A fractional $q$-integral operator associated with a certain class of $q$-Bessel functions and $q$-generating series

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Abstract

This paper deals with Al-Salam fractional $q$-integral operator and its application to certain $q$-analogues of Bessel functions and power series. Al-Salam fractional $q$-integral operator has been applied to various types of $q$-Bessel functions and some power series of special type. It has been obtained for basic $q$-generating series, $q$-exponential and $q$-trigonometric functions as well. Various results and corollaries are provided as an application to this theory.

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1 Introduction

The theory of $q$-calculus is an old subject centered on the idea of deriving $q$-analogous results without using limits. Jackson was the first to develop the $q$-calculus theory in systematic way [1]. He defined the concept of the $q$-integral and the concept of the $q$-difference operator in a generic manner. In excellence, the theory of $q$-calculus allows to deal with sets of non-differentiable functions, different classes of orthogonal polynomials, integral operators, and various classes of special functions including $q$-hypergeometric functions, $q$-Bessel functions, $q$-gamma and $q$-beta functions, and many others, to mention but a few. It connects mathematics and physics and plays a significant role in various fields of physical sciences such as cosmic strings [2], conformal quantum mechanics [3], and nuclear physics of high energy [4]. It, further, applies to topics in number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics, and the theory of relativity.

The $q$-integrals from 0 to $\xi$ and from 0 to $\infty$ are, resp., defined by Jackson as [1]

$$\int_{0}^{\xi} f(t) \, dq_t = \xi (1 - q) \sum_{j=0}^{\infty} q^{j} f(\xi q^j)$$

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and
\[ \int_{0}^{\infty} f(t) \, dq t = (1 - q) \sum_{j \in \mathbb{Z}} \frac{q^j f \left( \frac{q^j}{A} \right)}{A}. \] (2)

The \( q \)-analogue of the Bessel function
\[ J_\mu(\xi) = \sum_{j=0}^{\infty} \frac{(-1)^j (\xi)^{\mu+j}}{j! \Gamma(\mu+j+1)} \] (3)
of the first type, which was studied later by Hahn [5] and Ismail [6], is defined by [7] as
\[ J_{\mu}^{(1)}(\xi; q) = \left( \frac{\xi}{2} \right)^\mu \sum_{j=0}^{\infty} \frac{(-\xi^2)^j}{(q;q)_{\mu+j}(q; q)}, \quad |\xi| < 2. \] (4)

Jackson defines the \( q \)-analogue of the Bessel function of the second type as [7]
\[ J_{\mu}^{(2)}(\xi; q) = \left( \frac{\xi}{2} \right)^\mu \sum_{j=0}^{\infty} \frac{q^{(\mu+j)}(\xi^2)^j}{(q;q)_{\mu+j}(q; q)}, \quad \xi \in \mathbb{C}. \] (5)

Hahn [8] and Exton [9] introduced the third type \( q \)-Bessel function (called Hahn–Exton \( q \)-Bessel function) as
\[ J_{\mu}^{(3)}(\xi; q) = \xi^\mu \sum_{j=0}^{\infty} \frac{(-1)^j q^{(\mu+j)}(\xi^2)^j}{(q;q)_{\mu+j}(q; q)}, \quad \xi \in \mathbb{C}. \] (6)

The \( q \)-shifted factorials are defined, in literature, by fixing \( \xi \in \mathbb{C} \) as
\[ (\xi; q)_0 = 1; \quad (\xi; q)_n = \prod_{j=0}^{n-1} (1 - q^j \xi), \quad n = 1, 2, \ldots; \quad (\xi; q)_\infty = \lim_{n \to \infty} (\xi; q)_n. \] (7)

This indeed gives
\[ (\xi; q)_x = \frac{(\xi; q)_\infty}{(\xi q^x; q)_\infty}, \quad x \in \mathbb{R}. \] (8)

For \( \xi \in \mathbb{C} \), we mean
\[ [\xi]_q = \frac{1 - q^x}{1 - q}. \]

Hence, for \( n \in \mathbb{N} \), we obtain
\[ ([n]_q)! = \frac{(q; q)_n}{(1 - q)^n}. \]

