FIXED-CIRCLE PROBLEM ON $S$-METRIC SPACES WITH A GEOMETRIC VIEWPOINT

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Abstract. Recently, a new geometric approach called the fixed-circle problem has been introduced to fixed-point theory. The problem has been studied using different techniques on metric spaces. In this paper, we consider the fixed-circle problem on $S$-metric spaces. We investigate existence and uniqueness conditions for fixed circles of self-mappings on an $S$-metric space. Some examples of self-mappings having fixed circles are also given.

Keywords: fixed-circle problem; self-mapping; $S$-metric space.

1. Introduction

The existence and uniqueness theorems of fixed points of self-mappings satisfying some contractive conditions have been extensively studied since the time of Stefan Banach (see [1, 2]). Many authors have investigated new fixed-point theorems on metric spaces or generalizations of metric spaces. For example, Sedghi, Shobe and Aliouche obtained Banach’s contraction principle on $S$-metric spaces [12]. We studied some generalizations of Banach’s contraction principle on an $S$-metric space [8] and investigated new fixed-point theorems for the following contractive condition (which is called Rhoades’ condition [11]) (see [6, 14]):

$$(S25) \quad S(Tx,Tx,Ty) < \max\{S(x,y), S(Tx,Tx,x), S(Ty,Ty,y), S(Ty,Ty,x), S(Tx,Tx,y)\},$$

for each $x, y \in X$, $x \neq y$. We then gave the concept of diameter and obtained a new contractive condition using this notion as follows [6]:

$$(S25a) \quad S(Tx,Tx,Ty) < \text{diam}\{U_x \cup U_y\},$$

for each $x, y \in X$ ($x \neq y$), where $U_x = \{T^n x : n \in \mathbb{N}\}$, $U_y = \{T^n y : n \in \mathbb{N}\}$, $\text{diam}\{U_x\} < \infty$ and $\text{diam}\{U_y\} < \infty$. 

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Although the existence of fixed points of functions has been studied on various metric spaces, there is no study on the existence of fixed circles. Therefore, the fixed-circle problem arises naturally. There are some examples of functions with a fixed circle on some special metric spaces. For example, let $C$ be an $S$-metric space with the $S$-metric $S(z, w, t) = \frac{|z - t| + |w - t|}{2}$, for all $z, w, t \in C$. Let the mapping $T$ be defined as $Tz = \frac{1}{z}$ for all $z \in C\setminus\{0\}$. The mapping $T$ fixes the unit circle $C_{0,1}^S = \{ x \in X : S(x, x, 0) = 1 \}$.

Recently, Özdemir, İskender and Özgür used new types of activation functions having a fixed circle for a complex valued neural network [5]. The usage of these types activation functions leads us to guarantee the existence of fixed points of the complex valued Hopfield neural network (see [5] for more details).

Hence it is important to investigate some fixed-circle theorems on various metric spaces. In [9], we obtained some fixed-circle theorems on metric spaces. We studied some existence theorems for fixed circles with a geometric interpretation and gave necessary conditions for the uniqueness of fixed circles. Also, we provided some examples of self-mappings with fixed circles. On the other hand, we proved new fixed-circle results and applied the obtained results to the discontinuity problem and discontinuous activation functions [10].

Motivated by the above studies, our aim in this paper is to obtain some fixed-circle theorems for self-mappings on $S$-metric spaces. In Section 2., we recall some necessary definitions, lemmas and basic facts. In Section 3., we introduce the notion of a fixed circle on an $S$-metric space and then obtain some existence and uniqueness theorems for self-mappings having fixed circles via different techniques. We investigate the case in which the number of fixed circles is infinitely many. Some examples of self-mappings with fixed circles are given with a geometric viewpoint. Using Mathematica (Wolfram Research, Inc., Mathematica, Trial Version, Champaign, IL (2016)), we draw some figures related to the given examples.

2. Preliminaries

**Definition 2.1.** [12] Let $X$ be a nonempty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$.

1. $S(x, y, z) = 0$ if and only if $x = y = z$,
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$. 


Then $S$ is called an $S$-metric on $X$ and the pair $(X, S)$ is called an $S$-metric space.

The following lemma can be considered as the symmetry condition and it will be used in the proofs of some theorems.

