On the basic properties of $GC_n$ sets

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Abstract

A planar node set $\mathcal{X}$, with $\#\mathcal{X} = \binom{n+2}{2}$, is called $GC_n$ set if each node possesses fundamental polynomial in form of a product of $n$ linear factors. We say that a node uses a line if the line is a factor of the fundamental polynomial of the node. A line is called $k$-node line if it passes through exactly $k$-nodes of $\mathcal{X}$. At most $n + 1$ nodes can be collinear in any $GC_n$ set and an $(n + 1)$-node line is called a maximal line. The Gasca-Maeztu conjecture (1982) states that every $GC_n$ set has a maximal line. Until now the conjecture has been proved only for the cases $n \leq 5$.

Here, for a line $\ell$ we introduce and study the concept of $\ell$-lowering of the set $\mathcal{X}$ and define so called proper lines. We also provide refinements of several basic properties of $GC_n$ sets regarding the maximal lines, $n$-node lines, the used lines, as well as the subset of nodes that use a given line.

Key words: Polynomial interpolation, Gasca-Maeztu conjecture, $n$-correct set, $GC_n$ set, $\ell$-lowering, maximal line, proper line.

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1 Introduction

An $n$-correct set $\mathcal{X}$ in the plane is a node set for which the Lagrange interpolation problem with bivariate polynomials of total degree not exceeding $n$ is unisolvent. A line is called $k$-node line if it passes through exactly $k$ nodes of $\mathcal{X}$. It is a simple fact that at most $n + 1$ nodes can be collinear in an $n$-correct set. An $(n + 1)$-node line is called a maximal line. In $n$-correct node sets with geometric characterization: $GC_n$ sets, introduced by Chung and Yao [11], the fundamental polynomial of each node is a product of $n$ linear factors. The conjecture of M. Gasca and J. I. Maeztu [12] (GM conjecture) states that every $GC_n$ set possesses a maximal line. Until now the conjecture has been proved only for the cases $n \leq 5$ (see [2] and [13]). We say that a node uses a line if the line is a factor of the fundamental polynomial of this node. Denote the subset of nodes of $\mathcal{X}$ that use the line $\ell$ by $\mathcal{X}^\ell$. We have that

$$\mathcal{X}^\ell = \mathcal{X} \setminus \ell \text{ if } \ell \text{ is a maximal line.} \quad (1.1)$$
Recently V. Vardanyan and H.H. proved (see Theorem 3.1, [16], see also [1] and [15]) that if the Gasca-Maeztu conjecture is true, then for any GC\(_n\) set \(X\) and any \(k\)-node line \(\ell\) the following statements hold:

The line \(\ell\) is not used at all, or it is used by exactly \(\binom{s}{2}\) nodes of \(X\), where \(s\) satisfies the condition \(k - \delta \leq s \leq k\) with \(\delta = n + 1 - k\). Next, the set \(X^{\ell}\) forms a GC\(_{n-1}\) subset of \(X\). Moreover, to get \(X^{\ell}\), at the first step one obtains a node set \(X_1\) by removing the nodes in a maximal line of \(X\), which does not intersect the line \(\ell\) at a node of \(X\) or by removing the nodes in a pair of maximal lines of \(X\) which are concurrent together with \(\ell\). Thus \(X_1\) is a GC\(_{n-1}\) or GC\(_{n-2}\) set. Second step is similar to the first one, the only difference is that instead of the maximal lines of \(X\) now the maximal lines of \(X_1\) are used. By continuing same way, at the last step \(\omega\), the line \(\ell\) becomes a maximal line in a GC\(_{s-1}\) set \(X_\omega\), and hence, according to (1.1), we get

\[ X^{\ell} = X_\omega \setminus \ell, \quad \#X^{\ell} = \binom{s}{2}. \]

In this paper we characterize the cases when the above described procedure can be done by means of using only the original maximal lines of \(X\). We characterize the lines for which this takes place.

More specifically, we introduce the concept of \(\ell\)-lowering of a GC\(_n\) set \(X\), and denote it by \(\hat{X} := \hat{X}(\ell)\). This set plays an important role here. We get \(\hat{X}\) by removing, at a single step, the nodes in all maximal lines of \(X\) which do not intersect the line \(\ell\) at a node of \(X\) and by removing the nodes in all pairs of maximal lines of \(X\) which are concurrent together with \(\ell\). Then we specify cases when \(\ell\) is a maximal line in the obtained GC set \(\hat{X}\) and hence

\[ X^{\ell} = \hat{X} \setminus \ell. \]

We call such lines proper lines. We prove in forthcoming Theorem 4.1 that, all lines \(\ell\) used by more than 3 nodes, are proper, provided that the number of maximal lines of \(X\) differs from three. Moreover, in the general case one needs to complete just one or two steps of the maximal lines removal in \(\hat{X}\) till \(\ell\) becomes a maximal line.

Next, in this paper we specify exactly by how many nodes a given line is used, provided that GM conjecture is true. Let us call a node a \(2_m\) node in \(X\) if it is a point of intersection of two maximal lines in \(X\). In the forthcoming Theorem 4.2 we prove that a \(k\)-node line \(\ell\) is not used at all, or it is used by exactly \(\binom{k-r-\hat{r}}{2}\) nodes of \(X\), where \(r\) is the number of \(2_m\) nodes in \(\ell \cap X\) and \(\hat{r}\) is the number of \(2_m\) nodes in \(\ell \cap \hat{X}\).

Moreover, we have that always \(\hat{r} \leq 2\), and \(\hat{r} = 0\) if \(#X^{\ell} > 3\). Furthermore, for each \(2_m\) node in \(\ell \cap \hat{X}\) we have that one of the two maximal lines in \(\hat{X}\) to which it belongs is a maximal line in \(X\) and another is a proper line in \(X\). In this paper we consider and improve several other basic properties of GC\(_n\) sets.
Let us mention that Carnicer and Gasca first studied the set $X^\ell$ and proved that a $k$-node line $\ell$ can be used by at most $\binom{k}{2}$ nodes of a $GC_n$ set $X$ and in addition there are no $k$ collinear nodes that use $\ell$, provided that GM conjecture is true (see [5], Theorem 4.5).

### 1.1 Correct sets

Denote by $\Pi_n$ the space of bivariate polynomials of total degree at most $n$:

$$\Pi_n = \left\{ \sum_{i+j \leq n} c_{ij} x^i y^j \right\}.$$ 

We have that $N := \dim \Pi_n = \binom{n+2}{2}$.

Let $X$ be a set of $N$ distinct nodes: $X = \{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$.

**Definition 1.1.** A set of nodes $X$, $\#X = N$, is called $n$-correct if for any given numbers $c_1, c_2, \ldots, c_N$, there exists a unique polynomial $p \in \Pi_n$ satisfying the following conditions.

$$p(x_i, y_i) = c_i, \quad i = 1, 2, \ldots, N. \quad (1.2)$$

A polynomial $p \in \Pi_n$ is called an $n$-fundamental polynomial of a node $A \in X$, if

$$p(A) = 1 \quad \text{and} \quad p(B) = 0 \quad \forall B \in X \setminus \{A\}.$$ 

**Definition 1.2.** Given an $n$-correct set $X$. We say that a node $A \in X$ uses a line $\ell \in \Pi_1$, if

$$p^*_A, X = \ell q, \quad \text{where} \quad q \in \Pi_{n-1}.$$ 

The following proposition is well-known (see, e.g., [13] Proposition 1.5):

**Proposition 1.3.** Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line $\ell$. Then we have that

$$p = \ell r, \quad \text{where} \quad r \in \Pi_{n-1}.$$ 

Thus at most $n + 1$ nodes of an $n$-correct set $X$ can be collinear. A line $\lambda$ passing through $n + 1$ nodes of the set $X$ is called a maximal line. Note that we readily get (1.1) from Proposition 1.3.

Below we bring basic properties of maximal lines:

**Proposition 1.4 ([3], Prop. 2.1).** Let $X$ be an $n$-correct set. Then

(i) any two maximal lines of $X$ intersect necessarily at a node of $X$;

(ii) any three maximal lines of $X$ cannot be concurrent;

(iii) $X$ can have at most $n + 2$ maximal lines.
2 \( GC_n \) sets and the Gasca-Maeztu conjecture

Now let us consider a special type of \( n \)-correct sets satisfying a geometric characterization (GC) property introduced by K.C. Chung and T.H. Yao:

**Definition 2.1** ([1]). An \( n \)-correct set \( \mathcal{X} \) is called \( GC_n \) set (or GC set) if the \( n \)-fundamental polynomial of each node \( A \in \mathcal{X} \) is a product of \( n \) linear factors.

Thus, each node of a \( GC_n \) set uses exactly \( n \) lines.

**Proposition 2.2** ([5], Prop. 2.3). Let \( \lambda \) be a maximal line of a \( GC_n \) set \( \mathcal{X} \). Then the set \( \mathcal{X} \setminus \lambda \) is a \( GC_{n-1} \) set.

Next we present the Gasca-Maeztu (GM) conjecture:

**Conjecture 2.3** ([12], Sect. 5). Any \( GC_n \) set possesses a maximal line.

Till now, this conjecture has been confirmed for the degrees \( n \leq 5 \) (see [2], [13]). For a generalization of the Gasca-Maeztu conjecture to maximal curves see [14].

Denote by \( M(\mathcal{X}) \) the set of maximal lines of the node set \( \mathcal{X} \).

The concept of the defect introduced by Carnicer and Gasca in [3] is an important characteristic of \( GC_n \) sets.

**Definition 2.4** ([3]). The defect of an \( n \)-correct set \( \mathcal{X} \) is the number \( \text{def}(\mathcal{X}) := n + 2 - \#M(\mathcal{X}) \).

In view of Proposition 1.4 we have that \( 0 \leq \text{def}(\mathcal{X}) \leq n + 2 \).

