A remark on Rickard complexes

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Abstract In this paper, we characterize a Rickard complex, which induces a Rickard equivalence between the block algebras of a block \( b \) and its Brauer correspondent and whose vertices have the same order as defect groups of the block \( b \). The homology of such a Rickard complex vanishes at all degree but degree \( q \), and the homology at degree \( q \) induces a basic Morita equivalence between the block algebras in the sense of Puig.

Keywords: Finite group; Block; Rickard complex; Vertex

1. In [5], J. Rickard exhibits a splendid Rickard complex, which induces a splendid Rickard equivalence between the block algebra of a \( p \)-block of a finite \( p \)-nilpotent group and the group algebra of its defect group, which is isomorphic to a Morita equivalence not induced by a \( p \)-permutation module. Then in [1], Harris and Linckelmann extend Rickard’s technique and show a splendid Rickard complex, which induces a splendid Rickard equivalence between the block algebras of a block for finite \( p \)-solvable groups with abelian defect groups and its Brauer correspondent, which is also isomorphic to a Morita equivalence not induced by a \( p \)-permutation module. The two splendid Rickard complexes have vertex \( \Delta(Q) \) in terms of Puig (see Paragraph 7 below), where \( Q \) is a defect group of the blocks and \( \Delta(Q) \) is the diagonal subgroup of \( Q \times Q \). In this paper, we characterize a Rickard complex, which induces a Rickard equivalence between the block algebras of a block \( b \) and its Brauer correspondent and whose vertices have the same order as defect groups of the block \( b \).

2. We recollect some notation in [4]. Let \( p \) be a prime number and let \( k \) be an algebraically closed residue field \( k \) of characteristic \( p \). From the point of view of [4], complexes are considered as \( D \)-modules where, denoting by \( \mathfrak{F} \) the commutative \( k \)-algebra of all the \( k \)-valued functions on the set \( \mathbb{Z} \) of all rational integers, \( D \) is the \( k \)-algebra containing \( \mathfrak{F} \) as a unitary \( k \)-subalgebra and an element \( d \) such that \( D = \mathfrak{F} \oplus \mathfrak{F}d, \ d^2 = 0 \) and \( df = \text{sh}(f)d \neq 0 \) for any \( f \in \mathfrak{F} \) \( \setminus \{0\} \)

where sh denotes the automorphism on the \( k \)-algebra \( \mathfrak{F} \) mapping \( f \in \mathfrak{F} \) onto the \( k \)-valued function sending \( z \in \mathbb{Z} \) to \( f(z + 1) \); moreover, we denote by \( s \) and \( i_z \), for any \( z \in \mathbb{Z} \), the \( k \)-valued functions mapping \( z' \in \mathbb{Z} \) on \((-1)^{z'}\) and \( \delta_z^{z'} \) respectively. Except for all the group algebras over \( D \), we assume that all the modules and the algebras over \( k \) are finitely generated. If \( A \) is a \( k \)-algebra we denote by \( A^* \) the group of invertible elements of \( A \), and by \( A^0 \) the opposite \( k \)-algebra. Note that we have an isomorphism \( t: D \cong D^0 \) mapping \( f \in \mathfrak{F} \) on the \( k \)-valued function sending \( z \in \mathbb{Z} \) to \( f(-z) \), and \( d \) on \( sd \).

3. A \( D \)-interior algebra is a \( k \)-algebra \( A \) endowed with a unitary \( k \)-algebra homomorphism \( g: D \rightarrow A \); for any \( x, y \in D \) and any \( a \in A \), we write \( x \cdot a \cdot y \) instead of \( g(x)a\phi(y) \). Note that the isomorphism \( t: D \cong D^0 \) then determines a \( D \)-interior algebra structure for \( A^0 \). Moreover, we have a \( k \)-algebra homomorphism \( D \rightarrow k \) mapping \( f + f'd \) on \( f(0) \) for any \( f, f' \in \mathfrak{F} \), so that any \( k \)-algebra
admits a trivial structure of \( \mathcal{D} \)-interior algebra. The \( \mathcal{D} \)-interior algebra structure on \( A \) induces a \( \mathcal{D} \)-module structure on \( A \) by the equalities
\[
f(a) = \sum_{z,z' \in \mathbb{Z}} f(z) i_{z'} \cdot a \cdot i_{z'-z} \quad \text{and} \quad d(a) = (d \cdot a - a \cdot d) \cdot s
\]
for any \( a \in A \) and any \( f \in \mathcal{F} \). The \( k \)-algebra \( A \) endowed with this \( \mathcal{D} \)-module is a \( \mathcal{D} \)-algebra in the sense of Puig (see [4, 11.2.4]).

4. Let \( G \) be a finite group; recall that a \( kG \)-interior algebra is a \( k \)-algebra endowed with a unitary \( k \)-algebra homomorphism from \( kG \). Similarly, a \( \mathcal{D}G \)-interior algebra is a \( k \)-algebra \( A \) endowed with a unitary \( k \)-algebra homomorphism \( \rho : \mathcal{D}G \to A \) (but \( A \) is always finitely generated); for any \( x \in \mathcal{D}G \) and \( a \in A \), we write \( x \cdot a \) and \( a \cdot x \) instead of \( \rho(x)a \) and \( ap(x) \) respectively. If \( A \) and \( A' \) are \( \mathcal{D}G \)-interior algebras, the tensor product \( A \otimes_k A' \) admits a \( \mathcal{D}G \)-interior algebra structure given by
\[
f \cdot (a \otimes a') = \sum_{z,z' \in \mathbb{Z}} f(z + z') i_z \cdot a \otimes i_{z'} \cdot a', \quad d \cdot (a \otimes a') = d \cdot a \otimes s \cdot a' + a \otimes d \cdot a'
\]
and \( g \cdot (a \otimes a') = g \cdot a \otimes g \cdot a' \) for any \( f \in \mathcal{F} \), any \( g \in G \) and any \( a, a' \in A \). Here the first equality makes sense since in the sum above all but a finite number of terms vanish and since we have \( \text{sh}(s) = -s \).

