Exact Calculation of the Capacitance and the Electrostatic Potential Energy for a Nonlinear Parallel-Plate Capacitor in a Two-Parameter Modification of Born-Infeld Electrodynamics

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Abstract

The nonlinear capacitors are important devices in modern technologies and applied physics. The aim of this paper is to calculate exactly the capacitance and the electrostatic potential energy of a nonlinear parallel-plate capacitor by using a two-parameter modification of Born-Infeld electrodynamics. Our calculations show that the capacitance and the electrostatic potential energy of a nonlinear parallel-plate capacitor in modified Born-Infeld theory have the weak field expansions

\[ C = \varepsilon_0 A d + O(q^2) \]

and

\[ U = -q^2 \frac{\varepsilon_0 A d}{2} + O(q^4), \]

where \( q \) is the amount of electric charge on each plate of the capacitor. It is demonstrated that the results of this paper are in agreement with the results of Maxwell electrodynamics for weak electric fields. Numerical evaluations show that the nonlinear electrodynamical effects in modified Born-Infeld theory are negligible in the weak field regime.

Keywords: Classical field theories; Applied classical electromagnetism; Other special classical field theories; Nonlinear or nonlocal theories and models; Nonlinear Capacitor

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1 Introduction

The nonlinear capacitors have found a wide range of applications in circuit theory in electrical engineering and different branches of applied physics [1-7]. In a nonlinear capacitor, in contrast with the ordinary capacitors, the capacitance is a function of voltage, i.e., \( C = f(\Delta \phi) \) where \( \Delta \phi \) is the potential difference between the plates of the capacitor and the function \( f(\Delta \phi) \) can be determined empirically or from a fundamental electromagnetic theory like Maxwell electrodynamics or Born-Infeld theory [1,2,8-10]. In Ref. [2], it has been shown that for a nonlinear capacitor which its capacitance depends linearly on the voltage, the usual relation \( U = \frac{1}{2} C(\Delta \phi)^2 \) is not satisfied. The authors of Ref. [6] have shown that the electrostatic potential energy of a nonlinear capacitor can be expanded as follows:

\[ U = \alpha q^2 + \beta q^4 + \gamma q^6 + ..., \]

where \( \alpha, \beta, \gamma, \) and ... are material dependent constants. Today we know that the interaction between the charged bodies can be described by Maxwell equations classically [8]. On the other hand, Maxwell electrodynamics suffers

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from serious difficulties such as infinite self-energy of the point charges [8]. In 1934 Born and Infeld introduced the following Lagrangian density [9]:

\[
L_{BI} = \epsilon_0 \beta^2 \left\{ 1 - \sqrt{1 + \frac{c^2}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{c^4}{16\beta^4} (F_{\mu\nu} \star F^{\mu\nu})^2} \right\},
\]

(2)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the Faraday tensor, \( A^\mu = (\frac{1}{c} \phi, A) \) is the gauge potential, \( \star F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \) is the dual field tensor, and \( \beta \) is the maximum value of the electric field in Born-Infeld theory. In string theory the dynamics of electromagnetic fields on D-branes can be represented by a Born-Infeld type theory [10]. The solutions of Born-Infeld equations for an infinite charged line and an infinitely long cylinder have been obtained in Ref. [11]. In 1930s, Heisenberg, Euler, and Kockel studied the scattering of light by light according to Dirac’s hole theory [12-14]. They showed that Maxwell electrodynamics should be corrected by adding nonlinear terms due to the quantum electrodynamical effects [12-14]. It must be emphasized that, for weak electromagnetic fields, the Lagrangian density and the energy density of nonlinear electrodynamics have the following explicit expressions:

\[
L = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} F^i G^j,
\]

(3)

\[
u = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \left( 2\epsilon_0 i F^{i-1} G^j E^2 + (j - 1) F^i G^j \right),
\]

(4)

where \( c_{i,j} \) are field-independent parameters, \( F := (\epsilon_0 E^2 - \frac{B^2}{\mu_0}) \), and \( G := \sqrt{\frac{\epsilon_0}{\mu_0}} (E, B) \) [15,16]. Note that in the weak field regime (2) is a particular example of (3). The effect of nonlinear corrections on the electric field between the plates of a parallel-plate capacitor has been studied in the framework of Heisenberg-Euler-Kockel electrostatics [17]. In a recent paper, the capacitance and the electrostatic potential energy for a parallel-plate and spherical capacitors have been computed in ordinary Born-Infeld theory [18]. Iacopini and Zavattini suggested and developed a \((p, \tau)\)-two-parameter modification of Born-Infeld electrodynamics, in which the electrostatic self-energy of a point charge becomes a finite value for \( p < 1 \) [19]. The most important aim of this paper is to calculate the capacitance of a nonlinear parallel-plate capacitor from the viewpoint of Iacopini-Zavattini modification of Born-Infeld electrodynamics [19]. Another aim is to show that the nonlinear phenomena in electrodynamics are negligible for weak electric fields. This paper is organized as follows. In Section 2, the formulation of modified Born-Infeld electrodynamics coupled to an external current is presented. In Section 3, the symmetric energy-momentum tensor for modified Born-Infeld electrodynamics is constructed from the canonical energy-momentum tensor by using Belinfante’s procedure. In Section 4, we show that the capacitance and the electrostatic potential energy for a nonlinear parallel-plate capacitor in modified Iacopini-Zavattini electrodynamics can be calculated exactly when the parameter \( p \) takes the values \( \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \} \). It is verified that the results of Section 4 for \( p = \frac{1}{2} \) are compatible with those obtained previously in [18]. Numerical evaluations in summary and conclusions indicate that the nonlinear corrections to the electrostatic potential energy of a parallel-plate capacitor in modified Born-Infeld electrodynamics are not important in the weak field regime.

