Symmetry-breaking transition between defect-free and defect-laden turbulence in polar active matter

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Coherent flows of self-propelled particles are characterized by vortices and jets that sustain chaotic flows, referred to as active turbulence. Here, we reveal a transition between defect-free active turbulence and active turbulence laden with defects. Interestingly, we show that concurrent to the transition from defect-free active turbulence to defect-laden turbulence is the restoration of the previously broken $U(1)$-symmetry and fast decay of the two-point correlations which suggests a symmetry-breaking transition. By stability analyses of the topological charge density field, we provide an analytical prediction of the activity threshold for the transition to the defect-laden active turbulent state. Moreover, despite the distinct symmetry features between the defect-free and defect-laden active turbulence, we demonstrate universal behaviors for the self-similarity of the flow fluctuations at large scales, and exponential decay of polarity power spectrum, at small length scales compared to the active energy injection length. It is further shown that increasing the self-propulsion of the active particles results in exponential rarefaction of topological defects and a transition from active defect-laden turbulence to a flocking state. These findings reveal new dynamical transitions between distinct spatiotemporal organization patterns in polar active matter.

I. INTRODUCTION

Recent developments in active matter have shed light on a form of chaotic flows sustained by living microswimmers at low Reynolds numbers, often known as active turbulence [1]. A distinctive feature of active turbulence, compared to classic inertial turbulence, is the continuous injection of energy at the scale of individual constituents of the active material. Striking examples are dense bacterial suspensions [2, 3], cellular monolayers [4, 5], and assemblies of subcellular filaments [6, 7], that are all composed of active elements - i.e., individual bacterium, single cell, or motor proteins walking on single filaments - each capable of converting chemical energy to mechanical work [8–10]. Although, most of the work so far has been limited to visual resemblance with inertial turbulence, recent works suggest the existence of universal scaling features in active turbulence [7, 11] and even show [12] that transition to active turbulence belongs to the same directed percolation universality class as in the inertial turbulence [13].

Nevertheless, a majority of studies of active turbulence so far, including the works suggesting universal scaling laws and universality classes of turbulence transition, have focused on a subclass of active materials known as active nematics [14–18], which models flows generated by dense assemblies of elongated particles [19]. Within the active nematic framework, the particles are essentially characterized as shakers [20]: they neither have any polarity or ability to propel themselves, and instead generate head-tail symmetric (nematic) active stresses in the fluid. The active stresses in turn drive hydrodynamic instabilities in the flow and the orientation field of particles, resulting in a chaotic flow state that is interleaved with topological defects - singular points in the orientation field where the order breaks down [8, 10].

On the other hand, studies of self-propulsion effects on active turbulence have mostly neglected orientational couplings. Instead, the focus was put on generalized Navier-Stokes equations, often at the inertial regime (high Reynolds number), where activity is introduced through addition of phenomenological higher-order terms in the momentum equation, corresponding to a second order negative viscosity and a fourth-order hyper-viscosity, to give characteristic vortex length to active turbulence [2, 21–23]. As such, characteristics of active turbulence in models of polar active fluids accounting for both velocity and orientation fields (associated with the polarity of particles) at the same time were only marginally explored in the context of polar active emulsions [24] and polar flocks with inertia [25]. Additionally, notwithstanding the interesting recent characterisation of the annihilation of topological charges in colloidal flocks [26] and despite the emerging roles of topological defects in various biological fluids [27–30], the majority of research so far has focused on nematic topological defects [10, 29, 31] and studies of active turbulence in polar fluids in the presence of polar topological defects are lacking.

