On a Riemann–Liouville Type Implicit Coupled System via Generalized Boundary Conditions†

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Abstract: We study a coupled system of implicit differential equations with fractional-order differential boundary conditions and the Riemann–Liouville derivative. The existence, uniqueness, and at least one solution are established by applying the Banach contraction and Leray–Schauder fixed point theorem. Furthermore, Hyers–Ulam type stabilities are discussed. An example is presented to illustrate our main result. The suggested system is the generalization of fourth-order ordinary differential equations with anti-periodic, classical, and initial boundary conditions.

Keywords: Riemann–Liouville fractional derivative; coupled system; fractional order boundary conditions; green function; existence theory; Ulam stability

MSC: 26A33; 34B27; 45M10

1. Introduction

The generalization of ordinary derivatives leads us to the theory of fractional derivatives. The concept of fractional derivatives was established in 1695, after the well-known conversation of Leibniz and L’Hospital [1]. Mathematicians like Riemann, Liouville, Cauchy, Hadamard, Fourier, and Laplace contributed a lot and made the area more interesting for researchers. A fractional-order derivative is a global operator, which may act as a tool to modify or modernize different physical phenomena like control theory [2], dynamical process [3], electro-chemistry [4], mathematical biology [5], image and signal processing [6], etc. For more applications of the fractional differential equations (FDEs), we refer the reader to the works in [7–11]. Furthermore, the theory of coupled systems of differential equations is referred to as an important theory in the applied sciences envisaging different areas of biochemistry, ecology, biology, and classical fields of physical sciences and engineering. For details see in [12–14].

The theory regarding the existence of solutions of FDEs, drew significant attention of the researchers working on different boundary conditions, e.g., classical, integral, multipoint, non-local, periodic, and anti-periodic [15–18]. Among the qualitative properties of FDEs, the stability property of the solution is the central one, particularly the Hyers–Ulam (HU) stability [19–26]. Stability theory in the sense of HU was first discussed by Ulam [27] in the form of a question in 1940 and the following year, Hyers [28] answered his question...
in the context of Banach spaces. Recently, generalized HU stability was discussed by Alqifiary et al. [29] for linear differential equations. Razaei et al. [30] presented Laplace transform and HU stability of linear differential equations. Wang et al. [31] studied HU stability for two types of linear FDEs. Shen et al. [32] worked on the HU stability of linear FDEs with constant coefficients using Laplace transform method. Liu et al. [33] proved the HU stability of linear Caputo–Fabrizio FDEs. Liu et al. [34] studied the HU stability of linear Caputo–Fabrizio FDEs with the Mittag-Leffler kernel by Laplace transform method.

The above work motivate us to study the coupled implicit FDEs with fractional-order differential boundary conditions:

$$\begin{align*}
&D^αv(t) - \chi_1(t,u(t),D^αv(t)) = 0; \quad t \in \mathfrak{J}, \\
&D^αu(t) - \chi_2(t,v(t),D^αu(t)) = 0; \quad t \in \mathfrak{J}, \\
&D^{α-4}v(0) = η_1D^{α-4}v(σ), \quad D^{α-3}v(0) = η_2D^{α-3}v(σ), \quad D^{α-2}v(0) = η_3D^{α-2}v(σ), \quad D^{α-1}v(0) = η_4D^{α-1}v(σ), \\
&D^{α-4}u(0) = η_5D^{α-4}u(σ), \quad D^{α-3}u(0) = η_6D^{α-3}u(σ), \quad D^{α-2}u(0) = η_7D^{α-2}u(σ), \quad D^{α-1}u(0) = η_8D^{α-1}u(σ),
\end{align*}$$

(1)

where $3 < α, κ ≤ 4$, $\mathfrak{J} = [0,σ]$, $σ > 0$ and $η_i ≠ 1$ for $i = 1,2,\ldots,8$. $D^α, D^κ$ be the $α,κ$ order denotes Riemann–Liouville fractional derivatives and $χ_1,χ_2 : \mathfrak{J} × R × R → R$ be continuous functions.

Higher-order ordinary differential equations (ODEs) can be used to model problems arising from the field of applied sciences and engineering [35,36]. The generalization of fourth-order ODEs are FDEs (1) if $α = κ = 4$. Fourth-order differential equations have important applications in mechanics, thus have attracted considerable attention over the last three decades. The problem of static deflection of a uniform beam, which can be modeled as a fourth-order initial value problem is a good example of a real problem in engineering [37,38].

This problem has been extensively analyzed, some new techniques were developed and numerous general and impressive results regarding the existence of solutions were established in [39–42]. Sometimes, mathematical modeling of the various physical phenomena may arise as a coupled system of the foregoing ODEs. Furthermore, for $η_i = -1$ ($i = 1,2,\ldots,8$), we can obtain anti-periodic boundary conditions which are applicable in several mathematical models, some are given in [43,44].

The manuscript is categorized as follows. For our main results, we establish some basic notations, definitions, and lemma in Section 2. In Section 3, we present existence, uniqueness, and at least one solution of system (1) by applying the Banach contraction fixed point theorem and Leray–Schauder fixed point theorem. In Section 4, we discuss definitions of HU type stabilities, which help us to show that system (1) has HU type stabilities by two different approaches. In Section 5, by a particular example of the system (1), we show that our results are applicable.

2. Background Materials

In this fragment, we present basic notations with Banach spaces, definitions of the considered derivative and integral, and lemma, which will be utilized in the next sections. Suppose $C(\mathfrak{J})$ is a Banach space with a norm defined as $\|v\| = \sup_{t \in \mathfrak{J}} |v(t)|$. For $t \in \mathfrak{J}$, we define $v_r(t) = t^r v(t)$, $r ≥ 0$. Suppose that $S_1 = C_r(\mathfrak{J}) ⊂ C(\mathfrak{J})$ be the space of all functions $v$ such that $v_r \in S_1$ which yields to be a Banach space when endowed with the norm

$$\|v\|_{S_1} = \max\{\sup_{t \in \mathfrak{J}} |t^r v(t)|, \sup_{t \in \mathfrak{J}} |D^α v(t)|\}.$$

Similarly, $\|(v,u)\|_S = \|v\|_{S_1} + \|u\|_{S_2}$ is the norm defined on the product space, where $S = S_1 × S_2$. Obviously $(S,\|(v,u)\|_S)$ is a Banach space.
Definition 1. [45] For a continuous function \( v : \mathbb{R}^+ \rightarrow \mathbb{R} \), the Riemann–Liouville integral of order \( \alpha > 0 \) is defined as
\[
\mathcal{I}^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau,
\]
such that the integral is pointwise defined on \( \mathbb{R}^+ \).

Definition 2. [45] For a continuous function \( v : \mathbb{R}^+ \rightarrow \mathbb{R} \), the Riemann–Liouville derivative of order \( \alpha > 0 \) is defined as
\[
\mathcal{D}^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{v(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau,
\]
where \([\alpha]\) represents the integer part of \( \alpha \) and \( n = [\alpha] + 1 \). We note that for \( \varrho > -1, \varrho \neq \alpha - 1, \alpha - 2, \ldots, \alpha - n \), we have
\[
\mathcal{D}^\alpha t^\varrho = \frac{\Gamma(\varrho + 1)}{\Gamma(\varrho - \alpha + 1)} t^{\varrho - \alpha}
\]
and
\[
\mathcal{D}^\alpha t^{\varrho - i} = 0, \quad i = 1, 2, 3, \ldots, n.
\]

Lemma 1. [45] Solution of the following Riemann–Liouville FDE of order \( n - 1 < \alpha \leq n \)
\[
\mathcal{D}^\alpha v(t) = \vartheta(t),
\]
is
\[
\mathcal{I}^\alpha \mathcal{D}^\alpha v(t) = \mathcal{I}^\alpha \vartheta(t) + k_0 t^{\alpha-n} + k_1 t^{\alpha-n-1} + \cdots + k_{n-2} t^{\alpha-2} + k_{n-1} t^{\alpha-1},
\]
where \( k_i (i = 1, 2, 3, \ldots, n) \) are unknowns.

3. Existence Theory
This section is devoted to the equivalent integral form of the proposed problem.

Lemma 2. Let \( \vartheta \in C(J) \), the following \( \alpha \in (3, 4] \) order FDE with boundary conditions
\[
\begin{align*}
\mathcal{D}^\alpha v(t) &= \vartheta(t); \quad t \in J, \\
\mathcal{D}^{\alpha-4} v(0) &= \eta_1 \mathcal{D}^{\alpha-4} v(\sigma), \quad \mathcal{D}^{\alpha-3} v(0) = \eta_2 \mathcal{D}^{\alpha-3} v(\sigma), \\
\mathcal{D}^{\alpha-2} v(0) &= \eta_3 \mathcal{D}^{\alpha-2} v(\sigma), \quad \mathcal{D}^{\alpha-1} v(0) = \eta_4 \mathcal{D}^{\alpha-1} v(\sigma),
\end{align*}
\]
have the solution
\[
v(t) = \int_0^\sigma g_\alpha(t, \tau) \vartheta(\tau) \, d\tau,
\]
where
Using Lemma 1 on FDE (2), we have

$$
\nu(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \theta(\tau) d\tau + k_3 t^{\alpha - 1} + k_2 t^{\alpha - 2} + k_1 t^{\alpha - 3} + k_0 t^{\alpha - 4}. 
$$

(4)