For any subgroup \( H \) of \( G \), we denote by \( A^H \) the centralizer of \( \rho(H) \) in \( A \); obviously \( \rho(x) \in A^H \) for any \( x \in \mathcal{D}C_G(H) \) and thus the restriction of \( \rho \) to \( \mathcal{D}C_G(H) \) induces a \( \mathcal{D}C_G(H) \)-interior algebra structure on \( A^H \). Let \( B \) and \( C \) be two \( kG \)-interior algebras. A \( k \)-algebra homomorphism \( f : B \to C \) is a \( kG \)-interior algebra homomorphism if \( f \) preserves the \( kG \)-interior algebra structures on \( B \) and \( C \); furthermore, if \( f \) is injective and \( f(B) = f(1)Cf(1) \), then \( f \) is a \( kG \)-interior algebra embedding. Similarly \( \mathcal{D}G \)-interior algebra homomorphisms and \( \mathcal{D}G \)-interior algebra embeddings are defined.

5. Let us denote by \( \mathcal{C}_0(A) \) the centralizer of the image of \( \mathcal{D} \) in \( A \); since the images of \( \mathcal{D} \) and \( G \) centralize each other, \( \mathcal{C}_0(A) \) inherits a \( kG \)-interior algebra structure and, according to the terminology in [4], the pointed groups, their inclusions, the local pointed groups, etc. over the \( \mathcal{D}G \)-interior algebra \( A \) are nothing but the pointed groups, their inclusions, the local pointed groups, etc. over the \( kG \)-interior algebra \( \mathcal{C}_0(A) \). However, if \( H_{\beta} \) is a pointed group over \( A \), so that \( \beta \) is a conjugacy class of primitive idempotents in \( \mathcal{C}_0(A)^H \), the \( k \)-algebra \( A_{\beta} = iA_{i} \) for any \( i \in \beta \) inherits a \( \mathcal{D}H \)-interior algebra structure mapping \( y \in \mathcal{D}H \) on \( y \cdot i = i \cdot y \); and the \( k \)-algebra \( A_{\beta} \) is called an embedded algebra associated with \( H_{\beta} \). For any subgroup \( H \) of \( G \), we call contractible any point contained in the two-sided ideal
\[
\mathcal{E}_0(A^H) = \mathcal{C}_0(A)^H \cap \{ d \cdot a + a \cdot d \mid a \in A^H \}
\]
and we set \( \mathcal{H}_0(A^H) = \mathcal{C}_0(A)^H / \mathcal{E}_0(A^H) \), which still inherits a \( kC_G(H) \)-interior algebra structure; whenever \( \mathcal{H}_0(A^G) = \{ 0 \} \) we say that \( A \) is contractible. It is clear that if \( M \) is a \( \mathcal{D}G \)-module then \( \text{End}_k(M) \) is a \( \mathcal{D}G \)-interior algebra and we say that \( M \) is contractible whenever \( \text{End}_k(M) \) is so [4, Corollary 10.9]; moreover, we say that \( M \) is 0-split if it is \( \mathcal{D}G \)-isomorphic to the direct sum of a contractible \( \mathcal{D}G \)-module and a \( kG \)-module endowed with the trivial \( \mathcal{D} \)-structure defined above.

6. Let \( G \) and \( G' \) be two finite groups and let \( b \) and \( b' \) be respective blocks of \( G \) and \( G' \). Clearly the \( k \)-linear map \( kG' \to kG' \) sending \( x \) onto \( x^{-1} \) for any \( x \in G' \) is an opposite ring isomorphism. We denote by \( b'^{o} \) the image of \( b' \) through this opposite ring isomorphism. The \( k \)-linear map \( k(G \times G') \to kG \otimes kG' \) sending \( (x, y) \) onto \( x \otimes y \) is a \( k \)-algebra isomorphism, through which we identify both sides so that \( b \otimes b'^{o} \) is a block of \( G \times G' \). Let \( \tilde{M} \) be an indecomposable \( \mathcal{D}(G \times G') \)-module associated with \( b \otimes b'^{o} \) such that the restrictions of \( \tilde{M} \) to \( G \times \{ 1 \} \) and to \( \{ 1 \} \times G' \) are
both projective. We denote by $\tilde{M}^*$ the $k$-dual of $\tilde{M}$ which, via the isomorphism $t$ (see [4, 10.1.3]), still has a $D(G \times G')$-module structure. Following [4, 18.3.2], we say that $\tilde{M}$ defines a Rickard equivalence between $kGb$ and $kG'b'$ if, for suitable contractible $D(G \times G')$- and $D(G' \times G)$-modules $C$ and $C'$, we have respective $D(G \times G)$- and $D(G' \times G)$-module isomorphisms

$$\tilde{M} \otimes_{kG'} \tilde{M}^* \cong kGb \oplus C \quad \text{and} \quad \tilde{M}^* \otimes_{kG} \tilde{M} \cong kG'b' \oplus C'$$

where $kGb$ and $kG'b'$ have the trivial $D$-interior structure defined above.

7. Let $\tilde{P}_j$ be a maximal local pointed group over the $D(G \times G')$-interior algebra $\text{End}_k(\tilde{M})$ or, equivalently, over the $k(G \times G')$-interior algebra $C_0(\text{End}_k(\tilde{M}))$. Then $\tilde{P}$ is a vertex of the $D(G \times G')$-module $\tilde{M}$ and a $D:\tilde{P}$-module $\tilde{N} = j \cdot \tilde{M}$ for some $j \in \tilde{\gamma}$ is a source of the $D(G \times G')$-module $\tilde{M}$.

According to Theorem 18.8 in [4], the images $P \subset G$ and $P' \subset G'$ of $\tilde{P}$ through the canonical projections $\pi: G \times G' \to G$ and $\pi': G \times G' \to G'$ are defect groups of $b$ and $b'$ respectively.

