RICCI-FLAT KÄHLER METRICS ON TANGENT BUNDLES OF RANK-ONE
SYMMETRIC SPACES OF COMPACT TYPE

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ABSTRACT. We give an explicit description of all complete $G$-invariant Ricci-flat Kähler
metrics on the tangent bundle $T(G/K) \cong G^C/K^C$ of rank-one Riemannian symmetric spaces
$G/K$ of compact type, in terms of associated vector-functions.

1. INTRODUCTION

Over the latest decades there has been considerable interest in Ricci-flat Kähler metrics
whose underlying manifold is diffeomorphic to the tangent bundle $T(G/K)$ of a Riemannian
symmetric space $G/K$ of compact type. For instance, a remarkable class of Ricci-flat Kähler
manifolds of cohomogeneity one was discovered by M. Stenzel [18]. This has originated a
great deal of papers. To cite but a few: M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope [5]
studied certain harmonic forms on these manifolds and found an explicit formula for the
Stenzel metrics in terms of hypergeometric functions. Earlier, T. C. Lee [11] gave an explicit
formula of the Stenzel metrics for classical spaces $G/K$ but in another vein, using the ap-
proach of G. Patrizio and P. Wong [17]. Remark also that in the case of the standard sphere
$S^2$, the Stenzel metrics coincide with the well-known Eguchi-Hanson metrics [7]. On the
other hand, and as it is well known, Stenzel metrics continue being a source of results both
in physics and differential geometry. We cite here only to G. Oliveira [15] and M. Ionel and
T. A. Ivey [10].

In the present paper we give an explicit description of all complete $G$-invariant Ricci-flat
Kähler metrics on the tangent bundle $T(G/K)$ of rank-one Riemannian symmetric spaces
$G/K$ of compact type or, equivalently, on the complexification $G^C/K^C$ of $G/K$. To this end,
reached in our main assertions (Theorem 4.1 and its Corollary 4.3), we use the method of
our article [8], giving the result in terms of associated vector-functions (see below in this
introduction). In this article it is also shown that this set of metrics contains a new family
of metrics which are not $\partial \bar{\partial}$-exact if $G/K \in \{\mathbb{C}P^n, n \geq 1\}$, and coincides with the set of
$\partial \bar{\partial}$-exact Stenzel metrics for any of the latter spaces $G/K$.

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Remark here that until now, in the case of the space $\mathbb{CP}^n \ (n \geq 1)$, all known Ricci-flat Kähler metrics were Calabi metrics, so being hyper-Kählerian and thus automatically Ricci-flat (see O. Biquard and P. Gauduchon [2, 3] and E. Calabi [4]). Since by A. Dancer and M.Y. Wang [6, Theorem 1.1] any complete $G$-invariant hyper-Kählerian metric on $G/K = \mathbb{CP}^n \ (n \geq 2)$ coincides with the Calabi metric, our new metrics are not hyper-Kählerian.

Note also, that in [6] the Kähler-Einstein metrics on manifolds of $G$-cohomogeneity one were classified but only under one additional assumption: It is assumed that the isotropy representation of the space $G/H$ (see our notation below) splits into pairwise inequivalent sub-representations. This condition is crucial for the fact that the Einstein equation can be solved (see [6, Theorem 2.18]). But this assumption fails, for instance, for the symmetric space $\mathbb{CP}^n \ (n \geq 2)$.

Let $G/K$ be a rank-one symmetric space of a compact connected Lie group $G$. The tangent bundle $T(G/K)$ has a canonical complex structure $J^K_c$ coming from the $G$-equivariant diffeomorphism $T(G/K) \to G^C/K^C$. The latter space is the above-mentioned complexification of $G/K$. In our paper [8] we described, for such a $G/K$, all $G$-invariant Kähler structures $(\mathfrak{g}, J^K_c)$ which are moreover Ricci-flat on the punctured tangent bundle $T^+(G/K)$ of $T(G/K)$. This description is based on the fact that $T^+(G/K)$ is the image of $G/H \times \mathbb{R}^+ \subset T(G/K)$ under certain $G$-equivariant diffeomorphism. Here $H$ denotes the stabilizer of any element of $T(G/K)$ in general position. Such $G$-invariant Kähler and Ricci-flat Kähler structures are determined completely by a unique vector-function $a : \mathbb{R}^+ \to \mathfrak{g}_H$ satisfying certain conditions, $\mathfrak{g}_H$ being the subalgebra of $\text{Ad}(H)$-fixed points of the Lie algebra of $G$.

As for the contents, we recall in Section 2 some definitions and results on the canonical complex structure on $T(G/K)$. In Section 3 we recall the general description given in [8] of invariant Ricci-flat Kähler metrics on tangent bundles of Riemannian symmetric spaces of compact type, especially in Theorems 3.2 and 3.5 below, given here without proof. In Section 4 we state and prove Theorem 4.1 and its Corollary 4.3 giving the invariant Ricci-flat Kähler metrics on the punctured tangent bundles $T^+(G/K)$ of the rank-one Riemannian symmetric spaces of compact type and then the complete invariant Ricci-flat Kähler metrics on $T(G/K)$.

2. THE CANONICAL COMPLEX STRUCTURE ON $T(G/K)$

Consider a homogeneous manifold $G/K$, where $G$ is a compact connected Lie group and $K$ is some closed subgroup of $G$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$ respectively. There exists a positive-definite $\text{Ad}(G)$-invariant form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.

Denote by $\mathfrak{m}$ the $\langle \cdot, \cdot \rangle$-orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$, that is, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ is the $\text{Ad}(K)$-invariant vector space direct sum decomposition of $\mathfrak{g}$. Consider the trivial vector bundle $G \times \mathfrak{m}$ with the two Lie group actions (which commute) on it: the left $G$-action, $l_h : (g, w) \mapsto (hg, w)$ and the right $K$-action $r_k : (g, w) \mapsto (gk, \text{Ad}_{k^{-1}}w)$. Let

$$\pi : G \times \mathfrak{m} \to G \times_K \mathfrak{m}, \quad (g, w) \mapsto [(g, w)],$$
be the natural projection for this right $K$-action. This projection is $G$-equivariant. It is well known that $G \times_K m$ and $T(G/K)$ are diffeomorphic. The corresponding $G$-equivariant diffeomorphism

$$\phi : G \times_K m \to T(G/K), \quad [(g, w)] \mapsto \frac{d}{dt} \bigg|_0 g \exp(tw) K,$$

and the projection $\pi$ determine the $G$-equivariant submersion $\Pi = \phi \circ \pi : G \times m \to T(G/K)$.

Let $G^C$ and $K^C$ be the complexifications of the Lie groups $G$ and $K$. In particular, $K$ is a maximal compact subgroup of the Lie group $K^C$ and the intersection of $K$ with each connected component of $K^C$ is not empty (cf. A.L. Onishchik and E.V. Vinberg [16, Ch. 5, p. 221] and note that $G^C$, $K^C$, $G$ and $K$ are algebraic groups). Let $g^C = g \oplus ig$ and $\mathfrak{k}^C = \mathfrak{k} \oplus i\mathfrak{k}$ be the complexifications of the compact Lie algebras $g$ and $\mathfrak{k}$.

Since $G$ and $K$ are maximal compact Lie subgroups of $G^C$ and $K^C$, respectively, by a result of G.D. Mostow [12, Theorem 4], we have that $K^C = K \exp(it)$, $G^C = G \exp(im) \exp(it)$, and the mappings

$$G \times m \times \mathfrak{k} \to G^C, \quad (g, w, \zeta) \mapsto g \exp(iw) \exp(i\zeta),$$

$$K \times \mathfrak{k} \to K^C, \quad (k, \zeta) \mapsto k \exp(i\zeta),$$

are diffeomorphisms. Then the map

$$f^*_K : G^C/K^C \to G \times_K m, \quad g \exp(iw) \exp(i\zeta) K^C \mapsto [(g, w)],$$

is a $G$-equivariant diffeomorphism [13, Lemma 4.1]. It is clear that

$$f_K : G^C/K^C \to T(G/K), \quad g \exp(iw) \exp(i\zeta) K^C \mapsto \Pi(g, w),$$

is also a $G$-equivariant diffeomorphism. Since $G^C/K^C$ is a complex manifold, the diffeomorphism $f_K$ supplies the manifold $T(G/K)$ with the $G$-invariant complex structure which we denote by $J^C_K$.

3. Invariant Ricci-flat Kähler metrics on tangent bundles of compact Riemannian symmetric spaces. General description

We continue with the previous notations but in this section and the next one it is assumed in addition that $G/K$ is a rank-one Riemannian symmetric space of a connected, compact semisimple Lie group $G$.

3.1. Root theory of Riemannian symmetric spaces of rank one. Here we will review a few facts about Riemannian symmetric spaces of rank one [9, Ch. VII, §2, §11] and results of our paper [8] adapted to the case of these (rank one) spaces.

We have then

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}, \quad \text{where} \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad \text{and} \quad \mathfrak{k} \perp \mathfrak{m}.$$

In other words, there exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that

$$\mathfrak{k} = (1 + \sigma)\mathfrak{g} \quad \text{and} \quad \mathfrak{m} = (1 - \sigma)\mathfrak{g}.$$

Moreover, the scalar product $\langle \cdot, \cdot \rangle$ is $\sigma$-invariant.
Let \( a \subset m \) be some Cartan subspace of the space \( m \). There exists a \( \sigma \)-invariant Cartan subalgebra \( t \) of \( g \) containing the commutative subspace \( a \), i.e.

\[
t = a \oplus t_0, \quad \text{where} \quad a = (1 - \sigma)t, \quad t_0 = (1 + \sigma)t.
\]

Then the complexification \( t^C \) is a Cartan subalgebra of the reductive complex Lie algebra \( g^C \) and we have the root space decomposition

\[
g^C = t^C \oplus \sum_{\alpha \in \Delta} \tilde{g}_\alpha.
\]

Here \( \Delta \) is the root system of \( g^C \) with respect to the Cartan subalgebra \( t^C \). For each \( \alpha \in \Delta \) we have

\[
\tilde{g}_\alpha = \{ \xi, \tilde{\xi} \in g^C : \text{ad}_i \tilde{\xi} = \alpha(i) \tilde{\xi}, \tilde{\xi} \in t^C \} \quad \text{and dim}_\mathbb{C} \tilde{g}_\alpha = 1.
\]

It is evident that the centralizer \( \tilde{g}_0 \) of the space \( \alpha^C \) in \( g^C \) is the subalgebra

\[
(3.1) \quad \tilde{g}_0 = t^C \oplus \sum_{\alpha \in \Delta_0} \tilde{g}_\alpha,
\]

where \( \Delta_0 = \{ \alpha \in \Delta : \alpha|_{\alpha^C} = 0 \} \) is the root system of the reductive Lie algebra \( \tilde{g}_0 \) with respect to its Cartan subalgebra \( t^C \).

The set \( \Sigma = \{ \lambda \in (\alpha^C)^* : \lambda = \alpha|_{\alpha^C}, \alpha \in \Delta \setminus \Delta_0 \} \) is the set of restricted roots of the triple \((g, t, a)\), which is independent of the choice of the \( \sigma \)-invariant Cartan subalgebra \( t \) containing the Cartan subspace \( a \).

Since \( G/K \) is a rank-one Riemannian symmetric space, \( \dim a = 1 \). Then the restricted root system is either \( \Sigma = \{ \pm \epsilon \} \) or \( \Sigma = \{ \pm \epsilon, \pm \frac{1}{2} \epsilon \} \), where \( \epsilon \in (\alpha^C)^* \). There exists a unique (basis) vector \( X \in a \) such that \( \epsilon(X) = 1 \), where, since the algebra \( g \) is compact, \( \alpha(t) \subset i\mathbb{R} \) for each \( \alpha \in \Delta \). It is clear that multiplying our scalar product \( \langle \cdot, \cdot \rangle \) by a positive constant we can suppose that \( \langle X, X \rangle = 1 \). For each \( \lambda \in \Sigma \) define the linear function \( \lambda' : a \to \mathbb{R} \), by the relation \( i\lambda' = \lambda \). Note that then

\[
\langle X, X \rangle = 1, \quad i\epsilon' = \epsilon, \quad \text{and} \quad \epsilon'(X) = 1.
\]

Since the algebra \( \tilde{g}_0 \) coincides with the centralizer of the element \( X \in a \) in \( g^C \), there exists a basis \( \Pi \) of \( \Delta \) (a system of simple roots) such that \( \Pi_0 = \Pi \cap \Delta_0 \) is a basis of \( \Delta_0 \). Indeed, the element \( -iX \in it \) belongs to the closure of some Weyl chamber in \( it \) determining the basis \( \Pi \). Then \( \Pi_0 = \{ \alpha \in \Pi : \alpha(-iX) = 0 \} \). The bases \( \Pi \) and \( \Pi_0 \) determine uniquely the subsets \( \Delta^+ \) and \( \Delta_0^+ \) of positive roots of \( \Delta \) and \( \Delta_0 \), respectively. It is evident that

\[
\Delta^+ \setminus \Delta_0^+ = \{ \alpha \in \Delta : \alpha(-iX) > 0 \}.
\]

The following decomposition

\[
g^C = \tilde{g}_0 \oplus \sum_{\lambda \in \Sigma^+} (\tilde{g}_\lambda \oplus \tilde{g}_{-\lambda}), \quad \text{where} \quad \tilde{g}_\lambda = \sum_{\alpha \in \Delta \setminus \Delta_0, \alpha|_{\alpha^C} = \lambda} \tilde{g}_\alpha
\]

and \( \Sigma^+ \) denotes the subset of positive restricted roots in \( \Sigma \) determined by the set of positive roots \( \Delta^+ \), gives us a simultaneous diagonalization of \( \text{ad}(\alpha^C) \) on \( g^C \). Remark that in our
case either $\Sigma^+ = \{ \varepsilon \}$ or $\Sigma^+ = \{ \varepsilon, \frac{1}{2} \varepsilon \}$. Denote by $m_\lambda$ the multiplicity of the restricted root $\lambda \in \{ \pm \varepsilon, \pm \frac{1}{2} \varepsilon \}$, that is, $m_\lambda = \text{card}\{ \alpha \in \Delta : \alpha|_{\delta \varepsilon} = \lambda \}$.

For each linear form $\lambda$ on $\mathfrak{a}^C$ put

$$m_\lambda \overset{\text{def}}{=} \{ \eta \in \mathfrak{m} : \text{ad}_w^2(\eta) = \lambda^2(\eta) \},$$

$$\mathfrak{e}_\lambda \overset{\text{def}}{=} \{ \zeta \in \mathfrak{e} : \text{ad}_w^2(\zeta) = \lambda^2(\zeta) \}.$$

Then $m_\lambda = m_{-\lambda}$, $\mathfrak{e}_\lambda = \mathfrak{e}_{-\lambda}$, $m_0 = \mathfrak{a}$ and $\mathfrak{e}_0 = \mathfrak{h}$, where

$$\mathfrak{h} = \{ u \in \mathfrak{e} : [u, \mathfrak{a}] = 0 \} = (\ker \text{ad}_X) \cap \mathfrak{e}$$

is the centralizer of $\mathfrak{a}$ in $\mathfrak{e}$.

