ON CURVATURES OF HOMOGENEOUS SUB-RIEMANNIAN MANIFOLDS

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ABSTRACT. The author proved in the late 1980s that any homogeneous manifold with an intrinsic metric is isometric to some homogeneous quotient space of a connected Lie group by its compact subgroup with an invariant Finslerian or sub-Finslerian metric. In a case of trivial compact subgroup, invariant Riemannian or sub-Riemannian metrics are singled out from invariant Finslerian or sub-Finslerian metrics by their one-to-one correspondence with special one-parameter Gaussian convolutions semigroups of absolutely continuous probability measures. Any such semigroup is generated by a second order hypoelliptic operator. In connection with this, the author discusses briefly the operator definition of Ricci lower bounds for sub-Riemannian manifolds by Baudoin-Garofalo. Earlier, Agrachev defined a notion of curvature for sub-Riemannian manifolds. As an alternative, the author discusses in some detail the old definitions of the curvature tensors for rigged metrized distributions on manifolds given by Schouten, Wagner, and Solov’ev. To calculate the Solov’ev sectional and Ricci curvatures for homogeneous sub-Riemannian manifolds, the author suggests to use in some cases special riggings of invariant completely nonholonomic distributions on manifolds. As a justification, we find a foliation on the cotangent bundle $T^*G$ over a Lie group $G$ whose leaves are tangent to invariant Hamiltonian vector fields for the Pontryagin-Hamilton function. This function was applied in the Pontryagin maximum principle for the time-optimal problem. The foliation is entirely described by the co-adjoint representation of the Lie group $G$. Also we use the canonical symplectic form on $T^*G$ and its values for the above mentioned invariant Hamiltonian vector fields. In particular, the above rigging method is applicable to contact sub-Riemannian manifolds, sub-Riemannian Carnot groups, and homogeneous sub-Riemannian manifolds possessing a submetry onto a Riemannian manifold. At the end, some examples are presented.

Key words and phrases: co-adjoint representation, contact form, cotangent bundle, Hamiltonian vector field, homogeneous sub-Riemannian manifold, left-invariant sub-Riemannian metric, Lie algebra, Lie group, Pontryagin-Hamilton function, submetry, sub-Riemannian curvature, symplectic form.

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INTRODUCTION

With the help of the results by Iwasawa-Gleason-Yamabe on the structure of connected locally compact topological groups, the author proved in the late 1980s that every locally compact homogeneous space with intrinsic metric is a projective limit of a sequence of homogeneous manifolds with an intrinsic metric [1], [2]. In turn, any homogeneous manifold with an intrinsic metric is isometric to some homogeneous quotient space $G/H$ of a connected Lie group $G$ by its compact subgroup $H$ with $G$-invariant Finslerian or sub-Finslerian metric $d$ [3], [4]. The metric $d$ is defined by some $G$-invariant complete nonholonomic (vector) distribution $D$ on $G/H$ and norm $F$ on $D$. In the Finsler case $D = T(G/H)$. The distance $d(x, y)$ between any points $x, y \in G/H$ is equal to the infimum of lengths of piece-wise smooth paths tangent to $D$ and joining these points. By definition, the length of any such path $\gamma = \gamma(t)$, $a \leq t \leq b$ is equal to the integral $\int_a^b F(\gamma(t))dt$.

If $F$ is equal to the square root of scalar square with respect to the inner product $\langle \cdot, \cdot \rangle$, then $d$ is a Riemannian or sub-Riemannian metric. For any $G$-invariant (sub-)Finslerian (respectively, (sub-)Riemannian) metric $d$ on $G/H$ there exists a $G$-left-invariant and $H$-right-invariant (sub-)Finslerian (respectively, (sub-)Riemannian) metric $\delta$ on $G$ such that the canonical projection $p : (G, \delta) \to (G/H, d)$ is a submetry [5]. In the Riemannian case this submetry is a Riemannian submersion.

In [3] the case $H = \{e\}$ is considered; then $d$ is a left-invariant Finslerian or sub-Finslerian (or more special Riemannian or sub-Riemannian) metric on the Lie group $G$. In this case the smallest Lie algebra, containing the vector subspace $D(e)$ of the Lie algebra $\mathfrak{g} = (T_eG, [\cdot, \cdot])$ of the Lie group $G$, coincides with $\mathfrak{g}$, and $D$ is a left-invariant vector subbundle of the tangent bundle $TG$. As a consequence of the left-invariance of $D$ and norm $F$, it is enough to assign $D(e)$ and a value $F$ on $D(e)$.

These results together with the Pontryagin maximum principle for the corresponding left-invariant time-optimal problems [6], [1] on Lie groups permit in many cases to find (locally) shortest arcs of homogeneous intrinsic metrics on manifolds.

It is difficult to study general homogeneous sub-Finslerian manifolds and there are a few works on them. One can mention papers by Berestovskii [7] and G.A. Noskov [8]; there are found geodesics, i.e. locally shortest (curves), and shortest arcs of arbitrary left-invariant sub-Finslerian metrics on three-dimensional Heisenberg group.

Recently, A.A. Agrachev defined a curvature of sub-Riemannian manifolds [9]. For this he applied a thorough, natural, justified, and universal approach. However, in the general case, at least now, there is no available formula to calculate this curvature by the Agrachev method. It is possible to do this for contact sub-Riemannian manifolds [10], [11].

On the other hand, more than thirty years ago, my former colleague at Omsk State University A.F. Solov’ev defined and suggested how to calculate easily the sectional and Ricci curvatures of any rigged and metrized distribution $(D, \langle \cdot, \cdot \rangle)$ in manifolds.

In order to apply the Solov’ev method to the case of homogeneous sub-Riemannian manifolds, it is necessary to solve only one (generally difficult) problem, namely, to find a justified invariant rigging $D^\perp$ of $D$, i.e. a complementary distribution in $G/H$. 
We show that it is possible to apply the Solov’ev method in the following cases:
1) for any smooth contact sub-Riemannian manifold,
2) for any three-dimensional Lie group with left-invariant sub-Riemannian metric,
3) when there is a submetry from \((G/H, d)\) onto a Riemannian manifold,
4) for sub-Riemannian Carnot groups,
5) for sub-Riemannian \((G, d)\) when there is a unique rigging \(D^\perp\) of \(D\) such that \(D^\perp(e)\) is a Lie subalgebra \(\mathfrak{k}\) of Lie algebra of the Lie group \(G\) and \([\mathfrak{k}, D(e)] \subset D(e)\).

It is possible to show that for any homogeneous sub-Riemannian manifold \((M, d)\) there is a connected Lie group \(G\) with a left-invariant sub-Riemannian metric \(\delta\) such that there is a submetry from \((G, \delta)\) onto \((M, d)\) and, moreover, the problem of calculation of Solov’ev curvatures for \((M, d)\) is fully reduced to the case of \((G, \delta)\).

To justify the cases of the sub-Riemannian Lie group \(G\), we shall find a special foliation on the cotangent bundle \(T^*G\). Its leaves are tangent to invariant Hamiltonian vector fields for the Pontryagin-Hamilton function, applied in the Pontryagin maximum principle for the time-optimal problem. This foliation is transversal to the fibres of the canonical projection from \(T^*G\) onto \(G\). This projection maps any leaf of the foliation onto all \(G\). If a leaf contains covectors \(\xi_0\) and \(\xi_1\) over \(e\) and \(g \in G\) respectively, then \(\xi_0 = \text{Ad}^* g(\xi_1)\). These properties entirely characterize the foliation. Also we use the canonical symplectic structure on \(T^*G\) which is really tightly connected with another well-known canonical symplectic structure on orbits of the co-adjoint representation \(\text{Ad}^*\) of the Lie group \(G\). Notice that these considerations do not depend on a choice of a left-invariant Riemannian or sub-Riemannian metric on \(G\).

In the last chapter of the paper we shall consider some examples. It is interesting that every Hopf bundle presents a particular case of situation 3) above.

The author thinks that applications of the Solov’ev method to sub-Riemannian manifolds deserve attention because there appeared different notions of curvatures for these manifolds. Therefore it is useful to compare these notions and single out the “correct and applicable” one between them.

