Mean-field backward stochastic differential equations with subdifferential operator and its applications *

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Abstract

In this paper, we deal with a class of mean-field backward stochastic differential equations with subdifferential operator corresponding to a lower semi-continuous convex function. By means of Yosida approximation, the existence and uniqueness of the solution is established. As an application, we give a probability interpretation for the viscosity solutions of a class of nonlocal parabolic variational inequalities.

Keywords: Mean-field backward stochastic differential equation; Subdifferential operator; Yosida approximation; McKean-Vlasov equation; Viscosity solution

MSC 60H10, 60G40, 60H30

1 Introduction

The general nonlinear case of backward stochastic differential equations (BSDEs, in short) was first introduced by Pardoux and Peng [24]. Since then, a lot of works have been devoted to the study of the theory of BSDEs as well as their applications. This is due to the connections of BSDEs with mathematical finance, stochastic optimal control as well as stochastic games and partial differential equations (PDEs, in short) (see e.g. [10], [11], [12], [22], [23], [25], [27], [28] and so on).

Among the BSDEs, El Karoui et al. [16] introduced a special class of reflected BSDEs, which is a BSDE but the solution \( Y \) is forced to stay above a given lower barrier. By means of this kinds of BSDEs, they provided a probabilistic formula for the viscosity solution of an obstacle problem for a class of parabolic PDEs. In addition, Pardoux and Răşcanu [18] proved the existence and uniqueness of the solutions of BSDEs, on a random (possibly infinite) time interval, involving a subdifferential operator (which are also called backward stochastic variational inequalities) in order to give a probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Its extension

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to the probabilistic interpretation of the viscosity solution of the parabolic variational inequality (PVI, in short) with a mixed nonlinear multivalued Neumann-Dirichlet boundary condition was recently given in Maticiuc and Răşcanu [20].

Mathematical mean-field approaches have been used in many fields, not only in physics and Chemistry, but also recently in economics, finance and game theory, see for example, Lasry and Lions [17] and the references therein.

On the other hand, McKean-Vlasov stochastic differential equation of the form
\[
dX_t = b(X_t, \mu_t)dt + dW_t, \quad t \in [0, T], \quad X_0 = x,
\]
where
\[
b(X_t, \mu_t) = \int \Omega b(X_t(\omega), X_t(\omega')) P(d\omega') = E[b(\xi, X_t)]|_{\xi=x_t},
\]
b : \R^n \times \R^n \to \R being a (locally) bounded Borel measurable function and \(\mu(t; \cdot)\) being the probability distribution of the unknown process \(X(t)\), was suggested by Kac [13] as a stochastic toy model for the Vlasov kinetic equation of plasma and the study of which was initiated by Mckean [21]. Since then, many authors made contributions on McKean-Vlasov type SDEs and their applications, see for example, Ahmed [1], Ahmed and Ding [2], Borkar and Kumar [3], Chan [7], Crisan and Xiong [9], Kotelenez [14], Kotelenez and Kurtz [15] and the references therein.

Recently, Buckdahn et al. [4] introduced a new kind of BSDEs called as mean-field BSDEs. Furthermore, Buckdahn et al. [5] deepened the investigation of mean-field BSDEs in a rather general setting. They proved the existence and uniqueness as well as a comparison principle of the solutions for mean-field BSDEs. Moreover, they established the existence and uniqueness of the viscosity solution for a class of nonlocal PDEs with the help of the mean-field BSDE and a McKean-Vlasov forward equation.

Since the works [4] and [5] on the mean-field BSDEs, there are many efforts devoted to its generalization. Shi et al. [29] introduced and studied mean-field backward stochastic Volterra integral equations. Xu [30] obtained the existence and uniqueness of solutions for mean-field backward doubly stochastic differential equations, and gave the probabilistic representation of the solutions for a class of stochastic partial differential equations by virtue of mean-field BDSDEs. Li and Luo [18] proved the existence and uniqueness for reflected mean-field BSDEs. Li [19] studied reflected mean-field BSDEs in a purely probabilistic method, and gave a probabilistic interpretation of the obstacle problems of the nonlinear and nonlocal PDEs by means of the reflected mean-field BSDEs.

Motivated by the above works, the present paper aims to deal with a class of mean-field BSDEs with subdifferential operator corresponding to a lower semi-continuous convex function with the following form
\[
\begin{cases}
-dY_t + \partial \phi(Y_t)dt \supseteq E'[f(t, Y_t', Z_t', Y_t, Z_t)]dt - Z_t dW_t, \quad 0 \leq t \leq T, \\
Y_T = \xi_t,
\end{cases}
\]
where \(\partial \phi\) is the subdifferential operator of a proper, convex and lower semicontinuous function \(\phi\). \(\xi_t\) is called as the terminal condition. More details refer to Section 2.

The first goal of this paper is to find a triple of adapted processes \((Y, Z, U)\) in an appropriate space such that mean-field BSDE (1.1) hold (see Definition [24]). Then, it allow us to establish the unique viscosity solution of the following nonlocal parabolic variational inequality
\[
\begin{cases}
\frac{\partial u(t, x)}{\partial t} + Au(t, x) + E[f(t, X_t^{0, x_0}, x, u(t, X_t^{0, x_0}), u(t, x), Du(t, x) \cdot E[\sigma(t, X_t^{0, x_0}, x)]]} \in \partial \phi(u(t, x)), \\
u(T, x) = E[h(X_T^{0, x_0}, x)], x \in \R^n
\end{cases}
\]

\(1.2\)
Let the assumption

\[ A(u(t,x)) := \frac{1}{2} tr (E[\sigma(t,X_{t}^{0,x},x)] E[\sigma(t,X_{t}^{0,x},x)]^T D^2 u(t,x)) + \langle E[b(t,X_{t}^{0,x},x)], Du(t,x) \rangle. \]

The paper is organized as follows. In Section 2, we introduce some notations, basic assumptions and preliminaries. Section 3 is devoted to the proof of the existence and uniqueness of the solution to the mean-field BSDEs with subdifferential operator by means of Yosida approximation. In Section 4, we give a probability interpretation for the viscosity solution of a class of nonlocal parabolic variational inequalities by means of the mean-field BSDEs with subdifferential operator.

2 Notations, preliminaries and basic assumptions

Let \( T > 0 \) be a fixed deterministic terminal time. Let \( \{W_t\}_{t \geq 0} \) be a \( d \)-dimensional standard Brownian motion defined on some complete probability space \((\Omega, \mathcal{F}, P)\). We denote by \( \mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\} \) the natural filtration generated by \( \{W_t\}_{0 \leq t \leq T} \) and augmented by all \( P \)-null sets, i.e.,

\[ \mathcal{F}_s = \sigma\{W_r, r \leq s\} \vee \mathcal{N}_P, s \in [0, T], \]

where \( \mathcal{N}_P \) is the set of all \( P \)-null subsets. For any \( n \geq 1 \), \(|z|\) denotes the Euclidean norm of \( z \in \mathbb{R}^n \).

In what follows, we need the following spaces.

