An analog of the Iwasawa conjecture for a complete hyperbolic threefold of finite volume

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Abstract

For a unitary local system of rank one on a complete hyperbolic threefold of a finite volume with only one cusp, we will compare the order of its Alexander invariant at \( t = 1 \) and one of the Ruelle L function at \( s = 0 \). Our results may be considered as a solution of a geometric analogue of the Iwasawa main conjecture in the algebraic number theory.

1 Introduction

In recent days, it has been recognized there are many similarities between the theory of a number field and one of a topological threefold. In this note, we will show one more evidence, which is "a geometric analog of the Iwasawa conjecture".

At first let us recall the original Iwasawa conjecture (\[15\]). Let \( p \) be an odd prime and \( K_n \) a cyclotomic field \( \mathbb{Q}(\zeta_{p^n}) \). The Galois group \( \text{Gal}(K_n/\mathbb{Q}) \) which is isomorphic to \( \mathbb{Z}/(p^n-1) \times \mathbb{F}_p^* \) by the cyclotomic character \( \omega \) acts on the \( p \)-primary part of the ideal class group \( A_n \) of \( K_n \). By the action of \( \text{Gal}(K_1/\mathbb{Q}) \simeq \mathbb{F}_p^* \), it has a decomposition

\[
A_n = \bigoplus_{i=0}^{p-2} A_n^{\omega^i},
\]

where we set

\[
A_n^{\omega^i} = \{ \alpha \in A_n \mid \gamma \alpha = \omega(\gamma)^i \alpha \text{ for } \gamma \in \text{Gal}(K_1/\mathbb{Q}) \}.
\]

For each \( i \) let us take the inverse limit with respect to the norm map:

\[
X_i = \lim_{\longrightarrow} A_n^{\omega^i}.
\]

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If we set $K_\infty = \cup_n K_n$ and $\Lambda_\infty = \text{Gal}(K_\infty/K_1)$, each $X_i$ becomes a $\mathbb{Z}_p[[\Lambda_\infty]]$-module. Since there is an (noncanonical) isomorphism $\mathbb{Z}_p[[\Lambda_\infty]] \simeq \mathbb{Z}_p[[s]]$, each $X_i$ may be considered as a $\mathbb{Z}_p[[s]]$-module. Iwasawa has shown that it is a torsion $\mathbb{Z}_p[[s]]$-module and let $L^{\text{alg},i}_p$ be its generator, which will be referred as the Iwasawa power series.

On the other hand, let $\mathbb{Z}_p[[s]] \simeq \mathbb{Z}_p[[\Lambda_\infty]]$ be the ring homomorphism induced by $\omega$. For each $0 < i < p - 1$, using the Kummer congruence of the Bernoulli numbers, Kubota-Leopoldt and Iwasawa have independently constructed an element of $L^{\text{ana},i}_p$ which satisfies

$$\chi^r(L^{\text{ana},i}_p) = (1 - p^r)\zeta(-r),$$

for any positive integer $r$ which is congruent $i$ modulo $p - 1$. Here $\zeta$ is the Riemann zeta function. We will refer $L^{\text{ana},i}_p$ as the $p$-adic zeta function. The Iwasawa main conjecture, which has been solved by Mazur and Wiles ([8]) says that ideals in $\mathbb{Z}_p[[s]]$ generated by $L^{\text{alg},i}_p$ and $L^{\text{ana},i}_p$ are equal.

Now we will explain our geometric analog of the Iwasawa main conjecture.

It is broadly recognized a geometric substitute for the Iwasawa power series is the Alexander invariant. Let $X$ be a connected finite CW-complex of dimension three and $\Gamma_g$ its fundamental group. In what follows, we always assume that there is a surjective homomorphism

$$\Gamma_g \twoheadrightarrow \mathbb{Z}.$$ Let $X_\infty$ be the infinite cyclic covering of $X$ which corresponds to $\text{Ker} \epsilon$ by the geometric Galois theory and $\rho$ a finite dimensional unitary representation of $\Gamma_g$. Then $H^i(X_\infty, \mathbb{C})$ and $H^i(X_\infty, \rho)$ have an action of $\text{Gal}(X_\infty/X) \simeq \mathbb{Z}$, which make them $\Lambda$-modules. Here we set $\Lambda = \mathbb{C}[\mathbb{Z}]$ which is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Suppose that each of them is a torsion $\Lambda$-module. Then due to the results of Milnor ([10]), we know $H^i(X_\infty, \rho)$ is also a torsion $\Lambda$-module for all $i$ and vanishes for $i \geq 3$. Let $\tau^*$ be the action of $t$ on $H^i(X_\infty, \rho)$. Then the Alexander invariant is defined to be the alternating product of the characteristic polynomials:

$$A_p^*(t) = \frac{\det[t - \tau^* | H^0(X_\infty, \rho)] \cdot \det[t - \tau^* | H^2(X_\infty, \rho)]}{\det[t - \tau^* | H^1(X_\infty, \rho)]}.$$

We will take the Ruelle L-function as a geometric substitute for the $p$-adic zeta function. Let $\Gamma$ be a torsion free cofinite discrete subgroup of $PSL_2(\mathbb{C})$. It acts on the three dimensional Poincaré upper half space

$$\mathbb{H}^3 = \{(x, y, r) \mid x, y \in \mathbb{R}, r > 0\}$$
endowed with a metric
\[ ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}, \]
whose sectional curvature \( \equiv -1 \). Let \( X \) be the quotient, which is a complete hyperbolic threefold of finite volume. We will assume that it has only one cusp. Let \( \rho \) be a unitary character of \( \Gamma \). It defines a unitary local system on \( X \) of rank one, which will be denoted by the same symbol. By the one to one correspondence between the set of loxiodromic conjugacy classes of \( \Gamma \) and one of closed geodesics of \( X \), the Ruelle \( L \)-function is defined as
\[ R_{\rho}(z) = \prod_{\gamma} \det[1 - \rho(\gamma)e^{-z l(\gamma)}], \]
where \( \gamma \) runs through primitive closed geodesics. Here \( z \) is a complex number and \( l(\gamma) \) is the length of \( \gamma \). It is known \( R_{\rho}(z) \) is absolutely convergent if Re \( z \) is sufficiently large. We will show its square is meromorphically continued on the whole plane. (If the restriction \( \rho|_{\Gamma_\infty} \) of \( \rho \) to the fundamental group \( \Gamma_\infty \) of the cusp is nontrivial, \( R_{\rho}(z) \) will be meromorphically continued itself.) Let us define the order of \( R_{\rho}(z) \) at \( z = 0 \) to be
\[ \text{ord}_{z=0} R_{\rho}(z) = \frac{1}{2} \text{ord}_{z=0} R_{\rho}(z)^2. \]
We will compute it in terms of the dimension \( h^i(\rho) \) of \( H^i(X, \rho) \).

**Theorem 1.1.** Suppose \( \rho|_{\Gamma_\infty} \) is trivial. Then we have
\[ \text{ord}_{z=0} R_{\rho}(z) = 2(2h^0(\rho) - h^1(\rho) + 1). \]
On the contrary if \( \rho|_{\Gamma_\infty} \) is nontrivial,
\[ \text{ord}_{z=0} R_{\rho}(z) = -2h^1(\rho). \]

Note that in the latter case, \( h^0(\rho) \) vanishes by the assumption. We will find the “error term” 2 in the first identity is caused by the Hodge theory.

Suppose there is a surjective homomorphism from \( \Gamma \) to \( \mathbb{Z} \) and let \( X_\infty \) be the corresponding infinite covering of \( X \). Moreover suppose that all of the dimensions of \( H^1(X_\infty, \mathbb{C}) \) and \( H^1(X_\infty, \rho) \) are finite. Let \( g \) be a generator of the infinite cyclic group. In \([14]\) we have shown

**Theorem 1.2.** Suppose that \( H^0(X_\infty, \rho) \) vanishes. Then
\[ \text{ord}_{t=1} A^X_1(\rho) \leq -h^1(\rho). \]
Moreover if the action of \( g \) on \( H^1(X_\infty, \rho) \) is semisimple, they are equal.

