A New Correlation Coefficient for Comparing and Aggregating Non-strict and Incomplete Rankings

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Abstract

We introduce a correlation coefficient that is designed to deal with a variety of ranking formats including those containing non-strict (i.e., with-ties) and incomplete (i.e., unknown) preferences. The new measure, which can be regarded as a generalization of the seminal Kendall tau correlation coefficient, is proven to be equivalent to an axiomatic ranking distance specifically designed to treat individual rankings equitably when solving the consensus ranking problem. In an effort to further unify and enhance both robust ranking methodologies this work proves the equivalence of an additional axiomatic-distance and correlation-coefficient pairing in the space of non-strict incomplete rankings. The bridging of these complementary theories reinforces the singular suitability of the featured correlation coefficient to solve the general consensus ranking problem. The latter premise is bolstered by an accompanying set of experiments on random instances, which are generated via a herein developed sampling technique connected with the classic Mallows distribution of ranking data. To carry out the featured experiments we devise a specialized branch and bound algorithm that provides the full set of alternative optimal solutions, when applicable. Applying the algorithm on the generated random instances reveals that the featured correlation coefficient yields relative fewer alternative optimal solutions as data becomes noisier (i.e., as the input rankings get further from a ground truth).

Keywords: Group decisions and negotiations; robust ranking aggregation; Kendall tau correlation; axiomatic distances; non-strict incomplete rankings
1. Introduction

The consensus ranking problem (i.e. ranking aggregation) is at the center of many group decision-making processes. It entails finding an ordinal vector or ranking of a set of competing objects that minimizes disagreement with a profile of preferences (represented as ranking vectors). Common examples include corporate project selection, research funding processes, and academic program rankings [24]. Moreover, the mathematical measures and aggregation algorithms devised to solve the consensus ranking problem often find ready application in many other fields. For example, in Information Retrieval these fundamental tools have been used to compare, aggregate, and evaluate the accuracy of metasearch engine lists [22]. Hence, although this work considers the group decision-making context of ranking aggregation for ease of interpretability, many of its results could be readily adaptable to many other contexts such as Artificial Intelligence [5] and Biostatistics [34].

Although the mathematical roots of consensus ranking trace back to the development of voting systems of de Borda [15] and Condorcet [9], significant work remains to deal with real-world situations that upend many of the problem’s rigid and long-running assumptions. These issues have garnered renewed interest from the Operations Research community owing largely to the general intractability engendered by the more robust ranking aggregation systems [46]. Even so, it may be beneficial to consider transdisciplinary efforts in solving close variants of this problem. This work incorporates concepts from the statistical literature where the analog median ranking problem has been used for classification, prediction, and several other applications [23, 14]. The fundamental goal for intertwining these viewpoints is to reinforce and advance theoretical and computational aspects of consensus ranking when dealing with indispensable forms of ranking data. Additionally, this unison is intended to yield insights and perspectives that are generalizable to many other areas where similar problems have been considered.

This work deals with the consensus ranking problem in which the set of input rankings may contain ties and may be incomplete. This variety of preferences is the rule rather than the exception in group decision-making [18], making it imperative to utilize frameworks that possess this flexibility; otherwise, judges are implicitly forced to make arbitrary and/or careless choices. Moreover, since there is typically a finite budget or set of benefits that is to be allocated commensurate with the competitors’ positions in the consensus ranking, the chosen frameworks must employ robust measures that align with the given context. Specifically, this work assumes a neutral treatment of incomplete rankings, meaning a judge’s preferences over
her unranked objects are unknown—because in the most general case it is assumed she does not evaluate them—and, therefore, no inferences should be made about her preferences of these objects relative to other unranked or ranked objects. Such a treatment is particularly pertinent in situations where the evaluation of a large object set can be realistically accomplished only via the allocation of smaller subsets—which may differ both in content and size—to various judges. Practical reasons for this include time, expertise, and conflict of interest constraints [25]. It is also prudent to partition such a large undertaking into smaller tasks given that objectivity deteriorates and frustration grows as the number of alternatives evaluated increases [45, 4]. The theory and algorithms developed in this work are tailored to deal with these and other commonplace considerations.

The herein assumed neutral treatment of incomplete rankings clearly differs from the top-\(k\) treatment in which unranked objects are assumed to be tied for ordinal position \(k+1\), making them all strictly less-preferred than the \(k\) explicitly ranked objects (e.g., see [39, 30]). Although the featured correlation coefficient is not specially designed for the latter context, it can be provisionally (though not ideally) adapted for this purpose (see [41]).

This work makes the following novel contributions. First, it proves that the \(\tau_x\) ranking correlation coefficient devised in [18] is inadequate for dealing with incomplete rankings when the unranked objects do not connote any preferential information. Second, it develops the \(\hat{\tau}_x\) ranking correlation coefficient for dealing with a wide variety of ranking inputs; this measure is equivalent to \(\tau_x\) when the input rankings are restricted to be complete and it is equivalent to the seminal Kendall \(\tau\) correlation coefficient [28] when they are further restricted to not contain ties. Third, it proves that the featured correlation coefficient is equivalent to the axiomatic ranking distance recently developed in [42]. Thus, as a whole, the first three contributions refine and unify distance and correlation-based ranking aggregation. Fourth, it devises a customized branch and bound algorithm for solving the non-strict incomplete ranking aggregation problem and for obtaining all alternative optimal solutions efficiently. Fifth, it extends the repeated insertion model of [16] to sample non-strict incomplete rankings from statistical distributions linked with the classic Mallows \(\phi\)-distribution of ranking data [38]. As motivated by the foregoing discussion, it is reasonable to expect that these contributions will find useful interpretations within other fields.

The remainder of the paper is organized as follows. §2 introduces the notation and conventions utilized throughout this work. §3 reviews the pertinent literature on axiomatic distances and correlation coefficients for quantifying differences and similarities between rankings. §4 presents key theo-
retical results that strengthen the correlation-based framework for handling incomplete rankings: §4.1 demonstrates the inadequacy of Emond and Mason’s correlation coefficient; §4.2 derives the correlation coefficient featured in this work; §4.3 establishes the equivalence between two key axiomatic-distance and correlation-coefficient pairings and derives additional analytical insights from these connections. §5 presents a set of new algorithmic tools and experiments to compare the usefulness of these measures to solve the non-strict incomplete ranking aggregation problem: §5.1 develops an exact branch and bound algorithm; §5.2 devises an efficient statistical sampling framework used to construct nontrivial instances motivated by real-world scenarios; §5.3 §5.5 employ these scenarios to test two properties desired of ranking aggregation measures: decisiveness and electoral fairness. Lastly, §6 concludes the work and discusses future avenues of research.

2. Notation and Preliminary Conventions

Denoting $V = \{v_1, \ldots, v_n\}$ as a set of $n$ competing objects, a judge’s ranking or ordinal evaluation of $V$ is characterized by a vector $a$ of dimension of $n$, whose $ith$ element denotes the ordinal position assigned to object $v_i$. If $a_i < a_j$, $a$ is said to prefer $v_i$ to $v_j$ (or to disprefer $v_j$ to $v_i$), and when $a_i = a_j = p$, $a$ is said to tie $v_i$ and $v_j$ for position $p \in \{1, \ldots, n\}$, where $1 \leq i, j \leq n$ and $i \neq j$. Additionally, when $a_i$ is assigned a null value—heretofore signified by the symbol “•”—$v_i$ is said to be unranked within $a$; the objects explicitly ranked in $a$ are denoted by the subset $V_a \subseteq V$ (i.e., $a_i \neq •$ for $v_i \in V_a$). For example, in the 5-object ranking $a = (1, 2, 2, •, 4)$, $v_1$ is preferred over $v_2, v_3$, and $v_5$; $v_2$ and $v_3$ are tied for the second position but both are preferred over $v_5$; $v_4$ is left unranked; and $V_a = V \backslash \{v_4\}$. It is important to emphasize that, although any object $v_i \in V$ that is unranked within $a$ receives the same assignment $a_i = •$, it is not considered tied with other unranked objects or better/worse than the ranked objects.

Later sections also refer to the object-ordering $a^{-1}$ induced by a mapping $\Psi_b : a \in \{1, \ldots, n\}^n \rightarrow a^{-1} \in W(\{1, \ldots, n\})$, where $W(\{1, \ldots, n\})$ denotes the set of weak orders (or complete preorders) on $n$ objects. That is, $\Psi_b(a)$ sorts the objects in $V_a$ from best to worst, according to their ranks in $a$. For example, for $a = (1, 5, 2, 4, 3)$, $\Psi_b(a) = a^{-1} = (v_1, v_3, v_5, v_4, v_2)$. Extending this notation, $a^{-1}(i)$ specifies the $ith$-highest ranked object in when $a$ does not contain ties—that is, $\Psi_b$ is a bijection in this case and the inverse function $\Psi_b^{-1}$ returns a linear order. When $a$ contains ties, $\Psi_b$ sorts the objects into preference equivalence classes. For example, for $a = (1, 3, 3, 1, 5)$, $\Psi_b(a) = a^{-1} = ((v_1, v_4), (v_2, v_3), v_5)$. In the case of ties,
the inverse mapping $\Psi_b^{-1}(a^{-1})$ returns the ranking obtained by labeling each object with its corresponding equivalence class position in $a^{-1}$. For example, for $a^{-1} = (v_1, (v_2, v_4), v_5, v_3)$, $\Psi_b^{-1}(a^{-1}) = a = (1, 2, 5, 2, 4)$.

Various ranking aggregation systems may not be defined or equipped to properly handle the full variety of ranking data formats alluded to in the preceding paragraphs. For this reason, the following definitions highlight three primary ranking spaces by which they can be categorized.

**Definition 1.** Let $\Omega = \{\bullet, 1, \ldots, n\}^n$ denote the broadest ranking space consisting of all possible strict, non-strict, complete, and incomplete rankings—corresponding to rankings without ties, with and without ties, full, and partial and full, respectively. Since non-strict rankings encompass strict rankings and incomplete rankings encompass complete rankings, $\Omega$ is denoted alternatively as the space of non-strict incomplete rankings.

