Exact Constants in the Rosenthal Moment Inequalities for Sums of independent centered Random Variables.

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Abstract. We study the exact constants in the moment inequalities for sums of centered independent random variables: improve their asymptotics, low and upper bounds, calculate more exact asymptotics, elaborate the numerical algorithm for their calculation, study the class of smoothing etc.

Key Words: Rosenthal moment inequalities, Exact constants, Bessel’s and Bell’s functions, Bell numbers, Stirling’s formula and numbers, Banach spaces of random variables.

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1 Introduction. Statement of problem.

Let \( p = const \geq 2, \{\xi(i)\}, i = 1, 2, \ldots, n \) be a sequence of independent centered: \( E\xi(i) = 0 \) random variables belonging to the space \( L_p \), i.e. such that

\[
\forall i \quad \|\xi(i)\|_p \overset{def}{=} E^{1/p}|\xi(i)|^p < \infty. \tag{0}
\]

We denote \( \sum a(i) = \sum a(i), L(p) = C^p(p) \), where

\[
C(p) = \sup_{\{\xi(i)\}, \sum a(i)} \sup_n \frac{\|\sum \xi(i)\|_p}{\max(\|\sum \xi(i)\|_2, (\sum \|\xi(i)\|_p)^{1/p})}, \tag{1}
\]

(”\( C \)” denotes the centered case), where the external ”sup” is calculated over all the sequences of independent centered random variables satisfies the condition (0).

In the case if in (1) all the variables \( \{\xi(i)\} \) are symmetrically distributed, independent, \( \|\xi(i)\|_p < \infty \), we will denote the correspondent constants (more exactly, functions of \( p \)) \( S(p) \) (”\( S^{\prime}(\cdot) \)” denotes the symmetrical case) instead \( C(p) \) and \( K(p) \overset{def}{=} S^p(p) \) instead \( L(p) \). It is obvious that \( S(p) \leq C(p), K(p) \leq L(p) \). In the article [12] is proved that \( C(p) \leq 2S(p), L(p) \leq 2^p K(p) \).

The constant \( C(p), S(p) \) are called the exact constants in the moment inequalities for the sums of independent random variables and play very important role in the classical theory of probability ([1], 522 - 523, [2], p. 63;) theory of probability on the Banach spaces [4], in the statistics and theory of Monte - Carlo method ([19], section 5) etc.

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There are many publications on the behavior of constants \( C(p), S(p) \) at \( p \to \infty \). The first estimations are obtained in [6]; Rosenthal [7] proved in fact that \( C(p) \leq C_1^p; C_1 = \text{const} > 1 \); here and further \( C_j, j = 1, 2, \ldots \) are some positive finite absolute constants, \( \log = \ln \). In the article [17] is proved that \( C(p) \leq 9.6 \frac{p}{\log p} \). In the works [8], [9], [22] are obtained the non-asymptotical bide-sides estimations for \( S(p) \):

\[
(e\sqrt{2})^{-1} p / \log p \leq S(p) \leq 7.35 \frac{p}{\log p}, \quad p \geq 2,
\]

and there are some moment estimations for the sums independent nonnegative random variables. See also Latala [12], Utev [17], [18]; Pinelis and Utev [20] and so one.

In the articles of Ibragimov R. and Sharachmedov Sh. [10], [11] and Utev [17], [18] is obtained the explicit formula for \( S(p) \): \( S(2) = 1 \); at \( p \in (2, 4] \)

\[
S(p) = \left( 1 + \sqrt{\frac{2p}{\pi}} \Gamma \left( \frac{p+1}{2} \right) \right)^{1/p};
\]

\[
p \geq 4 \Rightarrow S(p) = \|\tau_1 - \tau_2\|_p,
\]

where \( \Gamma(\cdot) \) is the Gamma function, a random variables \( \tau_j \) are independent and have the Poisson distribution with parameters 0.5: \( E\tau_j = D\tau_j = 1/2 \).

As a consequence was obtained that at \( p \to \infty \)

\[
S(p) = \frac{p}{e \cdot \log p} \left( 1 + o \left( \frac{\log^2 \log p}{\log p} \right) \right).
\]

In the article [11] is obtained the following representation for the values \( L(2m), m = 2, 4, 6, \ldots \):

\[
C^{2m}(2m) = L(2m) = E(\theta - 1)^{2m} = e^{-1} \sum_{n=0}^{\infty} (n - 1)^{2m} / n!,
\]

where the random variable \( \theta \) has the Poisson distribution with parameter 1, and there is a hypothesis that for all the values \( p \geq 4 \) \( C^p(p) = L(p) \).

We will denote also for all values \( p \geq 4 \)

\[
L(p) = E|\theta - 1|^p = e^{-1} \sum_{n=0}^{\infty} |n - 1|^p / n!, \quad G(p) = L^{1/p}(p).
\]

In the report [13] are obtained the estimations for \( C(p) \) in the case if the sequence \( \{\xi(i)\} \) is the sequence of martingale - differences, in the article [18] there are some generalizations for weakly dependent random variables \( \{\xi(i)\} \).

In this article we improve the bide-side estimations and asymptotics for \( S(p), G(p) \) at \( p \to \infty \), find the exact boundaries for the different approximation of \( S(p), G(p) \); describe the algorithm for the numerical calculation of \( K(p), L(p) \); study the analytical properties of \( K(p), L(p) \) etc.