- \( S^2_F([0,T];\mathbb{R}) \): the space of \( \mathbb{F} \)-adapted processes \( Y : \Omega \times [0,T] \to \mathbb{R} \) such that \( E\left[ \sup_{t \in [0,T]} |Y_t|^2 \right] < +\infty \);

- \( H^2_F(0,T;\mathbb{R}^n) \): the space of \( \mathbb{F} \)-progressively measurable processes \( \psi : \Omega \times [0,T] \to \mathbb{R}^n \) such that \( ||\psi||^2 := E \int_0^T |\psi_t|^2 dt < +\infty \).

Now given a measurable function \( f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) which satisfies that \( (f(t,y,z))_{t \in [0,T]} \) is \( \mathbb{F} \)-progressively measurable for all \( (y,z) \in \mathbb{R} \times \mathbb{R}^d \). We give the following assumption (H0):

(i) There exists some constant \( k > 0 \) such that for all \( t \in [0,T] \), \( y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d \), it holds that

\[ |f(\omega, t, y, z) - f(\omega, t, y', z')| \leq k \left( |y - y'| + |z - z'| \right), \quad dP \times dt - a.s. \]

(ii) \( E \int_0^T |f(s,0,0)|^2 ds < +\infty \).

The following result is an immediate consequence of Theorem 1.1 in Pardoux and Răşcanu [26].

**Theorem 2.1** Let the assumption (H0) be satisfied. Then, for the given terminal value \( \xi \) satisfying \( E[|\xi|^2 + \varphi(\xi)] < +\infty \), the BSDE with subdifferential operator

\[
\begin{cases}
-dY_t + \partial \varphi(Y_t) dt \geq f(t,Y_t,Z_t)dt - Z_t dW_t, \quad 0 \leq t \leq T, \\
Y_T = \xi,
\end{cases}
\]

has a unique solution \((Y,Z,U)\) satisfies that

(i) \((Y,Z,U) \in S^2_F(0,T;\mathbb{R}) \times H^2_F(0,T;\mathbb{R}^d) \times H^2_F(0,T;\mathbb{R})\).
Lemma 2.1

Under the assumptions (H1) and (H2), for any given $\xi \in L^2(\Omega; F_T; P; R)$, the mean-field BSDE (2.1) has a unique solution $(Y, Z) \in S^2_\mathbb{P}(0, T; R) \times H^2_\mathbb{P}(0, T; R^d)$. 

Remark 2.1

We emphasize that, due to our notations, the driving coefficient $f$ of (2.1) has to be interpreted as follows

$$E'[f(s, Y'_s, Z'_s, Y_s, Z_s)](\omega) = E[f(s, Y'_s, Z'_s, Y_s, Z_s)](\omega|\mathcal{F}_s) = \int_{\Omega} f(s, Y'_s(\omega), Z'_s(\omega), Y_s(\omega), Z_s(\omega))P(d\omega').$$
In Buckdahn et al. [5], the authors also presented the following comparison result.

**Lemma 2.2** Let \( f_i = f_i(\omega, \omega', t, y', z, y, z) \), \( i = 1, 2 \) be two drivers satisfying the assumptions (H1) and (H2). Moreover, we suppose that

(i) one of the two coefficients is independent of \( z' \);

(ii) one of the two coefficients is nondecreasing in \( y' \).

Let \( \xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \) and denote by \((Y^1, Z^1)\) and \((\xi_1, f_1)\) respectively. Then if \( \xi_1 \geq \xi_2, \mathbb{P}\text{-a.s.}, \) it holds that also \( Y_1 \geq Y_2, \ t \in [0, T], \mathbb{P}\text{-a.s.} \)

### 2.2 Mean-field BSDEs with subdifferential operator

In this subsection, we introduce some preliminaries of mean-field BSDEs with subdifferential operator.

Consider the mean-field BSDE as the form

\[
\begin{cases}
-dY_t + \partial \phi(Y_t) dt \ni E'[f(t, Y'_t, Z'_t, Y_t, Z_t)] dt - Z_t dW_t, & 0 \leq t \leq T, \\
Y_T = \xi,
\end{cases}
\]  

(2.2)

where \( \xi \) is the terminal value and satisfies that

(H3) \( E[|\xi|^2 + \phi(\xi)] < +\infty \).

Moreover, \( \partial \phi \) in mean-field BSDE (2.2) is the subdifferential operator of the function \( \phi : \mathbb{R} \rightarrow [0, +\infty] \) which satisfies the following assumptions:

(A1) \( \phi \) is a proper \( (\phi \neq +\infty) \), convex and lower semicontinuous function,

(A2) \( \phi(y) \geq \phi(0) = 0 \).

Let us define

\[
\text{Dom} \phi = \{u \in \mathbb{R} : \phi(u) < +\infty\}, \\
\partial \phi(u) = \{u^* \in \mathbb{R} : \langle u^*, v - u \rangle + \phi(u) \leq \phi(v), \forall v \in \mathbb{R}\}, \\
\text{Dom}(\partial \phi) = \{u \in \mathbb{R} : \partial \phi(u) \neq \emptyset\}, \\
(u, u^*) \in \partial \phi \iff u \in \text{Dom}(\partial \phi), u^* \in \partial \phi(u).
\]

**Remark 2.2** It is well known that the subdifferential operator \( \partial \phi \) is a maximal monotone operator, i.e., is maximal in the class of operators which satisfy the condition

\[
\langle u^* - v^*, u - v \rangle \geq 0, \forall (u, u^*), (v, v^*) \in \partial \phi.
\]

We end this section by introduce the definition of the solution for the mean-filed BSDE (2.2).

**Definition 2.1** The triple \((Y, Z, U)\) is called as the solution of mean-filed BSDE (2.2) with subdifferential operator if
(i) \((Y,Z,U) \in S_2^F(0,T;R) \times H_2^F(0,T;R^d) \times H_2^F(0,T;R)\).

(ii) \(E \int_0^T \varphi(Y_t)dt < +\infty\).

(iii) \((Y_t,U_t) \in \partial \varphi, dP \times dt\text{-a.e. on } \Omega \times [0,T]\).

(iv) \(Y_t + \int_t^T U_sds = \xi + \int_t^T E'[f(s,Y'_s,Z'_s,Y_s,Z_s)]ds - \int_t^T Z_s dW_s, 0 \leq t \leq T\).

3 Existence and uniqueness of the solution

This section is devoted to prove the existence and uniqueness of the solution for (2.2). Firstly, let us propose the main result of this section.

**Theorem 3.1** Assume that the assumptions (H1)--(H3) hold. Then there exists a unique solution for the mean-field BSDE (2.2).

We mention that our proof is based on the Yosida approximations. For this purpose, let’s introduce an approximation of the function \(\varphi\) by a convex \(C^1\)-function \(\varphi_\varepsilon, \varepsilon > 0\), defined by

\[
\varphi_\varepsilon(u) = \inf \left\{ \frac{1}{2\varepsilon} |u-v|^2 + \varphi(v) : v \in R \right\} = \frac{1}{2\varepsilon} |u - J_\varepsilon u|^2 + \varphi(J_\varepsilon u),
\]

where \(J_\varepsilon u = (I + \varepsilon \partial \varphi)^{-1}(u)\) is called the resolvent of the monotone operator of \(\partial \varphi\). For reader’s convenience, we illustrate some properties of this approximation, one can see Brezis \([6]\) for more details.

