On dense subspaces in a class of Fréchet function spaces on $\mathbb{R}^n$

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Abstract. When dealing with concrete problems in a function space on $\mathbb{R}^n$, it is sometimes helpful to have a dense subspace consisting of functions of a particular type, adapted to the problem under consideration. We give a theorem that allows one to write down many of such subspaces in commonly occurring Fréchet function spaces. These subspaces are all of the form \( \{ pf_0 \mid p \in \mathcal{P} \} \) where \( f_0 \) is a fixed function and \( \mathcal{P} \) is an algebra of functions. Classical results like the Stone-Weierstrass theorem for polynomials and the completeness of the Hermite functions are related by this theorem.

1. Introduction.

Dense subspaces are an important tool in analysis. There are general theorems guaranteeing the density of subspaces consisting of functions possessing the greatest possible regularity, such as the density of \( C^\infty_c(\mathbb{R}^n) \) in \( L^p(\mathbb{R}^n, dx) \) (\( 1 \leq p < \infty \)) and in the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \). In spite of the significance of these theorems in the general theory, they are sometimes of little help in more concrete situations. As an example, when studying the harmonic oscillator on the real line, one wants to know that the Hermite functions span a dense subspace of \( L_2(\mathbb{R}, dx) \). This is of course a classical result, but it does not follow from the theorem we mentioned; it requires a separate proof. We exhibit a theorem that allows a quick conclusion that some “special” subspaces are dense. To illustrate the idea of the proof, let us sketch it in a particular case.

Let \( \mathcal{S}(\mathbb{R}) \) be the space of rapidly decreasing functions on the real line, endowed with its usual Fréchet topology. Let \( \psi(x) = \exp(-x^2/2) \) and consider the subspace \( L = \{ P\psi \mid P \text{ a polynomial} \} \) of \( \mathcal{S}(\mathbb{R}) \). Then it is known that \( L \) is dense in \( \mathcal{S}(\mathbb{R}) \); this follows e.g. from the results in [S, p. 263]. The results in [loc.cit.] are based on recurrence relations for the Hermite functions; we give a more intuitive proof. To this end, fix \( T \in \mathcal{S}(\mathbb{R})' \) and consider the map \( H_T : \mathbb{C} \mapsto \mathbb{C} \), defined by \( H_T(\lambda) = \langle T, e^{-i\lambda}\psi \rangle \) where \( e^{-i\lambda}(x) = \exp(-i\lambda x) \). Then \( H_T \) is in fact holomorphic and \( (\frac{d}{d\lambda})^k H_T(0) = \langle T, (-ix)^k \psi \rangle \) (\( k = 0, 1, \ldots \)). Thus, if \( T \) vanishes on \( L \), then \( \langle T, e^{-i\lambda}\psi \rangle = 0 \) for all \( \lambda \in \mathbb{C} \), in particular for all \( \lambda \in \mathbb{R} \). Now due to
the completeness of $\mathcal{S}(\mathbb{R})$, the weak integral

$$I_f = \int_{\mathbb{R}} f(\lambda)e^{-i\lambda \psi}d\lambda$$

exists in $\mathcal{S}(\mathbb{R})$ for all $f \in \mathcal{S}(\mathbb{R})$, and is in fact equal to $\mathcal{F}(f)\psi$ (where $\mathcal{F}$ denotes Fourier transform). Hence

$$\langle T, \mathcal{F}(f)\psi \rangle = 0 \quad (\forall f \in \mathcal{S}(\mathbb{R})).$$

But the Fourier transform maps $\mathcal{S}(\mathbb{R})$ onto itself, so

$$\langle T, f\psi \rangle = 0 \quad (\forall f \in \mathcal{S}(\mathbb{R})).$$

Now observe that $\{f\psi \mid f \in \mathcal{S}(\mathbb{R})\}$ is dense in $\mathcal{S}(\mathbb{R})$: it contains $C^\infty_c(\mathbb{R})$ since $\psi$ has no zeros. We conclude that $T = 0$ and, finally, that $L$ is dense in $\mathcal{S}(\mathbb{R})$ by the Hahn-Banach theorem.

The above proof is based on a combination of function theory and Fourier analysis. The application of this combination in density problems has a long history: it goes at least back to Hamburger’s work in 1919 on $L^2((0, \infty), dx)$ ([Hi]). This paper fits into this tradition: it turns out that the combination of function theory and Fourier analysis can be put to good use in a more general context to supply dense subspaces in function spaces on $\mathbb{R}^n$, provided that the topology is defined in a certain way (to be explained in Section 2). We (must) assume that the space is Fréchet, since the existence of vector-valued integrals as in the example above is essential.

The main theorem is Theorem 2.13, stating that the annihilators of certain subspaces are equal. In particular, one of them is dense if and only if the other is. E.g., in the above example it follows from Theorem 2.13 that $\{P\psi \mid P \text{ a polynomial}\}$ and $\{f\psi \mid f \in \mathcal{S}(\mathbb{R})\}$ have the same annihilator, and we happen to know that the latter subspace is dense in $\mathcal{S}(\mathbb{R})$ since it contains $C^\infty_c(\mathbb{R})$. Thus it is the combination of Theorem 2.13 below and “general” density theorems that allows one to conclude that some “special” subspaces are dense as well.

This paper is organized as follows. In Section 2, we start with the observation that the way in which a number of well-known function spaces are topologized can be described in a uniform manner. This being done, we prove the main theorem. Section 3 contains applications in three cases. These cases do not exhaust the possible applications of the method in this paper; they rather serve as an illustration, leaving it to the reader to apply the method to situations of his interest. We conclude in Section 4 with remarks on possible variations of the method and connections with representation theory.

2. Main theorem.

The proof in the Introduction works for a whole class of function spaces on $\mathbb{R}^n$, provided that the topology is defined in a certain manner. After establishing the conventions and
notation, we start with an attempt to formalize the way in which a number of function spaces are topologized, and we give some examples. We then work towards Theorem 2.13 on equality of annihilators.

