Unification of Galileon Dualities

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Abstract: We study dualities of the general Galileon theory in $d$ dimensions in terms of coordinate transformations on the coset space corresponding to the spontaneously broken Galileon group. The most general duality transformation is found to be determined uniquely up to four free parameters and under compositions these transformations form a group which can be identified with $GL(2,\mathbb{R})$. This group represents a unified framework for all the up to now known Galileon dualities. We discuss a representation of this group on the Galileon theory space and using concrete examples we illustrate its applicability both on the classical and quantum level.
1. Introduction

The Galileons represent a theory of real scalar field $\phi$ with derivative interactions with interesting properties. It emerges in its simplest form as an effective theory of the Dvali-Gabadadze-Porrati model [1, 2] as well as of the de Rham-Gabadadze-Tolley massive gravity theory [3] in the decoupling limit. Generalization of the Galileon Lagrangian was proposed by Nicolis, Rattazzi and Trincherini [4] as the long distance modifications of General relativity. In the seminal paper [4] also the complete classification of possible terms of the Galileon Lagrangian has been made and some of the physical consequences have been studied in detail, i.a. it was demonstrated that such theories exhibit the so-called Vainshtein
mechanism [5]. In fact the general structures appearing in the Galileon Lagrangian have been already discovered in the 70’ as a building blocks of the Horndeski Lagrangian [6], which is the most general Lagrangian built from no more than the second order derivatives of the scalar field and leading to the second-order Euler-Lagrange equations. From another point of view the Galileon Lagrangian can be obtained as a special non-relativistic limit of the Dirac-Born-Infeld Lagrangian describing the fluctuations of the $d$-dimensional brane in the $d + 1$ dimensional space-time [7]. For a pedagogical introduction into the Galileon physics as well as for the complete list of literature see e.g. [8].

Putting aside very important cosmological aspects, the Galileon theory itself has an amazing structure which has been studied intensively in the literature (for pedagogical introductions into the technical aspects see e.g. [9, 10]). For instance on the quantum level it exhibits the so-called non-renormalization theorem which prevents the tree-level Galileon couplings from obtaining the quantum corrections stemming from loops [11, 12, 13, 14]. Another interesting feature is the existence of dualities, i.e. such transformations of fields and coordinates which preserve the form of the Galileon Lagrangian, though it changes its couplings. The duality transformations therefore interrelate different Galileon theories on the contrary to the symmetry transformations which leave the action invariant. The first such a duality has been recognized already in the paper [4] where it was shown that the transformation $\phi \rightarrow \phi + \frac{1}{4}H^2 x^2$ converts one form of the Lagrangian into another one. The latter then describes the fluctuations of the Galileon field about the de Sitter background solution. Another example of duality was mentioned and studied in [9] and it corresponds to the dual Legendre transform of the field. The most interesting duality has been discovered in [15] in the context of massive gravity and bigravity and has been further studied in [16].

In this paper we study these dualities from the unified point of view. We make use of the fact that the general Galileon theory can be understood as a low-energy effective theory describing the Goldstone bosons corresponding to the spontaneously broken symmetry according to the pattern $GAL(d, 1) \rightarrow ISO(d − 1, 1)$ where $GAL(d, 1)$ is the so-called Galileon group and its Lagrangian can be identified with generalized Wess-Zumino-Witten terms [17]. This allows us to classify the most general duality transformation and identify it as non-linear coordinate transformations on the coset space $GAL(d, 1)/SO(d − 1, 1)$. As we will show such duality transformations form a four-parametric group which can be identified with $GL(2, \mathbb{R})$ and which contains all the above mentioned dualities as special cases. We will also study the representation of this duality group on the Galileon theory space and using concrete examples we will demonstrate the possible applicability of dualities both in classical and quantum cases.

This paper is organized as follows. First, in Section 2 we introduce the Galileon symmetry and Lagrangian, discuss the Feynman rules and as an illustration we calculate the tree-level amplitudes up to the five-point one. In Section 3 we review the coset construction of the Galileon Lagrangian. Section 4 and 5 contain the main results of this work. In Section 4 we construct the most general duality transformations and in Section 5 we discuss their group structure. Several applications then follow in Section 6. Some technical details and alternative approaches are postponed in Appendices.
2. The Galileon Lagrangian

The Galileon represents the most general theory of a real scalar field $\phi$ in flat $d-$ dimensional space-time which satisfies the following two conditions. First, the action $S[\phi]$ is invariant with respect to the Galilean symmetry

$$\delta_{a,b}\phi = a + b \cdot x,$$

(2.1)

where $a$ and $b_{\mu}$ are real parameters, and therefore the Lagrangian changes under this symmetry by a total derivative

$$\delta_{a,b}L = \partial \cdot V_{a,b}.$$  

(2.2)

The second requirement is that the classical equations of motion contain at most second order derivatives of the field. As it has been proven in the seminal paper [4], (see also [9] and [10] for detailed pedagogical introduction and many useful formulae), these conditions allow just $d+1$ possible terms in the Lagrangian

$$L = \sum_{n=1}^{d+1} d_n L_n = \sum_{n=1}^{d+1} d_n \phi L_{n-1}^{\text{der}}$$  

(2.3)

where $d_n$ are real coupling constants and $L_{n}^{\text{der}}$ can be constructed from $d-$dimensional Levi-Civita tensor $\varepsilon^{\mu_1 \ldots \mu_d}$, the flat-space metric tensor $\eta_{\mu\nu}$ and the matrix of the second derivatives of the field $\partial \partial \phi$ as follows\(^1\)

$$L_{n}^{\text{der}} = \varepsilon^{\mu_1 \ldots \mu_d} \varepsilon^{\nu_1 \ldots \nu_d} \prod_{i=1}^{n} \partial_{\mu_i} \partial_{\nu_i} \phi \prod_{j=n+1}^{d} \eta_{\mu_j \nu_j} = -(d - n)! \det \{ \partial^{\nu_i} \partial_{\nu_j} \phi \}_{i,j=1}^{n}.$$  

(2.4)

In four dimensions we have explicitly

$$L_{0}^{\text{der}} = -d!$$

$$L_{1}^{\text{der}} = -6 \Box \phi$$

$$L_{2}^{\text{der}} = -2 \left[ (\Box \phi)^2 - \partial \partial \phi : \partial \partial \phi \right]$$

$$L_{3}^{\text{der}} = - \left[ (\Box \phi)^3 + 2 \partial \partial \phi : \partial \partial \phi : \partial \partial \phi - 3 \Box \phi \partial \partial \phi : \partial \partial \phi \right]$$

$$L_{4}^{\text{der}} = - \left[ (\Box \phi)^4 - 6 (\Box \phi)^2 \partial \partial \phi : \partial \partial \phi - 8 \Box \phi \partial \partial \phi : \partial \partial \phi : \partial \partial \phi - 3 (\partial \partial \phi : \partial \partial \phi)^2 \right].$$  

(2.5)

The equation of motion is then

$$\frac{\delta S[\phi]}{\delta \phi} = \sum_{n=1}^{d+1} n d_n L_{n-1}^{\text{der}} = 0$$  

(2.6)

and involves just the second derivatives of the Galileon field.

\(^1\)We use the convention $\eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)$, $\varepsilon^{0,1,\ldots,d-1} = 1$.  

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Let us note that the operator basis $L_n$ is not unique, we can choose also another set which differs from (2.4) by a total derivative and possible re-scaling. One of the many equivalent forms of the Lagrangian which can be obtained from (2.4) by means of the integration by parts and simple algebra is

$$\tilde{L} = \sum_{n=1}^{d+1} c_n \tilde{L}_n = \sum_{n=1}^{d+1} c_n (\partial \phi \cdot \partial \phi) \mathcal{L}^{\text{der}}_{n-2}.$$ (2.7)

Let us mention useful formulae (for derivation see e.g. [10])

$$\left(\partial \phi \cdot \partial \phi\right) \mathcal{L}^{\text{der}}_{n-1} = -\frac{2(d - n + 1)}{n + 1} \left[\phi \mathcal{L}^{\text{der}}_n - \partial \mu \left(H_n^\mu + \frac{n - 1}{d - n + 1} G_n^\mu\right)\right]$$ (2.8)

where

$$H_n^\mu = \phi \partial_{\alpha_1} \phi \varepsilon^{\alpha_2 \ldots \alpha_d \beta_1 \ldots \beta_d} \sum_{i=2}^{n} \partial_{\mu_i} \partial_{\nu_i} \phi \prod_{j=n+1}^{d} \eta_{\mu_j \nu_j}$$ (2.9)

$$G_n^\mu = \frac{1}{2} \left(\partial \phi \cdot \partial \phi\right) \partial^\mu \phi \varepsilon^{\mu_2 \ldots \mu_d \alpha_1 \ldots \alpha_{d-n+1} \beta_{d-n+1}} \sum_{j=1}^{d-n+1} \eta_{\alpha_j \beta_j} \prod_{k=2, k \neq i}^{n} \partial_{\mu_k} \partial_{\nu_k}$$ (2.10)

and thus after integration and omitting the surface terms we get

$$\int d^d x \left(\partial \phi \cdot \partial \phi\right) \mathcal{L}^{\text{der}}_{n-1} = -\int d^d x \frac{2(d - n + 1)}{n + 1} \phi \mathcal{L}^{\text{der}}_n.$$ (2.11)

For further convenience let us also write down explicitly the Feynman rule for $n$–point vertex

$$V_n(1, 2, \ldots, n) = (-1)^n d_n(d - n + 1)!(n - 1)! \sum_{\sigma \in S_n} G(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n-1)})$$ (2.12)

where we have introduced the Gram determinant $G(p_1, \ldots, p_{n-1})$

$$G(p_1, \ldots, p_{n-1}) = -\frac{1}{(d - n + 1)!} \varepsilon^{p_1 \ldots p_{n-1} \mu_2 \ldots \mu_d} \sum_{j=n}^{d} \eta_{\mu_j \nu_j}$$ (2.13)

and where the sum is over the cyclic permutations only\(^2\).

Using this Feynman rules, one can in principle calculate any tree-level $n$–point amplitude in the pure Galileon theory, however, due to the complicated structure of the vertices it is not an easy task. The most economic way how to organize the rather lengthy and untransparent calculation is the machinery of the Berends-Giele like recursion relations\(^3\) [18] which allows for an efficient computer algorithmization of the problem. In this way we can systematically calculate in principle any tree-level amplitude. What does it involve for $n = 3, 4, 5$ in the language of Feynman diagrams is depicted in Fig. 1. Note that crossing

\(^2\)Note that, the Gramm determinant is independent on the ordering of the vector arguments.

\(^3\)For an application to a similar problem see e.g. [19].
is tacitly assumed for these graphs which finally leads to four diagrams for 4-pt scattering and 26 for 5-pt scattering. In four dimension (in the theory with $d_1 = 0$) the calculation leads to the following results

$$M(1, 2, 3) = 6d_3 G(1, 2) = \frac{3}{2} d_3 p_3^4 = 0 \quad (2.14)$$

$$M(1, 2, 3, 4) = 12(2d_4 - 9d_3^2)G(1, 2, 3) \quad (2.15)$$

$$M(1, 2, 3, 4, 5) = -24 \left(72d_3^4 - 24d_3d_4 + 5d_3\right) G(1, 2, 3, 4) \quad (2.16)$$

(we were also capable to calculate the 6-pt diagrams which involves 235 Feynman diagrams).

Without the deeper understanding of the structure of the Galileon theory these results look suspiciously simple;\(^4\) in fact it was our main motivation for starting to study this model more systematically. In what follows we shall i.a. show how to understand these results and how they can be obtained almost without calculation on a single sheet of paper.

### 3. Coset construction of the Galileon action

The Galilean symmetry is a prominent example of the so called non-uniform symmetry, i.e. a symmetry which does not commute with the space-time translations [20, 21]. Indeed, denoting the infinitesimal translations and Galilean transformations of the Galileon field $\delta_c \phi$ and $\delta_{a,b} \phi$ respectively,

$$\delta_c \phi = c \cdot \partial \phi \quad (3.1)$$

$$\delta_{a,b} \phi = a + b \cdot x, \quad (3.2)$$

we get

$$[\delta_c, \delta_{a,b}] \phi = c \cdot b = \delta_{c-b,0} \phi \quad (3.3)$$

\(^4\)Note that, while the four- and five-point amplitudes are sums of Feynman graphs including those with one and two propagators naively generating pole terms (see Fig. 1), the resulting amplitude is represented by a purely contact term.
Let us add to this transformations also the Lorentz rotations and boosts $\delta_\omega$

$$\delta_\omega \phi = \frac{1}{2} \omega^{\mu\nu} (x_\mu \partial_\nu - x_\nu \partial_\mu) \phi$$  \hspace{1cm} (3.4)$$

we get then

$$[\delta_\omega, \delta_{a,b}] \phi = -b \cdot \omega \cdot x = \delta_{0,-b \cdot \omega} \phi = \delta_{0,-b \cdot \omega} \phi$$ \hspace{1cm} (3.5)$$

Therefore the infinitesimal transformations $\delta_c, \delta_\omega, \delta_{a,b}$ form a closed algebra with generators $P_a, J_{ab} = -J_{ba}, A$ and $B_a$ respectively. In terms of these generators

$$\delta_c = -ic^a P_a$$ \hspace{1cm} (3.6)$$
$$\delta_\omega = \frac{i}{2} \omega^{ab} J_{ab}$$ \hspace{1cm} (3.7)$$
$$\delta_{a,b} = -ia A - ib^a B_a$$ \hspace{1cm} (3.8)$$

and the commutator algebra can be rewritten in the form of the Galileon algebra $\text{gal}(d,1)$

$$[P_a, P_b] = [P_a, A] = [B_a, A] = [J_{ab}, A] = 0$$
$$[P_a, B_b] = i\eta_{ab} A$$
$$[J_{ab}, P_c] = i (\eta_{bc} P_a - \eta_{ac} P_b)$$
$$[J_{ab}, B_c] = i (\eta_{bc} B_a - \eta_{ac} B_b)$$
$$[J_{ab}, J_{cd}] = i (\eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac})$$ \hspace{1cm} (3.9)$$

which corresponds to the Galileon group $\text{GAL}(d,1)$ (see [17]).

Within the Galileon theory this group is realized non-linearly on the fields $\phi$ and space-time coordinates $x^\mu$. Indeed, for the generators $P_a, A$ a $B_a$ we have

$$-i P_a x^\mu = \delta^a_\mu$$
$$-i A \phi = 1$$
$$-i B_a \phi = x_a.$$ \hspace{1cm} (3.10)$$

The generators $A$ a $B_a$ are spontaneously broken, the order parameter can be identified with

$$\langle 0 | \delta_{a,b} \phi | 0 \rangle = a + b \cdot x.$$\hspace{1cm} (3.11)$$

This corresponds to the symmetry breaking pattern $\text{GAL}(d,1) \rightarrow \text{ISO}(d-1,1)$. Let us note that the above transformations are not completely independent in the sense of refs. [22, 21]. Indeed, their localized forms with space-time dependent parameters $a(x)$ and $b_\mu(x)$ yield the same local transformation

$$\delta_{a(x),b(x)} \phi = a(x) + b(x) \cdot x$$ \hspace{1cm} (3.11)$$

which corresponds to the local shift of the Galileon field $\phi$. More precisely, writing $a_b(x) = x \cdot b(x)$, we can identify

$$\delta_{a_b(x),0} = \delta_{0,b(x)}.$$ \hspace{1cm} (3.12)$$

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Physically this means that the local fluctuations of the order parameter which correspond to the Goldstone modes are not independent. As a result the particle spectrum does not contain the same number of Goldstone bosons as is the number of the broken generators (i.e. \(d + 1\)) but just one zero mass mode which can be identified with the Galileon field \(\phi\). (see [21, 23] for recent discussion of this issue).

