Noncommutative Geometry, Quantum Hall Effect and Berry Phase

B. Basu∗ and P. Bandyopadhyay†
Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata-700108

Abstract

Taking resort to Haldane’s spherical geometry we can visualize fractional quantum Hall effect on the noncommutative manifold $M_4 \times \mathbb{Z}_N$ with $N > 2$ and odd. The discrete space leads to the deformation of symplectic structure of the continuous manifold such that the symplectic area is given by $\Delta p \cdot \Delta q = 2\pi \hbar m$ with $m$ an odd integer which is related to the Berry phase and the filling factor is given by $\frac{1}{m}$. We here argue that this is equivalent to the noncommutative field theory as prescribed by Susskind and Polychronakos which is characterized by area preserving diffeomorphism. The filling factor $\frac{1}{m}$ is determined from the change in chiral anomaly and hence the Berry phase as envisaged by the star product.

I. INTRODUCTION

It is well known that the first quantized system of particles in a strong magnetic field naturally realizes a noncommutative space. We also know that a system of electrons moving in a strong magnetic field exhibits certain interesting properties. When the electron density lies at certain rational fractions of the density corresponding to the fully filled Landau level, the electrons condense into an incompressible fluid-like states whose excitations are characterized by fractional charges and are described by Laughlin wave functions. When we fill as many electrons as possible in a two dimensional phase space, each electron occupy an area $\Delta p \cdot \Delta q = 2\pi \hbar$ which is responsible for the maximum electron density $n_e = \frac{eB}{2\pi\hbar}$ in each Landau level. In the noncommutative manifold $M_4 \times \mathbb{Z}_N$ with $N > 2$ and odd, the symplectic area is deformed [1]. In this case, $\Delta p \cdot \Delta q = 2\pi \hbar m$ with $m$ an odd integer and the Laughlin states are achieved with $\nu = \frac{1}{m}$. This deformation of symplectic structure leads to the Berry phase as it corresponds to the association of magnetic flux with an electron.

On the other side, Susskind [5] conjectured that the second quantized noncommutative field theory can describe the Laughlin states when we take into account the noncommutative version of the $U(1)$ Chern-Simons theory. This also reveals the connection between D brane physics with quantum Hall effect. The space noncommutative condition requires infinite dimensional matrices which was modified by Polychronakos [6] to realize appropriate finite matrix model. However, in this framework involving noncommutative field theory the system is characterized by area preserving diffeomorphism.

In this note, we shall show that taking resort to spherical geometry the star product deformation of gauge fields effectively leads to the change in chiral anomaly and hence induces the Berry phase compatible with the fractional filling factor. This suggests that the deformation of the symplectic structure of the noncommutative manifold $M_4 \times \mathbb{Z}_N$ with $N > 2$ and odd represents an equivalent description of the noncommutative version of the $U(1)$ Chern-Simons theory which is characterized by area preserving diffeomorphism. Thus we realize a dual relationship between noncommutative manifold and noncommutative field theory.

II. NONCOMMUTATIVE MANIFOLD $M_4 \times \mathbb{Z}_N$, DEFORMATION OF SYMPLECTIC STRUCTURE AND BERRY PHASE

Noncommutative geometry is characterized by the fact that space-time structure acquires some fuzziness i.e points are ill defined. The fuzziness of phase space variables may be viewed such that the mean

∗Electronic address: banasri@isical.ac.in
†Electronic address: pratul@isical.ac.in
position of a particle at \( q_\mu \) in the external observable space has a stochastic extension given by a stochastic variable \( \hat{Q}_\mu \). The position and momentum of a relativistic quantum particle in complexified space-time is then given by

\[
Q_\mu = q_\mu + i\hat{Q}_\mu \\
P_\mu = p_\mu + i\hat{P}_\mu
\]  

(1)

Introducing the dimensionless variable \( \omega = \frac{\hbar}{mc} \) it is shown that these relativistic canonical commutation relations admit the following representations of \( Q_\mu/\omega \) and \( P_\mu/\omega \) where the latter are considered as acting on functions defined in phase space

\[
\frac{Q_\mu}{\omega} = -i(\frac{\partial}{\partial p_\mu} + \phi_\mu) \\
\frac{P_\mu}{\omega} = -i(\frac{\partial}{\partial q_\mu} + \psi_\mu)
\]  

(2)

where \( \phi_\mu(\psi_\mu) \) are some matrix valued functions in the noncommutative case.

