Convolution-Weight-Distribution Assumption: Rethinking the Criteria of Channel Pruning

Huang Zhongzhan\textsuperscript{1,2} Wang Xinjiang\textsuperscript{2} Luo Ping\textsuperscript{3}

Abstract
Channel pruning is one of the most important techniques for compressing neural networks with convolutional filters. However, in our study, we find strong similarities among some primary pruning criteria proposed in recent years. The sequence of filters importance in a convolutional layer according to these criteria are almost the same, resulting in similar pruned structures. This finding can be explained by our assumption that the trained convolutional filters approximately follow a Gaussian-alike distribution, which is demonstrated through systematic and comprehensive statistical tests. Under this assumption, the similarity of these criteria is theoretically proved. Moreover, we also find that if the network has too much redundancy (exists a large number of filters in each convolutional layer), then these criteria can not distinguish the importance of the filters. This phenomenon is due to that the convolutional layer will form a special geometric structure when redundancy is large enough and our assumption holds: for every pair of filters in one layer, (1) Their $\ell_2$ norm are equivalent; (2) They are equidistant; (3) and they are orthogonal. The full appendix is released at https://github.com/dedekinds/CWDA.

1. Introduction
Pruning (Han et al., 2015; Li et al., 2016; He et al., 2019) a trained neural network is commonly seen in network compression. In particular, for neural networks with convolutional filters, channel pruning (Liu et al., 2019; Ding et al., 2019a; Frankle & Carbin, 2019) refers to the pruning of the filters in the convolutional layers. Specifically,

\begin{equation}
F_{i,j} \in \mathbb{R}^{N_i \times k \times k}
\end{equation}

represent the $j_{th}$ filter of the $i_{th}$ convolutional layer, where $N_i$ is the number of input channels for $i_{th}$ layer and $k$ denotes the kernel size of the convolutional filter. In $i_{th}$ layer, there are $N_{i+1}$ filters. There are many pruning methods following three standard procedures, called one-shot pruning: (Train): Train a network

\begin{table}[h]
\centering
\begin{tabular}{|l|l|l|}
\hline
Criteria & Model & Pruned Filters’ Index (Top 8) \\
\hline\hline
$\ell_1$ & ResNet18 & [111, 212, 33, 61, 68, 152, 171, 45] \\
$\ell_2$ & ResNet18 & [111, 33, 212, 61, 171, 42, 243, 129] \\
GM & ResNet18 & [111, 212, 33, 61, 68, 45, 171, 42] \\
Fermat & ResNet18 & [111, 212, 33, 61, 68, 45, 171, 42] \\
\hline
$\ell_1$ & VGG16 & [102, 28, 9, 88, 66, 109, 86, 45] \\
$\ell_2$ & VGG16 & [102, 28, 88, 9, 109, 66, 86, 45] \\
GM & VGG16 & [102, 28, 9, 88, 109, 66, 45, 86] \\
Fermat & VGG16 & [102, 28, 88, 9, 109, 66, 45, 86] \\
\hline
\end{tabular}
\caption{The pruned filters’ index ordered by the filters’ importance from given pruning criteria (ascending order), taking Pytorch pre-trained model VGG16 (3\textsuperscript{rd} Conv) and ResNet18 (12\textsuperscript{th} Conv) as examples.}
\end{table}

Figure 1. The Spearman’s rank correlation coefficient (Sp) for different criteria. (a-c) are Sp between $\ell_1$ and $\ell_2$, GM and $\ell_2$, Fermat and $\ell_2$ from Pytorch pre-trained ResNet18 (12\textsuperscript{th} Conv), respectively. The results of VGG16 (3\textsuperscript{rd} Conv) are shown in (d-f). If the Sp of two pruning criteria is close to 1, then the sequence of their pruned filters may have strong similarity.

\begin{enumerate}
\item Let $F_{i,j} \in \mathbb{R}^{N_i \times k \times k}$ represent the $j_{th}$ filter of the $i_{th}$ convolutional layer, where $N_i$ is the number of input channels for $i_{th}$ layer and $k$ denotes the kernel size of the convolutional filter.
\item In $i_{th}$ layer, there are $N_{i+1}$ filters. There are many pruning methods following three standard procedures, called one-shot pruning: (Train): Train a network
\item Work in progress.
\end{enumerate}
from scratch; (Prune) Use a certain criterion to calculate filters importance, and prune the filters which have small importance; (Fine-tune) After additional training, the pruned network can recover its accuracy to some extent. Similarly, there is iterative pruning (He et al., 2018; Frankle & Carbin, 2019; Renda et al., 2020), which uses Prune and Fine-tune alternately. In this paper, in order to focus on the pruning criteria, the relatively simple one-shot pruning (layer-wise) is used in all pruning experiments.