Due to [10, (1.5), (1.6)], we, resp., write
\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \] (9)
\[ \begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-1)^k q^{\alpha k - \frac{k(k-1)}{2}} = \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(k+1) \Gamma_q(\alpha - k)}. \tag{10} \]

The \(q\)-analogue of the exponential function of the second type is given by

\[ e_q(\xi) = \sum_{j=0}^{\infty} \frac{\xi^j}{(q; q)_j} = \frac{1}{(\xi; q)_\infty}, \quad |\xi| < 1, \tag{11} \]

whereas the \(q\)-analogue of the exponential function of the first type is given by

\[ E_q(\xi) = \sum_{j=0}^{\infty} \frac{(-1)^j q^{\frac{j(j-1)}{2}} \xi^j}{(q; q)_j} = (\xi; q)_\infty, \quad \xi \in \mathbb{C}. \]

Consequently, the following formula holds:

\[ (q^{\xi + m}; q)_\infty = \frac{(q^\xi; q)_\infty}{(q^\xi; q)_m}, \quad m \in \mathbb{N}. \tag{12} \]

For real arguments \(t\), the \(q\)-analogues of the gamma function are given by [11]

\[ \Gamma_q(t) = \int_0^1 x^{t-1} E_q(-qx) \, dq \quad \text{and} \quad \hat{\Gamma}_q(t) = \int_0^\infty x^{t-1} e_q(-x) \, dx. \tag{13} \]

Henceforth, for \(t \in \mathbb{R}\) and \(n \in \mathbb{N}\), the following auxiliary results hold:

\[ \Gamma_q(t + 1) = [t]_q \Gamma_q(t), \quad \Gamma_q(n + 1) = [n]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(t + 1) = \frac{1 - q^t}{1 - q} \Gamma_q(t). \tag{14} \]

The theory of fractional calculus was born in early 1695 due to a very deep question raised in a letter of L’Hospital to Leibniz [12–16]. During a long period of time (300 years), the fractional calculus has kept the attention of top level mathematicians. It has become a very useful tool for tackling dynamics of complex systems from various branches of science and engineering. The fractional \(q\)-calculus is the \(q\)-extension of the ordinary fractional calculus. Integral operators have attained their popularity due to their wide range of applications in various fields of science and engineering [17–22] and [23–34]. In [35, 36] Al-Salam and Agarwal studied certain \(q\)-fractional integrals and derivatives. Recently, perhaps due to explosion in research within the fractional calculus setting, new developments in the theory of fractional \(q\)-difference calculus, specifically, the \(q\)-analogues of the integral and the differential fractional operator properties were made, see, e.g., [37–39]. In [36, p. 966], Al-Salam defines a fractional \(q\)-integral operator in the form of the basic integral

\[ K_q^{\alpha} f(x) = \frac{q^{-\alpha} x^\alpha}{\Gamma_q(\alpha)} \int_x^\infty (y-x)^{\alpha-1} y^{-\alpha} f(y q^{1-\alpha}) \, d(y; q), \tag{15} \]
provided $\alpha \not\equiv 0, -1, -2, \ldots$. With the aid of series definition (1), the above equation can be expressed as

$$K^\alpha_q f(x) = (1 - q)^\alpha \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(\eta + 1)}{2(k + 1)}} \left[ \frac{-\alpha}{k} \right] f(xq^{-a-k}).$$

(16)

Consequently, by applying (9), (2) can be expressed as

$$K^\alpha_q f(x) = (1 - q)^\alpha \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(\eta + 1)}{2(k + 1)}} \left( \frac{(q; q)_a}{(q; q)_q(q; q)_{a-k}} \right) f(xq^{-a-k}).$$

Therefore, it follows that

$$K^\alpha_q f(x) = \frac{(q; q)_a}{(1 - q)^\alpha} \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(\eta + 1)}{2(k + 1)}} \left( \frac{(q; q)_a}{(q; q)_q(q; q)_{a-k}} \right) f(xq^{-a-k}).$$

(17)

In what follows, we discuss the Al-Salam fractional $q$-integral (15) on some special functions. We apply it to various types of $q$-Bessel functions and some power series of special type. In Sect. 1, we already recalled some definitions and notations from the fractional $q$-calculus theory. In Sect. 2, we apply the Al-Salam fractional $q$-integral to a finite product of $q$-Bessel functions. In Sect. 3, we apply the Al-Salam fractional $q$-integral to a power series. We also include some new applications. In Sect. 4, we apply the Al-Salam $q$-integral operator to some $q$-generating series.