Applying boundary conditions of (2) on (4), we get unknowns

$$
k_0 = \frac{\eta_1}{(1 - \eta_1)\Gamma(\alpha - 3)} \left[ \frac{1}{6} \int_0^\sigma (\sigma - \tau)^3 \theta(\tau) d\tau + \frac{\eta_2 \sigma}{2(1 - \eta_2)} \int_0^\sigma (\sigma - \tau)^2 \theta(\tau) d\tau \right],
$$

$$
k_1 = \frac{\eta_2}{(1 - \eta_2)\Gamma(\alpha - 2)} \left[ \frac{1}{2} \int_0^\sigma (\sigma - \tau)^2 \theta(\tau) d\tau + \frac{\eta_3 \sigma}{(1 - \eta_3)} \int_0^\sigma (\sigma - \tau) \theta(\tau) d\tau \right],
$$

$$
k_2 = \frac{\eta_3}{(1 - \eta_3)\Gamma(\alpha - 1)} \left[ \int_0^\sigma (\sigma - \tau) \theta(\tau) d\tau + \frac{\eta_4 \sigma}{(1 - \eta_4)} \int_0^\sigma \theta(\tau) d\tau \right],
$$

$$
k_3 = \frac{\eta_4}{(1 - \eta_4)\Gamma(\alpha)} \int_0^\sigma \theta(\tau) d\tau.
$$

Put the values of $k_0, k_1, k_2$ and $k_3$ in Equation (4), we obtain
\[ v(t) = \frac{1}{\Gamma(a)} \int_0^t (t-\tau)^{a-1} \theta(\tau)d\tau + \frac{\eta_1 t^{a-4}}{6(1-\eta_1)\Gamma(a-3)} \int_0^\sigma (\sigma-\tau)^3 \theta(\tau)d\tau \]
\[
+ \left[ \frac{\eta_2 t^{a-3}}{2(1-\eta_2)\Gamma(a-2)} + \frac{\eta_1 \eta_2 t^{a-4}}{2(1-\eta_1)(1-\eta_2)\Gamma(a-3)} \right] \int_0^\sigma (\sigma-\tau)^2 \theta(\tau)d\tau 
\]
\[
+ \left[ \frac{\eta_3 t^{a-2}}{(1-\eta_3)\Gamma(a-1)} + \frac{\eta_1 \eta_2 \eta_3 t^{a-3}}{(1-\eta_2)(1-\eta_3)\Gamma(a-2)} + \frac{\eta_1 (1+\eta_2) \eta_3 t^{a-4}}{2(1-\eta_1)(1-\eta_2)(1-\eta_3)\Gamma(a-3)} \right] 
\times \int_0^\sigma (\sigma-\tau) \theta(\tau)d\tau
\]
\[
+ \frac{\eta_2 (1+\eta_2) \eta_4 t^{a-3}}{2(1-\eta_2)(1-\eta_3)(1-\eta_4)\Gamma(a-2)} 
+ \frac{\eta_1 \eta_2 (1+\eta_3) + \eta_2 + \eta_3 + \eta_4 t^{a-4}}{(1-\eta_1)(1-\eta_2)(1-\eta_3)(1-\eta_4)\Gamma(a-3)} \int_0^\sigma \theta(\tau)d\tau
\]
\[
= \int_0^\sigma G_a(t, \tau) \theta(\tau)d\tau,
\]
where \(G_a(t, \tau)\) is given by (3). \(\square\)

**Remark 1.** Let \(\mu \in C(3)\), the following \(\kappa \in [3,4]\) order FDE with boundary conditions
\[
\begin{align*}
\mathcal{D}^{\kappa} u(t) &= \mu(t); \ t \in \mathcal{I}, \\
\mathcal{D}^{\kappa-1} u(0) &= \eta_5 \mathcal{D}^{\kappa-1} u(\sigma), \ \mathcal{D}^{\kappa-2} u(0) = \eta_6 \mathcal{D}^{\kappa-2} u(\sigma), \\
\mathcal{D}^{\kappa-2} u(0) &= \eta_7 \mathcal{D}^{\kappa-2} u(\sigma), \ \mathcal{D}^{\kappa-1} u(0) = \eta_8 \mathcal{D}^{\kappa-1} u(\sigma)
\end{align*}
\]
has the solution
\[
u(t) = \int_0^\sigma G_\kappa(t, \tau) \mu(\tau)d\tau,
\]
where \(G_\kappa(t, \tau)\) is given by
\[
G_\kappa(t, \tau) = \begin{cases}
\frac{t-\tau}{\Gamma(\kappa)} + \frac{\eta t^{\kappa-4}(\sigma-\tau)^3}{6(1-\eta)(1-\eta_3)\Gamma(\kappa-3)} + \frac{(1-\eta_3)\eta_4 t^{\kappa-3}+\eta_5 \eta_4 t^{\kappa-4}(k-3)}{2(1-\eta_3)(1-\eta_4)\Gamma(k-2)}(\sigma-\tau)^2 \\
+ \frac{\eta t^{\kappa-3}(\sigma-\tau)}{(1-\eta)(1-\eta_3)\Gamma(\kappa-2)} - \frac{\eta t^{\kappa-2}(\sigma-\tau)}{(1-\eta)(1-\eta)(1-\eta_4)\Gamma(\kappa-1)} + \frac{\eta_4 t^{\kappa-4}(\sigma-\tau)}{(1-\eta)(1-\eta)(1-\eta_3)\Gamma(\kappa-2)} \\
+ \frac{\eta t^{\kappa-2}(\sigma-\tau)}{(1-\eta)(1-\eta)(1-\eta_4)\Gamma(\kappa-1)} - \frac{\eta t^{\kappa-1}(\sigma-\tau)}{(1-\eta)(1-\eta)(1-\eta_3)\Gamma(\kappa-2)} + \frac{\eta_4 t^{\kappa-3}(\sigma-\tau)}{(1-\eta)(1-\eta)(1-\eta_3)\Gamma(\kappa-1)} \Gamma(\kappa-3) \\
\end{cases}
\]
for \(0 \leq t < \tau \leq \sigma,
\]

**Remark 2.** Putting \(\alpha = 4\) and \(\eta_1 = \eta_2 = \eta_3 = \eta_4 = -1\) in (3), gives Green's function \(G_\alpha(t, \tau)\) of fourth-order ODE with anti-periodic boundary conditions.

**Remark 3.** Putting \(\alpha = 4\) and \(\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0\) in (5), gives the solution of fourth-order ODE having initial conditions.
For the reason of advantage, we set the following notations:

\[
\Omega_\alpha = \max \left\{ \frac{\sigma^4}{\Gamma (\alpha + 1)} + \frac{\eta_1 \sigma^4}{24 (1 - \eta_1) \Gamma (\alpha - 3)} + \frac{\eta_2 (1 - \eta_1) \sigma^4 + \eta_1 \eta_2 \sigma^6 (\alpha - 3)}{6 (1 - \eta_1) (1 - \eta_2) \Gamma (\alpha - 2)} \right\}
\]

\[
\Omega_\kappa = \max \left\{ \frac{\sigma^4}{\Gamma (\kappa + 1)} + \frac{\eta_5 \sigma^4}{24 (1 - \eta_5) \Gamma (\kappa - 3)} + \frac{\eta_6 (1 - \eta_5) \sigma^4 + \eta_5 \eta_6 \sigma^6 (\kappa - 3)}{6 (1 - \eta_5) (1 - \eta_6) \Gamma (\kappa - 2)} \right\}
\]

and

\[
\nu (t) = \frac{1}{\Gamma (\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
+ \frac{\eta_1 \sigma^4}{6 (1 - \eta_1) \Gamma (\alpha - 3)} \int_0^\sigma (\tau - \tau)^3 \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
+ \frac{\eta_2 \sigma^4}{2 (1 - \eta_2) \Gamma (\alpha - 2)} + \frac{2 (1 - \eta_1) (1 - \eta_2) \Gamma (\alpha - 3)}{24 (1 - \eta_1) \Gamma (\alpha - 3)} \int_0^\sigma (\tau - \tau)^2 \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
+ \frac{\eta_3 \sigma^4}{(1 - \eta_3) \Gamma (\alpha - 1)} + \frac{\eta_2 \eta_4 \sigma^6 (\alpha - 3)}{2 (1 - \eta_1) \Gamma (\alpha - 2)} + \frac{\eta_3 \eta_4 \sigma^6 (\alpha - 3)}{2 (1 - \eta_1) (1 - \eta_2) \Gamma (\alpha - 2)} \int_0^\sigma (\tau - \tau)^2 \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
+ \frac{\eta_2 \eta_4 \sigma^6 (\alpha - 3)}{6 (1 - \eta_1) (1 - \eta_2) \Gamma (\alpha - 2)} + \frac{\eta_3 (1 + \eta_3 + \eta_5 + \eta_6 \eta_7) \sigma^6 (\kappa - 3)}{2 (1 - \eta_1) \Gamma (\kappa - 1)} \int_0^\sigma (\tau - \tau)^2 \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
\times \int_0^\sigma (\tau - \tau)^2 \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
\times \int_0^\sigma (\tau - \tau)^2 \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau
\]

\[
= \int_0^\sigma G_\alpha (t, \tau) \chi_1 (\tau, u(\tau), D^\alpha \nu (\tau)) d\tau,
\]
and

\[ u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \chi_2(\tau, v(\tau), D^\alpha u(\tau)) d\tau + \frac{\eta_1 t^{\alpha-4}}{6(1 - \eta_1)\Gamma(\alpha - 3)} \int_0^\sigma (\sigma - \tau)^3 \chi_2(\tau, v(\tau), D^\alpha u(\tau)) d\tau \]

\[ + \left[ \frac{\eta_2 t^{\alpha-3}}{2(1 - \eta_2)\Gamma(\alpha - 2)} + \frac{\eta_2 \sigma t^{\alpha-4}}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha - 3)} \right] \int_0^\sigma (\sigma - \tau)^2 v(\tau) d\tau \]

\[ + \left[ \frac{\eta_3 t^{\alpha-2}}{(1 - \eta_3)\Gamma(\alpha - 1)} + \frac{\eta_3 \sigma t^{\alpha-3}}{(1 - \eta_3)(1 - \eta_4)\Gamma(\alpha - 3)} \right] \int_0^\sigma (\sigma - \tau) v(\tau) d\tau \]

\[ \times \int_0^\sigma (\sigma - \tau) \gamma(\tau) d\tau + \left[ \frac{\eta_4 t^{\alpha-1}}{(1 - \eta_4)\Gamma(\alpha - 2)} + \frac{\eta_4 \sigma t^{\alpha-2}}{(1 - \eta_4)(1 - \eta_3)\Gamma(\alpha - 3)} \right] \int_0^\sigma \gamma(\tau) d\tau \]

\[ = \int_0^\sigma G_{\alpha}(t, \tau) \chi_2(\tau, v(\tau), D^\alpha u(\tau)) d\tau. \]

