Misspecification Analysis of High-Dimensional Random Effects Models for Estimation of Signal-to-Noise Ratios

Xiaohan Hu\textsuperscript{1} and Xiaodong Li\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of California, Davis
\textsuperscript{2}Department of Statistics, University of California, Davis

Abstract

Estimation of signal-to-noise ratios and residual variances in high-dimensional linear models has various important applications including, e.g. heritability estimation in bioinformatics. One commonly used estimator, usually referred to as REML, is based on the likelihood of the random effects model, in which both the regression coefficients and the noise variables are respectively assumed to be i.i.d Gaussian random variables. In this paper, we aim to establish the consistency and asymptotic distribution of the REML estimator for the SNR, when the actual coefficient vector is fixed, and the actual noise is heteroscedastic and correlated, at the cost of assuming the entries of the design matrix are independent and skew-free. The asymptotic variance can be also consistently estimated when the noise is heteroscedastic but uncorrelated. Extensive numerical simulations illustrate our theoretical findings and also suggest some assumptions imposed in our theoretical results are likely relaxable.

1 Introduction

Estimation and inference for signal-to-noise ratios (SNR) as well as residual variances in high-dimensional linear models are fundamental statistical problems with various important applications. A notable application of SNR estimation is the heritability estimation (Falconer, 1961) in genome wide association studies (GWAS), which aims to study how much of the variance of certain phenotype can be explained by genetic variations. Another important application is regarding how to select tuning parameters in regularized regression such as Lasso and Ridge regression (Sun and Zhang, 2012; Dicker, 2014; Janson et al., 2017; Dicker and Erdogdu, 2016; Dobriban and Wager, 2018). One important method to estimate the SNR in high-dimensional linear models with modern applications is the random effects likelihood based estimator (Yang et al., 2011a; Gusev et al., 2014; De los Campos et al., 2015; Yang et al., 2017; Steinsaltz et al., 2018; Ma and Dicker, 2019), which is usually referred to as restricted maximum likelihood, or REML, in the literature (Jiang, 2007). Asymptotic analysis for REML under linear mixed effects models is a well-studied topic in the statistical literature; see e.g. Hartley and Rao (1967), Jiang (1996), Rao (1997), and Jiang (2007).

An interesting line of work in the literature investigates the asymptotic behavior of random effects likelihood estimators under misspecified models, i.e., the true model for the
coefficient vector does not follow the postulated i.i.d. Gaussian model. In fact, back to Jiang (1996), consistency and asymptotic normality have been established for Gaussian random effects likelihood estimators even if the coefficient vector consists of i.i.d. but non-Gaussian components. A recent notable paper Jiang et al. (2016) shows that such estimators can be consistent and asymptotically normal even the true model follows a sparse random effects model. Model misspecification analysis for REML has also been extended to the case where the coefficient vector can be a general fixed one, at the cost of the assumption that the design matrix consists of i.i.d. Gaussian entries (Dicker and Erdogdu, 2016). Their analysis relies crucially on the rotational invariance of the Gaussian design matrix, and also employs some general normal approximation tools developed in Dicker and Erdogdu (2017).

Note that beyond random effects likelihood estimates of the SNR or residual variances, other methods have also been proposed in the literature. Examples include the method of moments (Haseman and Elston, 1972; Dicker, 2014), EigenPrism (Janson et al., 2017), and Lasso and sparsity based methods (Sun and Zhang, 2012; Fan et al., 2012; Bayati et al., 2013). Under high-dimensional settings, unless the coefficient is very sparse, the empirical performance of estimation of SNR, heritability or noise variance by REML is in general comparable or much better than that of the above alternative methods; See extensive simulation studies conducted in Dicker and Erdogdu (2016).

1.1 High-dimensional Linear Models with Heteroscedastic and Correlated Noise

We focus on the following high-dimensional linear model throughout this paper

\[ y = Z\beta + \epsilon, \]  

where \( Z \) is an \( n \times p \) design matrix with \( p \) being allowed to be greater than \( n \), \( \beta \) is the vector of regression coefficients, and \( y \) is the response vector. For the noise vector \( \epsilon \), we assume it satisfies \( \epsilon \sim \mathcal{N}_n(0, \Sigma_\epsilon) \), where \( \Sigma_\epsilon \) is a positive definite matrix. This implies that we allow for correlated and heteroscedastic noise in the linear model. In particular, we denote the diagonal entries of \( \Sigma_\epsilon \) as \( \sigma_1^2, \ldots, \sigma_n^2 \). Also denote the average noise level as \( \sigma_0^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \). Our goal is to make inference about the signal-to-noise ratio (SNR) parameter

\[ \gamma_0 := \frac{\|\beta\|^2}{\sigma_0^2}. \]

1.2 REML Based on Homogeneous and Gaussian Random Effects

As aforementioned, one common SNR estimator in practice is based on the likelihood of the Gaussian random effects model, in which the coefficient vector is modeled as \( p^{-1/2} \alpha \), where \( \alpha \) is assumed to consist of i.i.d. \( \mathcal{N}(0, \sigma_0^2) \) variables. In addition, the noise terms are assumed to be independent and follow the same distribution \( \mathcal{N}(0, \sigma_\epsilon^2) \). Comparing the true model and the postulated model, it is clear that \( \sigma_0^2 \) corresponds to \( \sigma_\epsilon^2 \), \( \|\beta\|^2 \) corresponds to \( \sigma_\alpha^2 \), and \( \gamma_0 = \|\beta\|^2/\sigma_0^2 \) corresponds to \( \gamma := \sigma_\alpha^2/\sigma_\epsilon^2 \). Based on this postulated homogeneous and Gaussian random effects model, REML estimation, i.e. maximum likelihood estimation, can be derived for the variance components \( \sigma_\alpha^2 \) and \( \sigma_\epsilon^2 \) (Jiang, 2007). In fact, under the
above Gaussian random effects model, there holds $y \sim \mathcal{N}_n(0, \Omega)$, where

$$\Omega = \Omega(\sigma_\varepsilon^2, \sigma_\alpha^2) := \sigma_\varepsilon^2 I_n + \frac{\sigma_\alpha^2}{p} Z Z^\top := \sigma_\varepsilon^2 V_\gamma,$$

and

$$V_\gamma = I_n + \frac{\gamma}{p} Z Z^\top. \quad (2)$$

Then, the log-likelihood function for $(\sigma_\varepsilon^2, \sigma_\alpha^2)$ is given as below:

$$l(\sigma_\varepsilon^2, \sigma_\alpha^2) = c - \frac{1}{2} \log \det(\Omega) - \frac{1}{2} y^\top \Omega^{-1} y,$$

where $c$ is a constant. By taking the partial derivatives of the log-likelihood with respect to $\sigma_\varepsilon^2$ and $\sigma_\alpha^2$ to obtain the score functions, we got the following likelihood equations:

$$S_{\sigma_\varepsilon^2}(\sigma_\varepsilon^2, \sigma_\alpha^2) := \frac{1}{2} y^\top \Omega^{-1} y - \frac{1}{2} \text{trace}(\Omega^{-1}) = 0$$

$$S_{\sigma_\alpha^2}(\sigma_\varepsilon^2, \sigma_\alpha^2) := \frac{1}{2} y^\top \Omega^{-1} Z Z^\top \Omega^{-1} y - \frac{1}{2} \text{trace}\left(\Omega^{-1} Z Z^\top \Omega^{-1}\right) = 0.$$

By the fact that $\frac{1}{p} Z Z^\top = \frac{1}{p}(V_\gamma - I_n)$, the above set of equations can yield a single equation about the SNR $\gamma = \sigma_\alpha^2 / \sigma_\varepsilon^2$:

$$\Delta(\gamma) := y^\top B_\gamma y = 0. \quad (3)$$

where

$$B_\gamma = \frac{V_\gamma^{-1}}{n} - \frac{V_\gamma^{-2}}{\text{trace}(V_\gamma^{-1})}. \quad (4)$$

Let $\hat{\gamma}$ be a solution to (3), which is referred to the (misspecified) REML estimator of the true SNR $\gamma_0 = \|\beta\|^2 / \sigma_0^2$.

### 1.3 Misspecification Analysis of REML

We aim to study the consistency and asymptotic distribution of $\hat{\gamma}$ when the Gaussian random effects model is significantly misspecified, i.e., the actual coefficient vector $\beta$ is a general fixed one, and the actual noise $\varepsilon$ is heteroskedastic and correlated. Certainly, there is a trade-off between the misspecification on $\beta$ and $\varepsilon$, and the assumption on the design matrix $Z$. Our main results, which will be presented in the next section, assert that the consistency and asymptotic distribution of $\hat{\gamma}$ can be rigorously established as long as the entries in $Z$ are independent, symmetric and sub-Gaussian standardized random variables. Here the skew-free assumption is imposed basically for technical reasons, and we will employ numerical simulations to show that this assumption might be relaxable.

Our misspecification analysis is conducted under the asymptotically proportional setting $n, p \to \infty$ such that $n/p \to \tau > 0$, where $1/\tau$ is usually referred to as the limiting aspect ratio in the literature.

In our main results, we will show that the asymptotic variance of $\sqrt{n} \hat{\gamma}$ only relies on the aspect ratio $1/\tau$, the true SNR $\gamma_0$, and a parameter $\kappa$ that characterizes both heterogeneity and correlation of noise terms. In order to estimate the variance and thereby make inferences
on the true SNR $\gamma_0$, we also need to estimate both the average noise level $\sigma_0^2$ and the parameter $\kappa$. In fact, with the SNR estimate $\hat{\gamma}$, based on the postulated Gaussian random effects model, we can estimate $\sigma_0^2$ through

$$\hat{\sigma}^2 = \frac{1}{n} \mathbf{y}^\top \mathbf{V}_{\hat{\gamma}}^{-1} \mathbf{y}. \quad (5)$$

One intuition of this estimator is the following identity based on the postulated (and mis-specified) Gaussian and homogeneous random effects model

$$\mathbb{E}[\mathbf{y}^\top \mathbf{V}_{\gamma}^{-1} \mathbf{y}] = \mathbb{E}[\mathbf{V}_{\gamma}^{-1} \mathbf{y} \mathbf{y}^\top] = \mathbb{E}[\mathbf{V}_{\gamma}^{-1} \mathbf{\Omega}] = n\sigma_\varepsilon^2.$$

The estimation of $\kappa$ is in general difficult under the case of correlated noise. However, when the noise is heterogeneous but uncorrelated, there is a natural approach to estimating $\kappa$. We will elaborate on this estimation in the next section.

### 1.4 Organization of the Paper

This paper is organized as follows. In Section 2, we introduce our main results that characterize the conditions on the design matrix $\mathbf{Z}$, the fixed regression vector $\mathbf{\beta}$, and the noise vector $\varepsilon$, under which the consistency and asymptotic distribution of the REML $\hat{\gamma}$ can be derived. We will also show how to estimate the asymptotic variance when the noise is known to be heterogeneous but uncorrelated. In Section 3, extensive numerical simulations are conducted to demonstrate the consistency and sampling distribution of $\hat{\gamma}$ under different settings to justify our theoretical results empirically. Coverage properties of plug-in confidence intervals for the SNR under the case of independent heterogeneous noise will also be illustrated. The proofs of our main results are given in Section 4, while some preliminary tools as well as proofs of important supporting lemmas are deferred to the appendix. In Section 5, we give a summary of our contributions, and also discuss several remaining questions that we intend to study in future.

### 2 Main Results

Our main goal in this paper is to study the consistency and asymptotic distribution of the REML estimator of SNR, $\hat{\gamma}$, which is the solution to the estimating equation (3) derived from the Gaussian homogeneous random effects model, if the actual coefficient vector $\mathbf{\beta}$ is a general fixed one, and the actual noise $\varepsilon$ is heteroscedastic and correlated. In addition, when the noise is heteroscedastic and uncorrelated, we are also interested in estimating the asymptotic variance consistently, such that inference on the SNR $\gamma_0$ can be conducted. We first introduce our result on the consistency of $\hat{\gamma}$:

**Theorem 2.1.** Consider the linear model (1) with the asymptotic setting $n, p \to \infty$ such that $\sqrt{n} \frac{n}{p} - \tau \to 0$, where $\tau > 0$ is a fixed constant. Assume that the entries of the design matrix $\mathbf{Z}$ are independent, symmetric, sub-Gaussian, and unit-variance random variables, and their maximum sub-Gaussian norm is uniformly upper bounded by some numerical constant $C_0$. Let $\varepsilon$ be the vector of correlated and heteroscedastic Gaussian noise: $\varepsilon \sim \mathcal{N}_n(\mathbf{0}, \Sigma_\varepsilon)$ with variances (diagonal entries) $\sigma_1^2, \ldots, \sigma_n^2$, so that
(i) $\max_{i \in [n]} \sigma_i^2$ is uniformly bounded by $C_0$;

(ii) $\frac{1}{n} \sum_{i=1}^n \sigma_i^2 = \sigma_0^2$, where $\sigma_0^2$ is set to be fixed for all $n$;

(iii) $\|\Sigma_\epsilon\|_F = o(n)$.

Let $\beta$ be the coefficient vector with fixed two-norm $\|\beta\|^2 > 0$ for all $n$, which implies the SNR $\gamma_0 := \|\beta\|^2/\sigma_0^2$ is fixed for all $n$.

Under the above conditions, there is a sequence of estimates $\hat{\gamma}_n$ as solutions to (3) satisfying $\hat{\gamma}_n \xrightarrow{P} \gamma_0$ as $n \to \infty$. Moreover, the corresponding sequence of noise variance estimate in (5) satisfies $\hat{\sigma}^2 \xrightarrow{P} \sigma_0^2$.

Before we state our next result regarding the asymptotic distribution of $\hat{\gamma}$, we need to introduce the following probability density function of the Marčenko-Pastur law with the parameter $\tau > 0$:

$$f_\tau(x) = \frac{1}{2\pi \tau x} \sqrt{(b_+(\tau) - x)(x - b_-(\tau))} 1_{\{b_-(\tau) \leq x \leq b_+(\tau)\}},$$

where $b_\pm(\tau) = (1 \pm \sqrt{\tau})^2$. Note that the Marčenko-Pastur law also has a point mass $1 - \tau^{-1}$ at the origin when $\tau > 1$. With $f_\tau(x)$, and any $\tau, \gamma > 0$, we define the following quantities based on the Marčenko-Pastur law for any positive integer $k$:

$$h_k(\gamma, \tau) = \int_{b_-(\tau)}^{b_+(\tau)} \frac{1}{(1 + \gamma x)^k} f_\tau(x) + \left(1 - \frac{1}{\tau}\right) 1_{\{\tau > 1\}}. \tag{6}$$

With these quantities determined as integrals based on the Marčenko-Pastur law, we are able to obtain the following result on the asymptotic distribution of $\hat{\gamma}$ by imposing additional assumptions on the infinity norm of $\beta$, and both the Frobenius and operator norms of $\Sigma_\epsilon$:

**Theorem 2.2.** In addition to the assumptions in Theorem 2.1, we further assume $\|\beta\|_\infty = o(p^{-1/4})$. For the noise $\epsilon$, we make the following additional assumptions on its covariance matrix:

(i) $\|\Sigma_\epsilon\|$ is uniformly bounded;

(ii) $\kappa = \frac{1}{n\sigma_0^2} \|\Sigma_\epsilon\|_F^2$ is fixed for all $n$.

Then, with $h_k(\gamma_0, \tau)$ as in (6), as $n \to \infty$,

$$\sqrt{n} (\hat{\gamma} - \gamma_0) \xrightarrow{D} N\left(0, 2\gamma_0^2 \left(\frac{1}{h_2(\gamma_0, \tau)} - h_2^2(\gamma_0, \tau) + \kappa - \tau - 1\right)\right). \tag{7}$$

Note that the asymptotic variance of $\hat{\gamma}$ given in Theorem 2.2 relies solely on the limiting aspect ratio $1/\tau$, true SNR $\gamma_0$, and the parameter $\kappa$ that is determined by the correlation and heterogeneity of the noise $\epsilon$. Since $\gamma_0$ can be consistently estimated by $\hat{\gamma}$, to construct confidence intervals for $\gamma_0$, we need to estimate $\kappa$ in order to estimate the asymptotic variance of $\hat{\gamma}$. Though $\kappa$ is inestimable for general noise covariance $\Sigma_\epsilon$, it is estimable when
the noise is uncorrelated. In this case, \( \kappa \) can be simply referred to as the heterogeneity parameter, since

\[
\kappa = \frac{1}{n\sigma_0^4} \| \Sigma \|_F^2 = \frac{1}{n\sigma_0^4} \sum_{i=1}^{n} \sigma_i^4.
\]  

(8)

Under our assumptions on the design matrix, it is easy to get

\[
E[y_i^4] = \sum_{j=1}^{p} (E[z_{ij}^4] - 3) \beta_j^4 + 3\| \beta \|_2^4 + 6\| \beta \|_2^2 \sigma_i^2 + 3\sigma_i^4
\]

\[
\approx 3\| \beta \|_2^4 + 6\| \beta \|_2^2 \sigma_i^2 + 3\sigma_i^4,
\]

which implies \( (1/n) \sum_{i=1}^{n} E[y_i^4] \approx 3\| \beta \|_2^4 + 6\| \beta \|_2^2 \sigma_0^2 + 3\kappa\sigma_0^4 \). By this heuristic, we give the following estimate for the heterogeneity parameter

\[
\hat{\kappa} := \frac{1}{3n\sigma_0^4} \sum_{i=1}^{n} y_i^4 - (\hat{\gamma}_n^2 + 2\hat{\gamma}_n).
\]  

(9)

The following result guarantees the consistency of \( \hat{\kappa} \):

**Proposition 2.3.** Under the assumptions in Theorem 2.2, if \( \varepsilon \) consists of independent heteroscedastic variables, the estimate of the heterogeneity parameter given in (9) satisfies \( \hat{\kappa} \xrightarrow{P} \kappa \).

We give several remarks on Theorems 2.1 and 2.2 in order to highlight our contributions as well as some important features of these results:

**Remark 1** (Heterogeneity and correlation in noise). A prominent distinction between our result and previous work Jiang et al. (2016) and Dicker and Erdogdu (2016) is on noise modeling misspecification. Recall that the actual \( \beta \) is assumed to follow the sparse random effect model in Jiang et al. (2016), while assumed to be fixed in Dicker and Erdogdu (2016). However, both of them assume the noise is i.i.d. Gaussian, that is, no model misspecification on the noise. In contrast, we consider a very general setting for the noise to be heterogeneous and correlated. By introducing the parameter \( \kappa \) that summarizes the heterogeneity and correlation in the noise, the asymptotic variance of the REML estimator \( \hat{\gamma} \) can be also elegantly characterized. It is surprising to us that the effect of heterogeneity and correlation on the uncertainty of \( \hat{\gamma} \) can be neatly captured by \( \kappa \) defined in Theorem 2.2.

**Remark 2** (Non-Gaussian and skew-free entries in the design matrix). Both our work and Dicker and Erdogdu (2016) consider the setting that the actual \( \beta \) is fixed, but their analysis relies crucially on the assumption that the design matrix consists of i.i.d. Gaussian entries. We relax this condition to non-Gaussian and skew-free entries. An open question is whether this skew-free assumption is essential. In the next section, we will use numerical simulations to demonstrate that this condition is likely inessential.

**Remark 3** (Asymptotic variance). It is worth emphasizing that when the noise variables are independent and homogeneous, which implies that \( \kappa = 1 \), the asymptotic distribution given in (7) is consistent with the result derived from i.i.d. Gaussian design in Dicker and Erdogdu (2016). In fact, an explicit formula can be derived for the asymptotic variance...
based essentially on the Stieltjes transform of the Marchenko-Pastur distribution, see e.g. Lemma 3.11 in Bai and Silverstein (2010). Define

$$m_\tau(z) = \int_{b_-(\tau)}^{b_+(\tau)} \frac{1}{x+z} f_\tau(x) \, dx + \frac{1}{z} \left( 1 - \frac{1}{\tau} \right) \mathbf{1}_{\{\tau > 1\}} = \frac{(\tau - z - 1) + \sqrt{(\tau - z - 1)^2 + 4z\tau}}{2z\tau}.$$  

Then we can obtain

$$h_1(\gamma, \tau) = \frac{1}{\gamma} m_\tau \left( \frac{1}{\gamma} \right) = \frac{(\tau\gamma - 1 - \gamma) + \sqrt{(\tau\gamma - 1 - \gamma)^2 + 4\tau\gamma}}{2\tau\gamma},$$

and

$$h_2(\gamma, \tau) = -\frac{1}{\gamma^2} m'_\tau \left( \frac{1}{\gamma} \right) = -\frac{(\tau\gamma - \tau + \gamma + 1) \left( -\gamma - 1 + \sqrt{(\tau\gamma - 1 - \gamma)^2 + 4\tau\gamma} \right)}{2\gamma^2\tau^2 \sqrt{(\tau\gamma - 1 - \gamma)^2 + 4\tau\gamma}}.$$  

We illustrate the asymptotic variance in Figure 1 with $\kappa = 1$ and $n = 100$. From this figure, fixing the aspect ratio $1/\tau$, the variance of $\hat{\gamma}$ increases in the true SNR $\gamma_0$; while fixing $\gamma_0$, the variance of $\hat{\gamma}$ first decreases and then increases in the aspect ratio $1/\tau$.

![Figure 1: Asymptotic variance of $\hat{\gamma}$ with $\kappa = 1$ and $n = 100.$](image)

Remark 4 (Proof techniques). A key technical idea of the proof of our main results is to introduce two Rademacher sequences, one to flip signs of columns and one to flip signs of rows, to facilitate the asymptotic analysis of the estimating function $\Delta(\gamma)$ given in (3). This is the reason why we need the entries of $Z$ to be independent and skew-free. The benefit of this idea is three-pronged: First, consistency and asymptotic variance can be established and calculated by conditioning everything except for the double Rademacher sequences; Second, the effect of heterogeneity and correlation of the noise on the asymptotic variance can be clearly revealed by this conditioning; Third, the asymptotic normality of $\hat{\gamma}$ can also be established relatively easily by off-the-shelf normal approximation results for
quadratic forms of Rademacher variables, e.g. Chatterjee (2008). Here we highlight that the calculation of asymptotic variance relies on extending the “leave-k-column-out” argument in Jiang et al. (2016) to both rows and columns given we are dealing with more general noise.

3 Experiments

As aforementioned, comparisons between REML and other methods such as method of moments and sparseness/Lasso based approaches in estimating variance components have been well-investigated empirically in the literature; see e.g. Dicker and Erdogdu (2016). In this section, we aim to use numerical simulations to illustrate and complement our theoretical findings regarding how the asymptotic distribution of $\hat{\gamma}$ relies on the coefficient vector and the actual distribution of noise. In particular, we will verify the sampling distribution result we have obtained in Theorem 2.2 by numerical experiments under different settings on the design matrix, the coefficient vector, and the heterogeneous and correlated noise. In the case of uncorrelated heterogeneous noise, We will also demonstrate the coverage property of the plug-in confidence intervals with $\kappa$ estimated in (9). Throughout our numerical experiments, we use the Minorization-Maximization (MM) algorithm given in Zhou et al. (2019) to maximize (1.2) and hence obtain the random effects likelihood estimate $\hat{\gamma}$ and $\hat{\sigma}^2$.

