Pseudo-Hermitian Systems with $\mathcal{P}\mathcal{T}$-Symmetry: Degeneracy and Krein Space

B. Choutri · O. Cherbal · F. Z. Ighezou · M. Drir

Received: 25 October 2016 / Accepted: 27 January 2017 / Published online: 3 February 2017
© Springer Science+Business Media New York 2017

Abstract We show in the present paper that pseudo-Hermitian Hamiltonian systems with even $\mathcal{P}\mathcal{T}$-symmetry ($\mathcal{P}^2 = 1$, $\mathcal{T}^2 = 1$) admit a degeneracy structure. This kind of degeneracy is expected traditionally in the odd $\mathcal{P}\mathcal{T}$-symmetric systems ($\mathcal{P}^2 = 1$, $\mathcal{T}^2 = -1$) which is appropriate to the fermions (Scolarici and Solombrino, Phys. Lett. A 303, 239 2002; Jones-Smith and Mathur, Phys. Rev. A 82, 042101 2010). We establish that the pseudo-Hermitian Hamiltonians with even $\mathcal{P}\mathcal{T}$-symmetry admit a degeneracy structure if the operator $\mathcal{PT}$ anticommutes with the metric operator $\eta_\sigma$ which is necessarily indefinite. We also show that the Krein space formulation of the Hilbert space is the convenient framework for the implementation of unbroken $\mathcal{P}\mathcal{T}$-symmetry. These general results are illustrated with great details for four-level pseudo-Hermitian Hamiltonian with even $\mathcal{P}\mathcal{T}$-symmetry.

Keywords Pseudo-Hermiticity · $\mathcal{P}\mathcal{T}$-symmetry · Degeneracy · Krein space

1 Introduction

The research works which deal with pseudo-Hermitian and $\mathcal{P}\mathcal{T}$-symmetric Hamiltonians have received a great deal of interest over the last two decades [3–10, 12, 13]. In this
context, Sato et al. [14] established a generalization of the Kramers degeneracy to pseudo-Hermitian Hamiltonians admitting even time-reversal symmetry ($T^2 = 1$). This extension is achieved using the mathematical structure of split-quaternions\(^1\) instead of quaternions, usually adopted in the case of Hermitian Hamiltonians with odd time-reversal symmetry ($T^2 = -1$) [15]. In a recent paper [16], we have found that the metric operator for the pseudo-Hermitian Hamiltonian $H$ that allows the realization of the generalized Kramers degeneracy is necessarily indefinite. We have further shown that such $H$ with real spectrum also possesses odd antilinear symmetry induced from the existing odd time-reversal symmetry of its Hermitian counterpart $h$, so that the generalized Kramers degeneracy of $H$ is in fact crypto-Hermitian Kramers degeneracy [16].

On the other hand, as it is well known in $\mathcal{PT}$ quantum theory [5, 12], due to the anti-linearity of $\mathcal{PT}$, the eigenstates of a $\mathcal{PT}$-symmetric Hamiltonian $H$, may or may not be the eigenstates of $\mathcal{PT}$. If every eigenstate of $H$ is also an eigenstate of $\mathcal{PT}$, then we have an unbroken $\mathcal{PT}$-symmetry, which corresponds to real eigenvalues. Conversely, if some of the eigenstates of a $H$ are not simultaneously eigenstates of $\mathcal{PT}$, then we have a broken $\mathcal{PT}$-symmetry, which corresponds to complex conjugate eigenvalues. The $\mathcal{PT}$-broken and $\mathcal{PT}$-unbroken phases are separated by exceptional points (EPs) [17]. For $\mathcal{PT}$-symmetric systems, the exceptional points are an obligatory passage in the $\mathcal{PT}$-broken and $\mathcal{PT}$-symmetric phase transitions. In this context, Berry has shown [18] that non-Hermitian physics differs radically from Hermitian physics in the presence of degeneracies. This non-Hermitian behavior has been further illustrated in several physical examples [18]. Recently, the role of degeneracy in $\mathcal{PT}$-symmetry breaking has been also subject of investigations [19]. The degeneracy for odd $\mathcal{PT}$-symmetric systems ($P^2 = 1$, $T^2 = -1$) appropriate to the fermions was first proven by Scolarici and Solombrino in [1], and studied afterwards by Jones-Smith and Mathur [2] who extended $\mathcal{PT}$-symmetric quantum mechanics to the case of odd time-reversal symmetry. It has been established that an analog of Kramers degeneracy exists also for odd $\mathcal{PT}$-symmetric systems, and an unbroken $\mathcal{PT}$-symmetry can exist if we assemble the two column vectors of the $\mathcal{PT}$ doublets in a single quaternionic column [2].

The purpose of the present paper is to extend the non-Hermitian degeneracy behavior developed for odd $\mathcal{PT}$-symmetric systems [2] to even $\mathcal{PT}$-symmetric ones ($P^2 = 1$, $T^2 = 1$).

