Quantum error rejection code with spontaneous parametric down conversion

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Abstract

We propose a linear optical scheme to transmit an unknown qubit robustly over bit-flip-error channel. To avoid the technical difficulty of the standard quantum error correction code, our scheme is based on the concept of error-rejection. The whole scheme is based on currently existing technology.

I. INTRODUCTION

An unknown qubit can be sent to a remote party robustly through a noisy channel if we use the quantum error correction code (QECC) [1–3], which plays a very important role in quantum computation and information [4]. The main idea there is first to encode the unknown qubit to an entangled state of many qubits and after the remote party receives this quantum code, he first decodes it and then obtains the original state faithfully. This is very different from the classical error correction since the unknown qubit in principle cannot be copied [5] or observed exactly therefore the simple repetition code as used in classical coding is not applicable here.

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With the discovery of maximal polarization entangled state with the spontaneous parametric down conversion (SPDC) [6], linear optics method has been perhaps the most powerful tool for realizing the entanglement based quantum tasks. So far many of the tasks have been proposed or demonstrated with linear optics, such as quantum teleportation [7], universal quantum cloning [8], quantum U-NOT operation [9], quantum entanglement concentration and purification [10,11] and destructive quantum logic gate [12]. However, none of the quantum error correction code has been experimentally realized so far [13]. Realizing either Shor’s 9-qubit code, Steane’s 7-qubit code or the 5-qubit code [3] is technically challenging by our current technology. All of them are based on the quantum entangled state with more than 5 qubits. This requires at least 3 pairs to be emitted by SPDC [6]. In a paper two years ago [15], the optical realization of quantum error rejection code over the bit-flip-error channel is considered. It was shown there [15] that the controlled-NOT operation in quantum error correction can be done probabilistically by a polarizing beam splitter and one can transfer a qubit robustly over a bit flip channel by teleportation. However, that scheme is based on the resource of three-photon GHZ state which is thought of as a type of impractical resource by our currently existing technology [15]. In particular, it was pointed in Ref. [15] that the post selection method given by [16,17] cannot be applied to the scheme proposed in [15]. In this paper, we propose a realization of quantum error rejection code over bit-flip-error channel with currently existing devices and resources in linear optics.

II. 2-BIT BIT-FLIP ERROR REJECTION CODE

To test the main points of the quantum error correction code we shall consider a simpler case here: transmitting an unknown qubit robustly over the bit flip channel using a smaller quantum code. We assume no phase flip noise for channel. Note that even in such a case there is no trivial way to complete the task: a repetition code is not allowed by the non-cloning principle.

To further simplify the experimental realization, instead of correcting the error, here we
shall only reject the corrupted qubits by using an *quantum error rejection code* (QERC). Suppose Alice is given the following unknown qubit

\[ |u\rangle = (\cos(\gamma/2)|0\rangle + e^{i\phi}\sin(\gamma/2)|1\rangle). \]  

(1)

If the qubit is directly sent through the channel, the qubit state after passing through the bit flip channel will be

\[ \rho_a = (1 - \eta)|u\rangle\langle u| + \eta|u_f\rangle\langle u_f| \]  

(2)

and

\[ |u_f\rangle = (\cos(\gamma/2)|1\rangle + e^{i\phi}\sin(\gamma/2)|0\rangle). \]  

(3)

For all possible initial states on the Bloch sphere, the average error rate caused by the bit flipping channel is

\[ E_0 = 1 - \frac{1}{\pi} \int_0^2 \pi \int_0^\pi \langle u|\rho_a|u\rangle d\gamma d\phi = \frac{2}{3} \eta. \]  

(4)

To send the unknown state robustly to the remote party Bob, Alice first encodes it into

\[ |q\rangle = (\cos(\gamma/2)|00\rangle + e^{i\phi}\sin(\gamma/2)|11\rangle). \]  

(5)

To make this encoding she does not need any information of the given state. What she needs to do is simply the conditional unitary transformation of

\[ |00\rangle \longrightarrow |00\rangle; |10\rangle \longrightarrow |11\rangle, \]  

(6)

where the first state is the unknown given qubit and the second one is the ancilla qubit.

