Abstract

In recent years, finding new satisfiability algorithms for various circuit classes has been a very active line of research. Despite considerable progress, we are still far away from a definite answer on which circuit classes allow fast satisfiability algorithms. This survey takes a (far from exhaustive) look at some recent satisfiability algorithms for a range of circuit classes and highlights common themes. A special focus is given to connections between satisfiability algorithms and circuit lower bounds. A second focus is on reductions from satisfiability algorithms to a range of polynomial time problems, such as matrix multiplication and the Vector Domination Problem.

1 Introduction

Ever since Cook and Levin [3 16] proved the NP-completeness of 3-SAT, satisfiability problems played a central role within complexity theory. Their result puts satisfiability algorithms in the center of the arguably most important question in computer science, P vs. NP.

To show P = NP with a satisfiability algorithm, the algorithm has to run in polynomial time, which is a long way from the trivial $\tilde{O}(2^n)$ exhaustive search algorithm. This raises the question if we can improve over exhaustive search at all. In particular, we want to know if we can find satisfiability algorithms that improve over exhaustive search by an exponential factor, i.e. run in time $\tilde{O}(2^{(1-\mu)n})$ for some constant $\mu > 0$. For such a runtime we refer to $\mu$ as the savings of the algorithm.

For $k$-SAT the answer is yes due to Monien and Speckenmeyer [17]. Since then, several faster algorithms for $k$-SAT have been found, e.g. [19 21 20 8]. Unfortunately it is not possible to give an extensive overview for $k$-SAT algorithms here. We will however discuss the algorithm by Paturi, Pudlák and Zane in further detail.

For other circuit classes this question is still wide open. In particular, for general polynomial size circuits no fast satisfiability algorithms are known. If one believes that there is such an algorithm, then one approach is to tackle more and more general classes of circuits. If one believes that there are no fast algorithms for general polynomial size circuits, then where is the border between circuit classes that have fast satisfiability algorithms and classes that don’t? The Strongly Exponential Time Hypothesis (SETH) [13] conjectures that, while $k$-SAT does have a fast satisfiability algorithm for any constant $k$, many variants of CNF-SAT do not. The hypothesis says that for every constant $\mu > 0$, there is a $k$ such that $k$-SAT cannot be solved in time $O(2^{(1-\mu)n})$. This would put CNF-SAT with no restriction on the width of the clauses or the size of the formula on the side with no fast satisfiability algorithm.
For CNF-SAT, there is a duality between the width of the clauses and the size of the circuit [2]. As a consequence, a corollary of SETH is that there are no satisfiability algorithms with constant savings for CNF formulas with superlinear size [2]. It is therefore not surprising that even beyond CNF formulas, many satisfiability algorithms with constant savings require the circuit to be linear size.

In this survey we highlight several circuit classes that allow satisfiability algorithms with constant savings. For a survey on fast algorithms for $\mathbf{NP}$-complete problems that goes beyond satisfiability problems see the articles by Woeginger [33, 34].

Most restricted circuit classes were first defined in the context of circuit lower bounds. In some sense, algorithms and lower bounds are two sides of the same question. Algorithms try to find the most efficient way to compute something, while lower bounds give limitations on what is possible. This survey highlights several results that show how advances on one problem can lead to advances in the other. Intuitively, any satisfiability algorithm has to use the structure of the circuit somehow to improve over exhaustive search. The same structural properties can be used to argue which functions a circuit class fails to represent.

Section 3 discusses algorithms that rely on such a property. Section 4 discusses a more direct connection between satisfiability algorithms and lower bounds. In this result by Williams [30, 31], the lower bounds are derived from satisfiability algorithms, where the algorithm is treated as a black box. As a consequence, the connection between the satisfiability algorithm and the lower bound does not directly rely on a property of the circuit class, although the satisfiability algorithm does. This result not only provides a blueprint for potentially finding new lower bounds, but also describes the relationship between algorithms and lower bounds in a more formal framework.

In Section 5 we discuss satisfiability algorithms that use results on various polynomial time algorithms such as matrix multiplication to get constant savings. Those algorithms exploit links between satisfiability algorithms for circuits and problems within computer science beyond circuits.

Santhanam [23] wrote a survey with a similar focus as this survey, discussing, among others, the results on $k$-CNF, DeMorgan Formulas and $\mathsf{ACC}^0$ in greater detail.

2 Preliminaries

In this paper we consider different circuit classes. For a set of functions $B$, a circuit over basis $B$ is a sequence $q_0, \ldots, q_m$ such that $q_0$ and $q_1$ are the constants 0 and 1, $q_2$ to $q_{n+1}$ are the $n$ input variables $x_1, \ldots, x_n$. For every index $j \geq n + 2$ the circuit is defined by a gate, which is defined by a function from $B$ and the indices of the inputs to that function. The indices of the inputs are required to be less than $j$. The semantics of a circuit is that at every position, the function from $B$ is applied to the inputs, and given values to the input variables, we can compute the value of every gate from left to right. The last gate of a circuit is called the output gate. The size of a circuit is the number of gates other than the constants and inputs.

A restricted circuit class is a restriction of the definition of the circuit is some way. Depending on the context, we refer to the circuit class as either the set of circuits itself, or the class of problems that can be decided with such a circuit. All restricted circuit classes fix a basis $B$, and many restrict the size of the circuit. For example, $\mathbf{P}/\mathsf{Poly}$ is the class of problems that can by decided with a polynomial size circuit and basis $B_2$, which is the set of all 8 functions with fanin two. Some classes restrict the depth of a circuit, which is defined as the longest path from the output gate to an input gate. Intuitively, the depth is the time it takes to compute the result when running the computation with maximal parallelization. Other circuit classes may restrict the circuit to
formulas. In a formula, the output of a gate can be used at most once as input to other gates. We do however allow that literals and constants are used more than once. (Literals are variables or their negation.)

One important class of circuits are formulas in conjunctive normal form (CNF). A CNF is a circuit where the output gate is an AND of clauses, and every clause is an OR of literals. $k$-CNF further restricts the fanin of a clause to $k$.

$AC^0$ circuits allow AND and OR gates with arbitrary fanin, but only constant depth. We can assume without loss of generality that an $AC^0$ circuit consists of alternating layers of AND and OR gates, as we can otherwise merge two gates into one.

$ACC^0$ extends the basis of $AC^0$ circuits and also allows MOD$_m$ gates for any $m > 1$ for arbitrary fanin.

Another extension of $AC^0$ discussed in this paper is $TC^0$, the class of constant depth circuits where the gates are threshold gates, i.e. gates that take a weighted sum of the inputs and compare it to a threshold. For a threshold gate, both the threshold and the weights can be arbitrary real numbers. In this survey, we consider only threshold circuits of depth two. We call the output gate the top-level gate and the other threshold gates the bottom-level gates. We call the number of variable occurrences the number of wires.

