Definite and Indefinite Inner Product on Superspace (Hilbert-Krein Superspace)

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Abstract

We present natural (invariant) definite and indefinite scalar products on the $N = 1$ superspace which turns out to carry an inherent Hilbert-Krein structure. We are motivated by supersymmetry in physics but prefer a general mathematical framework.

1 Introduction

Supersymmetries generalize the notion of a Lie algebra to include algebraic systems whose defining relations involve commutators as well as anticommutators. Denoting by $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ the odd (anticommuting) generators, physical considerations require that (see [1]) the operators $Q_\alpha, \bar{Q}_{\dot{\alpha}} = (Q_\alpha)^+$ act in a bona fide Hilbert space $\mathcal{H}$ of states with positive definite metric. Here $(Q_\alpha)^+$ means the operator adjoint to $Q_\alpha$ in $\mathcal{H}$. From the commutation relations [1]

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^l_{\alpha\dot{\alpha}} P_l$$

where $\sigma^l, l = 0, 1, 2, 3$ are the Pauli matrices with $\sigma^0 = -1$ as in [1] and $P_l$ is the momentum, it follows that for any state $\Phi$ in $\mathcal{H}$ we have

$$\|Q_\alpha \Phi\|^2 + \|\bar{Q}_{\dot{\alpha}} \Phi\|^2 = (\Phi, \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \Phi) = 2\sigma^l_{\alpha\dot{\alpha}} (\Phi, P_l \Phi)$$
Summing over $\alpha = \dot{\alpha} = 1, 2$ and using $tr\sigma^0 = -2, tr\sigma^l = 0, l = 1, 2, 3$ yields for the Minkowski metric $(-1, 1, 1, 1)$

$$(\Phi, P_0\Phi) > 0$$

i.e. in a supersymmetric theory the energy $H = P_0$ is always positive. This positivity argument doesn't require any detailed knowledge of the Hilbert space $\mathcal{H}$ which is an imperative of any quantum theory. In this paper we present indefinite but also definite (invariant) inner products on $N = 1$ superspace, i.e. defined on supersymmetric functions on the $N = 1$ superspace and show that the inherent Hilbert space in supersymmetric theories appears in conjunction with an indefinite (Krein) scalar product. Roughly speaking each function on superspace can be decomposed in a chiral, antichiral and a transversal contribution. It turns out that in order to obtain positivity of the scalar product the transversal contribution has to be subtracted instead of adding it to the chiral/antichiral part.

Despite of the previous positivity argument leading to the energy positivity which relies on physical arguments, we prefer for this paper a general mathematical framework and even do not explicitly assume supersymmetry; in particular we do not assume Lorentz invariance. Comments on physics appear at the end of the paper. We use the notation and conventions of [1] with the only difference that from now on $\sigma^0, \bar{\sigma}^0$ are the identity instead of minus identity (our notations coincide with [2]). In particular our Minkowski metric $\eta^{lm}$ is $(-1, +1, +1, +1)$. The Fourier transform $\tilde{f}(p)$ of $f(x)$ is defined through

$$f(x) = \frac{1}{(2\pi)^2} \int e^{ipx} \tilde{f}(p) dp$$

where $px = p_l x^l = p_l \eta^{lm} x_m$.

We use the Weyl spinor formalism in the Van der Waerden notations as in the references cited above although for our purposes 4-component spinors would be better suited (see [3]). Working with Weyl spinors we have to assume for consistency reasons anticommutativity of their components which in our case are regular (test) functions (or distributions). This will be not the case at the point we define sesquilinear form (inner products) by integration on superspace connecting to the usual $L^2$-scalar product on functions. Certainly this is not a serious problem as it is clear to the reader (see also Section 3).

### 2 The supersymmetric functions

We restrict to the $N = 1$ superspace. We write the most general superspace (test) function $X = X(z) = X(x, \theta, \bar{\theta})$ as in [1, 2]
\[
X(z) = X(x, \theta, \bar{\theta}) = 
= f(x) + \theta \varphi(x) + \bar{\theta} \tilde{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) + 
\theta \sigma^l \bar{\theta} v_l(x) + \theta^2 \bar{\theta} \lambda(x) + \bar{\theta}^2 \theta \psi(x) + \theta^2 \bar{\theta}^2 d(x) \tag{2.1}
\]

where the coefficients are functions of \(x\) in Minkowski space of certain regularity which will be specified below (by the end of the paper we will admit distributions too). For the time being suppose that the coefficient functions are in the Schwartz space \(S\) of infinitely differentiable (test) functions with faster than polynomial decrease at infinity. For the vector component \(v\) we can write equivalently
\[
\theta \sigma^l \bar{\theta} v_l = \theta^\alpha \bar{\theta}^\dot{\alpha} v_{\alpha \dot{\alpha}}
\]
where
\[
v_{\alpha \dot{\alpha}} = \sigma^l_{\alpha \dot{\alpha}} v_l, v^l = \frac{1}{2} \bar{\sigma}^{l \dot{\alpha} \alpha} v_{\alpha \dot{\alpha}}
\]
which is a consequence of the "second" completeness equation
\[
\sigma^l_{\alpha \beta} \sigma^\dot{\beta}_{\dot{\alpha}} = -2 \delta^\dot{\alpha}_{\alpha} \delta^\dot{\beta}_{\beta}.
\]
Let us introduce the supersymmetric covariant (and invariant [1, 2]) derivatives \(D, \bar{D}\) with spinorial components \(D_{\alpha}, D^\alpha, \bar{D}_{\dot{\alpha}} , \bar{D}^{\dot{\alpha}}\) given by
\[
D_{\alpha} = \partial_{\alpha} + i \sigma^l_{\alpha \dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_l \tag{2.2}
\]
\[
D^\alpha = \epsilon^{\alpha \beta} D_{\beta} = -\partial^\alpha + i \sigma^l_{\alpha \dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_l \tag{2.3}
\]
\[
\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i \theta^\alpha \sigma^l_{\alpha \dot{\alpha}} \partial_l \tag{2.4}
\]
\[
\bar{D}^{\dot{\alpha}} = \epsilon^{\dot{\alpha} \dot{\beta}} \bar{D}_{\dot{\beta}} = \bar{\partial}^{\dot{\alpha}} - i \theta^\alpha \sigma^l_{\alpha \dot{\alpha}} \partial_l \tag{2.5}
\]
We accept on the way notations like
\[
\epsilon^{\alpha \beta} \sigma^l_{\beta \dot{\alpha}} = \sigma^l_{\alpha \dot{\alpha}}
\]
etc. but in the end we come back to the canonical index positions \(\sigma^l = (\sigma^l_{\alpha \dot{\alpha}}), \bar{\sigma}^l = (\bar{\sigma}^{l \dot{\alpha} \alpha})\).

