^2}{T^2} + \frac{1}{4\pi^2 N^2} \left[ (d\theta^2_1)^2 + (d\theta^3_1)^2 + (d\theta^3_2)^2 \right].
\]

Let us change variables

\[
x^a = \frac{x^a}{L}, \quad x^4 = \frac{t}{T}, \quad x^b = \frac{\theta^b_a}{2\pi N},
\]

where the indexes \( a \) and \( b \) numbering geometric coordinates take values 1, 2, 3. Then the interval square can be rewritten in the dimensionless form

\[
(d\sigma)^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 + (dx^1_1)^2 + (dx^3_1)^2 + (dx^3_2)^2.
\]

2.1. \( \Theta \Omega \)-motion of reference frame

We consider a body \( B \) in uniform rotation with respect to a frame \( K \) in the plane \( (dx^1, dx^2) \) about the angle \( \theta \) with the angular velocity \( \omega \). Let a frame \( K' \) rotate uniformly with respect to \( K \) in the same plane about the angle \( \theta' \) with the angular velocity \( \omega' \). The motion of body \( B \) will be characterized by the angular velocity \( \omega' \) and the angle \( \theta' \) with respect to \( K' \). In this case the dimensionless interval square is simplified

\[
(d\sigma)^2 = -(dx^1)^2 - (dx^2)^2 + (dx^4)^2 + (dx^1_1)^2.
\]

A turn in the space of differentials

\[
\|dx\| = U \|dx'\|
\]
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preserves the interval square. The turn of frame in the plane \((dx^1, dx^2)\) about the angle \(\theta\) is given by the matrix

\[
\Theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

The turn in the plane \((dx^4, dx^2_1)\) defined by the matrix

\[
\Omega = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\cos \xi & -\sin \xi \\
\sin \xi & \cos \xi
\end{bmatrix}
\]

corresponds uniform rotation of frame. The turns \(\Theta\) and \(\Omega\) are commutating to one another. The sequential realization of these turns gives the linear transformation

\begin{align*}
&dx^1 = \cos \theta (dx^1)' - \sin \theta (dx^2)', \\
&dx^2 = \sin \theta (dx^1)' + \cos \theta (dx^2)', \\
&dx^4 = \cos \xi (dx^4)' - \sin \xi (dx^2_1)', \\
&dx^2_1 = \sin \xi (dx^4)' + \cos \xi (dx^2_1)'.
\end{align*}

(2)

In order to find a relation between the angle \(\xi\) of turn matrix \(\Omega\) and the angular velocity of body \(B\), we consider the variations of coordinate differentials as functions of the turn angle variations:

\[
\delta ||dx|| = \delta U ||dx'||.
\]

Taking into account that

\[
||dx'|| = U^{-1} ||dx||,
\]

we get

\[
\delta ||dx|| = (\delta U U^{-1}) ||dx||.
\]

In our case \(U = \Theta \Omega = \Omega \Theta\) and

\[
\delta U U^{-1} = \delta \Theta \Theta^{-1} + \delta \Omega \Omega^{-1}.
\]

Using this matrix we obtain

\begin{align*}
\delta dx^1 &= -\delta \theta dx^2, \\
\delta dx^2 &= \delta \theta dx^1, \\
\delta dx^4 &= -\delta \xi dx^2_1, \\
\delta dx^2_1 &= \delta \xi dx^4.
\end{align*}

(3)

Consider the differential

\[
\frac{\delta dx^1_2}{dx^2} = \frac{\delta dx^2_2}{dx^4} = \frac{\delta dx^4_2}{(dx^4)^2}.
\]

From (3) follows

\[
\delta x^2_4 = \delta \xi \left[1 + (x^2_{14})^2\right],
\]

(4)

where the notation \(x^2_{14} = dx^2_1/dx^4\) was introduced.

