ELECTROMAGNETIC MASS MODELS IN GENERAL THEORY OF RELATIVITY:
Ph.D. Thesis

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Abstract

“Electromagnetic mass” where gravitational mass and other physical quantities originate from the electromagnetic field alone has a century long distinguished history. In the introductory chapter we have divided this history into three broad categories – classical, quantum mechanical and general relativistic. Each of the categories has been described at a length to get the detailed picture of the physical background. Recent developments on Repulsive Electromagnetic Mass Models are of special interest in this introductory part of the thesis. In this context we have also stated motivation of our work. In the subsequent chapters we have presented our results and their physical significances. It is concluded that the electromagnetic mass models which are the sources of purely electromagnetic origin “have not only heuristic flavor associated with the conjecture of Lorentz but even a physics having unconventional yet novel features characterizing their own contributions independent of the rest of the physics”.
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SYNOPSIS

The investigations carried out in the thesis “ELECTROMAGNETIC MASS MODELS IN GENERAL THEORY OF RELATIVITY” form seven chapters including the introductory and concluding ones.

Introduction
The study of “electromagnetic mass” where gravitational mass and other physical quantities originate from the electromagnetic field alone has a century long distinguished history. In the introductory chapter we have divided this history into three broad categories – classical, quantum mechanical and general relativistic. Each of the categories has been described at a length to get the detailed picture of the physical background. In the classical part starting from the Lorentz’s Theory of Electrons it includes Thomson’s Concept of Electromagnetic Mass and Abraham’s Model for the Electrons. We have also described the drawbacks of Lorentz’s Model for the Electrons. In this connection Poincaré’s Theory of Electrons has been put forward to eliminate the discrepancy in Lorentz’s Model. Einstein’s Special Relativistic Model of Electrons and various other Models including the General Relativistic one and Quantum Electron Models have been included in other parts of the history. Recent developments on Repulsive Electromagnetic Mass Models are of special interest in this introductory part of the thesis. In this context we have also stated motivation of our work.

CHAPTER II
Relativistic Electromagnetic Mass Models with Cosmological Variable $\Lambda$ in Spherically Symmetric Anisotropic Source
In the chapter II a class of exact solutions for the Einstein-Maxwell field equations, which are obtained by assuming the erstwhile cosmological constant $\Lambda$ to be a space-variable scalar, viz., $\Lambda = \Lambda(r)$. The source considered here is static, spherically symmetric and anisotropic charged fluid. The solutions obtained are matched continuously to the exterior Reissner-Nordström solution and each of the four solu-
tions represents an electromagnetic mass model.

CHAPTER III
Classical Electron Model with Negative Energy Density in Einstein-Cartan Theory of Gravitation
The experimental results regarding the maximum limit of the radius of the electron \( \sim 10^{-16} \) cm and a few of the theoretical works readily suggest that there might be some negative energy density regions within the particle as per General Theory of Relativity. It is argued in the chapter III of the present investigation that such a negative energy density can also be obtained with a better physical interpretation in the framework of the Einstein-Cartan theory.

CHAPTER IV
Energy Density in General Relativity: a Possible Role for Cosmological Constant
We consider a static spherically symmetric charged anisotropic fluid source of radius \( \sim 10^{-16} \) cm by introducing a variable \( \Lambda \) dependent on the radial coordinate \( r \) under general relativity in the chapter IV. From the solution sets a possible role of the cosmological constant is investigated which indicates the dependence of energy density on it.

CHAPTER V
Relativistic Electromagnetic Mass Models: Charged Dust Distribution in Higher Dimensions
Electromagnetic mass models are proved to exist in higher dimensional theory of general relativity corresponding to charged dust distribution. In the chapter V, along with the general proof, a specific example is also cited as a supporting candidate.
CHAPTER VI

Relativistic Anisotropic Charged Fluid Spheres with Varying Cosmological Constant

Static spherically symmetric anisotropic source has been studied for the Einstein-Maxwell field equations assuming the erstwhile cosmological constant $\Lambda$ to be a space-variable scalar, viz., $\Lambda = \Lambda(r)$. The solutions thus obtained are shown to be electromagnetic in origin in the sense that all the physical parameters including the gravitational mass originate from the electromagnetic field alone. Moreover, to construct the models it is also shown that the generally used pure charge condition, viz., $\rho + p_r = 0$ is not always required for constructing electromagnetic mass models. This is the main theme of the chapter VI.

CHAPTER VII

Conclusions

Electromagnetic mass models which are the sources of purely electromagnetic origin “have not only heuristic flavor associated with the conjecture of Lorentz but even a physics having unconventional yet novel features characterizing their own contributions independent of the rest of the physics” (Tiwari 2001). This is, as Tiwari (2001) guess “may be due to the subtle nature of the mass of the source (being dependent on the electromagnetic field alone)”. Therefore, in our whole attempt we have tried to explore “the subtle nature of the mass of the source”. However, to do this under the general relativistic framework, we have considered Einstein field equations in its general form, i.e., with cosmological constant $\Lambda$ which also acts as a source term to the energy-momentum tensor. If we consider that $\Lambda$ has a variable structure which is dependent on the radial coordinate of the spherical distribution, viz., $\Lambda = \Lambda(r)$ then it can be shown that $\Lambda$ is related to pressure and matter energy density. Hence it contributes to the effective gravitational mass of the system.

It is seen that equation of state has an important role in connection to electromagnetic mass model. Therefore, at first we have obtained electromagnetic mass model under the condition $\rho + p = 0$. However, later on it is shown that electromagnetic
mass model can also be obtained by using more general condition \( \rho + p \neq 0 \).

The model considered in our work, in general, corresponds to a charged sphere with cosmological parameter in such a way that it does not vanish at the boundary. The idea behind is that the cosmological parameter is related to the zero point vacuum energy it should have some finite non-zero value even at the surface of the bounding system. For this type of spherical system we can have a class of solutions related to charged as well as neutral configurations.

It can be shown that these models have positive energy densities everywhere. Their corresponding radii are always much larger than \( 10^{-16} \) cm. Furthermore, as the radii of these models shrink to zero, their total gravitational mass becomes infinite.

It have been shown by Bonnor and Cooperstock (1989) that an electron must have a negative energy distribution (at least for some values of the radial coordinate). In this connection we have shown that the cosmological parameter \( \Lambda \) has a definite role on the energy density of micro particle, like electron. At an early epoch of the universe when the numerical value of negative \( \Lambda \) was higher than that of the energy density \( \rho \), the later quantity became a positive one. In the case of decreasing negative value of \( \Lambda \) there was a smooth crossover from positive energy density to a negative energy density.

It is suggested by Bonnor and Cooperstock (1989) and Herrera and Varela (1994) that spin and magnetic moment can be introduced to electron through the Kerr-Newman metric. But, it has been seen that the Kerr-Newman metric cannot be valid for distance scales of the radius of a subatomic particle. We, therefore, tackled the problem in the frame work of Einstein-Cartan theory where torsion and spin are inherently present. In this case, the only way is to take the spin to be the ‘intrinsic angular momentum’ that is the spin of quantum mechanical origin. In our work considering the spins of all the individual particles are assumed to be oriented along the radial axis of the spherical systems we have obtained some interesting solutions with physical validity.

Another important point we would like to mention here that in all the previous investigations we have studied electromagnetic mass models in 4-dimensional Einstein-
Maxwell space-times only. Therefore, one can ask whether electromagnetic mass models also can exist in higher dimensional theory of General Relativity. We have presented a model which corresponds to spherically symmetric gravitational sources of purely electromagnetic origin in the space-time of \((n + 2)\) dimensional theory of general relativity.

We have also taken up the problem of anisotropic fluid sphere as studied earlier in a different viewpoint. By expressing \(\Lambda\) in terms of electric field strength \(E\) we have explored some possibilities to construct electromagnetic mass models using the constraint \(\rho + p \neq 0\). We would like to mention here that unlike the solutions of Grøn (1986a,b) and Ponce de Leon (1987a,b) in the present investigation, in general, the electric field (and hence the cosmological constant) does not vanish at the boundary. However, it is shown that the class of solutions obtained here are related to charged as well as neutral systems of Grøn (1986a,b) and Ponce de Leon (1987a,b) depending on the values of the parameter \(N\).
Chapter 1

Introduction

“...the world stands before us as a great, eternal riddle.”
– Albert Einstein

The first elementary particle was called corpuscle by J. J. Thomson (1881), later named as electron – the word proposed by G. Johnstone Stoney in 1891 as a unit of electronic charge. There were two schools of thought, one school favoring atomistic world view, and the other believing in the continuum. The debate continued for several years in absence of sufficient experimental findings. It is in this respect that J. J. Thomson’s work turns out to be epoch making, and the credit for the discovery of electron belongs to him. He succeeded in a determination of the charge to mass ratio and the elementary charge $e$. The physical world and all its phenomena in principle can be reduced to the problem of the interaction of the elementary particles. The four types of fundamental interacting forces are labeled as electromagnetic, gravitational, strong and weak. Of these four interactions the electromagnetic one thoroughly investigated and has been best understood in connection with the electrically charged particles, especially, the electrons. The study of electromagnetic mass with a century long distinguished history can be divided into three broad categories – classical, quantum mechanical and general relativistic. Here we shall first follow the tradition established by the classical electron
theory. Actually, there are two reasons why the classical electron is studied:

a) There is no quantum mechanical model for the electron;

b) It is quite natural to complete the classical electromagnetic mass theory first before making the transition to a quantum description of the inertial properties of an electron.

In the classical framework Lorentz tried to tackle the problem of the electrodynamics of moving bodies. To get an overview of the problem starting from Lorentz, we shall first provide a brief historical account of the work done by different authors in a sequence.

1.1 A Brief Historical Background of the Theories About the Structure of the Electrons

1.1.1 Lorentz’s Theory of Electrons

Lorentz’s (1892) apparent motivation was to solve the null result of the Michelson-Morley experiment keeping the existence of ether as it is. He wanted to represent an electromagnetic world view in comparison to the Maxwellian electromagnetic theory. Based on this philosophy he developed his Theory of Electron in 1892. The basic assumption made by him is that microscopic charged particles or ions in motion through absolutely resting ether were the source of the electromagnetic disturbances. He further assumed that ether permeates the electrical particles and that electrical particles were perfectly rigid bodies. He considered an electric field vector $\vec{E}$ and a magnetic field vector $\vec{H}$ in this absolute ether frame and obtained

$$\vec{F} = \sigma \left( \vec{E} + \frac{\vec{u} \times \vec{H}}{c} \right)$$

(1.1)

where $\sigma \rightarrow$ electric charge density, $u \rightarrow$ velocity of electric particles and $c \rightarrow$ velocity of light quanta, photons with velocity $c = 2.99 \times 10^{10} cm/sec$ in vacua.
He also considered the Maxwell equations

\[ \vec{\nabla} \cdot \vec{E} = 4\pi \rho, \quad (1.2) \]
\[ \vec{\nabla} \cdot \vec{B} = 0, \quad (1.3) \]
\[ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (1.4) \]

and

\[ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (1.5) \]

where \( \vec{E}, \vec{B}, \vec{J} \) and \( c \) respectively are the electric field, magnetic field, current density and velocity of light in vacua.

Now, assuming that the Lorentz-Maxwell equations (1.1-1.5) were valid in the rest frame of the ether, Lorentz considered the Galilean transformations

\[ \tilde{x} = x - vt, \quad \tilde{y} = y, \quad \tilde{z} = z, \quad \tilde{t} = t \quad (1.6) \]

for a reference frame \( \tilde{S} \) moving with an uniform velocity \( v \) in the direction of \( x \) with respect to the frame \( S \). To retain the form of the wave equation describing an electromagnetic disturbance in both the reference frames \( S \) and \( \tilde{S} \) Lorentz introduced new variables \( x', y', z' \) and \( t' \) connected with the equations

\[ x' = \tilde{x}, \quad y' = \tilde{y}, \quad z' = \tilde{z}, \quad t' = t - \frac{v}{c^2} \tilde{x} \quad (1.7) \]

(with upto first order of \( v/c \)).

Applying this theory Lorentz was able to explain the electrodynamics of moving bodies and various optical phenomena like the propagation of light in dielectrics at rest as well as in moving media and hence the Fresnel’s dragging effect. The first fruitful idea for explaining the null result of Michelson-Morley came from Lorentz. He suggested that material bodies contract when they are moving, and the shortening is only in the direction of the motion. He proposed that the length of the interferometer arm parallel to the direction of motion is shortened by a factor \( (1 - v^2/2c^2) \). This is known as Lorentz-Fitz Gerald contraction hypothesis. The more simplified and generalized theory of Lorentz (1895) on electrodynamics which involved the
concept of local time in contrast to the universal true time was put forward by him to ensure that the form of Maxwell equations for charge-free-space remains the same in a moving frame (at least up to first order in $v/c$). Lorentz assumed that there are several electrons in each atom which are elastically bound to an equilibrium position and thus are able to perform harmonic vibrations with given frequencies. In electric conductors additional electrons were assumed to move freely. With these fundamental theoretical tools it was possible to explain a great number of phenomena – the absorption, scattering and refraction of light by matter, the Zeeman effect and many more. During 1900 to 1903 Lorentz conjectured that a part of the electron mass might be of electromagnetic origin. However, for the sake of history we should mention that even before Lorentz there were other notable scientists, who had expressed the idea of electromagnetic mass in their works.

### 1.1.2 Thomson’s Concept of Electromagnetic Mass

While studying the interaction of charged particles Thomson (1881) found that the kinetic energy of a charged sphere increases by its motion through a medium of finite specific inductive capacity. He pointed out that the increase in the kinetic energy was due to the self induced magnetic field of the charged sphere and calculated the total mass to be

$$m = m_0 + \mu$$  \hspace{1cm} (1.8)

with $\mu = \alpha e^2/ac^2$ where $m_0$ is the mass of the charged sphere, $\mu$ is the increased mass, $e$ the electric charge, $a$ the radius of the sphere and $\alpha$ is a numerical factor of order unity. Thus he came to the conclusion that “the effect of electrification is the same as if the mass of the sphere were increased ...”. This increased mass which clearly is of electromagnetic origin was termed as electromagnetic mass. The works of Thomson on electromagnetic mass were improved upon by O. Heaviside (1889) showing that the mass of a uniformly moving charged body varied with velocity. G. F. C. Searle (1897) extended the work of Heaviside as that the energy of a
charged body and therefore its mass increases with velocity. Later on W. Kaufmann (1901 a,b,c; 1902 a,b) through a series of experiments on beta rays established the dependence of the electron mass on velocity. He (1901b) showed that one-third of the fast moving electron mass was of electromagnetic origin. Hence it was probably an easy task for M. Abraham (1902) to speculate that the electron mass was completely of electromagnetic origin. By reanalyzing his experimental data Kaufmann (1902b) found that the mass of the electron is purely of electromagnetic origin.

1.1.3 Abraham’s Model for the Electrons

M. Abraham (1902, 1905) proposed the first field theoretical model for the electron on the basis of Lorentz field equation (1.1) and the Maxwell field equations (1.2 - 1.5). He assumed the electron to be a rigid sphere with uniform surface charge distribution. Further, he considered the electron charge density as a fundamental quantity. From Kaufmann’s experimental work he already knew about the velocity dependence of electron mass. So, he undertook a detailed study of the dynamics of electron including a scheme to derive the electron mass entirely from its self-field. Proposing the equation of motion for electron as an analogy for Newtonian equations of motion and considering that the total force acting on the electron should always vanish, Abraham ultimately got two types of electron masses, one is the longitudinal mass \( m_\parallel \) and the other is transverse mass \( m_\perp \) as

\[
m_\parallel = \frac{e^2}{2ac^2\beta^3} \left[ \frac{2\beta}{1-\beta^2} \right] - ln \left( \frac{1+\beta}{1-\beta} \right) \tag{1.9}
\]

and

\[
m_\perp = \frac{e^2}{4ac^2\beta^3} \left[ (1+\beta^2)ln \left( \frac{1+\beta}{1-\beta} \right) - 2\beta \right]. \tag{1.10}
\]

For low velocities and in the limit \( \beta = 0 \), the electron mass was

\[
\mu = m_\parallel = m_\perp = \frac{2}{3} \left( \frac{e^2}{ac^2} \right). \tag{1.11}
\]
He also established the electron radius to be of the order of $10^{-13}$ cm. Finally, he claimed that the transverse mass (1.10) was in agreement with Kaufmann’s experimental data to a good approximation.

1.1.4 Lorentz’s Model for the Electrons

In 1903 Lorentz extended his electromagnetic theory of 1895. Here he included the equations of motion of free electron together with a review of Abraham’s model of the electrons. In the further development of the theory Lorentz speculated that the spherical electron would experience an ellipsoidal change in its shape while it is in motion and obtained the transverse mass (the relativistic mass) for electron as

$$m_{\perp} = \frac{e^2}{6\pi ac^2(1 - \beta^2)^{1/2}}. \quad (1.12)$$

It is then a straight forward way to obtain the relativistic electromagnetic mass of the field of the electron as

$$m_{elec}' = \frac{U_{elec}}{c^2} = \frac{1}{2} \left( \frac{e^2}{ac^2} \right) \quad (1.13)$$

which is not the same as non-relativistic electromagnetic mass

$$m_{elec} = \frac{2}{3} \left( \frac{e^2}{ac^2} \right). \quad (1.14)$$

Combining these two one can write

$$U_{elec} = \frac{3}{4} m_{elec} c^2. \quad (1.15)$$

Certainly this relation unifies gravitation with electromagnetism and thus through Lorentz the concept of electromagnetic mass was born. Lorentz then proposed a model of an electron as an extended body consisting of only pure charge and no matter and the charge is uniformly distributed on a spherical shell. In his work Lorentz dealt only with the inertial aspect of mass. Lorentz was certainly aware of the Newtonian gravitational aspect of mass, but he probably disregarded gravitational effects because Newtonian gravitational forces are smaller than electrical forces by many orders of magnitude.
1.1.5 Drawbacks of Lorentz’s Model

A discrepancy lies between the two formulas (equations 1.13 and 1.14) for the electromagnetic mass. This problem implies that the relationship between momentum and velocity for a particle in Newtonian mechanics differs from that for the completely electromagnetic electron. This defect can be corrected by merging this theory with special relativity. The anomaly of factor $\frac{4}{3}$ disappears since it is incompatible with the relativistic transformation properties. For the finite electron this was first pointed out by Fermi (1922), but his work did not receive its recognition till 1965. For point electrons the removal of this factor was later rediscovered. The inertia and mass of the classical electron originate from the unbalanced mutual repulsion of the volume elements of the charge caused by the distorted electric field of an accelerating electron. However, it is not clear what keeps the electron stable since the Lorentz’s model of the electron describes its charge as uniformly distributed on a spherical shell, which means that its volume elements tend to blow up by repelling one another. This difficulty can be removed by eliminating the electron structure and assuming the particle to be a point particle. This indeed produces a new difficulty, i.e., when the radius shrinks to zero, the electron’s mass becomes infinitely large. This is the famous self energy problem. It exists in the classical theory as well as in the quantum theory of the electron. Its satisfactory solution is not yet known. Thus the electron is then considered to have a finite extension. The difficulty with the Lorentz’s model was that it had no mechanism to overcome the electrostatic repulsion of the charge, so that the body was unstable.

