Abstract

A road interchange where \( n \) roads meet and in which the drivers are not allowed to change lanes can be modelled as an embedding of a 2-coloured (hence bipartite) multigraph \( G \) with equal-sized colour classes into an orientable surface such that there is a face bounded by a Hamiltonian cycle \([10]\). The case of \( G \) a complete bipartite graph \( K_{n,n} \) corresponds to a complete \( n \)-way interchange where drivers approaching from each of \( n \) directions can exit to any other direction. The genus of the underlying surface can be interpreted as the number of bridges in the interchange.

In this paper we study the minimum genus, or the minimum number of bridges, of a complete interchange with a restriction that it is symmetric under the cyclic permutation of its roads. We consider both (a) abstract combinatorial/topological symmetry, and (b) symmetry in the 3-dimensional Euclidean space \( \mathbb{R}^3 \). The proof of (a) is based on the classic voltage and transition graph constructions. For (b) we use, among other techniques, a simple new combinatorial lower bound.

1 Introduction

We call a tuple \( I = (G, H, M, S) \) a weaving-free interchange, or simply an interchange, if \( M \) is an embedding of a 2-coloured multigraph \( G \) into an orientable surface \( S \) such that \( G \) has the same number of black and white vertices and the embedding \( M \) has a face with a directed Hamiltonian cycle \( H \) as its boundary walk \([10]\). Here pairs of subsequent white and black vertices on \( H \) model the roadways (groups of lanes with traffic flow in the same direction) that enter and exit the interchange respectively. The images under \( M \) of the edges of \( G \) model the actual lanes inside the interchange. Finally, the Hamiltonian cycle \( H \) corresponds to the perimeter of the interchange and bounds the corresponding outer face. \( I \) is called \( t \)-way if \( G \) has \( t \) black and \( t \) white vertices. \( I \) is complete if there
is an edge \( uv \) for each white vertex \( u \) and each black vertex \( v \) (multiple edges are allowed).

For even \( n \), a construction for complete \( n \)-way interchanges with minimum genus was described in [10]. The problem is equivalent to minimizing the genus of an embedding of \( K_{n,n} \) such that one face is bounded by a Hamiltonian cycle. The solution implies one of the special cases of the conjecture on the minimum genus of a complete tripartite graph. After several decades of partial progress, see [1,2,9], the complete proof of the conjecture has been announced very recently by Ellingham, Stephens and Zha [4].

In this paper we impose an additional symmetry restriction. Symmetric embeddings or interchange layouts result by combining a number of identical pieces. This may be an advantage for aesthetical or practical reasons, especially when we have 3-dimensional realisations as in Theorem 1.2 below.

Our first result yields the minimum genus of interchanges where the cyclic permutation of roads (shifting vertices along the Hamiltonian cycle by two positions) preserves the rotation system. This kind of combinatorial/topological symmetry is classically modelled by the voltage graph construction; for definitions we follow Gross and Tucker [5], see also Section 3.

**Theorem 1.1** Let \( n \geq 3 \) be an integer. Let \( M \) be an embedding of \( K_{n,n} \) derived from a loopless embedded voltage graph on two vertices and voltage group \( \mathbb{Z}_n \). Suppose \( M \) has a face \( F_H \) bounded by a Hamiltonian cycle. Let \( p_1 \) be the smallest prime divisor of \( n \). The genus of \( M \) is at least \( L_C(n) \) where

\[
L_C(n) = \begin{cases} 
\frac{n(n-2)}{4}, & \text{if } n \text{ is even}; \\
\left\lfloor \frac{n(n-1)}{4} \right\rfloor + 1 - \frac{1}{2} \left( \frac{n}{p_1} + 1 \right), & \text{if } n \equiv 3 \pmod{4}, p_1 \neq n \text{ and } p_1^2 \nmid n; \\
\left\lfloor \frac{n(n-1)}{4} \right\rfloor + 1 - \frac{1}{2} \left( \frac{n}{p_1} + 1 \right), & \text{if } n \equiv 3 \pmod{4} \text{ and } p_1^2 \mid n; \\
\frac{n(n-1)}{4} - 1, & \text{if } n \equiv 1 \pmod{4}, 3 \mid n \text{ and } 9 \nmid n; \\
\left\lfloor \frac{n(n-1)}{4} \right\rfloor, & \text{otherwise}.
\end{cases}
\]

Furthermore, this lower bound is best possible, and the derived embedding \( M \) that achieves genus \( L_C(n) \) and \( F_H \) can be chosen so that \( F_H \) is generated by a face of size 2 in the base embedding.

The fact that \( L_C(n) \) depends on the prime divisors of \( n \) may be seen as a remote link to the well established theory of regular embeddings of complete bipartite graphs \( K_{n,n} \) where the symmetry requirement is much stronger [6].

Now let \( M \) be an embedding of a graph \( G \) into a closed connected orientable surface \( S \) where \( S \) is itself embedded into \( \mathbb{R}^3 \) (we will denote the latter fact by \( S \subset \mathbb{R}^3 \)). For our second result we will work in the category of piecewise linear embeddings both for the graph in the surface, and the surface in the 3-dimensional
space. The image \( M(G) \) of \( G \) is defined as the subset of \( S \) which consists of the union of the set \( M(V(G)) \) of all points that \( M \) maps \( V(G) \) to and the curves connecting these points which \( M \) maps the edges of \( G \) to. The isometry group \( \text{Iso}(M) \) of \( M \) consists of all Euclidean isometries \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) which map \( S \) to \( S \), \( M(G) \) to \( M(G) \) and \( M(V(G)) \) to \( M(V(G)) \). We say that an embedding \( M \) has \( n \)-fold rotational symmetry if both the embedded graph and the underlying surface are invariant under the rotation \( r \) by angle \( 2\pi/n \) about some axis in \( \mathbb{R}^3 \), i.e., if \( r \in \text{Iso}(M) \). We call a road interchange \( I = (G,H,M,S) \) 3-dimensional if \( S \subset \mathbb{R}^3 \). The symmetry group \( \text{Iso}(I) \) of \( I \) consists of those \( f \in \text{Iso}(M) \) that fix the face bounded by \( H \).

The next is our main result about 3-dimensional road interchanges.

**Theorem 1.2** Let \( n \geq 2 \) be an integer. Let \( M \) be a piecewise linear embedding of \( K_{n,n} \) into a closed connected orientable surface \( S \subset \mathbb{R}^3 \) such that there is a face \( F_H \) bounded by a Hamiltonian cycle. Suppose \( M \) has \( n \)-fold rotational symmetry that leaves the boundary of \( F_H \) invariant. Then the genus of \( S \) is at least \( L^*_C(n) \), where

\[
L^*_C(n) = \begin{cases} 
\frac{n^2 - 1}{4}, & \text{if } n \equiv 0 \pmod{4}; \\
\frac{n(n-1)}{4}, & \text{if } n \equiv 1 \pmod{4}; \\
\frac{n(n-2)}{4}, & \text{if } n \equiv 2 \pmod{4}; \\
\frac{n(n+1)}{4} - 1, & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Furthermore, if \( n \neq 4 \), this lower bound is best possible.

Of course, \( L_C(n) \leq L^*_C(n) \), but note that equality holds only in certain cases. A simple, but suboptimal lower bound for \( L^*_C(n) \) can be obtained from Theorem 1.1 and the fact that not every orientable surface can be embedded in \( \mathbb{R}^3 \) with a given symmetry group: in \( \mathbb{R}^3 \) we can only have embeddings that have symmetry groups of a \( t \)-prism for certain \( t > 0 \), the Platonic solids and their subgroups [11,14]. The correct lower bound follows from a new combinatorial argument that exposes a local obstruction that prevents a corresponding ‘quotient embedding’ into a smaller genus surface, see Section 4.2.

Our constructions that achieve genus \( L^*_C(n) \) can be called ring road interchanges. For each entering motorway we construct a block with some ‘simple bridges’ (handles). We connect the blocks into a ring. For \( n \mod 4 \in \{1,2\} \) the layout is particularly simple: in the inter-block connections the traffic always moves counter-clockwise and the lanes heading to each exit motorway are adjacent. For \( n \mod 4 \in \{0,3\} \) and \( n > 4 \), the optimal constructions are similar, but they use a star-shaped bridge and some lanes where traffic between the blocks goes in the opposite, clockwise, direction, see Section 4.3.
The case \( n = 4 \) is an exception in Theorem \[1.2\]. We show below that the minimum genus in this case is \( 4 \), and one of the embeddings of such genus is the 4-way Pinavia interchange \([7]\), shown in Figure \[8\] below.

The Hamiltonian cycle assumption can be relaxed in Theorem \[1.2\]. The result remains true if we only require that no vertex of \( K_{n,n} \) is mapped to a fixed point and at least one point of the surface is a fixed point, see Lemma \[4.6\] below.

2 Definitions

A directed multigraph \( G \) is a pair \((V, E)\), where \( V \) and \( E \) are disjoint sets together with a map \( E \to V \times V \) assigning to each edge \( e \) its endpoints \((u, v)\); we say that \( e \) goes from \( u \) to \( v \). We use \( V(G) = V \) and \( E(G) = E \). When there is only one edge \( e \) with endpoints \((u, v)\) we use notation \( uv \) for \( e \). Below when necessary we consider any 2-coloured bipartite multigraph \( G \) as directed, by implicitly assuming all edges go from white vertices to black vertices.

For a directed multigraph \( G \) we let \( U(G) \) denote the underlying undirected multigraph. We say that a directed multigraph \( G_1 \) and an undirected multigraph \( G_2 \) are isomorphic if \( G_2 \) is isomorphic to \( U(G_1) \).

Let \( M \) be an embedding of a directed or undirected multigraph \( G \) into a surface \( S \). We call a connected component of \( S \setminus M(G) \) a region. If a region of \( M \) is 2-cell, i.e. if it is homeomorphic to a disk, we call it a face of \( M \). A well known fact is that there is a bijection between 2-cell embeddings of \( G \) (up to homeomorphism) and rotation systems for \( G \), which are sets \( \{\pi_v : v \in V(G)\} \) where \( \pi_v \) is a cyclic ordering of edges incident to \( v \) \([5]\).

Let \( \Gamma = (\Gamma, \ast) \) be a group. Given a directed multigraph \( G \) and a function \( \alpha : E(G) \to \Gamma \), called a voltage function or a voltage assignment, the derived graph \( G' \) for voltage group \( \Gamma \) is the directed multigraph \( G' \) on the vertex set \( V(G) \times \Gamma \) and edges \( \{(x, a), (y, a \ast \alpha(e)) : e \in E(G), f(e) = (x, y)\} \). Here \( f \) maps \( e \) to its endpoints. \((G, \alpha)\) is called the voltage graph or the base graph for \( G' \), and we call \( \Gamma \) the voltage group. If \( G' \) is derived from \((G, \alpha)\) and \( \Gamma \), we will say that its undirected version \( U(G') \) also is. If \( M \) is a 2-cell embedding of \( G \) into some surface \( S \) then \((G, M, S, \alpha)\) is called an embedded voltage graph.

The main point of the voltage graph construction is that the embedding \( M \) of a base graph \( G \) can be used to obtain a derived embedding \( M' \) of the derived graph \( G' \) \([5]\). Each face of \( M' \) is generated by precisely one face of \( M \). Let \( F \) be a face of \( M \) with boundary \( e_1^{s_1} \cdots e_k^{s_k} \). Here \( s_i = 1 \) if the direction of \( e_i \) agrees with the direction of the boundary walk of \( F \) and \( s_i = -1 \) otherwise. The net voltage of \( F \) is \( \alpha(e_1)^{s_1} \ast \cdots \ast \alpha(e_k)^{s_k} \), where \( x^{-1} \) denotes the inverse of \( x \) in \( \Gamma \). We call the number of edges in the boundary walk of \( F \) its size. We call a face
Hamiltonian if its boundary is a Hamiltonian cycle. If $F$ has size $k$ and net voltage $g$, then there are $n/|g|$ faces in $M'$ generated by $F$, and each of these faces has size $k|g|$. Here $|g|$ is the order of $g$ in $\Gamma$. The main base graph we will meet in this paper is the dipole graph $D_n$, that is, the bipartite multigraph on two vertices $\{w_0, v_0\}$ and $n$ parallel directed edges from $w_0$ to $v_0$. Here $w_0$ and $v_0$ stand for ‘white’ and ‘black’ vertex respectively, as we sometimes treat $D_n$ as 2-coloured.

The only group $\Gamma$ we will meet below will be $\mathbb{Z}_n$ (the cyclic group of order $n$).

Embedded voltage graphs obtained from $D_n$ can be alternatively represented by transition graphs \cite{1-3}. For our purposes, a transition graph of order $n$ is a directed (multi-)graph with vertex set $\mathbb{Z}_n$ and edge set consisting of the union of the edges of two directed Hamiltonian cycles $C_1$ and $C_2$. The edges of $C_1$ are called \textit{solid} edges and the edges of $C_2$ are called \textit{dotted} edges of $G$. An edge from $u$ to $v$ is denoted $u \Rightarrow v$ if it is solid, and $u \rightarrow v$ if it is dotted. The edges of $G$ are a disjoint union of edges of simple directed even cycles (i.e. without repeated vertices) in $G$ with alternating edge type. The \textit{net transition} $\alpha(C)$ of one such alternating cycle (called boundary walk in \cite{1}) $C = (v_1, \ldots, v_k)$ with edges $v_1 \Rightarrow v_2 \rightarrow v_3 \Rightarrow \ldots v_{k-1} \rightarrow v_k$ is $\alpha(C) = -v_1 + v_2 - v_3 \ldots + v_k$.