The paper is organized as follows. In Section 2, we first analyze the existence of the degeneracy structure for pseudo-Hermitian Hamiltonian systems with even $\mathcal{PT}$-symmetry. We will establish that the degeneracy structure exists if the $\mathcal{PT}$ operator anticommutes with the metric operator $\eta_\sigma$, which is necessarily indefinite metric. Moreover, we emphasize in Section 3 the role of the Krein space formulation of the Hilbert space for restoring an unbroken $\mathcal{PT}$-symmetry. This degeneracy structure is thus well implemented in the Krein space formulation of the Hilbert space. In the purpose of illustration of the above general results, we study in great details in Section 4, the pseudo-Hermitian four-level Hamiltonian invariant under the even $\mathcal{PT}$-symmetry. The paper ends with conclusion and outlook.

\(^{1}\)The quaternion algebra is generated by the $2 \times 2$ unit matrix $\sigma_0$ and the pure imaginary complex number $i$ multiplied by the SU(2) Pauli matrices $(i\sigma_x, i\sigma_y, i\sigma_z)$, while the split-quaternion algebra is generated by the $2 \times 2$ unit matrix $\sigma_0$ and the pure imaginary complex number $i$ multiplied by the SU(1,1) Pauli matrices $(-\sigma_x, -\sigma_y, i\sigma_z)$.
2 Pseudo-Hermiticity, $\mathcal{PT}$-Symmetry and Degeneracy

We start our analysis by considering a diagonalizable non-Hermitian Hamiltonian $H$ with a real discrete spectrum. The associated complete biorthonormal eigenbasis $\{|\psi_{n,a}\rangle, |\phi_{n,a}\rangle\}$ satisfies by definition [6, 9, 10, 20, 21],

$$H|\psi_{n,a}\rangle = E_n|\psi_{n,a}\rangle, \quad \langle \phi_{m,b}|H^\dagger = E_n^\ast |\phi_{m,b}\rangle,$$

where $n$ and $a, b$ are, respectively, the spectral and degeneracy labels, $d_n$ is the multiplicity (degree of degeneracy) of $E_n$. The Hamiltonian $H$ is taken to be pseudo-Hermitian, this means that $H$ satisfies the relation [6, 9]

$$H^\dagger = \eta H \eta^{-1},$$

where $\eta$ is Hermitian, linear invertible operator. As we deal with real eigenvalues of $H$, for such $H$ there exists a positive definite metric operator $\eta_+$ given by [10],

$$\eta_+ = \sum_n d_n \sum_{a=1} \langle \phi_{n,a}|\phi_{n,a}\rangle = 1,$$

and its inverse

$$\eta_+^{-1} = \sum_n d_n \sum_{a=1} \langle \psi_{n,a}|\psi_{n,a}\rangle,$$

with the properties [6, 10]

$$|\phi_{n,a}\rangle = \eta_+ |\psi_{n,a}\rangle, \quad |\psi_{n,a}\rangle = \eta_+^{-1} |\phi_{n,a}\rangle.$$

As established in [10], besides $\eta_+$, there are Hermitian invertible operators $\eta_{\sigma}$ that are also associated with the same biorthonormal system (1)–(4) and satisfy the pseudo-Hermitian relation (5). $\eta_{\sigma}$ and its inverse are given by [10]:

$$\eta_{\sigma} = \sum_n d_n \sum_{a=1} \sigma_n^a |\phi_{n,a}\rangle \langle \phi_{n,a}|,$$

$$\eta_{\sigma}^{-1} = \sum_n d_n \sum_{a=1} \sigma_n^a |\psi_{n,a}\rangle \langle \psi_{n,a}|,$$

where $\sigma = (\sigma_n^a)$ is a sequence of signs $\sigma_n^a = \pm$. Thus, $\eta_{\sigma}$ and $\eta_+$ are both metric operators associated to the Hamiltonian $H$. Moreover, $\eta_{\sigma}$ and $\eta_+$ are linked via the following relation [11],

$$C_{\sigma} = \eta_+^{-1} \eta_{\sigma},$$

or

$$\eta_+ = \eta_{\sigma} C_{\sigma}.$$
where \( C_\sigma \) is the grading operator, which is a linear invertible involution operator\(^2\) associated with \( H \), given by [9, 10],

\[
C_\sigma = \sum_n \sum_{a=1}^{d_n} \sigma_n^a |\psi_{n,a}\rangle \langle \phi_{n,a}|, \quad \text{with} \quad C_\sigma^2 = 1,
\]

with the properties [9],

\[
C_\sigma |\psi_{n,a}\rangle = \sigma_n^a |\psi_{n,a}\rangle.
\]