For the MAX-\(k\)-SAT problem we are given a $k$-CNF and a threshold $t$, and need to decide if there is an assignment that satisfies at least $t$ clauses. We view it as the satisfiability problem on the circuit class that consists of $k$-clauses and a threshold gate as the output gate.

A DeMorgan formula is a formula on AND and OR gates of fanin two. The depth is unbounded for DeMorgan Formulas. Note that we do not need to allow NOT gates other than when negating literals as we can push any NOT gates to the literals using DeMorgan’s Laws. For some arguments it is more intuitive to view a DeMorgan formula as a binary tree where the literals are leaves and the gates are inner nodes. With this in mind, we refer to the gates that lead up to a gate $q_j$ as the subtree rooted at $q_j$. We measure the size of a DeMorgan formula in its leaf size, the number of input literals.

Given a circuit $C$ on a variable set $V$, an assignment is a function $V \rightarrow \{0, 1\}$. A restriction is an assignment to a subset of $V$. For a restriction $\rho$, we denote by $C|\rho$ the circuit $C$ where all variables restricted by $\rho$ are replaced by constants accordingly.

For exponential functions $f(n)$, we use $\tilde{O}(f(n))$ as a shorthand for $O(f(n))\text{poly}(n)$. For algorithms with runtime $O(2^{(1-\mu)n})$, we refer to $\mu$ as the savings of the algorithm. Typically, we are interested in algorithms with constant savings, although the results in Section 4 rely on algorithms with subconstant savings.

3 Properties of Circuit Classes

In this section we consider three satisfiability algorithms for different circuit classes that are closely linked to lower bounds for the same class. For $k$-CNF, we discuss a satisfiability algorithm and a lower bound by Pudlák, Paturi and Zane [19]. For $AC^0$ we discuss an algorithm and a lower bound by Impagliazzo, Matthews and Paturi [11]. Lastly, for DeMorgan Formulas we discuss a lower bound by Subbotovskaya [27] and a satisfiability algorithm by Santhanam [22] that rely on the same properties.

There is no clear relation if the algorithm follows from the lower bound or vice versa. For some of these results, the results were motivated by the need to find faster satisfiability algorithms, with the lower bounds a consequence of the proof technique. In other cases, the idea for the algorithm
stems from the lower bound. In the case of DeMorgan Formulas, the circuit lower bound precedes the algorithm by almost 50 years.

The three algorithms are similar in several aspects. In all three examples we can extract a property of the circuit class that neither talks about satisfiability algorithms nor lower bounds, but both are derived from the property without using any additional information about the circuit class. For $k$-CNF, the statement is about how to encode satisfying assignments, for $\mathbf{AC}^0$ it is about partitions of the hypercube and for DeMorgan Formulas it is about shrinkage under restriction.

Furthermore, all three algorithms follow the same general outline: Restrict a set of variables to constants, simplify the circuit, and repeat until the output of the circuit is a constant. Where the algorithms rely on properties of the underlying circuit is when simplifying the restricted circuit. For $k$-CNF, the simplification step removes variables beyond the ones already restricted. For $\mathbf{AC}^0$, the simplify step reduces the depth of the circuit, and for DeMorgan Formulas, the simplify step shrinks the size of the circuit by a nontrivial number of gates.

The basis of these algorithms is the random restriction technique, which was first used by Subbotovskaya in her lower bound for DeMorgan Formulas. In its basic form, one picks a random variable (or a random set of variables) and restricts the variable randomly to 0 or 1. However, not all algorithms that rely on the random restriction technique are randomized algorithms. Some algorithms like the algorithm for $k$-SAT are more natural as a randomized algorithms, and additional ideas are required to derandomize it. Other algorithms such as Santhanam’s algorithm for DeMorgan Formulas explore all possible random choices for a restriction. As a result, the algorithm is deterministic.

### 3.1 Satisfiability Coding Lemma

The Satisfiability Coding Lemma [19] provides a description of the space of satisfying assignments of a $k$-CNF in terms of how many bits are required to describe an assignment.

Let $F$ be a formula in conjunctive normal form on $n$ variables. A satisfying assignment $\alpha$ is isolated in direction $x$ if the assignment that only differs from $\alpha$ in variable $x$ does not satisfy $F$. We say an assignment is isolated, if it is isolated in all directions and $j$-isolated, if it is isolated in exactly $j$ directions. If $\alpha$ is isolated in direction $x$, then there must be a clause $C$, such that $\alpha$ sets all literals of this clause to false except the literal corresponding to $x$. We call such a clause a critical clause.

An encoding of a satisfying solution is an injective map from satisfying solutions to binary strings. The simplest such encoding is to fix a permutation of the variables, and then describe the assignment to every variable as an $n$-bit string. However, since $F$ is a CNF, we can do better. If there is a unit clause $x$ (i.e. a clause consisting only of the literal $x$), then every assignment satisfying $F$ must assign the value 1 to $x$. Likewise, if there is a clause $\overline{x}$, every satisfying assignment sets $x$ to 0. In either case we can simply omit $x$ in our description of the satisfying assignment, and describe the assignment as an $(n-1)$-bit string that still uniquely defines the satisfying assignment. Furthermore, once we assign a value to $x$, the formula simplifies. If we assign 1 to $x$, clauses containing the literal $x$ are satisfied and can be omitted. Clauses containing $\overline{x}$ can be simplified by omitting the literal $\overline{x}$, as this literal cannot be used to satisfy the clause anymore. Therefore, by repeatedly either giving the assignment of the next variable in the permutation or simplifying using a unit clause we can encode any satisfying assignment. Algorithm [11] gives the full procedure to encode an assignment. It takes as input a satisfying assignment $\alpha$ and a permutation $\pi$ on the variables and produces an encoding $\text{enc}_{\alpha,\pi}$. The encoding algorithm fails to produce a string if
Algorithm 1: Encoding satisfying assignments

Data: formula $F$, number of variables $n$, permutation $\pi$, satisfying assignment $\alpha$
Result: $\text{enc}_{\alpha, \pi}$ encoding $\alpha$

$\text{enc} = \lambda$

for $i = 1$ to $|V|$ do
  $x \leftarrow \pi(i)$
  if $\{x\} \in F$ and $\{\overline{x}\} \in F$ then
    return FALSE
  if $\{x\} \in F$ then
    $v \leftarrow 1$
  else if $\{\overline{x}\} \in F$ then
    $v \leftarrow 0$
  else if $\{x\} \not\in F$ and $\{\overline{x}\} \not\in F$ then
    $v \leftarrow \alpha(x)$
    $\text{enc} \leftarrow \text{enc} \circ v$
  $F \leftarrow \text{simplify}(F|_{x=v})$

return $\text{enc}$

the input assignment $\alpha$ does not satisfy $F$. Algorithm 2 provides a decoding algorithm for the opposite direction. It takes as input the string $\text{enc}_{\alpha, \pi}$ and the permutation $\pi$ and outputs the original assignment $\alpha$. It fails if the input is not a valid encoding of a satisfying assignment.