We use the following notations for convenience:

\[ \gamma(t) = \chi_1(t, u(t), D^\alpha v(t)) = \chi_1(t, u(t), \gamma(t)) \]

\[ \chi(t) = \chi_2(t, v(\tau), D^\alpha u(\tau)) = \chi_2(t, v(\tau), \chi(t)). \]

Now, transform system (1) to the fixed point problem, let \( F : S \rightarrow S \) is an operator defined by

\[ F(v, u)(t) = \left\{ \begin{array}{ll}
\int_0^t G_{\alpha}(t, \tau) \chi_1(\tau, u(\tau), \gamma(\tau)) d\tau
\int_0^t G_{\alpha}(t, \tau) \chi_2(\tau, v(\tau), \chi(\tau)) d\tau
\end{array} \right\} = \left( \begin{array}{c}
F_\alpha(u, \gamma)(t)
F_\alpha(v, \chi)(t)
\end{array} \right). \quad (8) \]

Then, the fixed point of \( F \) and the solution of system (1) coincided, i.e.,

\[ F_\alpha(v)(t) = \]

\[ \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \gamma(\tau) d\tau + \frac{\eta_1 t^{\alpha-4}}{6(1 - \eta_1)\Gamma(\alpha - 3)} \int_0^\sigma (\sigma - \tau)^3 v(\tau) d\tau \]

\[ + \left[ \frac{\eta_2 t^{\alpha-3}}{2(1 - \eta_2)\Gamma(\alpha - 2)} + \frac{\eta_2 \sigma t^{\alpha-4}}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha - 3)} \right] \int_0^\sigma (\sigma - \tau)^2 v(\tau) d\tau \]

\[ + \left[ \frac{\eta_3 t^{\alpha-2}}{(1 - \eta_3)\Gamma(\alpha - 1)} + \frac{\eta_3 \sigma t^{\alpha-3}}{(1 - \eta_3)(1 - \eta_4)\Gamma(\alpha - 3)} \right] \int_0^\sigma (\sigma - \tau) v(\tau) d\tau \]

\[ \times \int_0^\sigma (\sigma - \tau) \gamma(\tau) d\tau + \left[ \frac{\eta_4 t^{\alpha-1}}{(1 - \eta_4)\Gamma(\alpha - 2)} + \frac{\eta_4 \sigma t^{\alpha-2}}{(1 - \eta_4)(1 - \eta_3)\Gamma(\alpha - 3)} \right] \int_0^\sigma \gamma(\tau) d\tau \]

and
\[
F(x)(t) = \frac{1}{\Gamma(\kappa)} \int_0^t (t-\tau)^{\kappa-1} x(\tau) d\tau + \frac{\eta_3 t^{\kappa-4}}{6(1-\eta_5)\Gamma(\kappa-3)} \int_0^\sigma (\sigma-\tau)^3 x(\tau) d\tau
\]
\[
+ \left[ \frac{\eta_6 t^{\kappa-3}}{2(1-\eta_6)\Gamma(\kappa-2)} + \frac{\eta_8 \eta_6 \sigma t^{\kappa-4}}{2(1-\eta_5)(1-\eta_6)\Gamma(\kappa-3)} \right] \int_0^\sigma (\sigma-\tau)^2 x(\tau) d\tau
\]
\[
+ \left[ \frac{\eta_7 t^{\kappa-2}}{(1-\eta_7)\Gamma(\kappa-1)} + \frac{\eta_6 \eta_7 \sigma t^{\kappa-3}}{(1-\eta_6)(1-\eta_7)\Gamma(\kappa-2)} + \frac{\eta_5 (1+\eta_6) \eta_7 \sigma^2 t^{\kappa-4}}{(1-\eta_5)(1-\eta_6)(1-\eta_7)\Gamma(\kappa-3)} \right] \int_0^\sigma \tau x(\tau) d\tau
\]
\[
\times \int_0^\sigma (\sigma-\tau) x(\tau) d\tau + \left[ \frac{\eta_8 t^{\kappa-1}}{\Gamma(\kappa)} + \frac{\eta_6 \eta_8 \sigma t^{\kappa-2}}{(1-\eta_6)(1-\eta_8)\Gamma(\kappa-1)} \right] \int_0^\sigma x(\tau) d\tau.
\]

Using Banach contraction theorem in the following, we prove the uniqueness of solution of system (1).

**Theorem 1.** Let the functions \(\chi_1, \chi_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) are continuous and satisfy the hypothesis:

**H1:** For every \(t \in J\) and \(u, v, x, \bar{u}, \bar{v}, \bar{x} : J \to \mathbb{R}\), there are \(\mathcal{L}_{\chi_1}, \mathcal{L}_{\chi_2}, \mathcal{L}_{\chi_1}, \mathcal{L}_{\chi_2}\), such that

\[
|\chi_1(t, u(t), v(t)) - \chi_1(t, \bar{u}(t), \bar{v}(t))| \leq \mathcal{L}_{\chi_1} |u(t) - \bar{u}(t)| + \mathcal{L}_{\chi_1} |v(t) - \bar{v}(t)|,
\]
\[
|\chi_2(t, v(t), x(t)) - \chi_2(t, \bar{v}(t), \bar{x}(t))| \leq \mathcal{L}_{\chi_2} |v(t) - \bar{v}(t)| + \mathcal{L}_{\chi_2} |x(t) - \bar{x}(t)|.
\]

In addition, suppose that

\[
\frac{Q_a \mathcal{L}_{\chi_1}(1 - \mathcal{F}_{\chi_2}) + Q_a \mathcal{L}_{\chi_2}(1 - \mathcal{F}_{\chi_1})}{(1 - \mathcal{F}_{\chi_2})(1 - \mathcal{F}_{\chi_1})} < 1,
\]

where \(Q_a\) and \(Q_x\) are defined by Equations (6) and (7), respectively. Furthermore, \(0 \leq \mathcal{F}_{\chi_1}, \mathcal{F}_{\chi_2} < 1\) (through out the paper). Then, the solution of system (1) is unique.

**Proof.** Consider \(\sup_{t \in J} \chi_1(t, 0, 0) = \Phi < \infty\) and \(\sup_{t \in J} \chi_2(t, 0, 0) = \Psi < \infty\), such that

\[
r \geq \frac{2Q_a \Phi^*(1 - \mathcal{F}_{\chi_2}) + 2Q_x \Psi^*(1 - \mathcal{F}_{\chi_1})}{2(1 - \mathcal{F}_{\chi_1})(1 - \mathcal{F}_{\chi_2}) - Q_a \mathcal{L}_{\chi_1} - Q_x \mathcal{L}_{\chi_2}}.
\]

We show that \(F(B_r) \subset B_r\), where

\[
B_r = \{(v, u) \in S : \|(v, u)\|_S \leq r, \|v\|_S \leq \frac{r}{2}, \|u\|_S \leq \frac{r}{2}\}.
\]

For \((v, u) \in B_r\), we have
\[ t^{4-a} |F_a(u)(t)| \leq t^{4-a} \frac{e^t}{\Gamma(a)} \int_0^t (t - \tau)^{a-1} (|\chi_1(\tau, u(\tau), \gamma(\tau)) - \chi_1(\tau, 0, 0)| + |\chi_1(\tau, 0, 0)|) d\tau + \frac{\eta_1}{6(1 - \eta_1)\Gamma(a - 3)} \]

\[ \times \int_0^T (\sigma - \tau)^3 (|\chi_1(\tau, u(\tau), \gamma(\tau)) - \chi_1(\tau, 0, 0)| + |\chi_1(\tau, 0, 0)|) d\tau + \frac{\eta_2}{2(1 - \eta_2)\Gamma(a - 2)} \]

\[ + \left( \eta_1^2 + \eta_2^2 \right) \int_0^T (\sigma - \tau)^2 (|\chi_1(\tau, u(\tau), \gamma(\tau)) - \chi_1(\tau, 0, 0)| + |\chi_1(\tau, 0, 0)|) d\tau + \frac{\eta_1(1 + \eta_2)\eta_3\sigma^2}{(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha - 2)} \]

\[ \times \int_0^T (\sigma - \tau)^2 (|\chi_1(\tau, u(\tau), \gamma(\tau)) - \chi_1(\tau, 0, 0)| + |\chi_1(\tau, 0, 0)|) d\tau + \frac{\eta_2(1 + \eta_3)\eta_4\sigma^2\sigma^3}{(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha - 2)} \]

\[ + \frac{\eta_4^3}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha - 2)} \int_0^T (|\chi_1(\tau, u(\tau), \gamma(\tau)) - \chi_1(\tau, 0, 0)| + |\chi_1(\tau, 0, 0)|) d\tau. \]  

