TORSION IN BOUNDARY COINVARIANTS AND K-THEORY FOR AFFINE BUILDINGS

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Abstract. Let \((G, \mathcal{J}, N, S)\) be an affine topological Tits system, and let \(\Gamma\) be a torsion free cocompact lattice in \(G\). This article studies the coinvariants \(H_0(\Gamma; C(\Omega, \mathbb{Z}))\), where \(\Omega\) is the Furstenberg boundary of \(G\). It is shown that the class \([1]\) of the identity function in \(H_0(\Gamma; C(\Omega, \mathbb{Z}))\) has finite order, with explicit bounds for the order.

A similar statement applies to the \(K_0\) group of the boundary crossed product \(C^*\)-algebra \(C(\Omega) \rtimes \Gamma\). If the Tits system has type \(A_2\), exact computations are given, both for the crossed product algebra and for the reduced group \(C^*\)-algebra.

1. Introduction

This article is concerned with coinvariants for group actions on the boundary of an affine building. The results are most easily stated for subgroups of linear algebraic groups. Let \(k\) be a non-archimedean local field with finite residue field \(\mathbb{F}\) of order \(q\). Let \(G\) be the group of \(k\)-rational points of an absolutely almost simple, simply connected linear algebraic \(k\)-group. Then \(G\) acts on its Bruhat-Tits building \(\Delta\), and on its Furstenberg boundary \(\Omega\).

Let \(\Gamma\) be a torsion free lattice in \(G\). The abelian group \(C(\Omega, \mathbb{Z})\) of continuous integer-valued functions on \(\Omega\) has the structure of a \(\Gamma\)-module. The module of \(\Gamma\)-coinvariants \(\Omega_{\Gamma} = H_0(\Gamma; C(\Omega, \mathbb{Z}))\) is a finitely generated group. We prove that the class \([1]\) in \(\Omega_{\Gamma}\) of the constant function \(1 \in C(\Omega, \mathbb{Z})\) has finite order. If \(G\) is not one of the exceptional types \(\tilde{E}_8, \tilde{F}_4\) or \(\tilde{G}_2\), then the order of \([1]\) is less than \(\text{covol}(\Gamma)\), where the Haar measure \(\mu\) on \(G\) is normalized so that an Iwahori subgroup of \(G\) has measure 1. There is a weaker estimate for groups of exceptional type. If \(G\) has rank 2 then the estimates are significantly improved.

Date: January 15, 2005.

2000 Mathematics Subject Classification. 51E24, 46L80.

Key words and phrases. affine building, Furstenberg boundary, K-theory, \(C^*\)-algebra.

Typeset by \LaTeX.
The topological action of $\Gamma$ on the Furstenberg boundary is encoded in the crossed product $C^*$-algebra $A_{\Gamma} = C(\Omega) \rtimes \Gamma$. Embedded in $A_{\Gamma}$ is the reduced group $C^*$-algebra $C^*_r(\Gamma)$, which is the completion of the complex group algebra of $\Gamma$ in the regular representation as operators on $\ell^2(\Gamma)$. The action of $\Gamma$ on $\Omega$ is amenable, so the K-theory of $A_{\Gamma}$ is computable by known results, in contrast to that of $C^*_r(\Gamma)$, which rests on the validity of the Baum-Connes conjecture. The natural embedding $C(\Omega) \to A_{\Gamma}$ induces a homomorphism

$$\varphi : \Omega_{\Gamma} \to K_0(A_{\Gamma})$$

and $\varphi([1]) = [1]_{K_0}$, the class of 1 in the $K_0$-group of $A_{\Gamma}$. Therefore $[1]_{K_0}$ has finite order in $K_0(A_{\Gamma})$.

If $\Gamma$ is a torsion free lattice in $G = SL_3(k)$ then exact computations can be performed. The Baum-Connes Theorem of V. Lafforgue [La] is used to compute $K_*(C^*_r(\Gamma))$ and the results of [RS] are used to compute $K_*(A_{\Gamma})$. In particular $K_0(C^*_r(\Gamma)) = \mathbb{Z}^{|S|}$, a free abelian group, one of whose generators is the class $[1]$. The embedding of $C^*_r(\Gamma)$ into $A_{\Gamma}$ induces a homomorphism $\psi : K_*(C^*_r(\Gamma)) \to K_*(A_{\Gamma})$. This homomorphism is not injective, since $[1]$ has finite order in $K_0(A_{\Gamma})$. The computations at the end of the article suggest that the only reason for failure of injectivity of the homomorphism $\psi$ is the fact that $[1]$ has finite order in $K_0(A_{\Gamma})$.

Much of this article considers the more general case where $\Gamma$ is a subgroup of a topological group $G$ with a BN-pair, and $\Gamma$ acts on the boundary $\Omega$ of the affine building of $G$.

The results are organized as follows. Sections 2 and 3 state and prove the main result concerning the class $[1]$ in $\Omega_{\Gamma}$. Section 4 gives improved estimates in the rank 2 case. Section 5 studies the connection with the K-Theory of the boundary algebra $A_{\Gamma}$. Comparison with K-theory of the reduced $C^*$-algebra $C^*_r(\Gamma)$ is made in Section 6 which contains some exact computational results for buildings of type $\tilde{A}_2$.

2. Torsion in Boundary Coinvariants

Let $(G, \mathcal{J}, N, S)$ be an affine topological Tits system [Gr, Definition 2.3]. Then $G$ is a group with a BN-pair in the usual algebraic sense [Ti1, Section 2] and the Weyl group $W = N/(\mathcal{J} \cap N)$ is an infinite Coxeter group with generating set $S$. The subgroup $\mathcal{J}$ of $G$ is called an Iwahori subgroup. A subgroup of $G$ is parahoric if it contains a conjugate of $\mathcal{J}$. The topological requirements are that $G$ is a second countable locally compact group and that all proper parahoric subgroups of $G$ are open and compact [Gr, Definition 2.3].
Let $n + 1 = |S|$ be the rank of the Tits system. The group $G$ acts on the Tits complex $\Delta$, which is an affine building of dimension $n$. It will be assumed throughout that $\Delta$ is irreducible; in other words, the Coxeter group $W$ is not a direct product of nontrivial Coxeter groups. Denote by $\Delta^i$ the set of $i$-simplices of $\Delta$, $(0 \leq i \leq n)$. The vertices of $\Delta$ are the maximal proper parahoric subgroups of $G$, and a finite set of such subgroups spans a simplex in $\Delta$ if and only if its intersection is parahoric. The action of $G$ on $\Delta$ is by conjugation of subgroups. The building $\Delta$ is a union of $n$-dimensional subcomplexes, called apartments. Each apartment is a Coxeter complex, with Coxeter group $W$.

Associated with the Coxeter system $(W, S)$ there is a Coxeter diagram of type $\tilde{X}_n$ ($X = A, B, \ldots, G$), whose vertex set $I$ is a set of $n + 1$ types, which are in natural bijective correspondence with the elements of $S$. Each vertex $v \in \Delta^0$ has a type $\tau(v) \in I$. The type of a simplex in $\Delta$ is the set of types of its vertices. By construction, the action of $G$ on $\Delta$ preserves types. A type $t \in I$ is special if deleting $t$ and all the edges containing $t$ from the diagram of type $\tilde{X}_n$ results in the diagram of the corresponding finite Coxeter group. A vertex $v \in \Delta$ is said to be special if its type $\tau(v)$ is special [BT, 1.3.7].

A simplex of maximal dimension $n$ in $\Delta$ is called a chamber. Every chamber has exactly one vertex of each type. If $\sigma$ is any chamber containing the vertex $v$ then the codimension-1 face of $\sigma$ which does not contain $v$ has type $I - \{\tau(v)\}$.