3.1 Consistency of $\hat{\gamma}$, $\hat{\sigma}^2$ and $\hat{\kappa}$

In this subsection, we consider the linear model with heterogeneous but uncorrelated noise, and then demonstrate the consistency of REML $\hat{\gamma}$ and $\hat{\sigma}^2$, as well as $\hat{\kappa}$ defined in (9). Here we only consider uncorrelated noise since we need to show the behavior of $\hat{\kappa}$. For the case of correlated noise, we will illustrate the sampling distribution of $\hat{\gamma}$ in the next subsection.

We assume that the coefficient vector $\beta$ is generated in the form of

$$
\beta \propto (1, 2^{-g}, 3^{-g}, \ldots, p^{-g})^T,
$$

where $g \geq 0$ determines the rate of decay for the coefficients, and the norm of $\beta$ is determined by $\sigma_0^2$ and the SNR $\gamma_0$ by $||\beta||^2 = \gamma_0 \sigma_0^2$.

For heteroscedastic independent noise, we generate $\sigma_i^2$ by the geometric sequence by first generating $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ in the form of

$$
(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \propto (1, q, q^2, \ldots, q^n),
$$

where $q > 0$ and $\sum_{i=1}^n \sigma_i^2 = n \sigma_0^2$. Next, $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ is shuffled randomly to generate $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$. Throughout this section, we chose $q = 0.95$ and $\sigma_0^2 = 0.5$, which also gives $\kappa = 30.7692$ by fixing $n = 1200$.

We consider the following settings on the key parameters to investigate and illustrate how the performance of $\hat{\gamma}$ relies on the magnitude decay in $\beta$, the aspect ratio $p/n$, and the SNR $\gamma_0$.

(i) (Varying magnitude decay in $\beta$): Fix $n = 1200$, $p = 2000$, $\gamma_0 = 2$. Let $g$ be varied from 0 to 2.
Figure 2: Estimates of SNR and noise level for simulations (i)(ii)(iii) under the t5 design. Each simulation is conducted over 100 independent Monte Carlo samples. The true SNR $\gamma_0$, $\sigma_0^2$ and $\kappa_0$ are marked in dash line. The black diamonds represent average estimates by $\hat{\gamma}$, $\hat{\sigma}^2$ and $\hat{\kappa}$.

(ii) (Varying aspect ratio): Fix $n = 1200$, $g = 0.5$, $\gamma_0 = 2$. Let the aspect ratio $1/\tau = p/n$ be varied from $2/3$ to $3$.

(iii) (Varying SNR): Fix $n = 1200$, $p = 2000$, $g = 0.5$. Let $\gamma_0$ be varied from $0.5$ to $5$.

Each simulation consists of 100 independent Monte Carlo samples. The performances of $\hat{\gamma}$, $\hat{\sigma}^2$ and $\hat{\kappa}$ under simulation settings (i)(ii)(iii) are shown in Figure 2 for design matrices with i.i.d. $t_5$ entries. All of these estimators appear to be consistent under various circumstances. In particular, we can see that the variance of estimators $\hat{\gamma}$ keeps more or less the same over different magnitude decays in $\beta$, while increases with the aspect ratio $1/\tau \in [2/3, 3]$, and also increases with the true SNR $\gamma_0 \in [1/2, 5]$. These observations are in line with the asymptotic variance presented in (7), which has also been illustrated in Figure...
3.2 Distribution of $\hat{\gamma}$

Now let’s study the sampling distribution of $\hat{\gamma}$ empirically for heterogeneous and correlated noise. Here we consider the setting $n = 1200$, $p = 2000$, $\sigma_0^2 = 0.5$, and $\gamma_0 = 2$. For the coefficient vector, assume $\beta_0$ generated from (10) with $g = 0.5$. For the heterogeneous and correlated noise, in addition to the variances generated according to (11) with $q = 0.95$, we impose the pairwise covariances as $\Sigma_{ij} = \rho |i - j| \cdot \sigma_i \sigma_j$ with $\rho = 0.1$. The resulting $\kappa$ defined in Theorem 2.2 is $\kappa = \|\Sigma\|^2_{F}/(n\sigma_0^4) = 30.8188$. We conduct Monte Carlo simulations with 1000 independent samples under the following settings of design matrices:

(i) The entries of $Z$ are i.i.d. Rademacher random variables.

(ii) The entries of $Z$ are i.i.d. standardized $t_5$ random variables.

(iii) The standardized genotype model proposed in Jiang et al. (2016): First, let the allele frequencies for SNPs be generated from $f_i \sim \text{Unif}[0.05, 0.5]$ for $i = 1, \ldots, p$. Next, generate the entries of the genotype matrix $U$ by following a discrete distribution over $\{0, 1, 2\}$ with assigned probabilities $(1 - f_j)^2$, $2f_j(1 - f_j)$, and $f_j^2$, respectively. Finally, standardize each column of $U$ to have zero mean and unit variance to obtain the design matrix $Z$.

In Figure 3, we compare the Monte Carlo simulated distribution of $\hat{\gamma}$ and its asymptotic distribution given in (7) under the above setups. We can observe that the asymptotic distribution of $\hat{\gamma}$ given in (7) approximates the true sampling distribution very well in our settings. Note that the entries of the standardized genotype design could be significantly skewed, thereby violating the requirement of symmetric distributions in Theorem 2.1. This suggests that this requirement is likely to be able to be further relaxed. Moreover, The Q-Q plots for these three cases are also exhibited in Figure 3, which are basically linear, though the seemingly heavy-tailed distribution of $\hat{\gamma}$ may stem from the large $\kappa$ due to heterogeneity and correlation of the noise.

3.3 Confidence Intervals of $\gamma_0$

To derive plug-in confidence intervals for $\gamma_0$, we need to either know $\kappa$ a priori, or know how to estimate $\kappa$. As such, we consider two cases: (i) homogeneous and uncorrelated noise ($\kappa = 1$), and (ii) heterogeneous and uncorrelated noise, under which $\kappa$ can be estimated by a method of moments given in (9). The asymptotic variance of $\hat{\gamma}$ can be estimated consistently in both cases, since it only relies on $\gamma_0$ and $\tau$ and $\kappa$ in (7). In both cases, we consider the signal setting (10), and set the parameters $n = 1200$, $p = 2000$, $\gamma_0 = 2$, $\sigma_0^2 = 0.5$ and $g \sim \text{Unif}[0, 2]$. In the second case, we also consider the noise variance setting (11) with parameters $q = 0.95$ and correspondingly $\kappa_0 = 30.7692$. In each case, we generate 200 independent datasets, with the design matrices following the i.i.d. Rademacher model, the i.i.d. $t_5$ model, or the standardized genotype model. The plug-in confidence intervals are demonstrated in Figure 4 for both cases, which appear to enjoy desirable coverage properties.
Figure 3: Probability density of the estimated SNR $\hat{\gamma}$ and the normal Q–Q plot of corresponding $\hat{\gamma}$ sets. In the probability density graph, the purple curve shows the pdf of normal distribution with sample mean and sample variance and the red curve shows the pdf of our theoretical normal distribution when the features are independent.
Figure 4: Plug-in 95% CI for 200 independent datasets for both homogeneous and heterogeneous noise. The estimates \( \hat{\gamma} \) are marked as circles and the true SNRs \( \gamma_0 \) are marked by the red line. The purple bars indicate the cases when the 95% CI does not cover \( \gamma_0 \).

4 Proof of the Main Results

4.1 Supporting Lemmas

Lemma 4.1. Under the assumptions of Theorem 2.1, we have

\[
\max_{i \in [n]} \varepsilon_i^2 = O_P(\log n) \tag{12}
\]

and

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i^2 - \sigma_0^2 \right| = o_P(1). \tag{13}
\]

Moreover, under the assumptions of Theorem 2.2, there holds

\[
\frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} \varepsilon_i^2 - n\sigma_0^2 \right) \Rightarrow N(0, 2\kappa^2\sigma_0^4). \tag{14}
\]
Lemma 4.2 (Theorem 9.10 of Bai and Silverstein (2010)). Under the assumptions of Theorem 2.1, for $V_\gamma$ defined in (2) and any integer $k > 0$, it is obvious that $\|V_\gamma^{-k}\| \leq 1$. Moreover, we have
\[
\frac{1}{n} \text{trace} \left( V_\gamma^{-k} \right) - h_k(\gamma, \tau) = O_P \left( \frac{1}{n} \right),
\]
where $h_k(\gamma, \tau)$ is defined in (6).

A key technique in proving Theorem 2.1 is the “leave-$k$-out” argument developed in Jiang et al. (2016). Here we list some useful notations.

Definition 1. Denote $Z = [z_1, \ldots, z_p]$ as a concatenation of column vectors. For any subset $C \subset \{1, \ldots, p\}$, denote $V_{\gamma,-C} := V_\gamma - \frac{1}{p} \sum_{k \in C} z_k z_k^\top$. For example, for any $i \neq j$,
\[
V_{\gamma,-ij} := V_{\gamma,-\{ij\}} = V_\gamma - \frac{\gamma}{p} \left( z_i z_i^\top + z_j z_j^\top \right).
\]
Furthermore, for $1 \leq i, j \leq p$, define
\[
\eta^{(l)}_{ij,C} := z_i^\top V_{\gamma,-C}^{-l} z_j.
\]
(15)
Finally, in the case $C = \emptyset$, simply denote
\[
\eta^{(l)}_{ij} := z_i^\top V_{\gamma}^{-l} z_j.
\]
(16)

The proofs of the following five results, Lemma 4.3 to Lemma 4.7, essentially follow the arguments or ideas in Jiang et al. (2016), though there might be some small differences. For completeness, we provide self-contained proofs for these results in the appendix except for Lemma 4.6, which has been explicitly given in the supplement of Jiang et al. (2016) (See Proposition S.1 therein).

Lemma 4.3. Under the conditions of Theorem 2.1, we have
\[
\max_{k \in [p]} \left| \frac{1}{n} \text{trace} \left( V_\gamma^{-l} \right) - \frac{1}{n} \text{trace} \left( V_{\gamma,-k}^{-l} \right) \right| \leq 2^l - 1, \quad l = 1, 2, 3, 4,
\]
and for $\eta^{(l)}_{kk,k}$ defined in (15),
\[
\max_{k \in [p]} \left| \frac{1}{n} \eta^{(l)}_{kk,k} - \frac{1}{n} \text{trace} \left( V_\gamma^{-l} \right) \right| = O_P \left( \frac{\log n}{n} \right), \quad l = 1, 2.
\]

Lemma 4.4. Under the conditions of Theorem 2.1, for fixed $\gamma > 0$, we have
\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V_\gamma^{-1} z_k - \frac{1}{n} \frac{\text{trace}(V_\gamma^{-1})}{1 + \frac{2}{p} \text{trace}(V_\gamma^{-1})} \right| = O_P \left( \frac{\log n}{n} \right),
\]
(18)
\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V_\gamma^{-2} z_k - \frac{1}{n} \frac{\text{trace}(V_\gamma^{-2})}{(1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}))^2} \right| = O_P \left( \frac{\log n}{n} \right),
\]
(19)
\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V_\gamma^{-l} z_k - \frac{1}{np} \text{trace} \left( V_\gamma^{-l} ZZ^\top \right) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \quad l = 1, 2, \tag{20}
\]

\[
\max_{1 \leq k \leq p} \left| \left( z_k^\top B_\gamma z_k \right)^l - \left( \frac{1}{p} \text{trace} \left( B_\gamma ZZ^\top \right) \right)^l \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \quad l = 1, 2, \tag{21}
\]

\[
\frac{1}{np} \text{trace} \left( V_\gamma^{-1} ZZ^\top \right) - \frac{1}{n} \frac{\text{trace}(V_\gamma^{-1})}{1 + \frac{2}{p} \text{trace}(V_\gamma^{-1})} = O_P \left( \frac{1}{n} \right), \tag{22}
\]

and

\[
\frac{1}{np} \text{trace} \left( V_\gamma^{-2} ZZ^\top \right) - \frac{1}{n} \frac{\text{trace}(V_\gamma^{-2})}{\left( 1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}) \right)^2} = O_P \left( \frac{1}{n} \right). \tag{23}
\]

**Lemma 4.5.** Under the conditions of Theorem 2.1, for fixed \( \gamma > 0 \), we have

\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V_\gamma^{-1} z_k - \frac{1}{np} \text{trace} \left( V_\gamma^{-1} ZZ^\top \right) \right.
\]

\[
+ \frac{1}{\left( 1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}) \right)^2} \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right) \left| = O_P \left( \frac{\log n}{n} \right) \right., \tag{24}
\]

and

\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V_\gamma^{-2} z_k - \frac{1}{np} \text{trace} \left( V_\gamma^{-2} ZZ^\top \right) \right.
\]

\[
+ \frac{\text{trace}(V_\gamma^{-2})}{\left( 1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}) \right)^3} \left[ \frac{2}{p} \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right) \right.
\]

\[
- \frac{1}{\left( 1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}) \right)^2} \left( \frac{1}{n} \eta_{kk,k}^{(2)} - \frac{1}{n} \text{trace}(V_\gamma^{-2}) \right) \left| = O_P \left( \frac{\log n}{n} \right) \right. \tag{25}
\]

**Lemma 4.6.** Under the conditions of Theorem 2.1, for fixed \( \gamma > 0 \) and \( l = 1, 2 \), we have

\[
\max_{1 \leq k \leq p} \mathbb{E} \left[ \left( \frac{1}{n} \eta_{kk,k}^{(l)} - \frac{1}{n} \text{trace}(V_\gamma^{-l} \cdot i) \right)^2 \right] \leq \frac{C}{n}, \tag{26}
\]

and

\[
\max_{i \leq i' \leq p} \mathbb{E} \left[ \left( \frac{1}{n} \eta_{ii,i'}^{(l)} - \frac{1}{n} \text{trace}(V_\gamma^{-l} \cdot i) \right) \left( \frac{1}{n} \eta_{jj,j'}^{(l)} - \frac{1}{n} \text{trace}(V_\gamma^{-l} \cdot j) \right) \right] \leq \frac{C}{n^2}, \tag{27}
\]

where \( C \) is a constant independent of \( n \).
Lemma 4.7. Under the conditions of Theorem 2.1, for fixed $\gamma > 0$ and $l = 1, 2$, we have

$$
\max_{k \neq j} |z_k^T V^{-l} z_j|^2 = O_P(n \log n) \quad \text{and} \quad \max_{k \neq j} |z_k^T B \gamma z_j|^2 = O_P \left( \frac{\log n}{n} \right).
$$

Further, under the assumptions of Theorem 2.2, we have

$$
\left\{ \begin{array}{l}
\frac{1}{p(p-1)} \sum_{i \neq j} n \left( z_i^T B \gamma z_j \right)^2 = \bar{\theta}_1(\gamma, \tau) + o_P(1) \\
\sum_{i \neq j} \beta_i^2 \beta_j^2 n \left( z_i^T B \gamma z_j \right)^2 = \|\beta\|^4 \bar{\theta}_1(\gamma, \tau) + o_P(1),
\end{array} \right.
$$

where $\bar{\theta}_1(\gamma, \tau) > 0$ is a constant only depending on $\gamma$ and $\tau$.

The above lemmas rely crucially on the “leave-$k$-column-out” argument in Jiang et al. (2016). Given we are dealing with heteroscedastic and correlated noises, we also need the following results, which rely on a similar “leave-$k$-row-out” argument.

Lemma 4.8. For any fixed $\gamma > 0$, under the assumptions of Theorem 2.1, we have

$$
\max_{i \in [n]} \left| \left( V_{\gamma}^{-l} \right)_{ii} - \frac{1}{n} \text{trace} \left( V_{\gamma}^{-l} \right) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \quad l = 1, 2, 3, 4,
$$

which implies

$$
\max_{i \in [n]} \left| (B_{\gamma})_{ii} - \frac{1}{n} \text{trace} (B_{\gamma}) \right| = O_P \left( \sqrt{\frac{\log n}{n^3}} \right)
$$

and

$$
\max_{i \in [n]} \left| (B_{\gamma})_{ii}^2 - \left( \frac{1}{n} \text{trace} (B_{\gamma}) \right)^2 \right| = O_P \left( \sqrt{\frac{\log n}{n^5}} \right).
$$

Lemma 4.9. For any fixed $\gamma > 0$, under the assumptions of Theorem 2.1, we have

$$
\max_{1 \leq i < j \leq n} |(B_{\gamma})_{ij}| = O_P \left( \sqrt{\frac{\log n}{n^3}} \right)
$$

and under the assumptions of Theorem 2.2

$$
\left\{ \begin{array}{l}
n \sum_{i \neq j} (B_{\gamma})_{ij}^2 = \bar{\theta}_2(\gamma, \tau) + o_P(1) \\
n \sum_{i \neq j} \epsilon_i^2 \epsilon_j^2 (B_{\gamma})_{ij}^2 = \bar{\theta}_2(\gamma, \tau) \sigma_0^4 + o_P(1),
\end{array} \right.
$$

where $\bar{\theta}_2(\gamma, \tau) > 0$ is a constant only depending on $\gamma$ and $\tau$.

Lemma 4.10. For any fixed $\gamma > 0$, under the assumptions of Theorem 2.1, we have

$$
\max_{k \in [p]} \max_{i \in [n]} \left| \left( V_{\gamma}^{-l} \right)_{ii} - \frac{1}{n} \text{trace} \left( V_{\gamma}^{-l} \right) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \quad l = 1, 2, 3, 4.
$$
4.2 New Representation based on Rademacher Sequences

Since the entries of $Z$ are independent and symmetric, we can replace the original design matrix $Z$ with $\tilde{Z} = \Lambda_\zeta Z \Lambda_\xi$ with the diagonal matrices

$$\Lambda_\zeta = \text{diag}(\zeta_1, \ldots, \zeta_n), \quad \Lambda_\xi = \text{diag}(\xi_1, \ldots, \xi_p),$$

with $\zeta_i$’s and $\xi_j$’s are i.i.d. Rademacher random variables that are also independent of $Z$, since $Z$ and $\tilde{Z}$ have the same distribution. We also denote

$$\xi = (\xi_1, \ldots, \xi_n)^\top \quad \text{and} \quad \zeta = (\zeta_1, \ldots, \zeta_n)^\top.$$

Under this new representation of the design matrix, the linear model (1) becomes

$$y = \Lambda_\zeta Z \Lambda_\xi \beta + \varepsilon.$$  \hspace{1cm} (33)

We want to emphasize that under this new representation, we still define $V_\gamma$ and $B_\gamma$ as before:

$$V_\gamma = I_n + \frac{\gamma}{p} ZZ^\top, \quad \text{and} \quad B_\gamma = \frac{V_\gamma^{-1}}{n} - \frac{V_\gamma^{-2}}{\text{trace}(V_\gamma^{-1})}.$$

However, the representation of the estimating equation (3) should be changed. In fact, the original $ZZ^\top$ is replaced with $\Lambda_\zeta ZZ^\top \Lambda_\zeta$. Therefore, the original $V_\gamma$ defined in (2) should be replaced with

$$\tilde{V}_\gamma = I_n + \frac{\gamma}{p} \Lambda_\zeta ZZ^\top \Lambda_\zeta = \Lambda_\zeta \left( I_n + \frac{\gamma}{p} ZZ^\top \right) \Lambda_\zeta = \Lambda_\zeta V_\gamma \Lambda_\zeta.$$

Also, it is easy to see that the original $B_\gamma$ should be replaced with $\tilde{B}_\gamma = \Lambda_\zeta B_\gamma \Lambda_\zeta$. Therefore, the estimating equation (3) should be rewritten as

$$\Delta(\gamma) := y^\top \tilde{B}_\gamma y$$

$$= (\Lambda_\zeta Z \Lambda_\xi \beta + \varepsilon)^\top \Lambda_\zeta B_\gamma \Lambda_\xi \left( \Lambda_\zeta Z \Lambda_\xi \beta + \varepsilon \right)$$

$$= \xi^\top \Lambda_\beta Z^\top B_\gamma Z \Lambda_\beta \xi + 2 \xi^\top \Lambda_\beta Z^\top B_\gamma Z \Lambda_\epsilon \zeta + \zeta^\top \Lambda_\epsilon B_\gamma Z \Lambda_\epsilon \zeta$$

$$= [\xi^\top, \zeta^\top] \begin{bmatrix} \Lambda_\beta Z^\top B_\gamma Z \Lambda_\beta & \Lambda_\beta Z^\top B_\gamma \Lambda_\epsilon \\ \Lambda_\epsilon B_\gamma Z \Lambda_\beta & \Lambda_\epsilon B_\gamma \Lambda_\epsilon \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix},$$  \hspace{1cm} (34)

where

$$\Lambda_\beta = \text{diag}(\beta_1, \ldots, \beta_n) \quad \text{and} \quad \Lambda_\epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_n).$$

Note that now $\Delta(\gamma)$ is a random variable about $Z$, $\varepsilon$, $\xi$ and $\zeta$. Straightforward calculation gives the conditional mean of $\Delta(\gamma)$ on $Z$ and $\varepsilon$:

$$\tilde{\Delta}_*(\gamma) := \mathbb{E} [\Delta(\gamma) | Z, \varepsilon] = \sum_{k=1}^p \beta_k^2 z_k^\top B_\gamma z_k + \text{trace}(\Lambda_\epsilon^2 B_\gamma).$$  \hspace{1cm} (35)

Furthermore, the conditional variance of $\sqrt{n}(\Delta(\gamma))$ on $Z$ and $\varepsilon$ can also derived as in the following lemma, the proof of which is deferred to the appendix.
Lemma 4.11. The conditional variance of $\Delta(\gamma)$ given $Z$ and $\varepsilon$ has the formula

$$\text{Var} \left[ \sqrt{n}(\Delta(\gamma)) \mid Z, \varepsilon \right] = 2n \sum_{1 \leq k \neq j \leq p} \beta_k^2 \beta_j^2 \left( z_k^\top B_\gamma z_j \right)^2 + 4n \sum_{k=1}^p \beta_k^2 z_k^\top B_\varepsilon A_\varepsilon^2 B_\gamma z_k + 2n \sum_{1 \leq k \neq j \leq n} \varepsilon_k^2 \varepsilon_j^2 (B_\gamma)_{kj}^2, \quad (36)$$

4.3 Proof of Theorem 2.1

With $\tilde{\Delta}_*(\gamma)$ defined in (35) and for any fixed $\gamma > 0$, we first aim at showing

$$\Delta(\gamma) - \tilde{\Delta}_*(\gamma) \xrightarrow{P} 0. \quad (37)$$

First, by (25) in Lemma 4.7, we have

$$\sum_{k \neq j} \beta_k^2 \beta_j^2 \left( z_k^\top B_\gamma z_j \right)^2 \leq \left( \max_{k \neq j} |z_k^\top B_\gamma z_j|^2 \right) \|\beta\|_2^2 = O_P \left( \frac{\log n}{n} \right).$$

Second, since $z_k$'s are sub-Gaussian vectors, it is obvious that

$$\max_{1 \leq k \leq p} \|z_k\|^2 = O_P(n).$$

Also, a simple consequence of Lemma 4.2 gives $\|B_\gamma\| = O_P(1/n)$, and Lemma 4.1 implies $\|A_\varepsilon^2\| \leq O(\log n)$. Therefore,

$$\sum_{k=1}^p \beta_k^2 z_k^\top B_\gamma A_\varepsilon^2 B_\gamma z_k \leq \|\beta\|_2^2 \|B_\gamma\|^2 \|A_\varepsilon^2\| \left( \max_{1 \leq k \leq p} \|z_k\|^2 \right) = O_P \left( \frac{\log n}{n} \right).$$

Third, by (12) in Lemma 4.1 and (30) in Lemma 4.9,

$$\sum_{1 \leq k \neq j \leq n} \varepsilon_k^2 \varepsilon_j^2 (B_\gamma)_{kj}^2 = O_P \left( \frac{\log n}{n} \right).$$

Plug the above bounds to (36), for any $\delta > 0$, by the conditional Chebyshev’s inequality, we have

$$\mathbb{P} \left\{ \left| \Delta(\gamma) - \tilde{\Delta}_*(\gamma) \right| > \delta \mid Z, \varepsilon \right\} \leq \frac{\text{Var} \left[ \Delta(\gamma) \mid Z, \varepsilon \right]}{\delta^2} \xrightarrow{P} 0.$$

Then, by the dominated convergence theorem, we have proved (37).

