THE PARITY OF ANALYTIC RANKS AMONG QUADRATIC TWISTS OF ELLIPTIC CURVES OVER NUMBER FIELDS

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Abstract. The parity of the analytic rank of an elliptic curve is given by the root number in the functional equation $L(E, s)$. Fixing an elliptic curve over any number field and considering the family of its quadratic twists, it is natural to ask what the average analytic rank in this family is. A lower bound on this number is given by the average root number. In this paper, we investigate the root number in such families and derive an asymptotic formula for the proportion of curves in the family that have even rank. Our results are then used to support a conjecture about the average analytic rank in this family of elliptic curves.

1. Introduction

Associated to an elliptic curve $E$ over a field $K$ is a family of curves, its quadratic twists. Within this family, it is natural to consider the the distribution of the most intriguing invariant of an elliptic curve, its Mordell-Weil rank, which, via the Birch and Swinnerton-Dyer conjecture is equal to its analytic rank. This investigation began when Goldfeld [6] conjectured that the average analytic rank in such a family associated to an elliptic curve over $\mathbb{Q}$, is $\frac{1}{2}$. The motivation for this conjecture comes from considering the root number in the $L$-function of a twisted elliptic curve and the folklore conjecture that elliptic curves of rank $\geq 2$ are rare. The result of this paper is a generalization of the conjecture to elliptic curves over arbitrary number fields. In many cases we find that the twists of $E/K$ do not have even and odd analytic rank in equal proportion and therefore the conjectured average value of the analytic rank is not $\frac{1}{2}$.

Recently, such an analysis was undertaken by Klagsbrun, Mazur and Rubin [9], wherein they find the density of curves of even Selmer rank among a family of quadratic twists. Using their results, they formulate a generalization of Goldfeld’s conjecture for elliptic curves over number fields. Some of their techniques lend themselves to the analytic aspect as well and have been a major inspiration for this paper. We are able to verify their results in many cases, equating the average algebraic rank (mod 2) with the average analytic rank (mod 2). Thus, the result of this paper can be viewed as a parity conjecture on average in this family of elliptic curves. In other words, the proportion of elliptic curves with even algebraic rank is equal to the proportion of curves that have even analytic rank, in a family of quadratic twists.

Because of the convenience of dealing with automorphic representations of $GL(2)$, in this paper we only consider elliptic curves which are modular, a property which is defined as follows. Let $E$ be an elliptic curve with conductor $\mathfrak{M}$ over a number

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field $K$. Then $E$ is modular if the Hasse-Weil $L$-function of $E/K$ is equal to the $L$-function of a cuspidal Hilbert modular form of parallel weight 2, and level $\mathfrak{M}$ (see [5]). In particular, the Hasse-Weil $L$-function of $E$, denoted $L(E/K, s)$ or simply $L(E, s)$ is equal to the $L$-function of a cuspidal automorphic representation $\pi_E = \otimes \pi_{E, v}$ of $GL(2, \mathbb{A}_K)$ associated to a Hilbert modular form, and therefore $L(E, s)$ satisfies a functional equation. It is known that all elliptic curves over $K = \mathbb{Q}$ are modular [1] and recently it has been shown that most elliptic curves over totally real fields and real quadratic extensions of totally real fields, are modular [10] [4].

Let $\Lambda_E(s)$ denote the “completed” $L$-function of $\pi_E$, that is, including the factors at the archimedian places. Then the functional equation takes the form $\Lambda_E(s) = w\Lambda(1 - s)$ where $w \in \{\pm 1\}$, called the root number will play a pivotal role in this paper. The analytic rank of $E$, denoted $rk(E)$ is defined as

$$rk(E) = \text{order of vanishing of } \Lambda_E(s) \text{ at } s = \frac{1}{2}.$$ 

For an elliptic curve $E/K$, we study the quadratic twists of $E$: these are elliptic curves $E'$ which are isomorphic to $E$ over some quadratic extension $K'/K$. In order to make assertions about density in a family of curves, it is necessary to order them in some way. Suppose that $E'$ is isomorphic to $E$ over some quadratic field $K'$. Then via standard class field theory, there is a unique quadratic Hecke character $\chi$ associated to a Hilbert modular form, and therefore $E$ over the quadratic field $K'$ with associated Hecke character $\chi$ and let $C(K)$ be the set of all quadratic Hecke characters of $\mathbb{A}_K^\times$. For $\chi \in C(K)$, let $q_1, q_2, ..., q_n$ be the places where $\chi$ is ramified. Then we define the norm, $N\chi = \max_i \{Nq_i\}$ which gives an ordering of twists of an elliptic curve. Our main result is the following,