Notice that there are many other statements of this problem: for the nonnegative variables [8], [12]; for the Hilbert space valued variables [18] etc.
2 Main results.

Let us introduce the following functions at \( p \geq 4 \):

\[
g(p) = \frac{p}{e \log p}, \quad \delta(p) = \frac{1}{\log p}, \quad \Delta(p) = \log \log p / \log p;
\]

\[
h(p) = g(p) \left( 1 + \Delta(p) + \Delta^2(p) \right) =
[p/(e \log p)] \cdot \left( 1 + \log \log p / \log p + (\log \log p / \log p)^2 \right);
\]

\[
I_n(z) = 2^{-n} \sum_{k=0}^{\infty} 4^{-k} z^k / (k! (n+k)!) \]
is the usually modified Bessel’s function of order \( n \);

\[
W(p) = \left( \frac{2}{e} \right) \sum_{n=1}^{\infty} n^p I_n(1);
\]

\[
B(p) = e^{-1} \sum_{n=1}^{\infty} n^p / n!, \quad p > 0; \quad B(0) = 1;
\]

\( B(p) \) are the well-known Bell’s numbers; \( B(p) = E \tau^p \), where the random variable \( \tau \) has the usually Poisson distribution with parameter 1: \( E \tau = D \tau = 1 \).

The generalized Bell’s function \( B(a, p; z) \) may be defined as

\[
B(a, p, z) = \sum_{n=0}^{\infty} \frac{|n-a|^p z^n}{e \cdot n!}.
\]

For example, \( B(0, p, 1) = B(p) \).

**Theorem 1.**

\[
1 = \inf_{p \geq 4} G(p) / g(p) < \sup_{p \geq 4} G(p) / g(p) = C_3,
\]

where

\[
C_3 = \sup_{p \geq 4} B^{1/p}(1, p, 1) / g(p) = G(C_4) / g(C_4) \approx 1.77638,
\]

\[
C_4 = \arg \max_{p \geq 4} B^{1/p}(1, p, 1) / g(p) \approx 33.4610;
\]

(The equality \( C_3 \approx 1.77638 \) means that \( |C_3 - 1.77638| \leq 5 \cdot 10^{-6} \));

\[
1 = \inf_{p=4,6,8...} C(p) / g(p) < \sup_{p=4,6,8...} C(p) / g(p) = C_5,
\]

where \( C_5 =

\[
\inf_{p=4,6,8...} B^{1/p}(1, p, 1) / g(p) = G(C_6) / g(C_6) \approx 1.77637, \quad C_6 = 34;
\]

\[
1 = \inf_{p \geq 15} G(p) / h(p) < \sup_{p \geq 15} G(p) / h(p) = G(C_8) / h(C_8) = C_7,
\]
where

\[ C_7 = \sup_{p \geq 15} B^{1/p}(1, p, 1)/h(p) \approx 1.2054, \]

\[ C_8 = \arg\max_{p \geq 15} B^{1/p}(1, p, 1)/h(p) \approx 71.430; \]

\[ 1 = \inf_{p \geq 4} S(p)/g(p) < \sup_{p \geq 4} S(p)/g(p) = C_9, \]  \hspace{1cm} (6a)

where

\[ C_9 = \sup_{p \geq 4} W^{1/p}(p)/g(p) = S(C_{10})/g(C_{10}) \approx 1.53572, \]

\[ C_{10} = \arg\max_{p \geq 15} W^{1/p}(p)/g(p) \approx 22.311; \]

\[ 1 = \inf_{p \geq 15} S(p)/h(p) < \sup_{p \geq 15} S(p)/h(p) = S(C_{12})/h(C_{12}) = C_{11}, \]  \hspace{1cm} (6b)

where

\[ C_{11} = \sup_{p \geq 15} W^{1/p}(p)/h(p) \approx 1.03734, \]

\[ C_{12} = \arg\max_{p \geq 15} W^{1/p}(p)/h(p) \approx 138.149; \]

\[ 1 = \inf_{p=16,18,20,...} C(p)/h(p) < \sup_{p=16,18,20,...} C(p)/h(p) = \]

\[ C(72)/h(72) = \sup_{p=16,18,20,...} B^{1/p}(1, p, 1)/h(p) \approx 1.2053. \]  \hspace{1cm} (6c)

(We choose the value 15 as long as the function \( \log \log p / \log p \) monotonically decreases for the values \( p \geq \exp(e) \approx 15.15426 \).

Notice than our estimations and constants (5a, 5b, 5c) and (6a, 6b,6c) are exact and improve the constants and estimations of Rosenthal [7]; Johnson, Schechtman, Zinn [8]; Ibragimov, Sharachmedov [10],[11]; Latala [12]; Utev [17], [18] etc. For example, \( 1/(1/\sqrt{2}) \approx 1.41421, 7.35e/C_3 \approx 11.2472. \)

**Theorem 2.** At \( p \to \infty \) \( G(p) = [p/(e \cdot \log p)] \times \)

\[ \left(1 + \frac{\log \log p}{\log p} + \frac{1}{\log p} + \frac{\log^2 \log p}{\log^2 p} + \frac{\log \log p}{\log^2 p} (1+o(1)) \right); \]  \hspace{1cm} (7a)

\[ S(p) = [p/(e \cdot \log p)] \times \]

\[ \left(1 + \frac{\log \log p}{\log p} + \frac{1 - \log 2}{\log p} + \frac{\log^2 \log p}{\log^2 p} + o \left(\frac{\log \log p}{\log^2 p}\right) \right). \]  \hspace{1cm} (7b)
Let us denote for the values $p \geq 4$ by $N = N(p)$, $M = M(p)$ the (unique) solutions of equations

$$M(p) \log M(p) = p, \quad N(p) \log(2N(p)) = p,$$

such that $N(p) = 0.5M(2p)$.