The relations (3) and (4) determine the coordinate transformations and the rule of composition of angular velocities in the case of uniform rotation of reference frame.
2.1.1. Rule of composition of angular velocities As the rotations \( \Theta \) and \( \Omega \) are commutating, the rule of composition of angular velocities does not depend on turn angle of frame. From (4) one can obtain

\[
x_{14}^2 = \tan(\xi + \xi'),
\]

where \( \xi' \) is integration constant. Let us take the relations \( x_{14}^2 = 0 \) for \( \xi = 0 \), \( (x_{14}^2)' = 0 \) for \( \xi' = 0 \) as initial conditions. Then

\[
x_{14}^2 = \tan \xi
\]

and \( (x_{14}^2)' = \tan \xi' \). Thus we find the rule of composition of angular velocities

\[
x_{14}^2 = \frac{x_{14}^2 + (x_{14}^2)'}{1 - x_{14}^2 (x_{14}^2)'}.
\]

If we change for dimensional values in correspondence with

\[
x_{14}^2 = \frac{dx_1^2}{dx^4} = \frac{T}{2\pi N} \frac{d\theta^2}{dt} = \frac{\omega}{\Omega},
\]

where the notation

\[
\Omega = \frac{2\pi N}{T} = \frac{c}{R}
\]

was introduced, we obtain

\[
\omega = \frac{\omega + \omega'}{1 - \frac{\omega\omega'}{\Omega^2}}.
\]

As we see, the constant \( \Omega \) is a fundamental angular velocity like the light velocity in the rule of composition of velocities. The relation for composition of angular velocities has a singularity when

\[
\omega \omega' = \Omega^2.
\]

It is possible that the presence of this singularity has a profound physical meaning.

2.1.2. Transformations of differentials of coordinates From (3) follows

\[
\cos \xi = \frac{1}{\sqrt{1 + \frac{\omega^2}{\Omega^2}}}, \quad \sin \xi = \frac{\omega}{\Omega \sqrt{1 + \frac{\omega^2}{\Omega^2}}}.
\]

If we substitute the above expressions in (2) and change for dimensional values, we obtain the transformations of coordinate differentials:

\[
\begin{align*}
dx^1 &= \cos \theta (dx^1)' - \sin \theta (dx^2)', \\
dx^2 &= \sin \theta (dx^1)' + \cos \theta (dx^2)', \\
dt &= \frac{1}{\sqrt{1 + \frac{\omega^2}{\Omega^2}}} (dt)' - \frac{\omega}{\Omega \sqrt{1 + \frac{\omega^2}{\Omega^2}}} (d\theta)', \\
d\theta &= \frac{\omega}{\Omega \sqrt{1 + \frac{\omega^2}{\Omega^2}}} (dt)' + \frac{1}{\sqrt{1 + \frac{\omega^2}{\Omega^2}}} (d\theta)'.
\end{align*}
\]
3. Relativistic mechanics

The necessity of the correction of relativistic mechanics stems from the fact that the velocities $v^a$ and the angles $\theta^a_b$ of geometrical turns are introduced as additional coordinates being independent of geometrical coordinates and time. The angles $\theta^a_b$ can be identified with particle interior freedom degrees responsible for a particle spin. Let us construct the relativistic mechanics generalized to accelerated motions and uniform rotations by analogy to the relativistic mechanics to be invariant with respect to uniform velocity motions.

3.1. 10-velocity

The differentials included in the interval square
\[(ds)^2 = c^2(dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - T^2(du^1)^2 - T^2(du^2)^2 - T^2(du^3)^2 + R^2(d\theta^3_2)^2 + R^2(d\theta^1_3)^2 + R^2(d\theta^2_1)^2\]
can be considered as the vector coordinates in 10-dimensional space. We have for the contravariant coordinates
\[dx^\mu = \{c dt, dx^1, dx^2, dx^3, T du^1, T du^2, T du^3, R d\theta^3_2, R d\theta^1_3, R d\theta^2_1\}\]
and for the covariant coordinates
\[dx_\mu = \{c dt, -dx^1, -dx^2, -dx^3, -T du^1, -T du^2, -T du^3, R d\theta^3_2, R d\theta^1_3, R d\theta^2_1\}.

