Deformed Traces and Covariant Quantum Algebras for Quantum Groups $GL_{qp}(2)$ and $GL_{qp}(1|1)$

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Abstract: The $q$-deformed traces and orbits for the two parametric quantum groups $GL_{qp}(2)$ and $GL_{qp}(1|1)$ are defined. They are subsequently used in the construction of $q$-orbit invariants for these groups. General $qp$-(super)oscillator commutation relations are obtained which remain invariant under the co-actions of groups $GL_{qp}(2)$ and $GL_{qp}(1|1)$. The $GL_{qp}(2)$-covariant deformed algebra is deduced in terms of the bilinears of bosonic $qp$-oscillators which turn out to be the central extension of the Witten-type deformation of $sl(2)$ algebra. In the case of supergroup $GL_{qp}(1|1)$, the corresponding covariant algebras contain the $N = 2$ supersymmetric quantum mechanical subalgebras.

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Symmetry groups and symmetry algebras have played a very important role in the development of modern theoretical physics. Quantum groups, which are deformations (and, in some sense, a generalization) of usual groups have attracted a great deal of interest since the seminal papers by Drinfeld [1], Jimbo [2], Faddeev et al. [3], Woronowicz [4] and Manin [5]. These deformed (super)groups present the examples of Hopf algebras and have found application in as diverse areas of physics and mathematics as nonlinear integrable models, statistical mechanics, conformal field theory, knot theory and solutions of Yang-Baxter equations, etc., (see, e.g., refs. [6-10] and references therein).

The general quantum deformations of Lie (super)groups or (super)algebras are the multi-parameter deformations. For instance, the general quantum deformation of $GL(N)$ has $\frac{1}{2}N(N-1)$ deformation parameters [5]. The simple example of the two-parameter quantum deformation of $GL(2)$ and its differential calculus have been considered in ref. [11]. Following the method of graded tensor product [12], the two-parameter deformation of the supergroup $GL(1|1)$ has been discussed [13].

Recently the idea of quantum orbits, the one-parameter deformed quantum trace, its subsequent application to the construction of $q$-deformed algebras and the formulation of $q$-deformed Yang-Mills theory have been developed in ref. [14]. The purpose of our present paper is to define the quantum trace and quantum orbits for the two parameter groups $GL_q(2)$ and $GL_q(1|1)$. Further, we obtain the invariants of the orbits of these groups and demonstrate that these can be succinctly expressed in terms of the deformed traces. Following the approach of ref. [15], we construct the (super)oscillator algebras which are covariant under the action of the $GL_q(2)$ and $GL_q(1|1)$ groups and show that the bilinears of these (super)oscillators form the one-parameter deformed covariant algebra in the case of the $GL_q(2)$ group and the covariant extensions of the $N = 2$ supersymmetric quantum mechanical algebras for the quantum group $GL_q(1|1)$. The more interesting case of $GL_q(2)$ leads to the construction of the central extension of the Witten-type $q$-algebra $U_q(sl(2))$ [16]. This algebra can be considered as the “adjoint representation” of the quantum group $GL_q(2)$. Note that, in the paper [21], the algebra $U_{pq}(gl(2))$ has been found as the dual to the quantum group $GL_q(2)$.

Following Manin’s quantum hyperplane approach [5] to the general construction of quantum groups, it can be shown that the $2 \times 2$ $GL_{q,p}(2)$ matrix $T_{ij} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the noncommuting elements $a, b, c$ and $d$ exhibits different braiding relations in rows and columns as given by [11]

$$ab = pba, \quad cd = pdc, \quad ac = qca, \quad bd = qdb, \quad \text{(1a)}$$

and other relations are

$$bc = \frac{q}{p} cb, \quad ad - da = (p - q^{-1}) bc = (q - p^{-1}) cb, \quad \text{(1b)}$$

where $q, p \in \mathbb{C}/\{0\}$. It is easy to note that the one-parameter quantum group $GL_q(2)$ corresponds to the special case of (1a) and (1b) when $q = p$. The inverse quantum matrix
(T_{ij}^{-1}) is defined as follows [11]
\[ T_{ij}^{-1} = D^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \equiv \begin{pmatrix} d & -p^{-1}b \\ -pc & a \end{pmatrix} D^{-1}, \] (2)

where the quantum determinant \( D = ad - pbc = da - q^{-1}bc = ad - qcb = da - p^{-1}cb \) is not the central element of the algebra (1a), (1b) when \( p \neq q \), but it obeys the following relations [11]:
\[ a (D, D^{-1}) = (D, D^{-1}) a, \quad b (D, D^{-1}) = (q, D^{-1}) b, \]
\[ c (D, D^{-1}) = (\frac{p}{q}, D, D^{-1}) c, \quad d (D, D^{-1}) = (D, D^{-1}) d. \] (3)

