Detecting entanglement through correlations between local observables

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Abstract. We propose a measure of two-qubit entanglement that is invariant under local unitary transformations, and which is based on local measurements covariances. It measures the Hilbert-Schmidt distance between the state and the product state obtained by multiplying the local density matrices. The measure has the benefit that the experimentalist need not have any a priori knowledge of the state to make the measurements. For pure states, the measure provides the state’s concurrence directly, without resorting to state tomography. For statistically mixed states, the measure provides bounds for the concurrence. The two-qutrit case is also studied.

1. Introduction
During the last years substantial efforts have been spent to investigate, and in particular to quantify, entanglement [1]. The two-qubit case has been extensively studied and entanglement of formation [2] and concurrence [3] are widely accepted as good entanglement measures.

Even if good measures exist, they require state tomography, that is, a full knowledge of the density matrix for a given state to be determined. A way to avoid these inconveniences is to use so-called entanglement witnesses [4–6], which can detect specific entanglement, but are not able to quantify it. An alternative to entanglement witnesses are local uncertainty relations (LUR) [7], but unfortunately no known LUR can detect all entangled states, and in general they do not give quantitative measure of entanglement. Assuming that one has an unknown state, it is therefore desirable to quantify its entanglement as well as possible, with the smallest experimental effort.

Several years ago Schlienz and Mahler proposed a general description of entanglement using the density matrix formalism [8]. For the bipartite case they show that an entanglement tensor, whose components are the covariances between pairs of generators of the respective algebra for each particle, is the difference between the composite density matrix and the tensor product of the reduced density matrices for each subsystem. By taking the square form of this tensor one obtains a distance which vanishes for any product state and is positive otherwise. This distance is maximal for maximally entangled states, and it is invariant under local unitary transformations.

In this paper we will connect the measure proposed in [8] with the local observable covariance relation for two qubits [9] and two qutrits. The measure quantifies entanglement for all pure states and, in some range, for mixed states. An advantage is that the measure only requires local measurements, facilitating the experimental effort. The measurement projectors can be cast in terms of mutually unbiased bases (MUB), that optimize the measurement process in-so-far that the expected measurement error is minimized [10].

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2. Two-qubit entanglement measure

In this section we take the entanglement measure for two qubits proposed in [8] and we extend the work done there for mixed states. For two qubits \(A\) and \(B\) the measure [8] reads

\[
G = \sum_{i,j=1}^{3} C^2 \left( \hat{\sigma}_i^A, \hat{\sigma}_j^B \right),
\]

where

\[
C \left( \hat{\sigma}_i^A, \hat{\sigma}_j^B \right) = \langle \hat{\sigma}_i^A \otimes \hat{\sigma}_j^B \rangle - \langle \hat{\sigma}_i^A \otimes \hat{1}^B \rangle \langle \hat{1}^A \otimes \hat{\sigma}_j^B \rangle,
\]

for \(i, j = 1, \ldots, 3\). Here, \(\hat{\sigma}_i^A\) denotes the \(i\)th Pauli matrix (operator) for system \(A\), and similar for \(B\), whose eigenvalues are \(\pm 1\). The Pauli matrices are generators of the \(su(2)\) algebra. They are traceless and orthogonal:

\[
\text{Tr} (\hat{\sigma}_i) = 0, \quad \text{Tr} (\hat{\sigma}_i \hat{\sigma}_j) = 2\delta_{ij}, \quad i = 1, \ldots, 3.
\]

The measure in Eq. (1) is easy to implement because one only has to count singles- and coincidence-rates. The measurements can be performed locally on each system and a total of nine measurement settings are sufficient to get all the results (\(\langle \hat{\sigma}_i \otimes \hat{1} \rangle\) and \(\langle \hat{1} \otimes \hat{\sigma}_j \rangle\) can be calculated from the other measurements).

As it was pointed out in [8], the measure (1) is proportional to the Hilbert-Schmidt norm measure of distance. That is,

\[
G = 4\text{Tr} \left\{ (\hat{\rho} - \hat{\rho}_A \otimes \hat{\rho}_B)^2 \right\},
\]

where \(\hat{\rho}_A\) denotes the density matrix of system \(A\) after tracing over system \(B\) and similar for \(\hat{\rho}_B\). Writing the measure in the form (4), the measure is easy to manipulate theoretically. Additionally, some desirable features are noticeable, such as the measure’s invariance under local unitary transformations, and that it vanishes for pure separable states and is positive otherwise.