Now, define

$$\Delta_{ss}(\gamma) = \sigma_0^2 \text{trace} (B_\gamma V_m) = \sigma_0^2 \text{trace} \left( B_\gamma \left( I_n + \frac{\gamma_0}{p} ZZ^\top \right) \right). \quad (38)$$

By (21) in Lemma 4.4, we can easily obtain

$$\left| \sum_{k=1}^p \beta_k^2 \left( z_k^\top B_\gamma z_k \right) - \frac{\|\beta\|^2}{p} \text{trace}(B_\gamma ZZ^\top) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right). \quad (39)$$
On the other hand, by Lemma 4.1 and (28) in Lemma 4.8, we have

\[ \left| \text{trace} (\Lambda_z B_\gamma) - \frac{1}{n} \text{trace} (\Lambda_z^2) \text{trace} (B_\gamma) \right| = O_P \left( \sqrt{\frac{\log^3 n}{n}} \right). \]

Furthermore, by Lemmas 4.1 and 4.2, we have

\[ \left| \frac{1}{n} \text{trace} (\Lambda_z^2) \text{trace} (B_\gamma) - \sigma_0^2 \text{trace} (B_\gamma) \right| = o_P(1). \]

Combine the above two inequalities,

\[ \left| \text{trace} (\Lambda_z^2 B_\gamma) - \sigma_0^2 \text{trace} (B_\gamma) \right| = o_P(1). \]

Then, by (35), (38), (39), and (40), we have

\[ \left| \Delta_s(\gamma) - \Delta_{**}(\gamma) \right| = o_P(1). \]

Combined with (37), we have

\[ \Delta(\gamma) - \Delta_{**}(\gamma) \xrightarrow{P} 0. \]

Finally, we have the following result that characterizes the limit of \( \Delta_{**}(\gamma) \) for any \( \gamma > 0 \).

**Lemma 4.12** (Jiang et al. (2016)). *Under the assumption of Theorem 2.1, we have*

\[ \Delta_{**}(\gamma) \xrightarrow{a.s.} c_\gamma, \]

*where \( c_\gamma > 0 \) for \( \gamma < \gamma_0 \), \( c_{\gamma_0} = 0 \), and \( c_{\gamma} < 0 \) for \( \gamma > \gamma_0 \).*

This result is basically given in Jiang et al. (2016), and we give a detailed proof in the appendix for self-containment.

Then, for any \( \gamma > 0 \), there holds \( \Delta(\gamma) \xrightarrow{P} c_\gamma \), which is positive, zero, or negative, depending on whether \( \gamma \) is smaller than, equal to, or greater than \( \gamma_0 \). Then by the argument of Theorem 3.7 in Lehmann and Casella (2006), with probability tending to one, the equation \( \Delta(\gamma) = 0 \) has a root \( \hat{\gamma}_n \) such that it converges to \( \gamma_0 \) in probability.

**Consistency of \( \hat{\sigma}^2 \)**

Let’s turn to show \( \hat{\sigma}_z^2 \xrightarrow{P} \sigma_0^2 \), where the noise variance estimate is defined in (5). Let \( s_n(\gamma) = \frac{1}{n} y^\top V_{\gamma}^{-1} y \). The noise variance estimate is then \( \hat{\sigma}^2 = s_n(\hat{\gamma}) \). From the previous sections, we know \( s_n(\gamma) \) converges to a continuous function \( s(\gamma) \) in probability. For example, if \( \tau < 1 \), we have

\[ s(\gamma) = \sigma_0^2 \int_{b_-(\tau)}^{b_+(\tau)} \left( \frac{1 + \gamma_0 x}{1 + \gamma x} \right) f_\tau(x) dx, \]

which gives \( s(\gamma_0) = \sigma_0^2 \).
An important observation is that \( s_n(\gamma) \) is decreasing. For any small \( \delta > 0 \) and \( \epsilon > 0 \), we know
\[
s_n(\gamma - \delta) \leq \bar{s}(\gamma - \delta) + \epsilon \quad \text{and} \quad s_n(\gamma + \delta) \geq \bar{s}(\gamma + \delta) - \epsilon
\]
with probability tending to 1. On the other hand, \( \hat{\gamma}_n \rightarrow \gamma_0 \) in probability implies that \( \gamma_0 - \delta < \hat{\gamma}_n < \gamma_0 + \delta \) with probability tending to 1. Therefore, we have
\[
\bar{s}(\gamma_0 - \delta) + \epsilon \geq s_n(\gamma_0 - \delta) \geq s_n(\hat{\gamma}_n) \geq s_n(\gamma_0 + \delta) \geq \bar{s}(\gamma_0 + \delta) - \epsilon
\]
with probability tending to 1. Since \( \delta \) and \( \epsilon \) can be arbitrarily small, we have
\[
\hat{\sigma}^2 = s_n(\hat{\gamma}) \xrightarrow{P} \bar{s}(\gamma_0) = \sigma_0^2.
\]

4.4 Proof of Theorem 2.2

Through the analysis of asymptotic distribution, we use the shorthand \( h_k = h_k(\gamma_0, \tau) \) for \( k = 1, 2, 3, 4 \), where \( h_k(\gamma_0, \tau) \) is defined in (6).

4.4.1 Decomposition of \( \Delta(\gamma_0) \)

The following lemma essentially given in Jiang et al. (2016) (without a detailed proof) reduces the asymptotic distribution of \( \hat{\gamma} \) to that of \( \Delta(\gamma_0) \). For the sake of completeness, we give a detailed proof for it in Appendix:

Lemma 4.13. Under the conditions of Theorem 2.1, assume \( \hat{\gamma}_n \) is a sequence of roots of \( \Delta(\gamma) = 0 \), which converges to \( \gamma_0 \) in probability. Then
\[
\sqrt{n}(\hat{\gamma} - \gamma_0) = -\frac{\sqrt{n}\Delta(\gamma_0)}{\Delta'_{\infty}(\gamma_0)} + o_P(1), \tag{41}
\]
where \( \Delta'_{\infty}(\gamma_0) \) is the limit of \( \Delta'(\gamma) \) as \( \gamma = \gamma_0 \) and has the formula
\[
\Delta'_{\infty}(\gamma_0) = \frac{\sigma_0^2 h_1^2 - h_2}{h_1}.
\]

To investigate the asymptotic distribution of \( \sqrt{n}\Delta(\gamma_0) \), consider the following orthogonal decomposition:
\[
\Delta(\gamma_0) = (\Delta(\gamma_0) - \bar{\Delta}_*(\gamma_0)) + \bar{\Delta}_*(\gamma_0).
\]
In other words, the expectation is taken with respect to Rademacher random variables \( \xi_i \)'s and \( \zeta_i \)'s. We aim to derive the asymptotic joint distribution of
\[
\left( \sqrt{n}(\Delta(\gamma_0) - \bar{\Delta}_*(\gamma_0)), \sqrt{n}\bar{\Delta}_*(\gamma_0) \right).
\]
4.4.2 Conditional Variance of $\Delta(\gamma_0) - \tilde{\Delta}_*(\gamma)$

In order to derive the asymptotic joint distribution of $\left(\sqrt{n}(\Delta(\gamma_0) - \tilde{\Delta}_*(\gamma)), \sqrt{n}\tilde{\Delta}_*(\gamma)\right)$, we first need to study the conditional distribution of $\Delta(\gamma_0) - e\Delta^*(\gamma_0)$ given $Z$ and $\varepsilon$. This consists of two steps: conditional variance and conditional normality. Let’s first study its conditional variance. Note we have

$$\text{Var} \left[ \sqrt{n}(\Delta(\gamma_0) - e\Delta^*(\gamma_0)) \right] = \text{Var} \left[ \sqrt{n}(\Delta(\gamma_0)) \right].$$

**Lemma 4.14.** Under the condition of Theorem 2.2, with $V_1$, $V_2$ and $V_3$ defined in (36), we have

$$V_1 \overset{P}{\to} 2\sigma_0^4 \left( (1 - 2h_1 + h_2) + \frac{4h_2 - 2h_3}{h_1} + \frac{h_2 - 2h_3 + h_4}{h_1^2} - \tau \left( h_1 - \frac{h_2}{h_1} \right)^2 \right),$$

$$V_2 \overset{P}{\to} 4\sigma_0^4 \left( (h_1 - h_2) - \frac{2h_2 - h_3}{h_1} + \frac{h_3 - h_4}{h_1^2} \right),$$

$$V_3 \overset{P}{\to} 2\sigma_0^4 \left( h_2 - \frac{2h_3}{h_1} + \frac{h_4}{h_1^2} - \left( h_1 - \frac{h_2}{h_1} \right)^2 \right).$$

Consequently,

$$\text{Var} \left[ \sqrt{n}(\Delta(\gamma_0)) \right] \overset{P}{\to} 2\sigma_0^4 \left( h_2 - \frac{h_1^2}{h_1^2} - (\tau + 1) \left( h_1 - \frac{h_2}{h_1} \right)^2 \right).$$

**Proof. Limit of $V_1$**

Since $\sum_{j \neq i} \beta_i^2 \beta_j^2 = \|\beta\|^4 - \sum_{k=1}^p \beta_k^4$ and $\sum_{k=1}^p \beta_k^4 = o(1)$ (by the assumption $\|\beta\|_\infty = o(p^{1/4})$), by (26) in Lemma 4.7, we have

$$\sum_{j \neq i} \beta_i^2 \beta_j^2 \left| \frac{1}{p(p-1)} \sum_{j \neq i} n \left( z_i^T B_{\gamma_0} z_j \right)^2 - \theta_3 \right| = o_P(1),$$

and

$$\sum_{j \neq i} \beta_i^2 \beta_j^2 \left( z_i^T B_{\gamma_0} z_j \right)^2 - \|\beta\|^4 \theta_3 = o_P(1).$$

These bounds imply that

$$\left| 2n \sum_{j \neq i} \beta_i^2 \beta_j^2 \left( z_i^T B_{\gamma_0} z_j \right)^2 - 2n \left( \|\beta\|^4 - \sum_{k=1}^p \beta_k^4 \right) \frac{1}{p(p-1)} \sum_{j \neq i} \left( z_i^T B_{\gamma_0} z_j \right)^2 \right| = o_P(1).$$

(45)
Then for $\sum_{j \neq i} (z_i^\top B_{\gamma_0} z_j)^2$, since
\[
\text{trace} \left( \left( B_{\gamma} \frac{1}{p} ZZ^\top \right)^2 \right) = \frac{1}{p^2} \left( \sum_{i \neq j} (z_i^\top B_{\gamma} z_j)^2 + \sum_{k=1}^{p} (z_k^\top B_{\gamma} z_k)^2 \right),
\]
by (21) in Lemma 4.4, we can have that
\[
\left| \frac{1}{p(p-1)} \sum_{j \neq i} (z_i^\top B_{\gamma_0} z_j)^2 \right|
= \frac{1}{p-1} \left| \frac{1}{p} \sum_{k=1}^{p} (z_k^\top B_{\gamma_0} z_k)^2 - \left( \text{trace} \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right) \right)^2 \right|
\leq \frac{1}{p-1} \max_{1 \leq k \leq p} \left| (z_k^\top B_{\gamma_0} z_k)^2 - \left( \frac{1}{p} \text{trace} \left( B_{\gamma_0} ZZ^\top \right) \right)^2 \right| = o_P \left( n^{-1} \right). \quad (46)
\]

Finally, combining (45) and (46), we can have
\[
\left| 2n \left( \|\beta\|^4 - \sum_{k=1}^{p} \beta_k^4 \right) \left( \frac{1}{p-1} \text{trace} \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right)^2 \right) - \frac{1}{p-1} \left( \text{trace} \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right) \right)^2 \right|
- 2n \sum_{j \neq i} \beta_i^2 \beta_j^2 \left( z_i^\top B_{\gamma_0} z_j \right)^2 = o_P \left( 1 \right). \quad (47)
\]

Further, Lemma 4.2 implies
\[
\begin{align*}
\left\{ \begin{array}{l}
\text{n trace} \left( \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right)^2 \right) \xrightarrow{P} \frac{1}{\gamma_0^2} \left( 1 - 2h_1 + h_2 \right) - 2 \frac{h_1 - 2h_2 + h_3}{h_1} + \frac{h_2 - 2h_3 + h_4}{h_1^2}, \\
\left( \text{trace} \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right) \right)^2 \xrightarrow{P} \frac{1}{\gamma_0^2} \left( \frac{h_1^2 - h_2}{h_1} \right)^2.
\end{array} \right.
\end{align*}
\]

Combine these limits, the bound in (47), the fact $\gamma_0 = \|\beta\|^2 / \sigma_0^3$, and the fact $\sum_{k=1}^{p} \beta_k^4 = o(1)$, we finish the proof of (42).

**Limit of $V_2$**

For any fixed $\gamma > 0$, straightforward calculation gives
\[
n z_k^\top B_{\gamma} \Lambda_{\gamma}^2 B_{\gamma} z_k = \frac{1}{n} z_k^\top V_{\gamma}^{-1} \Lambda_{\gamma}^2 V_{\gamma}^{-1} z_k - \frac{1}{n} \frac{z_k^\top V_{\gamma}^{-1} \Lambda_{\gamma}^2 V_{\gamma}^{-1} z_k}{\text{trace} \left( V_{\gamma}^{-1} \right)} + \frac{1}{n} \frac{z_k^\top V_{\gamma}^{-2} \Lambda_{\gamma}^2 V_{\gamma}^{-2} z_k}{\left( \frac{1}{n} \text{trace} \left( V_{\gamma}^{-1} \right) \right)^2}. \quad (48)
\]
By Lemma 4.1, we have

\[ z_k^\top V_{\gamma}^{-1} \Lambda_{\varepsilon}^2 V_{\gamma}^{-1} z_k = \frac{z_k^\top V_{\gamma}^{-1} \Lambda_{\varepsilon}^2 V_{\gamma}^{-1} z_k}{\left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^2}, \]

\[ z_k^\top V_{\gamma}^{-1} \Lambda_{\varepsilon}^2 V_{\gamma}^{-2} z_k = \frac{z_k^\top V_{\gamma}^{-1} \Lambda_{\varepsilon}^2 V_{\gamma}^{-2} z_k}{\left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^2} - \frac{\gamma}{p} \eta_{kk,k}(1) z_k^\top V_{\gamma}^{-1} \Lambda_{\varepsilon}^2 V_{\gamma}^{-2} z_k, \]

\[ z_k^\top V_{\gamma}^{-2} \Lambda_{\varepsilon}^2 V_{\gamma}^{-2} z_k = \frac{z_k^\top V_{\gamma}^{-2} \Lambda_{\varepsilon}^2 V_{\gamma}^{-2} z_k}{\left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^2} - \frac{2 \gamma}{p} \eta_{kk,k}(2) z_k^\top V_{\gamma}^{-2} \Lambda_{\varepsilon}^2 V_{\gamma}^{-2} z_k, \]

\[ + \frac{\gamma}{p} \eta_{kk,k}(2) z_k^\top V_{\gamma}^{-2} \Lambda_{\varepsilon}^2 V_{\gamma}^{-1} z_k \left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^4. \] (49)

By Lemma 4.1, we have

\[ \| V_{\gamma}^{-l} \Lambda_{\varepsilon}^2 V_{\gamma}^{-m} \| \leq \| \Lambda_{\varepsilon}^2 \| = O_P(\log n) \]

and hence

\[ \| V_{\gamma}^{-l} \Lambda_{\varepsilon}^2 V_{\gamma}^{-m} \|_F = O_P(\sqrt{n \log n}). \]

Then, by Hanson-Wright inequality and taking the uniform bound, we can easily get

\[ \max_{k \in [p]} \left| \frac{1}{n} z_k^\top V_{\gamma}^{-l} \Lambda_{\varepsilon}^2 V_{\gamma}^{-m} z_k - \frac{1}{n} \text{trace} \left( \Lambda_{\varepsilon}^2 V_{\gamma}^{-1} \left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^4 \right) \right| = o_P(1). \] (50)

By Lemma 4.10 and Lemma 4.1, we have

\[ \max_{k \in [p]} \left| \frac{1}{n} \text{trace} \left( \Lambda_{\varepsilon}^2 V_{\gamma}^{-1} \left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^4 \right) \right| = o_P(1). \] (51)

By Lemma 4.1 again, we have

\[ \max_{k \in [p]} \left| \frac{1}{n^2} \text{trace} \left( \Lambda_{\varepsilon}^2 \right) \text{trace} \left( V_{\gamma}^{-l} \left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^4 \right) \right| = o_P(1). \] (52)

Combining (50), (51) and (52) gives

\[ \max_{k \in [p]} \left| \frac{1}{n} z_k^\top V_{\gamma}^{-l} \Lambda_{\varepsilon}^2 V_{\gamma}^{-m} z_k - \frac{1}{n} \text{trace} \left( V_{\gamma}^{-l} \left(1 + \frac{\gamma}{p} \eta_{kk,k}(1)\right)^4 \right) \right| = o_P(1). \] (53)

Combining (53), (49), (48), Lemma 4.3 and Lemma 4.2, there exists some constant \( C(\gamma, \tau) \), such that

\[ \max_{k \in [p]} \left| nz_k^\top B_{\gamma} A_{\varepsilon}^2 B_{\gamma} z_k - C(\gamma, \tau) \right| = o_P(1). \]
This further implies both

\begin{align*}
&\left| C(\gamma, \tau) - n \text{trace} \left( B_\gamma A^2_\delta B_\gamma \frac{1}{p} ZZ^\top \right) \right| \\
&\leq \left| C(\gamma, \tau) - \frac{n}{p} \sum_{k=1}^p z^\top_k B_\gamma A^2_\delta B_\gamma z_k \right| \\
&\leq \max_{k \in [p]} \left| C(\gamma, \tau) - n z^\top_k B_\gamma A^2_\delta B_\gamma z_k \right| = o_P(1),
\end{align*}

and

\begin{align*}
&\left| n \sum_{k=1}^p \beta^2_k z^\top_k B_\gamma A^2_\delta B_\gamma z_k - \|\beta\|^2 C(\gamma, \tau) \right| \\
&\leq \|\beta\|^2 \max_{k \in [p]} \left| C(\gamma, \tau) - n z^\top_k B_\gamma A^2_\delta B_\gamma z_k \right| = o_P(1).
\end{align*}

Combining the above inequalities, there holds

\begin{align*}
\left| n \sum_{k=1}^p \beta^2_k z^\top_k B_\gamma A^2_\delta B_\gamma z_k - n \|\beta\|^2 \text{trace} \left( A^2_\delta B_\gamma \frac{1}{p} ZZ^\top B_\gamma \right) \right| = o_P(1). 
\end{align*}

Finally, note that

\begin{align*}
B_\gamma \frac{1}{p} ZZ^\top B_\gamma &= \frac{1}{\gamma} \left( \frac{V^{-1}}{n^2} - \left( \frac{2}{n \text{trace}(V^{-1})} + \frac{1}{n^2} \right) V^{-2} \\
&\quad + \left( \frac{2}{n \text{trace}(V^{-1})} + \frac{1}{(\text{trace}(V^{-1}))^2} \right) V^{-3} - \frac{1}{(\text{trace}(V^{-1}))^2} V^{-4} \right).
\end{align*}

By (27) in Lemma 4.8, there holds

\begin{align*}
\max_{i \in [n]} \left| n \left( B_\gamma \frac{1}{p} ZZ^\top B_\gamma \right)_{ii} - \text{trace} \left( B^2_\gamma \frac{1}{p} ZZ^\top \right) \right| = O_P \left( \sqrt{\frac{\log n}{n^3}} \right).
\end{align*}

Similar to (40), we have

\begin{align*}
\left| n \text{trace} \left( A^2_\delta B_\gamma \frac{1}{p} ZZ^\top B_\gamma \right) - n\sigma_0^2 \text{trace} \left( B^2_\gamma \frac{1}{p} ZZ^\top \right) \right| = o_P(1). 
\end{align*}

Combining (54) and (55) and letting \( \gamma = \gamma_0 \), we have

\begin{align*}
\left| 4n \sum_{k=1}^p \beta^2_k z^\top_k B_{\gamma_0} A^2_\delta B_{\gamma_0} z_k - 4\sigma_0^2 \|\beta\|^2 n \text{trace} \left( B^2_{\gamma_0} \frac{1}{p} ZZ^\top \right) \right| = o_P(1).
\end{align*}
Notice that
\[
4\sigma_0^2 \|\beta\|^2 n \text{trace} \left( B_{\gamma_0}^2 \frac{1}{p} ZZ^\top \right) = 4\sigma_0^2 \|\beta\|^2 n \text{trace} \left( B_{\gamma_0}^2 (V_{\gamma_0} - I_n) \right) = 4\sigma_0^4 n \text{trace} \left( \frac{V_{\gamma_0}^{-1} - V_{\gamma_0}^{-2}}{n^2} - \frac{2 V_{\gamma_0}^{-2} - V_{\gamma_0}^{-3}}{n \text{trace}(V_{\gamma_0}^{-1})} + \frac{V_{\gamma_0}^{-3} - V_{\gamma_0}^{-4}}{(\text{trace}(V_{\gamma_0}^{-1}))^2} \right).
\]

Therefore, we got (43) by Lemma 4.2.

