Deep Learning-Aided Trainable Projected Gradient Decoding for LDPC Codes

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Abstract—We present a novel optimization-based decoding algorithm for LDPC codes that is suitable for hardware architectures specialized to feed-forward neural networks. The algorithm is based on the projected gradient descent algorithm with a penalty function for solving a non-convex minimization problem. The proposed algorithm has several internal parameters such as step size parameters, a softness parameter, and the penalty coefficients. We use a standard tool set of deep learning, i.e., back propagation and stochastic gradient descent (SGD) type algorithms, to optimize these parameters. Several numerical experiments show that the proposed algorithm outperforms the BP decoding in some cases.

I. INTRODUCTION

Low-density parity-check (LDPC) codes have been adopted in numerous practical communication and storage systems, e.g., digital satellite broadcasting, wireless mobile communications, hard disks and flash memories, and the 5G standard as a key component to increase the reliability of the information exchange. Since the combination of LDPC codes and belief propagation (BP) decoding is capacity approaching (or achieving in some cases) with practical computation complexity, such wide use of the LDPC codes can be seen as a natural consequence. If we are allowed to use a code of long length, this trend would continue in future.

Recently, interest to machine to machine (M2M) communications is increasing in the context of internet of things (IoT). In M2M communications, latency of communications has critical importance in order to achieve harmonized real time operations of a number of machines. In 5G wireless system, ultra low latency (up to 1 millisecond) mode will be prepared for M2M communications. For error correction of such a system, a code with short code length, e.g., order of hundred, is preferable choice in order to reduce the latency. In such a situation, it is not clear whether BP decoding is the best candidate in terms of the decoding performance.

There have been several decoding algorithms that outperform the BP performance. Ordered Statistics Decoding (OSD) for LDPC codes is one of the most known algorithms in such a category. Based on the BP decoding, an OSD decoder iteratively produces candidate codewords by a re-encoding process. An OSD decoding process contains internal processes to find the $k$-largest values among $n$-candidates and Gaussian elimination. These operations are not suitable for hardware implementation and difficult to be parallelized.

Another stream of studies trying to find decoding algorithms surpassing the BP performance is the optimization-based decoding for LDPC codes. The origin of the optimization-based coding is the work by Feldman that presents a linear programming (LP) formulation of decoding for LDPC codes. Since the Feldman’s work, a number of works for optimization-based decoding have been presented in this decade. One notable work is the Alternative Direction Method of Multipliers (ADMM)-based decoding with the penalty function by Liu and Draper [7]. They presented that their decoder provides smaller bit error rate (BER) performances than those of BP decoding with reasonable computational complexity.

In this paper, we present a novel optimization-based decoding algorithm for LDPC codes that is suitable for hardware architectures specialized to feed-forward neural networks (NN). This is because the proposed algorithm mostly consists of matrix-vector products and coordinate-wise non-linear map operations. Based on recent interests in NN-oriented hardware architectures, it is expected that such an architecture will be included in future CODEC-chips. The algorithm is based on the projected gradient algorithm for solving the non-convex minimization problem based on the objective function that is a composition of a linear combination of received symbols and the penalty functions corresponding to the parity constraints. The decoding process consists of two steps, i.e., the gradient step and the projection step. The gradient step is the gradient descent process such that a search point moves to the direction of negative gradient of the objective function. The projection step is based on a soft projection operator for binary values.

Since the optimization problem we dealt is a non-convex problem, we cannot expect the convergence of a gradient descent type algorithm to the global minimum. A remarkable point of the proposed algorithm is that the initial point of a projected gradient process is randomly chosen. Since a trajectory of the search point depends on the initial point, the output of the algorithm may vary for each decoding trial, i.e., the output of the algorithm becomes a random variable. Executing the multiple trials with randomly initial points is a common technique in non-convex optimization to find the global minimum. We can expect that multiple decoding trials, called restarting, can improve the decoding performance.

The proposed algorithm has several internal parameters such as step size parameters for the gradient descent step, a softness parameter controlling the softness of the soft projection function, and penalty coefficients controlling the strength of the penalty term in the objective function. The appropriate choice of these parameters are of critical importance for the
algorithm to work properly. In this paper, we use a standard tool set of deep learning (DL), i.e., back propagation and stochastic gradient method (SGD) type algorithms, to optimize these parameters. By unfolding the signal-flow of the proposed algorithm, we can obtain a multilayer signal-flow graph that is similar to the multilayer neural network. Since all the internal processes are differentiable, we can adjust the internal trainable parameters via a training process based on DL techniques. This approach, data-driven tuning, is becoming a versatile technique especially for signal processing algorithms based on numerical optimization \[2\] \[3\].