**Definition 2.** Let $\Omega_C = \{1, \ldots, n\}^n$ denote the space of complete rankings over $n$ objects, which consists of all possible non-strict (and strict) rankings in which every object must be explicitly ranked (i.e., partial evaluations are disallowed).

**Definition 3.** Let $\Omega_S = \{\bullet, 1, \ldots, n\}^n$ denote the space of strict rankings over $n$ objects, which consists of all possible incomplete (and complete) rankings in which no objects may be tied.

Notice that Definition 3 explicitly excludes the subset of unranked objects within a ranking $a$ (i.e. $V \setminus V'_a$) as being tied even though every unranked object $v_i$ receives the same assignment $a_i = \bullet$. This convention is adopted to convey a null or unknown preference over all unranked objects within a ranking in accordance with the neutral treatment of incomplete rankings assumed throughout this paper.

From the above definitions, it is evident that $\Omega_C \subset \Omega$, $\Omega_S \subset \Omega$, and $\Omega_C$ and $\Omega_S$ are mutually incomparable with respect to set containment. To describe the ranking aggregation problem addressed in this work, let $\hat{\tau}(\cdot) : \Omega^2 \to [-1, 1]^3$ denote an arbitrary ranking correlation function. The correlation-based non-strict incomplete ranking aggregation problem (NI-RAP) is stated formally as:

$$\arg \max_{r \in \Omega_C} \sum_{k=1}^K \hat{\tau}(r, a^k),$$

where $a^k \in \Omega$ for $k = 1, \ldots, K$ (i.e., $k$ is the index of each judge or ranking). Alternatively, adopting the same notation and denoting $\hat{d}(\cdot) : \Omega^2 \to \mathbb{R}_{+\cup\{0\}}$
as an arbitrary ranking distance function, the distance-based NIRAP is stated formally as:

\[
\arg \min_{r \in \Omega_C} \sum_{k=1}^{K} \hat{d}(r, a^k).
\]

Expression (1) can be intuitively interpreted as the problem of finding a ranking \( r \) that maximizes agreement—quantified according to \( \hat{\tau} \)—with a set of \( K \) non-strict incomplete rankings; Expression (2) can be intuitively interpreted as the problem of finding a ranking \( r \) that minimizes disagreement—quantified according to \( \hat{d} \)—with the same inputs. As shown in [4, 2], for select axiomatic-distance and correlation-coefficient pairings, the two respective optimization problems are equivalent. It is imperative to point out that, although the input rankings are allowed to be incomplete to allow flexibility of preference expression, the consensus ranking is required to lie in the space of complete rankings—that is, \( r \in \Omega_C \) is a constraint in both problems. This condition is enforced because obtaining an evaluation for all considered objects is relevant for many group decision-making situations, but it may be modified to suit other contexts (e.g., top-\( k \) ranking aggregation [19]).

3. Literature Review

The principal focus of this work is on deterministic metric-based methods for comparing and aggregating rankings, which are regarded as the most robust methodologies within Operation Research and Social Choice [8]. The reader is directed to [10] for a review of score or utility based methods, which are relative more computationally efficient but cannot fulfill certain fundamental social choice properties associated with voting fairness (e.g., the Condorcet criterion [9] and its extensions [48, 49]). Additionally, there is a rich body of literature on nondeterministic or model-based ranking aggregation methods (e.g., see [20, 38, 41]). While these often rely on axiomatic distances, they are incomparable with the featured context in various notable respects including their assumptions, aggregation processes, and outputs.

3.1. Axiomatic Distances

Several axiomatic distances have been proposed to derive a consensus ranking, with each method solving a different variant of this classic problem. Most prominently, these distances are established in [42, 27, 6, 7, 13, 17, 11]. Within each work, the respective distance is typically advocated as the most suitable for aggregating inputs drawn from a specific ranking space through a set of axioms it uniquely satisfies. As [42] demonstrated, however, all
but the last distance on the list are either unable to deal with the broadest ranking space $\Omega$ or they inadvertently induce significant systematic biases associated with unfairness when applied (or extended) to solve the distance-based NIRAP. Thus, in pursuit of the fundamental goals of allowing a wide variety of ranking data and upholding a fair or equitable process, this subsection focuses on this recently developed distance and on the precursor distances upon which it is founded.

The first axiomatic distance was introduced by Kemeny and Snell in [27] to aggregate non-strict complete rankings; the distance function, written here succinctly as $d_{KS}$, quantifies the disagreement between a pair of ranking vectors as follows:

$$d_{KS}(a, b) = \frac{1}{\gamma} \sum_{i=1}^{n} \sum_{j=1}^{n} |\text{sign}(a_i - a_j) - \text{sign}(b_i - b_j)|,$$

(3)

where $a, b \in \Omega_C$ and $\gamma$ is a positive constant associated with a chosen minimum positive distance unit. In [27] $\gamma$ is set to 2 corresponding to a minimum distance unit of 1 (since each object pair is counted twice in the above expression), but henceforth it is fixed to 4 corresponding to a minimum positive distance unit of 1/2, which does not affect the solution to Problem (2) but has a convenient interpretation for handling ties [42]. Put simply, $d_{KS}(a, b)$ measures the number of pairwise rank reversals required to turn $a$ into $b$; when the rankings do not contain ties, this is also known as the bubble-sort distance [24]. The distance is synonymous with robust ranking aggregation in space $\Omega_C$ [1] owing to the combination of social choice properties for mitigating manipulation, enforcing fairness, and reducing individual human biases that it uniquely satisfies (see [48, 49]).

Although $d_{KS}$ was not originally defined to handle incomplete rankings, a seemingly straightforward extension was devised in [11] and [17]. The underlying axioms for this distance, referred to as the Projected Kemeny Snell distance and written here succinctly as $d_{P-KS}$, are provided in [32]. The corresponding distance function is defined as:

$$d_{P-KS}(a, b) = d_{KS}(a|_{V_a \cap V_b}, b|_{V_a \cap V_b}),$$

(4)

where $a, b \in \Omega$ in this case and where $a|_{V_a \cap V_b}$; $b|_{V_a \cap V_b}$ denote the projections of each ranking onto the subset of objects evaluated in both rankings. In other words, $d_{P-KS}$ enforces the intuitive interpretation that ranking disagreements should be based only on the objects ranked in common by $a$ and $b$ rather than in the differences of which objects were ranked (or not ranked) by each judge. That said, when $d_{P-KS}$ is utilized to solve the NIRAP, the
consensus ranking tends to favor judges who rank a higher number of objects since distances in higher dimensional spaces tend to be comparatively larger. This means that when unequal numbers of items are evaluated, $d_{P-KS}$ induces systematic *inegalitarianism*, by which unequal voting power is imposed on each input ranking. Consequently, some judges may cast disproportionate influence in the aggregation process, which can lead to unfair and quasi-dictatorial outcomes—i.e., a judge can dominate the aggregate ranking despite the oppositely aligned preferences of a large majority.

To overcome the principal drawback of $d_{P-KS}$, [42] developed the normalized projected Kemeny Snell distance, written here succinctly as $d_{NP-KS}$. The distance is equivalent to $d_{KS}$ when inputs are restricted to space $\Omega_C$ and, thus, it can be regarded as a generalization of the Kemeny Snell distance to the broadest space $\Omega$. The authors proved that $d_{NP-KS}$ uniquely satisfies an intuitive set of axioms desired of any distance defined in space $\Omega$ (see Appendix 7.1). The corresponding distance function is defined as:

$$d_{NP-KS}(a, b) = \begin{cases} 
\frac{d_{KS}(a|_{V_a \cap V_b}, b|_{V_a \cap V_b})}{\bar{n}(\bar{n}-1)/2} & \text{if } \bar{n} \geq 2, \\
0 & \text{otherwise},
\end{cases}$$

(5)

where $\bar{n} := |V_a \cap V_b| \geq 2$ since the Kemeny-Snell framework relies on pairwise comparisons. The above denominator ensures $0 \leq d_{NP-KS}(a, b) \leq 1$, regardless of how many objects are ranked or unranked by $a$ or $b$. In essence, this gives equal voting power to each input ranking or judge in the aggregation process. A pragmatic benefit of achieving egalitarianism irrespective of the different numbers of objects ranked is the elimination of the often unenforceable/unrealistic requirement of having to allocate an equal number of objects for each judge to evaluate. Indeed, in many cases a uniform assignment of objects may be impossible to implement due to differing expertise, disagreeing schedules, unplanned exemptions (e.g., conflicts of interest with the evaluated alternatives), and plenty of other practical reasons.

### 3.2. Correlation Coefficients

Correlation coefficients are an alternative methodology for quantifying differences in rankings with an extensive history and wide array of applications in statistical literature—e.g., see [47, 32, 31, 35]. Naturally, the agreement between judges $a$ and $b$ is measured on the interval $[-1, 1]$, where the minimum and maximum values indicate complete disagreement and complete agreement, respectively. The most prominent is the Kendall $\tau$ (tau) correlation coefficient [28], which was subsequently adapted into the $\tau_b$ correlation coefficient in [29] to handle non-strict rankings. However, [18] gave
compelling evidence that $\tau_b$ exhibits serious flaws when handling non-strict rankings; for example, $\tau_b$ yields an undefined correlation value of 0/0 when comparing the all-ties ranking to itself or to any other non-strict ranking. To replace it, the authors introduced the $\tau_x$ (tau-extended) ranking correlation coefficient, which relies on an alternative score matrix representation of $\mathbf{a} \in \Omega_C$ denoted as $[a_{ij}]$ whose individual elements are defined as:

$$a_{ij} = \begin{cases} 
1 & \text{if } a_i \leq a_j, \\
-1 & \text{if } a_i > a_j, \\
0 & \text{if } i = j
\end{cases} \quad (6)$$

where $1 \leq i, j \leq n$. Here, a tie connotes a positive statement of agreement; conversely, the $\tau_b$ score matrix (not shown) treats a tie as a declaration of indifference by assigning it a score of 0. Thus, adopting the former interpretation, the $\tau_x$ correlation between rankings $\mathbf{a}$ and $\mathbf{b}$ (with underlying score matrices $[a_{ij}]$ and $[b_{ij}]$) is given by the function:

$$\tau_x(\mathbf{a}, \mathbf{b}) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij}}{n(n-1)}. \quad (7)$$

Furthermore, [18] proved that $\tau_x$ and $d_{KS}$ are equivalent representations of the unique measure [27] satisfying the axioms in space $\Omega_C$, by connecting them via the equation:

$$\tau_x(\mathbf{a}, \mathbf{b}) = 1 - \frac{\gamma d_{KS}(\mathbf{a}, \mathbf{b})}{n(n-1)}, \quad (8)$$

where $\gamma > 0$ is a constant associated with the minimum $d_{KS}$ distance unit (see Equation (3))—since this work adopts a minimum distance unit of 1/2, it fixes $\gamma = 4$ in Equation (8). This connection renders $\tau_x$ with the intuitive axiomatic foundation of $d_{KS}$. At the same time, it suggests that the inadequacies of the latter to handle incomplete rankings described in [42] carry over to the former. This premise is further explored in the ensuing section.

