Fast and Optimal Laplacian Solver for Gradient-Domain Image Editing using Green Function Convolution

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Abstract

In computer vision, the gradient and Laplacian of an image are used in many different applications, such as edge detection, feature extraction and seamless image cloning. To obtain the gradient of an image, it requires the use of numerical derivatives, which are available in most computer vision toolboxes. However, the reverse problem is more difficult, since computing an image from its gradient requires to solve the Laplacian differential equation. The problem with the current existing methods is that they provide a solution that is prone to high numerical errors, and that they are either slow or require heavy parallel computing. The objective of this paper is to present a novel fast and robust method of computing the image from its gradient or Laplacian with minimal error, which can be used for gradient-domain editing. By using a single convolution based on Green’s function, the whole process is faster and easier to implement. It can also be optimized on a GPU using fast Fourier transforms and can easily be generalized for an n-dimension image. The tests show that the gradient solver takes around 2 milliseconds (ms) to reconstruct an image of 801x1200 pixels compared to between 6ms and 3000ms for competing methods. Furthermore, it is proven mathematically that the proposed method gives the optimal result when a perturbation is added, meaning that it always produces the least-error solution for gradient-domain editing. Finally, the developed method is validated with examples of Poisson blending, gradient removal, edge preserving blurring and edge-preserving painting effect.

Keywords: Computer vision; Poisson image editing; seamless cloning; Green function convolution; Gradient Laplacian Solver; gradient-domain editing.

Nomenclature and Acronyms

\begin{itemize}
\item $V_E$: Potential, often represents the signal or computed image
\item $V_{E,corr}$: Potential corrected to preserve colours and contrast
\item $E$: Electric field, being the gradient of $V_E$
\item $E_c$: Reconstructed conservative electric field
\item $E_p$: Perturbed non-conservative electric field
\item $V_{\text{mono}}$: Green’s function, equivalent to electric potential by a single monopole
\item $V_{\text{mono}}^F$: Optimal Green’s function in the Fourier domain
\item $V_{\text{dip}}$: Gradient of $V_{\text{mono}}$
\item $V_{\text{dip}}^F$: Complex dipole potential
\item $\nabla U$: Any possible conservative field
\item $x_m$: Cartesian coordinate in the dimension $m$
\item $r$: Euclidean distance
\item $n$: Number of dimensions of the data ($n = 2$ for an image)
\end{itemize}
1. Introduction

In science and engineering, many different signals can be interpreted as scalar potentials \( V \), especially when there is an interest in their gradient \( \nabla V \). For example, Newtonian gravity has its own equation for the gravitational potential and field [1]. Another example is how computer vision scientists can treat an image as a potential and compute numerically its gradient and Laplacian [2–7] with a simple convolution kernel such as Sobel [8–10]. However, computing the image from its gradient is a non-trivial task since it requires to solve a differential equation [2–4]. By solving it numerically, it can be used for many gradient-domain editing applications, such as gradient erasing, seamless cloning and vectorization with diffusion curves [2–4,11,12]. Furthermore, Bhat et al. presented a whole framework of gradient-domain image editing with unique and useful applications such as color filtering and edge sharpening [13].

The first method to solve the Laplacian (also called Poisson’s equation) was presented in 2003 by Perez et al. [3], which proposed to solve the differential equation by iteratively minimizing the variational problem. Other research followed by optimizing the computation speed and error [3,11], while others used the Jacobi method [2]. Both approaches converge to the right solution, but they are harder to implement since they are iterative, which also makes them slower to compute. An alternate way of solving the Poisson equation is proposed by Tanaka [4] by modifying the Poisson problem into a closed-form problem using cosine transforms.
In the research work presented here, a novel method is proposed, called Green Function Convolution (GFC), which allows to reconstruct an image from its modified gradient. The gradient of an image always produces a conservative field, meaning that the field can be integrated to obtain a potential. When editing the image in the gradient domain, a non-conservative perturbation can be added, meaning that the resulting field cannot be integrated to an exact solution. However, the proposed GFC method can find the best possible approximation for the even when a non-conservative perturbation is added. In fact, we mathematically prove in section 2.3 that GFC is the optimal possible solver for any perturbation added to the gradient of the image, meaning that gradient domain editing can be done with minimal error.

Another advantage of the GFC is that it is simple to implement, since it only requires to build a Green convolution kernel, and to compute the convolution between the Laplacian and the kernel in the Fourier domain. Since Fourier transforms are heavily optimized and present in most computer-vision libraries, such as MATLAB® (Mathworks, USA) [10] and OpenCV [9], the code is simple to implement. Furthermore, computing the Fourier transform can be used on a Graphic Processing Unit (GPU) to further accelerate the computation. Finally, the GFC approach can be used directly on the gradient or on the Laplacian depending on the user preference, contrarily to the other methods.

The first validation of the method is done by analyzing the error on the image reconstructed from its Laplacian in section 3.1. Furthermore, the current work provides many application-based examples are provided as validation that the method works and to show that it is more robust than other standard methods. The presented applications are given in section “3.2 Gradient-domain image editing” and are listed below. Additionally, the method is not limited to the given examples and should work with most gradient-domain image editing applications.