**Limit of V₃**

By (31) in Lemma 4.9, (29) in Lemma 4.8, and Lemma 4.2, we have
\[
V₃ = (2n) \sum_{i \neq j} \varepsilon_i^2 \varepsilon_j^2 (B_{\gamma_0})_{ij}^2 = 2\bar{\theta}_2(\gamma, \tau)\sigma_0^4 + o_P(1) = \sigma_0^4 \left( 2n \sum_{i \neq j} (B_{\gamma_0})_{ij}^2 \right) + o_P(1) = \sigma_0^4 \left( 2n \text{trace} (B_{\gamma_0}^2) - 2n \sum_{i=1}^n (B_{\gamma_0})_{ii}^2 \right) + o_P(1) = \sigma_0^4 \left( 2n \text{trace} (B_{\gamma_0}^2) - 2 (\text{trace} (B_{\gamma_0}))^2 \right) + o_P(1) \xrightarrow{P} 2\sigma_0^4 \left( \left( h_2 - \frac{2h_3}{h_1} + \frac{h_4}{h_1^2} \right) - \left( h_1 - \frac{h_2}{h_1} \right)^2 \right).
\]

\[\square\]

**4.4.3 Conditional Distribution of Δ(γ₀) − Δₗ(γ₀)**

Recall from (34) that
\[
\sqrt{n} \Delta(\gamma_0) = (\xi^\top \ \zeta^\top) \sqrt{n} \left[ \begin{array}{c} \Lambda_\beta Z^\top B_{\gamma_0} Z \Lambda_\beta \ \Lambda_\beta Z^\top B_{\gamma_0} \Lambda_\epsilon \\ \Lambda_\epsilon B_{\gamma_0} Z \Lambda_\beta \ \Lambda_\epsilon B_{\gamma_0} \Lambda_\epsilon \end{array} \right] \left( \xi \ \zeta \right) = Q \sqrt{n} \Delta(\gamma_0) \left( \begin{array}{c} \Lambda_\beta \ 0 \\ 0 \ \frac{1}{\sqrt{n}} \Lambda_\epsilon \end{array} \right) \left( \begin{array}{c} Z^\top I_n \ \sqrt{n} B_{\gamma_0} \ Z \\ \sqrt{n} I_n \ \sqrt{n} I_n \end{array} \right) \left( \begin{array}{c} \Lambda_\beta \ 0 \\ 0 \ \frac{1}{\sqrt{n}} \Lambda_\epsilon \end{array} \right) \left( \begin{array}{c} \xi \ \zeta \end{array} \right).
\]

Note that we can represent Q as
\[
Q = \left( \begin{array}{c} \Lambda_\beta \ 0 \\ 0 \ \frac{1}{\sqrt{n}} \Lambda_\epsilon \end{array} \right) \left( Z^\top I_n \right) \left( \begin{array}{c} \Lambda_\beta \ 0 \\ 0 \ \frac{1}{\sqrt{n}} \Lambda_\epsilon \end{array} \right).
\]

Since \(\bar{\Delta}_\epsilon(\gamma_0) = \mathbb{E}[\Delta(\gamma_0)|Z, \varepsilon]\), we have
\[
\sqrt{n}(\Delta(\gamma_0) − \bar{\Delta}_\epsilon(\gamma_0)) = (\xi^\top \ \zeta^\top) \left( Q - Q \right) \left( \begin{array}{c} \xi \ \zeta \end{array} \right).
\]

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where \( \hat{Q} \) is a diagonal matrix that maintains the diagonal part of \( Q \). In other words, conditional on \( Z \) and \( \varepsilon \), \( \sqrt{n}(\Delta(\gamma_0) - \hat{\Delta}_*(\gamma_0)) \) is a quadratic form about \( \xi \) and \( \zeta \).

Here we aim to use Theorem A.7 in the appendix to establish a normal approximation of the conditional distribution of \( \sqrt{n}(\Delta(\gamma_0) - \hat{\Delta}_*(\gamma_0)) \) given \( Z \) and \( \varepsilon \). In other words, we intend to show

\[
\frac{\| Q - \hat{Q} \|}{\| Q - \hat{Q} \|_F} = o_P(1). \tag{57}
\]

To establish a lower bound \( \| Q - \hat{Q} \|_F \), consider the block \( \sqrt{n} \Lambda \beta Z^\top B_{\gamma_0} Z \Lambda \beta \) of \( Q \), there holds

\[
\| Q - \hat{Q} \|_F \geq \sqrt{\sum_{i \neq j} \beta_i^2 \beta_j^2 (z_j^\top B_{\gamma_0} z_i)^2},
\]

the right-hand side of which converges to some nonvanishing limit based on (42).

Let’s now establish an upper bound of \( \| Q - \hat{Q} \|_F \). First, we have

\[
\| Q - \hat{Q} \| \leq \| Q \| + \| \hat{Q} \| \leq 2 \| Q \|,
\]

where the last inequality is due to the fact that all diagonal entries are bounded by the operator norm in magnitude. On the other hand,

\[
\| Q \| \leq \left\| \begin{pmatrix} \Lambda \beta & 0 \\ 0 & \frac{1}{n} \Lambda \varepsilon \end{pmatrix} \right\| \left\| \begin{pmatrix} Z^\top \\ \sqrt{n} I_n \end{pmatrix} \right\|^2 \| \sqrt{n} B_{\gamma_0} \|
\leq \max_{k \in [p], i \in [n]} \left\{ \beta_k^2, \frac{1}{n} \varepsilon_i^2 \right\} \cdot (\| Z \| + \sqrt{n})^2 \cdot \| \sqrt{n} B_{\gamma_0} \|.
\]

By the assumption \( \| \beta \|_\infty^2 = o_P(n^{-1/2}) \) as well as the bound given in (12), we have

\[
\max_{k \in [p], i \in [n]} \left\{ \beta_k^2, \frac{1}{n} \varepsilon_i^2 \right\} = o_P(n^{-1/2}).
\]

Note that we have \( \| Z \| = O_P(\sqrt{n}) \) by Theorem A.9. Also, Lemma 4.2 implies \( \| \sqrt{n} B_{\gamma_0} \| = O_P(n^{-1/2}) \). Combining the above, we have \( \| Q \| = o_P(1) \). This completes the proof of (57).

Therefore, by Theorem A.7, we have

\[
P \left\{ \frac{\sqrt{n}(\Delta(\gamma_0) - \hat{\Delta}_*(\gamma_0))}{\sqrt{\operatorname{Var}[\sqrt{n} \Delta(\gamma_0) | Z, \varepsilon]}} \leq t \right| Z, \varepsilon \right\} \overset{P}{\rightarrow} \Phi(t), \tag{58}
\]

where \( \Phi(t) \) is the c.d.f of standard normal distribution.

### 4.4.4 Asymptotic Distribution of \( \sqrt{n}\hat{\Delta}_*(\gamma_0) \)

This subsection is intended to show the following result that characterizes the asymptotic distribution of \( \hat{\Delta}_*(\gamma_0) \) defined in (35). Note that by the definition of \( B_{\gamma} \), we always have \( \operatorname{trace}(B_{\gamma} V_{\gamma}) = 0 \). Therefore,

\[
\| \beta \|^2 \operatorname{trace} \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right) + \sigma_0^2 \operatorname{trace} (B_{\gamma_0}) = \sigma_0^2 \operatorname{trace} (B_{\gamma_0} V_{\gamma_0}) = 0.
\]
Then, we can represent $\Delta_*(\gamma_0)$ as
\[
\Delta_*(\gamma_0) = \sum_{k=1}^p \beta_k^2 z_k^\top B_{\gamma_0} z_k - \|\beta\|^2 \text{trace} \left( B_{\gamma_0} \frac{1}{p} ZZ^\top \right) \\
+ \text{trace} \left( \Lambda_s^2 B_{\gamma_0} - \sigma_0^2 \text{trace} (B_{\gamma_0}) \right) \\
= \text{trace} \left( (\Lambda_s^2 - \sigma_0^2 I_n) B_{\gamma_0} \right) \\
+ \sum_{k=1}^p \beta_k^2 \frac{1}{n} \left( z_k^\top V_{\gamma_0}^{-1} z_k - \text{trace} \left( V_{\gamma_0}^{-1} \frac{1}{p} ZZ^\top \right) \right) \\
+ \frac{n}{\text{trace}(V_{\gamma_0}^{-1})} \sum_{k=1}^p \beta_k^2 \frac{1}{n} \left( z_k^\top V_{\gamma_0}^{-2} z_k - \text{trace} \left( V_{\gamma_0}^{-2} \frac{1}{p} ZZ^\top \right) \right).
\]

By Lemma 4.5 and Lemma 4.3, we have
\[
\sum_{k=1}^p \beta_k^2 \frac{1}{n} \left( z_k^\top V_{\gamma_0}^{-1} z_k - \text{trace} \left( V_{\gamma_0}^{-1} \frac{1}{p} ZZ^\top \right) \right) \\
= - \frac{1}{\left(1 + \frac{\gamma}{p} \text{trace}(V_{\gamma}^{-1})\right)} \sum_{k=1}^p \beta_k^2 \left( \frac{1}{n} \eta_k^{(1)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-k}^{-1} \right) \right) + O_P \left( \frac{\log n}{n} \right)
\]
and
\[
\sum_{k=1}^p \beta_k^2 \frac{1}{n} \left( z_k^\top V_{\gamma_0}^{-2} z_k - \text{trace} \left( V_{\gamma_0}^{-2} \frac{1}{p} ZZ^\top \right) \right) \\
= - \frac{2\gamma \text{trace}(V_{\gamma}^{-2})}{p \left(1 + \frac{\gamma}{p} \text{trace}(V_{\gamma}^{-1})\right)} \sum_{k=1}^p \beta_k^2 \left( \frac{1}{n} \eta_k^{(2)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-k}^{-2} \right) \right) \\
+ \frac{1}{\left(1 + \frac{\gamma}{p} \text{trace}(V_{\gamma}^{-1})\right)} \sum_{k=1}^p \beta_k^2 \left( \frac{1}{n} \eta_k^{(2)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-k}^{-2} \right) \right) + O_P \left( \frac{\log n}{n} \right).
\]

Then, for $l = 1, 2,$
\[
\mathbb{E} \left( \sum_{k=1}^p \beta_k^2 \left( \frac{1}{n} \eta_k^{(l)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-k}^{-l} \right) \right) \right)^2 \\
= \sum_{k=1}^p \beta_k^4 \mathbb{E} \left[ \left( \frac{1}{n} \eta_k^{(l)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-k}^{-l} \right) \right)^2 \right] \\
+ \sum_{i \neq j} \beta_i^2 \beta_j^2 \mathbb{E} \left[ \left( \frac{1}{n} \eta_i^{(l)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-i}^{-l} \right) \right) \left( \frac{1}{n} \eta_j^{(l)} - \frac{1}{n} \text{trace} \left( V_{\gamma,-j}^{-l} \right) \right) \right] \\
\leq \frac{C}{n} \|\beta\|^4 + \frac{C}{n^2} \|\beta\|^4 = o \left( \frac{1}{n} \right),
\]
26
for which we have used Lemma 4.6 as well as the fact $\|\beta\|_4 = o(1)$ (by the assumption $\|\beta\|_\infty = o(p^{-1/4})$). Then by (60), (61), and (62), in connection with Lemma 4.2, there holds

$$\sum_{k=1}^{p} \beta_k^2 \frac{1}{n} \left( z_k^T V_{\gamma_0}^{-1} z_k - \text{trace} \left( V_{\gamma_0}^{-1} \frac{1}{p} ZZ^T \right) \right) = o_P \left( \frac{1}{\sqrt{n}} \right), \quad l = 1, 2. \quad (63)$$

Then, equation (59) implies

$$\tilde{\Delta}_\gamma(\gamma_0) = \text{trace} \left( (\Lambda^2 - \sigma_0^2 I_n) B_{\gamma_0} \right) + o_P \left( \frac{1}{\sqrt{n}} \right). \quad (64)$$

Before deriving the asymptotic distribution of $\tilde{\Delta}_\gamma(\gamma_0)$, we first introduce a lemma, which is essentially an analogy to (63):

**Lemma 4.15.** *Under the conditions of Theorem 2.2, for any fixed $\gamma > 0$, there holds*

$$\sum_{i=1}^{n} (\varepsilon_i^2 - \sigma_0^2) (B_{\gamma_0})_{ii} - \frac{1}{n} \text{trace} (B_{\gamma_0}) = o_P \left( \frac{1}{\sqrt{n}} \right). \quad (65)$$

With this lemma, equation (64) gives

$$\tilde{\Delta}_\gamma(\gamma_0) = \frac{1}{n} \left( \sum_{i=1}^{n} (\varepsilon_i^2 - \sigma_0^2) \right) \text{trace} (B_{\gamma_0}) + o_P \left( \frac{1}{\sqrt{n}} \right).$$

Recall that Lemma 4.2 implies $\text{trace} (B_{\gamma_0}) \overset{P}{\to} h_1 - \frac{h_2}{n_1}$. Moreover, (14) in Lemma 4.1 gives

$$\frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} \varepsilon_i^2 \right) - n\sigma_0^2 \Rightarrow N \left( 0, 2\kappa\sigma_0^4 \right).$$

Then, by the Slutsky’s theorem, we get

$$\sqrt{n} \tilde{\Delta}_\gamma(\gamma_0) = \sqrt{n} \mathbb{E}[\Delta(\gamma_0)|Z, \varepsilon] \Rightarrow N \left( 0, 2\kappa\sigma_0^4 \left( \frac{h_2}{h_1} - h_1 \right)^2 \right). \quad (66)$$

### 4.4.5 Asymptotic Distribution of $\hat{\gamma}$

Denote

$$\begin{cases} \nu_1 = 2\kappa\sigma_0^4 \left( \frac{h_2}{h_1} - h_1 \right)^2 \\ \nu_2 = 2\sigma_0^4 \left( \frac{h_2-h_1^2}{h_1^2} - \left( \frac{h_2-h_1}{h_1} \right)^2 \right) \end{cases},$$

which are the asymptotic variances given in (66) and Lemma 4.14.

To establish the asymptotic distribution of $\hat{\gamma}$, we only need to find that of $\sqrt{n} \Delta(\gamma_0)$ by Lemma 4.13. Furthermore, it suffices to find the asymptotic joint distribution of

$$\left( \sqrt{n}(\Delta(\gamma_0) - \tilde{\Delta}_\gamma(\gamma)), \sqrt{n} \tilde{\Delta}_\gamma(\gamma) \right).$$
For any \((t, s) \in \mathbb{R}^2\), we have

\[
\mathbb{P} \left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t, \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \leq s \right\}
= \mathbb{E} \left[ \mathbb{P} \left\{ \frac{\sqrt{n} \Delta(\gamma_0)}{\sqrt{\nu_1}} \leq t, \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \leq s \right\} | Z, \varepsilon \right] \right]
= \mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t \right\}} \mathbb{P} \left\{ \frac{\sqrt{n} \Delta(\gamma_0)}{\sqrt{\nu_1}} \leq t, \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \leq s \right\} | Z, \varepsilon \right] \right].
\]

Note that

\[
\left| \mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t \right\}} \mathbb{P} \left\{ \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \leq s \right\} | Z, \varepsilon \right] \right| - \mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t \right\}} \Phi(t) \right] \leq \mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t \right\}} \mathbb{P} \left\{ \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \leq s \right\} | Z, \varepsilon \right] - \Phi(s) \leq 0,
\]

where the last inequality is due to (58). By (66), we have

\[
\mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t \right\}} \Phi(s) \right] = \mathbb{P} \left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t \right\} \Phi(s) \rightarrow \Phi(t)\Phi(s).
\]

Thus we can have

\[
\mathbb{P} \left\{ \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} \leq t, \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \leq s \right\} \rightarrow \Phi(t)\Phi(s),
\]

which implies that

\[
\left( \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}}, \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \right) \Rightarrow (X_1, X_2),
\]

where \([X_1, X_2] \sim \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)\). By Lemma 4.14, we have

\[
\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right] \overset{P}{\rightarrow} \nu_2.
\]

Then, the Slutsky’s theorem implies

\[
\left( \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}}, \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \right) \Rightarrow (X_1, X_2, \sqrt{\nu_2}).
\]

Letting \(g(x, y, z) = \sqrt{\nu_1}x + yz\), by the continuous mapping theorem, we can have

\[
\sqrt{n} \Delta(\gamma_0) = \sqrt{\nu_1} \frac{\sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\nu_1}} + \frac{\sqrt{n} \Delta(\gamma_0) - \sqrt{n} \Delta_*(\gamma_0)}{\sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]}} \cdot \sqrt{\text{Var} \left[ \frac{\sqrt{n} \Delta(\gamma_0)}{Z, \varepsilon} \right]} \Rightarrow \sqrt{\nu_1}X_1 + \sqrt{\nu_2}X_2.
\]
Finally, by Lemma 4.13 and the Slutsky’s theorem, we have

\[ \sqrt{n} (\hat{\gamma} - \gamma_0) \Rightarrow N \left( 0, \frac{\nu_1 + \nu_2}{(\Delta_\infty'(\gamma_0))^2} \right) \]

By \( \Delta_\infty'(\gamma_0) = \frac{\sigma_0^2 h_2^2 - h_1^2}{h_1} \) and the expressions in (67), simplifying the formula, we have

\[ \sqrt{n} (\hat{\gamma} - \gamma_0) \Rightarrow N \left( 0, 2\gamma_0^2 \left( \frac{1}{h_2 - h_1^2} + \kappa - \tau - 1 \right) \right) \]

### 4.5 Proof of Proposition 2.3

Straightforward calculation gives

\[ \mathbb{E}[y_i^4] = \sum_{j=1}^{p} \left( \mathbb{E}[z_{ij}^4] - 3 \right) \beta_j^4 + 3 \|\beta\|_2^4 + 6 \|\beta\|_2^2 \sigma_i^2 + 3 \sigma_i^4. \tag{68} \]

Then, when the noise is uncorrelated,

\[ \kappa = \frac{1}{n\sigma_0^4} \sum_{i=1}^{n} \sigma_i^4 = \frac{1}{3n\sigma_0^4} \sum_{i=1}^{n} \mathbb{E}[y_i^4] - (\gamma_0^2 + 2\gamma_0) + \frac{1}{3n\sigma_0^4} \sum_{i=1}^{n} \sum_{j=1}^{p} (\mathbb{E}[z_{ij}^4] - 3) \beta_j^4. \]

By the assumption \( \|\beta\|_\infty = o(p^{-1/4}) \) and \( z_{ij} \) is sub-Gaussian, it is obvious that

\[ \frac{1}{3n\sigma_0^4} \sum_{i=1}^{n} \sum_{j=1}^{p} (\mathbb{E}[z_{ij}^4] - 3) \beta_j^4 \leq \max_{1 \leq i \leq n} \mathbb{E}[z_{ij}^4] - 3 \leq \frac{1}{3n\sigma_0^4} \sum_{j=1}^{p} \beta_j^4 = o(1). \]

Furthermore, similar to (68), straightforward calculation implies \( \max_{1 \leq i \leq n} \mathbb{E}[y_i^8] = O(1) \), which implies

\[ \text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} y_i^4 \right] = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}[y_i^4] \leq \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}[y_i^8] = O \left( \frac{1}{n} \right), \]

Then we can have \( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[y_i^4] = \frac{1}{n} \sum_{i=1}^{n} y_i^4 + O_P(n^{-1/2}). \) Combining the above, we have

\[ \frac{1}{3n\sigma_0^4} \sum_{i=1}^{n} y_i^4 - (\gamma_0^2 + 2\gamma_0) \xrightarrow{P} \kappa. \]

In Theorem 2.1 we have already shown that \( \hat{\sigma}^2 \xrightarrow{P} \sigma_0^2 \) and \( \hat{\gamma}_n \xrightarrow{P} \gamma_0 \). By the Slutsky’s theorem, we obtain \( \hat{\kappa} := \frac{1}{3n\sigma_0^4} \sum_{i=1}^{n} y_i^4 - (\gamma_0^2 + 2\gamma_0) \xrightarrow{P} \kappa. \)
5 Discussion

This paper is concerning the estimation of signal-to-noise ratios and other related quantities in high-dimensional linear models. In particular, consistency and asymptotic distribution are derived for the REML estimator of the SNR under general fixed coefficient vector as well as heteroscedastic and correlated noise.

In future work, we first aim to relax the skew-free assumption imposed on the design entries as required in our main results. Recall that the assumption of symmetric distributions is essential in our asymptotic analysis, which enables us to introduce the double Rademacher sequence technique and then both consistency and asymptotic distribution can be obtained by dealing with conditional mean and variance through leave-\(k\)-out analysis. However, it would be interesting to relax this condition in future, given skew-free assumption might be violated in practice of high-dimensional linear models.

Several extensions of our work can be investigated in future. One interesting extension is SNR/heritability estimation with multi-variate outcomes without explicitly modeling the correlations between the responses. Along with REML, methods of moments (Haseman and Elston, 1972; Dicker, 2014) can also be considered as candidate estimators. Besides asymptotic analysis of SNR estimators, standard error estimation under this setting is challenging due to the correlations between the response variables, due to which plug-in estimators of the standard error may be unobtainable. It would be interesting to devise some resampling method such as jackknife to estimate the standard error of the SNR estimator.

Another related question is group SNR estimation in high-dimensional linear models with feature groups, which has important applications in heritability estimation in GWAS since genes can be naturally grouped based on chromosomes; see e.g. Yang et al. (2011b). Furthermore, group SNR estimation is highly related with group ridge regression (Ignatiadis and Lolas, 2020). Theoretically speaking, asymptotic analysis for linear mixed effects models with feature groups is also well-studied in the literature (Jiang, 1996). It would be interesting to extend our misspecification analysis to this case. If the design matrix is assumed to consist of i.i.d. Gaussian entries, combining the rotational invariance technique in Dicker and Erdogdu (2016) and the nonasymptotic analytical framework proposed in Dicker and Erdogdu (2017) in analyzing REML can be in principle applied to this extension. However, this approach is not sufficient for the case of heterogeneous and correlated noise. Moreover, we are also interested in removing the assumption of rotational invariance. We are interested in addressing these issues in future work.

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Appendix A  Preliminaries

Let’s first recall the famous Marčenko-Pastur law in random matrix theory.

**Theorem A.1** (Marčenko-Pastur law, V. A. Marchenko (1967)). Let $Z$ be an $n \times p$ random matrix whose entries are i.i.d. random variables with mean 0 and variance 1 in which $n/p \to \tau \in (0, \infty)$ as $n, p \to \infty$. Then the empirical spectral distribution (ESD) of $S = p^{-1}ZZ^\top$, which is defined as $F_S$, converges almost surely (a.s.) in distribution to $F_\tau$, whose p.d.f. is given by

$$f_\tau(x) = \begin{cases} \max\{\tau - 1, 0\}\delta_0(x) + \frac{1}{2\pi x} \sqrt{(b_+(\tau) - x)(x - b_-(\tau))} & b_-(\tau) \leq x \leq b_+(\tau) \\ 0 & \text{elsewhere} \end{cases}$$

where $b_\pm(\tau) = (1 \pm \sqrt{\tau})^2$ and $\delta_0(x)$ is a point mass $\tau^{-1}$ at the origin.

Note that in our settings, the entries of the design matrix are not necessarily identically distributed. To this end, we consider the following extension of Marčenko-Pastur law.