**Proposition 2.5** ([5], Crl. 3.5). Let \( \lambda \) be a maximal line of a \( GC_n \) set \( \mathcal{X} \) such that \( \#(\lambda \cap \mathcal{X}) \geq 3 \). Then we have that

\[
\text{def}(\mathcal{X} \setminus \lambda) = \text{def}(\mathcal{X}) \quad \text{or} \quad \text{def}(\mathcal{X}) - 1.
\]

This equality means that \( \#M(\mathcal{X} \setminus \lambda) = \#M(\mathcal{X}) - 1 \) or \( \#M(\mathcal{X}) \).

**Definition 2.6** ([4]). Given an \( n \)-correct set \( \mathcal{X} \) and a line \( \ell \). Then \( \mathcal{X}^\ell \) is the subset of nodes of \( \mathcal{X} \) which use the line \( \ell \).

Let \( \mathcal{X} \) be an \( n \)-correct set and \( \ell \) be a line with \( \#(\ell \cap \mathcal{X}) \leq n \). Then

(i) a maximal line \( \lambda \) is called \( \ell \)-disjoint if \( \lambda \cap \ell \cap \mathcal{X} = \emptyset \);

(ii) two maximal lines \( \lambda', \lambda'' \) are called \( \ell \)-adjoint if \( \lambda' \cap \lambda'' \cap \ell \in \mathcal{X} \).

The following two lemmas of Carnicer and Gasca play an important role in the sequel.

**Lemma 2.7** ([5], Lemma 4.4). Let \( \mathcal{X} \) be an \( n \)-correct set and \( \ell \) be a line with \( \#(\ell \cap \mathcal{X}) \leq n \). Suppose also that a maximal line \( \lambda \) is \( \ell \)-disjoint. Then we have that

\[
\mathcal{X}^\ell = (\mathcal{X} \setminus \lambda)^\ell.
\]
The set $\mathcal{X} \setminus \lambda$ is called $\ell_d$-reduction of $\mathcal{X}$.

**Lemma 2.8** ([5], Proof of Thm. 4.5). Let $\mathcal{X}$ be an $n$-correct set and $\ell$ be a line with $\#(\ell \cap \mathcal{X}) \leq n$. Suppose also that two maximal lines $\lambda', \lambda''$ are $\ell$-adjoint. Then we have that

$$\mathcal{X}^\ell = [\mathcal{X} \setminus (\lambda' \cup \lambda'')]^\ell.$$  

The set $\mathcal{X} \setminus (\lambda' \cup \lambda'')$ is called $\ell_a$-reduction of $\mathcal{X}$.

Next, by the motivation of above two lemmas, we introduce the concept of $\ell$-lowering of a GC$_n$ set.

**Definition 2.9.** Let $\mathcal{X}$ be a GC$_n$ set, $\ell$ be a $k$-node line, $2 \leq k \leq n$. We say that the set $\hat{\mathcal{X}} = \hat{\mathcal{X}}(\ell)$ is the $\ell$-lowering of $\mathcal{X}$, and briefly depict this by $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$, if

$$\hat{\mathcal{X}} = \mathcal{X} \setminus (U_1 \cup U_2),$$

where $U_1$ is the union of the $\ell$-disjoint maximal lines of $\mathcal{X}$, and $U_2$ is the union of the (pairs of) $\ell$-adjoint maximal lines of $\mathcal{X}$.

We get immediately from Lemmas 2.7 and 2.8 that

$$\mathcal{X} \downarrow_\ell \hat{\mathcal{X}} \Rightarrow \mathcal{X}^\ell = \hat{\mathcal{X}}^\ell.$$  

**Definition 2.10.** A node $A \in \mathcal{X}$ is called $k_m$-node if it belongs to exactly $k$ maximal lines.

In view of Proposition 1.4 (ii), there are type $0_m$, $1_m$, and $2_m$ nodes only.

It is easily seen that $\ell \cap \hat{\mathcal{X}}(\ell)$ contains no type $2_m$ node of $\mathcal{X}$.

Suppose that a line $\ell$ passes through exactly $k$ type $1_m$ nodes. Then there are exactly $\#M(\mathcal{X}) - k$ maximal lines which are either $\ell$-disjoint or $\ell$-adjoint. Hence, in view of Proposition 2.2, we readily get

$$\hat{\mathcal{X}}$$

is GC$_s$ set with $s = \text{def} \mathcal{X} + k - 2$.  \hspace{1cm} (2.1)

**Definition 2.11.** Let $\mathcal{X}$ be a GC$_n$ set, $\ell$ be a $k$-node line, $2 \leq k \leq n$ and $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$. Then the line $\ell$ is called proper if it is a maximal line in the set $\hat{\mathcal{X}}$.

The line $\ell$ is called proper ($-r$) if it becomes a maximal line after $r$ steps of application of $\ell_d$ or $\ell_a$-reductions, in all, starting with the set $\hat{\mathcal{X}}$.

We call the $\ell$-disjoint maximal line, or the union of the pair of $\ell$-adjoint maximal lines, used at the above mentioned $k$th step, the ($-k$) $d/a$ (disjoint/adjoint) item, $k = 1, \ldots, r$.

In forthcoming Theorem 4.1 we show that any used line is either proper, proper ($-1$), or proper ($-2$).

Denote by $Pr(\mathcal{X})$ the set of proper lines of $\mathcal{X}$.

We get immediately from [1.1] that $\mathcal{X}^\ell = \hat{\mathcal{X}} \setminus \ell$ if $\ell \in Pr(\mathcal{X})$.

In Proposition 4.7 we show that $\#Pr(\mathcal{X}) \in \{0, 3\}$ if def($\mathcal{X}$) $\neq 1$, provided that the Gasca-Maeztu conjecture is true.
3 Classification of GC sets and characterizations

Here we will consider the results of Carnicer, Gasca, and Godés, concerning the classification of GC$_n$ sets according to the defect of the sets.

**Theorem 3.1** ([9]). Let $\mathcal{X}$ be a GC$_n$ set. Assume that GM Conjecture holds for all degrees up to $n$. Then $\text{def}(\mathcal{X}) \in \{0,1,2,3,n-1\}$.

Of course this implies that $\#M(\mathcal{X}) \in \{3,n-1,n,n+1,n+2\}$.

### 3.1 The Chung-Yao natural lattices: defect 0 sets

Let a set $\mathcal{M}$ of $n+2$ lines be in general position, i.e., no two lines are parallel and no three lines are concurrent, $n \geq 0$. Then the Chung-Yao set is defined as the set $\mathcal{X}$ of all $\binom{n+2}{2}$ intersection points of these lines (see Fig. 3.1). We have that the $n+2$ lines of $\mathcal{M}$ are maximal for $\mathcal{X}$. Any particular node here is of $2_m$ type, i.e., lies in exactly 2 maximal lines. Observe, that the product of the remaining $n$ maximal lines gives the fundamental polynomial of the particular node. Thus $\mathcal{X}$ is a GC$_n$ set. Let us mention that any $n$-correct set $\mathcal{X}$, with $\text{def}(\mathcal{X}) = 0$, i.e., $\#M(\mathcal{X}) = n+2$, in view of Proposition [1.4] (i) and (ii), forms a Chung-Yao lattice. Recall that there are no $n$-correct sets with more maximal lines (Proposition [1.4] (iii)).

#### The used lines in the Chung-Yao lattice.

Evidently the set of used lines of Chung-Yao lattice coincides with the set of $n+2$ maximal lines. For each maximal line $\ell$, in view of (1.1), we have that $\mathcal{X}^\ell = \mathcal{X} \setminus \ell$, hence $\#\mathcal{X}^\ell = \binom{n+1}{2}$.

Thus the total number of line-usages equals: $(n+2)\binom{n+1}{2} = \frac{1}{2}(n+2)(n+1)n = n\binom{n+2}{2}$.

![Figure 3.1: A Chang-Yao lattice.](image)

![Figure 3.2: A Carnicer-Gasca lattice.](image)

### 3.2 The Carnicer-Gasca lattices: defect 1 sets

Let a set $\mathcal{M}$ of $n+1$ lines be in general position, $n \geq 2$. Then the Carnicer-Gasca lattice $\mathcal{X}$ is defined as $\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)}$, where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n+1$ lines, and $\mathcal{X}^{(1)}$ is a set of other $n+1$ non-collinear nodes, one in each line, to make the line maximal (see Fig.
We have that \( \#X = \binom{n+1}{2} + (n+1) = \binom{n+2}{2} \). It is easily seen that \( X \) is a \( GC_n \) set and has exactly \( n+1 \) maximal lines, i.e., the lines of \( \mathcal{M} \). The set \( X^{(2)} \) consists of type 2\( m \) nodes and the set \( X^{(1)} \) consists of type 1\( m \) nodes. Let us mention that any \( n \)-correct set \( X \), with \( \text{def}(X) = 1 \), i.e., \( \#M(X) = n + 1 \), forms a Carnicer-Gasca lattice (see [3], Proposition 2.4).

**The used lines in the Carnicer-Gasca lattice.**

It is easily seen that the set of used lines in the Carnicer-Gasca lattice consists of two classes:

1) The set of \( n+1 \) maximal lines;
2) The set of lines passing through at least two \( 1_m \) points.

Next we show that the lines of the class 2) are proper:

**Proposition 3.2.** Suppose that \( X \) is a Carnicer-Gasca lattice and \( \ell \) is a line passing through exactly \( k \) type \( 1_m \) nodes, where \( k \geq 2 \). Suppose also that \( X \downarrow \ell \). Then \( \hat{X} \) is a Chung-Yao lattice of degree \( k - 1 \), where \( \ell \) is a maximal line. Hence \( X_{\ell} = \hat{X} \setminus \ell \) and \( \#X_{\ell} = \binom{k}{2} \).

**Proof.** In view of (2.1) we have that \( \hat{X} \) is a \( GC_{k-1} \) set. Therefore the line \( \ell \), with \( k \) type \( 1_m \) nodes, as well as the \( k \) maximal lines of \( X \) intersecting the line \( \ell \) at these nodes are maximal lines of \( \hat{X} \).

