Extremal problems for ordered hypergraphs: small patterns and some enumeration

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Abstract

We investigate extremal functions \( ex_e(F, n) \) and \( ex_i(F, n) \) counting maximum numbers of edges and maximum numbers of vertex-edge incidences in simple hypergraphs \( H \) which have \( n \) vertices and do not contain a fixed hypergraph \( F \); the containment respects linear orderings of vertices. We determine both functions exactly if \( F \) has only distinct singleton edges or if \( F \) is one of the 55 hypergraphs with at most four incidences (we give proofs only for six cases). We prove some exact formulae and recurrences for the numbers of hypergraphs, simple and all, with \( n \) incidences and derive rough logarithmic asymptotics of these numbers. Identities analogous to Dobiński’s formula for Bell numbers are given.

1 Introduction and definitions

In this article we consider problems on hypergraphs of the following type. Suppose that \( H \) is a simple hypergraph with \( n \) vertices, which means that \( H \) is a finite set of finite nonempty subsets of \( \mathbb{N} = \{1, 2, \ldots\} \) with \( | \bigcup H | = n \), such that for no three vertices \( a < b < c \) in \( \bigcup H \) and for no two distinct edges \( A \) and \( B \) in \( H \) one has the four incidences \( a, b \in A \) & \( b, c \in B \). What are, in terms of \( n \), the maximum possible size \( |H| \) and the maximum possible number of incidences \( \sum_{A \in H} |A| \) of \( H \)? What are the maxima if the
forbidden incidence pattern is, for example, $a \in A \& a \in B \& a, b \in C$ ($a < b$ are vertices and $A, B, C$ are distinct edges)? How many distinct hypergraphs with linearly ordered vertices and $n$ incidences, simple and all, are there? The first two questions, and quite a few similar ones, are answered in Section 3. The third question is addressed in Section 4. This article is a continuation of Klazar [7]. We refer the reader to [7] for further results and for motivation of our extremal problems.

We denote $N = \{1, 2, 3, \ldots\}$ and work with the standard linear order $<$ on $N$. If $a, b, n \in N$ with $a \leq b$, we write $[a, b]$ for the interval $\{a, a+1, \ldots, b\}$ and $[n] = \{1, n\}$ for $\{1, 2, \ldots, n\}$. A hypergraph $H = (E_i : i \in I)$ is a finite list of finite nonempty subsets $E_i$ of $N$, called edges. $H$ is simple if $E_i \neq E_j$ for every $i, j \in I$, $i \neq j$. The elements of $\bigcup H = \bigcup_{i \in I} E_i \subset N$ are called vertices. Note that our hypergraphs have no isolated vertices. The simplification of $H$ is the simple hypergraph obtained from $H$ by keeping from each family of mutually equal edges just one edge. The deletion of $E_i$, $j \in I$, from $H = (E_i : i \in I)$ yields the hypergraph $(E_i : i \in I')$ where $I' = I \setminus \{j\}$. The deletion of $a \in \bigcup H$ from $H$ yields the hypergraph $(E_i \setminus \{a\} : i \in I)$ where the $\emptyset$'s arising from $E_i = \{a\}$ are omitted; this operation in general destroys simplicity. We may also delete $a$ only from some specified edges. The degree $\deg_H(v) = \deg_H(v)$ of a vertex $v$ of $H$ is the number of the edges $E \in H$ such that $v \in E$. The order $\nu(H)$ of $H = (E_i : i \in I)$ is the number of vertices $\nu(H) = |\bigcup H|$, the size $e(H)$ is the number of edges $e(H) = |H| = |I|$, and the weight $i(H)$ is the number of incidences between vertices and edges.

Two hypergraphs $H = (E_i : i \in I)$ and $H' = (E'_i : i \in I')$ are isomorphic if there are an increasing bijection $F : \bigcup H' \to \bigcup H$ and a bijection $f : I' \to I$ such that $F(E'_i) = E_{f(i)}$ for every $i \in I'$. $H'$ is a reduction of $H$ if $I' \subset I$ and $E'_i \subset E_i$ for every $i \in I'$. $H'$ is contained in $H$, in symbols $H' \preceq H$, if $H'$ is isomorphic to a reduction of $H$. We call that reduction of $H$ an $H'$-copy in $H$. For example, if $H' = (\{1\}_1, \{1\}_2)$ ($H'$ is a singleton edge repeated twice) then $H' \preceq H$ if and only if $H$ has two intersecting edges. Another example: If $H' = (\{1, 4\}, \{2, 3\})$ then $H'$ is contained in $H$ if and only if $H$ has four vertices $a < b < c < d$ such that $a$ and $d$ lie in one edge of $H$ while $b$ and $c$ lie in another edge. If $H' \not\preceq H$, we say that $H$ is $H'$-free. Let $F$ be any hypergraph. We associate with $F$ the extremal functions $\text{ex}_e(F, \cdot)$, $\text{ex}_i(F, \cdot) : N \to N$, defined by

$$\text{ex}_e(F, n) = \max\{e(H) : H \not\simeq F \& H \text{ is simple} \& v(H) = n\}$$

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\[ \text{ex}_i(F, n) = \max\{i(H) : H \not\sim F \text{ and } H \text{ is simple } \& \ v(H) = n\}. \]

In [7] we defined both functions with the requirement \( v(H) \leq n \). Here we are more interested in their precise values and therefore we require \( v(H) = n \).

Obviously, for every \( n \in \mathbb{N} \) and \( F \), \( \text{ex}_e(F, n) \leq 2^n - 1 \) and \( \text{ex}_i(F, n) \leq n2^{n-1} \), but much better bounds can be usually given. The reversal of a hypergraph \( H = (E_i : i \in I) \) with \( N = \max(\bigcup H) \) is the hypergraph \( \overline{H} = (E_i : i \in I) \) where \( E_i = \{N-x+1 : x \in E_i\} \). Reversals are obtained by reverting the linear ordering of vertices. It is clear that \( \text{ex}_e(F, n) = \text{ex}_e(\overline{F}, n) \) and \( \text{ex}_i(F, n) = \text{ex}_i(\overline{F}, n) \) for every \( F \) and \( n \).

In this article we complement the results of [7], where we derived some asymptotic upper bounds, and determine precise values of \( \text{ex}_e(F, n) \) and \( \text{ex}_i(F, n) \) for several hypergraphs \( F \). Then we address some naturally arising enumerative questions. The present article is a revised version of about one half of the technical report [6]; the other half appears in [7]. Sections 2 and 3 contain extremal results. In Theorems 2.1 and 2.3 we determine \( \text{ex}_e(F, n) \) and \( \text{ex}_i(F, n) \) exactly if \( F = S_k = (\{1\}, \{2\}, \ldots, \{k\}) \) consists only of distinct singleton edges. Then both functions are not nondecreasing: \( \text{ex}_e(S_k, k-1) > \text{ex}_e(S_k, k) \) and \( \text{ex}_i(S_k, k-1) > \text{ex}_i(S_k, k) \) \((k \geq 3)\). In Theorem 2.2 we prove that if \( F \) is nonisomorphic to \( S_k \), then \( \text{ex}_e(F, n) < \text{ex}_e(F, n+1) \) for every \( n \in \mathbb{N} \). Since all hypergraphs obtained from \( S_k \) by permuting its vertices are mutually isomorphic, in Theorems 2.1 and 2.3 the ordering of vertices is irrelevant. In Section 3 we determine both extremal functions exactly for every of the 55 hypergraphs \( F \) with \( 1 \leq i(F) \leq 4 \). In Propositions 3.1–3.5 we present proofs only for six cases (other three cases are subsumed in Theorems 2.1 and 2.3). Section 4 is enumerative. In Proposition 4.1 we enumerate simple hypergraphs with order \( n \). Theorem 4.2 enumerates both simple and all hypergraphs with prescribed numbers of edges of each cardinality. Corollary 4.3 enumerates both simple and all hypergraphs with weight \( n \) by a sum over integer partitions. Proposition 4.4 does the same less elegantly but more efficiently by recurrences. In Corollary 4.5 we give identities for hypergraphs which are analogous to the Dobiński’s formula for set partitions. In Proposition 4.6 we bound the numbers of hypergraphs with weight \( n \) by the Bell numbers.