Now let us introduce a 2 \( \times \) 2 quantum matrix \( E_{ij} \) with noncommuting elements. The following transformations of \( E_{ij} \)
\[ E_{ij} \rightarrow T_{ik} E_{kl} T_{lj}^{-1}, \] (4)
define, for all possible \( T_{ij} \in GL_{qp}(2) \), the quantum orbit in the space of 2 \( \times \) 2 \( q \)-matrices \( E_{ij} \) if the elements of this matrix commutes with that of \( T_{ij} \) (i.e. \([T_{ij}, E_{kl}] = 0\)). Since the elements of \( T \) are noncommuting objects, the usual trace of the matrix \( E \) is not invariant under transformations (4). However, it turns out that the expression (with \( r = \sqrt{(qp)} \))
\[ tr_{qp}(E) = tr_{qp}(TE T^{-1}) = r^{-1} E_{11} + r E_{22}, \] (5)
remains invariant under (4). We will call this the quantum \( (qp) \)-trace. It is straightforward to see that the case \( q = p \) in (5) yields the one-parameter trace of ref. [14] and \( q = p = 1 \) corresponds to the usual undeformed trace. It may be noticed that all other invariants for the orbit (4) can be written as \( c^n = Tr_{qp}(E^n) \).

Now let us construct the two parametric covariant quantum oscillators for \( GL_{qp}(2) \). With this end in mind, we introduce two sets of \( q \)-oscillators \( A_i \) and \( \tilde{A}_i \) \((i = 1, 2)\). In the language of differential geometry on the quantum hyperplane [17], these operators correspond to coordinates and derivatives. It is clear that the following relations:
\[ A_1 A_2 - q A_2 A_1 = 0, \quad \tilde{A}_1 \tilde{A}_2 - p^{-1} \tilde{A}_2 \tilde{A}_1 = 0, \] (6)
remain invariant under the \( GL_{qp}(2) \) transformations
\[ A_i \rightarrow T_{ij} A_j, \quad \tilde{A}_i \rightarrow \tilde{A}_j T_{ji}^{-1}, \] (7)
where the column matrix \( A_i = (A_1, A_2)^T \) and row matrix \( \tilde{A}_i = (\tilde{A}_1, \tilde{A}_2) \). Consistent with relations (6), the following oscillator algebra also remains invariant under transformations (7):
\[ A_2 \tilde{A}_1 - \frac{(\alpha - \beta)}{p} \tilde{A}_1 A_2 = 0, \quad A_1 \tilde{A}_2 - \frac{(\alpha - \beta)}{q} \tilde{A}_2 A_1 = 0, \quad A_2 \tilde{A}_2 - \alpha \tilde{A}_2 A_2 = 1 + \left(\alpha - \frac{(\alpha - \beta)}{r^2}\right) \tilde{A}_1 A_1, \] (8a)
if we postulate the validity of the following general bilinear oscillator relation

\[ A_1\ddagger A_1 - \alpha \ddagger A_1 = 1 + \beta \ddagger A_2. \]  

(8b)

Here \( \alpha \) and \( \beta \) are the c-number parameters which can be fixed by requiring associativity of the oscillator algebra (6) and (8). In fact, the oscillator algebra (6) and (8) give us the possibility to reorder a product of oscillators \( A_i \) and \( \ddagger A_i \) such that all the waved operators can be brought to the left side of the products. For example, let us consider the product \( A_1 \ddagger A_1 \ddagger A_2 \). There are two possible ways to reorder this expression, namely;

\[ (A_1 (\ddagger A_1 \ddagger A_2)) = \frac{\alpha - \beta}{r^2} [\ddagger A_2 + \alpha \ddagger A_1 A_1 + \beta \ddagger A_2 \ddagger A_2], \]  

(9a)

\[ ((A_1 \ddagger A_1) \ddagger A_2)) = (1 + \beta) \ddagger A_2 + \left(\frac{\alpha - \beta}{r^2} + \alpha \beta\right) \ddagger A_2 A_1 + \alpha \beta \ddagger A_2 \ddagger A_2. \]  