Since \( C_\sigma \) is an involution operator (\( C_\sigma^2 = 1 \)), in view of (14) one deduce that \( \sigma_n^a = 1 \). From the equations (11) and (14), one can express the relations between \( |\psi_{n,a}\rangle \) and \( |\phi_{n,a}\rangle \) given in (8) as follow,

\[
|\phi_{n,a}\rangle = \eta a C_\sigma |\psi_{n,a}\rangle = \sigma_n^a \eta \sigma |\psi_{n,a}\rangle, \quad |\psi_{n,a}\rangle = \sigma_n^a \eta^{-1} |\phi_{n,a}\rangle.
\]

Furthermore, we assume that \( H \) is invariant under the even \( PT \)-symmetry, i.e \([ H, PT ] = 0\), where \( P \) and \( T \) are parity and even time-reversal operators respectively, (\( P^2 = 1 \), \( T^2 = 1 \)). Here we define the parity operator \( P \) by following the definition given in [2]. The action of \( P \) to any state \( \psi \) is to multiply \( \psi \) by the matrix \( S \) which is a real \( N \)-dimensional matrix given by [2],

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( I \) denotes the \((N/2)\)-dimensional identity matrix. For the time-reversal operator, one follows the definition given in [14] as \( T = ZK \), with \( K \) being a complex conjugation operator, \( Z \) is an unitary matrix which can be chosen as a real matrix with all the diagonal terms equal to a \( 2 \times 2 \) Pauli matrix \( \sigma_x \) and all the off-diagonal terms equal to zero [14], namely

\[
Z = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We have \( T^2 = 1 \). It is useful to note that as established [2], the non-Hermitian Hamiltonians which are invariant under the odd \( PT \)-symmetry, admit Kramers degeneracy implemented by the mathematical structure of quaternions [15]. This degeneracy is expected because the time-reversal operator is odd (\( T^2 = -1 \)).

We propose to show in the following that a degeneracy exists also for pseudo-Hermitian Hamiltonians with even \( PT \)-symmetry. In order to show the degeneracy in the eigenvalues of \( H \), we show that the eigenstates \( |\psi_{n,a}\rangle \) and \( |\phi_{n,a}\rangle \) which correspond to the same eigenvalue \( E_n \) are linearly independent. As shown in [22], any antiunitary operator \( \theta \) can be written as \( \theta = UK \), where \( U \) is an unitary operator and \( K \) is the conjugation operator. Here we have

\[
\theta = PT = SZK = UK,
\]

where \( U = SZ \) is the unitary operator. Moreover, according to [14], one deduce from the antiunitary property\(^3\) of \( PT \), that pseudo-Hermiticity is consistent with the \( PT \)-symmetry if the metric operator commutes or anticommutes with the \( PT \) operator. Thus, as in [14],

---

\(^2\)The operator \( C_\sigma \) is denoted by \( S_\sigma \) in reference [10]. See equation (43) of [10] and the proposition 4 just before this equation.

\(^3\)The antiunitary property means that for any operator antiunitary denoted \( \theta \), we have \( \langle \theta x | \theta y \rangle = \langle y | x \rangle \).
Here we have the term $\sigma_a\eta_\sigma$ with the metric operator $\mathcal{P}\mathcal{T}$ in the odd $\eta_\sigma$-symmetry case. Now we show that in the above described scheme the indefiniteness feature is said to be unbroken if the states $|\psi_{n,a}\rangle$ are linearly independent but they are not orthogonal as in the odd $\eta_\sigma$-symmetry. In this aim, we shall show that the $\eta_\sigma$-norm related to the $\eta_\sigma$-inner product $\langle . | . \rangle_{\eta_\sigma} = \langle . | \eta_\sigma . \rangle$ of the eigenstates of $H$ is indefinite [26–29]. Thus we compute the norms $\langle \psi_{n,a} | \psi_{n,r} \rangle_{\eta_\sigma}$ and $\langle \mathcal{P}\mathcal{T}\psi_{n,a} | \mathcal{P}\mathcal{T}\psi_{n,a} \rangle_{\eta_\sigma}$. We have

$$\langle \mathcal{P}\mathcal{T}\psi_{n,a} | \mathcal{P}\mathcal{T}\psi_{n,a} \rangle_{\eta_\sigma} = \langle \mathcal{P}\mathcal{T}\psi_{n,a} | \eta_\sigma \mathcal{P}\mathcal{T}\psi_{n,a} \rangle_{\eta_\sigma}. \quad (22)$$

