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Exactly Solvable Potentials and Romanovski Polynomials in Quantum Mechanics

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Preface

This thesis for the University Degree of Master of Science is an account of my research at the Institute of Physics at the Autonomous University of San Luis Potosi, Mexico, during the time period September 2005—December 2006. It is devoted to the exact solutions of the Schrödinger equation with the hyperbolic Scarf potential (Scarf II) which finds various applications in physics ranging from soliton physics to electrodynamics with non-central potentials. In the current literature the exact solutions to Scarf II are written in terms of Jacobi polynomials of purely imaginary arguments and parameters that are complex conjugate to each other. The fact is that the above Jacobi polynomials are proportional to real orthogonal polynomials by the purely imaginary phase factor, \((-i)^n\), much like the phase relationship between the hyperbolic–and the trigonometric functions, i.e. \(\sin(ix) = i\sinh(x)\). These real polynomials, to be referred to as the Romanovski polynomials, have been largely ignored by the standard mathematical textbooks on polynomials and mathematical methods of physics texts although they are required in exact solutions of several physics problems ranging from supersymmetric quantum mechanics and quark physics to random matrix theory. It is one of the virtues of the present thesis to draw attention to the Romanovski polynomials and their importance in the physics with Scarf II.

I begin with reviewing the five possible real polynomial solutions of the generalized hypergeometric differential equation. Three of them are the well known classical orthogonal polynomials of Hermite, Laguerre and Jacobi, but the other two are different with respect to their orthogonality properties. The family of polynomials which exhibits finite orthogonality (meaning that only a finite number of them are orthogonal) are the Romanovski polynomials.

Next, I solve the one-dimensional Schrödinger equation with Scarf II in terms of the Romanovski polynomials. Then I focus on the problem of an electron within a particular potential that appears non-central in the polar angle coordinate while preserving the rotational invariance with respect to the azimuthal angle. I report the new observation that the (one-dimensional) Schrödinger equation with the hyperbolic Scarf potential defines the polar angle part of the respective wave functions. The latter define new non-spherical angular functions. I furthermore establish a new non-linear relationship between the Romanovski polynomials to the associated Legendre functions and employ it to obtain a non-standard orthogonality integral between infinite series of polynomials. This infinite orthogonality does not contradict the finite one because in the latter case the parameters change with the polynomial degree and same does the associated weight function. This circumstance allows to satisfy the orthogonality condition for an infinite number of polynomials. Finally, I also solve the Klein-Gordon equation with scalar and vector potentials of equal magnitudes and given by Scarf II. I conclude that the Romanovski polynomials are the most adequate degrees of freedom in the mathematics of the Scarf II potential.
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Chapter 1

Introduction.

The exactly solvable Schrödinger equations (SE) occupy a pole position in quantum mechanics in so far as most of them directly apply to relevant physical systems. Prominent examples are the quantum Kepler-, or, Coulomb-potential problem and its importance in the description of the discrete spectrum of the hydrogen atom [1], the harmonic-oscillator–, the Hulthen–, and the Morse potentials with their relevance to vibrational spectra [2], [3]. Another good example is given by the Pöschl-Teller potential [4] which appears as an effective mean field in many-body systems with $\delta$-interactions [5]. Knowing the exact SE solutions is furthermore of interest in testing approximative iteration methods [6].

There are various methods of finding the exact solutions of a Schrödinger equation for the bound states, an issue on which we shall focus in the present work.

- The traditional method [7] consists in reducing SE by an appropriate substitution of the variables to that very form of the generalized hypergeometric equation [8] whose solutions are polynomials, the majority of which, and especially the classical ones, being well studied.

- The second method suggests to first unveil the dynamical symmetry of the potential problem and then employ the relevant group algebra in order to construct the solutions as the group representation spaces [9]–[11].

- Finally, there is also the most recent and powerful method of the super-symmetric quantum mechanics (SUSYQM) which considers the special class of Schrödinger equations (in units of $\hbar = 1 = 2m$) that allows for a factorization according to [12], [13]

$$
(H(z) - e_n)\psi_n(z) = \left(-\frac{d^2}{dz^2} + v(z) - e_n\right)\psi_n(z) = 0,
$$

$$
H(z) = A^+(z)A^-(z) + e_0,
$$

$$
A^\pm(z) = \left(\pm\frac{d}{dz} + U(z)\right).
$$

Here, $H(z)$ stands for the (one-dimensional) Hamiltonian, $U(z)$ is the so called super-potential, and the operators $A^\pm(z)$ ladder between neighboring solutions. The super-potential allows to recover the ground state wave function, $\psi_{gst}(z)$, as

$$
\psi_{gst}(z) \sim e^{-\int U(y)dy}.
$$

The excited states are then built up on top of $\psi_{gst}(z)$ through the repeated action of the $A^+(z)$ operators. The supersymmetric quantum mechanics manages the family of exactly solvable potentials presented in Table 1.1. Though this Table contains 10 potentials, the variety is in fact not as big. The reason is that these potentials split into two classes such that the potentials belonging to same class can be mapped onto each other by means
of appropriate point transformations of the coordinates (so called “shape invariance”, see Ref. [14, 16] and references therein). The Coulomb, oscillator, and Morse potentials belong to the first class, while the second class contains the Rosen-Morse, Eckart, Pöschl-Teller and Scarf potentials. There are more exactly solvable potentials. The very popular in nuclear physics Woods-Saxon potential,

$$V_{WS}(r) = -\frac{V_0}{1 + e^{\frac{r-R_0}{a}}} - \frac{ce^{\frac{r-R_0}{a}}}{(1 + e^{\frac{r-R_0}{a}})^2}, \quad (1.3)$$

where $V_0$, $R_0$, and $a$ are constant parameters, has been resolved recently in Refs. [15] in terms of the Jacobi polynomials.

- The above methods refer to potentials with unconstrained parameters. However, there are also potentials which are exactly solvable only when the parameters obey certain constraints. Take as example the sextic anharmonic oscillator, $ax^6 + bx^4 + cx^2$ with $a > 0$ which allows for exact solvability [17] only if the parameters are restricted by the condition, $3\sqrt{a - \frac{b^2}{4a}} + c = 0$. In this case, any wave function corresponding to a level of energy $E$ acts as the generating function of a set of polynomials in the energy variable which differ from the solutions of the generalized hypergeometric equation. For the first time such polynomials have been reported by Bender and Dunne [18] while solving the one dimensional (1D) Schrödinger equation with the potential, $x^6 - (4s - 4J - 2)x^2$, for $J$ a positive integer and $s = 1/4$, or $s = 3/4$. The two choices of $s$ refer to even- and odd-parity of the eigen-wave functions, respectively. One can find a “weight function” with respect to which the Bender–Dunner polynomials appear “orthogonal” though the “weight function” neither needs to be positive.

While the non-relativistic Schrödinger equation is exactly solvable for the potentials listed in Table 1.1 (and for a few more), exact solutions to the relativistic Klein-Gordon– and Dirac equations are quite scarce. Besides the relativistic Coulomb problem, the Dirac equation seems so far to be amenable to an exact solution only for the relativistic oscillator, a result due to Moshinsky and Szczepaniak [19]. Klein-Gordon equations with scalar and vector potentials of equal magnitudes reduce to Schrödinger equations and allow for exact solvability (see Chpt. IV below). Finally, very recently, low-power potentials of the type, $\sim \beta/r^{1+\beta}$ with $-1 \neq \beta \neq -2$, have been shown to be quasi-solvable in the sense that only the zero energy solutions are exact [20]. The present study is devoted to the first item above.

1.1 The goal.

Table 1.1 shows that the hyperbolic-Scarf– and the trigonometric Rosen-Morse potentials (termed to as Scarf II, and Rosen-Morse I, respectively) are solved in terms of Jacobi polynomials of imaginary arguments and parameters that are complex conjugate to another.

The goal of this thesis is to solve the Schrödinger equation with the hyperbolic Scarf potential [14], [21]–[23] anew and to make the case that it reduces in a straightforward manner to a particular form of the generalized real hypergeometric equation whose solutions are given by a finite set of real orthogonal polynomials. In this manner, the finite number of bound states within the hyperbolic Scarf potential is brought in correspondence with a finite system of orthogonal polynomials of a new class. In due course, various new properties of the above polynomials are encountered.

These polynomials have been discovered in 1884 by the English mathematician Sir Edward John Routh [24] and rediscovered 45 years later by the Russian mathematician Vsevolod Ivanovich Romanovski in 1929 [25] within the context of probability distributions. Though they have been studied on few occasions in the current mathematical literature where they are termed to as
| Potential          | Variable                                                                 | Wave Function                                                                 |
|--------------------|--------------------------------------------------------------------------|-------------------------------------------------------------------------------|
| Shifted oscillator | $\frac{1}{2} \omega^2 (x - \frac{\lambda}{\omega})^2 - \frac{\lambda}{\omega}$ | $y = \sqrt{\frac{\lambda}{\omega}} (x - \frac{\lambda}{\omega})$ exp $(-\frac{1}{2} y^2) H_n(y)$ |
| 3D-oscillator      | $\frac{1}{2} \omega^2 r^2 + \frac{1}{2} (l + \frac{3}{2}) \omega$ | $y = \frac{1}{2} \omega r^2$ $y^{l+1} \exp \left(-\frac{1}{2} y\right) L_n^{(l+\frac{1}{2})}(y)$ |
| Coulomb            | $-\frac{e^2}{r} + \frac{1}{2} (l + \frac{1}{2}) \omega + \frac{e^2}{2r^2}$ | $y = \frac{e^2}{r} \exp \left(-\frac{1}{2} r^2\right)$ $\exp \left(-\frac{1}{2} y\right) L_n^{(2l+1)}(y)$ |
| Morse              | $A^2 + B^2 \exp(-2\alpha x)$ $-2B(A+\frac{\mu}{\omega}) \exp(-\alpha x)$ | $y = \frac{2B}{\omega} \exp(-\alpha x)$ $y^{s-n} \exp \left(-\frac{1}{2} y\right) L_n^{(s-n)}(y)$ |
| Scarf I            | $-A + (A^2 + B^2 - Ao) \sec^2(\alpha x)$ | $y = \sin(\alpha x)$ $i^2 (1 + y^2) \frac{\beta}{2} \exp(-\lambda \tan^{-1}(y)) \times P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(iy)$ |
| (Pöschl-Teller 1)  | $+B(2A + \alpha) \sec(\alpha x) \tan(\alpha x)$ | $y = \tan(\alpha x)$ | $(1-y) \frac{\beta^2}{2} (1+y) \frac{\beta}{2} P_n^{(s_1,s_2)}(y)$ |
| Scarf II           | $A^2 + (B^2 - A^2 - Ao) \sech^2(\alpha x)$ | $y = \coth(\alpha x)$ | $(y-1) \frac{\beta^2}{2} (y+1) \frac{\beta}{2} P_n^{(s_3,s_4)}(y)$ |
| Rosen-Morse       | $A^2 + B^2 \csc^2(\alpha x)$ | $y = \cot(\alpha x)$ | $i^n (y^2 + 1) \frac{\beta}{2} \exp(a \cot^{-1}(y)) \times P_n^{(-a-n,a-n-s-\frac{1}{2})}(iy)$ |
| Eckart             | $A^2 + B^2 \sech^2(\alpha x)$ | $y = \coth(\alpha x)$ | $(y-1) \frac{\beta^2}{2} (y+1) \frac{\beta}{2} P_n^{(s_1,s_2)}(y)$ |
| Pöschl-Teller 2    | $A^2 + (B^2 + A^2 + Ao) \csc^2(\alpha x)$ | $y = \csc(\alpha x)$ | $(y-1) \frac{\beta^2}{2} (y+1) \frac{\beta}{2} P_n^{(s,s)}(y)$ |
| Rosen-Morse I      | $A^2 + B^2 \cot(\alpha x) \csc(\alpha x)$ | $y = \cot(\alpha x)$ | $i^n (y^2 + 1) \frac{\beta}{2} \exp(a \cot^{-1}(y)) \times P_n^{(-a-n,a-n-s-\frac{1}{2})}(iy)$ |

Table 1.1: Exactly solvable shape invariant potentials in one dimension (according to [13][22]). All potential parameters are supposed to be positive. Other notations are: $s = A/\alpha$, $\lambda = B/\alpha$, $a = \lambda/(s-n)$, $s_1 = s - n + a$, $s_2 = s - n - a$, $s_3 = a - n - s$, and $s_4 = -(s + n + a)$. The orthogonal polynomials of Hermite, Laguerre, and Jacobi have been denoted in the standard way as $H(x)$, $L_n^{(\alpha)}(x)$, and $P_n^{(\mu,\nu)}(x)$, respectively.
“finite Romanovski” [26–29], or, “Romanovski-Pseudo-Jacobi” polynomials [30], they have been completely ignored by the textbooks on mathematical methods in physics, and surprisingly enough, by the standard mathematics textbooks as well [8], [31–34]. The notion “finite” refers to the observation that for any given set of parameters (i.e. in any potential), only a finite number of polynomials (finite number of bound states) appear orthogonal.

The Romanovski polynomials happen to be equal (up to a phase factor) to that very Jacobi polynomials with imaginary arguments and parameters that are complex conjugate to each other [35–37], much like the sinh(z) = i sin(iz) relationship. Although one may (but has not to) deduce the local characteristics of the latter such as generating function and recurrence relations from those of the former, the finite orthogonality theorem is qualitatively new. It does not copy none of the properties of the Jacobi polynomials but requires an independent proof. For this reason, in random matrix theory [39] the problem on the gap probabilities in the spectrum of the complex circular Jacobi ensemble is preferably treated in terms of the real Cauchy random ensemble, a venue that conducts once again to the Romanovski polynomials because, as it will be shown below, the weight function of the Romanovski polynomials for certain values of the parameters equals the Cauchy distribution.

The thesis contributes two new examples to the circle of the typical quantum mechanical problems [40]. The techniques used here extend the teachings on the Sturm-Liouville theory of ordinary differential equations beyond their textbook presentation.

Before leaving the introduction, a comment is in place on the importance of the Scarf II potential. It finds various applications in physics ranging from electrodynamics and solid state physics to particle theory. In solid state physics Scarf II is used in the construction of more realistic periodic potentials in crystals [41] than those built from the trigonometric Scarf potential (Scarf I) [42]. In electrodynamics Scarf II appears in a class of problems with non-central potentials [43, 44] (also see section IV below for more details). In particle physics Scarf II finds application in studies of the non-perturbative sector of gauge theories by means of toy models such as the scalar field theory in (1+1) space-time dimensions. Here, one encounters the so called “kink-like” solutions which are no more but the static solitons. The spatial derivative of the kink-like solution is viewed as the ground state wave function of an appropriately constructed Schrödinger equation which is then employed in the calculation of the quantum corrections to first order. In Ref. [45] it was shown that specifically Scarf II is amenable to a stable renormalizable scalar field theory.

The thesis is organized as follows.

- In the next chapter I first highlight in brief the basics of the generalized hypergeometric equation, review the classification of its polynomial solutions in various schemes, and construct for completeness of the presentation, the polynomials of Hermite, Laguerre, Jacobi, Romanovski, and Bessel.

- In chapter III I bring examples for potentials whose exact solutions are given in terms of the polynomials presented in the previous chapter.

- The original contribution of thesis is presented in chapter IV. There I

  - first provide the exact solution of the 1D-Schrödinger equation with Scarf II in terms of the finite Romanovski polynomials.

  - Next I consider the problem of an electron within a particular non-central potential along the line of Ref. [43, 44]. I made the new observation that the polar angle part of the electron wave function in this problem happens to be defined by Romanovski polynomials whose parameters depend on the polynomial degree. Within this context, the Romanovski polynomials act as designers of new non-spherical angular functions.

  - I explicitly construct and display graphically the lowest five non-spherical-functions and compare them with the standard $Y_l^m(\theta, \varphi)$ harmonics.
– I furthermore establish a new non-linear relationship between the Romanovski polynomials to the associated Legendre functions and employ it to obtain a non-standard orthogonality integral between infinite series of polynomials. This infinite orthogonality does not contradict the finite one because in the latter case the parameters change with the polynomial degree and same does the associated weight function. This circumstance allows to satisfy the orthogonality condition for an infinite number of polynomials.