**Proposition 3.1** For all \(\varepsilon, \delta > 0, u,v \in R\), it holds that

(i) \(\varphi_\varepsilon\) is a convex function with the gradient being a Lipschitz function;

(ii) \(\varphi_\varepsilon(u) \leq \varphi(u)\);

(iii) \(\nabla \varphi_\varepsilon(u) = \varphi_\varepsilon(u) = \frac{u - J_\varepsilon u}{\varepsilon} \in \partial \varphi(J_\varepsilon u)\);

(iv) \(|J_\varepsilon(u) - J_\varepsilon(v)| \leq |u - v|\);

(v) \(0 \leq \varphi_\varepsilon(u) \leq \langle \nabla \varphi_\varepsilon(u), u \rangle\);

(vi) \(\langle \nabla \varphi_\varepsilon(u) - \nabla \varphi_\varepsilon(v), u - v \rangle \geq -(\varepsilon + \delta) \langle \nabla \varphi_\varepsilon(u), \nabla \varphi_\varepsilon(v) \rangle\).

Since our method is based on the Yosida approximations, let us consider the following mean-field BSDE

\[
Y^\varepsilon_t + \int_t^T \nabla \varphi_\varepsilon(Y^\varepsilon_s)dt = \xi + \int_t^T E'[f(s,Y'_s,Z'_s,Y'_s,Z'_s)]dt - \int_t^T Z^\varepsilon_s dW_s, 0 \leq t \leq T.
\]

Since \(\nabla \varphi_\varepsilon\) is Lipschitz continuous, it is known from a recent result of Buchdahn et al. \([5]\) that the mean-field BSDE (3.2) has a unique solution \((Y^\varepsilon,Z^\varepsilon) \in S_2^F(0,T;R) \times H_2^F(0,T;R^d)\).
Setting
\[ U_t^e = \nabla \varphi_e(Y_t^e), \quad 0 \leq t \leq T, \]
we shall prove the convergence of the sequence \((Y^e, U^e, Z^e)\) to a process \((Y, U, Z)\), which is the desired solution of the mean-field BSDE (3.2).

Firstly, we establish some properties of the solution of mean-field BSDE (3.2). In what follows,\( C > 0 \) denotes a constant whose value may change from line to line.

**Lemma 3.1** Assume that the assumptions (H1)–(H3) hold. Then there exist two positive constants \( \lambda \) and \( C \) such that
\[
E \left[ \sup_{t \in [0,T]} e^{\lambda t} |Y_t^e|^2 + \int_0^T e^{\lambda s} |Z_s^e|^2 \, ds \right] \leq CM_1, \tag{3.3}
\]
where \( M_1 := E[|\xi|^2] + \tilde{E} \int_0^T e^{\lambda s} |f(s, 0, 0, 0)|^2 \, ds. \)

**Proof.** Itô’s formula yields that
\[
e^{\lambda t} |Y_t^e|^2 + \int_t^T e^{\lambda s} (\lambda |Y_s^e|^2 + |Z_s^e|^2) \, ds + 2 \int_t^T e^{\lambda s} \langle \nabla \varphi_e(Y_s^e), Y_s^e \rangle \, ds
\]
\[
= e^{\lambda T} |\xi|^2 + 2 \int_t^T e^{\lambda s} \langle Y_s^e, E'[f(s, Y_s^e, Z_s^e, Y_e^e, Z_e^e)] \rangle \, ds
\]
\[
- 2 \int_t^T e^{\lambda s} \langle Y_s^e, Z_s^e dW_s \rangle. \tag{3.4}
\]

By Young’s inequality and (H1), we have, for \( \gamma > 0 \)
\[
2 \langle Y_s^e, E'[f(s, Y_s^e, Z_s^e, Y_e^e, Z_e^e)] \rangle \leq \gamma |Y_s^e|^2 + \frac{1}{\gamma} |E'[f(s, Y_s^e, Z_s^e, Y_e^e, Z_e^e)]|^2
\]
\[
\leq \gamma |Y_s^e|^2 + \frac{8k^2}{\gamma} \left[ E'|Y_s^e|^2 + E'|Z_s^e|^2 \right]
\]
\[
+ \frac{8k^2}{\gamma} \left[ |Y_s^e|^2 + |Z_s^e|^2 \right] + \frac{2}{\gamma} E'|f(s, 0, 0, 0)|^2. \tag{3.5}
\]

Since \( \langle \nabla \varphi_e(y), y \rangle \geq 0 \), and hence,
\[
e^{\lambda t} |Y_t^e|^2 + \left( \lambda - \gamma - \frac{8k^2}{\gamma} \right) \int_t^T e^{\lambda s} |Y_s^e|^2 \, ds + \left( 1 - \frac{8k^2}{\gamma} \right) \int_t^T e^{\lambda s} |Z_s^e|^2 \, ds
\]
\[
\leq e^{\lambda T} |\xi|^2 + \frac{8k^2}{\gamma} \int_t^T e^{\lambda s} \langle E'|Y_s^e|^2 + E'|Z_s^e|^2 \rangle \, ds
\]
\[
+ \frac{2}{\gamma} \int_t^T e^{\lambda s} E'|f(s, 0, 0, 0)|^2 \, ds - 2 \int_t^T e^{\lambda s} \langle Y_s^e, Z_s^e dW_s \rangle. \tag{3.6}
\]

Choosing \( \gamma = 16k^2 \) and \( \lambda > \gamma + \frac{16k^2}{\gamma} \), then there exists a constant \( C > 0 \), depending on \( \lambda, \gamma \) and \( k \), such that
\[
E \left[ \int_0^T e^{\lambda s} |Y_s^e|^2 \, ds + \int_0^T e^{\lambda s} |Z_s^e|^2 \, ds \right] \leq C \left[ E e^{\lambda T} |\xi|^2 + \tilde{E} \int_0^T e^{\lambda s} |f(s, 0, 0, 0)|^2 \, ds \right]. \tag{3.7}
\]
On the other hand, combining (3.6) and (3.7), we get
\[
\sup_{t \in [0,T]} e^{\lambda t} |Y^t|^2 \leq C \left[ e^{\lambda T} |\xi|^2 + \int_0^T e^{\lambda s} |f(s,0,0,0)|^2 ds \
+ E e^{\lambda T} |\xi|^2 + \bar{E} \int_0^T e^{\lambda s} |f(s,0,0,0)|^2 ds \right] \\
+ 2 \sup_{t \in [0,T]} \left| \int_t^T e^{\lambda s} \langle Y^s, Z_s^t dW_s \rangle \right|. \tag{3.8}
\]

Thus, from Burkholder-Davis-Gundy's inequality, we have
\[
E \left[ \sup_{t \in [0,T]} e^{\lambda t} |Y^t|^2 \right] \leq C \left[ E e^{\lambda T} |\xi|^2 + \bar{E} \int_0^T e^{\lambda s} |f(s,0,0,0)|^2 ds \right] \\
+ \frac{1}{2} E \left[ \sup_{t \in [0,T]} e^{2\lambda t} |Y^t|^2 \right] + CE \int_0^T e^{\lambda s} |Z^s|^2 ds. \tag{3.9}
\]

We then complete the proof by (3.7).