As one of the simplest and most effective channel pruning criteria, ℓ² pruning (Li et al., 2016) is widely used in the industry. The core idea of this criterion is to sort the ℓ² norm of convolutional filters in one layer and then prune the filters, which have a small ℓ² norm. Similarly, there is ℓ₁ pruning (Frankle & Carbin, 2019; He et al., 2018). These pruning criteria are called “norm-based pruning” as they include norm in their design. Through the study of the distribution of norm, (He et al., 2019) demonstrates that “norm-based pruning” should satisfy two conditions when we consider the absolute “importance” of filters: (1) the variance of the norm of the filters cannot be too small; (2) the minimum norm of the filters should be small enough. However, these two conditions do not always hold. Given this situation, a new criterion considering the relative importance of the filter is proposed (He et al., 2019). Since this criterion uses the Fermat point in Euclidean space (i.e., geometric median (Cohen et al., 2016)), We call it Fermat . Later, due to the high calculation cost of Fermat point, (He et al., 2019) relaxed it and then got another criterion “GM” method. The details are shown in Table 2. F denotes the Fermat point¹ of 𝐹𝑖𝑗 in Euclidean space.

Table 2. Some channel pruning criteria.

| Criterion       | Details of importance                                      |
|-----------------|------------------------------------------------------------|
| ℓ₁ (Li et al., 2016) | ||𝐹𝑖1||₁               |
| ℓ₂ (Frankle & Carbin, 2019) | ||𝐹𝑖2||₂               |
| Fermat (He et al., 2019) | ||𝐹 − 𝐹𝑖||₂         |
| GM (He et al., 2019)                  | ∑[k=1]^N ||𝐹𝑖𝑘 − 𝐹𝑖𝑗||₂ |

In previous work (Liu et al., 2017; Han et al., 2015; Ding et al., 2019b; Aghasi et al., 2017; Dong et al., 2017; Renda et al., 2020), including the criteria mentioned above, they usually focused on how much the model was compressed, how much performance was restored, the inference efficiency of the pruned network and the cost of finding the pruned network, etc. However, there is little work to discuss such a problem:

What are actually the differences among these criteria?

We observe the differences from three perspectives.

¹http://en.wikipedia.org/wiki/Fermat_point.

²https://pytorch.org/docs/stable/torchvision/models.html

(1) The sequence of filters importance. Using Pytorch pre-trained model (VGG16 and ResNet18), we show the sequence of filters importance under different criteria in Table 1. It is easy to find that they have almost the same sequence. To further verify this observation, we use a statistical method, Spearman’s rank correlation coefficient (Sedwick, 2014), to quantify the similarity.

(2) Spearman’s rank correlation coefficient (Sp). Sp is a nonparametric measurement of ranking correlation and it assesses how well the relationship between two variables can be described using a monotonic function, i.e., filters ranking sequence in the same layer under two criteria in this paper. The results are shown in Fig. 1. In general, if the Sp is above 0.8, it means that there is a strong similarity between the sequence of two variables.

(3) The overlap of the pruned filters. Note that, although the sequences of filters importance under different criteria are not exactly the same, as the Sp is high enough, the actual pruned filters may have a high probability of overlap, which means that the pruned filters of different pruning criteria may have the same indexes in a layer. We use overlap degree (OD) to represent this kind of overlap:

\[ \text{OD}(C_1, C_2, p) = \frac{\#(C_1 \cap C_2)}{\#C_2} \times 100\%, \]

where \( C_1 \) and \( C_2 \) denote the set of pruned filters from two pruning criteria with \( p\% \) pruning ratio (the percentage of filter should be pruned in one layer); \# denotes the cardinality of a set. In Fig. 2, the results of OD when \( p = 0.5 \) are shown. These high OD throughout the network make the structure of the network after pruning very similar.

From three verifications mentioned above, the criteria about absolute importance of filters (like ℓ₁, ℓ₂) and the criteria about relative importance of filters (like Fermat, GM) may not be significantly different. Therefore we want to further
verify this similarity. In Section 2, First, we come up with an assumption about the distribution of the weights of the convolutional filters, called Convolution-Weight-Distribution Assumption (CWDA), with systematic and comprehensive statistical tests. Next, in Section 3, under the CWDA, the similarity of these criteria is theoretically proved. At the same time, more experiments are provided to verify this similarity. Last but not least, in Section 4, we discuss the conditions for CWDA to be satisfied and study the situation that the network has too much redundancy.

