Quantum Surveying: How Entangled Pairs Act as Measuring Rods on Manifolds of Generalized Coherent States.

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Abstract

Generalized coherent states arise from reference states by the action of locally compact transformation groups and thereby form manifolds on which there is an invariant measure. It is shown that this implies the existence of canonically associated Bell states that serve as measuring rods by relating the metric geometry of the manifold to the observed EPR correlations. It is further shown that these correlations can be accounted for by a hidden variable theory which is non-local but invariant under the stability group of the reference state.

Quantum dynamics provides the mathematical machinery for computing the orbit in Hilbert space $\mathcal{H}$ of an initially given state vector. But to experimentally identify the state vector $s'$ in $\mathcal{H}$ into which an initial state $s$ has evolved with arbitrary precision knowing only that it lies in a neighborhood $N$ of $s$, we must be able to choose for any $\epsilon > 0$ a finite set of states $s_1, s_2, \cdots, s_{J(\epsilon)}$ in $N$ such that $|s' - s_j| < \epsilon$ for some $j$. This implies that the space of states must be a locally compact subset of $\mathcal{H}$, i.e. every point has a neighborhood with compact closure\(^1\).
While the infinite dimensional Hilbert space $\mathcal{H}$ required for particle dynamics is not a locally compact space, the groups of transformations such as the Poincaré and Weyl-Heisenberg groups by which we relate particle detectors to one another are locally compact groups, and so then is the space of states that arise from a reference state by their action. In fact, all of the relevant groups are Lie groups$^2$ and hence the states lie on finite dimensional manifolds in $\mathcal{H}$.

By restricting the set of allowed states to such manifolds we have the required locally compact space, but it will not be a linearly closed space, i.e. the superposition principle will not hold. Since the Dirac-von Neumann interpretation of measurement which identifies observable properties with eigenvalues of self-adjoint operators is not implementable without the superposition principle, the restriction to locally compact spaces requires an alternative interpretation of the measurement process. Since the so-called measurement problem of quantum mechanics arises from the Dirac-von Neumann interpretation$^3$, we have an additional motivation for seeking such an alternative.

Because the set of allowed states is a manifold, we are led to interpret quantum measurement as a means of determining the metric geometry of that manifold. There is a useful analogy which clarifies the relationship of this interpretation with that of Dirac-von Neumann. When Gauss undertook the survey of Hanover he pointed out that the properties of a surface that can be determined by “small, flat bugs” that live on its surface are of intrinsic interest, and these do not require the assignment of Cartesian coordinates to its points relative to a reference point in space$^4$. The Dirac-von Neumann measurement paradigm which seeks to characterize a state by its membership in eigenspaces of observables is analogous to the latter,
while the interpretation we shall describe below corresponds to the Gaussian approach. In view of the analogy I shall call this *quantum surveying.*

The restriction to locally compact groups has a profound implication: For each such group G there will be an invariant measure\(^5\) which allows us to integrate over the group and thereby, as we shall see, construct pair states canonically related to G which will serve as measuring rods for quantum surveying. To this end we first introduce suitable kinematics for the manifold of states.

Let U be an irreducible, unitary representation on \(H\) of a Lie group G, and let \(|0\rangle \in H\) be a reference unit vector. We shall use \(g\) to indicate both a group element and its representation in U. The allowed states will be the images of \(|0\rangle\) by a transformation \(g \in G\). To label a state by the \(g\) which produces it is ambiguous because two elements \(g_1, g_2\) can produce equivalent states, i.e. states that differ by a phase factor. This means that \(h = g_1^{-1}g_2\) has \(|0\rangle\) as an eigenstate. Such elements form a subgroup \(G_o\) called the stability subgroup of the reference state. If we select one element \(g\) from each coset \(gG_o\) and define \(|g\rangle = g|0\rangle\) we shall have a one-one correspondence between states and labels. The set of states so labeled is referred to as a set of *generalized coherent states*\(^6\) and can be identified with the coset space \(F = G/G_o\). It is a homogeneous space, i.e. every pair of states is related by a transformation in the group.

