Calculations in induced gravity from higher-derivative field theories

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In this paper, we investigate Einstein’s gravity induced from higher-derivative scalar field theories. We develop an approach utilizing an effective theory of multiple fields for the higher-derivative theory. The expressions for induced cosmological constant and the induced gravitational constant are obtained in the present scenario of induced gravity in $D$ dimensions. We also show that finite values for the induced constants can be extracted in certain infinite-derivative theories.

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I. INTRODUCTION

In quantum field theory, the issue of ultraviolet (UV) divergence has been repeatedly discussed by many authors. In particular, various models in quantum theory including the gravitational field have been proposed as approaches to avoid the UV divergence, including attempts beyond the field theory.

The simplest idea is seen in a scenario where the theory with higher derivative of fields manages to improve the power counting for the UV divergences [1–6]. This is owing to the milder behavior of the Green’s function\(^1\) at a short distance in the higher-derivative theory than that in the canonical field theory. It is recently found that the vacuum expectation value of the scalar field squared in a certain higher-derivative theory can be reduced to be finite and free from divergence at a short distance from a conical spatial defect [7], whereas contrary to naive expectations, the quantum fluctuation of the stress tensor in the higher-derivative theory still suffers from UV divergences [8]. Although higher-derivative theories also have a difficult problem of causality [9–13] in addition, it is worth studying for effectiveness of higher derivatives in resolving general UV behavior of quantum field theory of higher-order gravity [14]\(^2\) and that with infinite derivatives [20, 21] and arbitrarily high derivatives [22–24].

There is a concept of induced gravity [25–40] as a completely different prospective description of the theory of gravity. It is based on the idea that the effective theory of the gravitational field is derived from the quantum effect of matter fields. However, because the gravitational constant has a dimension of the inverse square of mass in four dimensions, it is affected by the quadratic divergence in quantum field theory, so many models cannot predict even a fixed amount of the induced gravitational constant [27, 34]. The estimation of the induced cosmological constant resulting from quantum effects, which is order of the quartic divergence, is more difficult to compare with the astrophysical knowledge [41].

In this paper, we consider the gravity induced from the quantum theory with higher derivatives of a field. As already mentioned above, higher derivatives do not necessarily suppress divergences which appear in stress tensors. Our interest in higher-derivative the-

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1 We use the term “Green’s function” instead of “propagator”, because we use the Euclidean metric in this paper.

2 The authors of Refs. [15–18] considered induced gravity from the phase transition in the theory with higher derivatives and non-minimally coupled scalar fields. The author of Ref. [19] considered induced gravity from higher-order gravity with matter fields. Note that the Adler–Zee formulas (1.2) and (1.3) cannot be applied to these approaches in their original form.
ory lies in the other two features of the theory. First, any higher-derivative theory has a fundamental length scale, as seen from the dimensional counting. Second, higher-derivative theories are expressed by multiple fields. In higher-derivative scalar field theory, it is already known that the degree of freedom of a field increases as the number of derivatives increases [6, 42–46].

Recently, Kehagias et al. [47] investigated induced gravity in higher dimensional theories not only by the heat kernel method [32–38, 48] but also by the original methods [26–31, 39]. In their model, infinitely many excited states à la Kaluza–Klein theory play an important role in calculability of the cosmological constant and the gravitational constant without ambiguities. In their model, the dimensionful constants are proportional to the appropriate powers of the compactification scale. Therefore, we come to the idea that the induced gravity from higher-derivative field theories also may possess predictability in gravitational physics at least in a lowest order of perturbative quantum effects.

In this paper, we present a new model of induced gravity, which arises from the quantum effects in higher-derivative scalar field theory. We emphasize that calculable examples of induced gravity can be provided with certain higher-derivative theories. Interestingly, the calculations in those examples result in similar calculations to those in the model of Ref. [47] based on the Kaluza–Klein theory. A discussion of unitarity and causality falls outside the scope of this paper.

The main results we find are obtained by early standard methods of Adler–Zee formula [26–31, 39]. We denote the induced gravitational action as

$$ S = \int d^D x \sqrt{-g} \frac{1}{16\pi G_{ind}} (R - 2\Lambda_{ind}) , \quad (1.1) $$

where $R$ is the scalar curvature. Then, the induced cosmological constant $\Lambda_{ind}$ is given by

$$ \frac{\Lambda_{ind}}{8\pi G_{ind}} = -\frac{1}{D} \langle T(0) \rangle , \quad (1.2) $$

and the formula yields the induced gravitational constant $G_{ind}$ as

$$ \frac{1}{16\pi G_{ind}} = -\frac{1}{4D(D-1)(D-2)} \int d^D w |w|^2 \langle \bar{T}(x) T(y) \rangle , \quad (1.3) $$

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3 A similar scenario has been demonstrated earlier by using dimensional deconstruction [36–38] with the method of heat kernel. The condition for cancellation in UV divergences is identical to what was stated by Frolov and Fursaev [33].
where $w \equiv x - y$, the brackets $\langle \rangle$ indicate a vacuum expectation value, and $T \equiv T_\mu^\mu$ is the trace of the stress tensor of matter fields and $T \equiv T - \langle T \rangle$. It should be noted that we are working in $D$-dimensional Euclidean space [39]. In the following sections, we will apply the formulas above to the higher-derivative scalar field theories. A brief review of the derivation of (1.2) and (1.3) can be found in Appendix A.

This paper is organized as follows. The Section II gives a brief overview of higher-derivative theory and its effective action with multiple fields. The stress tensor of the theory is obtained here and the induced cosmological and gravitational constants are formulated by using the expression of the stress tensor. Section III deals with a case study of a certain higher-derivative theory with an infinite number of derivatives. Some conclusions are drawn in the final section.

