Heat capacity of holographic screen inspires MOND theory

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Abstract

It is argued that Planck mass may be considered as a candidate for the mass content of each degree of freedom of holographic screen. In addition, employing the Verlinde hypothesis on emergent gravity and considering holographic screen degrees of freedom as a $q$-deformed fermionic system, it is obtained that the heat capacity per degree of freedom inspires the MOND interpolating function. Moreover, the MOND acceleration is achieved as a function of Planck acceleration. Both ultra-relativistic and non-relativistic statistics are studied. We, therefore, believe that our results can at least mathematically be employed to write the MOND theory corresponding to various samples.

1. Introduction

The idea of Modified Newtonian dynamics (MOND) is simple, Newtonian dynamics breaks down at low accelerations, and therefore, a new constant is added to physics, usually called MOND acceleration ($\equiv a_0$), [1–6]. Based on MOND theory, the second law of Newton ($F = ma$) is modified as $F = ma\tilde{\mu}(a/a_0)$, where $\tilde{\mu}(a/a_0(\equiv x))$ is called the interpolating function which should satisfy these conditions [1–4]:

$$\tilde{\mu}(x) \rightarrow \begin{cases} 1, & x \gg 1 \text{ (Newtonian limit)} \\ x, & x \ll 1 \text{ (deep MOND limit (DML))} \end{cases}$$

$$\frac{d \ln \tilde{\mu}}{d \ln x} > -1.$$  \hfill (1)

In this manner, the Poisson equation is also modified as [4]

$$\nabla \left[ \tilde{\rho} \left( \frac{\nabla \phi}{a_0} \right) \nabla \phi \right] = 4\pi G\tilde{\rho}.$$  \hfill (2)

As a consequence, MOND theory has the ability to explain the rotational curves of spiral galaxies and their constant luminosity, which are related to each other based on the Renzo Sancisi’s prediction [4, 6]. The first approximations, made by Milgrom, say $a_0$ is of an order of $10^{-8}$ cm s$^{-2}$ [2], and more studies claim that $a_0 = 1.2 \times 10^{-8}$ cm s$^{-2}$ [5, 6].

Moreover, there is not any systematic way to find interpolating functions. The only things respected are equation (1) and compatibility with observations. On the other hand, debates on the possible values of MOND acceleration are still ongoing [7–11]. Indeed, while Medium-richness galaxy groups can help us test MOND at very low accelerations, it seems that this theory can not solve the problem of mass in galaxy clusters and also rich galaxy groups [6]. The latter may be considered as a sign for this hypothesis that the mass content of the sample is important in getting the true interpolating function. In summary, it would mathematically be worthwhile to introduce a systematic way for building interpolating functions. In this regard, it is addressed that the existence of a cutoff for acceleration ($a_0$) may be due to the quantum statistics satisfied by the degrees of freedom distributed on holographic screen [12], or even, may be considered as a sign of Debye gravity [13]. On the other hand, it is also shown that MOND can be obtained if generalized statistics is obeyed by the degrees of freedom of holographic screen [14].
There are two distinct methods for investigating the statistics and thermodynamics of intermediate states. The first method is based on Tsallis non-extensive statistics [15], and in general, the generalized entropies [16]. In the second method, the deformed quantum algebras are employed [17] leading to deformed thermostatistical functions. Recently, the physical meaning of deformation parameters, generalized statistics, and their implications have been tried to be understood by applying these deformed functions to various phenomena such as condensed matter physics, solid-state physics, nuclear physics, gravity, and related topics [14, 18–31].

Here, adopting the Verlinde hypothesis on the emergence of gravity and spacetime [32], and additionally, by relying on the above arguments, we are looking at the holographic screen as a system obeying $q$-statistics and study the predictions of this view on MOND and address some related topics. It is also worthwhile to emphasize that we do not want to study the probable shortcomings of the Verlinde hypothesis [33, 34] or even its strengths and debates against its probable weaknesses [20–22, 35–38]. Our aim is to investigate its predictions about the MOND theory. In order to achieve our goal, we begin by providing introductory notes on the Verlinde approach and $q$-statistics in the next two sections, respectively. The effects of applying $q$-deformed fermionic statistics to holographic screens on MOND theory are also studied in the fourth section. The last section includes a summary.

