Notes on A Superlinear Elliptic Problem

Haoyu Li

Center of Applied Mathematics, Tianjin University
Tianjin 300072, China
E-mail: hyli1994@hotmail.com

Abstract. Based on the theory of invariant sets of descending flow, we give a new proof of the existence of three nontrivial solutions and some remarks on it.

Keywords: Positive solution, sign-changing solution, superlinear elliptic problem.

2010 MSC: 34C23, 34K18, 35B32, 37G99.

Contents

1 Introduction 1

2 Preliminaries 3
2.1 Notations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
2.2 Definitions and lemmas . . . . . . . . . . . . . . . . . . . . . . . 3

3 Proof of Theorem 1.1 7
3.1 Verification of (3.1) . . . . . . . . . . . . . . . . . . . . . . . . . . 8
3.2 Verification of (3.3) . . . . . . . . . . . . . . . . . . . . . . . . . 8

4 Remarks 11
4.1 More constructions of the sign-changing minimax values . . . . . 11
4.2 A remark on the linking sets . . . . . . . . . . . . . . . . . . . 14

1 Introduction

The existence and multiplicity of solutions for superlinear elliptic problem are classical problems in the field of variational methods. It is well-known that under some general assumptions, we can obtain a positive and a negative solution. If in addition, let the nonlinear term be odd, we will have infinitely many solutions. These are classical results which can be found in [1] and [9]. In 1991, Wang discovered a third solution in [11] via the local linking and Morse theory. In 2001, Liu and Sun proved the existence of four solutions in [6] under a general framework of invariant sets of descending flow. One positive solution, one negative
solution, one sign-changing solution and one possible trivial solution. The effort on refining this result is never ceased. A classical improved version is \cite{7}, which can obtain more solutions with prescribed sign with some additional assumptions. In a recent work, \cite{4}, a result for four nontrivial solutions is given with a finer analysis.

In this note we give a new proof of the classical theorem on the existence of three solutions and some remarks on it. Let us make the following assumptions:

\((f1)\) The function \(f \in C(\mathbb{R})\) satisfies

\[
\begin{align*}
    f(u) &= o(|u|), \quad \text{as } u \to 0; \\
    f(u) &= O(|u|^{p-1}), \quad \text{as } u \to \infty,
\end{align*}
\]

where \(p \in (2, 2^*)\);

\((f2)\) There is a number \(\mu > 2\) with \(0 < \mu F(u) \leq uf(u)\), where

\[
F(u) = \int_0^u f(s)ds.
\]

Consider the superlinear elliptic problem

\[
\begin{cases}
    -\Delta u = f(u), & \text{in } \Omega, \\
    u = 0, & \text{on } \partial\Omega.
\end{cases}
\]  

(1.1)

where \(\Omega \subset \mathbb{R}^N\) is a domain whose boundary regular enough for the Sobolev embeddings and \(N \geq 3\).

**Theorem 1.1.** Suppose that the assumptions \((f1)\) and \((f2)\) are satisfied, the equation (1.1) possesses at least three solutions, a positive solution, a negative solution and a sign-changing solution.

**Remark 1.2.** Although this is a classical theorem with a large number of proofs, to the best of the author’s knowledge, the proof we give in this note is new. We refer the \cite{5}, \cite{6}, and \cite{11} for a historical reference. For more results on superlinear elliptic problems, readers can find them in \cite{3}.

The structure of this paper is as follows: In the second section, we will give some notations, definitions and lemmas. And we will prove the existence of critical points in the third section. In the last section, we give some remarks on the more constructions of the sign-changing minimax values and the linking sets.
2 Preliminaries

2.1 Notations

- An open \( \delta \)-neighbourhood of a set \( A \) is denoted by \( A_\delta \);
- The complementary set of the set \( A \) is denoted by \( A^c \);
- For a subset \( A \) of a linear space, let \( -A := \{-x | x \in A\} \);
- \( E := H^1_0(\Omega) \) denotes the usual Sobolev space. By \( \| \cdot \| \), we denote the Sobolev norm and \( | \cdot |_p \) the \( L^p \) norm. The closure in this note is always taken with respect to the Sobolev norm;
- We denote the open ball in the Sobolev space \( H^1_0(\Omega) \) with radii \( \rho \) and center at \( 0 \in H^1_0(\Omega) \) by \( B_\rho \), by \( \partial B_\rho \) its boundary;
- \( P^\pm = \{ u \in H^1_0(\Omega) | u \geq (\leq) 0 \text{ a.e. in } \Omega \} \) is positive (negative) cone in the Sobolev space \( H^1_0(\Omega) \);
- \( W_\varepsilon := P^+_\varepsilon \cup P^-_\varepsilon \);
- \( I^a = \{ u \in E | I(u) \leq a \} \) and \( I^a_b := I^a \setminus I^b \) are level sets;
- \( K = \{ u \in E | I'(u) = 0 \} \), \( K_a = \{ u \in E | I'(u) = 0 \text{ and } I(u) = a \} \) and \( K^a_b := K \cap I^a_b \) are the sets of critical points of \( I \) in \( E \).