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1 We use SI units in this paper. The space-time metric has the signature \((+, -, -, -)\).
2 A Brief Review of Modified Born-Infeld Electrodynamics

The modified Born-Infeld electrodynamics in a (3+1)-dimensional Minkowski space-time is described by the following Lagrangian density (see Eq. (B.10) in Ref. [19]):

\[ \mathcal{L}_{\mu\nu} = \frac{1}{2p} \epsilon_0 \beta^2 \left\{ 1 - \left[ 1 + \frac{c^2}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \tau \frac{c^4}{16\beta^4} (F_{\mu\nu} \times F^{\mu\nu})^2 \right]^p \right\} - J^\mu A_\mu, \]  

(5)

where \( p < 1 \) is a real dimensionless constant, \( \tau \) is another dimensionless constant, and \( J^\mu = (c\rho, J) \) is an external current for the \( U(1) \) gauge field \( A^\mu \). The parameter \( \beta \) in Eq. (5) is called the Born-Infeld parameter and shows the upper limit of the electric field in modified Born-Infeld electrodynamics. It is necessary to note that for \( p = \frac{1}{2}, \tau = 1 \) the Lagrangian density in Eq. (5) becomes the standard Born-Infeld Lagrangian density [9], while for \( p = 1, \tau = 0 \) we obtain the Maxwell Lagrangian density. The equation of motion for the vector field \( A_\mu \) is

\[ \frac{\partial \mathcal{L}_{\mu\nu}}{\partial A_\lambda} - \partial_\mu \left( \frac{\partial \mathcal{L}_{\nu\lambda}}{\partial (\partial_\nu A_\mu)} \right) = 0. \]  

(6)

If we put Eq. (5) into Eq. (6), we will get the inhomogeneous modified Born-Infeld equations as follows:

\[ \partial_\mu \left( \frac{F^{\sigma\lambda} - \tau \frac{c^2}{2\beta^2} (F_{\mu\nu} \times F^{\mu\nu}) F^{\sigma\lambda}}{\left[ 1 + \frac{c^2}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \tau \frac{c^4}{16\beta^4} (F_{\mu\nu} \times F^{\mu\nu})^2 \right]^{1-p}} \right) = \mu_0 J^\lambda. \]  

(7)

The dual field tensor \( *F^{\mu\nu} \) satisfies the following Bianchi identity:

\[ \partial_\mu *F^{\mu\nu} = 0. \]  

(8)

In (3+1)-dimensional space-time, the Faraday 2-form \( F \) and its dual \( *F \) have the following expressions [20]:

\[ F = F_{0i} dx^0 \wedge dx^i + \frac{1}{2} F_{ij} dx^i \wedge dx^j \]

\[ = E^i dt \wedge dx^i - \frac{1}{2} \epsilon^{ijk} B^j dx^i \wedge dx^k, \]  

(9)

\[ *F = -B^i dt \wedge dx^i + \frac{1}{2} \epsilon^{ijk} E^j dx^i \wedge dx^k, \]  

(10)

where \( i, j, k = 1, 2, 3 \), and \( \{ E^i \} = \{ E_x, E_y, E_z \}, \{ B^i \} = \{ B_x, B_y, B_z \} \).

Using Eqs. (9) and (10), Eqs. (7) and (8) take the following vector forms:

\[ \nabla \cdot \mathbf{D}(x, t) = \rho(x, t), \]  

(11)

\[ \nabla \times \mathbf{H}(x, t) = \mathbf{J}(x, t) + \frac{\partial \mathbf{D}(x, t)}{\partial t}, \]  

(12)

\[ \nabla \cdot \mathbf{B}(x, t) = 0, \]  

(13)

\[ \nabla \times \mathbf{E}(x, t) = -\frac{\partial \mathbf{B}(x, t)}{\partial t}, \]  

(14)

where \( \mathbf{D}(x, t) \) and \( \mathbf{H}(x, t) \) are given by

\[ \mathbf{D}(x, t) = \epsilon_0 \frac{\mathbf{E}(x, t) + \tau \frac{c^2}{2\beta^2} (\mathbf{E}(x, t) \cdot \mathbf{B}(x, t)) \mathbf{B}(x, t)}{\Omega_{\mu\nu}^{1-p}}, \]  