Contribution.

(1) We find strong similarities among some primary pruning criteria proposed in recent years.

(2) We propose and verify an assumption CWDA, which reveals that the trained convolutional filters approximately follow a Gaussian-alike distribution. And the similarity of these criteria can be theoretically proved by CWDA.

(3) We propose a geometric structure of convolutional filters in weight space when the network exists too much redundancy. Using this structure, we prove that these criteria can not distinguish the importance of the filters when redundancy is large enough.

2. Weight Distribution-Assumption

In this section, to explain the similarity among the pruning criteria shown in Table 2, we propose and verify an assumption about the distribution of convolutional filters.

(Convolution-Weight-Distribution-Assumption) Let \( F_{ij} \in \mathbb{R}^{N_i \times k \times k} \) be the \( j \text{th} \) trained filter of the \( i \text{th} \) convolutional layer. In general\(^3\), \( F_{ij} \) in \( i \text{th} \) layer are i.i.d and approximately follow a Gaussian-like distribution:

\[
F_{ij} \sim N(0, c^2 \cdot I_{N_i \times k \times k}),
\]

where \( c \) is a constant and \( I_{N_i \times k \times k} \) is an identity matrix.

2.1. Statistical test

In fact, CWDA is not easy to verify. For example, for ResNet164 (On Cifar100), the number of filters in the first stage is only 16, which is too small to be used to estimate statistics accurately. More other objective reasons are shown in Section 4.1. In response to these problems, we consider to verify three necessary conditions of CWDA,

(1) **Gaussian.** (i.e., in order to verify if \( F_{ij} \) approximately follow a Gaussian-like distribution.) In \( i \text{th} \) layer, we use Kolmogorov-Smirnov (KS) test (Lilliefors, 1967) to check if all the weights in the same layer follow a normal distribution \( N(0, c^2) \).

(2) **Standard Deviation.** (i.e., to verify if the standard deviation of each filter in any layers tends to be a constant \( c \).) Let \( \sigma_j \) denotes the standard deviation of all the weights of filter \( F_{ij} \) in \( i \text{th} \) layer. We use Student’s t test (Efron, 1969) to check if the variance of these \( \sigma_j \) is small enough. Note that, Section 4.1 shows that there are objective reasons (bad training, numbers, dimensions, etc.) that make CWDA sometimes untenable. So we emphasize that what we have tested are only necessary conditions and that these standard deviation are small enough instead of being very small.

(3) **Mean.** (i.e., to verify if the mean of \( F_{ij} \) is 0.) Let the mean of all the weights in the same layer is \( \mu \). We use Student’s t test (Efron, 1969) to check if \( \mu \leq \epsilon \cdot S \) and \( \mu \geq -\epsilon \cdot S \), where \( \epsilon \) is a small constant and \( S \) is sample variance.

In Table 3, we list a series of experiments to verify CWDA, and the details of these statistical tests are shown in each description of the experiment in Appendix G. In Fig. 3, taking Pytorch pre-trained model (VGG16 and ResNet18) as examples, we visualize the distribution of the convolutional filters.

![Figure 3. Visualization of the distribution of convolutional filters.](image)

3. Experiment and theory

In this section, we further verify the conclusion that the pruning criteria in Table 2 are highly similar from two perspectives. From an experimental point of view, in section 3.1, we used more experiments to investigate the similarity of the image classification accuracy and network structure of the model after pruning by different criteria. From a
Table 3. The experiments for having the comprehensive statistical tests on CWDA.

| NETWORK STRUCTURE (G.1) | OPTIMIZER (G.2) | REGULARIZATION (G.3) |
|-------------------------|----------------|----------------------|
| ResNet(He et al., 2016a) | SGD(Sutskever et al., 2013) | L1 norm |
| VGG(Simonyan & Zisserman, 2014) | ASGD(Polyak & Juditsky, 1992) | L2 norm |
| AlexNet(Krizhevsky, 2014) | Adam(Kingma & Ba, 2014) | RReLU(Xu et al., 2015) |
| DenseNet(Huang et al., 2017) | Adadelta(Zeiler, 2012) | GAN |
| PreResNet(He et al., 2016b) | Adamax(Kingma & Ba, 2014) | RRelu(Xu et al., 2015) |
| WRN(Zagoruyko & Komodakis, 2016) | Adadelta(Zeiler, 2012) | MATTING |
| ResNext(Xie et al., 2017) | - | - |

