Decoherent histories approach to tunneling times and its implication

Norifumi Yamada
Department of Information Science, University of Fukui, 3-9-1 Bunkyo, Fukui, Fukui 910-8507, Japan
E-mail: yamada@inws1.fuis.fukui-u.ac.jp

Abstract. Decoherent histories approach to quantum mechanics assigns probabilities to events that are not restricted to a single moment of time, provided that the decoherence condition holds for the set of events. The condition is however hardly satisfied in general, which has been considered to limit the usefulness of the histories approach. The present paper applies the histories approach to the tunneling time problem to show that, although the decoherence condition does not hold, the histories approach is still useful in predicting the central tendency and the dispersion of resident times. We consider such quantities that would become the average and the standard deviation of resident times if the decoherence condition held. Since the condition does not actually hold, they are not regarded as the true average and the true standard deviation. Nevertheless, a quantum clock approach allows us to interpret these quantities as a central value and a dispersion of resident times within the time resolution of the clock.

1. Introduction
Let us consider a particle moving on the $x$ axis, and ask two related questions: (Q1) Where do we find the particle at a specified time $t_0$? (Q2) When do we find the particle at a specified position $x_0$? If we try to answer these questions in terms of probabilities, our interest is in the probability distribution $P_{t_0}(x)$ of particle’s position $x$ at $t_0$ for Q1, and in the probability distribution $P_{x_0}(t)$ of particle’s arrival time $t$ at $x_0$ for Q2. In classical stochastic process (e.g., classical Brownian motion), the two distributions are definable and their expressions have been well studied. The situation is different in quantum mechanics. As is well known, $P_{t_0}(x)$ is definable and is given by $P_{t_0}(x) = |\Psi(x, t_0)|^2$, where $\Psi$ is the wave function. As to $P_{x_0}(t)$, the conventional quantum mechanics does not provide a clear prescription for calculating it. The difference between Q1 and Q2 is that in Q1 all the events whose occurrence is predicted by $P_{t_0}(x)$ are, in space-time picture, lying on the constant-time surface defined by $t = t_0$, while in Q2 all the events lie on the surface of constant position defined by $x = x_0$. In classical stochastic process, the concept of trajectory (in spacetime) does make sense, so that the events on the surface of constant time and those on the surface of constant position can be understood on equal footing as the intersections of trajectories with the surfaces. In this sense, the events considered in Q1 and the events considered in Q2 are in the same status in classical stochastic process. By contrast, they are not in the same status in the conventional quantum mechanics, because the conventional theory defines probabilities for the events considered in Q1 but does
not for those considered in Q2, thus giving a special status to the former, or more generally, to those events that are restricted to a moment of time.

The decoherent histories (or the consistent histories) approach to quantum mechanics [1, 2, 3, 4] is a modest generalization of the conventional quantum mechanics. Working with the histories approach, we can examine in what circumstances probabilities are definable for events that are not necessarily restricted to a moment of time and, if definable, what the probabilities are. The histories approach is thus suitable for studying the problem of arrival time and other time-related problems in quantum mechanics. Among them, the tunneling time problem [5, 6] would be of special importance not only as a fundamental topic in quantum mechanics but also as a practical issue that would have relevance to how fast tunneling devices can operate. The tunneling time problem asks “How long does it take for a particle to tunnel through a potential barrier?” Many ideas have been proposed to answer this seemingly simple-minded question. It has turned out, however, that the different ideas give differing answers, i.e., various tunneling times. The main question in the 80’s was then “Which is the correct tunneling time?” Today, however, a certain consensus seems to exist that the various tunneling times obtained from different models should be understood in a broad sense that they characterize different or complementary aspects of tunneling process, rather than in the narrow sense as the time taken by the particle to tunnel through the barrier region. It would then be interesting to explore the possibility of understanding the various tunneling times in terms of the moments of a probability distribution of tunneling times. To do this exploration, we need a distribution, but its existence is a matter of investigation, because, as in the case of the arrival time problem, the events whose occurrence is to be predicted by the distribution do not lie on the surface of constant time.