II. PRELIMINARIES

A. Notation

In this paper, a vector \( z \in \mathbb{R}^d \) is regarded as a row vector of dimension \( d \). For \( z = (z_1, \ldots, z_n) \in \mathbb{R}^n \) and a scalar \( c \in \mathbb{R} \), we will use the following notation:

\[
c + z := (c + z_1, c + z_2, \ldots, c + z_n)
\]

for simplicity. For a real-valued function \( f : \mathbb{R} \to \mathbb{R} \), \( f(z) \) means the coordinate-wise application of \( f \) to \( z = (z_1, \ldots, z_n) \) such that \( f(z_1), f(z_2), \ldots, f(z_n) \). The \( i \)-th element of \( f(z) \), i.e., \( f(z_i) \) is also denoted by \( (f(z))_i \).

The set of consecutive integers from 1 to \( n \) is denoted by \([n] := \{1, 2, \ldots, n\}\). An \( n \)-dimensional unit cube is represented by

\[
[0,1]^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall i \in [n], 0 \leq x_i \leq 1\}.
\]

The cardinality or size of a finite set \( A \) is represented by \(|A|\). The indicator function \( \mathbb{I}[\text{cond}] \) takes the value 1 if the condition is true; otherwise it takes the value 0.

B. Channel model

Let \( H \) be an \( n \times m \) sparse parity check matrix over \( \mathbb{F}_2 \) where \( n > m \). The binary LDPC code defined by \( H \) is denoted by

\[
C(H) := \{x \in \mathbb{F}_2^n \mid Hx^T = 0\}.
\]

In the following discussion, we consider that 0 and 1 in \( \mathbb{F}_2 \) are embedded in \( \mathbb{R} \) as 0 and 1, respectively. The design rate of the code is defined by \( \rho := 1 - n/m \).

In this paper, we assume additive white Gaussian noise (AWGN) channels with binary phase shift keying (BPSK) signaling. A transmitter chooses a codeword \( c \in C(H) \) according to the message fed into the encoder. A binary to bipolar mapping is applied to \( c \) to generate a bipolar codeword

\[
x := 1 - 2c \in \mathbb{R}^n.
\]

The bipolar codeword \( x \) is sent to the AWGN channel and the receiver obtains a received word \( y = x + w \) where \( w \) is an \( n \)-dimensional Gaussian noise vector with mean zero and the variance \( \sigma^2/2 \). The signal-to-noise ratio \( SNR \) is defined by \( SNR := 10\log_{10}(1/(2\sigma^2\rho)) \) (dB). The log likelihood ratio vector corresponding to \( y \) is given by \( \lambda := 2y/\sigma^2 \). The decoder’s task is to estimate the transmitted word from a given received word \( y \) as correct as possible. The maximum likelihood (ML) estimation can be expressed by a non-convex optimization form:

\[
\hat{x} := \arg\min_{c \in C(H)} \|y - (1 - 2c)\|_2^2.
\]

It is hopeless to solve the problem naively and directly because of its computational complexity. We need to rely on an approximate algorithm to tackle the problem.

C. Fundamental polytope

Feldman \[8\] proposed a continuous relaxation of the ML rule \( \hat{x} \) based on the fundamental polytope. The fundamental polytope is a polytope in \( \mathbb{R}^n \) such that any codeword of \( C(H) \) is a vertex of the polytope. In other words, the feasible region of \( \hat{x} \), i.e., \( C(H) \), is relaxed to the fundamental polytope in the Feldman’s formulation. The fundamental polytope is defined by the simple box constraints and a set of linear inequalities derived from the parity check constraints. In the following, we will review the definition of the fundamental polytope.