She then sends the two-qubit code to the remote party Bob over bit flip channel, i.e., there is a small probability \( \eta (\eta < \frac{1}{2}) \) that a qubit is flipped during the transmission. After Bob receives the code, he first takes a parity check on the two qubits: if their bit values are different, he gives up both of them; if the bit values are same, he decodes the code by measuring the first qubit in code \( |q\rangle \) in the basis \( |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \). If the result
is $|+\rangle$, he stores the second qubit; if the result is $|−\rangle$, he takes unitary transformation of $(|0\rangle, |1\rangle) → (|0\rangle, −|1\rangle)$ to the second qubit and then stores it. The parity check does not damage the code itself, since the collective measurement only shows whether the two qubits have the same bit value rather than the bit value information of each qubit. Note that with the normalization factor being omitted,

$$|q\rangle = |+\rangle|u\rangle + |−\rangle(\cos(\gamma/2)|0\rangle - e^{i\phi}\sin(\gamma/2)|1\rangle).$$

(7)

In the case that they have the same bit value, with a relative probability of $(1 - \eta)^2$ that neither quit in the code is flipped, i.e. the code state with Bob is still $|q\rangle$. With a relative probability in of $\eta^2$ that both of the qubits are flipped, i.e., the code state with Bob is

$$|e_1\rangle = (\cos(\gamma/2)|11\rangle + e^{i\phi}\sin(\gamma/2)|00\rangle).$$

(8)

The cases that one qubit is flipped and one qubit is unchanged will always lead to different bit values of the two qubits therefore are all discarded by Bob after the parity check. It can be calculated that the average fidelity between the finally stored state and the initial unknown state is $F = \frac{(1 - \eta)^2 + \eta^2}{3(1 - \eta)^2 + \eta^2}$. This shows that the error rate after decoding is

$$E_c = \frac{2}{3} \frac{\eta^2}{(1 - \eta)^2 + \eta^2}.$$  

(9)

However, if Alice directly sends the original qubit without entanglement based quantum coding, the error rate will be in the magnitude order of $\eta$, which is one order higher than that with quantum rejection code.

Note that the above scheme works for any unknown state including the case that the initial qubit is entangled with a third party.

### III. EXPERIMENTAL PROPOSAL

We now show the main result of this work: how to experimentally test the idea above with practically existing technology in linear optics. We propose the quantum error rejection scheme in figure 1. As we are going to show, our scheme works successfully whenever beam
I0, x0 and y0 each contains exactly one photon. We are now working in the polarization space, we replace the state notation $|0\rangle, |1\rangle$ by $|H\rangle, |V\rangle$ respectively.

1. Initial state preparation.

When one pair is emitted on each side of the nonlinear crystal, beam 0,1 and beam 2,3 are both in the entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$. With the clicking of D0, the initial unknown state $\left(\cos(\gamma/2)|H\rangle + e^{i\phi}\sin(\gamma/2)|V\rangle\right)$ is prepared on beam 1.

2. Encoding.

After step 1, the state of beam 1',2,3 is $\left(\cos \frac{\gamma}{2}\, |H\rangle + e^{i\phi}\sin \frac{\gamma}{2}\, |V\rangle\right) |\Phi\rangle_{23}$. The omitted subscripts are 1', 2, 3 from left to right to each term. With the combination of beam 1' and beam 2 by the PBS, the state for beam 2', 1'', 3 is

$$\frac{1}{\sqrt{2}} \left(\cos \frac{\gamma}{2}\, |H\rangle|H\rangle|H\rangle + e^{i\phi}|V\rangle \sin \frac{\gamma}{2}|V\rangle|V\rangle + \cos \frac{\gamma}{2}\, |0\rangle|HV\rangle|V\rangle + e^{i\phi}\sin \frac{\gamma}{2}|HV\rangle|0\rangle|H\rangle\right)$$

Here the subscripts are implied by $|w\rangle|s\rangle|t\rangle = |w\rangle_{2'}|s\rangle_{1'}|t\rangle_{3}$. Note that neither vacuum state $|0\rangle_{2'}$ nor two photon state $|HV\rangle_{2'}$ will cause the event of exactly one photon on beam x0. State $|HV\rangle = a_H^\dagger a_V^\dagger |0\rangle$. After a Hadamard transformation by HWP2, beam 2' is changed to the state $\frac{1}{\sqrt{2}}(|2H\rangle - |2V\rangle)$ on beam I2. This show that beam x0 contains either 2 photons or nothing, if beam 2' is in the state $|HV\rangle$. Therefore we need only consider the first two terms above. The first two terms above can be rewritten in the equivalent form of

$$|+\rangle_{2'} \left(\cos \frac{\gamma}{2}\, |HH\rangle_{1',3} + e^{i\phi}\sin \frac{\gamma}{2}|VV\rangle_{1',3}\right) + |-\rangle_{2'} \left(\cos \frac{\gamma}{2}\, |HH\rangle_{1',3} - e^{i\phi}\sin \frac{\gamma}{2}|VV\rangle_{1',3}\right).$$

This shows that the state of beam 1' is indeed encoded onto beam 1'' and beam 3 with the entangled state $\left(\cos(\gamma/2)|H\rangle_{1',3} + e^{i\phi}\sin(\gamma/2)|V\rangle_{1',3}\right)$, if beam 2' is projected to single photon state $|+\rangle$.