Note that \(D_{\alpha}\) does not contain the variable \(\theta\) and \(\bar{D}^{\dot{\alpha}}\) does not contain the variable \(\bar{\theta}\) such that we can write at the operatorial level:
\[
D^2 = D^\alpha D_{\alpha} = -(\partial^\alpha \partial_{\alpha} - 2i \bar{\theta} \sigma^l_{\alpha \dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_l + \bar{\theta}^2 \Box) \tag{2.6}
\]
\[
D^2 = D_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} = -(\bar{\partial}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} + 2i \theta \sigma^l_{\alpha \dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_l + \theta^2 \Box) \tag{2.7}
\]
where
\[ \Box = \eta^{lm} \partial_l \partial_m \]
is the d’Alembertian, \( \eta \) is the Minkowski metric tensor and
\[ \partial_{\alpha \dot{\alpha}} = \sigma^l_{\alpha \dot{\alpha}} \partial_l \]
Here we used the ”first” completeness relation for the Pauli matrices \( \sigma, \bar{\sigma} \):
\[ \text{Tr}(\sigma^l \bar{\sigma}^m) = \sigma^l_{\alpha \dot{\beta}} \bar{\sigma}^m_{\dot{\alpha} \beta} = -2\eta^{lm} \]  \hspace{1cm} (2.8)
We make use of the operators \([1, 2]\)
\[ c = \bar{D}^2 D^2, \quad a = D^2 \bar{D}^2, \quad T = D^\alpha \bar{D}^2 D_\alpha = D_\dot{\alpha} D^2 \bar{D}^{\dot{\alpha}} = -8\Box + \frac{1}{2}(c + a) \]  \hspace{1cm} (2.9)
which are used to construct formal projections
\[ P_c = \frac{1}{16\Box} c, \quad P_a = \frac{1}{16\Box} a, \quad P_T = -\frac{1}{8\Box} T \]  \hspace{1cm} (2.10)
on chiral, antichiral and transversal supersymmetric functions. These operators are, at least for the time being, formal because they contain the \( \text{d’Alembertian} \) in the denominator. Problems with the \( \text{d’Alembertian} \) in (2.10) in the denominator will be explained later. Chiral, antichiral and transversal functions are linear subspaces of general supersymmetric functions which are defined by the conditions \([1, 2]\)
\[ \bar{D}^{\dot{\alpha}} X = 0, \quad \dot{\alpha} = 1, 2; \quad D^\alpha X = 0, \quad \alpha = 1, 2; \quad D^2 X = \bar{D}^2 X = 0 \]
respectively. It can be proven that these relations are formally equivalent to the relations
\[ P_c X = X, \quad P_a X = X, \quad P_T X = X \]
(we mean here that \( P_i, i = c, a, T \) are applicable to \( X \) and the relations above hold).
We have formally
\[ P_i^2 = P_i, \quad P_i P_j = 0, \quad i \neq j; \quad i, j = c, a, T \]
and \( P_c + P_a + P_T = 1 \). Accordingly each supersymmetric function can be formally decomposed into a sum of a chiral, antichiral and transversal contribution (from a rigorous point of view this statement may be wrong and has to be reconsidered because of the problems with the \( \text{d’Alembertian} \) in the
denominator; fortunately we will not run into such difficulties as this will be made clear later in the paper).
Let us specify the coefficient functions in (2.1) for the chiral, antichiral and transversal supersymmetric functions.
For the chiral case \( X_c \) we have:
\[
\bar{\chi} = \psi = n = 0, v_l = \partial_l(if) = i\partial_l f, \\
\bar{\lambda} = -\frac{i}{2} \partial_l \varphi \sigma^l = \frac{i}{2} \sigma^l \partial_l \varphi, d = \frac{1}{4} \Box f
\] (2.11)
Here \( f, \varphi \) and \( m \) are arbitrary functions. For notations and relations see (2.23)-(2.27).
For the antichiral \( X_a \) case:
\[
\varphi = \bar{\lambda} = m = 0, v_l = \partial_l(-if) = -i\partial_l f, \\
\psi = \frac{i}{2} \sigma^l \partial_l \bar{\chi} = -\frac{i}{2} \partial_l \bar{\chi} \sigma^l, d = \frac{1}{4} \Box f
\] (2.12)
Here \( f, \bar{\chi} \) and \( n \) are arbitrary functions.
For the transversal case \( X_T \) \[2\] :
\[
m = n = 0, \partial_l v^l = 0, \\
\bar{\lambda} = \frac{i}{2} \partial_l \varphi \sigma^l = -\frac{i}{2} \sigma^l \partial_l \varphi, \psi = \frac{i}{2} \partial_l \bar{\chi} \sigma^l = -\frac{i}{2} \sigma^l \partial_l \bar{\chi}, d = \frac{1}{4} \Box f
\] (2.13)
Here \( f, \varphi, \bar{\chi} \) are arbitrary and \( v \) satisfies \( \partial_l v^l = 0 \).
Later on we will need the \( \theta^2 \bar{\theta}^2 \) – coefficients \([X_i X_i](x_1, x_2)\) of the quadratic forms \( X_i(x_1, \theta, \bar{\theta})X_i(x_2, \theta, \bar{\theta})\) for \( i = c, a, T \) where \( X_0 = X \) is arbitrary supersymmetric. They coincide with the Grassmann integrals
\[
\int d^2 \theta_1 d^2 \bar{\theta}_1 d^2 \theta_2 d^2 \bar{\theta}_2 X_i(x_1, \theta_1, \bar{\theta}_1) \delta^2(\theta_1 - \theta_2) \delta^2(\bar{\theta}_1 - \bar{\theta}_2)X_i(x_2, \theta_2, \bar{\theta}_2)
\] (2.14)
where \( \delta^2(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2, \delta^2(\bar{\theta}_1 - \bar{\theta}_2) = (\bar{\theta}_1 - \bar{\theta}_2)^2 \), \( d^2 \theta = \frac{1}{2} d\theta^1 d\theta^2, d^2 \bar{\theta} = -\frac{1}{2} d\bar{\theta}^1 d\bar{\theta}^2 \) and are listed below in the order of \( i = c, a, T \) :
\[
[X_c X_c](x_1, x_2) = \bar{f}(x_1)(\frac{1}{4} \Box f(x_2)) - \varphi(x_1)(\frac{i}{2} \sigma^l \partial_l \varphi(x_2)) + \bar{m}(x_1)m(x_2) - \\
-\frac{1}{2} \partial_l \bar{f}(x_1) \partial_l f(x_2) - (-\frac{i}{2} \partial_l \bar{\varphi}(x_1) \sigma^l) \varphi(x_2) + (\frac{1}{4} \Box \bar{f}(x_1)) f(x_2)
\] (2.15)
\begin{equation}
[X_a X_a](x_1, x_2) = \bar{f}(x_1)\left(\frac{1}{4} \Box f(x_2)\right) - \chi(x_1)\left(\frac{i}{2} \sigma^l \partial_l \bar{\chi}(x_2)\right) + \bar{n}(x_1) n(x_2) - \\
- \frac{1}{2} \partial^l \bar{f}(x_1) \partial_l f(x_2) - \left(-\frac{i}{2} \partial_l \chi(x_1) \sigma^l \right) \bar{\chi}(x_2) + \left(\frac{1}{4} \Box \bar{f}(x_1)\right) f(x_2) \tag{2.16}
\end{equation}

\begin{equation}
[X_T X_T](x_1, x_2) = \bar{f}(x_1)\left(-\frac{1}{4} \Box f(x_2)\right) - \bar{\varphi}(x_1)\left(-\frac{i}{2} \bar{\sigma}^l \partial_l \varphi(x_2)\right) - \\
- \left(\frac{i}{2} \partial_l \bar{\varphi}(x_1)\right) \sigma^l \varphi(x_2) - \frac{1}{2} \bar{v}^l(x_1) v_l(x_2) - \chi(x_1)\left(-\frac{i}{2} \sigma^l \partial_l \bar{\chi}(x_2)\right) - \\
- \left(\frac{i}{2} \partial_l \chi(x_1) \sigma^l \right) \bar{\chi}(x_2) + \left(-\frac{1}{4} \Box \bar{f}(x_1)\right) f(x_2) \tag{2.17}
\end{equation}

where we have used relations quoted in (2.23)-(2.27). The conjugate $\bar{X}$ is given in (2.34).

