DISCRETE INGHAM TYPE INEQUALITIES WITH A WEAKENED GAP CONDITION

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Abstract. We establish discrete Ingham type and Haraux type inequalities for exponential sums satisfying a weakened gap condition. They enable us to obtain discrete simultaneous observability theorems for systems of vibrating strings or beams.

1. Introduction

Harmonic analysis is an efficient tool in control theory. For example, various generalizations of a classical theorem of Ingham proved to be very helpful in establishing crucial observability theorems in many recent studies; see, e.g., [8]. The purpose of this work is to prove a new generalization, by weakening a usual gap condition and by replacing the observed integrals by Riemann sums, more realistic from a practical point of view. The theorem is then applied for the solution of two observability problems concerning systems of strings or beams.

Given a strictly increasing sequence \((\omega_k)_k\) of real numbers, we consider functions of the form

\[
x(t) = \sum_{k=-\infty}^{\infty} x_k e^{i\omega_k t}
\]

with complex coefficients \(x_k\). The following generalization of Parseval’s equality, which improved an earlier result of Jaffard, Tucsnak and Zuazua [7], was established in [2]:

**Theorem 1.** Assume that there exists a positive number \(\gamma\) satisfying

\[
\omega_{k+2} - \omega_k \geq 2\gamma \quad \text{for all} \quad k.
\]

Fix \(0 < \gamma_0 \leq \gamma\) arbitrarily and and set

\[
A_1 := \{k \in \mathbb{Z} : \omega_k - \omega_{k-1} \geq \gamma_0 \quad \text{and} \quad \omega_{k+1} - \omega_k \geq \gamma_0\};
\]

\[
A_2 := \{k \in \mathbb{Z} : \omega_k - \omega_{k-1} \geq \gamma_0 \quad \text{and} \quad \omega_{k+1} - \omega_k < \gamma_0\}.
\]

Then for every bounded interval \(I\) of length \(|I| > 2\pi/\gamma\) there exist two positive constants \(c_1\) and \(c_2\) such that

\[
c_1 Q(x) \leq \int_I |x(t)|^2 \, dt \leq c_2 Q(x)
\]

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for all sums of the form (1.1) with square summable coefficients, where we use the notation

\[ Q(x) := \sum_{k \in A_1} |x_k|^2 + \sum_{k \in A_2} |x_k + x_{k+1}|^2 + (\omega_{k+1} - \omega_k)^2(|x_k|^2 + |x_{k+1}|^2). \]

Remark. Under the stronger assumption

\[ \omega_{k+1} - \omega_k \geq \gamma \text{ for all } k \]

our result reduces to a classical theorem of Ingham [6]:

\[ c_1 \sum_{k=-\infty}^{\infty} |x_k|^2 \leq \int_I |x(t)|^2 \, dt \leq c_2 \sum_{k=-\infty}^{\infty} |x_k|^2. \]

We shall establish the following discrete version of Theorem 1:

**Theorem 2.** Assume that there exists a positive number \( \gamma \) satisfying (1.2), and introduce the sets \( A_1, A_2 \) as above. Given \( 0 < \delta \leq \pi/\gamma \) arbitrarily, fix an integer \( J \) such that \( J\delta > \pi/\gamma \). Then there exist two positive constants \( c_1 \) and \( c_2 \), depending only on \( \gamma \) and \( J\delta \), such that

\[ c_1 Q(x) \leq \delta \sum_{j=-J}^{J} |x(j\delta)|^2 \leq c_2 Q(x) \]

for all functions (1.1) whose coefficients satisfy the condition

\[ x_k = 0 \text{ whenever } |\omega_k| > \frac{\pi}{\delta} - \frac{\gamma}{2}. \]

Remarks.

- Under the stronger gap condition (1.3), Theorem 2 reduces to an earlier result proved in [9]:

\[ c_1 \sum_{k=-\infty}^{\infty} |x_k|^2 \leq \delta \sum_{j=-J}^{J} |x(j\delta)|^2 \leq c_2 \sum_{k=-\infty}^{\infty} |x_k|^2 \]

instead of (1.4).

- In view of Remarks 2.6 and 2.7 in [2], (1.4) implies that, more generally,

\[ c_1 Q(x) \leq \delta \sum_{j=-J}^{J} |x(t' + j\delta)|^2 \leq c_2 Q(x) \]

for every \( t' \in \mathbb{R} \), for all functions (1.1) whose coefficients satisfy the condition (1.5). The constants \( c_1, c_2 \) depend only on \( t', \gamma \) and \( J\delta \).

- Theorem 1 follows from Theorem 2. Indeed, fix a bounded interval \( I = [t' - R, t' + R] \) with \( R > \pi/\gamma \), choose \( \delta = R/J \) for every sufficiently large positive integer \( J \), and let \( J \to \infty \) in the resulting estimates.

