Differential equations from null vectors of the Ramond algebra

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Abstract. We consider chiral blocks of four Ramond fields of the $N=1$ super Virasoro algebra where one of the fields is in the (1,2) representation. We show how the null vector in the (1,2) representation determines the chiral blocks as series expansions. We then turn to the Ising model to find an algebraic method to determine differential equations for the blocks of four spin fields. Extending these ideas to the super Virasoro case, we find a first order differential equation for blocks of four Ramond fields. We are able to find exact solutions in many cases. We compare our blocks with results known from other methods.
1. Introduction

Conformal field theories in two dimensions can be used to describe string theory from the worldsheet perspective and statistical systems at a second order phase transition. The study of these field theories is often manageable because the infinite dimensional symmetry algebras which exist can reduce the field content to a finite number of representations. In such minimal cases, there are ‘null vectors’ which should decouple from all correlation functions. As a result the non-zero correlation functions satisfy differential equations which enable one to solve the theory completely.

Each symmetry algebra has its own series of minimal models for which this procedure works. The simplest case is that of minimal models of the Virasoro algebra. In [1], Belavin et al showed how to relate correlation functions of arbitrary fields in a Virasoro minimal model to those of a particular class of fields, called primary fields, and in turn showed how differential equations for four-point functions of primary fields could be found from the singular vectors that are present in these models.

The Virasoro algebra can be extended by extra generators to include supersymmetry, and the simplest of these extensions is the $N=1$ superconformal algebra. Although the $N=1$ superconformal algebra includes the Virasoro algebra, there is an infinite set of minimal models of the $N=1$ superconformal algebra which only includes a few minimal models of the Virasoro algebra, and the methods of [1] need to be generalised to find the differential equations satisfied by the correlation functions of primary fields of the $N=1$ superconformal algebra.

The first thing to note is that the superconformal algebra comes in two forms, called the Neveu-Schwarz (NS) and Ramond (R) algebras, and accordingly has two classes of representations and fields associated to these representations. The differential equations for correlations functions of four NS–type fields were worked out in [2, 3], but the extension to correlation functions of four Ramond fields has problems [4, 5] and the solution to these problems presented here requires an extension of ideas in [2, 5]. In this paper we perform this extension and find differential equations for correlation functions of four R–type fields.

The structure of the paper is as follows: we first introduce the NS and R algebras and describe their representations and singular vectors. We discuss their three point functions and describe the chiral blocks and how the singular vectors allow them to be found level-by-level. Next we describe the toy model of the Ising model and show how one can obtain a differential equation for the four-spin field correlation function, as in [5]. Next we apply these ideas to the Ramond correlation functions. This lets us write down a matrix-differential equation for the correlation functions. We check these by comparison with known results and present the exact solution for a variety of cases.
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2. Representation theory

The algebra of chiral superconformal transformations in the plane is generated by two fields, \( L(z) = \sum_m L_m z^{-m-2} \) and \( G(z) = \sum_m G_m z^{-m-3/2} \). According to the choice of boundary conditions on \( G(z) \), one may choose the labels \( m \) on \( G_m \) to be integer or half-integer; the algebra in these two sectors are called the Ramond and Neveu Schwarz algebras respectively. In both sectors the generators obey

\[
[L_m, L_n] = \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} + (m-n)L_{m+n},
\]

\[
\{G_m, G_n\} = \frac{c}{3} (m^2 - \frac{1}{4}) \delta_{m+n,0} + 2L_{m+n},
\]

\[
[L_m, G_n] = \left(\frac{m^2}{2} - n\right)G_{m+n}.
\]

It is usual to adjoin the operator \((-1)^F\) to the superconformal algebra, where \( F \) is the fermion number operator which anti-commutes with \( G_m \) and commutes with \( L_m \). This allows one to consider states of definite parity and is essential in many constructions. In a superconformal field theory the physically relevant representations of the superconformal algebra are irreducible highest weight representations. These are graded by \( L_0 \) eigenvalue, or level. See [2, 9] for more details.

2.1. NS representations

Highest weight representations of the Neveu-Schwarz algebra have a state of least \( L_0 \) eigenvalue \(|h\rangle\) such that

\[
L_m|h\rangle = h\delta_{m,0}|h\rangle, \quad m \geq 0,
\]

\[
G_m|h\rangle = 0, \quad m \geq 1/2.
\]

If one considers the extended algebra then it is usual to take the highest weight state bosonic \((-1)^F|h\rangle = |h\rangle\) or fermionic \((-1)^F|h\rangle = -|h\rangle\) in which case the addition of the fermion number operator does not alter the representation theory. It is useful to parametrise \( h \) and \( c \) as

\[
c(t) = 15/2 - 3/t - 3t,
\]

\[
h_{p,q}(t) = (1 - pq)/4 + (q^2 - 1)t/8 + (p^2 - 1)/(8t),
\]

since there is a singular vector of \( L_0 \) eigenvalue \( h + pq/2 \) whenever \( h = h_{p,q}(t) \) and \( c = c(t) \) and \( p, q \in \mathbb{Z}, \ p + q \in 2\mathbb{Z} \) (see refs. [6, 7, 8]). Note that the highest weight state can be either bosonic or fermionic, for example, the NS vacuum state \(|0\rangle\) is usually regarded as bosonic and so the NS highest weight state \( G_{-1/2}|0\rangle \) is fermionic. The highest weight representation \( \mathcal{H}_h \) is spanned by states of the form

\[
L_{i_1} \ldots L_{i_m} G_{j_1} \ldots G_{j_n} |h\rangle,
\]

and the level of such a state is \((\sum i_p + \sum j_q)\). States with even numbers of modes of \( G \) have integer level and states with odd numbers of modes have half-integer level and hence \( \mathcal{H}_h = \mathcal{H}^\text{even}_h + \mathcal{H}^\text{odd}_h \) where \( \mathcal{H}^\text{even}_h \) is spanned by integer level states and \( \mathcal{H}^\text{odd}_h \) by half-integer level states.
2.2. R representations

