OPTIMAL DESIGN OF WINDOW FUNCTIONS FOR FILTER WINDOW BANK

Xueling Zhou
Bingo Wing-Kuen Ling*, Hai Huyen Dam and Kok-Lay Teo
Faculty of Information Engineering
Guangdong University of Technology, Guangzhou 510006, China

Abstract. This paper considers the designs of the periodic window functions in the filter window banks. First, the filter window bank with the constant synthesis periodic window functions is considered. The total number of the nonzero coefficients in the impulse responses of the analysis periodic window functions is minimized subject to the near perfect reconstruction condition. This is an $L_0$ norm optimization problem. To find its solution, the $L_0$ norm optimization problem is approximated by the $L_1$ norm optimization problem. Then, the column of the constraint matrix corresponding to the element in the solution with the smallest magnitude is removed. Next, it is tested whether the feasible set corresponding to the new $L_0$ norm optimization problem is empty or not. By repeating the above procedures, a solution of the $L_0$ norm optimization problem is obtained. Second, the filter window bank with the time varying synthesis periodic window functions is considered. Likewise, the design of the periodic window functions in both the analysis periodic window functions and the synthesis periodic window functions is formulated as an $L_0$ optimization problem. However, this $L_0$ norm optimization problem is subject to a quadratic matrix inequality constraint. To find its solution, the set of the synthesis periodic window functions is initialized. Then, the set of the analysis periodic window functions is optimized based on the initialized set of the synthesis periodic window functions. Next, the set of the synthesis periodic window functions is optimized based on the found set of the analysis periodic window functions. Finally, these two procedures are iterated. It is shown that the iterative algorithm converges. A design example of a filter window bank with the constant synthesis periodic window functions and a design example of a filter window bank with the time varying synthesis periodic window functions are illustrated. It is shown that the near perfect reconstruction condition is satisfied, whereas this is not the cases for the nonuniform filter banks with the conventional samplers and the conventional block samplers.

1. Introduction. It is well known that uniform filter banks [22]-[7] are widely used in many applications [31]-[19]. However, a uniform filter bank only decomposes an input signal into the subband components in the uniform frequency grid. Since all the subband components are with the same bandwidth, this limitation reduces

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Filter bank design, window design, incompatible nonuniform filter bank, nonuniform block filter bank, $L_0$ norm optimization, $L_1$ norm optimization problem.
This paper was supported partly by the National Nature Science Foundation of China.
* Corresponding author: Bingo Wing-Kuen Ling.
the effectiveness for representing the input signal. In [18]-[2] a tree structure filter bank is employed. Here, an input signal is decomposed into the subband components in a specific nonuniform frequency grid. The most common tree structure filter bank is the wavelet based filter bank [20]-[23]. It is also widely used in many practical applications [4], [32]. However, even though the bandwidths of different subband components are different, the bandwidths of the subband components are constrained by the corresponding tree structure. Consequently, the subband components cannot be in an arbitrary nonuniform frequency grid [12], [13]. In [19], a nonuniform filter bank without the tree structure constraint imposed on the samplers is considered. Here, it decomposes an input signal into the subband components in an arbitrary nonuniform frequency grid. Since the subband components are in an arbitrary nonuniform frequency grid, it is also widely used in many practical applications [25]-[11]. However, in general a nonuniform filter bank does not achieve the exact perfect reconstruction condition [12]. This means, both the aliasing distortion and the transfer functional distortion could exist. As the aliasing distortion is caused by the loss of information, this linear time varying distortion is irreversible. Similarly, if there is a singularity in the transfer functional distortion, then the linear time invariant distortion is also irreversible.

To address this issue, a block nonuniform filter bank is proposed [24]. In particular, the conventional samplers in the conventional nonuniform filter bank are replaced by the corresponding block samplers in the block nonuniform filter bank. Since the sampling structure in the analysis block nonuniform filter bank is symmetric to that in the synthesis block nonuniform filter bank, the aliasing distortions can only be suppressed for some block nonuniform filter banks and the exact perfect reconstruction cannot be achieved. Recently, a filter window bank is proposed [14], [29]. It consists of a bank of analysis filters, a bank of analysis periodic windows, a bank of synthesis filters and a bank of synthesis periodic windows. When the synthesis periodic window functions are constant, the filter window bank becomes the generalizations of the conventional nonuniform filter bank and the block nonuniform filter bank. This is because the joint conventional downsampling functions and the conventional upsampling functions of the conventional nonuniform filter bank as well as the joint block downsampling functions and the block upsampling functions of the block nonuniform filter bank are the particular types of the analysis periodic window functions of the filter window bank. However, it is shown in [14], [29] that this filter window bank still does not achieve the exact perfect reconstruction and only the near perfect reconstruction can be achieved. As a result, several fundamental issues are required to be addressed. In particular, for the given sets of analysis filters and synthesis filters, what are the properties of the analysis periodic window functions of the filter window bank such that the near perfect reconstruction condition is satisfied? With such properties, how to formulate the design of the analysis periodic window functions as an optimization problem such that the performance of the filter window bank is maximized? Also, how to find a solution of the formulated optimization problem?

To achieve the exact perfect reconstruction, the block nonuniform filter bank with an asymmetric structure is proposed [10]. Nevertheless, the frequency selectivities of the filters could be very poor. Since the filter window bank with the time varying synthesis periodic window functions is the generalization of the block nonuniform filter bank with the asymmetric structure, the exact perfect reconstruction condition of the filter window bank with the time varying synthesis periodic window functions
can be achieved. However, as the frequency selectivities of the filters could be very poor, a near perfect reconstruction is preferred. Therefore, the similar fundamental issues are required to be addressed. That is, for the given sets of analysis filters and synthesis filters, what are the properties of both the analysis periodic window functions and the synthesis periodic window functions of the filter window bank such that the near perfect reconstruction condition is satisfied? With such properties, how to formulate the design of both the analysis periodic window functions and the synthesis periodic window functions as an optimization problem such that the performance of the filter window bank is maximized? Also, how to find a solution of the formulated optimization problem?

To answer the above fundamental questions, it is worth noting that the total number of the nonzero coefficients of the impulse responses of the analysis periodic window functions is related to the total number of the nonzero subband coefficients. For the filter window bank with the constant synthesis periodic window function, the total number of the subband coefficients is minimized. That is, the $L_0$ norm of the impulse responses of the analysis periodic window functions is minimized subject to the near perfect reconstruction condition. To find a solution of this $L_0$ norm optimization problem, an iterative approach is proposed. For the design of both the analysis periodic window functions and the synthesis periodic window functions, a similar $L_0$ norm optimization problem is formulated. In particular, the total number of nonzero coefficients of the impulse responses of both the analysis periodic window functions and the synthesis periodic window functions is minimized subject to the near perfect reconstruction condition. However, this $L_0$ norm optimization problem is subject to a quadratic matrix inequality constraint. To find a solution of this optimization problem, an iterated approach is proposed. It is shown that the iterated algorithm converges.

The remainder of this paper is as follows. In Section 2, the notations used throughout this paper are introduced. Also, the exact perfect reconstruction condition of the filter window bank is reviewed. In Section 3, the design of the analysis periodic window functions of the filter window bank with the constant synthesis periodic window functions is formulated as an $L_0$ norm optimization problem and an iterative approach is proposed to find its solution. In Section 4, similar works are presented for the designs of both the analysis periodic functions and the synthesis periodic window functions of the filter window bank. Similarly, the solution of this optimization problem is found by an iterated approach. In Section 5, the illustrative examples of the filter window banks with both the constant synthesis periodic functions and the time varying synthesis periodic window functions are presented. Finally, some concluding remarks are drawn in Section 6.

2. Notations and review on exact perfect reconstruction condition of filter window bank.

2.1. Notations. A block diagram of a filter window bank is shown in Figure 1. Let $M$ be the total number of channels in the filter window bank. For $i = 0, \cdots, M-1$, let $H_i(z)$, $\omega_i[n]$, $F_i(z)$ and $\nu_i[n]$ be, respectively, the transfer function of the $i^{th}$ single input single output analysis linear time invariant filter, the impulse response of the $i^{th}$ single input single output analysis linear time invariant filter, the impulse response of the $i^{th}$ analysis periodic window, the transfer function of the $i^{th}$ single input single output synthesis linear time invariant filter and the impulse response of the $i^{th}$ synthesis periodic window. Let the periods of $\omega_i[n]$ and $\nu_i[n]$ be, respectively, $\tilde{N}_i$ and $\hat{N}_i$ for $i = 0, \cdots, M-1$. Consider the case that the least common multiple
of $\tilde{N}_i$ is equal to that of $\hat{N}_i$ for $i = 0, \cdots, M - 1$. Let this least common multiple be $N$. For $i = 0, \cdots, M - 1$, let the discrete Fourier series of $\omega_i[n]$ and $\nu_i[n]$ be, respectively, $a_{i,p}$ for $p = 0, \cdots, \tilde{N}_i - 1$ and $b_{i,p}$ for $q = 0, \cdots, \hat{N}_i - 1$. Let

$$j = \sqrt{-1}$$

and $\delta[n]$ be the discrete time delta function. Let $x[n]$ and $\hat{x}[n]$ be the input and the output of the filter window bank, respectively.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{block_diagram.png}
\caption{Block diagram of a filter window bank.}
\end{figure}

2.2. Exact perfect reconstruction condition of a filter window bank. Let

$$W = e^{-\frac{2\pi j}{N}}$$

and $\text{mod}(a, b)$ be the remainder of $\frac{a}{b}$. For the filter window bank with the constant synthesis periodic window functions, we assume that

$$\nu_i[n] = 1$$

for $i = 0, \cdots, M - 1$.

Let $X(z)$ and $\hat{X}(z)$ be the $z$ transform $x[n]$ and $\hat{x}[n]$, respectively. Then, we have

$$\hat{X}(z) = \sum_{i=0}^{M-1} \sum_{p=0}^{\tilde{N}_i-1} a_{i,p} X(zW^{N_p \tilde{N}_i}) H_i(zW^{N_p \tilde{N}_i}) F_i(z).$$

Let

$$X_e(z) = \begin{bmatrix} X(z) & \cdots & X(zW^{N_i-1}) \end{bmatrix}^T,$$

$$H_{i,p}(z) = \begin{bmatrix} a_{i,p} H_i(zW^{N_p \tilde{N}_i}) & 0 & \cdots & 0 \end{bmatrix}^T$$

for $p = 0, \cdots, \tilde{N}_i - 1$ and for $i = 0, \cdots, M - 1,$

$$H_i(z) = \begin{bmatrix} H_{i,0}(z) & \cdots & H_{i,\tilde{N}_i-1}(z) \end{bmatrix}^T,$$

for $i = 0, \cdots, M - 1,$

$$H(z) = \begin{bmatrix} H_0(z) & \cdots & H_{M-1}(z) \end{bmatrix}$$
be the aliasing matrix and
\[ F(z) = \begin{bmatrix} F_0(z) & \cdots & F_{M-1}(z) \end{bmatrix}^T \] (9)
be the vector of the transfer functions of the synthesis filters. Then, we have
\[ \hat{X}(z) = X_e^T(z)H(z)F(z). \] (10)

Here, there are \( \frac{N}{N_c} \) zeros in \( H_{i,p}(z) \) for \( p = 0, \cdots, \tilde{N}_i-1 \) and for \( i = 0, \cdots, M-1 \). It is worth noting that \( H(z) \) is an \( N \times M \) matrix. Its first row is related to the transfer functional distortion and the rest rows are related to the aliasing distortion. For the compatibility of the filter window bank, it requires the existence of at least two nonzero elements in each row of \( H(z) \) in order to have a possibility to cancel the transfer functional distortion or the aliasing distortion. Therefore, we have the following results on the compatibility of the filter window bank [14], [29].

