On Hopf anomaly

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Abstract

This is a very brief but self-contained review of the concept of quantum group symmetries and their anomalies. Remarkably, general constructions can be very simply illustrated on the standard harmonic oscillator which is shown to possess a non-commutative and non-cocommutative anomalous quantum group symmetry.

1 Introduction.

The goal of these notes is to illustrate some Hopf (or quantum group) symmetry story on a simple example. For the latter, we have chosen the standard harmonic oscillator. General quantum group facts presented here are mostly known (though they are often presented in the literature in a somewhat scattered way) but the harmonic oscillator example of Hopf anomaly is original.

In the second section, we define the notion of the Hopf symmetry of quantum dynamical systems and in the third section, we develop the concept of Hopf moment maps and Hopf anomalies. In the fourth section, we illustrate all those notions on the simple example: the scaling symmetry of the harmonic oscillator. Some definitions and basic properties of Hopf algebras can be found in Appendix.
2 Hopf symmetry

Recall that an abstract quantum dynamical system is a pair \((A, \alpha_t)\) consisting of a noncommutative algebra \(A\) of observables and a time-evolution, i.e. a one-parameter group \(\alpha_t\) of automorphisms of \(A\). We say that a Hopf algebra \(H\) is the Hopf symmetry of \((A, \alpha_t)\) if:

1. \(A\) is a \(H\)-module-algebra.
2. Given \(x \in H\), it exists \(x(t) \in H\) such that there is the following equality of maps \(A \to A\): \(\alpha_t \circ x = x(t) \circ \alpha_t\).

Clarifications: 1. The algebra \(A\) is the \(H\)-module-algebra if \(H\) acts on \(A\) (i.e. there is the morphism of algebras \(H\) and \(\text{End}(A)\)) and this action respects the following Hopf-Leibniz rule:

\[
x(f * g) = (x'f) * (x''g), \quad x \in H, \quad f, g \in A.
\]

Here we used the Sweedler notation for the coproduct \(\Delta x = x' \otimes x''\) and \(*\) denotes the noncommutative product in \(A\). The Hopf-Leibniz rule can be interpreted as follows: the coproduct on \(H\) encodes the compatibility of the \(H\)-action with the product \(*\) in \(A\). From the point of view of noncommutative geometry, the Hopf symmetry thus acts by "noncommutative vector fields" which are NOT necessarily the derivations of the noncommutative algebra \(A\). This may seem surprising but it is the fact which has been since long time established either in the noncommutative geometry literature by [Connes and Moscovici 1] or on the quantum group side, see e.g. the book of [Majid]. Actually, only when we take \(H\) to be the (cocommutative) enveloping algebra \(U(\mathcal{G})\) of some Lie algebra \(\mathcal{G}\), we can see from (A1) that the Hopf-Leibniz rule (1) implies the action of \(\mathcal{G}\) on \(A\) by derivations. Indeed, Eqs.(1) and (A1) then give

\[
x(f * g) = (xf) * g + f * (xg), \quad x \in \mathcal{G}, \quad f, g \in A.
\]

In fact, this special case \(H = U(\mathcal{G})\) corresponds to the situation when the Hopf symmetry becomes the ordinary symmetry known from the textbooks on quantum mechanics.

2. The second condition of the Hopf symmetry is sometimes formulated in a more restrictive way by claiming that the action of \(H\) on \(A\)
should commute with the evolution $\alpha_t$. However, we prefer our more general formulation which, by the way, is necessary to describe important symmetries as e.g. the Poincare symmetry of relativistic systems.

3  Hopf moment maps and anomalies

The Hopf symmetry $\mathcal{H}$ of $(A, \alpha_t)$ is called Noetherian if it admits the Hopf moment map. If it does not admit it, it is called anomalous. To our best knowledge, the concept of the Hopf moment map was introduced by [Korogodsky]. It is an algebra homomorphism $m : \mathcal{H} \to A$ such that the action of $\mathcal{H}$ on $A$ can be written as follows

$$xf = m(x') * f * m(S(x'')) , \quad x \in \mathcal{H}, \quad f \in A. \quad (2)$$

