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Perturbing Eisenstein polynomials over local fields

par Kevin KEATING

Abstract. Let $K$ be a local field whose residue field has characteristic $p$ and let $L/K$ be a finite separable totally ramified extension. Let $\pi_L$ be a uniformizer for $L$ such that $\tilde{\pi}_L \equiv \pi_L + r\pi_L^{\ell+1} \pmod{\pi_L^{\ell+2}}$ for some $\ell \geq 1$ and $r \in \mathcal{O}_K$. Let $f(X)$ be the minimum polynomial of $\pi_L$ over $K$. In this paper we give congruences for the coefficients of $\tilde{f}(X)$ in terms of $\ell$, $r$, and the coefficients of $f(X)$. These congruences improve work of Krasner [8].

1. Introduction

Let $K$ be a field which is complete with respect to a discrete valuation $v_K$. Let $\mathcal{O}_K$ be the ring of integers of $K$ and let $\mathcal{P}_K$ be the maximal ideal of $\mathcal{O}_K$. Assume that the residue field $\bar{K} = \mathcal{O}_K/\mathcal{P}_K$ of $K$ is a perfect field of characteristic $p$. Let $K^{sep}$ be a separable closure of $K$ and let $L/K$ be a finite totally ramified subextension of $K^{sep}/K$. Let $\pi_L$ be a uniformizer for $L$ and let

\[ f(X) = X^n - c_1X^{n-1} + \cdots + (-1)^{n-1}c_{n-1}X + (-1)^nc_n \]

be the minimum polynomial of $\pi_L$ over $K$. Let $\ell \geq 1$, let $r \in \mathcal{O}_K$, and let $\tilde{\pi}_L$ be another uniformizer for $L$ such that $\tilde{\pi}_L \equiv \pi_L + r\pi_L^{\ell+1} \pmod{\mathcal{P}_L^{\ell+2}}$. Let

\[ \tilde{f}(X) = X^n - \tilde{c}_1X^{n-1} + \cdots + (-1)^{n-1}\tilde{c}_{n-1}X + (-1)^n\tilde{c}_n \]

be the minimum polynomial of $\tilde{\pi}_L$ over $K$. In this paper we use the techniques developed in [7] to obtain congruences for the coefficients of $\tilde{f}(X)$ in terms of $\ell$, $r$, and the coefficients of $f(X)$. 

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The Hasse–Herbrand function \( \varphi_{L/K} : [-1, \infty) \to [-1, \infty) \) of \( L/K \) is defined in Chapter IV of [10] for finite Galois extensions, and in the appendix to [1] for finite separable extensions. Krasner [8, p. 157] showed that for \( 1 \leq h \leq n \) we have \( \tilde{c}_h \equiv c_h \pmod{\mathcal{P}^{\rho_h(\ell)}_K} \), where \( \kappa_h(\ell) = [\varphi_{L/K}(\ell) + \frac{h}{n}] \). In Theorem 4.3 we prove that \( \tilde{c}_h \equiv c_h \pmod{\mathcal{P}^{\rho_h(\ell)}_K} \) for certain integers \( \rho_h(\ell) \) such that \( \rho_h(\ell) \geq \kappa_h(\ell) \). Let \( h \) be the unique integer such that \( 1 \leq h \leq n \) and \( n \) divides \( n\varphi_{L/K}(\ell) + h \). Krasner [8, p. 157] gave a formula for the congruence class modulo \( \mathcal{P}^{\rho_h(\ell)}_K \) of \( \tilde{c}_h - c_h \). In Theorem 4.5 we give similar formulas for up to \( v_p(n) + 1 \) values of \( h \).

Heiermann [4] gave formulas which are analogous to the results presented here. Let \( S \subset \mathcal{O}_K \) be the set of Teichmüller representatives for \( K \). Let \( \pi_K \) be a uniformizer for \( K \) and let \( \mathcal{F}(X) \) be the unique power series with coefficients in \( S \) such that \( \pi_K = \pi_K^r \mathcal{F}(\pi_L) \). Suppose \( \tilde{\pi}_L \) is another uniformizer for \( L \) such that \( \pi_L \equiv \pi_L + r \tilde{\pi}_L^{\ell+2} \pmod{\mathcal{P}_L^{\ell+2}} \) for some \( \ell \geq 1 \) and \( r \in S \). Let \( \tilde{\mathcal{F}} \) be the series with coefficients in \( S \) such that \( \pi_K = \tilde{\pi}_L^{r} \tilde{\mathcal{F}}(\tilde{\pi}_L) \). Using Theorem 4.6 of [4] one can compute some coefficients of \( \tilde{\mathcal{F}} \) in terms of \( r \) and the coefficients of \( \mathcal{F} \).

In Section 2 we recall some facts about symmetric polynomials from [7]. The main focus is on expressing monomial symmetric polynomials in terms of elementary symmetric polynomials. In Section 3 we define the indices of inseparability of \( L/K \) and some generalizations of the Hasse–Herbrand function \( \varphi_{L/K} \). In Section 4 we prove our main results. In Section 5 we give some examples which illustrate how the theorems from Section 4 are applied.

2. Symmetric polynomials and cycle digraphs

Let \( n \geq 1 \), let \( w \geq 1 \), and let \( \mu \) be a partition of \( w \). We view \( \mu \) as a multiset of positive integers such that the sum of the elements of \( \mu \) is equal to \( w \). The number of parts of \( \mu \) is called the length of \( \mu \), and is denoted by \( |\mu| \). For \( \mu \) such that \( |\mu| \leq n \) we let \( m_\mu(X_1, \ldots, X_n) \) be the monomial symmetric polynomial in \( n \) variables associated to \( \mu \); see [11, Section 7.3] for the definition and general facts about monomial symmetric polynomials. For \( 1 \leq h \leq n \) let \( e_h(X_1, \ldots, X_n) \) denote the elementary symmetric polynomial of degree \( h \) in \( n \) variables. By the fundamental theorem of symmetric polynomials there is a unique polynomial \( \psi_\mu \in \mathbb{Z}[X_1, \ldots, X_n] \) such that \( m_\mu = \psi_\mu(e_1, \ldots, e_n) \). In this section we use a theorem of Kulikauskas and Remmel [9] to compute certain coefficients of \( \psi_\mu \).

The formula of Kulikauskas and Remmel can be expressed in terms of tilings of a certain type of digraph. We say that a directed graph \( \Gamma \) is a cycle digraph if it is a disjoint union of finitely many directed cycles of length \( \geq 1 \). We denote the vertex set of \( \Gamma \) by \( V(\Gamma) \), and we define the sign of \( \Gamma \) to
be $\text{sgn}(\Gamma) = (-1)^w c$, where $w = |V(\Gamma)|$ and $c$ is the number of cycles that make up $\Gamma$.

