Probabilistic generation of entanglement in optical cavities

Anders S. Sørensen and Klaus Mølmer
QUANTOP, Danish National Research Foundation Center for Quantum Optics
Department of Physics and Astronomy, University of Aarhus, DK-8000 Aarhus C, Denmark

We propose to produce entanglement by measuring the transmission of an optical cavity. Conditioned on the detection of a reflected photon, pairs of atoms in the cavity are prepared in maximally entangled states. The success probability depends on the cavity parameters, but high quality entangled states may be produced with a high probability even for cavities of moderate quality.

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Quantum Non-Demolition (QND) measurements on samples of atoms have been proposed as a means to entangle atoms in the samples [1, 2]. By QND detection atoms can be projected into an entangled state by the measurement and following these proposals experiments have produced and verified entanglement of large atomic samples [3, 4]. Here we propose to use a similar QND detection to entangle pairs of atoms inside an optical cavity.

By entangling pairs of atoms the present scheme could be used to connect small scale quantum information processors, e.g. trapped ion crystals, and thereby facilitate the construction of a large scale quantum computer.

To our knowledge, all existing proposals for quantum computation with cavity QED require the strong coupling regime \( g^2/\kappa \gamma \gg 1 \), where \( g \) is the atom-cavity coupling constant, \( \kappa \) is the cavity decay rate and \( \gamma \) is the decay rate of the atoms. Here we avoid this condition by using a probabilistic scheme, which is only successful with a probability \( P_s < 1 \), but where we are certain that we have produced an entangled state when we have a click in a detector. Optical transmission has already been used in Ref. [3] to probe atoms inside a cavity, and here we show that a similar setup can be used to entangle two atoms in the cavity. Probabilistic generation of entanglement has also been suggested in Refs. [4, 5, 6].

Our proposal combines advantages of each of these proposals: as in Refs. [4, 5, 6] we entangle pairs of atoms (not samples) and by using cavities the outgoing photons are in a well confined mode as in Refs. [3, 4, 5], but unlike Refs. [4, 5, 6] we do not require the strong coupling limit \( g^2/\kappa \gamma \gg 1 \).

The experimental setup is sketched in Fig. 1(a). We consider a single mode of an optical cavity with annihilation operator \( \hat{c} \). In the figure we show a ring cavity, but a standing wave cavity could also be used. The field in the cavity can decay through two leaky mirrors with decay rates \( \kappa_a \) and \( \kappa_b \). At the mirrors we have incoming fields described by \( \hat{a}_{\text{in}}, \hat{b}_{\text{in}} \) and outgoing fields described by \( \hat{a}_{\text{out}}, \hat{b}_{\text{out}} \). We assume that light is shined onto the cavity in the mode described by \( \hat{a}_{\text{in}} \) and that the other incoming mode is in the vacuum state. If the two cavity mirrors have the same decay rates \( \kappa_a = \kappa_b \) and the incident light is on resonance with the cavity it is transmitted through the cavity without any reflection. If, however, anything disturbs the field, the cavity no longer transmits all the light. By detecting scattered photons we thus register the atomic interaction with the cavity field, and this measurement can be used to entangle atoms in the cavity.

We consider two atoms with two stable ground states \( |0\rangle \) and \( |1\rangle \) and an excited state \( |e\rangle \). The cavity field couples the state \( |1\rangle \) to the excited state \( |e\rangle \) with a coupling strength \( g_k \) for the \( k \)th atom. See Fig. 1(b). The excited state \( |e\rangle \) decays with a rate \( \gamma \) and we assume a closed transition such that atoms always end up in the state \( |1\rangle \) after a decay. Initially each of the atoms is prepared in the state \( \cos(\phi)|0\rangle + \sin(\phi)|1\rangle \) so that the combined state of the two atoms is given by

\[
\cos^2(\phi)|00\rangle + \sqrt{2} \cos(\phi) \sin(\phi) \frac{|01\rangle + |10\rangle}{\sqrt{2}} + \sin^2(\phi)|11\rangle.
\]

Since the state \( |0\rangle \) does not couple to the cavity field, the cavity transmits the field with certainty if both atoms are in this state. If a photon is detected in \( \hat{a}_{\text{out}} \) the \( |00\rangle \) component of the state (1) is projected out, and if \( \phi \ll 1 \) we are left with a good approximation of the maximally entangled state \( |\Psi_{\text{EPR}}\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \).

