Primordial electric fields before recombination in early Universe

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November 17, 2020

Abstract

This work is a supplement on the previous research about primordial electromagnetic fields. In this work, three important problems are discussed: The evolution of primordial electric fields, the electric solitons in plasma before decoupling and their influences on the power spectra of cosmic microwave background. Detailed computations show that the primordial electric fields dissipate by Landau damping effect on both large scale and small scale and there is no impact on the spectrum. While, before recombination, there exist solitary waves stably propagating in plasma whose speed is significantly slower than that of baryonic acoustic oscillations which works only at extremely scale. On the other hand, the amplitude of solitons is significantly small, so there merely exist the messages about such electric solitary waves on the spectrum. In a word, as relevant monographs on cosmology, neglecting the electromagnetic fields (electric fields at least) is a reasonable treatment on the calculations of cosmic microwave background. However, these electric solitary waves have preserved after the recombination, which may be the origin of initial electromagnetic fields to promote generation and evolution of galaxies.

1. Introduction

Primordial electromagnetic fields are often considered as the essential components in early Universe [1,2]. They relate to the electromagnetic dynamical process in the primordial plasma, cosmological perturbation dynamics and galaxies’ generation and evolution [2–4]. Precious studies focus mainly on the primordial magnetic problems. The origin of the magnetic fields have been widely studied and may generate in different ways, like inflation [5–7], quark confinement [8,9] and neutrino decoupling [10–12]. Such a field plays an important role in the electrodynamic properties [1], baryon acoustic oscillations [13,14] and tensor perturbations [15–17]. It has attracted a lot of interests on the study of the tensor perturbed mode induced by the cosmic magnetic fields, since it they give rise to a anisotropy signal on B-mode polarizations of cosmic microwave background (CMB) [18–24]. However, the average amplitude of magnetic field should be small enough to be consistent with the observations on spatial large-scale isotropy. This is the reason why linear nonequilibrium or instability method is applied to study the perturbation dynamics [3,25–30].

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But unfortunately, the importance of the study on primordial cosmological electric fields has not been realized till now. It may be partly because the previous work did not show any amazing results about electric fields or because the attractive from electric effects in primordial plasma are not so strong compared with it from magnetic effects. But it needs to point out, in plasmonics, the electric properties have an equally important position as magnetic ones [31,32].

Vlasov equation, as the mathematical tool on the plasma statistical properties in electromagnetic fields, is deeply studies during the last century and widely used in the nuclear industry and other fields [33–38]. Relevant theoretical results, like Landau damping effect, acoustic waves and electric solitary wave, have been solidly tested by modern experiments. Therefore it’s reasonable to deduce that the corresponding phenomena should also occur in primordial plasma. Now extend the relevant results to the early Universe and several questions not discussed earlier are going to propose now. The first question is the generation of magnetic field must be accompanied with the generation of electric field. So it’s necessary to analyse the evolution of electric fields and discuss the waves propagating in the plasma. The second question is that whether there is a special way to persist electric field after recombination, since it seems to be not just magnetic fields that work on the formation of first generation galaxies and electric fields should get involved as well. The last question is whether primordial cosmological electric fields influence on the CMB power spectra and how significantly they are.

In this paper, I mainly concentrate on answering these three problems. The method, of course, is the Vlasov equation, which is a Boltzmann equation describing a charged particle’s state in phase space of moving in an electromagnetic field. The optimal scenario to obtain the power spectra of CMB is the Boltzmann equation, as illustrated in series of monographs [39–41]. So a potentially feasible method to study the effects on perturbed dynamics, evolutions for large-scale structure and CMB from the electromagnetic fields is the Vlasov equation. As the critical part of this work, the first step is obtaining the Vlasov equation in perturbed FRW metric which could be accomplished by applying the geodesic equation. Based on this equation, the problems introduced previous could solved via the existing mature solutions. Relevant conclusions are quite brief but persuasive. The initial electric fields dissipate out via Landau damping, intrinsic electric solitary waves propagate with a speed something like average thermal velocity but not exactly and there is almost no influences on CMB power spectra except at extremely small scale.

The following contents of this paper is organized as follows. In Sec. 2, the Vlasov equation in perturbed FRW Universe is obtained via the geodesic equation, which is the key step of this work. In next section, it’s proofed the initial electric fields generated before recombination are gong to dissipate, during which it is assumed the electrons state under a thermal equilibrium and obey the Maxwell distribution. In Sec. 4, the existence of electric solitary waves in primordial plasma is proofed via a series of nonlinear equations, which could retain after recombination to help the generations and evolutions of the galaxies and galaxy cluster. In Sec. 5, the relation between the acoustic oscillations and cosmic electric fields is analytically computed, and the influences on CMB are also discussed. In the final section, brief conclusions and further research programmes are given.
2. Vlasov Equation in Perturbed FRW Universe

The scaler perturbation in perturbed FRW Universe is described by the metric

\[ g_{00}(x, t) = -1 - 2\Psi(x, t), \]
\[ g_{0i}(x, t) = 0, \]
\[ g_{ij}(x, t) = a^2 \delta_{ij}[1 + 2\Phi(x, t)], \]  \(1\)

where \(a\) is scale factor dependent only on cosmic time and \(\Psi, \Phi\) are two variables describing the scaler perturbation under conformal gauge. The distribution function of a particle under equilibrium state depends on position coordinate \(x\), momentum \(p\) and time \(t\) whose complete differential on \(t\) reads

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt}. \]  \(2\)

The four-momentum is defined as

\[ P^\mu = m U^\mu \]  \(3\)

which relates to the four-velocity

\[ U^\mu = \frac{dx^\mu}{d\lambda}, \]  \(4\)

where \(m\) is the rest mass of a particle and \(\lambda\) is the proper time. The second term on the right-hand in Eq. (2) could be reexpress in terms of four-momentum by applying the definitions in Eq. (3). Therefor,

\[ \frac{dx^i}{dt} = \frac{dx_i}{d\lambda} \frac{d\lambda}{dt} = \frac{P^i}{P^0}. \]  \(5\)

The four-momentum of a massive particle is also derived in terms of particle’s energy and momentum:

\[ P^\mu = \left[ E(1 - \Psi), p \hat{p} \frac{1 - \Phi}{a} \right], \]  \(6\)

where \(p^2 = g_{ij} P^i P^j\). Similarly, it’s also available to get the complete differential of a massive particle’s, like electron, distribution function

\[ \frac{df_e}{dt} = \frac{\partial f_e}{\partial t} + \frac{\partial f_e}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f_e}{\partial E} \frac{dE}{dt}. \]  \(7\)

where the term disappears in Eq. (2) with the derivative respect to orientation vector of momentum \(\hat{p}^i\). The electrodynamics equation in curved space-time is written as

\[ \frac{DP^\mu}{D\lambda} = \frac{q}{c} F^{\mu\nu} U_\nu, \]  \(8\)
with $F^{\mu\nu}$ denoting the electromagnetic tensor. Setting $\mu = 0$, Eq. (8) describing a electron’s motion is rewritten in terms of Christoffel symbol as

$$m \frac{dU^0}{d\lambda} + m \Gamma_{0\mu}^{0} P^\mu = -e_0 E^i U_i.$$  \hfill (9)

Repeating the computations in Ref. [39], the derivative of energy to cosmic time gives

$$\frac{dE}{dt} = - \left( H \frac{p^2}{E} + \Phi \frac{p^2}{a} \frac{\partial \Psi}{\partial x^i} - e_0 \frac{1}{a} \frac{p^2}{E} \frac{\partial \phi}{\partial x^i} \right).$$  \hfill (10)

In the equation above, the dot $\dot{}$ denotes the derivative with respect to cosmic time $t$, $H = \dot{a}/a$ represents the Hubble parameter and $\phi$ means electric potential whose gradient relates to the electric field $E = -\nabla \phi$ appeared in Eq. (9).

Instituting Eqs. (3), (5) and (10) into Eq. (7), there is

$$\frac{\partial f}{\partial \lambda} + \frac{\dot{v}^i}{a} \frac{\partial f}{\partial x^i} - \left[ H \frac{p^2}{E} + \Phi \frac{p^2}{a} \frac{\partial \Psi}{\partial x^i} - e_0 \frac{1}{a} \frac{p^2}{E} \frac{\partial \phi}{\partial x^i} \right] \cdot \frac{\partial f}{\partial v} = 0.$$  \hfill (11)

This is the Vlasov equation that describes a electron’s statistical state in phase space forced by a electric field moving in a perturbed FRW Universe. This equation in flat spacetime has been widely and deeply studied in the area of plasma physics and mathematics.