**Theorem A.2** (Bai (1999), Theorem 2.8). Let $Z$ be an $n \times p$ random matrix whose entries are independent random variables with mean 0 and variance 1. Assume that $n/p \to \tau \in (0, \infty)$ and that for any $\delta > 0$,

$$\frac{1}{\delta^2 np} \sum_{i,j} \mathbb{E} \left[ |z_{ij}^{(n)}|^2 I_{(|z_{ij}^{(n)}| \geq \delta \sqrt{n})} \right] \to 0.$$

Then $F_S$, defined as in Theorem A.1, tends almost surely to the Marčenko-Pastur law with ratio index $\tau$.

**Corollary A.1.** Under the assumption of Theorem A.1 or A.2, for any integer $l$, we have

$$\frac{1}{n} \text{trace}(S^l) \xrightarrow{a.s.} \int_{b_-(\tau)}^{b_+(\tau)} x^l f_\tau(x)dx \quad \text{as} \quad n, p \to \infty.$$

Define the sub-Gaussian norm of a random variable $\zeta$ as

$$\|\zeta\|_{\psi_2} \equiv \sup_{q \geq 1} \{q^{-1/2} (\mathbb{E}|\zeta|^q)^{1/q} \}.$$

A random variable $\zeta$ is sub-Gaussian if and only if its sub-Gaussian norm $\|\zeta\|_{\psi_2} < \infty$. We have the following equivalent characterizations on the sub-Gaussianity of a random variable:

**Lemma A.1** (Vershynin (2010), Lemma 5.5). A random variable $\zeta$ is sub-Gaussian if and only if

1) $\|\zeta\|_{\psi_2} < \infty$;  

2) $\mathbb{P}\{|\zeta| > t\} \leq \exp(1 - t^2/K^2)$ for some parameter $K > 0$ and all $t > 0$.  

Part 2) actually implies that the design matrix under the setting of Theorem 2.1, in which the entries have sub-Gaussian norms that are uniformly upper bounded, satisfies the conditions in Theorem A.2. In fact, if \( \zeta \) is sub-Gaussian random variable, then by the identity \( E[X] = \int_0^\infty P(X > t)dt \) for any nonnegative random variable \( X \), we have

\[
E\left[|\zeta|^2 I_{(|\zeta| \geq \delta \sqrt{n})}\right] = \int_{\delta \sqrt{n}}^\infty P\{|\zeta| > t\}2tdt + \delta^2 n P\{|\zeta| > \delta \sqrt{n}\} \\
\leq 2 \int_{\delta \sqrt{n}}^\infty e^{1-\frac{t^2}{\kappa^2}}tdt + \delta^2 ne^{1-\frac{\delta^2 n}{K^2}} \\
= (K^2 + \delta^2 n)e^{1-\frac{\delta^2 n}{\kappa^2}}.
\]

This implies that for \( n \times p \) random matrices \( Z \) whose entries have uniformly upper bounded sub-Gaussian norms,

\[
\frac{1}{\delta^2 np} \sum_{i,j} E\left[|z_{ij}^{(n)}|^2 I_{(|z_{ij}^{(n)}| \geq \delta \sqrt{n})}\right] \to 0,
\]
as \( n, p \to \infty \), for any \( \delta > 0 \).

Our proof also relies crucially on the following fundamental concentration inequalities.

**Proposition A.3** (Hanson–Wright inequality, Rudelson and Vershynin (2013)). Let \( \zeta = (\zeta_1, \cdots, \zeta_n)^T \), where the \( \zeta_i \)'s are independent random variables satisfying \( \mathbb{E}(\zeta_i) = 0 \) and \( \|\zeta_i\|_{\psi_2} \leq K < \infty \). Let \( A \) be an \( n \times n \) deterministic matrix. Then we have for any \( t > 0 \),

\[
P\{|\zeta^T A\zeta - \mathbb{E}(\zeta^T A\zeta)| > t\} \leq 2 \exp\left\{-c \min\left(\frac{t^2}{K^4 \|A\|_F^2}, \frac{t}{K^2 \|A\|}\right)\right\},
\]

where \( c > 0 \) is an absolute constant. Here \( \|A\| \) and \( \|A\|_F \) denote the operator and Frobenius norms of \( A \), respectively.

**Proposition A.4** (Hoeffding-type inequality, Vershynin (2010), Proposition 5.10). Let \( \zeta = (\zeta_1, \cdots, \zeta_n)^T \), where the \( \zeta_i \)'s are independent centered sub-Gaussian random variables. Let \( K = \max_{1 \leq i \leq n} \|\zeta_i\|_{\psi_2} \) and \( a = (a_1, \cdots, a_N)^T \in \mathbb{R}^N \). Then we have for any \( t \geq 0 \),

\[
P\{|a^T \zeta| > t\} \leq e \exp\left\{-c \frac{t^2}{K^2 \|a\|_2^2}\right\},
\]

where \( c > 0 \) is an absolute constant.

**Proposition A.5** (Bernstein-type inequality, Vershynin (2010), Proposition 5.16). Let \( \zeta = (\zeta_1, \cdots, \zeta_n)^T \), where the \( \zeta_i \)'s are independent centered sub-exponential random variables. Let \( K = \max_{1 \leq i \leq n} \|\zeta_i\|_{\psi_2} \) and \( a = (a_1, \cdots, a_N)^T \in \mathbb{R}^N \). Then we have for any \( t \geq 0 \),

\[
P\{|a^T \zeta| > t\} \leq 2 \exp\left\{-c \min\left(\frac{t^2}{K^2 \|a\|_2^2}, \frac{t}{K \|a\|_\infty}\right)\right\},
\]

where \( c > 0 \) is an absolute constant.

The next result, the famous Sherman-Morrison-Woodbury formula in matrix analysis is repeatedly used in our proofs, as the corner stone of leave-one-out analysis.
Theorem A.6 (Sherman-Morrison-Woodbury formula, Horn and Johnson (1990), Page 19). Let $P$ and $Q$ be $n$-dimensional non-singular matrices such that $Q = P + UV^\top$, where $U, V \in \mathbb{R}^{n \times q}$. Then

$$Q^{-1} = (P + UV^\top)^{-1} = P^{-1} - P^{-1}U(I_q + V^\top P^{-1}U)^{-1}V^\top P^{-1}.$$  

The following results, implied by Chatterjee (2008) and Chatterjee (2009), are conditions for the normality of quadratic forms.

Theorem A.7 (Chatterjee (2008), Proposition 3.1). Let $X = (X_1, \ldots, X_n)$ be i.i.d. Rademacher random variables and $A = (a_{ij})_{1 \leq i, j \leq n}$ be a real symmetric matrix. Let $W = X^\top AX$ and

$$\sigma^2 = \text{Var}(W) = \frac{1}{2} \text{trace}(A^2).$$

Let $\mu$ be the law of $(W - \mathbb{E}(W))/\sqrt{\text{Var}(W)}$ and let $\nu$ be the standard Gaussian law. We define

$$d_W := \mathcal{W}(\mu, \nu),$$

where $W$ is the Kantorovich–Wasserstein distance between two probability measures with

$$\mathcal{W}(\mu, \nu) = \sup \left\{ \left| \int hd\mu - \int hd\nu \right| : h \text{ Lipschitz, } \|h\|_{Lip} \leq 1 \right\}.$$  

Then,

$$d_W \leq \left( \frac{\text{trace}(A^4)}{2\sigma^4} \right)^{1/2} + \frac{5}{2\sigma^3} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}^2 \right)^{3/2} \leq 6\sqrt{2} \frac{\|A\|_2^2}{\|A\|_F^2}.$$  

Theorem A.8 (Chatterjee (2009)). Suppose $x$ is a gaussian random vector with mean 0 and covariance matrix $\Sigma$. Take any $g \in C^2(\mathbb{R})$ and let $\nabla g$ and $\nabla^2 g$ denote the gradient and Hessian of $g$. Let

$$\varsigma_1 = \left( \mathbb{E} \|\nabla g(x)\|^4 \right)^{1/4}, \quad \varsigma_2 = \left( \mathbb{E} \|\nabla^2 g(x)\|^4 \right)^{1/4}.$$  

Then let $W = g(x)$ have a finite fourth moment and $U$ be a normal random variable having the same mean and variance as $W$,

$$d_{TV}(W, U) \leq \frac{2\sqrt{5} \|\Sigma\|_{\frac{3}{2}}^{\frac{3}{2}} \varsigma_1 \varsigma_2}{\text{Var}[W]}.$$  

Here $d_{TV}$ is the total variation distance between random variables $u$ and $v$,

$$d_{TV}(u, v) = \sup_{B \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(u \in B) - \mathbb{P}(v \in B)|,$$

where $\mathcal{B}(\mathbb{R})$ denotes the collection of Borel sets in $\mathbb{R}$.

Next, there is a famous result for the bounds of eigenvalues of the sub-gaussian random matrix.

Theorem A.9 (Theorem 5.39, Vershynin (2010)). Let $Z$ be an $n \times p$ matrix whose rows are independent sub-gaussian isotropic random vectors. Then for every $t \geq 0$, with probability at least $1 - 2\exp(-ct^2)$ one has

$$\sqrt{n} - C\sqrt{p} - t \leq \lambda_{\min}(Z) \leq \lambda_{\max}(Z) \leq \sqrt{n} + C\sqrt{p} + t$$

Here $C = C_K$, $c = c_K > 0$ depend only on the subgaussian norm $K$ of the rows.
Appendix B  Proofs of Lemmas in Section 4

In this section we give detailed proofs of the technical lemmas that appear in Section 4. As mentioned earlier, the proofs of Lemmas 4.3, 4.4, 4.5, 4.7 and 4.12 basically follow the proof ideas in Jiang et al. (2016), but we provide self-contained proofs here for completeness. Interested readers are recommended to read Jiang et al. (2016) for deeper insights.

B.1 Proof of Lemma 4.1

Since \( \varepsilon_i \sim N(0, \sigma_i^2) \), there holds \( \varepsilon_i^2/\sigma_i^2 \sim \chi_1^2 \). By the standard Laurent-Massart bound (Laurent and Massart (2000)), there holds that

\[
P\{ \varepsilon_i^2/\sigma_i^2 - 1 \geq 2\sqrt{t} + 2t \} \leq \exp(-t) \quad \text{and} \quad P\{1 - \varepsilon_i^2/\sigma_i^2 \geq 2\sqrt{t} \} \leq \exp(-t).
\]

Taking \( t = 2\log n \), we can have for any \( i = 1, \ldots, n \),

\[
P\{ \max_{1 \leq i \leq n} \varepsilon_i^2/\sigma_i^2 - 1 \geq 2\sqrt{2\log n} + 4\log n \} \leq \frac{1}{n},
\]

which implies (12).

Since \( \varepsilon \sim N_n(0, \Sigma_{\varepsilon}) \), we can rewrite \( \sum_{i=1}^n \varepsilon_i^2 \) as \( \varepsilon^T \varepsilon \), then

\[
E[\varepsilon^T \varepsilon] = \text{trace}(\Sigma_{\varepsilon}) = \sum_{i=1}^n \sigma_i^2 = n\sigma_0^4,
\]

and

\[
\text{Var}[\varepsilon^T \varepsilon] = \text{trace}(\Sigma_{\varepsilon}^2) = 2\|\Sigma_{\varepsilon}\|^2_F.
\]

By the assumption that \( \|\Sigma_{\varepsilon}\|_F = o(n) \),

\[
\text{Var}\left[\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2\right] = o(1).
\]

Then we can have (13).

By applying Theorem A.8 directly, we can take

\[
g(\mathbf{x}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i^2,
\]

then

\[
d_{TV}(g(\varepsilon), U) \leq \sqrt{5}\|\Sigma_{\varepsilon}\|_{2,1}^2 \leq \frac{\sqrt{5}}{\frac{1}{4}\|\Sigma_{\varepsilon}\|_F^2} \|\Sigma_{\varepsilon}\|^2,
\]

where \( U \sim N(\sqrt{n}\sigma_0^2, \frac{1}{n}\|\Sigma_{\varepsilon}\|^2_F) \).

Since

\[
\frac{\partial g}{\partial x_i} = \frac{2}{\sqrt{n}} x_i,
\]

\[
\frac{\partial g}{\partial x_i \partial x_j} = \begin{cases} \frac{2}{\sqrt{n}} & i = j, \\ 0 & i \neq j, \end{cases}
\]

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by the assumption that \( \| \Sigma_\varepsilon \|_F^2 = n \kappa \sigma_0^4 \), it follows that

\[
\varsigma_1 = \left( \mathbb{E} \left\| \nabla g(\varepsilon) \right\|_4^4 \right)^{\frac{1}{4}} = \left( \mathbb{E} \left( \sum_{i=1}^{n} \left( \frac{2}{\sqrt{n}} \varepsilon_i \right)^2 \right)^2 \right)^{\frac{1}{4}} = \frac{2}{\sqrt{n}} \left( \mathbb{E} \left( \sum_{i=1}^{n} \varepsilon_i^2 \right)^2 \right)^{\frac{1}{4}}
\]

\[
= \frac{2}{\sqrt{n}} \left( \text{Var} [\varepsilon^\top \varepsilon] + \left( \mathbb{E} [\varepsilon^\top \varepsilon] \right)^2 \right)^{\frac{1}{4}}
\]

\[
= O(1),
\]

and \( \varsigma_2 = \left( \mathbb{E} \left\| \nabla^2 g(\varepsilon) \right\|_4^4 \right)^{\frac{1}{4}} = O(1/\sqrt{n}) \). Then by the assumption that \( \| \Sigma_\varepsilon \| \) is uniformly bounded, we can have that as \( n \to \infty \)

\[
d_{TV}(g(\varepsilon), U) = O(1/\sqrt{n}) = o(1),
\]

which implies (14).

**B.2 Proof of Lemma 4.3**

For convenience, define

\[
\begin{align*}
\rho_k &:= \eta_{kk,k}^{(1)} := z_k^\top V_{\gamma,-k}^{-1} z_k, \\
\phi_k &:= \eta_{kk,k}^{(2)} := z_k^\top V_{\gamma,-k}^{-2}, \\
\psi_k &:= \eta_{kk,k}^{(3)} := z_k^\top V_{\gamma,-k}^{-3}.
\end{align*}
\]

First, there is a simple relationship: \( \psi_k \leq \phi_k \leq \rho_k \). In fact, since \( I_n - V_{\gamma,-k}^{-1} \succeq 0 \), we know that

\[
\rho_k - \phi_k = z_k^\top V_{\gamma,-k}^{-1/2} (I - V_{\gamma,-k}^{-1}) V_{\gamma,-k}^{-1/2} z_k \geq 0.
\]

i.e., \( \phi_k \leq \rho_k \). We can similarly obtain \( \psi_k \leq \phi_k \).

Using Sherman-Morrison-Woodbury formula (Theorem A.6), we have

\[
V_{\gamma}^{-1} = V_{\gamma,-k}^{-1} - \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} V_{\gamma,-k}^{-1} z_k z_k^\top V_{\gamma,-k}^{-1},
\]

and

\[
V_{\gamma}^{-2} = \left( V_{\gamma,-k}^{-1} - \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} V_{\gamma,-k}^{-1} z_k z_k^\top V_{\gamma,-k}^{-1} \right)^2
\]

\[
= V_{\gamma,-k}^{-2} - \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} V_{\gamma,-k}^{-2} z_k z_k^\top V_{\gamma,-k}^{-1} - \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} V_{\gamma,-k}^{-1} z_k z_k^\top V_{\gamma,-k}^{-2}
\]

\[
+ \left( \frac{\gamma}{p} \right)^2 (1 + \frac{\gamma}{p} \rho_k)^{-2} \phi_k V_{\gamma,-k}^{-1} z_k z_k^\top V_{\gamma,-k}^{-1}.
\]

By (71) and (72), we can also have

\[
\text{trace}(V_{\gamma}^{-1}) = \text{trace}(V_{\gamma,-k}^{-1}) - \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} \phi_k,
\]
and
\[
\text{trace}(V^{-2}) = \text{trace}(V^{-2})_{\gamma, -k} = \frac{2\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} \psi_k + \frac{(\gamma)^2}{p} (1 + \frac{\gamma}{p} \rho_k)^{-2} \phi_k^2.
\]

Then
\[
\left| \text{trace}(V^{-1})_{\gamma, -k} - \text{trace}(V^{-1})_{\gamma, -k} \right| = \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} \phi_k \leq \frac{\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} \rho_k < 1,
\]
and
\[
\left| \text{trace}(V^{-2})_{\gamma, -k} - \text{trace}(V^{-2})_{\gamma, -k} \right| = \frac{2\gamma}{p} (1 + \frac{\gamma}{p} \rho_k)^{-1} \rho_k + \frac{(\gamma)^2}{p} (1 + \frac{\gamma}{p} \rho_k)^{-2} \rho_k^2 < 3.
\]

Similarly, we can also prove that
\[
\left| \text{trace}(V^{-3})_{\gamma, -k} - \text{trace}(V^{-3})_{\gamma, -k} \right| \leq 7 \text{ and } \left| \text{trace}(V^{-4})_{\gamma, -k} - \text{trace}(V^{-4})_{\gamma, -k} \right| \leq 15.
\]

Since the entries of $Z$ are independent sub-Gaussian and $E(z_{ik}) = 0$, using Proposition A.3, we have, for any $1 \leq k \leq p$ and $t > 0$:
\[
\mathbb{P}\left\{ |\rho_k - \text{trace}(V^{-1})_{\gamma, -k}| > t \mid V_{\gamma, -k} \right\} \leq 2 \exp\left\{ -c \min\left( \frac{t^2}{K^4 \| V^{-1}_{\gamma, -k} \|_F^2}, \frac{t}{K^2 \| V^{-1}_{\gamma, -k} \|} \right) \right\},
\]
where $c$ and $K$ are positive constants. If we set
\[
t = t_k = K^2 \max\left( \sqrt{\frac{2 \log p}{c}} \| V^{-1}_{\gamma, -k} \|_F, \frac{2 \log p}{c} \| V^{-1}_{\gamma, -k} \| \right),
\]

it follows that
\[
\mathbb{P}\left\{ |\rho_k - \text{trace}(V^{-1})_{\gamma, -k}| > t \right\} \leq 2/p^2.
\]

Then
\[
\mathbb{P}\left\{ \max_{1 \leq k \leq p} |\rho_k - \text{trace}(V^{-1})_{\gamma, -k}| > 1 \right\} \leq \frac{2}{p}.
\]

By Lemma 4.2, $\| V^{-1}_{\gamma, -k} \| \leq 1$, and $\| V^{-1}_{\gamma, -k} \|_F \leq \sqrt{n} \| V^{-1}_{\gamma, -k} \| \leq \sqrt{n}$, we can obtain that
\[
t_k \leq K^2 \max\left( \sqrt{\frac{2}{c} \sqrt{n} \log p}, \frac{2}{c} \log p \right),
\]
which implies
\[
\mathbb{P}\left\{ \max_{1 \leq k \leq p} |\rho_k - \text{trace}(V^{-1})_{\gamma, -k}| > C \sqrt{n \log p} \right\} \leq 2/p
\]
for some constant $C > 0$. Then, it follows that
\[
\max_{1 \leq k \leq p} |\rho_k - \text{trace}(V^{-1})_{\gamma, -k}| = O_P(\sqrt{n \log n}). \tag{75}
\]

By a similar argument, we have
\[
\max_{1 \leq k \leq p} |\phi_k - \text{trace}(V^{-2})_{\gamma, -k}| = O_P(\sqrt{n \log n}). \tag{76}
\]

Combining (73), (75), (74) and (76), we have
\[
\max_{1 \leq k \leq p} |\rho_k - \text{trace}(V^{-1})_{\gamma}| = O_P(\sqrt{n \log n}), \tag{77}
\]
and
\[
\max_{1 \leq k \leq p} |\phi_k - \text{trace}(V^{-2})_{\gamma}| = O_P(\sqrt{n \log n}). \tag{78}
\]
B.3 Proof of Lemma 4.4

Based on (71) and (72), there holds

\begin{align}
\mathbf{z}_k^\top \mathbf{V}_\gamma^{-1} \mathbf{z}_k &= (1 + \frac{\gamma}{p} \rho_k)^{-1} \rho_k, \\
\mathbf{z}_k^\top \mathbf{V}_\gamma^{-2} \mathbf{z}_k &= (1 + \frac{\gamma}{p} \rho_k)^{-2} \phi_k,
\end{align}

(79)

(80)

where \( \rho_k \) and \( \phi_k \) are defined in (70).

Let’s now come back to find approximations of \( \mathbb{E}[A_1|\mathbf{Z}] \) and \( \mathbb{E}[A_2|\mathbf{Z}] \). We define the following intermediate quantities

\[ \theta_1 = \frac{1}{n} \text{trace}(\mathbf{V}_\gamma^{-1}) \quad \text{and} \quad \theta_2 = \frac{1}{n} \frac{\text{trace}(\mathbf{V}_\gamma^{-2})}{\left(1 + \frac{\gamma}{p} \text{trace}(\mathbf{V}_\gamma^{-1})\right)^2}. \]

(81)

Then by (79) and (77), we can have

\[ \max_{1 \leq k \leq p} \left| \mathbf{z}_k^\top \mathbf{V}_\gamma^{-1} \mathbf{z}_k \right| \leq \max_{1 \leq k \leq p} \left| \frac{\text{trace}(\mathbf{V}_\gamma^{-1}) - \rho_k}{n} \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \]

(82)

which implies (18). Similarly, by (80) and (78), there holds

\[ \max_{1 \leq k \leq p} \left| \mathbf{z}_k^\top \mathbf{V}_\gamma^{-2} \mathbf{z}_k \right| \leq \max_{1 \leq k \leq p} \left| \frac{\text{trace}(\mathbf{V}_\gamma^{-2})}{n} \right| \left(1 + \frac{\gamma}{p} \text{trace}(\mathbf{V}_\gamma^{-1})\right)^2 \phi_k \]

\[ \leq \frac{1}{n} \left[ \max_{1 \leq k \leq p} |\text{trace}(\mathbf{V}_\gamma^{-2}) - \phi_k| + \max_{1 \leq k \leq p} \frac{2\gamma}{p} \rho_k |\text{trace}(\mathbf{V}_\gamma^{-2}) - \phi_k| \right. \]

\[ + \left. \max_{1 \leq k \leq p} \frac{2\gamma}{p} \phi_k |\rho_k - \text{trace}(\mathbf{V}_\gamma^{-1})| + \max_{1 \leq k \leq p} \frac{\gamma^2}{p^2} \rho_k^2 |\text{trace}(\mathbf{V}_\gamma^{-2}) - \phi_k| \right] \]

\[ + \max_{1 \leq k \leq p} \frac{\gamma^2}{p^2} \phi_k \left( |\rho_k - \text{trace}(\mathbf{V}_\gamma^{-1})| + |\rho_k + \text{trace}(\mathbf{V}_\gamma^{-1})| \right). \]

(83)

It follows, by the facts \( \text{trace}(\mathbf{V}_\gamma^{-1}) = O_P(n) \), \( \text{trace}(\mathbf{V}_\gamma^{-2}) = O_P(n) \), \( \rho_k = O_P(n) \) and \( \phi_k = O_P(n) \) (Lemma 4.2 and Lemma 4.3), (19) is true.