(9)

Consider

\[ |v(t)| \leq |\chi_1(t, u(t), \gamma(t)) - \chi_1(t, 0, 0)| + |\chi_1(t, 0, 0)| \]

\[ \leq |\chi_1(t, 0, 0)| + \mathcal{L}_{\chi_1}|u(t)| + \int_{\chi_1} |v(t)| \]

\[ \leq |\chi_1(t, 0, 0)| + \mathcal{L}_{\chi_1}|u(t)|. \]  

(10)

Substituting (10) in (9), we get

\[ \|F_a(u)\| \leq \left[ \frac{\sigma^4}{\Gamma(a + 1)} + \frac{\eta_1\sigma^4}{24(1 - \eta_1)\Gamma(a - 3)} + \frac{\eta_2(1 - \eta_1)\sigma^4 + \eta_1\eta_2\sigma^4(\alpha - 3)}{6(1 - \eta_1)(1 - \eta_2)\Gamma(\alpha - 2)} \right] \]

\[ + \left( \frac{\eta_2(1 - \eta_3)\sigma^4 + \eta_2\eta_4\sigma^4(\alpha - 2)}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha)} + \frac{\eta_1(1 + \eta_2)\eta_3\sigma^4}{2(1 - \eta_1)(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha - 3)} \right) \]

\[ + \frac{\eta_2(1 + \eta_3)\eta_4\sigma^4(\alpha - 1)}{(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha)} + \frac{\eta_1(1 + \eta_2)(1 + \eta_3) + \eta_2 + \eta_3 + \eta_4\sigma^4(\alpha - 1)}{6(1 - \eta_1)(1 - \eta_2)(1 - \eta_3)(1 - \eta_4)\Gamma(\alpha - 3)} \]

\[ \int_{\chi_1} |\psi^* + \mathcal{L}_{\chi_1}r| \]

Therefore,

\[ \|F_a(u)\| \leq \mathcal{Q}_x \frac{2\psi^* + \mathcal{L}_{\chi_1}r}{2(1 - \mathcal{I}_{\chi_1}).} \]  

(11)

On the same way, we can write

\[ \|F_a(u)\| \leq \mathcal{Q}_x \frac{2\psi^* + \mathcal{L}_{\chi_2}r}{2(1 - \mathcal{I}_{\chi_2})}. \]  

(12)

Inequalities (11) and (12) combined give

\[ \|F(v, u)\| \leq r. \]
For any \( t \in \mathcal{J} \), and \((v_1, u_1), (v_2, u_2) \in S\), we get
\[
\left| t^{\lambda-a} \left[ F_\alpha(v_1)(t) - F_\alpha(v_2)(t) \right] \right| \\
\leq t^{\lambda-a} \int_0^t (t - \tau)^{\alpha-1} \left| \chi_1(\tau, u_1(\tau), Y_1(\tau)) - \chi_1(\tau, u_2(\tau), Y_2(\tau)) \right| d\tau + \frac{\eta_1}{6(1 - \eta_1) \Gamma(\alpha - 3)} + \frac{\eta_2 t}{2(1 - \eta_2) \Gamma(\alpha - 2)} \\
+ \frac{\eta_1 \eta_2 \sigma}{2(1 - \eta_1)(1 - \eta_2) \Gamma(\alpha - 3)} \left| \int_0^\sigma (\sigma - \tau)^2 \left| \chi_1(\tau, u_1(\tau), Y_1(\tau)) - \chi_1(\tau, u_2(\tau), Y_2(\tau)) \right| d\tau \right| \\
+ \frac{\eta_1 \eta_2 \sigma}{(1 - \eta_3) \Gamma(\alpha - 1)} + \frac{\eta_1^2 \sigma^2}{6(1 - \eta_1)(1 - \eta_2)(1 - \eta_3)(1 - \eta_4) \Gamma(\alpha - 3)} \\
+ \frac{\eta_2 \eta_3 \sigma^2}{2(1 - \eta_2)(1 - \eta_3)(1 - \eta_4) \Gamma(\alpha - 2)} \left| \int_0^\sigma \left| \chi_1(\tau, u_1(\tau), Y_1(\tau)) - \chi_1(\tau, u_2(\tau), Y_2(\tau)) \right| d\tau \right| \\
\leq \left[ \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} + \frac{\eta_1 \sigma^4}{6(1 - \eta_1)(1 - \eta_2) \Gamma(\alpha - 3)} + \frac{\eta_2 (1 - \eta_1) \sigma^4}{6(1 - \eta_1)(1 - \eta_2) \Gamma(\alpha - 2)} \right] \\
+ \frac{\eta_3 (1 - \eta_2) \sigma^4 + \eta_2 \eta_3 \sigma^4 (a - 2)}{2(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 1)} + \frac{\eta_1 (1 + \eta_2) \eta_3 \sigma^4}{2(1 - \eta_1)(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 3)} \\
+ \frac{\eta_3 (1 + \eta_2) \eta_3 \sigma^4 (a - 1)}{(1 - \eta_3) \Gamma(\alpha - 1)} + \frac{\eta_1 ((1 + \eta_2)(1 + \eta_3) + \eta_2 + \eta_3) \eta_4 \sigma^4}{2(1 - \eta_1)(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 3)} \\
+ \frac{\eta_1 ((1 + \eta_2)(1 + \eta_3) + \eta_2 + \eta_3) \eta_4 \sigma^4}{6(1 - \eta_1)(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 3)} \left| \frac{\zeta_1}{1 - \zeta_1} \right| \| u_1 - u_2 \| \\
\text{and thus we get} \\
\| F_\alpha(v_1) - F_\alpha(v_2) \| \leq \frac{Q_\alpha L_\chi_1}{1 - T_\chi_1} \| u_1 - u_2 \|. \tag{13}
\]

Similarly,
\[
\| F_\alpha(u_1) - F_\alpha(u_2) \| \leq \frac{Q_\alpha L_\chi_2}{1 - T_\chi_2} \| v_1 - v_2 \|. \tag{14}
\]

From the inequalities (13) and (14), we get that
\[
\| F(v_1, u_1) - F(v_2, u_2) \|_S \leq \frac{Q_\alpha L_\chi_1 (1 - T_\chi_2) + Q_\alpha L_\chi_2 (1 - T_\chi_1)}{(1 - T_\chi_2)(1 - T_\chi_1)} \| (v_1, u_1) - (v_2, u_2) \|_S.
\]

Therefore, \( F \) is a contraction operator. Therefore, by Banach’s fixed point theorem, \( F \) has a unique fixed point, so the solution of the problem (1) is unique. \( \Box \)

The next result is based on the following Leray–Schauder alternative theorem.

**Theorem 2.** \([46]\) Let \( F : S \to S \) be an operator which is completely continuous (i.e., a map that restricted to any bounded set in \( S \) is compact). Suppose

\[ B(F) = \{ v \in S : v = \lambda F(v), \lambda \in [0, 1] \}. \]

Then, either the operator \( F \) has at least one fixed point or the set \( B(F) \) is unbounded.
Theorem 3. Suppose the functions $\chi_1, \chi_2 : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the following hypothesis:

H$_2$: For every $t \in \mathbb{J}$ and $u, v : \mathbb{J} \to \mathbb{R}$, there are $\phi_i (i = 1, 2, 3) : \mathbb{J} \to \mathbb{R}^+$, such that

$$|\chi_1 (t, u(t), v(t))| \leq \phi_1 (t) + \phi_2 (t) |u(t)| + \phi_3 (t) |v(t)|.$$ 

Similarly, for every $t \in \mathbb{J}$ and $v, x : \mathbb{J} \to \mathbb{R}$, there are $\varphi_i (i = 1, 2, 3) : \mathbb{J} \to \mathbb{R}^+$, such that

$$|\chi_2 (t, v(t), x(t))| \leq \varphi_1 (t) + \varphi_2 (t) |u(t)| + \varphi_3 (t) |x(t)|,$$

with $\sup_{t \in \mathbb{J}} \phi_i (t) = \phi_i^*, \sup_{t \in \mathbb{J}} \varphi_i (t) = \varphi_i^* (i = 1, 2, 3)$.

In addition, it is assumed that

$$\Omega_0 = \max \left\{ \frac{\Omega_2 \phi_2^*}{1 - \phi_3^*}, \frac{\Omega_4 \phi_4^*}{1 - \phi_3^*} \right\} < 1 \text{ and } 0 \leq \phi_3^*, \varphi_3^* < 1. \quad (15)$$

Then, the system (1) has at least one solution.