(15)

\[ \mathbf{H}(x, t) = \frac{1}{\mu_0} \frac{\mathbf{B}(x, t) - \tau \frac{c^4}{16\beta^4} (\mathbf{E}(x, t) \cdot \mathbf{B}(x, t)) \mathbf{E}(x, t)}{\Omega_{\mu\nu}^{1-p}}, \]  

(16)
and $\Omega_{p,\tau}$ is defined as follows:

$$\Omega_{p,\tau} := \left[1 + \frac{c^2}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{c^4}{16\beta^4} \left(F_{\mu\nu} \star F^{\mu\nu}\right)^2 \right]^p.$$  \hspace{1cm} (17)  

Now, let us study the electrostatic case where $B = J = 0$ and all other physical quantities are time independent. In this case the modified Born-Infeld equations (11)-(14) are

$$\nabla \cdot \left(\frac{E(x)}{\left[1 - \frac{E^2(x)}{\beta^2}\right]^{1-p}}\right) = \frac{\rho(x)}{\epsilon_0},$$  \hspace{1cm} (18)  

$$\nabla \times E(x) = 0.$$  \hspace{1cm} (19)  

The above equations are basic equations of modified Born-Infeld electrostatics [19]. By using the divergence theorem in vector calculus, we get the integral form of Eq. (18) as follows:

$$\oint_{C_2} \frac{1}{\left[1 - \frac{E^2(x)}{\beta^2}\right]^{1-p}} E(x) \cdot \hat{n} \, da = \frac{1}{\epsilon_0} \int_{C_3} \rho(x) d^3x,$$  \hspace{1cm} (20)  

where $C_2$ is a 2-chain which is the boundary of a 3-chain $C_3$, i.e., $C_2 = \partial C_3$ [20]. Equation (20) is Gauss's law in modified Born-Infeld electrostatics.

3 The Symmetric Energy-Momentum Tensor for Modified Born-Infeld Electrodynamics

In this section, we obtain the symmetric energy-momentum tensor for modified Born-Infeld electrodynamics. According to Eq. (5), the Lagrangian density for modified Born-Infeld electrodynamics in the absence of external current $J^\mu$ is

$$L_{p,\tau} = \frac{1}{2\mu_0 \epsilon_0 \beta} \left\{|1 - \left[1 + \frac{c^2}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{c^4}{16\beta^4} \left(F_{\mu\nu} \star F^{\mu\nu}\right)^2 \right]^p\right\}. \hspace{1cm} (21)$$  

From (21), we derive the following classical field equation:

$$\partial_\tau \left(\frac{F^{\sigma\lambda} - \frac{c^2}{2\beta^2} \left(F_{\mu\nu} \star F^{\mu\nu}\right) \star F^{\sigma\lambda}}{\Omega_{p,\tau}^{\frac{1}{p-1}}}\right) = 0.$$  \hspace{1cm} (22)  

The canonical energy-momentum tensor for Eq. (21) is [21-23]

$$\Theta^{\sigma}_{\eta} = \frac{\partial L_{p,\tau}}{\partial (\partial_\tau A_\lambda)} \left(\partial_\eta A_\lambda\right) - \delta^{\sigma}_{\eta} L_{p,\tau}. \hspace{1cm} (23)$$  

If we substitute (21) into (23) and use (22), we will obtain the following expression for the canonical energy-momentum tensor $\Theta^{\sigma}_{\eta}$:

$$\Theta^{\sigma}_{\eta} = \frac{1}{\mu_0} \frac{F^{\sigma\lambda} - \frac{c^2}{2\beta^2} \left(F_{\mu\nu} \star F^{\mu\nu}\right) \star F^{\sigma\lambda}}{\Omega_{p,\tau}^{\frac{1}{p-1}}} F_{\lambda\eta} + \frac{1}{2\mu_0 \epsilon_0 \beta^2} (\Omega_{p,\tau} - 1) \delta^{\sigma}_{\eta} + \partial_\lambda \mathcal{R}^{\lambda\sigma}_{\eta}, \hspace{1cm} (24)$$
where
\[ R^{\lambda\sigma}_{\eta} := \frac{1}{\mu_0} \frac{F^\lambda\sigma - \tau \frac{e^2}{4\pi} (F_{\mu\nu} \ast F_{\mu\nu}) \ast F^\lambda\sigma}{\Omega_{\eta,\tau}^{\frac{1}{2} - 1}} A_{\eta}, \quad (25) \]
\[ R^{\sigma\lambda}_{\eta} = -R^{\lambda\sigma}_{\eta}. \quad (26) \]