Here again we use the Sweedler notation for the coproduct and $S$ stands for the antipode of $\mathcal{H}$. In particular, if $\mathcal{H} = U(\mathcal{G})$ and $x \in \mathcal{G}$, then the formulae for the coproduct and the antipode (A1) permit to rewrite (2) in the following familiar form

$$xf = m(x) * f - f * m(x) \equiv [m(x), f].$$

The Noetherian Hopf symmetry descends to the space of states $V$. Let us explain what this means: we call a triple $(A, \alpha_t, V)$ a concrete realization of the abstract quantum dynamical system $(A, \alpha_t)$ if $V$ is the representation space of the algebra $A$ such that the evolution is given by unitary operators $U(t)$ on $V$. The Hopf moment map $m$ gives immediately the representation of $\mathcal{H}$ on $V$. On the other hand, the anomalous Hopf symmetry exists only on the level of the algebra of observables and cannot be implemented on the space of states.

4  Scaling the harmonic oscillator

Consider the anihilation and creation operators $a, \bar{a}$ fulfilling the standard commutation relation

$$[a, \bar{a}] = 1.$$

The algebra $A$ consists of polynomials in $a, \bar{a}$ and the evolution automorphism $\alpha_t$ on $A$ is defined by

$$\alpha_t(a) = e^{-i\omega t}a, \quad \alpha_t(\bar{a}) = e^{i\omega t}\bar{a},$$
where $\omega$ is a parameter.

Then consider a Hopf algebra $H$ (with a unit element $\eta$) generated by $T_0, T_1, T_2$ and by the following relations

$$[T_0, T_1] = 2T_1, \quad [T_0, T_2] = 2T_2, \quad [T_1, T_2] = 0.$$ 

The coproduct $\Delta$ is defined as

$$\Delta T_\alpha = \eta \otimes T_\alpha + T_\alpha \otimes \eta, \quad \alpha = 1, 2; \quad (3)$$

$$\Delta T_0 = \eta \otimes T_0 + T_0 \otimes \eta - 4T_1 \otimes T_2. \quad (4)$$

The antipode $S$ reads

$$S(\eta) = \eta, \quad S(T_0) = -T_0 - 4T_1T_2, \quad S(T_1) = -T_1, \quad S(T_2) = -T_2 \quad (5)$$

and the counit $\varepsilon$ is

$$\varepsilon(\eta) = 1, \quad \varepsilon(T_0) = \varepsilon(T_1) = \varepsilon(T_2) = 0.$$

**Note**: Although our Hopf algebra $\mathcal{H}$ does not coincide with that of [Connes and Moscovici 2] there are nevertheless some similarities between them.

Now we define the action of $\mathcal{H}$ on $A$ as follows

$$T_0(\bar{a}^m a^n) = -2(m+n)\bar{a}^m a^n, \quad T_1(\bar{a}^m a^n) = -n\bar{a}^m a^{n-1}, \quad T_2(\bar{a}^m a^n) = m\bar{a}^{m-1} a^n.$$ 

Is this $\mathcal{H}$-action the Hopf symmetry of $(A, \alpha_t)$? It is easy to check the Hopf-Leibniz rule (1) with the help of the following formulae:

$$T_0 f = -2[\bar{a}a, f] + 4[\bar{a}, f]a, \quad T_1 f = [\bar{a}, f], \quad T_2 f = [a, f], \quad f \in A. \quad (6)$$

It is also easy to verify the second condition on p.2. Indeed, the action of $T_0$ clearly commutes with the evolution $\alpha_t$, while for the action of $T_1$ and $T_2$, we have

$$\alpha_t \circ T_1 = (e^{i\omega t} T_1) \circ \alpha_t, \quad \alpha_t \circ T_2 = (e^{-i\omega t} T_2) \circ \alpha_t.$$ 

Thus $\mathcal{H}$ is the Hopf symmetry of the harmonic oscillator.