Highest weight representations of the extended Ramond algebra have, in general, a two-dimensional highest weight space spanned by states $|\lambda^\pm\rangle$ of definite fermion parity $\pm 1$ satisfying

\begin{align}
L_m |\lambda^\pm\rangle &= h_\lambda \delta_{m,0} |\lambda^\pm\rangle, \quad m \geq 0, \\
G_m |\lambda^\pm\rangle &= 0, \quad m > 0, \\
G_0 |\lambda^\pm\rangle &= b_{\lambda}^\pm |\lambda^\mp\rangle,
\end{align}

where $h_\lambda = \lambda^2 + c/24$ and $b_\lambda^\pm = y_\lambda$, $b_\lambda^\mp = \lambda/y$ and $y$ can be chosen freely; representations with different $y$ are equivalent. Such a representation has two singular vectors at level $pq/2$ whenever $p, q \in \mathbb{Z}$, $p - q$ an odd integer, $c = c(t)$ and $\lambda = \lambda_{p,q}(t)$,

$$\lambda_{p,q}(t) = \frac{p - qt}{2\sqrt{2t}}.$$  \hfill (10)

We shall mostly be concerned with the constraints arising from the vanishing of the singular vectors at level 1 in the representation $(1, 2)$ which take the form

$$(L_{-1} + \frac{2t}{1 - 2t} G_{-1} G_0) |\lambda^\pm_{1,2}\rangle = 0.$$ \hfill (11)

3. Ramond chiral blocks

We would like to find the chiral blocks of four Ramond fields

$$F(z) = \langle \lambda^\pm_{\infty} | \phi_{\lambda}^\pm (1) \phi_{\lambda}^\mp (z) | \lambda^\mp \rangle,$$  \hfill (12)

where the intermediate states lie in a particular NS representation. We can do this step-by-step by first calculating the operator product expansions (opes)

$$\phi_{\lambda}^\pm (z) |\mu\rangle,$$ \hfill (13)

and then forming the chiral block by summing over intermediate states. To do this we need to define the vertex operators $\phi_{\lambda}^\pm (z)$ of fixed fermion parity and we also need to specify whether the NS representation in the intermediate channel is bosonic or fermionic.

For the field operators, we shall use the definition in [2]:

$$K_m^+(w) \phi_{\lambda}^\pm (w) \pm i \phi_{\lambda}^\pm (w) K_{m-1/2}(w) w^{1/2} = a_{\lambda}^\pm w^m \phi_{\lambda}^\mp (w),$$ \hfill (14)

where the combinations $K_m^+(w)$ and $K_m^-(w)$ are

$$K_m^+(w) = \oint_{0,|z|>|w|} G(z) z^m \sqrt{z - w} \frac{dz}{2\pi i} = G_m - \frac{w}{2} G_{m-1} - \frac{w^2}{8} G_{m-2} + \ldots,$$

$$K_m^-(w) = \oint_{0,|w|>|z|} G(z) z^m \sqrt{w - z} \frac{dz}{2\pi i} = G_m - \frac{1}{2w} G_{m+1} - \frac{1}{8w^2} G_{m+2} + \ldots$$  \hfill (15)

$a_{\lambda}^+ = \lambda x$, $a_{\lambda}^- = \lambda/x$ and $m$ can be integral or half-integral as circumstances dictate (here $a^\pm$ play the same role as $b^\pm$ in [9] and $x$ as $y$).
It turns out that the four opes in (13) depend on only two quantities. It is convenient to form the following four linear combinations:

\[
F^\pm_{\lambda\mu}(z) = \phi^\pm_{\lambda}(z) |\mu^\pm\rangle \pm \text{i}xy\phi^\pm_{\lambda}(z) |\mu^\mp\rangle,
\]

\[
G^\pm_{\lambda\mu}(z) = x\phi^\pm_{\lambda}(z) |\mu^\pm\rangle \pm \text{i}y\phi^\pm_{\lambda}(z) |\mu^\mp\rangle,
\]

(16)

We shall first consider the case where the NS highest weight state is bosonic and the opes depend on the two constants \( f^\pm_{\lambda\mu} = \langle h|F^\pm_{\lambda\mu}(1) \). Using (15) and the standard result for the Virasoro algebra (see (27)), we can calculate the overlaps of \( F^\pm_{\lambda\mu}(z) \) and \( G^\pm_{\lambda\mu}(z) \).

We list here the first few, taking \( z = 1 \) for convenience:

\[
\langle h|G_1/2G^\pm_{\lambda\mu}(1) = (\lambda \pm \mu) f^\pm_{\lambda\mu},
\]

\[
\langle h|L_1F^\pm_{\lambda\mu}(1) = (h + h_\lambda - h_\mu) f^\pm_{\lambda\mu},
\]

\[
\langle h|G_{3/2}G^\pm_{\lambda\mu}(1) = \frac{1}{2}(3\lambda \pm \mu) f^\pm_{\lambda\mu},
\]

\[
\langle h|L_1G_{1/2}G^\pm_{\lambda\mu}(1) = (\lambda \pm \mu)(h + \lambda^2 - \mu^2 + \frac{1}{2}) f^\pm_{\lambda\mu}.
\]

If we take the NS representation to be fermionic rather than bosonic, then \( F^\pm \) and \( G^\mp \) just swap roles in these equations, so without loss of generality we consider henceforth the intermediate channel to be bosonic. Given these combinations \( F^\pm, G^\pm \), it is possible to define the chiral blocks and calculate them to be

\[
\lambda_1, 1 \lambda_2, z \quad \lambda_\infty \quad h^{\text{even}} \quad \epsilon \quad \epsilon' \quad \lambda_0 = F^\pm_{\lambda_1\lambda_\infty}(1) F^\prime_{\lambda_2\lambda_0}(z)
\]

\[
= z^{h-h_\lambda-h_\mu} \left\{ 1 + z \frac{(h+h_\lambda-h_0)(h_\lambda+h_\mu-h_\infty)}{2h} + \ldots \right\},
\]

(18)

\[
\lambda_1, 1 \lambda_2, z \quad \lambda_\infty \quad h^{\text{odd}} \quad \epsilon \quad \epsilon' \quad \lambda_0 = G^\pm_{\lambda_1\lambda_\infty}(1) G^\prime_{\lambda_2\lambda_0}(z)
\]

\[
= z^{h-h_\lambda-h_0+1/2} \left\{ \frac{(\lambda_\infty + \epsilon'\lambda_0)(\lambda_1 + \epsilon\lambda_\infty)}{2h} + \ldots \right\},
\]

(19)

where ‘even’ denotes projection of the intermediate states onto \( H^\text{even}_h \) and ‘odd’, projection onto \( H^\text{odd}_h \), where we again note that we have assumed that \( |h\rangle \) is bosonic.