Let \( A_i(i \in [m]) \) be an index set defined by

\[
A_i := \{j \in [n] \mid h_{i,j} = 1\}
\]

where \( h_{i,j} \) denotes the \((i,j)\)-element of \( H \). Let \( T_i \) be the family of subsets in \( A_i \) with odd size, i.e.,

\[
O_i := \{S \subset A_i \mid |S| \text{ is odd}\}.
\]

The parity polytope \( Q(H) \) defined based on the parity check matrix \( H \) is defined by

\[
Q(H) := \{x \in \mathbb{R}^n \mid x \text{ satisfies the parity constraint } (9)\},
\]

where the parity constraints are given by

\[
\forall i \in [m], \forall S \in O_i, 1 + \sum_{t \in S} x_t - 1 - \sum_{t \in A_i \setminus S} x_t \leq 0. \tag{9}
\]

A parity constraint defines a half-space and the intersection of the half-spaces induced by all the parity constraints is the parity polytope. These parity constraints introduced by Feldman \[8\] come from the convex hull of the single parity check codes.

The fundamental polytope \( \mathcal{F}(H) \) corresponding to \( H \) is the intersection of \( Q(H) \) and \( \{0,1\}^n \):

\[
\mathcal{F}(H) := Q(H) \cap \{0,1\}^n.
\]

Since the number of the parity constraints for a given \( i \in [m] \) is \( 2^{A_i} - 1 \), the total number of all the constraints becomes \( n + \sum_{i \in [m]} 2^{A_i} - 1 \). In the case of LDPC codes, the maximum size of the row weight, i.e., \( \max_{i \in [m]} |A_i| \) is constant to \( n \) and thus the total number of constraints is

\[
n + \sum_{i \in [m]} 2^{A_i} - 1 = n + \rho n O(1) = O(n). \tag{11}
\]

LP decoding \[8\] is based on the following minimization problem:

\[
\min_{x \in \mathbb{R}^n} \lambda x^T \text{ subject to } x \in \mathcal{F}(H), \tag{12}
\]
where $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Since the objective function and all the constraints are linear, this problem is an LP problem. The fundamental polytope includes vertices that are not contained in $C(H)$. This means that LP decoding may produce a non-integral (or fractional) solution. It is known that, if we have an integral solution, it coincides with the ML estimate. This property is called the **ML certificate property** of LP decoding.

### III. Trainable Projected Gradient Decoding

This section describes the proposed decoding algorithm in detail. Firstly, a basic idea is briefly explained. The following subsections are devoted to describe the details of the proposed algorithm.

#### A. Overview

We start from an unconstrained optimization problem closely related to the LP decoding [8]:

$$\text{minimize}_{x \in \{0, 1\}^n} \lambda x^T + \beta P(x), \quad (13)$$

where $P(x)$ is a penalty function satisfying $P(x) = 0$ if $x \in Q(H)$; otherwise $P(x) > 0$. The scalar parameter $\beta$ called the penalty coefficient that adjusts the strength of the penalty term. From the ML certificate property, it is clear the solution of (13) coincides with the ML estimate if $\beta$ is sufficiently large. Although the optimization problem in (13) is a non-convex problem, it can be a start point of a numerical optimization algorithm for solving (5).

Let

$$f_\beta(x) := \lambda x^T + \beta P(x), \quad (14)$$

which is our objective function to be minimized. We here use the projected gradient descent algorithm for solving (13) in an approximate manner. The projected gradient descent algorithm consists two steps, the **gradient step** and the **projection step**.

The gradient descent step moves the search point along the negative gradient vector of the objective function. The projection step moves the search point into a feasible region. The two steps are alternatively performed.

In the gradient step, a search point is updated in a gradient descent manner, i.e.,

$$r_t := s_t - \gamma_t \nabla f_\beta(s_t), \quad (15)$$

where $\nabla f_\beta(x)$ is the gradient of $f_\beta(x)$. The index $t$ represents the iteration index. A scalar $\gamma_t \in \mathbb{R}$ is the step size parameter. If the step size parameter is appropriate, a search point moves to a new point having a smaller value of the objective function. The parameter $\beta_t \in \mathbb{R}$ is an iteration-dependent penalty coefficient.

The projection step is given by

$$s_{t+1} := \xi(\alpha(r_t - 0.5)), \quad (16)$$

where $\xi$ is the sigmoid function defined by

$$\xi(x) := 1/(1 + \exp(-x)). \quad (17)$$

The parameter $\alpha$ controls the softness of the projection. Precisely speaking, the function $\xi$ is not the projection to the binary symbols $\{0, 1\}$. The projection step exploits soft-projection based on the shifted sigmoid function (See Fig. 1) because the true projection to discrete values results in insufficient convergence behavior in a minimization process.