![FIG. 1: Experimental setup and energy levels of the atoms. (a) Light described by the annihilation operator \( \hat{a}_{\text{in}} \) is shined onto a cavity containing two atoms. By detecting the reflected light, described by \( \hat{a}_{\text{out}} \), we perform a QND detection of the presence of atoms interacting with the field. (b) The atoms have two ground states \( |0\rangle \) and \( |1\rangle \) and an excited state \( |e\rangle \) which decays to the ground state \( |1\rangle \) with a rate \( \gamma \). The cavity field couples the state \( |1\rangle \) to the state \( |e\rangle \) with a coupling strength \( g \) and a detuning \( \delta \).](image-url)
Taking into account realistic imperfections such as spontaneous emission and imperfect detectors, we calculate the success probability $P_s$ and the fidelity $F = \langle \Psi_{\text{EPR}} | \rho | \Psi_{\text{EPR}} \rangle$, where $\rho$ is the density matrix for the atoms. To calculate these quantities we first calculate the transmission properties of the cavity by assuming that the strength of the field is below saturation, and we then discuss the effect of photo detection.

In the rotating frame with respect to the cavity frequency, the interaction of the atoms with the cavity field is described by the Hamiltonian ($\hbar = 1$)

$$H = \sum_k g_k |e\rangle\langle 1| \hat{c} + g_k^* \hat{c}^\dagger |1\rangle\langle e| + \delta |e\rangle\langle e|,$$  

where $\delta$ is the detuning of the cavity from the atomic resonance, and where the sum is over all atoms in the cavity. To describe the effect of spontaneous emission we introduce Lindblad relaxation operators $\hat{d}_k = \sqrt{\gamma_k} |1\rangle\langle e|$, for each of the atoms. It is convenient to adopt different normalization conditions for the free fields and for the field in the cavity. In the cavity we normalize $\hat{c}$ to the number of photons whereas the free fields are normalized to the flux of photons, i.e., the commutation relations are $[\hat{c}(t), \hat{c}(t')^\dagger] = 1$ whereas $[\hat{a}_\text{in}(t), \hat{a}_\text{in}(t')^\dagger] = \delta(t-t')$ and similarly for the other free fields. Taking the Fourier transform of the Heisenberg equations of motion we find

$$\dot{\hat{c}}(\omega) = \sum_k g_k^* \sigma_{+k}(\omega) + i \sqrt{\kappa_a} \hat{a}_\text{in}(\omega) + i \sqrt{\kappa_b} \hat{b}_\text{in}(\omega) \sqrt{\omega^2 + \delta^2},$$

$$\sigma_{+k}(\omega) = \frac{-i}{\omega - \omega_k^2} \left[ i \gamma_k F_k(\omega) + \frac{2}{\sqrt{\pi}} \int d\omega' |g_k^* \hat{c}^\dagger(\omega') + i \sqrt{\gamma_k F_k^\dagger(\omega')} \sigma_{+k}(\omega - \omega') - \text{H.C.} \right],$$

where we have introduced $\sigma_{+k} = |1\rangle\langle e|_k$ and $\sigma_{zk} = |1\rangle\langle 1|_k - |e\rangle\langle e|_k$ and the projector onto the $\{|e\rangle, |1\rangle\}$ subspace $P_k = |1\rangle\langle 1|_k + |e\rangle\langle e|_k$. $F_k$ is the vacuum noise operator associated with the decay of the $k$th atom, and $\kappa = \kappa_a + \kappa_b$. Since the atoms cannot decay out of the $\{|e\rangle, |1\rangle\}$ subspace the projectors $P_k$ are independent of time and we find $P_k(\omega) = \sqrt{2\pi} |1\rangle\langle 1|_k(t = 0) \delta(\omega)$, where we have used that at $t = 0$ all atoms are in the ground states.

