WEIGHTED SUM FORMULAS OF MULTIPLE ZETA VALUES WITH EVEN ARGUMENTS

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ABSTRACT. We prove a weighted sum formula of the zeta values at even arguments, and a weighted sum formula of the multiple zeta values with even arguments and its zeta-star analogue. The weight coefficients are given by (symmetric) polynomials of the arguments. These weighted sum formulas for the zeta values and for the multiple zeta values were conjectured by L. Guo, P. Lei and J. Zhao.

1. Introduction

For a positive integer $n$ and a sequence $k = (k_1, \ldots, k_n)$ of positive integers with $k_1 > 1$, the multiple zeta value $\zeta(k)$ and the multiple zeta-star value $\zeta^*(k)$ are defined by the following infinite series

$$\zeta(k) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

and

$$\zeta^*(k) = \zeta^*(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

respectively. The number $n$ is called the depth. In depth one case, both $\zeta(k)$ and $\zeta^*(k)$ are special values of the Riemann zeta function at positive integer arguments. The study of these values may be traced back to L. Euler. Among other things, L. Euler found the following sum formula

$$\sum_{i=2}^{k-1} \zeta^*(i, k-i) = (k-1)\zeta(k), \quad k \geq 3,$$

or equivalently,

$$\sum_{i=2}^{k-1} \zeta(i, k-i) = \zeta(k), \quad k \geq 3.$$

There are many generalizations and variations of the sum formula, among which we mention some weighted sum formulas at even arguments. In [2], the following
formula
\[ \sum_{i=1}^{k-1} \zeta(2i, 2k - 2i) = \frac{3}{4} \zeta(2k) \]
was proved by using the regularized double shuffle relations of the double zeta values. M. E. Hoffman considered the sum
\[ \sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} \zeta(2k_1, \ldots, 2k_n) \]
in [7], and we showed in [9] that the formulas given by M. E. Hoffman in [7] are consequences of the regularized double shuffle relations of the multiple zeta values. Later in [3], new families of weighted sum formulas of the forms
\[ \sum_{k_1 + k_2 = k \atop k_j \geq 1} F(k_1, k_2) \zeta(2k_1) \zeta(2k_2), \sum_{k_1 + k_2 = k \atop k_j \geq 1} G(k_1, k_2) \zeta(2k_1) \zeta(2k_2) \zeta(2k_3) \]
and
\[ \sum_{k_1 + k_2 = k \atop k_j \geq 1} F(k_1, k_2) \zeta(2k_1, 2k_2), \sum_{k_1 + k_2 + k_3 = k \atop k_j \geq 1} G(k_1, k_2, k_3) \zeta(2k_1, 2k_2, 2k_3) \]
were given, where \( F(x, y) \) and \( G(x, y, z) \) are (symmetric) polynomials with rational coefficients. And in the end of [3], L. Guo, P. Lei and J. Zhao proposed the following general conjecture.

**Conjecture 1.1** ([3, Conjecture 4.7]). Let \( F(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n] \) be a symmetric polynomial of degree \( r \). Set \( d = \deg_x F(x_1, \ldots, x_n) \). Then for every positive integer \( k \geq n \) we have
\[ \sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta(2k_1) \cdots \zeta(2k_n) = \sum_{l=0}^{T} e_{F,l}(k) \zeta(2l) \zeta(2k - 2l), \quad (1.1) \]
\[ \sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta(2k_1, \ldots, 2k_n) = \sum_{l=0}^{T} e_{F,l}(k) \zeta(2l) \zeta(2k - 2l), \quad (1.2) \]
where \( T = \max\{[(r + n - 2)/2], [(n - 1)/2]\}, e_{F,l}(x), c_{F,l}(x) \in \mathbb{Q}[x] \) depend only on \( l \) and \( F \), \( \deg e_{F,l}(x) \leq r - 1 \) and \( \deg c_{F,l}(x) \leq d \).

Here as usual, for a real number \( x \), we denote by \( [x] \) the greatest integer that does not exceed \( x \).

The purpose of this paper is to give a proof of Conjecture 1.1. In fact, we also prove the zeta-star analogue of (1.2). To prove (1.1), as in [3], we first establish a weighted sum formula of the Bernoulli numbers. In [3], L. Guo, P. Lei and J. Zhao used certain zeta functions to study the Bernoulli numbers. Here we just use the generating function of the Bernoulli numbers. Hence our method seems more elementary. After getting the weighted sum formula of the Bernoulli numbers, we obtain the weighted sum formula (1.1) by using Euler’s evaluation formula of the zeta values at even arguments. Finally, applying the symmetric sum formulas of M. E. Hoffman [4], we obtain the weighted sum formula (1.2) and its zeta-star analogue from the formula (1.1).
The paper is organized as follows. In Section 2, we deal with the weighted sum of the Bernoulli numbers. In Section 3, we prove the weighted sum formula (1.1). And in Section 4, we prove the weighted sum formula (1.2) and its zeta-star analogue. Finally, in Section 5, we show that the weighted sum formulas obtained in this paper can be deduced from the regularized double shuffle relations of the multiple zeta values.

2. A weighted sum formula of Bernoulli numbers

The Bernoulli numbers \( \{B_i\} \) are defined by
\[
\sum_{i=0}^{\infty} \frac{B_i}{i!} t^i = \frac{t}{e^t - 1}.
\]
It is known that \( B_0 = 1 \), \( B_1 = -\frac{1}{2} \) and \( B_i = 0 \) for odd \( i \geq 3 \). We set
\[
f(t) = \frac{t}{e^t - 1} - 1 + \frac{1}{2} \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} t^{2i},
\]
\[
g(t) = \frac{t}{e^t - 1} + \frac{1}{2} \sum_{i=0}^{\infty} \frac{B_{2i}}{(2i)!} t^{2i}.
\]
We compute the derivatives of the even function \( f(t) \). Let \( D = t \frac{d}{dt} \) and \( h(t) = \frac{t}{e^t - 1} \).
Then using the formula
\[
h'(t) = \frac{1}{e^t - 1} - \frac{t}{(e^t - 1)^2},
\]
we find that for any nonnegative integer \( m \),
\[
D^m f(t) = \sum_{i=0}^{m+1} f_{mi}(t) h(t)^i. \quad (2.1)
\]
Here \( f_{mi}(t) \) are polynomials determined by \( f_{00}(t) = \frac{1}{2} t - 1 \), \( f_{01}(t) = 1 \) and the recursive formulas
\[
\begin{aligned}
f_{m0}(t) &= tf'_{m-1,0}(t) & \text{for } m \geq 1, \\
f_{m,m+1}(t) &= -mf_{m-1,m}(t) & \text{for } m \geq 1, \\
f_{mi}(t) &= tf'_{m-1,i}(t) + i(1-t)f_{m-1,i}(t) - (i-1)f_{m-1,i-1}(t) & \text{for } 1 \leq i \leq m.
\end{aligned} \quad (2.2)
\]
In particular, for any integers \( m, i \) with \( 1 \leq i \leq m+1 \), we have \( f_{mi}(t) \in \mathbb{Z}[t] \). From (2.2), it is easy to see that for any nonnegative integer \( m \), we have
\[
f_{m0}(t) = \frac{1}{2} - \delta_{m,0}, \quad f_{m,m+1}(t) = (-1)^m m!.
\]

Lemma 2.1. For any integers \( m, i \) with \( 1 \leq i \leq m+1 \), we have \( \deg f_{mi}(t) = m + 1 - i \), and the leading coefficient \( c_{mi} \) of \( f_{mi}(t) \) satisfies the condition \( (-1)^m c_{mi} > 0 \).

