QUATERNIONIC-CONTACT HYPERSURFACES

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Abstract. In this text, we prove that every quaternionic-contact structure can be embedded in a quaternionic manifold.

1. Introduction

There has been a great deal of interest recently in the study of special classes of complete Einstein metrics whose behavior at infinity looks like the hyperbolic $\mathbb{K}$-space with $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Among these, one finds some negatively curved Kähler-Einstein metrics constructed by Cheng and Yau ([Che80]) on bounded strictly pseudo-convex domains of $\mathbb{C}^n$ and whose conformal infinity is a strictly pseudo-convex CR-manifold with the strictly pseudo-convexe CR-structure induced by the amiant complex structure of $\mathbb{C}^n$: one says that this CR-structure is realizable.

The standard example of a such metric is the complex hyperbolic metric $g_\mathbb{C}$ on the unit ball of $\mathbb{C}^n$ that is explicitly given by

$$g_\mathbb{C} = \frac{4\text{euc}}{\rho} + \frac{1}{\rho^2}(d\rho^2 + (Id\rho)^2)$$

with euc being the euclidean metric of $\mathbb{C}^n$, $\rho = 1 - |x|^2$ and $I$ being the complex structure of $\mathbb{C}^n$. The conformal infinity is the standard CR-structure of $S^{2n+1}$ with contact distribution the maximal $I$-invariant subspace $H = \ker Id\rho$ of $TS^{2n+1}$ and where the almost complex structure on $H$ is the restriction of $I$.

One knows that all strictly pseudoconvex CR-structures of dimensions at least 7 are locally realizable in $\mathbb{C}^n$ ([Kur82], [Aka87]) but there are strictly pseudo-convexe CR 3-manifolds that are not realizable, even locally ([Nir73]).

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In this note, we are interested in a similar problem concerning the conformal infinity of metrics modelled on the quaternionic-hyperbolic metric $g_{\mathbb{H}}$ with Levi-Civita connection $\nabla^\mathbb{H}$ and defined on the $4(n+1)$-ball of $\mathbb{H}^{n+1}$ by the formula

$$g_{\mathbb{H}} = \frac{4 \text{euc}}{\rho} + \frac{1}{\rho^2}(d\rho^2 + (I_1d\rho)^2 + (I_2d\rho)^2 + (I_3d\rho)^2),$$

where $\text{euc}$ is the euclidean metric on $\mathbb{H}^{n+1} \simeq \mathbb{R}^{4(n+1)}$, $\rho = 1 - |x|^2$ and $I_1, I_2, I_3$ are the endomorphisms obtained by right-multiplication by the purely imaginary quaternions $i, j$ and $k$. The metric $\rho^2 g_{\mathbb{H}}|_{TS^{4n+3}}$ is degenerate, and its kernel $H^{\text{can}} = \cap_{i=1}^3 \ker(I_i d\rho)$ satisfies

$$d(I_i d\rho)|_{H^{\text{can}}} = 4 \text{euc}|_{H^{\text{can}}}(I_i', \cdot)$$

and defines what we call a quaternionic contact distribution:

**Definition 1.1** \cite{Biq00}. Let $M$ be a smooth manifold of dimension $4n + 3$. A codimension 3 distribution $H$ on $M$ is quaternionic contact (QC) if there exists a metric $g_H$ on $H$ such that one can find locally defined 1-forms $\eta_1$, $\eta_2$ and $\eta_3$ vanishing on $H$ and an induced pointwise quaternionic structure $(I_1, I_2, I_3)$ on $H$ (i.e. $I_1^2 = -1$ and $I_1 I_2 = -I_2 I_1 = I_3$) with

$$d\eta_i|_H = g_H(I_i', \cdot).$$

In this case, the conformal class of $g_H$ is totally determined by the distribution $H$. Remark that in \cite{Ale05}, Alekseevsky and Kamishima introduced a notion of quaternionic CR-structure that can be defined as a QC-distribution with an induced quaternionic structure $(I_1, I_2, I_3)$ of integrable almost-complex structures.

Let $N$ be a smooth manifold with boundary $M$ admitting a quaternionic contact distribution $H$. A metric $g$ defined on a neighborhood $M \times [0, a] \subset N$ of $M$ with coordinates $(x, \rho)$ is called asymptotically hyperbolic quaternionic with conformal infinity $H$ if

$$g \sim \frac{1}{\rho} g_H + \frac{1}{\rho^2}(d\rho^2 + \eta_1^2 + \eta_2^2 + \eta_3^2)$$

when $\rho$ goes to zero. One can prove that if the dimension of $M$ is greater than 7, then every QC-distribution $H$ is the conformal infinity of an asymptotically hyperbolic quaternionic-Kähler metric (AHQK metric) \cite{Biq00}. If $\text{dim}(M) = 7$, the author found a criterion for
a given QC-distribution be the conformal infinity of a quaternionic-Kähler manifold, [Duc04]. This condition corresponds to the existence of a CR-integrable twistor space and one says that the QC-distribution is integrable in this case.

The space \( \mathbb{H}^{n+1} \) is endowed with a quaternionic structure, i.e. a \( GL(n+1, \mathbb{H})Sp(1) \)-structure with a torsion-free connection which can be chosen to be the canonical flat connection \( \nabla^0 \) in the case of \( \mathbb{H}^n \). The distribution \( H^{\text{can}} \) is the maximal \( \{ I_i \}_{i=1,2,3} \)-invariant subspace of \( TS^{4n+3} \) and the pointwise quaternionic structure of \( H^{\text{can}} \) is the restriction of the ambient pointwise quaternionic structure of \( \mathbb{H}^{n+1} \), one says that \( S^{4n+3} \) is a quaternionic contact hypersurface of \( H^{n+1} \).

**Definition 1.2.** Let \( M \) be a real hypersurface in a quaternionic manifold \( (N, Q) \) and let \( H \) be the maximal \( Q \)-invariant subspace of \( TN \). The hypersurface \( M \) is called a QC-hypersurface of \( N \) if \( H \) is quaternionic-contact with induced pointwise quaternionic structure that coincides with the restriction to \( H \) of the elements of \( Q \).

The aim of this note is to investigate to what extent a given QC manifold can be realized as a real hypersurface of a quaternionic manifold and thus extending the canonical example of the realization of \( H^{\text{can}} \) in \( \mathbb{H}^{n+1} \). In fact, the analogy can be made more precise at the level of the connections. Indeed, the connection \( \nabla^H \) is quaternionic on the unit ball with a pole of order 1 along \( S^{4n+3} \). The difference \( \nabla^H - \nabla^0 \) is a tensor \( a \frac{d\rho}{\rho} \) where \( a : (\mathbb{H}^{n+1})^* \to (\mathbb{H}^{n+1})^* \otimes \text{End}(\mathbb{H}^{n+1}) \) is linear and depends only of the \( GL(n+1, \mathbb{H})Sp(1) \)-structure of \( \mathbb{H}^{n+1} \), i.e. \( a \) can be defined on any quaternionic manifold \( M \) and goes from \( T^*M \otimes \text{End}(TM) \) (see the next section for an explicit description of \( a \)).

Building on results of [Biq00], we prove the following theorem:

**Theorem 1.1.** Let \((M^{4n+3}, H)\) be an integrable QC-manifold, \( n \geq 1 \). There exists a quaternionic manifold \((N_0^{4n+4}, Q)\) such that:

(i) \( M \) is a QC-hypersurface of \( N_0 \).