As an useful exercise let us put $x_1 = x_2$ in $[\bar{X}_i X_i](x_1, x_2), i = c, a, T$ and compute the integral

\[\int d^4 x [\bar{X}_i X_i](x)\]

We want to make clear that this computation is done only for pedagogical reasons; we perform it because we will need a similar computation in momentum space (!) at a later stage in this paper. We integrate by parts and use the faster than polynomial decrease of the coefficient functions and of their derivatives to obtain for the chiral case:

\begin{equation}
\int d^4 x [\bar{X}_c X_c](x) = \\
= \int d^4 x \bar{f}(x) \Box f(x) - \int d^4 x \bar{\varphi}(x) i \bar{\sigma}^l \partial_l \varphi(x) + \int d^4 x \bar{m}(x) m(x) \tag{2.18}
\end{equation}

For the antichiral case:

\begin{equation}
\int d^4 x [\bar{X}_a X_a](x) = \\
\int d^4 x \bar{f}(x) \Box f(x) - \int d^4 x \bar{\chi}(x) i \sigma^l \partial_l \bar{\chi}(x) + \int d^4 x \bar{n}(x) n(x) \tag{2.19}
\end{equation}

and for the transversal case:
We have used the following notations and relations (see for instance the standard references mentioned above):

\[ \int d^4x [X_T X_T](x) = -\frac{1}{2} \int d^4x \bar{f}(x) \Box f(x) + \int d^4x \bar{\varphi}(x)i\sigma^l \partial_l \varphi(x) + \int d^4x \varphi(x)i\sigma^l \partial_l \bar{\varphi}(x) - \frac{1}{2} \int d^4x \bar{v}^l(x)v_l(x) \quad (2.20) \]

Certainly the best we can expect in our paper is to find a Hilbert space structure on supersymmetric functions such that the decomposition formally suggested by \( P_c + P_\alpha + P_T = 1 \) is a direct orthogonal sum of chiral, antichiral and transversal functions, but this is definitely not the case as will be clear soon. In this paper we are going to uncover the exact mathematical structure of this decomposition in its several variants. This will be done by explicit computations. We start computing the action of the operators \( D_\alpha, D^\alpha, \bar{D}_\alpha, \bar{D}^\alpha, D^2, \bar{D}^2, c, a, T \) on \( X \). Usually in physics one doesn’t need the results of all these elementary but long computations in an explicit way and this is the reason they are not fully recorded in the literature. It turns out that for our purposes we need at least some of them.

For a given \( X \) as in (2.1) the expressions \( D_\beta X, D^\gamma X, \bar{D}^\delta X, \bar{D}^\gamma X \) are easily computed but are not given explicitely here because they are in fact not necessary in order to compute higher covariant derivatives used in this paper (in order to compute higher derivatives we use (2.6) and (2.7)). We start recording the results for \( D^2, \bar{D}^2 \) applied on \( X \):

\[ \bar{D}^2 X = -4n + \theta(-4\psi - 2i\sigma^l \partial_l \bar{\chi}) + \theta^2(-4d - 2i\partial_l v^l - \Box f) + \theta \sigma^l \bar{\theta}(-4i\partial_l n) + \theta^2 \bar{\theta}(-2i\bar{\sigma}^l \partial_l \bar{\psi} - \Box \bar{\chi}) + \theta^2 \bar{\theta}^2(-\bar{\Box} n) \]

\[ D^2 X = -4m + \bar{\theta}(-4\bar{\lambda} - 2i\bar{\sigma}^l \partial_l \varphi) + \bar{\theta}^2(-4d + 2i\partial_l v^l - \Box f) + \theta \sigma^l \bar{\theta}(4i\partial_l m) + \theta^2 \bar{\theta}(-2i\sigma^l \partial_l \bar{\lambda} - \Box \varphi) + \theta^2 \bar{\theta}^2(-\bar{\Box} m) \]

or in a more suggestive way taking into account the chirality/antichirality of \( \bar{D}^2 X, D^2 X \) (see (2.11), (2.12)):

\[ \bar{D}^2 X = -4n + \theta(-4\psi - 2i\sigma^l \partial_l \bar{\chi}) + \theta^2(-4d - 2i\partial_l v^l - \Box f) + \theta \sigma^l \bar{\theta}(-4i\partial_l n) + \theta^2 \bar{\theta}(\frac{1}{2}i\sigma^l \partial_l)(-4\psi - 2i\sigma^l \partial_l \bar{\chi}) + \theta^2 \bar{\theta}^2(-\bar{\Box} n) \quad (2.21) \]

\[ D^2 X = -4m + \bar{\theta}(-4\bar{\lambda} - 2i\bar{\sigma}^l \partial_l \varphi) + \bar{\theta}^2(-4d + 2i\partial_l v^l - \Box f) + \theta \sigma^l \bar{\theta}(4i\partial_l m) + \bar{\theta}^2 \theta(\frac{1}{2}i\sigma^l \partial_l)(-4\bar{\lambda} - 2i\bar{\sigma}^l \partial_l \varphi) + \theta^2 \bar{\theta}^2(-\bar{\Box} m) \quad (2.22) \]

We have used the following notations and relations (see for instance the standard references mentioned above):
(ψσ^l)_β = ψ^o_σ σ^l_β, (σ^l_X)_β = σ^l_β χ_β, (χσ^l)_α = χ_α σ^l_β, (σ^l ψ)̄ = σ^l_̄α ψ_β \tag{2.23}

with (σ^l X)^α = - (χ̄σ^l)^α etc. as well as

ψσ^l X = ψ^o σ^l_α χ_β = -χ̄σ^l ψ = -χ̄σ^l_̄α β_β \tag{2.24}

where σ^l_α_β = σ^l_β_α.

As far as the complex conjugation is concerned we have:

(ψσ^l)^*_α = (σ^l_̄)α, (χ̄σ^l)^*_α = (̄ᾱβ)_α, (ψσ^l_̄)_* = χσ^l_̄ \tag{2.25}

where * is the complex conjugation defined such that

(ψ_α)^* = ψ_̄α \tag{2.26}

(ψ^o_α)^* = ψ^o_̄α \tag{2.27}

The unusual properties of the Grassmann derivative were taken into account; in particular \(∂^*_α = -̄∂_α\) etc.

As expected \(\overline{D^2 X}\) and \(D^2 X\) are chiral and antichiral functions respectively.