1.1.6 Poincaré’s Theory of Electrons

Poincaré (1905, 1906), with the aim of overcoming the instability and inconsistency of Abraham’s model with respect to the special relativistic Lorentz transformations proposed that an attractive force of cohesive type and consequently non-electromagnetic in nature can be added so as to just balance the stresses and establish stability. He first considered the coordinate transformations in the proper form
and called these as Lorentz transformations which are

\[ x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - vx/c^2). \]  

(1.16)

Then he proposed a new Lagrangian which was composed of two parts

\[ L = L_{\text{field}} + L_{\text{stress}} = -\frac{4}{3} u_0^{em}(1 - \beta^2)^{1/2} \]  

(1.17)

where \( L_{\text{field}} \to \) the Lagrangian for the electron’s self electromagnetic fields, \( L_{\text{stress}} \to \) the Lagrangian related with the so-called Poincaré stress and \( u_0^{em} \to \) electromagnetic energy of the spherical body.

This new Lagrangian not only solved the instability problem but also removed the discrepancy in the calculation of mass of the electron. In fact, one can show that in a relativistic theory in which the electron self energy is not infinite, the self stress will vanish and the particle will be stable.

1.1.7 Einstein’s Special Relativistic Model of Electrons

Einstein’s special theory of relativity is based on two unique postulates, that is, the principle of relativity and the constancy of velocity of light in vacuum. Using these postulates he could derive the Lorentz transformations and explain the length contraction and time dilation as a kinematical consequence of these transformations. He further showed the covariance of Maxwell-Lorentz electromagnetic field equations under Lorentz transformation and that the Lorentz force equation (1.1) is a consequence of the principle of relativity. The longitudinal and transverse masses of the electron as given by Einstein (1905) are

\[ m_\parallel = \gamma^3 m_0 \]  

(1.18)

and

\[ m_\perp = \gamma m_0 \]  

(1.19)
where $m_0$ is the rest mass of the electron. These results are exactly identical to those of Lorentz. Of course, the dependence of mass on velocity, as given by equation (1.19) does not mean that only the electromagnetic mass of an electron is a privileged mass to vary with velocity; rather it is just like any other kind of mass.

### 1.1.8 Various Other Type of Models Including General Relativistic Electron Models

In 1912 G. Mie (1912 a,b) tried to build charged particle models based on electromagnetic fields alone, so that the mass of the charged particle like electron could be completely of electromagnetic origin, and suggested a modification of the Maxwell-Lorentz field equations. He assumed that the complete electromagnetic field is determined by ten universal quantities which are functions of the four-potentials $A^i$ and the Maxwell tensors $F^{ij}$. But this unitary field theory ultimately failed. The other workers on the theory of unitary field, with different view points, were M. Born and L. Infeld (1934), B. Hoffmann (1935 a,b) and F. Bopp (1940, 1943). The search for a solution to the problem of the electron structure were made by H. Weyl (1918a,b; 1919), T. Kaluza (1921) and O. Klein (1926a,b; 1928) in the realm of unified field theory where gravitational and electromagnetic fields have been unified into a single theory. In 1919, the first general relativistic approach towards an electromagnetic mass was put forward by Einstein. To overcome the drawbacks of Mie’s theory Einstein proposed a model where gravitational forces would provide the necessary stability to the electron and the contribution to the mass would also come from it. The discovery of quantum mechanics in 1925 – 1926 called for an extension of the classical theory of the electron to the atomic and subatomic domain. A very important and unexpected property of electrons was found almost simultaneously with the establishment of this new mechanics. It was discovered that the electron has an intrinsic angular momentum, a spin and a magnetic moment associated with it. Essential progress towards a relativistic quantum mechanical description of electrons was made by Dirac (1928). It was Dirac who was able to
devise a wave equation for the electron which fulfilled the relativistic requirements. Actually quantum mechanics treats electron as a point-like charged particle with spin and hence extended electron could not be accommodated within it. Thus it seems that for a better description of the electron structure with a spin in the General Relativistic framework it is already seen that the Einstein-Cartan-Maxwell or Einstein-Maxwell-Dirac space-times is preferable rather than the Einstein-Maxwell space-times (Ray and Bhadra 2004a). In the various classical point charge theories (Mehra 1973) the electron is treated as a point charge having a pure charge without any structure. But these theories cannot overcome the self energy problem of the electron which becomes infinite at its location. This infinite self energy problem can be solved by considering an extended charge distribution for the electron. But these theories have no satisfactory quantum versions. In the framework of Einstein’s general theory of relativity (which was proposed in 1916) a lot of work has been carried out by different authors on charged body. Some models which were developed to study the structure of electron are due to Kyle and Martin (1967), Cohen and Cohen (1969) and Baylin and Eimerl (1972). Katz and Horwitz (1971) and López (1984) have developed classical extended electron models from the general relativistic point of view. In all these models an electron has been considered to be a microscopic sphere of charged perfect fluid or as a spherical shell of matter embodied by charge.

1.2 A Short Account of the Recent Developments in the Electromagnetic Mass Models

So long we have seen in the theories that in the Special Relativity and even in the General Relativity the mass of the electron has been considered to contain two parts, that is, the non-electromagnetic part and the electromagnetic part. The exceptions are Lorentz’s and Abraham’s theories which have independently considered the mass of electron to be completely of electromagnetic origin. In the following section we shall give a brief description of the above idea showing how the conjectures of
Lorentz-Abraham may be revived on the ground of the General Relativity.

1.2.1 Repulsive Electromagnetic Mass Models: Electron Type

Repulsive gravitation is produced by the negative mass of the polarized vacuum. The vacuum fluid obeying an equation of state $\rho = -p$ was taken by most of the workers for the construction of electromagnetic mass model. By considering the relation between the metric coefficients i.e., $g_{00}g_{11} = -1$ (which for both the Schwarzschild and the Reissner-Nordström matrices equivalent to the relation $\nu + \lambda = 0$) to be valid inside a charged perfect fluid distribution, it is shown by Tiwari et al. (1984) that the mass energy density and the pressure of the distribution are of electromagnetic origin. In the absence of charge, however, there exists no interior solutions. A particular solution which confirms the same and matches smoothly with the exterior Reissner-Nordström was obtained by them. This solution represents a charged particle whose mass is entirely of electromagnetic origin. The pressure being negative here the model is under tension and hence the source is of repulsive nature. In the approach taken by Gautreau (1985), following Tiwari et al. (1984), the electron’s mass is associated with the Schwarzschild gravitational mass given by general relativity and not with the inertial mass used by Lorentz (1904). In this case the Schwarzschild mass of an extended charged body as seen at infinity arises from the charge as well as the matter possessed by it. Here the field equations for a Lorentz type pure charge extended electron are obtained by setting the matter terms equal to zero in the field equations for a spherically symmetric charged perfect fluid. An explicit solution to the pure charge field equations are examined by Gautreau (1985). López (1984) proposed a classical model of the spinning electron in which the particle is the source of the Kerr-Newman field. The electron here is regarded as a charged rotating shell with surface tension. The phenomenon of repulsive vacuum gravitation proved to be of importance in cosmology with appearance of the inflationary universe models. Grøn (1985) pointed out the possibility that repulsive gravitation may be of importance also in connection with elementary particle models. This possibility was realized by Tiwari et al. (1984) and by López (1984). Poincaré stresses
were explained by them as being due to vacuum polarization in connection with a recently presented class of electromagnetic mass models in general relativity. The gravitational blue shift of light is explained as being due to repulsive gravitation produced by the negative gravitational mass of the polarized vacuum. Grøn (1985) pointed out that the electron model of López (1984), which includes spin, and which is a source of the Kerr-Newman field gives rise to repulsive gravitation.

Assuming an implicit relation among the unknown physical parameters viz., the pressure $p$, the charge density $\sigma$ and the electromagnetic potential $\phi$, it has been shown by Tiwari et al. (1986) that $\phi$ satisfies the well known Lane-Emden equation. Electromagnetic mass models corresponding to the exact solutions of the Lane-Emden equation were obtained by them. The radii of some of the models were compared by them with the “classical electron radius”. López (1986) analyzed the stability of a classical ellipsoidal electron model. The model was found to be stable under oscillations which change the size of the ellipsoid without altering its shape. It is further shown by López (1986) that angular momentum conservation does not allow the existence of other oscillation models. Ponce de Leon (1987a) investigated the relation $g_{00}g_{11} = -1$ in the case in which the interior is filled with imperfect fluid. He found that the core of such a distribution is gravitationally repulsive provided the energy density is positive. Ponce de Leon (1988) also investigated the different aspect of the phenomenon of gravitational repulsion in static sources of the Reissner-Nordström field. He found that in the case of perfect fluid spheres there exists a close relation between the gravitational repulsion and the Weyl curvature tensor. Ponce de Leon (1988) proved that the static source of the Reissner-Nordström field gives rise to gravitational repulsion only if the pure gravitational field energy inside the sphere is negative. It is also proved that although the gravitational repulsion always takes place in the interior of a charged perfect fluid sphere when its radius is less than the classical electron radius, this is not necessarily so either in the case of anisotropic charged spheres or if the net charge of the body is concentrated at its boundary only. He further found that the charge contributes negatively to the effective gravitational mass, in the sense that an increase in the charge causes a decrease
in gravitational mass. He explained the gravitational repulsion as being due to this negative contribution rather than the strain of vacuum because of vacuum polarization. Bonnor and Cooperstock (1989) found, by modelling the electron as a charged sphere obeying Einstein-Maxwell theory, that it must contain some negative rest mass. The total gravitational mass within this sphere is negative which is one of the assumptions made in singularity theorems of general relativity. López (1992) constructed a classical model of the spinning electron in general relativity consisting of a rotating charge distribution with Poincaré stresses. Obviously he obtained a class of interior solutions of the Kerr-Newman field. The negative pressures or tensions obtained here are identified with the cohesive forces introduced by Poincaré (1905, 1906) to stabilize the Lorentz electron model. They are shown by López (1992) to be the source of a negative gravitational mass density and thereby of the violation of the energy conditions inside the electrons. Herrera and Varela (1994) pointed out the role played by the negative rest mass as mentioned in the work of Bonnor and Cooperstock (1989). Here the electron is modeled as a spherically symmetric charged distribution of matter deprived of spin and magnetic moment. Since the electrostatic energy of a point charge is infinite, the only way to produce a finite total mass is the presence of an infinite amount of negative energy at the center of symmetry. They (Herrera and Varela 1994), by analyzing some extended electron models, showed that negative energy distributions result from the requirement that the total mass of these models remains constant in the limit of a point particle. Tiwari and Ray (1996) dealt with a model which is the charged generalization of static dust sphere in Einstein-Cartan theory. They obtained a set of solutions with torsion and spin which represents an electromagnetic mass model. Blinder (2001) proposed a model for the classical electron as a point charge with finite electromagnetic self energy. Modified form of the Reissner-Nordström and Kerr-Newman solutions of the electromagnetic equations were derived. Moreover, the self interaction of a charged particles with its own electromagnetic field was shown to be equivalent to its reaction to the vacuum polarization.
1.2.2 Repulsive Electromagnetic Mass Models: Stellar Type

The work of Ray and Das (2002) which is concerned with the charged analogue of Bayin’s work (1978) related to Tolman’s type and presents astrophysically interesting aspects of stellar structure. However, in a static spherically symmetric Einstein-Maxwell space-time this class of astrophysical solution found out by Pant and Sah (1979) and Ray and Das (2002) has been revisited in connection with the phenomenological relationship between the gravitational and electromagnetic fields (Ray and Das 2004). Considering Riccati equation with known value of charge $q$ for the total charge on the sphere in the following form

$$q(a) = Ka^n$$  \hspace{1cm} (1.20)

they have shown in one of the cases that the gravitational mass for $n = 1$ can be given as

$$m = q^2 + a_0 a_1 \left( \frac{q}{K} \right)^2 + a_1^2 \left( \frac{q}{K} \right)^3.$$  \hspace{1cm} (1.21)

It is thus qualitatively shown that the charged relativistic stars of Tolman (1939) and Bayin (1978) type are of purely electromagnetic origin. Obviously, the existence of this type of astrophysical solutions is a probable support to the extension of Lorentz’s conjecture that electron-like extended charged particle possesses only ‘electromagnetic mass’ and no ‘material mass’.

In this connection some known static charged fluid spheres of Tolman-VI type solutions have been reexamined and the gravitational masses are shown to be of electromagnetic origin by Ray and Das (2007a). They have considered a more general form of the gravitational mass as follows:

$$m = \frac{n(2-n)a^2 + 2q^2}{2(1 + 2n - n^2)a}$$  \hspace{1cm} (1.22)

where for physical viability the values on $n$ to be assigned are $0 \leq n \leq 2$.

For the specific choice $K = 1/\sqrt{2}$ and $n = 1$ of these parameters, the ansatz expressed in equation (1.20) reduces to $q(a)/a = 1/\sqrt{2}$, where $a$ is the radius of the sphere. It is interesting to note that for this charge-radius ratio all the perfect
fluid equations of state reduce to the form $\rho + p = 0$ which is known as the ‘pure charge condition’ (Gautreau 1985) and also imperfect-fluid equation of state in the literature for the matter distribution under consideration is in tension and hence the matter is named as a ‘false vacuum’ or ‘degenerate vacuum’ or ‘$\rho$-vacuum’ (Davies 1984; Blome and Priester 1984; Hogan 1984; Kaiser and Stebbins 1984).

Ray and Das (2007b) have again considered the Einstein-Maxwell space-time in connection with some of the astrophysical solutions previously obtained by Tolman (1939) and Bayin (1978). The effect of charge inclusion in these solutions has been investigated thoroughly and the nature of fluid pressure and mass density throughout the sphere have also been discussed. Mass-radius and mass-charge relations have been found out for various cases of the charged matter distribution. Two cases are obtained where perfect fluid with positive pressures gives rise to electromagnetic mass models such that gravitational mass is of purely electromagnetic origin. The stability conditions have been investigated for all these Tolman-Bayin type static charged perfect fluid solutions in connection with the stellar configurations.

### 1.2.3 Lorentz’s Electromagnetic Mass: a Clue for Unification?

Ray (2007) in his review of the electromagnetic mass model by Lorentz has described the philosophical perspectives and given a historical account of this idea, especially, in the light of Einstein’s Special Relativistic formula $E = mc^2$. It is known that, at distances below $10^{-32}$ m, the strong, weak and electromagnetic interactions are “different facets of one universal interaction” (Georgy, Quinn and Weinberg 1974, Wilczek 1998). This is already confirmed by (i) the theories of the unification of electricity and magnetism by Maxwell, (ii) that of earth’s gravity and universal gravitation by Newton and (iii) “...the unified weak and electromagnetic interaction between elementary particles...” by S. Glashow, A. Salam and S. Weinberg for which Nobel Prize was awarded to them in 1979 . Therefore, as regards unification scheme, Ray (2004) has argued that though there has been much progress towards a unification of all the other forces – strong, electromagnetic and weak – in the
Grand Unified Theory (GUT), gravity has not been included in the scheme. In this context it has also been mentioned that there are some problems with gravity: (i) the strength of the gravitational interaction is enormously weaker than any other force (the hierarchy problem) and (ii) General Theory of Relativity does not consider gravity a force, rather a kind of field for which a body rolls down along the space-time curvature (the field theoretical problem). As a probable alternative solution to this problem Ray (2004) has put forward Lorentz’s conjecture of ‘electromagnetic mass’ and suggested that this may be a competent candidate of the long desired unification.

1.3 Motivation and Discussion of Our Investigation

Our motivation to work on relativistic electromagnetic mass models is based on a slightly different approach. We have studied the role of the cosmological constant in constructing the electromagnetic mass model. This is the central part of our work. It has been observed that Tiwari et al. (1984) and other authors constructed electromagnetic mass models without considering any $\Lambda$ term. So, we have considered here $\Lambda$ in the Einstein field equations by assuming it to be a scalar rather than a constant and hence re-examined the work of Ray and Ray (1993) and Tiwari and Ray (1996). We obtained a class of exact solutions for the Einstein-Maxwell field equations by assuming the cosmological constant to be a space variable scalar, i.e., $\Lambda = \Lambda(r)$. The source considered in the chapter II is static, spherically symmetric and anisotropic charged fluid type. The solutions obtained are matched continuously to the exterior Reissner-Nordström solution and each of the four solutions represents an electromagnetic mass model.