The action of $G$ on $\Delta$ is strongly transitive, in the sense that $G$ acts transitively on the set of pairs $(\sigma, A)$ where $\sigma$ is a chamber contained in an apartment $A$ of $\Delta$. The building $\Delta$ is locally finite, in the sense that the number of chambers containing any simplex is finite, and it is thick, in the sense that each simplex of dimension $n - 1$ is contained in at least three chambers. If $\tau$ is a simplex in $\Delta$ of dimension $n - 1$ and type $I - \{t\}$, then the number of chambers of $\Delta$ which contain $\tau$ is $q_t + 1$ where $q_t \geq 2$. The integer $q_t$ depends only on $t$; not on $\tau$.

Associated with the group $G$ there is also a spherical building, the building at infinity $\Delta_\infty$. The boundary $\Omega$ of $\Delta$ is the set of chambers of $\Delta_\infty$, endowed with a natural compact totally disconnected topology, which we shall describe later on. Since $G$ acts transitively on the chambers of $\Delta_\infty$, $\Omega$ may be identified with the topological homogeneous space $G/B$, where the Borel subgroup $B$ is the stabilizer of a chamber of $\Delta_\infty$. 
**Example 2.1.** A standard example is \( G = \text{SL}_{n+1}(\mathbb{Q}_p) \), where \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers. In this case \( B \) is the subgroup of upper triangular matrices in \( G \), and \( \Omega \) is the Furstenberg boundary of \( G \).

If \( \Gamma \) is a subgroup of \( G \), then \( \Gamma \) acts on \( \Omega \), and the abelian group \( C(\Omega, \mathbb{Z}) \) of continuous integer-valued functions on \( \Omega \) has the structure of a \( \Gamma \)-module. The module of \( \Gamma \)-coinvariants, \( C(\Omega, \mathbb{Z})_\Gamma \), is the quotient of \( C(\Omega, \mathbb{Z}) \) by the submodule generated by \( \{ g \cdot f - f : g \in \Gamma, f \in C(\Omega, \mathbb{Z}) \} \). Recall that \( C(\Omega, \mathbb{Z})_\Gamma \) is the homology group \( H_0(\Gamma; C(\Omega, \mathbb{Z})) \). For the rest of this article, \( C(\Omega, \mathbb{Z})_\Gamma \) will be denoted simply by \( \Omega_\Gamma \). Define \( c(\Gamma) \in \mathbb{Z}_+ \cup \{ \infty \} \) to be the number of \( \Gamma \)-orbits of chambers in \( \Delta \).

If \( \Gamma \) is a torsion free cocompact lattice in \( G \), then \( c(\Gamma) \) is the number of \( n \)-cells of the finite cell complex \( \Delta \setminus \Gamma \). Suppose that the Haar measure \( \mu \) on \( G \) has the Tits normalization \( \mu(\mathcal{I}) = 1 \) \[ Ti2, \S 3.7 \]. Then \( c(\Gamma) = \text{covol}(\Gamma) \).

We shall see below that if \( \Gamma \) is a torsion free cocompact lattice in \( G \) then \( \Omega_\Gamma \) is a finitely generated abelian group. Note that such a torsion free lattice \( \Gamma \) acts freely and properly on \( \Delta \) \[ Cd \, \text{Lemma 2.6, Lemma 3.3} \]. If \( f \in C(\Omega, \mathbb{Z}) \) then \( [f] \) will denote its class in \( \Omega_\Gamma \). Also, \( 1 \) will denote the constant function defined by \( 1(\omega) = \omega \) for \( \omega \in \Omega \).

**Theorem 2.2.** Let \((G, \mathcal{I}, N, S)\) be an affine topological Tits system and let \( \Gamma \) be a torsion free lattice in \( G \). Then \( \Omega_\Gamma \) is a finitely generated abelian group and the following statements hold.

1. The element \([1]\) has finite order in \( \Omega_\Gamma \).
2. If \( s \in \mathcal{I} \) is a special type, then the order of \([1]\) in \( \Omega_\Gamma \) satisfies
   \[ \text{ord}([1]) < q_s \cdot \text{covol}(\Gamma). \]
3. If, in addition, \( G \) is not one of the exceptional types \( \tilde{G}_2, \tilde{F}_4, \tilde{E}_8 \), then
   \[ \text{ord}([1]) < \text{covol}(\Gamma). \]

**Remark 2.3.** A torsion free lattice in \( G \) is automatically cocompact \[ Sc2, \Pi.1.5 \].

**Remark 2.4.** Suppose that \( \Gamma \) is isomorphic to a subgroup of a group \( \Gamma' \) and that the action of \( \Gamma \) on \( \Omega \) extends to an action of \( \Gamma' \) on \( \Omega \). Then there is a natural surjection \( \Omega_\Gamma \to \Omega_{\Gamma'} \). It follows that Theorem 2.2 remains true if \( \Gamma \) is replaced by any such group \( \Gamma' \).

**Remark 2.5.** The group \( \Omega_\Gamma \) depends only on \( \Gamma \) and not on the ambient group \( G \). This follows from the rigidity results of \[ KL \], if \( n \geq 2 \), and from \[ Gr \] if \( n = 1 \).
We now describe briefly how Theorem 2.2 applies to algebraic groups. Let $k$ be a non-archimedean local field and let $G$ be the group of $k$-rational points of an absolutely almost simple, simply connected linear algebraic $k$-group: e.g. $k = \mathbb{Q}_p$, $G = \text{SL}_{n+1}(\mathbb{Q}_p)$. Associated with $G$ there is a topological Tits system of rank $n + 1$, where $G$ has $k$-rank $n$ \[\text{[IM]}\]. Now $G$ acts properly on the corresponding Bruhat-Tits building $\Delta$ \[\text{[TI2, §2.1]},\] and on the boundary $\Omega = G/B$, where $B$ is a Borel subgroup \[\text{[BM, Section 5]}\].

Let $q$ be the order of the residue field $\overline{k}$. For each type $t \in I$ there is an integer $d(t)$ such that $q_t = q^{d(t)}$. That is, any simplex $\tau$ of codimension one and type $I - \{t\}$ is contained in $q^{d(t)} + 1$ chambers \[\text{[TI2, §2.4]}\]. If $G$ is $k$-split (i.e. there is a maximal torus $T \subset G$ which is $k$-split) then $d(t) = 1$ for all $t \in I$ \[\text{[TI2, §3.5.4]}\].

If $k$ has characteristic zero, then the condition that $\Gamma$ is torsion free can be omitted from Theorem 2.2. Recall that a non-archimedean local field of characteristic zero is a finite extension of $\mathbb{Q}_p$, for some prime $p$.

**Corollary 2.6.** Let $k$ be a non-archimedean local field of characteristic zero. Let $G$ be the group of $k$-rational points of an absolutely almost simple, simply connected linear algebraic $k$-group. If $\Gamma$ is a lattice in $G$, then the class $[1]$ has torsion in $\Omega_\Gamma$.

**Proof.** A lattice $\Gamma$ in $G$ is automatically cocompact \[\text{[M], Proposition IX, 3.7]}\]. By Selberg’s Lemma \[\text{[G]}, \text{Theorem 2.7]}\], $\Gamma$ has a torsion free subgroup $\Gamma_0$ of finite index. Now Theorem 2.2 implies that $[1]$ has finite order in $\Omega_{\Gamma_0}$. The result follows from the observation that there is a natural surjection $\Omega_{\Gamma_0} \to \Omega_\Gamma$. \[\Box\]

3. **Proof of Theorem 2.2**

Throughout this section, the assumptions of Theorem 2.2 are in force. Before proving Theorem 2.2, we require some preliminaries. Recall that a gallery of type $i = (i_1, \ldots, i_k)$ is a sequence of chambers $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ such that each pair of successive chambers $\sigma_{j-1}, \sigma_j$ meet in a common face of type $I - \{i_j\}$. Choose a special type $s \in I$, which will remain fixed throughout this section. Fix once and for all the following data.