| ATTENTION MECHANISM (G.4) | INITIALIZATION (G.5) | DATASET (G.6) |
|---------------------------|----------------------|---------------|
| SENet(Hu et al., 2018) | Kaiming-normal(He et al., 2015) | CIFAR10(Krizhevsky et al., 2009) |
| DIA Net(Huang et al., 2019) | Kaiming-uniform(He et al., 2015) | CIFAR100(Krizhevsky et al., 2009) |
| SRMNet(LEE et al., 2019) | Xavier-normal(Glorot & Bengio, 2010) | ImageNet(Russakovsky et al., 2015) |
| CBAM(Woo et al., 2018) | Xavier-uniform(Glorot & Bengio, 2010) | MNIST(LeCun et al., 1998) |
| IEBN(Li et al., 2019) | Orthogonal(Saxe et al., 2013) | - |
| SGENet(Li et al., 2019) | - | - |

| SEGMENTATION (G.7) | DETECTION (G.7) | BATCH NORMALIZATION (G.8) |
|-------------------|----------------|--------------------------|
| SegNet(Badrinarayanan et al., 2017) | Faster RCNN(Ren et al., 2015) | SGD |
| PSPNet(Zhao et al., 2017) | VGG | VGG-bn |
| - | - | - |

| PYTORCH PRETRAIN (G.9) | MATING (G.7) | LEARNING RATE (G.10) |
|------------------------|-------------|---------------------|
| ResNet18/34/50 | Deep image matting(Xu et al., 2017) | Schedule150-255 |
| VGG11/16/19 | AlphaGAN matting(Lutz et al., 2018) | Schedule82-164 |

| STYLE TRANSFER (G.7) | GAN(G.7) | - |
|----------------------|---------|---|
| Fast neural style(Johnson et al., 2016) | DCGAN(Radford et al., 2015) | - |

3.1. Experiment

In Table 4, we provide additional image classification experiments, which are repeated three times for each criterion. In the "Pruned" and "Fine-tuned" stage, the classification accuracies (acc.) are similar to each other from different criteria in each group of experiments. This phenomenon further shows the similarity of pruning criteria. The similar acc. may also imply those pruned network from different pruning criteria may be very similar.

In Fig. 4, we show the Sp between different pruning criteria on different datasets. The Sp in most convolutional layers is more than 0.9, which means the network structures after pruning are almost the same. Note that the Sp in transitional area (i.e., the layer where the dimensions of the filter change, as the layer between stage 1 and stage 2 of ResNet164. The number of channels in stage 1 and stage 2 are 16 and 32 respectively.) are relatively small. It is interesting and will not have a great impact on the structural similarity of the whole network pruned. The reason for this phenomenon may be that the layers in these areas are sensitive. The similar observations are shown in Figure.2 in (Ding et al., 2019b) and Figure.6 and Figure.10 in (Li et al., 2016). In Section 4, we have a further discussion about this phenomenon.

3.2. Theoretical analysis

In this section, the similarities among the pruning criteria in Table 2 are proved theoretically. Let $C_1$ and $C_2$ be two pruning criteria to calculate the importance for all convolutional filters of one layer. If they can make the similar sequence of importance, we define $C_1$ and $C_2$ are approximately monotonic to each other and use $C_1 \cong C_2$ to represent this relationship. In Section 1 and Fig 4, we use the Spearman’s rank correlation coefficient (Sp) to describe this relationship. However, the Sp is not easy to be analyzed theoretically. Therefore, we consider about a stronger condition. Let $X = (x_1, x_2, ..., x_k)$ and $Y = (y_1, y_2, ..., y_k)$ are two given sequences. we first should normalize their magnitude, i.e., let $\hat{X} \sim X/\sqrt{\text{Var}(X)}$ and $\hat{Y} \sim Y/\sqrt{\text{Var}(Y)}$. Because, in these situations, the ratio $\hat{X}/\hat{Y}$ and $\hat{Y}/\hat{X}$ will be close to two constants $a, b$. Note that, for any $1 \leq i \leq k$, $\hat{x}_i \approx a \cdot \hat{y}_i$ and $\hat{y}_i \approx b \cdot \hat{x}_i$ and we have $ab \approx 1$ and $a, b \neq 0$. So there exists an approximately monotonic mapping from $\hat{y}_i$ to $\hat{x}_i$ and it makes the Sp between $X$ and $Y$ is close to 1.