3. Damping of primordial electric field before decoupling

3.1. Linearized Vlasov equation

Eq. (11) describes an electron’s Boltzmann equation in electric field with respect of cosmic time $t$, but in this section the derivative to proper time $\lambda$ is quite convenience on the calculations. Define the three-velocity with a new symbol

$$v^i = \frac{dx^i}{d\lambda}.$$  \hfill (12)

The complete differential of electron’s state in phase space reads

$$\frac{df_e}{d\lambda} = \frac{\partial f_e}{\partial \lambda} + \frac{\partial f_e}{\partial x^i} \frac{dx^i}{d\lambda} + \frac{\partial f_e}{\partial v} \frac{dv}{d\lambda} = 0.$$  \hfill (13)

For convenience and easy understanding, from now on in this section, the symbol of proper time $\lambda$ is replaced by the symbol $t$, and it means proper time instead of cosmic time. Applying the geodesic equation (8) by setting the component $\mu = i$ and inserting it into the Boltzmann equation (13), it arrives

$$\frac{\partial f}{\partial \lambda} + v \cdot \frac{\partial f}{\partial x} - \left[ Hv + \frac{\dot{\Phi}}{\partial t} + \frac{1}{a^2} \Delta \Psi + e_0 \frac{1}{m} E \right] \cdot \frac{\partial f}{\partial v} = 0,$$  \hfill (14)

where the the dot in this section represents the derivative to proper time $\lambda$. In Eq. (14), the following equations or relations have been used: $\Gamma_{\alpha\beta}^{\gamma} = \partial_{\alpha} \Psi / a^2$, $\Gamma_{\gamma\beta}^{\alpha} = \delta_{\beta\gamma} (H + \Phi / \partial t)$ and $dt/d\lambda \approx 1$ since nonrelativistic limit.

The amplitude of electric field is so weak that the deviation to the equilibrium about electron’s distribution function
is quite slight:

\[
f_e(x, v, t) = n_{e0}(v, t) + \delta n_e(x, v, t),
\]

where \(n_{e0}\) is the probability density function under equilibrium state in absence of scaler perturbations in Eq. (1). Substituting Eq. (15) into Eq. (14) and consider only the first order, together with the Maxwell equation, it shows the equations

\[
\frac{\partial \delta n}{\partial t} + v \cdot \frac{\partial \delta n}{\partial x} - \left[ H v + \frac{1}{a^2} \nabla \Psi + \frac{e_0}{m} E \right] \cdot \frac{\partial n_{e0}}{\partial v} = 0,
\]

\[
\rho(x, t) = -e_0 \int \delta n(x, v, t) d^3v.
\]

\[
\mathbf{j}(x, t) = -e_0 \int \delta n(x, v, t)v d^3v,
\]

\[
\nabla \cdot \mathbf{E} = 4\pi \rho(x, t),
\]

\[
\frac{\partial \mathbf{E}}{\partial t} = -4\pi \mathbf{j}(x, t), \quad \nabla \times \mathbf{E} = 0.
\]

\(\rho\) in Eq. (17) is the charge density while \(\mathbf{j}\) in Eq. (18) is the current density.

Set the perturbed probability density function shows an oscillating form

\[
\delta n = \delta n(v)e^{i[k \cdot x - \int \omega(t) dt]}.
\]

Therefore, Eq. (16) becomes by neglecting \(\Phi\) and \(\Psi\)

\[
-i\omega \delta n + i v \cdot k \delta n = \left[ H v + \frac{e_0}{m} E \right] \cdot \frac{\partial n_{e0}}{\partial v}.
\]

Then, Eq. (22) gives the fractional solution to perturbed probability density function:

\[
\delta n = \frac{1}{i[k \cdot v - \omega(t)]} \left[ H v + \frac{e_0}{m} E \right] \cdot \frac{\partial n_{e0}}{\partial v}.
\]

### 3.2. Polarization of Plasma Gas

Polarization vector indicates the deviation from electric neutrality, and it defines

\[
\mathbf{D}_0 = \mathbf{E}_0 + 4\pi \mathbf{P},
\]

where \(\mathbf{D}\) is the electric displacement vector which connects the electric field \(\mathbf{E}\) with permittivity tensor \(\varepsilon_{ij}\):

\[
D_{0i} \exp \left[ i \left( k \cdot x - \int t \omega(t') dt' \right) \right] = \\
\varepsilon_{ij}(\omega, k, t) E_{0j} \exp \left[ i \left( k \cdot x - \int t' \omega(t') dt' \right) \right].
\]

The permittivity tensor could be decomposed into two components, the transverse one and longitudinal one. In this paper, the only component, transverse permittivity \(\varepsilon_1\), is used which means the direction of electric field \(\mathbf{E}_0\) is parallel to the
direction of waves $\mathbf{k}$, or $\mathbf{E}_0 \parallel \mathbf{k}$. So, polarization vector is

$$4\pi \mathbf{P}_0 = (\varepsilon_1 - 1)\mathbf{E}_0.$$  \hfill (26)

According to Eq. (80), polarization vector is able to define in such a way:

$$\nabla \cdot \mathbf{P} = -a^3 n_e \int a^3 \Phi \, d\mathbf{r}',$$  \hfill (27)

$$\frac{\partial \mathbf{P}}{\partial t} = a^3 n_e \mathbf{v}.$$  \hfill (28)

Of course, the definitions above indicate $\mathbf{P}$ propagates as a form of oscillating wave

$$\mathbf{P} = \mathbf{P}_0 \exp \left[ i \left( \mathbf{k} \cdot \mathbf{x} - \int \mathbf{\omega}(\mathbf{t}) \, d\mathbf{t}' \right) \right].$$  \hfill (29)

Thus, inserting Eq. (23) and (29) into Eq. (27) gives

$$i \mathbf{k} \cdot \mathbf{P} = e_0 a^3 \int \frac{d^3 \mathbf{v}}{i(\mathbf{k} \cdot \mathbf{v} - \omega)} \frac{\partial n_0}{\partial \mathbf{v}} \left( H \mathbf{v} + \frac{e_0}{m} \mathbf{E} \right),$$

$$= \frac{e_0^2}{m} a^3 \mathbf{E} \cdot \int \frac{d^3 \mathbf{v}}{i(\mathbf{k} \cdot \mathbf{v} - \omega)} \frac{\partial n_0}{\partial \mathbf{v}} + e_0 H \mathbf{\alpha} \int \frac{d^3 \mathbf{v}}{i(\mathbf{k} \cdot \mathbf{v} - \omega - i0)} \frac{\partial n_0}{\partial \mathbf{v}} \cdot \mathbf{v}. \hfill (30)$$

The equation above is rewritten in a short form $i \mathbf{k} \cdot \mathbf{P} = \mathbf{E} \cdot \mathbf{K} + \mathbf{K}_0$, compared which with Eq. (26) gives the expression to the transverse permittivity tensor

$$\varepsilon_1(\omega, \mathbf{k}, t) = 1 - \frac{4\pi e_0^2}{mk^2 a^3} \int \frac{d^3 \mathbf{v}}{i(\mathbf{k} \cdot \mathbf{v} - \omega - i0)} \frac{\partial n_0}{\partial \mathbf{v}} \cdot \mathbf{k}.$$  \hfill (31)

By applying Eq. (28), it gives the expression of the longitudinal permittivity tensor

$$\varepsilon_0(\omega, \mathbf{k}, t) = 1 + \frac{4\pi e_0^2}{m\omega a^3} \int \frac{n_0(\mathbf{v}) d^3 \mathbf{v}}{i(\mathbf{k} \cdot \mathbf{v} - \omega - i0)},$$  \hfill (32)

but this formula will not be used any longer, so there’s no need to show any more interpretations.

3.3. Expression of electric fields

As discussed above, density function of electron could contain a slight perturbation as a function of $t$

$$n(\mathbf{x}, \mathbf{k}, t) = n_0(\mathbf{v}, t) + \delta n(\mathbf{r}, \mathbf{v}, t),$$  \hfill (33)

whose initial condition is

$$n(\mathbf{x}, \mathbf{k}, 0) = n_0(\mathbf{v}, 0) + g(\mathbf{r}, \mathbf{v}, 0).$$  \hfill (34)
Start with Eq. (16), (17) and (20), perturbed density function $\delta n$ and electric potential $\phi$ satisfy the equations

$$\frac{\partial \delta n}{\partial t} + v \cdot \frac{\partial \delta n}{\partial x} = \left[ Hv + \Phi v + \frac{e_0}{\alpha m} \nabla \Psi + \frac{e_0}{m} E \right] \cdot \frac{\partial n_0}{\partial v} = 0,$$

(35)

$$\nabla^2 \phi = 4\pi e_0 \int \delta n(x, v, t) d^3v.$$

(36)

$\delta n$ and $\phi$ are transformed into wave number vector space by Fourier transformation

$$\delta n_k(v, t) = \int_{-\infty}^{+\infty} \delta n(x, v, t) e^{ik \cdot x} dx,$$

(37)

and

$$\phi_k(t) = \int_{-\infty}^{+\infty} \phi(x, t) e^{ik \cdot x} dx.$$

(38)