- Poisson blending: Used to seamlessly copy an element from one image to another.
- Gradient thresholding: Used to remove textures, blurry elements and low contrast background.
- Edge contrast and blurring: Used to enhance contrast between objects but blur the region without clear edges.
- Edge painting effect: Similar to the last point but with a paint-style blurring.

The GFC method is part of a bigger project of image analysis called CAMERA-I (Convolution Approach of Magnetic and Electric Repulsion to Analyze an Image) [5–7], meaning that the full name of the proposed method is CAMERA-I-GFC.

2. Computing the potential associated to a field

To understand how to compute the image from its gradient field or Laplacian, we first need to understand the mathematical equations that allow the computation of any potential from its Laplacian using Green’s function [14]. This section will show how to find the appropriate Green’s function and how to use it for either the gradient or the Laplacian. Then, we will demonstrate mathematically that using Green’s function is the optimal tool when there is a non-conservative perturbation that is added to the gradient field.

2.1. Green’s function to solve the Laplacian

One of the problems that we face with image or signal processing is the reconstruction of the signal \( V_E \) based on its gradient field \( E \), given by the scalar product integration as given by equation (1), where \( dl \) is the path of integration. The reverse problem, given by equation (2), is trivial since the gradient \( V \) is computed using numerical derivative in each direction \( x_n \), such as those given by the Sobel method [8,15]. However, the problem with equation (1) is that it works only with conservative fields, meaning that the field is the gradient of a function and that the line integral on different paths between the same points return the same value [1,14]. This means that any non-conservative noise or perturbation added to \( E \) will prevent us from applying equation (1) which limits the scope of gradient domain editing.

\[
V_E = -\int_C E \cdot dl
\]  
\[E = \nabla V_E (x_1, x_2, ..., x_n)
\]
Fortunately, another way of computing $V_E$ is using a convolution with a Green’s function [14]. This convolution noted with the star symbol $\ast$ is presented in equation (3), where the potential of a single monopole $V_{\text{mono}}$ is Green’s function in $n$-dimensions [14], which is at the heart of the GFC method presented in this paper.

$$V_E = (\nabla^2 V_E) \ast V_{\text{mono}}$$  \hspace{1cm} (3)

The value of $V_{\text{mono}}$ is given in equation (4), with the constant $S_{n-1}$ given in equation (5) where $\Gamma$ is the gamma function and $r$ is the Euclidean distance given in equation (6) [14].

$$V_{\text{mono}} = \frac{-1}{S_{n-1}} \int r^{(1-n)} dr$$

$$V_{\text{mono}} = \frac{-1}{S_{n-1}} \begin{cases} \frac{\ln(r)}{n} & n = 2 \\ \frac{r^{2-n}}{n-2} & n \neq 2 \end{cases}, \quad n \in \mathbb{N}^*$$  \hspace{1cm} (4)

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$  \hspace{1cm} (5)

2.2. Green’s function derivative to solve the gradient

The following section will show how to compute $V_E$ from its gradient, instead of its Laplacian. Although they are mathematically similar, there are some advantages of using the gradient. For example, our previous work demonstrated that modifying locally the orientation of the gradient allows computing the probability of being inside a partial contour [6,7].

To find the equation for the gradient, the first step is to take equation (3) and substitute $\nabla V_E$ by $\mathbf{E}$, as given by equation (2). This gives the divergence of $\mathbf{E}$ (noted $\nabla \cdot \mathbf{E}$) in equation (7). Then, using the properties of the convolution (noted $\ast$), the derivation terms are moved to the monopole potential in equation (8).

$$V_E = (\nabla \cdot \mathbf{E}) \ast V_{\text{mono}}$$  \hspace{1cm} (7)

$$V_E = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} E_x^i \ast V_{\text{mono}} = \sum_{i=1}^{n} E_x^i \ast \frac{\partial}{\partial x_i} (V_{\text{mono}})$$  \hspace{1cm} (8)

Based on our previous work [5–7], we know that the derivative term of the potential in (8) is equivalent to the dipole potential given in equation (9).

$$\mathbf{V}_{\text{dip}} = \nabla V_{\text{mono}} = (V_{\text{dip}}^1, V_{\text{dip}}^2, \ldots, V_{\text{dip}}^n)$$  \hspace{1cm} (9)

Hence, we find in equation (10) that the potential $V_E$ is the sum of different convolutions, which is an identical equation to the one we developed for image analysis [5,7] and the spatial probability of belonging to partial contours [6].

$$V_E = \sum_{i=1}^{n} (E_x^i \ast V_{\text{dip}}^i)$$  \hspace{1cm} (10)

It is important to note that we can add any integration constant to the value of $V_E$ without changing the validity of the previous equations. This constant is set to 0 in all the computer vision applications presented in the current work but can be set to any value to shift the luminosity of the image.
2.3. Proof of optimal result for any perturbations in the gradient

The above-presented mathematical equations showed how to re-compute the image $V_E$ from its gradient or Laplacian using the convolutions in equations (3) and (10). However, there are many applications that require adding non-conservative perturbations to the field, such as those presented in gradient thresholding and edge blurring presented in section 3.2. The perturbed field is noted $E_p$, while the computed field and potentials are respectively $E_c$ and $V_c$. As shown in section 3.2, the perturbations can be any gradient-domain editing application, which modifies the gradient of the image.

