Universal scaling laws and density slope for dark matter halos from rotation curves and energy cascade

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ABSTRACT
Smallscale challenges suggest some missing pieces in our current understandings of dark matter. A cascade theory for dark matter flow is proposed to provide extra insights, similar to the cascade in hydrodynamic turbulence. The energy cascade from small to large scales with a constant rate $\varepsilon \mu$ ($\approx -4.6 \times 10^{-7} m^2/s^3$) is a fundamental feature of dark matter flow. Energy cascade leads to a two-thirds law for kinetic energy $v^2_r$ on scale $r$ such that $v^2_r \propto (\varepsilon \mu r)^{2/3}$, as confirmed by N-body simulations. This is equivalent to a four-thirds law for mean halo density $\rho_s$ confined in the scale radius $r_s$ such that $\rho_s \propto \varepsilon^{2/3} \mu^{-1} r_s^{-4/3}$, as confirmed by data from galaxy rotation curves. By identifying relevant key constants, critical scales of dark matter might be obtained. The largest halo scale $r_L$ can be determined by $-u^3_0/\varepsilon \mu$, where $u_0$ is the velocity dispersion. The smallest scale $r_T$ is dependent on the nature of dark matter. For collisionless dark matter, $r_T \propto (-G\mu/\varepsilon \mu)^{1/3} \approx 10^{-1/3} m$, where $h$ is the Planck constant. A uncertainty principle for momentum and acceleration fluctuations is also postulated. For self-interacting dark matter, $\rho_{\eta} \propto \varepsilon^{3/4} \mu^{-3/4} (\sigma/m)^{3/4}$, where $\sigma/m$ is the cross-section. On halo scale, the energy cascade leads to an asymptotic slope $\gamma = -4/3$ for fully virialized halos with a vanishing radial flow, which might explain the nearly universal halo density. Based on continuity equation, halo density is analytically shown to be closely dependent on the radial flow and mass accretion such that simulated halos can have different limiting slopes. A modified Einasto density profile is proposed accordingly.

Key words: Dark matter flow; N-body simulations; Rotation curve; Collisionless; Self-interacting; Core-cusp

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1 INTRODUCTION
Standard CDM (cold dark matter) paradigm of cosmology has many successes in the formation and evolution of large scale structures and the contents and states of our universe (Peebles 1984; Spergel et al. 2003; Komatsu et al. 2011; Frenk & White 2012). Despite great successes, serious theoretical and observational difficulties still exist (Perivolaropoulos & Skara 2022; Bullock & Boylan-Kolchin 2017). Especially, CDM model predictions of structures on small scales (<1 Mpc) are inconsistent with some observations. Examples are the core-cusp problem (Flores & Primack 1994; de Blok 2010), the missing satellite problem (Klypin et al. 1999; Moore et al. 1999), the too-big-to-fail problem (Boylan-Kolchin et al. 2011, 2012). In addition, the origin of Baryonic Tully-Fisher relation (BTFR) and MOND (modified Newtonian dynamics) (Milgrom 1983; McGaugh et al. 2000; Famaey & McGaugh 2013) is still not clear. These small scale challenges might be related to each other (Garrison-Kimmel et al. 2014; Bullock & Boylan-Kolchin 2017; Del Popolo et al. 2014) and suggest missing pieces in our current understandings. First, the cusp-core problem describes the discrepancy between the cuspy halo density predicted by cosmological CDM only N-body simulations and the cored density inferred from observational data for dwarf galaxies. The predicted halo density exhibits a cuspy profile with inner density $\rho(r) \propto r^{-\gamma}$, where slope $\gamma$ persistently exceeds different observations (de Blok & Bosma 2002; de Blok et al. 2003; Swaters et al. 2003; Kuzio de Naray & Kauffmann 2011). Even for the cuspy profile predicted by cosmological simulations, there seems no consensus on the exact value of asymptotic slope $\gamma$, but with a wide range between -1.0 to -1.5. Since the first prediction of $\gamma = -1.0$ in NFW profile (Navarro et al. 1997), the inner density slope of simulated halos seems to have different values from $\gamma > -1.0$ (Navarro et al. 2010) to $\gamma = -1.2$ (Diemand & Moore 2011), and $\gamma = -1.3$ (Governato et al. 2010; McKeown et al. 2022). To summarize, some key questions are: is there an asymptotic slope for dark matter halos? why there exists a nearly universal density profile? and why different inner slopes $\gamma$ exist in simulations?