All topological function spaces under consideration are assumed to be complex. This is essential during the proof (since holomorphic functions are involved), but in applications it is usually an easy matter to derive a result for the real case from the result for the complex case. In order to guarantee the existence of certain vector-valued integrals, we assume that the spaces are Fréchet (which includes local convexity by convention) from the start, although the results up to and including Corollary 2.11 also hold without this assumption.

By convention, Borel measures take finite values on compact sets, which implies that any extension of a Borel measure on an open subset of $\mathbb{R}^n$ is $\sigma$-finite (a technical condition that allows application of Fubini’s theorem).

The argument of functions is usually omitted; (in)equalities involving functions should always be read pointwise almost everywhere.

We write $(\cdot, \cdot)$ for the usual bilinear form on $\mathbb{C}^n \times \mathbb{C}^n$: if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are in $\mathbb{C}^n$, then $(x, y) = \sum_{j=1}^n x_j y_j$. The standard two-norm on $\mathbb{C}^n$ is denoted by $\| \cdot \|$: we have $|(x, y)| \leq \|x\| \|y\|$. The standard multi-index notation is used throughout: if $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $|\alpha| = \sum_{i=1}^n \alpha_i$ and $D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}$. We write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ ($i = 1, \ldots, n$).

For $\epsilon > 0$, let $C^n_\epsilon = \{ \lambda \in \mathbb{C}^n | \| \Im \lambda \| < \epsilon \}$. If $E$ is a topological vector space, then $E'$ denotes its dual. For $S \subset E$ we let $S^\perp = \{ T \in E' | Ts = 0 \ \forall s \in S \}$ be the annihilator of $S$.

The Fourier transform, finally, is denoted by $\mathcal{F}$.

We observe that the topology of many locally convex function spaces on $\mathbb{R}^n$ is defined by seminorms involving only one or more of the following ingredients:

- amount of differentiability,
- amount of integrability with respect to a measure,
- behaviour on compacta, and
- weight functions.

Moreover, many of these spaces are in fact metrizable and complete. The following definition introduces an ad hoc terminology for this kind of spaces and formalizes a common method of topologizing function spaces on $\mathbb{R}^n$.

**Definition 2.1.** A complex Fréchet space $E$ is a common Fréchet function space on $\mathbb{R}^n$ if there is a sextuple $(U, \{U_k\}_{k=1}^\infty, \mu, p, m, \{\nabla_N^\alpha\}_{N=0,1,2,\ldots; |\alpha| \leq m})$ such that:

1. $U \subset \mathbb{R}^n$ is open and non-empty.
2. $U_k \subset \mathbb{R}^n$ is open for all $k$, and $U = \bigcup_{k=1}^\infty U_k$.
3. $\mu$ is the completion of a Borel measure on $U$. Let $M(U)$ denote the vector space of $\mu$-measurable functions on $U$, where we agree to identify two elements if they are equal a.e. ($\mu$).
4. $E$ is a subspace of $M(U)$.
5. $1 \leq p \leq \infty$.
6. $m \in \{0, 1, 2, \ldots \} \cup \{\infty\}$.
7. The $\nabla_N^\alpha$ are linear maps $\nabla_N^\alpha : E \mapsto M(U)$, indexed by a non-negative integer $N$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ subject to the condition $|\alpha| \leq m$, such that:
   - $\nabla_N^0$ is the inclusion of $E$ in $M(U)$;
   - $\| \chi_k \nabla_N^\alpha e \|_p < \infty \quad (\forall e \in E, \forall k, \forall N, \forall \alpha (|\alpha| \leq m))$. Here $\chi_k$ denotes the characteristic function of $U_k$ and the $p$-norm corresponds to the measure $\mu$.
8. The topology on $E$ is defined by seminorms $\{p_{k,\alpha,N}\}_{k,N=0,1,2,\ldots;|\alpha|\leq m}$ on $E$, where $p_{k,\alpha,N}$ is defined by $\| \chi_k \nabla_N^\alpha e \|_p$ ($e \in E$).
9. For all $\alpha (|\alpha| \leq m)$ and $\beta, \gamma \leq \alpha$ there exist constants $c_{\beta,\gamma;\alpha}$ such that
   \[
   \nabla_N^\alpha (ge) = \sum_{\beta + \gamma = \alpha} c_{\beta,\gamma;\alpha} (D^\beta g)(\nabla_N^\gamma e) \quad (\forall N)
   \]
   whenever $g \in C^\infty(U)$ and $e \in E$ are such that $ge \in E$.

The integer $m$ should be thought of as describing that the distributional derivatives up to order $m$ are “regular” (this usually means that the function is $C^m$, but it has a different meaning in the context of Sobolev spaces), the $\nabla_N^\alpha$ can be interpreted as distributional differentiation followed by multiplication by a weight function, the $U_k$ allow incorporation of behaviour on compacta, and $p$ of course expresses integrability (essential boundedness if $p = \infty$). The following examples illustrate this.

**Example 2.2.** Let $\mu$ be the completion of a Borel measure on an open subset $U$ of $\mathbb{R}^n$. Then $L_p(U, \mu) (1 \leq p \leq \infty)$ is a common Fréchet function space: put $U_k = U \ (\forall k)$, $m = 0$, and let $\nabla_N^0$ be the inclusion for all $N$.

**Example 2.3.** Let $U \subset \mathbb{R}^n$ be open. The space $C^m(U) (1 \leq m \leq \infty)$ is canonically embedded in $M(U)$ (here Lebesgue measure is tacitly understood). Choose a sequence $\{U_k\}_{k=1}^\infty$ of open subsets of $U$ such that $U = \bigcup_{k=1}^\infty U_k$, $\overline{U_k}$ is compact for all $k$, and $U_k \subset U_{k+1}$ for all $k$. Put $\nabla_N^\alpha = D^\alpha (|\alpha| \leq m, N \geq 0)$ (note that this a legitimate definition on the embedding of $C^m(U)$, since an equivalence class contains exactly one continuous representative). Let $p = \infty$. Then the usual topology on $C^m(U)$ is obtained, showing that $C^m(U)$ is a common Fréchet function space.

