we present a systematic classification and analysis of possible pairing instabilities in graphene-based moiré superlattices. Motivated by recent experiments on twisted double-bilayer graphene showing signs of triplet superconductivity, we analyze both singlet and triplet pairing separately, and describe how these two channels behave close to the limit where the system is invariant under separate spin rotations in the two valleys, realizing an SU(2)$_+ \times$ SU(2)$_-$ symmetry. Further, we discuss the conditions under which singlet and triplet can mix via two nearly degenerate transitions, and how the different pairing states behave when an external magnetic field is applied. We find that an approximate SU(2)$_+ \times$ SU(2)$_-$ symmetry can generically account for the linear increase of the critical temperature with small magnetic fields, and we map out the possible forms of the phase diagram as a function of temperature and magnetic field. We examine which of the pairing states can arise in mean-field theory and the type of pairing favored in the presence of strong ferromagnetic fluctuations, which are expected to be present in twisted double-bilayer graphene. Finally, we detail the differences in the classification when the additional microscopic or emergent symmetries relevant for twisted bilayer graphene and ABC trilayer graphene on hexagonal boron nitride are taken into account. Our study illustrates that graphene superlattices provide a rich platform for exotic superconducting states, and could allow for the admixture of singlet and triplet pairing even in the absence of spin-orbit coupling.

I. INTRODUCTION

Experiments on twisted bilayers of graphene have recently revealed interaction-induced insulating phases and superconductivity when the relative angle between the layers is fine-tuned to yield almost flat moiré bands, which enhances the impact of electronic correlations [1–4]. Due to the strong-coupling nature of the problem, which is corroborated by tunneling spectroscopy measurements [5–7], the form and mechanism of the insulating and superconducting phases are still under debate, despite considerable theoretical effort [8–42]. Another graphene-based moiré system that displays both superconducting and correlated insulating behavior is ABC-stacked trilayer graphene on hexagonal boron nitride [43, 44]. In this case, the moiré pattern results from the difference in lattice constants and bandwidths, and it can be controlled by application of a vertical electric field [45, 46].

The most recent member of the family of strongly correlated graphene superlattice systems is twisted double-bilayer graphene [47–49], where two individually aligned AB-stacked bilayer graphene bilayers are twisted with respect to one another. As theoretical calculations show [50–55], flat electronic bands can be realized by tuning the twist angle and a vertical electric field. Similar to the above-mentioned graphene moiré systems, both correlated insulating [47–49] and superconducting [47, 48] phases are observed in experiment. However, in stark contrast to twisted bilayer and trilayer graphene, the superconducting transition temperature is found to increase linearly with a weak in-plane magnetic field [48], which is a strong indication of triplet pairing [39, 52]. Furthermore, the gap of the correlated insulating phase is also seen to increase with an applied magnetic field, indicating ferromagnetic order [47–49].

In this paper, we study the possible pairing states in graphene moiré superlattices. Motivated by the recent experimental signatures of triplet pairing, we pay special attention to the triplet channel, and possible mixed singlet and triplet phases. While the weak spin-orbit coupling in graphene seems to disfavor the latter class of phases, projections of the Coulomb interaction on the relevant moiré bands evince that the interaction terms that couple the spin degrees of freedom of the two valleys, $v = \pm$, of the system are much weaker than other interaction terms that do not [11, 46, 52]. Together with the nearly valley-diagonal band structure, this hints that the system is approximately invariant under independent spin rotations in the two valleys. As has been pointed out before [17, 41], the associated SU(2)$_+ \times$ SU(2)$_-$ symmetry renders the singlet and triplet pairing channels degenerate. This paper will address the questions: (i) under which conditions can singlet and triplet mix when the SU(2)$_+ \times$ SU(2)$_-$ symmetry is only weakly broken, and (ii) which triplet state transforms into which singlet upon reversing the sign of the symmetry-breaking interactions? In this way, we map out all possible phase diagrams close to the SU(2)$_+ \times$ SU(2)$_-$-invariant limit.

We point out that loop corrections can significantly modify the form of the interactions at the energies relevant to superconductivity. Whether these corrections strongly break the SU(2)$_+ \times$ SU(2)$_-$ symmetry is presently not known, which is why we also classify and discuss pairing in the absence of this symmetry.

We will primarily concentrate on pairing in twisted
double-bilayer graphene, as it is the only system where signs of triplet pairing have been reported so far. While the lattice is invariant under three-fold rotation, \( C_3 \), perpendicular to the graphene sheets, and under a two-fold in-plane rotation, \( C_{2y} \), the latter is broken due to the vertical electric field that is applied to tune the bandstructure and to induce superconductivity. It seems currently unclear whether the superconducting state coexists with the likely ferromagnetic correlated insulator and whether, at least in part of the phase diagram, there is a thermal transition directly from the (paramagnetic) normal metal to superconductivity without any ferromagnetic order. For this reason, we will analyze two scenarios separately: (I) there is no ferromagnetic order around the critical temperature, \( T_c \), of superconductivity, and (II) there is ferromagnetic order already at \( T > T_c \) that coexists microscopically with superconductivity for \( T < T_c \) (or at the minimum, the associated ferromagnetic moments couple significantly to the superconducting order parameter). To probe both of these cases, we will begin with the analysis of the superconducting states transforming under the irreducible representations (IRs) of the point group \( C_3 \) assuming time-reversal symmetry in the high-temperature phase. This is relevant for case (I) above. In order to address scenario (II), we will later add the coupling to the time-reversal-symmetry breaking magnetic moments and examine how it affects the superconducting transition.

Studying the coupling of the superconducting states to the magnetic field, \( B \), also allows us to determine which of the pairing states are compatible with the linear increase of the critical temperature with small \( B \), and to describe the possible phase diagrams in the temperature-magnetic field, \( T-B \). In particular, the possible phase diagrams in the presence of a magnetic field for pairing states, which have been subject to intense scrutiny in twisted bilayer graphene, we are mainly interested in pairing states with a finite triplet component, prompted by the more recently discovered correlated physics of twisted double-bilayer graphene. In this context, Ref. 56 mainly focuses on the correlated insulating phase in this system, whereas Ref. 52 also discusses pairing. We extend the work of Ref. 52 by allowing for momentum-dependent pairing states, contrasting weakly and significantly broken \( SU(2)_+ \times SU(2)_- \) symmetry, investigating admixed singlet and triplet phases, analyzing fluctuation corrections to mean-field theory, and mapping out the phase diagram in the presence of a magnetic field.

The paper is organized as follows: as described above, we start with twisted double-bilayer graphene. In Sec. II, we introduce the model and the action of the relevant symmetries. We first discuss pairing in the trivial IR of the point group of the system in Sec. III and then generalize to the complex IR \( \mathcal{E} \) in Sec. IV. Sec. V demonstrates how strong fluctuations can yield significant corrections to mean-field theory. The consequences of the additional symmetries potentially relevant to pairing in twisted bilayer graphene and ABC trilayer graphene are explored in Sec. VI. A summary and discussion of our results can be found in Sec. VII.

II. MODEL AND SYMMETRIES

We focus on the (nearly flat) conduction band of twisted double-bilayer graphene which appears to host the superconducting phase observed experimentally [47, 48]. Owing to the presence of a gap to other bands in the relevant parameter regime [51–54], it is reasonable to describe the superconducting instability in a single-band picture. We stress, however, that most of our conclusions are symmetry-based and thus, also apply when several bands are taken into account.

Denoting the corresponding electronic creation and annihilation operators by \( c_{\mathbf{k} \sigma v} \), where \( \mathbf{k} \) is crystal momentum, \( \sigma \) spin, and \( v = \pm \) represents the valleys, the general pairing term can be written as

\[
\mathcal{H}_{\text{SC}} = \sum_{\mathbf{k}} c_{\mathbf{k} \sigma v}^\dagger (\Delta_{\mathbf{k}} \gamma_0 \tau_z)^{\sigma_v,\sigma'_{v'}} c_{-\mathbf{k} \sigma' v'}^\dagger + \text{H.c.} \quad (1)
\]

Here and in the following, \( \sigma_j \) and \( \tau_j \) are Pauli matrices in spin and valley space, respectively, and the \( 4 \times 4 \) matrix \( \Delta_{\mathbf{k}} \) is the superconducting order parameter. In
where $C_3$ is the crystalline point group, $\text{SU}(2)_\pm$ is spin rotation in valley $v = \pm$, and $U(1)_v$ corresponds to valley charge conservation. As argued in Ref. 52, the intervalley “Hund’s” coupling $J$ is much smaller than the intravalley-density interaction $V$. In combination with the fact that the noninteracting band structure only has very small valley mixing, the system is invariant under Eq. (2) to a good approximation. In the presence of a finite Hund’s coupling, Eq. (2) is reduced to

$$G_2 = C_3 \times \text{SU}(2)_s \times U(1)_v,$$  \hspace{1cm} (3)

where $\text{SU}(2)_s$ is global spin rotation. To define these symmetries more precisely, we specify their representation on the electronic field operators:

$$C_3 : c_k \rightarrow c_{C_3 k},$$  \hspace{1cm} (4a)

$$\text{SU}(2)_s : c_k \rightarrow e^{i\varphi \sigma} c_k,$$  \hspace{1cm} (4b)

$$\text{SU}(2)\pm : c_k \rightarrow (P_\pm e^{i\varphi \sigma} + P_{\mp}) c_k,$$  \hspace{1cm} (4c)

$$U(1)_v : c_k \rightarrow e^{i\varphi \tau_z} c_k,$$  \hspace{1cm} (4d)

with $P_k = (\tau_0 \pm \tau_z)/2$ being the valley projection operators. Furthermore, time-reversal is represented by the antiunitary operator $\Theta$ with

$$\Theta c_k \Theta^\dagger = i \sigma_y \tau_z c_{-k}.$$  \hspace{1cm} (5)

To classify superconductivity, we proceed as usual [57] and express $\Delta_k$ in Eq. (1) in terms of the IRs $n$ (with dimension $d_n$) of the point group as

$$\Delta_k = \sum_n \sum_{\mu=1}^{d_n} \eta_\mu^n \chi_\mu^n(k), \quad \eta_\mu^n \in \mathbb{C},$$  \hspace{1cm} (6)

where $\chi_\mu^n(k)$ are partner functions transforming under the IR $n$. Within the minimal description of pairing in Eq. (1), which only involves one band per valley, $\chi_\mu^n(k) \in \mathbb{C}^{4 \times 4}$ are matrices in spin and valley space.

In our case, the point group has the form $G_j = C_3 \times U(1)_v \times G_j^s$ with $G_j^s = \text{SU}(2)_+ \times \text{SU}(2)_- \simeq SO(4)$ and $G_2^s = \text{SU}(2)_s$. As a consequence, the IRs of $G_j$ have the form $n = n_{C_3} \times n_v \times n_s$ where $n_{C_3}$, $n_v$, and $n_s$ are IRs of $C_3$, $U(1)_v$, and $G_j^s$ respectively. We can thus rewrite Eq. (6) more explicitly as

$$\Delta_k = \sum_{n_{C_3}} \sum_{n_v} \sum_{n_s} \eta_{\mu_1 \mu_2 \mu_3}^{n_{C_3}}(k) \chi_{\mu_1}^{n_v} \chi_{\mu_2}^{n_s}.$$  \hspace{1cm} (7)

In order to classify superconducting states, we need to consider the different IRs of $C_3$, $U(1)_v$, and $G_j^s$.

Let us begin our discussion of IRs with $U(1)_v$. While it has, in general, countably infinite IRs (one-dimensional and with character $\epsilon^{im_v \varphi}$, $m_v \in \mathbb{Z}$), only three are relevant here as all representations with $|m_v| > 1$ cannot be realized with only two valleys. First, there is the trivial representation, $m_v = 0$, with $\chi_{\mu_0}^{m_v=0} = a \tau_0 + b \tau_z$ with $a, b$ Recalling the extra factor of $\tau_z$ in Eq. (1), this translates to purely intervalley pairing. Secondly, the pair of complex conjugate representations with $m_v = \pm 1$ has to be considered. Note that due to time-reversal symmetry, the complex representations cannot be discussed separately. Here, the basis functions read as $\chi_{\mu_0}^{m_v=\pm 1} = \tau_s \pm i \tau_g$; as such, this corresponds to purely intravalley pairing.

We thus see that $U(1)_v$ prohibits the mixing of inter- and intervalley pairing. As time-reversal (5) interchanges the valleys along with sending $k \rightarrow -k$ and we assume zero-momentum Cooper pairs, we will restrict our discussion to intervalley pairing, i.e., $m_v = 0$ for the rest of the paper.

As is well known [58], $C_3$ has the following IRs, both of which are one-dimensional: the trivial one, $A$, and the complex representation $E$ (and its complex conjugate partner). We analyze each of these IRs in Secs. III and IV, and in both cases, discuss the differences between $G_j$ and $G_j^s$. We will also see how the states “connect” once $G_j^s$ is weakly broken to $G_j^s$ due to a small but finite value of the Hund’s coupling.

### III. TRIVIAL REPRESENTATION OF THE CRYSTALLINE POINT GROUP

For simplicity, we begin with the trivial representation $A$ of $C_3$, which is real and one-dimensional. In fact, the following discussion will not be modified as long as the IR is real and one-dimensional and there is no crystalline symmetry relating the two valleys. Interestingly, the last assumption is violated in twisted bilayer graphene and trilayer graphene on boron nitride, so we treat these systems exclusively in Sec. VI below.

As already mentioned, we consider only intervalley pairing which corresponds to a real and one-dimensional IR as well. This means that the order parameter in
Eq. (7) has the form

$$\langle \Delta_k \rangle_{\sigma,v,\sigma'} = \delta_{v',v} \chi^A(k,v) \sum_{\mu=1}^{d_n} \eta^\mu_n \left( \chi^{\mu}_n(v) \right)_{\sigma \sigma'},$$

(8)

where $\chi^A(k,v)$ is invariant under $k \rightarrow gk$ for all generators $g$ of the crystalline point group (here we only have $g = C_3$).

A. Limit of exact SU(2)$_+ \times$ SU(2)$_-$ symmetry

To proceed further, we have to inspect the scenarios for both $G_1^\dagger$ and $G_2^\dagger$. We start with the former, i.e., we assume that the Hund’s coupling is zero. Inserting Eq. (8) in the general pairing Hamiltonian (1), we obtain a pairing term of the form

$$\mathcal{H}_{\text{SC}} = \sum_{k,v} i_{kv} \left( \mathcal{M}_{kv} i_y \mathcal{G}_{y,v} + \text{H.c.} \right),$$

(9)

where $\mathcal{M}_{kv} = \sum_{\mu=1}^{d_n} \eta^\mu_n \chi^{\mu}_n(v)$ is matrices in spin space. Fermi-Dirac statistics implies

$$\mathcal{M}_{kv} = \sigma_y \mathcal{M}^T_{k\bar{v}} \sigma_y.$$  

(10)

Rewriting pairing in terms of singlet and triplet as $\mathcal{M}_{kv} = \sigma_0 \mathcal{M}^\dagger_{kv} + \mathcal{D}_{kv}$, Eq. (10) is equivalent to $\Delta^\dagger_{kv} = \Delta_{k\bar{v}}$ and $\mathcal{D}_{kv} = \mathcal{D}_{k\bar{v}}$, as expected.

We now study the stable superconducting phases in this channel by writing down the most general Ginzburg-Landau expansion constrained by the symmetries

$$\Theta : M_{kv} \rightarrow M^\dagger_{kv},$$

SU(2)$_+ \times$ SU(2)$_- : M_{kv} \rightarrow e^{-i\varphi \cdot \mathcal{V}} M_{kv} e^{i\varphi \cdot \mathcal{V}}.$$  

(11a)

(11b)

Due to the constraint (10) stemming from Fermi-Dirac statistics, we express the free energy in terms of one valley only (say $v = +$) as $\mathcal{F} = \mathcal{F}[M_{k+} = \chi^A(k,+)\Delta_+]$, and the pairing in the other valley just follows from Eq. (10).