(9b)

As can be seen, we obtain two different results on the right hand sides of (9a) and (9b) which are found to coincide only in two cases

(i) \( \alpha = 1/r^2; \quad \beta = 0, \)  

(ii) \( \alpha = 1/r^2; \quad \beta = \frac{(1 - r^2)}{r^2}. \)  

(10a)

(10b)

The case (10a) leads to the following oscillator algebra

\[ A_2 \ddagger A_1 - q \ddagger A_2 = 0, \quad A_1 \ddagger A_2 - p \ddagger A_2 A_1 = 0, \quad A_2 \ddagger A_2 - pq \ddagger A_2 A_2 = 1 + (pq - 1) \ddagger A_1 A_1, \]

\[ A_1 \ddagger A_1 - pq \ddagger A_1 A_1 = 1, \quad \ddagger A_1 A_2 - q \ddagger A_2 A_1 = 0, \quad \ddagger A_2 - p^{-1} \ddagger A_2 \ddagger A_2 = 0, \]  

(11)

while the algebra corresponding to (10b) can be obtained from eqns. (11) by the replacements \( i = 1 \leftrightarrow i = 2 \) and \( q, p \leftrightarrow q^{-1}, p^{-1} \). It is worth noting that the algebras of the covariant pair of q-oscillators [15] can be obtained from (11) by the substitution \( q = p, \ddagger A_i = A_i^\dagger \) and the replacements corresponding to the solution (10b). Here we stress that the procedure of obtaining conditions (10) is equivalent to the procedure of deducing and solving Yang-Baxter equations.

It can now be seen from eqn. (7) that the quantum matrix

\[ E_{ij} = A_i \ddagger A_j, \]  

(12)

satisfies the transformation law (4) of the quantum orbit. Furthermore, the invariance of the trace (5) under transformations (4) leads to the following \( GL_{qp}(2) \) invariant Hamiltonian \( (H_{qp}) \) in the bilinears of the covariant oscillators

\[ H_{qp} = r^{-1} A_1 \ddagger A_1 + r A_2 \ddagger A_2. \]  

(13)

From the q-oscillator algebra (6), (8), one can see that the Hamiltonian (13) is related to the trivially \( GL_{qp}(2) \) invariant Hamiltonian \( (\ddagger H) \) by the following equation

\[ \ddagger H = \sum_{i=1}^{2} \ddagger A_i A_i = \frac{1}{r \alpha + r^{-1} \beta} [H_{pq} - (r + r^{-1})], \]  

(14)
where $\alpha$ and $\beta$ are defined in eqns. (10).

Now we would like to define the “adjoint representation” of the quantum group $GL_{qp}(2)$ and establish that the corresponding space of representation can be realized as a central extension of the Witten-type $U_q(sl(2))$ algebra. For this purpose, the $q$-matrix $E_{ij}$ can be rewritten in the following form

$$E = \frac{1}{r + r^{-1}} \begin{pmatrix} Tr_{qp}(E) & 0 \\ 0 & Tr_{qp}(E) \end{pmatrix} + \begin{pmatrix} rE_0 & (r + r^{-1})E_{12} \\ (r + r^{-1})E_{21} & -r^{-1}E_0 \end{pmatrix}$$

(15)

where $E_0 = E_{11} - E_{12}$ and

$$\bar{E} = \begin{pmatrix} rE_0 & (r + r^{-1})E_{12} \\ (r + r^{-1})E_{21} & -r^{-1}E_0 \end{pmatrix}.$$  

From expression (15), it is clear that transformations (4) lead to the following “four-dimensional representation” of $GL_{qp}(2)$

$$\begin{pmatrix} E_0' \\ E_{12}' \\ E_{21}' \\ H_{qp}' \end{pmatrix} = \frac{1}{D} \begin{pmatrix} D + (p + q^{-1})bc & -(q + p^{-1})ac & (p + q^{-1})db & 0 \\ -pq^{-1}ba & a^2 & -pq^{-2}b^2 & 0 \\ qp^{-1}cd & -q^2p^{-1}c^2 & d^2 & o \\ 0 & 0 & 0 & D \end{pmatrix} \begin{pmatrix} E_0 \\ E_{12} \\ E_{21} \\ H_{qp} \end{pmatrix}.$$  

(16)