- **(A1)** An apartment $A$ in $\Delta$.
- **(A2)** A sector $S$ in $A$ with base vertex $v$ of type $s$ and base chamber $C$.
- **(A3)** The unique vertex $v' \in S$ of type $s$, obtained by reflecting $v$ in a codimension-1 face of $C$.
- **(A4)** The unique chamber $C'$ containing $v'$ which is the base chamber of a subsector of $S$. 
\( (A5) \) A minimal gallery of type \( i = (i_1, \ldots, i_k) \) from \( C \) to \( C' \), where \( i_1 = s \). This minimal gallery necessarily lies inside \( S \).

These data are illustrated by Figure 11 which shows part of an apartment in a building of type \( \tilde{G}_2 \) and a minimal gallery from \( C \) to \( C' \). Special vertices are indicated by large points.

![Figure 1](image_url)

**Figure 1.** Part of an apartment \( A \) in a building of type \( \tilde{G}_2 \).

Now let \( \mathcal{D} = \Delta^n / \Gamma \), the set of \( \Gamma \)-orbits of chambers of \( \Delta \). Since \( \Gamma \) acts freely and cocompactly on \( \Delta \), \( \mathcal{D} \) is finite and elements of \( \mathcal{D} \) are in 1-1 correspondence with the set of \( n \)-cells of the finite cell complex \( \Delta / \Gamma \).

If \( x, y \in \mathcal{D} \), let \( M_i(x, y) \) denote the number of \( \Gamma \)-orbits of galleries of type \( i \) which have initial chamber in \( x \) and final chamber in \( y \). If \( \sigma_0 \in x \) is fixed then \( M_i(x, y) \) is equal to the number of galleries \( (\sigma_0, \sigma_1, \ldots, \sigma_k) \) of type \( i \) with final chamber \( \sigma_k \in y \). To see this, note that any gallery of type \( i \) with initial chamber in \( x \) and final chamber in \( y \) lies in the \( \Gamma \)-orbit of such a gallery \( (\sigma_0, \sigma_1, \ldots, \sigma_k) \). Moreover, two distinct galleries of this form lie in different \( \Gamma \)-orbits. For suppose that \( (\sigma_0, \sigma_1, \ldots, \sigma_j, \tau_{j+1}, \ldots, \tau_k) \) is another such gallery, with \( \tau_{j+1} \neq \sigma_{j+1} \), the first chamber at which they differ. Then \( \tau_{j+1} \) and \( \sigma_{j+1} \) have a common face of codimension one, and so lie in different \( \Gamma \)-orbits, since the action
of $\Gamma$ is free. (If $g\tau_{j+1} = \sigma_{j+1}$, then $g$ must fix every point in the common codimension one face and so $g = 1$.) A similar argument shows that if $\sigma_k \in y$ is fixed then then $M_i(x, y)$ is equal to the number of galleries $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of type $i$ with initial chamber $\sigma_0 \in x$. Cocompactness of the $\Gamma$-action implies that $M_i(x, y)$ is finite.

If $\sigma$ is a chamber in $\Delta$, then the number $N_i$ of galleries $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of type $i$, with final chamber $\sigma_k = \sigma$, is independent of $\sigma$. This follows, since $G$ acts transitively on the set $\Delta^s$ of chambers of $\Delta$. Note that $N_i > 1$, by thickness of the building $\Delta$.

Two different galleries of type $i$ which have final chamber $\sigma$ are necessarily in different $\Gamma$-orbits, by freeness of the action of $\Gamma$. It follows that if $y \in \mathcal{D}$, then the number of $\Gamma$-orbits of galleries $(\sigma_1, \ldots, \sigma_k)$ of type $i$, with $\sigma_k \in y$, is equal to $N_i$. In other words, for each $y \in \mathcal{D}$,

\begin{equation}
\sum_{x \in \mathcal{D}} M_i(x, y) = N_i.
\end{equation}

Recall that if $\tau$ is a simplex of $\Delta$ of codimension one and type $I - \{t\}$, then the number of chambers of $\Delta$ which contain $\tau$ is $q_t + 1$ where $q_t \geq 2$. Thus the number of galleries $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of type $i$, with final chamber $\sigma_k = \sigma$ (fixed, but arbitrary), is equal to $q_{i_1} q_{i_2} \cdots q_{i_k}$, where $i_1 = s$. On the other hand, this number is also equal to the number of galleries $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of type $i$, with initial chamber $\sigma_0 = \sigma$ (fixed, but arbitrary). It follows that, for each $x \in \mathcal{D}$,

\begin{equation}
\sum_{y \in \mathcal{D}} M_i(x, y) = N_i.
\end{equation}

**Definition 3.1.** Fix a type $s \in I$. Let $\alpha_s$ denote the number of chambers of $\Delta$ which contain a fixed vertex $u$ of type $s$. Since $G$ acts transitively on the set of vertices of type $s$, $\alpha_s$ does not depend on the choice of the vertex $u$.

**Remark 3.2.** The Iwahori subgroup $I$ is a chamber of $\Delta$. Let the parahoric subgroup $P_s < G$ be the vertex of the type $s$ of $I$. Then $P_s$ is a maximal compact subgroup of $G$ containing $I$ and $\alpha_s = |P_s : I|$. (In [G04 Section 3], $\alpha_s$ is denoted $\tau_{\{s\}}$.)

**Lemma 3.3.** Let $s \in I$ be a special type. Then

\begin{equation}
N_i < q_s \cdot \alpha_s.
\end{equation}

**Proof.** Fix a chamber $\sigma_0$. We must estimate the number of galleries $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of type $i$ (with initial chamber $\sigma_0$).

There are $q_s$ possible choices for $\sigma_1$. Suppose that $\sigma_1$ has been chosen and let $u$ be the vertex of $\sigma_1$ not belonging to $\sigma_0$. By construction, $\sigma_k$
also contains \( u \) (Figure 2) and so there are less than \( \alpha_s \) possible choices for \( \sigma_k \). (Note the \( \sigma_k \neq \sigma_1 \).) Once \( \sigma_k \) has been chosen, there is a unique (minimal) gallery of type \((i_2, \ldots, i_k)\) with initial chamber \( \sigma_1 \) and final chamber \( \sigma_k \). In other words, the gallery \((\sigma_0, \sigma_1, \ldots, \sigma_k)\) is uniquely determined, once \( \sigma_1 \) and \( \sigma_k \) are chosen. There are therefore at most \( q_s(\alpha_s - 1) \) choices for this gallery. \( \square \)

Remark 3.4. An easy calculation in \( \tilde{A}_2 \) buildings shows that the estimate (3) cannot be improved to \( N_i \leq \alpha_s \).

Definition 3.5. Let \( \Gamma \) be a torsion free cocompact lattice in \( G \). If \( s \in I \), let \( n_s(\Gamma) \) (or simply \( n_s \), if \( \Gamma \) is understood) denote the number of \( \Gamma \)-orbits of vertices of type \( s \) in \( \Delta \).

Recall that \( \text{covol}(\Gamma) \) is equal to the number of \( \Gamma \)-orbits of chambers in \( \Delta \).

Lemma 3.6. Fix a type \( s \in I \). Then \( \text{covol}(\Gamma) = n_s(\Gamma) \cdot \alpha_s \).

Proof. Choose a set \( S \) of representative vertices from the \( \Gamma \)-orbits of vertices of type \( s \) in \( \Delta \). Thus \(|S| = n_s(\Gamma)\). For \( v \in S \), let \( R_v \) denote the set of chambers containing \( v \). Each \( R_v \) contains \( \alpha_s \) chambers. We claim that the number of chambers in \( R = \bigcup_{v \in S} R_v \) equals \( \text{covol}(\Gamma) \).

