A new Einstein-nonlinear electrodynamics solution in 2+1-dimensions

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We introduce a class of solutions in 2 + 1-dimensional Einstein-Power-Maxwell theory for circularly symmetric electric field. The electromagnetic field is considered with an angular component given by \( F_{\mu\nu} = E_0 \delta^\mu_\nu \delta^\theta_0 \) for \( E_0 = \) constant. First, we show that the metric for zero cosmological constant and the Power-Maxwell Lagrangian of the form \( \sqrt{|F_{\mu\nu}F^{\mu\nu}|} \), coincides with the solution given in 2 + 1-dimensional gravity coupled with a massless, self interacting real scalar field. With the same Lagrangian and a non-zero cosmological constant we obtain a non-asymptotically flat wormhole solution in 2 + 1-dimensions. The confining motions of massive charged and chargeless particles are investigated too. Secondly, another interesting solution is given for zero cosmological constant together with conformal invariant condition. The formation of timelike naked singularity for this particular case is investigated within the framework of the quantum mechanics. Quantum fields obeying the Klein-Gordon and Dirac equations are used to probe the singularity and test the quantum mechanical status of the singularity.

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I. INTRODUCTION

There has always been benefits in studying lower dimensional field theoretical spacetimes such as 2 + 1-dimensions in general relativity. This is believed to be the projection of higher dimensional cases to the more tractable situations that may inherit physics of the intricate higher dimensions. Recent decades proved that the cases of lower dimensions are still far from being easily understandable and in fact entails its own characteristics. The absence of gravitational degree of freedom such as Weyl tensor or pure gravitational waves necessitates endowment of physical sources to fill the blank and create its own curvatures. Among these the most popular addition has been a negative cosmological constant which makes anti-de Sitter spacetimes to the extent that it makes even black holes [1]. Addition of electromagnetic [2] and scalar fields [3, 4] also are potential candidates to be considered in the same context. Beside minimally coupled massless scalar fields which has little significance to add, non-minimally self-coupled scalar fields has also been considered. In particular, the real, radial, self-interacting scalar field with a Liouville potential among others seems promising [1]. The distinctive feature of the source in such a study is that the radial pressure turns out to be the only non-zero (i.e., \( T_r^r \neq 0 \)) component of the energy-momentum tensor [4]. In effect, such a radial pressure turns out to make a naked singularity but not a black hole. Being motivated by the self-intracting scalar field in 2 + 1-dimensional gravity we attempt in this paper to do similar physics with a non-linear electromagnetic field which naturally adds its own non-linearity. Our choice for the Lagrangian in nonlinear electrodynamics (NED) is the square root of the Maxwell invariant, i.e. \( \sqrt{|F_{\mu\nu}F^{\mu\nu}|} \), where as usual the field tensor is defined by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). This Lagrangian belongs to the class of NED with \( k^{-}\)power law Maxwell invariant \( |F_{\mu\nu}F^{\mu\nu}|^k \) [5]. Similar Lagrangian in 3 + 1-dimensions with magnetic field source has been considered in [6] and with electric field as well as electromagnetic fields together in [7]. A combination of linear Maxwell with the square root term has been investigated in [8].

Such a Lagrangian naturally breaks the scale invariance, i.e. \( x_\mu \rightarrow \lambda x_\mu, A_\mu \rightarrow \lambda A_\mu \) for \( \lambda = \)constant, even in 3 + 1-dimensions so that interesting results are expected to ensue. Let us add that such a choice of NED has the feature that it doesn’t attain the familiar linear Maxwell limit unless \( k = 1 \). One physical consequence beside others, of the square-root Maxwell Lagrangian is that it gives rise to confinement for geodesic particles. For a general discussion of confinement in general relativistic field theory the reader may consult [8].

Contrary to the previous considerations [4] in this study our electric field is not radial, instead our field tensor is expressed in the form \( F_{\mu\nu} = E_0 \delta^\mu_\nu \delta^\theta_0 \) for \( E_0 = \)constant. This amounts to the choice for the electromagnetic vector potential \( A_\mu = E_0 \left( a_0 \theta, 0, b_0 t \right) \), where our spacetime coordinates are labelled as \( x^\mu = \{ t, r, \theta \} \) and the constants \( a_0 \) and \( b_0 \) satisfy \( a_0 + b_0 = 1 \). The particular choice \( a_0 = 0, b_0 = 1 \) leaves us with the vector potential \( A_\mu = \delta^\theta_0 E_0 t \), which yields a uniform field in the angular direction. The only non vanishing energy momentum tensor component is \( T_r^r \) which accounts for the radial pressure. Our ansatz electromagnetic field in the circularly symmetric static metric gives a solution with zero cosmological constant that is identical with the spacetime obtained from an entirely different source, namely the self-intracting real scalar field.
II. FIELD EQUATIONS AND THE NEW SOLUTION

