Variational Study of SU(3) Gauge Theory by Stationary Variance

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Abstract. The principle of stationary variance is advocated as a viable variational approach to gauge theories. The method can be regarded as a second-order extension of the Gaussian Effective Potential (GEP) and seems to be suited for describing the strong-coupling limit of non-Abelian gauge theories. The single variational parameter of the GEP is replaced by trial unknown two-point functions, with infinite variational parameters to be optimized by the solution of a set of coupled integral equations. The stationary conditions can be easily derived by the self-energy, without having to write the effective potential, making use of a general relation between self-energy and functional derivatives that has been proven to any order. The low-energy limit of pure Yang-Mills SU(3) gauge theory has been studied in Feynman gauge, and the stationary equations are written as integral equations for the gluon and ghost propagators. A physically sensible solution is found for any strength of the coupling. The gluon propagator is finite in the infrared, with a dynamical mass that decreases as a power at high energies. At variance with some recent findings in Feynman gauge, the ghost dressing function does not vanish in the infrared limit and a decoupling scenario emerges as recently reported for the Landau gauge.

1. Introduction

There is a growing consensus on the utility of variational methods as analytical tools for a deeper understanding of the infrared (IR) limit of non-Abelian gauge theories. The IR slavery of these theories makes the standard perturbation theory useless below some energy scale, and our theoretical knowledge of the IR limit relies on lattice simulation and on non-perturbative techniques like functional renormalization group and Dyson-Schwinger equations. Variational methods have been developed [1, 2, 3, 4, 5, 6, 7] as a complement to these analytical approaches and quite recently the method of stationary variance[8, 9] has been advocated as a powerful second order extension of the Gaussian Effective Potential (GEP)[10, 11, 12, 13]. The GEP is a genuine variational method and has been successfully applied to many physical problems in field theory, from scalar and electroweak theories[13, 14, 15, 16, 17, 18, 19, 20] to superconductivity[21, 22, 23] and antiferromagnetism[24], but turns out to be useless for gauge interacting fermions[25]. Actually, since the GEP only contains first order terms, it is not suited for describing the minimal coupling of gauge theories that has no first-order effects.

Several methods have been explored for including fermions[20] and higher order corrections[26], sometimes spoiling the genuine variational character of the method. By a formal higher order extension of the GEP[27] the method of stationary variance has been developed as
a genuine variational method that keeps in due account second order effects and seems to be suited to deal with the minimal coupling of gauge theories. While the method has been shown to be viable for the simple Abelian case of QED[28], its full potentialities have not been explored yet. As a non-perturbative tool that can deal with fermions in gauge theories, the method seems to be very useful for exploring the IR limit of QCD, and its natural field of application is the non-Abelian SU(3) gauge theory[29].

As a first step, in Ref.[29] we explored the solution of the stationary equations for pure Yang-Mills SU(3) theory. While the method is a genuine variational tool that does not require any small parameter, the technique is based on standard Feynman rules of perturbation theory. The single variational parameter of the GEP is replaced by trial unknown two-point functions, with infinite variational parameters to be optimized by the solution of a set of integral equations, the stationary equations. However, these equations can be easily derived by the self-energy, without having to write the effective potential, making use of a general relation between self-energy and functional derivatives that has been proven to any order[27]. For pure Yang-Mills theory the method of stationary variance provides a set of non-linear coupled integral equations whose solutions are the propagators for gluons and ghosts. Therefore the work has a double motivation: the technical aim of showing that the method is viable and a solution does exist (which was not obvious nor proven in general), and the physical interest on the gluon propagator in the IR limit, where its properties seem to be related to the important issue of confinement.

On the technical side, having shown that a sensible untrivial solution does exist is a major achievement that opens the way to a broader study of QCD by the same method. Inclusion of quarks would be straightforward as some fermions, the ghosts, are already present in the simple Yang-Mills theory, and they already seem to play well their role of canceling the unphysical degrees of freedom.