**Lemma 2.1.** [12] Let $(X, S)$ be an $S$-metric space. Then we have

\[ S(x, x, y) = S(y, y, x). \]

The relationships between a metric and an $S$-metric was given in what follows.

**Lemma 2.2.** [4] Let $(X, d)$ be a metric space. Then the following properties are satisfied:

1. $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an $S$-metric on $X$.
2. $x_n \to x$ in $(X, d)$ if and only if $x_n \to x$ in $(X, S_d)$.
3. $\{x_n\}$ is Cauchy in $(X, d)$ if and only if $\{x_n\}$ is Cauchy in $(X, S_d)$.
4. $(X, d)$ is complete if and only if $(X, S_d)$ is complete.

The metric $S_d$ was called an $S$-metric generated by $d$ [7]. We know some examples of an $S$-metric which are not generated by any metric (see [4, 7, 14] for more details).

On the other hand, Gupta claimed that every $S$-metric on $X$ defines a metric $d_S$ on $X$ as follows:

\begin{equation}
(2.1) \quad d_S(x, y) = S(x, x, y) + S(y, y, x),
\end{equation}

for all $x, y \in X$ [3]. However, the function $d_S(x, y)$ defined in (2.1) does not always define a metric because the triangle inequality is not satisfied for all elements of $X$ everywhere (see [7] for more details).

The notions of an open ball, a closed ball and diameter were introduced on $S$-metric spaces as the following definitions.

**Definition 2.2.** [12] Let $(X, S)$ be an $S$-metric space. The open ball $B_S(x_0, r)$ and closed ball $B_S[x_0, r]$ with a center $x_0$ and a radius $r$ are defined by

\[ B_S(x_0, r) = \{ x \in X : S(x, x_0) < r \} \]

and

\[ B_S[x_0, r] = \{ x \in X : S(x, x_0) \leq r \}, \]

for $r > 0$ and $x_0 \in X$. 

Definition 2.3. [6] Let \((X, S)\) be an \(S\)-metric space and \(A\) be a nonempty subset of \(X\). The diameter of \(A\) is defined by
\[
diam(A) = \sup \{S(x, x, y) : x, y \in A\}.
\]
If \(A\) is \(S\)-bounded, then we will write \(diam(A) < \infty\).

Now we define the notion of a circle on an \(S\)-metric space.

Definition 2.4. Let \((X, S)\) be an \(S\)-metric space and \(x_0 \in X, r \in (0, \infty)\). We define the circle centered at \(x_0\) with the radius \(r\) as
\[
C_{x_0, r}^S = \{x \in X : S(x, x, x_0) = r\}.
\]

3. Some Fixed-Circle Theorems on \(S\)-Metric Spaces

In this section, we introduce the notion of a fixed circle on an \(S\)-metric space. Then we investigate some existence and uniqueness theorems for self-mappings having fixed circles.

Definition 3.1. Let \((X, S)\) be an \(S\)-metric space, \(C_{x_0, r}^S\) be a circle on \(X\) and \(T : X \to X\) be a self-mapping. If \(Tx = x\) for all \(x \in C_{x_0, r}^S\) then the circle \(C_{x_0, r}^S\) is said to be a fixed circle of \(T\).

3.1. The existence of fixed circles

We obtain some existence theorems for fixed circles of self-mappings.

Theorem 3.1. Let \((X, S)\) be an \(S\)-metric space and \(C_{x_0, r}^S\) be any circle on \(X\). Let us define the mapping
\[
\varphi : X \to [0, \infty), \varphi(x) = S(x, x, x_0),
\]
for all \(x \in X\). If there exists a self-mapping \(T : X \to X\) satisfying
\[
S(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r
\]
and
\[
S(x, x, Tx) + S(Tx, Tx, x_0) \leq r,
\]
for all \(x \in C_{x_0, r}^S\), then \(C_{x_0, r}^S\) is a fixed circle of \(T\).
Proof. Let $x \in C_{x_0, r}^S$. Then using the conditions (3.2), (3.3), Lemma 2.1 and the triangle inequality, we get