We start with the following action for the Einstein’s theory of gravity coupled with a NED Lagrangian

\[ I = \frac{1}{2} \int dx^3 \sqrt{-g} \left( R - 2\Lambda + \alpha |\mathcal{F}|^k \right). \quad (1) \]

Here \( \mathcal{F} = F_{\mu\nu}F^{\mu\nu} \) is the Maxwell invariant with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( \alpha \) is a real coupling constant, \( k \) is a rational number and \( \Lambda \) is the cosmological constant. Our line element is circularly symmetric given by

\[ ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2d\theta^2, \quad (2) \]

where \( A(r) \) and \( B(r) \) are unknown functions of \( r \) and \( 0 \leq \theta \leq 2\pi \). Also we choose the field ansatz as

\[ \mathbf{F} = E_0 dt \wedge d\theta \quad (3) \]

in which \( E_0 = \text{constant} \), is a uniform electric field and its dual can be found as \( \star \mathbf{F} = E_0 \sqrt{\frac{2}{\pi}} dr \). Naturally, the integral of \( \star \mathbf{F} \) gives the total charge. This electric field is derived from an electric potential one-form given by

\[ A = E_0 (a_0 t d\theta - b_0 \theta dt) \quad (4) \]

in which \( a_0 \) and \( b_0 \) are constants satisfying \( a_0 + b_0 = 1 \). The nonlinear Maxwell’s equation reads

\[ d \left( \star \mathbf{F} \frac{\mathcal{F}^k}{|\mathcal{F}|} \right) = 0, \quad (5) \]

which upon substitution

\[ \mathcal{F} = 2F_{\theta\phi}F^{\theta\phi} = -\frac{2E_0^2}{A(r)r^2} \quad (6) \]

is trivially satisfied. Next, the Einstein-NED equations are given by

\[ G^\nu_\mu + \frac{1}{3}\Lambda \delta^\nu_\mu = T^\nu_\mu \quad (7) \]

in which

\[ T^\nu_\mu = \frac{\alpha}{2} |\mathcal{F}|^k \left( \delta^\nu_\mu - \frac{4k}{|\mathcal{F}|} F_\mu \Lambda \star F^{\nu \lambda} \right). \quad (8) \]

Having \( \mathcal{F} \) known one finds

\[ T^t_t = T^\theta_\theta = \frac{\alpha}{2} |\mathcal{F}|^k (1 - 2k), \quad (9) \]

and

\[ T^r_r = \frac{\alpha}{2} |\mathcal{F}|^k \quad (10) \]

as the only non-vanishing energy-momentum component. To proceed further, we must have the exact form of the Einstein tensor components given by

\[ G^t_t = \frac{B'}{2r} \quad (11) \]

\[ G^r_r = \frac{BA'}{2rA} \quad (12) \]

and

\[ G^\theta_\theta = \frac{2A''AB - A'^2B + A'B'A}{4A^2}, \quad (13) \]
in which a 'prime' means $\frac{d}{dr}$. The field equations then read as follows

$$\frac{B'}{2r} + \frac{1}{3} \Lambda = \frac{\alpha}{2} |F|^k (1 - 2k),$$

(14)

and

$$\frac{B A'}{2r A} + \frac{1}{3} \Lambda = \frac{\alpha}{2} |F|^k (1 - 2k).$$

(15)

and

$$\frac{2A''AB - A'^2 B + A'B'A}{4A^2} + \frac{1}{3} \Lambda = \frac{\alpha}{2} |F|^k (1 - 2k).$$

(16)

III. AN EXACT SOLUTION FOR $k = \frac{1}{2}$

Among the values which $k$ may take the most interesting one is $k = \frac{1}{2}$. In this specific case $T^i_t = T^\theta_\theta = 0$ and $T^r_r = \frac{5}{2} \sqrt{|F|}$ is the only non-zero component of the energy momentum tensor. The field equations admit the general solutions for $A(r)$ and $B(r)$ given by

$$B(r) = D - \frac{\Lambda}{3} r^2$$

(17)

and

$$A(r) = \left( D - \frac{\Lambda}{3} r^2 \right) \left( C + \frac{r \alpha |E_0|}{D \sqrt{2} \sqrt{D - \frac{4}{3} r^2}} \right)^2.$$ 