(ii) \( M \) separates \( N \) into two quaternionic manifolds \( N_0^+ \) and \( N_0^- \) such that \( N_0^+ \) has a definite-positive quaternionic-Kähler metric with conformal infinity \( H \), Levi-Civita connection \( \nabla^H \) and \( N_0^- \) has a quaternionic-Kähler metric with signature \((4, 4n)\) and conformal infinity \( H \).

(iii) If \( \rho \) is a defining function of \( M \), then \( \nabla^H - a \frac{d\rho}{\rho} \) extends to a smooth quaternionic connection on \( N \).
When $n = 0$, $M$ is a conformal 3-manifold which is the conformal infinity of a unique self-dual Einstein metric. The conformal class of this metric admits a prolongation in such a way that $M$ becomes an hypersurface in a self-dual conformal 4-manifold. In this setting, the theorem proved in this paper appears to be a generalization of this fact.

On the other hand, an hypersurface $M$ in a 4-dimensional conformal manifold $(N, [g])$ defines the data of a conformal metric and a second fundamental form that vanish iff there exists an Einstein metric on $N - M$ in the conformal class of $[g]$ with conformal infinity $M$. LeBrun proved that with the data of a conformal metric $[h]$ and a second fundamental form $\Omega$ on a 3-manifold $M$, one can construct an embedding of $M$ into a self-dual 4-manifold inducing $([h], \Omega)$ on $M$. In this paper, we generalize the notion of conformal second fundamental form to the case of a QC-hypersurface and prove that it vanishes for the embedding given by Theorem 1.1. In particular, if $M$ is a QC-hypersurface with non-vanishing second fundamental form in a quaternionic manifold $N$ with boundary $M$, there does not exist any AHQK-metric on $N - M$, compatible with the quaternionic structure of $N - M$ and with conformal infinity $M$.

We now describe briefly the organization of the paper. In the first section, we recall some basic facts about quaternionic manifolds and give the local description of a QC-hypersurface and thus define the notion of weakly quaternionic contact manifolds.

In the next section, we define the integrability of a QC-distribution and show that the QC-distribution of a QC-hypersurface is integrable.

In the third section, we give the definition of the twistor space of a QC-structure and prove Theorem 1.1.

The aim of the last section is to define the second fundamental forms of a QC-hypersurface and prove that they vanish under the hypothesis of Theorem 1.1.

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2. HYPERSURFACES IN QUATERNIONIC MANIFOLDS

In this preliminary section, I describe some basic facts in quaternionic geometry, see [Bes87,p.410] and [Sal86] for more details. I give also the local description of a quaternionic contact hypersurface.
2.1. Quaternionic manifolds.

**Definition 2.1.** An almost quaternionic manifold is a $4n$-dimensional manifold endowed with a $GL(n, \mathbb{H})Sp(1)$-structure. A quaternionic manifold is an almost-quaternionic manifold admitting a torsion-free $GL(n, \mathbb{H})Sp(1)$-connection.

**Remark 1.** An almost quaternionic manifold $M$ is the data of a sub-bundle $Q \subset End(TM)$, locally spanned by a pointwise quaternionic structure $(I_1, I_2, I_3)$. The manifold $(M, Q)$ is quaternionic if there exists a torsion-free connection that preserves $Q$.

Let $(M, Q)$ be a quaternionic manifold, and let $\nabla$ be a torsion-free connection preserving $Q$. If $\nabla'$ is another torsion free connection that preserves $Q$, then $\nabla' = \nabla + a$ where $a$ is in the kernel $\ker \partial$ of the torsion map

$$\partial : \Lambda^1 \otimes (gl(n, \mathbb{H}) \oplus sp(1)) \to \Lambda^2 \otimes \Lambda^1.$$ 

Let $g$ be the Lie algebra $gl(n, \mathbb{H}) \oplus sp(1)$.

**Lemma 2.1 (Sal86).** The kernel of the torsion map $\partial$ is the set $\{a^\theta \in \Lambda^1 \otimes g, \theta \in \Lambda^1\}$, where $a^\theta$ is defined by

$$a^\theta_X Y = \theta(X)Y + \theta(Y)X - \sum_{i=1}^3 (\theta(I_iX)I_iY + \theta(I_iY)I_iX).$$

**Proof.** It is well known that the restriction of $\partial$ to $\Lambda^1 \otimes sp(1)$ is injective (this follows from the unicity of a Levi-Civita connection). Moreover, the only common irreducible summand in the $GL(n, \mathbb{H})Sp(1)$ decomposition of $\Lambda^1 \otimes sp(1)$ and $\Lambda^1 \otimes gl(n, \mathbb{H})$ is $\Lambda^1$, where the embedding of $\Lambda^1$ in $\Lambda^1 \otimes sp(1)$ is given by $\theta \mapsto \overline{\theta} = \sum_i \theta \circ I_i \otimes I_i$. The torsion map $\partial$ is $GL(n, \mathbb{H})Sp(1)$-equivariante, hence if $\partial a = 0$ the $\Lambda^1 \otimes sp(1)$-part $\overline{\theta}$ of $a$ must live in $\Lambda^1$. Now, if $a'$ is the $\Lambda^1 \otimes gl(n, \mathbb{H})$-part of $a$, we obtain

$$a'_X I_3 Y = I_1 a'_X I_2 Y = I_1 a'_{I_2 Y} X - I_1 \partial \overline{\theta}_X I_2 Y$$
$$= a'_{I_1 Y} I_2 Y - \partial \overline{\theta}_{I_1 Y} I_1 X - I_1 \partial \overline{\theta}_X I_2 Y$$
$$= I_2 a'_Y I_1 X - I_2 \partial \overline{\theta}_{I_1 Y} X - \partial \overline{\theta}_{I_2 Y} I_1 X - I_1 \partial \overline{\theta}_X I_2 Y$$
$$= -I_3 a'_X Y + I_3 \partial \overline{\theta}_Y X - I_2 \partial \overline{\theta}_{I_1 X} Y - \partial \overline{\theta}_{I_2 Y} I_1 X - I_1 \partial \overline{\theta}_X I_2 Y.$$ 

Developing the last line of the previous computation gives the lemma. 

$\square$
If \((M, \mathcal{Q})\) is an almost quaternionic manifold and \(\nabla\) is a connection preserving \(\mathcal{Q}\), one defines the torsion \(T(\mathcal{Q})\) of \(\mathcal{Q}\) to be the projection of the torsion \(T^\nabla\) onto \(\Lambda^2 \otimes \Lambda^1 / \partial(\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))\). It does not depend of the choice of connection \(\nabla\) preserving \(\mathcal{Q}\) and it vanishes iff \(\mathcal{Q}\) is quaternionic.

2.2. The flat model. In this section, we describe the link between the flat hyperkähler metric \(euc\) on \(\mathbb{H}^{n+1}\) and the standard quaternionic-Kähler hyperbolic metric \(g_H\) on the unit ball \(B^{4(n+1)} \subset \mathbb{H}^{n+1}\). Let \(\nabla^0\) be the flat (hyperkähler) connection on \(\mathbb{H}^{n+1}\) and \(\nabla^H\) be the Levi-Civita connection of \(g_H\).

**Lemma 2.2.** The connections \(\nabla^H\) and \(\nabla^0\) are quaternionic and related by the formula 

\[
\nabla^H = \nabla^0 + a \frac{d\rho}{\rho}.
\]

**Proof.** Let \(\rho = 1 - |x|^2\) and let \(\partial_\rho\) be the dual vector field of \(d\rho\) respectively to \(euc\).