**Example 2.4.** Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all smooth functions of rapid decrease at infinity, canonically embedded in $M(\mathbb{R}^n)$ (where Lebesgue measure is again understood). Let $U_k = \mathbb{R}^n \ (\forall k)$, and put $(\nabla_N^\alpha f)(x) = (1 + \|x\|^N)D^\alpha f(x)$ ($x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n)$). Let $p = m = \infty$. Thus $\mathcal{S}(\mathbb{R}^n)$ is a common Fréchet function space.

The reader will have no trouble verifying that other spaces (e.g., $L_p^{loc}(U)$ and the Sobolev spaces $W^{m,p}(U)$) are common in the sense of the above definition.

We now embark on the proof of the main result, Theorem 2.13. The proof consists of two steps, as in the Introduction. The first step consists of showing that certain $E$-valued
maps are holomorphic, and in the second one an $E$-valued integral is identified. The two results together then easily yield the theorem.

As it turns out, the polynomials in the example in the Introduction can be replaced by polynomial functions of the components of an arbitrary smooth map $\Phi : U \mapsto \mathbb{R}^n$ (a diffeomorphism in the applications in Section 3) without complicating the proof. Since derivatives up to order $m$ also appear in the topology of $E$ (notably in Definition 2.1.9), we are led to the following definition.

**Definition 2.5.** Let $\Phi : U \mapsto \mathbb{R}^n$ be of class $C^\infty$ with components $\Phi_1, \ldots, \Phi_n$, let $\epsilon > 0$ and let $m$ be a non-negative integer. Then:

1. $\mathcal{P}_\Phi$ is the unital algebra of functions on $U$ generated by $\{\Phi_j | 1 \leq j \leq n\}$.
2. $\mathcal{P}_{D\Phi,m}$ is the unital algebra of functions on $U$ generated by $\{D^\alpha \Phi_j | 1 \leq j \leq n, \ |\alpha| \leq m\}$.

In order to avoid repetitions, we make the following assumption for the remainder of this section.

**Assumption 2.6.** We assume that we are given a common complex Fréchet function space $E$ with defining sextuple $(U, \{U_k\}_{k=1}^\infty, \mu, p, m, \{\nabla_N^\alpha\}_{N=0,1,2,\ldots;|\alpha|\leq m})$, a $C^\infty$-map $\Phi : U \mapsto \mathbb{R}^n$, $e_0 \in E$, and $\epsilon > 0$ such that:

1. $e^{(j\lambda,\Phi)} e_0 \in E$ ($\forall \lambda \in \mathcal{C}_\epsilon^n$, $\forall g \in \mathcal{P}_\Phi$).
2. $\|\chi_k e^{\epsilon\lambda,\Phi} \|_{g\nabla_N^\alpha e_0} < \infty$ ($\forall k, \forall \alpha (|\alpha| \leq m)$, $\forall N$, $\forall g \in \mathcal{P}_{D\Phi,m}$).

To obtain the example in the Introduction, one takes $E = \mathcal{S}(\mathbb{R}^n)$, $\Phi$ the identity, $e_0(x) = \exp(-x^2/2)$ and any $\epsilon > 0$.

For $p \in \mathcal{P}_\Phi$, define the map $\mathcal{H}_p : \mathcal{C}_\epsilon^n \mapsto E$ by:

$$\mathcal{H}_p(\lambda) = e^{-i(\lambda,\Phi)} p e_0 \quad (\lambda \in \mathcal{C}_\epsilon^n).$$

We start by proving that $\mathcal{H}_1$ is weakly holomorphic and identifying its derivatives (Proposition 2.10).

**Lemma 2.7.** $\mathcal{H}_p$ is continuous on $\mathcal{C}_\epsilon^n$ for all $p \in \mathcal{P}_\Phi$.

**Proof.** Fix $\lambda^0 \in \mathcal{C}_\epsilon^n$. We are to prove that

$$\lim_{h \to 0} p_{k,N,\alpha} \left( e^{-i(\lambda^0+h,\Phi)} p e_0 - e^{-i(\lambda^0,\Phi)} p e_0 \right) = 0 \quad (\forall k, \forall N, \forall \alpha (|\alpha| \leq m)).$$

Fix $k, N$ and $\alpha$. We have

$$p_{k,N,\alpha} \left( e^{-i(\lambda^0+h,\Phi)} p e_0 - e^{-i(\lambda^0,\Phi)} p e_0 \right) =$$

$$= \left\| \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} \chi_k D^\beta \left( e^{-i(\lambda^0,\Phi)} \left( e^{-i(h,\Phi)} - 1 \right) p \right) \nabla_N^\gamma e_0 \right\|_p.$$
The summation can be written as a finite sum of terms of two types (corresponding to zero and one or more differentiations acting on the expression in square brackets, respectively):

\[ \chi_k e^{-i(\lambda^0, \Phi)} \left( e^{-i(h, \Phi)} - 1 \right) g \nabla_N^\delta e_0 \]  
(type 1)

(where \( g \in P_{D\Phi, m} \) does not depend on \( h \), and \(|\delta| \leq m \)), and

\[ \chi_k Q_{g, \delta}(h) e^{-i(\lambda^0, \Phi)} e^{-i(h, \Phi)} g \nabla_N^\delta e_0 \]  
(type 2)

(where again \( g \in P_{D\Phi, m} \) does not depend on \( h \), \(|\delta| \leq m \), and \( Q_{g, \delta} \) is a polynomial on \( \mathbb{C}^n \) such that \( Q_{g, \delta}(0) = 0 \)). To estimate terms of the first type, note that

\[ e^{-iz} - 1 = -\int_0^1 iz e^{-itz} dt \quad (z \in \mathbb{C}), \]

hence

\[ |e^{-i(h, \Phi)} - 1| \leq \|h\| \|\Phi\| e^{\|\text{Im}\,h\| \|\Phi\|}. \]

This readily implies that terms of the first type are dominated by

\[ \|h\| \left| \chi_k e^{\|\Phi\|} \|\Phi\| g \nabla_N^\delta e_0 \right| \leq \|h\| \sum_{j=1}^n \left| \chi_k e^{\|\Phi\|} \Phi_j g \nabla_N^\delta e_0 \right| \]

if \( \|\text{Im}\,h\| + \|\text{Im}\,\lambda^0\| \leq \epsilon \).