**Lemma 3.2** Assume that the assumptions (H1)--(H3) hold. Then there exists a constant \( C > 0 \) such that for all \( t \in [0,T] \),

(i) \( E \int_0^T e^{\lambda s} |\nabla \varphi_e(Y^s_t)|^2 ds \leq CM_2 \),

(ii) \( E e^{\lambda s} \varphi(J_e(Y^s_t)) + E \int_t^T e^{\lambda s} \varphi(J_e(Y^s)) ds \leq CM_2 \),

(iii) \( E (e^{\lambda t} |Y^s_t - J_e Y^s_t|^2) \leq CE M_2 \),

where \( M_2 := E[e^{\lambda T} \varphi(\xi) + e^{\lambda T} |\xi|^2] + E \int_0^T e^{\lambda s} |f(s,0,0,0)|^2 ds \).

**Proof.** The stochastic subdifferential inequality in Pardoux and Răşcanu [26] gives that
\[
e^{\lambda T} \varphi_e(\xi) \geq e^{\lambda s} \varphi_e(Y^s_t) + \int_t^T e^{\lambda s} \langle \nabla \varphi_e(Y^s), dY^s \rangle + \int_t^T \varphi_e(Y^s_t) d(e^{\lambda s}),
\]

and hence
\[
e^{\lambda T} \varphi_e(Y^s_t) + \lambda \int_t^T e^{\lambda s} \varphi_e(Y^s) ds + \int_t^T e^{\lambda s} |\nabla \varphi_e(Y^s)|^2 ds \\
\leq e^{\lambda T} \varphi_e(\xi) + \int_t^T e^{\lambda s} \langle \nabla \varphi_e(Y^s), E'[f(s,Y^s_t,Y^s_t',Z^s_t)] \rangle ds \\
- \int_t^T e^{\lambda s} \langle \nabla \varphi_e(Y^s), Z^s_t dW_s \rangle. \tag{3.10}
\]

By Young's inequality and (H2), we have
\[
\int_0^T e^{\lambda s} \langle \nabla \varphi_e(Y^s), E'[f(s,Y^s_t,Y^s_t',Z^s_t)] \rangle ds \\
\leq \frac{1}{2} \int_0^T e^{\lambda s} |\nabla \varphi_e(Y^s)|^2 ds + \int_0^T e^{\lambda s} E'[f(s,0,0,0)]^2 ds \\
+ 4k^2 \int_0^T e^{\lambda s} |Y^s|^2 + E'[Y^s_t']^2 + |Z^s_t|^2 + E'[Z^s_t']^2 ds. \tag{3.11}
\]

8
This together with (3.10) yields
\[
\frac{1}{2} E \int_0^T e^{\lambda s} |\nabla \phi_{\varepsilon} (Y^\varepsilon_s) |^2 ds \leq e^{\lambda T} \phi_{\varepsilon} (\xi) + \bar{E} \int_0^T e^{\lambda s} |f(\varepsilon,0,0,0)|^2 ds + 4k^2 E \int_0^T e^{\lambda s} (|Y^\varepsilon_s|^2 + E'|Y^{\varepsilon'}_s|^2 + |Z^\varepsilon_s|^2 + E'|Z^{\varepsilon'}_s|^2) ds. \quad (3.12)
\]
Thus, (i) is hold from Lemma 3.1 and the fact that \( \phi_{\varepsilon} (u) \leq \phi(u) \).

On the other hand, combining (3.10) and (3.11), we get from Lemma 3.1 that
\[
E e^{\lambda s} \phi_{\varepsilon} (Y^\varepsilon_s) + E \int_0^T e^{\lambda s} \phi_{\varepsilon} (Y^\varepsilon_s) ds \leq CM_2.
\]
Since \( \phi(J_{\varepsilon} (y)) \leq \phi_{\varepsilon} (y) \), it follows that
\[
E e^{\lambda s} \phi (J_{\varepsilon} (Y^\varepsilon_s)) + E \int_0^T e^{\lambda s} \phi (J_{\varepsilon} (Y^\varepsilon_s)) ds \leq CM_2.
\]
For (iii), since
\[
\frac{1}{2} E e^{\beta t} |Y^\varepsilon_t - J_{\varepsilon} (Y^\varepsilon_t)|^2 \leq e^{\beta t} \phi_{\varepsilon} (Y^\varepsilon_t),
\]
it follows from (3.13) that
\[
E \left[ e^{\beta t} |Y^\varepsilon_t - J_{\varepsilon} (Y^\varepsilon_t)|^2 \right] \leq CeM_2.
\]

The proof is complete.

**Lemma 3.3** Assume that the assumptions (H1)–(H3) hold. Then
\[
E \left[ \sup_{t \in [0,T]} e^{\lambda t} |Y^\varepsilon_t - Y^{\delta}_t|^2 \right] + E \int_0^T e^{\lambda s} (|Y^\varepsilon_s - Y^\delta_s|^2 + |Z^\varepsilon_s - Z^\delta_s|^2) ds \leq C(\varepsilon + \delta)M_2.
\]

**Proof.** By Itô’s formula, we have
\[
e^{\lambda s} |Y^\varepsilon_s - Y^\delta_s|^2 + \lambda \int_t^T e^{\lambda r} |Y^\varepsilon_r - Y^\delta_r|^2 dr + \int_t^T e^{\lambda r} |Z^\varepsilon_r - Z^\delta_r|^2 dr
\]
\[
+ 2 \int_t^T e^{\lambda r} \langle Y^\varepsilon_r - Y^\delta_r, \nabla \phi_{\varepsilon} (Y^\varepsilon_r) - \nabla \phi_{\delta} (Y^\delta_r) \rangle dr
\]
\[
= 2 \int_t^T e^{\lambda r} \langle Y^\varepsilon_r - Y^\delta_r, E' [f(s,Y^{\varepsilon'}_s,Z^{\varepsilon'}_s,Y^\varepsilon_s,Z^\varepsilon_s) - f(s,Y^\delta_s,Z^\delta_s,Y^\delta_s,Z^\delta_s)]dr
\]
\[
- 2 \int_t^T e^{\lambda r} \langle Y^\varepsilon_r - Y^\delta_r, (Z^\varepsilon_r - Z^\delta_r) dW_s \rangle. \quad (3.14)
\]
Using Young’s inequality and (H2), we get for \( \gamma > 0 \)
\[
2 \int_t^T e^{\lambda r} \langle Y^\varepsilon_r - Y^\delta_r, E' [f(s,Y^{\varepsilon'}_s,Z^{\varepsilon'}_s,Y^\varepsilon_s,Z^\varepsilon_s) - f(s,Y^\delta_s,Z^\delta_s,Y^\delta_s,Z^\delta_s)]dr
\]
\[
\leq \gamma \int_t^T e^{\lambda r} |Y^\varepsilon_r - Y^\delta_r|^2 ds + \frac{4k^2}{\gamma} \int_t^T e^{\lambda r} (|Y^\varepsilon_r - Y^\delta_r|^2 + E'|Y^{\varepsilon'}_r - Y^\delta_r|^2
\]
\[
+ |Z^\varepsilon_r - Z^\delta_r|^2 + E'|Z^{\varepsilon'}_r - Y^\delta_r|^2) ds. \quad (3.15)
\]
Consequently, Lemma 3.2 implies that combining (3.14) and (3.15), by the same procedure as the proof of Lemma 3.1, there exists a constant $C > 0$ such that