Proof. We use induction on \( m \). Assume that \( m \geq 1 \). The result for \( i = m + 1 \) follows from \( f_{m,m+1}(t) = (-1)^m m! \). Now assume the integer \( i \) satisfies the condition \( 1 \leq i < m \), and
\[
f_{m-1,i}(t) = c_{m-1,i} t^{m-1} + \text{lower degree terms}
\]
with \((-1)^{m-1}c_{m-1,i} > 0\). Let \(c_{m0} = \frac{1}{2}\). Then we have

\[ f_{mi}(t) = (-ic_{m-1,i} - (i-1)c_{m-1,i-1})t^{m+1-i} + \text{lower degree terms}. \]

As

\[ (-1)^m(-ic_{m-1,i} - (i-1)c_{m-1,i-1}) = i(-1)^{m-1}c_{m-1,i} + (i-1)(-1)^{m-1}c_{m-1,i-1} > 0, \]

we get the result.

Therefore we have

\[ f_{m0}(t) = c_{m0}t - \delta_{m,0} \]

with \(c_{m0} = \frac{1}{2}\), and for any integers \(m, i\) with the condition \(1 \leq i \leq m + 1\), we have

\[ f_{mi}(t) = c_{mi}t^{m+1-i} + \text{lower degree terms}, \]

with the recursive formula

\[ c_{mi} = -ic_{m-1,i} - (i-1)c_{m-1,i-1}, \quad (1 \leq i \leq m) \]

and \(c_{m,m+1} = (-1)^m m!\).

**Corollary 2.2.** For any nonnegative integer \(m\), we have \(c_{m1} = (-1)^m\).

For later use, we need the following lemma.

**Lemma 2.3.** For any nonnegative integer \(m\), we have

\[ \sum_{i=1}^{m+1} (-1)^{i-1} f_{mi}(t) t^{i-1} = 1. \] (2.3)

In particular, we have

\[ \sum_{i=1}^{m+1} (-1)^{i-1} c_{mi} = \delta_{m,0}. \] (2.4)

**Proof.** We proceed by induction on \(m\) to prove (2.3). The case of \(m = 0\) follows from the fact \(f_{01}(t) = 1\). Now assume that \(m \geq 1\), using the recursive formula (2.2), we have

\[
\sum_{i=1}^{m+1} (-1)^{i-1} f_{mi}(t) t^{i-1} = \sum_{i=1}^{m} (-1)^{i-1} f'_{m-1,i}(t) t^{i} + \sum_{i=1}^{m} (-1)^{i-1} if_{m-1,i}(t) t^{i-1} \\
+ \sum_{i=1}^{m} (-1)^i i f_{m-1,i}(t) t^{i} + \sum_{i=1}^{m} (-1)^i (i-1)f_{m-1,i-1}(t) t^{i-1} + m!t^m \\
= \sum_{i=1}^{m} (-1)^{i-1} (f_{m-1,i}(t)t^i)' + (-1)^m mf_{m-1,m}(t)t^m + m!t^m \\
= \frac{d}{dt} \sum_{i=1}^{m} (-1)^{i-1} f_{m-1,i}(t)t^i.
\]

Then we get (2.3) from the induction assumption. Finally, comparing the coefficients of \(t^m\) of both sides of (2.3), we get (2.4).
Now we want to express \( h(t)^i \) by \( D^m g(t) \). For this purpose, we use matrix computations. For any nonnegative integer \( m \), let \( A_m(t) \) be a \((m + 1) \times (m + 1)\) matrix defined by

\[
A_m(t) = \begin{pmatrix}
  f_{01}(t) & f_{12}(t) & & \\
  f_{11}(t) & f_{12}(t) & & \\
  \vdots & \vdots & \ddots & \\
  f_{m1}(t) & f_{m2}(t) & \cdots & f_{m,m+1}(t)
\end{pmatrix}.
\]

Note that for \( m \geq 1 \), we have

\[
A_m(t) = \begin{pmatrix}
  A_{m-1}(t) & 0 \\
  \alpha_m(t) & (-1)^m m!
\end{pmatrix}
\]

with \( \alpha_m(t) = (f_{m1}(t), \ldots, f_{mm}(t)) \). From linear algebra, we know that the matrix \( \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \) is invertible with

\[
\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix},
\]

provided that \( A \) and \( B \) are invertible square matrices. Therefore by induction on \( m \), we find that for all nonnegative integer \( m \), the matrices \( A_m(t) \) are invertible, and the inverses satisfy the recursive formula

\[
A_m(t)^{-1} = \begin{pmatrix} A_{m-1}(t)^{-1} & 0 \\ (-1)^m m! \alpha_m(t)A_{m-1}(t)^{-1} & (-1)^m m! \end{pmatrix}, \quad (m \geq 1) \tag{2.5}
\]

For any nonnegative integer \( m \), set

\[
A_m(t)^{-1} = \begin{pmatrix}
  g_{01}(t) & g_{12}(t) & & \\
  g_{11}(t) & g_{12}(t) & & \\
  \vdots & \vdots & \ddots & \\
  g_{m1}(t) & g_{m2}(t) & \cdots & g_{m,m+1}(t)
\end{pmatrix}.
\]

**Lemma 2.4.** Let \( m \) and \( i \) be integers.

1. For any \( m \geq 0 \), we have \( g_{m,m+1}(t) = (-1)^m \frac{1}{m!} \); 
2. If \( 1 \leq i \leq m \), we have the recursive formula

\[
g_{mi}(t) = (-1)^{m+1} \frac{1}{m!} \sum_{j=i}^{m} f_{mj}(t)g_{j-1,i}(t); \quad (2.6)
\]

3. If \( 1 \leq i \leq m+1 \), we have \( g_{mi}(t) \in \mathbb{Q}[t] \) with \( \deg g_{mi}(t) \leq m + 1 - i \); 
4. For \( 1 \leq i \leq m + 1 \), set

\[
g_{mi}(t) = d_{mi} t^{m+1-i} + \text{lower degree terms}.
\]

Then we have \( d_{m,m+1} = (-1)^m \frac{1}{m!} \) and

\[
d_{mi} = (-1)^{m+1} \frac{1}{m!} \sum_{j=i}^{m} c_{mj} d_{j-1,i} \tag{2.7}
\]

for \( 1 \leq i \leq m \).
Corollary 2.5. For any nonnegative integer \( m \), we have \( d_{m1} = (-1)^m \).

