Incomplete quantum process tomography and principle of maximal entropy

Mário Ziman

Research Center for Quantum Information, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia

The main goal of this paper is to extend and apply the principle of maximum entropy (MaxEnt) to incomplete quantum process estimation tasks. We will define a so-called process entropy function being the von Neumann entropy of the state associated with the quantum process via Choi-Jamiolkowski isomorphism. It will be shown that an arbitrary process estimation experiment can be reformulated in a unified framework and MaxEnt principle can be consistently exploited. We will argue that the suggested choice for the process entropy satisfies natural list of properties and it reduces to the state MaxEnt principle, if applied to preparator devices.

PACS numbers: 03.65.Wj,03.67.-a,03.65.Ta

I. INTRODUCTION

Physical objects and processes are described by parameters that are directly, or indirectly, accessible experimentally and represent the maximal knowledge about physical systems (according to physical theory used). In quantum theory (see for instance [1–3]) the complete information (knowledge) is represented by the concepts of quantum state (normalized positive operator), quantum observable (normalized positive operator valued measure) and quantum channel (completely positive linear trace-preserving map). One of the main characteristics of quantum system is its dimension and one usage of the unknown channel acting on the Hilbert space, \( \rho : \mathcal{H} \rightarrow \mathcal{H} : \rho \geq 0, \text{Tr}\rho = 1 \), i.e. a density operator, or a density matrix. The number of independent real parameters determining the quantum states scales as \( N_{\text{state}} = d^2 - 1 \). The quantum processes/operations correspond to completely positive trace-preserving linear maps defined on the set of all linear operators including the set of all states \( S(\mathcal{H}) \). The number of independent real parameters determining the particular quantum process equals \( N_{\text{process}} = d^2(d^2 - 1) \).

Quantum measurements give us probability distributions over the set of all possible outcomes \( \{x_1, \ldots, x_L\} \), where \( L \) is some positive integer. In theory, the measured probabilities \( p_j \) are determined by the Born’s rule \( p_j = \text{Tr}\rho F_j \), where \( F_j \) is a positive operator (quantum effect) corresponding to outcome \( x_j \). These operators form the so-called positive operator valued measure (POVM), i.e. \( F_j \) are positive (\( F_j \geq 0 \)) and they sum up to identity operator \( \sum_j F_j = I \). The number of parameters specifying POVM depends on the total number of outcomes \( L \) and equals \( N_{\text{measurement}} = (L - 1)d^2 \).

The goal of quantum tomography is to estimate and fix all these parameters [4–6]. However, already for small systems (in dimension) the number of parameters is increasing rapidly, especially for quantum channels [7–11]. It seems that the complete knowledge about quantum objects is not a very realistic dream and experimentally we will not be able to perform all the desired tests [12–14]. Fortunately, there are situations in which even the knowledge of only few parameters enables us to make reasonable and nontrivial predictions about the behavior and properties of the system. A typical (classical) example is the equilibrium thermodynamics in which only few parameters are used to describe the complex behavior of a system of approximately \( 10^{23} \) degrees of freedom. Our aim is to describe the properties of quantum objects as honestly as possible even in cases when the complete information is not available. In particular, in this paper we will focus on incomplete quantum process tomography.

In Section II we will define the concept of process measurement and shortly describe the idea of quantum process tomography. The maximum entropy principle is described in Section III and also the idea is extended to process estimation problems by introducing the concept of process entropy. Finally, in Section IV the MaxEnt procedure is applied to particular examples of incomplete ancilla-free estimation of qubit channels.

II. QUANTUM PROCESS MEASUREMENT

A general quantum process tomography experiment consists of a test state \( \rho \) that is transformed in some specific procedure \( P \) involving the unknown channel \( \mathcal{E} \) into a state \( \rho' \), and a measurement (POVM) \( \mathcal{M} \) performed on the state \( \rho' \). This framework includes all the possible strategies [5, 6, 14] via which the parameters of quantum channels \( \mathcal{E} \) are accessible. We will define a process measurement as a particular choice of the test state \( \rho \), of the procedure \( P \) and of the measurement \( \mathcal{M} \). Generally, the procedure \( P \) is composed of an application of some known quantum channels on the test state and one usage of the unknown channel \( \mathcal{E} \) acting on \( d \)-dimensional quantum system (qudit). That is \( P \) itself is a quantum channel that can be written as a product \( P = P_{\text{in}} \circ (F_{\text{anc}} \otimes \mathcal{E}) \circ P_{\text{out}} \), where \( P_{\text{in}}, P_{\text{out}} \) can be understood as being parts of the preparation of the initial state \( \rho \), and of the final measurement \( \mathcal{M} \), respectively. Con-
sionally, without loss of generality we can assume that $P = F_{\text{anc}} \otimes \mathcal{E}$, where $F_{\text{anc}}$ is a known quantum channel acting on some ancillary system and also can be included as being a part of either preparation of $\varrho$, or measurement performed. A process measurement is called ancilla-free if either the initial state $\varrho$ is factorized, or the ancillary system is trivial. Otherwise the process measurement is ancilla-assisted and $\varrho' = F_{\text{anc}} \otimes \mathcal{E}[\varrho]$.

For example, consider $\mathcal{E}$ is a qudit quantum channel and the test state $\varrho = \Psi_+$ is a maximally entangled state of two qudits $(\Psi_+ = \frac{1}{\sqrt{n}} \sum |j\rangle \otimes |j\rangle |k\rangle)$. The unknown channel is applied only on second of the qudits while the first one is transformed trivially, i.e. $\varrho' = \omega_{\varrho} = I \otimes \mathcal{E}[\Psi_+]$ [6]. Performing the informationally complete POVM (resulting in complete specification of $\omega_{\varrho}$) the channel $\mathcal{E}$ can be uniquely identified, because the mapping $\mathcal{E} \mapsto \omega_{\varrho} = \mathcal{J}[\mathcal{E}]$ is the well known Choi-Jamiolkowski isomorphism [15, 16] between the set of quantum qudit channels and set of quantum states of two qudits. Hence, via general POVM measurements of the output state we can acquire either complete, or partial, knowledge on the channel. Let us note that individual ancilla-free process measurements cannot be informationally complete, but they can be combined together to gain the complete information. In the following sections we will concentrate onto situations in which the collection of process measurements provides us with partial information, only.