2 Main results

**Theorem 1** Let $\{j_{2\mu}^{(1)}(2\sqrt{\delta_1}t; q), \ldots, j_{2\mu}^{(1)}(2\sqrt{\delta_1}t; q)\}$ be a set of first kind $q$-Bessel functions and

$$f(t) = t^{\Delta-1} \prod_{j=1}^{n} j_{2\mu}^{(1)}(2\sqrt{\delta_1}t; q).$$

(18)

Then, for some $B = q^{-(\Delta-1)} \frac{(q; q)_a}{(1-q)^\alpha} x^{\Delta-1}$, we have

$$K^\alpha_q f(x) = B \prod_{j=1}^{n} \left( \delta_j x q^{-a} \right) \sum_{m=0}^{\infty} \delta_j x^{m} q^{am} \frac{(q^{2j+1}; q)_\infty}{\Gamma_q(m + 1)}$$

$$\times \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(\eta + 1)}{2(k + 1)}} \frac{(q^{2j+1}; q)_\infty}{\Gamma_q(k + 1)} q^{1 - \alpha - k}.$$

**Proof** By employing (18), the fractional $q$-integral (17) reveals

$$K^\alpha_q f(x) = \frac{(q; q)_a}{(1 - q)^\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(\eta + 1)}{2(k + 1)}}}{(q; q)_q(q; q)_{a-k}} f(xq^{-a-k})$$

$$= \frac{(q; q)_a}{(1 - q)^\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(\eta + 1)}{2(k + 1)}}}{(q; q)_q(q; q)_{a-k}} \frac{(xq^{-a-k})^{\Delta-1}}{\Delta-1}$$

$$\times \prod_{j=1}^{n} j_{2\mu}^{(1)}(2\sqrt{\delta_1}xq^{-a-k}; q)$$
Remark 2 Let \( f^{(1)}_{2m} \left( 2 \sqrt{q} \right) \), \( f^{(1)}_{2m+1} \left( 2 \sqrt{q} \right) \) be a set of first kind \( q \)-Bessel functions and \( f(t) = t^{\alpha-1} \prod_{j=1}^{n} f^{(1)}_{2m_j} \left( 2 \sqrt{q} \right) \). Then, for some \( B = q^{-(\alpha-1)} (q_\alpha)_\infty x^{\Delta-1} \), we have
\[
K_q^{n,\alpha} f(x) = B \prod_{j=1}^{n} (\delta_j x q^{-\alpha})^{\mu_j} \sum_{m=0}^{\infty} (\delta_j x q^{-\alpha})^{\mu_j} \sum_{m=0}^{\infty} \frac{(q_\alpha)_\infty}{\Gamma_q(m+1)} (-1)^k \frac{q^{k-1+\alpha+1+\frac{1}{2} k+\frac{1}{2} \Delta}}{\Gamma_q(k+1) \Gamma_q(1-\alpha-k)}.
\]
Proof Indeed, from Theorem 1 and (21), we have

\[ K^\alpha_{q,\nu} f(x) = \frac{q^{n(\Delta-1)\delta \Delta-1}}{(q; q)_\infty (1-q)^{-\alpha}} \Gamma_q(1-\alpha) \prod_{j=1}^{\infty} (\delta_j x^{\alpha-\alpha_j})^{\mu_j} \times \sum_{m=0}^{\infty} \frac{(\delta_j x^{\alpha-\alpha_j})^m (q^m)_{\infty} \Gamma_q(m+1)}{\Gamma_q(1-\alpha-k)(1) \Gamma_q(1-\alpha-k)} \]

This completes the proof of the remark. \(\square\)

