There exist several approaches that investigate the connectedness of spacetime events through solutions of the Lorentz force equation. These approaches separate into three categories, that consider different equations. We clarify the physical meaning of each equation showing that only one method is based on the Lorentz force equation. The other two approaches lead respectively to a less restrictive equation that defines an electromagnetic flow on the cotangent fiber bundle, or to an unphysical constraint between charge-to-mass ratio and proper length of the solution. We outline the physical meaning of each approach studying the variational formulations and clarifying the results obtained in the explored directions.

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I. INTRODUCTION

Recently, a work by E. Caponio and A. Masiello, in some articles already cited as [16], and announced in [17, 19] has appeared, with minor modifications, in the pages of the Journal of Mathematical Physics [18]. The article deals with the problem of connectedness of a globally hyperbolic spacetime through solutions of the Lorentz force equation. This problem attracted some attention in the last years since a positive answer, i.e. the proof of the statement \textit{in a globally hyperbolic spacetime, given two chronologically related events, there is a solution of the Lorentz force equation passing through them}, would generalize to charged particles a well-known theorem by Avez and Seifert [2, theorem 3.18], [10, proposition 6.7.1]. Caponio and Masiello’s work, now published, had the merit of introducing a Kaluza-Klein approach to deal with this geometrical problem. Unfortunately, it shares a problem, with some other works on the same subject, concerning the physical interpretation of the results obtained. This problem arises since while in the Lorentz force equation the 4-velocity should be \textit{a priori} normalized, usually with conditions as \(u^\alpha u_\alpha = 1\) or \(p^\alpha p_\alpha = m^2\), in Caponio and Masiello’s article this condition is dropped leading to different physical and mathematical problems (see subsections A and C of the bibliography). Indeed, dropping the normalization condition the space of solutions becomes infinitely larger than in the usual Lorentz force equation and it becomes easier to prove the connectedness through solutions of the modified equation. While these modified problems have a mathematical interest in their own right and can be studied as such [22, 25], results on them have been sometimes improperly ascribed as results on the more restrictive Lorentz force equation [18, 24, 27, 31, 32, 34] leading to a condition where, apparently, the same claims and theorems are published twice in different journals (compare [18, 21]). In fact some of those results are related to the Lorentz force equation (subsection B of the bibliography) while others are related to less restrictive equations or different constraints (subsections A and C of the bibliography). In this comment we try to clarify this complex situation putting into perspective the results obtained in the different directions.

II. LORENTZ FORCE EQUATION (LFE)

The Lorentz force equation describes the motion of a charged particle over spacetime in presence of an electromagnetic field. It describes the motion of ideal spinless, pointlike particles when the radiative corrections and all the quantum effects can be ignored.

Consider a spacetime \(M\), that is a time-oriented \(n\)-dimensional Lorentzian manifold endowed with a metric \(g\) with signature \((+,-,\cdots,-)\), and an electromagnetic field \(F\) on \(M\), that is, a skew symmetric closed 2-form. Let \(c = 1\). A point particle of rest mass \(m > 0\) and electric charge \(q \in \mathbb{R}\), moving under the action of the field \(F\), has a worldline which satisfies the \textit{Lorentz force equation} (LFE) (cf. [6, 8])

\[
D_s \left( \frac{dx}{ds} \right) = \frac{q}{m} \hat{F}(x) \left[ \frac{dx}{ds} \right].
\]

Here \(x = x(s)\) is the worldline of the particle parameterized with respect to the proper length, \(\frac{dx}{ds}\) is the \(n\)-velocity, \(D_s \left( \frac{dx}{ds} \right)\) is the covariant derivative of \(\frac{dx}{ds}\) along \(x(s)\) associated to the Levi-Civita connection of \(g\), and \(\hat{F}(x)[\cdot]\) is the linear map on \(T_x M\) obtained raising the left-hand index of \(F\). It is understood that the Lorentz force equation is determined only once the parameter \(q/m\) has been given. The solutions of the Lorentz force equation are future-oriented timelike curves parametrized with respect to proper length. They are
interpreted as trajectories of the particle with the given ratio $q/m$ on spacetime. These solutions may be regarded as $C^2$ mappings $x$ from an interval of the real line to $M$. Given such mappings $x(\lambda)$ all the others obtained from this one by an orientation-preserving reparametrization, $\lambda = f(X')$, $f' > 0$, are regarded as physically equivalent. In other words the parametrization assures only the differentiability of the curve, what matters physically is the image of the application, that is, the trajectory. Note, indeed, that any mathematical model willing to describe the motion of a particle should face with what can be actually observed. The principal observable in the motion of a charge is its trajectory on spacetime. Indeed, knowing the spacetime under consideration and the electromagnetic field, from the trajectory one usually recovers the charge-to-mass ratio (using (1)) and the proper time of the particle (integrating the line element over the trajectory). The proper time so obtained can sometimes be compared with the one measured directly from the decay of a charged particle previously created, showing that the previous method is consistent with observations.