This paper shows that, although the histories approach does not define a probability distribution of tunneling times, it still gives such quantities that can be interpreted as a measure of central tendency and a measure of dispersion of resident times (there are two types of tunneling times, the resident time and the passage time, as explained below).

2. Decoherent histories approach

The present author applied the histories approach to the tunneling time problem to find out that a probability distribution of tunneling times is not definable [7, 8]. This is concluded from the calculation of the decoherence functional $D(\tau', \tau)$ [3], which plays the central role in the histories approach. The decoherence functional $D(\tau', \tau)$ is a functional on the space of Feynman paths, representing the interference between the class of Feynman paths that take time $\tau'$ to traverse the barrier region and the class of Feynman paths that take time $\tau$ to do so. The basic criterion in the histories approach is the decoherence condition [3], which is now $\text{Re}D(\tau', \tau) \propto \delta(\tau' - \tau)$, requiring that there be no interference between different classes of paths. If this condition holds, the histories approach defines a probability distribution of tunneling times and is given by the proportionality coefficient by $P(\tau)$, the decoherence condition can be written as

$$\text{Re}D(\tau', \tau) = \delta(\tau' - \tau)P(\tau). \quad (1)$$

It was shown in [7, 8] that this condition does not hold. Therefore the idea of using a probability distribution of tunneling times to characterize the various tunneling times does not work, because the distribution is not definable. A modest strategy that can be taken in this situation is to consider such quantities that would become the moments of tunneling times if the decoherence condition held and explore the relationship between the quantities and the various tunneling times. In particular, if the decoherence condition held,

$$\tau = \int d\tau' \int d\tau \frac{\tau' + \tau}{2} D(\tau', \tau) \quad (2)$$
would become \( \int d\tau \tau P(\tau) \), the average tunneling time, and
\[
\overline{\tau^2} \equiv \int d\tau' \int d\tau \tau' \tau D(\tau', \tau)
\] (3)
would become \( \int d\tau \tau^2 P(\tau) \), so that
\[
\sigma^2 \equiv \overline{\tau^2} - (\overline{\tau})^2
\] (4)
would be the variance of tunneling times. Although \( \overline{\tau} \) and \( \sigma^2 \) are not the true average and the true variance because the decoherence condition does not actually hold, they are still well defined and have the dimensions of time and time squared. We call \( \overline{\tau} \) and \( \sigma^2 \) the quasiaverage and the quasivariance, respectively, and \( \sigma \) the quasideviation. Note that \( \overline{\tau} \) and \( \overline{\tau^2} \) are real valued due to the Hermiticity of the decoherence functional, i.e., \( D(\tau', \tau) = D(\tau, \tau') \).

To calculate \( \overline{\tau} \) and \( \sigma \), we need to know \( D(\tau', \tau) \), which is given as an inner product of branch wave functions \( \Psi(x,t|\tau) \):
\[
D(\tau', \tau) = \frac{1}{P_T} \lim_{t \to \infty} \int dx \Psi(x,t|\tau') \Psi^*(x,t|\tau),
\] (5)
where \( P_T \) is the tunneling probability. The long time limit is to ensure the completion of the scattering process, and the range of integration is restricted to the transmitted region (the right side of the barrier, provided that the particle is incident from the left). The branch wave function is defined as
\[
\Psi(x,t|\tau) = \int dx_0 K(x,t,x_0|\tau) \Psi(x_0,0),
\] (6)
where \( \Psi(x_0,0) \) is the initial state and \( K(x,t,x_0|\tau) \) is the sum of \( e^{iS/\hbar} \) (\( S \) being the action) over those paths that connect \((x_0,0)\) and \((x,t)\) under the restriction that they take time \( \tau \) to traverse the barrier region. Here we must notice that, since a Feynman path crosses the barrier region many times in general, the time \( \tau \) that a Feynman path takes to traverse the barrier region may be defined in several ways. The quantities \( K, \Psi, D, \) and therefore \( \overline{\tau} \) and \( \sigma \) depend on how \( \tau \) is defined. We use two definitions: (i) \( \tau \) is defined as the resident time, which is the sum of the times during which the path is inside the barrier region, (ii) \( \tau \) is defined as the passage time, which is the difference between the last time the path leaves the barrier region and the first time it enters the region.