In chapter III we have examined whether even without employing the vacuum fluid equation of state $\rho + p = 0$ a stable model with electromagnetic mass can be constructed. Here we have considered a charged anisotropic static spherically symmetric fluid source of finite radius. The field equations thus obtained under certain mathe-
matical assumptions yield a set of solutions which are shown to be electromagnetic in origin. Electromagnetic mass models have been studied by several authors under the special assumption $\rho + p = 0$. Here we have shown that even for $\rho + p \neq 0$ electromagnetic mass model can be constructed. This is one of the motivations of the present investigation. However, the same question whether there exists any electromagnetic mass models where this condition $\rho + p \neq 0$ is violated was addressed by Tiwari et al. (1991) and obtained electromagnetic mass model in the isotropic and axially symmetric matter distribution or charged dust case only, whereas in chapter IV we have searched a solution by employing a relation between the radial and tangential pressures as $p_\perp = p_r + \alpha q^2 r^2$. Our aim is to see if there is any effect of space dependent $\Lambda$ on the energy density of classical electron. Here we have considered an extended static spherically symmetric distribution of an elementary particle like electron having the radius of the order of $10^{-16}$ cm. It is already suggested that there might be some negative energy density regions within the particle in the general theory of relativity (Bonnor and Cooperstock 1989, Herrera and Varela 1994). It is, therefore, argued in the present investigation that such a negative energy density also can be obtained with a better physical interpretation in the framework of Einstein-Cartan theory.

In many theories higher dimensions play an important role, specially in superstring theory which demands more than usual four dimensional space time. This is also true in studying the models regarding unification of gravitational force with other fundamental forces in nature. So long electromagnetic mass model has been studied extensively in the four dimensional space-times of the General Relativity. Here, in chapter VI, we have presented electromagnetic mass model in the space-time of higher dimensional theory of general relativity. Under this motivation we have considered here a static spherically symmetric charged dust distribution corresponding to higher dimensional theory of general relativity.

In chapter VI we have studied static spherically symmetric anisotropic source for the Einstein-Maxwell space-times assuming the erstwhile cosmological constant $\Lambda$ dependent on the spatial coordinate, viz., $\Lambda = \Lambda(r)$. It is shown that the solutions
thus obtained are of electromagnetic in origin in the sense that all the physical parameters including the gravitational mass originate from the electromagnetic field alone. It is also shown that the generally used pure charge condition, viz., $\rho + p_r = 0$ is not always required for constructing electromagnetic mass models.

The concluding chapter VII offers a general discussion on the whole work along with the future scope of the field of electromagnetic mass models in General Relativity.
Chapter 2

Relativistic Electromagnetic Mass Models with Cosmological Variable $\Lambda$ in Spherically Symmetric Anisotropic Source

“Whether one or the other of these methods will lead to the anticipated “world law” must be left to future research.”

– Max Born (1962)

2.1 Introduction

A very important problem in cosmology is that of the cosmological constant the present value of which is infinitesimally small ($\Lambda \leq 10^{-56} \text{cm}^{-2}$). However, it is believed that the smallness of the value of $\Lambda$ at the present epoch is because of the Universe being so very old (Beesham 1993). This suggests that the $\Lambda$ can not
be a constant. It will rather be a variable, dependent on coordinates – either on space or on time or on both (Sakharov 1968; Gunn and Tinslay 1975; Lau 1985; Bertolami 1986a,b; Özer and Taha 1986; Reuter and Wetterich 1987; Freese et al. 1987; Peebles and Ratra 1988; Wampler and Burke 1988; Ratra and Peebles 1988; Weinberg 1989; Berman et al. 1989; Chen and Wu 1990; Berman and Som 1990; Abdel-Rahman 1990; Berman 1990a,b; Berman 1991a,b; Sistero 1991; Kalligas et al. 1992; Carvalho et al. 1992; Ng 1992; Beesham 1993 and Tiwari and Ray 1996).

Now, once we assume $\Lambda$ to be a scalar variable, it acquires altogether a different status in Einstein’s field equations and its influence need not be limited only to cosmology. The solutions of Einstein’s field equations with variable $\Lambda$ will have a wider range and the roll of scalar $\Lambda$ in astrophysical problems will be of as much significance as in cosmology.

It is this aspect that motivated us to reexamine the work of Ray and Ray (1993) and Tiwari and Ray (1996) with the generalization of anisotropic and charged source respectively. One can realize from the present investigations how the variable $\Lambda$ generates different types of solutions which are physically interesting as they provide a special class of solutions known as electromagnetic mass models (EMMMM).

In section 2.2, the Einstein-Maxwell field equations with variable $\Lambda$ are derived. Solutions corresponding to different cases for anisotropic system are obtained in section 2.3. All the solutions obtained are matched with the exterior Reissner-Nordström (RN) solution on the boundary of the charged sphere. Finally, some concluding remarks are made in section 2.4.

### 2.2 Field Equations

The Einstein-Maxwell field equations for the spherically symmetric metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(2.1)

corresponding to charged anisotropic fluid distribution are given by

$$e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi\rho + E^2 + \Lambda,$$

(2.2)
\[ e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 = 8\pi p_r - E^2 - \Lambda, \]  
(2.3)

\[ e^{-\lambda}[\nu''/2 + \nu'^2/4 - \nu'\lambda'/4 + (\nu' - \lambda')/2r] = 8\pi p_\perp + E^2 - \Lambda \]  
(2.4)

and

\[ (r^2E)' = 4\pi r^2\sigma e^{\lambda/2}. \]  
(2.5)

The equation (2.5) can equivalently be expressed in the form,

\[ E(r) = \frac{1}{r^2} \int_0^r 4\pi r^2\sigma e^{\lambda/2}dr = \frac{q(r)}{r^2}. \]  
(2.6)

where \( q(r) \) is total charge of the sphere under consideration.

Also, the conservation equation is given by

\[ \frac{d}{dr}(p_r - \Lambda/8\pi) + \rho + (p_r)\nu'/2 = \frac{1}{8\pi r^4} \frac{d}{dr}(q^2) + 2(p_\perp - p_r)/r. \]  
(2.7)

Here, \( \rho, \ p_r, \ p_\perp, \ E, \ \sigma \) and \( q \) are respectively the matter-energy density, radial and tangential pressures, electric field strength, electric charge density and electric charge. The prime denotes derivative with respect to radial coordinate \( r \) only.

Equations (2.2) and (2.3) yield

\[ e^{-\lambda}(\nu' + \lambda') = 8\pi r(\rho + p_r). \]  
(2.8)

Again, equation (2.2) can be expressed in the general form as

\[ e^{-\lambda} = 1 - 2M(r)/r, \]  
(2.9)

where

\[ M(r) = 4\pi \int_0^r [\rho + (E^2 + \Lambda)/8\pi]r^2dr \]  
(2.10)

is the active gravitational mass of a charged spherical body which is dependent on the cosmological parameter \( \Lambda = \Lambda(r) \).
2.3 Solutions

A number of solutions can be obtained depending on different suitable conditions on equation (2.7). Here we assume the relation $g_{00}g_{11} = -1$, between the metric potentials of metric (2.1), which, by virtue of equation (2.8), is equivalent to the equation of state \(^\dagger\)

$$\rho + p_r = 0. \tag{2.11}$$

The equations (2.2) – (2.5) being underdetermined, we further assume the following conditions

$$\sigma e^{\lambda/2} = \sigma_0 \tag{2.12}$$

and

$$p_\perp = np_r, \quad (n \neq 1), \tag{2.13}$$

where $\sigma_0$ is a constant (which from (2.6) can be interpreted as the volume density of the charge $\sigma$ being constant) and $n$ is the measure of anisotropy of the fluid system. Equation (2.6), with equation (2.12), provides the electric field and charge as

$$E = q/r^2 = 4\pi \sigma_0 r/3. \tag{2.14}$$

Using equations (2.11), (2.13) and (2.14), in equation (2.7), we get

$$\frac{d}{dr}(p_r - \Lambda/8\pi) - 2(n - 1)p_r/r = 2Ar, \quad A = 2\pi \sigma_0^2/3, \tag{2.15}$$

which is a linear differential equation of first order.

Since the equation (2.15) involves two dependent variables, $p_r$ and $\Lambda$, to solve this equation, we consider the following four simple cases.

\(^\dagger\)In terms of energy-momentum tensor this can be expressed as $T^0_0 = T^1_1$. 

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2.3.1 $\Lambda = \Lambda_0 - 8\pi p_r, \quad (\Lambda_0 \text{ = constant})$

The solutions in this case are obtained as

$$e^\nu = e^{-\lambda} = 1 - 2M(r)/r, \quad (2.16)$$

$$\rho = -p_r = -p_{\perp}/n = (\Lambda - \Lambda_0)/8\pi = A a^{-(n-3)}r^2[a^{n-3} - r^{n-3}]/(n - 3) \quad (2.17)$$

and

$$M(r) = \frac{4\pi A a^{-(n-3)}r^5}{15(n - 3)(n + 2)}[(n + 2)(n + 3)a^{n-3} - 30r^{n-3}] + \Lambda_0 r^3/6, \quad (2.18)$$

where $a$ is the radius of the sphere.

Some general features of these solutions are as follows:

1. As we want, customarily, $\rho > 0$ (and hence $p_r < 0$), we must have, from (2.17), $n > 3$. However, we can choose $n < 3$ (and certainly $n \neq 1$). In that case also $\rho$ becomes positive. This result, viz., the positivity of matter-energy density is obvious as the electron radius for the present model is $10^{-13}$ cm, which is much larger than the experimental upper limit $10^{-16}$ cm (Quigg 1983). Within this limit the charge distribution of matter must contain some negative rest mass (Bonnor and Cooperstock 1989; Herrera and Varela 1994). This is the reason why we cannot consider $\rho \leq 0$ and hence $p_r \geq 0$ in the present model.

2. Similarly, we can observe that the effective gravitational mass (which we get after matching of the interior solution to the exterior RN solution on the boundary),

$$m = M(a) + q^2(a)/2a - \Lambda_0 a^3/6 = 8\pi A(n + 3)a^5/5(n + 2), \quad (2.19)$$

is positive for both the choices, $n > 0$ and $n < 0$. In this respect, the Tolman-Whittaker mass,

$$m_{TW} = \int_V (T_0^0 - T_\alpha^\alpha)\sqrt{-g}dV, (\alpha = 1, 2, 3 \quad \text{and} \quad g \to 4D)$$

$$= -\frac{8\pi A a^{-(n-3)}r^5}{15(n - 3)(n + 2)}[2(n + 2)(n + 3)a^{n-3} - 15(n + 1)r^{n-3}] - \Lambda_0 r^3/3, \quad (2.20)$$
can also be examined. In general, this is negative and also equal to modified Tolman-Whittaker mass (Devitt and Florides 1989),

\[ m_{DF} = e^{-(\nu + \lambda)/2}m_{TW}, \]  

(2.21)
as \( \nu + \lambda = 0 \), by virtue of the condition \( g_{00}g_{11} = -1 \) in the present paper.

(3) Pressure being negative the model is under tension. This repulsive nature of pressure is associated with the assumption (2.11), where matter-energy density is positive. This negativity of the pressure corresponds to a repulsive gravitational force (Ipser and Sikivie 1983; López 1988).

(4) The cosmological parameter \( \Lambda \), which is assumed to vary spatially, can be shown to represent a parabola having the equation of the form \( \Lambda = 8\pi A[(a/2)^2 - (r - a/2)^2] + \Lambda_0 \) for a particular case \( n = 2 \). The value of \( \Lambda \) increases from 0 to \( a/2 \) and then decreases from \( a/2 \) to \( a \) and hence it is maximum at \( a/2 \). The vertex of the parabola is at \( r = a/2 \) whereas the values of \( \Lambda \) at \( r = 0 \) and at \( r = a \) are \( \Lambda_0 \), the erstwhile cosmological constant. The same result can also be obtained from equation (2.17) as at the boundary of the sphere \( r = a \), \( p_r = p_\perp = 0 \) (and hence \( \Lambda = \Lambda_0 \)).

(5) The solution set provides electromagnetic mass model (EMMM) (Feynman et al. 1964; Tiwari et al. 1984, 1986, 1991; Gautreau 1985; Gron 1985, 1986a, 1986b; Ponce de Leon 1987a, 1987b, 1988; Tiwari and Ray 1991a, 1991b, 1997; Ray et al. 1993; Ray and Ray 1993). This means that the mass of the charged particle such as an electron originates from the electromagnetic field alone (for a brief historical background, see Tiwari et al. 1986).

(6) The present model corresponds to Ray and Ray (1993) for \( n = 1 \), under the assumption \( p_r = -\Lambda/8\pi \). It can be observed that the other simple possibility, \( p_r = \Lambda/8\pi \), does not exist for this case (equation (23) of Ray and Ray (1993)).
$2.3.2 \quad \Lambda = \Lambda_0 + 8\pi p_r$

In this case we have the following set of solutions:

\[ e^\nu = e^{-\lambda} = 1 - 2M(r)/r, \quad (2.22) \]

\[ \rho = -p_r = -p_\perp/n = -\frac{(\Lambda - \Lambda_0)}{8\pi} = Ar^2/(n - 1) \quad (2.23) \]

and

\[ M(r) = 4\pi Ar^5/15 + \Lambda_0 r^3/6. \quad (2.24) \]

Here some simple observations are as follows:

(1) In this case also the electron radius being $\sim 10^{-13}$ cm the matter-energy density should be positive (Bonnor and Cooperstock 1989; Herrera and Varela 1994). This positivity condition requires that $n$ must be greater than unity.

(2) The effective gravitational mass,

\[ m = 8\pi Aa^5/5, \quad (2.25) \]

is always positive whereas the Tolman-Whittaker mass which is also equal to the modified Tolman-Whittaker mass, i.e.,

\[ m_{TW} = m_{DF} = -16\pi Ar^5/15 - \Lambda_0 r^3/3, \quad (2.26) \]

is always negative in the region $0 < r \leq a$. The gravitational mass in this case is independent of anisotropic factor $n$.

(3) The pressures $p_r$ and $p_\perp$ are repulsive for $n > 1$ (as in the previous case).

(4) The equation (2.23) for $n = 2$ can be written in the form $\Lambda = -8\pi Ar^2 + \Lambda_0$. This yields a half-parabola whose vertex is at $r = 0$ and the parabola lies in the fourth-quadrant of the coordinate systems $(r, \Lambda)$.

(5) The effective gravitational mass as obtained in (2.25) is of electromagnetic origin.

(6) The matter-energy density $\rho$ as well as the pressures $p_r$ and $p_\perp$ are all zero at the centre of the spherical distribution and increase radially being maximum at the
boundary. This situation is somewhat unphysical though not at all unavailable in
the literature (Som and Bedran 1981).

2.3.3 \[ \Lambda = \Lambda_0 - 8\pi \int \frac{\rho r}{r} dr \]

The solution set for this case is given by

\[ e^\nu = e^{-\lambda} = 1 - 2M(r)/r, \quad (2.27) \]

\[ \rho = -p_r = -p_\perp/n = 2Aa^{-((2n-5)r^2)/(2n-5)}[a^{2n-5} - r^{2n-5}], \quad (2.28) \]

\[ \Lambda = \frac{8\pi Aa^{-((2n-5)r^2)}}{(2n-3)(2n-5)}[(2n-3)a^{2n-5} - 2r^{2n-5}] - 8\pi Aa^2/(2n-3) + \Lambda_0 \quad (2.29) \]

and

\[ M(r) = \frac{8\pi Aa^{-((2n-5)r^5)}}{15n(2n-3)(2n-5)}[n(n+2)(2n-3)a^{2n-5} - 15(n-1)r^{2n-5}] \]

\[ -4\pi Aa^2r^3/[n(2n-3)] + \Lambda_0 r^3/3. \quad (2.30) \]

Some general features of the above set of solution are as follows:
(1) The matter-energy density is positive and pressures are negative for \( n > 5/2 \).
(2) The effective gravitational mass,

\[ m = 8\pi A(3n + 1)a^5/15n, \quad (2.31) \]

is positive for \( n > 1 \). On the other hand, the Tolman-Whittaker mass and the
modified Tolman-Whittaker mass, being equal, are given by

\[ m_{TW} = m_{DF} = -\frac{32\pi Aa^{-((2n-5)r^5)}}{15n(2n-3)(2n-5)}[n(n+2)(2n-3)a^{2n-5} \]

\[ -15(n-1)(2n-1)r^{2n-5} + 8\pi Aa^2r^3/[3(2n-3)] - \Lambda_0 r^3/3. \quad (2.32) \]

Depending on the different values of \( n \) these masses may be negative or positive.

(3) The matter-energy density and the pressures, as usual, are zero at the centre
\( r = 0 \) as well as at the boundary \( r = a \). Thus the maximum value must be in the
region \(0 < r < a\). This can be confirmed from equation (2.28) which, for the value \(n = 2\), represents a parabola of the form \(\Lambda = 2A[(a/2)^2 - (r - a/2)^2]\), the vertex being at \(r = a/2\).

(4) The value of \(\Lambda\) at the centre \(r = 0\) is \([\Lambda_0 - 8\pi Aa^2/(2n - 3)]\). It acquires maximum value \(\Lambda_0\) at the boundary \(r = a\).

(5) The solution set represents EMMM.

### 2.3.4 \(\Lambda = \Lambda_0 + \int \frac{p_r}{r} dr\)

The solutions in this case are given by

\[ e^\nu = e^{-\lambda} = 1 - 2M(r)/r, \quad (2.33) \]

\[ \rho = -p_r = -p_\perp/n = \frac{2Aa^{-(2n-3)}r^2}{(2n-3)}[a^{2n-3} - r^{2n-3}], \quad (2.34) \]

\[ \Lambda = -\frac{8\pi Aa^{-(2n-3)}r^2}{(2n-1)(2n-3)}[(2n-1)a^{2n-3} - 2r^{2n-3}] + 8\pi Aa^2/(2n-1) + \Lambda_0 \quad (2.35) \]

and

\[ M = \frac{8\pi Aa^{-(2n-3)}r^5}{15(n+1)(2n-1)(2n-3)}[n(n+1)(2n-1)a^{2n-3} - 15(n-1)r^{2n-3}] \]

\[ + \frac{4\pi Aa^2r^3}{3(2n-1)} + \frac{\Lambda_0 r^3}{6}. \quad (2.36) \]

Here, the observations are as follows:

(1) The matter-energy density is positive and pressures are negative for \(n > 3/2\).