Since the perturbation can be non-conservative, the field $E_p$ it does not have an associated potential. Hence, there is a need to find the conservative field $E_c$ that is the best possible approximation of $E_p$. This section will prove that equations (3) and (10) give the optimal $V_c$ and $E_c$ for any possible perturbation. Thus, it will prove that the GFC method gives the most optimal solution, meaning that it is robust to perturbation and that it will converge to the least error solution.

Using Hilbert projection theorem, we know that the minimum-error solution is given when $E_p - E_c$ is orthogonal to any conservative field $\nabla U$ at any point [16]. Hence, we need to prove that $F = 0$ (equation (11)), where $d\mu$ is the infinitesimal hyper-volume for the integration.

$$F = \int_{\mathbb{R}^n} \left[ (E_p - E_c) \cdot \nabla U \right] d\mu = 0 \quad (11)$$

To prove it, we first replace the value of $E_c$ by its correspondence $E_p$, as given in equation (12). Then, we substitute $V_c$ by $\left( (\nabla \cdot E_p) \ast V_{\text{mono}} \right)$ according to equation (7). We also define the variable $A$ as a temporary variable to make it easier to follow the proof.

$$F = \int_{\mathbb{R}^n} \left[ (E_p - \nabla (V_c)) \cdot \nabla U \right] d\mu$$

$$F = \int_{\mathbb{R}^n} \left[ E_p - \nabla (\left( \nabla \cdot E_p \right) \ast V_{\text{mono}}) \right] \cdot \nabla U\right] d\mu = \int_{\mathbb{R}^n} [A \cdot \nabla U] d\mu \quad (12)$$

By adding and subtracting the term $(\nabla \cdot A)U$ inside the integral, we obtain equation (13). Then, we use the divergence properties in equation (14) to regroup the positive terms inside an integral and the negative terms in another.

$$F = \int_{\mathbb{R}^n} [(A \cdot \nabla U) + (\nabla \cdot A)U - (\nabla \cdot A)U] d\mu$$

$$F = \int_{\mathbb{R}^n} [\nabla \cdot (AU)] d\mu - \int_{\mathbb{R}^n} (\nabla \cdot A)U d\mu \quad (13)$$

In equation (14), the term noted $B$ has a value of 0 and is canceled. This is due to Gauss’s theorem which states that the integral of a divergence is the integral of the flux outside the surface [1,14]. However, as it is explained later in section 3.1.2, since a zero padding is added around the image, then the flux is 0 at every point of the boundaries of the surface. Therefore, equation (15) is the remaining term of equation (14), where the value of $A$ is substituted by its definition in equation (11).

$$F = -\int_{\mathbb{R}^n} \nabla \cdot \left[ E_p - \nabla \left( (\nabla \cdot E_p) \ast V_{\text{mono}} \right) \right] U d\mu \quad (15)$$

Then, equation (16) distributes the derivative operators according to the properties of the sum and the convolutions.
\[ F = -\int_{\mathbb{R}^n} \left[ \nabla \cdot E_p - \nabla^2 \left( (\nabla \cdot E_p) \ast V_{\text{mono}} \right) \right] U \, d\mu \]

\[ = -\int_{\mathbb{R}^n} \left[ \nabla \cdot E_p - \left( (\nabla \cdot E_p) \ast \nabla^2 V_{\text{mono}} \right) \right] U \, d\mu \]  

(16)

Finally, since \( V_{\text{mono}} \) is a Green’s function, then by definition \( \nabla^2 V_{\text{mono}} \) is a Dirac’s delta \([14]\). Knowing that for any function \( f \) convoluted with a Dirac’s delta \( \delta \), we have \( f \ast \delta = f \) \([14]\), equation (17) gives us the final result \( F = 0 \). Hence, \( E_p - E_c \) is orthogonal to any other field, meaning that the conservative field \( E_c \) has the least error when compared to the perturbed field \( E_p \).

\[ F = -\int_{\mathbb{R}^n} \left[ \nabla \cdot E_p - \left( (\nabla \cdot E_p) \ast \delta \right) \right] U \, d\mu \]

\[ F = -\int_{\mathbb{R}^n} \left[ \nabla \cdot E_p - \nabla \cdot E_p \right] U \, d\mu \]

(17)

\[ F = 0 \]

This completes the proof that the GFC method allows to compute the field \( E_c \) and the potential \( V_p \) which are the optimal conservative approximation for any perturbed field \( E_p \). Hence, the GFC method will always converge to the least-error possible solution, meaning that it is robust to any change or perturbation added to the field. This proof is also valid in the case of an n-dimension image or signal, not just in 2D.

