The Intersection R-Torsion for Finite Cone

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1 Introduction

Torsion invariants were originally introduced in the 3-dimensional setting by K. Reidemeister [25] in 1935 who used them to give a homeomorphism classification of 3-dimensional lens spaces. The Reidemeister torsions (R-torsions for short) are defined using linear algebra and combinatorial topology. The salient feature of R-torsions is that it is not a homotopy invariant but rather a simple homotopy invariant; hence a homeomorphism invariant as well. From the index theoretic point of view, R-torsion is a secondary invariant with respect to the Euler characteristic. For geometric operators such as the Gauss-Bonnet and Dolbeault operator, the index is the Euler characteristic of certain cohomology groups. If these groups vanish, the Index Theorem has nothing to say, and secondary geometric and topological invariant, e.g., R-torsion, appears. R-torsions were generalized to arbitrary dimensions by W. Franz [13] and later studied by many authors (Cf. [20]).

Analytic torsion (or Ray-Singer torsion), which is a certain combinations of determinants of Hodge Laplacians on k-forms, is an invariant of Riemannian manifolds defined by Ray and Singer [24] as an analytic analog of R-torsions. The Ray-Singer conjecture, that the analytic and Reidemeister torsion agree on closed manifolds, was proved independently by Cheeger [4] and Müller [21] using different techniques. Later, Vishik [30] gave a cutting and pasting proof from the viewpoint of topological quantum field theory, and Bismut and Zhang [1] used Witten deformation to generalize it to arbitrary flat bundles, see also [22] for the unimodular case.

Further significant work includes that of Müller [22], which extended the theorem to unimodular representations, that of Bismut and Zhang [1], which treated general representations (in which interesting secondary invariants come in), and that of Burghelea-Friedlander-Kappeler-McDonald [3], which dealt with infinite dimensional representations.

It is a natural question whether the Ray-Singer conjecture/Cheeger-Müller theorem extends to singular manifolds. For manifolds with isolated conical singularity, both the R-torsion and analytic torsion have been defined by Dar [12], using respectively, the intersection homology of Goresky-MacPherson [14, 15] and Cheeger’s theory of heat kernels for conical singularity [5]. There are several possible approaches to this question, among which the most natural one is to reduce the problem to three parts. One concerns manifolds with boundary, for which the question has been extensively studied [4, 18, 19, 9, 2]. The second part would be a finite cone. The last part deals with the Mayer-Vietories principle.

In this paper we concentrate on the intersection R-torsion side of the story. (We note that there has been extensive work on the analytic torsion of cones and conical manifolds, see, for example, [27, 29, 16, 23].) We will first study the intersection R-torsion of a finite cone. Our main result expresses it as a combination of determinants of the combinatorial Laplacian on the cross section

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of the cone. We then study an analytic invariant which is obtained by replacing the combinatorial Laplacian by the Hodge Laplacian.

More specifically, consider the finite cone \( X = C(Y) \) with the cross section \( Y \) a closed \((n-1)\)-dimensional manifold. Let \( I\tau^p(X) \) denote the intersection R-torsion of \( X \), where \( p = (p_2, p_3, \ldots, p_n) \) is a given perversity. Then,

**Theorem 1.1** Let \( \Delta^{(c)} \) denote the combinatorial Laplacian of the cross section \( Y \). Then

\[
\ln I\tau^p(X) = \sum_{p=0}^{n-p_n-1} (-1)^{p+1} p \ln \det \Delta^{(c)}_p + (n-p_n) \sum_{p=p_n}^{n-1} (-1)^{p+1} \ln \det \Delta^{(c)}_p.
\]

This leads us to an analogous analytic invariant for an even dimensional manifolds. Thus, let \( Y \) be an even dimensional closed manifold with \( m = \dim Y \). Let \( p \) be an integer such that \( 0 \leq p \leq m-1 \) (\( p \) corresponds to \( p_n \) which is determined by a given perversity). Given an orthogonal representation \( \rho : \pi_1(Y) \to O(N) \), one has an associated flat vector bundle \( E_{\rho} \) with compatible metric on \( Y \). Let \( \Delta_k \) be the Laplacian acting on differential forms on \( Y \) with coefficients in \( E_{\rho} \). Then we define

\[
\ln T_p(Y, \rho) = \frac{1}{2} \left[ \sum_{k=0}^{m-p} (-1)^k \ln \det(\Delta_k) + (m-p) \sum_{k=m-p+1}^{m} (-1)^k \ln \det(\Delta_k) \right].
\]

For \( p = 0 \) this gives the usual analytic torsion which is trivial for dimensional reasons. Other values of \( p \) give nontrivial and more interesting analytic invariants that bear close resemblance to the so called Cheeger’s half torsion, see §4. To investigate what kind of invariant \( \ln T_p(Y, \rho) \) defines, we now look at its variation under metric change. Let \( g(u) \) be a family of Riemannian metrics on \( Y \) and \( \Delta_k(u) \) the corresponding Laplacian (when there is no ambiguity we will often write \( \Delta_k \) instead of \( \Delta_k(u) \). Let \( * = d \star / du \) and \( \alpha = \star^{-1} \alpha \). Denote by \( E_k(t) = e^{-t\Delta_k(u)} \) the heat kernel and let \( E_k = E_k^{\text{ex}} + E_k^{\text{ce}} + E_k^{\text{h}} \) denote the Hodge decomposition of \( E_k \) into its exact, coexact and harmonic parts. We have the following result regarding the variation of \( \ln T_p(Y, \rho) \).

**Theorem 1.2** The variation of \( \ln T_p(Y, \rho) \) is given by

\[
\frac{d}{du} \ln T_p(Y, \rho) = \phi_{12} \sum_{k=0}^{m-p-1} (-1)^{k+1} \text{Tr}(P_{H^k} \alpha) + \phi_{12} \sum_{k=0}^{m-p-1} (-1)^k \lim_{t \to 0} \text{Tr}(e^{-t\Delta_k} \alpha) + (-1)^{m-p+1} \phi_{12} \lim_{m-p \to 0} \text{Tr}(E_{m-p}^{\text{ce}}(t) \alpha),
\]

where \( P_{H^k} \) denote the projection onto the cohomology \( H^k \) and \( \lim_{m-p \to 0} \text{Tr}(E_{m-p}^{\text{ce}}(t) \alpha) \) denotes the constant term in the asymptotic expansion of \( \text{Tr}(E_{m-p}^{\text{ce}}(t) \alpha) \).

In particular, for \( p = m/2 \), the variation of this invariant under conformal changes is local:

\[
\frac{d}{du} \ln T_{m/2}(Y, \rho) = \phi_{12} \sum_{k=0}^{m-1} (-1)^k \text{Tr}(P_{H^k} \alpha) + \phi_{12} \sum_{k=0}^{m-1} (-1)^k \lim_{t \to 0} \text{Tr}(e^{-t\Delta_k} \alpha).
\]

Finally, we examine the R-torsion of the Mayer-Vietoris sequence.