2 Singleton hypergraphs

Note that functions \( \text{ex}_e(\{1\}, n) \) and \( \text{ex}_i(\{1\}, n) \) are undefined.
Theorem 2.1 Let $S_k = (\{1\}, \{2\}, \ldots, \{k\})$. Then, for $k \geq 2$,
\[
ex_e(S_k, n) = \begin{cases} 
2^n - 1 & \text{if } 1 \leq n < k \\
2^{k-2} & \text{if } n \geq k.
\end{cases}
\]
In particular, for $k \geq 3$ the function $ex_e(S_k, n)$ has the unique global maximum $ex_e(S_k, k - 1) = 2^{k-1} - 1$.

Proof. The case $1 \leq n < k$ is clear. For $n \geq k \geq 2$ we have $ex_e(S_k, n) \geq 2^{k-2}$ because $\{[n]\} \cup (E : \emptyset \neq E \subset [k-2]) \neq S_k$. We prove by induction on $k$ that for $n \geq k$ also $ex_e(S_k, n) \leq 2^{k-2}$. For $k = 2$ this holds because $ex_e(S_2, n) = 1$ for every $n \in \mathbb{N}$. Let $n \geq k \geq 3$ and let $H$ be a simple $S_k$-free hypergraph with $\bigcup H = [n]$. We show that we can assume that (i) $\deg(v) \geq 2$ for every $v \in \bigcup H$ and (ii) there is an $E \in H$ with $|E| \geq 2$ and an $a \in E$ such that $E \backslash \{a\} \not\in H$.

If (i) is false, there is a vertex contained in a unique edge. We delete the edge from $H$ and obtain a hypergraph $H'$ which must be $S_{k-1}$-free. We are done by induction: $e(H) = e(H') + 1 \leq (2^{(k-1)-1} - 1) + 1 = 2^{k-2}$. Suppose that (ii) is false. Let $a \in \bigcup H$ be arbitrary and $E \in H$, $a \in E$, be such that $|E|$ is as small as possible. If $|E| > 1$, there is a $b \in E$, $b \neq a$. By the negation of (ii), $E \backslash \{b\} \in H$, contradicting the minimality of $|E|$. Thus $|E| = 1$ and $\{a\} \in H$. Hence $\{a\} \in H$ for every $a \in \bigcup H$. But this implies the contradiction $H \succ S_k$ (since $n \geq k$).

Thus (i) and (ii) hold. Let $a$ and $E$ be as in (ii). Let $E' \in H$ be such that $a \in E'$, $E' \neq E$, and, if possible, $|E'| = 1$. We obtain $H'$ by deleting $E'$ from $H$ and then deleting $a$ from $H \backslash \{E'\}$. Some edges may get duplicated and therefore we set $H''$ to be the simplification of $H'$. By (i), $v(H'') = v(H) - 1 = n - 1 \geq k - 1$. Since any $S_{k-1}$-copy in $H''$ can be extended by $E'$ and $a$ to an $S_k$-copy in $H$, $H'' \not\in S_{k-1}$. Also, $e(H') \leq 2e(H'') - 1$ because, by (ii), $E \backslash \{a\}$ is not duplicated in $H'$. Notice that $\emptyset \not\in H''$ because we have deleted $\{a\}$ as $E'$. By induction (now we use the stronger upper bound on $e(H'')$),
\[
e(H) = e(H') + 1 \leq (2e(H'') - 1) + 1 = 2e(H'') \leq 2 \cdot 2^{(k-1)-2} = 2^{k-2}.
\]

The function $ex_e(S_k, n)$ has the strange feature of being independent of $n$. We show that other extremal functions $ex_e(F, n)$ are increasing, as one expects.
Theorem 2.2 If $F$ is not isomorphic to any $S_k = (\{1\}, \{2\}, \ldots, \{k\})$, then

$$\text{ex}_e(F, n) < \text{ex}_e(F, n+1)$$

for every $n \in \mathbb{N}$.

Proof. Let $\bigcup F = [m]$, $m \geq 2$, and $F \neq S_m$. We say that $\{i\} \in F$ is an isolated singleton of $F$ if $\deg(i) = 1$. Let $l$ be the maximum number such that $\{1\}, \{2\}, \ldots, \{l\}$ are isolated singletons of $F$. Since $F \neq S_m$, we have $0 \leq l < m$. Any other isolated singleton of $F$ is preceded by at least $l+1$ vertices. We proceed by induction on $n < m - F$. Let $H \subseteq \bigcup F$, $n \geq 1$ because then $\text{ex}_e(F, n) = 2^n - 1$. Let $n \geq m - 1$ and let $H$, $\bigcup H = [n]$, attain the value $\text{ex}_e(F, n)$. If $a \in E \subseteq H$ and $\{a\} \not\subseteq H$, we replace $E$ by $\{a\}$. The new hypergraph is simple, $F$-free, and has the same size as $H$. By the inductive assumption, it must have also the same order. Repeating the replacements, we obtain a simple $F$-free hypergraph $H'$ such that $e(H') = e(H) = \text{ex}_e(F, n)$, $\bigcup H' = \bigcup H = [n]$, and $\{a\} \subseteq H'$ for every $a \in [n]$. We define $H''$ by inserting in $H'$, between the vertices $l$ and $l+1$, a new singleton edge $\{u\}$. $H''$ is simple and satisfies $v(H'') = n + 1$ and $e(H'') = e(H') + 1 = \text{ex}_e(F, n) + 1$. We show that $H''$ is $F$-free. This gives $\text{ex}_e(F, n+1) \geq e(H'') > \text{ex}_e(F, n)$. If $H'' \succ F$, the new edge $\{u\}$ would have to participate in every $F$-copy in $H''$ as an isolated singleton. It cannot play the role of any of the initial $l$ isolated singletons of $F$ because $\{i\} \in H'$ for every $i \in [n]$ and $n \geq m - 1 \geq l$; we would have already $F \succ H'$. It cannot play the role of any other isolated singleton of $F$ either because those are preceded in $F$ by at least $l+1$ vertices but $\{u\}$ is preceded in $H''$ by only $l$ vertices. Thus $H'' \not\supseteq F$. \hfill \Box

Theorem 2.3 Let $S_k = (\{1\}, \{2\}, \ldots, \{k\})$. Then, for $k \geq 2$,

$$\text{ex}_i(S_k, n) = \begin{cases} n2^{n-1} & \text{for } 1 \leq n < k \\ n + (k-2)2^{k-3} & \text{for } k \leq n \leq 2^{k-3} + 1 \\ (k-1)n - (k-2) & \text{for } n \geq \max(k,2^{k-3}+1). \end{cases}$$

In particular, $\text{ex}_i(S_k, k-1) > \text{ex}_i(S_k, n)$ for $k \leq n \leq \max(k,2^{k-2})$ ($k \geq 3$).

Proof. The first case is clear. We suppose that $n \geq k \geq 2$ and that $H$ is a simple hypergraph with $\bigcup H = [n]$. We consider its dual $H^*$:

$$H^* = \{E_i^* : i \in [n]\}$$

where $E_i^* = \{E \in H : i \in E\}$. (continued on next page)
Thus $e(H^*) = v(H) = n$. Let $\Gamma(X) = \Gamma_H(X)$ be for $X \subset [n]$ defined by

$$\Gamma(X) = \left| \bigcup_{i \in X} E^*_i \right| = |\{E \in H : E \cap X \neq \emptyset\}|.$$ 

By the defect form of P. Hall’s theorem (Lovász \[8, Problems 7.5 and 13.5\]) applied on $H^*$, $H$ is $S_k$-free if and only if

$$\max_{X \subset [n]} |X| - \Gamma(X) \geq n - k + 1.$$ 

Thus if $H$ is $S_k$-free, there exists a set $X \subset [n]$ of cardinality $l$, $n - k + 2 \leq l \leq n$ ($\Gamma(X) \geq 1$), intersected by only at most $l - n + k - 1$ edges of $H$. And contrariwise, every such a hypergraph is (trivially) $S_k$-free. Hence

$$i(H) \leq (l - n + k - 1)n - (l - n + k - 2) + (n - l)2^{n-l-1} = f(l, k, n)$$

and this bound is attained.