Denote by \( n_i \) the number of line usages in the class \( i \), \( i = 1, 2 \). Then we have that the total number of line-usages equals:

\[
n_1 + n_2 = (n+1)\binom{n+1}{2} + \binom{n+1}{2} = \frac{1}{2}(n+1)(n+1)n + \frac{1}{2}(n+1)n = n\binom{n+2}{2}.
\]

### 3.3 Defect 2 sets

Let a set \( \mathcal{M} \) of \( n \) lines be in general position, \( n \geq 3 \). Then consider the lattice \( \mathcal{X} \) defined as

\[
\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)},
\]

where \( \mathcal{X}^{(2)} \) is the set of all intersection nodes of these \( n \) lines, \( \mathcal{X}^{(1)} \) is a set of other \( 2n \) nodes, two in each line, to make the line maximal and \( \mathcal{X}^{(0)} \) consists of a single node, denoted by \( O \), which does not belong to any line from \( \mathcal{M} \) (see Fig. 3.3). Correspondingly, we have that \( \#\mathcal{X} = \binom{2}{2} + 2n + 1 = \binom{n+2}{2} \).

Note that all the nodes of \( \mathcal{X}^{(k)} \) belong to exactly \( k \) maximal lines and thus are \( k_m \)-nodes, \( k = 0, 1, 2 \).

In the sequel we will need the following characterization of \( GC_n \) set \( \mathcal{X} \), with \( \#M(\mathcal{X}) = n \), due to Carnicer and Gasca:

**Proposition 3.3 ([3], Prop. 2.5).** A set \( \mathcal{X} \) is a \( GC_n \) set with exactly \( n \) maximal lines: \( \mathcal{M} = \{\lambda_1, \ldots, \lambda_n\} \), where \( n \geq 3 \), if and only if, the representation (3.1) holds with the following additional properties (see Fig. 3.3):

(i) There are 3 lines \( \ell_1^O, \ell_2^O, \ell_3^O \), called \( O \)-lines, concurrent at the node \( O : O = \ell_1^O \cap \ell_2^O \cap \ell_3^O \), such that \( \mathcal{X}^{(1)} \subset \ell_1^O \cup \ell_2^O \cup \ell_3^O \);
(ii) No line $\ell^o_i$ contains $n + 1$ nodes of $X$, $i = 1, 2, 3$.

Notice that each above line $\ell^o_i$ contains at least two type 1 $m$ nodes.

**The used lines in defect 2 sets.**

Suppose that $M(X) = \{\lambda_1, \ldots, \lambda_n\}$. Consider a pair of maximal lines $\lambda_i, \lambda_j$, $1 \leq i < j \leq n$. We have that the node $A_{ij} := \lambda_i \cap \lambda_j$ uses $n - 2$ maximal lines, i.e., all maximal lines except $\lambda_i, \lambda_j$. Next let us identify the remaining two used lines. We have two $1_m$ nodes in each of the two maximal lines, which, in view of above item (i), lie also in the three $O$-lines. Denote by $n_{ij}$, the number of $O$-lines containing at least one of these four $1_m$ nodes. Evidently we have that $n_{ij} = 2$ or $3$.

In the case $n_{ij} = 2$ (the case of the node $A_{34}$ in Fig. 3.3) the four $1_m$ nodes belong to two $O$-lines, which are the two remaining used lines. In the case $n_{ij} = 3$ (the case of the node $A_{12}$ in Fig. 3.3) two of four $1_m$ nodes belong to the same $O$-line, denoted by $\ell^o_{k(i,j)}$, where $1 \leq k(i,j) \leq 3$. Denote also by $\ell_{ij}$ the line passing through the other two $1_m$ nodes. It is easily seen that in this case the lines $\ell^o_{k(i,j)}$ and $\ell_{ij}$, are the remaining two lines used by the node $A_{ij}$.

The set of used lines in $X$ consists of the following three classes:

1) The set of $n$ maximal lines;
2) The set of three $O$-lines $\{\ell_{12}^o, \ell_{34}^o, \ell_{33}^o\}$;
3) The set of lines $\ell_{ij}$, $1 \leq i < j \leq n$, with $n_{ij} = 3$.

Let us verify that these classes are disjoint, i.e., lines from different classes cannot coincide. It suffices to show that the classes 2) and 3) are disjoint. Indeed, any line $\ell_{ij}$ passes through two $1_m$ nodes which belong to two different $O$-lines. Hence $\ell_{ij}$ cannot coincide with an $O$-line.

Note also that $\ell_{ij} = \ell_{i'j'}$ implies that $(i, j) = (i', j')$. Indeed there is an $O$-line which intersects both lines at nodes of $X$. Hence without loss of generality we may assume that $\lambda_i = \lambda'_i$, i.e., $i = i'$. Now suppose by way of contradiction that $j \neq j'$. Then the other two $O$-lines do not intersect the maximal line $\lambda_i$ at nodes of $X$, which contradicts Proposition 3.3 (i).
Next let us show that the lines in the class 2) are proper:

**Proposition 3.4.** Suppose that $\mathcal{X}$ is a GC$_n$ set of defect 2 and the line $\ell := \ell_i$, $i \in \{1, 2, 3\}$, passes through exactly $k$ type 1$_m$ nodes, where $k \geq 2$. Suppose also that $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$. Then $\hat{\mathcal{X}}$ is a GC$_k$ set, of defect 0 or 1, where $\ell$ is a maximal line. Hence $\mathcal{X}_\ell = \hat{\mathcal{X}} \setminus \ell$ and $\#\mathcal{X}_\ell = \frac{(k+1)^2}{2}$.

**Proof.** In view of Proposition 2.2, we have that $\hat{\mathcal{X}}$ is a GC$_k$ set. Then notice that the line $\ell$ is a $(k+1)$-node line in $\hat{\mathcal{X}}$, with the node $O$ and $k$ type 1$_m$ nodes. Hence the line $\ell$, as well as the $k$ maximal lines of $\mathcal{X}$ intersecting the line $\ell$ at $k$ type 1$_m$ nodes are maximal lines of $\hat{\mathcal{X}}$.

Notice that $\hat{\mathcal{X}}$ is a defect 0 set if and only if a line $\ell_j$, $j \in \{1, 2, 3\} \setminus \{i\}$ is a maximal line, too. Then the other one passes through the node $O$ only. □

Note that the above line $\ell_i$ is used by $\binom{k}{2}$ type 2$_m$ nodes and $k$ type 1$_m$ nodes. Thus, in view of Proposition 3.3, (i), altogether the three $O$-lines are used by $2n$ type 1$_m$ nodes.

**Definition 3.5.** Suppose $\ell$ is an used line which is not proper. A node $A \in \ell \cap \mathcal{X}$ is called $\ell$-special if it is 2$_m$-node in the set $\hat{\mathcal{X}} := \hat{\mathcal{X}}(\ell)$. In Propositions 3.6 and 3.11 we show that if a node $A$ is an $\ell$-special then $A = \lambda \cap \ell^o$, where $\lambda \in M(\mathcal{X})$ and $\ell^o \in \Pr(\mathcal{X})$.

Thus, $A$ is necessarily an 1$_m$ node for $\mathcal{X}$. We also show that each used line $\ell$ may have at most two special nodes. We call the maximal line of $\mathcal{X}$ at an $\ell$-special node: $\ell$-special maximal.

Next let us show that the lines in the class 3) are used by only one node, and are proper (−1).

**Proposition 3.6.** Suppose that $\mathcal{X}$ is a GC$_n$ set of defect 2, $\ell := \ell_{ij}$, $1 \leq i < j \leq n$, with $n_{ij} = 3$ and $\mathcal{X} \downarrow_\ell \hat{\mathcal{X}}$. Then $S := \ell \cap \ell_k^{o(i,j)}$ is the only possible $\ell$-special node and

$$S \in \hat{\mathcal{X}} \Leftrightarrow S \text{ is an 1}_m \text{ node in } \mathcal{X} \Leftrightarrow S \text{ is a 2}_m \text{ node in } \hat{\mathcal{X}} \Leftrightarrow S = \ell^o \cap \lambda^*$,

where $\ell^o \in \Pr(\mathcal{X}) \cap M(\hat{\mathcal{X}})$ and $\lambda^* \in M(\mathcal{X})$.

Next we have that $\text{def}\hat{\mathcal{X}} = 1$, $\#\mathcal{X}_\ell = 1$, and the line $\ell$ is proper (−1), i.e.,

$$\mathcal{X}_\ell = \hat{\mathcal{X}} \setminus (C \cup \ell), \text{ where } C \text{ is the d/a item.}

Moreover if $S \notin \hat{\mathcal{X}}$ then $\hat{\mathcal{X}}$ is a GC$_2$ set, $C = \ell_k^{o(i,j)} \in M(\hat{\mathcal{X}})$. While if $S \in \hat{\mathcal{X}}$, then $\hat{\mathcal{X}}$ is a GC$_3$ set, $C = \ell_k^{o(i,j)} \cup \lambda^*$, where $\lambda^* \in M(\mathcal{X})$ and $S \in \lambda^*$. 


Proof. Notice that, in view of Proposition 3.3 (i), all type 1\textsubscript{n} nodes of X belong to the three O-lines. Assume first that S \notin X. Then the line \ell intersects only two O-lines in nodes of X. Hence it has just two 1\textsubscript{m} nodes. Thus, in view of \eqref{2.1}, we have that \tilde{\mathcal{X}} is a GC\textsubscript{2} set. It is easily seen that 

\[ M(\tilde{\mathcal{X}}) = \{\lambda_i, \lambda_j, \ell_{k(i,j)}^o\} \]

hence def \tilde{\mathcal{X}} = 1. Also, \ell is a 2-node maximal line in \tilde{\mathcal{X}} \setminus \ell_{k(i,j)}^o and \#X^\ell = 1. Clearly, in view of Lemma 2.8 \ell has no special node.