In connection with this, it is appropriate to give the following extensive quotation from the end of the Introduction to the paper [12] by F. Baudoin and N. Garofalo:
“For general metric measure spaces, a different notion of lower bounds on the Ricci tensor based on the theory of optimal” (Kantorovich-Monge mass) “transportation has been recently proposed independently by Sturm [13], [14] and by Lott-Villani [15] (see also [16]). However, as pointed out by Juillet [17], the remarkable theory developed in those papers does not appear to be suited for sub-Riemannian manifolds. For instance, in that theory the flat Heisenberg group \(\mathbb{H}^1\) has curvature \(-\infty\). . . An analysis shows that, interestingly, our notion of the Ricci tensor, coincides, up to a scaling factor, with” one given in [9], [10], [11].

1. Preliminaries

The following statements are true [18]:
1) A locally compact homogeneous space with an intrinsic metric \((M, d)\) is isometric to a homogeneous Riemannian manifold of sectional curvature \(\leq K\) for a number \(K \in \mathbb{R}\) if and only if \((M, d)\) has Alexandrov curvature \(\leq K\);

2) there exist infinite-dimensional locally compact homogeneous spaces with an intrinsic metric of Alexandrov curvature \(\geq K > -\infty\);

3) a finite-dimensional locally compact homogeneous space with intrinsic metric \((M, d)\) is isometric to a homogeneous Riemannian manifold of sectional curvature \(\geq K\) if and only if \((M, d)\) has Alexandrov curvature \(\geq K\);

4) if a locally compact homogeneous space with intrinsic metric \((M, d)\) has curvature \(\geq K > 0\), then \((M, d)\) is isometric to some homogeneous Riemannian manifold with sectional curvature \(\geq K\).

On the other hand, the author does not know any natural geometric characterization of sub-Riemannian metrics in the class of homogeneous sub-Finslerian metrics. Possibly, there is no such characterization.

Contemporary methods of probability theory, functional analysis, and partial differential equations permit to set a one-to-one correspondence between Riemannian or sub-Riemannian metrics on any given Lie group \(G\) and symmetric in the sense of H. Heyer [19] and E. Siebert [20] one-parameter convolution Gaussian semigroups of absolutely continuous (with respect to a left-invariant Haar measure \(\omega\) on the group \(G\)) probability measures \(\mu_t = u_t \omega\) with densities \(u_t, t > 0 [19, 20]\). Moreover, the function \(u : \mathbb{R}_+ \times G \to \mathbb{R}_+, u(t, g) = u_t(g)\), is a smooth (i.e. infinitely differentiable) solution of a linear hypoelliptic parabolic homogeneous partial differential equation \((\partial/\partial t - L)u = 0\) similar to the heat equation [20]. Here \(L = \sum_{i=1}^{m} X_i^2\), where \(X_1, \ldots, X_m\) are left-invariant vector fields on \(G\) generating the Lie algebra \(g\) [19, 20]. Therefore \(L\) is a left-invariant linear hypoelliptic operator in the sense of Hörmander [21]. The operator \(L\) naturally corresponds to the left-invariant (sub-)Riemannian metric \(d\) on \(G\) defined by a distribution \(D\) with orthonormal basis \(X_1, \ldots, X_m\). Conversely, let a left-invariant (sub-)Riemannian metric \(d\) on \(G\) be defined by a distribution \(D\) with orthonormal basis \(X_1, \ldots, X_m\). Then the operator \(L = \sum_{i=1}^{m} X_i^2\) assigns a unique smooth solution \(u = u(t, \cdot)\), \(t > 0\), of the differential equation \((\partial/\partial t - L)u = 0\) such that \(\mu_t = u_t \omega\) is a Gaussian convolution semigroup of absolutely continuous probability measures on \(G\).

One can easily see that \(L^1 = 0\), \(L\) is a symmetric and non-positive operator relative to \(\omega\), i.e. for any \(f, g \in \mathcal{C}_0^\infty(G)\),

\[
\int_G f L g d\omega = \int_G g L f d\omega, \quad \int_G f L f d\omega \leq 0.
\]

2. ON THE OPERATOR DEFINITIONS OF CURVATURES

In the paper [12], F. Baudoin and N. Garofalo introduced a generalized curvature-dimension inequality. We shall apply (only) definitions of this work to the case we are interested in, namely, the Lie group \(G = G^n\) with a left-invariant sub-Riemannian metric \(d\) and a hypoelliptic operator \(L = \sum_{i=1}^{m} X_i^2, 2 \leq m < n\).
They associate with such $L$ the following symmetric differential bilinear form

$$\Gamma(f, g) = \frac{1}{2} \{ L(fg) - fLg - gLf \}.$$ (1)

The expression $\Gamma(f) := \Gamma(f, f) = \sum_{i=1}^{k} (X_i f)^2$ is le carré du champ \cite{12} since

$$d(x, y) = \sup \{|f(x) - f(y)| f \in C^\infty(G), \|\Gamma(f)\|_\infty \leq 1\},$$

where $\|f\|_\infty = \text{ess sup}_{G} |f|$ for a smooth function $f$ on $G$.

In addition, in \cite{12} is given some symmetric bilinear differential form of the first order $\Gamma^Z : C^\infty(G) \times C^\infty(G) \to \mathbb{R}$, such that

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h), \quad \Gamma^Z(f) := \Gamma^Z(f, f) \geq 0, \quad \Gamma^Z(1) = 0.$$ (2)

In the case of a Lie group, it is natural to define it in the following manner. Let us assume that there are chosen a left-invariant rigging of distribution $D$, i.e. a left-invariant distribution $D^\perp$ on $G$, complementary to $D$ such that $D \oplus D^\perp = TG$, a left-invariant scalar product $(\cdot, \cdot)$ on $TG$ such that $(\cdot, \cdot)_D = \langle \cdot, \cdot \rangle$ and $(D, D^\perp) = 0$, and also a left-invariant basis of vector fields $Z_1, \ldots, Z_l$ in $D^\perp$, orthonormal relative to $(\cdot, \cdot)_{D^\perp}$, so that $m + l = n$. Notice that the English term “rigging” was suggested by V. V. Wagner in \cite{22}. We define $\Gamma^Z(f, g) := \sum_{j=1}^{l} Z_j f \cdot Z_j g$. Then in \cite{12} are defined second order differential forms

$$\Gamma_2(f, g) = \frac{1}{2} \{ L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf) \},$$ (3)

$$\Gamma^Z_2(f, g) = \frac{1}{2} \{ L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma(g, Lf) \}.$$ (4)

**Definition 1.** It is said that in $G$ is satisfied a generalized curvature-dimension inequality $\text{CD}(\rho_1, \rho_2, \kappa, r)$ relative to $L$ and $\Gamma^Z$, if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$, and $0 < r \leq \infty$ such that the inequality

$$\Gamma_2(f) + \nu \Gamma^Z_2(f) \geq \frac{1}{r}(Lf)^2 + \left( \rho_1 - \frac{\kappa}{r} \right) \Gamma(f) + \rho_2 \Gamma^Z(f)$$ (5)

is satisfied for all $f \in C^\infty(G)$ and every $\nu > 0$.

It should be pointed out that if in Definition 1 we take $r = n = \dim(M)$, $L = \Delta$, $\Gamma^Z \equiv 0$, $\rho_1 = \rho$, $\kappa = 0$ for a smooth Riemannian manifold $M$, then we obtain the inequality $\text{CD}(\rho, n)$ of Bakry-Emery. Bakry showed (see quotations in \cite{12}) that $\text{Ric}(M) \geq \rho \Leftrightarrow \text{CD}(\rho, n)$. Precisely this equivalence served as the motivation for the work \cite{12} by Baudoin-Garofalo. The parameter $\rho_1$ plays the main role in the inequality (1) since in geometric examples, considered in \cite{12}, it represents the lower bound for the sub-Riemannian generalization Ricci curvature.