For a square barrier, the detailed calculations yield the following results [9]:
\[
\tau_r = -\frac{\hbar}{V} \frac{\partial \theta}{\partial V}, \quad \sigma_r = -\frac{\hbar}{V} \frac{\partial \ln |T|}{\partial V}, \quad \tau_p = \frac{\hbar}{E} \frac{\partial \theta}{\partial E}, \quad \sigma_p = \left| \frac{\hbar}{2E_0} \right|,
\] (7)
where \( T = |T|e^{i\theta} \) is the transmission amplitude, which depends on the energy \( E \) of the particle, the height \( V \) and the width \( d \) of the square barrier, and the mass of the particle. We have attached subscript \( r \) to the results obtained for resident time and subscript \( p \) to those obtained for passage time. In Eq. (7), the \( V \) derivatives are evaluated at the height \( V_0 \) of the barrier under consideration, and the \( E \) derivatives are evaluated at the incident energy \( E_0 \) of the particle. Having the \( E \) or \( V \) derivative of \( |T| \) or \( \theta \) is not new to the tunneling time community. In fact, the Larmor time \( \tau_{LM} \equiv -\frac{\hbar}{V} \frac{\partial \theta}{\partial V} \), the Büttiker-Landauer time \( \tau_{BL} \equiv -\frac{\hbar}{E} \frac{\partial \ln |T|}{\partial E} \), the Bohm-Wigner time (the phase time) \( \tau_{BW} \equiv \hbar \frac{\partial \theta}{\partial E} \), and the Pollak-Miller time \( \tau_{PM} \equiv \hbar \frac{\partial \ln |T|}{\partial E} \) are the well-known time scales that have been repeatedly obtained from various methods [5, 6] as the tunneling time. Unlike those traditional methods, the derivation of the four tunneling times outlined above does not rely on any specific models and gives the four times in a unified manner (\( \sigma_p \) contains an additional time constant \( \hbar/2E_0 \)). Moreover, the obtained results raise an interesting question that is unlikely to be put forward as long as the traditional methods are used: Is it possible to understand the four tunneling times in some sense as central tendency and dispersion? For the case of resident time, the answer is affirmative as shown in the next section.
3. Quantum clock approach

A natural idea to determine the time that a particle takes to traverse a spatial region is to couple the particle to a degree of freedom that plays the role of a clock and find out how much the clock gains in the region.

Peres [10] considered a quantum clock whose Hamiltonian is given by

\[ H_c = -i\hbar \omega \frac{\partial}{\partial \theta}, \]

where \( \omega \) is the angular velocity of the clock hand and \(-i\hbar \partial/\partial \theta \) \((0 \leq \theta < 2\pi)\) is its angular momentum. A physical situation that leads to \( H_c \) is a magnetic moment placed in a uniform magnetic field directed along the \( z \) axis. The eigenstates of \( H_c \) are given by

\[ u_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}, \quad (m = -j, \ldots, j) \]

with eigenvalues \( m\hbar \omega \). The \( N \equiv 2j + 1 \) eigenstates are plane waves and not localized on the \( \theta \) axis.

By superposing the \( N \) eigenstates, we can construct \( N \) localized “dial” states as follows:

\[ v_l(\theta) = \frac{1}{\sqrt{N}} \sum_{m=-j}^{j} e^{-i2\pi m/l} u_m(\theta) \]

\[ = \frac{1}{\sqrt{2\pi N}} \sum_{m=-j}^{j} e^{im(\theta-2\pi l/N)}, \quad (l = -j, \ldots, j), \]

where each \( v_l(\theta) \) is peaked at \( \theta = \theta_l \equiv 2\pi l/N \). The peak positions of the \( N \) dial states, which constitute a localized orthonormal basis, are equally spaced in the range \( 0 \leq \theta < 2\pi \), corresponding to the numbers marked with equal spacing on the face of a classical clock. Since \( \omega \) is the angular velocity, \( \theta = \theta_l \) corresponds to \( t = t_l \equiv \theta_l/\omega \). When the clock state coincides with one of the dial state \( v_l(\theta) \), the time indicated by the clock is \( t_l \), but with the uncertainty \( \pm \pi/N\omega \) because the adjacent peaks are separated by \( 2\pi/N \) in \( \theta \). The time resolution of the clock is thus \( 2\pi/N\omega \).