Now assume that S \in \tilde{\mathcal{X}}. Then, since S \notin O and S is not 2\textsubscript{m} node in X, we conclude that S is an 1\textsubscript{m} node in X. Thus S \in \lambda^*, where \lambda^* \in M(X). Therefore the line \ell has exactly three 1\textsubscript{m} nodes. Next, in view of \eqref{2.1}, we have that \tilde{\mathcal{X}} is a GC\textsubscript{3} set. In this case we have 

\[ M(\tilde{\mathcal{X}}) = \{\lambda_i, \lambda_j, \lambda^*, \ell_{k(i,j)}^o\} \]

and def \tilde{\mathcal{X}} = 1. It is easily seen that \ell is a 3-node line in \tilde{\mathcal{X}} with only one special node: S = \ell_{k(i,j)}^o \cap \lambda^*. Moreover, we have that \ell is a maximal line in \tilde{\mathcal{X}} \setminus (\ell_{k(i,j)}^o \cup \lambda^*) and \#X^\ell = 1. \hfill \Box

Denote by \(n_i\) the number of line usages in the class \(i\), \(i = 1, 2, 3\). Then we have that the total number of line usages equals:

\[ n_1 + n_2 + n_3 = n\binom{n+1}{2} + 2\binom{n}{2} + 2n = \frac{1}{2}n(n+1)n + n(n-1) + 2n = n\binom{n+2}{2}. \]

Here we take into account the remark after Proposition 3.3 and the fact that each 2\textsubscript{m} node uses exactly two lines from the classes 2) and 3).

3.4 Defect 3 sets

Let a set \(\mathcal{M} = \{\lambda_1, \ldots, \lambda_{n-1}\}\) of \(n - 1\) lines be in general position, \(n \geq 4\). Then consider the lattice \(\mathcal{X}\) defined as

\[ \mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)}, \]

where \(\mathcal{X}^{(2)}\) is the set of all intersection nodes of these \(n - 1\) lines, \(\mathcal{X}^{(1)}\) is a set of other \(3(n - 1)\) nodes, three in each line, to make the line maximal and \(\mathcal{X}^{(0)}\) consists of exactly three non-collinear nodes, denoted by \(O_1, O_2, O_3\), which do not belong to any line from \(\mathcal{M}\) (see Fig. 3.4). Correspondingly, we have that 

\[ \#\mathcal{X} = \binom{n-1}{2} + 3(n-1) + 3 = \binom{n+2}{2}. \]

Note that all the nodes of \(\mathcal{X}^{(k)}\) belong to exactly \(k\) maximal lines and thus are \(k_m\)-nodes, \(k = 0, 1, 2\).

Denote by \(\ell_{i}^{\text{po}}\), \(1 \leq i \leq 3\), the line passing through the two 0\textsubscript{m}-nodes in \(\{O_1, O_2, O_3\} \setminus \{O_i\}\). We call these lines OO-lines. Suppose that \(\mathcal{X}^{(1)} = \{A_1^1, A_2^1, A_3^1 \in \lambda_i : 1 \leq i \leq n - 1\}\).

In the sequel we will need the following characterization of GC\textsubscript{n} set \(\mathcal{X}\), with \(\#M(\mathcal{X}) = n - 1\), due to Carnicer and Godés (see [10], Section 5, Case d=3, for a proof detail):
Proposition 3.7 ([8], Thm. 3.2). A set $X$ is a GC$_n$ set with exactly $n - 1$ maximal lines $\lambda_1, \ldots, \lambda_{n-1}$, where $n \geq 4$, if and only if, with some permutation of the indexes of the maximal lines and 1$_m$-nodes, the representation (3.2) holds with the following additional properties (see Fig. 3.4):

(i) $X^{(1)} \setminus (\ell_1^oo \cup \ell_2^oo \cup \ell_3^oo) = \{D_1, D_2, D_3\}$, where $D_i := A_{ij}$;

(ii) Each line $\ell_i^oo, i = 1, 2, 3$, passes through exactly $n - 2$ type 1$_m$-nodes (and through two O-nodes). Moreover, $\ell_i^oo \cap \lambda_i \notin X, \ i = 1, 2, 3$;

(iii) The triples $\{O_1, D_2, D_3\}, \{O_2, D_1, D_3\}, \{O_3, D_1, D_2\}$ are collinear.

Let us denote by $\ell_{1dd}, \ell_{2dd}, \ell_{3dd}$, the lines passing through the latter triples, respectively, and call them DD-lines. Also, the nodes $D_i$ are called D-nodes.

The used lines in the defect 3 sets.

Suppose that $M(X) = \{\lambda_1, \ldots, \lambda_{n-1}\}$. Consider a pair of maximal lines $\lambda_i, \lambda_j$, where $1 \leq i < j \leq 3$. Assume that $\{i, j, k\} = \{1, 2, 3\}$. We have that the node $A_{ij} := \lambda_i \cap \lambda_j$ uses $n - 3$ maximal lines, i.e., all the maximal lines except $\lambda_i$ and $\lambda_j$ (the case of the node $A_{12}$ in Fig. 3.4). Next let us identify the remaining three used lines. We have three 1$_m$ nodes in each of two maximal lines, six in all. Two of these nodes belong to the first used line: $\ell_{1dd}^k$. Two belong to the second used line: $\ell_{2oo}^k$. Finally, the last used line is denoted by $\ell_{3dd}$, which passes through the remaining two 1$_m$ nodes in OO-lines.

Now consider a pair of maximal lines $\lambda_i, \lambda_j$, where $1 \leq i \leq 3$ and $4 \leq j \leq n - 1$. Assume that $\{i, k_1, k_2\} = \{1, 2, 3\}$. As above the node $A_{ij} := \lambda_i \cap \lambda_j$ uses $n - 3$ maximal lines, i.e., all maximal lines except $\lambda_i$ and $\lambda_j$ (the case of the node $A_{34}$ in Fig. 3.4). Next we identify the remaining three used lines. Again we have three 1$_m$ nodes in each of two maximal lines, six in all. Now
four of these six nodes lie in the two used \(OO\)-lines \(\ell_{i_1}^{oo}, \ell_{i_2}^{oo}\). Finally, denote by \(\ell_i^j\) the third used line passing through \(D_i\) and the node \(\lambda_j \cap \ell_i^{oo}\.

The set of used lines in \(X\) consists of the following five classes:

1) The set of \(n - 1\) maximal lines;
2) The set of three \(OO\)-lines \(- \{\ell_i^{oo}, 1 \leq i \leq 3\}\);
3) The set of three \(DD\)-lines \(- \{\ell_i^{dd}, 1 \leq i \leq 3\}\);
4) The set of three lines \(\{\ell_{ij}, 1 \leq i < j \leq 3\}\);
5) The three sets of \(n - 4\) lines \(\{\ell_i^j, 4 \leq j \leq n - 1\}, i = 1, 2, 3\).

\textbf{Proposition 3.8.} The classes of lines 1)-5) are disjoint.

\textit{Proof.} It is enough to show that the classes 2)-5) are disjoint.

Let us start with a line \(\ell_2 := \ell_m^{oo}\) from the class 2).

This line cannot coincide with a line \(\ell_i^{dd}\) from the class 3). Indeed, the latter line passes through two \(D\)-nodes, which, according to Proposition 3.7, do not belong to any line from the class 2).

Then \(\ell_2\) cannot coincide with a line \(\ell_{ij}\) from the class 4), since the latter line passes through two \(1_m\) nodes which belong to two different \(OO\)-lines.

Next \(\ell_2\) cannot coincide with a line \(\ell_i^j\) from the class 5). Indeed, the latter line passes through a \(D\)-node, which, as was mentioned above, does not belong to any line from class 2).

Now consider a line \(\ell_3 := \ell_m^{dd}\) from the class 3). This line cannot coincide with a line \(\ell_{ij}\) from the class 4), since the latter line certainly does not pass through two \(D\)-nodes: \(D_i\) and \(D_j\).

Next let us show that \(\ell_3\) cannot coincide with a line \(\ell_i^j\) from the class 5). Set \(\{m, k_1, k_2\} = \{1, 2, 3\}\). We have that the line \(\ell_m^{dd}\) intersects the lines \(\ell_i^{oo}\) and \(\ell_k^{oo}\) at their intersection node \(O_m\). On the other hand the line \(\ell_i^j\) intersects \(\ell_i^{oo}\) at an \(1_m\) node. Thus, if the given two lines coincide, then we readily conclude that \(i \neq k_1, k_2\), hence \(i = m\). Now the line \(\ell_m^{dd}\) passes through the node \(D_m\), while \(\ell_3\) does not pass through \(D_m\). Hence they cannot coincide.

Finally, let us show that a line \(\ell_{ij}\) from the class 4) cannot coincide with a line \(\ell_{i'}^{j'}\) from the class 5). Set \(\{i, j, k\} = \{1, 2, 3\}\). We have that the line \(\ell_{ij}\) intersects the maximal lines \(\lambda_i\) and \(\lambda_j\) at nodes belonging to \(OO\)-lines and the line \(\ell_{i'}^{j'}\) passes through the node \(D_{i'}\). Thus, if the given two lines coincide, then we conclude that \(i' \neq i, j\), hence \(i' = k\). Thus it suffices to show that line \(\ell_{23}\) does not pass through the node \(D_1\):

\textbf{Lemma 3.9.} Let \(X\) be a \(GC_n\) set of defect 3. Then the line \(\ell_{ij}\), \(1 \leq i < j \leq 3\), does not pass through the node \(D_k\), where \((i, j, k) = (1, 2, 3)\).