Construction of the low energy effective Lagrangian describing the dynamics of the Goldstone bosons corresponding to the spontaneous breakdown of the non-uniform symmetries is a generalization of the coset construction of Callan, Coleman, Wess and Zumino [24, 25] and has been formulated by Volkov [26] and Ogievetsky [27]. Applied to the Galileon case, where the only linearly realized generators of the Galileon group are the Lorentz rotations and boosts \(J_{ab}\), the coset space is \(GAL(d,1)/SO(d−1,1)\) the elements of which are the left cosets \(\{gSO(d−1,1)\}\) where \(g \in GAL(d,1)\). The coordinates on this coset space can be chosen in a standard way by means of a unique choice of the representant \(U\) of each left coset. Such a representant can be written in terms of the coset coordinates \(x^a\), \(\phi\) and \(L^a\) as

\[
U \equiv U(x, \phi, L) = \exp(i x^a P_a) \exp(i \phi A + i L^a B_a) \quad (3.13)
\]

The general element of the galileon group \(g \in GAL(d,1)\)

\[
g = \exp \left( \frac{i}{2} \omega^{ab} J_{ab} \right) \exp (i c^a P_a) \exp (i a A + i b^a B_a) \quad (3.14)
\]

acts on the cosets by means of the left multiplication and consequently the coset coordinates transform according to

\[
U' \equiv U(x', \phi', L') = g U h^{-1} \quad (3.15)
\]

where \(h \equiv h(g, x, \phi, L) \in SO(d−1,1)\) is the compensator arranging \(U'\) to be of the form \((3.13)\). As usual, the stability group, which is the Lorentz group \(SO(d−1,1)\) here, is realized linearly (\(\phi\) transformed as a scalar and \(x\) and \(L\) are vectors), and the general element \((3.14)\) of the Galileon group \(g \in GAL(d,1)\) acts on \(U\) as follows

\[
g U(x, \phi, L) = \exp( i x'^a P_a ) \exp( i \phi' A + i L'^a B_a ) \exp \left( \frac{i}{2} \omega^{ab} J_{ab} \right) \quad (3.16)
\]

where

\[
x'^a = \Lambda(\omega)^a_b (x^b + c^b), \quad \phi' = \phi + a + b \cdot x, \quad L'^a = \Lambda(\omega)^a_b \left( L^b + b^b \right)
\]

and where

\[
\Lambda(\omega) = \exp \left( \frac{i}{2} \omega^{ab} M_{ab} \right), \quad (M_{ab})^c_d = \delta^c_a \delta_{bd} - \delta^c_b \delta_{ad}.
\]

As a result, for the general element of the Galileon group \(g \in GAL(d,1)\) we have the following compensator

\[
h(g, x, \phi, L) = \exp \left( \frac{i}{2} \omega^{ab} J_{ab} \right) \quad (3.17)
\]

Note that, the compensator does not depend on the coset coordinates \((x, \phi, L)\) and therefore treating \(\phi\) and \(L^a\) as space-time dependent fields, the compensator has no explicit or
implicit $x$ dependence. This simplifies the application of the general recipe [26, 27] significantly, because the requirement of the invariance with respect to the local stability group can be replaced by much simpler requirement of global invariance.

The basic object for the construction of the effective Lagrangian is the Maurer-Cartan form, which can be expressed in the coordinates $x^a$, $\phi$ and $L^a$ as

$$
\frac{1}{i} U^{-1} dU = \exp \left( -i \phi A - i L^b B_b \right) \exp (i x^d P_d) \exp (i \phi A + i L^a B_a) = \exp \left( -i L^b B_b \left( dx^c P_c + d \phi A + d L^d B_d \right) \exp (i L^a B_a) \right)
$$

where in the second line we have used the fact that $A$ commutes with all the other generators. Using further

$$
\exp \left( -i L^b B_b \right) P_c \exp (i L^a B_a) = P_c - L^a \eta_{bc} A
$$

we get finally

$$
\frac{1}{i} U^{-1} dU = dx^c P_c + \left( d \phi - L^a \eta_{bc} dx^c \right) A + d L^d B_d = \omega^c P_c + \omega_A A + \omega^d B_d
$$

The form $\omega^c_P$ is particularly simple. In the general case we get $\omega^c_P = e^{a}_{\mu} (x) dx^\mu$ and $e^{a}_{\mu}$ plays a role of $d$—bein, intertwining the abstract group indices $a, \ldots$ with space-time indices $\mu, \ldots$ and the flat metric $\eta_{ab}$ with the effective space-time metric $g_{\mu\nu}$ according to

$$
g_{\mu\nu} = \eta_{ab} e^{a}_{\mu} e^{b}_{\nu}
$$

In our case $e^{a}_{\mu} = \delta^{a}_{\mu}$, the space-time metric is therefore flat

$$
g_{\mu\nu} = \eta_{\mu\nu}
$$

and the abstract group indices are identical with the space-time ones. This also ensures that the volume element $d^d x$ is invariant with respect to the non-linearly realized Galileon group. Note also that there is no term of the form $\omega^{ab}_J J_{ab}$ on the right hand side of (3.20). This implies that the usual group covariant derivative is in our case identical with ordinary partial derivatives $\partial_{\alpha}$. The forms $\omega^c_P$, $\omega_A$ and $\omega^d_B$ transform under a general element of the Galileon (3.14) group $g \in GAL(d, 1)$ (cf. (3.17)) according to

$$
\omega^c_P' = \Lambda(\omega)^c_B \omega^B_P, \\
\omega_A' = \Lambda(\omega)^a_B \omega^B_A, \\
\omega^d_B' = \omega^d_B.
$$

These forms span three irreducible representations of the stability group $SO(d - 1, 1)$ (namely two vectors and one scalar) and can be therefore used separately as the basic building blocks for the construction of the effective Lagrangian. The general recipe requires to use this building block and their (covariant) derivatives to construct all the possible terms which are invariant with respect to local stability group. As we have mentioned above, in
our case we make do with ordinary partial derivatives and the last requirement can be rephrased as the global $SO(d-1,1)$ invariance when we identify the abstract group and space-time indices with help of the trivial $d$-bein $\delta^a_\mu$. Therefore, writing

$$d\phi(x) = \partial_\mu \phi(x) dx^\mu, \quad dL'(x) = \partial_\mu L'(x) dx^\mu,$$

the most general invariant term of the Lagrangian is the Lorentz invariant combinations of the fields $\partial_\mu L'$ and $D_\mu \phi$, where

$$D_\mu \phi \equiv \partial_\mu \phi - L_\mu$$

and their derivatives.

Apparently we have ended up with $d+1$ Goldstone fields $\phi$ and $L_\mu$ however this is not the final answer. In fact these fields are not independent. The standard possibility how to eliminate the unwanted degrees of freedom is to require an additional constraint \cite{17, 21}

$$\omega_A = 0 \iff L_\mu = \partial_\mu \phi, \quad (3.25)$$

which is invariant with respect to the group $GAL(d,1)$ and which is known as the inverse Higgs constraint (IHC) \cite{28}. Then the only remaining nontrivial building blocks are $\partial_\mu \partial_\nu \phi$ and its derivatives\footnote{Another possibility how to treat the problem of additional degrees of freedom is based on the field redefinition}

$$L = \mathcal{L}_{\text{inv}}(\partial_\mu \partial_\nu \phi, \partial_\lambda \partial_\mu \partial_\nu \phi, \ldots). \quad (3.26)$$

The Galileon Lagrangian represents a different type of possible terms contributing to the invariant action, namely those which are not strictly invariant on the Lagrangian level, but are invariant only up to a total derivative. Such terms can be identified as the generalized Wess-Zumino-Witten (WZW) terms \cite{29, 30} , as was proved and discussed in detail in \cite{31, 32}. From the point of view of the coset construction, the WZW terms originate in the integrals of the closed invariant $(d+1)$-forms\footnote{More precisely we integrate the pull-back of the form $\omega_{d+1}$ with respect to the map $B_{d+1} \to GAL(d,1)/SO(d-1,1)$ which maps $(x^\alpha, x^d) \to (\delta_a^\alpha x^\alpha, \phi(x^d), L'(x^d, x^d))$.} $\omega_{d+1}$ on $GAL(d,1)/SO(d-1,1)$ (these correspond to the variation of the action) over the $d+1$ dimensional ball $B_{d+1}$ the boundary of which is the compactified space-time $S_d = \partial B_{d+1}$

$$S_{\text{WZW}} = \int_{B_{d+1}} \omega_{d+1} \quad (3.27)$$

The invariant term $M^2 \partial_\mu \phi \partial^\mu \phi$, which was responsible for the kinetic term of the field $\phi$ in the original Lagrangian goes within the new parametrization in terms of $\phi$ and $\psi^\mu$ into the mass term of the field $\psi^\mu$. This field then does not correspond more to the Goldstone boson and can be integrated out from the effective Lagrangian, provided we are interested in the dynamics of the field $\phi$ only. We end up again with the just one nontrivial building block $\partial_\mu \partial_\nu \phi$. See \cite{21} for detailed discussion of this aspect of the spontaneously broken non-uniform symmetries.
In order to prevent these contributions to the action to degenerate into the strictly invariant Lagrangian terms discussed above it is necessary that the form $\omega_{d+1}$ is not an exterior derivative of the invariant $d$-form on $GAL(d,1)/SO(d−1,1)$. This means that $\omega_{d+1}$ has to be a nontrivial element of the cohomology $H^{d+1}(GAL(d,1)/SO(d−1,1),\mathbb{R})$ (see [33] and also [34] for a recent review on this topic). In the case of Galileon such forms can be constructed out of the covariant 1-forms $\omega_\mu^P$, $\omega_\mu^A$, $\omega_\mu^B$ with indices contracted appropriately to get Lorentz invariant combinations. As was shown in [17], there are $d + 1$ such $\omega_{d+1}$, namely

$$\omega_{d+1}^{(n)} = \varepsilon_{\mu_1...\mu_d} (d\phi - L_\mu dx^\mu) \wedge dL^{\mu_1} \wedge ... \wedge dL^{\mu_n−1} \wedge dx^{\mu_n} \wedge ... \wedge dx^{\mu_d}, \quad (3.28)$$

where $n = 1,2,...,d+1$. These forms are closed

$$\omega_{d+1}^{(n)} = d\beta_{d}^{(n)} \quad (3.29)$$

where\(^7\) [17]

$$\beta_{d}^{(n)} = \varepsilon_{\mu_1...\mu_d} \phi \; dL^{\mu_1} \wedge ... \wedge dL^{\mu_n−1} \wedge dx^{\mu_n} \wedge ... \wedge dx^{\mu_d} + \frac{n−1}{2(d−n+2)!} \varepsilon_{\mu_1...\mu_d} L^2 dL^{\mu_1} \wedge ... \wedge dL^{\mu_n−2} \wedge dx^{\mu_n−1} \wedge ... \wedge dx^{\mu_d} \quad (3.30)$$

and therefore

$$\int_{B_{d+1}} \omega_{d+1}^{(n)} = \int_{\partial B_{d+1}} \beta_{d}^{(n)} = \int_{S_{d}} \beta_{d}^{(n)} \quad (3.31)$$

Note that the $d$-forms $\beta_{d}^{(n)}$ are not invariant, and therefore $\omega_{d+1}^{(n)}$ are nontrivial elements of $H^{d+1}(GAL(d,1)/ISO(d−1,1),\mathbb{R})$. Imposing now the IHC constraint (3.25), we can finally identify

$$\int_{S_{d}} \beta_{d}^{(n)} = \frac{1}{n} \int_{S_{d}} d^d x L_n \quad (3.32)$$

4. Galileon duality as a coset coordinate transformation

The canonical coordinates $(x,\phi,L)$ on the coset space $GAL(d,1)/SO(d−1,1)$ which we have defined according to (3.13) are not the only possible ones. We can freely use any other set of coordinates connected with them by a general coordinate transformation of the form

$$x^\mu = \xi^\mu(x',L',\phi')$$
$$L^\mu = \Lambda^\mu(x',L',\phi')$$
$$\phi = f(x',L',\phi'). \quad (4.1)$$

Not all such new coordinates are of any use, e.g. those transformations (4.1) which are not covariant with respect to the $SO(d−1,1)$ symmetry will hide this symmetry in the effective

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\(^7\)The opposite sign of the second term in comparison with [17] stems from different convention for the metric tensor $\eta_{\mu\nu}$. 
Lagrangian. Even if the covariance is respected, in the general case the resulting Lagrangian might be difficult to recognize as a Galileon theory. In this section we shall make a classification of those coordinate changes which preserve the general form of the Galileon action as a linear combination of the $d+1$ terms discussed in the previous sections (though we allow for change of the couplings). Such a transformation of the coset coordinates can be then interpreted as a Galileon duality.

It is obvious from (3.27) and (3.28) that, provided the forms $\omega_\mu^P, \omega_A$ and $\omega_B^\mu$ can be expressed in the primed coordinates as a (covariant) linear combination (with constant coefficients) of the primed forms $\omega'_\mu^P, \omega'_A$ and $\omega'_B^\mu$ where

$$
\begin{align*}
\omega'_\mu^P &= dx'^\mu \\
\omega'_A &= d\phi' - L'_\mu dx'^\mu \\
\omega'_B^\mu &= dL'^\mu,
\end{align*}
$$

the coordinate transformation corresponds to a duality transformation of the Galileon action. Indeed, provided

$$
\begin{align*}
\omega'_A &= \alpha_A A \omega'_A \\
\omega'_B^\mu &= \alpha_B B \omega'_B^\mu + \alpha_B P \omega'_P^\mu \\
\omega'_P^\mu &= \alpha_P B \omega'_B^\mu + \alpha_P P \omega'_P^\mu,
\end{align*}
$$

we have

$$
\omega^{(n)}_{d+1} = \varepsilon_{\mu_1...\mu_d} \omega_A \wedge \omega_B^{\mu_1} \wedge ... \wedge \omega_B^{\mu_{n-1}} \wedge \omega_P^{\mu_n} \wedge ... \wedge \omega_P^{\mu_d}
$$

$$
= \alpha_{AA} \sum_{k=0}^{d-n+1} \sum_{l=0}^{n-1} \binom{d-n+1}{k} \binom{n-1}{l} \alpha_B^k \alpha_P^l \alpha_B^{d-n+1-k} \alpha_P^{n-1-l} \omega^{(l+k+1)}_{d+1},
$$

and, after imposing the IHC constraint (3.25), the corresponding term in the action satisfies

$$
\int S_d \beta_{d+1}^{(n)} = \alpha_{AA} \sum_{k=0}^{d-n+1} \sum_{l=0}^{n-1} \binom{d-n+1}{k} \binom{n-1}{l} \alpha_B^k \alpha_P^l \alpha_B^{d-n+1-k} \alpha_P^{n-1-l} \int S_d \beta_{d+1}^{(l+k+1)}.
$$

This means that the coordinate transformation maps linear combination of the $d+1$ basic building block of the Galileon action onto different linear combination of the same building blocks and the two apparently different Galileon theories are in fact dual to each other.

The conditions (4.3) constrain the form of the duality transformation (4.1) strongly. We have in the primed coordinates (here and in what follows the superscript at the symbol

---

8Note that, the constants $\alpha_{IJ}$ cannot be decorated with any Lorentz index because the only invariant tensors at our disposal are $\eta_{\mu\nu}$ and $\varepsilon_{\mu_1...\mu_d}$.

