ON COMPONENTS OF A KERDOCK CODE AND THE DUAL OF THE BCH CODE $C_{1,3}$

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Abstract. In the paper we investigate the structure of $i$-components of two classes of codes: Kerdock codes and the duals of the primitive cyclic BCH code with designed distance 5 of length $n = 2^m - 1$, for odd $m$. We prove that for any admissible length a punctured Kerdock code consists of two $i$-components and the dual of BCH code is $i$-component for any $i$. We give an alternative proof for the fact that the restriction of the Hamming scheme to a doubly shortened Kerdock code is an association scheme [12].

Keywords: Kerdock code, shortened Kerdock code, punctured Kerdock code, Reed-Muller code, uniformly packed code, dual code, association scheme, t-design

1. Introduction

Let $\mathbb{F}^n$ be the vector space of dimension $n$ over the Galois field $GF(2)$. Denote by $0^n$ and $1^n$ the all-zero and all-one vectors in $\mathbb{F}^n$ respectively. The Hamming distance $d(x, y)$ between vectors $x, y \in \mathbb{F}^n$ is the number of positions at which the corresponding symbols in $x$ and $y$ are different. The Hamming weight $w(x)$ of a vector $x$ is $d(x, 0^n)$. A code of length $n$ is a subset of $\mathbb{F}^n$. Vectors of a code are called codewords. The size of a code is the number of its codewords. The code distance (or minimum distance) of a code is the minimum value of the Hamming distance between two different codewords from the code. The kernel $Ker(C)$ of a code $C$ is $\{x : x + C = C\}$. Obviously, the code $C$ is a union of cosets of $Ker(C)$. The code obtained from a code $C$ by deleting one coordinate position is called the punctured code. Such code we denote by $C^*$ and doubly punctured code by $C^{**}$. The shortened code of $C$ is obtained by selecting the subcode of $C$ having zeros at a certain position and deleting this position. We denote such code by $C'$. Doubly shortened code we denote by $C''$. For a code $C$ denote by $I(C)$ the set of distances between its codewords: $I(C) = \{d(x, y) : x, y \in C\}$ and by $C_i$ denote the set of its codewords of weight $i$: $C_i = \{x \in C : w(x) = i\}$. All other necessary definitions and notions can be found in [2].

Given a code $C$ with minimum distance $d$ consider the graph $G_i(C)$ with the set of codewords as the set of vertices and the set of edges $\{(x, y) : d(x, y) = d, x_i \neq y_i\}$. A connected component of the graph $G_i(C)$ is called the $i$-component of the code. If the minimum distance $d$ is greater then 2 then changing the value in $i$th coordinate position in all vectors of any $i$-component by the opposite one in the code leads to a code with the same parameters: length, size and code distance. Therefore, we can obtain an exponential number (as a function of the number of $i$-components in...
the code) of different codes with the same parameters. Such an approach was earlier successfully developed for the class of perfect codes. The method of \( i \)-components allowed to construct a large class of pairwise nonequivalent perfect codes and was used to study various code properties, see the survey [9].

Punctured Preparata codes, perfect codes with code distance 3 and the primitive cyclic BCH code \( C_{1,3} \) with designed distance 5 of length \( 2^m - 1 \), odd \( m \) are known to be uniformly packed [11], [5]. Therefore, the fixed weight codewords of the extensions of these codes form 3-designs, which was proved by Semakov, Zinoviev and Zaitsev in [11]. An analogous property holds for duals of codes from these classes. Let \( C^\perp \) be a formally dual code to a code \( C \) with code distance \( d \), i.e. their weight distributions are related by McWilliams identities [2]. In Theorem 9, Ch. 9, [2] it was shown that the set of codewords of any fixed weight in \( C^\perp \) is \( (d - \tilde{s}) \)-design, where \( \tilde{s} \) denotes the number of different nontrivial (not equal to 0 and \( n \)) weights of the codewords of \( C^\perp \). It is well-known that a Kerdock code and a Preparata code of the same length are formally dual. Therefore, the fixed weight codewords of a Kerdock code are 3-designs and the code \( \overline{C}_{1,3} \) orthogonal to \( C_{1,3} \) of length \( 2^m - 1 \), \( m \)-odd, are 2-designs respectively.