To see the equivalence between Eq. (1) and Eq. (4) one can expand the density matrix \(\hat{\rho}_A\) as

\[
\hat{\rho}_A = \frac{1}{2} \sum_{n=0}^{3} \text{Tr} (\hat{1} \otimes \hat{\sigma}_n \hat{\rho}) \hat{\sigma}_n
\]

and similar for \(\hat{\rho}_B\). The two qubit density operator \(\hat{\rho}\) can also be expanded in the basis defined by the operators \(\hat{\sigma}_i\), v.i.z.

\[
\hat{\rho} = \frac{1}{4} \sum_{n=0}^{3} \sum_{m=0}^{3} \text{Tr} (\hat{\sigma}_m^A \otimes \hat{\sigma}_n^B \hat{\rho}) \hat{\sigma}_m^A \otimes \hat{\sigma}_n^B.
\]

Inserting the expansion (5) and its \(\hat{\rho}_B\) counterpart, and (6) into (4), and using the Pauli matrix relations (3), it is not difficult to rewrite the ensuing equation in the form (1).

Note that, because of the local unitary invariance counterpart, and (6) into (4), and using the Pauli matrix relations (3), it is not difficult to rewrite the ensuing equation in the form (1).

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2.1. Pure states

We will relate our measure to the well known concurrence in the case of pure states. Let us expand an ordinary pure two-qubit state into the eigenvectors $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ of the $\hat{\sigma}_A^3 \otimes \hat{\sigma}_B^3$ operator, that is

$$|\psi\rangle = \sum_{k,l=0}^1 \alpha_{kl} |k\rangle^A \otimes |l\rangle^B.$$  \hspace{1cm} (7)

Evaluating (1) and performing considerable trivial, but tedious, algebra, the result is

$$G = 4I (2 + 4I),$$ \hspace{1cm} (8)

where $I = (I_1^2 - I_2^2)/2$, and $I_1$, $I_2$ are the only two invariants under local unitary transformation for a system composed by two qubits [12]:

$$I_1 = \sum_{k,l=0}^1 \alpha_{kl} \alpha_{kl}^*, \quad I_2 = \sum_{k,l,m,n=0}^1 \alpha_{km} \alpha_{kn}^* \alpha_{ln} \alpha_{lm}^*.$$  \hspace{1cm} (9)

Observing that $4I = C^2$ [13], where $C$ denotes the well-known concurrence [3] for pure states, we finally get [9]

$$G = C^2 (2 + C^2).$$ \hspace{1cm} (10)

The concurrence is related to entanglement of formation and therefore one can directly quantify the entanglement of pure states by measuring $G$. For a pure state $G > 0$ implies that the state is entangled. This is an intuitive result because a separable, pure state cannot display any covariance between local measurements. Is worth noticing that the relation (10) has already been tested experimentally [11].

2.2. Mixed States

The relation between $G$ and the concurrence that held for the pure states in Eq. (10) is no longer valid for mixed states. Instead, $G$ can, in general, take any value in the shaded region plotted in Fig. 1. That is, $0 \leq G \leq 3$ or, if we write it in relation to the concurrence, one has

$$C^2 (2 + C^2) \leq G \leq 1 + 2C^2.$$ \hspace{1cm} (11)

The lower bound of this inequality is given by Eq. (10). That is, any pure state has the lowest possible values of $G$ for a given amount of entanglement. The reason for that is that correlations for pure states can only be due to entanglement.