**Theorem 3.** At $p \to \infty$, $m = 2, 3, 4, \ldots \to \infty$

$$G(p) = M(p)^{1-M(p)/p} \exp(M(p)/p) \left(1 + O(\log p/p)\right),$$

$$C(2m) = M^{1-M(2m)/2m(2m)} \exp(M(2m)/(2m)) \left(1 + O(\log m/m)\right),$$

$$S(p) = N \left(\frac{e}{2N}\right)^{N/p} \left(1 + O(\log p/p)\right).$$

Denote by $s(n, r)$ the usually Stirling’s numbers of a second kind appeared in the combinatorics ([14], p. 117):

$$x^n = \sum_{r=0}^{n} s(n, r)x(r); \quad x(r) \overset{\text{def}}{=} x(x-1)(x-2)\ldots(x-r+1), \quad x(0) = 1.$$

**Theorem 4.** Let $p$ be even: $p = 2m$, $m = 2, 3, 4, \ldots$. Then

$$K(2m) = \sum_{l=0}^{2m} (-1)^l \binom{2m}{l} \sum_{q=0}^{2m-l} \sum_{r=0}^{l} 2^{-r-q}s(2m-l, q)s(l, r),$$

$$C^{2m}(2m) = L(2m) = \sum_{l=0}^{2m} (-1)^l \binom{2m}{l} \sum_{r=0}^{2m-l} s(2m-l, r).$$

For the integer odds values $p = 5, 7, 9, \ldots$ we have the representation

$$G^p(p) = L(p) = \frac{2}{e} + \sum_{k=0}^{p} (-1)^k \binom{p}{k} B(p-k).$$

3 Auxiliary results.

1. In the symmetrical case for all the values $p \in [4, \infty)$ we have:

$$K(p) = (2/e) \sum_{n=1}^{\infty} n^p I_n(1) = W(p).$$

Namely, for the values $\tau_1, \tau_2$ from (3) we receive for the values $n = 1, 2, \ldots$:

$$P(\tau_1 - \tau_2 = n) = e^{-1} \sum_{k=0}^{\infty} \frac{2^{-k}2^{-(n+k)}}{k!(k+n)!} = I_n(1)/e.$$

2. On the basis of the equality (12) we can offer the numerical algorithm for $K(p)$ investigation, calculation and estimation. For the improvement of speed of convergence of series (12) we can write:
\[ 2\pi I_n(1) = \int_{-\pi}^{\pi} \exp(\cos(\theta)) \cos(n\theta) \, d\theta, \]

(see, for example, [16], p. 958, formula 5.) We obtain after the integration by parts

\[ 2\pi I_n(1) = (-1)^m n^{-2m} \int_{-\pi}^{\pi} (\exp(\cos(\theta)))^{(2m)} \cos(n\theta) \, d\theta, \]

\[ m = 1, 2, \ldots \]

Using the method of mathematical induction we conclude:

\[ (\exp \cos(\theta))^{(2m)} = \exp(\cos(\theta)) \, P_2m(\cos(\theta)), \]

where \( P_{2m}(x) \) are a polynomials of degree \( 2m \) which may be calculated by means of the recursion

\[ P_{2m+2}(x) = (1 - x^2) \left( P_{2m}' + 2P_{2m}'(x) + P_{2m}(x) \right) - x \left( P_{2m}'(x) + P_{2m}(x) \right) \]

with initial condition \( P_0(x) = 1 \). Therefore, we get the following representation for \( K(p) \):

\[ \pi e \, K(p) = \sum_{n=1}^{\infty} n^{p-2m} \int_{-\pi}^{\pi} \exp(\cos(\theta)) \, P_{2m}(\theta) \cos(n\theta) \, d\theta. \]  

(13)

3. Corollary. For the even numbers \( p = 2m, \ m = 1, 2, 3, \ldots \), all the numbers \( K(p) = K(2m), L(p) = L(2m) \) are integer.

In fact, it follows from formula (12) that

\[ K(2m) = (\pi e)^{-1} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} g^{(2m)}(\theta) \cos(n\theta) \, d\theta = \]

\[ e^{-1}(\exp(\cos(\theta)))^{(2m)}(0) = (-1)^m P_{2m}(1). \]

It is easy to verify that all the coefficients of polynomials \( P_{2m}(x) \) are integer; thus, the number \( P_{2m}(1) \) is integer.

The second conclusion of our corollary follows from the formula (10), as long as all the Stirling’s numbers are integer.