The aforementioned codes are related to association schemes. Let \( X \) be a set, and there are \( n + 1 \) relations \( R_t, t \in T \) that partition \( X \times X \). The pair \( (X, \{R_t\}_{t \in T}) \) is called an association scheme, if there are \( \delta_{i,j,k}(X) \), such that

- The relation \( \{(x,x):x \in X\} \) is \( R_j \) for some \( j \in I \).
- For any \( i \), the relation \( R_i^{-1} = \{(y,x): (x,y) \in R_i \} \) is \( R_j \) for some \( j \in I \).
- For any \( i,j,k \in I \) and \( x, y \) in \( X \), \( (x,y) \in R_i \) the following holds:

\[
\delta_{i,j,k}(X) = |\{ z : z \in X, (x,z) \in R_j, (y,z) \in R_k \}|.
\]

The numbers \( \delta_{i,j,k}(X), i,j,k \in I \) are called intersection numbers of the association scheme.

Let \( C \) be a binary code. Consider the partition of the cartesian square \( C \times C \) into distance relations, i.e. two pairs of codewords are in the same relation if and only if the Hamming distances between the pairs coincide. Such partition is called the restriction of the Hamming scheme to the code \( C \), see [7]. There are several cases where the restriction gives an association scheme. In this case, the code with this property is called distance-regular, see [10]. Using linear programming bound, Delsarte in [7] showed that the restriction of the Hamming scheme to a shortened Kerdock code is an association scheme. An analogous fact for Kerdock codes was proved in [10] by finding the intersection numbers of the restricted scheme directly. In work [12], see also [13], it is shown that the restriction to a doubly shortened Kerdock code is also an association scheme. The latter fact contributes to a significant part of the current paper concerning components of a Kerdock code, however we give an alternative combinatorial proof for this fact as we essentially need a convenient way of finding the intersection numbers of the scheme. Delsarte (Theorem 6.10, [7]) proved that the restriction of the Hamming scheme to the dual of any linear uniformly packed code (in particular, the code \( \overline{C}_{1,3} \), which is dual of the BCH code \( C_{1,3} \)) is an association scheme.

In this paper we show that the punctured Kerdock code have two \( i \)-components for any coordinate position \( i \), while the dual of a linear uniformly packed code with parameters of BCH code \( C_{1,3} \) is \( i \)-component for any coordinate position \( i \).
2. Components of Kerdock code

In the section we fix \( n \) to be \( 2^m \), for even \( m \), \( m \geq 4 \). A Kerdock code \( K \) is a binary code of length \( n \), and minimum distance \( d = (n - \sqrt{n})/2 \), consisting of the first order Reed–Muller code \( \text{RM}(1, m) \) and \( 2^{m-1} - 1 \) its cosets such that the weights of the codewords in a coset are \( d \) or \( n - d \). These codes were firstly constructed in \cite{3}, and further generalizations were obtained in \cite{4, 8}.

The weight distribution of a Kerdock code is well-known and is related with the weight distribution of a Preparata code via McWilliams identities \cite{2}.

\[
\begin{array}{c|c|c}
 i & \text{The number of codewords of weight } i & \\
 0 & 1 & \\
 d & n(n-2)/2 & \\
 n-d & 2n-2 & \\
 n & n(n-2)/2 & \\
\end{array}
\]

In order to prove that a Kerdock code consists of two \( i \)-component we use the following properties of the code, that come from its definition. Without loss of generality, \( 0^n \) is in a Kerdock code.

(K1) Any code \( K \) is a union of \( n/2 \) cosets of \( \text{RM}(1, m) \).

(K2) It is true that \( K_{n/2} \cup \{0^n, 1^n\} = \text{RM}(1, m) \).

(K3) The distance between codewords from different cosets of \( \text{RM}(1, m) \) in the code \( K \) is either \( d \) or \( n - d \).

(K4) Nonzero distances between codewords in any coset are either \( n/2 \) or \( n \).

(K5) \( \text{RM}(1, m) \subseteq \text{Ker}(K) \).

The property below follows from (K2)-(K5):

(K6) If for \( x, y \in K \) we have \( w(x + y) = n/2 \) then \( x + y \in K \).