4. Singular vector decoupling

The vanishing of the state (11) imposes constraints on the allowed fusions of the fields \( \phi^\pm_{1,2} \) and in fact allows one to determine the operator products \( F^\pm_{\lambda_1,2}(z) \) and \( G^\pm_{\lambda_1,2}(z) \) recursively. We take \( \lambda = \lambda_1, 1 \) as this simplifies the constraints on \( h \), and we shall denote \( F^\pm_{\lambda_1,1,2}(1) \) by \( F^\pm \) and \( G^\pm_{\lambda_1,1,2}(1) \) by \( G^\pm \) for simplicity.

If we use the equation (valid for any Virasoro primary field of weight \( h \))

\[
\phi_h(1)(L_{-1} - L_0) = (L_{-1} - L_0 + h)\phi_h(1),
\]

(20)
and the relations (23), we find that $\mathcal{F}^\pm$ and $\mathcal{G}^\pm$ satisfy

\begin{align}
(h_{1,s\pm1} - L_0 + L_{-1}) \mathcal{F}^\pm &= \mp \sqrt{\frac{L}{2}} K_{1/2}^\pm \mathcal{G}^\pm, \\
(h_{1,s\mp1} - L_0 + L_{-1}) \mathcal{G}^\pm &= \pm \sqrt{\frac{L}{2}} K_{1/2}^\pm \mathcal{F}^\pm,
\end{align}

where we assume the representation $h$ to be bosonic. Taking $h$ fermionic simply swaps $\mathcal{F}^\pm$ with $\mathcal{G}^\pm$. Considering the contribution of the highest weight state $|h\rangle$ to $\mathcal{F}^\pm$, we see that

\begin{equation}
(h_{1,s\pm1} - h) \langle h | \mathcal{F}^\pm = 0, \tag{23}
\end{equation}

so that $\mathcal{F}^\pm$ is only non-zero if $h = h_{1,s\pm1}$. The two allowed fusions and the operator products are thus

\begin{align}
&+ \text{ channel:} \quad (1, 2) \times (1, s) \longrightarrow (1, s+1) \\
\mathcal{F}^+ &= |h\rangle + \frac{s+1}{s} L_{-1}|h\rangle + \ldots \\
\mathcal{G}^+ &= -\frac{1}{s} \sqrt{\frac{2}{L}} G_{-1/2}|h\rangle + \sqrt{\frac{L}{2}} \left( G_{-3/2} + \frac{2(t-2+st)}{st} L_{-1} G_{-1/2} \right) |h\rangle + \ldots
\end{align}

\begin{align}
&- \text{ channel:} \quad (1, 2) \times (1, s) \longrightarrow (1, s-1) \\
\mathcal{F}^- &= |h\rangle + \frac{s+t-2}{st-2} L_{-1}|h\rangle + \ldots \\
\mathcal{G}^- &= -\frac{\sqrt{2t}}{2st-2} G_{-1/2}|h\rangle + \sqrt{\frac{L}{2}} \left( G_{-3/2} - \frac{2(st+t-4)}{st-2} L_{-1} G_{-1/2} \right) |h\rangle + \ldots
\end{align}

Using these expressions it is then possible in principle to calculate the chiral blocks order by order and turn the recursion relations for the operator product expansions into differential equations on the chiral blocks. Rather than carry this out in detail we will instead turn to a method to derive differential equations directly.

4.1. Differential equations from Singular vector decoupling

We would like to find differential equations for a chiral block of the form

\begin{equation}
F(z) = \langle \phi_\infty | \phi_1(1) \phi_2(z) | \lambda_{12} \rangle, \quad \tag{26}
\end{equation}

coming from the vanishing of (11), where $\phi_\infty$, $\phi_1$ and $\phi_2$ are possibly matrix-valued Ramond fields. As a first step, we consider the contribution coming from the mode $L_{-1}$. This is easy to calculate using the relations for a Virasoro primary field of weight $h$

\begin{equation}
[L_{m}, \phi(z)] = z^{m+1} \phi'(z) + z^m h(m+1) \phi(z). \quad \tag{27}
\end{equation}

The only complication is the need to remove the term in $\phi'_1(1)$ that would normally arise. This can be done by considering

\begin{equation}
L_{-1} |\lambda_{12} \rangle = (L_{-1} - L_0 + h_{1,2}) |\lambda_{12} \rangle \quad \tag{28}
\end{equation}

to find

\begin{equation}
\langle \phi_\infty | \phi_1(1) \phi_2(z) L_{-1} | \lambda_{12} \rangle = \left[ (z - 1) \frac{\partial}{\partial z} + h_1 + h_z + h_{1,2} - h_\infty \right] F(z). \quad \tag{29}
\end{equation}
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The difficulty now is to treat $G_{-1}$. We shall adapt some ideas used in the construction of the co-invariant spaces that are used to classify the fusion algebra of Virasoro and superconformal algebra representations.

In this method, one finds linear combinations of modes which annihilate the state $\langle \phi_\infty | \phi_1(1) \phi_z(z) \rangle$. The simplest way to find these is to find polynomials $p(w)$ such that $\langle \phi_\infty | \phi_1(1) \phi_z(z) p(w) T(w) \rangle$ has at most a simple pole at 1, $z$ and $\infty$. The modes of $p(w) T(w)$ then take simple values when acting on $\langle \phi_\infty | \phi_1(1) \phi_z(z) \rangle$. For example, for $m < 0$

$$l_m = \oint_0 T(w)(1 - w)(z - w)w^{m+1} \frac{dw}{2\pi i}$$

(30)

$$= zL_m - (1 + z)L_{m+1} + L_{m+2}$$

(31)

satisfies

$$\langle h_\infty | \phi_1(1) \phi_z(z) l_m = (1 - z)(z^{m+1}h_z - h_1) \langle h_\infty | \phi_1(1) \phi_z(z) \rangle .$$

(32)

As a toy example, we shall first apply this idea to the Ising model to reproduce the result of [5] before applying it to the superconformal algebra and Ramond fields.

