VOLUMES AND LIMITS OF MANIFOLDS WITH RICCI CURVATURE AND MEAN CURVATURE BOUNDS

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ABSTRACT. We consider smooth Riemannian manifolds with nonnegative Ricci curvature and smooth boundary. First we prove a global laplacian comparison theorem in the barrier sense for the distance to the boundary. We apply this theorem to obtain volume estimates of the manifold and of regions of the manifold near the boundary depending upon an upper bound on the area and on the inward pointing mean curvature of the boundary. We prove that families of oriented manifolds with uniform bounds of this type are compact with respect to the Sormani-Wenger Intrinsic Flat (SWIF) distance.

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1. Introduction

In the past few decades many important compactness theorems have been proven for families of smooth manifolds without boundary. Gromov has proven that families of manifolds with nonnegative Ricci curvature and uniformly bounded diameter are precompact in the Gromov-Hausdorff (GH) sense [9]. Cheeger-Colding have proven many beautiful properties of the GH limits of these manifolds including rectifiability of the GH limit spaces [7].

Little is known about the precompactness of families of manifolds with boundary. In particular, it is unknown whether sequences of manifolds with nonnegative Ricci curvature and uniformly bounded mean curvature and area of the boundary are precompact in the GH sense. Nor is it known whether the GH limits of such sequences are rectifiable.

Kodani [12] has proven GH precompactness of families with uniform bounds on sectional curvature. Wong [23] has proven GH precompactness of families with uniform bounds for the Ricci curvature, the second fundamental form and the diameter. Neither Kodani nor Wong study the rectifiability of the GH limit spaces of manifolds in the families they study. Anderson-Katsuda-Kurdy-Kwok-Taylor [2] and Knox [11] have proven $C^{1,\alpha}$ and rectifiability of the limit spaces assuming one has sequences with significant additional bounds on their manifolds. See [15] for a survey of the precompactness theorems for manifolds with boundary.

We prove precompactness theorems, Theorems 1.4 and 1.5, for families of oriented Riemannian $n$-manifolds $(M,g)$ with nonnegative Ricci curvature and uniform upper bounds on the area and the inward pointing mean curvature of the boundary:

\[(1.1) \quad \text{Vol}(\partial M) \leq A \quad \text{and} \quad H_{\partial M}(q) \leq H.\]

Our precompactness is with respect to the Sormani-Wenger Intrinsic Flat (SWIF) distance [21], in which the limit spaces are countably $H^n$ rectifiable, where $H^n$ denotes the $n$-dimensional Hausdorff measure.

One important feature of the SWIF distance is that if a sequence of oriented manifolds with volume and area uniformly bounded converges in GH sense then a subsequence converges in SWIF sense. Moreover, if the SWIF limit space is not the zero current space then it can be seen as a subspace of the GH limit (See Theorem 3.20 in [21]). Nonetheless, SWIF convergence does not imply GH subconvergence as can be seen in Example 3.5.

In [20] Sormani-Wenger have shown that for manifolds with nonnegative Ricci curvature (without boundary) the GH and SWIF limits agree. This is not necessarily true for manifolds with boundary. For example, consider a sequence of $n$-closed round balls of the same radii with one increasingly thin tip. The sequence converges in SWIF sense to a round closed ball but it converges to a round closed ball with a segment attached in the GH sense (cf. Example A.4 in [21]).

In order to prove our precompactness theorem we need to prove theorems for manifolds with boundary that were previously proven for manifolds with no boundary. One of the key tools in the work of Gromov is the Bishop-Gromov Volume Comparison Theorem [9]. A key tool in the work of Cheeger-Colding is the Laplacian Comparison Theorem (See [17]). In fact, this Laplacian Comparison Theorem may be applied to prove the Bishop-Gromov Volume Comparison Theorem (c.f. [8]).

We consider connected Riemannian manifolds $(M^n, g)$ with smooth boundary $\partial M$. We denote by $d : M \times M \to \mathbb{R}$ the metric on $(M,g)$ given by $g$ and say that $(M,g)$ is metrically complete if $(M,d)$ is complete as a metric space. Define $r : M \to \mathbb{R}$ by

\[(1.2) \quad r(p) := d(p, \partial M).\]
The laplacian of $r$ is denoted by $\Delta r$. The mean curvature of $\partial M$ with respect to the normal inward pointing direction is denoted by $H_{\partial M} : \partial M \to \mathbb{R}$.

**Theorem 1.1.** Let $n \geq 2$ and $M^n$ be a connected and metrically complete Riemannian manifold with smooth boundary with $\text{Ric}(M \setminus \partial M) \geq 0$. Then for all $p \in M$

\begin{equation}
\Delta r(p) \leq \frac{(n-1)H_{\partial M}(q)}{H_{\partial M}(q)r(p) + n-1}
\end{equation}

holds in the barrier sense, where $q \in \partial M$ is such that $r(p) = d(p, q)$.

In Theorem 1.1 we get $\Delta r(p) \leq 0$ when $H_{\partial M} = 0$. If $H_{\partial M} = (n-1)/H$ then $\Delta r(p) \leq (n-1)/(r(p) + H)$.

Sakurai has recently proven a laplacian comparison theorem for the same distance function whenever $r$ is smooth. In our paper we also include points where $r$ is not smooth obtaining a global laplacian comparison theorem in the barrier sense. The original laplacian comparison theorem is also proven globally in the barrier sense for distance functions on manifolds without boundary. This global comparison allows one to apply the maximum principle and has much stronger consequences than a laplacian comparison theorem which only holds where the function is smooth.

In Subsection 2.3 we apply Theorem 1.1 to obtain volume and area estimates for $M^{\delta_1} \setminus M^{\delta_2}$, $\delta_1 \geq \delta_2$ and $\partial M^\delta$, respectively, where

\begin{equation}
M^\delta := \{ p \in M | r(p) > \delta \}
\end{equation}

and $\partial M^\delta$ is the boundary (as a metric subspace of $M$) of $M^\delta$. Note that $\partial M^\delta \subset r^{-1}(\delta)$ but these sets are not necessarily equal as can be seen by taking $M = D(0, 1)$, the flat disc in the 2-dimensional euclidean space centered at the origin 0. There, $r^{-1}(1) = \{0\}$ and $M^1 = \emptyset$. Different volume estimates were obtained by Heintz and Karcher in [10] using Jacobi fields. In our paper, $A_{n,H} : [0, \infty) \to \mathbb{R}$ is the function given by

\begin{equation}
A_{n,H}(\delta) = \begin{cases} (H\delta + n - 1)^{n-1}/(n-1)^{n-1} & \text{if } H\delta + n - 1 \geq 0, \\ 0 & \text{otherwise,} \end{cases}
\end{equation}

where $n \geq 2$ and $H \in \mathbb{R}$.