II. INDUCED GRAVITY FROM THE EFFECTIVE LAGRANGIAN AND THE STRESS TENSOR

Recently, Gibbons et al. [46] proposed an effective Lagrangian of multiple fields for a general higher-derivative scalar field theory. They considered the Lagrangian with an arbitrary number of d’Alembert operators $\Box$ on a real scalar field $\phi$:

$$\mathcal{L} = -\frac{1}{2C} \phi(x) \prod_{i=1}^{n} A_i \phi(x),$$  \hspace{1cm} (2.1)

where $A_i \equiv -\Box + m_i^2$ and $C$ is a constant. The effective Lagrangian corresponding this Lagrangian reads [46]

$$\mathcal{L} = -\frac{1}{2C} \sum_{k=0}^{n} \eta_k(x) A_{k+1} \chi_{k+1}(x) + \frac{1}{2C} \sum_{k=1}^{n} \eta_k(x) \chi_k(x),$$  \hspace{1cm} (2.2)

where $\eta_0 \equiv \phi$ and $\chi_n \equiv \phi$. We obtain the relations, $\eta_k = [\prod_{i=1}^{k} A_i] \phi$ (1 $\leq$ $k$ $\leq$ $n$), and $\chi_k = [\prod_{i=k+1}^{n} A_i] \phi$ (1 $\leq$ $k$ $\leq$ $n-1$) from the iterative use of the equations of motion. If the expression for $\eta_i$ and $\chi_i$ constructed from $\phi$ is substituted into (2.2), the original Lagrangian (2.1) is recovered.

Then, the stress tensor of the theory is given by [46]

$$T_{\mu\nu} = \frac{1}{C} \sum_{k=0}^{n} \left[ (\partial_\mu \eta_k \partial_\nu \chi_{k+1}) - \frac{1}{2} g_{\mu\nu} (\partial_\rho \eta_k \partial^\rho \chi_{k+1} + m_{k+1}^2 \eta_k \chi_{k+1}) \right] + \frac{1}{2C} g_{\mu\nu} \sum_{k=1}^{n} \eta_k \chi_k,$$  \hspace{1cm} (2.3)
where $g_{\mu\nu}$ is the metric tensor of the background (flat) $D$ dimensional spacetime. At a glance, it seems inconsistent with the effective description of the Lee–Wick scalar field theory [6] with apparently indefinite signs of kinetic terms. This is consistent, however, as illustrated in Appendix B for $n = 2$.

Now, returning to (2.3), the trace of the stress tensor $T \equiv T_\mu^\mu$ can be written by

$$T = \frac{1}{C} \sum_{k=0}^{n} \left[ -\frac{D-2}{2} \partial_\mu \eta_k \partial^\mu \chi_{k+1} - \frac{D}{2} m^2_{k+1} \eta_k \chi_{k+1} \right] + \frac{D}{2C} \sum_{k=1}^{n} \eta_k \chi_k . \quad (2.4)$$

We use this form of the trace of the stress tensor in the formulas (1.2) and (1.3). Further, we only use the two-point functions such as $\langle \eta(x) \chi_j(y) \rangle$, which is evaluated from the Green’s function $\langle \phi(x) \phi(y) \rangle$. Then, we regard all fields not independent of $\phi$. We can simplify (2.4) as

$$T = \frac{1}{C} \sum_{k=0}^{n} \left[ -\frac{D-2}{2} \partial_\mu \eta_k \partial^\mu \chi_{k+1} - \frac{D}{2} \eta_k \Box \chi_{k+1} \right] , \quad (2.5)$$

where we used $\chi_k = \left( \prod_{i=k+1}^{n} A_i \right) \phi = A_{k+1} \chi_{k+1}$. The essential advantages of this form (2.5) is that we have only to concentrate ourselves on evaluations of vacuum expectation values of fields since the contributions of masses are implicit in the expression.

To obtain the quantum quantities, the fundamental basis we use is the Green’s function

$$\langle \phi(x) \phi(x') \rangle = G(x, x') = \frac{C}{\prod_{i=1}^{n} A_i} 1_{xx'} , \quad (2.6)$$

where $1_{xx'}$ denotes a covariant delta function $\frac{1}{\sqrt{|g|}} \delta^D(x, x')$ in this symbolic expression. From the Green’s function and the relations to the original scalar field $\phi$, it turns out to be

$$\langle \eta_k(x) \chi_{k+1}(x') \rangle = \left[ \prod_{i=1}^{k} A_i \right]_x \left[ \prod_{i=k+2}^{n} A_i \right]_{x'} \langle \phi(x) \phi(x') \rangle = \frac{C}{A_{k+1}} 1_{xx'} = C \Delta_{k+1}(x, x') , \quad (2.7)$$

where $\Delta_{k+1}(x, x')$ is the Green’s function of a canonical scalar field with mass $m_{k+1}$. This result is consistent if we consider $\eta$’s and $\chi$’s form a set of free fields governed by the Lagrangian (2.2). Note that the induced cosmological and gravitational constants are evaluated by the Adler–Zee formulas with these Green’s functions.

Consequently, we find a fairly simple expression for the induced cosmological constant

$$\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = -\frac{1}{D} \left( \frac{D}{2D} \right) \sum_{k=1}^{n} \left[ \frac{D-2}{2D} \partial_\mu \partial^\mu + \frac{1}{2} \Box_x \right] \Delta_k(x, x') . \quad (2.8)$$

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4 Here, the d’Alembertian should be considered to act on both $\eta_k$ and $\chi_{k+1}$ symmetrically.
On the other hand, the application of the Adler–Zee formula for the induced gravitational constant to the present theory yields (for a proof, see Appendix C)

\[
\frac{1}{16\pi G_{\text{ind}}} = -\frac{1}{4D(D-1)(D-2)} \int d^Dw |w|^2 \\
\times 2 \sum_{k=1}^{n} \left[ \frac{(D-2)^2}{4} \partial_{\mu} \partial_{\mu'} \Delta_k(x, x') \partial^\rho \partial'^\rho \Delta_k(x, x') + \frac{D(D-2)}{2} \partial_\mu \Delta(x, x') \partial^\rho \Box \Delta(x, x') \\
+ \frac{D^2}{4} [\Box x \Delta(x, x')]^2 \right],
\]

(2.9)

where \( w = x - x' \). The equations (2.8) and (2.9) are our main results in this paper.

The Green's function of a canonical free massive scalar field can be written by an integral form with the so-called Schwinger parameter

\[
\Delta_k(x, x') = \int_0^\infty ds \frac{d}{(4\pi s)^{D/2}} \exp \left[ -\frac{|w|^2}{4s} - m_k^2 s \right],
\]

(2.10)

where the symmetry \( \Delta(x, x') = \Delta(x', x) \) is apparent. Using this form, we can simplify the formulas (2.8) and (2.9). The details are shown in Appendix D.

