ON THE TWO-VARIABLE DRICHLET $q$-$L$-SERIES

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Abstract. In this study, we construct the two-variable Dirichlet $q$-$L$-function and the two-variable multiple Dirichlet-type Changhee $q$-$L$-function. These functions interpolate the $q$-Bernoulli polynomials and generalized Changhee $q$-Bernoulli polynomials. By using the Mellin transformation, we give an integral representation for the two-variable multiple Dirichlet-type Changhee $q$-$L$-function. We also obtain relations between the Barnes-type $q$-zeta function and the two-variable multiple Dirichlet-type Changhee $q$-$L$-function.

1. Introduction

In his paper [2], Barnes defined multiple zeta function. Barnes’ multiple zeta function $\zeta_r(s, w \mid a_1, \ldots, a_r)$ depends on parameters $a_1, \ldots, a_r$ that will be taken positive throughout this paper. It is defined by the series

$$\zeta_r(s, w \mid a_1, \ldots, a_r) = \sum_{m_1, \ldots, m_r}^\infty (w + m_1 a_1 + \ldots + m_r a_r)^{-s},$$

where $\text{Re}(w) > 0, \text{Re}(s) > r$.

The Barnes’ multiple Bernoulli polynomials $B_n(x, r \mid a_1, \ldots, a_r)$, cf. [2], are defined by

$$t^r e^{xt} \prod_{j=1}^r (e^{a_j t} - 1) = \sum_{n=0}^\infty B_n(x, r \mid a_1, \ldots, a_r) \frac{t^n}{n!},$$

for $|t| < 1$.

By [14] and [15], it is easy to see that

$$\zeta_r(-m, w \mid a_1, \ldots, a_r) = \frac{(-1)^r m!}{(r + m)!} B_{r+m}(w, r \mid a_1, \ldots, a_r),$$

for $w > 0$ and $m$ is a positive integer (for detail see [1], [5], [20], [29], [10], [17], [18], [19]).

Recently, many mathematicians and physicians have studied on zeta functions, multiple zeta functions, multiple $L$-series and multiple Bernoulli numbers and polynomials due to their importance. These functions and numbers are used in Number Theory, Complex Analysis and Mathematical Physics, $p$-adic Analysis and other areas (for detail see [2], [32], [11], [31], [11], [7], [8], [10], [23], [25], [28], [22], [30]).

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In [20], Matsumoto studied on general multiple zeta functions of several variables, involving both Barnes multiple zeta functions and Euler-Zagier sums as special cases.

In his paper, Ota[24] gave Kummer-type congruences for derivatives of Barnes’ multiple Bernoulli polynomials. He also generalized these congruences to derivatives of Barnes’ multiple Bernoulli polynomials by an elementary method and gave a $p$-adic interpolation of them.

By using non-Archimedean $q$-integration, Kim [8] defined multiple Changhee $q$-Bernoulli polynomials which form a $q$-extension of Barnes’ multiple Bernoulli polynomials. He also constructed the Changhee $q$-zeta functions (which gives $q$-analogue of Barnes’ multiple zeta functions). He found relations between the Changhee $q$-zeta function and Daehee $q$-zeta function.

In [30], Young gave some $p$-adic integral representation for the two-variable $p$-adic $L$-functions. For powers of the Teichmüller character, he used the integral representation to extend the $L$-function to the large domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials.

In [19], by using $q$-Volkenborn integration and uniform differentiable on $\mathbb{Z}_p$, Kim, Simsek and Srivastava constructed $p$-adic $q$-zeta functions. These functions interpolate the $q$-Bernoulli numbers and polynomials. The value of $p$-adic $q$-zeta functions at negative integers were given explicitly. They also defined new generating functions of $q$-Bernoulli numbers and polynomials. By using these functions, they proved analytic continuation of $q$-$L$-series. These generating functions also interpolate the Barnes-type Changhee $q$-Bernoulli numbers with attached to Dirichlet character as well. By applying the Mellin transformation, they obtained relations between the Barnes-type $q$-zeta function and new Barnes-type Changhee $q$-Bernoulli numbers.

In [16], Kim and Rim constructed two-variable $L$-function, $L(s, x | \chi)$. They showed that this function interpolates the generalized Bernoulli polynomials associated with $\chi$. By the Mellin transforms, they gave the complex integral representation for the two-variable Dirichlet $L$-function. They also found some properties of the two-variable Dirichlet $L$-function.