3. Transmission through the bit flip channel.

Beam 1'' and beam 3 then each pass through a dashed line rectangular boxes which work as bit flip channels. We shall latter show how the rectangular box can work as the bit flip channel.

4. Parity check and decoding.
After the code has passed through the noisy channel, one first take a parity check to decide whether to reject it or accept it. To do so one just observe beam $3''$. If it contains exactly 1 photon, the code is accepted otherwise it is rejected. Further, in decoding, one measures beam $3''$ in $|\pm\rangle$ basis (In our set-up this is done by first taking a Hadamard transformation to beam $3''$ and then measuring beam I3 in $|H\rangle,|V\rangle$ basis). If no qubit in the code has been flipped after passing through the channel, the state for beam $1'''$ and beam $3'$ is \[
\left(\cos(\gamma/2)|HH\rangle_{1'''3'} + e^{i\phi}\sin(\gamma/2)|VV\rangle_{1'''3'}\right)\] and this state keeps unchanged after passing through the PBS. Again the state of beam $3''$ and $I_1$ can be rewritten into \[
|+\rangle_{3''} \left(\cos\frac{\gamma}{2}|H\rangle_{I_1} + e^{i\phi}\sin\frac{\gamma}{2}|V\rangle_{I_1}\right) + |-\rangle_{3''} \left(\cos\frac{\gamma}{2}|H\rangle_{I_1} - e^{i\phi}\sin\frac{\gamma}{2}|V\rangle_{I_1}\right). \tag{12}\]
If beam $3''$ is projected to state $|+\rangle$, the original state is recovered in beam $I_1$. Note that if one of the beam in $1'',3$ is flipped, the polarization of beam $1'',3'$ will be either $H,V$ or $V,H$. This means beam $3''$ will be either in vacuum state or in the two photon state $|HV\rangle$. Beam I3 will be in the state $\frac{1}{\sqrt{2}}(2|H\rangle - 2|V\rangle)$ given state $|HV\rangle$ for beam $3''$. In either case, beam $3''$ or beam y0 shall never contain exactly 1 photon. This shows that the code will be rejected if one qubit has been flipped. The code with both qubits having been flipped can also be accepted, but the probability of 2-flipping is in general very small. Therefore the error rate of all those states decoded from the accepted codes is greatly decreased.

5. Verification of the fault tolerance of QERC.

To verify the fault tolerance property, we should observe the error rate of all the accepted qubits. The devices $Pv-,RPBS,D1$ and $D4$ are used to measure beam $I_1$ in basis of \[
|\psi\rangle = \left(\cos(\gamma/2)|H\rangle + e^{i\phi}\sin(\gamma/2)|V\rangle\right),
|\psi\rangle = \left(e^{-i\phi}\sin(\gamma/2)|H\rangle - \cos(\gamma/2)|V\rangle\right). \tag{13}\]
We shall only check the error rate to the accepted beams. For this we need check whether beam $I_0,x_0$ and $y_0$ each contains exactly one photon in our scheme. The 4-fold clicking $(D0,D2,D3,D1)$ or $(D0,D2,D3,D4)$ guarantees this. For simplicity, we shall call the 4-fold clicking $(D0,D2,D3,D1)$ as event $C_1$ and 4-fold clicking $(D0,D2,D3,D4)$ as event $C_4$ hereafter.
As we have shown, given the bit flip rate $\eta$, the average error rate without QERC is $E_0 = \eta/2$. The error rate for the accepted qubits with QERC is $E_c$ given by eq. (9). The experimental motivation is to observe the error rate with our QERC and to demonstrate this error rate is much less than $E_0$. The value $E_c$ is obtained by the experiment. We shall count the error rate based on the number of each type of four fold events, i.e., $C_1$ and $C_4$. Denoting $N_1$, $N_4$ as the observed number of $C_1$ and $C_2$ respectively. The value $E_c/(N_1 + N_4)$ is just the error rate for those accepted qubits with QERC.