The interval square is rewritten in the new coordinates as
\[(ds)^2 = dx^\mu dx_\mu, \quad (\mu = 1, \ldots, 10)\]

Let us introduce a generalized Lorentz factor
\[\gamma = \left(1 - \frac{v^2}{c^2} - \frac{a^2}{A^2} + \frac{\omega^2}{\Omega^2}\right)^{-1/2}.

We express the interval as
\[ds = \frac{c dt}{\gamma}\]
and define a 10-velocity as
\[u^\mu \equiv \frac{dx^\mu}{ds} = \frac{\partial s}{\partial x_\mu} = \left\{\gamma, \gamma \frac{v^b}{c}, \gamma \frac{a^b}{A}, \gamma \frac{\omega^b}{\Omega}\right\}\]
in contravariant coordinates, and
\[u_\mu \equiv \frac{dx_\mu}{ds} = \frac{\partial s}{\partial x^\mu} = \left\{\gamma, -\gamma \frac{v^b}{c}, -\gamma \frac{a^b}{A}, -\gamma \frac{\omega^b}{\Omega}\right\}\]
in covariant coordinates. For this $v^b = v_b$, $a^b = a_b$, $\omega^b = \omega_b$, $v^b v_b = v^2$, $a^b a_b = a^2$, $\omega^b_a \omega^a_b = \omega^2$. It is obvious that
\[u^\mu u_\mu = 1.\]
3.2. Operation for free particle

Let us define an operation for free particle with mass and moment of inertia as

\[ S = -\frac{E_0}{c} \int_{s_1}^{s_2} ds, \]

where the integration is over a line in 10-space, \( s_1, s_2 \) are points of the specified line,

\[ E_0 = mc^2 = J \Omega^2 \]

is the rest energy of particle, and \( J = mR^2 \). The quantity \( J \) will be called proper moment of inertia of particle. The operation can be expressed as an integral over time

\[ S = -E_0 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2} + \frac{\omega^2}{\Omega^2}} dt. \]

3.3. Energy, impulse, force, moment

We define a 10-impulse as

\[ p^\mu \equiv -\frac{\partial S}{\partial x_\mu} = \frac{E_0}{c} u^\mu \quad \text{and} \quad p_\mu \equiv -\frac{\partial S}{\partial x^\mu} = \frac{E_0}{c} u_\mu \]

in contravariant and covariant coordinates, respectively. Let us introduce a relativistic energy

\[ E = \gamma mc^2, \]

a relativistic impulse

\[ p = \gamma mv, \]

a relativistic kinetic force

\[ f = \gamma ma, \]

and a relativistic moment

\[ l = \gamma J \omega. \]

According to our concept, the relativistic moment being independent of particle motion in geometrical space can be identified with the spin of particle.

Through quantities introduced the component of 10-impulse can be written as

\[ p^\mu = \left\{ \frac{E}{c}, p^b, T f^b, \frac{l^a}{R} \right\} \quad \text{and} \quad p_\mu = \left\{ \frac{E}{c}, -p_b, -T f_b, \frac{l^a}{R} \right\}. \]

From (7) follows

\[ p^\mu p_\mu = \left( \frac{E_0}{c} \right)^2. \]

This can be written as a relation between energy, impulse, force, moment and rest energy in the relativistic mechanics generalized to accelerated motions and uniform rotations:

\[ \frac{E^2}{c^2} - p^2 - f^2 T^2 + \frac{l^2}{R^2} = \left( \frac{E_0}{c} \right)^2. \]
For particles with zero rest energy, we have
\[
\frac{E^2}{c^2} - p^2 - f^2 T^2 + \frac{l^2}{R^2} = 0.
\] (8)