Let $k$ and $n$ be fixed. The first difference of $f(l, k, n)$ with respect to $l$ is the increasing function

$$f(l + 1, k, n) - f(l, k, n) = n - 1 - (n - l + 1)2^{n-l-2}.$$ 

Therefore $f(l, k, n)$ attains its maximum in one of the endpoints $l = n - k + 2$ and $l = n$ or in both. The corresponding values are

$$f(n - k + 2, k, n) = n + (k - 2)2^{k-3}$$

and

$$f(n, k, n) = (k - 1)n - (k - 2).$$

These values are equal for $n = 2^{k-3} + 1$. For $n < 2^{k-3} + 1$ the former value dominates and for $n > 2^{k-3} + 1$ the latter. We obtain the values of $\text{ex}_e(S_k, n)$ in the remaining two cases. Maximum weights are attained by the hypergraph $H_1$ or by $H_2$, where the edges of $H_1$, respectively of $H_2$, are $[n]$ and all nonempty subsets of some $(k - 2)$-element set $Y \subset [n]$, respectively $[n]$ and some $k - 2$ distinct $(n - 1)$-element subsets of $[n]$. \hfill \Box

For $1 \leq n < k$ the maximum weight is attained only by the complete hypergraph. The proof shows that for $n \geq k$ the only types of extremal hypergraphs are $H_1$ and $H_2$. Thus the number of simple $S_k$-free hypergraphs having order $n$ and the maximum weight equals 1 if $1 \leq n < k$ and equals $\eta^\prime(n, k)$ if $n \geq k$, where for $k = 2, 3, 4$ always $\eta = 1$ and for $k \geq 5$ we have $\eta = 1$ if $n \neq 2^{k-3} + 1$ and $\eta = 2$ if $n = 2^{k-3} + 1$.

One can use P. Hall’s theorem to give another proof of Theorem 2.1. The number of hypergraphs $H$ attaining the value $\text{ex}_e(S_k, n)$ is seen to be 1 for
\( n < k \) and \( 2^{k-2} \binom{n}{k-2} \) for \( n \geq k \). The latter hypergraphs are all \( H \) of the form \( H = \{Y\} \cup (X : \emptyset \neq X \subset Z) \) where \( Z \) is a \( k - 2 \)-element subset of \([n]\) and \([n] \setminus Z \subset Y \subset [n] \).

We conjecture that if \( F \) is not isomorphic to any of the singleton hypergraphs \( S_k = (\{1\}, \{2\}, \ldots, \{k\}) \), then
\[
\text{ex}_i(F, n) < \text{ex}_i(F, n + 1)
\]
for every \( n \in \mathbb{N} \).

## 3 Forbidden hypergraphs of weight at most 4

In this section we give precise formulae for \( \text{ex}_e(F, n) \) and \( \text{ex}_i(F, n) \) for every \( F \) with \( 1 \leq i(F) \leq 4 \). There are 55 such nonisomorphic hypergraphs but due to the reversals it suffices to consider 39 of them. The proofs are usually straightforward and often repetitive. Lest the reader be bored and tired, we present here only a sample consisting of six cases. The proofs for all of the 39 cases can be found in [6]. First we list the hypergraphs \( F \), then we review the results in a table, and in the rest of the section we give proofs for six cases.

**Weight 1 and 2:**
\[
\begin{align*}
F_1 &= (\{1\}), \\
F_2 &= (\{1\}, \{1\}_2), \\
F_3 &= (\{1\}, \{2\}), \quad \text{and} \\
F_4 &= (\{1, 2\}).
\end{align*}
\]

**Weight 3:**
\[
\begin{align*}
F_5 &= (\{1\}_1, \{1\}_2, \{1\}_3), \\
F_6 &= (\{1\}_1, \{1\}_2, \{2\}), \\
\overline{F_6}, \\
F_7 &= (\{1\}, \{2\}, \{3\}), \\
F_8 &= (\{1\}, \{2\}), \\
\overline{F_8}, \\
F_9 &= (\{1\}, \{2, 3\}), \\
\overline{F_9}, \\
F_{10} &= (\{1, 3\}, \{2\}), \\
\text{and} \\
F_{11} &= (\{1, 2, 3\}).
\end{align*}
\]

**Weight 4:**
\[
\begin{align*}
F_{12} &= (\{1\}_1, \{1\}_2, \{1\}_3, \{1\}_4), \\
F_{13} &= (\{1\}_1, \{1\}_2, \{1\}_3, \{2\}), \\
\overline{F_{13}}, \\
F_{14} &= (\{1\}_1, \{1\}_2, \{2\}_1, \{2\}_2), \\
F_{15} &= (\{1\}_1, \{1\}_2, \{2\}, \{3\}), \\
\overline{F_{15}}, \\
F_{16} &= (\{1\}, \{2\}_1, \{2\}_2, \{3\}), \\
F_{17} &= (\{1\}, \{2\}, \{3\}, \{4\}).
\end{align*}
\]

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\(F_{18} = (\{1\}, \{1\}, 2, 1, 2)\), \(\overline{F_{18}}\), \(F_{19} = (\{1\}, \{1\}, 2, 3)\), \(\overline{F_{19}}\),
\(F_{20} = (\{1, 3\}, \{2\}, 1, 2)\), \(F_{21} = (\{1\}, \{2\}, 2, 3)\), \(\overline{F_{21}}\),
\(F_{22} = (\{1\}, \{2\}, 3)\), \(\overline{F_{22}}\), \(F_{23} = (\{1\}, \{2\}, 1, 3)\), \(\overline{F_{23}}\),
\(F_{24} = (\{1\}, \{2\}, 1, 2)\), \(F_{25} = (\{1\}, \{2\}, 3, 4)\), \(\overline{F_{25}}\),
\(F_{26} = (\{1\}, \{2\}, 3)\), \(\overline{F_{26}}\), \(F_{27} = (\{1\}, \{2\}, 1, 3)\), \(\overline{F_{27}}\),
\(F_{28} = (\{1, 4\}, \{2\}, 3)\), \(F_{29} = (\{1, 2\}, \{1, 3\})\), \(\overline{F_{29}}\),
\(F_{30} = (\{1, 2\}, \{2, 3\})\), \(F_{31} = (\{1, 2\}, \{1, 2\})\), \(F_{32} = (\{1, 2\}, \{3, 4\})\),
\(F_{33} = (\{1, 4\}, \{2, 3\})\), \(F_{34} = (\{1, 3\}, \{2, 4\})\), \(F_{35} = (\{1\}, \{1, 2, 3\})\),
\(\overline{F_{35}}\), \(F_{36} = (\{1, 2, 3\}, \{2\})\), \(F_{37} = (\{1\}, \{2, 3, 4\})\), \(\overline{F_{37}}\),
\(F_{38} = (\{1, 3, 4\}, \{2\})\), \(\overline{F_{38}}\), and \(F_{39} = (\{1, 2, 3, 4\})\).

The formulae in the table below hold for every \(n \in \mathbb{N}\) if it is not written else. The omitted values are: \(\text{ex}_e(F_k, 1) = \text{ex}_i(F_k, 1) = 1\) for every \(k\), \(\text{ex}_e(F_7, 2) = 4\), \(\text{ex}_e(F_{12}, 2) = 3\), \(\text{ex}_i(F_{12}, 2) = 4\), \(\text{ex}_i(F_{17}, 3) = 12\), \(\text{ex}_i(F_{18}, 2) = 4\), \(\text{ex}_i(F_{18}, 3) = 8\), \(\text{ex}_i(F_{18}, 4) = 11\), \(\text{ex}_i(F_{18}, 5) = 15\), and \(\text{ex}_i(F_{30}, 3) = 8\). In the first column, numbers \(\overline{F}_k\) with bar indicate that \(F_k\) is nonisomorphic to \(\overline{F}_k\) and thus the formulae in the \(k\)-th row apply to two hypergraphs.

| \(k\) | \(\text{ex}_e(F_k, n)\) | \(\text{ex}_i(F_k, n)\) |
|------|-------------------|-------------------|
| 1    | not defined       | not defined       |
| 2    | \(n\)             | \(n\)             |
| 3    | 1, 1, 1, \ldots   | \(n\)             |
| 4    | \(n\)             | \(n\)             |
| 5    | \([3n/2]\)         | \(2n\) \((n > 1)\) |
| 6    | \(n\)             | \(2n - 1\)        |
| 7    | 1, 3, 2, 2, \ldots | \(2n - 1\) \((n \neq 2)\) |
| 8    | \(n\)             | \(2n - 1\)        |
| 9    | 2\(n - 1\)        | 3\(n - 2\)        |
| 10   | 2\(n - 1\)        | 3\(n - 2\)        |
| 11   | \((n^2 + n)/2\)    | \(n^2\)           |
| 12   | 2\(n\) \((n > 2)\) | 3\(n\) \((n > 2)\) |
| 13   | 2\(n - 1\)        | \([7(n - 1)/2] + 1\) |
| 14   | \(n + 1\) \((n > 1)\) | 3\(n - 2\) |
| 15   | \(n + 1\) \((n > 1)\) | 3\(n - 2\) |
| 16   | \(n + 1\) \((n > 1)\) | 3\(n - 2\) |
The results for $k = 3, 7, 17$ are particular cases of Theorems 2.1 and 2.3. Cases $k = 33$ and 34 were proved already in Klazar [5].