Performing straightforward differentiations (see Appendix D) and taking a naive limitation \( x' \to x \) i.e., \( |w| \to 0 \), we obtain

\[
\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = -\int_0^\infty \frac{ds}{2(4\pi)^{D/2} s^{D/2+1}} \varrho(s),
\]

(2.11)

where

\[
\varrho(s) \equiv \sum_{k=1}^{n} e^{-m_k^2 s},
\]

(2.12)

is a useful abbreviation.

The calculation of the induced gravitational constant is obtained after lengthy but straightforward calculations (see Appendix D). The result is

\[
\frac{1}{16\pi G_{\text{ind}}} = \frac{1}{12(4\pi)^{D/2}} \int_0^\infty \frac{ds}{s^{D/2}} \varrho(s).
\]

(2.13)

These results confirm the equivalence of the heat kernel method and the Adler–Zee formula previously used in the literature [26–39].

The integral forms (2.11) and (2.13) are compact but include divergences as is known in the literature. The finite part of both the cosmological constant and the gravitational constant has often been reported in the model of the Kaluza–Klein theory and its generalization [52, 53], as known calculable examples. In practice, we should manage to cancel
the divergent part by introducing massive scalar, fermion, and vector fields \([33, 37, 38, 47]\) and even other compensating fields \([47]\). In this paper, though we focused on the scalar field theory, the higher-derivative generalization of spinor or vector field theories is feasible à la the generalized Lee–Wick theory \([6, 42, 43]\). The concrete example of cancellation of divergences is not attempted in this study.

In the next section in our present paper, we examine the induced gravity from certain scalar models with an infinite number of derivatives using the expression obtained above, and show that the induced cosmological and gravitational constants are calculable in the models.

### III. INDUCED COSMOLOGICAL AND GRAVITATIONAL CONSTANTS FROM AN INFINITE DERIVATIVE SCALAR FIELD THEORY

In this section, we introduce certain examples of higher-derivative theory. We first consider the following Lagrangian with higher derivatives on a real scalar field \(\phi\):

\[
L = -\frac{1}{2} \phi(x) \frac{\sqrt{-\Box} \sinh[\pi l \sqrt{-\Box}]}{\pi l} \phi(x) = -\frac{1}{2} \phi(x) \frac{1}{l^2 |\Gamma(il\sqrt{-\Box})|^2} \phi(x). \tag{3.1}
\]

Although this concise form of the Lagrangian seems to include the square root of the differential operator, there is actually no singular operator, as seen from the series expansion. The Green’s function in momentum space of this model first appeared in Refs. \([49–51]\). This falls in a type of the Pauli–Villars regularized function, since a formula of an alternating series shows

\[
\frac{\pi l}{\sqrt{-\Box} \sinh[\pi l \sqrt{-\Box}]} = \frac{1}{-\Box} + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{-\Box + \frac{k^2}{l^2}}. \tag{3.2}
\]

Therefore, we can expect better UV behavior of physical quantities connected to this function, so it is suitable as the first model to be studied. Incidentally, the Green’s function in flat configuration space can be written as \([7]\)

\[
G(x, x') = \frac{l^{D-1}}{\pi \frac{D-1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{D-1}{2} \right)}{|x - x'|^2 + 4\pi^2 l^2 \left( k + \frac{1}{2} \right)^2} \tag{3.3}
\]

It can be confirmed that the limit \(l \to 0\) reduces this expression to the canonical massless Green’s function.
The Lagrangian (3.1) can be written by using the infinite product as

$$\mathcal{L} = -\frac{1}{2C} \phi(x)(-\Box) \prod_{k=1}^{\infty} \left[ -\Box + \frac{k^2}{l^2} \right] \phi(x), \quad (3.4)$$

where $C = \prod_{k=1}^{\infty} \frac{k^2}{l^2}$. Further, in order to apply the treatment according to Ref. [46] and the formulation in the previous section, we rewrite this in the form

$$\mathcal{L} = -\frac{1}{2C} \phi(x) \prod_{i=1}^{\infty} A_i \phi(x), \quad (3.5)$$

where $A_i = -\Box + m_i^2$, $m_i^2 = (i - 1)^2/l^2$ ($i = 1, 2, 3, \ldots$). Note that $m_1 = 0$ in the present model.

A primitive calculation with the formula (2.11) gives, after the integration on $s$,

$$\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = -\frac{\Gamma(-D/2)}{2(4\pi)^{D/2}} \sum_{k=1}^{\infty} m_k^D = -\frac{\Gamma(-D/2)}{2(4\pi)^{D/2}l^D} \sum_{k=1}^{\infty} k^D = -\frac{\Gamma((D + 1)/2)\zeta_R(D + 1)}{2\pi^{(D+1)/2}(4\pi)^{D/2}l^D},$$

where $\zeta_R(z)$ is the Riemann’s zeta function. In the last equality, we used the mathematical formula $\zeta_R(z)\Gamma(z/2) = \pi^{z-1/2}\zeta(1 - z)\Gamma((1 - z)/2)$. In this way, the infinite sum can often be evaluated as a finite value, as in the Kaluza–Klein theories [52, 53], despite there being a divergence.

In this case, it is possible to consider the sum in $\varrho(s)$ first. Then, the mathematical formula leads to

$$\varrho(s) \equiv \sum_{k=1}^{\infty} e^{-m_k^2s} = \frac{1}{2} \left[ 1 + \vartheta_3(0, e^{-s/l^2}) \right] = \frac{1}{2} \left[ 1 + \sqrt{s} \vartheta_3(0, e^{-\pi^2l^2/s}) \right], \quad (3.7)$$

where $\vartheta_a(v, q)$ is the Jacobi theta function. If we naively (but as in the zeta function regularization [54]) discard the divergent contribution, we get

$$\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = -\int_0^{\infty} \frac{ds}{2(4\pi)^{D/2}l^{D+1}} \varrho(s)$$

$$\Rightarrow -\int_0^{\infty} \frac{l\sqrt{s}ds}{2(4\pi)^{D/2}l^{D+1}} \sum_{k=1}^{\infty} \exp \left( -\frac{\pi^2k^2l^2}{s} \right) = -\frac{\Gamma((D + 1)/2)\zeta_R(D + 1)}{2\pi^{D+1/2}(4\pi)^{D/2}l^D}, \quad (3.8)$$

which reproduces exactly same result as (3.6). For $D = 4$, the numerical value is found to be $\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = -2.5279 \times 10^{-5}l^{-4}$. 