This representation is reducible because we have one-dimensional ($H_{qp}$) and three-dimensional $E_0, E_{12}, E_{21}$ invariant subspaces. We can redefine the operators $E_+ = E_{12} = A_1\bar{A}_2$ and $E_- = E_{21} = A_2\bar{A}_1$ and show that the oscillator relations (11) lead to

$$[H_{qp}, E_{\pm}] = [H_{pq}, E_0] = 0,$$

(17)

$$[E_-, E_+] = \left(\frac{r^2 - 1}{r^2 + 1}\right) E_0^2 + \left(\frac{1}{r^2} + \frac{r^2 - 1}{r^3 + r} \right) H_{qp} E_0.$$  

(18)

This algebra turns out to be invariant under adjoint rotations (16).

For a given algebra, it is of utmost importance to obtain the Casimir operator(s) because the eigen values of such operator(s) designate the representation of the algebra. The $qp$-deformed quadratic Casimir operator for the algebra (18) turns out to be related to the invariant

$$c_2 = Tr_{qp}(E^2).$$

(19)

In fact, it can be expressed in terms of of the generators of (18) as given by

$$c = c_2 - \frac{H_{qp}^2}{r + r^{-1}} = (r^{-1} E_+ E_- + r E_- E_+ + \frac{E_0^2}{r + r^{-1}}).$$

(20)
We can rewrite the commutation relations (18) in a concise form as follows

\[ \tilde{E}_{ij} \tilde{E}_{jk} = (r + r^{-1}) c \delta_{ik} - \kappa \tilde{E}_{ik}, \]

(21)

where \( \kappa = [(r^2 - 1)/r^2] H_{qp} + (r^2 + 1)/r^3 \), \( c \) is the Casimir operator (20) and matrix \( \tilde{E} \) is defined in (15). Now the invariance of algebra (18) under adjoint rotation (16) becomes transparent in view of the transformation law \( \tilde{E} \to T\tilde{E}T^{-1} \).

It is worth noting that the algebra (18) depends only on a single parameter \( r = \sqrt{pq} \) and, therefore, is consistent with Drinfeld’s uniqueness theorem (see, e.g., ref. [11]). The algebras (17) and (18), in terms of the generators \( E_\pm, E_0 \) and \( H_{qp} \), give us the \( q \)-deformation of \( gl(2) \). This covariant algebra is the central extension of the Witten-type deformation of \( sl(2) \) algebra if we consider \( H_{qp} \) as the central element. We see, therefore, that the “adjoint-representation” of the group \( GL_{qp}(2) \) leads to the generalization of the algebra obtained in ref. [16] (see also ref. [14]).

Now we shall discuss the deformed trace and the orbits for the supersymmetric case of \( GL_{qp}(1|1) \). It is known that the 2 \( \times \) 2 quantum matrix \( T^s_{ij} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \) describes the the \( GL_{qp}(1|1) \) group, if the noncommuting odd elements \( \beta \) and \( \gamma \) and the even elements \( a \) and \( d \) satisfy different braiding relations in rows and columns as given by [13]

\[
\begin{align*}
    a\beta &= p\beta a, & a\gamma &= q\gamma a, & \beta\gamma &= -(q/p) \gamma\beta, & d\gamma &= q\gamma d, \\
    d\beta &= p\beta d, & \beta^2 &= \gamma^2 &= 0, & ad - da &= -(p - q^{-1})\beta\gamma = (q - p^{-1})\gamma\beta,
\end{align*}
\]

(22)

These relations reduce to the one-parameter case of \( GL_q(1|1) \) in the limit \( p = q \). The antipode \( (S_{ij} = (T^s)^{ij})^{-1} \) and the quantum super determinant \( D^s \) (q-berezinian) are obtained by applying the Borel-Gauss decomposition on the matrix \( T^s \). These are given as follows [13,18]

\[
\begin{align*}
    (T^s)^{ij} &= \begin{pmatrix} a^{-1}(1 + \beta d^{-1}\gamma a^{-1}) & -a^{-1}\beta d^{-1} \\
    -d^{-1}\gamma a^{-1} & d^{-1}(1 - \beta a^{-1}\gamma d^{-1}) \end{pmatrix}, \\
    D^s &= ad^{-1} - \beta d^{-1}\gamma d^{-1} = d^{-1}a - d^{-1}\beta d^{-1}\gamma.
\end{align*}
\]

(23)

