Efficient and fair trading algorithms in market design environments

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Abstract

We propose a new method to define trading algorithms in market design environments. Dropping the traditional idea of clearing cycles in generated graphs, we use parameterized linear equations to define trading algorithms. Our method has two advantages. First, our method avoids discussing the details of who trades with whom and how, which can be a difficult question in complex environments. Second, by controlling parameter values in our equations, our method is flexible and transparent to satisfy various fairness criteria. We apply our method to several models and obtain new trading algorithms that are efficient and fair.

Keywords: market design; trading algorithm; ordinal efficiency; fairness

JEL Classification: C71, C78, D71

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1 Introduction

The past decades have witnessed the rapid development of market design and its applications. In various market design environments, trading algorithms are used to find efficient outcomes. The fundamental idea is that, whenever an outcome is not efficient, there exists a chance for some agents to trade their assignments to enhance their welfare. So by exploiting welfare-enhancing trading chances, efficiency is obtained. Our contribution is to develop a new method to define trading algorithms, and to apply the method to several environments and obtain new algorithms that are both efficient and fair.

The most well-known trading algorithm in market design is probably the top trading cycle (TTC) algorithm proposed by Shapley and Scarf (1974) and attributed to David Gale to solve the housing market model. In the model, finite agents own distinct objects and wish to trade their objects. TTC iteratively generates directed graphs in which agents and objects, represented by nodes, respectively point to their favorite objects and owners, and then agents who form cycles trade endowments. This idea of generating graphs and then clearing cycles becomes the foundation for many other trading algorithms in the literature (e.g., Abdulkadiroğlu and Sönmez, 1999, 2003; Pápai, 2000; Roth et al., 2004; Pycia and Ünver, 2017; Dur and Ünver, 2019).

However, many market design environments are more complex than the housing market model, and in some environments the idea of generating graphs and clearing cycles cannot be straightforwardly extended. For example, if there are ties among agents (because of co-ownership, coarse priorities, or other reasons), an object may point to several agents in a generated graph, so that an agent/object may be involved in several cycles and the cycles may be non-disjoint. Moreover, when there are ties among agents, fairness concern arises (which is absent in the housing market model). Fairness often imposes symmetry requirements such as “equals being treated equally”, and it is often satisfied by randomization in market design. How to clear cycles that may be non-disjoint and incorporate randomization into the procedure to generate an efficient and fair trading algorithm is a challenge.

We illustrate the challenge by two examples in Section 2.

Our method drops the idea of clearing cycles. Facing a directed graph generated at

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1 In priority-based allocation problems, fairness is also defined as the elimination of justified envy. But if priorities are coarse, symmetry requirements also apply.

2 In particular, a procedure of randomly selecting disjoint cycles and then clearing them may look attractive, but it is actually undesirable. In the house allocation model where agents co-own all objects, this procedure is equivalent to the random priority (RP) algorithm (Abdulkadiroğlu and Sönmez, 1998), which is not ordinally efficient and not envy-free (Bogomolnaia and Moulin, 2001).
any step of a trading algorithm, we use parameterized linear equations to describe how the nodes in the graph trade with each other. The unknowns in the equations denote the amount of each node that is traded at the step, and the equations describe an input-output equilibrium in which the amount each node obtains from the others equals the amount it gives to the others. The parameters in the equations control how a node divides its demand among the other nodes it points to. By choosing different parameter values, we can produce trading algorithms that satisfy different fairness criteria. Compared with the idea of clearing cycles, our method has two advantages. First, our method does not need to identify any graphical components from generated graphs and avoids discussing the details of who trades with whom and how. Second, by controlling parameters in our equations, our method is flexible and transparent to discuss fairness.

The key to our method is that the solutions to our equations exist, and their existence has been proved by existing studies on the classical Leontief input-output model (Leontief, 1941). This ensures that in our method every trading algorithm is well-defined. The Leontief model describes a simple economy in which the input-output relationships among finite industries are characterized by a set of linear equations. When the economy reaches an equilibrium, every industry makes zero profit, and every column of the coefficient matrix of the input-output equations sums to one. In our method, our equations also describe an input-output equilibrium, because every column of our coefficient matrix also sums to one. The solutions to our equations are not unique, but because every node is constrained by a maximum trading quota, we choose the maximum solution subject to such constraints, meaning that at each step nodes can trade as much as possible.

We define our method formally in Section 3 by explaining how it solves an abstract directed graph, which may be generated at any step of a trading algorithm. It is interesting that, although our method does not need to identify any graphical components, we explain that it actually admits a graphical interpretation: every trading algorithm is a procedure of trading absorbing sets. Absorbing sets are disjoint components that exist in any generated graph. The nodes in each absorbing set trade with themselves, whereas the nodes not belonging to any absorbing set do not trade with any others. In simple graphs such as those generated in the housing market model, absorbing sets coincide with cycles, but in general, absorbing sets may be much more complex than cycles.

We apply our method to several market design models and obtain new trading algorithms. These models are defined in Section 4.

1. (Section 5) In the fractional endowment exchange (FEE) model, we obtain a class of
balanced trading algorithms (BTA), which are individually rational, ordinally efficient and satisfy various fairness criteria. FEE is an extension of the housing market model by allowing agents to own fractional endowments. Its rich endowment structure admits complicated co-ownership and thus can exhibit the power of our method. It also has interesting applications, which we will explain in the next point.

The house allocation model is well studied in the literature. By regarding objects as equally owned by agents, we interpret the model as a special case of FEE. Then it turns out that in the house allocation model, every BTA coincides with a simultaneous eating algorithm introduced by Bogomolnaia and Moulin (2001), and every BTA that treats agents equally coincides with the probabilistic serial (PS) algorithm. This means that PS can be regarded as a trading algorithm.\footnote{This observation is first made by Kesten (2009), but his method is more complicated than ours. In his algorithm, Kesten needs to carefully select cycles to clear. This makes his algorithm equivalent to PS in outcomes, but they do not coincide step by step.} It is not surprising that in the housing market model, every BTA reduces to TTC.

2. (Section 6) We provide two applications of the FEE model and BTA. In the first application, we use FEE to model time exchange markets where agents exchange time and skills through serving each other. An agent may provide several types of services to the others and may demand several types of services from the others. BTA can be straightforwardly used to find efficient and fair assignments in such markets. In the second application, we develop an efficient and fair method to deal with ceiling and floor distributional constraints in the hospital-doctor match problem studied by Kamada and Kojima (2015). By assuming that hospitals prefer to hire more doctors subject to constraints, we first generate a feasible assignment in which doctors have equal access to all hospitals and the numbers of doctors assigned to hospitals are not Pareto improvable. We then treat this assignment as doctors’ endowments and run a BTA that treats doctors equally. The outcome is an efficient assignment that satisfies all constraints and ensures no envy among doctors.

3. (Section 7) In the priority-based allocation model, when priorities may be coarse, we obtain the priority trading algorithm (PTA). Abdulkadiroğlu and Sönmez (2003) extend TTC to the model by assuming that priorities are strict, and apply the algorithm to school choice. However, real-life schools widely use coarse priorities. PTA generalizes Abdulkadiroğlu and Sönmez’s TTC by treating agents in priority ties equally without exogenously (and randomly) breaking ties. The outcome of PTA is
an ordinally efficient random assignment.

4. (Section 8) In the house allocation with existing tenants model, we obtain the eating-trading algorithm (ETA), which is a hybrid and extension of PS and TTC. At any time in the algorithm, if a group of existing tenants demand each other’s endowments so that they form a cycle, they trade (fractional) endowments instantly; otherwise, agents consume favorite objects with rates satisfying a rule called “you request my house - I get your rate”.

5. (Section 9) In the school choice with coarse priorities model, we solve a difficulty in Kesten and Ünver (2015) and obtain a different algorithm than theirs to find ex-ante stable assignments. Stability (or no justified envy) is a desirable fairness notion in school choice. When priorities are coarse and lotteries are used, Kesten and Ünver define ex-ante stability, and propose a two-stage mechanism to find efficient ex-ante stable assignments that treat equals equally. The second stage is a trading algorithm that is used to improve the efficiency of the first-stage algorithm. Because there are complicated ties among agents in the outcome of the first stage, how to treat equals equally in the second stage is a difficult question. Kesten and Ünver use an operations research method to solve it. Our method can easily solves the difficulty.

We emphasize that the application of our method is not limited to the above models. We use the above models to illustrate that our method is useful in various environments, no matter endowments/priorities are exogenously given or endogenously generated, or trading algorithms are only used in certain stages of multi-stage mechanisms. Before concluding the paper in Section 11, we discuss related literature in Section 10. Appendix includes additional results and all proofs.

2 Two examples

We present two examples of the FEE model defined in Section 4. We use the examples to illustrate the difficulty of using the traditional idea of clearing cycles to define efficient and fair trading algorithms, and the advantage of our method.

2.1 Example 1

In Example 1, there are five agents \{1, 2, 3, 4, 5\} and five objects \{a, b, c, d, e\}. Agents have fractional endowments and strict preferences shown as follows.
In a fair trading algorithm, if two agents contribute equal resources to an economy, it is reasonable to expect that any differences in their assignments are explained solely by their different preferences. Therefore, each of them should be satisfied by his own assignment compared with the other’s, so that they do not envy each other. This fairness criterion is called equal-endowment no envy (EENE) and formally defined in Section 5.1. In Example 1, agents 1 and 2, and agents 3 and 4 respectively own equal endowments.

Suppose we imitate TTC to define a trading algorithm in which, at every step, we generate a directed graph by letting remaining agents point to favorite objects and remaining objects point to all of remaining owners. Then at the first step we will generate Figure 1. The weight of every edge $o \rightarrow i$ is the amount of object $o$ owned by agent $i$.

There are five cycles in this graph:

- cycle 1: $3 \rightarrow d \rightarrow 3$
- cycle 2: $5 \rightarrow c \rightarrow 5$
- cycle 3: $1 \rightarrow c \rightarrow 4 \rightarrow a \rightarrow 1$
- cycle 4: $2 \rightarrow d \rightarrow 4 \rightarrow a \rightarrow 2$
- cycle 5: $1 \rightarrow c \rightarrow 3 \rightarrow d \rightarrow 4 \rightarrow a \rightarrow 1
These cycles are non-disjoint. Cycle 3 and cycle 4 share the edge 4 → a, and cycle 1 and cycle 3 are nested by cycle 5.\footnote{A cycle nests another cycle if all nodes in the second cycle are involved in the first cycle.}

If we want to use the idea of clearing cycles to define a trading process in Figure 1, we need to answer (1) whether all cycles in the graph are cleared, or only a subset of cycles are cleared, and (2) what amount of objects is traded in every (selected) cycle. Below we discuss two possible answers to these questions, which may look attractive at first glance, but turn out to be undesirable. We then explain how our method solves the difficulty.

### 2.1.1 Clearing all cycles equally

Our first idea is to treat the cycles generated at every step equally by trading an equal amount of objects in all of the cycles. At every step, for every edge \( o \rightarrow i \) in the generated graph, we count the number of cycles that involve the edge and denote this number by \( n_{o\rightarrow i} \). Denote the weight of every edge \( o \rightarrow i \) by \( \omega_{i,o} \), then at most an amount \( \omega_{i,o} n_{o\rightarrow i} \) of objects can be traded in each of the \( n_{o\rightarrow i} \) cycles.\footnote{We do consider agents’ demands because in the FEE model agents’ endowments are always no more than their demands.} So at every step, we will trade an amount

\[
\min_{o \rightarrow i; n_{o \rightarrow i} > 0} \frac{\omega_{i,o}}{n_{o \rightarrow i}}.
\]

of objects in every cycle. The resulting trading algorithm looks intuitively fair. For example, if two agents are “equal” (in terms of endowments and preferences), they will be treated equally at every step. On the other hand, if an agent’s endowments are demanded by more of the others than another agent’s endowments so that the former agent is involved in more cycles, the former agent will get a better outcome in the algorithm.

|   | a  | b  | c  | d  | e  |
|---|----|----|----|----|----|
| 1 | 1/2 | 1/2 |    |    |    |
| 2 | 1/4 | 1/2 | 1/4 |    |    |
| 3 |    | 3/4 | 1/4 |    |    |
| 4 |    |    | 1/4 |    |    |
| 5 |    | 1/2 |    | 1/2 |    |

Table 2: Assignment found by the first idea

However, if we apply this algorithm to Example 1, we will see that the found assignment violates EENE. Specifically, in Figure 1, we will trade an amount 1/4 of objects in every cycle. So 1 will obtain 1/2c, 2 will obtain 1/4d, 3 will obtain 1/2d, 4 will obtain...
3/4a, and 5 will obtain 1/4c. The following steps are straightforward because the generated graphs will be simple. At step two, 3 and 5 each will obtain 1/4c. At step three, 2 will obtain 1/4a and 5 will obtain 1/2e. At step four, 1 and 2 each will obtain 1/2b, and 3 and 4 each will obtain 1/4e. So the algorithm will find the assignment in Table 2. This assignment is ordinally efficient, but 2 does not regard his outcome as better than 1’s.

2.1.2 Clearing non-redundant cycles

Given the failure of the first idea, our second idea is to select a subset of cycles to clear. We observe that when a long cycle nests several short cycles, the long cycle can be regarded as redundant because, by clearing the short cycles, every agent in the long cycle can obtain the object he points to. So our second idea is to select non-redundant cycles to clear. We call a cycle redundant if it nests several shorter cycles and every node in the cycle is involved in one of the shorter cycles. Otherwise it is non-redundant.

|   | a    | b    | c    | d    | e    |
|---|------|------|------|------|------|
| 1 | 1/4  | 1/2  | 1/4  |      |      |
| 2 | 1/2  |      | 1/2  |      |      |
| 3 |      | 1/4  | 1/2  | 1/4  |      |
| 4 | 3/4  |      |      | 1/4  |      |
| 5 |      | 1/2  |      | 1/2  |      |

Table 3: Assignment found by the second idea

To obtain an algorithm, we need to specify the amount of objects traded in every non-redundant cycle. In every cycle, the amount of objects that can be traded is no more than the smallest weight of the edges in the cycle. If several cycles share an edge from an object to an agent and the weight of the edge is not enough to satisfy all of the cycles, we need to decide how to divide the weight of the edge among the several cycles. In Figure 1, cycle 5 is redundant in the presence of cycle 1 and cycle 3. Cycle 3 and cycle 4 share an edge 4 → a, but the edge is from an agent to an object. So we will clear the four non-redundant cycles, and the amount of objects traded in the four cycles will be respectively 1/2, 1/2, 1/4, and 1/2. After trading the four cycles, 1 obtains 1/4c, 2 obtains 1/2d, 3 obtains 1/2d, 4 obtains 3/4a, and 5 obtains 1/2c. The following steps are straightforward because all cycles in generated graphs will be of the form i → o → i. At step two, 3 will obtain 1/4c. At step three, 1 will obtain 1/4a and 5 will obtain 1/2e. At step four, 1 and 2 each will

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6The idea of clearing non-redundant cycles is first used by Kesten (2009) to define his probabilistic variant of TTC in the house allocation model.
obtain \(1/2b\), and 3 and 4 each will obtain \(1/4e\). So the algorithm will find the assignment in Table 3. This assignment is ordinally efficient, but it violates EENE. Agent 1 does not regard his outcome as better than agent 2’s.

