The Polarised Photon $g_1^\gamma$ Sum Rule at the Linear Collider and High Luminosity B Factories *

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Abstract: The sum rule for the first moment of the polarised (virtual) photon structure function $g_1^\gamma(x, Q^2; K^2)$ is revisited in the light of proposals for future $e^+e^-$ colliders. The sum rule exhibits an array of phenomena characteristic of QCD: for real photons ($K^2 = 0$) electromagnetic gauge invariance constrains the first moment to vanish; the limit for asymptotic photon virtuality ($m_\rho^2 \ll K^2 \ll Q^2$) is governed by the electromagnetic $U_A(1)$ axial anomaly and the approach to asymptopia by the gluonic anomaly; for intermediate values of $K^2$, it reflects the realisation of chiral symmetry and is determined by the off-shell radiative couplings of the pseudoscalar mesons; finally, like many polarisation phenomena in QCD, the first moment of $g_1^\gamma$ involves the gluon topological susceptibility. In this paper, we review the original sum rule proposed by Narison, Shore and Veneziano and extend the relation with pseudoscalar mesons. The possibility of measuring the sum rule in future polarised $e^+e^-$ colliders is then considered in detail, focusing on the International Linear Collider (ILC) and high luminosity $B$ factories. We conclude that all the above features of the sum rule should be accessible at a polarised collider with the characteristics of SuperKEKB.

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1. Introduction

The sum rule for the first moment of the polarised photon structure function $g_1^\gamma(x, Q^2; K^2)$ provides a window into many features of QCD dynamics, including the gluonic axial anomaly and the realisation of chiral symmetry. This sum rule was first proposed by Narison, Shore and Veneziano in 1992 [1] as part of a series of investigations into gluonic and anomaly-dependent phenomena in QCD, notably the origin of the ‘spin of the proton’ suppression observed in the first moment of the polarised proton structure function $g_1^p$ [2, 3, 4, 5, 6, 7]. At that time, however, the details of the sum rule were out of reach of contemporary colliders since the spin asymmetries which need to be measured to determine $g_1^\gamma(x, Q^2; K^2)$ require exceedingly large luminosities.

Since that time, collider technology has moved on and plans are now well advanced for machines capable of integrated annual luminosities in the regime of inverse attobarns. It is therefore appropriate to revisit the $g_1^\gamma$ sum rule and investigate whether this new generation of colliders will be able to measure the full array of QCD phenomena encoded in it.

We focus on two future machines. First, the International Linear Collider (ILC) has recently passed the technology choice phase and agreement has been found to proceed with the ‘cold’, i.e. superconducting magnet, design. If international agreement is forthcoming, it is hoped that a linear $e^+e^-$ collider with a CM energy of at least 500GeV will be commissioned around 2015. The projected luminosity for the ILC is around $10^{34}\text{cm}^{-2}\text{s}^{-1}$, corresponding to an annual integrated luminosity of order 0.1ab$^{-1}$ [8, 9]. Of course, for our purposes, the collider would have to be run in polarised mode.

The second machine we consider in detail is SuperKEKB. It was already noted in ref.[1] that high-luminosity B factories were the colliders of choice for measuring the $g_1^\gamma$ sum rule, since high energy is not in itself an advantage but ultra-high luminosity is essential. The proposed upgrade of KEKB to SuperKEKB [10] envisages an 8GeV ($e^-$) on 3.5GeV ($e^+$) collider with a target luminosity $5\times10^{35}\text{cm}^{-2}\text{s}^{-1}$, corresponding to 5ab$^{-1}$ annual integrated luminosity.
luminosity. As we shall show, if this machine were run with polarised beams, this luminosity would allow the full details of the off-shell first moment \( g_1^\gamma \) sum rule to be measured.

We come to these experimental considerations in section 5. We begin though with a brief review of the derivation of the first moment sum rule itself, emphasising that, as appropriate for a measurement in \( e^+e^- \) collisions (as opposed to doing two-photon physics using real back-scattered laser photons as the target), we are determining the off-shell structure function \( g_1^\gamma(x,Q^2;K^2) \), where \( K^2 \) is the virtuality of the off-shell ‘target’ photon in DIS. Almost all the interesting QCD physics resides in the \( K^2 \) dependence of the sum rule. An important point is that we should therefore avoid an unnecessary use of the ‘equivalent photon’ formalism \([11, 12]\) in establishing the sum rule. Moreover, all our results will be formulated in QCD field-theoretic terms, focusing on the OPE and current correlation functions, rather than using parton language. This makes the important non-perturbative results far more transparent. (For a selection of reviews and recent papers on \( g_1^\gamma \) from a parton perspective, see e.g. \([12, 13, 14, 15, 16, 17, 18, 19]\).)

Having reviewed the basic sum rule, in section 3 we study its ‘asymptotic’ properties for momenta \( K^2 \approx 0 \) and \( K^2 \gg m_\rho^2 \). The first moment is known \([20, 1, 21]\) to vanish for real photons as a consequence of electromagnetic gauge invariance (current conservation). For photon virtualities well above the relevant hadronic scale of \( m_\rho^2 \) (but still of course in the DIS regime \( K^2 \ll Q^2 \)), \( \int dx \ g_1^\gamma(x,Q^2;K^2) \) tends to a value fixed by the electromagnetic axial \( U_A(1) \) anomaly. Moreover, the approach to this asymptotic region is governed by the gluonic contribution to the anomaly, so there is much of theoretical interest even in this essentially perturbative regime.

For intermediate virtualities, \( K^2 \sim O(m_\rho^2) \), the first moment depends on the explicit momentum dependence of the form factors specifying the three-current ‘AVV’ correlation function involving the hadronic axial \( U_A(1) \) current and two electromagnetic currents. This is an important non-perturbative object in QCD and the ability to measure it explicitly for a range of photon momenta would provide an interesting window into chiral symmetry breaking and associated QCD phenomena. The reasons for this are explored in considerable detail in ref.[22], a companion paper to ref.[1]. In particular, we can show that these form factors are essentially the off-shell couplings of the pseudoscalar mesons \( \pi, \eta, \eta' \) to photons, whose on-shell limits are determined by the radiative decays \( \pi, \eta, \eta' \rightarrow \gamma \gamma \). The radiative pion decay has played a distinguished role in establishing the reality of anomalies and the nature of QCD (in particular, by providing a direct measure of the number of colours). In the flavour singlet sector, the theory is even more interesting as it involves in an essential way the gluonic axial \( U_A(1) \) anomaly and the gluon content of the \( \eta' \) meson \([23]\). In section 4, therefore, we explore the connection between the \( g_1^\gamma \) sum rule and radiative pseudoscalar decays, extending the results of ref.[1] to incorporate the analysis developed in our papers \([24, 25, 26]\). One theoretically interesting feature is the link with the gluon topological susceptibility, which plays a key role in many polarised QCD phenomena, notably the ‘spin of the proton’. This section may be read in conjunction with another paper, ref.[27], in which we revisit our results \([24, 25, 26]\) for radiative pseudoscalar decays and their relation with the topological susceptibility and Witten-Veneziano formula \([28, 29]\), and derive explicit experimental values for the pseudoscalar decay constants which may be
(carefully) compared with large $N_c$ chiral Lagrangians [30, 31] (see also ref.[32]).