For $i_{th}$ layer, we use $v_j$ to represent $F_{ji}, j = 1, 2, ..., N$. And $v_j$ meets CWDA (i.e., $v_j$ are i.i.d and $v_j \sim N(0, e^c \cdot I)$).
Table 4. The classification accuracy(%) of several networks and datasets using different pruning criteria.

|                  | Experiment (1) | Experiment (2) | Experiment (3) |
|------------------|----------------|----------------|----------------|
|                  | Trained        | Pruned         | Fine-tuned     |
| CIFAR10          | 93.61          | 63.21          | 93.51          |
| VGG16            | 93.65          | 63.47          | 93.22          |
|                  | 93.21          | 54.61          | 93.22          |
| GM               | 93.65          | 63.47          | 93.22          |
|                  | 93.21          | 54.61          | 93.22          |
| CIFAR100         | 70.02          | 52.39          | 69.19          |
| VGG16            | 70.02          | 52.39          | 69.19          |
|                  | 70.02          | 52.39          | 69.19          |
| GM               | 70.02          | 52.39          | 69.19          |
|                  | 70.02          | 52.39          | 69.19          |
| ImageNet         | 92.97          | 77.91          | 92.72          |
| ResNet56         | 92.97          | 77.91          | 92.72          |
|                  | 92.97          | 77.91          | 92.72          |
| GM               | 92.97          | 77.91          | 92.72          |
|                  | 92.97          | 77.91          | 92.72          |
| CIFAR100         | 71.36          | 50.64          | 70.15          |
| ResNet56         | 71.36          | 50.64          | 70.15          |
|                  | 71.36          | 50.64          | 70.15          |
| GM               | 71.36          | 50.64          | 70.15          |
|                  | 71.36          | 50.64          | 70.15          |
| ImageNet         | 73.31          | 62.25          | 73.06          |
| ResNet34         | 73.31          | 62.25          | 73.06          |
|                  | 73.31          | 62.25          | 73.06          |
| GM               | 73.31          | 62.25          | 73.06          |
|                  | 73.31          | 62.25          | 73.06          |

Figure 4. Spearman’s rank correlation coefficient (Sp) between different pruning criteria on several networks and datasets. Each curve is the mean obtained by three experiments in Table 4. The details of these experiments are shown in Appendix H.

(1) For $\ell_2 \cong \ell_1$. In fact, $\ell_2 \cong \ell_1$ (their importance rankings are similar) is not trivial. Generally speaking, for convolutional filters, $\text{dim}(v_i)$ is large enough. Since $v_i$ satisfies CWDA, from Theorem 1, we know that the ratio between $\ell_1$ and $\ell_2$ have a bound $O(\text{dim}(v_i)^{-1})$, which means $\ell_2$ and $\ell_1$ are appropriate monotonic. Specific numerical validation is shown in Fig.9 of Appendix B.

**Theorem 1.** Let $X \sim N(0, c^2 \cdot I_n)$, we have

$$\max \left\{ \text{Var}_X \left( \frac{\ell_2(X)}{\ell_1(X)} \right), \text{Var}_X \left( \frac{\ell_1(X)}{\ell_2(X)} \right) \right\} \leq \frac{1}{n}. \quad (3)$$

where $\ell_1(X)$ denotes $\ell_1(X)/E(\ell_1(X))$ and $\ell_2(X)$ denotes $\ell_2(X)/E(\ell_2(X))$.

**Proof.** (See Appendix B).

(2) For $\ell_2 \cong \text{Fermat}$. Since $v_i$ satisfies CWDA, from Theorem 2, we know that the Fermat point of $v_i$ and the origin 0 approximately coincide. According to Table 2, $||\text{Fermat} - v_i||_2 \approx ||0 - v_i||_2 = ||v_i||_2$. Therefore, we have $\ell_2 \cong \text{Fermat}$. Note that, since CWDA, the centroid of $v_i$ is $G = \frac{1}{n} \sum_{i=1}^{n} v_i = 0$. Hence,

$$G = 0 \approx \text{Fermat}. \quad (4)$$

**Theorem 2.** Let random variable $v_i \in \mathbb{R}^k$ and they are i.i.d and follow normal distribution $N(0, \sigma I_k)$. For $F$ in $\mathbb{R}^k$, we have

$$\arg\min_F \left\{ \mathbb{E}_{v_i \sim N(0, \sigma I_k)} \sum_{i=1}^{n} ||F - v_i||_2 \right\} = 0. \quad (5)$$
Proof. (See Appendix C). □