Consider the tunneling of a particle of energy \( E \) through a square barrier of height \( V_0 \). The barrier region is assumed to be from \( x = 0 \) to \( x = d \). To determine the time taken by the particle to tunnel through the barrier region, we consider the Hamiltonian

\[ H = H_{V_0} + P(x)H_c \]

with \( P(x)=1 \) for \( 0 < x < d \) and \( P(x)=0 \) otherwise; \( H_{V_0} = p^2/2M + P(x)V_0 \) (\( p \) being the momentum and \( M \) the mass of the particle) is the particle Hamiltonian in the presence of the square barrier. Since the particle-clock interaction is limited to the region \( 0 < x < d \), the clock runs only in the barrier region to measure the time spend by the particle inside the barrier. Peres [10] introduced the above Hamiltonian for the case of a free particle (i.e., \( V_0 = 0 \)) to obtain a reasonable result that the clock initially pointing to \( t = 0 \) points to \( t = d/v \) after the transmission, where \( v = \sqrt{2E/M} \). Ohba [11] studied the case \( V_0 \neq 0 \) and showed that after the transmission the clock points to the pure imaginary time \(-id/v_E \), where \( v_E \) is the Euclidean velocity given by \( v_E = \sqrt{2(V_0 - E)/M} \). This led to a Lorentzian distribution that is peaked at \( t = 0 \) and has the width \( d/v_E \), which is the Büttiker-Landauer time for opaque barriers. In the following, we basically follow Ohba’s calculation but refine the approximations used to derive the final state of the clock. We will then find that the clock points to the complex time \( \tau_{LM} - i\tau_{BL} \) after the transmission.
Let us assume that at $t = 0$ the particle is in the state $e^{ikx}$ and the clock is in $v_0(\theta)$. The total wave function is thus $\Psi(x, \theta; t = 0) = e^{ikx}v_0(\theta)$. The final state of the system is given by

$$\Psi(x, \theta; t) = e^{-i\mathcal{H}t/\hbar}e^{ikx}v_0(\theta)$$

$$= \frac{1}{\sqrt{2\pi N}} \sum_{m=-j}^{j} e^{-i(H_{V_0} + P(x)\mathcal{H})t/\hbar}e^{ikx}e^{im\theta}. \quad (13)$$

Since $H_c e^{im\theta} = \hbar \omega e^{im\theta}$, $H_c$ in Eq. (13) can be replaced by $m\hbar \omega$, giving

$$\Psi(x, \theta; t) = \frac{1}{\sqrt{2\pi N}} \sum_{m=-j}^{j} e^{-i(H_{V_0} + P(x)m\hbar\omega)t/\hbar}e^{ikx}e^{im\theta}. \quad (14)$$

Note here that $H_{V_0} + P(x)m\hbar\omega = H_{V_0} + \hbar \omega$, because the interaction term $P(x)m\hbar\omega$ has the same form as the potential term $P(x)V_0$ involved in $H_{V_0}$. We are interested in the final state in the region $x > d$. As is familiar in the stationary scattering problem with plane waves,

$$e^{-iH_{V_0} + \hbar \omega t/\hbar}e^{ikx} = T(V_0 + m\hbar \omega)e^{ikx-iEt/\hbar} \quad (x > d), \quad (15)$$

where $E = \hbar^2 k^2/2M$, and $T(V)$ is the transmission amplitude as the function of barrier height $V$. Substituting Eq. (15) into Eq. (14), we have, for $x > d$,

$$\Psi(x, \theta; t) = e^{ikx-iEt/\hbar} \frac{1}{\sqrt{2\pi N}} \sum_{m=-j}^{j} T(V_0 + m\hbar \omega)e^{im\theta}. \quad (16)$$