Suppose $H$ is a simple hypergraph such that $H \not> F$ for some $F, E \in H$ is an edge, and $a \in E$ is a vertex such that $\{a\} \not\in H$. Replacing $E$ with $\{a\}$ we obtain a hypergraph $H'$ with the same size as $H$ and possibly smaller order. Moreover, $H'$ is simple and $H' \not> F$. Repeating the replacements, in the end we obtain a singleton completion $H'$ of $H$ with these properties: $H'$ is simple, $H' \not> F$, $e(H') = e(H)$, $v(H') \leq v(H)$, and $\{a\} \in H'$ for every $a \in \bigcup H'$. Singleton completion helps to determine $\text{ex}_e(F, n)$ if $F$ has at least one singleton edge; we used it already in the proof of Theorem 2.2.

**Proposition 3.1** For every $n \in \mathbb{N}$, $\text{ex}_e(F_6, n) = n$ and $\text{ex}_i(F_6, n) = 2n - 1$. 

| $k$ | $\text{ex}_e(F_k, n)$ | $\text{ex}_i(F_k, n)$ |
|-----|----------------------|----------------------|
| 17  | 1, 3, 7, 4, 4, ...   | $3n - 2 \ (n \neq 3)$|
| 18  | $2n - 1$              | $4n - 6 \ (n > 5)$   |
| 19  | $2n - 1$              | $3n - 2$             |
| 20  | $2n - 1$              | $3n - 2$             |
| 21  | $2n - 1$              | $3n - 2$             |
| 22  | $2n - 1$              | $3n - 2$             |
| 23  | $2n - 1$              | $3n - 2$             |
| 24  | $n$                   | $2n - 1$             |
| 25  | $4n - 5 \ (n > 1)$    | $8n - 12 \ (n > 1)$  |
| 26  | $4n - 5 \ (n > 1)$    | $8n - 12 \ (n > 1)$  |
| 27  | $4n - 5 \ (n > 1)$    | $8n - 12 \ (n > 1)$  |
| 28  | $4n - 5 \ (n > 1)$    | $8n - 12 \ (n > 1)$  |
| 29  | $2n - 1$              | $4n - 4 \ (n > 1)$   |
| 30  | $(n^2/4) + n$         | $2 \lfloor n^2/4 \rfloor + n \ (n \neq 3)$ |
| 31  | $(n^2 + n)/2$         | $n^2$                |
| 32  | $2 \lfloor (n + 1)^2/4 \rfloor - 1$ | $5 \lfloor (n + 1)^2/4 \rfloor - 2n - 2$ |
| 33  | $4n - 5 \ (n > 1)$    | $8n - 12 \ (n > 1)$  |
| 34  | $4n - 5 \ (n > 1)$    | $8n - 12 \ (n > 1)$  |
| 35  | $(n^2 + n)/2$         | $n^2$                |
| 36  | $(n^2 + n)/2$         | $n^2$                |
| 37  | $n^2 - n + 1$         | $(5n^2 - 9n + 6)/2$  |
| 38  | $n^2 - n + 1$         | $(5n^2 - 9n + 6)/2$  |
| 39  | $(n^3 + 5n)/6$        | $(n^3 - n^2 + 2n)/2$ |
Proposition 3.2 For every \( n \in \mathbb{N} \), \( \text{ex}_e(F_5, n) = \lfloor 3n/2 \rfloor \) and \( \text{ex}_i(F_5, n) = 2n \) \((n > 1)\). For every \( n > 2 \), \( \text{ex}_e(F_{12}, n) = 2n \) and \( \text{ex}_i(F_{12}, n) = 3n \).

Proof. The conditions \( H \not\supset F_5 \) and \( H \not\supset F_{12} \) are equivalent, respectively, with \( \deg_H(v) \leq 2 \) and \( \deg_H(v) \leq 3 \) for every \( v \in \bigcup H \). Thus the results for \( \text{ex}_i(F_5, n) \) and \( \text{ex}_i(F_{12}, n) \) are clear.

We have \( \text{ex}_e(F_5, n) \geq n + \lfloor n/2 \rfloor \) because \( \{\{i\}, \{2j - 1, 2j\} : i \in [n], j \in \lfloor [n/2] \rfloor \} \not\supset F_5 \). Let \( H \) be any simple hypergraph with \( H \not\supset F_5 \) and \( v(H) = n \) and let \( H' \) be its singleton completion, \( v(H') = m \leq n \). It follows that \( e(H) = e(H') \leq m + \lfloor m/2 \rfloor \leq n + \lfloor n/2 \rfloor \) because the nonsingleton edges of \( H' \) must be mutually disjoint.

We have \( \text{ex}_e(F_{12}, n) \geq 2n \) \((n > 2)\) because \( \{\{i\}, \{i, i + 1\} \pmod{n} : i \in [n] \} \not\supset F_{12} \). Let \( H \) be any simple hypergraph with \( H \not\supset F_{12} \) and \( v(H) = n \) and let \( H' \) be its singleton completion. If \(|E| \geq 3\) for an edge \( E \in H' \), then \( E_1 \not\in H' \) for some \( E_1 \subset E \) with \(|E_1| = 2\). Replacing, one by one, \( E \) with \( E_1 \), we get rid of all edges with three and more vertices. We obtain a simple \( H'' \) such that \( H'' \not\supset F_{12} \), \( v(H'') = m \leq n \), \( e(H'') = e(H') = e(H) \), \(|E| \leq 2\) for every \( E \in H'' \), and \( \{a\} \in H'' \) for every \( a \in \bigcup H' \). Hence \( e(H) = e(H'') \leq m + m \leq 2n \) because the 2-element edges of \( H'' \) must form disjoint paths and cycles (every vertex is contained in at most two 2-element edges). \(\square\)

The next result answers our second initial question.
Proposition 3.3  For every \( n \in \mathbb{N} \), \( \text{ex}(F_{18}, n) = 2n - 1 \). As for the other function, \( \text{ex}(F_{18}, 1) = 1 \), \( \text{ex}(F_{18}, 2) = 4 \), \( \text{ex}(F_{18}, 3) = 8 \), \( \text{ex}(F_{18}, 4) = 11 \), \( \text{ex}(F_{18}, 5) = 15 \), and \( \text{ex}(F_{18}, n) = 4n - 6 \) for \( n \geq 6 \).

Proof. We have \( \text{ex}(F_{18}, n) \geq 2n - 1 \) because \( (\{i\}, \{i, n\}, \{n\} : i \in [n-1]) \not\in F_{18} \). Let \( H \) be any simple hypergraph with \( H \not\supset F_{18} \) and \( v(H) = n \) and let \( H' \) be its singleton completion, \( v(H') = m \leq n \). In \( H' \), every two nonsingleton edges may intersect only in the common last vertex. Deleting from each nonsingleton edge of \( H' \) its last vertex, we obtain mutually disjoint subsets of \( [n-1] \). Hence \( e(H) = e(H') \leq m + n - 1 \leq 2n - 1 \).