Proof. First, we prove that $F$ is completely continuous. In view of continuity of $\chi_1, \chi_2$, the operator $F$ is also continuous. For any $(v, u) \in B_r$, we have

$$t^{4-a} |F_a (v)(t)| \leq t^{4-a} \int_0^t (t - \tau)^{a-1} |\chi_1 (\tau, u(\tau), v(\tau))| \, d\tau + \frac{\eta_1}{6(1 - \eta_1) \Gamma(a - 3)} \int_0^\sigma (\sigma - \tau)^3 |\chi_2 (\tau, v(\tau), x(\tau))| \, d\tau + \frac{\eta_2 t}{2(1 - \eta_2) \Gamma(a - 2)}$$

$$+ \frac{\eta_2 \eta_3 \sigma}{2(1 - \eta_1)(1 - \eta_2) \Gamma(a - 3)} \int_0^\sigma (\sigma - \tau)^2 |\varphi_1 (\tau)| \, d\tau + \frac{\eta_3^2}{(1 - \eta_3) \Gamma (a - 1)}$$

$$+ \frac{\eta_1 (1 + \eta_2) \eta_4 \sigma^2 t}{6(1 - \eta_1)(1 - \eta_2)(1 - \eta_3)(1 - \eta_4) \Gamma(a - 3)} \int_0^\sigma |\varphi_3 (\tau)| \, d\tau. \quad (16)$$

Now by $H_2$, we have

$$|\chi_1 (t, u(t), v(t))| \leq \phi_1 (t) + \phi_2 (t) |u(t)| + \phi_3 (t) |v(t)|$$

$$\leq \phi_1 (t) + \phi_2 (t) |u(t)| \leq \phi_1 (t) + \phi_2 (t) |u(t)|$$

Therefore, (16) implies

$$\|F_a (v)(t)\| \leq \frac{\sigma^4}{\Gamma (a + 1)} + \frac{\eta_1 \sigma^4}{24(1 - \eta_1) \Gamma(a - 3)} + \frac{\eta_2 (1 - \eta_1) \sigma^4 + \eta_1 \eta_2 \sigma^4 (a - 3)}{6(1 - \eta_1)(1 - \eta_2) \Gamma(a - 2)}$$

$$+ \frac{\eta_3 (1 - \eta_2) \sigma^4 + \eta_2 \eta_3 \sigma^4 (a - 2)}{2(1 - \eta_1)(1 - \eta_2) \Gamma(a - 1)} + \frac{\eta_3 (1 + \eta_2) \eta_4 \sigma^4 (a - 1)}{2(1 - \eta_1)(1 - \eta_2)(1 - \eta_3) \Gamma(a - 3)}$$

$$+ \frac{\eta_1 (1 - \eta_2)(1 + \eta_3) + \eta_2 + \eta_3 \eta_4 \sigma^4 (a - 1)}{(1 - \eta_3)(1 - \eta_4) \Gamma (a)} + \frac{\eta_2 (1 + \eta_3) \eta_4 \sigma^4 (a - 1)}{2(1 - \eta_2)(1 - \eta_3)(1 - \eta_4) \Gamma(a - 2)}$$

$$+ \frac{\eta_1 (1 + \eta_2)(1 + \eta_3) + \eta_2 + \eta_3 \eta_4 \sigma^4 (a - 1)}{6(1 - \eta_1)(1 - \eta_2)(1 - \eta_3)(1 - \eta_4) \Gamma(a - 3)} \|2\phi_1^* + \phi_2^*\| \leq \frac{2\phi_1^* + \phi_2^*}{2(1 - \phi_3^*)}. \quad (17)$$
which implies that
\[ \|F_a(v)\| \leq \Omega_a \frac{2\phi_1^* + \phi_2^* r}{2(1 - \phi_3^*)}. \] (18)

Similarly, we get
\[ \|F_a(u)\| \leq \Omega_a \frac{2\phi_1^* + \phi_2^* r}{2(1 - \phi_3^*)}. \] (19)

Thus, it follows from the inequalities (18) and (19) that \( F_a \) is uniformly bounded. Now, we prove that \( F_a \) is equicontinuous. Let \( 0 \leq t_2 < t_1 \leq t \). Then, we have
\[
\left| t_1^{\alpha-a}F_a(v)(t_1) - t_2^{\alpha-a}F_a(v)(t_2) \right| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[ t_1^{\alpha-a} - t_2^{\alpha-a} \right] v(\tau)d\tau - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} t_2^{\alpha-a}v(\tau)d\tau \right|
+ \left[ \eta_4(t_2^2 - t_1^2) + \frac{\eta_3\eta_4^2(t_2^2 - t_1^2)}{(1 - \eta_3)\Gamma(\alpha + 1)} + \frac{\eta_2(1 + \eta_3)\eta_4^2(t_1 - t_2)}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha + 2)} \right]
+ \left[ \frac{\eta_3^2(t_1 - t_2)}{2(1 - \eta_3)\Gamma(\alpha + 2)} + \frac{\eta_2\eta_3^2(t_1 - t_2)}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha + 2)} \right]
+ \left[ \eta_2^2(t_1 - t_2) \right]
+ \frac{6(1 - \eta_2)\Gamma(\alpha + 2)}{t_1^{\alpha-a} - t_2^{\alpha-a}} \left( \phi_1^* + \phi_2^* |u| \right) \to 0 \text{ as } t_1 \to t_2.
\]

Similarly
\[
\left| t_1^{\alpha-a}F_a(u)(t_1) - t_2^{\alpha-a}F_a(u)(t_2) \right| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[ t_1^{\alpha-a} - t_2^{\alpha-a} \right] v(\tau)d\tau - \frac{1}{\Gamma(\alpha)} \int_1^{t_2} t_2^{\alpha-a}v(\tau)d\tau \right|
+ \left[ \eta_4(t_2^2 - t_1^2) + \frac{\eta_3\eta_4^2(t_2^2 - t_1^2)}{(1 - \eta_3)\Gamma(\alpha + 1)} + \frac{\eta_2(1 + \eta_3)\eta_4^2(t_1 - t_2)}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha + 2)} \right]
+ \left[ \frac{\eta_3^2(t_1 - t_2)}{2(1 - \eta_3)\Gamma(\alpha + 2)} + \frac{\eta_2\eta_3^2(t_1 - t_2)}{2(1 - \eta_2)(1 - \eta_3)\Gamma(\alpha + 2)} \right]
+ \left[ \eta_2^2(t_1 - t_2) \right]
+ \frac{6(1 - \eta_2)\Gamma(\alpha + 2)}{t_1^{\alpha-a} - t_2^{\alpha-a}} \left( \phi_1^* + \phi_2^* |u| \right) \to 0 \text{ as } t_1 \to t_2.
\]

Therefore, \( F(v, u) \) is equicontinuous. Thus, we proved that the operator \( F(v, u) \) is continuous, uniformly bounded, and equicontinuous, concluding that \( F(v, u) \) is completely continuous. Now, by using Arzela–Ascoli theorem, the operator \( F(v, u) \) is compact.
Finally, we are going to check that $B = \{(v, u) \in S \mid (v, u) = \lambda F(v, u), \lambda \in [0, 1]\}$ is bounded. Suppose $(v, u) \in B$, then $(v, u) = \lambda F(v, u)$. For $t \in \mathbb{S}$, we have

$$v(t) = \lambda F_x(v)(t), \quad u(t) = \lambda F_x(u)(t).$$

Then,

$$t^{4-\alpha} |v(t)| \leq \left[ \frac{\sigma^4}{\Gamma(a + 1)} + \frac{\eta_1 \sigma^4}{24(1 - \eta_1) \Gamma(a - 3)} + \frac{\eta_2(1 - \eta_1) \sigma^4 + \eta_3 \eta_2 \sigma^4(\alpha - 3)}{6(1 - \eta_1)(1 - \eta_2) \Gamma(a - 2)} + \frac{\eta_3(1 - \eta_2) \sigma^4 + \eta_3 \eta_3 \sigma^4(a - 2)}{2(1 - \eta_2)(1 - \eta_3) \Gamma(a - 1)} + \frac{\eta_2(1 + \eta_3) \eta_4 \sigma^4}{2(1 - \eta_2)(1 - \eta_3)(1 - \eta_4) \Gamma(a - 2)} \right] \frac{\phi_1(t) + \phi_2(t)|v(t)|}{1 - \phi_3(t)}$$

and

$$t^{4-\kappa} |u(t)| \leq \left[ \frac{\sigma^4}{\Gamma(k + 1)} + \frac{\eta_5 \sigma^4}{24(1 - \eta_5) \Gamma(k - 3)} + \frac{\eta_6(1 - \eta_5) \sigma^4 + \eta_6 \eta_6 \sigma^4(\kappa - 3)}{6(1 - \eta_5)(1 - \eta_6) \Gamma(k - 2)} + \frac{\eta_7(1 - \eta_6) \sigma^4 + \eta_7 \eta_7 \sigma^4(\kappa - 2)}{2(1 - \eta_6)(1 - \eta_7) \Gamma(k - 1)} + \frac{\eta_6(1 + \eta_7) \eta_8 \sigma^4}{2(1 - \eta_6)(1 - \eta_7)(1 - \eta_8) \Gamma(k - 2)} + \frac{\eta_8(1 + \eta_7) \eta_8 \sigma^4}{2(1 - \eta_6)(1 - \eta_7)(1 - \eta_8) \Gamma(k - 3)} \right] \frac{\phi_1(t) + \phi_2(t)|v(t)|}{1 - \phi_3(t)}.$$ (20)

Therefore, from (20) and (21), we have

$$\|v\| \leq \frac{\Omega_\delta \phi_1^* + \phi_2^* |u|}{1 - \phi_3^*}$$

and

$$\|u\| \leq \frac{\Omega_\kappa \phi_1^* + \phi_2^* |v|}{1 - \phi_3^*},$$

which imply that

$$\|v\| + \|u\| = \frac{\Omega_\delta \phi_1^* + \Omega_\kappa \phi_1^*}{1 - \phi_3^*} + \frac{\phi_1^* |v|}{1 - \phi_3^*} + \frac{\phi_2^* |u|}{1 - \phi_3^*} + \frac{\Omega_\kappa \phi_2^*}{1 - \phi_3^*}.$$

Consequently, we get

$$\|(v, u)\| \leq \frac{\Omega_\delta \phi_1^* + \Omega_\kappa \phi_1^*}{(1 - \phi_3^*)(1 - \phi_3^*)(1 - \phi_0^*)},$$

for any $t \in \mathbb{S}$, where $\Omega_0$ is defined by (15), which infer that $B$ is bounded. Therefore, by Theorem 2, $F$ has at least one fixed point. Thus, the system (1) has at least one solution. □
4. Stability Results