4. For example, \( K(6) = 31, L(6) = 41 \). For the non-integer values \( p \) we can use the method described above. We obtained:
| p | K(p) | L(p) |
|---|---|---|
| 2 | 1 | 1 |
| 4 | 4 | 4 |
| 4.5 | 6.3358 | 6.6712 |
| 5 | 10.4118 | 11.7358 |
| 5.5 | 17.686 | 21.538 |
| 6 | 31 | 41 |
| 6.5 | 55.819 | 80.5508 |
| 7 | 103.22 | 162.7358 |
| 7.5 | 192.45 | 337.176 |
| 8 | 379 | 715 |
| 8.5 | 757.7 | 1549.28 |
| 9 | 1126.5 | 3425.7358 |
| 9.5 | 3015.0 | 7721.29 |
| 10 | 6556 | 17722 |

| p | K(p) | L(p) |
|---|---|---|
| 10.5 | 14000.4 | 41385.2 |
| 11 | 30403.2 | 98253.7 |
| 11.5 | 67091.3 | 236982 |
| 12 | 150349 | 580317 |
| 12.5 | 67091.3 | 236982 |
| 13 | 788891.0 | 3.63328E+006 |
| 13.5 | 341951.2 | 1.44191E+006 |
| 14 | 150349 | 580317 |
| 14.5 | 341951.2 | 1.44191E+006 |
| 15 | 150349 | 580317 |
| 15.5 | 341951.2 | 1.44191E+006 |
| 16 | 788891.0 | 3.63328E+006 |
| 16.5 | 341951.2 | 1.44191E+006 |
| 17 | 150349 | 580317 |
| 17.5 | 2.57666E+009 | 2.59791E+010 |
| 18 | 6.69849E+009 | 7.36008E+010 |
| 18.5 | 1.75916E+010 | 2.106E + 011 |
| 19 | 4.66582E+010 | 6.08476 + 011 |
| 19.5 | 1.24952E+011 | 1.56298E+008 |
| 20 | 3.37789E+011 | 5.22427E+012 |
| 20.5 | 9.21603E+011 | 1.55157E+013 |
| 21 | 2.53714E+012 | 4.64999E+013 |

5. Using the discrete analog of the saddle-point method ([24], p. 262 - 264), [10]), we find that

$$ M(p) = \left[ \frac{p}{\log p} \right] \cdot (1 + \varepsilon(p)), $$

where at $p \to \infty$

$$ \varepsilon(p) = \Delta(p) + \Delta^2(p) - \delta(p) \Delta(p) (1 + o(1)). \quad (14) $$

Hence

$$ N(p) = \left[ \frac{p}{\log(2p)} \right] \cdot (1 + \varepsilon(2p)) = $$

$$ \left[ \frac{p}{\log p} \right] \cdot \left[ 1 + \Delta(p) + \Delta^2(p) - \delta(p) \Delta(p) (1 + \log 2)(1 + o(1)) \right]. $$

Define for the values $p \geq P_0 = 700$ the following functions and constants:

$$ C_{14} = (1 - \log \log P_0 / \log P_0) \approx 1.402365, $$
\[ C_{15} = 2 \cdot \left[ (1 + 4\Delta^2(P_0))^{1/2} + 1 \right] \approx 0.928958, \]

\[ \zeta(p) = \log 2 / \log(2p), \]

\[ \varepsilon_+(p) = \Delta(p) + C_{14} \Delta^2(p), \quad \varepsilon_-(p) = \Delta(p) + C_{15} \Delta^2(p), \]

\[ M_+ = M_+(p) = [p / \log p] \cdot (1 + \varepsilon_+(p)), \]

\[ M_- = M_-(p) = [p / \log p] \cdot (1 + \varepsilon_-(p)), \quad (15a) \]

\[ N_+(p) = [p / (e \cdot \log(2p))] \cdot (1 + \varepsilon_+(2p)), \]

\[ N_-(p) = [p / (e \cdot \log(2p))] \cdot (1 + \varepsilon_-(2p)). \quad (15b) \]

More exact calculation show us that for all the values \( p \geq P_0 \)

\[ M_-(p) \leq M(p) \leq M_+(p), \quad N_-(p) \leq N(p) \leq N_+(p). \]

Namely, it is very simple to see that \( \forall p \geq P_0 \Rightarrow \)

\[ M_- \log M_- < p = M \log M < M_+ \log M_+. \]

6. Let us denote

\[ b_1(x, p) = x^p / \Gamma(x + 1), \]

where

\[ \Gamma(x) = \int_0^\infty y^{x-1} e^{-y} \, dy \]

is the usually gamma function;

\[ b_2(x, p) = x^p / (2^x \Gamma(x + 1)); \]

\[ V(x, p) = p \log x - x \log x + x, \]

\[ X(p) = V(M(p), p)/p = \sup_{x \geq 4} V(x, p)/p, \]

\[ W(x, p) = p \log x - x \log x + x(1 - \log 2), \]

\[ Y(p) = W(N(p), p)/p = \sup_{x \geq 4} W(x, p)/p. \]

We have using the equality (14): \( X(p) = \log[p/(e \cdot \log p)] + \)
\[ \Delta(p) + \delta(p) + [\log(1 + \varepsilon(p)) - \varepsilon(p) + \Delta(p)\varepsilon(p)] + \\
\{\delta(p)(\varepsilon(p) - \log(1 + \varepsilon(p))) - \delta(p)\varepsilon(p)\log(1 + \varepsilon(p))\} = \\
\log[p/(e \cdot \log p)] + X_0(p), \]

where for \( p \geq P_0 \) \( X_2(p) < X_0(p) < X_1(p) \), \( X_1(p) \overset{\text{def}}{=} \\
\Delta(p) + \delta(p) + \Delta(p)\varepsilon_+(p) + \delta(p)[\varepsilon_+(p) - \log(1 + \varepsilon_+(p))], \quad (16a) \]