Proof. We use induction on \( m \). If \( m \geq 1 \), using (2.7) and the induction assumption, we get
\[
d_{m1} = (-1)^{m+1} \frac{1}{m!} \sum_{j=1}^{m} (-1)^{j-1} c_{mj}.
\]

By (2.4), we have
\[
d_{m1} = (-1)^{m+1} \frac{1}{m!} (\delta_{m,0} - (-1)^m c_{m,m+1}),
\]
which implies the result. \( \square \)

To get \( h(t)^s \), we rewrite (2.1) as
\[
\begin{pmatrix}
g(t) \\
dg(t) \\
\vdots \\
d^mg(t)
\end{pmatrix} - \frac{1}{2} t
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix} = A_m(t)
\begin{pmatrix}
h(t) \\
h(t)^2 \\
\vdots \\
h(t)^{m+1}
\end{pmatrix}, \tag{2.8}
\]

and rewrite (2.3) as
\[
A_m(t)
\begin{pmatrix}
1 \\
-t \\
t^2 \\
\vdots \\
(-1)^{m} t^m
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}.
\]

Therefore we find
\[
\begin{pmatrix}
h(t) \\
h(t)^2 \\
\vdots \\
h(t)^{m+1}
\end{pmatrix} = A_m(t)^{-1}
\begin{pmatrix}
g(t) \\
dg(t) \\
\vdots \\
d^mg(t)
\end{pmatrix} - \frac{1}{2} t
\begin{pmatrix}
1 \\
-t \\
t^2 \\
\vdots \\
(-1)^{m} t^m
\end{pmatrix}.
\]
Then for any positive integer \(i\), we get
\[
h(t)^i = \sum_{j=1}^{i} g_{i-1,j}(t)D^{j-1}g(t) + \frac{1}{2}(-1)^i t^i. \tag{2.9}
\]

For the later use, we prepare a lemma.

**Lemma 2.6.** For a nonnegative integer \(m\), the functions \(1, g(t), Dg(t), \ldots, D^mg(t)\) are linearly independent over the rational function field \(Q(t)\).

**Proof.** Let \(p(t), p_0(t), p_1(t), \ldots, p_m(t) \in Q(t)\) satisfy
\[
p(t) + p_0(t)g(t) + p_1(t)Dg(t) + \cdots + p_m(t)D^mg(t) = 0.
\]
Using (2.8), we get
\[
p(t) + \frac{1}{2} t \sum_{j=0}^{m} p_j(t) + (p_0(t), \ldots, p_m(t))A_m(t) \begin{pmatrix} h(t) \\ \vdots \\ h(t)^{m+1} \end{pmatrix} = 0.
\]
Since \(e^t\) is transcendental over \(Q(t)\), we know \(e^t - 1\), and then \(h(t)\) is transcendental over \(Q(t)\). Hence we have
\[
p(t) + \frac{1}{2} t \sum_{j=0}^{m} p_j(t) = 0, \quad (p_0(t), \ldots, p_m(t))A_m(t) = 0,
\]
which implies that all \(p_j(t)\) and \(p(t)\) are zero functions as the matrix \(A_m(t)\) is invertible.

From now on let \(n\) be a fixed positive integer, and \(m_1, \ldots, m_n\) be fixed nonnegative integers. We want to compute \(D^{m_1}f(t) \cdots D^{m_n}f(t)\). On the one hand, using (2.1), we have
\[
D^{m_1}f(t) \cdots D^{m_n}f(t) = \sum_{i=0}^{m_1+\cdots+m_n+n} f_i(t)h(t)^i,
\]
with
\[
f_i(t) = \sum_{i_1 + \cdots + i_n = i \atop 0 \leq i_j \leq m_j + 1} f_{m_1i_1}(t) \cdots f_{m_ni_n}(t).
\]

**Lemma 2.7.** We have
\[
f_0(t) = \prod_{j=1}^{n} \left( \frac{1}{2} - \delta_{m_j,0} \right),
\]
and \(\text{deg } f_i(t) \leq m_1 + \cdots + m_n + n - i\) for any nonnegative integer \(i\).

**Proof.** For integers \(i_1, \ldots, i_n\) with the conditions \(i_1 + \cdots + i_n = i\) and \(0 \leq i_j \leq m_j + 1\), we have
\[
\text{deg}(f_{m_1i_1}(t) \cdots f_{m_ni_n}(t)) \leq \sum_{j=1}^{n} (m_j + 1 - i_j) = m_1 + \cdots + m_n + n - i,
\]
which deduces that \(\text{deg } f_i(t) \leq m_1 + \cdots + m_n + n - i\). \(\square\)

Then using (2.9), we get
\[
D^{m_1}f(t) \cdots D^{m_n}f(t)
\]
\[
\sum_{i=1}^{m_1+\cdots+m_n+n} f_i(t) \left( \sum_{j=1}^{i} g_{i-j}(t) D^{j-1} g(t) + \frac{1}{2}(-1)^i t^i \right) + f_0(t)
\]
\[
= \sum_{j=1}^{m_1+\cdots+m_n+n} F_j(t) D^{j-1} g(t) + F_0(t)
\]

with
\[
F_0(t) = f_0(t) + \frac{1}{2} \sum_{i=1}^{m_1+\cdots+m_n+n} (-1)^i f_i(t) t^i
\]

and
\[
F_j(t) = \sum_{i=j}^{m_1+\cdots+m_n+n} f_i(t) g_{i-j}(t), \quad (1 \leq j \leq m_1 + \cdots + m_n + n).
\]

**Lemma 2.8.** Let \( j \) be a nonnegative integer with \( j \leq m_1 + \cdots + m_n + n \). Then
1. the function \( F_j(t) \) is even;
2. we have
\[
F_0(t) = \frac{1}{2} \prod_{j=1}^{n} \left( \frac{1}{2} t - \delta_{m_j,0} \right) + \frac{1}{2} (-1)^n \prod_{j=1}^{n} \left( \frac{1}{2} t + \delta_{m_j,0} \right).
\]

In particular, \( \deg F_0(t) \leq n \);
3. if \( j > 0 \), we have \( \deg F_j(t) \leq m_1 + \cdots + m_n + n - j \). Moreover, we have \( \deg F_1(t) \leq m_1 + \cdots + m_n + n - 2 \) provided that \( n \) is even or \( m_1, \ldots, m_n \) are not all zero.

**Proof.** Since \( D^m f(t) \) and \( D^m g(t) \) are even, we have
\[
\sum_{j=1}^{m_1+\cdots+m_n+n} F_j(t) D^{j-1} g(t) + F_0(t) = \sum_{j=1}^{m_1+\cdots+m_n+n} F_j(-t) D^{j-1} g(t) + F_0(-t).
\]

Then by Lemma 2.6, we know all \( F_j(t) \) are even functions.

By the definition of \( f_i(t) \), we have
\[
\sum_{i=0}^{m_1+\cdots+m_n+n} (-1)^i f_i(t) t^i = \prod_{j=1}^{n} \sum_{i_j=0}^{m_j+1} (-1)^i f_{m_j,i_j}(t) t^{i_j}.
\]

Using (2.3), we find
\[
\sum_{i=0}^{m_1+\cdots+m_n+n} (-1)^i f_i(t) t^i = \prod_{j=1}^{n} (f_{m_j,0}(t) - t).
\]

Then we get (2) from the fact that \( f_{m_0}(t) = \frac{1}{2} t - \delta_{m,0} \) and the expression of \( f_0(t) \).

Since
\[
\deg f_i(t) g_{i-j}(t) \leq (m_1 + \cdots + m_n + n - i) + (i-j) = m_1 + \cdots + m_n + n - j,
\]
we get \( \deg F_j(t) \leq m_1 + \cdots + m_n + n - j \).