**Theorem 1.** \cite{2} [Theorem 9, Ch. 9] Let \( C \) be a code of length \( n \) and minimum distance \( d \), \( C^\perp \) be a code which is formally dual to \( C \), \( \bar{s} = |I(C^\perp) \setminus \{0,n\}|. \) Then the set of codewords of any fixed nonzero weight in \( C^\perp \) is \((d - \bar{s})\)-design.

Theorem 1 applied to Preparata and Kerdock codes implies the following:

(K7) \( K_d, K_{n/2}, K_{n-d} \) are 3-designs.

In order to proceed further we need the following lemma.

**Lemma 1.** Let \( x \) be a vector of weight \( i \), \( D \) be \( 1-(n,j,\lambda_1) \)-design. Let the distance between \( x \) and vectors of \( D \) take values \( k_1, \ldots, k_s \) with multiplicities \( \delta^{k_1}, \ldots, \delta^{k_s} \) respectively. Then the following formula holds:

\[
\sum_{i=1}^{s} \delta^{k_i} \cdot \frac{i + j - k_i}{2} = i\lambda_1
\]

and \( \delta^{k_1}, \delta^{k_2} \) are uniquely defined by \( \delta^{k_1}, \ldots, \delta^{k_s} \).

**Proof.** Let the distance between the vector \( x \) and an arbitrary vector \( y \) from \( D \) be \( k_l \), then there are

\[
\frac{i + j - k_l}{2}
\]
common unit coordinates for \( x \) and \( y, l = 1, 2, \ldots, k_s \). On the other hand, there are exactly \( \lambda_1 \) vectors of \( D \) that have a prefixed coordinate to be 1. Double counting of
\[
\sum_{y \in D} |\{i : x_i = y_i = 1\}|
\]
gives \( \sum_{i=1}^n \delta^{ki} \cdot \frac{i+j-k_l}{2} = i\lambda_1 \). Finally \( \delta^{k1}, \delta^{k2} \) are uniquely defined by \( \square \) taking into account that \( \sum_{i=1}^n \delta^{ki} = |D| \), where \( |D| = \lambda_1 \).

Note that \( I(K') = \{0, d, n/2, n - d\} \), as we exclude the all-one vector in \( K' \).

**Theorem 2.** The restriction of the Hamming scheme to a doubly shortened Kerdock code \( K'' \) is an association scheme.

**Proof.** In the proof of the current theorem we use the following convention. By \( \delta_{i,j}^k(x) \) we denote the number of codewords of weight \( j \) in \( K'' \) at distance \( k \) from the weight \( i \) codeword \( x \) in \( K'' \). Obviously, the restriction of the Hamming scheme to \( K'' \) is an association scheme if \( \delta_{i,j}^k(x) \) for all \( i, j, k \in I(K'') \) are shown to be independent on the choice of a codeword \( x \) of weight \( i \) regardless of translation of \( K'' \) by its codeword. The proof below relies only on properties (K1)-(K7) of a Kerdock code \( K \) that are independent on the translation of the code.

**Lemma 2.** The number \( \delta_{i,j}^k(x) \) does not depend on the choice of a codeword \( x \) in \( K'' \) if \( i \) or \( j \) equals to \( n/2 \).

**Proof.** The property (K4) implies that the distances between codewords from \( K_{n/2}'' \) and \( K_d'' \) or \( K_{n-d}'' \) cannot be \( n/2 \). Moreover (K7) implies that the sets of the fixed weight codewords of a doubly shortened Kerdock code are 1-designs, so by Lemma \( \square \) the intersection numbers \( \delta_{i,j}^d(x) \) and \( \delta_{i,j}^{n-d}(x) \) are uniquely determined and do not depend on a choice of \( x \) if \( i \) and \( j \) are not equal to \( n/2 \) simultaneously.

Finally, \( RM(1, m)' \) is a linear Hadamard code, so the set of nonzero codewords \( K_{n/2}'' \) of its shortening \( RM(1, m)_{n/2}' \) are also at distance \( n/2 \) apart pairwise, so \( \delta_{n/2,j}^k(x) \) is \( n/4 - 1 \) if and only if \( k = n/2 \) and is zero otherwise. \( \square \)

**Lemma 3.** Let \( n/2 \in \{i, j, k\} \). Then the number \( \delta_{i,j}^k(x) \) does not depend on the choice of a codeword \( x \) of weight \( i \).

