A SPACE LEVEL LIGHT BULB THEOREM FOR DISKS

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Abstract. Given a knot \( s : S^{k-1} \to \partial M \) with a framed dual sphere \( G : S^{d-k} \to \partial M \), we describe the homotopy type of the space of neat embeddings \( \mathbb{D}^k \to M^d \), with boundary \( s \), as the loop space of neatly embedded \((k-1)\)-disks in the manifold obtained from \( M \) by attaching a handle to \( G \). As a consequence, we conclude that the Dax invariant gives a complete isotopy classification of such disks in a 4-manifold. Moreover, we compute the resulting group structure on the set of these isotopy classes and show that it is nonabelian in most cases. Finally, we recover all previous results for spheres and prove that for them the Dax invariant reduces to the Freedman-Quinn invariant.

1 Introduction and survey of results

This paper started off as an attempt to understand a result of Dave Gabai [Gab20a] showing that an isotopy invariant going back to Dax makes the case of disks in 4-manifolds much richer than that of spheres. We realized that methods of Cerf and Dax can be applied to not only construct this invariant but show that it completely classifies isotopy, Theorem A, in the setting of the so called “light bulb trick”.

In Section 1.1.1 we show how our results imply all previous results for spheres [Gab20b; ST19], in Section 1.1.3 we discuss the resulting group structure on isotopy classes of disks, and in Section 1.2 we give an overview of our results regarding spaces of embedded disks in all dimensions.

1.1 The 4-dimensional light bulb theorem for disks with dual in the boundary

For a smooth, compact, connected 4-manifold \( M \) we study the space of neat\(^1\) embeddings \( \text{Emb}_0(\mathbb{D}^2, M) \) of a disk into \( M \) that restrict to a fixed knot \( s : S^1 \to \partial M \) on the boundary. Consider the free abelian group \( \mathbb{Z}[\pi \setminus 1] \) on the set of nontrivial elements in the fundamental group \( \pi = \pi_1 M \), with its usual involution \( \sigma \) induced by \( g \mapsto g^{-1} \). Dax [Dax72] discovered an interesting homomorphism

\[
dax : \pi_3 M \to \mathbb{Z}[\pi \setminus 1]^\sigma := \{ r \in \mathbb{Z}[\pi \setminus 1] : r = \tau \}
\]

of abelian groups, that was recently made explicit by Gabai [Gab20a]. The element \( \text{dax}(a) \) counts self-intersections of certain generic immersions \( \mathbb{D}^3 \hookrightarrow \mathbb{D}^2 \times M^4 \) obtained from \( a : S^3 \to M \), with fundamental group elements defined by a clever choice of sheets at each double point, avoiding the usual indeterminacy \( g = -g \) for self-intersection invariants in dimension 6.

This count of Dax also works for a homotopy \( H \) between \( K_0, K_1 \in \text{Emb}_0(\mathbb{D}^2, M) \), leading to the Dax invariant \( \text{Dax}(H) \in \mathbb{Z}[\pi \setminus 1]^\sigma \). This behaves additively under gluing homotopies and the homomorphism \( \text{dax} \) above results from the special case where \( H \) is a self-homotopy. As a consequence, the class of \( \text{Dax}(H) \) modulo the image of \( \text{dax} \) is independent of \( H \) and one has the following result, see also [Gab20a, Cor.0.5].

Lemma 1.1. The (relative) Dax invariant \( \text{Dax}(K_0, K_1) \in \mathbb{Z}[\pi \setminus 1]^\sigma / \text{dax}(\pi_3 M) \) is defined for homotopic embeddings \( K_0, K_1 \) and is an obstruction for the existence of an isotopy between them.

Even for \( M = \mathbb{D}^4 \) the vanishing of \( \text{Dax}(K_0, K_1) \) is not a sufficient condition for \( K_t \) to be isotopic, the fundamental group of the complements is a simple additional invariant. Moreover, in general not every element in \( \mathbb{Z}[\pi \setminus 1]^\sigma / \text{dax}(\pi_3 M) \) is realized as the Dax invariant relative to a given \( K_0 \).

However, we show that both of these statements hold true under the following assumption: the boundary condition \( s : S^1 \to \partial M \) has a geometric dual sphere \( G : S^2 \hookrightarrow \partial M \), meaning that \( G \) intersects \( s \) transversely and in a single positive point (it implies that removing \( K \) from \( M \) does not change the fundamental group).
This puts us in the realm of the “light bulb theorem” proven for spheres in 4-manifolds in [Gab20b; ST19]. As explained in Section 1.1.1 below, those results are special cases of the following theorem for disks.

**Theorem A.** In the presence of a dual sphere \( G \), two embeddings \( K_0, K_1 \in \text{Emb}_0(D^2, M) \) are isotopic if and only if they are homotopic and the resulting relative Dax invariant vanishes, \( \text{Dax}(K_0, K_1) = 0 \). Moreover, for a given \( K \) any \( r \in \mathbb{Z}[\pi \setminus 1]^* \) is realized in the following sense: There is an explicit disk \( K + \text{fm}(r)^G \) homotopic to \( K \), as in Figure 1, such that \( r \equiv \text{Dax}(K + \text{fm}(r)^G, K) \in \mathbb{Z}[\pi \setminus 1]^*/\text{dax}(\pi_3M) \).

The realization result for a given pair \((K, r)\) was previously obtained in [Gab20a, Thm.0.6(ii)] but using a different construction involving self-referential disks. Our geometric action \((r, K) \mapsto K + \text{fm}(r)^G\) of the group \( \mathbb{Z}[\pi \setminus 1] \) on the set \( \text{Emb}_0(D^2, M) := \pi_0 \text{Emb}_0(D^2, M) \) of isotopy classes of embeddings is as follows, see Figure 1: perform finger moves to \( K \) along the group elements that make up \( r \in \mathbb{Z}[\pi \setminus 1] \) (each finger move introduces a pair of double points), and then revert this back into an embedding by adding a tube for each double point (along distinct choices of sheets for the given double point pair) into a parallel copy of \( \pm G \). This tubing into the dual is often called the Norman trick, and this action was studied in [ST19] to obtain embedded Norman spheres. We will discuss a space version of this trick in Section 5.3.1.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Constructing all embeddings in the homotopy class of \( K \) by finger moves and then Norman tricks.

On (relative) homotopy classes of maps we have \([K + \text{fm}(r)^G] = [K \# (\pi - r) \cdot G]\), which equals \([K]\) precisely for \( r \in \mathbb{Z}[\pi \setminus 1]^* \). The next corollary implies that the stabilizer of this \( \mathbb{Z}[\pi \setminus 1] \)-action for any \( K \) is the same subgroup \( \text{dax}(\pi_3M) \leq \mathbb{Z}[\pi \setminus 1] \). There we denote by \( \mu_2 \) Wall’s (reduced) self-intersection invariant that counts double points (with signs and fundamental group elements) of a generic immersion \( J: D^2 \hookrightarrow M \) representing the given \([J] \in \text{Map}_0(D^2, M) := \pi_0 \text{Map}_0(D^2, M) \). The element \( \mu_2[J] \) is an obstruction for representing such a homotopy class by an embedding and it is the only obstruction in our setting.

**Corollary 1.2.** If \( s \) is null homotopic in \( M \), there is an exact sequence of groups/set

\[
\mathbb{Z}[\pi \setminus 1]^*/\text{dax}(\pi_3M) \xrightarrow{\text{fm}(r)^G/\text{Dax}} \text{Emb}_0(D^2, M) \xrightarrow{\text{Map}_0} \mathbb{Z}[\pi \setminus 1]/\langle r - \pi \rangle
\]

More explicitly, this means that \( \mu_2 \) is surjective, \( \text{im}(j) = \mu_2^{-1}(0) \), and for all \( J \in \text{im}(j) \) the group \( \mathbb{Z}[\pi \setminus 1]^* \) acts via \((K, r) \mapsto K + \text{fm}(r)^G\) transitively on the fibers \( j^{-1}(J) \), with stabilizer \( \text{dax}(\pi_3M) \). Finally, for any \( K \in j^{-1}(J) \), the relative Dax invariant \( \text{Dax}(\bullet, K) : j^{-1}(J) \xrightarrow{\sim} \mathbb{Z}[\pi \setminus 1]^*/\text{dax}(\pi_3M) \) inverts this action.

In particular, if \( \pi_1M = 1 \) we obtain a bijection \( j: \text{Emb}_0(D^2, M) \cong \text{Map}_0(D^2, M) \), so every homotopy class of disks with boundary \( s \) contains a unique embedding up to isotopy, reproving [Gab20a, Thm.0.6(i)].

**Example 1.3.** Consider the interior connected sum \( M := (S^2 \times D^2) \sharp M' \), with \( U = S_0 \times D^2 \) and \( G = S^2 \times 0 \) both lying in the \( S^2 \times D^2 \) part of \( M \). The separating 3-sphere \( S^3 \hookrightarrow M \) has \( \text{dax}(S) = 0 \) as it is embedded. However, the class \( g \cdot [S] \in \pi_3M \), obtained by the usual \( \pi \)-action on \( \pi_3M \), is represented by adding a tube along \( g \) to \( S \), and is no longer embedded: \( g \) must intersect \( S \) twice as \( \pi \) is concentrated on the other side of \( S \). We will compute \( \text{dax}(g \cdot [S]) = g + \pi \) in Lemma 5.22, and this will imply that the Dax invariant in this case reduces to the Freedman–Quinn invariant of [ST19], suitably adapted to disks in Definition 5.10.

For \( M' = S^1 \times D^3 \) this gives a non-simply connected example where \( \mathbb{Z}[\pi \setminus 1]^*/\text{dax}(\pi_3M) \) is trivial, so in \( M \) there is at most one isotopy class of embeddings in a given homotopy class. In contrast, we will see in Example 1.10 that the boundary connected sum of \( S^2 \times D^2 \) and \( S^1 \times D^3 \) has an interesting Dax invariant, utilized in [Gab20a, Thm.0.8] to construct in this manifold a 3-ball homotopic but not isotopic to \( p \times D^3 \).
1.1.1 Comparison with the light bulb theorem for spheres. Let $G$: $S^2 \hookrightarrow N$ be a framed sphere in the interior of an oriented 4-manifold $N$, with a meridian disk $m_G$. The boundary of $M := N \setminus \nu(G)$ contains the meridian circle $s = \partial m_G$ and also a dual sphere for it, namely a push-off of $G$ by the framing. Thus, we are in the setting of Theorem A and we can compare the two isotopy classifications at hand, assuming again that $s$ is null homotopic in $M$ (to make our sets nonempty).

Define the sets $\text{Emb}^G[S^2, N]$ of isotopy classes of embeddings $F$: $S^2 \hookrightarrow N$ that are dual to $G$, and $\text{Map}^G[S^2, N]$ of based homotopy classes $s$: $S^2 \rightarrow N$ with $\lambda(S, G) = 1$, based at any point of $G(S^2)$.

**Proposition 1.4.** There is a commutative diagram of exact sequences of groups/sets

\[
\begin{align*}
Z[\pi_1 M \setminus 1]^\sigma/\text{dax}(\pi_3 M) & \xrightarrow{\text{fn}(\cdot)^G/\text{Dax}(\pi_3 M)} \text{Emb}_0[D^2, M] \xrightarrow{j} \text{Map}_0[D^2, M] \xrightarrow{\mu_2} Z[\pi_1 M \setminus 1]/(r - \bar{r}) \\
\cong & \\
F_2 T_N/\text{mu}_3(\pi_3 N) & \xrightarrow{\text{fn}(\cdot)^G/\text{F}_Q} \text{Emb}^G[S^2, N] \xrightarrow{j} \text{Map}^G[S^2, N] \xrightarrow{\mu_2} Z[\pi_1 N \setminus 1]/(r - \bar{r})
\end{align*}
\]

Moreover, the first and last vertical maps are group isomorphisms and the second one is equivariant.

The leftmost group in the bottom is a quotient of the $F_2$-vector space generated by 2-torsion $T_N$ in $\pi_1 N$, and it is the target of the Freedman–Quinn invariant $FQ(F, \ast): j^{-1}(S) \xrightarrow{\cong} F_2 T_N/\mu_3(\pi_3 N)$. This was defined and shown to be an isomorphism in [ST19], together with the exactness of the lower sequence. We will show in Section 5.4 how those results follow from our upper sequence; a crucial step was sketched in Example 1.3 above, which applies to the manifold $N \setminus \nu(G)$ as it decomposes as $(S^2 \times D^2)\setminus M'$.

In fact, in Section 5.2.1 we will show that the Dax invariant always determines the Freedman–Quinn invariant for disks, and note that if the component of $\partial M$ containing $G$ is not diffeomorphic to $S^1 \times S^2$, then Dax is a strictly stronger invariant in general, taking values in a much larger abelian group (torsion higher than just 2-torsion and also torsion-free parts arise in many examples, see Section 1.1.2).

**Remark 1.5.** In [Gab20b] as well as [ST19], the authors only assume that the given spheres with dual $G$ are homotopic, not based homotopic. However, in both cases they argue at the beginning of the proofs why such a homotopy can also be found in the based setting, see [ST19, Lem.2.9] for an elementary argument.

1.1.2 Computing the homomorphism $\text{dax}$. To apply our results to particular 4-manifolds, we need a way to compute the homomorphism $\text{dax}: \pi_3 M \rightarrow Z[\pi_1 \setminus 1]^\sigma$. We prove the next result in Section 5.1.3, assuming only that $M$ is a compact, connected, oriented 4-manifold with non-trivial boundary.

**Theorem B.** The following is a commutative diagram of exact sequences of groups:

\[
\begin{align*}
\Gamma(\pi_2 M) & \xrightarrow{- \circ H} \pi_3 M \xrightarrow{\text{Hur}} H_3(\bar{M}; \mathbb{Z}) \\
\xrightarrow{\mu_2} & \xrightarrow{\text{dax}} Z[\pi_1 \setminus 1]/(g - \bar{g}) \xrightarrow{\mu_3} Z[\pi_1 \setminus 1]^\sigma/(1, g + \bar{g})
\end{align*}
\]

The first row is Whitehead’s “certain exact sequence” [Whi50], with a homomorphism out of $\Gamma(\pi_2 M)$ by definition the same as a quadratic map out of $\pi_2 M$. Precomposing with the Hopf map $H$: $S^3 \rightarrow S^2$ is one such map; another is Wall’s self-intersection invariant $\mu_3$ from Corollary 1.2. This is quadratic with bilinear form $[\lambda]$, the reduction of the intersection form $\lambda: \pi_2 M \times \pi_2 M \rightarrow Z[\pi_1 \setminus 1]$ of $M$ to $Z[\pi_1 \setminus 1]/(1, g - \bar{g})$.

In this target, the hermitian form $\lambda$ becomes symmetric: $[\lambda(a_1, a_2)] = [\lambda(a_2, a_1)] = [\lambda(a_2, a_1)]$ as required by the $\Gamma$-functor.

The map $\mu_3$ on the right is also a Wall invariant, counting self-intersections of $\pi_1$-trivial 3-manifolds immersed in $M \times \mathbb{R}^2$, where the $\mathbb{R}^2$-factor makes $\mu_3$ linear because the bilinear intersection term vanishes.

The target of $\mu_3$ is isomorphic to $F_2[T_M]$ and its cokernel is the target of the Freedman-Quinn invariant in Proposition 1.4. Precomposition with the Hopf map is injective because the next term in the Whitehead sequence is $H_4(\bar{M}; \mathbb{Z})$ which vanishes for 4-manifolds with nontrivial boundary.
For closed 4-manifolds \( M \), the entire diagram in Theorem B still exists, except for the map \( \dax \), where our definition requires a nontrivial boundary. In fact, there is an \( \mathbb{S}^2 \)-bundle \( M \) over \( \mathbb{R}P^2 \), Example 5.12, for which there cannot be a homomorphism \( \pi_3 M \rightarrow \mathbb{Z}[\pi \setminus 1]^\sigma \) making the left square commute.

**Corollary 1.6.** Given \( a, a_1, a_2 \in \pi_3 M \) and the Whitehead product \( [a_1, a_2]_W \in \pi_3 M \) we have

\[
\dax(a \circ H) = \tilde{\lambda}(a, a) \quad \text{and} \quad \dax([a_1, a_2]_W) = \tilde{\lambda}(a_1, a_2) + \tilde{\lambda}(a_2, a_1) \in \mathbb{Z}[\pi \setminus 1]^\sigma,
\]

where \( \tilde{\lambda} := \lambda - \lambda_1 \), for the intersection form \( \lambda : \pi_2 M \times \pi_2 M \rightarrow \mathbb{Z}[\pi] \) and its coefficient \( \lambda_1 \) at \( 1 \in \pi \).

**Proof.** The left commuting square in Theorem B says that \( \dax(a \circ H) = \mu_2(a) + \overline{\mu_2(a)} \), and this equals \( \tilde{\lambda}(a, a) \) modulo the coefficient of 1. The Whitehead product formula follows from the quadratic property 
\( (a_1 + a_2) \circ H = a_1 \circ H + a_2 \circ H + [a_1, a_2]_W \), already used in Whitehead’s sequence. \( \square \)

A useful tool for computing \( \dax \) is to consider the action of \( \pi = \pi_1 M \). It acts on all groups in Theorem B: on the bottom three by the conjugation action on \( \mathbb{Z}[\pi] \), and on homotopy groups of \( M \) by changing whiskers. The maps \( \mu_2, \mu_3 \) are \( \pi \)-equivariant, \( \mu_i(g \cdot a) = g \cdot \mu_i(a) \cdot \gamma \), but for \( \dax \) we have

\[
\dax(g \cdot a) = g \cdot \dax(a) \cdot \gamma + a_g + \overline{a_g},
\]

where the additional term \( a_g \in \mathbb{Z}[\pi \setminus 1] \) is the equivariant intersection number of the loop \( g \) and the 3-sphere \( g \cdot a \). Note that if \( a = b \circ H \) for \( b \in \pi_2 M \) then \( a_g = 0 \), since \( a(\mathbb{S}^3) = b(\mathbb{S}^2) \) is “2-dimensional” and hence generically disjoint from any 1-dimensional loop \( g \). Thus, the displayed formula is consistent with the equivariance of \( \mu_2 \). We will not use it, so we leave its proof to a future paper.

Instead, let us show the utility of Theorem B on a class of examples. For a framed, null homotopic knot \( c : \mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^2 \) define the 4-manifold \( M_c \) by attaching a 2-handle to the boundary of \( \mathbb{S}^1 \times \mathbb{D}^3 \) along \( c \), see Figure 2 for an example. The fundamental group is infinite cyclic \( \pi \cong \mathbb{Z} \) with generator \( t = \mathbb{S}^1 \times p \) and \( M_c \cong \mathbb{S}^1 \times \mathbb{S}^2 \) because the attaching circle \( c \) is null homotopic. As there is no 2-torsion in \( \pi \), the norm map \( g \mapsto g + \overline{g} \) in Theorem B is an isomorphism, so we canonically identify the target \( \mathbb{Z}[\mathbb{Z} \setminus 1]/\langle t^n - t^{-n}, n \in \mathbb{Z} \rangle \) of \( \dax \) with the ideal \( t \cdot \mathbb{Z}[t] \) in the polynomial ring \( \mathbb{Z}[t] \).

\[
\begin{align*}
\text{Figure 2. A Kirby diagram for } M_c & \text{ with } c \text{ on the right, one } k \text{-handle for } k = 0, 1, 2 \text{ and } \mu_2(S_c) = -t + t^2. \\
\text{Let } S_c \in \pi_2 M_c & \text{ be represented by the sphere built out of the core of the 2-handle, together with a null homotopy of } c \text{ in } \mathbb{S}^1 \times \mathbb{S}^2. \text{ To compute } \mu_2(S_c) \in t \cdot \mathbb{Z}[t] \text{ one decomposes the null homotopy into a sequence of finger moves, and then sums signed powers of } t \text{ for each self-intersection as in Figure 2.} \\
\text{Lemma 1.7. The group } \dax(\pi_2 M_c) & \text{ is the principal ideal } \mu_2(S_c) \cdot \mathbb{Z}[t] \subseteq t \cdot \mathbb{Z}[t]. \text{ Moreover, for any polynomial } f \in t \cdot \mathbb{Z}[t] \text{ there exists a framed null homotopic knot } c : \mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^2 \text{ with } \mu_2(S_c) = f. \\
\text{It follows that any quotient } t \cdot \mathbb{Z}[t]/f & \text{ arises as } \mathbb{Z}[\pi \setminus 1]^\sigma /\dax(\pi_3 M_c) \text{ for some knot } c. \text{ For example, } t \cdot \mathbb{Z}[t]/(nt^m) \cong \mathbb{Z}^{m-1} \times (\mathbb{Z}/n)^\infty. \text{ The group } t \cdot \mathbb{Z}[t]/f \text{ is finitely generated if and only if the leading coefficient of } f \text{ is } \pm 1; \text{ actually, any finitely generated abelian group can be realized using appropriate } f. \\
\text{This discussion of } \dax \text{ applies to any 4-manifold } M; \text{ to use it for Theorem A, we need a boundary condition } s : \mathbb{S}^1 \hookrightarrow \partial M \text{ with a dual sphere } G. \text{ For } M = M_c \text{ these exist only if } c \text{ is a 0-framed unknot, giving exactly Example 1.10. However, in the boundary connected sum } S = M_c \cup (\mathbb{S}^2 \times \mathbb{D}^2) \text{ we can take } s = p \times \mathbb{S}^1, G = \mathbb{S}^2 \times 0. \text{ Then from Theorem B it easily follows that } \dax(\pi_3 M) = \dax(\pi_3 M_c), \text{ so we have } \mathbb{Z}[\pi \setminus 1]^\sigma /\dax(\pi_3 M) \cong t \cdot \mathbb{Z}[t]/\mu_2(S_c), \text{ in which any element is realized as the relative Dax invariant.}
\end{align*}
\]
1.1.3 Group structure on \( \text{Emb}_0[\mathbb{D}^2, M] \). As above, \( \text{Emb}_0[\mathbb{D}^2, M] \) is the set of isotopy classes of neatly embedded disks with boundary circle \( s \), which has a geometric dual sphere \( G \) in \( \partial M \). We pick an arbitrary basepoint \( U \) (an “undisk”), and use it to equip this set with more structure as follows.

**Theorem C.** In the above setting there is an exact sequence of groups and homomorphisms

\[
\mathbb{Z}[\pi \setminus 1]^{\sigma}/\text{dax}(\pi_3 M) \xrightarrow{\text{U + fm}^{(\sigma)}_G} \text{Emb}_0[\mathbb{D}^2, M] \xrightarrow{j} \text{Map}_0[\mathbb{D}^2, M] \xrightarrow{\mu_2} \mathbb{Z}[\pi \setminus 1]/(r - r)
\]

where the first and last term have the usual abelian group structures. The group structures on the second and third term are new and in general nonabelian, with the chosen disk \( U \) as the trivial group element.

In [Gab20a, Prop.4.2] a group structure on \( j^{-1}[U] \) was given so that the relative Dax invariant is an epimorphism \( \text{Dax}(\cdot, U) : j^{-1}[U] \rightarrow \mathbb{Z}[\pi \setminus 1]^{\sigma}/\text{dax}(\pi_3 M) \). Our Theorem C not only says that this is an isomorphism of (abelian) groups, but also that the entire set of embeddings has a group structure that can be nonabelian, which we found quite surprising.

A group structure on \( \text{Emb}_0[\mathbb{D}^k, M^d] \) actually exists for all dimensions \( k \geq 1 \) and \( d \geq 2 \) in the presence of a framed dual sphere for \( s \) in \( \partial M \), and even at the space level: \( \text{Emb}_0[\mathbb{D}^k, M^d] \) is homotopy equivalent to a based loop space \( \Omega E \), so the group structure is the one of the fundamental group \( \pi_1 E \), see Section 1.2.

Gluing \(-U\) to a disk with the same boundary gives a bijection \(-U \cup \cdot : \text{Map}_0[\mathbb{D}^2, M] \xrightarrow{\cong} \pi_2 M \) where \( U \) corresponds to 0. However, the usual abelian group structure (\( \pi_2 M, + \)) does not translate to the one in Theorem C because of the quadratic property of Wall’s self-intersection invariant \( \mu_2 \). In fact, the correct group structure that turns \( \mu_2 \) into a homomorphism translates to the following multiplication on \( \pi_2 M \):

\[
F_1 \ast F_2 := F_1 + F_2 - \lambda(F_1, F_2) \cdot G \quad \text{for } F_i \in \pi_2 M.
\]

In particular, this group is abelian if and only if the equivariant intersection form \( \lambda \) on \( \pi_2 M \) is symmetric. Recall that \( \lambda \) is always hermitian, \( \lambda(F_1, F_2) = \overline{\lambda(F_2, F_1)} \in \mathbb{Z}[\pi] \), but rarely symmetric.

It follows that the inherited group structure on \( \pi_2 M/\mathbb{Z}[\pi] \cdot G \) is the usual abelian one. We can also describe the kernel of the map from \( \text{Emb}_0[\mathbb{D}^2, M] \) to this quotient using an extension of the \( \mathbb{Z}[\pi \setminus 1] \)-action on \( \text{Emb}_0[\mathbb{D}^2, M] \) to a \( \mathbb{Z}[\pi] \)-action. Namely, we let \( 1 \in \pi \) act by sending \( K \) to the disk \( K + \text{fm}(1)^G := K^G_\text{tw} \) obtained from \( K \) by adding one positive interior twist and tubing the resulting double point into \(-G\), see Figure 3. Moreover, we extend the involution \( \sigma(r) = r \) from \( \mathbb{Z}[\pi \setminus 1] \) to a self-map of \( \mathbb{Z}[\pi] \) by \( \sigma(1) = 0 \).

**Proposition 1.8.** The following is a commutative diagram of central extensions, with the connecting map from the upper right to the lower left given as the identity:

\[
\begin{array}{ccc}
\mathbb{Z}[\pi \setminus 1]^{\sigma}/\text{dax}(\pi_3 M) & \xrightarrow{\text{U + fm}^{(\sigma)}_G} & \text{Emb}_0[\mathbb{D}^2, M] \\
\downarrow & & \downarrow j \\
\mathbb{Z}[\pi]/\text{dax}(\pi_3 M) & \xrightarrow{U \# (\sigma - \text{Id}) \cdot G} & \mathbb{Z}[\pi \setminus 1]^{\sigma}/\mu_2^{-1}(0) \\
\downarrow & & \downarrow -U \cup \cdot \\
\mathbb{Z}[\pi]/\text{dax}(\pi_3 M) & \xrightarrow{U + \text{fm}(\sigma)^G} & \text{Emb}_0[\mathbb{D}^2, M] \\
& & \downarrow \pi_2 M/\mathbb{Z}[\pi] \cdot G \\
\downarrow & & \\
\mathbb{Z}[\pi]/\mathbb{Z}[\pi \setminus 1]^{\sigma} & \xrightarrow{\text{U + fm}^{(\sigma)}_G} & \mathbb{Z}[\pi \setminus 1]^{\sigma}/\mu_2^{-1}(0)
\end{array}
\]

We have thus written the unknown group \( \text{Emb}_0[\mathbb{D}^2, M] \) as an extension in two different ways, where the “difference group” \( \mathbb{Z}[\pi]/\mathbb{Z}[\pi \setminus 1]^{\sigma} \) is moved from the right to the left when we go from the upper to the lower extension. Since the extensions are central, the lower one is simpler, and its group commutator pairing is just a skew-symmetric map from two copies of that abelian group to the center \( \mathbb{Z}[\pi]/\text{dax}(\pi_3 M) \).

The next proposition identifies this pairing as the reduced intersection form \( \tilde{\lambda} \), but this is not skew-symmetric when considered in \( \mathbb{Z}[\pi] \). Fortunately, Corollary 1.6 says that modulo \( \text{dax}(\pi_3 M) \) it is!
Proposition 1.9. For the commutator of $K_1, K_2 \in \Emb_0[\mathbb{D}^2, M]$ we have
$$[K_1, K_2] = U + \text{fm}(\lambda)^G,$$
with $\lambda = \bar{\lambda}(-U \cup K_1, -U \cup K_2) \in \mathbb{Z}[\pi \setminus 1]$ the reduced intersection form as in Corollary 1.6. Moreover, there is an isomorphism of groups
$$\eta \times p : \Emb_0[\mathbb{D}^2, M] \cong \mathbb{Z} \times \Emb_0[\mathbb{D}^2, M] / (U_{tw}/).$$
Here $\eta(K) := (e_\nu(K, U) - W(-U \cup K))/2$, where $e_\nu(K, U) = \langle e(\nu K, \nu U), [\mathbb{D}^2] \rangle$ is the Euler number of the normal bundle $\nu K$, relative to the framing on the boundary given by $\nu U$, while the second term is $\langle W(-U \cup K), [S^2] \rangle$ for $W \in H^2(\mathbb{M}; \mathbb{Z}) \cong \text{Hom}(\pi_2 M, \mathbb{Z})$ with $W(G) = 0$, a $\mathbb{Z}[\pi]$-equivariant integral lift of the Stiefel-Whitney class $w_2(T\mathbb{M})$ of the tangent bundle of the universal covering.

Such a lift $W$ exists because any oriented 4-manifold has a spin$^c$ structure; we also choose $W = 0$ whenever $\mathbb{M}$ is spin. The difference in $\eta$ is indeed divisible by two thanks to the stability of $w_2$ and the property $e \equiv w_2 \pmod{2}$ for 2-dimensional vector bundles. We will see that $\eta$ is indeed a splitting, i.e. a homomorphism that takes $U_{tw}/$ to 1 in Section 5.3.

Example 1.10. Consider the boundary connected sum $M := (\mathbb{S}^2 \times \mathbb{D}^2)\#(\mathbb{S}^1 \times \mathbb{D}^3)$ with $U$ and $G$ as in Example 1.3 and denote by $z$ a generator of $\pi \cong \mathbb{Z}$. Then one easily computes $dax = 0$, see [Gab20a, Thm.3.10]. Thus, the lower exact sequence in Proposition 1.8 says that the action on $U$ induces an isomorphism $\mathbb{Z}[z, z^{-1}] \cong \Emb_0[\mathbb{S}^1, M]$, even though there are just $\mathbb{Z}[z, z^{-1}]/(z + z^{-1}) \cdot \mathbb{Z}[(z + z^{-1})]$ many homotopy classes. This result is completely new and very different from the spherical case.

In this example, the group $\Emb_0[\mathbb{D}^2, M]$ is free abelian of infinite rank. By our computation of the commutator pairing in Proposition 1.9, to find a nonabelian example it suffices we find classes $a_1, a_2 \in \pi_2 M$ such that $\lambda(a_1, a_2) \notin \mathbb{Z}[\pi]$. For example, take $M = M_0 \# M_1$ for any simply connected 4-manifold $M_0$ that has a nontrivial (Z-valued) intersection form $\lambda_0$ and where $\pi_1 M_1$ contains an element $g$ with $g^2 \neq 1$. Then for $a_i \in \pi_2 M_0$ we have $\lambda_M(g \cdot a_1, a_2) = g \cdot \lambda_0(a_1, a_2) \neq \lambda_0(a_1, a_2)$ as long as $\lambda_0(a_1, a_2) \neq 0$.