– Before closing by a brief summary, I consider a Klein-Gordon equation with scalar and vector potentials of equal magnitudes which are given by Scarf II.
Chapter 2

Generalized hypergeometric equation. Polynomial solutions.

All classical orthogonal polynomials appear as solutions of the so-called generalized hypergeometric equation (the presentation in this section closely follows Ref. [28]),

\[ \sigma(x)y''_n(x) + \tau(x)y'_n(x) - \lambda_n y_n(x) = 0, \]

\[ \sigma(x) = ax^2 + bx + c, \quad \tau(x) = xd + e, \quad \lambda_n = n(n-1)a + nd. \]

There are various methods for finding the polynomial solution, here denoted by

\[ y_n(x) \equiv P_n \left( \begin{array}{ccc} d & e \\ a & b & c \end{array} \right) x \].

The symbol \( P_n \left( \begin{array}{ccc} d & e \\ a & b & c \end{array} \right) x \) makes the equation parameters explicit and stands for a polynomial of degree \( n \), \( \lambda_n \) being the eigenvalue parameter, and \( n = 0, 1, 2, \ldots \).

2.1 Classification scheme of Koepf-Masjed-Jamei.

In Ref. [28] the solutions to Eq. (2.1) have been classified according to the five parameters \( a, b, c, d, \) and \( e \). Furthermore, a master formula for a generic monic polynomial solutions, \( P_n \), has been derived by Koepf and Masjed-Jamei, according to them one finds

\[ P_n \left( \begin{array}{ccc} d & e \\ a & b & c \end{array} \right) x = \sum_{k=0}^{n} \binom{n}{k} G_k^{(n)}(a, b, c, d, e) x^k, \]

\[ G_k^{(n)} = \left( \frac{2a}{b + \sqrt{b^2 - 4ac}} \right)^n \left( \begin{array}{ccc} (k - n), \\ 2 - \frac{d}{a} - 2n, \end{array} \right) \frac{2\sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}}. \]

The \( a = 0 \) case is handled as the \( a \to 0 \) limit of Eq. (2.4) and leads to the appearance of \( _2F_0 \) in place of \( _2F_1 \) (see Ref. [28] for details). Though the original derivation of this result is a bit cumbersome, its verification with the help of the symbolic software Maple is straightforward.
2.2 Classification according to Nikiforov-Uvarov.

In the method of Nikiforov and Uvarov [8] the (polynomial) solutions to the hypergeometric equation (2.1) are classified according to the so called weight function, \( w(x) \), and built up from the Rodrigues formula,

\[
\begin{align*}
\sigma(x) y_n''(x) + y_1(x) y_n'(x) - \lambda_n y_n(x) &= 0, \\
y_1(x) &= \frac{N_1}{w(x)} \frac{d}{dx} [w(x) \sigma(x)], \\
\lambda_n &= -n(y'_1(x) + \frac{1}{2}(n-1)\sigma''(x)),
\end{align*}
\]

with \( N_1 \) being the \( y_1(x) \) normalization constant. The weight function is calculated from integrating the so called Pearson differential equation,

\[
\frac{d}{dx} (\sigma(x) w(x)) = y_1(x) w(x),
\]

and then plugged into the Rodrigues formula to generate the polynomial solutions as

\[
y_n(x) = \frac{N_n}{w(x)} \frac{d^n}{dx^n} [w(x) \sigma^n(x)],
\]

with \( N_n \) being a normalization constant. The \( y_n(x) \)’s are normalized polynomials and are orthogonal with respect to the weight function \( w(x) \) within a given interval, \([l_1, l_2]\), provided \( \sigma(x) > 0 \), and \( w(x) > 0 \) holds true for all \( x \in [l_1, l_2] \), and \( \sigma(x) w(x) x^l \big|_{x=l_1} = \sigma(x) w(x) x^l \big|_{x=l_2} = 0 \) for any \( l \) integer, and with suitable normalization constants,

\[
\int_{l_1}^{l_2} w(x) y_n(x) y_{n'}(x) dx = \delta_{nn'}, \quad \forall \quad n, n' \in \{0, 1, 2, \ldots\}.
\]

The Nikiforov-Uvarov method combines well with the Koepf-Masjed-Jamei scheme. Indeed, the solution to the Pearson equation can be cast into the form

\[
\omega(x) \equiv W \begin{pmatrix} d & e \\ a & b \end{pmatrix} = \exp \left( \int \frac{(d-2a)x + (e-b)}{ax^2 + bx + c} dx \right).
\]

It shows how one can calculate any weight function associated with any parameter set of interest (the symbol used for the weight function makes again the equation parameters explicit). The best strategy in recovering the weight function corresponding to a given set of parameters is to consider its logarithmic derivative, \( \frac{\omega'(x)}{\omega(x)} \), and then match the parameters accordingly. In order to illustrate this procedure, we pick up one of the examples given in [28] and consider the weight function

\[
\omega(x) = (-x^2 + 3x - 2)^{10} = (1 - x)^{10}(2 - x)^{10}.
\]

The calculation of the logarithmic derivative gives

\[
\frac{\omega'(x)}{\omega(x)} = \frac{-20x + 30}{-x^2 + 3x - 2} = \frac{(d-2a)x + (e-b)}{ax^2 + bx + c},
\]

which matches with the parameters \( a = -1, b = 3, c = -2, d = -22, \) and \( e = 33 \). Therefore, the resulting weight function is

\[
\omega(x) = W \begin{pmatrix} -22 & 33 \\ -1 & 3 \end{pmatrix}.
\]
In the notations of Koepf–Masjed-Jamei the Rodrigues formula looks like
\[
P_n\left(\begin{array}{ccc} d & e & c \\ a & b & c \end{array} \bigg| x \right) = \prod_{k=1}^{n} (d + (n + k - 2)a) P_n\left(\begin{array}{ccc} d & e & c \\ a & b & c \end{array} \bigg| x \right) = \frac{1}{W\left(\begin{array}{ccc} d & e & c \\ a & b & c \end{array} \bigg| x \right)} \\
\times \frac{d^n}{dx^n} \left( (ax^2 + bx + c)^n W\left(\begin{array}{ccc} d & e & c \\ a & b & c \end{array} \bigg| x \right) \right).
\]
(2.14)

The polynomials associated with the weight function in Eq. (2.11) can be constructed explicitly by Eq. (2.14) and treated as independent entities and without even knowing that they are no more but Jacobi shifted to the interval \(x \in [1, 2]\). The great appeal of combining the master formulas in the respective Eqs. (2.4), and (2.14) is that they allow for the direct and pragmatic construction of all the polynomial solutions to the generalized hypergeometric equation.

One identifies as special cases
- the Jacobi polynomials with \(a = -1, b = 0, c = 1, d = -\gamma - \delta - 2,\) and \(e = -\gamma + \delta,\)
- the Laguerre polynomials with \(a = 0, b = 1, c = 0, d = -1,\) and \(e = \alpha + 1,\)
- the Hermite polynomials with \(a = b = 0, c = 1, d = -2,\) and \(e = 0,\)
- the Romanovski polynomials with \(a = 1, b = 0, c = 1, d = 2(1-p),\) and \(e = q\) with \(p > 0,\)
- the Bessel polynomials with \(a = 1, b = 0, c = 0, d = \alpha + 2,\) and \(e = \beta.\)

These parametrizations will be referred to as "canonical". Any other parametrization can be reduced to one of the above sets upon an appropriate shift of the argument.

The first three polynomials are the only ones that are traditionally presented in the standard textbooks on mathematical methods in physics such like [31] – [34], while the fourth and fifth seem to have escaped due attention. Notice, the Legendre–, Gegenbauer–, and Chebychev polynomials appear as particular cases of the Jacobi polynomials. The Bessel polynomials are not orthogonal in the conventional sense, i.e. over a real interval (see section 3.6 below for details).

Some of the properties of the fourth polynomials have been studied in the current mathematical literature such as Refs. [26], [27]– [30]. Their weight function is calculated from Eq. (2.10) as
\[
w^{(p,q)}(x) = (x^2 + 1)^{-p} e^{qt} \tan^{-1} x.
\]
(2.15)

This weight function has first been reported by Routh [24], and independently Romanovski [25]. Notice that for a vanishing \(q,\) \(w^{(p,q)}(x)\) becomes equal to the student’s \(t\) distribution [16] and acts as that very probability distribution. The \(w^{(1,0)}(x)\) case stands for the popular Cauchy (or, Breit-Wigner) distribution. Within the light of this discussion, the weight function of the Romanovski polynomials has been interpreted in Ref. [46] as the most natural extension of the student’s \(t\) distribution.

The polynomials associated with Eq. (2.15) are called after Romanovski and will be denoted by \(R^{(p,q)}_m(x)\). They have non-trivial orthogonality properties over the infinite interval \(x \in [-\infty, +\infty].\) Indeed, as long as the weight function decreases as \(x^{-2p},\) hence integrals of the type
\[
\int_{-\infty}^{+\infty} w^{(p,q)}(x) R^{(p,q)}_m(x) R^{(p,q)}_{m'}(x) dx,
\]
(2.16)
are convergent only if
\[
m + m' < 2p - 1,
\]
(2.17)
meaning that only a finite number of Romanovski polynomials are orthogonal. This is the reason
for which one speaks of “finite”Romanovski polynomials. The orthogonality theorem has been
proved in Refs. [27, 37]. The differential equation satisfied by the Romanovski polynomials reads
as
\[(1 + x^2)\frac{d^2R_n^{(p,q)}(x)}{dx^2} + (2(-p + 1)x + q)\frac{dR_n^{(p,q)}(x)}{dx} - (n(n - 1) + 2n(1 - p))R_n^{(p,q)}(x) = 0. \tag{2.18}\]
In the next section we shall show that the Schrödinger equation with the hyperbolic Scarf potential
reduces precisely to the very Eq. (2.18).

2.3 Bochner’s classification. Explicit polynomial construction.

The earliest classification of the solutions to the generalized hypergeometric equation is due to
Bochner [47] (see also Refs. [34, 36, 37, 38] for more recent works) and based on the form of \(\sigma(x)\)
which can be rephrased in terms of the \(\sigma(x)\) roots. The presentation in this section closely follows
Refs. [36, 37]. Within this scheme one finds the following five polynomial classes:

1. \(\sigma(x)\) is a constant:

The canonical form of the generalized hypergeometric equation is
\[
H''(x) - 2\alpha xH'(x) + \lambda H(x) = 0, \tag{2.19}
\]
where \(\alpha\) is a parameter. The solutions are the generalized Hermite polynomials \(\{H_\alpha^n(x)\}\) \((\alpha = 1\)
characterizes the ordinary Hermite polynomials), and their weight function is
\[
\omega^{(\alpha)}(x) = e^{-\alpha x^2}. \tag{2.20}
\]
The case \(\alpha = 1\) provides the standard orthogonality relation,
\[
\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_{n'}(x)dx = \delta_{nn'}, \quad \forall \quad n, n' \in \{0, 1, 2, \ldots\}. \tag{2.21}
\]
The lowest Hermite polynomials are then obtained as:
\[
\begin{align*}
H_0(x) &= 1, \\
H_1(x) &= 2x, \\
H_2(x) &= -2 + 4x^2, \\
H_3(x) &= -12x + 8x^3, \\
H_4(x) &= 12 - 48x^2 + 16x^3,
\end{align*}
\tag{2.22}
\]
and are represented in Fig. 2.1

2. \(\sigma(x)\) is of first degree:

The canonical form of the corresponding hypergeometric equation is obtained as
\[
xL''(x) + y_1(x)L'(x) + \lambda L(x) = 0. \tag{2.23}
\]
For \(y_1(x) = -\alpha x + \beta + 1, \alpha = 1,\) and \(\beta\) being a real number, the latter equation coincides with
the equation for the associated Laguerre polynomials, \(\{L_\alpha^{(1,\beta)}(x)\}\), while for \(\alpha = 1,\) and \(\beta = 0\) it
describes the ordinary Laguerre polynomials. The weight function is given by
\[
\omega^{(\alpha,\beta)}(x) = x^\beta e^{-\alpha x}. \tag{2.24}
\]
For \(\alpha, \beta > 0\) the orthogonality integral within the interval \(x \in [0, \infty)\) reads
\[
\int_0^\infty x^\beta e^{-\alpha x} L_n^{(\alpha,\beta)}(x) L_{n'}^{(\alpha,\beta)}(x) dx = \delta_{nn'}, \quad \forall \ n, n' \in \{0, 1, 2, \ldots\}. \tag{2.25}
\]
From now onwards we shall suppress the first upper index in \(L_n^{(1,\beta)}(x)\) for simplicity. The lowest five associated Laguerre polynomials for, say, \(\beta = 1\), are now calculated as
\[
L_0^{(1)}(x) = 1,
L_1^{(1)}(x) = 2 - x,
L_2^{(1)}(x) = \frac{1}{2}(6 - 6x + x^2),
L_3^{(1)}(x) = \frac{1}{6}(24 - 36x + 12x^2 - x^3),
L_4^{(1)}(x) = \frac{1}{24}(120 - 240x + 120x^2 - 20x^3 + x^4). \tag{2.26}
\]
They are shown in Fig. 2.2.

3. \(\sigma(x)\) is of the second degree, with two different real roots:

The canonical form of the corresponding hypergeometric equation is obtained as
\[
(1 - x^2) P''(x) + y_1(x) P'(x) + \lambda P(x) = 0. \tag{2.27}
\]
The latter equation coincides with the Jacobi one for \(y_1(x) = \alpha - \beta - (\alpha + \beta + 2)x\) where \(\alpha, \beta \in \mathbb{R}\) are the polynomial parameters. The Jacobi polynomials, \(\{P_n^{(\alpha,\beta)}(x)\}\), are then defined by the
Rodrigues formula in terms of the following weight function:
\[ \omega^{(\alpha,\beta)}(x) = (1 - x)^\alpha (1 + x)^\beta. \] (2.28)

Furthermore, one has to restrict the parameters to \( \alpha, \beta > -1 \) in order to ensure orthogonality in the interval \( x \in [-1, +1] \) according to
\[ \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta P_n^{(\alpha,\beta)}(x) P_{n'}^{(\alpha,\beta)}(x) dx = \delta_{nn'}, \quad \forall \ n, n' \in \{0, 1, 2, \ldots\}. \] (2.29)

Several particular cases have received their proper names. These are:
- Gegenbauer when \( \alpha = \beta \),
- Chebyshev I and II when \( \alpha = \beta = \pm 1/2 \),
- Legendre when \( \alpha = \beta = 0 \).

The lowest five Jacobi polynomials for the toy values \( \alpha = 1, \beta = 2 \) of the parameters are explicitly calculated as:
\[
\begin{align*}
P_0^{(1,2)}(x) &= 1, \\
P_1^{(1,2)}(x) &= \frac{1}{2}(-1 + 5x), \\
P_2^{(1,2)}(x) &= 3 + 9(-1 + x) + \frac{21}{4}(-1 + x)^2, \\
P_3^{(1,2)}(x) &= 4 + 21(-1 + x) + 28(-1 + x)^2 + \frac{21}{2}(-1 + x)^3, \\
P_4^{(1,2)}(x) &= 5 + 40(-1 + x) + 90(-1 + x)^2 + 75(-1 + x)^3 + \frac{165}{8}(-1 + x)^4.
\end{align*}
\] (2.30)

They are shown in Fig. 2.3.

The lowest Gegenbauer polynomials and for \( \alpha = 1 \) read:
\[
\begin{align*}
C_0^{(1)}(x) &= 1, \\
C_1^{(1)}(x) &= 2x, \\
C_2^{(1)}(x) &= -1 + 4x^2, \\
C_3^{(1)}(x) &= -4x + 8x^3, \\
C_4^{(1)}(x) &= -1 - 12x^2 + 16x^4.
\end{align*}
\] (2.31)
Figure 2.4: Gegenbauer polynomials for $\alpha = 1$.

Figure 2.5: Chebyshev I polynomials. They are graphically displayed in Fig. 2.4.

Next, the lowest Chebyshev II polynomials are:

$U_0(x) = 1,$
$U_1(x) = 2x,$
$U_2(x) = -1 + 4x^2,$
$U_3(x) = -4x + 8x^3,$
$U_4(x) = -1 - 12x^2 + 16x^4.$  \( \text{(2.33)} \)

They are graphically displayed in Fig. 2.5.
Finally, one finds the lowest Legendre polynomials as
\[ P_0(x) = 1, \]
\[ P_1(x) = x, \]
\[ P_2(x) = -\frac{1}{2}(1 - 3x^2), \]
\[ P_3(x) = -\frac{1}{2}(3 - 5x^3), \]
\[ P_4(x) = -\frac{1}{8}(-15x + 70x^3 - 63x^5). \]
\[ (2.34) \]

They are shown in Fig. 2.7.