Suppose, by way of contradiction, that the line \(\ell_{ij} = \ell_{23}\), say, passes through the node \(D_1\) (see Fig. 3.5). Let us apply the Pappus theorem for the pair of triple collinear nodes here: \(\{D_1, B, C\}\) and \(\{O_1, D_3, D_2\}\). Denote by \(\ell(P, Q)\) the line passing through the points \(P\) and \(Q\). Observe that
Figure 3.5: May the node $D_1$ belong to the line $\ell_{23}$?

Thus, according to the Pappus theorem we get that the triple of nodes \{O_2, O_3, A_{12}\} is collinear, i.e., the OO-line $\ell_1^\infty$ passes through the point of intersection of maximal lines $\lambda_1$ and $\lambda_2$, leading to contradiction (cf. the last part of Proposition 3.8’s proof, [16]).

Next, let us verify that within each class the differently denoted lines are different. This is evident for classes 1), 2), 3) and 5). Thus assume that two lines in class 4) coincide: $\ell_{ij} = \ell_{i'j'}$. We have that each of these lines passes through two 1m nodes belonging to different OO-lines. Hence there is an O-line which intersects both lines at nodes of $X$. Hence, without loss of generality, we may assume that $i = i'$. Now suppose by way of contradiction that $j \neq j'$. Then we have that two OO-lines do not intersect the maximal line $\lambda_i$ at nodes of $X$, which contradicts Proposition 3.7 (i).

Next we show that the lines in the class 2) are proper.

Proposition 3.10. Suppose that $X$ is a GC$_n$ set of defect 3 and $\ell = \ell_i^\infty$, where $i \in \{1,2,3\}$. Suppose also that $X \downarrow \ell X$. Then $\hat{X}$ is a GC$_{n-1}$ set of defect 2, where $\ell$ is a maximal line. Hence $X^\ell = X \setminus \ell$ and $\#X^\ell = \binom{n}{2}$. Moreover $X^\ell$ is a GC$_{n-2}$ set of defect 2.

Proof. We have that $\hat{X} = X \setminus \lambda_i$. It is easily seen that $\ell$ is a maximal line of $\hat{X}$. In view of Proposition 2.5 we conclude that def($X') = 2$. It remains to notice that the set $X_2$ is GC$_{n-1}$ set satisfying the conditions of Proposition 3.3, with the O-lines $\ell_j^\infty$, $\ell_k^\infty$, and $\ell_i^\dd$, where we have that $\{i,j,k\} = \{1,2,3\}$.

Next we show that the lines in the classes 3), 4), and 5), are proper $(-1), (-1)$, and $(-2)$, and are used by 3, 1, and 1 node, respectively. We also identify the special nodes in the lines.
Proposition 3.11. Suppose that \( X \) is a \( GC_n \) set of defect three. Set 
\[ \ell_1 := \ell_k^\text{dd}, \ell_2 := \ell_{ij}, \text{ and } \ell_3 := \ell_m^o, \quad \text{where } 1 \leq k \leq 3, \quad 4 \leq m \leq n - 1, \quad \text{and} \quad \{i, j, k\} = \{1, 2, 3\}. \]
Then the following are the only possible special nodes in the respective lines: 
\[ S_1 := \ell_1 \cap \ell_k^o, \quad S_2 := \ell_2 \cap \ell_k^o, \quad S_i := \ell_3 \cap \ell_{ki}^o, \quad i = 3, 4. \]

For each \( S = S_q \) and the respective set \( \hat{X} \) we have that 
\[ S \in \hat{X} \iff S \text{ is an } 1_m \text{ node in } \hat{X} \iff S \text{ is a } 2_m \text{ node in } \hat{X} \iff S = \ell^o \cap \lambda^*, \]
where \( \ell^o \in \Pr(X) \cap M(\hat{X}) \) and \( \lambda^* \in M(X) \).

Next, we have that \( \text{def} \hat{X}(\ell_1) = \text{def} \hat{X}(\ell_2) = 2, \quad \text{def} \hat{X}(\ell_3) = 1, \) and 
\[ \#\chi^{\ell_1} = 3, \quad \#\chi^{\ell_2} = \#\chi^{\ell_3} = 1. \]

Further, the lines \( \ell_1, \ell_2, \) and \( \ell_3 \), are proper \((-1), (-2), \) and \((-2)\), respectively:
\[ \chi^{\ell_1} = \hat{X} \setminus (C_1 \cup \ell_1), \quad \chi^{\ell_2} = \hat{X} \setminus (C_2 \cup \ell_m^d \cup \ell_2), \quad \chi^{\ell_3} = \hat{X} \setminus (C_3 \cup C_4 \cup \ell_3), \quad (3.3) \]
where \( C_i \) is the d/a item.

Moreover, the following assertions hold for the lines \( \ell_1 \) and \( \ell_2 \):
- If \( S_i \notin \hat{X}(\ell_1) \), then \( \hat{X}(\ell_1) \) is a \( GC_3 \) set, and \( C_i = \ell_k^o, \ i = 1, 2. \)
- If \( S_i \in \hat{X}(\ell_1) \), then \( \hat{X}(\ell_1) \) is a \( GC_4 \) set, \( C_i = \ell_k^o \cup \lambda_i^*, \) where \( \lambda_i^* \in M(X) \) and \( S_i \in \lambda_i^*, \ i = 1, 2. \)

The following assertions hold for the line \( \ell_3 \), where \( \{k, k_3, k_4\} = \{1, 2, 3\} : \)
- If \( S_3, S_4 \notin \ell \) then \( \hat{X}(\ell_3) \) is \( GC_3 \) set, and \( C_i = \ell_k^o, \ i = 3, 4. \)
- If \( S_3 \in \ell, \ S_4 \notin \ell \) then \( \hat{X}(\ell_3) \) is \( GC_4 \) set, \( C_3 = \ell_k^o \cup \lambda^*, \ C_4 = \ell_k^o. \)
- If \( S_3, S_4 \in \ell \) then \( \hat{X}(\ell_3) \) is \( GC_5 \) set, \( C_i = \ell_k^o \cup \lambda_i^*, \) where \( \lambda_i^* \in M(X) \) and \( S_i \in \lambda_i^*, \ i = 3, 4. \)

Proof. Notice that, in view of Proposition 3.7 (i), all type \( 1_m \) nodes of \( X \), except the three \( D \)-nodes, belong to the three \( OO \)-lines.

The line \( \ell_1 \) passes through two \( D \)-nodes and the point of intersection of two \( OO \)-lines: the node \( O_1 \). Thus \( S_1 = \ell_1 \cap \ell_k^o \) is the only candidate for a third \( 1_m \) node in \( \ell_1 \).

In view of Lemma 3.9 the line \( \ell_2 \) does not pass through \( D \)-nodes. It passes through two \( 1_m \) nodes which are intersection points with \( OO \)-lines. Thus \( S_2 = \ell_2 \cap \ell_k^o \) is the only candidate for a third \( 1_m \) node in \( \ell_2 \).

Suppose first that the lines \( \ell_1 \) and \( \ell_2 \) do not have a third type \( 1_m \) nodes.

The following holds for both lines \( \ell = \ell_1 \) and \( \ell = \ell_2 \).

Observe that \( \ell \) has exactly two type \( 1_m \) nodes coinciding with the intersection points with the maximal lines \( \lambda_i \) and \( \lambda_j \). Thus, in view of (2.1), we have that \( \hat{X} \) is a \( GC_5 \) set. It consists of the node \( A_{ij} \), six \( 1_m \) nodes: three by three lying in the lines \( \lambda_i \) and \( \lambda_j \), and the three \( 0_m \) nodes: \( O_1, O_2, O_3 \). Here the line \( \ell_k^o \) with 4 nodes, including two \( O \) nodes, is an \( \ell \)-disjoint maximal.

Next, the line \( \ell \) is a maximal line in the set \( \hat{X} \setminus \ell_k^o \). Hence we readily get that first two relations of (3.3) hold with \( C_q = \ell_k^o, \ q = 1, 2. \)

Now notice that \( M(\hat{X}) = \{\lambda_i, \lambda_j, \ell_k^o\} \) and hence \( \text{def} \hat{X} = 2 \). Indeed, if there is another maximal line then it is easily seen that it has to pass through
two $O$ nodes and 2 other nodes in $\lambda_i$ and $\lambda_j$. But clearly the two $OO$-lines: $\ell_i^{oo}$ and $\ell_j^{oo}$, pass through only 3 nodes of $X$. Notice also that $\ell_1$ is a 3-node line in $\hat{X}$ and $\ell_1$ is a 2-node line in $\hat{X}$, without any special node.

Next, suppose that each of the lines $\ell_1$ and $\ell_2$ has a third type $1_m$ node. Then we have that $S_q = \lambda^*_q \cap \ell_k^{oo}$, where $\lambda^*_q \in M(X)$, $q = 1, 2$. Now the number of type $1_m$ nodes in each of the lines equals to three. Thus, in view of (2.1), we have that $X(\ell_q)$ is a GC4 set, $q = 1, 2$. Here $\ell_k^{oo}$ and $\lambda^*_q$ form a pair of $\ell$-adjoint maximal lines and we readily get that the first two relations of (3.3) hold with $C_q = \ell_k^{oo} \cup \lambda^*_q$, $q = 1, 2$. Note that $S_q$ is the only special node in $\ell_q$, $q = 1, 2$.

Notice that $M(\hat{X}(\ell_q)) = \{\lambda_i, \lambda_j, \ell_k^{oo}, \lambda^*_q\}$ and hence def$\hat{X}(\ell_q) = 2$, $q = 1, 2$. Indeed, if there is more than four maximal lines then it is easily seen that $X \setminus \lambda^*_q$ would have more than three maximal lines, which contradicts the consideration in the previous case.

Now consider the line $\ell_3$. It passes through one $D$-node, i.e., $D_k$, and the point of intersection with one $OO$-line $\ell_k^{oo}$. Thus the points of intersections with the other two $OO$-lines, i.e., $S_3$ and $S_4$, are the only candidates for a third and forth $1_m$ nodes in $\ell_3$.