It turns out that the Hopf symmetry $\mathcal{H}$ does not admit the Hopf moment map $m$. Indeed, by inspecting the formulae (2), (3), (4), (5) and (6), we easily find that if $m$ existed then it would have to be given by

$$m(T_0) = -2\bar{a}a + c_0, \quad m(T_1) = \bar{a} + c_1, \quad m(T_2) = a. \quad (7)$$
Here $c_0, c_1$ are central in $A$. However, whatever choice of $c_0, c_1 \in A$ we make, the map $m$ defined by Eq.(7) is not the algebra homomorphism because

$$m([T_1, T_2]) \neq [m(T_1), m(T_2)], \quad m([T_0, T_1]) \neq [m(T_0), m(T_1)].$$

What is the physical interpretation of this Hopf symmetry? Clearly, $(T_1 + T_2)$ and $i(T_1 - T_2)$ generate the Galileo boost and the translation, respectively, while $T_0$ is just the scaling of $A$. We know that the classical Hamiltonian equations of the harmonic oscillator are linear; this is also true for the quantum equations of motion written in the Heisenberg picture:

$$\frac{d}{dt}a(t) = -i\omega a(t), \quad \frac{d}{dt}\bar{a}(t) = i\omega \bar{a}(t).$$

It is therefore obvious, that the scaling $a, \bar{a} \rightarrow \lambda a, \lambda \bar{a}$ commutes with the evolution $a_t(a) \equiv a(t), a_t(\bar{a}) \equiv \bar{a}(t)$. We have moreover established that this scaling is the anomalous non-cocommutative Hopf symmetry.

5 Appendix: A Hopf primer

**Definition:** A Hopf algebra $\mathcal{H}$ is an unital associative algebra equipped with algebra homomorphisms $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and $\varepsilon : \mathcal{H} \rightarrow \mathbb{R}$ and with an algebra antihomomorphism $S : \mathcal{H} \rightarrow \mathcal{H}$ (i.e $S(xy) = S(y)S(x), S(\eta) = \eta$) verifying the following axioms

$$\varepsilon(x')x'' = x'\varepsilon(x'') = x, \quad S(x')x'' = x'S(x'') = \varepsilon(x)\eta, \quad \varepsilon(S(x)) = \varepsilon(x),$$

$$S(x') \otimes S(x'') = (S(x))'' \otimes (S(x))', \quad \Delta x' \otimes x'' = x' \otimes \Delta x'', \quad x \in \mathcal{H}.$$

**Clarifications:**
1. We have denoted the unit of $\mathcal{H}$ by $\eta$.
2. The multiplication in $\mathcal{H}$ can be denoted as $\mu : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ but often we write only $xy$ instead of $\mu(x \otimes y)$.
3. We have used the Sweedler notation for the coproduct

$$\Delta x = \sum_j x'_j \otimes x''_j \equiv x' \otimes x'', \quad x \in \mathcal{H}.$$
4. We note that the product on \( \mathcal{H} \) canonically induces the product on \( \mathcal{H} \otimes \mathcal{H} \) whose unit element is \( \eta \otimes \eta \).

5. We say that \( \mathcal{H} \) is cocommutative if \( x' \otimes x'' = x'' \otimes x' \).

**Basic cocommutative example:** The enveloping algebra \( U(\mathcal{G}) \) of a Lie algebra \( \mathcal{G} \) is the cocommutative Hopf algebra whose structure is completely determined by the Lie bracket \([.,.]\) in \( \mathcal{G} \). Indeed, if \( T \) is the tensor algebra of the vector space \( \mathcal{G} \), then \( U(\mathcal{G}) = T/J \), where the two-sided ideal \( J \) is generated by tensors

\[
x \otimes y - y \otimes x - [x,y], \quad x, y \in \mathcal{G}.
\]

The coproduct \( \Delta : U(\mathcal{G}) \to U(\mathcal{G}) \otimes U(\mathcal{G}) \), the counit \( \varepsilon : U(\mathcal{G}) \to \mathbb{R} \) and the antipode \( S : U(\mathcal{G}) \to U(\mathcal{G}) \) are defined as

\[
\Delta x = \eta \otimes x + x \otimes \eta, \quad \varepsilon(x) = 0, \quad S(x) = -x, \quad x \in \mathcal{G}.
\]  

(A1)

We note that the definition of \( \Delta \), \( \varepsilon \) and \( S \) acting on whatever element of \( U(\mathcal{G}) \) follows from (A1) and from the (anti)homomorphism properties of \( \Delta \), \( \varepsilon \) and \( S \).

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**References:**

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