4. Handling Incomplete Rankings via Correlation Coefficients

Up to this point, there has not been a ranking correlation coefficient explicitly tailored for dealing with non-strict incomplete rankings (those belonging to the broadest ranking space $\Omega$), to the best of our knowledge. Indeed, although [18] suggested that $\tau_x$ could fulfill this extended role, this assertion has not been formally proved nor empirically validated. Hence, the first of the ensuing subsections examines this hypothesis. Afterward, [42]...
introduces the ranking correlation coefficient $\hat{\tau}_x$ for dealing with rankings in space $\Omega$, along with the properties and axioms it satisfies. This new ranking correlation coefficient is shown to be equivalent to $\tau_x$ and to $\tau$ when the input rankings are restricted to lie in spaces $\Omega_C$ and $\Omega_C \cap \Omega_S$, respectively. Then, §4.3 establishes the equivalence of $\hat{\tau}_x$ with the axiomatic distance $d_{NP-KS}$ as well as the equivalence of $\tau_x$ with $d_{P-KS}$ when the input rankings lie in space $\Omega$, and it elaborates on the consequences.

4.1. Inadequacy of the Kendall Tau-Extended Correlation Coefficient

This subsection provides cogent evidence that $\tau_x$ is not an adequate measure for quantifying and aggregating differences between incomplete rankings. Specifically, counter to what is claimed in [18], employing $\tau_x$ produces incongruous and counterintuitive results when a judge’s unranked objects should convey no preferential information. The veracity of these assertions is established via intuitive examples and the accompanying discussion.

Table 1 displays a simple instance consisting of ten incomplete rankings in space $\Omega$ over object set $V = \{v_1, \ldots, v_5\}$. Since, for every pair of objects $(v_i, v_{i+1})$ two or more judges strictly prefer $v_i$ over $v_{i+1}$ while only one judge ties them, for $1 \leq i \leq 4$, it is reasonable to expect the egalitarian outcome $(1, 2, 3, 4, 5)$ as the unique consensus ranking. However, using the $\tau_x$ correlation coefficient, the unique optimum is $(1, 1, 1, 1, 1)$, mirroring the preferences of $a^{10}$ exactly; most strikingly, this occurs even though $v_1$ is strictly preferred over $v_2$ by three of the four judges who evaluate $(v_1, v_2)$. Effectively, $a^{10}$ wields quasi-dictatorial influence due to the relatively higher number of items it ranks. This indicates that applying $\tau_x$ to aggregate incomplete rankings inadvertently imposes unequal voting power into the aggregation process, specifically benefiting rankings with higher completeness.

Conversely, the correlation coefficient introduced in §4.2 overcomes these key issues by effectively assigning equal ranking power to each judge—for the Table 1 example, $\hat{\tau}_x$ yields the unique optimal solution $(1, 2, 3, 4, 5)$.

| Objects | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $a^8$ | $a^9$ | $a^{10}$ | $r^*$ |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|------|
| $v_1$   | 1     | 1     | 1     | •     | •     | •     | •     | •     | •     | 1       | 1    |
| $v_2$   | 2     | 2     | 2     | 1     | 1     | •     | •     | •     | •     | 1       | 1    |
| $v_3$   | •     | •     | •     | 2     | 2     | 1     | 1     | •     | •     | 1       | 1    |
| $v_4$   | •     | •     | •     | •     | •     | •     | •     | 2     | 2     | 1       | 1    |
| $v_5$   | •     | •     | •     | •     | •     | •     | •     | •     | 2     | 2       | 1    |

Table 1: Judge $a^{10}$ appears to wield semi-dictatorial influence in the $\tau_x$ consensus ranking

The inadequacy of $\tau_x$ to handle incomplete rankings can be discerned at a more fundamental level from its inability to yield the expected values 1
and $-1$ when an incomplete ranking is correlated with itself and with its reverse ranking, respectively. In fact, the achievable correlation range shrinks as the number of ranked objects decreases, which translates into systematic inequalitarianism. For instance, \( \tau_x(a, a) = \frac{1}{3} \) when \( a = (1, 2, \bullet) \) and \( \tau_x(a, a) = \frac{1}{4} \) when \( a = (1, 2, 3, \bullet) \). Conversely, the new measure introduced in \$4.2\) prevents the correlation range contraction by discounting the impact of unranked objects through the inclusion of a scaling factor.

Although this work does not advocate applying \( \tau, \tau_x, \) or \( \hat{\tau}_x \) directly for top-\( k \) ranking aggregation, a provisional approach for this special context is as follows. First, fill in every unranked position with the value \( k+1 \); second, solve the modified instance utilizing \( \tau_x \) if the input contains ties and \( \tau \) otherwise. Alternatively, since \( \hat{\tau}_x \) is equivalent to \( \tau_x \) and \( \tau \) in the restricted spaces \( \Omega_C \) and \( \Omega_C \cap \Omega_S \), respectively (see \$4.2\), the ranking correlation coefficient introduced in this work can be employed in both cases.

### 4.2. Derivation of the Scaled Kendall Tau-Extended Correlation Coefficient

To quantify the agreement between non-strict incomplete rankings via correlation coefficients, a fundamental requirement is that the correlation between any pair of rankings \( a, b \in \Omega \) must lie within the interval \([-1, 1]\). Specifically, the \(-1\) and 1 values must be achieved whenever \( a \) and \( b \) completely agree and completely disagree, respectively; otherwise, a value lying strictly in the interior of the interval should be returned commensurate with the level of agreement between the rankings. These and other basic requirements that a given ranking correlation function \( \tau(\cdot) \) should satisfy in space \( \Omega \) are captured through the set of metric-like axioms exhibited in Table 2.

As explained in \$4.1\), \( \tau_x \) cannot fulfill some of these essential requirements in its current form. Hence, this subsection derives a new correlation coefficient, which will be proven to satisfy the Table 2 axioms in \$4.3\). As a first step, we define a corresponding score matrix \([a_{ij}]\) representation for \( a \) as:

\[
a_{ij} = \begin{cases} 
1 & \text{if } a_i \leq a_j, \\
-1 & \text{if } a_i > a_j, \\
0 & \text{if } i = j, \text{or } a_i = \bullet, \text{or } a_j = \bullet 
\end{cases} \tag{9}
\]

where \( 1 \leq i, j \leq n \). Note that this score matrix can be obtained by extending Equation (6) to also assign \( a_{ij} = 0 \) whenever object \( i \) or \( j \) (or both) is unranked in \( a \) and, thus, it is equivalent to the \( \tau_x \) score matrix when the input rankings are complete. This extension was cursorily proposed in [18], although it was neither explicitly stated nor implemented therein. It is chosen as the basis of the new correlation coefficient also because its treatment
of ties is equivalent to the Kemeny Snell “half-flip” metric, which assigns only half of a rank reversal between \( a \) and \( b \) whenever one ties \((v_i, v_j)\) but the other professes a strict preference for \( v_i \) over \( v_j \) or vice versa.

As a second step, consider score matrices \([a_{ij}]\) and \([b_{ij}]\) respectively defined according to Equation (9) and their associated matrix inner product:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij}.
\]

When \( a \) and \( b \) rank every object, the number of non-zeros in each score matrix and the maximum matrix inner product are both equal to \( n(n-1) \). The reasons are that the score matrix diagonal elements are all 0 and that \( a_{ij}b_{ij} = 1 \) for all \( i \neq j \) when \( b_{ij} = a_{ij} \). It is also straightforward to discern that a minimum matrix inner product of \(-n(n-1)\) can be achieved only if \( a \) does not contain ties and \( b_{ij} = -a_{ij} \) for all \( i \neq j \).

When \( a \) or \( b \) does not rank every object, for each \( v_i \) such that either \( a_i = \bullet \) or \( b_i = \bullet \) the \( i \)th score matrix row and column are set with all zeros, thereby decreasing the maximum/increasing the minimum matrix inner products by \( 2(n-1) \). Put otherwise, such a matrix inner product may be calculated as if the \( i \)th row and column of both score matrices do not exist. Hence, the maximum and minimum inner products of \([a_{ij}]\) and \([b_{ij}]\) are reduced to \( \bar{n}(\bar{n}-1) \) and \(-\bar{n}(\bar{n}-1)\), respectively, where \( \bar{n} = |V_a \cap V_b| \). Accordingly, a new correlation function can be derived to achieve the full expected correlation interval \([-1, 1]\). It is named the scaled Kendall tau-extended correlation coefficient, written here succinctly as \( \hat{\tau}_x \), and is defined as:

\[
\hat{\tau}_x(a, b) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij}}{\bar{n}(\bar{n}-1)},
\]

which may be rewritten in terms of \( \tau_x \) to explain the chosen nomenclature via the equation:

\[
\hat{\tau}_x(a, b) = \frac{n(n-1)}{\bar{n}(\bar{n}-1)} \tau_x(a, b),
\]

assuming the underlying score matrix of \( \tau_x \) is given by Equation (9). Expressly, this alternative expression emphasizes that, by scaling \( \tau_x(a, b) \) by the factor \( \frac{n(n-1)}{\bar{n}(\bar{n}-1)} \geq 1 \), \( \hat{\tau}_x \) removes the impact of irrelevant pairwise preference comparisons—the pairs of objects unranked by \( a, b \), or both—from the correlation. As a result, the correlation minimum and maximum values \(-1 \) and \( 1 \) may be achieved when a non-strict incomplete ranking is compared with a suitable ranking. Precise details of these guarantees are deferred
Table 2: Axioms for correlation coefficient-based comparison and aggregation of non-strict and incomplete rankings

