We consider the anisotropic Ising model on the triangular lattice with finite boundaries, and use Kaufman’s spinor method to calculate low-temperature series expansions for the partition function to high order. From these we can obtain 108-term series expansions for the bulk, surface and corner free energies. We extrapolate these to all terms and thereby conjecture the exact results for each.

Our results agree with the exactly known bulk free energy. For the isotropic case, they also agree with Vernier and Jacobsen’s conjecture for the 60° corners, and with Cardy and Peschel’s conformal invariance predictions for the dominant behaviour at criticality.

1 Introduction

Vernier and Jacobsen[1] considered a number of two-dimensional lattice models in statistical mechanics that are “exactly solved” in the sense that their bulk free energies (and where appropriate their order parameters) have been calculated exactly. They developed series expansions of typically twenty–five or so terms for the surface and corner free energies, and from these were able to conjecture the exact forms.

For the square lattice Ising model, the bulk free energy was obtained by Onsager in 1944,[2] and the surface free energy in 1967 by McCoy and Wu.[3, eqn.4.24b][4, p.126, eqn.4.24b] In 2017 the author[5] and Hucht[6, 7] derived the low-temperature form of the corner free energy and showed that Vernier and Jacobsen’s conjectures were indeed correct for all three free energies.

Here we consider the anisotropic ferromagnetic Ising model on the triangular lattice and develop low-temperature ($T < T_c$) series expansions for the ordered phase. The bulk free energy was obtained in 1950 Houtappel and others,[8],[9],[10], [11] and in 1964 quite elegantly by Stephenson.[12] Vernier and Jacobsen[1] were unable to obtain enough terms in their series to reliably conjecture the surface and the 120° corner free energies. We have used the
spinor method of Kaufman,[13] which greatly simplifies the problem. However, unlike ref.[5], we have not solved the problem algebraically, but have obtained the first 108 terms in the series expansions of the surface and various corner free energies of the triangular lattice. When we expand the results as infinite products in powers of the elliptic nome \( p \) that naturally enters the calculation, we observe patterns in the exponents of period 24, and from them extrapolate the full expansion.

We believe our results for the anisotropic surface and corner free energies to be new, as are those for the isotropic surface and 120° corner free energies. We find agreement with previous results, in particular the predictions of conformal invariance for the logarithmic divergence of the corner free energies.

2 The Ising model

Following Vernier and Jacobsen, we first consider the Ising model on the parallelogram of the first figure in Fig.1. This has \( M \) rows, \( N \) columns and \( MN \) sites (including the boundaries and corners). It also has \( 2N \) sites on the upper and lower horizontal boundaries, and \( 2M \) on the two sloping boundaries. On each site \( i \) we place a spin \( \sigma_i \) with value +1 or −1. The partition function is

\[
Z = \sum_{\{\sigma\}} \exp \left[ \sum_{i,j} K_1 \sigma_i \sigma_j + \sum_{i,k} K_2 \sigma_i \sigma_k + \sum_{i,l} K_3 \sigma_i \sigma_l \right],
\]

(2.1)

where the outer sum is over all \( 2^{MN} \) values of spins \( \sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{MN}\} \), the first inner sum is over all adjacent horizontal pairs of sites (i.e. edges) \( i,j \). Similarly, the second sum is over all edges \( i,k \) parallel to the left and right boundaries, and the third over all edges \( i,l \) in the remaining direction. we shall refer to these three types of edges as types 1, 2, 3, respectively.

When \( K_1, K_2, K_3 \) are all large and positive, the largest two contributions to the sum in (2.1) are from the cases when all the spins are equal, either to +1 or to −1. If we define \( \tilde{Z} \) by

\[
\tilde{Z} = 2 e^{M(N-1)K_1+N(M-1)K_2+(M-1)(N-1)K_3} \ ,
\]

(2.2)

then it follows that

\[
\tilde{Z} = 1 + \text{smaller terms} \ ,
\]

(2.3)

where for given \( M,N \) the smaller terms tend to zero as \( K_1, K_2, K_3 \to \infty \).

Considering the effect of changing the sign of just a few of the spins, defining the three Boltzmann weights

\[
z_j = e^{-2K_j} \quad \text{for } j = 1, 2, 3 ,
\]

(2.4)

and expanding in the combined powers of \( z_1, z_2, z_3 \), we find that

\[
\tilde{Z} = 1 + 2z_1 z_2 + 2z_1 z_2 z_3 + 3z_1^2 z_2^2 + (2M-2)z_1 z_2^2 z_3 + (2N-2)z_1^2 z_2 z_3 + 4z_1^2 z_2^2 z_3 + 2z_1^3 z_2^2 z_3^2 + 6z_1^3 z_2^2 z_3^2 + (4M-4)z_1^3 z_2^2 z_3^2 + (4N-4)z_1^3 z_2^2 z_3^2 + (MN-5)z_1^3 z_2^2 z_3^2 + 2z_1 z_2^3 z_3^2 + \cdots
\]

(2.5)
so for the isotropic case \( z_1 = z_2 = z_3 = z \):

\[
\hat{Z} = 1 + 2z^2 + 2z^3 + (2M + 2N - 1)z^4 + 8z^5 + (MN + 4M + 4N - 3)z^6 + O(z^7)
\]

These results are correct to this order \( z^6 \) for \( M, N \geq 4 \). To order \( z^4 \) they are correct for \( M, N \geq 3 \).

Developing these series further, and expanding \( \log \hat{Z} \) rather than \( \hat{Z} \), we find that to all the orders we have calculated, \( \log \hat{Z} \) is linear in \( M \) and \( N \), provided \( M, N \) are sufficiently large. This implies that

\[
\hat{Z} = \kappa_{b}^{MN} \kappa_{s1}^{2N} \kappa_{s2}^{2M} \kappa_{c}^{4}, \quad (2.6)
\]

where \( \kappa_{b}, \kappa_{s1}, \kappa_{s2}, \kappa_{c} \) are independent of \( M, N \). These are the bulk, surface and corner free energies partition functions discussed in [5]. They are the exponential of the free energies discussed by Vernier and Jacobsen.[1]

**Figure 1:** Examples of quadrilateral shapes on the triangular lattice

### 3 Transfer matrices

From now on, we consider \( \hat{Z} \) rather than \( Z \), i.e. we remove the leading factor \( e^{K_i} \) in each edge of type \( i \).

We can construct the partition function of the first lattice in Fig.1 in the usual way, using transfer matrices. We follow the notation of section 2 of [5], except that here we use local transfer matrices that each add only a single edge of the lattice, and we re-arrange the 2\( N \)-dimensional matrices so that rows \( 1, 2, 3, \ldots, N \) become \( 1, 3, 5, \ldots, 2N - 1 \) and rows \( N + 1, N + 2, \ldots, 2N \) become \( 2, 4, \ldots, 2N \), and similarly for the columns.
Let \( \sigma = \{\sigma_1, \ldots, \sigma_N\} \) be the spins on a row of the lattice, and \( \sigma' = \{\sigma'_1, \ldots, \sigma'_N\} \) be the spins on the row above. Then we can define the \( 2^N \)-dimensional row-to-row transfer matrices \( U_{j,j+1}, V_j, W_{j,j+1} \), with elements

\[
(U_{j,j+1})_{\sigma,\sigma'} = e^{K_1(\sigma_j \sigma_{j+1} - 1)} \delta_{\sigma,\sigma'}
\]

\[
(V_j)_{\sigma,\sigma'} = \left[ \delta_{\sigma_j,\sigma'_j} + e^{-2K_2} \delta_{\sigma_j,-\sigma'_j} \right] \prod_{k=1,k\neq j}^{N} \delta_{\sigma_k,\sigma'_k} \tag{3.1}
\]

\[
(W_{j,j+1})_{\sigma,\sigma'} = e^{K_3(\sigma_j \sigma_{j+1} - 1)} \delta_{\sigma,\sigma'}
\]

The matrices \( U_{j,j+1}, W_{j,j+1} \) are diagonal.

Let

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and define \( s_j, c_j \) to be the \( 2^N \)-dimensional matrices

\[
s_j = 1 \otimes \cdots \otimes 1 \otimes s \otimes 1 \otimes \cdots \otimes 1,
\]

\[
c_j = 1 \otimes \cdots \otimes 1 \otimes c \otimes 1 \otimes \cdots \otimes 1,
\]

\( s, c \) on the RHS being in position \( j \). Then

\[
U_{j,j+1} = \exp\{K_1(s_j s_{j+1} - 1)\}, \tag{3.4}
\]

\[
V_j = (1 - e^{-4K_2})^{1/2} e^{K_2 c_j}, \tag{3.5}
\]

\[
W_{j,j+1} = \exp\{K_3(s_j s_{j+1} - 1)\}, \tag{3.6}
\]

where

\[
tanh K_2^* = e^{-2K_2}. \tag{3.7}
\]

The usual row-to-row transfer matrix is

\[
T = U_{1,2} V_1 W_{1,2} \cdots U_{N-1,N} V_{N-1} W_{N-1,N} V_N \tag{3.8}
\]

(each \( U \) adds a horizontal edge in the lower row, each \( V \) a vertical edge, and each \( W \) a slanting edge). Let \( \xi \) be the \( 2^N \)-dimensional vector with all entries unity. Then the partition function is

\[
Z = \xi^T T^M U_1 U_2 \cdots U_N \xi \tag{3.9}
\]

(here the \( U \)'s add the bottom row of the lattice).