The most general free energy to quartic order in $\Delta_+$, invariant under Eq. (11a) and $\Delta_+ \rightarrow e^{i\varphi} \Delta_+$, reads as

$$\mathcal{F} \sim \frac{a(T)}{2} \left[ \Delta^\dagger_+ \Delta_+ \right] + \frac{b_1}{4} \left( \text{tr} \left[ \Delta^\dagger_+ \Delta_+ \right] \right)^2$$

$$+ \frac{b_2}{2} \left[ \Delta^\dagger_+ \Delta_+ \Delta^\dagger_+ \Delta_+ \right] + \frac{b_3}{4} \left( \text{tr} \left[ \sigma_y \Delta_+ \sigma_y \Delta^\dagger_+ \right] \right)^2.$$  

(12)

Note that $\text{tr} \left( \sigma_y \Delta_+ \sigma_y \Delta^\dagger_+ \right)^2 / 2 = \text{tr} \left[ \Delta^\dagger_+ \sigma_y \Delta^\dagger_+ \sigma_y \Delta_+ \right]$, so the latter is not an independent term to consider. It further holds that $\text{tr} \left[ \sigma_y \Delta_+ \sigma_y \Delta^\dagger_+ \right]^2 / 2 = \left( \text{tr} \left[ \Delta^\dagger_+ \Delta_+ \right] \right)^2 - \text{tr} \left[ \Delta^\dagger_+ \Delta_+ \Delta^\dagger_+ \Delta_+ \right]$, which allows us to set $b_3 = 0$ in the following without loss of generality.

Using the singular-value decomposition of $\Delta_+$, it is straightforward to find all symmetry-invariant minima of Eq. (12). There are two different states depending on the sign of $b_2$, which we label by $A_{m_v=0}(\Delta^\dagger; d)$, where $\Delta^\dagger$ and $d$ refer to the singlet and the triplet vector, respectively. $A$ indicates the trivial IR of $C_3$, and $m_v = 0$ connotes intervalley pairing (IR of $U(1)_v$ with $m_v = 0$). If $b_2 > 0$, we get $\Delta_+ \propto \sigma_0$, i.e., $M_{k\pm} = \lambda_{C_3} \sigma_0$, with $\lambda_{C_3} = \lambda_k$; according to the notation introduced above, this state will be labelled as $A_{m_v=0}(1; 0, 0, 0)$. There are (infinitely) many different equivalent representations of this state as, for instance, the transformations in Eq. (11b) mix the singlet and triplet components—as described by the isomorphism SU(2)$_+ \times$ SU(2)$_-$ so $SO(4)$. However, for the sake of notational clarity, we will henceforth only show one convenient representative of each state. $A_{m_v=0}(1; 0, 0, 0)$ state preserves time-reversal symmetry and breaks SU(2)$_+ \times$ SU(2)$_-$ down to SU(2)$_+$. Rotations of the total spin along a single axis.

B. Turning on the Hund’s coupling

In reality, there is, of course, a finite Hund’s coupling that reduces $G_1^\dagger = SU(2)_+ \times SU(2)_-$ to only global spin rotations, $G_2^\dagger = SU(2)_+$ already in the high-temperature phase. In Ref. 52, the Hund’s coupling $J$ has been estimated to be about 60 times smaller than the intravalley interaction $V$. However, as already pointed out in the introduction, $J$ might be enhanced due to loop corrections. For this reason, we first classify the possible instabilities in the absence of an approximate SU(2)$_+ \times$ SU(2)$_-$ symmetry and then, analyze how the different states “connect” for small values of $J$ and whether admixtures of singlet and triplet are possible.

To introduce our notation, we will begin with the classification for the reduced symmetry group $G_2$ in Eq. (3); in that case, we will have either singlet or triplet pairing:  

a. Singlet: This corresponds to the $d_{n_v} = 1$ one-dimensional IR of $G_2^\dagger$ with $\chi_{n_v}^\dagger = \sigma_0$ in Eq. (8). The pairing Hamiltonian simply has the form

$$\mathcal{H}_{\text{SC}} = \sum_{k,v} \lambda^\dagger_{kv} c_{kv}^\dagger (i\sigma_y)_{\sigma,\sigma'} c_{-kv}^{\dagger \sigma'} + \text{H.c.},$$  

(13)

with $\lambda^\dagger_{kv} = \lambda^\dagger_{-kv}$ and $\lambda_{C_3} = \lambda_{k\sigma}$. All symmetries of the high-temperature phase are preserved. We refer to this state as $A_{m=0}$ with the $I_s$ referring to spin singlet,
structure would favor breaks time-reversal symmetry and will be denoted by $\lambda d$ instead, $b$ is the electronic band energy in valley $v$. Here, $\omega$ this term preserves time-reversal symmetry and breaks $\sigma \rightarrow -\sigma$. One might wonder what kind of interaction or band structure would favor $A_{m=0}^3$, $A_{m=0}^5(1,0,0)$, and $A_{m=0}^7(1,1,0)$, connect to the two derived in the previous subsection with enhanced SU(2) $\times$ SU(2) symmetry, namely $A_{n=0}^2(1,0,0,0)$ and $A_{m=0}^3(1,1,0,0)$. To this end, we decompose the Ginzburg-Landau expansion, Eq. (12), of the previous section into singlet and triplet by writing $\Delta_+ = \Delta^s + \sigma d$. 

Observe that $|d^T d|^2$ is not an independent quartic term since $|d^T d|^2 = (d^2 d)^2 - |d^* d|^2$. The free energy in Eq. (14) has two stable minima. For $b_2 > 0$, we have $d \propto (1, i, 0)^T$, whence

$$\mathcal{H}_{SC} = \sum_{k,v} \lambda_{k,v}^s c^\dagger_{k\sigma \uparrow} c_{k\sigma \downarrow} + \text{H.c.},$$

(15)

with $\lambda_{k,v}^s = -\lambda_{-k,\bar{v}}^s$ and $\lambda_{-k,\bar{v}}^s = \lambda_{k,v}^s$. As is easily seen, this term preserves time-reversal symmetry and breaks SU(2) down to spin rotation along a single axis. This state will be referred to as unitary triplet and denoted by the symbol $A_{m=0}^3(1,0,0)$, where the components just indicate the direction of the triplet vector. If, instead, $b_2 < 0$, we obtain $d \propto (1, i, 0)^T$, whence

$$\mathcal{H}_{SC} = \sum_{k,v} \lambda_{k,v}^t c^\dagger_{k\sigma \uparrow} ((\sigma_x i \sigma_y) c_{k\sigma \downarrow} + \text{H.c.},$$

(16)

with $\lambda_{k,v}^t$ as above. This is a nonunitary triplet state. It breaks time-reversal symmetry and will be denoted by $A_{m=0}^3(1,1,0)$.

One might wonder what kind of interaction or band structure would favor $A_{m=0}^7(1,0,0)$ over $A_{m=0}^5(1,0,0)$ and vice versa. In mean-field theory, as detailed in Appendix A, it is straightforward to show by evaluation of a one-loop diagram that

$$b_2^2 = T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{|\lambda_{k}^s|^4}{(\omega_n^2 + \xi_{k+}^2)^2}. $$

(17)

Here, $\omega_n$ are fermionic Matsubara frequencies and $\xi_{k+}$ is the electronic band energy in valley $v = +$ of the nearly-flat band hosting superconductivity. We observe that $b_2^2 > 0$ holds irrespective of microscopic details and hence, $A_{m=0}^3(1,0,0)$ is generically favored if we neglect corrections beyond mean-field theory (such as residual interactions or frequency dependence of pairing). Intriguingly, there have been experimental reports [59] of intrinsically nonunitary pairing in LaNiC$_2$, i.e., nonunitary triplet pairing born out of a paramagnetic normal state. Thus, there is reason to believe that we cannot generically exclude this state, but we do not expect it to show up in any simple mean-field computation.

1. How do the states connect in the $J = 0$ limit?

Next, we establish how the three possible states, $A_{m=0}^3$, $A_{m=0}^5(1,0,0)$, and $A_{m=0}^7(1,1,0)$, connect to the two derived in the previous subsection with enhanced SU(2) $\times$ SU(2) symmetry, namely $A_{n=0}^2(1,0,0,0)$ and $A_{m=0}^3(1,1,0,0)$. To this end, we decompose the Ginzburg-Landau expansion, Eq. (12), of the previous section into singlet and triplet by writing $\Delta_+ = \Delta^s + \sigma d$. Since $\text{tr} \left[ \Delta_+^\dagger \Delta_+ \right] = |\Delta^s|^2 + |d^*|^2$ singlet and triplet are degenerate at quadratic order in $\mathcal{F}$ as a consequence of the enhanced SU(2) $\times$ SU(2) symmetry. For nonzero $J$, this degeneracy is lifted and we have

$$\mathcal{F} \sim a(T) \left( |\Delta^s|^2 + |d^*|^2 \right) + \delta a(T) \left( |\Delta^s|^2 - |d^*|^2 \right),$$

(18)

where $\delta a$ can be made arbitrarily small as $J \rightarrow 0$. Neglecting, for now, the “back action” of the superconducting order parameter that condenses first on the second one (as described by higher-order terms in the Ginzburg-Landau expansion), we conclude that there are two superconducting transitions at $T_{c0}^\pm = T_{c0}^1 \pm \Delta T_c^0$ with $\Delta T_c = |\delta a(T)|/\alpha$, taking $a(T) \sim \alpha(T - T_c^0)$ near $T_c^0$. The extra index 0 in $T_{c0}^0$ highlights the fact that the aforementioned higher-order terms in the Ginzburg-Landau expansion can significantly affect the lower transition temperature, $T_c^0 \neq T_{c0}^0$. Of course, this has no effect on the higher transition temperature, $T_c^+ = T_{c0}^+ \neq T_{c0}^0$.

Before analyzing these corrections, it is useful to estimate the temperature scale $\Delta T_c$. Using the expected result, $\Delta T_c^0 \sim \Lambda \exp(-1/|V|)$ of mean-field theory (from the linearized gap equations), where $\Lambda$ is the cutoff and $\nu$ the density of states at the Fermi level, this leads to

$$\frac{\Delta T_c}{T_c^0} \sim \frac{|J|}{V^2 \nu}. $$

(19)

The large density of states, taken together with the estimated value of $V$—which is larger than even the bandwidth [52] of the flat bands—and the relation $J \ll V$, implies that $\Delta T_c \ll T_c^0$ [60]; the temperature/energy scale $\Delta T_c$ is most likely too small to be visible in experiments. While the estimate above is only based on mean-field theory, it indicates at least that it is important to study the behavior of superconductivity in the limit of small $\Delta T_c/T_c^0$ (and hence, weakly broken SU(2) $\times$ SU(2) $\times$ SU(2) symmetry), accounting for the possibility of two transitions and mixing of singlet and triplet pairing (despite the absence of spin-orbit coupling). Moreover, we will see that nearly degenerate singlet and triplet pairing also has crucial consequences for the behavior of superconductivity in the presence of a magnetic field.

While we postpone the analysis of magnetic fields to Sec. III C, here, we investigate the possibility of an ad-
mixture of singlet and triplet in the presence of time-reversal symmetry [relevant to scenario (I) defined in the introduction]. As anticipated above, this requires also considering the quartic terms of Eq. (12). We find

\[ F \sim a(T) \left( |\Delta|^2 + d^\dagger d \right) + \delta a \left( |\Delta|^2 - d^\dagger d \right) + (b_1 + b_2)|\Delta|^4 + (b_1 + b_2) (d^\dagger d)^2 + b_2 |d^\dagger |d|d^\dagger |d|d^\dagger |d|^2 \]

+ 2(b_1 + 2b_2)|\Delta|^2 d^\dagger d + 2b_2 \mathrm{Re} \left( \langle \Delta \rangle^2 |d^\dagger |d \rangle^2 \right), \quad (20)

neglecting corrections to the quartic terms coming from finite \( J \).

Looking at the first transition with the higher transition temperature, we assess which of the two distinct triplet states, \( A_{m_\nu=0}(1, 0, 0) \) and \( A_{m_\nu=0}(1, i, 0) \), and the singlet state can be stabilized by starting from \( A_{m_\nu=0}(1; 0, 0, 0) \) or \( A_{m_\nu=0}(1; 1, 0, 0) \) and turning on a finite Hund’s coupling \( J \). For this purpose, we can neglect the coupling terms in the third line of Eq. (20). Clearly, if \( \delta a < 0 \) (“anti-Hund’s coupling”), we get a singlet state for both \( A_{m_\nu=0}(1; 0, 0, 0) \) and \( A_{m_\nu=0}(1; 1, 0, 0) \). A straightforward way of establishing which of the triplet states is realized when \( \delta a > 0 \) ("conventional" Hund’s coupling) proceeds by evaluating their respective free energy in Eq. (20). One finds that the state \( A_{m_\nu=0}(1, 0, 0) \) is realized if \( b_2 > 0 \); otherwise, \( A_{m_\nu=0}(1, i, 0) \) is favored. This brings us to the conclusion that

\[ A_{m_\nu=0}(1; 0, 0, 0) \rightarrow A_{m_\nu=0}^1(0, 0) \quad \text{or} \quad A_{m_\nu=0}^3(1, 0, 0), \quad (21a) \]

\[ A_{m_\nu=0}(1; 1, 0, 0) \rightarrow A_{m_\nu=0}^1(1, 0) \quad \text{or} \quad A_{m_\nu=0}^3(1, i, 0), \quad (21b) \]

at the first transition (see the schematic phase diagram in Fig. 1). This result is just a consequence of the fact that the form \( \Delta_+ \propto \sigma_0 \) for the \( A_{m_\nu=0}(1; 0, 0, 0) \) state we had chosen in the previous section can alternatively be written as \( \Delta_+ \propto \sigma_x \) due to the SU(2) \( _+ \times \) SU(2) \( _- \) symmetry and, thus, explicitly assumes the form of the unitary triplet state. Similarly, \( \Delta_+ \propto \sigma_0 + i\sigma_y \) used above for \( A_{m_\nu=0}(1; 1, 0, 0) \) can also be written as \( \Delta_+ \propto \sigma_x + i\sigma_y \). This is why it transitions into the nonunitary triplet state, upon turning on a nonzero Hund’s coupling.

In order to determine whether there is a second transition, we have to include the coupling terms between singlet and triplet in the third line of Eq. (20). To illustrate that these terms can be crucial, we consider the case \( \delta a > 0 \) and \( b_2 > 0 \), i.e., the triplet state \( A_{m_\nu=0}^3(1, 0, 0) \) condenses first. This leads to the coupling between singlet and triplet \( 2c|\Delta|^2|d(T)|^2, \ c = b_1 + b_2 \), in the free energy, where we have made use of the fact that a relative phase of \( \pi/2 \) between singlet and triplet is energetically most favorable. As a result of \( |d(T)|^2 = (\delta a - a(T))/2c \), which is valid as long as there is no additional singlet pairing, the growing triplet component induces the extra term

\[ 2c|\Delta|^2|d(T)|^2 = (\delta a - a(T))|\Delta|^2, \quad (22) \]

which is always larger than the “bare” quadratic term of singlet pairing [in the first line of Eq. (20)]. Accordingly, there is no second transition (at least close to \( T_{c,0} \) where our Ginzburg-Landau approach is valid) into a state that has a nonzero singlet component. We also checked that Eq. (20) does not allow for a first-order transition.

Similarly, all other cases can be scrutinized and one finds that if triplet dominates, there is no second transition. However, if singlet has a larger transition temperature (\( \delta a < 0 \)), there is a second transition into a phase with singlet and triplet pairing when \( b_2 < 0 \). This transition happens at the temperature

\[ T_{c}^{-} = T_{c,0} \left( 1 + \frac{c - |b_2|}{|b_2|} \right), \quad \delta \equiv \frac{\delta a}{\alpha T_{c,0}} \approx \frac{J}{\sqrt{2} \nu}. \quad (23) \]

The stability of the Ginzburg-Landau expansion only requires \( c > 0 \) and \( c > -b_2 \), so both \( T_{c}^{-} < T_{c,0} \) and \( T_{c}^{-} > T_{c,0} \) are possible. More importantly, unless \( |b_2|/c \) is fine-tuned to be of order \( \delta \), generally, \( T_{c}^{-} \rightarrow T_{c,0} \) as \( J \rightarrow 0 \) and the two transitions, if present, are likely too close to be experimentally discernible. Due to the term \( 2b_2 \mathrm{Re}[\langle \Delta \rangle^2 |d|^2] \) in the free energy, we obtain the unitary triplet vector \( d = d_0(1, 0, 0)^T \) with \( |\Delta|^2 d_0^\dagger d_0 \in \mathbb{R} \) (same phase). This is to be expected as \( \Delta_+ \propto \sigma_0 + i\sigma_z \) for the “parent” state \( A_{m_\nu=0}(1; 1, 0, 0) \).