The dashed boxes work as bit flip channels. For such a purpose, the phase shift $\theta(\theta)$ or $\theta_1(-\theta_1)$ to vertical photon created by P(-P) or P1(-P1) should be randomly chosen from $\pm \pi_2$. Note that here a $\Delta$ degree HWP is mathematically defined as the unitary

$$U = \begin{pmatrix} \cos \Delta & -\sin \Delta \\ \sin \Delta & \cos \Delta \end{pmatrix}$$

in the basis of $\{|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$. The dashed box changes the incoming state to outgoing state by the following rule:

$$(|H\rangle, |V\rangle) \rightarrow \sqrt{\frac{1}{1 + \epsilon}} \left( |H\rangle + \sqrt{\epsilon} e^{i\theta} |V\rangle, |V\rangle - \sqrt{\epsilon} e^{-i\theta} |H\rangle \right)$$ (14)

Given an arbitrary state $|u\rangle = (\cos \frac{\gamma}{2} |H\rangle + e^{i\phi} \sin \frac{\gamma}{2} |V\rangle)$, after it passes a dashed square box, the state is changed to

$$|u_a\rangle = \sqrt{\frac{1}{1 + \epsilon}} \left[ |u\rangle - e^{i\theta} \sqrt{\epsilon} (\cos \frac{\gamma}{2} |V\rangle - e^{i\phi} e^{-2i\theta} \sin \frac{\gamma}{2} |H\rangle) \right]$$ (15)

Note that $e^{-2i\theta} = -1$, since $\theta$ is either $\pi/2$ or $-\pi/2$. Since $e^{i\theta}$ takes the value of $\pm i$ randomly, the state $|u_a\rangle$ is actually in an equal probabilistic mixture of both $\sqrt{\frac{1}{1 + \epsilon}} (|u\rangle \pm i \sqrt{\epsilon} |u_f\rangle)$ therefore the output state of the dashed line square box is

$$\rho_a = \frac{1}{1 + \epsilon} (|u\rangle \langle u| + \epsilon |u_f\rangle \langle u_f|)$$ (16)

Here $|u_f\rangle$ is defined by Eq.(3) with $|0\rangle, |1\rangle$ being replaced by $|H\rangle, |V\rangle$ respectively. This shows that the flipping rate of the dashed box channel is

$$\eta = \frac{\epsilon}{1 + \epsilon}.$$ (17)
Taking average over all possible initial states on Bloch sphere, the average error rate after a successful decoding by our scheme is

$$E_c = \frac{2}{3} \frac{\epsilon^2}{1 + \epsilon^2}. \quad (18)$$

However, if beam 1’ is directly sent to the remote party through one dashed box in our figure, the average error rate is

$$E_0 = \frac{2\epsilon}{3(1 + \epsilon)},$$

which is much larger than that through the quantum error rejection code if $\epsilon$ is small.

Although we need a random phase shift of $\pm \frac{\pi}{2}$ for both $\theta$ and $\theta_1$ in each dashed line square box to create the bit flip channels, in an experiment motivated towards detecting the error rate of quantum error correction code with such a channel, we can simply choose $(\theta, \theta_1) = \{(-\frac{\pi}{2}, -\frac{\pi}{2}), (-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2})\}$ separately and run the set-up in each case for a same duration. The average error rate over the total four durations is just the error rate for the bit flip channel where $\theta$ is randomly chosen from $\pm \frac{\pi}{2}$.

The overall efficiency of the experiment can be increased by 4 times if we accept all cases of the initial state preparation and also use the error correction code of $(\cos(\gamma/2)|HH\rangle - e^{i\phi}\sin(\gamma/2)|VV\rangle)$. To do so we only need to replace the polarizer $P_h$ by a PBS and add one more photon detector there, and also detect beam x and beam y in the figure.

Our scheme can also be used on the entangled state. To do so we need remove the devices HWP1 and $P_{V+}$, $P_h$, $P_{V-}$, RPBS and D0, D1,D4, and measure the correlation of beam 0 and beam in $\{|H\rangle, |V\rangle\}$ basis and $\{|+,|\rangle \}$ basis.

**IV. EFFECTS CAUSED BY DEVICE IMPERFECTIONS**

Now we consider the effects caused by the imperfections including limited efficiency, dark counting of the photon detectors and multi-pair (3-pair) emission in SPDC process.

The limited efficiency of the photon detector only decreases the observable coincidence rate but does not affect the fault tolerance property of the code. Note that the purpose of the proposed experiment is to check the error rate to all states which have passed the parity.
check. This corresponds to the 4-fold coincidence observation. If the detecting efficiency is low, many events which should cause the coincidence would be rejected. That is to say, many good codes will be rejected. But the low detection efficiency will never cause a corrupted code to pass the parity check. So the net effect of the low detection efficiency is to reduce the total number of accepted states but it does not changes the error rate for the accepted qubits. In other words, an experiment with limited detector efficiency is equivalent to that with perfect detectors and a lossy channel. Dark counting can be disregarded here because during the coincidence time in the order of 10ns the dark counting probability is less than $10^{-6}$ [14]. This can always be ignored safely provided the photon detector efficiency is much larger than $10^{-6}$. Normally, the detector efficiency is larger than 10%, which is much larger than the dark counting rate.