As noted above, the assumptions made about G imply that F is a manifold upon which there exists an invariant measure \(d\mu\) by which we can integrate over F. The invariance of the measure implies that the operator

\[
I \equiv \int_F d\mu|g\rangle\langle g|
\]

commutes with every \(g\). Since U is irreducible, Schur’s Lemma informs us
that we can take $I$ to be the unit operator by suitably scaling $d\mu$. Observe that

$$Tr(I) = \int_F d\mu \equiv V_F$$

is the “volume” of the space of states and is finite if and only if $G$ is compact.

It follows from (1) that although $F$ is itself not linearly closed, its linear closure is all of $\mathcal{H}$. The states of $F$ are not mutually orthogonal, and in general $F$ is infinitely over-complete, i.e. one can delete an infinite subset, and the linear closure of the remaining states is still $\mathcal{H}$.

We next use the representation of $I$ to construct a pair state with remarkable properties: Consider the object

$$|B\rangle\rangle = C \int_F d\mu |g\rangle \otimes \langle g|,$$

where $C$ is a normalization constant. We can interpret this as a pair state in the following way: Let $\tau$ be any anti-unitary operator and observe that the map

$$\tau |g\rangle \rightarrow \langle g|$$

is unitary. Thus one can think of $\langle g|$ as the $\tau$-reversal of $|g\rangle$, i.e. we might just as well have written the state as

$$|B\rangle\rangle = C \int_F d\mu |g\rangle \otimes |g^{\tau}\rangle, \text{ with } |g^{\tau}\rangle \equiv \tau |g\rangle.$$  

In (3) the representation space is the tensor product of $\mathcal{H}$ with its dual, whereas in (5) it is the tensor product of $\mathcal{H}$ with itself. We shall use the form (3) which has a more transparent structure. Let us compute the probability in state $|B\rangle\rangle$ that one member is found in state $|g_1\rangle$ and its partner in state $\langle g_2|$, i.e. that the pair is found in the state

$$|g_1, g_2\rangle\rangle = |g_1\rangle \otimes \langle g_2|.$$
This is computed using (1) to be

\[ p(g_1, g_2) = |\langle g_1, g_2 | B \rangle|^2 = |C|^2 |\langle g_1 | g_2 \rangle|^2. \]  

(7)

It follows that there is equal probability \( |C|^2 \) for finding one member in any state if nothing is given about its partner, but that the conditional probability for finding one member in state \( |g_1 \rangle \) if its partner is found in the state \( \langle g_2 | \) is

\[ p(g_1 | g_2) = |\langle g_1 | g_2 \rangle|^2. \]  

(8)

When the measure is scaled to make \( I \) the unit operator we find that

\[ \langle \langle B | B \rangle \rangle = |C|^2 V_F, \]  

(9)

which means that \( |B \rangle \rangle \) can be normalized by choosing \( C = V_F^{-1/2} \) if and only if \( V_F \) is finite, i.e. \( G \) is compact. Since we are interested in groups that may be locally compact but not compact we shall have to make sense of \( |B \rangle \rangle \) in general through a limiting process. This creates no problems because \( C \) disappears in computing the conditional probability (8).

We see from (8) that \( |B \rangle \rangle \) exhibits perfect EPR correlation, i.e. one concludes with certainty that one member of the pair will be in state \( |g \rangle \) if its partner is in state \( \langle g | \). If \( U \) is is the spin-1/2 representation of \( SU_2 \) and \( \tau \) is the time-reversal operator, it is shown in the Appendix that \( |B \rangle \rangle \) is the familiar Bohm-Ahronov singlet. We shall refer to \( |B \rangle \rangle \) as the \textit{generalized Bell state canonically associated with the manifold} \( F \).