A summary of these results is provided by the schematic phase diagrams in Fig. 1. We observe that the proximity to the enlarged symmetry in spin space, SU(2) \( _+ \times \) SU(2) \( _- \), favors the possibility of having a nonzero triplet component: for \( b_2 < 0 \), even a negative Hund’s coupling (anti-Hund’s) allows for \( d \neq 0 \) and leads to the exotic possibility of significant \( d_0 \simeq \Delta^s \) for
We believe that the current status of experiments does not allow to exclude pairing phases that will not exhibit quasi-long-range order and a BKT transition in the limit of infinite system size.

\[ T_c - T > \Delta T_c \] singlet-triplet mixing in spite of the absence of spin-orbit coupling.

It is noteworthy that all the states are fully gapped (more precisely, they have no symmetry-enforced nodes) except for the nonunitary triplet with \( A^3_{m_z = 0}(1, i, 0) \), which is gapped for one spin species while the other is completely gappless. The admixture of singlet and unitary triplet has two unequal gaps for the two spin species both of which are finite as long as the magnitudes of singlet and triplet are not fine-tuned to be equal. All the states, along with their order parameters and properties, are summarized in Table I.

We finally comment on the nature of the thermal phase transition for the different superconducting states once fluctuations of the order parameter are taken into account. Neglecting stray fields, the transition into the singlet phase \( A^{1s} \) is expected to be a BKT transition with quasi-long-range order of the complex-valued order parameter \( \Delta^s \) below the transition temperature. For the triplet states, it is important to keep in mind that \( \mathbf{d} \) cannot even have quasi-long-range order as it transforms as a three-component vector under spin-rotation. For the unitary triplet state [with order parameter manifold \( S_2 \times S_1/\mathbb{Z}_2 \)] a BKT transition of the composite charge-4\( e \) order parameter \( \mathbf{d}^2 \mathbf{d} \) is possible and is associated with the (un)binding of half vortices. This is different for the nonunitary state [with order parameter manifold \( S_3/\mathbb{Z}_2 \cong SO(3) \)] where \( \mathbf{d}^2 \mathbf{d} = 0 \) and no BKT transition into a quasi-long-range ordered superconductor is expected. For the case of the two consecutive transitions in Fig. 1(b) with \( \delta < 0 \), we first expect a BKT transition into a singlet phase followed by a crossover at which the triplet vector becomes nonzero.

However, we point out that, even in the simplest case of the singlet \( A^{1s} \), there are significant corrections to the BKT transition resulting from stray fields and mirror vortices [61], which make the observation of a pristine BKT transition in a (charged) superconductor difficult. We believe that the current status of experiments does not allow to exclude pairing phases that will not exhibit quasi-long-range order and a BKT transition in the limit of infinite system size.

### 2. Expectations within mean-field theory

Lastly, we evaluate what a naïve mean-field computation is expected to yield. In fact, from Eq. (17), we already know that the prefactor of the term \( |\mathbf{d}^2 \times \mathbf{d}|^2 \) in Eq. (20) must be positive within mean-field theory and therefore, it holds that \( b_2 > 0 \). For completeness, we mention that in the mean-field approximation, \( b_1 = 0 \), as shown in Appendix A. Consequently, a single-band mean-field computation will generally favor Fig. 1(a) over (b); in other words, only half of the phases proposed in this section can be found in mean-field, which we also indicate in the last column of Table I.

However, there is no fundamental mechanism prohibiting the mixing of singlet and triplet via two transitions (see, e.g., Ref. 62) and there are multiple reasons why we can effectively have \( b_2 < 0 \) (and \( b_1 > 0 \) to ensure stability): for instance, strong residual interactions and fluctuations have been shown to modify the values of the quartic terms in the free energy significantly [40, 63], thereby stabilizing phases that are otherwise not possible in the mean-field approximation. Given the underlying strong-coupling features of the problem, it is plausible that there are sizable corrections to mean-field theory. In addition, disorder can dress the Ginzburg-Landau expansion and it is unclear whether adding frequency dependence to the gap function could be of relevance.

In Sec. V, we will analyze the impact of spin fluctuations, which are expected to be relevant for twisted-double bilayer graphene, and find that these generically decrease the value of \( b_2 \); if sufficiently strong, these fluctuations will favor the phase diagram in Fig. 1(b).

### Table I. Summary of the different intervalley pairing states transforming under the trivial representation of the point group \( C_3 \) in the absence of a magnetic field. For notational convenience, we neglect the extra label \( m_z = 0 \) to indicate intervalley pairing. \( \lambda_k \) is a real-valued and Brillouin-zone-periodic function that is invariant under \( C_3 \). To lowest order, we can take \( \lambda_k \) to be independent of \( k \). We also indicate the minimal number of nodes, which state it transforms to when setting \( J = 0 \) ["SO(4) parent"] and reversing the sign of \( J \) ("Hund’s partner"), and whether the state can be found in a single-band mean-field (MF) computation neglecting residual interactions. In the last line, \( \eta \) describes the temperature-dependent strength of admixing of the unitary triplet state.

| Pairing        | \( M_{k^+} \) | Nodes         | SO(4) parent | Hund’s partner | Possible in MF |
|----------------|--------------|---------------|--------------|----------------|----------------|
| \( A^{1s} \)   | \( \lambda_k \sigma_0 \) | none          | \( A(1; 0, 0, 0) \) | \( A^{3s}(1, 0, 0) \) | ✓              |
| \( A^3(1, 0, 0) \) | \( \lambda_k \sigma_z \) | none          | \( A(1; 0, 0, 0) \) | \( A^1 \)       | ✓              |
| \( A^3(1, i, 0) \) | \( \lambda_k (\sigma_z + i\sigma_y) \) | gapless/none  | \( A(1; 1, 0, 0) \) | \( A^{1s} + A^{3s}(1, 0, 0) \) | ✗              |
| \( A^{1s} + A^{3s}(1, 0, 0) \) | \( \lambda_k (\sigma_0 + \eta \sigma_z) \) | none          | \( A(1; 1, 0, 0) \) | \( A^{3s}(1, i, 0) \) | ✗              |
C. In the presence of a magnetic field

As asserted above, we now generalize the Ginzburg-Landau expansion to also include the coupling to a Zeeman field $M_Z$ and an (in-plane) orbital coupling $M_O$. Both of these terms can either be due to an applied external magnetic field or due to the correlated insulating state \([47–49]\). This enables us to discuss (i) the behavior of the superconducting critical temperature $T_c^+$ as a function of an external magnetic field in the absence of any ferromagnetic moments associated with the correlated insulating state [case (I) defined in the introduction]. At the same time, we can study (ii) how the transition temperature and the order parameter of superconductivity is affected by the potentially coexisting ferromagnetic order [case (II)].

1. Leading superconducting transition

We first turn our attention to the leading superconducting transition with the highest temperature $T_c^+$; potential subsequent superconducting transitions at lower temperatures are addressed later in Sec. III C 2. For the goal of studying the first transition, we can restrict ourselves to quadratic order in the order parameter. Only keeping terms up to quadratic order in the magnetic field as well, we obtain

$$F_M \sim a(T) \left( |\Delta|^2 + d^i d^j + \delta a \left( |\Delta|^2 - d^i d^j \right) \right)$$

$$+ 2\delta c_1 M_Z \cdot \text{Im} \left( d^i \Delta^j + i c_2 M_Z \cdot d^i \times d^j \right)$$

$$+ \left( c_3 M_Z^2 + c_4 M_O^2 \right) \left( |\Delta|^2 + d^i d^j \right)$$

$$+ (\delta c_6 M_Z^2 + \delta c_6 M_O^2) \left( |\Delta|^2 - d^i d^j \right)$$

$$+ 2c_7 M_O M_Z \cdot \text{Re} \left( d^i \Delta^j \right).$$

(24)

While the prefactors $\delta a$, $\delta c_1$, $\delta c_3$, and $\delta c_6$ are necessarily zero in the limit $J \to 0$, where the SU($2)_+ \times$ SU($2)_-$ symmetry becomes exact, all remaining terms can be nonzero (and different in their values) at $J = 0$. Notice that the third and last terms have not been considered in Ref. 52; these terms arise only when both singlet and triplet are allowed for and lead to the admixture of a unitary triplet state with a singlet superconductor. The vanishing of $\delta a$ and $\delta c_6$ at $J = 0$ is an obvious consequence of the enhanced SU($2)_+ \times$ SU($2)_-$ symmetry. To see that $\delta c_1$ also has to vanish as $J \to 0$, let us take $M_Z$ along the $z$ direction; this breaks SU($2)_+ \times$ SU($2)_-$ down to O($2)_+ \times$ O($2)_-$, i.e., the system is only invariant under $c_{kv} \to e^{i\varphi} c_{kv}$. Performing this transformation with $\varphi = 0$ and $\varphi = \pi/2$, we get $(\Delta^i, d^i) \to \left( i d^i, i \Delta^i \right)$ and hence, $\delta c_1 \to -\delta c_1$. With the same argument, it can be proven that $\delta c_3$ has to go to zero as $J \to 0$. In Appendix A, we show that $\delta c_1 = 0$ in mean-field theory within the single-band description, even when SU($2)_+ \times$ SU($2)_-$ is broken; this results from an emergent valley-exchange symmetry within the single-band mean-field approximation. However, $c_7$ is not constrained to vanish as a consequence of the emergent symmetry and we derive an explicit microscopic expression for it, see Eq. (A10).

In discussing the highest critical temperature and the corresponding order parameter for $M_Z, M_O \neq 0$, it is instructive to first look at the linear-in-field terms in Eq. (24). We find two different cases. If $|c_2 M_Z| + \delta a > \sqrt{(\delta c_1 M_Z)^2 + \delta a^2}$, one obtains a pure triplet state of the type $A^{(\nu)^{M_Z}}_{\nu v=0}(1, i, 0)$. Choosing $M_Z = M_Z e_x$ with $M_Z > 0$, the triplet vector is given by $d = (0, 1, \text{sign}(c_2) i)^T$ and the critical temperature is

$$T_c = T_{c,0} + (\delta a + |c_2 M_Z|) / \alpha.$$

(25)

![FIG. 2. Phase diagram as a function of temperature $T$ and Zeeman field $M_Z = M_Z e_x$ when (a,b) singlet dominates at low fields and (c,d) triplet dominates, which we obtain by minimizing Eq. (28). Thin (thick) black lines correspond to second (first) order transitions. The phases for $M_Z = 0$ are indicated in red and we recover the four different possible temperature dependences of Fig. 1. Recall from Sec. III B 2 that $b_2 > 0$ is expected in mean-field theory. However, as we will see in Sec. V, strong ferromagnetic fluctuations will favor $b_2 < 0$. As symmetry requires $\delta c_1$ to be proportional to the Hund’s coupling $J$, we have set $\delta c_1 = 0$ here. For nonzero $\delta c_1$, the singlet superconducting phases will contain an admixture of unitary triplet as described by Eq. (26) and a first-order transition into a singlet state (with unitary triplet admixture) will be possible at lower temperatures and nonzero Zeeman field in part (c).]
Else, if $|c_2 M_Z| + \delta a < \sqrt{\delta c_1 M_Z^2 + \delta a^2}$, one finds an admixture of singlet and triplet with order parameter

$$\Delta^s = \Delta_0, \quad d = i e x \Delta_0 \frac{\delta c_1 M_Z}{\sqrt{\delta c_1 M_Z^2 + \delta a^2} - \delta a}. \quad (26)$$

The transition temperature in this case is

$$T_c = T_{c,0} + \sqrt{\delta a^2 + (\delta c_1 M_Z)^2}/\alpha. \quad (27)$$

We see from Eq. 26) that there is an approximately equal mixing of singlet and triplet for $|\delta c_1 M_Z| \gg \delta a$ while in the opposite limit, $|\delta c_1 M_Z| \ll \delta a$, either singlet or triplet dominates depending on whether $\delta a < 0$ or $\delta a > 0$. The relative phase of $\pi/2$ between singlet and triplet makes the pairing state break time-reversal symmetry as is required in order to couple linearly to magnetic moments.

To understand how the approximate SU(2)$_{+}\times$SU(2)$_{-}$ symmetry can naturally explain the linear-in-magnetic field behavior, we first consider case (II), i.e., there is already microscopically coexisting ferromagnetic order (or there is at least a significant coupling between superconductivity and the ferromagnetic moments) at $T_c$. Then, $M_Z$ and $M_O$ should be thought of as the combination of the applied external magnetic field and the ferromagnetic order parameter. In this scenario, it is apt to assume $\delta a \ll \max(\delta c_1 M_Z, c_2 M_Z)$ and we generically obtain a linear increase of the critical temperature with magnetic field [see Eqs. 25 and 27]. If $c_2 > \delta c_1$, we obtain the nonunitary triplet state with $d \propto (1, i, 0)^T$, which we expect close to the $J = 0$ line, while $\delta c_1 > c_2$ leads to the admixture of singlet and triplet with $d \propto (1, 0, 0)^T$. As $\delta c_1$ vanishes for $J = 0$ and in single-band mean-field theory (even when $J \neq 0$), we expect the former scenario to be more likely, which will favor the nonunitary triplet state as the leading instability.

In the case of scenario (I), we should view $M_Z$ and $M_O$ in Eq. 24 as resulting entirely from the Zeeman and orbital coupling of the external magnetic field alone. For large magnetic fields where $\delta a \ll \max(\delta c_1 M_Z, c_2 M_Z)$, the same conclusions as above will apply and $T_c$ will generically vary linearly with the field. However, for sufficiently small magnetic fields, we have $\delta a \gg \max(\delta c_1 M_Z, c_2 M_Z)$. In this limit, only $\delta a > 0$ favoring the nonunitary triplet pairing $A_{n=0}^{d_{+}}(1, i, 0)$ is consistent with the transition temperature changing linearly with magnetic field. Alternatively, the system could ultimately be in a singlet state at $M_Z = 0$ (i.e., $\delta a < 0$) but the magnitude of $\delta a$ is sufficiently small such that the “rounding off” of $T_c^+(M_Z)$ at low $M_Z$ cannot be seen in experiment.

2. Quartic terms and sub-leading transitions

Having examined the first superconducting transition that takes place upon cooling the system down starting from the normal state, we now assess whether and what type of subsequent superconducting transitions can occur. In this context, we need to include terms quartic in the superconducting order parameter and extend Eq. (24) to

$$F_M \sim (a(T) + \delta a)|\Delta^s|^2 + (a(T) - \delta a) \sum_{s=\pm,0} |d_s|^2 \quad (28)$$

$$+ 2\delta c_1 M_Z \text{Im}(d_0^* \Delta^s) + c_2 M_Z (|d_+|^2 - |d_-|^2) \quad (29)$$

$$+ (b_1 + b_2) (|\Delta^s|^4 + |d_0|^4) + (b_1 + 2b_2) (|d_+|^4 + |d_-|^4) \quad (30)$$

$$+ 2(b_1 + 2b_2)|\Delta^s|^2 \sum_{s=\pm,0} |d_s|^2 - 4b_2 \text{Re} [d_0^* d_+^* d_-^*] \quad (31)$$

$$+ 2b_2 \text{Re} [|\Delta^s|^2 (|d_0|^2 + 2|d_+|^2|d_-|^2)] \quad (32)$$

$$+ 2b_1 |d_+|^2 |d_-|^2 + 2(b_1 + 2b_2)|d_0|^2 (|d_+|^2 + |d_-|^2),$$

where we kept only the terms linear in magnetic field, took $M_Z$ along the $z$-axis, and re-expressed the triplet in the form $d = d_0(1, i, 0)/\sqrt{2} + d_-(1, -i, 0)/\sqrt{2} + d_0(0, 0, 1)$. This parametrization is more convenient in the presence of a magnetic field than that used in Eq. (20). Additionally, we have neglected the impact of the magnetic field on the quartic terms.