Remark 1.11. Any $K_0, K_1 \in \Emb_0(\mathbb{D}^2, M)$ as above and with $e_\nu(K_0, K_1) = 0$ are related by a self-diffeomorphism of $M$ that is the identity on $\partial M$. After all, there is a diffeomorphism $M \nu K_1 \cong M \cup_G h^3$, since we are attaching a canceling 2-3 handle pair, so the composite $M \setminus \nu K_0 \cong M \setminus \nu K_1$ is the identity on $(\partial M) \setminus vs$. Since the framings agree, the map on the remaining $\mathbb{D}^2 \times \mathbb{S}^1 \cong \nu K_1$ is isotopic to the identity, so extends across $\nu K_1$. See [Sch19, Lem.2.3], [ST19, Lem.6.1] and [Gab20a, Lem.5.3].

1.2 A space level light bulb theorem for arbitrary dimensions

The following results are crucial steps in our proof of Theorem A and use ideas going back to Cerf [Cer68]. For a smooth, compact, connected, oriented $d$-dimensional manifold $M$, consider the space $\Emb_0(\mathbb{D}^k, M)$ of neat embeddings with fixed boundary value $s : \mathbb{S}^{d-1} \hookrightarrow \partial M$. Assume that $s$ has a framed geometric dual $G : \mathbb{S}^{d-k} \hookrightarrow \partial M$, i.e. $G$ comes with a trivialization of its normal bundle and $s \pitchfork G$ is a single point.
Let $M_G$ be the manifold obtained from $M$ by attaching to $\partial M$ a $d$-dimensional $(d-k+1)$-handle along $G$. Then $\partial M$ changes by a surgery on $G$ and since $s$ intersects $G$, only one part of it, called $u_-$, lies in $\partial M_G$. In fact, $s = u_- \cup_{\partial M} u_+$ with $u_- : \mathbb{D}^{k-1} \to \partial M_G$, whereas $u_+ : \mathbb{D}^{k-1} \to M_G$ is a neat embedding.

**Theorem D.** Any $U \in \text{Emb}_0(\mathbb{D}^k, M)$ leads to a a fibration sequence\(^2\)

$$\text{Emb}_0(\mathbb{D}^k, M) \xrightarrow{f_U} \Omega \text{Emb}_0(\mathbb{D}^{k-1}, M_G) \xrightarrow{\delta_{\nu_G}} \Omega^{k-1}\mathbb{S}^{d-k}$$

where the loop $f_U(K)$ based at $u_+$ is obtained by foliating $\mathbb{D}^k$ by $\mathbb{D}^{k-1}$.

One can try to define the map $f_U$ as follows. Consider the foliation of $K$ by $K(t) : \mathbb{D}^{k-1} \to M \subset M_G$, using a parametrization $\mathbb{D}^k \simeq \mathbb{D}^{k-1} \times I$; this is a path from $u_-$ to $u_+$, so to get a closed loop use the inverse of such a foliation of $U$. However, $u_-$ is not neat so does not lie in the space $\text{Emb}_0(\mathbb{D}^{k-1}, M_G)$. One way around this would be to enlarge this space to also include embeddings that lie in $\partial M_G$; after all, they are limits of neat embeddings and we believe the homotopy type of the space does not change.

We opted for a second way, making the space $\text{Emb}_0(\mathbb{D}^k, M)$ smaller: consider its subspace $\text{Emb}_{\nu}(\mathbb{D}^k, M)$ consisting of those neat embeddings that agree with $U$ on a fixed collar $G \times [1-\varepsilon, 1] \subset \mathbb{D}^k$. Pick a neat embedding $u_+ : \mathbb{D}^{k-1} \hookrightarrow U(\mathbb{S}^{k-1} \times [1-\varepsilon, 1])$ that is close to $u_-$, and define $f_U(K) = f(U) \cdot f(K)$ as the concatenation of the path $U(t, -) \cup U(t, +)$ from $u_+$ to $u_-$, with the path $K(t, -) \cup K(t, +)$ from $u_-$ back to $u_+$. Together with the inclusion induced homotopy equivalence $\text{Emb}_{\nu}(\mathbb{D}^k, M) \simeq \text{Emb}_0(\mathbb{D}^k, M)$ from Proposition 2.5, this gives the map $f_U$ in Theorem D above.

**Remark 1.12.** Let us point out that existence of a disk $U : \mathbb{D}^k \to M$ ensures that the diffeomorphism type of $M_G$ is independent of the choice of a framing $\psi$ of $G$, and in fact, of the choice of a dual $G$ to $s$ all together. First note that $M$ is obtained from the complement $M \setminus \nu U$ of an open tubular neighborhood of $U$ by attaching a $(d-k)$-handle with cocore $U$. As $G$ is dual to $\partial U$ and has trivial normal bundle, the standard handle cancellation of this $(d-k)$- and $(d-k+1)$-handle pair gives a diffeomorphism

$$M_G \simeq M \cup (G, \psi), h^{d-k+1} \simeq (M \setminus \nu U) \cup h^{d-k} \cup h^{d-k+1} \simeq M \setminus \nu U.$$ 

So far we have only used the collar near $u_-$, but there is a way to make $f_U$ into a homotopy equivalence by also using it near $u_+$. Namely, foliate $\mathbb{D}^k$ by a 1-parameter family of thickened disks $\mathbb{D}^{k-1} \times [0, \varepsilon]$, so that $K \in \text{Emb}_{\nu}(\mathbb{D}^k, M)$ gives a path from $u_-^t$ to $u_+^t$ in the space $\text{Emb}_{\nu}(\mathbb{D}^{k-1}, M_G)$ of such $\varepsilon$-augmented $(k-1)$-disks, and then use $U$ to complete it to a loop at $u_+^t$. This will lead to the following key result.

**Theorem E.** Any $U \in \text{Emb}_{\nu}(\mathbb{D}^k, M)$ leads to a pair of inverse homotopy equivalences

$$\text{Emb}_{\nu}(\mathbb{D}^k, M) \xrightarrow{\simeq}{\sim} \Omega \text{Emb}_{\nu}(\mathbb{D}^{k-1}, M_G),$$

where $f_U$ is the $\varepsilon$-augmented foliation map, and for $n \geq 0$, the value of $\pi_0 a_U$ on an $\mathbb{S}^n$-family of isotopies $\mathbb{S}^1 \to \text{Emb}_{\nu}(\mathbb{D}^{k-1}, M_G)$ is obtained by applying the family version of the ambient isotopy theorem.

We prove this carefully in Section 3.2 using Cerf’s trick: we check contractibility of a space of half-disks in $M_G$, i.e. embeddings that restrict to $u_-$ on one part of the boundary and whose other part of the boundary is free to move in the interior of $M_G$, and use that the restriction map to the free boundary is a fibration by Cerf (see Section 2.3). The fibration sequence in Theorem D then follows from Theorem E using 3.3: forgetting the $\varepsilon$-augmentation is a fibration $\text{Emb}_{\nu}(\mathbb{D}^{k-1}, M_G) \to \text{Emb}_0(\mathbb{D}^{k-1}, M_G)$ whose fiber $\Omega^{k-1}\mathbb{S}^{d-k}$ measures the normal derivative into the $\varepsilon$-augmentation.

**Remark 1.13.** The essence of Theorem E is that it increases the codimension of the embedding space and thus simplifies computations dramatically. For example, the work of Dax [Dax72] can be used to compute one homotopy group of $\text{Emb}_0(\mathbb{D}^k, M)$ further than expected, see Remark 3.5. This is exactly the strategy we use to prove our results for $k = 2$. More generally, for $k = d - 2$ the Goodwillie–Weiss embedding tower [GKW01] converges for $\text{Emb}_0(\mathbb{D}^{k-1}, M_G)$ but in general not for $\text{Emb}_0(\mathbb{D}^k, M)$.

\(^2\) A sequence of based maps $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence if $i$ factors through a weak equivalence to a homotopy fiber of $p$. In particular, it induces a long exact sequence on homotopy groups. We use $\Omega$ to denote based loop spaces.
One might hope to use Theorems D or E repeatedly. However, the boundary $u_0: S^{k-2} \hookrightarrow M_G$ of our new $(k-1)$-disks cannot have a geometric dual, as it bounds the disk $u_\infty$ in $\partial M_G$ (so it is null homotopic).

**Example 1.14.** For $k = 1$ and $d \geq 2$ we get a homotopy equivalence (the fibration splits by Lemma 4.7)

$$\text{Emb}_0(D^1, M) \cong \Omega S^{d-1} \times \Omega(M \cup_{\partial^{d-1}} D^d).$$

Here a dual for the boundary condition means that the two points $s(S^0)$ lie in distinct components of $\partial M$, one of which is diffeomorphic to $S^{d-1}$, and along which a $d$-handle is attached, giving $M_G := M \cup_{\partial^{d-1}} D^d$.

For $\dim(M) = d = 2$ it follows that the group $\mathbb{Z} \times \pi_1(M \cup_{\partial^1} D^2)$ acts simply and transitively on the set $\text{Emb}_0(D^1, M)$ of connected components of $\text{Emb}_0(D^1, M)$, which are all contractible. The action of $\mathbb{Z}$ is by Dehn twists around one boundary component, while $\pi_1(M \cup_{\partial^1} D^2)$ acts by the “point-push” map, well known in the surface community, see [FM11]. For $d > 2$ we get such an action of $\pi_1(M)$ on $\text{Emb}_0(D^1, M)$; for $d \geq 4$ this follows directly from general position, and for $d = 3$ it is the classical “light bulb trick”, see Section 2.1. For every $d > 2$ the connected components have nontrivial higher homotopy groups.

**Example 1.15.** If $k = d$ then any $(D^d, \partial D^d) \hookrightarrow (M, \partial M)$ is a diffeomorphism. Let $M_+: = M \cup D^d$ and $G: S^0 \hookrightarrow \partial M_+$ satisfy $G(1) \in \partial M$, $G(0) \in \partial D^d$. Then Theorem D gives

$$\text{Diff}_0(D^d) \cong \text{Emb}_0(D^d, M_+) \cong \Omega \text{Emb}_0(D^{d-1}, D^d)$$

since $(M_+) \cong \mathbb{D}^d$. This is the result of Cerf [Cer68, App.] which motivated our entire approach.

**Example 1.16.** For $k = d - 1$ and $M = S^1 \times D^{d-1}$, $s = p \times \partial D^{d-1}$, $G = S^1 \times q$, Theorem D implies

$$\pi_{d-1} \text{Emb}_0(D^{d-1}, S^1 \times D^{d-1}) \cong \pi_n \text{Emb}_0(D^{d-2}, D^d),$$

for all $n \geq 1$. This was discussed by Budney and Gabai [BG20] (using the same arguments of Cerf), who construct for $d \geq 4$ and $n = d - 4$ an infinitely generated subgroup of the left hand side.

### 1.3 Embedding spaces of 2-disks in higher dimensions

Focusing on $k = 2$ and $d \geq 4$, Theorem E gives for all $n \geq 0$ explicit isomorphisms

$$\pi_n \text{Emb}_0(D^2, M) \cong \pi_n \text{Emb}_0(D^2, M)^{\mathbb{R}^2} \cong \pi_{n+2} \text{Emb}_{0^+}(D^1, M_G),$$

where the boundary condition $s: S^1 \hookrightarrow \partial M$ has a dual $G: S^{d-2} \hookrightarrow \partial M$ and $M_G = M \cup G h^{d-1}$, as before. We will use the (high dimensional) Dax invariant (and a variant $dax_*$ of $dax$) to prove Theorem 4.16 which gives a description of the groups $\pi_{d-3} \text{Emb}_{0^+}(D^1, M_G)$ for all $n \leq d - 4$; in particular, for $d$ even we show that $\pi_{d-3} \text{Emb}_{0^+}(D^1, M_G) \cong \mathbb{Z} \oplus \pi_{d-3} \text{Emb}_{0^+}(D^1, M_G)$. This will imply the following generalization of our 4-dimensional results for all $d \geq 4$, see Section 5 for the proof.

**Theorem F.** There are isomorphisms $\pi_n \text{Emb}_0(D^2, M) \cong \pi_{n+2} M$ for $n \leq d - 5$ and a group extension

$$d \text{ even : } \mathbb{Z}[\pi] \langle 1, dax(\pi_{d-1} M) \rangle \rightarrow \pi_{d-4} \text{Emb}_0(D^2, M, U) \otimes (\mathbb{Z} \oplus \pi_{d-2} M / \mathbb{Z}[\pi] \cdot G$$

$$d \neq 5, 9 \text{ odd : } \mathbb{Z}[\pi] \langle dax(\pi_{d-1} M) \rangle \rightarrow \pi_{d-4} \text{Emb}_0(D^2, M, U) \otimes (\mathbb{Z}/2 \oplus \pi_{d-2} M / \mathbb{Z}[\pi] \cdot G$$

where to $K: S^{d-4} \rightarrow \text{Emb}_0(D^2, M)$ we assign the homotopy class modulo $\mathbb{Z}[\pi] \cdot G$ of the map $-U \cup K \in \text{Map}_*(S^{d-4}, \text{Map}_*(S^2, M)) \cong \text{Map}_*(S^{d-2}, M)$.

Moreover, if $d \neq 4$ is even, $\eta(K)$ is one half of the relative Euler number of the normal bundle of the immersion $S^{d-4} \times D^2 \hookrightarrow S^d \times M$, $(\tilde{t}, x) \mapsto (\tilde{t}, K(x))$, relative to the constant family $U$ (they agree on $\partial(S^{d-4} \times D^2)$). For $d = 4$ the map $\eta$ was given in Proposition 1.9.

**Acknowledgments.** We thank Dave Gabai for sharing his insight into the Dax invariant of Example 1.10 during his visit to Bonn. We also thank Hannah Schwartz and Rob Schneiderman for interesting discussions related to this work. Both authors cordially thank the Max Planck Institute for Mathematics in Bonn. The first author was also supported by the Fondation Sciences Mathématiques de Paris.
2 Preliminaries

2.1 A warm-up: the classical light bulb trick lifted to spaces

The classical Light Bulb Trick (LBT) is concerned with neat arcs of the form $K: \mathbb{D}^1 \times \{1\} \hookrightarrow \mathbb{D}^1 \times S^2$, that agree with the inclusion $U: \mathbb{D}^1 \times \{1\} \hookrightarrow \mathbb{D}^1 \times S^2$ on a collar $\partial^1 \mathbb{D}^3 : = \mathbb{D}_{\epsilon}^2 \cup \mathbb{D}_{\epsilon}^2 \subseteq \mathbb{D}^3$, where $\mathbb{D}_{\epsilon}^2 : = [-1, -1+\epsilon]$, $\mathbb{D}_{\epsilon}^2 : = [1-\epsilon, 1]$. Then the LBT shows that $K$ is isotopic to $U$: pick a sequence of “crossing changes” from $K$ to $U$, and realize them as isotopies using the swinging motion around the “light bulb” $G : = \{1\} \times S^2$. In other words, the space $\text{Emb}_0(\mathbb{D}^1, \mathbb{D}^1 \times S^2)$ of such arcs is connected.

In fact, one can completely compute the homotopy type of this space, as we now briefly explain; see Section 3 for more details. Firstly, attach a 3-ball to $G$, so that $U: \mathbb{D}^1 \hookrightarrow \mathbb{D}^1 \times S^2 \subseteq \mathbb{D}^3$ has $U(-1) \in \partial \mathbb{D}^3$, while $U(1)$ is a point in the interior of $\mathbb{D}^3$. We claim that

$$\text{Emb}_0(\mathbb{D}^1, \mathbb{D}^1 \times S^2) \simeq \text{Emb}_0(\mathcal{Q}^1, \mathbb{D}^3) : = \{A: \mathbb{D}^1 \hookrightarrow \mathbb{D}^3 \mid A \equiv U \text{ on } \partial \mathbb{D}^1\}.$$  

Indeed, since the interior of any $A \in \text{Emb}_0(\mathcal{Q}^1, \mathbb{D}^3)$ misses $A(1) = U(1) \in \mathbb{D}^3$, it also misses a neighborhood of it, and we can view it as a neat arc in $\mathbb{D}^1 \times \mathbb{D}_{\epsilon}^2 \cong \mathbb{D}^1 \times \mathbb{D}^2$.

Finally, we claim that $\text{Emb}_0(\mathcal{Q}^1, \mathbb{D}^3) \simeq \Omega(S^2 \times S^2)$, the space of based loops in $S^2 \times S^2$. Actually, the proof applies more generally, for any 3-manifold $M$ which has a boundary component $G: S^2 \hookrightarrow \partial M$, and an arc $U: \mathbb{D}^1 \hookrightarrow M$ with $U(-1) \in \partial M \setminus G$ and $U(1) \in G$. Then consider the space $\text{Emb}_0(\mathbb{D}^1, M)$ of $K: \mathbb{D}^1 \hookrightarrow M$ with $K|_{\partial \mathbb{D}^1} = U|_{\partial \mathbb{D}^1}$, and let $M_G := M \cup_G \mathbb{D}^3$, the result of attaching a 3-handle along $G$.

**Theorem 2.1.** In the situation above, there are explicit homotopy equivalences

$$\text{Emb}_0(\mathbb{D}^1, M) \simeq \text{Emb}_0(\mathcal{Q}^1, M_G) \xrightarrow{\Omega f_U} \Omega \text{Emb}(\mathcal{Q}^2, M_G) \xrightarrow{\Omega \Omega D_1} \Omega(S^2 \times M_G).$$

In particular, $\pi_0 \text{Emb}_0(\mathbb{D}^1, M) \cong \pi_1 M$, so two arcs are isotopic if and only if they are homotopic.
We refer to the next section for the proof (in all dimensions), and now give only an outline. The first homotopy equivalence was sketched above for \( M = \mathbb{D}^1 \times S^2 \). Next, we use Cerf’s “family version” of Ambient Extension Theorem, which says that the map \( \text{ev}_{+,}\cdot: \text{Emb}_{\mathbb{D}^1}(\mathbb{D}^1, M; U) \to \text{Emb}(\mathbb{D}^1, M_G) \), which restricts embeddings to the submanifold \( \mathbb{D}^1 = [1 - \varepsilon, 1] \subseteq \mathbb{D}^1 \), is a locally trivial fibration.

On one hand, the fiber of \( \text{ev}_{+,}\cdot \) over \( u^+_t = U|_{\mathbb{D}^1} \) is precisely \( \text{Emb}_{\mathbb{D}^1}(\mathbb{D}^1, M_G) \). On the other hand, the total space \( \text{Emb}_{\mathbb{D}^1}(\mathbb{D}^1, M_G; U) \) is contractible, since embeddings shrink arbitrarily close to their value at \(-1\), where their derivative is fixed. This implies that in the fibration sequence

\[
\Omega \text{Emb}(\mathbb{D}^1, M_G) \to \text{Emb}_{\mathbb{D}^1}(\mathbb{D}^1, M; U) \to \text{Emb}(\mathbb{D}^1, M_G)
\]

the connecting map \( \alpha_U \) is a homotopy equivalence; see Lemma 2.8. Moreover, arcs in \( \text{Emb}(\mathbb{D}^1, M_G) \) shrink arbitrarily close to their value at \(-1\), so the unit derivative map to the unit tangent bundle of \( M_G \)

\[
\mathcal{D}_1: \text{Emb}(\mathbb{D}^1, M_G) \xrightarrow{\sim} S(M_G)
\]

is a homotopy equivalence (see Theorem 3.3). Finally, \( S(M_G) \cong S^2 \times M_G \) as 3-manifolds are parallelizable.

Thus, to finish the proof it remains to explicitly identify \( \alpha_U \) and \( f_U \); note that they depend on the choice of a basepoint \( U \) in the total space \( \text{Emb}_{\mathbb{D}^1}(\mathbb{D}^1, M_G; U) \). The value of \( \alpha_U \) on \( \gamma \in \Omega \text{Emb}(\mathbb{D}^1, M_G) \) is an embedded disk \( \alpha_U(\gamma): \mathbb{D}^1 \hookrightarrow M_G \) obtained by dragging \( U \) along \( \gamma \in \Omega \text{Emb}(\mathbb{D}^1, M_G) \) as in Figure 4.

**Figure 4.** Moments \( t = 0, 0 < t < 1, t = 1 \) of an ambient isotopy \( \Phi_t \) defining \( \alpha_U \). A neighborhood of the intersection \( U \cap \gamma \) has to be “dragged along” all the way during the isotopy, whereas a neighborhood of the undercrossing in \( \gamma \) is dragged only for a while.

More precisely, \( \alpha_U(\gamma) := \Phi_t(U) \in \text{Emb}_{\mathbb{D}^1}(\mathbb{D}^1, M_G) \) for some extension of the isotopy of intervals \( \gamma \) to an ambient isotopy \( \Phi_t \) of \( M_G \) (so diffeomorphism \( \Phi_t \) takes \( u^+_t \) to itself). The homotopy inverse \( f_U \) of \( \alpha_U \) glues the given embedded arc \( K \) to \( U \) along their boundary \( U \) and foliates the resulting loop \(-U \cup_0 K\) by small intervals to give \( f_U(K) \in \Omega \text{Emb}(\mathbb{D}^1, M_G) \).

**Remark 2.2.** For neatly immersed arcs the unit derivative map \( \mathcal{D}(U)^{-1} \cdot \mathcal{D}(\cdot): \text{Imm}_0(\mathbb{D}^1, M) \to \Omega S M \) is a homotopy equivalence by Smale [Sma58]. Interestingly, the proof essentially follows the outline above, but here the key is to show that the restriction map for immersions is also a fibration.

### 2.2 Embeddings of manifolds with corners

Following Cerf [Cer61] a \( d \)-dimensional manifold with corners is a topological \( d \)-manifold \( X \) with boundary, together with (a maximal atlas of) charts around all \( x \in X \) with domain \( \mathbb{R}^d_{(q)} := \mathbb{R}^q \times [0, \infty)^{d-q} \) for some \( 0 \leq q \leq d \) that send the origin \( 0 \) to \( x \). One requires that each transition map, initially defined only on an open subset of \( \mathbb{R}^d_{(q)} \), extends to a smooth map on an open subset of \( \mathbb{R}^d \). This gives a notion of smooth maps \( Y \to X \) between manifolds with corners and there is a Whitney topology on the set \( C^\infty(Y, X) \).

A component of \( X_{(q)} := \{ x \in X \mid x \leftrightarrow 0 \in \mathbb{R}^d_{(q)} \} \) in some chart is called a \( q \)-face of \( X \); it is a \( q \)-dimensional smooth manifold (without boundary), and the set \( X \) is a disjoint union of its faces. Define the boundary \( \partial X \) as the union of codimension \( \geq 1 \) faces, and a corner of \( X \) as a face of codimension \( \geq 2 \) (e.g., one of the vertices in a square).

Restricting to \( q \in \{d - 1, d\} \), we recover the usual structure on \( X \) of a smooth manifold with boundary. Note that manifolds with corners are clearly closed under cartesian product. Moreover, Cerf constructs “prismatic neighbourhoods” of \( \partial X \) in \( X \) playing the role of collars (and restrict to them on faces in \( \partial X \)).

For a \( d \)-manifold with corners \( X \) a subset \( X' \subseteq X \) is a \( d' \)-dimensional submanifold for some \( 0 \leq d' \leq d \) if each point of \( X' \) admits a chart in \( X \) for which \( X' \) maps bijectively to some submodel \( \mathbb{R}^{d'}_{(q')} \subseteq \mathbb{R}^{d'}_{(q)} \). A
submodel is given by choosing $0 \leq q' \leq q$, $0 \leq k \leq \min\{q - q', d' - q\}$, and inserting $d - d'$ many zeroes:

$$R_{(q')}^d \ni (x_1, \ldots, x_{q'}, y_1, \ldots, y_{d' - q'}) \mapsto (x_1, \ldots, x_{q'}, y_1, \ldots, 0, 0, \ldots, 0, y_{k+1}, \ldots, y_{d' - q'}) \in R_{(q')}^d.$$ 

These relative charts induce the structure of a $d'$-manifold with corners on $X'$. For example, if $X$ is a smooth manifold with boundary, then for the corners of $X'$ of codimension $d' - q' \leq 2$, we have the following different cases, depending on the value of $q'$ listed on the left:

$d'$: For a top dimensional face $F \subseteq X'$ we have either $F \subseteq \partial X$ or $F \subseteq X(d) = X \setminus \partial X$.

$d' - 1$: For a small neighborhood $V_p \subseteq X'$ of $p \in X'(d' - 1)$ there are 3 possibilities: either $V_p \subseteq \partial X$ respectively $V_p \subseteq X \setminus \partial X$ as above, or $(V_p, \partial V_p) \subseteq (X, \partial X)$ is a neat submanifold.

$d' - 2$: A small neighborhood $V_p \subseteq X'$ of $p \in X'(d' - 2)$ looks like a neighborhood of $\bar{0} \in \mathbb{R}^{d' - 2} \times [0, \infty)^2$, and there are 4 possibilities for $V_p \subseteq X$. The case $(x_1, \ldots, x_{q'}, y_1, y_2) \mapsto (x_1, \ldots, x_{q'}, y_1, 0, \ldots, 0, y_2)$ is the most interesting: exactly "one half" of $\partial V_p$, the one corresponding to $\bar{0} \times [0, \infty) \times \{0\}$, lies in $\partial X$ while the rest of $V_p$ lies in $X \setminus \partial X$.

Thus, a smooth submanifold with boundary is either neat or contained in the interior of $X$, while the simplest next case is the local model listed last above, which we use for half-disks $X' = \mathcal{G}$, see Figure 5.

A smooth map $f$ of manifolds with corners is an embedding, written $f: Y \hookrightarrow X$, if $f(Y)$ is a submanifold of $X$ whose induced corner structure makes $f: Y \to f(Y)$ a diffeomorphism. An immersion of manifolds with corners is a smooth map that is locally an embedding. Spaces of embeddings and immersions inherit the Whitney $C^\infty$-topology.

**Definition 2.3.** For $y: Y \hookrightarrow X$ let $\text{Emb}(Y, X; y) \subseteq \text{Emb}(Y, X)$ consist of embeddings $f: Y \hookrightarrow X$ such that for each $p \in Y$ and each face $F \subseteq X$ we have $f(p) \in F_X$ if and only if $y(p) \in F_X$; we say that $y$ and $f$ have the same "incidence relations" [Cer61, p. 281].

Furthermore, for a closed subset $Y' \subseteq Y$, let $\text{Emb}_y(Y, X; y) \subseteq \text{Emb}(Y, X; y)$ consist of those embeddings $f$ that agree with $y$ on $Y'$, that is $f|_{Y'} = y|_{Y'}$. We say that "$y$ is the boundary condition along $Y'$" (note that $y$ at the same time determines the incidence relations).  

**2.2.1 Restriction maps for embeddings.** Consider compact manifolds with corners and embeddings

$$Z' \hookrightarrow Z \rightarrow Y \rightarrow X.$$ 

We say a subset $Y' \subseteq Y$ is a local normal tube to $Z \subseteq Y$ along $Z'$ if $Y' \cap Z = Z'$ and there is a tubular neighborhood $V \subseteq Y$ of $Z$ in $Y$ such that $Y' \cap V = pr^{-1}(Z')$, where $pr: V \to Z$ is the projection.

**Theorem 2.4.** With the above notation, the following restriction maps are both locally trivial:

I. $ev_Z: \text{Emb}(Y, X; y) \to \text{Emb}(Z, X; z)$ [Cer61, p.294 Cor.2, with notation $E \subseteq H \subseteq F$];

II. $ev_Z: \text{Emb}_Y(Y, X; y) \to \text{Emb}_Z(Z, X; z)$ [Cer61, p.298 Cor.2].

Here a map $p: E \to B$ is locally trivial if for each $b \in B$ there exists a neighborhood $x \in V \subseteq B$ and a homeomorphism $p^{-1}(V) \cong V \times p^{-1}(x)$. Palais [Pal60] showed 2.4.1 in the case when all manifolds have empty boundary and $Y$ is compact; Cerf extended this to manifolds with corners and quite general boundary conditions as in 2.4.1I. We also record the following fact, which will be used to replace neat embeddings by those fixed on a collar.

**Proposition 2.5** ([Cer61, Prop.9 p. 337]). In the above notation, if $Z' = Z \subseteq Y$ is the closure of a codimension 1 face, then the inclusion $\text{Emb}_{Y'}(Y, X; y) \hookrightarrow \text{Emb}_{Z'}(Y, X; y)$ is a weak homotopy equivalence.

### 2.3 Hurewicz fibrations

It is a standard fact that a locally trivial map over a paracompact base is a Hurewicz fibration. As $\text{Emb}_Y(Y, X; y)$ are metrizable infinite-dimensional manifolds [Mic80], they are paracompact by Stone's theorem, so Theorem 2.4 implies that the restriction maps $ev_Z$ are Hurewicz fibrations. Recall that
Lemma 2.6. Up to homotopy rel. $X \times \{0\}$ and over $B$, the lift $H$ as in the diagram

\[
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{H_0} & E \\
\downarrow & & \downarrow p \\
X \times [0,1] & \xrightarrow{h} & B \\
\end{array}
\]

is uniquely determined by the initial conditions $(h, H_0)$. As a consequence, its restriction $H_1: X \times \{1\} \to E$ is also unique up to homotopy.

Proof. If $\Lambda^{k-1} \subseteq \partial \Delta^k$ is a $(k-1)$-horn, i.e. it consists of all $(k-1)$-faces of the k-simplex, except for one, then a Hurewicz fibration also has the lifting property for pairs $(X \times \Delta^k, X \times \Lambda)$. This is the fact that these are acyclic cofibrations in the model structure on topological spaces for which the Hurewicz fibrations are the fibrations. The case $k=1$ is precisely the lifting problem (1), and we use the case $k=2$ to show the uniqueness of the lifted homotopy.

Namely, suppose we have two lifts $H$ and $H'$ in diagram (1). Since they agree on $X \times \{0\}$ we can glue them together at the vertex $v_0 \in \Delta^2$ to give a map $X \times \Lambda^1 \to E$ that lifts (the restriction of) the map $\hat{h}: X \times \Delta^2 \to B$ which equals $h$ on all rays from $v_0$ to the opposite edge $\{v_1, v_2\} \subseteq \Delta^2$. The lift $X \times \Delta^2 \to E$ is thus a homotopy from $H$ to $H'$ rel. $X \times \{0\}$ and over $B$. \qed

In particular, consider in (1) the space $X := \Omega B$ of loops in $B$ based at $b := p(e)$, for a basepoint $e \in E$, and the initial conditions $H_0 = \text{const}_e: \gamma \mapsto e$ and $h = \text{ev}: (\gamma, t) \mapsto \gamma(t)$. Any lift $\hat{\text{ev}} := H: \Omega B \times [0,1] \to E$ at time $t = 1$ takes values in the fiber $F := p^{-1}(b)$, hence gives a so-called connecting map $\delta := \hat{\text{ev}}_1: \Omega B \to F$.