Let $\Gamma$ be a cycle digraph with $w \geq 1$ vertices and let $\lambda$ be a partition of $w$. A $\lambda$-tiling of $\Gamma$ is a set $S$ of subgraphs of $\Gamma$ such that

1. Each $\gamma \in S$ is a directed path of length $\geq 0$.
2. The collection $\{V(\gamma) : \gamma \in S\}$ forms a partition of the set $V(\Gamma)$.
3. The multiset $\{|V(\gamma)| : \gamma \in S\}$ is equal to $\lambda$.

Let $\mu$ be another partition of $w$. A $(\lambda, \mu)$-tiling of $\Gamma$ is an ordered pair $(S, T)$, where $S$ is a $\lambda$-tiling of $\Gamma$ and $T$ is a $\mu$-tiling of $\Gamma$. Let $\Gamma'$ be another cycle digraph with $w$ vertices and let $(S', T')$ be a $(\lambda, \mu)$-tiling of $\Gamma'$. An isomorphism from $(\Gamma, S, T)$ to $(\Gamma', S', T')$ is an isomorphism of digraphs $\theta : \Gamma \to \Gamma'$ which carries $S$ onto $S'$ and $T$ onto $T'$. Say that the $(\lambda, \mu)$-tilings $(S, T)$ and $(S', T')$ of $\Gamma$ are isomorphic if there exists an isomorphism from $(\Gamma, S, T)$ to $(\Gamma', S', T')$. Say that $(S, T)$ is an admissible $(\lambda, \mu)$-tiling of $\Gamma$ if $(\Gamma, S, T)$ has no nontrivial automorphisms. Let $\eta_{\lambda\mu}(\Gamma)$ denote the number of isomorphism classes of admissible $(\lambda, \mu)$-tilings of $\Gamma$.

Let $w \geq 1$ and let $\lambda, \mu$ be partitions of $w$. Set

$$d_{\lambda\mu} = (-1)^{|\lambda|+|\mu|} \sum_{\Gamma} \text{sgn}(\Gamma) \eta_{\lambda\mu}(\Gamma),$$

where the sum is over all isomorphism classes of cycle digraphs $\Gamma$ with $w$ vertices. Since $\eta_{\lambda\mu} = \eta_{\mu\lambda}$ we have $d_{\mu\lambda} = d_{\lambda\mu}$. In Theorem 1(ii) of [9], Kulikauskas and Remmel proved the following:

**Theorem 2.1.** Let $n \geq 1$, let $w \geq 1$, and let $\mu$ be a partition of $w$ whose length is $\leq n$. Let $\psi_{\mu}$ be the unique element of $\mathbb{Z}[X_1, \ldots, X_n]$ such that $m_{\mu} = \psi_{\mu}(e_1, \ldots, e_n)$. Then

$$\psi_{\mu}(X_1, \ldots, X_n) = \sum_{\lambda} d_{\lambda\mu} \cdot X_{\lambda_1} X_{\lambda_2} \ldots X_{\lambda_k},$$

where the sum is over all partitions $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ of $w$ such that $1 \leq \lambda_i \leq n$ for $1 \leq i \leq k$.

We now recall some formulas from [7] for computing values of $\eta_{\lambda\mu}(\Gamma)$.

**Proposition 2.2.** Let $a, b, c, d, w$ be positive integers such that $a \neq c$, $b \neq d$, and let $r, s$ be nonnegative integers. Let $\Gamma$ be a directed cycle of length $w$.

1. Suppose $w = ra = sb + d$. Let $\lambda$ be the partition of $w$ consisting of $r$ copies of $a$, and let $\mu$ be the partition of $w$ consisting of $s$ copies of $b$ and one copy of $d$. Then $\eta_{\lambda\mu}(\Gamma) = a$.

2. Suppose $w = ra + c = sb + d$. Let $\lambda$ be the partition of $w$ consisting of $r$ copies of $a$ and one copy of $c$, and let $\mu$ be the partition of $w$ consisting of $s$ copies of $b$ and one copy of $d$. Then $\eta_{\lambda\mu}(\Gamma) = w$. 
Proof. Statement (1) follows from Proposition 2.5 of [7] if \( s = 0 \), and from Proposition 2.3 of [7] if \( s \geq 1 \). Statement (2) follows from Proposition 2.2 of [7]. □

Using these formulas we can compute \( d_{\lambda \mu} \) in some cases.

**Proposition 2.3.** Let \( a, b, c, d, w \) be positive integers such that \( a \neq c \) and \( b \neq d \). Let \( r, s \) be nonnegative integers such that \( w = ra + c = sb + d \) and \( a > sb \). Let \( \lambda \) be the partition of \( w \) consisting of \( r \) copies of \( a \) and \( 1 \) copy of \( c \), and let \( \mu \) be the partition of \( w \) consisting of \( s \) copies of \( b \) and \( 1 \) copy of \( d \). Then

\[
d_{\lambda \mu} = \begin{cases} (-1)^{r+s+w+1}w & \text{if } b \nmid c \text{ or } sb < c, \\ (-1)^{r+s+w+1}(w - ab) & \text{if } b \mid c \text{ and } sb \geq c. \end{cases}
\]

**Proof.** Let \( \Gamma \) be a cycle digraph which has an admissible \((\lambda, \mu)\)-tiling. Suppose \( \Gamma \) consists of a single cycle of length \( w \). Then by Proposition 2.2(2) we have \( \eta_{\lambda \mu}(\Gamma) = w \). Suppose \( \Gamma \) has more than one cycle. Since \( \Gamma \) has a \( \mu \)-tiling, \( \Gamma \) has a cycle \( \Gamma_1 \) such that \( |V(\Gamma_1)| \leq sb \). Since \( a > sb \) and \( \Gamma \) has a \( \lambda \)-tiling, it follows that \( |V(\Gamma_1)| = c = mb \) for some \( m \) such that \( 1 \leq m \leq s \). Hence if \( \Gamma \) has more than one cycle we must have \( b \mid c \) and \( c \leq sb \). Let \( \lambda_1 \) be the partition of \( c \) consisting of one copy of \( c \) and let \( \mu_1 \) be the partition of \( c \) consisting of \( m \) copies of \( b \). Then every \( \lambda \)-tiling of \( \Gamma \) restricts to a \( \lambda_1 \)-tiling of \( \Gamma_1 \), and every \( \mu \)-tiling of \( \Gamma \) restricts to a \( \mu_1 \)-tiling of \( \Gamma_1 \). It follows from Proposition 2.2(1) that \( \eta_{\lambda_1 \mu_1}(\Gamma_1) = b \).