By using the anticommutation relation between $\eta_\sigma$ and $\mathcal{P}\mathcal{T}$, the antiunitary property of $\mathcal{P}\mathcal{T}$ and $\langle \mathcal{P}\mathcal{T} \rangle^2 = 1$, the Hermiticity of $\eta_\sigma$, one obtains from the last equation that

$$\langle \mathcal{P}\mathcal{T}\psi_{n,a} | \mathcal{P}\mathcal{T}\psi_{n,a} \rangle_{\eta_\sigma} = -\langle \mathcal{P}\mathcal{T}\psi_{n,a} | \eta_\sigma \mathcal{P}\mathcal{T}\psi_{n,a} \rangle_{\eta_\sigma} = -\langle \mathcal{P}\mathcal{T}^2 \eta_\sigma \psi_{n,a} | \mathcal{P}\mathcal{T}^2 \psi_{n,a} \rangle$$

$$= -\langle \eta_\sigma \psi_{n,a} | \psi_{n,a} \rangle = -\langle \psi_{n,a} | \eta_\sigma \psi_{n,a} \rangle = -\langle \psi_{n,a} | \psi_{n,a} \rangle_{\eta_\sigma}. \quad (23)$$

Thereby, the norms $\langle \psi_{n,a} | \psi_{n,a} \rangle_{\eta_\sigma}$ and $\langle \mathcal{P}\mathcal{T}\psi_{n,a} | \mathcal{P}\mathcal{T}\psi_{n,a} \rangle_{\eta_\sigma}$ are of opposite signs, which confirm the indefiniteness feature $\eta_\sigma$. Due to this degeneracy behavior, a question arises: Is the $\mathcal{P}\mathcal{T}$-symmetry in the broken or unbroken phase? Traditionally, the $\mathcal{P}\mathcal{T}$-broken and $\mathcal{P}\mathcal{T}$-unbroken phases are separated by exceptional points which are non-Hermitian degenerate points with coalesced eigenvalues and eigenvectors. In our case, we deal with an other kind of degeneracy different from the exceptional points degeneracy.

In conclusion of this section, we have established that the pseudo-Hermitian Hamiltonians with even $\mathcal{P}\mathcal{T}$-symmetry, $\mathcal{P}\mathcal{T}$ is said to be unbroken if the states $|\psi_{n,a}\rangle$ are invariant under $\mathcal{P}\mathcal{T}$, i.e.

### 3 Krein Space Formulation

Let us show how we achieve an unbroken $\mathcal{P}\mathcal{T}$-symmetry. The idea is the passage to the Krein space formulation of the Hilbert space. We recall that in the case of even $\mathcal{P}\mathcal{T}$-symmetry, $\mathcal{P}\mathcal{T}$ is said to be unbroken if the states $|\psi_{n,a}\rangle$ are invariant under $\mathcal{P}\mathcal{T}$, i.e.
However, the situation is different here, although the $\mathcal{PT}$-symmetry is even, the eigenstates $|\psi_{n,a}\rangle$ are not invariant under $\mathcal{PT}$, i.e $\mathcal{PT}|\psi_{n,a}\rangle \neq |\psi_{n,a}\rangle$ because we have established previously that the $\mathcal{PT}$ doublets $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ are linearly independent. Then the $\mathcal{PT}$-symmetry is broken in the Hilbert space spanned by the eigenstates of $H$. The way out from this dead end is in fact to introduce the Krein space formulation of the Hilbert space [24, 25]. Indeed, let us define new eigenstate $|\chi_{n,a}\rangle$ spanned by the $\mathcal{PT}$ doublets $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ as follows,

$$|\chi_{n,a}\rangle = \frac{i}{\sqrt{2}} (|\psi_{n,a}\rangle - \mathcal{PT}|\psi_{n,a}\rangle), \quad (24)$$

with the following properties, (i) the $|\chi_{n,a}\rangle$ are eigenstates of $H$ with the same eigenvalue $E_n$, (ii) the states $|\chi_{n,a}\rangle$ are invariant under $\mathcal{PT}$, i.e

$$\mathcal{PT}|\chi_{n,a}\rangle = -\frac{i}{\sqrt{2}} \mathcal{PT}(|\psi_{n,a}\rangle - \mathcal{PT}|\psi_{n,a}\rangle) = |\chi_{n,a}\rangle, \quad (25)$$

we have used the anti-linearity and the involution properties of $\mathcal{PT}$. Now we show that the Hilbert space $\mathcal{K}$ spanned by the states $|\chi_{n,a}\rangle$ is a Krein space [24, 25]. $\mathcal{K}$ possess the following properties: (i) $\mathcal{K}$ is endowed with the indefinite inner product $\langle . | . \rangle_{\eta\sigma}$ defined in Section 2, (ii) $\mathcal{K}$ can be decomposed in a pair of vector subspaces $\mathcal{H}_\pm$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are spanned by the $\mathcal{PT}$ doublets $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ respectively; $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\oplus$ means direct sum which means that for all element $f \in \mathcal{K}$ there are unique $f_\pm \in \mathcal{H}_\pm$ such that $f = f_+ + f_-$. (iii) In the purpose of showing that $\mathcal{H}_+$ and $\mathcal{H}_-$ are orthogonal, we calculate the inner product $\langle \psi_{n,a} | \mathcal{PT}|\psi_{n,a}\rangle_{\eta\sigma}$ for any elements $|\psi_{n,a}\rangle \in \mathcal{H}_+$ and $\mathcal{PT}|\psi_{n,a}\rangle \in \mathcal{H}_-$, we have