The transformation of components of 10-impulse can be described by the formalism similar to that for transformation of coordinate differentials. For example, in the case of $\Theta \Omega$-motion of frame, the transformations (8) imply
\[
p^1 = \cos \theta (p^1)' - \sin \theta (p^2)',
p^2 = \sin \theta (p^1)' + \cos \theta (p^2)',
E = \frac{1}{\sqrt{1 + \frac{\omega^2}{\Omega^2}}} E' - \frac{\omega}{\sqrt{1 + \frac{\omega^2}{\Omega^2}}} l',
l = \frac{\omega}{\Omega^2} \sqrt{1 + \frac{\omega^2}{\Omega^2}} E' + \frac{1}{\sqrt{1 + \frac{\omega^2}{\Omega^2}}} l'.
\]

3.4. Wave equation

A 10-dimensional derivative operator is given by
\[
\frac{\partial}{\partial x_\mu} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x_b}, -\frac{1}{T} \frac{\partial}{\partial v_b}, \frac{1}{R} \frac{\partial}{\partial \theta_a} \right\}
\]
for contravariant components and by
\[
\frac{\partial}{\partial x^\mu} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^b}, \frac{1}{T} \frac{\partial}{\partial v^b}, \frac{1}{R} \frac{\partial}{\partial \theta^a} \right\}
\]
for covariant components. Using the derivative operator one can write a wave equation for light
\[
\frac{\partial^2 \psi}{\partial x^\mu \partial x_\mu} \equiv \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{T^2} \frac{\partial^2 \psi}{\partial v^2} + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0.
\] (9)

This wave equation is different from the conventional one by two later addends allowing to describe accelerated light motion as well as uniform rotary light motion.

Let $\psi(t, x, v, \theta)$ be an arbitrary function describing the wave field. We shall try for the solution of wave equation (9) in the form
\[
\psi = \psi_0 \exp \left[ i \left( \kappa_b x^b + \xi_b v^b - \nu t - n_a \theta^a \right) \right],
\] (10)
where $\kappa_b$ are the coordinates of wave vector, $\xi_b$ are the coordinates of wave velocity vector, $\nu$ is the wave circular frequency. The quantities $n_a^b$ will be called coordinates of vector of phase multiplicity. The length of this vector $n = \sqrt{n_b^a n^b_a}$ will be called phase multiplicity. After substitution (10) in the wave equation, we get
\[
\frac{\nu^2}{c^2} - \kappa^2 - \frac{\xi^2}{T^2} + \frac{n^2}{R^2} = 0.
\] (11)

If we multiply this equation by the Planck constant square and compare the result to (8) within the framework of corpuscular-wave duality, we obtain a set of relations for particles with zero rest energy:
\[
E = \hbar \nu, \quad p = \hbar \kappa, \quad f = \frac{\hbar}{T^2} \xi, \quad l = \hbar n.
\]
The first two are de Broglie’s relations; the third one is the relation between the wave vector of velocity $\xi$ and the relativistic kinetic force; the last one represents the Bohr postulate generalized to particles with zero rest energy.

By analogy with the traditional definition of wave velocity

$$v^b \equiv \frac{\partial \nu}{\partial \kappa^b} = \frac{\kappa^b}{\nu} c^2$$

and the wave acceleration defined in \[9\] by

$$a^b \equiv \frac{\partial \nu}{\partial \xi^b} = \frac{\xi^b}{\nu} A^2,$$

we define a wave angular velocity

$$\omega^b_a \equiv \frac{\partial \nu}{\partial n^a_b} = \frac{n^b_a}{\nu} \Omega^2.$$

From \[11\] we obtain the relation

$$c^2 - v^2 - a^2 T^2 + \omega^2 R^2 = 0,$$

which will be called equation of light motion.

