Profinite groups with restricted centralizers of \( \pi \)-elements

Cristina Acciarri and Pavel Shumyatsky

Abstract. A group \( G \) is said to have restricted centralizers if for each \( g \) in \( G \) the centralizer \( C_G(g) \) either is finite or has finite index in \( G \). Shalev showed that a profinite group with restricted centralizers is virtually abelian. Given a set of primes \( \pi \), we take interest in profinite groups with restricted centralizers of \( \pi \)-elements. It is shown that such a profinite group has an open subgroup of the form \( P \times Q \), where \( P \) is an abelian pro-\( \pi \) subgroup and \( Q \) is a pro-\( \pi' \) subgroup. This significantly strengthens a result from our earlier paper.

1. Introduction

A group \( G \) is said to have restricted centralizers if for each \( g \) in \( G \) the centralizer \( C_G(g) \) either is finite or has finite index in \( G \). This notion was introduced by Shalev in \cite{13} where he showed that a profinite group with restricted centralizers is virtually abelian. We say that a profinite group has a property virtually if it has an open subgroup with that property. The article \cite{3} handles profinite groups with restricted centralizers of \( w \)-values for a multilinear commutator word \( w \). The theorem proved in \cite{3} says that if \( w \) is a multilinear commutator word and \( G \) is a profinite group in which the centralizer of any \( w \)-value is either finite or open, then the verbal subgroup \( w(G) \) is virtually abelian.

In \cite{1} we study profinite groups in which \( p \)-elements have restricted centralizers, that is, groups in which \( C_G(x) \) is either finite or open for any \( p \)-element \( x \). The following theorem was proved.

2010 Mathematics Subject Classification. 20E18, 20F24.

Key words and phrases. Profinite groups, centralizers, \( \pi \)-elements, FC-groups.

This research was supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and Fundação de Apoio à Pesquisa do Distrito Federal (FAPDF), Brazil.
Theorem 1.1. Let $p$ be a prime and $G$ a profinite group in which the centralizer of each $p$-element is either finite or open. Then $G$ has a normal abelian pro-$p$ subgroup $N$ such that $G/N$ is virtually pro-$p'$. 

The present paper grew out of our desire to determine whether this result can be extended to profinite groups in which the centralizer of each $\pi$-element, where $\pi$ is a fixed set of primes, is either finite or open. As usual, we say that an element $x$ of a profinite group $G$ is a $\pi$-element if the order of the image of $x$ in every finite continuous homomorphic image of $G$ is divisible only by primes in $\pi$ (see [10], Section 2.3] for a formal definition of the order of a profinite group).

It turned out that the techniques used in the proof of Theorem 1.1 were not quite adequate for handling the case of $\pi$-elements. The basic difficulty stems from the fact that (pro)finite groups in general do not possess Hall $\pi$-subgroups.

In the present paper we develop some new techniques and establish the following theorem about finite groups.

If $\pi$ is a set of primes and $G$ a finite group, write $O^{\pi'}(G)$ for the unique smallest normal subgroup $M$ of $G$ such that $G/M$ is a $\pi'$-group. The conjugacy class containing an element $g \in G$ is denoted by $g^G$.

Theorem 1.2. Let $n$ be a positive integer, $\pi$ be a set of primes, and $G$ a finite group such that $|g^G| \leq n$ for each $\pi$-element $g \in G$. Let $H = O^{\pi'}(G)$. Then $G$ has a normal subgroup $N$ such that

1. The index $[G : N]$ is $n$-bounded;
2. $[H : N] = [H, H]$;
3. The order of $[H, N]$ is $n$-bounded.

Throughout the article we use the expression “$(a, b, \ldots)$-bounded” to mean that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, \ldots$.

The proof of Theorem 1.2 uses some new results related to Neumann’s BFC-theorem [8]. In particular, an important role in the proof is played by a recent probabilistic result from [2]. Theorem 1.2 provides a highly effective tool for handling profinite groups with restricted centralizers of $\pi$-elements. Surprisingly, the obtained result is much stronger than Theorem 1.1 even in the case where $\pi$ consists of a single prime.