**Theorem 1.3** Assume that the Witt condition \( H^H(Y) = 0 \) holds. Then the R-torsion of the Mayer-Vietoris sequence in intersection cohomology

\[
\cdots \to IH_{(2)}^{q}(Y) \to IH_{(2)}^{q+1}(X) \to IH_{(2)}^{q+1}(M) \oplus IH_{(2)}^{q+1}(C(Y)) \to IH_{(2)}^{q+1}(Y) \to \cdots
\]
is equal to the R-torsion of the truncated exact sequence of the pair $(M,Y)$

$$0 \rightarrow H^{\overline{n}}_r(M,Y) \rightarrow H^{\overline{n+1}}(M) \rightarrow H^{\overline{n+1}}(Y) \rightarrow H^{\overline{n+2}}(M,Y) \rightarrow \cdots$$

The rest of the paper is organized as follows. In the next section, we review the definition of R-torsion and recall the intersection cohomology of Goresky-MacPherson, leading to the definition of intersection R-torsion. In §3, we compute the intersection R-torsion of a finite cone. The analytic analog is studied in §4. Finally, §5 deals with the R-torsion of the Mayer-Vietoris sequence.

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## 2 The definition of Intersection R-torsion

We briefly recall the definition and characteristic properties of R-torsion. Roughly speaking, the R-torsion measures to what extent the boundary map of a chain complex can be made to preserve a preferred volume element. Let $C$ be a real vector space of dimension $n$ and let $b = (b_1, \cdots, b_n)$, $c = (c_1, \cdots, c_n)$ be two different bases for $C$. Then $c_i = a_{ij}b_j$ and $(a_{ij}) \in GL(n,R)$. We denote the volume change between two bases $det(a_{ij})$ by $|c/b|$. Let $(\partial, C) : 0 \rightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots C_1 \xrightarrow{\partial} C_0 \rightarrow 0$ be a chain complex of real vector spaces. Let $c_i$ be a preferred basis for $C$ and $h$ a preferred basis for the homology group $H^*(C)$. Denote by $B_i$ the image of the boundary map $\partial_i + 1 : C_{i+1} \rightarrow C_i$ and $Z_i$ its kernel. We choose a basis $b_i$ for $B_i$, which lifts to linearly independent set $b_i \in C_{i+1}$, i.e. $\partial b_i = b_i$. Using the inclusions $0 \subset B_i \subset Z_i \subset C_i$ where $Z_i/B_i \equiv H_i$, $C_i/Z_i \equiv B_{i-1}$ we see that $b_i, h_i, b_{i-1}$ combine to give a new basis for $C_i$. The R-torsion of the chain complex is the real number $\tau(c,h)$ defined by

$$\ln \tau(c,h) = \sum_{i=0}^{n} (-1)^i \ln ||[b_i h_i b_{i-1}/c_i]||. \tag{2.1}$$

The R-torsion $\tau(c,h)$ does not depend on the choice of $b_i, b_{i-1}$, but it depends on the preferred bases $c_i, h_i$. In fact, it depends only on the volume elements determined by these preferred bases. More precisely,

$$\ln \tau(c',h) = \ln \tau(c,h) + \sum_{i=0}^{n} (-1)^i \ln ||[c_i/c'_i]||, \tag{2.2}$$

and

$$\ln \tau(c,h') = \ln \tau(c,h) + \sum_{i=0}^{n} (-1)^i \ln ||[h_i'/h_i]||. \tag{2.3}$$

When the preferred basis of the homology is chosen according to the preferred basis of the chain complex, there is an elegant representation of the R-torsion in terms of the combinatorial Laplacian. The choice of a preferred basis for each $C_i$ represents $\partial_i : C_i \rightarrow C_{i-1}$ as a real matrix. Let $\partial_i^* : C_{i-1} \rightarrow C_i$ be the transpose matrix. The combinatorial Laplacian is $\Delta_i^{(c)} = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i : C_i \rightarrow C_i$. By the finite dimensional Hodge theory, $\ker \Delta_i^{(c)} \cong H_i(C,\partial)$. Effectively, the preferred basis $c_i$ determines an inner product on $C_i$ in which $c_i$ becomes orthonormal. If we choose the
preferred basis \( h \) on \( H_i(C, \partial) \) to correspond to an orthonormal basis of \( \ker \Delta_i^{(c)} \subset C_i, \) then,
\[
\ln \tau(c, h) = \frac{1}{2} \sum_{i=0}^{n} (-1)^{i+1} i \log \Delta_i^{(c)}.
\] (2.4)

Now if \( K \) be a finite CW complex, consider \( \tilde{K} \) the universal covering complex of \( K \). The fundamental group \( \pi \) of \( K \) acts on \( \tilde{K} \) as the group of covering transformations. This action makes \( C(\tilde{K}) \), the cellular chain complex associated with \( \tilde{K} \), a free \( \mathbb{R} \pi \)-module generated by the cells \( e_i \) of the complex \( K \). We pick a preferred basis for \( C_i(\tilde{K}) \) coming from the \( i \)-cells of \( K \), denoted \((e_i^1, e_i^2, \ldots, e_i^{k_i})\).

Let \( \epsilon : \pi \rightarrow O(n) \) be an orthogonal representation of the fundamental group. Then one can construct a chain complex of real vector spaces by setting \( C_i(K, \epsilon) = C_i(\tilde{K}) \otimes_{\mathbb{R} \pi} \mathbb{R}^n \). We have a preferred choice of basis for each vector space \( C_i(K, \epsilon) \) given by \( e_i^j \otimes x_k \) where \( x_k \) is an orthonormal basis for \( \mathbb{R}^n \). With a choice of preferred basis \( h \) in homology, the torsion of the complex of real vector spaces \( C_i(K, \epsilon) \) is a real number and will be denoted \( \tau(K, \epsilon, h) \).

The R-torsion is a combinatorial invariant i.e. invariant under subdivision of \( K \). It is a topological invariant when the chain complex is acyclic.

The R-torsion of a closed manifold \( M \) is the R-torsion of the cell complex determined by a cell structure of \( M \). In this case, the preferred base for the homology is obtained via Hodge theory through an orthonormal basis of the harmonic forms. With this choice of preferred basis in homology it was shown in celebrated work of Cheeger [4] and Müller [21] that \( \tau(M, \epsilon) \) equals the so called analytic torsion (Ray-Singer conjecture).

The intersection R-torsion is defined for pseudomanifolds by Dar [12] using the intersection homology theory of Goresky-MacPherson. We recall the basic facts of Intersection Homology Theory. A pseudomanifold \( X \) of dimension \( n \) is a compact PL space for which there exists a closed subspace \( Z \) with dimension \( Z \leq n - 2 \) such that \( X - Z \) is an \( n \)-dimensional oriented manifold which is dense in \( X \). A stratification of a pseudomanifold is a filtration by closed subspaces
\[
X = X_n = X_{n-1} \supset X_{n-2} = Z \supset \cdots \supset X_1 \supset X_0
\]
such that for each point \( p \in X_i - X_{i-1} \) there is a filtered space \( V = V_n \supset V_{n-1} \supset \cdots \supset V_i = \{ \text{a point} \} \) and a mapping \( V \times B^i \rightarrow X \) which takes \( V_j \times B^i \) (PL) homeomorphically to a neighborhood of \( p \) in \( X_j \). \( X_i - X_{i-1} \) is an \( i \)-dimensional manifold called the \( i \)-dimensional stratum. Every pseudomanifold admits a stratification.

We will actually be working with pseudomanifold with boundaries. However, in our situation, the boundaries do not intersect with the singular strata. Hence the above discussions can be easily modified to adapt to the corresponding situation.