Switching Eq. (35) and (36) into Fourier space by neglecting the perturbed scaler variables $\Phi$ and $\Psi$

$$\frac{\partial \delta n_k(v, t)}{\partial t} + ik \cdot v \delta n_k(v, t) - \left[ Hv - \frac{e_0}{m} \phi_k(t) i k \right] \cdot \frac{\partial n_0(v)}{\partial v} = 0,$$

(39)

and

$$a^{-2}k^2 \phi_k(t) = 4\pi e_0 \int \delta n_k(v, t) d^3v.$$

(40)

Then, Laplace transforms on $\delta n_k(v, t)$ and $\phi_k(t)$ about proper time $t$ give

$$\delta_{k,s}(v) = \int_0^{+\infty} e^{-st} \delta n_k(v, t) dt$$

(41)

and

$$\phi_{k,s} = \int_0^{+\infty} e^{-st} \phi_k(t) dt,$$

(42)

whose inverse transformation is

$$\phi_k(t) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} \phi_{k,s} e^{st} ds = \frac{1}{2\pi i} \sum_{i} \text{Res}\left(\phi_{k,s} e^{st}\right)$$

(43)

with $\text{Res}(\cdots)$ denoting the $i$'th residual of integrand $\phi_{k,s} e^{st}$. While the Laplace transform on the first term $\delta n_k(v, t)/\partial t$ including a time derivative in Eq. (35) reads

$$\int_0^{+\infty} \frac{\partial \delta n_k(v, t)}{\partial t} e^{-st} dt = s\delta_{k,s}(v) + g_k(v),$$

(44)

where $g_k(v)$ represents the Fourier transformation on $g(x, v, 0)$.

Multiply $e^{-st}$ on both sides of Eq. (39) and then integrate from 0 to $+\infty$. Using Eq. (44), then Eq. (39) becomes an
equation as a function of parameter $s$:

$$s\delta n_{k,s}(v) - g_k(v) + i k \cdot v \delta n_{k,s} - Hv \cdot \frac{\partial n_0}{\partial v} + \frac{4\pi e_0^2 a^3}{mk^2} k \cdot \frac{\partial n_0}{\partial v} \int \delta n_{k,s}(v')d^3v' = 0. \quad (45)$$

Collate the equation above, and integrate over all $v$ to find the solution

$$
\left\{ 1 - \frac{4\pi e_0^2 a^3}{mk^2} \int \frac{1}{k \cdot v - is - i0} k \cdot \frac{\partial n_0}{\partial v} d^3v' \right\} \int \delta n_{k,s}(v')d^3v' \\
= \int \frac{g_k(v)}{s + ik \cdot v} d^3v + \int \frac{1}{s + ik \cdot v} \frac{\partial n_0}{\partial v} d^3v. \quad (46)
$$

Multiplying $4\pi e_0/k^2$ on both sides of Eq. (46) and applying the definitions on transverse permittivity $\varepsilon_1(\omega, k)$ in Eq. (31) and electric potential $\varphi_k$ in Eq. (42), the electric potential is obtained

$$
\varphi_{k,s} = \frac{14\pi e_0 a^3}{k^2 \varepsilon_1(is, k)} \left\{ \int \frac{g_k(v)d^3v}{k \cdot v - is - i0} + \int \frac{Hv \cdot \frac{\partial n_0}{\partial v} d^3v}{k \cdot v - is - i0} \right\}. \quad (47)
$$

It notes the Laplace transformations above play only on the oscillating variables, like $\delta n$, but avoid the scalar factor $a$. This is because the intrinsic oscillating frequency $\Omega$, is extremely larger than the Hubble expansion rate $H$ as discussed at the of this section. Besides, the exactly analytic expression of $a$ is not going to be involved at all in this section.

### 3.4. Plasma with Maxwell Distribution

The electron’s mass is much lighter than the background particles, like proton, so it’s reasonable to assume electrons is on the equilibrium which follow the Maxwell distribution [42]

$$
n_e(v) = N_e \sqrt{\text{det} g_{ij}} \left( \frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left( -\frac{g_{ij} P^i v^j}{2k_B T} \right), \quad (48)
$$

here $g_{ij} = a^2 \delta_{ij}$ denoting the background metric and $N_e \propto a^{-3}$ denoting the particle number density. Define a new variable about velocity $\tilde{v} = \alpha v$, then the distribution functions follows

$$
\tilde{n}_0(\tilde{v}) = N_{e0} \left( \frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left( -\frac{m\tilde{v}^2}{2k_B T} \right), \quad (49)
$$

with $N_{e0} \equiv N_e a^3$ representing electron’s number density at present time. Integrating Eq. (49) over $\tilde{v}_x$ and $\tilde{v}_z$ generates the Maxwell distribution along $x$ direction

$$
\tilde{n}_0(\tilde{v}_x) = N_{e0} \left( \frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left( -\frac{m\tilde{v}_x^2}{2k_B T} \right), \quad (50)
$$

With the discussions above and setting $k$ parallel to $x$-axis, the transverse permittivity could be reexpressed as follow:

$$
\varepsilon_1 = 1 - \frac{4\pi e_0^2 a^3}{k^2 m} \int k \frac{\partial \tilde{n}_0}{\partial \tilde{v}_x} \tilde{v}_x d\tilde{v}_x - \omega - i0, \quad (51)
$$

\[8\]
with definition of physical wave number \( k_p = k/a \). Instituting Eq. (50) into Eq. (51), it arrives
\[
\varepsilon_1 = 1 - \frac{4\pi e_0^2}{k_p m} \int \frac{d\tilde{v}_x}{\tilde{v}_x - \frac{\tilde{v}_x}{k_p} - i0} N_0 \left( \frac{m}{2\pi k_B T} \right)^{\frac{1}{2}} \left( -\frac{m\tilde{v}_x^2}{k_B T} \right) \exp \left( -\frac{m\tilde{v}_x^2}{2k_B T} \right) .
\]
(52)

It’s convenient to define the following parameters for the subsequent computations
\[
\begin{align*}
\lambda_e^{-1} &= \sqrt{\frac{4\pi e_0^2 N_0}{k_B T}}, \\
z &= \frac{\tilde{v}_x}{\sqrt{2v_e}}, \\
x &= \frac{\omega}{\sqrt{2k_p v_e}}.
\end{align*}
\]
(53)

The parameters in the equations above used to reflect the plasma characteristics are \( \lambda_e \), which means the Debye radius at the moment of recombination, and \( v_c \), which corresponds to the average thermal velocity. Thus Eq. (52) reexpresses
\[
\varepsilon_1 = 1 + \frac{1}{k_p^2 \lambda_e^2} \int \frac{d\tilde{v}_x}{\tilde{v}_x - \frac{\tilde{v}_x}{k_p} - i0} \frac{1}{\sqrt{2\pi} v_c} \exp \left( -\frac{\tilde{v}_x^2}{2v_c^2} \right) \\
= 1 + \frac{1}{k_p^2 \lambda_e^2} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{z}{z - x - i0} e^{-z^2} dz \\
= 1 + \frac{1}{k_p^2 \lambda_e^2} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{(z - x) + x}{z - x - i0} e^{-z^2} dz \\
= 1 + \frac{1}{k_p^2 \lambda_e^2} \left[ 1 + \frac{x}{\sqrt{\pi}} \int_0^\infty \frac{z}{z - x - i0} e^{-z^2} dz \right] \\
= 1 + \frac{1}{k_p^2 \lambda_e^2} [1 + F(x)].
\]
(54)

In the last equality, integral of Gaussian type (142) has been used and \( F(x) \) is a function as a form of Cauchy’s type
\[
F(x) \equiv \frac{x}{\sqrt{\pi}} \int_0^\infty \frac{e^{-z^2}}{z - x - i0} dz.
\]
(55)