On the physical side, the properties of the gluon propagator in Feynman gauge are basically unexplored. In Coulomb gauge[2, 7, 3, 4, 5] and in Landau gauge[6, 30, 31, 32, 33, 34, 35, 36, 37, 38] there has been an intense theoretical work in the last years. In Landau gauge theoretical and lattice data are generally explained in terms of a decoupling regime, with a finite ghost dressing function and a finite massive gluon propagator. The more recent findings confirm the original prediction[39] of a dynamical mass generation for the gluon. In Feynman gauge we do not find a very different scenario.

2. Details of the calculation

The method has been described in detail in Ref.[29]. Here, we review the main steps for the derivation of the stationary equations.

The Lagrangian of pure Yang-Mills SU(3) gauge theory is

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{fix}$$  \hspace{1cm} (1)

where $\mathcal{L}_{YM}$ is the Yang-Mills term

$$\mathcal{L}_{YM} = -\frac{1}{2} \text{Tr} \left( \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right)$$  \hspace{1cm} (2)

and $\mathcal{L}_{fix}$ is the gauge fixing term.

The quantum effective action $\Gamma[A']$ can be written as

$$e^{i\Gamma[A']} = \int_{1PI} \mathcal{D}A e^{iS[A'+A]} J_{FP}[A' + A]$$  \hspace{1cm} (3)

where $A'$ is an external background field (to be set to zero), $S[A]$ is the action, $J_{FP}[A]$ is the Faddev-Popov determinant and the path integral represents a sum over one particle irreducible (1PI) graphs.
Expressing the determinant $J_{FP}$ as a path integral over ghost fields $\omega_n(x)$, the effective action can be written as
\[
e^{i\Gamma} = \int_{FP} \mathcal{D}A_{\omega}\mathcal{D}A_{\omega^*} e^{iS_0[A_{\omega},\omega^*]}e^{iS_I[A_{\omega},\omega^*]}\tag{4}
\]
where the total action is
\[
S_{tot} = S_0 + S_I = \int \mathcal{L}_{YM}d^4x + \int \mathcal{L}_{fix}d^4x + S_{gh}
\tag{5}
\]
and $S_{gh}$ is the ghost term. We can split the action in the two parts, the free action $S_0$ and the interaction $S_I$, by insertion of trial functions[27]. We define the free action $S_0$ as
\[
S_0 = \frac{1}{2} \int A^{\mu\nu}D^{-1\mu\nu}_{ab}(x,y)A_{ab}^{\nu\mu}(y)d^4xd^4y + \int \omega_{ab}^\nu(x)G^{-1}_{ab}(x,y)\omega_{ab}(y)d^4xd^4y
\tag{6}
\]
where $D_{\mu\nu}^{ab}(x,y)$ and $G_{ab}(x,y)$ are unknown trial matrix functions. The interaction then follows as the difference
\[
S_I = S_{tot} - S_0
\tag{7}
\]
and can be written as the sum of a two-point term and three local terms: the ghost vertex, the three-gluon vertex and the four-gluon vertex respectively
\[
S_I = S_2 + \int d^4x [\mathcal{L}_{gh} + \mathcal{L}_3 + \mathcal{L}_4].\tag{8}
\]

In a concise notation, the two-point interaction term can be written as
\[
S_2 = \frac{1}{2} \int A \left[ D_0^{-1} - D^{-1} \right] A + \int \omega^* \left[ G_0^{-1} - G^{-1} \right] \omega
\tag{9}
\]
where $D_0$ and $G_0$ are the standard free-particle propagators for gluons and ghosts respectively. The three local interaction terms are
\[
\mathcal{L}_3 = -g f_{abc} (\partial_{\mu}A_{\mu}^a)A_{\nu}^{b\mu}A_{\nu}^{c\mu}
\]
\[
\mathcal{L}_4 = -\frac{1}{4} g^2 f_{abc} f_{ade} A_{\mu}^a A_{\nu}^f A_{\rho}^{d\mu} A_{\rho}^{e\nu}
\]
\[
\mathcal{L}_{gh} = -g f_{abc} (\partial_{\mu}\omega_{\mu}^a)\omega_{ab}A_{\nu}^{c\mu}
\tag{10}
\]
where $g$ is the $SU(3)$ coupling constant and $f_{abc}$ are the structure constants of the group. The trial functions $G_{ab}$, $D_{\mu\nu}^{ab}$ cancel in the total action $S_{tot}$ which is exact and cannot depend on them. Thus this formal decomposition holds for any choice of the trial functions.