Suppose first that the line $\ell_3$ does not have a third and forth type $1_m$ nodes. Then it has exactly two type $1_m$ nodes. Thus, in view of (2.1), we have that $\hat{X}$ is a GC3 set. It consists of the node $A_{ij}$, six $1_m$ nodes: three by three lying in the lines $\lambda_i$ and $\lambda_j$, and the three $o_m$ nodes: $O_1, O_2, O_3$. Here the lines $\ell_k^{oo}$, $q = 1, 2$, with 4 nodes, including two $O$ nodes, are $\ell$-disjoint maximal lines. Next, the line $\ell_3$ is a maximal line in the set $X \setminus (\ell_1^{oo} \cup \ell_2^{oo})$. Hence we readily get that the third relation of (3.3) holds with $C_q = \ell_k^{oo}$, $q = 3, 4$.

Now notice that $M(\hat{X}) = \{\lambda_i, \lambda_j, \ell_k^{oo}, \ell_k^{oo}\}$ and hence def$X = 1$. Note that $\ell_3$ is a 2-node line in $\hat{X}$, without any special node.

Next, suppose that the line $\ell_3$ has a third type $1_m$ node, $S_3$, say. Then we have that $S_3 = \lambda^*_3 \cap \ell_k^{oo}$, where $\lambda^*_3 \in M(X)$. Now the number of type $1_m$ nodes in $\ell_3$ equals to three. Thus, in view of (2.1), we have that $X(\ell_q)$ is a GC4 set. Here $\ell_k^{oo}$ and $\lambda^*_3$ form a pair of $\ell$-adjoint maximal lines and we readily get that the third relation of (3.3) holds with $C_3 = \ell_k^{oo} \cup \lambda^*_3$ and $C_4 = \ell_k^{oo}$. Note that $S_3$ is the only special node in $\ell_3$.

Notice that $M(\hat{X}) = \{\lambda_i, \lambda_j, \ell_k^{oo}, \lambda^*_3\}$ and hence def$X = 1$. Indeed, if there is more than five maximal lines then it is easily seen that $X \setminus \lambda^*$ would have more than four maximal lines, which contradicts the consideration in the previous case.

Finally, suppose that the line $\ell_3$ has four type $1_m$ nodes. Then we have that $S_q = \lambda^*_q \cap \ell_k^{oo}$, where $\lambda^*_q \in M(X)$, $q = 3, 4$. Now the number of type $1_m$ nodes in $\ell_3$ equals to four. Thus, in view of (2.1), we have that $X(\ell_q)$ is a GC5 set. Here $\ell_k^{oo}$ and $\lambda^*_q$, $q = 3, 4$, form two pairs of $\ell$-adjoint
maximal lines and we readily get that the third relation of (3.3) holds with $C_q = \ell_{k_q} \cup \lambda_q^*$, $q = 3, 4$. Notice that in this case $S_3$ and $S_4$ are the only special nodes in $\ell_3$. Notice also that $M(\hat{X}) = \{\lambda_i, \lambda_j, \ell_{k_3}, \ell_{k_4}, \lambda_3^*, \lambda_4^*\}$ and hence $\text{def}\hat{X} = 1$.

Denote by $n_i$ the number of line usages in the class $i$), $i = 1, \ldots, 5$. Then we have that the total number of line-usages equals:

$$n_1 + \cdots + n_5 = (n - 1){n+1 \choose 2} + 3{n \choose 2} + 9 + 3(n - 4) = n{n+2 \choose 2}. $$

### 3.5 Generalized principal lattices: defect $n - 1$ sets

A principal lattice is defined as an affine image of the set (see Fig. 3.6)

$$PL_n := \{(i, j) \in \mathbb{N}_0^2 : \ i + j \leq n\}. $$

Let us set $I = \{0, 1, \ldots, n + 1\}$. Observe that the following 3 sets of $n + 1$ lines, namely $\{x = i : i \in I\}$, $\{y = j : j \in I\}$, and $\{x + y = k : k \in I\}$, intersect at $PL_n$. We have that $PL_n$ is a $GC_n$ set. Moreover, the following is the fundamental polynomial of the node $(i_0, j_0) \in PL_n$:

$$p^*_{i_0j_0}(x, y) = \prod_{0 \leq i < i_0, \ 0 \leq j < j_0, \ 0 \leq k < k_0} (x - i)(y - j)(x + y + n + k), \quad (3.4)$$

where $k_0 = n - i_0 - j_0$. Next let us bring the definition of the generalized principal lattice due to Carnicer, Gasca and Godès (see [6], [7]):

**Definition 3.12 ([7]).** A node set $\mathcal{X}$ is called a generalized principal lattice, briefly $GPL_n$, if there are 3 sets of lines each containing $n + 1$ lines

$$\ell^0_i := \ell_i(\mathcal{X}), \ i = 0, 1, \ldots, n, \ j = 0, 1, 2, \quad (3.5)$$

such that the $3n + 3$ lines are distinct,

$$\ell^0_i \cap \ell^1_j \cap \ell^2_k \cap \mathcal{X} \neq \emptyset \iff i + j + k = n$$

and

$$\mathcal{X} = \{x_{ijk} \mid x_{ijk} := \ell^0_i \cap \ell^1_j \cap \ell^2_k, 0 \leq i, j, k \leq n, i + j + k = n\}. $$
Observe that if \( 0 \leq i, j, k \leq n \), \( i + j + k = n \) then the three lines \( \ell_i^0, \ell_j^1, \ell_k^2 \), intersect at a node \( x_{ijk} \in \mathcal{X} \). This implies that each node of \( \mathcal{X} \) belongs to only one line of each of the three sets of \( n+1 \) lines. Therefore \( \#\mathcal{X} = (n+1)(n+2)/2 \).

One can find readily, as in the case of \( PL_n \), the fundamental polynomial of each node \( A = x_{i_0j_0k_0} \in \mathcal{X}, i_0 + j_0 + k_0 = n \):

\[
p_A^* = \prod_{0 \leq i < i_0, 0 \leq j < j_0, 0 \leq k < k_0} \ell_i^0 \ell_j^1 \ell_k^2.
\]

(3.6)

Thus \( \mathcal{X} \) is a \( GC_n \) set.

Let us bring a characterization for \( GPL_n \) set due to Carnicer and Godés:

**Theorem 3.13** ([7], Thm. 3.6). Assume that GM Conjecture holds for all degrees up to \( n-3 \). Then the following statements are equivalent:

(i) \( \mathcal{X} \) is generalized principal lattice of degree \( n \);

(ii) \( \mathcal{X} \) is a \( GC_n \) set with \( \#M(\mathcal{X}) = 3 \).

The used lines in the generalized principal lattice.

From (3.6) we have that the only used lines in in the the generalized principal lattice \( \mathcal{X} \) are

1) The three sets of \( n \) lines \( \ell_s^r(\mathcal{X}) \), where \( 0 \leq s \leq n-1, r = 0, 1, 2 \).

Let \( \ell := \ell_{n-k+1}^r(\mathcal{X}) \) be a \( k \)-node line. Then it is easily seen that

\[
\mathcal{X}^\ell = \mathcal{X} \setminus (\ell_0^r \cup \ell_1^r \cup \cdots \cup \ell_{n-k+1}^r).
\]

(3.7)

Hence \( \mathcal{X}^\ell \) is a \( GC_{k-2} \) subset of \( \mathcal{X} \). Hence \( \#\mathcal{X}^\ell = \binom{k}{2} \).

Moreover, if \( k \leq n \) and \( \mathcal{X}^\ell \neq \emptyset \) then for a maximal line \( \lambda_1 \) of \( \mathcal{X} \) we have that \( \lambda_1 \cap \ell \notin \mathcal{X} \) and \( \#(\lambda_1 \cap \mathcal{X}^\ell) = 0 \). For the remaining two maximal lines we have that \( \#(\lambda \cap \mathcal{X}^\ell) = k - 1 \).

Next we have that the total number of line-usages here equals:

\[
3 \left[ \frac{(n+1)}{2} + \frac{(n)}{2} + \cdots + \frac{(2)}{2} \right] = 3 \left( \frac{n+2}{3} \right) = n \left( \frac{n+2}{2} \right).
\]

4 Some applications

4.1 On the set \( \mathcal{X}^\ell \)

For \( GC_n \) sets of defect \( n-1 \) there is a simple formula for the set \( \mathcal{X}^\ell \), namely (3.7), which yields all its basic properties. For other \( GC_n \) sets we have the following

**Theorem 4.1.** Let \( \mathcal{X} \) be a \( GC_n \) set with \( \text{def}\mathcal{X} \neq n-1 \), where \( n \geq 5 \), \( \ell \) be a used line and \( \mathcal{X} \downarrow_\ell \mathcal{X} \). Assume that GM Conjecture holds for all degrees up
to n. Then $\text{def}\hat{X} \leq \text{def}\mathcal{X} - 1$ and we have that either $\ell$ is proper and hence $\mathcal{X}^{\ell} = \hat{X} \setminus \ell$, or otherwise it is just proper $(-k)$, where $k = 1$ or $2$. In the latter two cases we have that the set $\hat{X}$ is a $GC_k$ set with $k \leq 5$ and the set $\mathcal{X}^{\ell}$ is $GC_0$ or $GC_1$ set. Next, if $\ell$ is proper then it has no special node, and if it is proper $(-k)$ then it has at most $k$ special nodes, $k = 1, 2$. Moreover, if $(-1)$ or $(-2)$ $d/a$ items are disjoint lines, then they are proper in $\mathcal{X}$, and, if the items are pairs of adjoint lines then one in each pair is a proper line and another is a maximal line in $\mathcal{X}$.

\textbf{Proof.} The proof is divided into three cases.

\textbf{Case 1.} $\text{def}(\mathcal{X}) = 0$, or 1.

In this case, by Proposition 3.2 all used lines are maximal or proper.