The space of geometric chains \( C_*(X) \) is the collection of all simplicial chains with respect to some triangulation where one identifies the two chains if their images coincide under some common subdivision. The intersection homology theory is obtained by restricting to only \emph{allowable} chains, described by the so called perversity.

A perversity is a sequence of integers \( \bar{p} = (p_2, p_3, \ldots, p_n) \) such that \( p_2 = 0 \) and \( p_k+1 = p_k \) or \( p_{k+1} \). If \( i \) is an integer and \( \bar{p} \) is a perversity, a subspace \( Y \subset X \) is \((\bar{p}, i)\) allowable if \( \dim(Y) \leq i \) and \( \dim(Y \cap X_{n-k}) \leq i - k + p_k \) for \( k \geq 2 \). In other words, \( p_k \) describes how much \( X \) is allowed to deviate from intersecting the stratum \( X_{n-k} \) transversally. The intersection chains \( IC_i^\bar{p}(X) \) is the subspace of \( C_i(X) \) consisting of those chains \( \xi \) such that \( [\xi] \) is \((\bar{p}, i)\) allowable and \( |\partial \xi| \) is \((\bar{p}, i-1)\) allowable. The \( i \)-th Intersection Homology Group of perversity \( \bar{p} \), \( IH_i^\bar{p}(X) \) is the \( i \)-th homology group of the chain complex \( IC_i^\bar{p}(X) \).
The intersection chain complex as we defined is not finitely generated. In order to define the
Intersection R-torsion we need to work with finitely generated chain groups. To do this one uses the
basic sets $R_t^p$.

Let $X$ be a pseudomanifold with a fixed stratification. Let $T$ be a triangulation of $X$ subordinate
to the stratification i.e. such that each $X_k$ is a subcomplex of $T$. Define $R_t^p$ be the subcomplex of
$T'$, the first barycentric subdivision of $T$, consisting of all simplices which are $(p, i)$ allowable.

Let $R_t^p(X)$ be the chain complex whose $i$-th chain group consists of simplicial chains $e_i$ such that
$|e_i| \in R_t^p$ and $|\partial e_i| \in R_{t-1}^p$. It is a free abelian group generated by finitely many chains $\{e_i\}$. The
homology group $H_i(\bar{\mathcal{R}}^p(X))$ is canonically isomorphic to $IH_t^p(X)$.

Let $X$ be a combinatorial invariant and independent of the stratification.

Provided a preferred basis in homology is chosen. Dar [12] proved that the intersection $R$-torsion is
simplex of $Z_w, \bar{\mathcal{R}}$ where the cross section

Lemma 3.2 Write $\mathcal{R}^p(X)$ be the chain complex whose $i$-th chain group consists of simplicial chains $e_i$ such that
$|e_i| \in R_t^p$ and $|\partial e_i| \in R_{t-1}^p$. It is a free abelian group generated by finitely many chains $\{e_i\}$. The
homology group $H_i(\mathcal{R}^p(X))$ is canonically isomorphic to $IH_t^p(X)$.

In this section we restrict ourself to the finite cone. Let $X = C(Y)$ be a finite cone with $dim X = n,
where the cross section $Y$ is a closed manifold. We will also write $X = w * Y$ with $w$ the cone tip.

If $\sigma = [a_0, \ldots, a_p]$ is an oriented simplex of $Y$, then $[w, \sigma] = [w, a_0, \ldots, a_p] = w * \sigma$ is an oriented
simplex of $w * Y$. Similarly, if $\eta = \sum n_i \sigma_i$ is a $p$-chain of $Y$, then $[w, \eta] = \sum n_i[w, \sigma_i]$, and

$$\partial [w, \eta] = \begin{cases} 
\eta - w & \text{dim } \eta = 0 \\
\eta - [w, \partial \eta] & \text{dim } \eta > 0 
\end{cases}$$

We have the following results:

Lemma 3.1 If $Z_p$ is a $p$-cycle of $X$ for $p \geq 1$, then $Z_p = \partial [w, C_p]$ for some $p$-chain $C_p$ of $Y$.

Proof: Write $Z_p = C_p + [w, D_{p-1}]$, where $C_p$ and $D_{p-1}$ are both carried by $Y$. Then by the above
observation and using that $Z_p$ is closed, we have $Z_p = \partial [w, C_p]$.

Of course, this is compatible with the well known fact that

$$H_p(X) = \begin{cases} 0 & p \geq 1 \\
\mathbb{Z} & p = 0 
\end{cases}$$

Now let $\bar{p}$ be a perversity. Since $X$ has only strata of dimension $n$ and 0, the intersection chains
and homology will only depend on $p_n$. In fact, we have

Lemma 3.2 The intersection chains of $X$ are given by

$$IC^\bar{p}_i(X) = \begin{cases} 
C_i(Y), & i < n - p_n, \\
\{ \xi \in C_i(X) \mid \partial \xi \in C_{i-1}(Y) \}, & i = n - p_n, \\
C_i(X), & i > n - p_n. 
\end{cases}$$
In particular,
\[ IH^p_i(X) = \begin{cases} H_i(Y), & i < n - p_n - 1, \\ 0, & i \geq n - p_n - 1. \end{cases} \] (3.5)

From now on we will sometimes suppress the superscript \( \bar{p} \) here when there is no confusion. Let \( h_p(Y) = \{ h^p_{1}, \cdots, h^p_{k_p}(Y) \} \) be a preferred basis for \( H_p(Y) \). Any triangulation of \( Y \) gives rise to a preferred basis for \( C_p(Y) \). At first we will assume that the basis of \( IH^p_i(X) \) is chosen to be compatible with those of \( H_p(Y) \) with respect to (3.4).

**Theorem 3.3** Let \( \Delta^{(c)} \) denote the combinatorial Laplacian of the cross section \( Y \). Then
\[
\ln I\hat{\tau}(X) = \sum_{p=0}^{n-p_n-1} (-1)^{p+1} p \ln \det \Delta^{(c)}_p + (n - p_n) \sum_{p=p_n}^{n-1} (-1)^{p+1} \ln \det \Delta^{(c)}_p.
\]

**Proof:** The intersection R-torsion is defined in terms of the chain complex
\[
\cdots \to IC_{p+1}(X) \to IC_p(X) \to IC_{p-1}(X) \to \cdots.
\] (3.6)

We examine the terms of this complex according to their degrees.

Case 0. \( p = n \)

In this case, \( IC_n(X) = C_n(X) \), so we only need to consider the usual chains of \( X \).

Let \( c_n(Y) = \{ \sigma^0_n(Y), \cdots, \sigma^{n-1}_n(Y) \} \) be the preferred basis of \( (n-1) \)-chains of \( Y \). Then \( \{ [w, c_{n-1}(Y)] \} \) is the preferred basis of \( C_n(X) \). Choose a basis \( b_p(Y) = \{ b^0_p(Y), \cdots, b^0_{k_p}(Y) \} \) for \( B_p(Y) \), and their lifts \( \tilde{b}_p(Y) = \{ \tilde{b}^0_p(Y), \cdots, \tilde{b}^0_{k_p}(Y) \} \), and \( h_p(Y) = \{ h^0_p(Y), \cdots, h^0_{k_p}(Y) \} \) the basis for \( H_p(Y) \). Then by the fact that \( B_n(X) = Z_n(X) = 0 \) and \( B_{n-1}(X) = \partial[w, C_{n-1}(Y)] \), we can choose basis for \( b_{n-1}(X) \) as follows:

\[ \tilde{b}_{n-1}(X) = \{ [w, b_{n-1}(Y)], [w, h_{n-1}(Y)], [w, \tilde{b}_{n-2}(Y)] \} \]

So the determinant of the transition matrix \( D_n \) is:

\[ D_n = \begin{bmatrix} [w, b_{n-1}(Y)], [w, h_{n-1}(Y)], [w, \tilde{b}_{n-2}(Y)] \\ \omega, c_{n-1}(Y) \end{bmatrix} \]

which is \( D_n = A_{n-1} \) where \( A_{n-1} \) denotes the corresponding determinant of the transition matrix for \( Y \).