The wave equation generalizing the Klein-Gordon equation for massive particles to accelerated motion can be written as

$$\frac{\partial^2 \psi}{\partial x^\mu \partial x_\mu} + \frac{m^2 c^2}{\hbar^2} \psi = 0.$$

If the solution of this equation looks like the function \[10\], then

$$\frac{\nu^2}{c^2} - \frac{\kappa^2}{T^2} - \frac{\xi^2}{R^2} = \frac{m^2 c^2}{\hbar^2}.$$

(12)

Let us assume

$$\nu = \frac{m c^2}{\hbar} + \nu',$$

where $\nu' \ll \frac{m c^2}{\hbar}$. For frequency $\nu'$ the equation \[12\] is reduced to

$$\frac{2 m \nu'}{\hbar} - \frac{\kappa^2}{T^2} - \frac{\xi^2}{R^2} = 0.$$

(13)

By analogy to the traditional definition of wave packet group velocity

$$v^b_g \equiv \frac{\partial \nu'}{\partial \kappa^b} = \frac{\hbar}{m} \kappa^b$$

and the wave packet group acceleration defined in \[4\] by

$$a^b_g \equiv \frac{\partial \nu'}{\partial \xi^b} = \frac{\hbar}{m} \frac{\xi^b}{T^2},$$

we define a wave packet group angular velocity as

$$\langle \omega^b_a \rangle_g \equiv \frac{\partial \nu'}{\partial n^a_b} = \frac{\hbar}{m} \frac{n^b_a}{R^2}.$$

From \[13\] we obtain the expression

$$\hbar \nu' = \frac{m v^2_g}{2} + \frac{m T^2 a^2_g}{2} - \frac{m R^2 \omega^2_g}{2},$$

which is a generalization of the known de Broglie’s relation.
4. Special relativity generalization to kinematics of arbitrary order

4.1. Generalized space-time

In the previous Sections was shown that if turn angles in the planes \((dx^{i_1}, dx^{i_2})\) are considered as additional coordinates \(x^{i_1, i_2}\) to space-time coordinates \(x^{i_1}\), the relativistic kinematic for accelerated and rotary motions can be produced. In the space with such coordinates, the interval is represented by differentials \(dx^{i_1}, dx^{i_1, i_2}\), and the dimensionless interval square has the form

\[
(d\sigma)^2 = g_{i_1 k_1} dx^{i_1} dx^{k_1} + g_{i_1 i_2, k_1 k_2} dx^{i_1, i_2} dx^{k_1 k_2}.
\]

The complexity of the algebraic structure of space is determined by the maximal tensor degree of additional coordinates. It will be called kinematics order. To construct the kinematics of higher order, the above approach should be applied by a recurrence way.

In the general case, the kinematics of order \(n\) is considered in the space with coordinates \(x^{i_1, i_2, i_3, \ldots, i_k}\), where \(k = 1 \ldots n\). These coordinates are turn angles in the planes \((dx^{i_1, i_2, i_3, \ldots, i_m}, dx^{i_m, i_{m+1}, \ldots, i_k})\), where \(m = k/2, \ldots, k - 1\). The space itself will be called generalized space-time and will be denoted by \(X_N\), where \(N\) is its dimension. In the generalized space-time, the interval square is written as

\[
(d\sigma)^2 = g_{I K} dx^I dx^K.
\]

(14)

The interval square of this kind is a particular case of generalizations made in our book [10].

4.2. Structure equations of the generalized space-time

Let us consider a turn in the generalized space-time \(X_N\), i.e. a linear transformation

\[ ||dx|| = U ||dx'|| \]

preserving the interval square (\(\mathbb{4}\)). The turn matrix \(U\) depends on the turn angles \(x^\alpha\) which are the coordinates of \(X_{N+1}\). A set of turns is group. In other words, for this set a composition rule

\[
U = U_2 U_1
\]

(15)

can be introduced, an unity element can be defined, and an inverse element exists for any element from \(U\).