To solve the Eqs. we assume that the light intensity is so low that at most a single photon is in the cavity at a time. In this limit the equations of motion become linear and we can find the Fourier transforms of the fields by solving simple linear equations. If we neglect all terms involving more than a single $\hat{c}$ operator we can omit the integral in Eq. (3c) when we insert it into Eq. (3b), and by inserting the resulting equation into Eq. (3a) we obtain

$$\dot{\hat{c}}(\omega) = \sqrt{\kappa_a} \hat{a}_\text{in}(\omega) + \sqrt{\kappa_b} \hat{b}_\text{in}(\omega) + i \sqrt{\gamma_k} \sum_k g_k^* P_k(t=0) F_k(\omega) \sqrt{\omega^2 + \delta^2}. $$

Using the input/output relations $\hat{a}_\text{out} = \hat{a}_\text{in} - \sqrt{\kappa_a} \hat{c}$ and $\hat{b}_\text{out} = \hat{b}_\text{in} - \sqrt{\kappa_b} \hat{c}$ we calculate $R(\omega)$ and $T(\omega)$, the reflection and transmission probabilities for incident light at a frequency $\omega$. Here we shall only need the expressions for $R$ and $T$ in the situation where the light is resonant with the cavity, the cavity mode is resonant with the atoms, all coupling constants have the same magnitude $|g_k|^2 = g^2$ for all $k$, and the two mirrors have the same transmittance $\kappa_a = \kappa_b$. In this situation $R$ and $T$ are

$$R_N = \left( \frac{4 N g^2 / \kappa \gamma}{1 + 4 N g^2 / \kappa \gamma} \right)^2, \quad \quad \quad \quad (5a)$$

$$T_N = \left( \frac{1}{1 + 4 N g^2 / \kappa \gamma} \right)^2. \quad \quad \quad \quad (5b)$$

Note that $R$ and $T$ depend on $N$, the total number of atoms in state $|1\rangle$ at $t = 0$ ($N = \sum_k P_k$).

We assume that the transition between $|1\rangle$ and $|e\rangle$ is a closed optical transition so that the atoms always end up in the state $|1\rangle$ after a decay and the diagonal density matrix elements are therefore unaffected by the interaction with the cavity field. The off-diagonal density matrix elements, however, decay due to spontaneous emission from the atoms. From the Hamiltonian (2) and the relaxation operators we find the equations of motion

$$\frac{d}{dt} |0\rangle\langle 1|_k = -ig_k \hat{c}^\dagger + \sqrt{\gamma_k} F_k^\dagger |0\rangle\langle e|_k,$$

$$\frac{d}{dt} |0\rangle\langle 0|_k = |0\rangle\langle 1|_k ( -ig_k \hat{c} + \sqrt{\gamma_k} F_k ). \quad \quad \quad \quad (6a)$$

To solve the Eqs. we integrate Eq. (6b) with respect to time and substitute it for $|0\rangle\langle e|_k$ in Eq. (6a). Assuming that $|0\rangle\langle 1|_k$ does not change on a time scale $1/\gamma$ and introducing the Fourier transform of the cavity field we find

$$|0\rangle\langle 1|_k (t = \infty) = : |0\rangle\langle 1|_k(t = 0) e^{- \int dw |g_k|^2 (\omega) P_k(t=0) \sqrt{\omega^2 + \delta^2}} :,$$

where $: :$ denotes normal ordering, and where we have omitted the noise coming from $F_k$ and $F_k^\dagger$. To evaluate the quality of the produced entangled state for a pair of atoms we need the coherence $\xi = |01\rangle_2 (10\rangle_2$. By using Eq. (4) and the relation between $\hat{c}$ and $\hat{a}_\text{in}$ and by assuming $|g_1|^2 = |g_2|^2$ we find

$$\xi(t = \infty) = : \xi(t = 0) \exp \left( - \int dw \lambda(\omega) \hat{a}_\text{in}(\omega) \hat{a}_\text{in}(\omega) \right) :,$$

where $\lambda(\omega) = 1 - R_1(\omega) - T_1(\omega)$ is the probability that an incident photon with frequency $\omega$ leads to a spontaneous
emission from an atomic state with $\tilde{N} = 1$. Since only resonant light is incident on the cavity $\lambda = 1 - R_1 - T_1$ with $R_1$ and $T_1$ given by Eq. 6, the probability to have one and two atoms in state $|1\rangle$, and $R_1$ and $R_2$ are the reflection probabilities (55).