**Theorem 1.2.** Let $n \geq 2$ and $M^n$ be a connected and metrically complete Riemannian manifold with smooth boundary such that $\text{Ric}(M \setminus \partial M) \geq 0$, $H_{\partial M} \leq H$ and $\text{Vol}(\partial M) < \infty$. If $\delta_1 \geq \delta_2 \geq 0$ then

\begin{equation}
\text{Vol}(M^{\delta_2} \setminus M^{\delta_1}) \leq \text{Vol}(\partial M) \int_{\delta_2}^{\delta_1} A_{n,H}(t) dt,
\end{equation}

where $A_{n,H}$ is as in (1.5). If $\text{Diam}(M) \leq D$ then

\begin{equation}
\text{Vol}(M) \leq \text{Vol}(\partial M) \int_{0}^{\tilde{D}} A_{n,H}(t) dt,
\end{equation}

where $\tilde{D} = D$ if $H \geq 0$ and $\tilde{D} = \min\{D, -(n-1)/H\}$ if $H < 0$.

Explicitly, the integral of $A_{n,H}$ is the following

\begin{equation}
\int_{\delta_2}^{\delta_1} A_{n,H}(t) dt = \begin{cases} \frac{\delta_1 - \delta_2}{nH^{(n-1)/n}} & \text{if } H = 0 \\ \frac{n-1}{nH^{(n-1)/n}} \int_{\delta_2}^{\delta_1} & \text{if } H \neq 0, \end{cases}
\end{equation}

\footnote{His paper appeared on the arxiv after our original posting.}
where $\delta_1 = \delta_1$ if $H \geq 0$ and $\delta_1 = \min\{\delta_1, -(n-1)/H\}$ if $H < 0$.

We see that the equality of both, volume and area, estimates is achieved by all the Riemannian manifolds of the sequence given in Example 3.7. Also for the standard ball of radius $R$ in $n$-euclidean space.

**Theorem 1.3.** Let $n \geq 2$, $M^n$ be a connected and metrically complete Riemannian manifold with smooth boundary such that $\text{Ric}(M \setminus \partial M) \geq 0$, $H_{\partial M} \leq H$ and $\text{Vol}(\partial M) < \infty$. Then, $L^1$-almost everywhere,

\[
\text{Vol}(\partial M^\delta) \leq \text{Vol}(\partial M)A_{n,H}(\delta),
\]

where $L^1$ denotes the 1-dimensional Lebesgue measure and $A_{n,H}$ is as in (1.5).

In Section 2 we prove Theorems 1.1, 1.2 and 1.3. In Section 3 we review some basic definitions about SWIF distance such as Wenger’s compactness theorem. Wenger showed that given a sequence of complete oriented Riemannian manifolds of the same dimension with $\text{Vol}(M_j) \leq V$, $\text{Vol}(\partial M_j) \leq A$ and $\text{Diam}(M_j) \leq D$ a subsequence converges in the SWIF sense to an integral current space (See Theorem 1.2 in [22], cf. Theorem 4.9 in [21]). We use Wenger’s compactness theorem along with the area and volume estimates to prove convergence theorems 1.4, 1.5 and 1.6.

**Theorem 1.4.** Let $D, A > 0$, $H \in \mathbb{R}$ and $(M^n_j, g_j)$ be a sequence of oriented, connected and metrically complete Riemannian manifolds with smooth boundary that satisfy

\[
(1.10) \quad \text{Ric}(M_j \setminus \partial M_j) \geq 0,
\]

\[
(1.11) \quad \text{Vol}(\partial M_j) \leq A, \quad H_{\partial M_j} \leq H,
\]

and

\[
(1.12) \quad \text{Diam}(M_j) \leq D.
\]

Then there is an $n$-integral current space $(W, d, T)$ and a subsequence $(M_{\mu_j}, d_{\mu_j}, T_{\mu_j})$ that converges in SWIF sense

\[
(1.13) \quad (M_{\mu_j}, d_{\mu_j}, T_{\mu_j}) \xrightarrow{\mathcal{T}} (W, d, T),
\]

where $T_j(\omega) := \int_{M_j} \omega$.

The necessity of diameter and mean curvature uniform bounds in Theorem 1.4 can be seen in Example 3.7 and Example 3.6 respectively.

Myers proved that for a complete Riemannian manifold with Ricci curvature bounded from below geodesics past certain distance must have conjugate points (See Myers’ diameter estimate in [17]). Thus, the diameter of the manifold is bounded below by this distance. Li and Li-Nguyen in [13] and [14], respectively, proved that if $(M^n, g)$ is a complete connected Riemannian manifold with smooth boundary such that $\text{Ric}(M \setminus \partial M) \geq 0$ and $H_{\partial M} \leq H < 0$ then $r \leq -(n-1)/H$. Hence, $\text{Diam}(M)$ can be bounded in terms of $-(n-1)/H$ and $\text{Diam}(\partial M)$ (See Remark 2.6). We get the following compactness theorem.

**Theorem 1.5.** Let $D', A > 0$ and $(M^n_j, g_j)$ be a sequence of oriented, connected and metrically complete Riemannian manifolds with smooth boundary that satisfy

\[
(1.14) \quad \text{Ric}(M_j \setminus \partial M_j) \geq 0,
\]

\[
(1.15) \quad \text{Vol}(\partial M_j) \leq A, \quad H_{\partial M_j} \leq H < 0,
\]
and

\begin{equation}
\text{Diam}(\partial M_j) \leq D'.
\end{equation}

Then there is a subsequence \((M_{\delta_j}, d_{\delta_j}, T_{\delta_j})\) and an n-integral current space \((W, d, T)\) such that

\begin{equation}
(M_{\delta_j}, d_{\delta_j}, T_{\delta_j}) \xrightarrow{F} (W, d, T).
\end{equation}

In Example 3.7 we describe a sequence that satisfies all the hypotheses of Theorem 1.5 except that \(H_{\partial M_j} = 0\) for all \(j\). This sequence does not converge in SWIF sense. Hence a uniform negative bound on the mean curvature is needed.

When using GH distance the following can occur (See Example 4.10 of [16] and Example 3.5). There exists a sequence of oriented connected Riemannian manifolds with smooth boundary that satisfy \((M_j, d_j)\) is complete as metric space, \(\text{Ric}(M_j \setminus \partial M_j) \geq 0\), \(\text{Vol}(\partial M_j) \leq A\), \(H_{\partial M_j} \leq H\) and \(\text{Diam}(M_j) \leq D\) such that

\begin{equation}
(M_j, d_j) \xrightarrow{GH} (X, d_X)
\end{equation}

and for every decreasing sequence \(\delta_i \to 0\)

\begin{equation}
(M_{\delta_i}, d_{\delta_i}) \xrightarrow{GH} (Y_{\delta_i}, d_{Y_{\delta_i}}),
\end{equation}

but \((Y_{\delta_i}, d_{Y_{\delta_i}})\) does not converge in GH sense to \((X, d_X)\). In the theorem below we see that this situation does not happen if we replace GH distance by SWIF distance.