Consider the variations of coordinate differentials as functions of the turn angle variations:

\[
\delta ||dx|| = \delta U ||dx'|| = \Delta U ||dx||,
\]

(16)
where the notation $\Delta U = \delta U U^{-1}$ was introduced. These relations will be called
structure equations of the generalized space-time (for more details, see [10]). In the
expanded form they are written as
\[
\delta dx^I = \Delta U(x^\alpha)^I_K \, dx^K.
\]
In a special case the structure equations reduce to the equations (3) and
the equations (7), (12) from [9].

In the neighbourhood of the turn group unit, the structure equations take the form
\[
\delta dx^I = C^I_K \delta x^\alpha \, dx^K,
\]
where the notation
\[
C^I_K = \partial_\alpha U^I_K \Big|_{x^\alpha=0}
\]
is used. The values $C^I_K$ are called the structure constants. In particular, if the set of
turn angles is the same as the coordinate set of $X_N$, the structure equations is written as
\[
\delta dx^I = C^I_K \delta x^L \, dx^K.
\]
From the composition rule (13) it follows that
\[
\delta U = \delta U_2 U_1 + U_2 \delta U_1.
\]
If we multiply this expression on the right by $U^{-1} = (U_1)^{-1} (U_2)^{-1}$ and substitute the
result in (R), we obtain the structure equations for the turn angles involved in the
composition rule:
\[
\delta \|dx\| = \left[ \Delta U_2 + U_2 \Delta U_1 (U_2)^{-1} \right] \|dx\|.
\]

4.3. Relation between velocity of arbitrary order and generalized space-time coordinates

An arbitrary motion in the generalized space-time makes itself evident in the fact that
one coordinate set depends on other coordinate set. In the simplest case, for example, the
space coordinates are functions of time: $x^a = x^a(x^4)$.

Consider a set of arguments, $x^\alpha_1$, and a set of functions, $x^\alpha_2$, in the generalized
space-time $X_N$. In other words, it is proposed that functional relationships $x^\alpha_2(x^\alpha_1)$
exist. The differentials of these functions can be written as
\[
dx^\alpha_2 = V^\alpha_2_{\beta_1} \, dx^\beta_1.
\]
The quantity $V^\alpha_2_{\beta_1}$ will be called velocity of arbitrary order. We assume that
the remaining coordinates of $X_N$ are invariant. Then the interval is represented by the
differentials $dx^\alpha_2, dx^\alpha_1$, and the interval square has the form
\[
(d\sigma)^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta = g_{\alpha_2\beta_2} \, dx^\alpha_2 \, dx^\beta_2 + g_{\alpha_1\beta_1} \, dx^\alpha_1 \, dx^\beta_1.
\]
Introduce a turn in $X_N$
\[
dx^\alpha = U^\alpha_\beta (dx^\beta)'.
\]
We shall find a relation between velocity of arbitrary order and generalized space-time coordinates which are turn angles in the planes \((dx^\alpha, dx^\beta)\). For this purpose we consider the variations of coordinate differentials:

\[
\delta dx^\alpha = \Delta U^\alpha_\beta \, dx^\beta,
\]

where \(\Delta U^\alpha_\beta = (\delta U \, U^{-1})^\alpha_\beta\). This can be rewritten in the expanded form

\[
\delta dx^{\alpha_2} = \Delta U^{\alpha_2}_{\beta_2} \, dx^{\beta_2} + \Delta U^{\alpha_2}_{\beta_1} \, dx^{\beta_1},
\]

\[
\delta dx^{\alpha_1} = \Delta U^{\alpha_1}_{\beta_2} \, dx^{\beta_2} + \Delta U^{\alpha_1}_{\beta_1} \, dx^{\beta_1}.
\]

(18)