Here we use numerical simulations of continuum polar active matter together with analytical arguments to shed light on the turbulence characteristics of active fluids laden with topological defects, accounting for both hydrodynamic effects and polar ordering at zero Reynolds number. We first show how increasing active stresses results in a transition from defect-free active turbulence to active topological turbulence, where chaotic flows are interleaved with chaotic arrangement

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FIG. 1. Transition from defect-free to defect-laden active turbulence upon increasing strength of active stresses. Shown here are the averaged defect density (blue dots) and the rms-velocity (orange dots) in units of $V_p = \sqrt{AK/\eta}$, characterizing the passive relaxation velocity of the polar particles. Active turbulence arises as soon as the activity is non-zero, but topological defects form and persist in the polarity field only beyond a finite threshold both in contractile ($\zeta < 0$) and extensile systems ($\zeta > 0$). Results were averaged over time and for 10 simulations from different initial conditions, the errors bars standing for the corresponding standard deviation. The normalized kurtosis $\kappa_i$ of both quantities are also represented at the bottom: for large activity levels, $\kappa(\rho_d) \approx 0$, which indicates the population of defects is typically equilibrium-like and follows a normal distribution law. For small activity levels, there are no defects at all and $\kappa$ is set by convention to $-3$. In between, the transition is characterized by very rare formation and annihilation events, as evidenced by the peaks in $\kappa$. There are no such peaks for the kurtosis of the rms-velocity, which indicates there is no equivalent transition. Insets illustrate polarity field (left panels) and the corresponding velocity field (right panels) characterized by streamlines (solid black lines) colored by the normalized vorticity $\omega/\omega_{\text{max}}$. Only a small subset of the full domain (1/16th zoomed-in region) is shown. Within the polarity field positive and negative integer topological defects are marked by red asters and blue squares, respectively.

of full-integer topological defects. We provide an analytic argument to predict the activity threshold for the transition, based on the stability analyses of the topological charge density field. We further show that ordering and symmetry characteristics are different within these two dynamical regimes: while polar symmetry is broken within the defect-free active turbulence, the transition to defect-laden turbulence results in the restoration of the previously broken $U(1)$-symmetry. Additionally, we demonstrate the existence of universal scaling in enstrophy and polarity power spectra that are observed for all activities and within both defect-free and defect-laden active turbulence.

Finally, we find that increasing the self-propulsion force results in the exponential rarefaction of topological defects with activity, leading to a transition from active topological turbulence to a flocking state.
II. METHODS

A. Hydrodynamic model

We consider an incompressible polar active fluid in two spatial dimensions, with the local orientational order described by a two-component order parameter $\mathbf{p}$ pointing in the direction of self-propulsion. Let then $\mathbf{u}$ and $\rho$ be the velocity and density of the polar fluid, respectively. Coarse-grained hydrodynamic equations can be derived by phenomenological considerations [8, 25, 32] and take the form

$$\frac{Du_i}{Dt} = \frac{1}{\rho} \partial_j \sigma_{ij},$$

$$\frac{Dp_i}{Dt} = (\lambda E_{ij} + \Omega_{ij}) p_j - \frac{1}{\gamma} h_i,$$

along with the incompressibility condition $\partial_i u_i = 0$. Here $\frac{Du_i}{Dt} = \partial_t + \mathbf{u} \cdot \nabla$ is the usual advection derivative. The strain rate tensor $\dot{\mathbf{E}}_{ij} = (\partial_i u_j + \partial_j u_i)/2$ and the vorticity tensor $\Omega_{ij} = (\partial_i u_j - \partial_j u_i)/2$ are respectively the symmetric and anti-symmetric parts of the velocity gradient tensor. In the momentum balance equation (1a), the stress tensor $\sigma_{ij}$ is broken into a sum of three parts; viscous $\sigma_{ij}^{\text{vis}} = 2\eta \dot{\mathbf{E}}_{ij}$, passive $\sigma_{ij}^{\text{pas}} = -P \delta_{ij} + C_{ijkl} \dot{\mathbf{p}}_k \mathbf{p}_l$, and active stresses $\sigma_{ij}^{\text{act}} = -\zeta (p_ip_j - \frac{1}{2} \delta_{ij})$ [33, 34].

The first term in the passive stress is the usual hydrodynamic pressure. The second term accounts for elastic stresses through the anisotropic elastic stiffness tensor $C_{ijkl} = \frac{\lambda}{2} \delta_{ik} \delta_{jl} - \frac{\lambda}{2} \delta_{ij} \delta_{kl} + \frac{\lambda - 1}{2} \delta_{il} \delta_{jk}$, with $\lambda$ the flow alignment parameter and $\mathbf{h} = -\delta \mathcal{F} / \delta \mathbf{p}$ [35] the molecular field defined here from the free energy

$$\mathcal{F} = \int d^2 \mathbf{x} \left\{ A \left( -\frac{\mathbf{p}^2}{2} + \frac{p^4}{4} \right) + \frac{K}{2} \partial_i p_j \partial_i p_j \right\},$$

which contains a local energy density with an energy scale $A$ that controls the isotropic-polar transition favoring the emergence of finite polarity at $|\mathbf{p}| = 1$, and a non-local energy contribution from polar elasticity with an elastic constant $K$ that penalizes deformations in the polarity field.