We determine \( \text{ex}(F_{18}, n) \); this is not as easy as it might seem. We have \( \text{ex}(F_{18}, n) \geq 4n - 6 \) for \( n \geq 6 \) because \( (\{i, n-1\}, \{i, n\}, \{n-1\}, \{n\} : i \in [n-2]) \not\in F_{18} \). To prove the opposite inequality, consider a simple \( F_{18} \)-free \( H \) with \( \cup H = [n] \). Since \( H \not\supset F_{18} \), \( \deg(1) \leq 2 \). We delete 1 from \( H \) and obtain \( H_1 \); \( i(H_1) \leq i(H) + 2 \). \( H_1 \) has at most two duplicated edges. Let \( E_1 = E_2 \) be one of the duplications. If \( |E_1| = 1 \), we delete \( E_1 \) from \( H_1 \). If \( |E_1| \geq 2 \), we delete from \( E_1 \) its last vertex. This creates no new duplication (else \( H \supset F_{18} \)). In this way we remove from \( H_1 \) both possible duplications and obtain a simple \( H_2 \) with \( \cup H_2 = [2, n] \) and \( i(H) \leq 4 + i(H_2) \). We have the inductive inequality \( i(H) \leq 4 + \text{ex}(F_{18}, n - 1) \). Note that \( \deg_H(2) \leq 2 \) and thus for induction we may as well delete 2 instead of 1. If one of \( \{1\}, \{2\}, \) and \( \{1, 2\} \) is an edge of \( H \), then the deletion of \( \{1\} \) or \( \{2\} \) and the removal of at most one duplication give us the stronger bound \( i(H) \leq 3 + \text{ex}(F_{18}, n - 1) \). Note also that \( \deg_H(v) \geq 3 \) implies that \( v \) is the last vertex of every edge containing it.

We prove that for \( n = 1, 2, 3, 4, 5, \) and 6 one has \( \text{ex}(F_{18}, n) = 1, 4, 8, 11, 15, \) and 18, and that \( \text{ex}(F_{18}, n) \leq 4n - 6 \) for \( n \geq 6 \). The first two values are trivial. By the inductive inequality, \( \text{ex}(F_{18}, 3) \leq 4 + 4 = 8 \). Weight 8 is attained by \( (\{3\}, \{1, 3\}, \{2, 3\}, \{3\}) \). Let \( n = 4 \), \( H \) be simple and \( F_{18} \)-free, and \( \cup H = [4] \). Clearly, \( \deg(1), \deg(2) \leq 2 \). Let first \( \deg(3) \geq 3 \) and \( p \) be the number of edges in \( H \) intersecting both \([2]\) and \([3, 4]\). Clearly, \( p \leq \deg(1) + \deg(2) \leq 4 \). Since no edge can contain both 3 and 4, \( \deg(3) + \deg(4) \leq p + 2 \leq 6 \) and \( i(H) = \sum_1^4 \deg(i) \leq 2 \cdot 2 + 6 = 10 \). Now let \( \deg(3) \leq 2 \) and \( p \) be the number of edges \( E \in H \) such that \( 4 \in E \) and \( E \cap [3] \neq \emptyset \). Then \( p \leq \text{ex}(F_{15}, 3) = 4, \) \( \deg(4) \leq 1 + p \leq 5 \) and \( i(H) = \sum_1^4 \deg(i) \leq 3 \cdot 2 + 5 = 11 \). Weight 11 is attained by \( (\{4\}, \{i, 4\}, \{4\} : i \in [3]) \). Thus \( \text{ex}(F_{18}, 4) = 11 \). By the inductive inequality, \( \text{ex}(F_{18}, 5) \leq 4 + 11 = 15 \). Weight 15 is attained by \( (\{5\}, \{i, 5\}, \{2j - 1, 2j, 5\} : i \in [4], j \in [2]) \).
It remains to show that \( \text{ex}_i(F_{18}, 6) = 18 \) and not \( 4 + 15 = 19 \). Weight 18 is attained by \( \{6\}, \{i, 6\}, \{1, 2, 6\}, \{3, 4, 5, 6\} : i \in [5] \). We elaborate the argument that we used for \( n = 4 \). Let \( H, \cup H = [6] \), be simple and \( F_{18}\)-free. Clearly, \( \deg(1), \deg(2) \leq 2 \) and \( \deg(3) \leq 4 \). If \( \deg(3) = 4 \), no edge intersects both \( [3] \) and \( [4, 6] \) and \( i(H) \leq 2 \cdot \text{ex}_i(F_{18}, 3) = 16 \). If \( \deg(3) = 3 \), we delete 3 from \( H \). If this creates a duplication, one of \( \{1\}, \{2\} \) or \( \{1, 2\} \) is an edge of \( H \) and by the above remark, \( i(H) \leq 3 + \text{ex}_i(F_{18}, 5) = 18 \). If no duplication arises, again \( i(H) \leq \deg(3) + \text{ex}_i(F_{18}, 5) = 18 \). So \( \deg(3) \leq 2 \). Let \( k = \deg(4) \). Let first \( k \geq 3 \) and \( p \) be the number of edges intersecting both \( [4] \) and \( [5, 6] \) (none of them contains 4). If \( 4 \in E \in H \) then \( 4 = \max E \). Therefore edges incident with 4 contribute by at least \( k - 1 \) to \( \deg(1) + \deg(2) + \deg(3) \leq 6 \) and thus \( k \leq 7 \) and \( p \leq 6 - (k - 1) = 7 - k \). If \( \deg(5) \geq 3 \), \( \deg(5) + \deg(6) \leq p + 2 \leq 9 - k \) (no edge contains both 5 and 6) and \( i(H) = \sum_1^6 \deg(i) \leq 3 \cdot 2 + 9 - k = 15 \). If \( \deg(5) \leq 2 \), we have \( \deg(6) \leq 2 + p \leq 9 - k \) and \( i(H) \leq 4 \cdot 2 + 9 - k = 17 \). We may assume that \( k = \deg(4) \leq 2 \) and thus \( \deg(i) \leq 2 \) for every \( i \in [4] \). If \( \deg(5) \geq 3 \), we again set \( p \) to be the number of edges \( E \in H \) intersecting both \( [4] \) and \( [5, 6] \). We have \( p \leq 4 \cdot 2 = 8 \) and \( \deg(5) + \deg(6) \leq p + 2 \leq 10 \). Thus \( i(H) = \sum_1^6 \deg(i) \leq 4 \cdot 2 + 10 = 18 \). If \( \deg(5) \leq 2 \), let \( p \) be the number of edges \( E \in H \) intersecting \( [5] \) and containing 6. Then \( p \leq \text{ex}_e(F_5, 5) = 7 \) and \( \deg(6) \leq 1 + p \leq 8 \). We have again \( i(H) = \sum_1^6 \deg(i) \leq 5 \cdot 2 + 8 = 18 \). Thus \( \text{ex}_e(F_{18}, 6) = 18 \).

Finally, using induction starting at \( n = 6 \) and the inductive inequality, we see that for \( n \geq 6 \) we have \( \text{ex}_e(F_{18}, n) \leq 4n - 6 \).

The irregular initial behaviour of \( \text{ex}_e(F_{18}, n) \) permits to start the induction only from \( n = 6 \). This makes \( \text{ex}_e(F_{18}, n) \) the hardest function of the table to determine.

We have chosen to present the following case because its treatment in [5] contains errors.

**Proposition 3.4** For every \( n \in \mathbb{N} \), \( \text{ex}_e(F_{29}, n) = 2n - 1 \). For every \( n > 1 \), \( \text{ex}_e(F_{29}, n) = 4n - 4 \) (and \( \text{ex}_e(F_{29}, 1) = 1 \)).

**Proof.** We have \( \text{ex}_e(F_{29}, n) \geq 2n - 1 \). Let \( H \) be any simple \( F_{29}\)-free hypergraph with \( v(H) = n \). It follows that the first vertices of the nonsingleton edges of \( H \) must be all distinct. Thus \( e(H) \leq n + n - 1 = 2n - 1 \).

We have \( \text{ex}_e(F_{29}, n) \geq 4n - 4 \) (for \( n > 1 \)) because \( \{\{i\}, \{j, n - 1, n\}, \{n - 1, n\} : i \in [n], j \in [n - 2]\} \not\ni F_{29} \). Let \( H \) be any simple \( F_{29}\)-free hypergraph.
with $\bigcup H = [n]$, $n > 1$. We delete 1 from $H$ and obtain $H'$. From the previous argument we know that $\deg_H(1) \leq 2$. Thus $i(H) \leq i(H') + 2$. One duplication may appear in $H'$ if $A \subset H$ and $\{1\} \cup A \in H$ for some $A \subset [2,n]$. If this happens, we delete (one) $A$ from $H'$ and obtain $H''$. Else we set $H'' = H'$. $H''$ is simple, $F_{29}$-free and $\bigcup H'' = [2,n]$. $|A| \geq 3$ implies $H \succ F_{29}$ which is forbidden. Thus $|A| \leq 2$ and we have the inductive inequality $i(H) \leq i(H') + 2 \leq i(H'') + 4 \leq \text{ex}_i(F_{29}, n - 1) + 4$. Starting from $\text{ex}_i(F_{29}, 2) = 4$, induction shows that $\text{ex}_i(F_{29}, n) \leq 4n - 4$. \hfill $\Box$

The next result answers our first initial question.