To find the upper bound of Eq. (11) we look at the density matrix

$$\hat{\rho}_u = \frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|) + \gamma \left( e^{-i\theta} |11\rangle \langle 00| + e^{i\theta} |00\rangle \langle 11| \right)$$  \hspace{1cm} (12)

with $0 \leq \gamma \leq 1/2$ and $\theta$ an arbitrary real number. The class of states defined by this density matrix interpolates between maximally classically correlated states with no entanglement ($G = 1$, $C = 0$ when $\gamma = 0$) and maximally entangled (pure) states which have the highest correlations of any states ($G = 3$, $C = 1$ when $\gamma = 1/2$). Using the definition of $G$ in Eq. (1) and calculating the concurrence for the state (12), we have $G(\hat{\rho}_u) = 1 + 2C^2$. A simulation with many thousands of arbitrary states shows that, indeed, no state is outside the range given by (11). Hence, for mixed states, we can establish the separability criterion: If $G > 1$, the state is nonseparable.

In general, $G$ is an “entanglement witness” for mixed states, since it can detect entanglement for a class of states. But if one has a state with a large $G$, say 2.5, $G$ can give more information (see Fig. 1) because the concurrence is bound to a relatively small range around 0.9. Hence, even if one is not able to determine the concurrence exactly, $G$ is still able to limit a state to a certain range of the concurrence.
3. Two-qutrits

In this section we will extend the results in the previous section to two qutrits. Considering that MUB optimize the measurement process [10], we will take the Patera matrices [14] as generators of the su(3) algebra instead the well-know Gell-Mann matrices. For a qutrit, there exists four MUB with projectors \( \hat{\rho}_{i,j} \), where the subscript \( i \) denotes the basis and, every basis’ projectors sum to the identity. That is \( \sum_{j=1}^{3} \hat{\rho}_{i,j} = 1 \). In terms of the basis elements, the Patera matrices can be constructed in pairs for each basis as

\[
L_{i,1} = \frac{1}{\sqrt{6}} (2\hat{\rho}_{i,1} - \hat{\rho}_{i,2} - \hat{\rho}_{i,3}), \quad L_{i,2} = \frac{1}{\sqrt{2}} (\hat{\rho}_{i,2} - \hat{\rho}_{i,3}) \quad \forall \ i = 1, \ldots, 4.
\]  

(13)

Note that these matrices are traceless and orthonormal, \( \text{Tr}(L_{i,j}L_{i,j}') = \delta_{i,i'}\delta_{j,j'} \), in similarity with the Pauli matrices (3). Inserting the Patera matrices in the measure proposed in [8], one obtains a measure similar to (1):

\[
G_3 = \sum_{i,i'=1}^{4} \sum_{j,j'=1}^{2} C^2(L_{i,j}^A, L_{i,j'}^B).
\]  

(14)

Since four times the measure (14) is satisfying the relation (4) it has similar properties as (1), namely it is invariant under local unitary transformations, it vanishes for product states and is positive otherwise, and it is maximal for maximally entangled states. It is also bounded by \( 0 \leq G_3 \leq 8/9 \).

For three-level systems there are two nontrivial invariants under local unitary transformations [15]; the generalization of the concurrence, whose square is \( C^2 = 1 - \text{Tr} \hat{\rho}_A^2 \), and the three-concurrence \( C_3 = |c_1c_2c_3| \), where \( c_i, i = 1, 2, 3 \), are the two-qutrit Schmidt-decomposition probability amplitudes. Considering a pure state in the Schmidt decomposition and evaluating (14) using the generalized concurrence and \( C_3 \), one can derive the relation

\[
G_3 = C^2 + C^4 - 6C_3.
\]  

(15)

Similarly to the two-qubit case, the relation (15) gives a lower limit for the proposed measure in terms of the concurrences. Unfortunately is difficult to obtain an upper limit similar to the one given in (11), because \( C_3 \) is undefined for mixed states and there exists no systematic to way of parameterizing the class of qutrit bipartite states with a given \( C \). Nevertheless, a separability criterion is still valid. If \( G_3 > 1/4 \) the state is entangled.
4. Summary
We have shown that the Hilbert-Schmidt distance between a density matrix and the product matrix of the partially traced density matrix can detect, and roughly quantify, entanglement.

For two-qubits with large entanglement, the entanglement can be pinpointed with good accuracy in terms of the concurrence. For two qutrits the same measure and similar approach can be used. For pure states the measure $G_3$ provides a combination of the two entanglement monotones $C$ and $C_3$ in the bipartite qutrit space.

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