(3) For \( GM \cong \text{Fermat} \). First, two theorems are shown:

**Theorem 3.** For \( n \) random variables \( a_i \in \mathbb{R}^k \) follow \( N(0, c^2 \cdot I_k) \). When \( k \) is large enough and \( c \) is small enough, we have such an estimation:

\[
\text{Var}_i \left( \frac{F_1(a_i)}{F_2(a_i)} \right) \approx \frac{1}{2nk}, \quad \text{Var}_i \left( \frac{F_2(a_i)}{F_1(a_i)} \right) \approx \frac{1}{2nk}, \quad (6)
\]

where \( F_1(a_i) = \sum_{i=1}^{n} ||a_i||^2 / \mathbb{E}(\sum_{i=1}^{n} ||a_i||^2) \) and \( F_2(a_i) = \sum_{i=1}^{n} ||a_i||^2 / \mathbb{E}(\sum_{i=1}^{n} ||a_i||^2) \).

**Proof.** (See Appendix D). □

**Theorem 4.** Let \( v_0, v_1, \ldots, v_k \) be the \( k + 1 \) vectors in \( n \) dimensional Euclidean space \( \mathbb{E}^n \). For all \( P \in \mathbb{E}^n \),

\[
\sum_{i=0}^{k} ||P - v_i||^2 = \sum_{i=0}^{k} ||G - v_i||^2 + (k+1)||P-G||^2, \quad (7)
\]

where \( G \) is the centroid of \( v_i \), will hold if it satisfies one of the following conditions:

1) If \( k \geq n \) and \( \text{rank}(v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0) = n \).
2) If \( k < n \) and \( (v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0) \) are linearly independent.

if \( v_i \sim N(0, c^2 \cdot I_n) \), Eq.(7) holds with probability 1.

**Proof.** (See Appendix E). □

Let \( P \in \{v_1, v_2, ..., v_N\} \). Since \( v_i \sim N(0, c^2 \cdot I) \), we can obtain that \( a_i = P - v_i \sim N(0, 2c^2 \cdot I) \) if \( P \neq v_i \).

According to the analysis in Section 3.2 (1) and Theorem 3, we have

\[
\sum_{i=1}^{n} ||a_i||^2 \approx \sum_{i=1}^{n} ||v_i||^2. \quad (8)
\]

Next, we can prove \( (k+1)||P - F||^2 \) (Fermat) and \( \sum_{i=1}^{N} ||P - v_i||^2 \) (GM) are approximately monotonic, where \( P = \{v_1, v_2, ..., v_N\} \).

\[
(9)
\]

\[
\cong (k+1)||P - G||^2 \quad \text{Since Eq. (4)}
\]

\[
= \sum_{i=1}^{N} ||P - v_i||^2 - \sum_{i=1}^{N} ||G - v_i||^2 \quad \text{Since Theorem 4}
\]

\[
\cong \sum_{i=1}^{N} ||P - v_i||^2 - \sum_{i=1}^{N} ||G - v_i||^2 \quad \text{Since Eq. (8)}
\]

\[
\cong \sum_{i=1}^{N} ||P - v_i||^2 \quad (10)
\]

The reason for the last equation is that \( \sum_{i=1}^{N} ||G - v_i||^2 \) is a constant for given \( v_i \). Therefore, from Section 3.2 (1) to Section 3.2 (3), we have \( \ell_1 \cong \ell_2 \cong \text{Fermat} \cong \text{GM} \). (\( \cong \) has transitive Property) Hence, the similarities among these criteria are proved.

### 4. Discussion

In this section, we first discuss the reasons why CWDA sometimes does not hold; then, we consider a special situation that there is too much redundancy in the neural network.

#### 4.1. Why CWDA sometimes does not hold

The CWDA is not necessarily completely correct. As shown in the statistical tests from Appendix G, a small number of convolutional layers can not pass the statistical test. In this section, we try to analyze this phenomenon.

1) **Need to be trained well enough.** The distribution of weights can be discussed only when the network is trained well. If the network itself does not converge or its performance is poor, the analysis about the distribution of weights is meaningless. For example, the model is too strong and causes severe overfitting, or the model is too weak to fit the data effectively.