Under this same condition for \( h \), terms of the second type are dominated by

\[ |Q_{g, \delta}(h)| \left| \chi_k e^{\|\Phi\|} g \nabla_N^\delta e_0 \right|. \]

Now note that an inequality \( |f| \leq \sum_{j=1}^N |f_j| \) implies that \( \|f\|_p \leq \sum_{j=1}^N \|f_j\|_p \). Hence the lemma follows from Assumption 2.6.2 and the observation that each of the majorants contains a term that tends to zero as \( h \) tends to zero.

\[ \square \]

We need the following lemma to establish the existence of \( E \)-valued integrals later on. The easy proof is left to the reader.

**Lemma 2.8.** For all \( k, N \) and \( \alpha \) (\(|\alpha| \leq m\)) there exists a polynomial \( Q_{k,N,\alpha} \) in one variable such that

\[ p_{k,N,\alpha}(H_1(\lambda)) \leq Q_{k,N,\alpha}(\|\lambda\|) \quad (\forall \lambda \in \mathbb{C}^n_\epsilon). \]

**Lemma 2.9.** For all \( p \in P_{\Phi}, H_p \) is holomorphic on \( \mathbb{C}^n_\epsilon \) in each variable separately and

\[ \frac{d}{d\lambda_l} H_p = H_{-i\Phi,p} \quad (l = 1, \ldots, n). \]
Proof. Let us prove the lemma for the first variable. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$; fix $\lambda^0 \in \mathbb{C}_c^n$, $k, N$ and $\alpha$ and consider for $h \in \mathbb{C}$ ($h \neq 0$):

$$p_{k,N,\alpha} \left( \frac{e^{-i(\lambda^0 + he_1, \Phi)} p e_0 - e^{-i(\lambda^0, \Phi)} p e_0 + ie^{-i(\lambda^0, \Phi)} \Phi p e_0}{h} \right) =$$

$$= \left| \sum_{\beta + \gamma = \alpha} c_{\beta,\gamma;\alpha} \chi_k D^\beta \left\{ e^{-i(\lambda^0, \Phi)} \left[ \frac{e^{-i\Phi_1} - 1}{h} + i\Phi_1 \right] p \right\} \nabla^\gamma e_0 \right|_p.$$

Now a moment’s thought shows that the summation can be written as a finite sum of terms of three types (corresponding to zero, one, and two or more differentiations acting on the expression in square brackets, respectively):

$$\chi_k e^{-i(\lambda^0, \Phi)} \left\{ \frac{e^{-i\Phi_1} - 1}{h} + i\Phi_1 \right\} g \nabla^\delta e_0 \quad \text{(type 1)}$$

(where $g \in \mathcal{P}_{\Phi,m}$ does not depend on $h$, and $|\delta| \leq m$),

$$\chi_k e^{-i(\lambda^0, \Phi)} (e^{-i\Phi_1} - 1) g \nabla^\delta e_0 \quad \text{(type 2)}$$

(where again $g \in \mathcal{P}_{\Phi,m}$ does not depend on $h$, and $|\delta| \leq m$), and

$$\chi_k Q_{g,\delta}(h) e^{-i(\lambda^0, \Phi)} e^{-i\Phi_1} g \nabla^\delta e_0 \quad \text{(type 3)}$$

(where $g \in \mathcal{P}_{\Phi,m}$ does not depend on $h$, $|\delta| \leq m$, and $Q_{g,\delta}$ is a polynomial on $\mathbb{C}$ such that $Q_{g,\delta}(0) = 0$).

Now note that

$$e^{-iz} - 1 + iz = -\int_0^1 z^2 t e^{i(t-1)z} \, dt \quad (z \in \mathbb{C}),$$

hence

$$\left| \frac{e^{-i\Phi_1} - 1}{h} + i\Phi_1 \right| \leq |h| |\Phi_1|^2 e^{||\Phi||} \leq |h| |\Phi_1|^2 e^{||\Phi||}.$$

Thus, if $|\text{Im} \, h| + ||\text{Im} \, \lambda^0|| \leq \epsilon$, then each term of the first type is dominated by

$$|h| \left| \chi_k e^{\epsilon ||\Phi||} \Phi_1^2 g \nabla^\delta e_0 \right|.$$

If one uses (1) again, it is easy to see that under the same condition for $h$ the terms of the second type are dominated by

$$|h| \left| \chi_k e^{\epsilon ||\Phi||} \Phi_1 g \nabla^\delta e_0 \right|.$$

Terms of the third type are dominated by

$$|Q_{g,\delta}(h)| \left| \chi_k e^{\epsilon ||\Phi||} g \nabla^\delta e_0 \right|.$$

Each of the majorants contains a term that tends to zero if $h$ tends to zero, so the lemma follows as in the conclusion of the proof of Lemma 2.7. \qed
Proposition 2.10. Under Assumption 2.6 the map \( T \circ \mathcal{H}_1 : \mathbb{C}_c^n \to \mathbb{C} \) is holomorphic for all \( T \in \mathbb{E}' \), and
\[
D^\alpha (T \circ \mathcal{H}_1) = T \circ \mathcal{H}_{(-i)^{|\alpha|} \Phi_{1}^{\alpha_1} \ldots \Phi_{n}^{\alpha_n}} \quad (\forall \alpha).
\]

Proof. The weak holomorphy of \( \mathcal{H}_1 \) follows from Lemma 2.7 and Lemma 2.9 (or from Lemma 2.9 alone if one invokes Hartog’s theorem). The derivatives are identified by repeated application of Lemma 2.9.