5. The Ising model

The Ising model can be formulated as the theory of a single free fermion field $\psi(z)$. As with the superconformal algebra, this field can have half-integer (NS) or integer (R) modes

$$\psi(z) = \sum_r \psi_r z^{-r-1/2}$$

(33)

satisfying

$$\{\psi_m, \psi_n\} = \delta_{m+n,0} .$$

(34)

The Ramond algebra has a zero mode $\psi_0$ satisfying $(\psi_0)^2 = \frac{1}{2}$ and so there are two inequivalent irreducible highest weight representations of the Ramond algebra, with highest weights $|\pm\rangle$ satisfying

$$\psi_0 |\pm\rangle = \pm \frac{1}{\sqrt{2}} |\pm\rangle , \quad \psi_m |\pm\rangle = 0 \text{ for } m > 0 .$$

(35)

A unitary highest weight representation of the Ramond algebra will have a highest weight space on which $\psi_0$ is represented by a matrix with eigenvalues $\pm 1$. We shall denote the (vector-valued) highest weight of such a general representation by $|\sigma\rangle$ and the chiral fields corresponding to such a state by $\sigma(z)$.

The energy-momentum tensor can be written in terms of $\psi$ as

$$T(z) = \frac{1}{2} \psi'(z) \psi(z) ,$$

(36)

so that

$$L_{-2}|0\rangle = \frac{1}{2} \psi_{-3/2} \psi_{-1/2} |0\rangle , \quad L_{-1} |\pm\rangle = \frac{1}{2} \psi_{-1} \psi_0 |\pm\rangle = \pm \frac{1}{2\sqrt{2}} \psi_{-1} |\pm\rangle$$

(37)
This last equation can be viewed as a null-vector equation, the analogue of (11) and we can use it to find a differential equation for the correlation functions of the form

\[ F_\pm(z) = \langle \sigma | \sigma(1) \sigma(z) | \pm \rangle . \]  

(38)

Here, each \( \sigma \) is some vector-value representation as is the function \( F \). We will not actually need to specify the exact form of these representations, as we will see shortly.

We know that the space of Virasoro conformal blocks with the correct properties is two dimensional so we have to consider \( F_\pm(z) \) as a vector in some space of solutions and the differential equation we obtain will be a matrix differential equation.

According to the idea outlined above, we want to find combinations of modes of the Ramond algebra \( \Psi_\infty, \Psi_1 \) and \( \Psi_z \) so that

\[
\langle \sigma | (1) \sigma(1) \sigma(z) \Psi_\infty = \langle \sigma | (1) \sigma(z) \Psi_1 ,
\]

\[
\langle \sigma | (1) \sigma(1) \sigma(z) \Psi_z = \langle \sigma | (1) \psi(0)(z) \rangle .
\]

(40)  

We will require that these operators square to \( 1/2 \) and mutually anti-commute, that is satisfy the algebra

\[ \{ \Psi_\alpha, \Psi_\beta \} = \delta_{\alpha \beta} . \]  

(42)

We start by defining operators \( \Phi_\alpha \) as integrals,

\[ \Phi_\alpha = \oint_0 p_\alpha(w) \psi(w) \frac{dw}{2\pi i} , \]

(43)

where the contour encloses the origin but not the points 1 or \( z \) and \( p_\alpha(w) \) are functions which remove the unwanted singularities in \( w \) from the state

\[ \langle \sigma | (1) \sigma(1) \sigma(z) \psi(w) \rangle . \]  

(44)

Since the operator product of the field \( \psi \) with a Ramond field \( \sigma \) is of the form

\[ \psi(z) \sigma(w) \sim \frac{1}{\sqrt{z-w}} \hat{\psi}_0 \sigma(w) + O(\sqrt{z-w}) , \]  

(45)

suitable combinations are

\[
\Phi_\infty = \oint_0 (1-w)^{1/2}(z-w)^{1/2}w^{-3/2}\psi(w) \frac{dw}{2\pi i} = \sqrt{z} \left( \psi_{-1} - \frac{1+z}{2z} \hat{\psi}_0 - \frac{(1-z)^2}{8z^2} \psi_1 + \ldots \right) \]

(46)

\[
\Phi_1 = \oint_0 (1-w)^{-1/2}(z-w)^{1/2}w^{-3/2}\psi(w) \frac{dw}{2\pi i} = \sqrt{z} \left( \psi_{-1} - \frac{1-z}{2z} \hat{\psi}_0 - \frac{(1-z)(1+3z)}{8z^2} \psi_1 + \ldots \right) \]

(47)

\[
\Phi_z = \oint_0 (1-w)^{1/2}(z-w)^{-1/2}w^{-3/2}\psi(w) \frac{dw}{2\pi i} = \frac{1}{\sqrt{z}} \left( \psi_{-1} + \frac{1-z}{2z} \hat{\psi}_0 + \frac{(1-z)(3+z)}{8z^2} \psi_1 + \ldots \right) \]

(48)
It is easy to calculate the anti-commutators of these operators – either as contour integrals or directly in terms of the modes – to find

\[
(\Phi_\infty)^2 = \frac{1}{2}, \quad (\Phi_1)^2 = -\frac{1 - z}{2}, \quad (\Phi_z)^2 = \frac{1 - z}{2z^3},
\]

(49)

\{\Phi_\alpha, \Phi_\beta\} = 0, \quad \alpha \neq \beta.

(50)

Consequently we can define new combinations \(\Psi_\alpha\) which satisfy (42) as

\[
\Psi_\infty = \Phi_\infty, \quad \Psi_1 = i\frac{1}{\sqrt{1 - z}}\Phi_1, \quad \Psi_z = \frac{z^{3/2}}{\sqrt{1 - z}}\Phi_z.
\]

(51)

We can now use the combinations \(\Psi_\alpha\) to replace the mode \(\psi_{-1}\) in the singular vector (37). There remains a large degree of choice in how to do this as we can replace \(\psi_{-1}\) by any of the \(\Psi_\alpha\) as follows:

\[
\psi_{-1}|\pm\rangle = \left(\frac{1}{\sqrt{z}}\Psi_\infty + \frac{1 + z}{2z}\psi_0\right)|\pm\rangle
\]

\[
= \left(-i\sqrt{\frac{1 - z}{z}}\Psi_1 + \frac{1 - z}{2z}\psi_0\right)|\pm\rangle
\]

\[
= \left(\frac{\sqrt{1 - z}}{z}\Psi_z - \frac{1 - z}{2z}\psi_0\right)|\pm\rangle
\]

(52)

Without loss of generality, we will now just consider the correlation functions \(F_+(z)\). The general expression for the null vector relation (37) we can obtain in terms of \(\Psi_\alpha\) using (52) is of the form

\[
\left(L_{-1} - \sum_{\alpha=1}^3 h_\alpha(z)\Psi_\alpha - d(z)\right)|+\rangle = 0.
\]