As in Kaufman[13] and [5], we can replace the \( 2^N \)-dimensional matrices \( U, V, W \) by \( 2N \)-dimensional ones \( \tilde{U}, \tilde{V}, \tilde{W} \), with elements

\[
(\tilde{U}_{m+1})_{i,j} = (\tilde{V}_m)_{i,j} = (\tilde{W}_{m+1})_{i,j} = \delta_{i,j} \text{ for } i,j = 1, \ldots, 2N, \tag{3.10}
\]

except for the entries

\[
(\tilde{U}_{m+1})_{2m,2m} = (\tilde{U}_{m+1})_{2m+1,2m+1} = \cosh 2K_1,
\]

\[
(\tilde{U}_{m+1})_{2m,2m+1} = -(\tilde{U}_{m+1})_{2m+1,2m} = i \sinh 2K_1,
\]
\[(\bar{V}_m)_{2m-1,2m-1} = (\bar{V}_m)_{2m,2m} = \cosh 2K_2 / \sinh 2K_2 ,\]
\[(\bar{V}_m)_{2m-1,2m} = - (\bar{V}_m)_{2m,2m-1} = -i / \sinh 2K_2 ,\]
\[(\tilde{W}_{m,m+1})_{2m,2m} = (\tilde{W}_{m,m+1})_{2m+1,2m+1} = \cosh 2K_3 ,\]
\[(\tilde{W}_{m,m+1})_{2m,2m+1} = - (\tilde{W}_{m,m+1})_{2m+1,2m} = i \sinh 2K_3 .\]

Thus each of the \(\tilde{U}, \tilde{V}, \tilde{W}\) matrices has just four elements that are not 1 or 0.

If we also define the simple 2N by N matrix \(J\) with entries zero except for

\[J_{2j-1,j} = 1 , \quad J_{2j,j} = i \quad \text{for} \quad j = 1, \ldots, N ,\]

and, analogously to (3.8),

\[\bar{T} = \tilde{U}_{1,2} \tilde{V}_1 \tilde{W}_{1,2} \cdots \tilde{U}_{N-1,N} \tilde{V}_{N-1} \tilde{W}_{N-1,N} \tilde{V}_N , \quad (3.11)\]

then

\[\tilde{Z} = 2^{N/2} e^{-N_1 K_1 - N_3 K_3} (1 - e^{-4K_2})^{N_2/2} (\det Q)^{1/2} , \quad (3.12)\]

where \(N_2\) is the number of type-2 edges (in this case \(N_2 = N(M - 1)\)), and \(Q\) is the \(N\) by \(N\) matrix

\[Q = J^\dagger \tilde{T}^M \tilde{U}_{1,2} \tilde{U}_{2,3} \cdots \tilde{U}_{N-1,N} J \quad , \quad (3.13)\]

\(J^\dagger\) being the hermitian transpose of \(J\).

**Other shapes**

Each of the matrices \(\tilde{U}_{m,m+1}, \tilde{V}_m, \tilde{W}_{m,m+1}\) entering (3.13) via (3.11) can be associated with a particular edge of the first graph in Fig. 1. Other shapes can be obtained by simply deleting the corresponding matrices. For instance the second graph in Fig. 1 can be obtained by deleting the \(\tilde{U}_{N-2,N-1}, \tilde{U}_{N-1,N}\) before the \(J\) in (3.13), the \(\tilde{U}_{N-1,N}, \tilde{V}_{N-1}, \tilde{W}_{N-1,N}, \tilde{V}_N\) in the furthest-right matrix \(\bar{T}\) in (3.13), and the \(\tilde{V}_N\) in the next-furthest-right \(\bar{T}\).

This reduces the number of type-1 edges by 3, of type-2 edges by 3, and of type-3 edges by 1, so we must replace \(N_2\) in (3.12) by \(N(M - 1) - 3\).

4 The isotropic case

We first focus the isotropic case, when \(K_1 = K_2 = K_3\) and \(z_1 = z_2 = z_3 = z\).

We have performed calculations on various convex shaped graphs on the triangular lattice, expanding \(\tilde{Z}\) to orders as high as 108 in \(z\) (or in the elliptic parameter \(p\) defined below). In all cases we find that (2.6) generalizes to

\[\tilde{Z} = 2 (\kappa_b)^{n_b} (\kappa_s)^{n_s} (\kappa_c)^{n_c} (\kappa_e)^{\tilde{n}_c} \quad (4.1)\]

where \(n_b\) is the number of sites in the graph, \(n_s\) is the number of surface sites, \(n_c\) is the number of 60° corners, and \(\tilde{n}_c\) is the number of 120° corners.

When counting these numbers, \(n_b\) includes all sites, including those on the surfaces and corners, and \(n_s\) is the sum of all surface sites, including the adjacent corners. (So any individual surface must have at least two sites.) Hence the number of distinct boundary sites on a graph is 
\[n_s - n_c - \tilde{n}_c.\]
So for the particular graphs in Fig.1 (both with $M = 7$, $N = 5$), for the first graph
\[ \{n_b, n_s, n_c, \tilde{n}_c\} = \{35, 24, 2, 2\} \]
while for the second graph
\[ \{n_b, n_s, n_c, \tilde{n}_c\} = \{32, 23, 1, 4\} \]
and for the two graphs in Fig.2,
\[ \{n_b, n_s, n_c, \tilde{n}_c\} = \{28, 21, 3, 0\} \text{ and } \{37, 24, 0, 6\} \]

We expect (4.1) to be true in the sense that the ratio of the LHS to the RHS $\to 1$ when the area and all surfaces become large. Then this limit defines $\kappa_b, \kappa_s, \kappa_c, \tilde{\kappa}_c$.

**Figure 2:** The regular triangular and hexagonal shapes on the triangular lattice

It is convenient to expand, not in powers of the low-temperature variable $z$, but in the related variable $p$, defined by
\[ z = p \prod_{k=1}^{\infty} \frac{(1 - p^{24k-20})(1 - p^{24k-4})}{(1 - p^{24k-16})(1 - p^{24k-8})} . \] (4.2)

Then from (11.7.14) of [14], the bulk free energy is
\[ \kappa_b = \prod_{k=1}^{\infty} \frac{(1 - p^{24k-12})^2}{(1 - p^{24k-18})(1 - p^{24k-6})} \times \prod_{k=1}^{\infty} \left\{ \frac{(1 - p^{24k-14})}{(1 - p^{24k-10})} \right\}^{6k-3} \times \prod_{k=1}^{\infty} \left\{ \frac{(1 - p^{24k+8})(1 - p^{24k-2})(1 - p^{24k-4})}{(1 - p^{24k-8})(1 - p^{24k+2})(1 - p^{24k+4})} \right\}^{6k} , \] (4.3)
which is the result (56) of [1].

Presumably the surface free energy could be calculated from the eigenvalues of the row-to-row transfer matrix, as has been done for the square lattice,[3],[4],[5] We shall not attempt this here, but merely present our results obtained by extrapolation from series expansions.

The exponents in (4.3) are either constants or linear in \( k \), and have a repeat pattern of period 24. For both the surface and free energies of the parallelogram in the first diagram of Fig. 1, we observe that the corresponding exponents in the product expansions behave similarly (to the order \( p^{108} \) that we calculated), except that some of the corner free energy exponents are quadratic in \( k \). These observations enable us to confidently conjecture that

\[
\kappa_{s_1} = \kappa_{s_2} = \kappa_{s_3} = \prod_{k=1}^{\infty} \left( \frac{1 - p^{12k-2}}{1 - p^{12k+2}} \right)^2 \left( \frac{1 - p^{12k-4}}{1 - p^{12k-8}} \right)^{2k-1}
\]

\[
\times \prod_{k=1}^{\infty} \left( \frac{1 - p^{24k-2})(1 - p^{24k+16})}{(1 - p^{24k+2})(1 - p^{24k-10})} \right)^k \left( \frac{1 - p^{24k-16}}{1 - p^{24k-10}} \right)^{2k-1}
\]

\[
\times \prod_{k=1}^{\infty} \left( \frac{1 - p^{24k+2})(1 - p^{24k+4})(1 - p^{24k-8})}{(1 - p^{24k-2})(1 - p^{24k-4})(1 - p^{24k+8})} \right)^2 \left( \frac{1 - p^{24k-10}}{1 - p^{24k-14}} \right)^{2k-1}
\]

\[
\kappa_c = \prod_{k=1}^{\infty} \frac{(1 - p^{12k-2})^{2k-1}(1 - p^{24k-16})^{5k-3}(1 - p^{24k-4})^{3k}}{(1 - p^{24k-12})^{1/3}(1 - p^{24k-10})^{2k-1}(1 - p^{24k-20})^{3k-3}(1 - p^{24k-8})^{5k-2}}
\]

\[
\tilde{\kappa}_c = \prod_{k=1}^{\infty} \frac{(1 - p^{24k-14})^{1/2}(1 - p^{24k-10})^{1/2}}{(1 - p^{24k-12})^{1/6}}
\]

\[
\times \frac{1}{(1 - p^{12k-9})(1 - p^{12k-7})^2(1 - p^{12k-5})^2(1 - p^{12k-3})(1 - p^{24k-4})^{k^2-2k+2}}
\]

\[
\times \prod_{k=1}^{\infty} \frac{(1 - p^{24k-20})^{k^2-k+2}(1 - p^{24k-12})^{k^2-k-1}(1 - p^{24k-4})^{k^2-k+2}}{(1 - p^{24k-16})^{k^2-1}(1 - p^{24k-8})^{k^2-2k}(1 - p^{24k})^{k^2}}
\]

\[
\times \prod_{k=1}^{\infty} \frac{(1 - p^{24k-18})^{k+2}(1 - p^{24k-10})^{2k}}{(1 - p^{24k-14})^{2k-2}(1 - p^{24k-6})^{k-3}}
\]

These results are true to order \( p^{108} \) for all the four shapes listed here, as well as some others. Our formula for \( \kappa_c \) agrees with the conjecture (60) of [1].