Each chamber in \( \Delta \) is clearly in the \( \Gamma \)-orbit of some chamber in \( R \). Moreover, any two distinct chambers in \( R \) lie in different \( \Gamma \)-orbits. For suppose that \( \sigma_v \in R_v, \sigma_w \in R_w \) and \( g\sigma_v = \sigma_w \), where \( g \in \Gamma \). Then \( gv = w \), since the action of \( \Gamma \) is type preserving and every chamber contains exactly one vertex of type \( s \). Therefore \( v = w \), since distinct vertices in \( S \) lie in different \( \Gamma \)-orbits. Moreover \( g = 1 \), since the action of \( \Gamma \) is free. Thus \( \sigma_v = \sigma_w \). This shows that there are \( \text{covol}(\Gamma) \) chambers in \( R \). \( \square \)

Before proving Theorem 2.2, we provide more details of the structure of the boundary \( \Omega \). Let \( \sigma \) be a chamber in \( \Delta^n \) and let \( s \) be a special vertex of \( \sigma \). The codimension one faces of \( \sigma \) having \( s \) as a vertex determine roots containing \( \sigma \), and the intersection of these roots is a sector in \( \Delta \) with base vertex \( s \) and base chamber \( \sigma \). Two sectors are parallel if the Hausdorff distance between them is finite. This happens if and only if they contain a common subsector. The boundary \( \Omega \) of \( \Delta \) is the set of parallel equivalence classes of sectors in \( \Delta \) \([\text{Ron}, \text{Chap. 9.3}]\). If \( \omega \in \Omega \) and if \( s \) is a special vertex of \( \Delta \) then there exists a unique sector \([s, \omega]\) in \( \omega \) with base vertex \( s \) \([\text{Ron}, \text{Lemma 9.7}]\).

If \( \sigma \in \Delta^n \), let \( o(\sigma) \) denote the vertex of \( \sigma \) of type \( s \). Recall that vertices of type \( s \) are special. Let \( \Omega(\sigma) \) denote the set of boundary
point \( \omega \) whose representative sectors have base vertex \( o(\sigma) \) and base chamber \( \sigma \). That is,

\[
\Omega(\sigma) = \{ \omega \in \Omega : \sigma \subset [o(\sigma), \omega] \}.
\]

The sets \( \Omega(\sigma), \sigma \in \Delta^n \), form a basis for the topology of \( \Omega \). Moreover, each \( \Omega(\sigma) \) is a clopen subset of \( \Omega \). Let \( \gamma_i \) denote the set of ordered pairs \( (\sigma, \sigma') \in \Delta^n \times \Delta^n \) such that there exists a gallery of type \( i \) from \( \sigma \) to \( \sigma' \). Then for each \( \sigma \in \Delta^n \), \( \Omega(\sigma) \) can be expressed as a disjoint union

\[
(4) \quad \Omega(\sigma) = \bigcup_{(\sigma, \sigma') \in \gamma_i} \Omega(\sigma').
\]

For if \( \omega \in \Omega(\sigma) \), then the sector \([o(\sigma), \omega]\) is strongly isometric, in the sense of [Gr 15.5] to the sector \( S \) in the apartment \( A \), as described at the beginning of this section. Let \( \sigma' \) be the image under this strong isometry of the chamber \( C' \) in \( A \). Then \( (\sigma, \sigma') \in \gamma_i \) and \( \omega \in \Omega(\sigma') \). Thus \( \Omega(\sigma) \) is indeed a subset of the right hand side of (4). Conversely, each set \( \Omega(\sigma') \) on the right hand side of (4) is contained in \( \Omega(\sigma) \). For if \( (\sigma, \sigma') \in \gamma_i \) and \( \omega \in \Omega(\sigma') \) then the strong isometry from \([o(\sigma'), \omega]\)
onto $S'$ extends to a strong isometry from $[o(\sigma), \omega]$ onto $S$ \cite{G抄袭} §15.5 Lemma. Thus $\omega \in \Omega(\sigma)$.

To check that the union on the right of (4) is disjoint, suppose that $\omega \in \Omega(\sigma') \cap \Omega(\sigma'')$, where $(\sigma, \sigma'), (\sigma, \sigma'') \in \gamma_i$. Then the strong isometry from $[o(\sigma'), \omega]$ onto $[o(\sigma''), \omega]$ extends to a strong isometry from $[o(\sigma), \omega]$ onto itself, which is necessarily the identity map. In particular, $\sigma' = \sigma''$.

If $\sigma \in \Delta^n$, let $\chi_\sigma \in C(\Omega, \mathbb{Z})$ denote the characteristic function of $\Omega(\sigma)$. That is

$$\chi_\sigma(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi_\sigma - \chi_g = \chi_\sigma - g \cdot \chi_\sigma$ for each $g \in \Gamma$, the class $[\chi_\sigma]$ of $\chi_\sigma$ in $\Omega_\Gamma$ depends only on the $\Gamma$-orbit of $\sigma$ in $\Delta^n$. If $x = \Gamma \sigma \in \mathcal{D}$, it therefore makes sense to define

$$[x] = [\chi_\sigma] \in \Omega_\Gamma.$$ (5)

Now it follows from (4) that, for each $\sigma \in \Delta^n$,

$$\chi_\sigma = \sum_{(\sigma, \sigma') \in \gamma_i} \chi_{\sigma'} = \sum_{y \in \mathcal{D}} \sum_{\sigma' \in y} \chi_{\sigma'}.$$ (6)

Passing to equivalence classes in $\Omega_\Gamma$ gives, for each $x \in \mathcal{D}$,

$$[x] = \sum_{y \in \mathcal{D}} M_i(x, y)[y].$$ (7)

We can now proceed with the proof of the Theorem 2.2. If $s$ is a vertex of type $s$ of $\Delta$, then each element $\omega \in \Omega$ lies in $\Omega(\sigma)$ where $\sigma$s is the base chamber of the sector $[s, \omega)$. Moreover $\omega$ lies in precisely one such set $\Omega(\sigma)$, with $\sigma \in \Delta^n$, $o(\sigma) = s$. Therefore

$$1 = \sum_{\sigma \in \Delta^n, o(\sigma) = s} \chi_\sigma.$$ (8)

Since the action of $\Gamma$ on $\Delta$ is free and type preserving, no two chambers $\sigma \in \Delta^n$ with $o(\sigma) = s$ lie in the same $\Gamma$-orbit. To simplify notation, let $n_s = n_s(\Gamma)$, the number of $\Gamma$-orbits of vertices of type $s$ in $\Delta$. If we choose a representative set $\mathcal{S}$ of vertices of type $s$ in $\Delta$ then the chambers containing these vertices form a representative set of
chambers, by the proof of Lemma 3.6. It follows that in ΩΓ,
\[ n_s \cdot [1] = \sum_{\sigma \in \Delta^n} \sum_{o(\sigma) = s} \chi_{\sigma} \]  
(by (5))
\[ = \sum_{x \in D} [x]. \]
Therefore
\[ n_s \cdot [1] = \sum_{x \in D} \sum_{y \in D} M_i(x, y)[y] \]  
(by (7))
\[ = \sum_{y \in D} \left( \sum_{x \in D} M_i(x, y) \right)[y] \]
\[ = \sum_{y \in D} N_i \cdot [y] \]  
(by (11))
\[ = N_i n_s \cdot [1]. \]
It follows that
\[ n_s(N_i - 1) \cdot [1] = 0, \]
which proves the first assertion of Theorem 2.2.

Using Lemmas 3.3, 3.6 we can estimate the order of the element [1].

\[ n_s(N_i - 1) < n_s \cdot (q_s \alpha_s - 1) = q_s \cdot \text{covol}(\Gamma) - n_s. \]
This proves the second assertion of Theorem 2.2. The next Lemma proves the final assertion of Theorem 2.2 by showing that the estimate of the order of [1] can be improved if certain exceptional cases are excluded.

**Lemma 3.7.** Suppose that the Weyl group is not one of the exceptional types ˜E8, ˜F4, ˜G2. Then
\[ \text{ord}([1]) < \text{covol}(\Gamma). \]

**Proof.** An examination of the possible Coxeter diagrams [Bou, Chap VI, No 4.4, Théorème 4] shows that if the diagram is not one of the types ˜E8, ˜F4, ˜G2, then it contains at least two special types. Therefore every chamber of Δ contains at least two special vertices. Choose two such vertices and suppose that they have types s and t, say. In that case the condition (A3) on the apartment A in Δ can be changed to read:

- (A3′) The unique special vertex \( v' \in S \) of type \( t \) which lies in \( C \).
Assume that the remaining conditions (A1), (A2), (A4), (A5) are unchanged. Figure 3 illustrates the setup in the $\tilde{B}_2$ case.