It is well known that the canonical energy-momentum tensor \( \Theta^{\sigma\eta} \) in (23) is generally not symmetric [21-24]. Belinfante showed that the canonical energy-momentum tensor \( \Theta^{\sigma\eta} \) in Eq. (23) can be written as follows [21]:
\[ \Theta^{\sigma\eta} = T^{\sigma\eta} + \partial_\lambda R^{\lambda\sigma}_{\eta}, \quad (27) \]
where the second- and third-order tensors \( T^{\sigma\eta} \) and \( R^{\lambda\sigma}_{\eta} \) must satisfy the following conditions:
\[ T^{\sigma\eta} = T^{\eta\sigma}, \quad (28a) \]
\[ R^{\lambda\sigma}_{\eta} = -R^{\sigma\lambda}_{\eta}. \quad (28b) \]
The second-order tensor \( T^{\sigma\eta} \) in the above equations is called the symmetric energy-momentum tensor [22]. A comparison between Eqs. (24) and (27) clearly shows that the symmetric energy-momentum tensor for modified Born-Infeld electrodynamics is
\[ T^{\sigma\eta} = \frac{1}{\mu_0} \frac{F^{\sigma\lambda} - \tau \frac{e^2}{4\pi} (F_{\mu\nu} \ast F_{\mu\nu}) \ast F^{\sigma\lambda}}{\Omega_{\eta,\tau}^{\frac{1}{2} - 1}} F^{\lambda\eta} + \frac{1}{2p} \epsilon_0 \beta^2 (\Omega_{\eta,\tau}^{\frac{1}{2} - 1} - 1) \delta^{\sigma\eta}. \quad (29) \]

After straightforward but tedious calculations, one finds that in the presence of an external current the symmetric energy-momentum tensor \( T^{\sigma\eta} \) in (29) satisfies the following equation:
\[ \partial_\sigma T^{\sigma\eta} = J^\sigma F_{\sigma\eta}. \quad (30) \]
Using Eqs. (9) and (10) together with Eq. (29), the energy density of modified Born-Infeld electrodynamics is given by
\[ u(x, t) = T^0_0(x, t) = \frac{1}{2p} \epsilon_0 \beta^2 \left\{ \left( 2p - 1 \right) \frac{E^2(x, t)}{\beta^2} + \tau \frac{\left( E(x, t) \cdot E(x, t) \right)^2}{\beta^2} + \frac{\epsilon^2 B^2(x, t)}{\beta^2} + 1 \right\} \left[ \frac{1}{\beta^2} \left( \frac{E^2(x, t)}{\beta^2} - c B^2(x, t) \right)^{1-p} - 1 \right]. \quad (31) \]

According to Eq. (31), the energy density of an electrostatic field in modified Born-Infeld electrodynamics becomes
\[ u(x) = T^0_0(x) = \frac{1}{2p} \epsilon_0 \beta^2 \left\{ \left( 2p - 1 \right) \frac{E^2(x)}{\beta^2} + 1 \left( 1 - \frac{E^2(x)}{\beta^2} \right)^{1-p} - 1 \right\}. \quad (32) \]
For \( p = \frac{1}{2} \), the modified electrostatic energy density in Eq. (32) becomes the energy density of an electrostatic field in Born-Infeld electrodynamics, i.e.,
\[ u(x) = \epsilon_0 \beta^2 \left( \frac{1}{\beta^2} \left( 1 - \frac{E^2(x)}{\beta^2} \right)^{1-p} - 1 \right). \quad (33) \]

\[ ^2 \text{It is obvious that for source-free modified Born-Infeld theory the right-hand side of (30) vanishes, i.e., } \partial_\sigma T^{\sigma\eta} = 0. \]
4 Calculation of the Capacitance and the Electrostatic Potential Energy of a Nonlinear Parallel-Plate Capacitor in Modified Born-Infeld Theory

In order to calculate the capacitance of a nonlinear parallel-plate capacitor in modified Born-Infeld electrostatics, we assume a capacitor composed of two large parallel conducting plates with area \( A \) and separation \( d \) (see Figure 1).

![Figure 1: A parallel-plate capacitor. The Gaussian surface is represented by dashed lines. The symmetry of the problem implies that \( E(x) = E_z(-\hat{e}_z) \), where \( \hat{e}_z \) is the unit vector in the \( z \)-direction.](image)

By applying modified Gauss's law in (20) to the Gaussian surface in Figure 1, we obtain the following equation for \( E_z \)

\[
E_z^\Gamma + \left( \frac{q}{\beta^2 \epsilon_0 A} \right)^\Gamma E_z^2 - \left( \frac{q}{\epsilon_0 A} \right)^\Gamma = 0,
\]

where \( \Gamma := \frac{1}{1-p} \). Now, let us obtain the exact solutions of Eq. (34) for \( p \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \} \).

For \( p = \frac{1}{2} (\Gamma = 2) \), Eq. (34) becomes the quadratic equation

\[
\left[ 1 + \left( \frac{q}{\beta \epsilon_0 A} \right)^2 \right] E_z^2 - \left( \frac{q}{\epsilon_0 A} \right)^2 = 0.
\]

For \( p = \frac{2}{3} (\Gamma = 3) \), Eq. (34) becomes the cubic equation

\[
E_z^3 + \left( \frac{q}{\beta^2 \epsilon_0 A} \right)^3 E_z^2 - \left( \frac{q}{\epsilon_0 A} \right)^3 = 0.
\]

For \( p = \frac{3}{4} (\Gamma = 4) \), Eq. (34) becomes the following quartic equation

\[
E_z^4 + \left( \frac{q}{\beta^3 \epsilon_0 A} \right)^4 E_z^2 - \left( \frac{q}{\epsilon_0 A} \right)^4 = 0.
\]