**Theorem 8.** Let $G$ be a finite group, let $b$ be a block of $G$ with defect group $P$ and let $b'$ be the Brauer correspondent of $b$ in the normalizer $G'$ of $P$ in $G$. Let $\tilde{M}$ be a noncontractible indecomposable $D(G \times G')$-module inducing a Rickard equivalence between $kGb$ and $kG'b'$. Let $D:\tilde{P}$-module $\tilde{N}$ be a source of the $D(G \times G')$-module $\tilde{M}$. Then the following are equivalent:

8.1. The groups $\tilde{P}$ and $P$ have the same order.

8.2. The homology of the $D(G \times G')$-module $\tilde{M}$ vanishes at all degree but some degree $q$, the homology $H_q(\tilde{M})$ at degree $q$, a $k(G \times G')$-module, induces a basic Morita equivalence between $kGb$ and $kG'b'$ in the sense of Puig in [4], and the $D:\tilde{P}$-module $\text{End}_k(\tilde{N})$ determined by the $\tilde{P}$-conjugation and the $D$-algebra structure on $\text{End}_k(\tilde{N})$ is 0-split.

In this case, the homology of the $D:\tilde{P}$-module $\tilde{N}$ vanishes at all degree but degree $q$, and the homology $H_q(\tilde{N})$ at degree $q$, a $k:\tilde{P}$-module, is a source of the $k(G \times G')$-module $H_q(\tilde{M})$.

9. We recall Brauer quotients and Brauer homomorphisms in [6] and then prepare several lemmas. Let $G$ be a finite group and $V$ be a $kG$-module. For any subgroup $P$ of $G$, we denote by $V(P)$ the $k$-submodule of all $P$-fixed elements of $V$, by $V(P)$ the Brauer quotient

$$V(P) = V^P / \sum_R V^P_R,$$

where $R$ runs over the set of all proper subgroups of $P$ and $V^P_R$ is the image of the usual relative trace map $\text{Tr}^P_R: V^R \to V^P$, and by $\text{Br}^V_P$ the canonical surjective homomorphism $V^P \to V(P)$, which is the so-called Brauer homomorphism associated to $P$ and $V$. Obviously, the $kG$-module structure on $V$ induces $kN_G(P)$-module structures on both $V^P$ and $V(P)$, and $\text{Br}^V_P$ is a homomorphism of $kN_G(P)$-modules. Let $A$ be a $kG$-interior algebra. We apply the Brauer quotient $V(P)$ to the $kG$-module $A$ induced by the $G$-conjugation, and then get the Brauer quotient $A(P)$. It is easily checked that both $A^P$ and $A(P)$ are $k$-algebras and that the Brauer homomorphism $\text{Br}^V_P$ is a $k$-algebra homomorphism, whose kernel is the sum of all ideals $A^P_R$, where $R$ runs over the set of all proper subgroups of $P$.

Let $Q$ be a $p$-group. A $kQ$-interior algebra $A$ is a primitive $kQ$-interior algebra if $A^Q$ is a local algebra; furthermore, if $A(Q) \neq 0$, then $A$ is a local primitive $kQ$-interior algebra.

**Lemma 10.** Let $H$ be a finite group with a normal $p$-subgroup $Q$ and $T$ be a local primitive $kQ$-interior algebra. Then the kernel $\text{Ker}(\text{Br}^T_Q \otimes_{kH})$ is contained in the radical of $(T \otimes_k kH)^Q$. 


Proof. Set \( \bar{H} = H/Q \). The canonical homomorphism \( H \to \bar{H} \) induces a \( k \)-algebra homomorphism \( \varpi : kH \to k\bar{H} \). By tensoring both sides of the homomorphism \( \varpi \) with \( T \), we get a new \( k \)-algebra homomorphism
\[
1 \otimes \varpi : T \otimes_k kH \to T \otimes_k k\bar{H}
\]
mapping \( t \otimes a \) onto \( t \otimes \varpi(a) \) for any \( t \in T \), \( a \in kH \). Clearly \( 1 \otimes \varpi \) is a \( kQ \)-interior algebra homomorphism and it induces a \( k \)-algebra homomorphism
\[
(T \otimes_k kH)^Q \to (T \otimes_k k\bar{H})^Q = T^Q \otimes_k k\bar{H},
\]
which maps \( (T \otimes_k kH)^Q \) into \( T^Q \otimes_k k\bar{H} \) for any proper subgroup \( R \) of \( Q \). Since \( T \) is a local primitive \( kQ \)-interior algebra, \( T^Q \) is contained in \( J(T^Q) \) and thus the image of \( (T \otimes_k kH)^Q \) through Homomorphism 10.1 is contained in the radical of the image of Homomorphism 10.1. But obviously the kernel of Homomorphism 10.1 is contained in the radical of \( (T \otimes_k kH)^Q \). Therefore \( (T \otimes_k kH)^Q_R \) is contained in the radical of \( (T \otimes_k kH)^Q \). The proof is done.

Lemma 11. Let \( H \) be a finite group with a normal \( p \)-subgroup \( Q \) and \( T \) be a local primitive \( Q \)-interior algebra. If \( i \) is a primitive idempotent in \( (kH)^Q \) such that \( \text{Br}_{kH}^i(1) \neq 0 \), then \( 1 \otimes i \) is also a primitive idempotent in \( (T \otimes_k kH)^Q \) such that \( \text{Br}_{kH}^i(1 \otimes i) \neq 0 \).