The main process of the proposed decoding algorithm described later is the iterative process executing the gradient step and the projection step.

#### B. Penalty function and objective function

**The penalty function corresponding to the parity constraints** is defined by

$$P(x) := \frac{1}{2} \sum_{i \in [m]} \sum_{S \in O_i} \left[ \nu \left( 1 + \sum_{t \in S} (x_t - 1) - \sum_{t \in A_i \setminus S} x_t \right) \right]^2 \quad (18)$$

where the function $\nu$ is the ReLU function defined by $\nu(x) := \max\{0, x\}$. This penalty function is a standard penalty function corresponding to the parity polytope $Q(H)$ based on the quadratic penalty. From this definition of the penalty function $P(x)$, we immediately have $P(x) = 0$ if $x \in Q(H)$ and $P(x) > 0$ if $x \notin Q(H)$.

In the proposed decoding algorithm to be described later, the gradient of the penalty function is needed. The partial derivative of $P(x)$ with respect to the variable $x_k \ (k \in [n])$ is given by

$$\frac{\partial}{\partial x_k} P(x) = \sum_{i \in [m]} \sum_{S \in O_i} \nu \left( 1 + \sum_{t \in S} (x_t - 1) - \sum_{t \in A_i \setminus S} x_t \right) \times \left( \mathbb{I}[k \in S] - \mathbb{I}[k \in A_i \setminus S] \right). \quad (19)$$

As described before, the objective function to be minimized in a decoding process is given by

$$f_\beta(x) = \lambda x^T + \beta P(x). \quad (20)$$
The first term of the objective function prefer a point close to the received word. On the other hand, the second term prefer a point in the parity polytope. The partial derivative of the objective function with respect to the variable $x_k$ is thus given by

$$\frac{\partial}{\partial x_k} f_\beta(x) = \lambda_k + \beta \frac{\partial}{\partial x_k} P(x). \tag{21}$$

C. Concise representation of gradient vector

For the following argument, it is useful to introduce a concise representation of the gradient vector.

For the odd size family $O_i (i \in [m])$, we prepare a bijection $\phi_i : O_i \to [2^{A_i} - 1]$. For example, in the case of $A_i := \{1, 2, 3\}$, we have $O_i := \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. A possible choice of $\phi_i$ is as follows:

$$\phi_i(\{1\}) = 1, \quad \phi_i(\{2\}) = 2, \quad \phi_i(\{3\}) = 3, \quad \phi_i(\{1, 2, 3\}) = 4.$$

Let

$$L := \sum_{i=1}^{m} 2^{A_i - 1}, \tag{22}$$

which indicates the total number of parity constraints required to define $Q(H)$. The function $\ell(i, S) (i \in [m], S \subset O_i)$ defined by

$$\ell(i, S) := \phi_i(S) + \sum_{k=1}^{i-1} 2^{A_i - 1} \tag{23}$$

is a bijection from the set $\{(i, S) : i \in [m], S \subset O_i\}$ to $[L]$.

Let us see a simple example. Suppose that $H$ is given by

$$H = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}. \tag{24}$$

A set of bijections $\{\phi_i\}_{i \in [3]}$ defines the following $\ell(i, S)$:

\[
\begin{array}{c|c}
  i & S \mapsto \ell(i, S) \\
\hline
  1 & \{1\} \rightarrow 1 \\
  1 & \{2\} \rightarrow 2 \\
  1 & \{3\} \rightarrow 3 \\
  1 & \{1, 2, 3\} \rightarrow 4 \\
  2 & \{1, 2, 3\} \rightarrow 5 \\
  2 & \{4\} \rightarrow 6 \\
  3 & \{4\} \rightarrow 7 \\
  3 & \{5\} \rightarrow 8 \\
  3 & \{6\} \rightarrow 9 \\
  3 & \{4, 5, 6\} \rightarrow 10.
\end{array}
\]

The following matrices $Q$ and $R$ play a key role to derive a concise representation of the gradient vector. The matrix $Q \in \{0, 1\}^{n \times L}$ satisfies

$$Q_{i,j} = \begin{cases} 
1, & \text{if } i \in S \text{ and } j = \ell(i, S) \\
0, & \text{otherwise}
\end{cases} \text{ for any } i \in [m] \text{ and for any } S \subset O_i.$$

for any $i \in [m]$ and for any $S \subset O_i$. In a similar way, the matrix $R \in \{0, 1\}^{n \times L}$ satisfies

$$R_{i,j} = \begin{cases} 
1, & \text{if } i \in A_i \setminus S \text{ and } j = \ell(i, S) \\
0, & \text{otherwise}
\end{cases} \text{ for any } i \in [m] \text{ and for any } S \subset O_i.$$

for any $i \in [m]$ and for any $S \subset O_i$.