In a recent paper it has been pointed out that noncommutative geometry having the space-time manifold \( M_4 \times \mathbb{Z}_2 \) leads to the quantization of a fermion when the discrete space-time is incorporated as an internal variable. Indeed, this leads to the introduction of an anisotropic feature in the internal space so that we can consider the space-time coordinate in complex space-time as \( z_\mu = x_\mu + i\xi_\mu \) where \( \xi_\mu \) represents a \textit{direction vector} attached to the space-time point \( x_\mu \). The two orientations of the \textit{direction vector} give rise to two internal helicities corresponding to fermion and antifermion. The complex space-time exhibiting the internal helicity states can be written in terms of a two-component spinorial variable \( \theta(\bar{\theta}) \) when the \textit{direction vector} \( \xi_\mu \) is associated with \( \theta \) through the relation \( \xi_\mu = \frac{1}{2} \lambda_\mu^\alpha \theta_\alpha \) (\( \alpha = 1, 2 \)). This helps us to write the relevant metric as \( g_{\mu\nu}(x,\theta,\bar{\theta}) \). This metric gives rise to the \( SL(2,C) \) gauge theory where the gauge fields \( A_\mu \) are matrix valued having the \( SL(2,C) \) group structure and the curvature \( F_{\mu\nu} \) is given by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
\]  

(3)

The phase space variables may be written in terms of a gauge theoretical extension of a relativistic quantum particle. Indeed, from the relation we can now identify \( \phi_\mu(\psi_\mu) \) with gauge field \( A_\mu(B_\mu) \). That is, we can conceive of the position and momentum operators given by

\[
\frac{Q_\mu}{\omega} = -i(\frac{\partial}{\partial p_\mu} + A_\mu) \\
\frac{P_\mu}{\omega} = i(\frac{\partial}{\partial q_\mu} + B_\mu)
\]  

(4)

It is now noted that the curvature \( F_{\mu\nu} \) will deform the symplectic structure. Indeed, it will change the symplectic form of the phase space as

\[
\Omega = \frac{1}{2} g^{ij} dp_i \wedge dq_j
\]  

(5)

with

\[
g^{ij} = J^{ij} + \hbar \Delta^{ij}
\]  

(6)

where

\[
J^{ij} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}
\]  

(7)

is associated with the usual symplectic structure and \( \Delta^{ij} \) is the curvature tensor coming from the induced vector potential \( A_\mu \).
The Poisson bracket is modified as

\[ \{ f, g \} = g_{ij} \left( \frac{\partial f}{\partial q_i} \right) \left( \frac{\partial g}{\partial p_j} \right) = \{ f, g \} + \{ f, g \}' + ... \]  

(8)

where \( f, g \) are arbitrary functions on the phase space, \( g_{ij} \) is the inverse of \( g^{ij} \). The Poisson bracket will help to define the quantum field theory in the modified form if we replace this by the commutator.

It is noted that in the manifold \( M_4 \times Z_2 \) when the discrete space is considered as an internal extension, the direction vector \( \xi_\mu \) attached to the space-time point \( x_\mu \in M_4 \), effectively corresponds to a vortex line. Again, a vortex line is topologically equivalent to a magnetic flux. Thus in this picture, a fermion can be viewed as a boson attached with a magnetic flux associated with the gauge field. In the manifold \( M_4 \times Z_N \) with \( N > 2 \) and odd, we note that the gauge field theoretical extension will induce a density of vortices in the Bose field which corresponds to \( \frac{1}{N} \) of a vortex per boson. As an example, for \( N=3 \), we can view the system such that each boson carries \( \frac{1}{3} \) of magnetic flux quantum. That is, three bosons will share one magnetic flux quantum and the system will represent a three Bose particle composite having one magnetic flux. It may be remarked here that under duality relationship [10], we can consider these vortices as Bose particles and each hard core boson as a flux tube carrying one Dirac flux quantum. This suggests that the ratio between the number of new vortices and the number of new bosons now becomes 3 to 1 and each new boson swallows three vortices.