Similarly, the primitive calculation of the induced gravitational constant gives
\[
\frac{1}{16\pi G_{\text{ind}}} = \frac{\Gamma(1 - D/2)}{12(4\pi)^{D/2}} \sum_{k=1}^{\infty} m_k^{D-2} = \frac{\Gamma(1 - D/2)}{12(4\pi)^{D/2}l^{D-2}} \sum_{k=1}^{\infty} k^{D-2} = \frac{\Gamma((D-1)/2)\zeta_R(D-1)}{12\pi^{D-3/2}(4\pi)^{D/2}l^{D-2}}.
\]  

This result is also obtained by discarding the divergent contribution in \(\varrho(s)\) in (2.13), as
\[
\frac{1}{16\pi G_{\text{ind}}} = \frac{1}{12(4\pi)^{D/2}} \int_{0}^{\infty} \frac{ds}{s^{D/2}} \varrho(s) = \int_{0}^{\infty} \frac{l\sqrt{\pi ds}}{12(4\pi)^{D/2}l^{D+1/2/2}} \sum_{k=1}^{\infty} \exp \left( -\frac{\pi^2 k^2 s}{l^2} \right) = \frac{\Gamma((D-1)/2)\zeta_R(D-1)}{12\pi^{D-3/2}(4\pi)^{D/2}l^{D-2}}.
\]  

For \(D = 4\), the numerical value is \(\frac{1}{16\pi G_{\text{ind}}} = +3.21362 \times 10^{-5}l^{-2}\).

Another model with an infinite number of derivatives is
\[
\mathcal{L} = -\frac{1}{2} C' \phi(x) \cosh[\pi l \sqrt{-\Box} \phi(x)] = -\frac{1}{2} \phi(x) \frac{\pi}{\Gamma(\frac{1}{2} + il\sqrt{-\Box})^2} \phi(x).
\]  

Note that this model does not have a massless mode. The Lagrangian of this model can be written by using the infinite product as
\[
\mathcal{L} = -\frac{1}{2C'} \phi(x) \prod_{k=1}^{\infty} \left[ -\Box + \frac{(k - 1/2)^2}{l^2} \right] \phi(x),
\]  

where \(C' = \prod_{k=1}^{\infty} \frac{(k - 1/2)^2}{l^2}\). Then, the mass spectrum is given by \(m_i^2 = (i - 1/2)^2/l^2\) \((i = 1, 2, 3, \ldots)\) and we find
\[
\varrho(s) \equiv \sum_{k=1}^{\infty} e^{-m_k^2 s} = \frac{1}{2} \vartheta_2(0, e^{-s/l^2}) = \frac{1}{2} \sqrt{s} \vartheta_4(0, e^{-\pi^2 l^2/s}),
\]  

where \(\vartheta_a(v, q)\) is the Jacobi theta function.

Therefore, the finite part of the induced cosmological constant in this case can be obtained by
\[
\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = -\frac{\Gamma(-D/2)}{2(4\pi)^{D/2}} \sum_{k=1}^{\infty} m_k^D = -\frac{\Gamma(-D/2)}{2(4\pi)^{D/2}l^D} \sum_{k=1}^{\infty} \left( k - \frac{1}{2} \right)^D = \left( 1 - \frac{1}{2^D} \right) \frac{\Gamma((D+1)/2)\zeta_R(D+1)}{2\pi^{D+1/2}(4\pi)^{D/2}l^D},
\]  

(3.14)
\[
\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = - \int_0^\infty \frac{ds}{2(4\pi)^{D/2}s^{D/2+1}} g(s)
\]
\[
\Rightarrow - \int_0^\infty \frac{l\sqrt{\pi}ds}{2(4\pi)^{D/2}s^{(D+1)/2+1}} \sum_{k=1}^\infty (-1)^k \exp \left( -\frac{\pi^2k^2}{s} \right)
\]
\[
= \left( 1 - \frac{1}{2^{D-2}} \right) \frac{\Gamma((D+1)/2)\zeta(D+1)}{2\pi^{D+1/2}(4\pi)^{D/2}l^D}.
\]

(3.15)

For \(D = 4\), the numerical value is \(\frac{\Lambda_{\text{ind}}}{8\pi G_{\text{ind}}} = +2.36991 \times 10^{-5}l^{-4}\).

Similarly, the finite part of the induced gravitational constant can be obtained by

\[
\frac{1}{16\pi G_{\text{ind}}} = \frac{\Gamma(1 - D/2)}{12(4\pi)^{D/2}} \sum_{k=1}^\infty m_k^{D-2}
\]
\[
= \frac{\Gamma(1 - D/2)}{12(4\pi)^{D/2}l^{D-2}} \sum_{k=1}^\infty \left( k - \frac{1}{2} \right)^{D-2} = - \left( 1 - \frac{1}{2^{D-2}} \right) \frac{\Gamma(1 - D/2)\zeta(2 - D)}{12(4\pi)^{D/2}l^{D-2}}
\]
\[
= - \left( 1 - \frac{1}{2^{D-2}} \right) \frac{\Gamma((D - 1)/2)\zeta(D - 1)}{12\pi^{D-3/2}(4\pi)^{D/2}l^{D-2}},
\]

(3.16)

or

\[
\frac{1}{16\pi G_{\text{ind}}} = \frac{1}{12(4\pi)^{D/2}} \int_0^\infty \frac{ds}{s^{D/2}} g(s)
\]
\[
= \int_0^\infty \frac{l\sqrt{\pi}ds}{12(4\pi)^{D/2}s^{(D+1)/2}} \sum_{k=1}^\infty (-1)^k \exp \left( -\frac{\pi^2k^2}{s} \right)
\]
\[
= - \left( 1 - \frac{1}{2^{D-2}} \right) \frac{\Gamma((D - 1)/2)\zeta(D - 1)}{12\pi^{D-3/2}(4\pi)^{D/2}l^{D-2}}.
\]

(3.17)

For \(D = 4\), its numerical value is \(\frac{1}{16\pi G_{\text{ind}}} = -2.41021 \times 10^{-5}l^{-2}\).

