Closed-form expansions for the bivariate chromatic polynomial of paths and cycles

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Abstract. We establish closed-form expansions for the number of colorings of a path or cycle on \( n \) vertices with colors from the set \( \{1, \ldots, x\} \) such that adjacent vertices are colored differently or with colors from \( \{y+1, \ldots, x\} \).

Keywords. graph, chromatic polynomial, coloring, path, cycle, recurrence, closed-form

1 Introduction

Let \( G = (V, E) \) be a finite, simple graph having vertex-set \( V \) and edge-set \( E \). For any \( x \in \mathbb{N} \) and \( y = 0, \ldots, x \) we use \( P(G, x, y) \) to denote the number of vertex-colorings \( f : V \to \{1, \ldots, x\} \) such that for any edge \( \{v, w\} \in E \), either \( f(v) \neq f(w) \) or \( f(v) = f(w) > y \). This function has been introduced in [2] and is known to be a polynomial in the indeterminates \( x \) and \( y \), which is now referred to as the bivariate chromatic polynomial [1]. This polynomial generalizes the chromatic polynomial (in the particular case where \( x = y \)), the independence polynomial, and the matching polynomial.

Let \( P_n \) denote the path resp. cycle on \( n \) vertices. By considering the lattice of forbidden colorings, it is proved in [2] that

\[
P(P_n, x, y) = \sum_{0 \leq i + 2j \leq n} (-1)^{n-i-j} \binom{i+j}{i} \binom{n-i-j-1}{n-i-2j} x^i y^j,
\]

(1)

\[
P(C_n, x, y) = (-1)^n y + \sum_{0 \leq i + 2j \leq n} \frac{(-1)^{n-i-j}}{i+j} \binom{i+j}{i} \binom{n-i-j-1}{n-i-2j} x^i y^j.
\]

(2)

In this short note, we give closed-form expansions for the sums in (1) and (2). Until now, such a closed-form expansion is only known for stars [2].
2 The path

Our first result generalizes the fact that the chromatic polynomial of any path on \( n \) vertices is \( x(x - 1)^{n-1} \).

**Theorem 1.** For any path \( P_n \) on \( n \) vertices, any \( x \in \mathbb{N} \) and \( y = 0, \ldots, x \), except for \( x = y = 1 \), we have

\[
P(P_n, x, y) = \sqrt{(x + 1)^2 - 4y - x - 1} \cdot \left( \frac{x - 1 - \sqrt{(x + 1)^2 - 4y}}{2} \right)^n + \sqrt{(x + 1)^2 - 4y + x + 1} \cdot \left( \frac{x - 1 + \sqrt{(x + 1)^2 - 4y}}{2} \right)^n.
\]

The proof of Theorem 1 is based on a recent decomposition formula for \( P(G, x, y) \) by Averbouch et al. \[1\]. This formula involves three kinds of edge elimination:

- \( G - e \): The graph obtained from \( G \) by removing the edge \( e \).
- \( G/e \): The graph obtained from \( G \) by identifying the end points of \( e \), and then, in the resulting multigraph, replacing each pair of parallel edges by a single edge.
- \( G^\dagger_e \): The graph obtained from \( G \) by removing \( e \) and all incident vertices.

We use \( \bullet \) resp. \( \emptyset \) to denote the simple graph consisting of only one vertex, respectively the empty graph (which has no vertex).

**Lemma 2** \([1]\). For any finite simple graph \( G \) and any edge \( e \) of \( G \), the bivariate chromatic polynomial \( P(G, x, y) \) satisfies the recurrence relation

\[
P(G, x, y) = P(G-e, x, y) - P(G/e, x, y) + (x - y) \cdot P(G^\dagger_e, x, y)
\]

with initial conditions \( P(\bullet, x, y) = x \) and \( P(\emptyset, x, y) = 1 \).

We proceed with our proof of Theorem 1.

**Proof of Theorem 1.** Obviously, the statement holds if \( x = 1 \) and \( y = 0 \). Hence, we may assume that \( x > 1 \). By choosing an end edge of \( G \), Lemma 2 yields the recurrence

\[
P(P_n, x, y) = (x - 1)P(P_{n-1}, x, y) + (x - y)P(P_{n-2}, x, y) \quad (n \geq 3),
\]

with initial conditions

\[
P(P_0, x, y) = 1, \quad P(P_1, x, y) = x.
\]

Since this is a homogeneous linear recurrence of degree two with constant coefficients, its solution is of the form

\[
a_n = c_1 r_1^n + c_2 r_2^n
\]

where

\[
r_{1/2} = \frac{x - 1}{2} \pm \frac{1}{2} \sqrt{(x + 1)^2 - 4y}
\]

2
are the roots of the characteristic equation

\[ r^2 - (x - 1)r - x + y = 0. \]

Note that, since \( x > 1 \) and \( y \leq x \), the discriminant \((x + 1)^2 - 4y\) of the characteristic equation is positive, so there are exactly two different solutions.