In Table 3.1 we list all compact Riemannian symmetric spaces of rank one with their corresponding multiplicities $m_\varepsilon, m_{\varepsilon/2}$ and type of the algebra $\mathfrak{h}$.

| $G/K$ | $\dim m_\varepsilon$ | $m_{\varepsilon/2}$ | $\mathfrak{h}$ |
|-------|-------------------|-------------------|----------------|
| $\mathbb{S}^n, (n \geq 2)$ | $n$ | $n-1$ | 0 |
| $[\mathbb{R} \mathbb{P}^m]^*$ | $[\text{SO}(n+1)/\text{SO}(n)]^*$ | $n$ | $n-1$ | $\text{so}(n-1)$ |
| $\mathbb{C} \mathbb{P}^n, (n \geq 2)$ | $\text{SU}(n+1)/\text{SU}(1) \times \text{U}(n)$ | $2n$ | $1$ | $2n-2$ |
| $\mathbb{H} \mathbb{P}^n, (n \geq 1)$ | $\text{Sp}(n+1)/\text{Sp}(1) \times \text{Sp}(n)$ | $4n$ | $3$ | $4n-4$ |
| $\text{CaP}^2$ | $\text{F}_4/\text{Spin}(9)$ | $16$ | $7$ | $8$ |
| $\text{so}(1) = 0, \text{su}(2) = \text{su}(1) = 0, \text{sp}(0) = 0$. The symmetric spaces $G/K$ with non-connected $K$ are marked with $[\cdot]^*$ in Table 3.1. |

It is clear that $m_\lambda^C \oplus \mathfrak{e}_\lambda^C = \mathfrak{g}_\lambda^C \oplus \mathfrak{e}_{-\lambda}^C$ for $\lambda \in \Sigma^+$ and $\mathfrak{g}_0^C = m_0^C \oplus \mathfrak{e}_0^C = \mathfrak{a}^C \oplus \mathfrak{h}^C$ (the Cartan subspace $\mathfrak{a}^C$ is a maximal commutative subspace of $m_\varepsilon^C$). By [9] Ch. VII, Lemma 11.3, the following decompositions are direct and orthogonal:

$$\mathfrak{m} = \mathfrak{a} \oplus m_\varepsilon \oplus m_{\varepsilon/2}, \quad \mathfrak{e} = \mathfrak{h} \oplus \mathfrak{e}_\varepsilon \oplus \mathfrak{e}_{\varepsilon/2},$$

where to simplify the notation we suppose that $m_{\varepsilon/2} = 0$ and $\mathfrak{e}_{\varepsilon/2} = 0$ if $\frac{1}{2} \varepsilon \not\in \Sigma$. We shall put

$$m^+ \overset{\text{def}}{=} m_\varepsilon \oplus m_{\varepsilon/2}, \quad \mathfrak{e}^+ \overset{\text{def}}{=} \mathfrak{e}_\varepsilon \oplus \mathfrak{e}_{\varepsilon/2}.$$

Since the restriction of the operator $\text{ad}_X$ to the subspace $m^+ \oplus \mathfrak{e}^+$ is nondegenerate and $\text{ad}_X(\mathfrak{m}) \subset \mathfrak{e}$, $\text{ad}_X(\mathfrak{e}) \subset \mathfrak{m}$ for any vector $\xi_{\lambda} \in m_{\lambda} \subset \mathfrak{m}$, $\lambda \in \Sigma^+$, by (3.2) and (3.4) there exists a unique vector $\xi_{\lambda} \in \mathfrak{e}_{\lambda}$ such that

$$[X, \xi_{\lambda}] = -\lambda'(X) \xi_{\lambda}, \quad [X, \xi_{\lambda}] = \lambda'(X) \xi_{\lambda},$$

where, recall, $\varepsilon'(X) = 1$. In particular, $\dim m_{\lambda} = \dim \mathfrak{e}_{\lambda} = m_{\lambda}$ and there exists a unique endomorphism $T : m^+ \oplus \mathfrak{e}^+ \rightarrow m^+ \oplus \mathfrak{e}^+$ such that

$$\text{ad}_X \big|_{m_{\lambda} \oplus \mathfrak{e}_{\lambda}} = \lambda'(X) T \big|_{m_{\lambda} \oplus \mathfrak{e}_{\lambda}}, \quad T(\mathfrak{m}_{\lambda}) = \mathfrak{e}_{\lambda}, \quad T(\mathfrak{e}_{\lambda}) = m_{\lambda}, \quad \forall \lambda \in \Sigma^+.$$
This endomorphism is orthogonal because $T^2 = -\text{Id}_{m^+ \oplus t^+}$ and the endomorphism $\text{ad}_x$ is skew-symmetric. Note also here that by (3.1) the subspace

$$t_0 = (1 + \sigma)t$$

is a Cartan subalgebra of the centralizer $h$ and $t = a \oplus t_0$. Moreover, since $[t_0, m] \subset m$ and $[t_0, t \subset t, [a, t_0] = 0$, from definitions (3.2) and (3.6) we obtain that

$$[t_0, m_{\lambda}] \subset m_{\lambda} \quad \text{and} \quad [t_0, t_{\lambda}] \subset t_{\lambda} \quad \text{for each} \quad \lambda \in \Sigma^+,$$

$$[\text{ad}_x, T] = 0 \quad \text{on} \quad m^+ \oplus t^+ \quad \text{for each} \quad x \in t_0.$$

Fix the Weyl chamber $W^+$ in $a$ containing the element $X$:

$$W^+ = \{ w \in a : \varepsilon(-iw) > 0 \} = \{ w \in a : \varepsilon'(w) > 0 \} = \mathbb{R}^+ X.$$

The subspace $m \subset g$ is $\text{Ad}(K)$-invariant. Each nonzero $\text{Ad}(K)$-orbit in $m$ intersects the Cartan subspace $a$ and also the Weyl chamber $W^+$, that is, $\text{Ad}(K)(W^+) = m \setminus \{0\}$. The set $m^R = m \setminus \{0\}$ of all nonzero elements of $m$ is the set of regular points in $m$.

Consider the centralizer $H$ of the Cartan subspace $a$ in $\text{Ad}(K)$, i.e.

(3.7) \quad $H = \{ k \in K : \text{Ad}_k u = u \ \text{for all} \ \ u \in a \} = \{ k \in K : \text{Ad}_k X = X \}$.

It is clear that the algebra $h$ (see (3.5)), is the Lie algebra of $H$.

Our interest now centers on what will be shown to be an important subalgebra of $g$. Let $g_H \subset g$ be the subalgebra of fixed points of the group $\text{Ad}(H)$, i.e.

(3.8) \quad $g_H \overset{\text{def}}{=} \{ u \in g : \text{Ad}_h u = u \ \text{for all} \ h \in H \}$.

It is evident that $g_H \subset g_h$, where

(3.9) \quad $g_h \overset{\text{def}}{=} \{ u \in g : [u, \zeta] = 0 \ \text{for all} \ \zeta \in h \}$

is the centralizer of the algebra $h$ in $g$. Note that in the general case one has $g_H \neq g_h$ (see Example 4.6 in [8]).

To understand the structure of the algebra $g_H$ we consider more carefully the centralizer $g_h$. Since $h$ is a compact Lie algebra, $h = z(h) \oplus [h, h]$, where $z(h)$ is the center of $h$ and $[h, h]$ is a maximal semisimple ideal of $h$. It is clear that

$$z(h) \subset g_h \quad \text{and} \quad g_h \cap [h, h] = 0 \quad \text{because} \quad \langle g_h, [h, h] \rangle = \langle [g_h, h], h \rangle = 0.$$

Therefore $g_h \cap h = z(h)$ and $g_h \oplus [h, h] = g_h + h$ is a subalgebra of $g$.

By its definition, $z(h)$ is a subspace of the center of the algebra $g_h$. Moreover, by (3.3), $a \subset g_h$. The space $a \oplus z(h) \subset g_h$ is a Cartan subalgebra of $g_h$ (a maximal commutative subalgebra of $g_h$) because the centralizer of $a$ in $g$ equals $a \oplus h$, $a \oplus z(h)$ is the center of the algebra $a \oplus h$ and $g_h \cap (a \oplus h) = a \oplus z(h)$ by definition of $g_h$ (see also [8, Subsection 4.1]).

Since $a \subset g_h$ and $t_0 \subset h$, then $a \oplus t_0 \subset g_h + h$. But $a \oplus t_0 = t$ is a Cartan subalgebra of $g$. This means that the complex reductive Lie algebras $(g_h + h)^C$, $g_h^C$ and $h^C$ are $\text{ad}(t^C)$-invariant subalgebras of $g^C$. Taking into account that $t \cap g_h = a \oplus z(h)$ and $t \cap h = t_0$, we
obtain the following direct sum decompositions:
\begin{equation}
(3.10) \quad g_h^C = a^C \oplus \hat{\mathfrak{g}}(h)^C \oplus \sum_{\alpha \in \Delta_h} \tilde{g}_\alpha \quad \text{and} \quad h^C = t_0^C \oplus \sum_{\alpha \in \Delta_0} \tilde{g}_\alpha,
\end{equation}
where $\Delta_h$ is some subset of the root system $\Delta$. Since the spaces $a \oplus \hat{\mathfrak{g}}(h) \subset t$ and $t_0 \subset t$ are Cartan subalgebras of the algebras $g_h$ and $h$ respectively, the decompositions above are the root space decompositions of $(g_h^C, (a \oplus \hat{\mathfrak{g}}(h))^C)$ and $(h^C, t_0^C)$, respectively. In particular, the subset $\Delta_h \subset \Delta$ is the root system of $(g_h^C, (a \oplus \hat{\mathfrak{g}}(h))^C)$.

Since $\mathfrak{h} \subset \mathfrak{t}$, we see that $\sigma(h) = h$ and the centralizer $g_h$ of $h$ in $g$ is $\sigma$-invariant. By [8, Proposition 4.3],
\begin{equation}
\Delta_h = \{ \alpha \in \Delta : \alpha(t_0) = 0, \alpha + \beta \not\in \Delta \text{ for all } \beta \in \Delta_0 \}.
\end{equation}
But by [8, Lemma 4.1] this subset $\Delta_h$ of the set of roots $\Delta$ admits the following alternative description:
\begin{equation}
(3.11) \quad \Delta_h = \{ \alpha \in \Delta : \alpha(t_0) = 0, m_\lambda = 1, \text{ where } \lambda = \alpha|_{a^C} \}.
\end{equation}
As follows from Table 3.1, two such restricted roots $\{ \epsilon, -\epsilon \} \subset \Sigma$ of multiplicity 1 exist if and only if $G/K \in \{ CP^n(n \geq 1), \mathbb{RP}^2 \} (CP^1 \cong S^2)$. Hence for any of the latter rank-one symmetric spaces $g_h = a \oplus \hat{\mathfrak{g}}(h)$. Since for these latter spaces $\hat{\mathfrak{g}}(h) = 0$ (see Table 3.1), we obtain that
\begin{equation}
(3.12) \quad g_h = a \quad \text{if} \quad G/K \not\in \{ CP^n(n \geq 1), \mathbb{RP}^2 \}.
\end{equation}

Since $\sigma(h) = h$, the centralizer $g_h$ of $h$ in $g$ is $\sigma$-invariant, i.e.
\begin{equation}
g_h = m_h \oplus \mathfrak{k}_h, \quad \text{where} \quad m_h = g_h \cap m, \quad \mathfrak{k}_h = g_h \cap \mathfrak{k}
\end{equation}
and as $a \subset m_h$ is a maximal commutative subspace of $m$, the space $a$ is a Cartan subspace of $m_h$. Then the set
\begin{equation}
(3.13) \quad m_h = a \oplus \sum_{\lambda \in \Sigma_h \cap \Sigma^+} m_\lambda, \quad \mathfrak{k}_h = \hat{\mathfrak{g}}(h) \oplus \sum_{\lambda \in \Sigma_h \cap \Sigma^+} \mathfrak{k}_\lambda.
\end{equation}
To describe the algebra $g_H \subset g_h$ we consider now in more detail the subgroup $H \subset K$. By [8, Proposition 4.4], $H = (\exp(a) \cap K)H_0$, where $H_0 = \exp h$ is the identity component of the Lie group $H$ ($H_0 \subset K$ because $h \subset \mathfrak{k}$). Since the group $H \subset K$ is compact and $K$ is a subgroup of the group of fixed points of certain involutive automorphism of $G$ acting by $\exp(v) \mapsto \exp(-v)$ on $\exp(a)$, the discrete group $D_a \defeq \exp(a) \cap K$ is finite and
\begin{equation}
(3.14) \quad D_a = \{ \exp v : v \in a, \exp v = \exp(-v) \} \cap K.
\end{equation}
Since $[h, g_h] = 0$, the group $Ad(H_0)$ acts trivially on $g_h$ and therefore
\begin{equation}
g_H = \{ u \in g_h : Ad_{\exp u}u = u \text{ for all } v \in a \text{ such that } \exp v \in D_a \}. 
\end{equation}
Taking into account that \([a, t] = 0\), we conclude that the group \(\text{Ad}_{\exp a}\) acts trivially on the space \(a \oplus \mathfrak{z}(h) \subset t\) and consequently, by (3.12),

\[
(3.15) \quad g_H = g_0 = a \quad \text{if} \quad G/K \not\in \{\mathbb{C}P^n (n \geq 1), \mathbb{R}P^2\},
\]

and \(g_H\) contains \(a \oplus \mathfrak{z}(h)\) otherwise. For the space \(G/K = \mathbb{C}P^n (n \geq 1)\) we will calculate the algebra \(g_H\) in the next section using the matrix representation for \(g \cong \mathfrak{su}(n+1)\).

The algebra \(g_H\) is \(\sigma\)-invariant because by definition (3.7), \(\sigma \text{Ad}(H)\sigma = \text{Ad}(H)\). In particular,

\[
g_H = m_H \oplus \mathfrak{t}_H, \quad \text{where} \quad m_H = g_H \cap m, \quad \mathfrak{t}_H = g_H \cap \mathfrak{t},
\]

and \((g_H, \mathfrak{t}_H)\) is a symmetric pair. By maximality conditions the space \(a \subset g_H\) is a Cartan subspace of \(m_H \subset g_H\) and the space \(a \oplus \mathfrak{z}(h)\) is a Cartan subalgebra of \(g_H\).

For each \(\lambda \in \Sigma^+\) and \(g \in D_a \subset \exp a\) we have that \(\text{Ad}_g(m_\lambda + \mathfrak{t}_\lambda) = m_\lambda + \mathfrak{t}_\lambda\) because \(\text{Ad}_{\exp \nu} = e^{ad\nu}\). The set

\[
\Sigma_H = \{\lambda \in \Sigma_h : \text{Ad}_g|_{m_\lambda \oplus \mathfrak{t}_\lambda} = \text{Id}_{m_\lambda \oplus \mathfrak{t}_\lambda} \text{ for all } g \in D_a\}
\]

is the set of restricted roots of the triple \((g_H, \mathfrak{t}_H, a)\). By (3.11) each element \(\lambda \in \Sigma_H \subset \Sigma_h \subset \Sigma\) has multiplicity 1 as an element of \(\Sigma\), that is, \(\dim m_\lambda = \dim \mathfrak{t}_\lambda = 1\).