Having reviewed and developed the theory of the $g_1^\gamma$ first moment sum rule, we then return in section 5 to the experimental question of whether it can be measured, including the full range of $K^2$ dependence, in the forthcoming generation of high-luminosity colliders. Our conclusion is that the ILC is marginal for this purpose, but that a polarised collider with the energy and luminosity of SuperKEKB would be able to uncover the full dynamical richness of the sum rule.

2. The sum rule for $\int dx \ g_1^\gamma(x, Q^2; K^2)$

We are concerned with the process $e^+e^- \rightarrow e^+e^-X$, which at sufficiently high energy is dominated by the two-photon interaction shown in Fig.1. The deep-inelastic limit is characterised by $Q^2, \nu_e, \nu \rightarrow \infty$ with $x_e = Q^2/2\nu_e$ and $x = Q^2/2\nu$ fixed, where (see Fig.1 for definitions of the momenta) $Q^2 = -q^2$, $K^2 = -k^2$, $\nu_e = p_2.q$, $\nu = k.q$ and $s = (p_1 + p_2)^2$. We also consider the ‘target photon’ to be relatively soft, $K^2 \ll Q^2$.

Verifying the first moment sum rule for the polarised structure function $g_1^\gamma(x, Q^2; K^2)$ requires studying the spin asymmetries in cross-sections which are differential with respect to $Q^2, K^2$ and $x_e$. Experimentally, these are determined from

$$x_e = \frac{E'_1 \sin^2 \frac{\theta_1}{2}}{E - E'_1 \cos^2 \frac{\theta_1}{2}} \quad x = \frac{Q^2}{Q^2 + W^2}$$

$$Q^2 = 4EE'_1 \sin^2 \frac{\theta_1}{2} \quad K^2 \simeq EE'_1 \theta_1^2$$

(2.1)

Here, $E'_1(E'_2)$ and $\theta'_1(\theta'_2)$ are the energy and scattering angle of the hard-scattered (target) electron and $W$ is the invariant hadronic mass. For the values $K^2 \sim m_e^2$ of interest in the sum rule, the target electron is nearly-forward and $\theta'_2$ is very small. If it can be tagged, then the virtuality $K^2$ is simply determined from eq.(2.1); otherwise $K^2$ can be inferred indirectly from a measurement of the total hadronic energy.

A systematic presentation of the relations between cross-section moments and structure functions from first principles may be found in ref.[1], so here we shall only display some key formulae. ‘Electron structure functions’ $F_2^e(x_e, Q^2)$, $F_L(x_e, Q^2)$ and $g_1^\gamma(x_e, Q^2)$ are introduced in the analogous way to ordinary nucleon structure functions and are related to the spin-dependent cross-sections as follows:

$$\sigma = 2\pi\alpha^2 \frac{1}{s} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx_e}{x_e^2} \left[ F_2^e \left( \frac{x_e s}{Q^2} - 1 + \frac{1}{2} \frac{Q^2}{x_e s} \right) - F_L^e \frac{1}{2} \frac{Q^2}{x_e s} \right]$$

(2.2)

We have made a number of changes of notation compared to ref.[1]. The dictionary is $K^2 \leftrightarrow \kappa^2$, $\nu_e \leftrightarrow \nu$, $\nu \leftrightarrow \tilde{\nu}$, $x_e \leftrightarrow x$, $x \leftrightarrow y$. The standard DIS notation $(\nu, x)$ therefore refers here to the target photon, rather than the target electron as in ref.[1]
\[ \Delta \sigma = 2\pi \alpha^2 \frac{1}{s} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx_e}{x_e} g_1^e \left[ 1 - \frac{1}{2} \frac{Q^2}{x_e s} \right] \]  

(2.3)

where \( \sigma = \frac{1}{2}(\sigma_+ + \sigma_-) \) and \( \Delta \sigma = \frac{1}{2}(\sigma_+ - \sigma_-) \) with \(+, -\) referring to the electron helicities. The parameter \( Q^2/x_e s \ll 1 \) and only leading order terms are retained below. \( \alpha \) is the fine structure constant.

The photon structure functions themselves may be defined in the standard way in terms of the matrix elements of the off-shell matrix elements \( \langle \gamma(k, e^+) | J_{\mu}^{\text{em}}(q) J_{\nu}^{\text{em}}(-q) | \gamma(k, e) \rangle \) of electromagnetic currents in the DIS limit. We can readily show that the electron structure functions introduced above can be expressed as convolutions of the photon structure functions with appropriate Altarelli-Parisi splitting functions. In particular, we have

\[ F_2^e(x_e, Q^2) = \frac{\alpha}{2\pi} \int_0^\infty \frac{dK^2}{K^2} \int_0^1 \frac{dx_e}{x_e} P_{\gamma e}(x_e) \int_0^1 \frac{dx}{x} F_2^\gamma(x, Q^2; K^2) \]  

(2.4)

\[ g_1^e(x_e, Q^2) = \frac{\alpha}{2\pi} \int_0^\infty \frac{dK^2}{K^2} \int_0^1 \frac{dx_e}{x_e} \Delta P_{\gamma e}(x_e) \int_0^1 \frac{dx}{x} g_1^\gamma(x, Q^2; K^2) \]  

(2.5)

where

\[ P_{\gamma e}(z) = \frac{1}{z} (1 + (1 - z)^2) \quad \Delta P_{\gamma e}(z) = (2 - z) \]  

(2.6)

These results allow us to write relations that link the \( x \)-moments of the photon structure functions themselves to the \( x_e \)-moments of the cross-section. These key expressions, which we return to in section 5, are:

\[ \int_0^1 dx_e x_e^n \frac{d^3\sigma}{dQ^2 dx_e dK^2} = \alpha^3 \frac{1}{Q^4 K^2} \int_0^1 dz \ z^n P_{\gamma e}(z) \int_0^1 dx \ x^{n-1} F_2^\gamma(x, Q^2; K^2) \]  

(2.7)

\[ \int_0^1 dx_e x_e^n \frac{d^3\Delta \sigma}{dQ^2 dx_e dK^2} = \alpha^3 \frac{1}{s Q^2 K^2} \int_0^1 dz \ z^{n-1} \Delta P_{\gamma e}(z) \int_0^1 dx \ x^{n-1} g_1^\gamma(x, Q^2; K^2) \]  

(2.8)

The integrals factorise, so we see that the \( n \)th \( x \)-moments of the photon structure functions are given by the \((n + 1)\)st \( x_e \)-moments of the fully differential cross-sections.