From the initial conditions (4) we obtain the following expressions for the coefficients in (5):

\[
\begin{align*}
    c_1 &= \frac{\sqrt{(x + 1)^2 - 4y} - x - 1}{2\sqrt{(x + 1)^2 - 4y}}, \\
    c_2 &= \frac{\sqrt{(x + 1)^2 - 4y} + x + 1}{2\sqrt{(x + 1)^2 - 4y}}.
\end{align*}
\]

Now, by putting the expressions from (6) and (7) into (5) the statement of the theorem is proved. Alternatively, proceed by induction on \( n \), using (3) and (4).

**Remark 3.** For \( x = y \) the formula in Theorem 1 specializes to the chromatic polynomial of \( P_n \), which coincides with the chromatic polynomial of any tree on \( n \) vertices. Note, however, that the formula in Theorem 1 does not extend to trees. As an example, for the star \( S_4 \) on four vertices one easily finds that \( P(S_4, x, y) = 3xy - 3x^2y + x^4 - y \), whereas for the path on four vertices, we have \( P(P_4, x, y) = 2xy - 3x^2y + x^3 - y + y^2 \).

### 3 The cycle

Based on our preceding result on paths, we subsequently generalize the fact that the chromatic polynomial of any cycle on \( n \) vertices is \((x - 1)^n + (-1)^n(x - 1)\).

**Theorem 4.** For any cycle \( C_n \) on \( n \geq 3 \) vertices, any \( x \in \mathbb{N} \) and \( y = 0, \ldots, x \),

\[
P(C_n, x, y) = \left(\frac{x - 1 - \sqrt{(x + 1)^2 - 4y}}{2}\right)^n + \left(\frac{x - 1 + \sqrt{(x + 1)^2 - 4y}}{2}\right)^n + (-1)^n(y - 1).
\]

**Proof.** Since \( P(C_n, 1, 1) = 0 \), we may assume that not both \( x \) and \( y \) are equal to 1. By choosing an edge of \( G \), Lemma 2 yields the recurrence

\[
P(C_n, x, y) + P(C_{n-1}, x, y) = P(P_n, x, y) + (x - y)P(P_{n-2}, x, y) \quad (n \geq 4),
\]

with initial condition

\[
P(C_3, x, y) = x^3 - 3xy + 2y.
\]

Iterating (9) we obtain

\[
P(C_n, x, y) = (-1)^n \left( \sum_{i=4}^{n} (-1)^i \left( P(P_i, x, y) + (x - y)P(P_{i-2}, x, y) \right) - P(C_3, x, y) \right).
\]

(10)
Since not both $x$ and $y$ are equal to 1, we may apply Theorem 1 and write
\[ P(P_i, x, y) + (x - y)P(P_{i-2}, x, y) = cr^i + ds^i \] (11)
where
\[ r = \frac{x - 1 - \sqrt{(x + 1)^2 - 4y}}{2}, \quad c = \frac{\sqrt{(x + 1)^2 - 4y} - x - 1}{2 \sqrt{(x + 1)^2 - 4y}} \left(1 + \frac{x - y}{r^2}\right), \]
\[ s = \frac{x - 1 + \sqrt{(x + 1)^2 - 4y}}{2}, \quad d = \frac{\sqrt{(x + 1)^2 - 4y} + x + 1}{2 \sqrt{(x + 1)^2 - 4y}} \left(1 + \frac{x - y}{s^2}\right). \]

From (11) it follows that
\[
\sum_{i=4}^{n} (-1)^i \left(P(P_i, x, y) + (x - y)P(P_{i-2}, x, y)\right)
\]
\[ = c \left(\sum_{i=0}^{n} (-r)^i + r^3 - r^2 + r - 1\right) + d \left(\sum_{i=0}^{n} (-s)^i + s^3 - s^2 + s - 1\right)
\]
\[ = c \left(\frac{1 - (-r)^{n+1}}{1 - (-r)} + (r - 1)^3\right) + d \left(\frac{1 - (-s)^{n+1}}{1 - (-s)} + (s - 1)^3\right). \]

Putting this into (10) we obtain
\[ P(C_n, x, y) = c \frac{r^{n+1} + (-1)n r^4}{r + 1} + d \frac{s^{n+1} + (-1)n s^4}{s + 1} + (-1)^{n-1}P(C_3, x, y), \] (12)
which Sage [3] simplifies to (8).

References

[1] I. Averbouch, B. Godlin, and J.A. Makowsky, An extension of the bivariate chromatic polynomial, Europ. J. Combin. 31 (2010), 1–17.

[2] K. Dohmen, A. Pönitz, and P. Tittmann, A new two-variable generalization of the chromatic polynomial, Discrete Math. Theoret. Comput. Sci. 6 (2003), 69–90.

[3] W. Stein et al., Sage Mathematics Software (Version 4.7.1), The Sage Development Team, 2011, http://www.sagemath.org.