Corollary 2.11. Under Assumption 2.6 we have
\[
(\mathcal{P}_\Phi e_0)^\perp = (\text{Span}\{e^{i(\lambda,\Phi)}e_0 \mid \lambda \in \mathbb{C}_c^n\})^\perp
\]
\[
= (\text{Span}\{e^{i(\lambda,\Phi)}e_0 \mid \lambda \in \mathbb{R}^n\})^\perp.
\]

The completeness of \( \mathbb{E} \) (which has not been used until now) is brought into play in the following lemma.

Lemma 2.12. Under Assumption 2.6 the weak integral
\[
I_f = \int_{\mathbb{R}^n} f(\lambda) \mathcal{H}_1(\lambda) \, d\lambda
\]
exists in \( \mathbb{E} \) for all \( f \in \mathcal{S}(\mathbb{R}^n) \), and is equal to \((\mathcal{F}(f) \circ \Phi) e_0\).

Proof. We recall a basic existence theorem for weak integrals [R1, Theorem 3.27 and the remark preceding the theorem]: if \( \mathbb{E} \) is a Fréchet space, \( X \) is a compact Hausdorff space, \( \Psi : X \to \mathbb{E} \) is continuous and \( \mu \) is a bounded Borel measure \( X \), then the weak integral \( \int_X \Psi \, d\mu \) exists in \( \mathbb{E} \). Lemma 2.8 enables one to invoke this theorem, as follows. Let \( X = \mathbb{R}^n \cup \{\infty\} \) be the one-point compactification of \( \mathbb{R}^n \). As a consequence of Lemma 2.8 and the fact that \( f \in \mathcal{S}(\mathbb{R}^n) \), the map \( \lambda \mapsto (1 + \|\lambda\|)^N f(\lambda) \mathcal{H}_1(\lambda) \) from \( \mathbb{R}^n \) into \( \mathbb{E} \) extends to a continuous map \( \Psi_N : X \to \mathbb{E} \) by putting \( \Psi(\infty) = 0 \), for any \( N \geq 0 \). Choose \( N \) so large that \( \int_{\mathbb{R}^n} (1 + \|\lambda\|)^{-N} \, d\lambda < \infty \). Extend the measure \( (1 + \|\lambda\|)^{-N} \, d\lambda \) from \( \mathbb{R}^n \) to a bounded measure on \( X \) by declaring the measure of \( \{\infty\} \) to be zero. Now apply the existence theorem to \( \Psi_N \).

To identify the integral, fix \( k \) and note that \( \chi_k e \in L_p(\mathcal{U}_k, \mu) \) for all \( e \in \mathbb{E} \), since (in the notation of Definition 2.1) \( p_{k,0,0}(e) = \|\chi_k e\|_p \) is one of the seminorms defining the topology on \( \mathbb{E} \). Hence any \( g \in L_q(\mathcal{U}_k, \mu) \) (where \( q \) is the conjugate exponent of \( p \)) defines an element of \( \mathbb{E}' \). So by the very definition of a weak integral we have
\[
\langle g, I_f \rangle = \int_{\mathbb{R}^n} f(\lambda) \langle g, e^{-i(\lambda,\Phi)} e_0 \rangle \, d\lambda,
\]
where
\[
\langle g, e^{-i(\lambda,\Phi)} e_0 \rangle = \int_{\mathcal{U}_k} g(x) e^{-i(\lambda,\Phi(x))} e_0(x) \, d\mu(x).
\]
Now an application of Fubini's theorem shows that the integrals
\[ \int_{\mathbb{R}^n} f(\lambda) \left\{ \int_{U_k} g(x) e^{-i(\lambda, \Phi(x))} e_0(x) \, d\mu(x) \right\} \, d\lambda \]
and
\[ \int_{U_k} g(x) \left\{ \int_{\mathbb{R}^n} f(\lambda) e^{-i(\lambda, \Phi(x))} \, d\lambda \right\} e_0(x) \, d\mu(x) \]
are equal, i.e.
\[ \langle g, I_f \rangle = \int_{U_k} g(x) (\mathcal{F}(f) \circ \Phi)(x) e_0(x) \, d\mu(x) \quad (\forall g \in L_q(U_k, \mu)) \].

Now recall that an \( L_p \)-space is always canonically embedded in \( L_q' \) if \( 1 \leq p < \infty \), and that this embedding also holds for \( p = \infty \) if the measure is \( \sigma \)-finite ([Z, Lemma \( \beta \), p. 357]). We conclude that \( I_f = (\mathcal{F}(f) \circ \Phi) e_0 \) a.e. (\( \mu \)) on \( U_k \), which proves the lemma since the \( U_k \) cover \( U \).

We finally arrive at the theorem on equality of annihilators that was mentioned in the Introduction.

**Theorem 2.13.** Under Assumption 2.6 the following subspaces of \( E' \) are equal:
1. \( \{(\mathcal{F}(f) \circ \Phi) e_0 \mid f \in S(\mathbb{R}^n)\}^\perp \).
2. \( \{(f \circ \Phi) e_0 \mid f \in S(\mathbb{R}^n)\}^\perp \).
3. \( (\mathcal{P}_\Phi e_0)^\perp \).
4. \( (\text{Span}\{e^{i(\lambda, \Phi)} e_0 \mid \lambda \in \mathbb{C}^n\})^\perp \).
5. \( (\text{Span}\{e^{i(\lambda, \Phi)} e_0 \mid \lambda \in \mathbb{R}^n\})^\perp \).

**Proof.** The equality of 3, 4 and 5 is just Corollary 2.11. Since \( \mathcal{F} : S(\mathbb{R}^n) \mapsto S(\mathbb{R}^n) \) is a bijection, the equality of 1 and 2 follows trivially. The inclusion 5\( \subset 1 \) is a consequence of Lemma 2.12. As to the converse, suppose that \( T \in \{(\mathcal{F}(f) \circ \Phi) e_0 \mid f \in S(\mathbb{R}^n)\}^\perp \). Then Lemma 2.12 shows that
\[ \int_{\mathbb{R}^n} f(\lambda)(T \circ \mathcal{H}_1)(\lambda) \, d\lambda = 0 \quad (\forall f \in S(\mathbb{R}^n)). \]
Since \( T \circ \mathcal{H}_1 \) is continuous, it is identically zero, proving 1\( \subset 5 \).