\( X_2(p) \overset{\text{def}}{=} \Delta(p) + \delta(p) + \log(1 + \varepsilon_-(p)) - \varepsilon_-(p) - \\
-\delta(p)\varepsilon_-(p)\log(1 + \varepsilon_-(p)). \quad (16b) \]

The function \( p \to X_1(p), \ p \in [P_0, \infty) \) is monotonically decreasing and

\[ \exp(X_1(P_0)) < 1.7563, \ \lim_{p \to \infty} X_1(p) = 0. \quad (16c) \]

At the same manner we get: \( Y(p) = \log[p/(e \cdot \log p)] + Y_0(p) \), where

\[ Y_0(p) = \log(1 - \zeta(p)) - (1 + \varepsilon(2p)) \times \\
[1 - \zeta(p) - \Delta(2p) + \delta(2p)\log(1 + \varepsilon(2p))] + \delta(2p)(1 + \varepsilon(2p)) \overset{\text{def}}{=} \\
\log g(p) + Y_0(p), \ Y_2(p) \leq Y_0(p) \leq Y_1(p), \]

\[ Y_1(p) \overset{\text{def}}{=} \Delta(2p) + \delta(2p) + (1 + \varepsilon_+(2p)) \cdot \delta(p)\log 2/(1 + \delta(p)\log 2) + \\
\varepsilon_+(2p)[\Delta(2p) + \delta(2p)], \quad (16d) \]

\[ Y_2(p) \overset{\text{def}}{=} \Delta(2p) + \delta(2p) + \varepsilon_-(2p)[\Delta(2p) + \delta(2p)], \quad (16e) \]

where the function \( p \to Y_1(p), \ p \in [P_1, \infty), \ P_1 = 10^6 \) is monotonically decreasing and

\[ \exp(Y_1(P_1)) < 1.442, \ Y_1(p) \downarrow 0, \ p \to \infty, \ \lim_{p \to \infty} Y_2(p) = 0. \quad (17) \]

7. Upper bound for \( L(p) \). Assume in this section that \( p \geq P_0 = 700 \). We have for the values \( p \geq P_0 \), using the well-known Stirling’s formula:

\[ e \cdot L(p) - 1.5 = \sum_{n=3}^{\infty} b_1(n - 1, p) \leq \int_2^{\infty} b_1(x, p) \, dx + \sup_{x \geq 3} b_1(x, p) \leq \]
\[(2\pi)^{-1/2} \exp(p \cdot X(p)) + (2\pi)^{-1/2} \int_{2}^{\infty} \exp(V(x, p)) \, dx.\]

Split the last integral into three parts so that

\[J(p) \overset{\text{def}}{=} \int_{2}^{\infty} \exp(V(x, p)) \, dx = J_1 + J_2 + J_3, \quad J_j = J_j(p), \quad j = 1, 2, 3,\]

where

\[J_1(p) = \int_{2}^{M-\sqrt{p}} \exp(V(x, p)) \, dx, \quad J_2 = \int_{M-\sqrt{p}}^{M+\sqrt{p}} \exp(V(x, p)) \, dx, \quad J_3 = \int_{M+\sqrt{p}}^{\infty} \exp(V(x, p)) \, dx,\]

we have for the integral \(J_2\), taking into account the inequalities \(M_- < M < M_+\) and inequality: \(p \in [M - \sqrt{p}, M + \sqrt{p}] \Rightarrow\)

\[V(x, p) \leq pX(p) - 0.5(x - M)^2 \cdot (2p) \cdot (p + 2M_+^{-2}) <\]

\[pX(p) - 0.5(x - M)^2 \cdot p \cdot M_+^{-2}(p) :\]

\[J_2 \leq \exp(p \cdot X(p)) \cdot \int_{M-\sqrt{p}}^{M+\sqrt{p}} \exp \left(-0.5p(x - M)^2 \cdot M_+^{-2} \right) \, dx <\]

\[\exp(p \cdot X(p)) \cdot \int_{-\infty}^{\infty} \exp \left(-0.5p(x - M(p))^2 \cdot M_+^{-2} \right) \, dx =\]

\[\sqrt{2\pi} \cdot \exp(p \cdot X(p)) \cdot M_+ / \sqrt{p} \leq \exp(p \cdot X(p)) \cdot \Psi_1(p),\]

where

\[\Psi_1(p) = \sqrt{2\pi} \cdot p \cdot \left[1 + \Delta + C_{14} \Delta^2 \right] / \log p.\]

Now we estimate the integral \(J_3\). For the values \(x \geq M + \sqrt{p}\) are true the following inequalities:

\[V(x, p) \leq pX(p) - 0.5 \cdot (2p) \cdot (p/M_+^2(p)) \leq pX(p) -\]

\[\log^2 p \cdot (1 + \Delta + C_{14} \Delta^2)^{-2};\]

\[dV(x, p)/dx \leq -p/M_+^2(p) \cdot [x - M(p) - \sqrt{p}];\]

therefore \(J_3 \leq \exp(p \cdot X(p)) \cdot \Psi_2(p),\) where \(\Psi_2(p) =\)

\[\exp \left(- \log^2 p \cdot (1 + \Delta(p) + C_{14} \Delta^2(p))^{-2} \right) \times\]
\[
\int_{M+\sqrt{p}}^{\infty} \exp \left( -p M^2 (x - M - \sqrt{2p}) \right) \, dx =
\]
\[
\exp \left( -\log^2 p \cdot (1 + \Delta(p) + C_{14} \Delta^2(p))^{-2} \right) \times
\]
\[
p \cdot \left( 1 + \Delta + C_{14} \Delta^2 \right) \cdot \log^{-2}(p)
\]
and analogously we find the upper estimate for \( J_1 \).