**Summation formulae**

The above products can readily be converted to exponentials of sums by using the general formula, true for all \( r, s, a, b, c \) (\( r \) positive):

\[1\] The \( q \) of Vernier and Jacobsen is our \( p^7 \). We have corrected the exponent in the second product of their formula to \( 6k - 3 \).
\[
\log \prod_{k=1}^{\infty} (1 - p^{r_{k-s}})^{a+bk+ck^2} = \\
- \sum_{m=1}^{\infty} \frac{p^{r_{m}}}{m} \left[ \frac{a}{1 - p^{r_{m}}} + \frac{b - c}{(1 - p^{r_{m}})^2} + \frac{2c}{(1 - p^{r_{m}})^3} \right].
\]

Thus (4.3) becomes

\[
\log \kappa_b = \sum_{m=1}^{\infty} \frac{F(p^{m})}{m}, \quad \text{where} \quad F(p) = \frac{p^6(1 - p^2)^3(1 + p^2)}{(1 + p^{12})(1 - p^2 + p^4)^2}.
\]

Similarly, the logarithms of \(\kappa_s, \kappa_c, \tilde{\kappa}_c\) are given by (4.7), with the function \(F\) replaced by

\[
\log \kappa_s : \quad F(p) = \frac{p^4(1 - p^2)^2(1 - p^4)(1 + p^6 + p^{12})}{(1 + p^{12})(1 - p^2 + p^4)^2(1 - p^4 + p^8)},
\]

\[
\log \kappa_c : \quad F(p) = \frac{p^{12}}{3(1 - p^{24})} + \frac{p^2(1 + p^4)(1 + p^{12})}{(1 + p^4 + p^8)(1 - p^{12})} - \\
\frac{p^8(2 - p^8 + 3p^{12} - p^{16} + 2p^{24})}{(1 + p^8 + p^{16})(1 - p^{24})},
\]

\[
\log \tilde{\kappa}_c : \quad F(p) = \frac{p^{12}}{6(1 - p^{24})} - \frac{p^{10}}{2(1 - p^4)(1 + p^8 + p^{16})} + \\
\frac{p^3}{(1 - p^2)(1 - p^2 + p^4)} - \frac{p^6(3 + 6p^4 + 8p^8 + 9p^{12} + 8p^{16} + 6p^{20} + 3p^{24})}{(1 + p^3)(1 + p^4 + p^8)(1 - p^{24})} - \\
\frac{p^4(2 + 4p^4 + 3p^8 + 3p^{12} + 6p^{16} + 7p^{20} + 6p^{24} + 3p^{28} + 3p^{32} + 4p^{36} + 2p^{40})}{(1 + p^4)^2(1 + p^8 + p^{16})(1 - p^{24})},
\]

In every case, \(F(p^{-1}) = -F(p)\), which means that the sum over \(m\) in (4.7) can be extended from \(-\infty\) to \(\infty\) (excluding zero), which is a valuable check on our conjectures and makes it simple to use a Poisson transformation to expand about the critical point \(p = 1\).

**5 The general anisotropic case**

There are now three types of surfaces: we can say that those lying on edges with interaction coefficients \(K_i\) are of “type \(i\)” and associate with them a surface free energy \(\kappa_{s,i}\).

There are also three types of 60° corners. If the adjoining boundary edges are of types \(i - 1\) and \(i + 1 \pmod{3}\), we also say they are of “type \(i\)” and associate a corner free energy \(\kappa_{c,i}\). Similarly, for 120° corners, we associate a corner free energy \(\tilde{\kappa}_{c,i}\).
If there are \( n_{s,i}, n_{c,i}, \tilde{n}_{c,i} \) surfaces, 60° corners and 120° corners of type \( i \), respectively, then (4.1) generalizes to

\[
\tilde{Z} = 2 (\kappa_b)^{n_b} \prod_{i=1}^{3} (\kappa_{s,i})^{n_{s,i}} (\kappa_{c,i})^{n_{c,i}} (\tilde{\kappa}_{c,i})^{\tilde{n}_{c,i}} \tag{5.1}
\]

These \( \kappa_b, \kappa_{s,i}, \kappa_{c,i}, \tilde{\kappa}_{c,i} \) are the partition functions per site, edge-site, 60°-corner and 120° corner, respectively. We shall occasionally also use the corresponding free energies \( f_b, f_{s,i}, f_{c,i}, \tilde{f}_{c,i} \), defined by

\[
-\beta f_b = \log \kappa_b \ , \quad -\beta f_{s,i} = \log \kappa_{s,i} \ , \quad -\beta f_{c,i} = \log \kappa_{c,i} \ , \quad -\beta \tilde{f}_{s,i} = \log \tilde{\kappa}_{s,i} 
\]

where \( \beta = 1/(k_B T) \), \( k_B \) being Boltzmann’s constant and \( T \) the temperature.

Again, \( n_{s,i} \) is the sum of all sites on surfaces of type \( i \), including the adjacent corners. Hence the number of distinct boundary sites on a graph is \( \sum_{i=1}^{3} (n_{s,i} - n_{c,i} - \tilde{n}_{c,i}) \).

So for the first graph in Fig.1 (with \( M = 7, N = 5 \)),

\[
\{ n_b; n_{s,1}, n_{s,2}, n_{s,3}; n_{c,1}, n_{c,2}, n_{c,3}; \tilde{n}_{c,1}, \tilde{n}_{c,2}, \tilde{n}_{c,3} \} = \{ 35; 10, 14, 0; 0, 0, 2; 0, 0, 2 \}
\]

and for the second graph in Fig.1 and the two graphs in Fig.2, the corresponding numbers are

\[
\{ 32; 8, 12, 3; 0, 0, 1; 1, 1, 2 \} , \quad \{ 28; 7, 7, 7; 1, 1, 1; 0, 0, 0 \} , \quad \{ 37; 8, 8, 8; 0, 0, 0; 2, 2, 2 \}
\]

respectively.

We do now encounter a problem with defining the corner free energies. We can vary \( n_b \) and the \( n_{s,i} \) arbitrarily, so (5.1) does indeed define \( \kappa_b \) and the \( \kappa_{s,i} \). However, we are not free to choose any values of \( n_{c,i} \) and \( \tilde{n}_{c,i} \): there are only 11 distinct possible convex shapes and they all satisfy

\[
n_1 - \tilde{n}_1 = n_2 - \tilde{n}_2 = n_3 - \tilde{n}_3 .
\]

It follows that we can multiply each \( \kappa_{c,i} \) by any function \( \rho_i \), provided we also divide each \( \tilde{\kappa}_{c,i} \) by \( \rho_i \), and that \( \rho_1 \rho_2 \rho_3 = 1 \), i.e we are free to make the transformation

\[
\kappa_{c,i} \to \rho_i \kappa_{c,i} \ , \quad \tilde{\kappa}_{c,i} \to \tilde{\kappa}_{c,i}/\rho_i \ , \quad \rho_1 \rho_2 \rho_3 = 1 . \tag{5.2}
\]

Put another way, the \( \kappa_b, \kappa_{s,i}, \kappa_{c,i}, \tilde{\kappa}_{c,i} \) only enter (5.1) via the combined factors

\[
\kappa_b, \quad \kappa_{s,i}, \quad \kappa_{c,i}, \quad \tilde{\kappa}_{c,i} \quad \kappa_{c,1}, \kappa_{c,2}, \kappa_{c,3}, \quad \tilde{\kappa}_{c,1}, \tilde{\kappa}_{c,2}, \tilde{\kappa}_{c,3} , \tag{5.3}
\]

for \( i = 1, 2, 3 \).

We have made a number of calculations for various convex shapes (up to 108 terms in expansions in powers of \( p \)). They all support the conjecture that (5.1) is true in the limit when each surface is large, the \( \kappa_b, \kappa_{s,i}, \kappa_{c,i}, \tilde{\kappa}_{c,i} \) being independent of the shape.
Elliptic function parametrization

To handle the anisotropic case, it is convenient to introduce two “elliptic-type” functions

\[ G(w, q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n-3}/w)(1 - q^{2n+1}w)}{(1 - q^{2n-3}w)(1 - q^{2n+1}w)} , \tag{5.4} \]

\[ H(w, q) = \prod_{n=1}^{\infty} \left[ \frac{(1 - q^{2n-1}/w)}{(1 - q^{2n+1}w)} \right]^{2n-1} \left[ \frac{(1 - q^{2n}w)}{(1 - q^{2n}w)} \right]^{2n} \tag{5.5} \]

so

\[ \log G(w, q) = \sum_{m=1}^{\infty} \frac{q^m(w^m - w^{-m})}{m(1 + q^{2m})} , \tag{5.6} \]

\[ \log H(w, q) = \sum_{m=1}^{\infty} \frac{q^m(w^m - w^{-m})}{m(1 + q^m)^2} . \tag{5.7} \]

We also introduce parameters \( q, a_1, a_2, a_3 \) such that

\[ a_1 a_2 a_3 = q \tag{5.8} \]

and the \( z_1, z_2, z_3 \) of (2.4) are

\[ z_j = z(a_j, q) = a_j^{1/2} G(a_j, q) . \tag{5.9} \]