### 2.1.3 Our method

The idea of clearing cycles attempts to give the full details of the trading process at every step, that is, who trade with whom and how. As we have seen, for complicated graphs it is hard to give such details. Our method avoids such details by directly characterizing the outcome of the trading process at every step by a set of linear equations. For Figure 1, we will write the following equations:

\[
\begin{align*}
&x_a = x_4, \\
&x_b = 0, \\
&x_c = x_1 + x_5, \\
&x_d = x_2 + x_3, \\
&x_e = 0, \\
&x_1 = 1/2x_a + 1/2x_b, \\
&x_2 = 1/2x_a + 1/2x_b \\
&x_3 = 1/3x_c + 1/2x_d + 1/3x_e, \\
&x_4 = 1/3x_c + 1/2x_d + 1/3x_e, \\
&x_5 = 1/3x_c + 1/3x_e.
\end{align*}
\]

In the equations, for every agent \(i\), the unknown \(x_i\) denotes the amount of the favorite object obtained by \(i\), and for every object \(o\), the unknown \(x_o\) denotes the amount of \(o\) that is traded. The first five equations are obtained by definitions: every \(x_o\) equals the sum of \(x_i\) where \(i\) most prefers \(o\). The last five equations mean that to obtain \(x_i\) of his favorite object, every \(i\) needs to lose an equal amount of endowments. When an object \(o\) is owned by several agents, for fairness concern we require that every owner of \(o\) lose an equal amount of \(o\) in the trading process. Therefore, because \(a\) and \(b\) are owned by 1 and 2, each of the two agents will lose \(1/2x_a\) of \(a\) and \(1/2x_b\) of \(b\) in the trading process in Figure 1. This explains the equations for \(x_1\) and \(x_2\). The other equations are similar.

At every step, agents cannot trade more objects than their endowments. So for Figure
1, we have the following constraints:

\[
\frac{1}{2}x_a \leq \frac{1}{2}, \quad \frac{1}{2}x_b \leq \frac{1}{2}, \quad \frac{1}{3}x_c \leq \frac{1}{4}, \quad \frac{1}{2}x_d \leq \frac{1}{2}, \quad \frac{1}{3}x_e \leq \frac{1}{4}.
\]

By solving equations (1) subject to the above constraints, we obtain the following maximum solution

\[
x^* = \begin{pmatrix} x_1^* & x_2^* & x_3^* & x_4^* & x_5^* & x_a^* & x_b^* & x_c^* & x_d^* & x_e^* \\ 1/3 & 1/3 & 2/3 & 2/3 & 1/6 & 2/3 & 0 & 1/2 & 1 & 0 \end{pmatrix}.
\]

It means that in the trading process in Figure 1, 1 obtains 1/3c and loses 1/3a, 2 obtains 1/3d and loses 1/3a, 3 obtains 2/3d and loses 1/6c and 1/2d, 4 obtains 2/3a and loses 1/6c and 1/2d, and 5 obtains 1/6c and loses 1/6c. So d is exhausted, and b and e are not involved in the trading process. After the first step, we can similarly write equations and constraints for following steps. Our method will find the assignment in Table 4, which is ordinally efficient and satisfies EENE. It actually satisfies a stronger fairness criterion defined in Section 5.1.

|   | a   | b   | c   | d   | e   |
|---|-----|-----|-----|-----|-----|
| 1 | 1/8 | 1/2 | 3/8 |     |     |
| 2 | 1/8 | 1/2 | 1/24| 1/3 |     |
| 3 |     | 1/12| 2/3 | 1/4 |     |
| 4 | 3/4 |     | 1/4 |     |     |
| 5 |     | 1/2 | 1/2 |     |     |

Table 4: Assignment found by our method

2.2 Example 2

Example 2 shows that the idea of clearing cycles can be misleading even when the generated graph is simple. In Example 2, there are five agents \{1, 2, 3, 4, 5\} and three objects \{a, b, c\}. Endowments and preferences are shown below. Agents 1 and 5 respectively most prefer their endowments \(a\) and \(1/2c\). So it is intuitive to let them obtain their endowments, and then remove them. Among the remaining agents, 2 and 3 own equal amounts of \(b\), and both most prefer 4’s endowment \(c\). An intuitively fair assignment is to let 2 and 3 each give \(1/4b\) to 4 in exchange for \(1/4c\).

Now suppose we draw Figure 2 and use the idea of clearing cycles. Because the three cycles in the graph are disjoint, there is no difficulty to clear all of them. After clearing
the cycles, 1 will obtain his endowment $a$, 5 will obtain his endowment $1/2c$, and 3 and 4 will exchange endowments so that 3 will obtain $1/2c$ and 4 will obtain $1/2b$. After that, 2 will remain with his endowment $1/2b$. This assignment violates EENE, because although 2 and 3 own equal endowments and essentially have equal preferences (given that 1 will obtain his endowment $a$ anyway), they are treated differently. Agents 3 and 4 exchange endowments before 2 has the chance to reveal his preference over $c$.

In our method, for Figure 2 we will solve the following equations

\begin{align*}
\begin{cases}
x_a &= x_1 + x_2, \\
x_b &= x_4, \\
x_c &= x_3 + x_5, \\
x_1 &= x_a, \\
x_2 &= 1/2x_b, \\
x_3 &= 1/2x_b, \\
x_4 &= 1/2x_c, \\
x_5 &= 1/2x_c.
\end{cases}
\end{align*}

(2)
subject to the constraints

\[ x_a \leq 1, \quad 1/2 x_b \leq 1/2, \quad 1/2 x_c \leq 1/2. \]

We will obtain the following maximum solution

\[
x^* = \begin{pmatrix}
  x_1^* & x_2^* & x_3^* & x_4^* & x_5^* & x_a^* & x_b^* & x_c^* \\
  1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

So only agent 1 will obtain his endowment \( a \) at the first step. After they are removed, agent 2 will point to \( c \). Then we can write an equation system and get a solution in which 2 and 3 each will give \( 1/4 b \) to 4 in exchange for \( 1/4 c \), and 5 will obtain his endowment \( 1/2 c \). This is the intuitively fair assignment discussed at the beginning of the example.

### 3 Formal definition of our method

Our method is applicable to trading algorithms in which a directed graph is generated at every step, and the nodes in the graph trade with each other so that every node gives up an amount of something from itself in exchange for an equal amount of something from the others it points to. Our method does not explain how the directed graph is generated. In different environments the graph generation rule can be different.

To define our method in a general manner, let \( G = (\mathcal{V}, \mathcal{E}) \) denote a directed graph that is generated at any step of a trading algorithm, where \( \mathcal{V} \) is a set of nodes and \( \mathcal{E} \subset \mathcal{V} \times \mathcal{V} \) is a set of directed edges. Every edge \((u, v) \in \mathcal{V}\) means that node \( u \) points to node \( v \) (or that \( u \) demands \( v \)). Every node points to some other nodes, and no node points to itself. For every \( v \in \mathcal{V} \), define \( \mathcal{V} \to v = \{ u \in \mathcal{V} : (u, v) \in \mathcal{E} \} \) to be the set of nodes pointing to \( v \), and define \( \mathcal{V} \leftarrow v = \{ u \in \mathcal{V} : (v, u) \in \mathcal{E} \} \) to be the set of nodes pointed by \( v \). Every node \( u \) is available in a quota \( q_u \geq 0 \). Nodes trade with the others subject to their quotas. For every node \( u \), we use \( x_u \) to denote the amount of something that \( u \) obtains from the others in the trading process at the step. Because \( u \) may point to multiple nodes, we introduce parameters \( \{ \lambda_{v,u} \}_{v \in \mathcal{V} \to u} \) to denote how \( u \) divides its demand among the nodes it points to. For every \( v \in \mathcal{V} \leftarrow u \), the parameter \( \lambda_{v,u} \) denotes the proportion of \( u \)'s demand that is satisfied by consuming \( v \) at the step. In other words, \( u \) obtains an amount \( \lambda_{v,u} x_u \) of something from \( v \). So the parameters satisfy that every \( \lambda_{v,u} \in [0,1] \) and \( \sum_{v \in \mathcal{V} \rightarrow u} \lambda_{v,u} = 1 \). On the other hand, for every node \( v \), the total amount of something it gives to the others
is $\sum_{u \in \mathcal{V} \to v} \lambda_{v,u} x_u$, which shall equal the amount $x_v$ it obtains from the others. Therefore,

\[(3) \quad x_v = \sum_{u \in \mathcal{V} \to v} \lambda_{v,u} x_u \quad \text{for all } v \in \mathcal{V}.
\]

If a node $v$ is not pointed by any other nodes (i.e., $\mathcal{V} \to v = \emptyset$), then $x_v = 0$, because $v$ does not trade anything with the others.

Equations (3) are the equations we use to characterize the outcome of the trading process at any step of a trading algorithm. The parameters $\lambda_{v,u}$ are chosen by mechanism designers and $x_v$ are treated as unknowns. We will solve the equations subject to the contraints that $x_v \leq q_v$ for all $v \in \mathcal{V}$. By choosing the maximum solution that satisfies the constraints, we let nodes trade as much as possible at every step. For convenience, we write equations (3) in a matrix form. Let $x = (x_v)_{v \in \mathcal{V}}$ and $q = (q_v)_{v \in \mathcal{V}}$. Define a parameter matrix $\Lambda = (\lambda_{v,u})_{v,u \in \mathcal{V}}$ such that, for all $v, u \in \mathcal{V}$, $\lambda_{v,u} \in [0, 1]$, $\lambda_{v,u} = 0$ if $v \notin \mathcal{V}_u \to$, and $\sum_{v \in \mathcal{V}} \lambda_{v,u} = 1$. Then we will solve

\[(4) \quad \Lambda x = x, \quad \text{subject to } x \leq q.
\]

We say that a solution $x^*$ to (4) is the **maximum solution** if $x^* \leq q$, and for every other solution $x^\diamond$ that satisfies $x^\diamond \leq q$, $x^*_v \geq x^\diamond_v$ for all $v \in \mathcal{V}$ and $x^*_v > x^\diamond_v$ for some $v \in \mathcal{V}$.

Then, a trading algorithm in our method will have the following procedure.

**A trading algorithm in our method**

**Step** $d \geq 1$: Generate a directed graph $G(d) = (\mathcal{V}(d), \mathcal{E}(d))$. Choose a parameter matrix $\Lambda(d)$ and then solve the maximum solution $x^*$ to

\[
\Lambda(d)x = x, \quad \text{subject to } x \leq q(d).
\]

Let nodes trade according to $x^*$. Update the quotas of nodes and go to the next step. Stop when there is no more trading.

Different from trading algorithms in the literature that explicitly identify and clear cycles in the generated graphs, our method does not need to identify any graphical component. We even do not need to explicitly draw graphs, as long as the set of nodes that each node demands is well-defined. The key to our method is that, the maximum solution to the equation system (4) exists and is nonnegative.
Theorem 1. The maximum solution $x^*$ to the equation system (4) exists and is nonnegative.

The coefficient matrix $\Lambda$ of $\Lambda x = x$ does not have full rank, but all of its elements are in $[0,1]$ and its every column sums to one. In matrix theory, $\Lambda$ is called a stochastic matrix. This reminds us of the closed Leontief input-output model (Leontief, 1941), which is a classical model that provides a simple general equilibrium view of an economy. In some sense, our equation system $\Lambda x = x$ describes a kind of trading equilibrium between nodes at every step. The existing studies (Peterson and Olinick, 1982; Leon, 2015) on the Leontief model imply the existence of solutions to $\Lambda x = x$. The solutions are not unique, but the constraints $x \leq q$ pin down the unique maximum solution. In the maximum solution, some constraint $x_v \leq q_v$ will be binding. We present a self-contained proof of Theorem 1 in Appendix A.

As for computation, we note that $\Lambda x = x$ is a linear equation system. It can be easily solved by well-developed computation methods and softwares. In simple examples it can be solved by hand. On the other hand, the idea of identifying and clearing cycles needs not to be easier than solving linear equations, especially when the generated graphs are complicated.

Another advantage of our method is its flexibility. By choosing different parameters $\Lambda$, we can obtain different trading algorithms. The choice of $\Lambda$ is related to fairness. For example, we may choose $\lambda_{v,u} = \frac{1}{|V_u|}$ for all $v \in V_u$, meaning that every node equally divides its demand among the nodes it points to. In our applications we will discuss how to choose $\Lambda$ to obtain fair trading algorithms.

3.1 A graphical explanation of our method

Although trading algorithms in our method are defined by iteratively solving linear equations, they actually admit a graphical explanation that generalizes the usual descrip-

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7 In the Leontief model, a number of industries produce distinct products, and every industry requires input of the products from the other industries and possibly also of its own. Their input-output relationships can be described by a set of linear equations. Specifically, suppose there are $n$ industries. Let $a_{uv} \in [0,1]$ denote the amount of input from the $u$-th industry that necessary to produce one unit of output in the $v$-th industry. Here a unit means one dollar’s worth. Let $x_u$ denote the units of output produced by the $u$-th industry. So $x = (x_1, x_2, \ldots, x_n)$ satisfies the set of linear equations:

$$x_u = a_{u1}x_1 + a_{u2}x_2 + \cdots + a_{un}x_n \quad \text{for all } u = 1, 2, \ldots, n.$$ 

That is, all output of the $u$-th industry becomes input into the other industries and possibly itself. In an equilibrium, the cost of producing one unit of output in every $v$-th industry is no greater than one dollar. That is, $\sum_{u=1}^n a_{uv} \leq 1$. By definition, $\sum_{v=1}^n a_{uv}x_v = x_u$. Thus, $\sum_{u=1}^n \sum_{v=1}^n a_{uv}x_v = \sum_{u=1}^n x_u$. Rearranging terms, we have $\sum_{v=1}^n (\sum_{u=1}^n a_{uv})x_v = \sum_{u=1}^n x_u$. So $\sum_{u=1}^n a_{uv} = 1$. 

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tion of TTC. We show that every trading algorithm is an iterative procedure of clearing **absorbing sets.** In a generated graph, absorbing sets are disjoint components such that the nodes in each absorbing set trade only with themselves, while the nodes not in any absorbing set are not involved in any trading. In this sense, absorbing sets are generalizations of cycles in the procedure of TTC.

Formally, in a directed graph \((V, E)\), given a parameter matrix \(\Lambda\), we say that there is a **directed path** from a node \(v\) to another node \(u\) if there exists a sequence of nodes \(v_1, v_2, \ldots, v_z\) such that \(v_1 = v, v_z = u\), and for every \(\ell \in \{1, \ldots, z-1\}\), \(\lambda_{v_{\ell+1}, v_\ell} > 0\).

**Definition 1.** In a directed graph \((V, E)\), a subset \(V \subset V\) is an **absorbing set** if

1. (No outgoing edge) There is no directed path from any node in \(V\) to any node not in \(V\);
2. (Inside connected) Within \(V\) there is a directed path from every node to every other node.