The expressions (2.4,2.5) are derived from first principles using only Feynman diagram rules and the operator product expansion. Despite the appearance of the AP splitting functions, no parton technology is used. Most importantly, it is not necessary to resort to the equivalent photon approximation \([11, 12]\) (see ref.[1] for a careful discussion of this point), so we can be certain that our results are accurate throughout the required range of target photon momenta \( K^2 \). The relevant OPE is for the product of electromagnetic currents:

\[ i \ J_{\mu}^{\text{em}}(q) J_{\nu}^{\text{em}}(-q) \sim Q^2 \to \infty \sum_{n=1, \text{odd}} \sum_{a} \frac{2^n}{Q^{2n}} q_{\mu_2} \cdots q_{\mu_n} i \epsilon_{\nu \mu_1 \mu_2} q_{\alpha}^{\alpha} E_1^{\alpha n}(Q^2) R_{a, n}^{\mu_1 \cdots \mu_n}(0) \]

(2.9)

where the \( E_1^{\alpha n}(Q^2) \) are Wilson coefficients and \( R_{a, n}^{\mu_1 \cdots \mu_n}(0) \) are the complete set of odd-parity, twist 2 operators in QCD. (We only indicate here the odd-parity operators, which
contribute to $g_1^\gamma$, to establish notation; the even-parity operators contribute to $F_2^\gamma$ and $F_L^\gamma$. See ref.[1] for full details and the explicit forms of the relevant operators.) Their form factors in the 3-point correlation functions with the photon fields $A_\lambda(k)$ are:

$$\langle 0 | R_{a,n}^{\mu_1...\mu_n} (0) A_\lambda(k) A_\rho(-k) | 0 \rangle = \frac{1}{K^4} k^{\mu_1} ... k^{\mu_n} i \epsilon_{\lambda \rho \mu_1...\mu_n} k_\alpha \hat{R}_{a,n}(K^2) \quad (n \geq 1, \text{odd})$$

(2.10)

The key result now is the relation between the moments of the structure functions and the Wilson coefficients and form factors from the OPE. We can show [1]

$$\int_0^1 dx \ x^{n-1} g_1^\gamma(x, Q^2; K^2) = \sum_a E_{1,n}^a(Q^2) \hat{R}_{a,n}(K^2)$$

(2.11)

with similar expressions for $F_2^\gamma$ and $F_L^\gamma$. We are of course primarily concerned with the $n = 1$ moment of $g_1^\gamma(x, Q^2; K^2)$. In this case, the only relevant $R_{a,1}^{\mu}$ operator is the hadronic $U_A(1)$ axial current $J_{\mu 5}^r$,

$$R_{a,1}^{\mu} \rightarrow J_{\mu 5}^r = \bar{\psi} T^r \gamma_\mu \gamma_5 \psi$$

(2.12)

where $T^r$ are the $SU(3)$ generators (including the flavour singlet $T^0 = 1$), and $\psi$ are quark fields. Clearly, only the diagonal generators $a = 3, 8, 0$ contribute. The form factor is therefore defined as

$$\langle 0 | J_{\mu 5}^r (0) A_\lambda(k) A_\rho(-k) | 0 \rangle = \frac{1}{K^4} i \epsilon_{\lambda \rho \mu a} k_\alpha \hat{R}_{r,1}(K^2)$$

(2.13)

and the essential first moment sum rule is

$$\int_0^1 dx \ g_1^\gamma(x, Q^2; K^2) = \sum_{r=3,8,0} E_{1,1}^r(Q^2) \hat{R}_{r,1}(K^2)$$

(2.14)

To sum up this section, we have established in eq.(2.8) how to measure the first moment of the polarised photon structure function $g_1^\gamma(x, Q^2; K^2)$ in terms of the spin asymmetry of the differential cross-section, $d^3 \Delta \sigma/dQ^2 dx_e dK^2$. The kinematic variables $Q^2$ and $x_e$ are found from the energy and scattering angle of the hard-scattered electron, while the target photon virtuality $K^2$ is most easily found by tagging the nearly-forward electron. On the theory side, eq.(2.14) determines the first moment of $g_1^\gamma(x, Q^2; K^2)$ in terms of a perturbatively known Wilson coefficient $E_{1,1}^r(Q^2)$ and a non-perturbative form factor $\hat{R}_{r,1}(K^2)$ characterising the AVV 3-current correlation function involving the hadronic $U_A(1)$ axial current and two electromagnetic currents.

3. The AVV correlation function, anomalies and momentum dependence

The first element of the sum rule is the Wilson coefficient $E_{1,1}^r(Q^2)$. The $Q^2$-dependence is governed by the RGE and the solution is well known. For QCD with $N_c = 3$ and $N_f = 3$ active flavours (results for the arbitrary $n$th moments and general $N_c, N_f$ are quoted in ref.[1]), we have

$$E_{1,1}^r = c^{(r)} \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) \quad (r = 3, 8)$$

(3.1)
\[ E_{1}^{0,1} = c^{(0)} \exp \left[ \int_{0}^{t} dt' \gamma(\alpha_{s}(t')) \right] \left( 1 - \frac{\alpha_{s}(Q^{2})}{\pi} \right) \] (3.2)

where \( t = \frac{1}{2} \ln Q^{2}/\mu^{2} \). The coefficients are fixed by the quark charges and are \( c^{(3)} = 1/3 \), \( c^{(8)} = 1/3 \sqrt{3} \) and \( c^{(0)} = 2/9 \). Since the singlet axial current is not conserved because of the gluonic contribution to the \( U_{A}(1) \) anomaly, it is associated with an anomalous dimension \( \gamma \), which enters into the momentum dependence of the singlet Wilson coefficient. Explicitly, \( \gamma = -\gamma_{0} \frac{\alpha_{s}}{4\pi} - \gamma_{1} \frac{\alpha_{s}^{3}}{(4\pi)^{2}} + \ldots \) where \( \gamma_{0} = 0 \) and \( \gamma_{1} = 3/4 \). Notice the important result that the expansion begins only at \( O(\alpha_{s}^{2}) \). The beta function, which determines the running of the QCD coupling \( \alpha_{s}(Q^{2}) \), is similarly given by \( \beta = -\beta_{0} \frac{\alpha_{s}^{2}}{4\pi} - \beta_{1} \frac{\alpha_{s}^{3}}{(4\pi)^{2}} + \ldots \) with \( \beta_{0} = 18 \).