A fundamental aspect of Eq. (1) is that it contains a second derivative on the left-hand side and a first derivative on the right-hand side. This implies that the curve $x$ should be parametrized with respect to proper time $s$ in order to infer what is the corresponding charge-to-mass ratio.

Note that given a timelike solution $x(s)$ of (1) the charge-to-mass ratio $q/m$ is uniquely determined unless $x(s)$ satisfies

$$\dot{F}(x) \left[ \frac{dx}{ds} \right] = 0, \quad (2)$$

and then

$$D_s \left( \frac{dx}{ds} \right) = 0. \quad (3)$$

That is, $x(s)$ is a geodesic of $M$ whose tangent vectors stay in the $\ker \dot{F}$. If we restrict our analysis to solutions of (1) connecting two chronologically related events, $x_0 \ll x_1$ this situation occurs in very special cases and physically is not of primary interest. In fact given the geodesics connecting the two events, suppose that (2) is satisfied by one of them, we see that this condition is spoiled under a suitable small perturbation $\delta F$ i.e. condition (2), if regarded as a condition on the electromagnetic field, is unstable. However, we shall include this case in our study. Let $X$ be the set of $C^2$ timelike curves that satisfy (1) for a certain $q/m$ (each one with their own $q/m$). We define a charge-to-mass ratio function $\Delta_m : X \to \mathbb{R} \cup \{R\}$ where $R$ here is just a symbol in the following way: if the curve $x \in X$ does not satisfy (2), then we define (we use the roman letters for the function) $\Delta_m(x) = q/m$ where $q/m$ is derived from (1); if the curve $x$ satisfies (2), then we define $\Delta_m(x) = R$. In this way the function $\Delta_m(x)$ becomes an observable. The case $\Delta_m(x) = R$ happens when from the observation of $x$ the observer can not infer the real (in mathematical sense) value of the charge-to-mass ratio of the particle. If $\Delta_m(x) = R$, then $x$ solves (1) independently of the value of $q/m$.

Finally, let $F$ be an exact electromagnetic field $F = dA$. Recall that the Lorentz force equation is satisfied by any timelike stationary point of the charged-particle action [16] (we write for short $ds = \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$)

$$I_{x_0, x_1}[\gamma] = \int_\gamma (ds + \frac{q}{m} \omega), \quad (4)$$

defined on the set of $C^1$ causal curves connecting $x_0$ and $x_1$. Thus, in order to prove the existence of connecting solutions of the Lorentz force equation one can look for timelike stationary points of this functional.

### III. THE ELECTROMAGNETIC FLOW EQUATION (EFE)

In order to describe the motion of a charged particle we have to determine the set of trajectories of its motion on spacetime. It is clear from the previous equations that the set of solutions will differ only for particles having a different charge-to-mass ratio and that at least looking at the motion of the particle, neither the mass nor the charge are separately observable [14]. Consider the equation

$$D_\lambda \left( \frac{dx}{d\lambda} \right) = Q\dot{F}(x) \left[ \frac{dx}{d\lambda} \right], \quad (5)$$

with $\lambda$ a dimensional parameter (in some works [18, 23, 27, 32, 33] the letter $s$ is used in place of $\lambda$ but this does not mean that in those works $s$ is the proper time). Its dimension is chosen in such a way that $Q$ has the dimension of a charge. The solutions of this equation are mappings $x : (\lambda_0, \lambda_1) \to M$. This equation is referred, in [18, 23, 27, 31, 32, 33, 34, 35], as the Lorentz force equation of a particle of charge $Q$ (sometimes normalized). We wish to show that it is inappropriate to call it Lorentz force equation, as this terminology could lead to confusion both from the physical and mathematical side. In order to understand what represents a solution of Eq. (5) let us simply take a solution $x(\lambda)$, parametrize it with proper time and substitute it on (1). First of all, Eq. (5) implies that