For the ferromagnetic case, we must have \( 0 < z_j \leq 1 \), so each of \( a_1, a_2, a_3 \) must be less than one. Then (5.8) implies that for \( j = 1, 2, 3, \)

\[ q < a_j \leq 1 . \tag{5.10} \]

The anisotropic triangular Ising model is discussed in section 11.7 of [14]. Here we write the \( l, K'_1, K'_2, K'_3 \) therein as \( k, K_1, K_2, K_3 \). Then from eq. (11.7.5) therein, with \( v'_j = (1 - z_j)/(1 + z_j) \), we obtain

\[ k^2 = \frac{16 z_1^2 z_2^2 z_3^2}{(1 + z_2 z_3 + z_3 z_1 + z_1 z_2)(1 + z_2 z_3 - z_3 z_1 - z_1 z_2)} \times \]

\[ \frac{1}{(1 - z_2 z_3 + z_3 z_1 - z_1 z_2)(1 - z_2 z_3 - z_3 z_1 + z_1 z_2)} . \tag{5.11} \]

This \( k \) is the elliptic modulus corresponding to the nome \( q \). From (15.1.4) of [14],

\[ k = 4q^{1/2} \prod_{n=1}^{\infty} \left( \frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^4 = \prod_{n=1}^{\infty} \left( \frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^4 , \tag{5.12} \]

where \( q' \) is the nome conjugate to \( q \). If we define \( \lambda, u_1, u_2, u_3 \) so that for \( j = 1, 2, 3, \)

\[ q = e^{-\pi \lambda} , \quad z_j = e^{-\pi u_j} , \tag{5.13} \]

then, using (5.8),

\[ q' = e^{-\pi / \lambda} , \quad u_1 + u_2 + u_3 = \lambda . \tag{5.14} \]

The system is ferromagnetically ordered if \( z_1, z_2, z_3 \) are real and positive and

\[ 0 < k < 1 . \tag{5.15} \]

If \( a_1 = a_2 = a_3 = q^{1/3} \), then we regain the isotropic case, with \( q = p^6 \).
The bulk and surface free energies

Again we can use (11.7.14) of [14] to obtain

$$\kappa_b = \kappa_b(a_1, q) \kappa_b(a_2, q) \kappa_b(a_3, q) \ ,$$  \hspace{1cm} (5.16)

where

$$\kappa_b(a, q) = \prod_{k=1}^{\infty} \left\{ \frac{(1 - q^{4k-2})^{2/3}}{(1 - q^{4k-3})^{1/3}(1 - q^{4k-1})^{1/3}} \left[ \frac{(1 - q^{4k-2}/a)}{(1 - q^{4k-2})} \right]^{2k-1} \right. \\
\left. \left[ \frac{(1 - q^{4k+1})a(1 - q^{4k-1})a(1 - q^{4k})}{(1 - q^{4k-1}/a)(1 - q^{4k+1}/a)(1 - q^{4k})} \right] \right\} ,$$  \hspace{1cm} (5.17)

or equivalently

$$\kappa_b(a, q) = \left( \frac{4q^{1/2}}{k} \right)^{1/12} \left( \frac{G(a, q)}{H(a, q)} \right)^{1/2} \ ,$$ \hspace{1cm} (5.18)

When $$a_1 = a_2 = a_3 = q^{1/3}$$ and $$q = p^6$$, this is the same as (4.3) above.

We could presumably derive at least the surface free energies by generalizing the methods of [5]. We have not done so, but from our series expansions (up to order $$p^{108}$$) we conjecture that

$$\kappa_{s,i} = \kappa_s(a_i|a_{i+1}, a_{i-1}|q) \ ,$$ \hspace{1cm} (5.19)

where

$$\kappa_s(a_1|a_2, a_3|q) = \prod_{k=1}^{\infty} \left\{ \frac{(1 - q^{2k-1/2}/a_1^{1/2})(1 - q^{4k-3}a_1)}{(1 - q^{2k-3/2}a_1^{1/2})(1 - q^{4k-1}/a_1)} \right\}^{2k-1} \left( \frac{1 - q^{2k-1/2}a_1^{1/2}}{1 - q^{2k+1/2}a_1^{1/2}} \right)^{2k} \times \left( \frac{(1 - q^{4k}/a_1)(1 - q^{4k+2}/a_1)}{(1 - q^{4k}a_1)(1 - q^{4k-2}a_1)} \right)^{k/2} \times \left( \frac{(1 - q^{4k-1}/a_2)(1 - q^{4k+1}/a_2)(1 - q^{4k-1}/a_3)(1 - q^{4k+1}/a_3)}{(1 - q^{4k-1}a_2)(1 - q^{4k+1}a_2)(1 - q^{4k-1}a_3)((1 - q^{4k-1}a_3)} \right)^{k/2} \ ,$$ \hspace{1cm} (5.20)

or equivalently

$$\kappa_s(a_1|a_2, a_3|q) = H(a_1^{1/2}/q^{1/2}, q) \left( \frac{H(a_1/q, q)H(a_2, q)H(a_2, q)}{G(a_1/q, q)H(a_1/q, q^2)^2G(a_2, q)G(a_3, q)} \right)^{1/4} .$$

The corner free energies

For the 60° corner free energy we can choose

$$\kappa_{c,i} = \kappa_c(a_i, q) \ ,$$ \hspace{1cm} (5.21)

where

$$\kappa_c(a, q) = \prod_{k=1}^{\infty} \left( \frac{1 - q^{4k-2})^{1/6}}{(1 - q^{4k-2})^{5k-5/2}(1 - q^{4k})^{3k}} \right) \frac{1}{\prod_{m=1}^{\infty} \left( 1 - q^{m-1/2/a^{m-1/2}} \right)} \ .$$ \hspace{1cm} (5.22)
\[ \times \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} \frac{(1 - q^{2k+m/2-1}/a^{m/2})^{4k-2}}{(1 - q^{2k+m-1/2}/a^{m-1/2})^{4k}(1 - q^{4k+m-2}/a^{m})^{10k-5}(1 - q^{4k+m}/a^{m})^{6k}}. \]

The 120° corner free energy is the most cumbersome of all our formulae. With the above choice of \( \kappa_{c,i} \) it is

\[ \vec{\kappa}_{c,i} = \vec{\kappa}_{c}(a_i|a_{i+1},a_{i-1}|q), \]

where

\[ \vec{\kappa}_{c}(a_1|a_2,a_3|q) = P_0 P_1(a_1) P_2(a_1) P_3(a_1) P_4(a_1|a_2,a_3) P_5(a_1|a_2,a_3) \] (5.23)

and, defining

\[ \epsilon_{m,j} = 1/2 \text{ if } m = j \text{, } \epsilon_{m,j} = 1 \text{ if } m > j, \]

\[ P_4(a_1|a_2,a_3) = Q(a_1,a_2)Q(a_1,a_3), \quad P_5(a_1|a_2,a_3) = R(a_1,a_2)R(a_1,a_3), \]

we have

\[ P_0 = \prod_{k=1}^{\infty} (1 - q^{4k-2}), \]

\[ P_1(a) = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} \frac{(1 - a^{m-1/2}q^{2k+m-3/2})^{4k-2}(1 - a^{m-1/2}q^{2k+m-1/2})}{(1 - a^{m-1/2}q^{2k+m-5/2})^{4k-4}(1 - a^{m-1/2}q^{2k+m-2})^2} \]

\[ P_2(a) = \prod_{k=1}^{\infty} \prod_{m=0}^{\infty} \frac{(1 - a^m q^{2k+m-1/2})^{2\epsilon_{m,0}}}{(1 - a^m q^{2k+m-3/2})^{2\epsilon_{m,0}}} \]

\[ P_3(a) = \prod_{k=1}^{\infty} \frac{(1 - q^{2k}/a)^{k/2}}{(1 - q^{4k-1}/a)^k (1 - q^{4k+1}/a)^{k}} \times \frac{(1 - q^{4k-3})^{8k-5}(1 - q^{4k-1})^{8k-2}}{(1 - q^{4k-2})^{9k-7/2}(1 - q^{4k})^{7k}} \times \frac{(1 - a q^{4k-2})^{15k-21/2}(1 - a q^{4k})^{15k-5}}{(1 - a q^{4k-1})^{17k-9}(1 - a q^{4k+1})^{13k}} \times \prod_{m=2}^{\infty} \frac{(1 - a^m q^{4k+m-3})^{16k-11}(1 - a^m q^{4k+m-1})^{16k-5}}{(1 - a^m q^{4k+m})^{14k}(1 - a^m q^{4k+m-2})^{18k-9}}, \] (5.24)

\[ Q(a,b) = \prod_{k=1}^{\infty} \frac{(1 - b^{1/2}q^{2k-3/2})^{2k-1}}{(1 - b^{1/2}q^{2k-1/2})^{2k}} \times \prod_{m=1}^{\infty} \frac{(1 - b^{1/2}a^{m/2}q^{2k+m-3/2})^{(4k-2)\epsilon_{m,1}}(1 - b^{1/2}a^{m/2}q^{2k+m-2})}{(1 - b^{1/2}a^{m/2}q^{2k+m-2/1})^{4k\epsilon_{m,1}}(1 - b^{1/2}a^{m/2}q^{2k+m-2/2})}, \] (5.25)

\[ R(a,b) = \prod_{k=1}^{\infty} \frac{(1 - b q^{4k-4})^{k-1}(1 - b q^{4k-2})^k (1 - b a q^{4k-3})^{3k-2}}{(1 - b q^{4k-3})^{2k-1}(1 - b a q^{4k-2})^{4k-2}} \] (5.26)
\[ \times \frac{(1 - baq^{4k-1})^{3k-1}}{(1 - baq^{4k})^{2k}} \prod_{m=2}^{\infty} \frac{(1 - ba^m q^{4k + m - 1})^{4k - 3}(1 - ba^m q^{4k + m - 2})^{4k - 1}}{(1 - ba^m q^{4k + m - 3})^{6k - 3}(1 - ba^m q^{4k + m - 1})^{2k}}. \]