$$\langle \psi_{n,a} | \mathcal{PT}|\psi_{n,a}\rangle_{\eta\sigma} = \langle \psi_{n,a} | \eta\sigma \mathcal{PT}|\psi_{n,a}\rangle = -\langle \psi_{n,a} | \mathcal{PT}\eta\sigma|\psi_{n,a}\rangle = -\langle (\mathcal{PT})^2 \eta\sigma|\psi_{n,a}\rangle |\mathcal{PT}|\psi_{n,a}\rangle = -\langle \eta\sigma \psi_{n,a} | \mathcal{PT}|\psi_{n,a}\rangle = -\langle \psi_{n,a} | \eta\sigma \mathcal{PT}|\psi_{n,a}\rangle = -\langle \psi_{n,a} | \mathcal{PT}|\psi_{n,a}\rangle_{\eta\sigma}, \quad (26)$$

thereby $\langle \psi_{n,a} | \mathcal{PT}|\psi_{n,a}\rangle_{\eta} = 0$. This means that $\mathcal{H}_+$ and $\mathcal{H}_-$ are orthogonal. Here, we have again used the anticommutation relation between $\eta\sigma$ and $\mathcal{PT}$ and the antiunitary property of $\mathcal{PT}$ and $(\mathcal{PT})^2 = 1$. The Hilbert space $\mathcal{K}$ is therefore a Krein space. It should be noticed that the condition of unbroken $\mathcal{PT}$-symmetry in the odd case is different to our even $\mathcal{PT}$-symmetry case. Indeed, in the odd case the $\mathcal{PT}$ doublets are assembled in two column vectors $(|\psi_{n,a}\rangle, \mathcal{PT}|\psi_{n,a}\rangle)$ which form a single component column of quaternions [2]. It is not possible to assemble our $\mathcal{PT}$ doublets in two column vectors, because the single component column obtained is a split-quaternion which possesses a structure different from the quaternion [14].

4 Illustration

In this section we illustrate the above general results with an example. We consider the four-level model described by the following non-Hermitian Hamiltonian:

$$H = \begin{pmatrix} a & ib \\ ic & -a \end{pmatrix}, \quad (27)$$
where $b = b_0 \sigma_0 + b_1 \sigma_x + b_2 \sigma_y - ib_3 \sigma_z$ and $c = b_0 \sigma_0 - b_1 \sigma_x - b_2 \sigma_y + ib_3 \sigma_z$ are real split-quaternions$^4$, and $a = a_0 \sigma_0$ is the real split-quaternion proportional to the identity, the $\sigma_k (k = x, y, z)$ are the Pauli matrices. By setting $A = b_1 + ib_2$ and $B = b_0 + ib_3$, this Hamiltonian $H$ can also be written as a four-level Hamiltonian as follows:

$$H = \begin{pmatrix}
a_0 & 0 & iB^* & iA^* \\
0 & a_0 & iA & iB \\
iB & -iA^* & -a_0 & 0 \\
iA & iB^* & 0 & -a_0
\end{pmatrix}, \quad (28)$$

where $A^*$ and $B^*$ are complex conjugates of $A$ and $B$ respectively. The Hamiltonian $H$, in (28), satisfies the following properties: (i) $H$ is pseudo-Hermitian, this means that $H$ satisfies the relation $H^\dagger = \eta H \eta^{-1}$, (ii) $H$ is invariant under the even $\mathcal{PT}$-symmetry, i.e $[H, \mathcal{PT}] = 0$, with $\mathcal{P}^2 = 1$, $\mathcal{T}^2 = 1$, where $\mathcal{P}$ and $\mathcal{T}$ are given in the case of the four-level system by [2, 16],

$$\mathcal{P} = \begin{pmatrix} I_2 & 0 \\
0 & -I_2 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} \sigma_x & 0 \\
0 & \sigma_x \end{pmatrix}, \quad (29)$$

where $I_2$ is the $2 \times 2$ identity matrix, $\sigma_x$ is Pauli matrix, $K$ being the complex conjugation operator. Moreover, $H$ admits an indefinite metric operator $\eta_\sigma$ which anticommutes with $\mathcal{PT}$, $\eta_\sigma$ is given explicitly by,

$$\eta_\sigma = \begin{pmatrix} \sigma_z & 0 \\
0 & -\sigma_z \end{pmatrix}. \quad (30)$$

The eigenvalues of $H$ are

$$E_\pm = \pm \Omega, \quad \text{with} \quad \Omega = \sqrt{a_0^2 + |A|^2 - |B|^2}.$$  