3. Applications.

In this section, we obtain some density results for the spaces in the Examples 2.2-2.4 by combining Theorem 2.13 and the “standard” density theorems. As we remarked in the Introduction, the applications in this section merely serve to illustrate how this combination can be used to conclude that subspaces of a certain type are dense. We therefore emphasize
that we apply Theorem 2.13 rather crudely by requiring \( \Phi : U \mapsto \Phi(U) \) to be a diffeomorphism. Under this condition, the subspace of pull-backs \( \{ f \circ \Phi \mid f \in \mathcal{S}(\mathbb{R}^n) \} \) (figuring in Theorem 2.13) contains \( \mathcal{C}^\infty_c(U) \), which is usually “large”. This enables us to invoke standard density theorems. In situations where \( \Phi \) is not a diffeomorphism, the reader may still have some use for Theorem 2.13, depending on the particular circumstances in the problem under consideration.

As in the previous section, all topological vector space are assumed to be complex. The reader will have no trouble deriving theorems for the real case from the results for the complex case.

We must distinguish between cases at this stage. The reason for this is more or less obvious: if e.g. \( E = \mathcal{S}(\mathbb{R}^n) \), then the existence of a zero for \( e_0 \) implies that none of the annihilators in the Theorem 2.13 is zero, whereas a statement on the existence of a single zero is generally simply meaningless in the case of \( L^p(U, \mu) \). We therefore elaborate separately on the Examples 2.2-2.4.

For the convenience of the reader, we recall some notation from the previous section: if \( \epsilon > 0 \), then \( \mathcal{C}^\infty_{\epsilon} = \{ \lambda \in \mathbb{C}^n \mid \| \text{Im} \lambda \| < \epsilon \} \), and if \( \Phi : U \mapsto \mathbb{R}^n \) has components \( \Phi_1, \ldots, \Phi_n \) then \( \mathcal{P}_\Phi \) is the unital algebra of functions on \( U \) generated by \( \{ \Phi_j \mid 1 \leq j \leq n \} \).

Theorem 3.1. Let \( \mu \) be the completion of a Borel measure on an open subset \( U \) of \( \mathbb{R}^n \), and let \( 1 \leq p < \infty \). Let \( \Phi : U \mapsto \Phi(U) \) be a diffeomorphism of class \( \mathcal{C}^\infty \). Let \( f_0 \in L_p(U, \mu) \) and suppose that there exists \( \epsilon > 0 \) such that

\[
\left\| e^{\epsilon ||\Phi||} p f_0 \right\|_p < \infty \quad (\forall p \in \mathcal{P}_\Phi).
\]

Then the annihilators of the following subspaces of \( L_p(U, \mu) \):

1. \( \{ (f \circ \Phi) f_0 \mid f \in \mathcal{S}(\mathbb{R}^n) \} \).
2. \( \mathcal{P}_\Phi f_0 \).
3. \( \text{Span}\{ e^{i(\lambda, \Phi)} f_0 \mid \lambda \in \mathcal{C}^\infty_{\epsilon} \} \).
4. \( \text{Span}\{ e^{i(\lambda, \Phi)} f_0 \mid \lambda \in \mathbb{R}^n \} \).

are all equal to \( \{ g \in L_q(U, \mu) \mid gf_0 = 0 \text{ a.e. } (\mu) \} \), where \( q \) is the conjugate exponent of \( p \). In particular, these subspaces are dense in \( L_p(U, \mu) \) if and only if \( f_0(x) \neq 0 \) for almost all \( x \) (\( \mu \)).

Proof. The equality of the annihilators is just an application of Theorem 2.13. If \( gf_0 = 0 \) a.e. (\( \mu \)), then \( g \) is obviously in the annihilator. Conversely, let \( g \in L_q(U, \mu) \) be in the annihilator. The subspace in 1 contains \( \mathcal{C}^\infty_c(U) f_0 \), so in particular

\[
\int_U g h f_0 \, d\mu = 0 \quad (\forall h \in \mathcal{C}^\infty_c(U)).
\]

Since \( \mathcal{C}^\infty_c(U) \) is dense in \( \mathcal{C}_0(U) \) (the continuous functions on \( U \) vanishing at infinity) under the supremum-norm, the dominated convergence theorem shows that

\[
\int_U g f_0 h \, d\mu = 0 \quad (\forall h \in \mathcal{C}_0(U)).
\]
Hence \( \int_U |g f_0| \, d\mu = 0 \) by the Riesz representation theorem ([R3, Theorem 6.19]), i.e. \( g f_0 = 0 \) a.e. \((\mu)\).
The criterium for density is a direct consequence of the description of the annihilator.

The theorem also holds if \( \mu \) is simply a Borel measure, or any measure \( \nu \) that is intermediate (in the sense of domains of definition and extension) between a Borel measure and its completion. Indeed, suppose that \( \mu \) is a Borel measure on \( U \) with the \( \sigma \)-algebra \( \mathcal{B}(U) \) of Borel subsets of \( U \) as domain of definition, and let \( \mu^* \) be its completion with corresponding \( \sigma \)-algebra \( \mathcal{B}^*(U) \). Suppose that \( \Sigma \) is a \( \sigma \)-algebra such that \( \mathcal{B}(U) \subset \Sigma \subset \mathcal{B}^*(U) \), and let \( \nu \) be a measure on \( \Sigma \) such that the restriction of \( \nu \) to \( \mathcal{B}(U) \) is equal to \( \mu \). Then it follows from [AB, p. 92] that \( \nu \) is in fact equal to the restriction of \( \mu^* \) to \( \Sigma \). But then [R3, Lemma 1, p. 154] shows that the natural embedding of \( L_p(U, \nu) \) in \( L_p(U, \mu^*) \) is in fact surjective, hence an isometric isomorphism. Thus, since the theorem holds for \( \mu^* \), it also holds for \( \nu \).