B. Numerical method and the simulation parameters

We simulate equations (1a,1b) using a hybrid lattice-Boltzmann method, combining finite-difference method for the evolution of polarity vector Eq. (1b), and the lattice-Boltzmann method for solving the incompressible Navier-Stokes equation Eq. (1a) with $\rho = 40$ and $\eta = 3.6$ in lattice Boltzmann units, ensuring that the Reynolds number in the simulations is negligible (Re $\ll 1$) [12, 36]. The other relevant dimensionless numbers describing the system are: (i) the dimensionless ratio of the viscosities $\eta/\gamma$, (ii) ratio of micro to macro length scales $(\sqrt{K/A})/L$, (iii) the flow alignment parameter $\lambda$, and (iv) the dimensionless active stress $\zeta = \zeta/A$. Unless otherwise stated, we fix the viscosity ratio to $\eta/\gamma = 3.6$, micro to macro length scale to $(\sqrt{K/A})/L = 2 \times 10^{-4}$ (assuring that the coherence length $l_p = \sqrt{K/A}$ is significantly smaller than the domain size $L$), and the flow alignment parameter to $\lambda = 0.1$.

Simulations were initialized with quiescent velocity field and noisy polar alignments close to the uniformly oriented state $\mathbf{p} = \hat{\mathbf{e}}_x$ under periodic boundary conditions, on quadratic domains of linear dimension $L = 1024$, unless otherwise is stated.

III. RESULTS

A. Activity-induced transition from defect-free to defect-laden active turbulence

We begin by characterizing the flow and order parameter fields as functions of the activity parameter $\zeta$, through the dimensionless rms-velocity $V_{\text{rms}} = V_{\text{rms}}/v_p$ that is normalized by the characteristic velocity of passive relaxation of polarity $V_p = \sqrt{K/\eta}$, and the average density of topological defects $\rho_d = (N_d)/L^2$, respectively. To isolate the effect of active stress from self-propulsion force, we first set the self-propulsion force strength to zero, i.e. ($\alpha = 0$). As evident from Fig. 1, increasing activity beyond a certain threshold results in a continuous increase in the defect density for both extensile ($\zeta > 0$) and contractile ($\zeta < 0$) systems. We have checked that the threshold does not depend on the system size $L$, as simulations with the domain size of $512 \times 512$ and $2048 \times 2048$ lead to a similar dependence of defect density on the active stress. Interestingly, simultaneous measurement of the root mean squared velocity indicates that even below the activity required for defect nucleation, active stresses disturb the polarity field such that spontaneous flows are generated within the system, for any non-zero activity (Fig. 1C). This is in fact expected as a consequence of the hydrodynamic instability of the polar ordered state to any finite activity [37, 38]. A closer look at the velocity field of the system below the defect nucleation threshold shows that chaotic flows characterized by flow vortices and jets span the system (Fig. 1A; only a fraction of the entire domain is shown), although no topological defects are present in the polarity field. Qualitatively similar chaotic flows manifest at higher activities with a smaller typical length scale $L_{\text{act}}$, defined below, though with the distinctive feature that the flows are now laden with the presence of full-integer topological defects (vortex-antivortex pairs) in the polarity field (Fig. 1B). It is important to distinguish this transition from the recently observed dynamics in inertial polar active matter, where including the particle inertia was shown to result in a crossover from defect turbulence to phase turbulence [25]. The latter is characterized by an aligned, but noisy flock of polar...
particles, and is approached by increasing the inertia of active particles. Here, on the other hand, the defect-free turbulent state is characterized by a spatio-temporal arrangement of the polar particles in a chaotic and non-flocking state, and the transition is governed by active stress generation in the absence of any particle inertia.