Taking $\delta c_1 = 0$ (as it has to vanish for $J = 0$), the different possible phase diagrams are summarized in Fig. 2. The possibility illustrated in part (c) of Fig. 2 corresponds to the picture put forward by Ambegaokar and Mermin [64] for He$_3$ in the presence of a magnetic field, which might very well also apply to twisted double-layer graphene [48, 52]. The difference with Ref. 64 is that we do not get a third transition since we work with a one-dimensional IR.

However, there are three other options, depicted in Fig. 2(a), (b), and (d), that we cannot easily exclude given the experimental data: owing to the strong-coupling properties of the problem at hand or other reasons propounded above, a nonunitary triplet state might be dominant at $M_Z = 0$, as seems to be the case in LaNiC$_2$ [59] and is favored by our fluctuation approach of Sec. V under this condition, only one transition is expected even when $M_Z \neq 0$ [see Fig. 2(d)]. It could also be that singlet dominates without a magnetic field instead. We can see in Fig. 2(a) and (b) that, in these two cases, triplet shows up and $T_c$ increases linearly when $M_Z > 2|\delta a|/c_2$. The small value of $\Delta T_c/T_b^c$ estimated in Eq. (19) suggests that resolving this initial region, where $T_c$ is constant as a function of $M_Z$, is experimentally challenging.

3. Nonlinear couplings in a magnetic field

We finally come back to the nonlinear couplings in Eq. (24), which have two main effects: they lead to a suppression of superconductivity, associated with $c_{3,4}$ and...
δC_{5,6}, and can induce an admixture of singlet and triplet resulting from the term \( \propto C_7 \). As for the former effect, we notice that the suppression of singlet and triplet is enforced to be nearly identical for small \( J \) due to the SU(2)\(_+\) × SU(2)\(_-\) symmetry. Resultantly, if the effective \( J \) relevant for superconductivity is indeed small, the nonlinear terms \( \propto M_2^2, M_3^2 \) are not expected to affect the competition between singlet and triplet significantly. Contrarily, the term \( C_7 \) can indeed be relevant as it can modify the phase diagrams in Fig. 2: for instance, a nonzero value of \( M_{OCA} \) will lead to an admixture of unitary triplet pairing with the singlet phase in Fig. 2(a). If \( M_{OCA} > C_2 \), there will be an additional first-order transition into a singlet state with unitary triplet component at lower temperatures in Fig. 2(c).

IV. COMPLEX REPRESENTATION OF \( C_4 \)

In this section, we extend our previous analysis to the complex IR \( E \) of the spatial point group \( C_3 \). Time-reversal symmetry necessitates treating the representation and its complex-conjugate partner on an equal footing. Alternatively, one can think of a two-dimensional (reducible) representation with partner functions transforming as \( x \) and \( y \) under \( C_3 \).

Akin to our discussion earlier, we first study the case of nonzero Hund’s coupling, \( J \neq 0 \), with point group \( G_2 \) in Eq. (3), which enables us to distinguish between singlet and triplet pairing. After discussing all symmetry-allowed singlet and triplet states separately, we will derive the phase diagrams analogous to Fig. 1: we will examine how these states “connect” when adiabatically changing the Hund’s coupling from negative to positive values, and whether singlet and triplet can mix when changing the Hund’s coupling from negative to positive.

A. Nonzero Hund’s coupling

To proceed with singlet pairing, we parametrize \( M_{kv} \) in Eq. (9) according to

\[
M_{k+} = \sum_{\mu=\pm} \eta_{\mu} (X_k + i \mu Y_k) \sigma_0, \tag{29}
\]

while \( M_{k-} \) is determined by the Fermi-Dirac constraint (10); \( X_k \) and \( Y_k \) are real-valued functions that are continuous on the Brillouin zone and transform as \( k_x \) and \( k_y \) under \( C_3 \). A one-parameter family of possible choices for the lowest-order functions (i.e., with minimal number of sign changes in the Brillouin zone) is given by

\[
(X_k, Y_k)^T = R_{\phi} \left( X_k^{(1)}, Y_k^{(1)} \right)^T \tag{30a}
\]

with arbitrary \( \phi \in [0, 2\pi) \), where \( R_{\phi} \) is a 2 × 2 matrix describing rotations by angle \( \phi \), \( R_{\phi} = e^{i\phi \sigma_3} \), and

\[
X_k^{(1)} = \frac{2}{\sqrt{3}} \sin(\sqrt{3}k_x/2) \cos(k_y/2), \tag{30b}
\]

\[
Y_k^{(1)} = \frac{2}{3} \left( \sin k_y + \cos(\sqrt{3}k_x/2) \sin(k_y/2) \right). \tag{30c}
\]

Both \( X_k \) and \( Y_k \) have to vanish at \( \Gamma, K, \) and \( K’ \) as these momenta are invariant under \( C_3 \). Further, both \( X_k \) and \( Y_k \) must have lines of zeros going through these high symmetry points; the orientation of these lines is, however, not fixed due to the absence of additional reflection or in-plane rotation symmetries—this is different from the situation for twisted bilayer and trilayer graphene in Sec. VI. For Eq. (30), the orientation of these zeros changes with \( \phi \).

With the parametrization defined in Eq. (29), the relevant symmetries act as follows

\[
\begin{align*}
C_3: \quad (\eta_+, \eta_-) &\rightarrow (\omega \eta_+, \omega^* \eta_-), \quad \omega = e^{i\frac{2\pi}{3}}, \quad (31a) \\
\Theta: \quad (\eta^*, \eta_-) &\rightarrow (\eta_-, \eta^*). \quad (31b)
\end{align*}
\]

It readily follows from Eq. (31) that the most general free energy up to quartic order reads as

\[
\mathcal{F} \sim a(|\eta_+|^2 + |\eta_-|^2) + b_1^*(|\eta_+|^2 + |\eta_-|^2)^2 + b_2^*|\eta_+|^2|\eta_-|^2. \tag{32}
\]

The sign of \( b^*_2 \) therefore distinguishes between two different singlet phases: if \( b^*_2 > 0 \), we have \( (\eta_+, \eta_-) = (1,0) \) which corresponds to

\[
M_{k+} = (X_k + i Y_k) \sigma_0. \tag{33}
\]

Exactly as in Sec. III, we always show only one out of the many symmetry-equivalent representations of the order parameter—instead of using a general parametrization of a phase—to make the notation and the discussion of properties of the superconducting state more easily accessible. The state in Eq. (33) breaks time-reversal symmetry but preserves \( C_3 \) (and spin-rotation symmetry).

We refer to this state as a chiral singlet superconductor and denote it by \( E^{1s}(1, i) \) in the following. It is fully gapped (unless the Fermi surfaces go through the \( \Gamma, K, \) or \( K’ \) point) and has been investigated extensively in the recent literature on pairing in twisted bilayer graphene [10, 19, 24, 27, 30, 33, 34, 37, 39, 41].

Conversely, if \( b^*_2 < 0 \), we find that \( |\eta_+| = |\eta_-| \) at the minimum of Eq. (32). As the relative phase \( \varphi \) between \( \eta_+ \) and \( \eta_- = \eta_+ e^{i\varphi} \) is not fixed by Eq. (32), one might naively conclude that higher order terms have to be considered. In fact, in sixth order, there is indeed the contribution

\[
c_1 \text{Re} \left[ \eta_+^3 (\eta_-^*)^3 \right] + c_2 \text{Im} \left[ \eta_+^3 (\eta_-^*)^3 \right], \quad c_{1,2} \in \mathbb{R}, \tag{34}
\]

and the relative phase \( \varphi \) will depend on \( c_1/c_2 \). However, upon reinserting \( \eta_- = \eta_+ e^{i\varphi} \) into Eq. (29), we notice that...
\[ \varphi \neq 0 \] simply corresponds to rotating the basis functions \( X_k \) and \( Y_k \) into each other, which does not change their transformation behavior under \( C_3 \) \( \varphi \) is directly related to \( \phi \) in Eq. (30a)\). Consequently, we can set \( \varphi = 0 \) without loss of generality, which implies

\[ M_{kv} = \Delta_s X_k \sigma_0. \]  

This state, which we call \( E^{1s}(1, 0) \), breaks \( C_3 \) but preserves time-reversal symmetry; this is the nematic singlet phase.

Within a single-band mean-field description (see Appendix A), we find \( b_1^t = b_2^t/2 > 0 \). As such, mean-field theory generically favors the chiral singlet superconductor over the nematic state \( E^{1s}(1, 0) \); this has been noted before in the context of twisted bilayer graphene \[41] and Ref. \[40\] discusses how strong fluctuations can stabilize the nematic phase.

Turning to triplet pairing, we now modify the parametrization (29) to

\[ M_{k+} = \sum_{\mu = \pm} \sum_{\nu = 1}^{3} \eta_{\mu \nu} (X_k + i\mu Y_k) \sigma_{\nu}, \]  

where \( X_k \) and \( Y_k \) are defined exactly as before. For simplicity, we introduce the complex-vector notation, \( d_\mu = (\eta_{\mu,1}, \eta_{\mu,2}, \eta_{\mu,3})^T, \mu = \pm \). The representations of the symmetries now read as

\[
\begin{align*}
C_3: & \quad (d_+, d_-) \rightarrow (\omega d_+, \omega^* d_-), \\
\Theta: & \quad (d_+, d_-) \rightarrow (d_-, d_+), \\
SU(2)_s: & \quad (d_+, d_-) \rightarrow (R d_+, R d_-),
\end{align*}
\]  

with \( R \in \text{SO}(3) \) and \( \omega = e^{i\pi/2} \). The most general free-energy expansion is given by

\[
F \sim a \sum_{\mu = \pm} d_\mu^T d_\mu + b_1^t \left( \sum_{\mu = \pm} d_\mu^T d_\mu \right)^2 + b_2^t (d_+^T d_- + d_-^T d_+),
\]

\[ + b_3^t |d_+^T d_-|^2 + b_4^t |d_-^T d_+|^2 + b_5^t \sum_{\mu = \pm} |d_\mu^T d_\mu|^2 \]  

up to quartic order, where \( b_j^t \in \mathbb{R} \); the different symmetry-allowed phases follow from the stable minima of the free energy. When minimizing Eq. (38), we recognize that the relative phase between \( d_+ \) and \( d_- \) can always be absorbed into a redefinition of the basis functions \( X_k \) and \( Y_k \), as for the singlet above. In total, we find eight distinct triplet states which we label by \( E^{3t}(a) \) through \( E^{3t}(h) \). Phase diagrams describing which of these phases is realized for a given configuration of the quartic couplings \( b_j \) can be found in Appendix B; here, we merely list all the phases and describe their properties:

(a) This state can be represented by \( d_+ = d_- = (1, 0, 0)^T \) with associated order parameter \( M_{k+} = X_k \sigma_x \). It will be labelled as \( E^{3t}(a) \) and, more physically, corresponds to a nematic unitary triplet phase. It preserves time-reversal symmetry, but breaks both SU(2), spin-rotation symmetry [down to O(2)] and \( C_3 \) rotational symmetry. This state has two symmetry-enforced nodal points at each Fermi surface around the \( K, K' \), or \( \Gamma \) point. Owing to the lack of any reflection symmetry (cf. the discussion of \( D_3 \) in Sec. VI below), the positions of these nodal points are not pinned to any specific direction.

(b) One representative configuration of this phase is given by \( d_+ = (1, -i, 0)^T/2 \) and \( d_- = (1, i, 0)^T/2 \), corresponding to \( M_{k+} = X_k \sigma_x + Y_k \sigma_y \). This state, denoted by \( E^{3t}(b) \) in the following, only has point nodes at \( \Gamma, K, K' \), i.e., it is expected to exhibit a full gap for generic Fermi surfaces not going through these high-symmetry points. While this state breaks spin-rotation symmetry as well as \( C_3 \), the product of \( C_3 \) and a rotation in spin space along \( \sigma_z \) with angle \( 2\pi/3 \) is preserved; this can be viewed as the spontaneous formation of spin-orbit coupling.

(c) Here, we can write \( d_+ = d_- = (1, i, 0)^T/2 \) and, hence, \( M_{k+} = X_k (\sigma_x + i\sigma_y) \). This is a nematic nonunitary triplet state which breaks time-reversal symmetry and \( C_3 \). One spin-species will be gapless while the other will have nodal lines (i.e., point nodes on the Fermi surface).

(d) The triplet vectors in this phase can be written as \( d_+ = (1, 0, 0)^T, d_- = 0 \) leading to \( M_{k+} = (X_k + iY_k) \sigma_x \). This is the chiral unitary triplet state that breaks SU(2)s spin-rotation symmetry [down to O(2)] and time-reversal, but preserves \( C_3 \). Except for \( \Gamma, K, \) and \( K' \) this state has no symmetry-imposed nodal points.

(e) For this state, we have \( d_+ = (1, i, 0)^T, d_- = 0 \), i.e., \( M_{k+} = (X_k + iY_k) (\sigma_x + i\sigma_y) \), which is the chiral nonunitary triplet state. It preserves \( C_3 \), but breaks SU(2)s spin-rotation symmetry [down to O(2)] and time-reversal. Here, one of the spin components will be gapless while the other is fully gapped (as before, except for the high symmetry points \( \Gamma, K, \) and \( K' \) which are generically not on the Fermi surface).

(f) This phase is characterized by \( d_+ = (1, 0, 0)^T, d_- = (0, 1, 0)^T \), which translates to \( M_{k+} = (X_k + iY_k) \sigma_x + (X_k - iY_k) \sigma_y \). In this state, both time-reversal, \( C_3 \), and spin-rotation symmetry are broken. The excitation spectrum is given by \( E_{\pm}(k) = \sqrt{\xi_{k+}^2 + 2(X_k \pm Y_k)^2} \), so it is characterized by "two gaps", given by \( |X_k \pm Y_k| \), both of
which are forced to vanish at two points for each Fermi surface enclosing $K$, $K'$, and $\Gamma$. While the number of nodes of this state and of $E^3(b)$ are the same, the spin degrees of freedom on the Fermi surface have nodes at the same two momenta for $E^3(a)$. For $E^3(f)$, however, the two spin species have nodal points at different momenta.

\[ \text{(g) Denoted by } E^3(g), \text{ this phase has } d_+ = \cos(\alpha)(1, i, 0)/\sqrt{2}, d_- = \sin(\alpha)(0, 0, 1)^T, \text{ where the parameter } \alpha \text{ varies continuously with } \beta_i \text{ in the part of the phase diagram where this state is realized. The corresponding order parameter can be written as } M_{k+} = \cos(\alpha)(X_k + i Y_k)(\sigma_x + i \sigma_y)/\sqrt{2} + \sin(\alpha)(X_k - i Y_k)\sigma_z, \text{ where } 0 < \alpha < \pi, \text{ and can be viewed as a superposition of a chiral nonunitary triplet state and a unitary state with opposite chirality. This state breaks time-reversal symmetry, spin-rotation invariance, and } C_3 \text{ but preserves the product of } C_3 \text{ and spin rotation by angle } 2\pi/3 \text{ along } \sigma_z. \text{ So, similar to the state } E^3(b) \text{ above, this state spontaneously entangles rotations in spin and real space. It is fully gapped (again, as long as the Fermi surfaces do not go through } \Gamma, K, \text{ and } K' \text{), with two different gaps } [1 \pm g_a(X_k^2 + Y_k^2)]^{1/2}, \text{ where } g_a = \cos \alpha/1 + \sin^2 \alpha.

\[ \text{(h) Finally, the triplet phase } E^3(h) \text{ has } d_+ = (\cos \alpha, 0, i \sin \alpha)^T, d_- = (0, \cos \alpha, -i \sin \alpha)^T, \text{ which yields } M_{k+} = \cos(\alpha)|\langle X_k + i Y_k \rangle\sigma_x + \langle X_k - i Y_k \rangle\sigma_y| - 2\sin(\alpha)Y_k\sigma_z. \text{ It can be seen as a superposition of the states } E^3(a) \text{ and } E^3(f) \text{ to which it reduces for } \alpha = \pi/2 \text{ and } \alpha = 0; \text{ it will have two nodal points for } \alpha \text{ close to these limiting cases, but can be fully gapped for other values of } \alpha. \text{ For } \alpha \neq \pi/2, \text{ this state breaks time-reversal, } C_3, \text{ and spin-rotation symmetry.}

In Appendix A, we show that $b_1^3 = b_2^3/2 = -b_3^3/2 = -2b_4^3/2 > 0$ and $b_5^3 = 0$ within a single-band mean-field description. Minimizing Eq. (38) yields that the phases $E^3(b)$ and $E^3(d)$ have the lowest energy and are exactly degenerate for this configuration of quartic couplings. This degeneracy within mean-field theory, which was noted before in Ref. 17, will be lifted by corrections resulting, e.g., from residual interactions. In Sec. V, we will find that $E^3(b)$ is favored in the presence of ferromagnetic fluctuations. We will also see that significant fluctuations can stabilize phases other than the two, $E^3(b)$ and $E^3(d)$, favored in mean-field theory.