Proof. Since \( \text{Br}_{kH}^i(1) \) is primitive in \( (Q) \) and \( \text{Br}_{kH}^i(1) \) is primitive in \( (kH)^Q \), \( \text{Br}_{kH}^i(1) \otimes \text{Br}_{kH}^j(i) \) is primitive in \( (T \otimes_k kH)^Q \). On the other hand, by [3, Proposition 5.6], there is a \( k \)-algebra isomorphism
\[
(T \otimes_k kH)^Q \cong (T \otimes_k kH)^Q,
\]
mapping \( \text{Br}_{kH}^i(1) \otimes \text{Br}_{kH}^j(i) \) onto \( \text{Br}_{kH}^j(i \otimes j) \). In particular, this isomorphism maps \( \text{Br}_{kH}^i(1) \otimes \text{Br}_{kH}^j(i) \) onto \( \text{Br}_{kH}^j(i \otimes j) \). Therefore \( \text{Br}_{kH}^j(i \otimes j) \) is primitive in \( (T \otimes_k kH)^Q \). Since it follows from Lemma 10 that the kernel \( \text{Ker}(\text{Br}_{kH}^j(i \otimes j)) \) is contained in the radical of \( (T \otimes_k kH)^Q \), \( 1 \otimes i \) is a primitive idempotent in \( (T \otimes_k kH)^Q \).

Lemma 12. Let \( M \) be a \( \mathcal{D}P \)-module. Assume that the \( kP \)-interior algebra \( \mathbb{H}_0(\text{End}_k(M)) \) is a primitive \( kP \)-interior algebra. Then the homology of \( M \) vanishes at all degree but some degree \( q \) and there is a \( kP \)-interior algebra isomorphism \( \mathbb{H}_0(\text{End}_k(M)) \cong \text{End}_k(\mathbb{H}_q(M)) \).

Proof. By [2, Theorem 3.1], there is a short exact sequence of group homomorphisms
\[
0 \to \Pi_q \text{Ext}_k^1(\mathbb{H}_q(M), \mathbb{H}_{q+1}(M)) \to \mathbb{H}_0(\text{End}_k(M)) \to \Pi_q \text{End}_k(\mathbb{H}_q(M)) \to 0,
\]
where \( \xi \) is induced by the map \( \mathbb{C}_0(\text{End}_k(M)) \to \Pi_q \text{End}_k(\mathbb{H}_q(M)) \) sending a chain map \( f \) onto the induced family \( (f_q) \in \Pi_q \text{End}_k(\mathbb{H}_q(M)) \). Since \( \text{Ext}_k^1(\mathbb{H}_q(M), \mathbb{H}_{q+1}(M)) = 0 \) for each \( q \), the homomorphism \( \xi \) is a group isomorphism. Clearly the isomorphism \( \xi \) preserves the composition of maps and the \( kP \)-interior algebra structures on \( \mathbb{H}_0(\text{End}_k(M)) \) and \( \Pi_q \text{End}_k(\mathbb{H}_q(M)) \); that is to say, \( \xi \) is a \( kP \)-interior algebra isomorphism. Since the \( kP \)-interior algebra \( \mathbb{H}_0(\text{End}_k(M)) \) is a primitive \( kP \)-interior algebra, the isomorphism \( \xi \) forces that the homology of \( M \) vanishes at all degree but some degree \( q \); in particular, we have a \( kP \)-algebra isomorphism \( \mathbb{H}_0(\text{End}_k(M)) \cong \text{End}_k(\mathbb{H}_q(M)) \). The proof is done.

Remark. The differential \( d \) on the \( \mathcal{D} \)-algebra \( \text{End}_k(M) \) (see Paragraph 3) is different from a differential on \( \text{End}_k(M) \) defined in [2, Chapter V, 1.6], but by replacing \( d \) by \( sd \), this difference
will disappear. Since \( s \) does not affect the homology of \( \text{End}_k(M) \), \cite[Theorem 3.1]{2} can be applied to the \( \mathcal{D} \)-algebra \( \text{End}_k(M) \).

13. We begin to prove Theorem 8. We keep the notation in Theorem 8 and assume that Statement 8.1 holds. By \cite[Theorem 18.8]{4}, the images \( R \) and \( R' \) of \( \hat{P} \) through the canonical projections \( \pi: G \times G' \to G \) and \( \pi': G \times G' \to G' \) are defect groups of \( b \) and \( b' \) respectively. Since the orders of \( P \) and \( \hat{P} \) are the same, the two projections \( \pi \) and \( \pi' \) induces group isomorphisms \( \hat{P} \cong R \) and \( \hat{P} \cong R' \), through which we identify \( R \), \( R' \) and \( \hat{P} \). Since \( P \) is normal in \( G' \), \( P \) is the unique defect group of \( b' \) and thus \( R' \) is equal to \( P \). In particular, we have \( R = \hat{R} = \hat{R}' = P \). By \cite[Theorem 18.8]{4}, there are maximal local pointed groups \( P_{\gamma} \) on \( kGb \), \( P_{\gamma'} \) on \( kG'\gamma' \) and \( P_{\gamma} \) on \( \text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'} \) such that we have a \( kP \)-interior algebra isomorphism

\[
(\mathcal{D}P_{\gamma}) \cong \mathbb{H}_0((\text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'}))
\]

and such that the \( \mathcal{D}(P \times P) \)-module \( (\text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'})_{\hat{\gamma}} \) determined by the \( \mathcal{D} \)-algebra structure on \( \text{End}_k(\bar{N}) \) and by the left and right multiplications of \( P \) on \( \text{End}_k(\bar{N}) \) is 0-split. In this case, the embedded algebras \( (kG)_{\gamma} \) and \( (kG')_{\gamma'} \) associated to \( P_{\gamma} \) and \( P_{\gamma'} \) are source algebras (see \cite{6}) of the block algebras \( kGb \) and \( kG'\gamma' \) respectively.