(2) The effective gravitational mass,

\[ m = 8\pi A(3n+5)a^5/15(n+1), \quad (2.37) \]

for the condition \(n > 1\) is always positive, whereas the Tolman-Whittaker mass,

\[ m_{TW} = m_{DF} = -\frac{8\pi Aa^{-(2n-3)}r^5}{15(n+1)(2n-1)(2n-3)}[4n(n+1)(2n-1)a^{2n-3} \]

\[ -15(n-1)(2n+1)r^{2n-3}] - 8\pi Aa^2r^3/[3(2n-1)] - \Lambda_0 r^3/3 \quad (2.38) \]
may have positive or negative value depending on the choice of \( n \).

(3) The values related to \( \rho \) and \( p \) are zero both at \( r = 0 \) and \( r = a \).

(4) The effective gravitational mass as well as the other physical variables, including \( \Lambda \), are of purely electromagnetic origin.

### 2.4 Conclusions

(1) As mentioned in the introduction, the present work considers \( \Lambda \), the erstwhile cosmological constant, to be a variable dependent on space coordinates. The contribution of this variable \( \Lambda \) can be seen in the calculations given in the previous sections. It can be seen that \( \Lambda \) is related to pressure and matter-energy density, and therefore contributes to effective gravitational mass of the astrophysical system.

(2) The present EMMMs have been obtained under the condition \( \rho + p_r = 0 \) (equation (2.11)). This problem thus requires further investigation to see whether such models can be obtained even for the condition \( \rho + p_r \neq 0 \).
Chapter 3

Classical Electron Model with Negative Energy Density in Einstein-Cartan Theory of Gravitation

“If you can look into the seeds of time, and say which grain will grow and which will not...”
– Shakespear (Macbeth)

3.1 Introduction

Recently, Cooperstock and Rosen (1989), Bonnor and Cooperstock (1989), and Herrera and Varela (1994) have shown that within the experimentally obtained upper limit of the size of the electron (\( \sim 10^{-16} \text{ cm} \)) (Quigg 1983), when it is modeled as a charged sphere obeying Einstein-Maxwell theory, must contain some negative
gravitational mass density regions within the particle. According to Cooperstock, Rosen and Bonnor (CRB) (1989), the rest mass or active gravitational mass within this sphere, by virtue of the relation

\[ M = m - \frac{q^2}{2a}, \]  

is negative and about \( 10^{-52} \) cm (when the inertial mass or effective gravitational mass, charge and radius, respectively, of the electron, are \( m = 6.76 \times 10^{-56} \) cm, \( q = 1.38 \times 10^{-34} \) cm, and \( a = 10^{-16} \) cm in relativistic units). Further, Herrera and Varela (HV) (1994) have shown, in one of the cases of their paper, that the matter-energy density

\[ \rho = (\alpha q^2 + \frac{2}{3}\pi \sigma_0^2)(a^2 - r^2), \]  

for the constant \( \alpha = -4.77 \times 10^9 \) cm\(^{-6} \) (when radius \( a \sim 10^{-16} \) cm) is also negative, \( \sigma_0 \) being the constant charge density at the centre of the spherical distribution. These models, however, lack spin and magnetic moment and hence do not possess the actual physical characteristics required for an electron.

As an alternative way both the groups suggest the stationary Kerr-Newman (KN) metric (Newman et al. 1965) related to the solution of Einstein-Maxwell equations to be more appropriate than those described earlier. However, in this context it is also to be mentioned here that the KN metric cannot be valid for distance scales of the radius of a subatomic particle (Mann and Morris 1993; Herrera and Varela 1994).

We, therefore, feel that the problem can be tackled in the framework of Einstein-Cartan (EC) theory, where torsion and spin are inherently present in the formulation of the theory itself.

### 3.2 An Overview: The Negative Density Models

Before going into the Einstein-Cartan theory let us have a bird’s-eye view of the negative matter-energy density models which we have mentioned in the introduction.
3.2.1 The Cooperstock-Rosen-Bonnor (CRB) Model

Cooperstock and Rosen (1989) and Bonnor and Cooperstock (1989) in their papers have shown that any spherically symmetric distribution of charged fluid, irrespective of its equation of state, whose total mass, radius and charge correspond to the observed values of the electron, must have a negative energy distribution (at least for some values of the radial coordinate). Considering a static spherically symmetric charge distribution with the line-element

\[ ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \]  (3.3)

they have argued that when the Einstein-Maxwell equation

\[ R^0_0 - \frac{1}{2} \delta^0_0 R = 8\pi (T^0_0 (m) + T^0_0 (em)) \]  (3.4)

is written in the form

\[ e^{-2\lambda} = e^{2\nu} = 1 - \frac{1}{r} \int_0^r (8\pi \rho + e^{-(\nu+\lambda)} E^2) r^2 dr \]  (3.5)

and hence is equated with the Reissner-Nordström exterior metric on the boundary \( r = a \), which as usual gives

\[ 1 - \frac{2m}{a} + \frac{q^2}{a^2} = 1 - \frac{1}{a} \int_0^a (8\pi \rho + e^{-(\nu+\lambda)} E^2) r^2 dr. \]  (3.6)

Then for the previous specifications of mass, charge and radius of the electron it can be shown that

\[ \frac{q^2}{a^2} - \frac{2m}{a} \sim 2 \times 10^{-36} > 0. \]  (3.7)

So, the left hand side of the above equation (3.6) must be greater than unity and hence on the right hand side \( \rho < 0 \) for some values of \( r \) implying that the electron must contain some negative rest mass density though the net mass is as usual a positive quantity.
3.2.2 The Herrera-Varela (HV) Model

Following the CRB model (1989) Herrera and V arela (1994) have discussed the fact that the electron, when modeled as a relativistic spherically symmetric charged distribution of matter, must contain some negative rest mass if its radius is not larger than $\sim 10^{-16}$ cm. In this regard they have analyzed some extended electron models and have shown that negative energy density distributions result from the requirement that the total mass of these models remains constant in the limit of a point particle. Among all these extended electron models the model of Tiwari et al. (1984) demands special attention to us which will be seen very much relevant to our present work. Herrera and V arela (1994) generalize this model of Tiwari et al. (1984) by introducing a condition of anisotropy in the form

$$ p_\perp - p_r = \alpha q^2 r^2 \tag{3.8} $$

where $\alpha$ is a constant.

Thus the solution obtained by Herrera and V arela (1994) is as follows:

$$ e^{-2\lambda} = e^{2\nu} = 1 - \frac{16}{45} \pi^2 \sigma_0^2 r^2 (5a^2 - 2r^2) - \frac{8}{15} \pi \alpha q^2 r^2 (5a^2 - 3r^2), \tag{3.9} $$

$$ p_r = -(\alpha q^2 + \frac{2}{3} \pi \sigma_0^2)(a^2 - r^2), \tag{3.10} $$

$$ p_\perp = \alpha q^2 r^2 - (\alpha q^2 + \frac{2}{3} \pi \sigma_0^2)(a^2 - r^2), \tag{3.11} $$

$$ m = \frac{64}{45} \pi^2 \sigma_0^2 a^5 + \frac{8}{15} \pi \alpha q^2 a^5, \tag{3.12} $$

$$ \rho = (\alpha q^2 + \frac{2}{3} \pi \sigma_0^2)(a^2 - r^2) \tag{3.13} $$

and

$$ q = \frac{4}{3} \pi \sigma_0 a^3 \tag{3.14} $$

The value of $\alpha$ can be obtained from the equation (3.12) as $\alpha = -4.77 \times 10^{95} \text{ cm}^{-6}$ and hence the energy density, as given by the equation (3.13) is negative for the radius of the electron $a = 10^{-16}$ cm.

Now, from the equation (3.12) it can be seen that the effective gravitational mass,
$m$, is of purely electromagnetic origin and corresponds to the TRK model (1984) with $\alpha = 0$ case. This type of models where mass, including all the other physical parameters, originate from the electromagnetic field alone are known as the electromagnetic mass models [EMMM] in the literature [Feynman et al. 1964] and have been investigated by several authors (Florides 1962, 1983; Cooperstock and de la Cruz 1978; Tiwari et al. 1984, 1986, 1991, 2000; Gautreau 1985; Grøn 1985, 1986a,b; Ponce de Leon 1987a,b, 1988; Tiwari and Ray 1991a,b, 1997; Ray et al. 1993; Ray and Ray 1993). In the present paper we shall construct such a model within the framework of Einstein-Cartan theory with negative matter-energy density for some values of the radial coordinate.

3.3 The Field Equations of Einstein-Cartan Theory

The EC field equations are given by

$$R^i_j - \frac{1}{2} \delta^i_j R = -\kappa t^i_j \quad (3.15)$$

and

$$Q^i_{jk} - \delta^i_j Q^l_{lk} - \delta^i_k Q^l_{jl} = -\kappa S^i_{jk}, \quad (3.16)$$

where $t^i_j$ is the canonical energy-momentum tensor (asymmetric), $Q^i_{jk}$ is the torsion tensor and $S^i_{jk}$ is the spin tensor (with $\kappa = -8\pi$, $G$ and $c$ being unity in relativistic units).

The asymmetric energy-momentum tensor here is given by

$$t^i_j = T^i_j + \frac{1}{2} g^{ik} \nabla_m (S^m_{jk}), \quad (3.17)$$

$\nabla_m$ being covariant derivative with respect to the torsionless, symmetric Levi-Civita connection $\Gamma^i_{jk}$ and the symmetric energy-momentum tensor $T^i_j$ will consist of two
parts, viz., matter and electromagnetic tensors and which, respectively, are

\[ T_{ij}^{(m)} = (\rho + p)u^iu_j - pg^i_j \]  
(3.18)

and

\[ T_{ij}^{(em)} = \frac{1}{4\pi}(-F_{jk}F_{ik} + \frac{1}{4}\delta^i_j F_{kl}F^{kl}), \]  
(3.19)

where \( \rho \) is the matter-energy density, \( p \) is the fluid pressure, \( u^i \) is the velocity four-vector (with \( u^iu_i = 1 \)) and \( F_{ij} \) is the electromagnetic field tensor.

The conservation equations for the EC theory can be given through the Bianchi identities as

\[ \nabla_k[ (\rho + p)u^k - g^{ki}u^l \nabla_m(u^m S_{li})] = u^j \nabla_j p \]  
(3.20)

and

\[ [(\rho + p)u^k - g^{ki}u^l \nabla_m(u^m S_{li})] \nabla_k u_j = -\nabla_l (u^l u_j) + u^k S_{jm} R_{mk}^l - \frac{1}{2} u^k S_{tm} R_{lm}^k \]  
(3.21)

Now, electromagnetic fields not being coupled with torsion (Novello 1976; Raychaudhuri 1979) the Maxwell equations as usual take the form

\[ \nabla_j F^{ij} = J^i \]  
(3.22)

and

\[ (J^i \sqrt{-g})_{,i} = 0. \]  
(3.23)

The electromagnetic field tensor, \( F_{ij} \), in the above equation (3.22) is related to the electromagnetic potentials as \( F_{ij} = A_{i,j} - A_{j,i} \) which is equivalent to \( F_{[i,j,k]} = 0 \), \( A_i \) being the electrostatic potentials. Here and in what follows a comma denotes the partial derivative with respect to the coordinate indices involving the index.

Again, the spin tensor and the intrinsic angular momentum density tensor are related in the form

\[ S^i_{jk} = u^i S_{jk}, \]  
(3.24)

with

\[ u^i S_{ik} = 0. \]  
(3.25)
Now, assuming that the spins of the individual charged particles composing the fluid distribution are all aligned in the radial directions (Prasanna 1975; Raychaudhuri 1979; Tiwari and Ray 1997) and the matter is at rest with respect to the observer, the non-vanishing components of the spin tensor can be obtained, from equations (3.24) and (3.25), as

\[ S^0_{\ 23} = -S^0_{\ 32} = s(g_{00})^{-1/2}, \]  

whereas, from equation (3.16), we have the torsion tensor as

\[ Q^0_{\ 23} = -Q^0_{\ 32} = -\kappa s(g_{00})^{-1/2}, \]

\[ s = S_{23} \] being the only non-vanishing component of the intrinsic angular momentum density tensor. Here, we have followed the convention \((t, r, \theta, \phi) = (0, 1, 2, 3)\).

The Einstein-Cartan-Maxwell equations with source can be written as (Tiwari and Ray 1997)

\[ e^{-2\lambda}(\frac{2\lambda'}{r} - \frac{1}{r^2}) + \frac{1}{r^2} = 8\pi \tilde{\rho} + E^2, \]  

\[ e^{-2\lambda}(\frac{2\nu'}{r} + \frac{1}{r^2}) - \frac{1}{r^2} = 8\pi \tilde{p}_r - E^2, \]  

\[ e^{-2\lambda}[\nu'' + \nu' \lambda' + \frac{(\nu' - \lambda')}{r}] = 8\pi \tilde{p}_\perp + E^2 \]

and

\[ (r^2 E)' = 4\pi r^2 \sigma e^\lambda, \]

where \(\tilde{\rho}, \tilde{p}_r, \tilde{p}_\perp\) and \(E\) are the effective matter-energy density, effective pressures (radial and tangential) and electric field respectively, and are defined as

\[ \tilde{\rho} = \rho - 2\pi s^2, \]  

\[ \tilde{p}_r = p_r - 2\pi s^2, \]  

\[ \tilde{p}_\perp = p_\perp - 2\pi s^2 \]

and

\[ E = -\exp[-(\nu + \lambda)]\phi' = \frac{q}{r^2}, \]

\[ \rho, p_r, p_\perp, s, \phi \] and \(q\) being the ordinary matter-energy density, ordinary pressures (radial and tangential), spin density, electrostatic potential and electric charge.
respectively. Here, $\sigma$ represents the electric charge density and prime denotes differentiation with respect to the radial coordinate $r$.

Now, the conservation equations (3.20) and (3.21) with the help of equations (3.22) and (3.23) reduce to

$$\frac{d\tilde{p}_r}{dr} = -(\tilde{\rho} + \tilde{p}_r)\nu' + \frac{1}{8\pi r^4} \frac{d}{dr}(q^2) + \frac{2(\tilde{p}_\perp - \tilde{p}_r)}{r}.$$ (3.36)

This is the key equation which is to be solved for constructing EMMM.

### 3.4 The Solutions

Addition of (3.28) and (3.29), under the assumption $g_{00}g_{11} = -1$ (or equivalently, in terms of energy-momentum tensors $T^0_0 = T^1_1$), provides the pure charge condition

$$\tilde{\rho} + \tilde{p}_r = 0,$$ (3.37)

where, in general, $\tilde{\rho}$ is assumed to be positive and hence $\tilde{p}_r$ is negative. However, as is evident from equation (3.32), $\tilde{\rho}$, being the effective energy-density, can even be negative due to the positive second term related to the spin on the right hand side. Thus, the possibility of equation (3.33) being satisfied with $\tilde{p}_r$ being positive is not ruled out.

To make (3.31) and (3.36) solvable, we further assume that

$$\sigma e^\lambda = \sigma_0$$ (3.38)

and

$$p_\perp - p_r = \tilde{p}_\perp - \tilde{p}_r = \alpha q^2 r^2$$ (3.39)

following TRK Model (Tiwari et al. 1984) and HV model (Herrera and Varela 1994) respectively, where $\sigma_0$ and $\alpha$ are two constants as mentioned earlier.

By substituting (3.37) - (3.39) in (3.31) and (3.36), we get

$$E = \frac{q}{r^2} = \frac{4}{3} \pi \sigma_0 r,$$ (3.40)
\[ p_r = 2\pi s^2 - (\alpha q^2 + \frac{2}{3}\pi \sigma_0^2)(a^2 - r^2), \quad (3.41) \]
\[ p_\perp = 2\pi s^2 - \alpha q^2(a^2 - 2r^2) - \frac{2}{3}\pi \sigma_0^2(a^2 - r^2) \quad (3.42) \]

and

\[ \rho = 2\pi s^2 + (\alpha q^2 + \frac{2}{3}\pi \sigma_0^2)(a^2 - r^2). \quad (3.43) \]

The active gravitational mass

\[ M(r) = 4\pi \int_0^r (\rho - 2\pi s^2 + \frac{E^2}{8\pi})r^2 dr \quad (3.44) \]

takes the form, by virtue of (3.40) and (3.43), as

\[ M(r) = \frac{8}{135}\pi^2 \sigma_0^2 r^3[8\pi \alpha a^6(5a^2 - 3r^2) + 3(5a^2 - 2r^2)]. \quad (3.45) \]

Thus, the metric potentials \( \lambda \) and \( \nu \) are given by

\[ e^{-2\lambda} = e^{2\nu} = 1 - \frac{2M(r)}{r}, \quad (3.46) \]

whereas, the effective gravitational mass mentioned in (3.1), can be obtained as

\[ m = \frac{64}{45}\pi^2 \sigma_0^2 a^5(1 + \frac{2}{3}\pi \alpha a^6), \quad (3.47) \]

which corresponds to the second case (B) of HV model (1994) and is of purely electromagnetic origin. This corresponds to the TRK model (1984) with \( \alpha = 0 \) case.

It can be noted here that unlike the matter-energy density the effective gravitational mass is independent of spin.