In Figure 1 of Section 2, given the parameters chosen in equations (1), there is only one absorbing set and it consists of nodes other than \(b\) and \(e\). In Figure 2 of Section 2, given the parameters chosen in equations (2), there is only one absorbing set and it consists of \(1\) and \(a\). Note that \(1\) and \(a\) form a cycle in Figure 2. The other cycles in the figure are not absorbing sets because they have outgoing edges.

In the procedure of TTC, when we identify a cycle, the cycle directly tells us how the agents in the cycle should trade endowments. An absorbing set can be more complicated than a cycle (e.g., Figure 1). Identifying an absorbing set does not tell us how the nodes it includes should trade with themselves. We need to solve \(\Lambda x = x\) to get the answer. Actually, from the proof of Theorem 1 we can see that the maximum solution \(x^*\) to \(\Lambda x = x\) is obtained by solving the equation system restricted to each absorbing set. Specifically, suppose \(V_1, \ldots, V_K\) are the absorbing sets in a generated graph and \(U\) is the set of nodes not in any absorbing set. For each absorbing set \(V\), denote the maximum solution to \(\Lambda_V x = x\) by \(x^*_V\). Then the maximum solution \(x^*\) to \(\Lambda x = x\) is

\[
x^* = (x^*_V, \ldots, x^*_K, 0_U).
\]

The existence and disjointness of absorbing sets in any directed graph and that the nodes in every absorbing set trade with themselves are implied by the proof of Theorem 1 (see Lemma 3 in Appendix A). These properties can also be obtained directly from Definition 1.

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8 Recall that for any two nodes \(u\) and \(v\), \(\lambda_{v,u} > 0\) only if \(u\) points to \(v\), but we allow for that \(u\) points to \(v\) but \(\lambda_{v,u} = 0\).
• Existence: First, the whole graph has no outgoing edge. Second, if a subset of nodes has no outgoing edge but is not inside connected, it must include a strict subset that has no outgoing edge.\footnote{If $V \subset V$ has no outgoing edge but is not inside connected, then there exist $v, v' \in V$ such that there is no directed path from $v$ to $v'$. Let $V_1$ be the set of nodes in $V$ that can be reached from $v$ through directed paths. Then $v' \notin V_1$, and there is no directed path from any node in $V_1$ to any node in $V \setminus V_1$. Because there is neither directed path from any node in $V_1$ to any node outside of $V$, $V_1$ has no outgoing edge.} Last, every singleton set is inside connected. These facts imply the existence of absorbing sets.

• Disjointness: Suppose two absorbing sets share a node. Then from the shared node we can find a directed path to reach any other node in each of the two absorbing sets, meaning that no outgoing edge is violated.

• Inside trading: In any absorbing set, because there is no outgoing edge, the nodes in the absorbing set obtain nothing from outside nodes. Then outside nodes must obtain nothing from the absorbing set, because otherwise some node in the absorbing set must lose more than what he obtains in the trading process.

4 Market design environments

We apply our method to several market design models. In all of these models, a finite set of objects $O$ are assigned to a finite set of agents $I$. Each $o \in O$ has $q_o \in \mathbb{N}$ copies, and each $i \in I$ demands a copy of an object, and has a strict preference relation $\succ_i$ over objects. These models differ in ownership or priority structures.

• In the **housing market** model, there are equal numbers of agents and objects, every object has one copy, and every agent initially owns a distinct object.

• In the **house allocation** model, there are equal numbers of agents and objects, every object has one copy, and objects are collectively owned by all agents.\footnote{Following Hylland and Zeckhauser (1979) and Abdulkadiroğlu and Sönmez (1998) (and others), we present the simplest setup of the house allocation model. But there is no difficulty for our results if the numbers of agents and objects are unequal and objects have multiple copies.}

• In the **house allocation with existing tenants** (HET) model, every object has one copy, and there exists a subset of objects $O_E$ in which every $o$ is privately owned by a distinct agent. The other objects $O \setminus O_E$ are collectively owned by all agents. This model reduces to the house allocation model if $O_E = \emptyset$, and reduces to the housing market model if $O_E = O$.  

\[9\]
• In the fractional endowments exchange (FEE) model, every agent \( i \) owns an amount (probability share) \( \omega_{i,o} \in [0,1] \) of every object \( o \). For every agent \( i \), \( \sum_{o \in O} \omega_{i,o} \leq 1 \), and for every object \( o \), \( q_o = \sum_{i \in I} \omega_{i,o} \). The vector \( \omega_i = (\omega_{i,o})_{o \in O} \) is called \( i \)'s endowments, and \( \omega = (\omega_{i,o})_{i \in I, o \in O} \) is called the endowment matrix.

• In the priority-based allocation (PBA) model, each object \( o \) ranks agents by a weak order \( \succeq_o \), which is called a priority ranking. Let \( \sim_o \) denote the symmetric component of \( \succeq_o \). Two agents \( i, j \) are said to be in priority ties for \( o \) if \( i \sim_o j \). Agents in priority ties for an object are understood to have equal rights over the object.

All of the five models have been studied in the literature. We will obtain new trading algorithms in the last three models and discuss their relations with existing algorithms in the first two models. The FEE model is a direct extension of the housing market model, and in this paper we also regard the house allocation model as a special case of the FEE model by letting every agent own an equal division of every object.

In these models we will discuss how to assign probability shares of objects to agents. An assignment is represented by a matrix \( p = (p_{i,o})_{i \in I, o \in O} \in \mathbb{R}_{+}^{|I| \times |O|} \) where \( p_{i,o} \) denotes the amount/probability share of \( o \) assigned to \( i \). It satisfies that, for all \( o \in O \), \( \sum_{i \in I} p_{i,o} \leq q_o \), and for all \( i \in I \), \( \sum_{o \in O} p_{i,o} \leq 1 \). The Birkhoff-von Neumann theorem and its generalization (Birkhoff, 1946; Von Neumann, 1953; Kojima and Manea, 2010) ensure that every assignment is a randomization over deterministic assignments in which every \( p_{i,o} \in \{0,1\} \).

A vector \( l \in \mathbb{R}_{+}^{|O|} \) is a lottery if \( \sum_{o \in O} l_o \leq 1 \). Given a preference profile \( \succeq_I = \{\succeq_i\}_{i \in I} \), a lottery \( l \) (first-order) stochastically dominates another lottery \( l' \) for an agent \( i \), denoted by \( l \succeq^1_i l' \), if \( \sum_{o' \succeq_i o} l'_{o'} \geq \sum_{o' \succeq_i o} l_{o'} \) for all \( o \). If the inequality is strict for some \( o \), then \( l \) strictly stochastically dominates \( l' \) for \( i \), denoted by \( l >^1_i l' \). An assignment \( p \) strictly stochastically dominates another assignment \( p' \), denoted by \( p >^1 p' \), if \( p_i >^1 p'_i \) for all \( i \) and \( p_j >^1 p'_j \) for some \( j \). An assignment \( p \) is ordinally efficient if it is not strictly stochastically dominated.

5 The fractional endowment exchange model

Applying our method to the FEE model, we obtain a class of algorithms we call balanced trading algorithms (BTA). The outcome of BTA is an ordinally efficient assignment, and it satisfies various fairness criteria when the parameters in the algorithm are properly chosen. To formally define BTA, we first define some notations.

At the beginning of any step \( d \geq 1 \):
• $I(d)$ is the set of remaining agents;

• $O(d)$ is the set of remaining objects;

• for every $i \in I(d)$, $o_i(d)$ is $i$’s favorite object among $O(d)$;

• $\omega(d) = (\omega_{i,o}(d))_{i \in I, o \in O}$ represents agents’ remaining endowments;

• $p(d) = (p_{i,o}(d))_{i \in I, o \in O}$ represents the temporarily found assignment.

After the trading process at any step $d \geq 1$,

• for every $i \in I(d)$, $x_i(d)$ is the amount of $o_i(d)$ that $i$ obtains at step $d$;

• for every $o \in O(d)$, $x_o(d)$ is the amount of $o$ that is traded at step $d$.

By these definitions, for every $o \in O(d)$,

$$x_o(d) = \sum_{i \in I(d)} 1\{o_i(d) = o\} \cdot x_i(d).$$

Note that if an object $o$ is not demanded by any agents at step $d$, then $x_o(d) = 0$.

For every $o \in O(d)$, $x_o(d)$ is also the total amount of $o$ that its owners give up at step $d$. For every $i \in I(d)$ and $o \in O(d)$, we use a parameter $\lambda_{i,o}(d)$ to denote the fraction of $x_o(d)$ that is given up by $i$ from his endowments. So $\lambda_{i,o}(d) \in [0, 1]$, $\sum_{i \in I(d)} \lambda_{i,o}(d) = 1$, and $\lambda_{i,o}(d) = 0$ if $\omega_{i,o}(d) = 0$. Then the amount of endowments that every $i \in I(d)$ gives up at step $d$ is $\sum_{o \in O(d)} \lambda_{i,o}(d)x_o(d)$. Because agents trade endowments by giving up an amount in exchange for an equal amount, we obtain the equations that, for every $i \in I(d)$,

$$x_i(d) = \sum_{o \in O(d)} \lambda_{i,o}(d)x_o(d).$$

Because agents can never use more objects than their endowments to trade with the others, we have the constraints that, for every $i \in I(d)$ and $o \in O(d)$, $\lambda_{i,o}(d)x_o(d) \leq \omega_{i,o}(d)$.

**Balanced Trade Algorithm**

**Initialization:** $I(1) = I$, $O(1) = O$, $\omega(1) = \omega$, and $p(1) = 0$.

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11 By convention, $1\{o_i(d) = o\} = 1$ if $o_i(d) = o$ and otherwise $1\{o_i(d) = o\} = 0$. 

Step $d \geq 1$: Each $i \in I(d)$ reports his favorite remaining object $o_i(d)$. After choosing parameters $\Lambda(d) = (\lambda_{i,o}(d))_{i \in I(d), o \in O(d)}$, let $x^*(d) = (x^*_a(d))_{a \in I(d) \cup O(d)}$ denote the maximum solution to the equation system

\[
\begin{align*}
    x_{o}(d) &= \sum_{i \in I(d)} 1[o_i(d) = o] \cdot x_i(d) \quad \text{for all } o \in O(d), \\
    x_i(d) &= \sum_{o \in O(d)} \lambda_{i,o}(d)x_o(d) \quad \text{for all } i \in I(d),
\end{align*}
\]

subject to the constraints

\[
\lambda_{i,o}(d)x_o(d) \leq \omega_{i,o}(d) \quad \text{for all } i \in I(d) \text{ and } o \in O(d).
\]

For all $i \in I$ and all $o \in O$, let $p_{i,o}(d + 1) = p_{i,o}(d) + x^*_i(d)$ if $i \in I(d)$ and $o = o_i(d)$, and otherwise let $p_{i,o}(d + 1) = p_{i,o}(d)$; let $\omega_{i,o}(d + 1) = \omega_{i,o}(d) - \lambda_{i,o}(d)x_o(d)$ if $i \in I(d)$ and $o \in O(d)$, and otherwise let $\omega_{i,o}(d + 1) = \omega_{i,o}(d)$. Let $I(d + 1) = \{i \in I : \sum_{o \in O} \omega_{i,o}(d + 1) > 0\}$ and $O(d + 1) = \{o \in O : \sum_{i \in I(d + 1)} \omega_{i,o}(d + 1) > 0\}$. If $O(d + 1)$ is empty, stop; otherwise go to step $d + 1$.

The equation system (5) and the constraints (6) are special cases of those in Section 3. So the maximum solution $x^*(d)$ at every step exists. In the maximum solution, some constraint $\lambda_{i,o}(d)x_o(d) \leq \omega_{i,o}(d)$ will be binding. It means that at every step some agent will use up his endowment of some object. So BTA must stop in at most $|I| \times |O|$ steps.

Every step of BTA takes a parameter matrix $\Lambda(d)$ as an input. By choosing different $\Lambda(d)$, we obtain different algorithms. So BTA is a class of algorithms. The choice of $\Lambda(d)$ determines how the owners of every object divide the right of using the object to trade with the others. Its relation to fairness is explained in the next subsection.

5.1 Efficiency and fairness of BTA

In the FEE model, an assignment $p$ is individually rational (IR) if $p_i \succeq_i^{sd} \omega_i$ for all $i$. We say a BTA is individually rational or ordinally efficient if the assignments it finds always satisfy these properties. It is easy to prove that every BTA is individually rational and ordinally efficient. This is because at every step, the lottery of the favorite object that every agent obtains stochastically dominates the lottery of endowments that he loses.

**Proposition 1.** Every BTA is individually rational and ordinally efficient.
In market design, fairness often imposes symmetry requirements such as “equals being treated equally”, or elimination of justified envy. In the FEE model, two agents are equal if they have equal preferences and equal endowments. So we say an assignment \( p \) satisfies equal treatment of equals (ETE) if for all \( i, j \in I \) such that \( \omega_i = \omega_j \) and \( \succ_i = \succ_j \), \( p_i = p_j \). In an assignment \( p \), we say an agent \( i \) envies another agent \( j \) if \( p_i \succ_i \equiv p_j \) does not hold. That is, \( i \) does not regard his lottery as better than \( j \)’s. In the FEE model, \( i \)'s envy towards \( j \) may be justified if \( j \) owns more endowments than \( i \) does. But if the two agents have equal endowments, it is reasonable to expect that each of them makes the best choice for themselves in the trading process, so that any differences in their lotteries are explained by their different preferences. So we say an assignment \( p \) satisfies equal-endowment no envy (EENE) if for all \( i, j \in I \) such that \( \omega_i = \omega_j \), \( p_i \equiv p_j \) and \( p_j \equiv p_i \). It is clear that EENE implies ETE.

Although we cannot require no envy between two agents of unequal endowments, it does not mean that we cannot discuss fairness between them. If an agent \( i \) owns slightly more endowments but obtains a much better lottery than another agent \( j \) does, then the assignment can be argued as unfair for \( j \). We propose a new fairness notion called bounded envy, which requires that if \( i \) envies \( j \), then \( j \) must have some advantage relative to \( i \) in endowments, and \( i \)'s envy towards \( j \) must be bounded by \( j \)'s advantage in endowments. Formally, in an assignment \( p \), for any two agents \( i \) and \( j \), if \( i \) envies \( j \), we use

\[
\max_{o \in O} \left[ \sum_{o' \succ_i o} p_{j,o'} - \sum_{o' \succ_i o} p_{i,o'} \right]
\]

to measure \( i \)'s envy towards \( j \).