Theorem 1.3. Let $\pi$ be a set of primes and $G$ a profinite group in which the centralizer of each $\pi$-element is either finite or open. Then $G$ has an open subgroup of the form $P \times Q$, where $P$ is an abelian pro-$\pi$ subgroup and $Q$ is a pro-$\pi'$ subgroup.
Thus, the improvement over Theorem 1.1 is twofold – the result now covers the case of $\pi$-elements and provides additional details clarifying the structure of groups in question. Furthermore, it is easy to see that Theorem 1.3 extends Shalev’s result [13] which can be recovered by considering the case where $\pi = \pi(G)$ is the set of all prime divisors of the order of $G$.

We now have several results showing that if the elements of a certain subset $X$ of a profinite group $G$ have restricted centralizers, then the structure of $G$ is very special. This suggests the general line of research whose aim would be to determine which subsets of $G$ have the above property. At present we are not able to provide any insight on the problem. Perhaps one might start with the following question:

Let $n$ be a positive integer. What can be said about a profinite group $G$ such that if $x \in G$ then $C_G(x^n)$ is either finite or open?

Proofs of Theorems 1.2 and 1.3 will be given in Sections 2 and 3, respectively.

2. Proof of Theorem 1.2

The following lemma is taken from [1]. If $X \subseteq G$ is a subset of a group $G$, we write $\langle X \rangle$ for the subgroup generated by $X$ and $\langle X^G \rangle$ for the minimal normal subgroup of $G$ containing $X$.

**Lemma 2.1.** Let $i, j$ be positive integers and $G$ a group having a subgroup $K$ such that $|x^G| \leq i$ for each $x \in K$. Suppose that $|K| \leq j$. Then $\langle K^G \rangle$ has finite $(i, j)$-bounded order.

If $K$ is a subgroup of a finite group $G$, we denote by

$$Pr(K, G) = \frac{|\{(x, y) \in K \times G : [x, y] = 1\}|}{|K||G|}$$

the relative commutativity degree of $K$ in $G$, that is, the probability that a random element of $G$ commutes with a random element of $K$. Note that

$$Pr(K, G) = \frac{\sum_{x \in K} |C_G(x)|}{|K||G|}.$$

It follows that if $|x^G| \leq n$ for each $x \in K$, then $Pr(K, G) \geq \frac{1}{n}$.

The next result was obtained in [2] Proposition 1.2. In the case where $K = G$ this is a well known theorem, due to P. M. Neumann [9].

**Proposition 2.2.** Let $\epsilon > 0$, and let $G$ be a finite group having a subgroup $K$ such that $Pr(K, G) \geq \epsilon$. Then there is a normal subgroup $T \leq G$ and a subgroup $B \leq K$ such that the indexes $[G : T]$ and $[K : B]$, and the order of the commutator subgroup $[T, B]$ are $\epsilon$-bounded.
We will now embark on the proof of Theorem [1.2].

Assume the hypothesis of Theorem [1.2]. Let \( X \) be the set of all \( \pi \)-elements of \( G \). Clearly, \( H = \langle X \rangle \). Given an element \( g \in H \), we write \( l(g) \) for the minimal number \( l \) with the property that \( g \) can be written as a product of \( l \) elements of \( X \). The following result is straightforward from [4] Lemma 2.1.

**Lemma 2.3.** Let \( K \leq H \) be a subgroup of index \( m \) in \( H \), and let \( b \in H \). Then the coset \( Kb \) contains an element \( g \) such that \( l(g) \leq m-1 \).

Let \( m \) be the maximum of indices of \( C_H(x) \) in \( H \) where \( x \in X \). Obviously, we have \( m \leq n \).