(18)

in which $D$ and $C$ are two integration constants. One observes that setting $E_0 = 0$ gives the correct limit of BTZ black hole up to a constant $C^2$ which can be absorbed in time. The other interesting limit of the above solution is found when we set $C = 0$ which yields

$$A(r) = \left( \frac{r \alpha |E_0|}{D \sqrt{2}} \right)^2.$$ 

(19)

The reduced line element, therefore, becomes

$$ds^2 = -\left( \frac{\alpha |E_0|}{D \sqrt{2}} \right)^2 r^2 dt^2 + \frac{1}{D - \frac{4}{3} r^2} dr^2 + r^2 d\theta^2.$$ 

(20)

which for three different cases admits different geometries;

i) $\Lambda > 0$: When the cosmological constant is positive the solution becomes non-physical for $r^2 > \frac{3D}{\Lambda}$. For $D < 0$ the signature of the spacetime is openly violated.

ii) $\Lambda < 0$: In this case the solution is a black string for $D > 0$ whose Kretschmann scalar is given by

$$K = \frac{4 (r^4 \Lambda^2 - 2 \Lambda^2 D + 3 D^2)}{3 r^4},$$

(21)

with the singularity at the origin which is also the horizon. However, for $D < 0$ with negative cosmological constant the solution becomes a wormhole with the throat located at $r = r_0 = \sqrt{\frac{3 |D|}{\Lambda}}$ and

$$ds^2 = -\left( \frac{\alpha |E_0|}{D \sqrt{2}} \right)^2 r^2 dt^2 + \frac{1}{|D| \left( \frac{r^2}{r_0^2} - 1 \right)} dr^2 + r^2 d\theta^2.$$ 

(22)

In order to conceive the geometry of the wormhole in this case we introduce a new coordinate $z = z(\rho)$ such that

$$\frac{dr^2}{|D| \left( \frac{r^2}{r_0^2} - 1 \right)} = dr^2 + dz^2$$

(23)

where

$$z(r) = \pm \int \left( \frac{1}{|D| \left( \frac{r^2}{r_0^2} - 1 \right)} - 1 \right) dr.$$ 

(24)

It should be supplemented that the ranges of $r$ must satisfy

$$r_0 < r.$$ 

(25)

To cast our spacetime into the standard wormhole metric we express it as

$$ds^2 = -e^{2f} dt^2 + \frac{dr^2}{(1 - b(r))} + r^2 d\theta^2.$$ 

(26)

Here $f(r) \sim \ln r$ and $b(r) = r \left( 1 + |D| - \frac{|D| r^2}{r_0^2} \right)$ are known as the redshift and shape functions, respectively. The throat of our wormhole is at $r_0$ where $b(r_0) = r_0$, and the flare-out condition (i.e., $b'(r_0) < 1$) is satisfied by the choice of our parameters. We have also $\frac{b'(r)}{r} < 1$ for $r > r_0$. We must add that distinct from an asymptotically flat wormhole herein we have a range for $r$, given in (25).

iii) $\Lambda = 0$: for the case when cosmological constant is zero one finds (only $D > 0$ is physical)

$$ds^2 = -\left( \frac{\alpha |E_0|}{D \sqrt{2}} \right)^2 r^2 dt^2 + \frac{1}{D} dr^2 + r^2 d\theta^2.$$ 

(27)

which after a simple rescaling of time and setting $D = 1$ one gets

$$ds^2 = -r^2 dt^2 + dr^2 + r^2 d\theta^2.$$ 

(28)

We notice that although our solution is not a standard black hole there exists still a horizon at $r = 0$ which makes our solution a black point [9] [10]. In [9] such black points appeared in $3+1$-dimensional gravity coupled to the logarithmic $U(1)$ gauge theory and in [10] coupled to charged dilatonic fields. This is the conformally flat $2+1$-dimensional line element, and through the transformation $r = e^R$, which is given by

$$ds^2 = e^{2R} \left( -dt^2 + dR^2 + d\theta^2 \right)$$

(29)

obtained also in the self-intracting scalar field model [4].
A. Geodesic Motion for $\Lambda = 0$

1. Chargeless Particle

To know more about the solution found above one may study the geodesic motion of a massive particle (timelike). The Lagrangian of the motion of a unit mass particle within the spacetime (28) is given by (for simplicity we remove tildes over the coordinates)

$$L = -\frac{1}{2} r^2 i^2 + \frac{1}{2} r^2 + \frac{1}{2} r^2 \theta^2$$

(30)

where a 'dot' denotes derivative $\frac{d}{ds}$ with $s$ an affine parameter. The conserved quantities are

$$\frac{\partial L}{\partial t} = - r^2 i = - \alpha_0$$

(31)

$$\frac{\partial L}{\partial \theta} = r^2 \theta = \beta_0$$

(32)

with $\alpha_0$ and $\beta_0$ as constants of energy and angular momentum. The metric condition reads

$$-1 = - r^2 i^2 + r^2 + r^2 \theta^2$$

(33)

which upon using (31) and (32) yields

$$r^2 = \left( \frac{\alpha_0^2 - \beta_0^2}{r^2} - 1 \right).$$

(34)