The connections \(\nabla^H\) and \(\nabla^0\) are quaternionic hence there exist a 1-form \(\theta\) such that \(\nabla^H = \nabla^0 + a \theta\). Let \(X \in \mathbb{H}^{n+1}\) satisfying \(d\rho(I_1 X) = d\rho(I_2 X) = d\rho(I_3 X) = 0\). Then, the formula \((\nabla^0 g_H + a \theta g_H)(X, X) = \nabla^H g_H(X, X) = 0\) gives

\[
\frac{4}{\rho} euc(X, X) \theta(\cdot) + \frac{4}{\rho} \theta(X)(euc(X, \cdot) - \sum_{i=1}^3 euc(I_i X, \cdot)) = -\frac{4}{\rho^2} euc(X, X) d\rho(\cdot).
\]

Applying this to \(X \neq 0\) and to \(I_i \partial_\rho\) gives \(\theta(X) = 0\) and \(\theta(I_i \partial_\rho) = 0\) whereas applying this to \(\partial_\rho\) gives \(\theta(\partial_\rho) = -1/\rho\). \(\square\)

In this description, one sees that \(\nabla^H\) admits a prolongation \(\overline{\nabla} = \nabla^0 + a \frac{d\rho}{\rho}\) to \(\mathbb{H}^{n+1}\) with pole along \(S^{4n+3}\) whereas \(\nabla^0 = \nabla + a \frac{d\rho}{\rho}\) is smooth on all \(\mathbb{H}^{n+1}\). We will keep this description in mind in order to prove the theorem\[\square\]. Indeed, we will prove that if \(\nabla\) is the connection of the AHQK-metric with given boundary \((M, H)\) and \(\rho\) is a defining function of \(M\), then both the quaternionic structure and \(\nabla + a \frac{d\rho}{\rho}\) have a smooth prolongation into a neighbourhood of \(M\).

**Remark 2.** As a quaternionic manifold, \(\mathbb{H}^{n+1}\) compactifies to \(\mathbb{H}P^{n+1}\).
Remark 3. There is another quaternionic-Kähler metric defined on $\mathbb{H}^{n+1}$, with positive scalar curvature, which comes from the embedding $\mathbb{H}^{n+1} \hookrightarrow \mathbb{H}P^{n+1}$, $x \mapsto [1, x]$. This metric can be written as

$$g_+ = \frac{4 \text{euc}}{1 + |x|^2} - \frac{1}{(1 + |x|^2)^2}((d\rho)^2 + (I_1d\rho)^2 + (I_2d\rho)^2 + (I_3d\rho)^2))$$

on $\mathbb{H}^{n+1}$.

2.3. Local description. In this section, we give the fundamental property of a QC-hypersurface and discuss the general situation of a real hypersurface in a quaternionic manifold.

**Proposition 2.1.** Let $(N, Q)$ be a quaternionic manifold and $\nabla$ be a torsion-free connection preserving $Q$. Suppose $f : M \to \mathbb{R}$ is a smooth function with non-vanishing differential $df$ for all $x \in M = f^{-1}(0)$. Then $M$ is a QC hypersurface of $N$ iff $\nabla df$ defines a positive or negative definite metric on the maximal $Q$-invariant subspace $H$ of $TM$ and $\nabla df(IX, IY) = \nabla df(X, Y)$ for all $X, Y \in H$ and $I \in Q$.

**Proof.** Assume first that $M = f^{-1}(0)$ is a QC-hypersurface in $N$ and that $(I_1, I_2, I_3)$ is a local choice of quaternionic structure defined in a neighbourhood of $p \in M$. The QC distribution is the distribution $H = \cap_i \ker df \circ I_i$ in $TM$. By hypothesis, there exists a metric $g$ on $H$ and $J_i \in \text{Vect}(I_1, I_2, I_3)$ such that $d(df \circ I_i)|_U = g(J_i \cdot, \cdot)$. The connection $\nabla$ is torsion-free, hence $A = \nabla df$ is symmetric. We obtain for $X, Y \in H$

$$A(X, I_iY) - A(Y, I_iX) + df((\nabla_X I_i)Y) - df((\nabla_Y I_i)X) = g(J_iX, Y).$$

Because $\nabla$ preserves the quaternionic structure and $df(I_iX) = 0$ for $X \in H$, we get

$$A(X, I_iY) - A(I_iX, Y) = g(J_iX, Y).$$

Therefore, $I_i$ and $J_i$ commute. Using now the fact that $J_i \in \text{vect}(I_j)_j$, we obtain the existence of $\lambda_i \in \mathbb{R}$ such that $J_i = \lambda_i I_i$. On one hand, we have

$$A(X, I_3Y) - A(I_3X, Y) = \lambda_3 g(I_3X, Y),$$

and on the other hand,

$$A(X, I_3Y) = A(X, I_1 I_2 Y) = \lambda_1 g(I_1 X, I_3 Y) + A(I_1 X, I_2 Y) = \lambda_1 g(I_3 X, Y) - \lambda_2 g(I_3 X, Y) - A(I_3 X, Y).$$
and therefore
\[ 2A(X, I_3 Y) = (\lambda_1 - \lambda_2 - \lambda_3) g(I_3 X, Y). \]
We thus get \(-2A(X, Y) = (\lambda - 1 - \lambda_2 - \lambda_3) g(X, Y).\) Hence, there exists
a scalar \(\lambda\) such that \(\lambda_1 - \lambda_2 - \lambda_3 = \lambda\), and by circular permutation
\(\lambda_2 - \lambda_3 - \lambda_1 = \lambda_3 - \lambda_1 - \lambda_2 = \lambda\). This gives \(\lambda_i = -\lambda \neq 0\) and
\[ 2A(X, Y) = \lambda g(X, Y). \]
□

**Remark 4.** Assume that \((M, H, g_H)\) is a quaternionic hypersurface in a
quaternionic-Kähler 8-manifold \((N, Q, g)\) where \(Q\) is the quaternionic
structure and \(g\) is the riemannian metric. In that case, the conformal
class \(g_H\) is completely determined by the quaternionic structure of
\(H\) and is thus equal to the conformal class of \(g|_H\). This is not necessarily
the case in dimension greater than 8.

For the sake of completeness, we describe now the structure induced
on a general hypersurface in a quaternionic manifold.

**Definition 2.2.** Let \(M^{4n+3}\) be a smooth manifold. A weakly quater-
nionic contact structure on \(M\) is the data of a codimension 3 distribu-
tion \(H\) and a \(GL(n, \mathbb{H})Sp(1)\)-structure \(Q \subset \text{End}(H)\) on \(H\) such that
locally there exist 1-forms \((\eta_1, \eta_2, \eta_3)\) vanishing on \(H\) and a
\(SO(3)\)-basis \((I_1, I_2, I_3)\) of \(Q\) satisfying :

(i) \(d\eta_i(I_i X, I_i Y) = d\eta_i(X, Y)\);
(ii) the tensor \(g = d\eta_1(\cdot, I_1 \cdot) + d\eta_1(I_2 \cdot, I_3 \cdot)\) is non-degenerate on \(H\);
(iii) for all \(X, Y \in H\), one has the equalities :
\[ g(X, Y) = d\eta_2(X, I_2 Y) + d\eta_2(I_3 X, I_1 Y) = d\eta_3(X, I_3 Y) + d\eta_3(I_1 X, I_2 Y). \]

The tensor \(g\) is symmetric. If it is positive definite, we say that \((H, Q)\)
is a strictly pseudo-convex weakly quaternionic contact distribution.