To illustrate the theorem, consider the case \( n = 1 \). Take \( f_0(x) = e^{-x^2} \), let \( \mu \) be Lebesgue measure and let \( \Phi \) be the identity. Then Theorem 3.1 asserts that the Hermite functions span a dense subspace in \( L_p(\mathbb{R}, dx) \) \((1 \leq p < \infty)\). If we restrict the Lebesgue measure to \((0, \infty)\) and put \( f_0(x) = e^{\frac{\alpha}{2} x^2} \) \((\alpha > -1)\), then the density of the span of the Laguerre functions in \( L_p((0, \infty), dx) \) is obtained for \( 1 \leq p < \infty \) if \( \alpha \geq 0 \), and for \( 1 \leq p < -2/\alpha \) if \(-1 < \alpha < 0\). In particular, this span is dense in \( L_2((0, \infty), dx) \) for all \( \alpha > -1 \). The theorem also allows one to conclude that more exotic subspaces are dense, e.g. (with \([.]\) denoting the entire function)

\[
\{ P(x\sqrt{1 + x^2}) e^{-\sqrt{x^2 + \cos x^2} \left[ x^2 + 2 \right]} | P \text{ a polynomial} \}
\]
is dense in \( L_p(\mathbb{R}, dx) \) if \( 1 \leq p < \infty \).

For arbitrary \( n \), the polynomials are dense in \( L_p(U, \mu) \) \((1 \leq p < \infty)\) if there exists \( \epsilon > 0 \) such that \( e^{\epsilon \|x\|} \in L_p(U, \mu) \) (take \( f_0 = 1 \)).

**Theorem 3.2.** Let \( U \subset \mathbb{R}^n \) be open and let \( 0 \leq m \leq \infty \). Endow \( C^m(U) \) with its usual topology of uniform convergence of all derivatives of order \( \leq m \) on compact subsets of \( U \). Let \( f_0 \in C^\infty(U) \), and let \( \Phi : U \mapsto \Phi(U) \) be a diffeomorphism of class \( C^\infty \). Then the annihilators of the following subspaces of \( C^m(U) \) are equal:

1. \( \{(f \circ \Phi) f_0 | f \in \mathcal{S}(\mathbb{R}^n)\} \).
2. \( \mathcal{P}_\Phi f_0 \).
3. \( \text{Span}\{e^{i \langle \lambda, \Phi \rangle} f_0 | \lambda \in \mathbb{C}^n \} \).
4. \( \text{Span}\{e^{i \langle \lambda, \Phi \rangle} f_0 | \lambda \in \mathbb{R}^n \} \).

These subspaces are dense in \( C^m(U) \) if and only if \( f_0 \) has no zeros.

**Proof.** The equality is again an application of Theorem 2.13. The condition for density is necessary, since point evaluations are continuous. The sufficiency is immediate if one recalls that \( C^\infty_c(U) \) is dense in \( C^m(U) \) ([T, Theorem 15.3]), showing that the subspace in 1 is dense if \( f_0 \) has no zeros. 

\[\Box\]
If we take $f_0 = 1$ and $\Phi$ the identity, then we obtain the well-known density of the polynomials in $C^m(U)$ [T, Corollary 4, p. 160].

The Stone-Weierstrass theorem for polynomials also follows from the theorem (although this is admittedly not the shortest way to prove it). Indeed, let $K \subset \mathbb{R}^n$ be compact and non-empty. Choose an open neighbourhood $U$ of $K$. If $f \in C_c(K)$, then $f$ has a continuous extension $f^{ext} \in C(U)$ as a consequence of Tietze’s theorem [B, Theorem 10.4, p. 30]. Since the polynomials are dense in $C(U)$, $f^{ext}$ can be approximated uniformly by polynomials on any compact subset of $U$, in particular on $K$.

The same proof as for Theorem 3.2 yields:

**Theorem 3.3.** Let $f_0 \in S(\mathbb{R}^n)$ and let $\Phi : \mathbb{R}^n \mapsto \Phi(\mathbb{R}^n)$ be a diffeomorphism of class $C^\infty$. Suppose that there exists $\epsilon > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \|x\|^N \left| e^{\epsilon \|x\|} gD^\alpha f_0 \right| < \infty,$$

for all $N \geq 0$, all multi-indices $\alpha$ and all functions $g$ that are polynomials in arbitrary derivatives of the components of $\Phi$. Then the annihilators of the following subspaces of $S(\mathbb{R}^n)$ are equal:

1. $\{(f \circ \Phi) f_0 \mid f \in S(\mathbb{R}^n)\}$.
2. $\mathcal{P}_\Phi f_0$.
3. $\text{Span}\{e^{i\lambda \cdot \Phi} f_0 \mid \lambda \in C^n_\epsilon\}$.
4. $\text{Span}\{e^{i\lambda \cdot \Phi} f_0 \mid \lambda \in \mathbb{R}^n\}$.

These subspaces are dense in $S(\mathbb{R}^n)$ if and only if $f_0$ has no zeros.

The density of the span of the Hermite functions, as “proved” in the Introduction, follows from the application of the theorem with $f_0(x) = \exp(-x^2/2)$ and $\Phi$ the identity.

Let $G$ be a locally compact abelian group $G$ and $f_0 \in L_1(G)$. Then it is well known that the translates of $f_0$ span a dense subspace of $L_1(G)$ if and only if the Fourier transform of $f_0$ has no zeros [R2, Theorem 7.2.5.d]. The following corollary has the same flavour; it shows e.g. that the translates of the Gaussian $\exp(-|x|^2)$ span a dense subspace in $S(\mathbb{R}^n)$.

The corollary follows from Theorem 3.3 if one takes $\Phi$ the identity and recalls that the Fourier transform induces a homeomorphism in $S(\mathbb{R}^n)$.