Standard Feynman graphs can be drawn for this theory with the trial propagators $D_{\mu\nu}^{ab}$ and $G_{ab}$ that play the role of free propagators, and the vertices that can be read from the interaction action $S_I$ in Eq.(8). However, the expansion is not in power of the strength parameter $g$, but must be regarded as an expansion in powers of the optimized interaction $S_I$.

Genuine variational methods can be established by the functional derivative of the effective potential with respect to the trial propagators, in order to fulfill some given stationary conditions[27]. Moreover, as recently discussed in Ref.[27], the functional derivatives can be written in terms of self-energy graphs, without having to write the effective potential
\[
\frac{\delta V_n}{\delta D_{\mu\nu}^{ab}(p)} = \frac{i}{2} \left( \Pi_{\nu}^{\mu,ba}(p) - \Pi_{\nu}^{\mu,ba}(p-1) \right),\tag{11}
\]
\[
\frac{\delta V_n}{\delta G_{ab}(p)} = -i \left( \Sigma_{\nu}^{ba}(p) - \Sigma_{\nu}^{ba}(p-1) \right),\tag{12}
\]
where the nth-order gluon polarization function \( \Pi_{n}^{\mu,ab} \) and the nth-order ghost self-energy \( \Sigma_{n}^{ab} \) are the sum of all nth-order connected two-point graphs without tadpoles, while \( V_{n} \) is the nth-order term of the effective potential.

In this work we use the method of stationary variance\[8, 9\] that has been shown to be viable in simple Abelian gauge theories like QED\[28\]. According, the self-energy graphs are required up to second order, and the stationary conditions for the variance follow as\[29\]

\[
\Pi_{2}^{\nu,ab}(p) = \Pi_{1}^{\nu,ab}(p) \\
\Sigma_{2}^{ba}(p) = \Sigma_{1}^{ba}(p). \tag{13}
\]

The choice of Feynman gauge, \( \xi = 1 \), simplifies the calculation once we take

\[
P_{\mu\nu}^{ab}(p) = \delta_{ab}\eta_{\mu\nu}D(p) = \delta_{ab}\eta_{\mu\nu}\frac{f(p)}{-p^{2}} \tag{14}
\]

where \( D(p) \) is an unknown trial function and \( f(p) \) is a trial gluon dressing function. That choice is perfectly legitimate, but is equivalent to a variation of the trial propagator inside a more limited class of functions.

Defining the summed quantity

\[
\Pi_{n}(p) = \frac{1}{4(N^{2} - 1)} \sum_{ab,\mu\nu} \delta_{ab}\eta_{\mu\nu}(p)\Pi_{n}^{\mu,ab}(p) \tag{15}
\]

where \( \eta_{\mu\nu} \) is the metric tensor, we obtain the simple stationary equation

\[
\Pi_{2}(p) = \Pi_{1}(p) \tag{16}
\]

while in any case, color symmetry ensures that we can always take

\[
G_{ab}(p) = \delta_{ab}G(p) = \delta_{ab}\frac{\chi(p)}{-p^{2}} \tag{17}
\]

where \( \chi(p) \) is a trial ghost dressing function.

Despite their simple shape, the stationary equations are a set of coupled non-linear integral equations for the trial functions \( D, G \). In terms of 1PI proper functions and dressing functions, switching to the Euclidean formalism, the stationary equations can be written as

\[
\chi(p_{E}) = \left[ 1 + \frac{1}{p_{E}^{2}} \Sigma_{2}^{\ast}(p_{E}) \right] \\
f(p_{E}) = \frac{p_{E}^{2}}{p_{E}^{2} + M^{2}} \left[ 1 - \frac{\Pi_{2}^{\ast}(p_{E})}{p_{E}^{2} + M^{2}} \right]. \tag{18}
\]

Of course an iterative solution of these equations requires a numerical evaluation of the one- and two-loop graphs contributing to the second-order proper functions \( \Pi_{2}^{\ast}, \Sigma_{2}^{\ast} \) that we need at each step as functionals of the unknown trial dressing functions \( f, \chi \). The details of the numerical evaluation have been described in Ref.[29].