Let \( \Gamma_2 \) be another cycle of \( \Gamma \). Since \( \Gamma \) has a \( \lambda \)-tiling, \( |V(\Gamma_2)| \geq a > sb \). Hence every \( \mu \)-tiling of \( \Gamma_2 \) which includes a path \( \delta \) with \( |V(\delta)| = d \). Since \( \mu \) has only one part equal to \( d \), it follows that \( \Gamma = \Gamma_1 \cup \Gamma_2 \). Therefore we have \( |V(\Gamma_2)| = ra = (s - m)b + d \). Let \( \lambda_2 \) be the partition of \( ra \) consisting of \( r \) copies of \( a \) and let \( \mu_2 \) be the partition of \( (s - m)b + d = ra \) consisting of \( s - m \) copies of \( b \) and \( 1 \) copy of \( d \). Then every \( \lambda \)-tiling of \( \Gamma \) restricts to a \( \lambda_2 \)-tiling of \( \Gamma_2 \), and every \( \mu \)-tiling of \( \Gamma \) restricts to a \( \mu_2 \)-tiling of \( \Gamma_2 \). It follows from Proposition 2.2(1) that \( \eta_{\lambda_2 \mu_2}(\Gamma_2) = a \). Hence

\[
\eta_{\lambda \mu}(\Gamma) = \eta_{\lambda_1 \mu_1}(\Gamma_1) \cdot \eta_{\lambda_2 \mu_2}(\Gamma_2) = ba.
\]

Suppose \( b \nmid c \) or \( c > sb \). Then it follows from the above that the only cycle digraph which has a \((\lambda, \mu)\)-tiling consists of a single cycle of length \( w \). Hence by (2.1) we get

\[
d_{\lambda \mu} = (-1)^{(r+1)+(s+1)} \cdot (-1)^{w-1}w.
\]

Suppose \( b \mid c \) and \( sb \geq c \). Then \( c = mb \) with \( 1 \leq m \leq s \). Hence there are two cycle digraphs which have a \((\lambda, \mu)\)-tiling: a single cycle of length \( w \), and the union of two cycles with lengths \( c = mb \) and \( ra = (s - m)b + d \).
Therefore by (2.1) we get
\[ d_{\lambda \mu} = (-1)^{(r+1)+(s+1)}((-1)^{w-1}w + (-1)^{w-2}ab). \]
Hence the formula for \( d_{\lambda \mu} \) given in the theorem holds in both cases. \( \square \)

We recall some results from [7] regarding the \( p \)-adic properties of the coefficients \( d_{\lambda \mu} \). Let \( w \geq 1 \) and let \( \lambda \) be a partition of \( w \). For \( k \geq 1 \) let \( k \ast \lambda \) be the partition of \( kw \) which is the multiset sum of \( k \) copies of \( \lambda \), and let \( k \cdot \lambda \) be the partition of \( kw \) obtained by multiplying the parts of \( \lambda \) by \( k \).

**Proposition 2.4.** Let \( t \geq j \geq 0 \), let \( w' \geq 1 \), and set \( w = w'p^t \). Let \( \lambda' \) be a partition of \( w' \) and set \( \lambda = p^t \cdot \lambda' \). Let \( \mu \) be a partition of \( w \) such that there does not exist a partition \( \mu' \) with \( \mu = p^{j+1} \ast \mu' \). Then \( p^{t-j} \) divides \( d_{\lambda \mu} \).

**Proof.** This is proved in Corollary 3.4 of [7]. \( \square \)

**Proposition 2.5.** Let \( w' \geq 1 \), \( j \geq 1 \), and \( t \geq 0 \). Let \( \lambda', \mu' \) be partitions of \( w' \) such that the parts of \( \lambda' \) are all divisible by \( p^j \). Set \( w = w'p^t \), so that \( \lambda = p^t \cdot \lambda' \) and \( \mu = p^t \ast \mu' \) are partitions of \( w \). Then \( d_{\lambda \mu} \equiv d_{\lambda' \mu'} \pmod{p^{t+1}} \).

**Proof.** This is proved in Proposition 3.5 of [7]. \( \square \)

### 3. Indices of inseparability

Let \( L/K \) be a totally ramified extension of degree \( n = up^n \), with \( p \nmid u \). Let \( \pi_L \) be a uniformizer for \( L \) whose minimum polynomial over \( K \) is
\[ f(X) = X^n - c_1X^{n-1} + \cdots + (-1)^{n-1}c_{n-1}X + (-1)^n \]
For \( k \in \mathbb{Z} \) define \( \tau_p(k) = \min\{v_p(k), \nu\} \). For \( 0 \leq j \leq \nu \) set
\begin{align*}
\tilde{i}_j^{\pi_L} &= \min\{nv_K(c_h) - h : 1 \leq h \leq n, \tau_p(h) \leq j\} \\
&= \min\{v_L(c_h\pi_L^{n-h}) : 1 \leq h \leq n, \tau_p(h) \leq j\} - n.
\end{align*}
Then \( \tilde{i}_j^{\pi_L} \) is either a nonnegative integer or \( \infty \); if \( \text{char}(K) = p \) then \( \tilde{i}_j^{\pi_L} \) must be finite, since \( L/K \) is separable. Let \( e_L = v_L(p) \) denote the absolute ramification index of \( L \). We define the \( j \)th index of inseparability of \( L/K \) to be
\[ i_j = \min\{\tilde{i}_j^{\pi_L} + (j' - j)e_L : j \leq j' \leq \nu\}. \]
By Proposition 3.12 and Theorem 7.1 of [4], \( i_j \) does not depend on the choice of \( \pi_L \). Furthermore, our definition of \( i_j \) agrees with Definition 7.3 in [4]; for the characteristic-\( p \) case see also [2, p. 232–233] and [3, Section 2]. Write \( i_j = A_jn - b_j \) with \( 1 \leq b_j \leq n \).

**Remark 3.1.** If \( \tilde{i}_j^{\pi_L} \) is finite we can write \( \tilde{i}_j^{\pi_L} = a_jn - b_j \) with \( a_j \geq 1 \) (see [7, Section 4]). Thus if \( i_j = \tilde{i}_j^{\pi_L} + (j' - j)e_L \) then \( A_j = a_j + (j' - j)e_K \).
The following facts are easy consequences of the definitions:

1. \( 0 = i_0 < i_{\nu-1} \leq \ldots \leq i_1 \leq i_\nu < \infty \).
2. If \( \text{char}(K) = p \) then \( i_j = i_{\nu j}^p \).
3. Let \( m = \bar{\nu}_p(i_j) \). If \( m \leq j \) then \( i_j = i_m = i_{\nu j}^m = i_{\nu j}^m \). If \( m > j \) then \( \text{char}(K) = 0 \) and \( i_j = i_{\nu j}^m + (m - j)e_L \).