If we differentiate the equation (17) by the angles of turn matrix, we obtain

\[
\delta dx^{\alpha_2} = \delta V^{\alpha_2}_{\beta_1} \, dx^{\beta_1} + V^{\alpha_2}_{\beta_1} \delta dx^{\beta_1}.
\]

From this expression, (17), and (18) follows

\[
\delta V^{\alpha_2}_{\beta_1} = \Delta U^{\alpha_2}_{\beta_1} + \Delta U^{\alpha_2}_{\beta_2} V^{\beta_2}_{\beta_1} - V^{\alpha_2}_{\gamma_1} \Delta U^{\gamma_1}_{\beta_1} - V^{\alpha_2}_{\gamma_1} \Delta U^{\gamma_1}_{\beta_2} V^{\beta_2}_{\beta_1}.
\]

(19)

This differential equation establishes the desired relation between velocity of arbitrary order and generalized space-time coordinates.

In the case when turns are considered only in planes \((dx^{\alpha_2}, dx^{\beta_1})\),

\[
\Delta U^{\alpha_2}_{\beta_2} = 0, \quad \Delta U^{\alpha_1}_{\beta_1} = 0,
\]

and the differential equation obtained is simplified:

\[
\delta V^{\alpha_2}_{\beta_1} = \Delta U^{\alpha_2}_{\beta_1} - V^{\alpha_2}_{\gamma_1} \Delta U^{\gamma_1}_{\beta_1} - V^{\alpha_2}_{\gamma_1} \Delta U^{\gamma_1}_{\beta_2} V^{\beta_2}_{\beta_1}.
\]

(19)

The equation (14) and the equations (9), (13), (14) from [9] are the particular cases of the last differential relation.

In the case of 1-dimensional functional dependence \(x^\alpha = x^\alpha(x^\beta)\) for two arbitrary coordinates \(x^\alpha\) and \(x^\beta\), (19) reduces to

\[
\delta V^{\alpha}_{\beta} = \delta \psi \left[ 1 \pm (V^{\alpha}_{\beta})^2 \right],
\]

where \(\psi\) is turn angle in the plane \((dx^\beta, dx^\alpha)\), and the sign of \((V^{\alpha}_{\beta})^2\) is determined the signs of metric tensor components \(g_{\alpha\alpha}\) and \(g_{\beta\beta}\). Its solution is

\[
V^{\alpha}_{\beta} = \begin{cases} 
\tan \psi, & \text{for } g_{\alpha\alpha} = g_{\beta\beta}; \\
\tanh \psi, & \text{for } g_{\alpha\alpha} = -g_{\beta\beta}.
\end{cases}
\]

5. Conclusions

The expansion of the space-time to the generalized space-time is the consequence of uniform approach to the differentiation of kinematic variables. In order to increase the space dimension, the following recurrence rule is used: in the next step of generalization, additional coordinates of space-time are angles of turns in planes involving the coordinates which were introduced in the previous step. The Minkowskian space-time is used for starting the recurrence generalization procedure.

In context of the specified generalization, the new kinematic constants are introduced for application. For example, the only fundamental constant such as the
velocity of light $c$ is required in the first step, but the two additional constants such as the fundamental radius $R$ and time $T$ (or the fundamental angular velocity $\Omega$ and acceleration $A$) are required in the second step.

We propose that the motion of light (as particle or wave) is subject to the condition for the interval square in the generalized space-time

$$(d\sigma)^2 = 0.$$ 

The kinematic properties of light becomes more rich through the expansion of space-time. In the Einsteinian special relativity, the light is in linear uniform velocity motion with fundamental velocity $c$. For the second step of generalization, the light motion includes accelerated motion and proper rotation. This sequence can be continued. It is clear that the kinematics of higher order can describe not only motions of new kinds but any motions represented by the kinematics of the previous order as well.

The space generalization considered is naturally linked to a wide spectrum of theoretical conceptions like Kaluza-Klein theory where additional dimensions are invoked to describe different phenomena in the uniform context.

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