**Theorem.** Let $E$ be a modular elliptic curve over a number field $K$ such that no local supercuspidal representations occur in the factorization of $\pi_E$, then

$$\lim_{X \to \infty} \frac{\#\{\chi \in C(K) \mid N\chi \leq X \text{ and } rk(E^\chi) \text{ is even}\}}{\#\{\chi \in C(K) \mid N\chi \leq X\}} = \frac{1 + (-1)^{rk(E)}\kappa}{2}$$

where $\kappa = \prod \kappa_v$ is a product over the places of $K$ given by

$$\kappa_v = \begin{cases} 
0 & \text{if } K_v \cong \mathbb{R} \\
1 & \text{if } K_v \cong \mathbb{C} \\
2/|c_v| - 1 & E \text{ has split multiplicative reduction at } v \\
1 - 2/|c_v| & E \text{ has nonsplit multiplicative reduction at } v \\
1 - 2/|c_v| & E \text{ has multiplicative reduction in a quadratic extension at } v \\
1 & E \text{ has potentially multiplicative reduction (non-quadratic)} \\
1 & \text{otherwise} \\
\end{cases}$$

and $|c_v|$ is the number of degree 2 extensions of $K_v$ (if $v \mid 2$ then $|c_v| = 4 \cdot 2^{[K_v: \mathbb{Q}]}$) otherwise $|c_v| = 4$).

In particular the theorem holds unconditionally for semi-stable curves over a real quadratic field [10]. Note that if the field $K$ has a real embedding then the density of even analytic ranks is exactly $\frac{1}{2}$. 

2. Notation

Let $K$ be a number field and let $E$ be an elliptic curve over $K$ with conductor $\mathfrak{N}$. Let $\Lambda_E(s)$ be the completed $L$-function of $E$, which, under the assumption of modularity, is also the $L$-function of a cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$. Let $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \pm 1$ be a quadratic Hecke character with conductor $f$ and let $\Lambda_{E}(s, \chi) = \Lambda_{E\chi}(s)$ denote the $L$-function of the quadratic twist $E\chi$.

Let $w \in \{\pm 1\}$ be the “root number” in the functional equation, i.e. such that $\Lambda_{E}(s) = w\Lambda_{E}(1 - s)$ and let $n(\chi)$ be the “change” in the epsilon factor, i.e such that $\Lambda_{E}(s, \chi) = n(\chi) \cdot \omega \Lambda_{E}(1 - s, \chi)$ holds. Recall that $\text{rk}(E)$ denotes the analytic rank of $E$, as defined in the previous section.

Let $C(K, \infty)$ be the group of global quadratic Hecke characters and $C(K_v)$ be the group of local quadratic characters at a place $v$ of $K$. We define the norm of a Hecke character as $N\chi := \max_{\chi \text{ ramifies}} \{Nq\}$.

Let $C(K, X)$ be the group of global Hecke characters such that $N\chi \leq X$ and $\Gamma = \prod_{v \in \Sigma} C(K_v)$ where $\Sigma$ is a finite set of places including $\infty$, and all $v \mid \mathfrak{N}$.

Let $E/K$ be a modular elliptic curve, then the $L$-function $\Lambda_E(s)$ is equal to the $L$-function $\Lambda(\pi_E, s)$ attached to a global automorphic representation $\pi_E$ of $GL(2, \mathbb{A}_Q)$ arising from a Hilbert modular form. In particular, this implies that $\Lambda(\pi_{E\chi}, s) = \Lambda(\chi \otimes \pi_E, s)$.

Let $K_v$ be a local field. If $K_v$ is non-archimedian, let $\varpi$ be a uniformizing element and let $\mathfrak{o}_v$ be the ring of integers of $K_v$.

At a non-archimedian place $v$, the admissible representations of $GL(2, K_v)$ are classified into 3 basic types of representation [8]. The class of supercuspidal representations is excluded from this paper because their local epsilon factors present difficulties with respect to computation. The other two types, the principal series representations, denoted $\pi(\mu_1, \mu_2)$ and the special representation denoted $\sigma(\mu_1, \mu_2)$ are discussed in detail in [8] which is also a major reference for the computational aspects in this paper. Because elliptic curves are self-dual we have that $\mu_1 = \mu_2^{-1}$ in both the supercuspidal and the special representations occurring in the factorization of $\pi_E$.