These two theorems imply the following corollary, which may be considered as a geometric analogue of the Iwasawa main conjecture.
Corollary 1.1. Suppose that \( H^0(X_\infty, \rho) \) vanishes.

1. If \( \rho|_{\Gamma_\infty} \) is nontrivial, we have
   \[ \text{ord}_{z=0} R_\rho(z) \geq 2 \text{ord}_{t=1} A_X^* (\rho). \]

2. If \( \rho|_{\Gamma_\infty} \) is trivial, we have
   \[ \text{ord}_{z=0} R_\rho(z) \geq 2(1 + \text{ord}_{t=1} A_X^* (\rho)). \]

Moreover if the action of \( g \) on \( H^1(X_\infty, \rho) \) is semisimple, they are equal.

For a closed hyperbolic threefold, similar results have been proved in [13].

Although it seems curious there is a difference between two invariants, such a phenomenon also occurs in the Iwasawa theory of an elliptic curve ([3]). In their case the reason of the pathology is a ramification and non-semisimplicity of a Galois representation associated to an elliptic curve. This is quite similar to our case.

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2 A spectral decomposition and the Hodge theory

Let \( \Omega^j(\rho) \) be a vector bundle of \( j \)-forms on \( X \) twisted by \( \rho \) and the space of its square integrable sections will be denoted by \( L^2(X, \Omega^j(\rho)) \). The positive Hodge Laplacian has the selfadjoint extension to \( L^2(X, \Omega^j(\rho)) \), which will be denoted by \( \Delta \). Note that the Hodge star operator induces an isomorphism of Hilbert spaces:

\[
L^2(X, \Omega^j(\rho)) \simeq L^2(X, \Omega^{3-j}(\rho)), \quad j = 0, 1, \tag{1}
\]

which commutes with \( \Delta \).

Let \( L^2(X, \Omega^j(\rho))_d \) be the closure of a subspace of \( L^2(X, \Omega^j(\rho)) \) spanned by eigenvectors of the Laplacian and \( L^2(X, \Omega^j(\rho))_c \), its orthogonal complement. As we will see in §4, \( L^2(X, \Omega^j(\rho))_c \) may be 0 according to the behavior of \( \rho \) at the cusp. Also it is known that \( L^2(X, \Omega^j(\rho))_c \) is generated by the eigenpacket of the Eisenstein series.(See §4) The space

\[
\mathcal{H}^j(\rho) = \{ \varphi \in L^2(X, \Omega^j(\rho)) \mid \Delta \varphi = 0 \},
\]
will be referred as the space of harmonic forms. It is contained in $L^2(X, \Omega^j(\rho))_d$ and in particular its dimension is finite. We will call its dimension $L^2$-Betti number and write it by $\beta_j(\rho)(2)$. Note that by (1) we have

$$\beta_j(\rho)(2) = \beta_{3-j}(\rho)(2), \quad j = 0, 1.$$ 

We will explain the relation between the $L^2$-Betti numbers and the topological one.

Let $h^j(\rho)$ (resp. $h^j_c(\rho)$) be the dimension of $H^j(X, \rho)$ (resp. of the image of the compact supported cohomology group $H^j_c(X, \rho)$ in $H^j(X, \rho)$). Then Zucker has shown ([16], see also the introduction of [9]):

1. $\beta_0(\rho)(2) = h^0(\rho),$
2. $\beta_1(\rho)(2) = h^1_c(\rho).$

For $A > 0$ we set

$$\mathbb{H}^3_A = \{(x, y, r) \mid x, y \in \mathbb{R}, r \leq A\}$$

and let $X_A$ be its image by the natural projection:

$$\mathbb{H}^3 \cong X_A.$$

If $A$ is sufficiently large, the complement

$$Y_A = X \setminus X_A$$

is homeomorphic to a product of a two dimensional torus $T^2$ and an open interval $(A, \infty)$. In particular $X_A$ is a deformation retract of $X$ and we have a commutative diagram:

$$\begin{array}{ccc}
H^1(X_A, \partial X_A, \rho) & \longrightarrow & H^1(X_A, \rho) \\
\downarrow & & \downarrow \\
H^1_c(X, \rho) & \longrightarrow & H^1(X, \rho).
\end{array}$$

Here the vertical arrows are isomorphisms. The above morphism is completed by the exact sequence:

$$\cdots \rightarrow H^{i-1}(\partial X_A, \rho) \rightarrow H^i(X_A, \partial X_A, \rho) \rightarrow H^i(X_A, \rho) \rightarrow H^i(\partial X_A, \rho) \rightarrow \cdots.$$ 

Note that we have

$$H^i(\partial X_A, \rho) \simeq H^i(T^2, \rho).$$

**Lemma 2.1.** Suppose $H^0(\partial X_A, \rho)$ vanishes. Then we have

$$\beta_1(\rho)(2) = h^1(\rho).$$
Proof. With the Poincaré duality, the assumption implies $H^2(\partial X_A, \rho)$ also vanishes. On the other hand the index theorem tells us
$$\chi(\partial X_A, \rho) = 0.$$ Thus we have
$$H^1(\partial X_A, \rho) = 0$$ and
$$H^1(X_A, \partial X_A, \rho) \to H^1(X_A, \rho)$$ is an isomorphism. Now the desired result will follow from Zucker’s result.

Lemma 2.2. Suppose $H^0(\partial X_A, \rho)$ does not vanish. Then we have
$$\beta_1(\rho)_2 = h^1(\rho) - 1.$$

Proof. The assumption implies the restriction of $\rho$ to $\partial X_A$ is trivial and
$$H^0(\partial X_A, \rho) = H^2(\partial X_A, \rho) = \mathbb{C}, \quad H^1(\partial X_A, \rho) = \mathbb{C}^2.$$ We first suppose that $H^0(X, \rho)$ vanishes. By the Poincaré duality we know that $H^1(X_A, \partial X_A, \rho)$ also vanishes and have an exact sequence:
$$0 \to \mathbb{C} \to H^1(X_A, \partial X_A, \rho) \to H^1(X_A, \rho) \to \mathbb{C}^2 \to H^2(X_A, \partial X_A, \rho) \to H^2(X_A, \rho) \to \mathbb{C} \to 0.$$ This exact sequence and the identity
$$\dim H^j(X_A, \partial X_A, \rho) = h^{3-j}(\rho), \quad j = 0, 1.$$ will imply
$$h^1(\rho) = h^2(\rho).$$ In particular we have
$$0 \to \mathbb{C} \to H^1(X_A, \partial X_A, \rho) \to H^1(X_A, \rho) \to \mathbb{C} \to 0,$$ which shows
$$\beta_1(\rho)_2 = h^1(\rho) - 1.$$ Next suppose that $H^0(X, \rho)$ does not vanish. Then $\rho$ is the trivial representation and the restriction
$$H^0(X_A, \rho) \to H^0(\partial X_A, \rho)$$ is an isomorphism. The Poincaré duality implies that the connecting homomorphism
$$H^2(\partial X_A, \rho) \to H^3(X_A, \partial X_A, \rho)$$ is also isomorphic and thus
$$0 \to H^1(X_A, \partial X_A, \rho) \to H^1(X_A, \rho) \to \mathbb{C}^2 \to H^2(X_A, \partial X_A, \rho) \to H^2(X_A, \rho) \to 0.$$ is exact. Now we will obtain the desired result by the same argument as before.
Here is an example.

Let \( K \) be a hyperbolic knot in the three dimensional sphere and \( X \) its complement. By definition \( X \) admits a hyperbolic structure of a finite volume and it is known that the dimension of \( H^1(X, \mathbb{C}) \) is one. Thus Lemma 2.2 shows \( \beta_1(\mathbb{C})_{(2)} \) vanishes.