Then we can have for \( l = 1, 2 \),

\[ \left| \frac{\mathbf{z}_k^\top \mathbf{V}_\gamma^{-l} \mathbf{z}_k}{n} - \frac{1}{np} \text{trace} \left( \mathbf{V}_\gamma^{-1} \mathbf{Z} \mathbf{Z}^\top \right) \right| \]

\[ = \left| \frac{\mathbf{z}_k^\top \mathbf{V}_\gamma^{-l} \mathbf{z}_k}{n} - \theta_1 + \frac{1}{p} \sum_{i=1}^{p} \left( \theta_1 - \sum_{i=1}^{p} \mathbf{z}_i^\top \mathbf{V}_\gamma^{-1} \mathbf{z}_i \right) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \]

(84)

which implies (20). Then by the definition of \( \mathbf{B}_\gamma \), when \( l = 1 \) we can get (21) from (20).
When \( l = 2 \), for \((z_k^\top B_\gamma z_k)^2\),

\[
(z_k^\top B_\gamma z_k)^2 = \left( \frac{1}{n} z_k^\top V_\gamma^{-1} z_k - \frac{1}{n} z_k^\top V_\gamma^{-2} z_k \right)^2
= \left( \frac{1}{n} \eta_{kk}^{(1)} \right)^2 - 2 \frac{1}{n} \eta_{kk}^{(1)} \eta_{kk}^{(2)} + \frac{\left( \frac{1}{n} \eta_{kk}^{(2)} \right)^2}{\left( \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right)^2}. \tag{85}
\]

Then by triangle inequality, for any \( l, m = 1, 2 \),

\[
\frac{1}{n} \eta_{kk}^{(l)} \frac{1}{n} \eta_{kk}^{(m)} - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right)
\leq \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \frac{1}{n} \eta_{kk}^{(m)} - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) + \frac{1}{n} \eta_{kk}^{(l)} - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right)
+ \left| \frac{1}{n} \eta_{kk}^{(m)} - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \right| \left| \frac{1}{n} \eta_{kk}^{(l)} - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \right|.
\]

By (84) and the fact that

\[
\frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) = O_P(1), \quad \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) = O_P(1),
\]

we can have

\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} \eta_{kk}^{(l)} \frac{1}{n} \eta_{kk}^{(m)} - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right). \tag{86}
\]

Since

\[
\left( \text{trace} \left( B_\gamma \frac{1}{p} ZZ^\top \right) \right)^2
= \left( \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) - \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \right)^2
= \left( \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \right)^2 - 2 \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right)
+ \frac{\left( \frac{1}{n} \text{trace} \left( V_\gamma^{-1} \frac{1}{p} ZZ^\top \right) \right)^2}{\left( \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right)^2},
\]

then by (85) and (86), there holds that

\[
\max_{1 \leq k \leq p} \left| (z_k^\top B_\gamma z_k)^2 - \left( \text{trace} \left( B_\gamma \frac{1}{p} ZZ^\top \right) \right)^2 \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right). \tag{87}
\]
Finally, as we can know from (18), (20) and Lemma 4.2 that $(np)^{-1} \text{trace}(V_\gamma^{-1} ZZ^\top)$ converges to the same limit as 

\[
\frac{n^{-1} \text{trace}(V_\gamma^{-1})}{1 + \gamma p^{-1} \text{trace}(V_\gamma^{-1})},
\]

which means

\[
\frac{1}{\gamma} (1 - h_1(\gamma, \tau)) = \frac{h_1(\gamma, \tau)}{1 + \gamma \tau h_1(\gamma, \tau)}.
\]

And Lemma 4.2 shows that

\[
\left| \frac{1}{np} \text{trace}(V_\gamma^{-1} ZZ^\top) - \frac{1}{\gamma} (1 - h_1(\gamma, \tau)) \right| = O_P \left( \frac{1}{n} \right),
\]

and

\[
\left| \frac{1}{n} \text{trace}(V_\gamma^{-1}) \frac{1}{1 + \frac{2}{p} \text{trace}(V_\gamma^{-1})} - \frac{h_1(\gamma, \tau)}{1 + \gamma \tau h_1(\gamma, \tau)} \right| = O_P \left( \frac{1}{n} \right).
\]

Combine the above two inequalities, we can get (22). Similarly, by (19), (20) and Lemma 4.2 we can get (23).

### B.4 Proof of Lemma 4.5

By (22) and (79), we can know that

\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V_\gamma^{-1} z_k - \frac{1}{np} \text{trace}(V_\gamma^{-1} ZZ^\top) \right|
+ \frac{1}{(1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}))^2} \left| \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right|
= \max_{1 \leq k \leq p} \left| \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right|
+ \frac{1}{(1 + \frac{2}{p} \text{trace}(V_\gamma^{-1}))^2} \left| \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace}(V_\gamma^{-1}) \right| + O_P \left( \frac{1}{n} \right),
\]

(88)
and similarly by (23) and (80), there holds that

\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V^{-2} z_k - \frac{1}{np} \text{trace} \left( V^{-2} ZZ^\top \right) \right|
\]

\[
+ \frac{\text{trace}(V^{-2})}{\left(1 + \frac{\gamma}{p} \text{trace}(V^{-1})\right)^3} \frac{2\gamma}{p} \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V^{-1}) \right)
\]

\[
- \frac{1}{\left(1 + \frac{\gamma}{p} \text{trace}(V^{-1})\right)^2} \left( \frac{1}{n} \eta_{kk,k}^{(2)} - \frac{1}{n} \text{trace}(V^{-2}) \right)
\]

\[
= \max_{1 \leq k \leq p} \left| \frac{\eta_{kk,k}^{(2)}}{(1 + \frac{\gamma}{p} \eta_{kk,k}^{(1)})^2} - \frac{\text{trace}(V^{-2})}{\left(1 + \frac{\gamma}{p} \text{trace}(V^{-1})\right)^2} \right|
\]

\[
+ \frac{\text{trace}(V^{-2})}{\left(1 + \frac{\gamma}{p} \text{trace}(V^{-1})\right)^3} \frac{2\gamma}{p} \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V^{-1}) \right)
\]

\[
- \frac{1}{\left(1 + \frac{\gamma}{p} \text{trace}(V^{-1})\right)^2} \left( \frac{1}{n} \eta_{kk,k}^{(2)} - \frac{1}{n} \text{trace}(V^{-2}) \right) + O_p \left( \frac{1}{n} \right).
\]

Define

\[
z_\gamma(x) = \frac{x}{1 + \frac{\gamma}{p} x}, \quad w_\gamma(x, y) = \frac{x}{1 + \frac{\gamma}{p} y^2}.
\]

By the Taylor series expansion, as \( \frac{1}{n} \eta_{kk,k}^{(1)} \to \frac{1}{n} \text{trace}(V^{-1}) \)

\[
z_\gamma \left( \frac{1}{n} \eta_{kk,k}^{(1)} \right) = z_\gamma \left( \frac{1}{n} \text{trace}(V^{-1}) \right) + z_\gamma' \left( \frac{1}{n} \text{trace}(V^{-1}) \right) \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V^{-1}) \right)
\]

\[
+ R_1 \left( \frac{1}{n} \eta_{kk,k}^{(1)}, \frac{1}{n} \text{trace}(V^{-1}) \right).
\]

Here \( R_1 \) is the remainder term

\[
R_1 = \frac{1}{2} z_\gamma''(c_k) \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V^{-1}) \right)^2
\]

where \( c_k \) is some constant between \( \frac{1}{n} \eta_{kk,k}^{(1)} \) and \( \frac{1}{n} \text{trace}(V^{-1}) \). Then

\[
\frac{\frac{1}{n} \eta_{kk,k}^{(1)}}{1 + \frac{\gamma}{p} \eta_{kk,k}^{(1)}} - \frac{\frac{1}{n} \text{trace}(V^{-1})}{1 + \frac{\gamma}{p} \text{trace}(V^{-1})}
\]

\[
= z_\gamma \left( \frac{1}{n} \eta_{kk,k}^{(1)} \right) - z_\gamma \left( \frac{1}{n} \text{trace}(V^{-1}) \right)
\]

\[
= z_\gamma' \left( \frac{1}{n} \text{trace}(V^{-1}) \right) \left( \frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}(V^{-1}) \right) + R_1 \left( \frac{1}{n} \eta_{kk,k}^{(1)}, \frac{1}{n} \text{trace}(V^{-1}) \right),
\]

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where 
\[ z'_\gamma(x) = \frac{1}{(1 + \frac{\gamma n}{p} x)} \] \[ z''\gamma(x) = \frac{\gamma n}{p} \frac{1}{(1 + \frac{\gamma n}{p} x)^3}. \]

By (88), this implies that
\[
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V^{-1}_\gamma z_k - \frac{1}{np} \text{trace} \left( V^{-1}_\gamma ZZ^\top \right) \right|
\begin{align*}
&+ \frac{1}{\left(1 + \frac{2}{p} \text{trace}(V^{-1})\right)^2} \left( \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace}(V^{-1}) \right) \\
&= \max_{1 \leq k \leq p} \left| R_1 \left( \frac{1}{n} \eta^{(1)}_{kk,k}, \frac{1}{n} \text{trace} \left( V^{-1}_\gamma \right) \right) \right| + O_P \left( \frac{1}{n} \right),
\end{align*}
\]

and by Lemma 4.3,
\[
\begin{align*}
\max_{1 \leq k \leq p} R_1 \left( \frac{1}{n} \eta^{(1)}_{kk,k}, \frac{1}{n} \text{trace} \left( V^{-1}_\gamma \right) \right) \\
&= \max_{1 \leq k \leq p} \left| \frac{1}{2} z'' \gamma (c_k) \left( \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace} \left( V^{-1}_\gamma \right) \right)^2 \right| \\
&\leq \frac{1}{2} \max_{1 \leq k \leq p} \left| \frac{\gamma n}{p} \frac{1}{(1 + \frac{\gamma n}{p} c_k)^3} \right| \max_{1 \leq k \leq p} \left| \left( \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace} \left( V^{-1}_\gamma \right) \right)^2 \right| \\
&\leq \frac{\gamma n}{2p} \max_{1 \leq k \leq p} \left| \left( \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace} \left( V^{-1}_\gamma \right) \right)^2 \right| \\
&= O_P \left( \frac{\log n}{n} \right).
\end{align*}
\]

Combine the above two inequalities, we can get (24).

Similarly, by the Taylor series expansion, as \( \frac{1}{n} \eta^{(2)}_{kk,k} \rightarrow \frac{1}{n} \text{trace} \left( V^{-2}_\gamma \right), \) we can have
\[
\begin{align*}
\max_{1 \leq k \leq p} \left| \frac{1}{n} z_k^\top V^{-2}_\gamma z_k - \frac{1}{np} \text{trace} \left( V^{-2}_\gamma ZZ^\top \right) \right|
&+ \frac{\text{trace}(V^{-2})}{\left(1 + \frac{2}{p} \text{trace}(V^{-1})\right)^3} \frac{2\gamma}{p} \left( \frac{1}{n} \eta^{(1)}_{kk,k} - \frac{1}{n} \text{trace}(V^{-1}) \right) \\
&\quad - \frac{1}{\left(1 + \frac{2}{p} \text{trace}(V^{-1})\right)^2} \left( \frac{1}{n} \eta^{(2)}_{kk,k} - \frac{1}{n} \text{trace}(V^{-2}) \right) \\
&= \max_{1 \leq k \leq p} \tilde{R}_1 \left( \frac{1}{n} \eta^{(2)}_{kk,k}, \frac{1}{n} \text{trace} \left( V^{-2}_\gamma \right), \frac{1}{n} \eta^{(1)}_{kk,k}, \frac{1}{n} \text{trace} \left( V^{-1}_\gamma \right) \right) + O_P \left( \frac{1}{n} \right).
\end{align*}
\]

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Since
\[
\frac{\partial^2 w_\gamma}{\partial x^2}(x, y) = 0,
\]
\[
\left|\frac{\partial^2 w_\gamma}{\partial x \partial y}(x, y)\right| = \left|-2\frac{\gamma n}{p} \frac{1}{\left(1 + \gamma \frac{n}{p} y\right)^3}\right| \leq \frac{2 \gamma n}{p} \quad \text{for } y \geq 0,
\]
\[
\left|\frac{\partial^2 w_\gamma}{\partial y^2}(x, y)\right| = 6 \left(\frac{\gamma n}{p}\right)^2 \frac{x}{\left(1 + \gamma \frac{n}{p} y\right)^3} \leq 6 \left(\frac{\gamma n}{p}\right)^2 x \quad \text{for } x, y \geq 0,
\]
by Lemma 4.3 we can have
\[
\max_{1 \leq k \leq q} \left|\bar{R}_1\left(\frac{1}{n} \eta_{kk,k}^{(2)}, \frac{1}{n} \eta_{kk,k}^{(1)}, \frac{1}{n} \text{trace}\left(V_\gamma^{-2}\right)\right)\right| \leq \max_{1 \leq k \leq q} \left|\frac{\partial^2 w_\gamma}{\partial x \partial y}(c_{k1}, c_{k2}) \left(\frac{1}{n} \eta_{kk,k}^{(2)} - \frac{1}{n} \text{trace}\left(V_\gamma^{-2}\right)\right) \left(\frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}\left(V_\gamma^{-1}\right)\right)\right|
\]
\[
+ \max_{1 \leq k \leq q} \left|\frac{\partial^2 w_\gamma}{\partial y^2}(c_{k1}, c_{k2}) \left(\frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}\left(V_\gamma^{-1}\right)\right)^2\right|
\]
\[
\leq 2\frac{\gamma n}{p} \max_{1 \leq k \leq q} \left|\frac{1}{n} \eta_{kk,k}^{(2)} - \frac{1}{n} \text{trace}\left(V_\gamma^{-2}\right)\right| \max_{1 \leq k \leq q} \left|\frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}\left(V_\gamma^{-1}\right)\right|
\]
\[
+ 3 \left(\frac{\gamma n}{p}\right)^2 c_{k1} \max_{1 \leq k \leq q} \left|\frac{1}{n} \eta_{kk,k}^{(1)} - \frac{1}{n} \text{trace}\left(V_\gamma^{-1}\right)\right|^2
\]
\[
= O_p\left(\frac{\log n}{n}\right),
\]
where \(c_{k1}\) is some constant between \(\frac{1}{n} \eta_{kk,k}^{(2)}\) and \(\frac{1}{n} \text{trace}\left(V_\gamma^{-2}\right)\) and \(c_{k2}\) is some constant between \(\frac{1}{n} \eta_{kk,k}^{(1)}\) and \(\frac{1}{n} \text{trace}\left(V_\gamma^{-1}\right)\).

### B.5 Proof of Lemma 4.7

Note that
\[
(z_k^T V_\gamma^{-1} z_j)^2 = (1 + \frac{\gamma}{p} \rho_k)^{-2} (z_k^T V_\gamma^{-1} z_j)^2 \leq \left(z_k^T V_\gamma^{-1} z_j\right)^2
\]
and
\[
(z_k^T V_\gamma^{-2} z_j)^2 = \left(1 + \frac{\gamma}{p} \rho_k\right)^{-1} z_k^T V_\gamma^{-2} z_j + \left(-\frac{\gamma}{p} \phi_k (1 + \frac{\gamma}{p} \rho_k)^{-2} z_k^T V_\gamma^{-1} z_j\right)^2
\]
\[
\leq \frac{1}{2} (1 + \frac{\gamma}{p} \rho_k)^{-2} \left(z_k^T V_\gamma^{-2} z_j\right)^2 + \frac{1}{2} \left(\frac{\gamma}{p} \phi_k\right)^2 (1 + \frac{\gamma}{p} \rho_k)^{-4} \left(z_k^T V_\gamma^{-1} z_j\right)^2
\]
\[
\leq 2 \left(\left(z_k^T V_\gamma^{-2} z_j\right)^2 + \left(z_k^T V_\gamma^{-1} z_j\right)^2\right)
\]
where the last inequality is due to $0 < \phi_k \leq \rho_k$.

Denote $Z_{-k} = [z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_p]$. Note that the components of $z_k$ are independent mean-zero sub-Gaussian random variables, conditional on $Z_{-k}$, by Proposition A.4, we have, for any $k \neq j$ and $t \geq 0$,

$$\mathbb{P}\left\{ |z_k^T V_{\gamma, -k}^{-1} z_j| \geq t \mid Z_{-k} \right\} \leq e \exp\left\{ -c \frac{t^2}{K^2 \| V_{\gamma, -k}^{-1} z_j \|^2} \right\},$$

where $c$ and $K$ are some positive constants. By letting $t = K \sqrt{\frac{3 \log p}{c} \| V_{\gamma, -k}^{-1} z_j \|^2}$, it follows

$$\mathbb{P}\left\{ |z_k^T V_{\gamma, -k}^{-1} z_j| \geq C \sqrt{\log p} \| V_{\gamma, -k}^{-1} z_j \| \mid Z_{-k} \right\} \leq \frac{e}{p^3},$$

where $C$ is some positive constant. It further implies the unconditional probability inequality

$$\mathbb{P}\left\{ |z_k^T V_{\gamma, -k}^{-1} z_j| \geq C \sqrt{\log p} \| V_{\gamma, -k}^{-1} z_j \| \right\} \leq \frac{e}{p^3}. \quad (91)$$

By the fact $\| V_{\gamma, -k}^{-1} \| \leq 1$, we have $\| V_{\gamma, -k}^{-1} z_j \| \leq \| z_j \|$. Note that $z_{1j}^2 - 1, z_{2j}^2 - 1, \ldots, z_{nj}^2 - 1$ are independent centered sub-exponential random variables, by the Proposition A.5, we have

$$\mathbb{P}\left\{ \left| \sum_{i=1}^n (z_{ij}^2 - 1) \right| \geq t \right\} \leq 2 \exp\left\{ -c \min \left( \frac{t^2}{K^2 n}, \frac{t}{K} \right) \right\},$$

where $c$ and $K$ are some positive constants. Take $t = K \sqrt{3n \log p}$, then we get

$$\mathbb{P}\left\{ \| z_j \|^2 - n \geq C \sqrt{n \log p} \right\} \leq \frac{2}{p^3}, \quad (92)$$

for some constant $C$. Combining the above (91) and (92) together, with probability at least $1 - (2 + e)/p^3$, there holds

$$(z_k^T V_{\gamma}^{-1} z_j)^2 \leq (z_k^T V_{\gamma, -k}^{-1} z_j)^2 \leq C(n + C \sqrt{n \log p}) \log p.$$

By a similar argument with the fact that $\| V_{\gamma, -k}^{-2} \| \leq 1$, we can have with probability at least $1 - (2 + e)/p^3$

$$(z_k^T V_{\gamma}^{-2} z_j)^2 \leq 2 \left( (z_k^T V_{\gamma, -k}^{-2} z_j)^2 + (z_k^T V_{\gamma, -k}^{-1} z_j)^2 \right) \leq 4C(n + C \sqrt{n \log p}) \log p,$$

for some constant $C$.

Above inequalities imply that with probability at least $1 - (2 + e)/p$, there hold

$$\max_{k \neq j} |z_k^T V_{\gamma}^{-1} z_j|^2 \leq C(n + C \sqrt{n \log p}) \log p, \quad \text{and}$$

$$\max_{k \neq j} |z_k^T V_{\gamma}^{-2} z_j|^2 \leq 4C(n + C \sqrt{n \log p}) \log p.$$
Then, there follows that
\[ \max_{k \neq j} |z_k^\top V_{\gamma}^{-1} z_j|^2 = O_P(n \log p) \quad \text{and} \quad \max_{k \neq j} |z_k^\top V_{\gamma}^{-2} z_j|^2 = O_P(n \log p). \]

Next, define
\[ \tilde{\theta}_1(\gamma, \tau) = \kappa_{1,1}(\gamma, \tau) - 2\frac{\kappa_{1,2}(\gamma, \tau)}{h_1(\gamma, \tau)} + \frac{\kappa_{2,2}(\gamma, \tau)}{h_1^2(\gamma, \tau)}, \]
where
\[ \kappa_{m,i}(\gamma, \tau) = \sum_{q_1=1}^{l} \sum_{q_2=1}^{m} \bar{a}_{q_1}^{(l)}(\gamma, \tau) \bar{a}_{q_2}^{(m)}(\gamma, \tau) h_{q_1+q_2}(\gamma, \tau), \]
and
\[ \bar{a}_{1}^{(1)}(\gamma, \tau) = \frac{1}{(1 + \tau \gamma h_1(\gamma, \tau))^2}, \quad \bar{a}_{1}^{(2)}(\gamma, \tau) = \frac{-2\tau \gamma h_2(\gamma, \tau)}{(1 + \tau \gamma h_1(\gamma, \tau))^3}, \quad \bar{a}_{2}^{(2)}(\gamma, \tau) = \frac{1}{(1 + \tau \gamma h_1(\gamma, \tau))^2}. \]
Recall that \( h_1(\gamma, \tau) \) and \( h_2(\gamma, \tau) \) are defined in (6). Now, by the definition of \( \eta_{ij}^{(l)} \) in (16), we can rewrite \( (z_i^\top B_{\gamma} z_j)^2 \) as
\[
(z_i^\top B_{\gamma} z_j)^2 = \left( \frac{1}{n} z_i^\top V_{\gamma}^{-1} z_j - \frac{1}{n} \frac{1}{\text{trace}(V_{\gamma}^{-1})} \right)^2 \]
\[ = \left( \frac{1}{n} \eta_{ij}^{(1)} \right)^2 - 2 \frac{1}{n} \frac{1}{\text{trace}(V_{\gamma}^{-1})} \eta_{ij}^{(1)} \eta_{ij}^{(2)} + \left( \frac{1}{n} \frac{1}{\text{trace}(V_{\gamma}^{-1})} \right)^2. \tag{93} \]
The following results are implied by Jiang et al. (2016) in the supplementary material.