Now consider the amputated AVV correlation function in eq. (2.13), defined with the electromagnetic current \( J_{A}(em) \). We first allow the axial current momentum \( p \) to be non-zero and subsequently take the required limit \( p \to 0 \). There are two important Ward identities. Electromagnetic current conservation implies

\[ i k_{i}^{A} \langle 0|J_{\mu}^{A}(p) J_{\nu}^{A}(em)(k_{1}) J_{\rho}^{A}(em)(k_{2})|0 \rangle = 0 \quad \text{(sim for } k_{2}^{A}) \] (3.3)

The (anomalous) chiral Ward identity, which follows from the usual anomalous conservation law (for \( N_{f} = 3 \))

\[ \partial^{\mu}J_{\mu 5} = M_{\rho 5} \phi_{5}^{0} + 6Q\delta_{\rho 0} + a^{(r)} \frac{\alpha}{8\pi} \tilde{F}_{\mu \nu} F_{\mu \nu} \] (3.4)

where \( \phi_{5}^{r} = \bar{\psi} T^{r}\gamma_{5}\psi \) and \( Q = \frac{\alpha}{8\pi} \text{tr} G^{\mu \nu} G_{\mu \nu} \), with \( G_{\mu \nu} \) the gluon field strength and \( F_{\mu \nu} \) the electromagnetic field strength, is

\[ \delta^{\mu 0} 6 \langle 0|Q(p) J_{\lambda}^{A}(em)(k_{1}) J_{\rho}^{A}(em)(k_{2})|0 \rangle \delta_{\rho 0} + \frac{1}{8\pi} a^{(r)} \epsilon_{\lambda \rho \alpha \beta} k_{1}^{\alpha} k_{2}^{\beta} = 0 \] (3.5)

The notation used for the quark masses follows ref.[24]: \( \text{diag}(m_{u}, m_{d}, m_{s}) = \sum_{r=3,8,0} m_{r} T^{r} \), then \( M_{rt} = d_{rs} m_{s} \), where \( d_{rs} \) are the usual \( SU(3) \) \( d \)-symbols. The term involving the gluon topological density \( Q \) is the gluonic \( U_{A}(1) \) anomaly and only appears in the flavour singlet case, while the final term arises because of the electromagnetic \( U_{A}(1) \) anomaly whose strength depends on the quark charges. With our normalisations, \( a^{(3)} = 1 \), \( a^{(8)} = 1/\sqrt{3} \) and \( a^{(0)} = 4 \). (Notice that this notation differs from ref.[1]. \( a^{(r)} \) here is \( 2N_{c} \) times the \( a^{(r)} \) of ref.[1] but coincides with the \( a^{r}_{em} \) of refs.[24, 25, 26].)

Now define form factors for the correlation functions appearing above:

\[ -i \langle 0|J_{\mu 5}^{A}(p) J_{\lambda}^{A}(em)(k_{1}) J_{\rho}^{A}(em)(k_{2})|0 \rangle = A_{1}^{r} \epsilon_{\mu \lambda \rho \alpha} k_{1}^{\alpha} + A_{2}^{r} \epsilon_{\mu \lambda \rho \alpha} k_{2}^{\alpha} \]

\[ + A_{3}^{r} \epsilon_{\mu \lambda \rho \beta} k_{1}^{\beta} k_{2}^{\alpha} k_{2}^{\rho} + A_{4}^{r} \epsilon_{\mu \rho \alpha \beta} k_{1}^{\alpha} k_{2}^{\beta} \]

\[ + A_{5}^{r} \epsilon_{\mu \lambda \rho \beta} k_{1}^{\alpha} k_{2}^{\beta} k_{1}^{\rho} + A_{6}^{r} \epsilon_{\mu \rho \alpha \beta} k_{1}^{\alpha} k_{2}^{\beta} k_{2}^{\rho} \] (3.6)

where the six form factors are functions of the invariant momenta, i.e. \( A_{i}^{r} = A_{i}^{r}(p^{2}, k_{1}^{2}, k_{2}^{2}) \),

\[ M_{rt} \langle 0|\phi_{5}^{r}(p) J_{\lambda}^{A}(em)(k_{1}) J_{\rho}^{A}(em)(k_{2})|0 \rangle = D^{r}(p^{2}, k_{1}^{2}, k_{2}^{2}) \epsilon_{\lambda \rho \alpha \beta} k_{1}^{\alpha} k_{2}^{\beta} \] (3.7)
\[ 6 \langle 0| Q(p) J_{\lambda}^{(em)}(k_1) J_{\rho}^{(em)}(k_2) |0 \rangle = B(p^2, k_1^2, k_2^2) \epsilon_{\lambda\rho\alpha\beta} k_1^\alpha k_2^\beta \]  \hspace{1cm} (3.8)

With these definitions, it follows immediately from eq.(2.13) that the second element of the sum rule, i.e. the form factor \( \hat{R}_{r,1} \), is just

\[
\hat{R}_{r,1}(K^2) = 4\pi \alpha \left( A_1^r - A_2^r \right)(K^2)
\]

\[
= -4\pi \alpha \left( D^r(K^2) + \delta^0 B(K^2) - \frac{1}{8\pi^2} \alpha^{(r)} \right) \quad (3.9)
\]

where we have used the notation \( D^r(K^2) = D^r(0, k^2, k^2) \) etc. The non-perturbative QCD dynamics governing the first moment sum rule is therefore encoded in these 3-current form factors.

In the next section, we discuss the \( K^2 \) dependence of the sum rule in terms of these form factors. However, without any detailed knowledge of their non-perturbative features, we can already determine the first moment of \( g_1^r \) in the limit \( K^2 = 0 \), corresponding to real photons, and in the ‘asymptotic’ region \( K^2 \gg m_\rho^2 \).

The first observation is that electromagnetic current conservation requires \( \hat{R}_{r,1}(K^2) \) to vanish at \( K^2 = 0 \). Substituting the form factor expansion (3.6) into the Ward identity (3.3), and taking \( p = 0 \), we find

\[
A_1^r = A_3^r k_2^2 + A_5^r \frac{1}{2} (p^2 - k_1^2 - k_2^2)
\]

\[
A_2^r = A_4^r k_1^2 + A_6^r \frac{1}{2} (p^2 - k_1^2 - k_2^2) \quad (3.10)
\]

Provided none of the form factors have singularities at \( p^2 = 0 \) (or \( K^2 = 0 \)), as is the case away from the chiral limit, then it follows immediately that both \( A_1^r(K^2) \) and \( A_2^r(K^2) \) are of \( O(K^2) \) for small photon virtuality. (The chiral limit is subtle and is discussed in detail in ref.[22].) We therefore establish \( \hat{R}_{r,1}(0) = 0 \) and therefore [20, 1, 21]

\[
\int_0^1 dx \, g_1^r(x, Q^2; K^2 = 0) = 0 \quad (3.11)
\]

Next, we consider the asymptotic limit of large \( K^2 \), while still keeping in the DIS regime of \( K^2 \ll Q^2 \). For this, we need the large \( K^2 \) limit of the form factors \( A_1^r(K^2) \), \( D^r(K^2) \) and \( B(K^2) \), which can be obtained using the renormalisation group. The flavour non-singlet (\( r = 3, 8 \)) and singlet (\( r = 0 \)) cases are different. In the non-singlet case, since the axial current is conserved, the form factors \( A_1^r(K^2) \) satisfy a homogeneous RGE (see ref.[1] for explicit details), with the standard solution

\[
A_1^r(K^2; \alpha_s(\mu); m) = A_1^r(\mu^2; \alpha_s(t); e^{-t} m(t)) \quad (3.12)
\]

where \( \mu \) is an RG reference scale, \( t = \frac{1}{2} \ln \frac{K^2}{\mu^2} \) (contrast with the ‘\( t \)’ in the Wilson coefficient expressions, which refers to the scale \( Q^2 \)), \( \alpha_s(t) \) and \( m(t) \) are running couplings and \( m \) generically denotes the individual quark masses \( m^r \). The large \( K^2 \) limit of \( A_1^r - A_2^r \) is therefore obtained from the correlation function evaluated at weak coupling in the chiral...
limit. In this limit, \( D^r \) is clearly zero, so recalling eq.(3.9), we conclude that in the flavour
non-singlet sector,

\[
\hat{R}_{r,1}(K^2 \to \infty) = \frac{1}{2}a^{(r)} \frac{\alpha}{\pi} \quad (r = 3, 8) \tag{3.13}
\]

The asymptotic value of the form factor is therefore determined by the electromagnetic
\( U_A(1) \) anomaly coefficient.