Case 1. \( n - p_n < p < n \)

In this case, \( IC_p(X) = C_p(X) \), so we only need to consider the usual chains of \( X \).

Let \( c_p(Y) = \{ \sigma^0_p(Y), \cdots, \sigma^p_p(Y) \} \) be the preferred basis of \( p \)-chains of \( Y \). Then \( \{ c_p(Y), [w, c_{p-1}(Y)] \} \) is the preferred basis of \( C_p(X) \). Choose a basis \( b_p(Y) = \{ b^0_p(Y), \cdots, b^0_{k_p}(Y) \} \) for \( B_p(Y) \), and their lifts \( \tilde{b}_p(Y) = \{ \tilde{b}^0_p(Y), \cdots, \tilde{b}^0_{k_p}(Y) \} \), and \( h_p(Y) = \{ h^0_p(Y), \cdots, h^0_{k_p}(Y) \} \) the basis for \( H_p(Y) \). Then by the fact that \( B_p(X) = Z_p(X) = \partial[w, C_p(X)] \), we can choose a basis for \( b_p(X) \) as follows:

\[ b_p(X) = \{ \partial[w, b_p(Y)], \partial[w, h_p(Y)], \partial[w, \tilde{b}_{p-1}(Y)] \} \]

\[ = \{ b_p(Y), h_p(Y), \tilde{b}_{p-1}(Y) - [w, b_{p-1}(Y)] \} \]
and the lifts \( \tilde{b}_{p-1}(X) \) (even though the lifts \( \tilde{b}_{p-1}(X) \) depends on \( B_{p-1}(X) \subset IC_{p-1}(X) \) which may not be \( C_{p-1}(X) \) when \( p-1 = n - p_n \), \( B_{p-1}(X) \) still consists of the ordinary boundaries):

\[
\tilde{b}_{p-1}(X) = \{[w, b_{p-1}(Y)], [w, h_{p-1}(Y)], [w, \tilde{b}_{p-2}(Y)]\}
\]

So the determinant of the transition matrix \( D_p \) is:

\[
D_p = \begin{bmatrix}
    b_p(Y), h_p(Y), \tilde{b}_{p-1}(Y), [\omega, b_{p-1}(Y)], [\omega, h_{p-1}(Y)], [\omega, \tilde{b}_{p-2}(Y)] \\
    c_p(Y), [\omega, c_{p-1}(Y)]
\end{bmatrix}
\]

which is

\[
D_p = A_p A_{p-1}.
\] (3.7)

Case 2. \( p = n - p_n \).

In this case, we still have \( IH_p(X) = H_p(X) = 0 \). By Lemma 3.2, \( IC_p(X) = \{ \eta \in C_p(X) \mid \partial \eta \in C_{p-1}(Y) \} \).

For \( \eta \in IC_p(X) \), write \( \eta = C_p(Y) + [w, D_{p-1}(Y)] \). Then \( \partial \eta = \partial C_p(Y) + D_{p-1}(Y) - [w, \partial D_{p-1}(Y)] \).

Thus we must have \( \partial D_{p-1}(Y) = 0 \) for \( \partial \eta \in C_{p-1}(Y) \), which implies that \( D_{p-1}(Y) \in B_{p-1}(Y) \oplus H_{p-1}(Y) \).

Thus,

\[
\partial(IC_p(X)) = \partial(C_p(Y)) \oplus \partial[w, B_{p-1}(Y) \oplus H_{p-1}(Y)]
\]

\[
= \partial(C_p(Y)) \oplus B_{p-1}(Y) \oplus H_{p-1}(Y)
\]

\[
= B_{p-1}(Y) \oplus H_{p-1}(Y)
\]

Hence we can take \( \{[w, b_{p-1}(Y)], [w, h_{p-1}(Y)]\} \) as basis of \( \tilde{B}_{p-1}(X) \).

The fact that \( IH_p(X) = H_p(X) = 0 \) implies

\[
IC_p(X) = B_p(X) \oplus \tilde{B}_{p-1}(X).
\]

Then the determinant of the transition matrix \( D_p \) is

\[
D_p = \begin{bmatrix}
    b_p(Y), h_p(Y), \tilde{b}_{p-1}(Y), [\omega, b_{p-1}(Y)], [\omega, h_{p-1}(Y)] \\
    c_p(Y), [\omega, c_{p-1}(Y)]
\end{bmatrix}
\]

which yields

\[
D_p = A_p
\] (3.8)

Case 3: \( p = n - p_n - 1 \)

In this case

\[
IC_p(X) = C_p(Y)
\]

\[
IH_p(X) = \text{Im}(H_p(Y) \to H_p(X)) = 0
\]
Consider the following sequence:

\[
\ldots \rightarrow I C_{p+1}(X) \rightarrow I C_p(X) \rightarrow I C_{p-1}(X) \rightarrow \ldots .
\]

Then as before,

\[
\partial (I C_{p+1}(X)) = B_p(Y) \oplus H_p(Y),
\]

and

\[
\partial [I C_p(X)] = \partial [C_p(Y)] = B_{p-1}.
\]

Thus the determinant of the transition matrix is:

\[
D_p = \left[ \begin{array}{c} b_p(Y), H_p(Y), \tilde{b}_{p-1}(Y) \\ C_p(Y) \end{array} \right] = A_p
\]

(3.9)

Case 4: \( p < n - p_n - 1 \)

In this case, \( I C_p(X) = C_p(Y), \ I H_p(X) = H_p(Y) \). Since the choice of the preferred basis on \( I H_p(X) \) is compatible with that of \( H_p(Y) \), it is easy to see that \( D_p = A_p \).

Combining the above results, we have:

\[
\tau(I C) = \prod_{p=0}^{n} (D_p)^{(-1)^p}
\]

\[
= \prod_{p=0}^{n-p_n-1} (A_p)^{(-1)^p} \cdot \prod_{p=n-p_n+1}^{n-1} (A_p \cdot A_{p-1})^{(-1)^p} \cdot (A_{n-1})^{(-1)^n}
\]

\[
= \prod_{p=0}^{n-p_n-1} (A_p)^{(-1)^p}
\]

(3.10)

Thus,

\[
\ln I \tau^\bar{p}(X) = \ln \tau(I C) = \sum_{p=0}^{n-p_n-1} (-1)^p \ln A_p
\]

\[
= \sum_{p=0}^{n-p_n-1} (-1)^{p+1} p \ln \det \Delta_p^{(c)} + (n-p_n) \sum_{p=p_n}^{n-1} (-1)^{p+1} \ln \det \Delta^{(c)}_p.
\]

(3.11)

Here we have used the equation \( \ln A_p = -\frac{1}{2} \sum_{k=p}^{n-1} (-1)^{k-p} \ln \det \Delta^{(c)}_k \). \[\square\]

In Theorem [3.3] the basis of \( I H^p(X) \) is chosen to be compatible with those of \( H^p(Y) \) with respect to (3.5). As discussed in the previous section, the convention for Reidemeister torsion is that the choice of the cohomology basis is determined by the Hodge theory. Thus we will now consider the issue briefly.