2.2 Definitions and lemmas

The following definitions and lemmas are standard for modern variational methods for sign-changing critical points. We still state it here for the completeness. We refer \([5],[6] \) and \([8] \) for a general theoretic construction.

**Definition 2.1.** Let \( X \) be a Banach space and \( I \in C^1(X, \mathbb{R}) \). Then \( P \subset X \) is called an admissible invariant set with respect to \( I \) at level \( c \), if \( K_c \setminus P = \emptyset \) implies the existence of a positive constant \( \varepsilon_0 \) and a function \( \eta \in C(E, E) \) with

1. \( \eta(P) \subset P \),
2. \( \eta|_{I^c-\varepsilon} = id_{I^c-\varepsilon} \),
3. \( \eta(I^{c+\varepsilon} \setminus P) \subset I^{c-\varepsilon} \),

for any \( \varepsilon \in (0, \varepsilon_0) \).

**Remark 2.2.** There is a symmetric version of definition 2.1. Readers can find it in \([5],[6] \) and \([8] \).

The energy of the Problem (1.1) is

\[ I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u), \]

for \( u \in H^1_0(\Omega) = E \). It should be noted that the \( I \) satisfies the \((PS)\) condition. This is a classical result. We will claim Lemma 2.3 and refer \([11] \) for its proof.
Lemma 2.3. I satisfies the \((PS)\) condition.

In the following part, we shall state some basic properties of \(P_\pm\), \(P_{\pm}^+\) and \(W_\varepsilon\). Consider the operator \(A\) defined by

\[ v = (-\Delta)^{-1}(f(u)) =: A(u). \]

Then the operator \(A : E \to E\) is well-defined, continuous and compact. One can find an analogue in [8, Section 3.1].

Lemma 2.4. (1) \(I'(u)(u - A(u)) = \|u - A(u)\|^2\),
(2) \(\|I'(u)\| = \|u - A(u)\|\).

Proof.
(1) Since \(u, A(u) \in H_0^1(\Omega)\),

\[
I'(u)(u - A(u)) = \int_\Omega (-\Delta)(u - A(u))(u - A(u))
= \int_\Omega |\nabla(u - A(u))|^2
= \|u - A(u)\|^2.
\]

(2) For any \(\varphi \in H_0^1(\Omega)\),

\[
I'(u)\varphi = (u - A(u), \varphi)_H
\leq \|u - A(u)\| \cdot \|\varphi\|.
\]

Hence \(\|I'(u)\| \leq \|u - A(u)\|\). Similarly,

\[
\|I'(u)\| \geq \|u - A(u)\|.
\]

Since every solution can be considered as a fixed point of the mapping (and hence of the flow), we have the following lemma.

Lemma 2.5. There is an \(\varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0)\), \(A(P_{\pm}^\varepsilon) \subset P_{\pm}^\varepsilon\) and every nontrivial solution in \(P_{\pm}^\varepsilon\) is positive (negative) i.e. the solution belongs to \(P_{\pm}^\varepsilon\).

This lemma is similar to [8, Lemma 3.4]. For the completeness, we give the proof here.

4
Proof. We only proof the theorem for \( P^- \). By (g1) and (g2), for any \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that
\[
|f(u)| \leq \delta |u| + C_\delta |u|^{q-1}.
\]
Let \( u \in E \) and \( v = A(u) \). Since for any \( r \in [2, 2^*] \), there is a constant \( m_r > 0 \) such that
\[
|u^+|_r = \inf_{w \in P^\pm} |u - w|_r \leq m_r \inf_{w \in P^\pm} \|u - w\| = m_r d(u, P^\pm).
\]
Combining the fact that \( d(v, P^-) \leq \|v^+\| \) and (g2),
\[
d(v, P^-)\|v^+\| \leq \|v^+\|^2 = (v, v^+)_H
= \int_\Omega f(u)v^+ \leq \delta \int_\Omega |u^+| + C_\delta \int_\Omega |u^+|^{p-1}|v^+|
\leq \delta |u^+|_2|v^+|_2 + C_\delta |u_2^+|^{p-1}|v^+|_p
\leq C(\delta d(u, P^-) + C_\delta d(u, P^-)^{p-1})\|v^+\|.
\]
It follows that
\[
d(A(u), P^-) \leq C(\delta d(u, P^-) + C_\delta d(u, P^-)^{p-1}).
\]
Choosing \( \delta \) small enough, there exists a \( \varepsilon_0 > 0 \) such \( \varepsilon \in (0, \varepsilon_0) \),
\[
d(A(u), P^-) \leq \frac{1}{2} d(u, P^-)
\]
for any \( u \in P^- \), which implies that \( A(P^-) \subset P^- \). Then if there is any point \( u \in P^- \) such that \( A(u) = u \), we will have \( u \in P^- \). Using the maximum principle, if \( u \neq \theta \), then \( u < 0 \) a.e. in \( \Omega \).