Related to the Legendre polynomials are the associated Legendre functions which are defined as
\[ P^n_m(x) = (-1)^m(1 - x^2)\frac{d^m}{dx^m}P_n(x). \]
\[ (2.35) \]

4. \( \sigma(x) \) is of the second degree, with one double real root:

The canonical form of the corresponding differential equation is obtained as
\[ x^2y''(x) + y_1(x)y'(x) + \lambda y(x) = 0. \]
\[ (2.36) \]

For \( y_1(x) = \alpha x + \beta \), with \( \alpha, \beta \in \mathbb{R} \), the weight function is given by
\[ \omega^{(\alpha,\beta)}(x) = x^{2-\alpha}e^{-\frac{x}{2}}. \]
The polynomials \( \left\{ y_0^{(\alpha,\beta)}(x) \right\} \) with \( \alpha = \beta = 2 \), are called after Bessel [48]. They are not orthogonal in the conventional sense within a real interval. By considering \( x \in \mathbb{C} \) these polynomials have special characteristics. Only when the integration contour is the unit circle in the complex plane can one design orthogonality integrals and with respect to the “weight function” \( e^{-\frac{x}{2}} \). To see this notice that Eq. (2.36) with \( \alpha = \beta = 2 \) can be rewritten to give
\[
(x^2e^{-\frac{x}{2}}y'_n(x))' = n(n+1)e^{-\frac{x}{2}}y_n(x).
\]
(2.38)

An integration by parts counter-clockwise around the unit circle leads to
\[
n(n+1) \int_U y_m(x)y_n(x)e^{-\frac{x}{2}}dx = \int_U (x^2e^{-\frac{x}{2}}y'_n(x))y_m(x)dx = - \int_U x^2e^{-\frac{x}{2}}y'_m(x)y'_n(x)dx.
\]
(2.39)

Upon interchanging \( m \) against \( n \) and a subsequent subtraction, one arrives at the orthogonality relation
\[
\int_U y_m(x)y_n(x)e^{-\frac{x}{2}}dx = 0.
\]
(2.40)

The lowest Bessel polynomials are:
\[
y_0(x) = 1,
y_1(x) = 1 + x,
y_2(x) = 1 + 3x + 3x^2,
y_3(x) = 1 + 6x + 15x^2 + 15x^3,
y_4(x) = 1 + 10x + 45x^2 + 105x^3 + 105x^4.
\]
(2.41)

They are displayed in Fig. 2.8.

5. \( \sigma(x) \) is of the second degree, with two complex roots:

This is the case of prime interest to the thesis. The generalized hypergeometric equation for this choice of \( \sigma(x) \) takes the form
\[
(1 + x^2)R''(x) + y_1(x)R'(x) + \lambda R(x) = 0.
\]
(2.42)

Taking \( y_1(x) \) as \( y_1(x) = (2\beta + 1)x + \alpha \), and \( \alpha, \beta \in \mathbb{R} \), we encounter a family of polynomials which we denote by \( \left\{ R_n^{(\beta,\alpha)}(x) \right\} \). These are the Romanovski polynomials. The corresponding weight function (also mentioned before in Eq. (2.15)) reads
\[
\omega^{(\beta,\alpha)}(x) = (1 + x^2)^{\beta-\frac{1}{2}}e^{-\alpha \tan^{-1}(x)}.
\]
(2.43)
For $\beta = -p + 1/2$, and $\alpha = -q$ Eq. (2.43) coincides with Eq. (2.15). The latter polynomials have special orthogonal properties and are studied in greater detail in the next chapter.

The lowest Romanovski polynomials and for the toy values $\beta = -1$, and $\alpha = 1$ are:

\begin{align*}
R_{-1}^{(-1,1)}(x) &= 1, \\
R_{1}^{(-1,1)}(x) &= -1 - 9x, \\
R_{2}^{(-1,1)}(x) &= -6 + 16x + 56x^2, \\
R_{3}^{(-1,1)}(x) &= 16 + 84x - 126x^2 - 210x^3, \\
R_{4}^{(-1,1)}(x) &= 20 - 240x - 360x^2 + 480x^3 + 360x^4. 
\end{align*}

Some of them are shown in Fig. 2.9. In summary, the polynomial solutions of the generalized hypergeometric equation that permit for a Rodrigues representation fall into five classes. Three of them correspond to the celebrated classical orthogonal polynomials of Jacobi, Laguerre, and Hermite (recall that Gegenbauer, Chebyshev I and II, and Legendre appeared as special cases of the Jacobi polynomials). The principal characteristics of those five classes are collected in Table 2.1.

| Notion        | Symbol | Weight Function: $\omega(x)$ | Density: $\sigma(x)$ | Interval | Parameters |
|---------------|--------|-----------------------------|----------------------|----------|------------|
| Generalized Hermite | $H_n^{(\alpha)}(x)$ | $e^{-\alpha x^2}$ | 1 | $(-\infty, \infty)$ | |
| Generalized Laguerre | $L_n^{(\alpha,\beta)}(x)$ | $x^\beta e^{-\alpha x}$ | $x$ | [0, $\infty$) | |
| Jacobi        | $P_n^{(\alpha,\beta)}(x)$ | $(1-x)^\alpha (1+x)^\beta$ | $(1-x^2)$ | [-1,1] | $(\alpha, \beta > -1)$ |
| Legendre      | $P_n(x)$ | 1 | $(1-x^2)$ | [-1,1] |  |
| Gegenbauer    | $C_n^{(\lambda)}(x)$ | $(1-x^2)^{\lambda-\frac{1}{2}}$ | $(1-x^2)$ | [-1,1] | $(\lambda > -1/2)$ |
| Chebyshev I   | $T_n(x)$ | $1$ | $(1-x^2)^{\frac{1}{2}}$ | [-1,1] |  |
| Chebyshev II  | $U_n(x)$ | $(1-x^2)^{\frac{1}{2}}$ | $(1-x^2)$ | [-1,1] |  |
| Generalized Bessel | $y_n^{(\alpha,\beta)}(x)$ | $x^\alpha e^{-\frac{x^2}{2}}$ | $x^2$ | complex | $-\beta > n$ |
| Romanovski    | $R_n^{(\lambda,\alpha,\beta)}(x)$ | $(1+x^2)^{\lambda-\frac{1}{2}} e^{-\alpha \tan^{-1}(x)}$ | $(1+x^2)$ | $(-\infty, \infty)$ | $-\beta > n$ |

Table 2.1: Comparison among the different families of polynomials.
Chapter 3

Orthogonal polynomials in quantum mechanics.

As already mentioned in the introduction, the classical orthogonal polynomials shape the exact solutions of a variety of quantum mechanical potentials. This is so because the Schrödinger equation with one of those potentials can be transformed into Eq. (2.1) upon an appropriate change of the variables. This section is devoted to the one-dimensional Schrödinger equation (which equally well can be the radial part of an $S$ wave 3D problem). In the following, we shall assign each polynomial in Table 2.1 to a potential that is exactly solvable in terms of that very polynomial.

3.1 The Schrödinger equation with a radial potential.

Two particles within a potential that depends only on the relative distance, $r$,

$$V(r_1, r_2) = V(r), \quad r = |r_1 - r_2|,$$  \hspace{1cm} (3.1)

with $r_1$, and $r_2$ being the respective coordinates of first and second particle, are described by a time-independent Schrödinger equation of the form of a single particle within the center of mass frame,

$$-\frac{\hbar^2}{2\mu} \nabla^2 \Psi(r) + V(r)\Psi(r) = E\Psi(r),$$  \hspace{1cm} (3.2)

where $\mu$ stands for the reduced mass. This equation can be solved by standard techniques like separation of the variables, in which case the solution factorizes into a radial function, $u(r)$, and an angular part, $\Theta(\theta)\Phi(\phi)$, which for central potentials is given by the spherical harmonics, $\Theta(\theta)\Phi(\phi) = Y_{lm}(\theta, \phi)$, i.e.

$$\Psi(r) = u(r)Y_{lm}(\theta, \phi).$$  \hspace{1cm} (3.3)

In substituting the latter wave function in the three-dimensional Schrödinger equation, amounts to the well known one-dimensional radial equation

$$-\frac{\hbar^2}{2\mu} \frac{d}{dr} \left( r^2 \frac{d}{dr} u(r) \right) + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} u(r) + V(r)u(r) = Eu(r).$$  \hspace{1cm} (3.4)

In making the coordinate and the wave functions dimensionless upon rescaling,

$$z \equiv \frac{r}{d}, \quad u(r) \equiv \frac{\psi(z)}{d^{l+1/2}},$$  \hspace{1cm} (3.5)

where $d$ is an appropriate length scale, Eq. (3.4) takes the form

$$-\frac{1}{z^2} \frac{d}{dz} \left( z^2 \frac{d}{dz} \psi(z) \right) + \frac{l(l+1)}{z^2} \psi(z) + v(z)\psi(z) = \epsilon\psi(z).$$  \hspace{1cm} (3.6)
Here,

\[ v(z) \equiv V(r)/(\hbar^2/2\mu d^2), \quad \epsilon \equiv E/(\hbar^2/2\mu d^2), \]

and the normalization is defined by the integral

\[ \int_0^{\infty} z^2 \psi(z) \psi^*(z) \, dz = 1. \]

(3.8)

In terms of the new variable, \( R(z) \), introduced via

\[ \psi(z) \equiv \frac{R(z)}{z}, \]

(3.9)

Eq. (3.6) becomes

\[ -\frac{d^2}{dz^2} R(z) + \left( \frac{l(l+1)}{z^2} + v(z) - \epsilon \right) R(z) = 0. \]

(3.10)

With that the normalization integral changes to

\[ \int_0^{\infty} R(z) R^*(z) \, dz = 1. \]

(3.11)

Casting the three-dimensional Schrödinger equation into the form of Eq. (3.10), brings two important advantages. The first one is that all variables are dimensionless, which allows for an easier mathematical treatment. The second advantage is that Eq. (3.10) takes the form of 1D Schrödinger equation with an effective potential defined as

\[ v_{1d}(z) = v(z) + \frac{l(l+1)}{z^2}. \]

(3.12)

Finally, not to forget the \( R(0) = 0 \) boundary condition, telling that the wave function has to vanish at the origin if it is not to penetrate the centrifugal barrier. In this fashion, the 3D problem has been replaced by an equivalent 1D problem.

### 3.2 Orthogonal polynomials in exact wave functions.

The Schrödinger wave functions, \( \psi(z) \), in diagonalizing an Hermitian differential operator, are known to constitute a complete orthogonal set. One can try to reduce the radial Schrödinger equation (written in the variable \( z \)) to an appropriate polynomial equation (written in the variable \( x \)) by means of the following change of variables:

\[ z = f(x), \quad R(f(x)) = g(x), \quad g(x) = \sqrt{\omega(x)} F_n^{(\alpha,\beta)}(x) \frac{1}{\sqrt{d(f(x))/dx}}. \]

(3.13)

If this substitution turns out to be successful, the differential equation for \( F_n^{(\alpha,\beta)}(x) \) will be a version of the generalized hypergeometric equation whose solutions have been presented in the previous chapter. When expressed in terms of polynomials, the orthogonality of the Schrödinger wave functions translates into orthogonality between the involved polynomials according to

\[ \int_0^{+\infty} R_n(z) R_{n'}(z) \, dz = \int_{l_1}^{l_2} g_n(x) g_{n'}(x) \, df(x) \]

\[ = \int_{l_1}^{l_2} \omega^{(\alpha,\beta)}(x) F_n^{(\alpha,\beta)}(x) F_{n'}^{(\alpha,\beta)}(x) \, dx. \]

(3.14)

The parameters \( \alpha, \beta \) are some functions of the potential parameters. Occasionally, the price of casting the Schrödinger equation in the form of Eq. (2.5) is that \( \alpha \) and \( \beta \) acquire an
$n$-dependence. In such cases, as we shall see below, Eq. (3.13) may not hold valid and the orthogonality of the wave functions may not amount to the orthogonality integral of the polynomials with free parameters. In the following we employ the idea of Eq. (3.13) in order to solve the Schrödinger equations for a variety of potentials. We show that the Hermite polynomials shape the solutions of the 1D oscillator, the Laguerre polynomials define the wave functions of a particle within the 3D-oscillator– and the Coulomb wells. The Jacobi polynomials solve exactly the hyperbolic Rosen-Morse (Rosen-Morse II) potential, while the $\sim e^{\alpha r}$ barrier is solved in terms of the Bessel polynomials. The results listed in Table 1.1 above have been obtained pursuing same path.

3.3 Hermite polynomials and the harmonic oscillator.

The simple harmonic oscillator in quantum mechanics is a very well known example elaborated in all the standard textbooks. Its solutions are given in terms of the Hermite polynomials. In order to see this we write down the 1D Schrödinger equation, which reads:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dy^2} \phi(y) + V(y) \phi(y) = E \phi(y). \quad (3.15)$$

Here, $m$ is the particle’s mass, $V(x)$ stands for the potential, and $E$ is the energy of the system. The simple harmonic oscillation potential is given by

$$V_{HO}(y) = \frac{1}{2} m \omega^2 y^2, \quad (3.16)$$

where $\omega$ is the frequency of the oscillations in classical mechanics. Substitution of the latter equation into Eq. (3.15) gives

$$\frac{d^2}{dy^2} \phi(y) + 2 \frac{m}{\hbar^2} \left( E - \frac{1}{2} m \omega^2 y^2 \right) \phi(y) = 0. \quad (3.17)$$

The two parameters of the problem ($m, \omega$) provide a unit of length for the problem:

$$d \equiv \sqrt{\frac{\hbar}{m \omega}}, \quad z = \frac{y}{d}, \quad \phi(y = zd) = \frac{\psi(z)}{\sqrt{d}}. \quad (3.18)$$

In performing a substitution of the form in Eq. (3.13) and given by

$$z = f(x) = x, \quad \psi(z = x) = g(x), \quad g(x) = N_n \sqrt{e^{-x^2}} H_n(x), \quad (3.19)$$

where $N_n$ is a normalization constant, a new equation is obtained (after substitution and reordering):

$$\frac{d^2}{dx^2} g(x) + \left( \epsilon - x^2 \right) g(x) = 0, \quad (3.20)$$

where $\epsilon \equiv \frac{2m d^2}{\hbar^2} E$ now is dimensionless one finds

$$H''(x) - 2x H'(x) + (\epsilon - 1) H(x) = 0, \quad (3.21)$$

which is the Hermite equation

$$H''(x) - 2x H'(x) + \lambda H(x) = 0, \quad (3.22)$$

with $\lambda \in \mathbb{R}$ a constant to be determined. This equation admits a polynomial solution of degree $n$, $H_n(x)$, only if $\lambda = 2n$. Since we are interested in such solutions, we conclude

$$\epsilon_n - 1 = 2n, \quad n \in \{0, 1, 2, ...\}, \quad (3.23)$$
Figure 3.1: The simple 1D harmonic oscillator, \( V(x) = \frac{1}{2} m\omega^2 x^2 \). Here \( m\omega = 1 \).

![Diagram showing the simple 1D harmonic oscillator](image)

i.e.,

\[
E_n = \frac{\hbar^2}{m\ell^2} (n + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2}).
\]

(3.24)

The orthogonality integral between the wave functions amounts to the orthogonality integral of the Hermite polynomials

\[
\int_\infty^{-\infty} \psi_n(z) \psi_{n'}(z) dz = \int_\infty^{-\infty} N_n N_{n'} e^{-\frac{z^2}{4}} H_n \left( \frac{z}{\ell} \right) H_{n'} \left( \frac{z}{\ell} \right) d \left( \frac{z}{\ell} \right) = \delta_{nn'}.
\]

(3.25)

The harmonic oscillator potential finds application in the description of vibrational modes in nuclei, atoms, molecules, and crystal lattices. This potential is shown in Figure 3.1.