In the sequel we often write \( A \asymp B \) instead of double inequalities of the form \( c_1 A \leq B \leq c_2 A \) for brevity.

The preceding theorem will enable us to prove discrete observability theorems for systems of vibrating strings and beams. For the latter, we will also need to
investigate what happens when we add a new exponent to the system, i.e., when we consider sums of the form

\[(1.6)\]

\[x(t) = x' e^{i\omega' t} + \sum_{k=-\infty}^{\infty} x_k e^{i\omega_k t}\]

with complex coefficients \(x', x_k\), instead of \((1.1)\), where \(\omega'\) is a real number not belonging to the sequence \((\omega_k)\).

The following result is a discrete version of a generalization of a theorem of Haraux, allowing a weakened gap condition. In order to simplify its statement, let us introduce the quadratic form

\[Q'(x) := |x'|^2 + Q(x)\]

**Theorem 3.** Assume \((1.2)\) and introduce the sets \(A_1, A_2\) as in Theorem 2. Assume that for some positive \(\delta > 0\) and for some positive integer \(J\) there exist two positive constants \(c_1, c_2\), depending only on \(\gamma\) and \(J\delta\), such that

\[(1.7)\]

\[c_1 Q(x) \leq \delta \sum_{j=-J}^{J} |x(j\delta)|^2 \leq c_2 Q(x)\]

for all sums of the form \((1.1)\) with complex coefficients \(x_k\) satisfying \((1.5)\). If \(\omega'\) is a real number not belonging to the sequence \((\omega_k)\), then for every positive integer \(J'\) there exist two positive constants \(c_3, c_4\), depending only on \(\gamma, J\delta, J'\delta\) and

\[\gamma' := \inf_k |\omega_k - \omega'|\]

and another constant \(c'\), depending only on \(\gamma, J'\delta\), such that

\[(1.8)\]

\[c_3 Q'(x) \leq \delta \sum_{j=-J-J'}^{J+J'} |x(j\delta)|^2 \leq c_4 Q'(x)\]

for all sums of the form \((1.6)\) with complex coefficients \(x', x_k\) satisfying \((1.5)\) and

\[(1.9)\]

\[|\omega_k - \omega'| < 2c'/\delta.\]

We may deduce from the preceding theorem the following

**Corollary 4.** Assume \((1.2)\) and introduce the sets \(A_1, A_2\) as in Theorem 2. Assume that for \(R > 0\) there exist two positive constants \(c_1, c_2\), depending only on \(\gamma\) and \(R\), such that

\[c_1 Q(x) \leq \int_{-R}^{R} |x(t)|^2 \, dt \leq c_2 Q(x)\]

for all sums of the form \((1.1)\) with complex coefficients \(x_k\) satisfying \((1.5)\). If \(\omega'\) is a real number not belonging to the sequence \((\omega_k)\), then for every \(R' > R\) there exist two positive constants \(c_3, c_4\), depending only on \(\gamma, R, R'\) and

\[\gamma' := \inf_k |\omega_k - \omega'|\]

and another constant \(c'\), depending only on \(\gamma, R'\), such that

\[c_3 Q'(x) \leq \int_{-R'}^{R'} |x(t)|^2 \, dt \leq c_4 Q'(x)\]

for all sums of the form \((1.6)\) with complex coefficients \(x', x_k\) satisfying \((1.5)\) and \((1.9)\).
Indeed, we may assume without loss of generality that \( R' / R \) is a rational number. Then it suffices to apply Theorem 3 with arbitrarily large integers \( J \) for which \( J' := JR' / R \) is also integer, and with \( \delta := R / J \), and then letting \( J \to \infty \).

Theorems 2 and 3 are proved in the following two sections. They are applied in Section 4 for the solution of two observability problems.

2. Proof of Theorem 2

We proceed in three steps.

First step. We begin by recalling the summatory formula of Poisson: if \( G \) is a function belonging to \( H^1_{0}(-\gamma, \gamma) \) and its Fourier transform is given by the formula

\[
g(t) = \int_{-\infty}^{\infty} G(x) e^{-i tx} \, dx
\]

for all real \( t \), then all functions of the form (1.1) with finitely many nonzero coefficients satisfy the following identity:

\[
\delta \sum_{j=-\infty}^{\infty} g(j\delta)|x(j\delta)|^2 = 2\pi \sum_{k,n=-\infty}^{\infty} G(\omega_k - \omega_n)x_k \overline{x_n}.
\]