We can see that the column order of $Q$ and $R$ depends on the choice of the bijections $\{\phi_i\}_{i \in [m]}$, i.e., a different choice of $\{\phi_i\}_{i \in [m]}$ yields a column permuted version of $Q$ and $R$. However, in the following argument, the column order does not cause any influence for gradient computation. We thus can choose any set of bijections $\{\phi_i\}_{i \in [m]}$.

Suppose that $H$ is given by $[24]$. A set of bijections $\{\phi_i\}_{i \in [3]}$ is also given. From the definition of $Q$ and $R$, and $\ell(\cdot, \cdot)$, we have

$$Q = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}, \tag{25}$$

$$R = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}. \tag{26}$$

We are now ready to derive a concise expression of the gradient. By rewriting $[19]$ with the matrices $Q$ and $R$, we have

$$\nabla P(x) = \nu(1 + (x - 1)Q - xR)D^T, \tag{27}$$

where $D := Q - R$. By using this expression, the gradient vector $\nabla f_\beta(x)$ can be concisely rewritten by

$$\nabla f_\beta(x) = \lambda + \beta \nu (1 + (x - 1)Q - xR)D^T. \tag{28}$$

From this expression, the evaluation of the gradient vector is based on the evaluation of the matrix-vector products with sparse matrices $Q, R,$ and $D$. The computational complexity of the gradient vector is to be discussed in the next subsection.

Let us see a simple example of the $3 \times 6$ parity check matrix $[24]$. If $x$ is a codeword of $C(H)$, e.g., $x := (0, 1, 1, 1, 1, 0)$, the gradient of the penalty term $\nabla P(x)$ becomes the zero vector. This is because the value of the penalty function is constant (i.e., zero) in the parity polytope. Another example is that the center point of the parity polytope $x := (1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$ gives also the zero gradient. On the other hand, a non-codeword binary vector results in a non-zero gradient, e.g., $\nabla P(1, 1, 1, 1, 1, 0) = (1, 1, 1, 0, 0, 0)$. The gradient of the penalty term becomes non-zero for non-codeword binary vector. This property is useful for decoding processes because a search point repels non-codeword binary vectors in gradient descent processes.

D. Trainable Projected Gradient Decoding

The following decoding algorithm is based on the projected gradient descent algorithm described in the previous subsections.

Trainable Projected Gradient (TPG) Decoding

- Input: received word $y \in \mathbb{R}^n$
Output: estimated word \( \hat{c} \in C(H) \)

Parameters: \( t_{\text{max}} \): maximum number of the projected gradient descent iterations (inner loop), \( r_{\text{max}} \): maximum number of restarting (outer loop)

Step 1 (initialization for restarting) The restarting counter is initialized to \( r := 1 \).

Step 2 (random initialization) The initial vector \( s_1 \in \mathbb{R}^n \) is randomly initialized, i.e., each element in \( s_1 \) is chosen uniformly at random in \( \{ x \in \mathbb{R} | 0 \leq x \leq 1 \} \).

The iteration index is initialized to \( t := 1 \).

Step 3 (gradient step) Execute the gradient descent step

\[
r_t := s_t - \gamma_t (y + \beta_t \nu (1 + (s_t - 1)Q - s_t R) D^T).
\]  

(29)

Step 4 (projection step) Execute the projection step:

\[
s_{t+1} := \xi (\alpha (r_t - 0.5)).
\]  

(30)

Step 5 (parity check) Evaluate a tentative estimate \( \hat{c} := \theta(s_{t+1}) \) where the function \( \theta \) is the thresholding function defined by

\[
\theta(x) := \begin{cases} 
0, & x < 0.5, \\
1, & x \geq 0.5.
\end{cases}
\]  

(31)

If \( H \hat{c} = 0 \) holds, then output \( \hat{c} \) and exit.