When a direction vector \( \xi_\mu \) is attached to a space-time point \( x_\mu \), the wave function of the particle concerned will be characterized by three Euler angles \( \theta, \phi \) and \( \chi \). The angle \( \chi \) corresponds to the rotational orientation around the direction vector. The angular momentum relation is now given by

\[ J = r \times p - \mu \hat{r} \]  

(9)

Here \( \mu \) corresponds to eigenvalue of the operator \( i \frac{\partial}{\partial \chi} \) having values 0, \( \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2} \ldots \). In the ground state we have \( J = |\mu| \) and the wave functions become

\[ \psi^J_m = D^J_{m,\mu}(\theta, \phi, \chi) \]  

(10)

with \( J = |\mu| \) and \( m = -J, -J + 1, \ldots + J \). Indeed for \( J = m = |\mu| = \frac{1}{2} \), we can construct the following spinorial functions

\[ u = \cos \frac{\theta}{2} \exp \left( i \phi + i \frac{\chi}{2} \right) = D^\frac{1}{2}_{\frac{1}{2},-\frac{1}{2}}(\theta, \phi, \chi) \]

\[ v = \sin \frac{\theta}{2} \exp \left( -i \phi + i \frac{\chi}{2} \right) = D^\frac{1}{2}_{\frac{1}{2},\frac{1}{2}}(\theta, \phi, \chi) \]  

(11)

If we consider the geodesic projection coordinate

\[ \zeta = \frac{v}{u} = \tan \frac{\theta}{2} \exp(-i \phi), \]

for the Hopf fibration \( S^2 = SU(2)/U(1) \), the base space turns to be Kähler manifold and the symplectic structure is given by [11]

\[ \Omega = 2i \frac{d\zeta \wedge d\bar{\zeta}}{(1 + |\zeta|^2)^2} = 2i \frac{\partial^2 K}{\partial \zeta \partial \bar{\zeta}} d\zeta \wedge d\bar{\zeta} \]  

(12)

where \( K = \ln(1 + |\zeta|^2) \) is the Kähler potential. The Hilbert space \( \mathcal{H}_N \) on this Hopf fibration \( S^2 \) is composed by the \( N = 2S + 1 \) one particle wave functions \( \psi^J_m \) around the Dirac monopole \( \mu \) (\( J = |\mu| \)). Evidently, for the three particle system, we have \( |\mu| = \frac{1}{2} \) and the geodesic projection coordinate is given by \( \zeta' = (\frac{\zeta}{\bar{\zeta}})^{3/2} \). So from the relation [12] we find that the symplectic structure \( \Omega' \) in this case is given by \( \Omega' = 3\Omega \). Thus the symplectic structure for the noncommutative manifold \( M_4 \times Z_3 \) corresponds to the
relation \( \triangle p\triangle q = 2\pi m\hbar \) with \( m = 3 \). This can be generalized for the manifold \( M_4 \times Z_N \) with \( N > 2 \) and odd when we have \( \triangle p\triangle q = 2\pi m\hbar \) with \( m = N \).

It is noted that in angular momentum relation (9), for \( \mu \) an integer we can have

\[
J = r \times p - \mu \hat{r} = r' \times p'
\]

indicating the vanishing of magnetic field. This is the case for the manifold \( M_4 \times Z_N \) with \( N > 4 \) and even. In this case, the change in the symplectic structure will not be manifested unless the state is split into a pair.

It is to be noted that the vector potential \( A_\mu \) responsible for the deformation of the symplectic structure effectively gives rise to the geometrical phase of Berry. Indeed, when a system \( \hat{H}_0 \) interacts with certain internal Hamiltonian \( h_{\text{int}} \) to have \( \hat{H}_{\text{eff}} = \hat{H}_0 + h_{\text{int}} \), the response of the internal system corresponding to the change of the external canonical system becomes a part of the action as to associate the change of field as (12)

\[
S_{\text{eff}} = S_0 + \hbar \Gamma(c)
\]

The physical importance of this geometrical phase \( \Gamma(c) \) is that it becomes a magnetic flux associated with the induced vector potential \( A_\mu \). This implies that the factor \( m \) occurring in the deformed symplectic structure \( \triangle p\triangle q = 2\pi m\hbar \) is associated with the Berry phase of the system.