In [18], Kim constructed the two-variable $p$-adic $q$-$L$-function which interpolates the generalized $q$-Bernoulli polynomials associated with Dirichlet character. He also gave some $p$-adic integrals representation for this two-variable $p$-adic $q$-$L$-function and derived $q$-extension of the generalized formula of Diamond and Ferro and Greenberg for the two variable $p$-adic $L$-function in terms of the $p$-adic gamma and log gamma function.

In this paper, more precisely, we define and prove the following results:

We define the two-variable Dirichlet-type Changhee $q$-$L$-function as follows:

**Definition 1.** Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Then

$$L_q(s, x | \chi; w_1) = w_1 \sum_{n=0}^{\infty} \frac{\chi(n)q^{w_1n}}{[w_1n]^s}.$$
Theorem 1. Let $s, w_1 \in \mathbb{C}$, with $\text{Re}(w_1) > 0$.

$$L_{q,r}(s \mid \chi; w_1) = [f]^{-s} \sum_{a=1}^{f} \frac{\chi(a) \zeta_q(s, \frac{x + w_1 a}{f})}{f} |w_1|,$$

where $\zeta_q(s, w | w_1)$ is denoted by the Barnes-type Changhee $q$-zeta function, which is defined by (see [6, 7, 8, 10, 13, 19]): For $s \in \mathbb{C}$, we have

$$\zeta_q(s, w | w_1) = -\frac{1}{s-1} \log q + w_1 \sum_{n=0}^{\infty} \frac{q^{w_1 n} w^n}{[w_1 n + w]^s}.$$

A relationship between $L_{q,(s, x \mid \chi; w_1)}$ and generalized Changhee $q$-Bernoulli numbers are given as follows:

Theorem 2. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$, let $s, w_1 \in \mathbb{C}$, with $\text{Re}(w_1) > 0$ and let $n \in \mathbb{Z}^+$. Then

$$L_q(1-n, x \mid \chi; w_1) = -\beta_{n, \chi}(x \mid q | w_1),$$

where $\beta_{n, \chi}(x, q | w_1)$ is given by the following generating function

$$-tw_1 \sum_{n=1}^{\infty} \frac{\chi(n) q^{nw_1} e^{(n w_1 + x)t}}{n!} = \sum_{n=1}^{\infty} \frac{\beta_n(x, q | w_1) t^n}{n!}.$$

(For these polynomials see [6, 7, 8, 10, 13, 19].)

The two-variable Dirichlet-type multiple Changhee $q$-L-functions are defined as follows.

Definition 2. Let $s, w_1, \ldots, w_r \in \mathbb{C}$, with $\text{Re}(w_j) > 0$, $j = 1, 2, \ldots, r$. For a Dirichlet character $\chi$ with conductor $f \in \mathbb{Z}^+$, we define

$$L_{q,r}(s, x \mid \chi; w_1, \ldots, w_r) = \left( \prod_{j=1}^{r} w_j \right) \sum_{n_1, n_2, \ldots, n_r = 1}^{\infty} \frac{\prod_{k=1}^{r} \chi(n_k) q^{(\sum_{m=1}^{r} w_m n_m)^s}}{[x + \sum_{m=1}^{r} w_m n_m]^s}.$$

Theorem 3. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Let $s, w_1, \ldots, w_r \in \mathbb{C}$, with $\text{Re}(w_j) > 0$, $j = 1, 2, \ldots, r$. Then

$$L_{q,r}(s, x \mid \chi; w_1, \ldots, w_r) = [f]^{-s} \sum_{a_1, \ldots, a_r = 1}^{f} \left( \prod_{k=1}^{r} \chi(a_k) \right) \zeta_{q,r}(s, \frac{x + w_1 a_1 + \ldots + w_r a_r}{f} | w_1, \ldots, w_r),$$

where $\zeta_{q,r}(s, w | w_1, w_2, \ldots, w_r)$, the Barnes-type multiple Changhee $q$-zeta functions, is defined by

$$\zeta_{q,r}(s, w | w_1, w_2, \ldots, w_r) = \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} q^{w_n w_1 + w_n w_2 + \ldots + w_n w_r},$$

$\Re(w) > 0$, $q \in C$ with $| q | < 1$ (For the Barnes-type Changhee multiple $q$-zeta functions see [6, 7, 8, 10, 13, 19]).