**Definition 2.** An assignment \( p \) satisfies bounded envy if, for every distinct \( i, j \in I \),

\[
\max_{o \in O} \left[ \sum_{o' \succ_i o} p_{j,o'} - \sum_{o' \succ_i o} p_{i,o'} \right] \leq \sum_{o \in O} \mathbf{1}\{\omega_j,o > \omega_i,o\} \cdot (\omega_j,o - \omega_i,o).
\]

Note that the right-hand side of the inequality in Definition 2 only sums over objects \( o \) where \( \omega_j,o > \omega_i,o \). When \( i \) envies \( j \) in an assignment \( p \), \( j \) does not need to own more amount of every object than \( i \) does. For example, suppose that \( i \) owns \( a \), \( j \) owns \((1/2b,1/2c)\), and their preferences are \( c \succ_i a \succ_i b \) and \( c \succ_j b \succ_j a \). If they obtain their own endowments, then \( i \) envies \( j \), and this is bounded envy by Definition 2. It is clear that bounded envy implies EENE.

Now we present three conditions on the parameter matrix \( \Lambda(d) \) in BTA that can respectively ensure the above three fairness criteria.

**Definition 3.** A BTA satisfies
(1) **stepwise equal treatment of equals** (stepwise ETE) if at each step $d$, for every $i, j \in I(d)$, $\omega_i(d) = \omega_j(d)$ and $o_i(d) = o_j(d)$ imply $\lambda_i(d) = \lambda_j(d)$;

(2) **stepwise equal-endowment equal treatment** (stepwise EEET) if at each step $d$, for every $i, j \in I(d)$, $\omega_i(d) = \omega_j(d)$ implies $\lambda_i(d) = \lambda_j(d)$;

(3) **bounded advantage** if at each step $d$, for every $i, j \in I(d)$ and every $o \in O(d)$, $\omega_{i,o}(d) \geq \omega_{j,o}(d)$ implies that $\lambda_{i,o}(d) \geq \lambda_{j,o}(d)$ and $\omega_{i,o}(d+1) \geq \omega_{j,o}(d+1)$.

The first two conditions apply “equals being treated equally” to the procedure of BTA. At any step $d$, if we require $\lambda_i(d) = \lambda_j(d)$, it implies that $x_i(d) = x_j(d)$ and for every $o \in O(d)$, $\omega_{i,o}(d) - \omega_{i,o}(d+1) = \omega_{j,o}(d) - \omega_{j,o}(d+1)$. In words, $i$ and $j$ will obtain equal amounts of their respective favorite objects and lose equal amount of each endowment at step $d$. Stepwise ETE imposes this condition on any two agents of equal remaining endowments and equal favorite object at any step, while stepwise EEET imposes the condition on any two agents of equal remaining endowments at any step.

Bounded advantage is stronger than the first two conditions because it is applicable to any two agents. At any step $d$, if $\omega_{i,o}(d) \geq \omega_{j,o}(d)$, then bounded advantage requires that $\lambda_{i,o}(d) \geq \lambda_{j,o}(d)$ and $\omega_{i,o}(d+1) \geq \omega_{j,o}(d+1)$. They together imply that

$$0 \leq \lambda_{i,o}(d)x_o(d) - \lambda_{j,o}(d)x_o(d) = \left[\omega_{i,o}(d) - \omega_{i,o}(d+1)\right] - \left[\omega_{j,o}(d) - \omega_{j,o}(d+1)\right]$$

$$\leq \omega_{i,o}(d) - \omega_{j,o}(d).$$

In words, if $\omega_{i,o}(d) \geq \omega_{j,o}(d)$, then $i$ has an advantage than $j$ in using $o$ to trade with the others at step $d$, but this advantage is bounded by $i$’s advantage in the endowment of $o$.

**Proposition 2.** (1) Every BTA satisfying stepwise ETE satisfies ETE;

(2) Every BTA satisfying stepwise EEET satisfies EEEN;

(3) Every BTA satisfying bounded advantage satisfies bounded envy.

In the FEE model there can be stronger fairness criteria than the three we have defined. The merit of our method is that, by choosing the parameter matrix $\Lambda(d)$, we can precisely control fairness in the procedure of BTA. Below we present two BTA that satisfy bounded advantage, but they are motivated by more precise fairness ideas.
• **Equal-BTA**: At every step, the remaining owners of each remaining object use equal amounts of the object to trade with the others. That is, for every $i \in I(d)$ and $o \in O(d)$,

$$\lambda_{i,o}(d) = \frac{1\{\omega_{i,o}(d) > 0\}}{\sum_{j \in I(d)} 1\{\omega_{j,o}(d) > 0\}}.$$

• **Proportional-BTA**: At every step, the remaining owners of each remaining object use amounts of the object in proportion to their endowments of the object to trade with the others. That is, for every $i \in I(d)$ and $o \in O(d)$,

$$\lambda_{i,o}(d) = \frac{\omega_{i,o}(d)}{\sum_{j \in I(d)} \omega_{j,o}(d)}.$$

To see the difference between the two algorithms, consider two agents $i, j$ and an object $o$ such that $\omega_{i,o} > \omega_{j,o} > 0$. In Equal-BTA, as long as $i$ and $j$ both own $o$ at the beginning of any step, they will use equal amounts of $o$ to trade with the others at the step. So $j$ will use up his endowment of $o$ earlier than $i$ in the algorithm. In Proportional-BTA, at every step $i$ and $j$ will use amounts of $o$ that are proportional to their endowments of $o$ to trade with the others, and they will use up their endowments of $o$ at the same step.

### 5.2 The house allocation model and the housing market model

In the housing market model, it is clear that every BTA reduces to TTC. In the house allocation model, Bogomolnaia and Moulin (2001) propose the class of simultaneous eating algorithms (SEA) to find ordinally efficient assignments. In a SEA, agents consume probability shares of objects continuously in time. A SEA satisfies a fairness notion called *equal-division lower bound* (Thomson, 2011) if every agent’s assigned lottery weakly stochastically dominates the equal division lottery (where $p_{i,o} = 1/|I|$ for every $i$ and $o$). An intuitively fair SEA is the so-called probabilistic serial (PS) algorithm, which gives agents equal consumption rates.

We interpret the house allocation model as a special case of FEE where agents own equal divisions of all objects. In the house allocation model, we prove that every BTA is a SEA that satisfies equal-division lower bound, and every BTA that treats agents equally is PS. Therefore, our result transparently shows that PS can be interpreted as a trading algorithm, and BTA can be regarded as extensions of SEA to the FEE model.

**Proposition 3.** In the house allocation model:
1. Every BTA is a SEA satisfying equal-division lower bound;

2. Every BTA satisfying stepwise EEET is PS.

Kesten (2009) firstly shows that PS can be understood as a trading algorithm. Comparing Kesten’s algorithm with ours will reveal the advantage of our method. Also treating equal divisions of all objects as agents’ endowments, Kesten’s algorithm generates a directed graph in which every node is a pair of an agent \( i \) and an object \( o \). At every step, every node \((i, o)\) points to all of the nodes that hold \( i \)’s favorite object. This produces a complicated graph. Kesten carefully selects cycles from the generated graph to clear. The resulting trading algorithm is complicated and does not coincide with PS step by step, although they are equivalent in outcomes. By contrast, our method does not select cycles and highlights the essence of trading endowments.

6 Two applications of the FEE model and BTA

6.1 Time exchange problem

Many people around the world choose to exchange time and skills on centralized platforms without using transfers. Time banks are examples of such platforms (Andersson et al., 2021). Participants trade a wide range of services, such as legal assistance, babysitting, medical care, and many others. A time exchange market can be described as a FEE problem in which \( I \) is the set of participants, \( O \) is the set of service types, and for every \( i \) and \( o \), \( \omega_{i,o} \) is the amount of service \( o \) that \( i \) can provide. Conceptually, service types can be as general as contracts in matching theory (Hatfield and Milgrom, 2005). A service type can specify the time and location to finish a task, and other details if necessary. One unit of a service could mean one hour/day/week.

In a time exchange market, an agent may demand several services and may want to reserve some of his services for himself. For every \( i \) and \( o \), we introduce \( \overline{q}_{i,o} \in [\omega_{i,o}, 1] \) and \( \underline{q}_{i,o} \in [0, \omega_{i,o}] \) to denote the upper bound and lower bound on \( i \)’s demand over \( o \). So \( \underline{q}_{i,o} \) is the amount of service \( o \) that \( i \) reserves for himself. An assignment \( p \) is called feasible if, for every \( i \) and \( o \), \( p_{i,o} \in [\underline{q}_{i,o}, \overline{q}_{i,o}] \). Real-life agents may have complex preferences over bundles of services, but it is impractical to design a mechanism to elicit complex preferences. By applying BTA, we ask agents to report ordinal preferences on services.

Specifically, we will adapt BTA as follows. At each step \( d \):
1. Let each remaining agent $i$ report his favorite remaining service type $o$ that satisfies $p_{i,o}(d) < q_{i,o}$.

2. For all $i \in I(d)$ and $o \in O(d)$, we choose a parameter matrix such that $\lambda_{i,o}(d) > 0$ only if $\omega_{i,o}(d) > q_{i,o}$.

3. The constraints (6) are replaced by

$$
\begin{align*}
\lambda_{i,o}(d)x_o(d) & \leq \omega_{i,o}(d) - q_{i,o} \\
x_i(d) & \leq \frac{q_{i,o}(d)}{\lambda_{i,o}(d)} - p_{i,o}(d) 
\end{align*}
$$

for all $i \in I(d)$, $o \in O(d)$,

for all $i \in I(d)$,

4. An agent $i$ remains after step $d$ if $\omega_{i,o}(d + 1) > q_{i,o}$ for some $o \in O$, and a service type $o$ remains after step $d$ if $\omega_{i,o}(d + 1) > q_{i,o}$ for some $i \in I$.

We may choose a fair BTA, for example Equal-BTA or Proportional-BTA, to obtain a fair assignment. Because this application is straightforward, we omit details.

6.2 Two-sided matching with distributional constraints

In this subsection we consider the Japanese medical residency match with distributional constraints model (Kamada and Kojima, 2015). In the model hospitals are located in different geographic regions. To solve the shortage of doctors in rural regions, a market designer controls the distribution of doctors by imposing constraints on the number of doctors hired by the hospitals in each region. A ceiling/floor constraint is an upper/lower bound on the number of doctors assigned to a region. The literature has successfully dealt with ceiling constraints, but it demonstrates difficulty of dealing with floor constraints. We apply BTA to obtain a new solution that deals with both ceiling and floor constraints.

Given a set of doctors $I$ and a set of hospitals $O$, a constraint is a triple $(S, q_S, q_{S'})$ where $S \subseteq O$ is the set of hospitals in a region and $(q_S, q_{S'})$ are the upper and lower bounds on the number of doctors assigned to $S$ (so $0 \leq q_S \leq q_{S'}$). An assignment $p$ satisfies the constraint if $q_S \leq \sum_{i \in I, o \in S} p_{i,o} \leq q_{S'}$. We require the collection of the constraints $\{(S, q_S, q_{S'})\}$ form a hierarchy. That is, for any two constraint sets $S$ and $S'$, if $S \cap S' = \emptyset$, then either $S \subset S'$ or $S' \subset S$. An assignment that satisfies all constraints is called feasible. We assume that there exists at least one feasible assignment. Every doctor $i$ has a strict preference relation $\succ_i$ over hospitals. We assume that every hospital $o$ strictly prefers a feasible assignment $p'$ to another feasible assignment $p$ if $\sum_{i \in I} p'_{i,o} > \sum_{i \in I} p_{i,o}$, and regards them as indifferent.
if $\sum_{i \in I} p'_{i,o} = \sum_{i \in I} p_{i,o}$. We call a feasible assignment $p$ two-sided efficient if there does not exist a different feasible assignment $p'$ such that $p'$ strictly stochastically dominates $p$ for doctors and $\sum_{i \in I} p'_{i,o} \geq \sum_{i \in I} p_{i,o}$ for all $o$. We say $p$ is envy-free for doctors if, for every distinct $i, j \in I$, $p_i \geq_{sd} p_j$ and $p_j \geq_{sd} p_i$.

Our idea of applying BTA to the model is as follows. We first generate a feasible assignment $\omega$ such that for all $i, j \in I$ and all $o \in O$, $\omega_{i,o} = \omega_{j,o}$. For example, if $p$ is a feasible assignment, then we can generate $\omega$ by letting $\omega_{i,o} = \frac{\sum_{i \in I} p_{i,o}}{|I|}$ for all $i$ and $o$. By treating $\omega$ as doctors’ endowments, we give doctors equal rights/obligations of going to each hospital. We then let doctors exchange their rights/obligations by running Equal-BTA. Because the number of doctors assigned to each hospital does not change in the procedure of Equal-BTA, the found assignment must satisfy all constraints. The found assignment is envy-free for doctors because they start with equal endowments. Finally, if the numbers of assigned doctors in $\omega$ are on the Pareto frontier for hospitals, the found assignment is two-sided efficient.

Formally, we call the following procedure equal access exchange algorithm (EAE).

1. Arbitrarily choose a feasible assignment $\omega$ such that, for all $i, j \in I$ and all $o \in O$, $\omega_{i,o} = \omega_{j,o}$, and there is no feasible assignment $p'$ such that

$$\sum_{i \in I} p'_{i,o} \geq \sum_{i \in I} \omega_{i,o} \text{ for all } o \in O;$$

$$\sum_{i \in I} p'_{i,o'} > \sum_{i \in I} \omega_{i,o'} \text{ for some } o' \in O.$$  

2. Regard $\omega$ as doctors’ endowments and run Equal-BTA.

**Proposition 4.** In the Japanese medical residency match with distributional constraints model, EAE is feasible, envy-free for doctors, and two-sided efficient.

In EAE, the choice of $\omega$ determines the number of doctors assigned to each hospital. So it determines fairness among hospitals. We do not discuss how to choose $\omega$ and leave it to the market designer. Kamada and Kojima (2015) have some related discussions about how to assign hospital quotas fairly in their priority-based approach.

7 Priority-based allocation problem

In the priority-based allocation model, if all priorities are strict, Abdulkadiroğlu and Sönmez (2003) have presented a generalization of TTC: at every step, let every agent point
to his favorite object and every object point to the agent of highest priority, and then clear cycles. Abdulkadiroğlu and Sönmez apply this algorithm to school choice, but real-life schools often use coarse criteria (e.g., walk zones, sibling attendance, and socioeconomic status) to rank students, which lead to coarse priorities with large indifference classes (Erdil and Ergin, 2008). The standard approach to dealing with coarse priorities in the literature is to break priority ties randomly before running TTC (or the deferred acceptance algorithm). But this approach can result in efficiency loss.\footnote{For example, in the extreme case that all agents are in priority ties for all objects, running TTC after breaking ties uniformly at random is equivalent to running the random priority algorithm (Abdulkadiroğlu and Sönmez, 1998), which is not ordinally efficient.} Using our method, we define the priority trading algorithm (PTA), which is ordinally efficient, treats equal-priority agents equally, and does not explicitly and randomly break priority ties.

At every step of PTA, only agents of highest priority for an object can use the object to trade with the others. Because the agents in priority ties have equal rights over the object, we let the agents of highest priority use equal amounts of the object to trade with the others. In this sense, PTA is a generalization of Equal-BTA.

### Priority trading algorithm

**Notations:** Let $I(d)$, $O(d)$, $o_i(d)$, $p(d)$ and $x^*(d)$ have the same definitions as in BTA. Among $I(d)$, let $I_o(d)$ be the set of agents who have highest priority for each $o \in O(d)$.

