A Hamiltonian-based analytical approach for three-dimensional heat conduction of cylinders with specific mixed boundary conditions

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Abstract. A novel analytical symplectic method is introduced to investigate the three-dimensional steady-state heat conduction of cylinders with specific mixed boundary conditions (partial temperature and partial heat flux density). By defining the temperature and heat flux density as the mutually dual variables, the Hamiltonian form of governing equations are established. The original problem is reduced into a symplectic eigenproblem which can be solved by the method of separation of variables and a symplectic sub-system. Exact analytical solution is obtained and expressed in terms of symplectic eigensolutions. Comparison studies demonstrate the accuracy of the proposed method. Some new results are given also.

1. Introduction

Heat conduction problems in cylinders usually appear in engineering applications like pipes and vessels e.g. submarine pipelines. It is important to determine the temperature distributions within the cylinders. A considerable number of analytical examples have been provided by Carslaw and Jaeger [1] and Ozisik [2] in their books. More recently, various analytical methods have been applied to heat conduction problems, including integral transforms [3], Green’s function [4], expansion technique [5], state space approach [6]. Pyr’yev [3] investigated the two-dimensional transient heat conduction of two-layer cylinders with surface heat generation based on Laplace transform. Biswas et al. [7] proposed a multi-layer annulus composite model for heat conduction with time dependent boundary conditions. Tarn and Wang [6] studied the steady-state heat conduction in functionally graded cylinders with emphasis on the effects of end conditions. However, these studies belong to Lagrange formalism. In such methods, due to the limitations of one-variable Lagrange system, some complex problems are hard to solve, e.g. cylinders with specific mixed boundary conditions (partial temperature and partial heat flux density on one surface).

To overcome this drawback, Xu et al. [8] introduced the symplectic approach in the analysis of 2D transient heat conduction in a rectangle with specific mixed boundary conditions. The
symplectic elasticity was developed by Zhong and his collaborators [9], and has been successfully applied to solid mechanics and engineering. Motivated by the work of Xu et al. [8], this paper presents analytical solutions for steady-state heat conduction in three-dimensional homogeneous cylinders with specific mixed boundary conditions. The cylinders have homogeneous lateral BCs and prescribed temperature or heat flux are given as its end conditions. The temperature and the heat flux along the longitudinal direction are defined as mutually dual variables in the symplectic space. Based on the mutually dual variables, the Hamiltonian system for heat conduction problem is presented. The symplectic eigenvalues and eigensolutions are obtained with the symplectic sub-system and separation of variables. The solutions are expressed as series expansion form which may benefit the relevant numerical methods, and its coefficients are determined by the specific mixed boundary conditions of end surfaces.

2. The fundamental problem
Consider a 3D isotropic hollow circular cylinders in Figure 1. The boundary conditions (BCs) are as follows:

BC1: \( T|_{r=a} = 0 \) or \( -k \partial T / \partial r |_{r=a} = 0 \), BC2: \( T|_{r=b} = 0 \) or \( -k \partial T / \partial r |_{r=b} = 0 \) \( \text{(1a)} \)

BC3: \( T|_{z=b} = T_b(r, \theta) \) for \( \Omega_1 \), BC4: \( -k \partial T / \partial z |_{z=b} = \varphi_b(r, \theta) \) for \( \Omega_2 \) \( \text{(1b)} \)

BC5: \( T|_{z=-b} = \varphi_b(r, \theta) \) for \( \Omega_1 \), BC6: \( -k \partial T / \partial z |_{z=-b} = \varphi_b(r, \theta) \) for \( \Omega_2 \) \( \text{(1c)} \)

For a solid cylinder, the BC2 is written as

BC2: \( T|_{r=a} = \text{finite} \) or \( \partial T / \partial r |_{r=a} = \text{finite} \) \( \text{(1d)} \)

**Figure 1.** A hollow circular cylinder with specific mixed boundary conditions.

The Fourier’s law of heat conduction [2] in circular cylindrical coordinates \((r, \theta, z)\) is given as

\[
\varphi_r = -k \partial T / \partial r, \quad \varphi_\theta = -k \partial T / (r \partial \theta), \quad \varphi_z = -k \partial T / \partial z
\]  

where \(k\) is the thermal conductivity, \(\varphi_r\), \(\varphi_\theta\), and \(\varphi_z\) are the heat flux along the \(r\)-, \(\theta\)-, and \(z\)-directions, respectively.