This eigenvalues are indeed twofold degenerate. The Hamiltonian $H$ represents a new class of even $\mathcal{PT}$-symmetric Hamiltonians with degeneracy. We remark that $H$ is asymmetric. We deal with real eigenvalues, i.e.

$$a_0^2 + |A|^2 > |B|^2. \quad (31)$$

The $\mathcal{PT}$ doublets ($|\psi_{-+}\rangle, |\psi_{-\pm}\rangle$) and ($|\psi_{++}\rangle, |\psi_{+-}\rangle$) associated to the negative and positive eigenvalues respectively are given as follows:

For the negative eigenvalue $E_- = -\Omega$:

$$|\psi_{-+}\rangle = k \begin{pmatrix} iA^* \\
iB \\
0 \\
-\left(\Omega + a_0\right) \end{pmatrix}, \quad |\psi_{-\pm}\rangle = \mathcal{PT}|\psi_{-+}\rangle = k \begin{pmatrix} -iA^* \\
-iB \\
\left(\Omega + a_0\right) \\
0 \end{pmatrix}, \quad (32)$$

For the positive eigenvalue $E_+ = \Omega$:

$$|\psi_{++}\rangle = k \begin{pmatrix} \left(\Omega + a_0\right) \\
0 \\
iB \\
-iA \end{pmatrix}, \quad |\psi_{+-}\rangle = \mathcal{PT}|\psi_{++}\rangle = k \begin{pmatrix} 0 \\
\Omega + a_0 \\
-iA^* \\
iB^* \end{pmatrix}. \quad (33)$$

$^4$The split-quaternion algebra is generated by the $2 \times 2$ unit matrix $\sigma_0$ and the pure imaginary complex number $i$ multiplied by the $\text{SU}(1,1)$ Pauli matrices ($-\sigma_x$, $-\sigma_y$, $i\sigma_z$).
The eigenstates \(|\phi_{-+}\rangle, |\phi_{--}\rangle\) and \(|\phi_{++}\rangle, |\phi_{+-}\rangle\) associated to \(H^\dagger\) are obtained by the action of \(\eta_\sigma\) given in (30) to the eigenstates of \(H\), they are given for the negative eigenvalue \(E_- = -\Omega\) by:

\[
|\phi_{-+}\rangle = k \begin{pmatrix} iA^* \\ -iB \\ 0 \\ -\left(\Omega + a_0\right) \end{pmatrix}, \quad |\phi_{--}\rangle = k \begin{pmatrix} -iB^* \\ iA \\ 0 \\ -(\Omega + a_0) \end{pmatrix},
\]

and for the positive eigenvalue \(E_+ = \Omega\) by:

\[
|\phi_{++}\rangle = k \begin{pmatrix} \Omega + a_0 \\ 0 \\ -iB \\ -iA \end{pmatrix}, \quad |\phi_{+-}\rangle = k \begin{pmatrix} 0 \\ iA^* \\ iB^* \end{pmatrix},
\]

where \(k\) is the normalization constant fixed by the requirement that:

\[
2\Omega(\Omega + a_0)|k|^2 = 1.
\]

These states satisfy the abnormal relations which is a consequence of the indefinite metric \(\eta_\sigma\) given in (30),

\[
\langle \psi_{+,+} | \phi_{++,+} = 1, \langle \psi_{+,--} | \phi_{--,+} = -1, \langle \psi_{++,-} | \phi_{++,+} = 1, \langle \psi_{--,-} | \phi_{--,+} = -1,
\]

\[
\langle \psi_{m\alpha} | \phi_{\bar{m}\bar{\alpha}} = 0, \langle \psi_{m\alpha} | \phi_{\bar{m} \bar{\alpha}} = 0, \langle \psi_{m\alpha} | \phi_{\bar{m} \bar{\alpha}} = 0.
\]

where \(\alpha = \pm\), \(m = \pm\), \(\bar{m}\) and \(\bar{\alpha}\) are the opposite signs of \(m\) and \(\alpha\) respectively. These states satisfy also the relations

\[
|\psi_{++}\rangle \langle \psi_{++} | - |\psi_{+-}\rangle \langle \psi_{+-} | + |\psi_{--}\rangle \langle \psi_{--} | - |\psi_{-+}\rangle \langle \psi_{-+} | = 1.
\]

The \((\text{indefinite})\ \eta_\sigma\)-norms of the \(\mathcal{PT}\) doublets are given by

\[
\langle \psi_{++} | \psi_{++} \rangle_{\eta_\sigma} = \langle \psi_{++,+} | \phi_{++,+} \rangle = 1, \quad \langle \psi_{++,--} | \phi_{++,+} \rangle = -1, \quad \langle \psi_{++,--} | \phi_{++,+} \rangle = 1, \quad \langle \psi_{++,--} | \phi_{++,+} \rangle = -1.
\]