Corollary 2.7. Up to homotopy, the map $\delta: \Omega B \to F$ is independent of the choice of a lift $\hat{\text{ev}}$ and only depends on $(p, e)$. Moreover, it is natural: given two commuting squares on the right of the diagram

\[
\begin{array}{ccc}
\Omega B' & \xrightarrow{\delta'} & F' & \xrightarrow{p'} & B' \\
\downarrow & \downarrow & \downarrow r.F & & \downarrow & \downarrow g.b \\
\Omega B & \xrightarrow{\delta} & F & \xrightarrow{p} & B
\end{array}
\]

any choice of connecting maps $\delta, \delta'$ makes the square on the left commute up to homotopy.

Proof. The first part follows directly from Lemma 2.6. The naturality follows from it as well, this time applied to lifting $g.b \circ \text{ev'}: \Omega B' \times [0,1] \to B$ with the initial condition $g \circ \text{const}_{e'}$. \qed

If $B$ is well-pointed the above discussion holds in the pointed category; this is the case for our spaces of embeddings as they are locally contractible. Then $\Omega B$ is well-pointed at $\text{const}_e$ and there are pointed connecting maps $\delta: \Omega B \to F$, inducing the boundary maps in the long exact sequence of homotopy groups for the fibration $p$. In particular, if $E$ is contractible then any such $\delta$ is a weak homotopy equivalence. We will also need the following strengthening, whose proof we did not find in the literature.

Lemma 2.8. If $E$ is contractible, a connecting map $\delta$ is a homotopy equivalence with homotopy inverse $(p,R)|_F: F \to \Omega B$, where $R: E \to \mathcal{P}_b E := \{\gamma: [0,1] \to E \mid \gamma(0) = e\}$ is a contraction of $E$, $\text{ev}_1 \circ R = \text{Id}_E$.

Proof. Consider the following diagram of two fibrations and a lifting problem on top:

\[
\begin{array}{ccc}
\Omega B \times \{1\} & \xrightarrow{\delta} & \mathcal{P}_b B \times \{0\} & \xleftarrow{\text{const}_e} & \mathcal{P}_b B \times [0,1] \\
\downarrow & & \downarrow \text{ev} & & \downarrow \text{ev} \\
F & \xrightarrow{p.R|_F} & E & \xleftarrow{p.R} & B \\
\Omega B & \xrightarrow{\delta} & \mathcal{P}_b B & \xleftarrow{\text{ev}_1} & B
\end{array}
\]
As a map between contractible spaces, \( p_*R := (t \mapsto p \circ R_t) \) is a homotopy equivalence. Being over \( B \) (\( \text{ev}_1 p_*R = p \)), this is also a fiber homotopy equivalence [Hat02, 4.H]. For the same reason, the restriction \( q \coloneqq \hat{\text{ev}}_1 : \mathcal{P}_b B \to E \) of the lift \( \hat{\text{ev}} \) is a fiber homotopy equivalence over \( B \). We claim that there is a homotopy \( p_*R \circ q \simeq \text{Id} \) over \( B \), so uniqueness of inverses will imply that \( p_*R \) and \( q \) are inverse homotopy equivalences, as well as the desired restrictions \( p_*R|_F \) to \( F \subseteq E \) and \( \delta = q|_{\mathcal{P}_b B} \to \Omega B \subseteq \mathcal{P}_b B \).

To find the claimed homotopy we observe two solutions for the outer lifting problem on the right of (2) (for the fibration \( \text{ev}_1 \)). One lift is clearly \( p_*R \circ \hat{\text{ev}} \). We define the second by the formula

\[
(\gamma, t) \mapsto \begin{cases}
\Gamma_t(s), & s \in [0, 1-t] \\
\gamma(s - (1-t)), & s \in [1-t, 1]
\end{cases},
\]

where \( \Gamma_t \in \Omega B \) is a homotopy from \( \Gamma_0 = p_*R(e) \) to \( \Gamma_1 = \text{const}_b \), which exists by the argument below. This is indeed another solution, since \( \text{ev}_1 \) evaluates all paths at \( s = 1 \), so it gives \( \gamma(t) = \text{ev}(\gamma, t) \), while restricting to \( t = 0 \) gives exactly \( \Gamma_0 = p_*R(e) \). The first lift at \( t = 1 \) is \( p_*R \circ q \) whereas the second is \( \text{Id}_{\mathcal{P}_b B} \), so the uniqueness from Lemma 2.6 gives the desired homotopy over \( B \) between them.

To find a homotopy \( \Gamma_t \) from \( R(e) \in \Omega E \) to \( \text{const}_e \) rel. \( e \), first observe that these loops are homotopic (since \( E \) is contractible to \( e \)), and then by the usual argument they are also based homotopic: free homotopy classes in a path-connected space are in bijection with the conjugacy classes in the fundamental group, and for the unit this consists of a single element.

\[\square\]

### 3 Spaces of disks and half-disks

In this section we work in arbitrary dimensions and prove Theorem D. Disks with duals in \( M \) are reduced to half-disks in \( X = M_\partial \) in Section 3.2, while half-disks in \( X \) are described as loops in \( X \) of “\( \varepsilon \)-augmented” disks of lower dimension in Section 3.1. In Section 3.3 we discuss the map that forgets the \( \varepsilon \)-augmentation.

#### 3.1 From half-disks to loops of augmented disks

Given an embedding \( U : D \hookrightarrow X \) of compact manifolds with corners, and a closed subset \( b \subseteq D \), recall from Section 2.2 the space \( \text{Emb}_b(D, X; U) \) consisting of those embeddings \( K : D \hookrightarrow X \) which have the same incidence relations (for faces in \( D, X \)) as \( U \), and that agree with \( U \) on \( b \). We let \( U \) be its basepoint.

In particular, if \( D \) and \( X \) are smooth manifolds with boundary, the incidence relation \( U(D) \cap \partial X = U(\partial D) \) together with the boundary condition on \( b = \partial D \) reproduces the space \( \text{Emb}_\partial(D, X) \coloneqq \text{Emb}_\partial(D, X; U) \) of neat embeddings (see our first footnote). For such \( U \) we can expand the boundary condition to a closed collar \( b = (\partial D) \times [0, \varepsilon] \subseteq D \) and define \( \text{Emb}_{\partial r}(D, X) := \text{Emb}_{(\partial D) \times [0, \varepsilon]}(D, X; U) \). It follows from Proposition 2.5 that the inclusion \( \text{Emb}_{\partial r}(D, X) \hookrightarrow \text{Emb}_\partial(D, X) \) is a weak homotopy equivalence.

We will need the next simplest case of manifolds with corners, which also have codimension 2 faces.

We take the domain \( D \) to be \( Q^k := \{ x \in \mathbb{R}^k : ||x|| \leq 1, x_1 \leq 0 \} \), which is the west half of the unit \( k \)-dimensional disk, and consider subsets \( D_- := \{ x \in \mathcal{Q} : ||x|| = 1 \} \) and \( D_+ := \{ x \in \mathcal{Q} : x_1 = 0 \} \), which are \((k-1)\)-dimensional disks with \( \partial \mathcal{Q} = D_- \cup D_+ \) and \( S_0 := D_- \cap D_+ \), which is a \((k-2)\)-dimensional sphere. Then \( \mathcal{Q} \) is a \( k \)-manifold with corners with one \( k \)-face \( \mathcal{Q} \), two \((k-1)\)-faces \( D_{k-1} \) and one \((k-2)\)-face \( S_0 \), the unique corner of \( \mathcal{Q} \).

Moreover, consider subsets \( D_-^\varepsilon := \{ x \in \mathcal{Q} : ||x|| \geq 1 - \varepsilon \} \) and \( D_+^\varepsilon := \{ x \in \mathcal{Q} : x_1 \geq -\varepsilon \} \) (shaded strips in Figure 5), both diffeomorphic to \( \mathbb{D}^{k-1} \times [0, 1] \) and with \( D_-^\varepsilon \cap D_+^\varepsilon \cong S_0 \times [0, \varepsilon]^2 \).

Denote \( \partial^r \mathcal{Q} := D_-^\varepsilon \cup D_+^\varepsilon \) (an example of Cerf’s prismatic collar).

Next, we fix a smooth manifold with boundary \( X \) and an embedding \( U : \mathcal{Q} \hookrightarrow X \) of manifolds with corners such that \( U \) maps \( \mathcal{Q} \) to the interior of \( X \) and the other incidence relations are determined by the restrictions of \( U \) to \( D_{k-1} \) as follows (see Figure 5): the image of \( u_- := U|_{D_-} \) is contained in \( \partial X \), while \( u_+ := U|_{D_+} : D_+ \hookrightarrow X \) is a neat embedding, with \( u_0 := \partial(u_-) = \partial(u_+) : S_0 \hookrightarrow \partial X \).
We write $u^\varepsilon_\triangle := U|_{\triangle^\varepsilon}$ and $u^\varepsilon_\cdot := U|_{\triangle^\varepsilon \cap \partial \triangle^\varepsilon}$, and identify their domains with the corresponding products.

The elements of Cerf’s space $\text{Emb}(\mathcal{Q}^k, X; U)$ are called half-disk in $X$. We are interested in its subspace $\text{Emb}_{\partial^k}(\mathcal{Q}^k, X) := \text{Emb}_{\partial^k}(\mathcal{Q}^k, X; U)$ which by definition consists of those half-disk $K : \mathcal{Q}^k \hookrightarrow X$ that agree with $U$ on the prismatic collar $\partial^k \mathcal{Q}^k$. Equivalently, $K$ is a topological embedding that agrees with $U$ on $\partial^k \mathcal{Q}^k$ and restricts to an (ordinary) smooth embedding on interiors $\mathcal{Q}^k \hookrightarrow X \setminus \partial X$.

We saw in Proposition 2.5 that the space of neat disks $\text{Emb}_{\partial}(\mathcal{D}^k, X) = \text{Emb}_{\partial}(\mathcal{D}^k, X; u^\varepsilon)$ has a weakly equivalent subspace $\text{Emb}_{\partial^k}(\mathcal{D}^k, X) = \text{Emb}_{\partial^k}(\mathcal{D}^k, X; u^\varepsilon)$ given in [Cer68, App.], but we also identify the maps involved, needed for our geometric constructions.

Consider the fibration sequence

$$
\text{Emb}_{\partial^k}(\mathcal{D}^k, X) \xrightarrow{\epsilon} \text{Emb}(\mathcal{Q}^k, X; U) \xrightarrow{\epsilon} \text{Emb}(\mathcal{Q}^k, X),
$$

where $\epsilon$ is given on homotopy groups by the family ambient isotopy theorem, while $\epsilon$ maps a half-disk $K$ to the loop of $\epsilon$-augmented $(k-1)$-disks induced by appropriate foliation of the sphere $K \cup \partial - U$.

**Theorem 3.1.** For all $k \geq 1$ and $d \geq 1$ there are inverse homotopy equivalences

$$
\Omega \text{Emb}_{\partial^k}(\mathcal{D}^k, X) \xrightarrow{\epsilon} \text{Emb}(\mathcal{Q}^k, X),
$$

where $\epsilon_U$ is given on homotopy groups by the family ambient isotopy theorem, while $\epsilon_U^\varepsilon$ maps a half-disk $K$ to the loop of $\epsilon$-augmented $(k-1)$-disks induced by appropriate foliation of the sphere $K \cup \partial - U$.

**Proof.** Consider the fibreation sequence

$$
\text{Emb}(\mathcal{Q}, X) = \text{Emb}_{\partial^k}(\mathcal{D}^k, X) \xrightarrow{\epsilon} \text{Emb}_{\partial^k}(\mathcal{Q}, X; U) \xrightarrow{\epsilon} \text{Emb}_{\partial^k}(\mathcal{D}^k, X),
$$

where $\epsilon$ restricts $K : \mathcal{Q} \hookrightarrow X$ to the $\epsilon$-collar $\mathcal{D}^k \subseteq \mathcal{Q}$ of the unconstrained half of its boundary. This is a fibration by Cerf’s Theorem 2.4.11 for $Y' := S^d \subseteq \mathcal{Q} := Y \quad Y' := S^d \cap \mathcal{D}^k \subseteq \mathcal{D}^k := Z$. We will show that its total space $E^\varepsilon := \text{Emb}_{\partial^k}(\mathcal{Q}, X; U)$ admits an explicit contraction

$$
R : E^\varepsilon \times [0, 1] \rightarrow E^\varepsilon, \quad \text{with } R_0 = \text{const}_U, \ R_1 = \text{Id}. \quad (4)
$$

Then Lemma 2.8 implies that any connecting map

$$
\epsilon_U : \Omega \text{Emb}_{\partial^k}(\mathcal{D}^k, X) \rightarrow \text{Emb}(\mathcal{Q}^k, X),
$$

is a homotopy equivalence, and is by definition is given by lifting the loop in the base space to a path in the total space and taking the endpoint. In our setting this amounts to extending a loop $\gamma$ of $\epsilon$-augmented $(k-1)$-disks based at $u^\varepsilon_\triangle$ to an isotopy of half-disks starting with $U$ and ending with the desired half-disk $\epsilon_U(\gamma)$. By the same lemma, the restriction $\epsilon_U^\varepsilon(K)(t) = \epsilon_U^\varepsilon \circ R(t) = R_0(K)|_{\mathcal{D}^k}$ is a homotopy inverse to $\epsilon_U$, and we will see below that it is indeed given by a foliation, cf. Section 1.2.

To construct the retraction $R$ we start with a path of re-embeddings $\varphi_t : \mathcal{Q} \hookrightarrow \mathcal{Q}, \ t \in [\varepsilon, 1]$, such that

1. $\varphi_1 = \text{Id}_\mathcal{Q}$ and $\varphi_t|_{\mathcal{Q}_{\varepsilon/2}} = \text{Id}_{\mathcal{Q}_{\varepsilon/2}}$ for all $t$,
2. $\varphi_t(\mathcal{Q}) \subseteq \mathcal{D}_{\varepsilon}$, 
3. $\varphi_t(\mathcal{D}_{\varepsilon}) \subseteq \mathcal{D}_{\varepsilon}$ for all $t$.

It is not hard write down such an isotopy $\varphi_t$ using radial coordinates, see Figure 6 for $k = 2$. Then consider the homotopy $E^\varepsilon \times [\varepsilon, 1] \rightarrow E_{\varepsilon/2} := \text{Emb}_{\partial^k}(\mathcal{Q}, X; U)$, defined by $K \mapsto K \circ \varphi_t$. By Property (1) this indeed defines paths in the space $E_{\varepsilon/2}$ (note that $E^\varepsilon \subseteq E_{\varepsilon/2}$ is smaller as it has the stronger boundary condition), ending with $K\varphi_1 = K$, and starting with $K\varphi_1 = U\varphi_1$, using Property (2) and that $K \in E^\varepsilon$.

We next modify this homotopy to have image contained in the subspace $E^\varepsilon$. We fix an ambient isotopy $\Phi_t : X \xrightarrow{\Phi_t} X, \ t \in [\varepsilon, 1]$, supported in a collar of $\partial X$, such that $\Phi_0 = \text{Id}_X$ and $U \circ \varphi_t|_{\mathcal{Q}_{\varepsilon/2}} = \Phi_t^{-1} \circ U|_{\mathcal{D}_{\varepsilon/2}}$ for $t \in [\varepsilon, 1]$. This can be constructed explicitly in a collar $\partial X \times [\varepsilon, 1] \rightarrow X$ (or extend the isotopy of half-disks $U\varphi_t$ by the usual ambient isotopy theorem). Then for $t \in [\varepsilon, 1]$ let $R_t(K) := \Phi_t \circ K \circ \varphi_t$. 
Recall that the model half-disk $K\varphi$ is the inverse to the small set near the corner, so that $R_\varepsilon(K)$, for $t \in [\varepsilon, 1]$ is in $E^\varepsilon$, i.e. it agrees with $U$ on $D^\varepsilon_\times$. Indeed, by Property (3) and $K \in E^\varepsilon$ we have $K\varphi|_{D^\varepsilon_-} = U\varphi_t|_{D^\varepsilon_-}$, so $R_\varepsilon(K)|_{D^\varepsilon_-} := \Phi_tK\varphi_t|_{D^\varepsilon_-} = \Phi_tU\varphi_t|_{D^\varepsilon_-} = U|_{D^\varepsilon_-}$ by construction of $\Phi$.

This defines a path from $R_t(K) = \Phi_tK\varphi$ to $R_1(K) = \text{Id}_X K\text{Id}_\Omega = K$, and now each half-disk $R_t(K)$ for $t \in [\varepsilon, 1]$ is in $E^\varepsilon$, i.e. it agrees with $U$ on $D^\varepsilon_\times$. Finally, for $t \in [0, \varepsilon]$ we let $R_t(K) := R_{1+t-\varepsilon/\varepsilon}(U)$. This goes from $U$ to $\Phi \circ U\varphi$, so that $R_t, t \in [\varepsilon, 1]$ to a map $R: E^\varepsilon \times [0, 1] \to E^\varepsilon$ which is the desired contraction as in (4). Finally, the homotopy inverse to $\alpha U$ is defined by the formula

$$f^{-1}_t(K)(t) = R_t(K)|_{D^\varepsilon_-} = \begin{cases} \Phi_t \circ \varphi_{1+t/\varepsilon} \circ U \circ \varphi_{1+t-\varepsilon/\varepsilon}|_{D^\varepsilon_-}, & t \in [0, \varepsilon] \\ \Phi_t \circ K \circ \varphi_t|_{D^\varepsilon_-}, & t \in [\varepsilon, 1], \end{cases}$$

which we call the foliation: outside of a fixed collar of $X$ it agrees either with augmented arc $U\varphi_t|_{D^\varepsilon_-}$ or $K\varphi_t|_{D^\varepsilon_-}$, and in the collar uses their modifications by $\Phi$, making “a turn” at $\alpha \varepsilon := \Phi\circ U\varphi_t|_{D^\varepsilon_-}$. $\square$

### 3.2 From neat disks to half-disks

Recall that the model half-disk $\Omega \subseteq D^k$ has boundary decomposed into two $(k-1)$-disks $\partial \Omega = D_- \cup_{S_0} D_+$ intersecting along the $(k-2)$-sphere $S_0$, the corner of $\Omega$. Also recall that $U: \Omega \hookrightarrow X$ by definition restricts to a neat $(k-1)$-disk $u_+: D_+ \hookrightarrow X$, while the image of $D_- \subseteq \partial \Omega$ is contained in $\partial X$.

Using a Riemannian metric on $X$ we extend $u_+$ to an embedding $V: D_+ \times D^{d-k+1}_{\leq \varepsilon/2} \hookrightarrow X$ onto a closed tubular neighborhood $V u_+$. We may assume that the restriction $V|: D_+ \times [0, \varepsilon] \cong D^+ \hookrightarrow X$ to the first normal vector agrees with our preferred $\varepsilon$-augmentation $u_+] = U|_{D^+}$ and also, by decreasing $\varepsilon$ if necessary, that $\text{im}(u_+) = V u_+ \cap \text{im}(U)$, i.e. the half-disk $U$ does not return to this $\varepsilon$-neighborhood of $u_+$.

We can view $V(D_+ \times D^{d-k+1}_{\leq \varepsilon/2})$ as a $(d-k+1)$-handle attached to $X \setminus V(D_+ \times D^{d-k+1}_{\leq \varepsilon/2})$ along $V(D_+ \times S^{d-k}_{\leq \varepsilon/2})$, see Figure 7. Conversely, that complement is obtained from $X$ by removing a $(k-1)$-handle with core $u_+(D_+)$, and is a smooth manifold with boundary only if we first smoothen the corner $V(S_0 \times S^{d-k}_{\leq \varepsilon/2})$.

This is a standard procedure, used for example when attaching handles in the smooth category. In our context it amounts to picking an open subset $h_+ \subseteq \text{core } u_+$ which is the union of $V(D_+ \times D^{d-k+1}_{\leq \varepsilon/2})$ and a small set near the corner, so that $X \setminus h_+$ is a compact smooth manifold with boundary.

**Figure 6.** The image of $\varphi_t$ for $t = 2/3, 1/3, \varepsilon$. Dashed strips show where $\varphi_t$ is the identity; they are always contained in the blue-colored strip $D^\varepsilon_- \subseteq \Omega$. The black line is the image of $D^{k-1}_\times \times \{\varepsilon\} \subseteq D^+_\times \subseteq \Omega$.

**Figure 7.** Removing a handle $h_+ \subseteq X$ turns the half-disk $U$ in $X$ into a neat disk $U'$ in $X \setminus h_+$.

**Figure 8.** The model smoothening.
We make such a choice once and for all in $\mathbb{D}_+ \times \mathbb{D}^{d-k+1}$, and let $h_+$ be its image in $X$ under $V$, see Figure 8. Namely, the constant radius $\varepsilon/2$ along $\mathbb{D}_+$ increases near $S_0$ by a smooth function with fairly obvious properties (ensuring that the stretching function $s$ in the next proof is well defined on all of $h_+$).

Now let $\mathbb{D}' := \mathcal{Q} \setminus V^{-1}(h_+) \subseteq \mathcal{Q}$ and fix a diffeomorphism $\mathbb{D}^k \cong \mathbb{D}'$ that is the identity near $\mathbb{D}^k \setminus \mathbb{D}_+$. Then the restriction of $U$ to $\mathbb{D}'$ is a neat embedding $U' : \mathbb{D}' \to X \setminus h_+$, by our choice of $\varepsilon$. This is an element in the space of neat embeddings $\text{Emb}_{\mathbb{D}'/2}(\mathbb{D}', X \setminus h_+) \subseteq \text{Emb}(\mathbb{D}^k, X \setminus h_+)$, where the boundary condition is the remaining $\varepsilon/2$-part of $u^r_+$ in $\mathbb{D}'$, as well as the original $u^r_-$ along $u_-$. Note that we can reconstruct $U$ from $U'$ as $U = U' \cup u^r_-$. 

**Lemma 3.2.** The map $\text{Emb}_{\mathbb{D}'/2}(\mathbb{D}', X \setminus h_+) \to \text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X)$, $K' \mapsto K' \cup u^r_+$, is a homotopy equivalence.

**Proof.** The chosen boundary conditions for $K'$ make this map well-defined. It is continuous with image $\mathcal{E} \subseteq \text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X)$ consisting of those half-disks that meet $h_+$ only along $\text{im}(u^r_+ \nu^r_+)$. In fact, it is a homeomorphism onto $\mathcal{E}$ whose inverse is given by restricting embeddings from $\mathcal{Q}$ to $\mathbb{D}'$. It thus suffices to construct a homotopy inverse from $\text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X)$ back to this subspace $\mathcal{E}$.

![Figure 9. Stretching $K$ towards the dashed lines to avoid the smallest central disk $h_+$.](image1)

![Figure 10. Stretching functions $s(r, t) : [0, \varepsilon] \to [0, \varepsilon]$ for fixed $r < \varepsilon/2$ and three values of $t \in [0, 1]$.](image2)

To this end, use the Riemannian metric on $X$ to obtain the continuous map

$$r : \text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X) \to (0, 2\varepsilon)$$

so that $r(K)$ gives the minimal distance of $K(\mathbb{D}')$ to $K(\mathbb{D}_+) = U(\mathbb{D}_+)$. Possibly shrinking $\varepsilon$ further, we may assume that the geodesic distance to $U(\mathbb{D}_+)$ on the sphere bundle in $\nu^r u_+$ is $\delta$ for all $\delta < \varepsilon$. By compactness and the injectivity of $K$, $r(K)$ is strictly positive as claimed and we will now stretch it to $\frac{5}{2}\varepsilon$, in order to deform $K$ until it lies in $\mathcal{E}$. So we pick a smooth “stretching” function

$$s : (0, \varepsilon/2) \times [0, 1] \times [0, \varepsilon] \to [0, \varepsilon]$$

such that $s(r, t, x) = x$ whenever one of the following conditions is satisfied: $t = 0$ or $x = 0$ or $x \geq \frac{1}{5}\varepsilon$. Moreover, we require $s(r, 1, r) = \frac{2}{5}\varepsilon$ for all $r$ and that each function $s(r, t, -)$ is strictly increasing.

If $(x, v) \in [0, \varepsilon] \times S^{d-k}$ are polar coordinates, then we will refer to “stretching by $s(r, t)$” as the self-diffeomorphism $(x, v) \mapsto (s(r, t, x), v)$ of $\mathbb{D}_{2\varepsilon}^{d-k+1}$. The same formula applies to disk bundles in vector bundles if we stretch in a constant way along the base. Using the parametrization $V$, we can apply such diffeomorphisms also to our tubular neighborhood $\nu u_+$. The stretching near the smoothened corners along $S_0$ needs to be slightly modified but we leave the necessary variation to the reader.

We can then construct a homotopy $H : \text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X) \times [0, 1] \to \text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X)$ with $H_0 = \text{Id}$ and $\text{im}(H_1) \subseteq \mathcal{E}$, induced by a smooth family of diffeomorphisms $\phi_{(r, t)} : X \to X$ that are the identity outside $\nu^r u_+$ and on $\nu u_+$ they stretch by $s(r(K), t)$. More precisely, we define $H(K, t)$ to be the half-disk that equals $K$ on $\mathbb{D}'\mathbb{Q}$ but away from that collar is given by the composition $\phi_{s(r(K), t)} \circ K$. The properties of the stretching function $s$ show that each $H_t$ sends $\mathcal{E}$ to itself and that $H_1$ is the required homotopy inverse. In fact, homotopies for both compositions to the identity are constructed from $H_t$ as follows: If $j : \mathcal{E} \subseteq \text{Emb}_{\mathbb{D}^k}(\mathbb{Q}^k, X)$ is the inclusion then $H_1 \circ j : \mathcal{E} \to \mathcal{E}$ is homotopic via $H_t \circ j$ to $H_0 \circ j = \text{Id}_{\mathcal{E}}$. Similarly, $j \circ H_1 = H_1 \simeq H_0 = \text{Id}$.

$\square$
We can now prove Theorem E. This is about the space of neat $k$-disks in a $d$-manifold $M$, with the boundary $s: S^{k-1} \hookrightarrow \partial M$ which has a framed geometric dual $G: S^{d-k} \hookrightarrow \partial M$, that is, the normal bundle $\nu_{\partial M}(G)$ is trivial and $G \cap s = \{y\}$. Then the theorem says $\text{Emb}_\partial(D^k, M) \simeq \Omega \text{Emb}_\partial(D^{d-k}, M_G)$, where $M_G := M \cup_{(G, \psi)} \partial G$ is obtained by attaching a $(d-k+1)$-handle $h = D^{d-k+1} \times D^{k-1}$ along any framing $\psi: S^{d-k} \times D^{k-1} \hookrightarrow \nu_{\partial M}(G)$ (this choice is inessential because $M_G \simeq M \setminus \nu U$, see Remark 1.12).

Proof of Theorem E. Removing from $M$ an open $\varepsilon$-neighborhood $h_+$ of the cocore $\{0\} \times D^{k-1}$ of $h$ gives $M$ back because we are all together just attaching $S^{d-k} \times [\varepsilon, 1] \times D^{k-1}$ along $S^{d-k} \times \{1\} \times D^{k-1} \hookrightarrow \partial M$. Using this diffeomorphism $M_G \setminus h_+ \simeq M$ and Proposition 2.5 we have

$$\text{Emb}_\partial(D^k, M) \simeq \text{Emb}_\partial(D^k, M_G \setminus h_+) \simeq \Omega \text{Emb}_\partial(D^{d-k}, M_G).$$

Applying Lemma 3.2 and Theorem 3.1 to $X := M_G$ we obtain

$$\text{Emb}_\partial(D^k, M_G \setminus h_+) \simeq \text{Emb}_\partial(D^k, M_G) \simeq \Omega \text{Emb}_\partial(D^{d-k}, M_G).$$

The final statement of Theorem E, identifying homotopy equivalences, follows from Theorem 3.3. □

3.3 Forgetting augmentations

Recall that on one hand, $\text{Emb}_\partial(D^{d-k}, X) \subseteq \text{Emb}_\partial(D^{d-k}, X)$ consists of neat embeddings $D^{d-k} \hookrightarrow X$ that agree with one such disk $u_+$ on the collar $S^{d-k-2} \times [0, \varepsilon] \subseteq D^{d-k}$, so the $\varepsilon$-notation records a stronger boundary condition, along an entire collar. On the other hand, $\text{Emb}_\partial(D^{d-k}, X)$ consists of $\varepsilon$-augmented neat $(k-1)$-disks $D^{d-k-1} \times [0, \varepsilon] \hookrightarrow X$ that agree with one such $u_+^\varepsilon$ on $(S^{d-k-2} \times [0, \varepsilon]) \times [0, \varepsilon] \subseteq D^{d-k} \times [0, \varepsilon]$. So here the $\varepsilon$-notation reflects additional structure, together with a stronger boundary condition.

This additional structure has a fairly simply homotopy type that just reflects a normal vector field along a $(k-1)$-disk. Let $V_k(X)$ be the Stiefel bundle of orthonormal $k$-frames in $TX$. Theorem 3.3. Forgetting augmentations leads to a commutative diagram of fibration sequences

$$\begin{array}{ccc}
\Omega^{k-1}S^{d-k} & \longrightarrow & \text{Emb}_\partial(D^{d-k}, X) \\
\downarrow & & \downarrow e_{0}\varepsilon \longrightarrow \\
\Omega^{k-1}S^{d-k} & \longrightarrow & \Omega^{k-1}V_k(X) \\
\end{array}$$

Proof. The map $e_{0}\varepsilon$ is a fibration by Theorem 2.4.11. For a fixed Riemannian metric on $X$, the unit derivative along $D^{d-k} \times \{0\}$ in the direction of $[0, \varepsilon]$ is a map

$$\mathcal{D}_\varepsilon: e_{0}\varepsilon^{-1}(u_+) \rightarrow \Gamma_{u_+}^{\varepsilon}(S^n u_+).$$

The target here is the space of those sections of the unit sphere-bundle $S^n u_+$ of the normal bundle $nu_+$ of $u_+: D^{d-k} \hookrightarrow X$ that agree with $\mathcal{D}_\varepsilon(u_0)$ along $S^{d-k-2} \times [0, \varepsilon]$, where $u_0 = u_+^\varepsilon |_{S^{d-k-2} \times [0, \varepsilon]}$. We claim that $\mathcal{D}_\varepsilon$ is a homotopy equivalence. A homotopy inverse $\text{Exp}_{u_+}$ comes from identifying the total space of $\nu_{u_+}$, with a tubular neighborhood of $u_+$ via a scaled exponential map: for a unit normal vector field $\xi$ along $u_+$ define $\text{Exp}_{u_+}(\xi): D^{d-k} \times [0, \varepsilon] \rightarrow X$ by $\text{Exp}_{u_+}(\xi)(x, y) := \exp(y \cdot \xi(x))$. Here we may assume by compactness of $X$ that $\varepsilon$ is smaller than the injectivity radius of the chosen metric.