As a measure of the quality of the produced entanglement we use the fidelity $F = \langle \Psi_{\text{EPR}} | \rho | \Psi_{\text{EPR}} \rangle = p_{1,c}/2 + \text{Re}(\zeta)$, where $p_{1,c}$ is the probability to have one atom in state $|1\rangle$ conditioned on the detection of a photon. With a Fock state as input $\text{Re}(\zeta) = p_{1,c}/2$ because there can not have been spontaneous emission when the photon is detected by the detector, and conditioned on the detection of a photon the fidelity of the entangled state is $F = (p_{1,i} R_1)/(p_{1,i} R_1 + p_{2,i} R_2)$. In Fig. 2 we show the success probability $P_s$ as a function of $g^2/\kappa\gamma$ for $\eta = 1$ with fixed values of the fidelity $F = 0.8, 0.9,$ and 0.99. With non-ideal detector efficiency the probability should be multiplied by $\eta$.

If a high fidelity of the entangled state is required ($F \approx 1$) the success probability becomes very low, and it is advantageous to use a double detection scheme in which we prepare the atoms in the initial state $|1\rangle$ with $\phi = \pi/4$ ($p_{1,i} = 1/2$). If we detect a photon we interchange the states $|0\rangle$ and $|1\rangle$ in both atoms and we probe the transmission of the cavity once more. If another photon is detected we have excluded both the $|00\rangle$ and $|11\rangle$ components of Eq. 1 and we are left with an entangled state with a fidelity $F = 1$. The success probability for the double detection $P_s = \eta^2 R_1^2/2$ is also plotted in Fig. 2 (with $\eta = 1$).

It should be noted that imperfections such as absorption in the mirrors or a mismatch between the incident field and the cavity mode, may cause reflection of photons which will be mistaken for successful generation of the entangled state. If the imperfections cause a fraction $f$ of the incident photons to be reflected, the state conditioned on the two-photon detection will have its fidelity reduced by the ratio between the two-count probability in the desired state and the total two-count probability, $(R_1(1-f)+f)^2/(R_1(1-f)+f)^2+f(R_2(1-f)+f))$. For small values of $f$ the fidelity is also reduced by a small amount, i.e., for $g^2/\kappa\gamma = 1$ we get $F = 0.98$ (0.85) for $f = 1\%$ (10\%).

The photon number states are hard to produce experimentally, and we shall now investigate the more realistic situation where the incoming light is in a coherent state. With coherent light there is a probability to have more than one photon in the pulse, and thus a probability that the atoms have spontaneously emitted a photon when we detect a photon in the detector. We assume that the atoms are initially prepared in the state $|0\rangle$ and we then continuously monitor the reflection from the cavity with a continuous source of coherent light in the incoming mode. If we have a click in the detector the experiment is successful and we block the light to avoid spontaneous emission. If we have not registered a click in the detector after a certain mean photon number $n_{\text{max}}$ have been shined onto the cavity, the protocol is unsuccessful and we have to restart the experiment.

If the first click is observed after a mean photon number $n$, the probability to be in the subspace with one atom in state $|1\rangle$ is

$$p_{1,c} = \frac{p_{1,i} R_1 e^{-\eta R_1 n}}{p_{1,i} R_1 e^{-\eta R_1 n} + p_{2,i} R_2 e^{-\eta R_2 n}}. \tag{9}$$

According to Eq. 8 the atomic coherence is reduced by the factor $e^{-\lambda n}$ after the interaction with a coherent state with a mean photon number $n$, and we find $F_c = p_{1,c}/2 + \text{Re}(\zeta) = p_{1,c}(1 + e^{-\lambda n})/2$. By averaging this expression with the probability distribution for the first click $dP_1/dn = \eta p_{1,i} R_1 e^{-\eta R_1 n} + \eta p_{2,i} R_2 e^{-\eta R_2 n}$ we

$$FIG. 2: Success probability with a Fock state as input. Full lines assume a single detection, the dashed line is for double detection. Starting from above the full lines have the fidelities $F = 0.8, 0.9$ and 0.99. Within the present model the fidelity of the dashed line is unity. The probabilities assume a detector efficiency of unity. With non-ideal detectors the full (dashed) lines should be multiplied by $\eta (\eta^2)$.}
Again we prepare the atoms in the initial state (1) with detection scheme for high fidelities and/or good cavities. The upper (lower) full curves in the figure have \( n \) \( g^2/\kappa \gamma = 1 \) (for \( g^2/\kappa \gamma \leq 1 \)). The preparation angle \( \phi \) varies between 0.2 and 0.4 over the range of the figure, and \( n_{\text{max}} \) is on the order of unity for \( g^2/\kappa \gamma \sim 1 \) (to be precise: all the curves in the figure have \( n_{\text{max}} < 2 \) for \( g^2/\kappa \gamma < 2 \)).