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = (\frac{ds}{d\lambda})^2 = C^2 \quad (6)$$

that is the parameter $\lambda$ is related to proper length by the relation $C d\lambda = ds$ where $C$ is a constant. We restrict to the timelike solutions, the only ones that may receive the interpretation of massive particles, and therefore restrict our attention to the case $C \in \mathbb{R}, C > 0$. Replacing the...
previous equation in (1) we find
\[ D_s \left( \frac{dx}{ds} \right) = \frac{Q}{C} \tilde{F}(x) \left[ \frac{dx}{ds} \right] . \]  
and hence the charge-to-mass ratio is \( q/m = Q/C \). All the problems with Eq. (5) arise from the fact that the constant \( C \) is not fixed a priori leading to solutions of arbitrary charge-to-mass ratios. Solutions to this equation cannot be associated to trajectories of a given particle.

1. Problems involving the “charge” \( Q \)

If we characterize Eq. (5) with a parameter \( Q \) it should at least have some physical meaning in the solutions of that equation. Consider the equation (5) with \( Q \) replaced with \( Q' \neq Q \), \( \text{sgn}(Q) = \text{sgn}(Q') \). If \( x(\lambda) \) is a solution of (5), \( x(\frac{Q'}{Q} \lambda) \) is a solution of (5) with \( Q \) replaced with \( Q' \). However, the trajectories are exactly the same so the set of trajectories of (5) is independent of the value of \( Q \). It depends only on \( \text{sgn}(Q) \). Thus there is no reason to call Eq. (5) the Lorentz force equation of charge \( Q \) since its solutions when regarded as trajectories (ultimately the only observable) do not depend on \( |Q| \). It is preferable to replace \( Q \rightarrow \epsilon = \text{sgn}(Q) \) and change the dimensionality of \( \lambda \) to obtain, given \( \epsilon = \pm 1 \),
\[ D_\lambda \left( \frac{dx}{d\lambda} \right) = \epsilon \tilde{F}(x) \left[ \frac{dx}{d\lambda} \right] . \]  
We see therefore that Eq. (5) displays a coefficient that has in fact no clear physical meaning and that does not select different spaces of solutions although the notation could suggest the contrary. These observations point out that, in general, if a mathematical model is designed to describe a physical situation, as far as possible, only observables should enter the construction, and in particular the coefficients of the model should have some physical consequence in order to receive a physical interpretation. In this case the coefficient \( Q \) in (5) cannot receive an interpretation and should be better removed as suggested.

2. Problems involving the charge-to-mass ratios

Consider the motion of two whatever particles, with the same sign of their charge-to-mass ratios. Their trajectories solve the Lorentz force equation (11) each one with their respective charge-to-mass ratios. However, both trajectories can be easily parametrized to give a solution of the same equation (5) with \( Q \) of the same sign of their charge-to-mass ratios. It suffices to take as parameter \( d\lambda = \frac{q}{qm} ds \) where \( q \) and \( m \) are the charge and the mass of the particle that follows the trajectory considered. Thus Eq. (5) is unable to distinguish between solutions with different charge-to-mass ratios. Any trajectory solution of a Lorentz force equation relative to a certain charge-to-mass ratio with \( \text{sgn}(q/m) = \text{sgn}(Q) \) is solution of (5), while the converse is of course not true.

In order to solve these problems the only way out is to add a constraint that should be satisfied by every solution. Eq. (5) should be coupled with the equation
\[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \left( \frac{dx}{d\lambda} \right)^2 = m^2 . \]  
Note that while any solution of Eq. (5) implies that \( ds/d\lambda \) is a constant, here we are fixing its value \( a \) priori and thus are removing many solutions of the original equation (11). The system so obtained is clearly equivalent to the original Lorentz force equation but should be better avoided since there appear three unobservable quantities (at least looking at the motion of the particle) \( Q \), \( m \), and \( \lambda \) while in the Lorentz force equation all the coefficients appear in truly observable combinations. In the works mentioned no a priori constraint is imposed on the square of the 4-velocity. Some authors refer to (11) as a source for their terminology while Sachs and Wu define the Lorentz force equation correctly as a system of (5) with (11) definitions 3.1.1 and 3.8.1. Note that even regarding (5) as an equation of a system it remains misleading to call it Lorentz force equation of a charge \( Q \), as in this way one can easily forget that this definition is correct only inside the system.