The proof proceeds exactly as before, except that all the chambers in a gallery $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of type $i$ now contain a common vertex $u$ of type $t$. Therefore equation (3) becomes

$$N_i < \alpha_t.$$ 

Observe that one must be careful with the notation. For example in equation (6), the function $\chi_\sigma$ on the left is now defined in terms of sectors based at the vertex of type $s$ of $\sigma$, whereas the functions $\chi_{\sigma'}$ on the right will now be defined in terms of sectors based at the vertex of type $t$ of $\sigma'$. Equation (9) becomes

$$(11) \quad (n_t N_i - n_s) \cdot [1] = 0.$$ 

The order of the element $[1]$ is bounded by

$$(12) \quad n_t \cdot \alpha_t - n_s = \text{covol}(\Gamma) - n_s.$$ 

Finally, we verify that $\Omega_{\Gamma}$ is a finitely generated group. Sets of the form $\Omega(\sigma)$, $\sigma \in \Delta^n$, form a basis of clopen sets for the topology of $\Omega$. It follows that the abelian group $C(\Omega, \mathbb{Z})$ is generated by the set of characteristic functions $\{\chi_\sigma : \sigma \in \Delta^n\}$. We show that $\Omega_{\Gamma}$ is generated by $\{[x] : x \in \mathcal{D}\}$.

**Lemma 3.8.** Every clopen set $V$ in $\Omega$ may be expressed as a finite disjoint union of sets of the form $\Omega(\sigma)$, $\sigma \in \Delta^n$. 

![Figure 3. Part of an apartment $A$ in a building of type $\tilde{B}_2$, and a minimal gallery from $C$ to $C''$.](image-url)
Proof. Fix a special vertex $s$ of type $s$ in $\Delta$. For each $\omega \in \Omega$, sets of the form $\Omega(\sigma)$ with $\sigma \in \Delta^n$ and $\sigma \subset [s, \omega)$ form a basic family of open neighbourhoods of $\omega$. Therefore, for each $\omega \in V$, there exists a chamber $\sigma_\omega \in \Delta^n$ with $\sigma_\omega \subset [s, \omega)$ and $\omega \in \Omega(\sigma_\omega) \subseteq V$. The clopen set $V$, being compact, is a finite union of such sets:

$$V = \Omega(\sigma_{\omega_1}) \cup \ldots \cup \Omega(\sigma_{\omega_k}).$$

Fix a sector $Q$ in $\Delta$, with base vertex $s$. For each $j$, $1 \leq j \leq k$, let $C_j$ be the chamber in $Q$ which is the image of $\sigma_{\omega_j}$, under the unique strong isometry from $[s, \omega_j)$ onto $Q$. Let $Q_j$ be the subsector of $Q$ with base chamber $C_j$ ($1 \leq j \leq k$), and choose a chamber $C$ in $\bigcap_{j=1}^k Q_j$. Informally, $C$ is chosen to be sufficiently far away from the base vertex $s$.

For $1 \leq j \leq k$, let $\tau_j$ be the chamber in $[s, \omega_j)$ which is the image of $C$ under the strong isometry from $[s, \omega_j)$ onto $Q$. For each $\omega \in \Omega(\sigma_{\omega_j})$ there is a retraction from $[s, \omega)$ onto $[s, \omega_j)$ [Gu 4.2]. Let $\tau_j(\omega)$ be the inverse image of the chamber $\tau_j$ under this retraction. By local finiteness of $\Delta$, there are only finitely many such chambers $\tau_j(\omega)$, $\omega \in \Omega(\sigma_{\omega_j})$. Call them $\tau_{j,l}$, $1 \leq l \leq n_j$. Thus $\Omega(\sigma_{\omega_j})$ may be expressed as a finite disjoint union:

$$\Omega(\sigma_j) = \bigcup_l \Omega(\tau_{j,l}).$$

Moreover, if $\omega \in \Omega(\tau_{j,l})$ then the strong isometry from $[s, \omega)$ onto $Q$ maps $\tau_{j,l}$ to the chamber $C$. Finally, $V$ may be expressed as a disjoint union:

$$V = \bigcup_{j,l} \Omega(\tau_{j,l}).$$

To check that this union is indeed disjoint, suppose that $\omega \in \Omega(\tau_{j,l}) \cap \Omega(\tau_{r,s})$. Then, under the strong isometry from $Q$ onto $[s, \omega)$, the image of the chamber $C$ is equal to both $\tau_{j,l}$ and $\tau_{r,s}$. In particular, $\tau_{j,l} = \tau_{r,s}$. □

Proposition 3.9. Let $(G, \mathfrak{I}, N, S)$ be an affine topological Tits system, and let $\Gamma$ be a subgroup of $G$. Then

(a) The abelian group $C(\Omega, \mathbb{Z})$ is generated by the set of characteristic functions $\{\chi_\sigma : \sigma \in \Delta^n\}$.
(b) $\Omega_\Gamma$ is generated by $\{[x] : x \in \mathfrak{I}\}$.

Proof. (a) Any function $f \in C(\Omega, \mathbb{Z})$ is bounded, by compactness of $\Omega$, and so takes finitely many values $n_i \in \mathbb{Z}$. Now $V_i = \{\omega \in \Omega : f(\omega) = n_i\}$ is a clopen set in $\Omega$. It follows from the preceding Lemma that $f$ may be expressed as a finite sum $f = \sum_j m_j \chi_{\sigma_j}$, with $\sigma_j \in \Delta^n$. 
(b) This is an immediate consequence of (a).

4. Further calculations in the rank 2 case

This section is devoted to showing that the estimate for the order of $[1]$ given by Theorem 2.2 can be improved if the building $\Delta$ is 2-dimensional. The group $G$ has type $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$. Denote the type set by $I = \{s, a, b\}$, where $s$ is a special type of the corresponding Coxeter diagram, as indicated below. Note that in the $\tilde{B}_2$ case, the vertex $b$ is also special. In the $\tilde{A}_2$ case, all vertices are special and $q_t = q$ for all $t \in I$.

\[
\begin{array}{c}
\tilde{A}_2 & \tilde{B}_2 & \tilde{G}_2 \\
\bullet & \bullet & \bullet \\
s & a & s \\
\end{array}
\]

Proposition 4.1. Under the preceding assumptions, let $\Gamma$ be a torsion free lattice in $G$. Then

\[(q_s^2 - 1)n_s \cdot [1] = 0\]

in $\Omega_{\Gamma}$.

Proof. We prove the $\tilde{G}_2$ case. For the minimal gallery of type $i$ between $C$ and $C'$ described in Figure 4, we obtain $N_i = q_s q_a^3 q_b^2$, so that

\[(14) \quad q_s q_a^3 q_b^2 n_s \cdot [1] = n_s \cdot [1].\]

On the other hand, for a minimal gallery of type $j$ between $C$ and $C''$ described in Figure 5 below, we obtain $N_j = q_s q_a^4 q_b^4$, so that

\[(15) \quad q_s q_a^4 q_b^4 n_s \cdot [1] = n_s \cdot [1].\]

Equations (14), (15) imply that

\[q_s^2 n_s \cdot [1] = n_s \cdot [1],\]

thereby proving (13).

The $\tilde{B}_2$ and $\tilde{A}_2$ cases follow by similar calculations, using the configurations in Figure 5 below.\[\square\]
Figure 4. The $\tilde{G}_2$ case.

The $\tilde{B}_2$ case.

The $\tilde{A}_2$ case.