**B. Approximate SU(2)$_+ \times$ SU(2)$_-$**

After having classified singlet and triplet separately, we now focus on small Hund’s coupling for which SU(2)$_+$ and SU(2)$_-$ is an approximate symmetry and singlet and triplet are nearly degenerate at the quadratic level of the free energy. This requires studying them on an equal footing and generalizing the parametrization in Eqs. (29) and (36) to include both singlet and triplet, i.e., extending the summation over $\nu$ in Eq. (36) to $\nu = 0, 1, 2, 3$. In analogy with Sec. III A, we use $2 \times 2$ matrices and write

\[ M_{k\nu} = \sum_{\mu=\pm}(X_k + i \mu Y_k)\Delta_{\mu\nu}, \quad \Delta_{\mu\nu} = \sum_{\nu=0}^3 \eta_{\mu\nu}\sigma_{\nu}. \quad (39) \]

It is easy to see that the symmetries act according to

\[ C_3 : (\Delta_+, \Delta_-) \rightarrow (\omega\Delta_+, \omega^*\Delta_-), \quad (40a) \]

\[ \Theta : (\Delta_+, \Delta_-) \rightarrow (\Delta_-, \Delta_+), \quad (40b) \]

\[ G^a : \Delta_{\mu} \rightarrow e^{-i\phi_+}\Delta_{\mu} e^{i\phi_-}, \quad (40c) \]

where, recall, $G^a = SU(2)$_+$×SU(2)$_-$ as an exact symmetry, the most general free energy up to quartic order reads as

\[ \mathcal{F} \sim a \sum_{\mu=\pm} \text{tr}[\Delta^\mu\Delta_{\mu}] + b_1^4 \left( \sum_{\mu=\pm} \text{tr}[\Delta^\mu\Delta_{\mu}] \right)^2 + b_2^2 \sum_{\mu=\pm} \text{tr}[\Delta^\mu\Delta_{\mu}], \quad \Delta_{\mu} \rightarrow \delta a \sum_{\mu=\pm} \text{tr}[\Delta^\mu\Delta_{\mu}] \quad (41) \]

At first glance, one might think that there are additional terms with extra factors of $\sigma_y$, similar to the last term in Eq. (12). However, as before, all of them can be related to combinations of the terms already present in Eq. (41) as outlined in Appendix B.

Following the procedure applied in Sec. III to the one-dimensional IR A, we now add a small quadratic term, $\delta a \sum_{\mu=\pm} (\Delta^\mu_{\mu})^2 - d^2_{\mu}d^2_{\mu}$, where $\Delta^\mu_{\mu}$ and $d_{\mu}$ are the singlet and triplet component of $\Delta_{\mu}$ in Eq. (41), i.e., $\Delta_{\mu} = \sigma_0 \Delta^s_{\mu} + \sigma \cdot d_{\mu}$. This term breaks SU(2)$_+$×SU(2)$_-$ and hence, makes singlet and triplet inequivalent. It allows us to study which of the different singlet and triplet states defined above can mix, and to identify “Hund’s partners”, i.e., which states transform into each other when changing the sign of the Hund’s coupling $J$ and accordingly, of $\delta a$. This generalizes the phase diagrams in Fig. I and Table I to the complex representation.

We find that, out of the eight different triplet states $E^3(a)$ to $E^3(h)$, only two—$E^3(a)$ and $E^3(d)$—do not allow for a singlet-triplet admixture when reversing the sign of $J$ (or $\delta a$) so that singlet has the higher transition temperature. The reason for the absence of an admixture is the same as sketched by way of example in Sec. III B: besides pure singlet and pure triplet terms,
TABLE II. Summary of possible pairing states transforming under the complex representation $E$ of $C_3$. The labeling of the pairing states and their symmetry properties can be found in the main text. The states are ordered by pure singlet, triplet, and admixtures of singlet and triplet. The latter are only expected generically when the SU(2)$_{-}$ × SU(2)$_{+}$ symmetry is weakly broken. We use $X_{k}$ and $Y_{k}$ to denote real-valued continuous functions on the Brillouin zone that transform as $k_{x}$ and $k_{y}$ under $C_3$ [see, e.g., Eq. (30)]. The temperature-dependent coefficient $\eta$ describes the admixture of a triplet/singlet pairing at a second transition to a purely singlet/triplet one. Furthermore, $a, b \in \mathbb{R}$ vary continuously with system parameters. The minimal number of nodes on any Fermi surface enclosing the $\Gamma$, $K$, or $K^\prime$ point is indicated in the column “Nodes”. As before, two states are referred to as Hund’s partners if they transform into each other under reversing the sign of the Hund’s coupling, such as the pure singlet and unitary triplet in Fig. 1(a). As singlet and triplet mix for both $\delta > 0$ and $\delta < 0$, there are no Hund’s partners for $E^{3s}(g)$ and $E^{3s}(h)$; the corresponding mixed phases, contained in the last two lines of the table, are their own Hund’s partners.

| Pairing | $M_k+i$ | Nodes | Hund’s partner | MF |
|---------|---------|-------|---------------|----|
| $E^{1s}(1,0)$ | $X_k \sigma_0$ | 2 points | $E^{1s}(a)$ | × |
| $E^{1s}(1, i)$ | $(X_k + i Y_k) \sigma_0$ | 0 | $E^{1s}(d)$ | ✓ |
| $E^{3s}(a)$ | $X_k \sigma_x$ | 2 points | $E^{1s}(1,0)$ | × |
| $E^{3s}(b)$ | $X_k \sigma_x + Y_k \sigma_y$ | 0 | $E^{1s}(0, i) + E^{3s}(a)$ | ✓ |
| $E^{3s}(c)$ | $X_k(\sigma_x + i \sigma_y)$ | ↓ gapless/2 points | $E^{1s}(1,0) + E^{3s}(a)$ | × |
| $E^{3s}(d)$ | $(X_k + i Y_k) \sigma_x$ | 0 | $E^{1s}(1, i)$ | ✓ |
| $E^{3s}(e)$ | $(X_k + i Y_k) \sigma_z + (X_k - i Y_k) \sigma_y$ | ↓ gapless/0 | $E^{1s}(1, i) + E^{3s}(d)$ | × |
| $E^{3s}(f)$ | $(X_k + i Y_k) \sigma_x + (X_k - i Y_k) \sigma_y$ | 2 points | $E^{1s}(1, -i) + E^{3s}(d)$ | × |
| $E^{3s}(g)$ | $a(X_k + i Y_k)(\sigma_x + i \sigma_y) + b(X_k - i Y_k) \sigma_z$ | 0 | — | × |
| $E^{3s}(h)$ | $a[(X_k + i Y_k) \sigma_x + (X_k - i Y_k) \sigma_y] + b Y_k \sigma_z$ | 0 | — | × |

The quartic terms in Eq. (41) also contain couplings between singlet and triplet, as is readily seen by inserting the parametrization $\Delta_\mu = \eta_0 \Delta_\mu^0 + \sigma \cdot d_\mu$, $\mu = \pm$ (the full expansion can be found in Appendix B). At the first transition, one of either singlet or triplet becomes nonzero and hence, “renormalizes” the quadratic term of the other channel. In some cases, this renormalization can prohibit the presence of a second transition. In the case of phases $E^{3s}(a)$ and $E^{3s}(d)$, we just obtain the pure singlets $E^{1s}(1,0)$ and $E^{1s}(1, i)$, respectively, without a second transition. The easiest way to interpret why we do not have an admixtures in these cases is to look at the associated SO(4) parent states: the two triplets correspond to $(\eta_+; d_+) = (\eta_-; d_-) = (0; 1, 0, 0)$ and $(\eta_+; d_+) = (0; 1, 0, 0)$, $(\eta_-; d_-) = 0$, respectively. Both of these configurations can be “rotated” into the pure singlets $(\eta_+; d_+) = (\eta_-; d_-) = (1; 0, 0, 0)$ and $(\eta_+; d_+) = (1; 0, 0, 0)$, $(\eta_-; d_-) = 0$ via a SU(2)$_{+}$ × SU(2)$_{-}$ transformation.

For all other triplets, the Hund’s partner is an admixed phase. Specifically, as regards $E^{3s}(b)$ and $E^{3s}(e)$, the Hund’s partner is an admixture of a nematic singlet state and a nematic unitary triplet $E^{3s}(a)$, with different relative phases and spatial orientations: for the former, the order parameter can be written as $i Y_k \sigma_0 + \eta X_k \sigma_x$, where $\eta$ describes the temperature-dependent strength of mixing, while it is $X_k(\sigma_0 + \eta \sigma_x)$ for the latter. On any Fermi surface around one of the high-symmetry points $\Gamma$, $K$, or $K'$, these two states have zero and two nodal points, respectively. Again, the form of the admixed state can be understood from the representation of the triplet state in terms of $(\eta_0; d_\mu)$. For instance, we have $(\eta_+; d_+) = (0; 1, -i, 0)$, $(\eta_-; d_-) = (0; 1, i, 0)$ for $E^{3s}(b)$, which is equivalent to $(\eta_+; d_+) = (1; 1, 0, 0)$, $(\eta_-; d_-) = (-1; 1, 0, 0)$ after applying an appropriate SU(2)$_{+}$ × SU(2)$_{-}$ transformation.

Likewise, the Hund’s partners of $E^{3s}(e)$ and $E^{3s}(f)$ are admixtures of a chiral singlet and a unitary triplet state with the same and opposite chirality, respectively. The associated order parameters can be written as $(X_k + i Y_k)(\sigma_0 + \eta \sigma_x)$ and $(X_k + i Y_k) \sigma_0 + \eta (X_k - i Y_k) \sigma_x$. While the first of the two states has two fully established gaps, given by $(1 \pm \eta) \sqrt{X_k^2 + Y_k^2}$ (with ± referring to the spin species), the other has two gaps, $|X_k|$ and $|Y_k|$, with distinct momentum dependencies; it, thus, exhibits two point nodes per Fermi surface which occur at dif-
ferent positions for the two spin species, similar to the
associated triplet phase $E^{3s}(f)$.

In general, admixing a singlet component at a second
transition to a triplet state is less likely to occur as
a singlet state has less options to “adapt” (order parameter comprises two complex numbers for $E$) than a triplet
state (order parameter comprises six complex numbers).

While this is not possible for the one-dimensional representa-
tion $A$ (see Fig. 1), the IR $E$ does allow for this
scenario but only for the second states $E^{3s}(g)$ and $E^{3s}(h)$:
for small $\delta a < 0$, we find the second transition where an
additional chiral (nematic) singlet component is admixed to
$E^{3s}(g) (E^{3s}(h))$. As both pure triplet states can be fully
gapped, the same holds for the admixed phases. The
admixture of the extra singlet component does not change
the symmetries of $E^{3s}(g)$ and $E^{3s}(h)$ listed in Sec. IV A
above. Reversing the sign of $\delta a$ to small positive values,
we obtain the same admixed phase. The only difference
is that the first transition is a singlet transition into a
chiral (nematic) phase and the secondary triplet $E^{3s}(g)
[E^{3s}(h)]$ becomes nonzero at a lower transition temperature.

The key results of this section, the pure triplet/singlet
states and the possible admixed phases for small $J$ along
with their order parameters and properties, are summarized
in Table II.

Let us finally discuss the impact of fluctuations of the
order parameter on the thermal phase transitions. As
readily follows from the respective order parameter man-
ifolds, the singlet phases in Table II exhibit a conven-
tional BKT transition, the triplets $(a)$, $(b)$, $(d)$, $(f)$, $(g)$,
and $(h)$ will be charge-4$e$ superconductors where only
spin-rotation invariant combinations of the triplet vector
assume quasi-long-range order at finite temperature, and
the triplets $(c)$ and $(e)$ will only display a crossover. How-
ever, as pointed out above, none of these three classes of
transitions can currently be excluded based on the ex-
perimental data.

C. Behavior in a magnetic field

Finally, we survey the behavior of the pairing states of
the complex representation in the presence of a Zeeman
field, $M_Z$, and in-plane orbital coupling $M_O$, along the
same lines as Sec. III C. From Eqs. (31) and (37), it fol-
lows that there are three possible coupling terms linear
in the field and quadratic in the superconducting order
parameter given by

\[
\Delta F^E_M \sim M_Z \cdot \sum_\mu \left[ \delta c^E_1 \text{Im} \left( d^*_\mu \eta_\mu \right) + c^E_2 \mu \text{Re} \left( d^*_\mu \eta_\mu \right) \right] \\
+ i c^E_3 M_Z \cdot \sum_\mu d^*_\mu \times d^*_\mu, \tag{42}
\]

Notice that, exactly as for the IR $A$, there is no linear
coupling to the in-plane orbital field, which is prohibited
by time-reversal and $C_3$ rotation symmetry. While the
first term in Eq. (42) is again forced to vanish for $J \to 0$ [for the same reason as $\delta c_1$ in Eq. (24)], the second
singlet-triplet-mixing coupling, $c^E_2$, is not constrained to
be zero for $J = 0$. However, the emergent symmetry in
the single-band mean-field description of Appendix A,
leads to $c^E_2 = 0$, so it is natural to expect $c^E_2 < c^E_3$ such
that the last term in Eq. (42) describes the dominant
linear coupling to the magnetic field—even when $J$ is
small. As expounded in Appendix A, the expression for
$c^E_3$ is identical in form to that for $c_2$ in Eq. (24). As
such, the linear increase of the (first) superconducting
transition temperature with small magnetic fields seen in
experiment does not permit one to distinguish between
the IRs $A$ and $E$.

There is one difference between the pairing states of
the two IRs worth mentioning here: while the form of the
leading triplet vector in a magnetic field is completely
fixed to be $d \propto (1, i, 0)^T$ for the one-dimensional IR $A$,
the complex IR allows for either $E^{3s}(c)$ or $E^{3s}(e)$ pair-
ing for nonzero $M_Z$. Which of the two is realized, de-
dends on the value of the quartic terms in Eq. (38): if
$b^2_3 + b^3_3 > 0$, the state $E^{3s}(c)$ will be preferred while
the opposite sign corresponds to $E^{3s}(e)$. Within single-band
mean-field theory, we find $b^2_3 = 0$ and $b^3_3 > 0$, which leads
to phase $E^{3s}(e)$. In the next section, we will see that
additional ferromagnetic fluctuations will further enlarge
the positive value of $b^2_3 + b^3_3$ and consequently, not af-
fect the mean-field prediction that $E^{3s}(e)$ is the leading
triplet state with the highest transition temperature in
the presence of a magnetic field.

V. FLUCTUATION-INDUCED
SUPERCONDUCTIVITY

Among the plethora of possible superconducting
phases outlined in this paper, only a few can be real-
ized in mean-field theory (see Tables I, II, and IV). This
originates from the fact that, within mean-field theory,
the ratio of the quartic terms is fixed and, degeneracies
aside, only one state can occur for each IR. However, the
presence of sizable correlations in the nearly flat bands of
graphene moiré systems is expected to give rise to signif-
icant corrections to mean-field theory. This has recently
been demonstrated for the case of charge-density-wave
fluctuations in twisted bilayer graphene [40], and in
the context of nematic fluctuations in the iron-based super-
conductors [63].