### B. Symmetry-restoration by transition from defect-free to defect-laden active turbulence

We next investigate whether the emergence of topological defects leads to alterations in the global ordering features of the polar active turbulence. To this end, we first introduce the order parameter $\Psi$ as the spatiotemporal average value of the polarity $p$:

$$\Psi = \int \frac{dt}{T} \int \frac{d^2x}{L^2} p(t, x)$$

and measure its absolute value (Fig. 2A; orange solid line). Upon increasing the active stress from zero, the active turbulence state immediately manifests a broken $U(1)$-symmetry evident from a finite average polarity in the absence of topological defects. This global polarisation decreases with increasing the activity and, remarkably, drops to zero as the activity-threshold for defect nucleation is approached. As such, the transition to active defect-laden turbulence results in the restoration of the $U(1)$-symmetry in the system.

To further characterize the signature of this symmetry-breaking transition, we also measure the equal-time spatial pair-correlation function for the polarity field $\Gamma(r) = \langle \delta p_i(x) \delta p_i(x + r) \rangle$, where $\delta p_i(r) = p_i(r) - \langle p_i(r) \rangle$ is the deviation from the local mean value and the averaging is performed over 13 different realizations, with distinct initial conditions and over time, once the system has reached a statistical steady state. Similar to the global polarization, the correlation length $\xi = \xi/l_p$, normalized with respect to the coherence length $l_p = \sqrt{K/A}$, decreases with increasing activity and asymptotically approaches a small value of order of the coherence length, which is independent of activity. The transition to this constant correlation length is concomitant with the onset of defect nucleation and the decay of the global polar order as evidenced in Fig. (2A, B). This behavior is independent of the exact definition of $\xi$, as we have checked that the variation of an auxiliary correlation length corresponding to a drop to 95% of the maximum value leads to a same trend. Thus, while the correlation length $\xi$ depends on activity $\zeta$ in the defect-free turbulent state, it asymptotically approaches to a constant value after the transition to defect-laden turbulence. Together, these measurements of the global polarisation and the correlation length indicate that the transition from defect-free to defect-laden active turbulence is marked by significant alterations in both the global symmetry of the collective organisation and in the local correlations between the polarities of active constituents.

### C. Mechanism of defect pair nucleation in active turbulence of polar particles

Having established the significant impact of the topological defects on the collective organization of polar active matter and its spontaneous flows, we next explore the mechanism of how topological defects are formed within the active turbulence state as the activity approaches the threshold value. Figure 3 illustrates zoomed-in snapshots of the polarity field at the onset of one defect pair nucleation. Three snapshots are shown at simulation times corresponding to before, at, and after a single pair nucleation for both extensile and contractile active stresses. To identify and track topological defects, we make use of the diffuse charge density, defined in the next section as $D = \frac{1}{\pi} \epsilon_{ij} \epsilon_{kl} \partial_i \partial_k \partial_j \partial_l p_{ij}$. It carries the $\pm 1$ charge of the integer topological defect at the singularities in the polarity field, and is zero elsewhere. Shortly before the onset of defect pair nucleation, the polarity field is characterized by locally ordered domains separated by lines of kink walls (Fig. 3 and Supplementary Movies [39]). At the onset of defect pair nucleation the polarity within one of the kink walls flips locally, leading to spontaneous emergence of a pair of full-integer defects (Fig. 3). After the nucleation event, the defect pair gets separated by an ordered region of polarity alignment (Fig. 3). While the negatively charged antivortices ($-1$ topological defects) have similar structures in both extensile and contractile systems, the positively charged $+1$ topological defects take the form of asters and vortices in contractile and extensile systems, respectively (Fig. 3C,D), in agreement with earlier analytical predictions on the stability of defect structures in polar active matter [40].
FIG. 3. Defect pair nucleation process. (A,B) Temporal evolution of the average bulk free-energy $\bar{F}_{\text{bulk}} = (F_{\text{bulk}} + A/4)/AL_{\text{act}}$, (top) measured in units of $AL_{\text{act}}$ relative to the background field $-A/4$; along with (C,D) simulation snapshots depicting the polarity field and diffuse charge density (heat plot) before, at, and after the nucleation of one pair of oppositely charged topological defects (bottom). Column (A,C) contractile and (B,D) extensile. The blue regions in A, B mark the period of existence of the defect pair considered. Time $\bar{t}$ is measured in units of the passive relaxation time of polarity over the simulation domain $t_p = \gamma L/\sqrt{AK}$ (see also Supplementary Movies [39]).