The further study on other models which have various mass spectra and extension to the spinor and vector field theories is left for a future work. Please note that the sum of contributions from various matter fields determines the induced constants.\(^5\)

**IV. CONCLUSION**

In this paper, we have studied the theory of induced gravity derived from the higher-derivative theory. In the higher-derivative scalar field theory without self-interaction, it is concluded that the induced cosmological and gravitational constants are calculated as a

\(^5\) Very powerful methods for calculating effective actions, including background gravitational fields, can be found in Ref. [55].
sum of those obtained from free scalar fields, whose number corresponds to the number of d’Alembertian acting on the scalar field. In addition, we have confirmed that the Adler–Zee formula and the heat kernel method give the same result in arbitrary dimensions, and shown that a finite value for the induced constants can be extracted in the model with infinite derivatives. Unlike the Kaluza–Klein theory, we can consider models with arbitrary masses \(m_k\) in the higher-derivative theories\(^6\) and we can even use copies of multiple models which result in different signs and magnitudes of induced quantities. The higher-derivative theories can be found in modified gravity and string theory, though the structures of them are more complicated than the models considered in this paper. We would like to take this work as the first step in exploring quantum effects in more general models of the higher-derivative theory.

As a future issue, first of all, we will consider the introduction of fields with other spins, similar to the conventional induced gravity model including Ref. [47], for other fields can cancel divergences. Our investigations so far have only been on free higher-derivative theories. In addition, since the higher-derivative theory with self-interaction has a complicated structure even considering an effective theory, the calculation of the quantum effect becomes generally non-trivial. One may need to think about a theory with some useful symmetry. By the way, in general, in the theory of higher-derivative fields coupled to the curvature, the calculation of the induced gravitational constant becomes quite nontrivial.\(^7\) We would like to examine this case by constructing various models.

Extensions in different directions include mathematical applications to recent discussions of entropic field theory [56–58], induced gravity effect in continuous mass distribution theory [59–61], and so on. These topics are also reserved for future work.

**Appendix A: The derivation of (1.2) and (1.3)**

We briefly review the derivation of the Adler–Zee formulas (1.2) and (1.3) in this Appendix (for a review, see Refs. [31, 39]).

\(^6\) That is the fundamental scale \(l\) used in our models should not be the Planck length.

\(^7\) The induced gravity from the superrenormalizable models [20–24] has not been considered by anyone and this could also be decent work to do.
In the weak-field limit, the action is expanded as

\[
S[\phi, g_{\mu\nu}] = S[\phi, \eta_{\mu\nu}] + \int d^D x \frac{\delta S}{\delta g_{\mu\nu}(x)}(\phi, \eta_{\mu\nu}) \eta_{\mu\nu}(x) + \frac{1}{2} \int d^D x \int d^D y \frac{\delta^2 S}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y)}(\phi, \eta_{\mu\nu}) \eta_{\mu\nu}(x) \eta_{\rho\sigma}(y) + \cdots, \tag{A1}
\]

where \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \). Then, the effective action \( S[h] \) is given by

\[
S[h] = -\ln \int [D\phi] e^{-S[\phi, g_{\mu\nu}]} = \frac{1}{2} \int d^D x h^{\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle + \frac{1}{4} \int d^D x h^{\mu\nu}(x) h^{\rho\sigma}(x) \langle \delta T_{\mu\nu}(x) \rangle \delta g_{\rho\sigma}(y) - \frac{1}{8} \int d^D x \int d^D y h^{\mu\nu}(x) h^{\rho\sigma}(y) \langle \bar{T}_{\mu\nu}(x) \bar{T}_{\rho\sigma}(y) \rangle + \cdots, \tag{A2}
\]

where we used the stress tensor \( T^{\mu\nu}(x) = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \). \( \langle \cdots \rangle \) denotes a vacuum expectation value and \( \bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \langle T_{\mu\nu} \rangle \). Note that we are working with \( D \)-dimensional Euclidean space \( (\eta_{\mu\nu} = \delta_{\mu\nu}) \). For simplicity, we specialize the metric as \( h_{\mu\nu}(x) = \frac{1}{D} \eta_{\mu\nu} h(x) \) (Refs. [29, 31, 39]) and then we can express the effective action as a functional of \( h(x) \).

The part of the effective action which includes no derivative of \( h \) is given by

\[
\int d^D x \frac{1}{2D} h(x) \langle T(x) \rangle + O(h^2), \tag{A3}
\]

while the part of the effective action which includes two derivatives of \( h \) is

\[
- \int d^D x \frac{1}{16D^3} h(x) \Box h(x) \int d^D w |w|^2 \langle \bar{T}(x) \bar{T}(y) \rangle + \cdots, \tag{A4}
\]

where \( T \equiv \eta^{\mu\nu} T_{\mu\nu} \). Here we used the Taylor expansion \( h(y) = h(x) - w^\mu \partial_\mu h(x) + \frac{1}{2} w^\mu w^\nu \partial_\mu \partial_\nu h(x) \), where \( w = x - y \), and the isotropy of the space, i.e., \( w^\mu w^\nu \rightarrow \eta^{\mu\nu} |w|^2 / D \) in the integral.