3. Applications in computer vision

Solving the Laplacian and the gradient is a problem dating back many centuries, with applications in different areas of science and engineering \([1,14]\), but the current paper focuses on applications in the field of computer vision. There are many already-proven applications of the solvers in computer vision, including seamless cloning, seamless composite and animated diffusion curves \([2–4]\), etc. Those applications are part of a branch called gradient-domain image editing (GDIE) \([13]\). Since all those applications are already demonstrated, we will only focus mainly on showing the proof-of-concept of the GFC with some comparison to Perez \([3]\), Jeschke \([2]\) and Tanaka \([4]\) methods.

3.1. Image reconstruction

The mathematical proof of section 2.3 demonstrated that the GFC method should give the optimal result without any iterative computation, even when a perturbation is added to the gradient. The current section will show how to implement the optimal GFC solver numerically. This section follows by showing examples of the method’s results and by quantifying the errors associated to the reconstruction.

3.1.1. Process summary

Figure 1 shows a summary of the process used to reconstruct the image from its modified gradient or Laplacian. All those steps are simple to implement in OpenCV and Matlab since they mostly use already available functions in their respective computer vision toolboxes. Also, none of these steps are iterative or computationally expensive, except for the convolution, which is demonstrated to be fast to compute according to section “3.1.5 Computation speed”. The next section will discuss the construction of the \( V_{\text{mono}} \) and \( V_{\text{dip}} \) kernels and the convolution process. Some of the other steps such as the gradient editing and color correction will be discussed in later sections.
3.1.2. Computing the image from its Laplacian or Gradient

The previous sections 2.1 and 2.2 showed the mathematical formulations of the Green’s functions (3) and (10) that allow solving the Laplacian and the gradient problems analytically. This section will focus on how to use those equations numerically, for a computer vision application. We will observe how those equations are simply the consecutive application of convolution kernels. Using convolutions in computer vision is easy, since the convolution functions are heavily optimized on both CPU and GPU in many different libraries [9,10].

Based on equation (4) with \( n = 2 \) (since the image is in 2D), the \( V_{\text{mono}} \) convolution kernel must be built according to equation (18), with the center of the matrix being at \( r = 0 \), and the size being twice the size \( V_E \). Some examples are given in our previous work [5,6]. Also, the central point of \( V_{\text{mono}} \), where \( r = 0 \), will be set to an optimized value of \( 1.7/(2\pi) \). This value is almost 0.27, which allows to cancel the value 4 in the Laplacian kernel and compensate for numerical errors.

\[
V_{\text{mono}} = \begin{cases} \frac{1}{2\pi} \ln(r) , & r \neq 0 \\ 1.7 , & r = 0 \end{cases} \tag{18}
\]

The numerical Laplacian is defined as the convolution kernel given in equation (19) [8,9].

\[
K_{p2} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{19}
\]

Finally, we can retrieve \( V_E \) from its Laplacian by applying the consecutive convolutions given by (20) on the image \( I \). However, for the following equation to work according to the boundaries of the images, we must add a 5-pixel padding of repeated values around \( I \), followed by a 5-pixel padding with value 0. These paddings are removed after the computation, but they are essential to rebuild an exact copy of the image by removing boundary errors, because they make sure that the Laplacian kernel does not overflow the image. However, no padding should be used for blending applications such as seamless cloning since the goal is to seamlessly “blend” the border, while the padding is means for an exact border reconstruction.

Figure 1. Process diagram of the image reconstruction.
\[
V_E = I \ast K_{p^2} \ast V_{\text{mono}}
\]  \hspace{1cm} (20)

In case we are working with the gradient instead of the Laplacian, we use the right-sided derivative kernels given in (22). Then, we use the complex left-sided derivative kernel given in (23). A complex kernel is used since it allows reducing the number of convolution required [5,6], which reduces the computation time. Using complex numbers does not increase the computation time since convolutions with large kernels are done using a Fourier transform \( \mathcal{F} \) in the complex domain [17]. The Fourier-domain convolution is shown in equation (21) where \( A \) and \( B \) are any matrices and \( \mathcal{F}^{-1} \) is the reverse transform, and \( \circ \) is the element-wise product operator. Numerically, the Fourier transform is fast and easy to compute using Fast Fourier Transform (FFT) algorithms.

\[
A \ast B = \mathcal{F}^{-1}(\mathcal{F}(A) \circ \mathcal{F}(B))
\]  \hspace{1cm} (21)

Using a right-derivative kernel followed by a left-derivative kernel allows reducing the numerical error since the right-derivatives followed by left-derivatives gives the centered Laplacian kernel defined in (19). In equation (22), we define the norm of the gradient \(|E|\) and its orientation \(\theta\) [5–7]. Then, equation (23) defines the complex dipole kernel, which will later be used on the gradient [5–7].

\[
\begin{align*}
K_{\text{dx}} &= \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \\
E_x &= I \ast K_{\text{dx}} \quad , \quad E_y = I \ast K_{\text{dx}}^T \\
|E| &= \sqrt{E_x^2 + E_y^2} \quad , \quad \theta = \arctan \left( \frac{E_y}{E_x} \right) \\
V_{\text{dip}}^\theta &= V_{\text{mono}} \ast K_\theta \\
K_\theta &= \begin{bmatrix}
-1 + i & 1 & 0 \\
-i & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]  \hspace{1cm} (22)