The following decompositions are direct and orthogonal:

\[
m_H = a \oplus \sum_{\lambda \in \Sigma_H \cap \Sigma^+} m_\lambda, \quad \mathfrak{t}_H = \mathfrak{z}(h) \oplus \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \mathfrak{t}_\lambda.
\]

**Remark 3.1.** Put \(m_H^+ = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} m_\lambda\) and \(\mathfrak{t}_H^+ = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \mathfrak{t}_\lambda\). Consider the orthogonal decompositions: \(m^+ = m_H^+ \oplus m_H^+\) and \(\mathfrak{t}^+ = \mathfrak{t}_H^+ \oplus \mathfrak{t}_H^+\), where \(m_H^+ = \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m_\lambda\) and \(\mathfrak{t}_H^+ = \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \mathfrak{t}_\lambda\). Since the decompositions

\[
g_H = a \oplus m_H^+ \oplus \mathfrak{t}_H^+ \oplus \mathfrak{z}(h), \quad g = a \oplus m_H^+ \oplus \mathfrak{t}_H^+ \oplus m_H^+ \oplus \mathfrak{t}_H^+ \oplus \mathfrak{z}(h) = g_H \oplus (m_H^+ \oplus \mathfrak{t}_H^+) \oplus [h, h]
\]

are orthogonal and \([g_H, h] = 0\), one has that \(g_H \oplus [h, h]\) is a subalgebra of \(g\).

Moreover, because of its definition, \(T(m_\lambda) = \mathfrak{t}_\lambda, T(t_\lambda) = m_\lambda\) for all restricted roots \(\lambda \in \Sigma^+, \) we obtain that

\[
T(m_H^+) = \mathfrak{t}_H^+, \quad T(m_H^+) = m_H^+ \quad \text{and} \quad T(m_H^+) = m_H^+.
\]

Fix in each subspace \(m_\lambda, \lambda \in \Sigma^+,\) some basis \(\{\xi^j_\lambda, j = 1, \ldots, m_\lambda\}\), orthonormal with respect to the form \(\langle \cdot, \cdot \rangle\). In the case when \(\lambda \in \Sigma_0 \cap \Sigma^+, m_\lambda = 1\) we have a unique vector \(\xi^1_\lambda\).

As we remarked above, for each \(\lambda \in \Sigma^+\) there exists a unique basis \(\{\xi^j_\lambda, j = 1, \ldots, m_\lambda\}\) of \(\mathfrak{t}_\lambda\) such that for each pair \(\{\xi^j_\lambda, \xi^j_\lambda, j = 1, \ldots, m_\lambda\}\), condition (3.5) holds. The basis \(\{\xi^j_\lambda, j = 1, \ldots, m_\lambda\}, \lambda \in \Sigma^+,\) of \(\mathfrak{t}_\lambda,\) is also orthonormal due to the orthogonality of the operator \(T\) (see (3.6)). Fix also some orthonormal basis \(\{\xi^k_{\lambda}, k = 1, \ldots, \dim \mathfrak{h}\}\) of the centralizer \(\mathfrak{h}\) of \(a\) in \(\mathfrak{t}\). We will use the orthonormal basis

\[
X, \xi^j_\lambda, \xi^j_\lambda, j = 1, \ldots, m_\lambda, \lambda \in \Sigma^+; \xi^k_{\lambda}, k = 1, \ldots, \dim \mathfrak{h},
\]

of the algebra \(g\) in our calculations below.
3.2. The canonical complex structure on \( G/H \times W^+ \cong G/H \times \mathbb{R}^+ \). By definition (3.7) of the group \( H \), the map

\[
K/H \times W^+ \to \mathfrak{m}^R, \quad (kH,w) \mapsto \text{Ad}_k w,
\]

is a well-defined diffeomorphism because, recall, \( W^+ = \mathbb{R}^+ X \) and \( \mathfrak{m}^R = \mathfrak{m} \setminus \{0\} \). Thus the map

\[
f^+: G/H \times W^+ \to G \times_K \mathfrak{m}^R, \quad (gH,w) \mapsto [(g,w)],
\]

is a well-defined \( G \)-equivariant diffeomorphism of \( G/H \times W^+ \) onto the subset \( D^+ = G \times_K \mathfrak{m}^R \), which is an open dense subset of \( G \times_K \mathfrak{m} \).

It is clear that the diagram

\[
\begin{array}{ccc}
G \times W^+ & \xrightarrow{\text{id}} & G \times \mathfrak{m}^R \\
\downarrow \pi_H \times \text{id} & & \downarrow \pi \\
G/H \times W^+ & \xrightarrow{f^+} & G \times_K \mathfrak{m}^R \\
\end{array}
\]

(3.16)

where \( \pi_H: G \to G/H \) is the canonical projection, is commutative.

Denote by \( \xi^l \) the left \( G \)-invariant vector field on \( G \) corresponding to \( \xi \in \mathfrak{g} \). The submersion (projection) \( \pi: G \times \mathfrak{m} \to G \times_K \mathfrak{m} \) is (left) \( G \)-equivariant. Therefore, the kernel \( \mathcal{K} \subset T(G \times \mathfrak{m}) \) of the tangent map \( \pi_* : T(G \times \mathfrak{m}) \to T(G \times_K \mathfrak{m}) \) is generated by the global (left) \( G \)-invariant vector fields \( \xi^L \), for \( \xi \in \mathfrak{t} \), on \( G \times \mathfrak{m} \),

\[
(3.17) \quad \xi^L(g,w) = (\xi^l(g), [w, \xi]) \in T_g G \times T_w \mathfrak{m},
\]

where the tangent space \( T_w \mathfrak{m} \) is canonically identified with the space \( \mathfrak{m} \).

To describe the \( G \)-invariant Ricci-flat Kähler metrics on \( T(G/K) \) associated to the canonical complex structure \( J^K_e \), we first attempt to describe such metrics on the punctured tangent bundle \( T^+(G/K) \equiv T(G/K) \setminus \{ \text{zero section} \} \) of \( G/K \). It is clear that \( T^+(G/K) = \phi(G \times_K \mathfrak{m}^R) \) and therefore

\[
T^+(G/K) = (\phi \circ f^+)(G/H \times W^+),
\]

that is, \( T^+(G/K) \) is \( G \)-equivariantly isomorphic to the direct product \( G/H \times W^+ \), where the action of the group \( G \) on the first component is the natural one and that on the second component is the trivial one (see the commutative diagram (3.16)). This \( G \)-equivariant diffeomorphism determines a \( G \)-invariant complex structure on \( G/H \times W^+ \), which we denote also by \( J^K_e \).

Note also here that the tangent space \( T_o(G/H) \) at \( o = \{ H \} \in G/H \) can be identified naturally with the space \( \mathfrak{m} \oplus \mathfrak{t}^+ = \mathfrak{a} \oplus \mathfrak{m}^+ \oplus \mathfrak{k}^+ \), because by definition \( \mathfrak{t} = \mathfrak{h} \oplus \mathfrak{t}^+ \) and \( \mathfrak{h} \) is the Lie algebra of the group \( H \).

Considering the coordinate \( x \) on \( W^+ = \mathbb{R}^+ X \) associated with the basis vector \( X \) of \( \mathfrak{a} \), we identify naturally \( W^+ \subset \mathfrak{a} \) with \( \mathbb{R}^+ \) replacing \( w = xX \) by \( x \):

\[
G/H \times W^+ \to G/H \times \mathbb{R}^+, \quad (gH, xX) \mapsto (gH, x).
\]
By the $G$-invariance it suffices to describe the operators $J^K(o)$ only at the points $(o,x) \in G/H \times \mathbb{R}^+$, where $o = \{H\}$. By (4.47),

$$J^K(o,x)(X,0) = \left(0, \frac{\partial}{\partial x} \right),$$

(3.18)

$$J^K(o,x)\left(\xi^j, 0\right) = \left(\frac{-\cosh \lambda'_x}{\sinh \lambda'_x}, \xi^j, 0\right), \quad j = 1, \ldots, m_\lambda, \quad \lambda \in \Sigma^+,$$

where $\lambda'_x = \lambda'(x) \in \mathbb{R}$, that is, $\lambda'_x = x$ if $\lambda = e$ and $\lambda'_x = \frac{1}{2}x$ if $\lambda = \frac{1}{2}e$. Here $T_o(G/H)$ is identified naturally with the space $\mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} m_\lambda \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{t}_\lambda$, $\mathfrak{a} = \mathfrak{R}X$, and, in the first equation, we use naturally the usual basis vector \{\partial/\partial x\} of $T_o \mathbb{R}^+$.

The second relation in (3.18) can be represented in a more general form (see (4.27)): $J^K(o,x)\left(\xi^j, 0\right) = \left(\frac{-\cos \text{ad}_{xX}}{\sin \text{ad}_{xX}} \xi, 0\right)$, where $\xi \in \mathfrak{m}^+$.

Let $F = F(J^K)$ be the subbundle of $(1,0)$-vectors of the structure $J^K$ on the manifold $G/H \times \mathbb{R}^+$. Since the map $\pi_H \times \text{id}: G \times \mathbb{R}^+ \rightarrow G/H \times \mathbb{R}^+$ is a submersion, there exists a unique maximal complex subbundle $\mathcal{F}$ of $T^\mathbb{C}(G \times \mathbb{R}^+)$ such that $(\pi_H \times \text{id})_{*}\mathcal{F} = F$. As shown in (4.28),(4.29), $\mathcal{F}$ is generated by the kernel $\mathcal{H}$ of the submersion $\pi_H \times \text{id}$,

(3.19)

$$\mathcal{H}(g,x) = \{(\xi^j(g), 0), \xi \in \mathfrak{h}\}, \quad g \in G, \, x \in \mathbb{R}^+,$$

and the left $G$-invariant global vector fields on $G \times \mathbb{R}^+$:

$$Z^X(g,x) = (X^j(g), -i \frac{\partial}{\partial x}),$$

$$Z^{\xi^j}(g,x) = \left(\left(\frac{1}{\cosh \lambda'_x}, \xi^j, -i \frac{1}{\sinh \lambda'_x} \xi^j\right), (g), 0\right),$$

where $j = 1, \ldots, m_\lambda, \lambda \in \Sigma^+$.

To simplify calculations in the next subsection, for the vector fields of the second family we will use a more general expression

$$Z^\xi(g,x) = \left((R_\xi \xi - i S_\xi \xi)^j (g), 0\right), \quad \xi \in \mathfrak{m}^+,$$

in terms of the two operator-functions $R: \mathbb{R}^+ \rightarrow \text{End}(\mathfrak{g})$ and $S: \mathbb{R}^+ \rightarrow \text{End}(\mathfrak{g})$ on the set $\mathbb{R}^+$ such that

$$R_\xi \eta = \frac{1}{\text{cos \text{ad}_{xX}}\eta} \eta \quad \text{if} \quad \eta \in \mathfrak{m}^+ \oplus \mathfrak{t}^+, \quad R_\xi \eta = 0 \quad \text{if} \quad \eta \in \mathfrak{a} \oplus \mathfrak{h},$$

$$S_\xi \eta = \frac{-1}{\text{sin \text{ad}_{xX}}\eta} \eta \quad \text{if} \quad \eta \in \mathfrak{m}^+ \oplus \mathfrak{t}^+, \quad S_\xi \eta = 0 \quad \text{if} \quad \eta \in \mathfrak{a} \oplus \mathfrak{h},$$

where, recall, $xX \in W^+ \subset \mathfrak{a}$. Remark also that $\frac{1}{\text{cos \text{ad}_{xX}}\eta} = \eta$ if $\eta \in \mathfrak{a} \oplus \mathfrak{h}$ but $R_\xi \eta = 0$ in this case. Since the operator $\text{ad}_{xX}$ is skew-symmetric with respect to the scalar product on $\mathfrak{g}$,
each operator $R_x$ is symmetric and $S_x$ is skew-symmetric:

$$\langle R_x \eta_1, \eta_2 \rangle = \langle \eta_1, R_x \eta_2 \rangle, \quad \langle S_x \eta_1, \eta_2 \rangle = \langle \eta_1, -S_x \eta_2 \rangle, \quad x \in \mathbb{R}^+, \quad \eta_1, \eta_2 \in \mathfrak{g}.$$ 

Moreover, since $xX \in W^+ \subset \mathfrak{a}$, the restrictions $R_x|_{m^+ \oplus \mathfrak{t}^+}$ and $S_x|_{m^+ \oplus \mathfrak{t}^+}$ are nondegenerate and by Remark 3.1 the following relations hold:

$$R_x(m_s^+) = m_s^+, \quad R_x(\mathfrak{t}_s^+) = \mathfrak{t}_s^+, \quad S_x(m_s^+) = \mathfrak{t}_s^+, \quad S_x(\mathfrak{t}_s^+) = m_s^+, \quad s \in \{H, \ast\}.$$

It is clear also that

$$R_x|m_\lambda \oplus \mathfrak{t}_\lambda = \frac{1}{\cosh \lambda_x} \text{id}_{m_\lambda \oplus \mathfrak{t}_\lambda}, \quad S_x|m_\lambda \oplus \mathfrak{t}_\lambda = \frac{1}{\sinh \lambda_x} T|m_\lambda \oplus \mathfrak{t}_\lambda$$

for all $\lambda \in \Sigma^+$, and $[R_x, T] = [S_x, T] = 0$ on $m^+ \oplus \mathfrak{t}^+$ for all $xX \in W^+$, where, recall, the operator $T$ is defined by the expression (3.6).

3.3. Invariant Ricci-flat Kähler metrics on $G/H \times \mathbb{R}^+$. Let $\mathcal{K}(G/H \times \mathbb{R}^+) = \{(\mathfrak{g}, \omega, J^K)\}$ (resp. $\mathcal{R}(G/H \times \mathbb{R}^+) = \{(\mathfrak{g}, \omega, J^K)\}$) be the set of all $G$-invariant Kähler (resp. Ricci-flat Kähler) structures on $G/H \times \mathbb{R}^+$, identified also with the set $\mathcal{K}(T^+(G/K))$ (resp. $\mathcal{R}(T^+(G/K))$) of all $G$-invariant Kähler (resp. Ricci-flat Kähler) structures on the open dense subset $T^+(G/K)$ of $T(G/K)$, associated with $J^K$, via the $G$-equivariant diffeomorphism $\phi \circ f^+: G/H \times \mathbb{R}^+ \rightarrow T^+(G/K) (\mathbb{R}^+ \cong W^+)$.

Put

$$\{T_1, \ldots, T_n\} = \{Z^X\} \cup \{Z^\lambda_j, \lambda \in \Sigma^+, j = 1, \ldots, m_\lambda\}.$$

The following theorem is Theorem 4.8 from [8] (adapted to the rank one case) which describes the spaces $\mathcal{K}(G/H \times \mathbb{R}^+)$ and $\mathcal{R}(G/H \times \mathbb{R}^+)$ in terms of invariant forms on the space $G \times \mathbb{R}^+$:

**Theorem 3.2.** [8] Let $\mathcal{K}(G \times \mathbb{R}^+) = \{\tilde{\omega}\}$ be the set of all 2-forms $\tilde{\omega}$ on $G \times \mathbb{R}^+$ such that

1. the form $\tilde{\omega}$ is closed;
2. the form $\tilde{\omega}$ is left $G$-invariant and right $H$-invariant;
3. the kernel of $\tilde{\omega}$ coincides with the subbundle $\mathcal{H} \subset T(G \times \mathbb{R}^+)$ in (3.19);
4. $\tilde{\omega}(T_j, T_k) = 0$, $j, k = 1, \ldots, n$;
5. $i \tilde{\omega}(T, \overline{T}) > 0$ for each $T = \sum_{j=1}^n c_j T_j$, where $(c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\}$.

Let $\mathcal{R}(G \times \mathbb{R}^+) = \{\tilde{\omega}\}$ be the subset of the set $\mathcal{K}(G \times \mathbb{R}^+) = \{\tilde{\omega}\}$ consisting of all elements $\tilde{\omega}$ such that the following condition holds (in addition):

6. $\det(\tilde{\omega}(T_j, \overline{T_k}))$ is constant on $G \times \mathbb{R}^+$.