Lemma 2.6. There is a Lipschitz mapping \( B : E \setminus K_\varepsilon \rightarrow E \) such that
\begin{enumerate}
\item \( B(P^-) \subset P^- \) for \( 0 < \varepsilon < \varepsilon_0 \);
\item \( \frac{1}{2}\|u - B(u)\| \leq \|u - A(u)\| \leq 2\|u - B(u)\| \) for any \( u \in H_0^1(\Omega) \);
\item \( I'(u)(u - B(u)) \geq \frac{1}{2}\|u - A(u)\|^2 \) for any \( u \in E \setminus K \);
\end{enumerate}
This is [2, Lemma 2.1]. The main idea of this lemma is to replace the gradient field by its pseudo-gradient field. We refer [2, Lemma 2.1] for the proof.

Lemma 2.7. If for some \( \varepsilon > 0 \) small enough, \( K_\varepsilon \setminus W_\varepsilon = \phi \), then there is a \( \varepsilon_0 > 0 \), for \( 0 < \varepsilon < \varepsilon' < \varepsilon_0 \), there is a continuous mapping \( \eta : [0, 1] \times H_0^1(\Omega) \rightarrow H_0^1(\Omega) \) such that
\( (1) \ \eta(0, u) = u \) for any \( u \in H^1_0(\Omega) \).
\( (2) \ \eta(t, u) = u \) for all \( t \in [0, 1] \), \( u \notin I^{c+\varepsilon}_{c-\varepsilon} \).
\( (3) \ \eta(1, I^{c+\varepsilon}\setminus W_\varepsilon) \subset I^{c-\varepsilon} \).
\( (4) \ \eta(t, \overline{P}\pm_e^\varepsilon) \subset \overline{P}\pm_e^\varepsilon \) for all \( t \in [0, 1] \).

**Proof.** This proof is similar to the one of [8, Lemma 3.6]. But we give it here for the completeness. We denote \( W_\varepsilon \) by \( W \) for short.

Since \( K_c \setminus W = \emptyset \), there is a \( \delta > 0 \) such that \( (K_c)_{\delta} \subset W \). Since \( I \) satisfies the (PS)-condition, there exist \( \varepsilon_0, \alpha > 0 \) such that

\[ \|I'(u)\| \geq \alpha, \]

for \( u \in I^{-1}([c-\varepsilon_0, c+\varepsilon_0]) \setminus (K_c)\frac{\delta}{2} \). By Lemma 2.4 and Lemma 2.6 there is a constant \( \beta > 0 \) such that

\[ \|u - B(u)\| \geq \alpha, \]

for \( u \in I^{-1}([c-\varepsilon_0, c+\varepsilon_0]) \setminus (K_c)\frac{\delta}{2} \). Without loss of generality, assume that \( \varepsilon_0 \leq \frac{\beta}{32} \). Define

\[ V(u) := \frac{u - B(u)}{\|u - B(u)\|} \]

for \( u \in E_0 = E \setminus K \) and a Lipschitz function \( g : E \to [0, 1] \) such that

\[ g(u) = \begin{cases} 0, & \text{if } u \notin I^{c+\varepsilon}_{c-\varepsilon} \text{ or } u \in (K_c)\frac{\delta}{2}, \\ 1, & \text{if } u \in I^{c+\varepsilon}_{c-\varepsilon} \text{ or } u \notin (K_c)\frac{\delta}{2}. \end{cases} \quad (2.1) \]

Consider the Cauchy problem

\[ \begin{cases} \frac{d\tau}{dt} = -g(\tau)V(\tau), \\ \tau(0, u) = u. \end{cases} \quad (2.2) \]

For any \( u \in E \), the problem (2.2) admits a unique solution \( \tau(\cdot, u) \in C(\mathbb{R}^+, E) \). Define \( \eta(t, u) = \tau(\frac{16\varepsilon}{\beta}t, u) \). And hence (1) and (2) are proved.

Let \( u \in I^{c+\varepsilon}\setminus W \). \( \tau(t, u) \) is decreasing for \( t \geq 0 \). If there is a \( t_0 \in [0, \frac{16\varepsilon}{\beta}] \) with \( \tau(t_0, u) < c - \varepsilon \) then \( \tau(1, u) < c - \varepsilon \). Otherwise, for any \( t \in \left[0, \frac{16\varepsilon}{\beta}\right] \), \( \tau(t, u) \geq c - \varepsilon \). Then, \( \tau(t, u) \in \tau^{-1}[c - \varepsilon, c + \varepsilon] \) for \( t \in \left[0, \frac{16\varepsilon}{\beta}\right] \). We claim that for any \( t \in \left[0, \frac{16\varepsilon}{\beta}\right] \), \( \tau(t, u) \notin (K_c)\frac{\delta}{2} \). If \( \tau(t_1, u) \in (K_c)\frac{\delta}{2} \) for some \( t_1 \in \left[0, \frac{16\varepsilon}{\beta}\right] \), then we have