For the proof we begin by remarking that since \( \pi / \delta \geq \gamma \), \( G \) vanishes outside the interval

\[
I := \left(-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right),
\]

so that

\[
g(j\delta) = \int_I G(x) e^{-i j\delta x} \, dx
\]

for all integers \( j \). Since \( G \) is Hölder continuous with exponent \( 1/2 \), applying the Dini–Lipschitz theorem (see, e.g., [14]) to the trigonometric orthonormal basis

\[
\sqrt{\frac{\delta}{2\pi}} e^{i j\delta x}, \quad j \in \mathbb{Z}
\]

of \( L^2(I) \), we conclude that

\[
\delta \sum_{j=-\infty}^{\infty} g(j\delta)e^{i j\delta x} = 2\pi G_\delta(x)
\]

for all real \( x \), where \( G_\delta \) denotes the \( 2\pi / \delta \)-periodic function which is equal to \( G \) in the interval \( I \). Observe that

\[
G_\delta(x) = G(x) \quad \text{whenever} \quad |x| \leq \frac{2\pi}{\delta} - \gamma.
\]

Now we have

\[
\delta \sum_{j=-\infty}^{\infty} g(j\delta)|x(j\delta)|^2 = \delta \sum_{k,n=-\infty}^{\infty} x_k \overline{x_n} \sum_{j=-\infty}^{\infty} g(j\delta)e^{i(\omega_k - \omega_n)j\delta}
\]

\[
= 2\pi \sum_{k,n=-\infty}^{\infty} G_\delta(\omega_k - \omega_n)x_k \overline{x_n}
\]

\[
= 2\pi \sum_{k,n=-\infty}^{\infty} G(\omega_k - \omega_n)x_k \overline{x_n}.
\]
The last equality follows from (2.2) and from the fact that if $x_k \neq 0$ and $x_n \neq 0$, then by (1.5) we have necessarily

$$|\omega_k - \omega_n| \leq \frac{2\pi}{\delta} - \gamma.$$  

Second step. We prove the direct inequality (the second inequality in (1.4)). We are going to apply the identity (2.1) with the functions $H, G$ defined by

$$H(x) := \begin{cases} 
\cos^2 \frac{\pi x}{2\gamma} & \text{if } |x| \leq \gamma, \\
0 & \text{if } |x| > \gamma,
\end{cases}$$

the convolution product $G := H \ast H$, and their Fourier transforms $h$ and $g$. One can readily verify (see [2] for details) that there exist two positive constants $\alpha$ and $\beta$ such that

$$0 \leq G(0) - G(x) \leq \alpha x^2 \quad \text{for all } x;$$
$$G(x) = 0 \quad \text{whenever } |x| \geq \gamma;$$
$$g(t) \geq 0 \quad \text{for all } t;$$
$$g(t) \geq \beta \quad \text{whenever } |t| \leq \pi/(2\gamma).$$

We may assume without loss of generality that $\alpha \geq 1.$
Starting with (2.1) and using these relations we obtain the following estimates, where $J'$ denotes the (lower) integer part of $\pi/(2\gamma\delta)$:

$$\frac{\beta}{2\pi}\delta J' \sum_{j=-J'} |x(j\delta)|^2 \leq \frac{\delta}{2\pi} \sum_{j=-\infty}^\infty g(j\delta)|x(j\delta)|^2$$

$$= \sum_{k,n=-\infty}^\infty G(\omega_k - \omega_n)x_k \overline{x_n}$$

$$= \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} G(0)(|x_k|^2 + |x_{k+1}|^2)$$

$$+ \sum_{k \in A_2} G(\omega_{k+1} - \omega_k)(x_k \overline{x_{k+1}} + \overline{x_k}x_{k+1})$$

$$= \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} G(0)|x_k + x_{k+1}|^2$$

$$+ \sum_{k \in A_2} (G(\omega_{k+1} - \omega_k) - G(0))(x_k \overline{x_{k+1}} + \overline{x_k}x_{k+1})$$

$$\leq \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} G(0)|x_k + x_{k+1}|^2$$

$$+ \sum_{k \in A_2} (G(0) - G(\omega_{k+1} - \omega_k)) \cdot (|x_k|^2 + |x_{k+1}|^2)$$

$$\leq \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} G(0)|x_k + x_{k+1}|^2$$

$$+ \sum_{k \in A_2} \alpha(\omega_{k+1} - \omega_k)^2 \cdot (|x_k|^2 + |x_{k+1}|^2)$$

$$\leq \alpha Q(x).$$

We conclude that for $J = J'$ the direct inequality holds with

$$c_2 := \frac{2\pi\alpha}{\beta}.$$

A translation argument in [2], Remark 2.6 shows that we have, more generally,

$$\delta \sum_{j=-J'} |x(t' + j\delta)|^2 \leq \frac{4\pi\alpha}{\beta}(1 + |t'|^2)Q(x)$$

for every real number $t'$. The direct inequality for a general integer $J$ hence follows by covering the set $\{-J, \ldots, J\}$ of consecutive integers by $M$ translates of $\{-J', \ldots, J'\}$ where $M$ denotes the upper integer part of $(2J + 1)/(2J' + 1)$, and summing the $M$ corresponding inequalities.