The Lagrangian density function for the 3D steady state heat conduction is

\[
L(r, \theta, z) = -k / 2 \left[ (\partial^2 T / \partial r^2) + (\partial^2 T / \partial \theta^2) / r^2 + (\partial T / \partial z)^2 \right]
\]  

The principle of the minimum dissipation of heat quantity, as given by

\[
\delta \int \int \int L \, dr \, d\theta \, dz = 0
\]  

The governing equations in the Lagrange system can be derived from Eq. (4),

\[
\partial^2 T / \partial r^2 + \partial T / (r \partial r) + \partial^2 T / (r^2 \partial \theta^2) + \partial^2 T / \partial z^2 = 0
\]  

3. The Hamiltonian system

3.1. Establishment of Hamiltonian equations

Denoting \( \dot{z} = \partial( ) / \partial z \) as the differentiation with respect to \( z \), the original variable and its corresponding dual variable are defined as

\[ q = T, \quad p = \partial L / \partial \dot{q} = -k \partial T / \partial \dot{z} = \varphi. \]  

(6)

The Hamiltonian function can be introduced as

\[ H(q, p) = p\dot{q} - L(q, \dot{q}). \]  

(7)

Therefore, the Hamiltonian form of governing equation can be derived by

\[ \dot{q} = \partial H / \partial p, \quad \dot{p} = -\partial H / \partial q \]  

(8)

Denoting \( \Psi = \{q, p\}^T \) as a total unknown vector, Eq. (8) can be expressed in a matrix form of

\[ \Psi = H \Psi \]  

(9)

where \( H \) is the Hamiltonian operator matrix.

It is natural to employ the method of separation of variables to solve Eq. (9). Let

\[ \Psi(r, \theta, z) = \sum \psi(r, \theta) e^{\mu z} \]  

(10)

where \( \mu \) is the eigenvalue alone the \( z \)-axis and \( \psi(r, \theta) = \{\tilde{T}(r, \theta), \tilde{\varphi}(r, \theta)\}^T \).

Substituting Eq. (10) into Eq. (9) yields

\[ H \psi = \mu \psi \]  

(11)

where \( \mu \) and \( \psi \) are the eigenpair of the Hamiltonian operator matrix.

3.2. The Hamiltonian sub-system

Since the partial differential Eq. (11) cannot be directly solved, the Hamiltonian sub-system is introduced to change the partial differential Eq. (12) into an ordinary differential equation. The Lagrangian function for the sub-system is

\[ \tilde{L}(r, \theta) = -k / 2 \left[ (\partial T / \partial r)^2 + (\partial T / \partial \theta)^2 / r^4 + (\partial T / \partial z)^2 \right] \]  

(12)

Denoting \( \dot{\theta} = \partial( ) / \partial \theta \) as the differentiation with respect to \( \theta \), and \( \tilde{\varphi}_\theta = -k (\partial T / \partial \theta) / r \).

Similar to Eqs. (6)-(7), we have

\[ q_r = \tilde{T}, \quad p_r = \partial L / \partial \dot{q}_r = \tilde{\varphi}_\theta \]  

(13)

\[ H_s(q_r, p_r) = p_r \dot{q}_r - L(q_r, \dot{q}_r) \]  

(14)

where \( q_r \) and \( p_r \) are the original variable and corresponding dual variable for the sub-system, while \( H_s \) is the Hamiltonian function for the sub-system. Similar to Eq. (8), the governing equation of the Hamiltonian sub-system is

\[ \dot{q}_r = \partial H_s / \partial p_r, \quad \dot{p}_r = -\partial H_s / \partial q_r \]  

(16)

namely,

\[ \begin{bmatrix} 0 & -r / k \\ k (r \partial^2 / \partial r^2 + \partial / \partial r + r \mu^2) & 0 \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{\varphi}_\theta \end{bmatrix} = \begin{bmatrix} \dot{q}_r \\ \dot{p}_r \end{bmatrix}. \]  

(17)

3.3. Eigenvalues and eigensolutions
For $\mu = 0$: the zero eigensolutions are

$$\psi_0^\alpha = \left\{ q_{00}^\alpha, p_{00}^\alpha \right\}^T = \{1, 0\}^T, \quad \psi_0^\beta = \left\{ q_{00}^\beta, p_{00}^\beta \right\}^T = \{z, -k\}^T$$

It can be proved that the zero eigensolutions only exist when all the lateral BCs are prescribed heat flux.