3. Change in Global Root Number

The functional equation of an elliptic curve takes the form $\Lambda_E(s) = w\Lambda_E(1 - s)$ where $w$ is either $+1$ or $-1$. Thus, the order of vanishing at $s = \frac{1}{2}$ is even if and only if $w = 1$. Consider a quadratic twist of this $L$-function $\Lambda_E(s, \chi)$, we expect the root number to change by a prescribed amount

$$\Lambda_E(s, \chi) = n(\chi)w\Lambda_E(1 - s, \chi)$$

so that,

(3.1) $\text{rk}(E^\chi) \equiv \text{rk}(E) \mod 2 \iff n(\chi) = 1$

Next we compute $n(\chi)$ explicitly.
Proposition 1. Let $\chi = \prod_v \chi_v$ and let $\varpi$ denote a local uniformizer at $v$. Then

$$n(\chi) = \prod_v n_v(\chi_v)$$

where the $n_v$ are given explicitly by the following table:

| $i$ | Type of Representation | $n_v(\chi_v)$ |
|-----|------------------------|---------------|
| 1   | $\pi_v = \pi_v(\mu_v, \mu_v^{-1})$ , $\chi_v$ is unramified. | 1             |
| 2   | $\mu_v$ is unramified, $\chi_v$ is ramified. | $\chi_v(-1)$ |
| 3   | $\pi_v = \pi_v(\mu_v, \mu_v^{-1})$ , $\chi_v$ is ramified, $\mu_v \chi_v$ is unramified. | $\chi_v(-1)$ |
| 4   | $\mu_v$ is ramified, $\chi_v(-1)$ | $\chi_v(-1)$ |
| 5   | $\pi_v = \sigma_v(\mu_v, \mu_v^{-1})$ , $\chi_v$ is unramified, $\mu_v \chi_v$ is ramified. | $1$          |
| 6   | $\mu_v$ is unramified, $\chi_v(-1)$ | $\chi_v(-1)$ |
| 7   | $\pi_v = \sigma_v(\mu_v, \mu_v^{-1})$ , $\chi_v$ is ramified, $\mu_v \chi_v$ is ramified. | $\chi_v(-1)$ |
| 8   | $\mu_v$ is ramified, $\chi_v(-1)$ | $\chi_v(-1)$ |
| 9   | $\pi_v = \sigma_v(\mu_v, \mu_v^{-1})$ , $\chi_v$ is ramified, $\mu_v \chi_v$ is ramified. | $\chi_v(-1)$ |
| 10  | $\mu_v$ is ramified, $\chi_v(-1)$ | $\chi_v(-1)$ |

Proof. Let $\mathfrak{N} = \prod_v \eta_v$ be the conductor of the elliptic curve $E$, and $f = \prod_v f_v$, the conductor of $\chi$ as in section 2, and let $\varpi_v$ be a local uniformizer at $v$. It follows from the tensor-product theorem ([7] Chapter 10, [8]), that $n(\chi_v) = \prod_v n_v(\chi_v)$, where the local root number $n(\chi_v) \in \{\pm 1\}$ is such that

$$L_v(s, \chi \otimes \pi) = n(\chi_v) \cdot w_v L_v(1 - s, \chi \otimes \pi)$$

holds, where $w_v$ is the local root number of the (untwisted) local $L$-function. In order to find $n(\chi_v)$ we first compute

$$a_v(s) = \frac{e(s, \pi_v)}{e(s, \pi_v \otimes \chi_v)}$$

and then for any $s \in \mathbb{R}_{>0}$ we have that

$$n(\chi_v) = \frac{a_v(s)}{|a_v(s)|}$$

The essence of this manipulation is that the root number is the “sign” of the epsilon factor. The proposition then follows from local epsilon factor computations in each case as given in Jacquet-Langlands [8]. As an illustration we include the proof in one of the above cases.