Finally, for \( p = \frac{5}{6} (\Gamma = 6) \), (34) becomes the following sextic equation

\[
E_z^6 + \left( \frac{q}{\beta^5 \epsilon_0 A} \right)^6 E_z^2 - \left( \frac{q}{\epsilon_0 A} \right)^6 = 0.
\]

Note that the above sextic equation by a suitable change of variable reduces to a cubic equation. In Galois theory, the Abel-Ruffini theorem or Abel’s impossibility theorem says that: for all \( n \geq 5 \), there is a polynomial
Given by

\[ E_z^{p=\frac{1}{2}} = \frac{q}{\epsilon_0 A} \Pi^{p=\frac{1}{2}}(q), \]  
(35a)

\[ E_z^{p=\frac{-1}{2}} = \frac{q}{\epsilon_0 A} \Pi^{p=\frac{-1}{2}}(q), \]  
(35b)

\[ E_z^{p=\frac{2}{1}} = \frac{q}{\epsilon_0 A} \Pi^{p=\frac{2}{1}}(q), \]  
(35c)

\[ E_z^{p=\frac{-2}{1}} = \frac{q}{\epsilon_0 A} \Pi^{p=\frac{-2}{1}}(q), \]  
(35d)

where

\[ \Pi^{p=\frac{1}{2}}(q) := \frac{1}{\sqrt{1 + \left(\frac{q}{\beta \epsilon_0 A}\right)^2}}, \]  
(36a)

\[ \Pi^{p=\frac{-1}{2}}(q) := \frac{3}{2} \left[ \frac{1}{2} - \frac{1}{27\beta^6} \left(\frac{q}{\epsilon_0 A}\right)^6 \right] + \sqrt{\frac{1}{4} - \frac{1}{27\beta^6} \left(\frac{q}{\epsilon_0 A}\right)^6} \]  
(36b)

\[ \Pi^{p=\frac{2}{1}}(q) := \sqrt{1 + \frac{1}{4\beta^4} \left(\frac{q}{\epsilon_0 A}\right)^4 - \frac{1}{2\beta^2} \left(\frac{q}{\epsilon_0 A}\right)^2}, \]  
(36c)

\[ \Pi^{p=\frac{-2}{1}}(q) := \sqrt{\frac{3}{2} \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{27\beta^6} \left(\frac{q}{\epsilon_0 A}\right)^6} \right] + \frac{3}{2} \left[ \frac{1}{4} + \frac{1}{27\beta^6} \left(\frac{q}{\epsilon_0 A}\right)^6 \right]}, \]  
(36d)

When the Born-Infeld parameter \( \beta \) takes the large values, the behavior of the electric fields in (35a)-(35d) are given by

\[ E_z^{p=\frac{1}{2}} = \left(\frac{q}{\epsilon_0 A}\right) \left[ 1 - \frac{1}{2} \beta^{-2} \left(\frac{q}{\epsilon_0 A}\right)^2 + \frac{3}{8} \beta^{-4} \left(\frac{q}{\epsilon_0 A}\right)^4 + O(\beta^{-6}) \right], \]  
(37a)

\[ E_z^{p=\frac{-1}{2}} = \left(\frac{q}{\epsilon_0 A}\right) \left[ 1 - \frac{1}{3} \beta^{-2} \left(\frac{q}{\epsilon_0 A}\right)^2 + \frac{1}{9} \beta^{-4} \left(\frac{q}{\epsilon_0 A}\right)^4 + O(\beta^{-6}) \right], \]  
(37b)

\[ E_z^{p=\frac{2}{1}} = \left(\frac{q}{\epsilon_0 A}\right) \left[ 1 - \frac{1}{4} \beta^{-2} \left(\frac{q}{\epsilon_0 A}\right)^2 + \frac{1}{32} \beta^{-4} \left(\frac{q}{\epsilon_0 A}\right)^4 + O(\beta^{-6}) \right], \]  
(37c)

\[ E_z^{p=\frac{-2}{1}} = \left(\frac{q}{\epsilon_0 A}\right) \left[ 1 - \frac{1}{6} \beta^{-2} \left(\frac{q}{\epsilon_0 A}\right)^2 - \frac{1}{72} \beta^{-4} \left(\frac{q}{\epsilon_0 A}\right)^4 + O(\beta^{-6}) \right]. \]  
(37d)

It must be emphasized that in obtaining the above results, the Mathematica software has been used [27]. All of the electric fields (37a)-(37d), have the Maxwellian limit in the weak field regime. The first term on the right-hand side of (37a)-(37d) represent the electric field between the plates of a parallel-plate capacitor in Maxwell