Then (i) For any 2-form $\tilde{\omega} \in \mathcal{K}(G \times \mathbb{R}^+)$ there exists a unique 2-form $\omega$ on $G/H \times \mathbb{R}^+ \cong T^+(G/K)$ such that $(\pi_H \times \text{id})^* \omega = \tilde{\omega}$. The map $\tilde{\omega} \mapsto \omega$ is a one-to-one map from $\mathcal{K}(G \times \mathbb{R}^+)$ onto $\mathcal{K}(G/H \times \mathbb{R}^+) \cong \mathcal{K}(T^+(G/K))$.

(ii) If the group $G$ is semisimple then the restriction of this map to $\mathcal{R}(G \times \mathbb{R}^+)$ is a one-to-one map from $\mathcal{R}(G \times \mathbb{R}^+)$ onto $\mathcal{R}(G/H \times \mathbb{R}^+) \cong \mathcal{R}(T^+(G/K))$. 

Remark 3.3. Note that condition (5) of the previous theorem is equivalent to the following condition: the Hermitian matrix-function $w(x)$ on $\mathbb{R}^+$ with entries $w_{jk}(x) = i\bar{\omega}(T_j, \overline{T_k})(e,x)$, $j,k = 1, \ldots, n$, is positive-definite.

To prove that a Kähler structure on $T^+(G/K)$ admits a Kähler extension to the whole $T(G/K)$ we will use Corollary 4.10 from [8] (adapted to the rank one case):

Corollary 3.4. [8] Let $\omega \in \mathcal{K}(G/H \times \mathbb{R}^+)$ and $\bar{\omega} = (\pi_H \times \text{id})^* \omega$. Then $\omega = ((\phi \circ f^+)^{-1})^* \omega \in \mathcal{K}(T^+(G/K))$. Suppose that there exists a smooth form (extension) $\omega_0$ on the whole tangent bundle $T(G/K)$ such that $\omega_0 = \omega$ on $T^+(G/K)$. Then the form $\omega_0$ determines a $G$-invariant Kähler structure on $T(G/K)$ (associated to the canonical complex structure $J^K_\mathbb{R}$) if and only if for some sequence $x_m \in \mathbb{R}^+$, $m \in \mathbb{N}$, such that $\lim_{m \to \infty} x_m = 0$, the Hermitian matrix $w(0)$ with entries $w_{jk}(0) = \lim_{m \to \infty} w_{jk}(x_m) = \lim_{m \to \infty} i\bar{\omega}(T_j, \overline{T_k})(e,x_m)$, $j,k = 1, \ldots, n$, is positive-definite.

3.4. General description of the space $\mathcal{R}(G \times \mathbb{R}^+)$. For any vector $a \in \mathfrak{g}$, denote by $\theta^a$ the left $G$-invariant 1-form on the group $G$ such that $\theta^a(\xi^l) = \langle a, \xi \rangle$. Since $r^* \lambda^a = \theta^{Ad_x a}$, where $g \in G$, the form $\theta^a$ is right $H$-invariant if and only if $Ad_h a = a$ for all $h \in H \subset G$. Because
\[
d\theta^a(\xi^l, \eta^l) = -\theta^a([\xi^l, \eta^l]) = -\langle a, [\xi, \eta] \rangle,
\]
the $G$-invariant form $\omega^a$ on $G$,
\[
\omega^a(\xi^l, \eta^l) \overset{\text{def}}{=} \langle a, [\xi, \eta] \rangle, \quad \xi, \eta \in \mathfrak{g},
\]
is a closed 2-form on $G$.

Let $\text{pr}_1 : G \times \mathbb{R}^+ \to G$ and $\text{pr}_2 : G \times \mathbb{R}^+ \to \mathbb{R}^+$ be the natural projections. Choosing some orthonormal basis $\{e_1, \ldots, e_N\}$ of the Lie algebra $\mathfrak{g}$, where $e_1 = X$, put $\bar{\theta}^e_k \overset{\text{def}}{=} \text{pr}_1^*(\theta^{e_k})$ and $\bar{\omega}^e_k \overset{\text{def}}{=} \text{pr}_2^*(\omega^{e_k})$. For any vector-function $a : \mathbb{R}^+ \to \mathfrak{g}$, $a(x) = \sum_{k=1}^{N} a^k(x) e_k$, denote by $\bar{\theta}^a$ (resp. $\bar{\omega}^a$) the $G$-invariant 1-form $\sum_{k=1}^{N} \theta^a \cdot \bar{\theta}^e_k$ (resp. 2-form $\sum_{k=1}^{N} \omega^a \cdot \bar{\omega}^e_k$).

The following theorem [8] Theorem 5.1] (adapted to the rank one case) describes the spaces $\mathcal{K}(G \times \mathbb{R}^+)$ and $\mathcal{R}(G \times \mathbb{R}^+)$ in terms of some $\mathbb{R}^+$-parameter family of exact 1-forms on the Lie group $G$:

Theorem 3.5. [8] Let $\bar{\omega}$ be a 2-form belonging to $\mathcal{K}(G \times \mathbb{R}^+)$, where the compact Lie group $G$ is semisimple. Then there exists a unique (up to a real constant) smooth function $f : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto f(x)$, and a unique smooth vector-function $a : \mathbb{R}^+ \to \mathfrak{g}_H$ given by
\[
a(x) = a^0(x) + z_h + a^f(x) + a^m(x), \quad a^0(x) = f(x)X, \quad z_h \in \mathfrak{z}(h),
\]
\[
(3.21) \quad a^f(x) = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \frac{c^f_{\lambda}}{\cosh \lambda(xX)} e^l_{\lambda} \in \mathfrak{t}^H_+, \quad a^m(x) = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \frac{c^m_{\lambda}}{\sinh \lambda(xX)} e^l_{\lambda} \in \mathfrak{m}^H_+,
\]
where $c^m_{\lambda}, c^f_{\lambda} \in \mathbb{R}$, such that $\bar{\omega}$ is the exact form expressed in terms of $a$ as
\[
(3.22) \quad \bar{\omega} = d\bar{\theta}^a = dx \wedge \bar{\theta}^a - \bar{\omega}^a, \quad \bar{a}' = \frac{\bar{\theta}^a}{\bar{\omega}^a}.\]
Moreover, for all points $x \in \mathbb{R}^+$, the following conditions (1)–(3) hold:

1. the components $a^k(x) + z_h$ and $a^m(x)$ of the vector-function $a(x)$ in (3.21) satisfy the commutation relations

$$\begin{align*}
(R_x \cdot \text{ad}_{a^k(x)} \cdot R_x + S_x \cdot \text{ad}_{a^k(x)} \cdot S_x + (R_x^2 + S_x^2) \text{ad}_{z_h})(m^+) &= 0, \\
(R_x \cdot \text{ad}_{a^m(x)} \cdot S_x - S_x \cdot \text{ad}_{a^m(x)} \cdot R_x)(m^+) &= 0;
\end{align*}$$

moreover, if $G/K$ is an irreducible Riemannian symmetric space and $a^k(x) \equiv 0$, then $z_h = 0$;

2. the Hermitian $p \times p$-matrix-function $w_H(x) = (w_{k|j}(x))$, $p = \dim m_H = 1 + \text{card}(\Sigma_H \cap \Sigma^+)$, with indices $k, j \in \{1\} \cup \{\lambda, \mu \in \Sigma_H \cap \Sigma^+\}$ and entries

$$
w_{1|1}(x) = 2f''(x),
$$
$$
w_{1|1}(x) = 2\lambda'(x) \left( i \frac{c^k_{\lambda}}{\cosh^2 \lambda_x^j} - \frac{c^m_{\lambda}}{\sinh^2 \lambda_x^j} \right), \quad \lambda \in \Sigma_H \cap \Sigma^+,
$$
$$
w_{1|1}(x), \quad \lambda, \mu \in \Sigma_H \cap \Sigma^+, \quad \text{determined by (3.24)},
$$
is positive-definite;

3. if $m^+_s \neq 0$ then the Hermitian $s \times s$-matrix $w_*(x) = (w_{j|k}(x))$, where $s = \dim m^+_s = \Sigma_{\lambda \in \Sigma^+ \setminus \Sigma_H} m_\lambda$, with indices $j, k \in \{\lambda, \mu \in \Sigma^+ \setminus \Sigma_H, j = 1, \ldots, m_\lambda\}$ and entries

$$
w_{j|k}(x) = -\frac{2i}{\sinh \lambda_x^j \sinh \mu_x^k} \left< \left( \text{ad}_{a^k(x) + z_h} \right) \zeta_j^j, \zeta_k^k \right> - \frac{2}{\cosh \lambda_x^j \sinh \mu_x^k} \left< \left( \text{ad}_{a^m(x) + a^m(x)} \right) \zeta_j^j, \zeta_k^k \right>
$$
is positive-definite.

If in addition

4. either $\det w_H(x) \cdot \det w_*(x) = \text{const}$ when $m^+_s \neq 0$ or $\det w_H(x) \equiv \text{const}$ otherwise, then $\tilde{\omega} \in \mathcal{R}(G \times \mathbb{R}^+)$.

Conversely, any 2-form as in (3.22) determined by a vector-function $a : \mathbb{R}^+ \to g_H$ as in (3.21) for which conditions (1)–(3) hold, belongs to $\mathcal{K}(G \times \mathbb{R}^+)$ and if in addition (4) holds, it belongs to $\mathcal{R}(G \times \mathbb{R}^+)$. Also Theorem 3.2 immediately implies

**Corollary 3.6.** Let $G/K$ be a rank-one Riemannian symmetric space of compact type. Each $G$-invariant Kähler metric $g$, associated with the canonical complex structure $J^K_\mathfrak{g}$ on $G/H \times \mathbb{R}^+ \cong T^+(G/K)$, where $T^+(G/K)$ is an open dense subset of $T(G/K)$, is uniquely determined by the Kähler form $\omega(\cdot, \cdot) = g(-J^K_\mathfrak{g} \cdot, \cdot)$ on $G/H \times \mathbb{R}^+$ given by

$$(\pi_H \times \text{id})^* \omega = d\tilde{\theta}^a,$$
Choose (4.1) \( Z \) trace-form given by \[ \langle Z \rangle \]

\( \theta \) denotes the space of traceless skew-Hermitian (4.2) Jacobi identity, ideal \[ \langle \cdot \rangle \]

By the invariance of the form \( \omega \) on \( G \times \mathbb{R}^+ \)

These spaces are Hermitian symmetric spaces and therefore we will review a few facts about them. (3.21)

Corollary 3.7. Let \( \omega \) be a G-invariant symplectic form on \( G/H \times \mathbb{R}^+ \) such that \( (\pi_H \times \text{id})^* \omega = d\theta^a \), where \( a: \mathbb{R}^+ \rightarrow a \), \( a(x) = f'(x)X \), for some function \( f \in C^\infty(\mathbb{R}^+, \mathbb{R}) \). Then the pair \((\omega, J^K)\) is a Kähler structure on \( G/H \times \mathbb{R}^+ \) (equivalently \( \pi_H \times \text{id})^* \omega = \mathcal{K}(G \times \mathbb{R}^+) \)) if and only if \( f'(x) > 0 \) and \( f''(x) > 0 \) for all \( x \in \mathbb{R}^+ \). In this case, the G-invariant function \( Q: G/H \times \mathbb{R}^+ \rightarrow \mathbb{R}, Q(gH, x) = 2f(x) \), is a potential function of the Kähler structure \((\omega, J^K)\) on \( G/H \times \mathbb{R}^+ \).

The Kähler structure \((\omega, J^K)\) with G-invariant potential function \( Q \) is Ricci-flat Kähler (equivalently \( \pi_H \times \text{id})^* \omega \in \mathcal{R}(G \times \mathbb{R}^+) \)) if and only if

\[
f'' \cdot \prod_{\lambda \in \Sigma^+} \left( \frac{2\lambda'(a)}{\sinh 2\lambda'(a)} \right)^{m^\lambda} = f'' \cdot \left( \frac{2f'}{\sinh(2f')} \right)^{m_e} \cdot \left( \frac{f''}{\sinh(f''')} \right)^{m_{e/2}} \equiv \text{const.}
\]

4. Complete invariant Ricci-flat Kähler metrics on tangent bundles of rank-one Riemannian symmetric spaces of compact type

Let \( g \) be a compact Lie algebra and let \( \sigma, \kappa, m, a, X \in a, \Sigma, \) etc. be as in Section 3. We continue with the previous notations but in this section it is assumed in addition that the subgroup \( K \) is connected.

In this Section using Theorem 3.5 we describe all invariant Ricci-flat Kähler structures on the tangent bundles of the spaces under study, in terms of explicit expressions of the corresponding vector-valued functions \( a \).

To this end we give with more detail the facts concerning the case \( G/K = \mathbb{C}P^n (n \geq 1) \). These spaces are Hermitian symmetric spaces and therefore we will review a few facts about them (9) Ch. VIII, §§4–7). The compact Lie subalgebra \( \kappa \) of the semisimple Lie algebra \( g = \mathfrak{su}(n+1) \) is the direct sum \( \kappa = \mathfrak{z} \oplus [\mathfrak{z}, \mathfrak{z}] \) of the one-dimensional center \( \mathfrak{z} \) and the semisimple ideal \( [\mathfrak{z}, \mathfrak{z}] \cong \mathfrak{su}(n) \). The subalgebra \( \kappa \) coincides with the centralizer of \( \mathfrak{z} \) in \( g \). Here \( \mathfrak{su}(n+1) \) denotes the space of traceless skew-Hermitian \( (n+1) \times (n+1) \) complex matrices and \( \kappa = \{ (b_{jk}) \in \mathfrak{su}(n+1) : b_{1j} = b_{j1} = 0, j = 2, \ldots, n+1 \} \). Fix on \( g = \mathfrak{su}(n+1) \) the invariant trace-form given by \( \langle B_1, B_2 \rangle = -2 \text{Tr} B_1 B_2, B_1, B_2 \in \mathfrak{su}(n+1) \). There exists a unique (up to a sign) element \( Z_0 \in \mathfrak{z}(\mathfrak{k}) \) such that the endomorphism \( I = \text{ad}Z_0 |_m : m \rightarrow m \) satisfies \( I^2 = -\text{Id}_m \).

Choose \( Z_0 \) as

\[
Z_0 = \text{diag}(ib_0, i(b_0 - 1), \ldots, i(b_0 - 1)), \quad b_0 = n/(n+1).
\]

(4.1)

By the invariance of the form \( \langle \cdot, \cdot \rangle \) on \( g \), the form \( \langle \cdot, \cdot \rangle |_m \) is \( I \)-invariant. Moreover, by the Jacobi identity,

\[
[I\xi, I\eta] = [\xi, \eta], \quad I[\xi, \eta] = [\xi, I\eta] \quad \text{for all } \xi, \eta \in m, \zeta \in \kappa.
\]

(4.2)

Denote by \( E_{jk} \) the elementary \( (n+1) \times (n+1) \) matrix whose entries are 0 except for 1 at the entry in the \( j \)th row and \( k \)th column. Choose as basis vector \( X \in a \) the matrix \( X = \frac{1}{2}E_{12} - \frac{1}{2}E_{21} \in m \subset \mathfrak{su}(n+1) \). We will show below (using direct matrix calculations) that this
choice is consistent with the notation of the previous sections, i.e. in this case \( (X, X) = 1 \) and the restricted root system \( \Sigma \) of \( (g, \mathfrak{t}, a) \) coincides with the set \( \{ \pm \} \) if \( n = 1 \) and \( \{ \pm \} \) if \( n \geq 2 \). The center \( \mathfrak{z}(h) \) of the centralizer \( h = g_X \cap \mathfrak{t} \) of \( X \in a \) in \( \mathfrak{t} \) is trivial for \( n = 1 \) and one-dimensional for \( n \geq 2 \) (see Table 3.1). It is easy to verify that \( \mathfrak{z}(h) = \mathbb{R}Z_1 \), where

\[
(4.3) \quad Z_1 = \text{diag}(ib_1, ib_1, i(b_1 - 1), \ldots, i(b_1 - 1)), \quad b_1 = (n - 1)/(n + 1), \quad n \geq 2.
\]

Note that \( Z_1 = 0 \) for \( n = 1 \).