Thus, \( L(p) < e^{-1} \cdot \exp(p \cdot X(p)) \times \]
\[
\left[ 1.5 \exp(-p \cdot X(p)) + (2\pi)^{-1/2} + \Psi_1(p) + 2(2\pi)^{-1/2} \Psi_2(p) \right] =
\]
\[
\exp(p \cdot X(p)) \cdot \Psi_3^p(p), \tag{18a}
\]
where we find by the direct calculations: \( \Psi_3(P_0) \leq 1.00826 \) and at \( p \geq P_0 \)
\[
\Psi_3(p) \downarrow 1, \ p \to \infty; \ \Psi_3(p) \leq 1 + C_{18} \log p/p. \tag{18b}
\]

9. **Low bound for** \( L(p) \). Denote \( q = p - 1/2 \). We obtain using the Sonin’s estimate for factorials:

\[
eL(p) \geq \sum_{n=4}^{\infty} b_1(n-1,p) = \sum_{n=3}^{\infty} b_1(n,p) \geq \]
\[
\int_{\bar{4}}^{\infty} b_1(x,p) \, dx \geq (2\pi)^{-1/2} \exp(-1/12) \int_{4}^{\infty} \exp(V(x,q)) \, dx \geq \]
\[
(2\pi)^{-1/2} \exp(-1/12) \int_{M(q)}^{M(q)+\sqrt{q}} \exp(V(x,q)) \, dx.
\]
Since the following implication holds: \( q \in [M(q), M(q) + \sqrt{q}] \Rightarrow \)
\[
V(x,q) \geq q X(q) - 0.5(x - M(q))^2 qM_{-2}(q),
\]
we have:

\[
eL(p) \geq (2\pi)^{-1/2} \exp(-1/12) \exp(qX(q)) \times \]
\[
\int_{M(q)}^{M(q)+\sqrt{q}} \exp \left[ -0.5q M^{-2}(q) (x - M(q))^2 \right] \, dx \geq M_{-}(q) \times \]
\[
0.5 \exp(-1/12) \sqrt{q} \exp(qX(q)) \left[ 1 - \exp \left( -q^2/M_{-}^2 \right) \right] =
\]
\[
e \cdot \exp(p \cdot X(p)) \cdot \Psi_4^p(p),
\]
where
\[
\Psi_4(p) \downarrow 1, \ p \to \infty; \ \Psi_4(p) \geq 1 + C_{19} \log p/p.
\]
Thus,

$$\exp(p \cdot X(p)) \cdot \Psi_3^0(p) \leq L(p) \leq \exp(p \cdot X(p)) \cdot \Psi_3^3(p), \quad (19a)$$

$$\Psi_3(p) \leq 1 + C_{19} \log p/p, \ \Psi_4(p) \geq 1 + C_{20} \log p/p,$$

$$\Psi_3(p) \downarrow 1, p \rightarrow \infty; \Psi_3(P_0) \leq 1.00826. \quad (19c)$$

10. Upper and low bounds for $K(p)$ are provided analogously to the upper bound for $L(p)$, but we assume in this section that $p \geq P_1 = 10^6$. Briefly:

$$\sum_{k=0}^{\infty} \frac{4^{-k}}{k! (n+k)!} < \frac{1}{n!} \sum_{k=0}^{\infty} \frac{4^{-k}}{k!} = \frac{\sqrt{e}}{n!} < \frac{1.285}{n!},$$

hence

$$K(p) < 2e^{-3/4} \sum_{n=1}^{\infty} \frac{np}{n!} 2^{-n} = 2\sqrt{e} \cdot B(0, p, 1/2).$$

Further, we conclude, again using the Stirling estimate for factorials:

$$0.5 e^{3/4} K(p) = \sum_{n=1}^{\infty} b_2(n, p) \leq \int_2^{\infty} b_2(x, p) \, dx + \sup_{x \geq 2} b_2(x, p) \leq$$

$$(2\pi)^{-1/2} \exp(p \cdot Y(p)) + (2\pi)^{-1/2} \int_2^{\infty} \exp(W(x, p)) \, dx.$$

We have, again split the last integral:

$$I_4 \overset{\text{def}}{=} \int_2^{\infty} \exp(W(x, p)) \, dx =$$

$$\left( \int_2^{N(p) - \sqrt{p}} + \int_{N(p) + \sqrt{p}}^{N(p) + \sqrt{p}} + \int_{N(p) + \sqrt{p}}^{\infty} \right) \exp W(x, p) \, dx =$$