$$S(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r$$
$$= S(x, x, x_0) + S(Tx, Tx, x_0) - 2r$$
$$\leq S(x, x, Tx) + S(x, x, Tx) + S(Tx, Tx, x_0) + S(Tx, Tx, x_0) - 2r$$
$$= 2S(x, x, Tx) + 2S(Tx, Tx, x_0) - 2r$$
$$\leq 2r - 2r = 0$$

and so

$$S(x, x, Tx) = 0,$$

which implies $Tx = x$. Consequently, $C_{x_0, r}^S$ is a fixed circle of $T$. □

Remark 3.1. 1) Notice that the condition (3.2) guarantees that $Tx$ is not in the interior of the circle $C_{x_0, r}^S$ for $x \in C_{x_0, r}^S$. Similarly, the condition (3.3) guarantees that $Tx$ is not the exterior of the circle $C_{x_0, r}^S$ for $x \in C_{x_0, r}^S$. Hence $Tx \in C_{x_0, r}^S$ for each $x \in C_{x_0, r}^S$, and so we get $T(C_{x_0, r}^S) \subset C_{x_0, r}^S$.

2) If an $S$-metric is generated by any metric $d$, then Theorem 3.1 can be used on the corresponding metric space.

3) The converse statement of Theorem 3.1 is also true.

Now we give an example of a self-mapping with a fixed circle.

Example 3.1. Let $X = \mathbb{R}$ and the function $S : X^3 \to [0, \infty)$ be defined by

$$S(x, y, z) = |x - z| + |y - z|,$$

for all $x, y, z \in \mathbb{R}$ [13]. Then $(X, S)$ is called the usual $S$-metric space. This $S$-metric is generated by the usual metric on $\mathbb{R}$. Let us consider the circle $C_{0, 2}^S$ and define the self-mapping $T_1 : \mathbb{R} \to \mathbb{R}$ as

$$T_1x = \begin{cases} x & \text{if } x \in \{-1, 1\} \\ 10 & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping $T_1$ satisfies the conditions (3.2) and (3.3). Hence $C_{0, 2}^S = \{-1, 1\}$ is a fixed circle of $T_1$.

Notice that $C_{\frac{2}{2}, 11}^S = \{-1, 10\}$ is another fixed circle of $T_1$ and so the fixed circle is not unique for a giving self-mapping.

On the other hand, if we consider the usual metric $d$ on $\mathbb{R}$ then we obtain $C_{0, 2} = \{-2, 2\}$. The circle $C_{0, 2}$ is not a fixed circle of $T_1$.

Example 3.2. Let $X = \mathbb{R}^2$ and let the function $S : X^3 \to [0, \infty)$ be defined by

$$S(x, y, z) = \sum_{i=1}^{2} (|x_i - z_i| + |x_i + z_i - 2y_i|),$$
for all \( x = (x_1, x_2), y = (y_1, y_2) \) and \( z = (z_1, z_2) \). Then it can be easily seen that \( S \) is an \( S \)-metric on \( \mathbb{R}^2 \), which is not generated by any metric, and the pair \((\mathbb{R}^2, S)\) is an \( S \)-metric space.

Let us consider the unit circle \( C_{0,1}^S \) and define the self-mapping \( T_2 : \mathbb{R} \to \mathbb{R} \) as

\[
T_2 x = \begin{cases} 
  x & \text{if } x \in C_{0,1}^S \\
  (1,0) & \text{otherwise} 
\end{cases},
\]

for all \( x \in \mathbb{R}^2 \). Then the self-mapping \( T_2 \) satisfies the conditions (3.2) and (3.3). Therefore \( C_{0,1}^S \) is a fixed circle of \( T_2 \) as shown in Figure 3.1.

![Fig. 3.1: The fixed circle of \( T_2 \).](image)

In the following example, we give an example of a self-mapping which satisfies the condition (3.2) and does not satisfy the condition (3.3).