If we set
\[
\tilde{c}_{mi} = \begin{cases} 
\frac{1}{2} \delta_{m,0} & \text{if } i = 0, \\
\tilde{c}_{mi} & \text{if } i \neq 0,
\end{cases}
\]

then (1) follows from the fact that \( \tilde{c}_{mi} \) is even for \( i = 0 \) and \( \tilde{c}_{mi} \) is odd for \( i > 0 \).
then the coefficient of $t^{m+1-i}$ in $f_{mi}(t)$ is $\tilde{c}_{mi}$ for any integers $m, i$ with the condition $0 \leq i \leq m + 1$. Since

$$F_1(t) = \sum_{i=1}^{m_1+\cdots+m_n+n} \sum_{\substack{i_1+\cdots+i_n\equiv i \mod{m_j+1} \atop 0 \leq i_j \leq m_j+1}} f_{m_1i_1}(t) \cdots f_{m_ni_n}(t) g_{i-1,1}(t),$$

and $d_{i-1,1} = (-1)^{i-1}$, we find the coefficient of $t^{m_1+\cdots+m_n+n-1}$ in $F_1(t)$ is

$$\sum_{i=1}^{m_1+\cdots+m_n+n} \sum_{\substack{i_1+\cdots+i_n\equiv i \mod{m_j+1} \atop 0 \leq i_j \leq m_j+1}} (-1)^{i-1} \tilde{c}_{m_1i_1} \cdots \tilde{c}_{m_ni_n}$$

$$= \tilde{c}_{m_10} \cdots \tilde{c}_{m_00} - \prod_{j=1}^{n} \sum_{i_j=0}^{m_j+1} (-1)^{i_j} \tilde{c}_{m_ji_j},$$

which is

$$\tilde{c}_{m_10} \cdots \tilde{c}_{m_00} - \prod_{j=1}^{n} (\tilde{c}_{m_j0} - \delta_{m_j0})$$

by (2.4). Then the coefficient of $t^{m_1+\cdots+m_n+n-1}$ in $F_1(t)$ is

$$\left(\frac{1}{2}\right)^n (1 - (-1)^n) \delta_{m_10} \cdots \delta_{m_n0},$$

which is zero if $n$ is even or at least one $m_i$ is not zero. $\square$

Now for a positive integer $j$ with $j \leq m_1 + \cdots + m_n + n$, let $a_{jt} \in \mathbb{Q}$ be the coefficient of $t^{2l}$ in the polynomial $F_j(t)$. Then we have

$$F_j(t) = \sum_{l=0}^{\left[\frac{m_1+\cdots+m_n+n-j}{2}\right]} a_{jt} t^{2l}. \tag{2.10}$$

If $n$ is even or $m_1, \ldots, m_n$ are not all zero, we have

$$F_1(t) = \sum_{l=0}^{\left[\frac{m_1+\cdots+m_n+n-2}{2}\right]} a_{1lt} t^{2l}.$$

Hence we have

$$D^{m_1} f(t) \cdots D^{m_n} f(t) = \sum_{j=1}^{m_1+\cdots+m_n+n} \left[\frac{m_1+\cdots+m_n+n-j}{2}\right] \sum_{l=0}^{\left[\frac{m_1+\cdots+m_n+n-2}{2}\right]} a_{jt} t^{2l} D^{j-1}g(t) + F_0(t).$$

Changing the order of the summation, we have

$$D^{m_1} f(t) \cdots D^{m_n} f(t) = \sum_{l=0}^{T} \sum_{j=1}^{m_1+\cdots+m_n+n-2l} a_{jt} t^{2l} D^{j-1}g(t) + F_0(t),$$

where

$$T = \left\{ \begin{array}{ll}
\left[\frac{m_1}{2}\right] & \text{if } m_1 = \cdots = m_n = 0,
\left[\frac{m_1+\cdots+m_n+n-2}{2}\right] & \text{otherwise}.
\end{array} \right.$$ 

Since

$$D^{j-1} g(t) = \sum_{i=0}^{\infty} (2i)^{j-1} \frac{B_{2i}}{(2i)!} t^{2i},$$
we get
\[ D^{m_1} f(t) \cdots D^{m_n} f(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\min(T,k)} \left( \sum_{j=1}^{m_1 + \cdots + m_{n} + n - 2l} a_{j_l} (2k - 2l)^{j - 1} \right) \frac{B_{2k-2l}}{(2k-2l)!} t^{2k} + F_0(t). \]

Then the coefficient of \( t^{2k} \) in \( D^{m_1} f(t) \cdots D^{m_n} f(t) \) is
\[ \sum_{l=0}^{\min(T,k)} \left( \sum_{j=1}^{m_1 + \cdots + m_{n} + n - 2l} 2^{j-1} a_{j_l} (k - l)^{j - 1} \right) \frac{B_{2k-2l}}{(2k-2l)!}, \tag{2.11} \]

provided that \( k \geq n \).

On the other hand, since
\[ D^{m} f(t) = \sum_{i=1}^{\infty} (2i)^{m} \frac{B_{2i}}{(2i)!} t^{2i}, \]
we find the coefficient of \( t^{2k} \) in \( D^{m_1} f(t) \cdots D^{m_n} f(t) \) is
\[ \sum_{k_1 + \cdots + k_n = k} \frac{(2k_1)^{m_1} \cdots (2k_n)^{m_n} B_{2k_1} \cdots B_{2k_n}}{(2k_1)! \cdots (2k_n)!}. \tag{2.12} \]

Finally, comparing (2.11) with (2.12), we get a weighted sum formula of the Bernoulli numbers.

**Theorem 2.9.** Let \( n, k \) be positive integers with \( k \geq n \). Then for any nonnegative integers \( m_1, \ldots, m_n \), we have
\[ \sum_{k_1 + \cdots + k_n = k} \frac{k^{m_1} \cdots k^{m_n} B_{2k_1} \cdots B_{2k_n}}{(2k_1)! \cdots (2k_n)!} = \sum_{l=0}^{\min(T,k)} \left( \sum_{j=1}^{m_1 + \cdots + m_{n} + n - 2l} a_{j_l} (2k - 2l)^{j - 1} \right) \frac{B_{2k-2l}}{(2k-2l)!}, \tag{2.13} \]

where \( T = \max\left\{ \left( [m_1 + \cdots + m_{n} + n - 2]/2, [(n-1)/2] \right) \right\} \) and \( a_{j_l} \) are determined by (2.10).

Note that in [11, Theorem 1], A. Petoević and H. M. Srivastava had considered the case of \( m_1 = \cdots = m_n = 0 \). See also [1, Theorems 1 and 2].

In the end of this section, we list some explicit examples of \( n = 4 \). Note that some examples of \( n = 2 \) and \( n = 3 \) were given in [3].

**Example 2.10.** Let \( k \) be a positive integer with \( k \geq 4 \). Set
\[ \sum_{k_1 + k_2 + k_3 + k_4 = k} \frac{B_{2k_1} B_{2k_2} B_{2k_3} B_{2k_4}}{(2k_1)! (2k_2)! (2k_3)! (2k_4)!}. \]

We have
\[ \sum_{k_1 + k_2 + k_3 + k_4 = k} \frac{B_{2k_1} B_{2k_2} B_{2k_3} B_{2k_4}}{(2k_1)! (2k_2)! (2k_3)! (2k_4)!} = \frac{(k+1)(2k+1)(2k+3)}{3} \frac{B_{2k}}{(2k)!} + \frac{2k}{3} \frac{B_{2k-2}}{(2k-2)!}, \]
\[ \sum_{k_1 + k_2 + k_3 + k_4 = k} \frac{k^2 B_{2k_1} B_{2k_2} B_{2k_3} B_{2k_4}}{(2k_1)! (2k_2)! (2k_3)! (2k_4)!} = \frac{120}{k(k+1)(2k+1)(2k+3)(4k+3)} \frac{B_{2k}}{(2k)!} - \frac{k(4k^2 - 6k + 3)}{24} \frac{B_{2k-2}}{(2k-2)!} - \frac{2k-5}{160} \frac{B_{2k-4}}{(2k-4)!}. \]
where

\[ \sum_{k_1} B_{2k_1} B_{2k_2} B_{2k_3} B_{2k_4} = \frac{k(k+1)(2k+3)(4k^2 + 6k + 1)}{240} \frac{B_{2k}}{(2k)!} \]

\[ - \frac{k(12k^3 - 12k^2 - 11k + 9)}{96} B_{2k-2} - \frac{(2k-5)(13k - 9)}{960} B_{2k-4}. \]