The article \cite{12} is based on (4) and the general Hypothesis 1,2,3. Hypothesis 1 is equivalent to completeness of the metric $d$ which is satisfied in our case. Hypothesis 2 is the following commutation relation:

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)) \quad \text{for all} \quad f \in C^\infty(G).$$ (5)

Hypothesis 3 has a technical character. It is enough to say that it is valid for $G$ in consequence of the work \cite{23} E. Siebert.
3. On definitions of curvatures for rigged metrized distributions

In the papers [21] by Schouten and van Kampen and [22] by Wagner the authors introduced and studied curvature tensors of nonholonomic manifolds. It is not easy to read and understand these articles because of the coordinate presentation of notions and results. In the paper [25] by E.M. Gorbatenko there was given a modern coordinate-free presentation of parts of these papers which are interesting for us. We shall follow this presentation in the situation of a homogeneous quotient manifold \( M = G/H \) of a connected Lie group \( G \) by its compact subgroup \( H \) with \( G \)-invariant completely nonholonomic distribution \( D \) and Riemannian metric \((\cdot,\cdot)\) on \( M \).

Notice that we need the metric \((\cdot,\cdot)\) on all \( M \) only in order to define below shortly a rigging \( D^\perp \) of distribution \( D \).

Below \( T, H \) and \( V \) denote \( C^\infty \)-modules of vector fields on \( M \), tangent respectively to distributions \( TM, D \) and \( D^\perp \). Then \( T = H \oplus V \), i.e. any vector field \( X \in T \) is uniquely presented in a view \( X = HX + VX \), where \( HX \in H, VX \in V \), and \((HX, VX) = 0\). Let \( \nabla \) be the Levi-Civita connection of the Riemannian manifold \((M,(\cdot,\cdot))\) and \((\cdot,\cdot) = (\cdot,\cdot)_D\). One can easily check that the formula

\[
\nabla_X Y := H\nabla_X Y, \quad X, Y \in H
\]

defines a metric connection without torsion on \( H \), i.e. for \( X, Y, Z \in H \),

\[
X(Y,Z) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,
\]

\[
T\nabla(X,Y) := \nabla_X Y - \nabla_Y X - H[X,Y] = 0.
\]

Moreover \( \nabla \) depends on \((\cdot,\cdot)\) and the rigging \( D^\perp \) but does not depend on \((\cdot,\cdot)_{D^\perp} \); \( \nabla \) is the unique metric connection without torsion on \( H \) for \((\cdot,\cdot) \) and \( D^\perp \) [20].

The Schouten tensor for nonholonomic manifold \((D, (\cdot,\cdot))\) is an analogue of the curvature tensor for Riemannian manifolds and defined as follows

\[
K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - H[V[X,Y],Z], \quad X, Y, Z \in H.
\]

Wagner wrote in [22]: “The Schouten tensor does not justify his title “the curvature tensor” already because on the ground of his properties one cannot judge on the curvature of nonholonomic manifold, i.e. on the absence of absolute parallelism” (for the connection \( \nabla \)).

Before defining the curvature tensor by Wagner (or Wagner-Schouten as in [25]) one needs to introduce some mappings and corresponding notations.

There exists a strongly increasing sequence of \( C^\infty(M) \)-modules

\[
H_0 := H, H_i \subset H_{i+1} := H_i + [H_i, H_i], \ldots, H_r = T
\]

of vector fields on the \( M \) tangent to the corresponding distributions \( D_0 = D, D_i \subset D_{i+1}, D_r = TM \) on \( M \), \( r \) is the nonholonomy order of distribution \( D \). Using the scalar product \((\cdot,\cdot)\) on \( TM \), we get decompositions

\[
D_{i+1} = D_i \oplus \Theta_i, \quad \Theta_i := D_i^\perp \cap D_{i+1}, \quad TM = \bigoplus_{i=0}^{r-1} \Theta_i \oplus D
\]

and a unique morphism of vector bundles \( \theta_i : D_{i+1}/D_i \to \Theta_i \subset D^{i+1}, i = 0, \ldots, r-1 \), the right inverse to the canonical morphism \( \pi_i : D_{i+1} \to D_{i+1}/D_i \). There is also a
surjective morphism \( \delta_i : \Lambda^2 D_i \to D_{i+1}/D_i \), prescribed by linear combinations of mappings \( X \land Y \to [X,Y] \mod D_i \) for \( X,Y \in H_i \).

Further, following [22] and [25], it is defined canonically a new unique invariant Riemannian metric \( \{ \cdot, \cdot \} \) on \( M \), whose restriction \( \{ \cdot, \cdot \}_{|D} = \{ \cdot, \cdot \} \) and the last decomposition in (5) is orthogonal. For this it is enough to assign \( \{ \cdot, \cdot \} \) by induction on \( i = 0, \ldots, r - 1 \). A scalar product \( g \) on a vector space \( V \) defines non-degenerate linear mapping \( g : V \to V^* \) such that \( g(x,y) = g(x)(y) = g(y)(x) \), \( x,y \in V \), and it is defined by it. It is not difficult to check that \( g \) defines the scalar product

\[
g^\wedge : \Lambda^2 V \to (\Lambda^2 V)^* \mid g^\wedge = \phi \circ \Lambda^2 g,
\]

where \( \phi : \Lambda^2 V^* \to (\Lambda^2 V)^* \) is the canonical isomorphism:

\[
\phi(f \land h)(x \land y) = f(x)h(y) - f(y)h(x), \quad x,y \in V, \quad f,h \in V^*.
\]

More explicitly,

\[
(u_1 \land v_1, u_2 \land v_2) = (u_1, u_2)(v_1, v_2) - (u_1, v_2)(v_1, u_2).
\]

By definition,

\[
\{ \cdot, \cdot \}^{-1}_{|D} = \theta_i \circ \delta_i \circ ((\{ \cdot, \cdot \}_{|D_i})^\wedge)^{-1} \circ (\theta_i \circ \delta_i)^*.
\]

Let us define also a morphism of vector bundles

\[
\mu_i : D_{i+1} \to \Lambda^2 D_i \mid \mu_i = ((\{ \cdot, \cdot \}_{|D_i})^\wedge)^{-1} \circ \delta_i^* \circ \theta_i^* \circ \{ \cdot, \cdot \}_{|D_i} \circ \theta_i \circ \pi_i.
\]

Let us denote by \( \nabla^i \) a metric connection without torsion on \( (D_i, \{ \cdot, \cdot \}_{|D_i}) \) and \( H_i, V_i \) are projections playing the same role for \( D_i, D_i^\perp \), as \( H, V \) for \( D, D^\perp \). Now we are ready to introduce the Wagner covariant derivative.

Let us set \( \tilde{K} = K \) (the Schouten tensor) and define \( \tilde{\circ} : H_1 \times H_0 \to H_0 \) by the condition

\[
\tilde{\circ}_X Y = \nabla^0_{H_1} Y + \tilde{K}(\mu_0(X))(Y) + [V X, Y], \quad X \in H_1, \quad Y \in H_0
\]

and \( \tilde{K} : \Lambda^2 H_1 \to \text{End}(H_0) \) for \( X,Y \in H_1, Z \in H_0 \) by condition

\[
\tilde{K}(X \land Y)(Z) = \tilde{\circ}_X(\tilde{\circ}_Y Z) = \nabla^1_{H_1}[X,Y] Z - \tilde{\circ}_{H_1[X,Y]} Z - H[V[X,Y], Z].
\]

Further, by induction, \( \tilde{\circ} : H_{i+1} \times H_i \to H_i \),

\[
\tilde{\circ}_{X} Y = \nabla^i_{H_{i+1}} Y + \tilde{K}(\mu_i(X))(Y) + [V_i X, Y];
\]

\( \tilde{K} : \Lambda^2 H_{i+1} \to \text{End}(H_i) \);

\[
\tilde{K}(X \land Y)(Z) = \tilde{\circ}_{X} (\tilde{\circ}_{Y} Z) = \nabla^{i+1}_{H_{i+1}} Z - \tilde{\circ}_{H_{i+1}[X,Y]} Z - H_i[V_i[X,Y], Z].
\]
Let us call \( \Box, \ldots, \Box^{r-1} \) intermediate Wagner connections, \( \Box \) the Wagner connection, and the curvature tensor \( \tilde{K} \) as the Wagner curvature tensor of strongly rigged completely nonholonomic distribution \( D \). The distribution \( D \) possesses absolute parallelism with respect to \( \nabla \) if and only if the Wagner curvature tensor of the distribution \( D \) is equal to zero \([22], [25]\).