III. PRINCIPLE OF MAXIMUM ENTROPY

Maximum entropy (MaxEnt) principle was originally introduced in statistics in order to estimate a probability distribution providing that only partial information on that probability is available [17]. There are many probability distributions compatible with the given constraints and our aim is to choose one of them that in some sense represents our knowledge the most honestly. This choice cannot be logically derived and some additional principle must be introduced. Using the results of information theory [18] on the uniqueness of Shannon entropy, one can argue that [17, 19] the probability distribution maximizing the Shannon entropy is the best choice we can make. Such probability maximizes the uncertainty (measured by entropy) and, intuitively, also our predictions about the unspecified parameters are as uncertain as possible. That is, a conclusion based on MaxEnt principle is introducing as little additional information as it is possible [19].

This idea was generalized to the domain of quantum state tomography [20, 21] by using the concept of von Neumann entropy [1] $S(\varrho) = - \text{Tr} \varrho \log \varrho$, which is considered to be the quantum extension of Shannon entropy. A state observation level $O_n$ is defined as a set of $n$ ($n \le d^2$) mean values $\{f_1, \ldots, f_n\}$ of linearly independent operators $\{F_1, \ldots, F_n\}$ related to unknown state $\varrho$ via the trace rule $f_j = \text{Tr}_\omega F_j = \langle F \rangle_\varrho$. If the operators $\{F_1, \ldots, F_n\}$ form a POVM the numbers $f_j$ represent the measured probabilities. The answer to incomplete state tomography problem based on maximum entropy principle is given by the following equation

$$q = \arg \max \{S(\varrho) | \varrho \in S(H), f_j = \text{Tr}_\omega F_j, j = 1, \ldots, n\}.$$  

The following state is the formal solution of the MaxEnt problem [17, 21]

$$\varrho = \frac{1}{Z} \exp(-\sum_j \lambda_j F_j),$$  

where $Z = \text{Tr}[\exp(-\sum_j \lambda_j F_j)]$ and $\lambda_j$ are Lagrange multipliers fixed by the system of equations

$$f_j = \text{Tr}_\omega F_j = -\frac{\partial}{\partial \lambda_j} \ln Z(\lambda_1, \ldots, \lambda_n).$$  

For example, consider a two-level quantum system (qubit) and observation levels

$$O_1 = \{\langle \sigma_z \rangle_\varrho\}$$

$$O_2 = \{\langle \sigma_y \rangle_\varrho, \langle \sigma_z \rangle_\varrho\}$$

$$O_3 = \{\langle \sigma_x \rangle_\varrho, \langle \sigma_y \rangle_\varrho, \langle \sigma_z \rangle_\varrho\}.$$

A qubit state can be expressed in a so-called Bloch sphere picture as $\varrho = \frac{1}{2}(I + r \cdot d)$ with $r_j = \text{Tr}_\omega \sigma_j$. The MaxEnt principle applied for $O_1, O_2$ sets mean values $r_j$ of all the unobserved operators to zero. That is, for $O_1 = \{z\}$ we get $\varrho = \frac{1}{2}(I + z \sigma_z)$ and for $O_2 = \{y, z\}$ the MaxEnt estimation gives $\varrho = \frac{1}{2}(I + y \sigma_y + z \sigma_z)$. Observation level $O_3$ provides complete information about the quantum state, hence the principle of maximum entropy is not needed in this case.

Clearly, there is a problem if we consider similar incomplete estimation task for processes, namely, which entropy should be maximized? Unlike quantum states the quantum channels are lacking some concept of entropy, or uncertainty. In fact, what does it mean that a quantum process is uncertain? Our goal is to introduce a suitable concept of a channel/process entropy $S_{\text{proc}}(\mathcal{E})$ and investigate its properties. Before analyzing different choices let us discuss some (intuitive) properties of the process entropy.

1. Uncertainty of unitary channels. Without any doubts the unitary channels play a very specific role among all quantum processes. For unitary processes the interaction of the system with its environment is trivial. The physical invertibility is the unique and characteristic property of the unitary channels. In some sense the channel entropy should reflect how much noise the channel introduces. Unitary channels are noiseless and in what follows we will assume that the channel entropy is invariant under unitary preprocessing ($\mathcal{V}$) and unitary post-processing ($\mathcal{U}$), i.e., $S_{\text{proc}}(\mathcal{E}) = S_{\text{proc}}(\mathcal{U} \circ \mathcal{E} \circ \mathcal{V})$. It follows that all unitary channels have the same
value of uncertainty that can be set to zero. Moreover, we do require that $S_{\text{proc}}(\mathcal{E}) = 0$ implies that $\mathcal{E}$ is unitary.

2. Uniqueness of maximum. For the purposes of incomplete process estimation exploiting the MaxEnt principle it is necessary that the maximum is unique. Hence there must be a unique channel, for which the uncertainty is maximal. This channel should be the result of the incomplete estimation if no data are available, i.e., when the observation level is trivial, $\mathcal{O}_0 = \emptyset$. Because of the unitary invariance the channel must be invariant under unitary preprocessing and postprocessing, i.e., $\mathcal{E}_{\text{max}} = \mathcal{U} \circ \mathcal{E}_{\text{max}} \circ \mathcal{V}$. Only the channel mapping the whole state space into a total mixture ($\rho \mapsto \frac{1}{d} I$) is invariant in this sense. It is argued in [22] that this channel is indeed the average channel over all possible qubit channels. To guarantee that in any process measurement the maximum is unique it is sufficient that the process entropy is a concave function, i.e., $S_{\text{proc}}(\lambda \mathcal{E}_1 + [1-\lambda] \mathcal{E}_2) \geq \lambda S_{\text{proc}}(\mathcal{E}_1) + [1-\lambda] S_{\text{proc}}(\mathcal{E}_2)$.

3. Universality. This is not a condition on the concept of process entropy itself, but rather on the general possibility to employ such principle once we agree on a suitable measure of channel entropy. It is important that the maximum entropy principle is applicable for all process measurements. We shall discuss this issue later in more details.