Define
\[ S = \{0, \cdots, M-1\}. \] (11)

For each \( c \in S \) and for each \( p_c \in \{0, \cdots, \tilde{N}_c-1\} \), if \( \exists d_i \in S \) for some \( i \) where
\[ d_i \neq c \] (12)
\[ \forall i \]
\[ d_i \neq d_j \] (13)
for \( i \neq j \), such that, \( \exists p_{d_i} \in \{0,1,\cdots,\tilde{N}_{d_i}-1\} \) for each \( i \) where
\[ \frac{N_{p_c}}{N_c} = \frac{N_{p_{d_i}}}{N_{d_i}} \in \{0,1,\cdots,N-1\} \] (14)
\[ \forall i \]
then the filter window bank is called compatible. Here, \( c \) and \( d_i \) are the elements in \( S \). They refer to the column indices of \( H(z) \). On the other hand, \( \frac{N_{p_c}}{N_c} \) and \( \frac{N_{p_{d_i}}}{N_{d_i}} \) refer to the row indices of the nonzero elements in the \((c+1)^{th}\) column and the \((d_i+1)^{th}\) column of \( H(z) \), respectively. Hence, the above statement can be understood as follows. For each nonzero element in each column of \( H(z) \), if there exists at least one nonzero element in the same row but at different column of \( H(z) \), then the filter window bank is compatible. On the other hand, if \( \exists c \in S \) and \( \exists p_c \in \{0,\cdots,\tilde{N}_c-1\} \) but \( \nexists d \in S \) where
\[ d \neq c \] (15)
and \( \nexists p_d \in \{0,\cdots,\tilde{N}_d-1\} \), such that
\[ \frac{N_{p_c}}{N_c} = \frac{N_{p_{d}}}{{N_d}} \in \{0,1,\cdots,N-1\}, \] (16)
then the filter window bank is called incompatible. In other words, if there exists a nonzero element in a column of \( H(z) \) but there does not exist another nonzero element in the same row but at different column of \( H(z) \), then the filter window bank is incompatible.

Let \( E_0(z) \) and \( E_n(z) \) for \( n = 1, \cdots, M-1 \) be, respectively, the transfer functional distortion and the aliasing distortion of the filter window bank. Let \( \bar{c} \) and \( \bar{K} \) be the gain and the delay of the filter window bank, respectively. Let
\[ E(z) = \begin{bmatrix} E_0(z) & \cdots & E_{N-1}(z) \end{bmatrix}^T. \] (17)
Then, we have
\[ E_0(z) = \sum_{i=0}^{M-1} a_{i,0} H_i(z) F_i(z) - \bar{c}z^{-\mathcal{K}} \]  
(18)

and
\[ E_n(z) = \sum_{i=0}^{M-1} a_{i,\text{floor}\left(\frac{qN_i}{N}\right)} \delta\left[ \text{mod} \left( n, \frac{N}{N_i} \right) \right] H_i(zW^{\frac{N_i}{N}} \text{floor}(\frac{qN_i}{N})) F_i(z) \]  
(19)

for \( n = 1, \ldots, N - 1 \). Here, \( \text{floor}(a) \) denotes the small integer less than or equal to \( a \). Then, we have the following results [14], [29].

For the compatible filter window bank, the exact perfect reconstruction condition becomes:

(i) \( \exists \bar{c} > 0 \) and \( \exists \bar{K} \in \mathbb{Z}^+ \) such that
\[ \sum_{i=0}^{M-1} a_{i,0} H_i(z) F_i(z) = \bar{c}z^{-\mathcal{K}}, \]  
(20)

and (ii)
\[ \sum_{i=0}^{M-1} a_{i,\text{floor}\left(\frac{qN_i}{N}\right)} \delta\left[ \text{mod} \left( n, \frac{N}{N_i} \right) \right] H_i(zW^{\frac{N_i}{N}} \text{floor}(\frac{qN_i}{N})) F_i(z) = 0 \]  
(21)

for \( n = 1, \ldots, N - 1 \). For the incompatible filter window bank, suppose that \( \exists c \in S \) and \( p_c \in \{0, \ldots, \tilde{N}_c - 1\} \) but \( \nexists d \in S \) where
\[ d \neq c \]  
(22)

and \( \nexists p_d \in \{0, 1, \ldots, \tilde{N}_c - 1\} \) such that
\[ \frac{Np_c}{N_c} = \frac{Np_d}{N_d} \in \{0, 1, \ldots, N - 1\}. \]  
(23)

In this case, the exact perfect condition becomes
\[ a_{c,p_c} = 0. \]  
(24)

Here, since the filter window bank is incompatible, there is only one nonzero element in at least one row of \( H(z) \). In order to achieve the exact perfect reconstruction, the corresponding Fourier coefficients in these rows have to be zero in order to kill to the corresponding aliasing components.

For the filter window bank with the time varying synthesis periodic window functions, we have
\[ \sum_{i=0}^{M-1} \sum_{q=0}^{\tilde{N}_i-1} \sum_{p=0}^{\tilde{N}_i-1} b_{i,q,p} X(zW^{\frac{N}{N_i}} W^{\frac{N}{N_i}}) H_i(zW^{\frac{N}{N_i}} W^{\frac{N}{N_i}}) F_i(zW^{\frac{N}{N_i}}) = \bar{c}z^{-\mathcal{K}}. \]  
(25)

Let
\[ \tilde{H}_{i,p+\text{floor}\left(\frac{qN}{N_i}, \frac{N}{N_i}\right)}(z) = \begin{bmatrix} 0 & \cdots & 0 & a_{i,p} H_i(zW^{\frac{qN}{N_i}} W^{\frac{qN}{N_i}}) & 0 & \cdots & 0 \end{bmatrix}^T \]  
(26)

for \( p = 0, \ldots, \tilde{N}_i - 1 \), for \( q = 0, \ldots, \tilde{N}_i - 1 \) and for \( i = 0, \ldots, M - 1 \). Here, there are \( \text{mod}(\frac{qN}{N_i}, \frac{N}{N_i}) \) zeros and \( \frac{N}{N_i} - 1 - \text{mod}(\frac{qN}{N_i}, \frac{N}{N_i}) \) zeros before and after the nonzero
elements in \( \tilde{H}_{i,p+\text{floor}(\frac{N}{N_c} \cdot \frac{N_i}{N_c})}^q(z) \) for \( p = 0, \cdots, \hat{N}_i - 1 \), for \( q = 0, \cdots, \hat{N}_i - 1 \) and for \( i = 0, \cdots, M - 1 \), respectively. Denote

\[
\tilde{H}_{i,q}(z) = \left[ \tilde{H}_{i,0,q}(z) \cdots \tilde{H}_{i,N_i-1,q}(z) \right]^T \tag{27}
\]

for \( i = 0, \cdots, M - 1 \) and for \( q = 0, \cdots, \hat{N}_i - 1 \),

\[
\tilde{H}_i(z) = \left[ \tilde{H}_0(z) \cdots \tilde{H}_{M-1}(z) \right] \tag{28}
\]

for \( i = 0, \cdots, M - 1 \),

\[
\hat{H}(z) = \left[ \hat{H}_0(z) \cdots \hat{H}_{M-1}(z) \right] \tag{29}
\]

be the aliasing matrix of the analysis filter bank,

\[
\hat{F}_i(z) = \left[ b_{i,0}F_i(z) \cdots b_{i,N_i-1}F_i(zW^{N(\hat{N}_i-1)}) \right]^T \tag{30}
\]

for \( i = 0, \cdots, M - 1 \), and

\[
\hat{F}(z) = \left[ \hat{F}_0(z) \cdots \hat{F}_{M-1}(z) \right]^T \tag{31}
\]

be the vector of the modulated transfer functions of the synthesis filters. Then, we have

\[
\hat{X}(z) = X^T(z)\hat{H}(z)\hat{F}(z). \tag{32}
\]

It is worth noting that \( \hat{H}(z) \) is an \( N \times \sum_{i=0}^{M-1} \hat{N}_i \) matrix. Therefore, we have the following results on the compatibility of the filter window bank \([14, 29]\):

Define

\[
\hat{S} = \{0, \cdots, \hat{N}_0 - 1, \hat{N}_0, \hat{N}_0 + 1, \cdots, \hat{N}_0 + \hat{N}_1 - 1, \cdots, \sum_{i=0}^{M-2} \hat{N}_i, \sum_{i=0}^{M-2} \hat{N}_i + 1, \cdots, \sum_{i=0}^{M-1} \hat{N}_i - 1 \} \tag{33}
\]

\( \forall u \in \{0, 1, \cdots, \hat{N}_0 - 1 \} \), it is obvious to see that \( \exists q_0 \in \{0, \cdots, \hat{N}_0 - 1 \} \) such that the \((q_0 + 1)^{th}\) element in \( \hat{S} \) is \( u \). In this case, we define

\[
c = 0. \tag{34}
\]

Also,

\( \forall u \in \{ \hat{N}_0, \hat{N}_0 + 1, \cdots, \hat{N}_0 + \hat{N}_1 - 1, \cdots, \sum_{i=0}^{M-2} \hat{N}_i, \sum_{i=0}^{M-2} \hat{N}_i + 1, \cdots, \sum_{i=0}^{M-1} \hat{N}_i - 1 \} \), it is obvious to see that \( \exists c \in \{1, \cdots, M - 1 \} \) and \( \exists q_c \in \{0, \cdots, \hat{N}_c - 1 \} \) such that the \((\sum_{z=1}^{c} \hat{N}_z - 1 + q_c + 1)^{th}\) element in \( \hat{S} \) is \( u \). Hence, \( \forall u \in \hat{S} \), there exists a unique ordered pair \((c, q_c)\) where \( c \in \{0, 1, \cdots, M - 1 \} \) and \( q_c \in \{0, \cdots, \hat{N}_c - 1 \} \) such that the index of \( u \) in \( \hat{S} \) can be expressed in terms of \((c, q_c)\). For each \( u \in \hat{S} \) and for each \( p_c \in \{0, \cdots, \hat{N}_c - 1 \} \), if \( \exists (d_i, q_{d_i}) \) for some \( i \) where \( d_i \in \{0, 1, \cdots, M - 1 \} \), \( q_{d_i} \in \{0, 1, \cdots, \frac{N}{N_{d_i}} - 1 \} \),

\[
(d_i, q_{d_i}) \neq (c, q_c) \tag{35}
\]