This equation clearly shows a confinement in the motion for the particle geodesics in the form

$$r^2 \leq \alpha_0^2 - \beta_0^2.$$

(35)

Considering the affine parameter as the proper distance one finds from (41),

$$r = \sqrt{\alpha_0^2 - \beta_0^2 - (s - s_0)^2}$$

(36)

leading to a manifest confinement.

2. Charged Particle Geodesics

For a massive charged particle with unit mass and charge $q_0$ the Lagrangian is given by

$$L = -\frac{1}{2} \mu^2 i^2 + \frac{1}{2} i^2 + \frac{1}{2} \mu^2 \theta^2 + q_0 A_\mu \dot{x}^\mu$$

(37)

in which $A_\mu \dot{x}^\mu = A_0 \dot{x}^\theta = E_0 t \theta$ i.e. the choice $a_0 = 1$, $b_0 = 0$ in Eq. (4). The metric condition is as (33) and therefore the Lagrange equations yield

$$\frac{d}{ds} \left( r^2 i + q_0 E_0 t \right) = 0,$$

(38)

and

$$\ddot{r} = - q_0 E_0 \dot{\theta}$$

(39)

The first equation implies

$$r^2 \dot{\theta} + q_0 E_0 t = \gamma_0 = \text{const.}$$

(40)

while the second equation with a change of variable as $r^2 \frac{d}{ds} = \frac{d}{dz}$ and imposing (41) yields

$$\frac{d^2 t}{dz^2} = - q_0 E_0 \left( \gamma_0 - q_0 E_0 t \right).$$

(42)

This equation admits an exact solution for $t(z)$

$$t(z) = \frac{\gamma_0}{\omega} + C_1 e^{\omega z} + C_2 e^{-\omega z}$$

(43)

in which $C_1$ and $C_2$ are two integration constants and $\omega = q_0 E_0$. Next, the radial equation upon imposing the metric condition (33) becomes decoupled as

$$\ddot{r} + r^2 + 1 = 0.$$

(44)

The general solution for this equation is given by

$$r = \pm \sqrt{C_1 + 2C_2 s - s^2}$$

(45)

where we consider the positive root. Once more going one finds

$$r^2 \frac{dt}{dz} = \frac{dt}{dz}$$

(46)

which in turn becomes

$$\left( C_1 + 2C_2 s - s^2 \right) \frac{dt}{ds} = \omega \left( C_1 e^{\omega z} - C_2 e^{-\omega z} \right).$$

(47)

To proceed further we set $C_2 = 0$, $C_1 = b_0^2$, $C_2 = 0$ and $C_1 = 1$ so that (one must note that with this choice of integration constants $b_0^2 - s^2 \geq 0$)

$$(b_0^2 - s^2) \frac{dt}{ds} = \omega \left( t - \frac{\gamma_0}{\omega} \right).$$

(48)

This leads us to

$$t - \frac{\beta_0}{\omega} = \left( \frac{b_0 + s}{b_0 - s} \right)^{\frac{\omega}{2\gamma_0}} + \zeta,$$

(49)

in which $\zeta$ is an integration constant to be set to zero for simplicity. Having the latter relation one finds

$$r(t) = \frac{2b_0 |t - \frac{\gamma_0}{\omega}|^{\frac{b_0}{\omega}}}{|t - \frac{\gamma_0}{\omega}|^{\frac{2b_0}{\omega}} + 1}$$

(50)

which clearly shows a confining motion for the charged particles. This conclusion could also be obtained from Eq. (45) which implies $C_1 + 2C_2 s - s^2 \geq 0$ and consequently $r \leq \sqrt{C_1 + C_2^2}$. The angular variable $\theta(t)$ can also be reduced to an integral expression.
IV. A BRIEF ACCOUNT FOR A CONFORMALLY INVARIANT MAXWELL SOURCE WITH $\Lambda = 0$

In the first paper of Hassanine and Martinez [5] it was shown that the action given in (1) is conformally invariant if $k = \frac{3}{4}$. Here in this section we set $k = \frac{3}{4}$ and $\Lambda = 0$ such that the following solution is obtained for the field equations (14)-(16):

$$A(r) = \left(1 + \frac{\alpha E_0^3}{2^{3/4}} \sqrt{r}\right)^4$$  \hspace{1cm} (51)

and

$$B(r) = \frac{1}{\left(1 + \frac{\alpha E_0^3}{2^{3/4}} \sqrt{r}\right)^2}.$$  \hspace{1cm} (52)