We note that $\zeta_{q,r}(s, w | w_1, w_2, \ldots, w_r)$ is analytic continuation for $\text{Re}(s) > r$.

The numbers $L_{q,r}(-n, \chi | w_1, \ldots, w_r)$, ($n > 0$) are given explicitly by Theorem below.
Theorem 4. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Let $w_1, \ldots, w_r \in \mathbb{C}$, with $\operatorname{Re}(w_j) > 0$, $j = 1, 2, \ldots, r$ and let $n \in \mathbb{Z}^+$. Then

$$L_{q,r}(-n \mid \chi; w_1, \ldots, w_r) = (-1)^r \frac{n!}{(n+r)!} B_{n,\chi}^{(r)}(x \mid q; w_1, \ldots, w_r),$$

where $B_{n,\chi}^{(r)}(x, q; w_1, \ldots, w_r)$, the Barnes-type multiple Changhee $q$-Bernoulli polynomials, is defined by the following generating function

$$(-t)^r \left( \prod_{i=1}^r w_i \right) \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} q^{x+n_1 w_1 + n_2 w_2 + \cdots + n_r w_r} t^n \prod_{j=0}^{n-1} \left( x + \sum_{i=1}^r n_i w_i \right)$$

$$= \sum_{n=0}^{\infty} B_{n,\chi}^{(r)}(x \mid q; w_1, w_2, \ldots, w_r) t^n \frac{n!}{n!} \left( |t| < 2\pi \right)$$

(For $B_{n,\chi}^{(r)}(x, q; w_1, \ldots, w_r)$ see [6], [7], [8], [10], [13], [19]).

2. Definition and Notations

Let $\mathbb{C}$ be the set of complex numbers and $z \in \mathbb{C}$. The classical Bernoulli polynomials $B_n(z)$ are defined by means of the generating function (for details see [22], [27], [4], and [15], [19]):

$$F(t, x) = \frac{t e^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!} \left( |t| < 2\pi \right).$$

Putting $z = 0$ into (2.1), $B_n(0) = B_n$ is the usual $n$th classical Bernoulli number. The classical Bernoulli numbers are defined by means of the generating function:

$$F(t) = \frac{t e^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \left( |t| < 2\pi \right).$$

In [4], Carlitz defined $q$-extensions of these classical Bernoulli numbers and polynomials. Carlitz’s $q$-Bernoulli numbers, $\beta_n = \beta_n(q)$ are defined by [4]

$$\beta_0 = 1, q(q+1)^n - \beta_n = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing $\beta^n$ by $\beta_n$.

Carlitz’s $q$-Bernoulli polynomials $\beta_n(x \mid q)$ are defined as follows ([4], [8], [19])

$$\beta_n(x \mid q) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{k(x)} x^{n-k}.$$

Thus we note that

$$\lim_{q \to 1} \beta_n(q) = B_n,$$

and

$$\lim_{q \to 1} \beta_n(x \mid q) = B_n(x).$$

Let $\mathbb{Z}^+$ be the set of positive integer numbers. Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Then the generalized Bernoulli numbers $B_{n,\chi}$ are defined by means of the generating function

$$F_\chi(t) = \sum_{a=0}^{f-1} \frac{\chi(a) t e^{at}}{e^t - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} \left( |t| < 2\pi \right).$$
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The generalized Bernoulli polynomials $B_{n,\chi}(x)$ are defined by means of the generating function

$$
F_\chi(t, x) = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!} \quad (|t|<2\pi).
$$

Thus these polynomials, $B_{n,\chi}(x)$, are defined as follows:

$$
B_{n,\chi}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k,\chi x^{n-k}.
$$

The Riemann zeta function is defined by the series

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
$$

where $s \in \mathbb{C}$ with Re($s$) > 0.

The Hurwitz zeta function is defined by the series

$$
\zeta(s, x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^s},
$$

where $s \in \mathbb{C}$ with Re($s$) > 0 (1, 32, 31, 19, 16). We note that $\zeta(s, 1) = \zeta(s)$.