Theorem 3 Let \(J^{(2)} S_2 \sqrt{t; q}, \ldots, J^{(2)} S_3 \sqrt{t; q}\) and \(f(t) = t^{\Delta-1} \prod_{j=1}^{\infty} \Gamma_q(1-\alpha-k)\). Then, for some \(A = \frac{(q; q)_{\infty}}{(1-q)^{\alpha}} \), we have

\[ K^\alpha_{q,\nu} f(x) = A \prod_{j=1}^{\infty} \delta_j^{\mu_j} x^{\alpha-\alpha_j} \sum_{m=0}^{\infty} q^{m(m+1)} (\delta_j x^{\alpha-\alpha_j})^m (q^m)_{\infty} \Gamma_q(m+1) \Gamma_q(1-\alpha-k) \]

Proof Let the hypothesis of the theorem be satisfied. Then we have

\[ K^\alpha_{q,\nu} f(x) = \frac{(q; q)_{\infty}}{(1-q)^{\alpha}} \sum_{k=0}^{\infty} (\delta_j x^{\alpha-\alpha_j})^{\alpha-k} f(q^k) \]

Therefore, in view of (18) and (3), we write

\[ K^\alpha_{q,\nu} f(x) = (1-q)^{\alpha} \sum_{k=0}^{\infty} (\delta_j x^{\alpha-\alpha_j})^{\alpha-k} f(q^k) \]

\[ = \frac{(q; q)_{\infty}}{(1-q)^{\alpha}} \sum_{k=0}^{\infty} (\delta_j x^{\alpha-\alpha_j})^{\alpha-k} f(q^k) \]

\[ \times \prod_{j=1}^{\infty} J^{(2)} S_2 \sqrt{\delta_j x^{\alpha-\alpha_j}; q} \]

\[ = \frac{(q; q)_{\infty}}{(1-q)^{\alpha}} \sum_{k=0}^{\infty} (\delta_j x^{\alpha-\alpha_j})^{\alpha-k} f(q^k) \]

\[ \times \prod_{j=1}^{\infty} (\delta_j x^{\alpha-\alpha_j})^{\mu_j} \sum_{m=0}^{\infty} q^{m(m+1)} (\delta_j x^{\alpha-\alpha_j})^m \]
By the fact \((q; q)_k = \Gamma_q(1 + k)\) and the identity
\[
(\xi; q)_x = \frac{(q; q)_\infty}{(\xi; q)_\infty},
\]
we write
\[
K_q^{n,\alpha} f(x) = A \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{-\alpha \mu_j} \sum_{m=0}^{\infty} \frac{q^{m(m+\mu_j)} (-\delta_j x q^{-\alpha - k})^m}{(q; q)_\infty (q; q)_m} \frac{(q^{\mu_j + m+1}; q)_\infty}{\Gamma_q(m + \frac{1}{2} - \Delta)}
\]
\[
\times \sum_{k=0}^{\infty} (-1)^k q^{k(q+\alpha) + \frac{1}{2} k(k+1) + \frac{1}{2} k \mu_j \mu_j - \frac{1}{2} k \mu_j} \frac{(q; q)_k (q; q)_{-\alpha - k}}{(q; q)^{k(k+1)+\frac{1}{2} (\Delta - 1) + \frac{1}{2} (\mu_j + 1) - \frac{1}{2} \mu_j - \frac{1}{2} \Delta)}.
\]
This completes the proof of the theorem. \(\square\)