We have $\mathcal{D}_\varepsilon \circ \text{Exp}_{u_+} = \text{Id}$ by construction. To define a homotopy from $\text{Exp}_{u_+} \circ \mathcal{D}_\varepsilon$ to the identity in the space $e_{0}\varepsilon^{-1}(u_+)$ we observe that by continuously scaling the parameter and using the exponential map, it suffices to construct such a homotopy $K_t$ for $K: D^{d-k} \times [0, \varepsilon] \rightarrow \nu_{u_+}$. This is given by $K_t(x, y) := \frac{K(x, ty)}{t}$ for $t \in [0, 1]$, since $K_0$ is indeed the usual description of the normal derivative of $K = K_1$ at $(x, 0)$.

A trivialization of the sphere bundle $S^n u_+ \cong D^{d-k} \times S^{d-k}$ induces a homeomorphism between the space of its sections and the space $\text{Map}(D^{d-k}, S^{d-k})$; let $u_+': D^{d-k} \rightarrow S^{d-k}$ be the image of $\mathcal{D}_\varepsilon(u_0')$. Then the subspace $\Gamma_{u_0'}(S^n u_+)$ of sections which satisfy the boundary condition $\mathcal{D}_\varepsilon(u_0')$ is homeomorphic to

$$\Gamma_{u_0'}(S^n u_+) \cong \text{Map}_\partial(D^{d-k}, S^{d-k}; u_+) := \partial^{-1}(\partial u_+'),$$

the fiber over $\partial u_+'$ of the restriction map $\partial: \text{Map}(D^{d-k}, S^{d-k}) \rightarrow \text{Map}(S^{d-k}, S^{d-k})$. We identify this fiber with a $(k-1)$-fold loop space.
Lemma 3.4. Let $U: \mathbb{D}^n \to X$ be a based map with $U(\ast) = \ast_X$ for basepoints $\ast \in \partial \mathbb{D}^n$ and $\ast_X \in X$. Then there are inverse homotopy equivalences (pointed for $U$ and $-U \cup_{0} U$)

$$-U \cup_{0} \ast \colon \text{Map}_{0}(\mathbb{D}^n, X; U) \xrightarrow{\sim} \text{Map}_{0}(\mathbb{S}^n, X) =: \Omega X : U \cup \ast,$$

where $-U \cup_{0} K$ glues two disks along the boundary, while $U \cup S: \mathbb{D}^n \to \mathbb{S}^n \vee \mathbb{D}^n \to M$ is the wedge sum (pinch off a sphere from a neighborhood, say half-disk, of the basepoint $\ast \in \partial \mathbb{D}^n$ in $\mathbb{D}^n$).

Proof. For a homotopy from $-U \cup_{0} (U \cup \ast)$ to $\text{Id}_{\mathbb{D}^n \setminus X}$ we use the obvious null homotopy of the sphere $-U \cup_{0} U \simeq \text{const.}$. Similarly, for the homotopy from $U \cup (-U \cup_{0} \ast) = (U \cup -U) \cup_{0} \ast$ to the identity collapse the part $U \cup -U$ to $\ast$; use the foliation $\gamma_x \subseteq \mathbb{D}^n$ by the straight lines from $x \in \partial \mathbb{D}^n$ to $\ast \in \partial \mathbb{D}^n$ and then for each $x$ the obvious null homotopy of loop $U(\gamma_x)U(\gamma_x)^{-1}$ through loops based at $U(x)$. \hfill $\square$

Therefore, a homotopy equivalence claimed in the first part of the theorem is the composite

$$\Omega^{k-1}\mathbb{S}^{d-k} \xrightarrow{u_{+}^k \vee \ast} \text{Map}_{0}(\mathbb{D}^{k-1}, \mathbb{S}^{d-k}; u_{+}^k) \cong \Gamma_{u_{+}^k}(\mathbb{S}u_{+}^k) \xrightarrow{\text{Exp}_{u_{+}^k}} \text{ev}_{0}^{-1}(u_{+}^k), \tag{5}$$

which sends $\xi: \mathbb{S}^{k-1} \to \mathbb{S}^{d-k}$ to the augmented disk $\text{Exp}_{u_{+}^k}(u_{+}^k \vee \xi)$. This is given as $u_{+}^k$ except on a neighborhood $\ast \in \partial X$ where it comes from integrating the vector field $\xi$. More precisely, denote $(u_{+}^k) = u_{+}^k(\mathbb{D}^{k-1})$, where $\mathbb{D}^{k-1} = \mathbb{D}^{d-k} \vee \mathbb{D}^{d-k}$ and $\mathbb{D}^{-1}$ is the half containing $\ast$ as a pole, and let $(u_{+}^k)_{i}$ be $u_{+}^k$ scaled down to $(u_{+}^k)_{1}$. Then we can write $\text{Exp}_{u_{+}^k}(u_{+}^k \vee \xi) = (u_{+}^k)_{1} \cup \text{Exp}_{u_{+}^k}(u_{+}^k)_{i}(-\ast)$.

Consider the map $D: \text{Emb}_{0}\mathbb{D}^{k-1}, X \to \text{Map}_{0}(\mathbb{D}^{k-1}, V_{k-1}(X); D(u_{+}^k))$ taking derivatives in all $k-1$ tangent directions at each point of $\mathbb{D}^{k-1}$, and the map

$$D^c: \text{Emb}_{0}\mathbb{D}^{k-1}, X \to \text{Map}_{0}(\mathbb{D}^{k-1}, V_{k}(X); D(u_{+}^k))$$

taking derivatives in all $k$ tangent directions at each point of $\mathbb{D}^{k-1} \times \{0\}$. We turn targets of these maps into iterated loop spaces using Lemma 3.4, so that they fit into the diagram

$$\begin{array}{ccc}
\Omega^{k-1}\mathbb{S}^{d-k} & \xrightarrow{(u_{+}^k) \cup \text{Exp}_{u_{+}^k}(\cdot)} & \text{Emb}_{0}\mathbb{D}^{k-1}, X \xrightarrow{\text{ev}_{0}} \text{Emb}_{0}\mathbb{D}^{k-1}, X \\
\Omega^{k-1}\mathbb{S}^{d-k} & \xrightarrow{\text{Exp}_{u_{+}^k}(\cdot)} & \text{Emb}_{0}\mathbb{D}^{k-1}, X \xrightarrow{\text{exp}_{0}} \text{Emb}_{0}\mathbb{D}^{k-1}, X \\
\end{array}
\tag{6}$$

where $D^{c}_{u_{+}^k}(\cdot) := -D(u_{+}^k) \cup_{0} D^c(\cdot)$ and $D_{u_{+}^k}(\cdot) := -D(u_{+}^k) \cup_{0} D(\cdot)$. For basepoints at the bottom we use the images of $u_{+}^k$ under the vertical maps. The square on the right clearly commutes: forgetting the coordinate $[0, \xi]$ corresponds to forgetting the last vector in the $k$-frame. We define $i_k(\cdot) := -D(u_{+}^k) \cup_{0} D^c((u_{+}^k)) \cup_{0} D^{c}_{u_{+}^k}(\cdot)$ so that the middle square commutes, and observe this is canonically homotopic to the inclusion of the fiber of $p_k$ over the basepoint $D_{u_{+}^k}(\ast) = -D(u_{+}^k) \cup_{0} D(u_{+}^k)$. Namely, by the above lemma there is a homotopy from $i_k(\xi)$ to $D^{c}_{u_{+}^k}(\ast) \cup_{0} D_{u_{+}^k}(\ast) \cup_{0} i_k(\xi)$ which is clearly the claimed inclusion.

Note that we can choose the connecting map $\delta_{ev_{0}}$ for the top fibration sequence in (6) to be the composite of $\Omega D_{u_{+}^k}$ with the connecting map $\delta_{ev_{0}}$ for the bottom sequence. Combining Theorem E with Theorem 3.3 we obtain Theorem D: in the presence of a dual there is a fibration sequence

$$\Omega^{k-1}\mathbb{S}^{d-k} \xrightarrow{\text{Exp} \circ u_{U}} \text{Emb}_{0}\mathbb{D}^{k}, M \xrightarrow{f_{U}^i := \text{ev}_{0} \circ f_{U}^i} \text{Emb}_{0}\mathbb{D}^{k-1}, M_{G} \xrightarrow{\delta_{ev_{0}}} \Omega^{k-1}\mathbb{S}^{d-k}.$$ 

In particular, if $d = k$ or $d = k+1 \geq 3$, then $f_{U}$ is a homotopy equivalence. If $d > 2k$, then $\pi_{d-2k(1)}$ is a bijection.

Remark 3.5. The map $-D(u_{+}^k) \cup_{0} D(\cdot): \text{Imm}_{1}\mathbb{D}^{k-1}, X \to \Omega^{k-1} V_{k-1}(X)$ is a homotopy equivalence by Smale–Hirsch theory and Lemma 3.4, cf. Remark 2.2. One can next use the work of Dax [Dax72] (see also [GKW01]) to study the lowest homotopy group that distinguishes $\text{Imm}_{1}\mathbb{D}^{k-1}, X$ and $\text{Emb}_{0}\mathbb{D}^{k-1}, X$. Namely, Dax shows that whenever $k$ and $d$ are such that $d - 2(k - 1) \geq 1$ there is an isomorphism

$$\text{Dax: } \pi_{d-2(k-1)}(\text{Imm}_{1}\mathbb{D}^{k-1}, X), \text{Emb}_{0}\mathbb{D}^{k-1}, X; u) \to \mathbb{Z}[\pi_{1}X].$$

This sends $\mathbb{I}^{d-2(k-1)} \to \text{Imm}_{1}\mathbb{D}^{k-1}, X$ to the sum of “ordered” double point loops of a suitable “perfect” representative $\mathbb{I}^{d-2(k-1)} \times \mathbb{D}^{k} \to \mathbb{I}^{d-2(k-1)} \times M$. We treat the case $k = 2$ in detail in the next section, as this is the focus of the present paper; we plan to address the general case in future work.
4 Spaces of \( \varepsilon \)-augmented arcs

We restrict attention to the smallest domain dimensions. In Section 4.1 we study the space \( \text{Emb}_0(\mathbb{D}^1, X) \) of neat arcs in an oriented compact \( d \)-manifold \( X \) that agree on the boundary with a fixed “unknot” \( u: \mathbb{D}^1 \to X \), by comparing them to immersed arcs with the same boundary condition. In Section 4.2 we study the space of \( \varepsilon \)-augmented arcs \( \text{Emb}_{\varepsilon}^\gamma(\mathbb{D}^1, X) \), using the map which forgets the augmentation and the comparison to the Stiefel bundle from Section 3.3.

4.1 The Dax invariant for spaces of arcs

**Theorem 4.1.** The inclusion \( i: \text{Emb}_0(\mathbb{D}^1, X) \subset \text{Imm}_0(\mathbb{D}^1, X) \) is \( (d - 3) \)-connected and for \( d \geq 4 \) there is a short exact sequence of groups

\[
\mathbb{Z}[\pi_1 X] / \langle 1, \text{dax}_u(\pi_{d-1} X) \rangle \xrightarrow{\partial} \pi_{d-3}(\text{Emb}_0(\mathbb{D}^1, X), u) \xrightarrow{\pi_{d-1}} \pi_{d-3}(\text{Imm}_0(\mathbb{D}^1, X), u),
\]

where the homomorphism \( \text{dax}_u: \pi_{d-1} X \to \mathbb{Z}[\pi_1] \) is defined in (12) below in terms of the Dax invariant \( \text{Dax} \). We can make this more explicit using that the inclusion \( \text{Imm}_0(\mathbb{D}^1, X) \subset \text{Map}_0(\mathbb{D}^1, X; u) \) induces isomorphisms on homotopy groups in degrees \( \leq d - 3 \), see Lemma 4.7. Together with the homotopy equivalence \( \text{Map}_0(\mathbb{D}^1, X; u) \to \Omega X \) from Lemma 3.4 (where \( u(-1) \in X \) is the basepoint) this implies that the map \( p_u: \pi_n(\text{Imm}_0(\mathbb{D}^1, X), u) \to \pi_n(\Omega X) \cong \pi_{n+1} X \) is an isomorphism, given by \( p_u(f) = (\tilde{f} \mapsto f\tilde{t} - u^{-1}) \) union the canonical homotopy null of \( u \cdot u^{-1} \) on the boundary. Thus, we can reformulate the above theorem.

**Corollary 4.2.** For \( d \geq 4 \) we have \( \pi_0 \text{Emb}_0(\mathbb{D}^1, X) \cong \pi_0 \text{Imm}_0(\mathbb{D}^1, X) \) and for any \( u \in \text{Emb}_0(\mathbb{D}^1, X) \) and \( 1 \leq n \leq d - 4 \) isomorphisms \( p_u: \pi_n(\text{Emb}_0(\mathbb{D}^1, X), u) \cong \pi_{n+1} X \), and a group extension

\[
\mathbb{Z}[\pi_1 X] / \langle 1, \text{dax}_u(\pi_{d-1} X) \rangle \xrightarrow{\partial} \pi_{d-3}(\text{Emb}_0(\mathbb{D}^1, X), u) \xrightarrow{p_u} \pi_{d-2} X.
\] (7)

The first part of Theorem 4.1 follows by general position. Namely, double points of immersed arcs \( \Gamma(\tilde{t}): \mathbb{D}^1 \ni \tilde{t} \to X \) for \( \tilde{t} \in \mathbb{S}^k \), correspond to self-intersections of its track \( \Gamma \times \text{Id}: \mathbb{S}^k \times \mathbb{D}^1 \to \mathbb{S}^k \times X \), \( (\tilde{t}, \theta) \mapsto (\tilde{t}, \Gamma(\tilde{t})(\theta)) \). Thus, the set of double points has dimension \( k + d - 2(d - 1) \). This is negative if \( k \leq d - 3 \), so a generic \( k \)-family is embedded, i.e. gives a class in \( \pi_k \text{Emb}_0(\mathbb{D}^1, X) \). If \( k < d - 3 \) these lifts are also unique, by an analogous argument with one more parameter, implying the injectivity on \( \pi_k \).

Thus, it remains to determine the kernel of \( \pi_{d-3} \). This amounts to computing the relative homotopy group in degree \( d - 2 \) (i.e. \( \pi_{d-3} \) of the homotopy fiber of \( i \)) and the image of the (connecting) map

\[
\delta_{\text{Imm}}: \pi_{d-2} \text{Imm}_0(\mathbb{D}^1, X) \to \pi_{d-2}(\text{Imm}_0(\mathbb{D}^1, X), \text{Emb}_0(\mathbb{D}^1, X)).
\] (8)

4.1.1 The relative homotopy group. The computation of this group essentially follows from the work of Dax [Dax72], but we could not find it explicitly there (see also [GKW01]). It was also studied by Gabai in [Gab20a], who observed that the following is a consequence of the results of Dax. As usual, represent a relative homotopy class by a map \( F: \|d^2 - 2\| \times \|d^3 - 3\| \times I \to \text{Imm}_0(\mathbb{D}^1, X) \) satisfying

\[
F(\partial|d^3 - 3| \times \{1\}) = u \quad \text{and} \quad F(\|d^3 - 3\| \times \{1\}) \subset \text{Emb}_0(\mathbb{D}^1, X).
\] (9)

**Lemma 4.3.** After a small perturbation of \( F \) preserving boundary conditions the associated map

\[
\bar{F}: \|d^2 - 2\| \times \mathbb{D}^1 \to \|d^2 - 2\| \times X, \quad (\tilde{t}, \theta) \mapsto \bar{F}(\tilde{t}, \theta)(\bar{F}(\tilde{t})(\theta))
\]

is an immersion with only isolated transverse double points.

Thus, \( \bar{F} \) has finitely many double points, all of the form \( (\tilde{t}_i, x_i) \) with \( 1 \leq i \leq k \), for some \( \tilde{t}_i \in \|d^2 - 2\| \) and \( x_i = \bar{F}(\tilde{t}_i)(\theta_1^+) = \bar{F}(\tilde{t}_i)(\theta_1^-) \in X \) with \( \theta_1^- < \theta_1^+ \in \mathbb{D}^1 \). Let \( \varepsilon(\tilde{t}_i, x_i) \in \{\pm1\} \) be the relative orientation at \( (\tilde{t}_i, x_i) \), obtained by comparing orientations of the tangent space \( T_{(\tilde{t}_i, x_i)}(\|d^2 - 2\| \times X) \) and (in this order):

\[
d\bar{F}(\tilde{T}(\bar{F}(\tilde{t}_i, x_i))(\|d^2 - 2\| \times \mathbb{D}^1)) \oplus d\bar{F}(\tilde{T}(\bar{F}(\tilde{t}_i, x_i))(\|d^2 - 2\| \times \mathbb{D}^1)).
\] (10)
Again using the fact that $\theta_i^+$ come in a specified order along our interval, we define the group element $g_{x_i} \in \pi_1(X, u(-1))$ to be represented by the following loop based at $u(-1)$ (see Figure 11):

$$F(\tilde{t}_i)|_{[-1, \theta_i^-]} \cdot \frac{F(\tilde{t}_i)|_{[-1, \theta_i^+]}}{F(\tilde{t}_i)|_{[-1, \theta_i^-]}}$$  \hspace{1cm} (11)

Note how the order $\theta_i^- < \theta_i^+$ along the arc is crucial for defining these loops. In contrast, when computing associated loop for a self-intersection of an immersion $\mathbb{D}^{d-1} \hookrightarrow Y$, dim $Y = 2(d - 1)$, one has the indeterminacy $g_y = (-1)^{d-1}g_y^1$ coming from the choice of the order of sheets at $y \in Y$, see Section 5.1.2.

**Definition 4.4** ([Dax72], [Gab20a]). The formula $\text{Dax}[F] := \sum_i \varepsilon(\tilde{t}_i, x_i)g_{x_i},$ gives a well-defined map

$$\text{Dax}: \pi_{d-2}(\text{Imm}_0(\mathbb{D}^1, X), \text{Emb}_0(\mathbb{D}^1, X), u) \rightarrow \mathbb{Z}[\pi_1 X].$$

**Theorem 4.5.** For $d \geq 4$ the map $\text{Dax}$ is an isomorphism.

**Proof.** This is clearly additive since different maps $F$ get stacked in the $\mathbb{I}^{d-3}$-direction. We will construct an explicit inverse to $\text{Dax}$ called the realization map

$$\tau: \mathbb{Z}[\pi_1 X] \rightarrow \pi_{d-2}(\text{Imm}_0(\mathbb{D}^1, X), \text{Emb}_0(\mathbb{D}^1, X), u),$$

We first define for $g \in \pi_1 X$ a map $\tau(g): \mathbb{I}^{d-3} \times \mathbb{I} \rightarrow \text{Imm}_0(\mathbb{D}^1, X)$ satisfying the boundary conditions (9). Define for $\tilde{t} \in \mathbb{I}^{d-3} \times \{0\}$ the embedded arcs $\tau(g)_{\tilde{t}}$ by dragging $u$ along the group element $g$, then “swinging a lasso” around a meridian $\mu(\mathbb{S}^{d-2})$ to $u$ at $x \in u$, then drag again back to $u$. More precisely, foliate $\mu(\mathbb{S}^{d-2})$ by a family of arcs $\alpha_i$ based at two fixed points, see Figure 12. Use the pinch map $\mathbb{I}^{d-3} \rightarrow \mathbb{I} \cup \mathbb{S}^{d-3}$ and along $\mathbb{I}$ apply the finger move around $g$, ending with the connect sum into the arc $\alpha_N$, and for $\tilde{t} \in \mathbb{S}^{d-3}$ sum with the arc $\alpha_T$ instead. For $(\tilde{t}, s) \in \mathbb{I}^{d-3} \times \mathbb{I}$ the paths through immersed arcs $\tau(g)_{\tilde{t}, s}$ from $\tau(g)_{\tilde{t}, 0} = \tau(g)_{\tilde{t}}$ back to $\tau(g)_{\tilde{t}, 1} = u$ are obtained by similarly foliating the unique ball $\mu(\mathbb{S}^{d-1})$ bounded by $\mu(\mathbb{S}^{d-2})$.

Let us show that $\text{Dax} \circ \tau(g) = g$. In the family $\tau(g)_{\tilde{t}, s}$ all arcs are embedded except one, for which there is exactly one double point $\{x\} = u \cap \mu(\mathbb{S}^{d-1})$ and the associated loop is precisely $g_x = g \in \pi_1 X$, see the right part of Figure 12. To determine the sign choose coordinates $\mathbb{R}^d = \mathbb{R}^3 \times \mathbb{R}^{d-3}$ around $x \in X$ so that $\mathbb{R}^3 \times \{0\}$ is as depicted. The derivative at $\theta_4$-sheet of $\tau(g)$ applied to $\mathbb{I}^{d-3}$ is the sum of the positive $\mathbb{I}^{d-3}$ and $\mathbb{R}^{d-3}$ directions, applied to $\mathbb{I}$ it is the sum of the positive $\mathbb{I}$ direction and the upward pointing vector in our $\mathbb{R}^3$-chart, while applied to $\mathbb{D}^1$ it gives the vector pointing to the reader. At $\theta_3$-sheet we see the vector in the positive $\mathbb{I}^{d-2}$-direction and the vector pointing to the right in $\mathbb{R}^3$. Comparing to the canonical basis of $\mathbb{I}^{d-3} \times \mathbb{I} \times \mathbb{R}^{d-3} \times \mathbb{R}^3 \subseteq \mathbb{I}^{d-3} \times \mathbb{I} \times X$ we use $2(d - 2)$ transpositions, so $\varepsilon_x = +1$.
We extend \( \tau \) to \( \mathbb{Z}[\pi_1 X] \) linearly: for \( d \geq 4 \) the target is an abelian group, but for \( d = 4 \) we need to check that \( \tau(g) \) and \( \tau(h) \) commute; we do this in Remark \( 4.13 \) below. Thus, \( \text{Dax} \circ \tau = \text{Id}_{\mathbb{Z}[\pi_1 X]} \) by construction.

To show \( \tau \circ \text{Dax} = \text{Id} \) we use Goodwillie–Klein–Weiss fundamental theorem of embedding calculus [GKW01], which implies that the evaluation map \( e_2 \) from \( \text{Emb}_D(\mathbb{D}^1, X) \) to the second Taylor stage \( T_2 \) is \( 2(d - 3) \)-connected. Since \( T_1 \cong \text{Imm}_D(\mathbb{D}^1, X) \) and \( d \geq 4 \) we have

\[
\pi_{d-2}(\text{Imm}_D(\mathbb{D}^1, X), \text{Emb}_D(\mathbb{D}^1, X), u) = \pi_{d-3}\text{hfib}_u(\text{Emb}_D(\mathbb{D}^1, X) \xrightarrow{\tau} T_1) \xrightarrow{\pi_{d-3}\text{hfib}_u} \pi_{d-3}\text{hfib}_u(T_2 \to T_1)
\]

By the first author’s thesis [Kos20a; Kos20b] the last group is isomorphic to \( \mathbb{Z}[\pi_1 X] \) via an isomorphism \( \chi \). Moreover, by a slight generalization of the results there, we have \( \chi \circ \pi_{d-3} e_2 \circ \tau = \text{Id}_{\mathbb{Z}[\pi_1 X]} \); see [Kos20a, Rem. 1.10] and [Kos21]. Thus, \( \tau \) is an isomorphism, so its unique left inverse \( \text{Dax} \) is as well. \( \square \)

**Remark 4.6.** The given formulation of Theorem 4.5 was not stated by Dax or Gabai; however, [Gab20a] proves statements together equivalent to Theorem 4.1 in the case \( d = 4 \), with his spinning map as \( \partial \tau \).

### 4.1.2 Spaces of immersed arcs and the connecting map.

**Lemma 4.7.** There is a homotopy equivalence \( \text{Imm}_D(\mathbb{D}^1, X) \cong \Omega(\mathbb{S}^{d-1} \times X) \).

**Proof.** By Smale [Sma58] the derivative map gives a homotopy equivalence \( D_u : \text{Imm}_D(\mathbb{D}^1, X) \to \Omega(X) \), the loop space on the unit tangent bundle of \( X \), see Remark 2.2. Therefore, it suffices to prove that if \( X \) is a \( d \)-dimensional manifold with boundary, then \( \Omega(X) \cong \Omega(\mathbb{S}^{d-1} \times X) \). First note that the projection \( p_1 : \mathbb{S}(X) \to X \) has a section, since \( \partial X \neq \emptyset \) implies that the Euler class (which obstructs existence of a nonvanishing vector field) is trivial, as \( H^n(X; \mathbb{Z}_2) = 0 \). Then it is a standard fact (see e.g. [Gra71]) that if a fibration has a section, its looping \( \Omega \mathbb{S}^{d-1} \to \Omega(X) \to X \) is a trivial fibration. \( \square \)

Therefore, we have the mentioned isomorphisms \( p_u : \pi_n \text{Imm}_D(\mathbb{D}^1, X) \cong \pi_{n+1} X \) for all \( n \leq d - 3 \) and also a short exact sequence \( \mathbb{Z} \to \pi_{d-2} \text{Imm}_D(\mathbb{D}^1, X) \to \pi_{d-1} X \), which has a (noncanonical) splitting.

Recalling the connecting map \( \delta_{\text{imm}} \) from (8), we consider the composite

\[
\mathbb{Z} \xrightarrow{i_*} \pi_{d-2}(\text{Imm}_D(\mathbb{D}^1, X), u) \xrightarrow{\delta_{\text{imm}}} \pi_{d-2}(\text{Imm}_D(\mathbb{D}^1, X), u) \xrightarrow{\text{Dax}} \mathbb{Z}[\pi_1 X].
\]

**Proposition 4.8.** This composite takes \( 1 \in \mathbb{Z} \cong \pi_{d-1} \mathbb{S}^{d-1} \) to the unit 1 in the group ring \( \mathbb{Z}[\pi_1 X] \).

**Proof.** The map \( i_* \) for any \( X \) factors through the one for \( X = \mathbb{D}^d \), so it suffices to consider that case (in which both \( i_* \) and \( \text{Dax} \) are isomorphisms with \( \mathbb{Z} \)). By definition, \( i_*(1) \) is the class of any map \( \tau : \mathbb{S}^{d-2} \to \text{Imm}_D(\mathbb{D}^1, \mathbb{D}^d) \) whose Smale’s derivative \( D_u \circ \tau : \mathbb{S}^{d-2} \to \Omega \mathbb{S}^{d-1} \) corresponds to a degree 1 map.

Let us describe \( \tau \). Firstly, for parameters \((\vec{t}, s) \in \mathbb{D}^d - \{0\} \) of the upper hemisphere of \( \mathbb{S}^{d-2} \) let \( \tau(\vec{t}, s) := \tau(1)_{\vec{t}_s} \) be the path of immersed arcs from the previous proof. Recall that this drags a piece of \( u \) to the position \( a_N \), and then uses 2-disks coming from a foliation of the meridian ball \( \overline{\mu}(\mathbb{B}^{d-1}) \) to slide \( a_N \) into an arc on \( \mu(\mathbb{S}^{d-2}) \), giving an embedded arc \( \tau(1)_{\vec{t}} \). For parameters in the lower hemisphere of \( \mathbb{S}^{d-2} \), we will instead undo arcs \( \tau(1)_{\vec{t}} \) by an isotopy, so it will follow that \( \text{Dax}(i_*(1)) = \text{Dax}(\tau(\vec{t})) = \text{Dax} \circ \tau(1) = 1 \).

![Image](https://via.placeholder.com/150)

**Figure 13.** The present slice \( \mathbb{D}^3 \times \{0\} \subseteq \mathbb{D}^d \) contains the whole arc \( \tau(1)_{\vec{t}} \), and disk \( \overline{\mu}(\mathbb{B}^2) := \overline{\mu}(\mathbb{B}^{d-1}) \cap \mathbb{D}^3 \times \{0\} \).

In the second picture we see \( \tau(1)_{\vec{t}} \), whose subarc \( \alpha_S \) can be slid across \( \overline{\mu}(\mathbb{B}^2) \) without creating any double points.

First observe that in the foliation of \( \overline{\mu}(\mathbb{B}^{d-1}) \) there is a unique 2-disk \( \overline{\mu}(\mathbb{B}^2) \) which contains \( x \) (the only double point in the homotopy). We can pick coordinates so that \( \overline{\mu}(\mathbb{B}^2) \) is contained in the present slice \( \mathbb{D}^3 \times \{0\} \subseteq \mathbb{D}^d \) and \( \partial \overline{\mu}(\mathbb{B}^2) = \alpha_N \cup \partial \alpha_S \) as on the left of Figure 13. In particular, \( \tau(1)_{\vec{t}} \) contains \( \alpha_S \).
We can isotope \( \tau(1)_g \) as in Figure 13: we isotope the front guiding arc and a part of \( u \) in \( \tau(1)_g \) by “pulling them through” \( \mathbb{B}_2^2 \). Note that in the new position \( \alpha_S \) can be slid to \( \alpha_N \) across \( \mathbb{B}_2^2 \) without creating any double points, and that from there we have an obvious isotopy to \( u \) — namely, “by pulling tight”. More generally, the desired isotopy from each \( \tau(1)_g \) to \( u \) consists of the same isotopy as in Figure 13, then sliding across the corresponding 2-disk as before to get to position \( \alpha_N \), and then pulling this tight. □

We can use any section of \( p_u : \pi_{d-2}(\text{Imm}_d(D^1, X), u) \rightarrow \pi_{d-1}X \) to define a group homomorphism

\[
dax_u : \pi_{d-1}X \longrightarrow \pi_{d-2}(\text{Imm}_d(D^1, X), u)
\]

so that the last proposition implies \( \text{im}(\text{Dax} \circ \delta_{\text{Imm}}) = \langle 1, \text{dax}_u(\pi_{d-1}X) \rangle \). By construction the value \( \text{dax}_u([f]) \) is computed by lifting \( f : (\mathbb{I}^{d-1}, \partial \mathbb{I}^{d-1}) \rightarrow (X, u(-1)) \) to any family \( F : \mathbb{I}^{d-2} \rightarrow \text{Imm}_d(D^1, X) \) (in this case the entire boundary of \( F \) goes to \( u \)), calculating its Dax invariant \( \text{Dax}(F) = n(f) \cdot 1 + \text{dax}_u(f) \in \mathbb{Z}[\pi_1X] \), and disregarding the trivial group elements \( u(f) \). This finishes the proof of Theorem 4.1. □

**Remark 4.9.** When \( \partial u \) has a geometric dual \( G : S^{d-1} \hookrightarrow \partial X \) (so \( u \) has endpoints in different boundary components of \( X \), one of which is a sphere), the homomorphism \( \delta_{\text{Imm}} \) is surjective. Indeed, we saw by the classical LBT (Theorem 2.1) that \( \text{Emb}_0(D^1, X) \rightarrow \text{Emb}(D^1, X) \) is an isomorphism on \( \pi_{d-3} \) (whose inverse is induced by the inclusion \( X \hookrightarrow X_G \)).

Let us note that under the map \( \pi_{d-1}S^{d-1} \rightarrow \pi_{d-1}(\text{Emb}_0(D^1, X), u) \) the generator is sent to the family \( u^G_w \) of embedded arcs obtained from the interior twist family \( \tau \in \pi_{d-2}(\text{Imm}_d(D^1, X), u) \) (from the proof of Proposition 4.8) by tubing into \( G \). See Section 5 and Figure 15.

The next interesting case is when both endpoints are in the same spherical boundary component.