2) **The number of filters is insufficient.** In Appendix G, the layers that can not pass the statistical test are almost the layers that are in front of the network. A common feature of these layers is that they have very few numbers of filters, which are not enough to estimate statistics well. In Fig. 4, due to the sensibility of layers, a phenomenon that the Sp in transitional area are relatively small. Taking the second convolutional layer (64 filters) in VGG16 on CIFAR10 as example, we find that increasing the number of filters could alleviate this sensitivity. As shown in Fig. 5, we change the number of filters in this layer from 64 to 128 and 256. After that, the Sp increased significantly and it suggests that the enough number of filters is important.

3) **The dimensions of the filter are not enough.** The dimension of filter in the \( i_{th} \) layer is closely related to the number of filters in the \( (i-1)_{th} \) layer. To eliminate the factor about the number of filters, we can check the covariance matrix of each layer of filters, i.e., \( \text{Cov}(v_i, v_j)_{N \times N} \). If CWDA holds, we can expect that these covariance matrices should approximatively be a diagonal matrix with the same diagonal elements. From the results in Fig. 6, with the increase of dimensions (the dimensions of filters in these networks increase with the increase of depth), the covariance matrix is closer to a diagonal matrix with the same diagonal elements. This implies that the large enough dimensions of the filter is an important condition of CWDA.
4.2. Geometric structure of convolutional filters

**Theorem 5.** Let convolutional filters \( v_i \in \mathbb{R}^k \) of one layer are i.i.d and \( v_i \sim N(0, c^2 \cdot I_k) \). If \( k \to \infty \), then

1. \( \|v_i\|_2 \approx \|v_j\|_2 \to \sqrt{2c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)}, 1 \leq i, j \leq N; \)
2. \( \text{angle}(v_i, v_j) \to \frac{\pi}{2}, 1 \leq i, j \leq N; \)
3. \( \|v_i - v_j\|_2 \approx \|v_i - v_t\|_2, 1 \leq i, j, t \leq N; \)

**Proof.** (See Appendix F).

If CWDA holds, let’s consider a special situation that the network has too much redundancy (like VGG), which means that there exists a large number of filters in each convolutional layer. It’s easy to know that the dimensions of the filters are also large enough in this situation. As shown in Fig. 7, from Theorem 5 (1), the convolutional filters \( v_i \) of each layer locate approximately on the surface of \( k \)-dimensional sphere with \( 0 \) as the origin and \( \sqrt{2c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \) as the radius. Moreover, from Theorem 5 (2) and Theorem 5 (3), two vectors formed by any two convolutional filters in the same layer are orthogonal and equidistant. In fact, Fig. 6 provides another view of the geometric structure of convolutional filters. Since CWDA, \( E(v_i) = 0 \). So the covariance matrix \( (\text{Cov}(v_i, v_j))_{N \times N} = c \cdot (v_i^T v_j)_{N \times N} \), where \( c \) is a constant. That is to say, there is only one coefficient difference between covariance matrix and Gram matrix. Therefore, the diagonal elements of the matrix are \( \|v_i\|_2^2 \), and the non-diagonal elements are the dot product between the convolution filters. From the observation of Section 4.1 (3), on the one hand, these diagonal elements tend to be the same, which is consistent with the Theorem 5 (1); on the other hand, the non-diagonal elements of the matrix are almost 0, meaning that the filters are perpendicular to each other, which is consistent with the Theorem 5 (1). From Pythagoras theorem, the Theorem 5 (3) can be obtained.

In this situation, the pruning criteria mentioned in Table 2 cant distinguish the importance of the filters. (i.e. their motivations are not effective when there is enough redundancy) For example, the motivation of \( \ell_2 \) pruning is that the filters with small \( \ell_2 \) norm can be pruned (think that they provide less information(Ye et al., 2018)). But if this motivation...
Figure 7. The geometric structure of convolutional filters when the network has too much redundancy. For every pair of filters in one layer,
(1) Their $\ell_2$ norm are equivalent ($||v_1||_2 \approx ||v_2||_2 \approx ||v_3||_2$); (2) They are equidistant ($||v_1 - v_2||_2 \approx ||v_2 - v_3||_2 \approx ||v_3 - v_1||_2$); (3) and they are orthogonal ($v_1^T v_2 \approx v_2^T v_3 \approx v_3^T v_1 \approx 0$).

works, it should satisfy the condition that the norm deviation of the filters should be large (He et al., 2019). There is a toy example. Since this motivation, if the $\ell_2$ norm (regarded as "importance") of the filters in one layer are 0.9, 0.8, 0.4 and 0.01, its easy to know that we should prune the last filter. But if the norm are similar, like 0.91, 0.92, 0.93, 0.92, it’s not easy to know which filter should be pruned even though the first one is the smallest. Since Theorem 5 (1), with enough redundancy, the importance calculated by $\ell_2$ norm tend to be identical, i.e., its hard to distinguish these importance.