3.4 Laguerre polynomials in 3D oscillator and Coulomb wells.

3.4.1 The 3D oscillator.

Harmonic oscillations in three dimensions lead to a differential equation whose solutions are the associated Laguerre polynomials. One way to solve this problem is to consider the three-dimensional oscillation as the result of oscillators in \( x, y, \) and \( z \) directions. The solutions can then be expressed in terms of the product of three Hermite polynomials. However, given the rotational symmetry of the problem, the most natural coordinate choice are the spherical coordinates in which case the complete solution factorizes in a radial function and spherical harmonics in accord with the orthogonality integral.

Figure 3.2: Three-dimensional harmonic oscillator potential (dashed line), \( V(r) = \frac{1}{2} m\omega^2 r^2 \). Effective potential (solid line), \( V_{\text{eff}}(r) = \frac{1}{2} m\omega^2 r^2 + \frac{l(l+1)}{r^2} \) which includes the centrifugal barrier. Here \( m\omega = 1 \), and \( l = 1 \).
with Eq. (3.3). The 3D oscillator potential reads

\[ V(r) = \frac{1}{2}m\omega^2 r^2, \]

(3.26)

and is shown in Fig. 3.2 Changing variables to \( z = r/d \) and substituting into Eq. (3.26) results in

\[ \left(-\frac{d^2}{dz^2} - \frac{2}{z}\frac{d}{dz} + \frac{l(l+1)}{z^2} + z^2 \right)\psi(z) = \epsilon \psi(z). \]

(3.27)

The latter equation has to have well-behaved asymptotic solutions at the origin and infinity. For \( z \to \infty \) it reduces to

\[ \left(-\frac{d^2}{dz^2} + z^2 \right)\psi(z) = \epsilon \psi(z), \]

(3.28)

meaning that \( \psi \approx e^{-\frac{z^2}{2}} \) would be a good approximation. Next, we inspect the behavior of Eq. (3.27) at the origin, \( z \to 0 \), where it reduces to

\[ \left(-\frac{d^2}{dz^2} - \frac{2}{z}\frac{d}{dz} + \frac{l(l+1)}{z^2} \right)\psi(z) = 0. \]

(3.29)

Taking \( \psi(z) = z^\ell \) as a test function, we find that it would be a solution for \( t = l \), or, \( t = -l - 1 \). The former value is the correct one since it is finite at the origin. Now we shall find the exact expression for the solution applying the standard techniques we used before.

The above asymptotic behaviors suggest to try as an ansatz

\[ z\psi(z) = R(z) = K_n z^{\beta + \frac{1}{2}} e^{-\frac{z^2}{2}} F(z), \]

(3.30)

where \( K_n \) is a normalization constant. Insertion of the latter expression into Eq. (3.27) leads to

\[ (-1 + 2z^2 - 2\beta)F'(z) - zF''(z) + \left(\frac{1 + 4\ell + 4\beta^2}{4z} + z(2 + 2\beta - \epsilon)\right)F(z) = 0, \]

(3.31)

where \( \beta \) is a free parameter of a choice suitable for simplifying the resulting equation. Next we notice that in terms of the new variable, \( x \), introduced as \( z = f(x) = \sqrt{x} \), the ansatz in Eq. (3.30) can be converted to \( R(\sqrt{x}) = g(x) = N_n x^{\frac{\beta}{2} + \frac{1}{4}} e^{-\frac{x}{2}} L(x) \) with \( F(z = \sqrt{x}) = L(x) \), and \( N_n \) a new normalization constant. This expression is of the type in Eq. (3.19) in so far as

\[ g(x) = N_n \sqrt{x^\beta e^{-x}} L(x)/\sqrt{\frac{d}{dx} \sqrt{x}}, \]

(3.32)

where one recognizes the square root of the weight function of the associated Laguerre polynomials in front of \( L(x) \). The difference between \( N_n \) and \( K_n \) accounts for possible constants emerging from the inverse of the derivative of \( f(x) \). In effect, one arrives at

\[ xL''(x) + (\beta + 1 - x)L'(x) + \left(\frac{4\beta^2 - 4l(l+1) - 1}{4x} + \frac{\epsilon}{4} - \frac{\beta}{2} - \frac{1}{2}\right)L(x) = 0. \]

(3.33)

Now we make use of the freedom in \( \beta \) to nullify the singularity by setting \( \beta = l + \frac{1}{2} \) and thus, we are left with the equation for the associated Laguerre polynomials, equation (2.23). Therefore

\[ \frac{\epsilon}{4} - \frac{\beta}{2} - \frac{1}{2} = n, \quad 4\beta^2 - 4l(l+1) - 1 = 0, \]

(3.34)

allows to identify Eq. (3.33) with the equation for the associated Laguerre polynomials,

\[ xL''(x) + (l + \frac{1}{2} + 1 - x)L'(x) + nL_n(x) = 0, \]

(3.35)
whose solutions are \( L_n(x) = L_n^{(l + \frac{1}{2})}(x) \). In effect, the radial part of the Schrödinger wave function for the 3D oscillator is given as

\[
g_n(x) = N_n \sqrt{x^{l+\frac{1}{2}}e^{-x}} L_n^{(l + \frac{1}{2})}(x)/\sqrt{d} \sqrt{x}.
\]

(3.36)

Back to the \( z \) variable, the Schrödinger wave function is now obtained in its final form as

\[
\psi_n(z) = \frac{R(z)}{z} = K_n z^l e^{-z^2} L_n^{(l + \frac{1}{2})}(z^2).
\]

(3.37)

The orthogonality integral between the wave functions recovers the orthogonality between the Laguerre polynomials according to

\[
\int_0^{\infty} R_n(z) R_n(z) dz = \int_0^{\infty} K_n K_{n'} \sqrt{x^{l+\frac{1}{2}}e^{-x}} L_n^{(l + \frac{1}{2})}(x) \sqrt{x^{l+\frac{1}{2}}e^{-x}} L_{n'}^{(l + \frac{1}{2})}(x) dx = \delta_{nn'},
\]

(3.38)

where \( dz = d\sqrt{x} \). The energies are now given by \( \epsilon_n = 2(2n + l + \frac{3}{2}) \). The reason for which the associated, and not the ordinary Laguerre polynomials appeared in the solution of the 3D oscillator is that the angular momentum, \( l \), requires a polynomial parameter.

### 3.4.2 The hydrogen atom.

In this subsection we study the hydrogen atom, which is a two-body system consisting of one proton and one electron. The interaction is governed by the Coulomb potential,

\[
V(r) = -\frac{Ze^2}{r}, \quad Z = 1,
\]

(3.39)

as displayed in Fig. 3.3. Substituting the latter equation for the potential in Eq. (3.10) leads to

\[
-\frac{d^2}{dz^2} R(z) + \left( \frac{ll + 1}{z^2} - \frac{2}{z} - \epsilon \right) R(z) = 0, \quad z = rd,
\]

(3.40)

where the length scale \( d \), has been chosen as the Bohr radius, \( d = \hbar^2/\mu e^2 \), and \( \epsilon = 2Ed/e^2 \). We are going to solve this equation by means of an appropriate variable substitution. In order to find it we first study the asymptotic behavior of the solutions at both origin and infinity. For finite \( z \) the latter equation is equivalently rewritten as

\[
-\frac{d^2}{dz^2} R(z) + \left( \frac{ll + 1}{z} - 2 - \epsilon z \right) R(z) = 0,
\]

(3.41)

Figure 3.3: Coulomb potential (dashed line), \( V(r) = -\frac{Ze^2}{r} \). The effective potential (solid line), \( V_{eff}(r) = -\frac{Ze^2}{r} + \frac{l(l+1)}{r^2} \) includes the centrifugal barrier. Here \( Ze^2 = 10 \) and \( l = 1 \).
while near the origin, where \( R(0) = 0 \), and \( 1/z \ll 1/z^2 \) one finds

\[
\left( \frac{d^2}{dz^2} - \frac{(l+1)}{z^2} \right) R(z) \approx 0.
\]  

(3.42)

The latter equation can be solved by a function of the type, \( R(z) = z^t \). This would restrict \( t = l+1 \), and \( t = -l \), respectively. The second solution is not acceptable because it is not finite at the origin, and we are left with the first one. Next we study the \( z \to \infty \) asymptotic solution. In this case Eq. (3.40) becomes

\[
-\frac{d^2}{dz^2} R(z) = \epsilon R(z).
\]  

(3.43)

As long as we here are interested in the bound states, \( \epsilon < 0 \), which amounts to

\[
\frac{d^2}{dz^2} R(z) - |\epsilon| R(z) = 0.
\]  

(3.44)

The latter equation is solved by

\[
R(z) = Ce^{\sqrt{|\epsilon|}z} \quad \text{and} \quad R(z) = Ce^{-\sqrt{|\epsilon|}z}.
\]

As long as we want the wave function to go to zero for large \( z \), the first solution has to be dismissed. Next one changes variables in Eq. (3.40) according to

\[
z = f(x) = \frac{x}{\kappa}, \quad \kappa = 2\sqrt{|\epsilon|}, \quad R\left(\frac{x}{\kappa}\right) = g(x),
\]  

(3.45)

where the factor 2 has been taken for convenience. This yields

\[
\frac{d^2}{dx^2} g(x) + \left( -\frac{l(l+1)}{x^2} + \frac{1}{\sqrt{|\epsilon|}x} - \frac{1}{4} \right) g(x) = 0.
\]  

(3.46)

Taking into account the correct asymptotic behavior revealed above, the solution of the latter equation can be assumed as \( g(x) = x^{l+1} e^{-x/2} L(x) \). In result, one arrives at

\[
xL''(x) + (2l+1+1-x)L'(x) + L(x) \left( \frac{1}{\sqrt{|\epsilon|}} - l - 1 \right) = 0,
\]  

(3.47)

which coincides with the equation (3.35) for the associated Laguerre polynomials provided \( \alpha = 1 \), \( \beta = 2l+1 \), and \( \lambda = n \). With this in mind it is easy to verify that

\[
\epsilon_{nl} = -\frac{1}{(n + l + 1)^2}.
\]  

(3.48)

Recalling the definition of \( (\epsilon = 2Ed/e^2) \), we obtain the spectrum as

\[
E_{nl} = -\frac{e^4\mu}{2\hbar^2(n + l + 1)^2}.
\]  

(3.49)

At that stage one defines \( N = n + l + 1 \) as the principal quantum number, and realizes that while \( n \) can be any non-negative integer, the angular momentum is restricted to a finite number of values according to \( l = N - 1 - n \),

\[
l = 0, 1, 2, \ldots, N - 1.
\]  

(3.50)

In effect, one finds the well known degeneracy patterns in the spectrum of the hydrogen atom, \( E_{nl} = -\frac{e^4\mu}{2\hbar^2 N^2} \), meaning that \( \kappa \) depends on \( n \). The solution for the radial equation (3.47) is then concluded as

\[
\frac{g(x)}{x} \equiv \psi_{nl}(x) = C_{nl}x^l e^{-x/2} L_{N-l-1}^{(2l+1)}(x),
\]  

(3.51)

where use has been made of Eq. (3.9). The latter expression equivalently rewrites to

\[
\psi_{nl}(x) = C_{nl}x^{2l+1} e^{-x/2} L_n^{(2l+1)}(x)/\sqrt{x}.
\]  

(3.52)
Therefore, \( \psi_{n\ell}(x) \), is given in terms of the associated Laguerre polynomials, \( \{L_n^{(2\ell+1)}(x)\} \), with \( C_{n\ell} \) being the normalization constant. However, the wave function under consideration is not of the form in Eq. (3.13) because \( \sqrt{x} \neq (x/\kappa)' \). The consequence will be that the orthogonality integral of the wave functions

\[
\int_0^\infty C_{n\ell} C_{n'\ell} \sqrt{(\kappa_n z)^{2\ell+1} e^{-\kappa_n z}} L_n^{(2\ell+1)}(\kappa_n z) \sqrt{(\kappa_{n'} z)^{2\ell+1} e^{-\kappa_{n'} z}} L_{n'}^{(2\ell+1)}(\kappa_{n'} z) z^2 \frac{dz}{z} = \delta_{nn'},
\]

(3.53)
does not coincide with the orthogonality integral of the Laguerre polynomials with free parameters because it contains the additional first power of \( z \). It is important to be aware of the fact Eq. (3.53) describes orthogonality between states bound within different potentials corresponding to different factors, \( \frac{2z^2}{2\sqrt{|\kappa_n|}} \) versus \( \frac{2z^2}{2\sqrt{|\kappa_{n'}|}} \), of \( 1/x \) in Eq. (3.36). In the textbook on Mathematical Methods in Physics by Arfken and Weber (second reference in [31]) this phenomenon has been also attributed to the dependence of \( \kappa \) on the degree of the polynomial via the energy (see Exercise 13.2.11 there). As we shall see below, such a behavior is much more general and can occur also when \( x \) is neat but the parameters carry an \( n \) dependence.

### 3.5 Jacobi polynomials in Rosen-Morse II.

The Jacobi polynomials appear in several physics problems ranging from classical electrodynamics to quantum mechanics. We here focus on the hyperbolic Rosen-Morse potential given by

\[
v(z) = a^2 + \frac{b^2}{a^2} - a(a+1) \text{sech}^2(z) + 2b \tanh(z),
\]

(3.54)

and displayed in Fig. 3.4. The corresponding 1D Schrödinger equation is

\[
\frac{d^2 R(z)}{dz^2} + \left( \frac{b^2}{a^2} + a(a+1)(1 - \tanh^2(z)) - 2b \tanh(z) + e \right) R(z) = 0,
\]

(3.55)

with \( e = a^2 - \epsilon \), and \( \text{sech}^2(z) = 1 - \tanh^2(z) \). Changing variable to \( x = \tanh(z) \) meaning \( z = f(x) \) with \( f(x) = \tanh^{-1}(x) \), results in

\[
(1 - x^2) \frac{d^2 g(x)}{dx^2} - 2x \frac{dg(x)}{dx} + \left( \frac{b^2}{a^2(1 - x^2)} + a(a+1) - \frac{2bx + e}{1 - x^2} \right) g(x) = 0,
\]

(3.56)

Figure 3.4: The Rosen Morse II (hyperbolic) potential, \( V(r) = a^2 + \frac{b^2}{a^2} - a(a+1)\text{sech}^2(r) + 2b\tanh(r) \). Here \( a = 10 \) and \( b = 10 \).
with \( g(x) \) being defined as \( g(x) = R(\tanh^{-1} x) \). By means of the substitution, \( g(x) = (1 + x)^{\frac{3}{2}} (1 - x)^{\frac{1}{2}} P(x) \), the latter equation becomes

\[
(1 - x^2) \frac{d^2 P(x)}{dx^2} + \left( a(n + 1) - \frac{\alpha(1 + \beta)}{2} - \frac{\alpha^2}{4} - \frac{\beta}{2} - \frac{\beta^2}{4} + \frac{x(2b - \alpha^2 + \beta^2) + (\frac{\beta^2}{2} - \frac{\alpha^2}{2} - \frac{\beta^2}{4})}{x^2 - 1} \right) P(x) = 0,
\]

(3.57)

where \( \alpha, \beta \) are free parameters to be used to simplify the equation. Specifically, one makes use of the freedom in \( \alpha, \beta \) to nullify the singular term which restricts the parameters to:

\[
2b - \frac{\alpha^2}{2} + \frac{\beta^2}{2} = 0,
\]

(3.58)

\[
\frac{b^2}{a^2} - \frac{\alpha^2}{2} - \frac{\beta^2}{4} - e = 0.
\]

(3.59)

Next one requires the constant multiplying \( P(x) \) to be of the form

\[
a(n + 1) - \frac{\alpha(1 + \beta)}{2} - \frac{\alpha^2}{4} - \frac{\beta}{2} - \frac{\beta^2}{4} = \lambda_n = n(1 + n + \alpha + \beta).
\]

(3.60)

The latter equations are resolved by

\[
\beta = a - n + \frac{b}{n - a} \equiv \mu_n,
\]

(3.61)

\[
\alpha = a - n - \frac{b}{n - a} \equiv \nu_n,
\]

(3.62)

and

\[
e_n = \frac{b^2}{a^2} - (a - n)^2 - \frac{b^2}{(a - n)^2}.
\]

(3.63)

With that, equation (3.57) can be identified with the Jacobi form of the generalized hypergeometric equation,

\[
(1 - x^2) \frac{d^2 P(x)}{dx^2} + \left( a - \beta - x(2 + \alpha + \beta) \right) \frac{dP(x)}{dx} + n(n + \alpha + \beta + 1)P(x) = 0,
\]

(3.64)

whose solutions are the Jacobi polynomials \( \{ P_n^{(\alpha, \beta)}(x) \} \). In effect, the hyperbolic Rosen-Morse potential is solved exactly by

\[
g_n(x) = \sqrt{(1 - x)^{\mu_n}(1 + x)^{\nu_n}} P_n^{(\mu_n, \nu_n)}(x),
\]

(3.65)

and in accord with Table 1.1 (when translated to our notations). This wave function is not of the form in Eq. (3.13). As a consequence, the orthogonality integral between the wave functions does not recover the orthogonality between the Jacobi polynomials with free parameters as visible from

\[
\int_{-\infty}^{+\infty} N_n N_{n'} R_n(z) R_{n'}(z) dz = \int_{-1}^{+1} \sqrt{(1 - x)^{\mu_n}(1 + x)^{\nu_n}} P_n^{(\mu_n, \nu_n)}(x)
\]

\[
\times \sqrt{(1 - x)^{\mu_{n'}}(1 + x)^{\nu_{n'}}} P_{n'}^{(\mu_{n'}, \nu_{n'})}(x) d\tanh^{-1}(x).
\]

(3.66)

The culprits for this are the \( n \) dependent polynomial parameters. This is not to remain the only example for such an anomalous behavior.
3.6 The Bessel polynomials in spherical waves phenomena.