The case presented here is when $\pi_v$ is an unramified representation but $\chi_v \otimes \pi_v$ is not (line 2 in the chart). This is equivalent to the statement that $v \nmid \mathfrak{N}, v \mid f$ i.e. that the representation $\pi_v$ is an unramified principal series $\pi(\mu_1, \mu_2)$ and that $\chi_v$ is ramified with conductor $f_v$. Choose an additive character of $K_v$, say $\psi(x) = e^{2\pi i A(x/f_v)}$, as defined in Tate’s thesis [11]. With the formulas as in [8], we...
compute:
\[
\epsilon(s, \pi_v \otimes \chi_v, \psi)^{-1} = \epsilon(s, \chi_v \mu_1)^{-1} \epsilon(s, \chi_v \mu_2)^{-1} \\
= \prod_{i=1,2}^{N(d)} \mu_i \chi_v(\delta_i) N(\delta_i)^{s+1} N(\psi)^{-1} \sum \mu_i \chi_v(a) \psi(a) \\
= \frac{1}{\mu_1 \mu_2 (a)} \epsilon(s, \chi_v, \psi)^{-1} \\
= \epsilon(s, \chi_v, \psi)^{-1}
\]
the last formula following from the fact that \(\mu_1 \mu_2 = 1\) for all elliptic curves. We also have that
\[
\epsilon(s, \pi_v) = N(d)^{2s-1}
\]
whenever \(\pi_v\) is unramified. So that
\[
a_v = \frac{\epsilon(s, \chi_v, \psi)^{-2}}{N(d)^{2s-1}} \\
= \frac{N(d)^{2s-2} \tau(\chi_v)^2}{\chi_v(\delta_i)^2} \left( \sum_{a \mod f_v} \chi(a) \psi(a) \right)^2 \\
= N(d)^{2s-2} \tau(\chi_v)^2 \\
= N(d)^{2s-1} \chi_v(-1)
\]
Where \(\tau(\chi_v)\) is the Gauss sum of \(\chi_v\) with respect to the additive character \(\psi\). The fact that \(\tau(\chi_v)^2 = N(d) \chi(-1)\) can be shown using the epsilon factors for GL(1) (e.g. see [2], section 23.) We have that \(\epsilon_v(\chi_v) = \chi(-1)\).

Let \(S_1\) be the set of places that have the properties of the \(i\)th row in the Table. For example, \(S_1\) is the set of places where \(\pi_v\) is unramified and \(\chi_v\) is unramified.

**Proposition 2.** We have

\[
\text{rk}(E^\chi) \equiv \text{rk}(E) \mod 2 \iff \prod_{v \in S} \chi_v(-1) \prod_{v \in S} \chi_v(\infty) \prod_{v \in S} \chi_v(-\infty) \prod_{v \in S} -\chi_v \mu_v(w) = 1
\]

where \(S\) is the set of places where \(\pi_v\) is a special representation.

**Proof.** By equation 3.1 and Proposition 1 we have that

\[
\text{rk}(E^\chi) \equiv \text{rk}(E) \mod 2 \\
\prod_{v \in S_1, S_3, S_{10}} \chi_v(-1) \prod_{v \in S_2, S_4, S_5, S_8} \chi_v(\infty) \prod_{v \in S_7} -\chi_v(-\infty) \prod_{v \in S_9} -\chi_v \mu_v(w) = 1
\]

Now, since \(\chi\) is a Hecke character, \(\chi(-1, -1, \ldots, -1) = \prod \chi_v(-1) = 1\). Therefore, we may multiply the right hand side by \(\prod \chi_v(-1)\). Now since \(\chi_v\) is unramified in the sets \(S_1, S_3, S_{10}\) and \(S_6\), the result follows.

It will be useful to simplify the expression in Proposition 2. Let us denote \(\Sigma_1 = S_6 \cup S_7\) and \(\Sigma_2 = S_8 \cup S_9 \cup S_{10}\) and let \(m_v(\chi) = \chi_v(-1) a_v(\chi)\). More explicitly,

\[
m_v(\chi) = \begin{cases} 
\chi_v(w) & \text{if } \chi_v \text{ is unramified} \\
-\mu_v(w)^{-1} & \text{if } \chi_v \text{ is ramified}
\end{cases} \quad \text{for } v \in \Sigma_1
\]
We will rewrite Proposition 2 as
\[ \text{rk}(E^\chi) \equiv \text{rk}(E) \mod 2 \iff \prod_{v \mid \infty} \chi_v(-1) \prod_{\Sigma_1 \cup \Sigma_2} m_v(\chi) \]

note that \( \Sigma_1 \) are places where \( \pi_v \) is a special representation with unramified character and \( \Sigma_2 \) are places where \( \pi_v \) is a special representation with a ramified character. Thus, only the places where special representations occur (or real places) change the root number of a twisted curve.

4. The Density of Even Analytic Ranks

According to equation 3.2 the parity of the analytic rank doesn’t change upon twisting by \( \chi \) if and only a certain product of \(-1\)’s and \(+1\)’s occurring on the right-hand side of 3.2 is equal to 1. The next step involves computation the proportion of cases where the product is \(+1\). This amounts to an exercise in counting quadratic characters.