**Theorem 1.6.** Let \(D, A > 0\), \(H \in \mathbb{R}\) and \((M^n_j, g_j)\) be a sequence of oriented, connected and metrically complete Riemannian manifolds with smooth boundary that satisfy

\begin{equation}
\text{Ric}(M_j \setminus \partial M_j) \geq 0,
\end{equation}

\begin{equation}
\text{Vol}(\partial M_j) \leq A, \quad H_{\partial M_j} \leq H,
\end{equation}

and

\begin{equation}
\text{Diam}(M_j) \leq D.
\end{equation}

Suppose that there exist an integral current space \((W, d, T)\), a non increasing sequence \(\delta_i \to 0\) and integral current spaces \((W_{\delta_i}, d_{W_{\delta_i}}, T_{\delta_i})\) such that

\begin{equation}
(M_j, d_j, T_j) \xrightarrow{F} (W, d, T)
\end{equation}

and for all \(i\)

\begin{equation}
(M_{\delta_i}, d_{\delta_i}, T_{\delta_i}) \xrightarrow{F} (W_{\delta_i}, d_{W_{\delta_i}}, T_{\delta_i}).
\end{equation}

Then we have

\begin{equation}
(W_{\delta_i}, d_{W_{\delta_i}}, T_{\delta_i}) \xrightarrow{F} (W, d, T).
\end{equation}

In Subsection 3.3 we provide examples of sequences of manifolds with boundary. Example 3.5 defines a sequence (as in [21]) that converges in the SWIF sense but not in the GH sense. Example 3.6 defines a sequence (as in [19]) that shows the necessity of a uniform bound of the mean curvature required in Theorem 1.4. Example 3.7 shows the necessity of a uniform bound of the diameter required in Theorem 1.4 and that equality holds in both volume and area estimates given in Theorem 1.2 and Theorem 1.3. Moreover, this example shows that the assumption \(H < 0\) in Theorem 1.5 is needed.
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2. Volume, Area and Diameter Bounds

In this section we see that the function \( r \) is differentiable almost everywhere by showing that it is a Lipschitz map and invoking Rademacher’s theorem. We also give a proof that shows that \( r \) is bounded when \( H_{\partial M} \leq H < 0 \). This result is used to bound the diameter of \( M \) in terms of the diameter of \( \partial M \) and \( H \). We also prove Theorems \[13\] and \[14\].

**Lemma 2.1.** Let \( M^n \) be a connected and metrically complete Riemannian manifold with smooth boundary. Then \( r = d(\partial M) \) is a Lipschitz function with \( \text{Lip}(r) = 1 \).

**Proof.** Let \( p, q \in M \). There exists \( p' \in \partial M \) such that \( r(p) = d(p', p) \). Then, \( r(q) - r(p) \leq d(p', q) - d(p', p) \leq d(q, p) \).

In the same way it is proven that \( r(p) - r(q) \leq d(q, p) \). Thus, \( r \) is a Lipschitz function with \( \text{Lip}(r) = 1 \). \( \Box \)

The composition of \( r \) with a normal coordinate on a strongly convex ball is a Lipschitz function. Hence, \( r \) is differentiable except for a zero measure set.

2.1. Diameter Bounds for Manifolds with Negative Mean Curvature. We give the definitions of a focal point and a cut point of \( \partial M \) which assigns to each point \( p \) in the domain the unique point in the boundary that equals \( r(p) \). Then we prove the theorem of Li and Li-Nguyen, \[13\] and \[14\], respectively, that gives an upper bound on \( r \) when \( H_{\partial M} \leq H < 0 \). With that bound we get an upper estimate of the diameter of \( M \) in Remark \[2.6\].

**Definition 2.2.** The point \( q \in M \) is a focal point of \( \partial M \) if there exists a geodesic \( \gamma : [0, a] \to M \) such that \( \gamma(0) \in \partial M \), \( \gamma'(0) \in T_{\gamma(0)} \partial M^2 \) and \( \gamma(a) = q \), and a Jacobi field \( J \) along \( \gamma \) that vanishes at \( b \) and satisfies \( J(0) \in T_{\gamma(0)} \partial M \) and \( J'(0) + S_{\gamma(0)}(J(0)) \in T_{\gamma(0)} \partial M^2 \).

A cut point of \( \partial M \) is either a first focal point or a point with two geodesics back to the boundary of the same length achieving the distance to the boundary. Denote by \( \text{cut}(\partial M) \) the set of cut points of \( \partial M \).

**Remark 2.3.** If \((M, g)\) is a connected Riemannian manifold with boundary with \((M, d)\) complete as a metric space, then geodesics normal to the boundary are not minimizing past a focal point (See Section 11.4, Corollary 1 of Theorem 5 in \[3\]).

As a consequence the following function is well defined.

**Definition 2.4.** Let \( \pi : M \setminus \text{cut}(\partial M) \to \partial M \) be the function that assigns to \( p \in M \setminus \text{cut}(\partial M) \) the only point \( \pi(p) \in \partial M \) that satisfies \( r(q) = d(\pi(q), q) \).

**Lemma 2.5.** [Li, Li-Nguyen] Let \( M^n \) be a connected and metrically complete Riemannian manifold with smooth boundary such that \( \text{Ric}(M \setminus \partial M) \geq 0 \). Suppose that \( p' \in \partial M \) has \( H_{\partial M}(p') < 0 \). Then the geodesic \( \gamma \) that starts at \( p' \) with initial vector the unitary normal inward vector stops minimizing after time \( t_0 > -\frac{n-1}{H_{\partial M}(p')} \).
Proof. Let $E_i$ be an orthonormal basis of parallel vector fields along $\gamma$ such that $E_n = \gamma'$. Let $V_i(t) = (t_0 - t)E_i(t)$ be vector fields along $\gamma$. Then,

$$I_0(V_i) = \int_0^{t_0} \left( \langle V_i', V_i' \rangle - \langle R(\gamma', V_i)\gamma', V_i \rangle \right)(t)dt + \left\langle S_{\gamma'}V_i, V_i \right\rangle(0) - \left\langle S_{\gamma'}V_i, V_i \right\rangle(t_0)$$

(2.2)

$$= \int_0^{t_0} \left( 1 - (t_0 - t)^2 \right) \left\langle R(\gamma', E_i)\gamma', E_i \right\rangle(t)dt + t_0^2 \left\langle S_{\gamma'}E_i, E_i \right\rangle(0).$$

(2.3)

Now we add $I(V_i)$ to the left side of (2.12), and use the fact that $\text{Ric}(\mathbb{M} \setminus \partial \mathbb{M}) \geq 0$.