In this context it is to be mentioned here that the junction conditions in the EC theory are different from that of general theory of relativity and indeed read like this (Arkuszewski et al. 1975)

\[ n_i u^i \mid_\perp = 0 \quad (3.48) \]

and

\[ p \mid_\perp = 2\pi G(n_i S^i) \mid_\perp \quad (3.49) \]

where \( S^i \) is the spin density pseudo-vector. Here condition (3.48) is the same as in classical relativistic hydrodynamics and has already been incorporated by matching
the interior solution with the exterior Reissner-Nordström field at the boundary of the spherical distribution. The condition (3.49), however, is the additional condition to be satisfied in EC theory. In the present case, it is only the effective pressure (radial) that vanishes on the boundary and not the ordinary radial pressure which, by virtue of equation (3.41), equals $2\pi s^2$. The spin is aligned in the radial direction and hence the spin density pseudo-vector is hypersurface orthogonal. Thus, the boundary condition (3.49) will become

$$p_r|_{r=a} = 2\pi s^2|_{r=a},$$

which, depending on whether $s$ is a constant or function of coordinates, will automatically be satisfied.

In this connection it is to be mentioned here that the spin density, $s$, in the final solutions (3.41) – (3.43) remains arbitrary (function of $r$). An explicit functional form of this spin density can also be obtained by assuming some additional physically viable possibility, such as the one used by Prasanna (1975) by splitting the conservation equation into two parts, the second part relating to conservation of spin only, giving the functional form of spin density as $s = s_0 e^{-\nu}$ (where $s_0$ is the value of $s$ at $r = 0$ i.e. the central spin density). This can, using equations (3.38) and (3.46) and the condition $g_{00}g_{11} = 1$, (that is, $\nu + \lambda = 0$) equivalently be written as $s\sigma = s_0\sigma_0 = \text{constant}$. The functional form of spin density is, however, not relevant in our discussion as our problem is concerned with the properties related to the ‘electron’, an elementary particle whose radius is of the order of $10^{-16}$ cm. Indeed, as the spin function is arbitrary, there is no loss in generality, even if we assume it to be almost a constant (that is, the quantized value of the spin of the electron).

### 3.5 The Negative Energy Density Model

Let us have a closer observation of the results of the previous section 3.4. The equation (3.43) related to matter-energy density has the spin density part in the first term where spin density is defined as $s = 3S/4\pi a^3$, where $S$ is the spin of electron the quantized value of which is $S = \hbar/2$. Then substituting the standard values for
different parameters in the relativistic units, as mentioned in the introductory part, the numerical value for the matter-energy density (3.43) can be shown as

\[ \rho = 6.14 \times 10^{-37} - 6.81 \times 10^{-27} (10^{-32} - r^2). \]  

(3.51)

The first term related to spin, being of the order of $10^{-37}$, is too small compared to the last term and hence the spin contribution is negligible. Now, the equation (3.51) indicates that the central density at $r = 0$ is negative and its magnitude is about $10^{-5}$. On the other hand, the total density at the boundary, $r = a$, is positive as usual with the numerical value about $10^{-37}$. This change in the sign of the energy density is because of the presence of the spin term in equation (3.51) which, indeed, is the contribution of the EC theory. In the absence of spin, however, we could have negative and zero densities at the centre and boundary of the electron respectively. This change in the sign again indicates that the central negative value gradually increases along the radius and somewhere, in the region $0 < r_c < a$, it becomes zero, where $r_c$ is the critical radius. Obviously, the amount of negative energy density is less than its positive counterpart the balance of which ultimately provides the net density as the positive one.

It is already mentioned that, in general, for any spherical fluid distribution the density on the surface should be zero where we are getting some non-zero value for it. This finite value of density is solely coming from the spin contributed part $2\pi s^2$. Thus for the vanishing spin the situation corresponds to the general behaviour (vide equation (17) of Herrera and Varela 1994). In this context it is also to be noted here that up to the critical radius behaviour of our model is similar to that of Herrera and Varela (1994). Beyond this cut off radius the energy density is regulated by spin which makes the overall density of the model positive. This particular aspect lack in the model of Herrera and Varela (1994) where the total energy density is a negative quantity. This increase of matter-energy density due to spin density can probably be accounted for the kinetic energy through the angular motion of the electron here. Similar kind of examination is also possible for the pressures, radial and tangential, both. The radial pressure, in this case of equation (3.41), takes the following value

\[ p_r = 6.14 \times 10^{-37} + 6.81 \times 10^{-27} (10^{-32} - r^2). \]  

(3.52)
However, pressure is throughout positive here from the centre to boundary. These results, i.e. negative energy density and positive pressure, are in accordance with the pure charge condition (3.37) which reads as $\rho = -p_r + 4\pi s^2$.

### 3.6 Conclusions

(i) The possible origin of the intriguing negative matter-energy density in the work of Cooperstock and Rosen (1989), Bonnor and Cooperstock (1989), Herrera and Varela (1994) and present paper may be due to the finiteness of the total mass of the Reissner-Nordström solution (Visser 1989). Since the electrostatic energy of a point charge is infinite, the only way to produce a finite total mass is the presence of an infinite amount of negative energy at the center of symmetry. According to Bonnor and Cooperstock (1989) the negativity of the energy density and hence the active gravitational mass is consistent with the phenomenon, known as the Reissner-Nordström repulsion (de la Cruz and Israel 1967; Cohen and Gautreau 1979; Tiwari et al. 1984; Cooperstock and Rosen 1989). In this regards Bonnor and Cooperstock (1989) also have discussed about the singularity theorems of general relativity (Hawking and Ellis 1973). They have shown that the negative regions are liable to exist over distances of order $10^{-13}$ cm and as the proof of the singularity theorems depends on the manifold structure of space-time valid down to lengths of order $10^{-15}$ cm so might break down below this. On the other hand, in the context of Einstein-Cartan theory of gravitation the idea of negative mass is not a new one as stated by de Sabbata and Sivaram (1994): “...torsion provides a natural framework for the description the negative mass under extreme conditions of such as in the early universe, when a transition from positive to negative mass can take place.”

(ii) We have considered in the present chapter an extended static spherically symmetric distribution of an elementary particle like electron having the radius of the order of $10^{-16}$ cm, and even if for the finite size of the physical system the spin in Einstein-Cartan theory can be related to orbital rotation (which indeed is not the case), for systems of dimensions of subatomic particle the orbital rotation loses its
meaning. In this case, the only way is to take the spin to be the ‘intrinsic angular momentum’, that is, the spin of quantum mechanical origin (in our problem since $s$ is arbitrary, we can consider its quantized value or an average value). In this respect we would like to quote here from Hehl et al. (1974), “It is crucial to note that spin in $U_4$ theory is canonical spin, that is, the intrinsic spin of elementary particles, not the so-called spin of galaxies or planets.”

(iii) Following other authors (Prasanna 1975; Raychaudhuri 1979; Tiwari and Ray 1997), in the present work the spins of all the individual particles are assumed to be oriented along the radial axis of the spherical systems. As to how this alignment is brought about is not very much clear. We have discussed here only a few possible ways of realizing this situation. According to Raychaudhuri (1979), in general, there will be an interaction between the spins of the particles and the magnetic field. The overall effect is the alignment of the spins. In this context, Prasanna (1975) mentioned that such an alignment may be meaningful either in the case of spherical symmetry when magnetic field is present or else one has to consider axially symmetric field. As stated above, our viewpoint is that in the case of physical systems of the size of the electron, the radial alignment of the spin is not ruled out. The solution obtained supports this view.

(iv) Though our present approach via Einstein-Cartan theory to inject spin may be interesting, we feel even that there should have some room to discuss the relationship of our work with an alternative means to provide spin and magnetic moment. We think this may possible through Dirac-Maxwell theory where spin and magnetic moment are naturally incorporated through the Dirac spin (Bohm and Cooperstock 1999; Lisi 1995) and would like to pursue this problem in future investigations.

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Chapter 4

Energy Density in General Relativity: a Possible Role for Cosmological Constant

“The most insignificant thing contains some little unknown element. We must find it!”
– Maupassant

4.1 Introduction

The structure of electron was, for a long time, an intrigue problem to the researchers. Many scientists, like Lorentz (1904) and even Einstein (1919) tried to solve the problem to show that the electron mass is a electromagnetic field dependent quantity (for a detail account see the references Tiwari, Rao and Kanakamedala (1986) and Wilczek (1999)). Later on, under general relativity some models have been constructed by different authors describing extended electron with its mass entirely of
electromagnetic origin (Florides 1962; Cooperstock and de la Cruz 1978; Tiwari, Rao and Kanakamedala 1984; Gautreau 1985). Recently, based on the experimental upper limits on the size of the electron as $\sim 10^{-16}$ cm (1983) it is argued by Cooperstock and Rosen (1989), Bonnor and Cooperstock (1989) and Herrera and Varela (1994) that in the framework of general theory of relativity the electron-like spherically symmetric charged distribution of matter must contain some negative mass density. Being motivated by these results with historical and heuristic values we would like to explore a possible role for cosmological constant on the energy density of electron when it is modeled as a dependent on the radial coordinate $r$ of the charged spherical matter distribution.

The basic logic for considering variability of so called cosmological constant, which was introduced by Einstein in 1917 to obtain a static cosmological model, is related to the observational evidence of high redshift Type Ia supernovae (Perlmutter et al. 1998; Riess et al. 1998) for a small decreasing value of cosmological constant ($\Lambda_{\text{present}} \leq 10^{-56} \text{cm}^{-2}$) at the present epoch. This indicates that instead of a strict constant the $\Lambda$ could be a function of space and time coordinates. If the role of time-dependent $\Lambda$ is prominent in the cosmological realm, then space-dependent $\Lambda$ has an expected effect in the astrophysical context. It is, therefore, argued by Narlikar et al. (1991) that the space-dependence of $\Lambda$ cannot be ignored in relation to the nature of local massive objects like galaxies. Our aim, however, to see if there is any effect of space-dependent $\Lambda$ on the energy density of the classical electron. This is because cosmological constant is thought to be related to the quantum fluctuations as evident from the theoretical works by Zel’dovich (1967). Moreover, it is believed through indirect evidences that 65 % of the contents of the universe is to be in the form of the energy of vacuum (Martins 2002). Thus, the energy density of vacuum due to quantum fluctuation might have, in our opinion, some underlying relation to the energy density of Lorentz’s extended electron (1904) under general relativistic treatment.

In the present chapter IV we have tried to find out, through some specific case studies, that energy density of classical electron is related to the variable cosmological
constant and the gravitational mass of the electron is entirely dependent on the electromagnetic field alone.

### 4.2 The field equations

To carry out the investigation we have considered the Einstein-Maxwell field equations for the case of anisotropic charged fluid distribution (in relativistic units $G = c = 1$) which are given by

$$G^i_j \equiv R^i_j - g^i_j R/2 = -8\pi [T^i_j (m) + T^i_j (em) + T^i_j (vac)], \quad (4.1)$$

$$[(-g)^{1/2} F^{ij}]_{,ij} = 4\pi J^i (-g)^{1/2} \quad (4.2)$$

and

$$F_{[ij,k]} = 0 \quad (4.3)$$

where $F^{ij}$ is the electromagnetic field tensor and $J^i$, current four vector which is equivalent to $J^i = \sigma u^i$, $\sigma$ being the charge density and $u^i$ is the four-velocity of the matter satisfying the relation $u_i u^i = 1$.

The matter, electromagnetic and vacuum energy-momentum tensors are, respectively given by

$$T^i_j (m) = (\rho + p_\perp) u^i u_j - p_\perp g^i_j + (p_\perp - p_r) \eta^i \eta_j, \quad (4.4)$$

$$T^i_j (em) = -[F_{jk} F^{ik} - g^i_j F_{kl} F^{kl}/4]/4\pi \quad (4.5)$$

and

$$T^i_j (vac) = g^i_j \Lambda (r)/8\pi \quad (4.6)$$

where $\rho$, $p_r$ and $p_\perp$ are the proper energy density, radial and tangential pressures respectively and $\eta_i$ is the unit space-like vector on which the condition to be imposed is $\eta_i \eta^i = -1$. Here $p_r$ is the pressure in the direction of $\eta_i$ whereas $p_\perp$ is the pressure on the two-space orthogonal to $\eta_i$.

Now, for the spherically symmetric metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.7)$$
the Einstein-Maxwell field equations (4.1) - (4.6) corresponding to anisotropic charged fluid with spatially varying cosmological constant i.e. \( \Lambda = \Lambda(r) \), are given by

\[
e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi T^0_0 = 8\pi \tilde{\rho} + E^2, \quad (4.8)
\]

\[
e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 = -8\pi T^1_1 = 8\pi \tilde{p}_r - E^2, \quad (4.9)
\]

\[
e^{-\lambda}[\nu''/2 + \nu'/4 - \nu'\lambda'/4 + (\nu' - \lambda'/2r] \\
= -8\pi T^2_2 = -8\pi T^3_3 = 8\pi \tilde{p}_\perp + E^2 \quad (4.10)
\]

and

\[
[r^2 E]' = 4\pi r^2 \sigma e^{\lambda/2}, \quad (4.11)
\]

where \( E \), the intensity of electric field, is defined as \( E = -e^{-(\nu+\lambda)/2}\phi' \) and can equivalently be expressed, from equation (4.11), as

\[
E = \frac{1}{r^2} \int_0^r 4\pi r^2 \sigma e^{\lambda/2} dr. \quad (4.12)
\]

Here prime denotes differentiations with respect to the radial coordinate \( r \) only.

In the above equations (4.8) - (4.10) we have considered that

\[
\tilde{\rho} = \rho + \Lambda(r)/8\pi, \quad (4.13)
\]

\[
\tilde{p}_r = p_r - \Lambda(r)/8\pi \quad (4.14)
\]

and

\[
\tilde{p}_\perp = p_\perp - \Lambda(r)/8\pi, \quad (4.15)
\]

where \( \tilde{\rho} \), \( \tilde{p}_r \) and \( \tilde{p}_\perp \) are the effective energy density, radial and tangential pressures respectively.

The equation of continuity \( T^i_j; i = 0 \), is given by

\[
\frac{dp_r}{dr} - \frac{1}{8\pi} \frac{d\Lambda(r)}{dr} + \frac{1}{2}(\rho + p_r)\nu' = \frac{1}{8\pi r^4} \frac{dq^2}{dr} + \frac{2(p_\perp - p_r)}{r} \quad (4.16)
\]

where \( q \) is the charge on the spherical system.

We assume the relation between the radial and tangential pressures (Herrera and Varela 1994) as

\[
p_\perp - p_r = \alpha q^2 r^2, \quad (4.17)
\]
where $\alpha$ is a constant.

Hence, by use of equations (4.14) and (4.17), the equation (4.16) reduces to

$$\frac{d\tilde{p}_r}{dr} + \frac{1}{2}(\tilde{\rho} + \tilde{p}_r)\nu' = \frac{1}{8\pi r^4} \frac{dq^2}{dr} + 2\alpha q^2 r.$$  \hspace{1cm} (4.18)

Now, equation (4.8) can be expressed in the following form as

$$e^{-\lambda} = 1 - 2M/r,$$  \hspace{1cm} (4.19)

where the active gravitational mass, $M$, is given by

$$M = 4\pi \int_0^r \left[ \tilde{\rho} + \frac{E^2}{8\pi} \right] r^2 dr.$$  \hspace{1cm} (4.20)

### 4.3 The solutions

#### 4.3.1 Model for $\rho + p_r = 0$

Let us now solve the equation (4.18) under the assumption between the stress-energy tensors as $T^1_1 = T^0_0$, which implies that

$$\tilde{\rho} + \tilde{p}_r = \rho + p_r = 0.$$  \hspace{1cm} (4.21)

In order to make the equation (4.12) integrable we assume that

$$\sigma = \sigma_0 e^{-\lambda/2},$$  \hspace{1cm} (4.22)

where $\sigma_0$ is the charge density at $r = 0$ of the spherical distribution, i.e. the central density of charge.

Now, using condition (4.22) in equation (4.12), we get for the expression of electric charge and intensity of the electric field as

$$q = Er^2 = \frac{4}{3}\pi \sigma_0 r^3.$$  \hspace{1cm} (4.23)

With the help of equations (4.21) and (4.23), the equation (4.18) reduces to

$$\frac{d\tilde{p}_r}{dr} = \frac{4}{3}\pi \sigma_0^2 r + 2\alpha q^2 r.$$  \hspace{1cm} (4.24)
Thus the solution set is given by

\[ e^{-\lambda} = e^\nu = 1 - \frac{16}{45}\pi^2 \sigma_0^2 r^2 (5a^2 - 2r^2) - \frac{8}{15}\pi a q^2 r^2 (5a^2 - 3r^2), \quad (4.25) \]

\[ p_r = -(\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2) + \frac{\Lambda(r)}{8\pi}, \quad (4.26) \]

\[ p_\perp = \alpha q^2 r^2 - (\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2) + \frac{\Lambda(r)}{8\pi}, \quad (4.27) \]

and

\[ \rho(r) = (\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2) - \frac{\Lambda(r)}{8\pi}. \quad (4.28) \]

The active gravitational mass which is defined in the equation (4.20), then, by virtue of the equations (4.23) and (4.28), takes the form as

\[ M(r) = \frac{8}{135}\pi^2 \sigma_0^2 r^3 [8\pi \alpha a^6 (5a^2 - 3r^2) + 3(5a^2 - 2r^2)]. \quad (4.29) \]

Thus, the metric potentials \( \lambda \) and \( \nu \) are given by

\[ e^{-\lambda} = e^\nu = 1 - \frac{2M(r)}{r}. \quad (4.30) \]

The total effective gravitational mass can be obtained, after smoothly matching of the interior solution to the exterior Reissner-Nordström solution on the boundary, as

\[ m = M(a) + \frac{q^2(a)}{2a} = \frac{64}{45}\pi^2 \sigma_0^2 a^5 (\frac{2}{3}\pi a^6), \quad (4.31) \]

which corresponds to the second case (B) of Herrera-Varela model (1994) and represents “electromagnetic mass” model such that gravitational mass of a charged fluid sphere originates from the electromagnetic field alone (Lorentz 1904; Feynman et al. 1964). This again corresponds to the Tiwari-Rao-Kanakamedala model (1984) with \( \alpha = 0 \) case and thus the present model reduces to isotropic one.