**Lemma 4.10.** Assume \( u : D^1 \hookrightarrow X \) has both endpoints in the same component \( S \subseteq \partial X \) with \( S \cong S^{d-1} \) and \( u \) is homotopic into \( \partial X \). Then for all \( g \in \pi_1X \setminus 1 \) we have \( \text{dax}_u(g \cdot [S']) = g + (-1)^d \tau \), where \( S' \subseteq X \) is a push-off of \( S \) into the interior of \( X \), together with a short whisker to the basepoint \( u(-1) \in S \).

**Proof.** The class \( g \cdot [S'] \) is represented by adding a representative \( \gamma \) of \( g \) to the whisker of \( S' \). Since our basepoint \( u(-1) \) is in \( S \) and the collar between \( S \) and \( S' \) has trivial fundamental group, \( \gamma \) intersects \( S' \) transversely in two points \( x_+, x_- \) of opposite sign as in Figure 14. Note that \( \text{dax}([S']) = 0 \) since there is a \((d - 2)\)-family of embedded arcs based at \( u \) representing this class: connect-sum a piece of \( u \) into \( S' \) from the “inside”, and swing it around \( S' \). Then \( g \cdot [S'] \) is represented by the family obtained from the one for \([S']\) by conjugating all the arcs by \( \gamma \). This amounts to first sliding \( u \) along the finger guided by \( \gamma \), then pass through double points \( x_+, x_- \), then swing across the foliated \( S' \), and finally return along the finger.

The signed group element at \( x_+ \) is \(+g\), the computation being the same as in Figure 12. Similarly, at \( x_- \) the double point loop is given by \( g^{-1} \), and we claim the sign is \(-(-1)^{d-1} \). Namely, \((-1)^{d-1} \) arises since here the earlier time \( \theta_- \) occurs on the sheet which is moving, while additional minus sign is because that sheet has the opposite tangent vector (and unchanged for the other sheet). □
4.1.3 Fundamental group commutators. For $d = 4$ our extension from Theorem 4.1 still exists but only in the category of (not necessarily abelian) groups, with abelian normal subgroup and quotient:

$$Z[\pi_1 X] / \langle 1, \text{dax}_u(\pi_3 X) \rangle \xrightarrow{\partial x} \pi_1(\text{Emb}_0(\mathbb{D}^1, X), u) \xrightarrow{p_n} \pi_2 X.$$  \hspace{1cm} (13)

We can thus expect an interesting group commutator pairing, which is indeed the case.

**Proposition 4.11.** Extension (13) is central and the commutator pairing is given by

$$[f_1, f_2] = \partial x(\lambda(p_n f_1, p_n f_2)), \quad f_j \in \pi_1(\text{Emb}_0(\mathbb{D}^1, X), u),$$

where $\lambda: \pi_2 X \times \pi_2 X \to Z[\pi_1 X]$ is the equivariant intersection form. As a consequence, the group $\pi_1(\text{Emb}_0(\mathbb{D}^1, X), u)$ is abelian if and only if the image of $\lambda$ is contained in $\langle 1, \text{dax}_u(\pi_3 X) \rangle \subseteq Z[\pi_1 X]$.

For example, $\pi_1(\text{Emb}_0(\mathbb{D}^1, X), u)$ is not abelian whenever $u$ is homotopic to $\partial M$ and there are classes $a_1, a_2 \in \pi_2 X$ such that $\lambda(a_1, a_2)$ is not fixed under the involution $\sigma(g) = g^{-1}$. This will follow from Theorem 5.11 which implies that $\text{dax}_u(\pi_3 X) \subseteq Z[\pi_1 X]^\sigma$. For example, this happens for $X = M_0 \# M_1$ when $M_0$ is simply connected with nontrivial intersection form and $\pi_1 M_1$ has an element $g$ with $g^2 \neq 1$.

**Proof of Proposition 4.11.** Given $f_1, f_2 \in \pi_1(\text{Emb}_0(\mathbb{D}^1, X), u)$, by a preliminary homotopy we may assume that they are represented by maps $f_i: \mathbb{I} \to \text{Emb}_0(\mathbb{D}^1, X)$ supported on disjoint open subintervals $J_1, J_2 \subseteq \mathbb{D}^1$, in the sense that for $\theta \notin J_i$ we have $f_i(t)(\theta) = u(\theta)$ for all $t \in \mathbb{I}$. We may also assume that $f_i(J_i \cap u(\mathbb{I}^1)) = \emptyset$ for both $i$, as generic intersections of 2- and 1-dimensional sets in a 4-manifold.

**Definition 12.** The (parametrized connected) sum $f_1 \#_2 f_2: \mathbb{I}^2 \to \text{Im}_0(\mathbb{D}^1, X)$ is defined by

$$f_1 \#_2 f_2(t_1, t_2)(\theta) := \begin{cases} f_1(t_1)(\theta), & \text{for } \theta \in J_1, \\ f_2(t_2)(\theta), & \text{for } \theta \in J_2, \\ u(\theta), & \text{for } \theta \in \mathbb{I}^1 \setminus (J_1 \cup J_2). \end{cases} \hspace{1cm} (14)$$

The boundary $\partial (f_1 \#_2 f_2): \partial \mathbb{I}^2 \to \text{Emb}_0(\mathbb{D}^1, X)$ is equal to the commutator $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$ in $\Omega \text{Emb}_0(\mathbb{D}^1, X)$, which can be seen by schematically labeling the domain square:

$$\begin{array}{ccc} u & \overset{f_1}{\longrightarrow} & u \\ \downarrow f_2 & & \downarrow f_2 \\ u & \overset{f_1 \#_2 f_2}{\longrightarrow} & u \end{array} \quad \cong \quad \begin{array}{ccc} u & \overset{f_1}{\longrightarrow} & u \\ \downarrow f_2 & & \downarrow f_2 \\ u & \overset{f_1 \#_2 f_2}{\longrightarrow} & u \end{array} \hspace{1cm} (15)$$

Moreover, each $f_1 \#_2 f_2(t_1, t_2)$ is a local embedding (hence an immersion), so $f_1 \#_2 f_2$ represents an element in $\pi_2(\text{Im}_0(\mathbb{D}^1, X), \text{Emb}_0(\mathbb{D}^1, X), u)$ with boundary $[f_1, f_2]$. We saw above that this group is isomorphic to $Z[\pi_1 X]$ via the Dax invariant, computed using certain generic representatives in which only finitely many arcs have double points. We will arrange this for our map $f_1 \#_2 f_2$.

First note that the arc $f_1 \#_2 f_2(t_1, t_2)$ is embedded if and only if $f_1(t_1)(J_1 \cap f_2(t_2)(J_2) = \emptyset$. This is generically not true for all $t_1, t_2$, but by general position $f_1$ can be chosen so that their adjoints $\lim \mathbb{I} \times \mathbb{D}^1 \to X$ are transverse, so intersect in isolated points $x_j = f_1(t_j(\theta_j)) = f_2(t'_j(\theta'_j)) \in X$ for $(t_j, \theta_j) \in \mathbb{I} \times \mathbb{D}^1$, $j = 1, \ldots, r$. As a consequence, the only arcs $f_1 \#_2 f_2(t_1, t_2)$ that are not embedded are those with $t_1 = t'_1$, $t_2 = t'_2$ for some $j$, and $f_1 \#_2 f_2(t_1, t_2)(\theta'_j) = f_1 \#_2 f_2(t'_1, t'_2)(\theta'_j) = x_j$ is the only double point of this arc.

**Lemma 4.13.** If $J_1$ is before $J_2$ in the orientation of $\mathbb{D}^1$ then $\text{Dax}(f_1 \#_2 f_2) = \lambda(p_n f_1, p_n f_2) \in Z[\pi_1 X]$.

**Proof.** Recall that $p_n f_1: \mathbb{I}^2 \to \Omega X$ is the union of $(t \mapsto f_i(t) \cdot u^{-1})$ and canonical null homotopies of $u \cdot u^{-1}$ for $t \in \partial \mathbb{I}^2$ (recall that concatenates $\mathbb{I}^1$). As $f_i(J_i \cap u(\mathbb{I}^1)) = \emptyset$, the transverse intersection points \( \{x_1, \ldots, x_r\} = p_n f_1 \cap p_n f_2 \), counted by the pairing $\lambda$, correspond exactly to the double points of those arcs $f_1 \#_2 f_2(t_1, t_2)$ that are not embedded, counted in the Dax invariant. The Dax loop $g_{x_j}$ goes along a whisker on $f_1$ from $u(-1)$ to $x_j$ and then back on $f_2$, exactly as in the formula computing $\lambda(p_n f_1, p_n f_2)$.

The signs also agree. If $\text{sgn}_{x_j}(f_1, f_2) = +1$, then $df_1(t_1, \theta_j) + df_2(t_1, \theta_j)$ orients $T_{x_j} X$, so with the standard vectors $(dt_1, dt'_j)$ orients $T_{x_j} X$. For Dax, near $\theta_j$ the derivative $d(f_1 \#_2 f_2)$ is zero in the other
direction $t_{3-1}$, so $d(f_1 \# f_2)|_{(t'_1, t'_2, a'_1)}$ is oriented as $-(dt'_1 \circ df_1)|_{(t'_1, a'_1)}$, as we had to flip the first two vectors, whereas $d(f_1 \# f_2)|_{(t'_1, t'_2, a'_2)}$ is oriented as $dt'_1 \circ df_2|_{(t'_1, a'_2)}$. It follows that their sum orients $T_{(t'_1, t'_2)}I^2 + T_{a_1}X$ (since $dt'_1$ passes 3 vectors to become first), so $\varepsilon_{a_1} = 1$ by its definition (10).

To prove Proposition 4.11, we first show that the extension $E := \pi_1(\text{Emb}_0(\mathbb{D}^1, X), u)$ in (13) with normal subgroup $A = \mathbb{Z}[\pi_1 X]/\langle 1, \text{dax}_u(\pi_2 X) \rangle$ is central, i.e., that the elements $[f, \partial \tau(r)] = \partial(f \# \partial \tau(r))$ are trivial. First note that they lie in the image of $\partial \tau$ since $\pi_2 X$ is Abelian. Thus, they are detected by their Dax invariant, and this vanishes by Lemma 4.13:

$$\text{Dax}(f \# \partial \tau(r)) = \lambda(p_{u f_1}, p_u \partial \tau(r)) = \lambda((p_{u f_1}, 0)) = 0.$$  

The claimed commutator pairing also follows from Lemma 4.13, using that $r$ is the inverse to $\text{Dax}$:

$$[f_1, f_2] = \partial(f_1 \# f_2) = (\partial \tau \circ \text{Dax})(f_1 \# f_2) = \partial(\lambda(p_{u f_1}, p_u f_2)).$$

Note that in the above proof we could equally well choose the opposite order of $J_2 < J_1$ in $\mathbb{D}^1$. Equivalently, keep $J_1 < J_2$ but in (14) use a map $f'_1$ supported on $J_{1-1}$, and isotopic to $f_1$. Then $\partial(f'_1 \# f_2(t)) = |f'_1, f_2(t)|$ is isotopic to $\partial(f_1 \# f_2(t)) = |f_1, f_2(t)|$ continuously in $t \in \partial \mathbb{D}^2$. Using this isotopy on an annulus extends $f'_1 \# f_2$ to a map $f_1 \# f_2 : I^2 \to \text{Imm}_0(\mathbb{D}^1, X)$.

Then $\partial \tau(f_1 \# f_2) = |f_1, f_2|$, and as in Lemma 4.13 we find $\text{Dax}(f_1 \# f_2) = -\lambda(a_2, a_1) = -\lambda(a_1, a_2)$, since now arcs $f_2$ appear as the $\theta$-sheet. In particular, $\lambda \text{Dax}(f_1 \# f_2) - \text{Dax}(f_1 \# f_2)$ has trivial boundary, so must be in the kernel of $\partial \tau$, that is, in $\langle 1, \text{dax}_u(\pi_2 X) \rangle$. We can also identify the class in $\pi_2 X$ witnessing this; recall from Corollary 1.6 that $\lambda$ denotes $\lambda$ minus its coefficient at 1.

**Proposition 4.14.** For the Whitehead product $[a_1, a_2]_W \in \pi_3 X$ of $a_1, a_2 \in \pi_2 X$, we have

$$\text{dax}_u([a_1, a_2]_W) = \lambda \text{Dax}(a_1, a_2) + \lambda \text{Dax}(a_1, a_2).$$

**Proof.** Pick $f_1 : S^1 \to \text{Emb}_0(\mathbb{D}^1, X)$ so that $a_i = p_u f_1$ and $f_1 \# f_2$ generic as above. Then $f_1 \# f_2$ is also generic, and gluing them along $\partial \mathbb{D}^2$ gives a map $f_1 \# f_2 - f_1 \# f_2 : S^2 \to \text{Imm}_0(\mathbb{D}^1, X)$ for which we show

$$\text{dax}_u(p_{u f_1 f_2} - f_1 \# f_2) = [a_1, a_2]_W.$$

Let $F_1 : S^1 \to \Omega X$ be obtained from $p_u f_1$ by null homotoping its part which agrees with $u \cdot u^{-1}$, so $a_i = F_1$. Similar homotopies, continuous in $(t_1, t_2) \in I^2$ and agreeing on $\partial I^2$, show

$$(f_1 \# f_2(t, t_2)) \cdot u^{-1} \simeq F_1(t_1) \cdot F_2(t_2) \quad \text{and} \quad (f_1 \# f_2(t_1, t_2)) \cdot u^{-1} \simeq F_2(t_2) \cdot F_1(t_1) \in \Omega X.$$

These homotopies glue to a homotopy rel. boundary from $p_u(f_1 \# f_2 - f_1 \# f_2) : I^2 \to \Omega X$, which is defined by $(t_1, t_2) \mapsto (f_1 \# f_2 - f_1 \# f_2(t, t_2))$ union canonical null homotopies of $u \cdot u^{-1} \simeq c := \text{const}_{u^{-1}}$ on $\partial I^2$, to the map obtained by gluing squares $(t_1, t_2) \mapsto F_1(t_1) \cdot F_2(t_2) \cdot (t_1, t_2) \mapsto F_2(t_2) \cdot F_1(t_1)$. Schematically (on the left the canonical null homotopies are not drawn, but are used for the homotopy):

$$
\begin{array}{c|c|c|c|c}
  uu^{-1} & uu^{-1} & uu^{-1} & c \\
  uu^{-1} & \cdots & uu^{-1} & c \\
  uu^{-1} & -(f_1 \# f_2) \cdot u^{-1} & uu^{-1} & c \\
\end{array}
\simeq
\begin{array}{c|c|c|c|c}
  c & F_1 \cdot F_2 \\
  c & \cdots & |F_1, F_2| & \cdots & c \\
  c & -(F_1 \cdot F_2) & c & \cdots & c \\
\end{array}

(16)

The Whitehead product $[a_1, a_2]_W \in \pi_3 X$ is represented (in fact, its adjoint, the Samelson product, see e.g. [Whi78]) by the map $[F_1, F_2] : I^2 \to \Omega X$ which takes $(t_1, t_2)$ to the commutator loop $[F_1(t_1), F_2(t_2)] \in \Omega X$, union canonical null homotopies $[F_1(t), u] \simeq c$ for $t \in \partial I^2$. By cutting the loop direction $\mathbb{D}^1$ in half, we can view the adjoint $[F_1, F_2] : \mathbb{D}^2 \to X$ as glued from two cubes along their faces $I^2 \times \{0\}$. After a manipulation which is the 3-dimensional analogue of (15), these cubes become precisely the adjoints of the two squares on the right of (16), showing that $[F_1, F_2]$ is homotopic to $p_u(f_1 \# f_2 - f_1 \# f_2)$.

**Remark 4.15.** A similar argument shows that for $g, h \in \pi_1 X$ elements $v(g), v(h) : I^2 \to \text{Imm}_0(\mathbb{D}^1, X)$ commute in relative $\pi_2$: they can be constructed using disjoint supports $J_i$ and different meridian balls...
\( \mu_1(\mathbb{B}^3) \), so there is a null homotopy of their commutator \( \tau(g) \# \tau(h) : I \times I^2 \to \text{Imm}_0(\mathbb{D}^1, X) \) given at \((t_0, t_1, t_2)\) by applying the map \( \tau(g)(t_0, t_1) \) on \( J_1 \) and \( \tau(h)(t_0, t_2) \) on \( J_2 \), analogously to (14).

### 4.2 Spaces of \( \varepsilon \)-augmented arcs

We have the following analogue of Theorem 4.1 for augmented arcs.

**Theorem 4.16.** For \( n \leq d - 4 \) there are isomorphisms \( p_n : \pi_n(\text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X), u^\varepsilon) \cong \pi_{n+1}X \) and for \( d \geq 4 \) there are explicit short exact sequences of groups (with \( \text{dax}^\varepsilon_{ur} \) defined in (20) below)

\[
d \text{even}: \quad \mathbb{Z}[\pi_1X]/\langle 1, \text{dax}_{ur}(\pi_{d-1}X) \rangle \xrightarrow{\partial \varepsilon} \pi_{d-3}(\text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X), u^\varepsilon) \xrightarrow{(\eta \circ Dax^\varepsilon) \oplus p_n} \mathbb{Z} \oplus \pi_{d-2}X,
\]

\[
d \text{odd}, d \neq 5, 9: \quad \mathbb{Z}[\pi_1X]/\langle 1, \text{dax}_{ur}(\pi_{d-1}X) \rangle \xrightarrow{\partial \varepsilon} \pi_{d-3}(\text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X), u^\varepsilon) \xrightarrow{(\eta \circ Dax^\varepsilon) \oplus p_n} \mathbb{Z}/2 \oplus \pi_{d-2}X.
\]

Moreover, for \( d \text{ even} \) \((\eta \circ Dax^\varepsilon, \pi_{d-3} \text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X) \cong \mathbb{Z} \times \pi_{d-3} \text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X) \) is a group isomorphism, so any \( S^{d-3} \)-family of embedded arcs \( \mathbb{Z} \) many augmentations. For \( d \text{ odd} \), the number of augmentations is twice the order of the element 1 in \( \mathbb{Z}[\pi_1X]/\langle \text{dax}^\varepsilon_{ur}(\pi_{d-1}X) \rangle \).

We will define \( \eta \) in Theorem 4.20, and \( \partial \varepsilon \) will be the family of arcs \( \partial \) with suitable \( \varepsilon \)-augmentations; \( p_n \) is the same composite of a forgetful map and concatenation with \( u^{-1} \) as in (7). For even \( d \), a splitting \( \pi_{d-3}\text{Exp} : \mathbb{Z} \to \pi_{d-3} \text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X) \) is on 1 given by “integrating” the \((d - 3)\)-familiy of unit normal vector fields to \( u \), covering its meridian \( \mu(S^{d-2}) \), cf. the proof of Theorem 3.3 and Figure 15. This is true also for \( d = 4 \), even though we saw in Proposition 4.11 that \( \pi_1 \text{Emb}_0(\mathbb{D}^1, M_G) \) is usually not abelian.

To prove Theorem 4.16 we consider the derivative map \( Dax^\varepsilon_{ur} : \text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X) \to \Omega V_2(X) \) from Theorem 3.3, which said that \( Dax^\varepsilon_{ur} \) takes the fibration \( \text{ev}_0 : \text{Emb}(\mathbb{D}^1, X) \to \text{Emb}_0(\mathbb{D}^1, X) \), with fiber \( \Omega^{1-1}S^{d-k} \), to the looping of the fiber \( \pi_2 : V_2(X) \to V_1(X) \). Recall that \( V_k(X) \) denotes the space of \( k \)-frames of \( TX \) and we have homotopy equivalences \( \text{Imm}_0(\mathbb{D}^1, X) \cong \Omega V_1(X) = \Omega S^1 = \Omega(S^{d-1} \times X) \) (see Lemma 4.7).

The map between the long exact sequences in homotopy groups of these fibrations implies that \( Dax^\varepsilon_{ur} \) is also \((d - 3)\)-connected, and gives the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
\pi_{d-1}S^{d-2} & \xrightarrow{\pi_{d-1}V_2(X)} & \pi_{d-1}V_1(X) & \xrightarrow{\pi_{d-1}D_{ax}} & \pi_{d-1}V_1(X) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\pi_{d-2}(\Omega V_2(X), \text{Emb}^\varepsilon)} & \pi_{d-2}(\Omega V_1(X), \text{Emb}^\varepsilon) & \xrightarrow{\partial \varepsilon} & \pi_{d-2}(\Omega V_1(X), \text{Emb}^\varepsilon) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{d-2}\text{Emb} & \xrightarrow{\delta_{ev_0}} & \pi_{d-2}S^{d-2} & \xrightarrow{\pi_{d-2}\text{Exp}} & \pi_{d-2}\text{Emb} \\
\downarrow \pi_{d-2}D_{ax} & & \downarrow \pi_{d-2}D_{ax} & & \downarrow \pi_{d-2}D_{ax} \\
\pi_{d-1}V_1(X) & \xrightarrow{\delta_{p_2}} & \pi_{d-2}S^{d-2} & \xrightarrow{\pi_{d-2}D_{ax}} & \pi_{d-2}V_2(X) & \xrightarrow{\pi_{d-2}D_{ax}} & \pi_{d-2}V_1(X) \\
\end{array}
\]

(17)

Here we abbreviate \( \text{Emb} := \text{Emb}_{Dax}(\mathbb{D}^1, X) \) and \( \text{Emb}^\varepsilon := \text{Emb}_{Dax}^\varepsilon(\mathbb{D}^1, X) \) and \( \delta_{ev_0}, \delta_{p_2} \) are the corresponding connecting maps. Recall from the previous section the splitting extension \( \mathbb{Z} \leftarrow \pi_{d-1}V_1(X) \rightarrow \pi_{d-1}X \), the isomorphism \( \text{Dax} : \pi_{d-2}(\Omega V_1(X), \text{Emb}) \to \mathbb{Z}[\pi_1X] \) and its inverse \( \tau \), and \( \pi_{d-2}V_1(X) \cong \pi_{d-2}X \).

In order to determine the kernel of the surjection \( \pi_{d-3}D_{ax}^\varepsilon \) and prove Theorem 4.16, we use the auxiliary Theorem 4.20 below, which identifies \( \delta_{p_2} \) and splittings of \( \pi_{d-2}D_{ax} \) using the following geometric input.

**Definition 4.17.** Let \( \xi \to X \) be a \( d \)-dimensional vector bundle with inner product and \( \text{forb}_k : V_k(\xi) \to X \) the Stiefel bundle of orthonormal sequences \((v_1, \ldots, v_k)\) of vectors \( v_i \in \xi_x, \ x \in X \). Let \( \xi_k \to V_k(\xi) \) be the complementary \((d - k)\)-dimensional vector bundle to the \( k \) canonical sections of \( \text{forb}_k(\xi) \) to \( V_k(\xi) \). Denote by \( e(\xi_k) \in H^{d-k}(V_k(\xi) ; Z) \) its Euler class and define a homomorphism \( \text{Dax}^\varepsilon_k : \pi_{d-k}V_k(\xi) \to \mathbb{Z} \) by

\[
e_k^\varepsilon(f) := \langle e(\xi_k), f_*[S^{d-k}] \rangle = \langle e(\xi_k^*), [S^{d-k}] \rangle.
\]

**Lemma 4.18.** Let \( p_k : V_k(\xi) \to V_{k-1}(\xi) \) be the fibration which forgets the last vector in a frame, and \( i_k : S^{d-k} \hookrightarrow V_k(\xi) \) a fiber inclusion. We have \( e_k^\varepsilon(i_k) = \chi(S^{d-k}) \), the Euler characteristic of \( S^{d-k} \).
Proof. This follows since \(i^*(\xi_k) \cong TS^{d-k}\). Indeed, the fiber of \(\xi_k\) over \((x, v_1, \ldots, v_k) \in V_k(\xi)\) is the orthogonal complement \((v_1, \ldots, v_k) \perp \xi_x\), so if \(v_k = u \in S(V)\) for a fixed \(V := (v_1, \ldots, v_{k-1}) \perp \xi_x\), then the fiber is \(u^\perp \subseteq V\), which is exactly the tangent space to the \((d-k)\)-sphere \(S(V) \subseteq V\) at \(u\).

**Lemma 4.19.** The connecting map for the fibration \(p_{k+1}: V_{k+1}(\xi) \to V_k(\xi)\) induces on \(\pi_{d-k-1}\) precisely the homomorphism \(e^d_k\) \(\delta_{p_{k+1}}: \pi_{d-k-1}V_k(\xi) \to \pi_{d-k-1}S^{d-k-1} \cong \mathbb{Z}\).

Proof. Since the statement is inherited by pullbacks of vector bundles, it suffices to check it for \(\xi = \gamma\), the universal \(d\)-dimensional bundle \(\gamma \to BO_d\) over the Grassmannian \(BO_d\) of \(d\)-planes in \(\mathbb{R}^\infty\). Then \(EO_d := V_d(\gamma)\) is the contractible, free \(O_d\)-space and we have for \(0 \leq k \leq d\) homotopy equivalences

\[
V_k(\gamma) \cong EO_d/O_{d-k} \cong BO_{d-k}.
\]

The maps \(p_{k+1}\) are induced by inclusions \(O_{d-k-1} \hookrightarrow O_{d-k}\) and we have a 5-term fibration sequence, where the connecting map in question is induced by the action map \(act\) on a basepoint in \(S^{d-k-1}\):

\[
\begin{array}{cccc}
O_{d-k-1} & \longrightarrow & O_{d-k} & \longrightarrow \ S^{d-k-1} & \longrightarrow \ V_{k+1}(\gamma) & \overset{p_{k+1}}{\longrightarrow} \ V_k(\gamma),
\end{array}
\]

(18)

Recall that the Euler class is the unique (necessary and sufficient) obstruction for finding a nonvanishing section in an \(n\)-dimensional vector bundle over \(S^n\), and \(\pi_{d-k-1}O_{d-k}\) are isomorphism classes of such bundles for \(n = d - k\) (via clutching). Then the claim follows since \(\pi_{d-k-1}(act)\) is also the unique obstruction for such a section by the exact sequence on homotopy groups for the above fibration.

For \(\xi = TX\) the tangent bundle \(TX\) of a \(d\)-manifold \(X\) we denote \(V_k(X) := V_k(\xi)\) as before, and \(T_kX = \xi_k\). Recall that oriented 3-manifolds are spin and oriented 4-manifolds are spin*.

**Theorem 4.20.** The connecting map \(\delta_{p_2}: \pi_{d-1}V_1(X) \longrightarrow \pi_{d-2}S^{d-2} \cong \mathbb{Z}\) for \(p_2: V_2(X) \to V_1(X)\) is given by \(\delta_{p_2} = e^d_1\). If \(d\) is even then \(e^d_1 = 0\), whereas if \(d\) is odd the image of \(e^d_1\) is \(2 \cdot \mathbb{Z}\).

Moreover, for \(d \neq 5, 9\) there are splittings \(\eta: \pi_{d-2}V_2(X) \longrightarrow \mathbb{Z}/im(e^d_1)\) of \(\pi_{d-2}\).

If \(d \neq 4\) then one such splitting is given by \(e^d_2/2\), whereas for \(d = 4\) splittings are in natural 1-1 correspondence with spin* structures on the universal covering \(\tilde{X}\), for \(d = 3\) splittings are in natural 1-1 correspondence with spin structures on \(X\). For \(d \neq 3, 5, 9\) odd there are splittings that only depend on the isomorphism class of the vector bundle \(V_2(X) \to V_2(X)\).

For \(d \neq 3\) (mod 8) one can actually use this isomorphism class to get a concrete splitting \(\eta(f) := f^*(TX) \in \text{Vect}_{d-2}(S^{d-2}) \cong \pi_{d-2}BO_{d-2} \cong \mathbb{Z}/2\).

Proof. The first claim is Lemma 4.19 for \(\xi = TX\) and \(k = 1\). For the second sentence, if \(d\) is even we immediately have \(e^d_1 = 0\), since the Euler class vanishes on \((d-1)\)-dimensional bundles on \(S^{d-1}\). We next show that for \(d\) odd, the image of \(e^d_1\) is \(2 \cdot \mathbb{Z}\).

We saw in Lemma 4.18 that \(e^d_1([1]) = \chi(S^{d-1}) = 2\) and hence \(2 \cdot \mathbb{Z}\) is contained in the image. To show that this is everything\(^3\) we make use of the fact that our vector bundles arise as pullbacks \(\xi = f^*(T_1X)\). Recall that \(e(\xi) \equiv w_{d-1}(\xi)\) (mod 2) for \(\dim(\xi) = d - 1\), so it suffices to study this Stiefel–Whitney class which, unlike the Euler class, is a stable characteristic class. As a consequence

\[
e(\xi) \equiv w_{d-1}(\xi) = w_{d-1}(\xi \oplus \mathbb{R}) = w_{d-1}(f^*(T_1X \oplus \mathbb{R})) = w_{d-1}(f^*p_1^*(TX)).
\]

(19)

Thus, \(e^d_1(f) \equiv (w_{d-1}(TX), [p_1 \circ f], [S^{d-1}])\) (mod 2), and \(im(e^d_1) = 2 \cdot \mathbb{Z}\) if \(w_{d-1}(TX)\) vanishes on all spherical homology classes, which we proceed to prove. Writing \(v_j \in H^1(X; \mathbb{Z}/2)\) for Wu classes and \(Sq^i\) for Steenrod squares, the Wu formula says

\[
w_{d-1}(TX) = \sum_{i+j=d-1} Sq^i(v_j).
\]

Since the cohomology of \(S^{d-1}\) is concentrated in two dimensions, pulling back to it kills all these summands, except possibly \(Sq^0(v_{d-1}) + Sq^{d-1}(v_0)\). The second term vanishes as \(Sq^i\) is zero on cohomology classes in degrees \(i\). Finally, \(Sq^0(v_{d-1}) = v_{d-1}\) also vanishes on compact \(d\)-manifolds \(X\) for \(d \geq 3\) by its defining property \(\langle v_{d-1} \cup x, [X] \rangle = (\langle Sq^{d-1}x, [X] \rangle = 0\) for all \(x \in H^1(X, \partial X; \mathbb{Z}/2)\), proving the claim.

\(^3\) Alternatively, for \(d \neq 3, 5, 9\) the Euler class of any \((d-1)\)-dimensional vector bundle on \(S^{d-1}\) is even by Adam’s solution of the Hopf invariant 1 problem (but the universal complex, quaternion and octonion line bundles have Euler number 1).
It remains to construct a splitting of $\pi_{d-2}i_2$, for the fiber inclusion $i_2: S^{d-2} \hookrightarrow V_2 X$ of $p_2$. Putting $k = 2$ in Definition 4.17 gives us a homomorphism $e_2^d: \pi_{d-2}V_2(X) \to \mathbb{Z}$ which takes $[i_2]$ to $\chi(S^{d-2})$. Thus, if $d$ is even and we show that the image of $e_2^d$ equals $2 \cdot \mathbb{Z}$, then $e_2^d/2$ will be our desired splitting $\eta$. As above, since the Euler class equals the Stiefel-Whitney class $w_d$, one has
\[ e_2^d(f) \equiv \langle w_{d-2}(TX), (\text{for some } f), [S^{d-2}] \rangle \pmod{2}, \]
so $e_2^d$ is even if and only if $w_{d-2}(TX)$ vanishes on all spherical homology classes. This holds for $d \neq 4$ by the same argument using Wu classes as above.