For $\mu \neq 0$: assuming $\{\bar{T}(r, \theta), \bar{\varphi}_y(r, \theta)\} = \sum \{\bar{T}(r), \bar{\varphi}_y(r)\} e^{i\omega t}$ and substituting it into Eq. (17), we have

$$\begin{bmatrix}
-\omega & -r/k \\
\mu r & -\omega
\end{bmatrix} \begin{bmatrix}
\bar{T}(r) \\
\bar{\varphi}_y(r)
\end{bmatrix} = 0 \quad (18)$$

According circumferential continuity, we have

$$\omega = \pm ni \quad (n = 1, 2, 3, \ldots) \quad (19)$$

Substituting Eq. (19) into Eq. (18), it yields

$$\bar{\varphi}_y(r) \left( r \partial^2 \bar{T}(r) / \partial \theta^2 + \partial \bar{T}(r) / \partial \theta + r \mu \bar{T}(r) \right) = 0 \quad (20)$$

and the corresponding general solution is

$$\bar{T}_n(r) = U_n(\mu r) \phi_n(\mu r) = c_1 J_n(\mu r) + c_2 Y_n(\mu r) \quad (21)$$

where $J_n(\mu r)$ is the Bessel's function of the first kind of order $n$ while $Y_n(\mu r)$ is the Bessel's function of the second kind of order $n$. The eigenvalue $\mu$ and constants $c_1$ and $c_2$ can be determined by the lateral BCs (1a) and (1d).

The non-zero eigensolutions are

$$\psi_{nm}^{\alpha, \alpha} = \left\{ q_{nm}^{\alpha, \alpha}, p_{nm}^{\alpha, \alpha} \right\} = \left\{ U_n(\mu r) e^{i\omega t}, -kU_n(\mu r) e^{i\omega t} \right\}, \quad \psi_{nm}^{\alpha, \beta} = \left\{ q_{nm}^{\alpha, \beta}, p_{nm}^{\alpha, \beta} \right\} = \left\{ U_n(\mu r) e^{-i\omega t}, -kU_n(\mu r) e^{-i\omega t} \right\}, \quad (22a)$$

$$\psi_{nm}^{\beta, \alpha} = \left\{ q_{nm}^{\beta, \alpha}, p_{nm}^{\beta, \alpha} \right\} = \left\{ U_n(\mu r) e^{i\omega t}, kU_n(\mu r) e^{i\omega t} \right\}, \quad \psi_{nm}^{\beta, \beta} = \left\{ q_{nm}^{\beta, \beta}, p_{nm}^{\beta, \beta} \right\} = \left\{ U_n(\mu r) e^{-i\omega t}, kU_n(\mu r) e^{-i\omega t} \right\}, \quad (22b)$$

$$\psi_{nm}^{\alpha, \beta} = \left\{ q_{nm}^{\alpha, \beta}, p_{nm}^{\alpha, \beta} \right\} = \left\{ U_0(\mu r), -kU_0(\mu r) \right\}, \quad \psi_{nm}^{\beta, \alpha} = \left\{ q_{nm}^{\beta, \alpha}, p_{nm}^{\beta, \alpha} \right\} = \left\{ U_0(\mu r), kU_0(\mu r) \right\} \quad (22c)$$

where the first superscript of the eigensolution represents the type of $\mu$ while the second superscript of the eigensolution represents the type of $\omega$; $\psi_{nm}^{\alpha}$ and $\psi_{nm}^{\beta}$ are for $\omega = 0$.