**Lemma 2.4.** For any \( x \in X \) the subgroup \( [H, x] \) has \( m \)-bounded order.

**Proof.** Take \( x \in X \). Since the index of \( C_H(x) \) in \( H \) is at most \( m \), by Lemma [2.3] we can choose elements \( y_1, \ldots, y_m \) in \( H \) such that \( l(y_i) \leq m-1 \) and the subgroup \( [H, x] \) is generated by the commutators \( [y_i, x] \), for \( i = 1, \ldots, m \). For any such \( i \) write \( y_i = y_i^1 \cdots y_i^{m-1} \), with \( y_{ij} \in X \). Using standard commutator identities we can rewrite \( [y_i, x] \) as a product of conjugates in \( H \) of the commutators \( [y_{ij}, x] \). Let \( \{h_1, \ldots, h_s\} \) be the conjugates in \( H \) of all elements from the set \( \{x, y_{ij} \mid 1 \leq i, j \leq m\} \). Note that the number \( s \) here is \( m \)-bounded. This follows from the fact that \( C_H(x) \) has index at most \( m \) in \( H \) for each \( x \in X \). Put \( T = \langle h_1, \ldots, h_s \rangle \). Since \( [H, x] \) is contained in the commutator subgroup \( T' \), it is sufficient to show that \( T' \) has \( m \)-bounded order. Observe that the centre \( Z(T) \) has index at most \( m^s \) in \( T \), since the index of \( C_H(h_i) \) is at most \( m \) in \( H \) for any \( i = 1, \ldots, s \). Thus, by Schur’s theorem [11] 10.1.4, we conclude that the order of \( T' \) is \( m \)-bounded, as desired. \( \Box \)

Select \( a \in X \) such that \( |a^H| = m \). Choose \( b_1, \ldots, b_m \) in \( H \) such that \( l(b_i) \leq m-1 \) and \( a^H = \{a^{b_i} \mid i = 1, \ldots, m\} \). The existence of the elements \( b_i \) is guaranteed by Lemma [2.3]. Set \( U = C_G(\langle b_1, \ldots, b_m \rangle) \). Note that the index of \( U \) in \( G \) is \( n \)-bounded. Indeed, since \( l(b_i) \leq m-1 \) we can write \( b_i = b_{i1} \cdots b_{i(m-1)} \), where \( b_{ij} \in X \) and \( i = 1, \ldots, m \). By the hypothesis the index of \( C_G(b_{ij}) \) in \( G \) is at most \( n \) for any such element \( b_{ij} \). Thus, \( [G : U] \leq n^{(m-1)m} \).

The next result is somewhat analogous to [14] Lemma 4.5.

**Lemma 2.5.** If \( u \in U \) and \( ua \in X \), then \( [H, u] \leq [H, a] \).

**Proof.** Assume that \( u \in U \) and \( ua \in X \). For each \( i = 1, \ldots, m \) we have \( (ua)^{b_i} = ua^{b_i} \), since \( u \) belongs to \( U \). We know that \( ua \in X \) so taking into account the hypothesis on the cardinality of the conjugacy
class of \( ua \) in \( H \), we deduce that \((ua)^H\) consists exactly of the elements \( ua^b \), for \( i = 1, \ldots, m \). Thus, given an arbitrary element \( h \in H \), there exists \( b \in \{ b_1, \ldots, b_m \} \) such that \((ua)^h = ua^b\) and so \( u^h a^h = u a^b \). It follows that \([u, h] = a^h a^{-h} \in \langle H, a \rangle \), and the result holds. \( \square \)

**Lemma 2.6.** The order of the commutator subgroup of \( H \) is \( n \)-bounded.

**Proof.** Let \( U_0 \) be the maximal normal subgroup of \( G \) contained in \( U \). Recall that, by the remark made before Lemma 2.5, \( U \) has \( n \)-bounded index in \( G \). It follows that the index \([G : U_0]\) is \( n \)-bounded as well.