As discussed above, the choice of these functions \( \kappa_c(a_i), \tilde{\kappa}(a_i|a_{i+1}, a_{i-1}) \) is not unique, though the product \( \kappa_c(a_i) \tilde{\kappa}(a_i|a_{i+1}, a_{i-1}) \) is. We are free to multiply \( \kappa_c(a) \) by \( u(a_i|a_{i+1}, a_{i-1}) \), and divide \( \tilde{\kappa}(a_i|a_{i+1}, a_{i-1}) \) by \( u(a_i|a_{i+1}, a_{i-1}) \), provided only that

\[ u(a_1|a_2, a_3) u(a_2|a_3, a_1) u(a_3|a_1, a_2) = 1. \]

Apart from the square lattice Ising model result and the conjectures of Vernier and Jacobsen[1], the author knows of no argument that corner free energies should have simple product expansions in terms of elliptic variables such as \( a_1, a_2, a_3, q \). However, we note that the second argument \( b \) only enters the factors in the function \( Q \) linearly in \( b^{1/2} \), and \( R \) linearly in \( b \). This is a significant simplification and encourages us to believe in the correctness of these conjectures.

**Summation formulae for the anisotropic case**

As in the previous section, we can convert these products to sums by taking the logarithms, obtaining

\[ \log \kappa_b(a, q) = \sum_{m=1}^{\infty} \frac{F_b(a^m, q^m)}{m}, \quad (5.27) \]

where

\[ F_b(a, q) = \frac{q - q^2}{3(1 + q)(1 + q^2)} + \frac{q(a - 1/a)}{2(1 + q^2)} - \frac{q(a - 1/a)}{2(1 + q)^2}. \quad (5.28) \]

The series \((5.27)\) is convergent provided \( q^2 < |a| < q^{-2} \).

Similarly,

\[ \log \kappa_s(a_1|a_2, a_3|q) = \sum_{m=1}^{\infty} \frac{F_s(a_1^m|a_2^m, a_3^m|q^m)}{m}, \quad (5.29) \]

where

\[ F_s(a_1|a_2, a_3|q) = \frac{q^{1/2}(a_1^{1/2} - q/a_1^{1/2})}{(1 + q)^2} + \frac{a_1 - q^2/a_1}{4(1 + q)^2} - \frac{(1 + q)^2(a_1 - q^2/a_1)}{4(1 + q)^2} + \frac{q(a_2 - 1/a_2 + a_3 - 1/a_3)}{4(1 + q)^2} - \frac{q(a_2 - 1/a_2 + a_3 - 1/a_3)}{4(1 + q^2)}, \quad (5.30) \]

the series \((5.29)\) being convergent if \( q^3 < |a_1| < 1/q \) and \( q^2 < |a_2|, |a_3| < 1/q^2 \).

Also, for \( |a| > q \),

\[ \log \kappa_c(a, q) = \sum_{m=1}^{\infty} \frac{F_c(a^m, q^m)}{m}, \quad (5.31) \]
where
\[
F_c(a, q) = -\frac{q^2}{6(1-q^4)} + \frac{q^{1/2}(1+q^2)}{a^{1/2}(1-q/a)(1+q^2) - q(1+q/a)/(1-q/a)(1+q^2)} + \frac{q^2(1+q/a)}{2(1-q/a)(1+q^2)}. \tag{5.32}
\]

Finally, provided \(|a_1| < q^{-1}\) and \(|a_2|, |a_3| < \min[q^{-1}, q^{-1}|a_1|^{-1}]\),
\[
\log \tilde{\kappa}_c(a_i|a_{i+1}, a_{i-1}|q) = \sum_{m=1}^{\infty} \tilde{F}_c(a_1^m|a_2^m, a_3^m|q^m) , \tag{5.33}
\]
where
\[
\tilde{F}_c(a_1|a_2, a_3|q) = -\frac{q^2}{3(1-q^4)} + 2qa_1^{1/2}(1-q^{1/2}+q) + \frac{q^{1/2}(1+q_a)}{(1+q)(1-q a_1)} + \frac{q^2(1+q a_1)}{2(1+q^2)(1-q a_1)} - \frac{2q(1+qa_1)}{(1+q)^2(1-q a_1)} - \frac{q(1+qa_1)}{2(1+q^2)(1-q a_1)} - \frac{q(1-q a_1)(q+a_1)}{2a_1(1+q)^2(1+q^2)} - \frac{q^{1/2}(a_2^{1/2} + a_3^{1/2})(1-a_1^{1/2})(1-q a_1^{1/2})}{(1+q)^2(1-q^{1/2} a_1^{1/2})} + \frac{q(a_2 + a_3)(1+q+q^2)(1-a_1)(1-q^2 a_1)}{(1+q)^2(1+q^2)(1-q a_1)} . \tag{5.34}
\]
These formulae for \(F_b, F_a, F_c, \tilde{F}_c\) follow from the product forms (5.17) - (5.26). They all have the property [using (5.8)] that they are negated by inverting their arguments. When \(a_1 = a_2 = a_3 = q^{1/3}\) they of course agree with the isotropic formulae (4.7) - (4.10).

We are taking \(0 < q < 1\) throughout this paper, so the products in (5.17) - (5.26) are always convergent. The sums in (5.28) - (5.34) are convergent only if the specified restrictions are satisfied.

### 6 Inversion-type relations

The results of the previous two sections should apply throughout the physical ferromagnetic regime, where \(z_1, z_2, z_3\) are all real, positive, and less than one. This is when \(a_1, a_2, a_3\) are all real and
\[
p^4 < a_j < p^{-2} \quad \text{and} \quad a_1a_2a_3 = 1 . \tag{6.1}
\]
However, our results for \(\kappa_b(a), \ldots, \tilde{\kappa}_c(a_1|a_2, a_3)\) are meromorphic functions of \(a_1, a_2, a_3\) in the complex plane, so can immediately be extended to all complex values. It is these extensions (analytic continuations) that we consider here.

The edge matrices \(U_{j,j+1}, V_j, W_{j,j+1}\) depend on \(K_1, K_2, K_3\), or equivalently \(z_1, z_2, z_3\), respectively. From (3.1),
\[
[U_{j,j+1}(z_1)]^{-1} = U_{j,j+1}(z_1^{-1}) , \quad V_j(z_2)^{-1} = (1-z_2^2)^{-1} V_j(-z_2)
\]
\[
[W_{j,j+1}(z_3)]^{-1} = W_{j,j+1}(z_3^{-1}) . \tag{6.2}
\]

If we write the transfer matrix $T$ as $T(z_1, z_2, z_3)$, then from (3.8), its transposed inverse is
\[
T(z_1, z_2, z_3)^{-1} = (1 - z_2^2)^{-N} T(1/z_1, -z_2, 1/z_3) \quad (6.3)
\]

If $\Lambda(z_1, z_2, z_3)$ is the maximum eigenvalue of $T$ in the physical regime (when $z_1, z_2, z_3$ are real, positive and less than one), then this suggests that the analytic continuation of $\Lambda(z_1, z_2, z_3)$ satisfies
\[
\Lambda(z_1, z_2, z_3) \Lambda(1/z_1, -z_2, 1/z_3) = (1 - z_2^2)^N \quad (6.4)
\]

Since
\[
\Lambda(z_1, z_2, z_3) = \kappa_b^N \kappa_s,2^2 ,
\]
this implies, using the $z_1, z_2 \to z_2, z_1$ symmetry, that
\[
\kappa_b(z_1, z_2, z_3) \kappa_b(1/z_1, -z_2, 1/z_3) = 1 - z_2^2 ,
\]
\[
\kappa_s,1(z_1, z_2, z_3) \kappa_s,1(-z_1, 1/z_2, 1/z_3) = 1 . \quad (6.5)
\]

Using (2.4), we can verify that
\[
z(1/a, q) = 1/z(a, q) \quad , \quad (q^2/a, q) = -z(a, q) . \quad (6.6)
\]

Changing variables from $z_1, z_2, z_3$ to $a_1, a_2, a_3$, from (5.17), (5.19) and (6.5),
\[
\kappa_b(1/a, q) \kappa_b(a, q) = \chi \quad , \quad \kappa_b(q^2/a, q) \kappa_b(a, q) = \frac{1-z(a,q)^2}{\chi^2} , \quad (6.7)
\]
\[
\kappa_s(a_1|a_2,a_3) \kappa_s(q^2/a_1|a_2,1/a_3) = 1 , \quad (6.8)
\]

where $\chi$ is a constant, independent of $a_1, a_2, a_3$, but possibly dependent on $p$. (6.7) and (6.8) are “inversion relations”.[15, 16, 5] For the square lattice, we were able to obtain two more relations (one for $\kappa_{s,i}$ and one for $\kappa_c$), but unfortunately that method breaks down here.\(^2\)

**Observed identities**

Now we use our results of section (5) to see if they do in fact satisfy (6.7), (6.8) and any similar relations.