**Example 3.3.** Let \( X = \mathbb{R} \) and the function \( S : X^3 \to [0, \infty) \) be defined by

\[
S(x, y, z) = |x - z| + |x + z - 2y|,
\]

for all \( x, y, z \in \mathbb{R} \) [7]. Then \( S \) is an \( S \)-metric which is not generated by any metric and \((X, S)\) is an \( S \)-metric space. Let us consider the circle \( C_{0,3}^S \) and define the self-mapping \( T_3 : \mathbb{R} \to \mathbb{R} \) as

\[
T_3 x = \begin{cases} 
  -\frac{3}{7} & \text{if } x = -\frac{3}{7} \\
  \frac{5}{7} & \text{if } x = \frac{5}{7} \\
  7 & \text{otherwise} 
\end{cases},
\]
for all \( x \in \mathbb{R} \). Then the self-mapping \( T_3 \) satisfies the condition (3.2) but does not satisfy the condition (3.3). Clearly \( T_3 \) does not fix the circle \( C_{0,3}^S \).

In the following example, we give an example of a self-mapping which satisfies the condition (3.3) and does not satisfy the condition (3.2).

**Example 3.4.** Let \((X, S)\) be an \( S \)-metric space, \( C_{x_0,r}^S \) be a circle on \( X \) and the self-mapping \( T_4 : X \to X \) be defined as
\[
T_4x = x_0,
\]
for all \( x \in X \). Then the self-mapping \( T_4 \) satisfies the condition (3.3) but does not satisfy the condition (3.2). Clearly \( T_4 \) does not fix the circle \( C_{x_0,r}^S \).

Now we give another existence theorem for fixed circles.

**Theorem 3.2.** Let \((X, S)\) be an \( S \)-metric space and \( C_{x_0,r}^S \) be any circle on \( X \). Let the mapping \( \varphi \) be defined as (3.1). If there exists a self-mapping \( T : X \to X \) satisfying
\[
S(x, x, Tx) \leq \varphi(x) - \varphi(Tx) (3.4)
\]
and
\[
hS(x, x, Tx) + S(Tx, Tx, x_0) \geq r, (3.5)
\]
for all \( x \in C_{x_0,r}^S \) and some \( h \in [0,1) \), then \( C_{x_0,r}^S \) is a fixed circle of \( T \).

**Proof.** Let \( x \in C_{x_0,r}^S \). On the contrary, assume that \( x \neq Tx \). Then using the conditions (3.4) and (3.5), we obtain
\[
S(x, x, Tx) \leq \varphi(x) - \varphi(Tx) \\
= S(x, x, x_0) - S(Tx, Tx, x_0) \\
= r - S(Tx, Tx, x_0) \\
\leq hS(x, x, Tx) + S(Tx, Tx, x_0) - S(Tx, Tx, x_0) \\
= hS(x, x, Tx),
\]
which is a contradiction since \( h \in [0,1) \). Hence we get \( Tx = x \) and \( C_{x_0,r}^S \) is a fixed circle of \( T \). \( \square \)

**Remark 3.2.** 1) Notice that the condition (3.4) guarantees that \( Tx \) is not in the exterior of the circle \( C_{x_0,r}^S \) for \( x \in C_{x_0,r}^S \). Similarly, the condition (3.5) shows that \( Tx \) can lie on either the exterior or the interior of the circle \( C_{x_0,r}^S \) for \( x \in C_{x_0,r}^S \). Hence \( Tx \) should lie on the interior of the circle \( C_{x_0,r}^S \).

2) If an \( S \)-metric is generated by any metric \( d \), then Theorem 3.2 can be used on the corresponding metric space.

3) The converse statement of Theorem 3.2 is also true.

Now we give some examples of self-mappings which have a fixed-circle.
Example 3.5. Let $X = \mathbb{R}$ and $(X, S)$ be the usual $S$-metric space. Let us consider the circle $C_{1,2}^S = \{0, 2\}$ and define the self-mapping $T_5 : \mathbb{R} \to \mathbb{R}$ as

$$T_5x = \begin{cases} 
eq e^x - 1 & \text{if } x = 0 \\ 2x - 2 & \text{if } x = 2 \\ 3 & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping $T_5$ satisfies the conditions (3.4) and (3.5). Hence $C_{1,2}^S$ is a fixed circle of $T_5$.

On the other hand, if we consider the usual metric $d$ on $\mathbb{R}$ then we have $C_{1,2} = \{-1, 3\}$. The circle $C_{1,2}$ is not a fixed circle of $T_5$. But $C_{1,1} = \{0, 2\}$ is a fixed circle of $T_5$ on $(X, d)$.