**Proof.** We show that \( \delta_{i,j}^{n/2}(x) = \delta_{i,n/2}^j(x) \). Consider the set \( \{z \in K_j'' \}, d(z, x) = n/2 \} \). By definition it is of the size \( \delta_{i,j}^{n/2}(x) \). Consider the translation of the set by \( x \in K_{n/2}'' \). Since \( x \) is of weight \( n/2 \), the property (P6) implies that \( x + z \) is a codeword of the doubly shortened Kerdock code \( K'' \). The substitution \( z' = z + x \) gives the equality
\[
\{z + x : z \in K_j'', d(z, x) = n/2 \} = \{z' \in K_{n/2}'', d(z', x) = j \}.\]
The cardinality of the right hand side is \( \delta_{i,j}^{n/2}(x) \), so \( \delta_{i,j}^{n/2}(x) = \delta_{i,n/2}^j(x) \) and the number is independent on \( x \) by Lemma \( \square \).

**Lemma 4.** The number \( \delta_{i,j}^k(x) \) does not depend on the choice of a codeword \( x \) of weight \( i \) for \( i, j, k \in I(K'') \).
Proof. Since $I(K^m) = \{0, d, n/2, n - d\}$, the nonzero distances between codewords from $K_i^m$ and $K_j^m$ take not more than three nontrivial values. The property (K7) implies that $K_i^m$ is a 1-design and by Lemma 3 the number $\delta_{i,j}^{n/2}(x)$ of codewords at distance $n/2$ in $K_i^m$ from $x$ is independent on choice of $x$ in $K_i^m$, so the numbers $\delta_{i,j}^d(x)$ and $\delta_{i,j}^{n-d}(x)$ are independent on $x$ by Lemma 4.

The considerations in the beginning of the proof of the theorem and Lemma 4 imply that the restriction of the Hamming scheme to $K^m$ is an association scheme. □

In order to find components of the punctured Kerdock code, we need one more lemma.

Lemma 5. Let $C$ be a code of length $n'$ such that the restriction of the Hamming scheme to its codewords is an association scheme. Let $I(C)$ be such that $I(C) \cap \{n' - i : i \in I(C)\} = \emptyset$. Then the restriction of the Hamming scheme to the code $C = C \cup (1^{n'} + C)$ is an association scheme.

Proof. If $i$ is in $I(C)$, denote by $i'$ the number $n' - i$. If there are given three distances from $I(C)$ and even belonging to $I(C)$ is even then the corresponding intersection number of $C$ is zero:

$$\delta_{i,j}^{i'}(C) = \delta_{i,j}^k(C) = \delta_{i,j}^{i'}(C) = \delta_{i,j}^{i'}(C) = 0.$$ 

Otherwise, the intersection number of $C'$ coincides with that of $C$:

$$\delta_{i,j}^{i'}(C) = \delta_{i,j}^k(C) = \delta_{i,j}^{i'}(C) = \delta_{i,j}^k(C) = \delta_{i,j}^k(C).$$

□

Theorem 3. Let $K^*$ be a punctured Kerdock code, $i \in \{1, \ldots, n - 1\}$. The code $K^*$ consists of two $i$-components and codewords are in the same component if their puncturings in $i$th position have weights of the same parity.

Proof. Consider any two coordinates $i, j$ of a Kerdock code of length $n$. Proving that there are just two $i$-components in $K_i^*$ is equivalent to showing that the minimum distance graph of the doubly punctured Kerdock code $K_{ij}^{**}$ has two connected components (which are actually even and odd weight codewords). Recall that the minimum distance graph of a code is the graph with vertex set being codewords and edgeset being pairs of codewords at code distance.

The minimum distance of the code $K^{**}$ is even and equal to $d - 2$. The even weight codewords of $K_{ij}^{**}$ are obtained from codewords of $K$ having 0 or 1 simultaneously in $i$th and $j$th positions by puncturing in these positions and the odd weight codewords of $K_{ij}^{**}$ are obtained from the codewords of $K$ having both 0 and 1 in $i$th or $j$th positions by puncturing in these positions. Moreover, the odd weight subcode $K_{ij}^{**}$ is obtained as a translation of even weight subcode $K_{ij}^{**}$. Indeed, let $x$ be in $RM(1, m)$, having 0 in $i$th position and 1 in $j$th position (there is such vector in the code $RM(1, m)$ since codewords of $RM(1, m)$ of weight $n/2$ form 3-design). Since $x$ is in $Ker(K)$, the addition of even weight codewords of $K_{ij}^{**}$ with the codeword $x^{**}$ obtained from $x$ by puncturing in $i$th and $j$th position is the odd weight subcode of $K_{ij}^{**}$.