14. Clearly we have the equality \( \mathbb{C}_0(\text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'}) = \mathbb{C}_0(\text{End}_k(\bar{N})) \otimes_k (kG')_{\gamma'} \) which induces a \( kP \)-interior algebra isomorphism

\[
\mathbb{H}_0(\text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'}) \cong \mathbb{H}_0(\text{End}_k(\bar{N})) \otimes_k (kG')_{\gamma'}.
\]

Since \( P \) is normal in \( G' \) and the \( kP \)-interior algebra \( \mathbb{C}_0(\text{End}_k(\bar{N})) \) is a local primitive \( kP \)-interior algebra, by Lemma 11 the point \( \hat{\gamma} \) only contains the identity element of \( \text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'} \). Then by Isomorphisms 13.1 and 14.1 we have a \( kP \)-interior algebra isomorphism

\[
(\mathcal{D}P_{\gamma}) \cong \mathbb{H}_0(\text{End}_k(\bar{N})) \otimes_k (kG')_{\gamma'}.
\]

We consider the \( \mathcal{D}P \)-modules \( \text{End}_k(\bar{N}) \) and \( \text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'} \) determined respectively by their \( \mathcal{D} \)-algebra structures and the \( P \)-conjugations on them. Since \( (kG')_{\gamma'} \) has a \( P \times P \)-stable basis containing its unity (see \cite[Proposition 38.7]{6}), the \( \mathcal{D}P \)-module \( \text{End}_k(\bar{N}) \) is a direct summand of the \( \mathcal{D}P \)-module \( \text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'} \). Since the \( \mathcal{D}(P \times P) \)-module \( \text{End}_k(\bar{N}) \otimes_k (kG')_{\gamma'} \) is 0-split, so is the \( \mathcal{D}P \)-module \( \text{End}_k(\bar{N}) \).

15. Since \( (kG)_{\gamma} \) is a \( kP \)-primitive interior algebra, the isomorphism 14.2 forces that the \( kP \)-interior algebra \( \mathbb{H}_0(\text{End}_k(\bar{N})) \) is also a primitive \( kP \)-interior algebra. Then by Lemma 12, the homology of the \( \mathcal{D} \)-module \( \bar{N} \) vanishes at all degree but degree \( q \) and the map \( \mathbb{C}_0(\text{End}_k(\bar{N})) \to \text{End}_k(\mathbb{H}_q(\bar{N})) \) sending a chain map \( f : \bar{N} \to \bar{N} \) onto the induced \( k \)-module homomorphism \( f_q : \mathbb{H}_q(\bar{N}) \to \mathbb{H}_q(\bar{N}) \) induces a \( k \)-algebra isomorphism

\[
\mathbb{H}_0(\text{End}_k(\bar{N})) \cong \text{End}_k(\mathbb{H}_q(\bar{N}));
\]

moreover this \( k \)-algebra isomorphism actually is a \( kP \)-interior algebra isomorphism. So we have a \( kP \)-interior algebra isomorphism \( (kG)_{\gamma} \cong \text{End}_k(\mathbb{H}_q(\bar{N})) \otimes_k (kG')_{\gamma'} \). By \cite[Theorem 7.2]{4}, \( \mathbb{H}_q(\bar{N}) \) is an endo-permutation \( kP \)-module with vertex \( P \).

16. Clearly \( \{\text{id}_{\bar{M}}\} \) is a point of \( G \times G' \) on the \( \mathcal{D}(G \times G') \)-interior algebra \( \text{End}_k(\bar{M}) \), where \( \text{id}_{\bar{M}} \) is the identity map on \( \bar{M} \). Since the \( \mathcal{D}P \)-module \( \bar{N} \) is a source of the \( \mathcal{D}(G \times G') \)-module \( \bar{M} \), the \( \mathcal{D}P \)-module \( \bar{N} \) corresponds to a pointed group \( P_{\delta} \) on \( \text{End}_k(\bar{M}) \) so that \( P_{\delta} \) is a defect pointed group.
of the pointed group \((G \times G')_{(id,\tilde{M})}\); moreover the \(D\)-interior algebra \(\text{End}_k(\tilde{N})\) is an embedded algebra associated with \(P_\delta\). We denote by \(\text{Ind}_P^{G\times G'}(\text{End}_k(\tilde{N}))\) the injective induction of the \(D\)-interior algebra \(\text{End}_k(\tilde{N})\) from \(P\) to \(G \times G'\) (see [4, 12.2]), which is a \(\mathcal{D}(G \times G')\)-interior algebra. Then by [4, Corollary 14.11], there is a \(\mathcal{D}(G \times G')\)-interior algebra embedding

\[
\tilde{H} : \text{End}_k(\tilde{M}) \to \text{Ind}_P^{G\times G'}(\text{End}_k(\tilde{N})).
\]

On the other hand, by [4, 2.6.5], it is easily checked that there is a \(\mathcal{D}(G \times G')\)-interior algebra isomorphism \(\text{Ind}_P^{G\times G'}(\text{End}_k(\tilde{N})) \cong \text{End}_k(\text{Ind}_P^{G\times G'}(\tilde{N}))\). Therefore we get a \(\mathcal{D}(G \times G')\)-interior algebra embedding \(\text{End}_k(\tilde{M}) \to \text{End}_k(\text{Ind}_P^{G\times G'}(\tilde{N}))\). Then by [6, Example 13.4], it is easily checked that \(\tilde{M}\) is a direct summand of the \(\mathcal{D}(G \times G')\)-module \(\text{Ind}_P^{G\times G'}(\tilde{N})\).

17. Since the homology of the \(\mathcal{D}\)-module \(\tilde{N}\) vanishes at all degree but degree \(q\), so do the homology of the \(\mathcal{D}(G \times G')\)-module \(\text{Ind}_P^{G\times G'}(\tilde{N})\) and the homology of the \(\mathcal{D}(G \times G')\)-module \(\tilde{M}\); moreover we have \(H_q(\text{Ind}_P^{G\times G'}(\tilde{N})) = \text{Ind}_P^{G\times G'}(H_q(\tilde{N}))\). Then by [2, Theorem 3.1], the map

\[
H_0(\text{End}_k(\text{Ind}_P^{G\times G'}(\tilde{N}))) \to \text{End}_k(\text{Ind}_P^{G\times G'}(H_q(\tilde{N})))
\]

sending the image in \(H_0(\text{End}_k(\text{Ind}_P^{G\times G'}(\tilde{N})))\) of a chain map \(f : \text{Ind}_P^{G\times G'}(\tilde{N}) \to \text{Ind}_P^{G\times G'}(\til{N})\) onto the induced \(k\)-module homomorphism \(f_q : \text{Ind}_P^{G\times G'}(H_q(\til{N})) \to \text{Ind}_P^{G\times G'}(H_q(\til{N}))\) is a \(k\)-linear isomorphism, which actually is a \((G \times G')\)-interior algebra isomorphism. Since the \(\mathcal{D}(G \times G')\)-module \(\til{M}\) is a direct summand of \(\text{Ind}_P^{G\times G'}(\til{N})\), Isomorphism 17.1 induces a \((G \times G')\)-interior algebra isomorphism \(\Phi : H_0(\text{End}_k(\til{M})) \to \text{End}_k(H_q(\til{M}))\).