9Note that, the formula $\omega_A = \alpha_A A \omega'_A$ ensures a compatibility of the IHC constraint with the coordinate transformation.
of partial derivative indicates the corresponding primed variable, e.g. \( \partial^{(\alpha)} \equiv \partial / \partial \phi^{\alpha} \)

\[
\begin{align*}
\omega^\mu_P &= \partial^{(L)}_\nu \xi^\mu + \left( \partial^{(x)}_\nu \xi^\mu + L'_\nu \partial^{(\phi)} \xi^\mu \right) \omega^\mu_P + \partial^{(\phi)} \xi^\mu \omega^\mu_A \\
\omega^\mu_B &= \partial^{(L)}_\nu \Lambda^\mu + \left( \partial^{(x)}_\nu \Lambda^\mu + L'_\nu \partial^{(\phi)} \Lambda^\mu \right) \omega^\mu_B + \partial^{(\phi)} \Lambda^\mu \omega^\mu_A \\
\omega^\mu_A &= \left( \partial^{(\phi)} f - \Lambda_\mu \partial^{(\phi)} \xi^\mu \right) \omega^\mu_A + \left( \partial^{(L)}_\mu f - L'_\mu \partial^{(L)}_\nu \xi^\mu \right) \omega^\mu_B \\
&\quad + \left[ \partial^{(x)}_\nu f + L'_\nu \partial^{(\phi)} f - \Lambda_\mu \left( \partial^{(x)}_\nu \xi^\mu + L'_\nu \partial^{(\phi)} \xi^\mu \right) \right] \omega^\mu_P 
\end{align*}
\]

(4.6)

and comparing the coefficients at \( \omega^\mu_P, \omega^\mu_A \) and \( \omega^\mu_B \) in the expressions for \( \omega^\mu_P \) and \( \omega^\mu_B \) with the corresponding right hand sides of (4.3) we get the following set of differential equations for \( \xi^\mu \) and \( \Lambda^\mu \)

\[
\begin{align*}
\partial^{(\phi)} \xi^\mu &= 0, \quad \partial^{(L)}_\nu \xi^\mu = \delta^\nu_\alpha \alpha_{PB}, \quad \partial^{(x)}_\nu \xi^\mu + L'_\nu \partial^{(\phi)} \xi^\mu &= \delta^\nu_\alpha \alpha_{PP}, \\
\partial^{(\phi)} \Lambda^\mu &= 0, \quad \partial^{(L)}_\nu \Lambda^\mu = \delta^\nu_\alpha \alpha_{BB}, \quad \partial^{(x)}_\nu \Lambda^\mu + L'_\nu \partial^{(\phi)} \Lambda^\mu &= \delta^\nu_\alpha \alpha_{BP}.
\end{align*}
\]

(4.7)

Integration of these equations is trivial, we get (up to the additive constants\(^{10}\))

\[
\begin{align*}
\xi^\mu &= \alpha_{PB} L^\mu + \alpha_{PP} x'^\mu \\
\Lambda^\mu &= \alpha_{BB} L^\mu + \alpha_{BP} x'^\mu.
\end{align*}
\]

(4.8)

Comparisson of coefficients in both expressions for \( \omega_A \) gives, after using the explicit form (4.8) of \( \xi^\mu \) and \( \Lambda^\mu \), the following differential equations for \( f \)

\[
\begin{align*}
\partial^{(\phi)} f &= \alpha_{AA}, \quad \partial^{(L)}_\mu f = \alpha_{PB} \left( \alpha_{BB} L'_\mu + \alpha_{BP} x'^\mu \right), \quad \partial^{(x)}_\nu f + L'_\nu \partial^{(\phi)} f = \alpha_{PP} \left( \alpha_{BB} L'_\nu + \alpha_{BP} x'^\nu \right)
\end{align*}
\]

From the first equation it follows

\[
f = \alpha_{AA} \phi' + F(x', L')
\]

(4.9)

where the function \( F \) of two variables satisfies

\[
\begin{align*}
\partial^{(L)}_\mu F &= \alpha_{PB} \left( \alpha_{BB} L'_\mu + \alpha_{BP} x'^\mu \right) \\
\partial^{(x)}_\nu F &= \alpha_{PP} \left( \alpha_{BB} L'_\nu + \alpha_{BP} x'^\nu \right) - \alpha_{AA} L'_\nu.
\end{align*}
\]

(4.10)

Integration of this equation is possible only if the integrability conditions are satisfied

\[
\partial^{(x)}_\nu \partial^{(L)}_\mu F = \partial^{(L)}_\mu \partial^{(x)}_\nu F
\]

(4.11)

This constraints the possible values of the constants \( \alpha_{IJ} \)

\[
\alpha_{PB} \alpha_{BB} = \alpha_{PP} \alpha_{BB} - \alpha_{AA}
\]

(4.12)

which means

\[
\alpha_{AA} = \det (\alpha) \equiv \det \left( \begin{array}{cc} \alpha_{PP} & \alpha_{PB} \\ \alpha_{BP} & \alpha_{BB} \end{array} \right).
\]

(4.13)

\(^{10}\)We have set these additive constants equal to zero. The reason is that, if nonzero, they corresponds to the additional combination of the space-time translation and Galileon transformation. Both these additional contributions are exact symmetries of the Galileon theory and does not bring about anything new.
Imposing this constraint, the equations (4.10) transforms into the form
\begin{align*}
  \partial_{\mu}^{(L)} F &= \alpha_{PB} \alpha_{BB} L_{\mu}^{\prime} + \alpha_{PB} \alpha_{BP} x_{\mu}^{\prime} \\
  \partial_{\nu}^{(x)} F &= \alpha_{BP} \alpha_{PB} L_{\nu}^{\prime} + \alpha_{PP} \alpha_{BP} x_{\nu}^{\prime}.
\end{align*}
(4.14)
which can be easily integrated (again up to the additive constant corresponding to trivial shift of \( \phi \))
\begin{align*}
  F &= \int_{0}^{(x',L')} \left( d x' \cdot \partial^{(x)} F + d L' \cdot \partial^{(L)} F \right) \\
  &= \frac{1}{2} \left( \alpha_{PB} \alpha_{BB} L'^{2} + 2 \alpha_{PB} \alpha_{BP} x' \cdot L' + \alpha_{PP} \alpha_{BP} x'^{2} \right)
\end{align*}
(4.15)
As a result we get the most general formulae\(^{11}\) for the duality transformation of the coset coordinates
\begin{align*}
  x_{\mu} &= \alpha_{PP} x_{\mu}^{\prime} + \alpha_{PB} L_{\mu}^{\prime}, \quad L_{\mu} = \alpha_{BB} L_{\mu}^{\prime} + \alpha_{BP} x_{\mu}^{\prime} \\
  \phi &= \det (\alpha) \phi' + \frac{1}{2} \left( \alpha_{PB} \alpha_{BB} L'^{2} + 2 \alpha_{PB} \alpha_{BP} x' \cdot L' + \alpha_{PP} \alpha_{BP} x'^{2} \right)
\end{align*}
(4.16)
and of the basic building blocs of the Galileon Lagrangian
\begin{align*}
  \omega_{B}^{\mu} &= \alpha_{BB} \omega_{B}^{\mu} + \alpha_{BP} \omega_{P}^{\mu} \\
  \omega_{P}^{\mu} &= \alpha_{PB} \omega_{B}^{\mu} + \alpha_{PP} \omega_{P}^{\mu} \\
  \omega_{A} &= \det (\alpha) \omega_{A}.
\end{align*}
(4.17)
Imposing the IHC constraint (3.25) we get finally
\begin{align*}
  x &= \alpha_{PP} x^{\prime} + \alpha_{PB} \partial' \phi' \\
  \phi &= \det (\alpha) \phi' + \frac{1}{2} \left( \alpha_{PB} \alpha_{BB} \partial' \phi' \cdot \partial' \phi' + 2 \alpha_{PB} \alpha_{BP} x' \cdot \partial' \phi' + \alpha_{PP} \alpha_{BP} x'^{2} \right) \\
  \partial \phi &= \alpha_{BB} \partial' \phi' + \alpha_{BP} x' \cdot \partial' \phi'.
\end{align*}
(4.18)
Let us note that the last formula of (4.18) (the transformation of \( \partial \phi \)) is compatible with the first two as a result of the compatibility of the IHC constraint with the coordinate transformation mentioned above. We can also prove this easily by explicit calculation (see Appendix B).

Let us finally write down the explicit formula for the duality in terms of the Galileon action. It is expressed by the identity
\begin{align*}
  S[\phi] &= S_{\alpha}[\phi']
\end{align*}
(4.19)
where
\begin{align*}
  S[\phi] &= \int d^{d} x \sum_{n=1}^{d+1} d_{n} \mathcal{L}_{n} \\
  S_{\alpha}[\phi] &= \int d^{d} x \sum_{n=1}^{d+1} d_{n}(\alpha) \mathcal{L}_{n}
\end{align*}
(4.20)
(4.21)
\(^{11}\)Up to the remnants of the omitted additive constants, as discussed above.
and the couplings of the two dual action are interrelated as

\[ d_n(\alpha) = \sum_{m=1}^{d+1} A_{nm}(\alpha)d_m \]  

(4.22)

where the matrix \( A_{nm}(\alpha) \) has the following form

\[ A_{nm}(\alpha) = \det(\alpha) \frac{m}{n} \sum_{k=0}^{d-m+1} \sum_{l=0}^{m-1} \binom{d-m+1}{k} \binom{m-1}{l} \alpha^k_{PP} \alpha^l_{BB} \alpha^{d-m+1-k}_{PP} \alpha^{m-1-l}_{BP} \delta_{n,l+k+1} \]  

(4.23)

5. \( GL(2, \mathbb{R}) \) group of the Galileon dualities

The duality transformations introduced in the previous section has natural \( GL(2, \mathbb{R}) \) group structure under compositions. This is immediately seen from their action on the 1-forms \( \omega_A \), \( \omega_P \) and \( \omega_B \) (cf. (4.17)) and on the coset coordinates \( x^\mu \) and \( L^\mu \): the duality transformation is in one-to-one correspondence with the matrix

\[ \alpha = \begin{pmatrix} \alpha_{PP} & \alpha_{PB} \\ \alpha_{BP} & \alpha_{BB} \end{pmatrix} \]  

(5.1)

and composition of two duality transformations corresponding to the matrices \( \alpha \) and \( \beta \) is again a duality transformation described by matrix \( \alpha \cdot \beta \). The condition \( \det \alpha \neq 0 \) ensures regularity\(^{12}\) of the transformation of the coordinates on the coset space (4.16).

A little bit less obvious is the group property for the duality transformation of \( \phi \). To demonstrate it let us rewrite (4.16) in the form

\[ X = \alpha \cdot X' \]
\[ \phi = \det \alpha \phi' + \frac{1}{2} X'^T \cdot \tilde{\alpha} \cdot \alpha \cdot X' \]  

(5.2)

where

\[ X = \begin{pmatrix} x \\ L \end{pmatrix}, \quad \tilde{\alpha} = \begin{pmatrix} \alpha_{BP} & 0 \\ 0 & \alpha_{PP} \end{pmatrix} \]  

(5.3)

Then composition of two dualities means

\[ X = \alpha \cdot X' = \alpha \cdot (\beta \cdot X'') = (\alpha \cdot \beta) \cdot X'' \]
\[ \phi = \det \alpha \phi' + \frac{1}{2} X'^T \cdot \tilde{\alpha} \cdot \alpha \cdot X' \]
\[ = \det (\alpha \cdot \beta) \phi'' + \frac{1}{2} X''^T \cdot \left( \left( \tilde{\beta} \cdot \tilde{\alpha} \right) \det \alpha + \beta^T \cdot \tilde{\alpha} \cdot \alpha \cdot \beta \right) \cdot X'' \]  

(5.4)

However, as can be proved by direct calculation,

\[ \left( \tilde{\beta} \cdot \tilde{\alpha} \right) \det \alpha + \beta^T \cdot \tilde{\alpha} \cdot \alpha \cdot \beta = \left( \tilde{\alpha} \cdot \tilde{\beta} \right) \cdot (\alpha \cdot \beta) \]  

(5.5)

\(^{12}\)The Jacobian of the transformation (4.16) is \((\det(\alpha))^2\)
and therefore
\[ \phi = \det (\alpha \cdot \beta) \phi'' + \frac{1}{2} X''^T \cdot (\alpha \cdot \beta) \cdot (\alpha \cdot \beta) \cdot X'' \]  
(5.6)
as expected.

On the space $D_{d+1}$ of the Galileon theories, which can be treated as a $d+1$ dimensional real space with elements
\[ d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{d+1} \end{pmatrix}, \]  
(5.7)
corresponding to $d+1$-tuples of the couplings $d_n$, we have a linear representation of the duality group $GL(2, \mathbb{R})$ by the matrices $A_{nm}(\alpha)$ explicitly given by (4.23).

Let us now discuss some important special cases. The duality transformations corresponding to the one-parameter subgroup of matrices
\[ \alpha_{dS}(\zeta) = \begin{pmatrix} 1 & 0 \\ 2\zeta & 1 \end{pmatrix} \]  
(5.8)
which satisfy
\[ \alpha_{dS}(\zeta) \cdot \alpha_{dS}(\zeta') = \alpha_{dS}(\zeta') \cdot \alpha_{dS}(\zeta) = \alpha_{dS}(\zeta + \zeta'), \]  
(5.9)
result in the following explicit transformation
\[ x = x', \quad \phi = \phi' + \zeta x'^2, \quad \partial \phi = \partial' \phi' + 2\zeta x'. \]  
(5.10)
Fixing the parameter $\zeta = H^2/4$ the dual theory can be interpreted as an expansion of the original Galileon field about the de Sitter solution
\[ \phi_{dS} = \frac{1}{4} H^2 x^2 \]  
(5.11)
The fact, that the fluctuations $\phi'$ about such a background are described by a dual Galileon Lagrangian has been established already in the seminal paper [4]. For the transformation of the couplings we get explicitly
\[ d_n(\alpha_{dS}(\zeta)) = \sum_{m=n}^{d+1} \binom{m}{n} (2\zeta)^{m-n} d_m \]  
(5.12)
Another example concerns the following matrix
\[ \alpha_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  
(5.13)
It results in the duality transformation
\[ x = \partial' \phi', \quad \phi = -\phi' + x' \cdot \partial' \phi', \quad \partial \phi = x' \]
which can be rewritten in the more symmetric form as

\[ x \cdot x' = \phi(x) + \phi'(x') \quad (5.14) \]

and which corresponds to the Legendre transformation. Duality properties of the Galileon theory with respect to this transformation has been discussed in detail in [9]. Explicit form for the dual couplings reads

\[ d_n(\alpha_L) = -\frac{d - n + 2}{n} d_{d-n+2}. \quad (5.15) \]

Let us now assume the diagonal matrix

\[ \alpha_S(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-\Delta-1} \end{pmatrix} \quad (5.16) \]

corresponding to the scaling transformation (\(\Delta\) is the Galileon scaling dimension)

\[ x = \lambda x', \quad \phi = \lambda^{-\Delta} \phi', \quad \partial \phi = \lambda^{-\Delta-1} \partial' \phi' \quad (5.17) \]

for which the dual couplings simply scale according their dimension as

\[ d_n(\alpha_S(\lambda)) = \lambda^{d+2-n(\Delta+2)} d_n. \quad (5.18) \]

More general scaling is also possible, namely

\[ \alpha_S(\lambda, \kappa) = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa \end{pmatrix} \quad (5.19) \]

for which

\[ x = \lambda x', \quad \phi = \lambda \kappa \phi', \quad \partial \phi = \kappa \partial' \phi' \quad (5.20) \]

and in the dual theory

\[ d_n(\alpha_S(\lambda, \kappa)) = \kappa^n \lambda^{d-n+2} d_n. \quad (5.21) \]

Let us assume now duality transformations induced by the matrices of the form\(^\text{13}\)

\[ \alpha_D(\theta) = \begin{pmatrix} 1 & -2\theta \\ 0 & 1 \end{pmatrix} \quad (5.22) \]

which represents a one-parameter subgroup

\[ \alpha_D(\theta) \cdot \alpha_D(\theta') = \alpha_D(\theta') \cdot \alpha_D(\theta) = \alpha_D(\theta + \theta') \quad (5.23) \]

and the coordinate and field transformation reads

\[ x = x' - 2\theta \partial' \phi', \quad \phi = \phi' - \theta \partial' \phi' \cdot \partial' \phi', \quad \partial \phi = \partial' \phi'. \quad (5.24) \]

\(^\text{13}\)The rationale for the minus sign of the element \(\alpha_{PB}\) is that with this choice the infinitesimal form of this duality transformation is

\[ \phi(x) = \phi'(x') + \theta \partial' \phi' \cdot \partial' \phi'. \]

See Appendix A for bottom up construction of the finite duality transformation from the infinitesimal one.
Such a type of duality (with special value of the parameter $\theta$) has been discussed in the papers [15, 16, 35] and its one-parametric group structure has been recognized in a very recent paper [36]. The couplings transform according to

$$d_n(\alpha_D(\theta)) = \frac{1}{n} \sum_{m=1}^{n} m \left(\frac{d - m + 1}{n - m}\right)(-2\theta)^{n-m} d_m.$$  