Step 6 (end of inner loop) If \( t < t_{\text{max}} \) holds, then \( t := t + 1 \) and go to Step 3.

Step 7 (end of outer loop) If \( r < r_{\text{max}} \) holds, then \( r := r + 1 \) and go to Step 2; Otherwise, output \( \hat{c} \) and quit the process.

The trainable parameters \( \{ \gamma_t \}_{t=1}^{t_{\text{max}}} \) control the step size in the gradient descent step and \( \{ \beta_t \}_{t=1}^{t_{\text{max}}} \) defines relative strength of the penalty term. The trainable parameter \( \alpha \) controls the softness of the soft-projection. These parameters are adjusted in a training process described later. In the gradient step, we use the received word \( y \) instead of the log likelihood ratio vector \( \lambda \) since \( \lambda_i \propto y_i \) for \( i \in [n] \) under the assumption of the AWGN channel. The proportional constant can be considered to be involved in the step size parameter \( \gamma_t \). The parity check in Step 5 helps early termination that may reduce the expected number of decoding iterations.

The TPG decoding is a double loop algorithm. The inner loop (starting from Step 2 and ending at Step 6) is a projected gradient descent process such that a search point gradually approaches to a candidate codeword as the number of iterations grows. The outer loop (starting from Step 1 and ending at Step 7) is for executing multiple search processes with different initial point. The technique is called restarting. The initial search point of TPG decoding \( s_1 \) is randomly chosen in Step 2. In non-convex optimization, restarting with a random initial point is a basic technique to find a better sub-optimal solution.

The most time consuming operation in the TPG decoding is the gradient step (29). We here discuss the computational complexity of the gradient step. In order to simplify the argument, we assume an \((\ell, r)\)-regular LDPC code where \( \ell \) and \( r \) stands for the column weight and the row weight, respectively. In the following time complexity analysis, we will focus on the number of multiplications because it dominates the time complexity of the algorithm. Since the number of non-zero elements in \( Q \) and \( R \) is \( m2^{r-1} \), the number of required multiplications over real numbers for evaluating \((s_1 - 1)Q\) and \( s_1 R \) is \( m2^{r-1} \). On the other hand, the multiplication regarding \( D^T \) needs \( m2^{r} \) multiplications because the number of non-zero elements in \( Q - R \) is \( m2^{r} \). In summary, the computation complexity of the TPG decoding is \( O(m2^r) \) per iteration.

E. Training Process

As we saw in the previous subsection, TPG decoding contains several adjustable parameters. It is crucial to train and optimize these parameters appropriately for achieving reasonable decoding performance.

Let the set of trainable parameters be

\[
\Theta_t := \{ \alpha, \{ \beta_t \}_{t=1}^{t_{\text{max}}}, \{ \gamma_t \}_{t=1}^{t_{\text{max}}} \}(t \in [t_{\text{max}}]).
\]  

(32)

Based on a random initial point \( s_1 \) and \( \Theta_t \), we define the function \( g_{s_1}(y) := s_{t+1}(t \in [t_{\text{max}}]) \) where \( s_{t+1} \) is given by the recursion

\[
r_t := s_t - \gamma_t (y + \beta_t \nu (1 + (s_t - 1)Q - s_t R) D^T).
\]  

(33)

\[
s_{t+1} := \xi (\alpha (r_t - 0.5)).
\]  

(34)

In other words, \( g_{s_1}(y) \) represents the search point of a projected gradient descent process after \( t \) iterations. In the training process of \( \Theta_t \), we use mini-batch based training with a SGD-type parameter update.

Suppose that a mini-batch consists of \( K \)-triples:

\[
B := \{(c_1, y_1, s_{1,1}), (c_2, y_2, s_{1,2}), \ldots, (c_K, y_K, s_{1,K})\}
\]  

(35)

which is a randomly generated data set according to the channel model. The vector \( c_k \in C(H)[k \in [K]] \) is a randomly chosen codeword and \( y_k = (1 - 2c_k) + w_k \) is a corresponding received word where \( w_k \) is a Gaussian noise vector. The vector \( s_{1,k} \in \mathbb{R}^n[k \in [K]] \) is chosen from the \( n \)-dimensional unit cube uniformly at random, where these vectors are used as the random initial values.