\[ \frac{\delta}{2} \leq \|\tau(t_1, u)\| \leq \int_0^{t_1} \|\tau(s, u)\|ds \leq t_1 \leq \frac{16\varepsilon}{\beta}. \]
since \( u \notin (K_c)_{\frac{\delta}{2}} \). And we have a contradiction with \( \varepsilon < \varepsilon_0 \leq \frac{\delta \varepsilon}{2} \). So \( g(\tau(t,u)) = 1 \) for \( t \in [0, \frac{16\varepsilon}{\beta}] \). Then, by Lemma 2.6

\[
I\left(\tau\left(\frac{16\varepsilon}{\beta}, u\right)\right) = I(u) - \frac{1}{8} \int_0^{\frac{16\varepsilon}{\beta}} \|\tau(s, u) - B\tau(s, u)\| ds
\leq I(u) - \frac{1}{8} \int_0^{\frac{16\varepsilon}{\beta}} \|\tau(s, u) - B\tau(s, u)\| ds
\leq c - \varepsilon.
\]

And one can find the proof of (4) in [6]. \( \square \)

Let \( \eta(u) = \eta(1, u) \). And hence \( P^\pm \) are admissible invariant set with respect to \( I_\theta \) at level \( c \) when \( K_c \setminus P^\pm = \emptyset \). The same with \( W \).

Following the same idea, by modifying the equation (2.1), we can proof the existence of the next mapping.

**Lemma 2.8.** For some \( 0 < \varepsilon_1 < \varepsilon_2 \), if \( K_c \setminus W^\varepsilon_2 = \emptyset \), then there is a \( \varepsilon_0 > 0 \), for \( 0 < \varepsilon < \varepsilon' < \varepsilon_0 \), there is a continuous mapping \( \eta : [0, 1] \times H^1_0(\Omega) \to H^1_0(\Omega) \) such that

1. \( \eta(0, u) = u \) for any \( u \in H^1_0(\Omega) \).
2. \( \eta(t, u) = u \) for all \( t \in [0, 1] \), \( u \in (I^c_{c+\varepsilon})^c \cup W^\varepsilon_1 \).
3. \( \eta(1, I^{c+\varepsilon} \setminus W^\varepsilon_2) \subset I^{c-\varepsilon} \).
4. \( \eta(t, P^\pm_\varepsilon) \subset P^\pm_\varepsilon \) for all \( t \in [0, 1] \).

### 3 Proof of Theorem 1.1

Consider the following three minimax values

\[
c_+ = \inf_{g \in \Gamma_+} \sup_{t \in [0, 1]} I(g(tRe_1)),
\]

\[
c_- = \inf_{g \in \Gamma_-} \sup_{t \in [0, 1]} I(g(-tRe_1))
\]

and

\[
c_s = \inf_{g \in \Gamma_+} \sup_{t \in [0, 1]} I(u),
\]

where for the value \( c_+ \), we denote

\[
\Gamma_+ = \{g \in C(E, E) \mid g(0) = 0, g(Re_1) = Re_1 \text{ and } g(\overline{P^+_\varepsilon}) \subset \overline{P^+_\varepsilon}\},
\]

\[
\Gamma_- = \{g \in C(E, E) \mid g(0) = 0, g(\overline{P^-_\varepsilon}) \subset \overline{P^-_\varepsilon}\}.
\]
$e_1$ is the principle eigenfunction with positive sign and unit Sobolev norm and $R > 0$ is such a number with $I(Re) < 0$ and $I(-Re) < 0$; for the value $c_-$, we denote
\[
\Gamma_\sim = \{ g \in C(E, E) \mid g(0) = 0, g(-Re) = -Re_1 \text{ and } g(P_{\varepsilon}^-) \subset P_{\varepsilon}^- \},
\]
and for the value $c_+$, we denote
\begin{itemize}
  \item $\Gamma_s = \{ g \in C(E, E) \mid g|_{\partial B_2^+} = id \}$;
  \item $B_2^+ = B_R \cap E_2^+$;
  \item $E_2^+ = [0, +\infty)e_2 \oplus \mathbb{R}e_1$, where $e_2$ is a sign-changing function, say, the second eigenvector of $-\Delta$ with zero Dirichlet boundary value;
  \item $\partial_b B_2^+ = \partial B_R \cap E_2^+$, $\partial B_2^+ = \partial_b B_2^+ \cup (B_R \cap \mathbb{R}e_1)$;
  \item $W_{\varepsilon_2} = P_{\varepsilon_2}^+ \cup P_{\varepsilon_2}^-$ for some positive $\varepsilon_2$ small enough.
\end{itemize}

Let $R > 0$ be a number large enough such that $I|_{\partial_b B_2^+} < 0$. We shall prove that
\[
K_{c_+} \cap P^+ \neq \emptyset, \quad (3.1)
\]
\[
K_{c_+} \cap P^- \neq \emptyset, \quad (3.2)
\]
and
\[
K_{c_+} \setminus W_{\varepsilon_2} \neq \emptyset, \quad (3.3)
\]
for some suitable positive $\varepsilon_2$.