Following [4, (4.4)], for \( 0 \leq j \leq \nu \) we define functions \( \bar{\varphi}_j : [0, \infty) \to [0, \infty) \) by \( \bar{\varphi}_j(x) = i_j + p^j x \). The generalized Hasse–Herbrand functions \( \varphi_j : [0, \infty) \to [0, \infty) \) are then defined by

\[ \varphi_j(x) = \min\{\bar{\varphi}_{j_0}(x) : 0 \leq j_0 \leq j\}. \]

It follows that \( \varphi_j(x) \leq \varphi_{j'}(x) \) for \( 0 \leq j' \leq j \). By Corollary 6.11 of [4] we have \( \varphi_\nu(x) = n\varphi_{L/K}(x) \) for all \( x \geq 0 \).

For a partition \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) whose parts satisfy \( 1 \leq \lambda_i \leq n \) define \( c_\lambda = c_{\lambda_1}c_{\lambda_2} \cdots c_{\lambda_k} \). The following is proved in Proposition 4.2 of [7].

**Proposition 3.2.** Let \( w \geq 1 \) and let \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) be a partition of \( w \) whose parts satisfy \( 1 \leq \lambda_i \leq n \). Choose \( q \) to minimize \( \bar{\pi}_p(\lambda_q) \) and set \( t = \bar{\pi}_p(\lambda_q) \). Then \( v_L(c_\lambda) \geq \bar{\pi}_L + w \) if \( v_L(c_\lambda) = \bar{\pi}_L + w \) and \( \bar{\pi}_L < \infty \) then \( \lambda_q = b_t \) and \( \lambda_i = b_\nu = n \) for all \( i \neq q \).

4. **Perturbing \( \pi_L \)**

In this section we prove our main theorems. We begin by applying the results of Section 2 to the totally ramified extension \( L/K \). Write \( [L : K] = n = up^\nu \) with \( p \nmid u \). Let \( \pi_L, \bar{\pi}_L \) be uniformizers for \( L \), with minimum polynomials over \( K \) given by

\[ f(X) = X^n - c_1X^{n-1} + \cdots + (-1)^{n-1}c_{n-1}X + (-1)^nc_n \]
\[ \bar{f}(X) = X^n - \bar{c}_1X^{n-1} + \cdots + (-1)^{n-1}\bar{c}_{n-1}X + (-1)^n\bar{c}_n. \]

Let \( 1 \leq h \leq n \) and set \( j = \bar{\nu}_p(h) \). Define a function \( \rho_h : \mathbb{N} \to \mathbb{N} \) by

\[ \rho_h(\ell) = \left\lfloor \frac{\varphi_j(\ell) + h}{n} \right\rfloor. \]

Let \( \ell \geq 1 \). We say \( \bar{f} \sim_\ell f \) if \( \bar{c}_h \equiv c_h \pmod{\mathcal{P}_K^{\rho_h(\ell)}} \) for \( 1 \leq h \leq n \). Thus \( \sim_\ell \)

is an equivalence relation on the set of minimum polynomials over \( K \) for uniformizers of \( L \).

Let \( \sigma_1, \ldots, \sigma_n \) be the \( K \)-embeddings of \( L \) into \( L^{sep} \). For each partition \( \mu \) of length \( \leq n \) define \( M_\mu : L \to K \) by

\[ M_\mu(\alpha) = m_\mu(\sigma_1(\alpha), \ldots, \sigma_n(\alpha)). \]

For \( 1 \leq h \leq n \) define \( E_h : L \to K \) by

\[ E_h(\alpha) = e_h(\sigma_1(\alpha), \ldots, \sigma_n(\alpha)). \]

Then \( c_h = E_h(\pi_L) \) and \( \bar{c}_h = E_h(\bar{\pi}_L) \).
Proposition 4.1. Let \( \varphi(X) = r_1X + r_2X^2 + \cdots \) be a power series with coefficients in \( \mathcal{O}_K \) such that \( \tilde{\pi}_L = \varphi(\pi_L) \). Then for \( 1 \leq h \leq n \) we have

\[
E_h(\tilde{\pi}_L) = \sum_{\mu} r_{\mu_1}r_{\mu_2} \cdots r_{\mu_h}M_\mu(\pi_L),
\]

where the sum ranges over all partitions \( \mu = \{\mu_1, \ldots, \mu_h\} \) of length \( h \).

Proof. This follows from Theorem 2.1 by setting \( \lambda \) and \( \mu \).

Proposition 4.2. Let \( n \geq 1 \), let \( w \geq 1 \), and let \( \mu \) be a partition of \( w \) whose length is \( \leq n \). Then

\[
M_\mu(\pi_L) = \sum_{\lambda} d_{\lambda\mu}c_\lambda,
\]

where the sum is over all partitions \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) of \( w \) such that \( 1 \leq \lambda_i \leq n \) for \( 1 \leq i \leq k \).

Proof. This follows from Theorem 2.1 by setting \( X_i = E_i(\pi_L) = c_i \).

Let \( \ell \geq 1 \). Our first main result gives congruences between the coefficients of \( f(X) \) and the coefficients of \( \tilde{f}(X) \) under the assumption \( \tilde{\pi}_L \equiv \pi_L \) (mod \( \mathcal{P}_L^{\ell+1} \)).

Theorem 4.3. Let \( \pi_L, \tilde{\pi}_L \) be uniformizers for \( L \) and let \( f(X), \tilde{f}(X) \) be the minimum polynomials for \( \pi_L, \tilde{\pi}_L \) over \( K \). Suppose there are \( \ell \geq 1 \) and \( \sigma \in \text{Aut}_K(L) \) such that \( \sigma(\tilde{\pi}_L) \equiv \pi_L \) (mod \( \mathcal{P}_L^{\ell+1} \)). Then \( \tilde{f} \sim _\ell f \).