In view of the above, it is enough to show the connectedness of the minimum distance graph of the even weight subcode of $K^{**}$, whose codewords have weights
from \{0, d−2, d, n/2−2, n/2, n−d−2, n−d, n−2\}. The proof significatively relies on the fact that the restriction of the Hamming scheme to \(K\) is an association scheme which follows from Theorem \(2\) and Lemma \(3\). We show that certain intersection numbers of the restriction of the Hamming scheme to \(K\) are nonzeros.

**Lemma 6.** The following equalities hold:

\[
\delta_{d−2,n/2}^{n−d}(K) = \frac{n^2 + 6n - 2nd + 8d}{4(n−2d)}.
\]

\[
\delta_{d−2,n/2}^{n−d}(K) = \frac{n^2 + 2nd + 2n}{4(n−2d)}.
\]

**Proof.** By equality \(2\), we know that \(\delta_{n,d,\gamma/2}^{n−k−2}(K^\gamma) = \delta_{n−d,n/2}^{n−d}(K)\) for \(k = d, n−d\). It is easy to see that the nonzero codewords of the code \(RM(1, m)\) form \(1−(n−2, n/2, n/4)\)-design, since there are exactly \(2n−2\) nonzero codewords of \(RM(1, m)\) of weight \(n/2\) which form 3-design. From \((K3)\) we have that \(\delta_{n−d,n/2}^{n−d}(K) = \delta_{n−d,n/2}^{n−d}(K^\gamma)\) is the number of nonzero codewords of \(RM(1, m)\), so it is \(n−d−1\). Therefore, we obtain the following equality from Lemma \(1\)

\[
\delta_{n−d,n/2}^{n−d}(K^\gamma) \frac{n}{2} + \frac{n}{2}−1 − \frac{n−d}{n−d,n/2}(K^\gamma)(\frac{3n}{4}−d) = \frac{n}{4}(n−d).
\]

and we find that

\[
\delta_{n−d,n/2}^{n−d}(K^\gamma) = \frac{n^2 − 6n−2nd + 8d}{4(n−2d)}, \quad \delta_{n−d,n/2}^{n−d}(K^\gamma) = \frac{n^2 − 2nd + 2n}{4(n−2d)}.
\]

From the values given by \(3\) and \(4\) we see that \(\delta_{d−2,n/2}^{n−d}(K^\gamma)\) and \(\delta_{d−2,n/2}^{n−d}(K^\gamma)\) are nonzeros, which is equivalent to

\[
\delta_{d−2,n/2}^{d−2}(K^\gamma) \neq 0, \quad \delta_{n/2, n−d−2}^{d−2}(K^\gamma) \neq 0.
\]

Consider the codewords of \(K_{d−2}^\gamma\). Obviously, the codewords cannot be at distance \(n/2\) pairwise apart, which follows, for example, from the Plotkin bound. Therefore there are codewords of weight \(d−2\) at distance \(d\) apart and \(\delta_{d−2,d−2}^{d−2}(K^\gamma) \neq 0\), which is equivalent to

\[
\delta_{d−2,d−2}^{d−2}(K^\gamma) \neq 0.
\]

From \(6\) we see that any codeword of \(K_{n/2}^\gamma\) is at distance \(d−2\) from at least one codeword of \(K_{d−2}\) and a codeword of \(K_{n−d−2}^\gamma\) is at distance \(d−2\) from at least one codeword of \(K_{n/2}^\gamma\). Therefore, \(K_{d−2}^\gamma, K_{n/2}^\gamma, K_{n−d−2}^\gamma\) are in one connected component of the minimum distance graph of \(K^\gamma\). Taking into account the equality \(3\) this fact is equivalent to the fact that the codewords of \(K_{n−d}^\gamma, K_{n−d−2}^\gamma\) and \(K_{d−2}^\gamma\) belong to one component. Finally, the inequality \(6\) implies that \(K_{d−2}^\gamma\) and \(K_{n−d−2}^\gamma\) are in one component, which implies that the codewords of weights \(\{0, d−2, d, n/2−2, n/2, n−d−2, n−d, n−2\}\) are in one connected component, which is exactly the minimum distance graph of \(K^\gamma\).