2. Verlinde gravity

The essence of Verlinde’s idea is that gravity is no longer fundamental and it appears as an entropic force (i.e. it emerges as the tendency of systems to increase their entropy, in agreement with the second law of thermodynamics) [32]. We consider a test particle with mass $m_t$. When this particle approaches the holographic screen (it takes the distance $\Delta x = \frac{\hbar}{m_t c}$ from the holographic screen [32]), the change of entropy of the holographic screen is given by

$$\Delta S = 2\pi k_B \frac{m_t c}{\hbar} \Delta x,$$

where $c$ is the speed of light, and $k_B$ denotes Boltzmann constant. There is a relation between temperature and acceleration as [39]

$$T = \frac{1}{2\pi k_B c} \frac{\hbar a}{c}.$$  \hspace{1cm} (4)

For simplicity, we consider energy units such that $k_B = 1$ throughout this paper. The particle feels the force $F = T \frac{\Delta S}{\Delta x} = m_t a$. Moreover, since the holographic screen is a two-dimensional hypersurface with radius $R$ and area $A = 4\pi R^2$, the number of bits ($N$) distributed on it, is calculated by

$$N = \frac{A c^3}{G l_p} \equiv \frac{A}{l_p^2}.$$  \hspace{1cm} (5)

where $G$ and $l_p \equiv \sqrt{\frac{G l_p}{c^3}}$ denote the Newtonian gravitational constant and the Planck length, respectively. The total energy of the screen may be modeled by a one dimensional Boltzmann gas [32]

$$E = \frac{1}{2} N T,$$

which can also be found out by using the mass $M$ confined to it as

$$E = M c^2.$$  \hspace{1cm} (7)

Using the equations (4)–(7), the Newtonian gravity can finally be obtained as

$$m_t a = T \frac{\Delta S}{\Delta x} \rightarrow a = \frac{G M}{R^2}.$$  \hspace{1cm} (8)

The classical relation (6), indeed thermal energy, plays a crucial role in getting the above result. Therefore, according to the Verlinde approach [32], gravitational effects are related to the thermal excitations of the degrees of freedom of the holographic screen. In fact, any modification to thermal energy (6), corresponding to degrees of freedom, can affect equation (8). Because the origin of the holographic screen and its degrees of freedom are not known, physicists recently applied quantum statistics or generalized statistics to horizon degrees of freedom to obtain modifications to equations (6) and thus (8) [12, 14, 18–24].
In this study, we propose a q-deformed fermionic system, which has crucial application [40], and for example, in [27, 41] some thermostatistical properties are examined and the connection between fermionic q deformation and the thermal effective mass of a quasi-particle is found out.

In terms of creation operator $a^*$, annihilation operator $a$, and the number operator $\hat{N}$, the symmetric q-deformed fermionic oscillators algebra is defined as [25–27]

$$aa^* + qa^*a = q^{-\hat{N}}, \quad [\hat{N}, a^*] = a^*, \quad [\hat{N}, a] = -a, \quad a^*a = \hat{N}, \quad aa^* = 1 - \hat{N},$$

where we have

$$[x] = q^x - q^{-x},$$

for the basic q-deformed quantum number. Here, $q$ is the real deformation parameter, and the Jackson derivative and mean occupation number are as

$$D_q f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})},$$

and

$$f_{\lambda,q} = \frac{1}{2\ln q} \ln \left[ \frac{z^{-\lambda}e^{\lambda z} + q}{z^{-\lambda}e^{\lambda z} + q^{-1}} \right],$$

respectively, in which fugacity ($z$) is used instead of $\exp(\beta\mu)$. In this statistics, if the degrees of freedom of holographic screen meet the $\varepsilon = \alpha \eta^2$ relation, then one can reach [27]

$$N = \frac{gA}{\lambda^2} h_\eta(\varepsilon, q),$$

$$U = \frac{gA}{\lambda^2} T \eta^2(\varepsilon, q),$$

where $\lambda = \frac{h_0^{1/2}}{\pi^{1/2}} \left[ \frac{\Gamma(2)}{\Gamma(3/2 + 1)} \right]^{1/2}$ is the generalized thermal wavelength [27] and $\eta = \frac{z}{\varepsilon}$, for the number of degrees of freedom and their energy content in two dimensions, respectively.

Defining $y = \beta\varepsilon$ and $h_\eta(\varepsilon, q) = \frac{1}{\Gamma(n)} \int_0^{\infty} y^{n-1} \ln \left( \frac{\varepsilon + q^{1/2} e^{\lambda z} + q^{-1/2}}{\varepsilon + q^{1/2} e^{\lambda z} + q^{-1/2}} \right) dy$, which denotes the generalized Fermi integral, and by using equation (5), one obtains

$$h_\eta(\varepsilon, q) = \frac{\lambda^2}{g^2 T},$$

as an equation that gives us fugacity $z$ and thus chemical potential corresponding to the degrees of freedom of holographic screen. The same assumption is also obtainable in [12].