3 The derivative of the Laplace transform of the heat kernel

For a function \( f \) on \([0, \infty)\), we define its derivative of the Laplace transform as

\[
L'(f)(z) = 2z \int_0^\infty e^{-tz^2} f(t) dt,
\]

if RHS is absolutely convergent. It is convenient to introduce more general transformation:

\[
L'(f)(s, z) = 2z \int_0^\infty t^{s-1} e^{-tz^2} f(t) dt. \tag{2}
\]

Suppose that the integral is absolutely convergent for \( s \) and \( z \) sufficiently large and that it is continued to a meromorphic function on an open domain \( U \) of \( \mathbb{C}^2 \) whose pole does not contain \( \{(1, z) \mid z \in \mathbb{C}\} \). Then we define

\[
L'(f)(z) = L'(f)(s, z)|_{s=1},
\]
on \( U \cap \{(1, z) \mid z \in \mathbb{C}\} \). For example, let us take

\[
f(t) = t^\nu,
\]

where \( \nu \) is a half integer. Then the integral

\[
2z \int_0^\infty t^{\nu+s-1} e^{-tz^2} dt
\]
is absolutely convergent for \( z > 0 \) and \( s > -\nu \) and is computed as

\[
L'(t^\nu)(s, z) = 2z \int_0^\infty t^{\nu+s-1} e^{-tz^2} dt = 2z^{(1-2\nu)-2s}\Gamma(s + \nu).
\]

This is meromorphic function on \( U = \{(s, z) \mid s, z \in \mathbb{C}, -\pi < \text{Im}\ z < \pi\} \) and we obtain

\[
L'(t^\nu)(z) = 2z^{-(1+2\nu)}\Gamma(1 + \nu). \tag{3}
\]

Note that the RHS is analytically continued as a rational function on the whole plane.
We will use the following notation:

\[ \delta_{0, \rho}(t) = \text{Trace}[e^{-t\Delta} | L^2(X, \Omega^0(\rho))_d], \]
\[ \delta_{1, \rho}(t) = \text{Trace}[e^{-t\Delta} | L^2(X, \Omega^1(\rho))_d] - \delta_{0, \rho}(t). \]

Let \( \Sigma_{j, \rho} \) be the set of eigenvalues of the Laplacian on \( L^2(X, \Omega^j(\rho))_d \). For sufficiently large \( z \in \mathbb{R} \), the derivatives of Laplace transforms of \( e^t \delta_{0, \rho}(t) \) and \( \delta_{1, \rho}(t) \) are well defined. We will study their properties.

**Lemma 3.1.** Let \( L'_1(z) \) be the derivative of the Laplace transform of \( \delta_{1, \rho}(t) \). Then it is meromorphically continued on the whole plane and has only simple poles whose residues are integers. Moreover it satisfies the following properties:

1. \( L'_1(z) \) satisfies a functional equation:
   \[ L'(-z) = -L'(z). \]
2. The residue of \( L'_1(z) \) at \( z = 0 \) is \( 2(\beta_1(\rho)(2) - \beta_0(\rho)(2)) \).

**Proof.** For \( \lambda \geq 0 \) and \( z > 0 \), we have

\[
L'(e^{-t\lambda})(z) = 2z \int_0^\infty e^{-t(z^2+\lambda)} dt = \frac{2z}{z^2 + \lambda} = \frac{1}{z - i\lambda} + \frac{1}{z + i\lambda}.
\]

Thus the equation:

\[
\delta_{1, \rho}(t) = \sum_{\alpha \in \Sigma_{1, \rho}} e^{-t\alpha} - \sum_{\beta \in \Sigma_{0, \rho}} e^{-t\beta},
\]

shows

\[
L'_1(z) = L'(\delta_{1, \rho})(z) = \frac{2(\beta_1(\rho)(2) - \beta_0(\rho)(2))}{z} + \sum_{\alpha \in \Sigma_{1, \rho}, \alpha > 0} \left( \frac{1}{z - i\alpha} + \frac{1}{z + i\alpha} \right) - \sum_{\alpha \in \Sigma_{1, \rho}, \beta > 0} \left( \frac{1}{z - i\beta} + \frac{1}{z + i\beta} \right),
\]

which implies desired results.

\[ \square \]

The same computation will show

\[
L'(e^t \delta_{0, \rho})(z) = \beta_0(\rho)(2)\left( \frac{1}{z - 1} + \frac{1}{z + 1} \right)
+ \sum_{\beta \in \Sigma_{0, \rho}, 0 < \beta \leq 1} \left( \frac{1}{z - \sqrt{1 - \beta}} + \frac{1}{z + \sqrt{1 - \beta}} \right)
+ \sum_{\beta \in \Sigma_{0, \rho}, \beta > 1} \left( \frac{1}{z - \sqrt{\beta - 1}i} + \frac{1}{z + \sqrt{\beta - 1}i} \right),
\]

which implies the following lemma.
Lemma 3.2. Let us put:

\[ L'_0(z) = L'(e^{t \delta_0, \rho})(z - 1). \]

Then \( L'_0(z) \) is meromorphically continued on the whole plane and has only simple poles whose residue an integer. Moreover it satisfies a functional equation:

\[ L'_0(1 + z) = -L'_0(1 - z). \]

and

\[ \text{Res}_{z=0} L'_0(z) = \text{Res}_{z=2} L'_0(z) = \beta_0(\rho)(2), \]

4 Selberg trace formula

In this section, we will review the Selberg trace formula following [13]. (See also [5] and [12].)

Let \( A \) be a split Cartan subgroup of \( G = \text{PSL}_2(\mathbb{C}) \). The Lie algebras of \( G \) and \( A \) will be denoted by \( \mathfrak{g} \) and \( \mathfrak{a} \), respectively. The choice of \( A \) determines a positive root \( \alpha \) of \( \mathfrak{g} \) and let \( H \) be an element of \( \mathfrak{a} \) satisfying

\[ \alpha(H) = 1. \]

If we exponentiate a linear isomorphism:

\[ \mathbb{R} \xrightarrow{h} \mathfrak{a}, \quad h(t) = tH, \]

we know \( A \) is isomorphic to the multiplicative group of positive real numbers \( \mathbb{R}^+ \) and will identify them.

Let \( K \simeq SO_3(\mathbb{R}) \) be the maximal compact subgroup. According to the Iwasawa decomposition \( G = KAN \) an element \( g \) of \( G \) can be written as

\[ g = k(g)a(g)n(g). \]

We put \( r(g) = a(g)^{-1} \).

Let \( M \) be the centralizer of \( A \) in \( K \), which is isomorphic to \( SO_2(\mathbb{R}) \). It determines a parabolic subgroup with a Langlands decomposition:

\[ P = MAN. \]

Let \( D_K \) or \( D_M \) be the set of dominant integral forms on \( \mathfrak{h}_K \otimes \mathbb{C} \), or on \( \mathfrak{m} \otimes \mathbb{C} \) respectively. Here \( \mathfrak{h}_K \) (resp. \( \mathfrak{m} \)) is a Cartan subalgebra of \( K \) (resp. the Lie algebra of \( M \)). The there is a natural bijection between \( D_K \) (resp. \( D_M \)) and the set of nonnegative integers \( \mathbb{Z}_{\geq 0} \) (resp. \( \mathbb{Z} \)). For \( \sigma \in D_M \) (resp. \( \lambda \in D_K \)), let \( C(\sigma) \) (resp. \( \tau_{\lambda} \)) be the corresponding highest weight representation of \( M \) (resp. \( K \)). Concretely \( \tau_{\lambda} \) is the standard representation of \( K \) on the space of homogeneous...
polynomials of three variables of degree $\lambda$. In particular the cotangent bundle $\Omega_{H^3}^1$ of $H^3$ is a homogeneous vector bundle associated to $\tau_1$:

$$\Omega_{H^3}^1 = G \times_K \tau_1.$$  \hspace{1cm} (4)

Let $\Gamma_{\infty}$ be the intersection of $\Gamma$ and $N$. Since we have assumed that $\Gamma$ has no elliptic element it coincides with $\Gamma \cap P$.

Now we recall the principal and the Eisenstein series. Let $\mathfrak{A}_\mathbb{C}$ be the complexification of $\mathfrak{A}$. For $(\sigma, s) \in D_M \times \mathfrak{A}_\mathbb{C}^\sigma \simeq \mathbb{Z} \times \mathbb{C}$, let $\mathcal{H}_{\sigma, s}$ be the Hilbert space of Borel measurable functions on $G$ which satisfies

$$f(xmn) = a^{-1-s}\sigma(m)f(x), \quad x \in G, m \in M, a \in A,$$

and

$$\|f\|_K^2 = \int_K |f(k)|^2 dk < \infty.$$  

The integral is taken with respect to the Haar measure on $K$ of total volume one.