(53)

where \(h_\alpha(z)\) and \(d(z)\) are functions of \(z\). Acting on this equation on the left by \(\langle\sigma|\sigma(1)\sigma(z)\rangle\) leads to a matrix differential equations for the correlation functions of the form

\[
\left((z - 1)\frac{\partial}{\partial z} + \frac{1}{8} - \sum_{\alpha=1}^3 \frac{1}{\sqrt{2}}h_\alpha(z)\gamma_\alpha - d(z)\right)F_+(z) = 0,
\]

(54)

where \(\gamma_\alpha/\sqrt{2}\) are matrices representing the action of the zero modes on the fields at \(z, 1\) and \(\infty\) satisfying the Clifford algebra

\[
\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}.
\]

(55)

We must now choose a representation of this algebra. The smallest representations of this algebra are two dimensional and there are two inequivalent choices for which \(\gamma_\infty \gamma_1 \gamma_z = \pm i\). It is essential for constructing the correct differential equation for the correlation functions of the spin field that the correct representation is chosen. To fix the equivalence class of the representation, we note that the operators \(\Psi_\alpha\) satisfy further relations, for example

\[
(\Psi_\infty \Psi_1 - i\Psi_z \psi_0)|\sigma\rangle = 0, \quad \left(\Psi_\infty \Psi_1 \mp \frac{i}{\sqrt{2}}\Psi_z\right)|\pm\rangle = 0.
\]

(56)
Either of these is sufficient to fix the class of the representation. Since we are considering $F_+(z)$, we note that $\psi_0|+\rangle = (1/\sqrt{2})|+\rangle$ implies we must choose the representation of the Clifford algebra for which $\gamma_\infty \gamma_1 = i\gamma_z$, i.e., for which $\gamma_\infty \gamma_1 \gamma_z = i$.

Returning to equation (54), the simplest choices are for two of the functions $h_\alpha$ to be zero, the other non-zero, so that only a single matrix $\gamma_\alpha$ appears in the matrix differential equation (54). In this case we can take the matrix to be diagonal (or alternatively consider the one-dimensional representations of the algebra $(\Psi_\alpha)^2 = 1/2$) and we obtain a set of first order differential equations for the correlation functions as follows which give the two components of the function $F_+$ as the chiral blocks associated to a particular channel. We illustrate this below.

5.1. Using $\Psi_\infty$

Setting $h_1 = h_z = 0$, we find $h_\infty = \sqrt{z}/8$ and $d = (1 + z)/8z$, so that the differential equation (54) becomes

$$\left( (z - 1) \frac{\partial}{\partial z} - \frac{1}{8z} - \frac{1}{4z} \gamma_\infty \right) F_+(z) = 0 ,$$

(57)

At this stage the only requirement on $\gamma_\infty$ is that it squares to 1, so we can consider one-dimensional subspaces of the space of correlation functions on which $\gamma_\infty$ takes values $\pm 1$, leading to the two solutions

$$\gamma_\infty = 1 \quad F_+ = \frac{\sqrt{1 - \sqrt{z}}}{z^{1/8}(1 - z)^{1/8}} \quad (58)$$

$$\gamma_\infty = -1 \quad F_+ = \frac{\sqrt{1 + \sqrt{z}}}{z^{1/8}(1 - z)^{1/8}} \quad (59)$$

which are the well-known chiral blocks of the Virasoro algebra associated to the following choice of channel:

$\gamma_\infty = 1 : \quad \infty \quad \begin{array}{c} 1/2 \\ z \end{array} \quad 0 , \quad \gamma_\infty = -1 : \quad \infty \quad \begin{array}{c} 0 \\ z \end{array} \quad 0$

5.2. Using $\Psi_1$

The differential equation (54) becomes

$$\left( (z - 1) \frac{\partial}{\partial z} + \frac{2z - 1}{8z} - \frac{i}{4} \sqrt{\frac{1 - z}{z}} \gamma_1 \right) F_+(z) = 0 ,$$

(60)

Taking $\gamma_1$ diagonal with values $\pm 1$ leads to the two solutions

$$\gamma_1 = 1 \quad F_+ = \frac{\sqrt{\sqrt{z} + \sqrt{z} - 1}}{z^{1/8}(1 - z)^{1/8}} \quad (61)$$

$$\gamma_1 = -1 \quad F_+ = \frac{\sqrt{\sqrt{z} - \sqrt{z} - 1}}{z^{1/8}(1 - z)^{1/8}} \quad (62)$$
Differential equations from null vectors of the Ramond algebra

which are the chiral blocks of the Virasoro algebra associated to the following channel:

\[
\begin{align*}
\gamma_1 = 1 : & \quad \infty \quad 0 \quad 0 \quad 1 \\
\gamma_1 = -1 : & \quad \infty \quad 1/2 \quad 0
\end{align*}
\]

5.3. Using \( \psi_z \)

The differential equation (54) becomes

\[
\left( (z - 1) \frac{\partial}{\partial z} + \frac{1}{8z} - \frac{\sqrt{1-z}}{4z} \gamma_z \right) F_+(z) = 0 ,
\]

Taking \( \gamma_z \) diagonal with values \( \pm 1 \) leads to the two solutions

\[
\begin{align*}
\gamma_z = 1 & \quad F_+ = \frac{\sqrt{1+1-z}}{z^{1/8}(1-z)^{1/8}} \\
\gamma_z = -1 & \quad F_+ = \frac{\sqrt{1-1-z}}{z^{1/8}(1-z)^{1/8}}
\end{align*}
\]

which are the well-known chiral blocks of the Virasoro algebra associated to the channel:

\[
\begin{align*}
\gamma_z = 1 : & \quad 1 \quad 0 \quad 0 \quad \infty \\
\gamma_z = -1 : & \quad 1 \quad 1/2 \quad 0 \quad \infty
\end{align*}
\]

6. The Ramond chiral blocks

Taking inspiration from the case of the Ising model, we will try to find differential equations for the chiral blocks

\[
F(z) = \langle \lambda_\infty | \phi_{\lambda_1}(1) \phi_{\lambda_z}(z) | \lambda_{12} \rangle ,
\]