Proof. We first show that the theorem holds in the case where \( \tilde{\pi}_L = \pi_L + r\pi_L^{\ell+1} \), with \( r \in \mathcal{O}_K \). Let \( 1 \leq h \leq n \) and set \( j = \tilde{\pi}_p(h) \). For \( 0 \leq s \leq h \) let \( \mu_s \) be the partition of \( \ell s + h \) consisting of \( h - s \) copies of 1 and \( s \) copies of \( \ell + 1 \). Then by Proposition 4.1 we have

\[
(4.1) \quad \tilde{c}_h = E_h(\pi_L) = \sum_{s=0}^{h} M_{\mu_s}(\pi_L)r^s = c_h + \sum_{s=1}^{h} M_{\mu_s}(\pi_L)r^s.
\]

To prove that \( \tilde{c}_h \equiv c_h \) (mod \( \mathcal{P}_K^{\rho_\ell(\ell)} \)) it suffices to show that \( v_K(M_{\mu_s}(\pi_L)) \geq \rho_\ell(\ell) \) for \( 1 \leq s \leq h \). Therefore by Proposition 4.2 it suffices to show \( v_L(d_{\lambda\mu_s}c_\lambda) \geq \varphi_j(\ell) + h \) for all \( 1 \leq s \leq h \) and all partitions \( \lambda \) of \( \ell s + h \) whose parts are at most \( n \).

Let \( 1 \leq s \leq h \) and set \( m = \min\{j, v_p(s)\} \). Then \( m \leq j \) and \( s \geq p^m \). Let \( \lambda = \{\lambda_1, \ldots, \lambda_k\} \) be a partition of \( \ell s + h \) such that \( 1 \leq \lambda_i \leq n \) for \( 1 \leq i \leq k \). Choose \( q \) to minimize \( v_p(\lambda_q) \) and set \( t = v_p(\lambda_q) \). By Proposition 3.2 we have \( v_L(c_\lambda) \geq i_t^L + \ell s + h \). Suppose \( m < t \). Then \( m < \nu \), so we have \( p^{m+1} \mid \gcd(h - s, s) \). Hence by Proposition 2.4 we get \( v_p(d_{\lambda\mu_s}) \geq t - m \).
Thus
\[ v_L(d_{\lambda \mu_s}c_\lambda) = v_L(d_{\lambda \mu_s}) + v_L(c_\lambda) \]
\[ \geq (t - m)v_L(p) + i_t^\pi + \ell s + h \]
\[ \geq i_m + \ell p^m + h. \]

Suppose \( m \geq t \). Then
\[ v_L(d_{\lambda \mu_s}c_\lambda) \geq v_L(c_\lambda) \]
\[ \geq i_t^\pi + \ell s + h \]
\[ \geq i_t + \ell p^m + h \]
\[ \geq i_m + \ell p^m + h. \]

In both cases we get
\[ v_L(d_{\lambda \mu_s}c_\lambda) \geq \tilde{\varphi}_m(\ell) + h \geq \varphi_j(\ell) + h, \]
and hence \( \tilde{c}_h \equiv c_h \pmod{\mathcal{P}_K^{\rho_\ell}} \). Since this holds for \( 1 \leq h \leq n \) we get \( \tilde{f} \sim \ell f \).

We now prove the general case. Since \( \tilde{f} \) is the minimum polynomial of \( \sigma(\tilde{\pi}_L) \) over \( K \) we may assume without loss of generality that \( \tilde{\pi}_L \equiv \pi_L \pmod{\mathcal{P}_L^{\ell+1}} \). By repeated application of the special case above we get a sequence \( \pi_L^{(0)} = \pi_L, \pi_L^{(1)}, \pi_L^{(2)}, \ldots \) of uniformizers for \( L \) with minimum polynomials \( f^{(0)} = f, f^{(1)}, f^{(2)}, \ldots \) such that for all \( i \geq 0 \) we have \( \pi_L^{(i)} \equiv \tilde{\pi}_L \pmod{\mathcal{P}_L^{\ell+i+1}} \) and \( f^{(i+1)} \sim_{\ell+i} f^{(i)} \). It follows that \( f^{(i+1)} \sim_{\ell} f^{(i)} \), and hence that \( f^{(i)} \sim_{\ell} f \) for all \( i \geq 0 \). Since the sequence \((f^{(i)})\) converges coefficientwise to \( \tilde{f} \) it follows that \( \tilde{f} \sim_{\ell} f \). \( \square \)

Recall that the Hasse–Herbrand function \( \varphi_{L/K} : [-1, \infty) \to [-1, \infty) \) is defined for arbitrary finite separable extensions \( L/K \) (see for instance the appendix to [1]). We say that \( b \geq 0 \) is a lower ramification break of \( L/K \) if \( \varphi'_{L/K}(b) \) is undefined. This extends the usual definition of lower ramification breaks for Galois extensions.

**Remark 4.4.** It follows from Theorem 4.3 that if \( \sigma(\tilde{\pi}_L) \equiv \pi_L \pmod{\mathcal{P}_L^{\ell+1}} \) for some \( \sigma \in \text{Aut}_K(L) \) then \( \tilde{c}_h \equiv c_h \pmod{\mathcal{P}_K^{\rho_\ell}} \) for \( 1 \leq h \leq n \). Define functions \( \kappa_h : \mathbb{N} \to \mathbb{N} \) by
\[ \kappa_h(\ell) = \left\lceil \frac{\varphi_\nu(\ell) + h}{n} \right\rceil. \]
Krasner [8, p. 157] showed that \( \tilde{c}_h \equiv c_h \pmod{\mathcal{P}_K^{\kappa_\ell}} \). Since \( \kappa_h(\ell) \leq \rho_\ell(\ell) \) Krasner’s congruences are in general weaker than the congruences that follow from Theorem 4.3. If \( \ell \) is greater than or equal to the largest lower ramification break of \( L/K \) then for \( 0 \leq j \leq \nu \) we have \( \varphi_j(\ell) = \varphi_\nu(\ell) \), and
hence $\kappa_h(\ell) = \rho_h(\ell)$. Therefore Theorem 4.3 does not improve on Krasner’s results in these cases.

For certain values of $h$ we get a more refined version of the congruences given by Theorem 4.3.