3. Regularization and numerical solutions

For any choice of the bare coupling \( g \), Eqs.(18) can be iterated and show a fast convergence towards a stable solution. The existence of a stable and physically reasonable solution for the
Figure 1. The renormalized propagator $D_R(p)$ in physical units for the bare coupling $g = 0.35, 0.40, 0.65, 0.75, 0.90, 1$. The energy scale has been fixed in order to fit the lattice data of Ref.[36] ($g = 1.02$, L=96) that are displayed as filled circles.

method of stationary variance is one of the main achievements of the present work, since the method can be developed further as a non-perturbative tool for the study of QCD.

For a numerical solution we first need to regularize all the diverging integrals by a non-perturbative multiplicative renormalization scheme. Since the variational method is not gauge invariant, the regulator can even break gauge symmetry, as we expect that gauge invariance should be recovered in physical observables only approximately in the present approximation. The simple choice of an energy cutoff in the Euclidean space $p^2 < \Lambda^2$ has the merit of giving physical results that are directly comparable with lattice simulations where a finite lattice acts just like an energy cutoff. Moreover, lattice simulations are the most natural benchmark for any variational calculation in the low energy limit. Thus we borrow from lattice simulation the regulating scheme and its physical interpretation in terms of a bare interaction parameter $g = g(\Lambda)$ which is supposed to be dependent on the energy scale $\Lambda$. Renormalization Group (RG) invariance requires that the physical observables are left invariant by a change of scale $\Lambda \to \Lambda'$ that is accompanied by the corresponding change of the bare interaction $g(\Lambda) \to g(\Lambda')$. Then, renormalized physical quantities can be defined that do not depend on the cutoff. We fix the scale by a direct comparison with the available lattice data. It is important to point out that the present regularization scheme does not need the inclusion of any counterterm in the Lagrangian and especially mass counterterms that are forbidden by the gauge invariance of the Lagrangian.

The gluon propagator is reported in Fig.1 for several values of the bare coupling. By an appropriate change of scale, the renormalized propagator $D_R(p)$ becomes almost independent of $g$ and all the curves fall one on top of the other. Here the single curves are rescaled in order to fall on top of the $g = 1$ bare propagator. Scaling is rather good with the exception of the far infrared
The renormalized second-order propagator is shown in physical units for \( g = 0.4, 0.65, 0.9, 1.15, 1.4, 1.65, 1.9, 2.15 \). By scaling, all the curves fall one on top of the other. The energy scale is fixed by a rough fit of the Landau-gauge lattice data of Ref.\[36\] (\( g = 1.02, L=96 \)) that are displayed as filled circles.

A physical energy scale is fixed in order to give a rough fit of the lattice data of Ref.\[36\] that are included in the figure. We must warn that the data of the simulation are obtained in the Landau gauge, while the present calculation is in Feynman gauge. While the propagator is not expected to be gauge invariant, the physical mass should not be too much sensitive to the gauge choice, and we may extract a rough estimate of the energy scale by comparison of the data. Unfortunately we could not find any recent lattice data in Feynman gauge to compare with. Despite the use of a different gauge, the main features of the lattice propagator seem to be reproduced by the trial function, with a pronounced flat behavior in the infrared.

We can regard the optimized trial propagators \( D(p) \) and \( G(p) \) as the free propagators in an expansion in powers of the interaction \( S_I \). Thus, it is quite reasonable to think that the actual approximation could be improved by just adding higher-order Feynman graphs. Higher order functions \( D_{(n)}, G_{(n)} \) can be built by Dyson equations

\[
D_{(2)}^{-1}(p) = \eta_{\mu\nu}(p^2 + M^2) - \Pi_{2\mu\nu}^*(p) \\
G_{(2)}^{-1}(p) = G^{-1}(p) - \Sigma_1(p) - \Sigma_2^*(p)
\]