Finally, the proposed GFC method allows to retrieve \(V_E\) from its gradient \(E\) using the real part of the complex convolution of equation (24), which is similar to our previous work [5–7]. In this equation, \( \circ \) is the element-wise product operator and \( \ast \) is the convolution operator.

\[
V_E = \Re \left( (|E| \circ e^{i\theta}) \ast V_{\text{dip}}^\theta \right)
\]  \hspace{1cm} (24)

3.1.3. Optimal Green’s function

This sub-section will present the results of applying GFC on the ECSSD dataset [18,19] which contains 1000 images with a resolution \( \sim 400\times300 \) pixels. This is carried out to demonstrate that the theory works and that both equations (20) and (24) allow reconstructing an image from its Laplacian/gradient with minimal error.

When using equation (18) to reverse the Laplacian operator on an image, we can observe that there is a reconstruction error. We tested the image reconstruction on the 1000 images of the ECSSD dataset [18,19] to evaluate the average root-mean-squared-error (RMSE) compared to the original image. The equation (25) shows how to compute the RMSE, where \( p_i \) is a pixel from the image \( I \), \( p_i \) is the corresponding pixel from the reconstructed image \( V_E \) and \( N \) is the total number of pixels.

\[
\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N} (p_i - p_i)^2}{N}}
\]  \hspace{1cm} (25)

The resulting error gives an average RMSE of 0.98, meaning that there is around 1 level of gray of error per pixel (for 256 possible levels). The error is almost imperceptible to a human eye, since most of the error is concentrated at the regions with strong gradient. The reason for the error is that the kernel in equation (18) is for a continuous space,
not for a numerical grid. However, the current section shows how to significantly reduce the error by building an optimal numerical Green’s function.

Equation (26) shows how \( V_{\text{mono}} \) is used for image reconstruction on any image \( I \), which allows to state in equation (27) that the Laplacian kernel \( K_{p^2} \) convoluted with \( V_{\text{mono}} \) should give a Dirac’s delta \( \delta \).

\[
K_{p^2} * I * V_{\text{mono}} = (K_{p^2} * V_{\text{mono}}) * I = \delta * I = I
\]  

(26)

\[
(K_{p^2} * V_{\text{mono}}) = \delta
\]  

(27)

By using the convolution definition in the Fourier domain given in equation (21), with \( \mathcal{F} \) being the Fourier transform and \( \mathcal{F}^{-1} \) being the inverse Fourier transform, we obtain equation (28). Then, we can isolate \( V_{\text{mono}} \) in equation (29) to obtain a mathematical definition of the Green’s function \( V_{\text{mono}} \) in the Fourier domain, which we note \( V_{\text{mono}}^\mathcal{F} \).

\[
\mathcal{F}^{-1}\left(\mathcal{F}(K_{p^2}) * \mathcal{F}(V_{\text{mono}})\right) = \delta
\]  

(28)

\[
V_{\text{mono}}^\mathcal{F} \equiv \mathcal{F}(V_{\text{mono}}) = \frac{\mathcal{F}(\delta)}{\mathcal{F}(K_{p^2})}
\]  

(29)

For this definition to work in a discrete environment, we need the matrices to all be the same size as the image \( I \). Hence, we define the zero-padded matrices \( \tilde{K}_{p^2} \) and \( \tilde{\delta} \) in equations (30) and (31), where the top left corner are the 3 × 3 Laplacian and Dirac kernels and the rest of the matrices is 0-valued.

\[
\tilde{K}_{p^2} \equiv \begin{bmatrix}
0 & -1 & 0 & \cdots & 0 \\
-1 & 4 & -1 & \cdots & 0 \\
0 & -1 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}_{\text{size}(I)}
\]  

(30)

\[
\tilde{\delta} \equiv \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}_{\text{size}(I)}
\]  

(31)

Using the definitions (30) and (31) alongside equation (29), we can find the padded Fourier-domain green function \( \tilde{V}_{\text{mono}}^\mathcal{F} \) in equation (32). One can find an analytical solution to \( \tilde{V}_{\text{mono}}^\mathcal{F} \), but the FFT algorithm is fast enough to justify avoiding the trouble. It is to note that the division produces an infinite value on the upper-left corner of \( \tilde{V}_{\text{mono}}^\mathcal{F} \), which can simply be set to zero. This is because, in the Fourier domain, the upper-left corner represents the integration constant which can be added later.

\[
\tilde{V}_{\text{mono}}^\mathcal{F} = \mathcal{F}\left(\tilde{\delta}\right)\mathcal{F}\left(\tilde{K}_{p^2}\right)^{-1}
\]  

(32)