Now we will define an action of $G$ on $\mathcal{H}_{\sigma, s}$ to be

$$(\pi_{\sigma, s}(g)f)(x) = f(g^{-1}x),$$

which will be referred as the principal series representation. It is known $\pi_{\sigma, s}$ is unitary if and only if $s$ is pure imaginary. The Cartan involution $w$ yields an isomorphism:

$$\pi_{\sigma, s} \simeq \pi_{-\sigma, -s}.$$  

In our case since any nonzero element of $D_M$ is unramified, $\pi_{\sigma, s}$ is not isomorphic to $\pi_{\sigma, -s}$ for $\sigma \neq 0$. We set

$$\pi(\sigma, s) = \begin{cases}  
\pi_{0, s} & \text{if } \sigma = 0 \\
\pi_{\sigma, s} \oplus \pi_{-\sigma, s} & \text{if } \sigma \neq 0 
\end{cases}$$

and let $\mathcal{H}(\sigma, s)$ be its representation space.

For a $K$-finite vector $\varphi_\sigma \in \mathcal{H}(\sigma, s)$, we associate an Eisenstein series $E(\varphi_\sigma, \rho, s)$, which is a function on $G$ defined as

$$E(\varphi_\sigma, \rho, s)(x) = \sum_{\gamma \in \Gamma_{\infty}\backslash \Gamma} \rho(\gamma)^{-1}r(\gamma x)^{1+s}\varphi_\sigma(\gamma x).$$

It is known that an Eisenstein series satisfies the following properties:

1. $E(\varphi_\sigma, \rho, s)$ absolutely convergents to a $C^\infty$-function on $\{s \in \mathbb{C} | \text{Re } s > 1\} \times G$, which is holomorphic and real analytic in $s$ and $x$, respectively. Moreover it is meromorphically continued on the whole plane.

2. For $\gamma \in \Gamma$,

$$E(\varphi_\sigma, \rho, s)(\gamma x) = \rho(\gamma)E(\varphi_\sigma, \rho, s)(x).$$
3. Let $Z$ be the center of the universal enveloping algebra of $G$. Then $E(\varphi_\sigma, \rho, s)$ is $Z$-finite.

4. Let us fix $x \in G$. Then the function on $K$ which is defined to be:

$$k \in K \rightarrow E(\varphi_\sigma, \rho, s)(xk)$$

is a $C^\infty$-function.

5. If $s$ is pure imaginary, $E(\varphi_\sigma, \rho, s)$ is absolutely square integrable on $\Gamma \backslash G$.

Next we will consider the spectral decomposition of $L^2(X, \Omega^j(\rho))$. For $m \in \Sigma_{j, \rho}$, let $e_m$ be the corresponding eigenvector. Let $L_\infty$ be the torus $\Gamma_\infty \backslash \mathbb{C}$ and $|L_\infty|$ its volume. For each $\sigma \in D_M$, let $\varphi_\sigma \in \mathcal{H}_{\sigma, s}$ be a nonzero $K$-invariant vector. Note that by definition an element of $\mathcal{H}_{\sigma, s}$ is determined by its restriction to $K$ and, by the Frobenius reciprocity law, $K$-invariant part of $\mathcal{H}_{\sigma, s}$ is isomorphic to $\mathbb{C}(\sigma)$.

1. Suppose the restriction of $\rho$ to $\Gamma_\infty$ is trivial. We will treat the spectral decomposition according to the case $j = 0$ and $j = 1$ separately.

(a) The case of $j = 0$.

Suppose $f \in L^2(X, \Omega^0(\rho))$ is contained in the domain of $\Delta$. Then it has a spectral expansion:

$$f = \sum_{m \in \Sigma_{0, \rho}} (f, e_m)e_m + \frac{1}{4\pi|L_\infty|} \int_{-\infty}^{\infty} (f, E(\varphi_0, \rho, it))E(\varphi_0, \rho, it) dt.$$

(b) The case of $j = 1$.

Since the restriction of $\tau_1 \otimes \mathbb{C}$ to $M$ has a decomposition:

$$\tau_1 \otimes \mathbb{C}|_M \simeq \mathbb{C}(-1) \oplus \mathbb{C}(0) \oplus \mathbb{C}(1),$$

an element $f$ in the domain of the Laplacian has an expansion:

$$f = \sum_{m \in \Sigma_{1, \rho}} (f, e_m)e_m + \frac{1}{4\pi|L_\infty|} \sum_{\sigma = -1}^{0} \int_{-\infty}^{\infty} (f, E(\varphi_\sigma, \rho, it))E(\varphi_\sigma, \rho, it) dt.$$

2. Suppose the restriction of $\rho$ to $\Gamma_\infty$ is nontrivial. Then we will know every element of $C^\infty(X, \Omega^j(\rho))$ is cuspidal. In fact let us choose $\gamma \in \Gamma_\infty$ so that $\rho(\gamma) \neq 1$. For $f \in C^\infty(X, \Omega^j(\rho))$ we have

$$\int_{L_\infty} f(x) dx = \int_{L_\infty} f(\gamma x) dx = \rho(\gamma) \int_{L_\infty} f(x) dx.$$
which shows
\[ \int_{L_\infty} f(x)dx = 0. \]
This implies that, in the spectral expansion, we do not have terms of Eisenstein series. Thus we have the eigenfunction expansion:
\[ f = \sum_{m \in \Lambda_{j,\rho}} (f, e_m)e_m, \]
for \( j = 0 \) and \( 1 \).

Now we will explain the Selberg trace formula. We want to compute the trace of heat kernel:
\[ \text{Tr}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))]. \]
On the geometric side it is computed by the orbital integrals:
\[ \text{Tr}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))] = I_j(t) + H_j(t) + U_j(t), \]
where \( I_j(t) \), \( H_j(t) \) and \( U_j(t) \) are the identity, the hyperbolic and the unipotent orbital integral, respectively. Each term will be discussed in the following sections separately.

On the other hand, according to the type of spectrum, we have an orthogonal decomposition:
\[ L^2(X, \Omega^j(\rho)) = L^2_d(X, \Omega^j(\rho)) \oplus L^2_c(X, \Omega^j(\rho)), \]
It is known that \( \text{Tr}[e^{-t\Delta} | L^2_d(X, \Omega^j(\rho))] \) is computed as
\[ \text{Tr}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))_d] = -T_j(t) - S_j(t), \]
where \( T_j(t) \) and \( S_j(t) \) are the threshold and the scattering term, respectively. They are defined in terms of the Fourier coefficients of the Eisenstein series and we will also compute them in \( \S 8 \). As we have seen, if the restriction of \( \rho \) to \( \Gamma_\infty \) is nontrivial, they will not appear. Thus we have the Selberg trace formula:
\[ \text{Tr}[e^{-t\Delta} | L^2(X, \Omega^j(\rho))_d] = I_j(t) + H_j(t) + U_j(t) + \delta_\rho(T_j(t) + S_j(t)), \quad (5) \]
where
\[ \delta_\rho = \begin{cases} 1 & \text{if } \rho|_{\Gamma_\infty} = 1 \\ 0 & \text{if } \rho|_{\Gamma_\infty} \neq 1 \end{cases} \]

5 Ruelle L-function and hyperbolic terms

Let \( \Gamma_h \) be the set of conjugacy classes of loxidromic elements of \( \Gamma \). Since there is a natural bijection between closed geodesics of \( X \) and \( \Gamma_h \), we may identify them. Thus an element \( \gamma \) of \( \Gamma_h \) is written as
\[ \gamma = \gamma_0^{\mu(\gamma)}, \]
where $\gamma_0$ is a primitive closed geodesic and $\mu(\gamma)$ is a positive integer, which will
be referred as the multiplicity. The length of $\gamma \in \Gamma_h$ will be denoted by $l(\gamma)$. Let $\Gamma_{h,prim}$ be the set of primitive closed geodesics.