**Theorem 4** Let \(f_{2j+1}^{(3)}(2\sqrt{q^{-1}\delta_1 t}; q), \ldots, f_{2j+n}^{(3)}(2\sqrt{q^{-1}\delta_n t}; q)\) be \(n\) \(q\)-Bessel functions and
\[
f(t) = t^{\Delta - 1} \prod_{j=1}^{n} \delta_j^{\mu_j} f_{2j+1}^{(3)}(2\sqrt{q^{-1}\delta_j t}; q).
\]
Then we have
\[
K_q^{n,\alpha} f(x) = \frac{x^{\Delta - 1} \Gamma_q(1 - \alpha)(1 - q)^n}{(q; q)_\infty} \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{\alpha \mu_j} \sum_{m=0}^{\infty} \frac{q^{m(m+\mu_j)} (\delta_j x q^{-\alpha - k})^m}{(q; q)_\infty (q; q)_m} \frac{(q^{\mu_j + m+1}; q)_\infty}{\Gamma_q(m + \frac{1}{2} - \Delta)}
\]
\[
\times \sum_{k=0}^{\infty} (-1)^k q^{k(q+\alpha) + \frac{1}{2} k(k+1) + \frac{1}{2} k \mu_j \mu_j - \frac{1}{2} k \mu_j} \frac{(q; q)_k (q; q)_{-\alpha - k}}{(q; q)^{k(k+1)+\frac{1}{2} (\Delta - 1) + \frac{1}{2} (\mu_j + 1) - \frac{1}{2} \mu_j - \frac{1}{2} \Delta)} \frac{\Gamma_q(k + 1)}{\Gamma_q(1 - \alpha - k)}.
\]
Proof By (2) and (6), we obtain

\[
K_{q}^{\eta,\alpha} f(x) = (1 - q)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} q^{k(\eta + 1) + \frac{1}{2} k(k+1)} \left[ -\frac{\alpha}{k} \right] f(xq^{-\alpha - k}) \\
= (1 - q)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} q^{k(\eta + 1) + \frac{1}{2} k(k+1)} \left[ -\frac{\alpha}{k} \right] (xq^{-\alpha - k})^{\Delta - 1} \\
\times \prod_{j=1}^{n} q^{\mu_{j}f(3)_{2i_{j}}} \left( q^{-1} \delta_{j} xq^{-\alpha - k}; q \right) \\
= (1 - q)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} q^{k(\eta + 1) + \frac{1}{2} k(k+1)} \left[ -\frac{\alpha}{k} \right] (xq^{-\alpha - k})^{\Delta - 1} \\
\times \prod_{j=1}^{n} q^{\mu_{j}f(3)_{2i_{j}}} \left( q^{-1} \delta_{j} xq^{-\alpha - k}; q \right) \\
\times \sum_{m=0}^{\infty} (-1)^{m} q^{m(\eta - 1) + m(-\alpha - k)\mu_{j}} (q^{-1} \delta_{j} xq^{-\alpha - k})^{m} (q; q)_{m}. 
\]

Equations (10), (21), and simple simplifications reveal

\[
K_{q}^{\eta,\alpha} f(x) = x^{\Delta - 1} (1 - q)^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} q^{k(\eta + 1) + \frac{1}{2} k(k+1)} (\Delta - 1)(-\alpha - k) \Gamma_{q}(1 - \alpha) \\
\times \prod_{j=1}^{n} \delta_{j}^{\mu_{j}f(3)_{2i_{j}}} \left( q^{-1} \delta_{j} xq^{-\alpha - k}; q \right) \\
\times \sum_{m=0}^{\infty} (-1)^{m} q^{m(\eta - 1) + m(-\alpha - k)\mu_{j}} (q^{-1} \delta_{j} xq^{-\alpha - k})^{m} (q; q)_{m}. 
\]

This completes the proof of the theorem. □

3 The fractional $q$-integral of the power series

This section is briefly devoted to the application of the fractional $q$-integral to functions of a power series form. Some corollaries associated with polynomials and unit functions are also deduced.

Theorem 5 Let $g(x) = \sum_{i=0}^{\infty} r_{i} x^{i}$ be a power series and $\beta$ be a positive real number. If $f(x) = (x^{\beta+1} g(x))$, then we have

\[
K_{q}^{\eta,\alpha} f(x) = \frac{q^{\alpha \beta - 1} (q; q)_{\alpha - \beta - 1} \Gamma_{q}(1 - \alpha)}{(1 - q)^{\alpha}} \sum_{i=0}^{\infty} r_{i} q^{-\alpha i} x^{i} \sum_{k=0}^{\infty} (-1)^{k} q^{k(\eta + 1) + \frac{1}{2} k(k+1)} (q^{-1} \delta_{j} xq^{-\alpha - k})^{k} \Gamma_{q}(k) \Gamma_{q}(\alpha - k). 
\]
Proof. Let \( g(x) = \sum_{i=0}^{\infty} r_i x^i \) be a power series and \( \beta \) be a positive real number. From (26) it follows

\[
K_{q}^{\alpha, \beta} f(x) = \frac{(q; q)_\infty}{(1 - q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\beta+1) + \frac{1}{2}k(k+1)}}{(q; q)_k (q; q)_{-\alpha - k}} f(xq^{-\alpha - k})
\]