18. We consider the \(kP\)-interior algebra homomorphism \(C_0(\text{End}_k(\til{N})) \to \text{End}_k(H_q(\til{N}))\) obtained by composing the surjective homomorphism \(C_0(\text{End}_k(\til{N})) \to H_0(\text{End}_k(\til{N}))\) and Isomorphism 15.1. Since \(C_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N}))) = \text{Ind}_P^{G\times G'}(C_0(\text{End}_k(\til{N})))\), the homomorphism \(C_0(\text{End}_k(\til{N})) \to \text{End}_k(H_q(\til{N}))\) induces a surjective \((G \times G')\)-interior algebra homomorphism

\[
C_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N}))) \to \text{Ind}_P^{G\times G'}(\text{End}_k(H_q(\til{N})))
\]

with the kernel \(H_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N})))\), which induces a \((G \times G')\)-interior algebra isomorphism

\[
\Psi : H_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N}))) \cong \text{Ind}_P^{G\times G'}(\text{End}_k(H_q(\til{N}))).
\]

19. Clearly the embedding \(\tilde{H}\) induces a \((G \times G')\)-interior algebra embedding

\[
C_0(\text{End}_k(\til{M})) \to C_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N})))
\]

which maps \(C_0(\text{End}_k(\til{M}))\) into \(C_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N})))\). Thus the embedding \(\tilde{H}\) induces a \((G \times G')\)-interior algebra embedding \(H_0(\tilde{H}) : H_0(\text{End}_k(\til{M})) \to H_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N})))\). We set

\[
H = \Psi \circ H_0(\tilde{H}) \circ \Phi^{-1}.
\]

We denote by \(\phi\) the composition of the canonical surjective homomorphism \(C_0(\text{End}_k(\til{M})) \to H_0(\text{End}_k(\til{M}))\) with the isomorphism \(\Phi\) and by \(\psi\) the composition of the canonical surjective homomorphism \(C_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N}))) \to H_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N})))\) with the isomorphism \(\Psi\). Then we have the following commutative diagram

\[
\begin{array}{ccc}
C_0(\text{End}_k(\til{M})) & \longrightarrow & C_0(\text{Ind}_P^{G\times G'}(\text{End}_k(\til{N}))) \\
\phi \downarrow & & \downarrow \psi \\
\text{End}_k(H_q(\til{M})) & \xrightarrow{H} & \text{Ind}_P^{G\times G'}(\text{End}_k(H_q(\til{N}))).
\end{array}
\]
20. Now we claim that the homomorphism $\phi$ induces a surjective $kG$-interior algebra homomorphism $C_0(\text{End}_{k(1 \times G')}(\tilde{M})) \to \text{End}_{k(1 \times G')}((\mathbb{H}_q(\tilde{M})))$ with the kernel $\mathbb{B}_0(\text{End}_{k(1 \times G')}(\tilde{M}))$. Since we have the commutative diagram 19.1, it suffices to show that the homomorphism $\psi$ induces a surjective $kG$-interior algebra homomorphism $C_0(\text{Ind}_{P}^{G \times G'}(\text{End}_k(\tilde{N})))^{1 \times G'} \to \text{Ind}_{P}^{G \times G'}(\text{End}_k((\mathbb{H}_q(\tilde{N})))^{1 \times G'}$ with the kernel $\mathbb{B}_0(\text{Ind}_{P}^{G \times G'}(\text{End}_k(\tilde{N})))^{1 \times G'}$. By [4, Theorem 15.4], there are a unique $\mathfrak{D}G$-interior algebra homomorphism

$$H_{G, G'}^\mathbb{N} : \text{Ind}_{P}^{G \times G'}(\text{End}_k(\tilde{N}))^{1 \times G'} \cong \text{Ind}_{P}^{G}(\text{End}_k(\tilde{N}) \otimes_k kG')$$

mapping $\text{Tr}_{1 \times 1}^G(x \otimes a \otimes 1)$ onto $1 \otimes (a \otimes x^{-1}) \otimes 1$ for any $a \in \text{End}_k(\tilde{N})$ and any $x \in G'$. This isomorphism $H_{G, G'}^\mathbb{N}$ induces a $kG$-interior algebra isomorphism

$$C_0(H_{G, G'}^\mathbb{N}) : \text{Ind}_{P}^{G \times G'}(C_0(\text{End}_k(\tilde{N})))^{1 \times G'} \cong \text{Ind}_{P}^{G}(C_0(\text{End}_k(\tilde{N})) \otimes_k kG').$$