Let us now see that measurement of \( p(g_1 | g_2) \) using \( |B \rangle \rangle \) reveals the metric structure of the manifold \( F \): Because \( \langle g_1 | g_2 \rangle \) is a scalar-product, it follows that

\[ d(g_1, g_2) \equiv \sqrt{1 - |\langle g_1 | g_2 \rangle|^2} \]  

(10)
is a metric on \( F \), i.e. it is non-negative, symmetric, obeys the triangle inequality, and vanishes if and only if \( g_1 \approx g_2 \) where “\( \approx \)” means membership in the same coset of \( G_o \). Since

\[
\langle g_1 | g_2 \rangle = \langle 0 | g_1^\dagger g_2 | 0 \rangle = \langle 0 | g_1^{-1} g_2 | 0 \rangle,
\]

we have

\[
p(g_1 | g_2) = |\langle 0 | g | 0 \rangle|^2 \equiv p(g), \ g = g_1^{-1} g_2.
\]

Observe that this is invariant under the substitution

\[
g \rightarrow g_o g g'_o, \ g_o, g'_o \in G_o.
\]

Thus \( p(g) \) is constant on the double cosets \( G_o \backslash g / G_o \). The distance \( d(g_1, g_2) \) can be written

\[
d(g_1, g_2) = d(g) \equiv \sqrt{1 - p(g)}, \ g = g_1^{-1} g_2
\]

and is therefore a function of the double coset to which the relation \( g \) between \( g_1 \) and \( g_2 \) belongs. The double cosets partition \( G \) just as left and right cosets do\(^7\). We shall refer to the double cosets as coherence relations between the two states and refer to \( d(g) \) as the “diameter” of the coherence relation \( g \).

The function \( d(g) \) has remarkable properties which follow from the metric properties of \( d(g_1, g_2) \):

\[
0 \leq d(g) \leq 1,
\]

where \( d(g) = 0 \) if and only if \( g \) is the identity coherence relation;

\[
d(g) = d(g^{-1});
\]

\[
d(g) + d(h) \geq d(gh).
\]

Now observe that each \( g_o \in G_o \) determines an automorphism \( g \in G \rightarrow g' = g_o g g_o^{-1} \in G \) which leaves \( F \) invariant since \( G_o \rightarrow G_o \). Since \( d(g_1', g_2') = \)
$d(g_1, g_2)$ this is an isometry of $F$ regarded as a manifold. The one-parameter subgroups $t \in R \rightarrow g_o(t)$ of $G_o$ define dynamical processes that take $F$ into itself, i.e.

$$|g\rangle \rightarrow |g(t)\rangle, \quad g(t) = g_o(t) g g_o(t)^{-1}. \quad (16)$$

Since $G$ is a Lie group, we will be able to write

$$g_o(t) = e^{-itH} \quad (17)$$

with some Hermitian operator $H$ and so obtain a Schrödinger equation in the Heisenberg picture:

$$dg(t)/dt = -i[H, g(t)]. \quad (18)$$

Thus the linear dynamics is preserved even though the space of allowed states is not linearly closed.

Now let us calculate the diameter of the coherence relation between nearby points on an orbit generated by $H$. We find for small $\delta t$:

$$d_g(\delta t) \equiv d(g(-\delta t/2), g(\delta t/2)) = d(g^{-1} e^{-iH\delta t} g) = \delta t \Delta_g(H), \quad (19)$$

where

$$\Delta_g(H) \equiv (\langle g|H^2|g\rangle - \langle g|H|g\rangle^2)^{1/2} \quad (20)$$

is the dispersion of $H$ in the state $|g\rangle$. Thus we arrive at the useful conclusion that the dispersion of the generators of the one-parameter subgroups of $G_o$ determine the local differential geometry of the manifold, and this is expressed by the diameters of coherence relations in the neighborhood of the identity.