$$\sum_{i=1}^{n-1} I_0(V_i) = \int_0^{t_0} \left( (n-1) - (t_0 - t)^2 \right) \sum_{i=1}^{n-1} \left\langle R(\gamma', E_i)\gamma', E_i \right\rangle(t)dt + t_0^2 \sum_{i=1}^{n-1} \left\langle S_{\gamma'}E_i, E_i \right\rangle(0)$$

(2.4)

$$\leq t_0(n-1) + t_0^2 H_{\partial \mathbb{M}}(p')$$

(2.5)

$$= t_0((n-1) + t_0 H_{\partial \mathbb{M}}(p')).$$

(2.6)

Assuming that $t_0 > -\frac{n-1}{H_{\partial \mathbb{M}}(p')}$ and since $H_{\partial \mathbb{M}}(p') < 0$, we get $(n-1) + H_{\partial \mathbb{M}}(p')t_0 < 0$. Then $\sum_{i=1}^{n-1} I_0(V_i) < 0$. Thus, there is $i$ for which $I_0(V_i) < 0$. Hence, $\gamma$ is not minimizing. \hfill \square

Remark 2.6. The lemma implies that if $H_{\partial \mathbb{M}} \leq H < 0$ and $\text{Diam}(\partial \mathbb{M}) \leq D'$ then $\text{Diam}(M)$ is bounded. For $p, q \in \mathbb{M}$

$$d_M(p, q) \leq d_M(p, \pi(p)) + d_M(\pi(p), \pi(q)) + d_M(\pi(q), q)$$

(2.7)

$$\leq -\frac{n-1}{H} + d_{\partial \mathbb{M}}(\pi(p), \pi(q)) - \frac{n-1}{H}$$

(2.8)

$$= D' - 2\frac{n-1}{H},$$

(2.9)

where we use that the intrinsic metric $d_{\partial \mathbb{M}}$ on $\partial \mathbb{M}$ is greater or equal than the restricted metric $d_{\partial \mathbb{M}}$.  

2.2. Laplacian Comparison Theorems. For manifolds with no boundary, first a laplacian comparison theorem for the distance function to a point was proven for points outside the cut locus (See [17]). Later on, Calabi extended the result to the barrier sense [4]. Here we also prove a laplacian comparison theorem for $r$ for points outside cut(\partial \mathbb{M}), Theorem 2.7 Then we define upper barrier function and laplacian comparison in the barrier sense (see [4]) and at the end prove the laplacian comparison theorem in the barrier sense stated in Section 1. Theorem 2.1

Theorem 2.7. Let $M^n$ be an $n$-dimensional connected metrically complete Riemannian manifold with smooth boundary. If $\text{Ric}(\mathbb{M} \setminus \partial \mathbb{M}) \geq 0$ then for all $p \in \mathbb{M} \setminus \text{cut}(\partial \mathbb{M})$ the following inequality holds

$$\Delta r(p) \leq \frac{(n-1)H_{\partial \mathbb{M}}(\pi(p))}{H_{\partial \mathbb{M}}(\pi(p))r(p) + n - 1}.$$ 

(2.10)

Proof. If $p \in \partial \mathbb{M}$ then $\pi(p) = p$ and $r(p) = 0$. Thus, the right hand side of (2.10) reduces to $H_{\partial \mathbb{M}}(p) = \Delta r(p)$. Hence, the theorem holds. For points in $M \setminus (\text{cut}(\partial \mathbb{M}) \cup \partial \mathbb{M})$ we use Bochner-Weitzenböck’s formula

$$|\text{Hess } r|^2 \geq \frac{(\Delta r)^2}{n-1}$$

(2.11)

Since $|\text{Hess } r|^2 \geq \frac{(\Delta r)^2}{n-1}$ and $\text{Ric}(\mathbb{M}) \geq 0$,

$$0 \geq \frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}(\Delta r).$$

(2.12)
Take $p \in M \setminus (\text{cut}(\partial M) \cup \partial M)$. Let $\gamma$ be the minimizing geodesic from $\pi(p)$ to $p$. Suppose that $\Delta r(\gamma(t)) \neq 0$ for all $t$. Then by rearranging terms in (2.12) we obtain:

\begin{equation}
(2.13) \quad -\int_0^r \frac{\Delta r}{(\Delta r)^2} \geq \int_0^r \frac{1}{n - 1}.
\end{equation}

Integrating along $\gamma$ we get:

\begin{equation}
(2.14) \quad \frac{1}{\Delta r(p)} \geq \frac{r(p)}{n - 1} + \frac{1}{\Delta r(\pi(p))}.
\end{equation}

\begin{equation}
(2.15) \quad = \frac{r(p)}{n - 1} + \frac{H_{\partial M}(\pi(p))}{H_{\partial M}(\pi(p))}.
\end{equation}

\begin{equation}
(2.16) \quad \frac{H_{\partial M}(\pi(p))r(p) + n - 1}{(n - 1)H_{\partial M}(\pi(p))}.
\end{equation}

If $H_{\partial M}(\pi(p)) < 0$ then by Lemma 2.8 both sides of the inequality are negative. If $H_{\partial M}(\pi(p)) > 0$ then both sides of the inequality are positive. Thus

\begin{equation}
(2.17) \quad \Delta r(p) \leq \frac{(n - 1)H_{\partial M}(\pi(p))}{H_{\partial M}(\pi(p))r(p) + n - 1}.
\end{equation}

Finally, we deal with the case in which there is $t_0 \geq 0$ such that $\Delta r(\gamma(t_0)) = 0$. Suppose that $t_0 = \inf \{t \mid \Delta r(\gamma(t)) = 0\}$. By (2.14), $\Delta r(\gamma(t))$ is non increasing. Since $0 \leq t_0$ and $\Delta r(\gamma(t_0)) = 0$, then $H_{\partial M}(\pi(p)) = \Delta r(\gamma(0)) \geq 0$. This means that the right hand side of (2.17) is nonnegative for $t \in [0, t_0]$. Using again that $\Delta r(\gamma(t))$ is non increasing we see that $\Delta r(\gamma(t)) \leq 0$ for $t \geq t_0$. Thus, (2.17) holds for $t \geq t_0$. \qed

**Definition 2.8.** Let $f$ be a continuous real valued function. An upper barrier for $f$ at the point $x_0$ is a $C^2$ function $f_{x_0}$ defined in some neighborhood of $x_0$ such that $f \leq f_{x_0}$ and $f(x_0) = f_{x_0}(x_0)$.

**Definition 2.9.** Let $f$ be a continuous function. We say that $\Delta f(x_0) \leq a$ in the barrier sense if for all $e > 0$ there is an upper barrier $f_{x_0,e} : U_e \to \mathbb{R}$ for $f$ at $x_0$ with

\begin{equation}
(2.18) \quad \Delta f_{x_0,e} \leq a + e.
\end{equation}

Now we are ready to extend Theorem 2.7.