Example 3.6. Let $X = \mathbb{R}^2$ and let the function $S : X^3 \to [0, \infty)$ be defined by

$$S(x, y, z) = \sum_{i=1}^2 (|e^{x^i} - e^{z^i}| + |e^{x^i} + e^{z^i} - 2e^{y^i}|),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then it can be easily checked that $S$ is an $S$-metric on $\mathbb{R}^2$, which is not generated by any metric, and the pair $(\mathbb{R}^2, S)$ is an $S$-metric space.

Let us consider the circle $C^S_{x_0,r}$ centered at $x_0 = (0, 0)$ with the radius $r = 2$ and define the self-mapping $T_6 : \mathbb{R} \to \mathbb{R}$ as

$$T_6x = \begin{cases} x & \text{if } x \in C^S_{0,2} \\ (\ln 2, 0) & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$.
for all \( x \in \mathbb{R}^2 \). Then the self-mapping \( T_6 \) satisfies the conditions (3.4) and (3.5). Therefore \( C_{0,2}^S \) is the fixed circle of \( T_6 \) as shown in Figure 3.2.

In the following example, we give an example of a self-mapping which satisfies the condition (3.4) and does not satisfy the condition (3.5).

**Example 3.7.** Let \( (X, S) \) be an \( S \)-metric space and \( C_{x_0,r}^S \) be a circle on \( X \). If we consider the self-mapping \( T_4 x = x_0 \), then the self-mapping \( T_4 \) satisfies the condition (3.4) but does not satisfy the condition (3.5). It can be easily seen that \( T_4 \) does not fix a circle \( C_{x_0,r}^S \).

In the following example, we give an example of a self-mapping which satisfies the condition (3.5) and does not satisfy the condition (3.4).

**Example 3.8.** Let \( X = \mathbb{R} \) and \( (X, S) \) be an \( S \)-metric space with an \( S \)-metric defined as in Example 3.3. Let us consider the unit circle \( C_{0,1}^S \) and define the self-mapping \( T_7 : \mathbb{R} \to \mathbb{R} \) as

\[ T_7 x = 1, \]

for all \( x \in \mathbb{R} \). Then the self-mapping \( T_7 \) satisfies the condition (3.5) but does not satisfy the condition (3.4). It can be easily shown that \( T_7 \) does not fix the unit circle \( C_{0,1}^S \).

Let \( I_X : X \to X \) be the identity map defined as \( I_X(x) = x \) for all \( x \in X \). Notice that the identity map satisfies the conditions (3.2) and (3.3) (resp. (3.4) and (3.5)) in Theorem 3.1 (resp. Theorem 3.2) for any circle. Now we determine a condition which excludes the \( I_X \) from Theorem 3.1 and Theorem 3.2. For this purpose, we give the following theorem.

**Theorem 3.3.** Let \( (X, S) \) be an \( S \)-metric space, \( T : X \to X \) be a self mapping having a fixed circle \( C_{x_0,r}^S \) and the mapping \( \phi \) be defined as (3.1). The self-mapping \( T \) satisfies the condition

\[ (I_S) \quad S(x, x, Tx) \leq \frac{\varphi(x) - \varphi(Tx)}{h}, \]

for all \( x \in X \) and some \( h > 2 \) if and only if \( T = I_X \).

**Proof.** Let \( x \in X \) be an arbitrary element. Then using the inequality \((I_S)\), Lemma 2.1 and triangle inequality, we obtain

\[
hS(x, x, Tx) \leq \varphi(x) - \varphi(Tx) = S(x, x, x_0) - S(Tx, Tx, x_0) \\
\leq 2S(x, x, Tx) + S(Tx, Tx, x_0) - S(Tx, Tx, x_0) = 2S(x, x, Tx)
\]

and so

\[
(h - 2)S(x, x, Tx) \leq 0.
\]
Since \( h > 2 \) it should be \( S(x, x, Tx) = 0 \) and so \( Tx = x \). Consequently, we obtain \( T = I_X \).

Conversely, it is clear that the identity map \( I_X \) satisfies the condition \((I_S)\).

**Remark 3.3.** 1) If a self-mapping \( T \), which has a fixed circle, satisfies the conditions (3.2) and (3.3) (resp. (3.4) and (3.5)) in Theorem 3.1 (resp. Theorem 3.2) but does not satisfy the condition \((I_S)\) in Theorem 3.3 then the self-mapping \( T \) cannot be an identity map.