The Bessel polynomials are special in the sense that their orthogonality is achieved by integration over a contour in the complex plane. In the context of quantum mechanics an infinite barrier (Fig. 3.5) of the type $Ae^{az}$ ($A, a$ being constants) will lead to Bessel's differential equation. Using this barrier in Eq. (3.10) for the case $l = 0$ results into

$$\frac{-d^2R(z)}{dz^2} + (\epsilon + Ae^{az})R(z) = 0. \quad (3.67)$$

Changing variables to $x = e^{-az}$ leads to

$$x^2 \frac{d^2g(x)}{dx^2} + x \frac{dg(x)}{dx} - \left(\epsilon + \frac{b}{x}\right)g(x) = 0, \quad (3.68)$$

A substitution of the type $g(x) = e^{-\frac{x}{2}}x^{\frac{1}{2}}y(x)$ leads to

$$x^2 \frac{d^2y(x)}{dx^2} + 2(x + 1) \frac{dy(x)}{dx} + \left(\frac{1 - 4e}{4} + \frac{4 - 4b}{4x^2}\right)y(x) = 0. \quad (3.70)$$

If the singular term is to vanish, then

$$\frac{1 - 4e}{4} = -n(n + 1). \quad (3.72)$$

From the above conditions we find $b = 1$, $\epsilon = \frac{4}{n} + n(n + 1)$ and Eq. (3.70) becomes the Bessel differential equation:

$$x^2 \frac{d^2y(x)}{dx^2} + 2(x + 1) \frac{dy(x)}{dx} = n(n + 1)y(x). \quad (3.73)$$

Here, $y(x)$ stand for the Bessel polynomials. Take notice that contrary to the previous examples the barrier potential is exactly solvable only when the parameters have been fixed by the condition $\frac{d}{r} = 1$. This is an example for a potential that is exactly solvable when the parameters obey constraints. The Bessel polynomials have been brought to attention by Krall and Frink [48] and are related to the following wave equation (in spherical coordinates)

$$\frac{1}{r^2} \left( r \frac{\partial^2}{\partial r^2}(ru(r, \theta, \varphi, t)) \right) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial u(r, \theta, \varphi, t)}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 u(r, \theta, \varphi, t)}{\partial \varphi^2} = \frac{1}{c^2} \frac{\partial^2 u(r, \theta, \varphi, t)}{\partial t^2}. \quad (3.74)$$

Figure 3.5: A barrier $V(r) = Ae^{ar}$ that leads to a Schrödinger equation with solutions in terms of Bessel polynomial’s and for the toy values $a = -10$, and $A = 1$.  

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If the latter equation is solved by separation of variables, the radial part \( f(r) \) is found to satisfy the following differential equation:

\[
\frac{r^2 d^2 f(r)}{dr^2} + 2r \frac{df(r)}{dr} + k^2 r^2 f(r) = n(n+1)f(r).
\] (3.75)

Here \( kc = \omega \), and \( e^{i\omega t} \) is a plane wave. For \( r = ks \), Eq. (3.75) becomes

\[
s^2 \frac{d^2 \tilde{f}(s)}{ds^2} + 2s \frac{d\tilde{f}(s)}{ds} + s^2 \tilde{f}(s) = n(n+1)\tilde{f}(s), \quad \tilde{f}(s) = f \left( r = \frac{s}{k} \right).
\] (3.76)

The latter equation can be transformed into Bessel’s equation by the substitution \( f(s) = s^{-\frac{1}{2}} J(s) \), which yields:

\[
s^2 \frac{d^2 J(s)}{ds^2} + s \frac{dJ(s)}{ds} + s^2 J(s) = (n + 1/2)^2 J(s).
\] (3.77)

For integer \( n \), the solutions of Eq. (3.77) are the Bessel functions of half-integral order and are well known. However, Eq. (3.77) can also be subjected to the transformation \( f(s) = w(s)/s \), giving

\[
s^2 \left( \frac{d^2 w(s)}{ds^2} + w(s) \right) = n(n+1)w(s).
\] (3.78)

Upon introducing the variable, \( z = ikr = is \), imaginary and real parts of the solution represent traveling waves. Because of that we admit in Eq. \( 3.78 \) \( z = is = ikr \), and \( w(s) = e^{-is} y(z) = e^{-i\omega} y(is) \), and obtain

\[
z^2 \left( \frac{d^2 y(z)}{dz^2} - 2 \frac{dy(z)}{dz} \right) = n(n+1)y(z).
\] (3.79)

For integer values of \( n \), Eq. (3.79) has solutions which are polynomials in \( 1/z \). Therefore the final substitution should be, \( x = 1/z = 1/ikr \), which allows to obtain the differential equation for the Bessel polynomials \( \{y_n(x)\} \),

\[
x \frac{d^2 y_n(x)}{dx^2} + (2x+2) \frac{dy_n(x)}{x} = n(n+1)y_n(x).
\] (3.80)

The full solution of the spherical wave equation is then given by

\[
u(r, \theta, \varphi, t) = r^{-1} P_\alpha^m (\cos(\theta)) \sin(m\varphi - \alpha) e^{i(\omega t - kr)} y_n(1/ikr),
\] (3.81)

where \( y_n(x) = y_n(1/ikr) \) is a Bessel polynomial, and \( kc = \omega \). The real and imaginary parts of (3.81) describe waves traveling in the radial direction with velocity \( c \). In conclusion, spherical waves are equivalently described either in terms of a class of polynomials orthogonal over the unit circle, or in the standard way in terms of Bessel functions.

### 3.7 Romanovski polynomials in Rosen-Morse I.

As already mentioned in the introduction, the Table 1.1 contains one more potential whose exact solutions require the Romanovski polynomials and this is the trigonometric Rosen-Morse potential (Rosen-Morse I). This case has been considered in great detail in Refs. [39, 40] and will not be repeated here. Instead, I would prefer to briefly review the main properties of its solutions within the context of its relevance in quark physics (the presentation on this section closely follows Ref. [37]). This aspect appears especially important to me because the Romanovski polynomials were found for the first time in the solutions of that very potential while searching to construct a quark model that matches reality on nucleon excitations as part of the research project “Dynamics of baryon resonances” run by our group. The subject of the present thesis continues the study of the Romanovski polynomials started in Refs. [39, 40] and extends knowledge on their properties by various new observations.

The great appeal of Rosen-Morse I is that
besides the Coulomb potential, the trigonometric Rosen-Morse potential is to the best of our knowledge the only exactly solvable potential that relates to a fundamental massless gauge theory.

Recall that the Coulomb potential is no more but the image in coordinate space of the propagator, \(-1/q^2\), of the photon, the U(1) gauge boson of electrodynamics, as it appears in elastic scattering of charged particles. In a similar way, as it will be argued below, Rosen-Morse I can be viewed as the image in coordinate space of the propagator of the gluons, the \(SU(3)_c\) gauge fields of the fundamental field theory of strong interactions, the Quantum Chromodynamics (QCD), as it appears in elastic scattering of quarks. Within this context, the Romanovski polynomials acquire the special status of major ingredients of the wave functions of bound quarks.

The quarks, as is well known, are the constituents of strongly interacting particles, baryons, and mesons. In what follows, the presentation will be focused on the baryons constituted by the so called light flavors, \(u\), \(d\), and \(s\), as are the nucleon \((N)\), the \(\Delta\), the \(\Lambda\), and their resonant excitations \((\Sigma, \Xi)\). The nucleon is understood as a particle which can exist in two different states distinguished by their electric charges, the proton, \(p(\text{uud})\), and the neutron, \(n(\text{udd})\), and is said to be a charge-doublet. The \(\Delta\) stands for a particle that can exist in four different states distinguished by their charges which are \(\Delta^{++}(\text{uuu})\), \(\Delta^+ (\text{uud})\), \(\Delta^0 (\text{udd})\), and \(\Delta^- (\text{ddd})\) and is termed to as a charge-quadruplet. The \(\Lambda(uds)\) particle is neutral and a charge singlet. There are also charge triplets like \(\Sigma\) represented by \(\Sigma^+(\text{uus})\), \(\Sigma^0(\text{uds})\), and \(\Sigma^-(\text{dds})\), and one more charge-doublet given by the \(\{\Sigma^- (\text{dds}), \Sigma^0 (\text{uus})\}\) family. Compared to the \(N\)-, \(\Delta\)-, and \(\Lambda\)-charge-multiplets, the \(\Sigma\) and \(\Xi\) are less known and will be left out of consideration in the following.

The problem which one is facing with the baryon resonances is the lack of an adequate systematics and the resulting deficits as the prediction of a large number of unobserved states. In the standard quark models, different charge-multiplets are supposed to join to bigger families, the so called \(SU(6)_S \times O(3)_L\) super-multiplets as displayed in Fig. 3.6. This figure reveals a strong overlap between the different super-multiplets and an apparent lack of degeneracy between the states belonging to same multiplet thus questioning the adequacy of the underlying classification scheme. In Refs. \([52]\) the \(SU(6)_S \times O(3)_L\) baryon classification scheme has been given up, the super-multiplets have been decomposed into \(N\)-, \(\Delta\)-, and \(\Lambda\)-spectra and have then have been studied separately.

The result was that to a surprisingly good accuracy, the nucleon excitation levels carry the same degeneracies as the levels of the electron with spin in the hydrogen atom, though the splittings of the former are quite different from those of the latter. Namely, compared to the hydrogen atom, the baryon level splittings contain in addition to the Balmer term also its inverse but of opposite sign. Same patterns are repeated by the excitation spectrum of the \(\Delta(1232)\) particle, the most important baryon after the nucleon (see Figs. 3.7,3.8). In this way baryons have been classified according to \(SU(2)_I \times O(4)\) with \(I\) standing for isospin (a number that encodes the dimensionality, \(D\), of the charge multiplets as \(D = 2I + 1\)). The appeal of the new classification scheme lies in the fact that no states drop out of the systematics, on the one side, and that the number of unobserved (“missing”) states predicted by it is significantly smaller than those of all preceding schemes.

The observed degeneracies in the spectra of the light quark baryons have been attributed in Ref. \([52]\) to the dominance of a quark–antiquark configuration in baryon structure. Within the light of these findings, the form of the potential in configuration space acquires crucial importance.

In Refs. \([49],[53]\) a case was made that precisely the trigonometric Rosen-Morse potential provides the degeneracies and level splittings that are required by the light quark baryon spectra.

The success of the trigonometric Rosen-Morse potential in quark physics is not accidental. It is due to the property of the latter to interpolate between the Coulomb- and the infinite well potentials. In order to understand this virtue one has to recall the QCD basics.
Indeed, the strong interaction between quarks within QCD is governed by interchanges of massless gauge bosons between quarks of three different “colors”, (“red”, “blue”, “green”). The above gauge theory predicts that the quark interactions proceed over one- or many gluon exchanges including gluon self-interactions (so called “non-Abelian” gauge theory). The latter are believed to be responsible for the so-called quark confinement, where highly energetic quarks remain trapped but behave as (asymptotically) free at small distances. The QCD equations are nonlinear and complicated due to the gluonic self-interaction processes. Their solution requires employment of highly sophisticated techniques, such as discretization of space time, so-called lattice QCD. Lattice QCD calculations of the properties of hadrons associates the one-gluon exchange with a Coulomb like, \(1/r\), potential, and predicts a linear confinement potential with increasing energy brought about by gluon-self interactions. The trigonometric Rosen-Morse potential in the parametrization of Ref. [53] where the parameter has been identified with \(l\), the relative quark–di-quark angular momentum (in units of \(\hbar\)) as visible from polynomials with free parameters

\[
\int_0^{2\pi} \frac{1}{\sin^2(z)} dz = \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \pi.
\]

This expansion clearly reveals the proximity of the \(\cot(z)\) term to the Coulomb– plus linear confinement potential, and the proximity of the \(\csc^2(z)\) term to the standard centrifugal barrier.

In this sense, Rosen-Morse I can be viewed as the image of space-like gluon propagation in coordinate space.

The great advantage of the trigonometric Rosen-Morse potential over the linear– plus Coulomb potentials is that while the latter is neither especially symmetric, nor exactly soluble, the former is both, it has the dynamical \(O(4)\) symmetry (as the hydrogen atom) and is exactly soluble. The exact solutions of the, now three dimensional, Schrödinger equation with \(v(t_{RM}(z))\) from Eq. (3.82) have been constructed in [53] on the basis of the one-dimensional solutions found in [49] and read:

\[
\psi_n(\cot^{-1}x) = (1 + x^2)^{-\frac{n+1}{2}} e^{-\frac{b}{n+1} \cot^{-1}(x)} C_n^{(- n + l + 1)}(x)C_n^{(- n + 1)}(x),
\]

with \(x = \cot(z)\). The \(C\) polynomials from [49] are Romanovski polynomials but with running parameters attached to the degree of the polynomial. The notations of Ref. [49] translate to the present ones as:

\[
C_n^{(- n + l + 1)}(x) = R_n^{(p_n, q_n)}(x), \quad q_n = -\frac{2b}{n + l + 1}, \quad p_n = n + l + 1, \quad n = 0, 1, 2, ...
\]

The wave function in Eq. (3.83) is not of the type in Eq. (3.13), a reason for which the orthogonality integral between the wave functions will not recover the orthogonality of the Romanovski polynomials with free parameters as visible from

\[
\int_0^{\pi} \psi_n(z)\psi_{n'}(z) dz = \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx = \delta_{n,n'},
\]

where the factor \(\frac{1}{1 + x^2}\) in the integrand comes from \(\frac{d\cot^{-1}(x)}{dx} = -1/(1 + x^2)\). In this way, the Romanovski polynomials that enter the solutions of the Schrödinger equation with the trigonometric
Rosen-Morse potential seem to disobey the finite orthogonality prescription. A discussion of this behavior will be given below. Finally, the associated energy spectrum is found as

$$\epsilon_n = (n + l + 1)^2 - \frac{b^2}{(n + l + 1)^2}. \quad (3.87)$$

Therefore, the Romanovski polynomials have been shown in Refs. [49, 53] to be important ingredients of the wave functions of quarks designed in accord with QCD quark-gluon dynamics.
Figure 3.6: Baryon resonances in the traditional quark model. Circles, bricks, and triangles stand for nucleon, Λ, and ∆ states, respectively. Different colors mark different $SU(6)_F \times O(3)_L$ multiplets. Notice the strong multiplet intertwining and the large mass separation inside the multiplets. (Courtesy M. Kirchbach)

Figure 3.7: The nucleon excitation spectrum below 2 GeV. (Courtesy M. Kirchbach).
Figure 3.8: The $\Delta$ excitation spectrum below 2 GeV. (Courtesy M. Kirchbach)

Figure 3.9: The trigonometric Rosen-Morse potential (solid line) and its proximity to the Coulomb– plus lineal potential as predicted by lattice QCD (thin dashed line) for the toy values $l = 1, b = 50$ of the parameters.
Chapter 4

Romanovski polynomials in Scarf II.