\textbf{Case 2.} $\text{def}(\mathcal{X}) = 2$.

In view of Proposition 3.4 all lines in classes 1) and 2) are maximal and proper, respectively. Thus consider a line $\ell := \ell_{ij}$ belonging to class 3). According to Proposition 3.6 the line $\ell$ is proper $(-1)$, and $\#\mathcal{X}^{\ell} = 1$. If the line $\ell$ has no special node then $d/a$ item is a disjoint line, namely the proper line $\ell_k^{(i,j)}$ from class 2) and $\hat{X}$ is $GC_2$ set. If the line $\ell$ has a special node $S = \ell_k^{(i,j)} \cap \ell$ then $\hat{X}$ is $GC_3$ set and $d/a$ item is the union of two adjoint lines $\ell_k^{(i,j)} \cup \lambda^{*}$, where $\ell_k^{(i,j)} \in Pr(\mathcal{X})$, $\lambda^{*} \in M(\mathcal{X})$, and $S \in \lambda^{*}$.

\textbf{Case 3.} $\text{def}(\mathcal{X}) = 3$.

If $\mathcal{X}$ is a defect 3 set then, in view of Proposition 3.10 all lines in classes 1) and 2) are maximal and proper, respectively. Thus consider lines $\ell_1 := \ell_k^{oo}$ and $\ell_2 := \ell_{ij}$ belonging to classes 3) and 4), respectively, where $\{i, j, k\} = \{1, 2, 3\}$. According to Proposition 3.11 the line $\ell_i$ is proper $(-i)$, $i = 1, 2$. In all cases the $(-2)$ $d/a$ item for $\ell_2$ is $\ell_k^{dd}$.

We also have that $\#\mathcal{X}^{\ell_2} = 1$ and $\mathcal{X}^{\ell_1}$ is a $GC_1$ set, hence $\#\mathcal{X}^{\ell_1} = 3$.

If the lines $\ell_1$ and $\ell_2$ have no special node, then $\hat{X}$ is $GC_3$ set, and $(-1)$ $d/a$ item, for both $\ell_1$ and $\ell_2$, is the proper line $\ell_k^{oo}$ from class 3).

Now assume that each line $\ell_q$ has a special node $S_q = \ell_k^{oo} \cap \ell_q$, $q = 1, 2$. Then $\hat{X}$ is $GC_4$ set, and $(-1)$ $d/a$ item is the union of the lines $\ell_k^{oo} \cup \lambda_q^{*}$, where $\ell_k^{oo} \in Pr(\mathcal{X})$, $\lambda_q^{*} \in M(\mathcal{X})$, and $S_q \in \lambda_q^{*}$, $q = 1, 2$.

Next consider a line $\ell := \ell_i^j$ belonging to class 5), where $1 \leq i \leq 3$, $4 \leq j \leq n - 1$. Suppose that $\{i, k_3, k_4\} = \{1, 2, 3\}$. According to Proposition 3.11 the line $\ell$ is proper $(-2)$ and $\#\mathcal{X}^{\ell} = 1$.

Assume that $\ell$ has no special node, then $\hat{X}$ is $GC_3$ set, $(-1)$ and $(-2)$ $d/a$ items are proper $\ell$-disjoint lines $\ell_k^{oo}$ and $\ell_k^{oo}$, respectively, which belong to class 2).

If the line $\ell$ has a special node $S_3 = \ell_k^{oo} \cap \ell$, say, then $\hat{X}$ is $GC_4$ set, and $(-1)$ and $(-2)$ $d/a$ items are $\ell_k^{oo} \cup \lambda^{*}$ and $\ell_k^{oo}$, respectively. Here $\ell_k^{oo} \in Pr(\mathcal{X})$, $\lambda_3^{*} \in M(\mathcal{X})$, and $S_3 \in \lambda_3^{*}$.

Now assume that the line $\ell$ has two special nodes $S_3 = \ell_k^{oo} \cap \ell$ and $S_4 = \ell_k^{oo} \cap \ell$. 

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Then $\hat{X}$ is GC set, and $(-1)$ and $(-2)$ d/a items are $\ell_{kq}^\emptyset \cup \lambda_q^*$, $q = 3, 4$, respectively. Here $\ell_{kq}^\emptyset \in \Pr(X)$, $\lambda_q^* \in M(X)$, and $S_q \in \lambda_q^*$.

\[\text{Theorem 4.2.} \quad \text{Let } X \text{ be a } GC_n \text{ set and } \ell \text{ be a } k\text{-node used line. Assume that } \ell \text{ contains exactly } r \text{ type } 2_m \text{ nodes in } X \text{ and } \hat{r} \text{ type } 2_m \text{ nodes in } \hat{X}. \text{ Assume that GM Conjecture holds for all degrees up to } n. \text{ Then, } X^\ell \text{ is a } GC_{s-1} \text{ set and hence } \#X^\ell = \binom{s}{2}, \text{ where } s = k - r - \hat{r}. \text{ Moreover, for any used line } \ell \text{ we have that } \hat{r} \leq 2. \text{ Furthermore, } \hat{r} = 0 \text{ if } \#X^\ell > 3.\]

\text{Proof.} First notice that $\hat{r}$ is the number of special nodes in the line $\ell$. Therefore, Theorem 4.1 implies the statements in parts “Moreover” and “Furthermore.

Then, in view of relation (1.1) and Proposition 1.4, we get that Theorem holds with $r = \hat{r} = 0$, if the line $\ell$ is a maximal line.

Next, in view of Lemmas 2.7 and 2.8 we get that Theorem holds with $\hat{r} = 0$, if the line $\ell$ is a proper line.

Thus Theorem holds if $X$ is a Chung-Yao or Carnicer-Gasca lattice, since then, in view of Proposition 3.2 all used lines are maximal or proper lines.

If $X$ is a defect 2 set then, in view of Proposition 3.4, all lines in classes 1) and 2) are maximal and proper, respectively. Thus consider a line $\ell := \ell_{ij}$ belonging to class 3). According to Proposition 3.6 the line $\ell$, not counting a special node, is 2-node line in $\hat{X}$ and $\#X^\ell = 1$. Also we have that $s = k - r - \hat{r} = 2$.

If $X$ is a defect 3 set then, in view of Proposition 3.10, all lines in classes 1) and 2) are maximal and proper, respectively. Thus consider lines $\ell_1 := \ell_{k}$ and $\ell_2 := \ell_{ij}$ belonging to classes 3) and 4), respectively, where $\{i, j, k\} = \{1, 2, 3\}$. According to Proposition 3.11 the line $\ell_1$, not counting a special node, is 3-node line in $\hat{X}$ and $\#X^\ell_1 = 3$, while the line $\ell_2$, not counting a possible special node, is 2-node line in $\hat{X}$ and $\#X^\ell_2 = 1$. Also we have that $s = k - r - \hat{r} = 2$.

Finally consider a line $\ell := \ell_i^1$ belonging to class 5), where $1 \leq i \leq 3$, $4 \leq j \leq n - 1$. According to Proposition 3.11 the line $\ell$, not counting possible two special nodes, is a 2-node line in $\hat{X}$ and $\#X^\ell = 1$. Also we have that $s = k - r - \hat{r} = 2$. □

From Theorem 4.2 we readily get

\text{Corollary 4.3.} \quad \text{Let } X \text{ be a } GC_n \text{ set and } \ell \text{ be a } k\text{-node used line. Assume that } \ell \text{ contains exactly } r \text{ type } 2_m \text{ nodes. Assume that GM Conjecture holds for all degrees up to } n. \text{ Then } \#X^\ell \in \{0, 1, 3, \binom{k-r}{2}\}.

Below, we restate a result from Theorem 3.1, [16]. In the “Moreover” and “Furthermore” parts we complement it.
Theorem 4.4. Let $\mathcal{X}$ be a $GC_n$ set, $\ell$ be a line with $\#\mathcal{X}^\ell = \binom{s}{2}$, $s \geq 2$. Assume that GM Conjecture holds for all degrees up to $n$. Then for any maximal line $\lambda$ we have that $\#(\lambda \cap \mathcal{X}^\ell) = s - 1$, or 0. Moreover, the latter case:

$$\#(\lambda \cap \mathcal{X}^\ell) = 0,$$

holds, if and only if

(i) $\lambda$ is an $\ell$-disjoint maximal line, or

(ii) $\lambda$ is one of $\ell$-adjoint maximal lines, or

(iii) $\lambda$ is an $\ell$-special maximal line.

Furthermore, for any line $\ell$ (iii) may hold for at most two maximal lines $\lambda$.

Proof. The direction “if” follows from Lemma 2.7 and 2.8 applied to both $\mathcal{X}$ and $\hat{\mathcal{X}}$. For the direction “only if” consider a maximal line $\lambda$ and assume that it is not $\ell$-disjoint, $\ell$-adjoint and $\ell$-special. Then, according to Propositions 3.4, 3.10, and 3.11, $\lambda$ is a maximal line in the $GC_{s-2}$ set $\mathcal{X}^\ell$ and hence

$$\#(\lambda \cap \mathcal{X}^\ell) = s - 1. \quad (4.1)$$

4.2 On $n$-nodes and proper lines in $GC_n$ sets

Denote by $N(\mathcal{X})$ the set of all $n$-node lines of $GC_n$ set $\mathcal{X}$.

Proposition 4.5. Let $\mathcal{X}$ be a $GC_n$ set, $n \geq 4$. Assume that GM Conjecture holds for all degrees up to $n$. Then $\#N(\mathcal{X}) \in \{0, 1, 2, 3\}$. Moreover, if a line $\ell$ is $n$-node line then it intersects each maximal line, except possibly one, at a node of $\mathcal{X}$. Furthermore, any two $n$-node lines intersect at a node of $\mathcal{X}$. Also, any three $n$-node lines are coincident, if $\#M(\mathcal{X}) \neq 3$.