### 3 Proofs for combinatorial embeddings

In this section we prove Theorem 1.1. Let $I = (G, H, \mathcal{M}, S)$ be a complete $n$-way interchange. Write $H = (v_0, \ldots, v_{2n-1})$. Suppose

\begin{itemize}
  \item[(*)] $\mathcal{M}$ has an orientation preserving automorphism $\sigma$ that maps $v_i$ to $v_{i+2}$ for each $i \in \mathbb{Z}_n$.
\end{itemize}

Then $\mathcal{M}$ is derived from an embedded voltage graph with two vertices and $m$ parallel edges, $m \geq n$, voltage group $\Gamma = \mathbb{Z}_n$ and a voltage assignment $\alpha$ such that for each $x \in \mathbb{Z}_n$ there is $e \in E(B)$ with $\alpha(e) = x$, i.e. $\alpha$ is a surjection. It follows from (*) that the base embedding can be chosen so that the face bounded by $H$ is generated by a face $F_0$ of size two. We can remove edges with repetitive voltage that do not lie on $F_0$ without increasing the genus until there are exactly $n$ edges of different voltage left. So for the minimum genus problem, we may assume $G$ is $K_{n,n}$ and $m = n$. Conversely, each graph derived from $D_n$ with $\Gamma = \mathbb{Z}_n$ and a bijective voltage assignment $\alpha : E(D_n) \rightarrow \mathbb{Z}_n$ such that at least one face of the derived graph is bounded by a Hamiltonian cycle and generated by a face of size 2 has property (*).
3.1 Even \( n \)

**Proof of Theorem 1.1 even \( n \).** Let \((D_n, \mathcal{M}_0, S_0, \alpha)\) be the base embedded voltage graph that yields \( \mathcal{M} \). Let \( F_0, F_1, \ldots, F_k \) denote the faces of the \( \mathcal{M}_0 \) and for each face \( F_i \) let \( k_i \) and \( g_i \) denote the size and net voltage of \( F_i \). Consider the sums \( \sum_{i=0}^{k} k_i \) and \( \sum_{i=0}^{k} g_i \): in both sums each edge is taken into account twice (once in both directions), therefore

\[
\sum_{i=0}^{k} k_i = 2n \quad \text{and} \quad \sum_{i=0}^{k} g_i \equiv 0 \pmod{n}. \quad (1)
\]

For every \( i = 0, 1, \ldots, k \) the face \( F_i \) generates \( f_i = \frac{n}{|g_i|} \) faces in the derived graph, each of which has size \( k_i \cdot |g_i| \). In order to obtain the lower bound for the genus of this embedding, we will seek to maximize the sum \( f = \sum_{i=0}^{k} f_i \). Fix a face \( F_0 \) of \( \mathcal{M}_0 \) that generates Hamiltonian faces in \( \mathcal{M} \). We call a face \( F, F \neq F_0 \) of \( \mathcal{M}_0 \) optimal if it generates quadrangular faces in \( \mathcal{M} \). All other faces \( F, F \neq F_0 \) will be called non-optimal. Without loss of generality we assume that \( F_1, \ldots, F_l \) are non-optimal and \( F_{l+1}, \ldots, F_k \) are optimal.

\( F_i \) is optimal iff \( k_i \cdot |g_i| = 4 \). So we can split the optimal faces \( F_i \) into two cases:

1) \( g_i \equiv 0 (f_i = n) \) and \( k_i = 4 \);
2) \( g_i \equiv \frac{n}{2} (f_i = \frac{n}{2}) \) and \( k_i = 2 \).

We denote \( \sum_{j=1}^{l} k_j = 2m \). For non-optimal faces \( F_j \) we have \( k_j \cdot |g_j| \geq 6 \), hence we can split them into three cases:

1) \( |g_j| = 1, k_j \geq 6 \) and \( f_j = n \leq k_j \cdot \frac{n}{6} \);
2) \( |g_j| = 2, k_j \geq 4 \) and \( f_j = \frac{n}{2} \leq k_j \cdot \frac{n}{8} \);
3) \( |g_j| > 2 \) and \( f_j \leq \frac{n}{3} \leq k_j \cdot \frac{n}{6} \).

Therefore

\[
\sum_{j=1}^{l} f_j \leq \sum_{j=1}^{l} k_j \cdot \frac{n}{6} = \frac{mn}{3}. \quad (2)
\]

Now we rewrite (1) as follows:

\[
0 \equiv \sum_{i=0}^{k} g_i \equiv g_0 + \sum_{j=1}^{l} g_j + \frac{2n - k_0 - 2m}{2} \cdot \frac{n}{2}.
\]

\[
= \begin{cases} 
  g_0 + \sum_{j=1}^{l} g_j & \text{if } k_0/2 + m \text{ is even}, \\
  g_0 + \sum_{j=1}^{l} g_j + \frac{n}{2} & \text{if } k_0/2 + m \text{ is odd}. 
\end{cases} \quad (3)
\]
We can also rewrite the sum $f$:

$$f = \sum_{i=0}^{k} f_i = f_0 + \sum_{j=1}^{l} f_j + \frac{2n - k_0 - 2m}{2} \cdot \frac{n}{2}. \quad (4)$$

First assume that $k_0 = 2$. Then $|g_0| = n, f_0 = 1$ and from (3) we have that $g_0$ cannot equal to 0 or $n/2$ when $n > 2$, hence there is at least one non-optimal face in $M_0$ (i.e. $l > 0$). Let $\hat{M}_0$ be an embedded voltage graph which gives the lowest derived graph embedding genus among all graphs $M_0$ satisfying theorem conditions with $k_0 = 2$.

Now for even $n$ it is easy to get a transition graph that yields an embedding of $K_{n,n}$ with two Hamiltonian faces and optimal genus $n(n - 2)/4$. This can be done by generalizing examples of Ellingham [3]. In particular, we can take the cycles $(C_1, C_2)$ where $C_1$ is

$$\left(\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1, 0\right)$$

and $C_2$ is

$$\begin{cases} 
(1, 2, -2, 4, -4, \ldots, -\left(\frac{n}{2} - 1\right), 0, -1, 3, -3, 5, -5, \ldots, n) & \text{if } n \equiv 2 \pmod{4}, \\
(1, 2, -2, 4, -4, \ldots, \frac{n}{2}, 0, -1, 3, -3, 5, -5, \ldots, -\left(\frac{n}{2} - 1\right)) & \text{if } n \equiv 0 \pmod{4}. 
\end{cases}$$

Note that in each case the two Hamiltonian faces are generated by two alternating 2-cycles, or 2-faces in the corresponding voltage graph: they are $(2, 1)$ and $(-1, 0)$.

From this construction we have a lower bound $\hat{f}$ for the number of faces in the embedding derived from $\hat{M}_0$:

$$\hat{f} = 1 + \sum_{j=1}^{l} f_j + \frac{n^2 - n - mn}{2} \geq 1 + 1 + \frac{n^2 - n - mn}{2} \quad (5)$$

It suffices to prove that in (5) the equality holds. From (5) and (2)

$$1 + \frac{(m - 1)n}{2} \leq \sum_{j=1}^{l} f_j \leq \frac{mn}{3}, \text{ therefore}$$

$$6 \leq n(3 - m).$$

As $m > 0$, $m$ equals to 1 or 2. If $m = 1$, then $l = 1$ and from (3) we have that $g_1 \equiv -g_0 \pmod{n}$ so $f_1 = 1$ and the equality in (5) holds.
Similarly, if \( m = 2 \) and \( l = 1 \), then \( 1 + \frac{n}{2} \leq \sum_{j=1}^{l} f_j = 1 \), a contradiction.

We are left with the case when \( m = 2 \) and \( l = 2 \). We have that \( k_1 = k_2 = 2 \) and \( f_j \leq \frac{n}{2} \) for \( j = 1, 2 \). As \( 1 + \frac{n}{2} \leq f_1 + f_2 \), one of \( f_1 \) and \( f_2 \) must equal to \( \frac{n}{2} \) and the other one belongs to \( \{ \frac{n}{3}, \frac{n}{4}, \frac{n}{5} \} \). Without loss of generality let \( f_1 = \frac{n}{3} \).

If \( \gcd(\frac{n}{2}, f_1, f_2) = r > 1 \), then from (4) we get \( \gcd(n, f_0) \geq r > 1 \), a contradiction. Therefore \( \gcd(\frac{n}{2}, f_1, f_2) = 1 \) and when \( f_2 \) equals to \( \frac{n}{4} \) or \( \frac{n}{5} \), then \( n \) must be 6, 12 or 30 respectively. In all these cases the equality (5) holds.

This ends our proof when \( k_0 = 2 \). We have seen that in this case the maximum number of faces \( \hat{f} \) does not exceed the one of construction having all quadrangular faces except from two Hamiltonian faces which has a genus \( g = \frac{n(n-2)}{4} \). As any construction with \( k_0 > 2 \) yields an embedding with at least two Hamiltonian faces, it also has a genus not smaller than \( \frac{n(n-2)}{4} \).

\[\square\]

### 3.2 Odd \( n \)

The proof for odd \( n \) is a bit trickier; \( L_{C}(n) \) itself is not as simple as for even \( n \). We start with two lemmas that give examples of optimal genus.

**Lemma 3.1** For each odd \( n, n \geq 3 \), there is a transition graph that yields an embedding of \( K_{n,n} \) of genus \( \left\lfloor \frac{n(n-1)}{4} \right\rfloor \) with a Hamiltonian face.

**Proof** A construction for \( n \equiv 3 \mod 4 \) is as follows. For \( n = 3 \), we can trivially take a transition graph with cycles \( C_1 = (0, 1, 2) \) and \( C_2 = (0, 2, 1) \) which yields an embedding of \( K_{3,3} \) with three Hamiltonian faces. We generalize this to \( n \geq 7 \) by taking

\[
C_1 = \left(0, 1, \ldots, \frac{n-1}{2}, n-1, n-2, \ldots, \frac{n+1}{2}\right)
\]

\[
C_2 = \left(0, \frac{n+1}{2}, 2, \frac{n+1}{2} + 2, 4, \ldots, n-1, \frac{n-1}{2}, n-2, \frac{n-1}{2} - 2, n-4, \ldots, 1\right).
\]

This transition graph has three alternating cycles of length two: \((0, 1), ((n + 1)/2, 0)\) and \(((n - 1)/2, n - 1)\). Their net transition is \(1, (n - 1)/2\) and \((n + 1)/2\), respectively, and all these elements have order \( n \) in \( \mathbb{Z}_n \). The remaining alternating cycles are of the form \((x, x + 1, (n + 1)/2 + x + 2, (n + 1)/2 + x + 1)\) or \((x + 1, x + 2, (n + 1)/2 + x, (n + 1)/2 + x + 1)\), thus they have size 4 and net transition 0. The derived graph has 3 Hamiltonian faces and all other faces of size 4.
A transition graph for \( n \equiv 1 \pmod{4} \) is

\[
C_1 = \left( \frac{n-1}{2}, \ldots, 1, \frac{n+1}{2}, \ldots, n-1, 0 \right) \quad \text{and} \quad C_2 = \left( \frac{n+1}{2}, 0, n-1, \frac{n-1}{2} - 2, n-3, \frac{n-1}{2} - 4, \ldots, \frac{n+1}{2} + 1, \right.
\]

\[
1, \frac{n+1}{2} + 2, 3, \frac{n+1}{2} + 4, \ldots, \frac{n-1}{2} - 1, \frac{n-1}{2} \bigg). \]

\[ \]

Figure 1: Transition graphs for \( n = 7 \) and \( n = 9 \) as in the proof of Lemma 3.1.

Again it is easy to see that this transition graph has two alternating cycles \((n-1, 0)\) and \(((n-1)/2, (n-1)/2-1)\), an alternating cycle \(((n+1)/2, (n+1)/2+1, 1, (n+1)/2, 0, (n-1)/2)\) of length 6 and net transition \(-n \equiv 0 \pmod{n}\) and all other alternating cycles of length 4 and net transition 0. Thus the derived embedding has two Hamiltonian faces, \( n \) faces of size 6 and all other faces of size 4. By Euler’s formula its genus is \( L_C(n) = \lfloor n(n-1)/4 \rfloor \). Both cases are illustrated in Figure 1.

**Lemma 3.2** Let \( n \geq 25 \), \( n \equiv 3 \pmod{4} \). Suppose \( g_1, g_2 \in \mathbb{Z} \setminus \{0\} \) and \( g_1 + g_2 + 1 = 0 \). Then there exists a transition graph on \( n \) vertices with 3 alternating 2-cycles of net transition 1, \( g_1 \) and \( g_2 \) respectively, and all other alternating cycles of size 4 and net transition 0.