\[\square\]
Remark 1. Theorems 2 and 3 are true for some other Kerdock-related codes. In particular, by considerations similar to those in proof of Theorem 2 one can show that a Kerdock and a shortened Kerdock codes produce association schemes, which gives an alternative (combinatorial) proof for the well-known facts from [7] and [10]. Analogously to the proof of Theorem 3, one can prove that the \( i \)-components of a Kerdock code coincide with the Kerdock code or equivalently, the minimum distance graph of a punctured Kerdock code is connected.

Remark 2. According to Theorem 3, new Kerdock codes cannot be constructed by means of traditional switchings. For convenience we set \( i = n - 1 \). By the proof Theorem 3 we know that two codewords are in one \((n-1)\)-component of the punctured Kerdock code \( K_n^* \) if and only if their puncturings in \((n-1)\)th coordinate position have weights of the same parity. Therefore, the codewords of the Kerdock code \( K_n^\perp \) could be represented as \( K_{00}, K_{11}, K_{01}, K_{10} \), where \( K_{ab} = \{ x \in K_n : x_{n-1} = a, x_n = b \} \), with \( K_{00} \cup K_{11} \) corresponding to one \((n-1)\)-component of \( K_n^* \) and \( K_{01} \cup K_{10} \) to the other one. Moreover, the "odd weight" component is the translation of the "even weight" one, i.e. there is a codeword \( x'_{01} \) of \( R\!M(1, m) \) such that \( (K_{01} \cup K_{10}) + (x'_{01}) = K_{00} \cup K_{11} \). Now the switching \( K = K_{00} \cup K_{11} \cup ((x'_{01}) + (K_{00} \cup K_{11})) \) to \( K' = K_{00} \cup K_{11} \cup ((x'_{10}) + (K_{00} \cup K_{11})) \) gives an equivalent code which is obtained from \( K \) by permuting \((n-1)\)th and \( n \)th coordinate positions.

3. Components of codes dual to BCH codes

In the section we fix \( n = 2^m \), \( m \) odd. We investigate the \( i \)-components of the dual code \( C_n^\perp \) of a primitive cyclic BCH code \( C_{1,3} \) with zeros \( \alpha \) and \( \alpha^3 \) with designed distance 5 by \( i \)-components, of length \( n-1 = 2^m - 1 \), \( m \) odd, here \( \alpha \) is a primitive element of the Galois field \( GF(2^m) \). The code shares many similar properties with a Kerdock code. We prove that \( C_n^\perp \) is an \( i \)-component for any coordinate position \( i \).

Further we use the following properties of the code \( C_{1,3}^\perp \).

(B1) \[2\] The minimum distance of the code \( C_{1,3}^\perp \) is \( d = \frac{n - \sqrt{2n}}{2} \). The code \( C_{1,3}^\perp \) has the following weight distribution:

| \( i \) | The number of codewords of weight \( i \) |
|---|---|
| 0 | 1 |
| \( d \) | \((n-1)(\frac{n}{2} + \sqrt{\frac{n}{8}})\) |
| \( \frac{n}{2} - d \) | \((n-1)(\frac{n}{2} - \sqrt{\frac{n}{8}})\) |

The fact below follows from Theorem 1 and (B1).

(B2) Fixed weight codewords of \( C_{1,3}^\perp \) form a 2-design.

The code \( C_{1,3} \) is uniformly packed \[5\]. In [7], Theorem 6.10 it was shown that any code that is dual to a linear uniformly packed code gives an association scheme.

(B3)\[7\] The restriction of the Hamming scheme to \( C_{1,3}^\perp \) is an association scheme.