2) If an \( S \)-metric is generated by any metric \( d \), then Theorem 3.3 can be used on the corresponding metric space.

### 3.2. The uniqueness of fixed circles

We investigate the uniqueness conditions of fixed circles given in the existence theorems. For any given circles \( C_{x,0}^S, r \) and \( C_{x,1}^S, \rho \) on \( X \), we notice that there exists at least one self-mapping \( T \) of \( X \) such that \( T \) fixes the circles \( C_{x,0}^S, r \) and \( C_{x,1}^S, \rho \). Indeed, let us define the mappings \( \phi_1, \phi_2 : X \to [0, \infty) \) as

\[
\phi_1(x) = S(x, x, x_0)
\]

and

\[
\phi_2(x) = S(x, x, x_1),
\]

for all \( x \in X \). If we define the self-mapping \( T_8 : X \to X \) as

\[
T_8x = \begin{cases} 
    x & \text{if } x \in C_{x,0}^S \cup C_{x,1}^S, \\
    \alpha & \text{otherwise},
\end{cases}
\]

for all \( x \in X \), where \( \alpha \) is a constant satisfying \( S(\alpha, \alpha, x_0) \neq r \) and \( S(\alpha, \alpha, x_1) \neq \rho \), it can be easily seen that the self-mapping \( T_8 : X \to X \) satisfies the conditions (3.2) and (3.3) in Theorem 3.1 (resp. (3.4) and (3.5) in Theorem 3.2) for the circles \( C_{x,0}^S, r \) and \( C_{x,1}^S, \rho \) using the mappings \( \phi_1 \) and \( \phi_2 \), respectively. Hence \( T_8 \) fixes both of the circles \( C_{x,0}^S, r \) and \( C_{x,1}^S, \rho \). In this way, the number of fixed circles can be extended to any positive integer \( n \) using the same arguments.

In the following example, the self-mapping \( T_9 \) has two fixed circle.

**Example 3.9.** Let \( X = \mathbb{R} \) and \((X, S)\) be an \( S \)-metric space with the \( S \)-metric defined in Example 3.3. Let us consider the circles \( C_{0,2}^S, C_{0,4}^S \) and define the self-mapping \( T_9 : \mathbb{R} \to \mathbb{R} \) as

\[
T_9x = \begin{cases} 
    x & \text{if } x \in \{-2, -1, 1, 2\}, \\
    \alpha & \text{otherwise},
\end{cases}
\]

for all \( x \in X \) where \( \alpha \in X \). Then the conditions (3.2) and (3.3) are satisfied by \( T_9 \) for the circles \( C_{0,2}^S \) and \( C_{0,4}^S \), respectively. Consequently, \( C_{0,2}^S \) and \( C_{0,4}^S \) are the fixed circles of \( T_9 \).
Now we investigate the uniqueness conditions for the fixed circles in Theorem 3.1 using Rhoades’ contractive condition on $S$-metric spaces.

**Theorem 3.4.** Let $(X, S)$ be an $S$-metric space and $C^{S}_{x_{0}, r}$ be any circle on $X$. Let $T : X \to X$ be a self-mapping satisfying the conditions (3.2) and (3.3) given in Theorem 3.1. If the contractive condition

\[ S(Tx, Tx, Ty) < \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), S(Ty, Ty, x), S(Tx, Tx, y)\}, \]

(3.6)

is satisfied for all $x \in C^{S}_{x_{0}, r}$, $y \in X \setminus C^{S}_{x_{0}, r}$ by $T$, then $C^{S}_{x_{0}, r}$ is a unique fixed circle of $T$.

**Proof.** Suppose that there exist two fixed circles $C^{S}_{x_{1}, r_{1}}$ and $C^{S}_{x_{2}, r_{2}}$ of the self-mapping $T$, that is, $T$ satisfies the conditions (3.2) and (3.3) for each circles $C^{S}_{x_{i}, r_{i}}$ and $C^{S}_{x_{i}, r_{i}}$. Let $x \in C^{S}_{x_{0}, r}$ and $y \in C^{S}_{x_{1}, r}$ be arbitrary points with $x \neq y$. Using the contractive condition (3.6), we obtain

\[ S(x, x, y) = S(Tx, Tx, Ty) < \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), S(Ty, Ty, x), S(Tx, Tx, y)\} \]

\[ = S(x, x, y), \]

which is a contradiction. Hence it should be $x = y$. Consequently, $C^{S}_{x_{0}, r}$ is the unique fixed circle of $T$. \hfill $\Box$

The following example shows that the circle $C^{S}_{x_{0}, r}$ is not necessarily unique in Theorem 3.2.