In the flavour singlet case, however, the 3-current correlation function satisfies an inhomogeneous RGE with anomalous dimension \( \gamma \) because of the anomalous non-conservation of the singlet axial current. In this case, therefore,

\[
A^0_i(K^2; \alpha_s(\mu); m) = \exp\left[-\int_0^t dt' \gamma(\alpha_s(t'))\right] A^0_i(\mu^2; \alpha_s(t); e^{-t}m(t)) \tag{3.14}
\]

Here, we need both the form factors \( D^0 \) and \( F^0 \) to evaluate the r.h.s. At weak coupling,
the correlation function involving the topological charge \( Q \) is of \( O(\alpha_s^2) \), and so contributes
only at the same order as other neglected terms. So once again, the asymptotic limit
is controlled simply by the anomaly coefficient. However, this time we also need the
anomalous dimension term, and the final result is

\[
\hat{R}_{0,1}(K^2 \to \infty) = \frac{1}{2}a^{(0)} \frac{\alpha}{\pi} \exp\left[-\int_0^t dt' \gamma(\alpha_s(t'))\right] \tag{3.15}
\]

The asymptotic form for the \( g_{1\gamma} \) sum rule is finally obtained by putting together
eqs.(3.13),(3.15) for the form factors with eqs.(3.1),(3.2) for the Wilson coefficients. This
gives:

\[
\int_0^1 dx \ g_{1\gamma}(x, Q^2; K^2 \to \infty) = \sum_{r=3,8,0} E_{r}^{-1}(Q^2) \hat{R}_{r,1}(K^2 \to \infty) \tag{3.16}
\]

\[
= \frac{1}{2} \frac{\alpha}{\pi} \left(1 - \frac{\alpha_s(Q^2)}{\pi}\right) \left[c^{(3)} a^{(3)} + c^{(8)} a^{(8)} + c^{(0)} a^{(0)} \exp\left[\int_{t(K)}^{t(Q)} dt' \gamma(\alpha_s(t'))\right]\right] \tag{3.17}
\]

with the obvious notation \( t(Q) = \frac{1}{2} \ln \frac{Q^2}{\mu^2}, t(K) = \frac{1}{2} \ln \frac{K^2}{\mu^2} \). Substituting \( \frac{\alpha_s(t)}{4\pi} \approx \frac{1}{3\pi t} \) for the
running couplings and reorganising terms, we obtain the final form of the sum rule:

\[
\int_0^1 dx \ g_{1\gamma}(x, Q^2; K^2 \to \infty) = \frac{2}{3} \frac{\alpha}{\pi} \left[1 - \frac{4}{9} \frac{1}{\ln Q^2/\Lambda^2} + \frac{16}{81} \left(\frac{1}{\ln Q^2/\Lambda^2} - \frac{1}{\ln K^2/\Lambda^2}\right)\right] \tag{3.18}
\]

Notice that the overall normalisation factor is \( N_c \sum_f \hat{e}_f^4 \), proportional to the sum of the
fourth power of the quark charges \( \hat{e}_f \), corresponding to the lowest order box diagram
contributing to \( g_{1\gamma} \).

The key physics in eqs.(3.17),(3.18) is that the first moment of \( g_{1\gamma}(x, Q^2; K^2) \) in the
asymptotic limit \( m^2_\rho \ll K^2 \ll Q^2 \) for the target photon virtuality is governed by the
quark charges, with a flavour dependence reflecting the electromagnetic \( U_A(1) \) anomaly
coefficients. The approach to this asymptotic value depends on logarithmic corrections
given by the anomalous dimension arising from the gluonic $U_A(1)$ anomaly in the flavour singlet current.

In between the limits $K^2 = 0$ and $K^2 \gg m_{\rho}^2$, the sum rule depends on form factors $F^r(K^2)$ as follows (substituting for $c(r)$ and $a(r)$ compared to eq.(3.17)):

$$\int_0^1 dx \ g_1^r(x,Q^2;K^2) = \frac{1}{18 \pi} \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) \times \left[ 3F^3(K^2) + F^8(K^2) + 8F^0(K^2;\mu^2 = K^2) \exp \left[ \int_{t(K)}^{t(Q)} dt' \gamma(\alpha_s(t')) \right] \right]$$

(3.19)

The form factors $F^r(K^2)$ interpolate between 0 for $K^2 = 0$ and 1 for asymptotically large $K^2$. Notice that in the anomalous singlet sector, we have to specify the renormalisation scale – with the anomalous dimension factor as shown, the form factor $F^0(K^2)$ must be evaluated at $\mu^2 = K^2$.

These form factors are simply written in terms of those defined in eqs.(3.6),(3.7),(3.8). Explicitly,

$$F^r(K^2) = \left( \frac{1}{8\pi^2}a(r) \right)^{-1} (A_1^r - A_2^r)(K^2)$$

$$= 1 - \left( \frac{1}{8\pi^2}a(r) \right)^{-1} \left( D^r(K^2) + \delta^0 B(K^2) \right)$$

(3.20)

The full $K^2$ dependence of the sum rule is therefore governed by the AVV correlation function, or alternatively, the correlation functions in the corresponding Ward identity. The sum rule therefore gives an experimental measure of these correlators, which are sensitive to the realisation of chiral symmetry in QCD [22] and the gluonic $U_A(1)$ anomaly. A first principles calculation of these correlation functions in QCD, if it were possible, would therefore give a complete prediction for the first moment sum rule.

4. $U_A(1)$ PCAC, $\pi, \eta, \eta' \to \gamma\gamma$, and the gluon topological susceptibility

In order to gain some more insight into the non-perturbative behaviour of the first moment of $g_1^r(x,Q^2;K^2)$, we can use the ideas of PCAC and spontaneously broken chiral symmetry to rewrite the sum rule in terms of the off-shell couplings for radiative decays of the pseudo-Goldstone bosons $\pi, \eta$ and $\eta'$, since these are also controlled by the AVV correlation function in QCD. Of course, the gluonic anomaly makes the application of PCAC to the $U_A(1)$ sector both interesting and subtle. In particular, the sum rule is sensitive to the gluon topological susceptibility, which plays a key role in many polarisation-dependent phenomena in QCD.