We continue with \(c = \overline{D^2 D^2}, a = D^2 \overline{D^2}:\)

\[
\begin{align*}
cX &= \overline{D^2 D^2} X = 16d - 8i∂_t v^l + 4□f + \theta(8□\varphi + 16iσ^l d\bar{λ}) + \\
&\quad + \theta^2(16□m) + \thetaσ^l\bar{θ}(16i∂_t d + 8∂_t d v^m + 4i∂_l □f) + \\
&\quad + \bar{θ}^2(8□\bar{λ} + 4iσ^l d^l □f) + \theta^2\bar{θ}^2(4□d - 2i∂_l □v^l + □^2 f) \tag{2.28}
\end{align*}
\]

\[
\begin{align*}
aX &= D^2 \overline{D^2} X = 16d + 8i∂_t v^l + 4□f + \bar{θ}(8□\bar{λ} + 16iσ^l d_λ) + \\
&\quad + \bar{θ}^2(16□m) + \thetaσ^l\bar{θ}(-16i∂_t d + 8∂_t d v^m - 4i∂_l □f) + \\
&\quad + \bar{θ}^2(8□ψ + 4iσ^l d_λ □f) + \theta^2\bar{θ}^2(4□d + 2i∂_l □v^l + □^2 f) \tag{2.29}
\end{align*}
\]

and finally obtain \(T = -8□ + \frac{1}{2}(c + a)\) applied on \(X\) as follows:

\[
\begin{align*}
TX &= 16d - 4□f + θ(-4□ψ + 8iσ^l d_λ) + \bar{θ}(-4□\bar{λ} + 8iσ^l d_λ) + \\
&\quad + θσ^l\bar{θ}(8∂_t d v^m - 8□v^l) + \theta^2\bar{θ}(-4□\bar{λ} + 2iσ^l d_λ □ψ) + \\
&\quad + \bar{θ}^2(4□ψ + 2iσ^l d_λ □ψ) + \theta^2\bar{θ}^2(-4□d + □^2 f) \tag{2.30}
\end{align*}
\]
or

\[ TX = 16d - 4 \Box f + \theta(-4 \Box \varphi + 8i \sigma^l \partial_l \lambda) + \bar{\theta}(-4 \Box \bar{\chi} + 8i \bar{\sigma}^l \partial_l \psi) + \\
+ \theta \sigma^l \bar{\theta}(8 \partial_l \partial^m v_m - 8 \Box v_l) + \theta^2 \bar{\theta}(-\frac{i}{2} \sigma^l \partial_l)(-4 \Box \varphi + 8i \sigma^l \partial_l \lambda) + \\
+ \bar{\theta}^2 \theta(-\frac{i}{2} \bar{\sigma}^l \partial_l)(-4 \Box \bar{\chi} + 8i \bar{\sigma}^l \partial_l \psi) + \theta^2 \bar{\theta}^2(-4 \Box d + \Box^2 f) \]  

(2.31)

Here we have used the relations

\[(\sigma \partial)(\bar{\sigma} \partial) = (\bar{\sigma} \partial)(\sigma \partial) = -\Box 1_{2 \times 2} \]  

(2.32)

where we briefly write

\[ \sigma \partial = \sigma^l \partial_l, \bar{\sigma} \partial = \bar{\sigma}^l \partial_l \]  

(2.33)

The relation (2.32) as well as the relation (2.8) follows from

\[ \sigma^l \bar{\sigma}^m + \sigma^m \bar{\sigma}^l = -2 \eta^{lm} 1_{2 \times 2} \]

where \(1_{2 \times 2}\) is the unit \(2 \times 2\) matrix. Written in the spinor notation it reads

\[ \sigma^l_\alpha \bar{\sigma}^m \delta_{\alpha \beta} + \sigma^m_\alpha \bar{\sigma}^l \delta_{\alpha \beta} = -2 \eta^{lm} \delta^\beta_\alpha \]

As expected \(D^2 D^2 X\) is chiral, \(\bar{D}^2 D^2 X\) is antichiral and \(TX\) is transversal. The transversality (2.13) of \(TX\) was put in evidence in (2.31).

In order to construct inner products in integral form we also need the conjugates \(\bar{X}, \bar{D}^2 X, \bar{D}^2 \bar{X}\), etc. of \(X, D^2 X, D^2 \bar{X}\) etc. where the conjugation includes besides the usual complex conjugation the Grassman conjugation too. We have

\[ \bar{X} = \bar{X}(x, \theta, \bar{\theta}) = \\
= \bar{f}(x) + \theta \chi(x) + \bar{\theta} \bar{\varphi}(x) + \theta^2 \bar{n}(x) + \bar{\theta}^2 \bar{m}(x) + \\
+ \theta \sigma^l \bar{\theta} \bar{v}_l(x) + \theta^2 \bar{\theta} \bar{\psi}(x) + \bar{\theta}^2 \theta \lambda(x) + \theta^2 \bar{\theta}^2 \bar{d}(x) \]  

(2.34)

where \(\bar{f}, \chi, \bar{\varphi}\), etc. are the complex conjugate functions to \(f, \bar{\chi}, \varphi\), etc.. Note that if \(X\) is chiral then \(\bar{X}\) is antichiral and viceversa. If \(X\) is transversal than \(\bar{X}\) is transversal. Although not absolutely necessary we record here other expressions too which can be used to give alternative proofs of results to follow by making use of partial integration in superspace. They are (use \((\chi \sigma^l \bar{\psi})^* = \psi \sigma^l \bar{\chi}\) where * is the complex conjugation which could have been written as bar too):
\[ \overline{D^2X} = D^2X = -4\bar{n} + \bar{\theta}(-4\bar{\psi} - 2i\sigma^l\partial_l\chi) + \bar{\theta}^2(-4\bar{d} + 2i\partial_l\bar{v}^l - \Box\bar{f}) + \\
+ \theta\sigma^l\bar{\theta}(4i\partial_l\bar{n}) + \theta^2\bar{\theta}(-2i\sigma^l\partial_l\bar{\psi} - \Box\chi) + \theta^2\bar{\theta}^2(-\Box\bar{n}) \quad (2.35) \]
\[ \overline{D^2\bar{X}} = D^2\bar{X} = -4\bar{m} + \theta(-4\lambda - 2i\sigma^l\partial_l\bar{\varphi}) + \theta^2(-4\bar{d} - 2i\partial_l\bar{v}^l - \Box\bar{f}) + \\
+ \theta\sigma^l\bar{\theta}(-4i\partial_l\bar{m}) + \theta^2\bar{\theta}(-2i\sigma^l\partial_l\lambda - \Box\bar{\varphi}) + \theta^2\bar{\theta}^2(-\Box\bar{m}) \quad (2.36) \]

or in a more suggestive way as chiral and antichiral function respectively
\[ \overline{D^2X} = -4\bar{n} + \bar{\theta}(-4\bar{\psi} - 2i\sigma^l\partial_l\chi) + \bar{\theta}^2(-4\bar{d} + 2i\partial_l\bar{v}^l - \Box\bar{f}) + \theta\sigma^l\bar{\theta}(4i\partial_l\bar{n}) + \\
+ \theta^2\bar{\theta}(\frac{i}{2}\sigma^l\partial_l)(-4\bar{\psi} - 2i\sigma^l\partial_l\chi) + \theta^2\bar{\theta}^2(-\Box\bar{n}) \quad (2.37) \]
\[ \overline{D^2\bar{X}} = -4\bar{m} + \theta(-4\lambda - 2i\sigma^l\partial_l\bar{\varphi}) + \theta^2(-4\bar{d} - 2i\partial_l\bar{v}^l - \Box\bar{f}) + \theta\sigma^l\bar{\theta}(4i\partial_l\bar{m}) + \\
+ \theta^2\bar{\theta}(\frac{i}{2}\sigma^l\partial_l)(-4\lambda - 2i\sigma^l\partial_l\bar{\varphi}) + \theta^2\bar{\theta}^2(-\Box\bar{m}) \quad (2.38) \]