Lemma 7. Let \( C \) be the punctured (in any coordinate position) code of the code \( C_{1,3}^\perp \). Then any codeword of weight \( d \) is at distance \( d - 1 \) from at least one codeword of weight \( d - 1 \).
Proof. Let $C_{d-1}$ be the set of codewords of the punctured code of $C_{1,3}$ of weight $d-1$. Suppose that $x$ is a codeword of weight $d$ such that $d(x, C_{d-1}) > d - 1$. Then $d(x, C_{d-1}) \in \{\frac{n}{2} - 1, n - d - 1\}$. Since the vectors of $C_{d-1}$ form 1-design which follows from the property (B2), we can use Lemma 1 to count the number $\delta_{d-1}^n$ of the codewords of $C_{d-1}$ at distance $\frac{n}{2} - 1$ from $x$:

$$\delta_{d-1}^n(d - \frac{n}{4}) + (|C_{d-1}| - \delta_{d-1}^n)\frac{3d - n}{2} = \lambda_1 \cdot d,$$

where $|C_{d-1}| = \lambda_1 \frac{n}{2} - 1$.

It is easy to see that

$$\frac{\delta_{d-1}^n}{|C_{d-1}|} = \frac{2(n^2 - 2n + 8d - 3nd - 2)}{(n - 2)(n - 2d)} > 1,$$

a contradiction. \hfill \Box

Lemma 8. The minimum weight codewords of $C_{1,3}$ span the code.

Proof. The code $C_{1,3}$ is the direct sum of the Hadamard codes $C_1^+$ and $C_3^+$, both of which consist of $n - 1$ nonzero codewords having weight $n/2$. The number of codewords of weight $d$ in $C_{1,3}$ is greater than $n$ (see (B1)). Therefore one can find three codewords in codes $C_1^+$ and $C_3^+$ with distances $d$ or $n/2$ pairwise, e.g. $x, x' \in C_1^+$ and $y \in C_3^+$, such that $d(x, x') = n/2$ and $d(x, y) = d(x', y) = d$. Hence, by property (B3), we have that the intersection number $\delta_{d,n/2}^n(C_{1,3})$ is nonzero, i.e. any codeword of weight $n/2$ is at distance $d$ from at least one codeword of weight $d$ in $C_{1,3}$.

The number of codewords of weight $n - d$ is less than the number of codewords of weight $d$, therefore any codeword of weight $n - d$ is at distance $d$ from at least one codeword of weight $n/2$ or $d$. So, the codewords of weight $d$ generate the code $C_{1,3}$.

Theorem 4. A code $C_{1,3}$ of length $n = 2m - 1$, $m$ odd, consists of one $i$-component for any coordinate position $i$.

Proof. By Lemma 7 any codeword of $C_{1,3}$ of weight $d$ with 0 in the $i$th coordinate position is at distance $d$ from a codeword of weight $d$ with 1 in the $i$th coordinate position. By Lemma 8 this implies that the set of all codewords of weight $d$ having 1 in the $i$th coordinate position generates the code $C_{1,3}$, i.e. the code $C_{1,3}$ is an $i$-component for any $i \in \{1, 2, \ldots, n - 1\}$.

Note that the properties (B1)-(B3) and the proof of Theorem 4 are the same for any code that is dual to a linear uniformly packed code with the same parameters as the BCH code. In particular, the cyclic code $C_{1,2j+1}^+$, $(j, m) = 1$ corresponding to the Gold function, $n - 1 = 2^m - 1$, $m$ odd as well as the duals of other linear codes obtained from almost bent functions (AB-functions) are uniformly packed and therefore each of them is an $i$-component for any $i$.

Corollary 1. The dual of a linear uniformly packed code with parameters of BCH code $C_{1,3}$ of length $n - 1 = 2^m$, $m$-odd is an $i$-component for any coordinate position $i$. 

Conclusion. We considered duals of two such well-known classes of uniformly packed codes as Preparata and 2-error correcting BCH code. The dual codes have large minimum distance, few nonzero weights and are related to designs and association schemes. We proved that i-components of these codes are maximum. It would be natural to study the structure of i-components of Preparata codes that are formal duals of Kerdock codes. For \( n = 15 \) these classes meet in the self-dual Nordstrom-Robinson code that has two i-components for any coordinate position i. With the help of a computer, we showed that \( C_{1,3}^{⊥} \) of length \( 2^m - 1 \) is an i-component for any \( i \) for even \( m \) also for \( m = 6, 8, 10 \) and the BCH code \( C_{1,3} \) consists of two i-components for any coordinate position i for any \( m: 5 \leq m \leq 8 \). Another challenging problem is finding i-components of the BCH codes \( C_{1,3} \) for any \( m \) and their duals for even \( m \).

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