The probability of 3-pair emission is less than the probability of 2-pair emission. The probability for $C_4$ event caused by 2-pair emission is in the magnitude order of $\epsilon^2 p^2$. The 3-pair emission probability can be comparable with this if $p$ is not so small. Also the low detecting efficiency and the encoding-decoding process will make 3-pair emission more likely to be observed than 2-pair events. Now we consider the joint effects of limited detecting efficiency and 3-pair emission. To see the effects, we shall calculate the rate of 4-fold events $C_1, C_4$ caused by the 3-pair emission. Among all 3-pair emissions, the cases that all 3 pairs at the same side of the crystal will never cause the coincidence event. Three pair states

$$|l\rangle = \frac{1}{\sqrt{6}}(|H\rangle_0|H\rangle_1 + |V\rangle_0|V\rangle_1)(|2H\rangle_2|2H\rangle_3 + |HV\rangle_2|HV\rangle_3 + |2V\rangle_2|2V\rangle_3)$$

$$|r\rangle = \frac{1}{\sqrt{6}}(|2H\rangle_0|2H\rangle_1 + |HV\rangle_0|HV\rangle_1 + |2V\rangle_0|2V\rangle_1)(|H\rangle_2|H\rangle_3 + |V\rangle_2|V\rangle_3)$$

(19)

can cause the 4-fold coincidence. The emission probability of each of them is $3p^3/4$, $p$ is the one-pair emission probability. The emission probability for these states are much lower than that of 2-pair state, $p^2$. However, the 3-pair emission could still distort the observed value of $N_4/(N_1 + N_4)$ significantly, since the value $N_4$ itself in the ideal case is also small (in the magnitude order of $\epsilon^2 p^2$). We want to verify the fault tolerance property of the error rejection code. In the ideal case this can be verified by the fact that $N_4/(N_1 + N_4) << \eta/2$. 
To check the joint effect of 3-pair emission and the limited detector efficiency, we need simply to calculate the modification of the rate of event $C_4$ by the 3-pair emission and detector efficiency. (Since $N_1$ in the ideal case is much larger than $N_4$, the 3-pair emission modification to $N_1$ will be disregarded.) If the modified value of $N_4/(N_1 + N_4)$ is close to the ideal result therefore still much less than $E_0 = \eta/2$, then we conclude that those imperfections do not affect the main conclusion of the experiment and the fault tolerance property of the error-rejection code can be demonstrated even with those imperfections.

Since it will make no difference to the measurement results in average, for calculation simplicity, we postpone all measurements until the code has passed the channels. And we shall also omit those states which will never cause 4-fold clicking. Given state $|r\rangle$ or $|l\rangle$ we can write the corresponding state on beam $0, I_1, 2', 3''$. The probability of causing the 4-fold clicking event $C_1$ can then be calculated base on the state of beam $0, I_1, 2', 3''$. Note that the state should pass through the bit flip channels (the dashed rectangular boxes). Therefore given $|r\rangle$ or $|l\rangle$ there could be 4 different state on beam $0, I_1, 2', 3''$. Given state $|r\rangle$ initially, with probability $(1 - \eta)^2$ that no qubit is flipped when passing through the dashed boxes. In such a case, with those terms which will never cause 4-fold clicking being omitted, the state of beam $0, I_1, 2', 3''$ will be

$$|r\rangle_0 = \frac{1}{\sqrt{6}} \left( \sqrt{2} \alpha \beta |2H,H,V\rangle + (\beta^2 - \alpha^2) |HV,H,V\rangle \right)$$

$$+ |HV\rangle \left[ \sqrt{2} \alpha \beta (|2H,H,H\rangle - |V,V,V\rangle) + (\beta^2 - \alpha^2) (|H,H,V\rangle + |HV,V,V\rangle) \right]$$

where $|\alpha|^2 + |\beta|^2 = 1$ and for each term we have used the notation and subscripts implication as the following:

$$|s\rangle |u,v,w\rangle = |s,u,v,w\rangle = |s\rangle_0 |u\rangle_1 |v\rangle_2 |w\rangle_3$$

(21)

In the following, we always imply this order for the omitted subscripts and omit those components which will never cause 4-fold clicking. With a probability of $\eta(1 - \eta)$ beam $1''$ is flipped, the state is then

$$|r\rangle_{1''} = \frac{1}{\sqrt{6}} \left[ \sqrt{2} \alpha \beta |2H,H,V,V\rangle + (\beta^2 - \alpha^2) |HV,H,V,V\rangle \right].$$