$I_5 + I_6 + I_7$. As long as at $x > N(p) + \sqrt{p}$

$$W(x, p) \leq pY(p) - 0.5p^2N_+^2(p) = pY(p) - 0.5 \log^2 p \cdot (1 + \varepsilon_+(2p))^{-2},$$

$$dW/dx \leq -pN_+^{-2}(x - N - \sqrt{p}),$$

we obtain:

$$I_7 \leq \exp(pY(p)) \cdot p \log^{-2} p \cdot (1 + \varepsilon_+(2p))^2 \times$$

$$\exp \left( -0.5 \log^2 p \cdot (1 + \varepsilon_+(2p))^{-2} \right).$$

Further, if $x \in [N(p) - \sqrt{p}, N(p) + \sqrt{p}]$, then

$$W(x, p) \leq pY(p) - 0.5pN_+^{-2}(p) \cdot (x - N(p))^2.$$
Therefore
\[ I_6 \leq \exp(pY(p)) \cdot \sqrt{p} \left(1 + \varepsilon_+(2p)\right)/\log(2p) \]
and \( K(p) \leq 2e^{-3/4} \exp(pY(p)) \times \)
\[ \left[ (2\pi)^{-1/2} + \sqrt{p(1 + \varepsilon_+(2p))}/\log(2p) + 2(2\pi)^{-1/2}p \log^{-2} p \right] \times \]
\[ \left[ (1 + \varepsilon_+(2p))^2 \cdot \exp(-0.5 \log^2 p \left(1 + \varepsilon_+(2p)\right)^{-2} \right]. \]

(20)

**Low bound for** \( K(p) \). We have: \( 0.5 e K(p) > \)
\[ \sum_{n=1}^{\infty} n^p 2^{-n}/n! = B(0, p, 1/2) > \exp(-1/12) (2\pi)^{-1/2} \exp(qY(q)) \times \]
\[ \int_{N(q)}^{N(q)+\sqrt{q}} \exp \left[ -0.5 \ q \ N^{-2}(q) (x - N(q))^2 \right] dx \geq \]
\[ \exp(-1/12) \sqrt{\pi/2} q^{-1/2} \ N^{-2}(q) \left(1 - \exp \left(-q^2/N^2(q)\right)\right)/\log(2q) = \]
\[ \exp(-1/12) \sqrt{\pi/2} \exp(qY(q)) \sqrt{q} (1 + \varepsilon_-(2q)) \times \]
\[ \left(1 - \exp \left(-q^2/N^2(q)\right)\right)/\log(2q). \]

Further estimations are like to the estimation for \( L(p) \) and may be omitted.

Result:
\[ \exp(p \cdot Y(p)) \cdot \Psi_6^p(p) \leq K(p) \leq \exp(p \cdot Y(p)) \cdot \Psi_5^p(p), \]

(21a)
where at \( p \geq P_1 \)
\[ \Psi_5^p(p) \leq 1 + C_{19} \log p/p, \ \Psi_6^p(p) \geq 1 + C_{20} \log p/p; \]

(21b)
\[ 11. \] For the correct calculations (by computer) we need to estimate the derivatives of our functions \( L(p), K(p) \). We show here the estimation of derivatives \( L^{(m)}(p), m = 1, 2, \ldots \). Namely, \( e \cdot L^{(m)}(p) = \)
\[ \sum_{n=3}^{\infty} \frac{(n-1)^p \log^m(n-1)}{n!} \leq \sum_{n=3}^{\infty} \frac{(n-1)^p \log^m n}{(n-1)!} < \]
\[ \sum_{n=2}^{\infty} \frac{n^p}{n!} \cdot \left(\sup_{n \geq 3} \frac{\log^m n}{n}\right) = \]
\[ \left(\frac{m}{e}\right)^m \cdot \sum_{n=2}^{\infty} \frac{n^p}{n!} = \left(\frac{m}{e}\right)^m \cdot (eB(p) - 1). \]

(22)
The derivative \( K^{(m)}(p), m = 1, 2, \ldots \) we estimate analogously.
It follows from this estimation that the functions $L(p)$ and $K(p)$ are infinitely differentiable at the interval $p \in (4, \infty)$. As long as $L(4 - 0) = L(4 + 0) = 4$, $K(4 - 0) = K(4 + 0) = 4$, both the functions $K(\cdot)$, $L(\cdot)$ are continuos in the semiclosed interval $[2, \infty)$. But

$$\frac{dK}{dp}(4 - 0) = \frac{dL}{dp}(4 - 0) \approx 3.149195, \quad \frac{dK}{dp}(4 + 0) \approx 3.51934,$$

therefore both the functions $K(\cdot)$, $L(\cdot)$ are not continuous differentiable in the set $(2, \infty)$. In the open intervals $(2, 4)$ and $(4, \infty)$ all the functions $L(p), K(p), C(p), S(p)$ are infinitely differentiable (see (22) and [10], [17]).