**Third step.** For the proof of the inverse inequality let us introduce the same function $H$ as above, but define this time $G := R^2H * H + H' * H'$. Denoting by $h$ and $g$ the Fourier transforms of $H$ and $G$, now we have the following properties
with suitable positive constants \( \alpha \) and \( \beta \):

\[
G(0) - G(x) \geq \alpha x^2 \quad \text{if} \quad |x| \leq \gamma;
\]
\[
G(x) = 0 \quad \text{whenever} \quad |x| \geq \gamma;
\]
\[
G(0) > 0 \quad \text{and} \quad G(0) - G(x) > 0 \quad \text{for all} \quad x \neq 0;
\]
\[
g(t) \leq 0 \quad \text{whenever} \quad |t| \geq R;
\]
\[
g(t) \leq \beta \quad \text{for all} \quad t.
\]

We may assume without loss of generality that \( \alpha \leq G(0) \).

Applying (2.1) and using these relations we obtain the following estimates, where \( J' \) denotes the upper integer part of \( \pi / (2\gamma \delta) \):

\[
\frac{\beta}{2\pi} \sum_{j=-J'}^{J'} |x(j\delta)|^2 \geq \frac{\delta}{2\pi} \sum_{j=-\infty}^{\infty} g(j\delta)|x(j\delta)|^2 = \sum_{k,n=\infty} G(\omega_k - \omega_n)x_k \overline{x_n} = \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} G(0)(|x_k|^2 + |x_{k+1}|^2)
\]
\[
+ \sum_{k \in A_2} G(\omega_{k+1} - \omega_k)(x_k \overline{x_{k+1}} + \overline{x_k} x_{k+1}) = \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} G(0)(|x_k|^2 + |x_{k+1}|^2)
\]
\[
+ \sum_{k \in A_2} G(\omega_{k+1} - \omega_k) \cdot (|x_k + x_{k+1}|^2 - |x_k|^2 - |x_{k+1}|^2) = \sum_{k \in A_1} G(0)|x_k|^2 + \sum_{k \in A_2} (G(0) - G(\omega_{k+1} - \omega_k))(|x_k|^2 + |x_{k+1}|^2)
\]
\[
+ \sum_{k \in A_2} G(\omega_{k+1} - \omega_k) \cdot |x_k + x_{k+1}|^2.
\]

Putting \( y := \omega_{k+1} - \omega_k \) for brevity, it remains to show that

\[
|x_k + x_{k+1}|^2 + y^2(|x_k|^2 + |x_{k+1}|^2)
\]

is majorized by a constant multiple of

\[
G(y)|x_k + x_{k+1}|^2 + (G(0) - G(y))(|x_k|^2 + |x_{k+1}|^2)
\]

for all \( 0 < y < \gamma \). We show the stronger inequality

\[
|x_k + x_{k+1}|^2 + \frac{G(0) - G(y)}{\alpha}(|x_k|^2 + |x_{k+1}|^2)
\]
\[
\leq \frac{G(y)}{\alpha} |x_k + x_{k+1}|^2 + \frac{3(G(0) - G(y))}{\alpha}(|x_k|^2 + |x_{k+1}|^2)
\]

or equivalently, that

\[
(\alpha - G(y)) |x_k + x_{k+1}|^2 \leq 2(G(0) - G(y))(|x_k|^2 + |x_{k+1}|^2).
\]

This is obvious for \( G(y) \geq \alpha \) because the right-hand side is nonnegative. If \( G(y) < \alpha \), then the inequality follows from our assumption \( \alpha \leq G(0) \) and from the elementary estimate \( |x_k + x_{k+1}|^2 \leq 2(|x_k|^2 + |x_{k+1}|^2) \).
3. Proof of Theorem 3

Proof of the direct part of (1.8). Applying the second inequality of (1.7) to the function
\[ z(t) := x(t) - x'e^{i\omega t} \]
instead of \( x(t) \), we obtain that
\[ \delta \sum_{j=-J}^{J} |x(j\delta)|^2 \leq 2\delta \sum_{j=-J}^{J} |z(j\delta)|^2 + 2\delta \sum_{j=-J}^{J} |x'e^{i\omega' j\delta}|^2 \]
\[ \leq 2c_2 Q(x) + 2\delta(2J + 1)|x'|^2 \]
\[ \leq \max\{2c_2, 6J\delta\} Q'(x). \]
Using [2], Remark 2.6 this inequality implies that, more generally,
\[ \delta \sum_{j=m-J}^{m+J} |x(j\delta)|^2 \leq \max\{4c_2, 12J\delta\} (1 + |m\delta|^2) Q'(x) \] (3.1)
for every integer \( m \). (In order to use this remark, we also apply Remark 2.5 of that paper which enables us to choose \( 0 < \gamma_0 \leq \gamma \) sufficiently small so that \( |\omega' - \omega_k| < \gamma_0 \) for all \( k \). Then \( \omega' \) belongs to \( A_1 \) in the extended exponent set, so that the corresponding quadratic form is \( Q'(x) \).)