**Step $d \geq 1$:** Let each $i \in I(d)$ report his favorite object $o_i(d)$. Let $x^*(d)$ be the maximum solution to the equations

$$
\begin{align*}
  x_o(d) &= \sum_{i \in I(d)} 1\{o_i(d) = o\} \cdot x_i(d) \quad \text{for all } o \in O(d), \\
  x_i(d) &= \sum_{o \in O(d)} 1\{i \in I_o(d)\} \cdot \frac{x_o(d)}{|I_o(d)|} \quad \text{for all } i \in I(d),
\end{align*}
$$

subject to the constraints

$$
\begin{align*}
  x_o(d) &\leq q_o - \sum_{k=1}^{d-1} x^{*o}_o(k) \quad \text{for all } o \in O(d), \\
  x_i(d) &\leq 1 - \sum_{k=1}^{d-1} x^{*i}_i(k) \quad \text{for all } i \in I(d).
\end{align*}
$$

For every $i \in I$ and $o \in O$, let $p_{i,o}(d+1) = p_{i,o}(d) + x^*_i(d)$ if $i \in I(d)$ and $o = o_i(d)$, and otherwise let $p_{i,o}(d+1) = p_{i,o}(d)$. Let $I(d+1) = \{i \in I : \sum_{k=1}^{d} x^{*i}_i(k) < 1\}$ and $O(d+1) = \{o \in O : \sum_{k=1}^{d} x^{*o}_o(k) < q_o\}$. If $O(d+1)$ or $I(d+1)$ is empty, stop the procedure. Otherwise, go to step $d + 1$.\footnote{For example, in the extreme case that all agents are in priority ties for all objects, running TTC after breaking ties uniformly at random is equivalent to running the random priority algorithm (Abdulkadiroğlu and Sönmez, 1998), which is not ordinally efficient.}
The assignment found by PTA is ordinally efficient. To precisely show its fairness property, we need to define hierarchical endowments implied by coarse priorities. Yet because the intuition has been demonstrated by our results in the FEE model, we omit the details. An obvious result is that the found assignment \( p \) satisfies no envy towards lower priority: for any \( i, j \in I \), if \( i \succeq_o j \) for all \( o \in O \), then \( p_i \succeq_{sd} p_j \). In particular, \( p \) satisfies equal-priority no envy: for any \( i, j \) of equal priority for all objects, \( p_i \succeq_{sd} p_j \) and \( p_j \succeq_{sd} p_i \).

**Proposition 5.** In the priority-based allocation model, PTA is ordinally efficient and satisfies no envy towards lower priority.

### 8 House allocation with existing tenants

We have shown that in the house allocation model and the housing market model, our BTA reduces to existing algorithms in the literature. The HET model is a hybrid of the two models. In this section we use our method to derive a new algorithm for the HET model. It generalizes both PS and TTC.

In the HET model, there exists a subset of objects \( O_E \) in which every object is privately owned by a distinct agent. Denote the set of agents who own \( O_E \) by \( I_E \) and call them existing tenants. The object privately owned by \( i \in I_E \) is denoted by \( o_i \), and the agent who privately owns \( o \in O_E \) is denoted by \( i_o \). The other agents \( I \setminus I_E \) are called newcomers.

We call our new algorithm the eating-trading algorithm (ETA). At every step of ETA, after agents report their favorite objects, we check if there exist cycles among existing tenants. If several existing tenants demand each other’s private endowments so that they form a cycle, let them trade their endowments immediately. The traded amount in each cycle depends on the residual demands of the existing tenants in the cycle. If there do not exist cycles among existing tenants, we let agents consume their favorite objects with following rates. First, the eating rates of all newcomers are one. Second, every existing tenant’s eating rate is equal to one plus the sum of the eating rates of the agents who are consuming his private endowment (if any). We call this rule “you request my house - I get your rate”. Therefore, ETA incorporates the features of both TTC and SEA.

#### Eating-trading algorithm

**Notations:** \( I(d) \), \( O(d) \) and \( o_i(d) \) have the same definitions as before. At the beginning of step \( d \), \( r_i(d) \) is the residual demand of each \( i \in I(d) \), and \( r_o(d) \) is the residual amount of each \( o \in O(d) \).
**Initialization:** \( I(1) = I, O(1) = O, \) and \( r_a(1) = 1 \) for all \( a \in I(1) \cup O(1) \). Set \( t = 0 \).

**Step** \( d \geq 1 \): Let every \( i \in I(d) \) point to his favorite object \( o_i(d) \). Let every \( o \in O(d) \cap O_E \) point to its owner \( i_o \) if \( i_o \in I(d) \).

- If there are cycles among existing tenants, for every cycle \( c \), let \( I(c) \) and \( O(c) \) be the set of agents and the set of objects involved in the cycle. The agents in each cycle trade the amount \( \min\{r_a(d)\}_{a \in I(c) \cup O(c)} \) of private endowments instantly. Remove agents whose demands are satisfied and objects that are exhausted. If the remaining agents \( I(d+1) \) or the remaining objects \( O(d+1) \) becomes empty, stop; otherwise, go to step \( d + 1 \).

- If there are no cycles, let agents simultaneously consume their favorite objects with following rates. Denote each \( i \)'s consumption rate by \( s_i(d) \). For all \( i \in I(d) \setminus I_E \), let \( s_i(d) = 1 \); for all \( j \in I(d) \cap I_E \), let \( s_j(d) = s_{o_j}(d) + 1 \) where \( s_{o_j}(d) = \sum_{i \in I(d); o_i(d) = o_j} s_i(d) \). This step stops when some agent’s demand is satisfied or some object is exhausted. If \( I(d+1) \) or \( O(d+1) \) becomes empty, stop; otherwise, go to step \( d + 1 \).

ETA reduces to PS in the house allocation model, and reduces to TTC in the housing market model.\(^{13}\) To see why ETA is actually a trading algorithm defined by our method, we regard the HET model as a special case of the priority-based allocation model where all agents are in priority ties for every \( o \in O \setminus O_E \), every existing tenant \( i \in I_E \) has the highest priority for his private endowment \( o_i \), and the others are in priority ties for \( o_j \). Then we prove that ETA coincides with the application of PTA to the HET model. Therefore, ETA is ordinally efficient and satisfies no envy towards newcomers.\(^{14}\) In particular, newcomers will not envy each other. An assignment \( p \) is individually rational in the HET model if for every \( i \in I_E \) and \( o \in O \) with \( p_{i,o} > 0 \), \( o \succeq _i o_j \).

**Proposition 6.** ETA is the application of PTA to the HET model. So ETA is individually rational, ordinally efficient, and satisfies no envy towards newcomers.

In this paper we present the simplest setup of the house allocation model. For a general setup in which the numbers of agents and objects may be unequal and objects may

\(^{13}\)ETA is first presented in the third chapter of Zhang (2017). Using the technique of Che and Kojima (2010) who prove the asymptotic equivalence between PS and the random priority algorithm in large markets, Zhang (2017) proves that ETA is asymptotically equivalent to the “you request my house - I get your turn” algorithm of Abdulkadiroğlu and Sönmez (1999), which is a generalization of TTC to the HET model.

\(^{14}\)An assignment \( p \) satisfies no envy towards newcomers if for every \( i \) and \( j \) such that \( i \in I \setminus I_E \), \( p_j \succeq _j p_i \).
have multiple copies, we can regard the house allocation model as a special case of the priority-based allocation model where all agents are in priority ties for all objects. Then we can prove similarly as before that PTA coincides with PS in the house allocation model.

9 Fair trading algorithm in a two-stage mechanism

In the models we have discussed, trading algorithms are used as primary mechanisms to find efficient and fair assignments. But in some models, a different algorithm is used in the first stage and a trading algorithm is used in the second stage to improve efficiency of the first stage. In this section, we use the two-stage mechanism of Kesten and Ünver (2015) (KU for short) as an example to illustrate that our method is also useful to define an efficient and fair trading algorithm in certain stages of multiple-stage mechanisms. Our method easily solves a difficulty in KU’s paper.

Specifically, KU consider the school choice model, which is a real-life example of the priority-based allocation model. But different from Section 7 where we take efficiency as the primary goal, in school choice it is more important to obtain stability (i.e., elimination of justified envies). When priorities are strict, the deferred acceptance (DA) algorithm finds the most efficient stable assignment. But when priorities are coarse, which is often the case in real life, fairness requires usage of randomization. It is undesirable to randomly break priority ties and then run DA, because it can result in ex-ante efficiency loss. The traditional stability notion is neither appropriate for random assignments. So KU propose two stability notions for random assignments. Ex-ante stability requires elimination of ex-ante justified envies, and strong ex-ante stability further requires elimination of discrimination among equal-priority students. KU want to find probabilistic variants of DA without explicitly breaking priority ties to find efficient assignments that satisfy the two stability notions.

KU propose the fractional deferred acceptance (FDA) algorithm to find the most efficient strongly ex-ante stable assignment. Because strong ex-ante stability has an additional requirement of eliminating discrimination among equal-priority students, it induces more efficiency loss than ex-ante stability. To find an efficient ex-ante stable assignment, it is sufficient to improve the efficiency of the outcome of FDA. KU prove that this is equivalent to clearing a type of cycles in the graph generated from the outcome of FDA. How-

---

An assignment \( p \) is ex-ante stable if there do not exist \( i, j \in I \) and \( o \in O \) such that \( i \succ_o j, p_{j,o} > 0 \), and for some \( o \succ o', p_{i,o'} > 0 \). It is further strongly ex-ante stable if there do not exist \( i, j \in I \) and \( o \in O \) such that \( i \sim_o j, p_{i,o} < p_{j,o}, \) and for some \( o \succ o', p_{i,o'} > 0 \).
ever, KU do not want to totally give up fairness among equal-priority students, so they want to maintain ETE, which requires that two students of equal preferences and equal priorities for all schools obtain equal lotteries. To get ETE, the cycles in the generated graph cannot be cleared arbitrarily. It turns out that the problem is not easy, and KU resort to an operations research method to solve the difficulty. They solve a series of maximum network flow problems at each step of their algorithm. They call the two-stage mechanism *fractional deferred acceptance and trading* (FDAT). Below we show that our method can easily solve their difficulty, which leads to a mechanism different than FDAT. We call it *Alternative FDAT*.

Formally, for any ex-ante stable assignment $p$, KU generate a graph as follows. For every $i$ and $o$, create a node represented by $(i,o)$ if $p_{i,o} > 0$. A node $(i,o)$ is said to envy another node $(j,o')$ if $i \neq j$ and $o' \succ_i o$. Create a directed edge from $(i,o)$ to $(j,o')$ if $(i,o)$ envies $(j,o')$ and for every other $(i',o'')$ that envies $(j,o')$, $i \succeq_o i'$. A node may point to several other nodes, and a node may be pointed by several other nodes. KU prove that $p$ is constrained (ordinally) efficient among ex-ante stable assignments if and only if there are no cycles in the generated graph. If $p$ is not constrained efficient, it suffices to clear all cycles in the generated graph to rectify efficiency loss.

**Alternative FDAT**

**Step 0:** Run FDA. Let $p(1)$ denote the outcome of FDA, which is ex-ante stable.

**Step $d \geq 1$:** Given $p(d)$, generate a directed graph as described above. Remove all nodes in the generated graph that do not point to any other nodes or are not pointed by any other nodes. After the removal, there may exist new nodes that do not point to any other nodes or are not pointed by any other nodes. We iteratively remove all of such nodes until we obtain a directed graph in which every node is involved in at least one cycle, or the graph becomes empty. In the latter case, we stop the procedure and return $p(d)$.

Let $\mathcal{N}$ denote the set of the remaining nodes in the graph. For every node $(i,o)$, let $\mathcal{N}_{(i,o)\rightarrow}$ denote the set of nodes pointed by $(i,o)$ and $\mathcal{N}_{\rightarrow(i,o)}$ denote the set of nodes that point to $(i,o)$. Then we find the maximum solution $x^*$ to the following equations

\begin{equation}
    x_{(i,o)} = \sum_{(j,o') \in \mathcal{N}_{\rightarrow(i,o)}} \frac{x_{(j,o')}}{|\mathcal{N}_{(j,o')\rightarrow}|} \quad \text{for all } (i,o) \in \mathcal{N},
\end{equation}
subject to the constraints

\[ x_{(i,o)} \leq p_{i,o}(d) \quad \text{for all } (i,o) \in \mathcal{N}. \]

Here \( x_{(i,o)} \) denotes the consumption of the node \((i,o)\), which is also the amount of \((i,o)\) traded at the step. That is why we impose the constraint \( x_{(i,o)} \leq p_{i,o}(d) \). Every node equally divides its demand over the other nodes it points to.

Note that every agent \( j \) may be represented by several nodes. So \( j \)'s total consumption at the step is the sum of every representative node's consumption. Formally, for every \( j \in I \) and \( o \in O \), if there exist \((j,o'),(i,o) \in \mathcal{N}\) such that \((j,o')\) points to \((i,o)\), then let

\[
p_{j,o}(d + 1) = p_{j,o}(d) + \sum_{o' \in O \colon (j,o') \in \mathcal{N} \land (j,o') \rightarrow (i,o)} \frac{x_{(j,o')}}{|\mathcal{N}(j,o') \rightarrow (i,o)|} - 1\{ (j,o) \in \mathcal{N} \} \cdot x_{(j,o)}.
\]

Otherwise, let \( p_{j,o}(d + 1) = p_{j,o}(d) \). Go to step \( d + 1 \).

At every step \( d \), an agent will use up the amount of an object he obtains and become better off. So the above algorithm has at most \(|I| \times |O|\) steps after running FDA.

Alternative FDAT is different from FDAT not only in the procedure, but also in the outcome.\(^{16}\) Alternative FDAT is intuitively fair because, at every step, every node equally divides its demand over the nodes it points to. In particular, if two agents \( i \) and \( j \) have equal preferences and equal priority for all schools, then they must receive equal lotteries in the outcome of FDA, and at every step of our algorithm, they must point to the same set of nodes and be pointed by the same set of nodes.\(^{17}\) So the outcome of Alternative FDAT must satisfy ETE, and this solves the difficulty in KU’s paper. We present the following result without a proof.

**Proposition 7.** The outcome of Alternative FDAT is constrained efficient among ex-ante stable assignments, and satisfies ETE.

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\(^{16}\)If we apply Alternative FDAT to KU’s Example 3 in their paper, it will find a different assignment than the outcome of FDAT.