(22)
With a probability of $\eta(1 - \eta)$ beam 3 is flipped, the state is then

$$|r\rangle_3 = \frac{1}{\sqrt{6}} \left[ \sqrt{2} \alpha \beta |2H, H, V, HV\rangle + (\beta^2 - \alpha^2) |HV, H, V, HV\rangle \right].$$

(23)

With a probability of $\eta^2$ both beam 1$''$ and beam 3 are flipped, the state is then

$$|r\rangle_b = \frac{1}{\sqrt{6}} 2H \langle \alpha^2 |V, H, 2V\rangle + \beta^2 |H, 2V, H\rangle + \sqrt{2} \alpha \beta |V, HV, V\rangle + \sqrt{2} \alpha \beta |H, V, HV\rangle)$$

$$+ \frac{1}{\sqrt{6}} |HV\rangle \left[ \alpha \beta (|V, H, 2V\rangle - |H, 2V, H\rangle) + (\beta^2 - \alpha^2) (|V, HV, V\rangle + |H, V, HV\rangle) \right].$$

(24)

Similarly, given initial state $|l\rangle$, we shall also obtain 4 different states in beam 0,1,2',3''. If no beam is flipped we have

$$|l\rangle_0 = \frac{1}{\sqrt{6}} |H\rangle \left[ \alpha (|H, 2H, 2H\rangle + |HV, H, HV\rangle) + \beta |V, HV, HV\rangle + \beta |2V, V, 2V\rangle \right]$$

(25)

If beam 1$''$ is flipped we have

$$|l\rangle_1'' = \frac{1}{\sqrt{6}} |H\rangle (\alpha |HV, H, HV\rangle + \beta |HV, HV, H\rangle).$$

(26)

If beam 3 is flipped we have

$$|l\rangle_3 = \frac{1}{\sqrt{6}} |H\rangle (\alpha |HV, H, HV\rangle + \beta |H, HV, HV\rangle).$$

(27)

If both beam 1$''$ and beam 3 are flipped we have

$$|l\rangle_b = \frac{1}{\sqrt{6}} |H\rangle \left[ \alpha (|2V, 2H, V\rangle + |HV, H, HV\rangle) + \beta (|HV, HV, H\rangle + |2H, V, 2H\rangle) \right].$$

(28)

We have denoted the 4-fold clicking event (D0,D2,D3,D4) by $C_4$. To calculate the rate of the $C_4$ caused by 3-pair states, we just calculate the 4-fold clicking probability caused by each of the above states and then take a summation of them. Moreover, the probability is dependent on the parameters in the initial state, $\alpha, \beta$, one should take the average over the whole Bloch sphere. However, in a real experiment, instead of testing the average over all Bloch sphere, it’s more likely to test the code by the average effect of four state of

$$(\alpha, \beta) = (1, 0); (0, 1); \frac{1}{\sqrt{2}} (1, 1); \frac{1}{\sqrt{2}} (1, -1).$$

(29)
Here we just take the average over these 4 states instead of the whole Bloch sphere for simplicity. This obviously will not affect the main points.

Consider the state $|r\rangle_0$. In case of $\alpha = 1, \beta = 0$, only the state $|\sqrt{1/6}|2H\rangle|HV, V, V\rangle$ can cause event $C_4$. The probability upper bound is $\frac{1}{24}\xi^4$, where $\xi$ is the detecting efficiency of a photon detector. In the calculation, we have used the fact that $|HV\rangle$ is changed to $\frac{1}{\sqrt{2}}(|2H\rangle - |2V\rangle)$ after the Hadamard transformation. Also, with 2 incident photons, a photon detector will be clicked by probability $1 - (1 - \xi)^2 = 2\xi - \xi^2$. To calculate the upper bound, we simply use $2\xi$ and discard $-\xi^2$, this will over estimate the clicking probability. However, we shall finally show that even with such an overestimation, all the $C_4$ events caused by 3-pair emission will not affect the main results. Similarly, if $\beta = 1, \alpha = 0$, the probability to cause $C_4$ events by state $|r\rangle_0$ is also upper bounded by $\frac{1}{24}\xi^4$. Now we consider the case of $\alpha = \beta = \frac{1}{\sqrt{2}}$. After calculation we find that the probability of $C_4$ event caused by each term in $|r\rangle_0$ is upper bounded by the following table the 4-fold clicking. The probability to cause the 4-fold clicking by each term is listed in the following table:

| term | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Sum. |
|------|---|---|---|---|---|---|---|---|------|
| Prob.| $\xi^4/48$ | $\xi^4/24$ | $\xi^4/48$ | $\xi^4/96$ | $\xi^4/96$ | 0 | 0 | $7\xi^4/48$ |