A. The electromagnetic flow

As we said, most problems arise because the set of solutions of (5) is larger than the one of (11). We may say that the first equation is solved by the trajectory of every charged particle with a charge-to-mass ratio of the same sign of \( \epsilon \). It is therefore natural that it can be recast in a form where no coefficient \( Q \) appears
\[ D_\lambda \left( \frac{dx}{d\lambda} \right) = \epsilon \tilde{F}(x) \left[ \frac{dx}{d\lambda} \right] . \]  
This equation can be studied in its own right. It needs however a different name. We suggest electromagnetic flow equation (EFE) in analogy with the equation of the geodesic flow. Indeed, let us introduce the quantity \( p_\alpha = g_{\alpha\beta} dx^\beta /d\lambda \) (we use the letter \( p \) since this is a one-form i.e. it lives in \( T^*M \); it has not the dimensions of a momentum), then Eq. (11) can be rewritten
\[ D_\lambda \left( \frac{dx}{d\lambda} \right) = \epsilon \tilde{F}(x) \left[ \frac{dx}{d\lambda} \right] . \]  
This equation determines a flow in \( T^*M \). The trajectories of this flow when projected on \( M \) satisfy Eq. (11). There is, however, a relevant difference with respect to the geodesic flow. In fact in that case the trajectories
starting from two points in the same fiber \( T^*M \), say \((x,p),(x',p')\), with \( p' = ap \), \( a \in \mathbb{R} \) project on the same trajectory over \( M \) while this does not happen for the electromagnetic flow. For this reason the solutions of \( \Omega \) are infinitely more numerous than those of the Lorentz force equation. This difference is crucial if one tries to prove the connectedness of spacetime.

**Remark.** For instance we show that existence results for the Lorentz force equation \( \Omega \) are in fact multiplicity results for the electromagnetic flow equation \( \Omega \).

Consider two events \( x_0 \ll x_1 \) in a globally hyperbolic spacetime and let \( F \) be exact, in \([21]\) it is proved that there is an interval \( U = (-r,r) \) of the real line such that for each \( q/m \in U \) there is a connecting solution of the Lorentz force equation \( \Omega \). This result was improved in \([21]\) where it was shown that \( r = +\infty \) if there is no null geodesic connecting \( x_0 \) and \( x_1 \). We have already seen that, suitably parametrized, a solution of \( \Omega \) with \( \text{sgn}(q/m) = \text{sgn}(Q) \) becomes solution of \( \Gamma \), but we can say more: solutions of Lorentz force equations with different charge-to-mass ratios are distinct unless a very special case. Indeed, let \( x(s) \) be solution of \( \Omega \) for a charge-to-mass ratio \( q/m \) and let \( x'(s) \) be a solution of \( \Omega \) with coefficient \( q'/m' \neq q/m \). Suppose \( x' = x \) and subtract the two Lorentz force equations. Since \( \Delta(q/m) \neq 0 \) we easily find that \( x \) satisfy the system \([2], [3]\). The existence of such geodesics, as we already said, happens only if the electromagnetic field satisfies a very restrictive constraint. For all the other cases we conclude that there is an infinite degeneracy of connecting solutions of \( \Gamma \): if \( \text{sgn}(Q) > 0 \) (resp. \( \text{sgn}(Q) < 0 \)) there is a solution for each \( q/m \in (0,r) \) (resp. \( q/m \in (-r,0) \)). This represents the strongest result up to now available on the existence and multiplicity of connecting solutions of \( \Gamma \).

### IV. OTHER RELATIONS BETWEEN LFE AND EFE

We point out in this section some other problems in which the roles of the Lorentz force equation and the electromagnetic flow equation could be confused.