(5.25)

It is obvious, that any duality transformation can be obtained as a combination of the above elementary types of transformations. Indeed, for general matrix $\alpha$ we can write the following decomposition

$$
\begin{pmatrix}
\alpha_{PP} & \alpha_{PB} \\
\alpha_{BP} & \alpha_{BB}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_{PP} & 0 \\
0 & \alpha_{PP}^{-1} \det(\alpha)
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_{PP}^{-1} \alpha_{PB} & 0 \\
0 & \alpha_{PP}^{-1} \alpha_{BB}
\end{pmatrix}.
\tag{5.26}
$$

Let us give another simple example of such a type of decomposition. For instance, we have

$$\alpha_D(\theta) = \left[\alpha_S(1,(2\theta)^{-1}) \cdot \alpha_{dS}(-2^{-1})\right] \cdot \alpha_L \cdot \left[\alpha_{dS}(2^{-1}) \cdot \alpha_S(1,-2\theta)\right],$$ 

(5.27)

and therefore we can understand the one-parametric duality (5.24) as a Legendre transformation of the function $\psi'(x')$ into $\psi(x)$ where

$$\psi'(x') = \frac{1}{2} x'^2 - 2\theta \phi'(x'), \quad \psi(x) = \frac{1}{2} x^2 + 2\theta \phi(x)$$ 

(5.28)

which can be written in the symmetric form (cf. (5.14)) as

$$x \cdot x' = \psi(x) + \psi'(x').$$ 

(5.29)

As we will see in the following sections, the duality transformations (5.24) are the most interesting ones relevant from the point of view of physical applications. Let us briefly comment on some properties of its representation on the Galileon theory space (5.7). First, because the matrices $A(\theta) \equiv A_{nm}(\alpha_D(\theta))$ are lower triangular matrices, any subspace $D_d(\alpha^{d+1})$ spanned by the $d+1$-tuples with first $k$ couplings equal to zero (i.e. $D_d^{(k)} = \{d|d_n = 0 \text{ for } n \leq k\}$) is left invariant by $A(\theta)$. We can therefore restrict ourselves to some fixed $D_d^{(k)}$ in what follows.\(^\text{15}\)

Note also that $\alpha_D(\theta)$ is a one-parametric subgroup and thus the matrices $A(\theta)$ satisfy a differential equation

$$\frac{d}{d\theta} A(\theta) = T \cdot A(\theta)$$ 

(5.30)

where

$$T_{mn} = \frac{d}{d\theta} A_{nm}(\alpha_D(\theta))|_{\theta=0} = -2 \frac{n-1}{n} (d - n + 2) \delta_{n,m+1}$$ 

(5.31)

\(^\text{14}\)The function $\psi'(x')$ can be obtained by means of application of the dual transformation corresponding to the product of matrices in the second square brackets in (5.27), similarly for $\psi(x)$.

\(^\text{15}\)For the physical applications it is natural to set $d_1 = 0$ in order to avoid tadpoles and assume therefore the subspace $D_d^{(1)}$. 

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and consequently we get for \(d + 1 - k\)-tuple \(d(\theta) \equiv d_n(\alpha_D(\theta)) \in D_{d+1}^{(k)}\)

\[
\frac{d}{d\theta}d(\theta) = T \cdot d(\theta).
\] (5.32)

This is a system of \(d - k\) nontrivial ordinary differential equations (note that the first of the equations (5.32) is trivial)

\[
\frac{d}{d\theta}d_{k+1}(\theta) = 0,
\]
i.e. \(d_{k+1}\) can be taken as fixed\(^{16}\) once for ever) describing the “running” of the couplings with the change of the duality parameter \(\theta\). Of course, the solutions are just \(d_n(\alpha_D(\theta))\) given by (5.25) with \(d_n, n > k\) as the initial conditions at \(\theta = 0\). Such a system have in general \(d - k - 1\) functionally independent integrals of motion \(I_{k+3}, I_{k+4}, \ldots, I_{d+1}\) which do not depend explicitly on \(\theta\). Once these are known, any other such an integral of motion can be then expressed as

\[I = f(I_{k+3}, I_{k+4}, \ldots, I_{d+1}),\] (5.33)

where \(f\) is some function. The set \(I_{k+3}, I_{k+4}, \ldots, I_{d+1}\) represents therefore a basis of the \(\alpha_D(\theta)\) duality invariants on the subspace \(D_{d+1}^{(k)}\) of the Galileon theory space.

The set of independent invariants \(I_{k+3}, I_{k+4}, \ldots, I_{d+1}\) can be constructed by means of elimination of the initial conditions and \(\theta\) the from the solution (5.25). This can be done as follows. Note that (5.25) for \(n = k + 2\) and \(d_n = 0\) for \(n \leq k\) reads

\[d_{k+2}(\theta) = d_{k+2} - 2\theta\frac{(k + 1)(d - k)}{k + 2}d_{k+1},\] (5.34)

and thus we have unique solution \(\theta^*\) for \(\theta\) such that \(d_{k+2}(\theta^*) = 0\). According to the group property we can rewrite the solution of (5.32) in the form

\[d(\theta) = A(\theta - \theta^*) \cdot d(\theta^*)\] (5.35)

with new initial conditions \(d(\theta^*)\). Inverting (5.35) we get

\[d(\theta^*) = A(\theta^* - \theta) \cdot d(\theta)\] (5.36)

the right hand side of which is \(\theta\) independent. For \(d_{k+2}\) the equation (5.35) reads

\[d_{k+2}(\theta) = -2(\theta - \theta^*)\frac{(k + 1)(d - k)}{k + 2}d_{k+1}\] (5.37)

and thus we can easily eliminate \(\theta - \theta^*\) solely in terms of \(d_{k+2}(\theta)\). Inserting now this for the explicit \(\theta - \theta^*\) dependence into (5.36) for \(n = k + 3, \ldots, d + 1\) we get the desired integrals of motion \(I_l(d_{k+2}(\theta), \ldots, d_{d+1}(\theta))\). Their interpretation is clear, according to our construction \(I_l\) represents a value of couplings \(d_l\) in the theory dual with the original one such that in the dual theory the coupling \(d_{k+2}\) is zero. These integrals form the basis of

\(^{16}\)For instance, for \(k = 1\) it is natural to set \(d_2 = 1/12\) in order to normalize the kinetic term of the Galileon as usual.
the $\alpha D(\theta)$ duality subgroup invariants on the Galileon theory subspace $D_{d+1}^{(k)}$ we started with.

Let us illustrate this general construction of $\alpha D(\theta)$ duality invariants in the case of three and four dimensional Galileon theory. We will restrict ourselves to the theory subspaces $D_4^{(1)}$ and $D_5^{(1)}$, i.e. we set in both cases $d_1 = 0$, and we further fix $d_2$ for $d = 3, 4$ as $1/4$ and $1/12$ respectively. For $d = 3$ we get from (5.25)

$$d_3(\theta) = d_3 - \frac{2}{3} \theta \quad d_4(\theta) = d_4 - \frac{3}{2} d_3 \theta + \frac{1}{2} \theta^2$$

and according the general recipe, the only $\alpha D(\theta)$ duality invariant is $I_4 = d_4(\theta^*)$ with $\theta^* = 3d_3/2$, explicitly

$$I_4 = d_4 - \frac{9}{8} d_3^2.$$ (5.39)

For $d = 4$ the $\alpha D(\theta)$ duality transformation reads

$$d_3(\theta) = d_3 - \frac{1}{3} \theta \quad d_4(\theta) = d_4 - 3 \theta d_3 + \frac{1}{2} \theta^2$$

$$d_5(\theta) = d_5 - \frac{8}{5} \theta d_4 + \frac{12}{5} \theta^2 d_3 - \frac{4}{15} \theta^3,$$ (5.40)

and we have two independent duality invariants $I_{4,5} = d_{4,5}(\theta^*)$ where $\theta^* = 3d_3$, explicitly

$$I_4 = d_4 - \frac{9}{2} d_3^2$$

$$I_5 = d_5 - \frac{24}{5} d_3 d_4 + \frac{72}{5} d_3^3.$$ (5.41)

6. Applications

Because the duality relates different Galileon theories, its main benefit is based on the possibility to solve a given problem in the simplest exemplar of the set of theories connected by duality and then to translate the result back to the apparently more complex original theory for which the problem has been formulated. In order to realize this approach effectively it is necessary to establish the transformation properties of various physically relevant quantities under the duality transformations. The main role play those which are invariants of the duality. As we have mentioned above, the most useful duality is the one-parametric subgroup $\alpha D(\theta)$, which is (together with $\alpha L$) the only one for which the field and coordinate transformation is nontrivial. Therefore the invariants with respect to this subgroup are the most important ones. In this section we will discuss the above aspects of the duality using several examples both on classical and quantum levels.

In what follows we almost exclusively work in four dimensions with Minkowski metric and in the Galileon Lagrangian we set $d_1 = 0$ to avoid the tadpole and $d_2 = 1/12$ to get a canonical normalization of the kinetic term.
6.1 Classical static solution

As a warm up, we will illustrate the use of duality on a simple example of the static axial-symmetric solution of Galilean equations with an external source coupled to the Galileon field as

\[ S_{\text{int}} = \int d^4 x \phi(x) T(x) \quad (6.1) \]

The source \( T \) will be represented by an infinite “cosmic string” along the \( x_1 \)-axis with linear density \( \sigma > 0 \)

\[ T(x) = \sigma \delta(x^2) \delta(x^3) \quad (6.2) \]

Note that, for general external source, the part \( S_{\text{int}} \) of the complete action violates duality. Therefore we cannot in general case simply argue that the duality transformation of the classical solution in the original theory is also a solution of the dual theory with the same external source. However, our source term is very special being local and therefore it modifies the equations of motion only on the set of points of zero measure. As we shall explicitly see, for such a source the duality works, which supports the conclusions made in the very recent paper [37].

Our axial-symmetric ansatz is thus

\[ \phi(x) \equiv \phi(z \bar{z}) \quad (6.3) \]

where we have introduced the complex coordinates \( z \) and \( \bar{z} \):

\[ z = x^2 + i x^3 \]
\[ \bar{z} = x^2 - i x^3 \quad (6.4) \]

i.e.

\[ \partial_2 = \partial + \bar{\partial}, \quad \partial_3 = i \partial - i \bar{\partial} \]
\[ d^2 z = -i dz d\bar{z} = 2 dx^2 dx^3 \quad (6.5) \]

In order to obtain the explicit form for the classical equations of motion we will start with the following useful formula [9]

\[ \mathcal{L}_4^\text{der}[\eta + w \partial \partial \phi] = -4! \det[\eta + w \partial \partial \phi] = \sum_{k=0}^{4} w^k \left( \begin{array}{c} 4 \\ k \end{array} \right) \mathcal{L}_k^\text{der}[\partial \partial \phi], \quad (6.6) \]

where we can easily work out the left hand side because the matrix \( \eta + w \partial \partial \phi \) is block-diagonal,

\[ \det[\eta + w \partial \partial \phi] = -1 + 4w \partial \partial \phi + 4w^2 \left[ \partial^2 \phi \partial \bar{\partial}^2 \phi - (\partial \partial \bar{\partial} \phi)^2 \right]. \quad (6.7) \]

Comparing this with the right hand side of (6.6) we get

\[ \mathcal{L}_1^\text{der}[\partial \partial \phi] = -4! \partial \partial \bar{\partial} \phi \]
\[ \mathcal{L}_2^\text{der}[\partial \partial \phi] = -\frac{2}{3} 4! \left[ \partial^2 \phi \partial \bar{\partial}^2 \phi - (\partial \partial \bar{\partial} \phi)^2 \right] \]
\[ \mathcal{L}_3^\text{der}[\partial \partial \phi] = \mathcal{L}_4^\text{der}[\partial \partial \phi] = 0 \quad (6.8) \]
and therefore the equation of motion with an external source $T$ is
\[ \frac{\delta S}{\delta \phi} = \sum_n n d_n \mathcal{L}^{\text{der}}_{n-1} = 2d_2 \mathcal{L}^{\text{der}}_{1} [\partial \partial \phi] + 3d_3 \mathcal{L}^{\text{der}}_{2} [\partial \partial \phi] + T = 0, \] (6.9)
where we will set $d_2 = 1/12$ in the following. In our case $T = 2\sigma \delta^{(2)}(z, \bar{z})$ so the equation of motion becomes
\[ 4\partial \partial \phi + 48d_3 \left[ \partial^2 \phi \partial^2 \phi - (\partial \partial \phi)^2 \right] = 2\sigma \delta^{(2)}(z, \bar{z}) \] (6.10)
First we can easily solve the theory for $d_3 = 0$. The axial symmetric solution is (up to a constant term)
\[ \phi(z\bar{z}) = \frac{\sigma}{4\pi} \ln z\bar{z}. \] (6.11)
In the case when $d_3 \neq 0$ we can first rewrite the equation of motion to
\[ \frac{1}{z} \left[ z\bar{z} \phi'(z\bar{z}) - 12d_3 z\bar{z} \phi'(z\bar{z})^2 \right] = \frac{\sigma}{2} \sigma \delta^{(2)}(z, \bar{z}), \] (6.12)
where the prime means a derivative with respect to $z\bar{z}$. By further integration over the disc with $z\bar{z} \leq R^2$ and using the Gauss theorem in two dimension we will arrive to
\[ \phi'(R^2) - 12d_3 \phi'(R^2)^2 = \frac{\sigma}{4\pi R^2}, \] (6.13)
which can be algebraically solve to
\[ \phi'(R^2) = \frac{1 \pm \sqrt{1 - 12d_3 \sigma}}{24d_3}. \] (6.14)
The final result can be obtained by elementary integration. We have two solutions, which is for $d_3 > 0$ defined only for $R^2 > 12d_3 \sigma / \pi$ (for $R^2 < 12d_3 \sigma / \pi$ this solution has an imaginary part)\textsuperscript{17}. We will show in the following how this can be obtained using duality in a much simpler and pure algebraical way.

The transformation of duality under the subgroup $\alpha_D(\theta)$ can be expressed in our coordinates as
\[ \begin{align*}
z_\theta &= z + 4\theta \partial \phi(z, \bar{z}) \\
\bar{z}_\theta &= \bar{z} + 4\theta \partial \phi(z, \bar{z}) \\
\phi_\theta(z_\theta, \bar{z}_\theta) &= \phi(z, \bar{z}) + 4\theta \partial \phi(z, \bar{z}) \partial \phi(z, \bar{z})
\end{align*} \] (6.15)
\textsuperscript{17}For $d_3 < 0$ the solution $\phi_-$ exhibits the Vainshtein mechanism [5] with Vainshtein radius $R^2_V = -12d_3 \sigma / \pi$. Indeed, outside and inside the Vainshtein radius we have
\[ \frac{d}{dR} \phi = \begin{cases} \frac{\sigma}{4\pi R} + O(R^{-3}), & \text{for } R > R_V, \\
\left(-\frac{\sigma}{12d_3 \pi}\right)^{1/2} + O(R), & \text{for } R < R_V. \end{cases} \]
while the remaining coordinates $x^0$ and $x^1$ are left unchanged (cf. (5.24)). Let us assume that $\phi(z, \bar{z})$ is the solution of the theory (6.11) with $d_3 = 0$. The duality transformation of $\phi(z, \bar{z})$ is then given implicitly as

$$
\begin{align*}
&z_\theta = z + \frac{\sigma \theta}{\pi \bar{z}} \\
&\bar{z}_\theta = \bar{z} + \frac{\sigma \theta}{\pi z}
\end{align*}
$$

$$
\phi_\theta(z_\theta, \bar{z}_\theta) = \frac{\sigma}{4\pi} \left( \ln z \bar{z} + \frac{\sigma \theta}{\pi z \bar{z}} \right). \quad (6.16)
$$

We have therefore

$$
\begin{align*}
z_\theta \bar{z}_\theta &= z \bar{z} + \left( \frac{\sigma \theta}{\pi} \right)^2 \frac{1}{z \bar{z}} + 2 \frac{\sigma \theta}{\pi}. \quad (6.17)
\end{align*}
$$

Let us note that for $\theta > 0$ the transformation $z \to z_\theta$ double covers the complement of a circle $z_\theta \bar{z}_\theta < 4 \frac{\sigma \theta}{\pi}$; inside of this circle $\phi_\theta$ is not defined. For $\theta < 0$ this transformation double covers the whole complex plane, the circle $z \bar{z} = - \frac{\sigma \theta}{\pi}$ is mapped to the point $z_\theta = 0$. The inversion of (6.17) which shall be inserted to the right hand side of $\phi_\theta(z_\theta, \bar{z}_\theta)$ is then

$$
\begin{align*}
z \bar{z} &= \frac{1}{2} \left( z_\theta \bar{z}_\theta - 2 \frac{\sigma \theta}{\pi} \right) \pm \sqrt{z_\theta \bar{z}_\theta \left( z_\theta \bar{z}_\theta - 4 \frac{\sigma \theta}{\pi} \right)}.
\end{align*}
$$

(6.18)

Now the duality means that

$$
S[\phi_\theta] = S_\theta[\phi] \quad (6.19)
$$

where $S$ and $S_\theta$ are the actions (without the external source term $S_{\text{int}}$) of the general Galileon theory and its $\alpha_D(\theta)$ dual respectively. In our case we take the former to be the general interacting theory (with $d_3 \neq 0$) and the latter we identify with its dual chosen in such a way that $d_3(\theta) = 0$. As we know from (5.40) such a theory can be obtained from the general one by duality transformation with $\theta = 3d_3$ and thus for this value the eq.(6.16) is expected to represent the wanted solution of (6.10, 6.13). Let us now verify that it is indeed the case.