In order to bring out the remarkable properties of the $q$–deformed fermion gas at low temperatures, the generalized Fermi integral can be calculated by using the Sommerfeld expansion method. So, we find

$$h_\eta(\varepsilon, q) = \frac{(\ln z)^n}{\Gamma(n + 1)} \left[ 1 + n(n - 1) \frac{\pi^2}{6} \gamma_1(q) \frac{1}{(\ln z)^n} \right],$$

up to the first order, in which

$$\gamma_n(q) = \int_0^\infty \frac{dy}{2\ln q} \ln \left( \frac{\varepsilon + q^{-1/2} e^{\lambda z} + q^{1/2}}{\varepsilon + q^{-1/2} e^{\lambda z} + q^{1/2}} \right),$$

satisfying $\gamma_0(q) = 1$ when $q = 1$. At high temperatures and independent of the value of $q$, equation (13) leads to $U = \eta^2 NT$, and the Boltzmann gas is recovered [27].

The existence of minimum length (the Planck length denoted by $l_P$) signals us to think of the holographic screen degrees of freedom as two-dimensional entities [32] meaning that each degree of freedom carries energy.
Comparing with the MOND equation for acceleration recovering the result of

where \( \varepsilon \) denotes the Fermi energy defined as

\[
\varepsilon_F = \left[ \frac{4\pi\hbar^2\alpha^2}{gA} \right]^{1/\eta} \frac{\varepsilon_0}{g^{1/\eta}},
\]

in which \( \varepsilon_0 \) denotes the Fermi energy of the spin-less sample \((g = 1)\).

equation (18) can be inserted in equation (7), to reach

\[
T^2 = \frac{6MC^2\varepsilon_F}{\pi^2\eta\gamma_1(q)N},
\]

combined with Unruh temperature (4), to get

\[
a^2 = \frac{24MC^2\varepsilon_F}{R^2\eta\gamma_1(q)N}
\]

for acceleration recovering the result of [12] at the limit of \( \gamma_1(q) = 1 \). Now, using (5) and (21), one easily reaches at

\[
a^2 = \left( \frac{6c\varepsilon_F}{\hbar\pi\eta\gamma_1(q)} \right) G \frac{M}{R^2}.
\]

Comparing with the MOND equation [1–4, 12] and equation (1), we easily see this limit is indeed DML \((\bar{\mu}(x) = x)\) that leads to

\[
\left( \frac{a}{a_0q} \right) = \frac{G M}{R^2},
\]

\[
a_0q = \frac{6c\varepsilon_F}{\hbar\pi\eta\gamma_1(q)},
\]

4. MOND theory as the heat capacity of the holographic screen

In fact, equation (4) claims that high-temperature limit is equivalent to the high acceleration limit, i.e. the territory of Newtonian physics \((\bar{\mu} = 1)\) compared to the MOND physics \((\bar{\mu} \neq 1)\) which becomes dominant at low accelerations \((\text{Deep MOND limit or briefly DML})\). Therefore, a true theory (result) should cover the Newtonian gravity at high temperatures (accelerations).

At low temperatures, equation (13) implies [27]

\[
U = U_0 + \frac{\pi^2}{6} \eta\gamma_1(q)N \frac{T^2}{\varepsilon_F},
\]

where \( U_0 = \frac{n}{\eta + 1}N\varepsilon_F \) is the ground state energy, and thus the corresponding thermal energy is obtained as

\[
U_{\text{th}}(q) = \frac{\pi^2}{6} \eta\gamma_1(q)N \frac{T^2}{\varepsilon_F},
\]

where \( \varepsilon_F \) denotes the Fermi energy defined as

\[
\varepsilon_F = \left[ \frac{4\pi\hbar^2\alpha}{gA} \right]^{1/\eta} \frac{\varepsilon_0}{g^{1/\eta}},
\]

in which \( \varepsilon_0 \) denotes the Fermi energy of the spin-less sample \((g = 1)\).
and thus

\[ a_{\text{eq}} = \frac{6\epsilon_p}{\hbar \pi \gamma_1(q)}, \]  

(24)

for \( \eta = 1 \) (\( s = 2 \)), and

\[ a_{\text{eq}} = \frac{3\epsilon_p}{\hbar \pi \gamma_1(q)}, \]  

(25)

for \( \eta = 2 \) (\( s = 1 \)). In [12], considering \( N_{\text{Pazy}} = \frac{\Lambda}{2\pi} \) instead of equation (5) and applying a non-relativistic Fermi–Dirac statistics (\( s = 2 \)) to holographic screen, it has been obtained that \( a_{\text{eq}} = 12\epsilon_p/\hbar \) in agreement with the \( \gamma_1(q = 1) = 1 \) limit of equation (24) (apart from a factor 2 due to \( N_{\text{Pazy}} = \frac{N}{2} \)).