### 3.1 Verification of (3.1)
Since (3.1) and (3.2) are similar, we only prove (3.1) here. Due to the Lemma 2.5, we only need to prove that
\[
K_{c_+} \cap P_{\varepsilon}^+ \neq \emptyset
\]
for $\varepsilon > 0$ small enough. Let us assume that $K_{c_+} \cap P_{\varepsilon}^+ = \emptyset$ holds. Using the (PS) condition, we can claim that there is a small positive number $\delta$ such that
\[
K^{c_+ + 2\delta}_{c_+} \cap P_{\varepsilon}^+ = \emptyset.
\]
In the following part, we only need to verify that there is a descending flow $\eta$ as in Lemma 2.7 satisfies that
\[
\eta(I^{c_+ + \delta} \cap P_{\varepsilon}^+) \subset I^{c_+ - \delta} \cap P_{\varepsilon}^+.
\]
And the rest part of the proof is a routine, we refer [9] for details.
On one hand, since
\[ I^{c_+ + \delta} \cap \overline{P^+_\varepsilon} \subset \overline{P^+_\varepsilon}, \]
we have
\[ \eta(I^{c_+ + \delta} \cap \overline{P^+_\varepsilon}) \subset \overline{P^+_\varepsilon}. \]

On the other hand, we claim that the norm of the pseudo-gradient vector field has a positive lower bound on the closed set \( I^{c_+ - \delta} \cap \overline{P^+_\varepsilon} \). Otherwise, there is a sequence \((u_n)_n \subset I^{c_+ + \delta} \cap \overline{P^+_\varepsilon}\) such that \( I'(u_n)_n \to 0 \) and
\[ (I(u_n))_n \subset [c_+ - \delta, c_+ + \delta] \]
as \( n \to \infty \). Using the (PS) condition, there is a \( u^* \in I^{c_+ + 2\delta} \cap \overline{P^+_\varepsilon} \) with \( I'(u^*) = 0 \). Hence we have a contradiction with \( K^{c_+ + 2\delta} \cap \overline{P^+_\varepsilon} = \emptyset \). Therefore the homeomorphism \( \eta \) defined by the pseudo-gradient vector field satisfies
\[ \eta(I^{c_+ + \delta} \cap \overline{P^+_\varepsilon}) \subset I^{c_+ - \delta}, \]
which implies that
\[ \eta(I^{c_+ + \delta} \cap \overline{P^+_\varepsilon}) \subset I^{c_+ - \delta} \cap \overline{P^+_\varepsilon}. \]

### 3.2 Verification of (3.3)

**Step 1** Firstly, we verify a linking-type result, i.e. for any \( g \in \Gamma_s \)
\[ (g(B_2^+) \cap \partial B_\rho) \setminus W_{\varepsilon_2} \neq \emptyset. \]

For any \( g \in \Gamma_s \), there is a odd mapping \( \overline{g} : B_2 \to E \) defined as
\[ \overline{g}(u) = \begin{cases} g(u), & u \in B_2^+; \\ -g(-u), & -u \in B_2^+, \end{cases} \]
where \( B_2 = B_2^+ \cup (-B_2^+) \). Using a genus argument (c.f. [10] Lemma 6.4),
\[ \gamma(\partial B_\rho \cap \overline{g}(B_2)) = 2 \]
and
\[ \gamma(\partial B_\rho \cap W_{\varepsilon_2}) = 1, \]
for \( \varepsilon_2 \) small enough. So \( (\overline{g}(B_2) \cap \partial B_\rho) \setminus W_{\varepsilon_2} = \emptyset \), i.e. \( \partial B_\rho \cap \overline{g}(B_2) \subset \partial B_\rho \cap W_{\varepsilon_2} \), is not possible. Since \( \overline{g} \) is odd, we have
\[ (g(B_2^+) \cap \partial B_\rho) \setminus W_{\varepsilon_2} \neq \emptyset. \]
Step 2  Secondly, we prove that $K_{c_s} \setminus W_{\varepsilon_1} \neq \emptyset$. Since for any $g \in \Gamma_s$, we have $(g(B^+_2) \cap \partial B_\rho) \setminus W_{\varepsilon_1} \neq \emptyset$, using $(f_1)$, we have

\[
\inf_{g \in \Gamma_s} \sup_{g(B^+_2) \setminus W_{\varepsilon_2}} I(u) \geq \inf_{\partial B_\rho} I \geq \inf_{\|u\| = \rho} \frac{1}{2} \|u\|^2 - \int F(u) \geq \inf_{\|u\| = \rho} \frac{1}{4} \|u\|^2 - C\|u\|^p > \alpha > 0
\]

for some suitable positive numbers $\alpha$ and $\rho$. Suppose $K_{c_s} \setminus W_{\varepsilon_2} = \emptyset$, we have a continuous mapping $\eta : [0, 1] \times E \to E$ such that