**Example 3.10.** Let $(X, S)$ be an $S$-metric space and $C^{S}_{x_{1}, r_{1}}, \cdots, C^{S}_{x_{n}, r_{n}}$ be any circles on $X$. Let us define the self-mapping $T_{10} : X \to X$ as

\[ T_{10}x = \begin{cases} x & \text{if } x \in \bigcup_{i=1}^{n} C_{x_{i}, r_{i}} \setminus \bigcup_{i=1}^{n} C_{x_{i}, r_{i}}, \\ x_0 & \text{otherwise}. \end{cases} \]

for all $x \in X$, where $x_0$ is a constant in $X$. Then it can be easily checked that the conditions (3.4) and (3.5) are satisfied by $T_{10}$ for the circles $C^{S}_{x_{1}, r_{1}}, \cdots, C^{S}_{x_{n}, r_{n}}$, respectively. Consequently, the circles $C^{S}_{x_{1}, r_{1}}, \cdots, C^{S}_{x_{n}, r_{n}}$ are fixed circles of $T_{10}$. Notice that these circles do not have to be disjoint.

Now we give the following uniqueness theorem for the fixed circles in Theorem 3.2 using the notion of diameter on $S$-metric spaces.

**Theorem 3.5.** Let $(X, S)$ be an $S$-metric space, $C^{S}_{x_{0}, r}$ be any circle on $X$, $U_{x} = \{T^{n}x : n \in \mathbb{N}\}$, $U_{y} = \{T^{n}y : n \in \mathbb{N}\}$, $\text{diam}\{U_{x}\} < \infty$ and $\text{diam}\{U_{y}\} < \infty$. Let
At first, using the inequality (3.10) and Lemma 2.1, we show that

\[ T \times (3.10) \]

we get \( B \) closed ball which is a contradiction. Hence it should be \( x = y \). Consequently, \( C_{x_0,r}^S \) is the unique fixed circle of \( T \).

\[ \text{Proof.} \quad \text{Assume that there exist two fixed circles } C_{x_0,r}^S \text{ and } C_{x_1,r}^S \text{ of the self-mapping } T, \text{ that is, } T \text{ satisfies the conditions (3.4) and (3.5) for each circles } C_{x_0,r}^S \text{ and } C_{x_1,r}^S. \]

Let \( x \in C_{x_0,r}^S \) and \( y \in C_{x_1,r}^S \) be arbitrary points with \( x \neq y \). Using the contractive condition (3.7), we obtain

\[ S(x, x, y) = S(Tx, Tx, Ty) < \text{diam}\{U_x \cup U_y\} = S(x, x, y), \]

which is a contradiction. Hence it should be \( x = y \). Consequently, \( C_{x_0,r}^S \) is the unique fixed circle of \( T \).  

3.3. Infinity of fixed circles

We give a new approach to obtain fixed-circle results. To do this, let us denote by \( R_S(x, y) \) the right side of the inequality (S25). Using the number \( R_S(x, y) \), we obtain the following theorem. This theorem generates many (finite or infinite) fixed circles for a given self-mapping.

**Theorem 3.6.** Let \( (X, S) \) be an S-metric space, \( T : X \to X \) be a self-mapping and \( r = \inf\{S(Tx, Tx, x) :Tx \neq x\} \). If there exists a point \( x_0 \in X \) satisfying

\[ S(x, x, Tx) < R_S(x, x_0) \]

for all \( x \in X \) when \( S(Tx, Tx, x) > 0 \) and

\[ S(Tx, Tx, x_0) = r \]

for all \( x \in C_{x_0,r}^S \), then \( C_{x_0,r}^S \) is a fixed circle of \( T \). The self-mapping \( T \) also fixes the closed ball \( B_S[x_0, r] \).