We exploit a simple squared loss function given by

\[
h^t(\Theta_t) := \frac{1}{K} \sum_{k=1}^{K} ||c_k - g_{s_{1,k}}^t(y_k)||^2_2
\]  

(36)

for a mini-batch \( B \). A back propagation process evaluates the gradient \( \nabla h^t(\Theta_t) \) and it is used for updating the set of parameters as \( \Theta_t := \Theta_t + \Delta_{\Theta_t} \), where \( \Delta_{\Theta_t} \) is determined by a SGD type algorithm such as AdaDelta, RMSProp, or Adam. Note that a mini-batch of size \( K \) is randomly renewed for each parameter update. Figure 2 illustrates the signal-flow diagram of TPG decoding and the corresponding unfolded graph used for a training process.

In order to achieve better decoding performance and stable training processes, we exploit incremental training such that \( h^1(\Theta_1), h^2(\Theta_2), \ldots, h^{t_{\text{max}}}(\Theta_{t_{\text{max}}}) \) are sequentially minimized \( 3 \). The details of the incremental training is as follows. At first, \( \Theta_1 \) is trained by minimizing \( h^1(\Theta_1) \). After finishing the training of \( \Theta_1 \), the values of trainable parameters in \( \Theta_1 \)
are copied to the corresponding parameters in $\Theta_2$. In other words, the results of the training for $\Theta_1$ are taken over to $\Theta_2$ as the initial values. Then, $\Theta_2$ is trained by minimizing $h^2(\Theta_2)$. Such processes continue from $\Theta_1$ to $\Theta_{t,max}$. The number of iterations for training $\Theta_i$, which is referred to as a generation, is fixed to $J$ for all $i \in [t_{max}]$. In this work, the training process was implemented by PyTorch [6].

IV. EXPERIMENTAL RESULTS

In this section, we will show several experimental results indicating the behavior and the decoding performance of the TPG decoding.

A. Behavior of TPG decoding

We trained a TPG decoder with a $(3,6)$-regular LDPC code of $n = 204$, $m = 102$. Several hyper parameters assumed in the training process are as follows: The maximum number of iterations is set to $t_{max} = 25$. The number of parameter updates for a generation is $J = 500$ and the mini-batch size is set to $K = 50$. We employed Adam optimizer [5] with learning rate 0.005 for the parameter updates. In a training process, the SNR of channel is fixed to $SNR = 4.0$ dB.

Figure 3 indicates the result of the training, i.e., trained parameters $\{\gamma_i\}_{t=1}^{25}$ and $\{\beta_t\}_{t=1}^{25}$. At the first iteration, the step size parameter $\gamma_t$ takes the value around 1.2 and the value gradually decreases to the values around 0.2. On the other hand, the penalty term constant $\beta_t$ starts from the small value around 1 and increases to the values around 5.5 at the 9-th round. The softness parameter is a shared trainable variable for all the rounds takes the value $\alpha = 8.05$.

In a decoding process of TPG decoding, we expect that the search point approaches to the transmitted codeword. In order to observe the behavior of a TPG process based on the recursive formula (33) (34), we show the trajectories of the normalized squared error in Fig. 3. The normalized squared error is defined by $(1/n)||s_t - c^*||^2_2$ where $c^*$ is the transmitted codeword. Figure 3 includes the trajectories of 10 trials with random initial values. A received word $y := (1 - 2c^*) + w$ is fixed during the experiment. The code is the $(3,6)$-regular LDPC code with $n = 204$ and $m = 102$ and the trainable parameters $\{\gamma_t\}_{t=1}^{25}$ and $\{\beta_t\}_{t=1}^{25}$ are set to the values in Fig. 3 and $\alpha$ is set to 8.05 according to the above training result. The noise variance is corresponding to 4.0 (dB).

From Fig. 4 we can observe that each curve indicates rapid decrease of the normalized squared error (around 10 rounds for convergence) and it means that a search point $s_t$ actually approaches to the transmitted word in the recursive evaluation of (33) (34). With several iterations (5 to 15), the normalized squared error gets to the value around $10^{-4}$. This results implies that the penalty function representing the parity constraints are effective to direct the search point towards the transmitted word and that trained parameters provide intended behavior in minimization processes. Another observation obtained from Fig. 4 is that search point trajectories are different from each other and that are dependent on the initial value. The idea of restarting is based on the expectation that random initial values provide random outcomes. The experimental results support this expectation.