Solov’ev introduced in the paper \([20]\) the notion of a curvature tensor of distribution on the Riemannian manifold. In particular, he obtained some special properties of the curvature of horizontal distribution of the Riemannian submersion and left-invariant distributions on Lie groups with left-invariant Riemannian metric.

He considers a Riemannian manifold \( M \) with metric tensor \((\cdot, \cdot)\), its Levi-Civita connection \( \nabla \), smooth distribution \( D \), and distribution \( D^\perp \), orthogonal to \( D \) relative to \((\cdot, \cdot)\); \( T, H, V \) are corresponding \( C^\infty(M) \)-modules of smooth vector fields on \( M \), tangent to corresponding distributions \( TM, D, D^\perp; H, V \) are projections from \( T = H \oplus V \) to \( H, V \).

The induced connection of distribution \( D \) is \( \nabla_X Y = H \nabla_X HY + V \nabla_X VY \), and its second fundamental form is the tensor field \( h = \nabla - \nabla \) \([21], [28]\); \( h^+ \) and \( h^- \) are symmetric and skew-symmetric parts of the field \( h \) respectively. It is proved in \([28]\) that the distribution \( D \) on the Riemannian manifold \((M, (\cdot, \cdot)) \) is totally geodesic (respectively involutive) if and only if \( h^+(HX, HY) = 0 \) (respectively \( h^-(HX, HY) = 0 \)) for all \( X, Y \in T \). If \( T \) is the torsion tensor for \( \nabla \) then
\[
(16) \quad T(X, Y) = -V[X, Y] = -2h^-(X, Y), \quad X, Y \in D.
\]

A diffeomorphism \( f : M \to N \) of Riemannian manifolds is called a \( D \)-isometry \([27], [28]\), where \( D \) is some distribution on \( M \), if differential \( df \) preserves lengths of vectors \( v \in D \) and \( df(D) \perp df(D^\perp) \). In \([27]\) it is proved

**Theorem 1.** Every \( D \)-isometry “preserves” the expression of view \((\nabla_X Y, Z) - \frac{1}{2}(X, T(Y, Z)) \) if \( X \in T, Y, Z \in H \).

Therefore in \([26]\) Solov’ev defines on a Riemannian manifold \((M, (\cdot, \cdot)) \) with distribution \( D \) a new linear connection, setting
\[
(17) \quad (C_X HY, Z) = (\nabla_X HY, HZ) - (1/2)(X, T(HY, HZ))
\]
and \( C_X VY \) arbitrary for any \( X, Y, Z \in T \). We shall suppose that \( C_X VY = 0 \). Then \( C_X (V) = 0, C_X (H) \subset H \) and by \((16)\),
\[
C_{HX} HY = \nabla_{HX} HY = H \nabla_{HX} HY, \quad (C_{VX} HY, Z) = -(1/2)(VX, T(HY, HZ)).
\]

By definition, the curvature tensor of the distribution \( D \) is \( K = \tilde{K} \circ H \), where \( \tilde{K} \) is the curvature tensor of the connection \( C \). It is stated in \([26]\) without proof that this curvature tensor is the Schouten curvature tensor if \( D \) is totally geodesic. Let \( \overline{R} \) be the curvature tensor of the connection \( \nabla \). Then
\[
(18) \quad (K(X, Y) Z, W) = (\overline{R}(X, Y) Z, W) - (1/2)(T(X, Y), T(Z, W))
\]
for any $X, Y, Z, W \in \mathbf{H}$ and therefore for any such vector fields
\[ (K(X, Y)Z, W) = (R(X, Y)Z, W) - 2(h^-(X, Y), h^-(Z, W)) + \\
(\hbar(X, W), h(Y, Z)) - (h(Y, W), h(X, Z)), \]
where $R$ is the curvature tensor of the Riemannian manifold $(M, (\cdot, \cdot))$. The equation (19) defines completely the value $K(HX, HY)HZ$ since $D$ is parallel with respect to $C$ and therefore $K(X, Y)HZ \in \mathbf{H}$ for any $X, Y, Z \in \mathbf{T}$. It may be considered as an analogue of the Gauss equation for a submanifold.

On the base of formula (18) or (19) are given (completely analogous to the Riemannian case) definitions of sectional $K_{uv}$ and Ricci $k_w$ curvatures in the direction of two-dimensional subspace $\| u \wedge v \neq 0$ and one-dimensional subspace $\| w \neq 0$ for $u, v, w \in D(p)$ and scalar curvature $s$ at a point $p$. The sectional curvature of a two-dimensional distribution is called its Gaussian curvature. In consequence of the definitions, these curvatures of the distribution $D$ are invariant relative to any $D$-isometry.

The sectional torsion for $0 \neq u \wedge v \subset D(p)$ is defined in [27] by the equality $t_{uv} = \| T(u, v)\|^2/\| u \wedge v \|^2$, where $\| u \wedge v \|^2 = \| u \|^2 \| v \|^2 - (u, v)^2$. Let $U$ be the domain of exponential mapping $\exp_p$ of the connection $\nabla$. The submanifold $E(p) = \exp_p(U \cap D(p))$ is called the osculation geodesic surface [27] of the distribution $D$ at the point $p$. In Theorem 1.3 from [26] is established the following geometric interpretation: $K_{uv} = K^{(1)}_{uv} + (3/4)t_{uv}$, where $K^{(1)}$ is the sectional curvature of the surface $E(p) \subset (M, (\cdot, \cdot))$.

With the help of this interpretation, formula (16), known connection [29] of sectional curvatures in the total space and the base of Riemannian submersion, and the complete geodesic property of horizontal distribution of Riemannian submersion it is established the following (Theorem 2.4 from [26])

**Theorem 2.** Let $\pi : (M, (\cdot, \cdot)) \rightarrow (B, \{\cdot, \cdot\})$ be a Riemannian submersion, $D$ and $D^\perp$ respectively its horizontal and vertical distributions on $M$. Then for any non-collinear vectors $u, v \in D(p)$, $p \in M$, $K_{uv} = K^B_{d\pi(u)d\pi(v)}$, where $K^B$ is the sectional curvature of the Riemannian manifold $(B, \{\cdot, \cdot\})$.

**Remark 1.** Application of this theorem to sub-Riemannian manifolds includes the case of sub-Riemannian manifolds with transverse symmetries considered in [12].

The following theorem transmits the content of Lemma 4.1 in [26].

**Theorem 3.** Let $G$ be a Lie group with left-invariant Riemannian metric $(\cdot, \cdot)$ and distribution $D$; $e_i, i = 1, \ldots, m, m + 1, \ldots, n$ an orthonormal basis of left-invariant vector fields on $(G, (\cdot, \cdot))$, $m = \dim \mathbf{D}$, $c_{ijk} = ([e_i, e_j], e_k)$. Then for $a \neq b$, $1 \leq a, b \leq m$,
\[ K_{e_a e_b} = (1/2) \sum_{i=1}^{n} c_{ab}(c_{bia} + c_{iab}) + \sum_{j=1}^{m} \{ (1/4)(c_{jab} + c_{jba})^2 - (3/4)(c_{ab})^2 - c_{jaa}c_{jbb} \} . \]

The following Proposition 4.7 from [26] is valid:
Theorem 4. The curvature tensor of any left-invariant distribution \(D\) on the Lie group \(G\) with bi-invariant Riemannian metric is equal to
\[
K(X,Y)Z = (1/4)H[X,H[Y,Z]] + (1/4)H[Y,H[Z,X]] + (1/2)H[Z,H[X,Y]] + H[Z,V[X,Y]], \quad X, Y, Z \in \mathcal{H}.
\]

Other papers by Solov'ev on the same subject are [30], [31], [32].