#### A. Symplectic formulation

Equation \( \Sigma \) is sometimes improperly referred as the Lorentz force equation of a charge \( Q \) in works that involve the so called twisted symplectic form \([1],[2],[13]\). Consider a spacetime \( M \) having a metric \( g_{\mu\nu} \) of signature \((+\cdots-\cdots)\). On the cotangent space \( T^*M \) lives the canonical form \( \Omega \) that in local coordinates reads \( \Omega = dp_\mu \wedge dq^\mu \). Let \( \pi : T^*M \to M \) be the canonical projection. On \( T^*M \) we can define the twisted symplectic form \( \Omega_F = \Omega + Qx^*F \) where \( F \) is the electromagnetic two-form. Let the relativistic invariant (super)Hamiltonian be \( \mathcal{H} = \frac{1}{2} \frac{d}{dx^i} p_\mu p^\mu \). A straightforward calculation shows that the integral lines of the Hamiltonian flow, i.e. the integral lines of \( X = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} + \frac{dp_\mu}{dt} \frac{\partial}{\partial p_\mu} \) such that \( i_X \Omega_F = -d\mathcal{H} \), projected on \( M \) are solutions of \( \Omega \). This approach is surely fascinating in fact, contrary for instance to the variational methods, here there is no reference to the potential 1-form. However, the equation deduced is not the Lorentz force equation so the same criticisms can be repeated here. The true Lorentz force equation can be obtained from the Hamilton equations using as Hamiltonian the relativistic energy.

#### B. Jacobi fields

Attention should also be paid on the different results available for the study of the Jacobi fields for the two equations. Indeed both \([24]\) and \([1]\) present a calculation of the deviation equation for the Lorentz force equation. However, the two expressions differ since in the former case the derivatives are with respect to a generic parameter while in the latter case they are with respect to proper time. Indeed, in \([24]\) what was actually calculated is the deviation equation for the electromagnetic flow equation. Since solutions of the Lorentz force equation, even with different charge-to-mass ratios, are solutions of the electromagnetic flow equation, the Jacobi fields for the Lorentz force equation are Jacobi fields for the electromagnetic flow equation while the converse is not true. Moreover, in principle, there could exist a Jacobi field of the electromagnetic flow equation which is not a Jacobi field of the Lorentz force equation for no values of the parameter \( q/m \). This essentially because the Jacobi field may actually be, in the space of solutions of the electromagnetic flow equation, a tangent vector that connects solutions with different values of the charge-to-mass ratio. Whether this possibility could be indeed realized in some cases could be the subject of further investigation.

#### C. The non-relativistic case

We end the section with a digression on the non-relativistic Lorentz force equation. Let us begin in an Euclidean space \( E \) and a spacetime \( \Pi = E \times \mathbb{R} \) of coordinates \( \{x^i,t\} \). It has the form

\[
\frac{d^2 x^i(t)}{dt^2} = \frac{q}{m} (F^i + \epsilon_{ijk} \frac{dx^j(t)}{dt} B^k), \tag{13}
\]

which differ from the relativistic Lorentz force equation in Minkowski spacetime only for a factor \( 1/\sqrt{1-(dx/dt)^2} \) lacking at the left-hand side, between the two derivatives. Now suppose that the electric field vanishes, then in terms of the relativistic electromagnetic tensor the previous equation reads

\[
\frac{d^2 x^i(t)}{dt^2} = \frac{q}{m} \sum_j \frac{dx^j}{dt}, \tag{14}
\]
which admits a natural generalization in a curved space $S$

$$
D_t \frac{dx^j(t)}{dt} = \frac{q}{m} F^j_i \frac{dx^i}{dt},
$$

(15)

where $D$ is the covariant derivative on space compatible with the space metric. This equation can be used to determine a flow in $T^*S$ called magnetic flow \[10\], the projection of trajectories in the flow being solutions of Eq. \[11\]. Thus in this non-relativistic limit the magnetic flow equation \[10\] and the Lorentz force equation coincide.