**Theorem 4.5.** Let $L/K$ be a finite totally ramified extension of degree $n = up^\nu$. For $0 \leq m \leq \nu$ write the $m$th index of inseparability of $L/K$ in the form $i_m = A_m n - b_m$ with $1 \leq b_m \leq n$. Let $\pi_L, \tilde{\pi}_L$ be uniformizers for $L$ such that there are $\ell \geq 1$, $r \in \mathcal{O}_K$, and $\sigma \in \text{Aut}_K(L)$ with $\sigma(\tilde{\pi}_L) \equiv \pi_L + r\pi_L^{\ell+1} \pmod{\mathcal{P}_L^{\ell+2}}$. Let $0 \leq j \leq \nu$ satisfy $v_p(\varphi_j(\ell)) = j$, and let $h$ be the unique integer such that $1 \leq h \leq n$ and $n$ divides $\varphi_j(\ell) + h$. Set $k = \frac{\varphi_j(\ell) + h}{n}$ and $h_0 = h/p^j$. Then

$$\bar{c}_h \equiv c_h + \sum_{m \in S_j} g_m c_n^{k-A_m c_b_m} r^{p^m} \pmod{\mathcal{P}_K^{k+1}},$$

where

$$S_j = \{ m : 0 \leq m \leq j, \varphi_j(\ell) = \tilde{\varphi}_m(\ell) \}$$

$$g_m = \begin{cases} (-1)^{k+\ell+A_m} (h_0 p^j - m + \ell - up^{\nu-m}) & \text{if } b_m < h \\
(-1)^{k+\ell+A_m} (h_0 p^j + \ell) & \text{if } h \leq b_m < n \\
(-1)^{k+\ell+A_m} up^{\nu-m} & \text{if } b_m = n. \end{cases}$$

**Proof.** We first prove that the theorem holds for $\tilde{\pi}_L = \pi_L + r\pi_L^{\ell+1}$. Let

$$\hat{f}(X) = X^n - \hat{c}_1 X^{n-1} + \cdots + (-1)^{n-1} \hat{c}_{n-1} X + (-1)^n \hat{c}_n$$

be the minimum polynomial for $\tilde{\pi}_L$ over $K$. Let $1 \leq s \leq h$ and let $\lambda$ be a partition of $\ell s + h$ whose parts are at most $n$. Choose $q$ to minimize $\pi_p(\lambda_q)$ and set $t = \varphi_p(\lambda_t)$. Recall that $\mu_s$ is the partition of $\ell s + h$ consisting of $h-s$ copies of $1$ and $s$ copies of $\ell + 1$. Since $\pi_p(h) = \pi_p(\varphi_j(\ell)) = j$ it follows from the proof of Theorem 4.3 that $v_K(d_{\lambda \mu_s \lambda}(\varphi_j(\ell))) \geq k$. Suppose $v_K(d_{\lambda \mu_s \lambda}(\varphi_j(\ell))) = k$. Then the inequalities in the proof of Theorem 4.3 must be equalities. Hence there is $0 \leq m \leq j$ such that $s = p^m$, $v_L(c_\lambda) = t \pi_L + t p^m + h$, and $\varphi_j(\ell) = \tilde{\varphi}_m(\ell)$.

It follows that $m \in S_j$ and $\lambda$ is a partition of $w_m$, where

$$w_m = \ell p^m + h = \tilde{\varphi}_m(\ell) - i_m + h = \varphi_j(\ell) + h - i_m = (k - A_m)n + b_m.$$ 

Let $\kappa_m$ be the partition of $w_m$ consisting of $k - A_m$ copies of $n$ and 1 copy of $b_m$. By Proposition 3.2 we see that $\lambda$ has at most one element not equal to $n$. Therefore $\lambda = \kappa_m$. Hence $c_\lambda = c_{\kappa_m} = c_n^{k-A_m c_b_m}$ and $v_p(b_m) = v_p(\lambda_q) = t$.

Using equation (4.1) and Proposition 4.2 we get

$$\bar{c}_h \equiv c_h + \sum_{m \in S_j} d_{\kappa_m}^{p^m} c_n^{k-A_m c_b_m} r^{p^m} \pmod{\mathcal{P}_K^{k+1}}.$$
Let $m \in S_j$. Since
$$j = \overline{\nu}_p(\varphi_j(\ell)) = \overline{\nu}_p(\varphi_m(\ell)) = \overline{\nu}_p(i_m + \ell p^m)$$
and $m \leq j$ we get $m \leq \overline{\nu}_p(i_m) = \overline{\nu}_p(b_m)$. Hence $b_m' = b_m/p^m$ is an integer. Let $\kappa'_m$ be the partition of
$$w'_m = (k - A_m)u_p\nu^{-m} + b'_m = h_0p^{j-m} + \ell$$
consisting of $k - A_m$ copies of $u_p\nu^{-m}$ and 1 copy of $b'_m$. Let $\mu'_{p^m}$ be the partition of $w'_m$ consisting of $h_0p^{j-m} - 1$ copies of 1 and 1 copy of $\ell + 1$. Since $h \leq n$ we have $u_p\nu^{-m} > h_0p^{j-m} - 1$. Hence if $b'_m \neq u_p\nu^{-m}$ then we can compute $d_{\kappa'_m,\mu'_{p^m}}$ using Proposition 2.3.

Suppose $b_m < h$. Then $h_0p^{j-m} - 1 \geq b'_m$, so by Proposition 2.3 we get
$$d_{\kappa'_m,\mu'_{p^m}} = (-1)^{k+\ell+A_m}(h_0p^{j-m} + \ell - u_p\nu^{-m}).$$
Suppose $h \leq b_m < n$. Then $h_0p^{j-m} - 1 < b'_m$, so by Proposition 2.3 we get
$$d_{\kappa'_m,\mu'_{p^m}} = (-1)^{k+\ell+A_m}(h_0p^{j-m} + \ell).$$
Suppose $b_m = n$, so that $b'_m = u_p\nu^{-m}$. Since $u_p\nu^{-m} > h_0p^{j-m} - 1$, the only cycle digraph which admits a $(\kappa'_m,\mu'_{p^m})$-tiling consists of a single cycle $\Gamma$ of length $w'_m$. By Proposition 2.2(1) we get $n_{\kappa'_m,\mu'_{p^m}}(\Gamma) = u_p\nu^{-m}$. It then follows from (2.1) that
$$d_{\kappa'_m,\mu'_{p^m}} = (-1)^{k+\ell+A_m}u_p\nu^{-m}.$$ Hence in all three cases we have $d_{\kappa'_m,\mu'_{p^m}} = g_m$.

Since $p^t \mid b_m$ we have $p^{t-m} \mid b'_m$, Therefore by Proposition 2.5 we get
$$d_{\kappa_m,\mu_{p^m}} \equiv d_{\kappa'_m,\mu'_{p^m}} \pmod {p^{t-m+1}}.$$ (4.3)
Since $m \leq t \leq \nu$ it follows from (3.2) and (3.1) that
$$\begin{align*}
i_m &\leq i_t^\pi + (t - m)e_L \\
nA_m - b_m &\leq nv_K(c_{b_m}) - b_m + (t - m)e_L \\
A_m &\leq v_K(c_{b_m}) + (t - m)e_K \\
k + 1 &\leq k - A_m + v_K(c_{b_m}) + (t - m + 1)e_K.
\end{align*}$$ (4.4)
Using (4.3) and (4.4) we get
$$d_{\kappa_m,\mu_{p^m}}c_n^{k-A_m}c_{b_m} \equiv d_{\kappa'_m,\mu'_{p^m}}c_n^{k-A_m}c_{b_m} \pmod {\mathcal{P}_K^{k+1}} \equiv g_mc_n^{k-A_m}c_{b_m} \pmod {\mathcal{P}_K^{k+1}}.$$ Hence by (4.2) the theorem holds when $\tilde{\pi}_L = \tilde{\pi}_L$.