1. $\eta(0, u) = u$ for any $u \in E$.
2. $\eta(t, u) = u$ for all $t \in [0, 1]$, $u \in (I_{c_s+\varepsilon}^c \cup W_{\varepsilon_1})$.
3. $\eta(1, I_{c_s+\varepsilon} \setminus W_{\varepsilon_2}) \subset I_{c_s-\varepsilon}$.
4. $\eta(t, P_{\varepsilon_1}^+) \subset P_{\varepsilon_1}^+$ for all $t \in [0, 1]$.

due to Lemma 2.8. Let $\varepsilon = \frac{c_s - \alpha}{4} > 0$. Select a $g_0 \in \Gamma_s$ satisfies

\[
\sup_{g_0(B^+_2) \setminus W_{\varepsilon_2}} I < c_s + \varepsilon.
\]

Denote $g_1 = \eta \circ g_0$. Then we have

\[
g_1(B^+_2) \setminus W_{\varepsilon_2} = \eta \circ g_0(B^+_2) \setminus W_{\varepsilon_2} = \eta((g_0(B^+_2) \setminus W_{\varepsilon_2}) \cup W_{\varepsilon_2}) \setminus W_{\varepsilon_2} \subset \eta(g(B^+_2) \setminus W_{\varepsilon_2}) \subset \eta(I_{c_s+\varepsilon}) \subset I_{c_s-\varepsilon}
\]

and

- $g_1|_{\partial h B^+_2} = \eta \circ g_0|_{\partial h B^+_2} = id$ since $\eta = id$ in $I_{c_s+2\varepsilon}^c$;
- $g_1|_{B^+_2 \cap \mathbb{R} e_1} = \eta \circ g_0|_{B^+_2 \cap \mathbb{R} e_1} = id$ since $\eta = id$ in $W_{\varepsilon_1}$ and $B^+_2 \cap \mathbb{R} e_1 \subset W_{\varepsilon_1}$.

Therefore we can find a $g_1 \in \Gamma_s$ with

\[
\sup_{g_1(B^+_2) \setminus W_{\varepsilon_2}} I \leq c_s - \varepsilon.
\]

Then, we have a contradiction with the definition of $c_s$. 

\[\square\]
4 Remarks

4.1 More constructions of the sign-changing minimax values

In this section, we will give two more constructions of the sign-changing minimax values, and the second one is a slight modification of the one in [5]. Since we will not use Lemma 2.8 in this section, which means we do not use different neighbourhoods of the positive and negative cones, we denote $W_\varepsilon$ by $W$ for short.

And always assume that the parameter $\varepsilon$ is small enough. Define the following two minimax values. The first one is:

$$c'_s := \inf_{g \in \Gamma'_s} \sup_{g(B_2^+)} I,$$

where

- $\Gamma'_s = \{ g \in C(B_2^+, E) \mid g|_{\partial_0 B_2^+} = id, g(\partial_1 B_2^+) \subset \overline{P_+^e} \text{ and } g(\partial_2 B_2^+) \subset \overline{P_-^e} \};$
- $B_2^+ = \{ x e_1 + y e_2 \mid y \geq 0 \text{ and } x^2 + y^2 \leq R^2 \};$
- $\partial_0 B_2^+ = \{ x^2 + y^2 = R^2 \} \cap B_2^+, \partial_1 B_2^+ = \{ x \geq 0 \text{ and } y = 0 \} \cap B_2^+ \text{ and } \partial_2 B_2^+ = \{ x \leq 0 \text{ and } y = 0 \} \cap B_2^+$,

and the functions $e_1$, $e_2$ and the number $R$ are the same with Section 3. The second one is:

$$c''_s := \inf_{g \in \Gamma''_s} \sup_{g(D_2^{++})} I,$$

where

- $\Gamma''_s = \{ g \in C(D_2^{++}, E) \mid g|_{\partial_0 D_2^{++}} = id, g(\partial_1 D_2^{++}) \subset \overline{P_+^e} \text{ and } g(\partial_2 D_2^{++}) \subset \overline{P_-^e} \};$
- $D_2^{++} = \{ x \alpha_1 + y \alpha_2 \mid x, y \geq 0 \text{ and } x^2 + y^2 \leq R^2 \};$
- $\partial_0 D_2^{++} = \{ x^2 + y^2 = R^2 \} \cap D_2^{++}, \partial_1 D_2^{++} = \{ x \geq 0 \text{ and } y = 0 \} \cap D_2^{++} \text{ and } \partial_2 D_2^{++} = \{ x = 0 \text{ and } y \geq 0 \} \cap D_2^{++}$;
- $\alpha_1, \alpha_2 \in C_0^\infty(\Omega)$ are two nonzero functions with unit Sobolev norm and $\alpha_1 \geq 0, \alpha_2 \leq 0 \text{ on } \Omega \text{ and } \text{supp}(\alpha_1) \cap \text{supp}(\alpha_2) = \emptyset$.