**Proposition B.1** (Jiang et al. (2016)). For any \( i \neq j \) and \( i, j \geq 1 \), we have
\[ \eta_{ij}^{(1)} = \bar{a}_{1,ij}^{(1)} \eta_{ii,ij}, \]
\[ \eta_{ij}^{(2)} = \bar{a}_{1,ij}^{(2)} \eta_{ii,ij} + \bar{a}_{2,ij}^{(2)} \eta_{jj,ij}, \]
with
\[ \bar{a}_{1,ij}^{(1)} = \frac{1}{\left(1 + \frac{\gamma}{p} \eta_{ii,ij}\right) \left(1 + \frac{\gamma}{p} \eta_{jj,ij}\right)}, \]
\[ \bar{a}_{1,ij}^{(2)} = -\frac{\gamma/2}{p} \eta_{ii,ij} \left(1 + \frac{\gamma}{p} \eta_{ii,ij}\right)^2 + \frac{\gamma/2}{p} \eta_{jj,ij} \left(1 + \frac{\gamma}{p} \eta_{jj,ij}\right)^2, \]
\[ \bar{a}_{2,ij}^{(2)} = \bar{a}_{1,ij}^{(1)}. \]
And
\[ \max_{1 \leq i \neq j \leq p} \max_{1 \leq l \leq 2} \max_{1 \leq q_1 \leq l} \left| \bar{a}_{q_1,ij}^{(l)} - \bar{a}_{q_1}^{(l)}(\gamma, \tau) \right| = O_P \left( \sqrt{\frac{\log p}{n}} \right). \]
Furthermore,
\[ \max_{1 \leq i \neq j \leq p} \max_{1 \leq l,m \leq 2} \max_{1 \leq q_1,q_2 \leq l} \left| \bar{a}_{q_1,ij}^{(l)} \bar{a}_{q_2,ij}^{(m)} - \bar{a}_{q_1}^{(l)}(\gamma, \tau) \bar{a}_{q_2}^{(m)}(\gamma, \tau) \right| = O_P \left( \sqrt{\frac{\log p}{n}} \right). \tag{94} \]
Proposition B.2 (Jiang et al. (2016)). For any $l \geq 1$ and $1 \leq i \neq j$,
\[
\frac{1}{n} \left| \text{trace} \left( V_{\gamma}^{-l} \right) - \text{trace} \left( V_{\gamma,-ij}^{-l} \right) \right| \leq \frac{1}{n} 2^{l+1}.
\] (95)

Proposition B.3 (Jiang et al. (2016)). For any $1 \leq q_1, q_2 \leq 2$, define
\[
d^{(q_1,q_2)}_{ij} := \frac{1}{n} \eta^{(q_1)}_{ij} \eta^{(q_2)}_{ij} - \frac{1}{n} \text{trace} \left( V_{\gamma,-ij}^{-(q_1+q_2)} \right).
\]
Then the following statements are true.

1) For some constant $K_1 > 0$,
\[
\max_{1 \leq i \neq j \leq p} \mathbb{E} \left[ \left( d^{(q_1,q_2)}_{ij} \right)^2 \right] \leq K_1.
\]

2) For any $i \neq j \neq i'$ (either $j = j'$ or not) and some constant $K_2 > 0$,
\[
\max_{i \neq j \neq i', j' \neq i'} \left| \mathbb{E} \left[ d^{(q_1,q_2)}_{ij} d^{(q_1,q_2)}_{i'j'} \right] \right| \leq \frac{K_2}{n}
\]

By (93), let’s first show that for $1 \leq l, m \leq 2$,
\[
\frac{1}{p(p-1)} \sum_{i \neq j} \frac{1}{n} \eta^{(l)}_{ij} \eta^{(m)}_{ij} \xrightarrow{P} \kappa_{m,l}(\gamma, \tau), \quad \text{and} \quad \sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \eta^{(l)}_{ij} \eta^{(m)}_{ij} \xrightarrow{P} \| \beta \|^4 \kappa_{m,l}(\gamma, \tau).
\] (96)

Since $\eta^{(q_1)}_{ij} \eta^{(q_2)}_{ij} > 0$ for any $q_1, q_2 = 1, 2, \ldots$, by Proposition B.1 we can have
\[
\sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \eta^{(l)}_{ij} \eta^{(m)}_{ij} = \sum_{q_1=1}^l \sum_{q_2=1}^m \left( \sum_{i \neq j} \bar{a}^{(l)}_{q_1,ij} \bar{a}^{(m)}_{q_2,ij} \beta_i^2 \beta_j^2 \frac{1}{n} \eta^{(q_1)}_{ij} \eta^{(q_2)}_{ij} \right) = \sum_{q_1=1}^l \sum_{q_2=1}^m \bar{a}^{(l)}_{q_1,ij}(\gamma, \tau) \bar{a}^{(m)}_{q_2,ij}(\gamma, \tau) \left( \sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \eta^{(q_1)}_{ij} \eta^{(q_2)}_{ij} \right) + o_P(1),
\] (97)

and
\[
\frac{1}{p(p-1)} \sum_{i \neq j} \frac{1}{n} \eta^{(l)}_{ij} \eta^{(m)}_{ij} \sum_{q_2=1}^m \bar{a}^{(l)}_{q_1,ij}(\gamma, \tau) \bar{a}^{(m)}_{q_2,ij}(\gamma, \tau) \frac{1}{p(p-1)} \sum_{i \neq j} \frac{1}{n} \eta^{(q_1)}_{ij} \eta^{(q_2)}_{ij} + o_P(1).
\] (98)

We know that
\[
\mathbb{E} \left[ \sum_{i \neq j} \beta_i^2 \beta_j^2 d^{(q_1,q_2)}_{ij} \right] = \mathbb{E} \left[ \sum_{i \neq j} \beta_i^2 \beta_j^2 \left( \mathbb{E} \left[ \frac{1}{n} \eta^{(q_1)}_{ij} \eta^{(q_2)}_{ij} \right] - \frac{1}{n} \text{trace} \left( V_{\gamma,-ij}^{-(q_1+q_2)} \right) \right) \right] = 0,
\]

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and by Proposition B.3, we have that
\[
E \left( \sum_{i \neq j} \beta_i^2 \beta_j^2 d_{ij}^{(q_1, q_2)} \right)^2 = \sum_{i \neq j} \beta_i^4 \beta_j^4 E \left[ (d_{ij}^{(q_1, q_2)})^2 \right] + 2 \sum_{i \neq i' \neq j} \beta_i^2 \beta_j^2 \beta_i' \beta_j' E \left[ (d_{ij}^{(q_1, q_2)} d_{i'j}^{(q_1, q_2)}) \right] + \sum_{i \neq j \neq i' \neq j'} \beta_i^2 \beta_j^2 \beta_i' \beta_j' E \left[ (d_{ij}^{(q_1, q_2)} d_{i'j'}^{(q_1, q_2)}) \right] \leq K_1 \| \beta \|^4 + 2 \frac{K_2}{n} \| \beta \|_2 \| \beta \|^4 + \frac{K_2}{n} \| \beta \|^8 = o_P(1),
\]

This implies that
\[
\sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \eta_{ij}^{(q_1)} \eta_{ij}^{(q_2)} = \sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \text{trace} \left( V^{(q_1 + q_2)} \right) + o_P(1),
\]
where by Proposition B.2 and Lemma 4.2,
\[
\left| \frac{1}{n} \text{trace} \left( V^{(q_1 + q_2)} \right) - h_{q_1 + q_2}(\gamma, \tau) \right| = o_P(1),
\]
and
\[
\sum_{i \neq j} \beta_i^2 \beta_j^2 h_{q_1 + q_2}(\gamma, \tau) = \| \beta \|^4 h_{q_1 + q_2}(\gamma, \tau) + o_P(1) \quad \text{since } \sum_{i=1}^p \beta_i^4 = o_P(1).
\]
Thus
\[
\sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \eta_{ij}^{(q_1)} \eta_{ij}^{(q_2)} = \| \beta \|^4 h_{q_1 + q_2}(\gamma, \tau) + o_P(1).
\]

Thus by (97) and (100), we can have that
\[
\sum_{i \neq j} \beta_i^2 \beta_j^2 \frac{1}{n} \eta_{ij}^{(q_1)} \eta_{ij}^{(q_2)} = \| \beta \|^4 \kappa_m(\gamma, \tau) + o_P(1).
\]

Similarly,
\[
E \left[ \frac{1}{p(p - 1)} \sum_{i \neq j} \frac{1}{n} \eta_{ij}^{(q_1)} \eta_{ij}^{(q_2)} \right] = 0,
\]
and
\[
E \left( \frac{1}{p(p - 1)} \sum_{i \neq j} \frac{1}{n} \eta_{ij}^{(q_1)} \eta_{ij}^{(q_2)} \right)^2 = \frac{1}{p^2(p - 1)^2} \left( \sum_{i \neq j} E \left[ (d_{ij}^{(q_1, q_2)})^2 \right] + 2 \sum_{i \neq i' \neq j} E \left[ (d_{ij}^{(q_1, q_2)} d_{i'j}^{(q_1, q_2)}) \right] \right) + \sum_{i \neq j \neq i' \neq j'} E \left[ (d_{ij}^{(q_1, q_2)} d_{i'j'}^{(q_1, q_2)}) \right] \leq \frac{1}{p^2(p - 1)^2} \left( p(p - 1)K_1 + 2p(p - 1)(p - 2) \frac{K_2}{n} + p(p - 1)(p - 2)(p - 3) \frac{K_2}{n} \right) = o_P(1).
\]
Then by Proposition B.2 and Lemma 4.2, we can have that
\[
\frac{1}{p(p-1)} \sum_{i \neq j} \frac{1}{n} \eta_{ij;ij}^{(q_1)} h_{ij}^{(q_2)} = h_{q_1+q_2}(\gamma, \tau),
\]
which implies
\[
\frac{1}{p(p-1)} \sum_{i \neq j} \frac{1}{n} \eta_{ij;ij}^{(q_1)} h_{ij}^{(q_2)} = \kappa_{m,t}(\gamma, \tau) + o_P(1),
\]
by (98). Now we have proved (96), then by (93) and Lemma 4.2, there holds that
\[
\begin{cases}
\left( \frac{n}{p(p-1)} \sum_{i \neq j} \left( z_i^\top B_\gamma z_j \right)^2 \right)^{1/2} = \bar{\theta}_1(\gamma, \tau) + o_P(1), \\
n \sum_{i \neq j} \beta_i^2 \beta_j^2 \left( z_i^\top B_\gamma z_j \right)^2 = \|\beta\|^4 \bar{\theta}_1(\gamma, \tau) + o_P(1).
\end{cases}
\]

### B.6 Proof of Lemma 4.8

Let \( \tilde{z}_i^\top \) be the \( i \)-th row of \( Z \). By Sherman-Morrison-Woodbury formula, we have
\[
V_\gamma^{-1} = \left( I_n + \frac{\gamma}{p} ZZ^\top \right)^{-1} = I_n - \frac{\gamma}{p} Z \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} Z^\top,
\]
and
\[
V_\gamma^{-2} = (V_\gamma^{-1})^2 = I_n - 2\frac{\gamma}{p} Z \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} Z^\top + \left( \frac{\gamma}{p} \right)^2 \left( Z \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} Z^\top \right)^2.
\]
Combining (101) and (102) gives
\[
(V_\gamma^{-1})_{ii} = 1 - \frac{\gamma}{p} \tilde{z}_i^\top \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} \tilde{z}_i,
\]
and
\[
(V_\gamma^{-2})_{ii} = 1 - 2\frac{\gamma}{p} \tilde{z}_i^\top \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} \tilde{z}_i + \left( \frac{\gamma}{p} \right)^2 \tilde{z}_i^\top \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} Z^\top Z \left( I_p + \frac{\gamma}{p} Z^\top Z \right)^{-1} \tilde{z}_i.
\]
where \( \tilde{z}_i \) is the \( i \)-th column of \( Z^\top \). Define
\[
\tilde{V}_\gamma = I_p + \frac{\gamma}{p} Z^\top Z,
\]
(103)
then we can rewrite \((V^{-1})_{ii}\) and \((V^{-2})_{ii}\) as
\[
(V^{-1})_{ii} = 1 - \frac{\gamma}{p} \bar{z}_i \bar{V}^{-1} \bar{z}_i^\top,
\]
\[
(V^{-2})_{ii} = 1 - \frac{\gamma}{p} \bar{z}_i \bar{V}^{-1} \bar{z}_i^\top - \frac{\gamma}{p} \bar{z}_i \bar{V}^{-2} \bar{z}_i^\top.
\]
Furthermore, by a similar argument, it can be shown that for \(l = 1, 2, \ldots\)
\[
(V^{-l})_{ii} = I_n - \frac{\gamma}{p} \sum_{q=1}^{l} Z \bar{V}^{-q} Z^\top,
\]
(104)

with
\[
(V^{-l})_{ii} = 1 - \frac{\gamma}{p} \sum_{q=1}^{l} \bar{z}_i^\top \bar{V}^{-q} \bar{z}_i.
\]
(105)

Similar to (20) in Lemma 4.3, combining the leave-one-out technique and Hanson-Wright inequality, taking the uniform bound gives
\[
\max_{i \in [n]} \left| \frac{1}{p} \bar{z}_i^\top \bar{V}^{-q} \bar{z}_i - \frac{1}{np} \text{trace} (\bar{V}^{-q} Z^\top Z) \right| = O_p \left( \sqrt{\frac{\log n}{n}} \right).
\]
(106)

Together with (104) and (105) yields (27).

Note that we have
\[
(B_{\gamma})_{ii} = \frac{1}{n} (V^{-1})_{ii} - \frac{1}{n} (V^{-2})_{ii} - \frac{1}{n} \text{trace} (V^{-1}),
\]
\[
(B_{\gamma})_{ii} = \left( \frac{1}{n} \right)^2 (V^{-1})_{ii}^2 - \frac{2 \left( \frac{1}{n} \right)^2 (V^{-1})_{ii} (V^{-2})_{ii}}{\left( \frac{1}{n} \text{trace} (V^{-1}) \right)^2} + \frac{(\frac{1}{n})^2 (V^{-2})_{ii}^2}{\left( \frac{1}{n} \text{trace} (V^{-1}) \right)^2}.
\]

By (27), we have
\[
\max_{i \in [n]} \left| (B_{\gamma})_{ii} - \frac{1}{n} \text{trace} (B_{\gamma}) \right| \leq \max_{i \in [n]} \left| \frac{1}{n} (V^{-1})_{ii} - \frac{1}{n^2} \text{trace} (V^{-1}) \right| + \frac{\max_{i \in [n]} \left| \frac{1}{n} (V^{-2})_{ii} - \frac{1}{n} \text{trace} (V^{-2}) \right|}{\frac{1}{n} \text{trace} (V^{-1})}
\]
\[
= O_p \left( \frac{1}{n} \sqrt{\frac{\log n}{n}} \right),
\]

and consequently
\[
\max_{i \in [n]} \left| (B_{\gamma})_{ii}^2 - \left( \frac{1}{n} \text{trace} (B_{\gamma}) \right)^2 \right| \leq \max_{i \in [n]} \left| (B_{\gamma})_{ii} - \frac{1}{n} \text{trace} (B_{\gamma}) \right|^2
\]
\[
+ 2 \frac{1}{n} \text{trace} (B_{\gamma}) \max_{i \in [n]} \left| (B_{\gamma})_{ii} - \frac{1}{n} \text{trace} (B_{\gamma}) \right|
\]
\[
= O_p \left( \frac{1}{n^2} \sqrt{\frac{\log n}{n}} \right),
\]
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which yield (28) and (29).

From (104), we have
\[
\left( V_{\gamma}^{-l} \right)_{ij} = -\frac{\gamma}{p} \sum_{k=1}^{l} \tilde{z}_i^T \tilde{V}_{\gamma}^{-k} \tilde{z}_j.
\]
As with (25) in Lemma 4.7, we can have
\[
\max_{i \neq j} \left| \tilde{z}_i^T \tilde{V}_{\gamma}^{-k} \tilde{z}_j \right| = O_P(\sqrt{n \log n}).
\]
Then (30) can be easily obtained by the fact
\[
(B_{\gamma 0})_{ij} = \frac{1}{n} (V_{\gamma 0}^{-1})_{ij} - \frac{1}{n^2} \text{trace}(V_{\gamma 0}^{-1}).
\]

**B.7 Proof of Lemma 4.9**

To prove (31), we can know that
\[
(B_{\gamma})_{ij} = \frac{1}{n^2} (V_{\gamma 1}^{-1})_{ij} - \frac{2}{n^2} (V_{\gamma 1}^{-1})_{ij} (V_{\gamma 2}^{-1})_{ij} + \frac{1}{n^2} \text{trace}(V_{\gamma 2}^{-1})^2
\]
\[
= \frac{\gamma^2}{n^2} \frac{1}{p^2} \left( \eta_{ij}^{(1)} \right)^2 - \frac{\gamma^2}{n^2} \frac{1}{p^2} \left( \eta_{ij}^{(2)} \right)^2 + \frac{1}{n^2} \frac{1}{p^2} \text{trace}(V_{\gamma 1}^{-1})^2 \frac{1}{n^2} \text{trace}(V_{\gamma 2}^{-1})^2,
\]
where
\[
\eta_{ij}^{(k)} := \tilde{z}_i^T \tilde{V}_{\gamma}^{-k} \tilde{z}_j.
\]

Define
\[
\bar{\theta}_2 = \gamma^2 \tau \left( \tilde{\kappa}_{1,1}(\gamma, \tau) - 2 \frac{\tilde{\kappa}_{1,1}(\gamma, \tau) + \tilde{\kappa}_{1,2}(\gamma, \tau)}{h_1(\gamma, \tau)} + \frac{\tilde{\kappa}_{1,1}(\gamma, \tau) + 2 \tilde{\kappa}_{1,2}(\gamma, \tau) + \tilde{\kappa}_{2,2}(\gamma, \tau)}{(h_1(\gamma, \tau))^2} \right).
\]

where
\[
\tilde{\kappa}_{m,l}(\gamma, \tau) = \sum_{q_1=1}^{l} \sum_{q_2=1}^{m} \tilde{a}_q^{(l)}(\gamma, \tau) \tilde{a}_q^{(m)}(\gamma, \tau) \tilde{h}_{q_1+q_2}(\gamma, \tau),
\]
and
\[
\tilde{a}_1^{(1)}(\gamma, \tau) = \frac{1}{(1 + \gamma \tilde{h}_1(\gamma, \tau))^2}, \quad \tilde{a}_1^{(2)}(\gamma, \tau) = \frac{-2 \gamma \tilde{h}_2(\gamma, \tau)}{(1 + \gamma \tilde{h}_1(\gamma, \tau))^3}, \quad \tilde{a}_2^{(2)}(\gamma, \tau) = \frac{1}{(1 + \gamma \tilde{h}_1(\gamma, \tau))^2}.
\]

Similar to the definition of \(h_l(\gamma, \tau)\), \(\tilde{h}_l(\gamma, \tau)\) is the limit of \(\frac{1}{l} \text{trace}(V_{\gamma 1}^{-1})\).

Then similar to Proposition B.1, using the leave-two-out technique, there holds that
\[
\eta_{ij}^{(1)} = \tilde{a}_1^{(1)} \eta_{ij}^{(1)}, \quad \eta_{ij}^{(2)} = \tilde{a}_1^{(2)} \eta_{ij}^{(1)} + \tilde{a}_2^{(2)} \eta_{ij}^{(2)},
\]
with
\[
\tilde{a}^{(1)}_{1;ij} = \frac{1}{\left(1 + \frac{\gamma}{p} \bar{\eta}^{(1)}_{ii;ij}\right) \left(1 + \frac{\gamma}{p} \bar{\eta}^{(1)}_{jj;ij}\right)},
\]
\[
\tilde{a}^{(2)}_{1;ij} = \frac{-\gamma}{p} \bar{\eta}^{(2)}_{ii;ij} + \frac{-\gamma}{p} \bar{\eta}^{(2)}_{jj;ij},
\]
\[
\tilde{a}^{(2)}_{2;ij} = \tilde{a}^{(1)}_{1;ij}.
\]

And
\[
\max_{1 \leq i \neq j \leq l, 1 \leq k \leq q, 1 \leq l \leq 2} |\tilde{a}_{1;ij}^{(l)} - \tilde{a}_{2;ij}^{(l)}(\gamma, \tau)| = O_P \left(\sqrt{\frac{\log p}{n}}\right).
\]

Furthermore,
\[
\max_{1 \leq i \neq j \leq p, 1 \leq l, m \leq 2} \max_{1 \leq k \leq q, 1 \leq l \leq 2} |\tilde{a}_{1;ij}^{(l)} - \tilde{a}_{2;ij}^{(m)}(\gamma, \tau)| = O_P \left(\sqrt{\frac{\log p}{n}}\right).
\]

Similar to (100) and (99), we can have that
\[
\frac{1}{n(n-1)} \sum_{i \neq j} \frac{1}{p} \tilde{\eta}^{(q_1)}_{ii;ij} \tilde{\eta}^{(q_2)}_{jj;ij} = \frac{1}{p} \text{trace} \left(\tilde{V}_{\gamma}^{-(q_1+q_2)}\right) + o_P(1),
\]
and
\[
\frac{1}{n^2} \sum_{i \neq j} \tilde{\eta}^{(q_1)}_{ii;ij} \tilde{\eta}^{(q_2)}_{jj;ij} = \sigma_0^4 \frac{1}{p} \text{trace} \left(\tilde{V}_{\gamma}^{-(q_1+q_2)}\right) + o_P(1),
\]

by (13) in Lemma 4.1. Then similar to the proof of Lemma 4.7, (31) can be obtained from (107).

**B.8 Proof of Lemma 4.10**

For any \( k = 1, \cdots, p \), denote \( Z_{-k} = [z_1, \cdots, z_{k-1}, z_{k+1}, \cdots] \), then
\[
V_{\gamma, -k} = V_{\gamma} - \frac{\gamma}{p} z_k z_k^\top = I_n + \frac{\gamma}{p} Z_{-k} Z_{-k}^\top.
\]

Similar to the proof of (27) in Lemma 4.8, we can define
\[
\tilde{V}_{\gamma, -k} = I_{p-1} + \frac{\gamma}{p} Z_{-k} Z_{-k}^\top,
\]
and \( \tilde{z}_{i, -k}^\top \) is the i-th row of \( Z_{-k} \). Then we can have that
\[
\left(\tilde{V}_{\gamma, -k}\right)_{ii} = 1 - \frac{\gamma}{p} \sum_{q=1}^l \tilde{z}_{i, -k}^\top \tilde{V}_{\gamma, -k}^{-q} \tilde{z}_{i, -k}.
\]
Again, similar to (20) in Lemma 4.3, combining the leave-one-out technique and Hanson-Wright inequality (taking \( t = \sqrt{\frac{3\log p}{c} \| \mathbf{V}_{\gamma_k}^{-q} \|_F} \)), taking the uniform bound gives

\[
\max_{k \in [p]} \max_{i \in [n]} \left| \frac{1}{p} \mathbf{z}_i^\top \mathbf{V}_{\gamma_k}^{-q} \mathbf{z}_i - \frac{1}{np} \text{trace} \left( \mathbf{V}_{\gamma_k}^{-q} \mathbf{Z}_k \mathbf{Z}_k^\top \right) \right| = O_P \left( \sqrt{\frac{\log n}{n}} \right), \quad (111)
\]

Then (104) and (105) implies

\[
\max_{k \in [p]} \max_{i \in [n]} \left| \mathbf{V}_{\gamma_k}^{-l} \right|_{ii} - \frac{1}{n} \text{trace} \left( \mathbf{V}_{\gamma_k}^{-l} \right) = O_P \left( \sqrt{\frac{\log n}{n}} \right), \quad l = 1, 2, 3, 4. \quad (112)
\]

By (17) in Lemma 4.3 and (112) we can get (32).