These functions are analytic continuation for Re($s$) > 1. Relation between $\zeta(s, x)$ and Bernoulli polynomials, $B_n(x)$ is given as follows:

For $n \in \mathbb{Z}^+$,

$$
\zeta(1-n, x) = -\frac{B_n(x)}{n}
$$

(for detail see [1, 32, 15]).

Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. The Dirichlet $L$-function is defined by

$$
L(s, \chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{n^s}.
$$

Relation between $L(s, \chi)$ and the generalized Bernoulli numbers, $B_{n,\chi}$ is given as follows: For $n \in \mathbb{Z}^+$,

$$
L(1-n, \chi) = -\frac{B_{n,\chi}}{n}.
$$

The two-variable $L$-function is defined as follows:

**Definition 3.** (16) Let $\chi$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$ and $s \in \mathbb{C}$.

$$
L(s, \chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{(n+x)^s}.
$$

We note that $L(s, \chi) = L(s, \chi)$. $L(s, x | \chi)$ is analytic continuation in $\mathbb{C}$ with only simple pole at $s = 1$ (16).

Relation between $L(s, x | \chi)$ and the generalized Bernoulli numbers, $B_{n,\chi}$ is given as follows: for $n \in \mathbb{Z}^+$,

$$
L(1-n, x | \chi) = -\frac{B_{n,\chi}}{n}.
$$

(For detail see 16).

A sequence of $p$-adic rational numbers as multiple Changhee $q$-Bernoulli numbers and polynomials are defined as follows [8, 12].
Let $a_1, \ldots, a_r$ be nonzero elements of the $p$-adic number field and let $z \in \mathbb{C}_p$.

$$
\beta_n^{(r)}(w : q | a_1, \ldots, a_r) = \frac{1}{\prod_{j=1}^{r} a_j} \int_{\mathbb{Z}_p} [w + \sum_{j=1}^{r} a_j x_j]^n d\mu_q(x),
$$

where

$$
\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f(x) d\mu_q(x_1) d\mu_q(x_2) \cdots d\mu_q(x_r)
$$

(see [9], [11], [12]). It is easily observed from (2.6) that

$$
\beta_n^{(r)}(w : q | a_1, \ldots, a_r) = \frac{1}{(1 \!-\! q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{wl} \prod_{j=1}^{r} \left( l + \frac{1}{a_j} \right)
$$

(2.7) $\beta_n^{(r)}(w : q | a_1, \ldots, a_r)$ is defined analogously as follows:

$$
\beta_n^{(r)}(w : a_1, \ldots, a_r) = \beta_n^{(r)}(w : q),
$$

where $\beta_n^{(r)}(w : q)$ are the $q$-Bernoulli numbers as follows:

$$
\beta_n^{(r)}(w : q | a_1, \ldots, a_r), \text{ for } n \in \mathbb{Z}^+.
$$

By using (2.6) and (2.7), we note that

$$
\lim_{n \to \infty} \beta_n^{(r)}(w : q | a_1, \ldots, a_r) = B_n(w, r | a_1, \ldots, a_r).
$$

The generating functions $F_q(t)$ and $F_q(t, x)$ occurring in (2.2) and (2.1), respectively. These generating functions, $F_q(t)$ of $q$-Bernoulli numbers $\beta_n(q)$ and $F_q(t, x)$, respectively, are given as follows:

$$
F_q(t) = \frac{q - 1}{\log q} \exp\left( \frac{t}{1 - q} \right) - t \sum_{n=0}^{\infty} q^n e^{nt} = \sum_{n=0}^{\infty} \frac{\beta_n(q) t^n}{n!}.
$$

By using (2.2) and (2.8), we have

$$
\lim_{q \to 1} \beta_n(q) = B_n, \text{ and } \lim_{q \to 1} F_q(t) = F(t).
$$

The generating function $F_q(x, t)$ of the $q$-Bernoulli polynomials $\beta_n(x : q)$ is defined analogously as follows:

$$
F_q(x, t) = \frac{q - 1}{\log q} \exp\left( \frac{t}{1 - q} \right) - t \sum_{n=0}^{\infty} q^n x^n e^{nt} = \sum_{n=0}^{\infty} \frac{\beta_n(x : q) t^n}{n!}.
$$