The length of the encoding depends on the number of unit clauses we encounter while encoding. The Satisfiability Coding Lemma gives a bound on the expected length if we pick the permutation uniformly at random.

Lemma 1 (Satisfiability Coding Lemma). Let $F$ be a $k$-CNF and let $\alpha$ be a $j(\alpha)$-isolated assignment. For a uniformly chosen permutation $\pi$ we have

$$E[|\text{enc}_{\alpha, \pi}|] \leq n - j(\alpha)/k$$

Proof. Let $x$ be a variable such that $\alpha$ is isolated in direction $x$ and let $C$ be its critical clause. We can omit the bit describing the assignment for $x$ if $C$ is a unit clause when $x$ occurs in the permutation. This happens exactly when when $x$ is the last variable of $C$ in the permutation. Since we choose the permutation uniformly at random, we can omit $x$ with probability $1/k$. By linearity of expectation we omit $j(\alpha)/k$ bits in expectation.

We can turn the Satisfiability Coding Lemma into a lower bound. We call a finite set of strings $S \subseteq \{0, 1\}^*$ prefix-free, if there are no two strings in $S$ such that one is a prefix of the other. Note that for a fixed permutation, the strings encoding satisfying assignments are prefix-free.

Theorem 1. A $k$-CNF has at most $2^{n-n/k}$ isolated satisfying assignments.

Proof. For each isolated assignment, the average code length is at most $n - n/k$. The same bound holds for the average code length over all permutations and satisfying assignments. Therefore, there is some permutation $\pi$ such that the average code length over all isolated assignments is

$$E[|\text{enc}_{\alpha, \pi}|] \leq n - j(\alpha)/k$$
Algorithm 2: Decode a string to an assignment

**Data:** formula $F$, number of variables $n$, permutation $\pi$, encoding $\text{enc}_{\alpha, \pi}$

**Result:** $\alpha$

1. $\alpha \leftarrow \emptyset$
2. $j \leftarrow 0$
3. for $i = 1$ to $|V|$ do
   1. $x \leftarrow \pi(i)$
   2. if $\{x\} \in F$ and $\{\pi\} \in F$ then
      - return FALSE
   3. if $\{x\} \in F$ then
      - $v \leftarrow 1$
   4. else if $\{\pi\} \in F$ then
      - $v \leftarrow 0$
   5. else if $\{x\} \not\in F$ and $\{\pi\} \not\in F$ then
      1. if $j > |\text{enc}|$ then
          - return FALSE
      2. $v \leftarrow \text{enc}_j$
      3. $j \leftarrow j + 1$
   6. $\alpha \leftarrow \alpha \cup \{x = v\}$
   7. $F \leftarrow \text{simplify} \left( F_{x=v} \right)$
4. return $\alpha$

- Let $S$ be the set of all such codes given permutation $\pi$ and let $S' = \{s^{n-|s|} \mid s \in S\}$, i.e. extend all strings to length $n$ by adding $\ast$. We can interpret strings in $S'$ as restrictions in the obvious way, where $\ast$ represents the free variables. Since $S$ is prefix-free, the restrictions in $S'$ are not overlapping. Since a restriction that leaves $l$ variables free covers $2^l$ assignments, we have $\sum_{s \in S} 2^{n-|s|} \leq 2^n$ and hence $\sum_{s \in S} 2^{-|s|} \leq 1$. Therefore

$$n - \frac{n}{k} \geq \sum_{s \in S} \frac{|s|}{|S|} = \sum_{s \in S} \left( -\log 2^{-|s|} \right) \geq -\log \left( \frac{\sum_{s \in S} 2^{-|s|}}{|S|} \right) \geq \log(|S|)$$

using Jensen’s Inequality. Hence $|S| \leq 2^{n-n/k}$. \hfill $\square$

To get a satisfiability algorithm from the Satisfiability Coding Lemma, consider the following algorithm, which we call the PPZ algorithm: Guess a permutation and an $n$-bit string uniformly at random and try to decode the string using the decode algorithm. Note that the algorithm might guess a string that is longer than what is actually read while decoding.

**Lemma 2.** Let $F$ be a $k$-CNF and $\alpha$ be a $j(\alpha)$-isolated solution. The PPZ algorithm returns $\alpha$ with probability at least $2^{-n+j(\alpha)/k}$.

**Proof.** The main observation is that given a permutation $\pi$, PPZ returns $\alpha$ if and only if the algorithm guesses all bits according to $\text{enc}_{\alpha, \pi}$ (plus potentially some additional bits). The probability
for this event is $2^{-|\text{enc}_{\alpha, \pi}|}$. Hence
\[
P(\text{PPZ returns } \alpha) = \sum_{\pi} P(\text{PPZ returns } \alpha \mid \pi) \frac{1}{n!} = \sum_{\pi} 2^{-|\text{enc}_{\alpha, \pi}|} \frac{1}{n!} \geq 2^{-n + j(\alpha)/k}
\]
using Jensen’s Inequality.

If we have at least one isolated solution, the probability is at least $2^{-(1-1/k)n}$. We show that this success probability holds in general. Intuitively, if there are no isolated solution, then there must be many solutions.

**Theorem 2.** The PPZ algorithm finds a satisfying assignment with probability at least $2^{-(1-1/k)n}$.

**Proof.** Let $\text{sat}(F)$ be the set of satisfying assignments for $F$ and for $\alpha \in \text{sat}(F)$, let $j(\alpha)$ denote the degree of isolation. We first prove $\sum_{\alpha \in \text{sat}(F)} 2^{-n + j(\alpha)/k} \geq 1$.

Fix some permutation and for all $\alpha$ (satisfying or not), let $s(\alpha) \in \{0, 1\}^n$ be the string describing the assignment according to the permutation. Note that this is not the same as the encoding of a satisfying assignment. Further, for satisfying assignments $\alpha$, let $s'(\alpha) \in \{0, 1, *\}^n$ be $s(\alpha)$, where a position is replaced by $*$, if $\alpha$ is isolated in that direction. We interpret the string $s'(\alpha)$ as a restriction in the natural way and claim that every assignment $\beta \in \{0, 1\}^n$ is covered by some restriction.

Let $\beta$ be an arbitrary assignment and let $\alpha \in \text{sat}(F)$ be the satisfying assignment with the smallest Hamming distance to $\beta$, i.e. the two assignments differ on the smallest number of variables. The restriction $s'(\alpha)$ must have a $*$ on every position where $s(\alpha)$ and $s(\beta)$ differ, as otherwise there would be a satisfying assignment with a smaller Hamming distance to $\beta$. Hence every assignment $\beta$ is covered by at least one restriction, and therefore $\sum_{\alpha \in \text{sat}(F)} 2^{j(\alpha)} \geq 2^n$ and $2^{-n + j(\alpha)/k} \geq 1$.