Let us recall some definitions related to HU stabilities:
Suppose the functions $\Theta_a, \Theta_k : \mathbb{J} \to \mathbb{R}^+$ are nondecreasing and $\epsilon_a, \epsilon_k > 0$. Consider the inequalities given below.

\[
\begin{align*}
|D^a v(t) - \chi_1(t, u(t), D^a v(t))| & \leq \epsilon_a, \ t \in \mathbb{J}, \\
|D^a u(t) - \chi_1(t, v(t), D^a u(t))| & \leq \epsilon_k, \ t \in \mathbb{J}, \\
|D^a v(t) - \chi_2(t, u(t), D^a v(t))| & \leq \Theta_a(t) \epsilon_a, \ t \in \mathbb{J}, \\
|D^a u(t) - \chi_2(t, v(t), D^a u(t))| & \leq \Theta_k(t) \epsilon_k, \ t \in \mathbb{J}, \\
|D^a v(t) - \chi_1(t, u(t), D^a v(t))| & \leq \Theta_a(t), \ t \in \mathbb{J}, \\
|D^a u(t) - \chi_2(t, v(t), D^a u(t))| & \leq \Theta_k(t), \ t \in \mathbb{J}.
\end{align*}
\] (22)

Definition 3. \cite{47} System (1) is HU stable, if there are $C_{a,k} = \max(C_a, C_k) > 0$ such that for some $\epsilon = \max(\epsilon_a, \epsilon_k) > 0$ and for each solution $(v, u) \in S$ of the inequality (22). There is a solution $(w, \zeta) \in S$ with

\[
\|(v, u)(t) - (w, \zeta)(t)\| \leq C_{a,k} \epsilon, \ t \in \mathbb{J}.
\] (25)

Definition 4. \cite{47} System (1) is generalized HU stable, if there is $\Phi_{a,k} \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi_{a,k}(0) = 0$, such that for each solution $(v, u) \in S$ of (22), there is a solution $(w, \zeta) \in S$ of problem (1), which satisfies

\[
\|(v, u)(t) - (w, \zeta)(t)\| \leq \Phi_{a,k}(\epsilon), \ t \in \mathbb{J}.
\] (26)

Definition 5. \cite{47} System (1) is HU–Rassias stable with respect to $\Theta_{a,k} = \max(\Theta_a, \Theta_k) \in C(\mathbb{J}, \mathbb{R})$, if there are constants $C_{\Theta_a, \Theta_k} = \max(C_{\Theta_a}, C_{\Theta_k}) > 0$ such that for some $\epsilon = (\epsilon_a, \epsilon_k) > 0$ and for each solution $(v, u) \in S$ of the inequality (23). There is a solution $(w, \zeta) \in S$ with

\[
\|(v, u)(t) - (w, \zeta)(t)\| \leq C_{\Theta_a, \Theta_k} \Theta_{a,k}(t) \epsilon, \ t \in \mathbb{J}.
\] (27)

Definition 6. \cite{47} System (1) is generalized HU–Rassias stable with respect to $\Theta_{a,k} = \max(\Theta_a, \Theta_k) \in C(\mathbb{J}, \mathbb{R})$, if there is constant $C_{\Theta_a, \Theta_k} = \max(C_{\Theta_a}, C_{\Theta_k}) > 0$, such that for each solution $(v, u) \in S$ of the inequality (24). There is a solution $(w, \zeta) \in S$ of (1), which satisfies

\[
\|(v, u)(t) - (w, \zeta)(t)\| \leq C_{\Theta_a, \Theta_k} \Theta_{a,k}(t), \ t \in \mathbb{J}.
\] (28)

Remark 4. We say that $(v, u) \in S$ is a solution of the inequality (22), if there are $\Psi_{\chi_1}, \Psi_{\chi_2} \in C(\mathbb{J}, \mathbb{R})$, which depends on $v, u$, respectively, such that

\[
\begin{align*}
(A_1) \ |\Psi_{\chi_1}(t)| & \leq \epsilon_a, \ |\Psi_{\chi_2}(t)| \leq \epsilon_k, \ t \in \mathbb{J}; \\
(A_2) & \begin{cases}
D^a v(t) = \chi_1(t, u(t), D^a v(t)) + \Psi_{\chi_1}(t), \ t \in \mathbb{J}, \\
D^a u(t) = \chi_2(t, v(t), D^a u(t)) + \Psi_{\chi_2}(t), \ t \in \mathbb{J}.
\end{cases}
\end{align*}
\]

Lemma 3. Let $(v, u) \in S$ be the solution of inequality (22), then we have

\[
\begin{align*}
\|v - m_1\| & \leq \Omega_a \epsilon_a, \ t \in \mathbb{J}, \\
\|u - m_2\| & \leq \Omega_k \epsilon_k, \ t \in \mathbb{J}.
\end{align*}
\]
Proof. By (A2) of Remark 4 and for \( t \in J \), we have

\[
\begin{align*}
\mathcal{D}^a v(t) &= \chi_1(t, u(t), \mathcal{D}^a v(t)) + \Psi_{\chi_1}(t), \\
\mathcal{D}^x u(t) &= \chi_2(t, v(t), \mathcal{D}^x u(t)) + \Psi_{\chi_2}(t), \\
\mathcal{D}^{a-4} v(0) &= \eta_1 \mathcal{D}^{a-4} v(\sigma), \quad \mathcal{D}^{x-3} v(0) = \eta_2 \mathcal{D}^{x-3} v(\sigma), \\
\mathcal{D}^{a-2} v(0) &= \eta_3 \mathcal{D}^{a-2} v(\sigma), \quad \mathcal{D}^{x-1} v(0) = \eta_4 \mathcal{D}^{x-1} v(\sigma), \\
\mathcal{D}^{k-4} u(0) &= \eta_5 \mathcal{D}^{k-4} u(\sigma), \quad \mathcal{D}^{x-3} u(0) = \eta_6 \mathcal{D}^{x-3} u(\sigma), \\
\mathcal{D}^{x-2} u(0) &= \eta_7 \mathcal{D}^{x-2} u(\sigma), \quad \mathcal{D}^{x-1} u(0) = \eta_8 \mathcal{D}^{x-1} u(\sigma).
\end{align*}
\tag{29}
\]

By Lemma 1, the solution of (29) can be written as

\[
\begin{align*}
v(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[ \chi_1(\tau, u(\tau), \mathcal{D}^a v(\tau)) + \Psi_{\chi_1}(\tau) \right] d\tau + \frac{\eta_1 t^{\alpha-4}}{6(1 - \eta_1) \Gamma(\alpha - 3)} \\
&\quad \times \int_0^\sigma (\sigma - \tau)^3 \left[ \chi_1(\tau, u(\tau), \mathcal{D}^a v(\tau)) + \Psi_{\chi_1}(\tau) \right] d\tau + \frac{\eta_2 t^{\alpha-3}}{2(1 - \eta_2) \Gamma(\alpha - 2)} + \frac{\eta_1 \eta_2 t^{\alpha-4}}{2(1 - \eta_1)(1 - \eta_2) \Gamma(\alpha - 3)} \\
&\quad \times \int_0^\sigma (\sigma - \tau)^2 \left[ \chi_1(\tau, u(\tau), \mathcal{D}^a v(\tau)) + \Psi_{\chi_1}(\tau) \right] d\tau + \frac{\eta_3 t^{\alpha-2}}{2(1 - \eta_3)(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 1)} + \frac{\eta_2 \eta_3 t^{\alpha-3}}{(1 - \eta_3)(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 2)} \\
&\quad + \frac{\eta_1 (1 + \eta_2) \eta_3 \eta_4 t^{\alpha-4}}{2(1 - \eta_1)(1 - \eta_2)(1 - \eta_3) \Gamma(\alpha - 3)} \int_0^\sigma (\sigma - \tau) \left[ \chi_1(\tau, u(\tau), \mathcal{D}^a v(\tau)) + \Psi_{\chi_1}(\tau) \right] d\tau \\
u(t) &= \frac{1}{\Gamma(\kappa)} \int_0^t (t - \tau)^{\kappa-1} \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau + \frac{\eta_5 t^{\kappa-4}}{6(1 - \eta_5) \Gamma(\kappa - 3)} \\
&\quad \times \int_0^\sigma (\sigma - \tau)^3 \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau + \frac{\eta_6 t^{\kappa-3}}{2(1 - \eta_6) \Gamma(\kappa - 2)} + \frac{\eta_5 \eta_6 t^{\kappa-4}}{2(1 - \eta_5)(1 - \eta_6) \Gamma(\kappa - 3)} \\
&\quad \times \int_0^\sigma (\sigma - \tau)^2 \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau + \frac{\eta_7 t^{\kappa-2}}{(1 - \eta_7)(1 - \eta_6) \Gamma(\kappa - 1)} + \frac{\eta_6 \eta_7 t^{\kappa-3}}{(1 - \eta_6)(1 - \eta_7) \Gamma(\kappa - 2)} \\
&\quad + \frac{\eta_5 (1 + \eta_6) \eta_7 \eta_8 t^{\kappa-4}}{2(1 - \eta_5)(1 - \eta_6)(1 - \eta_7) \Gamma(\kappa - 3)} \int_0^\sigma (\sigma - \tau) \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau \\
&\quad + \frac{\eta_6 (1 + \eta_7) \eta_8 t^{\kappa-3}}{(1 - \eta_6)(1 - \eta_7)(1 - \eta_8) \Gamma(\kappa - 2)} \int_0^\sigma (\sigma - \tau)^3 \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau \\
&\quad + \frac{\eta_7 (1 + \eta_8) \eta_8 t^{\kappa-2}}{(1 - \eta_7)(1 - \eta_8)(1 - \eta_9) \Gamma(\kappa - 1)} \int_0^\sigma (\sigma - \tau)^2 \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau \\
&\quad + \frac{\eta_8 t^{\kappa-1}}{6(1 - \eta_8)(1 - \eta_9)(1 - \eta_10) \Gamma(\kappa - 1)} \int_0^\sigma (\sigma - \tau) \left[ \chi_2(\tau, v(\tau), \mathcal{D}^x u(\tau)) + \Psi_{\chi_2}(\tau) \right] d\tau.
\end{align*}
\tag{30}
\]
From first equation of (30), we have