In this case, and contrary to the quaternionic-contact case, the quater-
nionic structure \(Q\) on \(H\) is not determined by the distribution \(H\). In
order to see that, let us describe a simple linear algebra example. On
\(\mathbb{H}\), let \((I_i^+)\) (resp. \((I_i^-)\)) be the action of \(i, j\) and \(k\) on right (resp. on
left), and define \(w_i^+ = \text{euc}(I_i^+ \cdot, \cdot)\) and \(w_i^- = \text{euc}(I_i^- \cdot, \cdot)\). We put for
\(|\lambda| < 1\), \(w_i^\lambda = \frac{1}{1-\lambda^2}(w_i^+ + \lambda w_i^-)\). Then, one has \(I_i^+ w_i^\lambda = w_i^\lambda\) and
\[ w_i^\lambda(\cdot, I_i^+ \cdot) + w_i^\lambda(I_i^- \cdot, I_i^+ \cdot) = \frac{4}{1-\lambda^2}\text{euc}, \]
with the other relations obtained by circular permutations. But on the other hand, we have that $w^\lambda_i \wedge w^\lambda_j = 2\delta_{ij}\nu$ where $\nu$ is the volume form of $euc$ so that there exists a quaternionic triple $(I^\lambda_i)$ and a metric $g^\lambda$ not in the conformal class of $euc$ such that $w^\lambda_i(\cdot,\cdot) = g^\lambda(I^\lambda_i\cdot,\cdot)$.

The notion of weakly quaternionic-contact distribution is introduced in order to describe hypersurfaces in quaternionic manifolds. Indeed, we have:

**Proposition 2.2.** Let $M^{4n+3}$ be an hypersurface in a quaternionic manifold $(N,Q)$ and let $H$ be the maximal $Q$-invariant subspace of $TM$. We denote still by $Q$ the set of the restrictions to $H$ of elements of $Q$. Let $f$ be a defining function of $M$, let $(I_1, I_2, I_3)$ be a local $SO(3)$-trivialization of $Q$ and put

$$g = \nabla df|_H + \sum_i I_i \nabla df|_H.$$ 

If $g$ is non-degenerate on $H$, then $(H,Q)$ is a weakly quaternionic-contact structure on $M$.

**Proof.** Let $f$ be a function defining $M$ locally, and $\eta_i = -df \circ I_i$ on $M$. Then, one has $d\eta_1|_H = \nabla df(\cdot,\cdot) + \nabla df(I^1_1, I^1_1)$ so that it is $I_1$-invariant and on $H$, 

$$d\eta_1(\cdot, I_1\cdot) + d\eta_1(I_2\cdot, I_3\cdot) = \nabla df(\cdot,\cdot) + \sum_{i=1}^3 \nabla df(I^i_1, I^i_1)$$

is $I_i$-invariant for all $i$. The other relations are obtained by cyclic permutation.

3. QC Geometry

This section gives the necessary background about QC-distribution. In particular, we define the integrability of a QC-distribution and prove that the QC-distribution of a 7-dimensional QC-hypersurface is integrable. Then we describe the properties of the Biquard connection that are used in the fifth section to compare the Biquard connection of a QC-hypersurface with the ambient quaternionic connection.

3.1. The group $Sp(n)Sp(1)$. The group $Sp(1)$ of unit quaternions acts on $\mathbb{H}^n$ by right multiplication and has centralizer $Sp(n) \subset SO(4n)$. One of the particular features of the group $Sp(n)Sp(1)$ is that it arises in Berger’s list of possible holonomy groups of non locally symmetric
riemannian manifolds. In this paper section, we are mainly interested in describing some features of the representations of $Sp(n)Sp(1)$.

Let $V^{(a_1, \ldots, a_n)}$ be the irreducible representation of $Sp(n)$ with highest root $(a_1, \ldots, a_n)$. If $\sigma \simeq \mathbb{C}^2$ is the standard representation of $Sp(1)$, then the irreducible representations of $Sp(n)Sp(1)$ are the tensor products $V^{(a_1, \ldots, a_n)} \otimes \sigma^p$ where $a_1 + \cdots + a_n + p$ is even and $\sigma^p$ is the $p$-symmetric power of $\sigma$; the real irreducible representations are the real parts $[V^{(a_1, \ldots, a_n)} \otimes \sigma^p]$ of the previous ones.

Following Salamon [Sal89], we put $\lambda_s^r = V^{(a_1, \ldots, a_n)}$ where $s$ of the $a_i$ are equal to 2, $r - 2s$ are equal to 1 and the others are zero and we abbreviate $[\lambda_s^r \otimes \sigma^p]$ in $[\lambda_s^r \sigma^p]$.

With this notation, we have $\mathfrak{sp}(1) = [\sigma^2]$, $\mathfrak{sp}(n) = [\lambda_2^n]$ and $[\lambda_0^n]$ is the set of symmetric traceless $\mathbb{H}$-linear endomorphisms.

Moreover, we have the decompositions

$$\mathfrak{so}(4n) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus [\lambda_0^n \mathfrak{sp}(1)]$$

and

$$\mathfrak{gl}(4n, \mathbb{R}) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus [\lambda_0^n \mathfrak{sp}(1)] \oplus [\mathfrak{sp}(n) \mathfrak{sp}(1)] \oplus [\lambda_0^n] \oplus \mathbb{R}$$

where $\lambda_0^n = 0$ if $n = 1$.

3.2. Integrability of a QC structure. Let $(M, H, g)$ be a QC distribution and $g$ a compatible metric on $H$ so that one gets a $Sp(n)Sp(1)$-structure on $H$. Let $(\eta_1, \eta_2, \eta_3)$ be a $SO(3)$-trivialization of the set of 1-forms vanishing on $H$. If $W$ is a complementary to $H$ and $(R_1, R_2, R_3)$ is a dual basis of $(\eta_i|_W)$, we put $\alpha_{ij} = \iota_{R_i} d\eta_j|_H$. Remark that we have the natural identification $W \simeq \mathfrak{sp}(1)$, $R_i \mapsto I_i$ and one can verify that

$$T^W = \sum_{i,j=1}^3 (\alpha_{ij} + \alpha_{ji}) \otimes I_i \otimes I_j \in H^* \otimes \mathfrak{sp}(1) \otimes \mathfrak{sp}(1) \simeq [\lambda^1 \sigma^1] \oplus [\lambda^1 \sigma^3] \oplus [\lambda^1 \sigma^5]$$

is a tensor. Changing $W$ corresponds to changing $T^W$ by an element in $W^* \otimes H \simeq [\sigma^2] \otimes [\lambda^1 \sigma]$, so that one can prove that there exists a unique complementary $W^g$ of $H$ such that $T^{W^g} \in [\lambda^1 \sigma^5]$. The decomposition of $H^* \otimes \mathfrak{sp}(1) \otimes \mathfrak{sp}(1)$ is explicitly given by:

$$[\lambda^1 \sigma^1] = \left\{ \sum_i r \otimes I_i \otimes I_i , \; r \in H^* \right\} ,$$

$$[\lambda^1 \sigma^3] = \left\{ \sum_{i,j} (I_i r_j + I_j r_i) \otimes I_i \otimes I_j , \; r_i \in H^*, \; \sum_i I_i r_i = 0 \right\} ,$$

$$[\lambda^1 \sigma^5] = \left\{ \sum_{i,j} a_{ij} \otimes I_i \otimes I_j , \; a_{ij} = a_{ji} \text{ and } \sum_j I_i a_{ij} = 0 \right\} .$$
Remark 5. The vector fields $R_i$ in $W^g$ are called Reeb vector fields of $(\eta_1, \eta_2, \eta_3)$.