By using (2.2) and (2.9), we have

$$
\lim_{q \to 1} \beta_n(x : q) = B_n(x), \text{ and } \lim_{q \to 1} F_q(x, t) = F(x, t).
$$

The series on the right-hand side of (2.7) and (2.8) are uniformly convergent in the wider sense. Consequently, we shall explicitly determine the $q$-Bernoulli numbers as follows:

$$
\beta_0(q) = \frac{q - 1}{\log q}, \quad q(q\beta(q) + 1)^n - \beta_n(q) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}
$$

with the usual convention about replacing $\beta^n$ by $\beta_n$. 
Let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \). The generating function of generalized \( q \)-Bernoulli numbers attached to \( \chi \) is given as follows (for detail see [6], [7], [8], [10], [13], [19], [19]):

\[
F_{q,\chi}(t) = -t \sum_{a=1}^{f} \sum_{n=0}^{\infty} q^{fn+a} e^{[fn+a]t} \sum_{a=1}^{f} \chi(a) \sum_{n=0}^{\infty} q^{fn+a} e^{[fn+a]t} \]

\[
= -t \sum_{n=0}^{\infty} \chi(n) q^n e^{[n]t} \]

\[
= \sum_{n=0}^{\infty} \beta_{n,\chi}(q) \frac{t^n}{n!}.
\]

where the coefficients, \( \beta_{n,\chi}(q) \) (\( n \geq 0 \)) are called generalized \( q \)-Bernoulli numbers with a Dirichlet character. We note from the definitions in (2.5) and (2.10) that

\[
\lim_{q \to 1} \beta_{n,\chi}(q) = B_{n,\chi},
\]

and

\[
\lim_{q \to 1} F_{q,\chi}(t) = F_{\chi}(t) = \sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{tf} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.
\]

Generating function of generalized \( q \)-Bernoulli polynomials is associated with a Dirichlet character as follows (see [6], [7], [8], [10], [13], [19]):

\[
F_{q,\chi}(x, t) = q^{x} t e^{-[x]t} \sum_{n=0}^{\infty} \chi(n) q^n e^{[n]q^x t} \]

\[
= -t \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]t} \]

\[
= \sum_{n=0}^{\infty} \beta_{n,\chi}(x:q) \frac{t^n}{n!}.
\]

By using (2.9), (2.10) and (2.11), we obtain

\[
F_{q,\chi}(x, t) = -t \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]t} \]

\[
= -e^{[x]t} q^x t \sum_{a=1}^{f} \chi(a) \sum_{n=0}^{\infty} q^{fn+a} e^{[fn+a]q^x t}.
\]

Let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \). Then

\[
\beta_{n,\chi}(x:q) = \frac{1}{[f]_{1-n}} \sum_{a=0}^{f-1} \chi(a) \beta_n \left( \frac{a+x}{f} ; q^f \right)
\]

(For detail see [7], [8], [9], [10], [11], [12], [13], [14], [15], [19]).

In [19], T. Kim, Y. Simsek and H. M. Srivastava gave new generating functions which produce new definitions of the Barnes-type Changhee \( q \)-Bernoulli polynomials and the generalized Barnes-type Changhee \( q \)-Bernoulli numbers with attached to Dirichlet character. These generating functions are very important in case of multiple zeta function. Therefore, by using these generating functions, they proved
relation between the Barnes-type Changhee \( q \)-zeta function and the Barnes-type Changhee \( q \)-Bernoulli numbers.

Let \( w, w_1, w_2, \ldots, w_r \) be complex numbers such that \( w_i \neq 0 \) for \( i = 1, 2, \ldots, r \).

Kim (6, 8, 11) and Kim, Simsek and Srivastava (19) defined the Barnes-type of Changhee \( q \)-Bernoulli polynomials of \( w \) with parameters \( w_1 \) as follows:

\[
F_q(w, t \mid w_1) = \frac{q - 1}{\log q} e^{\frac{tw}{1-q}} - w_1 t \sum_{n=0}^{\infty} q^{w_1 n + w} e^{[w_1 n + w] t} = \sum_{n=0}^{\infty} \frac{\beta_n(w : q \mid w_1) t^n}{n!} (|t| < 2\pi),
\]

where the coefficients, \( \beta_n(w : q \mid w_1) \) \((n \geq 0)\) are called Barnes-type of Changhee \( q \)-Bernoulli polynomials in \( w \) with parameters \( w_1 \).