**Theorem 4.1.** Let \( G/K \) be a rank-one Riemannian symmetric space of compact type with \( K \) connected. A 2-form \( \omega \) on the punctured tangent bundle \( T^+(G/K) \) of \( G/K \) determines a \( G \)-invariant Kähler structure, associated to the canonical complex structure \( J^K \), and the corresponding metric \( g = \omega(J^K \cdot, \cdot) \) is Ricci-flat, if and only if the 2-form \( \tilde{\omega} = ((\phi \circ f^+ \circ (\pi_H \times id))^\star \omega \) on \( G \times \mathbb{R}^+ \) may be expressed as \( \tilde{\omega} = d\tilde{\theta}^a \), where

1. for \( G/K \in \{ \mathbb{H}^n(n \geq 3), \mathbb{C}P^n(n \geq 1), \mathbb{C}aP^2 \} \) the vector-function \( a(x) = f'(x)X \), where

\[
(4.4) \quad (f'(x))^{m_e+m_e/2+1} = C \cdot \int_0^x (\sinh 2t)^{m_e}(\sinh t)^{m_e/2} dt + C_1,
\]

\( C, C_1 \in \mathbb{R}, C > 0, C_1 \geq 0; \)

2. for \( G/K \in \{ \mathbb{C}P^n(n \geq 1) \} \) the vector-function is

\[
a(x) = f'(x)X + \frac{cz}{\cosh^x} [IX, X] - \frac{1}{2} cz Z_1,
\]

where \( cz \) is an arbitrary real number and

\[
f'(x) = \sqrt{(C_n \sinh 2^n x + C_1)^{1/n} + c_z^2 \sinh^2 X \cosh^{-2} x},
\]

\( C, C_1 \in \mathbb{R}, C > 0, C_1 \geq 0. \)

The corresponding \( G \)-invariant Ricci-flat Kähler metric \( g = g(C, C_1, cz) \) on \( T^+(G/K) \) is uniquely extendable to a smooth complete metric on the whole tangent bundle \( T(G/K) \) if and only if \( C_1 = 0 \) (that is, \( \lim_{x \to 0} f'(x) = 0 \)).

**Proof.** By Theorem 3.5 we have to describe all vector-functions \( a : \mathbb{R}^+ \to g_H \) satisfying conditions (1)–(4) of that theorem. Then the 2-form \( \tilde{\omega} = d\tilde{\theta}^a \) belongs to the space \( \mathcal{R}(G \times \mathbb{R}^+) \). We consider the following two cases:

1. \( G/K \in \{ \mathbb{H}^n(n \geq 3), \mathbb{H}P^n(n \geq 1), \mathbb{C}aP^2 \} \). In this case by (3.15) \( g_H = g_0 \equiv a \). One gets that \( m_H = a \) and \( \mathfrak{t}_H = 0 \). Then \( a(x) = f'(x)X, x \in \mathbb{R}^+ \). Let us describe the Hermitian matrix-functions \( w^H(x) \) and \( w^s(x) \) from Theorem 3.5. As it is easily seen, the first matrix \( w^H(x) \) contains a unique element \( w^H_11(x) = 2f''(x) \) and the second one, \( w^s(x) \), is diagonal with elements

\[
w_{i/\ell}^H(x) = \frac{2f'(x)}{\cosh x \sinh x}, \quad w_{\ell/2\ell}^s(x) = \frac{f'(x)}{\cosh x \sinh x}.
\]
where \( j = 1, \ldots, m_e \) and \( k = 1, \ldots, m_{e/2} \). These matrices are positive definite if and only if \( f'(x) > 0 \) and \( f''(x) > 0 \) for all \( x \in \mathbb{R}^+ \). Hence the vector-function \( \mathbf{a} : \mathbb{R}^+ \to \mathbb{R} \) satisfies conditions (1)–(4) of Theorem 3.5 (see also Corollary 3.7) if and only if

\[
f'(x) > 0, \quad f''(x) > 0, \quad f'''(x) \cdot \left( \frac{f'(x)}{\cosh x \sinh x} \right)^m \left( \frac{f''(x)}{\cosh^2 x \sinh x} \right) \equiv \text{const}, \quad x \in \mathbb{R}^+.
\]

It is clear that the unique possible solution of these equations is of form (4.4).

(2) \( G/K = \mathbb{C}P^n \) \((n \geq 2)\). Theorem 3.5 was shown for \( G/K = \mathbb{C}P^1 \cong \mathbb{S}^2 \) in our paper [8] Theorem 6.1. Therefore in this proof we will suppose that \( n \geq 2 \). Since we have chosen the matrix \( X = \frac{1}{2}E_{12} - \frac{1}{2}E_{21} \in \mathfrak{m} \subset \mathfrak{su}(n + 1) \) as the basis vector \( X \in \mathfrak{a} \), it follows that

\[
Y \overset{\text{def}}{=} IX = [Z_0, X] = \frac{i}{2}E_{12} + \frac{i}{2}E_{21} \in \mathfrak{m}
\]

(4.7)

\[
Z \overset{\text{def}}{=} [IX, X] = -\frac{i}{2}E_{11} + \frac{i}{2}E_{22} \in \mathfrak{k}.
\]

It is easy to verify that the set \( \{X, Y, Z\} \) is an orthonormal system of vectors in \( \mathfrak{g} \) and

\[
[X, Y] = -Z, \quad [X, Z] = Y, \quad [Z, Y] = X,
\]

i.e. the vectors \( \{X, Y, Z\} \) form a canonical basis of the Lie algebra isomorphic to \( \mathfrak{su}(2) \). By (4.8), \( \text{ad}_X^2(IX) = -IX \) and as it is easy to verify, \( \text{ad}_X^2(\xi) = -\frac{i}{4}\xi \) for any vector \( \xi \) from the set of vectors

\[
\xi_{e/2}^{2j-1} = \frac{i}{2}E_{1(2+j)} - \frac{i}{2}E_{(2+j)1}, \quad \xi_{e/2}^{2j} = \frac{i}{2}E_{1(2+j)} + \frac{i}{2}E_{(2+j)1}, \quad j = 1, \ldots, n-1.
\]

Defining the restricted root \( \varepsilon \in (\mathfrak{a}^\mathbb{C})^* \) by the relation \( \varepsilon'(X) = 1 \) (\( \varepsilon(X) = i \)), we obtain that \( m_{e} = \mathbb{R}(IX) \) and that the set (4.9) is an orthonormal basis of the space \( m_{e/2} \) of dimension \( 2n - 2 \) (the orthogonal complement to \( \mathfrak{a} \oplus m_e \) in \( \mathfrak{m} \)). Moreover, \( IX \xi_{e/2}^{2j-1} = \xi_{e/2}^{2j} \) for each \( j = 1, \ldots, n-1 \). Therefore \( \Sigma^+ = \{\varepsilon, \frac{i}{4}\varepsilon\} \) \((n \geq 2)\).

Let us calculate the subalgebras \( \mathfrak{g}_h \) and \( \mathfrak{g}_H \) of \( \mathfrak{g} \) determined by relations (3.8) and (3.9). By (3.10) the space \( \mathfrak{a} \oplus \mathfrak{z}(h) \), where \( \mathfrak{a} = \mathbb{R}X \) and \( \mathfrak{z}(h) = \mathbb{R}Z_1 \), is a Cartan subalgebra of the algebra \( \mathfrak{g}_h \). Since \( \mathfrak{z}(h) \) belongs to the center of \( \mathfrak{g}_h \) we see that \( \text{rank}[\mathfrak{g}_h, \mathfrak{g}_h] \leq \dim \mathfrak{a} = 1 \), that is, \( \mathfrak{g}_h \cong \mathfrak{su}(2) \oplus \mathfrak{z}(h) \) if the algebra \( \mathfrak{g}_h \) is not commutative.

Since by definition \( [X, h] = 0 \) and \( h \subset \mathfrak{k} \), by (4.2) one gets that \( [IX, h] = 0 \), that is, \( IX \in \mathfrak{g}_h \).

By (4.8), the subalgebra in \( \mathfrak{g} \) generated by the vectors \( X \) and \( IX = Y \) is not commutative. Thus \( \mathfrak{g}_h \) is not commutative and, consequently, \( \mathfrak{g}_h \cong \mathfrak{su}(2) \oplus \mathfrak{z}(h) \). The vectors \( \{X, Y, Z, Z_1\} \) form an orthonormal basis of \( \mathfrak{g}_h \). Therefore \( \Sigma_h = \{\pm\varepsilon\} \).

Let us find now the algebra \( \mathfrak{g}_H \). The finite group \( D_n \) defined by relation (3.13) is given by

\[
D_n = \{\exp tX : \exp tX = \exp(-tX)\} \cap K
\]

\[
= \{\exp 4\pi X, \exp 2\pi X\} = \{\text{diag}(1, \ldots, 1), \text{diag}(-1, -1, 1, \ldots, 1)\}.
\]

It is clear that the group \( \text{Ad}(D_n) \) acts trivially on the space generated by the vectors \( X, Y, Z \) and \( Z_1 \). Therefore by (3.14) we have that \( \mathfrak{g}_H = \mathfrak{g}_0 \) and, consequently, \( \Sigma_H = \Sigma_h = \{\pm\varepsilon\} \). Note that \( D_n \subset H_0 \cong U(1) \times \mathbb{U}(n-1), U(1) \cong \{\exp tZ_1, t \in \mathbb{R}\} \), i.e. the subgroup \( H \) is connected.
Using properties (4.2) of the automorphism $I$, we can describe the actions of the operators $ad_Y$ and $ad_Z$ on $m \oplus t$ in terms of the operators $I$ and $ad_X$. Specifically, for any vectors $\xi \in m$, $\zeta \in t$, we have

\begin{align}
(4.10) & \quad [Y, \xi] = [IX, \xi] = -ad_X I \xi, \\
(4.11) & \quad [Y, \zeta] = [IX, \zeta] = I[ad_X \zeta].
\end{align}

Similarly, for $Z = [Y, X]$, using the Jacobi identity and relations (4.10), (4.11) we obtain that

\begin{align}
(4.12) & \quad [Z, \xi] = [Y, [X, \xi]] - [X, [Y, \xi]] = Iad_X^2 \xi + ad_X I \xi, \\
(4.13) & \quad [Z, \zeta] = [Y, [X, \zeta]] - [X, [Y, \zeta]] = -2 ad_X I ad_X \zeta.
\end{align}

From the definitions of $Z_0$ and $Z_1$ in (4.1) and (4.3), respectively, it follows that $ad_{Z_0} |_{m_\epsilon/2} = ad_{Z_1} |_{m_\epsilon/2}$. Moreover, since $b_1 = 1 = (b_0 - 1)$, then from (4.1) and (4.7) we obtain that $Z_1 = Z_0 = 2Z$. In other words,

\begin{align}
(4.14) & \quad ad_{Z_1} |_{m_\epsilon/2} = I |_{m_\epsilon/2} \quad \text{and} \quad Z - \frac{1}{2} Z_1 = -Z_0.
\end{align}

The operator-functions in (3.20) are given here by

\begin{align}
(4.15) & \quad R_x |_{m_\epsilon/2 @ t_\epsilon} = \frac{1}{\cosh x} \text{Id}_{m_\epsilon/2 @ t_\epsilon}, \\
(4.16) & \quad S_x |_{m_\epsilon/2 @ t_\epsilon/2} = \frac{1}{\sinh x} ad_X |_{m_\epsilon/2 @ t_\epsilon/2}, \\
& \quad S_x |_{m_\epsilon/2 @ t_\epsilon/2} = \frac{2}{\sinh x} ad_X |_{m_\epsilon/2 @ t_\epsilon/2}.
\end{align}

Put $\zeta_1 \epsilon = Y \in m_\epsilon$. With the notation of the previous subsection, $\zeta_1 \epsilon = Z \in t_\epsilon$. Now we have to verify conditions (1)–(4) of Theorem 3.5 for the vector-function

\begin{align}
(4.17) & \quad a(x) = a^0(x) + a^\ell(x) + a^m(x) + z_\theta = f'(x)X + cZ \varphi(x)Z + cY \psi(x)Y + c_1 Z_1,
\end{align}

where

\begin{align}
& \quad f \in C^\infty(\mathbb{R}^+, \mathbb{R}), \quad \varphi(x) = \frac{1}{\cosh x}, \quad \psi(x) = \frac{1}{\sinh x}, \quad c, c_1, c_1 \in \mathbb{R}.
\end{align}

Consider now the first condition in (3.23). We have the splitting $m^+ = m_\epsilon \oplus m_\epsilon/2$. Taking into account that by its definition $[\theta(h), g_\theta] = 0$ and $m_\epsilon = \mathbb{R}Y \subset g_\theta$, using relations (4.15), we can rewrite the first condition in (3.23) for the vector $Y = \zeta_1 \epsilon$ as

\begin{align}
(4.18) & \quad \frac{1}{\cosh x} \cdot R_x [cZ \varphi(x)Z, Y] + \frac{1}{\sinh x} \cdot S_x [cZ \varphi(x)Z, ad_X Y] = 0.
\end{align}

The first term in (4.18) vanishes because $[Z, Y] = X \in \mathfrak{a}$ and $R_x(a) = 0$; the second term vanishes because $ad_X Y = -Z$.

Since in our case $m_\epsilon = m_\epsilon/2$ and $t_\epsilon = t_\epsilon/2$, then by Remark 3.1, $[\theta(h) \oplus [h, h], m_\epsilon/2 \oplus t_\epsilon/2] \subset m_\epsilon/2 \oplus t_\epsilon/2$. Let now $\xi \in m_\epsilon/2$. Using relations (4.12), (4.13) and (4.16), expression (4.14)
and the fact that $I_2 \xi \in m_{e/2}$, we can rewrite the first condition in (3.23) as

$$0 = \frac{c_2\varphi(x)}{\cosh^{1/2} x} \cdot R_\xi \left[ Z, \xi \right] + \frac{2c_2\varphi(x)}{\sinh^{1/2} x} \cdot S_x \left[ Z, \text{ad}_x \xi \right] + \left( \frac{1}{\cosh^{1/2} x} - \frac{1}{\sinh^{1/2} x} \right) \cdot \left[ c_1 Z, \xi \right],$$

$$= \frac{c_2\varphi(x)}{\cosh^{1/2} x} \cdot (I \text{ad}_\xi^2 \xi + ad_\xi I_2 \xi) + \frac{4c_2\varphi(x)}{\sinh^{1/2} x} \cdot \text{ad}_x (2\text{ad}_x I \text{ad}_\xi^2 \xi) - \frac{c_1}{\cosh^{1/2} x \sinh^{1/2} x} \cdot I_\xi.$$  

Then

$$0 = \left( \frac{1}{\cosh^{1/2} x} + \frac{1}{\sinh^{1/2} x} \right) \cdot \frac{-c_2\varphi(x)}{2\cosh x} \cdot I_2 \xi - \frac{c_1}{\cosh^{1/2} x \sinh^{1/2} x} \cdot I_2 \xi = \frac{1}{\cosh^{1/2} x \sinh^{1/2} x} \cdot \left( -\frac{1}{2} c_2 \xi - c_1 \right) I_2 \xi,$$

because $\text{ad}_\xi^2 |_{m_{e/2}} = -\frac{1}{2} \text{Id}_{m_{e/2}}$. Thus $c_1 = -\frac{1}{2} c_2$.