The halo density inferred from observational data exacerbates the problem. Even the smallest predicted inner density slope from simulations is still greater than that from observations. Many solutions have been suggested to solve the cusp-core problem (Del Popolo & Le Delliou 2017). Within the CDM framework of collisionless dark matter, the baryonic solutions focus on different mechanisms for energy exchange between baryons and dark matter to enable a flatter inner density (Navarro et al. 1996; Oh et al. 2011; Benítez-Llambay et al. 2019). Beyond the CDM framework, the self-interacting dark matter is proposed as a potential solution (Spergel & Steinhardt 2000; Rocha et al. 2013; Peter et al. 2013). The elastic scattering with a
The highly localized and over-dense halos are a major manifestation of nonlinear gravitational collapse (Neyman & Scott 1952; Cooray & Sheth 2002) and the building blocks of dark matter flow, a counterpart to "eddies" in turbulence. The halo-mediated inverse mass cascade is not present in turbulence, but exists as a local, two-way, and asymmetric process in dark matter flow (Xu 2021a). The net mass transfer proceeds in a "bottom-up" fashion from small to large mass scales (inverse cascade) to allow for structure formation. Halos pass their mass onto larger and larger halos, until halo mass growth becomes dominant over the mass propagation. From this description, mass cascade can be described by a similar poem with "eddies" (or "whirls") simply replaced by "halos":

"Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth."

Energy cascade across halo groups is facilitated by the mass cascade and also a fundamental feature. Even on the halo scale, since halos are non-equilibrium objects, energy cascade should also play a role in the abundance and internal structure of halos. In this paper, we focus on the energy cascade, its evidence from galaxy rotation curves, and its critical role in halo internal structure and dark matter properties.

2 THE CONSTANT RATE OF ENERGY CASCADE

Particle-based N-body simulations are widely used to study the nonlinear gravitational collapse of dark matter (Peebles 1980). The simulation data for this work was generated from N-body simulations by Virgo consortium (Frenk et al. 2000; Jenkins et al. 1998). One way to determine the constant rate of energy cascade $\varepsilon_u$ is from a cosmic energy equation for energy evolution of dark matter flow in expanding background (Irvine 1961; Layzer 1963; Xu 2022h),

$$\frac{dE_v}{dt} + H (2K_p + P_S) = 0,$$

which is a manifestation of energy conservation in expanding background. Here $K_p$ is the specific (peculiar) kinetic energy, $P_S$ is the specific potential energy in physical coordinate. $E_v = K_p + P_s$ is the total energy, $H = d\dot{a}/a$ is the Hubble parameter ($Ht = 2/3$ for matter dominant universe), and $a$ is the scale factor. In statistically steady state, Eq. (1) admits a linear solution of $K_p = -\varepsilon_u t$ and $P_S = 7/5\varepsilon_u t$ (see Fig. 2) such that $\varepsilon_u$ can be found as,

$$\varepsilon_u = \frac{K_p}{t} = -\frac{3}{2} \frac{u^2}{H_0} = -\frac{3}{2} \frac{H_0 u_0^2}{t_0} = -4.6 \times 10^{-7} \frac{m^2}{s^2},$$

where $u_0 \equiv u (t = t_0) \approx 354.6 km/s$ is the one-dimensional velocity dispersion of all dark matter particles and $t_0$ is the present time.