Similarly, in the case of $\alpha = -\beta = \frac{1}{\sqrt{2}}$ the probability to cause $C_4$ event is also $7\xi^4/48$. Therefore in average the probability of $C_4$ events caused by state $|r\rangle_0$ is $\frac{3\xi^4}{16}$. We list the upper bound of average probability contribution to $C_4$ events caused by each state from eq.(20) to eq.(28) in the following table:

| state | $|r\rangle_0$ | $|r\rangle_1^\nu$ | $|r\rangle_3$ | $|r\rangle_b$ | $|l\rangle_0$ | $|l\rangle_2^\nu$ | $|l\rangle_3$ | $|l\rangle_b$ |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Prob. | $\frac{3\xi^4}{16}$ | $\xi^4/16$ | $\xi^4/16$ | $\frac{17\xi^4}{96}$ | $\frac{5\xi^4}{24}$ | $\xi^4/12$ | $\xi^4/12$ | $\frac{5\xi^4}{24}$ |

Suppose the detecting efficiency of the photon detector is $\xi$. With two incident photons, the photon detector will be clicked with probability $R_2 = 2\xi - \xi^2 < 2\xi$. For calculation simplicity we shall use $2\xi$ to replace $R_2$. This will overestimate the effect caused by 3-pair emission. Also we shall count all 5-fold clicking events as $C_4$, this will further overestimate the 3-pair emission effect because in a real experiment one may discard all 5-fold clicking.
events. With these two overestimation, what we shall calculate is the upper bound of the detectable error rate with 3-pair emission and limited detector efficiency being taken into consideration. The total probability of $C_4$ event caused by all 3-pair emission is upper bounded by

$$\lambda_3 = \xi^4 \left[ (1 - \eta)^2 \cdot \frac{19}{48} + (1 - \eta)\eta \cdot \frac{7}{24} + \eta^2 \cdot \frac{17}{48} \right] \cdot \frac{3p^3}{4}. \tag{30}$$

We have known that the probability of $C_4$ events caused 2-pair emission is $\lambda_2 = \frac{1}{16} \cdot \xi^4 \eta^2 p^2$, which corresponds to the error rate in the idea case, i.e. eq.(18,9). Therefore the observed value for $N_4/(N_1 + N_4)$ will be upper bounded by

$$E'_c = (1 + \lambda_3/\lambda_2)E_c. \tag{31}$$

Note that for whatever detection efficiency, the observed error rate is always upper bounded by $E'_c$. The observed error rate is higher than that in the ideal case due to the joint effect of non-perfect detection efficiency and the 3-pair emission. However, this does not affect the main result in a real experiment. As one may see from Fig.(2), with the QERC, the observed upper bound of error rate $E'_c$ is very close to the ideal one if the one pair emission rate is not larger than 0.002. Given such an emission rate, one may collect dozens of 4-fold clicking data per hour.

V. CONCLUDING REMARKS

In summary, we have shown how to encode and decode a type of 2-qubit quantum error rejection code with spontaneous parametric down conversion. In our scheme, we require beam $3''$ and beam $I1$ each contain exactly 1 photon. To verify this by our current technology we have no choice but to detect both of them. This means that the result is tested by post selection. However, as it was pointed out in Ref. [15], even a post selection result here has a wide application background such as the quantum cryptography and quantum communication. The details of the application of the post-selection quantum error rejection
code in quantum cryptography with hostile channel has been studied in [15]. Obviously, if our scheme is used for quantum key distribution (QKD), the threshold of error rate [18] of noisy channel is improved. A modified scheme can be used to reject the phase flip error. This may help to improve the tolerable channel flip rates of Gottesman-Lo protocol [18]. Details of this have been reported elsewhere [19]. Note that for the purpose of QKD, the encoding process can be omitted. One directly produces and sends the 2-bit code. In such a way, we may transmit thousands of 2-bit codes per second by our currently existing technology.

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REFERENCES

[1] P. W. Shor, Phys. Rev. A 52, 2493(1995).

[2] A. M. Steane, Phys. Rev. Lett. 77, 793(1996); Proc. R. Soc. Lond. A452, 2551(1996).

[3] C. H. Bennett et al, Phys. Rev. A 54, 3825(1996), and R. Laflamme et al, Phys. Rev. Lett. 77, 198(1996).

[4] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2000.

[5] W. K. Wootters and W. H. Zurek, Nature(London), 299, 802(1982).

[6] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A.V. Sergienko, and Y. H. Shih, Phys. Rev. Lett. 75, 4337(1995); A. G. White et al,Phys. Rev. Lett. 83, 3103(1999).