Writing $T(V) = e^{i\ln T(V)}$ and expanding $\ln T(V)$ around $V = V_0$ to the first order in $V - V_0$, we have

$$T(V) = T(V_0) \exp \left[ \frac{T'(V_0)}{T(V_0)}(V - V_0) \right] \quad (17)$$

$$= T(V_0) \exp \left[ -(i\tau_{LM} + \tau_{BL})(V - V_0)/\hbar \right], \quad (18)$$

where the prime denotes the derivative with respect to the potential height $V$, and $\tau_{LM}$ and $\tau_{BL}$ are respectively the Larmor time and the Büttiker-Landauer time evaluated at $V = V_0$. Substituting Eq. (18) into Eq. (16), we find

$$\Psi(x, \theta; t) = T(V_0)e^{ikx-iEt/\hbar} \frac{1}{\sqrt{2\pi N}} \sum_{m=-j}^{j} e^{-(i\tau_{LM} + \tau_{BL})m\omega}e^{im\theta}, \quad (19)$$

which shows that the final state of the clock is

$$\psi_{\text{final}}(\theta) = \frac{1}{\sqrt{2\pi N}} \sum_{m=-j}^{j} e^{im(\theta - \omega(\tau_{LM} - i\tau_{BL}))} \quad (20)$$

$$= v_0(\theta - \omega(\tau_{LM} - i\tau_{BL})). \quad (21)$$

The result obtained in [11] is reproduced if $T'(V_0)/T(V_0)$ in Eq. (17) is replaced by $d/\hbar v_E$, which leads to Eq. (21) with $\tau_{LM}$ and $\tau_{BL}$ replaced by 0 and $d/v$, respectively. Peres’s result for a free particle [10] is reproduced from Eq. (21) if we note that $\tau_{LM} = d/v$ and $\tau_{BL} = 0$ when $V_0 = 0$.

Formally, Eq. (21) means that the clock points to the complex time $t = \tau_{LM} - i\tau_{BL}$ after the transmission. One way to see the physical meaning of this formal statement is to examine the
distribution $|\psi_{\text{final}}(\theta)|^2$. It was shown in [11] that, for $|\theta| \ll 1$ and $\omega d/v_E \ll 1$, the distribution is a Lorentzian that is peaked at $\theta = 0$ with width $\omega d/v_E$, where $v_0(\theta + i\omega d/v_E)$ was used as $\psi_{\text{final}}(\theta)$. Thus, for the final state given by Eq. (21), the distribution is a Lorentzian that is peaked at $\theta = \omega \tau_{LM}$ with width $\omega \tau_{BL}$ for $|\theta - \omega \tau_{LM}| \ll 1$ and $\omega \tau_{BL} \ll 1$, which means that $\tau_{LM}$ is a measure of central tendency of the times indicated by the clock and $\tau_{BL}$ is a measure of dispersion around the central value. Recall that the histories approach gives $\tau_{LM}$ and $\tau_{BL}$ as the quasiaverage and the quasideviation of resident times, respectively. It was not clear in the preceding section, however, whether these quasi values have any physical meanings that one expects from the words “average” and “deviation”. The quantum clock approach shows that $\tau_{LM}$ is indeed a measure of central tendency and $\tau_{BL}$ is indeed a measure of dispersion.

4. Concluding remarks
For the case of opaque barriers, $\tau_{LM}$ is smaller than the time resolution $2\pi/N\omega$ of the quantum clock [12]. Thus the shift of the clock hand from $t = 0$ to $t = \tau_{LM}$ caused by tunneling is not “detectable”. By contrast, $\tau_{BL}$ can be much larger than $2\pi/N\omega$ for opaque barriers, so that the dispersion is detectable. These facts warn that the interpretation of the quasi values is still a subtle issue even in the context of quantum clock. However, the significance of our result lies in the point that the quasi values given by the histories approach when the decoherence condition does not hold turn out to be physically interpretable to some extent as a central tendency and a dispersion. We have shown this only for the case of resident time. Whether the same thing can be said or not for the case of passage time is a remaining problem. A big open question is whether the histories approach in general context (apart from the tunneling time problem) gives useful predictions even when the decoherence condition does not hold. Although this is far beyond the scope of the present paper, our result could be seen as a first step toward the solution of the general question.

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