The constant $\varepsilon_u$ represents the rate of energy cascade across different scales. The negative value $\varepsilon_u < 0$ reflects the direction (inverse) from small to large mass scales. Since baryons and dark matter are coupled together through gravitational interaction, the flow of baryonic matter should share the same rate of cascade that can be demonstrated by galaxy rotating curves (see Xu 2022l, Fig. 10). This work focus on the energy cascade in the flow of dark matter.

3 THE 2/3 AND -4/3 LAWS FOR ENERGY AND DENSITY

To develop statistical theory of dark matter flow on all scales, different statistical measures can be introduced including the correlation, structure, dispersion functions, and power spectrum for density, velocity and potential fields (Xu 2022f,i). Among different measures,
Figure 2. The time variation of kinetic and potential energies from $N$-body simulation. Both exhibit a power-law scaling with scale factor $a$, i.e. $K_P$ and $P_V \propto a^{3/2} \propto e_{nu} t$. The proportional constant $e_{nu}$ is estimated in Eq. (2).

Figure 3. Sketch of longitudinal and transverse velocities, where $u_T$ and $u_L$ are transverse velocities at two locations $x$ and $x'$. $u_L$ and $u'_L$ are two longitudinal velocities.

structure functions are of particular interest that describes how energy is distributed and transferred across different length scales. For a pair of particles at two different locations $x$ and $x'$ with velocity $u$ and $u' = u$, the second order longitudinal structure function $S_2^{LP}(r,t)$ (pairwise velocity dispersion in cosmology terms) is defined as

$$S_2^{LP}(r,t) = \left\langle (\Delta u_L)^2 \right\rangle = \left\langle (u'_L - u_L)^2 \right\rangle,$$

(3)

where $u_L = u \cdot \hat{r}$ and $u'_L = u' \cdot \hat{r}$ are two longitudinal velocities. The distance $r = |r| = |x' - x|$ and the unit vector $\hat{r} = r/r$ (see Fig. 3). For a given scale $r$, all particle pairs with the same separation $r$ can be identified in N-body simulation. The particle position and velocity data were recorded to compute the structure function in Eq. (3) by averaging over all pairs with the same $r$ (i.e. a pairwise average).

In incompressible flow, the structure function has a small scale limit $S_2^{LP}(r \to 0) = 0$ because of $u_L \approx u'_L$ due to the viscous force. However, in dark matter flow, $S_2^{LP}(r \to 0) = 2u^2 \neq 0$ due to the collisionless nature (Xu 2022), where $u^2$ is velocity dispersion in Eq. (2). The pair of particles with a sufficiently small $r$ is more likely from the same halo, while different pairs can be from different halos. Kinetic energy of particle pairs on scale $r$ includes contributions from both the relative motion of two particles and the motion of halos that particle pair resides in. The kinetic energy from the motion of halos is relatively same for different pairs. Kinetic energy involved in the energy cascade should be the part due to the relative motion.

Since original pairwise dispersion $S_2^{LP}(r)$ includes the total kinetic energy on scale $r$, a reduced structure function $S_2^{LP}(r) = S_2^{LP}(r) - 2u^2$ can be introduced to take the common part out and include only the part from relative motion with the right limit $\lim_{r \to 0} S_2^{LP} = 0$. This description indicates that $S_2^{LP}(r)$ should be determined by and only by $e_{nu} (m^2/s^3)$, gravitational constant $G (m^3/kg \cdot s^2)$, and scale $r$. By simple dimensional analysis, this reduced structure function must follow a two-thirds law, i.e. $S_2^{LP}(r) \propto (e_{nu})^{2/3} r^{2/3}$.