According to the introductions of Cauchy’s type in Appendix A, the permittivity at high frequency limit \( \omega \gg k_p v_e \), or \( x \gg 1 \), approximates
\[
\varepsilon_1 \approx 1 - \frac{\Omega_e^2}{\omega^2} \left( 1 + \frac{3k_p^2 v_e^2}{\omega^2} \right) + i \sqrt{\pi} \frac{\omega \Omega_e^2}{2 (k_p v_e)^3} \exp \left( -\frac{\omega^2}{2k_p^2 v_e^2} \right),
\]
(56)

where
\[
\Omega_e \equiv \frac{v_e}{\lambda_e} = \sqrt{\frac{4\pi N_0 e_0^2}{m}}
\]
(57)

is defined as the Langmuir frequency. In the low frequency limit \( x \ll 1 \), it evaluates
\[
\varepsilon_1 \approx 1 + \left( \frac{\Omega_e}{k_p v_e} \right)^2 \left[ 1 - \left( \frac{\omega}{k_p v_e} \right)^2 - i \sqrt{\frac{\pi}{2k_p v_e}} \right], \quad (x \ll 1).
\]
(58)
Applying parameters in Eq. (53), the second part of electric potential in Eq. (47) arrives

\[
\varphi^{(2)}_{k, \lambda} = \frac{H}{k_p} \frac{i 4\pi e_0}{k^2 a e_1} \int \frac{d\tilde{v}}{\tilde{v} - \frac{\omega}{2} - i0} \cdot \frac{\partial \tilde{n}_0(\tilde{v}_c)}{\partial \tilde{v}_c}
\]

\[
+ \frac{2H}{k_p} \frac{i 4\pi e_0}{k^2 a e_1} \int \frac{d^2\tilde{v}}{\tilde{v}_c - \frac{\omega}{2} - i0} N_{e0} \left( \frac{m}{2\pi k_B T} \right)^{\frac{1}{2}}
\]

\[
\times \exp \left( -\frac{m(\tilde{v}_c^2 + \tilde{v}_c^2 + \tilde{\nu}_c^2)}{2k_B T} \right) \cdot \left( \frac{m\tilde{v}_c^2}{k_B T} \right)
\]

\[
= \frac{H}{k_p} \frac{i 4\pi e_0}{k^2 a e_1} \frac{4\pi e_0^2 N_{e0}}{k_B^2 k_B^2} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{z^2 e^{-z}}{z - x - i0} dz
\]

\[
+ 2 \int_0^{+\infty} \frac{e^{-z}}{z - x - i0} \sqrt{2\pi} dz \int_0^{+\infty} z^2 e^{-z} dz
\]

\[
= \frac{1}{i\epsilon_1(\omega, k)} \frac{1}{k_B^2 e_0} \frac{1}{k_B^2 e_0} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{z^2 e^{-z}}{z - x - i0} dz
\]

\[
+ 1 \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{e^{-z}}{z - x - i0} dz
\]

\[
= \frac{1}{i\epsilon_1(\omega, k)} \frac{1}{k_p^2 \lambda_H^2} \left[ 1 + xF(x) + 2F(x) + \frac{F(x)}{x} \right],
\]

(59)

where

\[
\lambda_H^{-1} = -\frac{\sqrt{2} H \nu_e m}{k_B a^2 e_0}
\]

(60)

is a parameter relates to Hubble expansion rate, and Eq. (142) has been used during the calculations above.

Finally, we get the exact expression of electric potential under FRW metric in absence of \( \Phi \) and \( \Psi \)

\[
\varphi_{k, \lambda} = \frac{i 4\pi e_0}{k_B^2 \epsilon_1(\omega, k)} \frac{1}{k_B^2 \lambda_H^2} \frac{G(is/\sqrt{2} \nu_e k_p)}{G(is/\sqrt{2} \nu_e k_p)}
\]

(61)

where

\[
\chi(is, k) = \int \frac{2k(v_c)dv_c}{k\nu_c - 1s - i0}
\]

(62)

and

\[
G(x) = 1 + xF(x) + 2F(x) + F(x)/x.
\]

(63)

### 3.5. Diffusion damping of primordial electric fields

During the epoch before decoupling, electrons state on a thermal equilibrium with photons, but the protons have already exited the thermal equilibrium state with photons for along period, so \( T_e \gg T_p \). Thus the high frequency limit means \( \omega \gg k_B T_e \gg k_B T_p \). Eq. (56) rewrites by setting \( \beta \equiv k_B T_e / \omega \ll 1 \)

\[
\frac{\omega^2}{\Omega_e^2} = 1 + 3\beta^2 - i \frac{3\beta}{\pi} \beta^3 e^{-1/2\beta}.
\]

(64)
The first iteration gives the first order zero point

$$\omega = \Omega_e$$  \hspace{1cm} (65)$$

Then the second time gives the solution where the term $\beta^2$ has taken into account:

$$\omega = \Omega_e(1 + 3k_p^2 \lambda_e^2)^{1/2}.$$  \hspace{1cm} (66)

Finally, inserting Eq. (66) into Eq. (64), we get a complex solution $\omega = \omega' + i\omega''$, with

$$\omega' = \Omega_e \left(1 + \frac{3}{2}k_p^2 \lambda_e^2\right),$$  \hspace{1cm} (67)

$$\omega'' = -\sqrt{\frac{\pi}{8}} \frac{\Omega_e}{\sqrt{k_p^3 \lambda_e^3}} \exp\left[-\frac{1}{2k_p^2 \lambda_e^2}\right].$$  \hspace{1cm} (68)

Similarly, the zero point at low frequency is

$$\omega' = k_p v_e, \quad \text{and} \quad \omega'' = -\sqrt{\frac{\pi}{8}} \omega'.$$  \hspace{1cm} (69)

Thus the root of equation $\varphi_{1k} = 0$ reads $s = i\omega' - \gamma$ with $\gamma = -\omega''$. Based on Eq. (43), it leads to the expression of electric potential

$$\varphi_k(t) = \frac{2\epsilon_0}{k_p^2} \sum_i R_i e^{i\omega t} - \frac{1}{2\pi k_p^3 \lambda_e^3} \sum_i R'_i e^{i\omega t}.$$  \hspace{1cm} (70)

Obviously, the potential is going to damp out by Landau damping, so the initial condition before electron’s recombination, like neutrino’s decoupling, affects almost nothing on cosmic microwave background. Although the potential may be a large amplitude on large scale, it still damps away as well with time increasing.

On the other hand, although the electric potential damps out with time increasing, the particle density $\delta n$ has an oscillating mode. Inserting Eq. (47) into Eq. (45), we have

$$\delta_{k,s} = \frac{1}{s + ik \cdot v} \left\{ g_k(v) + \left[Hv + \frac{4\pi e_0^2}{mk^2} \left( \frac{\gamma(is,k)}{\epsilon_1(is,k)} + \frac{\varsigma(is,k)}{\epsilon_1(is,k)} \right) \right] \frac{\partial n_0}{\partial v}\right\}.$$  \hspace{1cm} (71)

Except the singularities at $\epsilon = 0$, the point $s = -ik \cdot v$ is also a singularity. So the inverse Laplace transformation on Eq. (71) gives an oscillating mode without damping

$$\delta n_k(v,t) \propto e^{-ik \cdot v}.$$  \hspace{1cm} (72)

Furthermore, it’s necessary to characterize the attenuating scale of electric fields by estimating the order of $a^{3/2} \lambda_e H$ or $a^{-3/2} \Omega_e / H$:

$$\frac{H^2}{(\Omega_e a^{-3/2})^2} = \frac{\epsilon_0 m c^2 a^{3} H_0^2}{4\pi e_0^2 N e_0} \frac{c}{H_0^2}.$$  \hspace{1cm} (73)
This number is convenient to calculate under natural unit system: \( \varepsilon_0 / 4\pi c_0^2 = 2.22 \times 10^{15} \text{ kg}^{-1} \cdot \text{m}^{-3} \cdot \text{s}^2 \), \( m^2 = 0.51 \text{ MeV} = 8.16 \times 10^{-14} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \), \( N_0 = 11.244X_0 \Omega_m h^2 \text{ m}^{-3} \) and \( H_0 / c = 3.33 \times 10^{-4} h \text{ Mpc}^{-1} \). At early epoch of the Universe, the main component is either radiation or matter, so \( H^2 / H_0^2 = \Omega_m a^{-3} [1 + a_{eq}/a] \). Then Eq. (73) becomes

\[
\frac{a^3 H^2}{\Omega_e^2} = 2.11 \times 10^{-20} \frac{\Omega_m h^2}{X_0 \Omega_b h^2} \left( 1 + \frac{a_{eq}}{a} \right). \tag{74}
\]

Recent Planck data shows \( \Omega_m h^2 = 0.1424 \) and \( \Omega_b h^2 = 0.022447 \). \( 43, 44 \). Saha equation predicts \( X_e \sim 10^{-2} \) at present time. With the discussions above, Eq. (73) estimates

\[
a^3 H^2 / \Omega_e^2 = 1.91 \times 10^{-19} \left( 1 + \frac{a_{eq}}{a} \right) \sim 10^{-18}, \tag{75}
\]

thus \( a^{3/2} H / \Omega_e \sim 10^{-9} \). While, it also estimates \( a^{3/2} \lambda_e H \sim 1 \). This estimation indicates that the electric field entering Hubble horizon damps out immediately by Landau damping effect. So the primordial electric fields give not any effects on the power spectra of cosmic microwave background.

### 4. Soliton of electric fields inside plasma before decoupling

In this section, let’s discuss the nonlinear effect in the primordial plasma.