The comparison of these expression with

\[
\int d^D x \sqrt{|g|} = \int d^D x \left( 1 + \frac{1}{2} h(x) + O(h^2) \right), \tag{A5}
\]

and

\[
\int d^D x \sqrt{|g|R} = \frac{(D - 1)(D - 2)}{4D^2} \int d^D x h(x) \Box h(x) + \cdots, \tag{A6}
\]

(where \( R \) is the scalar curvature) gives (1.2) and (1.3).
Appendix B: The stress tensor in the $n = 2$ Lee–Wick scalar field theory

We can introduce an auxiliary field to study the $n = 2$ Lee–Wick scalar field theory, referring works on higher-derivative theories [6] motivated by the seminal Lee–Wick model [2–4]. We assume the Lagrangian

$$\mathcal{L} = -\frac{1}{2(m_2^2 - m_1^2)}\phi(-\Box + m_1^2)(-\Box + m_2^2)\phi.$$  \hspace{1cm} (B1)

We also consider the alternative Lagrangian

$$\mathcal{L}' = -\frac{1}{2}\phi(-\Box + m_1^2)\phi - \psi(-\Box + m_2^2)\phi + \frac{1}{2}(m_2^2 - m_1^2)\psi^2.$$  \hspace{1cm} (B2)

The equation of motion of the initially auxiliary field $\psi$ from the Lagrangian (B2) is

$$\psi = \frac{1}{m_2^2 - m_1^2}(-\Box + m_1^2)\phi.$$  \hspace{1cm} (B3)

Thus, substituting this equation to the alternative Lagrangian $\mathcal{L}'$ yields the original Lagrangian $\mathcal{L}$. On the other hand, defining the field $\chi$ as

$$\chi = \phi + \psi = \frac{1}{m_2^2 - m_1^2}(-\Box + m_2^2)\phi,$$  \hspace{1cm} (B4)

the Lagrangian (B2) is also written by

$$\mathcal{L}' = -\frac{1}{2}\chi(-\Box + m_1^2)\chi + \frac{1}{2}\psi(-\Box + m_2^2)\psi.$$  \hspace{1cm} (B5)

This effective Lagrangian describes two free scalar fields, whose masses are $m_1$ and $m_2$. The kinetic term for the scalar field $\psi$ with mass $m_2$ has a “wrong” sign. This field decouples from the physical spectrum if $m_2 \to \infty$, since its mass becomes infinitely large.

The stress tensor derived from the last Lagrangian (B5) reads

$$T_{\mu\nu} = \partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\chi\partial^\rho\chi + m_1^2\chi^2) - \partial_\mu\psi\partial_\nu\psi + \frac{1}{2}g_{\mu\nu}(\partial_\rho\psi\partial^\rho\psi + m_2^2\psi^2).$$  \hspace{1cm} (B6)

By substituting (B3) and (B4) into this, we obtain

$$T_{\mu\nu} = \frac{1}{m_2^2 - m_1^2} \left[-\partial_\mu\phi\partial_\nu\phi - \partial_\mu\phi\partial_\rho\phi\partial^\rho\phi + g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}g_{\mu\nu}(\Box\phi)^2ight.$$

$$+ (m_1^2 + m_2^2) \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right) - \frac{1}{2}g_{\mu\nu}m_1^2m_2^2\phi^2\phi^2\right].$$  \hspace{1cm} (B7)
which coincides with the result found in Refs. [44–46]. Incidentally, the trace of the stress tensor becomes

\[
T = \frac{1}{m_2^2 - m_1^2} \left[ (D - 2) \partial_\rho \phi \partial^\rho \Box \phi - (m_1^2 + m_2^2) \frac{D - 2}{2} \partial_\rho \phi \partial^\rho \phi + \frac{D}{2} (\Box \phi)^2 - \frac{D}{2} m_1^2 m_2^2 \phi^2 \right]
\]

\[
= \frac{1}{m_2^2 - m_1^2} \left\{ - \frac{D - 2}{2} \left[ \partial_\rho \phi \partial^\rho (\Box + m_1^2) \phi + \partial_\rho \phi \partial^\rho (\Box + m_2^2) \phi \right] - \frac{D}{2} \left[ (-\Box + m_1^2) \phi \Box \phi + \phi (\Box + m_2^2) \phi + \phi (-\Box + m_1^2) (-\Box + m_2^2) \phi \right] \right\}, \quad (B8)
\]

which agrees with (2.4) for \( n = 2 \) up to the last term that can be discarded by the field equation.

**Appendix C: The proof of (2.9)**

By rewriting (2.5), we find

\[
\tilde{T}(x) = \lim_{x' \to x} \sum_{k=0}^{n} C^{-1} \mathcal{O}_{xx'} : \eta_k(x) \chi_{k+1}(x') : ,
\]

(C1)

where\(^8\)

\[
\mathcal{O}_{xx'} = - \frac{D - 2}{2} \partial_\rho \partial^\rho - \frac{D}{2} \Box_x ,
\]

(C2)

and : : stands for a normal ordered product. Therefore, we would like to know \( \sum_{k=0}^{n} \sum_{l=0}^{n} \langle : \eta_k(x) \chi_{k+1}(x') : ; \eta_l(y) \chi_{l+1}(y') : \rangle \) for an evaluation from the Adler–Zee formula.

To this end, we start with

\[
\langle : \eta_0(x) \chi_1(x') + \eta_1(x) \chi_2(x') : ; \eta_0(y) \chi_1(y') + \eta_1(y) \chi_2(y') : \rangle
\]

\[
= \langle \eta_0(x) \eta_0(y) \rangle \langle \chi_1(x') \chi_1(y') \rangle + \langle \eta_0(x) \chi_1(y') \rangle \langle \eta_0(y) \chi_1(x') \rangle
\]

\[
+ \langle \eta_1(x) \eta_1(y) \rangle \langle \chi_2(x') \chi_2(y') \rangle + \langle \eta_1(x) \chi_2(y') \rangle \langle \eta_1(y) \chi_2(x') \rangle
\]

\[
+ 2 \langle \eta_0(x) \eta_1(y) \rangle \langle \chi_1(x') \chi_2(y') \rangle + 2 \langle \eta_0(x) \chi_2(y') \rangle \langle \eta_1(y) \chi_1(x') \rangle ,
\]

(C3)

where we used the symmetry under \( (x, x') \leftrightarrow (y, y') \). Now, we use \( \eta_0 = \phi, \eta_1 = A_1 \phi, \chi_1 = A_2 \cdots A_n \phi, \chi_2 = A_3 \cdots A_n \phi \), and \( \langle \phi(x) \phi(x') \rangle = \frac{c}{A_1 \cdots A_n} 1_{xx'} \), in the symbolic notation. Then, we find

\[
C^{-2} \langle \eta_0(x) \eta_0(y) \rangle \langle \chi_1(x') \chi_1(y') \rangle = \frac{1}{A_1 \cdots A_n} 1_{xy} \frac{1}{A_1} A_2 \cdots A_n 1_{x'y'}
\]