**Theorem 3.1** ([Duc04]). Let $(H, g)$ be a quaternionic contact distribution in a manifold $M$ of dimension $4n+3$. The tensor $T^{W^g}$ is called the vertical torsion of $H$. It is conformally invariant and vanishes automatically when $n > 1$. If $n = 1$ and $T^{W^g} = 0$, one says that $H$ is integrable.

The importance of the integrability condition is enhanced by the following proposition.

**Theorem 3.2.** A QC-hypersurface $M$ of a quaternionic manifold $N$ of dimension $8$ is integrable.

**Proof.** Assume that $f$ is a defining function for $M$. There exists a vector field $\xi$ defined up to a vector field in $H$ and such that $df(\xi) = 1$ and $df(I_i \xi) = 0$. Moreover, $\nabla df$ is non-degenerate on $H$, hence we can assume that $\nabla df(\xi, X) = 0$ for all $X \in H$. Let us define $\alpha_{ij} = -i_{I_i \xi} d(df \circ I_j)|_H$. We have for $X \in H$, 

$$\alpha_{ij}(X) = -\nabla df(I_i \xi, I_j X) + \nabla df(X, I_j I_i \xi) + df(\nabla_X I_j) I_i \xi$$

and therefore,

$$\alpha_{ij}(X) + \alpha_{ji}(X) = \alpha_i(I_j X) + \alpha_j(I_i X)$$

where $\alpha_i(X) = -\nabla df(I_i \xi, X)$. \hfill \Box

Remark 6. The vector field $\xi$ that appears in the previous proof is called the normal vector field of $f$ along $M$.

3.3. The Biquard connection. In this section, I describe the connection of Biquard for a QC distribution. The results I give come from [Big00] and [Duc04]. Let $\tilde{g}$ be the metric equals to $\sum_i \eta_i^2$ on $W^g$, $g$ on $H$ and such that $H$ and $W^g$ are orthonormal.

**Theorem 3.3.** Let $H$ be a QC distribution, integrable if $n = 1$ and let $g$ be an adapted metric. There exists a unique connection $\nabla$ preserving $W^g$, $H$ and $\tilde{g}$ and satisfying:

(i) $\nabla$ preserves the $Sp(n)Sp(1)$ structure on $H$.

(ii) if $R \in W$ and $X \in H$, then the torsion $T(R, X)$ is in $H$ and $X \mapsto T(R, X) \in (sp(n) \oplus sp(1))^\perp$. 
(iii) If $X, Y \in H$, then $T(X, Y) \in W$ and if $R, R' \in W$ then $T(R, R') \in H$.

Remark 7. The point (ii) follows from 3.1.

Remark 8. If $X$ and $Y$ are in $H$, then the torsion $T(X, Y)$ satisfies
\[ T(X, Y) = \sum_{i} \langle I_i X, Y \rangle R_i . \]

Remark 9. The connection given here differs slightly of the connection of [Biq00]. In fact, the only differences lies in the terms $\nabla R_i R'$ of the connection, so that the terms $\nabla X R_i$ and $\nabla X I_i$ still coincide when we identify $I_i$ and $R_i$.

One can prove the following stronger result for the torsion:

**Proposition 3.1.** Assume that the QC distribution $H$ is integrable if $n = 1$. The part $T^H$ in $W^* \otimes H^* \otimes H$ of the torsion of $\nabla$ satisfies $T(R, \cdot) \in [\lambda_0^2 \text{sp}(1)] \oplus [\text{sp}(n) \text{sp}(1)]$ for all $R \in W$ and
\[ T^H \in [\lambda_0^2] \oplus [\text{sp}(n) \text{sp}(1)] \subset \text{sp}(1) \otimes ([\lambda_0^2 \text{sp}(1)] \oplus [\text{sp}(n) \text{sp}(1)]). \]

Remark 10. If $n = 1$, then $\lambda_0^2 = 0$, so that if $H$ is integrable, then $T^H \in \text{sp}(1) \text{sp}(1)$.

Remark 11. The previous proposition implies the existence of $\tau \in [\lambda_0^2]$ and $\tau_k \in \text{sp}(n)$ such that
\[ T^H_{R_i} = I_i \tau + \sum_{k,j=1}^{3} \epsilon^{ijk} I_j \tau_k . \]

4. **Twistor spaces**

In this section, we will prove theorem 1.1. In a first part, we recall the definition of the twistor space of a QC-distribution $H$ (integrable in dimension 7) and the properties of the twistor space of the AHQK metric which has conformal infinity $H$.

4.1. **The twistor space of a QC-distribution.** Let $(M^{4n+3}, H)$ be a QC-contact distribution that is integrable if $n = 1$, and let $g$ be a compatible metric on $H$. Let $(I_1, I_2, I_3)$ be a local quaternionic structure on $H$ and $(\eta_i)_{i=1,2,3}$ such that $d\eta_i(\cdot, \cdot) = g(I_i \cdot, \cdot)$ on $H$. Define
\[ \mathcal{T} = \{ x_1 I_1 + x_2 I_2 + x_3 I_3, \ x_1^2 + x_2^2 + x_3^2 = 1 \} \]
the set of compatible almost complex structures on $H$ and let $\pi : T \to M$. The space $\mathcal{T}$ is a 2-sphere bundle over $M$, and carries a 1-form $\eta = x_1 \pi^* \eta_1 + x_2 \pi^* \eta_2 + x_3 \pi^* \eta_3$ defined up to a conformal factor. At a point $I \in T$, one defines an almost-complex structure $I$ on $T$ in the following way: the connection $\nabla$ splits $T_I T$ into the space $T_{\pi(I)} T$ tangent to the fibers and an horizontal space $\text{Hor}_I T \cong T_{\pi(I)} M = H_{\pi(I)} \oplus W_{\pi(I)}$.

Changing the basis the local basis $(I_1, I_2, I_3)$, one can assume that $I = I_1$, and the almost complex structure $\mathcal{I}$ is the natural one on $T_{I_1} T \cong TS^2$, whereas $\mathcal{I}|_{H} = I_1$ and $\mathcal{I}(R_2) = R_3$.

**Theorem 4.1** ([Biq00], [Duc04]). Let $(M^{4n+3}, H)$ be an integrable QC-manifold. The triple $(T, \eta, I)$ is a CR-integrable structure of signature $(4n + 2, 2)$ and called the twistor space of $M$.

One has the following result:

**Proposition 4.1** ([LeB82], [Biq00]). Let $(M, H)$ be an analytic quaternionic contact distribution and let $(T, \eta, I)$ be its twistor space. There exists a contact holomorphic manifold $(N^{2n+3}, \eta^c)$, a family $(C_m)_{m \in N^c}$ of dimension $2n + 2$ of smooth rational curves in $N^c$ and a hypersurface $M^c \subset N^c$ such that:

- The distribution $\ker \eta^c$ is transverse to the curves $C_m$ as soon as $m \in N^c - M^c$.
- The normal bundle of the curves $C_m$ is $\mathcal{O}(1) \otimes \mathbb{C}^{2n+2}$.
- There exists a compatible real structure $\sigma$ on $N$ such that $M$ is the real slice of $M^c$ and $T$ is a real hypersurface in $N$ with the induced CR-structure.
- There exists an holomorphic metric $g$ on $N^c - M^c$ with holonomy $Sp_{n+1}(\mathbb{C})Sp(1)$ and conformal infinity $H$.