| Axiom             | Description                                                                                                                                 |
|-------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| 1. Relevance      | $\tau(a, b) = \tau(a)(V_a \cap V_b)$, $b_i(V_a \cap V_b)$                                                                                 |
| 2. Commutativity  | $\tau(a, b) = \tau(b, a)$                                                                                                                  |
| 3. Relaxed Triangle Inequality | $\tau(a)(V_a \cap V_b), b_i(V_a \cap V_b)$ $+ \tau(b)(V_a \cap V_b), c_i(V_a \cap V_b)$ $\geq$ $\tau(a)(V_a \cap V_b), c_i(V_a \cap V_b)$ $+$ $\tau(b)(V_a \cap V_b), b_i(V_a \cap V_b)$ Equality holds if and only if $b_i(V_a \cap V_b)$ is between the other two projected rankings |
| 4. Anonymity      | If $a$ results from $a$ by a permutation of the objects in $V$, and $b'$ results from $b'$ by the same permutation, then $\tau(a, b) = \tau(a', b')$ |
| 5. Extension      | If rankings $a$ and $b$ agree except for a set $S$ of $k$ elements, which is a segment of both, then $\tau(a, b)$ may be computed as if these $k$ objects were the only objects being ranked |
| 6. Scaling        | $-1 \leq \tau(a, b) \leq 1$, $\tau(a, b) = 1$ if and only if $a_i(V_a \cap V_b) = b_i(V_a \cap V_b)$ and $\tau(a, b) = -1$ if and only if $b_i(V_a \cap V_b)$ is the reverse ranking of $a_i(V_a \cap V_b)$ $(a_i(V_a \cap V_b)$ must be a linear ordering in the latter case) |

Clearly, when $V_a \cap V_b = V$ the Equation (11) scaling factor equals 1, meaning $\tau_x$ is equivalent to $\tau_x$ in space $\Omega_C$; hence, in this restricted space Equation (10) becomes Equation (7). Furthermore, $\hat{\tau}_x$ is equivalent to $\tau$ in space $\Omega_C \cap \Omega_S$ due to the equivalence between $\tau_x$ and $\tau$ in said space [18]. Thus, since $\hat{\tau}_x$ possesses the same advantages as $\tau_x$ and $\tau$ when the rankings are restricted to spaces $\Omega_C$ and $\Omega_C \cap \Omega_S$, respectively, it remains to demonstrate why $\hat{\tau}_x$ is uniquely suited to deal with the broader space of non-strict incomplete rankings $\Omega$. Further theoretical foundations for this claim are established in the next subsection and additional practical reasons are given by the empirical results obtained in [5.3–5.5] (corollaries 3, 4).

4.3. Key Axiomatic-Distance and Correlation-Coefficient Pairings

This subsection proves that $\hat{\tau}_x$ is equivalent to the normalized projected Kemeny-Snell distance $d_{NP-KS}$, which is the unique distance satisfying the set of intuitive axioms exhibited in Appendix 7.1 [12]. Because $\hat{\tau}_x$ is equivalent to $d_{NP-KS}$, $\hat{\tau}_x$ inherits the axioms for distance-based aggregation in space $\Omega$. Table 2 provides the corresponding axioms for correlation coefficient-based aggregation, which are further contextualized in Appendix 7.4. This subsection also proves the equivalence of another key axiomatic-distance and correlation-coefficient pairing in space $\Omega$. Together these results fill a significant gap in the literature because although [18] made a connection between distance and correlation-based methods for aggregating complete rankings (see Equation (8)), they conjectured that a parallel connection could not be established when dealing with incomplete rankings.

**Theorem 1** (Linear transformation between $\hat{\tau}_x$ and $d_{NP-KS}$). Let $a$ and $b$ be two arbitrary rankings over $n = |V|$ objects drawn from the space of
non-strict incomplete rankings, $\Omega$. Then, the $\hat{\tau}_x$ correlation coefficient and the $d_{NP-KS}$ distance are connected through the following equation:

$$d_{NP-KS}(a, b) = \frac{1}{2} - \frac{1}{2} \hat{\tau}_x(a, b). \quad (12)$$

**Proof.** For succinctness, denote $\bar{a} = a|_{V_a \cap V_b}$ and $\bar{b} = b|_{V_a \cap V_b}$ as the rankings over $\bar{n} \leq n$ objects obtained by projecting $a$ and $b$ onto the subset of objects $\bar{V} = V_a \cap V_b$ ranked in common. Notice that $\bar{a}$ and $\bar{b}$ are complete rankings over the same reduced universe of $\bar{n}$ objects (i.e., they lie in space $\Omega_C$ relative to $\bar{V}$). As such, using $1/2$ as the minimum $d_{KS}$ distance unit, the corresponding $\tau_x$ and $d_{KS}$ values for $\bar{a}$ and $\bar{b}$ are equated as follows [18]:

$$\tau_x(\bar{a}, \bar{b}) = 1 - \frac{4}{\bar{n}(\bar{n} - 1)} d_{KS}(\bar{a}, \bar{b}),$$

which expressed in terms of $d_{KS}$ yields the equivalent relationship:

$$d_{KS}(\bar{a}, \bar{b}) = \frac{\bar{n}(\bar{n} - 1)}{4} - \frac{\bar{n}(\bar{n} - 1)\tau_x(\bar{a}, \bar{b})}{4}, \quad (13)$$

$$= \frac{\bar{n}(\bar{n} - 1)}{4} - \frac{\bar{n}(\bar{n} - 1) \sum_{i=1}^{\bar{n}} \sum_{j=1}^{\bar{n}} \bar{a}_{ij} \bar{b}_{ij}}{4\bar{n}(\bar{n} - 1)}, \quad (14)$$

$$= \frac{\bar{n}(\bar{n} - 1)}{4} - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}}{4}, \quad (15)$$

where Equation [14] applies the definition of $\tau_x$ (see Equation (7)) with respect to $\bar{a}$ and $\bar{b}$, and where Equation [15] cancels a common factor in the second term and utilizes the fact that unranked items in either ranking vector contribute nothing to the sum—that is the matrix inner products are identical in the original and projected spaces. Now, multiplying both sides of Equation [15] by $[\bar{n}(\bar{n} - 1)/2]^{-1}$ gives:

$$\frac{d_{KS}(\bar{a}, \bar{b})}{\bar{n}(\bar{n} - 1)/2} = \frac{1}{2} - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}}{2\bar{n}(\bar{n} - 1)}$$

$$\Rightarrow d_{NP-KS}(a, b) = \frac{1}{2} - \frac{1}{2} \hat{\tau}_x(a, b) \quad \Box$$

**Corollary 1.** The respective NIRAP optimization problems typified by $\hat{\tau}_x$ and $d_{NP-KS}$ are equivalent and, thus, provide identical consensus rankings.

**Proof.** This is established through the following series of equations:
\[
\arg \min_{r \in \Omega} \sum_{k=1}^{K} d_{NP-KS}(r, a^k) = \arg \max_{r \in \Omega} \sum_{k=1}^{K} -d_{NP-KS}(r, a^k)
\]
\[= \arg \max_{r \in \Omega} \sum_{k=1}^{K} -\left[ \frac{1}{2} - \frac{1}{2} \hat{\tau}_x(r, a^k) \right] \tag{17}\]
\[= \arg \max_{r \in \Omega} \sum_{k=1}^{K} \hat{\tau}_x(r, a^k), \tag{18}\]

where the last equation results from the fact that scalars common to every term in the sum and constant terms do not impact the optimal solution. □

It is also expedient to find an equivalent axiomatic distance corresponding to \(\tau_x\) in space \(\Omega\) (recall that [18] proved that \(\tau_x\) is equivalent to \(d_{KS}\) only in the restricted space \(\Omega_C\)).

**Theorem 2** (Linear transformation between \(\tau_x\) and \(d_{NP-KS}\)). Let \(a\) and \(b\) be two arbitrary rankings of \(n = |V|\) objects drawn from the space of non-strict incomplete rankings \(\Omega\). Then, the \(\tau_x\) correlation coefficient and the \(d_{NP-KS}\) distance are connected through the following equation:

\[
d_{NP-KS}(a, b) = \frac{\bar{n}(\bar{n} - 1)}{4} - \frac{n(n - 1)}{4} \tau_x(a, b) \tag{19}\]

where \(\bar{n} = |\bar{V}| = |V_a \cap V_b|\) (i.e., the number of objects explicitly ranked by both \(a\) and \(b\)).

**Proof.** See Appendix 7.2 □

**Corollary 2.** The respective NIRAP optimization problems typified by \(\tau_x\) and \(d_{NP-KS}\) are equivalent and, thus, provide identical consensus rankings.

**Proof.** See Appendix 7.3 □

The connection between \(\hat{\tau}_x\) and \(d_{NP-KS}\) (see Theorem 1 and Corollary 1) provides synergistic support for the appropriateness of each measure to handle a realistic variety of ranking data. The suitability of \(d_{NP-KS}\) is reinforced by the fact that \(\hat{\tau}_x\) was purposely designed to attain the expected correlation interval \([-1, 1]\) when dealing with rankings from space \(\Omega\) (see §4.1). Inversely, these connections equip \(\hat{\tau}_x\) with a corresponding axiomatic foundation, from which its theoretical guarantees can be formally established. This includes the occurrence of the extrema correlation values \(-1, 1\).
Corollary 3. Let \( a, b \in \Omega \). A maximum \( \hat{\tau}_x \) correlation value of 1 is achieved if and only if \( a|(V_a \cap V_b) \) and \( b|(V_a \cap V_b) \) are the same ranking, and a minimum value of \(-1\) is achieved if and only if \( a|(V_a \cap V_b) \) is a linear ordering (i.e., it contains no ties) and \( b|(V_a \cap V_b) \) is the reverse linear ordering of \( a|(V_a \cap V_b) \).