**Proof of Theorem 2.7.** For points in $M \setminus \text{cut}(\partial M)$ the result follows by applying Theorem 2.7. Suppose that $p \in \text{cut}(\partial M)$. Take $q \in \partial M$ such that $r(p) = d(p, q)$. If $H_{\partial M}(q)r(p) + n - 1 = 0$ there is nothing to prove. If $H_{\partial M}(q)r(p) + n - 1 \neq 0$, for all $e > 0$ we will define an upper barrier $r_{p,e} : U_e \to \mathbb{R}$ for $r$ at $p$ such that in $U_e$

\begin{equation}
(2.19) \quad \Delta r_{p,e} \leq \frac{(n - 1)H_{\partial M}(q)}{H_{\partial M}(q)r(p) + n - 1} + e.
\end{equation}

Let $U$ be an open set of $\partial M$ that contains $q$ such that the map $U \times [0, \delta_0) \to M$ given by $(z, t) \mapsto \exp(t\nabla r(z))$ is a diffeomorphism. Let $x = (x_1, \ldots, x_{n-1}) : B_{\delta_0}(q) \to \mathbb{R}^{n-1}$ be a coordinate chart centered at $q$ such that $\partial_i(q) = \frac{\partial}{\partial x_i}(q)$ are orthonormal and $B_{\delta_0}(q) \subset U$ is a closed ball centered at $q$ with radius $2\delta$.

For $\delta \leq \delta_0$, let $\alpha > 0$ be such that

\begin{equation}
(2.20) \quad \alpha/2 \max \left\{ \sum_{i=1}^{n-1} x_i^2(z) \mid z \in B_{\delta_0}(q) \right\} \leq \delta.
\end{equation}
Define $g^{\delta,\alpha} : B_2(q) \to \mathbb{R}$ be the smooth function given by

$$
(2.21) \quad g^{\delta,\alpha}(z) = \delta - \alpha/2 \sum_{i=1}^{n-1} x_i^2(z).
$$

Then $g^{\delta,\alpha}$ satisfies the following properties:

- At $q$, $g^{\delta,\alpha}(q) = \delta$
- At other points, $0 \leq g^{\delta,\alpha} < \delta$. Thus, $g^{\delta,\alpha}$ has a maximum at $q$
- Denoting by $g^{\delta,\alpha}_i$ the $i$-th coordinate function of $g^{\delta,\alpha} \circ x^{-1}$, we have $\sum_{i=1}^{n} (g^{\delta,\alpha}_i)'(0) = -\alpha$.

By the existence of partitions of unity, there is a smooth function $\varphi : \partial M \to \mathbb{R}$ that satisfies $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_\delta(q)$ and $\text{spt} \varphi \subset B_2(q)$. Hence, we can extend $g^{\delta,\alpha}$ to a smooth function $g^{\delta,\alpha} : \partial M \to \mathbb{R}$ that satisfy the properties listed in the previous paragraph.

We construct upper barrier functions by constructing distance functions to $(n - 1)$-dimensional submanifolds $N_{\delta,\alpha} \subset M$. Define

$$
(2.22) \quad N_{\delta,\alpha} := \{ \exp(g^{\delta,\alpha}(z)\nabla r(z)) : z \in \partial M \}.
$$

**Claim:** For $\varepsilon > 0$ there exist $\delta > 0$ and $\alpha > 0$ small enough such that there is a neighborhood $U_\varepsilon$ of $p$ for which $r_{p,\varepsilon} : U_\varepsilon \to \mathbb{R}$ given by,

$$
(2.23) \quad r_{p,\varepsilon} = d(\cdot, N_{\delta,\alpha}) + \delta,
$$

is an upper barrier of $r$ at $p$ that satisfies (2.19).

**Proof of the claim:** For $y \in M$,

$$
(2.24) \quad r_{p,\varepsilon}(y) = d(y, N_{\delta,\alpha}) + \delta \leq \inf_{z \in \partial M} [r(y) - g^{\delta,\alpha}(z)] + \delta = r(y) + \inf_{z \in \partial M} [\delta - g^{\delta,\alpha}(z)].
$$

By definition of $g^{\delta,\alpha}$, $\delta - g^{\delta,\alpha}$ is nonnegative and it is zero only when $z = q$. Thus, $r_{p,\varepsilon}(y) \leq r(y)$. By the same reasoning, $q = \exp(g^{\delta,\alpha}(q)\nabla r(q))$ satisfies $d(p, q) + \delta \leq r_{p,\varepsilon}(p)$. Thus, $r_{p,\varepsilon}(p) = r(p)$.

Observe that $p$ is not in the cut locus of $N_{\delta,\alpha}$. Hence, there is a neighborhood of $p$ in which $r_{p,\varepsilon}$ is $C^2$. Thus, by continuity of $\Delta r_{p,\varepsilon}$ at $p$ and applying Theorem 2.7 to $r_{p,\varepsilon}$, for all $y$ in a neighborhood of $p$ that depends on $\varepsilon$ the following holds:

$$
(2.25) \quad \Delta r_{p,\varepsilon}(y) < \frac{\varepsilon}{2} + \Delta r_{p,\varepsilon}(p)
$$

$$
(2.26) \quad < \frac{\varepsilon}{2} + \frac{(n - 1)H_{N_{\delta,\alpha}}(\exp(g^{\delta,\alpha}(q)\nabla r(q)))}{H_{N_{\alpha}}(\exp(g^{\delta,\alpha}(q)\nabla r(q)))r_{p,\varepsilon}(p) + n - 1},
$$

where $H_{N_{\delta,\alpha}}$ denotes the mean curvature of $N_{\delta,\alpha}$ in the inward normal direction.

Now we bound the nontrivial term of the right hand side of the previous inequality. To do so, we first show that the function,

$$
(2.27) \quad (\delta, \alpha) \mapsto H_{N_{\delta,\alpha}}(\exp(g^{\delta,\alpha}(q)\nabla r(q))),
$$

is continuous at $(0, 0)$, where $g^{0,0} = 0, N_{0,0} = \partial M$.

Let’s calculate $H_{N_{\delta,\alpha}}(\tilde{q})$ where $\tilde{q} = \exp(g^{\delta,\alpha}(q)\nabla r(q))$. Recall that the map $U \times [0, \delta_0) \to M$ given by $(z, t) \mapsto \exp(t\nabla r(z))$ is a diffeomorphism and that $(x_1, ..., x_{n-1}) : B(q) \subset U \to \mathbb{R}^{n-1}$ is a coordinate system centered at $q$ such that $\partial_i(q) = \frac{\partial}{\partial x_i}(q)$ are orthonormal. Thus, we can suppose that $\partial_i$ are vector fields defined on $U \times [0, \delta_0)$. Then, the tangent space of $N_{\delta,\alpha}$ at $\tilde{q}$ is spanned by $E_i(\tilde{q}) := \partial_i(\tilde{q}) + (g^{\delta,\alpha}_i)'(0)\partial_\alpha(\tilde{q})$, $i = 1, ..., n - 1$, where $\partial_\alpha = \nabla r$. 