The corresponding derived graph is \( K_{n,n} \) with one Hamiltonian face, \( n/|g_i| \) faces of size \( 2|g_i| \) for \( i \in \{1, 2\} \) and all other faces of size 4. This graph has genus

\[
\lfloor \frac{n(n-1)}{4} \rfloor + 1 - \frac{1}{2} \left( \frac{n}{|g_1|} + \frac{n}{|g_2|} \right). \]

**Proof** Consider solutions of

\[
\begin{aligned}
a &= g_1 - b \\
d &= g_2 - c \\
e &= b + c
\end{aligned}
\]

(6)
with unknowns $a, b, c, d, e \in \mathbb{Z}_n$, operations modulo $n$ and a restriction that $a, b, c, d, e, -a, -b, -c, -d, -e$ are all distinct and avoid $\{0, 1, -1\}$.

We claim that (6) always has a feasible solution. Indeed, let $b$ and $c$ be taken independently and uniformly at random from $\mathbb{Z}_n$. Define $a, d$ and $e$ according to (6). Let $A$ be the event that $-a, a, -b, b, 0, -1, 1$ are all distinct, and additionally $b \neq -g_2$. Since $n$ is odd, $x \in \mathbb{Z}_n \setminus \{0\}$ implies $x \neq -x$. Also for $y \in \mathbb{Z}_n$ there is a unique solution of $2x = y$, which we denote $y/2$.

$A$ occurs if and only if $b \not\in \{0, \pm 1, -g_2\}$ and $a \not\in \{0, \pm 1, \pm b\}$, which is equivalent to the event $b \not\in \{0, -1, 1, g_1, g_1 - 1, g_1 + 1, g_1/2, -g_2\}$. So

$$P(A) \geq 1 - \frac{8}{n}. $$

Similarly, let $B$ be the event that $a, b, c, d, e, 1$ and their negations are distinct and non-zero. For $a$ and $b$ such that $A$ occurs

$$P(B|a, b) = P \left( c \not\in \{0, \pm 1, \pm a, \pm b\} \cap d \not\in \{0, \pm 1, \pm a, \pm b, \pm c\} \cap e \not\in \{0, \pm 1, \pm a, \pm b, \pm c, \pm d\} \right)$$

$$= P \left( c \not\in \{0, \pm 1, \pm a, \pm b, g_2, g_2 \pm 1, g_2 \pm a, g_2 \pm b, g_2 - g_1 + b, g_2/2, -b, -b \pm 1, -b \pm a, -2b, -b/2, (g_2 - b)/2\} \right) \geq 1 - \frac{24}{n}. $$

So for $n \geq 25$

$$P(B) = P(B|A)P(A) \geq \left( 1 - \frac{8}{n} \right) \left( 1 - \frac{24}{n} \right) > 0. $$

We further consider $(a, b, c, d, e)$ as a fixed solution of (6).

Let $t = (n - 1)/2$. Pick (we can always do it; in fact, we can do it in $e^{\Omega(n \ln n)}$ ways) a sequence $s_1 = (v_0, v_1, \ldots, v_t)$, such that

1. For each $x \in \mathbb{Z}_n \setminus \{0\}$, $s_1$ contains exactly one of $\{-x, x\}$.
2. $v_0 = 0$, $v_1 = 1$, $v_{k} = a$, $v_{k+1} = b$, $v_{l} = c$, $v_{l+1} = d$ and $v_{t} = e$, where $k$ and $l$ are even and $1 < k < l < t$.

Define $s_2 = (-v_1, \ldots, -v_{t})$. Then the elements of $s_1$ and $s_2$ partition $\mathbb{Z}_n$. We will construct the transition graph by adding solid and dotted edges to an empty graph on vertex set $\mathbb{Z}_n$. First add directed paths of solid directed edges

$$P_1 = (v_0, v_1, \ldots, v_{k}),$$
$$P_2 = (-v_1, -v_2, \ldots, -v_{k}, v_{k+1}, v_{k+2}, \ldots, v_{t}),$$
$$P_3 = (-v_{k+1}, -v_{k+2}, \ldots, -v_{l}, v_{l+1}, v_{l+2}, \ldots, v_{t}) \quad \text{and}$$
$$P_4 = (-v_{l+1}, -v_{l+2}, \ldots, -v_{t}).$$
Next, add dotted edges \((v_{k+1}, -v_k), (v_{l+1}, -v_l)\) and \((v_1, v_0)\) (i.e., the edges \((b, -a), (d, -c)\) and \((1, 0))\). These edges complete three alternating 2-cycles of net transition \(a + b = g_1, c + d = g_2\) and \(1 + 0 = 1\) respectively, see Figure 2.

Now for every \(i \in \{1, \ldots, l-1\} \setminus \{k,l\}\) add dotted directed edges \((v_{i+1}, -v_i)\) and \((-v_{i+1}, v_i)\), this yields an alternating 4-cycle \(v_i \Rightarrow v_{i+1} \Rightarrow -v_i \Rightarrow -v_{i+1} \Rightarrow v_i\) of net transition \(v_{i+1} + v_i - v_{i+1} - v_i = 0\).

Since \(k\) and \(l\) are even and \(t\) is odd, all the dotted edges added so far yield the following paths that go backwards alternating between \(s_1\) and \(s_2\):

\[
P'_1 = (-v_t, v_{t-1}, -v_{t-2}, \ldots, -v_{t+1}),
P'_2 = (v_t, -v_{t-1}, v_{t-2}, \ldots, v_{t+1}, v_t),
P'_3 = (v_1, -v_1, \ldots, -v_{k+1}) \quad \text{and}
P'_4 = (v_k, -v_k, \ldots, -v_1).
\]

Finally, add two alternating 4-cycles:

\[
A_1 = c \Rightarrow -b \Rightarrow -e \Rightarrow 0 \Rightarrow c = (v_i, -v_{k+1}, -v_i, v_0) \quad \text{and}
A_2 = e \Rightarrow -d \Rightarrow a \Rightarrow -1 \Rightarrow e = (v_i, -v_{l+1}, v_k, -v_1).
\]

Notice that the edges of \(A_1\) and \(A_2\) complete a solid Hamiltonian cycle \((P'_1 P'_2 P_3 P'_4)\) and a dotted Hamiltonian cycle \((P'_1 P'_4 P'_2 P'_3)\), where for paths \(A, B\) we use notation \(AB\) to denote the concatenation of their vertices in the obvious way.

To complete the proof, we have to check that \(A_1\) and \(A_2\) both have net transition 0. The net transition of \(A_1\) is \(-c - b + e = 0\) since \(e = b + c\). The net transition of \(A_2\) is

\[-e - d - a - 1 = -(a + d + e + 1) = -(a + d + b + c + 1) = -(1 + 1) = 0.\]

In the last step we used \(a + b + c + d = g_1 + g_2 = -1\), which follows from the first two equations of (6) and the theorem’s assumption. \(\square\)

We are now ready to complete the proof of Theorem 1.1.
Proof of Theorem 1.1 odd $n$. We continue with notation from the proof for even $n$. We first prove the lower bound. In the end of this proof we will show that it is optimal.

Let $s_1, \ldots, s_f$ be the sizes of the faces of $M$. Let $v, e,$ and $g$ denote the number of vertices, the number of edges and the genus of $M$ respectively. Denote the excess of $M$ by

$$ex = \sum_i (s_i - 4) = 2e - 4f = 2n^2 - 4f.$$ 
By Euler’s formula $ex = 8g - 2e + 4v - 8 = 8g - 2n^2 + 8n - 8$. So using the definition of $LC(n)$, in order to prove the inequality stated in the theorem, it suffices to show

$$ex \geq \begin{cases} 
6n - 8 - 4 \left( p_1 + \frac{n}{p_1} \right), & \text{if } n \equiv 3 \pmod{4}, n \not\equiv p_1 \text{ and } p_1^2 \nmid n, \\
6n - 4 - 4 \left( 1 + \frac{n}{p_1} \right), & \text{if } n \equiv 3 \pmod{4} \text{ and } (p_1^2 | n \text{ or } n = p_1), \\
6n - 16, & \text{if } n \equiv 1 \pmod{4}, 3 | n \text{ and } 9 \nmid n, \\
6n - 8, & \text{if } n \equiv 1 \pmod{4} \text{ and } (3 \nmid n \text{ or } 9 | n). 
\end{cases} \quad (7)$$

Since $g$ is integer, we must have

$$\begin{cases} 
6n - 8 - ex \equiv 0 \pmod{8} & \text{for } n \equiv 1 \pmod{4}, \\
6n - 12 - ex \equiv 0 \pmod{8} & \text{for } n \equiv 3 \pmod{4}. 
\end{cases} \quad (8)$$

With voltage group $\mathbb{Z}_n$ a face $F$ of size $k$ and voltage $h$ contributes

$$ex(F) = \frac{n}{|h|} (|h| - 4) = nk - \frac{4n}{|h|}$$

(9) to the excess of the derived embedding. Note that since each face $F$ in $M$ has size at least 4, $ex(F) \geq 0$.

Suppose there exists an embedding $M$ of $K_{n,n}$ as in the statement of the theorem, but with genus smaller than $LC(n)$, i.e. suppose that (7) does not hold. Let $B = (D_n, M_0, \alpha_0, \alpha)$ be the corresponding base graph as before. Note that the faces of $M_0$ have even sizes and the orders of their voltages are divisors of $n$, in particular they are odd. Then $|g_0| = 2n/k_0$ and by (9) $ex \geq ex(F_0) = k_0(n-2)$. For $k_0 \geq 8$ (7) holds. It cannot be $k_0 = 6$ either, since in this case $3 | n$, so $n$ is not prime and by (8), (7) still holds. The only case where we could have equality in (7) for $k_0 = 6$ is $n \equiv 1 \pmod{4}$ and $9 \mid n$.

Also $k_0 \neq 4$, since $4 \nmid 2n$. Thus for $B$ to be a counterexample we must have $k_0 = 2$, $|g_0| = n$ and $ex(F_0) = 2n - 4$. Since orders of elements are
preserved under a group automorphism, we can assume without loss of generality that \( g_0 = 1 \).

For any face of \( \mathcal{M}_0 \), we have that \( \text{ex}(F) = 0 \) if and only if \( F \) has size 4 and voltage 0. As before, we call such faces \( F \) optimal, and the remaining faces \( F \), \( F \neq F_0 \) non-optimal. We assume that \( F_1, \ldots, F_l \) are all the non-optimal faces of \( B \), and using \( g_0 = 1 \) and (11) we have

\[
1 + g_1 + \cdots + g_l = 0. \tag{10}
\]

For a non-optimal face we have \( g_i = jn/|g_i| \) for some \( j \in \{0, |g_i| - 1\} \), so

\[
\gcd(n/|g_1|, \ldots, n/|g_l|) = 1. \tag{11}
\]

Let \( F_i \) be non-optimal. As \( K_{n,n} \) has no repeated edges, \( g_i \) can only be zero if \( k_i \geq 6 \). Using (9) we have \( \text{ex}(F_i) \geq 2n - 4n/3 = 2n/3 \). Thus if \( B \) is a counterexample to (7), then \( 2n - 4 + 2ln/3 \leq \text{ex} < 6n - 8 \) which implies \( l \leq 5 \). If \( F_i \) has size \( k_i \geq 8 \), then

\[
\text{ex} \geq \text{ex}(F_0) + \text{ex}(F_i) \geq 2n - 4 + k_i n - 4n \geq 6n - 4
\]

and again (7) holds. Similarly, there cannot be any face of size 6 and voltage \( g \) with \( g \mod n \neq 0 \), since in this case \( |g| \geq 3 \) and \( \text{ex} \geq 2n - 4 + 6n - 4n/3 > 6n - 8 \).

Using (10), the fact that the right side of (9) increases with \( |h| \) and the above observations we conclude that (7) holds with inequality unless perhaps

1. some non-optimal face, say \( F_1 \), has size 6, \( g_1 = 0, l = 3 \), the other non-optimal faces \( F_j, j \geq 2 \) have size 2 and \( |g_j| \in \{3, 5\} \) (in the case \( l = 2 \), \( |g_2| = n \) by (10), so (7) holds with equality) or

2. some non-optimal face, say \( F_1 \), has size 4, \( l = 2 \) and \( g_1, g_2 \in \{3, 5\} \) (if \( l = 1 \) and \( |g_1| = n \) and (7) holds with equality).

Note that (11) implies that \( n \leq \prod_{i=1}^{l} |g_i| \). In particular, if a counterexample \( B \) has a non-optimal face which has size 4 or 6, then \( n \leq 15 \) and it can be easily checked using (7) only that (7) holds for all such \( n \).

So we may further assume that all non-optimal \( F_i \) have \( k_i = 2 \). By (1) \( 2n = 2(l + 1) + 4n_{opt} \) where \( n_{opt} \) is the number of optimal faces in \( \mathcal{M}_0 \). So \( l = n - 1 - 2n_{opt} \) is even. By (10) \( l > 0 \), therefore \( l \in \{2, 4\} \).

Let us assume that \( n = |g_0| \geq |g_1| \geq \cdots \geq |g_l| \).