In order to gain more insight into the defect pair nucleation process, we measure the averaged bulk free energy of the system from Eq. (2) during the time leading to a single defect pair nucleation event, as depicted in (Fig. 3A-B). For both extensile and contractile systems the onset of defect nucleation coincides with a burst in the bulk free energy. This is consistent with the spontaneous flipping of the polarity within the kink walls that occurs when activity-induced fluctuations in the polarity overcome the energy barrier of local flipping, set by the bulk free energy. Once the defect pair is formed, the bulk free energy drops as the high energy stored in the kink walls is released. In the following section, we look more closely at this defect nucleation process and derive an analytical description of the activity threshold that is set by the competition between bulk and elastic energies of the system, and activity-induced flows.

D. Onset of active defect-laden turbulence

To gain more theoretical insights into the mechanics that controls the onset of defect pair nucleation, we perform a stability analysis of the topological charge density field. To this end, we employ the Halperin-Mazenko method [41–43], whereby topological defects are identified as zeros in the polarity vector field $p$. These are regions where the magnitude of polar order vanishes, making the orientation of the $p$ undefined. The strength of the phase singularity at the defect position is quantified by a quanta of phase jump called topological charge $q_\alpha$, which is measured by going around a closed loop

$$2\pi q_\alpha = \oint_{C_\alpha} d1 \cdot \nabla \theta,$$

where $C_\alpha$ is an arbitrary contour around the defect and $\theta$ is the orientation angle of the polarity vector $p$. Applying Stokes’ theorem, we see that topological defects correspond to phase singularities where the phase ceases to be irrotational.

$$2\pi q_\alpha \int_{A_\alpha} d^2x \delta(x - x_\alpha) = \int_{A_\alpha} d^2x \nabla \times \nabla \theta.$$  

Combining this topological property with diffuse defects owing to the smooth vanishing of the polar order, one can show that the topological defects can be accurately tracked by the Jacobi determinant of the map defined from the coordinate space to the order parameter space. For the polar order, this reads as

$$D = \frac{1}{2\pi} \epsilon_{ijkl} \partial_i p_k \partial_j p_l$$

$D(x,t)$ is a smooth field which vanishes in regions of smooth polar order and becomes non-zero around the defect cores, where the magnitude of $p$ decays to zero. The $D$-field has the physical interpretation of a non-singular topological charge density field which locates the position and charge of the defects with a finite-size core. The evolution of the $D$-field follows the topological charge conservation law [42]

$$\partial_t D + \partial_i J_i = 0,$$
with the corresponding topological current density

\[ J_t = \epsilon_{i j k l} \partial_i p_k \partial_j p_l, \quad (8) \]

which is fully determined by the evolution of the polarity vector field given in Eq. (1b).

We perform a stability analysis of the ground state of zero \( D \) and uniform polar order to estimate the critical activity for the onset of defect nucleation, identified as an instability where \( D \) becomes non-zero due to melting of polar order. In general, this is triggered by several destabilizing factors, such as advection, flow alignment and vorticity, competing with the stabilising contribution of the relaxation to the equilibrium order. Alas, the stability analysis is analytically tractable only in some limit cases where one destabilizing term is balanced against the relaxation to obtain an estimate on what influences the critical activity. For instance, advection \( u \cdot \nabla p \) is a destabilizing term competing with the relaxation to the uniform equilibrium state \( (A,p) \). Balancing these two contributions in the evolution of \( D \), we find that in the limit of compressible and friction-dominated flows with damping coefficient \( \mu \), the associated critical activity for the onset of defect nucleation triggered by advection is \( \zeta_c \approx \frac{2\eta}{\gamma A} \) which is symmetric to sign reversal. This sign symmetry is retained also in the opposite viscous-dominated limit, although the critical value might be different. Vorticity is also a destabilizing factor, but alas we are not able to determine analytically its associated critical activity. However, from the symmetry of the vorticity contribution in the \( D \)-evolution with respect to the sign of \( \zeta \), it is possible to show that the corresponding onset is the same for extensile/contractile systems, i.e. symmetric with respect to the sign of \( \zeta \). The asymmetry in the value of the critical activity for extensile and contractile systems is however attributed to an instability assisted by the flow alignment. The ordered state is destabilized by the flow alignment only when \( \zeta < 0 \), i.e. contractile systems. For such systems and in the limit of compressible, viscous-dominated flows, the onset of defect nucleation triggered by a flow alignment instability is estimated as