For a 4-manifold $X$ this fails since the Wu formula just gives $w_2(TX) = v_2$. For example, $\mathbb{CP}^2$ has nontrivial $w_2$ on the (spherical) class represented by $\mathbb{CP}^1$. Fortunately, there is a way out because any oriented 4-manifold has a spin$^c$ structure, implying that the map $w_2(TX)\colon \pi_2 X \to \mathbb{Z}/2$ on spherical classes lifts to a homomorphism $W\colon \pi_2 X \to \mathbb{Z}$. Then $\frac{1}{2}(e_2^d - W)$ is the required splitting $\eta$, as $W$ clearly vanishes on the fiber inclusion $i_2$ of $p_2$.

Note that choices of splittings correspond exactly to choices of lifts $W$ in this case, and are in bijective correspondence with spin$^c$ structures on the universal cover $\tilde{X}$.

It remains to find splittings $\eta\colon \pi_{d-2}V_2(X) \to \mathbb{Z}/2$ for odd $d$. Assume $d \neq 3, 5, 9$ from now on. We apply to the universal bundle (18) the Hopf invariant theorem as in the last footnote and get the exact sequence $\mathbb{Z}/2 \hookrightarrow \pi_{d-2}BO_{d-2} \twoheadrightarrow \pi_{d-3}BO_{d-1}$, where the left map sends the nontrivial element to $T^2S^{d-2}$. (Note that for $d = 3, 5, 9$ this $\mathbb{Z}/2$ does not arise since then the connecting map $\pi_{d-1}BO_{d-1} \to \pi_{d-2}S^{d-2}$ is onto. In these cases we only proved that these Euler numbers are even for tangent bundles!)

By the same exact sequences for larger $d$ we see that the group $\pi_{d-2}BO_{d-1} \cong \pi_{d-2}BO$ is already in the stable range. Since $d$ is odd, by Bott periodicity the stable group vanishes unless $d \equiv 3 \pmod{8}$ (in which case it is $\mathbb{Z}/2$). Then returning to $X$, and comparing to the universal bundle, we see that a desired splitting $\eta$ arises by the formula $\eta(f) := f^*(T_2X) \in \pi_{d-2}BO_{d-2} \cong \mathbb{Z}/2$. If $d \equiv 3 \pmod{8}$ that sequence splits, $\pi_{d-2}BO_{d-2} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, by Kervaire [Ker60] (this is the entry $r = -1$ and $m = 8s + 1 = d - 2$ in his second table, $s \geq 1$). In particular, we get our splitting that again only depends on the isomorphism class of $T_2 X \to V_2(X)$ (but in these cases we do not know an explicit formula).

Finally, since the tangent bundle of an oriented 3-manifold is trivial by the Wu formula, so are the Stiefel bundles $V_2(X) \to X$ and the induced map on $\pi_1$ of a section of this map gives our splitting. Since $V_2(X) \cong V_3(X) \cong X \times SO_3$ and every Spin$^c$-bundle on a 3-complex is trivial, it follows that our splittings correspond to spin structures on $TX$. For consistency, note that spin structures respectively splittings form affine spaces (or torsors) over isomorphic vector spaces, $H^1(X; \mathbb{Z}/2) \cong \text{Hom}(\pi_1 X, \mathbb{Z}/2)$.

Thus, the remaining open cases are $d = 5, 9$, where $T^2S^{d-2}$ is trivial and $\pi_{d-2}BO_{d-2} \cong \mathbb{Z}$ is already stable, so the comparison to the universal bundle kills the $\mathbb{Z}/2$ that we are trying to split for tangent bundles.

Proof of Theorem 4.16. In all cases $d \neq 3, 5, 9$ a splitting $\eta$ from Theorem 4.20 gives an isomorphism $\eta \oplus \pi_{d-2}p_2\colon \pi_{d-2}V_2(X) \to \mathbb{Z}/2 \cong \text{im}(e_2^d) / \text{ker}(e_2^d)$, which is the desired right hand side in Theorem 4.16. We now compute the kernel of $\pi_{d-3}D^0 u$, which is the quotient of $\mathbb{Z}[\pi_1 X]$ by $\text{im}(\delta_{\text{imm}})$.

Assume $d \geq 4$ is even. Since $e_1^d = 0$ the maps $\pi_{d-1}p_2$ is surjective. It follows that $\text{im}(\delta_{\text{imm}}) \cong \text{im}(\delta_{\text{imm}}^*)$, so $\pi_{d-3}ev_0$ induces an isomorphism $(ev_0)_*: \ker(\pi_{d-3}D^0 u) \cong \ker(\pi_{d-3}D u)$. In the previous section we saw that the latter is isomorphic to $\mathbb{Z}[\pi_1 X]/(1, \text{ax}_s(\pi_{d-1} X))$, so we get the first exact sequence in the theorem, with the maps $\partial^c := (ev_0)^{-1}\partial c$, and $\eta \circ \pi_{d-3}D^0 u$, and $p_u := \pi_{d-2}p_2 \circ \pi_{d-3}D^0 u = \pi_{d-3}(D u \circ ev_0)$.

Furthermore, in this case $\pi_{d-3}\text{Exp}$ is injective (since $\pi_{d-2}i_2$ is), so $\eta \circ D^0 u$ is a left splitting. Therefore, the map $\eta \circ D^0 u \oplus \pi_{d-3}ev_0\colon \pi_{d-3}\text{Emb}_0(D^1, X) \to \mathbb{Z} \oplus \pi_{d-3}\text{Emb}_0(D^1, X)$ is an isomorphism.

Now assume $d$ is odd. Since $\text{im}(e_2^d) = 2 \cdot \mathbb{Z}$ and $\ker(e_2^d) = \text{im}(\pi_{d-1}p_2)$ we have the commutative diagram with exact sequences
\[
\begin{array}{ccccccc}
\pi_{d-1}S^{d-1} & \cong & \pi_{d-1}V_1 X & \pi_{d-1}p_2 & \pi_{d-1}X \\
\downarrow & \pi_{d-1}p_2 & \downarrow & \pi_{d-1}p_2 & \\
2 \cdot \mathbb{Z} & & \pi_{d-1}p_2 & & \pi_{d-1}X \\
\end{array}
\]
This implies that \( p'' \) is an isomorphism, and we can define
\[
dax_{u''(\pi_{d-1})} : \pi_{d-1} X \xrightarrow{(p'')^{-1}} \im(\pi_{d-1} p_2) \subseteq \pi_{d-2} \Imm \langle D, X \rangle \xrightarrow{\Dax \circ \delta_{\imm}} \mathbb{Z}[\pi_1 X],
\]
(cf. the definition of \( \dax_u \) in (12)). Then by construction we have \( \im(\delta_{\imm}) = \im(\dax_{u''(\pi_{d-1})}) \subseteq \mathbb{Z}[\pi_1 X] \). This gives the second claimed short exact sequence in the theorem, with the maps as before.

Finally, note that in this case \((\ev_0) : \ker(\pi_{d-3} \Dax_{u''}) = \ker(\pi_{d-3} \D_u) \) is not an isomorphism in general, but has for the kernel the cyclic group generated by the class of 1 \( \in \mathbb{Z}[\pi_1 X] \) modulo \( \dax_{u''(\pi_{d-1})} \). Taking the kernels in the bottom of (17) we obtain the exact sequence \( \ker(\ev_0) \hookrightarrow \ker(\pi_{d-3} \ev_0) \to \mathbb{Z}/2 \), so the cardinality of \( \ker(\pi_{d-3} \ev_0) = \mathbb{Z}/\im(\delta_{\ev_0}) \) – which is precisely the number of augmentations of an arc \( u \in \pi_{d-3} \Emb \) – is equal to two times the mentioned order.

\( \square \)

### 4.2.1 The 3-dimensional case.

We have so far considered \( d \ge 4 \). However, for \( d = 3 \) we still have an exact sequence comparing embedded to immersed arcs, which using Lemma 4.7 translates to:
\[
\pi_1 \Emb \langle D, 1 \rangle, X \xrightarrow{\pi_1 \D} \pi_2 S(X) \xrightarrow{\delta_{\imm}} k(X) \xrightarrow{\delta_{\imm}} \pi_0 \Emb \langle D, 1 \rangle, X \xrightarrow{\partial u} \pi_1 X.
\]

An element of the set \( k(X) := \pi_1 (\Imm \langle D, 1 \rangle, X, \Emb \langle D, 1 \rangle, X; u) \) is a knot together with a path to \( u \) through immersed arcs. Moreover, one still has a well-defined surjective map \( \Dax : k(X) \to \mathbb{Z}/2 \) (this is the universal order 1 Vassiliev invariant for knots in \( X \), see [Kos20a]), with a set-theoretic section \( r \), given by doing crossing changes along group elements.

Furthermore, Remark 4.9 also applies for \( d = 3 \): if \( \partial u \) has a geometric dual, then \( \pi_0 r \) is a bijection (as we saw in Theorem 2.1) and there is a distinguished class \( u^G_{\ev_0} \in \pi_1 \Emb \langle D, 1 \rangle, X \), given by “swinging the lasso” around the parallel push-off of the dual \( G \) into \( X \).

For augmented arcs in dimension \( d = 3 \) similarly as in the proof of Theorem 4.16 there is a short exact sequence \( \ker(\ev_0) \hookrightarrow \ev_0^{-1}(u) \to \mathbb{Z}/2 \), where the set \( \ker(\ev_0) = \ker(k(X)/\im(\delta_{\imm})) \to k(X)/\im(\delta_{\imm})) \) is the cyclic group generated by the crossing change along \( 1 \in \pi_1 X \). Interestingly, in this dimension there are now only two distinct cases, depending on whether \( \partial u \) has a geometric dual or not: \( u \) has respectively either exactly two framings (that is, \( \ev_0^{-1}(u) \cong \mathbb{Z}/2 \)) or countably many (\( \ev_0^{-1}(u) \cong \mathbb{Z} \)).

The first case is not hard to see explicitly: extending the loop \( u^G_{\ev_0} \) to an isotopy of augmented arcs returns us back to the framing 2 on \( u \) (so we get \( \delta_{\ev_0}(u^G_{\ev_0}) = 2 \)) – this is precisely the well-known light bulb trick for framed knots. To prove the second claim, one approach would be to show that \( 1 \in \mathbb{Z}[\pi_1 X]/\dax_{u''(\pi_2 X)} \) has infinite order; see [CCS14] for another approach.

## 5 Spaces of 2-dimensional disks with dual in the boundary

We collect the results obtained so far in order to prove Theorem F. It concerns the space \( \Emb \langle D^2, M \rangle \) of neat embeddings of the 2-disk in a manifold \( M \) of dimension \( d \ge 4 \), with the boundary condition \( s : S^1 \hookrightarrow \partial M \), which has a framed geometric dual \( G : \mathbb{S}^{d-2} \hookrightarrow \partial M \).

Firstly, Theorem D gives for all \( n \ge 0 \) explicit “ambient isotopy” and “c-foliation” isomorphisms
\[
\pi_n \Emb \langle D^2, M \rangle \xrightarrow{\pi_n f^G_u} \pi_{n+1} (\Emb \langle D^2, M \rangle, u^G_\pi),
\]
depending on the choice of a basepoint \( U \in \Emb \langle D^2, M \rangle \) (recall \( M_G := M \cup_G \mathbb{D}^{d-1} \) and \( s = u_- \cup u_+ \)).

Secondly, Theorem 4.16 gives for \( n \le d-4 \) isomorphisms \( \pi_n (\Emb \langle D^2, M \rangle, u^G_{\pi}) = \pi_{n+1} X \) and an extension on \( \pi_{d-3} \) by a quotient of \( \mathbb{Z}[\pi_1 X] \).

Combining these two results we obtain for any \( n \leq d-3 \) the isomorphism \( \pi_n \Emb \langle D^2, M \rangle \cong \pi_{n+2} M_G \), and the extensions for respectively even \( d \) and odd \( d \neq 5, 9 \):
\[
\begin{align*}
\mathbb{Z}[\pi_1 M_G] / \langle 1, \dax(\pi_{d-1} M_G) \rangle & \xrightarrow{\pi_{d-4} a_0 \cup \partial \nu_{u^G}} \mathbb{Z}[\pi_1 M_G] / \dax(\pi_{d-1} M_G) \xrightarrow{\pi_{d-4} \Emb \langle D^2, M \rangle \to \pi_{d-4} \Emb \langle D^2, M \rangle} \mathbb{Z}[\pi_1 M_G] / \dax(\pi_{d-1} M_G) \xrightarrow{\pi_{d-4} \Emb \langle D^2, M \rangle \to \pi_{d-4} \Emb \langle D^2, M \rangle} \mathbb{Z}[\pi_1 M_G] / \dax(\pi_{d-1} M_G) \xrightarrow{\pi_{d-4} \Emb \langle D^2, M \rangle \to \pi_{d-4} \Emb \langle D^2, M \rangle} \mathbb{Z}[\pi_1 M_G] / \dax(\pi_{d-1} M_G) \xrightarrow{\pi_{d-4} \Emb \langle D^2, M \rangle \to \pi_{d-4} \Emb \langle D^2, M \rangle} \mathbb{Z} / \pi_{d-2} M_G. \\
\end{align*}
\]
Our \( u_+ \) is homotopic into \( \partial M \) in which case we simply write
\[
dax := dax_{u^+}, \quad \text{and} \quad dax^e := dax_{u_+^e}.
\] (21)
In Theorem F we instead have a description only in terms of our original manifold \( M \), so we now remove all appearances of \( M_G \).

**Lemma 5.1.** The inclusion \( M \subset M_G \) induces isomorphisms \( \pi_i M \cong \pi_i M_G \) for all \( 0 \leq i \leq d-3 \), a surjection \( \pi_{d-1} M \to \pi_{d-1} M_G \), and a split short exact sequence of \( \mathbb{Z}[\pi_1 M] \)-modules
\[
\xymatrix{ \mathbb{Z}[\pi_1 M] \ar[r]^G & \pi_{d-2} M \ar[r] & \pi_{d-2} M_G. }
\]
where \( \lambda_M^{rel}(\pi_2(M, \partial M) \times \pi_{d-2}(M) \to \mathbb{Z}[\pi_1 M] \) is the relative equivariant intersection form.

**Proof.** Since attaching a \((d-1)\)-handle is homotopy equivalent to attaching a \((d-1)\)-cell, we immediately get \( \pi_i M \cong \pi_i M_G \) below degree \( d-2 \). Moreover, the relative homotopy group \( \pi_{d-1}(M_G, M) \) is the free \( \mathbb{Z}[\pi_1 M] \)-module spanned by \( h^{d-1} \). Once we show that homomorphism \( \lambda_M^{rel}(U, *) \) is a splitting, the surjectivity on \( \pi_{d-1} \) will follow from the long exact sequence of a pair. Indeed, since \( G \) is the geometric dual for \( s = \partial U \), we have \( \lambda_M^{rel}(s, G) = 1 \), so a push-off of \( G \) intersects \( U \) in the interior with \( \lambda_M^{rel}(U, G) = 1 \). \( \Box \)

**Lemma 5.2.** We have \( dax(\pi_{d-1} M_G) = dax(\pi_{d-1} M) \) as subgroups of \( \mathbb{Z}[\pi_1 M] \).

**Proof.** We need show that \( \text{im}(dax_M) = \text{im}(dax_{M_G}) \) for the respective \( dax \) maps. The diagram
\[
\xymatrix{ \pi_{d-1} M \ar[r]^p & \pi_{d-1} M_G \ar[l]_{dax_M} \ar[r]^{dax_{M_G}} & Z[\pi_1 M] \ar[l]_{dax_M} }
\]
commutes, since attaching a handle to \( \partial M \) does not influence the calculation of \( dax_M(\{t\}) = Dax(F) \). Indeed, if \( f : S^{d-1} \to M \) is represented by a family \( F(t) : D^1 \to M \), the same family also computes \( dax_M(\{t\}) \). This immediately implies \( \text{im}(dax_M) \subset \text{im}(dax_{M_G}) \). The other inclusion follows since the map \( p \) is surjective (see Lemma 5.1): if \( r = dax_{M_G}(a) \) for \( a \in \pi_{d-1} M \), then \( r = dax_M(b) \) for \( b = p(a) \). \( \Box \)

**Proof of Theorem F.** It only remains to see that the maps are as claimed in Theorem F, namely
\[
((\eta \circ D^e_{\mathcal{K}}) \oplus p_{u^+}) \circ \pi_{d-4} f^U_{1} = \eta \circ (-) \circ \partial U \subset K \subset \mathbb{Z}[\pi_1 M] \text{G}.
\]
For \( K : S^{d-4} \to \text{Emb}_{\mathcal{D}^2, M} \) the map \( f^U_{1}(K) : S^{d-3} \to \text{Emb}_{\mathcal{D}^2, M} \text{G} \) maps \( t \in S^{d-4} \subset S^{d-3} \subset S^{d-3} \text{G} \), the time \( t \) of the foliation of the sphere \( -U \subset K \subset \text{Emb}_{\mathcal{D}^2, M} \text{G} \) (use the canonical null homotopy of \( -U \subset K \subset \text{Emb}_{\mathcal{D}^2, M} \text{G} \) to get the map on the smash product). Then, \( p_{u^+} \circ \pi_{d-4} f^U_{1}(K) \) is the homotopy class of the map which takes \( t \in S^{d-4} \subset S^{d-3} \subset S^{d-3} \text{G} \), \( u^+ \in \pi_{d-4} f^U_{1}(K) \) \( 1 \in \Omega M_G \text{G} \) (based at \( u^+(-1) \)), see the discussion after Theorem 4.16. It is not hard to see that \( u^+ \) is inessential, i.e. this is homotopic to \( -U \subset K : S^{d-4} \to \text{Map}_{\mathcal{D}^2, M} \text{G} \), \( (p_{u^+} \circ \pi_{d-4} f^U_{1}(K)) = (-U \subset K) \text{G} \).

For \( d \) even we identify the composite \( \eta \circ D^e_{\mathcal{K}} \circ \pi_{d-4} f^U_{1} \), for the splitting \( \eta : \pi_{d-2} V_2(M_G) \to Z \subset \mathbb{Z} \) constructed in Theorem 4.20. For \( d \neq 4 \) it was given by \( \eta = \mathcal{E}^2_{\mathcal{K}}(f) \text{G} \subset \mathbb{Z} \subset \Omega_2(M_G) \text{G} \). For \( d = 4 \) we also have the correction term \( W_{M_G}(f) \text{G} \subset \Omega_2(M_G) \text{G} \) to \( W_2(TM_G) \). Using Lemma 5.1 we can lift this to \( W : \pi_2 M \to Z \text{G} \) by defining \( W(Z[\pi_1 M] \cdot G) = 0 \), so that \( W_{M_G}(f^U_{1}(K)) = W_{M_G}(f^U_{1}(K)) \text{G} \subset -U \subset K \). Thus, the following lemma finishes the proof of Theorem F. \( \Box \)

**Lemma 5.3.** The number \( \mathcal{E}^2_{\mathcal{K}}(f^U_{1}(K)) \in \mathbb{Z} \) is equal to the relative Euler number \( c_2(\mathcal{K}, \mathcal{U}) \) of the normal bundle of the immersion \( K : S^{d-4} \times D^2 \to S^{d-4} \times M \) given by \( (i, x) \mapsto (i, K(x)) \), relative to the immersion \( \mathcal{U} \subset M \text{G} \) corresponding to the constant family \( U \subset M \).

**Proof.** Observe that the normal bundle to \( K \) consists of vectors \( (0, u) \) \( \mathcal{U} \in T \mathcal{K}^{d-4} \) and \( u \) is a normal direction to \( K \) at \( x \) in \( M \). We need to compute the Euler class of the class \( f^U_{1}(K) \) \( f^U_{1}(K) \subset V_2(M_G) \text{G} \). This is obtained by gluing together two maps \( f^U_{1}(K) \) \( f^U_{1}(K) \) \( -U \subset \Omega V_2(M_G) \text{G} \), namely \( f_K = D^e_{\mathcal{K}} \mathcal{F}(K) \) and the constant family \( f_K \) \( f_K \) \( -U \subset \Omega V_2(M_G) \text{G} \).
First observe we can disregard $u_+$ as before, so that $f_K: \mathbb{I}^{d-4} \times \mathbb{D}^2 \to V_\epsilon(M_G)$ is at $(\tilde{t}, x)$ given by the derivative at $x \in \mathbb{D}^2$ of the embedded disk $K_\tilde{t}$ in $M_G$. Then the bundle $f_K^* T_G(M_G)$ is by definition given at a point $(\tilde{t}, x)$ as the subspace of $T_p M_G$, for $p = K_\tilde{t}(x)$, orthogonal to the derivative $\mathcal{D}_{\epsilon} K_\tilde{t}$; so belongs to the normal bundle of $K_\tilde{t}$ in $M_G$. The same is true for the constant family $\tilde{t} \mapsto -U$ in place of $K$, and they agree on the boundary $\partial (\mathbb{I}^{d-4} \times \mathbb{D}^2)$. Moreover, these normal bundles can be taken in $\mathbb{I}^{d-4} \times M$ instead. Thus, the Euler number of $f^* T_G(M_G)$ is precisely the relative Euler number of the normal bundles to the immersions $K$ and $U$.

When $d$ is even we also have an explicit map $a_U \text{Exp}(1): S^{d-4} \to \text{Emb}_0(\mathbb{D}^2, M)$ that splits off the $\mathbb{Z}$ factor. Namely, $\text{Exp}(1)$ is the $(d - 3)$-family of augmentations of $u_+$ as in Figure 15 obtained by integrating the vector field given by its meridian $\mu(S^{d-2})$ at a point $p = u_+(x)$, see the paragraph after Theorem 4.16.

Applying $a_U$ to it gives the family $U^G_{tw}: S^{d-4} \to \text{Emb}_0(\mathbb{D}^2, M)$ with $U^G_{tw}(\tilde{t}) = a_U(\text{Exp}(1)(\tilde{t} \wedge -))$, supported in a neighborhood of $p = \partial U$. For each fixed $\tilde{t} \in S^{d-4}$ the ambient isotopy pushes the arc $u_+$ around the loop of meridian circles $\mu(\tilde{t} \wedge -)$, see Figure 15. We conclude that $U^G_{tw}$ is given by doing a “family interior twist” to $U$ and then tubing the unique double point into $G$ at $p$. For $d = 4$ this figure is the end result, the disk $U^G_{tw}$ while for $d \geq 5$ it is one of the disks in the family $U^G_{tw}$.

**Remark 5.4.** The map $\pi_{d-4} a_U \circ \partial u_+$ is explicit for all $d \geq 4$: its value on $g \in \mathbb{Z}[\pi_1 X]$ is the family of 2-disks in $M$ obtained by applying to the half-disk $U$ in $M_G$ the ambient isotopy extended from the family $\partial u_+(g)$, and it can be described geometrically. Alternatively, one can guess a geometric candidate $f_g \in \pi_{d-4} \text{Emb}_0(\mathbb{D}^2, M)$ and check that its foliation has the correct Dax invariant, that is, $\text{Dax} \circ \bar{f}_g(f_g) = g$. Then $f_g = \pi_{d-4} a_U \circ \partial u_+(g)$ as these maps are mutual inverses. This is exactly what we do for $d = 4$ and our action $f_g := U + \text{fm}(\bullet)^G$ in Section 5.2.

![Figure 15. Wiggly curves describe a family of push-offs Exp(1) of the dashed arc $u_+$, contained in the present $u_+ \subseteq \mathbb{D}^3 \times \{0\} \subseteq \mathbb{D}^4$. For the two vectors which are parallel to $u_+$ parts of the push-off are respectively in the past and future. Disk $U^G_{tw}(\tilde{t})$ is the union of the top right strip, the wiggly arcs, and the rest of $U$ in bottom left.](image)

**Remark 5.5.** The map $-U \cup \star$ factors through $\pi_{d-2} M \to \pi_{d-2} M_G$ and one can also compute the kernel and image of $\pi_{d-4} \text{Emb}_0(\mathbb{D}^2, M) \to \pi_{d-2} M$. This is done in Proposition 1.8 for $d = 4$, see below.

**Remark 5.6.** The equivalence class of the extension in Theorem F for $d$ even is determined as follows. First divide out the $\mathbb{Z}$ which we know splits off and pick a set theoretic section $\rho$ of the quotient extension. If $\star$ denotes the group structure on $\pi_{d-4} \text{Emb}_0(\mathbb{D}^2, M) \cong \pi_{d-4} \text{Emb}_{\rho p}(\mathbb{D}^1, M_G)$, then for $a_1 \in \pi_{d-2} M_G$ the element $\rho(a_1) \star \rho(a_2) \star \rho(a_1 + a_2)^{-1}$ is in the kernel of $p_u$, and on this the Dax invariant is defined and inverts $a_U \circ \partial u_+$. The group 2-cocycle (with cohomology class is independent of $\rho$) is thus given by $(a_1, a_2) \mapsto \text{Dax}(\rho(a_1) \star \rho(a_2) \star \rho(a_1 + a_2)^{-1}) \in \mathbb{Z}[\pi_1 M]/(1, \text{dax}(\pi_{d-1} M))$. 
Outline. In the rest of the section we provide proofs of the statements made in Section 1.1. Firstly, for $d = 4$ Theorem F gives the bottom exact sequence in Proposition 1.8, namely

$$Z[\pi] / (1, \text{dax}(\pi_3 M)) \xrightarrow{\pi_0 a \circ \partial^e} \text{Emb}_0 [\mathbb{D}^2, M] \xrightarrow{\eta \oplus (-U \cup \cdot)} Z \oplus \pi_2 M / Z[\pi] \cdot G$$

(22)

where $\pi = \pi_1 M$ and $\eta = \frac{1}{2} ({\epsilon}_2(K, U) - W(\Upsilon \cup K))$. It remains to identify the inclusion $\pi_0 a \circ \partial^e$ as the geometric action on the set $\text{Emb}_0 [\mathbb{D}^2, M] := \pi_0 \text{Emb}_0 (\mathbb{D}^2, M)$, which was described in the introduction and will be made precise in Section 5.2 below. For this it suffices to show that this action is the inverse to $\text{Dax} \circ \pi_0^f$, which we interpret as the relative Dax invariant for homotopic disks, defined in Section 5.1.

Recall from Lemma 3.2 that the inclusion $M \subseteq M_G$ induces a bijection $\text{Emb}_0 [\mathbb{D}^2, M] \cong \text{Emb}_0 [\mathbb{D}^2, M_G]$, with the set of isotopy classes of half-disks in $M_G$. In Section 5.2 we will equip the latter set with a $Z[\pi]-$action, so that the bijection is equivariant. In fact, we will define relative Dax invariants for half-disks and neat disks in any 4-manifold $M$, and construct $Z[\pi_1 M]-$actions on them, not only for half-disks in $M_G$ and neat disks in $M$ which have a dual sphere.

We then proceed to show that a group structure on $\text{Emb}_0 [\mathbb{D}^2, M] \cong \pi_1 \text{Emb}_0 (\mathbb{D}^2, M_G)$ and the interesting group structure on $\pi_2 M$ make the maps in Theorems C and Proposition 1.8 into homomorphisms, see Section 5.3. Finally, in Section 5.4 we apply our results to recover those about spheres.

5.1 The relative Dax invariant for disks in 4-manifolds

In this section we discuss the Dax invariant of neat disks and half-disks in an arbitrary 4-manifold $M$. We fix two embeddings $u_\pm : \mathbb{D}^1 \hookrightarrow M$ such that $u_-(\mathbb{D}^1) \subset \partial M$ and $m_\pm := u_-(\pm 1) = u_+(\pm 1)$ with $m_-$ as basepoint, and set $s := u_- \cup u_+$. We will discuss two separate cases for $u_+$ in parallel:

1. $u_+(\mathbb{D}^1) \subset \partial M$ and $s$ is a smooth embedding, a boundary condition for neat disks in $\text{Emb}_0 (\mathbb{D}^2, M)$,

2. $u_+$ is a neat embedding (with interior mapping to the interior of $M$) and $s$ is a boundary condition for half-disks in $\text{Emb}_0 (\mathbb{D}^2, M)$.

In the second case we refer to $u_+$ as the “free” boundary of a half-disk $K$. The essential difference is that intersection points with $K$ can be pushed-off across $u_+$ as in Definition 5.13.

The first order of business is to translate the Dax invariant for families of arcs to an invariant of disks. To this end, we fix a convenient parametrization of $\mathbb{D}^2$, by the rectangle $\mathbb{I} \times \mathbb{D}^1 \hookrightarrow \mathbb{D}^2$. This map collapses the upper face $I \times 1$ to the point $i \in \mathbb{D}^2$ and the lower face $I \times -1$ to the point $-i \in \mathbb{D}^2$, and restricts to a diffeomorphism $I \times (-1, 1) \cong \mathbb{D}^2 \setminus \{i\}$, $(t, \theta) \mapsto (2t - 1, T(t, \theta))$, similar to $\varphi_t$ in the proof of Theorem 3.1.

Similarly, we fix a parametrization $\mathbb{I} \times \mathbb{D}^1 \hookrightarrow \mathbb{Q}^2$. Thus, the coordinates do not tell the difference between the two settings, but the boundary condition $u_+$ does, being either as in (1) or (2). Then an embedding $K : \mathbb{D}^2 \hookrightarrow M$ or $K : \mathbb{Q}^2 \hookrightarrow M$ with boundary $s$ gives a map $K : \mathbb{I} \times \mathbb{D}^1 \hookrightarrow M$ with boundary

$$K|_{x=1} = m_+, \quad K|_{x=-1} = m_-, \quad K|_{0 \times \mathbb{D}^1} = u_-, \quad K|_{1 \times \mathbb{D}^1} = u_+$$

(23)

So for $t \in I$ the embeddings $K(t, -) : \mathbb{D}^1 \hookrightarrow M$ go from $m_-$ to $m_+$ and are neat for $t \in (0, 1)$, while the extreme value $K(0, -) = u_- : \mathbb{D}^1 \hookrightarrow \partial M$ is not, and $K(1, -) = u_+$ is neat exactly in the case (2). In a future paper we will also discuss the case where $u_-$ is neat, but here we will use that it lies in $\partial M$.

5.1.1 Dax invariant for half-disks. A homotopy between two half-disks $K_0, K_1 \in \text{Emb}_0 (\mathbb{Q}^2, M)$ can be parametrized by

$$H : \mathbb{I} \times (\mathbb{I} \times \mathbb{D}^1) \hookrightarrow M, \quad (t_1, t_2, \theta) \mapsto H(t_1, t_2, \theta),$$

with $H(0, - , -) = K_0$ and $H(1, - , -) = K_1$. If $H$ is regular then $(t_1, t_2) \mapsto H(t_1, t_2, -)$ gives a map $\mathbb{I}^2 \hookrightarrow \text{Imm}_0 (\mathbb{D}^1, M)$ to which we would like to apply the Dax invariant from Section 4.1.