Using the Langlands decomposition, $\gamma \in \Gamma_h$ may be written as

$$g \gamma g^{-1} = m(\gamma) \cdot a(\gamma) \in MA$$

for a certain $g \in G$. Here $m(\gamma)$ is nothing but the holonomy of a pararell transformation along $\gamma$. Note that elements of $GL_2(\mathbb{R})$:

$$A^u(\gamma) = e^{l(\gamma) m(\gamma)} \quad A^s(\gamma) = e^{-l(\gamma) m(\gamma)}$$

describe an unstable or a stable action of the linear Poincaré map, respectively.

For $\gamma \in \Gamma_h$ we set

$$\Delta(\gamma) = \det[I_2 - A^s(\gamma)]$$

and

$$a_0(\gamma) = \frac{\rho(\gamma) \cdot l(\gamma_0)}{\Delta(\gamma)}, \quad a_1(\gamma) = \frac{\rho(\gamma) \cdot \textrm{Tr}[m(\gamma)] \cdot l(\gamma_0)}{\Delta(\gamma)}.$$ 

Now Theorem 2 of [2] shows the hyperbolic terms are given by

$$H_0(t) = H_0(t), \quad H_1(t) = H_0(t) + H_1(t),$$

where

$$H_0(t) = \sum_{\gamma \in \Gamma_h} \frac{a_0(\gamma)}{\sqrt{4\pi t}} \exp[-\left(\frac{l(\gamma)^2}{4t} + t + l(\gamma)\right)]]$$

and

$$H_1(t) = \sum_{\gamma \in \Gamma_h} \frac{a_1(\gamma)}{\sqrt{4\pi t}} \exp[-\left(\frac{l(\gamma)^2}{4t} + l(\gamma)\right)].$$

We will explain a relation between these hyperbolic terms and the Ruelle L-function.

For $j = 0, 1$ we set

$$S_j(z) = \exp[- \sum_{\gamma \in \Gamma_h} \frac{a_j(\gamma)}{l(\gamma)} e^{-zl(\gamma)}],$$

and let $Y_j(z)$ be its logarithmic derivative:

$$Y_j(z) = \sum_{\gamma \in \Gamma_h} a_j(\gamma) e^{-zl(\gamma)}.$$ 

Then Fried has shown (p.532, the formula (RS)):

$$R_\rho(z) = \frac{S_0(z) \cdot S_0(z + 2)}{S_1(z + 1)}.$$ (6)
Remark 5.1. If we use terminologies of Fried [2] p.529, for a hyperbolic threefold we have
\[ \sigma_0 = \sigma_2, \]
which implies
\[ S_0 = S_2. \]

Let \( L'_{0, hyp}(z) \) and \( L'_{1, hyp}(z) \) denote \( L'(e^t H_0)(z-1) \) and \( L'(H_1)(z) \), respectively.

Proposition 5.1. 1. \( L'_{0, hyp}(z) = Y_0(z). \)

2. \( L'_{1, hyp}(z) = Y_1(z + 1). \)

Proof. Since each statement will proved by the same way, we will only prove the first. By analytic continuation we may assume that \( z > 1 \). The equation:
\[ e^t H_0(t) = \sum_{\gamma \in \Gamma} \frac{a_0(\gamma)}{\sqrt{4\pi t}} \exp[-\left(\frac{l(\gamma)^2}{4t} + l(\gamma)\right)], \]
and
\[ 2z \int_0^\infty \frac{1}{\sqrt{4\pi t}} \exp[-tz^2 - \frac{l(\gamma)^2}{4t}]dt = e^{-z(l(\gamma))} \]
implies
\[ L'(e^t H_0)(z) = \sum_{\gamma \in \Gamma} a_0(\gamma) e^{-l(\gamma)} \cdot 2z \int_0^\infty \frac{1}{\sqrt{4\pi t}} \exp[-(tz^2 + \frac{l(\gamma)^2}{4t})]dt \]
\[ = \sum_{\gamma \in \Gamma} a_0(\gamma) e^{-(z+1)l(\gamma)} \]
\[ = Y_0(z + 1). \]
\[ \square \]

Combining Proposition 5.1 with (6), we have obtained
\[ \frac{d}{dz} \log R_\rho(z) = L'_{0, hyp}(z) - L'_{1, hyp}(z) + L'_{0, hyp}(z + 2). \] (7)

6 Identity terms

The formula of the Planchrel measures [2] and Theorem 2 of [2] implies
\[ I_0(t) = I_0(t), \quad I_1(t) = I_0(t) + I_1(t), \]
where
\[ I_0(t) = \text{vol}(X) \int_{-\infty}^\infty e^{-t(x^2+1)}x^2dx, \]
and

\[ I_1(t) = 2\text{vol}(X) \int_{-\infty}^{\infty} e^{-tx^2}(x^2 + 1)dx. \]

We will compute the derivative of the Laplace transforms of \( e^t I_0 \) and \( I_1 \).

Taking a derivative of the identity:

\[ \int_{-\infty}^{\infty} e^{-tx^2} dx = \frac{\sqrt{\pi}}{2} t^{-\frac{1}{2}} \]

with respect to \( t \), we obtain

\[ \int_{-\infty}^{\infty} x^2 e^{-tx^2} dx = \frac{\sqrt{\pi}}{4} t^{-\frac{3}{2}}. \]

In particular we have

\[ \int_{-\infty}^{\infty} e^{-tx^2}(1 + x^2)dx = \frac{\sqrt{\pi}}{2} (t^{-\frac{1}{2}} + \frac{t^{-\frac{3}{2}}}{2}). \]

The identity

\[ \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = -\frac{1}{2} \Gamma\left(-\frac{1}{2}\right), \]

and the equation (3) will show

\[ L'(I_1)(z) = -2\pi\text{vol}(X)(z^2 - 1). \]

By the same computation we will see

\[ L'(e^t I_0)(z) = -\pi\text{vol}(X)z^2. \]

Thus we have obtained the following proposition.

**Proposition 6.1.**

1. \( L'(e^t I_0)(z) = -\pi\text{vol}(X)z^2. \)

2. \( L'(I_1)(z) = -2\pi\text{vol}(X)(z^2 - 1). \)

### 7 Unipotent terms

We will recall the Osborne and Warner’s formula. (\text{[11]}\text{)}

Since the nilpotent radical \( N \) is diffeomorphic to its Lie algebra \( \mathfrak{N} \) by the exponential map:

\[ n^{\exp} N, \]
the Killing form induces a norm \(|·|\) on \(N\). We define the Epstein L-function of \(\rho\) to be

\[ L(\rho, s) = \sum_{0 \neq \gamma \in \Gamma_\infty} \rho(\gamma) |\gamma|^{-2(1+s)}. \]

It is absolutely convergent for \(\text{Re } s > 0\) and is meromorphically continued on the whole plane. Let \(\rho|_{\Gamma_\infty}\) be the restriction of \(\rho\) to \(\Gamma_\infty\). Then it is known

1. If \(\rho|_{\Gamma_\infty}\) is not trivial, it is an entire function.
2. If \(\rho|_{\Gamma_\infty}\) is trivial, it has a simple pole only at \(s = 0\).

Let \(R_\rho\) be the its residue at \(s = 0\) and we put

\[ C_\rho = \lim_{s \to 0} \left\{ L(\rho, s) - \frac{R_\rho}{s} \right\}. \]

Following Osborne and Warner ([11]), for a function on \(G\) which belongs to a Schwartz space \(C^p(G)\) \((0 < p < 1)\), we will consider the functions:

\[ T(f, s) = \frac{1}{2\pi} \int_N dn |n|^{-2s} \int_K f(\kappa n k^{-1}) dk, \]

and

\[ I(f, s) = 2|L_\infty| L(\rho, s) T(f, s). \]

It is known that \(T(f, s)\) is regular at \(s = 0\). In p.297 of [11] they have shown the unipotent orbital integral of \(f\) is given by

\[ \mathcal{U}(f) = \lim_{s \to 0} \frac{d}{ds} [s \cdot I(f, s)]. \]

This implies the following corollary.