Using the duality transformation of the derivatives (c.f. the last equation of (5.24))

$$
\partial \phi(z \bar{z}) = \bar{z} \phi'(z \bar{z}) = \partial_\theta \phi_\theta(z_\theta \bar{z}_\theta) = \bar{z}_\theta \phi'_\theta(z_\theta \bar{z}_\theta)
$$

(6.20)

we obtain

$$
\phi'_\theta(z_\theta \bar{z}_\theta) = \frac{z}{z_\theta} \phi'(z \bar{z}) = \frac{\bar{z}}{\bar{z}_\theta} \phi'(z \bar{z}) = \left( 1 + \frac{\sigma \theta}{\pi z \bar{z}} \right)^{-1} \phi'(z \bar{z}). \quad (6.21)
$$

(6.21)

Inserting this in the left hand side of (6.13) we get

$$
\phi'_\theta(z_\theta \bar{z}_\theta) - 12d_3 \phi'_\theta(z_\theta \bar{z}_\theta)^2 = \left( 1 + \frac{\sigma \theta}{\pi z \bar{z}} \right)^{-1} \phi'(z \bar{z}) - 12d_3 \left[ \left( 1 + \frac{\sigma \theta}{\pi z \bar{z}} \right)^{-1} \phi'(z \bar{z}) \right]^2
$$

$$
= \frac{\sigma}{4\pi z \bar{z}} \left( 1 + \frac{\sigma \theta}{\pi z \bar{z}} \right)^{-2} \left[ 1 + \frac{\sigma(\theta - 3d_3)}{\pi z \bar{z}} \right]. \quad (6.22)
$$

(6.22)
where in the last line we used the explicit form of \( \phi(z \bar{z}) \). Therefore for \( \theta = 3d_3 \) and expressing back \( z \bar{z} \) in terms of \( z_0 \bar{z}_0 \) we get

\[
\phi_0'(z_0 \bar{z}_0) - 4\theta \phi_0'(z_0 \bar{z}_0)^2 = \frac{\sigma}{4\pi z \bar{z}} \left( 1 + \frac{\sigma \theta}{\pi z \bar{z}} \right)^{-2} = \frac{\sigma}{4\pi z \bar{z} z_0 \bar{z}_0} \tag{6.23}
\]

which means that \( \phi_{3d_3} \) is a solution of the equation (6.13).

### 6.2 Hidden symmetries

The Galileon duality often interrelates apparently very different theories. For instance, let us assume a Galileon theory with additional \( Z_2 \) symmetry which corresponds to the intrinsic parity, namely

\[
\phi \rightarrow \phi^P = -\phi. \tag{6.24}
\]

On the Lagrangian level this symmetry requires \( d_n = 0 \) for all \( n \) odd. Under the general duality transformation such a \( Z_2 \) invariant theory might be mapped onto a dual with some \( d_{2k-1} \neq 0 \) and therefore the \( Z_2 \) symmetry ceases to be manifest in the dual theory. In this section we will study the way how the symmetries of the original Lagrangian are realized on the dual one.

Let us remind the definition of the dual action corresponding to the matrix \( \alpha \)

\[
S_{\alpha}[\phi] = S[\phi_{\alpha}] \tag{6.25}
\]

where \( \phi_{\alpha} \) is the duality transformation of the field \( \phi \) given by (cf. (4.18))

\[
x_{\alpha} = \alpha_{PP} x + \alpha_{PB} \partial \phi(x) \\
\phi_{\alpha}(x_{\alpha}) = \det(\alpha) \phi(x) + \frac{1}{2} (\alpha_{PB} \alpha_{BB} \partial \phi(x) \cdot \partial \phi(x) + 2 \alpha_{PB} \alpha_{BP} x \cdot \partial \phi(x) + \alpha_{PP} \alpha_{BP} x^2) \\
(\partial \phi)_{\alpha}(x_{\alpha}) = \alpha_{BB} \partial \phi(x) + \alpha_{BP} x. \tag{6.26}
\]

Here we have denoted\(^{18}\) \( (\partial \phi)_{\alpha} \equiv \partial \phi_{\alpha}/\partial x_{\alpha} \). Within this notation the group property of the duality transformations can be formally expressed as

\[
(Y_{\alpha})_{\beta} = Y_{\beta \cdot \alpha}, \quad Y = (x, \phi, \partial \phi). \tag{6.27}
\]

The inverse of the transformation (6.26) has the same form with the exchange

\[
\alpha \rightarrow \alpha^{-1} = \det(\alpha)^{-1} \begin{pmatrix} \alpha_{BB} & -\alpha_{PB} \\ -\alpha_{BP} & \alpha_{PP} \end{pmatrix}
\]

and can be written symbolically as

\[
Y = (Y_{\alpha})_{\alpha^{-1}} \quad Y = (x, \phi, \partial \phi). \tag{6.28}
\]

\(^{18}\)In the previous formulae and in what follows we suppress the Lorentz indices, the index \( \alpha \) in \( x_{\alpha} \) refers to the matrix \( \alpha \).
Using this notation the formula (6.25) can be rewritten in the form

\[ S[\phi] = S_a[\phi_{a-1}] \]  

(6.29)

Now any transformation of the general form

\[ Y_a \to (Y_a)' = \left( F^x(Y_a), F^\phi(Y_a), F^{\partial \phi}(Y_a) \right), \quad Y_a = (x_a, \phi_a, (\partial \phi)_a), \]  

(6.30)

where \( F^Y \), \( Y = x, \phi, \partial \phi \), are local functions\(^9\) of \( Y_a \), is realized in terms of the variables \( Y \) as

\[ Y \to Y' = ((Y_a)')_{a-1} = \left( F^x(Y_a), F^\phi(Y_a), F^{\partial \phi}(Y_a) \right)_{a-1}. \]  

(6.31)

Provided the original action is symmetric with respect to the transformation (6.31) we have using (6.29)

\[ S_a[\phi'] = S_a[((\phi_a)')_{a-1}] = S[\phi_{a-1}] = S[\phi_{a}] = S_a[\phi] \]  

(6.32)

and the dual action is invariant with respect to (6.31).

Let us now give some explicit examples of these general formulae. The first example is the intrinsic parity transformation mentioned in the introduction to this section. In this case, the formula (6.31) simplifies considerably. Let us note that the intrinsic parity transformation (6.24) can be treated as a special case of the duality transformations (5.20) with a matrix

\[ \alpha_P \equiv \alpha_S(1,-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(6.33)

Therefore \( \phi^P = \phi_{\alpha_P} \) and (6.31) has the form

\[ Y^P = \left( (Y_a)^P \right)_{a-1} = (Y_{\alpha_P^{-1} a})_{a-1} = Y_{a-1 \cdot \alpha_P} \]  

(6.34)

and the \( Z_2 \) symmetry is realized in the dual theory with action \( S_a[\phi] \) as a duality transformation associated with the matrix

\[ \beta_P(\alpha) = \alpha^{-1} \cdot \alpha_P \cdot \alpha = \text{det}(\alpha)^{-1} \begin{pmatrix} a_{PP} a_{BB} + a_{PB} a_{BB} & 2a_{PB} a_{BB} \\ -2a_{BP} a_{PP} & -a_{PP} a_{BB} - a_{PB} a_{BB} \end{pmatrix}, \]  

(6.35)

or explicitly

\[ x^P = \text{det}(\alpha)^{-1} \left[ (a_{PP} a_{BB} + a_{PB} a_{BB}) x + 2a_{PB} a_{BB} \partial \phi(x) \right] \]  

(6.36)

\[ \phi^P(x^P) = -\phi(x) - \text{det}(\alpha)^{-2} \left[ a_{PB} a_{BB} (a_{PP} a_{BB} + a_{PB} a_{BB}) \partial \phi(x) \cdot \partial \phi(x) \right. \]  

\[ +4a_{PB} a_{BB} a_{PP} a_{BB} x \cdot \partial \phi(x) + a_{PP} a_{BP} (a_{PP} a_{BB} + a_{PB} a_{BB}) x^2 \]  

\[ \left. +a_{PB} a_{BB} (a_{PP} a_{BP} + a_{PB} a_{BB}) \partial \phi(x) \right]. \]  

(6.37)

The transformation corresponding to the intrinsic parity is therefore realized in the dual theory non-linearly and non-locally as a simultaneous transformation of both space-time coordinates and fields.

\(^9\)Here \( F^\phi(Y) \) is in fact not independent because it has to be compatible with the remaining two functions in such a way that \( F^\phi(Y) = \partial^\phi \phi \).
In the same way we can find the dual realization of the Galileon symmetry (2.1). The general formula (6.31) reads in this case

$$\left(x, \phi, \partial \phi \right)' = \left(x_\alpha, \phi_\alpha + a + b \cdot x_\alpha, \left(\partial \phi\right)_\alpha + b \cdot \alpha\right)_{\alpha - 1}, \quad (6.38)$$
or explicitly

$$x' = x - \text{det} \left(\alpha\right)^{-1} \alpha_{PB} b \cdot x$$
$$\phi' = \phi + a \cdot \text{det} \left(\alpha\right)^{-1} - \frac{1}{2} \text{det} \left(\alpha\right)^{-2} \alpha_{PB} \alpha_{PB} b^2 + \text{det} \left(\alpha\right)^{-1} \alpha_{PB} b \cdot x$$
$$\left(\partial \phi\right)' = \partial \phi + \text{det} \left(\alpha\right)^{-1} \alpha_{PB} b. \quad (6.39)$$

A dual Galileon transformation is therefore superposition of space-time translation and Galileon transformation with special values of parameters, as was recognized for the duality of the type (5.24) in [16].

### 6.3 Tree level amplitudes

As we have mentioned in Section 2, the tree level amplitudes up to the 5pt one have surprisingly simple structure though they are sums of a large number of nontrivial contributions stemming from individual Feynman graphs with different topologies. Therefore large cancelations between different contribution have to occur the reason of which is not transparent on the Lagrangian level. In this subsection we will show on an elementary level how these results can be understood better with the help of the duality.

For general tree amplitudes we have

$$I = V - 1, \quad \sum_n n \alpha_n = 2I + E, \quad (6.40)$$

where $I$ and $E$ represents number of internal and external lines respectively, $V$ is number of vertices and $\alpha_n$ is number of vertices with $n$ legs; putting this together we get

$$\sum_n \left(n \alpha_n - 2\right) = E - 2. \quad (6.41)$$

It is clear that any amplitude must be represented by a linear combination of the monomials $\prod_n d_n^{\alpha_n}$ with $d_n$- independent kinematical coefficients, which carry the information on the momentum dependence of the amplitudes, explicitly\(^{20}\)

$$M(1, \ldots, E; d_n) = \sum_{\{\alpha_n\}} M_{\{\alpha_n\}}(1, \ldots, E) \prod_n d_n^{\alpha_n}. \quad (6.42)$$

Here the sum is over the sequences $\{\alpha_n\}_{n=3}^{d+1}$ which satisfy the condition (6.41) and the coefficients $M_{\{\alpha_n\}}(1, \ldots, E)$ represent the sum of Feynman diagrams with $\alpha_n$ vertices with $n$ legs. In what follows we restrict ourselves to case $d = 4$, i.e. the sum in (6.42) is over the ordered triplets $\{\alpha_3, \alpha_4, \alpha_5\}$.\(^{20}\)

\(^{20}\)Her we abbreviate $M(p_1, p_2, \ldots, p_E; d_n)$ by $M(1, \ldots, E; d_n)$ and similarly for the $d_n$ independent kinematical coefficients.
As we will see in Subsection 6.5, the tree-level $S$ matrix is invariant with respect to the duality $\alpha_D(\theta)$, therefore the amplitudes have to satisfy the following condition

$$\frac{\partial}{\partial \theta} M(1, \ldots, E; d_n(\theta)) = 0,$$

(6.43)

where $d_n(\theta)$ are given by (5.40). This gives us non-trivial constraints on the form of the coefficients $M_{\{\alpha_3,\alpha_4,\alpha_5\}}(1, \ldots, E)$. Let us now study the impact of these constraint on individual amplitudes. For $E = 3$ the only allowed sequence in (6.42) is $\{1, 0, 0\}$. Inserting (5.40) into (6.42) we get

$$M(1, 2, 3; d_n(\theta)) = \left(d_3 - \frac{1}{3} \theta\right) M_{\{1,0,0\}}(1, 2, 3),$$

(6.44)

and thus from (6.43) we get without any calculation (cf. (2.14))

$$M_{\{1,0,0\}}(1, 2, 3) = 0.$$  

(6.45)

For $E = 4$ we have in the same way

$$M(1, 2, 3, 4; d_n(\theta)) = \left(d_3 - \frac{1}{3} \theta\right)^2 M_{\{2,0,0\}}(1, 2, 3, 4) + \left(d_4 - 3\theta d_3 + \frac{1}{2} \theta^2\right) M_{\{0,1,0\}}(1, 2, 3, 4)$$

(6.46)

and nullifying the coefficient at different powers of $\theta$ we get the constraint

$$M_{\{2,0,0\}}(1, 2, 3, 4) = \frac{9}{2} M_{\{0,1,0\}}(1, 2, 3, 4)$$

(6.47)

and thus

$$M(1, 2, 3, 4; d_n) = \left(d_4 - \frac{9}{2} d_3^2\right) M_{\{0,1,0\}}(1, 2, 3, 4).$$

(6.48)

As $M_{\{0,1,0\}}(1, 2, 3, 4)$ is just the Feynman rule for the four-point vertex (cf. (2.12)) we may immediately write

$$M_{\{0,1,0\}}(1, 2, 3, 4) = 4! G(1, 2, 3).$$

(6.49)

Together this yields

$$M(1, 2, 3, 4; d_n) = 12 \left(2d_4 - 9d_3^2\right) G(1, 2, 3),$$

in agreement with (2.15). We can continue with further amplitudes and find out that the duality simplifies significantly the calculation. For instance for $E = 5$

$$M(1, 2, 3, 4, 5; d_n(\theta)) = d_3^3 M_{\{3,0,0\}}(1, 2, 3, 4, 5) + d_3d_4 M_{\{1,1,0\}}(1, 2, 3, 4, 5) + d_5 M_{\{0,0,1\}}(1, 2, 3, 4, 5)$$

(6.50)

and the duality constraints are now

$$M_{\{1,1,0\}}(1, 2, 3, 4, 5) = -\frac{24}{5} M_{\{0,0,1\}}(1, 2, 3, 4, 5)$$

$$M_{\{3,0,0\}}(1, 2, 3, 4, 5) = \frac{72}{5} M_{\{0,0,1\}}(1, 2, 3, 4, 5).$$

(6.51)
As a consequence

\[ M(1, 2, 3, 4, 5; d_n) = \left( \frac{72}{5} d_3^3 - \frac{24}{5} d_3 d_4 + d_5 \right) M_{\{0,0,1\}}(1, 2, 3, 4, 5). \]  

(6.52)

Again \( M_{\{0,0,1\}}(1, 2, 3, 4, 5) \) is just the Feynman rule

\[ M_{\{0,0,1\}}(1, 2, 3, 4, 5) = -5! G(1, 2, 3, 4) \]  

(6.53)

and we conclude without calculations (cf. (2.16))

\[ M(1, 2, 3, 4, 5; d_n) = -24 \left( 72 d_3^3 - 24 d_3 d_4 + 5 d_5 \right) G(1, 2, 3, 4). \]  

(6.54)

As a last example we take \( E = 6 \), the computer calculation of which though possible gives rather lengthy and untransparent final output so it is difficult to reveal any regular structure hidden in it. As we will see also here the duality helps considerably.