We note that

\[
\lim_{q \to 1} \beta_n(w : q \mid w_1) = w_n \beta_n(w), \quad \text{and} \quad \lim_{q \to 1} F_q(w, t \mid w_1) = w_1 t e^{w t},
\]

where \( \beta_n(w) \) are the ordinary Barnes Bernoulli polynomials.

By using (2.12), we easily obtain (6, 8, 11, 19)

\[
\beta_n(w : q \mid w_1) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lw} (-1)^l \frac{lw_1}{[lw_1]}.
\]

If \( w = 0 \) in the above, then

\[
\beta_n(0 : q \mid w_1) = \beta_n(q \mid w_1),
\]

where \( \beta_n(q \mid w_1) \) are called the Barnes-type Changhee \( q \)-Bernoulli numbers with parameter \( w_1 \).

Let \( \chi \) be the Dirichlet character with conductor \( f \). Then the generalized Barnes-type Changhee \( q \)-Bernoulli numbers with attached to \( \chi \) are defined as follows (6, 8, 11, 19):

\[
F_{q,\chi}(t \mid w_1) = -w_1 t \sum_{n=1}^{\infty} \chi(n) q^{w_1 n + w} e^{[w_1 n + w] t} = \sum_{n=0}^{\infty} \frac{\beta_n(\chi \mid w_1) t^n}{n!} (|t| < 2\pi).
\]

**Theorem 5.** (19) Let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \). Then

\[
\beta_{n,\chi}(x : q \mid w_1) = \frac{1}{f^{1-n}} \sum_{a=0}^{f-1} \chi(a) \beta_n(\frac{x + aw_1}{f} : q \mid w_1).
\]

The Barnes-type Changhee \( q \)-zeta functions are defined as follows:

**Definition 4.** (19) For \( s \in \mathbb{C} \), we have

\[
\zeta_q(s, w \mid w_1) = -\frac{(1-q)^s}{s-1} \log q + w_1 \sum_{n=0}^{\infty} \frac{q^{w_1 n + w}}{[w_1 n + w]^s}.
\]
Theorem 6. ([19]) If \( n \in \mathbb{Z}^+ \), then
\[
\zeta_q(1 - n, w \mid w_1) = -\frac{\beta_n(w : q \mid w_1)}{n}.
\]

Remark 1. \( \zeta_q(s, w \mid w_1) \) is analytic continuation in \( \mathbb{C} \) with only simple pole at \( s = 1 \).

3. The two-variable Dirichlet-type Changhee \( q \)-\( L \)-function

Let \( [6], [7], [8], [10], [13], [19] \)
\[
F_{q, \chi}(t, x \mid w_1) = -tw_1e^{|x|t}t \sum_{n=1}^{\infty} \chi(n)q^{wn_1}e^{[nw_1+|x|]t}
\]
\[
= -tw_1 \sum_{n=1}^{\infty} \chi(n)q^{wn_1}e^{[nw_1+|x|]t}
\]
\[
= \sum_{n=1}^{\infty} \beta_n(x : q \mid w_1) t^n,
\]
where we use \([x + a] = [x] + q^a[x]\) in (3.1). We consider the following contour integral:
\[
\frac{\Gamma(1 - s)e^{-\pi is}}{2\pi i} \int_C t^{-s-2}F_{\chi,q}(-t, x \mid w_1)dt = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{-s-2}F_{\chi,q}(-t, x \mid w_1)dt
\]
\[
= w_1 \sum_{n=1}^{\infty} \chi(n)q^{wn_1} \int_0^{\infty} t^{-s-1}e^{-[x+w_1n]t}dt
\]
\[
= w_1 \sum_{n=0}^{\infty} \chi(n)q^{wn_1} \frac{1}{[x+w_1n]^s}
\]
where \( C \) denotes a positively oriented (counter-clockwise) circle of radius \( R \), centered at the origin. The function
\[
Y(t, x) = \frac{1}{\Gamma(s)} t^{-s-2}F_{\chi,q}(-t, x \mid w_1)
\]
has pole \( t = 0 \) inside the contour \( C \). Therefore, if we want to integrate \( Y(t, x) \) function, then we have modify the contour by indentation at this point. We take indentation as identical small semicircle, which has radius \( r \), leaving \( t = 0 \).