#### 4.1. Basic equations via moment method

Back to the Vlasov equation appeared in Eq. (11) in terms of derivative to cosmic time \( t \) instead of proper time in Sec. Define two variables

\[
n_e \equiv \int \frac{d^3 p}{(2\pi)^3} f_e. \tag{76}
\]

and

\[
n_\phi \equiv \int \frac{d^3 p}{(2\pi)^3} p \hat{p}^\phi f_e. \tag{77}
\]

Multiply \( d^3 p / (2\pi)^3 \) on both sides of Eq. (11) and integrate over all momentum space,

\[
\frac{\partial}{\partial t} \int \frac{d^3 p}{(2\pi)^3} f_e + \frac{1}{a} \frac{\partial}{\partial x} \int \frac{d^3 p}{(2\pi)^3} \frac{p \hat{p}^\phi}{E} f_e
\]

\[
- \left[ H + \Phi \right] \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{E} \frac{\partial}{\partial x} \frac{p \hat{p}^\phi}{E} f_e
\]

\[
+ \frac{e_0}{a} \frac{\partial}{\partial \phi} \int \frac{d^3 p}{(2\pi)^3} \frac{p \hat{p}^\phi}{E} f_e = 0, \tag{78}
\]
The third term on the left hand integrates by part

\[
\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E} \frac{\partial f_e}{\partial E} = \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_e}{\partial p} p = -3 \frac{4\pi}{(2\pi)^3} \int_0^{\infty} p^2 f_e dp = -3n_e,
\]  

(79)

where the relation \((p/E)(\partial f_e/\partial E) = \partial f_e/\partial p\) has been used in first equality. Because electrons during the matter dominated epoch are nonrelativistic particles, \(v'\) is regarded as a first order variable. Inserting Eqs. (76), (77) and (79) into Eq. (78) and neglecting the terms higher than first order, it reduces to

\[
\frac{\partial n_e}{\partial t} + 3[H + \Phi] n_e + \frac{1}{a} \partial_t (n_e v') = 0.
\]  

(80)

This is called the zero order moment equation.

Multiplying \(\frac{d^3p}{(2\pi)^3} \frac{p^j}{E}\) on both sides of Eq. (11) and integrating over \(p\), this gives the first order moment equation

\[
\frac{\partial}{\partial t} \int \frac{d^3p}{(2\pi)^3} \frac{p^j}{E} f_e + \frac{1}{a} \int \frac{d^3p}{(2\pi)^3} \frac{p^j}{E^2} f_e 
- [H + \Phi] \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_e}{\partial E} \frac{p^j}{E} + \frac{1}{a} \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_e}{\partial x_i} \frac{p^j}{E} 
+ e_0 \frac{1}{a} \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_e}{\partial x_i} \frac{p^j p^l}{E^2} = 0.
\]  

(81)

The second term in Eq. (81) reexpresses as

\[
\int \frac{d^3p}{(2\pi)^3} \frac{p^j \hat{p}^l}{E^2} f_e = \int \frac{d^3p}{(2\pi)^3} \frac{f_e}{E} \left( \frac{p^l}{E} - v^l \right) \left( \frac{p^j}{E} - v^j \right) + \int \frac{d^3p}{(2\pi)^3} f_e v^j v^l = P^{ij} + n_e v^j v^l,
\]  

(82)

where \(P^{ij}\) is called pressure tensor which corresponds to the anisotropy. The third term equals approximately to \(-4\partial_t (n_e v^l)/a\) by making use of the relation \((p/E)(\partial f_e/\partial E) = \partial f_e/\partial p\) and the fourth term equals to \(n_e \partial_j \Psi /a\). The last integration is calculated by part explicitly as previous integrations. This integral is

\[
\int \frac{d^3p}{(2\pi)^3} \frac{p^l \hat{p}^j}{E^2} \frac{\partial f_e}{\partial E} = \int d\Omega \hat{p} \hat{p}^j \int_0^{\infty} \frac{\partial f_e}{\partial p} \frac{p}{E} dp = - \int d\Omega \hat{p} \hat{p}^j \int_0^{\infty} dp \left( \frac{3p^2}{E^3} - \frac{p^4}{E^3} \right) = -3 \delta^{ij} \frac{n_e}{m},
\]  

(83)

where the additional fact is applied

\[
\int d\Omega \hat{p} \hat{p}^j = \delta^{ij} \frac{4\pi}{3}
\]  

(84)
Therefore, with the discussions above, Eq. (81) becomes

\[
\frac{\partial(n_e \nu_j)}{\partial t} + \frac{1}{a} \partial_j (n_e \nu^j) + \frac{1}{a} \partial_j p^j + 4H n_e \nu_j \\
+ \frac{n_e}{a} \frac{\partial \Psi}{\partial \chi^j} + \frac{1}{a} \frac{e_0}{m} \frac{\partial \phi}{\partial \chi^j} = 0.
\]  

(85)

Regroup the first two terms in the equation above

\[
\frac{\partial(n_e \nu_j)}{\partial t} + \frac{1}{a} \frac{\partial(n_e \nu^j)}{\partial x^i} = \left[ \frac{\partial n_e}{\partial t} + \frac{1}{a} \frac{\partial(n_e \nu^j)}{\partial t} \right] \nu^j + n_e \left[ \frac{\partial \nu^j}{\partial t} + \frac{1}{a} \frac{\partial \nu^j v^j}{\partial x^i} \right].
\]  

(86)

Instituting Eq. (80) and (86) into Eq. (85) and neglecting the terms higher than first order but retaining the term \(\nu^j \partial_j \nu^j\), it obtains the evolutionary equation of velocity

\[
\ddot{\nu}^j + H \nu^j + \frac{1}{a} \nu^j \partial_j \nu^j + \frac{1}{a} \partial_j \Psi - \frac{1}{a} \frac{e_0}{m} \frac{\partial \phi}{\partial x^j} = 0.
\]  

(87)

Note that the terms containing \(\nu^j \partial_j \nu^j\) is retained since the following computations will make use of it. Eq. (87) is also able to describe a proton’s velocity equation just by exchanging \(-e_0/m\) to \(e_0/M\), with, of course, \(M\) denoting the mass of photon.

### 4.2. Nonlinear equations for electric fields

Electrons stay on an equilibrium state with photons and they move evidently much faster than protons, so electrons obey the Maxwell distribution

\[
N_e = N_{e0} \exp \left( \frac{e_0 \Phi}{k_B T_e} \right)
\]  

(88)

with

\[
N_{e0} = \int n_e(x, v, t) d^3v.
\]  

(89)

Thus we have the following equations

\[
\begin{align*}
\nabla^2 \phi &= 4\pi e_0 \left[ N_{e0} \exp \left( \frac{e_0 \Phi}{k_B T_e} \right) - N_p \right], \\
\frac{\partial N_p}{\partial t} + \frac{1}{a} \frac{\partial (N_p \nu^j_p)}{\partial x^i} + 3[H + \Phi] N_p &= 0, \\
\frac{\partial \nu^j_p}{\partial t} + H \nu^j_p + \frac{1}{a} \partial_j \Psi + \nu^i_p \partial_i \nu^j_p &= -\frac{1}{a} \frac{e_0}{M} \frac{\partial \phi}{\partial x^j}.
\end{align*}
\]  

(90a, 90b, 90c)
To nondimensionalize the equations, it’s necessary to define such parameters as below:

\[ \tilde{\phi} = \frac{e_0}{k_B T_e}, \quad \tilde{N}_e = a^3 N_e, \quad \tilde{N}_p = a^3 N_p, \quad \tilde{T} = a T, \quad \tilde{\Omega}_p = \frac{4 \pi e_0^2 N_e}{M}, \quad \tilde{i} = \Omega_p, \]

\[ \tilde{\lambda}^{-1} = \sqrt{\frac{4 \pi e_0^2 N_e}{k_B T_e}}, \quad \tilde{\lambda} = \frac{x}{\lambda}, \quad \tilde{\Omega}_p = \sqrt{\frac{4 \pi e_0^2 N_e}{M}}, \quad \tilde{\Omega} = \Omega_p, \]

\[ \tilde{u} = \frac{v_p}{\sqrt{k_B T_e/M}} \quad \text{and} \quad d\tilde{\tau} = d\tilde{t}/a. \]

(91)

Multiply \( a^2/k_B T_e \), Eq. (90a) reduces to

\[ \frac{\partial^2 \tilde{e}_0 \phi}{\partial x^2} + \frac{4 \pi e_0^2 N_e}{k_B T_e} \exp \left( \frac{e_0 \phi}{k_B T_e} \right) = \frac{\tilde{N}_p}{N_e}. \]

(92)

Inserting relevant parameters in Eq. (91), it leads to a nondimensional equation

\[ \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} = \tilde{\phi} - \tilde{N}. \]

(93)