Now, based on Verlinde hypothesis [32], the force experienced by the test particle of mass \( m \), is calculable by using

\[ F = T\frac{\Delta S}{\Delta x}, \]  

(26)

combined with equation (4), \( F = m_\ast a \tilde{\mu}(a/a_{\text{eq}}) \), and \( T\Delta S = \Delta E = C\Delta T \) to reach

\[ \tilde{\mu}(a/a_{\text{eq}}) = \frac{C(a/a_{\text{eq}})}{2\pi}\frac{\Delta a}{a}, \]  

(27)

whenever, (using equation (4)), \( a \) and \( a_{\text{eq}} \) are defined as \( \frac{2\pi T}{\hbar} \) and \( \frac{2\pi T_0}{\hbar} \), respectively, and \( C \) subsequently denotes the heat capacity.

Bearing the first paragraph of this section in mind, and in order to satisfy the Newtonian limit for which \( C = \eta N \), one should note that equation (27) leads to \( \tilde{\mu} = \frac{6N}{2\pi}\frac{\Delta a}{a} \) that should become equal to 1 (Newtonian limit for which \( \tilde{\mu} = 1 \)). This expectation yields \( \frac{\Delta a}{a} = \frac{2\pi}{\eta N} \) and thus

\[ \tilde{\mu}(a/a_{\text{eq}}) = \frac{C(a/a_{\text{eq}})}{N\eta} = \left\{ \begin{array}{ll}
\frac{C(a/a_{\text{eq}})}{N}, & \eta = 1 \\
\frac{C(a/a_{\text{eq}})}{2N}, & \eta = 2 
\end{array} \right., \]  

(28)

a result in agreement with [12] that estimates \( \tilde{\mu} \approx \frac{C}{2N_{\text{Pazy}}} \).

As we mentioned in the introduction, Milgrom found out \( a_0 \) is of the order of \( 10^{-8} \) cm s\(^{-2} \) [2], \( a_0 = 1.2 \times 10^{-8} \) cm s\(^{-2} \) is the most probable case [5, 6], and indeed, the constancy (uniqueness) of the value of MOND acceleration is still controversial [7–10]. On the other hand, gravity is the result of the existence of mass (energy), and thus, the amount of mass is reflected in the power of gravity. Hence, one may expect different values of \( q \) for different samples with distinct mass contents.

### 4.1. ultra-relativistic case (\( \alpha = c, s = 1 \))

In order to continue, we consider the \( \eta = 2 \) case, and plot interpolating function for some values of \( q \) and \( g \) to show the efficiency of the idea. In this manner, from equations (3) and (19), we reach

\[ \epsilon_p^0 = \sqrt{4\pi \frac{l_p \hbar}{c^2}} = \sqrt{\frac{4\pi \hbar c^2}{G}} = \sqrt{\frac{4\pi m_p c^2}{E_p}}, \]  

(29)

where \( m_p \) and \( E_p \) denote the Planck mass and energy, respectively. Now, we can write

\[ a_{\text{eq}} = \frac{3c}{\hbar \pi \gamma_1(q) q^2} \epsilon_p^0 = \frac{6}{\pi^2 \gamma_1(q) G^2} a_p, \] 

\[ T_{\text{eq}} = \frac{\hbar a_{\text{eq}}}{2\pi c} = \frac{3}{\pi^2 \gamma_1(q) G^2} T_p, \]  