Figure 4 plots the variation of $S_2^{LP}$ with scale $r$ at different redshifts $z$ from N-body simulations. The range $S_2^{LP} \propto (e_{nu})^{2/3} r^{2/3}$ can be clearly identified below a critical length scale $r_1 = -u_0^2/e_{nu}$. This range is formed due to the formation of halos and inverse energy cascade. On small scale, $S_2^{LP}$ kinetic energy) should finally read

$$S_2^{LP}(r) = a^{3/2} \beta^2_2 (-e_{nu})^{2/3} r^{2/3},$$

(4)

where the proportional constant $\beta_2 = 9.5$ can be found from Fig. 4. Since $S_2^{LP}$ represents the kinetic energy of relative motion on scale $r$, a different form of the two-thirds law (Eq. (4)) can be obtained. By introducing a typical velocity $v_\tau$ on a given scale $r$,

$$v_\tau^2 = S_2^{LP}(r) \left( \frac{2}{3} \beta_2^2 r^{3/2} \right),$$

(5)

the two-thirds law in Eq. (4) can be equivalently written as,

$$-e_{nu} = 2v^2 \frac{r}{v_\tau} = \alpha_r v_\tau \frac{2v^2}{r/v_\tau} = \frac{2v^2}{r},$$

(6)

where $\alpha_r$ is the scale of acceleration. Equation (6) also describes the cascade of kinetic energy in the inner halo region ($r < r_s$, where $r_s$ is the scale radius). The kinetic energy $v_\tau^2$ on scale $r$ is cascaded to large scale during a turnaround time of $t_r = r/v_\tau$. Combining Eq. (6) with the virial theorem $Gm_r/r \propto v_\tau^2$ on scale $r$, we can easily obtain the typical mass $m_r$ (enclosed within $r$), density $\rho_r$, velocity...
\[ v_r, \text{ and time } t \text{ on scale } r, \text{ all determined by } \varepsilon_u, G, \text{ and } r:\]
\[ m_r = \alpha_r r^{-0.5} G^{-1.5}/3 \quad \text{and} \quad \rho_r = \beta_r \varepsilon_u G^{-1} r^{-4}/3,\]
\[ v_r \propto (-\varepsilon_u r)^{1/3} \quad \text{and} \quad t \propto (-\varepsilon_u)^{-1/3} r^{2/3}, \quad (7) \]
where \( \alpha_r \) and \( \beta_r \) are two numerical constants. The predicted four-thirds law \( \rho_r(r) \propto r^{-4/3} \) for mean density enclosed in scale \( r \) can be directly tested by data from galaxy rotation curves (see Fig. 5).

### 4 HALO DENSITY SLOPE AND MEAN RADIAL FLOW

On small scale, inner halo region is assumed to be fully virialized for Eq. (7) to be valid. A vanishing radial flow \( u_r \) is expected from the stable clustering hypothesis, i.e. no net stream motion in physical coordinate along radial direction (Mo et al. 2010). With \( u_r \equiv 0 \), there is no mass, momentum, and energy exchanges between different shells. However, halos are non-equilibrium dynamic objects, whose internal structure should be dependent on the radial flow. This relation we\'ll develop might be useful for core/cusp controversy.

Since halo density models often involve a scale radius \( r_s \) (the length scale of a halo), we may introduce a reduced spatial-temporal variable \( x = x(r (t) = c(t)/r (t), \text{where } c \equiv c(t) \text{ is halo concentration and } r_h(t) \text{ is the virial size for halo of mass } m_h(t). \) Derivatives with respect to \( t \) and \( r \) can be derived using the chain rule,
\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial x}{\partial t} = \frac{x}{t} \frac{\partial}{\partial t} \frac{\partial x}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial r} = \frac{\partial}{\partial x} \frac{\partial x}{\partial r}. \quad (8) \]