\(^{17}\)The only exception is that \( i \) and \( j \) may point to each other. But in that case they will have equal consumptions of each other. So their symmetry will still be maintained.
10 Related literature

This paper contributes to the market design literature by proposing a new method to define trading algorithms. To our knowledge, our method has not been used in the literature. The closest idea to ours appears in Leshno and Lo (forthcoming) who use equations to reformulate TTC in a continuum school choice model. To understand the role of strict priorities in the extension of TTC proposed by Abdulkadiroğlu and Sönmez (2003), Leshno and Lo study a framework with finite schools and a continuum of students. The usual definition of TTC has discrete steps, but it is not appropriate in a continuum model. So they define a continuous-time TTC by using equations to characterize aggregate trading behavior over multiple steps. Their paper and ours have following differences. First, our motivations are different. They want to provide a reformulation of TTC that is tractable in their continuum model, whereas we want to propose a general method to define trading algorithms in finite market design models. Second, the sources of our difficulties are different. The difficulty in their paper is solely caused by the continuum framework, which is an artificial assumption. The difficulty in our paper is caused by real-life elements such as co-ownerships and coarse priorities. Third, fairness is an important concern in the models we study. We add parameters to our equations to control fairness, which is a novel idea and has no parallels in their equations.

In Section 3.1, we use the concept of absorbing sets to provide a graphical explanation of our method. This concept is not new in the graph theory. In the market design literature, Quint and Wako (2004) use this concept to discuss the strict core in the housing market model with weak preferences. Alcalde-Unzu and Molis (2011) use this concept to propose an extension of TTC to the housing market model with weak preferences (also see Jaramillo and Manjunath (2012)).

In the rest of this section, we compare the new algorithms we derive in this paper with related ones in the literature (see Table 6).

In the HET model, the closest related algorithm to our ETA is the individually rational PS (IR-PS) proposed by Yılmaz (2010). IR-PS is an extension of PS by accommodating the IR constraints of existing tenants. Its idea is to imitate the procedure of PS as much as possible. At any time in its procedure, if the IR constraint of a group of agents is going to be violated, IR-PS isolates the group and their remaining acceptable objects as a sub-problem by not allowing the other agents to consume the objects in the subproblem.18 In

18 In our setup of the HET model, newcomers accept all objects. We say that an object is acceptable to an existing tenant if it is no worse than his private endowment.
| Model                             | This paper | Literature                          |
|-----------------------------------|------------|-------------------------------------|
| HET                              | ETA        | IR-PS                               |
|                                  |            | (Yılmaz, 2010)                      |
| FEE                              | BTA        | Controlled-consuming                |
|                                  |            | (Athanassoglou and Sethuraman, 2011)|
| Time exchange                     | BTA        | Priority algorithm                   |
|                                  |            | (Andersson et al., 2021)             |
|                                  |            | (Manjunath and Westkamp, 2021)       |
| Distributional constraints       | EAE        | Flexible DA                         |
|                                  |            | (Kamada and Kojima, 2015)           |
|                                  |            | DA+Serial Dictatorship              |
|                                  |            | (Akin, forthcoming)                 |
| Priority-based allocation        | PTA        | TTC with exogenously broken ties    |
| Stability-based school choice    | Alternative FDAT | DA with exogenously broken ties |
|                                  |            | (Erdir and Ergin, 2008)            |
|                                  |            | FDAT                                |
|                                  |            | (Kesten and Ünver, 2015)           |

Table 6: Comparison with the literature

particular, if an existing tenant views his private endowment as the worst object, IR-PS will ignore his private ownership and treat him as a newcomer. So IR-PS will not reduce to TTC in the housing market model. IR-PS is IR, ordinally efficient, and satisfies a fairness notion called no justified envy (NJE).\(^{19}\) Differently, ETA inherits the interpretation of private ownership from the housing market model. It gives existing tenants the right to consume private endowments as well as the right to trade private endowments. Actually, we show that ETA is essentially a trading algorithm. This perspective provides a unified interpretation of PS, TTC, and ETA. Also, this difference means that the two algorithms have different incentive properties. For random assignment mechanisms, Mennle and Seuken (2020) decompose strategy-proofness into upper invariance, lower invariance, and swap monotonicity.\(^{20}\) Both IR-PS and ETA are not strategy-proof because they subsume PS as a special case. PS is not strategy-proof because it violates lower invariance, but it satisfies upper invariance and swap monotonicity. This is also true for ETA. But IR-PS only satisfies swap monotonicity. Violating upper invariance means that existing tenants can manipulate IR-PS by manipulating the ranking of private endowments in

\(^{19}\) An assignment satisfies NJE if for every two agents \(i\) and \(j\), if \(i\) envies \(j\), then \(i\)'s lottery is not IR for \(j\).

\(^{20}\) Upper invariance means that agents cannot change their chances of obtaining more preferred objects by changing the preference order of less preferred objects, while lower invariance means the other way. Swap monotonicity means that if an agent swaps the preference order of two consecutively ranked objects, then either the agent's lottery is unaffected or he receives more of the object whose rank becomes higher.
their preferences (this manipulation is often called *truncation strategy*).

**Athanassoglou and Sethuraman (2011)** extend IR-PS to the FEE model and call their algorithm *controlled consuming* (CC). They show that CC satisfies NJE but violates EENE. They admit that EENE is a natural fairness notion in the FEE model because “two agents with identical endowments bring exactly the same resources to the group, so any differences in their final assignment should be explained solely by their preferences.” They ask the existence of an algorithm satisfying ordinal efficiency, IR and EENE, and claim that it is a challenge to extend TTC to the FEE model. Our BTA answers both questions.\(^{21}\)

We provide two applications of the FEE model and BTA. To our knowledge, since the FEE model is proposed by **Athanassoglou and Sethuraman (2011)**, there are no interesting applications of the FEE model in the literature. Related to our first application to time exchange markets, **Andersson et al. (2021)** and **Manjunath and Westkamp (2021)** propose two different time exchange models. In the former paper each agent provides a distinct service and has dichotomous preferences over services in the market. In the latter paper agents are endowed with disjoint sets of shifts and have dichotomous preferences over the other agents’ shifts. Both papers recommend priority algorithms because fairness is not their concern. Differently, in our time exchange model an agent can provide several services and several agents can provide the same service. Our algorithm lets agents trade services efficiently and fairly.

Our second application to the Japanese medical residency match is motivated by **Kamada and Kojima (2015)**. **Kamada and Kojima** propose a stability-based fairness notion and adapt DA to deal with only ceiling constraints. **Akin (forthcoming)** generalizes Kamada and Kojima’s fairness notion to accommodate floor constraints. She proposes a two-stage mechanism in which the first stage runs DA for a subset of doctors and the second stage runs the serial dictatorship algorithm for the remaining doctors. Both papers rely on exogenous priorities. Differently, our method abandons priorities, treats doctors ex-ante equally, and can deal with both ceiling and floor constraints. We take both doctors and hospitals’ preferences into account. The two-sided efficiency notion we use is similar to a notion used by **Combe et al. (2018)** in their teacher (re)assignment model.

To our knowledge, in the priority-based allocation model, the literature has not pre-

\(^{21}\) In an unpublished paper **Aziz (2015)** makes an attempt to generalize TTC to the FEE model. To solve the difficulty caused by non-disjoint cycles, Aziz selects cycles by using exogenous rankings of agents and objects to prioritize cycles. This makes his algorithm unfair (e.g., violation of ETE). Similarly, **Altuntas and Phan (2017)** define probabilistic variants of TTC by letting objects use strict priorities to rank agents. So their algorithms are straightforward extensions of TTC. Priorities introduce asymmetry and result in artificial unfairness.
presented an ordinal algorithm that finds ordinally efficient assignments and respects agents’ priorities. As explained, running TTC after exogenously breaking priority ties is not ordinally efficient. The other algorithms in the literature take stability-based fairness as an objective. We have discussed FDA and FDAT of Kesten and Ünver (2015) in Section 7. Our PTA and their FDA(T) are different in treatments of priorities, because they inherit the difference between DA and TTC. We emphasize that PTA and FDA(T) are also different in treatments of priority ties. In the extreme case that all agents are in priority ties for all objects, PTA coincides with PS, while FDA(T) are different than PS. In particular, in the outcome of FDAT some agent may envy the others (see an example in Kesten and Ünver (2015)). Han (2017) proposes a different ex-ante fairness notion and an algorithm that incorporates the features of both DA and PS. The algorithm is not ordinally efficient.

In Section 9 we discuss how our method is useful to define trading algorithms in two-stage mechanisms, and take the mechanism of Kesten and Ünver (2015) as an example. In the school choice with coarse priorities model, Erdil and Ergin (2008) are the first to point out that DA with exogenously broken ties is inefficient, and that to rectify the efficiency loss of DA it is sufficient to clear cycles in a graph generated from the outcome of DA. This approach is further developed by Erdil and Ergin (2017) and Erdil and Kumano (2019). If fairness is a concern in the second stage of the two-stage mechanisms in these papers, our method can be similarly used to define an efficient and fair trading algorithm.

Finally, we should mention that in many models, the cost of obtaining ordinal efficiency is the loss of strategy-proofness. In the house allocation model, Bogomolnaia and Moulin (2001) prove that every ordinally efficient algorithm satisfying ETE is not strategy-proof. In the FEE model, Athanassoglou and Sethuraman (2011) prove that every ordinally efficient algorithm satisfying IR is not strategy-proof, and Aziz (2018) further prove that every such algorithm is not strategy-proof even in the weak sense. Therefore, every BTA is not strategy-proof. However, it has been an established idea in the literature that strategy-proofness can be restored in large markets under regularity conditions. In Appendix C we show that the idea works in this paper. We prove that every BTA that satisfies a continuity property is asymptotically strategy-proof in large markets when agents’ preferences become diverse. It implies that Equal-BTA and Proportional-BTA (and many other BTA) are asymptotically strategy-proof.

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22 An algorithm is **strategy-proof** if every agent’s lottery from truth-telling stochastically dominates his lottery from misreporting preferences. An algorithm is **weakly strategy-proof** if every agent’s lottery from truth-telling is not strictly stochastically dominated by his lottery from misreporting preferences (Bogomolnaia and Moulin, 2001).
11 Conclusion

This paper presents a new method to define trading algorithms and provides applications to several market design environments. The graph-based definition of TTC presented by Shapley and Scarf (1974) is influential in the market design literature. Many follow-up studies stick to the idea of clearing cycles in generated graphs to define trading algorithms. As discussed in the paper, this idea needs to answer the details of who trade with whom and how, which can be a difficult question in complex environments. Our equation-based definition avoids such details and captures the essence of a trading algorithm. By including parameters in our equations, our method can control fairness in a convenient and transparent manner. In applications, we obtain new trading algorithms by using a common idea of trading endowments or priorities. We expect to find more applications in future research, and hope to see that our method can solve difficulties in other models.

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A Proof of Theorem 1

The equation system (4) can be written as $(I - \Lambda)x = 0$. Because every column of $\Lambda$ sums to one, the sum of all columns of $I - \Lambda$ is zero. So $I - \Lambda$ is singular. It means that $(I - \Lambda)x = 0$ has nonzero solutions. We want to prove that it has nonnegative solutions, and it has a maximum solution that satisfies the constraints $x \leq q$.

We introduce some definitions. In a directed graph $(V, E)$, for any $V \subset V$, let $\Lambda_V = (\lambda_{v,u})_{v,u \in V}$ be the restriction of $\Lambda$ to $V$. For any $V' \subset V \subset V$, $\Lambda_{V'}$ is called a submatrix of $\Lambda_V$. $\Lambda_{V'}$ is called isolated in $\Lambda_V$ if $\lambda_{v,u} = 0$ for all $v \in V \setminus V'$ and $u \in V'$. For any $V \subset V$, $\Lambda_V$ is called isolated if it is isolated in $\Lambda$; otherwise it is called unisolated. $\Lambda_V$ is called reducible if it has a strict nonempty submatrix that is isolated in $\Lambda_V$; otherwise, it is called irreducible. For convenience, we also say $V$ is (un)isolated or (ir)reducible if $\Lambda_V$ is (un)isolated or (ir)reducible.

**Lemma 1.** $V$ can be uniquely partitioned into disjoint sets $V_1, V_2, \ldots, V_k, V_{k+1}$ such that

1. for all $\ell = 1, \ldots, k$, $V_\ell$ is nonempty, isolated, and irreducible;

2. $V_{k+1}$, which can be empty, is either unisolated or reducible, and it does not contain any strict nonempty subset that is isolated and irreducible.

**Proof.** We prove the lemma by constructing the partition. We first construct $V_1$. If $V$ is irreducible, let $V_1 = V$, and we are done by letting $k = 1$ and $V_2 = \emptyset$. Otherwise, $V$ contains a strict nonempty subset $V$ that is isolated. If $V$ is irreducible, let $V_1 = V$. Otherwise, $V$ contains a strict nonempty subset $V'$ that is isolated in $V$. Since $V$ is isolated, $V'$ is isolated. If $V'$ is irreducible, let $V_1 = V'$. Otherwise, $V'$ also contains a strict nonempty subset $V''$ that is isolated in $V'$. So $V''$ is also isolated. Note that every submatrix that consists of a single element must be irreducible. So by repeating the above argument we must be able to find an irreducible and isolated nonempty subset. Let the subset be $V_1$.

We then construct $V_2$. If $V \setminus V_1$ is irreducible and isolated, we are done by letting $V_2 = V \setminus V_1$, $k = 2$, and $V_3 = \emptyset$. Otherwise, if $V \setminus V_1$ does not contain any strict nonempty subset that is irreducible and isolated, we are done by letting $k = 1$ and $V_2 = V \setminus V_1$. If $V \setminus V_1$ contains a strict nonempty subset $V$ that is irreducible and isolated, let $V_2 = V$. Then we construct $V_3$ and possibly $V_4, \ldots, V_{k+1}$ from $V \setminus (V_1 \cup V_2)$ in a similar way as above. Since $V$ is a finite set, we must stop in finite steps.

Lemma 1 implies that by permuting rows and columns, we can write $\Lambda$ as the following block form:
(8) \[
\Lambda = \begin{bmatrix}
\Lambda_{V_1} & 0 & \cdots & 0 \\
0 & \Lambda_{V_2} & \cdots & 0 \\
& \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \Lambda_{V_k} \\
& & \cdots & 0 \\
\Lambda_{V_{k+1}} & & & & & \\
\end{bmatrix}
\]

For every \( \ell = 1, \ldots, k+1 \), let \( I_{V_\ell} \) denote the identity matrix of dimension \( |V_\ell| \times |V_\ell| \).

**Lemma 2.** (1) \( \text{Rank}(I_{V_\ell} - \Lambda_{V_\ell}) = |V_\ell| - 1 \) for all \( \ell = 1, \ldots, k \);

(2) \( \text{Rank}(I_{V_{k+1}} - \Lambda_{V_{k+1}}) = |V_{k+1}| \).

**Proof.** (1) For every \( \ell = 1, \ldots, k \), since \( V_\ell \) is isolated, it must be that \( \sum_{v \in V_\ell} \lambda_{v,u} = 1 \) for every \( u \in V_\ell \). Therefore, the sum of all columns of \( I_{V_\ell} - \Lambda_{V_\ell} \) is zero. So \( \text{det}(I_{V_\ell} - \Lambda_{V_\ell}) = 0 \) and \( \text{Rank}(I_{V_\ell} - \Lambda_{V_\ell}) < |V_\ell| \).

**Claim 1.** For every nonempty \( V \subseteq V_\ell \), there exists \( u \in V \) such that \( \sum_{v \in V} \lambda_{v,u} < 1 \).