(30)

in which \( a_p = \sqrt{\frac{c^6}{6\hbar}} \approx 5.56 \times 10^{31} \) m/s\(^2 \) denotes the Planck acceleration, \( T_p = E_p \) is the Planck temperature, and equation (4) has been employed to get the last line. Thus, the value of MOND acceleration depends on the value of \( q \) and \( g \).
Let us consider $a_{0q} = a_0 = 1.2 \times 10^{-8}$ cm s$^{-2}$ [5, 6], and use (30) to obtain a constraint on the values of $g(\equiv g^* \times 10^{122})$ and $q$ as

$$\gamma_1(q)(g^*)^2 = \frac{6}{\pi^2 a_0} a_p \approx 15.69. \quad (31)$$

For the general case including unknown MOND acceleration $a_{0q}$ [7], the above equation takes the form

$$\gamma_1(q)(g^*)^2 = \frac{6}{\pi^2 a_{0q}} a_p. \quad (32)$$

Simple calculations also lead to

$$\frac{GM}{r} = v^2 \beta \left( \frac{v^2}{r a_{0q}} \right) \quad (33)$$

for the velocity $v$ of a particle moving under the effect of source $M$.

If the value of MOND acceleration is constant, then by fitting equation (33) to observational data, and also by respecting constraint (31), simultaneously, one can find proper values of $g^*$ and $q$. On the other hand, if the value of MOND acceleration is not constant and known, then this approach has three free parameters including $a_{0q}, g^*$ and $q$, found out by fitting (33) to observations and also using condition (32). Such analysis can give us worthwhile info about $g$, $q$, and thus, the holographic screen nature.

### 4.2. Non-relativistic case

Here, $\alpha = \frac{1}{2m}$ and $s = 2$ leading to $\epsilon_0^0 = 2\pi E_p/m$, and thus, if we accept our debate on the value of the mass of the degrees of freedom distributed on the holographic screen, presented in the introduction (or equally, $m = m_p$), then $\epsilon_0^0 = 2\pi E_p$. We can also see

$$a_{0q}^{s=2} = \frac{12}{\gamma_1(q) g} \left( \frac{m_p}{m} \right) a_p \Rightarrow a_{0q}^{s=2} = \frac{12}{\gamma_1(q) g} a_p \bigg|_{m=m_p},$$

$$T_{0q}^{s=2} = \frac{\hbar a_{0q}^{s=2}}{2\pi c} = \frac{6}{\pi \gamma_1(q) g} \left( \frac{m_p}{m} \right) T_p \Rightarrow T_{0q}^{s=2} = \frac{6}{\pi \gamma_1(q) g} T_p \bigg|_{m=m_p} \quad (34)$$

leading to

$$\gamma_1(q) g = \frac{12}{a_0} \left( \frac{m_p}{m} \right) a_p \Rightarrow \gamma_1(q) g = \frac{12 a_p}{a_0} \bigg|_{m=m_p} \quad (35)$$

as the counterparts of equations (31) and (32), respectively. The counterpart of equation (33) is also easily obtained by changing $a_{0q}$ with $a_{0q}^{s=2}$. Therefore, in comparison with the ultra-relativistic case, here, we have a new free parameter $m$, if we do not accept the $m = m_p$ hypothesis.

### 4.3. Some primary solutions

In figure 1 (figure 2), the ultra-relativistic (non-relativistic) interpolating function has been plotted for different values of $q$ as well as $g^*(G^* = g \times 10^{-61}$ in non-relativistic case) by considering $a_0^0 = 1.2 \times 10^{-8}$ cm/s$^2$. In the non-relativistic case, we have considered $m = m_p$. It is clear that the overall behavior of the interpolating function is in complete agreement with MOND conditions (1). Our calculations show that i) the existence of $N\eta$ in the denominator of (28) guarantees the satisfaction of the Newtonian limit ($\eta(x \gg 1) \rightarrow 1$), and moreover, ii) in the ultra-relativistic case and for $q \geq 10$, the interpolating function has a maximum at a certain value of $x$ and then tends to unity in such a way that the condition (1) is satisfied. The maximum point occurs at $q \gtrsim 5$ for the non-relativistic interpolating function. It is clear that, compared to the ultra-relativistic case, the Newtonian...
condition is satisfied at smaller values of $x$ meaning that the corresponding interpolating function tends to 1 faster than the ultra-relativistic one.

5. Conclusion

Applying $q$-deformed fermionic statistics to the degrees of freedom of holographic screen, and relying on the Verlinde hypothesis, we found out that the MOND interpolating function can be understood as the heat capacity per degree of freedom of holographic screen. Consequently, i) there are more free parameters than [12], useful if one wants to fit the theory with various observations, and ii) the classical statistics (the Boltzmann gas) does only lead to Newtonian gravity because its heat capacity per degree of freedom is constant.