A general function \( F(x) \) can be introduced such that the mass \( m_r(r, t) \) enclosed in radius \( r \) and halo density \( \rho_h(r, t) \) and can all be expressed in terms of function \( F(x) \),
\[ m_r(r, t) = m_h(t) \left( \frac{F(x)}{F(c)} \right), \quad (9) \]
\[ \rho_h(r, t) = \frac{1}{4\pi r^2} \frac{\partial m_r(r, t)}{\partial r} = \frac{m_h(t)}{4\pi r^2} \left( \frac{F(x)}{x^2 F(c)} \right). \]
The time derivative of \( \rho_h(r, t) \) can be obtained from Eq. (9),
\[ \frac{\partial \rho_h(r, t)}{\partial t} = \frac{1}{4\pi r^2} \frac{\partial^2 m_r(r, t)}{\partial r \partial t}. \quad (10) \]
The mass continuity equation for a spherical halo simply reads,
\[ \frac{\partial \rho_h(r, t)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho_h(r, t) u_r(r, t) \right) = 0, \quad (11) \]
where \( u_r(r, t) \) is the mean radial flow velocity. From Eqs. (10) and (11), the enclosed mass \( m_r \) related to the radial flow as
\[ \frac{\partial m_r(r, t)}{\partial t} = -4\pi r^2 u_r(r, t) \rho_h(r, t). \quad (12) \]

With \( m_r \) and \( \rho_h \) from Eq. (9), the radial flow \( u_r \) simply reads
\[ u_r = \frac{1}{4\pi r^2} \frac{\partial}{\partial t} \frac{m_r(r, t)}{\partial t}. \quad (13) \]

While from Eq. (9) for \( m_r \), we have
\[ \frac{\partial}{\partial t} \left( \frac{\partial m_r}{\partial \ln t} \right) = - \frac{\partial \ln m_r}{\partial \ln t} \left( x \frac{F(x)}{x^2 F(c)} \right) + \frac{c F'(c)}{F(x)} \frac{\partial}{\partial \ln t} \frac{\partial \ln c}{\partial \ln t}. \quad (14) \]
Substituting into Eq. (13), the normalized radial flow \( u_h \) reads
\[ u_h(x) = u_r \frac{t}{r} = \frac{1}{x} \frac{\partial \ln m_r}{\partial \ln t} + \left( \frac{\partial \ln m_r}{\partial \ln x} + \frac{\partial \ln m_h}{\partial \ln t} \right) \frac{F(x)}{F(x)}. \quad (15) \]
For fast growing halos in their early stage with a constant \( c = 3.5 \) and \( r_s(t) \propto m_h(t) \propto t \) (Xu 2021b, 2022e), the cored density (pISO, Einasto, etc.) leads to \( u_h = 2x/3 \), while NFW profile leads to \( u_h = x/2 \) (see Eqs. (15) and (19)). Taking the derivative in Eq. (15), the density slope \( \gamma \) can be obtained exactly as,
\[ \gamma = \frac{\partial \ln \rho_h}{\partial \ln x} + \frac{\partial \ln m_r(r, t)}{\partial \ln t} = - \frac{\partial \ln r_s}{\partial \ln t} - 2. \quad (16) \]
Clearly, the spatial variation of slope \( \gamma \) comes from the radial flow \( u_h \), while the time variation of \( \gamma \) comes from \( r_s(t) \) and \( m_r(t, t) \) due to mass accretion. For fully virialized halos or the virialized inner core, we should have \( u_h \equiv 0 \) such that the asymptotic slope \( \gamma \) reads
\[ \gamma = \frac{\partial \ln \rho_h}{\partial \ln x} = \frac{\partial \ln m_r(r, t)}{\partial \ln r_s} - 3. \quad (17) \]