For the 3D problem, the eigenvalues $\omega$ is categorized as two groups

$$\begin{cases}
(\alpha) & \omega_j^{(\alpha)}, \quad j = 1, 2, \ldots, \quad \text{Re}(\omega_j^{(\alpha)}) > 0 \quad \text{or} \quad \text{Re}(\omega_j^{(\alpha)}) = 0 \quad \text{and} \quad \text{Im}(\omega_j^{(\alpha)}) > 0 \\
(\beta) & \omega_j^{(\beta)}, \quad j = 1, 2, \ldots, \quad \omega_j^{(\beta)} = -\omega_j^{(\alpha)}
\end{cases} \quad (23a)$$

Defining the inner product as

$$\langle \psi_j, \psi_i \rangle = \int \int_\Omega \left( q_j p_i - q_i p_j \right) d\Omega, \quad (24)$$

the adjoint symplectic orthogonality can be expressed as,

$$\begin{align}
\langle \psi_{nm}^{\alpha, \alpha}, \psi_{n'n'}^{\beta, \beta} \rangle &= 0, \quad \langle \psi_{nm}^{\alpha, \alpha}, \psi_{n'n'}^{\beta, \alpha} \rangle = 0, \quad \langle \psi_{nm}^{\beta, \alpha}, \psi_{n'n'}^{\beta, \alpha} \rangle = 0, \quad \langle \psi_{nm}^{\beta, \beta}, \psi_{n'n'}^{\beta, \beta} \rangle = 0 \\
\langle \psi_{nm}^{\alpha, \beta}, \psi_{n'n'}^{\beta, \alpha} \rangle &= 0, \quad \langle \psi_{nm}^{\alpha, \beta}, \psi_{n'n'}^{\beta, \beta} \rangle = 0, \quad \langle \psi_{nm}^{\beta, \alpha}, \psi_{n'n'}^{\alpha, \beta} \rangle = 0, \quad \langle \psi_{nm}^{\beta, \beta}, \psi_{n'n'}^{\alpha, \beta} \rangle = 0
\end{align} \quad (25a)$$

$$\begin{align}
\langle \psi_{nm}^{\alpha, \beta}, \psi_{n'n'}^{\beta, \beta} \rangle &= -\psi_{n'n'}^{\alpha, \beta}, \psi_{nm}^{\alpha, \alpha} \rangle = 2k \mu_{\beta \beta} \cdot 2\pi \cdot \int_0^\alpha \left( \mu u \right)^2 r dr \cdot \delta_{n n}^i \delta_{m m}^j \\
\langle \psi_{nm}^{\beta, \alpha}, \psi_{n'n'}^{\alpha, \alpha} \rangle &= -\psi_{n'n'}^{\beta, \alpha}, \psi_{nm}^{\beta, \beta} \rangle = 2k \mu_{\beta \beta} \cdot 2\pi \cdot \int_0^\alpha \left( \mu u \right)^2 r dr \cdot \delta_{n n}^i \delta_{m m}^j
\end{align} \quad (25b)$$

$$\begin{align}
\langle \psi_{nm}^{\alpha, \beta}, \psi_{n'n'}^{\beta, \beta} \rangle &= -\psi_{n'n'}^{\alpha, \beta}, \psi_{nm}^{\alpha, \alpha} \rangle = 2k \mu_{\beta \beta} \cdot 2\pi \cdot \int_0^\alpha \left( \mu u \right)^2 r dr \cdot \delta_{n n}^i \delta_{m m}^j \\
\langle \psi_{nm}^{\beta, \alpha}, \psi_{n'n'}^{\alpha, \alpha} \rangle &= -\psi_{n'n'}^{\beta, \alpha}, \psi_{nm}^{\beta, \beta} \rangle = 2k \mu_{\beta \beta} \cdot 2\pi \cdot \int_0^\alpha \left( \mu u \right)^2 r dr \cdot \delta_{n n}^i \delta_{m m}^j
\end{align} \quad (25c)$$