B.9 Proof of Lemma 4.11

In this section, we focus on the conditional variance \( \text{Var}[\Delta(\gamma_0)|\mathbf{Z}, \varepsilon] \). With \( \mathbf{y} \) defined in (33), we have

\[
\Delta(\gamma_0) = \mathbf{\xi}^\top \mathbf{A}_\beta \mathbf{Z}^\top M_1 \mathbf{B}_{\gamma_0} \mathbf{Z} \mathbf{A}_\beta \mathbf{\xi} + 2 \mathbf{\xi}^\top \mathbf{A}_\beta \mathbf{Z}^\top M_2 \mathbf{B}_{\gamma_0} \mathbf{A}_\varepsilon \mathbf{\xi} + \mathbf{\zeta}^\top M_3 \mathbf{B}_{\gamma_0} \mathbf{A}_\varepsilon \mathbf{\zeta},
\]

Then it is obvious that

\[
\text{Var}[\Delta(\gamma_0)|\mathbf{Z}, \varepsilon] = \mathbb{E} \left[ \Delta^2(\gamma_0)|\mathbf{Z}, \varepsilon \right] - \hat{\Delta}^2_*(\gamma_0), \quad (113)
\]

where by (35)

\[
\hat{\Delta}^2_*(\gamma_0) = \left( \sum_{k=1}^{p} \beta_k^2 \mathbf{z}_k^\top M_1 \mathbf{B}_{\gamma_0} \mathbf{z}_k + \text{trace} \left( \mathbf{A}_\varepsilon^2 M_3 \right) \right)^2 \quad (114)
\]

and we can have

\[
\mathbb{E} \left[ \Delta^2(\gamma_0)|\mathbf{Z}, \varepsilon \right] = \mathbb{E} \left[ M_1^2 | \mathbf{Z}, \varepsilon \right] + \mathbb{E} \left[ M_2^2 | \mathbf{Z}, \varepsilon \right] + 2 \mathbb{E} \left[ M_1 M_2 | \mathbf{Z}, \varepsilon \right] + 2 \mathbb{E} \left[ M_1 M_3 | \mathbf{Z}, \varepsilon \right] + 2 \mathbb{E} \left[ M_2 M_3 | \mathbf{Z}, \varepsilon \right]. \quad (115)
\]

Define

\[
\tilde{g}_{k,j,l,m} = \beta_k \beta_j \beta_l \beta_m \xi_k \xi_j \xi_l \xi_m (\mathbf{z}_k^\top \mathbf{B}_{\gamma_0} \mathbf{z}_j)(\mathbf{z}_l^\top \mathbf{B}_{\gamma_0} \mathbf{z}_m).
\]

Since \( \xi_i \)’s are i.i.d. Rademacher random variables, if \((k, j, l, m)\) has an odd multiplicity, we have \( \mathbb{E} [\xi_k \xi_j \xi_l \xi_m] = 0 \). This implies that

\[
\mathbb{E} \left[ M_1^2 | \mathbf{Z}, \varepsilon \right] = \mathbb{E} \left[ M_1^2 | \mathbf{Z} \right] = \sum_{k=1}^{p} \mathbb{E} \left[ \tilde{g}_{k,k,k,k} | \mathbf{Z} \right] + \sum_{j \neq i} \mathbb{E} \left[ \tilde{g}_{i,i,j,j} | \mathbf{Z} \right] + \sum_{j \neq i} \mathbb{E} \left[ \tilde{g}_{i,j,i,j} | \mathbf{Z} \right] + \sum_{j \neq i} \mathbb{E} \left[ \tilde{g}_{i,j,j,i} | \mathbf{Z} \right],
\]

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\[
\begin{align*}
\mathbb{E}[M_2^2|Z,\varepsilon] &= \mathbb{E}\left[4\xi^\top \Lambda_\beta Z^\top B_{\gamma_0}\Lambda_\varepsilon \zeta \zeta^\top \Lambda_\varepsilon B_{\gamma_0} Z \Lambda_\beta \xi | Z, \varepsilon \right] \\
&= 4 \sum_{k=1}^{p} \beta_k^2 z_k^\top B_{\gamma_0} \Lambda_\varepsilon^2 B_{\gamma_0} z_k,
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}[M_3^2|Z,\varepsilon] &= \mathbb{E}\left[\zeta^\top \Lambda_\varepsilon B_{\gamma_0} \Lambda_\varepsilon \zeta^\top \Lambda_\varepsilon B_{\gamma_0} \Lambda_\varepsilon | Z, \varepsilon \right] = \text{trace} \left((\Lambda_\varepsilon^2 B_{\gamma_0})^2\right),
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}[M_1 M_2|Z,\varepsilon] &= \mathbb{E}\left[2\xi^\top \Lambda_\beta Z^\top B_{\gamma_0} Z \Lambda_\beta \xi \zeta^\top \Lambda_\varepsilon Z^\top B_{\gamma_0} \Lambda_\varepsilon | Z, \varepsilon \right] = 0,
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}[M_1 M_3|Z,\varepsilon] &= \mathbb{E}[M_1|Z] \mathbb{E}[M_3|Z,\varepsilon] = \sum_{k=1}^{p} \beta_k^2 z_k^\top B_{\gamma_0} z_k \cdot \sigma_0^2 \text{trace} \left(\Lambda_\varepsilon^2 B_{\gamma_0}\right),
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}[M_2 M_3|Z,\varepsilon] &= \mathbb{E}\left[2\xi^\top \Lambda_\beta Z^\top B_{\gamma_0} \Lambda_\varepsilon \zeta \zeta^\top \Lambda_\varepsilon B_{\gamma_0} \Lambda_\varepsilon | Z, \varepsilon \right] = 0.
\end{align*}
\]

Combining these results with (115), we can have

\begin{align*}
\mathbb{E}\left[\Delta^2(\gamma_0)|Z,\varepsilon\right] &= \sum_{k=1}^{p} \mathbb{E}[\tilde{g}_{k,k,k,k}|Z] + \sum_{j \neq i} \mathbb{E}[\tilde{g}_{i,i,j,j}|Z] + \sum_{j \neq i} \mathbb{E}[\tilde{g}_{i,j,i,j}|Z] + \sum_{j \neq i} \mathbb{E}[\tilde{g}_{i,j,j,i}|Z] \\
&+ 4 \sum_{k=1}^{p} \beta_k^2 z_k^\top B_{\gamma_0} \Lambda_\varepsilon^2 B_{\gamma_0} z_k + \text{trace} \left((\Lambda_\varepsilon^2 B_{\gamma_0})^2\right) + 2 \sum_{k=1}^{p} \beta_k^2 z_k^\top B_{\gamma_0} z_k \cdot \sigma_0^2 \text{trace} \left(\Lambda_\varepsilon^2 B_{\gamma_0}\right).
\end{align*}

Thus by (113), (115) and (116) we can have

\begin{align*}
\text{Var}\left[\sqrt{n}(\Delta(\gamma_0))|Z,\varepsilon\right] &= 2n \sum_{j \neq i} \beta_i^2 \beta_j^2 \left(z_i^\top B_{\gamma_0} z_j\right)^2 + 4n \sum_{k=1}^{p} \beta_k^2 z_k^\top B_{\gamma_0} \Lambda_\varepsilon^2 B_{\gamma_0} z_k + 2n \sum_{j \neq i} \varepsilon_i^2 \varepsilon_j^2 (B_{\gamma_0})_{ij}^2.
\end{align*}

**B.10 Proof of Lemma 4.12**

Recall that \(\Delta_{*}(\gamma)\) is of the form (38), so we need to study the asymptotics of \(\text{trace} (V_{\gamma}^{-1})\), \(\text{trace} (V_{\gamma}^{-2})\), \(\text{trace} (V_{\gamma}^{-1} ZZ^\top)\) and \(\text{trace} (V_{\gamma}^{-2} ZZ^\top)\).

Denoting by \(\lambda_k\) the eigenvalues of \(p^{-1} ZZ^\top\), by Corollary A.1 and the fact that \(\gamma_0 =\)
\[ \|\beta\|^2/\sigma_0^2, p^{-1}ZZ^\top = \gamma^{-1}(V_{\gamma} - I_n), \]
we have

\[
\frac{1}{n} \text{trace}\left(V_{\gamma}^{-1}\right) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \gamma \lambda_k} \int_{b_-(\tau)}^{b_+ (\tau)} \frac{f_\tau (x)}{1 + \gamma x} dx + \delta_0 = h_1(\gamma, \tau),
\]

\[
\frac{1}{n} \text{trace}\left(V_{\gamma}^{-2}\right) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1 + \gamma \lambda_k)^2} \int_{b_-(\tau)}^{b_+ (\tau)} \frac{f_\tau (x)}{(1 + \gamma x)^2} dx + \delta_0 = h_2(\gamma, \tau),
\]

\[
\frac{\sigma_0^2 \gamma_0}{np} \text{trace}\left(V_{\gamma}^{-1}ZZ^\top\right) = \frac{\sigma_0^2 \gamma_0}{n \gamma} \left(\text{trace}\left(I_n\right) - \text{trace}\left(V_{\gamma}^{-1}\right)\right) \xrightarrow{a.s.} \frac{\gamma_0 \sigma_0^2}{\gamma} (1 - h_1(\gamma, \tau)),
\]

\[
\frac{\sigma_0^2 \gamma_0}{np} \text{trace}\left(V_{\gamma}^{-2}ZZ^\top\right) = \frac{\sigma_0^2 \gamma_0}{n \gamma} \left(\text{trace}\left(V_{\gamma}^{-1}\right) - \text{trace}\left(V_{\gamma}^{-2}\right)\right) \xrightarrow{a.s.} \frac{\gamma_0 \sigma_0^2}{\gamma} (h_1(\gamma, \tau) - h_2(\gamma, \tau)),
\]

where

\[
\delta_0 = \begin{cases} 0, & \tau \leq 1, \\ 1 - \frac{1}{\tau}, & \tau > 1. \end{cases}
\]

Then, there holds

\[
\Delta_{\tau_*}(\gamma)
= \frac{\sigma_0^2}{n} \text{trace}\left(V_{\gamma}^{-1}\left(I_n + \frac{\gamma_0}{p}ZZ^\top\right)\right) - \frac{\sigma_0^2}{\text{trace}\left(V_{\gamma}^{-1}\right)} \text{trace}\left(V_{\gamma}^{-2}\left(I_n + \frac{\gamma_0}{p}ZZ^\top\right)\right)
\xrightarrow{a.s.} \left(\frac{\gamma_0 \sigma_0^2}{\gamma} (1 - h_1(\gamma, \tau)) + \sigma_0^2 h_1(\gamma, \tau)\right) - \left(\frac{1}{h_1(\gamma)} \left(\frac{\gamma_0 \sigma_0^2}{\gamma} (h_1(\gamma, \tau) - h_2(\gamma, \tau)) + \sigma_0^2 h_2(\gamma, \tau)\right)\right)
= \sigma_0^2 \left(\frac{\gamma_0}{\gamma} - 1\right) \left(\frac{h_2(\gamma, \tau) - h_1^2(\gamma, \tau)}{h_1(\gamma, \tau)}\right).
\]

When \(\tau \leq 1\),

\[
\frac{h_2(\gamma, \tau) - h_1^2(\gamma, \tau)}{h_1(\gamma, \tau)} = \frac{\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) dx - \left(\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) \frac{f_\tau (x)}{1 + \gamma x} dx\right)^2}{\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) \frac{f_\tau (x)}{1 + \gamma x} dx}.
\] (117)

Since on \([b_-(\tau), b_+(\tau)]\), \(f_\tau (x) > 0\) for both \(\tau \leq 1\) and \(\tau > 1\), \((1 + \gamma x)^{-1}\) are strictly decreasing \((\gamma > 0)\), we have, by monotone function inequalities [Jiang, (2010), pages 148-149],

\[
\left(\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) dx \right)^2 < \left(\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) (1 + \gamma x)^2 dx\right) \left(\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) dx\right) = \int_{b_-(\tau)}^{b_+ (\tau)} \frac{f_\tau (x)}{(1 + \gamma x)^2} dx,
\]

which implies

\[
\int_{b_-(\tau)}^{b_+ (\tau)} \frac{f_\tau (x)}{(1 + \gamma x)^2} dx - \left(\int_{b_-(\tau)}^{b_+ (\tau)} \frac{f_\tau (x)}{1 + \gamma x} dx\right)^2 > 0.
\]

Similarly, when \(\tau > 1\), since

\[
\int_{b_-(\tau)}^{b_+ (\tau)} f_\tau (x) dx = \frac{1}{\tau},
\]

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the inequality above becomes
\[
\left( \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x} dx \right)^2 < \left( \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x^2} dx \right) \left( \int_{b-(\tau)}^{b+(\tau)} f_\tau(x) dx \right) = \frac{1}{\tau} \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x^2} dx.
\]
Then
\[
\int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x^2} dx + \left(1 - \frac{1}{\tau}\right) \left( \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x} dx + \left(1 - \frac{1}{\tau}\right) \right)^2
\]
\[
= \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x^2} dx - \left( \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x} dx \right)^2
\]
\[
+ \left(1 - \frac{1}{\tau}\right) \left( \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x} dx - \left(1 - \frac{1}{\tau}\right) \right)
\]
\[
> \left(1 - \frac{1}{\tau}\right) \left( \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x^2} dx - 2 \int_{b-(\tau)}^{b+(\tau)} \frac{f_\tau(x)}{1 + \gamma x} dx + 1 \right)
\]
\[
= \left(1 - \frac{1}{\tau}\right) \left( \int_{b-(\tau)}^{b+(\tau)} \frac{\gamma^2 x^2}{(1 + \gamma x^2)} f_\tau(x) dx \right)
\]
\[
> 0.
\]
Also, for both \( \tau > 1 \) and \( \tau \leq 1 \), the denominator \( h_1(\gamma, \tau) \) is positive obviously. Thus
\[
\frac{h_2(\gamma, \tau) - h_1^2(\gamma, \tau)}{h_1(\gamma, \tau)} > 0.
\]
Then it is shown that for both \( \tau \leq 1 \) and \( \tau > 1 \), the limit of \( \Delta_{\ast*}(\gamma) \) is \( c_\gamma = \sigma_0^2 \left( \frac{\gamma_0}{\gamma} - 1 \right) d_{\gamma, \tau} \), which is \( > 0, = 0 \) or \( < 0 \) depending on whether \( \gamma \) is \( < \gamma_0, = \gamma_0 \) or \( > \gamma_0 \).

**B.11 Proof of Lemma 4.13**

Recall that \( \Delta(\gamma) = y^\top B_\gamma y \) and
\[
B_\gamma = \frac{V_{\gamma}^{-1} - \frac{V_{\gamma}^{-2}}{\text{trace}(V_{\gamma}^{-1})}}{n}.
\]
Since for \( l = 1, 2, \ldots, \)
\[
\frac{d}{d\gamma} V_{\gamma}^{-l} = -l V_{\gamma}^{-(l+1)} \left( \frac{1}{p} ZZ^\top \right) = -\frac{l}{\gamma} \left( V_{\gamma}^{-l} - V_{\gamma}^{-(l+1)} \right),
\]
and
\[
\frac{d}{d\gamma} \text{trace}(V_{\gamma}^{-1}) = -\frac{1}{\gamma} \text{trace} \left( V_{\gamma}^{-1} - V_{\gamma}^{-2} \right),
\]
we can have that
\[
\frac{d}{d\gamma} B_\gamma = -\frac{1}{\gamma} \left( \frac{V_{\gamma}^{-1} - V_{\gamma}^{-2}}{n} + \frac{-2V_{\gamma}^{-2} + 2V_{\gamma}^{-3}}{\text{trace}(V_{\gamma}^{-1})} + \frac{V_{\gamma}^{-2} \text{trace} \left( V_{\gamma}^{-1} - V_{\gamma}^{-2} \right)}{\left( \text{trace}(V_{\gamma}^{-1}) \right)^2} \right). \quad (118)
\]
By a similar argument from the proof of Theorem 2.1, it can be checked that for every fixed \( \gamma \),
\[
\Delta'(\gamma) = \mathbf{y}^\top \frac{d}{d\gamma} B_{\gamma} \mathbf{y}
\]
converges in probability to \( \Delta'_\infty(\gamma) \), where \( \Delta'_\infty(\gamma) \) is some constant only depends on \( \gamma \) and \( \tau \), i.e. \( \Delta'(\gamma_0) = \Delta'_\infty(\gamma_0) + o_P(1) \). More specifically, similar to the argument in Section 4.3, we have that for any \( l = 1, 2, \ldots \)
\[
\frac{1}{n} \mathbf{y}^\top V_{\gamma}^{-l} \mathbf{y} \rightarrow_{P} \sigma_0^2 \frac{\gamma_0}{\gamma} h_{l-1}(\gamma, \tau) - \sigma_0^2 \left( \frac{\gamma_0}{\gamma} - 1 \right) h_l(\gamma, \tau),
\]
and
\[
\frac{1}{n} \sigma_0^2 \text{trace} \left( V_{\gamma}^{-l} V_{\gamma_0} \right) = \frac{1}{n} \sigma_0^2 \gamma_0 \text{trace} \left( V_{\gamma}^{-l} V_{\gamma_0} \right) - \frac{1}{2} \left( \frac{\gamma_0}{\gamma} - 1 \right) h_l(\gamma, \tau).
\]
Combining (119) and (120), we have when \( \gamma = \gamma_0 \)
\[
\frac{1}{n} \mathbf{y}^\top V_{\gamma_0}^{-l} \mathbf{y} \rightarrow_{P} \sigma_0^2 \frac{\gamma_0}{\gamma} h_{l-1}(\gamma_0, \tau).
\]
Therefore, by (118) and (121), as \( n \rightarrow \infty \),
\[
\mathbf{y}^\top \frac{d}{d\gamma} B_{\gamma_0} \mathbf{y} \rightarrow_{P} \frac{\sigma_0^2}{\gamma_0} h_2(\gamma_0, \tau) - h_2(\gamma_0, \tau) \frac{h_1(\gamma_0, \tau)}{h_1(\gamma_0, \tau)},
\]
which means
\[
\Delta'_\infty(\gamma_0) = \frac{\sigma_0^2}{\gamma_0} h_2(\gamma_0, \tau) - h_2(\gamma_0, \tau) \frac{h_1(\gamma_0, \tau)}{h_1(\gamma_0, \tau)}.
\]
By the Taylor series expansion, we have
\[
\Delta(\hat{\gamma}) = \Delta(\gamma_0) + \Delta'(\gamma_0) (\hat{\gamma} - \gamma_0) + \frac{1}{2} (\hat{\gamma} - \gamma_0)^2 \Delta''(\gamma_\delta)
\]
where \( \gamma_\delta \) is a number between \( \gamma_0 \) and \( \hat{\gamma} \). Since \( \Delta(\hat{\gamma}) = 0 \), we can rewrite this as
\[
\sqrt{n} (\hat{\gamma} - \gamma_0) = -\frac{\sqrt{n} \Delta(\gamma_0)}{\Delta'(\gamma_0) + \frac{1}{2} (\hat{\gamma} - \gamma_0) \Delta''(\gamma_\delta)}.
\]
Since we have already shown that \( \Delta'(\gamma_0) = \Delta'_\infty(\gamma_0) + o_P(1) \) and \( \hat{\gamma} - \gamma_0 = o_P(1) \), once we establish that
\[
\Delta''(\gamma_\delta) = O_P(1),
\]
it follows that
\[
\Delta'(\gamma_0) + \frac{1}{2} (\hat{\gamma} - \gamma_0)^2 \Delta''(\gamma_\delta) = \Delta'_\infty(\gamma_0) + o_P(1).
\]
Since $\hat{\gamma} - \gamma_0 = o_P(1)$, we can have $\gamma - \gamma_0 = o_P(1)$, then by (119) and (120)

$$\frac{1}{n} y^\top V_{\gamma_0}^{-1} y \xrightarrow{p} \sigma_0^2 \gamma_0 h_{1-1}(\gamma, \tau).$$

(126)

Then by (126) and some algebra, we can have

$$\Delta''(\gamma_0) = y^\top \frac{d^2}{d\gamma^2} B_{\gamma_0} y$$

$$\xrightarrow{p} 2\sigma_0\gamma_0 \frac{\gamma_0}{\gamma_0} \left( -2h_1(\gamma_0) + h_2(\gamma_0) + \frac{2h_2(\gamma_0) - h_3(\gamma_0)}{h_1(\gamma_0)} + \frac{h_4(\gamma_0)}{h_1(\gamma_0)} \right).$$

(127)

with $h_l(\gamma_0) = h_l(\gamma, \tau)$. Thus $\Delta''(\gamma_0) = O_P(1)$, (125) is proved. Then by (123) we can have

$$\sqrt{n}(\hat{\gamma} - \gamma_0) = -\sqrt{n} \Delta(\gamma_0) \Delta''(\gamma_0) + o_P(1),$$

(128)

where $\Delta''(\gamma_0)$ is defined in (122).

B.12 Proof of Lemma 4.15

Recall that

$$(B_{\gamma})_{ii} = \frac{1}{n} (V_{\gamma}^{-1})_{ii} - \frac{1}{n} \frac{1}{\text{trace}(V_{\gamma}^{-1})},$$

and for $l = 1, 2$,

$$(V_{\gamma}^{-l})_{ii} = 1 - \frac{1}{p} \sum_{k=1}^{l} \tilde{z}_i^\top V_{\gamma}^{-k} \tilde{z}_i.$$

Then

$$\sum_{i=1}^{n} (\varepsilon_i^2 - \sigma_0^2) \left( (B_{\gamma})_{ii} - \frac{1}{n} \text{trace}(B_{\gamma}) \right)$$

$$= -\gamma \sum_{i=1}^{n} \frac{\varepsilon_i^2 - \sigma_0^2}{n} \frac{1}{p} \left( \tilde{z}_i^\top \tilde{V}_{\gamma}^{-1} \tilde{z}_i - \text{trace} \left( \tilde{V}_{\gamma}^{-1} \frac{1}{n} Z^\top Z \right) \right)$$

$$+ \frac{\gamma}{n} \text{trace}(V_{\gamma}^{-1}) \sum_{l=1}^{2} \sum_{i=1}^{n} \frac{\varepsilon_i^2 - \sigma_0^2}{n} \frac{1}{p} \left( \tilde{z}_i^\top \tilde{V}_{\gamma}^{-l} \tilde{z}_i - \text{trace} \left( \tilde{V}_{\gamma}^{-1} \frac{1}{n} Z^\top Z \right) \right).$$

(129)

Since by Lemma 4.1

$$\sum_{i=1}^{n} \left( \frac{\varepsilon_i^2 - \sigma_0^2}{n} \right)^2 \leq \frac{1}{n} \max_{i \in [n]} \varepsilon_i^4 = o_P(1),$$

similar to (63), we can have for $l = 1, 2$

$$\sum_{i=1}^{n} \frac{\varepsilon_i^2 - \sigma_0^2}{n} \frac{1}{p} \left( \tilde{z}_i^\top \tilde{V}_{\gamma}^{-1} \tilde{z}_i - \text{trace} \left( \tilde{V}_{\gamma}^{-1} \frac{1}{n} Z^\top Z \right) \right) = o_P \left( \frac{1}{\sqrt{n}} \right),$$

which implies (65) by (129).