Interchanging the order of summation in (24) leads to

\[
K_{q}^{\alpha, \beta} f(x) = \frac{(q; q)_\infty}{(1 - q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\beta+1) + \frac{1}{2}k(k+1)}}{(q; q)_k (q; q)_{-\alpha - k}} r_i (xq^{-\alpha - k})^i.
\]  

Employing (21) indeed gives

\[
K_{q}^{\alpha, \beta} f(x) = \frac{q^{-a\beta+\alpha} x^{\beta-1} (q; q)_{-\alpha}}{(1 - q)_{-\alpha}} \sum_{i=0}^{\infty} r_i q^{-ai} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\beta+1) + \frac{1}{2}k(k+1) - k}}{(q; q)_k (q; q)_{-\alpha - k}}.
\]

Hence, the proof of the theorem is completed. \( \square \)

Corollary 6. Let \( \beta > 0 \) be a real number. Then we have

\[
K_{q}^{\alpha, \beta} (x^{\beta-1}) = \frac{q^{-a\beta+\alpha} x^{\beta-1} (q; q)_{-\alpha}}{(1 - q)_{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\beta+1) + \frac{1}{2}k(k+1) - k}}{(q; q)_k (q; q)_{-\alpha - k}}.
\]

This result follows from setting \( r_0 = 1 \) and \( r_i = 0 \) for \( i = 1, 2, 3, \ldots \).

Corollary 7. We have

\[
K_{q}^{\alpha, \beta}(1) = \frac{(q; q)_{-\alpha}}{(1 - q)_{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\beta+1) + \frac{1}{2}k(k+1) - k}}{(q; q)_k (q; q)_{-\alpha - k}}.
\]  

4. \( K_{q}^{\alpha, \beta} \) of \( q \)-generating Heine series

The basic \( q \)-generating series of the first type is defined by \[41\] as

\[
r \phi (\delta_1, \ldots, \delta_i; b_1, \ldots, b_i, q, \xi) = \sum_{i \geq 0} \frac{(\delta_1; q)_i, \ldots, (\delta_i; q)_i}{(q; q)_i, (b_1; q)_i, \ldots, (b_i; q)_i} ((-1)^i q^{i(2^i)})^{x_{i+s-r} x_{i}^i},
\]

where

\[
(2^i) = \frac{i(i - 1)}{2}, \quad r > s + 1, \quad q > 0.
\]  

The basic \( q \)-generating series of the second type is given as

\[
r \psi (\delta_1, \ldots, \delta_i; \delta_1, \ldots, \delta_i; q, \xi) = \sum_{i \geq 0} \frac{(\delta_1; q)_i, \ldots, (\delta_i; q)_i}{(q; q)_i, (\delta_1; q)_i, \ldots, (\delta_i; q)_i} ((-1)^i q^{i(2^i)})^{x_{i+r} x_{i}^i}.
\]
The parameters \( b_1, \ldots, b_s \) are given so that the denominator factors in terms of the series are never zero, and the basic series terminates when one of its numerator parameters is of type \( q^{-n}, n = 0, 1, 2, \ldots \).