We can also rewrite the second condition in (3.23) for $\xi \in m_{e/2}$ as

$$\frac{2c_2\varphi(x)}{\sinh^{1/2} x \cosh^{1/2} x} \cdot R_\xi \left[ c_Y \varphi(x)Y, \text{ad}_x \xi \right] = \frac{1}{\cosh^{1/2} x} \cdot S_x \left[ c_Y \varphi(x)Y, \xi \right] = 0.$$  

Taking into account the relations (4.10), (4.11) and (4.16) we obtain that

$$\frac{2c_2\varphi(x)}{\sinh^{1/2} x \cosh^{1/2} x} \cdot (I \text{ad}_\xi^2 \xi + ad_\xi I_2 \xi) = -\frac{c_Y}{\sinh^{1/2} x \cosh^{1/2} x} \cdot I_2 \xi = 0.$$  

Thus $c_Y = 0$ and therefore the component $a^m(x)$ of $a(x)$ vanishes. The second condition in (3.23) holds.

Summarizing the results proved above, we obtain that for the vector-function (4.17), condition (1) of Theorem 3.5 for $G/K = \mathbb{C}P^n \ (n \geq 2)$ is equivalent to the conditions $\{ c_2 \in \mathbb{R}, c_1 = -\frac{1}{2} c_2, c_Y = 0 \}$.

Let us describe the $2 \times 2$ Hermitian matrix-function $w_H(x), \ 2 = \dim a + \dim m_e$, according to condition (2) of Theorem 3.5.

It is clear that $w_{1|1}(x) = 2 f''(x)$. The function $w_{1|1}(x)$ is determined by the relation (for $\lambda = \varepsilon$): $w_{1|1}(x) = 2i \frac{c_2\varphi(x)}{\cosh^{1/2} x} - \frac{2c_2\varphi(x)}{\sinh^{1/2} x}$.

The function $w_{1|e}(x)$ is determined by the relation (3.24) for $\xi_{1e} = Z$ and $\xi_{1e} = Y$. By relations (4.8) and the invariance of the form $\langle \cdot, \cdot \rangle$, $w_{1|e}(x) = -\frac{2i}{\sinh x} \langle [c_2 \varphi(x)Z + c_1 Z, Z] \rangle - \frac{2}{\cosh x \sinh x} \langle \left[ f'(x)X, Y \right], Z \rangle = f'(x) \frac{2}{\cosh x \sinh x}$.

Hence, we conclude that the entries of $w_H(x)$ are

$$w_{11}(x) = 2 f''(x), \quad w_{1|1}(x) = 2i \frac{c_2\varphi(x)}{\cosh^{1/2} x}, \quad w_{1|1}(x) = \frac{2f'(x)}{\cosh x \sinh x}.$$  

Let us describe the Hermitian $s \times s$-matrix $w_s(x) = (w_{jk}(x))_s = \dim m_{e/2} = 2n - 2$, with entries $w_{jk}(x) = w_{jk}(x)_{e/2}(x), j, k = 1, \ldots, 2n - 2$, determined by relations (3.24):

$$w_{jk}(x) = -\frac{2i}{\sinh x} \left[ [c_2 \varphi(x)Z + c_1 Z, -2 \text{ad}_x \xi_j, -2 \text{ad}_x \xi^k] \right] - \frac{2}{\cosh x \sinh x} \langle \left[ f'(x)X, \xi_j \right], -2 \text{ad}_x \xi^k \rangle.$$
where we put \( \xi^j = \xi^{j}_{e/2} \) to simplify notation. Taking into account relations (4.13), (4.10) and the commutation relation \([\text{ad}_{Z_1}, \text{ad}_X] = 0\), we obtain that
\[
w_{jk}(x) = -\frac{2i}{\sinh x} (\langle 8cz\varphi(x)\langle \text{ad}_X I \text{ad}_X^2 \xi^j, \text{ad}_X \xi^k \rangle + 4c_1 \langle \text{ad}_X I \xi^j, \text{ad}_X \xi^k \rangle)
- \frac{2}{\cosh \frac{x}{2} \sinh \frac{x}{2}} (\langle -2 f'(x) \langle \text{ad}_X \xi^j, \text{ad}_X \xi^k \rangle)
- \frac{i}{\sinh \frac{x}{2}} cZ \left(\frac{1}{\cosh x} - 1\right) \langle I \xi^j, \xi^k \rangle + \frac{1}{\cosh \frac{x}{2} \sinh \frac{x}{2}} f'(x) \langle \xi^j, \xi^k \rangle.
\]

But the orthonormal basis \( \{\xi_{e/2}^j\}_{j=1}^{2n-2} \) is chosen in such a way that \( \xi^j_{e/2} = I \xi^j_{e/2} \). Thus from the relations above it follows that the Hermitian matrix \( w_+(x) \) is a block-diagonal matrix, where each block is an Hermitian \( 2 \times 2 \)-matrix. Each such block is determined by the pair of vectors \( (\xi^{2j-1}_{e/2}, \xi^{2j}_{e/2}) \), \( j = 1, \ldots, n-1 \), and it is a \( 2 \times 2 \) Hermitian matrix with the entries
\[
w_{(2j-1)(2j-1)}(x) = w_{(2j)(2j)}(x) = \frac{f'(x)}{\cosh \frac{x}{2} \sinh \frac{x}{2}},
\]

\[
w_{(2j-1)(2j)}(x) = \frac{icZ}{\sinh \frac{x}{2}} \left(1 - \frac{1}{\cosh x}\right).
\]

It is easily checked (calculating determinants of order 2) that vector-function (4.17) satisfies conditions (2), (3) and (4) of Theorem [3.5] if and only if all the mentioned Hermitian \( 2 \times 2 \) matrices are positive-definite and \( \det w_+(x) \cdot \det w_+(x) = 2^{2n-2} \cdot C^n \), \( C > 0 \), i.e. for all \( x \in \mathbb{R}^+ \) the following relations hold:
\[
\text{(a)} \ f''(x) > 0, \quad \text{(b)} \ f'(x) \frac{f''(x)}{\cosh x \sinh x} - \frac{c_2^2}{\cosh^4 x} > 0,
\]

\[
\text{(c)} \ f'(x) > 0, \quad \text{(d)} \ f'(x) \frac{f''(x)}{\cosh^2 \frac{x}{2} \sinh^2 \frac{x}{2}} - \frac{c_2^2}{\sinh^4 \frac{x}{2}} \left(1 - \frac{1}{\cosh x}\right)^2 > 0,
\]

and
\[
f''(x) f''(x) \frac{f(x)}{\cosh x \sinh x} - \frac{c_2^2}{\cosh^4 x} \left(f'(x) \frac{f''(x)}{\cosh^2 \frac{x}{2} \sinh^2 \frac{x}{2}} - \frac{c_2^2}{\sinh^4 \frac{x}{2}} \left(1 - \frac{1}{\cosh x}\right)^2\right)^{n-1} = 2^{2n-2} C^n.
\]

However, there exists an exact general solution of equation (4.22). Indeed, taking into account some well-known identities for the functions \( \cosh x \) and \( \sinh x \), and using the substitution \( g_1(x) = (f'(x))^2 \) one can rewrite (4.22) as
\[
g_1'(x) = \frac{2 c_2^2 \sinh x}{\cosh^3 x} + 2^{2n-1} C^n \cosh x \sinh x \left(\frac{\cosh^2 x \sinh^2 x}{4 g_1(x) \cosh^2 x - 4 c_2^2 \sinh^2 x}\right)^{n-1}.
\]

Next, using the substitution \( g_2(x) = g_1(x) \cosh^2 x - c_2^2 \sinh^2 x \) we obtain the Bernoulli equation
\[
g_2'(x) = \frac{2 \sinh x}{\cosh x} \cdot g_2(x) + 2C^n \cosh^3 x \sinh x \left(\frac{\cosh^2 x \sinh^2 x}{g_2(x)}\right)^{n-1}.
\]
with solutions \( g_2(x) = \cosh^2 x (C^n \sinh^{2n} x + C_1) \), on the whole semi-axis, i.e. we obtain that

\[
(4.23) \quad f'(x) = \sqrt{(C^n \sinh^{2n} x + C_1)^{1/n} + c_2^2 \sinh^2 x \cosh^{-2} x}.
\]

and therefore

\[
(4.24) \quad f''(x) = (f'(x))^{-1} (C^n \cdot (C^n \sinh^{2n} x + C_1)^{1/n} - \sinh^{2n-1} x \cosh x
\]

\[
+ c_2^2 (\tanh x - \tanh^3 x)).
\]

For these functions on the whole semi-axis relations (4.21b) and (4.21a) hold since \( \sinh x > 0 \) and \( \tanh x > \tanh^3 x \) on this set \( 0 < \tanh x < 1 \); also (4.21b) hold, because \( \tanh x - \tanh^3 x = \sinh x \cosh^{-3} x \); and (4.21d) hold, as \( \sinh^2 x \cosh^{-2} x = \frac{\cosh^2 x}{\sinh^2 x} \left( 1 - \frac{1}{\cosh x} \right)^2 \).

The form \( \tilde{\omega} = d\tilde{\theta}^a \) on \( G \times \mathbb{R}^+ \) determines a unique form \( \omega \) on \( G/H \times \mathbb{R}^+ = G/H \times W^+ \) such that \( \tilde{\omega} = (\pi_H \times \text{id})^* \omega \) (see Corollary 3.6). Let us study when the form \( \omega \) on \( G/H \times W^+ \cong T^+(G/K) \) admits an smooth extension to the whole tangent space \( T(G/K) \). To this end we will find the expression of the form \( \omega^R = ((f^+)^{-1})^* \omega \) on the space \( G \times_K m^R \cong T^+(G/K) \), where, recall, \( f^+: G/H \times \mathbb{R}^+ \to G \times_K m^R \) is a \( G \)-equivariant diffeomorphism. However, by the commutativity of diagram (3.16) there exists a unique form \( \tilde{\omega}^R \) on \( G \times m^R \) such that

\[
(4.25) \quad \tilde{\omega}^R = \pi^* \omega^R \quad \text{and} \quad \tilde{\omega} = \text{id}^* \tilde{\omega}^R.
\]

Thus it is sufficient to calculate the form \( \tilde{\omega}^R \) on the space \( G \times m^R \), because the form \( \omega^R \) on the space \( G \times_K m^R \cong T^+(G/K) \) may be extended (in a unique way if the extension does exist) to the whole tangent space \( T(G/K) \) if and only if the form \( \tilde{\omega}^R \) is extendable (admits an extension to the whole space \( G \times m \)).

By the second expression in (4.26),

\[
(4.26) \quad \tilde{\omega}^R_{(g,x)}((\xi^1_1(g), t_1X), (\xi^2_2(g), t_2X)) = \tilde{\omega}_{(g,x)}((\xi^1_1(g), t_1\frac{\partial}{\partial x}), (\xi^2_2(g), t_2\frac{\partial}{\partial x})).
\]

To describe \( \tilde{\omega}^R \) we consider again the two cases (1) and (2):

(1) \( G/K \in \{S^n (n \geq 3), \mathbb{H}^n (n > 1), \mathbb{C} \mathbb{P}^1 \} \). Since \( a(x) = f'(x)X \), by (3.22) at the point \( (g, xX) \in G \times W^+ \) and from (4.26) we have that

\[
\tilde{\omega}^R_{(g, xX)}((\xi^1_1(g), t_1X), (\xi^2_2(g), t_2X)) = -\langle f'(x)X, [\xi_1, \xi_2] \rangle + f''(x)(t_1 \langle X, \xi_2 \rangle - t_2 \langle X, \xi_1 \rangle),
\]

where \( \xi_1, \xi_2 \in \mathfrak{g} = T_e G \) and \( t_1, t_2 \in \mathbb{R} \). Consider on the whole tangent space \( T_{(g,w)}(G \times m^R) \) \( (w \in m^R = m \setminus \{0\}) \), the bilinear form \( \Delta \) given by

\[
\Delta_{(g,w)}((\xi^1_1(g), u_1), (\xi^2_2(g), u_2)) = -\langle \frac{f'(w)}{r}, [\xi_1, \xi_2] \rangle
\]

\[
+ \frac{f'(w)}{r}(\langle u_1, \xi_2 \rangle - \langle u_2, \xi_1 \rangle) + \frac{1}{r} \left( \langle u_1, w \rangle \langle w, \xi_2 \rangle - \langle u_2, w \rangle \langle w, \xi_1 \rangle \right),
\]

where \( \xi_1, \xi_2 \in \mathfrak{g} = T_e G, u_1, u_2 \in m = T_w m^R \). Here \( r = r(w), r^2(w) \overset{\text{def}}{=} \langle w, w \rangle (r(xX) = x) \). It is clear that this form is skew-symmetric.
Since $\frac{f'(r)}{r}' = \frac{f''(r)}{r''}$, it is easy to verify that

$$\Delta_{(g,x)}((\xi_1^l(g), t_1X), (\xi_2^l(g), t_2X)) = \tilde{\omega}^R_{(g,x)}((\xi_1^l(g), t_1X), (\xi_2^l(g), t_2X)),$$

i.e. the restrictions of $\tilde{\omega}^R$ and $\Delta$ to $G \times W^+ \subset G \times m^R$ coincide. Now to prove that the differential forms $\tilde{\omega}^R$ and $\Delta$ coincide on the whole tangent bundle $T(G \times m^R)$ it is sufficient to show that the form $\Delta$ is left $G$-invariant, right $K$-invariant and its kernel contains (and therefore coincides with) the subbundle $\mathcal{H} = \ker \pi$.

Since for each $k \in K$ the scalar product $\langle \cdot, \cdot \rangle$ is $Ad_k$-invariant and $Ad_k$ is an automorphism of $g$, the following relations hold:

$$\Delta_{(g,w)}((\xi_1^l(g), u_1), (\xi_2^l(g), u_2)) = \Delta_{(e,w)}((\xi_1^l, u_1), (\xi_2^l, u_2)) = \Delta_{(e,Ad_k w)}((Ad_k \xi_1, Ad_k u_1), (Ad_k \xi_2, Ad_k u_2)).$$

Hence, $\Delta$ is left $G$-invariant and right $K$-invariant.

The kernel $\mathcal{H} \subset T(G \times m)$ of the tangent map $\pi_* : T(G \times m) \to T(G \times_K m)$ is generated by the (left) $G$-invariant vector fields $\xi^l$, $\zeta \in \mathfrak{k}$ (3.17) on $G \times m$. Then, since $m \perp \mathfrak{k}$ and $\langle w, [w, \zeta] \rangle = 0$, we obtain

$$\Delta_{(g,w)}((\xi_1^l(g), u_1), \zeta^l(g, w)) = -\langle \frac{f'(r)}{r} w, [\xi_1, \zeta] \rangle + \frac{f'(r)}{r} \langle (u_1, \zeta) - \langle [w, \zeta], \xi_1 \rangle \rangle + \frac{1}{r} \left( \frac{f'(r)}{r} \right)' (w, \zeta) - \langle [w, \zeta], w \rangle (w, \xi_1) \rangle = -\langle \frac{f'(r)}{r} [w, \zeta], \xi_1 \rangle - \langle \frac{f'(r)}{r} [w, \zeta], \xi_1 \rangle = 0.$$

This means that $\mathcal{H} \subset \ker \Delta$. Thus $\tilde{\omega}^R = \Delta$ on $G \times m^R$ ($\mathcal{H} = \ker \Delta$ because the form $\omega$ is nondegenerate).