\[ \zeta_c \approx -\frac{2\eta}{\gamma A} A. \quad (9) \]

It is important to note that in numerical simulations all these destabilizing forces orchestrate and compete with the relaxation to uniform order. Therefore, the value of the critical activity is different than the estimated one corresponding to an isolated triggering factor. However, this stability analysis is informative and predicts generic properties of \( \zeta_c \): i) that, in the viscous-dominated regime, it depend on the energy level \( A \) controlling alignment, and ii) that the threshold activity has an asymmetry with respect to extensile/contractile active stresses in the presence of flow alignment.

To test the theoretical predictions on the activity threshold for defect nucleation, we next explore the dependence of defect density on the flow-aligning parameter \( \lambda \) and the energy scale \( A \) for the bulk free energy. Figure 4A compares the variation of defect density \( \rho_d \) with activity for three values of the flow-aligning parameter \( \lambda \). Interestingly, and consistently with the theoretical prediction, for \( \lambda = 0 \) the activity threshold for defect nucleation is symmetric with respect to the sign of the activity. Additionally, as predicted from the theory, a positive (negative) flow-aligning parameter shifts the defect density curve such that the activity threshold becomes smaller - in absolute value - for extensile (contractile) activities compared to their contractile (extensile) counterparts, leading to a more asymmetric profile of the defect density with respect to the sign of the activity. Moreover, as predicted from the theory, decreasing the energy scale in the bulk free energy \( A \) lowers the activity threshold for defect nucleation (Fig. 4B). This is also consistent with the mechanism identified for the nucleation of pairs of defects, relying on the spontaneous flipping of the polarity within the kink walls (Fig. 3C,D and Supplementary Movies [39]): since the parameter \( A \) sets the depth of a double-well potential in the bulk free energy, decreasing its value amounts to lowering the energy barrier for flipping the polarity, and as such leads to a reduced activity threshold for defect pair nucleation.

E. Universal scaling in defect-free and defect-laden active turbulence

Having established the mechanism of defect nucleation and the difference between defect-free and defect-laden active turbulence, we now investigate the common features between these two active turbulent states by characterising the energetic features of the flow and order parameter (Fig. 5). To this end, we first measure the power spectrum of the polarity order parameter \( \tilde{E}_p(k) = \)
\( \frac{1}{2} \langle \tilde{p}_v(k)\tilde{p}_v(k) \rangle \), with \( k \) the wave number averaged over azimuthal direction, for a range of activities spanning both defect-free and defect-laden turbulent states. Interestingly, when the wave number is non-dimensionalized by the active length scale \( L_{\text{act.}} = \sqrt{K/\zeta} \), which characterizes the scale of energy injection into the system by active stresses, the power spectrum \( \tilde{E}_p \) for all activity values shows an exponential decay at wave numbers larger than the inverse active length scale, as evident from a semilog plot of the order parameter spectrum versus the normalized wave number (Fig. 5A). Notice that the exponential cutoff scale still depends on activity and approaches a constant value at sufficiently high activity. This is related to the initial exponential decay of the correlation function \( \Gamma(r) \), with a correlation length that is activity dependent as discussed earlier. On the other hand, robust exponential tails at large wave numbers are characteristic for the spectral properties of the spontaneous flow, measured by the kinetic energy spectrum \( \tilde{E}_v(k) = \frac{1}{2} \langle \tilde{v}_i(k)\tilde{v}_i(k) \rangle \), with \( \tilde{v}_i \) the Fourier transforms of flow velocity (Fig. 5B). However, for wave numbers below the inverse active length scale, the kinetic energy spectrum \( \tilde{E}_v(k) \) and enstrophy spectrum \( \tilde{E}_w(k) = \frac{1}{2} \langle \tilde{\omega}(k)\tilde{\omega}(k) \rangle \), both exhibit power-law behavior with a constant exponent (Fig. 5C,D). This is important, because these activity levels span both the defect-free and defect-laden active turbulence states, indicating that for a polar active matter, despite distinct symmetry features and correlation lengths, there exist universal scaling of flow and order parameter spectrum in active turbulence state with and without topological defects. In particular, the enstrophy power spectrum that relates to the flow field shows a universal power-law scaling at scales larger than the active length \( L_{\text{act.}} \), while the polarity power spectrum that relates to the order parameter field shows a universal exponential decay at length scales smaller than the active length scale. These two distinct scaling relate defect-free and defect-laden turbulence in polar active matter.