Here \( D^s \) is the center for the algebra (22). The quantum orbits for for the supergroup \( GL_{qp}(1|1) \) can be defined through the transformations (4) for a 2 \( \times \) 2 super \( q \)-matrix \( E_{ij} \) whose elements (anti-)commute with that of \( T^s_{ij} \). Even though the elements of \( T^s_{ij} \) follow the graded commutation relations (22), the supertrace

\[ Str_{qp}(E) = E_{11} - E_{22} = Str_{qp}[T^s E(T^s)^{-1}], \]

(24)

remains invariant under transformations (4). It is interesting to note that eqn. (24) coincides with the supertraces for the undeformed and the one-parameter deformed supergroup \( GL(1|1) \). It can be shown that all invariants of the quantum “superorbit” \( (E \to T^s E(T^s)^{-1}) \) can be expressed as

\[ c_n = Str_{qp}(E^n) = Str(E^n). \]

(24a)
We now introduce the the set of bosonic \((A, \tilde{A})\) and fermionic \((B, \tilde{B})\) variables to study the system of \(qp\)-superoscillators that are covariant under the co-action of the quantum supergroup \(GL_{qp}(1|1)\). It is straightforward to demonstrate that the relations

\[
AB = q BA, \quad \tilde{B}\tilde{A} = p \tilde{A}\tilde{B}, \quad B^2 = \tilde{B}^2 = 0, \tag{25}
\]

remain invariant under the \(GL_{qp}(1|1)\) transformations

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} \rightarrow \begin{pmatrix}
a & \beta \\
\gamma & d
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix} \equiv \left(T^s\right) \begin{pmatrix}
A \\
B
\end{pmatrix}, \tag{26a}
\]

\[
\begin{pmatrix}
\tilde{A} \\
\tilde{B}
\end{pmatrix} \rightarrow \begin{pmatrix}
\tilde{A} & \tilde{B}
\end{pmatrix} \begin{pmatrix}
a^{-1}(1 + \beta d^{-1}\gamma a^{-1}) & -a^{-1}\beta d^{-1} \\
-d^{-1}\gamma a^{-1} & d^{-1}(1 - \beta a^{-1}\gamma d^{-1})
\end{pmatrix}, \equiv \left(\tilde{T}^s\right)^{-1} \begin{pmatrix}
\tilde{A} \\
\tilde{B}
\end{pmatrix}. \tag{26b}
\]

Consistent with the oscillator algebra (25), the other general \(qp\)-superoscillator relations are as follows

\[
\begin{align*}
A\tilde{B} &= \frac{\lambda + \nu}{q} \tilde{B}A, \\
B\tilde{A} &= \frac{\lambda + \nu}{p} \tilde{A}B, \\
A\tilde{A} &= (\lambda + \frac{\lambda + \nu}{pq}) \tilde{A}A = 1 - \left(\nu - \frac{(\lambda + \nu)}{pq}\right) \tilde{B}\tilde{B},
\end{align*} \tag{27}
\]

if we postulate the validity of the deformed anticommutator

\[
\{B, \tilde{B}\}_{(1, \nu)} \equiv B\tilde{B} + \nu \tilde{B}\tilde{B} = 1 + \lambda \tilde{A}A, \tag{28}
\]

where \(\lambda\) and \(\nu\) are c-number parameters which can be determined by requiring associativity of the algebra (25), (27) and (28). Indeed, we can reorder the tilded oscillators to the left of the product \(A, \tilde{A}, \tilde{B}\) in two different ways which lead to two different results on the right-hand side. To obtain the unique result on the right-hand side, we have to take

\[
(i) \quad \nu = 1, \quad \lambda = r^2 - 1, \quad (ii) \quad \nu = 1, \quad \lambda = 0. \tag{29a, b}
\]

The two parametric quantum superoscillator algebras corresponding to the solution (29a) and consistent with eqns. (25) are as follows

\[
\begin{align*}
B\tilde{A} &= q\tilde{B}A, \\
\tilde{B}\tilde{A} &= p\tilde{A}B, \\
A\tilde{A} &= -pq \tilde{A}A = 1, \\
B\tilde{B} + \tilde{B}\tilde{B} &= 1 + \left(pq \tilde{B}\tilde{B} \right). \tag{30a}
\end{align*}
\]

Similarly, the solution (29b) yields the following superoscillator relations consistent with eqns. (25)

\[
\begin{align*}
B\tilde{A} &= p^{-1} \tilde{A}B, \\
\tilde{B}\tilde{A} &= q^{-1} \tilde{B}A, \\
A\tilde{A} - \frac{1}{pq} \tilde{A}A &= 1 + \left(\frac{1 - pq}{pq}\right) \tilde{B}\tilde{B}. \tag{30b}
\end{align*}
\]

The case \(q = p, \tilde{A} = A^\dagger, \tilde{B} = B^\dagger\) for algebras (30a) and (30b) gives us the known algebras of the covariant pair of \(q\)-oscillators of ref. [15]. We stress here again that the procedure
of obtaining conditions (29) is equivalent to the derivation and solution of the graded Yang-Baxter equations.