Note that although Eq. \[10\] is formally equivalent to \[11\] (indeed there is no constraint on the square of $dx^i/dt$ and $t$ may be regarded as an external parameter) in a relativistic context the electromagnetic flow equation and the Lorentz force equation differ. For this reason, while it is quite natural to consider the problem of connectedness of space points at a fixed times (that is the problem of connecting two spacetime events) in the non-relativistic purely magnetic limit, i.e. to look for parametrized solution of \[10\] that satisfy a constraint $x^i(t_0) = x^i_0$ and $x^i(t_1) = x^i_1$, the same formal problem for Eq. \[11\] is less interesting since it has a completely different interpretation (see the next section).

V. A VARIATIONAL PROBLEM

Consider the functional

$$
J_{x_0, x_1}[\gamma] = \int_{\lambda_0}^{\lambda_1} \left( \frac{1}{2} g(\gamma'(\lambda), \gamma'(\lambda)) + Q \omega[\gamma'(\lambda)] \right) d\lambda.
$$

(16)

on the space of all the (absolutely continuous) causal curves, which connect $x_0$ and $x_1$ in the interval $[\lambda_0, \lambda_1]$. It generalizes the “energy” functional of Lorentzian geometry \[2\] to include a vectorial potential. The energy functional contrary to the length functional is well defined even for connecting curves whose causal character changes with the parametrization. The connectedness of spacetime through energy extremals has been studied deeply in the mathematical literature providing an application of Morse and Ljusternik-Schnirelman theory (see the survey \[12\]). The problem was then generalized to include a vectorial potential as in \[19\] (the works in section C of the bibliography are related to this kind of problems). From the physical point of view the fact that the functional $J_{x_0, x_1}$ is defined independently of the causal character of the curve makes it difficult to establish the causal character of the extremals although it enlarges its domain of applicability. Unfortunately, it has been often claimed that if the extremal is timelike than it is a solution of the LFE or equivalently that the functionals $J$ and $\mathcal{J}$ are equivalent. However, a timelike stationary point $\eta(\lambda)$ of $J_{x_0, x_1}$ satisfies Eq. \[19\] and the constraint

$$
x(\lambda_0) = x_0, \quad x(\lambda_1) = x_1.
$$

(17)

Let $\Delta \lambda = \lambda_1 - \lambda_0$, and $ds/d\lambda = C$, then integrating $C = (\int_{\gamma} ds)/\Delta \lambda$, thus the stationary point $\eta$ is a solution of the Lorentz force equation \[11\] with charge-to-mass ratio $q/m$ that satisfies

$$
\frac{q}{m} \int_{\eta} ds = Q \Delta \lambda.
$$

(18)

Here $Q$ and $\Delta \lambda$ are fixed in the variational principle but the length of the extremal is not fixed a priori and therefore the charge-to-mass ratio of the extremal is not determined a priori: different extremals will have different charge-to-mass ratios. This happens because in order to fix the charge-to-mass ratio one needs the constraint \[19\] while the variational principle \[19\] imposes the condition \[17\] which implies that all the extremals have the same product between charge-to-mass ratio and length. Thus, whatever is the choice of the product $Q \Delta \lambda$, the existence of stationary points of the action \[16\] does not imply the existence of a connecting solution of the Lorentz force equation having a prescribed charge-to-mass ratio $q/m$. Using this approach it is for instance not possible to prove the existence of connecting trajectories for a charge-to-mass ratio like the one of the electron or the one of the proton. Suppose one proves that \[19\] admits a timelike extremal: it can actually have a charge-to-mass ratio that does not corresponds to an existing particle. More generally, the same happens if one proves that Eq. \[18\] has a connecting solution. For this reason the physical interpretation of Eq. \[11\] and Eq. \[19\], and the variational principles \[10\] and \[11\] are different and in general to have a strict contact to physical questions \[11\] or \[19\] should be used. As another example suppose we wish to study in how many ways an electron can leave an event $x_0$ to reach an event $x_1$, then we should clearly study how many extremals the charged-particle action has. On the contrary if one proves that the action \[18\] has say, four extremals, it could be that none of them has the charge-to-mass ratio of the electron.