This chapter is devoted to the presentation of the original results obtained in the thesis. In the first section 4.1 I present the hyperbolic Scarf potential. The corresponding Schrödinger equation is solved in terms of the Romanovski polynomials and presented in subsection 4.1.1. The finite orthogonality of the Romanovski polynomials is discussed in section 4.2. Section 4.3 is devoted to the problem of an electron within a non-central potential, which is solved in spherical coordinates by separation of the variables. There, in sub-section 4.3.2 it is shown that the Romanovski polynomials solve exactly the polar angle equation and define new non-spherical angular functions. In subsection 4.3.3 a non-linear relationship between Romanovski polynomials and associated Legendre functions is established. Finally, in section 4.4 I solve the Klein-Gordon equation with scalar and vector potentials of same magnitudes and given by Scarf II.

4.1 The hyperbolic Scarf potential.

The hyperbolic Scarf potential can be viewed as the extension of the \( \text{sech}^2(r) \) potential, or, better, of the original Pöschl-Teller potential \[4\]. Take notice that in the SUSYQM nomenclature (presented in Table 1.1) the names of Pöschl and Teller are rather associated with the extended \( \text{csch}^2(r) \) (here marked as Pöschl-Teller 2). It seems that for the first time Scarf II has been constructed in Ref. \[55\]. It also has been encountered independently within the framework of the supersymmetric quantum mechanics \[13, 22, 23\] while exploring the superpotential

\[
U_m(x) = a \tanh(\alpha x) + b \text{sech}(\alpha x).
\]

Figure 4.1: The trigonometric Scarf potential (Scarf I) for the toy values of the parameters, \( a = 10, b = 5, \) and \( \alpha = 1. \) The horizontal lines represent the discrete levels.
On the other side, it can equally well be approached from the perspective of the trigonometric Scarf potential, here denoted by $V_t(z)$ and given by

$$V_t(z) = -a^2 + (a^2 + b^2 - a\alpha)\sec^2(\alpha z) - b(2a + \alpha)\tan(\alpha z)\sec(\alpha z). \quad (4.2)$$

The exact solution of the Schrödinger equation with the trigonometric Scarf potential (displayed in Fig. 4.1) can be obtained along the line of the concepts of the previous chapter. It is well known and given in terms of the Jacobi polynomials, $P_n^{(\beta,\delta)}(z)$, as [13],

$$\psi_n(z) = (1 - \sin(\alpha z))\binom{\alpha + b}{\alpha - a} P_n^{(\frac{\alpha - \beta}{2},\frac{\alpha - \beta}{2})}(\sinh(z))^{\alpha - \beta} J_n^{(\alpha + b - \frac{\alpha - \beta}{2},\alpha + b + \frac{\alpha - \beta}{2})}(z). \quad (4.3)$$

The corresponding energy spectrum is obtained as

$$e_n = (a + \alpha n)^2 - a^2. \quad (4.4)$$

The trigonometric Scarf potential can be transformed into its hyperbolic partner, the so-called hyperbolic Scarf potential, here denoted by $V_h(z)$ and given by

$$V_h(z) = a^2 + (b^2 - a^2 - a\alpha)\sech^2(\alpha z) + b(2a + \alpha)\sech(\alpha z)\tanh(\alpha z). \quad (4.5)$$

Figure 4.2 visualizes the hyperbolic Scarf potential and its discrete spectrum. The $V_h(z)$ potential has been obtained from $V_t(z)$ in performing the following substitutions in Eq. (4.2):

$$a \rightarrow ia, \quad \alpha \rightarrow -i\alpha, \quad \frac{a}{\alpha} \rightarrow -\frac{a}{\alpha}, \quad b \rightarrow b. \quad (4.6)$$

Upon the above substitutions the energy changes to

$$e_n = a^2 - (a - n\alpha)^2. \quad (4.7)$$

In the following we shall show that $n = 0, 1, 2, ... < a$ meaning that the number of bound states is finite. Yet, the most profound changes are suffered by the wave functions. Substitution of Eqs. (4.6) into Eq. (4.3) results in

$$\psi_n(-i\sinh(z)) = (1 + i\sinh(z))^{-\frac{\alpha}{2}} (1 - i\sinh(z))^{-\frac{\alpha}{2}} c_n P_n^{(ib+a-\frac{\alpha}{2},ib+a+\frac{\alpha}{2})}(-i\sinh(z)), \quad (4.8)$$

Figure 4.2: The hyperbolic Scarf potential $V_h(z) = a^2 + (b^2 - a^2 - a\alpha)\sech^2(\alpha z) + b(2a + \alpha)\sech(\alpha z)\tanh(\alpha z)$ with $a = 10$, $b = 5$ and $\alpha = 1$. Energy levels, $e_n = a^2 - (a - n\alpha)^2$, are included.
where \( c_n \) is some state dependent complex phase, and where we took \( \alpha = 1 \) for simplicity. The latter expression can be cast into the form frequently mentioned in the literature \([13, 22, 36, 45]\),

\[
\psi_n(-ix) = (1 + x^2)^{-\frac{1}{2}} e^{-\sigma \tan^{-1}(x)} c_n P_n^{(ib+a-b)}(-ix),
\]

\[
x = \sinh(z), \quad \rho = a, \quad \sigma = b.
\]

(4.9)

The latter equation gives the impression that the exact solutions of the hyperbolic Scarf potential rely upon Jacobi polynomials with complex indices and arguments.

In this thesis the case is made that this needs not be so and that the above wave functions can be expressed in terms of the real Romanovski polynomials.

### 4.1.1 The polynomial equation.

The Schrödinger equation for the potential of interest when rewritten in a new variable, \( x \), introduced via an appropriate point canonical transformation \([56, 57]\), taken by us as \( z = f(x) = \sinh^{-1}x \), is obtained as:

\[
(1 + x^2) \frac{d^2g(x)}{dx^2} + x \frac{dg(x)}{dx} + \left( \frac{-b^2 + a(a + 1)}{1 + x^2} - \frac{b(2a + 1)}{1 + x^2}x + \epsilon_n \right) g(x) = 0,
\]

(4.10)

with \( g(x) = \psi(\sinh^{-1}(x)) \). Inspired by Eq. (4.9) we now test the following substitution in Eq. (4.10)

\[
g(x) = (1 + x^2)^{\frac{3}{2}} e^{\frac{-i}{2} \tan^{-1}(x)} D^{(\beta, \alpha)}(x), \quad x = \sinh(z),
\]

(4.11)

In effect, Eq. (4.11) reduces to the following equation for \( D^{(\beta, \alpha)}(x) \),

\[
(1 + x^2) \frac{d^2D^{(\beta, \alpha)}(x)}{dx^2} + ((2\beta + 1)x - \alpha) \frac{dD^{(\beta, \alpha)}(x)}{dx} + \left( \beta^2 + \epsilon_n + \frac{(a + a^2 + \beta - \beta^2 - b^2 + \frac{a^2}{4}) + x(-b - 2ab + \frac{a^2}{2} - \alpha\beta)}{1 + x^2} \right) D^{(\beta, \alpha)}(x) = 0.
\]

(4.12)

Making use of the freedom in \( \alpha \) and \( \beta \), the coefficient in front of \( 1/(1 + x^2) \) may nullify,

\[
a + a^2 - b^2 + \frac{a^2}{4} + \beta - \beta^2 = 0,
\]

(4.13)

\[-b - 2ab + \frac{a^2}{2} - \alpha\beta = 0.
\]

(4.14)

Then, Eq. (4.12) reduces to the Romanovski equation (2.18). Finally, the identification of the constants in Eqs. (4.12) and (2.18) leads to a condition that defines the energy spectrum of the hyperbolic Scarf potential as

\[
\beta^2 + \epsilon_n = -n(2\beta + n).
\]

(4.15)

Resolving the three equations (4.13), (4.14), and (4.15) for \( \alpha \), \( \beta \) and \( \epsilon_n \) results in

\[
\beta = -a, \quad \alpha = 2b, \quad \epsilon_n = -(a - n)^2.
\]

(4.16)

This expression for the energy coincides with Eq. (4.17), as it should be. In this way it is proved that the \( D^{(\beta, \alpha)}(x) \) functions that enter the solution of the Schrödinger equation are equal to the Romanovski polynomials. Therefore, the \( D \) functions are polynomials. As a result, the wave functions in \( x \) space take the form

\[
g_n(x) = (1 + x^2)^{-\frac{3}{2}} e^{-b \tan^{-1}(x)} D_n^{(\beta, \alpha)}(x), \quad dx = \sqrt{1 + x^2} dz.
\]

(4.17)
The weight function from which the $D$ polynomials are obtained via the Rodrigues formula is

$$w^{(a+\frac{1}{2},-2b)}(x) = (1 + x^2)^{a-\frac{1}{2}}e^{-2bx}\tan^{-1}x. \quad (4.18)$$

The wave function is now equivalently rewritten to

$$g_n(x) = \sqrt{(1 + x^2)^{-a+\frac{1}{2}}e^{-2b\tan^{-1}(x)}}D_n^{(-a,2b)}(x)\frac{1}{\sqrt{\sinh^{-1}(x)}}, \quad (4.19)$$

and is of the type in Eq. (3.13). As a consequence, the orthogonality integral between the wave functions will recover the orthogonality between the polynomials as shown in Eq. (4.24) below. In order to relate the $\alpha$ and $\beta$ parameters to those of the Romanovski polynomials one can compare the coefficients in front of the first derivatives in the respective Eqs. (4.12), and (2.18),

$$2(-p + 1)x + q = (2\beta + 1)x - \alpha, \quad (4.20)$$
giving

$$\beta = -a = -p + \frac{1}{2}, \quad -\alpha = q = -2b. \quad (4.21)$$

In this way, the polynomials that enter the solution of the Schrödinger equation will be

$$D_n^{\beta=-a,\alpha=2b}(x) \equiv R_n^{(p=a+\frac{1}{2},q=-2b)}(x). \quad (4.22)$$

They are obtained by means of the Rodrigues formula from the weight function $w^{(a+\frac{1}{2},-2b)}(x)$ as

$$R_n^{(a+\frac{1}{2},-2b)}(x) = \frac{1}{w^{(a+\frac{1}{2},-2b)}(x)}\frac{dn}{dx^n}(1 + x^2)^nw^{(a+\frac{1}{2},-2b)}(x). \quad (4.23)$$

In this fashion, the hyperbolic Scarf potential has been solved in terms of the real Romanovski polynomials.

The orthogonality integral of the Schrödinger wave functions gives rise to the following orthogonality integral of the Romanovski polynomials,

$$\int_{-\infty}^{+\infty}g_n(x)g_n'(x)dx = \int_{-\infty}^{+\infty}(1 + x^2)^{-a+\frac{1}{2}}e^{-2b\tan^{-1}(x)}R_n^{(a+\frac{1}{2},-2b)}(x)R_n^{(a+\frac{1}{2},-2b)}(x)dx, \quad (4.24)$$

which coincides in form with the integral in Eq. (2.16) and is convergent for $n < a$.

In order to relate the result obtained by us to the current literature, it is quite instructive to compare Eq. (2.18) to the Jacobi equation,

$$(1 - x^2)\frac{d^2P_n^{(\gamma,\delta)}(x)}{dx^2} + (\gamma - \delta - (\gamma + \delta + 2)x)\frac{dP_n^{(\gamma,\delta)}(x)}{dx} - n(n + \gamma + \delta + 1)P_n^{(\gamma,\delta)}(x) = 0. \quad (4.25)$$

Upon complexification of the argument, $x \to ix$, the latter equation transforms into

$$(1 + x^2)\frac{d^2P_n^{(\gamma,\delta)}(ix)}{dx^2} + i(\gamma - \delta - (\gamma + \delta + 2)x)\frac{dP_n^{(\gamma,\delta)}(ix)}{dx} + n(n + \gamma + \delta + 1)P_n^{(\gamma,\delta)}(ix) = 0. \quad (4.26)$$

From a formal point of view, Eq. (4.26) can be made to coincide with Eq. (2.18) for the following parameters:

$$\gamma = -p - \frac{iq}{2}, \quad \delta = \gamma^*. \quad (4.27)$$

As long as identical equations have solutions that differ by at most a phase factor, the Romanovski polynomials are related to the complex Jacobi polynomials via

$$R_n^{(p,q)}(x) = i^nP_n^{(-\frac{1}{2},p+\frac{1}{2},-q)}(ix). \quad (4.28)$$
In this sense one relates in the literature the Jacobi polynomials of complex arguments and indices to the solutions of the hyperbolic Scarf potential. However, this relation is in our opinion misleading because the finite orthogonality makes the real orthogonal Romanovski polynomials \( \{ R_n^{(p,q)}(x) \} \) to a species that is fundamentally different from the complex \( \{ p_n^{(-p - \frac{i}{2}, -p + \frac{i}{2})}(ix) \} \).

The orthogonality properties of the complex Jacobi polynomials depend on the interplay between the integration contour and the parameter values and need special care \[58\]. Equation (4.28) in combination with Eqs. (2.16), and (2.17) in fact states that the contour over which Jacobi polynomials of the type \( \{ P_n^{(\eta, \eta^*)}(ix) \} \) are orthogonal is the real axis and not, as one naively would have expected, the finite interval \([−i, i]\).

4.2 Polynomial construction and finite orthogonality.

The construction of the \( R_n^{(a + \frac{1}{2}, -2b)}(x) \) polynomials needed in the exact solutions of Scarf II is now straightforward and based upon the Rodrigues representation in Eq. (2.14) where we plug in the weight function from Eq. (4.18). In carrying out the differentiations we find the lowest four (unnormalized) polynomials as

\[
\begin{align*}
R_0^{(a + \frac{1}{2}, -2b)} &= 1, \quad (4.29) \\
R_1^{(a + \frac{1}{2}, -2b)}(x) &= -2b + (1 - 2a)x, \quad (4.30) \\
R_2^{(a + \frac{1}{2}, -2b)}(x) &= 3 - 2a + 4b^2 - 8b(1 - a)x + (6 - 10a + 4a^2)x^2, \quad (4.31) \\
R_3^{(a + \frac{1}{2}, -2b)}(x) &= -266 + 12ab - 8b^3 + [-3(-15 + 16a - 4a^2) + 12(3 - 2a)b^2]x \\
&\quad + (-72b + 84ab - 24a^2b)x^2 + 2(-2 + a)(-15 + 16a - 4a^2)x^3. \quad (4.32)
\end{align*}
\]

Figure 4.3: The wave function \( \psi_1(z) \).

The finite orthogonality of the above polynomials has been proved among others in Refs. \[27, \[57\]. We here rather shall illustrate this property in terms of the polynomial normalization constants which becomes especially transparent in the interesting limiting case of the \( \text{sech}^2(z) \) potential (it appears in the non-relativistic reduction of the sine-Gordon equation (c.f. \[54\])) where one easily
finds that the normalization constant, \( N_n^{(a+\frac{1}{2},0)} \), is given by

\[
\begin{align*}
(N_1^{(a+\frac{1}{2},0)})^2 &= \frac{(2a-1)^2\sqrt{\pi}\Gamma(a-1)}{2\Gamma(a+\frac{1}{2})}, \quad a > 1, \\
(N_2^{(a+\frac{1}{2},0)})^2 &= \frac{2\sqrt{\pi}(a-1)\Gamma(a-2)}{\Gamma(a-\frac{3}{2})}(3-2a)^2, \quad a > 2, \\
(N_3^{(a+\frac{1}{2},0)})^2 &= \frac{3\sqrt{\pi}(a-2)\Gamma(a-3)}{\Gamma(a-\frac{5}{2})}(4a^2-16a+15)^2, \quad a > 3 \text{ etc.}
\end{align*}
\]

(4.33)

Using symbolic softwares such as Maple and Mathematica is quite useful in verifying the results reported here. The latter expressions show that for positive integer values of the \( a \) parameter, \( a = n \), only the first \( (n-1) \) Romanovski polynomials are orthogonal, as it should be in accord with Eq. (2.17), and the comment after (4.7). The general expressions for the normalization constants of any Romanovski polynomial are defined by integrals of the type \( \int_{-\infty}^{\infty} (1+x^2)^{n-p} e^{q\tan^{-1}(x)} dx \) and are analytic for \( (n-p) \) integer or half-integer.
4.3 Romanovski polynomials and non–spherical angular functions.