Define
\[
\eta = \prod_{k=1}^{\infty} (1 - q^{4k-2})^2 (1 - q^{2k-1}) , \quad \mu = \prod_{k=1}^{\infty} (1 - q^{4k-2}) . \quad (6.9)
\]

Using the theory of elliptic functions, we can establish from (4.2) that
\[
1 - z(a, q)^2 = \prod_{k=1}^{\infty} \frac{(1 - q^{4k-2})^4(1 - q^{2k-2}a)(1 - q^{2k}/a)(1 - q^{4k}/a)(1 - q^{4k-1}a)^2(1 - q^{4k-1}/a)^2}{(1 - q^{2k-1})^2(1 - q^{4k-3}a)^2(1 - q^{4k-1}/a)^2} , \quad (6.10)
\]
\[
\log \eta = \sum_{m=1}^{\infty} \frac{q^m}{m(1 + q^m)} - \frac{q^{2m}}{m(1 + q^{2m})} , \quad (6.11)
\]

\(^2\)We cannot write $Z$ in the form (7.3) of [5], the vector $\xi$ being independent of $a_1, a_2, a_3$. 

15
\[ \log \mu = - \sum_{m=1}^{\infty} \frac{q^{2m}}{m(1-q^{4m})}, \]

and
\[ \log[1 - z(a, q)^2] = \sum_{m=1}^{\infty} \left( 2 - \frac{q^m}{a^m} - \frac{a^m}{q^m} \right) \left( \frac{q^m}{m(1 + q^m)} - \frac{q^{2m}}{m(1 + q^{2m})} \right), \] (6.12)

Now look for relations between the \( F \) functions defined above. We find three relations involving \( F_b \) and \( F_s \):
\[ F_b(a, q) + F_b(1/a, q) = \frac{2}{3} \left( \frac{q}{1+q} - \frac{q^2}{1+q^2} \right), \] (6.13)
\[ F_b(a, q) + F_b(q^2/a, q) = \left( \frac{2}{3} - \frac{q}{a} - \frac{a}{q} \right) \left( \frac{q}{1+q} - \frac{q^2}{1+q^2} \right), \] (6.14)
\[ F_s(a_1|a_2, a_3|q) + F_s(q^2/a_1|1/a_2, 1/a_3|q) = 0. \] (6.15)

The corresponding sums in (5.28), (5.29) are convergent in the physical region \( q < a, a_1, a_2, a_3 < 1 \) (and beyond). Together with the relations of section (5), these relations imply that
\[ \kappa_b(a, q) \kappa_b(1/a, q) = \eta^{2/3}, \quad \kappa_b(a, q) \kappa_b(q^2/a, q) = \frac{1 - z(a)^2}{\eta^{4/3}}, \] (6.16)
\[ \kappa_s(a_1|a_2, a_3|q) \kappa_s(q^2/a_1|1/a_2, 1/a_3|q) = 1. \] (6.17)

We see that these are indeed the expected inversion relations (6.7), (6.8), with \( \chi = \eta^{2/3} \).

For the corner free energies, we also note from (5.32) and (5.34) the following two formal identities
\[ F_c(a, q) + F_c(q^2/a, q) = -\frac{q^2}{3(1-q^4)}, \] (6.18)
\[ \bar{F}_c(a_1|a_2, a_3|q) + \bar{F}_c \left( \frac{1}{q^2 a_1 \left| q^2 a_2 \right| q} \right) = -\frac{2q^2}{3(1-q^4)} + \frac{(1-q)(1-qa_1)^2}{2a_1(1+q)(1+q^2)}. \] (6.19)

However, there is no domain within which the corresponding series (5.31), (5.33) are convergent for both terms in (6.18) and (6.19). (The inversion (overlap) points are \( a = q \) and \( a_1 = q^{-1}, a_2 = a_3 = q \), which are at the boundary of the convergence of (5.31), (5.33).)

If one does naively substitute these formulae into the series, one obtains
\[ \kappa_c(a, q) \kappa_c(q^2/a, q) = \mu^{1/3}, \] (6.20)
\[ \bar{\kappa}_c(a_1|a_2, a_3|q) \bar{\kappa}_c \left( \frac{1}{q^2 a_1 \left| q^2 a_2 \right| q} \right) = \frac{\mu^{2/3} z(a_1, q)}{[z(a_1, q)^2 - 1]^{1/2}}, \] (6.21)
for $1 < |a_1| < q^{-1}$, but these equations appear to be meaningless. $\kappa_c$ has an essential singularities at $a = q$, so cannot be analytically continued to smaller values. Similarly, $\tilde{\kappa}_c$ cannot be continued to $a > q^{-1}$.

To use the inversion relation technique given in section 7 of [5], we need two distinct inversion-type relations to define a function, together with its being analytically continuable from one side of each inversion point to the other. Hence (6.16) is sufficient to determine $\kappa_b(a,q)$, but (6.17) is not sufficient to determine $\kappa_s(a_1|a_2,a_3|q)$.

Nor would the problematic equations (6.20), (6.21) be sufficient to determine $\kappa_c(a,q)$ or $\tilde{\kappa}_c(a_1|a_2,a_3|q)$.

7 Behaviour near criticality

We define $\lambda, u_j, q'$ (for $j = 1, 2, 3$) by

$$q = e^{-\pi \lambda}, \quad a_j = e^{-\pi u_j}, \quad q' = e^{-\pi/\lambda},$$

and note from (5.10) that

$$0 \leq u_j < \lambda$$

for $j = 1, 2, 3$.

In eqns. (5.4) - (5.7) we set

$$w = e^{-\pi u}.$$  

The critical case is obtained by taking the limit

$$\lambda, u_j \to 0^+$$

keeping the ratios $u_j/\lambda$ fixed. (For the isotropic case they are 1/3.) Then $q, k \to 1$ and the system becomes critical.

In this limit the above products and sums all converge more and more slowly. However, fortunately we can transform them to forms that converge quickly by using either the conjugate modulus identities of elliptic functions, or more generally by using the Poisson transform[14, eq. 15.8,1].

Poisson transform

If $g(x)$ is analytic on the real axis and its Fourier transform

$$\hat{g}(y) = \int_{-\infty}^{\infty} e^{ixy} g(x) dx$$

exists for all real $y$, then for any positive $\delta$,

$$\sum_{n=\infty}^{\infty} g(n\delta) = \delta^{-1} \sum_{n=\infty}^{\infty} \hat{g}(2\pi n/\delta).$$

All the functions $g(x)$ that we shall deal with are even functions, with $g(-x) = g(x)$, so $\hat{g}(-y) = \hat{g}(y)$ and (7.6) can be written

$$\sum_{n=1}^{\infty} g(n\delta) = -g(0)/2 + \hat{g}(0)/(2\delta) + \delta^{-1} \sum_{n=1}^{\infty} \hat{g}(2\pi n/\delta).$$
The Boltzmann weights $z_1, z_2, z_3$

The function $u^{1/2}G(w, q)$, where $G(w, q)$ is defined by (5.4), is the ratio of two elliptic theta functions. Either using the conjugate function identities of (15.7.1) - (15.7.3) of [14], or using the Poisson transform above, we find that (for all $w, q$)

$$
\log G(w, q) = \frac{\pi u}{2} + \log \tan \left[ \frac{\pi(\lambda - u)}{4\lambda} \right] - 4 \sum_{m \text{ odd}} \frac{(-1)^{(m-1)/2} q^m \sin[\pi mu/(2\lambda)]}{m(1 - q^m)}
$$

where $q'$ is defined by (7.1). The sum is over all odd positive values of $m$, i.e. $m = 1, 3, 5, \ldots$ Hence from (5.9), for $j = 1, 2, 3$,

$$
\log z_j = \log \tan \left[ \frac{\pi(\lambda - u_j)}{4\lambda} \right] - 4 \sum_{m \text{ odd}} \frac{(-1)^{(m-1)/2} q^m \sin[\pi mu_j/(2\lambda)]}{m(1 - q^m)} .
$$

(7.8)

We see that in the critical limit, when $q' \to 0$,

$$
z_j = \tan \left[ \frac{\pi(1 - u_j/\lambda)}{4\lambda} \right] \times \left[ 1 - 4q' \sin(\pi u_j/2\lambda) + O(q'^2) \right] .
$$

(7.9)

So $q'$ is proportional to the deviation from criticality $T_c - T$. We shall regard $q'$ as that deviation, i.e. we take

$$
T_c - T = q' .
$$

(7.10)

The function $H(w, q)$

We shall also need the the critical behaviour of the function $H(w, q)$ defined by (5.7), i.e.

$$
\log H(w, q) = -\frac{\pi \lambda}{4} \sum_{n=1}^{\infty} g(n\pi\lambda/2) ,
$$

(7.11)

where

$$
g(x) = \frac{\sinh(2ux/\lambda)}{x \cosh^2(x)} .
$$

(7.12)

Defining $g(0) = \lim_{x \to \infty} g(x) = 2u/\lambda$, this $g(x)$ is analytic on the real axis and, provided $|u| < \lambda$, we can use the Poisson transform above.