On the other hand, it is also seen that the MOND acceleration is a function of Planck acceleration, and additionally, Planck mass may be accepted as a candidate for the mass content of each degree of freedom of holographic screen. Therefore, in general, our approach has two (three) free parameters in ultra-relativistic (non-relativistic) case including $q$ and $g (q, g, and m)$, if we do not limit ourselves to a special MOND acceleration [7–11] and $m$ (in non-relativistic case). In both cases, any restriction on the value of MOND acceleration (such as accepting a value for it [6]) leads to eliminate one of the parameters $q$ and $g$ with the help of equations (32) and (35) in favor of the other.
It is finally worthwhile to mention that since a fermionic system has been focused, $g$ should be even. Of course, this condition is satisfied in our plots, but proper curves are obtained for very huge amounts of $g$. Indeed, until we get a comprehensive (at least a better) understanding of holographic screens, we can not say anything about the allowed even values of $g$. On the other hand, if we look at the results only as the mathematical functions that meet the MOND requirements (1), then we can even consider odd values for $g$ to achieve a better fitting with observations.

Data availability statement

No new data were created or analysed in this study.

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References

[1] Milgrom M 1983 ApJ 270 365
[2] Milgrom M 1983 ApJ 270 371
[3] Milgrom M 1986 ApJ 302 617
[4] Famaey B and McGaugh S S 2012 Living Rev. Relativity 15 10
[5] Begeman K G, Broeils A H and Sanders R H 1991 MNRAS 249 523
[6] Milgrom M and Hot Stud 2020 Phil. Sci. B 71 179
[7] Rodrigues D C, Marra V, del Popolo A and Davari Z 2018 Nature Astron. 2 668–72
[8] Chang Z and Zhou Y 2019 Mon. Not. Roy. Astron. Soc. 486 1658
[9] Chan M H and Del Popolo A 2020 Mon. Not. Roy. Astron. Soc. 492 5865
[10] Marra V, Rodrigues D C and de Almeida A O 2020 Mon. Not. Roy. Astron. Soc. 494 2875
[11] Kroupa P, Banik I, Haghi H, Zonoozi A H, Dabringhausen J, Javanmardi B, Müller O, Wu X and Zhao H 2018 Nature Astron. 2 925
[12] Pazy E and Argaman N 2012 Phys. Rev. D 85 104021
[13] Navia C E arXiv:1706.02960
[14] Moradpour H et al 2018 Phys. Lett. B 783 82
[15] Tsallis C 1988 J. Stat. Phys. 52 479
[16] Masi M 2005 Phys. Lett. A 338 217
[17] Arik M and Coon D D 1979 J. Math. Phys. 17 524
[18] Neto J A 2011 Int. J. Theor. Phys. 50 3552
[19] Abreu E M C et al 2018 EPL 124 30003
[20] Abreu E M C and Neto J A 2013 Phys. Lett. B 727 524
[21] Abreu E M C, Neto J A, Barboza E M and Nunes R C 2017 Int. J. Mod. Phys. A 32 1750028
[22] Abreu E M C, Ananias Neto J, Mendes A C R and Souza D O 2017 EPL 120 20003
[23] Ourabah K, Barboza E M, Abreu E M C and Neto J A 2019 Phys. Rev. D 100 103516
[24] Ourabah K 2020 Phys. Scr. 95 055005
[25] Biedenharn L C 1989 J. Phys. A 22 873
[26] Macfarlane A J 1989 J. Phys. A, Math. Gen. 22 4581
[27] Cai S et al 2007 J. Phys. A: Math. Theor. 40 11245
[28] Shu Y et al 2002 Phys. Lett. A 292 399
[29] Zheng Q J et al 2002 Physica A 391 563
[30] Senay M and Kibaroğlu S 2018 Int. J. Mod. Phys. A 33 1850218
[31] Kibaroğlu S and Senay M 2019 Mod. Phys. Lett. A 34 1950249
[32] Verlinde E 2011 JHEP 04 029
[33] Kobakhidze A 2011 Phys. Rev. D 83 021502 (R)
[34] Gao S 2011 Entropy 13 936
[35] Chaichian M, Oksanen M and Tureanu A 2011 Phys. Lett. B 702 419
[36] Chaichian M, Oksanen M and Tureanu A 2012 Phys. Lett. B 712 272
[37] Visser M 2011 JHEP 10 140 M
[38] Lee J W 2012 Found. Phys. 42 1153
[39] Unruh W G 1976 Phys. Rev. D 14 870
[40] Algin A 2011 Int. J. Theor. Phys. 50 1554
[41] Algin A and Senay M 2016 Physica A 447 232