We now prove the theorem in the general case. We may assume that
$$\tilde{\pi}_L \equiv \pi_L + r\tilde{\pi}_L^{\ell+1} \pmod {\mathcal{P}_L^{\ell+2}}.$$
It follows that \( \tilde{\pi}_L \equiv \hat{\pi}_L \) (mod \( P_L^{\ell+2} \)), so by Theorem 4.3 we get \( \tilde{c}_h \equiv \hat{c}_h \) (mod \( P_K^{\ell+1} \)). Since

\[
\rho_h(\ell) = \left\lceil \frac{\varphi_j(\ell) + h}{n} \right\rceil = \frac{\varphi_j(\ell) + h}{n} = k
\]

and \( \varphi_j(\ell + 1) > \varphi_j(\ell) \) we get \( \rho_h(\ell + 1) > k \). Therefore \( \tilde{c}_h \equiv \hat{c}_h \) (mod \( P_K^{k+1} \)), so the theorem holds for \( \tilde{\pi}_L \). \( \square \)

**Remark 4.6.** Suppose \( \pi_p(\varphi_j(\ell)) = j' \leq j \). Then \( \varphi_j(\ell) = \varphi_{j'}(\ell) \). In particular, \( \varphi_\nu(\ell) = \varphi_{\nu'}(\ell) \) with \( j' = \nu_p(\varphi_\nu(\ell)) \). Hence if \( 1 \leq h \leq n \) and \( n \) divides \( \varphi_\nu(\ell) + h \) then Theorem 4.5 gives a congruence for \( \tilde{c}_h \) modulo \( P_K^{k+1} \), where \( k = (\varphi_\nu(\ell) + h)/n \). This is the congruence obtained by Krasner \([8, p. 157]\). If \( \ell \) is greater than or equal to the largest lower ramification break of \( L/K \) then \( \varphi_j(\ell) = \varphi_\nu(\ell) \) for \( 0 \leq j \leq \nu \). Therefore Theorem 4.5 does not extend \([8]\) in these cases.

## 5. Some examples

In this section we give two examples related to the theorems proved in Section 4. We first apply these theorems to a 3-adic extension of degree 9.

**Example 5.1.** Let \( K \) be a finite extension of the 3-adic field \( \mathbb{Q}_3 \) such that \( v_K(3) \geq 2 \). Let

\[
f(X) = X^9 - c_1 X^8 + \cdots + c_8 X - c_9
\]

be an Eisenstein polynomial over \( K \) such that \( v_K(c_2) = v_K(c_6) = 2 \), \( v_K(c_h) \geq 2 \) for \( h \in \{1, 3\} \), and \( v_K(c_h) \geq 3 \) for \( h \in \{4, 5, 7, 8\} \). Let \( \pi_L \) be a root of \( f(X) \). Then \( L = K(\pi_L) \) is a totally ramified extension of \( K \) of degree 9, so we have \( u = 1, \nu = 2 \). It follows from our assumptions about the valuations of the coefficients of \( f(X) \) that the indices of inseparability of \( L/K \) are \( i_0 = 16, i_1 = 12, \) and \( i_2 = 0 \). Therefore \( A_0 = 2, A_1 = 2, A_2 = 1, \) and \( b_0 = 2, b_1 = 6, b_2 = 9 \). We get the following values for \( \tilde{\varphi}_j(\ell) \) and \( \varphi_j(\ell) \):

| \( \ell \) | \( \tilde{\varphi}_0(\ell) \) | \( \tilde{\varphi}_1(\ell) \) | \( \tilde{\varphi}_2(\ell) \) | \( \varphi_0(\ell) \) | \( \varphi_1(\ell) \) | \( \varphi_2(\ell) \) |
|---|---|---|---|---|---|---|
| 0 | 16 | 12 | 0 | 16 | 12 | 0 |
| 1 | 17 | 15 | 9 | 17 | 15 | 9 |
| 2 | 18 | 18 | 18 | 18 | 18 | 18 |
| 3 | 19 | 21 | 27 | 19 | 19 | 19 |

Now let \( \hat{\pi}_L \) be another uniformizer for \( L \), with minimum polynomial

\[
\hat{f}(X) = X^9 - \tilde{c}_1 X^8 + \cdots + \tilde{c}_8 X - \tilde{c}_9.
\]
Suppose $\tilde{\pi}_L \equiv \pi_L \pmod{P_L^2}$. Then by Theorem 4.3 we get $\tilde{f} \sim_1 f$. Using the table above we find that

$$\tilde{c}_h \equiv c_h \pmod{P_K^2} \quad \text{for } h \in \{1, 3, 9\};$$
$$\tilde{c}_h \equiv c_h \pmod{P_K^3} \quad \text{for } h \in \{2, 4, 5, 6, 7, 8\}.$$ 

This is an improvement on [8], which gives $\tilde{c}_h \equiv c_h \pmod{P_K^2}$ for $1 \leq h \leq 9$. If $\tilde{\pi}_L \equiv \pi_L \pmod{P_L^3}$ we get $\tilde{f} \sim_2 f$, and hence $\tilde{c}_h \equiv c_h \pmod{P_K^3}$ for $1 \leq h \leq 9$. If $\tilde{\pi}_L \equiv \pi_L \pmod{P_L^4}$ we get $\tilde{f} \sim_3 f$, and hence

$$\tilde{c}_h \equiv c_h \pmod{P_K^3} \quad \text{for } 1 \leq h \leq 8,$$
$$\tilde{c}_9 \equiv c_9 \pmod{P_K^4}.$$ 

Since the largest lower ramification break of $L/K$ is 2, the congruences we get for $\ell \geq 2$ are the same as those in [8].