Figure 5.
Let \( k \) be a non-archimedean local field with residue field \( \mathbb{F} \) of order \( q \). Let \( L \) be a simple, simply connected linear algebraic \( k \)-group and assume that \( L \) is \( k \)-split and has \( k \)-rank 2. Let \( G \) be the group of \( k \)-rational points of \( L \) and let \( \Gamma \) be a torsion free lattice in \( G \). Then \( q_t = q \) for all \( t \in I \) \([\text{Ti2}, \S 3.5.4]\), and equation (13) becomes

\[
(q^2 - 1)n_s \cdot [1] = 0.
\]

A parahoric subgroup \( P_s \) corresponding to a hyperspecial vertex of type \( s \) has maximal volume among compact subgroups of \( G \) \([\text{Ti2}, \S 3.8.2]\). This volume is \([P_s : I]\), by Remark 3.2. In particular, all such subgroups have the same volume. It follows that \( n_s = \text{covol}(\Gamma)/[P_s : I] \) has the same value for all hyperspecial types \( s \).

Suppose, for example, that \( G \) is the symplectic group \( \text{Sp}_2(k) \), which has type \( \tilde{B}_2 \) (or, equivalently, \( \tilde{C}_2 \)). Examination of the tables at the end of \([\text{Ti2}]\) shows that the diagram of \( G \) has two hyperspecial types \( s \), \( t \). Thus \( n_s = n_t \), and it follows from (11) and Figure 3 that

\[
(q^3 - 1)n_s \cdot [1] = 0.
\]

Combining this with (10) gives the following improvement to (10), for the case \( G = \text{Sp}_2(k) \):

\[
(q - 1)n_s \cdot [1] = 0.
\]

**Remark 4.2.** An interesting problem is to find the exact value of the order of \([1]\). This is known in the case where the group \( G \) has \( k \)-rank 1, and \( \Delta \) is a tree. In that case a torsion free lattice \( \Gamma \) in \( G \) is a free group of finite rank \( r \), and it follows from \([\text{R1}, \text{R2}]\) that \([1]\) has order \( r - 1 = -\chi(\Gamma) \), where \( \chi(\Gamma) \) denotes the Euler-Poincaré characteristic of \( \Gamma \). If \( G = \text{SL}_2(k) \), then \(-\chi(\Gamma) = (q - 1)n_s(\Gamma)\).

In the rank 2 case, the order of \([1]\) is in general smaller than \( \chi(\Gamma) \). For by \([\text{Se1}, \text{p. 150}, \text{Théorème 7}]\), \( \chi(\Gamma) = (q - 1)(q^m - 1)n_s(\Gamma) \), where \( m = 2, 3, 5 \) according as \( G \) has type \( \tilde{A}_2, \tilde{B}_2, \tilde{G}_2 \). Note however that by (10), (17), we do have \( \chi(\Gamma) \cdot [1] = 0 \) if \( G = \text{SL}_3(k) \) or \( G = \text{Sp}_2(k) \).

### 5. K-Theory of the Boundary Algebra \( A_{\Gamma} \)

We retain the general assumptions of Theorem 2.2. Thus \( G \) is a locally compact group acting strongly transitively by type preserving automorphisms on the affine building \( \Delta \), and \( \Gamma \) is a torsion free discrete subgroup of \( G \).

As in \([\text{RS}, \text{R1}]\), the group \( \Gamma \) acts on the commutative \( C^* \)-algebra \( C(\Omega) \), and one can form the full crossed product \( C^* \)-algebra \( A_{\Gamma} = \)
The inclusion map \( C(\Omega) \to A_\Gamma \) induces a natural homomorphism from \( C(\Omega, \mathbb{Z}) = K_0(C(\Omega)) \) to \( K_0(A_\Gamma) \), which maps \( \chi_\sigma \) to the class of the corresponding idempotent in \( A_\Gamma \). The covariance relations in \( A_\Gamma \) imply that for each \( g \in \Gamma \) and \( \sigma \in \Delta^n \), the functions \( \chi_\sigma \) and \( g \cdot \chi_\sigma = \chi_{g\sigma} \) map to the same element of \( K_0(A_\Gamma) \). Thus there is an induced homomorphism \( \varphi : \Omega_\Gamma \to K_0(A_\Gamma) \). Moreover \( \varphi([1]) = [1]_{K_0} \), the class of 1 in the \( K_0 \)-group of \( A_\Gamma \). We have the following immediate consequence of Theorem 2.2.

**Corollary 5.1.** If \( \Gamma \) is a torsion free lattice in \( G \) then \([1]_{K_0}\) has finite order in \( K_0(A_\Gamma) \).

**Remark 5.2.** Clearly the bounds for the order of \([1]_{K_0}\) obtained in the preceding sections apply also to \([1]_{K_0}\). If \( G \) has type \( \tilde{A}_n \), Corollary 5.1 was proved in [RS], [R1]. In that case \( q_t = q \) for all \( t \in I \). For \( n = 1 \), it follows from [R1, R2] that the order of \([1]_{K_0}\) is actually

\[
\text{ord}([1]_{K_0}) = (q - 1) \cdot n_s.
\]

The computational evidence at the end of Section 6 below indicates that \([1]_{K_0}\) also holds for \( n = 2 \).

Return now to the general assumptions of Theorem 2.2. It is important that \( \Gamma \) is amenable at infinity in the sense of [AR, Section 5.2]. Since the action of \( G \) on \( \Delta \) is strongly transitive, its action on the boundary \( \Omega \) is transitive. Therefore \( \Omega \) may be identified, as a topological \( \Gamma \)-space, with \( G/B \), where the Borel subgroup \( B \) is the stabilizer of some point \( \omega \in \Omega \). The next result shows that the group \( B \) is amenable and so the action of \( \Gamma \) on \( \Omega \) is amenable [AR, Section 2.2]. Moreover the crossed product algebra \( A_\Gamma \) is unique: the full and reduced crossed products coincide.

**Proposition 5.3.** Let \( \omega \in \Omega \) and let \( B = \{ g \in G : g\omega = \omega \} \). Then \( B \) is amenable and so \( (\Gamma, \Omega) \) is amenable as a topological \( \Gamma \)-space, if \( \Gamma \) is a closed subgroup of \( G \).

**Proof.** Let \( s \in \Delta^0 \) be a special vertex and let \( A \) be an apartment in \( \Delta \) containing the sector \([s, \omega] \). Let \( N_{\text{trans}} \) denote the subgroup of \( G \) consisting of elements which stabilize \( A \) and act by translation on \( A \).

If \( g \in B \), then the sectors \([gs, \omega] \) and \([g^{-1}s, \omega] \) both have base vertex \( gs \) and both represent the same boundary point \( \omega \). Therefore \( g(s, \omega) = [gs, \omega] \). Now the sectors \([gs, \omega], [s, \omega] \) and \([g^{-1}s, \omega] \) are all equivalent, and so contain a common subsector \( S \). The sectors \( S \) and \( gS \), being subsectors of \([s, \omega] \), are parallel sectors in the apartment \( A \). Let \( \sigma \) be...
the base chamber of $S$. Since $G$ acts strongly transitively on $\Delta$, there exists an element $g' \in G$ such that $g'A = A$ and $g', \sigma = g \sigma$. In particular $g' \omega = \omega$.

Since the action of $G$ is type preserving, it follows from [Gr] Theorem 17.3 that $g' \in N_{trans}$. Moreover $gv = g'v$, for all $v \in S$. Let $\lambda_\omega(g) = g'|_A$, the restriction of $g'$ to $A$. Then $\lambda_\omega(g)$ is the unique translation of $A$ such that $gv = \lambda_\omega(g)v$, for all $v \in S$. As the notation suggests, $\lambda_\omega(g)$ depends on $g$ and $\omega$, but not on $S$.

It is easy to check that the mapping $\lambda_\omega : g \mapsto \lambda_\omega(g)$ is a homomorphism from $B$ onto the group $T_0$ of type preserving translations on $A$. Since $T_0 \cong \mathbb{Z}^n$ is an amenable group, it will follow that $B$ is amenable if $\ker \lambda_\omega$ can be shown to be amenable.