Set \( B_{k_1, k_2, k_3, k_4} = \frac{B_{2k_1} B_{2k_2} B_{2k_3} B_{2k_4}}{(2k_1)! (2k_2)! (2k_3)! (2k_4)!}. \) Using the formulas

\[
\begin{align*}
\sum k_1 B_{k_1, k_2, k_3, k_4} &= \frac{k}{4} \sum B_{k_1, k_2, k_3, k_4}, \\
\sum k_1 k_2 B_{k_1, k_2, k_3, k_4} &= \frac{k^3}{12} \sum B_{k_1, k_2, k_3, k_4} - \frac{1}{3} \sum k_1^2 B_{k_1, k_2, k_3, k_4}, \\
\sum k_1^2 k_2 B_{k_1, k_2, k_3, k_4} &= \frac{k}{3} \sum k_1^2 B_{k_1, k_2, k_3, k_4} - \frac{1}{3} \sum k_1^3 B_{k_1, k_2, k_3, k_4}, \\
\sum k_1 k_2 k_3 B_{k_1, k_2, k_3, k_4} &= \frac{k^3}{24} \sum B_{k_1, k_2, k_3, k_4} - \frac{k}{2} \sum k_1^2 B_{k_1, k_2, k_3, k_4} \\
&\quad + \frac{1}{3} \sum k_1^3 B_{k_1, k_2, k_3, k_4},
\end{align*}
\]

one can work out other weighted sum formulas of the Bernoulli numbers with the condition \( m_1 + m_2 + m_3 + m_4 \leq 3. \)

### 3. Weighted sum formulas of zeta values at even arguments

Euler’s formula claims that for any positive integer \( k, \)

\[ \zeta(2k) = (-1)^{k+1} \frac{B_{2k}}{2(2k)!} (2\pi)^{2k}. \quad (3.1) \]

Then from Theorem 2.9, we get the following weighted sum formula for zeta values at even arguments.

**Theorem 3.1.** Let \( n, k \) be positive integers with \( k \geq n. \) Then for any nonnegative integers \( m_1, \ldots, m_n, \) we have

\[
\sum_{\substack{k_1 + \cdots + k_n = k \\
k_j \geq 1}} k_1^{m_1} \cdots k_n^{m_n} \zeta(2k_1) \cdots \zeta(2k_n) = (-1)^n \sum_{l=0}^{\min(T,k)} \frac{(2l)!}{B_{2l}} \cdot \left( \sum_{j=1}^{m_1 + \cdots + m_n + n-2l} a_{jl} \right) \frac{(k-l)^{j-1}}{2^{m_1 + \cdots + m_n + n-j-1}} \zeta(2l) \zeta(2k - 2l), \quad (3.2)
\]

where \( T = \max\{ [(m_1 + \cdots + m_n + n - 2)/2], [(n - 1)/2] \} \) and \( a_{jl} \) are determined by (2.10).

Finally, we obtain the weighted sum formula (1.1).

**Theorem 3.2.** Let \( n, k \) be positive integers with \( k \geq n. \) Let \( F(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n] \) be a polynomial of degree \( r. \) Then we have

\[
\sum_{\substack{k_1 + \cdots + k_n = k \\
k_j \geq 1}} F(k_1, \ldots, k_n) \zeta(2k_1) \cdots \zeta(2k_n) = \sum_{l=0}^{\min(T,k)} e_{F,l}(k) \zeta(2l) \zeta(2k - 2l),
\]

where \( T = \max\{ [(r + n - 2)/2], [(n - 1)/2] \} \), \( e_{F,l}(x) \in \mathbb{Q}[x] \) depends only on \( l \) and \( F, \) and \( \deg e_{F,l}(x) \leq r + n - 2l - 1. \)
Note that the polynomial \( F(x_1, \ldots, x_n) \) in Theorem 3.2 need not be symmetric, and the upper bound for the degree of the polynomial \( e_{F,l}(x) \) is different from that in Conjecture 1.1. In Conjecture 1.1, the upper bound for \( \deg e_{F,l}(x) \) is \( r-1 \), which should be a typo. See the examples below.

**Example 3.3.** Set \( \sum_{k_1+k_2+k_3+k_4=k} \). For a positive integer \( k \) with \( k \geq 4 \), we have

\[
\sum \zeta(2k_1)\zeta(2k_2)\zeta(2k_3)\zeta(2k_4) = \frac{(k+1)(2k+1)(2k+3)}{24} \zeta(2k) - 2k \zeta(2) \zeta(2k-2),
\]
\[
\sum k^2_1 \zeta(2k_1)\zeta(2k_2)\zeta(2k_3)\zeta(2k_4) = \frac{k(k+1)(2k+1)(2k+3)(4k+3)}{960} \zeta(2k) - \frac{k(4k^2 - 6k + 3)}{8} \zeta(2) \zeta(2k-2) + \frac{9(2k-5)}{8} \zeta(4) \zeta(2k-4),
\]
\[
\sum k^3_1 \zeta(2k_1)\zeta(2k_2)\zeta(2k_3)\zeta(2k_4) = \frac{k(k+1)(2k+1)(2k+3)(4k^2 + 6k + 1)}{1920} \zeta(2k) - \frac{k(12k^3 - 12k^2 - 11k + 9)}{32} \zeta(2) \zeta(2k-2) + \frac{3(2k-5)(13k-9)}{16} \zeta(4) \zeta(2k-4),
\]

which can deduce all other weighted sums (3.2) under the conditions \( n = 4 \) and \( m_1 + m_2 + m_3 + m_4 \leq 3 \) as explained in Example 2.10.

4. **Weighted sum formulas of multiple zeta values with even arguments**

To treat the weighted sum of the multiple zeta values with even arguments and its zeta-star analogue, we recall the symmetric sum formulas of M. E. Hoffman [4, Theorems 2.1 and 2.2]. For a partition \( \Pi = \{P_1, P_2, \ldots, P_i\} \) of the set \( \{1, 2, \ldots, n\} \), let \( l_j = \sharp P_j \) and

\[
c(\Pi) = \prod_{j=1}^{i} (l_j - 1)!, \quad \tilde{c}(\Pi) = (-1)^{n-i} c(\Pi).
\]