By the hypothesis \( a \) has at most \( n \) conjugates in \( G \), say \( \{ a g_1, \ldots, a g_n \} \).

Let \( T \) be the normal closure in \( G \) of the subgroup \( [H, a] \). Note that the subgroups \( [H, a g_i] \) are normal in \( H \), therefore \( T = [H, a g_1] \cdots [H, a g_n] \). By Lemma 2.4 each of the subgroups \( [H, a g_i] \) has \( n \)-bounded order. We conclude that the order of \( T \) is \( n \)-bounded.

Let \( Y = Xa^{-1} \cap U \). Note that for any \( y \in Y \) the product \( ya \) belongs to \( X \). Therefore, by Lemma 2.5 for any \( y \in Y \), the subgroup \( [H, y] \) is contained in \( [H, a] \). Thus,

(1) \[ [H, Y] \leq T. \]

Observe that for any \( u \in U_0 \) the commutator \([u, a^{-1}] = a^u a^{-1}\) lies in \( Y \) and so

(2) \[ [H, [U_0, a^{-1}]] \leq [H, Y]. \]

Since \([U_0, a^{-1}] = [U_0, a]\), we deduce from (1) and (2) that

(3) \[ [H, [U_0, a]] \leq T. \]

Since \( T \) has \( n \)-bounded order, it is sufficient to show that the derived group of the quotient \( H/T \) has finite \( n \)-bounded order. We pass now to the quotient \( G/T \) and for the sake of simplicity the images of \( G, H, U, U_0, X \) and \( Y \) will be denoted by the same symbols. Note that by (1) the set \( Y \) becomes central in \( H \) modulo \( T \). Containment (3) shows that \([U_0, a] \leq Z(H)\). This implies that if \( b \in U_0 \) is a \( \pi \)-element, then \([b, a] \in Z(H)\) and the subgroup \( \langle a, b \rangle \) is nilpotent. Thus the product \( ba \) is a \( \pi \)-element too and so \( b \in Y \). Hence, all \( \pi \)-elements of \( U_0 \) are contained in \( Y \) and, in view of (1), we deduce that they are contained in \( Z(H) \).

Next we consider the quotient \( G/Z(H) \). Since the image of \( U_0 \) in \( G/Z(H) \) is a \( \pi' \)-group and has \( n \)-bounded index in \( G \), we deduce that the order of any \( \pi \)-subgroup in \( G/Z(H) \) is \( n \)-bounded. In particular, there is an \( n \)-bounded constant \( C \) such that for every \( p \in \pi \) the order of the Sylow \( p \)-subgroup of \( G/Z(H) \) is at most \( C \). Because of Lemma
for any $p \in \pi$ each Sylow $p$-subgroup of $G/Z(H)$ is contained in a normal subgroup of $n$-bounded order. We deduce that all such Sylow subgroups of $G/Z(H)$ are contained in a normal subgroup of $n$-bounded order. Since $H$ is generated by $\pi$-elements, it follows that the order of $H/Z(H)$ is $n$-bounded. Thus, in view of Schur’s theorem [11, 10.1.4], we conclude that $|H'|$ is $n$-bounded, as desired.

We will now complete the proof of Theorem 1.2.

**Proof.** Assume first that $H$ is abelian. In this case the set $X$ of $\pi$-elements is a subgroup, that is, $X = H$. By the hypothesis we have $|x^G| \leq n$ for any element $x \in H$ and so the relative commutativity degree $Pr(H, G)$ of $H$ in $G$ is at least $\frac{1}{n}$. Thus, by virtue of Proposition 2.2, there is a normal subgroup $T \leq G$ and a subgroup $B \leq H$ such that the indexes $[G : T]$ and $[H : B]$, and the order of the commutator subgroup $[T, B]$ are $n$-bounded.