The link between the AVV correlation function and radiative pseudoscalar decays arises by writing the close analogue of the axial Ward identity (3.5) involving photon states:

$$ip^\nu(0|J_{\mu 5}^r|\gamma\gamma) = M_{rt}(0|\phi_5^r|\gamma\gamma) + 6\delta_{t0}(0|Q|\gamma\gamma) + a(r)\frac{\alpha}{8\pi}(0|F^{\mu\nu}\tilde{F}_{\mu\nu}|\gamma\gamma)$$

(4.1)

and assuming pseudoscalar dominance of the matrix elements to rewrite them in terms of the radiative couplings $g_{\pi\gamma\gamma}$, $g_{\eta\gamma\gamma}$ and $g_{\eta'\gamma\gamma}$. However, because of the anomaly, the relation of the operators $\phi_5^r$ and $Q$ to the physical pseudoscalars $\pi, \eta, \eta'$ is not entirely
straightforward and it is best to make a change of variables to operators which are more appropriate as interpolating fields for the pseudoscalar particles. This approach to ‘UA(1) PCAC’ and radiative pseudoscalar decays is described in detail in refs.[24, 25, 26]. See also ref.[27] for an analysis of experimental values for the various decay constants which arise.

The result is the following set of expressions for the form factors $F^r(K^2)$ in terms of the off-shell radiative pseudoscalar couplings for photon virtuality $K^2$:

$$F^3(K^2) = 1 - \left(\frac{\alpha}{\pi}\right)^{-1} f_{\pi\gamma\gamma}(K^2)$$

$$F^8(K^2) = 1 - \left(\frac{1}{\sqrt{3}}\frac{\alpha}{\pi}\right)^{-1} \left( f^{8\eta\gamma\gamma}(K^2) + f^{8\eta'\gamma\gamma}(K^2) \right)$$

$$F^0(K^2; \mu^2) = 1 - \left(\frac{4\alpha}{\pi}\right)^{-1} \left( f^{0\eta\gamma\gamma}(K^2) + f^{0\eta'\gamma\gamma}(K^2) + 6A g_{G\gamma\gamma}(K^2; \mu^2) \right)$$

We now discuss what insight this new representation gives into the momentum dependence of the form factors $F^r(K^2)$. The first striking observation is that the first moment of $g_1(x, Q^2; K^2)$ for an off-shell photon target involves the gluon topological susceptibility, as is characteristic of many polarisation phenomena in QCD. This arises through the dependence of the singlet form factor $F^0(K^2)$ on the non-perturbative constant $A$ which controls the topological susceptibility. For non-vanishing quark masses [33]:

$$\chi(0) \equiv \langle QQ \rangle = -A \left( 1 - A \sum_q \frac{1}{m_q(q\bar{q})} \right)^{-1}$$

The corresponding radiative coupling $g_{G\gamma\gamma}$ has a clear theoretical interpretation as the coupling of the two-photon state to a glueball-like operator $G$ orthogonal to the physical $\eta'$. It does not, however, necessarily refer to a physical particle state (see e.g. refs.[24, 25, 26, 27] for a further discussion), so we do not have a clear intuition about its momentum dependence.

Although these anomalous contributions are interesting from a theoretical perspective, in practice they may not be so significant for the sum rule. Arguments based on the $1/N_c$ expansion or OZI rule, carefully applied to the flavour singlet channel, suggest that the contribution of the $6Ag_{G\gamma\gamma}$ term on the l.h.s. of eq.(4.2) is subdominant. An explicit fit [27] of the decay constants and couplings in eqs.(4.2) indeed confirms that, for on-shell photons, the relative contribution of this term is around 20%.

The main result implied by eqs.(4.2) is that the momentum dependence of the form factors in the sum rule is determined by the non-perturbative couplings $g_{\pi\gamma\gamma}(K^2), g_{\eta\gamma\gamma}(K^2)$ and $g_{\eta'\gamma\gamma}(K^2)$. The relevant mass scale determining the crossover from $F^r(0) = 0$ to $F^r(\infty) = 1$ is therefore given by the non-perturbative scale in the photon channel of the pseudoscalar radiative coupling. Well-established ideas invoking vector meson dominance (VMD) imply that for $F^3(K^2)$ this scale is $m_\rho^2$ (equivalently $m_\omega^2$, $m_\phi^2$ for the other flavours). Given that QCD spontaneously breaks chiral symmetry, we therefore expect that the form factors interpolate smoothly between 0 and 1 with a crossover scale characterised by the vector meson masses. This is in sharp contrast to a perturbative QCD picture, in which this scale would correspond instead to the light quark masses [34]. (See ref.[22] for an
extensive discussion of this point.\(^2\) The \(K^2\) dependence of the first moment sum rule for \(g^\gamma_1(x, Q^2; K^2)\) is therefore a clear signal of chiral symmetry breaking.

We can try to justify this VMD prediction directly from QCD field theory as follows. (See also the closely related analyses of the AVV correlation functions and radiative and leptonic pseudoscalar decays in refs.[35, 36, 37] (and references therein). Comprehensive reviews of relevant QCD sum rule results may be found in refs.[38, 39].) Once we have related the form factors to the pseudoscalar radiative couplings, we can use the OPE for the two electromagnetic currents in, for example, the matrix element 

\[
\langle \pi | J^{(em)}_\lambda (k) J^{(em)}_\rho (-k) | 0 \rangle
\]

for large \(K^2\) (compare eq.(2.9)) to write

\[
\langle \pi | J^{(em)}_\lambda (k) J^{(em)}_\rho (-k) | 0 \rangle = 2 \epsilon^{\mu \alpha \lambda \rho} k_\alpha K^2 E_3^1 (K^2) \langle \pi | J^3_{\mu 5}(0) | 0 \rangle + \ldots
\]

Comparing with the definitions of the form factors and couplings, we find to leading order,

\[
F^3(K^2 \rightarrow \infty) = 1 - \frac{(4\pi)^2}{3} f_\pi^2 \frac{1}{K^2} + \ldots
\]

Comparing this large \(K^2\) behaviour with a simple interpolation formula such as \(F^3(K^2) \sim K^2/(K^2 + M^2)\), we would identify the characteristic crossover mass scale as \(M^2 \sim \frac{(4\pi)^2}{3} f_\pi^2\), which is numerically \(\sim m_\rho^2\). This estimate is therefore consistent with the VMD picture.

This completes our discussion of the non-perturbative QCD dynamics behind the momentum dependence of the \(g^\gamma_1(x, Q^2; K^2)\) sum rule. In the next section, we move on to discuss the experimental question of whether all this can actually be directly measured in DIS experiments at \(e^+e^-\) colliders.