Now the second inequality of (1.8) follows easily by covering the set \( \{-J - J', \ldots, J + J'\} \) of consecutive integers by \( M := \lfloor (2J + 2J' + 1)/(2J + 1) \rfloor \) translates of \( \{-J, \ldots, J\} \), and summing the corresponding inequalities (3.1). Since \( |m| \leq J' \) in all these inequalities, we obtain that
\[ \delta \sum_{j=-J-J'}^{J+J'} |x(j\delta)|^2 \leq c_4 Q'(x) \]
with
\[ c_4 = \left(1 + \frac{2J + 2J' + 1}{2J + 1}\right) \max\{4c_2, 12J\delta\} (1 + |J'\delta|^2). \]

Proof of the inverse part of (1.8). For \( x \) given by (1.6), the formula
\[ y(t) := x(t) - \frac{1}{2J'} \sum_{n=-J'}^{J'-1} e^{-i\omega n\delta} x(t + n\delta) \]
defines a function \( y \) of the form (1.1): an easy computation shows that
\[ y(t) = \sum_{k=-\infty}^{\infty} \left[ 1 - \frac{1}{2J'} \sum_{n=-J'}^{J'-1} e^{i(\omega_k - \omega') n\delta} \right] x_k e^{i\omega_k t} =: \sum_{k=-\infty}^{\infty} y_k e^{i\omega_k t}. \]
Observe that
\[ \frac{1}{2J'} \sum_{n=-J'}^{J'-1} e^{i(\omega_k - \omega') n\delta} = \frac{1}{2J'} e^{i(\omega_k - \omega') J' \delta} - e^{-i(\omega_k - \omega') J' \delta} e^{i(\omega_k - \omega') J' \delta - 1} \]
\[ = \frac{\sin(\omega_k - \omega') J' \delta}{J' e^{i(\omega_k - \omega') \delta} - 1} \]
and therefore
\[
\left| \frac{1}{2 J'} \sum_{n=1}^{J'-1} e^{i(\omega_k - \omega')n\delta} \right| = \frac{\left| \sin(\omega - \omega')J'\delta \right|}{J'(e^{i(\omega - \omega')\delta} - 1)}
\]
\[
= \frac{\left| \sin(\omega_k - \omega')J'\delta \right|}{2J'\sin(\omega_k - \omega')\delta/2}
\]
\[
= \frac{\sin(\omega_k - \omega')J'\delta}{(\omega_k - \omega')J'\delta} \cdot \frac{(\omega_k - \omega')\delta/2}{\sin(\omega_k - \omega')\delta/2} =: \varepsilon_k.
\]

For the sequel we need the following

**Lemma 5.**

(a) There exists a constant \( c' \), depending only on \( \inf_k |\omega_k - \omega'| \) and \( J'\delta \), such that
\[
\varepsilon := \sup_k \varepsilon_k < 1
\]
where the supremum is taken over the indices \( k \) satisfying (1.9).

(b) The function
\[
\omega \mapsto f(\omega) := \frac{\sin(\omega - \omega')J'\delta}{J'(e^{i(\omega - \omega')\delta} - 1)}
\]
is Lipschitzian in the interval \((\omega' - 2c'/\delta, \omega' + 2c'/\delta)\) with some constant \( L \), depending only on \( J'\delta \).

**Proof.**

(a) Since
\[
\inf_k |\omega_k - \omega'| > 0,
\]
we have
\[
\varepsilon' = \varepsilon'(J'\delta) := \sup_k \left| \frac{\sin(\omega_k - \omega')J'\delta}{(\omega_k - \omega')J'\delta} \right| < 1.
\]
It suffices to choose \( c' > 0 \) sufficiently small so that
\[
\inf_{0<|x|<c'} \frac{\sin x}{x} > \varepsilon'.
\]