The Romanovski differential equation appears in the problem of a particle within a non-central scalar potential, a result that can be concluded form Ref. [43]. In denoting such a potential by \( V(r, \theta) \), one can make for it the specific choice of

\[
V(r, \theta) = V_1(r) + \frac{V_2(\theta)}{r^2}, \quad V_2(\theta) = -c \cot(\theta).
\]

An interesting problem is the electrostatic non-central potential in which case \( V_1(r) \) is the Coulomb potential. The corresponding Schrödinger equation,

\[
\left[-\frac{\hbar^2}{2\mu} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2}\right] + V(r, \theta) \right] \Psi(r, \theta, \varphi) = E \Psi(r, \theta, \varphi),
\]

is solved in the standard way by separating variables,

\[
\Psi(r, \theta, \varphi) = \mathcal{R}(r) \Theta(\theta) e^{im\varphi}.
\]

The radial and angular differential equations for \( \mathcal{R}(r) \) and \( \Theta(\theta) \) are then found as

\[
\frac{d^2 \mathcal{R}(r)}{dr^2} + \frac{2}{r} \frac{d\mathcal{R}(r)}{dr} + \left[ \frac{2\mu}{\hbar^2} (V_1(r) + E) - \frac{l(l+1)}{r^2} \right] \mathcal{R}(r) = 0, \tag{4.37}
\]

and

\[
\frac{d^2 \Theta(\theta)}{d\theta^2} + \cot(\theta) \frac{d\Theta(\theta)}{d\theta} + \left[ l(l+1) - \frac{2\mu V_2(\theta)}{\hbar^2} - \frac{m^2}{\sin^2(\theta)} \right] \Theta(\theta) = 0, \tag{4.38}
\]

with \( l(l+1) \) being the separation constant.

4.3.1 Radial equation.

The radial equation \( \frac{d^2 \mathcal{R}(r)}{dr^2} \), in the \( \rho \) variable introduced as \( r \equiv \sqrt{\frac{\hbar^2}{2\mu|E|}} \rho \) and for \( E < 0 \) (bound states) reads

\[
\frac{2}{\rho} \frac{d \mathcal{R}(\rho)}{d\rho} + \frac{d^2 \mathcal{R}(\rho)}{d\rho^2} - \frac{l(l+1)}{\rho^2} \mathcal{R}(\rho) + \left( \frac{k}{\rho} - \frac{1}{4} \right) \mathcal{R}(\rho) = 0, \tag{4.39}
\]

with

\[
k \equiv \frac{Ze^2}{\hbar} \sqrt{\frac{\mu}{2|E|}}. \tag{4.40}
\]

This differential equation is identical to the radial equation of the Schrödinger equation for the hydrogen atom. For this reason, the substitution \( \mathcal{R}(\rho) = e^{-\frac{\rho}{2}} \rho^k \mathcal{G}(\rho) \) seems convenient. It leads to

\[
\rho \frac{d^2 \mathcal{G}(\rho)}{d\rho^2} + [(2l+2) - \rho] \frac{d \mathcal{G}(\rho)}{d\rho} + (k-l-1) \mathcal{G}(\rho) = 0, \tag{4.41}
\]

which coincides with the associated Laguerre differential equation,

\[
x \frac{d^2 L(x)}{dx^2} + (\beta + 1 - x) \frac{dL(x)}{dx} + \lambda L(x) = 0, \tag{4.42}
\]

upon setting \( \beta = 2l+1 \). For \( \lambda \), the following condition holds valid,

\[
\lambda_{n_r} = -n_r \left( N \frac{dL_{n_r}(x)}{dx} + \frac{1}{2} (n_r - 1) \frac{d^2 \sigma(x)}{dx^2} \right), \tag{4.43}
\]

which gives \( \lambda_{n_r} = k - l - 1 = n_r \). This implies \( k = n_r + l + 1 \), and with \( k \) from Eq. (4.40), the energy is obtained as

\[
E_{n_r,l} = -\frac{Z^2 e^4 \mu}{2\hbar^2} \frac{1}{(n_r + l + 1)^2}. \tag{4.44}
\]
4.3.2 Angular equation.

Before proceeding further we wish to notice that for $V_2(\theta) = 0$, and upon changing variables from $\theta$ to $\cos(\theta)$, Eq. (4.38) transforms into the associated Legendre equation and correspondingly

$$\Theta(\theta) \xrightarrow{V_2(\theta)\rightarrow0} P_l^m(\cos(\theta)),$$

an observation that will become important below. Following Ref. [43] one begins by substituting the polar angle variable by a new variable, $z$, introduced via $\theta \equiv f(z)$. This leads to the new equation

$$\left[ \frac{d^2}{dz^2} + \left( -\frac{f''(z)}{f'(z)} + f'(z) \cot(f(z)) \right) \right] \frac{d}{dz} \left[ -\frac{2\mu}{\hbar^2} V_2(f(z)) + l(l+1) - \frac{m^2}{\sin^2(f(z))} \right] f'(z) \psi(z) = 0,$$

(4.45)

with $f'(z) \equiv \frac{df(z)}{dz}$, and $\psi(z)$ defined as $\psi(z) \equiv \Theta(f(z))$. Next one can require that $f'(z)$ approaches zero at $z = 0$ like $\sin(z)$, meaning, $\lim_{z \rightarrow 0} \frac{f'(z)}{\sin(z)} \rightarrow 1$, and define $f(z)$ via

$$\frac{f''(z)}{f'(z)} = f'(z) \cot(f(z)).$$

(4.46)

In considering the left hand side of the latter equation as the logarithmic derivative of $f'(z)$, and the right hand side as the derivative of $\ln(\sin(f(z)))$, Eq. (4.46) equivalently rewrites to

$$\left( \ln f'(z) \right)' = \left( \ln(\sin(f(z))) \right)' ,$$

(4.47)

leading to

$$\frac{df(z)}{dz} = \sin(f(z)).$$

(4.48)

From this equation one concludes

$$dz = \frac{df(z)}{\sin f(z)}.$$ 

(4.49)

The latter expression can be integrated by means of the so called $\lambda$ function and is given by [59],

$$z = \ln |\csc(f(z)) - \cot(f(z))| = \lambda \left( f(z) - \frac{\pi}{2} \right).$$

(4.50)

Finally, upon exponentiation one arrives at

$$e^z = \frac{|\csc(f(z)) - \cot(f(z))|}{1 - \cos(f(z))}.$$ 

(4.51)

In using $\sin(f(z)) = 2 \sin \left( \frac{f(z)}{2} \right) \cos \left( \frac{f(z)}{2} \right)$ and $1 - \cos(f(z)) = 2 \sin^2 \left( \frac{f(z)}{2} \right)$, Eq. (4.51) simplifies to

$$e^z = \tan \left( \frac{f(z)}{2} \right)$$

(4.52)

With that the function $f(z)$ is obtained as

$$\theta \equiv f(z) = 2 \tan^{-1}(e^z).$$

(4.53)

From this equation one obtains $e^z$, and $e^{-z}$ as

$$e^z = \tan \left( \frac{\theta}{2} \right), \quad e^{-z} = \cot \left( \frac{\theta}{2} \right) ,$$

(4.54)
and finds \( \cosh(z) \) as

\[
\cosh(z) = \frac{e^z + e^{-z}}{2} = \frac{\tan \left( \frac{f(z)}{2} \right) + \cot \left( \frac{f(z)}{2} \right)}{2} = \frac{1}{\sin(f(z))}.
\] (4.55)

Getting back to \( f(z) = \theta \) results in

\[
\cosh(z) = \frac{1}{\sin(\theta)}, \quad \sinh(z) = -\sqrt{\cosh^2(z) - 1} = -\cot(\theta).
\] (4.56)

Correspondingly,

\[
\cos(\theta) = -\tanh(z).
\] (4.57)

Now, the derivative of \( f(z) \) is calculated as \( f'(z) = \theta \)

\[
f'(z) = \sin(\theta) = \sech(z). \]

Upon substituting the last relations into Eqs. (4.34), and (4.45), one arrives at

\[
\frac{d^2\psi(z)}{dz^2} + \left[ l(l+1) - \frac{2\mu\hbar}{\tilde{c}} \tanh(z) \frac{1}{\cosh(z)} - m^2 \right] \psi(z) = 0.
\] (4.58)

Defining \( \tilde{c} \equiv \frac{2\mu\hbar}{\tilde{c}} \) the latter equation takes the form:

\[
\frac{d^2\psi(z)}{dz^2} + \left[ l(l+1) \sech^2(z) - \tilde{c} \sech(z) \tanh(z) - m^2 \right] \psi(z) = 0. \] (4.59)

This equation has same form as the radial Schrödinger equation with Scarf II, i.e.

\[
\frac{d^2\psi(z)}{dz^2} + \left[ (-b^2 + a(a+1)) \sech^2(z) - b(2a+1)\sech(z)\tanh(z) + \epsilon \right] \psi(z) = 0,
\] (4.60)

provided,

- \( l(l+1) \) plays the role of \( -(b^2 - a(a+1)) \),
- \( m^2 \) plays the role of \( -\epsilon \),
- \( \tilde{c} \) plays the role of \( -b(2a+1) \).

Recall that the solution to eq. (4.60) was obtained in subsection 4.1.1 of the present chapter as,

\[
\psi_n(z) = N_n(1 + \sinh^2(z))^{-\frac{1}{2}} e^{-\frac{b}{2} \tan^{-1}(\sinh(z))} R_n^{(a+\frac{1}{2}, -\tilde{c})}(\sinh(z)),
\]

\[
\epsilon_n = -(a - n)^2, \quad -\infty < x < +\infty,
\] (4.61)

with \( N_n \) being a normalization constant. Back to the \( \theta \) variable and in making use of the equality \( \sinh(z) = -\cot(\theta) \), we find

\[
\Theta(\theta) = \psi_n(\sinh^{-1}(-\cot(\theta))) = N_n(1 + \cot^2(\theta))^{-\frac{1}{2}} e^{-\frac{b}{2} \tan^{-1}(\cot(\theta))} R_n^{(a+\frac{1}{2}, -\tilde{c})}(-\cot(\theta)),
\]

(4.62)

showing that the angular part of the exact solution to the non-central potential under consideration is defined by the Romanovski polynomials. The two parameters of the Romanovski polynomials have to be determined from the system of three equations,

\[
-b^2 + a(a+1) = l(l+1),
\]

\[
-b(2a+1) = \tilde{c},
\]

(4.63)

(4.64)
\[
\epsilon_n = -(a - n)^2 = -m^2, \quad a > n, \quad m > 0,
\]
meaning that the \(l, m, \) and \(\tilde{c}\) constants can not be independent. There exist various choices for \(a\) and \(b\). If defined on the basis of the first two equations, one encounters
\[
(a + \frac{1}{2})^2 = \frac{1}{2} \left( (l + \frac{1}{2})^2 + \sqrt{(l + \frac{1}{2})^4 + c^2} \right),
\]
\[
b^2 = \frac{1}{2} \left( -(l + \frac{1}{2})^2 + \sqrt{(l + \frac{1}{2})^4 + c^2} \right).
\]
Substitution of \(a\) into the third equation imposes a constraint on \(l\) as a function of \(m, \tilde{c}, \) and \(n\). A second choice for \(a\) and \(b\) is obtained by expressing \(a\) from the third equation in terms of \(m, \) and \(n\) as \(a = m + n\) and substituting in the second equation to obtain \(b\) as
\[
b = -\frac{\tilde{c}}{2(m + n) + 1}.
\]
Then the first equation imposes the following restriction on \(l\)
\[
X := (b^2 - a(a + 1)), \quad l = -\frac{1}{4} + \sqrt{\frac{1}{4} + X}.
\]
This \(l\) value which is not necessarily integer, can be plugged into Eq. (4.44) leading to a (discrete) spectrum that no longer bears any resemblance to the \(O(4)\) degeneracy. This is the path pursued by Ref. 11. I here instead take a third chance and express \(a, \) \(b,\) and \(\tilde{c}\) as functions of \(l\) alone according to
\[
a = b = l(l + 1), \quad n = a - m = l(l + 1) - m, \quad \tilde{c} = -b(2a + 1).
\]
This choice allows to consider integer \(l\) values. In making use of Eqs. (4.44), (4.45), the polar angle part of the wave function in this case becomes
\[
\psi_{n=l(l+1)-m}(\sinh^{-1}(-\cot(\theta))) = (1 + \cot(\theta)^2)^{-\frac{l(l+1)}{2}} e^{-l(l+1)\tan^{-1}(-\cot(\theta))} R_{l(l+1) - m}^{(l(l+1) + \frac{1}{2})}(-\cot(\theta)).
\]
The complete angular wave function now can be labeled by \(l\) and \(m\) (as a tribute to the spherical harmonics) and is given by
\[
Z_l^m(\theta, \varphi) = \psi_{n=l(l+1)-m}(\sinh^{-1}(-\cot(\theta))) e^{im\varphi} =
\]
\[
(1 + \cot^2(\theta))^{-\frac{l(l+1)}{2}} e^{-l(l+1)\tan^{-1}(-\cot(\theta))} R_{l(l+1) - m}^{(l(l+1) + \frac{1}{2})}(-\cot(\theta)) e^{im\varphi}.
\]
It reduces to the spherical harmonics \(Y_l^m(\theta, \varphi)\) for \(a = b = 0\). In this way, the Romanovski polynomials shape the angular part of the wave function in the problem under consideration. In the following, we shall refer to \(Z_l^m(\theta, \varphi)\) as “non-spherical angular functions”. In the appendix I present a side by side comparison of \(| Y_l^m(\theta, \varphi) | \) and \(| Z_l^m(\theta, \varphi) | \). A comment is in order on \(| Z_l^m(\theta, \varphi) |\). In that regard, it is important to become aware of the fact already mentioned above that the Scarf II potential possesses \(su(1, 1)\) as a potential algebra, a result reported by Refs. 11 60 among others. There, it was pointed out that the respective Hamiltonian, \(H,\) equals \(H = -C - \frac{1}{2}, \) with \(C\) being the \(su(1, 1)\) Casimir operator, whose eigenvalues in our convention are \(j(j + 1)\) with \(j > 0\) versus \(j(j + 1)\) and \(j < 0\) in the convention of 11 60. As a consequence, the Scarf II solutions can be viewed as representation spaces of irreducible \(SU(1, 1)\) representations. Specifically, in the case under consideration the representation is discrete, unitary and of infinite dimensionality. It is the one denoted by \(| D_j^{(m')}((\theta, \varphi)) | \), with \(m' = j, j + 1, j + 2, \ldots \). The \(SU(1, 1)\) labels \(m'\), and \(j\) are mapped onto ours via
\[
m' = a + \frac{1}{2} = l(l + 1) + \frac{1}{2}, \quad j = m' - n, \quad m' = j, j + 1, j + 2, \ldots
\]
meaning that both \( j \) and \( n' \) are a half-integer. In terms of these labels the energy rewrites as \( \epsilon_n = -(j - \frac{1}{2})^2 \). The condition \( a > n \) translates now as \( j > \frac{1}{2} \). In result, \( \Theta(\theta) \) becomes

\[
\Theta(\theta) = \psi_{n,m'-j} \left( \sinh^{-1}(-\cot \theta) \right) = \sqrt{(1 + \cot^2 \theta)^{-m' + \frac{1}{2}} e^{-2b \tan^{-1}(-\cot \theta)} R^{(m',-2b)}_{m'-j}(-\cot \theta)}.
\]

(4.73)

Here we kept the parameter \( b \) general because its value does not affect the \( SU(1,1) \) symmetry. Within this context, the \(|\psi_{n,m'-j}(\sinh^{-1}(-\cot \theta))|\)'s can be viewed as absolute values of \( \{D^+_j(m')(\theta, \varphi)\} \) eigenvector components [9] and realized in terms of the Romanovski polynomials. The \(|Z^n_j(\theta, \varphi)|\) functions are then images in polar coordinate space of \(|\psi_{n,m'-j}(\sinh^{-1}(-\cot \theta))|\)'s and realized in terms of the Romanovski polynomials. The representations are infinite because for a fixed \( j \), \( n' \) is bound from below to \( n'_\min = j \), but it is not bound from above. For example, \(|Z^n_1(\theta, \varphi)|\) refers to \( D^+_j(\frac{m'}{2})(\theta, \varphi) \), \(|Z^n_2(\theta, \varphi)|\) refers to \( D^+_j(\frac{m'-1}{2})(\theta, \varphi) \) etc.