4. Contact and symplectic structures

A smooth differential 1-form \(\omega\) on a smooth manifold \(M = M^{2k+1}\) is called contact if \(\omega \wedge \Lambda^k d\omega \neq 0\) everywhere on \(M\). A manifold with a contact form is called contact [33]. By theorem of G. Darboux, any point of a contact manifold is contained in some neighborhood \(U\) with coordinates \(x^1, \ldots, x^k, x^{k+1}, y_1, \ldots, y_k\) such that in these coordinates \(\omega|_U = \sum_{i=1}^{k} y_i dx^i + dx^{k+1}\) [34].

A contact distribution on a contact manifold \((M, \omega)\) is the null set of its contact form, i.e.
\[
D = \bigcup_{x \in M} D(x), \quad D(x) = \{v \in T_x M | \omega_x(v) = 0\}.
\]

It is clear that \(D\) is a smooth vector hyperdistribution on \(M\).

Theorem 5. A contact distribution on any contact manifold is completely nonholonomic and has a canonical rigging.

Proof. Since the form \(\omega \wedge \Lambda^k d\omega\) is non-degenerate and \(M\) is odd-dimensional, the following statements are valid:
1) for any point \(x \in M\) there exists a unique vector \(w_x \in T_x M\) such that \(\omega_x(w_x) = 1\) and \(d\omega(w_x, v) = 0\) for all \(v \in T_x M\);
2) if \(u \in D(x)\) is a non-zero vector, then there exists \(v \in D(x)\): \(d\omega(u, v) \neq 0\).

A vector field \(W\) on \((M, \omega)\) such that \(W(x) = w_x\), is called a Reeb vector field. The distribution \(D^\perp\) on \(M\), spanned by the Reeb vector field is a canonical rigging of distribution \(D\).

Let \(U, V \in D\) be arbitrary vector fields on \(M\) such that \(U(x) = u\) and \(V(x) = v\) for vectors \(u, v \in D(x)\) from p. 2) above. Then [34]
\[
d\omega(U, V) = U(\omega(V)) - V(\omega(U)) - \omega([U, V]) = -\omega([U, V])
\]
and \([U, V](x) \notin D(x)\), which proves that the distribution \(D\) is completely nonholonomic. \(\square\)

Theorem 6. Any contact distribution is invariant with respect to the local one-parameter transformation group generated by the Reeb vector field.

Proof. This arises from the following inequalities for \(X\) tangent to \(D\)
\[
0 = d\omega(W, X) = W(\omega(X)) - X(\omega(W)) - \omega([W, X]) = \omega([-W, X]) = 0
\]
and the fact that \([-W, X]\) is the Lie derivative of the vector field \(X\) in the direction of the vector field \(W\) [34]. \(\square\)
Notice that a non-zero differential 1-form, proportional to a contact form, is itself a contact form. Therefore the Reeb vector field depends on the contact form.

A smooth closed differential 2-form $\sigma$ on a smooth manifold $M = M^{2k}$ is called \textit{symplectic}, if its $k$-th exterior degree $\Lambda^k \sigma = \sigma \wedge \cdots \wedge \sigma \neq 0$ everywhere on $M$. A smooth manifold with a symplectic form is called \textit{symplectic} \cite{33}. By theorem of Darboux, for any point of any symplectic manifold in some its neighborhood $U$ there exist coordinates $x^1, \ldots, x^k, y_1, \ldots, y_k$ such that $\sigma = \sum_{i=1}^k dy_i \wedge dx^i$ \cite{34}. A non-zero differential 2-form, proportional to a symplectic form, is itself symplectic.

We shall need the \textit{canonical symplectic form} on the cotangent bundle $T^*M$ over an arbitrary smooth manifold $M = M^n$ \cite{35}. Let $p_M : TM \to M$, $p_M^* : T^*M \to M$, and $p_{T^*M} : TT^*M \to T^*M$ be the canonical projections. There exists a unique Liouville form $\alpha$ on $T^*M$ such that

$$\alpha(\Lambda) = p_{T^*M}(\Lambda)(dp_M^*(\Lambda)), \quad \Lambda \in TT^*M.$$ 

Clearly $\alpha = \sum_{i=1}^n \xi_i dx^i$ in natural coordinates $x^i, \xi_i; i = 1, \ldots, n$ on $T^*M$. By definition, $\sigma = da$, i.e. $\sigma = \sum_{i=1}^n d\xi_i \wedge dx^i$ in natural coordinates.

In the case of a homogeneous (sub-)Riemannian manifold $(M^n = G/H, d)$ (and not only in this case), the search problem for shortest arcs and geodesics locally reduces to a time-optimal problem which is formulated as follows in local coordinates $x, \xi$ on $T^*M$ \cite{36}. We are given a smooth mapping $f : X \times E^m \to \mathbb{R}^n$, $2 \leq m < n$, such that $f(x, \cdot)$ is a linear monomorphism for any $x \in X$ and the Pontryagin-Hamilton function

$$H(x, \xi, u) = \sum_{i=1}^n \xi_i f^i(x, u), \quad (x, \xi, u) \in X \times \mathbb{R}^n \times E^m.$$ 

If $x = x(t)$ is a geodesic parametrized by arc length in $(M, d)$ then there exists a continuous function $\xi = \xi(t) \neq 0$ such that for almost all $t$ there exist $u(t)$, the derivatives

$$\dot{\xi}_j(t) = -\frac{\partial H((x, \xi, u)(t))}{\partial x_j} = -\sum_{i=1}^n \xi_i(t) \frac{\partial f^i(x(t), u(t))}{\partial x^j},$$

$$\dot{x}(t) = \frac{\partial H((x, \xi, u)(t))}{\partial \xi} = f(x(t), u(t)), \quad \|u(t)\| = 1,$$

(22)

and the following condition is fulfilled

$$H(x(t), \xi(t), u(t)) = \max\{H(x(t), \xi(t), u)\|u\| \leq 1\} \equiv M_0 \geq 0.$$ 

A geodesic $x = x(t)$ in $(M, d)$ is called \textit{normal} if $M_0 > 0$ and \textit{abnormal} if $M_0 = 0$. It is called \textit{strictly abnormal} if there is no covector function $\xi = \xi(t)$ for which it is normal extremal in $(G, d)$. As was shown in the paper \cite{50} by W. Liu and H. Sussman, geodesics of a left-invariant sub-Riemannian metric $d$ on a Lie group $G$ could be strictly abnormal. We shall consider their example at the end of this paper.
In any case the ODE (22) defines the Hamiltonian system (vector field)
\[
\dot{x} = f(x, u), \quad \dot{\xi}_j = -\sum_{i=1}^{n} \xi_i \frac{\partial f^i(x, u)}{\partial x^j}.
\]
Therefore \( H(x, \xi, u) = \alpha(\phi(x, \xi, u)) \), where \( \phi(x, \xi, u) = (\dot{x}, \dot{\xi}) (x, \xi, u) \).

In the case of a Lie group \( G \) with Lie algebra \( g \) we understand elements of the pair \((\xi, u) \in g^* \times g\) respectively as a left-invariant 1-form and a left-invariant vector field on \( G \). Then the Pontryagin-Hamilton function \( H(\xi, u) = \xi(u) \) is defined on \( g^* \times g \). In [36] we proved the following

**Theorem 7.** For any Lie group \( G \) with Lie algebra \( g \), the Hamiltonian system for the function \( H \) on \( g^* \times g \) takes the form
\[
\dot{g} = dl_g(u), \quad g \in G, \quad u \in g,
\]
\[
\dot{\xi}(w) = \xi([u, w]), \quad u, w \in g.
\]

For a fixed \( u \in g \), in correspondence with ODEs (24) and (25), there is defined a vector field \( U \) on \( T^*M \):
\[
U(g, \xi) = (dl_g(u), \xi([u, \cdot])) = (\alpha(U), \xi([u, \cdot]))(g, \xi).
\]
Let \( V \) be an analogous vector field on \( T^*M \), defined by an element \( v \in g \). Then
\[
\sigma(U, V)((g, \xi)) = d\alpha(U, V)((g, \xi)) = (U(\alpha(V)) - V(\alpha(U)) - \alpha([U, V]))((g, \xi)) = \\
\xi([u, v]) - \xi([v, u]) - \xi([u, v]) = \xi([u, v]).
\]
Thus,
\[
\sigma(U, V)((g, \xi)) = \xi([u, v]).
\]