**Proposition 3.5** For every $n \in \mathbb{N}$, $\text{ex}_i(F_{30}, n) = \lfloor n^2/4 \rfloor + n$. We have $\text{ex}_i(F_{30}, n) = 2 \lfloor n^2/4 \rfloor + n$ for $n \neq 3$ and $\text{ex}_i(F_{30}, 3) = 8$.

**Proof.** We have $\text{ex}_i(F_{30}, n) \geq \lfloor n^2/4 \rfloor + n$ because $B_n = (\{i\}, \{j, k\} : i \in [n], j \in [[n/2]], k \in [[n/2] + 1, n]) \not\succ F_{30}$. Let $H$ be any simple $F_{30}$-free hypergraph with $\bigcup H = [n]$. If $|E| \geq 3$ for some $E \in H$, we replace $E$ with the two-element set consisting of the first two vertices of $E$. The resulting hypergraph is $F_{30}$-free and, since $H \not\succ F_{30}$, it is simple. Repeating the replacements, we get rid of all edges with three and more elements and may assume that $|E| \leq 2$ for every $E \in H$. The two-element edges of $H$ form a triangle-free graph on at most $n$ vertices. By a special case of Turán’s theorem (see [8, Problem 10.30]), $e(H) \leq n + \lfloor n^2/4 \rfloor$.

The lower bound on $\text{ex}_i(F_{30}, n)$ is provided again by $B_n$. We show that the maximum weight is attained also by $B_n$ with the exception of $n = 3$ when $\text{ex}_i(F_{30}, 3) = 8$ and not 7. We take any simple $F_{30}$-free hypergraph $H$ with $\bigcup H = [n]$ and eliminate large edges. If $E = \{a_1, a_2, \ldots, a_t\} \in H$ with $t \geq 4$ and $a_1 < a_2 < \ldots < a_t$, we replace $E$ with the edges $\{a_1, a_{t-1}\}, \{a_2, a_{t-1}\}, \ldots, \{a_{t-2}, a_{t-1}\}$. The resulting hypergraph $H'$ is simple, $F_{30}$-free, and satisfies $v(H') \leq v(H)$ and $i(H') \geq i(H)$. In this way we eliminate all edges with four or more elements. If $t = 3$ and $a_3 < n$, we replace $E$ with $\{a_2, a_3\}$ and $\{a_2, n\}$. Similarly if $1 < a_1$. Thus for bounding $i(H)$ from above we may assume that $|E| \leq 3$ for every $E \in H$ and that every 3-element edge, say $H$ has $k$ of them, is of the form $\{1, a, n\}$. No two-element edge is incident with any of the $a$’s and they form a triangle-free graph on at most $n - k$ vertices. By Turán’s theorem, $i(H) \leq n + 2\lfloor (n-k)^2/4 \rfloor + 3k$ and the bound is attained. For $n \geq 4$ it is maximized for $k = 0$ and for $n = 3$ for $k = 1$. Indeed, $(\{1\}, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\})$ has weight 8 and $(\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\})$ has weight 7. \hfill $\Box$
For each $F$ with $i(F) \leq 4$ it was not too hard to determine its extremal functions but for $i(F) = 5$ or 6 difficult cases start to appear. For example, it would be interesting to know what are $\text{ex}_c(F, n)$ and $\text{ex}_s(F, n)$, or even the graph version of $\text{ex}_c(F, n)$, if $F = (\{1, 6\}, \{2, 5\}, \{3, 4\})$ or if $F = (\{1, 2\}, \{2, 4\}, \{3, 5\})$ or if $F$ is some other ordered graph with three edges (there are 75 of them, 62 simple, see the table in the next section).

4 Enumeration of hypergraphs

For a hypergraph $F$ and $n \in \mathbb{N}$, we let $h_n(F)$ denote the number of all simple nonisomorphic $F$-free hypergraphs $H$ with $v(H) = n$. Let $h'_n(F)$ and $h''_n(F)$ be the analogous counting functions with $v(H) = n$ replaced by $i(H) = n$ and with the simplicity of $H$ dropped in $h''_n(F)$. Remember that we work with the ordered isomorphism; e.g., $F_{29} = (\{1, 2\}, \{1, 3\})$ and $F_{30} = (\{1, 2\}, \{2, 3\})$ are nonisomorphic. The enumerative problems to determine or to bound these counting functions are already for $i(F) \leq 4$ much more difficult than the extremal problems. It suffices to note, for example, that if $F = F_2 = (\{1\}_1, \{1\}_2)$ then $h_n(F) = h'_n(F) = h''_n(F) = b_n$ where $b_n$ is the Bell number that counts the partitions of $[n]$.

In Klazar [5] we found the ordinary generating functions $G_1(x)$, $G_2(x)$, and $G_3(x)$ of $h_n(F_{34})$, $h'_n(F_{34})$, and $h''_n(F_{34})$, respectively. (Recall that $F_{34} = (\{1, 3\}, \{2, 4\})$.) $G_1$, $G_2$, and $G_3$ are algebraic over $\mathbb{Z}(x)$ of degrees 3, 4, and 4, respectively, and their coefficients grow roughly like $(63.97055\ldots)^n$, $(5.79950\ldots)^n$, and $(6.06688\ldots)^n$ where the bases of the exponentials are algebraic numbers of degrees 4, 15, and 23, respectively. We did not succeed in enumerating $F_{33}$-free hypergraphs ($F_{33} = (\{1, 4\}, \{2, 3\})$) and we think it is a problem that deserves interest.

Here we shall investigate the total numbers $h_n$, $h'_n$, and $h''_n$ of, respectively, all simple nonisomorphic hypergraphs with $n$ vertices, all simple nonisomorphic hypergraphs with weight $n$, and all nonisomorphic hypergraphs with weight $n$. The numbers $h_n$ have been considered before in the problem of set covers but the remaining two problems seem new. We review the known formulae for $h_n$, derive for them a new recurrence, and then proceed to $h'_n$ and $h''_n$.

**Proposition 4.1** The numbers $h_n$ of nonisomorphic simple hypergraphs with
$n$ vertices satisfy for every $n \geq 1$ the following formulae.

1. $h_n = 2^{2^n-1} - \sum_{j=0}^{n-1} \binom{n}{j} h_j \quad (h_0 = 1)$

2. $h_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} 2^{2^j-1}$

3. $h_n = 2 \sum_{k,l \geq 0} \frac{h_k h_l \cdot (n-1)!}{(k+l-n+1)! \cdot (n-1-k)! \cdot (n-1-l)!} - h_{n-1}$

where in 3 the summation range is $\max(k,l) \leq n-1 \leq k+l$.

**Proof.** 1. This recurrence is proved in Hearne and Wagner [4] and is a rearrangement of the identity $2^{2^n-1} = \sum_{j=0}^{n} \binom{n}{j} h_j$.

The identity follows by noting that every simple hypergraph with $j \leq n$ vertices is isomorphic to exactly $\binom{n}{j}$ hypergraphs $H$ with $v(H) = j$ and $\bigcup H \subset [n]$, and that the simple hypergraphs $H$ with $\bigcup H \subset [n]$ correspond bijectively to the elements of the power set of the set $\{X \subset [n] : X \neq \emptyset\}$.

2. This formula is proved in Comtet [2, p. 165] and also in Macula [9]. We note that the identity of 1 is equivalent to $F(x) = e^x H(x)$ where

$$
F(x) = \sum_{n \geq 0} \frac{2^{2^n-1} x^n}{n!} \quad \text{and} \quad H(x) = \sum_{n \geq 0} \frac{h_n x^n}{n!}
$$

are exponential generating functions of the involved quantities. Thus $H(x) = e^{-x} F(x)$ and the formula follows.