Now, considering the observed values of mass, charge and radius of the electron (in relativistic units) as \( m = 6.76 \times 10^{-56} \text{ cm}, \quad q = 1.38 \times 10^{-34} \text{ cm} \) and \( a = 10^{-16} \text{ cm} \) the value of \( \alpha \), from the equation (4.31), is given by

\[ \alpha = -4.77 \times 10^{95} \text{ cm}^{-6}. \quad (4.32) \]
For the above value of \( \alpha \), the energy density in equation (4.28) becomes

\[ \rho(r) = -6.81 \times 10^{27} (a^2 - r^2) - \frac{\Lambda(r)}{8\pi}. \]  

(4.33)

The central energy density, \( \rho_0 \), at \( r = 0 \), then can be calculated as

\[ \rho_0 = -6.81 \times 10^{-5} - \frac{\Lambda_0}{8\pi}. \]  

(4.34)

Thus, from the equation (4.34) one can see that for \( \Lambda_0 > 0 \) the energy density of the electron is a negative quantity. It is to be noted here that in the cosmological context \( \Lambda \) positive is related to the repulsive pressure and hence an acceleration dominated universe as suggested by the SCP and HZT project report (Perlmutter et al 1998; Riess et al. 1998; Filippenko 2001; Kastor and Traschen 2002). However, equation (4.34) indicates that this negativity of energy density is also obtainable for \( \Lambda_0 < 0 \) (which indicates a collapsing situation of the universe (Cardenas et al. 2002)) for its very small value. In this context it is also possible to show that at an early epoch of the universe when the numerical value of negative \( \Lambda \) is higher than that of the first term of \( \rho \) (i.e. \( \sim 10^{-5} \) at \( r = 0 \)) obviously energy density is a positive quantity. Thus, in the case of decreasing negative value of \( \Lambda \) it is clear that there is a smooth crossover from positive energy density to a negative energy density via a phase of null energy density! However, these results confirm the vacuum equation of state \( \rho + p_r = 0 \) (Davies 1984; Blome and Priester 1984; Hogan 1984; Kaiser and Stebbins 1984).

We can also see that on the boundary, \( r = a \), the total energy density becomes

\[ \rho_a = -\frac{\Lambda_a}{8\pi}, \]  

(4.35)

which shows its clear dependency on the cosmological constant. However, for \( \Lambda_a > 0 \), \( \rho_a \) is negative whereas for \( \Lambda_a < 0 \), \( \rho_a \) is as usual a positive quantity. Through a simple and interesting exercise (as all the parameters related to the electron are known) one can find out the numerical value of \( \Lambda_a \), at the boundary of the spherical system from the equation (4.35), which equals \( \sim 10^{-7} cm^{-2} \). This constant value of \( \Lambda_a \) is too large and might be related to an early epoch of the universe. Here for finding
out the total energy density $\rho_a$ it is considered that $\rho_a \leq \rho_{\text{average}}$, where $\rho_{\text{average}}$ is equal to $m/\frac{4}{3}\pi a^3$ as the energy density of the spherical distribution is decreasing from centre to boundary.

4.3.2 Model for $\rho + p_r \neq 0$

Now using equation (4.23) the equation (4.18) can be written as

$$\frac{d}{dr} \left[ \tilde{p}_r - \frac{E^2}{8\pi} \right] + \frac{1}{2}(\tilde{\rho} + \tilde{p}_r)\nu' = \frac{E^2}{2\pi r} + 2\alpha q^2 r. \quad (4.36)$$

Assuming that the radial stress-energy tensor $T_{11} = 0$, one gets

$$\nu' = \left( e^\lambda - 1 \right) \frac{r}{r}, \quad (4.37)$$

and

$$\tilde{p}_r = \frac{E^2}{8\pi}. \quad (4.38)$$

Using equations (4.37) and (4.38) in equation (4.36), we have

$$\tilde{\rho} + \tilde{p}_r = \rho + p_r = \frac{(4\alpha q^2 r^2 + E^2/\pi)}{e^\lambda - 1}. \quad (4.39)$$

Thus, equation (4.20) takes the form as

$$M = 4\pi \int_0^r \left[ \frac{(4\alpha q^2 r^2 + E^2/\pi)}{e^\lambda - 1} \right] r^2 dr. \quad (4.40)$$

To make equation (4.40) integrable we assume that

$$E^2 = \pi k(e^\lambda - 1)(1 - R^2) - 4\pi \alpha q^2 r^2, \quad (4.41)$$

where $k$ is a constant and $R = r/a$, $a$ being the radius of the sphere.

Thus, the solution set is given by

$$e^{-\lambda} = 1 - AR^2(5 - 3R^2), \quad (4.42)$$

$$e^\nu = (1 - 2A)^{5/4}e^{\lambda/4}exp[5B\tan^{-1}B(6R^2 - 5) - \frac{1}{2}\tan^{-1}B], \quad (4.43)$$
\begin{align*}
  p_r &= \frac{1}{8} k (e^\lambda - 1) (1 - R^2) - \frac{1}{2} \alpha q^2 r^2 + \frac{\Lambda(r)}{8\pi}, \quad (4.44) \\
  p_\perp &= \frac{1}{8} k (e^\lambda - 1) (1 - R^2) + \frac{1}{2} \alpha q^2 r^2 + \frac{\Lambda(r)}{8\pi}, \quad (4.45)
\end{align*}

and
\[ \rho = k (1 - R^2) [1 - \frac{1}{8} (e^\lambda - 1)] + \frac{1}{2} \alpha q^2 r^2 - \frac{\Lambda(r)}{8\pi}, \quad (4.46) \]

where the constant \( A = 8\pi ka^2/15 \).

By application of the matching condition at the boundary we again get the total effective gravitational mass, which in the present case takes the form
\[ m = \frac{8}{15} \pi ka^3 + \frac{q^2}{2a}. \quad (4.47) \]

In view of the equation (4.41), for vanishing charge the constant \( k \) vanishes and hence makes the gravitational mass in the equation (4.47) to vanish. Thus, the present case \( \rho + p_r = k (1 - R^2) \neq 0 \) also represents electromagnetic mass model.

Now, the constant \( k \) can be expressed in terms of the known values of the electric mass, radius and charge as
\[ k = \frac{15}{16\pi a^4} (2am - q^2). \quad (4.48) \]

At \( r = 0 \), the energy density, from the equation (4.46), is then given by
\[ \rho_0 = -5.68 \times 10^{-5} - \frac{\Lambda_0}{8\pi}. \quad (4.49) \]

As, in the case of electron, \( k \) is a negative quantity so for \( \Lambda_0 > 0 \) the central energy density \( \rho_0 \) is negative only. However, for \( \Lambda_0 < 0 \) the central energy density may respectively be negative and positive depending on the numerical value of \( k \) whether it is higher and lower than that of \( \Lambda_0 \).

At \( r = a \), the total energy density is given by
\[ \rho_a = -4.54 \times 10^{-5} - \frac{\Lambda_a}{8\pi}. \quad (4.50) \]

Similarly, for \( \Lambda_a > 0 \), the energy density is negative whereas for \( \Lambda_a < 0 \), it may either be negative or positive depending on the numerical value of \( \Lambda_a \) as discussed in the previous case.
4.3.3 A test model

In the previous two cases we have qualitatively discussed the effect of cosmological parameter $\Lambda(r)$ on the energy density $\rho(r)$ of the electron. Let us now explore some quantitative effect and hence treat the equation (4.24) in a different way. If we substitute the value of $\tilde{p}_r$, from equation (4.14), then integrating equation (4.24) we get

$$\Lambda_{\text{eff}} = \Lambda(a) - \Lambda(r) = 8\pi\rho(r) - 8\pi(\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)(a^2 - r^2),$$

(4.51)

where $\Lambda_{\text{eff}}$ is the effective cosmological parameter.

We study the following cases:

For the central value of the energy density of the spherical distribution, i.e. at $r = 0$, the effective cosmological parameter becomes

$$\Lambda_0^{\text{eff}} = \Lambda(a) - \Lambda(0) = 8\pi\rho_0 - 8\pi(\alpha q^2 + \frac{2}{3}\pi\sigma_0^2)a^2.$$  

(4.52)

Considering that $\rho_0 \geq \rho_{\text{average}}$ the effective cosmological parameter, at $r = 0$, for the proper numerical values of the charge and radius of the electron can be found out as

$$\Lambda_0^{\text{eff}} = 1.71 \times 10^{-3} \text{cm}^{-2}.$$  

(4.53)

On the other hand, at the boundary, $r = a$, of the spherical distribution the effective cosmological parameter becomes

$$\Lambda_a^{\text{eff}} = \Lambda(a) - \Lambda(a) = 0.$$  

(4.54)

Thus, from the equations (4.53) and (4.54) it is shown that the effective cosmological parameter has a finite value at the centre of the electron which decreases radially and becomes zero at the boundary.

4.4 Discussions

We see from the above analysis that the cosmological parameter $\Lambda$ has a definite role even on the energy density of micro-particle, like electron. We, therefore, feel
that it may also be possible to extrapolate the present investigation to the massive astrophysical bodies to see the effect of spatially varying cosmological parameter on their energy densities and vice versa.

The proper pressure $p_r$, in general, being positive as evident from the equation (4.26) is in accordance with the condition (4.21) which may be explained as due to vacuum polarization (Grøn 1985). In this connection it is mentioned by Bonnor and Cooperstock (1989) that the negativity of the active gravitational mass and hence negative energy density for electron of radius $a \sim 10^{-16}$ is consistent with the Reissner-Nordström repulsion. We would also like to mention here that the equation of state in the form $p + \rho = 0$ is discussed by Gliner (1966) in his study of the algebraic properties of the energy-momentum tensor of ordinary matter through the metric tensors and called it the $\rho$-vacuum state of matter. It is also to be noted that the gravitational effect of the zero-point energies of particles and electromagnetic fields are real and measurable, as in the Casimir Effect (1948). According to Peebles and Ratra (2002), like all energy, this zero-point energy has to contribute to the source term in Einstein’s gravitational field equation. This, therefore, demands inclusion of vacuum energy related term cosmological constant in the field equation. In this regard it is interesting to recall the comment made by Einstein (1919) where he stated that “... of the energy constituting matter three-quarters is to be ascribed to the electromagnetic field, and one-quarter to the gravitational field” and did “disregard” the cosmological constant in his field equation is in contradiction to the present result as shown in the equation (4.31) and (4.47).

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Chapter 5

Relativistic Electromagnetic Mass Models: Charged Dust Distribution in Higher Dimensions

“The most incomprehensible thing about the world is that it is comprehensible.”
– Albert Einstein

5.1 Introduction

In many theories higher dimensions play an important role, specially in super string theory (Schwarz 1985; Weinberg 1986) which demands more than usual four dimensional space-time. This is also true in studying the models regarding unification of gravitational force with other fundamental forces in nature. In the case of a simple solution to the vacuum field equations of general relativity in (4 + 1) space-time dimensions Chodos and Detweiler (1980) have shown that it leads to a cosmology
which at the present epoch has \((3+1)\) observable dimensions in which the Einstein-Maxwell equations are obeyed. Lorenz-Petzold (1984) has studied a class of higher dimensional Bianchi-Kantowski-Sachs space-times of the Kaluza-Klein type whereas Ibanèz and Verdaguer (1986) have examined radiative isotropic cosmologies with extra dimensions related to FRW models.

In this connection it is interesting to note that electromagnetic mass models where all the physical parameters, including the gravitational mass, are arising from the electromagnetic field alone have been extensively studied (Tiwari et al. 1984; Gautreau 1985; Grøn 1986; Ponce de Leon 1987; Tiwari and Ray 1991a; Ray and Bhadra 2004a,b) in the space-time of four dimensional general relativity. Thus it is believed that study of electromagnetic mass models in higher dimensional theory will be physically more interesting.

Under this motivation we have considered here a static spherically symmetric charged dust distribution corresponding to higher dimensional theory of general relativity.

It is proved, as a particular case, from the coupled Einstein-Maxwell field equations that a bounded and regular interior static spherically symmetric charged dust solution, if exists, can only be of purely electromagnetic origin. An example, which is already available, is examined in this context and is shown that the solution set satisfies the condition of being electromagnetic origin.

### 5.2 The Einstein-Maxwell Field Equations

The Einstein-Maxwell field equations for the case of charged dust distribution are given by

\[
G^i_j \equiv R^i_j - \frac{1}{2} g^i_j R = -8\pi [T^i_j (m) + T^i_j (em)],
\]

\[
[(-g)^{1/2} F^{ij}]_{j} = 4\pi J^i (-g)^{1/2}
\]

and

\[
F_{[ij,k]} = 0
\]
where $F^{ij}$ is the electromagnetic field tensor and $J^i$ the current four vector which is equivalent to

$$J^i = \sigma u^i \quad (5.4)$$

$\sigma$ being the charge density and $u^i$ is the four velocity of the matter satisfying the relation

$$u_i u^i = 1. \quad (5.5)$$

The matter and electromagnetic energy momentum tensors are given by

$$T_{ij}^{(m)} = \rho u^i u^j \quad (5.6)$$

and

$$T_{ij}^{(em)} = \frac{1}{4\pi} [-F_{jk}F^{ik} + \frac{1}{4} g^{ij}F_{kl}F_{kl}], \quad (5.7)$$

where $\rho$ is the proper energy density.

Now we consider the $(n + 2)$ dimensional spherically symmetric metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 X_n^2 \quad (5.8)$$

where

$$X_n^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 + \ldots + \left[ \prod_{i=1}^{n-1} \sin^2 \theta_i \right] d\theta_n^2.$$

The convention adopted here for coordinates are

$$x^1 = r, \quad x^2 = \theta_1, \quad x^3 = \theta_2, \ldots, x^{n+1} = \theta_n, \quad x^{n+2} = t \quad (5.9)$$

and also

$$g_{11} = -e^\lambda, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta_1,$$

$$g_{44} = -r^2 \sin^2 \theta_1 \sin^2 \theta_2, \ldots, g_{(n+1)(n+1)} = -r^2 \left[ \prod_{i=1}^{n-1} \sin^2 \theta_i \right],$$

$$g_{(n+2)(n+2)} = e^\nu. \quad (5.10)$$
As we have considered here a static fluid, so

\[ u^i = [0, 0, 0, \ldots \, (n+1)\text{times}, \, e^{-\nu/2}] \]  

(5.11)

and

\[ J^1 = J^2 = J^3 = \ldots J^{n+1} = 0, \quad J^{n+2} \neq 0 \]  

(5.12)

so that the only non-vanishing components of \( F_{ij} \) of equations 5.2 and 5.3 are \( F_{1(n+2)} \) and \( F_{(n+2)1} \).

In view of the above, the Einstein-Maxwell field equations for static spherically symmetric charged dust corresponding to the metric (5.8) are

\[ e^{-\lambda} \left[ \frac{n\nu'}{2r} + \frac{n(n-1)}{2r^2} \right] - n(n-1)/2r^2 = -E^2, \]  

(5.13)

\[ e^{-\lambda} \left[ \frac{n\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + (n-1)(\nu' - \lambda')/2r + \frac{(n-1)(n-2)/2r^2}{2r^2} - \frac{(n-1)(n-2)/2r^2}{2r^2} \right] = E^2, \]  

(5.14)

\[ e^{-\lambda} \left[ \frac{n\lambda'}{2r} - \frac{n(n-1)}{2r^2} \right] + \frac{n(n-1)}{2r^2} = 8\pi\rho + E^2 \]  

(5.15)

and

\[ [r^nE']' = 4\pi r^n \sigma e^{\lambda/2} \]  

(5.16)

where \( E \), the electric field strength, is defined as \( E = -e^{-(\nu +\lambda)/2} \phi' \), the electrostatic potential \( \phi \) being related to the electromagnetic field tensor as \( F_{(n+2)1} = -F_{1(n+2)} = \phi' \).

### 5.3 Higher Dimensional Electromagnetic Mass Models

From the field equations (5.13) and (5.15), we have

\[ e^{-\lambda} (\nu' + \lambda') = 16\pi r^2 \rho / n. \]  

(5.17)
Now, from the conservation equations $T^i_j; i = 0$ one obtains

$$\rho \nu' = \left[ q^2 \right]/4\pi r^4 + (n - 2)E^2/2\pi r$$

(5.18)

where the charge, $q$, is related with the electric field strength, $E$, through the integral form of the Maxwell’s equation (5.16), which can be written as

$$q = Er^n = 4\pi \int_0^r \sigma e^{\lambda/2} r^n dr.$$  

(5.19)

Again, equation (5.15) can be expressed in the following form as

$$e^{-\lambda} = 1 - 4M/nr^{n-1}$$

(5.20)

where the active gravitational mass, $M$, is given by

$$M = 4\pi \int_0^r \left[ \rho + E^2/8\pi \right] r^n dr.$$  

(5.21)

Hence, following the technique of Tiwari and Ray (1991a) we see that for the vanishing charge density, $\sigma$, of the equation (5.19) one can arrive, through the equation (5.18), at the unique relation

$$\rho \nu' = 0.$$  

(5.22)

Thus, we have the following two cases:

**Case I: $\rho \neq 0$, $\nu' = 0$**

For this case, from equations (5.13) and (5.14), we have $\lambda$ as a constant quantity. This in turn makes $\rho$ equal to zero and hence by virtue of equation (5.22) space-time becomes flat.

**Case II: $\rho = 0$, $\nu' \neq 0$**

In this case, from equations (5.20) and (5.21), $\lambda$ becomes zero. Then from equation (5.17), we have the metric potential as a constant and again the space-time becomes flat.

We are not considering here the third case, viz. $\rho = \nu' = 0$, which is quite a trivial one. However, from the above cases (I) and (II) it is evident that, at least at a
particular case, all the charged dust models are of electromagnetic origin, viz., all the physical parameters are originating purely from electromagnetic field. This type of models are known as electromagnetic mass models in the literature (Lorentz 1904; Feynman et al. 1964).