There are four kinematical factors in this case corresponding to the sequences \{4, 0, 0\}, \{0, 2, 0\}, \{2, 1, 0\} and \{1, 0, 1\}. The duality constraints are

\[
\begin{align*}
M_{\{4,0,0\}}(1,...,6) &= \frac{81}{4} M_{\{0,2,0\}}(1,...,6) \\
M_{\{2,1,0\}}(1,...,6) &= -9 M_{\{0,2,0\}}(1,...,6) \\
M_{\{1,0,1\}}(1,...,6) &= 0,
\end{align*}
\]  

(6.55)

and thus when inserted to the formula (6.42) we get finally

\[
M(1,...,6; d_n) = \left( \frac{81}{4} d_3^4 + d_4^2 - 9 d_3 d_4 \right) M_{\{0,2,0\}}(1,...,6) \\
= \left( d_4 - \frac{9}{2} d_3 \right)^2 M_{\{0,2,0\}}(1,...,6).
\]  

(6.56)

Here \( M_{\{0,2,0\}} \) is the sum of graphs with two four-point vertices connected by one propagator and can be therefore written in the form

\[
M_{\{0,2,0\}}(1,...,6) = -16 \sum_{\sigma \in S_6} G(\sigma(1), \sigma(2), \sigma(3)) G(\sigma(4), \sigma(5), \sigma(6)) (p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)})^2,
\]  

(6.57)

where we sum over the permutations of the external momenta.

Of course these results are not surprising. The tree-level \( S \) matrix being an invariant of the duality subgroup \( \alpha_D(\theta) \) can depend on \( d_n \) only as a function of the the basic \( \alpha_D(\theta) \) duality invariants \( I_4, I_5 \) given by (5.41). Because these invariants can be interpreted as \( d_4 \) and \( d_5 \) in a dual theory with \( d_3 = 0 \), the tree-level amplitudes must have the form

\[
M(1,...,E; d_k) = \sum_{\{m,n\} \geq 0} M_{\{0,m,n\}}(1,...,E) I_4^m I_5^n
\]  

(6.58)

where the summation over \( m \) and \( n \) must fulfill (6.41), i.e.

\[
2m + 3n = E - 2.
\]  

(6.59)
This general structure can be easily recognized in all the above examples. Let us note that in general case
\[ I_4^m I_5^n = \sum_{\{\alpha_k\}} c_{\{\alpha_k\}}^{mn} \prod_k d_k^{\alpha_k} \]
(6.60)
where \( c_{\{\alpha_k\}}^{mn} \) are rational numbers. Then comparing the coefficients (6.42) and (6.58) we get the above discussed constraints on the individual contributions to the amplitude in a general form
\[ M_{\{\alpha_k\}}(1, \ldots, E) = \sum_{m,n} c_{\{\alpha_k\}}^{mn} M_{\{0,m,n\}}(1, \ldots, E). \]
(6.61)

As we have illustrated above, the tree-level amplitudes are invariants of the subgroup \( \alpha_D(\theta) \) but also their transformation properties with respect to the scalings \( \alpha_S(\lambda) \) and more generally \( \alpha_S(\lambda, \kappa) \) are transparent. Let us remind that, under the \( \alpha_S(\lambda) \) the couplings \( d_n \) scale according its dimension (cf. (5.18) with \( \Delta = (d - 2)/2 \), which is the canonical dimension of the field \( \phi \))
\[ d_n(\alpha_S(\lambda)) \equiv d_n(\lambda) = \lambda^{-1/2(d+2)(d-2)} d_n, \]
(6.62)
which just corresponds to the re-scaling of the units. Note that, for \( d \) even we can generalize the above scaling also to \( \lambda < 0 \). From the homogeneity of the tree\(^{21}\) amplitudes
\[ M(\lambda p_1, \ldots, \lambda p_n; \lambda^{\dim d_k} d_k) = \lambda^{\dim M(p_1, \ldots, p_n; d_k)} M(p_1, \ldots, p_n; d_k) \]
\[ = \lambda^{d-n(d-2)/2} M(p_1, \ldots, p_n; d_k), \]
(6.63)
it follows that two amplitudes with \( d_n \) and \( d_n(\lambda) \) are connected by
\[ M(p_1, \ldots, p_n; d_k(\lambda)) = \lambda^{d-n(d-2)/2} M(\lambda^{-1} p_1, \ldots, \lambda^{-1} p_n; d_k). \]
(6.64)
At tree\(^{22}\) level on the other hand
\[ M(\lambda^{-1} p_1, \ldots, \lambda^{-1} p_n; d_k) = \lambda^{-2(n-1)} M(p_1, \ldots, p_n; d_k), \]
(6.65)
as we will show in the Section 6.6 and therefore we get finally
\[ M(p_1, \ldots, p_n; d_k(\lambda)) = \lambda^{\frac{1}{2}(d+2)(2-n)} M(p_1, \ldots, p_n; d_k). \]
(6.66)
Therefore not only that it is sufficient to know the amplitudes for some representant of the group orbit of \( \alpha_D(\theta) \) in the theory subspace \( D^{(1)}_{d+1} \) but we can also travel between different (but qualitatively similar) orbits using the formula (6.66).

\(^{21}\)Note that, at the loop level, we have additional dependence of the amplitudes on additional dimensionfull parameters, namely on the counterterm couplings as well as on the renormalization scale.

\(^{22}\)As we will see in the subsequent sections, the loop amplitudes have higher degree of homogeneity with respect to re-scaling the momenta.
6.4 Classification of the Galileon theories

As we have shown, some physical consequences of the Galileon theories are not directly visible from the Galileon Lagrangian. This concerns e.g. the cancelations of the various contributions to the tree-level amplitudes as well as the hidden $\mathbb{Z}_2$ symmetry of the Galileon action discussed in the previous sections. However, as was seen in the latter case, such properties are usually shared by the theories which are connected by the group of duality transformations (or by some of its subgroup). It is therefore important to describe the equivalence classes of the Galileon theories with respect to the duality.

In what follows we will classify in this sense the Galileon theories in $d = 3$ and 4. We will restrict ourselves to the theory subspace $D^{(1)}_d$ with $d_2 = 1/4$ and $1/12$ respectively and we will consider only the dualities corresponding the upper triangular matrices $\alpha$ which make sense also in the quantum case.

6.4.1 Galileons in $d = 4$

The properties of the theory which belongs to the theory subspace $D^{(1)}_5$ with constants $d_3, d_4$ and $d_5$ are governed by the invariants of the duality transformation $I_4$ and $I_5$ given by (5.41). Let us remind that $I_n$ represents the value of the constant $d_n$ in the theory which is dual to the original one but satisfies the condition $d_3 = 0$. We have the following cases (see Fig. 2):

- $I_4 = I_5 = 0$, in this case the theory is dual to free theory
- $I_5 = 0$, in this case the theory is $\mathbb{Z}_2$ invariant (it is dual to the theory with $d_3 = d_5 = 0$). The $\mathbb{Z}_2$ invariance is realized by (6.36) with $\alpha = \alpha_D (-3d_3)$. The only non-zero amplitudes are those with even number of legs
- $I_4 = 0$, the theory is a dual to quintic Galileon (where $d_3 = d_4 = 0$)
- both $I_4, I_5 \neq 0$, the theory is dual to $d_3 = 0, d_{4,5} \neq 0$

Let us now summarize the cases for which a concrete coupling $d_n$ can be removed by duality transformation. The following conditions are easily derived as the conditions for the existence of the solutions of the equations $d_n(\theta) = 0$ with respect to $\theta$ (cf. (5.40))

- Every theory is dual to theory with $d_3 = 0$
- $I_4 < 0$, then theory is dual to just two theories with odd interactions where $d_4 = 0$
  (this can be achieved by duality transformation corresponding to $\alpha_D (\theta_\pm)$ for $\theta_\pm = 3d_3 \pm \sqrt{-2I_4}$)
- $I_4 > 0$, then there is no dual with $d_4 = 0$
- For $(8I_4)^3 + (15I_5)^2 > 0$ theory is dual to exactly one theory with $d_5 = 0$
- For $(8I_4)^3 + (15I_5)^2 < 0$ theory is dual to exactly three theories with $d_5 = 0$
The invariants $I_4$ and $I_5$ scale as $d_4$ and $d_5$, namely

$$I_4(\lambda) = \lambda^{-6} I_4, \quad I_5(\lambda) = \lambda^{-9} I_5.$$  \hfill (6.67)

Therefore by means of the scaling $\alpha_S(\lambda)$ we can always arrange either $I_4 = \pm 1$ or $I_5 = 1$ (with $\lambda < 0$ when necessary). To summarize, non-trivial theories (i.e. those which are not connected by dualities $\alpha_D(\theta)$ or $\alpha_S(\lambda)$) are

- $I_4 = I_5 = 0$ — free theory
- $I_4 = \pm 1$, $I_5 = 0$ — $Z_2$ invariant theory (only even amplitudes are non-zero)
- $I_5 = 1$, $I_4 = 0$ — quintic Galileon
- $I_5 = 1$, $I_4 \neq 0$ — general case

### 6.4.2 Galileons in $d = 3$

The situation in three dimension is even simpler. There is only one invariant of the $\alpha_D(\theta)$ duality

$$I_4 = d_4 - \frac{9}{8} d_3^2$$  \hfill (6.68)

From the previous it follows readily

$$d_3(\lambda) = \lambda^{-\frac{2}{5}} d_3, \quad d_4(\lambda) = \lambda^{-5} d_4, \quad I_4(\lambda) = \lambda^{-5} I_4$$  \hfill (6.69)

For $I_4 < 0$ we can remove $d_4$ by a duality $\alpha_D(\theta_\pm)$ with

$$\theta_\pm = \frac{3}{4} d_3 \pm \frac{1}{2} \sqrt{-2I_4}$$  \hfill (6.70)

Provided $I_4 > 0$ there is no $\alpha_D(\theta)$ dual with $d_4(\theta) = 0$ and naively there is no possibility to change the sign of $I_4$ by simple scaling because $\lambda < 0$ is not allowed in odd dimension for theory with $d_{2k-1} \neq 0$. However, we can first remove $d_3$ by $\alpha (3d_3/2)$ and only then scale with $\lambda < 0$ to arrange $I_4 = 1$, because there is no odd vertex in the dual theory. This leads to the following classification

- $I_4 = 0$ — free theory
- $I_4 = 1$ — $Z_2$ invariant theory

To summarize, up to the above described $\alpha_D(\theta)$ and $\alpha_S(\lambda)$ dualities there is only one non-trivial Galileon theory in three dimension the only nonzero amplitudes of which are the even ones.
Figure 2: The surfaces $I_2 = 0$ (cylindrical one corresponding to duals of the quintic Galileon) and $I_3 = 0$ ($Z_2$ symmetric Galileons) in the theory space $D_5^{(1)}$ with $d_2 = 1/12$ fixed. The intersection of these surfaces corresponds to the duals of a free theory. Both surfaces are invariant with respect to the scaling.

6.5 Duality of the $S$ matrix

On the quantum level the most important object is the $S$ matrix. In this section we will discuss its properties with respect to the Galileon duality.

Let us first briefly remind the well known equivalence theorem which makes a statement about the invariance of the $S$ matrix with respect to the field redefinitions (see e.g. [24]). The $S$ matrix can be obtained by means of LSZ formulae from the generating functional $Z[J]$ of the Green functions which can be expressed in terms of the functional integral.

$$Z[J] = \int D\phi \exp \left( \frac{i}{\hbar} S[\phi] + \frac{i}{\hbar} \langle J\phi \rangle \right).$$

In this formula we tacitly assume appropriate regularization which preserves the properties of the action with respect to the Galileon symmetry and duality transformations. The action can be expanded in powers of $\hbar$

$$S[\phi] = \sum_{n=0}^{\infty} \hbar^n S_n[\phi] \equiv S_0[\phi] + S_{CT}[\phi],$$

where $S_0[\phi]$ is the Galileon Lagrangian (2.3). The higher order terms $S_n[\phi]$ in the expansion (6.72) summed up in $S_{CT}[\phi]$ represent the counterterms which are necessary in order to renormalize the UV divergences stemming from the $n$-loop graphs. The discussion of these
counterterms we postpone to the Section 6.6, here we only stress that, because of the derivative structure of the Galileon interaction vertices, the counterterms $S_n[\phi]$ have more derivatives per field than the basic action $S_0[\phi]$, and that under our assumptions on the regularization the counterterms should respect the Galileon symmetry.

In the functional integral the field $\phi$ is a dummy variable and can be freely changed by means of the field redefinition $\phi \rightarrow F[\phi]$ according to

$$Z[J] = \int \mathcal{D}\phi \det \left( \frac{\delta F[\phi]}{\delta \phi} \right) \exp \left( \frac{i}{\hbar} S[F[\phi]] + \frac{i}{\hbar} \langle JF[\phi] \rangle \right),$$

(6.73)

where we have abreviated

$$2^{23} \langle \cdot \rangle \equiv \int d^d x \langle \cdot \rangle.$$

This should be compared with the generating functional $Z_F[J]$ in the theory with the action $S_F[\phi] \equiv S[F[\phi]]$

$$Z_F[J] = \int \mathcal{D}\phi \exp \left( \frac{i}{\hbar} S_F[\phi] + \frac{i}{\hbar} \langle J\phi \rangle \right)$$

(6.74)

Ignoring the Jacobian on the right hand side of (6.73) for a moment, the sufficient condition for the perturbative equivalence of the $S$ matrices in the theories with actions $S[\phi]$ and $S_F[\phi]$ is that the Fourier transforms of the Green functions of the operators $\phi(x)$ and $F[\phi](x)$ have the same one-particle poles at $p^2 = 0$ up to a simple re-scaling of the residues. This is achieved provided $F[0] = 0$ and

$$\langle 0| F[\phi](0) | p \rangle = Z_F \langle 0| \phi(0) | p \rangle$$

(6.75)

with $Z_F \neq 0$. This requirement is respected by the Galileon duality transformation which are represented by the upper triangular matrices with $\alpha_{PP} = 1$. To prove this, it is sufficient to investigate the dualities corresponding to $\alpha_D(\theta)$ and $\alpha_S(1, \kappa)$ separately because of the decomposition (5.26). In the former case we have

$$F[\phi](x) \equiv \phi_\theta(x) = \phi(x) + \theta \partial \phi(x) \cdot \partial \phi(x) + O(\theta^2, \phi^4)$$

(6.76)

and therefore

$$Z_F = 1 + O(\hbar)$$

(6.77)

while in the latter case we trivially get $Z_F = \kappa$. Thus the only obstruction which prevents us to make a statement on the equivalence of the on-shell $S$ matrices in both theories also at the loop level is the possible nontrivial functional determinant on the right hand side of

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23 In what follows we will often use this notation without further comments unless it shall lead to misinterpretation.