On halo scale, energy cascade with a constant rate \( \varepsilon_u \) is valid for all scales \( r \leq r_s \). Taking the enclosed mass \( m_r(r, t) \) as the mass scale in Eq. (7), we found \( m_r(r, t) \propto r_s^{5/3} \). For fully virialized halos with \( u_h \equiv 0 \), slope \( \gamma = -4/3 \) or a cuspy density \( \rho_h(r) \propto r^{-4/3} \) can be obtained from Eq. (16). Therefore, fully virialized halos should have universal cuspy density profiles due to energy cascade. In other words, simulated halos might have different slope \( \gamma \) due to nonzero radial flow and different mass accretion rate (Eq. (16)). The baryonic feedback provides potential mechanisms to enhance the gradient of \( u_h \) (deformation rate) in Eq. (16) and flatten the inner density. Finally, since \( u_h(x = 0) \equiv 0 \), there exists an asymptotic slope \( \gamma \) at \( x = 0 \) that is dependent on the local gradient of \( u_h \) around \( x = 0 \). Therefore, a better density profile (modified Einasto) can be proposed
\[ \rho_h(r) = \rho_0 x^\gamma \exp \left( -\frac{2}{\alpha} \frac{x^\gamma}{\alpha} \right), \quad (18) \]
\[ F(x) = \Gamma \left( \frac{3 + \gamma}{\alpha} \right) - \Gamma \left( \frac{3 + \gamma}{\alpha} - \frac{2}{\alpha} \right) \]
Here \( \rho_0 \) and \( \alpha \) are density and shape parameters and slope \( \gamma < 0 \).

### 5 TESTING -4/3 LAW AGAINST ROTATION CURVES

Next, the predicted four-thirds law in Eq. (7) \( (\rho_r(r) \propto r^{-4/3}) \) is tested against galaxy rotation curves that contain important information for dark matter halos. In practice, rotational curves can be first decomposed into contributions from different mass components. Model parameters for halo density (scale radius \( r_s \) and density scale \( \rho_0 \)) are obtained by fitting to the decomposed rotation curve. Here we use three sources of galaxy rotation curves,

(i) SPARC (Spitzer Photometry & Accurate Rotation Curves) including 175 late-type galaxies (Lelli et al. 2016; Li et al. 2020);
(ii) DMS (DiskMass Survey) including 30 spiral galaxies (Martinsson et al. 2013);
(iii) SOFUE (compiled by Sofue) with 43 galaxies (Sofue 2016).

For pseudo-isothermal (pISO) and NFW density models, we have
\[ \rho_{pISO} = \frac{\rho_0}{1 + x^2}, \quad F(x) = \log(1 + x) \approx \frac{x^2}{2}, \quad (19) \]
\[ \rho_{NFW} = \frac{\rho_0}{(x + 1)^2}, \quad F(x) = \log(1 + x) - \frac{x}{(1 + x)} \approx \frac{x^2}{2}, \quad (19) \]
where \( \rho_0 \) is a density parameter. From Eq. (9), halo density at \( r_s \)
\[ \rho_h(r_s) = \rho_h(x = 1) = \frac{\rho_0}{3 F'(c)} = \frac{\rho_0}{3 F'(c)} \]
where \( \rho_0 \) is the mean halo density. In this work, \( \Delta_e = 200 \) and critical density \( \bar{\rho}_{crit} = 3H_0^2/8\pi G = 10^{-26} \text{kg/m}^3 \). Using Eqs. (19) and (20),
concentration $c$ can be obtained from fitted model parameter $\rho_0$. The mean density within $r_s$ (density scale radius $r_s$) now reads

$$ \rho_s(r_s) = \frac{m_r(r_s)}{4\pi r_s^3/3} = \frac{F(1)c^3/\rho}{F(c)} = \frac{F(1)c^3}{F(c)} \Delta c \rho_{\text{crit}}, $$

(21)

Figure 5 presents the variation of typical density $\rho_s$ with scale $r_s$ obtained from three sources of galaxy rotation curves. The four-thirds law (Eq. 7) is also plotted for comparison with coefficients $\beta_r = 1.26$ and $\alpha_t = 5.28$ obtained from these data. Clearly, dark matter halos obtained from rotation curves follow the four-thirds law across 6 order of size and density. Equivalently, the inner limiting density $\rho_h = \rho_s^{3/4}$ for virialized halos with a vanishing radial flow is also confirmed by this plot using Eq. (17). Finally, this plot confirms the existence of a constant rate of cascade $\varepsilon_u$ below the largest scale $r_t$. Other relevant quantities (density, pressure, energy etc.) on scale $r_t$ are similarly determined by constants $\varepsilon_u$, $G$, and $u_0$ (Xu 2022k).