\( \forall i \) and

\[
(d_i, q_{d_i}) \neq (d_j, q_{d_j}) \tag{36}
\]

for \( i \neq j \), such that, \( \exists q_{d_i} \in \{0, 1, \cdots, \hat{N}_{d_i} - 1 \} \) for each \((d_i, q_{d_i})\) where

\[
\text{mod}(\frac{Np_c}{N_c} + \frac{Nq_c}{N_c}, N) = \text{mod}(\frac{Np_{d_i}}{N_{d_i}} + \frac{Nq_{d_i}}{N_{d_i}}, N) \in \{0, 1, \cdots, N - 1 \} \tag{37}
\]
∀i, then the filter window bank is called compatible. On the other hand, ∃u ∈ S and ∃v ∈ {0, ..., Nc - 1} but ̃d ∈ {0, 1, ..., M - 1} and ̃pd ∈ {0, 1, ..., Nc - 1} where

\[(d, q_d) \neq (c, q_c)\]  

(38)

and ̃pd ∈ {0, 1, ..., Nc - 1} such that

\[\text{mod} \left( \frac{Np_c + Nq_c}{Nc}, N \right) = \text{mod} \left( \frac{Npd + Nqd}{Nd}, N \right) \in \{0, 1, \cdots, N - 1\}\]  

(39)

then the filter window bank is called incompatible.

Let ̃E_0(z) and ̃E_n(z) for n = 1, 1, ..., N - 1 be, respectively, the transfer functional distortion and the aliasing distortion of the filter window bank, respectively. Let

\[\tilde{E}(z) = \left[ \tilde{E}_0(z) \cdots \tilde{E}_{N-1}(z) \right]^T.\]  

(40)

Then, it can be seen that

\[\tilde{E}_0(z) = \sum_{i=0}^{M-1} \left( a_{i,0}b_{i,0}H_i(z)F_i(z) + \sum_{q \in \{0, \cdots, N_i-1\}} \frac{a_i}{\tilde{N}_i - q/N_i} b_{i,q} H_i(zW^{N-iN_q/N_i}W^{N_q/N_i})F_i(zW^{N_q/N_i}) \right) \]  

(41)

and

\[\tilde{E}_n(z) = \sum_{i=0}^{M-1} \sum_{\tilde{N}_i + q/N_i \in \{0, \cdots, \tilde{N}_i-1\}} a_i \frac{\tilde{N}_i + q/N_i - q/N_q}{\tilde{N}_i} b_{i,q} H_i(zW^{N+iN_q/N_i+n-N_q/N_q}W^{N_q/N_q})F_i(zW^{N_q/N_q}) + \sum_{i=0}^{M-1} \sum_{q \in \{0, \cdots, \tilde{N}_i-1\}} \frac{a_i}{\tilde{N}_i} b_{i,q} H_i(zW^{N+iN_q/N_q})F_i(zW^{N_q/N_q}) \]  

(42)

for n = 0, 1, ..., N - 1. Then, we have the following results [14], [29]:

For the compatible filter window bank, the exact perfect reconstruction condition becomes:

(i) ∃c > 0 and ∃K ∈ Z^+ such that

\[\sum_{i=0}^{M-1} \left( a_{i,0}b_{i,0}H_i(z)F_i(z) + \sum_{\tilde{N}_i - q/N_i \in \{0, \cdots, \tilde{N}_i-1\}} \frac{a_i}{\tilde{N}_i - q/N_i} b_{i,q} H_i(zW^{N+iN_q/N_i-n-N_q/N_q}W^{N_q/N_q})F_i(zW^{N_q/N_q}) \right) \]  

\[= \tilde{c}z^{-K}\]  

(43)
Define the discrete Fourier series of the analysis periodic window functions as follows. Distortion can be expressed as a multiplication of a matrix and a vector containing periodic window functions, both the transfer functional distortion and the aliasing window bank are linear with respect to the discrete Fourier series of the analysis bank with constant synthesis periodic window functions.

In order to design and (\(n = 0, \ldots, N - 1\)). For the incompatible filter window bank, suppose that \(u \in \tilde{S}\) and \(p_c \in \{0, \ldots, \tilde{N}_c - 1\}\) but \(\tilde{d}d \in \{0, 1, \ldots, M - 1\}\) and \(\tilde{d}q_d \in \{0, 1, \ldots, \frac{N}{N_{d_i}} - 1\}\) where

\[
(d, q_d) \neq (c, q_c)
\]

(45)

and \(\tilde{d}p_d \in \{0, 1, \ldots, \tilde{N}_d - 1\}\) such that

\[
\mod \left(\frac{N_{p_c}}{N_c} + \frac{N_{q_c}}{N_c}, N\right) = \mod \left(\frac{N_{p_d}}{N_d} + \frac{N_{q_d}}{N_d}, N\right) \in \{0, 1, \ldots, N - 1\}.
\]

(46)

In this case, the exact perfect condition becomes either

\[
a_{c, p_c} = 0
\]

(47)

or

\[
b_{c, p_c} = 0
\]

(48)

for \(\mod \left(\frac{N_{p_c}}{N_c} + \frac{N_{q_c}}{N_c}, N\right) = n\).

3. Optimal design of analysis periodic window functions of filter window bank with constant synthesis periodic window functions. In order to design the analysis periodic window functions, first both the transfer functional distortion and the aliasing distortion of the filter window bank are expressed in terms of the Fourier coefficients of the analysis periodic window functions as discussed in Section 2.2. Then, the design problem is formulated as an optimization problem based on the obtained expressions.

As both the transfer functional distortion and the aliasing distortion of the filter window bank are linear with respect to the discrete Fourier series of the analysis periodic window functions, both the transfer functional distortion and the aliasing distortion can be expressed as a multiplication of a matrix and a vector containing the discrete Fourier series of the analysis periodic window functions as follows. Define

\[
a_i = \begin{bmatrix} a_{i,0} & \cdots & a_{i,\tilde{N}_i-1} \end{bmatrix}^T
\]

(49)

for \(i = 0, \ldots, M - 1\),

\[
a = \begin{bmatrix} a_0^T & \cdots & a_{M-1}^T \end{bmatrix}^T,
\]

(50)

\[
\Psi_{i,n}(z) = \begin{bmatrix} 0 & \cdots & 0 & \delta \left[ \mod \left(n, \frac{N}{N_i} \right) \right] H_{i}(zW_{\frac{N}{N_i}}^n) F_i(z) & 0 & \cdots & 0 \end{bmatrix}^T
\]

(51)

for \(n = 0, \ldots, N - 1\) and \(i = 0, \ldots, M - 1\),

\[
\Psi_n(z) = \begin{bmatrix} \Psi_{0,n}(z) & \cdots & \Psi_{M-1,n}(z) \end{bmatrix}^T
\]

(52)
for \( n = 0, \ldots, N - 1 \), and

\[
\Psi(z) = \begin{bmatrix} \Psi_0(z) & \cdots & \Psi_{N-1}(z) \end{bmatrix}^T. \tag{53}
\]

Here, there are \( \lfloor \frac{n}{N_i} \rfloor \) zeros and \( \tilde{N}_i - 1 - \lfloor \frac{n}{N_i} \rfloor \) zeros before and after

\[
\delta \left[ \text{mod} \left( n, \frac{N}{N_i} \right) \right] H_i(zW \frac{N}{N_i} \text{floor}(\frac{N}{N_i})) F_i(z) \text{ in } \Psi_{i,n}(z) \text{ for } n = 1, \ldots, N - 1 \text{ and for } i = 0, \ldots, M - 1, \text{ respectively. Then, we have}
\]

\[
E(z) = \Psi(z)a - \begin{bmatrix} \bar{c}z^{-\mathbf{K}} & 0 & \cdots & 0 \end{bmatrix}^T. \tag{54}
\]

In other words, we have

\[
E_0(z) = \Psi_0^T(z)a - \bar{c}z^{-\mathbf{K}} \tag{55}
\]

and

\[
E_n(z) = \Psi_n^T(z)a \tag{56}
\]

for \( n = 1, \ldots, N - 1 \). Define

\[
w_i = \begin{bmatrix} w_i[0] & \cdots & w_i[N_i - 1] \end{bmatrix}^T \tag{57}
\]

for \( i = 1, \ldots, M - 1 \) and

\[
w_i = \begin{bmatrix} w_0^T & \cdots & w_{M-1}^T \end{bmatrix}^T. \tag{58}
\]

Define

\[
U_i = \begin{bmatrix}
e^{\frac{2\pi i 0}{N_i}} & \cdots & e^{\frac{2\pi i (N_i - 1)}{N_i}} \\
e^{\frac{2\pi i 1}{N_i}} & \cdots & e^{\frac{2\pi i (N_i - 1)}{N_i}} \\
\vdots & \ddots & \vdots \\
e^{\frac{2\pi i (\tilde{N}_i - 1)}{N_i}} & \cdots & e^{\frac{2\pi i (\tilde{N}_i - 1)}{N_i}}
\end{bmatrix} \tag{59}
\]

for \( i = 1, \ldots, M - 1 \) and

\[
U = \text{diag}(U_0, \cdots, U_{M-1}). \tag{60}
\]

Here, \( \text{diag}(Z_0, \cdots, Z_{M-1}) \) denotes the diagonal matrix with the diagonal submatrices being \( Z_0 \cdots Z_{M-1} \). Then, we have

\[
w = Ua. \tag{61}
\]

Since the total number of the nonzero coefficients of the impulse responses of the analysis periodic window functions is related to the total number of the nonzero subband coefficients, the total number of the nonzero coefficients of the impulse responses of the analysis periodic window functions should be minimized. To achieve this objective, the \( L_0 \) norm of the analysis periodic window functions is minimized subject to the near perfect reconstruction condition. That is: Problem \( (P_c) \)

\[
\begin{align*}
\min_w & \quad \|w\|_0, \tag{62a} \\
\text{subject to} & \quad |E_n(w)| \leq \varepsilon, \quad \text{for } n = 0, \ldots, N - 1 \text{ and } \exists w \in [-\pi, \pi]. \tag{62b}
\end{align*}
\]