In summary, a channel entropy is some concave function (defined on the set of quantum channels) achieving its maximum for the complete contraction to the total mixture and vanishing only for unitary channels. A natural choice of process entropy seems to be related to the concepts of quantum channel capacity [23–26]. Quantum capacity quantifies the degree of preservation of quantum states during the transmission and this value is different from the classical capacity. Because of these properties, the capacities are not appropriate candidates for the definition of process entropy usable in incomplete process tomography tasks.

A. Choi-Jamiolkowski process entropy

The Choi-Jamiolkowski isomorphism provides us naturally with a notion of channel entropy. It uniquely associates a quantum state $\omega_F = (I \otimes \mathcal{E})[\Psi_+]$ with a quantum channel $\mathcal{E}$, hence we can adopt the von Neumann entropy of $\omega_F$ as being the channel entropy of $\mathcal{E}$ [27, 28]. Consider a quantum channel on $d$-dimensional system (qudit). Providing that for each process measurement we are able to define uniquely a state observation level $\mathcal{O}_n$ given by mean values $x_j = \text{Tr} X_j \omega_F$ of $n$ linearly independent Hermitian operators $X_j$, the MaxEnt problem for processes can be formalized as follows

$$\mathcal{E} = \arg \max_{\omega_F} S(\omega_F)$$

where the maximum of von Neumann entropy $S(\omega_F)$ is taken over all states $\omega_F \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ satisfying the constraints $\text{Tr}_2 \omega_F = \frac{1}{d} I$ and $x_j = \text{Tr} \omega_F X_j$ for all $X_1, \ldots, X_n \in \mathcal{O}_n$. The resulting state $\omega_F$ determines the quantum operation $\mathcal{E}$ uniquely via the inverse relation

$$\mathcal{E}[\rho] = d\text{Tr}_\text{anc}[(\rho^T \otimes I)\omega_F].$$

In what follows we will investigate the process entropy given as the von Neumann entropy of the state $\omega_F$ associated with the channel $\mathcal{E}$ via Choi-Jamiolkowski formalism.

It is straightforward to see that only for unitary channels the states $\omega_F$ are pure and hence $S_{\text{proc}}(\mathcal{E}) = S(\omega_F) = 0$ only for unitary channels, $\mathcal{E} = \mathcal{U}$. Moreover, unitary channels do not change the entropy of $\omega_F$, i.e., $S(\omega_F) = S(\omega_{F}\circ\mathcal{E})$. The concavity of $S_{\text{proc}}$ follows from the concavity of von Neumann entropy and the maximum is achieved for $\omega_F = \frac{1}{d} I$ that is associated with the channel mapping the whole state space into the maximally mixed state, $\mathcal{E} : \rho \mapsto \frac{1}{d} I$. In summary, the process entropy

$$S_{\text{proc}}(\mathcal{E}) = -\text{Tr} \omega_F \log \omega_F$$

satisfies all the desired properties we have discussed previously. The only open issue is its applicability in general process measurement.

Choi-Jamiolkowski isomorphism is associated with a specific process measurement using as the test state a maximally entangled state of two qudits. Second qudit is sent through the unknown channel while the first one is evolving trivially to obtain the state $\mathcal{E}[\Theta] = \omega_F$, that is estimated in some state measurement described by POVM. In this case the process observation level can be defined as the following set of mean values

$$\mathcal{O}_n^{\text{proc}} = \{x_1, \ldots, x_n\},$$

where

$$x_j = \text{Tr} F_j \omega_F = (F_j)[\Theta] = \langle F_j \rangle \mathcal{E}.$$  

Because of the identity $\text{Tr}_2 \omega_F = \frac{1}{d} I$ the process observation level is equivalent to a state observation level

$$\mathcal{O}_n^{\text{proc}} = \{x_1, \ldots, x_n, 0, \ldots, 0\}.$$  

The added zeros represent the mean values of $d^2 - 1$ operators $I \otimes \Lambda_j$, where $\Lambda_j$ are traceless Hermitian qudit operators forming a basis of the set of traceless Hermitian qudit operators, i.e., the general qudit state can be written as $\rho = \frac{1}{d} (I + r \Lambda)$. To be more precise we assume...
that the operators \( F_1, \ldots, F_n \) are linearly independent of operators \( I \otimes \Lambda_1, \ldots, I \otimes \Lambda_{d-1} \).

What if the test state is not the maximally entangled one? Is it possible to interpret the measured values as linear constraints on the state \( \Omega \) defined by Choi-Jamiolkowski isomorphism? Let us note that the linearity is crucial, because we implicitly assume that the constraints representing the incomplete information are linear, which guarantees that the set of possible solutions is convex and, hence, the entropy has a unique maximum.

**B. General quantum process experiment vs Choi-Jamiolkowski isomorphism**

Consider a general test state \( \Omega \) of the qudit and an arbitrary ancilla system. We will show that there exist a completely positive linear map \( A_\Omega : \mathcal{B}(\mathcal{H}_d) \to \mathcal{B}(\mathcal{H}_{\text{anc}}) \) such that \( A_\Omega \otimes I[\Psi_\alpha] = \Omega \). A general pure state \( |\Phi \rangle = \sum_{a,j} \Phi_{aj} |a\rangle |\alpha\rangle \) can be written as \( |\Phi \rangle = A_\Phi |\Psi_\alpha \rangle \), where the operator \( A_\Phi : \mathcal{H} \to \mathcal{H}_{\text{anc}} \) is defined as \( A_\Phi = \sqrt{\sum_{a,j} \Phi_{aj} |a\rangle \langle j|} \). A general mixed state \( \Omega \) can be written as convex combination of pure states \( \Omega = \sum_k \lambda_k |\Phi_k \rangle \langle \Phi_k| \). Linear maps resulting in the mean value of the operator \( \Omega \) is defined. It follows that for general factorized state \( \Omega = \sum_{a,l} \lambda_{al} |\varphi_a \rangle |\alpha_l \rangle \langle \alpha_l| \langle \varphi_a| \langle \alpha_l| \langle \varphi_a| \langle \alpha_l| \langle \varphi_a| \langle \alpha_l| \), the transformation \( A_\Psi \) is convex and, hence, the entropy has a unique maximum.

Therefore, according to Eq. (3.7) the ancilla-free process measurement is equivalent to measuring \( d^2 T \otimes F \) in the process measurement with maximally entangled state \( \Psi^+ \), where \( d^2 T \) is the transposed matrix \( \rho \) with respect to basis \( \{ |k\rangle \} \).