We also denote by \( \mathcal{P}_n \) the set of all partitions of the set \( \{1, 2, \ldots, n\} \). Then the symmetric sum formulas are

\[
\sum_{\sigma \in S_n} \zeta(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \sum_{\Pi \in \mathcal{P}_n} \tilde{c}(\Pi) \zeta(k, \Pi) \quad (4.1)
\]

and

\[
\sum_{\sigma \in S_n} \zeta^*(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \sum_{\Pi \in \mathcal{P}_n} c(\Pi) \zeta(k, \Pi), \quad (4.2)
\]

where \( k = (k_1, \ldots, k_n) \) is a sequence of positive integers with all \( k_i > 1 \), \( S_n \) is the symmetric group of degree \( n \) and for a partition \( \Pi = \{P_1, \ldots, P_i\} \in \mathcal{P}_n \),

\[
\zeta(k, \Pi) = \prod_{j=1}^{i} \zeta \left( \sum_{l \in P_j} k_l \right).
\]
Now let \( k = (2k_1, \ldots, 2k_n) \) with all \( k_i \) positive integers. Using (4.1) and (4.2), we have

\[
\sum_{\sigma \in S_n} \zeta(2k_{\sigma(1)}, \ldots, 2k_{\sigma(n)}) = \sum_{i=1}^{n} (-1)^{n-i} \sum_{l_1 + \cdots + l_i = n} \prod_{j=1}^{i} (l_j - 1)! \sum_{\Pi = (P_1, \ldots, P_i) \in \mathcal{P}_n} \zeta(k, \Pi) \tag{4.3}
\]

and

\[
\sum_{\sigma \in S_n} \zeta^*(2k_{\sigma(1)}, \ldots, 2k_{\sigma(n)}) = \sum_{i=1}^{n} \sum_{l_1 + \cdots + l_i = n} \prod_{j=1}^{i} (l_j - 1)! \sum_{\Pi = (P_1, \ldots, P_i) \in \mathcal{P}_n} \zeta(k, \Pi). \tag{4.4}
\]

From now on, let \( k, n \) be fixed positive integers with \( k \geq n \), and let \( F(x_1, \ldots, x_n) \) be a fixed symmetric polynomial with rational coefficients. It is easy to see that

\[
\sum_{k_1 + \cdots + k_n = k} F(k_1, \ldots, k_n) \sum_{\sigma \in S_n} \zeta(2k_{\sigma(1)}, \ldots, 2k_{\sigma(n)}) = n! \sum_{k_1 + \cdots + k_n = k} F(k_1, \ldots, k_n) \zeta(2k_1, \ldots, 2k_n)
\]

and

\[
\sum_{k_1 + \cdots + k_n = k} F(k_1, \ldots, k_n) \sum_{\sigma \in S_n} \zeta^*(2k_{\sigma(1)}, \ldots, 2k_{\sigma(n)}) = n! \sum_{k_1 + \cdots + k_n = k} F(k_1, \ldots, k_n) \zeta^*(2k_1, \ldots, 2k_n).
\]

On the other hand, for a partition \( \Pi = \{P_1, \ldots, P_i\} \in \mathcal{P}_n \) with \( \sharp P_j = l_j \), we have

\[
\sum_{k_1 + \cdots + k_n = k} F(k_1, \ldots, k_n) \zeta(k, \Pi) = \sum_{t_1 + \cdots + t_i = k} \sum_{k_1 + \cdots + k_i = t_1} F(k_1, \ldots, k_n) \zeta(2t_1) \cdots \zeta(2t_i). \tag{4.5}
\]

To treat the inner sum about \( F(k_1, \ldots, k_n) \) in the right-hand side of (4.5), we need the following lemmas.

**Lemma 4.1.** For any positive integer \( k \) and any nonnegative integers \( p_1, p_2 \), we have

\[
\sum_{i=1}^{k-1} i^{p_1} (k - i)^{p_2}
\]
Using the definition of the Bernoulli numbers, we get

\[ (i + j) \left( \begin{array}{c} p_2 \\ i + j - p_1 \end{array} \right) B_{i+j} \frac{B_{i+j}}{j+1} (k - 1)^{j+1} k^{p_1 + p_2 - i - j}. \]  

(4.6)

In particular, the right-hand side of (4.6) is a polynomial of \( k \) with rational coefficients of degree \( p_1 + p_2 + 1 \).

Proof. Let \( S_{p_1, p_2}(k) = \sum_{i=1}^{k-1} i^{p_1}(k-i)^{p_2} \) and let

\[ G_k(t_1, t_2) = \sum_{p_1, p_2 \geq 0} S_{p_1, p_2}(k) \frac{t_1^p t_2^q}{p_1! p_2!} \]

be the generating function. We have

\[ G_k(t_1, t_2) = \sum_{i=1}^{k-1} e^{it_1 + (k-i)t_2} = \frac{(1 - e^{(k-1)(t_1-t_2)}) e^{kt_2}}{e^{t_2} - 1}. \]

Using the definition of the Bernoulli numbers, we get

\[ G_k(t_1, t_2) = \sum_{i, j \geq 0, t \geq 0} (-1)^i \frac{B_i}{i! j! l!} (k - 1)^i k^l (t_1 - t_2)^{i+j} t_2^l \]

\[ = \sum_{i, j \geq 0} (-1)^i \frac{B_i}{i! j! l!} (k - 1)^j k^l (t_1 - t_2)^{i+j} t_2^l. \]

Finally, we obtain the expansion

\[ G_k(t_1, t_2) = \sum_{i, j, l \geq 0} (-1)^i \left( \begin{array}{c} p_1 + p_2 \\ i \end{array} \right) \frac{B_i}{i! j! l!} (k - 1)^j k^l t_1^i t_2^l. \]

Comparing the coefficient of \( \frac{t_1^p t_2^q}{p_1! p_2!} \), we get (4.6).

Then as a polynomial of \( k \), the degree of the right-hand side of (4.6) is less than or equal to \( p_1 + p_2 + 1 \), and the coefficient of \( k^{p_1 + p_2 + 1} \) is

\[ \sum_{j=p_1}^{p_1 + p_2} (-1)^j \left( \begin{array}{c} p_2 \\ j - p_1 \end{array} \right) \frac{1}{j+1}, \]

which is

\[ \sum_{j=0}^{p_2} (-1)^j \left( \begin{array}{c} p_2 \\ j \end{array} \right) \frac{1}{j + p_1 + 1}. \]

Since

\[ x^{p_1}(1 - x)^{p_2} = \sum_{j=0}^{p_2} (-1)^j \left( \begin{array}{c} p_2 \\ j \end{array} \right) x^{j+p_1}, \]

we find the coefficient of \( k^{p_1 + p_2 + 1} \) is

\[ \int_0^1 x^{p_1}(1 - x)^{p_2} dx = B(p_1 + 1, p_2 + 1) = \frac{p_1! p_2!}{(p_1 + p_2 + 1)!}, \]

which is nonzero.

More generally, we have
Lemma 4.2. Let $k$ and $n$ be integers with $k \geq n \geq 1$, and let $p_1, \ldots, p_n$ be non-negative integers. Then there exists a polynomial $f(x) \in \mathbb{Q}[x]$ of degree $p_1 + \cdots + p_n + n - 1$, such that

$$
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} k_1^{p_1} \cdots k_n^{p_n} = f(k).
$$

Proof. We proceed by induction on $n$. If $n = 1$, we may take $f(x) = x^{p_1}$. For $n > 1$, since

$$
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} k_1^{p_1} \cdots k_n^{p_n} = \sum_{k_1 + k_2 = k \atop k_j \geq 1} \left( \sum_{l_1 + \cdots + l_{n-1} = k_1} l_1^{p_1} \cdots l_{n-1}^{p_{n-1}} \right) k_n^{p_n},
$$

using the induction assumption we have

$$
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} k_1^{p_1} \cdots k_n^{p_n} = \sum_{k_1 + k_2 = k \atop k_j \geq 1} g(k_1) k_2^{p_n},
$$

where $g(x) \in \mathbb{Q}[x]$ is of degree $p_1 + \cdots + p_{n-1} + n - 2$. Then the result follows from Lemma 4.1.