Finally, using the Laplacian solver of equation (20) we can reconstruct the optimal potential \( V_E \) from its Laplacian \( (I * K_{p^2}) \). The result is given in equation (33), where \( V_E = I \) if the right integration constant is used. Since the values of \( V_E \) is constant everywhere on its borders but must be zero according to section 2.3, we add the integration constant \( c \) is added to ensure the potential on the borders in zero.

\[
V_E = R\left(\mathcal{F}^{-1}\left(\mathcal{F}(I * K_{p^2}) * \tilde{V}_{\text{mono}}^\mathcal{F}\right)\right) + c
\]  

(33)
In practice, we tested equation (33) on 1000 images of the ECSSD dataset and computed the average error with equation (25), which gives RMSE = 0.011. Hence, the error is 90 times lower with the optimal kernel of equation (29) than the one given by our previous work in equation (18). Furthermore, the error is low enough that a rounding to 8 bit reduces the error to 0 at every pixel.

The equation (33) also reduces the computation time since equations (20) and (24) required \( V_{\text{mono}} \) to be twice the size of the image, but equation (33) requires \( \tilde{V}_{\text{mono}}^F \) to be the same size as the image.

In summary, the equation (33) allows to optimally solve the Laplacian with negligible error using the Fourier-domain Green’s function \( \tilde{V}_{\text{mono}}^F \) of equation (32). Furthermore, \( \tilde{V}_{\text{mono}}^F \) can be pre-computed once and used for all the Laplacian that we require to solve.

3.1.4. Reversing any convolution

In theory, the equations developed in the previous section allow to reverse any convolution kernel that has been applied previously, since equation (32) finds the optimal functions that reversed the operator on the denominator of the division. However, in a numerical world, the error increases when the kernels are bigger and apply more blurring to the image since there is an information loss during blurring. For example, it is possible to reverse the popular Sobel gradient operator [9,17], but the tests on the ECSSD dataset shows an RMSE of 8.6, with most error concentrated on high gradient regions. This is because the Sobel operator causes a small blurring to the image, which cannot be reversed completely. Hence, the resulting image \( V^f \) from equation (33) for a Sobel gradient is a blurred version of \( I \).

3.1.5. Computation speed

By using equation (33), the computation time is low since FFT are highly optimized on CPUs and GPUs [9,10]. For example, the computation time is around 18ms on MATLAB® with an Intel® i7-6700K processor for a gray image (single channel) of resolution 801x1200. Also, using MATLAB’s GpuArray with the GPU nvidia® GTX 1080 Ti, the computation time when the overhead is eliminated is around 0.8ms. This is impressive since the RMSE is minimized and that no further iterations are needed. Such performance is reached using the functions \( \text{ifft}2 \) and \( \text{fft}2 \) in MATLAB to compute the convolution with Fourier transforms of equation (33).

In Figure 2, the total time for the GFC is noted 2ms, which includes 1.2ms for the preparation such as verifying the parameters and sending the matrices to the GPU. The remaining 0.8ms is used for the computation of the convolution.

For the GFC method, the computation time in Figure 2 does not include the computation of the optimal Green’s function \( \tilde{V}_{\text{mono}}^F \), since it can be pre-computed with equation (32). The time to build it is 5ms on the GPU and 36ms on the CPU. Even if \( \tilde{V}_{\text{mono}}^F \) is not pre-computed, the method is still fast enough to out-perform any competing algorithm, since \( \tilde{V}_{\text{mono}}^F \) is computed only once for the 3 channels of an image.

Figure 2 also allows us to compare the computing time of different algorithm for the reconstruction of a single channel image of resolution 801x1200. As we can observe, the logarithmic scale implies that the proposed GFC method is orders of magnitude faster than competing algorithms, such as Perez et al. [3] or Jeschke et al. [2].

The Perez [3] algorithm is downloaded from MathWorks [20], and later optimized to use the full capacities of MATLAB, but the matrix inversion alone takes 1770ms with another 1270ms to build the sparse matrix. It has no GPU implementation.

The Jeschke algorithm is downloaded directly from the code they provided in [2]. However, with the provided code did not include an image reconstruction method, but only a diffusion curve implementation. Hence, the comparison with their algorithm is not perfect. Their algorithm is known to converge in 400 iterations [2], which gives a total of 54 frames per second (fps) on GPU and 0.8 fps on CPU for a color image of resolution 801x1200. This translates into 6.2 ms (GPU) and 476.2 ms (CPU) for a single channel computation.

The Tanaka [4] algorithm is written by the author himself and is downloaded from MathWorks [21]. The computation time on the CPU to perform the cosine-transforms required to solve the Laplacian is around 292ms with a preparation time of 2ms. Furthermore, an additional time of 85ms is added to compute the cosine-transform solver, but it is not included in Figure 2 since it can be pre-computed.

Other methods such as the one proposed by Bhat et al. does not perform real-time image editing as stated in their paper [13], which means that it is definitely slower than the proposed approach.
In summary, the proposed GFC algorithm runs more than 3 times the speed the fastest competing algorithms on a single channel image. Since the method is really fast, a big part of the running time in a real application might be due to overheads and verifications.