To find these solutions we have to set the integration constant in such a way that the flat limit easily comes after one imposes $E_0 = 0$. The Kretschmann scalar of the spacetime is given by

$$K = \frac{3|E_0|^3 \alpha^2 \left(r\alpha^2 |E_0|^3 + 2\sqrt{2}(E_0^3)^{3/4} \sqrt{r\alpha} + \sqrt{2}\right)}{r^3 \left(1 + \frac{\alpha E_0^3}{2^{3/4}} \sqrt{r}\right)^8}.$$  \hspace{1cm} (53)

As one can see from the action (1) $\alpha < 0$ implies a ghost field which is not physical. Therefore we assume $\alpha > 0$ and consequently the only singularity of the spacetime is located at the origin $r = 0$. For $\alpha > 0$ the solution represents a non-black hole and non-asymptotically flat spacetime with a naked singularity at $r = 0$. In order to find the nature or the character of the singularity at $r = 0$, we perform a conformal compactification. The conformal radial / tortoise coordinate

$$r_* = \int \frac{dr}{1 + a\sqrt{r}} = \frac{2}{a^2} \left\{ a\sqrt{r} - \ln |1 + a\sqrt{r}| \right\},$$  \hspace{1cm} (54)

with $a = \frac{\alpha E_0^3}{2^{3/4}}$ helps us to introduce the retarded and advanced coordinates i.e. $u = t - r_*$ and $v = t + r_*$. The Kruskal coordinates are defined following $u$ and $v$ coordinates as

$$u' = -e^{a^2u}, \quad v' = e^{-a^2v},$$  \hspace{1cm} (55)

which consequently the line element in $u'$ and $v'$ coordinates becomes

$$ds'^2 = -\frac{e^{4a\sqrt{r}}}{a^4} du' dv' + r^2 d\theta^2,$$  \hspace{1cm} (56)

with the constraint

$$u'v' = -e^{-4a\sqrt{r}} \left(1 + a\sqrt{r}\right)^4.$$  \hspace{1cm} (57)

The final change of the coordinate

$$u'' = \arctan u', \quad 0 < u'' < \pi/2,$$

$$v'' = \arctan v', \quad 0 < v'' < \pi/2$$

brings infinity into a finite coordinate. In Fig. 1 we plot Carter-Penrose diagrams of the solutions (51) and (52). One observes in this diagram that the singularity located at $r = 0$ is timelike. The corresponding energy momentum tensor

$$T_{\mu\nu} = -\frac{1}{2\alpha} \left( \frac{2E_0^2}{A(r) r^2} \right)^{3/4} (1, -1, 1)$$  \hspace{1cm} (59)

implies that

$$\rho = -T_{t}^{t} = \frac{1}{2\alpha} \left( \frac{2E_0^2}{A(r) r^2} \right)^{3/4},$$  \hspace{1cm} (60)

$$p = T_{r}^{r} = \rho = \frac{1}{2\alpha} \left( \frac{2E_0^2}{A(r) r^2} \right)^{3/4},$$  \hspace{1cm} (61)

$$q = T_{\theta}^{\theta} = -\rho = -\frac{1}{2\alpha} \left( \frac{2E_0^2}{A(r) r^2} \right)^{3/4}$$  \hspace{1cm} (62)

and therefore all energy conditions including the weak, strong and dominant are satisfied.

V. SINGULARITY ANALYSIS

A. Quantum Singularities

One of the important predictions of the Einstein’s theory of relativity is the formation of the spacetime singularities in which the evolution of timelike or null geodesics is not defined after a proper time. Deterministic nature of general relativity requires that the spacetime singularities must be hidden by horizon(s), as conjectured by Penrose’s weak cosmic censorship hypothesis (CCH). However, there are some cases that the spacetime singularity is not covered by horizon(s), and it is...
called naked singularity. Hence, naked singularities violates the CCH and their resolution becomes extremely important. The most powerful candidate theory in resolving the singularities is the quantum theory of gravity. However, there is no consistent quantum theory of gravity yet. String theory [14] [15] and loop quantum gravity [16] are the two study field along the resolution of singularities. Another method that we shall employ in this paper is the criterion proposed by Horowitz and Marolf (HM) [17] that incorporates "self - adjointness" of the spatial part of the wave operator. In this criterion, the classical notion of geodesics incompletness with respect to point-particle probe will be replaced by the notion of quantum singularity with respect to wave probes. This criterion can be applied only to static spacetimes having timelike singularities. To understand better, let us consider, the Klein-Gordon equation for a free particle that satisfies $\frac{d\psi}{dt} = \sqrt{A_F}\psi$, whose solution is $\psi(t) = \exp[-it\sqrt{A_F}] \psi(0)$ in which $A_F$ denotes the extension of the spatial part of the wave operator.