If \( l = 4 \), then \( (|g_1|, \ldots, |g_l|) = (5, 5, 3, 3) \) cannot equal \( (5, 5, 3, 3) \) or \( (7, 5, 3, 3) \) as in these cases by (11) \( n \) is 15 or 105 respectively, so by (9) the inequality (7) holds. For \( (|g_1|, \ldots, |g_l|) = (q, 5, 3, 3) \) with \( q \geq 9 \), we have by (9), that \( \text{ex} > 6n - 4 \). Since
the right side of (9) is increasing in $|h|$, it follows by (11) that $|g_1| = q$ and $|g_2| = |g_3| = |g_4| = 3$, where either $q = n$ or $n = 3q$ for some $q$ not a multiple of 3. In the former case $ex = 2(2n - 4) + 3(2n - 4n/3) = 6n - 8$, so (7) holds with equality. In the latter case
\[ ex = 2n - 4 + 2n - 4n/q + 3(2n - 4n/3) = 6n - 16. \]

Using (8) this is only possible for $n \equiv 1 \mod 4$, $3 \mid n$ and $9 \nmid n$, so in both cases (7) holds with equality as well.

It remains to consider the case $l = 2$. Suppose $|g_1| = q$ and $|g_2| = p$ with $p \leq q$. Recall that $p \geq 3$. By (11) we have $pq = n \gcd(p, q)$ and
\[ ex = 2n - 4 + 2n - \frac{4n}{q} + 2n - \frac{4n}{p} = 6n - 4 - 4\gcd(p, q). \]
where $a(p, q) = n/q + n/p$. As $a(p, q) = (p + q)/\gcd(p, q)$, it is even, so by (8) such excess is only possible for $n \equiv 3 \mod 4$.

Thus
\[ ex \geq 6n - 4 - 4 \max_{(p, q) \in S} a(p, q) \]
where the set $S$ consists pairs $(p, q)$ of divisors of $n$ with $\gcd(n/p, n/q) = 1$ and $p, q > 1$.

For a prime $y$ and a positive integer $x$, let $\text{mul}(x, y)$ denote the largest integer $k$ such that $y^k \mid x$. (11) implies that for each prime factor $r$ of $n$ either $\text{mul}(p, r) = \text{mul}(n, r)$ or $\text{mul}(q, r) = \text{mul}(n, r)$.

First note that $a(p, n) \leq a(p_1, n)$ for any divisor $p$ of $n$ such that $p > 1$. $(p, q) = (p_1, n)$ will be our first candidate to minimize $ex$. For such $(p, q)$ we have $ex = 6n - 4 - 4(1 + n/p_1)$ as in the second case of (7).

Now suppose $q < n$. If $\gcd(p, q) > 1$, let $r$ be a prime that divides both $q$ and $p$. If $\text{mul}(q, r) = \text{mul}(n, r)$ define $p' = p/r^{\text{mul}(p, r)}$ and notice that $a(p', q) > a(p, q)$ and $(p', q) \in S$. (To see that $p'$ cannot equal 1, note that this would imply $\text{mul}(q, r) = \text{mul}(n, r)$ for all $r \mid n$, so $q = n$.) Similarly, if $\text{mul}(p, r) = \text{mul}(n, r)$ we can define $q' = q/r^{\text{mul}(q, r)}$ and get $a(p, q') > a(p, q)$. Repeating this for all $r$ that divide $\gcd(p, q)$ we obtain $(p', q') \in S$ such that $a(p', q') > a(p, q)$, $\gcd(p', q') = 1$ and so $p'q' = n$.

Without loss of generality we can further focus on the set $S_1$ of pairs $(p, q) \in S$ with $p \leq q$, $pq = n$ and $\gcd(p, q) = 1$. When $p$ has at least two unique prime divisors, say $p = r^kb$ where $r$ is prime, $b > 1$ and $r \nmid b$, we can define $p' = p/r^k$ and $q' = qr^k$. Then $a(p', q') = p/r^k + qr^k > p + q = a(p, q)$. Thus if $q < n$
\[ a(p, q) \leq \max_p a(r^{\text{mul}(n, r)}, n/r^{\text{mul}(n, r)}) \]
where the maximum is over all prime divisors of \( n \).

Suppose the smallest prime divisor \( p_1 \) of \( n \) has multiplicity 1. If \( n \) is prime, then \( p_1 = p = q = n \) and \( a(p, q) = 2 \) is the only possibility, in which case (7) holds with equality. Otherwise, for \((p, q) \in S_1\), we have \( p_1 \leq p \leq \sqrt{n} \). Since \( f(s) = a(s, n/s) = s + n/s \) is decreasing in the interval \((0, \sqrt{n}]\), we have \( a(p_1, n/p_1) > a(p, n/p) \) when \( p > p_1 \). So when \( \text{mul}(p_1, n) = 1 \) and \((p, q) \in S_1\)

\[
a(p, q) \leq \max(a(p_1, n/p_1), a(p_1, n)) = a(p_1, n/p_1).
\]

Now suppose \( p_1^2 \) divides \( n \). Then a maximum \( a(p, q) \) over \((p, q) \in S_1\) could be attained either with \( p = r^t \) or \( q = r^t \) for some prime divisor \( r \) of \( n \), \( t = \text{mul}(n, r) \) and \( q = n/p \) or with \( p = p_1 \) and \( q = n \). We claim that it is always the latter. To see this, first note that this is true for \( n \) a prime power. If \( n \) is not a prime power, take \( r \) that maximizes \( a(r^t, n/r^t) \). Note that \( r^t < b \), where \( b = n/r^t \), otherwise we could replace \( r \) with another prime divisor \( r_2 \) of \( n \) and get a larger value of \( a \). So \( r < b \) and for \( t \geq 2 \)

\[
a(r, n) - a(r^t, n/r^t) = br^{t-1} + 1 - b - r^t
\]

and \( a(r^t, n/r^t) < a(r, n) \leq a(p_1, n) \). For \( t = 1 \), writing \( n = p_1^2rx \) we get

\[
a(p_1, n) - a(r, n/r) = p_1rx + 1 - p_1^2x - r
\]

\[
= p_1(r - p_1)x + 1 - r \geq p_1(r - p_1) + (1 - r)
\]

\[
= (p_1 - 1)(r - p_1 - 1) > 0.
\]

In both cases \( a(r^t, n/r^t) < a(p_1, n) \). This finishes the proof of (7).

Let us show that (7) is best possible. If \( n \) is prime or \( n \equiv 1 \pmod{4} \) and \( 3 \nmid n \) or \( 9 \mid n \), Lemma 3.1 gives a construction of genus \( L_C(n) \).

Suppose \( n \equiv 3 \pmod{4} \) and \( p \) is not prime. The first few such \( n \) are 15, 27, 35. If \( p_1^2 \nmid n \), let \((p, q) = (p_1, n/p_1)\), otherwise let \((p, q) = (p_1, n)\). Since \( n/p \) and \( n/q \) are relatively prime, by Euclid’s algorithm we can find \( g_1, g_2 \in \mathbb{Z}_n \) such that

\[
g_1 + g_2 + 1 \equiv 0 \pmod{n}, \quad |g_1| = p \quad \text{and} \quad |g_2| = q.
\]

For \( n = 15 \) the transition graph corresponding to genus \( L_C(15) = 49 \) embedding is given by

\[
C_1 = (0, 1, 2, -1, -2, 3, 4, -3, -4, 5, 6, 7, -5, -6, -7)
\]

\[
C_2 = (7, -6, 5, -4, 3, -2, 1, 0, 4, -3, -7, 6, -5, 2, -1).
\]
It was found by computer and inspired Lemma 3.2. It corresponds to choices 
\( g_1 = 5, g_2 = 9 \) \( a = (g_1 - 1)/2, b = a + 1, c = (g_2 - 1)/2, d = c + 1 \) and 
\( s_1 = (0, 1, \ldots, 7) \) in that lemma. For other \( n \), since they are larger than 25, we 
can apply Lemma 3.2.

Finally, suppose \( n \equiv 1 \pmod{4}, 3 \mid n \) but \( 9 \nmid n \). The smallest such \( n \) are 
21, 33, 57. Let \((p, q) = (3, n/3)\) and define \( g_1, g_2 \) by solving (12). Define \( g_3 = n/3 \) 
and \( g_4 = -n/3 \). Then

\[
g_1 + g_2 + g_3 + g_4 + 1 = 0.
\]

Construct a transition graph as in the proof of Lemma 3.2 but with the following 
modification: after choosing suitable \( a, b, c, d \) and \( e \), choose \( f \in \mathbb{Z}_n \), such 
that \( f \not\in \{x, x\} \) and \( f \not\in \{x-n/3, -x-n/3\} \) for \( x \in \{0, 1, a, b, c, d, e\} \), and 
\( -f \neq f + n/3 \). We have at least \( n - 27 \) possibilities for \( f \). Assuming \( n > 27 \), we 
construct a transition graph as in the proof of Lemma 3.2 but additionally require that 
\( v_m = f \) and \( v_{m+1} = f + n/3 \) for some \( m, l+1 < m < t \), where \( l, m \) and \( t \) are 
as in Lemma 3.2. This construction has the same properties as the construction for 
\( n = 4k+3 \), except now \( t \) is even and the paths \( P_3' \) and \( P_4' \) have their endpoints 
switched, so the dotted edges make up two cycles instead of a single Hamiltonian cycle. 
We can remedy this by replacing the dotted edges \((v_{m+1}, -v_m)\) and 
\((-v_{m+1}, v_m)\) with \((v_{m+1}, -v_m)\) and \((-v_{m+1}, -v_m)\). This eliminates one 
alternating 4-cycle and introduces two new 2-cycles \((v_m, v_{m+1})\) and 
\((-v_m, -v_{m+1})\) with net transition \( n/3 \) and \(-n/3 \) respectively. Thus in the corresponding 
voltage graph we have 2-faces of orders \( n/3, 3, 3, 3 \) and genus \( \mathcal{L}_C(n) \) as required. 
Finally, a similar construction also exists for \( n = 21 \). The transition graph is given by

\[
C_1 = (0, 1, 5, -1, -5, 9, 7, 6, 8, -9, -7, -6, -8, -2, 3, 10, -4, 2, -3, -10, 4)
\]
\[
C_2 = (4, 10, 3, 2, 5, -1, -4, -10, -3, -2, -8, 6, -7, 9, -5, 1, 0, 8, -6, 7, -9)
\]

\[
\square
\]

4 Proofs for 3-dimensional embeddings

4.1 Proof of the lower bound: the geometric part

All surfaces in this section will be embedded in \( \mathbb{R}^3 \), compact and piecewise linear. 
A piecewise linear curve, or simply a curve, on a surface \( S \) is a piecewise linear map 
\( f : [0, 1] \to S \). We call a curve a simple arc, or simply an arc, if it is injective. We 
call it a simple closed curve, or simply a closed curve, if it is injective on \( [0, 1] \) with 
an exception that \( f(0) = f(1) \). We call a curve simple if it is an arc or a closed
curve. The endpoints of a (closed or non-closed) curve $f$ are $f(0)$ and $f(1)$, and we say that $f$ goes from $f(0)$ to $f(1)$. We say that a curve $f$ is \textit{internally disjoint} from a set $X$ if $f$ is closed or $f(x) \notin X$ for all $x \in (0,1)$. For convenience, we make no distinction between a curve $f$ and its image $f([0,1])$ when it is clear from the context.

Let $a$ and $b$ be two simple curves on a surface $S$ such that no endpoint belongs to the other curve and $a \cap b$ is finite. Then each point in $x \in a \cap b$ can be classified as a touching or crossing, see [8,12] and references therein. Fix an orientation of $S$. Define the \textit{intersection sign} $\sigma(x,a,b) \in \{-1,0,1\}$ as shown in Figure 3. More formally, in a small neighbourhood $U \subset S$ of $x$, we can represent $a$ as a segment (i.e., an arc) $a_{in}$ that ends at $x$ and a segment $a_{out}$ that starts at $x$, so that $a_{in} \cap a_{out} = \{x\}$ and $a_{in} \cup a_{out} = a \cap U$. Similarly we can define $b_{in}$ and $b_{out}$. We set $\sigma(x,a,b) = -1$, if the cyclical clockwise ordering around $x$ of the segments incident to $x$ is $(a_{in}, b_{in}, a_{out}, b_{out})$. We set $\sigma(x,a,b) = 1$ if the ordering is $(b_{out}, a_{out}, a_{in}, b_{in})$. Otherwise, we set $\sigma(x,a,b) = 0$ (i.e. the intersection is a touching). In the case $x$ is an endpoint of either curve we may define $\sigma(x,a,b) = 0$; this case can always be avoided by a small local modification.