We can now see that the super $q$-matrix

$$ E_{ij} = \begin{pmatrix} A\tilde{A} & A\tilde{B} \\ B\tilde{A} & B\tilde{B} \end{pmatrix}, \tag{31} $$

satisfies the transformation laws (4) if the covariant superoscillators obey the $GL_{qp}(1|1)$ transformation laws (26). The invariance of $qp$-supertrace (24) leads to the emergence of the following bilinear representation of the invariant Hamiltonian ($H^s_{qp}$) in terms of the the super $qp$-oscillators

$$ H^s_{qp} = A\tilde{A} - B\tilde{B} = \frac{\nu + \lambda}{qp} (\tilde{A}A + \tilde{B}B). \tag{32} $$

The right-hand side of eqn. (32) is trivially supercovariant in view of the transformations (26).

It is now obvious that the super transformations (4) lead to the following four-dimensional “adjoint representation” of $GL_{qp}(1|1)$

$$ \begin{pmatrix} Y' \\ E'_{12} \\ E'_{21} \\ H^s_{qp} \end{pmatrix} = \begin{pmatrix} 1 & \gamma d^{-1} & \beta a^{-1} & \beta d^{-1} \gamma a^{-1} \\ 0 & D^s & 0 & -\beta d^{-1} \\ 0 & 0 & (D^s)^{-1} & \gamma a^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ E_{12} \\ E_{21} \\ H^s_{qp} \end{pmatrix}. \tag{33} $$

Here $Y = (E_{11} + \mu \ E_{22})/(1 + \mu)$, where $\mu \neq -1$ is a c-number. The representation (33) is reducible because we have a one-dimensional invariant subspace with coordinate $H^s_{qp} = E_{11} - E_{22}$. To obtain the covariant algebra for the coordinates of this representation the key ingredient is to represent these coordinates in terms of covariant oscillators using the defining equation (31). Considering the operators

$$ Y = \frac{A\tilde{A} + \mu B\tilde{B}}{1 + \mu}, \quad H^s_{qp} = A\tilde{A} - B\tilde{B}, \quad Q = E_{12} = A\tilde{B}, \quad \bar{Q} = E_{21} = B\tilde{A}, \tag{34} $$

the covariant superalgebra for the case of (30a) is written as

$$ \begin{align*}
{[H^s_{qp},Q]} &= 0, &{[H^s_{qp},\bar{Q}]} &= 0, &{[H^s_{qp},Y]} &= 0, &Q^2 &= \frac{1}{2} \{Q,Q\} = 0, \\
\{Q,\bar{Q}\} &= [1 + (r^2 - 1) \ H^s_{qp}] \ H^s_{qp} = \mathcal{H}, &Q^2 &= \frac{1}{2} \{Q,\bar{Q}\} = 0, \\
[Q,Y] &= + [1 + (r^2 - 1) \ H^s_{qp} ] \ Q, &[Q,\bar{Q}] &= - [1 + (r^2 - 1) \ H^s_{qp} ] \ \bar{Q}. \tag{35a} \end{align*} $$

while for the case (30b), we have the following covariant superalgebra

$$ \begin{align*}
{[H^s_{qp},Q]} &= [H^s_{qp},\bar{Q}] = [H^s_{qp},Y] = 0, &Q^2 &= \bar{Q}^2 = 0, \\
\{Q,\bar{Q}\} &= H^s_{qp}, &[Q,Y] &= + Q, &[Q,\bar{Q}] &= - \bar{Q}. \tag{35b}. \end{align*} $$