For a given cavity the fidelity and success probability depend on the preparation angle \( \phi \) and the maximum mean photon number after which the preparation is restarted \( n_{\text{max}} \). We have numerically optimized the success probability with a fixed fidelity. The optimal probability for \( F = 0.9 \) is shown with full curves in Fig. 3. The upper (lower) full curve is for a detector efficiency \( \eta = 1 \) (\( \eta = 0.5 \)). The preparation angle \( \phi \) varies between 0.2 and 0.4 over the range of the figure, and \( n_{\text{max}} \) is on the order of unity for \( g^2/\kappa \gamma \sim 1 \) (to be precise: all the curves in the figure have \( n_{\text{max}} < 2 \) for \( g^2/\kappa \gamma < 2 \)).

Like for Fock states, it is an advantage to use a double detection scheme for high fidelities and/or good cavities. Again we prepare the atoms in the initial state |1\rangle with \( \phi = \pi/4 \). If we have a click after a mean number of photons \( n_1 \leq n_{1,\text{max}} \) have been sent in, we interchange the states |0\rangle and |1\rangle and wait for a second click. The scheme is successful if a second photon is detected after a mean number of photons \( n_2 \leq n_{2,\text{max}} \) have been shined onto the cavity. We find that the optimal strategy is to take the scheme to be successful if \( n_1 + n_2 \leq n_{\text{max}} \). With this strategy we find the fidelity

\[
F = \frac{1}{2} + \frac{n_1^2 R_1^2}{(1 - (\eta R_1 + \lambda) n_{\text{max}}) \eta R_1 + \lambda)^2 P_s} \frac{e^{-(\eta R_1 + \lambda) n_{\text{max}}}}{2 P_s} \tag{12}
\]

where the success probability \( P_s \) is given by

\[
P_s = \frac{1}{2} \left( 1 - e^{-\eta R_1 n_{\text{max}} (1 + \eta R_1 n_{\text{max}})} \right) \tag{13}
\]

In Fig. 3 the dashed curves show the success probability \( P_s \) for the double detection when we require \( F \geq 0.9 \) (for the flat part of the curves \( F \) is actually larger than 0.9).

Strong coupling is usually attempted in standing wave cavities but recently a ring cavity with a finesse of \( 1.8 \times 10^5 \) has been reported in [11], and we estimate this cavity to be in the regime with \( g^2/\kappa \gamma \sim 1 \) if the beam waist is reduced to 30 \( \mu \)m. For the ring cavity our scheme has the additional advantage that it only depends on the magnitude of the coupling constant \( |g|^2 \) and not on the phase, and hence, the atoms only need to be confined within the cavity waist and not within a wavelength of the field. In the ring cavity we must, however, take the mode traveling in the opposite direction into account. The precise effect of a photon leaving the cavity from this mode depends on the separation between the atoms and unless we restrict this separation to a half-integer number of wavelengths of the radiation, such an event will be as harmful as spontaneous emission. Treating photons leaving from the counter propagating mode on equal footing with spontaneous emission events, they have the effect of replacing \( g^2/\kappa \gamma \) by \( g^2/(\kappa \gamma + 4 \eta R^2 \kappa / \kappa) \) in the above expressions, where \( \tilde{g} \) and \( \kappa \) are the coupling constant and decay rate for the counter propagating mode. If \( g = \tilde{g} \) and \( \kappa = \tilde{\kappa} \) this limits the effective \( g^2/\kappa \gamma \) to values less than 1/4. Inserting an optical diode, however, may significantly increase \( \tilde{\kappa} \) and thereby eliminate the role of the counter propagating mode.

In conclusion we have proposed a realistic scheme for entanglement of atoms inside an optical cavity. The proposed scheme does not require the cavity to be in the strong coupling regime \( g^2/\kappa \gamma \gg 1 \) and high quality entangled pairs can be produced for cavities with \( g^2/\kappa \gamma \sim 1 \). The entanglement protocol has a finite success probability, and hence one can with certainty produce the entangled state by simply trying sufficiently many times. This implies that we can carry out quantum gates by using the entanglement as a channel for teleportation of the qubit contents which we assume to be stored in other quantum mechanical degrees of freedom, e.g., another ion in an ion trap.

* Present address: ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138

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