Below, the differential of any smooth mapping \( f \) of smooth manifolds is denoted by \( df \). Let us define the following mappings for the Lie group \( G \) [37]:
\[
I(g) : G \to G; \quad I(g)(g') = gg'g^{-1},
\]
\[
\text{Ad}_g = dI(g)_e : g = T_eG \to T_eG = g,
\]
\[
\text{ad} = (d\text{Ad})_e, \quad \text{ad} u(v) = [u, v], \quad u, v \in g,
\]
\[
\text{Ad}^* g : g^* \to g^*, \quad \text{Ad}^* g\xi(v) = \xi((\text{Ad} g)^{-1}(v)), \quad v \in (g), \quad \xi \in g^*,
\]
\[
\text{ad}^* u\xi(v) = \xi(-\text{ad} u(v)) = \xi([u, v]), \quad u, v \in g, \quad \xi \in g^*.
\]

**Theorem 8.** Let \( G \) be a Lie group with Lie algebra \( g \) and unit \( e \), \( \xi \in g^* = T^*_eG \) co-vector, \( \text{Ad}^* \xi(g) := \text{Ad}^* g(\xi), g \in G \), the action of co-adjoint representation of the Lie group \( G \) to the co-vector \( \xi \). Then
\[
(d(\text{Ad}^* \xi)(w))(v) = \text{ad}^* u(\text{Ad}^* g_0 \xi(v)),
\]
if
\[
\text{if} \quad (28) \quad u, v \in g, \quad w = dl_{g_0}(u) \in T_{g_0}G, \quad g_0 \in G.
\]
Proof. By Ado theorem on existence of exact matrix representation of any Lie algebra, the third theorem of Lie is valid (Theorem 2.9 in [37]). Then every Lie group is locally isomorphic to a matrix Lie group, possibly, with a strengthened topology (see details in Theorem 1 from [36]). Therefore we can suppose that $G$ is a matrix Lie group. Then $\text{Ad} g(v) = g v g^{-1}, dl_g(u) = gu$ if $u, v \in \mathfrak{g}$ $u, v \in G$.

**Lemma 1.** Let $g = g(t), t \in (a, b)$ be a smooth path in the Lie group $G$. Then
\begin{equation}
(\text{Ad} g(t)^{-1})'(0) = -g(t)^{-1} \cdot g'(t) \cdot g(t)^{-1}.
\end{equation}

**Proof.** Differentiating $g(t) \cdot g(t)^{-1} = e$ by $t$, we get
\[
0 = (g(t) \cdot g(t)^{-1})' = g'(t) \cdot g(t)^{-1} + g(t) \cdot (g(t)^{-1})',
\]
whence immediately follows (29). 

To prove Theorem 8 we choose a smooth path $g = g(t), t \in (-\varepsilon, \varepsilon)$, in the Lie group $G$ such that $g(0) = g_0, g'(0) = w$. Then by Lemma 1
\[
(d(\text{Ad}^* \xi))(w)(v) = (\xi((g(t)^{-1} \cdot v \cdot g(t)))'(0) =
\]
\[
\xi((g(t)^{-1} \cdot v \cdot g(t))'(0)) = \xi((g(t)^{-1})'(0) \cdot v \cdot g_0 + g_0^{-1} \cdot v \cdot g'(0)) =
\]
\[
\xi(-g_0^{-1} g'(0) g_0^{-1} \cdot v \cdot g_0 + g_0^{-1} \cdot v \cdot g'(0)) =
\]
\[
\xi(-u \cdot (g_0^{-1} \cdot v \cdot g_0) + (g_0^{-1} \cdot v \cdot g_0) \cdot u) =
\]
\[
\xi(- \text{ad} u((\text{Ad} g_0)^{-1}(v)) = \text{ad}^* u \xi((\text{Ad} g_0)^{-1}(v)) = \text{ad}^* u (\text{Ad}^* g_0 \xi(v)).
\]

Theorems 7 and 8 immediately imply

**Corollary 1.** For any connected Lie group $G$ and $\xi_0 \in T_e^* G$ the mapping
\[
(\text{Ad}^* (\cdot))^{-1}(\xi_0) : g \in G \rightarrow (\text{Ad}^* g)^{-1} \xi_0 \in T_g^* G
\]
is a unique section of the bundle $p_G^* : T^* G \rightarrow G$, which is a solution of the Hamiltonian system (22), (23) with an initial value $\xi_0$ at $e \in G$.

**Definition 2.** The mapping $(\text{Ad}^* (\cdot))^{-1} : G \rightarrow (\mathfrak{g}^* \rightarrow \mathfrak{g}^*)$ is called the co-adjoint representation of the Lie group $G$; $\text{ad}^* (\cdot) : \mathfrak{g} \rightarrow (\mathfrak{g}^* \rightarrow \mathfrak{g}^*)$ is the co-adjoint representation of the Lie algebra $\mathfrak{g}$. The image of the mapping $(\text{Ad}^* (\cdot))^{-1}(\xi_0)$ is called the orbit of element $\xi_0$ relative to the co-adjoint representation of the Lie group $G$.

Every non-trivial orbit of the co-adjoint representation of the Lie group admits a *canonical symplectic structure* [38]. Let $O(\xi)$ be the orbit of an element $\xi \in \mathfrak{g}^*$ relative to the co-adjoint representation of the Lie group $G$ with Lie algebra $\mathfrak{g}$. Then $T_k O(\xi) = \{ \eta = \text{ad}^* u \xi | u \in \mathfrak{g} \}$. By definition,
\begin{equation}
\sigma(\eta_1, \eta_2) = \xi([u_1, u_2]), \quad \text{if} \quad \eta_1 = \text{ad}^* u_1 \xi, \eta_2 = \text{ad}^* u_2 \xi.
\end{equation}

It is not difficult to check that this definition does not depend on the presentation of the elements $\eta_1, \eta_2$. Obviously, $\sigma$ is skew-symmetric. One can easily check also that it is non-degenerate. It follows from the Jacobi identity in the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ that the differential form $\sigma$ is closed [38].
Theorem 8 implies the invariance of the form (30) relative to $\operatorname{Ad}^* G$. Theorems 7, 8 and Corollary 1 show that the coincidence of right parts in formulae (26) and (30) is not occasional. In particular, one can define the canonical symplectic form on orbits of the co-adjoint representation otherwise with the help of the symplectic form $\sigma$ on $T^* G$; and the exactness of the second form implies that the first one is closed.

Remark 2. Theorems 2 and 3 from Lecture 7.3 in [39] give yet another alternative approach to construct the canonical symplectic structure on co-adjoint orbits, based on the notion of Poisson manifold. A. Weinstein remarked that Theorem 2 was formulated by Lie approximately in 1890. A.A. Kirillov supposed that Lie in no way used this result. F.A. Berezin reopened this theorem in 1968 when he investigated universal enveloping algebras (see quotations in [39]).

5. Rigging of left-invariant distributions on Lie groups

Each left-invariant sub-Riemannian metric on the Lie group $G$ is defined by a left-invariant completely nonholonomic vector distribution $D$ and left-invariant scalar product $(\cdot, \cdot)$ on $D$. The Solov'ev method, for a given rigging $D^\perp$ of distribution $D$, gives the unique curvatures of metrized distribution $(D, (\cdot, \cdot))$.

Let $D$ be a left-invariant completely nonholonomic distribution of dimension $m \geq 2$ and codimension $k \geq 1$ on the Lie group $G^n$. Then there are left-invariant differential 1-forms $\omega_{m+1}, \ldots, \omega_n$ such that $D = \{ v \in TG | \omega_j(v) = 0, j = m+1, \ldots, n \}$. It is clear that the forms $\omega_j$ are not defined uniquely by distribution $D$, but they constitute a basis over $\mathbb{R}$ of the unique vector space $\text{Nul}(D)$ of all left-invariant 1-forms on $G$, which annihilate the distribution $D$. Let us fix such forms $\omega_{m+1}, \ldots, \omega_n$ and some basis $X_1, \ldots, X_m$ of left-invariant vector fields on $G$ tangent to $D$.