**Theorem 8** Let \( \beta \) and \( \gamma \) be real numbers. Then, provided \( \beta > 0 \), we have

\[
K_{q^{\alpha}}(x^{\beta-1}) \phi_i(\delta_1, \ldots, \delta_i, \tilde{\delta}_i; q, \gamma x)
= \frac{q^{-\alpha \beta + \alpha} x^{\beta-1}(q; q)_{-\alpha}}{(1-q)^{\alpha}}
\times \sum_{i=0}^{\infty} r_i q^{-\alpha i} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\delta_i(q \alpha + \frac{1}{2} k + \frac{1}{2} - i)}}{(q; q)_k(q; q)_{-\alpha-k}}
\times (xq^{-\alpha-k})(xq^{-\alpha-k})^{\delta_i-1} \phi_i(\delta_1, \ldots, \delta_i, \tilde{\delta}_1, \ldots, \tilde{\delta}_i; q, \gamma x q^{-\alpha-k}).
\]

*Proof* Let \( \beta \) and \( \gamma \) be real numbers. Then, by (17), write

\[
K_{q^{\alpha}}(x^{\beta-1}) \phi_i(\delta_1, \ldots, \delta_i, \tilde{\delta}_i; q, \gamma x)
= \frac{(q; q)_{-\alpha}}{(1-q)^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\delta_i(q \alpha + \frac{1}{2} k + \frac{1}{2} - i)}}{(q; q)_k(q; q)_{-\alpha-k}}
\times f(xq^{-\alpha-k})(xq^{-\alpha-k})^{\delta_i-1} \phi_i(\delta_1, \ldots, \delta_i, \tilde{\delta}_1, \ldots, \tilde{\delta}_i; q, \gamma x q^{-\alpha-k}).
\]

On the other hand, we have

\[
, \phi_i(\delta_1, \ldots, \delta_i, \tilde{\delta}_1, \ldots, \tilde{\delta}_i; q, \gamma x q^{-\alpha-k}) \sum_{i=0}^{\infty} \frac{(\delta_1; q)_i, \ldots, (\delta_i; q)_i, (\tilde{\delta}_1; q)_i, \ldots, (\tilde{\delta}_i; q)_i}{(q; q)_i, (\delta_1; q)_i, \ldots, (\delta_i; q)_i} ((-1)^{i} q^{(2i)})^{i-r}
\times (\gamma x q^{-\alpha-k})^{i}
= \sum_{i=0}^{\infty} r_i x^i,
\]

where

\[
r_i = \frac{(\delta_1; q)_i, \ldots, (\delta_i; q)_i, (\tilde{\delta}_1; q)_i, \ldots, (\tilde{\delta}_i; q)_i}{(q; q)_i, (\delta_1; q)_i, \ldots, (\delta_i; q)_i} ((-1)^{i} q^{(2i)})^{i-r} \gamma x^{i} q^{-\alpha-k}. \tag{28}
\]

Therefore, by Theorem 5 we get

\[
K_{q^{\alpha}}(x^{\beta-1}) \phi_i(\delta_1, \ldots, \delta_i, \tilde{\delta}_1, \ldots, \tilde{\delta}_i; q, \gamma x)
= \frac{q^{-\alpha \beta + \alpha} x^{\beta-1}(q; q)_{-\alpha}}{(1-q)^{\alpha}} \sum_{i=0}^{\infty} r_i q^{-\alpha i} x^i
\times \sum_{k=0}^{\infty} (-1)^k \frac{q^{\delta_i(q \alpha + \frac{1}{2} k + \frac{1}{2} - i)}}{(q; q)_k(q; q)_{-\alpha-k}}. \tag{29}
\]

This completes the proof of the theorem. \( \square \)
Theorem 9  Let $\beta > 0$ and $r$ be real numbers. Then we have

$$K^{\alpha,\beta}_{q,a}(x^{\beta-1};y)_q = \frac{q^{-a\beta+\alpha}x^{\beta-1}(q;q)_{-a}}{(1-q)^{-a}} \sum_{i=0}^{\infty} r_i q^{-ai} x^i$$

$$\times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(q\alpha+1/2) + 1/2}}{\Gamma_q(k)\Gamma_q(-a-k)}.$$  

Proof By taking into account (20), we write

$$K^{\alpha,\beta}_{q,a}(x^{\beta-1};y)_q = \frac{(q;q)_{-a}}{(1-q)^{-a}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(q\alpha+1/2) + 1/2}}{(q;q)_{-a,k}} f(xq^{-a-k})^{\beta-1} \times f(xq^{-a-k})^{\beta-1} r_i \psi_q(x_1, \ldots, x_r; \hat{x}_{1}, \ldots, \hat{x}_{r}; q^{-a-k}).$$ (30)