**Proof.** Let \( x \in C_{x_0,r}^S \) and \( Tx \neq x \). Then using the inequality (3.8) and Lemma 2.1, we get

\[ S(x, x, Tx) < R_S(x, x_0) \]

\[ = \max \left\{ S(x, x, x_0), S(Tx, Tx, x), S(Tx_0, Tx_0, x_0), S(Tx, Tx, x_0) \right\}. \]

At first, using the inequality (3.10) and Lemma 2.1, we show \( Tx_0 = x_0 \). Suppose that \( Tx_0 \neq x_0 \). For \( x = x_0 \), we obtain

\[ S(x_0, x_0, Tx_0) < R_S(x_0, x_0) \]

\[ = \max \left\{ S(x_0, x_0, x_0), S(Tx_0, Tx_0, x_0), S(Tx_0, Tx_0, x_0) \right\} \]

\[ = S(Tx_0, Tx_0) = S(x_0, x_0, Tx_0), \]
a contradiction. It should be $T x_0 = x_0$. Then by the inequality (3.10), the condition (3.9), definition of $r$ and Lemma 2.1, we have

$$S(x, x, Tx) \leq \max \left\{ S(x, x, x_0), S(Tx, Tx, x), S(x_0, x_0, x_0), S(x, x_0, x_0), S(Tx, Tx, x_0) \right\}$$

$$= \max \{ r, S(Tx, Tx, x) \} = S(Tx, Tx, x) = S(x, x, Tx),$$

a contradiction. Therefore we get $Tx = x$, that is, $C_{x_0,r}^S$ is a fixed circle of $T$.

Finally we prove that $T$ fixes the closed ball $B_S[x_0, r]$. To do this, we show that $T$ fixes any circle $C_{x_0,\rho}^S$ with $\rho < r$. Let $x \in C_{x_0,\rho}^S$ and $Tx \neq x$. From the similar arguments used in the above, we have $Tx = x$.  

We give the following example.

**Example 3.11.** Let $X = \mathbb{R}$ be the usual $S$-metric space. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} x & \text{if } |x| < 3 \\ x + 2 & \text{if } |x| \geq 3 \end{cases},$$

for all $x \in \mathbb{R}$. The self-mapping $T$ satisfies the conditions of Theorem 3.6 with $x_0 = 0$.

Indeed, we get

$$S(x, x, Tx) = 2|x - Tx| = 4 > 0,$$

for all $x \in \mathbb{R}$ such that $|x| \geq 3$. Then we have

$$R_S(x, 0) = \max \{ S(x, x, 0), S(Tx, Tx, x), S(0, 0, 0), S(0, 0, x), S(Tx, Tx, 0) \}$$

$$= \max \{ 2|x|, 4, 0, 2|x|, 2|x + 2| \}$$

and so

$$S(x, x, Tx) < R_S(x, 0).$$

Therefore the condition (3.8) is satisfied. We also obtain

$$r = \min \{ S(Tx, Tx, x) : Tx \neq x \} = 4.$$

It can be easily seen that the condition (3.9) is satisfied by $T$. Consequently, $T$ fixes the circle $C_{0,4}^S = \{ x \in \mathbb{R} : |x| = 2 \}$ and the closed ball $B_S[0, 4] = \{ x \in \mathbb{R} : |x| \leq 2 \}$.

**Remark 3.4.**

1) Notice that the condition (3.9) guarantees that $Tx \in C_{x_0,r}^S$ for each $x \in C_{x_0,r}^S$ and so $T(C_{x_0,r}^S) \subset C_{x_0,r}^S$.

2) The self-mapping $T$ defined in Example 3.11 has other fixed circles. Theorem 3.6 gives us some of these circles.

3) A self-mapping $T$ can fix infinitely many circles (see Example 3.11).

The converse statement is not always true as seen in the following example.

**Example 3.12.** Let $x_0 \in X$ be any point. If we define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x & \text{if } x \in B_S[x_0, \mu] \\ x_0 & \text{if } x \notin B_S[x_0, \mu] \end{cases},$$

for all $x \in X$ with $\mu > 0$, then $T$ does not satisfies the condition (3.8), but $T$ fixes every circle $C_{\rho,\rho}^S$ with $\rho \leq \mu$. 

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