21. By [4, Theorem 4.4], there is a unique $kG$-interior algebra isomorphism

$$H_{G, G'}^{\mathbb{H}_q(\tilde{N})} : \text{Ind}_{P}^{G \times G'}(\text{End}_k((\mathbb{H}_q(\tilde{N})))^{1 \times G'} \cong \text{Ind}_{P}^{G}(\text{End}_k((\mathbb{H}_q(\tilde{N}))) \otimes_k kG')$$

mapping $\text{Tr}_{1 \times 1}^G(x \otimes a \otimes 1)$ onto $1 \otimes (a \otimes x^{-1}) \otimes 1$ for any $a \in \text{End}_k((\mathbb{H}_q(\tilde{N})))$ and any $x \in G'$. Clearly the $kP$-interior algebra homomorphism $C_0(\text{End}_k(\tilde{N})) \to \text{End}_k((\mathbb{H}_q(\tilde{N})))$ (see Paragraph 17) induces a surjective $kG$-interior algebra homomorphism

$$\text{Ind}_{P}^{G}(C_0(\text{End}_k(\tilde{N})) \otimes_k kG') \to \text{Ind}_{P}^{G}(\text{End}_k((\mathbb{H}_q(\tilde{N}))) \otimes_k kG'),$$

with the kernel $\text{Ind}_{P}^{G}(\mathbb{B}_0(\text{End}_k(\tilde{N})) \otimes kG') = \mathbb{B}_0(\text{Ind}_{P}^{G}(\text{End}_k(\tilde{N}) \otimes kG'))$, and this induced $kG$-interior algebra homomorphism makes the following diagram commutative.

$$\begin{array}{ccc}
\text{Ind}_{P}^{G \times G'}(C_0(\text{End}_k(\tilde{N})))^{1 \times G'} & \xrightarrow{C_0(H_{G, G'}^{\mathbb{H}_q(\tilde{N})})} & \text{Ind}_{P}^{G}(C_0(\text{End}_k(\tilde{N})) \otimes_k kG') \\
\downarrow \psi & & \downarrow \\
\text{Ind}_{P}^{G \times G'}(\text{End}_k((\mathbb{H}_q(\tilde{N})))^{1 \times G'} & \xrightarrow{H_{G, G'}^{\mathbb{H}_q(\tilde{N})}} & \text{Ind}_{P}^{G}(\text{End}_k((\mathbb{H}_q(\tilde{N}))) \otimes_k kG').
\end{array}$$

Therefore the $kG$-interior algebra homomorphism

$$C_0(\text{Ind}_{P}^{G \times G'}(\text{End}_k(\tilde{N})))^{1 \times G'} \to \text{Ind}_{P}^{G \times G'}(\text{End}_k((\mathbb{H}_q(\tilde{N})))^{1 \times G'}$$

induced by $\psi$ is surjective with the kernel $\mathbb{B}_0(\text{Ind}_{P}^{G \times G'}(\text{End}_k(\tilde{N})))^{1 \times G'}$. Then the claim is done.

22. In particular, we have a $kG$-interior algebra isomorphism

$$\mathbb{H}_0(\text{End}_{k(1 \times G')}(\tilde{M})) \cong \text{End}_{k(1 \times G')}((\mathbb{H}_q(\tilde{M}))).$$

On the other hand, since the $\mathfrak{D}(G \times G')$-module $\tilde{M}$ induces a Rickard equivalence between $kGb$ and $kG'b'$, by [4, Theorem 18.4] we have a $kG$-interior algebra isomorphism $kGb \cong \mathbb{H}_0(\text{End}_{k(1 \times G')}(\tilde{M}))$. Therefore we have a $kG$-interior algebra isomorphism

$$kGb \cong \text{End}_{k(1 \times G')}((\mathbb{H}_q(\tilde{M}))).$$
Then by [4, Proposition 6.5] the $k(G \times G')$-module $\mathbb{H}_q(\tilde{N})$ induces a Morita equivalence between $kB$ and $kG'$. We claim that this Morita equivalence is basic in the sense of Puig in [4]. There is a $(k(G \times G'))$-interior algebra isomorphism

$$\text{Ind}_{P}^{G \times G'}(\text{End}_k(\mathbb{H}_q(\tilde{N}))) \cong \text{End}_k(\text{Ind}_{P}^{G \times G'}(\mathbb{H}_q(\tilde{N}))).$$

By composing $H$ with this $(k(G \times G'))$-interior algebra isomorphism, we get a $(k(G \times G'))$-interior algebra embedding $\text{End}_k(\mathbb{H}_q(M)) \to \text{End}_k(\text{Ind}_{P}^{G \times G'}(\mathbb{H}_q(\tilde{N})))$. Then by [4, Example 13.4], $M$ is a direct summand of the $(k(G \times G'))$-module $\text{Ind}_{P}^{G \times G'}(\mathbb{H}_q(\tilde{N}))$. Since the $\mathcal{D}P$-module $\mathbb{H}_q(\tilde{N})$ is a direct summand of the restriction $\text{Res}_{P}^{G \times G'}(M)$ and has vertex $\Delta(P)$, the $\mathcal{D}P$-module $\mathbb{H}_q(\tilde{N})$ is a source of the $\mathcal{D}(G \times G')$-module $M$. The claim is done.

Now it remains to prove Statement 8.1 from Statement 8.2. We continue to keep the notation in Theorem 8, and assume that Statement 8.2 holds.

**23.** Since the $\mathcal{D}\tilde{P}$-module $\text{End}_k(\tilde{N})$ is 0-split, the $\mathcal{D}\tilde{P}$-module $\text{End}_k(\tilde{N})$ is a direct sum of a contractible $\mathcal{D}\tilde{P}$-module and the $\tilde{P}$-module $\mathbb{H}_0(\text{End}_k(\tilde{N}))$ determined by the $\tilde{P}$-conjugation. That is to say, there is a contractible $\mathcal{D}\tilde{P}$-module $C$ such that we have a direct sum decomposition $\text{End}_k(\tilde{N}) = \mathbb{H}_0(\text{End}_k(\tilde{N})) \oplus C$, from which, we easily conclude a direct sum decomposition

$$23.1 \quad (\text{End}_k(\tilde{N}))^\tilde{P} = \mathbb{H}_0(\text{End}_k(\tilde{N}))^\tilde{P} \oplus C^\tilde{P},$$