Now we return to the computation of the right-hand side of (4.5). Using Lemma 4.2, there exists a polynomial $f_{t_1, \ldots, t_i}(x_1, \ldots, x_i) \in \mathbb{Q}[x_1, \ldots, x_i]$ of degree $\deg F + n - i$, such that

$$
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta(k, \Pi) = \sum_{t_1 + \cdots + t_i = k \atop t_j \geq 1} f_{t_1, \ldots, t_i}(t_1, \ldots, t_i) \zeta(2t_1) \cdots \zeta(2t_i).
$$

Therefore we get

$$
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta(2k_1, \ldots, 2k_n) = \frac{1}{n!} \sum_{i=1}^{n} (-1)^{n-i} \sum_{l_1 + \cdots + l_i = n \atop l_j \geq 1, l_1 \geq l_2 \geq \cdots \geq l_i} \prod_{j=1}^{i} (l_j - 1)! n(l_1, \ldots, l_i) \sum_{t_1 + \cdots + t_i = k \atop t_j \geq 1} f_{t_1, \ldots, t_i}(t_1, \ldots, t_i) \zeta(2t_1) \cdots \zeta(2t_i)
$$

and

$$
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta^*(2k_1, \ldots, 2k_n) = \frac{1}{n!} \sum_{i=1}^{n} \sum_{l_1 + \cdots + l_i = n \atop l_j \geq 1, l_1 \geq l_2 \geq \cdots \geq l_i} \prod_{j=1}^{i} (l_j - 1)! n(l_1, \ldots, l_i) \sum_{t_1 + \cdots + t_i = k \atop t_j \geq 1} f_{t_1, \ldots, t_i}(t_1, \ldots, t_i) \zeta(2t_1) \cdots \zeta(2t_i),
$$

where

$$
n(l_1, \ldots, l_i) = \frac{n!}{\prod_{j=1}^{i} l_j! \prod_{j=1}^{i} \# \{ m \mid 1 \leq i \leq j, k_m = j \}!}.
$$
is the number of partitions \( \Pi = \{ P_1, \ldots, P_i \} \in \mathcal{P}_n \) with the conditions \( \sharp P_j = l_j \) for \( j = 1, 2, \ldots, i \).

Applying Theorem 3.2, we then prove the weighted sum formula (1.2) and its zeta-star analogue.

**Theorem 4.3.** Let \( n, k \) be positive integers with \( k \geq n \). Let \( F(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n] \) be a symmetric polynomial of degree \( r \). Then we have

\[
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta(2k_1, \ldots, 2k_n) = \sum_{l=0}^{\min\{T,k\}} c_{F,l}(k) \zeta(2l) \zeta(2k - 2l)
\]

and

\[
\sum_{k_1 + \cdots + k_n = k \atop k_j \geq 1} F(k_1, \ldots, k_n) \zeta^*(2k_1, \ldots, 2k_n) = \sum_{l=0}^{\min\{T,k\}} c_{F,l}^*(k) \zeta(2l) \zeta(2k - 2l),
\]

where \( T = \max\{(r + n - 2)/2, [(n - 1)/2]\} \), \( c_{F,l}(x), c_{F,l}^*(x) \in \mathbb{Q}[x] \) depend only on \( l \) and \( F \), and \( \deg c_{F,l}(x), \deg c_{F,l}^*(x) \leq r + n - 2l - 1 \).

Note that in Theorem 4.3, the upper bound for the polynomial \( c_{F,l}(x) \) is different from that in Conjecture 1.1. In Conjecture 1.1, the upper bound for \( \deg c_{F,l}(x) \) is \( \deg x_1 F(x_1, \ldots, x_n) \). It seems that one may obtain this upper bound but need more efforts.

**Example 4.4.** After getting the weighted sum formulas (3.2) with \( n = 2 \) and \( n = 3 \), we can obtain the weighted sum formulas of the multiple zeta values (resp. the multiple zeta-star values) of depth four. Here are some examples. For multiple zeta values, we have

\[
\sum \zeta(2k_1, 2k_2, 2k_3, 2k_4) = \frac{35}{64} \zeta(2k) - \frac{5}{16} \zeta(2) \zeta(2k - 2),
\]

\[
\sum (k_1^4 + k_2^4 + k_3^4 + k_4^4) \zeta(2k_1, 2k_2, 2k_3, 2k_4) = \frac{7k(10k - 3)}{128} \zeta(2k)
\]

\[
- \frac{10k^2 + 9k - 30}{32} \zeta(2) \zeta(2k - 2) + \frac{3(2k - 5)}{16} \zeta(4) \zeta(2k - 4),
\]

\[
\sum (k_1^3 + k_2^3 + k_3^3 + k_4^3) \zeta(2k_1, 2k_2, 2k_3, 2k_4) = \frac{7k(40k^2 - 18k + 3)}{512} \zeta(2k)
\]

\[
- \frac{40k^3 + 54k^2 - 174k + 15}{128} \zeta(2) \zeta(2k - 2) + \frac{3(2k - 5)(3k + 2)}{32} \zeta(4) \zeta(2k - 4),
\]

and for multiple zeta-star values, we have

\[
\sum \zeta^*(2k_1, 2k_2, 2k_3, 2k_4) = \frac{(4k - 5)(8k^2 - 20k + 3)}{192} \zeta(2k)
\]

\[
- \frac{4k - 7}{16} \zeta(2) \zeta(2k - 2),
\]

\[
\sum (k_1^2 + k_2^2 + k_3^2 + k_4^2) \zeta^*(2k_1, 2k_2, 2k_3, 2k_4)
\]

\[
= k \left( \frac{128k^4 - 600k^3 + 920k^2 - 600k + 227}{1920} \right) \zeta(2k)
\]

\[
- \frac{(2k - 3)(16k^2 - 63k + 68)}{96} \zeta(2) \zeta(2k - 2) - \frac{2k - 5}{16} \zeta(4) \zeta(2k - 4),
\]

\[16\]
\[
\sum (k_1^3 + k_2^3 + k_3^3 + k_4^3) \zeta(2k_1, 2k_2, 2k_3, 2k_4) = \\
\frac{1}{k(256k^5 - 1440k^4 + 2760k^3 - 2400k^2 + 1664k - 435)} \zeta(2k) \\
- \frac{32k^4 - 184k^3 + 318k^2 - 136k - 51}{128} \zeta(2) \zeta(2k - 2) \\
+ \frac{15(k - 4)(2k - 5)}{32} \zeta(4) \zeta(2k - 4).
\]

Here \( k \) is a positive integer with \( k \geq 4 \) and \( \sum = \sum_{k_1+k_2+k_3+k_4=k} \).

5. REGULARIZED DOUBLE SHUFFLE RELATIONS AND WEIGHTED SUM FORMULAS

In this section, we briefly explain that the weighted sum formulas in Theorems 3.1, 3.2 and 4.3 can be deduced from the regularized double shuffle relations of the multiple zeta values (For the details of the regularized double shuffle relations, one can refer to [8, 12] or [9]).

We get Theorem 3.1 and hence Theorem 3.2 just from (2.13) and Euler’s formula (3.1). While (2.13) is an equation about the Bernoulli numbers and Euler’s formula can be deduced from the regularized double shuffle relations ([9]). Hence we get Theorems 3.1 and 3.2 from the regularized double shuffle relations.