$$E \int_0^T e^{\lambda s}(\vert Y^e_s - Y^\delta_s \vert^2 + \vert Z^e_s - Z^\delta_s \vert^2)ds \leq C(\varepsilon + \delta)M_2, \tag{3.16}$$

and consequently, we can conclude the proof by Burkholder–Davis–Gundy’s inequality and (3.16).

Proof. Existence. From Lemma 3.3 we can deduce that there exist stochastic processes $Y \in S^2(0, T; R)$ and $Z \in H^2(0, T; R^d)$ such that

$$\lim_{\varepsilon \to 0}(Y^\varepsilon, Z^\varepsilon) = (Y, Z).$$

Consequently, Lemma 3.2 implies that

$$\lim_{\varepsilon \to 0} J_\varepsilon(Y^\varepsilon) = Y \text{ in } H^2(0, T; R)$$

and

$$\lim_{\varepsilon \to 0} E[|e^{\beta t}J_\varepsilon(Y^\varepsilon) - Y_t|^2] = 0, \quad 0 \leq t \leq T.$$

Moreover, Fatou’s lemma, (ii) of Lemma 3.2 Proposition 3.1 and the lower semicontinuity of $\varphi$ shows that (ii) of Definition 2.1 is satisfied.

On the other hand, (i) of Lemma 3.2 shows that $U^\varepsilon := \nabla \varphi(Y^\varepsilon_t)$ are bounded in the space $H^2(0, T; R)$, so there exists a subsequence $\varepsilon_n \to 0$ such that

$$U^{\varepsilon_n} \to U, \text{ weakly in } H^2(0, T; R).$$

Furthermore, we have

$$E \int_0^T |U_t|^2ds \leq \liminf_{n \to \infty} E \int_0^T |U^{\varepsilon_n}_t|^2ds \leq CM_2.$$

In virtue of (H1), by passing limit in mean-filed BSDE (3.2), we deduce that the triple $(Y, Z, U)$ satisfies (iv) of Definition 2.1.

Finally, since $U^\varepsilon_t \in \partial \varphi(J^\varepsilon(Y^\varepsilon_t))$, $t \in [0, T]$, it follows that, for all $V \in H^2(0, T; R),

$$e^{\beta t}\langle U^\varepsilon_t, V_t - J^\varepsilon(Y^\varepsilon_t) \rangle + e^{\beta t}\varphi(J^\varepsilon(Y^\varepsilon_t)) \leq e^{\beta t}\varphi(V_t), \quad dP \times dt \text{ a.e.}$$

Taking the liminf in the probability in the above inequality, then (iii) of Definition 2.1 holds.

Uniqueness. Let $(Y^i_t, Z^i_t, U^i_t), \ i = 1, 2$ be two solutions of mean-filed BSDE (3.2). We denote by

$$(\Delta Y_t, \Delta Z_t, \Delta U_t) := (Y^1_t - Y^2_t, Z^1_t - Z^2_t, U^1_t - U^2_t).$$

Applying Itô’s formula to $e^{\beta t}|\Delta Y_t|^2$ yields that

$$e^{\beta t}|\Delta Y_t|^2 + \int_t^T e^{\beta s}(\beta |\Delta Y_s|^2 + |\Delta Z_s|^2)ds + 2\int_t^T e^{\beta s}\langle \Delta Y_s, \Delta U_s \rangle ds$$

$$= 2\int_t^T e^{\beta s}\langle \Delta Y_s, E[f(s, Y^1_s, Z^1_s, Y^1_t, Z^1_t) - f(s, Y^2_s, Z^2_s, Y^2_t, Z^2_t)] \rangle ds$$

$$- 2\int_t^T e^{\beta s}\langle \Delta Y_s, \Delta Z_s dW_s \rangle.$$
Since

\[ \langle \Delta Y_s, \Delta U_s \rangle \geq 0, \quad dP \times dt \text{ a.e.} \]

Thus, as the same procedure as the proof of Lemma 3.1 we can derive the uniqueness of the solution. The proof is complete.

## 4 Viscosity solution of a nonlocal parabolic variation inequality

In this part, we will give a probability interpretation for the viscosity solutions of nonlocal parabolic variational inequalities via mean-field BSDEs with subdifferential operator studied before.

Let us consider the following McKean-Vlasov SDE parameterized by the initial condition \((t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)\):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\mathrm{d}X^t_{s, \zeta} = E'[b(s, (X^0_{s,x})', X^t_{s,\zeta})] \mathrm{d}s + E'[\sigma(s, (X^0_{s,x})', X^t_{s,\zeta})] \mathrm{d}W_s, s \in [t, T], \\
X^t_{t, \zeta} = \zeta,
\end{array} \right. \\
&\text{where } b : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \text{ and } \sigma : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times d} \text{ are two measurable functions satisfying the following assumptions:}
\end{aligned}
\]

(H4) \(b(\cdot, 0, 0)\) and \(\sigma(\cdot, 0, 0)\) are \(\bar{\mathbb{F}}\)-progressively measurable continuous processes and there exists some constant \(C > 0\) such that

\[ |b(t, x', x) + |\sigma(t, x', x)| \leq C(1 + |x|), \text{ a.s.}, \ \text{for all } 0 \leq t \leq T, x, x' \in \mathbb{R}^n; \]

(H5) \(b\) and \(\sigma\) are Lipschitz in \(x, x', \) i.e., there is a constant \(C > 0\) such that

\[ |b(t, x'_1, x_1) - b(t, x'_2, x_2)| + |\sigma(t, x'_1, x_1) - \sigma(t, x'_2, x_2)| \leq C(|x'_1 - x'_2| + |x_1 - x_2|), \text{ a.s., for all } 0 \leq t \leq T, x_1, x_2, x'_1, x'_2 \in \mathbb{R}^n. \]

It is known that, under the assumptions (H4) and (H5), SDE (4.1) has a unique strong solution (see, e.g, [5]). Moreover, it holds true that, for any \(p \geq 2\), there exists \(C_p \in R\) such that, for any \(t \in [0, T]\) and \(\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^d)\),

\[
\begin{aligned}
E \left[ \sup_{t \leq s \leq T} |X^t_{s, \zeta} - X^t_{s, \zeta'}|^p |\mathcal{F}_t \right] &\leq C_p |\zeta - \zeta'|^p, \quad P \text{-a.s.,} \\
E \left[ \sup_{t \leq s \leq T} |X^t_{s, \zeta}|^p |\mathcal{F}_t \right] &\leq C_p(1 + |\zeta|^p), \quad P \text{-a.s.}
\end{aligned}
\]  

(4.2)

Let us give two real-valued functions \(f(t, x', x, y', y, z)\) and \(h(x', x)\), which are assumed to satisfy the following conditions (H6).