(b) First we note that under the condition (1.9) we have
\[
\left| \frac{e^{i(\omega - \omega')\delta} - 1}{\delta} \right| = \left| \frac{\sin(\omega - \omega')\delta/2}{\delta/2} \right| > \varepsilon' |\omega - \omega'| \geq \varepsilon' \gamma'.
\]
Therefore
\[
|f'(\omega)| = \left| J'\delta \cos(\omega - \omega')J'\delta \right| \left| J'(e^{i(\omega - \omega')\delta} - 1) \right| - \left| \frac{\sin(\omega - \omega')J'\delta iJ'\delta e^{i(\omega - \omega')\delta}}{[J'(e^{i(\omega - \omega')\delta} - 1)]^2} \right|
\]
\[
\leq \left| \frac{\delta}{e^{i(\omega - \omega')\delta} - 1} \right| + \frac{1}{J'\delta} \left| e^{i(\omega - \omega')\delta} - 1 \right|^2
\]
\[
\leq \frac{1}{\varepsilon' \gamma'} + \frac{1}{(J'\delta)(\varepsilon' \gamma')^2}.
\]
The Lipschitz property follows by the mean value theorem because the constant on the right-hand depends only on \( \gamma' \) and \( J'\delta \).
It follows from part (a) of the lemma that

\[(3.2) \quad |x_k - y_k| \leq \varepsilon |x_k| \quad \text{for all } k \text{ satisfying (1.9).}\]

We claim that

\[(3.3) \quad Q(x) \leq c_0 Q(y)\]

with a suitable constant \(c_0\) (depending on \(\varepsilon\)). If \(k \in A_1\), then we deduce from (3.2) that

\[(3.4) \quad |x_k|^2 \leq (1 - \varepsilon)^{-2} |y_k|^2.\]

using part (b) of the above, in case \(k \in A_2\) we have

\[
| (x_k - y_k) + (x_{k+1} - y_{k+1}) | \leq |f(\omega_k)| \cdot |x_k + x_{k+1}| + |f(\omega_k) - f(\omega_{k+1})| \cdot |x_{k+1}|
\leq \varepsilon |x_k + x_{k+1}| + L|\omega_k - \omega_{k+1}| \cdot |x_{k+1}|.
\]

Hence

\[
(1 - \varepsilon)|x_k + x_{k+1}| \leq |y_k + y_{k+1}| + L|\omega_k - \omega_{k+1}| \cdot (1 - \varepsilon)^{-1} |y_{k+1}|
\]

and therefore

\[
|x_k + x_{k+1}| \leq (1 - \varepsilon)^{-1} |y_k + y_{k+1}| + L|\omega_k - \omega_{k+1}| \cdot (1 - \varepsilon)^{-2} |y_{k+1}|.
\]

Using this relation we obtain that

\[(3.5) \quad |x_k + x_{k+1}|^2 + (\omega_k - \omega_{k+1})^2 (|x_k|^2 + |x_{k+1}|^2)
\leq 2(1 - \varepsilon)^{-2} |y_k + y_{k+1}|^2
+ 2L^2(1 - \varepsilon)^{-4} + (1 - \varepsilon)^{-2}) \cdot (\omega_k - \omega_{k+1})^2 (|y_k|^2 + |y_{k+1}|^2).
\]

Finally, (3.3) follows from (3.4) and (3.5).

Next we show that

\[(3.6) \quad \sum_{j=-J}^{J} |y(j\delta)|^2 \leq 4 \sum_{m=-J'}^{J+J'-1} |x(m\delta)|^2.
\]

Indeed, using the Cauchy–Schwarz inequality we have

\[
|y(t)|^2 \leq 2|x(t)|^2 + 2\left| \frac{1}{2J'} \sum_{n=-J'}^{J'-1} e^{-i\omega' n\delta} x(t + n\delta) \right|^2
\leq 2|x(t)|^2 + \frac{2}{4(J')^2} \left( \sum_{n=-J'}^{J'-1} |e^{-i\omega' n\delta}|^2 \right) \cdot \left( \sum_{n=-J'}^{J'-1} |x(t + n\delta)|^2 \right)
= 2|x(t)|^2 + \frac{1}{J'} \sum_{n=-J'}^{J'-1} |x(t + n\delta)|^2
\]
for every $t$. Hence

$$
\sum_{j=-J}^{J} |y(j\delta)|^2 \leq 2 \sum_{j=-J}^{J} |x(j\delta)|^2 + \frac{1}{J'} \sum_{j=-J}^{J} \sum_{m=-J'}^{J'-1} |x(j\delta + n\delta)|^2
$$

$$
\leq 2 \sum_{j=-J}^{J} |x(j\delta)|^2 + 2 \sum_{m=-J-J'}^{J'+J'-1} |x(m\delta)|^2
$$

$$
\leq 4 \sum_{m=-J-J'}^{J+J'-1} |x(m\delta)|^2.
$$

Now applying the first inequality of (1.7) for $y$ instead of $x$ and using (3.3) and (3.6) we obtain that

$$
(3.7) \quad Q(x) \leq c_0 Q(y) \leq \frac{c_0}{c_1} \delta \sum_{j=-J}^{J} |y(j\delta)|^2 \leq \frac{4c_0}{c_1} \delta \sum_{m=-J-J'}^{J+J'-1} |x(m\delta)|^2.
$$