We see that the eigenstates \(|\psi_{n,a}\rangle\) are not invariant under \(\mathcal{PT}\), i.e \(\mathcal{PT} |\psi_{n,a}\rangle \neq |\psi_{n,a}\rangle\), the \(\mathcal{PT}\)-symmetry is therefore broken. In order to restore an unbroken \(\mathcal{PT}\)-symmetry, one introduces the Krein space formulation of the Hilbert space. The Krein space \(\mathcal{K}\) is spanned by the states \(|\chi_{-+}\rangle\) and \(|\chi_{++}\rangle\) which are linear combination of the The \(\mathcal{PT}\) doublets \(|\psi_{--}\rangle, \mathcal{PT} |\psi_{++}\rangle\) and \(|\psi_{++}\rangle, \mathcal{PT} |\psi_{++}\rangle\) associated to the negative and positive eigenvalue respectively. Thus, in the Krein space \(\mathcal{K}\), the eigenstates are given by,

For the negative eigenvalue \(E = -\Omega\):

\[
|\chi_{-+}\rangle = \frac{i}{\sqrt{2}} (|\psi_{-+}\rangle - \mathcal{PT} |\psi_{-+}\rangle),
\]

for the positive eigenvalue \(E_+ = \Omega\):

\[
|\chi_{++}\rangle = \frac{i}{\sqrt{2}} (|\psi_{++}\rangle - \mathcal{PT} |\psi_{++}\rangle).
\]

We see that states \(|\chi_{-+}\rangle\) and \(|\chi_{++}\rangle\) are invariant under \(\mathcal{PT}\), i.e

\[
\mathcal{PT} |\chi_{-+}\rangle = -\frac{i}{\sqrt{2}} \mathcal{PT} (|\psi_{-+}\rangle - \mathcal{PT} |\psi_{-+}\rangle) = |\chi_{-+}\rangle,
\]

and

\[
\mathcal{PT} |\chi_{++}\rangle = -\frac{i}{\sqrt{2}} \mathcal{PT} (|\psi_{++}\rangle - \mathcal{PT} |\psi_{++}\rangle) = |\chi_{++}\rangle.
\]
One has therefore achieved an unbroken $\mathcal{PT}$-symmetry in the Krein space $\mathcal{K}$.

# 5 Conclusion and Outlook

We have established in the present paper a new kind of degeneracy structure which is due to the non-Hermitian behavior of the system. We have shown that the pseudo-Hermitian Hamiltonians with real eigenvalues and even $\mathcal{PT}$-symmetry admit a degeneracy structure if the operator $\mathcal{PT}$ anticommutes with the metric operator $\eta_\sigma$ which is necessarily indefinite. We have also shown that the Krein space formulation of the Hilbert space is the convenient framework for the implementation of unbroken $\mathcal{PT}$-symmetry for our system. Let us discuss some implications and outlook of our analysis.

- (1): It is useful to point out that the $\mathcal{PT}$-symmetric four-level Hamiltonian $H$ given in (27) represents a new class of $\mathcal{PT}$-symmetric Hamiltonians written in split- quaternionic form which represent the most general traceless Hamiltonian matrix in the case of even $\mathcal{PT}$-symmetry by generalizing the time reversal operator to include a matrix multiplying the complex conjugation operator. The study of further features of this Hamiltonian will be interesting. For instance the construction of the $\mathcal{CPT}$ inner product in the form of Mostafazadeh $\eta_+$ inner product [30], with $\eta_+ \equiv \mathcal{P}\mathcal{C}$.

- (2): The generalization to even $\mathcal{PT}$-symmetry of the $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric representations of fermionic algebras developed by Bender [31] and one of the actual authors [23] in the odd $\mathcal{PT}$-symmetric case, will be also a part of our future investigations.

- (3): It is useful to point out that the graded time-reversal operator in fermion Fock space is also an interesting outlook. In this case, the Hilbert space is neither even nor odd with respect to time-reversal symmetry but rather has a graded structure with regard to time reversal. The action of time reversal operator $\mathcal{T}$ may as usual be represented by $\mathcal{T}\varphi = L\varphi^*$. where the unitary operator has however a block diagonal structure

$$L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},$$

with $L_+^2 = 1$, $L_-^2 = -1$. Thus time-reversal is even in the bosonic subspace; it is odd in the fermionic subspace. It is a general feature of the graded Fock space of fermions that decomposes into two subspaces which are even and odd with respect to time-reversal. Thus it will be interesting to extend our analysis and all the works in the literature which deal with time-reversal symmetry to the case of time-reversal symmetry in Fock space of fermions with a graded form which can be a subject of interest for $\mathcal{PT}$ community.