Let $a$ be a simple curve on an oriented surface $S$ and $C$ a finite set of pairwise disjoint simple curves on $S$. Suppose $a$ shares with each curve in $C$ a finite number of points. We define

$$\sigma(a,C) = \sum_{b \in C} \sum_{x \in a \cap b} \sigma(x,a,b).$$

Now let $\mathcal{M}$ be an embedding of a directed multigraph $G$ into a connected oriented surface $S \subset \mathbb{R}^3$, and let $C$ be a set of pairwise disjoint simple curves on $S$. For $e \in E(G)$ let $\mathcal{M}(e)$ be the simple curve on $S$ assigned to $e$ by $\mathcal{M}$. Given a positive integer $n$, the \textit{voltage assignment} function $\alpha_{\mathcal{M},C,n} : E(G) \to \mathbb{Z}_n$ is well defined if for each edge $e \in E(G)$, $\mathcal{M}(e)$ shares with each curve in $C$ a finite number of points and $\mathcal{M}(G)$ is disjoint from the endpoints of each curve in $C$. In that case for $e \in E(G)$ we set

$$\alpha_{\mathcal{M},C,n}(e) = \sigma(\mathcal{M}(e),C) \mod n.$$  

Figure 3: The sign $\sigma(x,a,b)$ of the intersection point $x$ between a solid black curve $a$ and a dotted red curve $b$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3}
\caption{The sign $\sigma(x,a,b)$ of the intersection point $x$ between a solid black curve $a$ and a dotted red curve $b$.}
\end{figure}
To study symmetric embeddings, and more generally immersions, of a surface into $\mathbb{R}^3$ we can use the following approach, see [11]. Given a surface $S' \subset \mathbb{R}^3$ which is invariant under the action of a finite subgroup $\Gamma$ of orientation preserving Euclidean isometries in $\mathbb{R}^3$, we can define the quotient mapping $q$ from $\mathbb{R}^3$ to the orbifold $\mathbb{R}^3/\Gamma$, the quotient space obtained by identifying each orbit of $\Gamma$ to a single point and equipped with a suitable topology. For $n$-fold rotational symmetry the corresponding quotient surface, or orbifold, $S'/\Gamma$ is a closed and orientable surface, and the map $q$ restricted to $S$ is a branched covering of $S'/\Gamma$ with an even number of branch points.

We additionally have an embedding of a graph $G'$ into $S'$ and we ask that this embedding is invariant under $\Gamma$. As above, we can then define an embedding of a quotient graph $G$ in $S'/\Gamma$. We will give a proof of the next rather straightforward proposition as we were unable to find a suitable equivalent in the literature.

**Proposition 4.1** Let $n \geq 3$ and $g' \geq 0$ be integers, and let $G$ be a directed connected multigraph and $G'$ a connected multigraph. The two statements below are equivalent.

(a) There is a piecewise linear embedding $M'$ of $G'$ into a closed connected oriented surface $S' \subset \mathbb{R}^3$ of genus $g'$ such that $M'$ has $n$-fold rotational symmetry, some vertex of $S'$ is a fixed point under the rotation, but no vertex of $G'$ is mapped to a fixed point.

(b) There is a piecewise linear embedding $M$ of $G$ into a closed connected oriented surface $S \subset \mathbb{R}^3$ together with a set $C$ of pairwise disjoint arcs or closed curves on $S$, such that the number of arcs (non-closed curves) in $C$ is $t \geq 1$, the genus $g$ of $S$ satisfies $ng + (n - 1)(t - 1) = g'$, $\alpha_{M,C,n}$ is well defined and $G'$ is isomorphic to the graph derived from the voltage graph $(G, \alpha_{M,C,n})$ and voltage group $\mathbb{Z}_n$.

For a simple example with $G'$ a complete bipartite graph $K_{3,3}$ and $G$ a dipole $D_3$ see Figure 6. In the special case when $M$ is 2-cell, $M'$ is just the derived embedding from the embedded voltage graph $(G, M, S, \alpha_{M,C,n})$ realised in $\mathbb{R}^3$ and voltage group $\mathbb{Z}_n$. When $S'$ has no fixed points under the rotation, our proof below shows that (a) still implies (b) with $t = 0$, however in the opposite direction we can end up with a disconnected surface $S'$ even if $S$ is connected.

**Proof** In this proof we only deal with oriented surfaces $S \subset \mathbb{R}^3$ where each boundary component can be represented by a closed curve $c$ on $S$. An arbitrary point $x \in c$ can be chosen as both the start and the end point. $c$ can be given two possible directions. The intersection of $c$ with a sufficiently small neighbourhood
around $x$ (which is homeomorphic to a half-disk) in $S$ can be represented by a segment $c_{\text{out}}$ outgoing from $x$ and a segment $c_{\text{in}}$ incoming to $x$. The clockwise ordering of these segments at $x$ does not depend on $x$. If the ordering is $(c_{\text{out}}, c_{\text{in}})$ we say that the points of $S$ are on the right of $c$. Otherwise, if the ordering is $(c_{\text{in}}, c_{\text{out}})$, we say that the points of $S$ are on the left of $c$.

**Proof of (a) $\implies$ (b).** Let $a$ be the $n$-fold rotation symmetry axis of $M'$. Let $r : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation around $a$ by $2\pi/n$. Fix any $n$ half-planes $P_0, \ldots, P_{n-1}$ containing $a$ as their common boundary, such that $r(P_j) = P_{(j+1) \mod n}$. Denote $X_j = P_j \cap S'$. $P_0, \ldots, P_{n-1}$ make up the boundaries of $n$ infinite prisms $\Pi_0, \ldots, \Pi_{n-1}$ with pairwise disjoint interiors, where $\Pi_j$ is bounded by (and includes the points of) $P_j$ and $P_{(j+1) \mod n}$. Let $S_j = \Pi_j \cap S'$.

Since $M'$ is piecewise linear and $S'$ is a surface, we can choose $P_0, \ldots, P_{n-1}$ so that

- For $j \in \{0, \ldots, n-1\}$, $X_j$ is a union of a finite number of (images of) pairwise disjoint simple curves in $S_j$ (and $S'$). We assume the direction of each curve $c \in X_j$ is chosen so that the points of $S_j$ are on the right of $c$.

- Each arc in $X_0$ starts and ends in $a$, and each curve in $X_0$ is internally disjoint from $a$. Define $B = S \cap a = X_0 \cap a$. Then $|B|$ is even and $|B| = 2t$ where $t$ is the number of arcs in $X_0$.

- The curves in $X_0$ are internally disjoint from $M'(V(G'))$.

- For each $e \in E(G')$, the curve $\mathcal{M}'(e)$ shares with $X_0$ a finite number of points.

Now for each $e \in E(G')$, the curve $c_{e} = \mathcal{M}'(e)$ is disjoint from $B$. Indeed, suppose on the contrary that $a$ and $\mathcal{M}'(e)$ contain a point $x$ in common. Then $x$ must belong to the interior of $c_{e}$, since $\mathcal{M}'(V(G'))$ contains no fixed points. If $c_{e}$ is tangent to $a$ at $x$, then, by considering the orbit (under the rotation) of the region incident to $e$, we see that since $n \geq 3$, $S$ is not locally homeomorphic to the Euclidean plane at $x$ (see, e.g., proof of Theorem 78.1 of [13]). Otherwise, by considering the orbit of $c_{e}$ in the neighbourhood of $x$, we see that $\mathcal{M}'$ is not a proper embedding, a contradiction.

Identifying each point $x \in X_0$ with $r(x) \in X_1$ we obtain from $S_0$ a quotient surface $S$. $S$ can be represented as a piecewise linear embedding in $\mathbb{R}^3$, for example, as the image of $S_0$ under a piecewise linear approximation $f$ of a function $f : \Pi_0 \to \mathbb{R}^3$, where $f$ rotates $x$ around $a$ by angle $n\theta$ if $x$ is at angle $\theta$ with $P_0$. We further assume that $S = f(S_0)$. Note that $C = f(X_0) := \{f \circ c : c \in X_0\}$ is then a finite set of arcs (we call them cuts) and closed curves (we call them cut loops) on $S$.  

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Let $p$ map $x \in \mathbb{R}^3$ to the unique point $p(x) \in \Pi_0 \setminus P_1$ in the orbit of $x$ under the $n$-fold rotation and let $h = f \circ p$. Restricted to $S'$, $h$ is a branched covering, the covered surface is $\tilde{S}$ and $B$ is the set of branch points, each of order $n$. It is easy to see that $\tilde{S}$ is closed and orientable. Since $S'$ contains a fixed point under the rotation, it follows that $\tilde{S}$ is connected.

Since $\mathcal{M}'(V(G'))$ avoids fixed points, we have that this set is mapped to a set $V$ of size $|V(G')|/n$ by $h$.

For any $e \in E(G')$, $\mathcal{M}'(e)$ can be considered a simple curve, so we can define a function $m_e$ by $m_e(x) = h(\mathcal{M}'(e)(x)) x \in [0, 1]$. Since $\mathcal{M}'(e)$ avoids $B$, $m_e$ is a simple curve.

As $n \geq 3$, the orbit of $\mathcal{M}'(e)$ under the rotation corresponds to $n$ distinct edges of $G'$, all of which are mapped to $m_e$ by $h$ (switch the direction of each such $\mathcal{M}'(f)$ to match the direction of $m_e$). Thus the images $m_e$ for $e \in E(G)$ define an embedding $\mathcal{M}$ of a connected directed graph $G$ with $|V(G')|/n$ vertices mapped to $V$ and with $|E(G)|/n$ edges into $S$.

Now for any $e \in E(G')$ with $e = st$ and $m_e$ going from $h(s)$ to $h(t)$ follow the curve $\mathcal{M}'(e)$ from $s$ to $t$. By the definition of $\sigma$, for any $i \in \mathbb{Z}_n$, $s \in \{-1, 1\}$ we cross from $S_i$ to $S_{(i+s) \mod n}$ at a point $x \in S'$ if and only if $\sigma(h(x), m_e, h \circ c) = s$, where $c$ is the curve of $X_i$ containing $x$. Summing over all points of intersection of $m_e$ with $X_0$ in $S$, we get that an edge $e \in E(G')$ with $\mathcal{M}'(e)$ an arc from $r^\gamma(u)$ to $r^\gamma(v)$ corresponds to an arc $m_e$ from $u$ to $v$ in $G'$ such that $\gamma = \beta + \sigma(m_e, C) \mod n$. So $G'$ is the graph derived from $(G, \alpha, \mathcal{M}, C, n)$ and the voltage group $\mathbb{Z}_n$.

Since $h$ has $2t$ branch points of order $n$, by the Riemann-Hurwitz formula, see, e.g., [14], the genus $g$ of $\tilde{S}$ satisfies $g' = ng + (n - 1)(t - 1)$.

Proof of (b) $\implies$ (a). Cut $S$ at each $c \in C$. We obtain a possibly disconnected surface $\tilde{S}$ with boundary, where each closed curve $c$ corresponds to two boundary components and each arc in $C$ to one boundary component. More precisely, by the surface classification theorem (e.g., [13]), there exists a piecewise linear map $h : \mathbb{R}^3 \to \mathbb{R}^3$ and a piecewise linear surface $\tilde{S}$ with boundary such that $h(\tilde{S}) = S'$, restricted to $\tilde{S} \setminus \cup_{c \in C} c$, $h$ is a homeomorphism and $\tilde{S}$ is a sphere with some number $a \leq g$ of simple handles and $b = 2(|C| - t) + t = 2|C| - t$ boundary components.

Fix an orientation of $\tilde{S}$ which agrees with the orientation of $S$ in the sense that rotations of any embedded graph are preserved under $h$. The direction of a curve $c \in C$ induces the direction of the boundary cycles of $\tilde{S}$: for a closed curve we obtain one boundary cycle $c_L$, $h(c_L) = c$ where the points of $\tilde{S}$ are on the right of $c_L$. Similarly we have another cycle $c'_R$, $h(c_R) = c$, where points of $\tilde{S}$ are on the left of $c_R$. For an arc $c$ we have only one boundary cycle $c'$ in $\tilde{S}$ such that
Theorem 4.2 Let $\mathcal{M}$ be a piecewise linear embedding of a graph $G$ into a closed connected orientable surface $S \subset \mathbb{R}^3$ such that there is a face $F_H$ bounded by a Hamiltonian cycle. Suppose $n = |V(G)| \geq 3$ and $\mathcal{M}$ has $n$-fold rotational symmetry that leaves the boundary of $F_H$ invariant. Then no point in $\mathcal{M}(V(G))$ lies on the rotational symmetry axis $a$.

Proof If $x = \mathcal{M}(v)$ is a fixed point, then consider the orbit of the curve $c_e = \mathcal{M}(e)$ such that $e$ is on the boundary of $F_H$. We get that each of $n$ edges incident
to \( v \) belongs to the boundary of \( H \). This is a contradiction, since \( H \) contains \( v \) only once.

### 4.2 Proof of the lower bound: the combinatorial part

Recall that a region of an embedding \( \mathcal{M} \) of \( G \) to \( S \) consists of all elements of an equivalence class of points that can be connected by a simple arc in \( S \setminus \mathcal{M}(G) \). A face is a region that is homeomorphic to an open disk in \( \mathbb{R}^2 \). We call the sum of lengths of all boundary walks of a region \( R \) the (boundary) size of \( R \). We call \( R \) a \( k \)-region, if its size is \( k \).

**Lemma 4.3** Let \( n \geq 3 \) be an integer. Let \( \mathcal{M} \) be a piecewise linear embedding of \( D_n \) into a closed connected oriented surface \( S \subset \mathbb{R}^3 \) of genus \( g \). Let \( C \) be a finite set of pairwise disjoint simple curves in \( S \), \( t \) of which are arcs, and \( |C| - t \) of which are closed curves. Suppose \( t \geq 1 \) and the voltage function \( \alpha_{\mathcal{M},C,n} : E(D_n) \to \mathbb{Z}_n \) is well defined and bijective.

Then \( ng + (n - 1)(t - 1) \geq L_C^*(n) \).