Since $H$ is a normal $\pi$-subgroup and $[G : H]$ is a $\pi'$-number, by the Schur–Zassenhaus Theorem [5, Theorem 6.2.1] the subgroup $H$ admits a complement $L$ in $G$ such that $G = HL$ and $L$ is a $\pi'$-subgroup. Set $T_0 = T \cap L$. Observe that the index $[L : T_0]$ is $n$-bounded since it is at most the index of $T$ in $G$. Thus we deduce that the index of $HT_0$ is $n$-bounded in $G$, as well.

We claim that the order of $[H, T_0]$ is $n$-bounded. Indeed, the $\pi'$-subgroup $T_0$ acts coprimely on the the abelian $\pi$-subgroup $B_1 = B[B, T_0]$, and so we have $B_1 = C_{B_1}(T_0) \times [B_1, T_0]$ ( [7, Corollary 1.6.5]). Note that $[B_1, T_0] = [B, T_0]$. Since the order of $[B, T_0]$ is $n$-bounded (being at most the order of $[T, B]$), we deduce that the index $[B_1 : C_{B_1}(T_0)]$ is $n$-bounded. In combination with the fact that $[H : B]$ is $n$-bounded, we obtain that the index $[H : C_{B_1}(T_0)]$ is $n$-bounded and so in particular $[H : C_H(T_0)]$ is $n$-bounded. Since $T_0$ acts coprimely on the abelian normal $\pi$-subgroup $H$, we have $H = C_H(T_0) \times [H, T_0]$. Thus we obtain that the order of the commutator subgroup $[H, T_0]$ is $n$-bounded, as claimed. Let $T_1 = C_{T_0}([H, T_0])$ and remark that the index $[T_0 : T_1]$ of $T_1$ in $T_0$ is $n$-bounded too. Set $N = HT_1$. From the fact that the indexes $[T_0 : T_1]$ and $[G : HT_0]$ are both $n$-bounded, we deduce that the index of $N$ in $G$ is $n$-bounded, as well.

Note that $N$ is normal in $G$ since the image of $N$ in $G/H \cong L$ is isomorphic to $T_1$ which is normal in $L$. Furthermore, we have $[H, T_1, T_1] = 1$, since $T_1 = C_{T_0}([H, T_0])$. Hence by the standard properties of coprime actions we have $[H, T_1] = 1$ ( [7, Corollary 1.6.4]). Therefore $[H, N] = 1$. This proves the theorem in the particular case where $H$ is abelian.
In the general case, in view of Lemma 2.6, the commutator subgroup $[H, H]$ is of $n$-bounded order. We pass to the quotient $\overline{G} = G/[H, H]$. The above argument shows that $\overline{G}$ has a normal subgroup $\overline{N}$ of $n$-bounded index such that $\overline{H} \leq Z(\overline{N})$. Here $Z(\overline{N})$ stands for the centre of $\overline{N}$. Let $N$ be the inverse image of $\overline{N}$. We have $[H, N] = [H, H]$ and so $N$ has the required properties. The proof is now complete. □

3. Proof of Theorem 1.3

We will require the following result taken from \cite{1}, Lemma 4.1.

**Lemma 3.1.** Let $G$ be a locally nilpotent group containing an element with finite centralizer. Suppose that $G$ is residually finite. Then $G$ is finite.

Profinite groups have Sylow $p$-subgroups and satisfy analogues of the Sylow theorems. Prosoluble groups satisfy analogues of the theorems on Hall $\pi$-subgroups. We refer the reader to the corresponding chapters in \cite{10}, Ch. 2] and \cite{15}, Ch. 2].