However,

$$r_i \psi_q(x_1, \ldots, x_r; \hat{x}_{1}, \ldots, \hat{x}_{r}; q^{-a-k}) = \sum_{i \geq 0} \frac{(\hat{x}_{1};q)_i \ldots (\hat{x}_{r};q)_i}{(q;q)_i (\hat{x}_{1};q)_i \ldots (\hat{x}_{r};q)_i} \left((-1)^i q^{(i^2/2)}ight)^{s-r} \times (y q^{-a-k})^i$$

$$= \sum_{i \geq 0} r_i x^i,$$

where

$$r_i = \frac{(\hat{x}_{1};q)_i \ldots (\hat{x}_{r};q)_i}{(q;q)_i (\hat{x}_{1};q)_i \ldots (\hat{x}_{r};q)_i} \left((-1)^i q^{(i^2/2)}ight)^{s-r} \times (y q^{-a-k})^i.$$ (31)

Hence, by Theorem 5 it follows

$$K^{\alpha,\beta}_{q,a}(x^{\beta-1};y)_q = \frac{q^{-a\beta+\alpha}x^{\beta-1}(q;q)_{-a}}{(1-q)^{-a}} \sum_{i=0}^{\infty} r_i q^{-ai} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(q\alpha+1/2) + 1/2}}{\Gamma_q(k)\Gamma_q(-a-k)}.$$  

This completes the proof of the theorem.  

Corollary 10 Let $\gamma$ be a real number. Then we have

$$K^{\alpha,\beta}_{q,a}(E_q(yx)) = \frac{(q;q)_{-a}}{(1-q)^{-a}} \sum_{i=0}^{\infty} r_i q^{-ai} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(q\alpha+1/2) + 1/2}}{\Gamma_q(k)\Gamma_q(-a-k)}.$$  

Proof By setting $\beta = 0, r = 0,$ and $s = 0,$ the result easily follows from Theorem 8. The proof is completed.  

Corollary 11 Let $\gamma$ be a real number. Then we have

$$K^{\alpha,\beta}_{q,a}(E_q(yx)) = \frac{(q;q)_{-a}}{(1-q)^{-a}} \sum_{i=0}^{\infty} r_i q^{-ai} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(q\alpha+1/2) + 1/2}}{\Gamma_q(k)\Gamma_q(-a-k)}.$$
Proof. By setting $\beta = 1$, $r = 0$, and $s = 0$, Theorem 8 completes the proof of the corollary. □

The proof of the following corollary is straightforward. Details are therefore deleted.

**Corollary 12** Let $\gamma$ be a real number. Then we have

\[
(i) K_{\eta}^{\alpha,\beta} \left( \sinh_{q}(\gamma x) \right) = K_{\eta}^{\alpha,\beta} \left( \frac{E_{q}(\gamma x) - E_{q}(-\gamma x)}{2} \right)
\]

\[
= \frac{(q; q)_{\infty}}{(1 - q)_{\infty}} \sum_{i=0}^{\infty} r_{i} q^{-ai} \sum_{k=0}^{\infty} (-1)^{k} \frac{q^{k(\eta + \alpha + \frac{1}{2} - i)}}{\Gamma_{q}(k) \Gamma_{q}(\alpha - k)} x^{i} (1 + (-1)^{i+1})
\]

\[
(ii) K_{\eta}^{\alpha,\beta} \left( \cosh_{q}(\gamma x) \right) = K_{\eta}^{\alpha,\beta} \left( \frac{E_{q}(\gamma x) - E_{q}(-\gamma x)}{2} \right)
\]

\[
= \frac{(q; q)_{\infty}}{(1 - q)_{\infty}} \sum_{i=0}^{\infty} r_{i} q^{-ai} \sum_{k=0}^{\infty} (-1)^{k} \frac{q^{k(\eta + \alpha + \frac{1}{2} - i)}}{\Gamma_{q}(k) \Gamma_{q}(\alpha - k)} x^{i} (1 + (-1)^{i})
\]

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