4.2. Embedding a QC structure in a quaternionic manifold. Let $(M^{4n+3}, H)$ be an integrable QC-manifold. We use the notations of section 4.1. If $x \in N^c$, then $N_x$ stands for the normal fiber bundle of $C_x$. We put

$$
E_x = H^0(C_x, L^{-\frac{1}{2}} \otimes N_x),
$$

$$
A_x = H^0(C_x, L^{\frac{1}{2}})
$$

so that $T_x N^c = H^0(C_x, N_x) = E_x \otimes A_x$. We get a $GL(2n+2, \mathbb{C})GL(2, \mathbb{C})$-structure on $N^c$ and an almost-quaternionic structure on $N$ such that
on $N^c - M^c$, the $Sp_{n+1} \mathbb{C}Sp(1)$-structure is a reduction of this $GL(2n+2, \mathbb{C})GL(2, \mathbb{C})$-structure. In particular, one sees that the almost quaternionic structure of $N - M$ admits a smooth prolongation to $M$. Because the almost-quaternionic on $N - M$ admits a quaternionic-Kähler metric, its torsion vanishes on $N - M$, and so on $N$ by continuity.

If $x \in M^c$, the tangent space of the curve $C_x$ lies in the kernel of $\eta^c$. It follows that the hyperplane $H^c_x = H^0(C_x, \ker \eta^c / TC_x) \subset T_x M^c$ is well defined. In fact, if $x \in M$, then $H^c_x$ is the real part of $H^c_x$. In the decomposition $T_x N^c = E_x \otimes A_x$, we see that $H^c_x$ is the kernel of the 1-form $\eta = \eta^c \otimes 1 \in (E \otimes A)^* \otimes L$. We deduce that:

**Lemma 4.1.** The hyperplane $H^c_x$ is invariant under the action of the subgroup $GL(A)$ of $GL(T_x N^c)$.

To summarize, we have proved the following result:

**Theorem 4.2.** Let $(M^{4n+3}, H)$ be an integrable QC-manifold. There exists a quaternionic manifold $(N^{4n+4}, Q)$ such that:

(i) $M$ is a QC-hypersurface of $N$ and $H$ is the $Q$-invariant subspace of $TN$.

(ii) $M$ separates $N$ into two quaternionic manifolds $N_+$ and $N_-$ such that $N_+$ has a definite-positive quaternionic-Kähler metric with conformal infinity $H$ and $N_-$ has a quaternionic-Kähler metric with signature $(4, 4n)$ and conformal infinity $H$.

We give now an explicit torsion-free connection on $N^c$ that preserves the quaternionic structure. Let $\nabla$ be the Levi-Civita connection of $g$. This is a meromorphic connection on $N^c$. Let $\rho$ be an holomorphic function defined on a neighbourhood of a point $p \in N^c$ and vanishing up to order one on $M^c$. One knows that we can write

$$g = \frac{1}{\rho^2}((d\rho)^2 + \eta^2 + \eta_2^2 + \eta_3^2) + \frac{1}{\rho}g_{H} + \cdots.$$

**Proposition 4.2.** The connection $\nabla + \frac{\partial \rho}{\rho}$ defined on a neighbourhood of $p \in M^c$ is holomorphic and its restriction to $N$ gives a torsion-free connection preserving the quaternionic structure of $N$.

**Proof.** We write

$$g = \frac{1}{\rho^2}((d\rho)^2 + \eta^2 + \eta_2^2 + \eta_3^2) + \frac{1}{\rho}g_{H} + g_0 + \cdots.$$
where $g_{-1}|H = g_H$ and $g_i$ is a covariant 2-tensor which does not depend of $\rho$. Because $g_{-1}|H$ is non degenerate, one can define the holomorphic orthogonal $W$ of $H$ with respect to $g_{-1}$. The dots will indicate terms of order strictly inferior in $\rho$ when $\rho$ goes to zero.

Let $(X_i)_{i \geq 4}$ be an orthonormal basis of $H$ respectively to $g_H$, $(X_i)_{1 \leq i \leq 3}$ be an orthonormal basis of $W \cap \ker d\rho$ for $\sum_i \eta_i^2$ and $X_0$ such that $d\rho(X_0) = 1$, $\eta_i(\partial_\rho) = 0$ and $g_{-1}(X_0, H) = 0$. The $Sp_{n+1}(\mathbb{C})Sp_1(\mathbb{C})$-structure is sent holomorphically to the $Sp_{n+1}(\mathbb{C})Sp_1(\mathbb{C})$-structure of the complexification of $g_H$, up to first order in $\rho$. Moreover, writing the equalities of the form

$$ g([X_i, X_j], X_k) = -\frac{1}{\rho^2} \sum_p d\eta_p(X_i, X_j)X_k + \cdots $$

if $i, j \geq 4$ and $1 \leq k \leq 3$, one sees that it sends the 1-jet of the $Sp_{n+1}(\mathbb{C})Sp_1(\mathbb{C})$ structure holomorphically to the 1-jet of the $Sp_{n+1}(\mathbb{C})Sp_1(\mathbb{C})$-structure of the complexification of $g_H$, up to first order in $\rho$. Therefore $\nabla$ has a pole of order 1 along $M^c$ and $\nabla + a \frac{d\rho}{\rho}$ admits an holomorphic continuation to $N^c$.

4.3. Examples. In this section, we describe an illustration of this theorem with the help of a family quaternionic-Kähler metrics obtained by Bogdan Alexandrov in [Bog01]. The construction begins with the data of a hyper-Kähler manifold $(M', I'_1, I'_2, I'_3, g)$ with Kähler forms $w'_i$ satisfying the hypothesis that there exists 1-forms $\alpha_i$ such that $d\alpha_i = w'_i$.

Let $(\rho, x_1, x_2, x_3)$ be the coordinates of $\mathbb{H}$, $N = \mathbb{H} \times M'$ and $M$ be the hypersurface defined by $\rho = 0$ and let $\pi : N \to M'$ be the projection. One defines an hypercomplex structure $(I_1, I_2, I_3)$ by the formula

$$ I_i d\rho = -dx_i + \pi^* \alpha_i $$

and $I_i \pi^* \beta = \pi^* I'_i \beta$ if $\beta \in T^* M'$.

Then, $M$ is a quaternionic contact hypersurface in $N$ with contact distribution $H = \cap \ker I_i d\rho$ and it is the conformal infinity or the quaternionic-Kähler metric

$$ g = -\frac{1}{\rho^2}(d\rho^2 + \sum_i (I_i d\rho)^2) + \frac{1}{\rho} \pi^* g'. $$

Moreover, the compatible metric on $H$ is $g' \circ \pi_*|_H$. 

5. SECOND FUNDAMENTAL FORMS OF A QC-HYPERSONFAC E

One may ask the following question: Does every embedding of a quaternionic-contact structure arises in the way of Theorem 1.1? The aim of this section is to give a first step toward an answer to this question. In particular, I will define second fundamental forms for QC-hypersurfaces and prove that it vanishes for the embedding given by Theorem 1.1.

Let \((N, Q, \nabla)\) be a quaternionic manifold, \(f\) a smooth function on \(N\) and assume that \(M\) is a QC-contact hypersurface with defining function \(f\), QC-distribution \(H\) and compatible definite positive metric \(g = 2\nabla df|_H\). Let \((I_1, I_2, I_3)\) be a local choice of quaternionic structure, \(\xi\) be the normal vector field of \(f\) along \(M\), \(\eta_i = -df \circ I_i\) and \(R_i = I_i \xi + r_i\) be the Reeb vector field of \(\eta_i\). From the proof of theorem 3.2, we get that \(r_i \in H\) is determined by \(\nabla d f(I_i \xi + 2r_i, X) = 0\) for \(X \in H\). The subvector bundle spanned by \(I_1 \xi, I_2 \xi\) and \(I_3 \xi\) is called \(W^\xi\), and \(p_H\) is the projection onto \(H\) with kernel \(W^\xi \oplus \mathbb{R}^\xi\).