Multiply \( a\tilde{\lambda}/(\tilde{N}_e \sqrt{k_B T_e}) \) on both sides of Eq. (90b):

\[ \frac{\tilde{\lambda} a}{\sqrt{k_B T_e/M}} \frac{\partial}{\partial \tilde{t}} \left( \frac{\tilde{N}_p}{\tilde{N}_e} \right) + \frac{\tilde{\lambda}}{a} \frac{\partial}{\partial \tilde{x}} \left( \frac{\tilde{N}_p}{\tilde{N}_e} \right) \frac{\tilde{v}_p}{\sqrt{k_B T_e/M}} = 0, \]

(94)

which leads to

\[ \frac{\partial \tilde{N}}{\partial \tilde{t}} + \frac{1}{a} \frac{\partial}{\partial \tilde{x}} \left( \tilde{N}_p \right) = 0. \]

(95)

Similarly, Eq. (90c) becomes

\[ \frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\tilde{u}}{a} \frac{\partial \tilde{u}}{\partial \tilde{x}} = -\frac{\partial \tilde{\phi}}{\partial \tilde{x}}. \]

(96)

4.3. Nonlinear solution

The electrical neutrality condition provides \( \tilde{N} = 1 + \tilde{n} \) with \( \tilde{n} \ll 1 \). The linearized equations to Eq. (93), (95) and (96) read

\[ \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} = \tilde{\phi} - \tilde{n}, \]

(97a)

\[ \frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{1}{a} \frac{\partial \tilde{u}}{\partial \tilde{x}} = 0, \]

(97b)

\[ \frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{\phi}}{\partial \tilde{x}} = 0. \]

(97c)

Take derivative of Eq. (97a) to \( \tilde{\tau} \) and insert Eq. (97b) into it,

\[ \frac{\partial^3 \tilde{\phi}}{\partial \tilde{x}^2 \partial \tilde{t}} = \frac{\partial \tilde{\phi}}{\partial \tilde{t}} - \frac{\partial \tilde{n}}{\partial \tilde{t}} = \frac{\partial \tilde{\phi}}{\partial \tilde{t}} + \frac{1}{a} \frac{\partial \tilde{u}}{\partial \tilde{x}}. \]

(98)
Another derivative of Eq. (98) to \( \tilde{\tau} \) gives

\[
\frac{\partial^4 \tilde{\phi}}{\partial \tilde{x}^4 \partial \tilde{\tau}^2} = \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x} \partial \tilde{\tau}^2}
\]

\[
= \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} - \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{\tau}^2} - \frac{\partial a^{-1} \partial \tilde{u}}{\partial \tilde{\tau}}.
\]

(99)

Since the Langmuir frequency \( \tilde{\Omega}_p \) is much larger than the Hubble expansion rate \( H \), i.e. \( \tilde{\Omega}_p \gg H \), as discussed in Sec. 3.5.

\[
\frac{da^{-1}}{d\tilde{\tau}} = \frac{-1}{a^2} \frac{da}{dt} \frac{dt}{d\tilde{\tau}} = -\frac{H}{\tilde{\Omega}_p} \ll 1.
\]

(100)

Thus we finally obtain the nonlinear partial differential equation

\[
\frac{\partial^2 \tilde{\phi}}{\partial \tilde{\tau}^2} - \frac{1}{a} \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} = \frac{\partial^4 \tilde{\phi}}{\partial \tilde{x}^4 \partial \tilde{\tau}^2}.
\]

(101)

This equation describes a solitary wave meaning its profile would not change as it propagates. Setting \( \tilde{\phi} \propto e^{i [k \tilde{x} - \int \tilde{\omega}(\tilde{\tau}) d\tilde{\tau}]} \)

(102)

with \( k = k_e \) and \( \tilde{\omega} = \omega/\tilde{\Omega}_p \), inserting which into Eq. (101), we get the dispersion relation

\[
\omega^2 = \frac{\tilde{\Omega}_p^2}{a} \frac{k^2 \lambda_e^2}{1 + k^2 \lambda_e^2}
\]

(103)

by applying the limit condition \( H \ll (d\omega/d\tau)/\omega \ll \tilde{\Omega}_p \). If consider only the linear part of Eq. (101) and return to natural unit, we have

\[
\frac{\partial^2 \tilde{\phi}}{\partial \tilde{\tau}^2} - \frac{k_B \tilde{T}_e}{aM} \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} = 0.
\]

(104)

Define the average thermal velocity

\[
c_s = \sqrt{k_B \tilde{T}_e/M}.
\]

(105)

which also stands for the phase velocity \( c_s = a^{-1/2} \lambda_e \Omega_p \). But note that \( c_s \) is neither the thermal velocity of electrons nor protons, instead it’s the collective group velocity of them since they are coupling tightly with each other before recombination. So solitary equation (101) propagates with velocity \( c_s \) approximately.

Now give further analysis on Eq. (101). Introduce the Mach number which represents the ratio of actual velocity to phase velocity

\[
M_a = \frac{\frac{dx}{d\tau}}{a^{1/2} c_s} = \frac{dx}{d\tau} \frac{a}{\lambda_e \Omega_p} \propto a^{1/2} \tilde{\tau}.
\]

(106)

\( \tilde{\phi}, \tilde{N} \) and \( \tilde{u} \) only dependent on a combination variable

\[
\tilde{\eta} = \bar{x} - \int_{\tau_0}^\tau M_a d\tilde{\tau}.'
\]

(107)
Then equations in (1) becomes a new set of dynamic equations in terms of $\eta$:

\[
\begin{cases}
\frac{d^2 \tilde{\phi}}{d\tilde{\eta}^2} = e^\tilde{\phi} - \tilde{N}, \\
-M_a \frac{d\tilde{N}}{d\tilde{\eta}} + \frac{1}{a} \frac{d(\tilde{N}\tilde{u})}{d\tilde{\eta}} = 0, \\
-M_a \frac{d\tilde{u}}{d\tilde{\eta}} + \frac{\tilde{u} d\tilde{u}}{a d\tilde{\eta}} = -\frac{d\tilde{\phi}}{d\tilde{\eta}},
\end{cases}
\]  
(108a)

(108b)

(108c)

with boundary condition $\tilde{\phi} \to 0$, $\frac{d\tilde{\phi}}{d\tilde{\eta}} \to 0$, $\tilde{u} \to 0$, $\tilde{N} \to 1$ and $\frac{d\tilde{N}}{d\tilde{\eta}} \to 0$ when $|x| \to \infty$, $|\tilde{\eta}| \to \infty$.

Integrate Eq. (108b) over $\tilde{\eta}$ from $\tilde{\eta}$ to $\infty$

\[-M_a \int_{\tilde{\eta}}^{+\infty} \frac{d\tilde{N}}{d\tilde{\eta}'} d\tilde{\eta}' = \frac{1}{a} \int_{\tilde{\eta}}^{+\infty} \frac{d(\tilde{N}\tilde{u})}{d\tilde{\eta}'} d\tilde{\eta}'
\]  
(109)

Integrating on Eq. (108c), we have

\[
\left(M_a a^\frac{1}{2} - a^\frac{1}{2} \tilde{u}\right)^2 = M_a^2 a - 2\tilde{\phi}.
\]  
(110)

Integrating on Eq. (108c), we have

\[
\left(M_a a^\frac{1}{2} - a^\frac{1}{2} \tilde{u}\right)^2 = M_a^2 a - 2\tilde{\phi}.
\]  
(111)

Instituting Eq. (110) into the equation above, it arrives

\[
\tilde{N} = \frac{M_a a^\frac{1}{2}}{\sqrt{(M_a a^\frac{1}{2})^2 - 2\tilde{\phi}}}
\]  
(112)

Then instituting this equation into Eq. (108a), there is

\[
\frac{d^2 \tilde{\phi}}{d\tilde{\eta}^2} = e^{\tilde{\phi}} - \frac{M_a a^\frac{1}{2}}{\sqrt{(M_a a^\frac{1}{2})^2 - 2\tilde{\phi}}}
\]  
(113)

Next multiply $\frac{d\tilde{\phi}}{d\tilde{\eta}}$ and integral over $\tilde{\eta}$

\[
\frac{1}{2} \left(\frac{d\tilde{\phi}}{d\tilde{\eta}}\right)^2 = e^{\tilde{\phi}} + M_a a^\frac{1}{2} \sqrt{(M_a a^\frac{1}{2})^2 - 2\tilde{\phi}} + \left(M_a a^\frac{1}{2} + 1\right).
\]  
(114)

As discussed previous, $M_a a^\frac{1}{2}$ is number or variable that deviates slightly from the unit, so define $\epsilon \equiv M_a a^\frac{1}{2} - 1 \ll 1$. Thus Eq. (114) rewrites

\[
\left(\frac{d\tilde{\phi}}{d\tilde{\eta}}\right)^2 = \frac{2}{3} \tilde{\phi}^2 (3\epsilon - \tilde{\phi}).
\]  
(115)
The solution to Eq. (115) is finally expressed as

\[
\tilde{\phi} = 3\epsilon \sech^{2} \left( \frac{\epsilon}{2} \left( \tilde{x} - \int_{\tilde{\tau}}^{\tau} M_{a} d\tilde{\tau} \right) \right)
\]

\[
\approx 3\epsilon \sech^{2} \left( \frac{\epsilon}{2} \left( \tilde{x} - \int_{\tilde{\tau}}^{\tau} c_{s} d\tilde{\tau} \right) \right),
\]

(116)

where \( \sech(x) \) is the hyperbolic secant function.