The expectation is that we can express the singular vector (11) in terms of combinations of the modes \( G_m \) which (inside the correlation function) lead to a matrix representation of the algebra of the zero modes acting on the primary fields inserted at 1, \( z \) and \( \infty \). That is, we will try to find combinations \( \Gamma_\alpha \) of the modes \( G_m \) (for \( \alpha \in \{ \infty , 1, z \} \)) which, when taken inside a four-point function (66) lead to a matrix representation \( g_\alpha \) of the zero-mode algebra

\[
\{ g_\alpha , g_\beta \} = (h_\alpha - c \frac{24}{\lambda_\alpha^2}) \delta_{\alpha\beta} = \lambda_\alpha^2 \delta_{\alpha\beta} ,
\]

and which will lead to a matrix differential equation for (66) in the form

\[
\left( \frac{d}{dz} + p(z) + \sum_\alpha q_\alpha(z) g_\alpha \right) F(z) = 0 .
\]
In terms of these operators, the $\Psi_\alpha$ modes of the Virasoro algebra:

$$G(z) \phi_\alpha(w) = \frac{\hat{G}_0 \phi_\alpha(w)}{(z-w)^{3/2}} + \frac{\hat{G}_{-1} \phi_\alpha(w)}{(z-w)^{1/2}} + O(\sqrt{z-w}), \quad (69)$$

where $\hat{G}_0$ is a matrix representation of the zero mode satisfying $\hat{G}_0^2 = \lambda^2 = h - c/24$, $\{\hat{G}_0, \hat{G}_{-1}\} = 0$ and $\hat{G}_{-1}^2 = L_{-2}$. Furthermore,

$$\langle \lambda | G(z) = \langle \lambda | (z^{-3/2}G_0 + z^{-1/2}G_1 + \ldots) \quad (70)$$

Consequently, we are motivated to consider the three combinations which remove all the singularities at two of the points $\infty$, $z$ and $1$, and turn the leading singularity at the remaining point into a simple pole:

$$\Psi_\infty = \oint_0 (1-w)^{3/2}(z-w)^{3/2}w^{-5/2}G(w)\frac{dw}{2\pi i},$$

$$\Psi_1 = \oint_0 (1-w)^{1/2}(z-w)^{3/2}w^{-5/2}G(w)\frac{dw}{2\pi i},$$

$$\Psi_z = \oint_0 (1-w)^{3/2}(z-w)^{1/2}w^{-5/2}G(w)\frac{dw}{2\pi i}. \quad (71)$$

These operators, however, do not square to constants as in (49), nor do they simply anti-commute. Their algebra is more complicated, and only simplifies inside the four-point functions (69). We can express their algebra in terms of suitable combinations of modes of the Virasoro algebra:

$$e_\infty = \oint_0 \frac{(1-w)^2(z-w)^2}{w^3}T(w)\frac{dw}{2\pi i},$$

$$e_1 = \frac{1}{(1-z)^2} \oint_0 \frac{(1-w)(z-w)^2}{w^3}T(w)\frac{dw}{2\pi i},$$

$$e_z = \frac{z^3}{(1-z)^2} \oint_0 \frac{(1-w)^2(z-w)}{w^3}T(w)\frac{dw}{2\pi i},$$

$$l_m = \oint_0 \frac{(1-w)^2(z-w)^2}{w^{3+m}}T(w)\frac{dw}{2\pi i}, \quad m \geq 1. \quad (72)$$

These combinations have the following properties when acting on the state

$$\langle \chi | = \langle h_\infty | \phi_{h_1}(1) \phi_{h_z}(z) ,$$

$$\langle \chi | (e_\infty - h_\infty) = \langle \chi | (e_1 - h_1) = \langle \chi | (e_z - h_z) = 0 ,$$

$$\langle \chi | l_m = 0 , \quad m \geq 1 \quad (73)$$

In terms of these operators, the $\Psi_\alpha$ satisfy

$$\Psi_\infty^2 = (e_\infty - \frac{c}{24}) - (1+z)l_2 + zl_1 ,$$

$$\Psi_1^2 = - (1-z)^3(e_1 - \frac{c}{24}) + zl_2 - (1-z)l_1 ,$$

$$\Psi_z^2 = \frac{(1-z)^3}{z^5}(e_z - \frac{c}{24}) + \frac{1}{z}l_2 + \frac{1-z}{z^2}l_1 . \quad (76)$$
We thus define the combinations $\Gamma_\alpha$ as
\[
\Gamma_\infty = \Psi_\infty, \quad \Gamma_1 = -i(1-z)^{-3/2}\Psi_1, \quad \Gamma_z = \frac{z^{5/2}}{(1-z)^{3/2}}\Psi_z.
\] (77)

Inside the four-point function (66), the terms in $l_2$ and $l_1$ vanish and $(e_\alpha - \frac{z}{24}) = \lambda_\alpha^2$, so that the action of the operators $\Gamma_\alpha$ inside (66) is given by matrices $g_\alpha$ satisfying the algebra (67). To use these, we have to express the singular vector in terms of the $\Psi_\alpha$. If we act with any of the $\Psi_\alpha$ in the highest weight state $|\lambda_{12}\rangle$, the leading term is a multiple of $G_{-3/2}|\lambda_{12}\rangle$, so that it is not possible to express the singular vector in terms of just one of the $\Psi_\alpha$. Instead, it is necessary to use all three and one finds that
\[
G_{-1}|\lambda_{12}\rangle = \left(\frac{1-2}{z}\lambda_{12} - i\left(\frac{1-z}{z}\right)^{1/2}\Gamma_1 - \frac{(1-z)^{1/2}}{z}\Gamma_z + z^{-1/2}\Gamma_\infty\right)|\lambda_{12}\rangle.
\] (78)

Combining (78) with (29) leads to the following matrix differential equation for (66), the main result of this article:
\[
(z - 1)F' + (h_{12} + h_1 + h_z - h_\infty)F(z) + \sqrt{\frac{1}{z}}\left[\frac{1+\frac{z}{2}}{2}\lambda_{12} - i\left(\frac{1-z}{z}\right)^{1/2}g_1 - \frac{(1-z)^{1/2}}{z}g_z + z^{-1/2}g_\infty\right]F(z) = 0.
\] (79)

We now turn to the analysis of this equation and its solutions. It will also be convenient to parametrise $\lambda_\alpha$ according to (10) as
\[
\lambda_\infty = \lambda_{1,q}, \quad \lambda_1 = \lambda_{1,r}, \quad \text{and} \quad \lambda_z = \lambda_{1,s}.
\] (80)

The first thing we can do is to check the indices of the solutions around the points $\infty, 1$ and 0, and then we can find the explicit solutions in various cases. Finally we compare these to solutions known by other methods.