If $A$ is not essentially self-adjoint, in other words if $A$ has an extension, the future time evolution of the wave function $\psi(t)$ is ambiguous. Then the HM criterion defines the spacetime as quantum mechanically singular. However, if there is only a single self-adjoint extension, the operator $A$ is said to be essentially self-adjoint and the quantum evolution described by $\psi(t)$ is uniquely determined by the initial conditions. According to the HM criterion, this spacetime is said to be quantum mechanically non-singular. The essential self-adjointness of the operator $A$, can be verified by considering solutions of the equation

$$A^*\psi \pm i\psi = 0,$$  

and showing that the solutions of Eq. (63), do not belong to Hilbert space $\mathcal{H}$ (we refer references; [18] for detailed mathematical analysis and [19] for applications of HM approach in different spacetimes). This will be achieved by defining the function space on each $t$ =constant hypersurface $\Sigma$ as $\mathcal{H} = \{ R | \| R \| < \infty \}$ with the following norm given for the metric (2) as,

$$\| R \|^2 = \int_0^{\text{constant}} \frac{r}{\sqrt{A(r)B(r)}} |R|^2 \, dr.$$  

### 1. Klein-Gordon Fields

Klein-Gordon equation for the metric (2) can be written by splitting temporal and spatial part and given by

$$\frac{\partial^2 \psi}{\partial t^2} = -A \psi$$  

in which $A$ denotes the spatial operator of the massless scalar wave given by

$$A = - (1 + a\sqrt{r})^2 \frac{\partial^2}{\partial r^2} - \frac{(1 + a\sqrt{r}) (1 + \frac{3a\sqrt{r}}{2})}{r} \frac{\partial}{\partial r} - \frac{(1 + a\sqrt{r})^4}{r^2} \frac{\partial^2}{\partial \theta^2}.$$  

Applying separation of variables $\psi = \psi(r)Y(\theta)$, the radial part of the Eq. (63) becomes

$$\frac{\partial^2 \psi(r)}{\partial r^2} + \frac{(1 + \frac{3a\sqrt{r}}{2})}{r (1 + a\sqrt{r})} \frac{\partial \psi(r)}{\partial r} + \left( \frac{c (1 + a\sqrt{r})^2}{r^2} \pm \frac{i}{(1 + a\sqrt{r})} \right) \psi(r) = 0,$$  

where $c$ stands for the separation constant. The spatial operator $A$ is essentially self adjoint if neither of two solutions of Eq. (67) is square integrable over all space $L^2(0, \infty)$. Because of the complexity in finding exact analytic solution to Eq. (67), we study the behavior of $R(r)$ near $r \to 0$ and $r \to \infty$. The behavior of the Eq. (67), near $r = 0$ is given by

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} + \frac{c}{r^2} R(r) = 0,$$  

whose solution is

$$R(r) = C_1 \sin \left( \sqrt{c} \ln(r) \right) + C_2 \cos \left( \sqrt{c} \ln(r) \right).$$  

The square integrability is checked by calculating the norm given in Eq. (64). Calculation has shown that $R(r)$ is square integrable near $r = 0$ and hence, it belongs to the Hilbert space and the operator $A$ is not essentially self-adjoint. As a result, the timelike naked singularity remains quantum mechanically singular with respect to the spinless wave probe. This result is in conform with the analysis in [13]. This result seem to show that irrespective of the direction of electric field, the singularity remains quantum singular with respect to waves obeying the Klein-Gordon equation.

### 2. Dirac Fields

The Dirac equation in $2+1$ dimensional curved spacetime for a free particle with mass $m$ is given by,

$$i\gamma^\mu (x) [\partial_\mu - \Gamma_\mu (x)] \Psi (x) = m \Psi (x),$$  

where $\Gamma_\mu (x)$ is the spinorial affine connection given by

$$\Gamma_\mu (x) = \frac{1}{4} g_{\lambda\alpha} \left[ e^{(i)}_{\nu\mu}(x)e^{(i)}_{\nu\lambda}(x) - \Gamma_{\nu\lambda}^\alpha (x) \right] s^\lambda \nu (x).$$
with

\[ s^{\lambda\nu}(x) = \frac{1}{2} \left[ \sigma^{\lambda}(x), \sigma^{\nu}(x) \right]. \] (72)