We have shown that measuring the outcome associated with \( F \) in the ancilla-free process measurement is equivalent to measuring \( \rho^T \otimes F \) in the process measurement with maximally entangled state \( \Psi^+ \), where \( \rho \) is the ancilla-free test state. It means that the ancilla-free process observation level consisting of mean values \( \langle F_1 \rangle_{\Omega}, \ldots, \langle F_n \rangle_{\Omega} \) is equivalent to \( \mathcal{E}_\rho^\text{proc} = \{ \langle d^j_1 \otimes F_1 \rangle_{\xi}, \ldots, \langle d^j_n \otimes F_n \rangle_{\xi} \} \).

**C. States as preparation channels**

Preparation devices play a completely different role than quantum channels. However, formally, they can be understood as mappings that transform an arbitrary input state into a fixed output state \( \xi \). In this sense preparation channels \( \mathcal{E}_\xi \) form a very specific convex subset of quantum channels. Let us apply the proposed maximum entropy based process tomography to preparation channels, i.e., to preparators. The process measurement is ancilla-free consisting of all linearly independent test states \( g_j \) and \( \rho \) of measurement of the mean value of Hermitian operator \( F \). According to previous paragraph the process observation level is described as \( \mathcal{E}_\rho^\text{proc} = \{ \langle d^j_1 \otimes F \rangle_{\xi_0}, \ldots, \langle d^j_n \otimes F \rangle_{\xi_n} \} \). The Choi-Jamiolkowski entropy of the channel \( \mathcal{E}_\xi \) equals (up to a constant) to the von Neumann entropy of the state \( \xi \), because \( I \otimes \mathcal{E}_\xi[\Psi_\alpha] = \frac{1}{d^2} I \otimes \xi \) implies \( S(\omega_{\mathcal{E}_\xi}) = \log_2 d + S(\xi) \). Moreover, because of the identity

\[
\langle d^j_1 \otimes F \rangle_{I \otimes \mathcal{E}_\xi[\Psi_\alpha]} = \langle d^j_1 \otimes F \rangle_{\frac{1}{d^2} I \otimes \xi} = \langle F \rangle_{\xi} ,
\]

it follows that finding a channel with the maximal Choi-Jamiolkowski entropy is equivalent to finding a state maximizing the von Neumann entropy. As a result we get that the process MaxEnt procedure, if applied to preparators, reduces to the state MaxEnt procedure. That is, the proposed Choi-Jamiolkowski process entropy is a consistent extension of the von Neumann entropy. In particular, the MaxEnt principle for states can be considered as being a special case of the MaxEnt principle for channels.
IV. EXAMPLES

In this section we shall present few examples of incomplete quantum process estimation for ancilla-free process measurements of a qubit channel.

A. \( O^{\text{proc}}_I = \{ \langle 2 \sigma^T \otimes \sigma_z \rangle_x \} \)

In this case the collected data provides us about information on the mean value of an observable \( \sigma_z \), hence the experiment gives us single value \( m = \langle \sigma_z \rangle \). Unfortunately, even in this simplest case we cannot give (see Appendix A) an analytic solution in its whole generality. In particular, we found the solutions in following cases

\[
\begin{align*}
\varrho &= \frac{1}{2} I : \mathcal{E}[\xi] = \frac{1}{2} (I + m \sigma_z), \\
\varrho &= |\psi\rangle \langle \psi| : \mathcal{E}[\xi] = \frac{1}{2} (I + \frac{1}{2} m (1 + (\vec{t} \cdot \vec{r}) \sigma_z),
\end{align*}
\]

(4.1)

where \( \varrho = \frac{1}{2} (I + \vec{r} \cdot \vec{\eta}) \) and \( \xi = \frac{1}{2} (I + \vec{t} \cdot \vec{\eta}) \). It is interesting that for pure test state the estimated channel is not unital even if there are unital channels satisfying the constraints.

An alternative method for incomplete process estimation was described in [22]. It is based on a different ad hoc rule demanding that no additional information about unobserved measurements (those completing the incomplete process observational level) is introduced. In particular, for states \( \varrho \) orthogonal (in Hilbert-Schmidt sense, i.e., Tr\( \varrho \varrho^* \) = 0) to given test states, the mean values are completely random, i.e., they are transformed into the total mixture (\( \eta \rightarrow \frac{1}{2} I \)). Hence the entropy of output states for unmeasured inputs is maximal. It means that if possible (meaning there is no contradiction with the data, or theory) the total mixture is preserved. Otherwise an optimization procedure minimizing the average distance from the total mixture is needed. This method was analyzed only for qubit channels and for ancilla-free process measurements. The extension of the method to all process measurements will require introduction of additional rules. Let us compare the method proposed in [22] and the one proposed in this paper.

If measuring \( \sigma_z \) and finding \( m = \pm 1 \) the output state must be pure and it corresponds to an eigenvalue of \( \sigma_z \).

In both mentioned scenarios we know the solution providing our knowledge consists of complete information of the action of the channel on the pure test state, thus, we know that \( \mathcal{E} : |\psi\rangle \mapsto \pm |z\rangle \), respectively. As it was argued in the work [22] the estimated transformation should map the whole Bloch sphere into the line connecting north and south pole, i.e., \( \vec{t} \rightarrow \vec{t}' = (0, 0, \pm t_z) \). However, the proposed MaxEnt estimation procedure gives different result. In particular, \( \vec{t} \rightarrow \vec{t}' = (0, 0, \pm (1 + t_z)/2) \). This transformation is not unital, but the total mixture is mapped to the state \( \vec{r}' = (0, 0, 1/2) \). A state \( \vec{t} = -\vec{r} \) orthogonal to the test state \( \vec{r} \) is transformed as follows

\[
\begin{align*}
\text{Scheme in [22]} : & \quad m = 1 \quad \mathcal{E}_{\text{est}} : -\vec{r} \mapsto -\vec{r}' \\
\text{MaxEnt} : & \quad m = 1 \quad \mathcal{E}_{\text{est}} : -\vec{r} \mapsto \vec{0}'
\end{align*}
\]

(4.2)

As we see in this case both methods transform orthogonal (in Hilbert-Schmidt sense) states to \( |\psi\rangle \) into the total mixture, but for MaxEnt procedure also the orthogonal (in Hilbert space sense) state \( |\psi'\rangle \) is mapped into the total mixture. In our opinion this feature (except the universality) justifies the usage of MaxEnt procedure in comparison with the scheme described in [22]. In fact, the uncertainty introduced by the estimation procedure on perfectly distinguishable (orthogonal) states from the test states should be as maximal as possible. And this is not the case for the method used in [22], for which the estimated channel preserves the orthogonal state.