In the following paragraph, it only need to check the linking results. The proof of the existence of sign-changing critical point is the same as we did in Section 3. We only verify the linking result for the first minimax value since the second one is similar.

**Proposition 4.1.** For any $g \in \Gamma'_s$,

$$(g(B_2^+) \cap \partial B_\rho) \setminus W \neq \emptyset.$$

**Proof.** The proof is divided into the following steps.
Step 1. Modification near the origin. In the whole process, we always assume that \( \epsilon > 0 \) is small enough. Since the mapping \( g \in \Gamma'_\epsilon, g(0) \in P^+_{\epsilon} \cup P^-_{\epsilon} \), then there are \( \rho_1, \rho_2 > 0 \) such that

\[
g(B^+_2 \cap B_{\rho_1}) \subset B_{\rho_2}
\]

and \( \rho_2 \) are small enough, say \( 10\rho_2 < \rho \). Then we can modify \( g \) on the set \( B^+_2 \cap B_{\rho_1} \) by

\[
\overline{g}(u) = \begin{cases} 
g(u) & u \in B^+_2 \setminus B_{\rho_1}; \\
\frac{||u||}{\rho_1} g(\rho_1 \frac{u}{||u||}) & u \in (B^+_2 \cap B_{\rho_1}) \setminus \{0\}; \\
0 & u = 0.
\end{cases}
\]

It is easy to verify that \( \overline{g}(0) = 0 \),

\[
\overline{g}(B^+_2) \cap \partial B_{\rho} = g(B^+_2) \cap \partial B_{\rho},
\]

and

\[
\overline{g}(\partial_1 B^+_2) \subset \overline{P^+_{\epsilon}},
\]

and

\[
\overline{g}(\partial_2 B^+_2) \subset \overline{P^-_{\epsilon}},
\]

since \( \overline{P^+_{\epsilon}} \) and \( \overline{P^-_{\epsilon}} \) are convex.

Step 2. Modification in \( \overline{P^-_{\epsilon}} \). To use the genus argument, we need to modify \( \overline{g} \) by a mapping whose restriction on \( \partial_1 B^+_2 \cup \partial_2 B^+_2 \) is odd. To do this, we define a homeomorphism

\[
\phi : B^+_2 \rightarrow \overline{B^+_2 \setminus C} =: D,
\]

where

\[
C = \{ -R \leq x \leq 0 \text{ and } 0 \leq y \leq \frac{R}{3} |\sin \frac{\pi x}{R}| \} \cap B^+_2,
\]

and

- \( \phi|_{\partial_1 B^+_2} = id; \)
- \( \phi|_{\partial_2 B^+_2} = id; \)
- On \( \partial_2 B^+_2 \), \( \phi(xe_1) = xe_1 + \frac{R}{3} |\sin \frac{\pi x}{R}| e_2 \) and hence \( \phi(\partial_2 B^+_2) = D \cap C \).

Then we define the following mapping \( \overline{\overline{g}} \). For the sake of convenient, we express it into coordinate form.

\[
\overline{\overline{g}}(xe_1 + ye_2) = \begin{cases} 
\overline{g} \circ \phi^{-1}(xe_1 + ye_2), & x, y \in D; \\
- \overline{g}(-xe_1), & u \in B^+_2 \cap \{x \leq 0 \text{ and } y = 0\}; \\
- \left(1 - \frac{3y}{R|\sin \frac{\pi x}{R}|}\right)g(-xe_1) + \\
\frac{3y}{R|\sin \frac{\pi x}{R}|} \overline{g} \circ \phi^{-1}(xe_1 + \frac{R}{3} |\sin \frac{\pi x}{R}| e_2), & u \in C \setminus (D \cup \{y = 0\}).
\end{cases}
\]
It is easy to verify that
- \( g \) is continuous on \( B_2^+ \);
- \( \overline{g}|_{\partial_0 B_2^+} = id \);
- \( g \) is odd on \( B_2^+ \cap \{ y = 0 \} = \partial_1 B_2^+ \cup \partial_2 B_2^+ \).