Since the fermions have only one spin polarization in 2 + 1 dimension [20], the Dirac matrices \( \gamma^{(j)} \) can be expressed in terms of Pauli spin matrices \( \sigma^{(j)} \) [21] so that

\[ \gamma^{(j)} = \left( \sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)} \right), \] (73)

where the Latin indices represent internal (local) frame. In this way,

\[ \{ \gamma^{(i)}, \gamma^{(j)} \} = 2\eta^{(ij)}I_{2 \times 2}, \] (74)

where \( \eta^{(ij)} \) is the Minkowski metric in 2+1 dimension and \( I_{2 \times 2} \) is the identity matrix. The coordinate dependent metric tensor \( g_{\mu\nu}(x) \) and matrices \( \sigma^{\mu}(x) \) are related to the triads \( e^{(i)}_{\mu}(x) \) by

\[ g_{\mu\nu}(x) = e^{(i)}_{\mu}(x) e^{(j)}_{\nu}(x) \eta^{(ij)}, \]

\[ \sigma^{\mu}(x) = e^{(i)}_{(\mu} \gamma^{(i)}, \]

where \( \mu \) and \( \nu \) stand for the external (global) indices. The suitable triads for the metric () are given by,

\[ e^{(i)}_{\mu}(t, y, \theta) = \text{diag} \left( (1 + a\sqrt{r})^{2}, (1 + a\sqrt{r}), r \right), \] (76)

The coordinate dependent gamma matrices and the spinorial affine connection are given by

\[ \sigma^{\mu}(x) = \left( \sigma^{(3)} (1 + a\sqrt{r})^{-2}, i (1 + a\sqrt{r})^{-1} \sigma^{(1)}, i\sigma^{(2)} \frac{i}{r} \right), \] (77)

\[ \Gamma_{\mu}(x) = \left( \frac{2a\sigma^{(2)}}{2\sqrt{r}}, 0, 0 \right). \]

Now, for the spinor

\[ \Psi = \left( \psi_{1}, \psi_{2} \right), \]

the Dirac equation can be written as

\[ -i \frac{\partial \psi_{2}}{(1 + a\sqrt{r})^{2}} - \frac{ai}{2\sqrt{r} (1 + a\sqrt{r})^{2}} \psi_{1} - \frac{1}{(1 + a\sqrt{r})} \frac{\partial \psi_{2}}{r \partial r} + \frac{1}{r} \frac{\partial \psi_{2}}{\partial \theta} - m\psi_{2} = 0 \] (79)

The following ansatz will be employed for the positive frequency solutions:

\[ \Psi_{n, E}(t, x) = \left( R_{1n}(r \partial \theta) e_{i\theta} \right) e^{i\theta} e^{-iEt}. \] (81)

The radial part of the Dirac equation becomes,

\[ \frac{\partial R_{2n}(r)}{\partial r} - e^{-i\theta} \left( \frac{E}{1 + a\sqrt{r}} - m (1 + a\sqrt{r}) \right) R_{1n}(r) - i \left[ \frac{(n + 1)(1 + a\sqrt{r})}{r} - \frac{c}{2\sqrt{r} (1 + a\sqrt{r})} \right] R_{2n}(r) = 0, \] (82)

\[ \frac{\partial R_{1n}(r)}{\partial r} - e^{i\theta} \left( \frac{1}{1 + a\sqrt{r}} - m (1 + a\sqrt{r}) \right) R_{2n}(r) + i \left[ \frac{n (1 + a\sqrt{r})}{r} + \frac{c}{2\sqrt{r} (1 + a\sqrt{r})} \right] R_{1n}(r) = 0. \] (83)

The behavior of the Dirac equations near \( r = 0 \) reduces to

\[ \frac{\partial^{2} R_{2n}(r)}{\partial r^{2}} + i \frac{\partial R_{2n}(r)}{r \partial r} + \left\{ \frac{n + 1}{r^{2}} (n + i) - \beta \right\} R_{2n}(r) = 0, \] (84)

\[ \frac{\partial^{2} R_{1n}(r)}{\partial r^{2}} + i \frac{\partial R_{1n}(r)}{r \partial r} + \left\{ \frac{n}{r^{2}} (n + 1 - i) - \beta \right\} R_{1n}(r) = 0, \] (85)

in which \( \beta = m^{2} - m(E + 1) + E \). These two equations (84) and (85), must be investigated for essential self-adjointness by using the Eq. (63). Hence, we have

\[ \frac{\partial^{2} R_{2n}(r)}{\partial r^{2}} + i \frac{\partial R_{2n}(r)}{r \partial r} + \left\{ \frac{n + 1}{r^{2}} (n + i) - \beta \pm i \right\} R_{2n}(r) = 0, \] (86)