3. This recurrence follows from the combinatorial definition of $h_n$. Any simple hypergraph $H$ with $\bigcup H = [n]$ decomposes uniquely into two hypergraphs $H_1$ and $H_2$: $H_1$ consists of the sets $E \setminus \{1\}$ such that $1 \in E \in H$ (we omit the $\emptyset$ if $\{1\} \in H$) and $H_2$ consists of the remaining edges of $H$ not containing 1. We relabel the vertices by an increasing injection so that $\bigcup H_1 = [k]$ and $\bigcup H_2 = [l]$. It is clear that $H_1$ and $H_2$ are simple and that $k, l \leq n-1$. To invert the decomposition, we first select two simple hypergraphs $H_1$ and $H_2$ with $\bigcup H_1 = [k]$ and $\bigcup H_2 = [l]$, which can be done in
We relabel their vertices and unite the vertex sets so that the set $[2, n]$ arises. This can be done in exactly 

$$\binom{n-1}{k+l-n+1, n-1-k, n-1-l}$$

ways by partitioning $[2, n]$ in $k+l-n+1, n-1-k,$ and $n-1-l$ vertices lying in $C = \bigcup H_1 \cap \bigcup H_2, \bigcup H_2 \setminus C,$ and $\bigcup H_1 \setminus C,$ respectively. We append to every edge in $H_1$ the new least vertex 1 and obtain a simple hypergraph $H$ with $n$ vertices. Finally, the possible addition of $\{1\}$ to $H$ (we always loose the edge $\{1\}$ when decomposing) gives two further options, with the exception of $H_1 = \emptyset$ when $\{1\}$ must be always added. This explains the factor 2 and the subtraction of $h_{n-1}$. The stated recurrence follows.

Either of the recurrences 1 and 3 or the explicit formula 2 give 

$$(h_n)_{n \geq 1} = (1, 5, 109, 32297, 21473223970362989, \ldots).$$

This quickly growing sequence is entry A003465 of Sloane [14].

We proceed to the problem of counting hypergraphs, simple and all, by their weight. The enumeration of all hypergraphs $F$ with $i(F) \leq 4$ in Section 3 shows that $(h'_n)_{n \geq 1} = (1, 2, 7, 28, \ldots)$ and $(h''_n)_{n \geq 1} = (1, 3, 10, 41, \ldots)$. We derive some formulae and algorithms which produce further terms of these sequences. Recall that a partition $\lambda = 1^{a_1}2^{a_2}\ldots l^{a_l}$ of $n \in \mathbb{N}$, where $a_i \geq 0$ are integers and $a_l > 0$, is the decomposition $n = 1+1+\cdots+1+2+\cdots+2+\cdots+l+\cdots+l$ with the part $i$ appearing $a_i$ times. Thus $\sum_i ia_i = n$. We write briefly $\lambda \vdash n$. If the hypergraph $H$ has weight $n$ and $a_i$ edges of cardinality $i$, the maximum edge cardinality being $l$, then $\lambda = 1^{a_1}2^{a_2}\ldots l^{a_l} \vdash n$ and we say that $H$ has edge type $\lambda$. We begin with counting hypergraphs with a fixed edge type.

**Theorem 4.2** Let $\lambda = 1^{a_1}2^{a_2}\ldots l^{a_l} \vdash n$ where $a_l > 0$. The number of nonisomorphic simple hypergraphs with weight $n$ and edge type $\lambda$ is

$$\sum_{j=l}^{n} \binom{j}{a_1} \binom{j}{a_2} \cdots \binom{j}{a_l} \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j},$$

and the number of nonisomorphic hypergraphs with weight $n$ and edge type $\lambda$ is

$$\sum_{j=l}^{n} \binom{j}{a_1} + 1 \binom{j}{a_2} + 1 \cdots \binom{j}{a_l} \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j}. $$

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Proof. Consider the polynomials
\[ W_n = W_n(x_1, x_2, \ldots, x_n) = \sum_{H} \prod_{i=1}^{n} x_{e(i,H)} \]
where we sum over all simple \( H \) with \( \bigcup H = [n] \), and \( e(i, H) \) is the number of \( i \)-element edges in \( H \). We refine the identity from the proof of 1 of Proposition 4.1 (which corresponds to \( x_1 = x_2 = \cdots = x_n = 1 \)) and obtain
\[ \prod_{i=1}^{n} (1 + x_i)^{\binom{n}{i}} = \sum_{j=0}^{n} \binom{n}{j} W_j. \]

In terms of exponential generating functions,
\[ \sum_{n \geq 0} \frac{y^n}{n!} \cdot \prod_{i=1}^{n} (1 + x_i)^{\binom{n}{i}} = e^y \cdot \sum_{n \geq 0} W_n y^n / n!. \]
We invert this relation as in the proof of 2 of Proposition 4.1 and get
\[ W_n(x_1, \ldots, x_n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \prod_{i=1}^{j} (1 + x_i)^{\binom{j}{i}}. \]
The number of nonisomorphic simple hypergraphs \( H \) with \( i(H) = n \) and edge type \( \lambda = 1^{a_1} 2^{a_2} \ldots l^{a_l} \vdash n \) is the coefficient at \( x_1^{a_1} \ldots x_l^{a_l} \) in \( W_l + W_{l+1} + \cdots + W_n \) which equals
\[ \sum_{m=l}^{n} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \prod_{i=1}^{j} \binom{j}{a_i} = \sum_{j=l}^{n} \prod_{i=1}^{l} \binom{j}{a_i} \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j}. \]
The derivation of the second formula is similar, only \( W_n \) becomes a power series and \( 1 + x_i \) is replaced by \((1 - x_i)^{-1}\) because now any \( i \)-element edge may come in arbitrary many copies. \( \square \)

We give for illustration the distribution of hypergraphs with weight 6 by their edge types. The first entry is the number of simple hypergraphs and the second, given only if different, is the number of all hypergraphs:

| \( \lambda \) | \( #H \) | \( 6^1 \) | \( 1^1 5^1 \) | \( 2^1 4^1 \) | \( 1^2 4^1 \) | \( 3^2 \) | \( 1^1 2^1 3^1 \) | \( 1^3 3^1 \) |
|---|---|---|---|---|---|---|---|---|
| \( 2^3 \) | \( 62, 75 \) | \( 1^2 2^2 \) | \( 1^4 2^1 \) | \( 6^1 \) |
| \( 1^1 \) | \( 11 \) | \( 41 \) | \( 41, 50 \) | \( 31, 32 \) | \( 239 \) | \( 63, 120 \) |

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Collecting the numbers over all edge types, we obtain formulae for the numbers \( h'_n \) and \( h''_n \).

**Corollary 4.3** The numbers of nonisomorphic hypergraphs with weight \( n \), simple and all, are (\( \lambda = 1^{a_1}2^{a_2} \ldots l^{a_l} \) with \( a_l > 0 \))

\[
\begin{align*}
    h'_n &= \sum_{\lambda \vdash n} \prod_{j=l=1}^{n} \binom{j}{i} a_i \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j} \\
    h''_n &= \sum_{\lambda \vdash n} \prod_{j=l=1}^{n} \left( \binom{j}{i} + a_i - 1 \right) \sum_{m=j}^{n} (-1)^{m-j} \binom{m}{j}.
\end{align*}
\]

Using these formulae and computer algebra system MAPLE, we have found the following values.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|
| \( h'_n \) | 1 | 2 | 7 | 28 | 134 | 729 | 4408 | 29256 | 210710 | 1633107 | 13528646 | 119117240 |
| \( h''_n \) | 1 | 3 | 10 | 41 | 192 | 1025 | 6087 | 39754 | 282241 | 2159916 | 17691161 | 154192692 |

Each of the three formulae in Proposition 4.1 gives an algorithm that calculates \( h_n \) in \( O(n^c) \) arithmetical operations. In fact, formula 2 requires only \( O(n) \) operations. In contrast, Corollary 4.3 gives algorithms that calculate \( h'_n \) and \( h''_n \) in roughly \( n^c p(n) \) operations, where \( p(n) = \left\lfloor \frac{n}{\sqrt[3]{3}} \right\rfloor \), which is a superpolynomial number because \( p(n) \sim (n \cdot 4\sqrt{3})^{-1} \cdot \exp(\pi(2n/3)^{1/2}) \) as found by Hardy and Ramanujan [3] (see also Andrews [11] and Newman [11, 12]). From the complexity point of view, Corollary 4.3 is much less effective than Proposition 4.1. On the other hand, it is superior to the trivial way of calculating \( h'_n \) and \( h''_n \) because these numbers grow superexponentially (see Proposition 4.6) but \( p(n) \) is subexponential. The number of operations required by Corollary 4.3 is therefore still substantially smaller than the number of objects enumerated by \( h'_n \) and \( h''_n \). We show that \( h'_n \) and \( h''_n \) can be calculated more effectively, again in \( O(n^c) \) arithmetical operations, by the approach that we used in the recurrence 3 of Proposition 4.1.
the hypergraphs $H$ any simple hypergraph We begin with the case of simple hypergraphs. We decompose where the summation range of the second sum is in both formula $e$ that $\sum_{i=1}^{d} (\sum_{j=1}^{d} a_{ij})$. These sums have $n^2$ summands. To obtain an effective algorithm for calculating $h'_n$ and $h''_n$, it suffices to establish effective recurrent relations for $h'_n,m,l$ and $h''_n,m,l$.