![Figure 2. Computation time (ms) in logarithmic scale for a single channel image reconstruction of resolution of 801x1200. The green results are for the CPU and the blue are for the GPU. The Perez [3] and Tanaka [4] methods have no GPU implementation.](image)

3.2. Gradient-domain image editing

From the mathematical proof presented in section 2.3, the GFC proved to be the optimal way to reconstruct an image from a perturbed gradient. The gradient perturbation is mostly voluntary, such as thresholding, local gradient erasing or non-edge removal.

With those results, we demonstrated that the GFC is a simple yet fast and robust method for gradient-domain image editing (GDIE). GDIE means that the modifications are done on the gradient of the image, instead of the image itself. It is mainly used for applications such as Poisson blending and diffusion curves [2–4]. This section will show the performance of the method for Poisson blending, as well as additional possible gradient-domain applications such as texture removal and edge preserving contrast enhancement with blurring, based on the work of Bhat et al. [13]. Those applications can potentially be used in image/video editing software, as well as image pre-processing.

3.2.1. Poisson blending

Poisson blending is a type of GDIE that allows to merge the gradient of 2 different images, such that the blending is seamless. Since the proposed GFC approach has a low computation time for large images, as demonstrated in section 3.1.5, our implementation of the Poisson blending uses a blend region that is bounded by the total size of the image. This means that if the cropping region passes through a high gradient region, our method is better at compensating the error. This is shown inside the blue circle of Figure 3 where GFC approach reconstructs smoothly the cropped edge. Also, the GFC blending appears more natural since the left side of the stamp is more transparent.
Figure 3. Example of Poisson Blending application; (a) Stamp to copy, with the red-dotted lines being the cropping region and the blue dotted circle being a region of the stamp that is accidentally cropped; (b) destination image; (c) Poisson blending from Perez algorithm [3,20]; (d) Proposed GFC blending.

In other cases where the cropping region does not pass through a high gradient, the results of the proposed GFC method is identical to the Perez method.

3.2.2. Gradient thresholding

The equations (33) presented an optimal Laplacian solver in the Fourier domain. In comparison, the literature proposes mostly iterative methods on the Laplacian [2–4], which gives an advantage for our method by making it faster and easier to implement. However, computing the Laplacian from the perturbed gradient requires an additional computing step to preserve the brightness and contrast.

The problem when editing the gradient in an image is that the desired reconstruction is not necessarily the result given by $V_E$ since we want to preserve the color information. Hence, we define a new corrected reconstruction $V_{E,corr}$ in equation (34), where $\sigma$ indicates the standard deviation and the top bar “$$\bar{}$$” indicates the average. As stated in section “2.2”, it is possible to add any constant to $V_E$ without changing the validity of the equation, which means that the addition and subtraction of equation (34) do not affect the potential. For the $\sigma$ ratio, it is meant to preserve the initial contrast of the image, and it simply changes the norm of the gradient by a constant factor.

$$V_{E,corr} = (V_E - \bar{V}_E) \frac{\sigma(I)}{\sigma(V_E)} + \bar{I} \quad (34)$$

In case no perturbation is added to the gradient, we have $V_E \approx 1$. This means that almost no correction will be added to $V_E$, resulting in $V_{E,corr} \approx V_E$.

Figure 4 shows an example of the process of computing the gradient with equation (22), thresholding it at 10% of the highest possible gradient and computing the reconstructed image with equations (24) and (34). We can see that most features of the initial image are preserved, but that there is less texture and fine elements.
Figure 4. Example of gradient thresholding and reconstruction steps. (a) Original image; (b) Gradient $|\mathbf{E}|$; (c) Thresholded gradient at 10%; (d) Reconstructed and corrected image $V_{E,corr}$.

Figure 5 shows 2 more examples of image reconstruction with a 10% gradient thresholding. In those images, the castle reflection is completely erased, along with the clouds. For the leopard picture, almost all the background information is erased except for the leopard.

Figure 5. Examples of reconstructed images after gradient threshold at 10%. (a) Image of a castle; (b) Reconstructed castle image $V_{E,corr}$ after 10% gradient threshold; (c) Image of a leopard; (d) Reconstructed leopard image $V_{E,corr}$ after 10% gradient threshold.

The reconstruction differences between our proposed GFC method and the Perez method [3] are negligible in the case of gradient removal. Hence, we do not present comparison images since the differences are imperceptible to the human eye.

3.2.3. Gradient eraser

The last section showed that the thresholding can blindly eliminate low gradient features, but it could also be used in a supervised way. For example, it is possible to create a gradient erasing tool that allows to manually slide an eraser on an image to remove local gradients. This tool would work in real-time, since our solver requires only 15ms on the CPU and 2ms on the GPU for each channel of an 801x1200 image. Furthermore, a smaller convolution kernel can be used to improve the computation time, with low added error.

3.2.4. Edge contrast and blurring

In some cases, we would like to modify the image in order to remove the textures but retain the most important object delimitations. In those cases, the gradient needs to be modified to be enhanced on the edges but reduced elsewhere. This will result in a higher contrast between different objects and a blurring effect inside the given objects.