Thus we arrive at the two-variable Dirichlet-type Changhee \( q \)-\( L \)-function, which is given in Definition 1:

Let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \),
\[
L_q(s, x \mid \chi; w_1) = w_1 \sum_{n=0}^{\infty} \frac{\chi(n)q^{wn_1}}{[x+w_1n]^s}.
\]
If we take \( q \to 1 \) and \( w = 1 \) in (3.3), then Definition 1 reduces to Definition 3, that is
\[
\lim_{q \to 1} L_q(s, x \mid \chi; 1) = \sum_{n=0}^{\infty} \frac{\chi(n)}{(x+n)^s}.
\]

The Dirichlet-type Changhee \( q \)-\( L \)-function and the Hurwitz-type Changhee \( q \)-\( zeta \) function are closely related, too. We give proof of this relation below.
Proof of Theorem 1. By setting \( n = a + kf \), where \( (k = 0, 1, 2, \ldots, \infty; a = 1, 2, \ldots, f) \) in (3.3), we have

\[
L_q(s, x | \chi; w_1) = w_1 \sum_{a=1}^{f} \chi(a) \sum_{k=0}^{\infty} \frac{q^{(aw_1 + kf w_1)}}{|x + aw_1 + kf w_1|^s}.
\]

By using (2.14) in the above equation, we easily arrive at the desired result.

First proof of Theorem 2. If we take \( s \to 1 - n \) in Theorem 1, where \( n \) is a positive integer, then we have

\[
L_q(1 - n, x | \chi; w_1) = \left[ f \right]^{n-1} \sum_{a=1}^{f} \chi(a) \zeta_q f(1 - n, x + aw_1 f | w_1).
\]

By using (2.13) in the above, we easily arrive at the following Theorem.

Second proof of Theorem 2. Proof of this Theorem similar to that of Theorem 8 in [19]. Let

\[
B(s, x) = \int_C z^{s-2} F_{\chi,q}(-z, x | w_1) dz,
\]

where \( C \) is Hankel's contour along the cut joining the points \( z = 0 \) and \( z = \infty \) on the real axis, which starts from the point at \( \infty \), encircles the origin \( (z = 0) \) once in the positive (counter-clockwise) direction, and returns to the point at \( \infty \) (see for details, [32] p. 245). Here, as usual, we interpret \( z^s \) to mean \( \exp(s \log z) \), where we assume \( \log \) to be defined by \( \log t \) on the top part of the real axis and by \( \log t + 2\pi i \) on the bottom part of the real axis. We thus find from the definition (3.4) that

\[
B(s, x) = (e^{2\pi i s} - 1) \int_{t}^{\infty} t^{s-2} F_{\chi,q}(-t, x | w_1) dt
\]

\[
+ \int_{C_\varepsilon} z^{s-2} F_{\chi,q}(-z, x | w_1) dz,
\]

where \( C_\varepsilon \) denotes a circle of radius \( \varepsilon > 0 \) (and centred at the origin), which is described in the positive (counter-clockwise) direction. Assume first that \( \text{Re}(s) > 1 \). Then

\[
\int_{C_\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

so we have

\[
B(s, x) = (e^{2\pi i s} - 1) \int_{0}^{\infty} t^{s-2} F_{\chi,q}(-t, x | w_1) dt,
\]
by using \(1\) and \(3\) in the above equation, after some elementary calculations, we obtain
\[
B(s, x) = (e^{2\pi is} - 1)\Gamma(s)L_q(s, x \mid \chi; w_1).
\]
Consequently,
\[
L_q(s, x \mid \chi; w_1) = \frac{B(s, x)}{(e^{2\pi is} - 1)\Gamma(s)},
\]
which, by analytic continuation, holds true for all \(s \neq 1\). This evidently provides us with an analytic continuation of \(L_q(s, x \mid \chi; w_1)\).