B. \( O^{\text{proc}}_4 = \{ \langle I \otimes \sigma_z \rangle_x, \langle I \otimes \sigma_y \rangle_x, \langle I \otimes \sigma_z \rangle_x, \langle I \otimes \sigma_x \rangle_x \} \)

Consider a situation that the unknown qubit channel is tested by the total mixture and the complete tomography of the output state is performed, i.e., mean values of \( \sigma_x, \sigma_y, \sigma_z \) are known. The corresponding state observation level is \( \mathcal{O}_0 = \{ \langle \vec{\sigma} \otimes I \rangle_x, \langle I \otimes \vec{r} \rangle_x = \{ \vec{0}, \vec{m} \} \} \), for which the solution is presented in Appendix B. In such case the proposed MaxEnt process tomography procedure leads us to the channel

\[
\mathcal{E}_{\text{est}} : \varrho \mapsto \varrho_0 = \frac{1}{2} (I + \vec{m} \cdot \vec{\sigma}),
\]

(4.3)

hence, the whole Bloch sphere is contracted into a single point \( \vec{m} \). As a result we get that if the total mixture is used to probe the channel action then according to MaxEnt procedure all the states are mapped into the output state \( \varrho_0 = \mathcal{E} [\frac{1}{2} I] \). In this case both the discussed procedures are giving the same estimation.

C. \( O^{\text{proc}}_4 = \{ \langle I \otimes \sigma_z \rangle_x, \langle 2 \langle |x\rangle \langle x| \rangle^T \otimes \sigma_z \rangle_x, \langle 2 \langle |y\rangle \langle y| \rangle^T \otimes \sigma_z \rangle_x, \langle 2 \langle |z\rangle \langle z| \rangle^T \otimes \sigma_z \rangle_x \} \)

In this case the process is probed with four test states (total mixture and positive eigenvectors of \( \sigma_x, \sigma_y, \sigma_z \) forming a vector of pure states \( \vec{\eta} \)), but only 2th component of the Bloch vector of the output state is measured. \( O_7 = \{ \langle I \otimes \sigma_z \rangle_x, \langle 2 \langle |x\rangle \langle x| \rangle^T \otimes \sigma_z \rangle_x, \langle \vec{\sigma} \otimes I \rangle_x \} = \{ z, \vec{z}, \vec{0} \} \) is the corresponding state estimation problem and \( z, \vec{z} \) are the experimentally identified mean values. Information encoded in these parameters can be equivalently rewritten into the form \( O_7 = \{ \langle I \otimes \sigma_z \rangle_x, \langle \vec{\sigma} \otimes \sigma_z \rangle_x, \langle \vec{\sigma} \otimes I \rangle_x \} \rightarrow \{ z, \vec{z}, \vec{0} \} \), where \( z, \vec{z} = \vec{z} - z \).

It is shown in [21] that for such state observation level the estimated density matrix reads

\[
\omega = \frac{1}{4} \left( I \otimes I + z I \otimes \sigma_z + \vec{z} \cdot \vec{\sigma} \otimes \sigma_z \right).
\]

(4.4)

Hence, the process is described by the following state transformation \( \varrho \mapsto \mathcal{E}_{\text{est}} [\varrho] \)

\[
\vec{t} \rightarrow \vec{t}' = (0, 0, z + \vec{z} \cdot \vec{t}),
\]

(4.5)
i.e., \( q' = \frac{1}{4}[I + (z + \xi^2 \cdot t)\sigma_z] \). As in all previous cases, also in this case the whole state space is mapped onto a subset of the line connecting states \(| + z \rangle\) and \(| - z \rangle\). However, in this case the final state depends also on parameters \( t_x, t_y \).

V. CONCLUSION AND DISCUSSION

We have addressed the problem of incomplete process estimation based on maximum entropy principle [19]. In general the maximum entropy principle is an intuition-based \textit{ad hoc} principle related to quantification of ignorance contained in probability distributions that seems to agree with our experience. This ignorance measured in entropy can be extended to domain of quantum states by introducing the von Neumann entropy. Our attempt here was to develop similar approach for processes. We argued that capacities are not good candidates for quantifying the uncertainty of quantum channels and we exploited the Choi-Jamiolkowski process entropy defined as

\[
S_{\text{proc}}(E) = -\text{Tr}[\rho_E \log \rho_E], \quad \rho_E = (I \otimes E)[\Psi_+], \quad (5.1)
\]

where \( \Psi_+ \) is the maximally entangled state. In particular, we showed that the suggested concept can be universally applied in all possible process measurements. The procedure is demonstrated on three incomplete ancilla-free estimation problems of a qubit channel: i) pure test state and projective measurement, ii) the total mixture as the test state and complete tomography of the output state, and iii) four test states and the same projective measurement.

We have shown that unlike the concepts of capacity of quantum channels the process entropy defined above is compatible with the following properties:

1. \textit{Uniqueness of maximum}: unique maximum for the channel contracting whole state space into the total mixture.
2. \textit{Minimum}: minimum is achieved only for unitary processes.
3. \textit{Unitary invariance}: invariant under unitary transformations, i.e., \( S_{\text{proc}}(U \circ E \circ V) = S_{\text{proc}}(E) \) for all unitary transformations \( U, V \).
4. \textit{Concavity}: function \( S_{\text{proc}}(E) \) is concave, i.e., \( S_{\text{proc}}(\rho E + q F) \geq q S_{\text{proc}}(E) + p S_{\text{proc}}(F) \).