# References

1. Scolarici, G., Solombrino, L.: Pseudo-Hermitian Hamiltonians, time-reversal invariance and Kramers degeneracy. Phys. Lett. A 303, 239 (2002)
2. Jones-Smith, K., Mathur, H.: Non-Hermitian quantum Hamiltonians with PT symmetry. Phys. Rev. A 82, 042101 (2010)
3. Bender, C.M., Boettcher, S.: Real spectra in non-Hermitian Hamiltonians having PT symmetry. Phys. Rev. Lett. 80, 5243 (1998)
4. Bender, C.M., Brody, D.C., Jones, H.F.: Complex extension of quantum mechanics. Phys. Rev. Lett. 89, 270401 (2002)
5. Bender, C.M.: Making sense of non-Hermitian Hamiltonians. Rep. Prog. Phys. 70, 947 (2007)
6. Mostafazadeh, A.: Pseudo-Hermiticity versus PT symmetry: the necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. J. Math. Phys. 43, 205 (2002)
7. Mostafazadeh, A.: Pseudo-Hermiticity versus PT-symmetry: II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum. J. Math. Phys. 43, 2814 (2002)
8. Mostafazadeh, A.: Pseudo-Hermiticity versus PT-Symmetry III: Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries. J. Math. Phys. 43, 3944 (2002)
9. Mostafazadeh, A.: Pseudo-Hermitian representation of quantum mechanics. Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010)
10. Mostafazadeh, A.: Pseudo-Hermiticity and generalized PT- and CPT-symmetries. J. Math. Phys. 44, 974 (2003)
11. See the relation (95) of Ref. [9]. In the case of real spectrum where $\eta_1 = \eta_+$ as given by relation (99) of Ref. [9]
12. Croke, S.: PT-symmetric Hamiltonians and their application in quantum information. Phys. Rev. A 91, 052113 (2015)
13. Brody, D.C.: Consistency of PT-symmetric quantum mechanics. J. Phys. A: Math. Theor. 49, 10LT03 (2016)
14. Sato, M., Hasebe, K., Esaki, K., Kohmoto, M.: Time-reversal symmetry in non-Hermitian systems. Prog. Theo. Phys. 127, 937 (2012)
15. Avron, J.E., Sadun, L., Segert, J., Simon, B.: Chern numbers, quaternions, and Berry’s phases in Fermi systems. Commun. Math. Phys. 124, 595 (1989)
16. Choutri, B., Cherbal, O., Ighezou, F.Z., Trifonov, D.A.: On the time-reversal symmetry in pseudo-Hermitian systems. Prog. Theor. Exp. Phys. 113A02, 9 (2014)
17. Heiss, W.D.: Exceptional points of non-Hermitian operators. J. Phys. A: Math. Gen. 37, 2455 (2004)
18. Berry, M.V.: Physics of nonhermitian degeneracies. Czechoslovak J. Phys. 54, 1039 (2004)
19. Ge, L., Stone, A.D.: Parity-time symmetry breaking beyond one dimension: The role of degeneracy. Phys. Rev. X 4, 031011 (2014)
20. Wong, J.: Results on Certain Non-Hermitian Hamiltonians. J. Math. Phys. 8, 2039 (1967)
21. Faisal, F.H.M., Moloney, J.V.: Time-dependent theory of non-hermitian Schrodinger equation: Application to multiphoton-induced ionisation decay of atoms. J. Phys. B 14, 3603 (1981)
22. Sakurai, J.J.: In: Tuan, S. F. (ed.) Modern Quantum Mechanics. Addison-Wesley Publishing Company, p. 269 (1994)
23. Cherbal, O., Trifonov, D.A.: Extended PT-and CPT-symmetric representations of fermionic algebras. Phys. Rev. A 85, 052123 (2012)
24. Mostafazadeh, A.: Krein-space formulation of PT symmetry, CPT-inner products, and pseudo-Hermiticity. Czech J. Phys. 56, 919 (2006)
25. Azizov, T.Y., Iokhvidov, I.S.: Linear Operators in Spaces with Indefinite Metric. Wiley, Chichester (1989)
26. Mostafazadeh, A.: Is pseudo-Hermitian quantum mechanics an indefinite-metric quantum theory? Czech J. Phys. 53, 1079 (2003)
27. Sudarshan, E.C.G.: Quantum mechanical systems with indefinite metric. I. Phys. Rev. 123, 2183 (1961)
28. Pauli, W.: On Dirac’s new method of field quantization. Rev. Mod. Phys. 15, 175 (1943)
29. Lee, T.D., Wick, G.C.: Negative metric and the unitarity of the S-matrix. Nucl. Phys. B 9, 209 (1969)
30. Mostafazadeh, A.: Exact PT-symmetry is equivalent to Hermiticity. J. Phys. A 36, 7081 (2003)
31. Bender, C.M., Klevansky, S.P.: PT-symmetric representations of fermionic algebras. Phys. Rev. A 84, 024102 (2011)