The Casimir operator $c^s$ for the covariant algebras (35a), (35b) is related to the invariant $c_2$ (24a) and can be expressed as

$$ c^s = c_2 - \frac{\mu - 1}{\mu + 1} (H^s_{qp})^2 = Q\bar{Q} - \bar{Q}Q + 2Y H^s_{qp}. $$
It is interesting to note that in the case of the super quantum group $GL_{qp}(1|1)$, we obtain $N = 2$ supersymmetric quantum mechanical algebras (with super charges $Q, \bar{Q}$, and supersymmetric Hamiltonian $H$ (35a) for the case (29a) and $H_{qs} \sum$ (32) for the case (29b)) as subalgebras of the covariant algebras (35a), (35b). The generator $Y$ gives the extensions of these quantum mechanical superalgebras. The $q$-superalgebra (35b) (and superalgebra (35a) after rescaling of the generators $Q, \bar{Q}, Y$ is isomorphic to the undeformed Lie superalgebra $gl(1|1)$. This is in agreement with the one-parameter case [18].

The emergence of the algebras with generators $\{Q, \bar{Q}, H(\widetilde{H}_{qs})\}$ from the $GL_{qp}(1|1)$ covariant superalgebras (35a), (35b) tells about the “hidden $q$-symmetry” in supersymmetric quantum mechanical systems with the Hamiltonians constructed above. It may be a strong physical motivation for the study of quantum supergroup.

To conclude, it worthwhile to note that the (super)traces (5) (for $p = q$) and (24) can be obtained as special cases from the general supertrace defined for the one-parameter deformed supergroup $GL_q(N|M)$. To obtain such a general supertrace the essential ingredients are the relations between deformed (super)traces and invariant Hamiltonians that are deduced in eqns. (14) and (32). Moreover, it is also essential to take into account the results of ref. [15], where one-parameter $q$-oscillator algebras and corresponding invariant Hamiltonians, in terms of the bilinears, have been obtained explicitly. Such a general supertrace for the quantum group $GL_q(N|M)$ is as follows

$$Str_q(E) = Str_q(TET^{-1}) = (q^{(M-N)/2}) \sum_{i=1}^{N} (q^{-(N+1)+2i}E_{ii}) - (q^{(N-M)/2}) \sum_{s=N+1}^{N+M} (q^{(M+1)-2(s-N)}E_{ss}),$$

where $E_{ij}$ is an $(N + M) \times (N + M)$ $q$-supermatrix and the elements of the quantum matrix $T_{ij}(i, j = 1, 2,...,N + M)$ generate the quantum group $GL_q(N|M)$. The matrix $T_{ij}$ acts on the coordinates $(A_1, A_2, ..., A_N, B_1, B_2, ..., B_M)$ defined on the quantum hyperplane. The basic relations for these coordinates are as follows

$$A_i A_j = q A_j A_i, \quad (i < j), \quad A_i B_s = q B_s A_i, \quad B_r B_s = -q B_s B_r, \quad (r < s).$$

(37)

We would like to emphasize at this juncture that the substitutions $N = 2, M = 0$ and $N = 1, M = 1$ lead to the emergence of the expressions for the (super)traces obtained in (5) (for $p = q$) and (24). Furthermore, $M = 0$ in eqn. (36) yields the expression for for the $q$-trace presented in ref. [14] for the quantum group $GL_q(N)$. The special case of this $q$-trace $(N = 3, M = 0)$ was used in ref. [19] in the context [3, 20] of the construction of the central elements for the quantum algebras.

It is possible to extend the covariant (super)algebras (18) and (35) which correspond to “adjoint representation” (16), (33) and construct the higher dimensional “representations” for the quantum groups $GL_{qp}(2)$, $GL_{qp}(1|1)$. For this purpose, one has to consider the following tensors which are elements of the enveloping algebra of $qp$-oscillators

$$E_{i_1, i_2, ..., i_n} B_1, B_2, ..., B_m = \tilde{A}_{i_1} \tilde{A}_{i_2} B_1, B_2, ..., B_m, \quad (38)$$
The transformations presented in (7) and (26) define the action of $GL_q(1|1)$ and $GL_q(2)$ on such tensors and give rise to the higher dimensional “representations” of these quantum groups. The interesting non-trivial problem here would be to separate (38) into irreducible representations and investigate corresponding covariant infinite-dimensional algebras. We hope that the notion of the quantum trace will help in the resolution of this problem.

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Note added:

After we communicated this paper to the arXiv, we were informed by V. K. Dobrev about his paper [21] where the algebra $U_{qp}(gl(2))$, as the dual to the quantum group $GL_{qp}(2)$, has been constructed.

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