4.3.3 Romanovski polynomials and associated Legendre functions.

Next, it is quite instructive to consider the case of a vanishing \( V_2(\theta) \), i.e. \( c = 0 \), and compare Eq. (4.58) to Eq. (4.60) for \( b = 0 \). In this case, and in accordance with Eq. (4.21)

\[
l = a = p - \frac{1}{2}, \quad m^2 = (l - n)^2, \quad q = -2b = 0,
\]

(4.74)

which allows one to relate \( n \) to \( l \) and \( m \) as \( m = l - n \). As long as the two equations are equivalent, their solutions differ at most by a constant factor. This allows to establish a relationship between the associated Legendre functions and the Scarf II wave functions. In taking into account Eqs. (4.11) and (4.17) together with Eqs. (4.59), one finds \( \cot(\theta) = -\sinh(z) \) which produces the following new relationship between the associated Legendre functions and the Romanovski polynomials

\[
P_l^m(\cos(\theta)) \sim (1 + \cot^2(\theta))^{-\frac{1}{2}} R_{l-m}^{(l+\frac{1}{2},0)}(-\cot(\theta)), \quad l - m = n = 0, 1, 2, \ldots l.
\]

(4.75)

In substituting the latter expression into the orthogonality integral between the associated Legendre functions,

\[
\int_{-1}^{1} P_l^m(\cos(\theta)) P_{l'}^{m'}(\cos(\theta)) d\cos(\theta) = 0, \quad l \neq l',
\]

(4.76)

results in the following integral

\[
\int_{-1}^{1} (1 + \cot^2(\theta))^{-\frac{1}{2}} R_{l-m}^{(l+\frac{1}{2},0)}(-\cot(\theta)) R_{l'-m}^{(l'+\frac{1}{2},0)}(-\cot(\theta)) d\cos(\theta) = 0, \quad l \neq l'.
\]

(4.77)

When rewritten to conventional notations, the latter expression becomes

\[
\int_{-\infty}^{+\infty} \sqrt{w(l+\frac{1}{2},0)(x)} R_{l-m}^{(l+\frac{1}{2},0)}(x) \sqrt{w(l'+\frac{1}{2},0)(x)} R_{l'-m}^{(l'+\frac{1}{2},0)}(x) \frac{dx}{1 + x^2} = 0, \quad l \neq l',
\]

\[
x = \sinh(z), \quad l - n = l' - n' = m \geq 0.
\]

(4.78)

This integral describes orthogonality between an infinite set of Romanovski polynomials with different polynomial parameters (they would define wave functions of states bound in different potentials). This new orthogonality relationship does not contradict the finite orthogonality in Eq. (2.17) which is valid for states belonging to same potential (equal polynomial parameters).

Rather, for different potentials, Eq. (2.17) can be fulfilled for an infinite number of states. To see this let us consider, for simplicity, \( n = n' = l - m \), i.e. \( l = l' \). Given \( p = l + \frac{1}{2} \), the condition in Eq. (2.17) defines normalizability and takes the form

\[
2(l - m) < 2(l + 1) - 1 = 2l,
\]

(4.79)
which is automatically fulfilled for any \( m > 0 \). The presence of the additional factor of \((1 + x^2)^{-1}\) guarantees convergence also for \( m = 0 \). Equation (4.78) reveals that for parameters attached to the degree of the polynomial, an infinite number of Romanovsky polynomials can appear orthogonal, although not precisely with respect to the weight function that defines their Rodrigues representation. The study presented here is kindred to Ref. \( 49 \) and Eq. (3.86) from above. Also there, the exact solutions of the Schrödinger equation with the trigonometric Rosen-Morse potential (employed as quark-di-quark interaction) have been expressed in terms of Romanovsky polynomials (not identified as such at that time) and also with parameters that depended on the degree of the polynomial. Also in this case, the \( n \)-dependence of the parameters, and the corresponding varying weight function allowed to fulfill Eq. (2.17) for infinitely many polynomials.

### 4.4 Hyperbolic Scarf potential in the Klein-Gordon equation.

The final example to be considered is the case of the relativistic Klein-Gordon equation,

\[
(P^2 - \mu^2)\Psi = 0, \quad P_v = i\partial_v,
\]

where \( \mu \) is the mass. One can introduce two different potentials in this equation. The first is a vector potential, \( A_v \), introduced via minimal coupling as \( P_v - gA_v \), with \( g \) being a constant, and the second is a scalar potential, \( S \), introduced via \( m \rightarrow \mu + S \). The vector potential in the so-called Coulomb gauge satisfies \( \nabla \cdot A = 0 \) but on many occasions one simplifies the problem in choosing \( \tilde{A} = 0 \) in which case only the time-like component of the vector potential, \( gA_0 \), denoted by \( V \) in the following, enters the equation. In effect, the Klein-Gordon equation with vector and scalar potential (in units of \( c = \hbar = 1 \)) takes the form

\[
[(i\frac{\partial}{\partial t} - V(r))^2 + \nabla^2 - (S(r) + \mu)^2]\psi(r) = 0. \tag{4.81}
\]

The latter equation simplifies significantly when \( S \) and \( V \) are equal. It has been shown in Refs. \( 61, 62 \) that in this case the solution of Eq. (4.81) can be found from those of an associated Schrödinger equation. Indeed, for time-independent potentials, the total wave function can be written as \( \Psi(r, t) = e^{-iEt}\psi(r) \), with \( E \) being the relativistic energy. This substitution results in:

\[
[\nabla^2 + (V(r) - E)^2 - (S(r) + \mu)^2]\psi(r) = 0. \tag{4.82}
\]

From now onward I shall focus on the special case of equal scalar and vector potentials, i.e. \( V(r) = S(r) \). In the following we change variable to \( V(r) \rightarrow \frac{\sqrt{2}V(r)}{2} \). In this case, the Klein-Gordon equation then rewrites to

\[
[\nabla^2 + \left( E - \frac{v(r)}{2} \right)^2 - \left( \frac{v(r)}{2} + \mu \right)^2]\psi(r) = 0. \tag{4.83}
\]

In the following \( v(r) \) is taken as the central hyperbolic Scarf potential. Separating variables in polar coordinates, \( \psi(r) = R(r)H(\theta)K(\varphi) \), leads to

\[
K''(\varphi) + m^2 K(\varphi) = 0, \tag{4.84}
\]

\[
H''(\theta) + \cot(\theta)H'(\theta) - [m^2 \csc^2(\theta) - s(s + 1)]H(\theta) = 0, \tag{4.85}
\]

and

\[
(r^2 R'(r))^2 - [s(s + 1) + (E + \mu)r^2 v(r) - (E^2 - \mu^2)r^2]R(r) = 0, \tag{4.86}
\]

where \( s \) and \( m \) are the separation constants. The angular equation for the azimuthal coordinate, \( \varphi \), has solutions satisfying periodical conditions: \( K(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi} \), \( m = 0, \pm 1, \pm 2, \ldots \). The equation for the polar angle, \( \theta \), can be transformed by means of \( x = \cos(\theta) \) and becomes

\[
(1 - x^2) \frac{d^2 f(x)}{dx^2} - 2x \frac{df(x)}{dx} + \left[ -\frac{m^2}{1 - x^2} + s(s + 1) \right]f(x) = 0. \tag{4.87}
\]
It can be identified with the associated Legendre differential equation whose solutions are \( f(\cos(\theta)) \equiv P^m_n(\cos(\theta)) \). The radial equation can be solved for \( s = 0 \). For \( R(r) = D(r)/r \) it takes the form

\[
\frac{d^2 D(r)}{dr^2} - \left( (E + \mu)A^2 + (E + \mu)(B^2 - A^2 - A) \text{sech}^2(r) + (E + \mu)B(2A + 1) \text{sech}(r) \text{tanh}(r) - (E + \mu)(E - \mu) \right) D(r) = 0.
\]

(4.88)

Upon naming

- \((E + \mu)(-A^2 + E - \mu)\) as \( \epsilon \),
- \((E + \mu)A^2\) as \( a^2 \),
- \((E + \mu)(B^2 - A^2 - A)\) as \( b^2 - a^2 - a \),
- \((E + \mu)B(2A + 1)\) as \( b(2a + 1) \),

and in changing variables to \( x = \sinh(r) \), \( D(r) \longrightarrow f(x) \), amounts to

\[
(1 + x^2) \frac{d^2 f(x)}{dx^2} + x \frac{df(x)}{dx} + \left( \frac{-b^2 + a(a + 1)}{1 + x^2} + \frac{-b(2a + 1)}{1 + x^2} x + \epsilon \right) f(x) = 0.
\]

(4.89)

The latter equation is equal to the polynomial form of the 1d-Schrödinger equation for the Scarf II in Eq. (4.10) whose solutions have been explicitly constructed in Eq. (4.19) above. Matching parameters leads to

\[
\alpha = 2b, \quad \beta = -a, \quad \epsilon_n = -(a - n)^2.
\]

(4.90)

The energies are then found as

\[
E^1_n = \frac{A^2 + 2An - 2A^2\mu + \sqrt{A^4 + 4A^4n - 4n^2 + 4A(A + 2n)\mu + 4\mu^2}}{2(1 + A^2)},
\]

\[
E^2_n = \frac{A^2 + 2An - 2A^2\mu - \sqrt{A^4 + 4A^4n - 4n^2 + 4A(A + 2n)\mu + 4\mu^2}}{2(1 + A^2)}.
\]

(4.91)

The two values for the energies correspond to particles, and antiparticles, as expected from the relativistic Klein-Gordon equation.
Figure 4.6: The non-central potential $V(r, \theta)$, for $c = -5$, here displayed in its intersection with the $x = 0$ plane, i.e. for $r = \sqrt{y^2 + z^2}$, and $\theta = \tan^{-1} \frac{y}{z}$. The polar angle part of its exact solutions is expressed in terms of the Romanovski polynomials.
Chapter 5

Conclusions and outlooks.

In this thesis I presented

• the classification of the (orthogonal) polynomial solutions to the generalized hypergeometric equation in the respective schemes of Koepf–Masjed-Jamei [28], Nikiforov-Uvarov [8], and Bochner [17],

• the explicit construction of the Hermite, Laguerre, Jacobi, Bessel and Romanovski polynomials,

• the solutions of the Schrödinger equations with the respective one- and three dimensional oscillator, the Coulomb- and the hyperbolic Rosen-Morse potentials, and the $\sim e^{\alpha x}$ barrier in terms of one of the above polynomials.

As new results I report

• the exact wave functions of the bound states within the hyperbolic Scarf potential in terms of the Romanovski polynomials,

• the finite orthogonality of the Romanovski polynomials in terms of their normalization constants, a property that allowed to map the finite number of bound states within Scarf II onto a finite set of polynomials,

• the Romanovski polynomials as main designers of non–spherical angular functions of a new type, which we identified with components of the eigenvectors of the infinite discrete unitary SU(1,1) representation, \( \{D^+_{j=m+\frac{1}{2}, l(l+1)+\frac{1}{2}}(\theta, \varphi)\} \),

• a non-linear relationship between Romanovski polynomials with parameters attached to the degree of the polynomial and the associated Legendre functions which lead to a new orthogonality integral for an infinite series of such Romanovski polynomials,

• the solution of the Klein-Gordon equation with equal scalar and vector potentials, taken as the hyperbolic Scarf potential.

I conclude that the Romanovski polynomials represent the most adequate degrees of freedom in the mathematics of the hyperbolic Scarf potential.

Further conclusions are:

• The orthogonality integral of Schrödinger wave function of the form given in Eq. 3.13 and free parameters always recovers the orthogonality of the involved polynomials.

• Equation 3.13 did not hold valid on several occasions in which the parameters happened to depend on the degree of the polynomials, in which case the orthogonality of the wave functions failed in recovering the orthogonality of the polynomials with the free parameters.
• As a rule, polynomials with parameters running with the degree of the polynomial, appear orthogonal with respect to a weight function which is altered in comparison with the one that enters the Rodrigues formula.

• The case of the hydrogen atom was special in so far as there the anomalous orthogonality integral between Laguerre polynomials was observed for polynomial parameters which did not depend on the degree of the polynomial, however the variable $x$ did.

In future research one can

• exploit the relationship between the Romanovski polynomials and the associated Legendre functions to write down various new recurrence relations for the former,

• employ their weight function as an extension of the student’s $t$ distribution and test it in statistical problems of estimating standard deviations from data,

• study symmetry relationships between the non-spherical angular functions which lead to equality between such functions of different parameters as visible by inspection from the Appendix.

The Romanovski polynomials are interesting mathematical entities in their own and future research is expected to shed more light on their properties and physics applications.
Chapter 6

Appendix.

In this Appendix I present a graphical side by side comparison of the absolute values of the spherical harmonics $|Y^m_l(\theta, \varphi)|$ and the $|Z^m_l(\theta, \varphi)|$ functions.

Figure 6.1: $|Y^0_0(\theta, \varphi)|$ vs $|Z^0_0(\theta, \varphi)|$.
Figure 6.2: $|Y_1^{0}(\theta,\varphi)|$ vs $|Z_1^{0}(\theta,\varphi)|$

Figure 6.3: $|Y_1^{\pm 1}(\theta,\varphi)|$ vs $|Z_1^{\pm 1}(\theta,\varphi)|$
Figure 6.4: $|Y_2^0(\theta, \varphi)|$ vs $|Z_2^0(\theta, \varphi)|$

Figure 6.5: $|Y_2^{\pm 1}(\theta, \varphi)|$ vs $|Z_2^{\pm 1}(\theta, \varphi)|$
Figure 6.6: $|Y_{\frac{3}{2}}^{\pm2}(\theta, \varphi)|$ vs $|Z_{\frac{3}{2}}^{\pm2}(\theta, \varphi)|$
Figure 6.7: $|Y^0_3(\theta, \varphi)|$ vs $|Z^0_3(\theta, \varphi)|$
Figure 6.8: $|Y_{3/2}^{\pm 1}(\theta, \varphi)|$ vs $|Z_{3/2}^{\pm 1}(\theta, \varphi)|$

Figure 6.9: $|Y_{3/2}^{\pm 2}(\theta, \varphi)|$ vs $|Z_{3/2}^{\pm 2}(\theta, \varphi)|$
Figure 6.10: \( |Y_{\pm 3}^3(\theta, \varphi)| \) vs \( |Z_{\pm 3}^3(\theta, \varphi)| \)
Figure 6.11: $|Y_4^0(\theta, \varphi)|$ vs $|Z_4^0(\theta, \varphi)|$
Figure 6.12: $| Y_{4}^{11}(\theta, \varphi) |$ vs $| Z_{4}^{11}(\theta, \varphi) |$

Figure 6.13: $| Y_{4}^{22}(\theta, \varphi) |$ vs $| Z_{4}^{22}(\theta, \varphi) |$
Figure 6.14: $|Y_{4}^{\pm 3}(\theta, \varphi)|$ vs $|Z_{4}^{\pm 3}(\theta, \varphi)|$

Figure 6.15: $|Y_{4}^{\pm 4}(\theta, \varphi)|$ vs $|Z_{4}^{\pm 4}(\theta, \varphi)|$
Figure 6.16: $|Y_0^0(\theta, \varphi)|$ vs $|Z_0^0(\theta, \varphi)|$
Figure 6.17: \( |Y_{5}^{\pm 1}(\theta, \varphi)| \) vs \( |Z_{5}^{\pm 1}(\theta, \varphi)| \)
Figure 6.18: $|Y_3^{2,2}(\theta, \phi)|$ vs $|Z_3^{2,2}(\theta, \phi)|$

Figure 6.19: $|Y_3^{3,3}(\theta, \phi)|$ vs $|Z_3^{3,3}(\theta, \phi)|$
Figure 6.20: $|Y^{\pm4}_{\ell}(\theta, \varphi)|$ vs $|Z^{\pm4}_{\ell}(\theta, \varphi)|$

Figure 6.21: $|Y^{\pm5}_{\ell}(\theta, \varphi)|$ vs $|Z^{\pm5}_{\ell}(\theta, \varphi)|$
Figure 6.22: $|Z_2^1(\theta, \varphi)|$, and $|Z_2^{-5}(\theta, \varphi)|$ as examples for non-spherical angular functions that don’t have a spherical counterpart.
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2.1 Comparison among the different families of polynomials.

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