**Proposition 4.4** Let $T(a,b,c) = (b+c-a,a-a-b,c)$. We have $h'_{n,0,0} = h''_{n,0,0} = 1$, $h'_{n,m,l} = h''_{n,m,l} = 0$ if $nml = 0$ but $n + m + l > 0$, and, for $n \geq 1$ and $1 \leq m, l \leq n$,

$$h'_{n,m,l} = \sum_{p=1}^{m} \sum_{l_1,l_2} T(l-1,l_1,l_2) \cdot (h'_{n_1,p,l_1} + h'_{n_1,p-1,l_1}) \cdot h'_{n_2,m-p,l_2}$$

$$h''_{n,m,l} = \sum_{p=1}^{m} \sum_{l_1,l_2} T(l-1,l_1,l_2) \sum_{q=0}^{p} h''_{n_1,p-q,l_1} \cdot h''_{n_2,m-p,l_2},$$

where the summation range of the second sum is in both formulae $n_i \geq 0$, $l_i \geq 0$, $n_1 + n_2 = n - p$, and $\max(l_1, l_2) \leq l - 1 \leq l_1 + l_2$.

**Proof.** We begin with the case of simple hypergraphs. We decompose any simple hypergraph $H$ with $i(H) = n$, $e(H) = m$, and $\cup H = [l]$ in the hypergraphs $H_1$ and $H_2$, where $H_1 = (E \setminus \{1\} : 1 \in E \in H)$ and $H_1 = (E : 1 \not\in E \in H)$. If $\{1\} \not\in H$, we remove $\emptyset$ from $H_1$. We denote $p = \deg_H(1)$, $i(H_1) = n_1$, $i(H_2) = n_2$, $v(H_1) = l_1$, and $v(H_2) = l_2$. It is clear that $e(H_2) = m - p$ and that the conditions of the second sum are met. If $\{1\} \not\in H$ then $e(H_1) = p$ else $e(H_1) = p - 1$. The decomposition is inverted as in the proof of 3 of Proposition 4.1. The cases $\{1\} \not\in H$ and $\{1\} \in H$ are reflected by the terms $h'_{n_1,p,l_1}$ and $h''_{n_1,p-1,l_1}$, respectively. The trinomial $T(l-1,l_1,l_2)$ counts the number of ways in which the set $[2, l]$ can be written as a union of two sets with $l_1$ and $l_2$ elements. We obtain the first recurrence. The proof of the recurrence for all hypergraphs is similar, the only difference being that now $\{1\}$ may have in $H$ multiplicity $q$, $0 \leq q \leq p = \deg_H(1)$. □

The recurrences give algorithms that calculate $h'_n$ in $O(n^6)$ operations and $h''_n$ in $O(n^7)$ operations.
For every rational polynomial \( P(m) \in \mathbb{Q}[m] \) it is true that
\[
\sum_{m=0}^{\infty} \frac{P(m)}{m!} = e \cdot q
\]
where \( e = 2.71828 \ldots \) is Euler number and \( q \in \mathbb{Q} \). This follows by expressing \( P(m) \) as the \( \mathbb{Q} \)-linear combination in the basis \( \{1, m, m(m-1), m(m-1)(m-2), \ldots\} \). One subfamily of this family of identities is Dobiński’s formula ([8, Problems 1.9a and 1.13] and [2, p. 210])
\[
\sum_{m=0}^{\infty} \frac{m^n}{m!} = e \cdot b_n
\]
in which \( b_n \) is the \( n \)-th Bell number (the number of partitions of \( [n] \)). We present two combinatorial subfamilies which are related to hypergraphs.

**Corollary 4.5** For every \( n \in \mathbb{N} \) we have the identities (\( \lambda = 1^{a_1}2^{a_2} \ldots l^{a_l} \) with \( a_l > 0 \))
\[
\sum_{m=0}^{\infty} \frac{1}{m!} \cdot \sum_{\lambda \vdash n} \prod_{i=1}^{l} \left( \binom{m}{i} a_i \right) = e \cdot \sum_{i(H)=n} \frac{1}{v(H)!} \sum_{i(H)=n} \prod_{i=1}^{l} \left( \binom{m}{i} + a_i - 1 \right)
\]
where \( e = 2.71828 \ldots \) and the star indicates that the sum is over simple hypergraphs \( H \) only.

**Proof.** Let \( n \in \mathbb{N} \). In the proof of Theorem 4.2 we used for simple hypergraphs the equation
\[
\sum_{m \geq 0} \frac{y^m}{m!} \cdot \prod_{i=1}^{l} (1 + x_i)^{\binom{m}{i}} = e^y \cdot \sum_{m \geq 0} \frac{W_m y^m}{m!}.
\]
The first stated identity now follows by setting \( x_i = x^i, i \in \mathbb{N} \), comparing the coefficients at \( x^n \) on both sides, and setting \( y = 1 \). The second identity follows by the same way from the analogous equation for all hypergraphs. \( \square \)

For \( n = 1, 2, 3, \) and \( 4 \) the factors at \( e \) in the first identity are, respectively, \( 1, 1, \frac{11}{6}, \) and \( \frac{25}{8} \), and in the second identity they are \( 1, 2, \frac{23}{6}, \) and \( \frac{89}{8} \).

It is natural to ask about the asymptotics of \( h'_n \) and \( h''_n \). We give a simple estimate in terms of the Bell numbers \( b_n \).
Proposition 4.6  For every $n \in \mathbb{N}$, one has the inequalities

$$b_n \leq h'_n \leq h''_n \leq 2^{n-1} b_n.$$ 

For $n \to \infty$,

$$\log h''_n = \log b_n + O(n) = n(\log n - \log \log n + O(1))$$ 

and the same holds for $h'_n$.

**Proof.** The first two inequalities are trivial. To prove the third inequality, we assign to every hypergraph $H$, where $i(H) = n$ and $\bigcup H = [m]$ with $m \leq n$, a pair $(Q, P)$ of partitions of $[n]$ as follows. We set $Q = (I_1, I_2, \ldots, I_m)$ where $I_1 < I_2 < \ldots < I_m$ are intervals such that $|I_i| = \deg_H(i)$. Thus $Q$ is a partition of $[n]$ into intervals. For every $E \in H$ we select a set $A_E \subset [n]$, $|A_E| = |E|$, such that (i) for every $i \in [m]$, $A_E \cap I_i \neq \emptyset$ iff $i \in E$ and (ii) the sets $A_E$ are mutually disjoint. This can be done and generally in more than one way. We set $P = (A_E : E \in H)$. It is clear that, regardless of the freedom in selecting $P$, distinct hypergraphs $H$ produce distinct pairs $(Q, P)$. The number of pairs $(Q, P)$ does not exceed $2^{n-1} b_n$ because there are exactly $2^{n-1}$ interval partitions of $[n]$. Thus we have the inequality $h''_n \leq 2^{n-1} b_n$. The logarithmic asymptotics follows from the asymptotics of $b_n$ that was found by Moser and Wyman [10], see [8, Problem 1.9b] or Odlyzko [13].

It is an interesting question how tight is each of the three above inequalities. The previous argument made no use of the fact that the partitions $Q$ and $P$ are “orthogonal” in the sense that $|I \cap A| \leq 1$ for every $I \in Q$ and $A \in P$. Using this, we can narrow the gap in the estimate $b_n \leq h''_n \leq 2^{n-1} b_n$. We shall treat this topic elsewhere.

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