Bell’s EPR Theorem informs us that it is not possible to construct a local hidden variable theory that will reproduce the function $p(g_1|g_2)$. Such
a theory would provide a map from states $|g\rangle$ to sets $\Lambda(g)$ with measure $\mu$ such that

$$p(g_1|g_2) = \mu(\Lambda(g_1) \cap \Lambda(g_2)).$$

(21)

In fact the incompatibility between the right and left sides of (21) is known to be a consequence of the difference between the metric structures implied by them. A new possibility emerges, however, from our restriction of allowed states to $\mathcal{F}$ which makes $p(g_1|g_2)$ a function only of the combination $g = g_1^{-1}g_2$. For let us suppose that in each run of a correlation experiment with $|B\rangle\rangle$ a random element $h$ of $\mathcal{F}$ serving as a hidden variable is generated in such a way that the probability of an $h$ with $d(h) < r$ is $r^2$. Suppose further that a correlation between detectors for $|g_1\rangle$ and $\langle g_2|$ is observed when the diameter $d(g)$ of the relation $g = g_1^{-1}g_2$ between them is smaller than that of $h$ and otherwise not. Then the probability of a correlation will be $1 - d(g)^2 = p(g)$, i.e. we will reproduce the quantum mechanical result. Since $d(g)$ is invariant when $g_1$ and $g_2$ evolve under the dynamical transformation (17), this type of hidden variable theory preserves the $G_o$ symmetry of the theory. In particular in relativistic theories where $G_o$ includes the Poincaré group the non-locality does not destroy the covariance of the theory. Because the hidden variable $h$ is non-local there is no violation of Bell’s Theorem.

We now illustrate the general ideas above by an important example, namely the quantum surveying of the electromagnetic field of a laser. We present the analysis for a single mode laser and then note its generalization to any number of modes.

The relevant group $G$ is the Weyl-Heisenberg (WH) group. This group has, according to the Stone-von Neumann theorem, a unique unitary representation (up to equivalence) known as the Fock representation obtained as follows: Let $a, a^\dagger$ be the Bose operators with the commutation rules
\[ [a, a^\dagger] = I. \] The group elements are then
\[ U(\theta, \lambda) = e^{i\theta} u(\lambda), \quad u(\lambda) = e^{\lambda^\dagger a^\dagger - \lambda a}. \] (22)

Here \( \lambda \) ranges over the finite complex plane and \( 0 \leq \theta < 2\pi \). The composition law is obtained from
\[ u(\lambda) u(\mu) = e^{i\theta} u(\lambda + \mu), \quad \theta = \text{Im}(\lambda^\ast \cdot \mu). \] (23)

Taking the reference state \( |0\rangle \) to be the Fock vacuum, i.e. the state annihilated by \( a \), the stability subgroup is \( G_o = U(\theta, 0), \ 0 \leq \theta < 2\pi \). Hence the manifold \( F = G/G_o \) of coherent states is in one-one correspondence with the set \( \lambda \), i.e. with the complex plane. The area element \( d^2 \lambda \) is an invariant measure on \( F \). We may thus take \( F \) to be the set of states of the form
\[ |\lambda\rangle \equiv u(\lambda)|0\rangle = e^{-|\lambda|^2/2} e^{\lambda^\dagger a^\dagger}|0\rangle \] (24)

which we recognize as the familiar Glauber\(^9\) coherent states of optics, and we then have
\[ p(\lambda|\mu) = |\langle \lambda|\mu \rangle|^2 = e^{-|\lambda-\mu|^2} = p(\lambda - \mu), \] (25)

and (1) becomes:
\[ I = \pi^{-1} \int d^2 \lambda |\lambda\rangle \langle \lambda|. \] (26)

To construct a generalized Bell state we observe that (3) gives
\[ |B\rangle\rangle = C\pi^{-1} \int d^2 \lambda |\lambda\rangle \otimes \langle \lambda| = C \int d^2 \lambda e^{-|\lambda|^2} (e^{\lambda^\dagger a^\dagger} \otimes e^{\lambda^\ast a})(|0\rangle \otimes \langle 0|) = \]
\[ C(e^{a^\dagger \otimes a})(|0\rangle \otimes \langle 0|). \] (27)