We illustrate here these corrections to mean-field the-
ory assuming strong ferromagnetic spin fluctuations.
This is prompted by experiments [47–49], which indicate
a spin-polarized correlated insulating state in twisted
Upon making the association Eq. (43) is the main symmetry-breaking perturbation. Representing the ferromagnetic spin moment in valley $v = \pm$ by $m_v$, we parametrize its contribution to the free energy as

$$F_m = \frac{1}{2} \sum_{v,v'} \langle \tilde{\chi}^{-1} \rangle_{vv'} m_v \cdot m_{v'}, \quad \tilde{\chi} = \left( \begin{array}{c} \chi \\ \delta \chi \\ \chi \end{array} \right).$$  (43)$$

In this expression, $\tilde{\chi}$ plays the role of the spin susceptibility (with $|\delta \chi| < \chi$ to ensure stability) and we expect $\delta \chi > 0$ close to a phase where the spin moments in the two valleys are aligned. The ratio $\delta \chi / \chi$ controls how strongly the SU(2)$_+ \times$ SU(2)$_-$ symmetry is broken down to SU(2)$_z$.

Focusing first on the one-dimensional IR $A$ of $C_3$, the magnetic moments couple to the superconducting order parameter in Sec. III according to

$$F^A_m = c_2 \sum_{v=\pm} m_v \cdot [i d^* \times d - 2v \text{Re}(d^* \Delta^s)],$$  (44)$$

where we have retained only the couplings invariant under SU(2)$_+ \times$ SU(2)$_-$ and assumed that $\delta \chi \neq 0$ in Eq. (43) is the main symmetry-breaking perturbation. Upon making the association $M_Z = \sum_v m_v$, we notice that $c_2$ is the same prefactor as in Eq. (24). In the same vein as Ref. 40, we integrate out the massive fluctuations of $m_v$. As a consequence of the coupling (44), this yields corrections to the terms quartic in the superconducting order parameters in Eq. (20), which can be conveniently split into two categories. First, there are corrections that preserve the SU(2)$_+ \times$ SU(2)$_-$ symmetry; these can be restated as renormalizations of the coefficients $b_1$ and $b_2$ in Eq. (20). Corrections of the second type break this symmetry, violating the form of the free-energy expansion (20). More explicitly, the renormalization of the free energy $F$ in Eq. (20) due to the presence of ferromagnetic spin fluctuations can be compactly stated as

$$F \rightarrow F|_{b_1 \rightarrow b_1 + \delta b_1, b_2 \rightarrow b_2 + \delta b_2} = \frac{1}{2} \sum_{\mu} \sum_{v=\pm} c_{v,\mu} m_v \cdot [i d^*_\mu \times d_\mu - 2v \text{Re}(d^*_\mu \Delta^s_\mu)].$$  (46)$$

Integrating out $m_v$, we again obtain corrections to the free energy which are quartic in the superconducting order parameter. In the limit of SU(2)$_+ \times$ SU(2)$_-$ invariance, $\delta \chi = 0$, these corrections can be represented by renormalizations of the couplings, $b_j \rightarrow b_j + \delta b_j$, in Eq. (41) with

$$\delta b_1 = -\delta b_2 = \chi (c_+^2 + c_-^2) / 2 > 0,$$

$$\delta b_3 = -\chi (c_+ - c_-)^2 < 0,$$

$$\delta b_4 = 0,$$

$$\delta b_5 = -\chi c_+ c_-.$$  (47)$$

To study the ramifications of this result, we first consider the limit of weak fluctuations, for which $\delta b_5$ in Eq. (47) are much smaller in magnitude than the mean-field value of $b_2$. Albeit small, the corrections $\delta b_j$ are crucial here due to the exact degeneracy of the states $E^{3+}(b)$ and $E^{3-}(d)$ in mean-field theory observed earlier. From Eq. (41) with the replacement $b_j \rightarrow b_j + \delta b_j$, we favor the phase diagram in part (b) of Fig. 1 over part (a). We point out that naively taking Eq. (45) alone would render the quartic free-energy expansion unstable for large enough $\chi$. However, denoting the mean-field value of $b_2$ by $b_0^2$, there exists a regime, $b_0^2 / 2 < c_+^2 \chi < b_0^2$, for which $b_2 < 0$ due to fluctuation corrections and the free energy in Eq. (20) is stable. For larger values of $\chi$, we can imagine adding the sextic term $c(\text{tr} [\Delta_+^3 \Delta_-^3])^3$ to the free energy to restore stability.

When $\delta \chi$ is of order $\chi$, the ferromagnetic fluctuations described by Eq. (43) induce considerable SU(2)$_-$-symmetry-breaking interactions. The presumed sign $\delta \chi > 0$ brings about a further enhancement of the term $-|d^* \times d|^2$ [as is obvious from Eq. (45)], which favors nonunitary triplet pairing relative to the SU(2)$_+ \times$ SU(2)$_-$-invariant form of the free energy in Eq. (20). Given that $\delta \chi < \chi$, strong ferromagnetic fluctuations are still expected to change the sign of $b_2$ relative to mean-field theory. The additional effect of $\delta \chi$ lies in effecting an additional first-order transition to a nonunitary triplet state in a third transition at lower temperatures for anti-Hund’s coupling in Fig. 1(b).

We have thus shown that significant ferromagnetic fluctuations can reverse the predictions of mean-field theory, and favor the nonunitary triplet state $A^{3+}(1, i, 0)$ and the admixed singlet-triplet phase $A^{1+} + A^{1-}(1, 0, 0)$ in Table I.

The same analysis can be performed for the complex IR $E$ of Sec. IV. In this case, the most general SU(2)$_+ \times$ SU(2)$_-$-invariant coupling between the superconducting order parameter and the spin fluctuations allows for two independent coupling constants, $c_\pm \in \mathbb{R}$, and has the form

$$F^E_m = \sum_{\mu=\pm} \sum_{v=\pm} c_{v,\mu} m_v \cdot [i d^*_\mu \times d_\mu - 2v \text{Re}(d^*_\mu \Delta^s_\mu)].$$  (46)$$

The new IR $E$ corrections to the free energy can be represented by renormalizations of the couplings, $b_j \rightarrow b_j + \delta b_j$, with

$$\delta b_1 = -\delta b_2 = \chi (c_+^2 + c_-^2) / 2 > 0,$$

$$\delta b_3 = -\chi (c_+ - c_-)^2 < 0,$$

$$\delta b_4 = 0,$$

$$\delta b_5 = -\chi c_+ c_-.$$  (47)$$

We start with the limit $|\delta \chi| \ll \chi$, where the structure of Eq. (20) is asymptotically preserved and the form of the two possible phase diagrams in Fig. 1 is unchanged. Since $\delta_2 < 0$, strong ferromagnetic fluctuations will change the sign of $b_2$ from its positive mean-field value to negative and, as opposed to mean-field theory,
find the free-energy difference of these two states to be
\[ \mathcal{F}_{E^3(b)} - \mathcal{F}_{E^3(d)} = -\frac{1}{4} \chi (c_+ - c_-)^2 \leq 0, \] (48)
thereby generically favoring \( E^3(b) \) along with its Hund’s partner \( E^3(0, i) + E^3(a) \), defined in Table II. In the one-band description of Appendix A, it always holds that \( c_+ = c_- \), which is, in turn, a consequence of an emergent valley-exchange symmetry. However, multiband effects are expected to lead to nonzero \( |c_+ - c_-| \ll |c_+| \), which is enough to lift the degeneracy according to Eq. (48).

Next, we turn to the limit of strong ferromagnetic fluctuations, where the mean-field values of \( b_j \) have to be treated as perturbations to the large \( \delta b_j \) in Eq. (47). As \( \chi \to \infty \), we find that, out of the triplet states in Table II, \( E^3(e) \) has the lowest energy unless \( c_+ = c_- \) or \( c_+ = -c_- \). We know that \( c_+ \approx c_- \) and hence, can safely neglect the latter. For the former, \( c_+ = c_- \), \( E^3(e) \) is found to be degenerate with \( E^3(c) \); however, for large but finite \( \chi \), the additional contribution to \( b_j \) from mean-field theory lifts this degeneracy, always selecting \( E^3(e) \).

Out of the multitude of possible pairing states in Table II, strong ferromagnetic fluctuations thus favor the chiral nonunitary triplet state \( E^3(c) \) and the mixed singlet-triplet phase \( E^3(1, i) + E^3(d) \). Which of these two states is realized, depends on whether singlet or triplet has the highest transition temperature (the sign of \( \delta a \)).

Finally, we come back to the impact of fluctuation corrections to the leading triplet phase in the presence of a magnetic field. As we have seen in Sec. IV C, the superconducting state with the highest transition temperature in the presence of a sufficiently strong magnetic field will be a triplet phase due to the linear coupling in the second line of Eq. (42). At the mean-field level, \( b_2^2 + b_3^2 > 0 \), which prefers \( E^3(e) \) over \( E^3(c) \) as the order parameter of this phase. Using the relations in Eq. (B6), it is straightforward to rephrase the fluctuation corrections (47) of \( b_j \) in terms of \( b_j^2 \to b_j^2 + \delta b_j^2 \) in Eq. (38). This yields
\[ b_2^2 + b_3^2 \to b_2^2 + b_3^2 + \chi (c_+ - c_-)^2; \] (49)
as expected, ferromagnetic fluctuations do not change the mean-field prediction in this case and \( E^3(e) \) is the dominant triplet order parameter in the presence of a magnetic field, for both strong and weak ferromagnetic fluctuations.

VI. ADDING FURTHER SYMMETRIES

So far, our cynosure has been twisted double-bilayer graphene in the presence of a vertical electric field as it has the fewest number of symmetries and since it is the only graphene moiré system for which experiments have shown indications of triplet pairing to date. In this section, we will see how our previous classification of superconducting instabilities changes once the additional symmetries, two-fold rotation, \( C_2 \), perpendicular to the plane of the system, and in-plane rotation symmetry, \( C_{2y} \), are added. These symmetries are relevant as either exact microscopic or approximate emergent symmetries of twisted bilayer graphene and ABC trilayer graphene on hexagonal boron nitride, both of which exhibit superconductivity [2, 44].

A. Consequences of a \( C_2 \) rotation symmetry

One crucial difference in twisted bilayer graphene is that the system has an approximate \( C_2 \) symmetry [13] that mixes the two valleys, i.e., the system is (approximately) invariant under
\[ C_2 : \ b_k \to \tau_x b_{-k}. \] (50)
To relate to our notation used above for the twisted double-bilayer system, we assume that it is sufficient to focus on a single band for describing superconductivity in twisted bilayer graphene as well. This is quite a natural assumption and, unless stated otherwise, we expect our conclusions to hold when additional bands are taken into consideration.

This (approximate) symmetry has attracted a lot of attention in the recent theory literature [11, 12, 15, 20] of the system since it, combined with time-reversal and \( C_3 \), leads to a \( C_6 \Theta \) symmetry, which is responsible for not only the presence of (nearly gapless) Dirac cones at \( K \) and \( K' \) but also the (approximate) vanishing of Berry curvature in twisted bilayer graphene. If the twist axis goes through the center of a hexagon, the system has \( C_6 \) rotation even as a microscopic symmetry. We note in passing that the (nearly) flat bands obtained in Refs. 52 and 53 for double-bilayer graphene do not feature any Dirac cones but have well-separated conduction and valence bands that are characterized by nonzero Chern numbers (at least in some parameter regime); this strongly indicates that \( C_2 \) is not an approximate symmetry in twisted double-bilayer graphene since \( C_2 \Theta \) would enforce zero Berry curvature.

In a similar fashion, based on the discussion in Ref. 46, the two-fold symmetry (50) could also be an important approximate symmetry for ABC trilayer graphene on hexagonal boron nitride.

All things considered, it is currently not known whether an approximate \( C_2 \) symmetry is relevant for superconductivity in twisted bilayer and ABC trilayer graphene. Therefore, we will now discuss what changes for the possible superconducting instabilities once we assume that the Hamiltonian is also invariant under the
The $C_2$ transformation plays a special role in two dimensions as it is equivalent to $k \rightarrow -k$ and can, thus, significantly affect superconducting instabilities [66]. In graphene moiré superlattices, it also relates the two valleys and “interferes” with the Fermi-Dirac constraint (10): decomposing the pairing into singlet and triplet,

$$M_{kv} = \lambda^s_{kv} \sigma_0 \Delta^s + \lambda^t_{kv} \sigma \cdot d,$$

Eq. (10) implies that $\lambda^s_{kv} = \lambda^{*}_{-kv}$ and $\lambda^t_{kv} = -\lambda^{*}_{-kv}$. Consequently, it holds (as long as the pairing matrix elements between different bands can be neglected) that

$$C_2 : (\Delta^s, d) \rightarrow (\Delta^s, -d),$$

i.e., all representations even (odd) in $C_2$ must be pure singlet (triplet) states and vice versa. This has a few implications worth mentioning. First, even if $C_2$ is not a good symmetry (say, it is significantly broken by interactions), SU(2)$_s$ spin-rotation invariance requires that the first transition must be into a pure singlet or triplet state and hence, the pairing must be either even or odd under $C_2$. In this sense, we can still distinguish between $p$-wave and $d$-wave pairing despite the presence of $C_2$-symmetry-breaking interactions. We emphasize that mixing will only be possible via multiple superconducting transitions (fostering admixtures of singlet and triplet) or interband pairing. The latter is expected to be quite weak given the typical splitting between the bands at half-filling (about 0.15 meV according to Ref. 2).

Secondly, if we do have an enhanced SU(2)$_+ \times$ SU(2)$_-$ symmetry (or are close to it), singlet and triplet are (nearly) degenerate. This forces the corresponding IRs of the spatial point group $D_6$ of the system, which behave identically under the subgroup $D_3$ but are even and odd under $C_2$, to be (nearly) degenerate at the quadratic level of the Ginzburg-Landau expansion. For instance, $A_1$ and $B_1$ of $D_6$ have to be degenerate, as summarized in Table III. Without a Zeeman field, an extra $C_2$ symmetry with action in Eq. (32) also has no consequences for the higher-order terms in the free energy since spin-rotation invariance necessitates that all of these terms are even in the triplet vector. The only difference arises in the presence of a Zeeman field or magnetic fluctuations: with $C_2$ symmetry, it must hold that $\delta \Gamma_1 = 0$ in Eq. (24) and $\delta \Gamma_2 = 0$ in Eq. (42) even when the SU(2)$_+ \times$ SU(2)$_-$ symmetry is broken. Furthermore, a $C_2$ symmetry implies $c_+ = c_-$ in Eq. (46). For this reason, weak ferromagnetic fluctuations do not lift the degeneracy of mean-field theory if we impose an exact $C_2$ symmetry and other types of fluctuations have to be considered.

In summary, when classifying superconducting states in twisted bilayer graphene or ABC trilayer graphene on hexagonal boron nitride in the absence of a Zeeman field, it is unimportant whether an approximate $C_2$ symmetry is relevant or not: singlet and triplet will always be even and odd under it. We can thus work with $D_3$ (instead of $D_6$) without loss of generality in the following. The only difference with twisted double-bilayer graphene (with finite displacement field) is an extra two-fold rotation symmetry, $C_{2y}$, along the $y$-axes with action

$$C_{2y} : c_k \rightarrow \tau_x c_{C_{2y} k},$$

where $C_{2y} k = (-k_x, k_y)$. While this is an exact microscopic symmetry for twisted bilayer graphene, it is not so for ABC trilayer graphene; nevertheless, $C_{2y}$ emerges as an approximate symmetry in the continuum description of the system [46]. The upshot of this additional symmetry for the possible superconducting instabilities is clarified in the next subsection.

### B. $D_3$ versus $C_3$

Due to the additional $C_{2y}$ symmetry, $D_3$ is a non-Abelian group and has three IRs—two one-dimensional and one two-dimensional representation (refer to the character table in Table III). It is convenient to begin with the one-dimensional IRs $A_1$ and $A_2$ and take $J \neq 0$. Since $C_{2y}$ interchanges the valleys, its action on the intravalley pairing order parameter (51) can be written as

$$C_{2y} : \left(\lambda^s_{kv}, \lambda^t_{kv}\right) \rightarrow \left(\lambda^s_{C_{2y} k v}, -\lambda^t_{C_{2y} k v}\right).$$

So, we see that a singlet (triplet) state transforming under $A_1$ ($A_2$) has no nodes while a singlet (triplet) in the $A_2$ ($A_1$) channel has symmetry-imposed nodes on the line $k_y = 0$ and along the directions rotated by $\pm \pi/3$. This creates six nodal points on any surface enclosing the $\Gamma$ point.