5. Cross-sections and spin asymmetries at the ILC and SuperKEKB

The spin-dependent cross-sections for the two-photon DIS process \(e^+e^- \rightarrow e^+e^- X\) defined in section 2 are given in ref.[1] as

\[
\sigma \simeq \frac{\alpha^4}{2\pi} \tilde{a} \frac{1}{Q_{\text{min}}^2} \log \frac{Q^2_{\text{min}}}{\Lambda^2} \log \frac{K^2_{\text{max}}}{K^2_{\text{min}}} \log \frac{x^e_{\text{max}}}{x^e_{\text{min}}} \log \frac{x^e_{\text{max}}}{\langle x^e \rangle}
\]

and

\[
\Delta\sigma \simeq \frac{\alpha^4}{2\pi} \frac{1}{s} \log \frac{Q^2_{\text{max}}}{Q^2_{\text{min}}} \log \frac{\langle Q^2 \rangle}{\Lambda^2} \log \frac{K^2_{\text{max}}}{K^2_{\text{min}}} \log \frac{x^e_{\text{max}}}{x^e_{\text{min}}} \log \frac{x^e_{\text{max}}}{\langle x^e \rangle}
\]

In these expressions, we have included experimental cuts on the maximum and minimum values of the kinematical variables \(Q^2, K^2, x^e\) and \(x\). \(\langle Q^2 \rangle\) is the geometric mean of \(Q^2_{\text{max}}\) and \(Q^2_{\text{min}}\) (similarly for \(\langle x^e \rangle\)). The constants \(\tilde{a}\) and \(\tilde{b}\) are approximations to the functions \(a(x)\) and \(b(x)\) given by inverse Mellin transforms of the moments \(a_n\) and \(b_n\) corresponding to the higher spin operators in the OPEs for \(F^\gamma_2\) and \(g^\gamma_1\). Numerically, \(\tilde{a} \simeq \tilde{b} \simeq 1.5\).

Of course, for final states characteristic of heavy quarks (c, b) with mass \(> \Lambda_{\text{QCD}}\), the crossover scale would simply be the quark mass itself, \(m_{c,b}^2\). Similarly for leptonic final states which probe the QED structure of the photon.
The spin asymmetry is therefore

\[
\frac{\Delta \sigma}{\sigma} \simeq \frac{1}{2} Q_{\text{min}}^2 \frac{Q_{\text{max}}^2}{\frac{Q_{\text{min}}^2}{s}} \log \frac{Q_{\text{max}}^2}{\frac{Q_{\text{min}}^2}{s}} \left[ 1 + \log \frac{Q_{\text{max}}^2}{\Lambda^2} \left( \log \frac{Q_{\text{min}}^2}{\Lambda^2} \right)^{-1} \right]
\] (5.3)

In order to extract information on the $g_1^\gamma$ structure function from $\Delta \sigma$, we need this spin asymmetry to be large. More precisely, for a statistically significant result, we require $\Delta \sigma / \sigma \geq 1 / \sqrt{L\sigma}$, where $L$ is the integrated luminosity [1].

Experimentally, the accelerator design specifies the CM energy $s$ and luminosity $L$, but we can then choose the cuts on the kinematic variables, subject of course to detector constraints, in order to maximise the measured cross sections and spin asymmetries necessary to determine $g_1^\gamma$. The relevant cuts are on $Q^2, K^2, \nu_e$ and $\nu$. The upper cut on $Q^2$ is limited by the detector acceptance and we take $Q_{\text{max}}^2 \simeq s/4$. For the lower cut, $Q_{\text{min}}^2$, we have to be within the DIS region but otherwise will keep this as a free parameter to be varied to try and obtain the most statistically significant measurement. For $K^2$, we set a lower cut at $K_{\text{min}}^2 \simeq m_e^2$ and vary to an upper limit well above $m_\rho^2$, taking $K_{\text{max}}^2 = 1\text{GeV}^2$ in the total cross-section estimates. Since $\nu = \frac{1}{2}(Q^2 + W^2)$, and $W_{\text{min}}^2$ is small, we choose the following cuts on the Bjorken variables: $\nu_{e_{\text{min}}} = \nu_{\text{min}} = \frac{1}{2} Q_{\text{min}}^2$ and $\nu_{e_{\text{max}}} = \nu_{\text{max}} = \frac{1}{2} s$.

Inserting these cuts into the formulae for the total cross-section and spin asymmetry, we have

\[
\sigma \simeq 0.5 \times 10^{-8} \frac{1}{Q_{\text{min}}^2} \log \frac{Q_{\text{min}}^2}{\Lambda^2} \left( \log \frac{s}{Q_{\text{min}}^2} \right)^2
\] (5.4)

and

\[
\frac{\Delta \sigma}{\sigma} = \frac{1}{2} \frac{Q_{\text{min}}^2}{s} \log \frac{s}{4Q_{\text{min}}^2} \left[ 1 + \log \frac{s}{4\Lambda^2} \left( \log \frac{Q_{\text{min}}^2}{\Lambda^2} \right)^{-1} \right]
\] (5.5)

As noted in section 2, the $n$th $x$-moment of $g_1^\gamma$ is determined by the $(n+1)$st $x_e$-moment of $\Delta \sigma$. In particular, from eq.(2.8) we have

\[
\int_0^1 dx_e x_e \frac{d^3\Delta \sigma}{dQ^2dx_e dK^2} = \frac{3}{2} \alpha^3 \frac{1}{s Q^2 K^2} \int_0^1 dx g_1^\gamma(x, Q^2; K^2)
\] (5.6)

Comparing with the first moment sum rule (3.19), we can therefore determine the form factors $F^r(K^2)$ if we can measure the $K^2$-dependence of the fully differential cross-section $d^3\Delta \sigma / dQ^2 dx_e dK^2$.

We now discuss whether such measurements are feasible at present and future $e^+e^-$ colliders. For this purpose, we consider two accelerators in detail – the International Linear Collider (ILC) and the proposed high-luminosity $B$ factory SuperKEKB. The contrasting machine parameters illustrate clearly the main issues involved in measuring the first moment sum rule.

At the time the sum rule was proposed in ref.[1], the luminosity available from the then current accelerators was inadequate for measuring the sum rule. For example, for a polarised version of LEP operating at $s = 10^4\text{GeV}^2$ with an annual integrated luminosity of $L = 100\text{pb}^{-1}$, and choosing the cut at $Q_{\text{min}}^2 = 10\text{GeV}^2$, we find $\sigma \simeq 35\text{pb}$ and $\Delta \sigma / \sigma \simeq 0.01$. However, the corresponding annual event rate would be $3.5 \times 10^3$ and the statistical
significance only $\sqrt{L}\Delta\sigma/\sigma \simeq 0.5$, so even a reliable measurement of the spin asymmetry could not be made.

Clearly, a hugely increased luminosity is required and this has now become available with proposals for machines with projected annual integrated luminosities measured in inverse attobarns. However, as noted in ref.[1], if this increased luminosity is associated with increased CM energy, then the $1/s$ factor in the spin asymmetry (5.2) sharply reduces the possibility of extracting a measurement of $g_1^\gamma$. For this reason, it was already recognised in ref.[1] that the best future colliders for studying the sum rule would be high-luminosity $B$ factories.