6.1. Indicial equations

These are the equations for the leading behaviour of the solution around a singular point. The equation (79) has singular points 0, 1 and $\infty$. We first consider the point 0 around which the solution will have an expansion of the form
\[
F(z) = \sum_{n=0}^{\infty} a_n z^{\alpha + n/2},
\] (81)

where $a_n$ are vectors. Note that the expansion will be in half-integer powers of $z$ as the differential equation explicitly contains $\sqrt{z}$. Around $z = 0$, the leading behaviour is determined by substituting (81) in (79) and examining the coefficient of $z^{\alpha - 1}$:
\[
(-\alpha + \sqrt{\frac{1}{2}[\frac{1}{2}\lambda_{12} - g_z])}a_0 = 0.
\] (82)

Since this equation only involves $g_z$, we can take $g_z$ to be diagonal with eigenvalues $\pm \lambda_z$. This leads to two solutions for $\alpha$,
\[
\alpha_\pm = \sqrt{\frac{1}{2}[\frac{1}{2}\lambda_{12} \pm \lambda_z]}.
\] (83)
If we use the parametrisation $\lambda_z = \lambda_{1,s}$, we find
\[
\alpha_+ = \frac{1}{8} (2t(s-1) - 1) = h_{1,s+1} - h_{1,s} - h_{1,2},
\]
\[
\alpha_- = \frac{1}{8} (3 - 2t(s+1)) = h_{1,s-1} - h_{1,s} - h_{1,2}.
\]
These values are exactly the expected exponents for the chiral blocks shown below:

\[
\alpha_+ : \lambda_{1,q} \begin{array}{c|c}
\lambda_{1,r} & \lambda_{1,s} \\
\hline
h_{1,s+1}^{\text{even}} & \lambda_{1,2}
\end{array}, \quad \alpha_- : \lambda_{1,q} \begin{array}{c|c}
\lambda_{1,r} & \lambda_{1,s} \\
\hline
h_{1,s-1}^{\text{odd}} & \lambda_{1,2}
\end{array}
\]

It is easy to check that similar results hold for the other two channels, corresponding to expanding the solution $F$ around $\infty$ in powers of $1/z$ and around 1 in powers of $(z-1)$.

6.2. Solution to the matrix differential equation

To solve the full equation (79) we must choose a matrix representation for $g_\alpha$. Up to now we have not had to specify the action of the zero modes on the primary fields, and as in the Ising model, we do not need to do it now. There are only two inequivalent representations for which $g_\infty g_1 g_z = \pm i \lambda_\infty \lambda_1 \lambda_z$, and the choice of representation is invariant under monodromy around $z = 0$ and $z = 1$. We shall take $g_\alpha$ to be given in terms of the Pauli matrices as
\[
g_\infty = \lambda_\infty \sigma^1 = \lambda_\infty \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
g_1 = \eta \lambda_1 \sigma^2 = \eta \lambda_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]
\[
g_z = \lambda_z \sigma^3 = \lambda_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where $\eta = \pm 1$. This choice turns the equation for $F$ into a real matrix differential equation and taking $g_z$ diagonal also means that the two components of $F$ have expansions in $z$ (and not $\sqrt{z}$). The two components of $F$ can be identified by comparison with (18) and (19) and are the odd and even chiral blocks (up to a sign)
\[
\alpha = \alpha_+ : F = \begin{pmatrix}
\lambda_{1,r} & \lambda_{1,s} \\
\hline
\lambda_{1,q}^{\text{even}} & h_{1,s+1}^{\text{even}} \\
\hline
\lambda_{1,r} & \lambda_{1,s} \\
\hline
\lambda_{1,q}^{\text{odd}} & h_{1,s+1}^{\text{odd}} \\
\hline
\lambda_{1,q} & \lambda_{1,2}
\end{pmatrix}
\]

(88)
7. Exact solutions

In some cases it is possible to find exact solutions to \(79\). The simplest case to consider is where all the \(\lambda_{\alpha}\) are equal to \(\lambda_{1,2}\). In this case

\[
h_{1,2} = \frac{3}{16} (2t - 1), \quad \lambda_{1,2} = \frac{1 - 2t}{2\sqrt{2t}}, \quad \sqrt{\frac{t}{2}} \lambda_{1,2} = \frac{4}{3} h_{1,2},
\]

and the fusion rules \((24)\) force \(\eta = +\). Writing the components of \(F\) as \((f_1, f_2)\), with the representation \((87)\), the differential equation \((79)\) becomes

\[
(z-1) \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} + 2h \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \frac{4h}{3} \begin{pmatrix} \frac{1+z}{2z} - \sqrt{\frac{1-z}{z}} & \frac{1}{2} - \sqrt{\frac{1+z}{z}} \\ \frac{1+z}{2z} + \sqrt{\frac{1-z}{z}} & \frac{1}{2} + \sqrt{\frac{1+z}{z}} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

The coefficients in the equation as presented have branch cuts at \(z = 0\) and \(z = 1\), and it is convenient to remove these by changing variables to

\[
u = \frac{1 - \sqrt{1-z}}{z} = \frac{\sqrt{z}}{2} + \ldots, \quad z = \frac{4u^2}{(1 + u^2)^2}.
\]

With this change and a redefinition \(F(z) = z^{-2h}G(u)\) to remove the leading singularity in \(z\), the equations simplify dramatically to

\[
- \frac{(1-u^4)}{8u} \frac{dG}{du} + \frac{2h}{3} \left( \frac{1}{u^2} - \frac{2u}{u^2} \right) G = 0
\]

with solutions

\[
G^+ = u^{-1+2t} (1-u^4)^{\frac{3-2t}{2}} \left( \frac{1-2t}{t} \cdot u \cdot F \left( \frac{1}{2} - t, \frac{1}{2} - \frac{t}{2}; \frac{t}{2}; u^4 \right) \right)
\]

\[
G^- = (1-u^4)^{-\frac{3-2t}{2}} \left( \frac{2t-1}{t-2} \cdot u^3 \cdot F \left( \frac{3}{2} - t, \frac{3}{2} - \frac{3t}{2}; \frac{2-t}{2}; u^4 \right) \right)
\]

where \(F\) is the standard Hypergeometric function. For special values of \(t\), the blocks with identity intermediate channel simplify further. For example at \(t = 5/3\), that is for four-point blocks of the field with \(h = 7/16\) in the tri-critical Ising model, \(G^-\) simplifies to

\[
G^- = (1-u^4)^{-7/4} \left( -u^3 (7 + u^4) \right)
\]