Using this vector fields we obtain
\[
\nabla_E E_i(q) = \nabla_{\partial_i}(\partial_i + (g_i^\delta)^\alpha(0)\partial_n)(\tilde{q})
\]
\[
= (\nabla_{\partial_i}\partial_i + (g_i^\delta)^\alpha(0)\nabla_{\partial_i}\partial_n + (g_i^\delta)^\alpha(0)\partial_n)(\tilde{q})
\]
\[
= (\nabla_{\partial_i}\partial_i + (g_i^\delta)^\alpha(0)\partial_n)(\tilde{q}).
\]

Now we can calculate \(H_{N_{\alpha}}(\tilde{q})\)
\[
(2.29)
\]
\[
H_{N_{\alpha}}(\tilde{q}) = -\sum_{i=1}^{n-1} \langle \nabla_{p_i,\partial_i} \nabla_{E_i} E_i \rangle (\tilde{q})
\]
\[
= -\sum_{i=1}^{n-1} \langle \partial_n, \nabla_{\partial_i} \partial_i + (g_i^\delta)^\alpha(0)\partial_n \rangle (\tilde{q})
\]
\[
= \Delta r(\tilde{q}) - \sum_{i=1}^{n-1} (g_i^\delta)^\alpha(0)
\]

(by linearity and definition of \(\Delta r\)).

Since \(\sum (g_i^\delta)^\alpha(0) = -\alpha\) and \(H_{N_{\alpha}} = H_{\partial M} = \Delta r\), the calculation above shows that \((\alpha, \delta) \mapsto H_{N_{\alpha}}(\exp(g^\delta q)(\nabla r(q)))\) is continuous at \((0,0)\).

By the previous paragraph and recalling that \(H_{\partial M}(q)r(p) + n - 1 \neq 0\), we know that the function,
\[
(\delta, \alpha) \mapsto \frac{(n-1)H_{N_{\alpha}}(\exp(g^\delta q)(\nabla r(q)))}{H_{N_{\alpha}}(\exp(g^\delta q)(\nabla r(q)))r_{p,\alpha}(p) + n - 1}
\]
is continuous at \((0,0)\). Thus, for \(\delta\) and \(\alpha\) small that depend on \(\varepsilon\),
\[
(2.30)
\frac{(n-1)H_{N_{\alpha}}(\exp(g^\delta q)(\nabla r(q)))}{H_{N_{\alpha}}(\exp(g^\delta q)(\nabla r(q)))r_{p,\alpha}(p) + n - 1} < \frac{(n-1)\Delta r(\exp(g^\delta q)(\nabla r(q)))}{\Delta r(\exp(g^\delta q)(\nabla r(q)))r(p) + n - 1}.
\]

The last step is to bound the nontrivial term of the right hand side of equation (2.31).

Note that the function
\[
(2.32)
\tilde{q} \mapsto -\frac{(n-1)\Delta r(\tilde{q})}{\Delta r(\tilde{q})r(p) + n - 1}
\]
is continuous at \(q\) because \(H_{\partial M}(q)r(p) + (n-1) \neq 0\). That means that in a neighborhood of \(q\),
\[
(2.33)
\frac{(n-1)\Delta r(\exp(g^\delta q)(\nabla r(q)))}{\Delta r(\exp(g^\delta q)(\nabla r(q)))r(p) + n - 1} < \tilde{z} + \frac{(n-1)H_{\partial M}(q)}{H_{\partial M}(q)r(p) + n - 1}.
\]

Putting together equations (2.25), (2.31) and (2.33) we get
\[
(2.34)
\Delta r_{p,\partial}(y) < \tilde{z} + \tilde{z} + \tilde{z} + \frac{(n-1)H_{\partial M}(q)}{H_{\partial M}(q)r(p) + n - 1}.
\]
This establishes Theorem 1.1. \(\square\)

2.3. Volume and Area Estimates. Recall that for \(\delta > 0\),
\[
(2.35)
M^\delta = \{p \in M | r(p) > \delta\}
\]
and \(\partial M^\delta\) is the boundary (as a metric subspace of \(M\)) for \(M^\delta\). In this subsection, area and volume estimates of \(\partial M^\delta\), and annular regions, \(M^{\delta_1} \setminus M^{\delta_2}\), \(\delta_1 \geq \delta_2\), are proven.

Using the normal exponential map we can write the volume form of \(M\) at a point \(p = \exp(r\nabla r(x))\) as \(A(x, r)dm(x)dt\), where \(x \in \partial M\) and \(dm(x)\) is the volume form of \(\partial M\). In the following lemma we bound \(A(x, t)\).
Lemma 2.10. Let $M^n$ be a connected and metrically complete Riemannian manifold with smooth boundary such that $\text{Ric}(M \setminus \partial M) \geq 0$ and $H_{\partial M} \leq H$. In $M \setminus \text{cut}(\partial M)$ write the volume form of $M$ as $A(x,t)dm(x)dt$, where $dm(x)$ is the volume form of $\partial M$. Then,

\begin{equation}
A(x, \delta) \leq A(x, 0)A_{n,H}(\delta),
\end{equation}

where $A_{n,H}$ is the function defined in equation (1.5).

Remark 2.11. Note that under the conditions of Lemma 2.10 if $H = 0$ then $A_{n,H} = 1$. Thus, equation (2.36) reduces to $A(x, \delta) \leq A(x, 0)$.

Proof. Let $p \in M \setminus (\text{cut}(\partial M) \cup \partial M)$. Then there is $(x, \delta) \in \partial M \times \mathbb{R}$ such that $r(p) = \delta = d(x, p)$. Let $\gamma$ be the minimizing geodesic from $x$ to $p$ so that $r(\gamma(t)) = t$. Note that if $H_{\partial M}(x) \leq H$ then

\begin{equation}
\frac{(n-1)H_{\partial M}(x)}{H_{\partial M}(x)r(\gamma(t)) + n-1} \leq \frac{(n-1)H}{Ht + n-1}.
\end{equation}

Thus, by Theorem 2.7 and since $A' = A'/A$

\begin{equation}
\frac{A'(x, t)}{A(x, t)} \leq \frac{(n-1)H}{Ht + n-1}.
\end{equation}

By Lemma 2.5 $Ht + n-1 \neq 0$ for $0 \leq t \leq \delta$. Thus, integrating (2.38) with respect to $t$ from 0 to $\delta$ we get

\begin{equation}
\ln \left( \frac{A(x, \delta)}{A(x, 0)} \right) \leq (n-1) \ln \left( \frac{H\delta + n-1}{n-1} \right).
\end{equation}

Taking exponentials in both sides of the inequality and arranging terms:

\begin{equation}
A(x, \delta) \leq A(x, 0) \left( \frac{H\delta + n-1}{n-1} \right)^{\delta_1 - 1} = A(x, 0)A_{n,H}(\delta).
\end{equation}

Using this estimate we obtain bounds for the volume of annular regions, $M^{\delta_2} \setminus M^{\delta_1}$ where $\delta_1 \geq \delta_2$.