The proposed Choi-Jamiolkowski process entropy serves as a very valuable tool in incomplete process tomography deserving future testing and investigation. Moreover, as it is shown in Section III.B, the proposed process entropy principle, if applied to state preparator devices, is equivalent to the state entropy principle based on von Neumann entropy. The key feature discussed in this manuscript is the universality of the proposed procedure following from the unification of all process measurements described in Section III.B. This idea goes beyond the applications in incomplete process estimation and is further developed in [30].

Recently, Olivares et al. [29] proposed and analyzed a state estimation problem combining incomplete information with some non-trivial apriori knowledge. In their approach the maximization of entropy is replaced by minimizing of Kullback relative entropy \( S(\rho' \| \rho_0) = \text{Tr}[\rho' \log \rho' - \log \rho_0] \) with a bias \( \rho_0 \) representing the prior knowledge. This approach can be directly extended to the case of channels by introducing the quantity \( S(\rho_E \| \rho_0) = -\text{Tr}[\rho_E (\log \rho_E - \log \rho_0)] \) with \( \rho_0 \) playing the role of prior information. If we set \( \rho_0 \) to be the state space contraction into the total mixture (i.e. \( \rho_0 = \frac{1}{d} I \)), then \( \text{Tr}[\rho_E \log \rho_0] = -\log d^2 \text{Tr} \rho_E = -2 \log d \). Consequently, \( S(\rho_E \| \rho_0) = 2 \log d - S(\rho_E) \) and the biased estimation problem reduces to the unbiased maximum process entropy estimation.

Let us give a simple example based on the observation level discussed in Section IV.A. Suppose that out of the performed measurement we acquire the information \( |0 \rangle \mapsto |0 \rangle \). We shall consider three different priors: i) identity channel \( \rho_0 = I \); ii) diagonalisation channel \( \rho_0 = \text{diag} \) transforming each state into its diagonal form in the basis \(|0\rangle, |1\rangle \); iii) \( \rho_0 = \frac{1}{2}(I + U_0) \), where \( U_0[\xi] = \sigma_z \xi \sigma_x \). Let us note that \( S(\rho \| \rho_0) \) is finite only if the support of \( \rho \) is included in the support of \( \rho_0 \). For the case of identity channel \( \rho_0 \) is a pure state, hence \( S(\rho_E \| \rho_0) < \infty \) only if \( \rho_E = \rho_0 \), i.e. \( \mathcal{E}_{\text{est}} = I \). Fortunately, the identity channel is in accordance with the constraint \(|0\rangle \mapsto |0\rangle \), hence the estimation gives the identity channel. In the second case it is straightforward to verify that the channel diag fulfills the constraints. Since \( S(\text{diag} \| \rho_0) = 0 \) is the minimal possible value we get \( \mathcal{E}_{\text{est}} = \text{diag} \). In the third case the support of \( \rho_0 \) is a linear span of vectors \(|\phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \) and \(|\phi_-\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \). Therefore, only for channels with \( \rho_E = a|\phi_+\rangle \langle \phi_+| + b|\phi_+\rangle \langle \phi_+| \) the relative entropy is finite. However, the constraint requires that \(|0\rangle \langle 0| = (aI + bU_0)|0\rangle \langle 0| = a|0\rangle \langle 0| + b|1\rangle \langle 1| \), i.e. necessarily \( b = 0 \). In such case \( a = 1 \), because otherwise \( aI \) is not a valid quantum channel. That is, only the identity channel satisfies the measured constraint, thus it minimizes the Kullback relative entropy. The estimation gives \( \mathcal{E}_{\text{est}} = I \). In all these cases we find different estimations as in the unbiased maximum entropy approach (see Eq.(4.2)). The role of prior information in incomplete process estimation deserves much deeper analysis than it is presented in these simple examples. However, such task is beyond the scope of this manuscript.

Acknowledgments

This work was supported by in part by the European Union projects QAP, by the Slovak Academy of Sciences via the project CE-PI and by the Slovak grant agency APVV and VEGA.
According to Section III this process observation level is equivalent to the following state observation level
\[ O_4 = \{ (2g^T \otimes \sigma_z)_{\psi}, (\sigma_x \otimes I)_\psi, (\sigma_y \otimes I)_\psi, (\sigma_z \otimes I)_\psi \} = \{ m, 0, 0, 0 \}. \]

Maximum entropy estimation gives us the following state
\[ \omega = \frac{1}{Z} \exp[-\vec{\lambda} \cdot (\vec{\sigma} \otimes I) - 2dg^T \otimes \sigma_z] \tag{A1} \]
where \( Z = \text{Tr}[\exp[-\vec{\lambda} \cdot (\vec{\sigma} \otimes I) - 2dg^T \otimes \sigma_z]] \) and \( \vec{\lambda}, d \) are Lagrange multipliers that can be determined by solving the system of algebraic equations
\[ \vec{0} = -\frac{\partial}{\partial \lambda} \ln Z \quad m = -\frac{\partial}{\partial d} \ln Z. \tag{A2} \]

Using the expression \( g^T = \frac{1}{2}(I + \vec{r}_T \cdot \vec{\sigma}) (\vec{r}_T = (r_x, -r_y, r_z)) \) the state can be written in the form \( \omega = \frac{1}{Z} e^{-R} \) with
\[ R = A \otimes |0\rangle \langle 0| + B \otimes |1\rangle \langle 1| \tag{A3} \]
\[ A = [\vec{\lambda} + \vec{r}_T] \cdot \vec{\sigma} + dI \]
\[ B = [\vec{\lambda} - \vec{r}_T] \cdot \vec{\sigma} - dI. \]

Since the operators \( A \otimes |0\rangle \langle 0| \) and \( B \otimes |1\rangle \langle 1| \) commute we can write
\[ e^{-R} = e^{-A \otimes |0\rangle \langle 0|} e^{-B \otimes |1\rangle \langle 1|} \]
\[ = e^{-A} \otimes |0\rangle \langle 0| + e^{-B} \otimes |1\rangle \langle 1|. \tag{A4} \]

Having in mind the operator identity
\[ e^{zI + \vec{y} \cdot \vec{\sigma}} = e^z [\cosh |\vec{y}| + \frac{1}{2} \sinh |\vec{y}| \vec{y} \cdot \vec{\sigma}] \tag{A5} \]
we obtain
\[ Z = \text{Tr} e^{-R} = 2(e^{-d \cosh |\vec{\lambda} + \vec{r}_T|} + e^d \cosh |\vec{\lambda} - \vec{r}_T|). \]