We get Theorem 4.3 from Theorem 3.2 and the symmetric sum formulas. While the symmetric sum formulas are consequences of the harmonic shuffle products ([6, Theorem 2.3]). In fact, let \( Y = \{ z_k \mid k = 1, 2, \ldots \} \) be an alphabet with noncommutative letters and let \( Y^* \) be the set of all words generated by letters in \( Y \), which contains the empty word \( 1_Y \). Let \( h^1 = \mathbb{Q} \langle Y \rangle \) be the noncommutative polynomial algebra over \( \mathbb{Q} \) generated by \( Y \). As in [5, 10], we define two bilinear commutative products \( * \) and \( \bar{*} \) on \( h^1 \) by the rules

\[
1_Y * w = w * 1_Y = w, \\
z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_{k+l}(w_1 * w_2); \\
1_Y \bar{*} w = w \bar{*} 1_Y = w, \\
z_k w_1 \bar{*} z_l w_2 = z_k(w_1 \bar{*} z_l w_2) + z_l(z_k w_1 \bar{*} w_2) - z_{k+l}(w_1 \bar{*} w_2),
\]

where \( w, w_1, w_2 \in Y^* \) and \( k, l \) are positive integers. Let

\[
h^0 = \mathbb{Q} 1_Y + \sum_{n \geq 1, k_1, \ldots, k_n \geq 1} \mathbb{Q} z_{k_1} \cdots z_{k_n}
\]

be a subalgebra of \( h^1 \), which is also a subalgebra with respect to either the product \( * \) or the product \( \bar{*} \). Let \( Z : h^0 \to \mathbb{R} \) and \( Z^* : h^0 \to \mathbb{R} \) be the \( \mathbb{Q} \)-linear maps determined by \( Z(1_Y) = Z^*(1_Y) = 1 \) and

\[
Z(z_{k_1} \cdots z_{k_n}) = \zeta(k_1, \ldots, k_n), \quad Z^*(z_{k_1} \cdots z_{k_n}) = \zeta^*(k_1, \ldots, k_n),
\]

where \( n, k_1, \ldots, k_n \geq 1 \) with \( k_1 \geq 2 \). It is known that both the maps \( Z : (h^0, *) \to \mathbb{R} \) and \( Z^* : (h^0, \bar{*}) \to \mathbb{R} \) are algebra homomorphisms. Hence from the following lemma, we know that the symmetric sum formulas are consequences of the harmonic shuffle products. And therefore Theorem 4.3 is also deduced from the regularized double shuffle relations.
Lemma 5.1. Let \( n \) be a positive integer and \( \mathbf{k} = (k_1, \ldots, k_n) \) be a sequence of positive integers. We have

\[
\sum_{\sigma \in S_n} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n)}} = \sum_{P = \{P_1, \ldots, P_j\} \in \mathcal{P}_n} c(\Pi) z_{k_{P_1}} \cdots z_{k_{P_j}} \quad \text{(5.1)}
\]

and

\[
\sum_{\sigma \in S_n} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n)}} = \sum_{P = \{P_1, \ldots, P_j\} \in \mathcal{P}_n} c(\Pi) z_{k_{P_1}} \cdots z_{k_{P_j}}, \quad \text{(5.2)}
\]

where \( z_{k_i} = z \sum_{i \in P_j} k_i \).

Proof. To be self contained, we give a proof here. We prove (5.1) and one can prove (5.2) similarly. We proceed by induction on \( n \). The case of \( n = 1 \) is obvious. Now assume that (5.1) is proved for \( n \). Let \( \mathbf{k} = (k_1, \ldots, k_n) \) and \( \mathbf{k}' = (k_1, \ldots, k_{n+1}) \). Since

\[
\sum_{\sigma \in S_{n+1}} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n+1)}} = \sum_{\sigma \in S_n} \sum_{j=1}^{n+1} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(j-1)}} z_{k_{\sigma(j)} + k_{n+1}} z_{k_{\sigma(j+1)}} \cdots z_{k_{\sigma(n)}}
\]

\[
+ \sum_{\sigma \in S_n} \sum_{j=1}^{n} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n+1)}}
\]

\[
= \sum_{\sigma \in S_{n+1}} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n+1)}} + \sum_{j=1}^{n} \sum_{\sigma \in S_n} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n+1)}}
\]

with \( \mathbf{k}^{(j)} = (k_1, \ldots, k_{j-1}, k_j + k_{n+1}, k_{j+1}, \ldots, k_n) = (k_1^{(j)}, \ldots, k_n^{(j)}) \), using the induction assumption on \( \mathbf{k} \) and \( \mathbf{k}^{(j)} \) with \( j = 1, \ldots, n \), we have

\[
\sum_{\sigma \in S_{n+1}} z_{k_{\sigma(1)}} \cdots z_{k_{\sigma(n+1)}} = \sum_{P = \{P_1, \ldots, P_j\} \in \mathcal{P}_n} c(\Pi) z_{k_{P_1}} \cdots z_{k_{P_j}} \ast z_{k_{n+1}}
\]

\[
- \sum_{j=1}^{n} \sum_{P = \{P_1, \ldots, P_j\} \in \mathcal{P}_n} c(\Pi) z_{k_{P_1}} \cdots z_{k_{P_j}} \ast z_{k_{n+1}}
\]

Because any \( \Pi \in \mathcal{P}_{n+1} \) must satisfy and can only satisfy one of the following two conditions:

(i) there exists one \( P \in \Pi \), such that \( P = \{k_{n+1}\} \);

(ii) for any \( P \in \Pi \), \( P \neq \{k_{n+1}\} \),

we see that the right-hand side of the above equation is just

\[
\sum_{\Pi = \{P_1, \ldots, P_j\} \in \mathcal{P}_{n+1}} c(\Pi) z_{k_{P_1}} \cdots z_{k_{P_j}}.
\]

Hence we get (5.1). \( \square \)

References

[1] K. Dilcher, Sums of products of Bernoulli numbers, J. Number Theory 60 (1996), 23-41.
[2] H. Gangl, M. Kaneko and D. Zagier, Double zeta values and modular forms, In Automorphic Forms and Zeta Functions, Proceedings of the Conference in memory of Tsuneo Arakawa. S. Böcherer, T. Ibukiyama, M. Kaneko, F. Sato (eds.), World Scientific, New Jersey, 71-106, 2006.
[3] L. Guo, P. Lei and J. Zhao, Families of weighted sum formulas for multiple zeta values, *Int. J. Number Theory* **11** (3) (2015), 997-1025.
[4] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (2) (1992), 275-290.
[5] M. E. Hoffman, The algebra of multiple harmonic series, *J. Algebra* **194** (2) (1997), 477-495.
[6] M. E. Hoffman, Quasi-symmetric functions and mod $p$ multiple harmonic sums, *Kyushu J. Math.* **69** (2015), 345-366.
[7] M. E. Hoffman, On multiple zeta values of even arguments, *Int. J. Number Theory* **13** (2017), 705-716.
[8] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, *Compos. Math.* **142** (2) (2006), 307-338.
[9] Z. Li and C. Qin, Some relations deduced from regularized double shuffle relations of multiple zeta values, preprint, arXiv:1610.05480.
[10] S. Muneta, Algebraic setup of non-strict multiple zeta values, *Acta Arith.* **136** (1) (2009), 7-18.
[11] A. Petojević and H. M. Srivastava, Computation of Euler’s type sums of the products of Bernoulli numbers, *Appl. Math. Lett.* **22** (2009), 796-801.
[12] G. Racinet, Doubles melanges des polylogarithmes multiples aux racines de l’unite, *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 185-231.

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