In (7.5), when $y > 0$ we can close the contour of integration round the upper half x-plane. The only singularities within the contour are double poles at

$$
x = \pi m i/2 ,
$$

where $m$ is an odd integer. The associated residue of $e^{ixy}g(x)$ is

$$
R_m(y) = -2i e^{-\pi my/2} \frac{[2S_m - 2\pi mu C_m/\lambda + \pi my S_m]}{\pi^2 m^2} ,
$$

where

$$
S_m = \sin(\pi mu/\lambda) , \quad C_m = \cos(\pi mu/\lambda) .
$$
so for \( y > 0 \),
\[
\hat{g}(y) = 2\pi i \sum_{m \text{ odd}} R_m(y) .
\] (7.13)

Substituting into (7.7) and performing the summation over \( n \), we obtain
\[
\log H(w, q) = \frac{\pi u}{4} - \frac{\hat{g}(0)}{4} - \frac{1}{2} \sum_{n=1}^{\infty} \hat{g}(4n/\lambda) \] (7.14)

\( \kappa_b \) and \( \kappa_s \)

From (15.1.4b) of [14],
\[
k = \prod_{m \text{ odd}} \left( \frac{1 - \frac{q'^2}{m}}{1 + \frac{q'^2}{m}} \right)^4
\] (7.15)

and from the equation before (7.8),
\[
G(w, q) = e^{\pi u/2} f_1(u/\lambda, q') ,
\] (7.16)

where we shall take \( u/\lambda \) to be fixed and the function \( f_1 \) is analytic in \( q' \) at \( q' = 0 \). Also, from (7.14),
\[
H(w, q) = e^{\pi u/4} g_2(u/\lambda, q'^2) \left[ 1 - \frac{8q'^2 \sin(\pi u/\lambda)}{\lambda} + O(q'^4/\lambda) \right] ,
\] (7.17)

where this function \( g_2(u/\lambda, q'^2) \) is analytic in \( q'^2 \) at \( q' = 0 \).

Substituting these formulae into (5.18), we obtain
\[
\kappa_b(e^{-\pi u}, e^{-\pi \lambda}) = 2^{1/6} e^{-\pi(\lambda-3u)/24} \left[ 1 + \frac{4q'^2 \sin(\pi u/\lambda)}{\lambda} + O(q'^4/\lambda) \right] g_3(u/\lambda, q') ,
\] (7.18)

where \( g_3(u/\lambda, q') \) is non-zero and analytic in \( q' \) at \( q' = 0 \).

Now from (5.16),
\[
\kappa_b = \prod_{j=1}^{3} \kappa_b(e^{-\pi u_j}) ,
\] (7.19)

and from (5.8),
\[
u_1 + u_2 + u_3 = \lambda .
\] (7.20)

It follows that the contributions of the factor \( e^{-\pi(\lambda-3u)/24} \) in (7.18) cancel out of (7.19), leaving the dominant singular contribution to the bulk free energy given by
\[
- \beta (f_b)_{\text{sing}} = -\frac{4q'^2 \log q'}{\pi} \sum_{j=1}^{3} \sin(\pi u_j/\lambda) .
\] (7.21)

The surface partition function per site \( \kappa_s \) is conveniently given by the equation after (5.20). Substituting the above forms of \( G(w, q) \) and \( H(w, q) \), we
find, again using (7.20), that the leading factors $e^{\pi uj/2}, e^{\pi uj/4}$ cancel one another out.

The other factors are analytic (Taylor expandable) in $q'$, except for the singular parts of the $H$ functions, given by (7.17). The dominant one of these is the factor coming from $H(a_1, q^2)$, which is (replacing $\lambda, q'$ in (7.17) by $2\lambda, q'^{1/2}$)
\[
1 - 4q' \sin[\pi(1 + u_1/\lambda)/2] + O(q'^2/\lambda)
\]
and gives the dominant singular contribution to the surface free energy:
\[
- \beta (f_{s,i})_{\text{sing}} = - \frac{q' \log q'}{\pi} \sin[\pi(1 + u_1/\lambda)/2] .
\]

$k_c$ and $\tilde{k}_c$

From (5.31) and (5.32),
\[
\log k_c(e^{-\pi u}, e^{-\pi \lambda}) = \sum_{n=1}^{\infty} \tilde{g}(n) ,
\]
where
\[
\tilde{g}(n) = F_c(e^{-\pi un}, e^{-\pi \lambda n})/n .
\]
This function $\tilde{g}(n)$ is even, but it is not analytic at $n = 0$. Rather, it has the expansion about $n = 0$:
\[
\tilde{g}(n) = \frac{5\lambda + u}{24\pi\lambda(\lambda - u)n^2} + \frac{\pi(3u - \lambda)}{72} + O(n^2) ,
\]
so we cannot use the Poisson transform of (7.5) - (7.7).

This difficulty is easily solved by defining
\[
\tilde{g}(n) = \frac{5\lambda + u}{24\pi\lambda(\lambda - u)n^2} + g(n) .
\]

Using the formula
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]
[17, 48.2], we see that
\[
\log k_c(e^{-\pi u}, e^{-\pi \lambda}) = -\beta f_c = \frac{\pi(5\lambda + u)}{144\lambda(\lambda - u)} + \sum_{n=1}^{\infty} g(n)
\]
\[
= - \frac{(5 + u/\lambda) \log q'}{144(1 - u/\lambda)} + \sum_{n=1}^{\infty} g(n) .
\]

Taking $\delta = 1$, the last term on the RHS is again given by (7.7), with $\tilde{g}(0)$ a function only of $u/\lambda$. As $\lambda, u \to 0$, $g(0) \to 0$, $\tilde{g}(0)$ is a constant, and the sum of the RHS of (7.7) tends exponentially to zero. The dominant singularity is

\footnote{Note that because of (7.20), the constant term $\pi(3u - \lambda)/72$ cancels out of the product $k_c, k_c, k_c$.}
therefore given by the first term on the RHS of (7.28) and is proportional to 
$q'$, i.e. to $T_c - T$.

The calculation for $\log \tilde{\kappa}_c$ follows closely that for $\log \kappa_c$. From (5.33) and (5.34),
\[
\log \tilde{\kappa}_c(e^{-\pi u_1} e^{-\pi u_2} e^{-\pi u_3} e^{-\pi \lambda}) = \sum_{n=1}^{\infty} g(n),
\]
where $g(n)$ is the RHS of (5.34), with $a_j = e^{-\pi u_j n}$ and $q = e^{-\pi \lambda n}$, divided by $n$. Again, using (7.20), $g(n)$ is an even function, but has a double pole at $n = 0$:
\[
\tilde{g}(n) = \frac{2\lambda - u_1}{12\pi \lambda (\lambda + u_1)n^2} + \frac{\pi(\lambda - 3u_1)}{72} + O(n^2),
\]

so
\[
\log \tilde{\kappa}_c = -\beta f_c = \frac{(2 - u_1/\lambda) \log q'}{72(1 + u_1/\lambda)} + \sum_{n=1}^{\infty} g(n),
\]
and the dominant singularity is given by the first term on the RHS.

The bulk, surface and corner free energies therefore have singularities proportional to $(T_c - T)^2 \log(T_c - T), (T_c - T) \log(T_c - T)$, and $\log(T_c - T)$, respectively, corresponding to the critical exponents being 2, 1, 0.

We observe the modularity property that the $g(0)$ terms in (7.7) cancel one another out of all the allowed products of the $\kappa$’s in (5.3).

**Predictions of conformal invariance**

For the isotropic triangular lattice, $u = u_1 = \lambda/3$ and (7.28), (7.31) give
\[
-\beta f_c = \frac{1}{18} \log q', \quad -\beta \tilde{f}_c = \frac{5}{288} \log q'.
\]

We see in the next section that we can obtain the square lattice from the triangular by setting $K_3 = 0$. This turns off the $K_3$ interactions. This square lattice model is isotropic if $K_1 = K_2$, i.e. if $a_1 = a_2 = q^{1/2}$, $a_3 = 1$ and hence $u_1 = u_2 = \lambda/2$, $u_3 = 0$. The corners are now of type 3, so we should replace $u$ and $u_1$ in (7.28), (7.31) by $u_3 = 0$. Taking the arithmetic mean of those equations to remove the ambiguity discussed at (5.2), we obtain
\[
-\beta f_{c, sq} = \frac{1}{2} \left( \frac{5}{144} + \frac{1}{36} \right) \log q' = \frac{1}{32} \log q'.
\]

These $f_c, \tilde{f}_c, f_{c, sq}$ are the free energies of the corners of the isotropic $60^\circ$ triangular, $120^\circ$ triangular, and $90^\circ$ square lattices, respectively. In 1988 Cardy and Peschel[18, eqn. 4.4] predicted that at the critical temperature the divergent logarithmic term in the corner free energy of any planar isotropic model of size $L$ would be
\[
\Delta F' = \frac{c}{24} \left( \frac{\gamma}{\pi} - \frac{\pi}{\gamma} \right) \log L.
\]
where $\gamma$ is the internal angle of the corner and $c$ is the conformal anomaly number. For the planar spin 1/2 XY-model and the Ising model, $c = 1/2$.\[19\]

Taking $\gamma = \pi/3, 2\pi/3$ and $\pi/2$ in this formula, we obtain agreement with the three equations in (7.32), (7.33), provided we replace $-\log q'$ by $\log L$.\[4\]

8 Comparison with the square lattice

The square lattice may be regarded as a special case of the triangular lattice, when one of the three interactions is turned off. If we take

$$a_3 = 1,$$

then from (5.4) and (5.9), (2.4)

$$z_3 = 1, \quad K_3 = 0.$$  

This is equivalent to removing the SE-NW lines in the first diagram in Fig. 1, so it just becomes a tilted version of the square lattice we discussed in [5]. The $K_1, K_2$ of this paper are the $H', H$ of [5], and from eqns. (2.7), (2.23), (6.8), (6.26) therein,

$$e^{-2H'} = \frac{q^{1/2}}{w} G(q/w^2, q), \quad e^{-2H} = w G(w^2, q),$$

where $G(a, q)$ is defined by (5.4) above.