Let us consider in more detail the action \[16\]. Note that the extremals when regarded as unparametrized curves depend only on the product $Q \Delta \lambda$ of \[16\]. In other words given $\beta = Q \Delta \lambda$ two choices of the action \[16\] with the same $\beta$ have the same extremals up to reparametrizations. We said that its timelike extremals, when regarded as trajectories, are solutions of the Lorentz force equation \[11\] for a charge-to-mass ratio that satisfies the constraint $\frac{q}{m} \int_{\eta} ds = \beta$. Conversely, given a timelike connecting solution of \[11\], $\eta(\lambda)$, that satisfies the constraint \[19\], a parametrization can be found so that $\eta(\lambda)$ becomes a timelike extremal of \[16\]; it suffices to choose the parametrization such that $d\lambda = \frac{m}{Q \Delta \lambda} ds$ and $\eta(\lambda_0) = x_0$.

We give now a variational principle that has the same unparametrized extremals of \[16\]. Consider the functional

$$
K_{x_0, x_1}[\gamma] = \frac{1}{2} \left( \int_{\gamma} ds \right)^2 + \beta \int_{\gamma} \omega
$$

(19)

defined on the set of $C^1$ causal curves connecting $x_0$ and $x_1$. A computation of the Euler-Lagrange equation im-
mediately shows that the timelike extremals of this functional are those timelike curves which satisfy the Lorentz force equation (14) having a charge-to-mass ratio and a length that satisfy the constraint \( \frac{q}{m} \int \eta \, ds = \beta \), i.e., they are the same, but this time unparametrized, extremals of (16). This functional removing the unobservable dependence on the parametrization could help to reveal more clearly the physical meaning of (16). For fixed \( q/m \) and \( \beta \) (that is (16) have in general different extremals, however let \( \eta \) be an extremal of the charged-particle action, and choose \( \beta = \frac{q}{m} \int \eta \, ds \) then (16) and therefore (10) will have \( \eta \) as extremal. In other words, for each connecting solution \( \eta \) of the Lorentz force equation of charge-to-mass ratio \( q/m \) there is a choice of \( \beta \) such that \( \eta \) is an extremal of \( K \) (or, which is the same, \( J \)) for that \( \beta \). Thus no connecting solution of the Lorentz force equation is left out considering the extremals of \( K \) for all the values of \( \beta \in \mathbb{R} \). The problem is that they are classified according to a parameter \( \beta \) which is not as interesting as the charge-to-mass ratio \( q/m \) is. It is interesting to note that since \( 0 \leq \int \eta \, ds \leq l(x_0, x_1) \), where \( l(x_0, x_1) \) is the Lorentzian distance function, Eq. (18) implies that

\[
\frac{q}{m} \geq \frac{\beta}{l(x_0, x_1)}
\]

that is, the variational principles (16), (19), for a given \( \beta \) have as timelike stationary points solutions of Lorentz force equations with charge-to-mass ratios having an absolute value bounded from below.

A. Erratas and other comments

It seems that some confusion regarding the use of the Lorentz force equation and its interpretation started from the work [30] where the authors introduced the functional (10). In this respect it is better to point out some erratas that may lead to improper interpretations. They show that an extremal point \( x(\lambda) \) of the action (the same as \( J \) but without the factor 1/2)

\[
J_{x_0, x_1}[\gamma] = \int_{x_0}^{x_1} \{ g(\gamma' (\lambda), \gamma' (\lambda)) + Q \omega [\gamma' (\lambda)] \} \, d\lambda
\]

has a constant square of the 4-velocity that they call \( m \) as in (14). A proof that \( x(\lambda) \) is also an extremal point of (4) with \( q/m = Q/m \) was also claimed, but unfortunately this statement is true only if \( J \) is replaced with \( J \). In fact, (we use our notation) knowing that the stationary point satisfies \( J \) for a certain \( m \), they use this in \( J \) to rewrite

\[
g(\gamma' (\lambda), \gamma' (\lambda)) = (\frac{ds}{d\lambda})^2 = m \frac{ds}{d\lambda}
\]

and replacing in (21) obtain \( m I_{x_0, x_1} \). Then they go on to calculate the Euler-Lagrange equation of (16) assuming that this should be equivalent to the initial one. However, it is well known that this way of working is incorrect and in fact the two variational principles \( J \) and \( I \) so constructed, do not necessarily share an extremal point. Indeed, using \( ds = m d\lambda \) and the Euler-Lagrange equation for \( J \) we find that the trajectory \( x \) satisfies the Lorentz force equation with charge-to-mass ratio \( Q/(2m) \) while in order to be an extremal of the obtained \( I \) it should satisfy it with charge-to-mass ratio \( Q/m \). In general it is incorrect to replace inside the variational principle information that follows from its Euler-Lagrange equation as the new variational principle so obtained does not have the same stationary points.