Proof. Assume that $\ell$ is an $n$-node line. Assume also that $\ell$ is type $(i, j, k)$, meaning that it passes through $i$ type 0$_m$ nodes, $j$ type 1$_m$ nodes, and $k$ type 2$_m$ nodes. Notice that to prove the “Moreover” part it suffices to verify the inequality $j + 2k \geq \#M(\mathcal{X}) - 1$. Consider the following cases:

**Case 1.** $\#M(\mathcal{X}) = n + 2$.

In this case all nodes are type 2$_m$ and there is no $n$-node line. Indeed, suppose by way of contradiction that $\ell$ is an $n$-node line. Then for the number of maximal lines intersecting $\ell$ we have $2n \leq n + 2$, i.e., $n \leq 2$. 

**Case 2.** $\#M(\mathcal{X}) = n + 1$.

In this case all nodes are type 1$_m$ or 2$_m$. Thus $\ell$ is type $(0, j, k) = (0, n - k, k)$. Then we have that $n - k + 2k \leq n + 1$. Hence $k = 0$ or 1.

It is easily seen that only one $n$-node line is possible if $n \geq 5$ : type $(0, n, 0)$ or type $(0, n - 1, 1)$. If $n = 4$ then either one type $(0, 4, 0)$ line is possible or two type $(0, 3, 1)$ lines.
Case 3. \( \#M(\mathcal{X}) = n \).

In this case we have one type 0\textsubscript{m} node: \( O \).
First suppose that the line \( \ell \) does not pass through \( O \), i.e., it is type \((0, j, k) = (0, n-k, k)\). Then we have that \( n-k+2k \leq n \), i.e., \( k = 0 \). Clearly in this case, in view of Proposition 3.3 (i), the number of type 1\textsubscript{m} nodes in \( \ell \) is less than or equal to 3. Hence \( n \leq 3 \).

Then suppose that the line \( \ell \) passes through \( O \), i.e., is type \((1, j, k) = (1, n-1-k, k)\). Also suppose that \( \ell \) is different from \( O \)-lines. In this case, by Proposition 3.3 (i), the number of type 1\textsubscript{m} nodes in \( \ell \) equals to 0. Hence \( k = n - 1 \) and we have that \( 2(n-1) \leq n \). Hence \( n \leq 2 \).

Next suppose that the line \( \ell \) is an \( O \)-line. For the number of type 1\textsubscript{m} nodes in the three \( O \)-lines we have \( n_1 + n_2 + n_3 = 2n \), where \( n_i \geq 2 \) is the number of 1\textsubscript{m} nodes in \( i \)th \( O \)-line. Suppose all three lines are \( n \)-node lines. Observe that then each of them has at most one 2\textsubscript{m} node. Therefore we have that \( 3(n-2) \leq 2n \) hence \( n \leq 6 \). Thus for \( n \geq 7 \) two \( n \)-node lines are possible in all: of types \((1, n-1, 0)\) or \((1, n-2, 1)\). It is easily seen that in the cases \( n = 4, 5, 6 \), all three \( O \)-lines can be \( n \)-node lines of types \((1, n-1, 0)\) or \((1, n-2, 1)\).

Case 4. \( \#M(\mathcal{X}) = n - 1 \).

In this case we have three non-collinear type 0\textsubscript{m} nodes: \( O_1, O_2, O_3 \).
First suppose that the line \( \ell \) does not pass through any \( O \) node, i.e., is type \((0, j, k) = (0, n-k, k)\). Then we get \( n-k+2k \leq n-1 \), i.e., \( k \leq -1 \).

Now suppose that the line \( \ell \) passes through one \( O \) node, i.e., is type \((1, j, k) = (1, n-1-k, k)\). Also suppose that \( \ell \) is different from \( O \)-\( O \)-lines. Then we have that \( n-1-k+2k \leq n-1 \). Hence \( k = 0 \). In this case, by Proposition 3.7 (iii), the number of type 1\textsubscript{m} nodes in \( \ell \) is less than or equal to 3 (the case of DD-line with one special node). Thus \( n-1 \leq 3 \) and \( n \leq 4 \). It is easily seen that in the case of \( n = 4 \) a DD-line can not have a special node and be a 4-node line.

Next suppose that the line \( \ell \) is an \( O \)-\( O \)-line. By Proposition 3.7 (ii), these three lines are \( n \)-node lines of type \((2, n-2, 0)\).

Case 5. \( \#M(\mathcal{X}) = 3 \).

According to Corollary 4.4 \[15\] in this case there are exactly three \( n \)-node lines and each of them intersects exactly two of three maximal lines at nodes of \( \mathcal{X} \), i.e., is type \((n-2, 2, 0)\).

Remark 4.6. Notice that all above \( n \)-node lines are either type \((i, j, 0)\) with \( i + j = n \), \( j + 2 \cdot 0 = \#M(\mathcal{X}) - 1 \), or type \((i, j, 1)\) with \( i + j + 1 = n \), \( j + 2 \cdot 1 = \#M(\mathcal{X}) \). Note that in the first case \( n \)-node line intersects all but one maximal line of \( \mathcal{X} \), and in the second case it intersects all maximal lines of \( \mathcal{X} \).

Recall that any line passing through at least two of \( n+1 \) type 1\textsubscript{m} nodes of a defect 1 set is a proper line. For other GC sets we have the following
Proposition 4.7. Let $X$ be a GC$_n$ set, def$X \neq 1$, $n \geq 4$. Assume that GM Conjecture holds for all degrees up to $n$. Then $\#Pr(X) \in \{0, 3\}$.

Proof. Indeed, we have that there is no proper line in Chung-Yao lattice. In the defect 2 and defect 3 sets the only proper lines are the three O-lines and three OO-lines, respectively.

Finally consider the case of defect $n - 1$ sets. It is easily seen that here no used line passes through a $2_m$ node and for each used line $\ell$ there is exactly one $\ell$-disjoint maximal line. Thus there are exactly three $\ell$-lowering sets, namely the sets $X \setminus \lambda$, where $\lambda \in M(X)$. Thus, we get readily that $Pr(X) = N(X)$. In view of the case 5) of the proof of Proposition 4.5 this completes the proof.

4.3 On the maximal and used lines

Next, we complement Proposition 2.5. We readily get from Proposition 4.5 and Remark 4.6 the following

Proposition 4.8. Let $X$ be a GC$_n$ set, $n \geq 4$. Assume that GM Conjecture holds for all degrees up to $n$. Then the relation

$$\text{def} (X \setminus \lambda) = \text{def}(X) - 1$$

(4.2)

holds for $\lambda \in M(X)$ if and only if there is type $(i, j, 0)$ $n$-node line $\ell$ such that $\ell \cap \lambda \notin X$. Hence (4.2) holds for at most three maximal lines $\lambda$.

Note that for all other maximal lines $\lambda$, in view of Proposition 2.5, we have that $\text{def}(X \setminus \lambda) = \text{def}(X)$.

For a defect $n - 1$ set there is a simple formula for the fundamental polynomials of nodes $A \in X^\ell$, namely (3.6). This formula gives the $n$ used lines of the node $A$. For other GC$_n$ sets we have the following

Theorem 4.9. Let $X$ be a GC$_n$ set with def$X \neq n - 1$, where $n \geq 5$, and $A \in X$ be a node. Then we have that among $n$ lines used by $A$ at most two lines are proper, at most one line is proper $(-1)$, at most one line is proper $(-2)$, and at least $n - 3$ lines are maximal.

Proof. Let us list the following possibilities:

1) All $n$ used lines of $A$ are maximal lines;
2) $n - 1$ used lines are maximal and one used line is proper;
3) $n - 2$ used lines are maximal and two used lines are proper;
4) $n - 2$ used lines are maximal, one is proper and one is proper $(-1)$;
5) $n - 3$ used lines are maximal and three used lines are proper;
6) $n - 3$ used lines are maximal, two are proper, and one is proper $(-2)$;
7) $n - 3$ used lines are maximal, one is proper, one is proper $(-1)$, and one is proper $(-2)$.
In view of the results in Section 3, one can readily verify the following:
If \( \mathcal{X} \) is a Chung-Yao lattice then for all nodes the item 1) holds.
If \( \mathcal{X} \) is a Carnicer-Gasca lattice then for all type 1\(_m\) nodes the item 1) holds. While for all 2\(_m\) nodes the item 2) holds.
If \( \mathcal{X} \) is defect 2 set then for the node \( O \) the item 1) holds. For all type 1\(_m\) nodes the item 2) holds. For all type 2\(_m\) nodes the item 3) or 4) holds.
If \( \mathcal{X} \) is defect 3 set then for the three \( O \)-nodes the item 2) holds. For the three \( D \)-nodes the item 3) holds. For the remaining type 1\(_m\) nodes the item 4) holds. Finally, for all 2\(_m\) nodes one of the items 5), 6) or 7) holds.

It was proved in Corollary 4.1 of [16] that for any \( k \) node line in a \( GC_n \) set \( \mathcal{X} \) there is either \( \ell \)-disjoint maximal line or a pair of \( \ell \)-adjoint maximal lines, if \( n \geq 6 \). Below we strengthen this result in the case of used lines.

**Corollary 4.10.** Let \( X_0 \) be a \( GC_n \) set, \( \ell \) be a \( k \)-node used line, \( 2 \leq k \leq n \). Assume that GM Conjecture holds for all degrees up to \( n \). Then there is either \( \ell \)-disjoint maximal line or a pair of \( \ell \)-adjoint maximal lines.

**Proof.** Observe that it suffices to verify that for any used line \( \ell \) we have that \( \mathcal{X} \neq \hat{X}(\ell) \). This, in the case of \( \text{def} \mathcal{X} \neq n - 1 \) follows from the statement \( \text{def} \hat{X} \leq \text{def} \mathcal{X} - 1 \) of Theorem 4.1. It remains to note that evidently, for any used line \( \ell \) in defect \( n - 1 \) set there is an \( \ell \)-disjoint maximal line.

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