**Corollary 7.1.**

1. If \(\rho|_{\Gamma_\infty}\) is nontrivial, we have

\[ \mathcal{U}(f) = 2|L_\infty| C_\rho T(f, 0). \]

2. If \(\rho|_{\Gamma_\infty}\) is trivial, we have

\[ \mathcal{U}(f) = 2|L_\infty| \{R_\rho T'(f, 0) + C_\rho T(f, 0)\}. \]

We will apply the corollary to our heat kernel.

Let \(o\) be a point of \(\mathbb{H}^3\) defined by

\[ o = (0, 0, 1), \]

and for \(x \in G\) we set

\[ [x] = x \cdot o. \]
Let $\tilde{K}_j(\cdot, \cdot, t)$ be the integral kernel of the heat operator on $\Omega^j_{\mathbb{H}^3}(\rho)$. Since $\Omega^j_{\mathbb{H}^3}$ is a homogeneous vector bundle:

$$\Omega^j_{\mathbb{H}^3} = G \times K \tau_j,$$

there is a $K$-biinvarinat function $K_{j,t} \in C^\infty(G, \text{End}(\tau_j))$ such that

$$K_{j,t}(x^{-1} \cdot y) = \tilde{K}_j([x], [y], t), \quad x, y \in G.$$

Its trace $k_{j,t}$ is contained in $C^p(G)$ for a certain $0 < p < 1$. We will define its nonabelian Fourier transform $\hat{k}_{j,t}$, which is a function on the set $D_M \times i\mathbb{R}^* \simeq \mathbb{Z} \times i\mathbb{R}$, to be:

$$\hat{k}_{j,t}(\sigma, i\lambda) = \text{Tr}[[\pi_{\sigma,i\lambda}(k_{j,t})]].$$

Then Fried has shown the following result ([2] Lemma 1).

**Fact 7.1.**

$$\hat{k}_{j,t}(\sigma, i\lambda) = \begin{cases} e^{-t\lambda^2} & \text{if } \sigma = \pm 1, j = 1 \\ e^{-t(1+\lambda^2)} & \text{if } \sigma = 0, j = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

This fact and the formula of Park ([12], p.12) will imply

$$T(k_0,t,0) = \frac{e^{-t}}{4\pi^2} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda$$

and

$$T(k_1,t,0) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda + \frac{e^{-t}}{4\pi^2} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda.$$  \hspace{1cm} (8)

We put

$$U_0(t) = U_0(t), \quad U_1(t) = U_1(t) - U_0(t).$$

The following proposition will follow from Corollary 7.1.

**Proposition 7.1.** Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then we have

$$U_0(t) = \frac{|L_\infty|C_\rho}{2\pi^2} \cdot e^{-t} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda,$$

and

$$U_1(t) = \frac{|L_\infty|C_\rho}{\pi^2} \cdot \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda.$$

We will compute the derivative of the Laplace transform of $e^tU_0$ and $U_1$.

For a nonzero real number $r$, let $\text{sgn}(r)$ be its sign. The following lemma will be proved by the Cauchy’s integral formula.
Lemma 7.1. Let $z$ be a positive number and $\alpha$ a complex number.

1. Suppose $\text{Im} \alpha \neq 0$. Then we have

$$\int_{-\infty}^{\infty} \frac{d\lambda}{(\lambda^2 + z^2)(\lambda - \alpha)} = \frac{\pi i}{z(\text{sgn}(\text{Im} \alpha) i \alpha)}.$$

2.

$$\int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 + z^2} = \frac{\pi}{z}.$$

For $z > 0$, using Lemma 7.1, we have

$$L'(e^tU_0)(z) = \frac{|L_{\infty}|C_{\rho}}{2\pi^2} \cdot 2z \int_0^\infty dte^{-t^2} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda$$

$$= \frac{|L_{\infty}|C_{\rho}}{\pi^2} \cdot z \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 + z^2}$$

$$= \frac{|L_{\infty}|C_{\rho}}{\pi}.$$

The same computation implies

$$L'(U_1)(z) = \frac{2|L_{\infty}|C_{\rho}}{\pi}.$$

Thus we have proved

Proposition 7.2. Suppose $\rho|_{\Gamma_{\infty}}$ is nontrivial. Then we have

$$L'(e^tU_0)(z - 1) - L'(U_1)(z) + L'(e^tU_0)(z + 1) = 0.$$

In the following, we will compute $2|L_{\infty}|R_{\rho}T'(k_{j,t}, 0)$. Since these terms appear only if $\rho|_{\Gamma_{\infty}}$ is trivial, we will assume its triviality. Under the assumption, $R_{\rho}$ is explicitly computed as

$$R_{\rho} = \frac{\pi}{|L_{\infty}|},$$

and we have

$$2|L_{\infty}|R_{\rho}T'(k_{j,t}, 0) = 2\pi T'(k_{j,t}, 0).$$

Let

$$SL_2(\mathbb{C}) \xrightarrow{\pi} G = PSL_2(\mathbb{C})$$

be the universal covering and $\tilde{K}$ (resp. $\tilde{M}$) the inverse image of $K$ (resp. $M$), which is isomorphic to $SU_2(\mathbb{C})$ (resp. $U(1)$). $SL_2(\mathbb{C})$ acts on the polynomial ring $\mathbb{C}[X, Y]$ by the linear transformation, which is decomposed into irreducible representations:

$$\mathbb{C}[X, Y] = \bigoplus_{\mu} \xi_{\mu}.$$

Here $\mu$ runs through nonnegative half integers and $\xi_{\mu}$ is the space of homogeneous polynomial of degree $2\mu$. In particular $\tau_1 \otimes \mathbb{C}$ is isomorphic to $\xi_1$ as
\(\tilde{K}\)-modules. Let \(D_{\tilde{K}}\) (resp. \(D_{\tilde{M}}\)) be the set of irreducible representations of \(\tilde{K}\) (resp. \(\tilde{M}\)), which will be parametrized by the set of nonnegative half integers \(\frac{1}{2}\mathbb{Z}_{\geq 0}\) (resp. half integers \(\frac{1}{2}\mathbb{Z}\)). Then the map

\[D_{\tilde{K}} \xrightarrow{\pi^\dagger} D_{\tilde{K}}\]

and

\[D_{\tilde{M}} \xrightarrow{\pi^\dagger} D_{\tilde{M}}\]

may be identified with the natural inclusions:

\[\mathbb{Z}_{\geq 0} \hookrightarrow \frac{1}{2}\mathbb{Z}_{\geq 0},\]

and

\[\mathbb{Z} \hookrightarrow \frac{1}{2}\mathbb{Z},\]

respectively.

Now here is the Hoffmann’s formula. (See the equation (8) and (49) in [4]. See also [12, §4]):

1. 
\[T'(k_0, t, 0) = \frac{1}{\pi} \{J^{(1)}_0(t) + \frac{1}{2}\pi \text{ p.v.} \int_{-\infty}^{\infty} J^{(2)}(0, i\lambda)(t) d\lambda\},\]

2. 
\[T'(k_1, t, 0) = \frac{1}{\pi} \sum_{\sigma = -1} J^{(1)}_{\sigma}(t) + \frac{1}{2}\pi \text{ p.v.} \int_{-\infty}^{\infty} J^{(2)}(\sigma, i\lambda)(t) d\lambda,\]

where p.v. means Cauchy’s principal value. We will explain each term.