Proof. See Appendix 7.3

In a nutshell, the above corollary results from a combination of \( d_{NP-KS} \) Axioms 2 and 7. Moreover, by directly substituting \( \hat{\tau}_x \) in place of \( d_{NP-KS} \) for the remaining five axioms listed in Appendix 7.1, a corresponding axiomatic foundation for \( \hat{\tau}_x \) is straightforward to obtain (see Table 2). Lastly, notice that since \( d_{NP-KS} \) uniquely satisfies the distance-based axioms, the one-to-one relationship established by Theorem 1 and Corollary 3 together establish that \( \hat{\tau}_x \) uniquely satisfies the corresponding correlation-based axioms.

5. Comprehensive Solution of the Correlation-Based NIRAP

Many competitive group decision-making scenarios entail the allocation of a limited budget among a subset of participants proportional to the consensus ranking solution. In these scenarios it may be prudent to obtain the full set of alternative optimal solutions when multiple consensus rankings exist or to conclusively determine that the optimal solution is unique to avoid unfair and/or arbitrary outcomes. However, obtaining even just one consensus or median ranking via correlation-based methods (or the equivalent axiomatic distance-based methods) is an \( NP \)-hard problem [3]. What is more, owing to the immensity of the solution space—there are approximately \( 0.5[(1.4)^{n+1}n!] \) possible non-strict complete rankings of \( n \) objects [21]—the NIRAP has been solved largely via specialized algorithms rather than general-purpose integer programming techniques. This section develops a customized branch and bound algorithm (B&B) for fulfilling the aforementioned objectives efficiently given input rankings from space \( \Omega \). In particular, B&B allows for an efficient exploration of the solution space by pruning unpromising branches (i.e., ordinal combinations) thereby avoiding full enumeration. It is important to highlight that B&B uses \( \hat{\tau}_x \) instead of \( d_{NP-KS} \) because the former enables a much quicker evaluation of the candidate solutions and modifications performed during execution. This computational edge comes from the fact that the definition of \( d_{NP-KS} \) contains non-linear terms (see Equations (3) and (5)) while that of \( \hat{\tau}_x \) is fully linear (see Equation (10)). Later in this section, B&B is implemented to compare the abilities of \( \tau_x \) and \( \hat{\tau}_x \) to achieve egaliterianism and decisiveness—characterized by the propensity to yield unique or few alternative
optima—when solving nontrivial NIRAP instances. To this end, a probabilistic approach for generating instances from space $\Omega$ is also introduced.

5.1. A Customized Branch and Bound Algorithm

Emond and Mason [18] devised a branch and bound algorithm for solving the general ranking aggregation problem that relies on a succinct function of cumulative agreement between the set of input rankings $\{a^k\}_{k=1}^K$ and an iteratively evolving candidate-solution vector $r \in \Omega_C$. In this respect $\tau_x$ offers a significant advantage over its distance counterpart $d_{P-KS}$ since:

$$\sum_{k=1}^K \tau_x(r, a^k) = \sum_{k=1}^K \sum_{i=1}^n \sum_{j=1}^n \frac{a^k_{ij} r_{ij}}{n(n-1)} = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1}^n A_{ij} r_{ij}, \quad (20)$$

where $[a^k_{ij}]$ and $[r_{ij}]$ represent the score matrices of $a^k$ and $r$, respectively; and where $[A_{ij}] = \sum_{k=1}^K a^k_{ij}$ is defined as the combined input (CI) matrix. In particular, once the CI matrix is computed, the number of matrix inner products required to calculate cumulative agreement relative to any candidate solution are reduced from $K$ to one. Furthermore, this expedient data structure enables a form of sensitivity analysis for determining the increase/decrease in cumulative agreement that would result if the preference or ordinal relationships of a few objects in the candidate ranking are altered. Conversely, the cumulative distance function for $d_{P-KS}$ (the axiomatic-distance counterpart of $\tau_x$) does not yield as wieldy of an expression due to the presence of nonlinear terms (see Equations (3) and (4)).

As Expression (20) indicates, the denominator $n(n-1)$ is common to each correlation coefficient term $\tau_x(r, a^k)$, for $k = 1, \ldots, K$. Therefore, it can be factored out of the cumulative correlation calculation and ignored in the corresponding optimization process. Conversely, each correlation coefficient term $\hat{\tau}_x(r, a^k)$ yields a different denominator equal to the number of objects ranked in common between $r$ and $a^k$, thereby rendering Expression (20) and the current version of Emond and Mason’s algorithm inapplicable for $\hat{\tau}_x$. To address this issue, the following theorem introduces and proves the validity of a corresponding function of cumulative agreement.

**Theorem 3** (Succinct function of cumulative agreement for $\hat{\tau}_x$). Let $r \in \Omega_C$, $a^k \in \Omega$, and $\bar{n}^k = |V_{a^k}|$ (the number of objects ranked by $a^k$), for $k = 1, \ldots, K$. Then, the $\hat{\tau}_x$ cumulative correlation between $r$ and $\{a^k\}_{k=1}^K$ can be computed according to the function:

$$\sum_{k=1}^K \hat{\tau}_x(r, a^k) = \sum_{i=1}^n \sum_{j=1}^n \hat{A}_{ij} r_{ij}, \quad (21)$$

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where \( \hat{A}_{ij} = \sum_{k=1}^{K} \frac{a_{ij}^k}{\bar{n}^k(\bar{n}^k - 1)} \) is the scaled combined input (SCI) matrix.

Proof. Since \( r \in \Omega_C \), the term \( \hat{\tau}_x(r, a^k) \) can be simplified as follows:

\[
\hat{\tau}_x(r, a^k) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^k r_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}^k}{\bar{n}^k(\bar{n}^k - 1)} r_{ij}.
\]

Thus, the denominator associated with each term is constant irrespective of the candidate-solution vector, thereby yielding the equivalent expressions:

\[
\sum_{k=1}^{K} \tau_x(r, a^k) = \sum_{k=1}^{K} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}^k}{\bar{n}^k(\bar{n}^k - 1)} r_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{A}_{ij} r_{ij}. \]

□

Figure 1: Flowchart of branch and bound algorithm (B&B)

Following the calculations of \( \hat{A}_{ij} \) and \( A_{ij} \), the steps of the branch and bound algorithms for \( \hat{\tau}_x \) and \( \tau_x \) are identical; for the reader’s convenience, Figure 1 provides a flowchart of our featured algorithm B&B, which is complemented by the outline in the remainder of this paragraph. First,
the absolute values of the SCI (CI, respectively) matrix entries are summed to yield an upper bound on the cumulative correlation achievable by any candidate-solution vector. An initial deviation penalty corresponding to a user-specified starting solution \( r_0 \in \Omega_C \) (obtained randomly or via a heuristic) is then calculated by subtracting its objective value from said upper bound. To describe the ensuing steps, recall that \( r_0^{-1} \) is the object-ordering induced by the mapping function \( \Psi_0(r_0) \) (see §2). For \( i = 2, \ldots, n \), the algorithm calculates incremental penalties of fixing object \( r_0^{-1}(i) \) (ranked \( i \)th in the reference starting solution) to every possible pairwise preference relative to a candidate sub-ranking of objects \( r_0^{-1}(1), \ldots, r_0^{-1}(i-1) \) by inspecting the respective SCI matrix entries. Three branches are created to reflect the possible ordinal relationships—i.e., preferred, tied, and dispreferred—between \( r_0^{-1}(i) \) and each \( r_0^{-1}(j) \in \{r_0^{-1}(1), \ldots, r_0^{-1}(i-1)\} \). If the incremental penalty of a branch exceeds the current minimum penalty, the branch is pruned; otherwise it is explored by considering the next object, \( r_0^{-1}(i+1) \). The algorithm prioritizes newly created branches when there are multiple branches to explore. Once a complete ranking is obtained, the minimum penalty is updated and the ranking is saved as a possible solution; at the end of the algorithm, all rankings with the final minimum penalty are returned as the set of optimal solutions (i.e., the median or consensus rankings).

5.2. Generation of Representative Instances from Space \( \Omega \)

In the computational study carried out in [42], \( d_{NP-KS} \) outperformed \( d_{P-KS} \) in yielding fewer alternative optima when solving instances with “predictable consensus rankings” of the distance-based NIRAP. Although these results seem to support the premise that \( \tilde{\tau}_x \) is better suited than \( \tau_x \) to solve the correlation-based NIRAP (due to Corollaries 1 and 2), their scope is limited based primarily on the restrictive types of instances therein considered. Specifically, since enumeration was employed to solve the problems exactly, each tested instance consisted of a maximum of 15 non-strict incomplete rankings of seven objects. Moreover, to generate the random instances used in their experiments, three simplistic templates were defined. The first two initialize every input ranking to a reference ranking \( a \in \Omega_C \)—the all-ties and the identity permutation, respectively—and a third differs slightly in that it initializes a minority of the inputs to the reverse ranking of \( a \) used in the second template—the inverse of the identity permutation. For all three templates incompleteness is then inserted to a random number and selection of objects within each initialized ranking. Such instances are not characteristic of most group decision-making settings mainly in that the incorporated level of individual disagreement (or lack thereof) is rather
An alternative experimental design that considers more diverse forms of agreement/disagreement is hereby presented to compare the relative decisiveness and electoral fairness of \( \tau_x \) and \( \hat{\tau}_x \) on a larger scope.