Here, \( \|w\|_0 \) denotes the \( L_0 \) norm of \( w \) and \( \varepsilon \) denotes the specification on the acceptable modulus bound for both the aliasing distortion and the transfer functional.
distortion. It can be seen easily that Problem \((P_c)\) is equivalent to the following optimization problem:

\[
\begin{align*}
\min_w & \quad \|w\|_0, \quad \text{(63a)} \\
\text{subject to} & \quad |\Psi_0^T (w) U^{-1} w - \bar{c} e^{-jwR}| \leq \varepsilon \quad \exists w \in [-\pi, \pi], \quad \text{(63b)} \\
& \quad |\Psi_n^T (w) U^{-1} w| \leq \varepsilon \quad \text{for } n = 0, \cdots, N - 1 \quad \text{and} \quad \exists w \in [-\pi, \pi]. \quad \text{(63c)}
\end{align*}
\]

Since both the aliasing distortion and the transfer functional distortion are complex valued functions, the constraints of this optimization problem are actually the quadratic matrix inequality constraints. In general, it is difficult to guarantee that these quadratic matrix inequality constraints are satisfied. To address this issue, first denote a complex number as \(z\) as well as its real part and imaginary part as \(z_r\) and \(z_i\), respectively. That is,

\[
z = z_r + jz_i. \quad \text{(64)}
\]

If

\[
|z_r| \leq \frac{\varepsilon}{\sqrt{2}} \quad \text{(65)}
\]

and

\[
|z_i| \leq \frac{\varepsilon}{\sqrt{2}}, \quad \text{(66)}
\]

then we have

\[
|z|^2 = |z_r|^2 + |z_i|^2 \leq \varepsilon^2 \quad \text{(67)}
\]

or

\[
|z| \leq \varepsilon. \quad \text{(68)}
\]

Hence, the optimization problem is approximated by the following optimization problem:

\[
\begin{align*}
\min_w & \quad \|w\|_0, \quad \text{(69a)} \\
\text{subject to} & \quad |\text{real}(\Psi_0^T (w) U^{-1} w - \bar{c} e^{-jwR})| \leq \frac{\varepsilon}{\sqrt{2}} \quad \exists w \in [-\pi, \pi], \quad \text{(69b)} \\
& \quad |\text{imag}(\Psi_0^T (w) U^{-1} w - \bar{c} e^{-jwR})| \leq \frac{\varepsilon}{\sqrt{2}} \quad \exists w \in [-\pi, \pi] \quad \text{(69c)} \\
& \quad |\text{real}(\Psi_n^T (w) U^{-1} w)| \leq \frac{\varepsilon}{\sqrt{2}} \quad \text{for } n = 0, \cdots, N - 1 \quad \text{and} \quad \exists w \in [-\pi, \pi], \quad \text{(69d)} \\
& \quad |\text{imag}(\Psi_n^T (w) U^{-1} w)| \leq \frac{\varepsilon}{\sqrt{2}} \quad \text{for } n = 0, \cdots, N - 1 \quad \text{and} \quad \exists w \in [-\pi, \pi]. \quad \text{(69e)}
\end{align*}
\]

Here, \(\text{real}(z)\) and \(\text{imag}(z)\) denote the real part and the imaginary part of \(z\), respectively. Note that the analysis periodic window functions are real valued,

\[
\text{real}(Z_1 Z_2) = \text{real}(Z_1)\text{real}(Z_2) - \text{imag}(Z_1)\text{imag}(Z_2) \quad \text{(70)}
\]

and

\[
\text{imag}(Z_1 Z_2) = \text{imag}(Z_1)\text{real}(Z_2) + \text{real}(Z_1)\text{imag}(Z_2). \quad \text{(71)}
\]
Therefore, Problem \((P_c)\) can be approximated by the following optimization problem:

\[
\begin{align*}
\min_w & \quad \|w\|_0, \\
\text{subject to} & \quad |\text{real}(\Psi^T_0(w)U^{-1})w - \text{real}(\bar{c}e^{-jw\pi})| \leq \frac{\varepsilon}{\sqrt{2}} \quad \forall w \in [-\pi, \pi], \\
& \quad |\text{imag}(\Psi^T_0(w)U^{-1})w - \text{imag}(\bar{c}e^{-jw\pi})| \leq \frac{\varepsilon}{\sqrt{2}} \quad \forall w \in [-\pi, \pi], \\
& \quad |\text{real}(\Psi^T_n(w)U^{-1})w| \leq \frac{\varepsilon}{\sqrt{2}} \quad \text{for} \; n = 0, \ldots, N-1 \quad \text{and} \quad \forall w \in [-\pi, \pi], \\
& \quad |\text{imag}(\Psi^T_n(w)U^{-1})w| \leq \frac{\varepsilon}{\sqrt{2}} \quad \text{for} \; n = 0, \ldots, N-1 \quad \text{and} \quad \forall w \in [-\pi, \pi].
\end{align*}
\]  

Since the above optimization problem is a functional inequality constrained optimization problem, it consists of an infinite number of constraints. In general, it is difficult to guarantee that those infinite numbers of constraints are satisfied. To address this difficulty, \(w\) is sampled in \([-\pi, \pi]\). In this case, there are only a finite number of constraints. Denote \(T\) and \(\tau\) as the corresponding matrix and the corresponding vector in the linear matrix inequality constraint, respectively. Then, the above optimization problem can be approximated by the following problem:

\[
\begin{align*}
\min_w & \quad \|w\|_0, \\
\text{subject to} & \quad Tw \leq \tau.
\end{align*}
\]  

It is worth noting that an orthogonal matching pursuit algorithm is the most common method for finding the solutions of the \(L_0\) norm optimization problems in the conventional compressive sensing applications [9], [28]. However, this \(L_0\) norm optimization problem is different from the \(L_0\) norm optimization problems in the conventional compressive sensing applications [17], [30]. For those \(L_0\) norm optimization problems in the conventional compressive sensing applications, the total number of the columns of the constraint matrix is more than that of its rows. However, it is not the case for this \(L_0\) norm optimization problem. Hence, it is not guaranteed that there is a solution for this \(L_0\) norm optimization problem. To address this problem, a simple optimization problem with the same feasible set is formulated. Then, we test whether the feasible set of the formulated optimization problem is empty or not. Since the analytical solutions exist for the quadratic programming problems while they do not exist for the linear programming problems, the required computational powers for finding the solutions of the quadratic programming problems are lower than that for finding the solutions of the \(L_1\) norm optimization problems. Therefore, the existence of the solution of the \(L_0\) norm optimization problem is tested via the following quadratic programming problem:

\[
\begin{align*}
\min_w & \quad \|Tw - \tau\|^2, \\
\text{subject to} & \quad Tw \leq \tau.
\end{align*}
\]  

However, the globally optimal solution of the \(L_0\) norm optimization problem may not be uniquely defined. In order to obtain a near globally optimal solution, the \(L_0\) norm optimization problem is approximated by a convex optimization problem. Since the \(L_1\) norm optimization problem is the nearest convex approximation of the \(L_0\) norm optimization problem, the following \(L_1\) norm optimization problem is evaluated:
Then, the magnitudes of the elements in the solution of the $L_1$ norm optimization problem are sorted in the ascending order. The column in $\mathbf{T}$ corresponding to the element in the solution with the smallest magnitude is removed. Denote $\tilde{\mathbf{T}}$ as the matrix with the removed column. Here, a new set of the constraints are obtained. It corresponds to the new $L_0$ norm optimization problem. Therefore, it is required to test whether the feasible set of the new $L_0$ norm optimization problem is empty or not. Similarly, the following quadratic programming problem is formulated and the existence of the solution of the new $L_0$ norm optimization problem is tested:

$$\min_{\tilde{\mathbf{w}}} \parallel \tilde{\mathbf{T}}\tilde{\mathbf{w}} - \tau \parallel^2,$$

subject to $\tilde{\mathbf{T}}\tilde{\mathbf{w}} \leq \tau$. (76a) (76b)

If the solution of this new quadratic programming problem exists, then the total number of the nonzero elements in $\mathbf{w}$ satisfying $\mathbf{T}\mathbf{w} \leq \tau$ can be reduced. In other words, $\|\mathbf{w}\|_0$ is reduced. Hence, we can repeat the above procedures for finding the solution of the corresponding $L_0$ norm optimization problem. On the other hand, if the feasible set of the new $L_0$ norm optimization problem is empty, then we cannot reduce the total number of the nonzero elements in $\mathbf{w}$. In this case, $\tilde{\mathbf{w}}$ is taken as the approximate solution of the original $L_0$ norm optimization problem.

4. Optimal design of both analysis periodic window functions and synthesis periodic window functions of filter window bank. Similar to the design method discussed in Section 3, both the transfer functional distortion and the aliasing distortion of the filter window bank with the time varying synthesis periodic window functions are expressed in terms of the Fourier coefficients of both the analysis periodic window functions and the synthesis periodic window functions. Then, the design of both the analysis periodic window functions and the synthesis periodic window functions is formulated as an $L_0$ norm optimization problem based on the obtained expressions. Let

$$\kappa_{0,i}(z) = \begin{bmatrix} \mathbf{H}_i(z) \mathbf{F}_i(z) & \mathbf{0}_{1 \times (\tilde{N}_i-1)} \\ \mathbf{0}_{(N_i-1) \times 1} & \mathbf{0}_{(\tilde{N}_i-1) \times (\tilde{N}_i-1)} \end{bmatrix}$$

for $i = 0, \ldots, M - 1$ and

$$\kappa_{1,i,q}(z) = \begin{bmatrix} \mathbf{0}_{(N_i-1) \times (\tilde{N}_i)} \\ \tilde{N}_i - q\frac{\tilde{N}_i}{N_i} \notin \{0, \ldots, \tilde{N}_i - 1\} \\ \mathbf{0}_{q \times (N_i - q\frac{\tilde{N}_i}{N_i})} \\ \mathbf{0}_{q \times 1} & \mathbf{0}_{q \times (\tilde{N}_i - q\frac{\tilde{N}_i}{N_i})} \\ \mathbf{0}_{1 \times (\tilde{N}_i - q\frac{\tilde{N}_i}{N_i})} & \kappa_{1,i,q}(z) \\ \mathbf{0}_{(N_i-1) \times (\tilde{N}_i - q\frac{\tilde{N}_i}{N_i})} & \mathbf{0}_{(N_i-1) \times (\tilde{N}_i - q\frac{\tilde{N}_i}{N_i})} \end{bmatrix}.$$
for $i = 0, \cdots, M - 1$ and for $q = 0, \cdots, \tilde{N}_i - 1$. Furthermore, let