5.1. Covariant derivatives in the direction of \(H\). One has a natural connection on \(TM\) defined by \(\nabla^H Y = p_H \nabla_X Y\) if \(X, Y \in TM\). We compare now this connection with the Biquard connection \(\nabla^g\) on \(H\).

The kernel of \(p_H\) is \(Q\)-invariant, thus \(\nabla^H\) preserves the \(GL(n, \mathbb{H})Sp(1)\)-structure of \(H\). Because the torsion of \(\nabla\) is zero, for \(X, Y \in H\), the torsion \(T^H\) of \(\nabla^H\) satisfies

\[
T^H(X, Y) = -\sum_i d\eta_i(X, Y)I_i \xi.
\]

Let \(a\) the skew-symmetrisation and \(X, Y\) and \(Z\) be in \(H\), so that if \(w_i = d\eta_i|_H\), then

\[
0 = a(\nabla^H w_i)(X, Y, Z)
\]

\[
+ d\eta_i(T^H(X, Y), Z) + d\eta_i(T^H(Z, X), Y) + d\eta_i(T^H(Y, Z), X),
\]

and therefore if we restrict now ourselves on \(H\),

\[
a(\nabla^H w_i) = -\sum_j i_{j\xi} d\eta_i|_H \wedge w_j.
\]

Let \(\Omega = \sum_i w_i^2\) be the fundamental form of the quaternionic structure of \(H\). We have obtained

\[a(\nabla^H \Omega) \in [\sigma^2] \otimes [\sigma^2] \otimes [\lambda^1 \sigma^1] \cap \Lambda^5 = [\lambda^1 \sigma^3] \oplus [\lambda^1 \sigma^1].\]
The skew-symmetrisation $a : \Lambda^1 \otimes \Lambda^1 \rightarrow \Lambda^5$ is injective if $n \geq 3$ and its kernel is $[\Lambda^3 \sigma^3]$ when $n = 2$ so that $\nabla \Omega \in [\Lambda^1 \sigma^3] \oplus [\Lambda^1 \sigma^1]$ if $n \geq 3$ and $\nabla \Omega \in [\Lambda^1 \sigma^3] \oplus [\Lambda^1 \sigma^1] \oplus [\Lambda^3 \sigma^3]$ if $n = 2$. On the other hand, $\nabla^H$ is quaternionic, so that

\begin{equation}
\nabla^H \Omega \in \Lambda^1 \otimes ((\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^\perp \cap (\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1))) ,
\end{equation}

i.e.

\begin{equation}
\nabla^H \Omega \in 2[\Lambda^1 \sigma^1] \oplus [\Lambda^3 \sigma^1] \oplus [\Lambda^3 \sigma^1].
\end{equation}

Comparing (3) and (4), we see that $\nabla^H \Omega$ must live in a factor isomorphic to $[\Lambda^1 \sigma^1]$. A change of quaternionic connection $\nabla \rightarrow \nabla + a^\theta$ changes $\nabla^H \Omega$ by a factor $\theta|_H \in [\Lambda^1 \sigma^1]$. Hence, one can choose $\theta$ in such a way that $\nabla^H \Omega$ vanishes.

**Proposition 5.1.** Assumes that $n \geq 2$. The adapted complementary vector bundle $W^g$ is equal to $W^\xi$. Moreover, one can choose $\nabla$ in such a way that $\nabla^H_X Y = \nabla_X^H Y$ for $X, Y \in H$.

**Corollary 5.1.** Assumes that $n \geq 2$ and that $\nabla$ is chosen as in [5.1]. The partial covariant derivatives $\nabla^H$ and $\nabla^g : \Gamma(Q) \rightarrow \Gamma(H^* \otimes Q)$ are equal.

5.2. **Covariente derivative in the direction of** $W^g$. In this subsection, we assume that $n \geq 2$ and that $\nabla$ is chosen as in proposition 5.1. Let $\varepsilon^{ijk}$ be the signature of the permutation $(123) \rightarrow (ijk)$, and let $(e_k)$ be an orthonormal basis of $H$, $(e^*_k)$ be its dual basis and $w_i = \frac{1}{2} \sum_k e^*_k \wedge I_i e_k \in \Lambda^2 H^*$ be the restriction $d_{H|H}$. We put $w_i^* = \frac{1}{2} \sum_k e_i \wedge I_i e_k$.

**Lemma 5.1.** For all $i, j \in \{1, 2, 3\}$, one has the following formula giving the action of $\nabla^g$ on $Q$:

\begin{equation}
\nabla^g_{I_i \xi} I_j = \nabla^H_{I_i \xi} I_j + \sum_{p,k} \varepsilon^{ijk} df(R_{w_k}^{\nabla} I_i \xi))I_p - \sum_{p,k=1}^{3} \varepsilon^{jkp} \nabla df(I_k \xi, I_l \xi)I_p.
\end{equation}

**Proof.** Let $T$ be the torsion of the Biquard connection $\nabla^g$. If $X \in H$, the $H$-part of the torsion $T^H(I_i \xi, X)$ of $\nabla^H$ is equal to $\nabla^H_X I_i \xi$. Therefore, we get:

\[
(\nabla^g_{I_i \xi} I_j - \nabla^H_{I_i \xi} I_j)X = T(I_i \xi, I_j X) - I_j T(I_i \xi, X) - \nabla^H_{I_i \xi} I_i \xi + I_j \nabla^H_{I_i \xi} I_i \xi.
\]
Because both $\nabla^g$ and $\nabla^H$ preserve $Q$, we can now take the $\mathfrak{sp}(1)$-part of this expression. The projection onto $\mathfrak{sp}(1)$ of $(T(I\xi, Ij) - IjT(I\xi, \cdot))|_H$ vanishes (see section 3.3), hence
\[
\nabla^g_{I\xi}I_j - \nabla^H_{I\xi}I_j = (-\nabla^H_{I\xi}I_j \xi + I_j \nabla^H_{I\xi}I_j)_{\mathfrak{sp}(1)}.
\]
Let $p \neq j$ be in $\{1, 2, 3\}$. As $\nabla df(W^\xi \oplus \Re \xi, H) = 0$, one gets for $X \in H$,
\[
g(I_j \nabla^H_{I\xi}I_j - \nabla^H_{I\xi}I_j, I_pX) = \nabla df(I_j \nabla_{I\xi}I_j - \nabla_{I\xi}I_j, I_pX) = -\nabla df(I_j I_pX, \nabla_{I\xi}I_j) - \nabla df(I_pX, \nabla_{I\xi}I_j).
\]
One has
\[
\nabla df(I_j I_pX, \nabla_{I\xi}I_j) = (I_j I_pX).df(\nabla_{I\xi}I_j) - df(\nabla_{I\xi}I_j, I_pX) = (I_j I_pX).(-\nabla df(X, I\xi) + X. df(I_j)) - df(\nabla_{I\xi}I_j, \nabla_{I\xi}I_j)
\]
so that
\[
g(I_j \nabla^H_{I\xi}I_j - \nabla^H_{I\xi}I_j, I_pX) = df(\nabla_{I\xi}I_pX, \nabla_{I\xi}I_j + \nabla_{I_pX} \nabla_{I\xi}I_j).
\]
We replace now $X$ by $e_i$ and sum over the basis $(e_1, \ldots, e_{4n})$ of $H$ to obtain the lemma.