Further
\[ \overline{cX} = \overline{D^2D^2X} = 16\bar{d} + 8i\partial_l\bar{v}^l + 4\Box\bar{f} + \bar{\theta}(8\Box\bar{\varphi} + 16i\sigma^l\partial_l\lambda) + \\
+ \bar{\theta}^2(16\Box\bar{n}) + \theta\sigma^l\bar{\theta}(16i\partial_l\bar{d} + 8\partial_l\partial^m\bar{v}_m - 4i\partial_l\Box\bar{f}) + \\
\theta^2\bar{\theta}(8\bar{\psi} + 4i\sigma^l\partial_l\Box\bar{\varphi}) + \theta^2\bar{\theta}^2(4\Box\bar{d} + 2i\partial_l\Box\bar{v}^l + \Box^2\bar{f}) \quad (2.39) \]
\[ \overline{aX} = \overline{D^2D^2\bar{X}} = 16\bar{d} - 8i\partial_l\bar{v}^l + 4\Box\bar{f} + \bar{\theta}(8\Box\chi + 16i\sigma^l\partial_l\bar{\psi}) + \\
+ \bar{\theta}^2(16\Box\bar{m}) + \theta\sigma^l\bar{\theta}(16i\partial_l\bar{d} + 8\partial_l\partial^m\bar{v}_m + 4i\partial_l\Box\bar{f}) + \\
\theta^2\bar{\theta}(8\bar{\psi} + 4i\sigma^l\partial_l\Box\chi) + \theta^2\bar{\theta}^2(4\Box\bar{d} - 2i\partial_l\Box\bar{v}^l + \Box^2\bar{f}) \quad (2.40) \]

and finally
\[ \overline{T\bar{X}} = T\bar{X} = T\bar{X} = 16\bar{d} - 4\Box\bar{f} + \theta(-4\Box\chi + 8i\sigma^l\partial_l\bar{\psi}) + \bar{\theta}(-4\Box\bar{\varphi} + \\
+ 8i\sigma^l\partial_l\lambda) + \theta\sigma^l\bar{\theta}(8\partial_l\partial^m\bar{v}_m - 8\Box\bar{v}^l) + \theta^2\bar{\theta}(-4\Box\bar{\psi} + 2i\sigma^l\partial_l\Box\chi) + \\
+ \theta^2\bar{\theta}(-4\Box\bar{\chi} + 2i\sigma^l\partial_l\Box\varphi) + \theta^2\bar{\theta}^2(-4\Box\bar{d} + \Box^2\bar{f}) \quad (2.41) \]

or
\[ \overline{T\bar{X}} = 16\bar{d} - 4\Box\bar{f} + \theta(-4\Box\chi + 8i\sigma^l\partial_l\bar{\psi}) + \bar{\theta}(-4\Box\bar{\varphi} + \\
+ 8i\sigma^l\partial_l\lambda) + \theta\sigma^l\bar{\theta}(8\partial_l\partial^m\bar{v}_m - 8\Box\bar{v}_l) + \theta^2\bar{\theta}(-\frac{i}{2}\sigma^l\partial_l)(-4\Box\chi + 8i\sigma^l\partial_l\bar{\psi}) + \\
+ \theta^2\bar{\theta}(-\frac{i}{2}\sigma^l\partial_l)(-4\Box\bar{\varphi} + 8i\sigma^l\partial_l\lambda) + \theta^2\bar{\theta}^2(-4\Box\bar{d} + \Box^2\bar{f}) \quad (2.42) \]
We start to look for (invariant) supersymmetric kernel functions \( K(z_1, z_2) = K(x_1, \theta_1, \bar{\theta}_1; x_2, \theta_2, \bar{\theta}_2) \) which formally induce inner products on supersymmetric functions by

\[
(X_1, X_2) = \int d^8 z_1 d^8 z_2 \bar{X}_1(z_1) K(z_1, z_2) X_2(z_2) = \int \bar{X}_1 K X_2
\]

(2.43)

where the bar on the r.h.s means conjugation (including Grassmann), \( z_i = (x_i, \theta_i, \bar{\theta}_i) \) and \( d^8 z = d^4 x d^2 \theta d^2 \bar{\theta} \). In the r.h.s of the last equality we have used a sloppy but concise notation of the integral under study. The simplest choice for \( K \) would be the identity kernel \( K(z_1, z_2) = k(z_1 - z_2) = \delta^2(\theta_1 - \theta_2) \delta^2(\bar{\theta}_1 - \bar{\theta}_2) \delta^4(x_1 - x_2) \) but it turns out that this choice is not sound. We settle down soon for more appropriate choices. Formally we have if \( \bar{K} = K \):

\[
(X_1, X_2) = (\bar{X}_2, \bar{X}_1)
\]

where the bars include Grassmann conjugation. The action of \( K \) on \( X \) is defined formally as

\[
Y_K(z_1) = (KX)(z_1) = \int d^8 z_2 K(z_1, z_2) X(z_2)
\]

Note that the general dependence of \( K \) on \( z_1, z_2 \) we admit is not necessarily through the difference \( z_1 - z_2 \). We assume that the coefficient functions of the supersymmetric functions involved belong to the Schwartz function space \( S \) of infinitely differentiable rapidly decreasing functions.

Now we are starting to induce positivity of the inner product by a proper choice of the kernel \( K \). By positivity in this section we mean non-negativity. The first candidate is

\[
K(z_1, z_2) = K_0(z_1 - z_2) = \delta^2(\theta_1 - \theta_2) \delta^2(\bar{\theta}_1 - \bar{\theta}_2) D^+(x_1 - x_2)
\]

(2.44)

where \( \delta^2(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2, \delta^2(\bar{\theta}_1 - \bar{\theta}_2) = (\bar{\theta}_1 - \bar{\theta}_2)^2 \) are the supersymmetric \( \delta \)-functions and \( D^+(x) \) is the Fourier transform of a positive measure \( d\rho(p) \) supported in the backward light cone \( \bar{V}^- \) (not necessarily Lorentz invariant):

\[
D^+(x) = \frac{1}{(2\pi)^2} \int e^{ipx} d\rho(p)
\]

(2.45)

which is of polynomial growth i.e. there is an integer \( n \) such that

\[
\int \frac{d\rho(p)}{(1 + |p|^2)^n} < \infty
\]

(2.46)

where \( |p| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2} \). Here \( px \) is the "most positive" Minkowski scalar product. Usually (for instance in quantum field theory) the Minkowski
scalar product is "most negative" and as a consequence the measure \( d\rho(p) \) is concentrated in the forward light cone \( \bar{V}^+ \). For the time being we do not necessarily assume Lorentz invariance of the measure. The special kernel \( K_0 \) depends only on the difference \( z_1 - z_2 \). In order to understand the idea behind this choice note first that for \( f \) and \( g \) functions of \( x \) in \( S \) the integral

\[
(f, g) = \int d^4x d^4y \bar{f}(x) D^+(x - y) g(y)
\]

(2.47)

where \( D^+(x) \) is given by (2.45) induces a positive definite scalar product (certainly in order to exclude zero-vectors we have to require the support of \( f \) and \( g \) in momentum space in \( \bar{V}^- \) to be concentrated on the support of \( d\rho(p) \) which is equivalent with factoring out zero-vectors and completion in (2.47)). Indeed the right hand side of (2.47) equals in momentum space \( \int \tilde{f}(p) \tilde{g}(p) d\rho(p) \) where \( \tilde{f} \) is the Fourier transform of \( f \) given by \( f(x) = \frac{1}{(2\pi)^2} \int e^{ipx} \tilde{f}(p) dp \). Note further that positivity is preserved if we multiply the measure \( d\rho(p) \) by \(-p^2\) or for the case of two-spinor functions \( f \) and \( g \) by \( \sigma p \) or \( \bar{\sigma} p \). In configuration space it means that we can accommodate the operators \( \Box \) and \(-i\sigma \partial, -i\bar{\sigma} \partial \) in the kernel of the integral without spelling out the positivity (we have as usually \( \frac{i}{\gamma} \partial = p \)).