\(^3\)Q is the field of rational numbers (see page 116 in Ref. [26]).
theory, while the higher-order terms represent the effect of nonlinear corrections. Using (19), it is obvious that we can write \( E(x) \) in the following way:

\[
E(x) = -\nabla \phi(x),
\]

where \( \phi(x) \) is the electrostatic potential. From (38) we obtain the following relation:

\[
\triangle \phi = -\int_j^f E(x).dl,
\]

where \( \triangle \phi = \phi_f - \phi_i \) is the potential difference between the initial and final points, and \( dl \) is an infinitesimal displacement vector. If we use Eqs. (35) and (39), we will get the following expressions for the potential difference between the plates of a nonlinear parallel-plate capacitor for \( p \in \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \} \)

\[
\triangle \phi_p^{\frac{p}{2}} = -\int_\_\_^+ \frac{q}{\epsilon_0 A} \Pi^{p=\frac{p}{2}}(q)(-\hat{e}_z).(\hat{e}_z dz)
\]

\[
= \frac{qd}{\epsilon_0 A} \Pi^{p=\frac{p}{2}}(q),
\]

(40a)

\[
\triangle \phi_p^{\frac{p}{3}} = -\int_\_\_^+ \frac{q}{\epsilon_0 A} \Pi^{p=\frac{p}{3}}(q)(-\hat{e}_z).(\hat{e}_z dz)
\]

\[
= \frac{qd}{\epsilon_0 A} \Pi^{p=\frac{p}{3}}(q),
\]

(40b)

\[
\triangle \phi_p^{\frac{p}{4}} = -\int_\_\_^+ \frac{q}{\epsilon_0 A} \Pi^{p=\frac{p}{4}}(q)(-\hat{e}_z).(\hat{e}_z dz)
\]

\[
= \frac{qd}{\epsilon_0 A} \Pi^{p=\frac{p}{4}}(q),
\]

(40c)

\[
\triangle \phi_p^{\frac{p}{6}} = -\int_\_\_^+ \frac{q}{\epsilon_0 A} \Pi^{p=\frac{p}{6}}(q)(-\hat{e}_z).(\hat{e}_z dz)
\]

\[
= \frac{qd}{\epsilon_0 A} \Pi^{p=\frac{p}{6}}(q).
\]

(40d)

As is well known, in electrostatics the capacitance \( C \) of a capacitor is the ratio of the amount of charge on each plate of a capacitor to the potential difference between the plates of the capacitor, i.e.,

\[
C = \frac{q}{\triangle \phi}.
\]

(41)

It is necessary to note that Eq. (41) is also applicable for determination of the capacitance of nonlinear capacitors [2,17,18]. After inserting (40) into (41), the capacitance of a nonlinear parallel-plate capacitor in modified Born-Infeld electrostatics for \( p \in \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \} \) becomes

\[
C_p^{\frac{p}{2}} = \frac{\epsilon_0 A}{d} \Pi^{p=\frac{p}{2}}(q),
\]

(42a)

\[
C_p^{\frac{p}{3}} = \frac{\epsilon_0 A}{d} \Pi^{p=\frac{p}{3}}(q),
\]

(42b)

\[
C_p^{\frac{p}{4}} = \frac{\epsilon_0 A}{d} \Pi^{p=\frac{p}{4}}(q),
\]

(42c)

\[
C_p^{\frac{p}{6}} = \frac{\epsilon_0 A}{d} \Pi^{p=\frac{p}{6}}(q).
\]

(42d)
The above equations have the following weak field expansions:

\[
C_{\ell=\frac{1}{2}} = \frac{1}{\epsilon_0 A} \left[ 1 + \frac{1}{2} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 - \frac{1}{8} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + \mathcal{O}(\beta^{-6}) \right], \quad (42e)
\]

\[
C_{\ell=\frac{3}{2}} = \frac{3}{\epsilon_0 A} \left[ 1 + \frac{1}{3} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 - \frac{1}{81} \beta^{-6} \left( \frac{q}{\epsilon_0 A} \right)^6 + \mathcal{O}(\beta^{-8}) \right], \quad (42f)
\]

\[
C_{\ell=2} = \frac{2}{\epsilon_0 A} \left[ 1 + \frac{1}{4} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 - \frac{1}{32} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + \mathcal{O}(\beta^{-6}) \right], \quad (42g)
\]

\[
C_{\ell=\frac{5}{2}} = \frac{5}{6 \epsilon_0 A} \left[ 1 + \frac{1}{6} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 - \frac{1}{24} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + \mathcal{O}(\beta^{-6}) \right]. \quad (42h)
\]

Equations (42e)-(42h) show that the capacitance of a nonlinear parallel-plate capacitor in modified Born-Infeld electostatics is a function of the amount of charge on each plate of the capacitor. Now, let us compute the energy density between the plates of a nonlinear parallel-plate capacitor in modified Born-Infeld electostatics. If we put (35) into (32), we will get the following results:

\[
u(x)_{\ell=\frac{1}{2}} = \epsilon_0 \beta^2 \left[ \frac{1}{\Pi_{\ell=\frac{1}{2}}(q)} - 1 \right], \quad (43a)
\]

\[
u(x)_{\ell=\frac{3}{2}} = \epsilon_0 \beta^2 \left[ \frac{3}{4 \epsilon_0 A} \left( \frac{q}{\epsilon_0 A} \Pi_{\ell=\frac{3}{2}}(q) \right)^2 + 1 \right] \frac{1}{1/2 \Pi_{\ell=\frac{3}{2}}(q)} - 1 \right], \quad (43b)
\]