4.4. Numerical Simulation

As the calculations in Sec. 3, the electric characteristic length evaluates \( \tilde{\lambda}^{-1} = 7.637 \times 10^{-5} \text{ Mpc}^{-1} \). If the major contributions are radiation and matters, the conformal time is briefly expressed

\[
\tau = \frac{2}{\sqrt{\Omega_{m} H_{0}^{2}}} \left[ \sqrt{a + a_{eq}} - \sqrt{a_{eq}} \right].
\]

(117)

Applying the recent Planck data, relevant cosmic data are best fitted as \( a_{e} = 1/1090, a_{eq} = 1/3400 \), where \( a_{eq} \) and \( a_{e} \) denote the scale factor at the present of radiation and matter equivalent and recombination. We have \( c\tau_{e} = 280.8 \text{ Mpc} \) and \( c\tau_{eq} = 112.9 \text{ Mpc} \) and \( \tau_{eq}/\tau_{e} = 0.4021 \). Then get the expression for scale factor as a function of conformal time via the inverse transform on Eq. (117)

\[
a = \left[ \frac{\tau \sqrt{\Omega_{m} H_{0}^{2}}}{2} + \sqrt{a_{eq}} \right]^{2} - a_{eq}
\]

(118)

with \( \eta = \tau/\tau_{e} \). Thus, the formula in hyperbolic secant function becomes

\[
\tilde{x} - \int_{\tilde{\tau_{eq}}}^{\tilde{\tau}} M_{a} d\tilde{\tau} = \tilde{x} - \int_{\tilde{\tau_{eq}}}^{\tilde{\tau}} (1 + \epsilon)c_{s} d\tilde{\tau}
\]

\[
\approx \tilde{\lambda}^{-1}_{e} \left[ x - (1 + \epsilon)c_{s} \sqrt{\frac{k_{B} T_{0}}{M c^{2}}} \int_{\tau_{eq}}^{\tau} a^{-1/2}(\tau/\tau_{s}) d\tau \right]
\]

(119)

\[
\approx \tilde{\lambda}^{-1}_{e} \left[ x - (1 + \epsilon)c_{s} \sqrt{\frac{k_{B} T_{0}}{M c^{2}}} \int_{\tau_{eq}/\tau_{s}}^{\eta} a^{-1/2}(\eta) d\eta \right]
\]

Fig. 1 and Fig. 2 plot the solitary evolutions with \( \epsilon = 0.01 \) and \( \epsilon = 0.005 \). The differences between them appear only the amplitude and width of the profile. The width of solitons’ profile are around \( D \sim 0.01 \text{ Mpc} \), which exactly approximate to the galaxies’ diameters. So this may indicate the primordial electric fields after recombination promote the formation of galaxies.

Before the ending of this section, two points need to be explained. First, in Sec. 3 the primordial electric fields damp away by Landau damping, but in this section, the fields propagate as solitons instead of attenuation. Are they paradoxical? Of course not. The former propagating as oscillating waves, or transverse wave precisely, these waves absorb by plasma during propagating. While the latter being a solitary wave, or longitudinal wave precisely, these waves remain the stability during their propagations, which is an intrinsic property inside the plasma matters. Second, this phenomenon successfully
Figure 1: The profiles of solitary waves in Eq. (116) with $\epsilon = 0.01$ at the moments $\tau/\tau_* = 0.5, 0.6, 0.7, 0.8, 0.9$ and 1.0 from left to right respectively.

Figure 2: The profiles of solitary waves with $\epsilon = 0.005$. 
explains the origin of initial electromagnetic fields which are often regarded as the necessary condition of galaxies’ generation and evolution [3].

5. Acoustic Oscillations

Before the recombination epoch, the electrons and photons coupled tight with each other caused by Compton scattering, so the free path is much shorter than the Hubble horizon.

The unintegrated equations for proton and electron read

\[
\frac{df_e(x, q, t)}{dt} = \langle C_{ep} \rangle_{QQ'} q' + \langle C_{e\gamma} \rangle_{pp'} q', \tag{120}
\]

\[
\frac{df_p(x, Q, t)}{dt} = \langle C_{ep} \rangle_{qq'} Q' + \langle C_{p\gamma} \rangle_{pp'} Q', \tag{121}
\]

where \(C_{ep}\) denotes the Coulomb scattering, \(C_{e\gamma}\) represents the Compton scattering between electron and photon which is inversely proportional to \(m_e^2\), and \(\langle \cdots \rangle_{QQ'}\) means the integral over momentum \(Q, Q', q, q'\). Multiplying \(\frac{d^3p}{(2\pi)^2} \frac{m_p \hat{p}}{E}\) on both sides of Eq. (120) and \(\frac{d^3p}{(2\pi)^2} \frac{m_p \hat{p}}{E}\) on both sides of Eq. (121), and integrating on momentum \(q\), it finally arrives at the baryonic velocity equation in Fourier space:

\[
v'_b + Hv_b' + ik\Psi - i\frac{e_0}{m_e} k\phi = \eta' \frac{4\rho_r}{3\rho_b} [3i\Theta_1 + v_b], \tag{122}
\]

where \(v_b\) denotes the baryonic velocity, \(\Theta_1\) denotes the dipole moment of perturbed photon’s temperature, \(\eta\) means the optic depth and prime ‘ is the derivative to conformal time \(\tau\). The Boltamann equations of photon for first two momentum give

\[
\Theta_0' + k\Theta_0 = -\Theta', \tag{123}
\]

\[
\Theta_1' - \frac{k}{3}\Theta_0 = \frac{k\Psi}{3} + i \left[ \Theta_1 - \frac{iv_b}{3} \right]. \tag{124}
\]

Insert Eq. (122) into itself and remain until the first order about baryonic velocity,

\[
v_b = -3i\Theta_1 + R \left( \frac{R}{\eta'} \left[ v'_b + \frac{a'}{a} v_b + ik\Psi - i\frac{e_0}{m_e} k\phi \right] \right) \tag{125}
\]

with \(R \equiv 3\rho_r^{(0)}/4\rho_t^{(0)}\). Then Eq. (124) becomes by inserting Eq. (125)

\[
\Theta_1 - \frac{k}{3}\Theta_0 = \frac{k\Psi}{3} + R \left( -\Theta_1 - \frac{a'}{a} \Theta_1 + \frac{1}{3} k\Psi - \frac{1}{3} \frac{e_0}{m_e} k\phi \right). \tag{126}
\]

It could be rewritten as

\[
\Theta_1' + \frac{a'}{a} R \Theta_1 - \frac{k}{3(1+R)} \Theta_0 = \frac{k}{3} \Psi - \frac{k}{3} \frac{e_0}{m_e} \phi. \tag{127}
\]

Eliminating \(\Theta_1\) by inserting Eq. (123) into Eq. (127), we finally get the photons coupled with gravity containing the
electric fields

\[
\left( \frac{d^2}{d\tau^2} + \frac{R'}{1+R} \frac{d}{d\tau} + k^2 c_s^2 \right) (\Theta_0 + \Phi) = -\frac{k^2}{3} \Psi + \frac{k^2}{3(1+R)} (\Phi + \dot{\phi}),
\]

(128)

with

\[ c_s = \sqrt{\frac{1}{3(1+R)}} \]

(129)

and

\[ \dot{\phi} = \frac{e_0 \phi}{m_e c^2}. \]

(130)

If neglecting the damping term and considering only the oscillating term, the two homogeneous solutions are

\[ s_1(k, \tau) = \sin[kr_s(\tau)], \text{ and } s_2(k, \tau) = \cos[kr_s(\tau)], \]

(131)

where the sound horizon is defined as

\[ r_s(\tau) = \int_{\tau_0}^{\tau} d\tau' c_s(\tau'). \]

(132)

According to the Green’s function theory, the photons’ temperature coupled with gravity can be constructed via the homogeneous solutions in Eq. (131)

\[
\Theta_0 + \Phi = c_1 s_1(\tau) + c_2 s_2(\tau)
+ \frac{k^2}{3} \int_{\tau_0}^{\tau} \left[ \Phi + \dot{\Phi} - \Psi \right] s_1(\tau') s_2(\tau) - s_1(\tau) s_2(\tau') \, d\tau'
+ \frac{k}{\sqrt{3}} \int_{\tau_0}^{\tau} \left[ \Phi + \dot{\Phi} - \Psi \right] (\tau') \sin[k(r_s(\tau) - r_s(\tau'))] \, d\tau'.
\]