Proof of Theorem 1.2. We compute

\begin{equation}
\text{Vol}(M^{\delta_2} \setminus M^{\delta_1}) = \int_{\delta_1}^{\delta_2} \int_{x \in \partial M} A(x, t)dm(x)dt
\end{equation}

\begin{equation}
\leq \int_{\delta_1}^{\delta_2} \int_{x \in \partial M} A_{n,H}(t)dm(x)dt
\end{equation}

\begin{equation}
= \text{Vol}(\partial M) \int_{\delta_1}^{\delta_2} A_{n,H}(t)dt.
\end{equation}

For $H = 0$ we have that

\begin{equation}
\int_{\delta_1}^{\delta_2} A_{n,H}(t)dt = \delta_1 - \delta_2.
\end{equation}

For $H \neq 0$ we get

\begin{equation}
\int_{\delta_1}^{\delta_2} A_{n,H}(t)dt = \frac{n-1}{Hn} \text{Vol}(\partial M) \left( \frac{H\delta + n-1}{n-1} \right)^{\delta_2 - \delta_1},
\end{equation}

where $\delta_i = \delta_i$ if $H > 0$. If $H < 0$, by definition $A_{n,H}(t) = 0$ for $t \geq -(n-1)/H$. Hence, $\delta_i = \min[\delta_i, -(n-1)/H]$. 

To get volume estimates we just have to evaluate the above integrals. We pick $\delta_2 = 0$. If $H \geq 0$, $r \leq D$. If $H < 0$, by Lemma 2.5 $r \leq -(n - 1)/H$. Thus, choose $\delta_1 = -(n - 1)/H$ when $H < 0$ and $\delta_1 = D$ otherwise.

**Remark 2.12.** For a manifold with no boundary, it is proven in Chavel [5] that $\text{cut}(p)$ has zero $n$-dimensional measure. The proof can be easily extended to prove that $\text{cut}(r)$ has zero $n$-dimensional Hausdorff measure (See [18]). Thus, we can get estimates of the volume of $M^{t_2} \setminus M^{t_1}$ in a straightforward way. But when calculating estimates of the volume of $\partial M^{t_1}$ we can encounter that $\text{cut}(r)$ has nonzero $n - 1$-dimensional Hausdorff measure or that $\partial M^{t_1}$ is not a submanifold. For example, consider a solid hyperboloid in 3-dimensional euclidean space. For an appropriate $\delta$, $\partial M^{t_1}$ is exactly two cones that intersect each other at the tip. Hence, the volume of $\partial M^{t_1}$ is not defined for all $\delta$.

**Proof of Theorem 1.3.** Recall that $L^1$ denotes the 1-dimensional Lebesgue measure. By Theorem 5.3 in [1] we know that $L^1$-almost everywhere

$$V \text{ol} (\partial M^{t_1}) = \frac{d}{dt} \left. V \text{ol} (M \setminus M') \right|_{t=\delta}$$

and by Theorem 1.2 that

$$V \text{ol} (M \setminus M') \leq V \text{ol} (\partial M^{t_1}) \int_0^\delta A_{n,H}(s) ds,$$

where $A_{n,H}$ is the continuous function given in (1.5). Thus, $V \text{ol} (\partial M^{t_1}) \leq V \text{ol} (\partial M) A_{n,H}(\delta)$.

**Remark 2.13.** Theorem 5.3 in [1] holds for metric spaces. This exact theorem is the Euclidean Slicing Theorem for the euclidean and manifold setting, Theorem 4.3.2 in [8].

3. **Convergence Theorems**

In the first subsection we state Wenger Compactness Theorem and Lemma 3.1 that gives an estimate of the SWIF distance between a manifold and a subset of it. These results are used in the second subsection to prove Theorem 1.4 about SWIF convergence of sequences of $\delta$-inner regions; when $\delta = 0$ we get Theorem 1.1. Then we prove Theorem 1.6. At the end of this section we discuss the SWIF convergence, if any, of some sequences of manifolds.

3.1. **Sormani-Wenger Intrinsic Flat Convergence.** Federer-Fleming introduced the term “integral current” (lying in Euclidean space) and extended Whitney’s notion of flat distance to integral currents. Ambrosio-Kirchheim in [1] extended Federer-Fleming’s integral currents to integral currents lying in arbitrary metric spaces. Later on Sormani-Wenger in [21] motivated by both Gromov-Hausdorff distance and flat distance defined intrinsic flat distance between $n$-integral current spaces $(X, d_X, T)$.

In general, $X$ is a countably $\mathcal{H}^n$-rectifiable metric space, $d_X$ is the metric on $X$ and $T$ an integral current in $\mathbf{I}_n(X)$ (See Definition 2.44 in [21]). In the setting of manifolds, the $n$-integral current space associated to an oriented manifold $(M^n, g)$ is just $(M, d, T)$, where $d$ is the metric on $M$ induced by $g$ and $T$ is integration over $M$ of top-dimensional differential forms of $M$, $T(\omega) = \int_M \omega$.

**Lemma 3.1.** Let $(M^n, g)$ be an oriented Riemannian manifold and $U$ an open set of $M$. Then

$$d_T ((M^n, d, T), (U, d', T')) \leq V \text{ol} (M \setminus U),$$
where $d' = d|_U$ and $T'$ is integration over $U$ of top-dimensional differential forms of $U$.

Proof. Since $U \subset M$, by definition of SWIF convergence (see Section 3 in [21]), we have

\[
(3.2) \quad d_F((M^n, d, T), (U, d', T')) \leq M(T - T') = M(\omega \mapsto \int_{M^U} \omega) = \text{Vol}(M \setminus U).
\]

\[\square\]

**Theorem 3.2.** [Wenger, Theorem 1.2 in [22]] Given a sequence of complete oriented Riemannian manifolds, $M_j$, of the same dimension with $\text{Vol}(M_j) \leq V$, $\text{Vol}(\partial M_j) \leq A$ and $\text{Diam}(M_j) \leq D$, then a subsequence converges in the SWIF sense to an integral current space.