(C4)

\(^8\) As previously mentioned in the text, \( \Box_x \) can read \( (\Box_x + \Box_x)/2 \).
\[ C^{-2} \langle \eta_0(x) \chi_1(y') \rangle \langle \eta_0(y) \chi_1(x') \rangle = \frac{1}{A_1} 1_{xy'} \frac{1}{A_1} 1_{x'y} \quad (C5) \]
\[ C^{-2} \langle \eta_1(x) \eta_1(y) \rangle \langle \chi_2(x') \chi_2(y') \rangle = A_1 \frac{1}{A_2 \cdots A_n} 1_{xy} \frac{1}{A_1 A_2} A_3 \cdots A_n 1_{x'y'} \quad (C6) \]
\[ C^{-2} \langle \eta_1(x) \chi_2(y') \rangle \langle \eta_1(y) \chi_2(x') \rangle = \frac{1}{A_2} 1_{xy'} \frac{1}{A_1} 1_{x'y} \quad (C7) \]
\[ C^{-2} \langle \eta_0(x) \eta_1(y) \rangle \langle \chi_1(x') \chi_2(y') \rangle = \frac{1}{A_2 \cdots A_n} 1_{xy} \frac{1}{A_1 A_2} A_3 \cdots A_n 1_{x'y'} \quad (C8) \]
\[ C^{-2} \langle \eta_0(x) \chi_2(y') \rangle \langle \eta_1(y) \chi_1(x') \rangle = \frac{1}{A_1 A_2} 1_{xy'} 1_{x'y}'. \quad (C9) \]

Further, after using \( \frac{A_2}{A_1} = 1 + \frac{m_2^2 - m_1^2}{m_1^2} \) in \( C4 \), \( \frac{A_3}{A_2} = 1 - \frac{m_2^2 - m_3^2}{m_2^2} \) in \( C6 \), and \( \frac{1}{A_1 A_2} = \frac{1}{m_2^2 - m_3^2} \left( \frac{1}{A_1} - \frac{1}{A_2} \right) \) in \( C4 \) and \( C6 \), the result for \( C3 \) turns out to be

\[
C^{-2} \langle \eta_0(x) \chi_1(x') + \eta_1(x) \chi_2(x') : \eta_0(y) \chi_1(y') + \eta_1(y) \chi_2(y') : \rangle
= \frac{1}{A_1} 1_{xy'} \frac{1}{A_1} 1_{x'y} + \frac{1}{A_1 A_3} 1_{xy} \frac{1}{A_1 A_3} A_1 A_2 1_{x'y'} + \frac{1}{A_2} 1_{xy'} \frac{1}{A_2} 1_{x'y} + \frac{1}{A_2 A_3} 1_{xy} \frac{1}{A_2 A_3} A_1 A_2 1_{x'y'}
+ 2 \frac{1}{A_1 A_2} 1_{xy'} 1_{x'y} + \frac{1}{A_1 A_2} 1_{xy} A_1 A_2 A_3 1_{x'y'}
= \frac{1}{A_1} 1_{xy'} \frac{1}{A_1} 1_{x'y} + \frac{1}{A_2} 1_{xy'} \frac{1}{A_2} 1_{x'y} + \frac{1}{A_2 A_3} 1_{xy} \frac{1}{A_2 A_3} A_1 A_2 1_{x'y'}
\]

where \( A_{3n} = A_3 \cdots A_n \) and we used the symmetry under \( y \leftrightarrow y' \). Here, we first note that if masses \( m_3, m_4, \ldots, m_n \) become infinity, this reduces exactly to, with the symmetry under \( x \leftrightarrow x' \) and \( y \leftrightarrow y' \),

\[
C^{-2} \langle \eta_0(x) \chi_1(x') + \eta_1(x) \chi_2(x') : \eta_0(y) \chi_1(y') + \eta_1(y) \chi_2(y') : \rangle
= 2 \frac{1}{A_1} 1_{xy'} \frac{1}{A_1} 1_{x'y} + 2 \frac{1}{A_2} 1_{xy'} \frac{1}{A_2} 1_{x'y} + 4 \frac{1}{A_1 A_2} 1_{xy'} 1_{x'y}'. \quad (C11) \]

It is worthwhile noting that the term like \( \frac{1}{A_1} 1_{xy'} \frac{1}{A_2} 1_{x'y'} \) does not appear in the expression. The last term in the above expression gives no contribution in the Adler–Zee formula defined by the integration \( (1.3) \) including \( |w|^2 \). We can naively regard that the partial integration makes the expression \( C10 \) to be \( C11 \) through moving the operator \( A_{3n} \), since the difference in \( x \) and \( x' \) is only due to the distinction of operation of derivatives in \( O_{xx'} \). If you have some worries on the integration by part in the Adler–Zee formula, let us proceed as follows.

The deviation from \( C11 \), if exists, should be a function of \( m_3, \ldots, m_n \). We extracted the first two terms included in \( \sum_{k=0}^n \sum_{l=0}^n \langle \eta_k(x) \chi_{k+1}(x') : \eta_l(y) \chi_{l+1}(y') : \rangle \), but the extraction can be arbitrary, and we can select the terms as \( 1 \rightarrow i \) and \( 2 \rightarrow j \). Then, the expression has a parallel form and the deviation is a function of \( \{ m_k \}, k \neq i, j \). Totally because all the number should appear in the expression of \( \sum_{k=0}^n \sum_{l=0}^n \langle \eta_k(x) \chi_{k+1}(x') : \eta_l(y) \chi_{l+1}(y') : \rangle \), the deviation function can be at most a constant. Since the case \( n = 2 \) is confirmed exactly, we
conclude that
\[
\langle \hat{T}(x)\hat{T}(y) \rangle = \lim_{x' \to x} \lim_{y' \to y} \mathcal{O}_{xx'} \mathcal{O}_{yy'} \sum_{k=1}^{n} 2\Delta_k(x, y)\Delta_k(x', y')
\] (C12)
is used in the Adler–Zee formula (1.3).