B. BER performances

Several hyper parameters assumed in the training process are as follows: The number of parameter updates for a generation is $J = 500$ and the mini-batch size is set to $K = 50$. We employed Adam optimizer [5] with learning rate 0.005 for the parameter updates. In a training process, the SNR of channel is fixed to $SNR = 4.0$ dB.

The decoding performances of TPG decoding for the rate $1/2$ $(3,6)$-regular LDPC code with $n = 204$ are shown in Fig. 5. Figure 5 includes the BER curves of TPG decoding with $r_{max} = 1, 10, 100$. As the baseline performance, the BER curve of the belief propagation (BP) decoding where the maximum number of iterations is set to 100. The BER performance of TPG decoding $(r_{max} = 1)$ is inferior to that of BP. On the other hand, we can observe that restarting significantly improves the decoding performance of the proposed.
algorithm. In the case of $r_{max} = 10$, the proposed algorithm shows around 0.2 dB gain over the BP at BER $= 10^{-5}$. In the case of $r_{max} = 100$, the proposed algorithm outperforms the BP and yields impressive improvement in BER performance. For example, it achieves 0.5 dB gain at BER $= 10^{-5}$. These results indicate that restarting works considerably well as we expected. This means that we can control trade-off between decoding complexity and the decoding performance in a flexible way.

Figure 4 shows the BER curves of TPG decoding for the rate $1/2$ (3,6)-regular LDPC code with $n = 504$. We can observe that the proposed algorithm again provides superior BER performance in the high SNR regime.

The average time complexity of the proposed decoding algorithm is closely related to the average number of iterations in the TPG decoding processes. Early stopping by the parity check (Step 5) reduces the number of iterations. The number of iterations means the number of execution of Step 3 (gradient step) for a given received word. The average number of iterations for (3,6)-regular LDPC code ($n = 204, m = 102$) are plotted in Figure 7. When SNR is 3.75 dB, the average number of iterations is around 30 for all the cases ($r_{max} = 1, 10, 100$).

V. CONCLUDING SUMMARY

In this paper, we present a novel decoding algorithm for LDPC codes, which is based on a non-convex optimization algorithm. The main processes in the proposed algorithm are the gradient and projection steps that have intrinsic massive parallelism that fits forthcoming deep neural network-oriented hardware architectures. Some of internal parameters can be optimized with data-driven training process with back propagation and a stochastic gradient type algorithm. Although we focus on the AWGN channels in this paper, we can apply TPG
decoding for other channels such as linear vector channels just by replacing the objective function.

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**REFERENCES**

[1] E. Nachmani, Y. Beery, and D. Burshtein, “Learning to decode linear codes using deep learning,” 2016 54th Annual Allerton Conf. Comm., Control, and Computing, 2016, pp. 341-346.

[2] K. Gregor and Y. LeCun, “Learning fast approximations of sparse coding,” in Proc. 27th Int. Conf. Machine Learning, pp. 399–406, 2010.

[3] D. Ito, S. Takabe, and T. Wadayama, “Trainable ISTA for sparse signal recovery,” IEEE Int. Conf. Comm., Workshop on Promises and Challenges of Machine Learning in Communication Networks, Kansas city, May. 2018. (arXiv:1801.01978)

[4] D. Divsalar, M. K. Simon, and D. Raphaeli, “Improved parallel interference cancellation for CDMA,” IEEE Trans. Commun., vol. 46, no. 2, pp. 258-268, Feb. 1998.

[5] D. P. Kingma and J. L. Ba, “Adam: A method for stochastic optimization,” [arXiv:1412.6980] 2014.

[6] PyTorch, [https://pytorch.org](https://pytorch.org)

[7] X. Liu and S. C. Draper, “The ADMM penalized decoder for LDPC codes,” IEEE Transactions on Information Theory, Vol. 62, Issue: 6, pp. 2966 - 2984, 2016.

[8] J. Feldman, “Decoding error-correcting codes via linear programming,” Massachusetts Institute of Technology, Ph. D. thesis, 2003.

[9] M. Fossorier and S. Lin, “Soft-decision decoding of linear block codes based on ordered statistics,” IEEE Transactions on Information Theory, Vol. 41, Issue:5, pp.1379 - 1396, 1995.