Finally, we can choose the circular velocity at $r_s$ as the typical velocity $v_s = \sqrt{G m_r(r_s)/r_s}$ such that $-\varepsilon_u = v_s^2/(\gamma_s r_s)$ (from Eq. 7). Figure 6 presents the rate of energy cascade $\varepsilon_u$ obtained from three sources of galaxy rotation curves with $\gamma_s = 6.83$. The dispersion in data might come from the spatial intermittence of energy cascade such that in different local environment may have slightly different $\varepsilon_u$ (also see Xu 2022i). Dwarf galaxies tend to have smaller $\varepsilon_u$ due to tidal stripping.

6 SCALES FOR SELF-INTERACTING DARK MATTER

To solve the core-cusp problem, the first option is the self-interacting dark matter (SIDM) model, where the cascade theory can be used to determine relevant scales. The cross-section $\sigma/m$ of self-interaction should introduce additional scales $\rho_{\gamma\gamma}$, $\rho_{\gamma\eta}$, and $\rho_{\eta\eta}$ (see Fig. 5), beyond which no structure can be formed due to self-interaction. These scales can be obtained by requiring at least one scatter per particle during the typical time $t_r$, i.e. $\rho_r(\sigma/m) v_r t_r = 1$ in Eq. (7). Combine this with the virial theorem and constant energy cascade in

$$ \varepsilon_u = \text{SPARC NFW fitting} $$
$$ \varepsilon_u = \text{SPARC pISO fitting} $$
$$ \varepsilon_u = \text{DMS NFW fitting} $$
$$ \varepsilon_u = \text{DMS pISO fitting} $$
$$ \varepsilon_u = \text{SOFUE NFW fitting} $$

Figure 6. The constant rate of inverse energy cascade $\varepsilon_u$ from galaxy rotation curve data. The dispersion in data might come from the spatial intermittence of energy cascade. Dwarf galaxy tends to have smaller $\varepsilon_u$. The flow of baryons shares the same rate of energy cascade (see Xu 2022j, for details). In cross-section $\sigma/m = 0.01 m^2/kg$ used in SIDM cosmological simulations (Rocha et al. 2013), halo core size is between $r_t$ and the maximum core size $r_{cmax} \approx 1kpc$. In other words, maximum core size $r_{cmax}$ formed from self-interaction (not any other mechanisms) can be used to identify the cross-section of postulated self-interaction.

In hydrodynamic turbulence, the smallest length scale $\eta = (\nu^2/\epsilon)^{1/4}$ (Kolmogoroff 1941) is determined by the fluid viscosity $\nu$ and rate of energy cascade $\epsilon$. The kinetic energy is injected at large scale and cascaded down to small scales. Below scale $\eta$, structures (eddies) are destroyed by viscous force and kinetic energy is dissipated into heat to increase system entropy (Fig. 1). Here $\epsilon$ can be a variable manually controlled or adjusted by the rate of energy injection on large scale. For faster mixing, thinking about stirring the coffee-milk fluid harder (the larger $\epsilon$), the scale $\eta$ would be smaller and mixing would be faster on that scale. However, for dark matter flow in our universe, inverse (NOT direct) energy cascade is required for structure formation on large scales. The rate of energy cascade $\varepsilon_u$ is a constant on small scales $r < r_t$, which should be a fundamental property of dark matter. Therefore, for self-interacting dark matter, it should be interesting and might be challenging to identify the potential mechanism on how the halo structure is generated on the smallest scale $r_t$ along with the increasing system entropy, as this is exactly opposite to what we see in turbulence.