Recall that the conical metric on \( X = C(Y) = [0, 1] \times Y \) is \( dr^2 + r^2 g \), where \( g \) is a metric on \( Y \). There is a natural decomposition of smooth \( i \)-forms on \( X \):

\[
\alpha_i = g(r) \phi_i + f(r) dr \wedge \omega_{i-1},
\]

(3.12)
where \( \phi_i, \omega_{i-1} \) are \( i \)-forms resp. \( (i-1) \)-forms on \( Y \). Following the notations from [5], we will decorate operators on \( Y \) with “\( \ast \)”. For example \( d \) will denote the exterior derivative on \( X \), while \( \tilde{d} \) will denote the exterior derivative on \( Y \). Similarly for the adjoints \( \tilde{\delta}, \tilde{\delta} \), and the Hodge Laplacian \( \Delta, \tilde{\Delta} \).

The following formula for the action of Laplacian is from [5] (3.8).

\[
\Delta \alpha_i = (-g'' - [n - 2i - 1]r - 1g')\phi_i + r^{-2}g\tilde{\Delta}\phi_i - 2r^{-3}g dr \wedge \tilde{\delta}\phi_i \\
+ \tilde{\delta} \tilde{\delta} \phi_i + r^{-2}f dr \wedge \tilde{\Delta} \omega_{i-1} - 2r^{-1}f d\omega_{i-1}.
\]

(3.13)

In particular, when \( g(r) = 1, f(r) = 0 \) and \( \phi \) is harmonic on \( Y \), \( \alpha_i = \phi_i \) is also harmonic on \( X \). Since \( H^i_{(2)}(X) \) (with absolute boundary condition) is isomorphic to \( (IH^{(p)}_{(2)}(X))^\ast \) where the perversity \( \bar{p} \) is taken to be the lower middle perversity, we see that for \( i < n/2 \) the isomorphism \( H^i(Y) \approx H^i_{(2)}(X) \) is given by \( \phi_i \rightarrow \phi_i \) for harmonic forms \( \phi_i \) on \( Y \). This isomorphism is not an isometry. Indeed \( \|\phi_i\|_{L^2(Y)}^2 = \frac{1}{\pi^{2i}}\|\phi_i\|_{L^2(Y)}^2 \) (\( i < n/2 \)). Correcting this, say via [20], we have (for the lower middle perversity)

\[
\ln I\tau(X) = \sum_{p < n/2} \frac{(-1)^p}{2} \ln(n - 2p) b_p(Y) + \sum_{p < n/2} (-1)^{p+1} p \ln \det \Delta_p^{(c)} \\
+ (n - \left\lfloor \frac{n}{2} \right\rfloor) \sum_{p \geq n/2} (-1)^{p+1} \ln \det \Delta_p^{(c)}.
\]

Here \( b_p(Y) \) denotes the \( p \)-th Betti number of \( Y \). In particular, when \( n = 3 \) we have

\[
\ln I\tau(X) = \frac{\ln 3}{2} + \sum_{p=0}^{2} (-1)^{p+1} p \ln \det \Delta_p^{(c)}.
\]

4 An analytic analogue

Following Ray-Singer’s idea of defining analytic torsion as a formal analog of the R-torsion on closed manifolds, we now study the formal analytic analog of the intersection R-torsion (1.2), which is intrinsic to the even dimensional cross section. That is, by replacing the combinatorial Laplacian by the Hodge Laplacian, we define an analytic invariant for an even dimensional closed manifold.

More precisely, let \( Y \) be an even dimensional closed manifold with \( m = \dim Y \). Let \( p \) be an integer such that \( 0 \leq p \leq m-1 \) (\( p \) corresponds to \( p_n \) which is determined by a given perversity). Given an orthogonal representation \( \rho : \pi_1(Y) \rightarrow O(N) \), one has an associated flat vector bundle \( E_\rho \) with compatible metric on \( Y \). Let \( \Delta_k \) be the Laplacian acting on differential \( k \) forms on \( Y \) with coefficients in \( E_\rho \). Then we define

\[
\ln T_p(Y, \rho) = \frac{1}{2} \left[ \sum_{k=0}^{m-p} (-1)^{k+1} k \ln \det(\Delta_k) + (m-p) \sum_{k=m-p+1}^{m} (-1)^{k+1} \ln \det(\Delta_k) \right].
\]

(4.14)

For \( p = 0 \), which corresponds to the minimum perversity,

\[
\ln T_0(Y, \rho) = \frac{1}{2} \sum_{k=0}^{m} (-1)^{k+1} k \ln \det(\Delta_k) = 0
\]
is the usual analytic torsion which is trivial for even dimensional manifolds. On the other hand, for \( p = m - 1 \) corresponding to the maximum perversity,

\[
\ln T_{m-1}(Y, \rho) = \frac{1}{2} \sum_{k=1}^{m} (-1)^{k+1} \ln \det(\Delta_k).
\]

The more interesting cases are given by \( p = \frac{m}{2} - 1 \) and \( p = \frac{m}{2} \) corresponding to the lower and upper middle perversity, respectively. In these cases, we have

\[
\ln T_{\frac{m}{2}-1}(Y, \rho) = \frac{1}{2} \left[ \sum_{k=0}^{m} (-1)^{k+1} k \ln \det(\Delta_k) + \left( \frac{m}{2} + 1 \right) \sum_{k=\frac{m}{2}+2}^{m} (-1)^{k+1} \ln \det(\Delta_k) \right]
\]

and

\[
\ln T_{\frac{m}{2}}(Y, \rho) = \frac{1}{2} \left[ \sum_{k=0}^{m} (-1)^{k+1} k \ln \det(\Delta_k) + \left( \frac{m}{2} + 1 \right) \sum_{k=\frac{m}{2}+1}^{m} (-1)^{k+1} \ln \det(\Delta_k) \right].
\]

When \( Y \) is oriented, we can actually use Poincaré duality to write it in terms of the Laplacians on half of the degrees. For example, for \( p = \frac{m}{2} - 1 \) corresponding to the lower middle perversity, we have

\[
\ln T_{\frac{m}{2}-1}(Y, \rho) = \frac{1}{2} \left[ \sum_{k=0}^{m-1} (-1)^{k+1}(k + \frac{m}{2} + 1) \ln \det(\Delta_k) + (-1)^{\frac{m}{2}+1} \frac{m}{2} \ln \det(\Delta_{\frac{m}{2}}) \right]. \tag{4.15}
\]

These bear close resemblance to the so called Cheeger’s half torsion [1].