(133)

Compared with the condition in absence of electric field, Eq. (133) does not show any differences except a phase in terms of a sine integral

\[ \Delta \phi = \frac{k}{\sqrt{3}} \int_{\tau_0}^{\tau} \dot{\phi}(\tau') \sin[k(r_s(\tau) - r_s(\tau'))] \, d\tau'. \]

(134)

As discussed in Sec. 3 and Sec. 4, the primordial electric fields damp out since Landau damping, so no need to calculate the influence on the acoustic oscillations. However the intrinsic electric fields in plasma propagates as solitary waves, so it needs to evaluate the amplitude of this contribution. \( \dot{\phi} \) has been achieved in Eq. (116), the temperature before recombination approximates \( k_B T \sim 0.1eV \), the energy of rest electron \( m_e c^2 \sim 0.5MeV \) and the amplitude of soliton \( e \sim 0.01 \). Thus the additional term’s amplitude approximates \( \dot{\phi} \sim 10^{-8} \). While the gravitational perturbation is observed at order \( \Phi, \Psi \sim 10^{-5} \). So the electric fields merely increase the amplitude of acoustic oscillations by a order of \( 10^{-3} \), which means the CMB’s power spectra almost change nothing at acoustic peak. Besides, the electrical acoustic wave,
Figure 3: The monopole at recombination in a standard CDM model with $\epsilon = 0.01$ and $\epsilon = 100$.

appeared in Eq. (105), propagates much slower than the gravitational acoustic wave, appeared in Eq. (129), the ratio between electrical acoustic horizon and gravitational acoustic horizon is

$$r = \frac{r^{(e)}}{r^{(g)}} = \frac{\int_0^\tau \epsilon_s^{(e)} d\tau'}{\int_0^\tau \epsilon_s^{(g)} d\tau'} \sim \sqrt{\frac{k_B T_0}{M c^2}} \frac{\Omega_b h^2}{\Omega_r h^2} \sim 10^{-4}. \quad (135)$$

This means the electrical acoustic peaks in CMB’s power spectra locate at extremely large $\ell$’s, which roughly corresponds to the scale of galaxies or galaxy cluster, consistent to the discussions in Sec. 4.

The simulation on Eq. (128) needs a further treatment on solitary solution in (116). Multiplying $\tau_*$ on both sides of Eq. (128) and defining $\eta = \tau/\tau_*$, then it rewrites

$$\left[ \frac{d^2}{d\eta^2} + \frac{dR/d\eta}{1 + R} \frac{d}{d\eta} + \left( \frac{\tau_*}{\tau_0} \right)^2 \frac{(k \tau_0)^2 c_s^2}{3} \right] (\Theta_0 + \Phi) =$$

$$- \left( \frac{\tau_*}{\tau_0} \right)^2 \frac{2 (k \tau_0)^2}{3(1 + R)} (\Phi + \bar{\phi}), \quad (136)$$

here $\tau_0 = 14161.5$ Mpc denoting the conformal time at present time. Based on the Fourier transform

$$\mathcal{F}[\text{sech}(x)] = k \pi \text{sech} \left( \frac{k \pi}{2} \right), \quad (137)$$

the transformation on $e_0 \bar{\phi}/mc^2$ reads

$$\mathcal{F}[\bar{\phi}] = \frac{k_B T_0}{mc^2 a} \frac{3(2\epsilon)^{1/2} \lambda_e}{c^2 t_0} \times \exp \left( \int \sqrt{\frac{E_c t_*}{2}} \frac{k_B T_0}{M c^2} \int_0^\eta a^{-1} \sqrt{\epsilon_s} d\eta' \right) \times \pi k c t_0 \text{sech} \left( \sqrt{\frac{2 \pi \lambda_e}{\epsilon^2 c t_0}} \right), \quad (138)$$
where $\text{scsh}(x)$ is the hyperbolic cosecant function. The Bardeen potentials follow the convenient fits [45]:

$$
\Phi(k, y) = \Phi \left[ (1 - T(k)) \exp[-0.11(ky/k_{eq})^{1.6}] + T(k) \right], \tag{139}
$$

$$
\Psi(k, y) = \Psi \left[ (1 - T(k)) \exp[-0.097(ky/k_{eq})^{1.6}] + T(k) \right], \tag{140}
$$

where $T(k)$ is the BBKS transfer function and $y \equiv a/a_{eq}$. Inserting the relevant equations into Eq. (136), then the numerical results are simulated in Fig. 3. The black solid line represents the exact solution with $\varepsilon = 0.01$ which almost totally coincides with the line of $\varepsilon = 0$. While, the red dashed line represents the condition with $\varepsilon = 100$. This condition goes against the assumption $\varepsilon \ll 1$, but it has already illustrated that the electric potential contributes a phase on the CMB’s power spectrum.

This section could be included in a brief statement: the CMB’s power spectra could merely distinguish the contributions from primordial electric fields.

### 6. Conclusions and further discussions

In this paper, three problems are discussed: the damping of primordial electric fields, electric solitons and their effects on acoustic oscillations. As calculated in relevant cosmological monographs [39], the effects on CMB from electromagnetic fields are often ignored, although, in plasma, such effects always play extremely significant roles.

First, the evolutions of primordial electric fields are semi-precisely calculated, but the results are similarly with the ones in plasma physics, which show the primordial fields dissipate out no matter at sub-horizon scale or at super-horizon scale by Landau damping effect, especially the fields having entered the electric acoustic horizon. It means the effects of neutrino’s recombination don’t need take into account since the initial perturbations have already damped out till the electron’s recombination. In a word, there is no need to consider the electromagnetic initial conditions but just only consider the gravitational initial conditions from inflation. Second, I proof that the electric fields propagate as solitary waves instead of oscillating waves and their speed is much slower than baryonic acoustic oscillations. And third, it illustrates the electric solitons merely affect the shape of CMB’s power spectra. But these solitons continue to exist after recombination and help generate the first generations of galaxies or galaxy cluster.

This paper illustrates the origin of electromagnetic fields after recombination, and offers an effective method to compute the electric influences on CMB’s power spectra. In Eq. (8), the mechanic equation containing only the electric fields is obtained by setting the component $\mu = 0$. Similarly, the equation including the electric and magnetic fields could be obtained by setting $\mu = i$. It’s not hard to imagine the Vlasov equation of the second moment will introduce the higher order tensors, like magnetic fields, anisotropic tensor (or pressure tensor) and tensor perturbations. Proper cutoff on higher moment will simplify the calculations. But the computations will be quite complex and relevant results must be attractive.

### Acknowledgment

This project is supported by the National Natural Science Foundation of China (Grants No. 11864030) and Scientific Research Funding Project for Introduced High Level Talents of IMNU (Grants No. 2020YJRC001). I also thank my undergraduate students Jian-Kia Yin, Jia-Xin Ma, Jia-Xian Liu and Lu-Jie Shi help me input the formulas.
A Integral of Cauchy’s type

$F(x)$ defined in Eq. (55) could analytically calculated under two special conditions. The first is high frequency limit, i.e. $x \gg 1$, then it approximates

$$F(x) = \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-z^2}}{z-x-i0} dz$$

$$= P. V. \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-z^2}}{z-x} dz + \frac{x}{\sqrt{\pi}} i \pi e^{-x^2}$$

$$= P. V. \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-z^2}}{x-i} dz + i \sqrt{\pi} x e^{-x^2}$$

$$\approx -P. V. \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dz \ e^{-z^2} \sum_{n=0}^{\infty} \left( \frac{z}{x} \right)^n + i \sqrt{\pi} x e^{-x^2}$$

$$\approx -1 - \frac{1}{2x^2} - \frac{3}{4x^4} + i \sqrt{\pi} x e^{-x^2}, \quad (141)$$

where P. V. means the Cauchy’s principal value [46], and it has been employed two Gaussian integrals [47]:

$$\int_{-\infty}^{\infty} x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{(2p)^n} \sqrt{\frac{\pi}{p}}, \quad (p > 0, \ n = 0, 1, \cdots) \quad (142)$$

and

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-px^2} dx = 0, \quad (p > 1, \ n = -1, 0, 1, \cdots). \quad (143)$$

The low frequency limit condition is evaluated as

$$F(x) = \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-z^2}}{z-x-i0} dz$$

$$= P. V. \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} dz \ \frac{e^{-z^2}}{z} \left[ 1 - 2zx + \frac{1}{2} (4z^2 - 2)x^2 - \frac{1}{3!} (8z^3 + 12z^2)x^3 + \cdots \right] + i \sqrt{\pi} x$$

$$\approx -2x^2 + \frac{4}{3} x^4 + i \sqrt{\pi} x, \quad (144)$$

where, of course, Eq. (142) and (143) have also been applied.

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