4 Proof of the main results.

**Proof of theorem 1.** We find by the direct calculations that

$$G(C_4)/g(C_4) \approx 1.77638,$$

but for we conclude from (17) that for the values $P_0 = 700$

$$G(p)/g(p) \leq 1.00826 \cdot 1.75913 = 1.77366,$$

hence

$$\arg \max_{p \in [4, 700]} G(p)/g(p) \in [4, 700].$$

We obtain by direct calculations using usually numerical methods and by means of computer:

$$\max_{p \in [4, 700]} G(p)/g(p) = G(C_4)/g(C_4) \approx 1.77638.$$

Further,

$$\inf_{p \geq 4} G(p)/g(p) = \min \left\{ \min_{p \in [4, 700]} G(p)/g(p), \inf_{p > 700} G(p)/g(p) \right\}.$$

We obtain by computer calculations that

$$\min_{p \in [4, 700]} G(p)/g(p) \approx 1.332,$$

and it follows from (18a), (18b), (18c) and (19) that

$$\inf_{p > 700} G(p)/g(p) = \lim_{p \to \infty} G(p)/g(p) = 1.$$

Thus,

$$\inf_{p \geq 4} G(p)/g(p) = \lim_{p \to \infty} G(p)/g(p) = 1.$$

Analogously, $S(C_{10})/g(C_{10}) \approx 1.53572$, but we have for the values $p \geq P_1$
\[ \frac{S(p)}{g(p)} \leq 1.0008333 \cdot 1.443 < 1.4444. \]

Therefore
\[ \arg\max_{p \in [4, \infty)} \frac{S(p)}{g(p)} \in [4, 1000000]. \]

We obtained after some technical difficulties using computer:
\[ \max_{p \in [4, P_1]} \frac{S(p)}{g(p)} = \frac{S(C_{10})}{g(C_{10})} \approx 1.53572. \]

Further,
\[ \inf_{p \geq 4} \frac{S(p)}{g(p)} = \min \left\{ \min_{p \in [4, P_1]} \frac{S(p)}{g(p)}, \inf_{p \geq P_1} \frac{S(p)}{g(p)} \right\} = \lim_{p \to \infty} \frac{S(p)}{g(p)} = 1. \]

The another assertions of theorem 1 are obtained analogously.

**Proof of theorems 2 and 3.** It follows from inequalities (19a), (19b), (19c) and (21a), (21b), (21c) that
\[ \exp(X(p)) \cdot (1 + C_{20} \log p/p) \leq \frac{G(p)}{g(p)} \leq \exp(X(p)) \cdot (1 + C_{19} \log p/p), \quad (23a) \]
\[ \exp(Y(p)) \cdot (1 + C_{20} \log p/p) \leq \frac{S(p)}{g(p)} \leq \exp(Y(p)) \cdot (1 + C_{19} \log p/p). \quad (23b) \]

Substituting the expression (16a) and (16b) into the formula (23a), we obtain after some simple calculation our assertions (9b); (9a) provided analogously.

Finally, substituting expressions (16d,e) into (23b), we obtain (7a), (7b).

**Proof of theorem 4.** Since \( E_{\theta(r)} = 1, r = 0, 1, 2, \ldots, \) (see (4)), we conclude
\[ E_{\theta^k} = E \sum_{l=0}^{k} s(k, l) \theta_{(r)} = \sum_{l=0}^{k} s(k, l). \]

Formula (11a) follows from the binomial formula. Equality (10) proved analogously.

Let us prove (11b). Since
\[ \sum_{n=2}^{\infty} \frac{n^p}{n!} = e \cdot B(p) - 1, \quad p = 1, 2, 3, \ldots; \quad \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2, \]
we conclude for the values \( p = 5, 7, 9, \ldots : \) \( e \cdot L(p) = \)
\[ 1 + \sum_{n=2}^{\infty} \frac{(n-1)^p}{n!} = 1 + \sum_{n=2}^{\infty} (n!)^{-1} \cdot \sum_{k=0}^{p} (-1)^k \binom{p}{k} n^{p-k} = \\
1 + \left[ \sum_{k=1}^{p} (-1)^k \binom{p}{k} (eB(p-k) - 1) \right] - \]
\[ (eB(0) - 2) = 2 + e \cdot \sum_{k=0}^{p} (-1)^{k} \binom{p}{k} B(p - k) - \sum_{k=0}^{p} (-1)^{k} \binom{p}{k} = 2 + e \sum_{k=0}^{p} (-1)^{k} \binom{p}{k} B(p - k). \]

5 Concluding Remark.

Our results allow us to obtain some generalizations on the Hilbert space symmetrical distributed random variables \( \{ \eta(i) \} \), \( i = 1, 2, 3, \ldots \). Let \((H, ||| \cdot |||)\) be a separable Hilbert space with the norm \(||| \cdot |||\), \( P(\eta(i) \in H) = 1, \forall i = 1, 2, 3 \ldots \ |||\eta|||_p \stackrel{\text{def}}{=} \mathbb{E}^{1/p} (|||\eta(i)|||^p) < \infty, p \geq 4, \)

\[ Z(p) = \sup_{\{\eta(i)\}} \sup_n \frac{||| \sum \eta(i) |||_p}{\max (||| \sum \eta(i) |||_2, (\sum |||\eta(i)|||_p^p)^{1/p})}. \]

Utev ([17], [18]) proved that \( Z(p) = S(p), p \geq 4 \) (in our notations). Therefore

\[ 1 = \inf_{p \geq 4} Z(p)/g(p) < \sup_{p \geq 4} Z(p)/g(p) = C_9 \approx 1.53572, \]

\[ 1 = \inf_{p \geq 15} Z(p)/h(p) < \sup_{p \geq 15} Z(p)/h(p) = C_{11} \approx 1.03734. \]

Probably, it is interest to obtain the exact constants in the moment inequalities for sums of independent nonnegative random variables in the spirit of articles [7], [8], [12] etc.

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