For each vertex $v$ of $[s, \omega)$, let $B_v = \{g \in B : gv = v\}$. Then

$$\ker \lambda_\omega = \bigcup_{v \in [s, \omega)} B_v.$$

Each of the groups $B_v$ is compact, being a closed subgroup of a parahoric subgroup. The group $\ker \lambda_\omega$ may thus be expressed as the inductive limit of the family of compact groups $\{B_v : v \in [s, \omega)\}$, directed by inclusion. Therefore $\ker \lambda_\omega$ is amenable. \hfill \Box

**Remark 5.4.** If $G$ is the group of $k$-rational points of an absolutely almost simple, simply connected linear algebraic $k$-group, this result is well known. For then the Borel subgroup $B$ is solvable, hence amenable.

The amenability of the $\Gamma$-space $\Omega$ has the consequence that the Baum-Connes conjecture, with coefficients in $C(\Omega)$ has been verified [Tu, Théorème 0.1]. Consequently $K_* (A_\Gamma)$ can be calculated by means of the Kasparov-Skandalis spectral sequence [KaS 5.6, 5.7]. This has initial terms

$$E^2_{p,q} = H_p(\Gamma, K_q(C(\Omega)))$$

$$= \begin{cases} H_p(\Gamma, C(\Omega, \mathbb{Z})), & \text{if } 0 \leq p \leq n \text{ and } q \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Note that $H_p = 0$ for $p > n$, since $\Gamma$ has homological dimension $\leq n$. Moreover $K_1(C(\Omega)) = 0$, since $\Omega$ is totally disconnected.
Suppose now that $\Delta$ has dimension $n = 2$. Some of the nonzero terms in the first quadrant are shown in (19).

\[
\begin{array}{ccccccc}
E^2_{04} & E^2_{14} & E^2_{24} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
E^2_{02} & E^2_{12} & E^2_{22} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
E^2_{00} & E^2_{10} & E^2_{20} & 0 & 0 & \cdots \\
\end{array}
\]

Recall that for $r \geq 2$ there are differentials $d^r_{p,q} : E^r_{p,q} \to E^{r-r+1}_{p-r,q+r-1}$, and $E^{r+1}_{p,q}$ is the homology of $E^r_{p,q}$. Since the differentials $d^2$ go up one row, it is clear that $d^2 = 0$ and $E^3_{p,q} = E^2_{p,q}$. Since the differentials $d^3$ go three units to the left, $d^3 = 0$ and $E^4_{p,q} = E^3_{p,q}$. Continuing in this way we see that $E^\infty_{p,q} = E^2_{p,q}$. Therefore the spectral sequence degenerates with $E^\infty_{p,q} = E^2_{p,q}$. Convergence of the spectral sequence to $K_\ast(\mathcal{A}_\Gamma)$ means that

\[K_1(\mathcal{A}_\Gamma) = H_1(\Gamma, C(\Omega, \mathbb{Z}))\]

and that there is a short exact sequence

\[0 \to H_0(\Gamma, C(\Omega, \mathbb{Z})) \to K_0(\mathcal{A}_\Gamma) \to H_2(\Gamma, C(\Omega, \mathbb{Z})) \to 0.\]

In particular, $\Omega_{\Gamma} = H_0(\Gamma, C(\Omega, \mathbb{Z}))$ is isomorphic to a subgroup of $K_0(\mathcal{A}_\Gamma)$.

6. $\tilde{A}_2$ BUILDINGS AND REDUCED GROUP $C^*$-ALGEBRAS

The reduced group $C^*$-algebra of a group $\Gamma$ is the completion $C^*_r(\Gamma)$ of the complex group algebra of $\Gamma$ in the regular representation as operators on $\ell^2(\Gamma)$. Let $\Gamma$ be a discrete torsion free group acting properly on the affine building $\Delta$, satisfying the hypotheses of Theorem 2.2. By Proposition 5.3, $\Gamma$ acts amenably on the compact space $\Omega$. It follows that the Baum-Connes assembly map is injective [Hig] and so the Novikov conjecture is true. (This also follows from [KaS].) Therefore the class $[1]$ in $K_0(C^*_r(\Gamma))$ does not have finite order.

Since $C^*_r(\Gamma)$ embeds in $\mathcal{A}_\Gamma$, there is a natural homomorphism

\[K_\ast(C^*_r(\Gamma)) \to K_\ast(\mathcal{A}_\Gamma).\]

This homomorphism is not injective, by Theorem 2.2, since $[1]$ does have finite order in $K_0(\mathcal{A}_\Gamma)$. It is therefore worth comparing the $K$-theories of these two algebras. If the building is type $\tilde{A}_2$, everything can be calculated explicitly.
The computation required is a corollary of [La], which states that the Baum-Connes conjecture holds for any discrete group \( \Gamma \) satisfying the following properties.

1. \( \Gamma \) acts continuously, isometrically and properly with compact quotient on a uniformly locally finite affine building or on a complete riemannian manifold of nonpositive curvature;
2. \( \Gamma \) has property (RD) of Jolissaint.

For a group \( \Gamma \) satisfying these conditions, \( K_\ast(C_r^\ast(\Gamma)) \) is isomorphic to the geometric group \( K_\ast(\Gamma, 1) = KK_\ast(C_0(\Delta), \mathbb{C}) \). (The notation is consistent with [BCH], because \( \Delta \) is \( \Gamma \)-compact.) This provides a way of calculating the groups \( K_\ast(C_r^\ast(\Gamma)) \).

Assume therefore that all the conditions of Theorem 2.2 hold, together with the condition that \( \Delta \) has type \( \tilde{A}_2 \). This is the case, for example, if \( \Gamma \) is a torsion free lattice in \( G = \text{SL}_3(k) \).

Condition (1) is clearly satisfied and condition (2) is also satisfied by the main result of [RRS]. The finite cell complex \( B\Gamma = \Delta/\Gamma \) is a \( K(\Gamma, 1) \) space [Br I.4], so the group homology \( H_\ast(\Gamma, \mathbb{Z}) \) is isomorphic to the usual simplicial homology \( H_\ast(B\Gamma) \) [Br Proposition II.4.1]. Thus \( H_0(\Gamma, \mathbb{Z}) = \mathbb{Z} \) and \( H_1(\Gamma, \mathbb{Z}) = \Gamma_{ab} \), the abelianization of \( \Gamma \). Moreover, since \( B\Gamma \) is 2-dimensional, the group \( \Gamma \) has homological dimension at most 2 [Br VIII.2 Proposition (2.2) and VIII.6 Exercise 6]. It follows that \( H_2(\Gamma, \mathbb{Z}) \) is free abelian and \( H_p(\Gamma, \mathbb{Z}) = 0 \) for \( p > 2 \). Since \( B\Gamma \) is 2-dimensional, the group \( \Gamma \) has homological dimension at most 2 [Br VIII.2 Proposition (2.2) and VIII.6 Exercise 6]. It follows that \( H_2(\Gamma, \mathbb{Z}) \) is free abelian and \( H_p(\Gamma, \mathbb{Z}) = 0 \) for \( p > 2 \). Since \( \Gamma \) satisfies the Baum-Connes conjecture, \( K_\ast(C_r^\ast(\Gamma)) \) coincides with its “\( \gamma \)-part” [Ka, Definition-corollary 3.12]. Therefore \( K_\ast(C_r^\ast(\Gamma)) \) may be computed as the limit of a spectral sequence \( E_{p,q}^r \) [KaS, Theorem 5.6 and Remark 5.7(a)]. Since \( \Gamma \) is torsion free, \( \Gamma \) acts freely on \( \Delta \). According to [KaS] Remarks 5.7(b)] the initial terms of the spectral sequence are

\[
E_{p,q}^2 = H_p(\Gamma, K_q(C)) = \begin{cases} H_p(\Gamma, \mathbb{Z}) & \text{if } p \in \{0, 1, 2\} \text{ and } q \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}
\]