An example:

The solution set obtained by Khadekar et al. (2001) for the static spherically symmetric charged dust is as follows:

\[ e^{\nu} = A r^{2N}, \]  
\[ 5.23 \]

\[ e^{-\lambda} = \left[ \frac{(n - 1)}{N + (n - 1)} \right]^2, \]  
\[ 5.24 \]

\[ \rho = \frac{N n(n - 1)^2}{8\pi r^2[N + (n - 1)]^2} \]  
\[ 5.25 \]

and

\[ \sigma = \frac{N(n - 1)^2[n(n - 1)]^{1/2}}{4\pi 2^{1/2}r^2[N + (n - 1)]^2}. \]  
\[ 5.26 \]

where \( A \) and \( N \) both are constants with the restriction that \( N \geq 0 \).

The total charge and mass of the sphere in terms of its radius, \( a \), are respectively given by

\[ q = \frac{N[n(n - 1)]^{1/2}a^{n-1}}{2^{1/2}[N + (n - 1)]} \]  
\[ 5.27 \]

and

\[ m = \frac{Na^{n-1}}{N + (n - 1)}. \]  
\[ 5.28 \]

The charge and mass densities in the present case take the relationship

\[ \sigma = [2(1 - 1/n)]^{1/2}\rho. \]  
\[ 5.29 \]
Therefore, the charge and mass densities are proportional to each other with the constant of proportionality \([2(1 - 1/n)]^{1/2}\) which takes the value unity for \(n = 2\) i.e. in the four dimensional case and the relation (5.29) reduces to the usual form

\[
\sigma = \pm \rho
\]

which is known as the De-Raychaudhuri (1968) condition for equilibrium of a charged dust fluid.

Thus, from equations (5.23) – (5.29), it is evident that all the physical quantities including the effective gravitational mass vanish and also the spherically symmetric space-time becomes flat when the charge density vanishes implying \(N = 0\). The solution here, therefore, satisfies the criteria of being of purely electromagnetic origin.

### 5.4 Discussions

The present paper is, in general, higher dimensional analogue of the work of Tiwari and Ray (1991a) whereas the example given here (Khadekar et al. 2001) is the higher dimensional analogue of the paper of Pant and Sah (1979). Thus, we have presented here a model which corresponds to spherically symmetric gravitational sources of purely electromagnetic origin in the space-time of higher dimensional theory of general relativity. It has been already proved in the four dimensional case of the present paper (Tiwari and Ray 1991a) that a bounded continuous static spherically symmetric charged dust solution, if exists, can only be of electromagnetic origin. Hence, this is also true in the higher dimensional case in general theory of relativity.

In this regard we would like to discuss briefly the role of higher dimensions in different context. It have been shown by Ibanéz and Verdagner (1986) that for the open models related to FRW cosmologies the extra dimensions contract as a result of cosmological evolution whereas for flat and closed models they contract only when there is one extra dimension. Fukui (1987) recovers Chodos-Detweiler (1980) type solutions, as mentioned in the introduction, where the Universe expands as \(t^{1/2}\) by the percolation of radiation into 4D space-time from the fifth dimension,
mass, although the 5D space-time-mass Universe itself is in vacuum as a whole. It is interesting to note that considering mass as fifth dimension a lot of other works also have been done by several researchers (Wesson 1983; Banerjee, Bhui and Chatterjee 1990; Chatterjee and Bhui 1990) which contain Einstein’s theory embedded within it. In one of such investigations it is argued that a huge amount of entropy can be produced following shrinkage of extra-dimension which may account for the very large value of entropy per baryon observed in 4D world (Chatterjee and Bhui 1990). Kaluza-Klein type higher dimensional inflationary scenario have been discussed by Ishihara (1984) and Gegenberg and Das (1985) where it is shown that the contraction of the internal space causes the inflation of the usual space.

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Chapter 6

Relativistic Anisotropic Charged Fluid Spheres with Varying Cosmological Constant

“What are man’s truths ultimately?
Merely his irrefutable errors!”
– Nietzsche

6.1 Introduction

The cosmological constant Λ, related to the energy of space, introduced by Einstein in general relativity has become very significant from the viewpoint of cosmology. Though Einstein ultimately abandoned it stating that it was a “blunder” in his life but Tolman keeps it as a constant quantity in his field equations even in 1939’s famous work related to astrophysical system. It is also, in favor of keeping Λ, argued by Peebles and Ratra (2002) that like all energy, the zero-point energy related to
space has to contribute to the source term in Einstein’s gravitational field equations. However, it is being gradually felt that the erstwhile cosmological constant \( \Lambda \) is indeed a scalar variable dependent on time rather than a constant as was being believed earlier. Recently, this variation in cosmological constant is also observationally confirmed due to the evidence of high redshift Type Ia supernova (Perlmutter et al. 1998; Riess et al. 1998) for a small decreasing value which is \( \leq 10^{-56} \text{cm}^{-2} \) at the present epoch. Obviously, once the \( \Lambda \) becomes a scalar its dependence need not be limited only to time coordinate (as in cosmology). Since it enters in the field equations as a variable, it must be dependent on space coordinates as well. Therefore, in general, the \( \Lambda \) is a scalar variable depending on, either or both, space and time coordinates. It is argued that just as in cosmology the dependence of \( \Lambda \) on time has been found to be of vital importance playing a significant role now, its dependence on space coordinates is equally important for astrophysical problems (Chen and Wu 1990; Narlikar, Pecker and Vigier 1991; Ray and Ray 1993; Tiwari and Ray 1996).

With this viewpoint, we consider here an anisotropic charged static fluid sphere by introducing a scalar variable \( \Lambda \) dependent only on the radial coordinate \( r \). The field equations thus obtained, under certain mathematical assumptions, yield a set of solutions which has another historical importance, known in the name of Electromagnetic Mass Models (EMMM) in the literature (Lorentz 1904; Hoffmann 1935; Feynman, Leighton and Sands 1964; Tiwari, Rao and Kanakamedala 1986; Wilczek 1999). The effective gravitational mass of these models depends on the electromagnetic field alone, viz., the effective gravitational mass vanishes when the charge density vanishes. Such models have been studied by several authors (Ray and Ray 1993; Tiwari, Rao and Kanakamedala 1984; Tiwari, Rao and Ray 1991; Gautreau 1985; Grøn 1985, 1986a,b; Ponce de Leon 1987a,b, 1988; Tiwari and Ray 1991a,b, 1997; Ray, Ray and Tiwari 1993). All these EMMMs, however, have been obtained under a special assumption \( \rho + p = 0 \), where \( \rho \) is the matter-energy density and \( p \) is the fluid pressure under the general condition that \( \rho > 0 \) and \( p < 0 \). This type of equation of state, viz., \( p = \gamma \rho \) with \( \gamma = -1 \), implies that the matter distribution is
in tension and hence the matter is known, in the literature, as a 'false vacuum' or 'degenerate vacuum' or '$\rho$-vacuum' (Davies 1984; Blome and Priester 1984; Hogan 1984; Kaiser and Stebbins 1984). A natural question arises whether there exists any EMMM where this condition is violated, i.e., when $\rho + p \neq 0$. This is the main motivation of the present investigation and here we have shown that even for $\rho + p \neq 0$ EMMM can be constructed. However, the same question was addressed by Tiwari, Rao and Ray (1991) and obtained EMMM in the isotropic and axially-symmetric matter distribution for charged dust case only whereas Ray and Bhadra (2004b) searched for solution to the problem by employing a relation between the radial and tangential pressures as $p_\perp = p_r + \alpha q^2 r^2$ where $\alpha$ is a non-zero constant factor and $q$ is electric charge of the spherical system of radius $r$. We shall consider in the present investigation different relation and would like to observe that how this helps us to find out several class of solutions related to EMMM.

6.2 The Einstein-Maxwell Field Equations

Let us consider a spherically symmetric line element

$$ds^2 = g_{ij}dx^idx^j = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (i, j = 0, 1, 2, 3) \quad (6.1)$$

where $\nu$ and $\lambda$ are the metric potentials.

Now, the Einstein field equations for the case of charged anisotropic source are

$$G^i_j \equiv R^i_j - \frac{1}{2}g^i_j R = -\kappa [T^i_j \text{(m)} + T^i_j \text{(em)} + T^i_j \text{(vac)}], \quad (6.2)$$

where $T^i_j \text{(m)}$, $T^i_j \text{(em)}$ and $T^i_j \text{(vac)}$ are respectively the energy-momentum tensor components for the anisotropic matter source, electromagnetic field and vacuum. The explicit forms of these tensors are given by

$$T^i_j \text{(m)} = (\rho + p_\perp)u^iu_j - p_\perp g^i_j + (p_\perp - p_r)\eta^i\eta_j, \quad (6.3)$$

$$T^i_j \text{(em)} = -\frac{1}{4\pi}[F_{jk}F^{ik} - \frac{1}{4}g^i_j F_{kl}F^{kl}] \quad (6.4)$$

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and

\[ T_{ij}^{(\text{vac})} = \frac{1}{8\pi} g^i_j \Lambda(r) \]  

(6.5)

with \( u_i u^i = -\eta_i \eta^i = 1 \). The Maxwell electromagnetic field equations are given by

\[ [(-g)^{1/2} F_{ij}],_j = 4\pi J^i (-g)^{1/2} \]  

(6.6)

and

\[ F_{[ij,k]} = 0, \]  

(6.7)

where the electromagnetic field tensor \( F_{ij} \) is related to the electromagnetic potentials through \( F_{ij} = A_{i,j} - A_{j,i} \) which, obviously, is equivalent to the equation (6.7).

Further, \( u^i \) is the 4-velocity of a fluid element, \( J^i \) is the 4-current satisfying \( J^i = \sigma u^i \), where \( \sigma \) is the charge density, and \( \kappa = 8\pi \) (in relativistic unit \( G = C = 1 \)). Here and in what follows a comma denotes the partial derivative with respect to the coordinates (involving the index).

The Einstein-Maxwell field equations (6.2) – (6.7) corresponding to static anisotropic charged source with cosmological variable, are then given by

\[ e^{-\lambda}(\lambda'/(r^2) - 1/r^2) + 1/r^2 = 8\pi T^0_0 = 8\pi \rho + E^2 + \Lambda, \]  

(6.8)

\[ e^{-\lambda}(\nu'/(r^2) + 1/r^2) - 1/r^2 = -8\pi T^1_1 = 8\pi p_r - E^2 - \Lambda, \]  

(6.9)

\[ e^{-\lambda}[\nu''/2 + \nu'/2 - 4\nu' \lambda'/4 + (\nu' - \lambda')/2r] = -8\pi T^2_2 = -8\pi T^3_3 = 8\pi p_\perp + E^2 - \Lambda \]  

(6.10)

and

\[ (r^2 E)^{'} = 4\pi r^2 \sigma e^{\lambda/2} \]  

(6.11)

The equation (6.11) can equivalently, in terms of the electric charge \( q \), be expressed as

\[ q(r) = r^2 E(r) = \int_0^r 4\pi r^2 \sigma e^{\lambda/2} dr \]  

(6.12)

where \( p_r, p_\perp \) and \( E \) are the matter-energy density, radial and tangential pressures and intensity of the electric field respectively. Here prime denotes derivative with respect to radial coordinate \( r \) only.

The equation of continuity, \( T^i_{;j} = 0 \) is given by

\[ \frac{d}{dr} \left[ p_r - \{E^2 + \Lambda(r)\}/8\pi \right] + (\rho + p_r) \nu'/2 = E^2/2\pi r + 2(p_\perp - p_r)/r. \]  

(6.13)
Now, our spherically symmetric fluid distribution under investigation is anisotropic in nature. This means that the radial and tangential pressures are, in general, unequal so that the simplest relation between them we assumed here as

\[ p_{\perp} = np_r, \quad (n \neq 1). \tag{6.14} \]

Assuming further that the radial stress \( T_{11} = 0 \) (Florides 1977; Kofinti 1985; Grøn 1986a; Ponce de Leon 1987b; Tiwari and Ray 1996) one gets

\[ \nu' = (e^\lambda - 1)/r \tag{6.15} \]

and

\[ p_r = [E^2 + \Lambda(r)]/8\pi. \tag{6.16} \]

Using equations (6.14) – (6.16), in equation (6.13), we get

\[ \rho + p_r = [(n + 1)E^2 + (n - 1)\Lambda(r)]/2\pi(e^\lambda - 1). \tag{6.17} \]

Similarly, equations (6.8) and (6.9) yield

\[ e^{-\lambda}(\nu' + \lambda') = 8\pi r(\rho + p_r). \tag{6.18} \]

Again, equation (6.8) through the equation (6.16), gives

\[ e^{-\lambda} = 1 - 2m(r)/r, \tag{6.19} \]

where \( m(r) \), called the effective gravitational mass, takes the form

\[ m(r) = M(r) + \mu(r) = 4\pi \int_0^r [\rho + p_r]r^2dr, \tag{6.20} \]

the active gravitational mass of Schwarzschild type and the mass equivalence of electromagnetic field, respectively, being defined as

\[ M(r) = 4\pi \int_0^r \rho r^2dr, \quad \mu(r) = 4\pi \int_0^r p_r r^2dr. \tag{6.21} \]

Now, from the above equation (6.20) it is easily observed that the condition \( \rho + p_r = 0 \), yields a flat space-time through the equations (6.19) and (6.15) and has been considered by Tiwari, Ray and Bhadra (2000) in another context. Hence, the non-trivial solutions exits here for the case \( \rho + p_r \neq 0 \) only.
6.3 Solutions for the Static Charged Fluid Spheres

Now we solve the equation (6.17) under the constraint $\rho + p_r \neq 0$, assuming different mathematical conditions. As the equation (6.17) is involved with two physical parameters, $E$ and $\Lambda$, so unless we specify one parameter in terms of other it is not possible to solve the equation (6.20). Therefore, a plausible straightforward relation between these parameters may be of the form $\Lambda(r) = \pm E(r)^2$. Physically this means that vacuum energy has contribution from the electrostatic field energy and proposal of this kind, in an implicit way, is not at all unavailable in the literature (Ray and Bhadra 2004b). Let us, therefore, assume the following two cases $\Lambda(r) = E^2 - N\Lambda_0$ and $\Lambda(r) = -E^2 + N\Lambda_0$ which will yield solutions with physically interesting features as the analysis of the next section demonstrates it clearly. However, in both the cases we have taken the assumptions in a way so that the cosmological variable $\Lambda$ does not vanish rather may be at most equal to $\Lambda_0$, the erstwhile cosmological constant having a finite non-zero value. Without considering $\Lambda_0$ we will have $\Lambda = 0$ at the boundary $r = a$ which is a bit unphysical and may create difficulties, such as to entropy like problems (Beesham 1993).

Case I

Let us assume that the cosmological constant is a function of radial distance such that

$$\Lambda(r) = E^2 - N\Lambda_0$$

(6.22)

where $N$ is a free parameter and $\Lambda_0$ is the erstwhile non-zero cosmological constant. With the help of equations (6.17) and (6.22), the equation (6.20) takes the form

$$m(r) = 2 \int_0^r [2nE^2 - (n-1)N\Lambda_0]r^2 dr/(e^\lambda - 1).$$

(6.23)

To make equation (6.23) integrable we further assume that

$$E^2 = q^2/r^4 = [k^2(e^\lambda - 1)(1 - R^2) + (n-1)N\Lambda_0]/2n,$$

(6.24)

where $k$ is a constant and $R = r/a$, $a$ being the radius of the sphere. This particular choice for the electric intensity generates a model for charged sphere which is
physically very interesting as it is related to EMMM as will be seen later on. Thus, the solution set is given by

\[ e^{-\lambda(r)} = 1 - AR^2(5 - 3R^2), \]
\[ e^{\nu(r)} = (1 - 2A)^{5/4} e^{\lambda(r)/4} \exp[5B\tan^{-1} B(6R^2 - 5) - \tan^{-1} B/2], \]
\[ p_r(r) = \frac{p_\perp(r)}{n} = [k^2\{e^{\lambda(r)} - 1\}(1 - R^2) - N\Lambda_0]/8\pi n \]

and

\[ \rho(r) = [k^2\{1 + 4n - e^{\lambda(r)}\}(1 - R^2) + N\Lambda_0]/8\pi n, \]

where

\[ A = 4k^2a^2/15, \quad B = [A/(12 - 25A)]^{1/2}. \]

Now, the exterior field of a spherically symmetric static charged fluid distribution described by the metric (6.1) is the unique Reissner-Nordström solution given by

\[ ds^2 = \left[ 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right] dt^2 - \left[ 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right]^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \]

Then, application of the matching condition on the boundary \( r = a \) yields the total effective gravitational mass in the following form

\[ m(a) = \frac{4}{15} k^2 a^3 + \frac{q(a)^2}{2a}. \]

Hence, in terms of the total gravitational mass, the total electric charge and the radius of the sphere, all the constants \( k, A \) and \( B \) can be expressed as

\[ k^2 = 15(2am - q^2)/8a^4, \]
\[ A = (2am - q^2)/2a^2 \]

and

\[ B^2 = (2am - q^2)/[24a^2 - 25(2am - q^2)]. \]

Comparing the equation (6.20) (and subsequently via the equation (6.21)) with the above equation (6.31) we can easily recognize the first term \((4k^2a^3/15)\) in the right hand side as the total Schwarzschild mass whereas the second term \((q^2/2a)\) is the
total mass equivalence of the electromagnetic field. Now, \( k \), in the first term of equation (6.31), is implicitly related to the charge \( q \) which is evident from the equation (6.24). Therefore, in principle, the gravitational mass \( m \) expressed in equation (6.31) is of purely electromagnetic in origin.