$$\begin{align}
\langle \psi_{nm}^{\alpha, \beta}, \psi_{n'n'}^{\beta, \beta} \rangle &= -\psi_{n'n'}^{\alpha, \beta}, \psi_{nm}^{\alpha, \alpha} \rangle = 2k \mu_{\beta \beta} \cdot 2\pi \cdot \int_0^\alpha \left( \mu u \right)^2 r dr \cdot \delta_{n n}^i \delta_{m m}^j \\
\langle \psi_{nm}^{\beta, \alpha}, \psi_{n'n'}^{\alpha, \alpha} \rangle &= -\psi_{n'n'}^{\beta, \alpha}, \psi_{nm}^{\beta, \beta} \rangle = 2k \mu_{\beta \beta} \cdot 2\pi \cdot \int_0^\alpha \left( \mu u \right)^2 r dr \cdot \delta_{n n}^i \delta_{m m}^j
\end{align} \quad (25d)$$
\[
\langle \psi_{0m}^a, \psi_{0l}^a \rangle = -\left\{ \psi_{0m}^a, \psi_{0l}^a \right\} = 2k \mu_0 \cdot 2\pi \cdot \int_0^\infty U_0 \left( \mu_0 r \right)^2 rdr \cdot \delta_{ml} \quad (25e)
\]

\[
\langle \psi_{0m}^a, \psi_{0l}^b \rangle = -\left\{ \psi_{0m}^a, \psi_{0l}^b \right\} = k\pi \cdot \left( a_1^2 - a_2^2 \right) \quad (25f)
\]

Finally, the solution of the problem can be expressed as
\[
\Psi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \left( A_{nm} \psi_{nm}^{a,\alpha} + B_{nm} \psi_{nm}^{b,\beta} \right) e^{\mu_{nm}z} + \left( C_{nm} \psi_{nm}^{\beta,\alpha} + D_{nm} \psi_{nm}^{\beta,\beta} \right) e^{-\mu_{nm}z} \right] + \sum_{n=1}^{\infty} \left( A_{nm} \psi_{nm}^{a,0} e^{\mu_{nm}z} + B_{nm} \psi_{nm}^{b,0} e^{-\mu_{nm}z} \right) + A_{00} \psi_{00}^a + B_{00} \psi_{00}^b \quad (26)
\]

where \( A_{nm}, B_{nm}, C_{nm}, D_{nm}, A_{0m}, B_{0m}, A_{00}, B_{00} \) are unknown coefficients to be determined in the next section. The zero eigensolutions \( \psi_{00}^a, \psi_{00}^b \) represent the part of solutions that are excluded from end effects, which is similar to the Saint-Venant's solution in the elasticity problem. The non-zero eigenvalues \( \mu_{nm}, \mu_{0n} \) and corresponding eigensolutions \( \psi_{nm}^{a,\alpha}, \psi_{nm}^{a,\beta}, \psi_{nm}^{\beta,\alpha}, \psi_{nm}^{\beta,\beta} \), \( \psi_{00}^a, \psi_{00}^b \) represent decay coefficients and solutions caused by end effect respectively.

### 4. Determination of the unknown coefficients

To obtain the unknown coefficients in Eq. (26), the specific mixed boundary conditions at the end of cylinder are expressed as,
\[
\Psi|_{z=\pm b} = \begin{cases}
T_{zb} = \\
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \left( A_{nm} P_{nm}^{a,\alpha} + B_{nm} P_{nm}^{b,\beta} \right) e^{\mu_{nm}z} + \left( C_{nm} P_{nm}^{\beta,\alpha} + D_{nm} P_{nm}^{\beta,\beta} \right) e^{-\mu_{nm}z} \right] + \sum_{m=1}^{\infty} \left( A_{0m} P_{0m}^a e^{\mu_{nm}z} + B_{0m} P_{0m}^b e^{-\mu_{nm}z} \right) + A_{00} P_{00}^a + B_{00} P_{00}^b \} \\
\Omega_1 \\
\end{cases}
\]

\[
\varphi_{zb} = \begin{cases}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \left( A_{nm} Q_{nm}^{a,\alpha} + B_{nm} Q_{nm}^{b,\beta} \right) e^{\mu_{nm}z} + \left( C_{nm} Q_{nm}^{\beta,\alpha} + D_{nm} Q_{nm}^{\beta,\beta} \right) e^{-\mu_{nm}z} \right] + \sum_{m=1}^{\infty} \left( A_{0m} Q_{0m}^a e^{\mu_{nm}z} + B_{0m} Q_{0m}^b e^{-\mu_{nm}z} \right) + A_{00} Q_{00}^a + B_{00} Q_{00}^b \} \\
\Omega_2 \\
\end{cases}
\]

where \( \Omega_1 \) and \( \Omega_2 \) represent the prescribed temperature and prescribed heat flux domains, respectively. Substituting Eq. (27) into Eq. (25) yields a set of algebraic equations
\[
X \left( A^T, B^T, C^T, D^T, A_0^T, B_0^T, A_0, B_0 \right)^T = S. \quad (28)
\]