This work generated some confusion in subsequent literature. For instance in [31] the authors include the 1/2 factor but then they state [31, Remark 1.1] (see also [32, Remark 1.2] and [26, p. 128]) that the functionals \( I \) with \( q/m = 1 \) and \( J \) with \( \beta = 1 \) have the same stationary points up to reparametrizations. They refer for a proof to [30]. However, this statement is incorrect since, as we said above, it is true that each extremal of \( J \), with \( \beta = 1 \), is extremal of \( I \) for a certain, unknown a priori, \( q/m \), and it is true that an extremal of \( I \), with \( q/m = 1 \), is an extremal of \( J \) for a certain, unknown a priori, \( \beta \), but this does not imply that \( J \) with \( \beta = 1 \) and \( I \) with \( q/m = 1 \) have the same stationary points. The corrected proof of the modified statement was given in [22]. We stress that in any case this problem did not affect the mathematical conclusions of those works although it severely restricts the physical implications.

VI. EXISTENCE RESULTS AND CONCLUSIONS

Let us come to the existence results available. A first result was obtained for the existence of connecting solutions of (5) in [30]. In [16, 17] it was proved that (5) has always a connecting solution in a globally hyperbolic spacetime. An analogous result for Eq. (16) was given in [21] (first relevant advances in [20]). This implied in particular the existence of a maximum for the charged-particle action (4) and could be read as a multiplicity result for Eq. (5).

The work on the action \( J \) began in [22, 24], the action \( J \) being a natural generalization of the “energy” functional of Lorentzian geometry to include a vectorial potential. This allows one to consider geometrical questions that otherwise could not be implemented using \( I \), for instance the spacetime connectedness through spacelike extremals of \( J \). From the physical point of view, however, most interesting are timelike extremals and in this respect existence results for \( J \) are up to now weaker than those for \( I \) (in a globally hyperbolic spacetime, for instance, as far as we know there could be no timelike extremals for certain values of \( \beta \)), although related results have been obtained for stationary spacetimes [24, 25, 26, 27, 32, 33], time periodic potentials and metrics [34], or under other assumptions [25, 28].
Table I: Different existence problems for the connecting solutions. Case B is the one of the Lorentz force equation. Let $\beta$ and $q/m$ be given. A solution of B is a solution of A but not necessarily of C. A solution of C is a solution of A but not necessarily of B. A solution of A is not necessarily a solution of B or C.

Although action $J$ has a good behavior under standard variational methods and Morse theory, and it gives rise to some interesting mathematical problems, we believe that it should not be studied as a substitute for $I$. Indeed, the variational difficulties for $I$ are now circumvented using a geometrical interpretation [21] that makes it possible to use causal techniques. Moreover, even if the results for $J$ and $I$ were comparable, the physical interpretation of $J$’s timelike extremals, that we previously pointed out, would not allow to make contact with realistic charge-to-mass ratios.

Finally, in order to clarify the relation between different articles we consider three existence problems for trajectories connecting the events $x_0$ and $x_1$. We regard each one as the problem of finding a connecting solution of

A. The electromagnetic flow equation (10) (or, which is the same 5).
B. The Lorentz force equation (1).
C. The equation (5) with the constraint (17).

Table II presents the three different existence problems pointing out if they have a variational Lagrangian formulation, on which parameters the functional depends, what is the physical constraint on the charge-to-mass ratio and what are the works that dealt or that are related with that problem.

In conclusion we believe to have clarified the mathematical and physical aspects of different problems considered in the literature. Although each one has something related to the Lorentz force equation, attention should be paid since the results available have different mathematical and physical meanings.

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14. Of course, only as long as the physics of the Lorentz force equation is concerned. At the variational level one should add more terms to the charged-particle action in order to make both charge and mass observables. This, however, would lead us to a different mathematical problem.

15. Since we regard the solution always as future-oriented trajectories we have in fact two flow equations depending on the value of $\epsilon$.

**Connecting solutions: Part A (Electromagnetic flow equation)**

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