A possible way to combine the gradient with the edge information is to compute their product. However, the product of 2 identical gradients must be invariant to avoid enhancing the contrast everywhere. This leads to the equation

$$\mathbf{E}_p \mathbf{C} \alpha$$

where $\mathbf{E}_p$ is the perturbed gradient, $\mathbf{C}$ is the intensity of the edge detection and $\alpha = [0,1]$ is a factor that preserves the invariance and allows to attribute a weight to the edge.
\[ |E_p| = |E|^{1-\alpha} \circ C^\alpha \] (35)

Figure 6 shows the generated gradient $|E|$ followed by the gradient $|E_p|$ generated using equation (35) with $\alpha = 0.5$ and $C$ computed from a structured forest edge detection [22]. Then, it shows the reconstructed image $V_{E,\text{corr}}$ using equation (24) with the color correction (34). We can observe that there is a loss of texture, but that the contrast is improved between the objects separated with strong edges.

![Figure 6. Example of reconstruction from mixed gradient and edge with equation (35). (a) Original image; (b) Gradient |E|; (c) Gradient mixed with structured forest edge detection [22] and $\alpha = 0.5$; (d) Reconstructed and corrected image $V_{E,\text{corr}}$.](image)

In Figure 7, we can see the effect of using different $\alpha$ parameters. The higher the parameter $\alpha$ is chosen, the stronger is the contrast between objects. However, a higher $\alpha$ creates noise and discoloration in the image. This is because a higher $\alpha$ produces a field that is too different from the original field, which yields in undesired coloring and brightness artefacts.

![Figure 7. Example of reconstruction from mixed gradient and edge with equation (35) and varying parameter $\alpha$ and the edges computed from structured forest edge detection [22].](image)

In Figure 8, we can observe more examples of reconstruction using $\alpha = 0.5$. For the castle image, most of the changes are in the reduction of texture and lighting variations. For the leopard, most of the background is blurred, except the tree that appears clearer. Also, the pattern inside the leopard is blurred, but its boundaries remain with a strong contrast.
Figure 8. Examples of reconstructed images with perturbed gradient from equation (35) with edges information from structured forest [22] and $\alpha = 0.5$. (a) Image of a castle; (b) Reconstructed castle image; (c) Image of a leopard; (d) Reconstructed leopard image.

3.2.5. Painting effect

One modification that can be done on the edges before the reconstruction is a thinning, which allows to remove any gradient that is not on a maximal line. This thinning is often called non-maximal suppression (NMS). Some implementations of edge detection such as the structured forest have a native gradient NMS option [22]. However, applying NMS to the edge information removes almost completely the texture information, meaning that the reconstructed image resembles a painting. Hence, the proposed method can be used for a painting filter with a high edge accuracy, as observed in Figure 9. This is similar to other applications such as diffusion curves [11], but without using a vectorization.

Since the thin edges do not necessarily intersect the gradient, the image reconstruction will not work directly with equation (35). To make sure that they always intersect, we simply add a Gaussian filter to the image with a standard deviation $\sigma = 1$. This filter is applied to the image before computing its gradient $E$, but it is not used for the edges $C$.

In Figure 9, we can see a comparison between our GFC method and the one proposed by Perez [3,20]. First, we notice that the GFC approach has a better color preservation than the Perez method. For example, we can see on the person image that the sky has a gradient of different colors. We also see that the castle image has many small coloration artefacts inside the castle and at the top of the sky. For the leopard image, both methods yield to similar results. In summary, Figure 9 shows that the proposed GFC appears more beautiful and the painting effect is more natural since the method is more accurate on fine details and color restoration than the competing Perez algorithm [3].
16

Figure 9. Examples of reconstructed images with perturbed gradient from equation (35) with edges information from structured forest [22] with NMS and $\alpha = 0.5$. (a) Original image; (b) Reconstructed image using Perez method [3,20]; (c) Reconstructed image using our GFC method.

4. Conclusion

This study detailed the development of the CAMERA-I-GFC method, which allowed to compute an image $V_E$ based on a conservative or perturbed field or Laplacian. To do so, it explained the theory behind the Green function convolution and was mathematically proven in section “2.3” that it is the optimal solution to generate a potential from a perturbed field, and we demonstrated in section “3.1.3” that the error is negligible with an RMSE of 0.011 on 256 levels. Then, we showed in Figure 2 that the computation speed is almost instantaneous compared to other methods, since it took around 2.0ms including the preparation time on a GPU for a 1200x801 image. It also showed that the reconstruction error is negligible on a dataset of 1000 images of resolution ~400x300, and even lower when the image resolution is higher. Finally, it showed different use-cases of perturbations added to the gradient, such as the thresholding, the edge preserving blurring, the seamless cloning and the painting effect.

This study allowed to build a robust and fast way to edit an image from its gradient, which can be used in many applications without significant impact on the computation time. Future works could focus on more concrete applications, such as supervised image/video editing and diffusion curves.

Acknowledgment

The authors are grateful to NSERC, through the discovery grant program RGPIN-2014-06289, and FRQNT/INTER for their financial support.

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