**Case II**

Here our choice is

\[
\Lambda(r) = -E^2 + N\Lambda_0. \tag{6.35}
\]

Then, from the equations (6.17) and (6.35), the gravitational mass (equation (6.20)) reduces to

\[
m(r) = 2\int_0^r [2E^2 + (n - 1)N\Lambda_0]r^2 dr / (e^{\lambda(r)} - 1). \tag{6.36}
\]

For the further assumption

\[
E^2 = q^2 / r^4 = [k^2(e^{\lambda(r)} - 1)(1 - R^2) - (n - 1)N\Lambda_0]/2, \tag{6.37}
\]

we obtained the solution set as

\[
e^{-\lambda(r)} = 1 - AR^2(5 - 3R^2), \tag{6.38}
\]

\[
e^{\nu(r)} = (1 - 2A)^{5/4}e^{\lambda(r)/4}exp[5Btan^{-1}B(6R^2 - 5) - tan^{-1}B/2], \tag{6.39}
\]

\[
p(r) = N\Lambda_0 / 8\pi \tag{6.40}
\]

and

\[
\rho(r) = [4k^2(1 - R^2) - N\Lambda_0] / 8\pi. \tag{6.41}
\]

We see that \( \lambda \) and \( \nu \) retain the same form as in the case I and hence the total gravitational mass is also given by the equation (6.31). Further, it is observed that the present case automatically reduces to an isotropic one as the pressure \( p \) does not associated with the anisotropic factor \( n \).
6.4 Physical Properties of the Static Charged Fluid Spheres

Case I: $\Lambda(r) = E^2 - N\Lambda_0$

Subcase (1): The equation (6.24) indicates that for getting a direct dependence of $k$ upon $q$ one can admit the following relaxation such that (i) $N = 0$ when $\Lambda_0 \neq 0$ and $(n - 1) \neq 0$, (ii) $(n - 1) = 0$ when $N \neq 0$ and $\Lambda_0 \neq 0$ and (iii) $\Lambda_0 = 0$ when $(n - 1) \neq 0$ and $N \neq 0$. The third possibility seems to contradict the observational results related to the Supernova type Ia where the cosmological constant is found to be a non-zero positive value whereas the second one provides an isotropic case. Then, suitably opting for $N = 0$ one obtains

$$q(r)^2 = k^2 r^4 (e^{\lambda(r)} - 1)(1 - R^2)/2n,$$

(6.42)

$$p_r(r) = p_\perp(r)/n = k^2 (e^{\lambda(r)} - 1)(1 - R^2)/8\pi n$$

(6.43)

and

$$\rho(r) = k^2 \{1 + 4n - e^{\lambda(r)}\}(1 - R^2)/8\pi n.$$  

(6.44)

Thus, for vanishing electric charge the gravitational mass in the equation (6.31), including all the physical parameters (viz., pressures and density), vanishes and one obtains EMMM.

Subcase (2): The central and the boundary pressures are found to be equal here i.e., $p_r(0) = p_\perp(0)/n = -N\Lambda_0/8\pi n$ and $p_r(a) = p_\perp(a)/n = -N\Lambda_0/8\pi n$ respectively whereas the respective densities are $(4nk^2 + N\Lambda_0)/8\pi n$ and $N\Lambda_0/8\pi n$. So, the present model has a constant pressure throughout the sphere though the density decreases from centre to boundary. For the value $N = 0$, however, we have zero pressure, both at the centre and boundary. The density decreases from the non-zero central value $k^2/2\pi$ and then smoothly decreases to zero at the boundary. Thus, with $N = 0$ the present model goes to a physically well-behaved static charged dust case.
Subcase (3): In the above analysis for the general value of $N$ we have seen that the fluid pressure is altogether negative whereas the density is a positive quantity. Now, it can be observed from the equations (6.27) and (6.28) that $p + p_r \neq 0$ except at the boundary where it is equal to $p_r(a) = -\rho(a)$ such that $\rho(a) > 0$ and $p(a) < 0$ as seen earlier. Otherwise, it will have a general value $k^2(1 - R^2)/2\pi$. The central value is then $p_r(0) = -\tilde{\rho}(0)$ where $\tilde{\rho}(0) = \rho(0) - k^2/2\pi$. As the model demands for $\rho > 0$ and $p < 0$, so the condition to be satisfied here is $\rho(0) > k^2/2\pi$. The general condition for the negative pressure and positive density is then $\rho > k^2(1 - R^2)/2\pi$ for all $r \leq a$. These results are also true for the sub-case $N = 0$.

Case II: $\Lambda(r) = -E^2 + N\Lambda_0$

Subcase (1): Here for $N = 0$ the solution set, as obtained in the equations (6.38), (6.40) and (6.41), reduces to

$$p(r) = 0 \quad (6.45)$$

and

$$\rho(r) = k^2(1 - R^2)/2\pi, \quad (6.46)$$

when the electric charge is given by

$$q(r)^2 = k^2(e^{\Lambda(r)} - 1)(1 - R^2)r^4/2. \quad (6.47)$$

Thus, as in the previous case, for $q = 0$ we get $k = 0$ which in turn makes mass, pressure and density to vanish and also the space-time becomes flat. Thus, the model presented here is an EMMM.

Subcase (2): In the present case also the central and the boundary pressures are equal with a value $N\Lambda_0/8\pi$ and the respective densities are $(4k^2 - N\Lambda_0)/8\pi$ and $-N\Lambda_0/8\pi$. The pressures become zero for $N = 0$, at the centre and boundary, and densities have the positive central value $k^2/2\pi$ whereas the boundary value is zero. Thus, we again get a physically interesting charged dust case with $N = 0$ which now corresponds to a case of isotropic fluid sphere. Here also the behavior is regular and
well defined.

Subcase (3): In the present case also, by virtue of equations (6.40) and (6.41), \( \rho + p_r \neq 0 \) which reads here as \( \rho + p_r = \frac{k^2(1 - R^2)}{2\pi} \). Here the central value, at \( r = 0 \), is \( \rho = -\bar{p} \) where \( \bar{p} = \left( p - \frac{k^2}{2\pi} \right) \) and the boundary one is \( \rho = -p \). Due to negative value of the density here the condition on the pressure to be imposed is \( p > \frac{k^2}{2\pi} \).

Thus, the present situation, viz. Case II, clearly provides an EMMM even with a positive pressure and therefore contradicts the comment made by Ivanov (2002) that “... electromagnetic mass models all seem to have negative pressure.” The same result, i.e. the positivity of pressures are also available in some cases of the work done by Ray and Das (2002) related to EMMM. However, the explanation given here is valid for any positive value of \( N \) and so the situation could completely be opposite if one assigns on \( N \) any negative value. At this stage, we should not put any restriction on the choice of the value of \( N \). This is because, in general, for a fluid sphere we should have \( p \geq 0 \) and \( \rho \geq 0 \) so that the weak energy conditions are satisfied. But there are also some special situations available within the spherical system (particularly in the case of electron with the radius \( \sim 10^{-16} \)) where the energy condition is violated due to negative energy density (Cooperstock and Rosen 1989; Bonnor and Cooperstock 1989; Ray and Bhadra 2004b). Thus, choosing the proper signatures of \( N \), we can have a class of models with diverse characters.

6.5 Role of \( \Lambda \): Previous and Present Status

The cosmological constant was introduced by Einstein in his field equation to obtain a static cosmological solution because of the fact that due to gravitational pull everything will collapse to a point and hence a un-wanting situation of singularity will take place. However, he was not satisfied with this new physical quantity as it seemed to violate Machian principle which he tried to incorporate in the framework of his General Relativity. He thus, ultimately rejected it mainly for two reasons: (i) that the theoretical work of de Sitter showing that the Einstein’s field equations
admitted a solution for empty Universe and (ii) that the experimental discovery of expanding Universe by Hubble.

As stated in the introduction, the concept of cosmological constant has been revived recently in the case of early Universe scenario and even in particle physics. It is gradually being felt that \( \Lambda \), the erstwhile cosmological constant is available rather than a constant, as was being believed earlier, varying with space or time or both (Tiwari and Ray 1996; Ray, Ray and Tiwari 1993; Tiwari, Ray and Bhadra 2000). Further, this varying \( \Lambda \) may be positive or negative (by imposing the condition that its value is not equal to zero). For instance, according to Zel’dovich (1968) the effective gravitational mass density of the polarized vacuum is negative. Similarly, the equation of state \( \rho + p = 0 \), employed by Tiwari, Rao and Kanakamedala (1984) to construct EMMM as a solution of Einstein-Maxwell field equations, provides negative pressure. It may be emphasized here that positive density has significant, rather major role in inflationary cosmology whereas negative density has influence on elementary particle models. The gravitational mass inside the spherical charged body is negative for \( r < 5a/4 \), where \( r \) is the radial coordinate and \( a \) is the radius of the sphere. It is argued by Grøn (1986a,b) that this negative mass and the associated gravitational repulsion is due to the strain of the vacuum because of vacuum polarization. He also argued that if a vacuum has a vanishing energy, then its gravitational mass will be negative and the observed expansion of the universe may be explained as a result of repulsive gravitation. Now, if we consider a negative \( \Lambda \) having a repulsive nature as was considered by Einstein then this gets the same status of negative pressure and also can be identified with the Poincaré stress. This repulsive gravitation associated with negative \( \Lambda \) can also be explained as the source of gravitational blue shift (Grøn, 1986a). On the contrary, positive \( \Lambda \) will be related to gravitational red shift. It may also be pointed out that according to Ipser and Sikivie (1984) domain walls are sources of repulsive gravitation and a spherical domain wall will collapse. To overcome this situation the charged “bubbles” with negative mass keep the wall static and hence in equilibrium. In this regard, we may also add that \( \Lambda \), via repulsive gravitation, is related to domain walls and playing an
important physical role.

Very recent observations conducted by the SCP and HZT (Perlmutter et al. 1998; Riess et al. 1998; Filippenko 2001; Kastor and Traschen 2002) show that the present value of Λ is positive one and hence related to the repulsive pressure. It is believed that the present state of acceleration dominated Universe is due to the driven force of this Λ. It is, therefore, to be noted that the negative Λ corresponds to a collapsing situation of the Universe (Cardenas et al. 2002).

6.6 Conclusions

(i) In both the above cases I and II, it is possible to show that EMMM can be obtained, in principle, using the constraint ρ + p ≠ 0. This particular point remained unnoticed by Grøn (1986a,b) and Ponce de Leon (1987a,b) both.

(ii) It can be noted that in terms of energy-momentum tensor of the fluid the condition ρ + p = 0 implies $T_{11} = T_{00}^{43}$ whereas ρ + p ≠ 0 constraint may be expressed as $T_{11} = 0$ as we have adopted in the present approach. It is also interesting to note that ρ + p = 0 and hence $T_{11} = T_{00}$ can be expressed in terms of the metric tensors (vide equation (6.1)) as $g_{00}g_{11} = −1$. A coordinate-independent statement of this relation is obtained by Tiwari, Rao and Kanakamedala 1984) by using the eigen values of the Einstein tensor $G^i_j$.

(iii) We would like to mention here that the solutions obtained by Grøn (1986a,b) and Ponce de Leon (1987a,b) represent a neutral system, viz., though the net charge is not zero but the charge on the surface of the spherical system vanishes. The models of the present paper, in general, do not correspond to this situation because of the fact that the electric field and hence cosmological constant does not vanish at the boundary. In both the Case I and Case II, the values of electric field, respectively, are $(n − 1)NΛ_0/2n$ and $−(n − 1)NΛ_0/2$ whereas those for cosmological parameters are $−(n + 1)NΛ_0/2n$ and $(n + 1)NΛ_0/2$. Therefore, the present solutions correspond
to a charged fluid sphere. Of course, for $N = 0$, like Grøn (1986a,b) and Ponce de Leon (1987a,b), we have neutral spheres (equation (6.42) of the Case I and equation (6.47) of the Case II). Thus, we have a class of solutions related to charged as well as neutral systems depending on the values of $N$.

The contents of this chapter has communicated to journal for publication.
Chapter 7

Conclusions

“So we come back again to the original idea of Lorentz – may be all the mass of an electron is purely electromagnetic, may be the whole 0.511 Mev is due to electrodynamics.”

– Feynman et al. (1964)

Electromagnetic mass models which are the sources of purely electromagnetic origin “have not only heuristic flavor associated with the conjecture of Lorentz but even a physics having unconventional yet novel features characterizing their own contributions independent of the rest of the physics” (Tiwari 2001). This is, as Tiwari (2001) guess “may be due to the subtle nature of the mass of the source (being dependent on the electromagnetic field alone)”. Therefore, in our whole attempt we have tried to explore “the subtle nature of the mass of the source”. However, to do this under the general relativistic framework, we have considered Einstein field equations in its general form, i.e., with cosmological constant $\Lambda$ which also acts as a source term to the energy-momentum tensor. If we consider that $\Lambda$ has a variable structure which is dependent on the radial coordinate of the spherical distribution, viz., $\Lambda = \Lambda(r)$ then it can be shown that $\Lambda$ is related to pressure and matter energy
density. Hence it contributes to the effective gravitational mass of the system.

It is seen that equation of state has an important role in connection to electromagnetic mass model. Therefore, at first we have obtained electromagnetic mass model under the condition $\rho + p = 0$. However, later on it is shown that electromagnetic mass model can also be obtained by using more general condition $\rho + p \neq 0$.

The model considered in our work, in general, corresponds to a charged sphere with cosmological parameter in such a way that it does not vanish at the boundary. The idea behind is that the cosmological parameter is related to the zero point vacuum energy it should have some finite non-zero value even at the surface of the bounding system. For this type of spherical system we can have a class of solutions related to charged as well as neutral configurations.

It can be shown that these models have positive energy densities everywhere. Their corresponding radii are always much larger than $10^{-16}$ cm. Furthermore, as the radii of these models shrink to zero, their total gravitational mass becomes infinite.

It have been shown by Bonnor and Cooperstock (1989) that an electron must have a negative energy distribution (at least for some values of the radial coordinate). In this connection we have shown that the cosmological parameter $\Lambda$ has a definite role on the energy density of micro particle, like electron. At an early epoch of the universe when the numerical value of negative $\Lambda$ was higher than that of the energy density $\rho$, the later quantity became a positive one. In the case of decreasing negative value of $\Lambda$ there was a smooth crossover from positive energy density to a negative energy density.

So far we have referred electron to be a spherically symmetric distribution of matter deprived of spin and magnetic moment. As an alternative way both Bonnor and Cooperstock (1989) as well as Herrera and Varela (1994) suggest that both spin and magnetic moment can be introduced at classical level through the Kerr-Newman metric. However in this context it is to be mentioned here that the Kerr-Newman metric cannot be valid for distance scales of the radius of a subatomic particle. We, therefore, thought that the problem can be tackled in the frame work of Einstein-Cartan theory where torsion and spin are inherently present. In this case, the only
way is to take the spin to be the ‘intrinsic angular momentum’ that is the spin of quantum mechanical origin. In our work considering the spins of all the individual particles are assumed to be oriented along the radial axis of the spherical systems we have obtained some interesting solutions with physical validity. However, though our present approach via Einstein-Cartan theory to inject spin may be interesting it, at once, demands some alternative means to provide spin and magnetic moment. This may be possible through Dirac-Maxwell theory where spin and magnetic moment are naturally incorporated through the Dirac spin. We would like to pursue this problem in future investigations.

Another important point we would like to mention here that in all the previous investigations we have studied electromagnetic mass models in 4-dimensional Einstein-Maxwell space-times only. Therefore, one can ask whether electromagnetic mass models also can exist in higher dimensional theory of General Relativity. We have presented a model which corresponds to spherically symmetric gravitational sources of purely electromagnetic origin in the space-time of \((n + 2)\) dimensional theory of general relativity.

We have also taken up the problem of anisotropic fluid sphere as studied earlier in a different view point. By expressing \(\Lambda\) in terms of electric field strength \(E\) we have explored some possibilities to construct electromagnetic mass models using the constraint \(\rho + p \neq 0\). We would like to mention here that unlike the solutions of Grøn (1986a,b) and Ponce de Leon (1987a,b) in the present investigation, in general, the electric field (and hence the cosmological constant) does not vanish at the boundary. However, it is shown that the class of solutions obtained here are related to charged as well as neutral systems of Grøn (1986a,b) and Ponce de Leon (1987a,b) depending on the values of the parameter \(N\).

It is to be mentioned here that other than Dirac-Maxwell theory where spin and magnetic moment are naturally incorporated through the Dirac spin, some other possibilities are awaiting to be investigated under the scheme of electromagnetic mass models. One of such possibilities is to study the relationship between the structures of soliton which have been identified with the electromagnetic field to that of elec-
trons which are also identified with the electromagnetic field (Tiwari 2001). This can be done by the use of Zakharov-Belinsky method to solve the Einstein-Maxwell equations. Another possibility is to conjecture that Weyl line-mass solutions and cosmic strings are identical entities, because it has been shown by Linet (1985) and Hiscock (1985) that the Weyl line-mass solutions can be identified with the cosmic strings. On the other hand Weyl line-mass solutions have been identified with the electromagnetic mass models (Tiwari et al. 1991).
List of Publications

1. R. N. Tiwari, Saibal Ray and Sumana Bhadra, “Relativistic Electromagnetic Mass Models with Cosmological Variable $\Lambda$ in Spherically Symmetric Anisotropic Source”

*Indian Journal of Pure and Applied Mathematics* (2000) 31 1017.

2. Saibal Ray and Sumana Bhadra, “Classical Electron Model with Negative Energy Density in Einstein-Cartan Theory of Gragitation”

*International Journal of Modern Physics D* (2004) 13 555.

3. Saibal Ray and Sumana Bhadra, “Energy Density in General Relativity: a Possible Role for Cosmological Constant”

*Physics Letters A* (2004) 322 150.

4. Saibal Ray, Sumana Bhadra and G. Mohanty, “Relativistic Electromagnetic Mass Models: Charged Dust Distribution in Higher Dimensions”

*Astrophysics and Space Science* (2006) 302 153.

5. Saibal Ray, Sumana Bhadra and G. Mohanty, “Relativistic Anisotropic Charged Fluid Spheres with Varying Cosmological Constant”

*Communicated to journal.*
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