**Proof:** Suppose \( \sum_{v \in V} \lambda_{v,u} = 1 \) for all \( u \in V \). Since \( \sum_{v \in V_\ell} \lambda_{v,u} = 1 \) for every \( u \in V_\ell \), it means that \( \lambda_{v,u} = 0 \) for all \( v \in V_\ell \setminus V \) and \( u \in V \). So \( \Lambda_V \) is isolated in \( \Lambda_{V_\ell} \), which contradicts the fact that \( \Lambda_{V_\ell} \) is irreducible.

Corollary 3.3 of Peterson and Olinick (1982) states that for a general matrix \( D = (d_{ij})_{i,j=1}^k \) such that \( d_{ij} \in [0,1] \) and \( \sum_{j=1}^k d_{ij} \leq 1 \) for all \( j \in X = \{1, \ldots, k\} \), if \( \text{det}(I_{k \times k} - D) = 0 \) and \( \text{det}(I_{(k-1) \times (k-1)} - D_{X \setminus j}) \neq 0 \) for all \( j \in X \), then every column of \( D \) sums to one. This result and Claim 1 imply the following claim.

**Claim 2.** For every nonempty \( V \subseteq V_\ell \), if \( \text{det}(I_V - \Lambda_V) = 0 \), then there exists some \( u \in V \) such that \( \text{det}(I_{V \setminus \{u\}} - \Lambda_{V \setminus \{u\}}) = 0 \).

**Proof:** Suppose \( \text{det}(I_{V \setminus \{u\}} - \Lambda_{V \setminus \{u\}}) \neq 0 \) for all \( u \in V \). By Corollary 3.3 of Peterson and Olinick (1982), every column of \( \Lambda_V \) sums to one. But it contradicts Claim 1.

Now we prove that for all \( u \in V_\ell \), \( \text{det}(I_{V_\ell \setminus \{u\}} - \Lambda_{V_\ell \setminus \{u\}}) \neq 0 \). Suppose towards a contradiction that \( \text{det}(I_{V_\ell \setminus \{u\}} - \Lambda_{V_\ell \setminus \{u\}}) = 0 \) for some \( u \in V_\ell \). By Claim 2, there exists \( u_1 \in V_\ell \setminus \{u\} \) such that \( \text{det}(I_{V_\ell \setminus \{u,u_1\}} - \Lambda_{V_\ell \setminus \{u,u_1\}}) = 0 \). By Claim 2 again, there further exists \( u_2 \in V_\ell \setminus \{u,u_1\} \) such that \( \text{det}(I_{V_\ell \setminus \{u,u_1,u_2\}} - \Lambda_{V_\ell \setminus \{u,u_1,u_2\}}) = 0 \). By repeatedly using Claim 2, we must find a submatrix consisting of only one element \( v \in V_\ell \) such that \( 1 - \lambda_{v,v} = 0 \), which contradicts the fact that \( \lambda_{v,v} = 0 \) for all \( v \in V \). So \( \text{det}(I_{V_\ell \setminus \{u\}} - \Lambda_{V_\ell \setminus \{u\}}) \neq 0 \) for all \( u \in V_\ell \). This implies that \( \text{Rank}(I_{V_\ell} - \Lambda_{V_\ell}) = |V_\ell| - 1 \).
(2) We first prove that $\Lambda V_{k+1}$ is unisolated. Suppose it is isolated, then by definition it must be reducible. So $\Lambda V_{k+1}$ contains a strict nonempty submatrix $\Lambda V$ that is isolated in $\Lambda V_{k+1}$. Since $\Lambda V_{k+1}$ is isolated, $\Lambda V$ is also isolated. Still by the definition of $\Lambda V_{k+1}$, $\Lambda V$ must be reducible. So $\Lambda V$ also contains a strict nonempty submatrix $\Lambda V'$ that is isolated in $\Lambda V$. Since $\Lambda V$ is isolated, $\Lambda V'$ is also isolated. By the definition of $\Lambda V_{k+1}$ again, $\Lambda V'$ must be reducible. By repeating this argument, we will finally find a submatrix consisting of a single element and conclude that it is reducible, which is a contradiction. So $\Lambda V_{k+1}$ is unisolated. It means that not every column of $\Lambda V_{k+1}$ sums to one. Then by Corollary 3.3 of Peterson and Olinick (1982) and same arguments as in Claim 1 and Claim 2, if $\det(I V_{k+1} - \Lambda V_{k+1}) = 0$, then there must exist some $v \in V_{k+1}$ such that $1 - \lambda_{v,v} = 0$, which is a contradiction. So $\det(I V_{k+1} - \Lambda V_{k+1}) \neq 0$, which implies that $\text{Rank}(I V_{k+1} - \Lambda V_{k+1}) = |V_{k+1}|$. \qed

Given the block form (8), Lemma 2 implies that

$$\text{Rank}(I - \Lambda) = \sum_{\ell=1}^{k+1} \text{Rank}(I V_{\ell} - \Lambda V_{\ell}) = |V| - k.$$  

So $(I - \Lambda)x = 0$ has $k$ linearly independent solutions. Below we construct the $k$ solutions.

For all every $\ell = 1, \ldots, k$, we consider the equation system $(I V_{\ell} - \Lambda V_{\ell})x_{V_{\ell}} = 0$. Since $\det(I V_{\ell} - \Lambda V_{\ell}) = 0$, 1 is an eigenvalue of $\Lambda V_{\ell}$. Since $\Lambda V_{\ell}$ is irreducible, by Frobenius Theorem (see Section 6.8 of Leon (2015)), 1 has a positive eigenvector $\tilde{x}_{V_{\ell}}$ that is a solution to $(I V_{\ell} - \Lambda V_{\ell})x_{V_{\ell}} = 0$. Recall that $\Lambda$ can be written in the block form (8). So

$$\tilde{x}_{\ell} = (0, \ldots, 0, \tilde{x}_{V_{\ell}}, 0, \ldots, 0)$$

is a nonnegative solution to $(I - \Lambda)x = 0$. It is clear that the $k$ solutions $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k$ are linearly independent. Therefore, every solution to $(I - \Lambda)x = 0$ is a linear combination of the $k$ solutions. That is, there exist $y_1, \ldots, y_k \in \mathbb{R}$ such that

$$x = y_1 \begin{bmatrix} \tilde{x}_{V_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ \tilde{x}_{V_2} \\ \vdots \\ 0 \end{bmatrix} + \cdots + y_k \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \tilde{x}_{V_k} \end{bmatrix}.$$
For every \( \ell = 1, \ldots, k \), define \( y^*_\ell = \min_{v \in V_\ell} \frac{d_v}{x_v} \). Then, \( \mathbf{x}^* = \sum_{\ell=1}^k y^*_\ell \mathbf{x}_\ell \) is the maximum solution that satisfies the constraint \( \mathbf{x} \leq \mathbf{q} \).

This finishes the proof of Theorem 1.

Lemma 3 proves that \( \{V_\ell\}_{1 \leq \ell \leq k} \) is the set of absorbing sets in the graph \((V, E)\).

**Lemma 3.** A subset of nodes \( V \subset V \) is an absorbing set in the graph \((V, E)\) if and only if \( \Lambda_V \) is irreducible and isolated.

**Proof.** (Only if) Let \( V \) be an absorbing set. Because \( V \) has no outgoing edges, for every \( u \in V \) and \( v \in V \setminus V, \lambda_{v,u} = 0 \). It means that \( \Lambda_V \) is isolated. Because \( V \) is inside connected, for any strict subset \( V' \) of \( V \), there exist \( v \in V \setminus V' \) and \( u \in V' \) such that \( \lambda_{v,u} > 0 \). It means that \( \Lambda_V \) is irreducible.

(If) Suppose \( \Lambda_V \) is irreducible and isolated. Being isolated directly means that \( V \) has no outgoing edges. Suppose there exist two nodes \( u, v \in V \) such that there is no directed path from \( u \) to \( v \). Let \( V_1 \) be the subset of \( V \) such that there is a directed path from \( u \) to every \( u' \in V_1 \). Let \( V_2 \) be the set of remaining nodes in \( V \). In particular, \( u \in V_1 \) and \( v \in V_2 \). Then there must be no directed path from every node in \( V_1 \) to every node in \( V_2 \); otherwise, there would be a directed path from \( u \) to every node in \( V_2 \), which contradicts the definition of \( V_2 \). It means that \( \lambda_{v,u} = 0 \) for all \( v \in V_2 \) and \( u \in V_1 \). So \( \Lambda_{V_1} \) is isolated in \( \Lambda_V \), which contradicts that \( \Lambda_V \) is irreducible. Therefore, \( V \) must be inside connected. \( \square \)

## B Proofs of Propositions 1-6

**Proof of Proposition 1.** (Individual rationality) At every step \( d \), for every \( i \in I(d) \) and \( o \in O(d) \) with \( \omega_{i,o}(d) > 0 \), it is clear that \( o_i(d) \succeq_i o \). Suppose the assignment \( p \) found by some BTA is not individually rational. Then there exist \( o^* \in O \) and \( i \in I \) such that \( \sum_{o \succeq_i o^*} p_{i,o} < \sum_{o \succeq_i o^*} \omega_{i,o} \). Let \( d \) be the earliest step after which all objects in \( \{o \in O : o \succeq_i o^*\} \) are exhausted. That is, \( \{o \in O : o \succeq_i o^*\} \cap O(d + 1) = \emptyset \) and \( \{o \in O : o \succeq_i o^*\} \cap O(d) \neq \emptyset \). Then \( i \)'s favorites objects from step one to step \( d \) must belong to \( \{o \in O : o \succeq_i o^*\} \), and \( \sum_{o \succeq_i o^*} p_{i,o} \) is the total amount of objects obtained by \( i \) at the end of step \( d \). But because \( \sum_{o \succeq_i o^*} p_{i,o} < \sum_{o \succeq_i o^*} \omega_{i,o} \), there must exist \( o' \succeq_i o^* \) such that \( \omega_{i,o'}(d + 1) > 0 \). This is a contradiction.

(Ordinal efficiency) Still let \( p \) denote the assignment found by BTA. Define a binary relation \( \succ \) on \( O \) such that for every \( o, o' \in O \), \( o \succ o' \) if there exists \( i \in I \) such that \( o \succ_i o' \) and \( p_{i,o'} > 0 \). Bogomolnaia and Moulin (2001) and Che and Kojima (2010) have shown that \( p \) is ordinally efficient if and only if \( \succ \) is acyclic. Since at any step of BTA an agent
reports an object only if better objects are exhausted, for any \( o \succ_i o' \) and \( p_{i,o'} > 0 \), \( o \) must be exhausted earlier than \( o' \). So \( \triangleright \) must be acyclic.

**Proof of Proposition 2.** (1) For any BTA that satisfies stepwise ETE, if any two agents \( i, j \) have equal endowments and equal preferences, then at step one they report the same favorite object and \( \lambda_i(1) = \lambda_j(1) \). So they obtain equal amounts of the favorite object and their remaining endowments after step one are equal. By induction it is easy to see that this holds for all remaining steps. So the two agents must obtain equal lotteries.

(2) Similarly, for any BTA that satisfies stepwise EEET, if any two agents \( i, j \) have equal endowments, then at each step \( d \) it must be that \( \lambda_i(d) = \lambda_j(d) \). It means that the two agents obtain equal amounts of their respective favorite objects at each step. Suppose \( i \) envies \( j \) in the found assignment \( p \). Then there exists \( o^* \in O \) such that \( \sum_{o \succeq_i o^*} p_{i,o} < \sum_{o \succeq_i o^*} p_{j,o} \). Let \( d \) be the earliest step after which all objects in \( \{ o \in O : o \succeq_i o^* \} \) are exhausted. Then \( i \)'s favorite objects from step one to step \( d \) must belong to \( \{ o \in O : o \succeq_i o^* \} \), and \( \sum_{o \succeq_i o^*} p_{i,o} \) is the total amount of objects obtained by \( i \) at the end of step \( d \). So we should have \( \sum_{o \succeq_i o^*} p_{j,o} \leq \sum_{d'=1}^d x_j(d') = \sum_{o \succeq_i o^*} p_{i,o} \). This is a contradiction.

(3) Let \( p \) be the assignment found by any BTA satisfying bounded advantage. For any distinct \( i, j \in I \), if \( \max_{o \in O} \left[ \sum_{o \succeq_i o^*} p_{j,o} - \sum_{o \succeq_i o^*} p_{i,o} \right] \leq 0 \), then \( i \) does not envy \( j \) and there is nothing to prove. So suppose that \( i \) envies \( j \), and let \( o^* \) be the solution to \( \max_{o \in O} \left[ \sum_{o \succeq_i o^*} p_{j,o} - \sum_{o \succeq_i o^*} p_{i,o} \right] \). Let \( d \) be the earliest step after which all \( o \succeq_i o^* \) are exhausted. That is, \( \{ o \in O : o \succeq_i o^* \} \cap O(d+1) = \emptyset \) and \( \{ o \in O : o \succeq_i o^* \} \cap O(d) \neq \emptyset \). Then,

\[
\sum_{o \succeq_i o^*} p_{i,o} \leq \sum_{o \in O} \left( \omega_{i,o} - \omega_{i,o}(d+1) \right) \quad \text{and} \quad \sum_{o \succeq_i o^*} p_{j,o} \leq \sum_{o \in O} \left( \omega_{j,o} - \omega_{j,o}(d+1) \right).
\]

For every \( o \in O \) such that \( \omega_{i,o} \geq \omega_{j,o} \), bounded advantage implies that for all \( 1 \leq d' \leq d \), \( \omega_{i,o}(d'+1) \geq \omega_{j,o}(d'+1) \) and \( \lambda_{i,o}(d') \geq \lambda_{j,o}(d') \). So

\[
\omega_{j,o}(d+1) = \sum_{d'=1}^d \lambda_{j,o}(d') x_o(d') \leq \sum_{d'=1}^d \lambda_{i,o}(d') x_o(d') = \omega_{i,o} - \omega_{i,o}(d+1).
\]

Equivalently,

\[
\left( \omega_{j,o}(d+1) \right) - \left( \omega_{i,o} - \omega_{j,o}(d+1) \right) \leq 0.
\]

For every \( o \in O \) such that \( \omega_{i,o} < \omega_{j,o} \), bounded advantage implies that for all \( 1 \leq d' \leq d \),
\[ \omega_i(o(d' + 1)) \leq \omega_j(o(d' + 1)) \text{ and } \lambda_i(o(d') \leq \lambda_j(o(d')). \text{ In particular, } \omega_i(o(d + 1)) \leq \omega_j(o(d + 1)). \]

So
\[
\omega_j(o(d + 1)) - \omega_i(o(d + 1)) \leq \omega_j(o) - \omega_i(o).
\]

Therefore,
\[
\sum_{o \in O} \omega_j(o) \leq \sum_{o \in O} \omega_i(o).
\]

So \( p \) satisfies bounded envy. \( \square \)

**Proof of Proposition 3.** For any house allocation problem, we run a BTA and suppose that the BTA has \( n \) steps. At every step \( d \in \{1, 2, \ldots, n\} \), every \( i \) reports the favorite object \( o_i(d) \) and obtains the amount \( p_i(o(d + 1)) - p_i(o(d)) \) of \( o_i(d) \). We arbitrarily choose \( n \) rational numbers \( t_1, t_2, \ldots, t_n \in [0, 1] \) such that \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1 \). We will describe the procedure of the BTA as a SEA procedure in which step \( d \) starts from \( t_{d-1} \) to \( t_d \). For every \( i \in I \), arbitrarily choose an eating rate function \( s_i : [0, 1] \rightarrow \mathbb{R}_+ \) such that for every \( d \in \{1, 2, \ldots, n\} \), \( \int_{t_{d-1}}^{t_d} s_i(t) = p_i(o(d + 1)) - p_i(o(d)) \). By choosing these rate functions, it is clear that at any \( t \in [t_{d-1}, t_d] \), \( i \) will consume \( o_i(d) \). So the SEA procedure will coincide with the BTA procedure. Because we endow agents with equal divisions of objects and every BTA is IR, the SEA procedure satisfies equal-division lower bound.