Comparing these equations with (5.9) above, the $w$ of [5] is given in terms of our $a_1, a_2$ by

$$w = a_2^{1/2} = (q/a_1)^{1/2}$$

and we can compare our results for $a_1 = q/w^2, a_2 = w^2, a_3 = 1$ with eqn (6.36) of [5].

From (5.28) above,

$$F_b(q/w^2, q) + F_b(w^2, q) + F_b(1, q) = \frac{q(1-q)(w-q/w)(w^{-1}-w)}{(1+q)^2(1+q^2)},$$

so from (5.16) and (5.27),

$$- \beta f_b = \log \kappa_b = \sum_{m=1}^{\infty} \frac{q^m(1-q^m)(w^m - q^m/w^m)(w^{-m}-w^{-m})}{m(1+q^m)^2(1+q^{2m})}. \quad (8.4)$$

Allowing for the fact that we are working with the partition function $\hat{Z}$, whereas [5] works with the full partition function $Z$, related by (2.2), this is the same as the first of the equations (6.36) of [5].

From (5.30),

$$F_s(q/w^2|w^2, 1|q) = \frac{q(w^{-1}-w)}{(1+q)^2} - \frac{q^2(w^{-2}-w^2)}{2(1+q^2)^2}, \quad (8.5)$$

\[4\] See equations 93 and 94 of Vernier and Jacobsen.[1]
so

\[ \log \kappa_{s,1} = \log \kappa_s(a_1, a_2, a_3|q) = \sum_{m=1}^{\infty} \frac{q^m (w^m - w^{2m})}{m(1 + q^m)^2} - \sum_{m=1}^{\infty} \frac{q^m (w^{2m} - w^{2m})}{2m(1 + q^{2m})^2} \]

\[ = \sum_{m=1}^{\infty} \frac{q^m (w^m - w^{2m})}{m(1 + q^m)^2} - \sum_{m \text{ even}} \frac{q^m (w^{2m} - w^{2m})}{m(1 + q^m)^2} \]

\[ = \sum_{m \text{ odd}} \frac{q^m (w^m - w^{2m})}{m(1 + q^m)^2} . \quad (8.6) \]

From the first diagram of Fig. 1, we see that \( n_{c,1} = 2N \), so comparing this with (1.1) of [5], our \( \log \kappa_{s,1} \) should be one-half of the \( H' - \beta f' \) therein. From the third equation of (6.36), we see that this is indeed so.

The equivalence of our result for \( \log \kappa_{s,2} \) with the \( H' - \beta f' \) of [5] follows immediately by interchanging \( a_1 \) with \( a_2 \) and \( w \) with \( q/2w \).

Finally, we consider the corner free energies in the square lattice case. From the first diagram in Fig. 1, all four corners are of type 3, so to compare our results with those of [5] we must now take \( a_1, a_2, a_3 = q/w^2, w^2, 1 \). Defining

\[ F_{c, sq}(q, w) = F_c(a_3, q) + \tilde{F}_c(a_3|a_1, a_2|q) \]

then

\[ F_{c, sq}(q, w) = F(q) - F(q^2)/2 , \quad (8.7) \]

where

\[ F(q) = \frac{2q^{1/2}(1 + q^2)}{(1 + q)(1 - q^2)} - q/(1 - q^2) . \quad (8.8) \]

We note that \( F_{c, sq}(q, w) \) is independent of \( w \). From (5.31) and (5.33), proceeding similarly to (8.6),

\[ \log |\kappa_c(a_1, q) \tilde{\kappa}_c(a_3|a_1, a_2|q)| = \sum_{m=1}^{\infty} \frac{F_{c, sq}(q^m, w^m)}{m} \]

\[ = \sum_{m=1}^{\infty} \frac{F(q^m)}{m} - \sum_{m=1}^{\infty} \frac{F(q^{2m})}{2m} = \sum_{m \text{ odd}} \frac{F(q^m)}{m} \]

\[ = \log k' + 2 \sum_{m \text{ odd}} \frac{q^m (1 + q^{2m})}{m(1 + q^m)(1 - q^{2m})} , \quad (8.10) \]

where we have used the relation

\[ \log k' = -8 \sum_{m \text{ odd}} \frac{q^m}{m(1 - q^{2m})} . \]

Allowing for the fact that (8.10) is the free energy for two corners, of 60° and 120°, whereas the RHS of (6.36) of [5] is the sum of all the four corners in Fig. 1 and includes the logarithm of the factor 2 in 2.2, (8.10) agrees with (6.36) of [5].
9 Summary

We have used Kaufman’s spinor method to simplify the exact calculation of the partition function $Z$ of the ferromagnetically ordered Ising model for an arbitrary convex polygon drawn on the triangular lattice. From this we have been able to calculate series expansions (to 108 terms) of the bulk, surface and the two corner free energies. We observe repeat patterns of period 24, enabling us to extrapolate and thereby conjecture the exact results. Our result for the bulk free energy agrees with the known result of Houtappel and others.[8]-[12]

We first consider the low-temperature isotropic case, when $T < T_c$ and all edges of the lattice have interaction coefficient $K$ and Boltzmann weight $z = e^{-2K}$. We conjecture the surface and corner free energies by extrapolation of the series. Our conjectures agree with that for the $60^\circ$ corners of Vernier and Jacobsen.[1]

We then consider the more general case when the three types of edges have different interaction coefficients $K_1, K_2, K_3$ and again $T < T_c$. The results are then more complicated (particularly so for the $120^\circ$ corner free energy), but we do see sufficient patterns in the series to confidently predict the the free energies. They agree with those for the isotropic case.

In all the cases we have studied, we find the general formula (5.1) holds, with $\kappa_b, \kappa_{b,i}, \kappa_{c,i}, \kappa_{c,i}$ given by (5.16), (5.17), (5.20) and (5.21) - (5.26).

As a further check on our conjectures, we find that if we take the third interaction coefficient $K_3$ to be zero, and hence $a_3 = 1, u_3 = 0$, we do indeed regain the exactly known square lattice results of [5].

We obtain the critical behaviour of the bulk, surface and the two corner free energies, and find logarithmic singularities corresponding to the exponents $\alpha$ being 2, 1, 0, 0, respectively, in agreement with the square lattice results. The results for the corner free energies on the isotropic triangular and square lattices agree with the predictions of conformal invariance[18, eqn. 4.4] and later numerical results.[20, 21]

10 Acknowledgement

The author is grateful to Jesper Jacobsen for helpful comments on the relation of these results to Cardy’s predictions from conformal invariance, and to Jacques Perk for pointing out ref.[20]

References

[1] Vernier E and Jacobsen J L 2012 J. Phys. A: Math. Theor. 45 045003 (41 pages)
[2] Onsager L 1944 Crystal statistics. I. A two-dimensional model with an order-disorder transition Phys. Rev 65 117–149
[3] McCoy B M, Wu T T 1967 Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model. IV Phys. Rev. 162 436 – 469
[4] McCoy B M, Wu T T 1973 The two-dimensional Ising model (Harvard University Press, Cambridge, Mass., reprinted Dover 2014, NY)
[5] Baxter R J 2017 The bulk, surface and corner free energies of the square-lattice Ising model. J. Phys. A: Math. Theor. 50 014001 (40 pages)
[6] Hucht A 2017 The square lattice Ising model on the rectangle I: finite systems J. Phys. A: Math. Theor. 50 065201 (23 pages)
[7] Hucht A 2017 The square lattice Ising model on the rectangle II: finite-size scaling limit J. Phys. A: Math. Theor. 50 265205 (23 pages)
[8] Houtappel R M F 1950 Order-disorder in hexagonal lattices.
[9] Wannier G H 1950 Antiferromagnetism. The triangular Ising net. Phys. Rev. 79 357 – 364
[10] Husimi K, Syozi I 1950 The statistics of honeycomb lattice. I Progr. Theoret. Phys. 5 177 – 186
[11] Syozi I 1950 The statistics of honeycomb lattice. II Progr. Theoret. Phys. 5 341–351
[12] Stephenson J 1964 Ising model on the triangular lattice J. Math. Phys. 5 1009 –1024
[13] Kaufman B 1949 Crystal statistics. II. partition function evaluated by spinor analysis Phys. Rev 76 1232–1243
[14] Baxter R J 1982 Exactly solved models in statistical mechanics, (Academic, London, re-printed 1989; Dover N.Y. 2007)
[15] Stroganov Yu G 1979 A new calculation method for partition functions in some lattice models Phys. Lett. 74A 116 – 118
[16] Baxter R J 1982 The inversion relation for some two-dimensional exactly solved models in lattice statistics J. Stat. Phys. 28 1–41
[17] Dwight H B 1957 Tables of Integrals and other mathematical data, (Macmillan, New York)
[18] internal energCardy J L and Peschel I 1988 Finite-size dependence of the free energy in two-dimensional critical systems J. Nucl. Phys. B 300 377–392
[19] Blöte H W J, Cardy J L and Nightingale M P 1986 Conformal invariance, the central charge, and universal finite-size amplitudes at criticality Phys. Rev. Lett. 56 742–745
[20] Wu X, Izmailan N and Guo W 2013 Shape-dependent finite-size effect of the critical two-dimensional Ising model on a triangular lattice Phys. Rev. E 87 022124 (7 pages)
[21] Wu X and Izmailan N 2015 Critical 2-dimensional Ising model with free, fixed ferromagnetic, fixed anti-ferromagnetic and double anti-ferromagnetic boundaries Phys. Rev. E 91 012102 (9 pages)