![Figure 2](image1.png)

**Figure 2:** The first graph shows the fall in total cross-section $\sigma$ (in pb) at the ILC as the experimental cut $Q_{\text{min}}^2$ is varied from 1 to 100GeV$^2$. The second graph shows the spin asymmetry $\Delta\sigma/\sigma$ rising over the same range of $Q_{\text{min}}^2$.

Now consider the accelerator parameters of the ILC. This will operate initially at a CM energy of 500GeV ($s = 2.5 \times 10^5\text{GeV}^2$) with a projected luminosity of $10^{34}\text{cm}^{-2}\text{s}^{-1}$, corresponding to an annual integrated luminosity of $0.1\text{ab}^{-1}$ [8, 9]. For simplicity, we assume here that this luminosity could be achieved with the ILC running with polarised beams. We now analyse how to optimise a measurement of $g_1^\gamma$ by varying the experimental cuts, in particular $Q_{\text{min}}^2$. In Fig. 2, we have plotted the cross-section $\sigma$ (in pb) and spin asymmetry $\Delta\sigma/\sigma$ for values of $Q_{\text{min}}^2$ varying from 1 to 100GeV$^2$. The sharp fall-off in the cross-section is determined by the $1/Q_{\text{min}}^2$ factor in eq.(5.4). On the other hand, in this range, the spin asymmetry rises with increasing $Q_{\text{min}}^2$ because of the corresponding factor in eq.(5.5). The absolute value of $\Delta\sigma/\sigma$ is kept relatively small because of the $1/s$ dependence of the spin asymmetry on the CM energy. Optimising the cut on $Q_{\text{min}}^2$ is therefore a balance between keeping the total event rate high and maximising the spin asymmetry. If we plot $\sqrt{L}\Delta\sigma/\sigma$ for this same range of $Q_{\text{min}}^2$ (Fig. 3), we see that it rises monotonically – in fact it reaches a maximum only at $Q_{\text{min}}^2 \sim 10^3\text{GeV}^2$ where the cross-section has fallen to a mere 0.5pb.

![Figure 3](image2.png)

**Figure 3:** Plot of the statistical measure $\sqrt{L}\Delta\sigma/\sigma$ for $Q_{\text{min}}^2$ between 1 and 100GeV$^2$ at the ILC.
As a reasonable compromise between event rate and spin asymmetry, we could choose to take the cut at \( Q_{\text{min}}^2 \approx 50\text{GeV}^2 \). This corresponds to \( \sigma \simeq 15\text{pb} \) and \( \Delta \sigma/\sigma \simeq 0.002 \). The annual event rate is \( 1.5 \times 10^6 \) with \( \sqrt{L\sigma\Delta\sigma/\sigma} \simeq 3 \), which would allow a measurement of the spin asymmetry itself. However, as we see from eq.(2.8), this determines only the \( n = 0 \) moment of \( g_1^\gamma \), integrated over \( K^2 \). A detailed study of the first moment sum rule itself would require a much greater \( \Delta \sigma/\sigma \).

This leads us to consider instead the new generation of ultra-high luminosity \( e^+e^- \) colliders. Although these are envisaged as \( B \) factories, these colliders operating with polarised beams would, as we now show, be extremely valuable for studying polarisation phenomena in QCD. As an example of this class, we take the proposed SuperKEKB collider. (The analysis for PEPII is very similar, the main difference being the additional ten-fold increase in luminosity in the current SuperKEKB proposals.)

**Figure 4:** The first graph shows the total cross-section \( \sigma \) (in pb) at SuperKEKB as the experimental cut \( Q_{\text{min}}^2 \) is varied from 1 to 10\( \text{GeV}^2 \). The second graph shows the spin asymmetry \( \Delta \sigma/\sigma \) over the same range of \( Q_{\text{min}}^2 \).

SuperKEKB is an asymmetric \( e^+e^- \) collider with \( s = 132\text{GeV}^2 \), corresponding to electron and positron beams of 8 and 3.5\( \text{GeV} \) respectively. The design luminosity is \( 5 \times 10^{35} \text{ cm}^{-2}\text{s}^{-1} \), which gives an annual integrated luminosity of 5\( \text{ab}^{-1} \) [10]. To see the effects of the experimental cut on \( Q_{\text{min}}^2 \) in this case, we again plot the total cross-section and the spin asymmetry in Fig. 4, this time for the range of \( Q_{\text{min}}^2 \) from 1 to 10\( \text{GeV}^2 \). As before, in this range \( \sigma \) is falling like \( 1/Q_{\text{min}}^2 \) while \( \Delta \sigma/\sigma \) rises to what is actually a maximum at \( 1/Q_{\text{min}}^2 = 10\text{GeV}^2 \). On the other hand, the statistical significance \( \sqrt{L\sigma\Delta\sigma/\sigma} \) falls monotonically, though is orders of magnitude improved on the corresponding plot for even the ILC (Fig. 5).

Taking \( Q_{\text{min}}^2 = 5\text{GeV}^2 \), we find \( \sigma \simeq 12.5\text{pb} \) with spin asymmetry \( \Delta \sigma/\sigma \simeq 0.1 \). The annual event rate is therefore \( 6.25 \times 10^7 \), with \( \sqrt{L\sigma\Delta\sigma/\sigma} \simeq 750 \). This combination of a very high event rate and the large 10\% spin asymmetry means that SuperKEKB has the potential not only to measure \( \Delta \sigma \) but to access the full first moment sum rule for \( g_1^\gamma \).
itself. Recall from eq.(5.6) that to measure \( \int_0^1 dx \, g_1^\gamma (x,Q^2;K^2) \) we need not just \( \Delta \sigma \) but the fully differential cross-section w.r.t. not only \( x_e \) and \( Q^2 \), but also \( K^2 \) if the interesting non-perturbative QCD physics is to be accessed. To measure this, we need to divide the data into sufficiently fine \( K^2 \) bins in order to plot the explicit \( K^2 \) dependence of \( g_1^\gamma \), while still maintaining the statistical significance of the asymmetry. The ultra-high luminosity of SuperKEKB ensures that the event rate is sufficient, while its moderate CM energy means that the crucial spin asymmetry is not overly suppressed by its \( 1/s \) dependence.

Our conclusion is that the new generation of ultra-high luminosity, moderate energy \( e^+e^- \) colliders, currently conceived as \( B \) factories, could also be uniquely sensitive to important QCD physics if run with polarised beams. In particular, they appear to be the only accelerators capable of accessing the full physics content of the sum rule for the first moment of the polarised structure function \( g_1^\gamma (x,Q^2;K^2) \). The richness of this physics, in particular the realisation of chiral symmetry breaking, the manifestations of the axial \( U_A(1) \) anomaly and the role of non-perturbative gluon dynamics, provides a strong motivation for giving serious consideration to an attempt to measure the \( g_1^\gamma \) sum rule at these new colliders.

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