The success probability of the PPZ algorithm is therefore lower bounded by
\[
P(\text{PPZ returns some } \alpha \in \text{sat}(F)) = \sum_{\alpha \in \text{sat}(F)} P(\text{PPZ returns } \alpha) \geq \sum_{\alpha \in \text{sat}(F)} 2^{-n + j(\alpha)/k} \geq 2^{-(1-1/k)n} \sum_{\alpha \in \text{sat}(F)} 2^{-n + j(\alpha)/k} \geq 2^{-(1-1/k)n} \sum_{\alpha \in \text{sat}(F)} 2^{-n + j(\alpha)} \geq 2^{-(1-1/k)n}
\]

By repeating the PPZ algorithm we get an algorithm that runs in time $\tilde{O}\left(2^{(1-1/k)n}\right)$ and has an arbitrarily small one-sided error.

### 3.2 AC$^0$ Circuits

Impagliazzo, Matthews and Paturi \[11\] give a characterization of AC$^0$ circuits based on restrictions. For every AC$^0$ circuit on $n$ variables with size $cn$ and depth $d$ there is a partition of the hypercube into restrictions such that the function described by the circuit is constant for each restriction. The size of the partition is $\tilde{O}\left(2^\left(1-\mu_{c,d}\right)n\right)$ for $\mu_{c,d} = \frac{1}{c (\log c + d \log d)^{3/2}}$ and can be constructed with only polynomial overhead over its size.
The main idea behind the proof is a depth reduction technique based on Håstad’s Switching Lemma [7]. The Switching Lemma says if you take a CNF formula (or, symmetrically, a DNF) and apply a random restriction to its inputs, then with high probability you can represent the resulting function as a small DNF formula (or CNF). This method can be applied for depth reduction.

Given an $\text{AC}^0$ circuit with alternating AND and OR gates, apply the Switching Lemma to the lowest two levels of the circuit. After applying a random restriction, with high probability we can swap the bottom two layers. We then have two consecutive AND (or OR) layers, which can be combined. The result is a circuit with reduced depth. The main technical obstacle here is that Håstad’s original Switching Lemma is not sufficient for the required savings. Instead, they prove the Extended Switching Lemma, which deals with the case of switching several CNF formulas on the same variables together.

A satisfiability algorithm follows immediately. Construct the partition as above and check each partition if it is the constant 0 or 1. The savings of the resulting algorithm is then $\mu_{c,d}$.

Another immediate consequence of such a partition is a bound on either the depth or the size required for parity. The only partition of the parity function into restriction where the function is constant has to restrict all $n$ variables. Hence the size of such a partition is $2^n$. Solving the inequality $\tilde{O}(2^{(1-\mu_{c,d})n}) \geq 2^n$ for either the size or the depth gives that every polynomial size $\text{AC}^0$ circuit requires at least depth $\log n \log \log n - o(\log n)$ and any depth $d$ circuit requires at least $2^n\left(\frac{n}{\log \log n}\right)^d$ gates. These bounds match lower bounds derived from the original Switching Lemma by Håstad [7], which is not surprising given that the techniques are strongly related.

Another lower bound that follows from this partition is a bound on the correlation of parity with an $\text{AC}^0$ circuit. For a circuit $C$ and a function $f$, the correlation is defined as

$$P(C(\alpha) = f(\alpha)) - P(C(\alpha) \neq f(\alpha))$$

where we choose the assignment $\alpha$ uniformly at random.

Consider an arbitrary $\text{AC}^0$ circuit $C$ and its partition into $\tilde{O}(2^{(1-\mu_{c,d})n})$ restrictions. Any restriction that contains more than one assignment agrees with parity on exactly half of all values. Hence its contribution to the correlation is 0. On the other hand, a restriction to a single assignment contributes only $2^{-n}$ to the correlation. Hence the correlation between $C$ and parity is at most

$$2^{-n}\tilde{O}\left(2^{(1-\mu_{c,d})n}\right) = \tilde{O}\left(2^{-\mu_{c,d}n}\right)$$

### 3.3 DeMorgan Formulas

In 1961, Subbotovskaya [27] gave a lower bound on the size of DeMorgan Formulas computing parity based on a random restriction technique. Her result is credited as the first use of a random restriction technique, which is used in many results after her, including the satisfiability algorithm for $k$-CNF in Section 3.1 for $\text{AC}^0$ circuits in Section 3.2 and for depth two threshold circuits in Section 5.2.

Consider an arbitrary DeMorgan formula on $n$ variables and size $s$. Pick a random set of $n - k$ variables and set them uniformly at random to either 0 or 1, how many gates are still required for the resulting function? Subbotovskaya’s argument given below proves that the number of gates shrinks to at most $\left(\frac{k}{n}\right)^{3/2} s$ in expectation, giving a shrinkage exponent of at least 1.5. It immediately follows that parity requires $O(n^{1.5})$ gates, as the number of gates required after fixing $n - 1$ variables to
constants is 1. Andreev [1] (see also [15]) later constructed a function that requires $O(n^{2.5})$ gates. It follows from this construction that the same function requires at least $n^{\omega+1}$ gates where $\omega$ is the shrinkage exponent. Ending a line of research to improve the lower bound on the shrinkage exponent [12, 18], Hästad [9] proves that the shrinkage exponent is 2.

The main observation is the following. Consider an AND gate $q_i$ such that the literal $x$ (i.e. the non-negated variable) is a direct input. Let the gate $q_j$ be the other input, called the neighbor. If the random restriction sets $x$ to 0, then the AND gate always evaluates to false. Since we have a formula, the output of the circuit does not depend on $q_j$ anymore.

As a consequence, if $q_j$ depends on the variable $x$, then we can simplify the circuit. Since the output only depends on the value of $q_j$ if $x$ is 1, we can replace every occurrence of $x$ in the subtree rooted at $q_j$ with 1. For the rest of this section we can therefore assume w.l.o.g. that the neighbor of a literal $x$ or $\bar{x}$ does not contain the variable $x$. In a lower bound argument, we can argue that if the neighbor depends on $x$, then the formula cannot be minimal. From an algorithmic standpoint, we can argue that if there are occurrences of the variable in the neighbor, then we can simplify the circuit in polynomial time.

If the gate is an OR gate or the direct literal is negated then the case is symmetric. Note that for this observation it crucial that DeMorgan Formulas do not allow XOR gates, as we can otherwise fix a direct input to either constant and the output still depends on the neighbor.

Using the fact that we pick the restriction randomly we get the following lemma.