### Table 3: Characterization of rankings generated from one ground truth and one dispersion $\phi$

| $\phi$ (Dispersion) | General group characterization | Example scenarios |
|---------------------|-------------------------------|-------------------|
| $\phi \in (0, 0.25]$ | Strong collective similarity | Federal grant proposal reviews, Olympic events with a style component, Standardized test essay grading |
|                      | “Subject-matter experts”      |                   |
| $\phi \in (0.25, 0.50]$ | Weak collective similarity | University rankings, Paid movie critiques, Official World Cup ranking forecasts |
|                      | “Seasoned body of objective observers” |                   |
| $\phi \in (0.50, 0.75]$ | Weak collective dissimilarity | Unpaid movie recommendations, Unsponsored top travel lists, Car brand preferences |
|                      | “Public with background information” |                   |
| $\phi \in (0.75, 1]$ | Strong collective dissimilarity | Favorite colors, Luckiest numbers |
|                      | “Jumble of heterogeneous opinions” |                   |

To generate test instances that are representative of different group decision-making scenarios, we devise a sampling approach based on Mallow’s $\phi$-model of ranking data \[38\], which is tailored to distance-based methods \[40\]. The standard $\phi$-model is parameterized by a reference or “ground truth” ranking $\mathbf{a} \in \Omega_C$ and dispersion $\phi \in (0, 1]$, which in conjunction with the $d_{KS}$ distance quantify the probability of observing a ranking $\mathbf{a} \in \Omega_C$ as:

$$P(\mathbf{a}) = P(\mathbf{a}|\mathbf{a}, \phi) = \frac{1}{Z} e^{d_{KS}(\mathbf{a}, \mathbf{a})},$$

where $Z = \sum_{\mathbf{r} \in \Omega_C} e^{d_{KS}(\mathbf{r}, \mathbf{a})} = (1 + \phi) \times (1 + \phi + \phi^2) \times \ldots \times (1 + \ldots + \phi^{n-1})$ is the normalization constant. Since setting $\phi$ to 1 yields the uniform distribution over space $\Omega_C$ and setting it nearer to 0 centers the distribution mass closer to $\mathbf{a}$, the dispersion parameter effectively controls the proximity of each generated ranking to the reference ranking \[36\]. This means nontrivial instances with objectively defined degrees of collective similarity/dissimilarity can be obtained by sampling from this distribution using different dispersion values. Table 3 defines four types of simple real-world group decision-making scenarios according to different ranges of $\phi$.

While Table 3 describes scenarios where every ranking is drawn using the same dispersion value, it is plausible to encounter situations where evaluations can be said to come from different ground truths or ranges of $\phi$ (i.e., where more than one type of group participates). Table 4 gives two types of complex scenarios to reflect such combinations. For the first scenario, two dispersions $\phi_1$ and $\phi_2$ exist over the same ground truth; $\phi_1$ is the dispersion of a majority of experts and $\phi_2$ is the dispersion for a minority of jumbled heterogeneous opinions (i.e., spammers). For the second scenario, two opposing ground truths exist, $\mathbf{a} = (1, 2, 3, 4, \ldots, |\mathbf{a}|)$ and
Table 4: Complex scenarios constructed from multiple ground truths or dispersions

| Sampling parameters | Generated instance description |
|---------------------|-------------------------------|
| \((1 - \alpha) \times 100\%\) of input rankings generated with \((a, \phi_1 \in (0, 0.25))\)
| \(\alpha \times 100\%\) of input rankings generated with 
| \((a, \phi_2 \in (0.75, 1))\) | A [\((1 - \alpha) \times 100\%\)]-majority hold nearly identical opinions, which are close to ground truth \(a\), while a [\(\alpha \times 100\%\)]-minority have nearly arbitrary opinions (i.e. with very low collective similarity), where \(0 < \alpha < 0.5\) |
| \((1 - \alpha) \times 100\%\) of input rankings generated with 
| \((a', \phi \in (0, 0.25))\) | A [\((1 - \alpha) \times 100\%\)]-majority hold nearly identical opinions, which are close to ground truth \(a\), while a [\(\alpha \times 100\%\)]-minority seeks to distort the outcome through cohesive contrarian opinions (i.e., close to ground truth \(a'\)), where \(0 < \alpha < 0.5\). |

\[a' = (|a'|, |a'| - 1, |a'| - 2, ..., 3, 2, 1)\] (i.e., the reverse ranking of \(a\)); a majority of expert evaluations is drawn from \(a\) and a minority of contrarian evaluations is drawn from \(a'\), using the same dispersion \(\phi\).

Since sampling directly from the \(\phi\)-distribution can be very inefficient, we adapt the repeated insertion model (RIM) developed in [10] for the efficient unconditional sampling of non-strict incomplete rankings. On a related note, we do not use the Generalized RIM model [36] since it is a conditional approach that relies on assumptions that are more appropriate for implicitly gathered choice data. In RIM, a set of specified insertion probabilities \(p_{ij}\) for \(1 \leq j \leq i \leq n\) is used to efficiently construct a random ranking \(a\) from the reference ranking \(a \in \Omega_c\). Specifically, assuming objects \(a^{-1}(1), \ldots, a^{-1}(i - 1)\) have been assigned certain ranking positions within \(a\), object \(a^{-1}(i)\) is then inserted at rank \(j \leq i\) (i.e., assigned position \(j\) in \(a\)) with probability \(p_{ij}\), for \(i = 1, \ldots, n\). The choice of insertion probabilities \(p_{ij} = \phi^{i-j}/(1 + \phi + \cdots + \phi^{i-1})\) guarantees that \(\sum_{j=1}^{i} p_{ij} = 1\) for all \(i\), and it induces the standard Mallows \(\phi\)-distribution [16]. To generate incomplete rankings by a similarly efficient procedure, we propose two natural extensions for RIM. With this intent in mind, assume that the object subset to be ranked by \(a\), written as \(V_a\), is known a priori. The pseudocodes of these insertion models, abbreviated as RIME1 and RIME2, are shown in Algorithms 1 and 2.

**Algorithm 1** Repeated Insertion Model Extension #1 (RIME1)

**Input:** Object set evaluated by \(a : V_a\), projected reference ranking: \(a|V_a\), dispersion: \(\phi\)

**Output:** Incomplete ranking sample from RIME1

1. Start with the empty ranking \(a = \{\bullet\}\)
2. for \(i = 1, 2, \ldots, |V|\) do
3.   for \(j = 1, 2, \ldots, i\) do
4.     if \(v_j \in V_a\) then
5.       \(a_i \leftarrow a_j|V_a\) with probability: \(p_{ij} = \phi^{i-j}/(1 + \phi + \cdots + \phi^{i-1})\)
6.     else
7.       \(a_i \leftarrow \bullet\)
Algorithm 2 Repeated Insertion Model Extension #2 (RIME2)

**Input:** Object set evaluated by \( a : V_a \), reference ranking: \( a \), dispersion: \( \phi \)

**Output:** Incomplete ranking sample from RIME2

1: Start with the empty rankings \( a = \{ \bullet \} \); \( a' = \{ \bullet \} \)

2: for \( i = 1, 2, ..., |V| \) do

3:   for \( j = 1, 2, ..., i \) do

4:     \( i' = i - \sum_{k=1}^{i-1} 1_{a_k = \bullet} \)

5:   for \( j' = 1, 2, ..., i' \) do

6:     if \( v_i \in V_a \) then

7:       \( a_{i'} = a_j \) with probability: \( p_{i'j'} = \phi^{i' - j'}/(1 + \phi + \cdots + \phi^{i' - 1}) \)

8: for \( i = 1, 2, ..., |V| \) do

9:   if \( v_i \in V \setminus V_a \) then

10:     \( a_i = \bullet \)

11: if \( v_i \in V_a \) then

12:   \( a_i = a_{i'} \)

13: \( i' = i + 1 \)

Explored in greater detail, RIME1 applies RIM to generate a ranking over \( V_a \) belonging to the Mallows \( \phi \)-distribution parameterized by \((a | V_a, \phi)\), which is then expanded to \( V \) by incorporating the unranked objects (i.e., setting their positions to \( \bullet \)). RIME2 applies RIM to generate a complete ranking over \( V \) belonging to the Mallows \( \phi \)-distribution parameterized by \((a, \phi)\), which is then converted into an incomplete ranking by keeping the numerical values of only the objects in \( V_a \) (i.e., replacing the positions of all other objects with \( \bullet \)). To explain the differences between RIME1 and RIME2 more intuitively, assume that in \( a \) object \( v_i \in V \) is strictly preferred over all objects in a nonempty subset \( V' \subset V \), all of which are in turn strictly preferred over object \( v_j \in V \). That is, in the ground truth, \( v_i \) is strictly preferred over \( v_j \) and \( V' \) are objects with intermediary ordinal positions. The core distinction between the two insertion models is that, as the subset of unranked objects (i.e., \( V' \)) for a given judge increases in size, RIME2 proportionally decreases the probability that the reference ordinal positions for \( v_i \) and \( v_j \) will be reversed in \( a \). Conversely, RIME1 determines the probability of this event as if \( v_i \) and \( v_j \) have consecutive ordinal positions in \( a \), thus ignoring the reference positions of unranked objects. In other words, unlike RIME1, RIME2 implicitly incorporates a relative “intensity of preference” \([10] [12]\) between \( v_i \) and \( v_j \) that reflects the intermediary ordinal positions \( a \) assigns to \( V' \) even though this subset is not explicitly considered by \( a \). Hence, the two insertion models generate non-strict incomplete ranking instances from opposing viewpoints regarding preferences over unranked objects. Although these models do not capture every possible assumption for such preferences,
they can be utilized to generate nontrivial random instances with controllable degrees of collective similarity from which general conclusions about the behavior of individual ranking measures can be drawn. It is important to mention that alternative sampling approaches and specialized distributions of ranking data could also be extended to generate random instances of incomplete rankings (e.g., the Plackett-Luce model \[37, 44\]). A thorough comparison of these alternatives and a formal verification of the statistical distributions induced by RIME1 and RIME2 are left for future work.

5.3. Assessing Decisiveness Given One Ground Truth and One Dispersion

This subsection concentrates on assessing the desired practical property of decisiveness. That is, since the incidence of multiple consensus rankings cannot be completely averted when aggregating rankings via the robust measures herein discussed, the propensity of a ranking measure to yield a unique or very few alternative optimal solutions deserves special attention. This is especially relevant owing to the fact that ranking aggregation algorithms are typically designed to yield only one of an instance’s possible multiple rankings, for the sake of efficiency. (For instance, \[42\] devised an exact algorithm for \(d_{NP-KS}\) based on the implicit hitting set approach \[43\] that returns exactly one optimal solution in \(\Omega_C \cap \Omega_S\). Thus, even though obtaining more than one optimal solution or certifying unique optimality may be infeasible in practice, it can be reassuring for decision makers to know a priori that certain measures intrinsically mitigate the occurrence of numerous alternative optimal solutions more than others. Accordingly, the decisiveness of \(\tau_x\) and \(\hat{\tau}_x\) is assessed by how each measure curtails the growth in the number of alternative optimal solutions as data becomes noisier. To this end, this section performs experiments on random nontrivial instances generated according to the scenario templates outlined in Tables 3 and 4.