Furthermore, using the function $z$ as introduced in the proof of the direct inequality,

$$
|x'|^2 = \frac{1}{2J+2J'} \sum_{m=-J-J'}^{J+J'-1} |x'e^{i\omega' m\delta}|^2
$$

$$
\leq \frac{1}{J+J'} \sum_{m=-J-J'}^{J+J'-1} |x(m\delta)|^2 + |z(m\delta)|^2.
$$

Applying to $z$ the already proved direct inequality and then the inequality (3.7), we obtain that

$$
\delta \sum_{m=-J-J'}^{J+J'-1} |z(m\delta)|^2 \leq c_4 Q'(z) = c_4 Q(x) \leq \frac{4c_0c_4}{c_1} \delta \sum_{m=-J-J'}^{J+J'-1} |x(m\delta)|^2.
$$

Combining this with (3.8) we get

$$
(3.9) \quad |x'|^2 \leq \left(1 + \frac{4c_0c_4}{c_1}\right) \frac{1}{J+J'} \sum_{m=-J-J'}^{J+J'-1} |x(m\delta)|^2.
$$

Finally, we conclude from (3.7) and (3.9) that

$$
Q'(x) \leq c\delta \sum_{m=-J-J'}^{J+J'-1} |x(m\delta)|^2
$$

avec

$$
c := \max \left\{ \frac{4c_0}{c_1}, \left(1 + \frac{4c_0c_4}{c_1}\right) \frac{1}{J\delta + J'\delta} \right\}.
$$
4. Simultaneous observability of strings and beams

Fix a number $0 < a < 1$ arbitrarily and consider the following problem:

$$\begin{align*}
& u_{tt} - u_{xx} = 0 \quad \text{in} \quad (0, a) \times \mathbb{R}, \\
& u_{tt} - u_{xx} = 0 \quad \text{in} \quad (a, 1) \times \mathbb{R}, \\
& u(0, \cdot) = u(a, \cdot) = u(1, \cdot) = 0 \quad \text{in} \quad \mathbb{R}, \\
& u(\cdot, 0) = u_{0,a} \quad \text{and} \quad u_t(\cdot, 0) = u_{1,a} \quad \text{in} \quad (0, a), \\
& u(\cdot, 0) = u_{0,1-a} \quad \text{and} \quad u_t(\cdot, 0) = u_{1,1-a} \quad \text{in} \quad (a, 1).
\end{align*}$$

(4.1)

This problem is well posed for

$$u_{0,a} \in H^1_0(0, a), \quad u_{1,a} \in L^2(0, a), \quad u_{0,1-a} \in H^1_0(a, 1), \quad \text{and} \quad u_{1,1-a} \in L^2(a, 1).$$

The following result follows at once from the proof of part (a) Theorem 5.1 in [2] if we apply Theorem 2 above instead of its continuous version. Set

$$\gamma = \frac{\pi}{2} \min\left\{ \frac{1}{a}, \frac{1}{1-a} \right\}.$$

Theorem 6. The following estimate holds for almost every $0 < a < 1$ and for every $\varepsilon > 0$. Given $0 < \delta \leq \frac{\pi}{\gamma}$ arbitrarily, fix an integer $J$ such that $J\delta > 2 \max\{a, 1-a\}$. For every $t' \in \mathbb{R}$, the inequality

$$\|u_0\|_{H^{-\varepsilon}(0,1)}^2 + \|u_1\|_{H^{-\varepsilon}(0,1)}^2 \leq C\delta \sum_{j=-J}^J |u_x(a - 0, t' + j\delta) - u_x(a + 0, t' + j\delta)|^2$$

is satisfied for all solutions of (4.1) whose initial data are $u_{0,a}$, $u_{1,a}$ linear combinations of the basis functions

$$\sin(n\pi a^{-1}x), \quad n \leq \frac{a}{\delta} - \frac{1}{4} \min\left\{ \frac{1}{a}, \frac{1}{1-a} \right\}$$

and whose initial data are $u_{0,1-a}$, $u_{1,1-a}$ linear combinations of the basis functions

$$\sin(m\pi(1-a)^{-1}x), \quad m \leq \frac{1-a}{\delta} - \frac{1}{4} \min\left\{ \frac{1}{a}, \frac{1}{a} \right\}.$$