**Lemma 3.** Let $F$ be a minimal DeMorgan formula on $n$ variables with size $s$. If we restrict a random set of $n-k$ variable to 0 or 1 uniformly at random, then the resulting formula has size at most $\left(\frac{k}{n}\right)^{3/2} s$ in expectation.

**Proof.** First consider the special case where we restrict only one randomly chosen variable $x$. At the very least, all occurrences of $x$ disappear. However, with a probability of $\frac{1}{2}$, we can also remove the neighbor of $x$. Since the neighbor does not depend on $x$ by assumption, it must contain at least one occurrence of another variable. In expectation, $x$ feeds into $\frac{s}{n}$ gates. Hence the expected number of leafs that we remove is $3/2 \frac{s}{n}$ and the expected size of the remaining tree is at most

$$s - \frac{3s}{2n} = \left(1 - \frac{3}{2n}\right)s \leq \left(n - 1\right)^{3/2} \frac{s}{n}$$

Restricting a random variable $n-k$ times therefore gives a formula with size at most

$$\left(\frac{n-1}{n}\right)^{3/2} \cdot \left(\frac{n-2}{n-1}\right)^{3/2} \cdots \left(\frac{k}{k+1}\right)^{3/2} s = \left(\frac{k}{n}\right)^{3/2} s$$

Santhanam uses the same ideas for a satisfiability algorithm. Let $F$ be a DeMorgan formula with size $cn$ for some constant $c$. While the lower bound result considers random restrictions, the satisfiability algorithm is deterministic. First of all, we simplify the formula as before. We can remove gates with at least one constant input. For some constants, we can remove the neighbor of the constant. Furthermore, for every variable $x$ feeding directly into a gate, replace all occurrences of $x$ in its neighbor with the appropriate constant. After simplification, instead of restricting a random variable, the algorithm restricts the variable that occurs the most often. Since there are $cn$ leaves in total, we can always find a variable $x$ that occurs at least $c$ times. Then recursively
find a satisfying assignment for $F|_{x=1}$ and $F|_{x=0}$ until the formula has no inputs, i.e. is constant. The result is a partition into restrictions where the circuit is constant, similar to the satisfiability algorithm for $\text{AC}^0$ circuits.

To analyze the runtime of this algorithm, the shrinkage as given by Subbotovskaya is not sufficient, as it only gives an expected shrinkage. Instead, Santhanam gives a concentration bound for the shrinkage. Consider the recursion tree of the algorithm, where every vertex is labeled by some formula $F$. The children of the node labeled $F$ are two nodes labeled $F|_{x=1}$ and $F|_{x=0}$, simplified as described above. The leaves are labeled by constant functions. The runtime of the algorithm is given, within a polynomial factor, by the size of this tree. For every non-leaf node, for at least one of its children, the size of the formula shrinks by $1.5c$ gates, while for the other child, the formula shrinks by at least $c$ gates. We call the earlier child the “good child” and the other one the “bad child”. If we consider a randomly chosen path from the root of the tree to a leaf, with high probability this path picks the “good child” several times. Any path that picks the “good child” often cannot be very long (i.e. close to $n$), as the tree must arrive at a leaf when the number of gates shrinks to $1$.

As a result, one can derive a bound on the number of leaves of this tree.

The details of the calculation are omitted here. The savings of the algorithm take the form $\frac{1}{\text{poly}(c)}$.

4 From Satisfiability to Lower Bounds

In the examples of connections between satisfiability algorithms and lower bounds discussed so far, the connection was implicit in nature. There is no blueprint for extracting circuit lower bounds from satisfiability algorithms or vice versa that follows directly from these results.

Williams [30, 31] gives a more formal connection between satisfiability algorithms for a circuit class and lower bounds for the same class. Given a satisfiability algorithm that improves over brute force by only a superpolynomial amount, he constructs a lower bound against $\text{NEXP}$ (nondeterministic exponential time). Not only is the satisfiability algorithm used as a black box, the result applies to a large set of natural circuit classes. By giving a satisfiability algorithm for $\text{ACC}^0$, Williams completes an (unconditional) proof for $\text{NEXP} \not\subseteq \text{ACC}^0$. Since the connection between satisfiability algorithms and circuit bounds is more general than just $\text{ACC}^0$ circuits, his result opens up a possible path to prove further lower bounds in the future.

The technique by Williams achieves a similar goal as the examples in the previous section, as the result is both a satisfiability algorithm for $\text{ACC}^0$ and a lower bound for the same circuit class. The satisfiability algorithms relies on properties of the circuit class. However, instead of deriving a circuit lower bound directly from the same properties, Williams adds another layer of abstraction. The proof of the circuit lower bound does not depend on the properties of the circuit directly, but only on the derived satisfiability algorithm. As a consequence of this abstraction, he is able to formalize a connection between algorithms and lower bounds. While it is difficult to characterize what properties of circuit classes lead to both satisfiability algorithms and lower bounds, the abstraction allows a quantitative statement on the required satisfiability algorithm.

In the first paper [30], Williams proves that if there is an algorithm for general circuit satisfiability that improves over exhaustive search by a superpolynomial amount, then $\text{NEXP} \not\subseteq \text{P/Poly}$. The proof is an indirect diagonalization argument. Assuming $\text{NEXP} \subseteq \text{P/Poly}$ and the existence of a fast satisfiability algorithm for general $\text{P/Poly}$ circuits, it gives an algorithm to solve an arbitrary problem $L \in \text{NTIME}(2^n)$ in nondeterministic time $O(2^n/\omega)$ for some superpolynomial
As a result, there are no problems in $\text{NTIME}(2^n)$ that are not in $\text{NTIME}(2^n/\omega)$, which contradicts the nondeterministic time hierarchy theorem \[25\].

For a rough outline of the proof, suppose there is a satisfiability algorithm for general circuits that improves over exhaustive search by a superpolynomial factor and $\text{NEXP} \subseteq \text{P}/\text{Poly}$. Then pick an arbitrary problem $L$ in $\text{NTIME}(2^n)$ and reduce it to the $\text{Succinct-3-SAT}$ problem, which is $\text{NEXP}$-complete. The Succinct-3-SAT problem is a variation on 3-SAT for exponential formulas. Instead of having the 3-CNF as an direct input, the input is a polynomial size circuit, such that on input $i$ in binary, the output is the $i$th bit of the encoding of the 3-CNF. The Succinct-3-SAT problem is then to decide if the implied 3-CNF is satisfiable. By the $\text{NEXP}$-completeness of Succinct-3-SAT we can, given an input $x$ to $L$, construct a polynomial size circuit $C$ with $n + O(\log n)$ inputs such that on input $i$ in binary, the output is the $i$th bit of a 3-CNF that is satisfiable if and only if $x \in L$. The number of variables of this 3-CNF formula is exponential in $n$.