1. \(J^{(1)}_{\sigma}(t)\)

Let \(\psi\) be the di-gamma function:

\[\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.\]

It is known that it has a logarithmic growth as \(|\lambda| \to \infty:\)

\[|\psi(i\lambda)| \sim \log |\lambda|.\]

We define a function \(\Omega(\sigma, -i\lambda)\) for \(\lambda \in \mathbb{R}\) as

\[\Omega(0, i\lambda) = 2\psi(1) - (\psi(1 + i\lambda) + \psi(1 - i\lambda))\]

and

\[\Omega(-1, i\lambda) = \Omega(1, i\lambda) = 2\psi(1) - \frac{1}{2}(\psi(-i\lambda) + \psi(i\lambda) + \psi(2 - i\lambda) + \psi(2 + i\lambda)).\]
Corollary of p.96 in [4] and the computation of §4 of [12] shows

\[ J^{(1)}_{\sigma}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega(\sigma, -i\lambda) \hat{k}_{j,t}(\sigma, i\lambda) d\lambda. \]

Note that the integral is absolutely convergent because \( \psi(z) + \psi(-z) \) is regular at \( z = 0 \). We will compute \( L'(e^tJ^{(1)}_{0})(z) \) and \( L'(J^{(1)}_{\sigma})(z) \) for \( \sigma = \pm 1 \).

Lemma 7.2. Let \( z \) and \( \alpha \) be positive numbers. Then we have the following identities.

(a) \[ L'(\int_{-\infty}^{\infty} \psi(\alpha + i\lambda)e^{-t\lambda^2} d\lambda) = L'(\int_{-\infty}^{\infty} \psi(\alpha - i\lambda)e^{-t\lambda^2} d\lambda) = 2\pi \psi(z + \alpha). \]

(b) \[ L'(\int_{-\infty}^{\infty} \frac{e^{-t\lambda^2}}{\alpha + i\lambda} d\lambda) = L'(\int_{-\infty}^{\infty} \frac{e^{-t\lambda^2}}{\alpha - i\lambda} d\lambda) = \frac{2\pi}{z + \alpha}. \]

(c) \[ L'(\int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda) = 2\pi. \]

Proof. Changing the order of integration, we have

\[ L'(\int_{-\infty}^{\infty} \psi(\alpha + i\lambda)e^{-t\lambda^2} d\lambda) = 2z \int_{0}^{\infty} dt e^{-tz^2} \int_{-\infty}^{\infty} \psi(\alpha + i\lambda)e^{-t\lambda^2} d\lambda = 2z \int_{-\infty}^{\infty} \frac{\psi(\alpha + i\lambda)}{\lambda^2 + z^2} d\lambda. \]

Note that \( \psi(\alpha + i\lambda) \) is regular on \( \text{Im} \lambda < \alpha \) and that \( |\psi(\alpha + i\lambda)| \sim \log |\lambda| \) as \( |\lambda| \to \infty \) (cf. [12], p.22). Making a curvilinear integral along a contour which first goes along the real axis from \(-\infty\) to \(\infty\) then turns around the under semi-circle by the clock-wise direction, we will obtain

\[ \int_{-\infty}^{\infty} \frac{\psi(\alpha + i\lambda)}{\lambda^2 + z^2} d\lambda = -2\pi i \cdot \text{Res}_{\lambda=-i\alpha} \frac{\psi(\alpha + i\lambda)}{\lambda^2 + z^2} d\lambda = \frac{\pi}{z} \psi(z + \alpha), \]

which implies the first identity. The remaining identities can be proved by the same way.

\[ \square \]
Using Lemma 7.2, we will obtain

\[ L'(e^t J_0^{(1)})(z) = 2(\psi(1) - \psi(z + 1)). \]

Also, for a positive number \( w \), Lemma 7.2 shows

\[ L'(\frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-tz^2} \{\psi(w + i\lambda) + \psi(w - i\lambda)\}d\lambda)(z) = \psi(z + w). \]

Making \( w \to 0 \), we obtain

\[ L'(\frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-tz^2} \{\psi(i\lambda) + \psi(-i\lambda)\}d\lambda)(z) = \psi(z), \]

which implies

\[ L'(J_{-1}^{(1)})(z) = L'(J_1^{(1)})(z) = 2\psi(1) - \psi(z) - \psi(z + 2). \]

Thus we have proved the following proposition.

**Proposition 7.3.** (a)

\[ L'(e^t J_0^{(1)})(z) = 2(\psi(1) - \psi(z + 1)). \]

(b)

\[ L'(J_1^{(1)})(z) = L'(J_1^{(1)})(z) = 2\psi(1) - \psi(z) - \psi(z + 2). \]

In particular

\[ L'(e^t J_0^{(1)})(z - 1) + L'(e^t J_0^{(1)})(z + 1) - \{L'(J_{-1}^{(1)})(z) + L'(J_1^{(1)})(z)\} = 0. \]

2. \( J^{(2)}(\sigma, i\lambda)(t) \)

Using the logarithmic derivative of the Harish-Chandra’s C-function \( C(\sigma, \nu) \), Park computed the term as (see §4 of [12]):

\[ J^{(2)}(\sigma, i\lambda)(t) = -k_{J_t}(\sigma, i\lambda) \cdot \frac{d}{d\nu} \log C(\sigma, \nu)|_{\nu = i\lambda}. \]

[7] Theorem 8.2 and the functional equation:

\[ \Gamma(z + 1) = z \cdot \Gamma(z) \]

shows

\[ C(\sigma, \nu) = \begin{cases} \frac{1}{\nu} & \text{if } \sigma = 0 \\ \nu & \text{if } \sigma = \pm 1. \end{cases} \]

Therefore \( J^{(2)}(\sigma, i\lambda)(t) \) is computed as

\[ J^{(2)}(\sigma, i\lambda)(t) = \begin{cases} \frac{e^{-t(1+\lambda^2)}}{-i\lambda} & \text{if } \sigma = 0 \\ -\frac{e^{-t(1+\lambda^2)}}{i\lambda} & \text{if } \sigma = \pm 1. \end{cases} \]
Therefore
\[
\frac{1}{2\pi} \text{p.v.} \int_{-\infty}^{\infty} e^{tJ^{(2)}(0,i\lambda)}(t) d\lambda = \frac{1}{2\pi} \text{p.v.} \int_{-\infty}^{\infty} J^{(2)}(-1,i\lambda)(t) d\lambda = \frac{1}{2\pi} \text{p.v.} \int_{-\infty}^{\infty} J^{(2)}(1,i\lambda)(t) d\lambda = 0.
\]

Together with Corollary 7.1, Proposition 7.2 and Proposition 7.3, these computation shows

**Proposition 7.4.** Suppose \( \rho|_{\Gamma_\infty} \) is trivial. Then
\[
L'(e^tU_0)(z-1) - L'(U_1)(z) + L'(e^tU_0)(z+1) = 0.
\]

### 8 Scattering terms

In this section we assume that the restriction of \( \rho \) to \( \Gamma_\infty \) is trivial. We first recall the basic facts of the scattering matrix. (See [5] and [13])

Let \( E_\infty(\varphi_\sigma, \rho, s) \) be the constant term of the Fourier expansion of \( E(\varphi_\sigma, \rho, s) \) at the cusp:
\[
E_\infty(\varphi_\sigma, \rho, s)(x) = \frac{1}{|L_\infty|} \int_{L_\infty} E(\varphi_\sigma, \rho, s)(xn) \, dn.
\]

Then there is an intertwining operator called the scattering matrix:
\[
\mathcal{H}(\sigma, s) \xrightarrow{C_{\rho,\sigma}(s)} \mathcal{H}(\sigma, s),
\]
which satisfies
\[
E_\infty(\varphi_\sigma, \rho, s)(x) = r(x)^{1+s} \varphi_\sigma(x) + r(x)^{1-s}(C_{\rho,\sigma}(s)\varphi_\sigma)(x).
\]

In fact \( C_{\rho,\sigma}(s) \) is a scalar if \( \sigma = 0 \) and is a \( 2 \times 2 \)-matrix if \( \sigma = \pm 1 \). In the latter case, since \( \mathcal{H}_{1,s} \) is not isomorphic to \( \mathcal{H}_{-1,s} \), we know (see [12] §3):
\[
C_{\rho,\sigma}(s) = \begin{pmatrix} 0 & C_+(\sigma, s) \\ C_-(\sigma, s) & 0 \end{pmatrix}.
\]

The argument of [13] §3.7 (see also [5] §6 and [13] §1, §2) shows the scattering matrix satisfies the following properties:

1. \( C_{\rho,\sigma}(0) = I \).