It is clear that in spite of the positivity properties induced by the kernel \( D^+ \) the scalar product in (2.43) with kernel (2.44) cannot be positive definite in superspace for general coefficient functions (for \( X_1 = X_2 = X \)) because the coefficient functions are mixed up in the process of Grassmann integration in an uncontrolled way. Fortunately there are other kernels deduced from \( K_0 \) which do the job. In order to keep the technicalities aside for the moment let us assume that the measure \( d\rho(p) \), besides being of polynomial growth, satisfies the stronger condition

\[
| \int \frac{1}{p^2 (1 + |p|^2)^n} d\rho(p) | < \infty
\]

(2.48)

with the integer \( n \) appearing in (2.46).

Certainly the condition above is relatively strong; it allows measures like \( d\rho(p) = \theta(-p_0) \delta(p^2 + m^2) dp \) with \( m > 0 \) but excludes the case \( m = 0 \) (in physics the massive and massless case respectively). The case \( m = 0 \) will be studied at the end of this section.

We arrived at the level of explaining our message. For this we introduce besides \( K_0(z_1 - z_2) \) three other kernels as follows

\[
K_c(z_1, z_2) = P_c K_0(z_1 - z_2)
\]

(2.49)

\[
K_a(z_1, z_2) = P_a K_0(z_1 - z_2)
\]

(2.50)

\[
K_T(z_1, z_2) = -P_T K_0(z_1 - z_2)
\]

(2.51)
with actions

\[ Y_i(z_1) = (K_iX)(z_1) = \int d^8z_2 K_i(z_1, z_2) X(z_2) \]

In (2.49)-(2.51) the operators \( P_i \) are understood to act on the first variable \( z_1 \) (see also (2.52)-(2.57) to follow). The condition (2.48) makes the formal definition \( Y = \int KX \) (with \( K \) replaced by one of the derived kernels \( K_i, i = c, a, T \) as written above) safe from a rigorous point of view because it takes care of the d’alembertian in the denominators introduced by the formal projections \( P_i, i = c, a, T \). We will remove this condition soon by slightly restricting the set of supersymmetric (test) functions but let us keep it for the time being. Note that the projections destroy the translation invariance in the Grassmann variables but not in the space coordinates. Because \( P_i, i = c, a, T \) contain Grassmann variables and derivatives thereof we have to specify on which variables they act in \( K_0(z_1 - z_2) \). By convention let us define by \( D^2_1K_0(z_1 - z_2), \bar{D}^2_1K_0(z_1 - z_2), T_1K_0(z_1 - z_2) \) the action of the operators \( D^2 \) and \( D^2, T \) on \( K_0(z_1 - z_2) \) on the first variable respectively and by \( D^2_2K_0(z_1 - z_2), \bar{D}^2_2K_0(z_1 - z_2), T_2K_0(z_1 - z_2) \) the action of these operators on the second variable. If indices are not specified, we understand the action on the first variable.

It can be proven (for similar computations see for instance [2]) that

\[
\begin{align*}
D^2_1K_0(z_1 - z_2) &= D^2_2K_0(z_1 - z_2) \\
\bar{D}^2_1K_0(z_1 - z_2) &= \bar{D}^2_2K_0(z_1 - z_2) \\
D^2_1\bar{D}^2_1K_0(z_1 - z_2) &= \bar{D}^2_2D^2_2K_0(z_1 - z_2) \\
\bar{D}^2_1D^2_1K_0(z_1 - z_2) &= D^2_2\bar{D}^2_2K_0(z_1 - z_2) \\
T_1K_0(z_1 - z_2) &= T_2K_0(z_1 - z_2) \\
\bar{T}_1K_0(z_1 - z_2) &= \bar{T}_2K_0(z_1 - z_2)
\end{align*}
\]

where in fact the relations (2.56), (2.57) coincide because \( \bar{T} = T \). We have used

\[ [D^2_1, D^2_2] = 0, \quad [\bar{D}^2_1, \bar{D}^2_2] = 0 \]

Note the minus sign in front of \( P_T \) in (2.51) which will be of utmost importance for us. Because of it the kernels \( K_i, i = c, a, T \) do not sum up to \( K \). This is at the heart of the matter being at the same time not too much embarrassing. We will prove by direct computation that the kernels \( K_i, i = c, a, T \) produce, each for itself, a positive definite scalar product in
the space of supersymmetric functions (at this stage we prove only nonnegativity; the problem of zero vectors is pushed to Section 3). Whereas this assertion is to be expected for $K_i$ for $i = c, a$, the minus sign in $K_T$ comes as a surprise. It will be the reason for the natural Krein (more precisely Hilbert-Krein) structure of the $N = 1$ supersymmetry which we are going to uncover (first under the restrictive condition (2.48) on the measure).

Denoting by $(\cdot, \cdot)$, $i = 0, c, a, T$ the inner products induced by the kernels $K_i$, $i = 0, c, a, T$:

$$(X_1, X_2)_i = \int \bar{X}_1 K_i X_2$$

(2.58)

we could compute them by brute force using the expressions (2.28)-(2.30) but it is not easy to get the positive definiteness of these inner products in the cases $i = c, a, T$. Alternatively we will proceed as follows. Let us start with the cases $i = c, a$. We use (2.52)-(2.57) and integrate by parts in superspace (see for instance [2]). This gives (in the sloppy integral notation) by partial integration in superspace

$$(X_1, X_2)_c = \int \bar{X}_1 K_c X_2 = \int \bar{X}_1 P_c K_0 X_2 = (\frac{1}{16} D^2 X_1, D^2 X_2)_0$$

(2.59)

$$(X_1, X_2)_a = \int \bar{X}_1 K_a X_2 = \int \bar{X}_1 P_a K_0 X_2 = (\frac{1}{16} D^2 X_1, D^2 X_2)_0$$

(2.60)

where we have also used $\bar{D}^2 X = D^2 \bar{X}$ etc. The last equality follows from obvious ones supplemented by $i\sigma^l \partial_l \bar{\varphi} = (i\sigma^l \partial_l \varphi)^* = -i\partial_l \varphi \sigma^l = i\sigma^l \partial_l \varphi$, etc.

In (2.59) $D^2$ from $P_c$ was moved to $\bar{X}_1$, the remaining $D^2$ (acting on $K_0$ on the first variable) was transferred by (2.52) to the second variable on $K_0$, and then moved on $X_2$ such that finally we get the last expression. The same procedure was applied for (2.60). The d’alembertian in the denominator can be absorbed in Fourier space in the measure $d\rho(p)$ which is supposed to satisfy condition (2.48). Using the $\delta$-function property in the Grassmann variables in $K_0$ we see that for instance in the antichiral case we get for $X_1 = X_2 = X$

$$(X, X)_a = \int \bar{X} K_a X = \int d^4x_1 d^4x_2 (\bar{D}^2 X)(D^2 X)(x_1, x_2) \frac{1}{16} D^+(x_1 - x_2)$$

(2.61)

where $[\cdot, \cdot]$, as before, gives the coefficient of the highest power in the Grassmann variables.
Note that $\bar{D}^2X$ is chiral such that for $[\bar{D}^2X](\bar{D}^2X)(x_1, x_2)$ we can apply (2.15). We integrate by parts in the usual coordinates using the faster than polynomial decrease of the involved functions and their derivatives and obtain in momentum space

$$\int \bar{X}K_aX =$$

$$= \int [\bar{f}_c(p)f_c(p) + \bar{\varphi}_c(p)(\sigma p)\bar{\varphi}_c + \bar{m}_c(p)m_c(p)](\frac{1}{p^2})dp(p) \quad (2.62)$$

where $f_c, \varphi_c, m_c$ are the coefficients of the chiral $\bar{D}^2X$ given by (2.21). We have used the translation invariance of $D_a^+(x)$ which enables us to read up the result in momentum space from the computation conducting to (2.18) which was performed in coordinate space (this is an unusual way to keep track of the $\delta$-function in momentum space generated by translation invariance which quickly gives the result).