In Lemma 4.3 and Definition 4.4 the base arc $u_-$ occurs on all but one boundary face of $\mathbb{I}^2$ (on which it lies in embeddings), while here we have for $t_1 \in \{0, 1\}$ embeddings $K(t_2, -)$ and for $t_2 \in \{0, 1\}$ the constant arcs $u_-$ respectively $u_+$. The former setup is convenient for defining a group structure on relative $\pi_2$ (glue maps of squares along one constant face), but one easily translates between the two as in (15): in the following diagram each point $(t_1, t_2)$ corresponds to the immersed arc $H(t_1, t_2)$, and double lines...
denote constant faces.

\[
\begin{align*}
\text{u}_- & \quad \xrightarrow{K_1} \quad \text{u}_+ \\
\text{u}_- & \quad \xrightarrow{H} \quad \text{u}_+ \quad \cong \quad \text{u}_+ \quad \xrightarrow{H} \quad \text{u}_+ \\
\text{u}_- & \quad \xrightarrow{K_0} \quad \text{u}_+ \\
\end{align*}
\]

(24)

Thus, \( \text{Dax}(H) \in \mathbb{Z}[\pi \setminus 1] \), where \( \pi = \pi_1 M \) is defined by counting double points of a generic representative of the map (from a 3-manifold to a 6-manifold) \( \text{pr}_{1,2} \times H : \mathbb{I}^2 \times \mathbb{D}^3 \to \mathbb{I}^2 \times M, (t_1, t_2, \theta) \mapsto (t_1, t_2, H(t_1, t_2, \theta)) \). Recall these are given by finitely many values of \((t_1, t_2)\) where the arcs \( H(t_1, t_2) \) have double points whose nontrivial fundamental group elements (together with their signs) we add up.

The homomorphism \( \text{dax} = \text{dax}_{u_-} : \pi_3 M \to \mathbb{Z}[\pi \setminus 1] \) was defined in (12) by \( \text{dax}(a) = \text{Dax}(S_a) \), where for \( a \in \pi_3 M := \pi_3(M, m_-) \) we pick any map \( S_a : \mathbb{I}^2 \to \text{Imm}_0(\mathbb{D}^3, M) \) taking the entire \( \partial \mathbb{I}^2 \) to \( u_+ \) and such that the union \( S_0 \cup_{\partial \mathbb{I}^2} S_a \) gives a representative of \( a \). Here \( S_0(t_1, t_2) = u_+ \) for every \((t_1, t_2) \in \mathbb{I}^2\).

**Remark 5.7.** Recall that any generic homotopy between embeddings of a surface in a 4-manifold has an algebraic zero number of cusps that can be canceled in pairs. In particular, any homotopy can be replaced by a generic regular homotopy \( H \) as we shall do from now on.

The above convention not to count the trivial fundamental group element in \( \text{Dax}(H) \) and \( \text{dax}(a) \) comes from (22) where a quotient of \( \mathbb{Z}[\pi] / \langle 1 \rangle \) appears on the left. This originates in Proposition 4.8, showing that one can locally change the homotopy \( H \) by adding a single double point with the associated loop \( \pm 1 \). We often write \( \mathbb{Z}[\pi \setminus 1] \) instead of \( \mathbb{Z}[\pi] / \langle 1 \rangle \) since there is an element \( 1 \in \mathbb{Z} \) coming from the right of (22), which is detected by the normal Euler number, instead of the Dax invariant (and realized by \( U_{tw} \)).

We can now prove one part of Lemma 1.1, namely that \( \text{Dax}(H) \) is independent of the choice of a homotopy \( H \) between half-disks \( K_0, K_1 \), modulo \( \text{dax}(\pi_3 M) \). In the next section we will show the remaining part, that for *neat disks* this actually takes values in the fixed point set of the involution, see Theorem 5.11.

For a regular homotopy \( H \) from \( K_0 \) to \( K_1 \) and a regular homotopy \( H' \) from \( K_1 \) to \( K_2 \) let \( H \cup_{K_1} H' \) be their concatenation along the top face \( K_1 \), as in the left diagram below. This is a regular homotopy with \( \text{Dax}(H \cup_{K_1} H') = \text{Dax}(H) + \text{Dax}(H') \), since each point in the glued rectangle lies in exactly one of the two squares, so corresponds to an immersed arc either in \( H \) or \( H' \). This is analogous to the additivity of the Dax invariant, where two squares were instead glued along a vertical \( u_+ \) face.

\[
\begin{align*}
\text{u}_- & \quad \xrightarrow{K_2} \quad \text{u}_+ \\
\text{u}_- & \quad \xrightarrow{H'} \quad \text{u}_+ \\
\text{u}_- & \quad \xrightarrow{K_1} \quad \text{u}_+ \\
\text{u}_- & \quad \xrightarrow{H} \quad \text{u}_+ \\
\text{u}_- & \quad \xrightarrow{K_0} \quad \text{u}_+ \\
\end{align*}
\]

\[
\begin{align*}
\text{u}_+ & \quad \xrightarrow{\sim} \quad \text{u}_+ \\
\text{u}_+ & \quad \xrightarrow{H} \quad \text{u}_+ \\
\text{u}_+ & \quad \xrightarrow{K_{1}^{-1} \cup_{K_0} H' \cup_{K_1} K_0} \quad \text{u}_+ \\
\text{u}_+ & \quad \xrightarrow{K_0} \quad \text{u}_+ \\
\end{align*}
\]

**Lemma 5.8.** If \( H, H'' \) are two regular homotopies from \( K_0 \) to \( K_1 \) then \( \text{Dax}(H) - \text{Dax}(H'') \in \text{dax}(\pi_3 M) \). In particular, the relative Dax invariant \( \text{Dax}(K_0, K_1) := \text{Dax}(H) \in \mathbb{Z}[\pi \setminus 1] / \text{dax}(\pi_3 M) \) is a well-defined isometry invariant of half-disks. By definition, it satisfies \( \text{Dax}(K_1, K_0) = -\text{Dax}(K_0, K_1) \)

**Proof.** We can view \(-H''\) as running backwards from \( K_1 \) to \( K_2 = K_0 \), so precisely one time direction is reversed, cf. (24). This implies \( \text{Dax}(-H'') = -\text{Dax}(H'') \). Then by the last paragraph we have

\[
\text{Dax}(H) - \text{Dax}(H'') = \text{Dax}(H) + \text{Dax}(-H'') = \text{Dax}(H \cup_{K_1} K_0, -H'') = \text{Dax}(H \cup_{K_{1}^{-1} \cup_{K_0} K_0} K_0, -H'').
\]

The last equality follows by (24), see the right part in the above diagram. Since \( S := H \cup_{K_{1}^{-1} \cup_{K_0} K_0} -H'' \) has \( u_+ \) along all of \( \partial \mathbb{I}^2 \), we have \( \text{Dax}(S) \in \text{dax}(\pi_3 M) \) by definition. If \( H \) is an isopy we clearly have \( \text{Dax}(K_0, K_1) = 0 \), so this is an obstruction to isopy.

**5.1.2 Dax and Freedman–Quinn invariants for neat disks.** We recall the very much related Freedman–Quinn invariant for embedded spheres developed by the second author and Schneiderman in [ST19], and adapt it to the setting of neat disks.
This uses Wall’s self-intersection invariant $\mu_3(B) \in \mathbb{Z}[\pi_1 P]/\langle 1, g + \overline{g} \rangle$ for a 3-ball $B: (\mathbb{D}^3, \mathbb{S}^2) \rightarrow (P, \partial P)$ generically immersed in a 6-manifold $P$, with boundary $\mathbb{S}^2$ embedded in $\partial P$. It is given as the sum of signed group elements $g_p := B(w_x)B(w_y)^{-1}$ over all double points $p = B(x) = B(y)$ of $B$, where $w_x, w_y: \mathbb{I} \rightarrow \mathbb{D}^3$ are arbitrary whiskers from $-i \in \mathbb{S}^2$ to $x, y$. Note that there is no preferred order between $x, y \in \mathbb{D}^3$ so we have to mod out $g + \overline{g}$ to remove this ambiguity. Then $\mu_3(B)$ is invariant under homotopies which are isotopies on the boundary. Moreover, by taking the boundary of $B$ to be a small trivial sphere one obtains a map

$$\mu_3: \pi_3 P \rightarrow \mathbb{Z}[\pi_1 P]/\langle 1, g + \overline{g} \rangle.$$  

**Lemma 5.9.** If $P^6 = \mathbb{I} \times Q^5$ then $\mu_3(B) = \mu_3(B)$ for any $B: \mathbb{D}^3 \rightarrow P$ with $B(\mathbb{S}^2) \subseteq (\frac{1}{2}) \times \partial Q \subseteq \partial P$.

**Proof.** This property is a consequence of the formula $\mu_3(B) - \mu_3(B) = \lambda_P(B, B')$ for a 3-ball in any 6-manifold $P$ and $B'$ any push-off of $B$. Under our assumption, we can find an extension $\beta: \mathbb{D}^3 \rightarrow \{\frac{1}{2}\} \times Q$ of $\partial B: \mathbb{S}^2 \rightarrow \{\frac{1}{2}\} \times \partial Q$ that is regularly homotopic (rel. boundary) to $B$. Therefore, we only need to prove the property for $\beta$ in place of $B$. However, we can arrange that $\beta$ and its push-off $\beta'$ have distinct $\mathbb{I}$-coordinates and hence are disjoint, so their intersection number $\lambda_P(\beta, \beta') = \lambda_P(B, B')$ vanishes.

We now extend this to the case of neat disks in $M$, namely, we apply $\mu_3$ to a generic regular homotopy $H: \mathbb{I} \times \mathbb{D}^3 \rightarrow M$ between neat disks $K_0, K_1 \in \text{Emb}_0(\mathbb{D}^2, M)$. As in the previous section, we use our rectangle $\mathbb{I} \times \mathbb{D}^1 \rightarrow \mathbb{D}^2$ for the coordinates and study a generic representative of the track of $\frac{1}{2} \times H$:

$$\tilde{H}: \mathbb{I} \times (\mathbb{I} \times \mathbb{D}^1) \rightarrow \mathbb{D}^2 \times M, \quad (t_1, t_2, \theta) \mapsto (t_1, \frac{1}{2}, H(t_1, t_2, \theta)).$$

So here $P = \mathbb{I}^2 \times M$ and the constant value in the second coordinate constrains this homotopy into 5 dimensions, showing that $\tilde{H}$ satisfies the assumption of Lemma 5.9, so $\mu_3(\tilde{H})$ is fixed under the involution.

**Definition 5.10.** The relative Freedman–Quinn invariant of homotopic neat disks $K_0, K_1$ is defined by

$$\text{FQ}(K_0, K_1) := \{\mu_3(\tilde{H})\} \in \mathbb{Z}[\pi]^\sigma/\langle 1, g + \overline{g}, \mu_3(\pi_3 M) \rangle \cong \mathbb{F}_2[T_M]/\langle \pi_3(\pi_3 M) \rangle.$$  

Recall that we denote $\pi := \pi_1 M$ and $\mathbb{Z}[\pi]^\sigma$ is the fixed point set of $\sigma(g) = \overline{g}$. The isomorphism on the right hand side comes from the isomorphism $\mathbb{F}_2[T_M] \cong \mathbb{Z}[\pi]^\sigma/\langle 1, g + \overline{g} \rangle$, induced by the inclusion $T_M \subseteq \pi$ of the set of nontrivial 2-torsion elements in $\pi$. The pair $(\mathbb{I}, \frac{1}{2})$ plays the same exact role as the pair $(\mathbb{R}, 0)$ used in [ST19], but is easier to compare with the Dax invariant.

**Theorem 5.11.** In the above setting, the projection map $q: \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi]/\langle 1, g + \overline{g} \rangle$ sends the Dax invariant $\text{Dax}(H)$ to the Wall invariant $\mu_3(\tilde{H})$. As a consequence, $q \circ \text{dax} = \mu_3$ on $\pi_3 M$ and the relative Dax invariant determines the relative Freedman–Quinn invariant via

$$\text{FQ}(K_0, K_1) = q(\text{Dax}(K_0, K_1)).$$

It also follows that $\text{Dax}(K_0, K_1)$ takes values in the fixed set of the involution $\mathbb{Z}[\pi]^\sigma/\langle 1, \text{dax}(\pi_3 M) \rangle$.

**Proof.** Observe that $\tilde{H}$ remains an immersion of a 3-ball into the 6-manifold $\mathbb{I}^2 \times M$ restricting to an embedding on the boundary, with any value in place of $1/2$ in the second component. Therefore, $\tilde{H}$ is homotopic to the map $\tilde{H}(t_1, t_2, \theta) = (t_1, t_2, H(t_1, t_2, \theta))$ by linearly connecting the values $1/2$ and $t_2$, $s \mapsto s/2 + t_2(1 - s)$, with an additional parameter $s \in \mathbb{I}$. This homotopy restricts to an isotopy on the boundary sphere. Thus, $\mu_3(\tilde{H}) = \mu_3(\tilde{H})$.

The map $\tilde{H}$ is also used for computing $\text{Dax}(H)$, for which the fundamental group element at a double point $p = (\tilde{t}, H(x)) = (\tilde{t}, H(y)) \in \mathbb{I}^2 \times M$ with $\tilde{t} = (t_1, t_2)$, $x = (\tilde{t}, \theta, +)$, $y = (\tilde{t}, \theta, -)$ is defined as the concatenation $g_p = w_xw_y^{-1}$ for $w_x = H(\tilde{t}, -)[|_{-1, \theta, +}]$ and $w_y = H(\tilde{t}, -)[|_{-1, \theta, -}]$, see (11). Observe that these two arcs are a particular choice for whiskers used for computing the double point loop for $\mu_3$. Moreover, the signs are in both cases computed as in (10). This implies $q \circ \text{Dax}(H) = \mu_3(\tilde{H})$, and immediately shows the other claimed relations $q \circ \text{dax}(a) = \mu_3(a)$ for $a \in \pi_3 M$, and $q \circ \text{Dax}(K_0, K_1) = \text{FQ}(K_0, K_1)$.  


Finally, by Lemma 5.9 the value \( \mu_3(\tilde{H}) \) is fixed under the involution, so \( \text{Dax}(H) = q\mu_3(\tilde{H}) \) must be as well, since the property \( r \neq \Psi \) is preserved by \( q \).

We note that the Freedman–Quinn invariant is a priori not defined for a homotopy \( H : I \times \mathcal{O}^2 \to M \) between half-disks because \( \tilde{H} : I^2 \times D^1 \to I^2 \times M \) does not take the entire boundary to \( \partial(I^2 \times M) \): the face \((t_1, 1, \theta)\) is taken to \((t_1, 1/2, \pi)\) which lies in the interior of \( I^2 \times M \) whenever \( t_1 \in (0, 1) \). However, we saw that the Dax invariant is perfectly well defined and Theorem 5.11 can be viewed as a more general definition of the Freedman–Quinn invariant.

5.1.3 Computing the homomorphism \( \text{dax} \). Let us prove Theorem B, which relates the homomorphism \( \text{dax} : \pi_3 M \to \mathbb{Z}[\pi \setminus 1]^\sigma \) to Wall’s self-intersection invariants \( \mu_2 \) and \( \mu_3 \) (recall that \( \text{dax} = \text{dax}_u \) for \( u \) homotopic into \( \partial M \), by our convention \((21))\).

**Proof of Theorem B.** The upper sequence is exact by a classical result of Whitehead [Whi50]: the kernel of the quotient map is homotopic into an abelian group. Moreover, since \( \text{ker}(\text{dax}) \) have values in \( \mathbb{Z} \), it was shown that \( \text{ker}(\text{dax}) \) have values in \( \mathbb{Z} \) for all \( a \in \pi_3 M \). Thus, using the fact \( \text{dax}(\pi_3 M) = \mathbb{Z}\), we have

\[
2 \cdot \text{dax}(a \circ H) = \text{dax}(2(a \circ H)) = \text{dax}((a, a)_W) = \lambda(a, a) + \lambda(a, a) = 2 \cdot \lambda(a, a).
\]

Notice that this lives in the abelian group \( \mathbb{Z}[\pi \setminus 1]^\sigma \), which is torsion-free as a subgroup of the free abelian group \( \mathbb{Z}[\pi \setminus 1] \). Therefore, we can divide both sides of the last equation by 2.

**Proof of Lemma 1.7.** By construction, \( \pi_2 M_c \) is a free \( \mathbb{Z}[\pi]-\text{module} \) with one generator \( S_c \). For a free abelian group \( A \), Whitehead’s group \( \Gamma(A) \) is generated by symmetric tensors \( a_i \otimes a_i \) and \( a_i \otimes a_j + a_j \otimes a_i \), where \( a_i \) runs through a basis for \( A \). For \( A = \pi_2 M_c \) we have \( a_i = t^i \cdot S_c, i \in \mathbb{N} \) and we get the equations in \( t \cdot \mathbb{Z}[t] \) of the form \( \Gamma(\mu_2)(a_i \otimes a_i) = \mu_2(a_i) = \mu_2(S_c) \) and for \( i > j \)

\[
\Gamma(\mu_2)(a_i \otimes a_j + a_j \otimes a_i) = \mu_2(a_i + a_j) - \mu_2(a_j) = \lambda(a_i, a_j) \equiv t^{i-j} \cdot \mu_2(S_c)
\]

where the last equation uses our identification of the targets of these maps with \( t \cdot \mathbb{Z}[t] \). It follows that the image of \( \Gamma(\mu_2) \) is exactly the ideal generated by \( \mu_2(S_c) \).

To realize a given polynomial \( f \), we start with the unknot in \( S^1 \times S^1 \) and for each \( \pm b_i t^i \) in \( f \) we do \( b_i \) finger moves along the group element \( t^i \), with the crossing change of correct sign \( \pm \), see Figure 2.

For closed 4-manifolds, the outer homomorphisms \( \mu_2, \mu_3 \) in Theorem B still work but there cannot exist a homomorphism \( \text{dax} \) making the left square commute.

**Example 5.12.** Take \( M = S^2 \times S^2 \) with generators \( a_1, a_2 \in \pi_2(S^2 \times S^2) \) coming from the factors. Then the Whitehead product \([a_1, a_2]_W\) vanishes by definition and so \( a_1 \otimes a_2 + a_2 \otimes a_1 \in \Gamma(\pi_2(S^2 \times S^2)) \) is in the kernel of \( \partial \circ H \). It is also the image of the generator coming from \( H_4(S^2 \times S^2; \mathbb{Z}) \).

The group \( \pi = \mathbb{Z}/2 \to (t) \) acts freely on \( S^2 \times S^2 \) via \( t \cdot (x_1, x_2) = (-x_1, -x_2) \) and if we define \( M \) to be its quotient then \( \mu_2(a_1 \otimes a_2 + a_2 \otimes a_1) = \mu_2(a_1 + a_2) - \mu_2(a_1) - \mu_2(a_2) = [\lambda(a_1, a_2)] = [1 - t] = [-t] \neq 0 \).

Hence a homomorphism \( \text{dax} \) making the left diagram in Theorem B commute cannot exist.

If we remove a 4-ball from \( M \) to create boundary then \( \partial H \) becomes an isomorphism and \( \text{dax}(a_1, a_2)_W \) is twice the generator in \( \mathbb{Z}[\pi \setminus 1]^\sigma \cong \mathbb{Z} \) as required by our diagram. Since the projections \( S^2 \hookrightarrow M \) of \( a_i \) are embeddings, it follows that the target of the Dax invariant \( \mathbb{Z}[\pi \setminus 1]^\sigma / \text{dax}(\pi_3 M) = \mathbb{Z}/2 \) is nontrivial in this case (and agrees with the target of the Freedman–Quinn invariant).


5.2 Geometric actions on 2-disks in 4-manifolds and the Dax invariant

We define a $\mathbb{Z}[\pi]$-action on half-disks which will under the bijection $\Emb_0(\mathbb{D}^2, M) \cong \Emb_0(\mathbb{G}^2, M_G)$ correspond to the $\mathbb{Z}[\pi]$-action on neat disks discussed in the introduction.

**Definition 5.13.** The action $(J, r) \mapsto J + \text{fm}(r)$ of $\mathbb{Z}[\pi]$ on $\Emb_0(\mathbb{G}^2, M)$ is given by writing $r = \sum_i \varepsilon_i g_i$, with $\varepsilon_i \in \{\pm 1\}$, $g_i \in \pi$, and performing the following maneuvers for each signed group element $\varepsilon_i g_i$.

To define the half-disk $J + \text{fm}(g)$ for $g \neq 1$ first do a finger move along $g$ on $J$, creating a generic immersion $J_g$ that has a single pair of transverse double points $p_{\pm} = J_g(x_{\pm}) = J_g(y_{\pm})$ with opposite signs (let the $x$-sheet be on the finger), see Figure 16. Then push off the points $x_+$ and $y_-$ across the free boundary $u_+$ of $J_g$ to create a new embedded half-disk $J + \text{fm}(g)$, as in Figure 1. Define $J + \text{fm}(-g)$ for $g \neq 1$ similarly, but using the opposite sheet choice: push off $x_-$ and $y_+$ instead.

Finally, define $J_{tw} = J + \text{fm}(\pm 1)$ by first doing an interior twist to $J$ with sign $\pm$, then push the unique double point off the free boundary (use either sheet).

![Figure 16. Finger move drawn in present. Only two open disks from the tip of the finger are in past and future.](image1)

![Figure 17. Push-off. The vertical disk is pushed along the dashed arc across $u_+$.](image2)

Note that [ST19, Lemma 3.9] implies that the first construction $J \pm \text{fm}(r)$ is isotopic to $J$ for $r = 1$, hence we need the second case $g = 1$ above. Using distinct sheet choices is also essential: pushing off $x_+$ and $x_-$ instead (or $y_+$ and $y_-$), gives a half-disk isotopic to the original $J$ (use an additional parameter to push off the tangency in the finger move). By construction, the homotopy class of the half-disk remains unchanged; in fact, the description above gives a canonical homotopy from $J$ to $J + \text{fm}(r)$.

For readers not familiar with 4-dimensional maneuvers, we give a precise meaning to the above constructions. They are best understood in terms of a stratification of the space of all immersions $\Imm_0(\mathbb{G}^2, M)$ of half-disks. The open, dense (codimension 0) stratum are the generic immersions whose singularities are finitely many interior double points that are transverse. A path in $\Imm_0(\mathbb{G}^2, M)$, i.e. a regular homotopy of half-disks, is generically a finite composition of paths of generic immersions (that can be implemented by self-isotopies of domain and range) and the following two types of paths (or their reverses) that meet the codimension 1 strata transversely in a single point.

- A **finger move** is a regular homotopy $J_t$ that is a generic immersion except for one time $t = t_0$, when $J_{t_0}$ has one double point that is not transverse but where the tangent spaces meet in a 1-dimensional subspace. Thus, one sheet moves by an ambient isotopy along a path $g$ before it exhibits a self-tangency and right after creates a pair of additional transverse double points as in Figure 16.

- Pushing off $x \in \mathbb{G}^2$ across the free boundary $\mathbb{D}^+$ is a regular homotopy $J_t$ that is a generic immersion except for one time $t = t_0$, when $J_{t_0}$ has one intersection of the interior point $x \in \mathbb{G}^2$ with a free boundary point $y_+ \in \mathbb{D}^+ \subseteq \partial \mathbb{G}^2$. Thus, the sheet around $x$ moves as in Figure 17 along the dotted arc from $y$ to $y_+$; for all $t < t_0$ we have $J_t(x) = J_t(y)$ whereas this double point disappears for $t > t_0$.

We leave it to the reader to show geometrically that the action in Definition 5.13 is well defined, i.e. that the isotopy class of $J + \text{fm}(r)$ does not depend on the order of maneuvers or the various choices.
in the construction. It follows that \((J + \text{fm}(r)) + \text{fm}(-r) = J\) on isotopy classes, so we can write \(J - \text{fm}(r) := J + \text{fm}(-r)\). Using the group extension (22) this independence of all choices also follows from the following result, which is the crucial ingredient for the proof of Theorem A below.

**Proposition 5.14.** For all \(r \in \mathbb{Z}[\pi \setminus 1]\) we have \(\text{Dax}(J + \text{fm}(r), J) = [r] \in \mathbb{Z}[\pi \setminus 1]/\text{dax}(\pi_3M)\).

**Proof.** By definition, \(\text{Dax}(J + \text{fm}(g), J) = \text{Dax}(J_t)\) is computed by counting double points of arcs that foliate some regular homotopy \(J_t: 
\mathbb{G}^2 \leftrightarrow M\) from \(J_0 = J + \text{fm}(r)\) to \(J_1 = J\). The mentioned canonical homotopy is one such choice and we can foliate it as follows, see Figure 18.

Let \(J_{t_0}\) be the moment of self-tangency, when the finger touches the disk. We foliate this half-disk \(J_{t_0}\) so that a single arc, call it \(\alpha_0\), has this point as a transverse self-intersection. Then \(J_t\) for some nearby \(t < t_0\) has two double points \(p_-, p_+\) which must occur on two distinct arcs \(\alpha_-, \alpha_+\) of the foliation, so that \(\alpha_+\) is closer to \(u_-\) than \(\alpha_0\) and \(\alpha_-\) is further from it. We can pick the guiding arc for the pushing-across operation for \(p_+\) to be the shortest path from \(p_+\) to \(u_+\), so that pushing will never produce self-intersecting arcs. For the guiding arc for \(p_-\) we pick part of \(\alpha_0\) from \(p_-\) all the way close to \(m_+\), and then a short arc from there to \(u_+\). Similarly, all arcs stay embedded during this push.

![Figure 18. Arcs \(\alpha_-, \alpha_0, \alpha_+\) in our foliation of a homotopy for the action, and dashed arcs guiding push-offs.](image)

Thus, it is clear that there is a single immersed arc in this foliation, namely \(\alpha_0\) in \(J_{t_0}\), and we claim that its unique double point has group element \(+g\), so that \(\text{Dax}(J_t) = g\). Indeed, this arc looks precisely like the one in the realization map family for \(g\), see the right picture in Figure 12.

**Remark 5.15.** For \(J + \text{fm}(\pm 1)\) instead of a finger move we perform the interior twist, and then the obvious homotopy \(J_t\) can be foliated so that all arcs are embedded. This shows that \(\text{Dax}(J + \text{fm}(\pm 1), J) = 0\), which is consistent with the discussion at the beginning of Section 5, see Figure 15.

**Proof of Theorem A.** Using Proposition 5.14 for the case \(X = M_G\) and the correspondence of half-disks in \(M_G\) and neat disks in \(M\) we obtain the analogous statement for neat disks from Theorem A: the action \(+\text{fm}(\bullet)^G\) of the group \(\mathbb{Z}[\pi \setminus 1]\) on \(\text{Emb}_0(\mathbb{D}^2, M)\) satisfies

\[
\text{Dax}(J + \text{fm}(r)^G, J) = r \in \mathbb{Z}[\pi \setminus 1]/\text{dax}(\pi_3M).
\]

Indeed, the bijection between half-disks in \(M_G\) and disks in \(M\) takes the operation of “pushing across the free boundary” into “tubing into the dual \(G\)”. Comparing Definition 5.13 for the \(\mathbb{Z}[\pi \setminus 1]\)-action on half-disks with the \(\mathbb{Z}[\pi \setminus 1]\)-action on disks from the introduction (see Figure 1), we see that they correspond to one another, as well as their Dax invariants.

Now suppose that \(K_0, K_1 \in \text{Emb}_0(\mathbb{D}^2, M)\) are homotopic neat disks whose relative Dax invariant \(\text{Dax}(K_0, K_1) = \mathbb{Z}[\pi \setminus 1]^\sigma/\text{dax}(\pi_3M)\) is trivial. Then the exact sequence (22) with \(U = K_1\) says that the disks \(K_0, K_1\) are isotopic. Indeed, \(\text{Dax}(\bullet, U)\) is precisely the inverse of the inclusion map \(U + \text{fm}(\bullet)^G\).

Finally, Lemma 5.16 below shows that \(r \in \mathbb{Z}[\pi \setminus 1]\) preserves the homotopy class if and only if \(r \in \mathbb{Z}[\pi \setminus 1]^\sigma\), i.e. is fixed under the involution.
Lemma 5.16. For a neat disk $K$ in $M$ the $\mathbb{Z}[\pi]$-action of $r$ changes its homotopy class by a connected sum with $(\sigma(r) - r)$ copies of $G$, that is, $K + \text{fn}(g)^{G}$ is homotopic to $K#(\sigma(r) - r) \cdot G \text{ rel. boundary}$.

Recall that $\sigma(1) = 0$, so for $r = 1$ this means $K^G_{tw} \simeq K#(-G)$ for the $+1$-interior twist $tw$.

Proof. Under the above correspondence with half-disks, during the homotopy from $K$ to $K + \text{fn}(g)$ we cross $u_+$ twice, which adds two copies of $G$. For the positive double point $p_+$ we use the $x$-sheet so the group element guiding the tube into $G$ is clearly $g$, but the sign is $-1$ as we. Indeed, we have $\text{sign}(u_+)= (\mathbb{D}_e^2, \mathbb{D}_t^2) = +1$ but also $\lambda(K, G) = 1$, so we can see in the 3-dimension model in Figure 17 we have to use negatively oriented $G$. The argument is analogous for the negative double point $p_-$, giving $\pm G$. □

Proof of Corollary 1.2. Recall that $\mu_2(J)=0$ if and only if the disk $J$ is homotopic to a generic immersion that admits a collection of disjointly embedded, framed Whitney disks. In the presence of a dual sphere $G$, the interiors of these Whitney disks can in addition be made disjoint from copies of $G$. As a consequence, Whitney moves give a regular homotopy from $J$ to an embedding and $\text{im}(j) = \mu_2^{-1}(0)$. Surjectivity of $\mu_2$ follows from $\mu_2(J#r \cdot G) = \mu_2(J) + [r]$, and all other statements follow from Theorem A. □

5.3 Nonabelian group structures on isotopy classes of disks

In this section we construct the group structures claimed in Theorem C and show that the two maps $j$ and $\mu_2$ are homomorphisms, completing its proof. We then prove Proposition 1.8. Recall that we study 2-disks in a 4-manifold $M$ with the boundary $s: S^1 \simeq \partial M$ that splits as $s = u_- \cup u_+$ and has a geometric dual $G$ with $\lambda(s, G) = 1$. The $\epsilon$-augmented boundary conditions are denoted by $u_\epsilon^a: \mathbb{D}^1 \times [0, \epsilon] \hookrightarrow M$.

5.3.1 A space version of the Norman trick. Our results have a nice geometric interpretation that goes beyond the action maps in the previous section. Namely, we will define a space $\text{Imm}^{f_{\alpha}}(\mathbb{D}^2, M)$ of “foliation generic” immersions, each of which will admit a canonical foliation by a path in $\text{Emb}^{\alpha}_{\partial M}(\mathbb{D}^1, M_G)$.

Using the “ambient isotopy” map $\alpha$ from Theorem E, such a path can then be turned into an embedding of a disk, and Theorem 5.18 below says that the composite is given by an iterated Norman trick.