As noted above the fact that the WH group has infinite volume (i.e. is not compact) means that \( |B\rangle\rangle \) is not normalizable. However, we can make sense of it as follows. For \( 0 \leq r < 1 \) one verifies that the norm of the state
\[ |B, r\rangle\rangle \equiv (1 - r^2)^{1/2} e^{r(a^\dagger \otimes a)}(|0\rangle \otimes \langle 0|) \] (28)
is unity. Although the state becomes improper as \( r \to 1 \), the singular normalization factor will, as noted above, disappear when conditional probabilities are computed. We may thus perform our surveying by studying correlations in the state \( |B, r\rangle \rangle \), and compute the metric properties from the limit of the correlations as \( r \to 1 \) at which they become perfect EPR correlations.

It is instructive to observe that \( |B, r\rangle \rangle \) can be regarded as a “twisted” form of the two particle vacuum \( |0\rangle \otimes \langle 0| \). For observe that the operator

\[
a_r \equiv (1 - r^2)^{-1/2}(a - ra^\dagger)
\]  

(29)
satisfies

\[
[a_r, a_r^\dagger] = 1, \ (a_r \otimes I)|B, r\rangle \rangle = 0 = (I \otimes a_r^\dagger)|B, r\rangle \rangle.
\]  

(30)

Thus the states \( |B, r\rangle \rangle \) for \( 0 \leq r < 1 \) are normalizable two-particle states in which there is a correlation that approaches perfect EPR correlation as \( r \to 1 \). The fact that the states \( |B, r\rangle \) are twisted vacua removes some of the mystery from the non-locality of EPR correlations.

Observe that the unitary operator

\[
V(t) \equiv e^{-itH}, \ H = \omega a^\dagger a
\]  

(31)

has the properties

\[
V(t)aV^{-1}(t) = e^{-i\omega t}a, \ V(t)|0\rangle = |0\rangle.
\]  

(32)

Hence we can enlarge the WH group by adjoining \( V(t) \). This simply enlarges the stability subgroup of \( |0\rangle \) so that it now contains the one parameter subgroup \( g_0(t) = V(t) \). The dispersion of \( H \) in the state \( |\lambda\rangle \) is readily computed to be

\[
\Delta_\lambda(H) = \omega|\lambda|.
\]  

(33)
Hence the diameter of the coherence relation between two nearby points on an orbit generated by $V(t)$ passing through $|\lambda\rangle$ will be

$$d_\lambda(\delta t) = |\lambda|\omega\delta t = \sqrt{N_\lambda\omega\delta t},$$

(34)

where $N_\lambda = |\lambda|^2$ is the mean photon number in the state $\lambda$.

Let us now see how the non-local hidden variable theory described above works for this example. Suppose that on each run of an experiment a random element labeled by $\lambda$ of the Weyl-Heisenberg group is generated such that the probability of choosing $\lambda$ in an area $d^2\lambda$ centered on $\lambda$ is $\rho(\lambda)d^2\lambda$ with

$$\rho(\lambda) = \pi^{-1}e^{-|\lambda|^2}. \quad (35)$$

Then since $d(\lambda) = \sqrt{1 - e^{-|\lambda|^2}}$, the probability for $d(\lambda) \geq d(\lambda_o)$ is the probability for $|\lambda| \geq |\lambda_o|$, which is

$$\pi^{-1}\int_{|\lambda| \geq |\lambda_o|} d^2\lambda e^{-|\lambda|^2} = e^{-|\lambda_o|^2} = 1 - d(\lambda_o)^2. \quad (36)$$

Thus the probability for $d(\lambda) < r$ is $r^2$. Thus if a correlation between $|\lambda_1\rangle$ and $|\lambda_2\rangle$ is recorded whenever $|\lambda_1 - \lambda_2| \leq |\lambda|$ we will reproduce the quantum mechanical prediction that the probability of correlation is $e^{-|\lambda_1 - \lambda_2|^2}$. Thus the required distribution for the non-local hidden variable is the Maxwellian distribution (35).