Here, \( X \) is a \((4NM + 2M + 2) \times (4NM + 2M + 2)\) constant matrix, \( A, B, C \) and \( D \) are \((NM) \times 1\) vectors. \( A_n \) and \( B_0 \) are \(M \times 1\) vectors. These vectors contain the undetermined coefficients \( A_{nm}, B_{nm}, C_{nm}, D_{nm}, A_{0m}, B_{0m}, A_{00}, B_{00} \), respectively. \( S = \{ S^A, S^B, S^C, S^D, S^a, S^b, S_a, S_b \}^T \) is a \((4NM + 2M + 2) \times 1\) vector. \( N \times M \) is the number of symplectic eigensolutions. After solving Eq. (27), the undetermined coefficients are obtained.

### 5. Numerical results and discussion

In this section, several examples are presented to demonstrate the validity of the proposed method. The thermal conductivity is taken as \( k = 1 \).
Firstly, a hollow cylinder with flux-preserved lateral BCs is taken into consideration to verify the presented method. The flux-preserved lateral BCs are taken as:

\( \partial T / \partial r \bigg|_{r=a} = 0; \quad \partial T / \partial r \bigg|_{r=b} = 0; \)

Comparisons of eigenvalues are shown in Tables 1. It is obviously that the eigenvalues of current solution agree well with the existing eigenvalues from Tarn and Wang [6].

| Table 1. Eigenvalues for hollow cylinders under flux-preserved BCs |
|---------------------------------------------------------------|
| Result of \( \mu_{0m} \) | \( m \) | 1 | 2 | 3 | 4 | 5 |
| \( a_2/a_1=0.1 \) Present | 3.94094 | 7.33057 | 10.74838 | 14.18864 | 17.64330 |
| Ref | 3.9409 | 7.3306 | 10.7484 | 14.1886 | 17.6433 |
| 0.5 Present | 6.39316 | 12.62470 | 18.88893 | 25.16241 | 31.43971 |
| Ref | 6.3931 | 12.6247 | 18.8889 | 25.1624 | 31.4397 |
| 0.9 Present | 31.42916 | 62.83848 | 94.25220 | 125.66702 | 157.08230 |
| Ref | 31.4292 | 62.8385 | 94.2522 | 125.6670 | 157.0823 |

Subsequently, a solid cylinder of radius \( a_1 = 1 \) and length \( h = 2b = 2 \) with specific mixed conditions at the end is investigated. The BCs are given as:

BC1: \(-k \partial T / \partial r \bigg|_{r=a} = 0; \quad \partial T / \partial r \bigg|_{r=b} = \text{finite}\)

BC3: \( T \bigg|_{z=b} = r \sin \theta \) for \( \Omega_1 \), BC4: \(-k \partial T / \partial z \bigg|_{z=b} = \varphi_b \) for \( \Omega_2 \)

BC5: \( T \bigg|_{z=b} = r \) for \( \Omega_1 \), BC6: \(-k \partial T / \partial z \bigg|_{z=b} = \varphi_b \) for \( \Omega_2 \)

where

\[
\varphi_a = 0.33 \sin(\theta) \left[ -0.83 + 0.64 \cos(wr) + 0.64 \sin(wr) + 0.38 \cos(2wr) \right] + 0.35 \sin(2wr) - 0.2 \cos(3wr) - 0.25 \sin(3wr) \]

\[
\varphi_b = -0.39 - 0.06 \sin(\theta) - 1.17 \cos(Wr) - 1.06 \sin(Wr) - 0.94 \cos(2Wr) - 0.04 \sin(2Wr) - 0.31 \cos(3Wr) + 0.29 \sin(3Wr) \]

In Table 2, the present results are compared with those obtained by ANSYS. It is clear that the proposed method is accurate enough for the 3D heat conduction problem involving specific mixed boundary conditions. Table 3 gives the eigenvalues and undetermined coefficients which could serve as benchmark solutions for future studies. In addition, the distributions of temperature and heat flux along the longitudinal direction is given by Fig. 2.