$$
\kappa_{n,0,i,q}(z) = \begin{cases}
\begin{bmatrix}
0_{(\tilde{N}_i) \times (\tilde{N}_i)} & 0 \\
0_{\tilde{N}_i} & 1 \\
\tilde{N}_i & 0
\end{bmatrix}, & \tilde{N}_i \notin \{0, \cdots, \tilde{N}_i - 1\} \\
\begin{bmatrix}
0_{q \times (\tilde{N}_n - \tilde{N}_i)} & 0_{q \times 1} & 0_{q \times (\tilde{N}_i + \tilde{N}_q - 1)} \\
0_{\tilde{N}_i} & 1 \\
\tilde{N}_i & 0
\end{bmatrix}, & \tilde{N}_i \notin \{0, \cdots, \tilde{N}_i - 1\}
\end{cases}
$$

and

$$
\kappa_{n,1,i,q}(z) = \begin{cases}
\begin{bmatrix}
0_{(\tilde{N}_i) \times (\tilde{N}_i)} & 0 \\
0_{\tilde{N}_i} & 1 \\
\tilde{N}_i & 0
\end{bmatrix}, & \tilde{N}_i \notin \{0, \cdots, \tilde{N}_i - 1\} \\
\begin{bmatrix}
0_{q \times (\tilde{N}_n - \tilde{N}_i)} & 0_{q \times 1} & 0_{q \times (\tilde{N}_i + \tilde{N}_q - 1)} \\
0_{\tilde{N}_i} & 1 \\
\tilde{N}_i & 0
\end{bmatrix}, & \tilde{N}_i \notin \{0, \cdots, \tilde{N}_i - 1\}
\end{cases}
$$

for $i = 0, \cdots, M - 1$ and for $q = 0, \cdots, \tilde{N}_i - 1$ and for $n = 1, \cdots, N - 1$. Define

$$
\kappa_0(z) = \text{diag}\left(\kappa_{0,0}(z), \cdots, \kappa_{0,M-1}(z)\right),
$$

$$
\kappa_1(z) = \text{diag}\left(\sum_{q=0}^{\tilde{N}_0-1} \kappa_{1,0,q}(z), \cdots, \sum_{q=0}^{\tilde{N}_{M-1}-1} \kappa_{1,M-1,q}(z)\right),
$$

$$
\bar{\kappa}_{n,0}(z) = \text{diag}\left(\sum_{q=0}^{\tilde{N}_0-1} \bar{\kappa}_{n,0,q}(z), \cdots, \sum_{q=0}^{\tilde{N}_{M-1}-1} \bar{\kappa}_{n,0,M-1,q}(z)\right),
$$

and

$$
\bar{\kappa}_{n,1}(z) = \text{diag}\left(\sum_{q=0}^{\tilde{N}_0-1} \bar{\kappa}_{n,1,q}(z), \cdots, \sum_{q=0}^{\tilde{N}_{M-1}-1} \bar{\kappa}_{n,1,M-1,q}(z)\right),
$$

for $n = 1, \cdots, N - 1$. Let

$$
b_i = [b_{i,0} \cdots b_{i,\tilde{N}_i-1}]^T
$$

for $i = 1, \cdots, M - 1$. Let

$$
b = [b_0^T \cdots b_{M-1}^T]^T.
$$

Then, we have

$$
\bar{E}_0(z) = b^T(\kappa_0(z) + \kappa_1(z))a - \bar{c}z^{-\bar{\kappa}}
$$

and

$$
\bar{E}_n(z) = b^T(\bar{\kappa}_{n,0}(z) + \bar{\kappa}_{n,1}(z))a
$$

for $n = 1, \cdots, N - 1$. Denote

$$
v_i = [v_{i0} \cdots v_{i[\tilde{N}_i-1]}]^T
$$
for \( i = 1, \ldots, M - 1 \). Let

\[
\mathbf{v} = \begin{bmatrix} \mathbf{v}_0^T & \cdots & \mathbf{v}_{M-1}^T \end{bmatrix}^T,
\]

(91)

\[
\tilde{\mathbf{U}}_i = \begin{bmatrix}
\begin{array}{ccc}
e^{j2\pi \times 0} & \cdots & \ne^{j2\pi \times (S_i-1)} \\
\vdots & \ddots & \vdots \\
e^{j2\pi \times (S_i-1)\times 0} & \cdots & \ne^{j2\pi \times (S_i-1)\times (S_i-1)}
\end{array}
\end{bmatrix}
\]

(92)

for \( i = 1, \ldots, M - 1 \), and

\[
\hat{\mathbf{U}} = \text{diag}(\hat{\mathbf{U}}_0, \ldots, \hat{\mathbf{U}}_{M-1}).
\]

(93)

Then, it follows that

\[
\mathbf{v} = \hat{\mathbf{U}} \mathbf{b}.
\]

(94)

In this paper, the design of both the analysis periodic window functions and the synthesis periodic window functions is formulated as the following \( L_0 \) norm optimization problem:

Problem (\( \tilde{P}_d \))

\[
\min_{(\mathbf{w}, \mathbf{v})} \|\mathbf{w}\|_0 + \|\mathbf{v}\|_0,
\]

subject to \(|\tilde{E}_q(w)| \leq \varepsilon, \) for \( q = 0, \ldots, N - 1 \) and \( \exists w \in [-\pi, \pi] \). (95b)

For real valued analysis periodic window functions, Problem (\( \tilde{P}_d \)) can be approximated by the following optimization problem:

\[
\min_{(\mathbf{w}, \mathbf{v})} \|\mathbf{w}\|_0 + \|\mathbf{v}\|_0
\]

subject to \(|\text{real}(\mathbf{v}^T \tilde{\mathbf{U}}^{-T}(\kappa_0(w) + \kappa_1(w))\mathbf{U}^{-1}\mathbf{w}) - \text{real}(\bar{c}e^{-jw\pi})| \leq \frac{\varepsilon}{\sqrt{2}} \), \( \exists w \in [-\pi, \pi] \), (96b)

\[\exists w \in [-\pi, \pi],\]

(96c)

\[\exists w \in [-\pi, \pi],\]

(96d)

\[\exists w \in [-\pi, \pi],\]

(96e)

\[\exists w \in [-\pi, \pi],\]

(96f)

\[\exists w \in [-\pi, \pi],\]

(96g)

\[\exists w \in [-\pi, \pi],\]

(96h)

\[\exists w \in [-\pi, \pi],\]

(96i)

However, the approximate Problem (\( \tilde{P}_d \)) still consists of four quadratic matrix inequality constraints. The conventional orthogonal matching pursuit method [13], [25] cannot be directly applied for finding the solution of this approximate problem. To address this difficulty, the following iterative algorithm is proposed.

**Algorithm 1**

**Step1:** Initialize \( \mathbf{w}_0^* \) and \( \mathbf{v}_0^* \). Denote an iteration index as \( k = 0 \). Let

\[
\gamma > 0
\]

be an acceptable value for terminating the algorithm. Define

\[
\Gamma_0 = \|\text{real}(\mathbf{w}_0^*)\|_1 + \|\text{real}(\mathbf{v}_0^*)\|_1.
\]

(97)
**Step 2:** Find an optimal solution of the following conventional $L_0$ norm optimization problem:

Problem $(\hat{P}_{v,k})$

$$\begin{align*}
\min_w & \quad \|w\|_0 + \|v^*_k\|_0, \\
\text{subject to} & \quad |\text{real}(v_k^T \hat{U}^{-T}(\kappa_0(w) + \kappa_1(w))U^{-1})w - \text{real}(\bar{e}e^{-jwK})| \leq \frac{\varepsilon}{2} \\
& \quad \exists w \in [-\pi, \pi], \\
& \quad |\text{imag}(v_k^T \hat{U}^{-T}(\kappa_0(w) + \kappa_1(w))U^{-1})w - \text{real}(\bar{e}e^{-jwK})| \leq \frac{\varepsilon}{2}, \\
& \quad \text{for } n = 0, \ldots, N-1, \text{ and } \exists w \in [-\pi, \pi], \\
& \quad |\text{imag}(v_k^T \hat{U}^{-T}(\kappa_0(w) + \kappa_1(w))U^{-1})w| \leq \frac{\varepsilon}{2}, \\
& \quad \text{for } n = 0, \ldots, N-1, \text{ and } \exists w \in [-\pi, \pi],
\end{align*}$$

Denote the obtained solution as $w^*_k$.

**Step 3:** Find an optimal solution of the following conventional $L_0$ norm optimization problem:

Problem $(\hat{P}_{v,k})$

$$\begin{align*}
\min_v & \quad \|w^*_k\|_0 + \|v\|_0, \\
\text{subject to} & \quad |v^T \text{real}(\hat{U}^{-T}(\kappa_0(w) + \kappa_1(w))U^{-1}w^*_k) - \text{real}(\bar{e}e^{-jwK})| \leq \frac{\varepsilon}{2} \\
& \quad \exists w \in [-\pi, \pi], \\
& \quad |v^T \text{imag}(\hat{U}^{-T}(\kappa_0(w) + \kappa_1(w))U^{-1}w^*_k) - \text{real}(\bar{e}e^{-jwK})| \leq \frac{\varepsilon}{2}, \\
& \quad \text{for } n = 0, \ldots, N-1, \text{ and } \exists w \in [-\pi, \pi], \\
& \quad |v^T \text{imag}(\hat{U}^{-T}(\kappa_0(w) + \kappa_1(w))U^{-1})w^*_k| \leq \frac{\varepsilon}{2}, \\
& \quad \text{for } n = 0, \ldots, N-1, \text{ and } \exists w \in [-\pi, \pi],
\end{align*}$$

Denote the obtained solution as $v^*_k$. Compute

$$\Gamma_{k+1} = \|\text{real}(w^*_k)\|_1 + \|\text{real}(v^*_k)\|_1.$$  

**Step 4:** If

$$\Gamma_k - \Gamma_{k+1} \leq \gamma,$$
then the algorithm terminates and \((\mathbf{w}_{k+1}^*, \mathbf{v}_{k+1}^*)\) is taken as an approximate solution of the approximated Problem \((\tilde{P}_d)\). Otherwise, increment the value of \(k\) and go back to Step 2.