From (2.62) we obtain by inspection the positivity of $\int \bar{X}K_aX = (X, P_aX)_0$. We use the positivity of $-p^2, \sigma p$ and $\bar{\sigma}p$. The same argument works for the chiral integral $\int \bar{X}K_aX = (X, P_cX)_0$.

Now we go over to the transversal integral $\int \bar{X}K_TX_2$. Here we cannot split the kernel in a useful way as we did in the chiral and antichiral cases but the following similar procedure can be applied.

We write using $P_T^2 = P_T$, the relation (2.56) and integration by parts in superspace

$$-(X_1, X_2)_T = -\int \bar{X}_1K_TX_2 = \int \bar{X}_1P_TK_0X_2 = \int \bar{X}_1P_T^2K_0X_2 =$$

$$= (P_TX_1, P_TX_2)_0 = \frac{1}{64}(\Box TX_1, \Box TX_2)_0 \quad (2.63)$$

Here, as in the antichiral case above, one of $P_T$ in $P_T^2$ acting on the first variable was moved to $\bar{X}_1$ and the second one was pushed through $K_0$ (modulo changing the variable) to $X_2$. In (2.63) we take $X_1 = X_2 = X$, integrate the $\theta, \bar{\theta}$-variables and use for $[(TX)(TX)](x_1, x_2)$ the expression (2.17). We can use now (2.20) by analogy in momentum space as above too. Note that by integration by parts we have enough derivatives in the numerator in order to cancel one of the two inverse d’alembertians in (2.63). By (2.48) the second d’alembertian is under control and the computation is safe. We propose to the reader to go this way in order to explicitely convince himself that the integral $-\int \bar{X}_1K_TX_2 = \int \bar{X}_1P_TK_0X_2$ (in contradistinction to the chiral/antichiral case) is negative for $X_1 = X_2$! A hint is necessary. Indeed
the only contribution which has to be looked up beyond the chiral/antichiral case is the vector contribution stemming from \( v\)-coefficients of the transversal supersymmetric function and this produces a negative contribution. In fact the negativity of the transversal contribution rests on the following property in momentum space. Let \( v(p) = (v_l(p)) \) be a vector function (not necessary real) such that \( p_l v^l(p) = 0 \). It means that the vector with components \( v_l(p) \) is orthogonal (in the euclidean meaning) to the (real) vector \( p_l \). But the momentum vector \( p \) is confined to the light cone (it must be in the support of \( d\rho(p) \)) such that the vector function \( v(p) \) must satisfy \( \bar{v}^l(p) v_l(p) \geq 0 \). Moreover if \( d\rho \) intersects the light cone \( p^2 = 0 \) the equality may be realized. We repeat here an old argument which was recognized in the frame of the rigorous version of the Gupta-Bleuler quantization in physics [4, 5]. But there is a new aspect: whereas in [4, 5] the free divergence condition \( p_l v_l(p) = 0 \) was introduced ad hoc in order to force the Gupta-Bleuler definite metric, it comes in for free here as a consequence of supersymmetry.

The last part of this section is dedicated to the more delicate question of abolishing the unpleasant restrictive condition (2.48) such that we can include in our analysis, from a physical point of view, the interesting "massless" case. From the consideration above it is clear that this is generally not possible. More precisely, if we want to retain the interpretation of supersymmetric quantum fields as operator-valued (super)distributions as this is the case for the usual quantum fields [6] (an interpretation which we subscribe to) we are forced to restrict the set of allowed test functions such that the d’alambertian in the denominator is annihilated. Restricting the set of test functions in quantum field theory is not a problem and is not at all new; it appeared even long time ago in the rigorous discussion of the Gupta-Bleuler quantization [4, 5].

Suppose that the coefficient functions in (2.1) satisfy the following restrictive conditions:

\[
\begin{align*}
d(x) &= \Box D(x) \quad (2.64) \\
\bar{\lambda}(x) &= i \sigma^l \partial_l \Lambda(x) \quad (2.65) \\
\psi(x) &= i \sigma^l \partial_l \bar{\Psi}(x) \quad (2.66) \\
v(x) &= \text{grad}_x \rho(x) + \omega(x), \text{div}_x \omega(x) = 0 \quad (2.67)
\end{align*}
\]

where \( D(x), \Lambda(x), \Psi(x), \rho(x), \omega(x) \) are arbitrary functions (in \( S \)). In the last equation \( \text{grad}_x \rho = (\partial_x \rho), \text{div}_x \omega = \partial_x \omega \).

The functions \( \rho(x), \omega(x) \) can be constructed as follows: let \( \rho \) be a solution of \( \Box \rho = \text{div} v \) and let \( \omega = v - \text{grad} \rho \). Then \( v = \text{grad} \rho + \omega \) with \( \text{div} \omega = \text{div} v - \Box \rho = 0 \).
We claim that under these conditions the results above concerning the positivity in the chiral/antichiral sectors and negativity in the transversal sector remain valid without the restrictive condition (2.48) on the measure $d\rho$. The conditions (2.64) to (2.67) produce the missing d’alembertian in

$$\int \bar{X}_1 K_i X_2, \ i = c, a, T$$

(2.68)
such that the condition (2.48) becomes superfluous. Indeed let us consider for example the chiral case (with the antichiral kernel $K_a$). From (2.21) we see that the following expressions appear in the integral (2.62):

$$(-4\bar{n})\Box(-4n)
$$

$$(-4\bar{\psi} - 2i\bar{\sigma}^l\partial_l\chi)(i\sigma^m\partial_n(-4\psi - 2i\sigma^m\partial_m\bar{\chi}))$$

$$(-4\bar{d} + 2i\partial_l\bar{v}^l - \Box\bar{f})(-4\bar{d} - 2i\partial_m v^m - \Box f)$$

It is clear that under the conditions (2.64) to (2.67) the missing d’alembertian in the integral (2.60) can be factorized such that the condition (2.48) on the measure $d\rho$ is no longer needed. The result remains positive. Similar arguments work for the chiral and transversal case. In the transversal case the interference between $\rho$ and $\omega$ in $\bar{v}^l v_l$ disappears (because $\text{div} \omega = 0$) and one can use (besides the positivity of the d’alembertian) again the Gupta-Bleuler argument with $\text{div} \omega = 0$.

The problem of possible zero-vectors for the non-negative inner products induced by the kernels $K_i, \ i = c, a, T$ will be discussed in the next section. For the moment note that there are plenty of them in each sector from the adjacent ones. The ”massless” case in which the measure is $d\rho(p) = \theta(-p_0)\delta^2(p^2)$ i.e. it is concentrated on the light cone deserves special attention. By putting together the non-negative inner products $(\cdot, \cdot), \ i = c, a, T$ all zero vectors simply disappear (see Section 3). We will construct the natural unique supersymmetric positive definite scalar product and obtain in the next section our rigorous Hilbert-Krein decomposition of the set of supersymmetric functions where the conditions (2.64) to (2.67) will play a central role.