Recall that an automorphism $\phi$ of a group $G$ is called fixed-point-free if $C_{G}(\phi) = 1$, that is, the fixed-point subgroup is trivial. It is a well-known corollary of the classification of finite simple groups that if $G$ is a finite group admitting a fixed-point-free automorphism, then $G$ is soluble (see for example \cite{12} for a short proof). A continuous automorphism $\phi$ of a profinite group $G$ is coprime if for any open $\phi$-invariant normal subgroup $N$ of $G$ the order of the automorphism of $G/N$ induced by $\phi$ is coprime to the order of $G/N$. It follows that if a profinite group $G$ admits a coprime fixed-point-free automorphism, then $G$ is prosoluble. This will be used in the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Recall that $\pi$ is a set of primes and $G$ is a profinite group in which the centralizer of every $\pi$-element is either finite or open. We wish to show that $G$ has an open subgroup of the form $P \times Q$, where $P$ is an abelian pro-$\pi$ subgroup and $Q$ is a pro-$\pi'$ subgroup.

Let $X$ be the set of $\pi$-elements in $G$. Consider first the case where the conjugacy class $x^G$ is finite for any $x \in X$. For each integer $i \geq 1$ set

$$S_i = \{x \in X; |x^G| \leq i\}.$$  

The sets $S_i$ are closed. Thus, we have countably many sets which cover the closed set $X$. By the Baire Category Theorem \cite{6}, Theorem 34] at least one of these sets has non-empty interior. It follows that there is a positive integer $k$, an open normal subgroup $M$, and an element $a \in X$ such that all elements in $X \cap aM$ are contained in $S_k$.  

Note that \( \langle a^G \rangle \) has finite commutator subgroup, which we will denote by \( T \). Indeed, the subgroup \( \langle a^G \rangle \) is generated by finitely many elements whose centralizer is open. This implies that the centre of \( \langle a^G \rangle \) has finite index in \( \langle a^G \rangle \), and by Schur’s theorem [11, 10.1.4], we conclude that \( T \) is finite, as claimed.

Let \( x \in X \cap M \). Note that the product \( ax \) is not necessarily in \( X \). On the other hand, \( ax \) is a \( \pi \)-element modulo \( T \). This is because \( \langle a^G \rangle \) becomes an abelian normal \( \pi \)-subgroup modulo \( T \) and the image of \( ax \) in the quotient \( G/\langle a^G \rangle \) is a \( \pi \)-element. In other words, there are \( y \in X \cap aM \) and \( t \in T \) such that \( ax = ty \). Observe that \( t \) has an open centralizer in \( G \) since \( t \in T \). In fact \( |G : C_G(t)| \leq |T| \). From the equality \( ax = ty \) deduce that \( |x^G| \leq k^2|T| \). This happens for any \( x \in X \cap M \). Using a routine inverse limit argument in combination with Theorem 1.2 we obtain that \( M \) has an open normal subgroup \( N \) such that the index \( |M : N| \) and the order of \( [H, N] \) are finite. Here \( H \) stands for the subgroup generated by all \( \pi \)-elements of \( M \). Choose an open normal subgroup \( U \) in \( G \) such that \( U \cap [H, N] = 1 \). Then \( U \cap M \) is an open normal subgroup of the form \( P \times Q \), where \( P \) is an abelian pro-\( \pi \) subgroup and \( Q \) is a pro-\( \pi' \) subgroup. This proves the theorem in the case where all \( \pi \)-elements of \( G \) have open centralizers.

Assume now that \( G \) has a \( \pi \)-element, say \( b \), of infinite order. Since the procyclic subgroup \( \langle b \rangle \) is contained in the centralizer \( C_G(b) \), it follows that \( C_G(b) \) is open in \( G \). This implies that all elements of \( X \cap C_G(b) \) have open centralizers (because they centralize the procyclic subgroup \( \langle b \rangle \) ). In view of the above \( C_G(b) \) has an open subgroup of the form \( P \times Q \), where \( P \) is an abelian pro-\( \pi \) subgroup and \( Q \) is a pro-\( \pi' \) subgroup and we are done.

We will therefore assume that \( G \) is infinite while all \( \pi \)-elements of \( G \) have finite orders and there is at least one \( \pi \)-element, say \( d \), such that \( C_G(d) \) is finite. The element \( d \) is a product of finitely many \( \pi \)-elements of prime power order. At least one of these elements must have finite centralizer. So without loss of generality we can assume that \( d \) is a \( p \)-element for a prime \( p \in \pi \).