Similarly, we can define any rigging $D^\perp$ of the distribution $D$ if we choose some left-invariant differential 1-forms $\omega_1, \ldots, \omega_m$ on $G$, which are linearly independent with $\omega_{m+1}, \ldots, \omega_n$, and set $D^\perp = \{ w \in TG | \omega_i(w) = 0, i = 1, \ldots, m \}$.

The considerations above, especially Relations (21) and (26), prompt three possible cases of naturally assigning left-invariant rigging $D^\perp$ of left-invariant distribution $D$ on the Lie group $G$:

1) The Jacobi identity implies that the set of left-invariant vector fields $X$ on $G$ such that $[X, D] \subset D$, is a Lie algebra. Therefore the corresponding $D^\perp(e)$ is a Lie subalgebra in $\mathfrak{g}$.

2) Since $D$ is a completely nonholonomic distribution, the Jacobi identity implies that the corresponding $D^\perp(e)$ is an ideal in $\mathfrak{g}$. 


Clearly, 3) is a partial case of 2). There are analogues of conditions 1), 2), 3) for homogeneous manifolds $G/H$. It is necessary to note that conditions 1)–3) are very general in two senses: they do not take into account a particular structure of homogeneous manifolds $G/H$ with invariant completely nonholonomic distribution $D$ as well as a sub-Riemannian metric connected with them. Maybe it would be possible to find other natural conditions to choose a rigging of $D$ in partial cases of $G/H$ and invariant sub-Riemannian metrics on $G/H$. For example, one could use some Killing vector fields on homogeneous sub-Riemannian manifolds.

6. Examples

By Theorem 5, any contact distribution has a canonical rigging, so in this case we can apply the Solov'ev definition of curvatures.

We shall show that any completely nonholonomic left-invariant rank two distribution on any three-dimensional Lie group $G$ is contact, so we can apply Theorem 8 to calculate the sectional curvature for any left-invariant sub-Riemannian metric on $G$. This curvature coincides with Ricci and Gaussian curvatures.

**Proposition 1.** A three-dimensional Lie group $G$ admits a left-invariant contact form $\omega$ with a contact distribution $D$: $\omega(D) = 0$ if and only if there exists a completely nonholonomic left-invariant rank two distribution $D$ on $G$ satisfying condition 1) from Section 5; moreover, there exists a non-zero left-invariant 1-form $\omega$ on $G$ such that $\omega(D) = 0$.

**Proof.** Necessity follows from Theorems 5 and 6.

Sufficiency. Suppose that a left-invariant completely nonholonomic rank two distribution $D$ on $G$ together with a unique left-invariant distribution $D^\perp$ satisfy condition 1) from Section 5, i.e. $[D^\perp(e), D(e)] \subset D(e)$. Then there exists a non-zero left-invariant 1-form $\omega$ on $G$ which is unique up to multiplication by a constant such that $\omega(D) = 0$ and a unique left-invariant vector field on $G$ tangent to $D^\perp$ such that $\omega(W) = 1$. Hence for any linearly independent left-invariant vector fields $X, Y$ on $G$, tangent to $D$, similarly to the proofs of Theorems 5 and 6 we get

$$d\omega(X, Y) = -\omega([X, Y]) \neq 0,$$

$$d\omega(W, X) = -\omega(W, X) = 0, \quad d\omega(W, Y) = -\omega(W, Y) = 0.$$

This means that $\omega$ is a contact form on $G$ and $W$ is the Reeb vector field for $\omega$. \(\square\)

**Proposition 2.** 1) There is no left-invariant completely nonholonomic rank two distribution on a three-dimensional Lie group $G$ if and only if $G$ is commutative or its Lie algebra $\mathfrak{g}$ admits a basis $e_1, e_2, e_3$: $[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$, $[e_2, e_3] = 0$.

2) There are four types of mutually non-isomorphic connected commutative Lie groups, and they are unimodular. There exists only one connected Lie group with the Lie algebra of second form; it is simply connected, solvable, non-unimodular and characterized by the property that, supplied by an arbitrary left-invariant Riemannian metric, it is isometric to the Lobachevsky space.

3) Every left-invariant completely nonholonomic rank two distribution on any three-dimensional Lie group is contact.
Proof. 1) The sufficiency in the first statement is clear. The necessity follows from formula (4.2), table on p. 307, and Lemma 4.10 in the paper [40] by Milnor. There are given the Lie brackets for special Milnor bases $e_1, e_2, e_3$ in the Lie algebras and a full classification respectively of unimodular and non-unimodular Lie algebras. There are six types of unimodular Lie algebras and a continuous connected one-parameter family of non-unimodular Lie algebras.

2) The statement about commutative groups is trivial; concerning another statement, see [40].

3) The same formula (4.2), table on p. 307, and Lemma 4.10 in [40], together with Proposition 1, imply that for any other three-dimensional Lie group $G$, the left-invariant distribution $D$ on $G$ with basis $e_1, e_2$ for $D(e)$ is completely nonholonomic and contact with respect to a left-invariant contact 1-form $\omega$ on $G$ with the left-invariant Reeb vector field $W$ such that $W(e) = e_3$. In Lemma 4.10, one needs to take $\alpha = 2, \delta = 0, \beta \neq 0$.

The proof is completed by a remark from the paper [41] by A. Agrachev and D. Barilari. It states that in each of the cases under consideration but one, all left-invariant bracket generating distributions are equivalent by an automorphism of the Lie algebra. The excluded cases are Lie groups $G$ with Lie algebra $\mathfrak{sl}(2)$. Besides the one considered above, the so-called elliptic distribution $D$ for $G$, there is a non-equivalent to it, the so-called hyperbolic distribution $D_h$ for $G$ such that the restriction of the Killing form onto $D_h(e)$ is sign-indefinite. We can take for $D_h(e)$ the basis $e_2, e_3$. Then formula (4.2), table on p. 307 in [40], and Proposition 1 imply that $D_h$ is hyperbolic, bracket generating, and contact with respect to a left-invariant contact 1-form $\omega$ on $G$ with left-invariant Reeb vector field $W$ such that $W(e) = e_1$.

The Reeb vector field $W$ could generate a local one-parameter subgroup of isometries for $(G, d)$ if and only if $G$ is locally isomorphic to the Heisenberg group $H^1$, $SO(3)$ or $SL(2)$. In the last case the corresponding distribution $D$ must be elliptic. Also $W$ will be tangent to a closed one-dimensional subgroup $H \subset G$ acting on the right by isometries in $(G, d)$. Then $G/H$ admits an invariant Riemannian metric $\delta$ such that the canonical projection $p : (G, d) \to (G/H, \delta)$ is a submetry. Therefore by Theorem 2 the Gaussian curvature of $(G, d)$ is equal to the constant Gaussian curvature of $(G/H, \delta)$. These groups with such metric $d$ were studied in [12] — [17]. There the corresponding Gaussian curvatures were equal respectively to 0, 1, −1 what agrees with statements in [12].

Notice that any two sub-Riemannian metrics on $H^1$ give isometric spaces. The corresponding distribution $D$ also satisfies both conditions 2) and 3) from section 5.

Proposition 3. Assume that a left-invariant sub-Riemannian metric $d$ on a Lie group $G$ is defined by the scalar product $\langle \cdot, \cdot \rangle$ on distribution $D$ with the rigging $D^\perp$ satisfying condition 3). Then all curvatures of $(G, d)$ are equal to zero.

Proof. Let $\langle \cdot, \cdot \rangle$ be a left-invariant Riemannian metric on $G$ such that $\langle \cdot, \cdot \rangle|_D = \langle \cdot, \cdot \rangle$, $(D, D^\perp) = 0, \{e_1, \ldots, e_m\}, \{e_{m+1}, \ldots, e_n\}$ an orthonormal bases in $D$ and $D^\perp$. Then
The group $\mathbb{H}^1$ is a partial and the simplest case of the so-called Carnot groups.