The experiments were performed on machines equipped with 22GB of RAM memory shared by two 2.8 GHz quad core Intel Xeon 5560 processors; code was written in Python. For parameter configuration detailed below, the number of alternative optima obtained with each measure is individually recorded for 10 corresponding instances and summarized via average (AVG) and standard deviation (SD) values. Test problems are solved exactly via B&B (see Figure 1) until the full solution space is fathomed, and afterwards the numbers of alternative optimal solutions are recorded. Since B&B follows a nearly identical logic when executed with \(\tau_x\) or \(\hat{\tau}_x\) and the respective run-time decisions are instance-specific, differences in solution times were insufficiently favorable in either direction and are unrecorded. For a more
efficient version of B&B, which has been shown to provide good quality solutions to the NIRAP empirically, but which does not return alternative optimal solutions nor guarantee global optimality theoretically, see [2].

Figure 2: \( \tau_a \) evinces much less decisiveness than \( \tilde{\tau}_a \) as the input rankings become noisier.

The first set of instances is generated according to the single-dispersion preference data characterizations outlined in Table 3 and various settings of additional parameters. In particular, to obtain data with differing noise levels, each instance generates \( K = 100 \) total rankings via RIME1 or RIME2 using single-dispersion values \( \phi \in \{0.05, 0.1, \ldots, 0.95, 1.00\} \) (five \( \phi \) values from each general group characterization described in Table 4); without loss of generality, the ground truth \( \mathbf{a} \) is set to \((1, 2, \ldots, n)\) in all these instances. Prior to generating a set of input rankings \( \{\mathbf{a}^k\}_{k=1}^K \), the object subset ranked by each \( \mathbf{a}^k \), \( V_{\mathbf{a}^k} \subseteq V \), is determined randomly along with its cardinality.
\[ |V_{a^k}| \leq n = |V|, \] which is drawn from the uniform distribution \( U(l, u) \). Different cardinality combinations of \( V \) and \( V_{a^k} \) are tested for \( k = 1, \ldots, K \), namely \( |V| = 8 \) with \( |V_{a^k}| \sim U(2, 6) \), \( |V| = 12 \) with \( |V_{a^k}| \sim U(3, 9) \), and \( |V| = 16 \) with \( |V_{a^k}| \sim U(4, 12) \). These combinations are chosen to reflect realistic scenarios in which judges evaluate uneven but reasonable numbers of objects. For instance, it is unreasonable for most human judges to objectively evaluate a large fraction of objects when \( |V| = 16 \).

Figures 2a, 2c, 2e for RIME1 and Figures 2b, 2d, 2f for RIME2 summarize the results graphically. Within each graph, respective alternative optima AVG values for \( \tau_x \) and \( \hat{\tau}_x \) are plotted for each \( \phi \)-value along with corresponding SD values, represented via error bars (to distinguish the measures visually, \( \tau_x \) lines are gray and thick and \( \hat{\tau}_x \) lines are blue and dotted). We point out that when \( |V| \geq 16 \) and \( \phi > .75 \), some instances could not finish solving within 24 hours or their B&B trees exceeded memory. Therefore, the horizontal axes of Figures 2c and 2d stop at \( \phi = .75 \). The plots for both RIME1 and RIME2 demonstrate that \( \hat{\tau}_x \) attained a lower AVG number of alternative optima than \( \tau_x \) for nearly all of the tested \( \phi \)-values. When \( \phi > .35 \), which comprises instances with weak collective similarity and instances with strong and weak collective dissimilarity, \( \tau_x \) AVG and SD values began to increase considerably. For example, in Figure 2c and 2d, they reached respective values of 42.2 and 115.7 for RIME1 and 5.8 and 6.1 for RIME2 respectively; conversely, \( \hat{\tau}_x \) values were never more than 2 and 2.5 in all experiments. Thus, while the ability of both measures to obtain a unique optimal solution expectedly decreased as \( \phi \) and \( |V| \) increased, the growth in magnitude and variability in the number of alternative optimal solutions was markedly more pronounced with \( \tau_x \) than with \( \hat{\tau}_x \). These results suggest that \( \hat{\tau}_x \) exhibits more decisiveness than \( \tau_x \) for these types of scenarios.

5.4. Assessing Decisiveness Given Multiple Ground Truths and Dispersions

A second set of realistic instances is generated with multiple ground truths and dispersions according to the complex scenario templates described in Table 4. The first portion of these instances is generated with ground truth \( a \) and dispersion parameters \( \phi_1 \in \{0.05, 0.1, 0.15, 0.2, 0.25\} \) and \( \phi_2 \in \{0.8, 0.85, 0.9, 0.95, 1\} \) associated with a majority of experts and a minority of spammers, respectively. The second portion of these instances is generated via a single dispersion parameter \( \phi \in \{0.05, 0.1, 0.15, 0.2, 0.25\} \), which is used to draw both a majority of expert rankings with ground truth \( a \) and a minority of contrarian rankings with ground truth \( a' = (|V|, |V| - 1, \ldots, 2, 1) \) (i.e., the reverse of \( a \)). The experiments test the effect
Figure 3: $\tau_x$ evinces less decisiveness, particularly when a minority of contrarian exists and when the total number of input rankings (judges) decreases of varying the number of judges $K \in \{25, 50, 75, 100\}$, and the minority proportion $\alpha \in \{0.05, 0.10, 0.15, 0.20\}$. For both types of instances, $|V| = 12$ with $|V_{\alpha k}| \sim U(6, 9)$ for $k = 1, \ldots, \lceil \alpha K \rceil$ (objects ranked by the minority of contrarians/spammers), $|V_{\alpha k}| \sim U(3, 6)$ for $k = \lceil \alpha K \rceil + 1, \ldots, K$ (objects
ranked by experts). The latter pair of settings is motivated by the inherent goal of spammers or contrarians to affect or manipulate the outcome by providing more evaluations than average users.

The results once again show that \( \hat{\tau}_x \) significantly outperformed \( \tau_x \) in terms of decisiveness. For instance, in the RIME1 experiments with \( K = 25 \), the \( \tau_x \) AVG and SD values were 32.7 and 67.9 (see Figure 3e), respectively, whereas the corresponding \( \hat{\tau}_x \) values were only 1.6 and 1.8 (see Figure 3f). It is important to point out that the discrepancy between the \( \tau_x \) and \( \hat{\tau}_x \) values was more pronounced for instances with contrarians than those with spammers because the latter can be said to inject a less extreme form of noise. That is, spammers distort the aggregate ranking in arbitrary directions, while contrarians cohesively pull away in one direction that is the opposite of the experts’ ground truth. Interestingly, as the Figure 3 line graphs indicate, the abilities of \( \tau_x \) and \( \hat{\tau}_x \) to obtain unique optimal solutions decreased as \( K \) decreased, but the value of \( \alpha \) did not seem to have a significant effect, except for small \( K \). Moreover, as Figure 2 and 3 show, there were no conclusive performance differences between insertion models RIME1 and RIME2, although RIME2 performed better than RIME1 in a slightly higher number of experiments. For this reason, we use RIME2 in the remaining experiment described in the ensuing subsection.

5.5. Assessing Electoral Fairness

As discussed in §4.2, \( \hat{\tau}_x \) is designed to assign equal voting power to each individual judge (i.e., to enforce electoral fairness) in the aggregation, whereas \( \tau_x \) gives increased representation to judges who evaluate more objects. To gauge the effect of this fundamental difference, the third set of instances is generated according to the complex scenario templates described in Table 4 combining multiple ground truths and dispersions. Similar to §5.4, we generate two types of instances: the first uses ground truth \( a \) and dispersion parameters \( \phi_1 \) and \( \phi_2 \) associated with a majority of experts and a minority of spammers, respectively (see §5.4); the second uses a dispersion parameter \( \phi \) to draw both a majority of expert rankings with a ground truth \( a \) and a minority of contrarians rankings with a ground truth that is the reverse of \( a \). The number of judges is set to \( K = 50 \) for all these instances.

The experiment tests two factors. First, it tests gradually increasing the uniform distribution minimum (\( l \)) and maximum (\( u \)) parameters from 3 to 6 and from 6 to 9, respectively, for drawing the number of objects ranked by the minority. That is, the distribution of \( |V_{a^k}| \) changes from \( U(3,6) \) to \( U(6,9) \) for \( k = 1, \ldots, \lceil \alpha K \rceil \) (objects ranked by contrarians/spammers), while it is fixed to \( |V_{a^k}| \sim U(3,6) \) for \( k = \lceil \alpha K \rceil + 1, \ldots, K \) (objects ranked
Figure 4: The similarity between the $\tau_x$ optimal rankings and the expert’s ground truth decreases sharply when a fixed minority of contrarians ranks more objects by experts). Second, it tests the effect of composing the minority with spammers versus contrarians, where the minority proportions tested are $\alpha \in \{0.05, 0.10, 0.15, 0.20\}$. To test these factors, the experiment calculates the average Solution to Ground-truth Similarity (AVG SGS) via the $\hat{\tau}_x$ correlation coefficient; it can be confidently used for this purpose since the aggregate rankings and the ground truth are complete—in fact, $\tau_x$ and $\hat{\tau}_x$ are interchangeable for this calculation (see §4.2).

Figure 4 shows that the aggregate rankings obtained with $\tau_x$ yield smaller AVG SGS for minorities of contrarians. This occurs because the contrarians hold the reverse ground truth, which makes it difficult to get close to the majority’s ground truth. Moreover, as the number of objects that contrarians evaluate increases, the average SGS decreases in Figure 4. In the worst case observed, AVG SGS becomes as low as 0.65, which occurs when $\alpha = 0.20$ and $U(6,9)$ for the number of objects ranked by the minority. Whereas a minority of contrarians causes strictly smaller average SGS value when $\tau_x$ coefficient is used to obtain the optimal solution, a minority of spammers causes similar AVG SGS for both $\tau_x$ and $\hat{\tau}_x$. This is because spammers do not have strong cohesiveness among them, meaning they cannot affect the aggregate ranking as strongly. On the other hand, even if the percentage
of contrarians is small, this causes the \( \tau_x \) aggregate rankings to get further from the experts’ ground truth. Thus, due to the implicit inegalitarianism induced by \( \tau_x \), a higher number of objects ranked by a minority of contrarians can significantly affect fairness in the aggregation process as the resulting solutions get unreasonably far from the majority of experts’ ground truth.