**Proof** Let \( \alpha = \alpha_{\mathcal{M},C,n} \). Suppose the claim is false, and take \( \mathcal{M} \) and \( C \) such that \( t \geq 1 \), \( \alpha \) is bijective and \( ng + (n - 1)(t - 1) < L_C^*(n) \). Then since \( (n - 1)(t - 1) \geq 0 \) and \( t, g, n \) and \( L_C^*(n) \) are integers

\[
g \leq \left\lfloor \frac{L_C^*(n) - 1}{n} \right\rfloor = \begin{cases} 
\frac{n-4}{4} & \text{for } n \equiv 0 \pmod{4}, \\
\frac{n-5}{4} & \text{for } n \equiv 1 \pmod{4}, \\
\frac{n-6}{4} & \text{for } n \equiv 2 \pmod{4}, \\
\frac{n-3}{4} & \text{for } n \equiv 3 \pmod{4}.
\end{cases} \tag{14}
\]

Since \( D_n \) is bipartite, each region of \( \mathcal{M} \) must have an even size. Let \( k \) be the total number of 2-regions. \( D_n \) has \( n \) edges and 2 vertices. It follows by Euler’s formula that the number \( r \) of regions is

\[
r \geq 2 - 2g - 2 + n = n - 2g.
\]

Assume first \( k < n - 2g \). Each edge either belongs to the boundary walks of two regions or to the boundary walks of the same region twice. Thus, the average size of those regions that have size at least 4 is

\[
\frac{2n - 2k}{r - k} \leq \frac{2n - 2k}{n - 2g - k}.
\]
On the other hand, this average must be at least 4. Thus

\[ 2n - 2k \geq 4(n - 2g - k) \]
\[ 2k \geq 2n - 8g \]
\[ k \geq n - 4g. \] \hspace{1cm} (15)

The last inequality trivially holds also when \( k \geq n - 2g \).

By our initial assumption, the number of arcs in \( C \) satisfies

\[ t < 1 + \frac{L^*_C(n) - ng}{n - 1}. \] \hspace{1cm} (16)

Let \( R \) be a 2-region with boundary consisting of arcs \( b = M(e_1) \) and \( c = M(e_2) \) from the white vertex to the black vertex. \( e_1 \) and \( e_2 \) must be different edges, since \( n \geq 2 \). We use the following simple property: if a curve \( a \in C \) does not have an endpoint in \( R \), then \( \sigma(M, b, a) = \sigma(M, c, a) \). If \( a \) has just one endpoint in \( R \), then \( |\sigma(M, b, a) - \sigma(M, c, a)| = 1 \). This property is a simple consequence of the definition of a region, the definition of \( \sigma \) and the fact that \( \alpha \) is well-defined (recall that we work in the piecewise-linear setting).

Thus if no arc in \( C \) has an endpoint in \( R \), then \( \alpha(e_1) = \alpha(e_2) \), which contradicts the assumption that \( \alpha \) is bijective. This implies that each 2-region must have at least one of \( 2t \) total endpoints. Since regions are pairwise disjoint sets, by the pigeonhole principle we must have \( 2t \geq k \).

In the case \( n \equiv 3 \pmod{4} \), suppose \( g = (n - 3)/4 \). Then (16) implies \( t = 1 \), so there are just two endpoints, but by (15) the number of 2-regions is at least 3, a contradiction. Thus we can further assume

\[ g \leq \frac{n - 3}{4} - 1 \text{ for } n \equiv 3 \pmod{4}. \] \hspace{1cm} (17)

From (15) and (16) we have

\[ n - 4g \leq k \leq 2t < 2 + \frac{2L^*_C(n) - 2ng}{n - 1} \]

or

\[ g > \frac{(n - 2)(n - 1) - 2L^*_C(n)}{2n - 4} = \begin{cases} 
\frac{n - 4}{4} & \text{for } n = 0 \pmod{4}, \\
\frac{n - 5}{4} + \frac{1}{2} - \frac{1}{2(n - 2)} & \text{for } n = 1 \pmod{4}, \\
\frac{n - 4}{4} + 1, & \text{for } n = 2 \pmod{4}, \\
\frac{n - 3}{4} - \frac{1}{2} - \frac{1}{2(n - 2)} & \text{for } n = 3 \pmod{4}.
\end{cases} \]

In all cases we have a contradiction to (14) and (17). \( \square \)
4.3 Proof of the upper bound: optimal ring road constructions

We represent rotations of an embedding of a directed multigraph as cyclic sequences of symbols $e_{i_1}^{s_1} \ldots e_{i_k}^{s_k}$. We have $s_j = -1$ if the edge $e_{i_j}$ is incoming and $s_j = 1$ otherwise, also we shorten $e_i = e_1^1$. Similarly we denote faces of embedded directed multigraphs as cyclic permutations of the same form. In this case the symbol $e^{-1}$ indicates that the corresponding arc $M(e)$ in the counter-clockwise facial walk is traversed in the opposite direction.

For a degree 2 vertex $x$ in a multigraph smoothing $x$ means connecting the neighbours of $x$ by an edge and removing $x$; this is extended to embedded or directed multigraphs in the obvious way.

**Lemma 4.4** For each integer $n \geq 2$ there is a 3-dimensional piecewise linear complete interchange that has $n$-fold rotational symmetry and genus at most $L_C^*(n)$.

**Proof** Consider a compact piecewise-linear orientable surface $B = B(n)$ embedded in $\mathbb{R}^3$, obtained by adding some number $g$, $g \geq 0$, handles to a rectangle with boundary $abcd$. Place a black point $v_b$ and a white point $v_w$ on the boundary segment $da$, so that $d, v_w, v_b, a$ are all distinct and occur in this order when going from $d$ to $a$.

Take a permutation $\pi = \pi(n) = (\pi_1, \ldots, \pi_{n-1})$ of $\{1, \ldots, n-1\}$ and place points $\pi_{n-1}, \ldots, \pi_1$ on the (interior of the) segment $cd$ in this order. Similarly, take a permutation $\pi' = \pi'(n) = (\pi'_1, \ldots, \pi'_{n-1})$ of $\{0, \ldots, n-2\}$ which is related to the permutation $\pi$ through

$$\pi'_i = \pi_i - 1 \quad \text{for } i \in \{1, n-1\}. \quad (18)$$

Place points $\pi'_1, \ldots, \pi'_{n-1}$ on the segment $ab$ in this order.

The point $\pi_i$, if $\pi_i = k$, will model $n - k$ lanes incoming from motorways $K, K-1, \ldots, K-(n-k)+1$ and exiting to motorway $K+k$. Here we think of motorways as indexed by $\mathbb{Z}_n$ and arranged in the counter-clockwise order, so that $K$ is (the index of) the current motorway, $K+1$ is the motorway to its right, etc.

Suppose there exists an embedding $\mathcal{M} = \mathcal{M}(n)$ of a graph $G = G(n)$ with vertices mapped to $\pi_1, \ldots, \pi_n, \pi'_1, \ldots, \pi'_{n-1}, v_w, v_b$ and edges mapped to the following arcs in $B$:

(a) $\pi'_iv_b$, where $\pi'_i = 0$ (i.e., the unique group of $n-1$ lanes exiting at the current motorway $K$),

\[1\]In order to keep the notation simpler, we use symbols $\pi_1, \ldots, \pi_n$ and $\pi'_1, \ldots, \pi'_n$ to refer both to two disjoint sets of points and to orderings of two overlapping sequences. The precise meaning should be interpreted from the context.
(b) $v_w \pi_i$, for each $i \in \{1, \ldots, n-1\}$ (i.e., the lanes from the current motorway $K$ to each of the $n - 1$ groups of lanes on the ring road with different destinations),

c) for every $k \in \{1, \ldots, n-2\}$ the unique arc $\pi'_i \pi_j$ such that $\pi'_i = \pi_j = k$ (i.e. the group of lanes from motorways $K - 1, \ldots, K - (n-k) + 1$ that continue on the ring road passing the current motorway $K$).

Expand each arc $\pi'_i \pi_j$ defined in (c), if, say, $\pi'_i = \pi_j = k$ into $n - k - 1$ disjoint 'parallel' arcs. Also expand the point $\pi'_i$ into $n - k - 1$ points, the starting points of these arcs, also placed on the boundary segment $ab$. By our definition, $\pi_i = \pi'_i + 1 = k + 1$. Replace the point $\pi_i$ by $n - k - 1$ new points on the segment $cd$ (so that $n - (k + 1) - 1$ of them are endpoints of the parallel arcs, and one point is the endpoint of the arc $v_w \pi_i$), so that the order of these points corresponds to the rotation at $\pi_i$. Next, add an edge $v_w v_b$ with its image subset of the segment $da$.

We get an embedding $\mathcal{M}'$ of a new (expanded) graph $G'$ into $B$ with degree 1 vertices corresponding to $n - 1 + \cdots + 1 = n(n - 1)/2$ points on each of the boundary segments $ab$ and $cd$, 2 vertices of degree $n$ corresponding to $v_w$ and $v_b$, and the arcs consisting of the union of $v_w v_b$ and the arcs (a), (b) and (c) replaced using the above modification.

Now make $n$ copies of $(B, \mathcal{M}', G')$, $B'_{0}, \ldots, B'_{n-1}$ and arrange them (using rotations and translations) counterclockwise into a 'ring', so that each block is disjoint and we have $n$-fold rotational symmetry. For $i \in \mathbb{Z}_n$ identify the segment $b(i+1) a(i+1)$ of block $B'_{i+1}$ and the segment $d(i) c(i)$ of $B'_i$ with the corresponding sides of a new polygon $P_i = a(i+1) b(i+1) c(i) d(i)$; also connect the $k$-th point on $d(i) c(i)$ with the $k$-th point of $a(i) b(i)$, using a straight arc inside $P_i$ for each $k \in \{1, \ldots, n(n-1)/2\}$. Make the surface closed by gluing a genus 0 surface with boundary $b(0) c(0) \ldots b(n-1) c(n-1)$ and a genus 0 surface with boundary $d(n-1) a(n-1) \ldots d(0) a(0)$. It easy to ensure the n-fold symmetry is preserved and newly added faces are piecewise linear.

For any $K, k \in \mathbb{Z}_n$ and $k \neq 0$ follow the path $P_{K,k}$ from $v_w^{(K)}$ in $B'_K$, such that the first edge on this path corresponds to $v_w \pi_i$, with $\pi_i = k$ and ending at the first copy of $v_b$, say $v_b^{(j)}$ in $B'_j$. Each internal vertex on $P_{K,k}$ has degree two. Using (18) and simple induction we see that $j \equiv K + k \pmod n$. We get a 3-dimensional embedding of a subdivision of $K_{n,n}$ with $n$-fold rotation symmetry into a surface of genus $ng$. To ensure the underlying embedding of $K_{n,n}$ has a face bounded by a Hamiltonian cycle $v_w^{(n-1)} v_b^{(n-1)} \ldots v_w^{(0)} v_b^{(0)}$ that contains a fixed point, for each $i \in \mathbb{Z}_n$ remove the path connecting $v_w^{(i)}$ with $v_b^{(i+1)}$ and add a new arc $v_w^{(i)} v_b^{(i+1)}$ that coincides with parts of the boundary segments $a(i) d(i)$ and $a(i+1) d(i+1)$ of $B'_i$.
Figure 4: The solution for $n = 6$.

and $B_{i+1}$ respectively.

Smooth all the degree 2 vertices in the resulting graph that are not copies of $v_w$ or $v_b$. We have obtained an embedding of $K_{n,n}$ with an $n$-fold rotation symmetry into a surface of genus $gn$. We call this embedding the ring road embedding derived from $(B, G, M)$.

To finish the proof for $n \mod 4 \in \{1, 2\}$ it suffices to show that the surface $B(n)$ of genus $g = g(n)$, the graph $G(n)$ and its embedding $M(n)$ into $B(n)$ exist with

$$g \leq \frac{L^*(n)}{n} = \begin{cases} \frac{n-1}{4} & \text{for } n \equiv 1 \pmod{4}; \\ \frac{n-2}{4} & \text{for } n \equiv 2 \pmod{4}. \end{cases}$$

(19)

Case $n \equiv 2 \pmod{4}$. The construction for $n = 2$ is trivial, so we assume $n \geq 6$. The building block of our construction is a (the?) solution for $n = 6$ with $\pi = \pi^{(6)} = (1, 4, 3, 2, 5), \pi' = \pi^{(6)'} = (0, 3, 2, 1, 4)$, and the rotations:

$$\begin{align*}
\pi_1 : v_w & v_4' & \pi_2 : \pi_1' v_w & \pi_3 : v_w \pi_2' & \pi_4 : \pi_3' v_w \\
\pi_2' : \pi_3 & \pi_3' : \pi_4 & \pi_4' : \pi_1 & \pi_5' : \pi_2 \\
\pi_5 : v_w & \pi_1' : \pi_b & v_w : \pi_3 \pi_2 \pi_5 \pi_4 \pi_1 & v_b : \pi_1'.
\end{align*}$$

This embedding has genus $(6 - 2)/4 = 1$ as required. It is shown in Figure 4. We construct a solution for arbitrary $n, n \equiv 2 \pmod{4}$ by concatenating $(n-2)/4$ copies of the solution for $n = 6$, see Figure 5 right.