Let \( P_0 \) be a Sylow \( p \)-subgroup of \( G \) containing \( d \). Since \( P_0 \) is torsion, we deduce from Zelmanov’s theorem [16] that \( P_0 \) is locally nilpotent. The centralizer \( C_G(d) \) is finite and so in view of Lemma 3.1 the subgroup \( P_0 \) is finite. Choose an open normal pro-\( p' \) subgroup \( L \) such that \( L \cap C_G(d) = 1 \). Note that any finite homomorphic image of \( L \) admits a coprime fixed-point-free automorphism (induced by the coprime action of \( d \) on \( L \)). Hence \( L \) is prosoluble. Let \( K \) be a Hall \( \pi \)-subgroup of \( L \). Since any element in \( K \) has restricted centralizer, Shalev’s result [13] shows that \( K \) is virtually abelian. We therefore can choose an
open normal subgroup $J$ in $L$ such that $J \cap K$ is abelian. If $J \cap K$ is finite then $G$ is virtually pro-$\pi'$ and we are done. If $J \cap K$ is infinite, then all $\pi$-elements of $J$ have infinite centralizers. This yields that all $\pi$-elements of $J$ have open centralizers in $J$ and in view of the first part of the proof, $J$ has an open normal subgroup of the form $P \times Q$, where $P$ is an abelian pro-$\pi$ subgroup and $Q$ is a pro-$\pi'$ subgroup. This establishes the theorem. □

References

[1] C. Acciarri, P. Shumyatsky, A stronger form of Neumann’s BFC-theorem, Isr. J. Math. 242, 269–278 (2021). https://doi.org/10.1007/s11856-021-2133-1.
[2] E. Detomi, P. Shumyatsky, On the commuting probability for subgroups of a finite group, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–14 (2021). doi:10.1017/prm.2021.68.
[3] E. Detomi, M. Morigi, P. Shumyatsky, Profinite groups with restricted centralizers of commutators, Proceedings of the Royal Society of Edinburgh, Section A: Mathematics, 150(5) (2020), 2301–2321. doi:10.1017/prm.2019.17.
[4] G. Dierings, P. Shumyatsky, Groups with boundedly finite conjugacy classes of commutators, Quarterly J. Math. 69(3) (2018), 1047–1051.
[5] D. Gorenstein, Finite Groups, Chelsea Publishing Company, New York, 1980.
[6] J. L. Kelley, General topology, Grad. Texts in Math., vol. 27, Springer, New York, 1975.
[7] E. I. Khukhro, Nilpotent groups and their automorphisms, Berlin-New York, de Gruyter, 1993.
[8] B. H. Neumann, Groups covered by permuting subsets, J. London Math. Soc. (3) 29 (1954), 236–248.
[9] P. M. Neumann, Two combinatorial problems in group theory, Bull. Lond. Math. Soc. 21 (1989), 456–458.
[10] L. Ribes, P. Zalesskii, Profinite Groups, 2nd edition, Springer Verlag, Berlin, New York, 2010.
[11] D. J. S. Robinson, A course in the theory of groups, Second edition. Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996.
[12] P. Rowley, Finite groups admitting a fixed-point-free automorphism group, J. Algebra, 174 (1995) 724–727.
[13] A. Shalev, Profinite groups with restricted centralizers. Proc. Amer. Math. Soc. 122 (1994), 1279–1284.
[14] J. Wiegold, Groups with boundedly finite classes of conjugate elements, Proc. Roy. Soc. London Ser. A 238 (1957), 389–401.
[15] J.S. Wilson, Profinite Groups, Clarendon Press, Oxford, 1998.
[16] E. I. Zelmanov, On periodic compact groups. Israel J. Math. 77, 83–95 (1992).