The nonzero terms in the first quadrant are shown in [19]. Exactly as for [19], the spectral sequence degenerates with \( E_{p,q}^\infty = E_{p,q}^2 \). Convergence of the spectral sequence to \( K_\ast(C_r^\ast(\Gamma)) \) means that

\[
K_1(C_r^\ast(\Gamma)) = H_1(\Gamma, \mathbb{Z})
\]

and that there is a short exact sequence

\[
0 \to H_0(\Gamma, \mathbb{Z}) \to K_0(C_r^\ast(\Gamma)) \to H_2(\Gamma, \mathbb{Z}) \to 0.
\]
Since $H_2(\Gamma, \mathbb{Z})$ is free abelian, \( \mathbb{Z} \) splits and we have
\[
\begin{align*}
K_0(C^*_r(\Gamma)) &= H_0(\Gamma, \mathbb{Z}) \oplus H_2(\Gamma, \mathbb{Z}), \\
K_1(C^*_r(\Gamma)) &= H_1(\Gamma, \mathbb{Z}).
\end{align*}
\]

Now $H_1(\Gamma, \mathbb{Z})$ is a finite group, because $\Gamma$ has Kazhdan’s property (T) \cite{BS, Corollary 1}. It follows that $K_0(C^*_r(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)}$, where $\chi(\Gamma)$ is the Euler-Poincaré characteristic of $\Gamma$. This proves

**Theorem 6.1.** Let $\Gamma$ be a torsion free cocompact lattice in $G$, where $(G, \mathcal{I}, N, S)$ is an affine topological Tits system of type $\tilde{A}_2$. Then
\[
K_0(C^*_r(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)} \quad \text{and} \quad K_1(C^*_r(\Gamma)) = \Gamma_{ab}.
\]

The value of $\chi(\Gamma)$ is easily calculated \cite[p. 150, Théorème 7]{Se1}, \cite[Section 4]{R1}. It is
\[
\chi(\Gamma) = (q - 1)(q^2 - 1) \cdot n_s(\Gamma),
\]
where $q$ is the order of the building $\Delta$ and $n_s(\Gamma)$ is the number of $\Gamma$-orbits of vertices of type $s$, where $s \in I$ is fixed.

In \cite{CMSZ} a detailed study was undertaken of groups of type rotating automorphisms of $\tilde{A}_2$ buildings, subject to the condition that the group action is free and transitive on the vertex set of the building. For $\tilde{A}_2$ buildings of orders $q = 2, 3$, the authors of that article give a complete enumeration of the possible groups with this property. These groups are called $\tilde{A}_2$ groups. Some, but not all, of the $\tilde{A}_2$ groups are cocompact lattices in $\operatorname{PGL}_3(k)$ for some local field $k$ with residue field of order $q$. It is an empirical fact that either $k = \mathbb{Q}_p$ or $k = \mathbb{F}_q((X))$ in all the examples constructed so far.

For each $\tilde{A}_2$ group $\tilde{\Gamma} < \operatorname{PGL}_3(k)$, consider the unique type preserving subgroup $\Gamma < \tilde{\Gamma}$ of index 3. Each such $\Gamma$ is torsion free and acts freely and transitively on the set of vertices of a fixed type $s$. That is $n_s = 1$. Therefore
\[
\chi(\Gamma) = (q - 1)(q^2 - 1) = 1 + \operatorname{rank} H_2(\Gamma, \mathbb{Z}).
\]

**Remark 6.2.** There are eight such groups $\Gamma$ if $q = 2$, and twenty-four if $q = 3$. Using the results of \cite{RS} and the MAGMA computer algebra package, one can compute $K_0(\mathcal{A}_\Gamma)$. One checks that in all these examples,
\[
\operatorname{rank} K_0(\mathcal{A}_\Gamma) = 2 \cdot \operatorname{rank} H_2(\Gamma, \mathbb{Z}) = \begin{cases} 
4 & \text{if } q = 2, \\
30 & \text{if } q = 3.
\end{cases}
\]
Furthermore, the class of \([1]\) in the \(K_0(\mathcal{A}_\Gamma)\) has order \(q - 1\). Note that for \(q = 2\) this means that \([1]\) = 0.

These values also appear to be true for higher values of \(q\). In particular, they have been verified for a number of groups with \(q = 4, 5, 7\). Here is an example with \(q = 4\).

**Example 6.3.** Consider the Regular \(\tilde{A}_2\) group \(\Gamma_r\), with \(q = 4\). This is a torsion free cocompact subgroup of \(\text{PGL}_3(\mathbb{K})\), where \(\mathbb{K}\) is the Laurent series field \(F_4((X))\) with coefficients in the field \(F_4\) with four elements. It is described in [CMSZ] Part I, Section 4, and its embedding in \(\text{PGL}_3(F_4((X)))\) is essentially unique, by the Strong Rigidity Theorem of Margulis. The group \(\Gamma_r\) is torsion free and has 21 generators \(x_i, 0 \leq i \leq 20\), and relations (written modulo 21):

\[
\begin{align*}
x_j x_{j+7} x_{j+14} &= x_j x_{j+14} x_{j+7} = 1 & &0 \leq j \leq 6, \\
x_j x_{j+3} x_{j-6} &= 1 & &0 \leq j \leq 20.
\end{align*}
\]

Let \(\Gamma < \text{PSL}_3(\mathbb{K})\) be the type preserving index three subgroup of \(\Gamma_r\). The group \(\Gamma\) has generators \(x_j x_0^{-1}, 1 \leq j \leq 20\). Using the results of [RS] one obtains

\[
K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{88} \oplus (\mathbb{Z}/2\mathbb{Z})^{12} \oplus (\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/7\mathbb{Z})^4 \oplus (\mathbb{Z}/9\mathbb{Z}),
\]

and the class of \([1]\) in \(K_0(\mathcal{A}_\Gamma)\) is \(3 + \mathbb{Z}/9\mathbb{Z}\), which has order \(q - 1 = 3\). It also follows from [RS] Theorem 2.1 that \(K_0(\mathcal{A}_\Gamma) = K_1(\mathcal{A}_\Gamma)\).

According to Theorem 6.1

\[
K_0(\mathcal{C}_r^*(\Gamma)) = \mathbb{Z}^{15} = \mathbb{Z}^{14} \oplus ([1]) \quad \text{and} \quad K_1(\mathcal{C}_r^*(\Gamma)) = (\mathbb{Z}/2\mathbb{Z})^6 \oplus (\mathbb{Z}/3\mathbb{Z}).
\]

The second equality was obtained using the MAGMA computer algebra package. This, and similar, examples suggest that the only reason for failure of injectivity of the natural homomorphism

\[
K_0(\mathcal{C}_r^*(\Gamma)) \to K_0(\mathcal{A}_\Gamma)
\]

is the fact that \([1]\) has finite order in \(K_0(\mathcal{A}_\Gamma)\).

**Example 6.4.** For completeness, here are the results of the computations for one of the groups with \(q = 3\). The Regular group 1.1 of [CMSZ], with \(q = 3\), has 13 generators \(x_i, 0 \leq i \leq 12\), and relations (written modulo 13):

\[
\begin{align*}
x_j^3 &= 1 & &0 \leq j \leq 13, \\
x_j x_{j+8} x_{j+6} &= 1 & &0 \leq j \leq 13.
\end{align*}
\]

Let \(\Gamma\) be the type preserving index three subgroup. The group \(\Gamma\) has generators \(x_j x_0^{-1}, 1 \leq j \leq 12\). Note that the group 1.1 has torsion, but
its type preserving subgroup $\Gamma$ is torsion free. One obtains

$$K_0(A_\Gamma) = \mathbb{Z}^{30} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^6 \oplus (\mathbb{Z}/13\mathbb{Z})^4,$$

and the class of $[1]$ in $K_0(A_\Gamma)$ is $1 + \mathbb{Z}/2\mathbb{Z}$, which has order $q - 1 = 2$. It also follows from Theorem 6.1 that

$$K_0(C^*_r(\Gamma)) = \mathbb{Z}^{16} \quad \text{and} \quad K_1(C^*_r(\Gamma)) = (\mathbb{Z}/3\mathbb{Z})^3 \oplus (\mathbb{Z}/13\mathbb{Z}).$$

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