**F. Exponential rarefaction of topological defects with increasing self-propulsion**

All our results so far were focused on the transition from defect-free to defect-laden active turbulence through increasing active stresses. In addition to the active stresses, another generic source of activity in polar materials is the ability of individual particles to self-propel by generating traction forces on their underl-
ing substrate, which locally puts momentum into the system. Although such traction forces are ubiquitously present and well-documented for both eukaryotic cell layers [44, 45] and twitching bacteria [3], their interplay with active stresses in governing the patterns of spatiotemporal organization in active materials is not well understood. To this end, we now turn into exploring the impact of having a non-zero self-propulsion force on the defect-laden turbulence state. We therefore introduce the traction force in the momentum equation (Eq. 1a) as [28, 46–49]

\[ \rho \frac{D u_i}{Dt} = \partial_j \sigma_{ij} + \alpha p_i - \mu u_i, \]  

(10)

The added last two terms on the right-hand side of Eq. (10), respectively govern the strength of the polar force generated by each active constituent \( \alpha \), and any existing friction with the underlying substrate \( \mu \). To vary the self-propulsion force in simulation, we introduce the dimensionless self-propulsion force \( \tilde{\alpha} = (\alpha/\zeta)/(\sqrt{K/A}) \) characterising the ratio of the length scale \( \alpha/\zeta \), set by the competition of self-propulsion force and active stress, to the micro length scale \( l_p = \sqrt{K/A} \), which characterizes the typical size of topological defects. We further retain the incompressibility condition \( (\partial_i u_i = 0) \) and fix the ratio of the velocities set by the traction force \( \alpha/\mu \) and the passive orientational relaxation velocity \( \sqrt{AK/\eta} \) to \( (\alpha/\mu)/(\sqrt{AK/\eta}) = 1.6 \), ensuring that the Reynolds number of the system remains negligible \( (Re << 1) \) and neither the compressible Toner-Tu [50] nor the dry limits [51] are approached upon varying \( \tilde{\alpha} \).

We start at a fixed level of active stress, \( \zeta = 2.0 \), leading to a defect-laden turbulence \( (|\zeta| > \zeta_c) \), and incrementally increase the strength of self-propulsion force \( \tilde{\alpha} \) from zero. Interestingly, upon increasing the self-propulsion strength the defect density drops sharply to zero (Fig. 6; blue curve): the active material is laden with fewer and fewer defects. Close inspection of the variation of the log defect density shows that this rarefaction process is exponential in the strength of self-propulsion (Fig. 6; top inset). Simultaneous measurement of the average magnitude of the global polarisation \( \Psi \) evidences a concurrent fast enhancement of the polarisation (Fig. 6; orange curve), which follows a logarithmic growth with increasing strength of self-propulsion (Fig. 6; bottom inset). This emergent polarisation is a hallmark of the transition from an active, turbulent state to a flocking phase. Interestingly, a recent study of inertial polar active matter has shown a transition from defect-turbulence to phase turbulence with increasing the particle inertia [25]. Although the particle inertia and traction force relate to different sources of activity in an active material - with the inertia of the particle appearing in the (self-)advection term in the polarity equation, while the traction force is a forcing term in the momentum equation - their common feature is the breaking of \( p \rightarrow -p \) symmetry that is absent in the active stress \( \sigma_{ij}^{\text{active}} = -\zeta \left( p_i p_j - \frac{1}{2} \delta_{ij} \right) \). As such this explicit breaking of the \( U(1) \) symmetry in the source of activity is the common feature between the traction force and particle inertia that transitions the active system towards a flocking phase. However, the dynamics of transition from defect-laden turbulence to flocking and the criticality of the related defect rarefaction process could be substantially different, and further studies will be needed to establish these similarities and differences in more detail.