(i) \(f : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \to R\) is a \(\bar{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable random variable and \(f : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to R\) is a measurable process such that \(f(\cdot, x', x, y', y, z)\) is \(\bar{\mathbb{F}}\)-adapted, for all \((x', x, y', y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d\).
(ii) There exists a constant $C > 0$ such that
\[
|f(t,x',x_1,x'_1,y_1,y_2,z_1) - f(t,x',x_2,x'_2,y_2,z_2)| + |h(x',x_1) - h(x',x_2)| \\
\leq C(|x'_1 - x'_2| + |x_1 - x_2| + |y'_1 - y'_2| + |y_1 - y_2| + |y_1 - y_2|), \text{ a.s.,}
\]
for all $0 \leq t \leq T, x_1, x'_1, x_2, x'_2 \in R^n, y_1, y'_1, y_2, y'_2 \in R$ and $z_1, z_2 \in R^d$.

(iii) $f$ and $h$ satisfy a linear growth condition, i.e., there exists some $C > 0$ such that, for all $x', x \in R^n$,
\[
|f(t,x',x,0,0,0)| + |h(x',x)| \leq C(1 + |x| + |x'|), \text{ a.s.}
\]

(iv) $f(\bar{\phi}, t, x', y', y, z)$ is continuous in $t$ for all $(x', x, y', y, z)$, $P(d\bar{\phi})$-a.s.

(v) $f(t, x', x, y', y, z)$ is nondecreasing with respect to $y'$.

(vi) There exists some $m \in \mathbb{N}^*$ such that $|\phi(E[h|X_{T}^{0,x}|, x])| \leq C(1 + |x|^m), \forall x \in R^n$.

Now, we consider the following coupled nonlocal parabolic variation inequality (PVI in short):
\[
\begin{align*}
\partial_t u(t,x) + & A(t,x) + E \left[ f(t, X_{T}^{0,x}, u(t,x), D u(t,x) \cdot E[\sigma(t, X_{T}^{0,x}, x)] \right] \\
& = E[h(X_{T}^{0,x}, x)], x \in R^n.
\end{align*}
\]  
(4.4)

where $Y_{T, x}$ is the solution of mean-field BSDE (4.3) with $x \in R^n$ at the place of $\zeta \in L^2(\Omega, \mathcal{F}, P; R^n)$.

In this section, we aim to study the following nonlocal parabolic variation inequality (PVI in short):
\[
\begin{align*}
\frac{\partial u(t,x)}{\partial t} + & A(t,x) + E \left[ f(t, X_{T}^{0,x}, u(t,x), D u(t,x) \cdot E[\sigma(t, X_{T}^{0,x}, x)] \right] \\
& = E[h(X_{T}^{0,x}, x)], x \in R^n.
\end{align*}
\]  
(4.5)

where
\[
A(t,x) := \frac{1}{2} tr(aD^2 u(t,x)) + (b, Du(t,x))
\]

with $a := E[\sigma(t, X_{T}^{0,x}, x)]E[\sigma(t, X_{T}^{0,x}, x)]^T$, $b := E[h(t, X_{T}^{0,x}, x)]$. Here the functions $b, \sigma, f$ and $h$ are supposed to satisfy (H4), (H5) and (H6), respectively, and $X_{T}^{0,x}$ is the solution of the SDE (4.1). Below, we denote by $S(n)$ the set of $n \times n$ symmetric non-negative matrices.
Definition 4.1 Let $u \in C_p([0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T] \times \mathbb{R}^n$. We denote by $\mathcal{P}^{1,2,+} u(t,x)$: the set of triples $(p,q,X) \in R \times R^n \times S(n)$ which are such that

$$u(s,y) \leq u(t,x) + p(s-t) + \langle q, y-x \rangle + \frac{1}{2} \langle X(y-x), y-x \rangle + o(|s-t| + |y-x|^2).$$

$\mathcal{P}^{1,2,-} u(t,x)$ is defined similarly as the set of triples $(p,q,X) \in R \times R^n \times S(n)$ which are such that

$$u(s,y) \geq u(t,x) + p(s-t) + \langle q, y-x \rangle + \frac{1}{2} \langle X(y-x), y-x \rangle + o(|s-t| + |y-x|^2).$$

Remark 4.1 Here $C_p([0,T] \times \mathbb{R}^n) = \{ u \in C([0,T] \times \mathbb{R}^n) : \text{There exists some constant } p > 0 \text{ such that } \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \frac{|u(t,x)|}{1+|x|^p} < +\infty \}.$

Next, we want to prove that $u(t,x)$ introduced by (4.4) is the unique viscosity solution of PVI (4.5). Before this, we first introduce the definition of viscosity solution of PVI (4.5), one can see Crandall, Ishii and Lions [13] for more details.

Definition 4.2 A random field $u \in C_p([0,T] \times \mathbb{R}^n)$ which satisfies $u(T,x) = E[h(X_{T,x_0}^t,x)]$.

(i) $u$ is a viscosity subsolution of PVI (4.5) if

$$u(t,x) \in \text{Dom} \phi, \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

and at any point $(t,x) \in [0,T] \times \mathbb{R}^n$, for any $(p,q,X) \in \mathcal{P}^{1,2,+} u(t,x)$, it holds that

$$-p - \frac{1}{2} \text{tr} (aX) - \langle b,q \rangle - E[f(t,X_{t,x_0}^t,x,u(t,X_{t,x_0}^t),u(t,x),q \cdot \sigma(t,X_{t,x_0}^t,x))] \leq -\phi'_-(u(t,x)).$$

(ii) $u$ is a viscosity supersolution of PVI (4.5) if

$$u(t,x) \in \text{Dom} \phi, \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

and at any point $(t,x) \in [0,T] \times \mathbb{R}^n$, for any $(p,q,X) \in \mathcal{P}^{1,2,-} u(t,x)$, it holds that

$$-p - \frac{1}{2} \text{tr} (aX) - \langle b,q \rangle - E[f(t,X_{t,x_0}^t,x,u(t,X_{t,x_0}^t),u(t,x),q \cdot \sigma(t,X_{t,x_0}^t,x))] \geq -\phi'_+(u(t,x)).$$

(iii) $u$ is a viscosity solution of PVI (4.5) if it is both a viscosity subsolution and a supersolution of PVI (4.5).

We can now state the main results of this section.