In the study of quantum Hall effect, we may visualize that the external magnetic field realizes the noncommutative manifold \( M_4 \times Z_N \) when we consider the electrons lie on the surface of a 3 dimensional sphere with a monopole at the center. In this geometry, the deformation of the symplectic structure \( \triangle p\triangle q = 2\pi m\hbar \) leads to the fractional quantum Hall effect when the filling factor is given by \( \nu = \frac{1}{m} = \frac{1}{2\mu} \) where \( \mu \) is the monopole strength. It may be remarked here that the deformation causes to increase the occupied area of an electron and hence changes the electron density to lie at certain fractions of the density corresponding to the density of the completely filled Landau level. When these electrons condense into an incompressible fluid we realize fractional quantum Hall effect.

### III. NONCOMMUTATIVE FIELD THEORY, CHIRAL ANOMALY AND BERRY PHASE

The noncommutative generalization for the free Maxwell Lagrangian density involves the star product of the noncommutative field strength \( \hat{F}_{\mu\nu} \) constructed from the potential \( \hat{A}_\mu \) and can be written as

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig (\hat{A}_\mu \ast \hat{A}_\nu - \hat{A}_\nu \ast \hat{A}_\mu)
\]

and

\[
\hat{L} = -\frac{1}{4} \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu}
\]

In terms of the conventional Maxwell tensor

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

it turns out that we can express \( \hat{A}_\mu \) and \( \hat{F}_{\mu\nu} \) as follows

\[
\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu})
\]

\[
\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu}
\]

where \( g \) is absorbed in \( \theta \). The tensor \( \theta^{\alpha\beta} \) is associated with the star product which is defined as

\[
(f \ast g)(x) = e^{ig \theta^{\alpha\beta} \partial_\alpha \partial_\beta f(x)g(x')}|_{x=x'}
\]
Apart from the total derivative term, the Lagrangian $\hat{L}$ is given by

$$\hat{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} + O(\theta^2) \quad (20)$$

It is now well known that the star product effectively involves a background magnetic field and so the second and third terms in eqns. (18) and (20) correspond to the interaction of this background field with the Maxwell field.

We may recall here that when a fermionic chiral current interacts with a gauge field it gives rise to chiral anomaly which is given by

$$\partial_{\mu} J_5^\mu = \frac{1}{8\pi^2} Tr F_{\mu\nu} F^{\mu\nu} \quad (21)$$

where $J_5^\mu$ is the axial vector current $\overline{\psi}\gamma^\mu\gamma^5\psi$ and

$$^*F_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

is the Hodge dual. The chiral anomaly is related to the Berry phase and we have

$$q = 2\mu = -\frac{1}{2} \int (\partial_{\mu} J_5^\mu) \ d^4x = -\frac{1}{16\pi^2} \int Tr \ ^*F_{\mu\nu}F_{\mu\nu} \ d^4x \quad (22)$$

Here, $q$ is the Pontryagin index and $\mu$ corresponds to the monopole strength.

When we consider the interaction of the chiral current with the noncommutative gauge field having the gauge field strength

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_{\alpha} \partial_{\beta} F_{\mu\nu}$$

we note that the chiral anomaly will be modified as

$$\frac{1}{8\pi^2} \left[ ^*F_{\mu\nu}F_{\mu\nu} + ^*F_{\mu\nu} \bar{F}_{\mu\nu} + ^*\bar{F}_{\mu\nu} \bar{F}_{\mu\nu} \right]$$

It is noted that we will have a change in the factor $\mu$ which will induce a change in the Berry phase. The modified value $\mu_{eff}$ will be given by

$$\mu_{eff} = \mu + \mu' + \bar{\mu} = \mu + \bar{\mu} \quad (24)$$

when we identify

$$\mu = -\frac{1}{32\pi^2} \int Tr \ ^*F_{\mu\nu}F_{\mu\nu} \ d^4x, \quad (25)$$

$$\mu' = -\frac{1}{32\pi^2} \int Tr \ ^*F_{\mu\nu} \bar{F}_{\mu\nu} \ d^4x, \quad (26)$$

$$\bar{\mu} = -\frac{1}{32\pi^2} \int Tr \ ^*\bar{F}_{\mu\nu} \bar{F}_{\mu\nu} \ d^4x$$

This implies that the induced background magnetic field associated with the star product effectively changes the number of magnetic flux quanta in a specified area through its interaction with the external gauge field strength which subsequently changes the chiral anomaly and hence the Berry phase.
Susskind [5] has conjectured that quantum Hall effect can be well described by the Chern-Simons action involving the noncommutative version of the $U(1)$ gauge theory on a two dimensional space. The action is given by

$$ S = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} (\hat{A}_\mu \ast \partial_\nu \hat{A}_\lambda + \frac{2}{3} \hat{A}_\mu \ast \hat{A}_\nu \ast \hat{A}_\lambda) $$  (28)

with $k = \frac{1}{e^2}$.