The concatenation is defined as follows. We use permutations $\pi = \pi^{(n)}$ and $\pi' = \pi'^{(n)}$ as in (18), where

$$\begin{align*}
\pi_{4t+1} &= 4t + 1, & \pi_{4t+2} &= 4t + 4, & \pi_{4t+3} &= 4t + 3, \\
\pi_{4t+4} &= 4t + 2 & & \text{for } t = 0, \ldots, [n/4] - 1 \text{ and} \\
\pi_{n-1} &= n - 1.
\end{align*}$$

(20)
Figure 5: Construction of a block $\mathcal{B} = \mathcal{B}(n)$ from block $\mathcal{B}(6)$, $n \equiv 1 \pmod{4}$ (left) and $n \equiv 2 \pmod{4}$ (right).
We aim to construct an embedding with the following rotations for \( t \in \{0, \ldots, (n-2)/4 - 1\} \):

\[
\begin{align*}
\pi_{4t+1} & : v_w \pi_{4t+4}^t \\
\pi_{4t+2} & : \pi_{4(t+1)+1}^t v_w \\
\pi_{4t+3} & : \pi_{4t+2}^t v_w \\
\pi_{4t+4} & : \pi_{4t+3}^t v_w \\
\pi_{4t+2}^t & : \pi_{4t+3} \\
\pi_{4t+3} & : \pi_{4t+4} \\
\pi_{4t+4} & : \pi_{4t+1} \\
\pi_{4(t+1)+1} & : \pi_{4t+2} \\
\end{align*}
\]

and

\[
\begin{align*}
\pi_1 & : v_b \\
v_b & : \pi_1^t \\
\pi_{n-1} & : v_w.
\end{align*}
\]

We claim that for each \( n \) such that \( n = 4k + 2 \), \( B(n) \) and \( M(n) \) can be chosen so that (19), (20) and (21) are satisfied. To prove this, we use induction on \( k \). We have already shown the claim for \( k = 1 \) with embedding \( M(6) \) into a surface \( B(6) \) of genus 1 with boundary rectangle \( abcd \). Let \( n = 4k + 2 \) with \( k \geq 2 \) and suppose the claim is true for \( n = 4(k-1) + 2 \), i.e., there is a required embedding \( M(n-4) \) into a surface \( B(n-4) \) with boundary rectangle \( a'b'c'd' \) with permutations \( \pi = \pi^{(n-4)} \), \( \pi' = \pi'^{(n-4)} \) as in (20), rotations (21), rotation at \( v_w = v_w^{(n-4)} \):

\[
v_w : \pi_{\sigma_1}^{(n-4)} \cdots \pi_{\sigma_{n-4-1}}^{(n-4)}
\]

and genus at most \( L^*_c(n-4)/(n-4) = L^*_c(n)/n - 1 \). Now consider \( \pi^{(n-4)} \) and \( \pi'^{(n-4)} \) as permutations of \( \{5, \ldots, n-1\} \) and \( \{4, \ldots, n-2\} \) respectively, i.e., simply shift the sets permuted by \( \pi^{(n-4)} \) and \( \pi'^{(n-4)} \) by 4.

Identify \( bc \) with \( a'd' \) and identify the points (and corresponding vertices) \( v_w^{(n-4)} \) with \( \pi_5^{(6)} \) and \( v_b^{(n-4)} \) with \( \pi_5^{(6)} \). Here \( v_w^{(t)} \) and \( v_b^{(t)} \) denote the points \( v_w \) and \( v_b \) respectively in the corresponding embedding \( M(t) \). Contract the edge mapped to \( \pi_5^{(6)} \pi_2^{(6)} \) and the edge mapped to \( \pi_5^{(6)} v_w^{(n-4)} \) (smoothing the point \( \pi_5^{(6)} \) and pulling the point \( \pi_5^{(6)} \) to \( v_b \)). We get a construction with boundary rectangle \( ab'c'd' \), the permutation

\[
\tilde{\pi} = (\pi_1^{(6)}, \ldots, \pi_4^{(6)}, \pi_1^{(n-4)}, \ldots, \pi_{n-4-1}^{(n-4)})
\]

on the segment \( dc' \), and the permutation

\[
\tilde{\pi}' = (\pi_1^{(6)}, \ldots, \pi_4^{(6)}, \pi_1^{(n-4)}, \ldots, \pi_{n-4-1}^{(n-4)})
\]

on the segment \( ab' \) and the following rotation at \( v_w^{(t)} \):

\[
v_w : \pi_3^{(6)} \pi_2^{(6)} \pi_1^{(n-4)} \cdots \pi_{\sigma_{n-4-1}}^{(n-4)} \pi_4^{(6)} \pi_1^{(6)}.
\]

The rotation system of the resulting embedding \( \tilde{M} \) satisfies (20) and (21) with \( \pi = \tilde{\pi} \) and \( \pi' = \tilde{\pi}' \). Finally, the genus of \( \tilde{M} \) is at most \( 1 + L^*_c(n)/n - 1 = L^*_c(n)/n. \)

\(^2\)Note that for points on the boundary, rotations are not identical modulo cyclic shifts.
Figure 6: (a) The embedding $\mathcal{M}_1(3)$ of $D_3$ and a set $C$ of two cut arcs (dashed), such that $\alpha_{\mathcal{M}_1,C,3} : E(D_3) \to \mathbb{Z}_3$ is bijective. (b) The resulting 3-fold symmetric embedding of $K_{3,3}$ with a face bounded by a Hamiltonian cycle.

Case $n \equiv 1 \pmod{4}$. We define $\pi = \pi^{(n)}$ as in (20), except we drop the last equality $\pi_{n-1} = n - 1$. We define the rotations for $\pi_1, \ldots, \pi_{n-1}, \pi'_1, \ldots, \pi'_{n-1}$ and $v_b$ as in (21), except we drop the last equality, and for $t = (n - 1)/4 - 1$, we replace the rotation at $\pi_{4t+2} = \pi_{n-3}$ with:

$\pi_{4t+2} : v_w$.

The rest of the proof is analogous to the proof in the case $n \equiv 2 \pmod{4}$, so we omit it. Figure 5, left, illustrates the construction.

Case $n \equiv 3 \pmod{4}$. For $n = 3$ we use Proposition 4.1 with the specific embedding $\mathcal{M}_1(3)$ of $D_3$ into a closed surface $S_1(3)$ of genus 0 shown in Figure 6. For $n \geq 7$, start with the embedding $\mathcal{M}(n - 1)$ given above. Expand it into the embedding $(B, \mathcal{M}', G')$ with boundary $abcd$ as described in the first part of the proof. Identify the segment $ab$ with the segment $dc$. Also identify the $k$th point on $ab$ with the $k$th point of $dc$ for $k \in \{1, \ldots, (n - 1)(n - 2)/2\}$, and smooth each of the identified points. Cap the two holes with boundaries $cb$ and $da$ respectively by gluing a new 1-face to each of them. We obtain an embedding $\mathcal{M}_1 = \mathcal{M}_1(n - 1)$ of a copy $G_1$ of $D_{n-1}$ into the surface $S_1 = S_1(n - 1) \subset \mathbb{R}^3$ of genus $g \leq L^*_C(n - 1)/(n - 1) = (n - 3)/4$. It is easy to ensure that $\mathcal{M}_1$ is piecewise-linear: for example, if we bend the rectangle $abcd$ into a cylinder and then take its piecewise-linear approximation. Let $C_1 = \{X_1\}$ where $X_1$ is a cut (an arc in $S_1$) that coincides with the segment $dc$. If we apply Proposition 4.1 with $S_1, \mathcal{M}_1$ and the group $\mathbb{Z}_{n-1}$, we obtain the solution for $n - 1 \equiv 2 \pmod{4}$ roads that we described above. Let us instead apply Proposition 4.1 with the group $\mathbb{Z}_n$. The corresponding voltage function $\alpha_{\mathcal{M}_1,C_1,n} : E(G_1) \to \mathbb{Z}_n$ is injective; indeed by our construction the outgoing arc $e_k$ that corresponds to $v_w \pi_i$, $\pi_i = k$ crosses...
X_1 exactly k times, each time with sign 1, so \( \sigma(e_k, C_1) = k \). Thus the image of \( \alpha_{M_1, C_1, n} \) is \( \{0, \ldots, n - 2\} \).

Insert a new edge \( e_{-1} \) to \( G_1 \) and map it to an arc from \( v_w \) to \( v_b \) parallel to \( M_1(e_0) \), so that the rotation at \( v_w \) is \( e_0e_{-1} \ldots \) and the rotation at \( v_b \) is \( \ldots e_{-1}^{-1}e_0^{-1} \ldots \). Now add a new cut \( X_2 \) disjoint from the image of \( G_1 \), that has only one point in common with \( e_{-1} \) and \( \sigma(e_{-1}, X_2) = -1 \). We obtained an embedding \( M \) of \( D_n \) into the surface \( S_1 \) of genus \( g \leq (n - 3)/4 \) and \( t = 2 \) cuts, so that \( \alpha_{M_1(X_1, X_2), n} : E(D_n) \to \mathbb{Z}_n \) is bijective. Now the proof follows by Proposition [4.1].

Case \( n \equiv 0 \pmod{4} \). The solution for \( n = 4 \) is exceptional, so we assume \( n \geq 8 \). We first determine the rotation at \( v_b \) in the embedding \( M_1(n - 2) \) of the copy \( G_1 \) of \( D_n-2 \) into the ‘quotient’ surface \( S_1(n - 2) \).

Claim 4.5 For \( n \equiv 2 \pmod{4} \) in the embedding \( M_1(n) \) the rotation at \( v_b \) is
\[
v_b : e_0^{-1} e_3^{-1} \ldots e_{n-3}^{-1} e_{n-1}^{-1} e_{n-4}^{-1} \ldots e_2^{-1} e_0^{-1};
\]
Furthermore, \( e_0^{-1} e_3^{-1} e_2^{-1} e_1^{-1} e_4^{-1} \) and \( e_2^{-1} e_1^{-1} e_3^{-1} e_4^{-1} \) are among the faces of \( M_1(n) \).

Proof Let \( B(n), G(n), M(n) \) be obtained with \( \pi = \pi^{(n)}_1, \pi' = \pi^{(n)}_2 \) as above. We say that \( k \in \{0, \ldots, n - 2\} \) is joined from the left on the ring road if either \( k = 0 \) or in \( M(n) \) the rotation at \( \pi_i \) where \( \pi_i = k \) is \( \pi_i' \) \( v_w \). Here \( j \) is such that \( \pi_j' = k \). Otherwise we say that \( k \) is joined from the right.

Consider travelling from the motorway 0 to the motorway \( n - 1 \) in the ring road embedding derived from \( B(n), G(n) \) and \( M(n) \). At block 0 we exit at the group of lanes corresponding to \( \pi_j^{(0)} \) with \( \pi_j = n - 1 \) (i.e. \( j = n - 1 \)). Next, we enter block 1 at the point \( \pi_j^{(1)} = n - 2 \) and exit at \( \pi_j^{(1)} \) such that \( \pi_j = n - 2 \). We exit block 2 at \( \pi_j^{(2)} \) such that \( \pi_j = n - 3 \), and so on. It follows that the rotation at \( v_b^{(n-1)} \) in the ring road embedding is \( r_{n-1} \) where \( r_i, i \in \{0, \ldots, n - 1\} \) are defined by
\[
r_i = \begin{cases} v_{0}, & \text{if } i = 0; \\ r_{i-1}v_i, & \text{if } i \geq 1 \text{ and } n - i - 1 \text{ is joined from the left}; \\ v_ir_{i-1}, & \text{if } i \geq 1 \text{ and } n - i - 1 \text{ is joined from the right}. \end{cases}
\]
Here for a sequence \( y = (y_1, \ldots, y_k) \) and an element \( x \) we use notation \( yx = (y_1, \ldots, y_k, x) \) and \( xy = (x, y_1, \ldots, y_k) \). Using [21] we see that \( n - i - 1 \) is joined from the right if \( i \equiv 0 \pmod{2} \) and joined from the left otherwise. Thus the rotation at \( v_b^{(n-1)} \) in the ring road embedding is
\[
v_b^{(n-1)} : v_w^{(n-2)}v_w^{(n-4)} \ldots v_w^{(4)}v_w^{(2)}v_w^{(0)}v_w^{(1)}v_w^{(3)} \ldots v_w^{(n-1)}.
\]

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Figure 7: Left: faces $e_0^{-1}e_3e_5^{-1}e_2$ and $e_2^{-1}e_1e_3^{-1}e_4$ in the embedding $\mathcal{M}_1$ and the intersection of the cut arc $X_1$ with these faces (red dashes). Right: adding a new cut arc, replacing the arc $e_2$ and adding new arcs $e_{-1}$ and $e_{-2}$ to solve the case $n \equiv 0 \pmod{4}$.