Now consider the following problem:

$$\begin{align*}
& u_{tt} + u_{xxxx} = 0 \quad \text{in} \quad (0, a) \times \mathbb{R}, \\
& u_{tt} + u_{xxxx} = 0 \quad \text{in} \quad (a, 1) \times \mathbb{R}, \\
& u_{xx}(0, \cdot) = u_{x}(a, \cdot) = u_{xx}(1, \cdot) = 0 \quad \text{in} \quad \mathbb{R}, \\
& u(\cdot, 0) = u_{0,a} \quad \text{and} \quad u_t(\cdot, 0) = u_{1,a} \quad \text{in} \quad (0, a), \\
& u(\cdot, 0) = u_{0,1-a} \quad \text{and} \quad u_t(\cdot, 0) = u_{1,1-a} \quad \text{in} \quad (a, 1).
\end{align*}$$

(4.2)

This system models two vibrating beams with simply supported endpoints, one of which is common to both beams. It is well posed for initial data satisfying

$$u_{0,a} \in H^1_0(0, a), \quad u_{1,a} \in H^{-1}(0, a), \quad u_{0,1-a} \in H^1_0(a, 1), \quad \text{and} \quad u_{1,1-a} \in H^{-1}(a, 1).$$

The following result follows at once from the proof of part (a) Theorem 6.1 in [2] if we apply Theorem 2 above instead of its continuous version:
Theorem 7. The following estimate holds for almost every $0 < a < 1$ and for every $\varepsilon > 0$. Given $\gamma > 0$ and $0 < \delta \leq \pi/\gamma$ arbitrarily, fix an integer $J$ such that $J\delta > \pi/\gamma$. For every $t' \in \mathbb{R}$, the inequality
\[ \|u_0\|_{H^{1-\varepsilon}(0,1)}^2 + \|u_1\|_{H^{-1+\varepsilon}(0,1)}^2 \leq C\delta \sum_{j=-J}^{J} |u_x(a-0,t'+j\delta) - u_x(a+0,t'+j\delta)|^2 \]
is satisfied for all solutions of (4.2) whose initial data are $u_{0,a}$, $u_{1,a}$ linear combinations of the basis functions
\[ \sin(n\pi a^{-1}x), \quad n \leq \frac{a}{\pi} \sqrt{\frac{\pi}{\delta} - \frac{\gamma}{2}} \]
and whose initial data are $u_{0,1-a}$, $u_{1,1-a}$ linear combinations of the basis functions
\[ \sin(m\pi (1-a)^{-1}x), \quad m \leq \frac{1-a}{\pi} \sqrt{\frac{\pi}{\delta} - \frac{\gamma}{2}}. \]

References

[1] C. Baiocchi, V. Komornik and P. Loreti, Théorèmes du type Ingham et application, à la théorie du contrôle, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), 453–458.
[2] C. Baiocchi, V. Komornik and P. Loreti, Ingham type theorems and applications to control theory, Bol. Un. Mat. Ital. B (8) 2 (1999), no. 1, 33–63.
[3] C. Baiocchi, V. Komornik and P. Loreti, Généralisation d’un théorème de Beurling et application à la théorie du contrôle, C. R. Acad. Sci. Paris Sér. I Math. 330 (4) (2000) 281–286.
[4] C. Baiocchi, V. Komornik and P. Loreti, Ingham-Beurling type theorems with weakened gap conditions, Acta Math. Hungar. 97 (1-2) (2002), 55–95.
[5] A. Haraux, Séries lacunaires et contrôle semi-interne des vibrations d’une plaque rectangulaire, J. Math. Pures Appl. 68 (1989), 457–465.
[6] A. E. Ingham, Some trigonometrical inequalities with applications in the theory of series, Math. Z. 41 (1936), 367–379.
[7] S. Jaffard, M. Tucsnak and E. Zuazua, On a theorem of Ingham, J. Fourier Anal. Appl. 3 (1997), no. 5, 577–582.
[8] V. Komornik and P. Loreti, Fourier Series in Control Theory, Springer-Verlag, New York, 2005.
[9] V. Komornik and P. Loreti, Semi-discrete Ingham type inequalities, Appl. Math. Optim., to appear.
[10] J.-L. Lions, Exact controllability, stabilizability, and perturbations for distributed systems, Siam Rev. 30 (1988), 1–68.
[11] J.-L. Lions, Contrôlabilite exacte et stabilisation de systèmes distribués I-II, Masson, Paris, 1988.
[12] P. Loreti, On some gap theorems, European women in mathematics—Marseille 2003, 39–45, CWI Tract, 135, Centrum Wisk. Inform., Amsterdam, 2005.
[13] P. Loreti and M. Mehrenberger, An Ingham type proof for a bigrid observability theorem, Prépublication de l’IRMA, no. 2005–012, Université Louis Pasteur, Strasbourg.
[14] A. Zygmund, Trigonometric Series I-II, Cambridge University Press, 1959.

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