\begin{align}
& t^{4-a} |v(t) - m_1(t)| \leq t^{4-a} \frac{\eta_1}{\Gamma(a)} \int_0^t (t - \tau)^{a-1} |\Psi_{x_1}(\tau)| \, d\tau \\
+ & \left| \frac{\eta_1}{6(1-\eta_1)\Gamma(a-3)} \right| \int_0^\sigma (\sigma - \tau)^3 |\Psi_{x_1}(\tau)| \, d\tau \\
+ & \left| \frac{\eta_1}{2(1-\eta_2)\Gamma(a - 2)} \right| + \left| \frac{\eta_1 \eta_2 \sigma}{2(1-\eta_1)(1-\eta_2)\Gamma(a-3)} \right| \int_0^\sigma (\sigma - \tau)^2 |\Psi_{x_1}(\tau)| \, d\tau \\
+ & \left| \frac{\eta_2^t}{(1-\eta_3)\Gamma(a - 1)} \right| + \left| \frac{\eta_2 \eta_3 \sigma^2}{(1-\eta_2)(1-\eta_3)\Gamma(a - 2)} \right| \\
+ & \left| \frac{\eta_3 \eta_4 \sigma^2}{6(1-\eta_1)(1-\eta_2)(1-\eta_3)(1-\eta_4)\Gamma(a-3)} \right| \int_0^\sigma |\Psi_{x_1}(\tau)| \, d\tau. 
\end{align}

(31)

where \( m_1(t) \) are those terms which are free of \( \Psi_{x_1} \). Using (6) and (A1) of Remark 4, (31) becomes

\[ \|v - m_1\| \leq \Omega_a \epsilon_a. \]

Similarly for second equation of (30), we obtain

\[ \|u - m_2\| \leq \Omega_c \epsilon_c. \]

\[ \square \]

4.1. Method (I)

**Theorem 4.** If hypothesis \( H_1 \) and

\[ \Lambda = 1 - \frac{\Omega_a \Omega_k L_{x_1} L_{x_2}}{(1 - \Omega_a \mathcal{L}_{x_1})(1 - \Omega_k \mathcal{L}_{x_2})} > 0 \]

(32)

hold, with \( 0 \leq \Omega_a \mathcal{L}_{x_1}, \Omega_k \mathcal{L}_{x_2} < 1 \). Then system (1) is HU stable.

**Proof.** Let \((v, u) \in S\) be the solution of (22) and \((w, \zeta) \in S\) be the solution of following system:

\begin{align}
\mathcal{D}^a w(t) - \chi_1(t, \zeta(t), \mathcal{D}^a w(t)) &= 0, \quad t \in \mathfrak{I}, \\
\mathcal{D}^\zeta w(t) - \chi_2(t, w(t), \mathcal{D}^\zeta w(t)) &= 0, \quad t \in \mathfrak{I}, \\
\mathcal{D}^{a-4} w(0) &= \eta_1 \mathcal{D}^{a-4} w(\sigma), \quad \mathcal{D}^{a-3} w(0) = \eta_2 \mathcal{D}^{a-3} w(\sigma), \\
\mathcal{D}^{a-2} w(0) &= \eta_3 \mathcal{D}^{a-2} w(\sigma), \quad \mathcal{D}^{a-1} w(0) = \eta_4 \mathcal{D}^{a-1} w(\sigma), \\
\mathcal{D}^{k-4} \zeta(0) &= \eta_5 \mathcal{D}^{k-4} \zeta(\sigma), \quad \mathcal{D}^{k-3} \zeta(0) = \eta_6 \mathcal{D}^{k-3} \zeta(\sigma), \\
\mathcal{D}^{k-2} \zeta(0) &= \eta_7 \mathcal{D}^{k-2} \zeta(\sigma), \quad \mathcal{D}^{k-1} \zeta(0) = \eta_8 \mathcal{D}^{k-1} \zeta(\sigma). 
\end{align}

(33)

Then in view of Lemma 1, for \( t \in \mathfrak{I} \) the solution of (33) is given by:
\[
\begin{align*}
\mathcal{w}(t) &= \frac{1}{\Gamma(a)} \int_0^t (t-\tau)^{a-1} \chi_1(\tau, \zeta(\tau), D^a w(\tau)) d\tau + \frac{\eta_1 t^{a-4}}{6(1-\eta_1) \Gamma(a-3)} \\
&\times \int_0^\sigma (\sigma - \tau)^3 \chi_1(\tau, \zeta(\tau), D^a w(\tau)) d\tau + \left[ \frac{\eta_2 t^{a-3}}{2(1-\eta_2) \Gamma(a-2)} + \frac{\eta_1 \eta_2 t^{a-4}}{2(1-\eta_1) (1-\eta_2) \Gamma(a-3)} \right] \\
&\times \int_0^\sigma (\sigma - \tau)^2 \chi_1(\tau, \zeta(\tau), D^a w(\tau)) d\tau + \left[ \frac{\eta_3 t^{a-2}}{2(1-\eta_3) \Gamma(a-1)} + \frac{\eta_2(1+\eta_3) t^{a-3}}{2(1-\eta_2)(1-\eta_3) \Gamma(a-2)} \right] \\
&\frac{(1-\eta_4) \Gamma(\alpha) + (1-\eta_3)(1-\eta_4) \Gamma(\alpha-1)}{6(1-\eta_1)(1-\eta_2)(1-\eta_3)(1-\eta_4) \Gamma(\alpha-3)} \int_0^\sigma (\sigma - \tau) \chi_1(\tau, \zeta(\tau), D^a w(\tau)) d\tau, \\
\zeta(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \chi_2(\tau, w(\tau), D^\alpha \zeta(\tau)) d\tau + \frac{\eta_5 t^{\alpha-4}}{6(1-\eta_5) \Gamma(\alpha-3)} \\
&\times \int_0^\sigma (\sigma - \tau)^3 \chi_2(\tau, w(\tau), D^\alpha \zeta(\tau)) d\tau + \left[ \frac{\eta_6 t^{\alpha-3}}{2(1-\eta_6) \Gamma(\alpha-2)} + \frac{\eta_5 \eta_6 t^{\alpha-4}}{2(1-\eta_5)(1-\eta_6) \Gamma(\alpha-3)} \right] \\
&\times \int_0^\sigma (\sigma - \tau)^2 \chi_2(\tau, w(\tau), D^\alpha \zeta(\tau)) d\tau + \left[ \frac{\eta_7 t^{\alpha-2}}{2(1-\eta_7) \Gamma(\alpha-1)} + \frac{\eta_6(1+\eta_7) t^{\alpha-3}}{2(1-\eta_6)(1-\eta_7) \Gamma(\alpha-2)} \right] \\
&\frac{\eta_5(1+\eta_6) \eta_7 t^{\alpha-4}}{2(1-\eta_5)(1-\eta_6)(1-\eta_7) \Gamma(\alpha-3)} \int_0^\sigma (\sigma - \tau) \chi_2(\tau, w(\tau), D^\alpha \zeta(\tau)) d\tau. \\
\end{align*}
\]

Consider

\[
t^{4-\alpha}|v(t) - w(t)| \leq t^{4-\alpha}|v(t) - m_1(t)| + t^{4-\alpha}|m_1(t) - w(t)|.
\]

Applying Lemma 3 in (35), we get
where $\mathcal{L}_{\chi_{1}} \parallel u - \zeta \parallel \mathcal{D}^{a}v - \mathcal{D}^{a}w \parallel$.

Using $H_{1}$ of Theorem 1 and (6) in (36), we have

$$
\|v - w\| \leq \frac{\Omega_{a}e_{a}}{1 - \Omega_{a}\mathcal{L}_{\chi_{1}}} + \frac{\Omega_{a}\mathcal{L}_{\chi_{1}}}{1 - \Omega_{a}\mathcal{L}_{\chi_{1}}} \| u - \zeta \|.
$$

Similarly, we can get

$$
\|u - \zeta\| \leq \frac{\Omega_{a}e_{x}}{1 - \Omega_{a}\mathcal{L}_{\chi_{2}}} + \frac{\Omega_{a}\mathcal{L}_{\chi_{2}}}{1 - \Omega_{a}\mathcal{L}_{\chi_{2}}} \| v - w\|.
$$