for any subgroup $R$ of $\tilde{P}$. Since the $\mathcal{D}\tilde{P}$-module $C$ is contractible, so is the $\mathcal{D}$-module $C^\tilde{P}$. Thus the $\mathcal{D}$-module $(\text{End}_k(\tilde{N}))^\tilde{P}_R$ is 0-split and we have $\mathbb{C}_0(C^\tilde{P}_R) = \mathbb{B}_0(C^\tilde{P}_R) = \mathbb{B}_0((\text{End}_k(\tilde{N}))^\tilde{P}_R)$. From the decomposition 23.1, we get a new direct sum decomposition

$$\mathbb{C}_0((\text{End}_k(\tilde{N}))^\tilde{P}_R) = \mathbb{H}_0(\text{End}_k(\tilde{N}))^\tilde{P}_R \oplus \mathbb{C}_0(C^\tilde{P}_R),$$

for any subgroup $R$ of $\tilde{P}$. This new decomposition implies that the inclusion map $\text{End}_{k\tilde{P}}(\tilde{N}) \subset \text{End}_k(\tilde{N})$ induces a surjective $k$-algebra homomorphism

$$23.2 \quad \mathbb{C}_0(\text{End}_{k\tilde{P}}(\tilde{N})) \to \mathbb{H}_0(\text{End}_k(\tilde{N}))^\tilde{P}$$

with the kernel $\mathbb{B}_0(\text{End}_{k\tilde{P}}(\tilde{N}))$, which induces a surjective $k$-algebra homomorphism

$$\mathbb{C}_0(\text{End}_{k\tilde{P}}(\tilde{N})) \to \mathbb{H}_0(\text{End}_k(\tilde{N}))(\tilde{P})$$

with the kernel $\mathbb{B}_0(\text{End}_{k\tilde{P}}(\tilde{N})) + \sum_R(\mathbb{C}_0(\text{End}_k(\tilde{N})))^\tilde{P}_R$, where $R$ runs over proper subgroups of $\tilde{P}$.

**24.** Since $\tilde{N}$ is a noncontractible $\mathcal{D}(G \times G')$-module, so is the $\mathcal{D}\tilde{P}$-module $\tilde{N}$. So $\mathbb{C}_0(\text{End}_{k\tilde{P}}(\tilde{N}))$ is a primitive $k$-algebra with the identity map outside $\mathbb{B}_0(\text{End}_{k\tilde{P}}(\tilde{N}))$. Then it follows from Homomorphism 23.2 that $\mathbb{H}_0(\text{End}_k(\tilde{N}))$ is a primitive $k\tilde{P}$-interior algebra. Then by Lemma 12, there is a $k\tilde{P}$-interior algebra isomorphism $\mathbb{H}_0(\text{End}_k(\tilde{N})) \cong \text{End}_k(\mathbb{H}_q(\tilde{N}))$; moreover, the $k\tilde{P}$-module $\mathbb{H}_q(\tilde{N})$ is indecomposable. We claim that the $k\tilde{P}$-module $\mathbb{H}_q(\tilde{N})$ has vertex $\tilde{P}$. Otherwise, by [6, Proposition 18.11] we have $\text{End}_k(\mathbb{H}_q(\tilde{N}))(\tilde{P}) = 0$ and then $\mathbb{H}_0(\text{End}_k(\tilde{N}))(\tilde{P}) = 0$; thus by Rosenberg’s Lemma, $\text{id}_{\mathbb{H}_q(\tilde{N})}$ belongs to either $\mathbb{B}_0(\text{End}_{k\tilde{P}}(\tilde{N}))$ or $(\mathbb{C}_0(\text{End}_k(\tilde{N})))^\tilde{P}_R$ for some proper subgroup $R$ of $\tilde{P}$; this contradicts with the $\mathcal{D}\tilde{P}$-module $\tilde{N}$ being a source of the noncontractible $\mathcal{D}(G \times G')$-module $\tilde{M}$. The claim is done.
25. Next we claim that the $k\hat{P}$-module $\mathbb{H}_q(\hat{N})$ is a source of the $k(G \times G')$-module $\mathbb{H}_q(\hat{M})$. Since the $\mathfrak{D}\hat{P}$-module $\hat{N}$ is a direct summand of the restricton $\text{Res}_{\hat{P}}^{G \times G'}(\hat{M})$, the $kP$-module $\mathbb{H}_q(\hat{N})$ is a direct summand of the restriction $\text{Res}_{\hat{P}}^{G \times G'}(\mathbb{H}_q(\hat{M}))$. Since the $\mathfrak{D}\hat{P}$-module $\hat{N}$ is a source of the $\mathfrak{D}(G \times G')$-module $\hat{M}$, $\hat{M}$ is a direct summand of the induced $\mathfrak{D}(G \times G')$-module $\text{Ind}_{\hat{P}}^{G \times G'}(\mathbb{H}_q(\hat{N}))$ (see Paragraph 16), thus $\mathbb{H}_q(\hat{M})$ is a direct summand of the $k(G \times G')$-module $\mathbb{H}_q(\text{Ind}_{\hat{P}}^{G \times G'}(\mathbb{H}_q(\hat{N})))$. Clearly there is an obvious $k(G \times G')$-module isomorphism $\mathbb{H}_q(\text{Ind}_{\hat{P}}^{G \times G'}(\mathbb{H}_q(\hat{N}))) \cong \text{Ind}_{\hat{P}}^{G \times G'}(\mathbb{H}_q(\hat{N}))$. So $\mathbb{H}_q(\hat{M})$ is a direct summand of the $k(G \times G')$-module $\text{Ind}_{\hat{P}}^{G \times G'}(\mathbb{H}_q(\hat{N}))$. Since the indecomposable $k\hat{P}$-module $\mathbb{H}_q(\hat{N})$ has vertex $\hat{P}$, the $k\hat{P}$-module $\mathbb{H}_q(\hat{N})$ is a source of the $k(G \times G')$-module $\mathbb{H}_q(\hat{M})$.

Finally, since the module $\mathbb{H}_q(\hat{M})$ induces a basic Morita equivalence between $kGb$ and $kG'b'$, by [4, Corollary 7.4] the groups $\hat{P}$ and $P$ have the same order.

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