| Table 2. Temperature of the solid cylinder (z=0) |
|------------------------------------------------|
| Result of \( T \) | \( \theta \) | 0° | 60° | 120° | 180° | 240° | 300° |
| \( r=0.2 \) Present | 0.32099 | 0.35574 | 0.35574 | 0.32099 | 0.28622 | 0.28622 |
| FEM | 0.32063 | 0.35538 | 0.35538 | 0.32063 | 0.28587 | 0.28587 |
| 0.4 Present | 0.32550 | 0.39164 | 0.39164 | 0.32550 | 0.25933 | 0.25933 |
| FEM | 0.32520 | 0.39129 | 0.39129 | 0.32520 | 0.25902 | 0.25902 |
| 0.6 Present | 0.33097 | 0.42232 | 0.42232 | 0.33097 | 0.23956 | 0.23956 |
| FEM | 0.33071 | 0.42202 | 0.42202 | 0.33071 | 0.23925 | 0.23925 |
| 0.8 Present | 0.33525 | 0.44295 | 0.44295 | 0.33525 | 0.22749 | 0.22749 |
| FEM | 0.33505 | 0.44271 | 0.44271 | 0.33505 | 0.22717 | 0.22717 |
| 1 Present | 0.33681 | 0.45015 | 0.45015 | 0.33681 | 0.22340 | 0.33681 |
| FEM | 0.33671 | 0.44988 | 0.44988 | 0.33671 | 0.22308 | 0.33671 |
Table 3. Coefficients and eigenvalues of the solid cylinder

|     | 0  | 1      | 2      | 3      | 4      | 5      |
|-----|----|--------|--------|--------|--------|--------|
| A_{0m} | 0.3310 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| B_{0m} | -0.3313 | -0.0128 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| A_{nm} n=1 | Re | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.1140 | 0.0005 | 0.0000 | 0.0000 | 0.0000 |
|        | Re | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| B_{nm} n=1 | Re | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.1140 | -0.0005 | 0.0000 | 0.0000 | 0.0000 |
|        | Re | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| C_{nm} n=1 | Re | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.0014 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Re | -0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| D_{nm} n=1 | Re | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | -0.0014 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Re | -0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
|        | Im | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| \mu_{nm} n=0 | - | 3.8317 | 7.0156 | 10.1735 | 13.3237 | 16.4706 |
|        | n=1 | - | 1.8412 | 5.3314 | 8.5363 | 11.7060 | 14.8636 |
|        | n=2 | - | 3.0542 | 6.7061 | 9.9695 | 13.1704 | 16.3475 |

(a) Temperature  (b) Heat flux along the longitudinal direction

Figure 2. The distributions of temperature and heat flux of a solid cylinder with specific mixed boundary conditions in end surfaces.

6. Conclusion

In this paper, an analytical solution based on symplectic approach is presented for steady-state heat conduction in three-dimensional cylinders with specific mixed boundary conditions. The Hamiltonian system is established in the symplectic space and the Hamiltonian sub-system is introduced to proceed with re-separation of variables so that the eigenvalues and eigensolutions are available. The temperature and heat flux along the longitudinal direction are solved simultaneously once the coefficients of the eigensolutions are determined by using the adjoint symplectic orthogonality together with end conditions. The eigenvalues and eigensolutions have specific physical meaning; the zero eigensolutions represent the part of solutions that are excluded.
from end effects, which is similar to the Saint-Venant's solution in the elasticity problem; the non-zero eigenvalues and corresponding eigensolutions represent decay coefficients and solutions caused by end effect respectively. In addition, the Hamiltonian sub-system has the potential to be of more extensive use in solving three dimensional problems with symplectic methods.

Acknowledgments
In this research work, the supports of National Natural Science Foundation of China (11402050, 11672054); Dalian Innovation Foundation of Science and Technology (No. 2018J11CY005); the Key Program of Natural Science Foundation of Liaoning Province of China (No. 20170540186); High Level Talents Support Plan of Dalian of China (No. 2017RQ111) and the Fundamental Research Funds for the Central Universities (No. DUT17LK57) are gratefully acknowledged.

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