Since $v_{(i)}^{(n-1)}$ in the ring road embedding is mapped by the covering projection to the arc $e_{n-1-i}$ in $\mathcal{M}_1$, it follows that the rotation at $v_b$ in $\mathcal{M}_1$ is (23). By (20), (22) and our construction, the rotation at $v_w$ in $\mathcal{M}_1$ is

$$v_w : e_0e_3e_4 \ldots e_5e_2e_1.$$  

Thus using (23), $\mathcal{M}_1$ has a face $e_0^{-1}e_3e_5^{-1}e_2$ and a face $e_2^{-1}e_1e_3^{-1}e_4$. □

Let $X_1$ be the cut arc in $S_1 = S_1(n-2)$ defined above. Using Claim 4.5 the fact that each crossing of an arc $e_k$ and $X_1$ has sign 1 and a simple geometric argument (Jordan-Schoenflies theorem in the piecewise-linear case), we obtain the order $X_1$ enters and leaves the faces $e_0^{-1}e_3e_5^{-1}e_2$ and $e_2^{-1}e_1e_3^{-1}e_4$. The faces and the segments of $X_1$ intersecting them are shown, up to homeomorphism in Figure 7(a).

Now in the face $e_0^{-1}e_3e_5^{-1}e_2$ add a new arc $e_{-1}$ from $v_w$ to $v_b$ parallel to $e_0$ and a new arc $e_2'$ parallel to $e_3$. Introduce a new cut arc $X_2$ that starts inside the newly created face $e_{-1}e_0^{-1}$, ends at the new face $e_3e_2'^{-1}$ such that $X_2$ shares exactly one point with each of $e_2'$ and $e_{-1}$, $\sigma(e_2', X_2) = \sigma(e_{-1}, X_2) = -1$ and it is disjoint from the image of $G_1$ and $X_1$. Finally, remove the arc $e_2$ and add a new arc $e_{-2}$ from $v_w$ to $v_b$, so that $\sigma(e_2', X_2) = -1$ and the rotation at $v_b$ becomes

$$v_b : \ldots e_0e_{-1}e_{-2}e_2'e_3,$$

see Figure 7(b). Let $C = \{X_1, X_2\}$. We have $\sigma(e_{-1}, C) = -1, \sigma(e_{-2}, C) = -2$ and $\sigma(e_2', C) = 2$. By the earlier argument, $\sigma(e_k, C) = \sigma(e_k, \{X_1\}) = k$ for all $k \in \{0, \ldots, n-3\} \setminus \{2\}$. Thus for the new embedding $\mathcal{M}_2$ of $D_n$ with arcs $e_{-2}, e_{-1}, e_0, e_1, e_2', e_3, \ldots, e_{n-3}$ the voltage function $\alpha_{\mathcal{M}_2, C, n} : D_n \to \mathbb{Z}_n$ is bijective and the proof follows by Proposition 4.1 □
4.4 Completing the proof for 3-dimensional embeddings

An embedding as in Theorem 1.2 has a face invariant under the \( n \)-fold rotational symmetry, and hence at least one point in that face is a fixed point under the rotation. Thus Theorem 1.2 follows from Proposition 4.2 and the next result.

**Lemma 4.6** Let \( n \geq 2 \) be an integer. Let \( \mathcal{M} \) be a piecewise linear embedding of \( K_{n,n} \) into a closed connected orientable surface \( S \subset \mathbb{R}^3 \). Suppose \( \mathcal{M} \) has \( n \)-fold rotational symmetry such that no vertex of the graph is mapped to a fixed point but

\[ \bullet \) some vertex of \( S \) is a fixed point.

Then the genus of \( S \) is at least \( L^*_C(n) \).

Furthermore, if \( n \neq 4 \), this lower bound is best possible and there exists a genus \( L^*_C(n) \) embedding satisfying the assumption of Theorem 1.2.

**Proof** The lemma is trivial for \( n = 2 \).

Suppose \( n \geq 3 \) and the embedding \( \mathcal{M} \) of a complete bipartite graph \( G \) into \( S \) satisfies all the assumptions of the statement. Use Proposition 4.1 to obtain an embedding \( \mathcal{M}_B \) of a base graph \( G_B \) in a base surface \( S_B \) together with a set of curves \( C \) in \( S_B \), such that \( G \) is isomorphic to the graph derived from the voltage graph \( (G_B, \alpha_{M_B,C,n}) \). Since \( G \) is bipartite it follows that \( |V(G_B)| = 2 \), \( |E(G_B)| = n \), and no edge is a loop. By inverting voltages if necessary, we can assume each edge of \( G_B \) is oriented from one vertex, call it \( v_w \), to another vertex, call it \( v_b \). Next, since \( G \) is derived from \( G_B \), it must be that the image \( \{ \alpha_{M_B,C,n}(e), e \in E(G_B) \} \) contains every element of \( \mathbb{Z}_n \). By Proposition 4.1 the genus of \( S \) is \( ng + (n-1)(t-1) \), where \( g \) is the genus of \( S_B \) and \( t, t \geq 1 \), is the number of arcs in \( C \). Now Lemma 4.3 implies that the genus of \( S \) is at least \( L^*_C(n) \).

Finally, the upper bound, or the fact that for each \( n \geq 3, n \neq 4 \) there is a symmetric embedding of genus \( L^*_C(n) \) follows by Lemma 4.4.

It remains to consider the special case \( n = 4 \).

**Lemma 4.7** In the case \( n = 4 \) of both Lemma 4.6 and Theorem 1.2 the minimum genus is 4.

**Proof** Suppose the minimum genus is \( g' < 4 \). Consider the genus \( g \) embedding \( \mathcal{M} \) of \( D_4 \) and the set \( C \) with \( t \) cut arcs obtained from a genus \( g' \) counterexample using Proposition 4.2 and Proposition 4.1.

By Lemma 4.3 we must have \( g = 0 \) and \( t = 2 \). That is, we have a \( D_4 \), two cut arcs and some number of cut loops embedded into the sphere, such that each
arc of the embedded graph makes a different number (modulo 4) of intersections with the cut arcs, as defined by (13).

Let the rotation at the white vertex be \( v_w : e_0 e_1 e_2 e_3 \). By Euler's formula, the set of regions of \( M \) consists of four 2-faces, let \( F_i \) be bounded by \( e_i \) and \( e_{(i+1) \mod 4} \). Write \( \sigma_X = (\sigma(e_0, X), \sigma(e_1, X), \sigma(e_2, X), \sigma(e_3, X)) \). Using the argument of the proof of Lemma 4.3, each of the four endpoints of the cut arcs must be contained in a different face. By the same argument, the contributions of a single cut arc \( c \) are \( \sigma_c = (x, x + s, x, x) \) if the endpoints of \( c \) are in \( F_0 \) and \( F_1 \), and \( \sigma_c = (x, x + s, x + s, x) \) if the endpoints of \( c \) are in \( F_0 \) and \( F_2 \). Here \( x \) is some integer, \( s \in \{1, -1\} \) and we considered two possible ways to pick the first edge for an intersection from each endpoint and two ways to direct \( C \). Similarly, for a cut loop \( c_1 \in C, \sigma_{c_1} = (z_1, z_1, z_1, z_1) \) for some integer \( z_1 \).

Summing all contributions, the image of \( \sigma_C \) is either \( \{y, y + s_1 + s_2\} \), or \( \{y, y + s_1, y + s_2\} \) or \( \{y, y + s_1, y + s_1 + s_2, y + s_2\} \), for some \( y \in \mathbb{Z} \) and \( s_1, s_2 \in \{-1, 1\} \). Therefore the voltage function \( \alpha_{M,C,4} \) is not bijective, a contradiction.

Finally, one of the solutions of genus 4 is the Pinavia interchange, see Figure 8. Another solution could be obtained by using the construction of Lemma 4.4.

4.5 Other observations

One may ask what happens if we drop the assumption that \( S \) contains a fixed point.

**Lemma 4.8** Let \( n, M, S \) be as in Lemma 4.6 but drop the assumption (●). Then the
genus of $S$ is at least

$$\tilde{L}_C(n) = \begin{cases} \frac{(n-2)^2}{4}, & \text{if } n \equiv 0 \pmod{4}; \\ \frac{n(n-1)}{4}, & \text{if } n \equiv 1 \pmod{4}; \\ \frac{n(n-2)}{4}, & \text{if } n \equiv 2 \pmod{4}; \\ \frac{n^2-3n+4}{4}, & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

and this bound is best possible.

**Proof** Suppose $S$ has no fixed points. As remarked in the comment after Proposition 4.1 the implication (a) $\implies$ (b) still holds yielding an embedding $\mathcal{M}_B$ of $D_n$ into a surface $S_B$ as in the proof of Lemma 4.6 but now the family $C$ of curves contains $t = 0$ arcs, so all curves in $C$ are loops. Using the same argument as in Lemma 4.3 $\mathcal{M}_B$ can have no regions with boundary of size 2. Writing $r$ for the number of regions and $g$ for the genus of $S_B$, we have by Euler’s formula

$$g \geq \left\lceil \frac{2 - 2 - r + n}{2} \right\rceil \geq \left\lceil \frac{n - \frac{2n}{4}}{2} \right\rceil = \left\lceil \frac{n}{4} \right\rceil.$$

So by the Riemann-Hurwitz formula the genus of $\mathcal{M}$ is

$$ng + (n - 1)(t - 1) = ng - (n - 1) \geq \tilde{L}_C^*(n).$$

Note that for $n \equiv 0 \pmod{4}$, $n \geq 8$ we construct an embedding $\mathcal{M}_B$ of $D_n$ into a torus with a set $C$ consisting of a single cut loop and genus $n/4$ as follows (see also Figure 9(a)). Start with the construction $B(n - 2)$ of the proof of Lemma 4.4 and let $\pi^{(n-2)} = (\pi_1, \ldots, \pi_{n-3})$ and $\pi'^{(n-2)} = (\pi'_1, \ldots, \pi'_{n-3})$ be the respective permutations. Extend these permutations by defining $\bar{\pi} = (\pi_1, \ldots, \pi_{n-3}, n - 2, n - 1)$ and $\bar{\pi}' = (\pi'_1, \ldots, \pi'_{n-3}, n - 3, n - 2)$ and add points $n - 3, n - 2$ on the segment $ab$ and the points $n - 2, n - 1$ on the segment $dc$ of $B(n - 2)$. Now identify $da$ with $cb$ in $B(n - 2)$ to get a cylinder with $(n - 4)/4$ handles. Add edges $\bar{\pi}_{n-2}, \bar{\pi}_{n-3}, \bar{\pi}'_{n-2}, v_w \bar{\pi}_{n-2}, v_w \bar{\pi}_n$ and $v_w v_b$ so that the rotations are

$$\bar{\pi}_{n-2} : \bar{\pi}_{n-3}, \quad \bar{\pi}'_{n-2} : \bar{\pi}_{n-3}, \quad \bar{\pi}_{n-2} : v_w \bar{\pi}_{n-2}, \quad \bar{\pi}_{n-3} : v_w \bar{\pi}'_{n-2}.$$

Now let $\mathcal{M}'$ be the embedding obtained by expanding the vertices $\bar{\pi}_i, \bar{\pi}'_i$, $i \in \{1, \ldots, (n - 1)\}$ similarly as in the proof of Lemma 4.4 If we identify the edges
Figure 9: (a) The green dotted edges are added in the construction of $M'$ for $n \equiv 0 \pmod{4}$; (b) a genus 1 base embedding that gives a solution (c) for $n = 4$ with symmetry $C_8$.

For $n = 4$ take $\tilde{\pi} = (1, 2, 3)$ and the rotations $v_w : (v_b, \tilde{\pi}_1, \tilde{\pi}_3, \tilde{\pi}_2)$, $\tilde{\pi}_1 : v_w \tilde{\pi}_2'$, $\tilde{\pi}_2 : \tilde{\pi}_1' v_w$, $\tilde{\pi}_3 : v_w$, $\tilde{\pi}_1' : v_b$, $\tilde{\pi}_2' : \tilde{\pi}_2$ and $\tilde{\pi}_3' : \tilde{\pi}_3$. There is also an interesting genus 1 construction with rotational symmetry group $C_{2n} = C_8$ (not colour-preserving) and genus 1, see Figure 9(b-c).

Finally, the construction for $n = 3 \pmod{4}$ of $L_C^*(n)$ is obtained in a similar way as the construction for $n = 1 \pmod{4}$, except that we do not need to add the points $\tilde{\pi}_{n-2}$ and $\tilde{\pi}'_{n-2}$.

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A Symmetric solutions for $5 \leq n \leq 9$

Above we presented complete symmetric interchanges of minimum genus for $n \in \{3, 4\}$ and simple procedure to construct rotationally symmetric interchanges with the minimum number of bridges for arbitrary $n \geq 5$. Below we present complete examples obtained as in the proof of Lemma 4.4 for $n$ up to 9. Note, that we omitted the edges $v_w^{(i)}, v_b^{(i)}$, $i \in \mathbb{Z}_n$, which are easy to add and which correspond to lanes for turning around. Also we show the vertices $\pi_k^{(i)} = \pi_k^{(i+1)}$, $k \in \{1, \ldots, n\}$, see the proof of Lemma 4.4.

Figure 10: $n = 5$. 

\[37\]
Figure 11: $n = 6$.

Figure 12: $n = 7$. The orbit of $e_{-1}$ is shown in blue.
Figure 13: $n = 8$. The orbits of $e_{-1}$, $e_{-2}$ and $e_2'$ are shown in blue, green and red respectively, and the orbit of the removed edge $e_2$ shown dotted.

Figure 14: $n = 9$. 

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