Inserting this expression into Eqs.(A2) we get
\[ \vec{0} = S_+ \frac{(\vec{\lambda} + \vec{r}_T)}{|\vec{\lambda} + \vec{r}_T|} + S_- \frac{(\vec{\lambda} - \vec{r}_T)}{|\vec{\lambda} - \vec{r}_T|} \]
\[ m(C_+ + C_-) = C_+ - C_- - 2S_+ \frac{(\vec{\lambda} + \vec{r}_T) \cdot \vec{r}_T}{|\vec{\lambda} + \vec{r}_T|} \tag{A6} \]
where \( S_\pm = e^{\mp d} \sinh |\vec{\lambda} \pm \vec{r}_T| \) and \( C_\pm = e^{\mp d} \cosh |\vec{\lambda} \pm \vec{r}_T| \). From the first of these equations it follows that \( \vec{\lambda} + \vec{r}_T \) and \( \vec{\lambda} - \vec{r}_T \) are collinear, i.e., \( \vec{\lambda} + \vec{r}_T = k(\vec{\lambda} - \vec{r}_T) \).

The case \( d = 0 \) requires that \( \vec{\lambda} = 0, m = 0 \) whatever test state \( \vec{r}_T \) is used. Thus, measuring the mean value \( \langle \sigma_x \rangle = m = 0 \) the MaxEnt results in the state \( \omega = \frac{1}{2} I \otimes I \) and, consequently, the estimated channel acts as follows
\[ \xi \rightarrow \xi' = 2\text{Tr}_1[(\xi^T \otimes I)\omega] = \frac{1}{2} I, \tag{A7} \]
i.e., the whole Bloch sphere is transformed into the total mixture.

For the case \( \vec{\lambda} = 0 \) the first equation implies that either \( d = 0 \) (leads to same solution as before), or \( \vec{r}_T = 0 \). If the test state is chosen to be in total mixture \( (\vec{r}_T = 0) \), the second equation leads to \( d = -\arctanh(m) \), hence the estimated state reads
\[ \omega = \frac{1}{4 \cosh d} e^{-dI \otimes \sigma_z} \]
\[ = \frac{1}{4 \cosh d} I \otimes (\cosh d - \sinh d \sigma_z) \]
\[ = \frac{1}{2} I \otimes (I + m \sigma_z). \tag{A8} \]

The corresponding process \( \mathcal{E}_{\text{est}} \) is given by the identity
\[ \mathcal{E}_{\text{est}}[\xi] = d\text{Tr}_1[(\xi^T \otimes I)\omega], \]
i.e.,
\[ \mathcal{E}_{\text{est}}[\xi] = \frac{1}{2} \text{Tr}_1[(\xi^T \otimes I)(I \otimes (I + m \sigma_z))] \tag{A9} \]
\[ = \frac{1}{2} \text{Tr}_1[\xi (I + m \sigma_z)] \tag{A10} \]
\[ = \frac{1}{2} (I + m \sigma_z). \tag{A11} \]

This transformation maps the whole state space into the single point \( \xi = \frac{1}{2} (I + m \sigma_z) \).

The last family of solutions of MaxEnt conditions is that the vectors \( \vec{\lambda} \) and \( \vec{r}_T \) are collinear. In this case we reduced the number of unknown parameters to \( \lambda = |\vec{\lambda}| \) and \( d \). The first condition out of Eqs. (A6) then reads
\[ 0 = e^{-d \sinh |(\lambda + d)r|} \frac{1}{|\lambda + d|} (\lambda + d) + e^d \sinh |(\lambda - d)r| \frac{1}{|\lambda - d|} (\lambda - d) \]
where we used \( r = |\vec{r}| = |\vec{r}_T| \). Analyzing all possible values for \( \lambda \pm d \) it follows that the absolute values can be omitted and the equations simplify to
\[ 0 = e^{-d} c_+ + e^d c_- \tag{A12} \]
\[ m = \frac{e^{-d} c_+ - e^d c_- - 2re^{-d} s_+}{e^{-d} c_+ + e^d c_-}, \tag{A13} \]
where \( s_\pm = \sinh[(\lambda \pm d)r], c_\pm = \cosh[(\lambda \pm d)r] \). After a short algebra these equations can be rewritten into the form
\[ e^{\lambda r} \cosh[d(1 - r)] = e^{-\lambda r} \cosh[d(1 + r)] \tag{A14} \]
and
\[ m(e^{\lambda r} \cosh[d(1 - r)] + e^{-\lambda r} \cosh[d(1 + r)]) = -e^{\lambda r} \sinh[d(1 - r)] - e^{-\lambda r} \sinh[d(1 + r)]) - 2re^{-d} \sinh[(\lambda + d)r]. \tag{A15} \]
Unfortunately, we cannot give a general solution in a closed form. Consider therefore a special case and let us assume that the test state is pure, i.e., \( r = 1 \). In such case the solution reads

\[
d = \frac{1}{2} \tanh(-m) = \frac{1}{4} \ln \frac{1 - m}{1 + m} \tag{A16}
\]

\[
\lambda = \frac{1}{2} \ln \cosh(2d) \, . \tag{A17}
\]

As a result we get

\[
\omega = \frac{1}{Z} e^{-d}[\cosh(\lambda + d)I - \sinh(\lambda + d)\vec{r}_T \cdot \vec{\sigma}] \otimes |0\rangle\langle 0| + \frac{1}{Z} e^{d}[\cosh(\lambda - d)I - \sinh(\lambda - d)\vec{r}_T \cdot \vec{\sigma}] \otimes |1\rangle\langle 1| \tag{A18}
\]

with \( Z = 2(e^\lambda + e^{-\lambda} \cosh(2d)) \). Let us denote by \( \vec{t} \) the Bloch vector corresponding to a general input state \( \xi \), then the estimated operation is given by the following prescription

\[
\xi \rightarrow \xi' = 2 \text{Tr}_1[(\xi^T \otimes I) \omega]
\]

\[
= \frac{1}{2} \left( 1 + \frac{1}{2} m(1 + \vec{t}_T \cdot \vec{r}_T) \right) |0\rangle\langle 0| + \frac{1}{2} \left( 1 - \frac{1}{2} m(1 + \vec{t}_T \cdot \vec{r}_T) \right) |1\rangle\langle 1| \tag{A19}
\]

where we have used \( \xi = \frac{1}{2}(1 + \vec{t} \cdot \vec{\sigma}) \). Taking into account that \( \vec{r}_T \cdot \vec{r}_T = \vec{r} \cdot \vec{r} \) we can write

\[
\xi \rightarrow \xi' = \frac{1}{2} \left( I + \frac{1}{2} m(1 + \vec{t} \cdot \vec{r}) \vec{\sigma}_z \right) \, . \tag{A20}
\]