2. \( C_{\rho,\sigma}(s) \) is absolutely convergent for \( \text{Re } s > 1 \) and is continued on the whole plane as a meromorphic function of order four.
It satisfies the functional equation:

$$C_{\rho,\sigma}(-s) \cdot C_{\rho,\sigma}(s) = I,$$

and its transpose conjugate $C_{\rho,\sigma}(s)^\ast$ is equal to $C_{\rho,\sigma}(s)$.

In particular $C_{\rho,\sigma}(i\lambda)$ is a unitary matrix for $\lambda \in \mathbb{R}$.

**The scattering terms.**

Let us put

$$\Psi_{\rho,\sigma}(s) = \text{Tr}[C_{\rho,\sigma}(-s) \frac{d}{ds} C_{\rho,\sigma}(s)].$$

Note that by definition we have

$$\Psi_{\rho,\sigma}(s) = \Psi_{\rho,-\sigma}(s), \quad \sigma = -1, 0, 1. \quad (11)$$

Using Fact 7.1 and the formula in p.16 of [13] together with (11), we will see

$$S_0(t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \Psi_{\rho,0}(i\lambda)e^{-t(1+\lambda^2)}d\lambda,$$

and

$$S_1(t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \Psi_{\rho,0}(i\lambda)e^{-t(1+\lambda^2)}d\lambda + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_{\rho,1}(i\lambda)e^{-t\lambda^2}d\lambda.$$

We put

$$S_0(t) = S_0(t),$$

$$S_1(t) = S_1(t) - S_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_{\rho,1}(i\lambda)e^{-t\lambda^2}d\lambda.$$

Following the argument of §6.4 of [5] we will obtain

$$\Psi_{\rho,0}(s) = c_{\rho,0} - \sum_k \left( \frac{1}{s - \alpha_k} - \frac{1}{s + \alpha_k} \right),$$

where $c_{\rho,0}$ is a constant and $\{\alpha_k\}_k$ is the set of poles of $C_{\rho,0}(z)$. Let $\{\beta_l\}_l$ be the set of poles of $\det C_{\rho,1}(s)$. Then the same argument as p.33 of [12] also implies

$$\Psi_{\rho,1}(s) = c_{\rho,1} - \sum_k \left( \frac{1}{s - \beta_l} - \frac{1}{s + \beta_l} \right),$$

where $c_{\rho,1}$ is a constant. Since both $C_{\rho,0}(z)$ and $\det C_{\rho,1}(s)$ are regular on the imaginary axis, real part of any $\alpha_k$ and $\beta_l$ are nonzero. Thus

$$\Psi_{\rho,0}(i\lambda)d\lambda = \{c_{\rho,0} - \frac{1}{i} \sum_k \left( \frac{1}{\lambda + i\alpha_k} - \frac{1}{\lambda - i\alpha_k} \right) \}d\lambda$$

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and
\[ \Psi_{\rho,1}(i\lambda)d\lambda = \{c_{\rho,1} - \frac{1}{i} \sum_k \left( \frac{1}{\lambda + i\beta_l} - \frac{1}{\lambda - i\beta_l} \right) \}d\lambda \]
are 1-forms regular on the real axis.

For \( z > 0 \) and a complex number \( \alpha \) whose imaginary part nonzero, we obtain by Lemma 7.1
\[
2z \int_0^\infty dt e^{-tz^2} \int_{-\infty}^\infty e^{-t\lambda^2} d\lambda = 2z \int_0^\infty d\lambda \frac{1}{\lambda - \alpha} \int_0^\infty e^{-t(\lambda^2 + z^2)} dt
\]
\[
= 2z \int_{-\infty}^\infty \frac{d\lambda}{(\lambda^2 + z^2)(\lambda - \alpha)}
\]
\[
= \frac{2\pi i}{z - \text{sgn(Im}\alpha)\alpha},
\]
which implies
\[
L'(e^tS_0)(z) = \frac{1}{2} \{c_{\rho,0} - \sum_k \left( \frac{1}{z + \text{sgn(Re}\alpha_k)\alpha_k} - \frac{1}{z + \text{sgn(Re}\alpha_k)\alpha_k} \right) \}
\]
\[
\]
and
\[
L'(S_1)(z) = c_{\rho,1} - \sum_{\text{Im } \alpha_k \neq 0} \left( \frac{1}{z + \text{sgn(Re}\beta_l)\beta_l} - \frac{1}{z + \text{sgn(Re}\beta_l)\beta_l} \right)
\]
\[
\]
Thus we have proved the following proposition.

**Proposition 8.1.** \( L'(e^tS_0)(z) \) (resp. \( L'(S_1)(z) \)) is meromorphically continued on the whole plane with only simple poles whose residues are half integers (resp. integers). Moreover it is regular on the real axis.

**The threshold terms**

If \( \sigma = \pm 1 \), we know the trace of \( C_{\rho,\sigma}(s) \) vanishes by (10) and the formula of p.16 of [13] implies
\[
T_0(t) = T_1(t) = -\frac{1}{4} e^{-t}.
\]
Thus for \( z > 0 \) we have
\[
L'(e^tT_0)(z) = L'(e^tT_1)(z)
\]
\[
= -\frac{1}{4} \cdot 2z \int_0^\infty e^{-tz^2} dt
\]
\[
= -\frac{1}{2z}.
\]
Now if we put

\[ L'_{0,sc}(z) = L'(e^t S_0)(z - 1) + L'(e^t T_0)(z - 1), \]

and

\[ L'_{1,sc}(z) = L'(S_1)(z), \]

we have proved the following proposition.

**Proposition 8.2.** \( L'_{0,sc}(z) \) and \( L'_{1,sc}(z) \) are continued on the whole plane as meromorphic functions with only simple poles whose residues are half integers and integers, respectively. Moreover we have

\[ \text{Res}_{z=0} L'_{0,sc}(z) = \text{Res}_{z=2} L'_{0,sc}(z) = \text{Res}_{z=0} L'_{1,sc}(z) = 0. \]

### 9 A proof the theorem

Now we are ready to prove Theorem 1.1. Up to the previous section we have seen the logarithmic derivative of the Ruelle L-function \( R_\rho(z) \) is continued on the whole plane as a meromorphic function which has only simple poles whose residues are at most half integers. Thus \( R_\rho(z)^2 \) is meromorphically continued on the whole plane. But if \( \rho|_{\Gamma_\infty} \) is nontrivial, since the contribution of the scattering terms does not exist, we find all the residues are integers. Thus \( R_\rho(z) \) is meromorphically continued on \( \mathbb{C} \) itself.

By the Selberg trace formula and (7) we will find the set of poles of \( \frac{d}{dz} \log R_\rho(z) \) is a union of two subsets, \( P_{\text{reg}} \) and \( P_{\text{sing}} \). Here \( P_{\text{reg}} \) (resp. \( P_{\text{sing}} \)) is the set of eigenvalues of \( \Delta \) on \( L^2(X, \Omega \cdot (\rho)) \) (resp. the poles of the derivative of the Laplace transforms of the unipotent orbital integrals an the scattering terms). Note that the derivative of the Laplace transform of the identity orbital integrals does not yield any pole. Proposition 7.4 and Proposition 8.2 implies 0 is not contained in \( P_{\text{sing}} \). Now we obtain

\[ \text{ord}_{z=0} R_\rho(z) = 2(2\beta_0(\rho)(2) - \beta_1(\rho)(2)). \]

(12)

by Lemma 3.1, Lemma 3.2 and the Selberg trace formula. Together with Lemma 2.1 and Lemma 2.2, (12) shows Theorem 1.1.

\[ \Box \]

Comparing the Riemann’s zeta function, we may consider \( P_{\text{sing}} \) or \( P_{\text{reg}} \) corresponds to the set of trivial or of essential zeros, respectively. As we have seen in §3, except for finitely many elements, it is contained in

\[ \{ z \in \mathbb{C} \mid \text{Re} z \in \{-1, 0, 1\} \}. \]

Thus we may say the Ruelle L-function satisfies the Riemann hypothesis. Note that if \( \rho|_{\Gamma_\infty} \) is nontrivial, Proposition 7.2 shows \( P_{\text{sing}} \) is empty.
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