Since **Algorithm1** is an iterative algorithm, the stability of the proposed algorithm is required to be analysed. In order to study whether the solutions of both Problem \((\tilde{P}_{w,k})\) and Problem \((\tilde{P}_{v,k})\) will converge to the solution of Problem \((\tilde{P}_d)\) or not, it is required to study whether both the functional values and the feasible sets of both Problem \((\tilde{P}_{w,k})\) and Problem \((\tilde{P}_{v,k})\) would converge to those of Problem \((\tilde{P}_d)\) or not. To study whether the functional values of both Problem \((\tilde{P}_{w,k})\) and Problem \((\tilde{P}_{v,k})\) would converge to that of Problem \((\tilde{P}_d)\) or not, we have the following result:

**Theorem1.** \(\forall \gamma > 0, \exists k' \geq 1\) such that

\[
\Gamma_{k'-1} - \Gamma_{k'} \leq \gamma.
\]  
(103)

**Proof.** It can be seen from Step 2 that

\[
\|\text{real}(\mathbf{v}_k^*)\|_1 + \|\text{real}(\mathbf{w}_{k+1}^*)\|_1 \leq \|\text{real}(\mathbf{v}_k^*)\|_1 + \|\text{real}(\mathbf{w}_k^*)\|_1 = \Gamma_k
\]  
(104)

for \(k \geq 0\). Also, it follows from Step 3 that

\[
\Gamma_{k+1} = \|\text{real}(\mathbf{v}_{k+1}^*)\|_1 + \|\text{real}(\mathbf{w}_{k+1}^*)\|_1 \leq \|\text{real}(\mathbf{v}_k^*)\|_1 + \|\text{real}(\mathbf{w}_{k+1}^*)\|_1
\]  
(105)

for \(k \geq 0\). Hence

\[
\Gamma_{k+1} \leq \Gamma_k
\]  
(106)

for \(k \geq 0\). As \(\Gamma_k\) is bounded for \(k \geq 0\), this implies that \(\forall \gamma > 0, \exists k' \geq 1\) such that

\[
\Gamma_{k'-1} - \Gamma_{k'} \leq \gamma.
\]  
(107)

This completes the proof.

To study whether the feasible sets of both Problem \((\tilde{P}_{w,k})\) and Problem \((\tilde{P}_{v,k})\) would converge to those of Problem \((\tilde{P}_d)\) or not, let the feasible sets of Problem \((\tilde{P}_{w,k})\), Problem \((\tilde{P}_{v,k})\) and Problem \((\tilde{P}_d)\) be \(\mathcal{F}_{v,k}^D\), \(\mathcal{F}_{w,k}^{\#D}\) and \(\mathcal{F}_D\), respectively. Denote \(\phi\) as the empty set. Then, we have the following result:

**Theorem2.** Suppose that \(\mathcal{F}_{v,k}^D \neq \phi\), \(\mathcal{F}_{w,k}^{\#D} \neq \phi\), \(\mathcal{F}_{v,k}^D \neq \phi\) and \(\mathcal{F}_D \neq \phi\). Then, \(\mathcal{F}_{v,k} \cap \mathcal{F}_{v,k+1} \neq \phi\). **Proof.** Since \(\mathbf{w}_{k+1}^* \in \mathcal{F}_{v,k}^D\), this implies that \((\mathbf{w}_{k+1}^*, \mathbf{v}_k^*) \in \mathcal{F}_D\). On the other hand, as \(\mathbf{v}_{k+1}^* \in \mathcal{F}_{w,k+1}^{\#D}\), this implies that \((\mathbf{w}_{k+1}^*, \mathbf{v}_{k+1}^*) \in \mathcal{F}_D\). This further implies that \(\mathbf{w}_{k+1}^* \in \mathcal{F}_{v,k+1}^D\) and \(\mathcal{F}_{v,k} \cap \mathcal{F}_{v,k+1}^D \neq \phi\). This completes the proof.

Although it is difficult to conclude whether \(\mathcal{F}_{v,k} \subset \mathcal{F}_{v,k+1}^D\) or \(\mathcal{F}_{v,k} \supset \mathcal{F}_{v,k+1}^D\), the proposed algorithm will still be terminated as the functional values of both Problem \((\tilde{P}_{w,k})\) and Problem \((\tilde{P}_{v,k})\) would converge.

5. Illustrative design examples.

5.1. Comparison between a nonuniform filter bank with conventional samplers and the corresponding filter window bank with constant synthesis periodic window functions. Consider a nonuniform filter bank with the conventional samplers as shown in Figure 2. Here, the set of sampling integers is
chosen as \( \{2, 3, 6\} \). The reason for choosing such set of samplers is because it corresponds to a well known incompatible nonuniform filter bank [2]. For an illustration purpose, consider

\[
H_0(z) = h_0 z^{-K_0}, \quad (108)
\]
\[
H_1(z) = h_1 z^{-K_1}, \quad (109)
\]
\[
H_2(z) = h_2 z^{-K_2}, \quad (110)
\]
\[
F_0(z) = f_0 z^{-K_0}, \quad (111)
\]
\[
F_1(z) = f_1 z^{-K_1}, \quad (112)
\]

and

\[
F_2(z) = f_2 z^{-K_2}. \quad (113)
\]

Then, it can be seen easily that

\[
E_0(z) = \frac{1}{2} h_0 z^{-K_0} f_0 z^{-K_0} + \frac{1}{3} h_1 z^{-K_1} f_1 z^{-K_1} + \frac{1}{6} h_2 z^{-K_2} f_2 z^{-K_2} - \frac{3h_0 f_0 + 2h_1 f_1 + h_2 f_2}{6} z^{-K}, \quad (114)
\]
\[
E_1(z) = \frac{1}{6} h_2 (ze^{-j\pi f_1})^{-K_2} f_2 z^{-K_1} - \frac{1}{6} h_2 (ze^{-j\pi f_2})^{-K_2} f_2 z^{-K_2}, \quad (115)
\]
\[
E_2(z) = \frac{1}{3} h_1 (ze^{-j\pi f_2})^{-K_2} f_1 z^{-K_2} + \frac{1}{6} h_2 (ze^{-j\pi f_2})^{-K_2} f_2 z^{-K_2}, \quad (116)
\]
\[
E_3(z) = \frac{1}{2} h_0 (z - 2^{-1})^{-K_0} f_0 z^{-K_0} + \frac{1}{6} h_2 (z - 2^{-1})^{-K_2} f_2 z^{-K_2}, \quad (117)
\]
\[
E_4(z) = \frac{1}{3} h_1 (ze^{-j\pi f_3})^{-K_2} f_1 z^{-K_2} + \frac{1}{6} h_2 (ze^{-j\pi f_3})^{-K_2} f_2 z^{-K_2}, \quad (118)
\]

and

\[
E_5(z) = \frac{1}{6} h_2 (ze^{-j\pi f_1})^{-K_2} f_2 z^{-K_2}. \quad (119)
\]

In this case, the exact perfect reconstruction condition cannot be satisfied for all realizable filters.

Now, this nonuniform filter bank with the conventional samplers is extended to a filter window bank with constant synthesis periodic window functions. Since the periods of the joint downsampling functions and the upsampling functions of the first channel, the second channel and the third channel of the nonuniform filter bank with the conventional samplers are equal to 2, 3 and 6, respectively, the periods of the analysis periodic window functions of the filter window bank are chosen accordingly. That is,

\[
\tilde{N}_0 = 2, \quad (120)
\]
\[
\tilde{N}_1 = 3 \quad (121)
\]

and

\[
\tilde{N}_2 = 6. \quad (122)
\]

For the real valued filters, the exact perfect reconstruction condition of the filter window bank becomes

\[
a_{0,1} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = a_{2,3} = a_{2,4} = a_{2,5} = 0 \quad (123)
\]

and

\[
a_{0,0} h_0 f_0 + a_{1,0} h_1 f_1 = 0. \quad (124)
\]
As the exact perfect reconstruction condition is satisfied \cite{14,29}, we choose
\[ \varepsilon = 0. \]
\hspace{1cm} (125)

As
\[ U_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \] 
and
\[ U_1 = \begin{bmatrix} 1 \\ 1 \\ e^{j2\pi/3} \\ e^{j4\pi/3} \end{bmatrix}, \]
we have
\[ \|w\|_0 = \left\| \begin{bmatrix} (U_0a_0)^T & (U_1a_1)^T & (U_2a_2)^T \end{bmatrix} \right\|_0 \]
\hspace{1cm} (128)

Therefore, Problem (Pc) is equivalent to the following optimization problem:
\[ \min_a \left\| \begin{bmatrix} a_0, 0 & a_0, 0 & a_1, 0 & a_1, 0 & 0 & \cdots & 0 \end{bmatrix} \right\|_0, \]
subject to
\[ a_{0,0}h_0f_0 + a_{1,0}h_1f_1 = \overline{c}, \]
\[ a_{0,0}, a_{1,0}, 0 \in \mathbb{R}, \]
and
\[ a_{0,1} = a_{1,1} = a_{2,1} = a_{2,0} = a_{2,2} = a_{2,3} = a_{2,4} = a_{2,5} = 0. \]
\hspace{1cm} (129)

The trivial solution of this optimization problem is
\[ a_{0,0}^* = \frac{\overline{c}}{h_0f_0} \]
\hspace{1cm} (130)

and
\[ a_{0,0}^* = a_{1,0}^* = a_{1,1}^* = a_{1,2}^* = a_{2,0}^* = a_{2,1}^* = a_{2,2}^* = a_{2,3}^* = a_{2,4}^* = a_{2,5}^* = 0. \] 
\hspace{1cm} (131)

This implies that
\[ w_0^* = \begin{bmatrix} \frac{\overline{c}}{h_0f_0} & \frac{\overline{c}}{h_0f_0} \end{bmatrix}^T, \]
\[ w_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, \]
\hspace{1cm} (132) \hspace{1cm} (133)

and
\[ w_2^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \]
\hspace{1cm} (134)

Since
\[ H_0(z)F_0(z) = h_0f_0z^{-1}, \]
\hspace{1cm} (135)

and
\[ w_0^* = \begin{bmatrix} \frac{\overline{c}}{h_0f_0} & \frac{\overline{c}}{h_0f_0} \end{bmatrix}^T, \]
\hspace{1cm} (136)

the first channel of the filter window bank behaves as a pure delay gain unit. This explains why the filter window bank can achieve the exact perfect reconstruction condition. However, as the second channel and the third channel of the filter window bank are disconnected as well as both \( H_0(z) \) and \( F_0(z) \) are the pure delay gain elements which do not perform any frequency selectivity function, this filter window bank is not useful for practical applications even though the exact perfect reconstruction condition is satisfied. This is the reason why the time varying synthesis
periodic windows functions are required. Alternative, the exact perfect reconstruction condition is required to be relaxed to a near perfect reconstruction condition. On the other hand, as

\[ \|w^*\|_0 = 2 \]  \hspace{1cm} (137)

this case corresponds to the minimum total number of nonzero coefficients of the impulse responses of the analysis periodic window functions. Obviously, this is a global optimal solution of Problem \((P_c)\). In the literature, it is a common practice to approximate an \(L_0\) norm optimization problem by the corresponding \(L_1\) norm optimization problem \([17],[30]\). Thus, the design of the periodic analysis window functions is also formulated as the following \(L_1\) norm optimization problem for the reason of comparison:

\[
\begin{align*}
\min_{a} & \quad \|Ua\|_1, \\
\text{subject to} & \quad a_{0,0}h_0f_0 + a_{1,0}h_1f_1 = \bar{c}, \\
& \quad a_{0,0}, a_{1,0} \in \mathbb{R} \\
& \quad a_{0,1} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = a_{2,3} = a_{2,4} = a_{2,5} = 0.
\end{align*}
\]  \hspace{1cm} (138a)