Let us now see how the results for a single mode field generalize to an n-mode field with bose operators $a = (a_1, \cdots, a_n)$ satisfying $[a_i, a_j^{\dagger}] = \delta_{ij}I$, $[a_i, a_j] = 0$. The group G is now the direct product of n copies of the Weyl-Heisenberg group and the coherent states are still described by (24) if we interpret $\lambda$ is an n-component complex vector with $|\lambda|^2 = |\lambda_1|^2 + \cdots + |\lambda_n|^2$, and $a = (a_1, \cdots, a_n)$. The reference state $|0\rangle$ is the n-mode Fock
vacuum. We can now describe an enormous group of dynamical processes by adjoining the group of transformations:

\[ V_H(t) = e^{-it\mathcal{H}}, \mathcal{H} = a^\dagger \cdot H \cdot a \]  

(37)

in which \( H \) ranges over the set of all \( n \otimes n \) Hermitian matrices. \( V_H(t) \) is unitary (for real \( t \)) because

\[ a^\dagger \cdot H \cdot a = \sum_{j,k=1}^{n} a_j^\dagger H_{jk} a_k \]  

(38)

is a Hermitian operator. Since it is quadratic in the bose operators one readily verifies that

\[ V_H(t)|\lambda\rangle = |\lambda(t)\rangle, \lambda(t) = e^{-iHt}\lambda. \]  

(39)

Thus the unitary evolution of the state vector in \( F \) is described by that of an \( n \)-component vector \( \lambda(t) \) which evolves under a unitary transformation \( e^{-iHt} \).

Let us now summarize our results: Any quantum mechanical system that can be characterized by transformations \( g \) belonging to a locally compact group \( G \) of a reference state \( |0\rangle \) can be identified with a manifold \( F \) of generalized coherent states. There will be a canonically associated generalized Bell state \( |B\rangle\rangle \) which serves to determine the metric geometry of the manifold through observable EPR correlations. Dynamical transformations are identified with elements of the stability subgroup \( G_o \) of the reference state, and such transformations take the states of \( F \) into one another. Thus the dynamics remains linear even though \( F \) is not a linear space. There is a canonically associated hidden variable interpretation of the stochastic behavior which, albeit non-local, preserves the symmetry of the theory under \( G_o \). In particular this will be the case in a covariant theory in which \( G_o \) includes the Poincaré group.
Appendix

Let us examine $|B\rangle\rangle$ when $G$ is $SU_2$. Choosing $|0\rangle$ as the north-pole on a sphere, a state which is stabilized by a $U_1$ subgroup, we see that $F$ can be identified with the points of the sphere (the Poincaré sphere). The invariant measure is the solid angle $d\Omega$ and we find with suitable $C$:

$$|B\rangle\rangle = C \int d\Omega (\cos \theta \ e^{-i\phi} \sin \theta) \otimes \left( \begin{array}{c} \cos \theta \\ e^{i\phi} \sin \theta \end{array} \right)$$

$$= 2^{-1/2} \left( (1,0) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + (0,1) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) = 2^{-1/2} (|\uparrow\rangle \otimes \langle \uparrow | + |\downarrow\rangle \otimes \langle \downarrow |) . \quad (A1)$$

Choosing $\tau$ to be the time-reversal operator for which $|\uparrow\rangle^\tau = |\downarrow\rangle$ and $|\downarrow\rangle^\tau = -|\uparrow\rangle$, the mapping $\langle x| \rightarrow |x^\tau\rangle$ gives

$$|B\rangle\rangle = 2^{-1/2} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) , \quad (A2)$$

which is recognized as the Bohm-Aharonov singlet.

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