We can also readily understand from Eq. (54) how the one-dimensional representations “connect” at the SU(2)$_+ \times$ SU(2)$_-$ point: at the high-symmetry point, $\lambda^s_{kv} = \lambda^t_{kv}$, ergo $A_1^{1s}$ and $A_2^{3t}$ or $A_1^{3s}$ and $A_2^{1t}$ must meet at the $J = 0$ line in Fig. 1. We summarize these observations in Table IV.
In addition, in the case of the two-dimensional representation $E$ of $D_3$, the $C_{2y}$ symmetry has nontrivial consequences. Once again, we take $J \neq 0$ which permits us to study singlet and triplet independently. As singlet pairing has already been analyzed in detail for twisted bilayer graphene (see, e.g., Ref. 27), we are chiefly concerned with the triplet states here. We parametrize the triplet pairing as in Sec. IV A with the sole distinction being that the basis functions $X_k$ and $Y_k$ are now constrained by the symmetries of $D_3$: we choose them to obey $X_{\text{C}_{2y} k} = -X_k$ and $Y_{\text{C}_{2y} k} = Y_k$, while transforming as $k_x$ and $k_y$ under $C_3$. A possible choice is given by Eq. (30) with $\phi = \pi/2$. With these conventions, the triplet vector transforms according to $(d_+, d_-) \to (d_-, d_+)$ under $C_{2y}$. This does not further constrain the quartic terms in the free energy (38), wherefore we can use the analysis of Sec. IV A for the point group $C_3$, bearing in mind the caveat that the relative phase, $\varphi$, between $d_+$ and $d_-$ cannot be absorbed in a redefinition of the basis functions $X_k$ and $Y_k$ any more due to the extra $C_{2y}$ symmetry. While $\varphi$ has no consequences for $E^{3s}(d)$ or $E^{3s}(e)$ and can be absorbed by performing a spin rotation for the phases $E^{3s}(b)$ and $E^{3s}(g)$, it describes different phases for all other stable minima of Eq. (38), and we have to go to higher order in the free-energy expansion to determine its value.

Consider $E^{3s}(a)$ for instance. Writing $d_+ = (1, 0, 0)^T$ and $d_- = e^{i\varphi}(1, 0, 0)^T$, it is easy to verify that the most general, $\varphi$-dependent sextic term to the free energy must have the form $c_1 \cos(3\varphi)$ with $c_1 \in \mathbb{R}$. This derives from Eq. (34) where the $C_{2y}$ symmetry forces $c_2$ to vanish. We thus find $\varphi = 2\pi n/3$, $n \in \mathbb{Z}$, for $c_1 < 0$ and $\varphi = \pi/3 + 2\pi n/3$ when $c_1 > 0$. These two minima correspond to two different states, which can be compactly represented by defining the “rotated” basis functions

$$
(X_k^e, Y_k^e)^T = R_{\varphi/2}(X_k, Y_k)^T, \quad R_\varphi = e^{i\varphi s},
$$

TABLE IV. Summary of the different intervalley pairing states classified by the IRs of the point group $D_3$. The notation closely parallels that of Table II. Here, we use $\lambda_1$ and $\lambda_2$ to denote continuous functions on the Brillouin zone that are even and odd under $(k_x, k_y) \to (k_x, -k_y)$, respectively, and are both invariant under $C_3$, $\lambda_1 = \lambda_2^C k$. Furthermore, $X_k^e$ and $Y_k^e$ are rotated basis functions defined in Eq. (55); a possible choice is given by $(X_k, Y_k)^T = R_{(s \varphi + \varphi/2)}(X_k^{(1)}, Y_k^{(1)})^T$ with $X_k^{(1)}, Y_k^{(1)}$ in Eq. (30a). To keep the notation short, each line with reference to $\varphi_1$ or $\varphi_2$ correspond to two distinct states with $\varphi_1 = 0, \pi/3$ and $\varphi_2 = 0, \pi/2$. The indicated number of nodal points refer to a Fermi surface enclosing the $\Gamma$ point.

| Pairing | $M_{kz}$ | Nodes around $\Gamma$ | Hund’s partner | MF |
|---------|----------|------------------------|----------------|----|
| $A_1^s$ | $\lambda_1^p \sigma_0$ | none | $A_2^s(1, 0, 0) \checkmark$ | $A_2^s(1, 0, 0)$ |
| $A_2^s$ | $\lambda_2^p \sigma_0$ | 6 points | $A_2^s(1, 0, 0)$ | $A_2^s(1, 0, 0)$ |
| $A_1^s(1, 0, 0)$ | $\lambda_2^p \sigma_x$ | 6 points | $A_2^s(1, 0, 0)$ | $A_2^s(1, 0, 0)$ |
| $A_3^s(1, 0, 0)$ | $\lambda_2^p \sigma_x$ | none | $A_2^s(1, 0, 0)$ | $A_2^s(1, 0, 0)$ |
| $A_1^s(1, i, 0)$ | $\lambda_2^p (\sigma_x + i \sigma_y)$ | gapless/6 points | $A_2^s + A_3^s(1, 0, 0)$ | $A_2^s + A_3^s(1, 0, 0)$ |
| $A_3^s(1, i, 0)$ | $\lambda_2^p (\sigma_x + i \sigma_y)$ | gapless/none | $A_2^s + A_3^s(1, 0, 0)$ | $A_2^s + A_3^s(1, 0, 0)$ |
| $A_1^s + A_2^s(1, 0, 0)$ | $X_k^e \lambda_2^p \sigma_0$ | 2 points | $E^{3s}(a)_{\varphi_1}$ | $E^{3s}(a)_{\varphi_1}$ |
| $A_3^s(1, 0, 0)$ | $X_k^e \lambda_2^p \sigma_0$ | 2 points | $E^{3s}(a)_{\varphi_1}$ | $E^{3s}(a)_{\varphi_1}$ |
| $E^{3s}(a)_{\varphi_1}$ | $(X_k^e + i Y_k^e)^p \sigma_0$ | 0 | $E^{3s}(d)$ | $E^{3s}(d)$ |
| $E^{3s}(b)$ | $X_k^e \lambda_2^p (\sigma_x + i \sigma_y)$ | 0 | $(E^{3s}(1, i, 0))^2$ | $(E^{3s}(1, i, 0))^2$ |
| $E^{3s}(c)_{\varphi_1}$ | $X_k^e (\sigma_x + i \sigma_y)$ | gapless/2 points | $(E^{3s}(1, 0, 0))^2 + (E^{3s}(a)_{\varphi_1})^2$ | $(E^{3s}(1, 0, 0))^2 + (E^{3s}(a)_{\varphi_1})^2$ |
| $E^{3s}(d)$ | $(X_k^e + i Y_k^e)^p \sigma_0$ | 0 | $E^{3s}(1, i)$ | $E^{3s}(1, i)$ |
| $E^{3s}(e)$ | $(X_k^e + i Y_k^e)^p (\sigma_x + i \sigma_y)$ | gapless/0 | $(E^{3s}(1, 0, 0))^2 + (E^{3s}(a)_{\varphi_1})^2$ | $(E^{3s}(1, 0, 0))^2 + (E^{3s}(a)_{\varphi_1})^2$ |
| $E^{3s}(f)_{\varphi_2}$ | $(X_k^e + i Y_k^e)^p (\sigma_x + i \sigma_y)$ | 2 points | $(E^{3s}(1, 0, 0))^2 + (E^{3s}(a)_{\varphi_1})^2$ | $(E^{3s}(1, 0, 0))^2 + (E^{3s}(a)_{\varphi_1})^2$ |
| $E^{3s}(g)$ | $a(X_k^e + i Y_k^e)^p (\sigma_x + i \sigma_y)$ | 0 | $E^{3s}(d)_{\varphi_2}$ | $E^{3s}(d)_{\varphi_2}$ |
| $E^{3s}(h)_{\varphi_2}$ | $a((X_k^e + i Y_k^e)^p (\sigma_x + i \sigma_y) + b(Y_k^e - i X_k^e)^p \sigma_x)$ | 0 | $E^{3s}(d)_{\varphi_2}$ | $E^{3s}(d)_{\varphi_2}$ |
with $X_k$ and $Y_k$ as introduced above. The order parameters are $M_{k+} = X_k^0 \sigma_x$ and $M_{k+} = X_k^0 \sigma_x = (\sqrt{3} X_k + Y_k) \sigma_x / 2$ for $c_1 < 0$ and $c_1 > 0$, respectively. We denote these two states by $E^{3s}(a)_0$ and $E^{3s}(a)_{\pm}$, respectively. The first state, $E^{3s}(a)_0$, preserves $C_{2y}$, but breaks $C_3$ rotation symmetry, and has a nodal line which is, as opposed to the states in Sec. IV A, pinned to $k_y = 0$. The other state, $E^{3s}(a)_{\pm}$, however, breaks $C_{2y}$ and the nodal line is not pinned to the $k_x$ axis.

The remaining triplet states, $E^{3t}(c)$, $E^{3t}(f)$, and $E^{3t}(h)$ of Sec. IV A can be analyzed in the same way. In all cases, we find two states corresponding to two different discrete values of the relative phase $\varphi$ between $d_+$ and $d_-$. For $E^{3t}(c)$ and $E^{3t}(h)$, we find $\varphi = 0$ or $\varphi = \pi/3$ as before, whereas $E^{3t}(f)$ requires even higher-order terms in the free energy expansion, yielding $\varphi = 0$ or $\varphi = \pi/2$.

In analogy to $E^{3s}(a)_0$, we label the states by $E^{3t}(c)_0$, $E^{3t}(f)_0$, $E^{3t}(h)_0$; their order parameters are the same as those of the corresponding states in Sec. IV A but with the rotated basis functions in Eq. (55) with the respective value of $\varphi$. Taken together, we obtain twelve triplet states for $D_3$, which are summarized in Table IV, instead of only eight for the point group $C_3$.

Finally, we can also ask how the different states behave for small $J$, i.e., whether singlet and triplet can mix and which phases are Hund’s partners. Exactly as illustrated above for the pure triplet phases, we have to consider higher-order terms that determine the relative phase between the chiral, $\mu = +$, and antichiral, $\mu = -$, basis functions. As this analysis closely parallels our previous discussions, we just present the result in Table IV. In total, there are ten symmetry-inequivalent mixed singlet and triplet phases. Seven of them are only possible if $\delta a < 0$ (singlet dominates); the remaining three can be realized for either sign of $\delta a$.

**VII. SUMMARY AND DISCUSSION**

In this work, we have presented a systematic classification and analysis of superconducting instabilities in graphene moiré systems. To this end, we have focused on zero-momentum Cooper pairs formed out of electrons in different valleys. Intervalley pairing is expected to be the dominant pairing channel as time-reversal relates the two valleys. We have first analyzed singlet and triplet pairing separately since spin-orbit coupling is expected to be very weak in graphene. However, theoretical estimates [11, 46, 52] of the interaction terms indicate that the system is approximately invariant under independent spin rotations in the two valleys, leading to an (approximate) $SU(2)_+ \times SU(2)_-$ symmetry and the (near) degeneracy of singlet and triplet pairing. For this reason, we have also classified the pairing instabilities close to this high-symmetry point, analyzing which triplet state transforms into which singlet phase upon changing the sign of the interactions breaking the $SU(2)_+ \times SU(2)_-$ symmetry. We have further derived the conditions under which singlet and triplet can mix despite the absence of spin-orbit coupling.

As experimental indications of triplet pairing have so far only been reported for twisted double-bilayer graphene [47, 48], we have dealt with pairing instabilities in mainly this system. Here, a displacement field is required to stabilize the superconducting state and reduces the point group to $C_3$. The pairing states and their properties associated with the real representation $A$ and the complex representation $E$ of $C_3$ are summarized in Tables I and II, respectively.

Being one-dimensional and real, $A$ only allows for one singlet, a unitary and a nonunitary triplet phase, and one mixed phase. The latter is expected to be relevant only if $SU(2)_+ \times SU(2)_-$ is weakly broken and the two consecutive transitions in the schematic phase diagram in Fig. 1(b) are very close. Using the values of the coupling constants in Ref. 52, we estimate the splitting to be about two orders of magnitude smaller than the critical temperature and hence, hard to see experimentally [60]. Whether renormalization-group corrections could enhance the impact of these weak symmetry-breaking perturbations at energies of order of the transition temperature is an open question, which we leave for future work. The gap structure of the four phases transforming under $A$ is quite different: while the nonunitary triplet is gapless for one of the spin species, the singlet and unitary triplet have a single, fully established gap, and the mixed phase has two finite but distinct gaps for the two spin species. We have further shown that single-band mean-field theory will generally favor the phase diagram in Fig. 1(a) over Fig. 1(b). However, the strong-coupling nature inherent in the problem makes the applicability of mean-field theory questionable and can lead to significant corrections which might eventually select other phases. We have illustrated these corrections for ferromagnetic fluctuations, expected to be relevant for twisted double-bilayer graphene, and find that the resulting corrections will, as opposed to mean field, generally favor the phase diagram in part (b) of Fig. 1 over that in part (a).

The complex representation allows for many more states: two pure singlets, eight triplets, and, if $SU(2)_+ \times SU(2)_-$ is only weakly broken, six distinct mixed phases. As compiled in Table II, all of these three classes of states allow for nodal points and fully gapped phases. However, only the triplets can have nodal lines (residual ungapped Fermi surfaces of one spin species). Only one out of the two different triplet states of the IR $A$ allow for an admixture of singlet and triplet for weak anti-Hund’s coupling but, in contrast, six out of the eight triplets transforming
under $E$ do so.

Out of the possible pairing states in Table II, single-band mean-field theory favors the two triplet states $E^{3v}(b)$ and $E^{3v}(d)$ along with their respective Hund’s partners—the mixed phase $E^{3v}(0, i) + E^{3v}(a)$ and the chiral singlet $E^{3v}(1, i)$. We have discussed how additional weak ferromagnetic fluctuations can lift the exact degeneracy of $E^{3v}(b)$ and $E^{3v}(d)$, generically favoring the former. In the limit of strong ferromagnetic fluctuations—expected to be most relevant to twisted double-bilayer graphene—we obtain the chiral nonunitary triplet superconducting state.

Motivated by the experimentally observed \cite{48} linear increase of the transition temperature with an in-plane magnetic field, we have also mapped out the possible phase diagrams in the presence of a magnetic field. As expected, if the SU(2)$_+ \times$ SU(2)$_-$ symmetry is significantly broken, the linear increase is only consistent with triplet pairing. For pairing in the $A$ channel, there are two possible phase diagrams, shown in Fig. 2(c) and (d), depending on which triplet state is realized in the absence of a magnetic field. The magnetic field fully determines the form of the leading triplet state to be $A^{3v}(1, i, 0)$ in the $A$ channel. For order parameters transforming under $E$, there are two possibilities for the leading triplet state, $E^{3v}(c)$ or $E^{3v}(e)$, in a magnetic field; which of the two is realized depends on the value of the quartic couplings in the free energy. Both mean-field theory and ferromagnetic fluctuations favor the $E^{3v}(e)$ state. If, however, SU(2)$_+ \times$ SU(2)$_-$ is only very weakly broken, singlet pairing as the dominant instability of the system is also consistent with the linear increase of the critical temperature; the two possible phase diagrams for the case of pairing in the IR $A$ are illustrated in Fig. 2(a) and (b).

We have also derived (within mean-field theory) the key couplings, $c_2$ in Eq. (24) and $c_b^2$ in Eq. (42), between the superconducting order parameter and the magnetic field $B$ that determine the slope of the increase, $\Delta T_c$, of the critical temperature with magnetic field. We found that they have the exact same mathematical form; as such, the behavior $\Delta T_c \simeq 2\mu_B B$, with Bohr magneton $\mu_B$, seen in experiment \cite{48}, is equally surprising for both pairing channels and does not favor one channel over the other. In both cases, this might either be accidental or due to quantum critical scaling \cite{52}.