\[
u(x)_{\ell=2} = \epsilon_0 \beta^2 \left[ \frac{1}{3 \epsilon_0 A} \left( \frac{q}{\epsilon_0 A} \Pi_{\ell=2}(q) \right)^2 + 1 \right] \frac{1}{1/2 \Pi_{\ell=2}(q)} - 1 \right], \quad (43c)
\]

\[
u(x)_{\ell=\frac{5}{2}} = \epsilon_0 \beta^2 \left[ \frac{3}{5 \epsilon_0 A} \left( \frac{q}{\epsilon_0 A} \Pi_{\ell=\frac{5}{2}}(q) \right)^2 + 1 \right] \frac{1}{1/2 \Pi_{\ell=\frac{5}{2}}(q)} - 1 \right]. \quad (43d)
\]
Using Eq. (43), the electrostatic potential energy of a nonlinear parallel-plate capacitor in modified Born-Infeld electrostatics according to Figure 1 becomes

\[ U^{p=\frac{1}{2}} = \int_{\text{area of a plate}} da \int_0^d dz u(x)^{p=\frac{1}{2}} \]

\[ = \epsilon_0 \beta^2 \left[ \sqrt{1 + \left( \frac{q}{\beta \epsilon_0 A} \right)^2} - 1 \right] Ad, \]  

(44a)

\[ U^{p=\frac{2}{3}} = \int_{\text{area of a plate}} da \int_0^d dz u(x)^{p=\frac{2}{3}} \]

\[ = \frac{3}{4} \epsilon_0 \beta^2 \left[ \frac{1}{3\beta^{2\frac{2}{3}}} \left( \frac{q}{\epsilon_0 A} \Pi^{p=\frac{2}{3}}(q) \right)^2 + 1 \right] \left[ \Pi^{p=\frac{2}{3}}(q) \right] - 1 \]  

(44b)

\[ U^{p=\frac{4}{5}} = \int_{\text{area of a plate}} da \int_0^d dz u(x)^{p=\frac{4}{5}} \]

\[ = \frac{2}{3} \epsilon_0 \beta^2 \left[ \frac{1}{2\beta^{\frac{4}{5}}} \left( \frac{q}{\epsilon_0 A} \Pi^{p=\frac{4}{5}}(q) \right)^2 + 1 \right] \left[ \Pi^{p=\frac{4}{5}}(q) \right] - 1 \]  

(44c)

\[ U^{p=\frac{6}{7}} = \int_{\text{area of a plate}} da \int_0^d dz u(x)^{p=\frac{6}{7}} \]

\[ = \frac{3}{5} \epsilon_0 \beta^2 \left[ \frac{2}{3\beta^{\frac{6}{7}}} \left( \frac{q}{\epsilon_0 A} \Pi^{p=\frac{6}{7}}(q) \right)^2 + 1 \right] \left[ \Pi^{p=\frac{6}{7}}(q) \right] - 1 \]  

(44d)

For the large values of the Born-Infeld parameter \( \beta \), the behavior of the electrostatic potential energies in (44a)-(44d) are given by

\[ U^{p=\frac{1}{2}}_{\text{large } \beta} = U_M \left[ 1 - \frac{1}{4} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 + \frac{1}{8} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + O(\beta^{-6}) \right], \]

(45a)

\[ U^{p=\frac{2}{3}}_{\text{large } \beta} = U_M \left[ 1 - \frac{1}{6} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 + \frac{1}{27} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + O(\beta^{-6}) \right], \]

(45b)

\[ U^{p=\frac{4}{5}}_{\text{large } \beta} = U_M \left[ 1 - \frac{1}{8} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 + \frac{1}{96} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + O(\beta^{-6}) \right], \]

(45c)

\[ U^{p=\frac{6}{7}}_{\text{large } \beta} = U_M \left[ 1 - \frac{1}{12} \beta^{-2} \left( \frac{q}{\epsilon_0 A} \right)^2 - \frac{1}{216} \beta^{-4} \left( \frac{q}{\epsilon_0 A} \right)^4 + O(\beta^{-6}) \right], \]

(45d)

where \( U_M = \frac{q^2}{2e_M} \) and \( C_M = \frac{\epsilon_0 A}{d} \) are the electrostatic potential energy and the capacitance of a parallel-plate capacitor in Maxwell electrostatics respectively. Equation (45) shows that the relation \( U_M = \frac{q^2}{2e_M} \) is not true for a parallel-plate capacitor in modified Born-Infeld electrostatics. In the limit of \( \beta \to \infty \), equations (45a)-(45d) reduce to the following equation:

\[ U^{p=\frac{1}{2}}_{\beta=\infty} = U^{p=\frac{2}{3}}_{\beta=\infty} = U^{p=\frac{4}{5}}_{\beta=\infty} = U^{p=\frac{6}{7}}_{\beta=\infty} = U_M. \]

