INTRODUCTION

The objective of this paper is to extend the use of the method of column analogy (Cross, 1930, Cross & Morgan 1945, Sözen 2002) to multi-span hinged frames. In a recent paper (Badir & Badir 2012) a column analogy procedure was presented for the analysis of multi-cell structures with fixed columns. The analysis presented herein involves consideration of conditions of rotations at the hinges. Together with the previous published work (Badir & Badir 2012), it constitutes a generalization of the column analogy method for the analysis of multi-cell structures.

METHOD OF ANALYSIS

For a description of the suggested method of analysis, consider the two-span hinged frame of Fig. 1(a) with moments of inertia of the columns: $I$ in exterior columns and $2I$ in interior column. This is a simplifying but not a necessary assumption.

1.2 Division of Multi-Span Frame: Case “0”

In Fig. 1(b) the multi-span hinged frame is divided into two isolated spans $a$ and $b$ with their inertia and loads by slicing column 1-2 into two halves; Case “0”. The moments $M_{k0}$ at the various sections $i$ of each frame including $M_{k0}$ and $M_{k0}'$ at column sections $k = 1$ and 1’ are computed by column analogy as usual. At hinges $k = 2$ and 2’ the rotations $r_{k0}$ and $r_{k0}'$ are also computed. These are simply the reactions at hinges 2 and 2’ of the elastic load $M_{0}/EI$, where $E$ is Young’s Modulus.

2.2 Correction Forces and Couples

Each isolated frame will now deform independently under the action of its external forces. In order to restore continuity of the multi-span frame, it is necessary to add corrections. These corrections are taken as the moments resulting from: two equal and opposite forces $X_1$, and two equal and opposite couples $X_2$, acting at the top of columns 1-2 and 1’-2’ as shown in Fig. 1(c).

2.3 Continuity Restoration

In the multi-span frame of Fig. 1(a), the two halves of the sliced column 1-2 must undergo identical deformations in order that they fit in together forming the original column. This situation will be satisfied only when the rotations at the two hinges and the bending moments in the two halves of the sliced column are identical. These two conditions of continuity may be stated as follows: (1) a condition dealing with moments at the top section $k$ of the column, Eq. (5) in the work of Badir & Badir (2012), namely:
\[ M_k^* = M_{k0}^* + \sum_{n} m_{kn}^* \cdot X_n = 0 \]  

(1)

in which \( M_{k0}^* = M_k + M'_{k0} = M_{k0} + M_{k0} \) and \( m_{kn}^* = m_{kn} + m_{kn} \) is the continuity moment-coefficient of case \( X_n = 1 \), and (2) another condition dealing with rotations at the bottom hinge \( k \) of the column, namely:

\[ r_k^* = r_{k0}^* + \sum_{n} r_{kn}^* \cdot X_n = 0 \]  

(2)

in which \( r_k^* = r_k + r'_k \), \( r_{k0}^* = r_{k0} + r'_k \), and \( r_{kn}^* = r_{kn} + r'_{kn} \) is the continuity rotation-coefficient of case \( X_n = 1 \). The subscripts of the rotation \( r \) have the same meaning as those in moments (Badir & Badir 2012). In general, for every column there are two unknowns and two conditions of continuity, giving as many equations as the number of unknown forces and couples \( X_n \). In the frame of Fig. 1(a) there are two equations and two unknowns \( X_1 \) and \( X_2 \). Solving Eqs. (1) and (2) simultaneously, the corrections \( X_n \) are obtained.

### 2.4 Bending Moment in Multi-Span Frame

In general, the moment at any section of the multi-span frame is determined by superposition as the sum of moment due to external loads in Case “0” and the moment due to correction forces and couples in Cases “n”. The final bending moment in the multi-span frame is given by Eqs. (8) and (9) of the work of Badir & Badir (2012); which are restated here for convenience

\[ M_i = M_{i0} + \sum m_m \cdot X_n \]  

(3)

at sections \( i \) not in columns; and

\[ \overline{M}_k = \overline{M}_{k0} + \sum m_{kn} \cdot X_n \]  

in columns sections \( k \) (4)