To test the satisfiability of this circuit without explicitly writing out the 3-CNF formula, we use the idea of a universal witness. Impagliazzo, Kabarnets and Wigderson \[10\] show that if $\text{NEXP} \subseteq \text{P}/\text{Poly}$, then for every satisfiable instance of a Succinct-3-SAT problem there is a polynomial size circuit such that on input $i$ in binary, it outputs the value of the $i$th variable in a satisfying assignment.

The nondeterministic algorithm proceeds as follows. First nondeterministically guess the universal witness for the given Succinct-3-SAT problem. Since the goal is to give an algorithm that runs in $\text{NTIME}(2^n/\omega)$ the algorithm is free to use nondeterminism at this point. Let this circuit be called $D$. From the Succinct-3-SAT instance $C$ we can construct a circuit $C'$ that takes as input a number $i$ in binary, and outputs the $i$th clause, consisting of three variables in binary (requiring $n + O(\log n)$ bits each) and three bits to indicate if the literals are negated. Each of these variables is then given as input to the circuit $D$. As a last step, we can check if the values that $D$ assigns to the variables satisfies the clause.

The circuit $D$ is a universal witness for the 3-CNF formula if and only if the constructed circuit is unsatisfiable, i.e. there is no input $i$ such that the universal witness does not give an assignment that satisfies the $i$th clause. Using the assumed fast algorithm for circuit satisfiability, we can decide this in time $O(2^n/\omega)$, resulting in an overall algorithm in $\text{NTIME}(2^n/\omega)$, contradicting the nondeterministic time hierarchy theorem.

In the second paper \[31\], Williams refines his result for restricted circuit classes. For any circuit class $\mathcal{C}$ that contains $\text{AC}^0$ and is closed under composition, if there is a satisfiability algorithm for $\mathcal{C}$ that improves over exhaustive search by a superpolynomial amount, then $\text{NEXP} \not\subseteq \mathcal{C}$. The main part of the proof is ensuring that the circuit constructed for the proof is in the class $\mathcal{C}$ so that we can apply the supposed algorithm for $\mathcal{C}$-SAT. In particular, the circuit $C'$ that takes as input a value $i$ and returns the $i$th clause is not necessarily in the class $\mathcal{C}$. The key idea to get around this is by guessing and checking an equivalent $\mathcal{C}$-circuit, and then building the whole circuit using the guessed component. By giving an algorithm for $\text{ACC}^0$-SAT in the same paper he completes the proof for $\text{NEXP} \not\subseteq \text{ACC}^0$.

If this approach is useful for other circuit classes than $\text{ACC}^0$ depends on if it is possible to find fast satisfiability algorithms for these classes. The result does certainly motivate the search for satisfiability algorithms for circuit classes that sit in expressive power somewhere between $\text{ACC}^0$ and $\text{P}/\text{Poly}$. 
5 Reductions to Polynomial Time Problems

Lower bounds and algorithms faster than the trivial approach are not something unique to circuits. For example, matrix multiplication has a trivial $O(n^3)$ algorithm. Until Strassen [26] gave a faster algorithm, it was unknown if this is the best we can do. Since then, several algorithms were discovered that improve on Strassen’s runtime, most notably Coppersmith and Winograd [5], and Williams [32]. Despite this progress, the exact value for the matrix multiplication exponent, the smallest $\omega$ such that matrix multiplication can be solved in time $O(n^\omega)$ is still unknown. It follows from Williams’ result that $\omega < 2.3727$, but it is not clear how far this can be improved.

The ingenuity that goes into these faster algorithm can be used for faster satisfiability algorithms. Williams [28] uses this idea directly and reduces MAX-2-SAT to matrix multiplication. Would one use the reduction to matrix multiplication and then use the trivial algorithm for the multiplication, the resulting algorithm would run in time $\tilde{O}(2^n)$. It is the faster matrix multiplication algorithm that results in constant savings.

The other examples discussed here reduce a satisfiability problem to other polynomial time problems. Impagliazzo, Paturi and Schneider [14] give a satisfiability algorithm for depth two threshold circuits that reduces the problem to the Vector Domination Problem, the problem of finding two vectors such that one dominates the other on every coordinate. The problem can be trivially solved in quadratic time. However, only using an algorithm faster than quadratic for the vector problem do we get a satisfiability algorithm with any savings.

Lastly we discuss a result by Pătraşcu and Williams [21], who reduce CNF-SAT to k-Dominating Set, the problem of finding a set of at most $k$ vertices in a graph such that every vertex is either in the set or adjacent to a vertex in the set. This reduction has a different flavor from the other reductions in the conclusions we can draw from the result. For both MAX-2-SAT and threshold circuits, the result is an algorithm with constant savings by using a fast algorithm for the polynomial time problem. For CNF-SAT, no algorithm with constant savings is known. If one subscribes to the belief that there are no algorithms with constant savings for CNF-SAT, then the reduction gives a lower bound for k-Dominating Set. If one does not believe that such a lower bound exists, then the reduction gives a mean to find a fast satisfiability algorithm.

All three algorithm follow a paradigm called “Split and List”: Split the variable set into several parts, and list every possible restriction of one of the parts. Using the list of (exponentially many) restrictions, the problem is then reduced to an exponentially large instance of the underlying polynomial time problem. As a consequence of the “Split and List” approach, all three algorithms require exponential space, which is a limiting factor for using these algorithms in practice.

The “Split and List” approach opens up a wide range of possible algorithms to explore. While matrix multiplication is a well studied problem with countless applications, the same is not true for the Vector Domination Problem. I am not aware of any applications outside of the literature on satisfiability algorithms, although it is not unlikely that it was used (under a different name) in a different context. This motivates looking for more problems that have not gathered a lot of attention but might have both a “Split and List” reduction from satisfiability problems and a nontrivial algorithm. A good place to start might be quadratic problems, problems whose trivial algorithm runs in quadratic time. The Vector Domination Problem is an example. There are many problems where the goal is to find a pair that satisfies some property and that have trivial quadratic runtime. Just as there is no known characterization of which circuit classes allow fast satisfiability algorithms, there is no characterization of which quadratic problems have subquadratic algorithms.

For more applications of the “Split and List” approach, see Chapter 6 of Williams’ Ph.D. thesis.
5.1 MAX-2-SAT

In this section we consider an algorithm for MAX-2-SAT by Williams [28]. Let $F$ be a 2-CNF on $n$ variables and $m$ clauses. The MAX-2-SAT problem asks if given a threshold $t$, is it possible to satisfy at least $t$ clauses. Williams gives an algorithm with constant savings that also generalizes to a weighted version of MAX-2-SAT, if the weights are small and integer. For the purpose of this paper, we will consider the unweighted case only.

This algorithm does not generalize directly to MAX-$k$-SAT. There are no known algorithms that achieve constant savings for MAX-$k$-SAT for $k \geq 3$. This marks a significant difference between MAX-$k$-SAT and $k$-SAT, as for $k$-SAT there are algorithms achieving constant savings for all constants $k$.