As can be seen, the two components are related by \(u \rightarrow 1/u\) which corresponds to \(z\) encircling the branch point at \(z = 1\) once.
8. Solutions known by other methods

We can check the differential equation and its solutions against solutions known by other methods. Principally, there are three values of the central charge in the superconformal minimal series which also appear in list of Virasoro minimal models, so that A-series of the superconformal minimal models can be identified with the following invariants of the Virasoro minimal models,

\[ SM(3, 5) = M(4, 5)_D, \]
\[ SM(2, 8) = M(3, 8)_D, \]
\[ SM(3, 7) = M(7, 12)_E. \]  

The representations in the superconformal models can be found as sums of representations in the Virasoro minimal models, and the chiral blocks of the superconformal models must be sums of Virasoro chiral blocks. Power series expansions of the Virasoro chiral blocks can be found easily, either by using one of the recursion relations of Zamolodchikov \cite{10} or by solving the differential equation from the singular vector. We present two examples here to show how this works.

8.1. \( SM(3, 5) \)

In this model, the tricritical Ising model, the Ramond representations of the superconformal algebra are \((1, 2)\) and \((2, 1) \equiv (1, 4)\) with conformal dimensions \(h_{1,2} = 3/80\) and \(h_{1,4} = 7/16\), and consequently all correlation functions of four Ramond fields can be found using the method in this paper.

If we consider just one case, the following two blocks can be found by series solution of the differential equation \(91\)

\[
\begin{array}{cccc}
\frac{3}{80} & \frac{7}{10} & \frac{7}{10} & \frac{3}{80} \\
\end{array}
\]

\[ = z^{-3/8} \left(1 + \frac{5z}{4} + \frac{75z^2}{64} + \frac{287z^3}{256} + \frac{8885z^4}{8192} + \ldots \right) \] (102)
Differential equations from null vectors of the Ramond algebra

\[
\begin{align*}
\frac{3}{80} - \frac{1}{10} - \frac{3}{80} &= z^{1/8} (1 + \frac{5z}{6} + \frac{149z^2}{192} + \frac{95z^3}{128} + \frac{5885z^4}{8192} + \ldots) \quad (103)
\end{align*}
\]

Since the Ramond representations of the unextended superconformal algebra and the even and odd sectors of the intermediate channel are each irreducible representations of the Virasoro algebra, these blocks can also be found using the representation theory of the Virasoro algebra. In this case, the irreducible Virasoro representation of weight \(3/80\) has Virasoro Kac labels \((2, 2)_V\) and so the odd and even chiral blocks are two solutions of a fourth order differential equation. This differential equation can also be solved for a series solution and two of the solutions are exactly those given in (102) and (103).

8.2. SM\((3, 7)\)

This model is related to the \(E_6\) invariant of the Virasoro minimal model \(M(7, 12)\). The irreducible super Virasoro representations of interest split into direct sums of irreducible Virasoro representations as follows

\[
\begin{align*}
\mathcal{H}_{\text{even}}^{(11)} &= \mathcal{H}^{Vir}_{(11)} \oplus \mathcal{H}^{Vir}_{(17)} , \\
\mathcal{H}^{(12)} &= \mathcal{H}^{Vir}_{(24)} \oplus \mathcal{H}^{Vir}_{(28)} , \\
\mathcal{H}_{\text{even}}^{(13)} &= \mathcal{H}^{Vir}_{(35)} \oplus \mathcal{H}^{Vir}_{(3, 11)} , \\
\mathcal{H}^{(14)} &= \mathcal{H}^{Vir}_{(34)} \oplus \mathcal{H}^{Vir}_{(38)} , \\
\mathcal{H}_{\text{even}}^{(15)} &= \mathcal{H}^{Vir}_{(25)} \oplus \mathcal{H}^{Vir}_{(21, 1)} , \\
\mathcal{H}^{(16)} &= \mathcal{H}^{Vir}_{(14)} \oplus \mathcal{H}^{Vir}_{(18)} .
\end{align*}
\]

(104)

Most of the superconformal chiral blocks are given by sums of Virasoro chiral blocks, but in some cases there is only a single Virasoro representation contributing to the intermediate channel and so the results of solving (91) and the Virasoro null vector equations can be compared directly. If we denote Virasoro representations by \((rs)_V\) then two such cases are

\[
\begin{align*}
(12) + (13)_{\text{even}} - (12) &= (24)_V \\
&= z^{-1/56} (1 + \frac{z}{28} + \frac{107z^2}{3136} + \frac{2523z^3}{87808} + \ldots) \quad (105)
\end{align*}
\]

\[
\begin{align*}
(12) + (15)_{\text{even}} - (12) &= (24)_V \oplus (25)_V \\
&= z^{11/28} (1 + \frac{9z}{28} + \frac{25309z^2}{122304} + \frac{536597z^3}{3424512} + \ldots)(106)
\end{align*}
\]

This time the blocks (105) and (106) can be calculated either as series solutions of (91) or as the series solutions of eighth order differential equations corresponding to the level eight null vector in the Virasoro representation \((2, 4)_V\), giving the same answers shown.
Differential equations from null vectors of the Ramond algebra

They can also be compared with the general series solution for Virasoro chiral blocks found by Zamolodchikov [10].

9. Conclusions

We have found a matrix differential equation for Ramond four-point chiral blocks. This can solved completely in some cases but as yet the full general solution is now known. These solutions were known already exactly in some other cases and as series solutions in a few more cases based on relations with Virasoro minimal models. The differential equations presented here reproduce these known results. Exact integral formulae based on free-field constructions are also known [11] and it remains to check that these satisfy the equations we have found.

Recently, recursive formulae generalising Zamolodchikov’s elliptic recursion formulae in [10] have been found [12] and again it remains to check that these give series expansions which satisfy our equations.

For the future, these equations and their solutions should enable one to extend calculations that have only fully been worked out in the Virasoro case to the superconformal case, such as the construction of the full set of boundary structure constants in [13] [14]

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