**Step 3. Odd extension.** Define
\[
\overline{g}(u) = \begin{cases} 
    g(u), & u \in B_2^+; \\
    -g(-u), & -u \in B_2^+.
\end{cases}
\]
Using the genus argument, we have
\[
(\overline{g}(B_2^+ \cap \partial B_\rho) \setminus W \neq \emptyset,
\]
which implies that
\[
(\overline{g}(B_2^+ \cap \partial B_\rho) \setminus W \neq \emptyset.
\]
On one hand, since
- \( \overline{g}(C) \subset \overline{P}_\varepsilon \) since \( \overline{g}(C \cap D) \subset \overline{P}_\varepsilon \), \( \overline{g}(C \cap \{ y = 0 \}) \subset \overline{P}_\varepsilon \) and \( \overline{P}_\varepsilon \) is convex;
- \( \phi \) is a homeomorphism,
we have
\[
\overline{g}(B_2^+) \setminus W = \overline{g}(D \cup C) \setminus W = \overline{g}(D) \setminus W = \overline{g}(\phi^{-1}(D)) \setminus W = \overline{g}(B_2^+) \setminus W.
\]
On the other hand, recall the result in Step 1,
\[
\overline{g}(B_2^+) \cap \partial B_\rho = g(B_2^+) \cap \partial B_\rho,
\]
we have
\[
(g(B_2^+) \cap \partial B_\rho) \setminus W = (g(B_2^+) \setminus W) \cap \partial B_\rho = (\overline{g}(B_2^+) \setminus W) \cap \partial B_\rho
\]
which is what we want.

Following a similar process, we can conclude the linking result for the second minimax value.
Proposition 4.2. For any \( g \in \Gamma_s'' \), we have

\[
(g(D_{2^+}^+) \cap \partial B_\rho) \setminus W \neq \emptyset.
\]

Then we can claim that \( K_{c_2} \setminus W \neq \emptyset \) and \( K_{c_2'} \setminus W \neq \emptyset \).

4.2 A remark on the linking sets

In the proceeding paragraph, when we deal with the linking structure of sign-changing critical points, we used the 1-codimensional set \( \partial B_\rho \). In fact, we can use some sets of higher codimension. For example, consider the set \( F = Y \cap \partial B_\rho \), where the subspace \( Y \) satisfies that \( \text{codim}(Y) = 1 \) and \( F \cap (E \setminus W) = \emptyset \). It is valid for all three kinds of critical values, we only consider the first kind as an example. Using the intersection lemma (c.f. [10, Lemma 6.4]), we have for any \( g \in \Gamma_s \),

\[
g(B^+_2) \cap F \neq \emptyset,
\]

and hence

\[
(g(B^+_2) \cap F) \setminus W \neq \emptyset.
\]

In this case, \( \text{codim}(F) = 2 \). It should be remarked that in [5], the linking set is \( \partial P^+ \cap \partial P^- \). These examples show that the set \( \partial B_\rho \) is not the 'smallest' linking set. It could be intersecting to look for a necessary and sufficient condition for a set to be a linking set for sign-changing mountain pass lemmas.

Acknowledgement

The author would like to express his sincere thanks to his advisor Professor Zhi-Qiang Wang for useful suggestions, infinitely patience and language help. This research is supported by CNSF(11771324).

References

[1] Ambrosetti A., Rabinowitz P. H.. Dual variational methods in critical point theory and applications[J]. Journal of Functional Analysis, 1973, 14(4):349-381.

[2] Bartsch T, Liu Z, Weth T. Nodal solutions of a p-Laplacian equation[J]. Proceedings of the London Mathematical Society, 2005, 91(1):129-152.

[3] Bartsch T, Wang Z.-Q., Willem M. The Dirichlet problem for superlinear elliptic equations[J]. Stationary partial differential equations. Vol. II, 2005, 2:1-55.
[4] Li C, Li S. Gaps of consecutive eigenvalues of Laplace operator and the existence of multiple solutions for superlinear elliptic problem[J]. Journal of Functional Analysis, 2016, 271(2):245-263.

[5] Liu J, Liu X, Wang Z.-Q.. Multiple mixed states of nodal solutions for nonlinear Schrodinger systems[J]. Calculus of Variations and Partial Differential Equations, 2015, 52(3-4):1-22.

[6] Liu Z, Sun J. Invariant Sets of Descending Flow in Critical Point Theory with Applications to Nonlinear Differential Equations [J]. Journal of Differential Equations, 2001, 172(2):257-299.

[7] Li S, Wang Z.-Q.. Mountain pass theorem in order intervals and multiple solutions for semilinear elliptic Dirichlet problems[J]. Journal Danalyse Mathmatique, 2000, 81(1):373-396.

[8] Liu Z, Wang Z.-Q., Zhang J. Infinitely many sign-changing solutions for the nonlinear Schrodinger-Poisson system[J]. Annali di Matematica Pura ed Applicata (1923 -), 2016, 195(3):775-794.

[9] Rabinowitz P. Minimax Methods in Critical Point Theory with Applications to Differential Equations[M]/ Minimax methods in critical point theory with applications to differential equations. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, 1986:100.

[10] Struwe M. Variational methods : applications to nonlinear partial differential equations and Hamiltonian systems[J]. Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, 1996, 34(2):1344C1360.

[11] Wang Z.-Q.. On a superlinear elliptic equation[J]. Annales De L'institut Henri Poincare, 1991, 8(1):43-57.

[12] Willem M. Minimax Theorems[M]. Birkhauser Boston, 1996.