24 The $O(\hbar)$ part stems from the contributions of the terms bilinear and higher in the derivatives of the field $\phi$ on the right hand side of (6.76). These terms start to contribute to $Z_F$ only at the loop level.

25 The case of $\alpha_S(\lambda, \kappa)$ is more complicated, because

$$F[\phi](x) = \lambda \kappa \phi(\lambda^{-1} x), \quad \int d^d x e^{ip \cdot x} \lambda \kappa \phi(\lambda^{-1} x) = \lambda^{d+1} \kappa \int d^d x e^{ip \cdot x} \phi(x)$$

(6.78)

and the behaviour of the Green functions under the re-scaling of the momenta is governed by the renormalization group. This is the rationale for the constraint $\alpha_{PP} = 1$. However, at the tree level this simplifies to scaling respecting the canonical dimensions.
Its actual value depends on the regularization. In what follows we will show that using dimensional regularization the functional determinant equals to one for the duality transformations $\alpha_D (\theta)$ (the case $\alpha_S (1, \kappa)$ is of course trivial).

For the infinitesimal $\theta$ we can expand the functional determinant according to (6.76) as

$$\det \left( \frac{\delta \phi(y)}{\delta \phi(x)} \right) = 1 + 2 \theta \text{Tr} (\partial \phi(x) \partial) .$$

The trace can be further expressed in a standard way (introducing the operators $\hat{X} \equiv x$, $\hat{K} \equiv -i \partial$ and their eigenvectors $|x\rangle$ and $|k\rangle$ respectively) as

$$\text{Tr} (\partial \phi(x) \partial) = i \int d^d z \hat{X} \cdot \hat{K} |x\rangle \langle k| \int d^d z \hat{K} |x\rangle \langle k| (k |x\rangle \langle k|) = i \int d^d z \hat{X} \cdot \hat{K} |x\rangle \langle k| (k |x\rangle \langle k|)$$

The first factor equals to zero for well behaved $\phi$ while the second one vanishes within the the dimensional regularization. We have thus

$$\det \left( \frac{\delta \phi(y)}{\delta \phi(x)} \right) = 1 + O(\theta^2) .$$

For the finite transformation we can use the fact that the transformation forms a one-parametric group and thus

$$\phi_{\theta+\xi} [\phi] = \phi_{\theta} [\phi_{\xi}] = \phi_{\xi} [\phi_{\theta}] ,$$

which implies

$$\det \left( \frac{\delta \phi_{\theta+\Delta \theta}}{\delta \phi_{\theta}} \right) = 1 + O((\Delta \theta)^2) .$$

Using the formula

$$\frac{\delta \phi_{\theta+\Delta \theta}(x)}{\delta \phi(y)} = \int d^d z \frac{\delta \phi_{\theta+\Delta \theta}(x)}{\delta \phi_{\theta}(z)} \frac{\delta \phi_{\theta}(z)}{\delta \phi(y)}$$

we get formally

$$\det \left( \frac{\delta \phi_{\theta+\Delta \theta}}{\delta \phi} \right) = \det \left( \frac{\delta \phi_{\theta+\Delta \theta}}{\delta \phi_{\theta}} \right) \det \left( \frac{\delta \phi_{\theta}}{\delta \phi} \right)$$

and therefore

$$\frac{\partial}{\partial \theta} \ln \det \left( \frac{\delta \phi_{\theta}}{\delta \phi} \right) = 0 ,$$

By means of integration from 0 to $\theta$ this leads to the desired statement\footnote{Let us note that without the knowledge that the Jacobian of this transformation is equal one, in a standard way, we can introduced the ghost fields which would reproduce the studied determinant. At the end, however, one would find that propagators of such ghosts are proportional to 1, and thus every integration over ghost loop with momentum $l$ is of the type:

$$\int d^{d-2} l \times \text{Polynomial}(l) = 0$$

which is true for the dimensional regularization.}.

$$\det \left( \frac{\delta \phi_{\theta}}{\delta \phi} \right) = 1 .$$
The on-shell $S$ matrices in theories with actions $S_F[\phi]$ and $S[\phi]$ are therefore formally equivalent for the above duality transformations. This statement, however, must be taken with care. The first reason is that the counterterm part $S_{CT}[\phi]$ of the action has not the form of the Galileon Lagrangian and transforms therefore highly nontrivially with respect to the duality (note that the duality transformation is in general non-local and involves infinite number of terms, cf. Appendix A). The second reason is that though we have formally established equivalence of the on-shell $S$ matrices, the off-shell Green functions stay to be different in both theories. Indeed, we have in fact only proved that Green functions of operators $\phi(x)$ in original theory and those of operators $F[\phi](x)$ in the dual theory coincide. Therefore, the recursive construction of the counterterms in the dual theory starting with the dual basic action $S_0[F[\phi]]$ will lead to counterterm action $S_{CT}^F[\phi]$ different from $S_{CT}[F[\phi]]$. On the other hand, the counterterms from $S_{CT}[F[\phi]]$ will be sufficient to cancel the divergences of the on-shell amplitudes in the dual theory.

### 6.6 Counterterms

From the results of the previous section it seems possible to use the duality relations also at the quantum level. However, this is true only provided the quantum level makes sense. Starting with the basic (i.e. the tree-level) Galileon Lagrangian and choosing an appropriate regularization prescription which preserves the Galileon symmetry (in what follows we use exclusively a dimensional regularization), we can construct one-loop diagrams. Such diagrams will be divergent and will thus need to be renormalized by the counterterms. From a simple dimensional consideration it is clear (and it will be explicitly shown below) that it is not possible to create such counterterms using the basic tree-level Lagrangian. We will thus have to add qualitatively new terms in the action constrained in their form only by the Galileon symmetry. In fact at any order of the loop expansion an infinite tower of new counterterms is necessary. This is of course nothing new, such a mechanism is well studied in many different effective theories e.g. in the Chiral perturbation theory (ChPT) \[38, 39\]. The problem of construction of higher order Lagrangians (e.g. next-to-leading-order and next-to-next-to-leading order as is the nowadays status in ChPT) is the problem by itself. Here we will merely classify the order (i.e. the degree of homogeneity in the external momenta or the number of derivatives) of the graphs and the corresponding counterterms at the given loop level.

Let us start with the Weinberg formula \[40\] in $d$-dimension for the number of derivatives in the counterterm for a given graph with $L$ loops and vertices $V$ with $d_V$ derivatives:\[27\]

$$D = 2 + (d - 2)L + \sum_V (d_V - 2). \tag{6.88}$$

The number of external legs $E$ and internal lines $I$ is connected via

$$\sum_V n_V = 2I + E, \tag{6.89}$$

---

\[27\] This formula holds provided the dimensional regularization or any other regularization without dimensionfull cutoff parameter is used to regulate the UV divergences. Note also that in our case of massless theory $D$ is also the superficial degree of divergence of the given graph.
where \( n_V \) is the number of legs for the given vertex \( V \). We can also simply extract number of loops

\[
L = I - V + 1.
\]  

(6.90)

Together with the previous relation this leads to

\[
E = 2 + \sum_V (n_V - 2) - 2L,
\]  

(6.91)

and thus

\[
D - 2(E - 1) = (d + 2)L + \sum_V (d_V - 2(n_V - 1)).
\]  

(6.92)

Let us now define an index of general vertex \( \delta_V \) as a surplus of the derivatives for the general vertex in comparison with the basic Lagrangian, namely

\[
\delta_V = d_V - 2(n_V - 1)
\]  

(6.93)

(i.e. for all the vertices of the basic Lagrangian \( \delta_V = 0 \)). In terms of such a defined index we can rewrite the formula (6.92) in the form

\[
\delta_{CT} = (d + 2)L + \sum_V \delta_V \equiv \delta_{\Gamma}
\]  

(6.94)

the right hand side of which defines the index \( \delta_{\Gamma} \) of a \( L \)-loop graph \( \Gamma \) built from the vertices with indices \( \delta_V \). This formula is in fact an Galileon analog of the Weinberg formula for ChPT and represents thus the connection between the loop expansion and expansion in the diagram index \( \delta_{\Gamma} \), which is the order of the diagram homogeneity in momenta (modulo logs) relative to tree-level diagrams constructed from the basic Lagrangian.

Note that according to the formula (6.94) each loop contributes with an additional \( d + 2 \) term in the counterterm index \( \delta_{CT} \). This means that the counterterms induced by the loops have \( \delta_{CT} > 0 \) and therefore (because for the vertices of the basic Lagrangian \( \delta_V = 0 \)) they must be different form the terms of the basic Lagrangian. In other words the basic Galileon Lagrangian is not renormalized by loops. This proves what is often meant in the literature as the non-renormalization theorem [11, 12, 13].

Let us note that similarly to the Weinberg formula for the ChPT, the formula (6.94) itself cannot be used for the proof of the generalized renormalizability. Note that the restriction \( \delta_{CT} = N = \text{const.} \) constrains only the number of derivatives \( d \) according to

\[
d = 2(n - 1) + N,
\]  

(6.95)

but it does not constrain the number \( n \) of fields. In the case of ChPT the additional principle is a chiral symmetry which ensures that the infinite number of counterterms differing by the number of fields at each order combine into a finite number of chiral invariant operators. In our case we have only the Galileon symmetry at our disposal. As we have discussed above, it tells us only that the most general Galileon invariant Lagrangian is built from
the building blocks $\partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_k} \phi$ where $k \geq 2$, therefore the general counterterm with $n$ legs satisfying (6.95) has the general form

$$\mathcal{L}_{CT}^{(n)} = \sum_{l, k_i \geq 2, \sum k_i = 2(n-1)+N} c^{(l)}_{k_1 k_2 \ldots k_n} T^{\mu_1 \mu_2 \ldots \mu_{k_1} \ldots \mu_{k_2} \ldots \mu_{k_n}} \prod_{j=1}^n \partial_{\mu_{l_1}} \partial_{\mu_{l_2}} \ldots \partial_{\mu_{l_k}} \phi. \quad (6.96)$$

with couplings $c^{(l)}_{k_1 k_2 \ldots k_n}$ and Lorentz invariant tensors $T^{\mu_1 \mu_2 \ldots \mu_{k_1} \ldots \mu_{k_2} \ldots \mu_{k_n}}$. Though for $n$ fixed we have finite number of terms, the number $n$ increases to infinity and already at the one loop level (where $N = d + 2$) we get infinite number of independent terms.

### 6.7 Examples of one-loop order duality

In the previous sections we have explicitly calculated the tree-level scattering amplitudes of the Galileon fields up to six particles. The non-trivial results start with the four-point scattering. In this section we will focus on this process and will study it at one-loop order. Of course, as mentioned above, such a full calculation would necessary need inclusion of so-far undefined Lagrangian $\mathcal{L}_{CT}^{(4)}$, which would play a role of counterterms in this process. However, our main motivation is to explicitly show that the duality is not spoiled at the quantum level (i.e. by loop contributions) at least for the graphs with the vertices from the basic Galileon Lagrangian. We will thus first calculate dimensionally regularized individual contributions to 4-pt scattering at one-loop order in one Galileon theory and show that the final result is connected with other Galileon theory connected by duality.

In the Table 6.7 we summarize the one-loop diagrams to be calculated and their corresponding divergent parts in $d = 4$ dimension (the full results in $d$ dimension for $A_1$–$A_6$ are summarized in Appendix C). Here we have used the standard Mandelstam variables for four-point scattering:

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2 \quad (6.97)$$

where all momenta are ingoing and on-shell so that $s + t + u = 0$. The singularity in $d = 4$ dimension is given by

$$\Lambda = \frac{1}{(4\pi)^2} \frac{1}{d - 4}. \quad (6.98)$$

Due to specific form of the of 3-pt vertex in the Galileon theory which can be rewritten in the form

$$\mathcal{V}_3(q_1, q_2, q_3) = 6d_3 \left[ (q_1 \cdot q_2)q_3^2 + (q_1 \cdot q_3)p_2^2 + (q_2 \cdot q_3)p_1^2 \right] \quad (6.99)$$

the contributions $A_7$ and $A_8$ (corresponding to graphs for which $\mathcal{V}_3$ is one of the two vertices of a bubble) are zero also for general $d$. Indeed, with external momenta on shell, the only term of (6.99) which could contribute is schematically $(p_{\text{ext}} \cdot l)(p_{\text{ext}} - l)^2$ where $p_{\text{ext}}$ is one of the external momenta and $l$ is the loop momentum. Therefore the $(p_{\text{ext}} - l)^2$ factor cancels one of the bubble propagators which thus degenerate in a massless tadpole and the latter is zero in dimensional regularization.
Table 1: One loop graphs contributing to the 4pt amplitude and their divergent parts.

Summing the diagrams together, we get that the divergent part of the amplitude for the 4-pt galileon-scattering at the one-loop order is

\[ A_{\text{div}} = \sum_i A_{\text{div}}^i = -\frac{3}{40} \Lambda (9d_3^2 - 2d_4)^2 (s^2 + t^2 + u^2)^3 = -\frac{3}{10} \Lambda I_4^2 (s^2 + t^2 + u^2)^3. \] (6.100)

Note that the degree of homogeneity in external momenta is in accord with the formula (6.94). As we have expected, the singular part (and in fact also the complete result (C.8), cf. Appendix C) depend on the \( \alpha_D (\theta) \) duality invariant \( I_4 \) which illustrates the conclusions of Section 6.5 that \( \alpha_D (\theta) \) dual theories produce the same \( S \)-matrices. This offers also another possibility how to use the duality relations similar to that we have discussed for the tree amplitudes in Section 6.3. Because \( I_4 \) is the coupling \( d_4 \) in the dual Galileon theory with new constants \( d_i (\theta^*) \) such as \( d_3 (\theta^*) = 0 \) we can effectively eliminate 3pt vertices by passing to this dual theory. The only diagram which is left to calculate in such a dual theory is \( A_4 \) which simplifies the calculations considerably.

Let us present another simple example of the one-loop calculation concerning the self-energy correction for the Galileon field. The relevant graph is depicted in Fig.3 and the
explicit result for the divergent part reads (see Appendix C for the complete result)

\[ \Sigma^{\text{div}}(p) = 9\Lambda d_3^4 \left( p^2 \right)^4, \]  

\[ \text{cf. also [41]. Therefore, on one hand, in Galileon theories with } d_3 \neq 0 \text{ we need a counterterm } L_{CT}^{(2)} \text{ of the general form (6.96) to renormalize this divergence. The corresponding Feynman rule reads} \]

\[ V_2(p_1) = \mu^{d-4} \left( -9\Lambda d_3^4 + C^r(\mu) \right) (p^2)^4, \]  

where \( \mu \) is the dimensional regularization scale which is necessary to restore the canonical dimension of the loop integration and where \( C^r(\mu) \) is a linear combination of the finite parts of the counterterm couplings \( c_{k_1k_2}^{(j)} \) in (6.96) renormalized at scale \( \mu \). On the other hand, in the above mentioned dual theory with \( d_3(\theta^*) = 0 \) such a divergence does not occur. This is consequence of the fact that off-shell Green functions are not invariants with respect to the duality as discussed in Section 6.5.

Extreme example of such non-invariance of the counterterms is the case of the free theory and some of its \( \alpha_D(\theta) \) duals. While the free theory does not need any counterterm of the above type, its dual always does. However, as far as the \( S \) matrix is concerned, no counterterms are needed for the graphs with the vertices from the basic dual Lagrangian because these graphs have to combine into the trivial \( S \) matrix of the original free theory which is trivially divergence-free. Therefore, the contribution of the divergent part of (6.102) and analogical counterterms (which are needed to renormalize the divergent subgraphs in the Table 6.7) has to cancel in the final result. This is, however, not true for the finite part of the counterterms the couplings which are in principle independent. E.g. the renormalization of the bubble subgraph in the graphs \( A_3 \) in Table 6.7 brings about the contribution (for \( d \to 4 \))

\[ A_3^{CT} = -9iC^r(\mu) \left( s^6 + t^6 + u^6 \right). \]  

Therefore, the only possibility how to recover the free theory \( S \) matrix also in the dual theory with counterterms is to set at some scale all the renormalized counterterm coupling constants equal to zero. Because the couplings run with the renormalization scale, this might seem to be insufficient, because at another scale the finite parts of the counterterms are in general nonzero. However, because in the amplitude all the contributions of the divergent parts of the counterterms cancel, in the same way are also canceled all the

\[ ^{28}\text{Note that any such dual has } d_3 = -\theta/3 \neq 0. \]
contributions stemming from the changes of the counterterms couplings with change of the renormalization scale\textsuperscript{29}.