**Appendix D: Details in calculation of the induced cosmological and gravitational constant using the Schwinger parameter**

The method using the integral form has been proposed by Kehagias et al.\[47\]. We here consider calculations for \( D \) dimensions. We start with
\[
\Delta_k(x, x') = \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \exp \left[ -\frac{|w|^2}{4s} - m_k^2 s \right],
\] (D1)
where the symmetry \( \Delta(x, x') = \Delta(x', x) \) is apparent. The sequential differentiation of this reveals
\[
\partial_\mu \Delta_k(x, x') = -\int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \frac{w_\mu}{2} \exp \left[ -\frac{|w|^2}{4s} - m_k^2 s \right],
\] (D2)
\[
\partial_\mu \partial_\nu \Delta_k(x, x') = \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \frac{\left(\frac{1}{2}g_{\mu\nu} - \frac{w_\mu w_\nu}{4s}\right)}{s^{D/2+1}} \exp \left[ -\frac{|w|^2}{4s} - m_k^2 s \right],
\] (D3)
\[
\partial_\mu \partial^\nu \Delta_k(x, x') = \int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \frac{\frac{D}{2} - \frac{|w|^2}{4s}}{s^{D/2+1}} \exp \left[ -\frac{|w|^2}{4s} - m_k^2 s \right],
\] (D4)
\[
\Box \Delta_k(x, x') = -\int_0^\infty \frac{ds}{(4\pi s)^{D/2}} \frac{\frac{D+2}{2} - \frac{|w|^2}{4s}}{s^{D/2+1}} \frac{w_\mu}{2s} \exp \left[ -\frac{|w|^2}{4s} - m_k^2 s \right].
\] (D5)

We have already obtained the necessary calculation for the induced cosmological constant.

Then, we consider the combinations which is necessary in calculating the induced gravitational constant. They are
\[
\partial_\mu \partial_\nu \Delta_k(x, x') \partial^\mu \partial^\nu \Delta_k(x, x')
= \frac{1}{(4\pi)^D} \int_0^\infty \int_0^\infty \int_0^\infty \frac{ds_1}{s_1^{D/2+1}} \frac{ds_2}{s_2^{D/2+1}} \left( \frac{D}{4} - \frac{|w|^2}{8s_1} - \frac{|w|^2}{8s_2} + \frac{|w|^4}{16s_1 s_2} \right)
\times \exp \left[ -\frac{|w|^2}{4s_1} - \frac{|w|^2}{4s_2} - m_k^2 (s_1 + s_2) \right],
\] (D7)
\[
\partial_\mu \Delta_k(x, x') \partial^\mu \Box \Delta_k(x, x')
= -\frac{1}{(4\pi)^D} \int_0^\infty \int_0^\infty \int_0^\infty \frac{ds_1}{s_1^{D/2+1}} \frac{ds_2}{s_2^{D/2+1}} \left( \frac{(D+2)|w|^2}{16s_1} + \frac{(D+2)|w|^2}{16s_2} - \frac{|w|^4}{32s_1^2} - \frac{|w|^4}{32s_2^2} \right)
\]
\times \exp \left[ -\frac{|w|^2}{4s_1} - \frac{|w|^2}{4s_2} - m_k^2(s_1 + s_2) \right], \quad (D8)
\left(\Box_x \Delta_k(x, x')\right)^2
= \frac{1}{(4\pi)^D} \int_0^\infty \int_0^\infty \frac{ds_1}{s_1^{D/2+1}} \frac{ds_2}{s_2^{D/2+1}} \left( \frac{D^2}{4} - \frac{D|w|^2}{8s_1} - \frac{D|w|^2}{8s_2} + \frac{|w|^4}{16s_1s_2} \right)
\times \exp \left[ -\frac{|w|^2}{4s_1} - \frac{|w|^2}{4s_2} - m_k^2(s_1 + s_2) \right]. \quad (D9)

The integration in the Adler–Zee formula can be performed by using the following Gaussian integral formulas:
\int d^D w |w|^2 e^{-\alpha|w|^2} = \frac{D \pi^{D/2}}{2} \frac{\pi^{D/2}}{\alpha^{D/2+1}} \quad (D10)
\int d^D w |w|^4 e^{-\alpha|w|^2} = \frac{D(D+2) \pi^{D/2}}{4} \frac{\pi^{D/2}}{\alpha^{D/2+3}} \quad (D11)
\int d^D w |w|^6 e^{-\alpha|w|^2} = \frac{D(D+2)(D+4) \pi^{D/2}}{8} \frac{\pi^{D/2}}{\alpha^{D/2+3}} \quad (D12)

Lastly, we transform the parameters as \( s \equiv s_1 + s_2 \) and \( u \equiv \frac{s_1}{s} \) \[47\]. Then the measure becomes \( ds_1ds_2 = sdsdu \). Finally, noticing \( \int_0^1 u(1-u)du = \frac{1}{6} \), we find
\int d^D w |w|^2 \partial_\mu \partial_{\nu'} \Delta_k(w) \partial^{\mu'} \partial^{\nu} \Delta_k(w) = \frac{1}{(4\pi)^D} \frac{D(D + 6D - 4)}{12} \int_0^\infty \frac{ds}{s^{D/2}} \exp \left[ -m_k^2 s \right] \quad (D13)
\int d^D w |w|^2 \partial_\mu \Delta_k(w) \partial^\mu \Delta_k(w) = -\frac{1}{(4\pi)^D} \frac{D(D - 2)(D + 2)}{12} \int_0^\infty \frac{ds}{s^{D/2}} \exp \left[ -m_k^2 s \right] \quad (D14)
\int d^D w |w|^2 (\Box_x \Delta_k(w))^2 = \frac{1}{(4\pi)^D} \frac{D(D - 2)(D - 4)}{12} \int_0^\infty \frac{ds}{s^{D/2}} \exp \left[ -m_k^2 s \right] \quad (D15)

Combining these results, we can find the induced gravitational constant. It may simply be verified that our results is in agreement for \( D = 4 \) with Ref. \[47\].

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