If a BTA satisfies stepwise EEET, then in any house allocation problem, agents must obtain equal amounts of their respective favorite objects at every step. So the BTA must coincide with PS. \( \square \)

**Proof of Proposition 4.** Let \( p \) be any assignment found by EAE. Because the generated endowment \( \omega \) is feasible, and \( \sum_{i \in I} p_i(o) = \sum_{i \in I} \omega_i(o) \) for every \( o \in O \), \( p \) is also feasible. Because all doctors have equal endowments in \( \omega \) and we use Equal-BTA, \( p \) is envy-free for doctors. Consider any feasible assignment \( p' \) such that \( \sum_{i \in I} p_i(o) \geq \sum_{i \in I} p_i(o) \) for all \( o \in O \). Because \( \omega \) is Pareto efficient for hospitals, \( p \) is also Pareto efficient for hospitals. There-
fore, we must have $\sum_{i \in I} p'_{i,o} = \sum_{i \in I} p_{i,o}$ for all $o \in O$. So $\sum_{i \in I} p'_{i,o} = \sum_{i \in I} \omega_{i,o}$ for all $o \in O$. It means that $p'$ can be regarded as an assignment in the FEE problem in which $\omega$ is doctors' endowments. But we know that $p$ is ordinally efficient for doctors in such a problem. So $p$ is two-sided efficient.

**Proof of Proposition 5.** Ordinal efficiency can be proved as in Proposition 1. To prove no envy towards lower priority, it is sufficient to note that if an agent $i$ has weakly higher priority than another agent $j$ for every object, then at every step of PTA, $i$ obtains weakly more amount of his favorite object than $j$ does. So $i$ does not envy $j$.

**Proof of Proposition 6.** We apply PTA to the HET model by regarding it as a priority-based allocation problem in which every $o \in O_E$ gives its owner highest priority and the others equal priority, and every $o \in O \setminus O_E$ gives all agents equal priority. Let $I(d)$, $O(d)$, and $o_i(d)$ be defined as in PTA. Among $O(d)$, let $O_E(d) = \{o \in O_E : o_i \in I(d)\}$ be the set of private endowments whose owners have not been removed at the beginning of step $d$. Then at each step $d$ of PTA, we find the maximum solution $x^*(d)$ to the equations

$$
\begin{aligned}
\begin{cases}
x_o(d) = \sum_{i \in I(d)} 1\{o_i(d) = o\} \cdot x_i(d) & \text{for all } o \in O(d), \\
x_i(d) = \frac{x_o(d)}{|I(d)|} & \text{for all } i \in I(d) \setminus I_E, \\
x_j(d) = \frac{x_o(d)}{|I(d)|} + x_{o_j}(d) & \text{for all } j \in I(d) \cap I_E.
\end{cases}
\end{aligned}
$$

subject to the constraints

$$
\begin{aligned}
\begin{cases}
x_o(d) \leq 1 - \sum_{k=1}^{d-1} x^*_o(k) & \text{for all } o \in O(d), \\
x_i(d) \leq 1 - \sum_{k=1}^{d-1} x^*_o(k) & \text{for all } i \in I(d).
\end{cases}
\end{aligned}
$$

To prove that ETA is the application of PTA to the HET model, we prove that agents' consumption at each step $d$ of ETA is exactly the maximum solution $x^*(d)$ to the above equations.

At each step $d$ of ETA, if there are cycles among existing tenants, for each existing tenant $j$ involved in a cycle, it is obvious that $x_j(d) = x_{o_j}(d)$, whereas for each agent $i$ and each object $o$ not involved in any cycle, $x_i(d) = x_o(d) = 0$. So $x(d)$ satisfies the above equations and constraints. If there are no cycles, let $t(d)$ be the duration of step $d$. Then for every $o \in O(d)$, $x_o(d) = \sum_{i \in I(d)} 1\{o_i(d) = o\} \cdot x_i(d)$, which holds by definition. For every $i \in I(d) \setminus I_E$, $x_i(d) = s_i(d)t(d) = t(d)$, whereas for every $j \in I(d) \cap I_E$, $x_j(d) = s_j(d)t(d) =$
\[ \sum_{i \in I(d)} 1 \{ o_i(d) = o_j \} \cdot s_i(d) t(d) + t(d) = \sum_{i \in I(d)} 1 \{ o_i(d) = o_j \} \cdot x_i(d) + t(d) = x_{o_j}(d) + t(d). \]

We say an agent \( i \) is linked to an object \( o \) if there exist distinct existing tenants \( j_1, j_2, \ldots, j_\ell \) such that \( i \) points to \( o_{j_1} \), \( j_1 \) points to \( o_{j_2} \), \( j_2 \) points to \( o_{j_3} \), \ldots, \( j_\ell \) points to \( o \). Because there are no cycles, every remaining agent must be linked to some object in \( O(d) \setminus O_E(d) \). Then the “you request my house - I get your rate” rule implies that the total rates of the agents who consume \( O(d) \setminus O_E(d) \) is equal to the number of remaining agents. So \( t(d) = \sum_{o \in O(d) \setminus O_E(d)} x_o(d) / |I(d)| \). It means that agents’ consumption at step \( d \) of ETA satisfies the above equations and constraints. Because at each step of ETA agents trade endowments or consume objects as much as possible until some agent is satisfied or some object is exhausted, agents’ consumption at step \( d \) of ETA coincides with the maximum solution \( x^*(d) \) to the above equations.

The ordinal efficiency and no envy towards newcomers of ETA are implications of Proposition 5. ETA is individually rational because every private endowment is never removed before its owner being removed so that the owner never obtains a positive amount of an object that is worse than his private endowment.

\( \square \)

### C Asymptotic strategy-proofness of BTA

In the FEE model, Athanassoglou and Sethuraman (2011) (and Aziz (2018)) prove that any IR and ordinally efficient algorithm is not strategy-proof (even in the weak sense). So every BTA is not strategy-proof. In a finite market, if an agent’s endowments are scarce or his preferences are very different from the others, he may have the power to manipulate the algorithm. But if many agents have identical endowments and preferences, the chance of manipulation could be small and vanish as the market size grows. This motivates us to evaluate the incentive property of BTA in large markets. In a large market, if any agent misreports preferences, the distribution of agent types in the population will change a little bit. If a BTA is insensitive to such a small change, which we will formalize as a continuity property, then the assignment found by the BTA will change a little bit. If the BTA further satisfies EENE, then the benefit from manipulation will be approximately zero as the market grows.

Formally, in any FEE market \( M = (I, O, \succsim_I, \omega) \), we define \( \Omega_M = \{ \hat{\omega} : \exists i \in I, \hat{\omega} = \omega_i \} \) to be the set of endowment types. We say an agent is type-\((\hat{\omega}, \succsim)\) if his endowment vector is \( \hat{\omega} \) and his preference relation is \( \succsim \). Let \( \mathcal{P} \) be the set of all preference relations. For every
We are concerned with large but finite markets. As a market grows, we assume that the number of agents increases but the set of object types $O$ and the set of endowment types are fixed. Formally, we say a sequence of markets $(M[n])_n^{\infty}$, where $M[n] = (O[n], I[n], \geq_l[n], \omega[n])$, is regular if

1. For all $n \geq 2$, $|I[n]| = n|I[1]|$, $O[n] = O[1]$, and $\Omega_{M[n]} = \Omega_{M[1]}$;
2. For all $(\hat{\omega}, \geq) \in \Omega_{M[1]} \times P$, $\exists A^{[\infty]}(\hat{\omega}, \geq) \in (0, 1)$ such that $\lim_{n \to \infty}A^{[n]}(\hat{\omega}, \geq) = A^{[\infty]}(\hat{\omega}, \geq)$.

Condition (2) says that as the market grows, for every $(\hat{\omega}, \geq) \in \Omega_{M[1]} \times P$, the proportion of type-$(\hat{\omega}, \geq)$ agents converges to a positive fraction $A^{[\infty]}(\hat{\omega}, \geq)$. We say two regular sequences $(M[n])_n^{\infty}$ and $(\tilde{M}[n])_n^{\infty}$ are asymptotically equivalent if $O[1] = \tilde{O}[1]$, $\Omega_{M[1]} = \Omega_{\tilde{M}[1]}$, and, for all $(\hat{\omega}, \geq) \in \Omega_{M[1]} \times P$, $A^{[\infty]}(\hat{\omega}, \geq) = \tilde{A}^{[\infty]}(\hat{\omega}, \geq)$.

For any BTA satisfying ETE and any regular $(M[n])_n^{\infty}$, let $p^{[n]}$ denote the found assignment for each $M[n]$, and for every $(\hat{\omega}, \geq)$, let $p^{[n]}_{(\hat{\omega}, \geq)}$ denote the lottery assigned to type-$(\hat{\omega}, \geq)$ agents in $M[n]$. Then we say a BTA satisfying ETE is continuous if, for any two regular sequences $(M[n])_n^{\infty}$ and $(\tilde{M}[n])_n^{\infty}$ that are asymptotically equivalent and for any $(\hat{\omega}, \geq) \in \Omega_{M[1]} \times P$, $\lim_{n \to \infty} p^{[n]}_{(\hat{\omega}, \geq)} = \lim_{n \to \infty} \tilde{p}^{[n]}_{(\hat{\omega}, \geq)}$.

Finally, we say a mechanism $\varphi$ is asymptotically strategy-proof if, for any regular $(M[n])_n^{\infty}$ and any $\varepsilon > 0$, there exists $n^* \in \mathbb{N}$ such that, for any $n > n^*$ and any $i \in I[n]$ in $M[n]$, $i$’s lottery obtained from truth-telling stochastically dominates his lottery obtained from misreporting any $\geq'_{i} \in P \setminus \{\geq_i\}$, with an error bounded by $\varepsilon$:

$$\sum_{\geq'_{i} \in P \setminus \{\geq_i\}} \varphi_{i, o}(\geq'_{i}) \geq \sum_{\geq'_{i} \in P \setminus \{\geq_i\}} \varphi_{i, o}(\geq_{i}) - \varepsilon \quad \text{for all } o \in O.$$

Our main result in this section is as follows.

**Proposition 8.** Any continuous BTA that satisfies EENE is asymptotically strategy-proof.

Equal-BTA and Proportional-BTA are examples of continuous BTA that satisfy EENE.

**Proof of Proposition 8.** Given any regular sequence of problems $(M[n])_n^{\infty}$, there exists $N > 0$ such that for all $n \geq N$ and all $(\hat{\omega}, \geq) \in \Omega_{M[1]} \times P$, the number of type-$(\hat{\omega}, \geq)$ agents
in \( M[n] \) are more than one. Suppose for some \((\hat{\omega}, \hat{\varepsilon}) \in \Omega_{M[1]} \times \mathcal{P} \), some type-(\hat{\omega}, \hat{\varepsilon}) agent \( i \) in the market \( M[n] \) reports \( \hat{\varepsilon}_i \in \mathcal{P} \setminus \{\varepsilon_i\} \). Then we obtain a new market \( \tilde{M}[n] \) such that

\[
|\{i \in \tilde{I}[n] : \hat{\omega}_i \equiv \hat{\omega} \land \hat{\varepsilon}_i \equiv \hat{\varepsilon}\}| = |\{i \in I[n] : \omega_i \equiv \hat{\omega} \land \varepsilon_i \equiv \hat{\varepsilon}\}| - 1,
\]

and the numbers of agents of other types in \( \tilde{M}[n] \) and \( M[n] \) are equal. Then \( (M[n])_{n \geq N} \) and \( (\tilde{M}[n])_{n \geq N} \) are two regular sequences that are asymptotically equivalent.

Let \( (p[n])_{n \geq N} \) and \( (\tilde{p}[n])_{n \geq N} \) be the sequences of assignments found by a continuous BTA satisfying EENE for the two sequences of markets. By continuity,

\[
\lim_{n \to \infty} p[n] = \lim_{n \to \infty} \tilde{p}[n].
\]

So for any \( \varepsilon > 0 \), there exists \( n^*(\hat{\omega}, \hat{\varepsilon}, \hat{\varepsilon}') > N \) such that, for all \( n > n^*(\hat{\omega}, \hat{\varepsilon}, \hat{\varepsilon}') \),

\[
\max_{o \in \mathcal{O}} \left| \sum_{o' \geq o} p[n]_{(\hat{\omega}, \hat{\varepsilon}), o'} - \sum_{o' \geq o} \tilde{p}[n]_{(\hat{\omega}, \hat{\varepsilon}'), o'} \right| < \varepsilon.
\]

Since the BTA satisfies EENE, \( p[n]_{(\hat{\omega}, \hat{\varepsilon})} \equiv_{sd} p[n]_{(\hat{\omega}, \hat{\varepsilon}')}. \) That is, for all \( o \in \mathcal{O} \),

\[
\sum_{o' \geq o} p[n]_{(\hat{\omega}, \hat{\varepsilon}), o'} \geq \sum_{o' \geq o} p[n]_{(\hat{\omega}, \hat{\varepsilon}'), o'}.
\]

Thus, for all \( o \in \mathcal{O} \),

\[
\sum_{o' \geq o} p[n]_{(\hat{\omega}, \hat{\varepsilon}), o'} \geq \sum_{o' \geq o} \tilde{p}[n]_{(\hat{\omega}, \hat{\varepsilon}), o'} - \varepsilon.
\]

Define \( n^* = \max\{n^*(\hat{\omega}, \hat{\varepsilon}, \hat{\varepsilon}') : (\hat{\omega}, \hat{\varepsilon}, \hat{\varepsilon}') \in \Omega_{M[1]} \times \mathcal{P} \times \mathcal{P} \land \hat{\varepsilon} \neq \hat{\varepsilon}'\} \). For all \( n > n^* \) and all \( i \in I[n] \) in \( M[n] \), \( i \)'s lottery obtained from truth-telling stochastically dominates his lottery obtained from misreporting any \( \varepsilon_i' \in \mathcal{P} \setminus \{\varepsilon_i\} \), with an error bounded by \( \varepsilon \). \( \Box \)