**Definition 3.** The Carnot group is a Lie group $G$, supplied by a 1-parameter multiplicative group of automorphisms $(\delta_s, \cdot)$, $s > 0$, such that the vector subspace $V := \{v \in g : d\delta_s(v) = sv\}$ generates $g$, i.e. the least subalgebra in $g$, containing $V$, coincides with $g$. The expression “the Carnot group with a left-invariant sub-Riemannian metric” means that $D(e) = V$.

**Corollary 2.** Any Carnot group $G$ with a left-invariant (sub-)Riemannian metric $d$ defined by left-invariant distribution $D$ and scalar product $(\cdot, \cdot)$ on $D$ (with mentioned rigging $D^\perp$ of distribution $D$ if $G$ is non-commutative) has zero curvatures.

**Proof.** Obviously, the statement is true if $G$ is a commutative Lie group because then $d$ is a Riemannian metric and $(G, d)$ is locally isometric to an Euclidean space. Otherwise, the Lie algebra of the Lie group $G$ is a graded nilpotent Lie algebra $g = \bigoplus_{k=1}^l g_k$ generated by $g_1$, where $l \geq 2$, $D(e) = g_1$. It is clear that $G$ satisfies condition 3) from Section 5 for $D^\perp(e) = \bigoplus_{k=2}^l g_k$, so we can apply Proposition 3. □

**Remark 3.** Corollary 2 is obvious for any Carnot group $G$ with a left-invariant (sub-)Riemannian metric $d$ because the members of one-parameter multiplicative group $(\delta_s, \cdot)$, $s > 0$, from Definition 3 are s-similarities of $(G, d)$. Calculations in the above proof of Proposition 3 demonstrate the correctness of adopted method for Carnot groups. The statement of Corollary 2 is given in [12] only for $G$ of step two.

There are the following Hopf bundles
\[
S^{2n+1} = U(n+1)/U(n) \rightarrow U(n+1)/(U(n) \times U(1)) = \mathbb{C}P^n, \quad n \geq 1,
\]
\[
S^{4n+3} = Sp(n+1)/Sp(n) \rightarrow Sp(n+1)/(Sp(n) \times Sp(1)) = \mathbb{H}P^n, \quad n \geq 1,
\]
\[
S^{4n+3} = Sp(n+1)/Sp(n) \rightarrow Sp(n+1)/(Sp(n) \times U(1)) = \mathbb{C}P^{2n+1}, \quad n \geq 1,
\]
\[
S^{15} = Spin(9)/Spin(7) \rightarrow Spin(9)/Spin(8) = \mathbb{C}aP^1 = S^8,
\]
\[
\mathbb{C}P^{2n+1} = Sp(n+1)/(Sp(n) \times U(1)) \rightarrow Sp(n+1)/(Sp(n) \times Sp(1)) = \mathbb{H}P^n, \quad n \geq 1.
\]

The fibres of these bundles are spheres of respective dimensions 1, 3, 1, 7, and 2.

If we supply all the total spaces (spheres) of the bundles of the first four types by the canonical Riemannian metrics of sectional curvature 1, then there are unique canonical Riemannian symmetric metrics on the bases of these bundles such that the corresponding projections are Riemannian submersions. After that there are unique canonical symmetric Riemannian metrics on the bases of the bundles of the last type such that the corresponding projections are Riemannian submersions.

Many details on these Riemannian submersions can be found in papers [48], [49]. The next to the last case is the most difficult, but at the same time the most interesting case, which involves essentially the Clifford algebras $C\ell^n$ and the Cayley algebra $Ca$ of octonions. The image of the Lie algebra $spin(7) = so(7)$ of the Lie subgroup $Spin(7)$ is not the standard inclusion into $spin(9) = so(9)$, but its image $\tau(so(7))$ under an outer automorphism $\tau$ of Lie algebra $so(8)$, with standard
inclusion $\mathfrak{so}(8) \subset \mathfrak{so}(9)$, the so-called \textit{triality automorphism} of order 3. In reality $\tau$ is induced by a rotation symmetry $s \in S_3$ of the Dynkin diagram $D_4$ (which is a tripod) of the Lie algebra $\mathfrak{so}(8)$.

Then the horizontal distributions $D$ of all these Riemannian submersions are completely nonholonomic in the total spaces of these bundles. We shall get homogeneous sub-Riemannian metrics on the total spaces with distributions $D$ if we supply $D$ by the induced scalar products from the previous Riemannian metrics. After this procedure, not changing the previous symmetric Riemannian metrics on the bases of the bundles, we get submetries from the sub-Riemannian manifolds onto the Riemannian symmetric spaces. In all cases analogues of condition 1) from Section 5 for horizontal and vertical distributions are satisfied. Therefore, by Theorem 2 we can calculate all curvatures of the total homogeneous sub-Riemannian manifolds, using the curvatures of the bases with symmetric Riemannian metrics. In the first three cases there are respective groups $U(1)$, $Sp(1)$, and $U(1)$ of \textit{transverse symmetries} studied in [12]. In the other cases this is impossible because the spheres $S^7$ and $S^2$ admit no structure of a Lie group.

Now we shall consider the Liu-Sussman example from Section 9.5 in [50]. Let $G$ be any four-dimensional Lie group whose Lie algebra $\mathfrak{g}$ has two generators $f$ and $g$ such that (1) $f$, $g$, $[f, g]$ and $[f, [f, g]]$ form a basis in $\mathfrak{g}$; (2) $[g, [f, g]]$ belongs to the linear span of vectors $f$, $g$, and $[f, g]$; (3) $[g, [f, g]]$ does not belong to the linear span of vectors $f$ and $[f, g]$. One can take $G = SO(3) \times \mathbb{R}$ with Lie algebra $\mathfrak{so}(3) \oplus \mathbb{R}$ and

\begin{equation}
    f = k_1 \oplus 1, \quad g = (k_1 + k_2) \oplus 2,
\end{equation}

where $k_1, k_2, k_3$ are generators of the Lie algebra $\mathfrak{so}(3)$ of the Lie group $SO(3)$ such that $[k_1, k_2] = k_3$, $[k_2, k_3] = k_1$, and $[k_3, k_1] = k_2$. One can easily check that

\begin{equation}
    [f, g] = k_3 \oplus 0, \quad [f, [f, g]] = -k_2 \oplus 0, \quad [g, [f, g]] = (k_1 - k_2) \oplus 0 = 2f - g.
\end{equation}

Therefore all conditions (1), (2), (3) are satisfied. The left-invariant sub-Riemannian metric $d$ on $G$ is defined by orthonormal basis $\{f, g\}$ on the vector subspace $D(e) \subset \mathfrak{g}$. By Theorems 5 and 6 in [50], the subgroups $g_1(t) = \exp(tg)$, $g_2(t) = \exp(-tg)$ and their left shifts are only strictly abnormal geodesics in $(G, d)$.

One can easily see from Relations (31) and (32) that \textit{the distribution $D$ does not satisfy any condition 1), 2), or 3) from Section 5.}

A simplest case is when $D^\perp(e)$ has orthonormal basis $k_3 \oplus 0$, $0 \oplus 1$. Then by (31), (32) and the notation of Theorem 3 the only non-zero constants are $c_{231} = -c_{321} = 2$,

\begin{equation}
    c_{123} = -c_{213} = c_{131} = -c_{311} = -c_{132} = c_{312} = -c_{134} = c_{314} = -c_{232} = c_{322} = 1.
\end{equation}

By Theorem 3 all curvatures of $(G, d)$ with this $D^\perp(e)$ are equal to $K_{fg} = 3/2$.

If we change $0 \oplus 1$ by $[f, [f, g]] = -k_2 \oplus 0$, then the only non-zero constants are

\begin{equation}
    c_{231} = -c_{321} = c_{341} = -c_{431} = 2,
\end{equation}

\begin{equation}
    c_{123} = -c_{213} = c_{134} = -c_{314} = -c_{232} = c_{232} = -c_{143} = 0
\end{equation}

\begin{equation}
    c_{413} = -c_{243} = c_{423} = -c_{342} = c_{432} = -c_{344} = c_{434} = 1.
\end{equation}

By Theorem 3 all curvatures of $(G, d)$ with this $D^\perp(e)$ are equal to $K_{fg} = 1$. 


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