5.3. Second fundamental forms. By a conformal change $\eta'_i = g^{2i} \eta_i$, the tensor $w_i$ becomes $(w^**_i)' = g^{-2} w^*_i$.

**Definition 5.1.** The conformal second fundamental form of the embedding $M \hookrightarrow N$ is the trace-free part of the tensor
\[
\sum_{i,j} (\nabla df(I_i \xi, I_j \xi) - 4 \nabla df(r_i, r_j)) \otimes w_i \otimes I_j.
\]
It does not depend of $f$ nor of the torsion-free connection $\nabla$.

**Remark 12.** When $n = 0$, we obtain the trace-free part of the second fundamental form of the embedding of a manifold $M$ into a conformal manifold $(N, [g])$.

**Proof.** Assume that $f' = g^{2}f$ is another defining function of $M$ where $g$ does not vanish, and that $\xi'$ is the normal vector field of $f'$ along $M$. Then, one has $df' = g^{2}df$ and $\nabla df' = 2gdg \otimes df + 2gdf \otimes dg + g^{2}\nabla df$ along $M$. Hence, if $\xi' = g^{-2} \xi + r$ with $df'(r) = 0$, then the formula $df'(I_i \xi') = df(I_i \xi) + g^{2}df(I_i r) = 0$ implies that $r \in H$. The Reeb vector
fields \( R'_i \) of the forms \( \eta'_i = g^2 \eta_i \) can be written \( R'_i = I_i \xi' + r'_i \) and satisfy \( \nabla df'(I_i \xi' + 2r'_i, X) = 0 \) for \( X \in H \), whence

\[
r'_i = g^{-2} r_i - \frac{1}{2} I_i r.
\]

We obtain

\[
\nabla df'(I_i \xi', I_j \xi') + 4 \nabla df'(r_i, r_j) = g^{-2} (\nabla df(I_i \xi, I_j \xi) - 4 \nabla df(r_i, r_j)) + \nabla df(I_i \xi, I_j r) + \nabla df(I_i r, I_j \xi) + 2 \nabla df(r_i, I_j r) + 2 \nabla df(r_j, I_i r).
\]

Therefore, the conformal curvature is independent of \( f \). A quick computation and the formula of lemma 2.1 give that it is independent of the connection \( \nabla \).

\[\square\]

**Definition 5.2.** Assume \( n = 1 \). The projection \( Q \) onto \([\lambda^1 \sigma^2] \) of the tensor

\[
\sum_i \nabla df(I_i \xi, \cdot)|_H \otimes I_i \in [\lambda^1_0 \sigma^1] \otimes [\sigma^2] \simeq [\lambda^1_0 \sigma^1] \oplus [\lambda^3_0 \sigma^3]
\]

is called the horizontal second fundamental form of the QC-hypersurface \( M \) in \( N \). It does not depend on the choice of torsion-free connection \( \nabla \) preserving \( Q \) nor of the choice of defining function \( f \) for \( M \).

**Proof.** Using the proof of the definition 3.1, we obtain that for \( X \in H \),

\[
\nabla df'(I_i \xi', X) = \nabla df(I_i \xi, X) + g^2 \nabla df(I_i r, X)
\]

and finally that only the factor in \([\lambda^1_0 \sigma^1]\) changes. \[\square\]

### 5.4. Second fundamental forms and AHQK-metrics.

**Theorem 5.1.** Assume that \((M, H)\) is a QC-hypersurface in a quaternionic manifold \((N, Q)\) such that there exists an AHQK-metric \( g \) defined on an open subset of \( N \), compatible with \( Q \) and with conformal infinity \((M, H)\). Then the second fundamental forms of \( M \) vanish.

**Proof.** Let \( \rho \) be a defining function of \( M \), let \( \nabla \) be a quaternionic connection such that \( \tilde{\nabla} = \nabla + a \frac{\partial}{\partial \rho} \) is the Levi-Civita connection of \( g \).

Let \((I_1, I_2, I_3)\) be a local trivialization of \( Q \) around a point \( p \in M \) such that the almost complex structures \( I_i \) are parallel at \( p \) respectively to \( \nabla \) and put \( \nabla I_i = \sum_{j,k} \varepsilon^{ijk} \beta_k \otimes I_j \).
We compare the curvature $\hat{R}^{\mathcal{V}}$ and $R^{\mathcal{V}}$ acting on $\mathcal{Q}$. We have the well known formula

$$\hat{R}^{\mathcal{V}} = R^{\mathcal{V}} + d\mathcal{V} a - \frac{da}{\mathcal{V}} + [a - \frac{da}{\mathcal{V}}, a - \frac{da}{\mathcal{V}}],$$

which gives

$$\hat{R}^{\mathcal{V}} - R^{\mathcal{V}} = \sum_{i=1}^{3} \frac{1}{\rho^2} \left( d(\rho \circ I_i) \wedge d\rho + \frac{1}{2} \sum_{j,k} \epsilon^{ijk} d\rho \circ I_j \wedge d\rho \circ I_k \right) \otimes I_i$$

$$+ \sum_{i=1}^{3} \frac{1}{\rho} d(d\rho \circ I_i) \otimes I_i.$$

Let $w_i$ be the Kähler form $w_i(\cdot, \cdot) = g(I_i \cdot, \cdot)$. The metric $g$ is quaternionic Kähler, hence $\hat{R}^{\mathcal{V}} = \sum_i w_i \otimes I_i$ on $\mathcal{Q}$. The term $R^{\mathcal{V}}$ in the previous formula extends smoothly on $M$, so that we obtain

$$w_i = \frac{1}{\rho^2} \left( d(\rho \circ I_i) \wedge d\rho + \frac{1}{2} \sum_{j,k} \epsilon^{ijk} d\rho \circ I_j \wedge d\rho \circ I_k \right) + \frac{1}{\rho} d(d\rho \circ I_i) + \gamma_i$$

where $\gamma_i$ extends smoothly on $M$. In particular, we get for all $i$,

$$g = \frac{1}{\rho^2} (d\rho + \sum_{j=1}^{3} (I_j d\rho)^2)$$

$$- \frac{1}{\rho} (\nabla d\rho + \nabla d\rho(I_i \cdot, I_i \cdot) + \sum_{j,k} \epsilon^{ijk} (\beta_k \otimes d\rho + I_i d\rho \otimes I_i \beta_k)) + \cdots$$

where the dots indicates terms that extend smoothly on $M$. We deduce that at $p$, one has $\nabla d\rho(X, Y) = \nabla d\rho(I_i X, I_i Y)$ for all $i$ and $X, Y$. Let $\xi$ be the normal vector field of $\rho$ along $M$. It turns out that it is equivalent to $W^\xi = W^g$ and to the vanishing of the second fundamental forms.

6. Final remarks

This paper is a first step in the direction of understanding the structures arising on hypersurfaces in quaternionic manifolds. In the case of quaternionic contact structures, it is interesting to know if one can construct an embedding of a QC-distribution into a quaternionic manifold with a given second fundamental form in the same way that in the 3-dimensional case ([LeB85]). We suspect that there are obstructions...
involving higher derivatives of the second fundamental forms, obstruc-
tions that one should be able to recognize in the construction of an
adapted twistor space.

In the general case of weakly QC-distribution, a CR-twistor space
should exist, but the construction of an adapted Biquard connection
remains to be done. We will adress these problems in a future work.

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