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7 SCALES FOR COLLISIONLESS DARK MATTER

Now let’s consider the second option: dark matter is still fully collisionless with flat halo core formed by other possible mechanisms. In this scenario, due to the collisionless nature, the four-thirds law should extend from galaxy scale to the smallest scale where the quantum effect becomes important (Fig. 7). This extension is more than 30 orders in size, which hopefully allows us to predict the mass, size and properties of dark matter particles (the $X$ particle) from three basic constants, i.e. $\dot{e}_u$, $G$, and Planck constant $\hbar$ (see Xu 2022), for details).

(i) For fully collisionless dark matter, there exists a unique “symmetry” between position and velocity in phase space. At any given location, collisionless particles can have multiple values of velocity (multi-stream regime). Similarly, particles with same velocity can be found at different locations. This “symmetry” in phase space is not possible for any non-relativistic baryonic matter.
(ii) Due to the long-range gravitational interaction, there exist fluctuations (uncertainty) not only in position ($x$) and velocity ($v = \dot{x}$), but also in acceleration ($a = \ddot{x}$), which provides a potential explanation for the origin of critical acceleration $\dot{e}_0$ in MOND (modified Newtonian dynamics) (see Xu 2022k, for details).

With $\psi(x)$, $\varphi(p)$, and $\mu(a)$ as wave functions for position, momentum, and acceleration, we can write:

$$\psi(x) = \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} \varphi(p) \cdot e^{i p x / h} dp,$$

$$\varphi(p) = \frac{1}{\sqrt{2\pi i \mu_x}} \int_{-\infty}^{\infty} \mu(a) \cdot e^{i a p / \mu_x} da,$$  \hspace{1cm} (24)

where constant $\mu_x = -m_x \dot{e}_u = 7.44 \times 10^{-22} \text{kg} \cdot \text{m}^2 / \text{s}^3$ (also discussed in Xu 2022j). An energy scale $\sqrt{\hbar \mu_x} \approx 10^{-9} \text{eV}$ can be obtained for the possible dark radiation due to dark matter annihilation or decay. Here we have two pairs of conjugate variables i) position $x$ and momentum $p$, and ii) momentum $p$ and acceleration $a$. By following the standard wave mechanics (will not repeat here), two uncertainty principles can be established for fluctuations of position, momentum, and acceleration for collisionless dark matter,

$$\sigma_x \sigma_p \geq \hbar / 2 \text{ and } \sigma_p \sigma_a \geq \mu_x / 2.$$  \hspace{1cm} (25)

More experiment data might be required to test this postulation.

8 CONCLUSIONS

Smalls scale challenges suggest some missing pieces in our current understandings of dark matter. A cascade theory for dark matter flow might provide extra insights. The energy cascade with a constant rate $\dot{e}_u$ across different scales is a fundamental feature of dark matter flow. N-body simulation suggests a two-thirds law, i.e. the kinetic energy $V_r^2 \propto (\dot{e}_u r)^{2/3}$ on scale $r$. This is equivalent to a four-thirds law for density on the same scale, i.e. $\rho_r \propto \dot{e}_u^{2/3} G^{-1} r^{-4/3}$, which can be directly confirmed by data from galaxy rotation curves. By identifying key constants on relevant scales, limiting scales for collisionless (determined by $\dot{e}_u$, $G$, $\hbar$) or self-interacting dark matter (by $\dot{e}_u$, $G$, $\sigma / m$) might be obtained. On halo scale, based on the continuity equation, halo density is shown to be closely dependent on the radial flow and mass accretion. The asymptotic density slope $\gamma = -4/3$ can be obtained and fully virtualized halos with a vanishing radial flow have a universal density profile. Simulated halos can have different limiting slopes due to finite radial flow and mass accretion, with a modified Einasto density profile proposed accordingly.

DATA AVAILABILITY

Two datasets for this article, i.e. a halo-based and correlation-based statistics of dark matter flow, are available on Zenodo (Xu 2022b,c), along with the accompanying presentation "A comparative study of dark matter flow & hydrodynamic turbulence and its applications" (Xu 2022a). All data are also available on GitHub (Xu 2022d).

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