Since

\[ \|Ua\|_1 = \| \begin{bmatrix} a_{0,0} & a_{0,0} & a_{1,0} & a_{1,0} & 0 & \cdots & 0 \end{bmatrix}^T \|_1 = 2|a_{0,0}| + 3|a_{1,0}|, \]  \hspace{1cm} (139)

this \(L_1\) norm optimization problem becomes the following optimization problem:

\[
\begin{align*}
\min_{a} & \quad 2|a_{0,0}| + 3|a_{1,0}|, \\
\text{subject to} & \quad a_{0,0}h_0f_0 + a_{1,0}h_1f_1 = \bar{c}, \\
& \quad a_{0,0}, a_{1,0} \in \mathbb{R} \\
& \quad a_{0,1} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = a_{2,3} = a_{2,4} = a_{2,5} = 0.
\end{align*}
\]  \hspace{1cm} (140a)

By constructing a straight line

\[ a_{0,0}h_0f_0 + a_{1,0}h_1f_1 = \bar{c} \]  \hspace{1cm} (141)

on the two dimensional Euclidean plane and evaluating the objective functional values at the intercept points \((\frac{\bar{c}}{h_0f_0}, 0)\) and \((0, \frac{\bar{c}}{h_1f_1})\), that is, comparing the magnitudes of \(2|\frac{\bar{c}}{h_0f_0}|\) and \(3|\frac{\bar{c}}{h_1f_1}|\), the solution of this \(L_1\) norm optimization problem is obtained as follows:

\[
(a_{0,0}^*, a_{1,0}^*) = \begin{cases} 
(\frac{\bar{c}}{h_0f_0}, 0) & 2|\frac{\bar{c}}{h_0f_0}| \leq 3|\frac{\bar{c}}{h_1f_1}| \\
(0, \frac{\bar{c}}{h_1f_1}) & 2|\frac{\bar{c}}{h_0f_0}| > 3|\frac{\bar{c}}{h_1f_1}| 
\end{cases} \]  \hspace{1cm} (142)

and

\[ a_{0,0}^* = a_{1,0}^* = a_{1,1}^* = a_{1,2}^* = a_{2,0}^* = a_{2,1}^* = a_{2,2}^* = a_{2,3}^* = a_{2,4}^* = a_{2,5}^* = 0. \]  \hspace{1cm} (143)

Note that the solution of this norm optimization problem is the same as that of norm optimization problem when

\[ 2|\frac{\bar{c}}{h_0f_0}| \leq 3|\frac{\bar{c}}{h_1f_1}|. \]  \hspace{1cm} (144)
However, it is not the case when
\[ 2\left| \frac{c}{h_0 f_0} \right| > 3\left| \frac{c}{h_1 f_1} \right|, \]  

This is due to the fact that the objective functions of these two optimization problems are different.

**Figure 2.** Nonuniform filter bank with the conventional samplers.

### 5.2. Comparison between a nonuniform block filter bank with the block samplers and the corresponding filter window bank with the time varying synthesis periodic window functions.

Now, consider a nonuniform block filter bank as shown in Figure 3. The block lengths of the samplers are all equal to 6. The set of the decimation ratios of the block downsamplers and the set of the upsampling ratios of the block upsamplers are chosen as \{3, 2, 6\}. This set of block sampling ratios is chosen because it corresponds to the same set of the sampling ratios of the incompatible nonuniform filter bank [12]. Suppose that both the analysis filters and the synthesis filters are causal and all the filter lengths are equal to 6. That is,

\[ H_i(z) = \sum_{m=0}^{5} h_{i,m} z^{-m} \]  
\[ F_i(z) = \sum_{m=0}^{5} f_{i,m} z^{-m} \]

for \( i = 0, 1, 2 \). For simplicity, the filters are obtained using the matlab function fir1. More specifically,

\[ H_0(w) = F_0(w) = 2\sqrt{3}e^{-jw^{2.5}} \left( 0.067 \cos(2.5w) + 0.1111 \cos(1.5w) + 0.3822 \cos(0.5w) \right), \]  

\[ H_1(w) = F_1(w) = 2\sqrt{2}e^{-jw^{2.5}} \left( -0.0041 \cos(2.5w) - 0.2417 \cos(1.5w) + 0.4537 \cos(0.5w) \right) \]  

and

\[ H_2(w) = F_2(w) = 2\sqrt{6}e^{-jw^{2.5}} \left( 0.0224 \cos(2.5w) - 0.1358 \cos(1.5w) + 0.3418 \cos(0.5w) \right). \]

It can be checked that this nonuniform block filter bank does not achieve the exact perfect reconstruction condition.

Now, this nonuniform block filter bank is generalized to a filter window bank with the time varying synthesis periodic window functions. As the block lengths of all the samplers in the block filter bank are equal to 6, the periods of both the
analysis periodic window functions and the synthesis periodic window functions of
the filter window bank are chosen as 6. For the block nonuniform filter bank, the
joint downsampling functions and the upsampling functions are

\[
\begin{align*}
  w_0[n] &= \begin{cases} 
    1 \mod (n, 6) = 0 \\
    1 \mod (n, 6) = 1 \\
    0 \mod (n, 6) = 2 \\
    0 \mod (n, 6) = 3 \\
    0 \mod (n, 6) = 4 \\
    0 \mod (n, 6) = 5 
  \end{cases} , \\
  w_1[n] &= \begin{cases} 
    1 \mod (n, 6) = 0 \\
    1 \mod (n, 6) = 1 \\
    1 \mod (n, 6) = 2 \\
    0 \mod (n, 6) = 3 \\
    0 \mod (n, 6) = 4 \\
    0 \mod (n, 6) = 5 
  \end{cases} , \\
  w_2[n] &= \begin{cases} 
    1 \mod (n, 6) = 0 \\
    0 \mod (n, 6) = 1 \\
    0 \mod (n, 6) = 2 \\
    0 \mod (n, 6) = 3 \\
    0 \mod (n, 6) = 4 \\
    0 \mod (n, 6) = 5 
  \end{cases} ,
\end{align*}
\]

and

\[
\begin{align*}
  \mathbf{w}^*_{0,0} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T , \\
  \mathbf{w}^*_{0,1} &= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T , \\
  \mathbf{w}^*_{0,2} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T , \\
  \mathbf{v}^*_{0,0} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T , \\
  \mathbf{v}^*_{0,1} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T ,
\end{align*}
\]

Here, we initialize the analysis periodic window functions of the filter window
bank as the same as the joint downsampling functions and the upsampling functions
of the block filter bank. Also, we initialize the synthesis periodic window functions
of the filter window bank as the constant functions. This is because this synthesis
part of the filter window bank corresponds to that of the block filter bank. That is,

\[
\begin{align*}
  \mathbf{w}^*_{0,0} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T , \\
  \mathbf{w}^*_{0,1} &= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T , \\
  \mathbf{w}^*_{0,2} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T , \\
  \mathbf{v}^*_{0,0} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T , \\
  \mathbf{v}^*_{0,1} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T ,
\end{align*}
\]

and

\[
\begin{align*}
  \mathbf{w}^*_{0,2} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T .
\end{align*}
\]

Next, we initialize

\[
\begin{align*}
  \mathbf{w}_0^* &= \begin{bmatrix} \mathbf{w}^*_{0,0} & \mathbf{w}^*_{0,1} & \mathbf{w}^*_{0,2} \end{bmatrix}^T ,
\end{align*}
\]

and

\[
\begin{align*}
  \mathbf{v}_0^* &= \begin{bmatrix} \mathbf{v}^*_{0,0} & \mathbf{v}^*_{0,1} & \mathbf{v}^*_{0,2} \end{bmatrix}^T .
\end{align*}
\]

Similarly, the set of the analysis filters and the set of the synthesis filters are
chosen the same as those of the block filter bank for the reason of comparison.
Furthermore,

\[
\begin{align*}
  \epsilon &= 0.1362 ,
\end{align*}
\]

and

\[
\begin{align*}
  \gamma &= 10^{-3} .
\end{align*}
\]
Table 1. Both the transfer functional distortions and the aliasing distortions of both the filter window bank and the block filter bank.

|                              | Filter window bank | Block filter bank | Improvement  |
|------------------------------|--------------------|-------------------|--------------|
| The maximum absolute value of | -17.3157 dB       | 0.9315 dB         | 18.2472 dB   |
| the transfer functional      |                    |                   |              |
| distortion                   |                    |                   |              |
| The maximum absolute value of | -24.8282 dB       | -2.0172 dB        | 22.8110 dB   |
| the first aliasing distortion|                    |                   |              |
| The maximum absolute value of | -18.6761 dB       | -5.4446 dB        | 13.2316 dB   |
| the second aliasing distortion|                    |                   |              |
| The maximum absolute value of | -27.5018 dB       | -9.3399 dB        | 18.1618 dB   |
| the third aliasing distortion|                    |                   |              |
| The maximum absolute value of | -18.7578 dB       | -5.1906           | 13.5672 dB   |
| the fourth aliasing distortion|                    |                   |              |
| The maximum absolute value of | -23.9515 dB       | -0.7191 dB        | 23.2325 dB   |
| the fifth aliasing distortion|                    |                   |              |

are chosen. This is because these values should be small enough for most practical applications. Once these parameters are initialized, **Algorithm 1** can be executed. Finally, by finding the solution of Problem (\( \tilde{P}_{w,k} \)) and the solution of Problem (\( \tilde{P}_{v,k} \)) iteratively, a solution of the approximate Problem (\( \tilde{P}_d \)) can be found.