3.2. **SWIF Compactness Theorems.** Given a $\delta$-inner region, $M^\delta = r^{-1}(\delta, \infty) \subset M^n$, we associate to it an $n$-integral current space: $(M^\delta, d_M^\delta, T^\delta)$, where $d_M$ is the metric of $M$ restricted to $M^\delta$ and $T^\delta$ is integration over $M^\delta$ of top differential forms of $M^\delta$, $T^\delta(\omega) = \int_{M^\delta} \omega$.

**Theorem 3.3.** Let $D, A > 0$ and $H \in \mathbb{R}$. If $(M^n_j, g_j)$ is a sequence of $n$-dimensional connected oriented and metrically complete Riemannian manifolds with smooth boundary that satisfy

\[
(3.3) \quad \text{Diam}(M_j) \leq D, \quad \text{Ric}(M_j \setminus \partial M_j) \geq 0, \quad \text{Vol}(\partial M_j) \leq A \quad \text{and} \quad H_{\partial M_j} \leq H,
\]

then for $\mathcal{L}^1$-a.e. $\delta \geq 0$ there is an $n$-integral current space $(W_\delta, d_W, T_\delta)$ and a subsequence that depends on $\delta$ such that $(M^n_j, d_M^j, T^\delta_j) \xrightarrow{\mathcal{F}} (W_\delta, d_W, T_\delta)$.

Proof. The result follows from Theorem 3.2. We just need to check that $(M^n_j, g_j)$ satisfies the hypotheses of that theorem. For $\delta = 0$, $\text{Diam}(M_j) \leq D$, $\text{Vol}(\partial M_j) \leq A$ and $\text{Vol}(M_j)$ is uniformly bounded by Theorem 1.2. For $\delta > 0$, $\text{Diam}(M^n_j) \leq \text{Diam}(M_j) \leq D$, where the metric of $M^n_j$ is the restricted metric. By Theorems 1.2 and 1.3 for $\mathcal{L}^1$-a.e. $\delta \geq 0$ $\text{Vol}(M^n_j) \leq \text{Vol}(M_j)$ and $\text{Vol}(\partial M^n_j)$ are uniformly bounded above. Hence, we can apply Theorem 3.2.

\[\square\]

Remark 3.4. The proof above consisted on showing that the given sequence satisfies the hypotheses of Theorem 3.2. Note that this could not be done if $H_{\partial M_j} \to \infty$ since by Theorem 1.2 we would get $\lim_{j \to \infty} \text{Vol}(M_j) = \infty$ (See Example 3.6). We also cannot apply Theorem 3.2 when $\text{Diam}(M_j) \to \infty$ (See Example 3.7).

**Proof of Theorem 1.6.** By the triangle inequality

\[
(3.4) \quad d_F(W_\delta, W) \leq d_F(W_\delta, M^n_{\delta j}) + d_F(M^n_{\delta j}, M_{\delta j}) + d_F(M_{\delta j}, W).
\]

Now, by Theorem 1.2

\[
(3.5) \quad d_F(M^n_{\delta j}, M_{\delta j}) \leq \text{Vol}(M_{\delta j} \setminus M^n_{\delta j}) \leq V(\delta, H, A, n),
\]

where $V(\delta, H, A, n)$ is a continuous function such that $\lim_{\delta \to 0} V(\delta, H, A, n) = 0$. Then, taking limits in (3.4) we get

\[
(3.6) \quad \lim_{i \to \infty} d_F(W_\delta, W) = \lim_{i \to \infty} \lim_{k \to \infty} d_F(W_\delta, W) = 0.
\]

\[\square\]
3.3. Examples. In this Subsection we present three examples of sequences of Riemannian manifolds. The first two examples presented are stated for compact manifolds with no boundary but can easily be generalized to manifolds with boundary. Example 3.3 defines a sequence that converges in the SWIF sense but not in the GH sense. Example 3.5 shows the necessity of a uniform bound of the mean curvature required in Theorem 1.4. Example 3.6 shows the necessity of a uniform bound of the diameter required in Theorem 1.4 and that equality holds in both volume and area estimates given in Theorem 1.2 and Theorem 1.3. Moreover, this example shows that the assumption \( H < 0 \) in Theorem 1.5 is needed.

Example 3.5 (Example A.7 in [21]). We define below a sequence of manifolds with positive scalar curvature that converges to a round \( n \)-sphere in the SWIF sense.

Let \( M^j \) be diffeomorphic to an \( n \)-sphere of volume \( V \). Suppose that \( M_j \) contains a connected open domain \( U_j \) isometric to a domain \( M_0 \setminus \bigcup_{i=1}^{N_j} B(p_{j,i}, R_j) \), where \( M_0 \) is a round sphere and \( B(p_{j,i}, R_j) \) are pairwise disjoint balls. Let each connected component of \( M_j \setminus U_j \) and each ball \( B(p_{j,i}, R_j) \) have volume bounded above by \( v_j/N_j \) where \( v_j \to 0 \). Then \( M_j \) converges as long as \( N_j R_j^{1/2} \to 0 \).

The sequence does not converge in the GH sense since, for \( \varepsilon \) small enough, the number of \( \varepsilon \)-balls needed to cover \( M_j \) goes to infinity as \( j \) goes to infinity.

Example 3.6 (Example 9.1 in [19]). Let \( M_j \) be the \( j \)-fold covering space of

\[
N_j = S^2 \setminus (B(p_+, 1/j) \cup B(p_-, 1/j)),
\]

where \((S^2, g_{S^2})\) is the 2-dimensional unit sphere, \( S^2 \), with the standard metric. The metric of \( M_j \) is the lifting of the metric of \( N_j \) and \( p_+, p_- \) are opposite poles. Then \( \text{Diam}(M_j) \leq 4\pi \), \( \text{Vol}(\partial M_j) \leq 4\pi \) and \( H_{\partial M_j} \to \infty \). No subsequence of \( M_j \) converges in SWIF sense, so \( H_{\partial M_j} < H \) for all \( j \) is necessary in Theorem 1.4.

Example 3.7. Let \( S^k \) be the \( k \)-dimensional unit sphere and \([0, j] \subset \mathbb{R} \) a closed interval with standard metrics. We endow \( S^k \times [0, j] \) with the product metric and define

\[
M_j := S^k \times [0, j]/\sim,
\]

where we identify antipodal points of \( S^k \times \{j\} \). Thus, \( \partial M_j = S^k \times \{0\}, H_{\partial M_j} = 0, \text{Vol}(\partial M_j) = \text{Vol}(S^k) \). Note that \( \text{Diam}(M_j) \to \infty \), \( \text{Vol}(M_j) = j \text{Vol}(S^k) \to \infty \). This sequence has no SWIF limit, so it proves the necessity of uniformly bounding the diameter of \( \{M_j\} \) in Theorem 1.4 and requiring \( H < 0 \) in Theorem 1.5.

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