Let \(s \to 1 - n\) in \(3.5\), where \(n\) is a positive integer. Since
\[
e^{2\pi is} = e^{2\pi i(1-n)} = 1\quad (n \in \mathbb{Z}^+)
\]
we have
\[
\lim_{s \to 1-n} \left\{(e^{2\pi is} - 1)\Gamma(s)\right\} = \lim_{s \to 1-n} \left\{(e^{2\pi is} - 1)\frac{\pi}{\sin(\pi s) \Gamma(1-s)}\right\} = 2\pi i(-1)^{n-1} \frac{1}{(n-1)!} \quad (n \in \mathbb{Z}^+)
\]
by means of the familiar reflection formula for \(\Gamma(s)\). Furthermore, since the integrand in \(3.4\) has simple pole order \(n+1\) at \(z = 0\), where also find from the definition \(3.4\) with \(s = 1 - n\) that
\[
B(1 - n, x) = \int_C z^{-n-1}F_{\chi, q}(-z, x \mid w_1)dz
\]
\[
= 2\pi i \text{Res}_{z=0} \left\{ z^{-n-1}F_{\chi, q}(-z, x \mid w_1) \right\} = (2\pi i)\frac{(-1)^n}{n!} \beta_n(x \mid q; w_1),
\]
where we have made of the power-series representation in \(3.4\). Thus by Cauchy Residue Theorem, we easily arrive at the desired result upon suitably combining \(3.6\) and \(3.7\) with \(3.5\).

Now, we define generalized multiple Changhee \(q\)-Bernoulli numbers attached to the Dirichlet character \(\chi\). We also construct the two-variable Dirichlet-type multiple Changhee \(q\)-L-functions. We then give relation between the two-variable Dirichlet-type multiple Changhee \(q\)-L-functions and the generalized multiple Changhee \(q\)-Bernoulli numbers as well.

The generalized multiple Changhee \(q\)-Bernoulli numbers attached to the Dirichlet character \(\chi\) are defined by means of the following generating function:
\[
F_{q, \chi}^{(r)}(t, x \mid w_1, \ldots, w_r) = (-t)^r \left( \prod_{j=1}^{r} w_j \right)
\times \sum_{n_1, \ldots, n_r=1}^{\infty} \left( \prod_{k=1}^{r} \chi(n_k) \right) q^{\left( \sum_{m=1}^{r} w_m n_m \right) t^{x+\sum_{m=1}^{r} w_m n_m}} (\mid t \mid < 2\pi),
\]
where \(w_1, \ldots, w_r \in \mathbb{R}^+, r \in \mathbb{Z}^+\).
Here, we can now construct the two-variable Dirichlet-type multiple Changhee $q$-$L$-function. By using the Mellin transformation and Residue Theorem in (3.8), we obtain

$$
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1-r} F_{q,r}(-t, x) \, dt \quad \mid w_1, \ldots, w_r)
$$

(3.9)

By using (3.9), we can arrive at the definition of the two-variable Dirichlet-type multiple Changhee $q$-$L$-functions, which is given in Definition 2, we also give this relation as follows:

For a Dirichlet character $\chi$ with conductor $f \in \mathbb{Z}^+$,

$$
L_{q,r}(s, \chi \mid w_1, \ldots, w_r) = \left( \prod_{j=1}^r w_j \right) \sum_{n_1, n_2, \ldots, n_r = 1}^{\infty} \left( \prod_{k=1}^r \chi(n_k) \right) q^{(\sum_{m=1}^r w_m n_m)}
$$

(3.10)

By using (3.8) to Theorem 3, the numbers $L_{q,r}(s, \chi \mid w_1, \ldots, w_r)$, $(n > 0)$ are given explicitly by Theorem 4 below.

**Proof of Theorem 3.** Proof of Theorem 3 runs parallel to that of Theorem 1 above, so we choose to omit the details involved. By setting $n_j = a_j + n_j f$, $(j \in \{1, 2, \ldots, r\}$, $n_j = 0, 1, \ldots, \infty$, and $a_j = 1, 2, \ldots, f$) in (3.11), we easily arrive at the following Theorem.

By using (3.8) to Theorem 3, the numbers $L_{q,r}(-n, \chi \mid w_1, ..., w_r)$, $(n > 0)$ are given explicitly by Theorem 4 below.

**Proof of Theorem 4.** Proof of Theorem 4 runs parallel to that of Theorem 2 above, so we choose to omit the details involved. If we take $s \to 1 - n$ in (3.9), where $n$ is a positive integer, then by using the Mellin transformation and Residue Theorem in (3.8), we easily arrive at the desired result.

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