\[ \frac{\partial^{2} R_{1n}(r)}{\partial r^{2}} + i \frac{\partial R_{1n}(r)}{r \partial r} + \left\{ \frac{n}{r^{2}} (n + 1 - i) - \beta \pm i \right\} R_{1n}(r) = 0. \] (87)
Since we are looking for a solution near \( r = 0 \), the constant terms inside the curly bracket can be ignored and the solutions are given by

\[
R_{2n}(r) = r^{\frac{1-i}{2}} \left[ C_{1n} r^{\sqrt{2i/2}} + C_{2n} r^{-\sqrt{2i/2}} \right],
\]
\[
R_{1n}(r) = r^{\frac{1-i}{2}} \left[ C_{3n} r^{\sqrt{2i/2}} + C_{4n} r^{-\sqrt{2i/2}} \right],
\]
in which \( C_{in} \) are arbitrary integration constants

\[
\chi_{1} = (n + 1) (n + i),
\]
and

\[
\chi_{2} = n (n + 1) - i.
\]
The square integrability of these solutions are checked by using the norm defined in Eq. (64). Based on the numerical calculation, both \( R_{1n}(r) \) and \( R_{2n}(r) \) are square integrable near \( r = 0 \). As a result, the arbitrary wave packet can be written as

\[
\Psi(t, x) = \sum_{n=-\infty}^{\infty} \left( \frac{R_{1n}(r)}{R_{2n}(r)} e^{i\theta} \right) e^{i\theta} e^{-iEt},
\]
and the initial condition \( \Psi(0, x) \) is enough to determine the time evolution of the wave. Hence, the initial value problem is well-posed and the spacetime remains nonsingular when probed with spinorial waves obeying the Dirac equation.

VI. CONCLUSION

We considered a specific form of NED Lagrangian in the form of a power law Maxwell invariant |\( F_{\mu\nu}F^{\mu\nu} \)|\( k \) with \( k = \frac{1}{2} \). It is known that a pure radial electric field with \( k = \frac{1}{2} \) does not satisfy the energy conditions \([11]\) so our field ansatz has been chosen differently to be a uniform angular electric field. One direct feature of this form of field ansatz yields the only non-zero component of the energy-momentum to be \( T_{rr} \). This indeed means that the energy density \( \rho = -T_{t}^{t} \) and the angular pressure \( p_{\theta} = T_{\theta}^{\theta} \) are both zero while the radial pressure is \( p_{r} = T_{r}^{r} = \frac{\xi}{2} \) with \( \xi = \sqrt{\frac{2q_{0}E_{0}}{r}} \). These make the weak energy conditions (WECs) to be satisfied, i.e., \( \rho \geq 0, \rho + p_{r} \geq 0 \) and \( \rho + p_{\theta} \geq 0 \). Even further, the strong energy conditions (SECs) are also satisfied which are WECs together with \( \rho + p_{r} + p_{\theta} \geq 0 \). Having radial pressure non-zero and divergence at \( r = 0 \) are features that can be seen from the nature of the resulting spacetime. For \( k = \frac{1}{2} \) we give two different classes of solutions, for \( \Lambda = 0 \) and \( \Lambda \neq 0 \). The one for \( \Lambda = 0 \) is not a black hole solution but is singular at \( r = 0 \) since the spacetime invariants \( R \sim \frac{1}{r}, R_{\mu\nu}R^{\mu\nu} \sim \frac{1}{r} \) and Kretschmann scalar \( \sim \frac{1}{r^{2}} \). This singularity is in the same order of divergence as the charged BTZ black hole with a radial singular electric field. As it has been found in this work, the singularity at the origin \( (r = 0) \) is also a zero for \( q_{0} \) that makes our solution a black point confines the radial motion of a massive particle (both charged and uncharged). This means that the particle can not go beyond a maximum radius. In the work of Schmidt and Singleton \([1]\) where a different matter source, namely a self-interacting scalar field is employed, the same solution has been found. This is perhaps an indication that the geometry found in the context of a real scalar field, sharing common metric with different sources in \( 2+1 \)-dimensions may apply to higher dimensional spacetimes. The case for \( \Lambda < 0 \) \( (k = \frac{1}{2}) \) corresponds to a non-asymptotically flat wormhole built entirely from the cosmological constant. Conformally invariant case \( k = \frac{3}{2} \) for the Maxwell field has also been considered briefly. Our timelike null singularity turns out to be in accordance with quantum regular / singular against the Dirac / Klein-Gordon probes, respectively.

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