Notice that as the minimum and maximum parameters increase from \( U(3, 6) \) to \( U(6, 9) \) and the minority proportion remains fixed, AVG SGS decreases. In particular, Figure 4c shows that AVG SGS for \( \tau_x \) decreased by more than 0.25 for a minority of contrarians with fixed \( \alpha \). In contrast, when the minority are spammers, the corresponding AVG SGS values remained to close to 1 and relatively stable in Figure 4d. This result indicates that \( \hat{\tau}_x \) is a more robust measure than \( \tau_x \) for such types of nontrivial instances since it is not as adversely affected by the differences in numbers of objects evaluated between individual judges.

6. Discussion

This work makes several noteworthy contributions to the area of robust ranking aggregation. Principally, it develops the \( \hat{\tau}_x \) ranking correlation coefficient, which fulfills the standard definitions of statistical correlation when dealing with non-strict incomplete rankings. In effect, unlike other comparable robust ranking measures, this enables \( \hat{\tau}_x \) to intrinsically assign equal voting power to each input ranking in the aggregation process, irrespective of the number of objects it ranks. Moreover, by establishing key theoretical connections with axiomatic ranking distances, the work unifies and enhances both robust methodologies for ranking aggregation. The paper also develops a customized branch and bound algorithm for obtaining the full set of optimal solutions efficiently, and it develops a statistical sampling framework for generating non-trivial instances of the non-strict incomplete ranking aggregation problem via a series of parameters.

In this research, ranking aggregation is applied to find the mathematical consensus between a set of subjective preferences. However, the ranking aggregation problem has wide-ranging applicability. In particular, the mathematical approach herein designed may be of use in numerous contexts where sets of incomplete ordinal data are compared to a “gold standard” or combined to remove noise or find a robust measure of central tendency. Indeed, although the Kemeny-Snell/Kendall-tau axiomatic framework may yield less intuitive semantic interpretations outside of group decision-making, its robustness has benefited various areas including Information Retrieval, Artificial Intelligence, and Biostatistics. For instance,
the neutral treatment of incomplete rankings is relevant for the integration of omics-scale data in Biostatistics, where rank-based methods have been utilized due to their invariance to transformation and normalization \cite{33}. Within this context, it is often necessary to integrate ranked lists of genes arising from multiple technology platforms or studies. To avoid obtaining suboptimal results and introducing noise/bias to the aggregation, one must factor in the different underlying spaces of each list and ensure no assumptions are made about the genes outside each respective space \cite{34}.

For future work, we plan to extend and tailor the present framework to some of these applications. Furthermore, the present work will be extended to aggregate top-$k$ incomplete rankings more efficiently than the provisional approach discussed in \S 4.1 and to analyze the properties of other plausible statistical distributions for sampling non-strict incomplete rankings.

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7. Appendix

7.1. Intuitive axioms for distance-based aggregation of non-strict and incomplete rankings

To understand the axioms, two basic concepts need defining. First, ranking \( b \) is said to be between rankings \( a \) and \( c \) if for every object-pair \((v_i, v_j)\) the preferences of \( b \) either agree with \( a \) or agree with \( c \) or \( a \) prefers \( v_j \), \( c \) prefers \( v_j \), and \( b \) ties them. Second, a nonempty object set \( V' \subseteq V \) is said to be a segment of \( a \) if \( V' := V \setminus V' \neq \emptyset \) and \( a \) either prefers every object in \( V' \) to every object in \( V' \) or vice versa.

| Axiom         | Description                                                                                                                                 |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------|
| 1. Relevance  | \( d(a, b) = d(a|V_a \cap V_b), b|V_a \cap V_b) \)                                                                                           |
| 2. Non-negativity | \( d(a, b) \geq 0 \) and \( d(a, b) = 0 \) if and only if \( a|V_a \cap V_b = b|V_a \cap V_b \)                                          |
| 3. Commutativity | \( d(a, b) = d(b, a) \)                                                                                                                   |
| 4. Relaxed Triangle Inequality | \( \frac{d(a|V_a \cap V_b), b|V_a \cap V_b)}{d(b|V_a \cap V_b), c|V_a \cap V_b)} + \frac{d(b|V_a \cap V_b), c|V_a \cap V_b)}{d(c|V_a \cap V_b), a|V_a \cap V_b)} \)  
|               | \( \geq \frac{d(a|V_a \cap V_b), c|V_a \cap V_b)}{d(b|V_a \cap V_b), c|V_a \cap V_b)} \) and equality holds if and only if \( b|V_a \cap V_b \) is between the other two projected rankings |
| 5. Anonymity  | If \( a' \) results from \( a \) by a permutation of the objects in \( V \), and \( b' \) results from \( b \) by the same permutation, then \( d(a, b) = d(a', b') \) |
| 6. Extension  | If rankings \( a \) and \( b \) agree except for a set \( S \) of \( k \) elements, which is a segment of both, then \( d(a, b) \) may be computed as if these \( k \) objects were the only objects being ranked |
| 7. Normalization | \( d(a, b) \leq 1 \) and \( d(a, b) = 1 \) if and only if \( b|V_a \cap V_b \) is the reverse ranking of \( a|V_a \cap V_b \) |

7.2. Proof of Theorem 2

(Linear transformation between \( \tau \) and \( d_{P-KS} \))

Theorem 2. Let \( a \) and \( b \) be two arbitrary rankings of \( n = |V| \) objects drawn from the space of non-strict incomplete rankings, \( \Omega \). Then, the \( \tau \) correlation coefficient and the \( d_{P-KS} \) distance are connected through the following equation:

\[
d_{P-KS}(a, b) = \frac{\bar{n}(\bar{n} - 1)}{4} - \frac{n(n - 1)}{4} \tau_x(a, b)
\]

where \( \bar{n} = |V| = |V_a \cap V_b| \) (i.e., the number of objects explicitly ranked by both \( a \) and \( b \)).

Proof. From Theorem 1, we have that:

\[
d_{NP-KS}(a, b) = \frac{1}{2} - \frac{1}{2} \hat{\tau}_x(a, b),
\]

which can be expanded via Equations 5 and 10 as:

\[
d_{KS}(a|V_a \cap V_b), b|V_a \cap V_b) = \frac{1}{2} - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \cdot b_{ij}}{2\bar{n}(\bar{n} - 1)}.\]

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Thus, multiplying both sides by \( \frac{n(n - 1)}{2} \) yields:

\[
d_{P-KS}(a, b) = \frac{n(n - 1)}{4} - \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij}
\]

which completes the proof since the bracketed expression matches the definition of \( \tau_x(a, b) \).

\[
d_{P-KS}(a, b) = \frac{n(n - 1)}{4} - \frac{n(n - 1)}{4} \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij} \right]
\]

7.3. Proof of Corollary 2

**Corollary 2.** The respective NIRAP optimization problems typified by \( \tau_x \) and \( d_{P-KS} \) are equivalent and, thus, provide identical consensus rankings.

**Proof.** This is established via the following series of equations:

\[
\arg \min_{r \in \Omega_C} \sum_{k=1}^{K} d_{P-KS}(r, a^k) = \arg \max_{r \in \Omega_C} \sum_{k=1}^{K} d_{P-KS}(r, a^k)
\]

\[
= \arg \max_{r \in \Omega_C} \sum_{k=1}^{K} \left[ (|V_r \cap V_{a^k}|)(|V_r \cap V_{a^k}| - 1) - \frac{n(n - 1)}{4} \tau_x(r, a^k) \right]
\]

\[
= \arg \max_{r \in \Omega_C} \sum_{k=1}^{K} \frac{n(n - 1)}{4} \tau_x(r, a^k) - \frac{(|V_{a^k}|)(|V_{a^k}| - 1)}{4}
\]

\[
= \arg \max_{r \in \Omega_C} \sum_{k=1}^{K} \tau_x(r, a^k)
\]

where Equation (24) ensues from Theorem 2 where Equation (25) results from the fact that, since \( r \) must be a complete ranking, \( |V_r \cap V_{a^k}| = |V_{a^k}| \) for every \( k \); and, where Equation (26) results from the fact that scalars common to every term in the sum as well as constant terms (i.e, the second term in Equation (25) is independent of any candidate solution) have no bearing on the optimal solution.
7.4. Proof of Corollary

**Corollary 3.** Let \( a, b \in \Omega \). The maximum \( \hat{\tau}_x \) correlation value of 1 is achieved when \( a|_{(V_a \cap V_b)} \) and \( b|_{(V_a \cap V_b)} \) are the same ranking, and the minimum \( \hat{\tau}_x \) correlation value of -1 is achieved when \( b|_{(V_a \cap V_b)} \) is the reverse ranking of \( a|_{(V_a \cap V_b)} \).

*Proof.* Reexpressing Equation (12) in terms of \( \hat{\tau}_x(a, b) \) gives:

\[
\hat{\tau}_x(a, b) = 1 - 2 d_{NP-KS}(a, b).
\]

By \( d_{NP-KS} \) Axiom 2, \( d_{NP-KS}(a, b) \) has a minimum value of 0, occurring if and only if \( a|_{(V_a \cap V_b)} \) and \( b|_{(V_a \cap V_b)} \) are the same ranking, thereby yielding \( \hat{\tau}_x(a, b) = 1 \). On the other hand, by \( d_{NP-KS} \) Axiom 7, \( d_{NP-KS}(a, b) \) has a maximum value of 1, occurring if and only if \( b|_{(V_a \cap V_b)} \) is the reverse ranking of \( a|_{(V_a \cap V_b)} \), thereby yielding \( \hat{\tau}_x(a, b) = -1 \). \( \square \)