Theorem 4.1 Under the assumptions (H4)–(H6), the function $u(t,x)$ defined by (4.4) is the viscosity solution of PVI (4.5).
Proof. For each \((t, x) \in [0, T] \times \mathbb{R}^n\), and \(\varepsilon \in [0, 1]\), let \((Y^{t,x,\varepsilon}_s, Z^{t,x,\varepsilon}_s), s \in [t, T]\), the solution of the mean-field BSDE

\[
\begin{align*}
Y^{t,x,\varepsilon}_s + \int_s^T \nabla \varphi_e(Y^{t,x,\varepsilon}_r) dr &= \int_s^T E[f(r, X^{0,x}_r, Y^{t,x,\varepsilon}_r, (Y^{0,x,\varepsilon}_r, Y^{t,x,\varepsilon}_r, Z^{t,x,\varepsilon}_r)] dr - \int_s^T Z^{t,x,\varepsilon}_r dW_r, \\
Y^{t,x,\varepsilon}_T &= E[h(X^{0,x}_T, X^{t,x}_T)].
\end{align*}
\]

(4.6)

It is known from Buckdahn et al. \([5]\) that

\[ u_\varepsilon(t, x) := Y^{t,x,\varepsilon}_t, \quad t \in [0, T], \quad x \in \mathbb{R}^n \]

is the unique viscosity solution (the authors, in \([5]\), gave an example to explain why the uniqueness is only in \(C_\rho([0, T] \times \mathbb{R}^n)\)) of the following nonlocal PDE:

\[
\begin{align*}
&\frac{\partial u_\varepsilon(t, x)}{\partial t} + Au_\varepsilon(t, x) + E[f(t, X^{0,x}_t, x, u_\varepsilon(t, X^{0,x}_t), u_\varepsilon(t, x), Du_\varepsilon(t, x) \cdot E[\sigma(t, X^{0,x}_t, x)])] = \nabla \varphi_e(u_\varepsilon(t, x)), \\
u_\varepsilon(T, x) &= E[h(X^{0,x}_T, x)], x \in \mathbb{R}^n.
\end{align*}
\]

(4.7)

Moreover, it follows from Lemma 3,3 that

\[ |u_\varepsilon(t, x) - u(t, x)| \to 0, \text{ as } \varepsilon \to 0 \]

(4.8)

for all \((t, x) \in [0, T] \times \mathbb{R}^n\).

Let’s first show that \(u\) is the subsolution of PVI (4.5). For \((t, x) \in [0, T] \times \mathbb{R}^n\) and \((p, q, X) \in \mathbb{P}^{1,2,+} u(t, x)\), it follows from Crandall, Ishii and Lions \([8]\) that there exist sequence

\[
\begin{align*}
\varepsilon_n &\to 0, \\
(t_n, x_n) &\in [0, T] \times \mathbb{R}^n, \\
(p_n, q_n, X_n) &\in \mathbb{P}^{1,2,+} u_\varepsilon(t_n, x_n)
\end{align*}
\]

(4.9)

such that

\[
(t_n, x_n, u_\varepsilon(t_n, x_n), p_n, q_n, X_n) \to (t, x, u(t, x), p, q, X) \quad \text{as} \quad n \to +\infty.
\]

But for any \(n\), we have

\[
-p_n - \frac{1}{2} \text{tr}(a_n X_n) - \langle b_n, q_n \rangle - E \left[ f \left( t_n, X^{0,x}_n, x_n, u_\varepsilon(t_n, X^{0,x}_n), u_\varepsilon(t_n, x_n), q_n \cdot E \left[ \sigma(t_n, X^{0,x}_n, x_n) \right] \right) \right] \\
\leq -\nabla \varphi_{u_\varepsilon}(u_\varepsilon(t_n, x_n)).
\]

(4.10)

where \(a_n := E[\sigma(t_n, X^{0,x}_n) E[\sigma(t, X^{0,x}_n)]], b_n := E[b(t_n, X^{0,x}_n, x_n)]\). Arguing as in Pardoux-Răşcanu \([26]\), we let \(y \in \text{Dom} \varphi\) such that \(y < u(t, x)\). Then by (4.3), the uniformly convergence \(u_\varepsilon \to u\) on compacts implies that there exists \(n_0 > 0\) such that \(y < u_\varepsilon(t_n, x_n), \forall n \geq n_0\). Thus, multiplying both sides of (4.10) by \(u_\varepsilon(t_n, x_n) - y\), we get

\[
\begin{align*}
-p_n - \frac{1}{2} \text{tr}(a_n X_n) - \langle b_n, q_n \rangle - E \left[ f \left( t_n, X^{0,x}_n, x_n, u_\varepsilon(t_n, X^{0,x}_n), u_\varepsilon(t_n, x_n), q_n \cdot E \left[ \sigma(t_n, X^{0,x}_n, x_n) \right] \right) \right] \\
(u_\varepsilon(t_n, x_n) - y) \leq \varphi(y) - \varphi(J_{\varepsilon_n}(u_\varepsilon(t_n, x_n))).
\end{align*}
\]

(4.11)
Passing to liminf$_{n 	o +\infty}$ on both sides of (4.11), we have that for all $y < u(t, x)$,

$$
\left\{ -p - \frac{1}{2} tr(aX) - \langle b, q \rangle - E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), q \cdot E[\sigma(t, X_t^{0,x_0}), x])] \right\} (u(t, x) - y) 
\leq \psi(y) - \psi(u(t, x)),
$$

(4.12)

it follows that

$$
-p - \frac{1}{2} tr(aX) - \langle b, q \rangle - E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), q \cdot E[\sigma(t, X_t^{0,x_0}), x])] 
\leq -\psi'_-(u(t, x)),
$$

(4.13)
i.e., $u$ is a viscosity subsolution of PVI (4.5). By the similar arguments we can show that $u$ is a viscosity supsolution of PVI (4.5), and thus, we complete the proof.

**Theorem 4.2** Under the assumptions (H4)–(H6), PVI (4.5) has a unique viscosity solution.

**Proof.** Here, we adopt the same arguments appeared in Ei Karoui et al. [16] and Pardoux and Răşcanu [26].

Suppose that $u$ is a subsolution and $v$ a supsolution of PVI (4.5) such that $u(T, x) = v(T, x) = E[h(X_T^{0,x_0}, x)], x \in \mathbb{R}^n$.