To investigate what kind of invariant \( \ln T_p(Y, \rho) \) defines, we now look at its variation under metric change. Let \( g(u) \) be a family of Riemannian metrics on \( Y \) and \( \Delta_k(u) \) the corresponding Laplacian (when there is no ambiguity we will often write \( \Delta_k \) instead of \( \Delta_k(u) \)). Let \( \ast = \ast / du \) and \( \alpha = \ast^{-1} \ast \). Denote by \( E_k(t) = e^{-t \Delta_k(u)} \) the heat kernel and let \( E_k = E_k^e + E_k^{ce} + E_k^h \) denote the Hodge decomposition of \( E_k \) into its exact, coexact and harmonic parts. We have the following result regarding the variation of \( \ln T_p(Y, \rho) \).

**Theorem 4.1** The variation of \( \ln T_p(Y, \rho) \) is given by

\[
\frac{d}{du} \ln T_p(Y, \rho) = \varphi 12 \sum_{k=0}^{m-p-1} (-1)^{k+1} \text{Tr}(P_{H_k} \alpha) + \varphi 12 \sum_{k=0}^{m-p-1} (-1)^{k+1} \text{LIM}_{t \to 0} \text{Tr}(e^{-t \Delta_k} \alpha)
\]

\[+ \ (-1)^{m-p+1} \varphi 12 \text{LIM}_{t \to 0} \text{Tr}(E_{m-p}^{ce}(t) \alpha), \]

where \( P_{H_k} \) denote the projection onto the cohomology \( H^k \) and \( \text{LIM}_{t \to 0} \text{Tr}(E_{m-p}^{ce}(t) \alpha) \) denotes the constant term in the asymptotic expansion of \( \text{Tr}(E_{m-p}^{ce}(t) \alpha) \).

Before we give the proof of our theorem, we need the following result from [4] (compare also with [24]) concerning the variation of heat kernel.
Theorem 4.2 (Cheeger) The variation of the trace of the heat kernel $E_k$ is given by

$$\frac{d}{du} \text{tr}(E_k(t)) = -t[\text{tr}(\Delta_{k+1}E^{ex}_{k+1}\alpha) - \text{tr}(\Delta_k E^{ex}_k \alpha) + \text{tr}(\Delta_k E^{ex}_k \alpha) - \text{tr}(\Delta_{k-1} E^{ex}_{k-1} \alpha)]$$

$$= t \frac{d}{dt}[\text{tr}(E^{ex}_{k+1}\alpha) - \text{tr}(E^{ex}_k \alpha) + \text{tr}(E^{ex}_k \alpha) - \text{tr}(E^{ex}_{k-1} \alpha)].$$

The following lemma is an immediate consequence of Cheeger’s result.

Lemma 4.3 For any integer $q$, $0 \leq q \leq m$, we have

$$\frac{\partial}{\partial u} \sum_{k=0}^{q} (-1)^k \text{tr}(E_k(t)) = t \frac{\partial}{\partial t} \sum_{k=0}^{q} (-1)^k \text{tr}(E_k(t)\alpha) + (-1)^q \text{tr}(E^{ex}_{q+1}(t)\alpha).$$

Similarly, for any integer $r$, $0 \leq r \leq m$,

$$\frac{\partial}{\partial u} \sum_{k=r}^{m} (-1)^k \text{tr}(E_k(t)) = t \frac{\partial}{\partial t} [(-1)^r \text{tr}(E^{ex}_r(t)\alpha) + (-1)^{r-1} \text{tr}(E^{ex}_{r-1}(t)\alpha)].$$

With these results at our disposal, we are now ready to prove the variational formula for our analytic invariant.

Proof of Theorem 4.1 Define for $\Re s$ sufficiently large

$$f(u, s) = \varnothing 12 \left[ \sum_{k=0}^{m-p} (-1)^k k \int_0^\infty t^{s-1} \text{Tr}(e^{-t[\Delta_k + P_{h_k}]}) dt + (m-p) \sum_{k=m-p+1}^{m} (-1)^k \int_0^\infty t^{s-1} \text{Tr}(e^{-t[\Delta_k + P_{h_k}]}) dt \right].$$

Then $f(u, s)$ has a meromorphic extension to the whole complex $s$-plane with a simple pole at $s = 0$. Indeed, since

$$\text{Tr}(e^{-t[\Delta_k + P_{h_k}]}) = \text{Tr}(e^{-t\Delta_k}) + e^{-t} \text{dim} H^k,$$

we have

$$\text{Res}_{s=0} f(u, s) = \varnothing 12 \left[ \sum_{k=0}^{m-p} (-1)^k k A_{m/2, k} + (m-p) \sum_{k=m-p+1}^{m} (-1)^k A_{m/2, k} \right],$$

where $A_{m/2, k}$ denotes the constant term in the asymptotic expansion of $\text{Tr}(e^{-t\Delta_k})$. Now let

$$\hat{f}(u, s) = f(u, s) - \Gamma(s) \text{Res}_{s=0} f(u, s).$$

Then $\hat{f}$ is holomorphic at $s = 0$ and we have

$$\hat{f}(u, 0) = \ln T_p(Y, \rho).$$

Now, for $\Re s$ sufficiently large

$$\frac{\partial}{\partial u} \int_0^\infty t^{s-1} \text{Tr}(e^{-t[\Delta_k + P_{h_k}]}) dt = \int_0^\infty t^{s-1} \frac{\partial}{\partial u} \text{Tr}(e^{-t[\Delta_k + P_{h_k}]}) dt$$

$$= \int_0^\infty t^{s-1} \frac{\partial}{\partial u} \text{Tr}(e^{-t\Delta_k}) dt.$$
Hence, using \( [4.10] \), \( [4.17] \), we derive

\[
\frac{\partial}{\partial u} f(u, s) = \phi 12 \left[ \sum_{k=0}^{m-p} (-1)^k \int_0^\infty t^k \frac{\partial}{\partial u} \text{Tr}(E_k(t)\alpha) \, dt + (-1)^{m-p+1} \int_0^\infty t^{k+1} \frac{\partial}{\partial u} \text{Tr}(E_{m-p}(t)\alpha) \, dt \right]
\]

\[
= \phi 12 \left[ \sum_{k=0}^{m-p} (-1)^{k+1} \int_0^\infty t^{k-1} \text{Tr}(E_k(t)\alpha) \, dt + (-1)^{m-p} \int_0^\infty t^{k} \text{Tr}(E_{m-p}(t)\alpha) \, dt \right]
\]

It follows then that

\[
\frac{\partial}{\partial u} \ln T_p(Y, \rho) = \phi 12 \left[ \sum_{k=0}^{m-p} (-1)^k \text{Tr}(P_{H^k} \alpha) + \phi 12 \sum_{k=0}^{m-p} (-1)^k \text{LIM}_{t \to 0} \text{Tr}(e^{-t\Delta^k} \alpha) \right]
\]

\[
+ \phi 12 \left[ (-1)^{m-p+1} \text{LIM}_{t \to 0} \text{Tr}(E_{m-p}(t)\alpha) \right].
\]

Just like Cheeger’s half torsion, we note the special property of the invariant for \( p = m/2 \) under conformal change.