The renormalized second-order propagator is shown in Fig.2 for several values of the bare coupling \( g \). By a proper scaling, all curves can be put one on top of the other, while a physical energy scale has been chosen in order to give a rough fit of the lattice data. The scaling is now very good, but the agreement with the (Landau gauge) lattice data of Ref.\[36\] is very poor and only a very loose energy scale can be fixed by this method. Once more, we expect that relevant
Figure 3. The second-order dynamical mass of Eq.(22) is reported in a log-log plot with the same scaling factors and energy scale of Fig.2, for $g = 0.9, 1.15$ and $1.4$. The straight dotted line shows the power behavior $\sim (p^2)^{-\eta}$ with the exponent $\eta = 1.5$. Differences may exist between propagators in different gauges and these differences may also depend on the formal definition of the propagator that is not an observable quantity but just an intermediate scheme-dependent step of the full calculation.

The renormalized second order propagator $D_{(2)}(p)$ in Fig.2 can be fitted quite well by the simple expression

$$D_{(2)}(p) \approx \frac{Z}{p^2 + m^2}$$  \hspace{1cm} (21)

yielding a physical mass parameter $m \approx 0.8$ GeV that is basically independent of $g$.

We can introduce a better definition for the dynamical mass if we take

$$D_{(2)}(p) = \frac{Z}{p^2 + m^2(p)}$$  \hspace{1cm} (22)

where now $m(p)$ is a function which is supposed to decrease as a power, $m^2 \sim (p^2)^{-\eta}$ for large energies.

The parameter $Z$ can be tuned in order to get a power-law behavior that would appear as a linear curve in a log-log plot. For any bare coupling we find $Z \approx 1$, as we expected by the knowledge of the exact asymptotic limit. For instance, in the case of $g = 0.9$ the asymptotic behavior of $m(p)$ is fitted by $Z = 0.9978$. The exponent turns out to be $\eta = 1.5$ for any coupling, as shown in Fig.3 where the function $m^2(p)$ is reported for some different values of the bare coupling, with the same scaling factors and energy scale of Fig.2. Scale dependent values of $\eta$, oscillating in the range $1.08 < \eta < 1.26$, have been reported in Landau gauge by Ref.[34]. While this kind of plot enhances minor deviations from the exact scaling, we find the
same high-energy power-law behavior for different couplings, with a dynamical mass $m(p)$ that saturates at $m(0) \approx 0.8$ GeV.

4. Concluding remarks
In summary, one of the major achievements of the present work is the proof that a physically consistent solution does exist for the coupled set of non-linear integral equations that arise from the condition of stationary variance. Since pure Yang-Mills theory already contains fermions (the ghosts), inclusion of quarks in the formalism is straightforward, and would open the way to a broader study of QCD by the same technique.

Feynman gauge is interesting because the IR behavior of the theory is basically unexplored yet in that gauge. The general picture that emerges from the calculation confirms a decoupling scenario, with a finite ghost dressing function, a finite gluon propagator in the IR limit, and a dynamical mass that decreases as a power in the UV limit.

The method can be improved in many ways. We did not bother about gauge invariance in this first approach, but the properties of the polarization function, namely the correct cancellations of the unphysical degrees of freedom by the ghosts, show that the constraints of gauge invariance can be satisfied, at least approximately, by the variational solution. Actually the polarization function is found approximately transverse up to a constant mass shift due to the dynamical mass generation. As far as the solution satisfies, even approximately, the constraints imposed by gauge invariance, the method is acceptable on the physical ground. While some attempts could be made for enforcing gauge invariance[1, 30], a physically motivated choice for the gauge would probably improve the approximation. Landau gauge would be a good candidate, as it would enforce the transversality in the polarization function from the beginning.

An other interesting further development would come from the extension of the formalism to the general case of a finite external background field. For a scalar theory that kind of approach allows a consistent definition of approximate vertex functions by the functional derivative of the effective action. For the GEP these functions can be shown to be the sum of an infinite set of bubble graphs[16]. A similar approach would give a more consistent approximation for the gluon propagator in the present variational framework. Eventually, the inclusion of quarks would lead to a direct comparison with the low energy phenomenology of QCD.

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