In the language of Bloch vectors the transformation reads

\[
\vec{t} \rightarrow \vec{t}' = (0, 0, \frac{1}{2} m[1 + \vec{t} \cdot \vec{r}]) \, . \tag{A20}
\]

**APPENDIX B: MAXENT SOLUTION FOR**

\( C_b = \{ L_I \otimes \vec{\sigma}, L_I \otimes I_0 \} = \{ \vec{m}, \vec{0} \} \)

According to maximum entropy principle the state maximizing the entropy has the form

\[
\omega = \frac{1}{Z} e^{-(\vec{X} \cdot \vec{\sigma}) \otimes -I - I \otimes (\vec{\mu} \cdot \vec{\sigma})}
\]

with

\[
Z = \text{Tr}e^{-(\vec{X} \cdot \vec{\sigma}) \otimes -I - I \otimes (\vec{\mu} \cdot \vec{\sigma})} = (\text{Tr}e^{-\vec{X} \cdot \vec{\sigma}})(\text{Tr}e^{-\vec{\mu} \cdot \vec{\sigma}}) = 4 \cosh(\vec{X}) \cosh(\vec{\mu})
\]

The values of \( \vec{X}, \vec{\mu} \) are given by the following system of equations

\[
\vec{0} = -\frac{1}{Z} \frac{\partial Z}{\partial \lambda} \Rightarrow \vec{0} = -\tanh(\vec{X}) \vec{X}
\]

\[
\vec{m} = -\frac{1}{Z} \frac{\partial Z}{\partial \mu} \Rightarrow \vec{m} = -\tanh(\vec{\mu}) \vec{\mu}
\]

and for the estimated state we get

\[
\omega = \frac{1}{Z} e^{-(\vec{X} \cdot \vec{\sigma}) \otimes -I - I \otimes (\vec{\mu} \cdot \vec{\sigma})} = \frac{1}{4} I \otimes (I + \vec{m} \cdot \vec{\sigma}) \, . \tag{B4}
\]

As a result we found that the estimated channel \( \xi_{\text{est}} \) maps the whole Bloch sphere into the point \( \frac{1}{2}(I + \vec{m} \cdot \vec{\sigma}) \).

---

[1] A. Perez: Quantum Theory: Concepts and Methods, (Kluwer, Dordrecht, 1993)
[2] M.A. Nielsen and I.L. Chuang: Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, 2000)
[3] I. Bengtsson and K. Zyczkowski: Geometry of quantum states: An introduction to quantum entanglement, (Cambridge University Press, Cambridge, 2006)
[4] Quantum State Estimation, edited by M. Paris and J. Reháček, (Springer, 2004)
[5] I. L. Chuang and M. A. Nielsen J. Mod. Phys. 44, 2455-2467 (1997), [arXiv: quant-ph/9610001]
[6] G. M. D’Ariano and P. L. Presti, Phys. Rev. Lett. 91 047902-1 (2003), [arXiv: quant-ph/0211133]
[7] J. B. Altepeter, et al., Phys. Rev. Lett. 90, 193601 (2003)
[8] A. M. Childs, I. L. Chuang, and D. W. Leung, Phys. Rev. A 64, 012314 (2001)
[9] J. O’Brien, et al., Phys. Rev. Lett. 93, 080502 (2004)
[10] Y. S. Weinstein, et al., J. Chem. Phys. 121, 6117-6133 (2004)
[11] M. Howard, et al., New J. Phys. 8, 33 (2006)
[12] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Phys. Rev. A 71, 062310 (2005)
[13] J. Emerson, R. Alicki, K. Zyczkowski, J. Opt. B: Quantum Semiclass. Opt. 7, S347 (2005)
[14] M. Mohseni, and D. A. Lidar, Phys. Rev. Lett. 97, 170501 (2006) [arXive:quant-ph/0601033]
[15] M. Choi, Linear Algebra and Its Applications, 285-290 (1975)
[16] A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972)
[17] E. T. Jaynes: Probability Theory: The Logic of Science, (Cambridge University Press, Cambridge, 2003)
[18] C. E. Shannon, Bell System Tech. J. 27, 379-423, 623-656 (1948)
[19] E. T. Jaynes, in Statistical Physics, K. Ford (ed.), Benjamin, New York, p. 181 (1963)
[20] R. Blankenbecler, and M. H. Partovi, Phys.Rev.Lett. 54, 373 (1985)
[21] V. Bužek, G. Drobný, G. Adam, R. Derka, and P. L. Knight, J. Mod. Optics 44, 2607 (1997)
[22] M. Ziman, M. Plesch, and V. Bužek : Foundations of Physics 36, 127-156 (2006), [arXive:quant-ph/0406088]
[23] B. Schumacher, M. A. Nielsen, Phys.Rev.A 54, 2629 (1996)
[24] B. Schumacher, M. D. Westmoreland, Phys.Rev.A 56, 131 (1997)
[25] A. S. Holevo, IEEE Trans.Inf.Theory 44, 269 (1998), [quant-ph/9611023]
[26] D. Kretschmann, R. Werner, New J. Phys. 6, 26 (2004),
(arXiv:quant-ph/0311037)

[27] Karol Zyczkowski, Ingemar Bengtsson, Open Sys. and Information Dyn. 11, 3-42 (2004)

[28] Wojciech Roga, Mark Fannes, and Karol Zyczkowski, J.Phys.A 41, 035305 (2008)

[29] S.Olivares, M.G.A.Paris, Phys.Rev.A 76, 042120 (2008)

[30] M.Ziman, Phys.Rev.A 77, 062112 (2008), [arXiv:0802.3862]