Let \( c_v \) be the set of local quadratic characters (if \( v \not| 2 \) then \( |c_v| = 4 \), and if \( v | 2 \) then \( |c_v| = 4 \cdot 2^{[K_v:Q]} \), and let \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_R \), where \( \Sigma_R \) is the set of real places of \( K \). For each element of \( \Sigma \), define
\[ \kappa_v = \begin{cases} \frac{1}{|c_v|} \sum_{\chi \in c_v} m_v(\chi), & \text{for } v \text{ in } \Sigma_1 \cup \Sigma_2 \\ \frac{1}{|c_v|} \sum_{\chi \in c_v} \chi_v(-1), & \text{for real primes} \end{cases} \]

Lemma 3. Let \( \Gamma = \prod_{v \in \Sigma} C(K_v) \). Then,
\[ \frac{|\{\chi \in \Gamma \mid n(\chi) = 1\}|}{|\Gamma|} = \frac{1 + \prod_{v \in \Sigma} \kappa_v}{2} \]

Proof. (This is lemma 7.5 in [2]) Let \( N = |\{\chi \in \Gamma \mid n(\chi) = 1\}| \). Then \( \Gamma \) can be written as \( \prod_{v \in \Sigma} c_v \), and we have that
\[ N - (|\Gamma| - N) = \sum_{\gamma \in \Gamma} \prod_{v \in \Sigma} n_v(\gamma_v) = \prod_{v \in \Sigma} \sum_{\gamma \in c_v} n_v(\gamma_v). \]

Now when we divide both sides by \( \Gamma = \prod_{v \in \Sigma} c_v \) we get that \( 2N/|\Gamma| - 1 = \prod_{v \in \Sigma} \kappa_v \) and the lemma follows.

Next, we compute the values of \( \kappa_v \).

Lemma 4. The local factors \( \kappa \) have the following values
- \( \kappa_v = 0 \) when \( K_v \cong \mathbb{R} \)
\[ \kappa_v = \begin{cases} 
\frac{2}{|c_v|} - 1 & \text{E has split multiplicative reduction at } v \\
1 - \frac{2}{|c_v|} & \text{E has nonsplit multiplicative reduction at } v \\
1 & \text{otherwise} \end{cases} \quad \text{for } v \in \Sigma_1 \]

\[ \kappa_v = \begin{cases} 
1 - 2/|c_v| & \text{E has mult. red. in a quad. extension at } v \\
1 & \text{otherwise} \end{cases} \quad \text{for } v \in \Sigma_2 \]

\[ \kappa_v = 1 \text{ if } E \text{ has good reduction, potentially good reduction, or } K_v \cong \mathbb{C} \]

Note that \( \Sigma_1 \) are the places where \( E \) has multiplicative reduction and \( \Sigma_2 \) are the places where \( E \) has potentially multiplicative reduction. Thus, \( \Sigma_1 \cup \Sigma_2 \cup \{ \text{real places} \} \) are the only places that affect the value of \( \kappa \).

**Proof.** For a real place, there are exactly two characters: \( \chi_{\text{triv}} \) and \( \chi_{\text{sign}} \) and

\[ \kappa_{\infty} = \frac{1}{2} (\chi_{\text{triv}}(-1) + \chi_{\text{sign}}(-1)) = 0. \]

For a non-archimedean place in \( \Sigma_1 \), there is one trivial character \( \chi_{\text{triv}} \) and an unramified character \( \chi_{u.r.} \). For the first two, \( m_v(\chi_v) = \chi_v(\varpi) \), so that \( m_v(\chi_{\text{triv}}) = 1 \) and \( m_v(\chi_{u.r.}) = -1 \). For the ramified characters \( \chi_{\text{ram}} \), \( m_v(\chi_{\text{ram}}) = -\mu_v(\varpi_v) \).

In total we have

\[ \kappa_v = \frac{1}{|c_v|} (1 - 1 + (|c_v| - 2) \mu_v(\varpi)) \]

It follows from a comparison of the \( L \)-function at multiplicative reduction and at a special representation with unramified character that \( \mu_v(\varpi) = 1 \) at split multiplicative reduction and \( \mu_v(\varpi) = -1 \) at non-split multiplicative reduction, which gives the result for \( v \in \Sigma_1 \).