Suppose $\tilde{\pi}_L \equiv \pi_L + r\pi_L^2 \pmod{P_L^2}$, with $r \in O_K$. By the table above we get $\overline{v}_3(\varphi_0(1)) = 0, \overline{v}_3(\varphi_1(1)) = 1, \overline{v}_3(\varphi_2(1)) = 2$ and $S_0 = \{0\}, S_1 = \{1\}, S_2 = \{2\}$. The corresponding values of $h$ are 1, 3, 9, and we have $h_0 = 1, k = 2$ in all three cases. By applying Theorem 4.5 with $\ell = 1, j = 0, 1, 2$ we get the following congruences:

$$\tilde{c}_1 \equiv c_1 + (-1)^{2+1+2}(1 + 1)c_2r \pmod{P_K^3}$$
$$\equiv c_1 - 2c_2r \pmod{P_K^3}$$
$$\tilde{c}_3 \equiv c_3 + (-1)^{2+1+2}(1 + 1)c_6r^3 \pmod{P_K^3}$$
$$\equiv c_3 - 2c_6r^3 \pmod{P_K^3}$$
$$\tilde{c}_9 \equiv c_9 + (-1)^{2+1+1}c_9^2r^9 \pmod{P_K^4}$$
$$\equiv c_9 + c_9^2r^9 \pmod{P_K^4}.$$ 

Only the congruence for $\tilde{c}_9$ follows from [8].

Suppose $\tilde{\pi}_L \equiv \pi_L + r\pi_L^3 \pmod{P_L^3}$. Then $\overline{v}_3(\varphi_2(2)) = 2$ and $S_2 = \{0, 1, 2\}$, which gives $h = 9, h_0 = 1, k = 3$. By applying Theorem 4.5 with $\ell = 2, j = 2$ we get the following congruence:

$$\tilde{c}_9 \equiv c_9 + (-1)^{3+2+2}(9 + 2 - 9)c_9c_2r$$
$$+ (-1)^{3+2+2}(3 + 2 - 3)c_9c_6r^3 + (-1)^{3+2+1}c_9^2c_9r^9 \pmod{P_K^4}$$
$$\equiv c_9 - 2c_2c_9r - 2c_6c_9r^3 + c_9^3r^9 \pmod{P_K^4}.$$ 

Suppose $\tilde{\pi}_L \equiv \pi_L + r\pi_L^4 \pmod{P_L^2}$. Then $\overline{v}_3(\varphi_0(3)) = 0$ and $S_0 = \{0\}$, so we get $h = 8, h_0 = 8, k = 3$. By applying Theorem 4.5 with $\ell = 3, j = 0$ we get the following congruence:

$$\tilde{c}_8 \equiv c_8 + (-1)^{3+3+2}(8 + 3 - 9)c_9c_2r \pmod{P_K^4}$$
$$\equiv c_8 + 2c_2c_9r \pmod{P_K^4}.$$
Again, since the largest lower ramification break of \( L/K \) is 2, the congruences we get for \( \ell \geq 2 \) are the same as those in [8].

One might hope to prove the following converse to Theorem 4.3: If \( \pi_L, \tilde{\pi}_L \) are uniformizers for \( L \) whose minimum polynomials satisfy \( \tilde{f} \sim_\ell f \), then there is \( \sigma \in \text{Aut}_K(L) \) such that \( \sigma(\tilde{\pi}_L) \equiv \pi_L \pmod{\mathcal{P}_L^{\ell+1}} \). The example below shows that this is not necessarily the case:

Example 5.2. Let \( \pi_L \) be a root of the Eisenstein polynomial \( f(X) = X^4 + 6X^2 + 4X + 2 \) over the 2-adic field \( \mathbb{Q}_2 \). Then \( L = \mathbb{Q}_2(\pi_L) \) is a totally ramified extension of \( \mathbb{Q}_2 \) of degree 4, with indices of inseparability \( i_0 = 5 \), \( i_1 = 2 \), and \( i_2 = 0 \). We get the following values for \( \tilde{\varphi}_j(\ell) \) and \( \varphi_j(\ell) \):

\[
\begin{array}{ccccccc}
\ell & \tilde{\varphi}_0(\ell) & \tilde{\varphi}_1(\ell) & \tilde{\varphi}_2(\ell) & \varphi_0(\ell) & \varphi_1(\ell) & \varphi_2(\ell) \\
0 & 5 & 2 & 0 & 5 & 2 & 0 \\
1 & 6 & 4 & 4 & 6 & 4 & 4 \\
2 & 7 & 6 & 8 & 7 & 6 & 6 \\
3 & 8 & 8 & 12 & 8 & 8 & 8 \\
\end{array}
\]

Set \( \tilde{\pi}_L = \pi_L + \pi_L^2 \), and let the minimum polynomial for \( \tilde{\pi}_L \) over \( \mathbb{Q}_2 \) be

\[
\tilde{f}(X) = X^4 - \tilde{c}_1X^3 + \tilde{c}_2X^2 - \tilde{c}_3X + \tilde{c}_4.
\]

By Theorem 4.3 we have \( \tilde{f} \sim_1 f \), and hence

\[
\begin{align*}
\tilde{c}_1 & \equiv 0 \pmod{4} \\
\tilde{c}_2 & \equiv 6 \pmod{4} \\
\tilde{c}_3 & \equiv -4 \pmod{8} \\
\tilde{c}_4 & \equiv 2 \pmod{4}.
\end{align*}
\]

Theorem 4.5 gives a refinement of the last congruence:

\[
\tilde{c}_4 \equiv 2 + (-1)^{2^1+1}(2 + 1 - 2) \cdot 2^{2^1 - 1} \cdot 6 + (-1)^{2^1+1} \cdot 2^{2^1 - 1} \cdot 2 \pmod{8} \\
\equiv 2 \pmod{8}.
\]

Using this refinement we get \( \tilde{f} \sim_2 f \).

Using [5] (see also [6, Table 4.2]) we obtain a list of the degree-4 extensions of \( \mathbb{Q}_2 \). Using the data in this list we find that \( L/\mathbb{Q}_2 \) is not Galois, and the only quadratic subextension of \( L/\mathbb{Q}_2 \) is \( M/\mathbb{Q}_2 \), where \( M = \mathbb{Q}_2(\sqrt{-1}) \). Hence \( \text{Aut}_{\mathbb{Q}_2}(L) = \text{Gal}(L/M) \). Since the lower ramification breaks of \( L/\mathbb{Q}_2 \) are 1, 3, and the lower ramification break of \( M/\mathbb{Q}_2 \) is 1, the lower ramification break of \( L/M \) is 3. Hence if \( \sigma \in \text{Aut}_{\mathbb{Q}_2}(L) \) then \( \sigma(\tilde{\pi}_L) \equiv \tilde{\pi}_L \pmod{\mathcal{P}_L^4} \). Since \( \tilde{\pi}_L = \pi_L + \pi_L^2 \) we get \( \sigma(\tilde{\pi}_L) \not\equiv \pi_L \pmod{\mathcal{P}_L^3} \).
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