**IV. DISCUSSION**

The results presented herein reveal distinct flow fields and patterns of collective self-organization of polar active particles that are primarily controlled by two main sources of activity with different symmetries: active stress with nematic symmetry, and self-propulsion force that possesses polar symmetry. In the absence of self-propulsion force, we find that increasing active stress in an polar active matter leads to a dynamical transitions from a defect-free to a defect-laden active turbulence state. Above a critical activity threshold, the turbulent flow is seeded with topological defects nucleated in the polar order parameter. Importantly, we find that the proliferation of defects screens the global polar order. Thus, in the defect-laden turbulent state the \( U(1) \)-symmetry is restored, whereas the defect-free active turbulence preserves the broken polar symmetry.
Interestingly, we show that in spite of their distinct topological content, the two active turbulence regimes share similar statistical properties with robust exponential tails for power spectra of polar order, flow velocity and vorticity on length scales smaller than the active length scale. On the other hand, over the length scales that are larger than the active length scale, entropy and energy spectra of flow fluctuations exhibit self-similarity with power-law scaling exponents. These statistical properties are also universal in the sense that they do not depend on the presence of topological defects, which may indicate that non-topological dissipative structures (e.g., kink walls) are the dominant driving force to fluctuations in order and flow of polar active fluids.

In addition to active stresses, the self-propulsion force favors flocking states and thus tempers with the stress-induced instability of the uniform state, leading eventually to the annihilation of all topological defects, thus suppressing flow fluctuations. Interestingly, we observe that an increase in the strength of the polar force in the defect-laden active turbulence regime leads to an exponential rarefaction of defects and enhancement of the global polarization. Although here we do not focus on the evolution of the properties of individual defects (such as their size or energy), it is clear that including the self-propulsion forces leads polar particles to consolidate their global aligning behavior, with defects becoming more sparse and the motion of the particles more coherent. This is reminiscent of the apparent violation of Mermin-Wagner theorem by two-dimensional flocks, that was recently elucidated [52]: the self-propulsive motion of particles within the flocks makes them better at aligning, lowering the critical dimension for long-range orientational order even in 2D. One difference is that the dynamics of our system is not noisy, but powered by active stresses that tend to destabilize the fully-aligned, homogeneous state, leading to the formation of kink walls and defects. Moreover, as the polarity vanishes in the area of the defects, so does the self-propulsion force: the reasoning for constant self-propulsion speed thus becomes obsolete. It is clear, however, that these two sources of activity ($\alpha$ and $\zeta$) compete to rule the symmetry of the polar field, with defects forming in proportion to the ratio of active stresses over active self-propulsion, therefore ultimately vanishing for large values of $\alpha$.

It is noteworthy that despite the ubiquitous presence of polarity in living and synthetic active materials [8, 9], majority of the current understanding of topological defects and flow features within the active turbulence is based on studies of active nematics. This is in part due to the growing number of biological systems that are identified with half-integer, nematic, topological defects [27, 29, 53] and discovery of potential biological functions for such defects [54–56]. It is important to note, however, that full-integer defects have also been identified as possible organization centers for mitotic spindles in microtubule-motor protein assemblies [57] and more recently as potential sites for mechanotransduction [58], and for cell differentiation in mouse myoblasts [28] and cartilage cells [59]. Our results provide a quantitative characterisation of the dynamical transitions, as well as the statistical imprints of flow and topological defects in polar active matter. Future work should focus on details of the dynamic distinctions in topological and flow features between the active turbulence in polar and nematic active materials, as well as generalized frameworks of the coupling between polar and nematic symmetries that have already been suggested theoretically [32, 60, 61] and observed experimentally [57, 59].

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