In this frame, one unknown correction force \( X_1 \) and one couple \( X_2 \) as shown in Fig. 1(c) are needed for continuity restoration. In Figs. 2(f) and (g) are given the statical moments in frames a and b in Case “1”, i.e. \( X_1 = 1 \). The straining actions and the indeterminate moment are given in the figure. The resulting moment-coefficients and rotation-coefficients are shown in columns (5), (6) and (7) of Table 1. A similar treatment of Case “2”, i.e. \( X_2 = 1 \) is shown in Figs. 2(h) and (i). The corresponding coefficients are entered in columns (8), (9) and (10) of Table 1. In columns (11) to (16) of Table 2 are found the values of \( M_{10}^*, r_{10}^* \) and the continuity moment and rotation-coefficients \( m_{11}, r_{11}^*, m_{12}, r_{12}^* \) readily obtained from Table 1. Now the two conditions of continuity (1) and (2) are easily written with the aid of Table 2 as follows

\[ -119.67992 + 12.524007X_1 - 0.576131X_2 = 0 \]  

(5)

and

\[ \overline{M}_{k0} = M_{k0} - M_{k0} \]

\[ m_{kn} = m_{kn} - m_{kn} \]
\[ -472.3423 + 86.368463X_1 - 1.414632X_2 = 0 \quad (6) \]

Therefore, \( X_1 = 6.5420506 \) and \( X_2 = -65.518481. \)

Finally, Eq. (3) is used to find the final bending moments \( M_i \) with the help of Table 1. The results are
Moments and Elastic Reactions are given in Columns (3) and (4) of Table 1.

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Table 1. Case “0”, and Moment and Rotation Coefficients; Cases “1 and 2”.

| Frame (1) | Section i, k, k’ (2) | Case “0” | Case “1” | Case “2” |
|-----------|----------------------|----------|----------|----------|
|           |                      | $M_{k0}$, $M'_{k0}$ | $r_{k0}$, $r'_{k0}$ | $m_{k1}$, $m'_{k1}$ | $r_{k1}$, $r'_{k1}$ | $m_{k2}$, $m'_{k2}$ | $r_{k2}$, $r'_{k2}$ |
| a         | B, C                 | -26.264287, 14.05762, -66.264287 | -353.54766 | -4.263831, -1.0082292, 5.736169 | 5.736169 | 40.35692 | -0.340649, 0.0534984, 0.659351 | 0         |
|           | 1, 2                | 119.67992, -472.3423 | 12.524007 | 86.358463 | -0.576131 | 1.4146322 |
| b         | 1’ E                 | -53.415637, 23.85186, -53.415639 |
|           | 2’                  | -118.79464 | 6.787838, 0, -2.4485583, -3.212162 | 6.787838 | 46.011543 | 0.764518, 0, -0.0905343, -0.235482 | 0.8147695 |

Table 2. Continuity Moment and Rotation Coefficients; $m_{kn}$ and $r_{kn}$

| Frame (1) | Section i, k, k’ (2) | Case “0” | Case “1” | Case “2” |
|-----------|----------------------|----------|----------|----------|
|           |                      | $M'_{k0}$ | $r'_{k0}$ | $m'_{k1}$ | $r'_{k1}$ | $m'_{k2}$ | $r'_{k2}$ |
| a         | 1                    | -119.67992 | -472.3423 | 12.524007 | 86.358463 | -0.576131 | 1.4146322 |
Table 3. Correction Moment Coefficients and Final Bending Moments

| Frame (1) | Section i, k, k' (2) | $\bar{M}_{k0}$ (17) | $\bar{m}_{k1}$ (18) | $\bar{m}_{k2}$ (19) | $M_i$ (20) | $\bar{M}_k$ (21) |
|-----------|---------------------|-------------------|-------------------|-------------------|----------|----------------|
| a         | B                  | -12.84865         | -1.061669         | -0.105167         | -31.83968| -12.838339     |
|           | C                  |                   |                   |                   | 3.9566   |                |
|           | I                  |                   |                   |                   | -71.937655|                |
|           |                    |                   |                   |                   | -31.83968|                |
| b         | 1'                 |                   |                   |                   | -59.099316|                |
|           | D                  |                   |                   |                   | 13.764937|                |
|           | E                  |                   |                   |                   | -59.001341|                |

given in column (20) of Table 3. Eq. (4) gives the final bending moment at section 1 of column 1-2 with the aid of columns (17), (18) and (19) of Table 3. The moment $\bar{M}_1$ is given in column (21). The final bending moment diagram is drawn in Fig. 3 on the tension side. Results are in excellent agreement with values obtained using classical structural methods.

Figure 3. Bending moment diagram (kN·m) and reactions

4 CONCLUSION
The forgotten method of column analogy used to analyze statically indeterminate single span and closed frames is extended to the analysis of multi-span frames with columns hinged to the ground. The procedure presented here, together with the previously published paper (Badir & Badir 2012), constitute a generalization to Professor Hardy Cross’s method of column analogy (Cross, 1930, Cross & Morgan 1945) commonly applied to “one cell” frames, arches, and curved beams.

5 REFERENCES
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