Using a “Split and List” technique, the algorithm reduces MAX-2-SAT to the problem of finding a triangle in a $2^{n/3} \times 2^{n/3} \times 2^{n/3}$ tripartite graph, which in turn can be solved by multiplying two $2^{n/3} \times 2^{n/3}$ matrices. Let $\omega$ denote the matrix multiplication exponent, i.e. the exponent of the fastest possible matrix multiplication algorithm.

**Theorem 3.** Let $F$ be a 2-CNF and let $t \in \mathbb{N}$. There is an algorithm that to find an assignment that satisfies at least $t$ clauses and runs in time $\tilde{O}(2^{\frac{7}{3}n})$, where $\omega$ is the matrix multiplication exponent.

**Proof.** We assume $n$ is divisible by 3. Separate the set of variables into three sets $A, B$, and $C$ all of size $\frac{n}{3}$. We can then distinguish six types of 2-clauses:

1. Both variables of the clause are in $A$.
2. Both variables are in $B$.
3. Both variables are in $C$.
4. Exactly one variable is in $A$ and exactly one variable is in $B$.
5. One variable is in $B$ and one variable is in $C$.
6. One variable is in $A$ and one variable is in $C$.

We say a clause is of type $T_a$, $T_b$, $T_c$, $T_{ab}$, $T_{bc}$ or $T_{ac}$ respectively.

For numbers, $s_a$, $s_b$, $s_c$, $s_{ab}$, $s_{bc}$ and $s_{ac}$ such that their sum is at least $t$, the algorithm decides if there is an assignment that satisfies exactly $s_d$ clauses of type $T_D$ for $D \in \{a, b, c, ab, bc, ac\}$. Since each number $s_D$ is a number between 0 and $m$, there are at most $m^6$ combinations of numbers. A 2-CNF has at most $4n^2$ clauses, hence solving each combination separately is only a polynomial overhead.

We now construct the following graph. Let $V_A$ be the set of assignments to the variables $A$ such that it satisfies exactly $s_a$ variables of type $T_a$. Likewise, let $V_B$ and $V_C$ be the set of assignments that satisfy exactly $s_b$ and $s_c$ clauses of their respective type. The vertex set of the graph is $V = V_A \cup V_B \cup V_C$. We have an edge between a vertex in $V_A$ and a vertex in $V_B$ if the two assignments together satisfy exactly $s_{ab}$ clauses of type $T_{a,b}$. We add edges between $V_B$ and $V_C$, and $V_A$ and $V_C$ in a similar fashion.
There is an assignment to the variables that satisfies exactly $s_D$ clauses of type $T_D$ for all $D$, if and only if there is a triangle in the constructed graph. The assignment corresponds to the three vertices in the triangle.

To find a triangle using matrix multiplication we construct a matrix $M_{ab}$ such that $M_{ab}[i,j] = 1$ if there is an edge between the $i$th element of $V_A$ and the $j$th element of $V_B$. We also construct matrices $M_{bc}$ and $M_{ac}$ in a similar fashion. Then there is a triangle if and only if there is an $i$ and a $j$ such that $(M_{ab} \cdot M_{bc})[i,j] \geq 1$ and $M_{ac}[i,j] = 1$. Since all matrices have size at most $2^{n/3} \times 2^{n/3}$ we can do the multiplication in time $O(2^{\omega n})$. We have a multiplicative overhead as we have to do a matrix multiplication for every possible combination of numbers $s_D$. However, this overhead only contributes a polynomial factor to the time of the whole algorithm.

5.2 Threshold Circuits

In this section we consider threshold circuits of depth two. The algorithm by Impagliazzo, Paturi, and Schneider [14] combines random restrictions as in Subbotovskaya’s lower bound for DeMorgan Formulas with the “Split and List” approach of Williams’ MAX-2-SAT algorithm.

We give an algorithm that decides satisfiability of a depth two threshold circuit on $n$ variables with $cn$ wires and arbitrary real weights that runs with savings of the form $O(\frac{1}{c^{O(1)} \cdot \omega})$. For this algorithm, the restriction on the size of the circuit is not on the number of gates, but on the number of literal occurrences.

The algorithm proceeds in two steps. First, we reduce the satisfiability problem on a depth two threshold circuit with $cn$ wires to (not too many) satisfiability problems on depth two threshold circuits on $n'$ variables and $\delta n'$ bottom-level gates, where $\delta$ is a small constant we can choose freely. For the circuit with few bottom-level gates, we need to allow direct wires, i.e. variables that directly feed into the top-level gate.

As a second step, we reduce the satisfiability problem on the remaining circuit to a problem we call the Vector Domination Problem. The Vector Domination Problem takes as inputs two sets of $d$-dimensional real vectors $A$ and $B$ with $|A| + |B| = N$ and the goal is to find a vector $a \in A$ and a vector $b \in B$ such that for every coordinate $i$, $a_i \leq b_i$. The reduction follows the “Split and List” paradigm.

The reduction from the satisfiability problem of depth two threshold circuits on $n'$ variables and $\delta n'$ bottom-level gates to the Vector Domination Problem gives an instance with $|A| = |B| = 2^{\delta n/2}$ and dimension $d = \delta n$. We could then solve the Vector Domination problem with the trivial $O(N^2)$ algorithm. Unfortunately, this would not give an algorithm faster than exhaustive search. Instead, we give an algorithm faster than quadratic for $\delta < 0.136$ which yields a satisfiability algorithm with constant savings.

For the reduction from a threshold circuit with a linear number of wires to a threshold circuit with few bottom-level gates, the idea is to select an (as large as possible) set $S$ such that restricting all variables not in $S$ results in a circuit with at most $\delta|S|$ bottom-level gates. The key observation is that gates that depend on at most one variable in $S$ simplify to a constant or a direct wire to the top-level gate after restriction, independent of the values the restriction assigns to the variables. We omit the details of the calculations here. It is possible to find a set $S$ with $|S| \geq \frac{\delta}{2^{\delta n/2}}$ such that the circuits have at most $\delta|S|$ bottom-level gates. The proof relies on random restrictions.