(46)
5 Summary and Conclusions

In 1930s Born-Infeld theory was introduced in order to remove the infinite self-energy of the electron in Maxwell electrodynamics [9]. In Born-Infeld electrodynamics the absolute value of the electric field has an upper limit \( \beta \), i.e., \( |E| \leq \beta \). In Born-Infeld paper the numerical value of \( \beta \) was [9,28]:

\[
\beta_{\text{Born-Infeld}} = 1.2 \times 10^{20} \frac{V}{m}.
\]  

(47)

Soff, Rafelski, and Greiner obtained the following lower bound on \( \beta \) [29]:

\[
\beta_{\text{Soff}} \geq 1.7 \times 10^{22} \frac{V}{m}.
\]  

(48)

In a paper about photon-photon scattering and photon splitting in a magnetic field in Born-Infeld theory, Davila and his coworkers obtained the following new lower bound on \( \beta \) [30]:

\[
\beta_{\text{Davila}} \geq 2.0 \times 10^{19} \frac{V}{m}.
\]  

(49)

In 1983, E. Iacopini and E. Zavattini introduced a \((p, \tau)\)-two-parameter modification of Born-Infeld electrodynamics, in which the self-energy of a point-like charge becomes finite for \( p < 1 \) [19]. In our paper, after a brief formulation of Born-Infeld-Iacopini-Zavattini electrodynamics (modified Born-Infeld electrodynamics) in the presence of an external current, the capacitance and the electrostatic potential energy of a nonlinear parallel-plate capacitor have been calculated exactly in the framework of modified Born-Infeld electrostatics for \( p \in \{ \frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \} \). In order to have a deeper understanding of nonlinear effects in modified Born-Infeld electrostatics, let us rewrite (45a) as follows:

\[
U_M \left. \right|_{\text{large } \beta} = U_M + U_M^{\text{first-order nonlinear correction}} + U_M^{\text{second-order nonlinear correction}} + O(\beta^{-6}),
\]  

(50)

where

\[
U_M^{\text{first-order nonlinear correction}} := -\frac{1}{2\beta^2 \epsilon_0 A d} U_M^2,
\]  

(51a)

\[
U_M^{\text{second-order nonlinear correction}} := \frac{1}{2\beta^4 \epsilon_0^2 A^2 d^2} U_M^3.
\]  

(51b)

Now, let us estimate the numerical values of \( U_M \), \( U_M^{\text{first-order nonlinear correction}} \), and \( U_M^{\text{second-order nonlinear correction}} \) in (50). For this aim, we use the following typical values for a parallel-plate capacitor (see page 804 in Ref. [31]):

\[
A = 100 \text{ cm}^2, \quad d = 1.0 \text{ mm}, \quad q = 1.06 \times 10^{-9} \text{ C}.
\]  

(52)

If we put Eqs. (47), (48), (49), and (52) into Eq. (50), we get

\[
U_M = 6.35 \times 10^{-9} \text{ J},
\]  

(53a)

\[
U_M^{\text{first-order nonlinear correction}} \left( \text{Born-Infeld} \right) = -1.58 \times 10^{-41} \text{ J},
\]  

(53b)

\[
U_M^{\text{second-order nonlinear correction}} \left( \text{Born-Infeld} \right) = 7.88 \times 10^{-74} \text{ J},
\]  

(53c)

\[
U_M^{\text{first-order nonlinear correction}} \left( \text{Soff} \right) = -7.88 \times 10^{-46} \text{ J},
\]  

(53d)

\[
U_M^{\text{second-order nonlinear correction}} \left( \text{Soff} \right) = 1.96 \times 10^{-82} \text{ J},
\]  

(53e)

\[
U_M^{\text{first-order nonlinear correction}} \left( \text{Davila} \right) = -5.69 \times 10^{-40} \text{ J},
\]  

(53f)

\[
U_M^{\text{second-order nonlinear correction}} \left( \text{Davila} \right) = 1.02 \times 10^{-70} \text{ J}.
\]  

(53g)
It must be noted that in (53d)-(53g) the minimum value of $\beta$ in Eqs. (48) and (49) has been used. Equations (53a)-(53g) tell us that the nonlinear corrections to electrostatic potential energy in a parallel-plate capacitor are not important in the weak field limit. For $p = \frac{7}{8}(\Gamma = 8)$, Eq. (34) becomes

$$E_z^8 + \left(\frac{q}{\beta^2 \varepsilon_0 A}\right)^8 E_z^2 - \left(\frac{q}{\varepsilon_0 A}\right)^8 = 0.$$  \hspace{1cm} (54)

Equation (54) is an eight-order equation which by a suitable change of variable reduces to a quartic equation. In future studies we want to obtain the exact solutions of (54) in order to calculate the capacitance and the electrostatic potential energy of a nonlinear parallel-plate capacitor in modified Born-Infeld electrostatics for $p = \frac{7}{8}$. More recently, a new modification of Born-Infeld electrodynamics has been developed which includes three independent parameters [32]. We hope to study the problems discussed in our work from the viewpoint of [32] in an independent research.

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