**Corollary 4.4** Under a family of conformal changes, for \( p = m/2 \), the variation of \( \ln T_p(Y, \rho) \) is local in the sense that

\[
\frac{d}{du} \ln T_p(Y, \rho) = \phi 12 \left[ \sum_{k=0}^{m-p-1} (-1)^k \text{Tr}(P_{H^k} \alpha) + \phi 12 \sum_{k=0}^{m-p-1} (-1)^k \text{LIM}_{t \to 0} \text{Tr}(e^{-t\Delta^k} \alpha) \right]
\]

**Proof.** If \( g = e^{2t \rho} g_0 \) is a conformal change of \( g_0 \), then its Hodge star on the \( p \)-forms is given by \( *_g = e^{(2p-m)t} *_{g_0} \) in terms of the star operator of \( g_0 \). It follows that \( \alpha = 0 \) on \( m/2 \)-forms. Our result follows from Theorem 4.1.

\[
\boxed{\Box}
\]

### 5 R-torsion of the Mayer-Vietoris sequences

Consider an \((m+1)\)-dimensional Riemannian manifold \( X \) with isolated conical singularity. Thus, \( X = C(Y) \cup M \), where \( M \) is a compact manifold with boundary and \( \partial M = Y \). It is understood in this section that the collar neighborhoods of the boundaries of \( M \) and \( C(Y) \) are extended so that they form an open cover of \( X \). We assume that \( m+1 \) is odd.

As we mentioned, the general Mayer-Vietoris Principle reduces the torsion of \( X \) to that of \( C(Y) \), \( M \) as well as the torsion of the Mayer-Vietoris sequence in the intersection cohomology. We now examine the torsion of the Mayer-Vietoris sequence.

We use the \( \mathcal{L}^2 \)-cohomology interpretation of the intersection cohomology in this setting \( \mathbb{C} \). The Mayer-Vietoris sequence goes

\[
\cdots \rightarrow H^q_{(2)}(Y) \xrightarrow{d^*} H^{q+1}_{(2)}(X) \rightarrow H^{q+1}_{(2)}(M) \oplus H^{q+1}_{(2)}(C(Y)) \rightarrow H^{q+1}_{(2)}(Y) \rightarrow \cdots.
\]

\[
(5.18)
\]

First, we have the following

\[
\boxed{\Box}
\]
Lemma 5.1 For the Mayer-Vietoris long exact sequence in cohomology \([5.13]\),
a). its part for \(q \leq m/2\) splits into the following short exact sequences:

\[
0 \rightarrow H^q_{(2)}(X) \rightarrow H^q_{(2)}(M) \oplus H^q_{(2)}(C(Y)) \rightarrow H^q_{(2)}(Y) \rightarrow 0 \tag{5.19}
\]

b). further,

\[
0 \rightarrow H^q_{(2)}(X) \rightarrow H^q_{(2)}(M) \oplus H^q_{(2)}(C(Y)) \rightarrow H^q_{(2)}(Y) \rightarrow 0 \tag{5.20}
\]
is a split short exact sequence.

c). the part of the Mayer-Vietoris sequence for \(q > m/2\) is naturally isomorphic to the truncated
exact sequence for the pair \((M, Y)\):

\[
H^{m/2}(Y) \rightarrow H^{m/2+1}(M, Y) \rightarrow H^{m/2+1}(M) \rightarrow H^{m/2+1}(Y) \rightarrow \cdots \rightarrow H^m(Y).
\tag{5.21}
\]

Proof: For a), we only need to show that, when \(q \leq m/2, \text{Im}(d^*) = 0\). Let \(\rho_1, \rho_2\) be
a partition of unity subordinate to the open cover of \(X\) by \(M, C(Y)\). That is, \(\rho_1, \rho_2 \in C^\infty(X), \ 0 \leq \rho_1, \rho_2 \leq 1, \rho_1 + \rho_2 = 1\) and \(\text{supp} \ \rho_1 \subset M, \ \text{supp} \ \rho_2 \subset C(Y)\). Then, for a closed \(q\)-form on \(Y\),

\[
d^*[w] = \begin{cases} 
-d(\rho_2 w) & \text{on } M, \\
-d(\rho_1 w) & \text{on } C(Y).
\end{cases}
\]

Here \(w\) is extended trivially along radial directions hence defines a \(q\)-form in a collared neighborhood
of \(Y\) in \(X\). In fact, \(d^*[w]\) is supported in this collared neighborhood and, interpreted properly, either
\([-d(\rho_2 w)]\) or \([d(\rho_1 w)]\) defines \(d^*[w]\). Now, \(d^*[w] = -d(\rho_2 w)\). By the result of \([5]\), for \(q \leq m/2, w\)
defines an \(L^2\) form on \(C(Y)\). This shows that \(d^*[w]\) is exact in \(L^2\) cohomology. Hence \(d^*[w] = 0\).

The statement b). is clear since these are short exact sequences of vector spaces. They can also
be seen directly as follows. We show that the composition \(p i^*\) in the following diagram

\[
\begin{array}{ccc}
H^q_{(2)}(X) & \xrightarrow{i^*} & H^q(M) \oplus H^q_{(2)}(C(Y)) \\
\downarrow p & & \downarrow \\
H^q(M) & & \\
\end{array}
\]
is an isomorphism. Here \(p\) is the projection onto the first factor. Indeed, for any \(w \in H^q_{(2)}(X)\),
\(p i^* w = p (i_M^* w, i_C^*(Y) w) = i_M^* w\). If \(i_M^* w\) is an exact form, \(i_M^* w = d\eta_2\) then \(i_C^* i_M^* w = i_C^* (d\eta_2) = d(i_C^* \eta_2)\) is exact on \(Y\). By \([5]\), for \(q \leq m/2, i_C^* \eta_2\) defines an \(L^2\) form on \(C(Y)\). Since the cohomology
class of a closed form on \(C(Y)\) is uniquely determined by its restriction on \(Y\) \([5]\), we see that
\(i_C^* (Y) w\) is exact. It follows then that \(i^* (w) = (i_M^* w, i_C^*(Y) w)\) is exact. Namely
\([i^* w] = 0\) on \(H^q_2(M) \oplus H^q_2(C^*_{0,1}(N))\). So \([w] = 0\) on \(H^q_{(2)}(X)\) by the injectivity of the short exact sequence. This shows that \(p i^*\) is injective.

For the surjectivity, take \(\eta \in H^q_{(2)}(M)\). Let \(\xi = i_C^* (\eta) \in H^q_{(2)}(Y)\). Then \(\xi\) extends to an \(L^2\) form
on \(C(Y)\) which is cohomologous with the restriction of \(\eta\) in a collared neighborhood of \(Y\). It follows that
\((\eta, \xi)\) is the image of some element of \(H^q_{(2)}(X)\), say \(w\). then \(p i^* (w) = \eta\)

Part c). follows from the natural isomorphisms \(H^q_{(2)}(X) \cong H^q(M, Y), H_{(2)}(C(Y)) \cong 0\) for
\(q > m/2\) \([5]\).

\[\square\]
Lemma 5.2  For a split short exact sequence

\[ 0 \rightarrow V_1 \xrightarrow{i} V_2 \xrightarrow{p} V_3 \rightarrow 0 \]

with preferred bases \( c_1, c_2, c_3 \), its \( R \)-torsion is determined by \( i(c_1), j(c_3) \) and \( c_2 \), where \( j \) is an homomorphism from \( V_3 \) to \( V_2 \) such that \( pj = \text{id} \). In fact, the \( R \)-torsion is given by

\[ \|i(c_1)j(c_3)/c_2\| \]

Proof: We choose \( b_1 = 0, b_2 = i(c_1) \), and \( b_3 = c_3 \) and set \( \tilde{b}_1 = c_1, \tilde{b}_2 = j(c_3) \) and \( \tilde{b}_3 = 0 \). The lemma follows.