[7] D. Bouwmeester, J-W Pan, K. Mattle, M. Eibl, H. Weinfurter and A. Zeilinger, Nature, 390, 575(1997); S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. 80, 869(1998); A. Furusawa et al, Science, 706(1998); E. Lombardi et al, Phys. Rev. Lett. 88, 070402(2002).

[8] C. Simon, G Weihs and A. Zeilinger, Phys. Rev. Lett. 84, 2993(2000); A. Lamas-Linares et al, Science 296, 712-714(2002).

[9] F. De Martini, V. Buzek, F. Sciarrino and C. Sias, Nature, 419, 815(2002).

[10] J. W. Pan et al, Nature, 410, 1067(2001); T. Yamamoto et al, Nature, 421, 343(2003); Z. Zhao, J. W. Pan and M. S. Zhan, Phys. Rev. A 64, 014301(2001); Z. Zhao, T. Yang, Y. A. Chen, A. N. Zhang and J. W. Pan, Phys. Rev. Lett. 90, 207901(2003); J. W. Pan et al, Nature 423, 417(2003).

[11] X. B. Wang, Phys. Rev. A 68, 060302(R) (2003).

[12] T. B. Pittman, B. C. Jacobs and J. D. Franson, Phys. Rev. Lett., 88, 257902(2002);
Phys. Rev. A, 64, 062311(2001); Phys. Rev. A, 66, 052305(2002); T. B. Pittman, M. J. Fitch, B.C. Jacobs, and J. D. Franson, arXiv:quant-ph/0303095 (2003).

[13] The NMR demonstration is not a strict demonstration since there is no quantum entanglement involved.

[14] T. Yamamoto, T. Koashi and N. Imoto, Phys. Rev. A 64, 012304(2001).

[15] D. Bouwmeester, Phys. Rev. A, 63, 040301(R), 2001.

[16] D. Bouwmeester, J-W Pan, M. Daniell, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 82, 1345(1999).

[17] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature(London) 403, 515(2000).

[18] D. Gottesman and H.-K. Lo, quant-ph/0105121.

[19] X. B. Wang, quant-ph/0306156, quant-ph/0308057.
FIG. 1. Realizing QERC with SPDC process. If beams I0, x0 and y0 each contain exactly one photon, beam I1 is accepted, otherwise it is rejected. The error rate of all the accepted beams is the $N_4/(N_1 + N_4)$, where $N_1$ and $N_4$ are the number of 4-fold clicking of (D0,D2,D3,D1) and (D0,D2,D3,D1) respectively. The dashed rectangular boxes play the role of bit flip channels. NC: nonlinear crystal used in SPDC process. M: mirror. Ph: horizontal polarizer. HWP2 and HWP3: $\pi/4$ half wave plates. HWP1: $\gamma/2$ half wave plate. Pv+,Pv−: $\phi, -\phi$ phase shifters to vertically polarized photons only. PBS: polarizing beam splitter which transmits the horizontally polarized photons and reflects the vertically polarized photons. D0,D1,D2,D3,D4: photon detectors. RPBS: rotated polarizing beam splitter which transmits the photon in the state $\cos\frac{\gamma}{2}|H\rangle + \sin\frac{\gamma}{2}|V\rangle$ and reflects the photon in state $(\sin\frac{\gamma}{2}|H\rangle - \cos\frac{\gamma}{2}|V\rangle)$. P, -P, P1 and -P1: phase shifters, each of takes a phase shift $\theta, -\theta, \theta_1, -\theta_1$ respectively to a vertically polarized photon only. $\theta, \theta_1$ each is a random value from $\pm \frac{\pi}{2}$. E: $\sin^{-1}\frac{\sqrt{\epsilon}}{\sqrt{1+\epsilon}}$ half wave plate.
FIG. 2. Effects of 3-pair emission and limited detection efficiency to practical experiments. The horizontal axis is for the bit flipping rate $\eta$ of the channel. The vertical axis is for the error rates. The top straight line is for $E_0$: the expected result in the case that all qubits are sent directly through the bit flip channel, without using QERC. The lowest curve is for $E_c$: the expected result in the idea case: sending the qubit with perfect QERC. The curve upper to the lowest curve is for $E'_c$, the upper bound of the expected result in the practical case of sending qubits with a non-perfect QERC with SPDC process. All calculations are done by taking average over the 4 states of eq.(29). The distortion comes from the 3-pair emission and the limited efficiency of the photon detectors. Fig. A,B,C,D are for the case of one pair emission probability $p = 1/100, 5/1000, 2/1000, 1/1000$ respectively in the SPDC process. Note that with whatever detection efficiency, the observed error rate is always upper bounded by $E'_c$. We can obviously see that the distortion caused by 3-pair emission and the limited detecting efficiency are negligible when $p \leq 2/1000$. 

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