For completeness, we have also studied, in Sec. VI, the changes in the classification when there is an extra in-plane rotation symmetry, $C_{2y}$, and a two-fold rotation, $C_2$, perpendicular to the plane. These two symmetries are relevant (either as exact or emergent symmetries) to twisted bilayer graphene and ABC trilayer graphene. We find that while the $C_2$ symmetry has no consequences for the classification, $C_{2y}$ not only pins the nodes of certain pairing states along high-symmetry lines but also leads to more pairing states as summarized in Table IV.

This work further illustrates that graphene moiré systems provide a very rich playground for novel strongly correlated superconducting phases and hope that our systematic analysis of pairing in the absence and presence of magnetic fields will help future theoretical and experimental studies to pinpoint the microscopic form of the superconducting state.

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**Appendix A: Microscopic Ginzburg-Landau expansion**

In this appendix, we derive the prefactors of the various free-energy expansions in the main text within mean-field theory. Unless stated otherwise, we use a single-band description.

1. **Without a magnetic field**

We imagine performing a mean-field decomposition in the Cooper channel and keeping only the singlet and triplet pairing of the dominant IR. The ensuing mean-field Hamiltonian for the one-band model has the form

$$H_{\text{MF}} = \sum_k \xi_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} + \sum_k \left( \Delta_k^+ + \sigma \cdot d_k \right) i \sigma_y \Gamma_{\sigma\sigma'} c_{-k\sigma'} \Gamma_{-k\sigma'}^{-1} \xi_{-k\sigma}^+,$$

where $\xi_{k\sigma}^+ = \xi_{-k\sigma}^-$ due to time-reversal symmetry. In Eq. (A1), we have omitted a constant term, which is quadratic in the superconducting order parameter and does not affect the quartic terms we derive below. Upon integrating out the fermions in Eq. (A1) and expanding the resulting free energy in the superconducting order parameter, the Ginzburg-Landau expansion coefficients can be obtained order by order.

Starting with the one-dimensional real IR $A$ of $C_3$, we write $\Delta_k = \lambda_k^\ell \Delta^\ell$, $d_k = \lambda_k^d d$, where $\lambda^\ell_k$ and $\lambda^d_k$ are momentum-dependent basis functions that are invariant under $C_3$. Using the generalization of Eq. (20) to
parametrize the free energy,
\[ F \sim a(T) \left( |\Delta|^2 + d^\dagger d \right) + \delta a \left( |\Delta|^2 - d^\dagger d \right) + \gamma_1 |\Delta|^4 \\
+ \gamma_2 \left( d^\dagger d \right)^2 + \gamma_3 |d^\dagger d| + \gamma_4 |\Delta|^2 d^\dagger d \\
+ \gamma_5 \text{Re} \left[ |\Delta|^2 d^\dagger d^\dagger \right], \]  
(A2)

which allows us to account for a nonzero \( J \) making singlet and triplet nonequivalent, we find
\[ \gamma_1 = \mathcal{F}[|\Delta|^4], \quad \gamma_2 = \gamma_3 = \mathcal{F}[|\Delta|^4], \]
\[ \gamma_4 = 4 \mathcal{F}[^{2} |\Delta|^2 |\Delta|^2], \quad \gamma_5 = 2 \mathcal{F}[(\lambda k)^2(\lambda^* k)^2]. \]  
(A3)

To keep the expressions compact, we have defined the functional
\[ \mathcal{F}[f_k] := T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{f_k}{(\omega_n^2 + \xi k^2)^2}. \]  
(A5)

When \( J = 0 \), we have \( \lambda k = \lambda k \) and hence, obtain
\[ \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4/4 = \gamma_5/2 > 0, \]
(A6)

which is compatible with the prefactors in Eq. (20) as required from the SU(2) \(_{+} \times \) SU(2) \(_{-} \) symmetry. On top, \( \gamma_1 = \gamma_3 \) is an additional constraint arising from the mean-field approximation (and not related to an exact symmetry). In terms of the prefactors in Eq. (20), it sets \( b_1 = 0 \), as stated in the main text. The positive sign of the coefficients in Eq. (A6) betokens that mean-field theory always favors part (a) in the phase diagram in Fig. 1.

Similarly, we can study the complex representation of \( \mathcal{C}_3 \) introduced in Sec. IV of the main text. Using the representation in Eq. (29) for the singlet pairing, \( \Delta_s = \sum \eta \eta (X_k + \imath y_k) \), it is straightforward to show that
\[ b^s_1 = b^s_2/2 = 2 \mathcal{F}[X k + Y k]^2 > 0 \]  
(A7)

for the coefficients \( b^s_1, b^s_2 \) in Eq. (32). Being positive, these coefficients favor the chiral superconductor \( E^{1+}(1, i) \) as was observed earlier [40, 41].

Finally, repeating this procedure for the triplet state with parametrization (36), \( d_k = \sum \eta \eta (x_k + \imath y_k) \), the coefficients in Eq. (38) evaluate to
\[ b^t_1 = b^t_2/2 = -b^t_2 = -2b^t_2 = 2 \mathcal{F}[X k + Y k]^2 > 0, \]
\[ b^t_2 = 0. \]  
(A8)

The triplet states \( E^{3+}(b) \) and \( E^{3+}(d) \) will have the lowest energy for this configuration of quartic coefficients as argued in the main text. The degeneracy between these two states will be lifted by corrections beyond the mean-field approximation, such as the ferromagnetic fluctuations of Sec. V. In the presence of a magnetic field, Eq. (A8) uniquely determines the chiral nonunitary triplet \( E^{3+}(e) \) as the leading instability (see Sec. IV C).

2. Coupling to a magnetic field

In this subsection, we will analyze several important coupling terms between the superconductor and the magnetic field from a weak-coupling perspective. The microscopic form of the coupling to the Zeeman, \( M_Z \), and in-plane orbital field, \( M_O \), reads as
\[ \mathcal{H}_B = \sum_k c^\dagger_{k\sigma v} \sigma \sigma \sigma \sigma c_{k\sigma v} \cdot M_Z + \sum_k g_v(k) c^\dagger_{k\sigma v} c_{k\sigma v} M_O, \]  
(A9)

where we have absorbed the \( g \)-factor of the Zeeman coupling into the definition of \( M_Z \). This is not possible for the orbital coupling, as its \( g \)-factor \( g_v(k) \) depends significantly on momentum. The form of \( g_v(k) \) is determined by microscopic details such as the Bloch states. All we need here is that \( g_v(k) = -g_v(-k) \), as follows from time-reversal symmetry (5), and we refer to Ref. 52 for a microscopic derivation of its momentum dependence.

To proceed further with Eq. (A9), we first set the orbital coupling to zero, \( M_O = 0 \). Even when the actual interacting multiband system is not invariant under \( C_2 \), the single-band mean-field Hamiltonian, \( \mathcal{H}_{MF} + \mathcal{H}_B \), is left invariant under the action of \( C_2 \) in Eq. (50) if we further set \( d_k \to -d_k \) in Eq. (A1). This emergent symmetry is a consequence of the special role of \( C_2 \) in two-dimensions as it acts on \( k \) in the same manner as time-reversal and, as such, can have crucial consequences for superconducting pairing [65].

In the present case, this symmetry implies that the coupling terms \( \delta c_1 \) in Eq. (24) and \( \delta c_1^E \) in Eq. (42) will vanish within single-band mean-field theory as is also readily confirmed by explicit calculation; we emphasize, however, that this is not an exact statement and we have checked that a multiband mean-field description allows for nonzero values. Nonetheless, we view the vanishing of these coupling in the weak-coupling single-band limit as an indication that they are likely small in the system.

Noting that the orbital coupling \( M_O \) breaks this emergent symmetry, we expect \( c_7 \) in Eq. (24) to be nonzero in single-band mean-field theory. Indeed, we obtain
\[ c_7 = 2T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} g_+(k)(k)^2 \left[ \frac{3(\omega_n^2 - \xi k^2)}{(\omega_n^2 + \xi k^2)^3} \right] \]  
(A10)

\[ = \frac{1}{3T^2} \int \frac{d^2 k}{(2\pi)^2} g_+(k)(k)^2 \left[ \frac{\tanh(\xi k^2/(2T))}{\xi k^2/(1 + \cosh(\xi k^2/T))} \right]. \]

Finally, the couplings of the Zeeman term to the triplet vector in Eqs. (24) and (42) are also not constrained by the emergent \( C_2 \) symmetry. We find these to be nonzero and given by
\[ c_2 = -4 \mathcal{F} \left[ \xi k^2 |\lambda k|^2 \right], \]  
(A11a)
\[ c_3 = -4 \mathcal{F} \left[ \xi k^2 (X k^2 + Y k^2) \right]. \]  
(A11b)
respectively. Our main observation here is that the forms of $c_2$ and $c_3^f$ are identical: the nonuniversal part is a momentum integral which, in both cases, is weighted by a function that is invariant under $C_3$ and has no symmetry-imposed nodes on the Fermi surface. Accordingly, it is not possible to distinguish between the IRs $A$ and $E$ based on the slope of the increase of $T_c^+$ in small magnetic fields.

3. Ferromagnetic fluctuations

In the last part of this appendix, we justify the expectation of $c_+ \simeq c_-$ in Eq. (46). Microscopically, the zero-momentum spin fluctuations, $m_v$, couple to the electrons as

$$H_m = \sum_{k,v} g_m^n ( k ) c_{\sigma \sigma v} \sigma c_{\sigma \sigma v} \cdot m_v,$$

(A12)

where $g_m^n ( k ) = g_m^n ( - k )$ as a consequence of time-reversal symmetry and we have, as before, assumed that we can focus on a single isolated electronic band. It is easy to see that $H_{MF} + H_m$ is again invariant under the $C_2$ symmetry in Eq. (50) if we further replace

$$d_k \rightarrow -d_k, \quad m_v \rightarrow m_0.$$

(A13)

While Eq. (44) is automatically invariant under Eq. (A13), the coupling for the two-dimensional representation in Eq. (46) is invariant only if $c_+ = c_-$. Consequently, multiband effects are required for nonzero $c_+ - c_-$, where we expect its value to be much smaller than $c_+ + c_-$, as posited in the main text. We also checked by explicit calculation that $c_+ \neq c_-$ is possible in a multiband description.

Appendix B: Details for the complex representation

In this appendix, we present additional details of the different phases transforming under the complex representation $E$ of $C_3$.

As a starting point, it is helpful to chart out a phase diagram describing which of the triplet phases $E^3(\alpha)$ to $E^3(h)$ is realized as a function of the quartic terms $b_{1,2,3,4,5}$ in Eq. (38). Upon recognizing that $b_1$ does not affect the form of the order parameter (but is assumed to be chosen so as to guarantee the stability of the expansion), we can conveniently display the phases as a function of $b_2^j / |b_2^j|$, $j = 3, 4, 5$, discussing the two possible signs of $b_2^j$ separately. Such a phase diagram is drawn in Fig. 3.

As the main text contends, there are no independent terms involving $\sigma_y$ to add to the $SU(2)_+ \times SU(2)_-$ invariant form of the free energy in Eq. (41). To see this, we note that it suffices to consider terms involving both $\Delta_+$ and $\Delta_-$ since terms with only $\Delta_+$ (or $\Delta_-$) have already been addressed in Sec. IIIA. Among the terms that mix $\Delta_+$ and $\Delta_-$, the following are consistent with time-reversal and $C_3$ symmetry:

$$\Delta F_1 = |\text{tr} [\sigma_y \Delta_+ \sigma_y \Delta_+^T]|^2,$$

$$\Delta F_2 = \text{tr} \left[ \Delta_+ \sigma_y \Delta_+^T \sigma_y \Delta_+^T \right] + \text{tr} \left[ \Delta_- \sigma_y \Delta_-^T \sigma_y \Delta_-^T \right],$$

$$\Delta F_3 = \text{tr} \left[ \Delta_+ \sigma_y \Delta_-^T \sigma_y \Delta_+^T \right] + \text{tr} \left[ \Delta_- \sigma_y \Delta_+^T \sigma_y \Delta_-^T \right].$$

(B1)

(B2)

(B3)

(B4)

However, all of these terms can be reformulated as

$$\Delta F_1 = |\text{tr} [\sigma_y \Delta_+ \sigma_y \Delta_+^T]|^2,$$

$$\Delta F_2 = \text{tr} \left[ \Delta_+ \sigma_y \Delta_+^T \sigma_y \Delta_+^T \right] + \text{tr} \left[ \Delta_- \sigma_y \Delta_-^T \sigma_y \Delta_-^T \right],$$

$$\Delta F_3 = \text{tr} \left[ \Delta_+ \sigma_y \Delta_-^T \sigma_y \Delta_+^T \right] + \text{tr} \left[ \Delta_- \sigma_y \Delta_+^T \sigma_y \Delta_-^T \right].$$

(B1)

(B2)

(B3)

(B4)

so they do not constitute independent terms to add to Eq. (41).

In concluding this appendix, we present the explicit form of the free-energy (41) in terms of singlet and triplet components. Inserting $\Delta_\mu = \sigma_0 \Delta_\mu^s + \sigma \cdot d_\mu$, $\mu = \pm$ in Eq. (41) and adding $SU(2)_+ \times SU(2)_-$ symmetry breaking only at the level of the quadratic terms, one arrives at

$$\mathcal{F} \sim a(T) \sum_\mu \left( |\Delta_\mu^s|^2 + d_\mu^T d_\mu \right) + \delta a \sum_\mu \left( |\Delta_\mu^s|^2 - d_\mu^T d_\mu \right) + \beta_1 \left( \sum_\mu |\Delta_\mu^s|^2 \right)^2 + \beta_2 |\Delta_\mu^s|^2 |\Delta_\mu^s|^2$$

$$+ \beta_3 \left( \sum_\mu d_\mu^T d_\mu \right) + \beta_4 (d_+^T d_-)(d_-^T d_+) + \beta_5 |d_+^T d_-|^2 + \beta_6 |d_+^T d_-|^2 + \beta_7 \sum_\mu |d_\mu^T d_\mu|^2$$

$$+ \beta_8 |d_+^T d_-|^2 + \beta_9 |d_+^T d_-|^2 + \beta_10 |d_+^T d_-|^2.$$
FIG. 3. Phase diagram for the free energy in Eq. (38). The different triplet states labelled (a) to (h) are defined in the main text in Sec. IVA.

\[
+ \beta_8 \sum_{\mu} |\Delta_{\mu}^x|^2 d_{\mu}^d \bar{d}_{\mu} + \beta_9 \sum_{\mu} |\Delta_{\mu}^y|^2 d_{\mu}^d \bar{d}_{\mu} + \beta_{10} \text{Re} \left[ \Delta_{\mu}^x \Delta_{\mu}^{\star} \bar{d}_{\mu}^d \bar{d}_{\mu}^d \right] \\
+ \beta_{11} \text{Re} \left[ \sum_{\mu} (\Delta_{\mu}^x)^2 d_{\mu} \bar{d}_{\mu} \right] + \beta_{12} \text{Re} \left[ (\Delta_{\mu}^x)^* \bar{d}_{\mu}^d \bar{d}_{\mu}^d \right],
\]

where $\bar{d} = -$ for $\mu = +$ and vice versa. Due to the fewer number of independent parameters in Eq. (41), there are many relations between the different coefficients $\beta_1, \ldots, \beta_{12}$, namely:

\[
\beta_1 = b_1 + b_2, \quad \beta_2 = b_3 + b_4 + 2(b_5 - b_2), \quad \beta_3 = b_1 + 2b_2, \quad \beta_4 = b_3 + 2b_5 - 4b_2, \\
\beta_5 = b_1 + 2b_5, \quad \beta_6 = -2b_5, \quad \beta_7 = -b_2, \quad \beta_8 = 2(b_1 + 2b_2), \\
\beta_9 = 2(b_1 + b_5) + b_3, \quad \beta_{10} = 2b_4 + 4b_5, \quad \beta_{11} = 2b_5, \quad \beta_{12} = 4b_5.
\]

It is not difficult to observe that the five different purely triplet quartic terms, $\beta_j = 3, 4, 5, 6, 7$, are all independent. Consequently, we can parametrize all twelve $\beta_j$ in terms of the five purely triplet terms and realize all of the triplet states of Sec. IVA.

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