For \( v \in \Sigma_2 \), we are in the situation where the local representation is of the form \( \sigma_v(\mu_v, \mu_v^{-1}) \) where \( \mu_v \) is ramified. If \( \chi_v \) is ramified, then \( m_v(\chi_v) = 1 \). If \( \chi_v \) is ramified and \( \chi_v \mu_v \) is also ramified, then \( m_v(\chi_v) = 1 \). If \( \chi_v \mu_v \) is ramified, then \( \mu_v |_{\mathbb{A}_K} \) must be a quadratic character and then there are exactly two characters \( \chi_{v,1} \) and \( \chi_{v,2} \) such that \( \chi_v \mu_v \) is unramified and furthermore, \( \chi_{v,1}(\varpi) = 1 \), and \( \chi_{v,2}(\varpi) = -1 \) so that \( m_v(\chi_{v,1}) = -\mu_v(\varpi) \) and \( m_v(\chi_{v,2}) = \mu_v(\varpi) \). In total, if \( \mu_v |_{\mathbb{A}_K} \) is not quadratic, then \( m_v(\chi_v) = 1 \) for all local characters \( \chi_v \) and if \( \mu_v |_{\mathbb{A}_K} \) is quadratic then

\[ \kappa_v = \frac{1}{|c_v|} (1 + (2 - |c_v|) \mu_v(\varpi) - \mu_v(\varpi)) = 1 - 2/|c_v| \]

Now, if \( \mu_v |_{\mathbb{A}_K} \) is quadratic, then the elliptic curve has potential multiplicative reduction at \( v \) (in particular, multiplicative reduction in a quadratic extension of \( K_v \)).

The following lemma deals with the problem that not all collections of local characters give rise to global characters.

**Lemma 5.** Recall that \( C(K) \) is the group of global quadratic characters, and \( \Gamma = \prod_{v \in \Sigma} C(K_v) \). The natural homomorphism \( \alpha : C(K) \to \Gamma \) is surjective.

**Proof.** Let \( \gamma = \prod_{v \in \Sigma} \gamma_v \in \prod_{v \in \Sigma} C(K_v) \) and let \( s \) be a place of \( K \), not in \( \Sigma \). Then if we set \( \gamma_v(q) = \gamma^{-1}(q) \), for all \( q \in K_v^\times \subset K_v^\times \), this defines a character on a dense subset of \( K_v^\times \), hence on all of \( K_v \) by continuity. Then \( \gamma' = (\prod \gamma_v) \cdot \gamma_s \) is trivial on \( K_v^\times \) and \( \alpha(\gamma') = \gamma \).
The lemma allows us to convert statements about the density of even analytic ranks among arbitrary products of local characters to statements about the density of even analytic ranks among Hecke characters.

To conclude, I will restate and prove the main theorem

**Theorem.** For all $X$ large enough,

$$\frac{\#\{\chi \in C(K, X) \text{ such that } \text{rk}(E^\chi) \text{ is even}\}}{|C(K, X)|} = \frac{1 + (-1)^{\text{rk}(E)}\kappa}{2}$$

where $\kappa = \prod_v \kappa_v$ are defined and computed above.

**Proof.** For $X$ large enough, the set of characters with norm less than $X$ surjects onto $\Gamma$ by lemma 5. Now by Proposition 2 the analytic rank only depends on a local product, and since the map $C(K) \rightarrow \Gamma$ is a homomorphism, all its fibers have the same size. Therefore,

$$\frac{\#\{\chi \in C(K, X) \text{ such that } \text{rk}(E^\chi) = \text{rk}(E)\}}{|C(K, X)|} = \frac{\#\{\chi \in \Gamma \text{ such that } n(\chi) = 1\}}{|\Gamma|}$$

And now the theorem follows from lemma 3.

Our theorem is about the expected parity of twists of $E$. The heuristic that elliptic curves have rank as low as possible implies that given the parity constraints, the curves in this family will have analytic rank 0 or 1 with far greater frequency than the higher order ranks. Thus together with the Birch and Swinnerton-Dyer conjecture, our results support the conjecture in [9].

**Conjecture.** (Klagsbrun-Mazur-Rubin)

$$\lim_{X \rightarrow \infty} \frac{\sum_{\chi \in C(K, X)} \text{rk}(E^\chi)}{|C(K, X)|} = \frac{1 + (-1)^{\text{rk}(E)}\kappa}{2}$$

Where the conjecture is adapted to analytic ranks and the local factors of $\kappa$ are explicitly computed in the cases of lemma 4.

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