A split short exact sequence can be written as

\[ 0 \rightarrow V_1 \xrightarrow{i} V_1 \oplus V_3 \xrightarrow{p} V_3 \rightarrow 0, \]

where \( i \) is not necessarily the natural inclusion, nor \( p \) the natural projection.

Lemma 5.3  For a split short exact sequence

\[ 0 \rightarrow V_1 \xrightarrow{i} V_1 \oplus V_3 \xrightarrow{p} V_3 \rightarrow 0 \]

with preferred bases \( c_1, c_1 \oplus c_3, c_3 \), consider the natural projection \( p_1 : V_1 \oplus V_3 \rightarrow V_1 \) onto the first factor and the natural inclusion \( i_2 : V_3 \rightarrow V_1 \oplus V_3 \) of the second factor. If \( p_1 i : V_1 \rightarrow V_1 \) is an isometry with respect to the inner product induced by the preferred basis \( c_1 \) and \( p_1 i_2 = \text{id} : V_3 \rightarrow V_3 \), then the \( R \)-torsion of the short exact sequence is trivial.

Proof: Using the lemma above we just need to compare the basis \( i(c_1) \oplus c_3 \) with \( c_1 \oplus c_3 \). Since \( p_1 i_1 \) is an isometry, we might as well replace \( c_1 \oplus c_3 \) with \( p_1 i_2 (c_1) \oplus c_3 \). Then clearly, the transition matrix from \( i(c_1) \oplus c_3 \) to \( p_1 i_2 (c_1) \oplus c_3 \) is an upper triangular matrix with all diagonal entries one. The lemma follows.

Combining the above results, we obtain the main result of this section on the \( R \)-torsion of the Mayer-Vietoris sequence.

Theorem 5.4  Assume that the Witt condition \( H^m(Y) = 0 \) holds. Then the \( R \)-torsion of the Mayer-Vietoris sequence in intersection cohomology

\[ \cdots \rightarrow IH^q_{(2)}(Y) \rightarrow IH^{q+1}_{(2)}(X) \rightarrow IH^{q+1}_{(2)}(M) \oplus IH^{q+1}_{(2)}(C(Y)) \rightarrow IH^{q+1}_{(2)}(Y) \rightarrow \cdots \]

is equal to the \( R \)-torsion of the truncated exact sequence of the pair \((M, Y)\)

\[ 0 \rightarrow H^{2+1}_{(2)}(M, Y) \rightarrow H^{2+1}_{(2)}(M) \rightarrow H^{2+1}_{(2)}(Y) \rightarrow H^{2+2}_{(2)}(M, Y) \rightarrow \cdots \]

References

[1] J.-M. Bismut and W. Zhang. An extension of a theorem by Cheeger and Müller. *Astérisque*, 205, 1992.

[2] J. Brüning, X. Ma. An anomaly formula for Ray-Singer metrics on manifolds with boundary. *Geom. Funct. Anal.* 16 (2006), no. 4, 767–837.
[3] D. Burghelea, L. Friedlander, T. Kappeler, and P. McDonald. Analytic and Reidemeister Torsion for Representations in Finite Type Hilbert Modules. *Geom. Funct. Anal.*, 6 (1996), pp. 751-859.

[4] J. Cheeger. Analytic torsion and the heat equation. *Ann. Math.* 109 (1979) 259-322.

[5] J. Cheeger. Spectral geometry of singular Riemannian spaces. *J. Diff. Geom.* 18, 575-657 (1983).

[6] J. Cheeger. $\eta$-invariants, the adiabatic approximation and conical singularities. I. The adiabatic approximation. *J. Differential Geom.* 26 (1987), no. 1, 175–221.

[7] X. Dai. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. *J. Amer. Math. Soc.*, 4:265–321, 1991.

[8] X. Dai. APS boundary conditions, eta invariants and adiabatic limits *Trans. AMS*, 354, pp. 107-122.

[9] X. Dai, H. Fang. Analytic torsion and $R$-torsion for manifolds with boundary. *Asian J. Math.* 4 (2000), no. 3, 695–714.

[10] X. Dai, D. Freed. Eta invariants and determinant lines. *J. Math. Phys.*, 35(1994), 5155-5194.

[11] X. Dai, D. Freed. Invariants $\eta$ et droites determinants. *C. R. Acad. Sci., Paris, Series I*, 320(1995), 585-591.

[12] A. Dar. Intersection $R$-torsion and analytic torsion for pseudomanifolds. *Math. Z.* 194 (1987), no. 2, 193–216.

[13] W. Franz. Über die Torsion einer überdeckung. *J. Reine Angew. Math.*, 173 (1935), pp. 245-254.

[14] M. Goresky, R. MacPherson. Intersection homology theory. *Topology* 19 (1980), no. 2, 135–162.

[15] M. Goresky, R. MacPherson. Intersection homology. II. *Invent. Math.* 72 (1983), no. 1, 77–129.

[16] L. Hartmann and M. Spreafico. The analytic torsion of a cone over an odd dimensional manifold. *J. Geom. Phys.*, 61 (2011), no. 3, 624657.

[17] Frances Kirwan. *An Introduction to Intersection Homology Theory*. Pitman Reasearch Notes in Mathematics Series., 187.

[18] J. Lott and M. Rothenberg. Analytic Torsion for Group Actions. *J. Diff. Geom.*, 34 (1991), pp. 431-481.

[19] W. Lück. Analytic and Topological Torsion for Manifolds with Boundary and Symmetry. *J. Diff. Geom.*, 37 (1993), pp. 263-322.

[20] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.* 72 1966 358–426.

[21] W. Müller. Analytic Torsion and $R$-torison of Riemannian Manifolds. *Adv. in Math.*, 28 (1978), pp. 233-305.

[22] W. Müller. Analytic Torsion and $R$-torison for Unimoduler Representations. *J. Amer. Math. Soc.*, 6 (1993), pp. 31-34.
[23] W. Müller and B. Vertman The metric anomaly of analytic torsion on manifolds with conical singularities. *Comm. Partial Differential Equations*, 39 (2014), no. 1, 146191.

[24] D. B. Ray and I. M. Singer. *R*-Torsion and the Laplacian on Riemannian Manifolds. *Advances in Mathematics* 7., 145-210 (1971).

[25] K. Reidemeister. Homotopieringe und Linsenräume. in *Hamburger Abhandl.*, 11, 1935, pp. 102-109

[26] S. Rosenberg. Nonlocal invariants in index theory. *Bull. Amer. Math. Soc.*, (N.S.) 34 (1997), no. 4, 423-433.

[27] M. Spreafico. Zeta function and regularized determinant on a disc and on a cone. *J. Geom. Phys.*, 54 (2005), no. 3, 355-371.

[28] Vladimir Turaev. *Introduction to Combinatorial Torsions*. Lectures in Mathematics., 2000.

[29] B. Vertman. Analytic torsion of a bounded generalized cone. *Comm. Math. Phys.*, 290 (2009), no. 3, 813-860.

[30] S. M. Vishik. Generalized Ray-Singer Conjecture I: a Manifold with a Smooth Boundary. *Comm. Math. Phys.*, 167 (1995), pp. 1-102.