The reduction from the satisfiability problem of a depth two threshold circuit on few bottom-level gates to the Vector Domination Problem is by a “Split and List” approach. First, for all remaining bottom-level gates, fix the output to either 0 or 1. There are $2^{|S|}$ such combinations.
For every threshold gate, we can express the condition that its output is 1 or 0 respectively as a linear inequality. Given output values for the bottom-level gate, we can also express the top-level gate as a linear inequality in the input variables. We now reduce the resulting system of linear inequalities to the Vector Domination Problem as follows. Split the set of remaining variables $S$ into two sets $S_1 = \{x_1, \ldots, x_{|S|/2}\}$ and $S_2 = \{x_{|S|/2+1}, \ldots, x_{|S|}\}$ of equal size. A linear inequality of the form $\sum_{i=1}^{|S|} a_i x_i \geq t$ is true if and only if $\sum_{i=1}^{|S|/2} a_i x_i \geq t - \sum_{i=|S|/2+1}^{|S|} a_i x_i$. Hence we can list all possible assignments to the variables in $S_1$ and calculate $\sum_{i=1}^{|S|/2} a_i x_i$ for each of the $|S| + 1$ inequalities. Likewise, calculate $t - \sum_{i=|S|/2+1}^{|S|} a_i x_i$ for each assignment to $S_2$ and each inequality. The system of inequalities is then satisfied by an assignment to both $S_1$ and $S_2$ if the vector of these values for the assignment to $S_1$ dominates the vector of values for the assignment $S_2$. The resulting Vector Domination Problem has $N = 2 \cdot 2^{|S|/2}$ vectors and dimension $\delta |S| + 1 \approx 2\delta \log N$.

The last part of the algorithm for depth two threshold circuits is an algorithm for the Vector Domination problem. Let $A$ and $B$ be the two sets vectors of dimension $d$. Let $N = |A| + |B|$. The algorithm is faster than the trivial $O(N^2)$ for $d \leq 0.272 \log N$ and works as follows. Let $m$ be the median of the first coordinates of both $A$ and $B$ and split the sets $A$ and $B$ into sets $A^+$, $A^-$, $B^+$, $B^-$, and $B^-$ depending on if the first coordinate is larger, equal, or smaller than the median. Then for a vector $a \in A$ to dominate a vector $b \in B$ either $a \in A^+$ and $b \in B^+$, or $a \in A^-$ and $b \in B^-$, or $a \in A^+ \cup A^-$ and $b \in B^- \cup B^+$. In the first two cases, the number of vectors can be at most half as we split at the median. In the last case, we know that the first coordinate of $a$ dominates the first coordinate of $b$. We can therefore recurse on vectors of dimension $d - 1$.

Furthermore, we require time $O(N)$ to calculate the median and split the sets $A$ and $B$. Hence the runtime of this algorithm for $N$ vectors of dimension $d$ is bounded by the recurrence relation

$$T(N, d) = 2T(N/2, d) + T(N, d - 1) + O(N)$$

which solves to

$$T(N, d) = \left(1 + \frac{\log N + 1}{d + 1}\right)O(N)$$

This runtime is $O\left(N^{2 - f(\delta)}\right)$ where $f(\delta) > 0$ for $\delta < 0.272$. This results in an algorithm for the satisfiability problem for depth two threshold circuits on $n'$ variables with $\delta n'$ bottom-level gates that runs in time $O\left(2^{(1 - g(\delta))n}\right)$, where $g(\delta) > 0$ if $\delta < 0.099$. The stronger requirement for $\delta$ comes from the additional overhead of guessing the output value for all bottom-level gates.

Choosing an arbitrary value $\delta$ smaller than 0.099 results in a satisfiability algorithm for depth two threshold circuits with $cn$ wires that runs in time $\tilde{O}\left(2^{(1 - 1/e^{O(n^2)})n}\right)$.

### 5.3 Reductions as Lower Bounds

In this section we discuss a reduction from a satisfiability problem to a polynomial time problem that is considerably different from the reductions discussed above in the conclusion it allows. In the previous sections we got satisfiability algorithms with constant savings by first reducing to a polynomial time problem and then solving that problem in a nontrivial way. In this section we discuss a result by Pătraşcu and Williams [21] which reduces CNF-SAT to $k$-Dominating Set. For $k$-Dominating Set, we are given a graph on $n$ vertices and $m$ edges and find a set $S$ of $k$ vertices, such that every vertex is either in $S$ or adjacent to a vertex in $S$. 

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There are two ways to interpret the result. If you subscribe to the belief that there are no algorithms for CNF-SAT with constant savings, then there are no algorithms faster than the currently known ones for k-Dominating Set. On the other hand, if you believe there to be faster CNF-SAT algorithms, then these reductions provide a possible way of finding such an algorithm.

The trivial algorithm for the k-Dominating Set problem is to enumerate all \( \binom{n}{k} \) sets of size \( k \) and test each of them in linear time. The resulting algorithm then runs in time \( O(n^{k+1}) \). Eisenbrand and Grandoni [6] give a faster algorithm that uses fast matrix multiplication.

**Lemma 4.** For \( k \geq 7 \), there is an algorithm for k-Dominating Set that runs in time \( n^{k+o(1)} \).

The details of the algorithm are omitted here.

While this algorithm improves over the trivial algorithm by almost a linear factor, it still requires that we consider every possible set of size \( k \).

Assuming that there is an algorithm for k-Dominating Set that avoids listing every possible set of size \( k \), we construct an algorithm for CNF-SAT that achieves constant savings.

**Theorem 4.** Assume there exists \( k \geq 3 \) such that k-Dominating Set has an algorithm that runs in time \( O(n^{k-\varepsilon}) \) for some \( \varepsilon > 0 \). Then there is an algorithm for CNF-SAT that runs in time \( \bar{O}(2^{(1-\varepsilon/k)n}) \).

**Proof.** Fix \( k \geq 3 \) to the smallest value such that there is a fast algorithm for k-Dominating Set. We assume that \( k \) divides \( n \).

Given a CNF on \( n \) variables and \( m \) clauses, we construct a graph in a “Split and List” fashion very similar to the construction of the graph for the MAX-2-SAT algorithm. Split the vertex set into \( k \) sets of size \( \frac{n}{k} \) each and for each assignment to one of sets, add a vertex to the graph. We further add edges such that each of the groups of \( 2^{n/k} \) vertices is a clique. For each clause we add one extra vertex which we connect to all vertices that correspond to partial assignments that satisfy the clause, i.e. the partial assignment assigns the value 1 to at least one literal in the clause. Lastly we add one extra vertex to every clique, not connected to any clause. We call this vertex the *dummy node*.

Consider a \( k \)-dominating set \( S \) for this graph. Since every dummy node is covered, there must be at least one vertex chosen from every clique. Since there are \( k \) cliques, each clique must have exactly one element in \( S \). Furthermore, for every clause, there must be vertex in \( S \) that is connected to the clause. Hence the partial assignment represented by that vertex satisfies the clause. Therefore, the union of the partial assignments in the set \( S \) is an assignment that satisfies every clause.

The number of vertices in the graph is \( k2^{n/k} + k + m \). By assumption we can solve the k-Dominating Set problem, and therefore the CNF-SAT problem, in time

\[
O \left( \left( k2^{n/k} + k + m \right)^{k-\varepsilon} \right) = O \left( 2^{(1-\varepsilon/k)n} \right) \text{poly}(m)
\]

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