The theory is invariant under the noncommutative gauge transformation

$$ \hat{A}_\mu \rightarrow U^{-1} \ast \hat{A}_\mu \ast U + i U^{-1} \ast \partial_\mu U $$  (29)

when $k$ is an integer.

The theory may be written in an equivalent form by choosing gauge $\hat{A}_0 = 0$ in which case the action becomes

$$ S = \frac{k}{4\pi} \int d^3x \epsilon^{ij} \hat{A}_i \partial_t \hat{A}_j $$  (30)

while the equation of motion for $\hat{A}_0$ must be imposed as a constraint

$$ F_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i (\hat{A}_i \ast \hat{A}_j - \hat{A}_j \ast \hat{A}_i) = 0 $$  (31)

It turns out that this action and constraint also arise from a matrix model given by

$$ S = \frac{k}{\theta} \int dt \text{Tr} \left( \frac{1}{2} \epsilon_{ij} DX^i X^j \right) + k \int dt \text{Tr}(A) $$  (32)

Where $X$ and $A$ are Hermitian matrices. The action is invariant under gauge transformation

$$ X^i \rightarrow U^{-1}X^iU $$  (33)
$$ A \rightarrow U^{-1}AU + iU^{-1}\partial_t U $$  (34)

as long as $U$ is taken to be trivial at $t = \pm \infty$ and $k$ is an integer. In the gauge $A = 0$, the action becomes

$$ S = \frac{k}{\theta} \int dt \text{Tr} \left( \frac{1}{2} \epsilon_{ij} X^i X^j \right) $$  (35)

while the equation of motion for $A$ is

$$ [X^1, X^2] = i\theta $$  (36)

which must be taken as a constraint. It is to be noted the commutator here is just the matrix commutator. This has no solution for finite dimensional matrices. A particular solution is $X^i = y^i$ and $y^1$ and $y^2/\theta$ are the usual matrices representing $x$ and $p$ in the harmonic oscillator basis. Expanding the action and the constraint about the classical solution $X^i = y^i + \theta e^{ij}A_j$ where $A_j$ are functions of the noncommutative coordinates $y^i$, gives precisely the Lagrangian (30) with the constraint (31).

Polychronakos [6] has modified this formalism such that the matrices $X^1$ and $X^2$ become finite dimensional. The quantization of the inverse filling fraction and of the quasiparticle number is shown to arise quantum mechanically and to agree with the Laughlin theory.

In spherical geometry, where electrons lie on the surface of a three dimensional sphere with a monopole at the centre, we can consider the relation

$$ \int_{M_3} F \wedge F = \int_{M_3} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) $$  (37)
This suggests that in 3 + 1 dimension we can replace the noncommutative Chern-Simons term with the term $\ast \hat{F}_{\mu \nu} \ast \hat{F}_{\mu \nu}$. In fact, the corresponding quantum Hall effect action in 3 + 1 dimension is given by

$$S = -\frac{\theta}{4} \int \ast \hat{F}_{\mu \nu} \ast \hat{F}_{\mu \nu} d^4 x$$

(38)

where $\theta$ is a coupling constant. So from the expression for $\hat{F}_{\mu \nu}$ as given by (23), we can write

$$\ast \hat{F}_{\mu \nu} \ast \hat{F}_{\mu \nu} = \ast F_{\mu \nu} F_{\mu \nu} + \ast F_{\mu \nu} \tilde{F}_{\mu \nu} + \ast \tilde{F}_{\mu \nu} \tilde{F}_{\mu \nu}$$

(39)

So the chiral anomaly is now changed and we have the associated Berry phase factor as given by eqn. (24). Indeed the expression for the Berry phase factor

$$\mu_{eff} = \mu + \mu' + \tilde{\mu} = \mu + \tilde{\mu}$$

(40)

implies that the term $\tilde{\mu}$ is associated with the modification caused by the background magnetic field inherent in the star product formalism.