We write (37) and (38) as

$$
\begin{bmatrix}
1 & \frac{\Omega_{a}\mathcal{L}_{\chi_{1}}}{1 - \Omega_{a}\mathcal{L}_{\chi_{1}}} \\
\frac{\Omega_{a}\mathcal{L}_{\chi_{2}}}{1 - \Omega_{a}\mathcal{L}_{\chi_{2}}} & 1
\end{bmatrix}
\begin{bmatrix}
\|v - w\| \\
\|u - \zeta\|
\end{bmatrix}
\leq
\begin{bmatrix}
\frac{\Omega_{a}e_{a}}{1 - \Omega_{a}\mathcal{L}_{\chi_{1}}} \\
\frac{\Omega_{a}e_{x}}{1 - \Omega_{a}\mathcal{L}_{\chi_{2}}}
\end{bmatrix}.
$$
From the above, we get
\[
\begin{bmatrix}
\|v - w\| \\
\|u - \zeta\|
\end{bmatrix} \leq \begin{bmatrix}
\frac{1}{\Lambda} \\
\frac{\Omega_a \Omega_k \mathcal{L}_1 \mathcal{L}_2}{\Lambda (1 - \Omega_a \mathcal{F}_1)} \frac{1}{\Lambda (1 - \Omega_a \mathcal{F}_2)}
\end{bmatrix}
\begin{bmatrix}
\frac{\Omega_a \epsilon_a}{1 - \Omega_a \mathcal{F}_1} \\
\frac{\Omega_a \epsilon_a}{1 - \Omega_a \mathcal{F}_2}
\end{bmatrix},
\]
where
\[\Lambda = 1 - \frac{\Omega_a \Omega_k \mathcal{L}_1 \mathcal{L}_2}{(1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)} > 0.\]

Further simplification gives
\[
\begin{align*}
\|v - w\| & \leq \frac{\Omega_a \epsilon_a}{\Lambda (1 - \Omega_a \mathcal{F}_1)} + \frac{\Omega_a \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_k \mathcal{F}_2)} + \frac{\Omega_a \Omega_k \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)}, \\
\|u - \zeta\| & \leq \frac{\Omega_a \epsilon_a}{\Lambda (1 - \Omega_k \mathcal{F}_2)} + \frac{\Omega_a \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_a \mathcal{F}_1)} + \frac{\Omega_a \Omega_k \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)}.
\end{align*}
\]
from which we have
\[
\|v - w\| + \|u - \zeta\| \leq \frac{\Omega_a \epsilon_a}{\Lambda (1 - \Omega_a \mathcal{F}_1)} + \frac{\Omega_a \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_k \mathcal{F}_2)} + \frac{\Omega_a \Omega_k \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)}
\]
\[
+ \frac{\Omega_a \Omega_k \mathcal{L}_1 \mathcal{L}_2 \epsilon_a}{\Lambda (1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)}, \tag{39}
\]
Let \(\epsilon = \max \{\epsilon_a, \epsilon_k\}\), then from (39) we have
\[
\|(v, u) - (w, \zeta)\|_5 \leq C_{a,k}\epsilon, \tag{40}
\]
where
\[
C_{a,k} = \frac{\Omega_a}{\Lambda (1 - \Omega_a \mathcal{F}_1)} + \frac{\Omega_a \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_k \mathcal{F}_2)} + \frac{\Omega_a \Omega_k \mathcal{L}_1 \epsilon_k}{\Lambda (1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)}
\]
\[
+ \frac{\Omega_a \Omega_k \mathcal{L}_1 \mathcal{L}_2 \epsilon_a}{\Lambda (1 - \Omega_a \mathcal{F}_1)(1 - \Omega_k \mathcal{F}_2)}.
\]

\[\square\]

Remark 5. By setting \(\Phi_{a,k}(\epsilon) = C_{a,k}\epsilon, \Phi_{a,k}(0) = 0\) in (40), then by Definition 4 the problem (1) is generalized HU stable.

H5: Let functions \(\Theta_a, \Theta_k : J \to \mathbb{R}^+\) be nondecreasing. Then, there are \(\xi \Theta_a, \xi \Theta_k > 0\), such that for every \(t \in J\), the inequalities
\[
J^a \Theta_a(t) \leq \xi \Theta_a(t) \quad \text{and} \quad J^k \Theta_k(t) \leq \xi \Theta_k(t)
\]
holds.

Remark 6. Lemma 3 and Theorem 4 gives that the system (1) is HU–Rassias and generalized HU–Rassias stable, if \(\epsilon_a = \Theta_a(t)\epsilon_a\) and \(\epsilon_k = \Theta_k(t)\epsilon_k\) with \(H_5\) and \(\Lambda > 0\).

4.2. Method (II)

Theorem 5. Under the hypothesis \(H_1\) and if \(\Lambda^* = 1 - \left[\frac{\Omega_a \mathcal{L}_1}{1 - \Omega_k \mathcal{F}_2} + \frac{\Omega_a \mathcal{L}_1}{1 - \Omega_a \mathcal{F}_1}\right] > 0\). Then system (1) is HU stable.
Remark 8. Lemma 3 and Theorem 5 gives that the system (1) is HU–Rassias and generalized HU–Rassias stable, if $\epsilon_\alpha = \Theta_\alpha(t)\epsilon_a$ and $\epsilon_\kappa = \Theta_\kappa(t)\epsilon_k$ with $H_3$ and $\Lambda^* > 0$.

Remark 7. With the help of Remark 5, we can obtain the generalized HU stability of system (1).

Remark 9. The results of coupled systems of fourth-order nonlinear FDES gives the results of fourth-order nonlinear system of ODES if $a, \kappa = 4$ with anti-periodic and initial conditions, if $\eta_i = -1$ ($i = 1, 2, \ldots, 8$) and $\eta_i = 0$ ($i = 1, 2, \ldots, 8$) respectively.

5. Example

Example 1. Consider the following coupled system of FDES:

$$\begin{align*}
\mathcal{D}^\alpha v(t) - \left[\frac{1}{4(t+2)^2} + \frac{1}{16} \sin^2 u(t)\right] \mathcal{D}^\alpha v(t) + \frac{1}{16} \mathcal{D}^\alpha u(t) + \frac{1}{16} \mathcal{D}^\alpha v(t) + \frac{1}{16} \mathcal{D}^\alpha u(t) = 0, \quad t \in [0, 1],
\mathcal{D}^\alpha u(t) - \left[\frac{1}{32\pi} \mathcal{D}^\alpha v(t) + \frac{1}{16(1 + \mathcal{D}^\alpha u(t))} + \frac{1}{2}\right] = 0, \quad t \in [0, 1],
\mathcal{D}^{\alpha-4} v(0) = \eta_1 \mathcal{D}^{\alpha-4} v(\sigma), \quad \mathcal{D}^{\alpha-3} v(0) = \eta_2 \mathcal{D}^{\alpha-3} v(\sigma),
\mathcal{D}^{\alpha-2} v(0) = \eta_3 \mathcal{D}^{\alpha-2} v(\sigma), \quad \mathcal{D}^{\alpha-1} v(0) = \eta_4 \mathcal{D}^{\alpha-1} v(\sigma),
\mathcal{D}^{\alpha-4} u(0) = \eta_5 \mathcal{D}^{\alpha-4} u(\sigma), \quad \mathcal{D}^{\alpha-3} u(0) = \eta_6 \mathcal{D}^{\alpha-3} u(\sigma),
\mathcal{D}^{\alpha-2} u(0) = \eta_7 \mathcal{D}^{\alpha-2} u(\sigma), \quad \mathcal{D}^{\alpha-1} u(0) = \eta_8 \mathcal{D}^{\alpha-1} u(\sigma).
\end{align*}$$

(43)

From system (43), we can see $\alpha = \kappa = \frac{10}{3}, \quad \sigma = 1, \quad \eta_1 = \eta_5 = \frac{1}{2}, \quad \eta_2 = \eta_6 = \frac{1}{3}, \quad \eta_3 = \eta_7 = -1$ and $\eta_4 = \eta_8 = -1$. Moreover, we have

$$\begin{align*}
| \chi_1(t, u_1(t), \mathcal{D}^\alpha v_1(t)) - \chi_1(t, u_2(t), \mathcal{D}^\alpha v_2(t)) | &\leq \frac{1}{16} | u_1(t) - u_2(t) | + \frac{1}{16} | \mathcal{D}^\alpha v_1(t) - \mathcal{D}^\alpha v_2(t) |,
| \chi_2(t, v_1(t), \mathcal{D}^\alpha u_1(t)) - \chi_2(t, v_2(t), \mathcal{D}^\alpha u_2(t)) | &\leq \frac{1}{16} | v_1(t) - v_2(t) | + \frac{1}{16} | \mathcal{D}^\alpha u_1(t) - \mathcal{D}^\alpha u_2(t) |.
\end{align*}$$

Therefore, we get $\mathcal{L}_{\chi_1} = \mathcal{I}_{\chi_1} = \mathcal{L}_{\chi_2} = \mathcal{I}_{\chi_2} = \frac{1}{16}$. Therefore,

$$\begin{align*}
\Omega_\alpha \mathcal{L}_{\chi_1} (1 - \mathcal{I}_{\chi_2}) + \Omega_\kappa \mathcal{L}_{\chi_2} (1 - \mathcal{I}_{\chi_1}) &\approx 0.75141 < 1.
\end{align*}$$

Thus, solution of (43) is unique. Moreover, system (43) is HU, generalized HU, HU–Rassias and generalized HU–Rassias stable by two different approaches under the conditions of Theorem 4 and Theorem 5, i.e., $\Lambda > 0$ and $\Lambda^* > 0$. 

6. Conclusions

This paper concluded that the solution of coupled implicit FDEs (1) is unique and exists by using the Banach contraction theorem and Leray–Schauder fixed point theorem. Under some assumptions, the aforesaid coupled system has at least one solution. Besides this, the considered coupled system is HU, generalized HU, HU–Rassias and generalized HU–Rassias stable. An example is presented to illustrate our obtained results. The proposed system (1) gives the following well-known system of ODEs, which has wide applications in applied sciences [5].

- \( \eta_i = -1 \) (\( i = 1, 2, \ldots, 8 \)) and \( \alpha, \kappa = 4 \), then we get fourth-order ODE system with anti-periodic boundary conditions.
- \( \eta_i = 0 \) (\( i = 1, 2, \ldots, 8 \)) and \( \alpha, \kappa = 4 \), then we get fourth-order ODE system with initial conditions.

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