To demonstrate the effectiveness of the filter window bank, both the transfer functional distortion and the aliasing distortion are employed as the performance indices. The results obtained are then compared with those obtained from the nonuniform block filter bank. Figure 4 shows both the transfer functional distortions and the aliasing distortions of both the filter window bank and the block filter bank. The numerical values are shown in Table 1. From Figure 4 and Table 1, we can see that the maximum absolute value of the transfer functional distortion of the filter window bank and that of the block filter bank are -17.3157 dB and 0.9315 dB, respectively. Here, there is 18.2472 dB improvement. On the other hand, the maximum absolute value of the first aliasing distortion of the filter window bank and that of the block filter bank are -24.8282 dB and -2.0172 dB, respectively. Here, there is 22.8110 dB improvement. The maximum absolute value of the second aliasing distortion of the filter window bank and that of the block filter bank are -18.6761 dB and -5.4446 dB, respectively. Here, there is 13.2316 dB improvement. The maximum absolute value of the third aliasing distortion of the filter window
bank and that of the block filter bank are -27.5018 dB and -9.3399 dB, respectively. Here, there is 18.1618 dB improvement. The maximum absolute value of the fourth aliasing distortion of the filter window bank and that of the block filter bank are -18.7578 dB and -5.1906 dB, respectively. Here, there is 13.5672 dB improvement. The maximum absolute value of the fifth aliasing distortion of the filter window bank and that of the block filter bank are -23.9515 dB and -0.7191 dB, respectively. Here, there is 23.2325 dB improvement. From these obtained results, we can conclude that the filter window bank has achieved significant improvements on both the transfer functional distortion and the aliasing distortion. Also, the required specification on both the transfer functional distortion and the aliasing distortion is satisfied. As the specification values on these distortions are small enough for practical applications, the designed filter window bank is useful for practical applications. On the other hand, both the transfer functional distortion and the aliasing distortion of the block filter bank are too large for practical applications. Also, it does not satisfy the required specification. It is worth noting that the total number of the degree of freedom of the Fourier coefficients of the filter window bank is increased, but the total number of the equations required to be satisfied is the same as that of the block filter bank. Therefore, the filter window bank can achieve both the lower transfer functional distortion and the lower aliasing distortion when compared with the block filter bank for the same set of analysis filters and synthesis filters.

Figure 3. Nonuniform block filter bank with the block samplers.

6. Conclusions. This paper proposes an optimal design of the filter window bank. For the design of the analysis periodic window functions of the filter window bank with the constant synthesis periodic window functions, the design problem is formulated as an $L_0$ norm optimization problem, where the total number of the nonzero coefficients of the analysis periodic window functions is minimized subject to the near perfect reconstruction condition. An iterative approach is employed for finding its optimal solution. Similarly, for the design of both the analysis periodic window functions and the synthesis periodic window functions of the filter window bank, the total number of the nonzero coefficients of both the analysis periodic window functions and the synthesis periodic window functions is minimized subject to the near perfect reconstruction condition. This is also an $L_0$ norm optimization problem. Likewise, an iterated algorithm is employed for finding its optimal solution. The obtained results show that the filter window bank can achieve significant improvements on both the transfer functional distortion and the aliasing distortion when compared with the nonuniform filter banks with the conventional samplers and the
Figure 4. (a) The transfer functional distortions, (b) the first aliasing distortion components, (c) the second aliasing distortion components, (d) the third aliasing distortion components, (e) the fourth aliasing distortion components and (f) the fifth aliasing distortion components of both the filter window bank and the block filter bank.

nonuniform block filter banks. The improvement on both the transfer functional distortion and the aliasing distortion is due to the increase in the total number of degree of freedom for designing the Fourier coefficients while maintaining the same total number of equations required to be satisfied.

Acknowledgments. This paper was supported partly by the National Nature Science Foundation of China (no. U1701266, no. 61372173 and no. 61671163), the Team Project of the Education Ministry of the Guangdong Province(2017KCXTD 011), the Guangdong Higher Education Engineering Technology Research Center for Big Data on Manufacturing Knowledge Patent (no. 501130144), and Hong Kong Innovation and Technology Commission, Enterprise Support Scheme (no. S/E/070/17).

REFERENCES

[1] K. D. Abdesselam, Design of stable, causal, perfect reconstruction, IIR uniform DFT filter banks, *IEEE Transactions on Signal Processing*, 48 (2000), 1110–1119.

[2] T. S. Bindiya and E. Elias, Design of totally multiplier-less sharp transition width tree structured filter banks for non-uniform discrete multitone system, *AEU-International Journal of Electronics and Communications*, 69 (2015), 655–665.

[3] M. Blanco-Velasco, F. Cruz-Roldán, E. Moreno-Martínez, J. I. Godino-Llorente and K. E. Barner, Embedded filter bank-based algorithm for ECG compression, *Signal Processing*, 88 (2008), 1402–1412.
[4] G. F. Choueiter and J. R. Glass, An implementation of rational wavelets and filter design for phonetic classification, *IEEE Transactions on Audio, Speech, and Language Processing*, 15 (2007), 939–948.

[5] F. Cruz-Roldán, P. Martín-Martín, J. Sáez-Landete, M. Blanco-Velasco and T. Saramäki, A fast windowing-based technique exploiting spline functions for designing modulated filter banks, *IEEE Transactions on Circuits and Systems I: Regular Papers*, 56 (2009), 168–178.

[6] H. H. Dam, S. Nordholm and A. Cantoni, Uniform FIR filter bank optimization with group delay specifications, *IEEE Transactions on Signal Processing*, 53 (2005), 4249–4260.

[7] G. Doblinger, A fast design method for perfect-reconstruction uniform cosine-modulated filter banks, *IEEE Transactions on Signal Processing*, 60 (2012), 6693–6697.

[8] B. Farhang-Boroujeny, Filter bank spectrum sensing for cognitive radios, *IEEE Transactions on Signal Processing*, 56 (2008), 1801–1811.

[9] C. Gu, J. Zhao, W. Xu and D. Sun, Design of linear-phase notch filters based on the OMP scheme and the chebychev window, *IEEE Transactions on Circuits and Systems II: Express Briefs*, 59 (2012), 592–596.

[10] C. Y. F. Ho, B. W. K. Ling and P. K. S. Tam, Representations of linear dual-rate system via single SISO LTI filter, conventional sampler and block sampler, *IEEE Transactions on Circuits and Systems II: Express Briefs*, 55 (2008), 168–172.

[11] A. Kumar, G. K. Singh and S. Anurag, An optimized cosine-modulated nonuniform filter bank design for subband coding of ECG signal, *Journal of King Saud University-Engineering Science*, 27 (2015), 158–169.

[12] B. W. K. Ling, C. Y. F. Ho, J. Cao and Q. Dai, Necessary and sufficient condition for a set of maximally decimated integers to be incompatible, *Necessary and Sufficient Condition for a Set of Maximally Decimated Integers to be Incompatible*, 9 (2013), 564–566.

[13] B. W. K. Ling, C. Y.-F. Ho, K. L. Teo, W. C. Siu, J. Z. Cao and Q. Y. Dai, Optimal design of cosine modulated nonuniform linear phase FIR filter bank via both stretching and shifting frequency response of single prototype filter, *IEEE Transactions on Signal Processing*, 62 (2014), 2517–2530.

[14] Q. Liu, X. Z. Zhang, W. K. Ling, M. Wang and Q. Dai, Exact perfect reconstruction of filter window bank with application to incompatible nonuniform filter banks, *IEEE/IET International Symposium on Communication Systems, Networks and Digital Signal Processing, CSNDSP*, (2016), 20–22.

[15] M. Narendar, A. P. Vinod, A. S. Madhukumar and A. K. Krishna, A tree-structured DFT filter bank based spectrum detector for estimation of radio channel edge frequencies in cognitive radios, *Physical Communication*, 9 (2013), 45–60.

[16] R. C. Nongpiur and D. J. Shpak, Maximizing the signal-to-alias ratio in non-uniform filter banks for acoustic echo cancellation, *IEEE Transactions on Circuits and Systems I: Regular Papers*, 59 (2012), 2315–2325.

[17] G. W. Ou, D. P. K. Lun and B. W. K. Ling, Compressive sensing of images based on discrete periodic Radon transform, *IET Electronics Letters*, 50 (2014), 591–593.

[18] A. Pandharipande and S. Dasgupta, On biorthogonal nonuniform filter banks and tree structures, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49 (2002), 1457–1467.

[19] S. Rahimi and B. Champagne, Oversampled perfect reconstruction DFT modulated filter banks for multi-carrier transceiver systems, *Signal Processing*, 93 (2013), 2942–2955.

[20] A. K. Soman and P. P. Vaidyanathan, On orthonormal wavelets and paraunitary filter banks, *IEEE Transactions on Signal Processing*, 41 (1993), 1170–1183.

[21] R. Soni, A. Jain and R. Saxena, An optimized design of nonuniform filter bank using variable-combinational window function, *AEU-International Journal of Electronics and Communications*, 67 (2013), 595–601.

[22] K. Swaminathan and P. Vaidyanathan, Theory and design of uniform DFT, parallel, quadrature mirror filter banks, *IEEE Transactions on Circuits and Systems*, 33 (1986), 1170–1191.

[23] G. Wang, Time-varying discrete-time signal expansions as time-varying filter banks, *IET Signal Processing*, 3 (2009), 353–367.

[24] X. G. Xia and B. W. Suter, Multirate filter banks with block sampling, *IEEE Transactions on Signal Processing*, 44 (1996), 484–496.

[25] X. M. Xie, S. C. Chan and T. I. Yuk, Design of perfect-reconstruction nonuniform combination filter banks with flexible rational sampling factors, *IEEE Transactions on Circuits and Systems I: Regular Papers*, 52 (2005), 1965–1981.
[26] H. Xiong, L. Zhu, N. Ma and Y. F. Zheng, Scalable video compression framework with adaptive orientational multiresolution transform and nonuniform directional filterbank design, *IEEE Transactions on Circuits and Systems for Video Technology*, 21 (2011), 1085–1099.

[27] Z. Xiong, K. Ramchandran, C. Herley and M. T. Orchard, Flexible tree-structured signal expansions using time-varying wavelet packets, *IEEE Transactions on Signal Processing*, 45 (1997), 333–345.

[28] W. Xu, J. X. Zhao and C. Gu, Design of linear-phase FIR multiple-notch filters via an iterative reweighted OMP scheme, *IEEE Transactions on Circuits and Systems II: Express Briefs*, 61 (2014), 813–817.

[29] C. Q. Yang, J. Xiao, Y. F. Zeng, B. W. Deng and W.-K. Ling, Design of periodic window functions in filter window filter banks for harsh environment, *International Conference on Industrial Informatics, INDIN*, (2016), 18–21.

[30] Z. Yang, B. W. K. Ling and C. Bingham, Approximate affine linear relationship between $L_1$ norm objective function values and $L_2$ norm constraint bounds, *IET Signal Processing*, 9 (2015), 670–680.

[31] K. C. Zangi and R. D. Koilpillai, Software radio issues in cellular base stations, *IEEE Journal on Selected Areas in Communications*, 17 (1999), 561–573.

[32] Y. Zhang, S. Negahdaripour and Q. Z. Li, Low bit-rate compression of underwater imagery based on adaptive hybrid Wavelets and directional filter banks, *Signal Processing: Image Communication*, 47 (2016), 96–114.

Received January 2019; revised May 2019.

E-mail address: zzhouxueling@gmail.com
E-mail address: yongquanling@gdut.edu.cn
E-mail address: H.Dam@curtin.edu.au
E-mail address: K.L.Teo@curtin.edu.au