Define

$$
\tilde{u}(t, x) := u(t, x)e^{\lambda t} \xi^{-1}(x),
$$
$$
\tilde{v}(t, x) := v(t, x)e^{\lambda t} \xi^{-1}(x) + \frac{\xi}{t},
$$
$$
\eta(x) := \xi^{-1}(x)D\xi(x) = p(1 + |x|^2)^{-1}x,
$$
$$
\kappa(t, x) := \xi^{-1}(x)D^2\xi(x) = p(1 + |x|^2)^{-1}I - p(p - 2)(1 + |x|^2)^{-2}x \otimes x,
$$

where $\xi(x) := (1 + x^2)^{\frac{\xi}{2}}$. Then, it is straightforward that $\tilde{u}$ and $\tilde{v}$ satisfy that (we write below $u, v$ instead of $\tilde{u}, \tilde{v}$)

$$
- \frac{\partial u(t, x)}{\partial t} - \tilde{A}u(t, x) = E[\tilde{f}(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), Du(t, x) \cdot E[\sigma(t, X_t^{0,x_0}), x])] 
\leq -e^{\lambda t} \xi^{-1}(x) \psi'_-(e^{-\lambda t} \xi(x) u(t, x))
$$

(4.14)

and

$$
- \frac{\partial v(t, x)}{\partial t} - \tilde{A}v(t, x) = E[\tilde{f}(t, X_t^{0,x_0}, x, v(t, X_t^{0,x_0}), v(t, x), Dv(t, x) \cdot E[\sigma(t, X_t^{0,x_0}), x])] 
\geq \frac{\xi}{t^2} - e^{\lambda t} \xi^{-1}(x) \psi'_{+} \left( e^{-\lambda t} \xi(x) \left( v(t, x) - \frac{\xi}{t} \right) \right),
$$

(4.15)

where

$$
\tilde{A} \psi := A \psi + \langle a \eta, D \psi \rangle + \left[ \frac{1}{2} tr(a \kappa) + \langle b, \eta \rangle - \lambda \right] \psi,
$$
$$
\tilde{f}(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), Du(t, x) \cdot E[\sigma(t, X_t^{0,x_0}), x])
:= e^{\lambda t} \xi^{-1}(x)f\left(t, X_t^{0,x_0}, x, e^{-\lambda t} \xi(x) u(t, X_t^{0,x_0}), e^{-\lambda t} \xi(x) u(t, x),
\right)$$
Since the right-hand side tends to zero as $K \to \infty$, we have
\[
e^{-\lambda_{\delta} \xi(x)} Du(t, x) \cdot E[\sigma(t, X_t^{0, \alpha}, x)] + e^{-\lambda_{\delta} \xi(x)} Du(t, x) \cdot E[\sigma(t, X_t^{0, \alpha}, x)]\]

Let
\[
F(t, X_t^{0, \alpha}, x, u(t, X_t^{0, \alpha}), u(t, x), Du(t, x), Du^2(t, x))
\]
\[
= -\tilde{u}(t, x) - E[\tilde{f}(t, X_t^{0, \alpha}, x, u(t, X_t^{0, \alpha}), u(t, x), Du(t, x) \cdot E[\sigma(t, X_t^{0, \alpha}, x)]]).
\]

Then, by the Lipschitz condition (ii) of Assumption (H6), for $y_1 > y_2$, we have
\[
\tilde{f}(t, x', x, r, y_1, \theta) - \tilde{f}(t, x', x, r, y_2, \theta)
\]
\[
= e^{-\lambda_{\delta} \xi^{-1}(x)} \left[ f(t, x', x, r, e^{-\lambda_{\delta} \xi(x)}y_1, \theta) - f(t, x', x, r, e^{-\lambda_{\delta} \xi(x)}y_2, \theta) \right]
\]
\[
\leq C(y_1 - y_2).
\]

Hence,
\[
F(t, x', x, r, y_1, \mu, \nu) - F(t, x', x, r, y_2, \mu, \nu) \geq \left[ \lambda - \frac{1}{2} tr(a\kappa) - \langle b, \eta \rangle - C \right] (y_1 - y_2).
\]

Since $a\kappa$ and $\langle b, \eta \rangle$ are bounded, then we can choose $\lambda$ large enough such that
\[
y \to F(t, x', x, r, y, \mu, \nu)
\]
is strictly increasing for any $(t, x', x, r, \mu, \nu) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times S(n)$, and thus $F$ is proper in the terminology of [3].

What we need to show is that for any $K > 0$, if $B_K := \{ |x| < K \}$,
\[
sup_{(0, T) \times B_K} (u - v)^+ \leq sup_{(0, T) \times \partial B_K} (u - v)^+.
\]

Since the right-hand side tends to zero as $K \to \infty$, we will prove this fact by contradiction.

Assume that there exists some $K > 0$ such that for some $(t_0, x_0) \in (0, T) \times B_K$
\[
\delta := u(t_0, x_0) - v(t_0, x_0) = sup_{(0, T) \times B_K} (u - v)^+ > sup_{(0, T) \times \partial B_K} (u - v)^+.
\]

We define $(\hat{t}, \hat{x}, \hat{y})$ as being a point in $[0, T] \times B_K \times B_K$ where the function
\[
\Phi_\alpha(t, x, y) = u(t, x) - v(t, x) - \frac{\alpha}{2} |x - y|^2
\]
achieves its maximum. Then by Lemma 8.7 in [16], we have:
(i) for $\alpha$ large enough, $(\hat{t}, \hat{x}, \hat{y}) \in [0, T] \times B_K \times B_K$,
(ii) $|\hat{x} - \hat{y}|^2 \to 0$ and $|\hat{x} - \hat{y}|^2 \to 0$,
(iii) $u(\hat{t}, \hat{x}) \geq v(\hat{t}, \hat{y}) + \delta$.

Then for $\alpha$ large enough,
\[
e^{-\lambda_{\delta} \xi(\hat{x})} u(\hat{t}, \hat{x}) \leq e^{-\lambda_{\delta} \xi(\hat{y})} \left( v(\hat{t}, \hat{y}) - \frac{\delta}{\hat{t}} \right)
\]
and, as a result,
\[
- \varphi'_-(e^{-\lambda_{\delta} \xi(\hat{x})} u(\hat{t}, \hat{x})) \leq - \varphi'_+ \left( e^{-\lambda_{\delta} \xi(\hat{y})} \left( v(\hat{t}, \hat{y}) - \frac{\delta}{\hat{t}} \right) \right).
\]

\[ (4.16) \]
Furthermore, from Theorem 8.3 in [8], we know that there exists

\((p, X, Y) \in R \times S(n) \times S(n)\)

such that

\((p, \alpha(\hat{x} - \hat{y}), X) \in P^{1,2} u(t, x),\)

\((p, \alpha(\hat{x} - \hat{y}), Y) \in P^{1,2} v(t, x).\)

Next, because \(u\) (resp. \(v\)) is a subsolution (resp. supsolution) and \(F\) is proper, then following the same line as in [16], it follows from (4.16) that

\[ F(\hat{t}, Y^0_{\hat{t}, x_0}, \hat{y}, u(\hat{t}, X^0_{\hat{t}, x_0}), v(\hat{t}, \hat{y}), \alpha(\hat{x} - \hat{y}), Y) - F(\hat{t}, X^0_{\hat{t}, \hat{x}}, \hat{x}, u(\hat{t}, X^0_{\hat{t}, \hat{x}}), v(\hat{t}, \hat{y}), \alpha(\hat{x} - \hat{y}), X) \geq \frac{\varepsilon}{t^2}.\]

Finally, by the Lipschitz condition (ii) of (H6) on \(f\), following the proof on Page 734 in [16], we can deduce a similar contradiction. The uniqueness is proved.

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