Expression (4.27) determines a smooth 2-form on the whole tangent bundle $T(G \times m)$ if and only if $\lim_{x \to 0} f'(x) = 0$, that is, $C_1 = 0$. Indeed, if $C_1 > 0$ it is easy to verify that

$$\lim_{x \to 0} f'(x) = C_1^{1/m} \text{ and } \lim_{x \to 0} f''(x) = 0, \text{ where } m = \dim m = m_\epsilon + m_{\epsilon/2} + 1. \text{ Therefore, by (4.27), } \lim_{x \to 0} \Delta_{(e,x)}((\xi_1^l, u_1), (\xi_2^l, u_2)) = \infty \text{ for some vectors } \xi_1, \xi_2 \in g, u_1, u_2 \in m \text{ such that }$$

$$\lim_{x \to 0} \frac{f'(x)}{x} (\langle u_1, \xi_2 \rangle - \langle u_2, \xi_1 \rangle - \langle u_1, X \rangle \langle X, \xi_2 \rangle + \langle u_2, X \rangle \langle X, \xi_1 \rangle) = \infty.$$

Let $C_1 = 0$. Since $\frac{\sinh x}{x} > 1$ for $x > 0$, there exists an even real analytic function on the whole axis, $\psi_2(x)$, such that

$$f'(x) = x \left( \frac{2 m \epsilon C + x^2 \psi_2(x)}{m} \right)^{1/m}, \quad \frac{2 m \epsilon C + x^2 \psi_2(x)}{m} > 0, \quad \forall x > 0.$$

In this case expression (4.27) determines a smooth 2-form on the whole space $G \times m$.

We will denote this form (extension) on $G \times m$ by $\tilde{\omega}^R_0$. There exists a unique 2-form $\omega^R_0$ on $G \times_K m \cong T(G/K)$ such that $\tilde{\omega}^R_0 = \pi^* \omega^R_0$. The forms $\tilde{\omega}^R_0$ and $\omega^R$ coincide, by construction, on the open submanifold $G \times_K m^R \cong T^+(G/K)$, that is, $\omega^R_0$ is a smooth extension of $\omega^R$.
Now we will prove, applying Corollary 3.4, that this extension is the Kähler form of the metric \( g_0 \) on the whole tangent bundle \( T(G/K) \). Indeed, by (4.6) and (4.29) for \( C_1 = 0 \),

\[
\begin{align*}
\lim_{x \to 0} w_{11}(x) &= 2 \left( \frac{2meC}{m} \right)^{1/m}, \\
\lim_{x \to 0} w_{ij}(x) &= \lim_{x \to 0} w_k \left( \frac{1}{m} \right)^{1/m},
\end{align*}
\]

that is, the corresponding limit diagonal Hermitian matrices \( \lim_{x \to 0} w_H(x) \) and \( \lim_{x \to 0} w_*(x) \) are positive-definite. Thus by Corollary 3.4, \( \omega^R_0 \) is the Kähler form of the metric \( g_0 \) (the extension of \( g \)) on \( G \times K \cong T(G/K) \).

(2) \( G/K = \mathbb{CP}^n \) \((n \geq 2)\). In this case the vector-function \( a \) takes the form

\[
a(x) = f'(x)X + c_Z \phi(x)Z + c_1 Z_1 = f'(x)X + c_Z (\phi(x) - 1) Z - c_Z Z_0,
\]

because \( Z_1 = 2(Z + Z_0) \) and \( c_1 = -\frac{1}{2}c_Z \). Here \( f'(x) \) is given in (4.3), \( \phi(x) = \frac{1}{\cos x} \) and \( c_Z \) is an arbitrary real number. Then, from (3.22), we have

\[
\omega^R_{(g,x)} \left( (\xi_1^I(g), t_1X), (\xi_2^I(g), t_2X) \right) = -\left( f'(x)X + c_Z (\phi(x) - 1) Z - c_Z Z_0, [\xi_1, \xi_2] \right) + f''(x) \left( t_1 \langle X, \xi_2 \rangle - t_2 \langle X, \xi_1 \rangle \right) + c_Z \phi'(x) \left( t_1 \langle Z, \xi_2 \rangle - t_2 \langle Z, \xi_1 \rangle \right),
\]

where \( \xi_1, \xi_2 \in g = T_eG \) and \( t_1, t_2 \in \mathbb{R} \).

Consider on the whole tangent space \( T_{(g,w)}(G \times m^R) \), the bilinear form \( \Delta_k \):

\[
\Delta_{(g,w)} \left( (\xi_1^I(g), u_1), (\xi_2^I(g), u_2) \right) = -\left( f'(x) w + \frac{c_Z}{r^2} \phi(x) - 1 \right) \left[ Iw, w \right] - c_Z Z_0, [\xi_1, \xi_2] \\
+ f''(x) \left( \langle u_1, \xi_2 \rangle - \langle u_2, \xi_1 \rangle \right) + \frac{1}{r^2} f''(x) \left( \langle u_1, w \rangle \langle w, \xi_2 \rangle - \langle u_2, w \rangle \langle w, \xi_1 \rangle \right) \\
+ c_Z \left( \frac{r^2 \phi'(x) - 2(\phi(x) - 1)}{r^2} \right) \left( \langle u_1, w \rangle \langle [Iw, w], \xi_2 \rangle - \langle u_2, w \rangle \langle [Iw, w], \xi_1 \rangle \right) \\
+ c_Z \left( \frac{r^2 \phi'(x) - 2(\phi(x) - 1)}{r^2} \right) \left( \langle u_1, Iw \rangle \langle w, u_2 \rangle - \langle u_2, Iw \rangle \langle w, u_1 \rangle \right) \\
+ c_Z \left( \frac{2r^2 \phi(x) - 1}{r^2} \right) \left( \langle [Iu_1, w], \xi_2 \rangle - \langle [Iu_2, w], \xi_1 \rangle + \langle u_1, Iu_2 \rangle \right),
\]

where \( \xi_1, \xi_2 \in g = T_eG, u_1, u_2 \in m = T_w m^R \). It is clear that this form is skew-symmetric. From the expression of \( \omega^R \) at the point \( (g, x) \) \in G \times W^+ \) given in (4.30) and taking into account that \( \langle IX, X \rangle = Z \) and \( \langle X, IX \rangle = 0 \), it is easy to verify that

\[
\Delta_{(g,x)} \left( (\xi_1^I(g), t_1X), (\xi_2^I(g), t_2X) \right) = \omega^R_{(g,x)} \left( (\xi_1^I(g), t_1X), (\xi_2^I(g), t_2X) \right).
\]

Since for each \( k \in K \) the scalar product \( \langle \cdot, \cdot \rangle \) is \( \text{Ad}_k \)-invariant, \( \text{Ad}_k \) is an automorphism of \( g \) and \( \text{Ad}_k(Z_0) = Z_0 \), \( \text{Ad}_k I = I \text{Ad}_k \), relations (4.28) hold now for this \( \Delta \), that is, \( \Delta \) is left \( G \)-invariant and right \( K \)-invariant. We now prove that \( \ker \Delta \supset \mathcal{H} \). By (3.17), since the form \( \Delta \) is left \( G \)-invariant, right \( K \)-invariant and \( \text{Ad}(K)(\mathbb{R}X) = m \), it is sufficient to show that the vectors \( (\zeta, x[X, \zeta]) \), \( \zeta \in \mathfrak{k} \), belong to the kernel of \( \Delta_{(e, xX)} \). Indeed, using the fact that
\[ [Z_0, ℓ] = 0, ℓ \perp m \text{ and } \langle \cdot, \cdot \rangle \text{ is } \text{Ad}(G)\text{-invariant, we have that} \]
\[ \Delta_{(e, x)} \left( \langle [X, X], [\xi_1, \xi] \rangle \right) = c_Z (\varphi(x) - 1) \langle -[IX, X], [\xi_1, \zeta] \rangle + 2c_Z (\varphi(x) - 1) \cdot \left( \langle [Iu_1, X], \zeta \rangle - \langle [IX, \zeta, X], \xi_1 \rangle + \langle u_1, [X, \zeta] \rangle \right). \]

This expression vanishes because the endomorphism \( I \) on \( m \) is skew-symmetric and
\[ -\langle [IX, X], [\xi_1, \xi] \rangle - 2\langle [X, [\xi_1, \zeta]], \xi_1 \rangle \overset{\text{(4.8)}}{=} \langle [Z, \zeta], \xi_1 \rangle + 2\langle [X, [X, \zeta]], \xi_1 \rangle \overset{\text{(4.13)}}{=} \langle -2\text{ad}_X I \text{ad}_X \zeta, \xi_1 \rangle + 2\langle [X, [X, \zeta]], \xi_1 \rangle = 0. \]

Thus the differential forms \( \tilde{ω}^R \) and \( \Delta \) coincide on the whole tangent bundle \( T(G \times m^R) \).

Our expression \( \text{lim}_{x \to 0} f'(x) = 0 \) determines a smooth 2-form on the whole tangent bundle \( T(G \times m) \) if and only if \( \lim_{x \to 0} f'(x) = 0 \), that is, \( c_1 = 0 \). Indeed, if \( c_1 > 0 \) it is easy to verify that \( \lim_{x \to 0} f'(x) = C_1^{1/2n} \) and \( \lim_{x \to 0} f''(x) = 0 \). Therefore by \( \text{lim}_{x \to 0} \Delta_{(e, x)} \left( \langle [X, X], [\xi_1, \xi] \rangle, \langle \xi_2, u_2 \rangle \right) = \infty \) for some vectors \( \xi_1, \xi_2 \in g, u_1, u_2 \in m \) such that
\[ \lim_{x \to 0} \frac{f'(x)}{x} \left( \langle u_1, \xi_2 \rangle - \langle u_2, \xi_1 \rangle - \langle u_1, X \rangle \langle X, \xi_2 \rangle + \langle u_2, [X, \xi_2] \rangle \right) = \infty. \]

Let \( c_1 = 0 \). In this case, the expression for the function \( f'(x) \) in \( \text{(4.5)} \) is independent of \( n \) and there exists an even real analytic function on the whole axis, \( \varphi_2(x) \), such that
\[ f'(x) = x \left( C + c_Z^2 + x^2 \varphi_2(x) \right)^{1/2}, \quad C + c_Z^2 + x^2 \varphi_2(x) > 0, \quad \forall x > 0. \]

Hence by \( \text{(4.5)} \) the functions \( \frac{f'(x)}{x} \) and \( \frac{1}{x} \left( \frac{f'(x)}{x} \right)' \) are even real analytic functions on the whole axis. Also taking into account that \( \varphi'(x) = \frac{\varphi(x)}{\cosh x} = -\varphi(x) \tanh x \) and \( \tanh' x = \varphi^2(x) \) we obtain that \( \varphi(x) = 1 - \frac{1}{2} x^2 + \frac{5}{24} x^4 + \varphi_6(x) x^6 \), where \( \varphi_6(x) \) is an even real analytic function on the whole axis \( \mathbb{R} \). Therefore the functions \( \varphi(x) - \frac{1}{x^2} \) and \( \frac{\varphi(x) - 2(\varphi(x) - 1)}{x^4} = \frac{5}{12} x^2 + 4\varphi_6(x)x^2 + \varphi_6'(x)x^3 \) are even real analytic functions defined on the whole axis. Therefore the expression \( \text{lim}_{x \to 0} \Delta_{(e, x)} \left( \langle [X, X], [\xi_1, \xi] \rangle, \langle \xi_2, u_2 \rangle \right) = \infty \) determines a smooth 2-form on the whole tangent space \( T(G/K) \). We will denote, as in the previous cases, this form (extension) on \( T(G \times m) \) by \( \tilde{ω}^R_0 \).

By continuity the form \( \tilde{ω}^R_0 \) is closed, left \( G \)-invariant, right \( K \)-invariant and \( \mathcal{H} \subset \ker \tilde{ω}^R_0 \). It is clear that there exists a unique (closed) 2-form \( ω^R_0 \) on \( T(G/K) \) such that \( \tilde{ω}^R = π^* ω^R_0 \).

Now we will prove, applying Corollary \( \text{(3.4)} \) that this extension is the Kähler form of the metric \( g_0 \) on the whole tangent bundle \( T(G/K) \). Indeed, the entries of \( w_H(x) \) are determined by expressions \( \text{(4.19)} \) and therefore by \( \text{(4.32)} \),
\[ \lim_{x \to 0} w_1 |_{1} = 2 \sqrt{C + c_Z^2}, \quad \lim_{x \to 0} w_1 |_{2} = 2ic_Z, \quad \lim_{x \to 0} w_1 |_{3} = 2 \sqrt{C + c_Z^2}. \]
Also from relations (4.20) it follows that for the block-diagonal Hermitian matrix \( w_s(x) \) for each its \( 2 \times 2 \) block we have

\[
\lim_{x \to 0} w_{(2j-1)(2j-1)}(x) = \lim_{x \to 0} w_{(2j)(2j)}(x) = 2\sqrt{C + c_Z^2},
\]

\[
\lim_{x \to 0} w_{(2j-1)(2j)}(x) = 2ic_Z.
\]

It is easy to check that the corresponding limit diagonal Hermitian matrices \( \lim_{x \to 0} w_H(x) \) and \( \lim_{x \to 0} w_s(x) \) are positive-definite. Thus by Corollary 3.4, \( \omega_0^R \) is the Kähler form of the metric \( g_0 \) (the extension of \( g \) on \( G \times K \)).

Let us prove that the metric \( g_0 \) determined by the form \( \omega_0^R \) on the whole tangent bundle \( T(G/K) \cong G \times K \) is complete.

First of all suppose that \( \omega_0^R \) is determined by the vector-function \( a(x) = f^r(x)X \). By Corollary 3.7, such a metric admits a \( G \)-invariant potential function \( 2f(r) \) on \( T(G/K) \setminus G/K \), where \( r \) is the norm function determined by a \( G \)-invariant metric on \( G/K \). Since in our cases \( f(x) \) is the restriction of an even smooth function on the whole axis \( \mathbb{R} \), there exist a smooth extension of \( 2f(r) \) to the whole tangent bundle \( T(G/K) \). By continuity, this extension is a potential function on \( T(G/K) \). Now, Stenzel described all \( G \)-invariant Kähler structures \( (\omega, J^K) \) on \( T(G/K) \), where \( G/K \) is a compact symmetric space of rank one admitting a \( G \)-invariant potential function [13]. Thus the set of metrics \( c g_0, c > 0 \), coincides with Stenzel’s set of metrics. The completeness of these metrics is proved in Stenzel’s paper [18] (see also another proof of this fact in Mykytyuk [14]).