7. Summary

In this paper we have studied the duality transformations of the general Galileon theories in \( d \) dimensions. According to the interpretation of the Galileon action as the generalized WZW term of the effective theory describing a Goldstone boson of the spontaneous breakdown of the Galileon group down to the \( d \)-dimensional Poincaré group we have studied the most general coordinate transformations on the corresponding coset space. The requirement that such a general transformation acts linearly on the basic building blocks of the Galileon Lagrangian (and therefore it represents a duality transformation) constraints the form of the transformation uniquely up to four free parameters. Under composition these duality transformations form a group which can be identified with \( GL(2, \mathbb{R}) \). All the up to now known Galileon dualities can be identified as special elements (or one-parametric subgroups) of this duality group. We have also studied its action on the space of the Galileon theories and found a basis of the independent invariants of one of its most interesting one-parametric subgroups denoted \( \alpha_D(\theta) \). This subgroup is represented in the space of fields as a field redefinition which can be understood both as a simultaneous space-time coordinates and field transformation or as a non-local change of the fields which includes infinite number of derivative dependent terms. We then have studied the applications of the duality group on concrete simple examples. We have shown that we can use it to generate classical solution of the interacting Galileon theory from the solution of the more simple one even when the Galileon is coupled to the local external source. As a second example we have demonstrated the usefulness of the Galileon duality for calculations of the tree on-shell scattering amplitudes and for finding the relations between the contributions of the apparently very different Feynman graphs with completely different topologies. We have also established the dual formulation of the additional symmetries of the Lagrangian and found that the \( \mathbb{Z}_2 \) symmetry is realized non-linearly and non-locally in the dual theory. As another example we have classified the equivalence classes (with respect to the duality subgroup \( \alpha_D(\theta) \) combined with scaling) of the Galileon theories in three and four dimensions and found e.g. that there is up to the above dualities only one nontrivial interacting theory in three dimensions which exhibits the \( \mathbb{Z}_2 \) symmetry. Then we have discussed the transformation properties of the \( S \) matrix on the loop level and established its formal invariance within the dimensional regularization, though only the tree-level part of the complete action with counterterms transforms nicely under the duality field redefinition and as we have discussed on a concrete example of the one-loop four-point on-shell amplitude, due to the counterterms the duality is not completely straightforward. It rather holds on the regularized level for the loop graphs with vertices form the basic tree-level Lagrangian. We have also touched the problem of the counterterms classification based on a generalization

\textsuperscript{29}Note that within dimensional regularization, the coefficient at the \( \ln(\mu/\mu') \) term in the formula for the running of the renormalized one-loop coupling coincide with the coefficient at the \( \Lambda \) in formula for the bare coupling.
of the Weinberg formula and with help of the latter we discussed the non-renormalization theorem.

Note added: After this work was completed two works [37, 36] closely connected with the topic studied in this paper appeared. Both these papers concern the properties of the one-parametric duality subgroup denoted as $\alpha_D(\theta)$ in our notation and partially overlap with our results.

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A. Bottom up construction of the duality subgroup $\alpha_D(\theta)$

In this Appendix we get more elementary treatment of the Galileon duality corresponding to the subgroup $\alpha_D(\theta)$. In fact this was the way we had started to think about the Galileon duality.

Let us assume an infinitesimal field transformation

$$\phi \rightarrow \phi + \theta \partial \phi \cdot \partial \phi,$$  \hspace{1cm} (A.1)

where $\theta$ infinitesimal parameter. The infinitesimal change of $\phi$ can be also understood as an action of the following operator (which is defined on the space of the functionals $F[\phi]$ of the field $\phi$)

$$\delta_{\theta} \equiv \theta \left( \delta_{\phi} \frac{\delta}{\delta_{\phi}} \right) = \theta \left( \partial_{\phi} \cdot \partial_{\phi} \frac{\delta}{\delta_{\phi}} \right),$$  \hspace{1cm} (A.2)

on a special functional $F[\phi](x) = \phi(x)$. Here and in what follows we again abbreviate $\langle \cdot \rangle \equiv \int d^d x \langle \cdot \rangle$.

Acting by the operator $\delta_{\theta}$ on the Galileon action (cf. (2.3))

$$S[\phi] = \sum_{n=1}^{d+1} d_n \left\langle \phi L_{n-1}^{\text{der}} \right\rangle,$$  \hspace{1cm} (A.3)

we get

$$\delta_{\theta} S[\phi] = \theta \left( \partial_{\phi} \cdot \partial_{\phi} \frac{\delta S[\phi]}{\delta_{\phi}} \right) = \sum_{n=2}^{d+1} \theta n d_n \left\langle \partial_{\phi} \cdot \partial_{\phi} L_{n-1}^{\text{der}} \right\rangle,$$

which can be rewritten with help of the formula (2.11) to the form

$$\delta_{\theta} S[\phi] = -2 \sum_{n=3}^{d+1} \theta \frac{n-1}{n} (d - n + 2) d_{n-1} \left\langle \phi L_{n-1}^{\text{der}} \right\rangle = \sum_{n=3}^{d+1} \theta \delta d_n \left\langle \phi L_{n-1}^{\text{der}} \right\rangle,$$  \hspace{1cm} (A.4)

with

$$\delta d_n = -2 \frac{n-1}{n} (d - n + 2) d_{n-1},$$  \hspace{1cm} (A.5)
Therefore to the first order in $\theta$ the transformation (A.1) conserves the Galileon structure of the Lagrangian and merely shifts the coupling constants $d_n$ by $\delta d_n$. Note that the transformation (A.1) with finite $\theta$ can be used to eliminate the cubic term from the interaction Lagrangian, however, the Galileon structure of the Lagrangian is spoiled with additional interaction terms which are generated by the transformations. The way how to eliminate the cubic term consistently without leaving the space of the Galileon theories is now clear. It suffices to construct the finite transformation by means of iteration of the infinitesimal one, i.e. to exponentialize it according to

$$
\phi_\theta = \exp(\delta_\theta) \phi = \exp \left( \theta \partial \phi \cdot \partial \phi \frac{\delta}{\delta \phi} \right) \phi = \phi + \theta \partial \phi \cdot \partial \phi + 2\theta^2 \partial \phi \cdot \partial \partial \phi \cdot \partial \phi + \ldots \quad (A.6)
$$

Appling this finite transformation to the Galileon action results in a dual action $S_{\theta}[\phi]$ defined as

$$
S_{\theta}[\phi] \equiv S[\phi] = \exp \delta_\theta S[\phi] = \sum_{n=2}^{d+1} d_n(\theta) \left\{ \phi L_n^{\text{der}} \right\}. \quad (A.7)
$$

It is not difficult to show that $d_n(\theta) = d_n(\alpha_D(\theta))$ where the right hand side is given by (5.25). From this construction it is clear that the transformatitons $\phi_\theta$ form a one parametric group.

In what follows we will give an alternative elementary derivation of the explicit form of $\phi_\theta$. Let us denote

$$
\phi(\theta, x) = \exp \left( \theta \partial \phi \cdot \partial \phi \frac{\delta}{\delta \phi} \right) \phi(x), \quad (A.8)
$$

then we get by derivative with respect to $\theta$

$$
\frac{\partial \phi(\theta, x)}{\partial \theta} = \exp \left( \theta \partial \phi \cdot \partial \phi \frac{\delta}{\delta \phi} \right) \left\{ \partial \phi \cdot \partial \phi \frac{\delta}{\delta \phi} \right\} \phi(x)
= \exp \left( \theta \partial \phi \cdot \partial \phi \frac{\delta}{\delta \phi} \right) \partial_\mu \phi(\theta, x) \partial^\mu \phi(\theta, x) = \partial_\mu \phi(\theta, x) \partial^\mu \phi(\theta, x). \quad (A.9)
$$

Therefore the function $\phi(\theta, x)$ is a solution of the following Cauchy problem for the partial differential equation of the first order

$$
\frac{\partial \phi(\theta, x)}{\partial \theta} = \partial_\mu \phi(\theta, x) \partial^\mu \phi(\theta, x), \quad \phi(0, x) = \phi(x). \quad (A.10)
$$

This problem can be solved by standard method of characteristics which are the solutions of a set of ordinary differential equations

$$
\frac{d\theta}{dt} = 1, \quad \frac{dP}{dt} = 0, \quad \frac{dx}{dt} = -2p, \quad \frac{dp}{dt} = 0, \quad \frac{d\phi}{dt} = -2p \cdot p + P^2. \quad (A.11)
$$

The solution of this equation is

$$
\theta = \theta_0 + t, \quad P = P_0, \quad p = p_0, \quad x = x_0 - 2t p_0, \quad \phi = \phi_0 + t \left( P_0^2 - 2 p_0 \cdot p_0 \right). \quad (A.12)
$$

The general recipe how to get the five-dimensional integral surface corresponding to the equation (A.10) consist of two steps. First we replace the integrations constants $\theta_0, \ldots, \phi_0$
with functions of four parameters \( a_i, i = 1, \ldots, 4 \) in such a way that the following conditions are satisfied
\[
P_0(a_i) - p_0(a_i) \cdot p_0(a_i) = 0, \quad \partial \phi_0(a_i) = p_0(a_i) \cdot \partial x_0(a_i) + P_0(a_i) \partial \theta_0(a_i)
\]
and subsequently we eliminate the parameters \( a_i \) and \( t \) from the equations
\[
\begin{align*}
\theta &= \theta_0(a_i) + t, \quad x = x_0(a_i) - 2t p_0(a_i) \\
\phi &= \phi_0(a_i) + t \left( P_0^2(a_i) - 2p_0(a_i) \cdot p_0(a_i) \right).
\end{align*}
\tag{A.13}
\]
Let us choose the parameters \( a_i \) to be just \( x_0 \), we get then
\[
t = \theta - \theta_0(x_0), \quad P_0(x_0) = p_0(x_0) \cdot p_0(x_0), \quad p_0(x_0) = \partial \phi_0(x_0) - P_0(x_0) \partial \theta_0(x_0),
\]
and
\[
\begin{align*}
x &= x_0 - 2t p_0(x_0) \\
&= x_0 - 2(\theta - \theta_0(x_0)) \left( \partial \phi_0(x_0) - p_0(x_0) \cdot \partial \theta_0(x_0) \right) \\
\phi &= \phi_0(x_0) + t \left( P_0^2(x_0) - 2p_0(x_0) \cdot p_0(x_0) \right) \\
&= \phi_0(x_0) - (\theta - \theta_0(x_0)) p_0(x_0) \cdot p_0(x_0). \tag{A.14}
\end{align*}
\]
A special choice \( \theta_0(x_0) = 0 \) gives
\[
x = x_0 - 2\theta \partial \phi_0(x_0), \quad \phi(\theta, x) = \phi_0(x_0) - \theta \partial \phi_0(x_0) \cdot \partial \phi_0(x_0). \tag{A.15}
\]
The initial condition of the Cauchy problem is \( \phi(0, x) = \phi(x) \) and therefore \( \phi_0(x_0) = \phi(x_0) \). Thus the final solution of the Cauchy problem is
\[
\begin{align*}
x &= x_0 - 2\theta \partial \phi(x_0), \quad \phi(\theta, x) = \phi(x_0) - \theta \partial \phi(x_0) \cdot \partial \phi(x_0) \tag{A.16}
\end{align*}
\]
which is nothing else but the duality transformation (5.24).

**B. Compatibility of duality and IHC constraint**

In this Appendix we demonstrate by explicit calculation the consistency of the duality transformation with the IHC constraint. For the derivative of the field \( \phi \) with respect to the unprimed coordinates we get with help of the second row of (4.18)
\[
\begin{align*}
\partial \phi &= \partial x' \cdot \partial' \phi \\
&= \partial x' \cdot \left[ \det (\alpha_{IJ}) \partial' \phi' \right. \\
&\quad \left. + \frac{1}{2} \left( 2\alpha_{PB}\alpha_{BB}\partial' \phi' \cdot \partial' \phi' + 2\alpha_{PB}\alpha_{BP}\partial' \phi' \cdot \partial' \phi' + 2\alpha_{PP}\alpha_{BP}x' \cdot \partial' \phi' + 2\alpha_{PP}\alpha_{BP}x' \right) \right]. \tag{B.1}
\end{align*}
\]
Differentiation of the first row of (4.18) we get
\[
\eta = \partial x' \cdot (\alpha_{PP} \eta + \alpha_{PB} \partial' \phi') \tag{B.2}
\]
\[ \partial \phi = \det (\alpha_{ij}) \partial x' \cdot \partial' \phi' + \alpha_{BB} \alpha_{PP} \partial x' \cdot \partial' \phi' \cdot \partial' \phi' \]
\[ + \alpha_{PP} \alpha_{BB} \partial x' \cdot \partial' \phi' \cdot x' + \alpha_{PP} \alpha_{BB} \partial x' \cdot x' \]
\[ = \det (\alpha_{ij}) \partial x' \cdot \partial' \phi' + \alpha_{BB} (\eta - \alpha_{PP} \partial x') \cdot \partial' \phi' + \alpha_{PP} \alpha_{BB} \partial x' \cdot \partial' \phi' \]
\[ + \alpha_{PP} (\eta - \alpha_{PP} \partial x') \cdot x' + \alpha_{PP} \alpha_{BB} \partial x' \cdot x' \]
\[ = \det (\alpha_{ij}) \partial x' \cdot \partial' \phi' + \alpha_{BB} \partial x' \cdot \partial' \phi' - (\alpha_{BB} \alpha_{PP} - \alpha_{PP} \alpha_{BB}) \partial x' \cdot \partial' \phi' + \alpha_{PP} \partial x' \]
\[ = \alpha_{BB} \partial' \phi' + \alpha_{PP} \partial x' \]

(B.3)

where we have used the integrability constraint (4.13) in the last line.

C. Full form of 4-pt scattering amplitude and self-energy

The corresponding contributions are

\[ A_1 = d_3^4 \frac{B(s) \ 81s^4[(d^2 + 6d + 32)s^2 - 72tu]}{32(d^2 - 1)} + \text{cykl} \]  
\[ A_2 = -d_3^4 \frac{B(s) \ 81(d + 2)s^6}{4(d - 1)} + \text{cykl} \]  
\[ A_3 = d_3^4 \frac{B(s) \ 81s^6}{2} + \text{cykl} \]  
\[ A_4 = d_3^4 \frac{B(s) \ 9s^4[(d^2 - 2d)s^2 - 8tu]}{8(d^2 - 1)} + \text{cykl} \]  
\[ A_5 = d_3^4 d_4^4 \frac{B(s) \ 27s^4[(d + 4)(d - 2)s^2 + 24tu]}{8(d^2 - 1)} + \text{cykl} \]  
\[ A_6 = -d_3^4 d_4^4 \frac{B(s) \ 27s^6(d - 2)}{2(d - 1)} + \text{cykl} \]

We have used the cyclic summation over all Mandelstam variables (e.g. \((s^2 + \text{cykl}) = s^2 + t^2 + u^2\)). The loop function is given by

\[ B(s) = \frac{1}{(4\pi)^{d/2-2}} \frac{1}{d - 3} \Gamma(2 - d/2)s^{d/2 - 2} \]  

(C.7)

Summing up all diagrams leads to

\[ A = (d_4 - \frac{3}{2}d_2^2) \frac{B(s) \ 9s^4[d(d - 2)s^2 - 8tu]}{8(d^2 - 1)} + \text{cykl.} \]  

(C.8)

The full result for the one-loop self-energy reads

\[ \Sigma(p) = -\frac{1}{(4\pi)^{d/2}} \frac{9}{2} d_3^2 (p^2)^4 B \left( \frac{p^2}{\mu^2} \right). \]  

(C.9)
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