Up to now we have used based loop spaces of embedded arcs rather then path spaces: in Theorem E we had the inverse homotopy equivalences $f_{\alpha}^{-1}$ (foliates $-U \cup K$ into a loop) and $\alpha_{\epsilon}$ (extends a loop to an ambient isotopy and applies it to $U$). We can instead consider inverse homotopy equivalences

$$\text{Emb}^{\alpha}_{\partial M}(\mathbb{D}^2, M) \xrightarrow{f_{\alpha}^{-1}} P^n_{\epsilon_{\alpha}} \text{Emb}^{\alpha}_{\partial M}(\mathbb{D}^1, M_G),$$

where $f_{\alpha}$ foliates a disk $K$ into a path $[\epsilon, \epsilon]\times S^1$ of $\epsilon$-augmented arcs from $u_-^{\epsilon}$ to $u_+^{\epsilon}$ using the parametrization $(t, \theta) \in I \times \mathbb{D}^1 \to \mathbb{D}^2$ (see Section 5.1), and $\alpha$ extends a path into an ambient isotopy, applies its endpoint to the half-disk $U_0$ to get a half-disk in $M_G$, and finally uses the homotopy equivalence from Lemma 3.2 to get a neat disk in $M$. Here we fix a small half-disk $U_0: \mathbb{D}^2 \hookrightarrow M$ that is the restriction of $u_-^{\epsilon}$, going from $u_-: \mathbb{D}^1 \hookrightarrow \partial M$ to a nearby neat arc $u_-: \mathbb{D}^1 \hookrightarrow M_G$, as discussed in the introduction below Theorem D.

It is easy to see that for $\gamma_U := f_{\alpha}(U)$ the map $(\gamma_U^{-1})_*: P^n_{\epsilon_{\alpha}} \text{Emb}^{\alpha}_{\partial M}(\mathbb{D}^1, M_G) \to \Omega \text{Emb}^{\alpha}_{\partial M}(\mathbb{D}^1, M_G)$ given by $\gamma \mapsto \gamma_U^{-1} \cdot \gamma$ (concatenate paths from left to right) is a homotopy equivalence which takes $f_{\alpha}, \alpha$ to $f_{\alpha}, \alpha_U$.

Definition 5.17. We denote by $\text{Imm}^{f_{\alpha}}(\mathbb{D}^2, M)$ the subspace of neat immersions $J: \mathbb{D}^2 \hookrightarrow M$ that equal $s^{\epsilon}$ near the boundary and are foliation generic, that is satisfy the following conditions:

1. $J$ is generic, i.e. its singularities are finitely many transverse double points, and
2. $J$ respects the $(t, \theta)$-foliation, in the sense that each $J(t, -): \mathbb{D}^1 \hookrightarrow M$ is an embedding.

Such a fog immersion $J$ gives a path $f_{\alpha}(J) \in P^n_{\epsilon_{\alpha}} \text{Emb}^{\alpha}_{\partial M}(\mathbb{D}^1, M_G)$, by using $J(t, -)$ and reparametrizing $[0, \epsilon]$ according to the minimal distance of two double points in the domain of $J$. Note that these arcs actually lie in $M$ but we consider them in the larger manifold $M_G = M \cup_G h^3 \supset M$. 

We define the *Norman map* $\mathcal{N}$ as the composite
\[
\text{Imm}^{\text{emb}}_{\partial}^f(\mathbb{D}^2, M) \xrightarrow{\nu} P^1_{\text{rel}} \text{Emb}_{\partial}^p(\mathbb{D}^3, M_G) \xrightarrow{a} \text{Emb}_{\partial}^p(\mathbb{D}^2, M).
\]
By (25) the restriction of $\mathcal{N}$ to $\text{Emb}_{\partial}^p(\mathbb{D}^2, M) \subset \text{Imm}^{\text{emb}}_{\partial}^f(\mathbb{D}^2, M)$ is homotopic to the identity. The following result says that $\mathcal{N}$ is a space version of the Norman trick.

**Theorem 5.18.** The embedded disk $\mathcal{N}(J)$ is isotopic to the “Norman disk” $J_Z$, which is obtained from $J$ by applying the Norman trick to all double points of $J$ along arbitrary arcs $Z$ that are transverse to our foliation of $\mathbb{D}^2$, see below.

If $J(x_i) = J(y_i)$ are the double points of $J$, $1 \leq i \leq n$, we know that each pair $x_i, y_i \in I \times \mathbb{D}^1$ has distinct $t$-coordinates and we assume $t(x_i) < t(y_i) \in I$. For the Norman trick on $J$, we choose disjointly embedded arcs $Z \subset \mathbb{D}^2 \setminus \{x_i\}$ from each $y_i$ to a point $y_i^+$ in $\{1\} \times \mathbb{D}^1$, the $+$-side of the boundary of $\mathbb{D}^2$. Then we choose a small radius $r$ and remove the disks $\mathbb{D}^2_i(x_i)$ from our domain $I \times \mathbb{D}^1$ and replace them by disjoint tubes of radii $r$ along the arcs $Z$ (these are the radius $r$ circle bundles of the normal bundle of $J$, restricted to $Z$), capping them off by disjoint copies of $G$ (with radius $r$ disks removed around the point $G \cap u_+$). This gives the Norman disk $J_Z$, compare [ST19, Def.3.2] for the case of spheres.

**Remark 5.19.** Theorem 5.18 implies that the isotopy class of $J_Z$ does not depend on the choice of $Z$, as long as these arcs are transverse to the leaves $(t, -)$ of our foliation of $I \times \mathbb{D}^1$. A key step in the proof of LBT in [ST19] is to show (in Lemma 3.5) that any choice of $Z$ gives isotopic Norman spheres, as long as we fix the sheet choices.

Here we are in a special case, only transverse arcs $Z$ will come from a vector field that implements the ambient isotopy (extending the isotopy $f^\varepsilon(J)$ of $\varepsilon$-augmented arcs in $M_G$). Hence, the independence (of the isotopy class of $J_Z$) from the choice of transverse arcs $Z$ is a consequence of the independence from the choice of ambient isotopy, which follows from the existence of $a$.

**Proof of Theorem 5.18.** By the continuity of the map $\mathcal{N}$, we can change $J$ along a path in $\text{Imm}^{\text{emb}}_{\partial}^f(\mathbb{D}^2, M)$ without changing the isotopy class of $\mathcal{N}(J)$. In particular, we can do an ambient isotopy of the domain $\mathbb{D}^2$ that arranges for all $x_i, y_i \in I \times \mathbb{D}^1 \rightarrow \mathbb{D}^2$ to have distinct $\theta$-coordinates and so that their $t$-coordinates satisfy $t(x_1) < t(y_1) < t(x_2) < t(y_2) < \cdots < t(y_n)$. Moreover, we can turn any choice of transverse arcs $Z$ into the collection of horizontal arcs $Z_t(t) := (t, \theta(y_j))$ for $t \in [t(y_j), 1]$. These are disjoint embeddings in $\mathbb{D}^2 \setminus \{x_i\}$ and we assume that they are more than a distance of $r$ apart.

The Norman disk $J_Z$ comes from the half-disk $HJ_Z$ in $M_G$ that is obtained from $J$ by pushing the double points off the free boundary $u_+ \subset M_G$ along the same arcs $Z$. When translated to neat disks in $M$, the tubing off the free boundary is replaced by tubing into $G$. So it suffices to show that there is an isotopy from $U_0$ to $HJ_Z \subset \text{Emb}_{\partial}^p(G^2, M)$ rel. $u_-$, which restricts to $f^\varepsilon(J)$ on each $\varepsilon$-augmented arc.

First assume that $n = 1$, so $J(x) = J(y)$ is the only double point. We may assume $t(x) < 1/2, 3/4 < t(y)$, so $J$ is the union of two embedded disks $J_x: [0, 3/4] \times \mathbb{D}^1 \hookrightarrow M$ and $J_y: [1/2, 1] \times \mathbb{D}^1 \hookrightarrow M$. There is an isotopy of domains as in the proof of Theorem 3.1 between $U_0$ and $J_x$ rel. $u_-$ which we embed into an ambient isotopy. By construction, the endpoint of this ambient isotopy takes $U_0$ to $J_x$ as required, and it restricts to $f^\varepsilon(J_x)$ on each $\varepsilon$-augmented arc.

Let $U_{1/2}$ be the restriction of $J$ to $[1/2, 3/4] \times \mathbb{D}^1 \hookrightarrow M$. Then the same step gives an ambient isotopy such that $\varphi_1$ takes $U_{1/2}$ to $J_y$. However, when we glue these isotopies together, we have to apply $\varphi_1$ to $J_x$, not just $U_{1/2}$. The intersection point $J(x) = J(y)$ means that a small disk $\mathbb{D}^2_t(x)$ will be moved by $\varphi_t$. In terms of the generating vector field of the isotopy, recall that it is $\partial/\partial t$ along the arc $Z(t) := (t, \theta(y))$ for $t \in [t(y), 1]$ and we have to taper its length off to zero near this arc (keeping it pointing to the right). We do this so that $r$ is small enough so that points of this distance from $Z$ still move in the direction $\partial/\partial t$ but the vector field vanishes everywhere else on $J_y$.

Thus, $\varphi_1$ replaces our disk $\mathbb{D}^2_t(x)$ by a tube of radius $r$ along $Z$ that extends just across $u_+$ because that is where the vector field is also zero. When we glue the disks $\varphi_1(J_x)$ with $J_y$ along $U_{1/2}$, we obtain by construction the Norman disk $J_Z$ and also an ambient isotopy to $U_0$. 
In general \( J \) is the union of disks \( J_1, J_2, \ldots, J_n \), with one double point \( x_i, y_i \) in the interior of each \( J_i : [(i-1)/n, i/n] \times \mathbb{D}^1 \rightarrow M \). By similar discussions as above, we see that the tubes along the arcs \( Z_i \) are extended to the right across all disks \( J_1, J_2, \ldots, J_n \) because when pushing off the free boundary of \( J_i \), one gets a new intersection with \( J_{i+1} \) that gets pushed across its free boundary, and so on. This means that the ambient isotopy takes \( U_0 \) to \( J_2 \) as required in our claim. \( \square \)

Let us describe the effect of \( \mathcal{N} \) under the forgetful map to \( \text{Map}_0[\mathbb{D}^2, M] \). Let \( \mu_{1,1} : \text{Imm}^{fog}_{0}\mathbb{D}^2, M] \rightarrow \mathbb{Z}[\pi] \) be a lift of Wall’s self-intersection invariant \( \mu_2 \) to fog immersions: it is the sum \( \mu_{1,1}(J) := \sum_{i=1}^n \varepsilon_i \cdot g_{x_i} \) over double points of \( J \), obtained using the order of sheets \( (x_i, y_i) \) from the definition of a fog immersion.

**Lemma 5.20.** The following diagram commutes

\[
\begin{array}{ccc}
\text{Imm}^{fog}_{0}\mathbb{D}^2, M] & \xrightarrow{\mathcal{N}} & \text{Emb}_{\partial} \mathbb{D}^2, M] \\
\downarrow j \times \mu_{1,1} & & \downarrow j \\
\text{Map}_0[\mathbb{D}^2, M] \times \mathbb{Z}[\pi] & \xrightarrow{(a, r) \mapsto a \# r \cdot [-G]} & \text{Map}_0[\mathbb{D}^2, M] \\
\end{array}
\]

In other words, modulo homotopy \( \mathcal{N}(J) \) is the ambient oriented connected sum of the disk \( J \) and copies of the sphere \(-G\) along the group elements \( \mu_{1,1}(J) \).

**Proof.** To obtain \( \mathcal{N}(J) = J_2 \) each double point \( p = J(x) = J(y) \) is tubed into a copy of \( G \), and we claim that the double point loop \( g_p \) becomes the fundamental group coefficient for that copy.

By definition, \( g_p \) starts at the basepoint \( m_- = u_-(−1) \) goes along a whisker to \( x \), then jumps to \( y \) and returns on a whisker to \( m_- \). The Norman trick takes the disk \( \mathbb{D}_r(x) \) and replaces it by a tube into \( G \) along the arc \( Z \), which can be taken as a whisker for \( y \). So we precisely add or subtract \( g_p \cdot G \) to the homotopy class of \( J \) in the sense of ambient oriented connected sum. The sign of \( G \) used in the connected sum is opposite to \( \varepsilon_p \), by a similar reason as in Lemma 5.16, using \( \lambda(J, G) = +1 \): the sum of \( \mathbb{D}_r(x) \) and \( G \) is determined by the arc \( Z \) (equivalently the \( y \)-sheet), so a positive copy of \( G \) is used if and only if \( x \)-and \( y \)-sheet at \( p \) intersect negatively. \( \square \)

**Remark 5.21.** The sign in this proof can also be seen from the fact that \( \mathcal{N}(J) \) is an embedding and hence when viewed as a fog immersion, it must have vanishing \( \mu_{1,1} \)-invariant, for which we have

\[
0 = \mu_{1,1}(\mathcal{N}(J)) = \mu_{1,1}(J) + \lambda(J, r \cdot [-G]) = \mu_{1,1}(J) - r \cdot \lambda(J, G) = \mu_{1,1}(J) - r \in \mathbb{Z}[\pi].
\]

Thus, \( \mu_{1,1}(J) \) \( r \) copies of \( [-G] \) are added to \( J \). We also get an alternative proof of Corollary 1.2. Namely, if a homotopy class has vanishing \( \mu_2 \), it is not hard to see (via isotopies of the domain \( \mathbb{D}^2 \)) that it can be represented by a fog immersion \( J \) with \( \mu_{1,1}(J) = 0 \). Then the Norman disk \( \mathcal{N}(J) \) is an embedding that is homotopic to \( J \) by Lemma 5.20.

**5.3.2 Group structures.** The group structure on \( \text{Emb}_0[\mathbb{D}^2, M] \) with trivial element \( U \) comes from our main isomorphisms, first with half-disks \( \text{Emb}_0[\mathbb{D}^2 \setminus \partial, M_G] \) and then with \( \pi_1(\text{Emb}_0(\mathbb{D}^1, M_G), u^\circ) \), using the foliation map \( f^*_0 \). We saw in (25) that this map factors through the foliation map \( f^* \) into the space of paths in \( \text{Emb}_0(\mathbb{D}^1, M) \) from \( u^\circ \) to \( u^+ \). The concatenation of loops corresponds to the H-space structure on this path space given by the formula

\[
(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot u^0 \cdot \gamma_2, \quad \text{for } \gamma_i \in \text{Emb}_0(\mathbb{D}^1, M). \tag{26}
\]

This translates to the group structure \( * \) on \( \text{Emb}_0[\mathbb{D}^2, M] \) in terms of Norman disks:

\[
K_1 \ast K_2 := a(f^*(K_1) \cdot u^-^{1} \cdot f^*(K_2)) = \mathcal{N}(K_1 \cdot -U \cdot K_2), \tag{27}
\]

where \( K_1 \cdot -U \cdot K_2 \) is the following fog immersion. We glue the three embedded disks \( K_1, K_2, U \) along boundary arcs \( u \pm \) in the order left to right and let \(-U\) run backwards from \( u_+ \) to \( u_- \). Doing this naively gives zeroes of the derivative along the two arcs, say for \( K_1 \cdot -U \), but we know that the result should be of the form \( a(f^*(K_1) \cdot u^-^{1}) \), so locally around the arc \( u_+ \) they in fact do glue into an embedding.
To see how, simply apply to $K_1$ the endpoint of the ambient isotopy coming from the isotopy of $\varepsilon$-augmented arcs $\gamma_u^{-1}$; near $u_+$ boundary of $K_1$ this isotopy runs $K_1$ backwards for a while and then veers of. So $K_1$ gets shrunk near $u_+$ then gets pushed off of $U$ and a short embedded annulus $u_+ \times [-\delta, \delta]$ gets added to smooth out the union of $K$ and $U$ near $u_+$. This happens locally near the fixed arcs $u_\pm$, and no new intersection points arise. The same process occurs near $u_- \to$ glue $-U$ and $K_2$ to a local embedding.

Since all three disks are embedded and we may assume that they meet transversely, the condition in Definition 5.17 is automatically satisfied: we have intersections $K_1(x) = U(y)$, where $x$ lies to the left of $y$ in our concatenation, similarly for intersection points $K_1(y) = K_2(y)$ and $U(x) = K_2(y)$. In particular,

$$\mu_{1,2}(K_1 \cdot -U \cdot K_2) = \lambda(K_1, -U) + \lambda(K_1, K_2) + \lambda(-U, K_2) = \lambda(-U \cup K_1, -U \cup K_2),$$

so Lemma 5.20 implies by our normalization $\lambda(U, G) = 1$:

$$j(K_1 * K_2) = j(N(K_1 \cdot -U \cdot K_2)) = j(K_1 \cdot -U \cdot K_2) \# \lambda(-U \cup K_1, -U \cup K_2) \cdot [-G]. \tag{28}$$

Analogously, on $\text{Map}_0(D^2, M) = P_{u+} \cdot \text{Map}_0(D^1, M)$ we have an $H$-space structure given by the same formula (26), and makes the bijection $-U \cup \ast = (\gamma_u^{-1})_*: \text{Map}_0(D^2, M) \overset{\cong}{\to} \pi_2 M, \gamma \mapsto a_\gamma := \gamma_u^{-1} \cdot \gamma$ into a group isomorphism under the usual $+ \ast \pi_2 M$.

However, we make the right hand side of (28) into the definition of a twisted $H$-space structure on $\text{Map}_0(D^2, M)$, that is, for paths $\gamma_i \in \text{Map}_0(D^2, M)$ we let

$$\gamma_1 \ast \gamma_2 = (\gamma_1 \cdot \gamma_u^{-1} \cdot \gamma_2) \# \lambda(\gamma_u^{-1} \cdot \gamma_1, \gamma_u^{-1} \cdot \gamma_2) \cdot (-G).$$

Under the bijection $-U \cup \ast$ this gives the twisted group structure on $a_i = a_{\gamma_i} \in \pi_2 M$, as in the introduction:

$$a_1 \ast a_2 := a_{\gamma_1} \ast a_{\gamma_2} = (\gamma_u^{-1} \cdot \gamma_1 \cdot \gamma_u^{-1} \cdot \gamma_2) - \lambda(a_1, a_2) \cdot [G] = a_1 + a_2 - \lambda(a_1, a_2) \cdot [G].$$

Proof of Theorem C. The map $j: \text{Emb}_0[D^2, M] \to \text{Map}_0[D^2, M]$ is a homomorphism for the above group structures by (28). To see that $\mu_2: \text{Map}_0[D^2, M] \to \mathbb{Z}[\pi \setminus 1]/(r - \tau)$ is a homomorphism as well, use

$$\mu_2(\gamma_1 \ast \gamma_2) = \mu_2(\gamma_1 \cdot \gamma_u^{-1} \cdot \gamma_2) - \lambda(\gamma_u^{-1} \cdot \gamma_1, \gamma_u^{-1} \cdot \gamma_2),$$

where similarly to the equation in (28) we find

$$\mu_2(\gamma_1 \cdot \gamma_u^{-1} \cdot \gamma_2) = \mu_2(\gamma_1) + \mu_2(\gamma_2) + \lambda(\gamma_1, \gamma_u^{-1}) + \lambda(\gamma_1, \gamma_2) + \lambda(\gamma_u^{-1}, \gamma_2).$$

Collecting the terms gives the required equation $\mu_2(\gamma_1 \ast \gamma_2) = \mu_2(\gamma_1) + \mu_2(\gamma_2)$. This finishes the proof of Theorem C, since the exactness of the 4-term sequence in that theorem follows from Corollary 1.2. □

Proof of Proposition 1.8. We first show that the right vertical sequences in the diagram in Proposition 1.8 is short exact. Recall $\sigma(g) = \overline{g}$ for $g \neq 1$ and $\sigma(1) = 0$ and let us extend that diagram to the right:

$$\begin{array}{c}
\begin{array}{c}
\mathbb{Z}[\pi]/\mathbb{Z}[\pi] \\
\downarrow \text{U}(\#(\sigma(\ast) - 1) \cdot G)
\end{array}
\xrightarrow{\sigma - \text{Id}}
\begin{array}{c}
\mathbb{Z}[\pi]/\text{im}(\sigma - \text{Id}) \\
\downarrow \text{U}(\#(\ast) \cdot G)
\end{array}
\xrightarrow{\mu_2} \\
\begin{array}{c}
\text{Emb}_0[D^2, M] \\
\downarrow j
\end{array}
\xrightarrow{\mu_2^{-1}(0)} \\
\begin{array}{c}
\text{Map}_0[D^2, M] \\
\downarrow \text{U}(\ast)
\end{array}
\xrightarrow{\mu_2} \\
\begin{array}{c}
\pi_2 M/\mathbb{Z}[\pi] \cdot G \\
\downarrow \text{U}(\ast)
\end{array}
\xrightarrow{\cong} \\
\begin{array}{c}
\pi_2 M_G
\end{array}
\end{array}$$

The new vertical sequence is exact by Lemma 5.1 and the two new horizontal 3-term sequences are exact by definition. It follows that the original right vertical sequence, now in the second column, is also exact.

The connecting map in the diagram in Proposition 1.8 is computed as follows. Start with $g \in \pi_1 M$ representing a generator of the group $\mathbb{Z}[\pi]/\mathbb{Z}[\pi]^G$ in the lower left corner. Acting by $g$ on $U$ gives $U \cup \text{fm}(g)^G \in \text{Emb}_0[D^2, M]$, which is by $j$ mapped to the homotopy class $[U \cup \text{fm}(g)^G] = [\text{U}(\#(\sigma(g) - g) \cdot G)]$.

by Lemma 5.16. This is clearly also the image of $g$ under the map from the upper right corner of the diagram in Proposition 1.8. Therefore, its connecting map is the identity.
The exactness of the lower horizontal sequence in Proposition 1.8 was shown in Theorem F, proven at the beginning of this section, see (22). The upper horizontal sequence is exact by a diagram chase, using the information proven previously.

\[ \square \]

**Proof of Proposition 1.9.** The second statement about the splitting comes from Theorem 4.16, where we also give a nice interpretation of the quotient group as the fundamental group group of non-augmented arcs in \( M_G \). It follows that \( U_G^2 \) is a central element in our group of embeddings that comes from \( 1 \in \mathbb{Z}[\pi] \) and maps to \( 1 \in \mathbb{Z} \) under our maps in Propositions 1.8 and 1.9.

The first statement, about the commutator pairing, follows from Proposition 4.11, in which we considered the group \( \pi_1 \operatorname{Emb}_0(\mathbb{D}^1, M_G) \), but not the \( \epsilon \)-augmented version that gives \( \operatorname{Emb}_0[\mathbb{D}^2, M] \). However, we also showed that those two differ by the \( \mathbb{Z} \) on the right hand side (22). Since the embedded disk \( U_G^2 \) is central and maps to 1 in this \( \mathbb{Z} \), it follows that the commutator pairing of \( \pi_1 \operatorname{Emb}_0(\mathbb{D}^1, M_G) \) gives that of \( \operatorname{Emb}_0[\mathbb{D}^2, M] \) after subtracting the coefficient of 1 in \( \lambda \).

\[ \square \]

Being used to the quadratic property of Wall’s self-intersection invariant, we were very surprised to find

\[ \lambda \] is hermitian and the coefficients of

\[ a \] cancel.

This can be compared with Proposition 1.9, in which we computed the commutator \( [K_1, K_2] = U + \text{fm}(r)^G \) with \( r = \lambda(a_1, a_2) \in \mathbb{Z}[\pi \setminus 1] \) for \( K_i \in \operatorname{Emb}_0(\mathbb{D}^2, M) \) and \( a_i = [-U \cup K_i] \in \pi_3 M \). Since we have \( U + \text{fm}(r)^G \simeq U \# (r - r) G \), it follows that \( -U \cup [K_1, K_2] \simeq -U \cup (U \# (r - r) \cdot G) \simeq (r - r) \cdot G \). This agrees with the previous paragraph because \( \lambda \) is hermitian and the coefficients of 1 cancel.

5.4 Comparison with previous results for spheres

There is one case in which our results for disks and previous results for spheres turn out to be identical. Namely, assume that \( \partial M \) has a connected component diffeomorphic to \( S^1 \times S^2 \) and let \( s \) and \( G \) be duals in it corresponding to \( S^1 \times p \) and \( q \times S^2 \) under that diffeomorphism. Given a neat disk \( U : \mathbb{D}^2 \hookrightarrow M \) with boundary \( s \) we can use it to ambiently surger this component to become a 3-sphere \( V : S^3 \hookrightarrow M \).

**Lemma 5.22.** For every \( 1 \neq g \in \pi_1 M \), we have \( \text{dax}(g \cdot V) = g + \overline{g} \in \mathbb{Z}[\pi_1 M \setminus 1] \). In particular, we get

\[ \mathbb{Z}[\pi]^\sigma/(1, \text{dax}(\pi_3 M)) = \mathbb{Z}[\pi]^\sigma/(1, g + \overline{g}, \text{dax}(\pi_3 M)) \simeq F_2 T M/\mu_3(\pi_3 M) \]

Recall from Section 5.1.2 that \( T_M := \{ g \in \pi_1 M \mid g^2 = 1 \neq g \} \) and \( \mu_3 \) is Wall’s (reduced) self-intersection invariant. So in this case the relative Dax and Freedman-Quinn invariants for neat disks take values in the same group, compare Definition 5.10.

**Proof.** Recall that we write \( \text{dax} \) for \( \text{dax}_u \) where \( u \) is an arc null homotopic into the boundary. The union of the ambient 2-handle \( vU \) and a collar in \( M \) of our boundary component containing \( s \) and \( G \) leads to a connected sum decomposition \( M \cong X \# (S^2 \times \mathbb{D}^2) \) along the separating sphere \( V \). Under this diffeomorphism \( G \) is taken to \( S^2 \times 0 \) and \( U \) to \( p \times \mathbb{D}^2 \). The inclusion \( I \subseteq M \) preserves Dax invariants and induces an isomorphism of fundamental groups. Therefore, the computation of \( \text{dax}(g \cdot V) \) follows from Lemma 4.10 applied to \( d = 4 \) and \( S' := V \subseteq \partial X \).

\[ \square \]
Proof of Proposition 1.4. We will show that all vertical maps are bijections and that the first square with Dax and FQ maps commutes, while the rest of the diagram commutes by construction. Observe that \( N \) is obtained from \( M := N \setminus \nu(G) \) by attaching a 2-handle with core \( m_G \) to \( s \) and a 4-handle. Since by assumption there exists a dual sphere \( S \) for \( G \), the circle \( s \) bounds the disk \( U := S \setminus m_G \) in \( M \), so it is null homotopic in \( M \). Thus, the inclusion \( i : M \hookrightarrow N \) induces a canonical isomorphism \( \pi = \pi_1 M = \pi_1 N \), so the rightmost vertical map is an isomorphism.

Furthermore, we have a short exact sequence \( 0 \to \pi_2 M \xrightarrow{i_*} \pi_2 N \xrightarrow{\lambda(G)} \mathbb{Z}[\pi] \to 0 \), and using the bijection \( \text{Map}_0[\mathbb{D}, M] \cong \pi_2 M, J \mapsto -U \cup J \), it follows that the second to right map is a bijection as well:

\[
\text{Map}_0[\mathbb{D}, M] \to \text{Map}_0^G[\mathbb{S}^2, N], \quad J \mapsto J \cup m_G = (-U \cup J) \# (U \cup m_G).
\]

The map \( q \) from Theorem 5.11, which takes the relative Dax invariant to the relative Freedman-Quinn invariant (for disks in \( M \)), now becomes an isomorphism thanks to Lemma 5.22:

\[
\text{Dax}(K_0, K_1) \in \mathbb{Z}[\pi]^\mu_f/(1, \text{dax}(\pi_3 M)) \cong \mathbb{F}_2 T_M / \mu_3(\pi_3 M) \cong \text{FQ}(K_0, K_1).
\]

The Freedman-Quinn invariant for disks \( K_i \) in \( M \) maps by construction under \( i \) to the Freedman-Quinn invariant \( \text{FQ}(F_0, F_1) \in \mathbb{F}_2 T_N / \mu_3(\pi_3 N) \) of spheres \( F_i = K_i \cup m_G \) in \( N \). Thus, once we show the equality \( i_* (\mu_3 (\pi_3 M)) = \mu_3 (\pi_3 N) \subseteq \mathbb{F}_2 T_N \), the leftmost vertical map will be an isomorphism and the first square will commute. This equality is similar to Lemma 5.2, except that \( \mu_3 \) factors through the (surjective) Hurewicz maps, so it suffices to show that the induced map \( i_* : H_3(M) \to H_3(N) \) is surjective. Indeed, the 2-handle is attached in a homotopically trivial way so does not change \( H_3(M) \) at all.

We are left with the most interesting vertical arrow, second from the left, \( \text{Emb}_0(\mathbb{D}, M) \to \text{Emb}^G(\mathbb{S}^2, N), K \mapsto K \cup m_G \). It is surjective because for any \( F \in \text{Emb}^G(\mathbb{S}^2, N) \) we can arrange that it is isotopic to \( m_G \) near \( G \). Namely, use local coordinates in which \( G \) and \( m_G \) are linear around \( F \cap G = m_G \cap G \), so the inverse function theorem applied to the restriction of \( F \) to an \( \mathbb{R}^2 \) neighborhood of the point \( F^{-1} (F \cap G) \), that is \( F_1 : \mathbb{R}^2 \to \nu(F \cap G) = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \), implies that \( F \) is locally a graph over \( m_G \) and hence can locally be isotoped to it; this isotopy can be extended to all of \( F \), keeping \( G \) fixed.

Finally, the injectivity of this map follows from the commutativity of the square with the Dax and FQ invariants, and the exactness of the top sequence: If \( K_0 \) and \( K_1 \) lead to isotopic spheres \( F_i \) the Dax maps are equal and hence the \( K_i \) are isotopic. \( \square \)

In this proof, we started with the 4-manifold \( N \) and removed \( \nu G \) to create \( M \) with a new boundary component diffeomorphic to \( S^1 \times S^2 \). Conversely, we may start with a 4-manifold \( M \) with such a boundary component (cf. Example 1.3) and add \( \mathbb{D}^2 \times S^2 \) to \( M \) along it. If there exists \( U \colon \mathbb{D}^2 \to M \) with boundary \( s = S^1 \times p \), then this larger 4-manifold \( N \) contains in its interior a framed sphere \( G = 0 \times S^2 \), dual to a sphere \( F_1 := U \cup_0 (\mathbb{D}^2 \times p) \). The normal Euler number of \( F_1 \) depends on the precise way we glue \( \mathbb{D}^2 \times S^2 \) to \( M \), our \( Z \) choices being parametrized by \( \pi_1 (SO_2) = \pi_1 (SO_3) \leq \text{Diff}(S^1 \times S^2) \), if we want our disks to match up along \( s \). Due to the factorization over \( \pi_1 (SO_3) = \mathbb{Z}/2 \), there are at most two diffeomorphism types of manifolds \( N \) that can arise, differing by a Gluck twist. Then Proposition 1.4 shows that the previous LBT for spheres in \( N \) implies our LBT for disks in \( M \).

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