## 3 Hilbert-Krein Superspace

In this section we present, on the basis of the results of Section 2, the generic Krein structure of supersymmetries. Let $V$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and $\omega$ an operator on $V$ with $\omega^2 = 1$ (do not confuse this $\omega$ with the one in (2.67)). If $(\phi, \psi) = \langle \phi, \omega \psi \rangle; \phi, \psi \in V$ is a (positive
definite) scalar product on $V$ than we say that $V$ has a Krein structure. By completing in the scalar product $(.,.)$ we obtain an associated Hilbert space structure (if $(.,.)$ has zero vectors we have in addition to factorize them before completing). We obtain what we call a Hilbert-Krein space (or Hilbert-Krein structure). Hilbert-Krein structures naturally appear in gauge theories (including the well understood case of electrodynamics; see for instance the book [7]).

Suppose the condition (2.48) on the measure $d\rho(p)$ is satisfied ans, as always, $X$ and $Y$ are concentrated on its support. We decompose $X = X_1 + X_2 + X_3$ where $X_1 = X_c = P_cX, X_2 = X_a = P_aX, X_3 = X_T = P_TX$. Then the simplest supersymmetric Hilbert-Krein structure which emerges from the considerations of the preceding section is given by

$$<X,Y> = \int d^8z_1 d^8z_2 X^T(z_1)K_0(z_1 - z_2)Y(z_2)$$

(3.1)

in the notations $X^T = (X_1, X_2, X_3), Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_T \end{pmatrix}, K_0(z) = K_0(z)I_3$. Here $I_3$ is the 3x3 identity matrix and $X^T$ is the transpose of $X$.

Now let

$$(X,Y) = <X,\omega Y>$$

(3.2)

with

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Certainly $(.,.)$ is positive definite on the basis of results obtained in Section 2. It is clear that although each inner product $(.,.)_i$ has zero vectors this will be no longer the case for (3.2).

Although very general the scalar product (3.4) is obstructed by the (from the point of view of applications) unnatural restriction (2.48) of the measure $d\rho$. It holds for the massive but fails for the massless case. Now the restrictions (2.62)-(2.67) on (test) supersymmetric functions come into play. Indeed, under these conditions we can always decompose a supersymmetric function into its chiral, antichiral and transversal part and write down the indefinite as well as the definite scalar products (3.1) and (3.2). Note that in the massless case there is an overlap between chiral/antichiral and transversal sectors which consists of zero vectors and has to be factorized. From (2.11)-(2.13) follows that a function $X$ belongs to this overlap if
$$X(z) = f(x) + \theta \varphi(x) + \bar{\theta} \bar{\chi}(x) \pm i \theta \sigma_l \bar{\partial}_l f(x)$$

with

$$\partial_l \varphi^l = \sigma^l \partial_l \bar{\chi} = 0, \quad \Box f = 0$$

The restrictive condition on the measure was transferred to a restrictive conditions on (test) functions, a procedure which is common for rigorous quantum gauge fields (see for instance [7]). In Section 2 we have seen that the content of the restrictive conditions on test functions in supersymmetry might be of less extent as compared to similar conditions in the usual case (remember the zero divergence condition which comes for free). The Hilbert-Krein structure on supersymmetric functions subjected or not to the conditions (2.64)-(2.67) is the main result of this paper.

We believe that it justified to call standard Hilbert-Krein supersymmetric space the space of supersymmetric functions with indefinite and (positive) definite inner products given as above by

$$<X,Y> = \int \bar{X}^T K_0 X$$

$$(X,Y) = <X,\omega Y> \quad (3.3)$$

It is exactly the supersymmetric analog of the relativistic Hilbert space used in quantum field theory in order to produce the Fock space of the free theory [6]. As a first application we mention here that the free chiral/antichiral supersymmetric quantum field theory (i.e. the quantum field formally generated by the free part of the Wess-Zumino Lagrangian) is characterized by the positive definite (at this stage only non-negative) two point function

$$\left( \frac{1}{16} D^2 D^2 \quad \frac{m}{16} D^2 \right) K_0$$

where $d\rho(p) = \theta(-p_0)\delta(p^2 + m^2)dp$ with $m > 0$. The correspondence to the two-point functions of the chiral $\Phi$ and antichiral $\bar{\Phi}$-quantum fields is indicated below

$$\left( \begin{array}{cc} \Phi \bar{\Phi} & \Phi \Phi \\ \bar{\Phi} \bar{\Phi} & \bar{\Phi} \Phi \end{array} \right) \sim \left( \frac{1}{16} D^2 D^2 \quad \frac{m}{16} D^2 \right) K_0 \quad (3.5)$$

The proof of non-negativity of (3.4) is by computation. The factorisation of the zero-vectors in (3.4) can be made explicit by imposing the equations of
motion $\bar{D}^2 \Phi = 4m\Phi, D^2 \Phi = 4m\Phi$ on the test functions [8].

The supersymmetric vacuum coincide with the function one and the supersymmetric Fock space is symmetric (note that following our reasoning all supersymmetric Fock spaces must be symmetric; we expect antisymmetric Fock spaces for ghost fields).

A non-interacting quantum (free) system consisting of a chiral/antichiral and a (massive) vector part is characterized by the positive definite operator in the standard Hilbert-Krein space (remember $T = -8\Box P_T = D^\alpha \bar{D}^2 D_\alpha = D_\dot{\alpha} D^2 D^\dot{\alpha}$)

$$
\begin{pmatrix}
\frac{1}{16} \bar{D}^2 D^2 & \frac{m}{4} \bar{D}^2 & 0 \\
\frac{m}{4} D^2 & \frac{1}{16} D^2 \bar{D}^2 & 0 \\
0 & 0 & \frac{1}{8} T
\end{pmatrix}
K_0
$$

(3.6)

The massless case is different; one has to take into account the conditions (2.64)-(2.67) [8]. Other applications include a supersymmetric Källen-Lehmann representation [8].

Before ending let us make two remarks. The first concerns the perspective of the present work. We succeeded to uncover the inherent Hilbert-Krein structure of the $N = 1$ superspace. It means that the formal decomposition of supersymmetric functions into chiral, antichiral and transversal components, which was common tool from the first days of superspace, was turned here into what we call the Hilbert-Krein structure of the $N = 1$ superspace or the standard supersymmetric Hilbert-Krein space. It shows that positivity (and as such unitarity) requires the substraction of the transversal part instead of its addition as this might be suggested by the above mentioned formal decomposition. Problems with the d’alembertian in the denominator of the projections $P_i, \ i = c, a, T$ have been discussed. The natural way to avoid singularities is to impose some restrictions on the (test) functions. There are other applications in sight to which we hope to come to (for some first modest steps see [8]).

The second remark is of technical nature. We worked in the frame of the van der Waerden calculus using Weyls spinors. This is very rewarding from the point of view of computations in supersymmetry but is not totally satisfactory from the rigorous point of view. Indeed the components of the Weyl spinors as coefficient functions for our supersymmetric (test) functions are supposed to anticommute and this is unpleasant when tracing back the supersymmetric integrals to usual $L^2$-integrals. Of course this is not a problem. A reformulation of the results using anticommuting Grassmann variables but
commuting fermionic components is possible. The net results remain un-
changed as it should be.

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