Let us prove that the metric \( g_0 \) determined by the form \( \omega_0^R \) on the whole tangent bundle \( T(G/K) \cong G \times K \) is complete if \( G/K = \mathbb{CP}^n, n \geq 2 \). To this end, consider again its description (4.30) on the space \( G/H \times \mathbb{R}^+ (G = SU(n+1), H \cong U(1) \times SU(n-1)) \). For our aim it is sufficient to calculate the distance \( \text{dist}(b, c) \) between the compact subsets \( G/H \times \{ b \} \) and \( G/H \times \{ c \} \), where \( \text{dist}(b, c) = \inf\{d(p_b, p_c), p_b \in G/H \times \{ b \}, p_c \in G/H \times \{ c \}\} \). Since the sets \( G/H \times \{ x \} \) are compact, it is clear that the metric \( g_0 \) is complete if and only if for some \( b > 0 \) one has \( \lim_{c \to 0} \text{dist}(b, c) = \infty \).

To calculate the function \( \text{dist}(b, c) \) note that the tangent bundle \( T(G/K) \cong G \times K \) is a cohomogeneity-one manifold, i.e. the orbits of the action of the Lie group \( G \) have codimension one. We will use only one fundamental fact on the structure of these manifolds [11]: A unit smooth vector field \( U \) on a \( G \)-invariant domain \( D \subset T(G/K) \) which is \( g_0 \)-orthogonal to each \( G \)-orbit in \( D \) is a geodesic vector field, i.e. its integral curves are geodesics of the metric \( g_0 \).

We now describe such a vector field \( U \) on the domain \( G \times \mathbb{R}^+ \cong T(G/K) \setminus G/K \). Put

\[
(4.33) \quad f_U(x) = \left( \frac{f''(x)}{f'(x)f''(x) + c_Z^2 \varphi(x) \varphi''(x)} \right)^{1/2}, \quad x \in \mathbb{R}^+,
\]

where, recall, \( \varphi(x) = \frac{1}{\cosh x} \).
Lemma 4.2. Such a unit vector field \( U \) on \( G/H \times \mathbb{R}^+ \) is \( G \)-invariant and at the point \((o,x)\), \( o = \{H\}, x \in \mathbb{R}^+ \), is determined by the expression

\[
U(o,x) = f_U(x) \cdot \left( -\frac{cz\phi'(x)}{f'(x)}, \frac{\partial}{\partial x} \right).
\]

For the coordinate function \( x \) on \( G/H \times \mathbb{R}^+ \) the following inequality holds

\[
|dx_{(o,x)}(\xi, t \frac{\partial}{\partial x})| \leq f_U(x) \cdot \| (\xi, t \frac{\partial}{\partial x}) \|_{(o,x)},
\]

where \((\xi, t \frac{\partial}{\partial x}) \in T_{(o,x)}(G/H \times \mathbb{R}^+) = (m \oplus t^+) \times \mathbb{R} \) and \( \| \cdot \| \) is the norm given by the metric \( g \).

Proof (of Lemma) Since the vector field \( U \) is unique (up to sign), it is sufficient to verify that each vector \( U(o,x) \) in (4.34) is \( g \)-orthogonal to the \( G \)-orbit through \( (o,x) \), i.e. to the subspace \( V(o,x) \subset T_{(o,x)}(G/H \times \mathbb{R}^+) \) generated by the vectors \((\xi, 0), \xi \in m \oplus t^+, \) and that \( \| U(o,x) \| = 1 \).

Using expression (4.30) for the form \( \tilde{\omega}^R \) we obtain the following expression for the form \( \omega \) at \((o,x)\):

\[
\omega_{(o,x)}\left( (\xi_1, t_1 \frac{\partial}{\partial x}), (\xi_2, t_2 \frac{\partial}{\partial x}) \right) = -\langle f'(x)X + cz(\phi(x) - 1)Z - czZ_0, [\xi_1, \xi_2]\rangle + f''(x)(t_1\langle X, \xi_2 \rangle - t_2\langle X, \xi_1 \rangle) + cz\phi'(x)(t_1\langle Z, \xi_2 \rangle - t_2\langle Z, \xi_1 \rangle),
\]

where \( \xi_1, \xi_2 \in m \oplus t^+, t_1, t_2 \in \mathbb{R} \).

Fix a point \((o,x) \in G/H \times \mathbb{R}^+ \) and consider a tangent vector \( \tilde{Y} = \left( bY, \frac{\partial}{\partial x} \right), b \in \mathbb{R} \), at \((o,x)\). This vector is \( g \)-orthogonal to \( V(o,x) \) if and only if this vector is \( \omega \)-orthogonal to the subspace \( J^K_c(V(o,x)) \) generated by the vectors \((\xi_1, t_1 \frac{\partial}{\partial x}), \xi_1 \in m^+ \oplus t^+, \) \((\langle \xi, X \rangle = 0)\), \( t_1 \in \mathbb{R} \), because by (3.18),

\[
J^K_c(o,x)(X, 0) = (0, \frac{\partial}{\partial x}), \quad J^K_c(o,x)(Y, 0) = (-\frac{cz\phi}{\sinh x}Z, 0)
\]

and \( J^K_c(o,x)(\xi_j, 0) = (-\frac{\cosh x R_j}{\sinh x}, 0), j = 1, \ldots, 2(n-1). \) By (4.36) for any \( \xi_1 \in m^+ \oplus t^+, \) \( t_1 \in \mathbb{R} \) we obtain

\[
\omega_{(o,x)}\left( ([\xi_1, t_1 \frac{\partial}{\partial x}], (bY, \frac{\partial}{\partial x})) = b\langle f'(x)X + cz(\phi(x) - 1)Z - czZ_0, [\xi_1]\rangle + f''(x)(t_1\langle X, bY \rangle - \langle X, \xi_1 \rangle) + cz\phi'(x)(t_1\langle Z, bY \rangle - \langle Z, \xi_1 \rangle)
\]

\[
= -(b f'(x) + cz\phi'(x)) \langle Z, \xi_1 \rangle,
\]

because \( \xi_1 \) \( \perp \) \( X \) and \( [Z_0, Y] = IY = -X \) (see also relations (4.8)). By (4.38) the vector \( U(o,x) = f_U(x) \tilde{Y} \) with \( b = -cz\phi'(x) / f'(x) \) is \( g \)-orthogonal to the subspace \( V(o,x) \).

But by (4.37), \( J^K_c(o,x) \left(-\frac{cz\phi'(x)}{f'(x)}Y, \frac{\partial}{\partial x}\right) = \left( -\frac{cz\phi'(x)}{f'(x)}Z - X, 0 \right) \) because \( \phi'(x) = -\phi(x) \tan h x \). Taking into account relations (4.36), (4.31), (4.8) and the fact that \( \langle Z_0, Z \rangle = -1 \) we obtain that \( \omega(J^K_c(U), U) = f^2_U \left( f'' + \frac{cz\phi'}{f} \right) \equiv 1, \) i.e. \( \| U \| \equiv 1. \)
To prove the inequality in the statement it is sufficient to find the Hamiltonian vector field $H^x$ of the function $x$. This vector field is $G$-invariant as so are the form $\omega$ and the function $x$. Let us show that $H^x(o,x) = (a(x)X + c(x)Z, 0)$, where $a, c$ are some functions of $x$. Indeed, using relations (4.35), (4.36), (4.7), (4.8), $Z_0, Z = 0$ and the invariance of the form $\langle \cdot, \cdot \rangle$, we obtain the following expression at the point $(o, x)$ for any $\xi_1 \in m \oplus t^+$, $t_1 \in \mathbb{R}$:

\[
\omega\left((\xi_1, t_1 \frac{\partial}{\partial x}), (aX + cZ, 0)\right) = -\langle f'X + cZ(\varphi - 1)Z - czZ_0, [\xi_1, aX + cZ]\rangle
\]

\[
+ f''t_1 \langle X, aX + cZ \rangle + cZ\varphi' t_1 \langle Z, aX + cZ \rangle
\]

\[
= (cf' - cz\varphi)\langle Y, \xi_1 \rangle + (af'' + czc\varphi')t_1,
\]

Now it is easy to see that $\omega((\xi_1, t_1 \frac{\partial}{\partial x}), H^x) \overset{\text{def}}{=} d\lambda((\xi_1, t_1 \frac{\partial}{\partial x}) = t_1$ at the point $(o, x)$ for arbitrary $t_1 \in \mathbb{R}$, $\xi_1 \in m \oplus t^+$ if and only if

\[
a = c \cdot \frac{f'}{cz\varphi} \quad \text{and} \quad c = \frac{cz\varphi}{f'' + czc\varphi'}.
\]

Since $J_c^K(H^x)(o, x) = \left( c \frac{\sinh x}{\cosh x} Y, a \frac{\partial}{\partial x} \right)$ and $a = f_U^2$, we obtain at the point $(o, x)$

\[
\|H^x\|^2 \overset{\text{def}}{=} \omega\left((c \frac{\sinh x}{\cosh x} Y, a \frac{\partial}{\partial x}), (aX + cZ, 0)\right) = d\lambda\left(c \frac{\sinh x}{\cosh x} Y, a \frac{\partial}{\partial x}\right) = a = f_U^2.
\]

Now, by the Cauchy-Schwarz inequality for metrics one has at the point $(o, x)$

\[
|d\lambda((\xi_1, t_1 \frac{\partial}{\partial x})| = \omega((\xi_1, t_1 \frac{\partial}{\partial x}), H^x) | = |g((\xi_1, t_1 \frac{\partial}{\partial x}), J_c^K(H^x))|
\]

\[
\leq \|J_c^K(H^x)\| \cdot \|(\xi_1, t_1 \frac{\partial}{\partial x})\| = \|H^x\| \cdot \|(\xi_1, t_1 \frac{\partial}{\partial x})\|,
\]

that is, we obtain (4.35). \hfill \Box

Using now the vector field $U$ we shall calculate the distance between the level sets $G/H \times \{b\}$ and $G/H \times \{c\}$ in $G/H \times \mathbb{R}^+$ with respect to the metric $g$. Let $\gamma(t) = (\hat{g}(t)H, \hat{x}(t))$, $t \in [0, T]$, be the integral curve of the vector field $U$ with initial point $p_b \in G/H \times \{b\}$, that is, $\hat{x}(0) = b$. There exists a function $h$ on $\mathbb{R}^+$ such that the function $h(\hat{x}(t))$ is linear in $t$. It is easy to verify that $h(x) = \int_b^x \frac{ds}{f_U(s)}$, because by (4.34)

\[
\frac{d}{dt}h(\hat{x}(t)) = h'(\hat{x}(t)) \cdot d\gamma(t) = h'(\hat{x}(t)) \cdot \hat{x}'(t) = h'(\hat{x}(t)) \cdot (f_U(\hat{x}(t))) = 1.
\]

Suppose that $p_c \in G/H \times \{c\}$, where $p_c = \gamma(t_c)$, $t_c \in [0, T]$. Since the curve $\gamma$ is a geodesic, the length of the curve $\gamma(t), t \in [0, t_c]$, from $p_b$ to $p_c$ is $t_c = h(x(p_c)) - h(x(p_b)) = (h(c) - h(b))$. Thus $\text{dist}(b, c) \geq h(c) - h(b)$.

For any other curve $\gamma_1(t) = (\hat{g}(t)H, \hat{x}(t))$, with $\|\gamma_1(t)\| = 1$, starting at the point $p_b$, and ending at a point $p_c \in G/H \times \{c\}$, $p_c = \gamma_1(t_c)$ (of length $t_c$), we obtain by Lemma 4.2

\[
\frac{d}{dt}h(\hat{x}_1(t)) = h'(\hat{x}_1(t)) \cdot d\gamma_1(t) \leq \frac{1}{f_U(\hat{x}_1(t))} \cdot f_U(\hat{x}_1(t)) \cdot \|\gamma_1(t)\| = 1.
\]
Thus \( h(c) - h(b) \leq t^1_c \) and the length \( t^1_c \) of the curve \( \gamma_t \) from \( p_{k} \) to \( p_{k}^1 \) is not less than the length of the curve \( \gamma(t) \), \( t \in [0, t_c] \). Thus, the distance between the level surfaces \( G/H \times \{c\} \) and \( G/H \times \{c\} \) is \( |h(c) - h(b)| \).

Now, since by (4.23) and (4.24) for \( C_1 = 0 \),

\[
f'(x) = \frac{C \sinh^2 x + c_2 \sinh^2 x \cosh^{-1} x}{f''(x)},
\]

we obtain that \( f'(x) \sim \sqrt{C} \sinh x \), \( f''(x) \sim \sqrt{C} \sinh x \) and, by (4.33), \( \frac{1}{f''(x)} \sim (\sqrt{C} \sinh x)^{1/2} \) as \( x \to \infty \). Therefore \( \lim_{x \to \infty} h(x) = \infty \). Hence the metric \( g_0 = g_0(C, c_2, 0) \) (that is, for \( C_1 = 0 \)) on the tangent bundle \( T(G/K) \) is complete for any \( C > 0 \), \( c_2 \in \mathbb{R} \). □

It is well known that \( \mathbb{R}P^n \cong S^n/\mathbb{Z}_2 \) as \( \mathbb{R}P^n = SO(n+1)/O(n) \) and \( S^n = SO(n+1)/SO(n) \) \((n \geq 2)\). Hence each \( SO(n+1) \)-invariant Ricci-flat Kähler structure on \( T \mathbb{R}P^n \) is uniquely determined by a \( \mathbb{Z}_2 \)-invariant Ricci flat Kähler structure on \( T \mathbb{R}P^n \).

**Corollary 4.3.** If \( n \geq 3 \), each \( G \)-invariant Ricci-flat Kähler structure \( (g(C, C_1), J^K) \) on the punctured tangent bundle \( T^+(G/K) = T^+(SO(n+1)/SO(n)) = T^+S^n \) determines an invariant Ricci-flat Kähler structure on \( T^+\mathbb{R}P^n \). If \( n = 2 \), the \( G \)-invariant Ricci-flat Kähler structure \( (g(C, C_1, c_2), J^K) \) on \( T^+(G/K) = T^+(SO(3)/SO(2)) = T^+S^2 \) determines an invariant Ricci-flat Kähler structure on \( T^+\mathbb{R}P^2 \) if and only if \( c_2 = 0 \). All these invariant Ricci-flat Kähler metrics on \( T^+\mathbb{R}P^n \) are uniquely extendable to complete metrics on the whole tangent bundle \( T\mathbb{R}P^n \), \( n \geq 2 \), if and only if \( C_1 = 0 \).

**Proof.** We will use the notations of the proof of Theorem 4.1. As it follows from its proof the Kähler structure \( (g(C, C_1), J^K) \) on \( T^+(G/K) = T^+(SO(n+1)/SO(n)) \) \((n \geq 3)\) is \( \mathbb{Z}_2 \)-invariant if and only if the form \( \mathring{\omega}^K = \Delta \) (see (4.27)) on \( G \times m \) is right \( K_1 \)-invariant, where \( K_1 = O(n) \) \((K \subset K_1 \subset G)\). The form \( \mathring{\omega}^R \) is right \( K_1 \)-invariant because \( \text{Ad}(K_1)(m) = m \) and \( \text{Ad}(K_1) \) is a subgroup of the group of inner automorphisms \( \text{Ad}(G) \) of \( g \). Similarly, if \( n = 2 \), the form \( \mathring{\omega}^R = \Delta \) (see (4.31)) is right \( K_1 \)-invariant if and only if \( c_2 = 0 \) because \( \text{Ad}(K_1)Z \not\in \{Z\} \) \((Z_0 = Z \) if \( n = 2 \) and \( \mathbb{R}P^2 \) is not a homogeneous complex manifold). Now the last assertion of the corollary immediately follows from the last assertion of Theorem 4.1. □

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