**Corollary 3.4.** Let $f_0 \in S(\mathbb{R}^n)$. Suppose that there exists $\epsilon > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \left| e^{\epsilon \|x\|} D^\alpha f_0 \right| < \infty$$

for all multi-indices $\alpha$. Then the translates of the Fourier transform of $f_0$ span a dense subspace of $S(\mathbb{R}^n)$ if and only if $f_0$ has no zeros.

4. Closing remarks.
The method that we used to exhibit dense subspaces has some flexibility. Let us take a closer look at the structure of the proof, indicating a possible method of proof in cases that are not covered by Theorem 2.13.

Let $E$ be some function space on $\mathbb{R}^n$. Fix $f \in E$, $T \in E'$, and consider the map $H_T : \mathbb{R}^n \to \mathbb{C}$ defined by $H_T(\lambda) = \langle T, e^{-i\langle \lambda, \cdot \rangle} f \rangle$ (we take the identity for $\Phi$ for convenience and assume that $e^{-i\langle \lambda, \cdot \rangle} f \in E$ for all $\lambda \in \mathbb{R}^n$). Note that the domain of $H_T$ is $\mathbb{R}^n$ and not $\mathbb{C}^n$. What one really wants to prove is that $\langle T, pf \rangle = 0$ for all polynomials $p$ is equivalent to $H_T$ being identically zero on $\mathbb{R}^n$ (rather than $\mathbb{C}^n$), since $\mathbb{R}^n$ is the domain of the integral in Lemma 2.12. In our case we proved the (formally obvious) fact that $\langle T, pf \rangle$ is in fact a multiple of a derivative of $H_T$, evaluated at zero — and $H_T$ happened to extend to a holomorphic map on $\mathbb{C}^n$. But there are other theorems that ascertain that a function is identically zero if all derivatives at a point vanish: e.g., if $n = 1$ then one might try to prove that $H_T$ is in a quasi-analytic class (see [R3]). Once this hurdle is taken one can consider the weak integrals as above, try to prove that they exist and (hopefully) conclude that $T = 0$, or at least obtain a useful description of $(\text{Span}\{pf | p \text{ a polynomial}\})^\perp$.

As illustration of another way of concluding that $H_T = 0$, let us prove the following proposition. The proof is a variation on Hamburger’s method in [H]. There is a holomorphic function involved, and it follows immediately from the hypotheses that many of its derivatives vanish in $0$. As suggested above, we use additional information to conclude that the function is in fact equal to zero.

**Proposition 4.1.** Let $1 \leq p < \infty$, and let $N \geq 0$, $l \geq 2$ be integers. Then the span of \{${x^n e^{-x} | n \geq N, \ l \mid n}$\} is dense in $L_p((0, \infty), dx)$.

**Proof.** Let $g \in L_q((0, \infty), dx)$ (where $q$ is the conjugate exponent of $p$) and suppose that $g$ is in the annihilator of $\text{Span}\{x^n e^{-x} | n \geq N, \ l \mid n\}$. Put $\Omega = \{\lambda \in \mathbb{C} | \text{Im} \lambda < \frac{1}{2}\}$, and let

$$H_g(\lambda) = \int_0^{\infty} g(x) e^{-i\lambda x} e^{-x} \, dx \quad (\lambda \in \Omega).$$

Then $H_g$ is holomorphic and bounded on $\Omega$, and $(\frac{d}{dx})^n H_g(0) = 0$ (n $\geq N$, l $\mid n$). Put $\omega = e^{\frac{2\pi i}{l}}$. There exists a polynomial $P(\lambda)$ such that $H_g(\omega^k \lambda) - P(\omega^k \lambda) = H_g(\lambda) - P(\lambda)$ for all $\lambda \in \Omega$ and all integers $k$ such that $\omega^k \lambda \in \Omega$ (take the first $N$ terms of the power series of $H_g$ around 0 for $P$). This implies that $H_g$ can be extended to an entire function. Indeed, let $\lambda \in \mathbb{C}$. Choose $k$ such that $\omega^k \lambda \in \Omega$ (which is possible since $l \geq 2$) and put $H_g^{ext}(\lambda) = H_g(\omega^k \lambda) - P(\omega^k \lambda) + P(\lambda)$, which is well-defined in view of the above. Since $H_g$ is bounded on $\Omega$, $H_g^{ext}$ is apparently entire and of polynomial growth, hence in fact equal to a polynomial. On the other hand, the restriction of $H_g^{ext}$ to $\mathbb{R}$ is the Fourier transform of $ge^{-x} \chi_{(0, \infty)}$, which tends to zero as $\lambda \to \pm \infty$ by the Riemann-Lebesgue lemma. We conclude that $H_g = 0$. But then $g = 0$ almost everywhere on $(0, \infty)$ by the injectivity of the Fourier transform and the fact that $e^{-x}$ has no zeros. \[\Box\]

We end with a reformulation of the results in Section 2 in terms of representation theory.
The local complex Lie group $\mathbb{C}_e^n$ acts on $e_0$ as if the action came from a representation:

$$e^{-i(\lambda_1, \Phi)}(e^{-i(\lambda_2, \Phi)}e_0) = e^{-i(\lambda_1 + \lambda_2, \Phi)}e_0,$$

provided that $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \mathbb{C}_e^n$ (a consequence of Assumption 2.6.1). But this action is in general not the restriction of the obvious candidate for a global action of $\mathbb{C}_e^n$ on $E$ since $e^{-i(\lambda, \Phi)}e$ simply need not be in $E$ for all $e \in E$ and all $\lambda \in \mathbb{C}_e^n$. One might call $e_0$ a local representation vector, where the term “local” has a double meaning: it expresses the fact that $\mathbb{C}_e^n$ is a local Lie group and also the fact that the action of this local Lie group is not necessarily globally defined on $E$. Assumption 2.6 implies more than just this: $e_0$ is in fact a holomorphic vector, and the action of the Lie algebra of $\mathbb{C}_e^n$ on $e_0$ can be identified. Finally, the density parts of the theorems in Section 3 are theorems stating that certain holomorphic local representation vectors are cyclic for the local action of $\mathbb{C}_e^n$.

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