In an earlier paper [15], we have shown that in the lowest Landau level, we can have the filling factor $\nu$ in terms of the Berry phase factor $\mu$ such that $\nu = \frac{1}{2\mu}$. In view of this, we can now write

$$\nu_{eff} = \frac{1}{2\mu_{eff}}$$

(41)

where $\mu_{eff}$ is given by (38).

This result can be compared with that of Dayi and Jellal [16]. They have shown that if there is noncommutativity of coordinates in two dimensional space given by

$$[X, Y] = i\theta$$

(42)

then there is a change in external magnetic field caused by the inherent field in geometry and the effective field is given by

$$B_{eff} = \frac{B}{1 - \frac{e\theta}{4hc}}$$

(43)

This causes the change in the filling factor

$$\nu_{eff} = \frac{\phi_0 \rho}{B} \left(1 - \frac{e\theta B}{4hc}\right)$$

(44)

where $\phi_0 = hc/e$ and $\rho$ denotes electron density.

In sec.2 we have argued that in the noncommutative manifold $M_4 \times Z_N$, the symplectic structure is modified such that $\Delta p \Delta q = 2\pi m h$ with $m$ an odd integer given by $m = 2\mu$. In relation to quantum Hall fluid, this implies that we have now the filling factor given by $\nu = \frac{1}{m} = \frac{1}{2\mu_{eff}}$. So from the relation (11) we note that we arrive at the equivalent result in terms of the noncommutative field theory. However, in the noncommutative manifold $M_3 \times Z_N$, the Berry phase is associated with the deformation of symplectic structure where in the framework of noncommutative field theory the system is characterized by area preserving diffeomorphism. Thus we have a duality relation between these two formalisms.

V. DISCUSSION

It has been pointed out here that the noncommutative manifold $M_4 \times Z_N$ induces deformation of symplectic structure which leads to the fractional statistics for $N > 2$ and odd. In fact, this deformation is related to the Berry phase factor which is responsible for the fractional filling factor of the incompressible Hall fluid. On the other hand, in spherical geometry where electrons reside on the surface of a sphere with a monopole at the center, the noncommutative field theory induces change in chiral anomaly leading to the Berry phase responsible for the fractional quantum Hall effect. However, the latter is characterized by
area preserving diffeomorphism. Thus we can arrive at a dual relation between noncommutative manifold $M_4 \times Z_N$ (with $N > 2$ and odd) and noncommutative field theory.

It may be recalled that there exists a correspondence between noncommutative geometry and quantum space which is characterized by certain quantum group symmetry and both represent a lattice structure. Kogan [17] has demonstrated the connection between quantum symmetry, magnetic translation and area preserving diffeomorphism in Landau problem and discussed the relevance of $U_q(SL(2))$ in quantum Hall system. In a recent paper [4] a possible link between the Berry phase factor $\mu$ with the deformation parameter $q$ of the deformed algebra $U_q(SL(2))$ has been suggested. Indeed, Kogan’s analysis suggests that in a quantum Hall system the deformation parameter $q$ of the symmetry $U_q(SL(2))$ is related to the filling factor $\nu$ of the form $1/m$, $m$ being an odd integer, through the relation $q = \exp(i2\pi\nu)$ which implies that $q$ is related to the Berry phase factor $\mu$ through the relation $q = \exp(i\frac{2\pi}{2m\epsilon})$. It may be observed that the association of fractional quantum Hall states with the $Z_p$ spin system [18] directly links up the quantum space and noncommutative geometry with the lattice structure. In fact for $q = \exp(i\frac{2\pi}{3})$, $q = \exp(i\frac{2\pi}{5})$, ...... we have the corresponding $Z_3$, $Z_5$, .... spin system representing fermion number $\frac{1}{3}$, $\frac{1}{5}$ which corresponds to the Berry phase factor $\mu_{eff} = \frac{3}{2}$, $\frac{5}{2}$ and so on. This evidently relates the deformation parameter of the quantum group $U_q(SL(2))$ when $q$ is a root of unity with noncommutative geometry which realizes these Berry phase factors either through the deformation of the symplectic structure in the noncommutative manifold $M_4 \times Z_N$ or through the change in chiral anomaly in the noncommutative field theory.

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