We can also use Theorem 5 to analyze other criteria. Due to Theorem 5 (3), for any given convolutional filter $v_i$ in one layer, $\sum_{j=1}^N ||v_i v_j||_2$ are approximate, which shows that the GM method also exist the same problem as $\ell_2$. Next, from Theorem 5 (1) and Theorem 1, we know that $\ell_1 \approx \ell_2$ and there is no significant difference in filters’ importance calculated by $\ell_1$. In addition, as the Eq. (4), the Fermat point coincides with the origin, so a similar conclusion can be obtained for Fermat method. In CVPR 2019 Oral, there is a pruning criterion using orthogonality of filters, called RePr (Prakash et al., 2019). Based on Theorem 5 (2) and Theorem 5 (1), we also know that the motivations of this criterion is not effective pruning when the redundancy is too large.

4.3. About weight distribution

In Fig. 8, we show the other learnable parameters, i.e., Gamma and Beta in Batch normalization (BN) and weights in fully connected neural network (FC) in Pytorch pre-trained model VGG16-BN. For BN, the distribution of its weights does not satisfy CWDA and the similar results are shown in (Liu et al., 2017; Tian et al., 2019). Moreover, the distribution of the weights of FC isn’t considered to be a normal distribution in previous work (Bellido & Fiesler, 1993; Neal, 1995; Go et al., 2004), which is consistent with the observation (can not pass the KS test in Section 2.1) in Fig. 8 (b). Hence, it seems that the convolutional filters are the only learnable parameters meet CWDA. Of course, the studies on the distribution of weights of FC should be explored in more detail and it is a future work. Next, despite passing the statistical test and the $p$-value is large enough, this does not mean that CWDA is correct. Based on our observation, we propose a more general conjecture:

(CWDA Plus) The convolution filters $F_{ij} \in \mathbb{R}^{N_i \times k \times k}$ in $i_{th}$ layer are i.i.d and $F_{ij} \sim P(0, \text{Symmetric})$, where $P(0, \text{Symmetric})$ is a symmetric distribution with mathematical expectation 0.

5. Conclusion

The convolutional filters approximatively follow a Gaussian-alike distribution, and it makes some primary pruning criteria proposed in recent years obtain similar pruning results. Moreover, we find that these criteria can not distinguish the importance of the filters when the redundancy of a neural network is large enough.

References

Aghasi, A., Abdi, A., Nguyen, N., and Romberg, J. Net-trim: Convex pruning of deep neural networks with per-
formance guarantee. In *Advances in Neural Information Processing Systems*, pp. 3177–3186, 2017.

Badrinarayanan, V., Kendall, A., and Cipolla, R. Segnet: A deep convolutional encoder-decoder architecture for image segmentation. *IEEE transactions on pattern analysis and machine intelligence*, 39(12):2481–2495, 2017.

Bellido, I. and Fiesler, E. Do backpropagation trained neural networks have normal weight distributions? In *International Conference on Artificial Neural Networks*, pp. 772–775. Springer, 1993.

Cohen, M. B., Lee, Y. T., Miller, G., Pachocki, J., and Sidford, A. Geometric median in nearly linear time. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pp. 9–21. ACM, 2016.

Crooks, G. E. Survey of simple, continuous, univariate probability distributions. Technical report, Technical report, Lawrence Berkeley National Lab, 2013., 2012.

Cubuk, E. D., Zoph, B., Mane, D., Vasudevan, V., and Le, Q. V. Autoaugment: Learning augmentation strategies from data. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 113–123, 2019.

DeVries, T. and Taylor, G. W. Improved regularization of convolutional neural networks with cutout. *arXiv preprint arXiv:1708.04552*, 2017.

Ding, X., Ding, G., Guo, Y., and Han, J. Centripetal sgd for pruning very deep convolutional networks with complicated structure. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pp. 4943–4953, 2019a.

Ding, X., Zhou, X., Guo, Y., Han, J., Liu, J., et al. Global sparse momentum sgd for pruning very deep neural networks. In *Advances in Neural Information Processing Systems*, pp. 6379–6391, 2019b.

Dong, X., Chen, S., and Pan, S. Learning to prune deep neural networks via layer-wise optimal brain surgeon. In *Advances in Neural Information Processing Systems*, pp. 4857–4867, 2017.

Duchi, J., Hazan, E., and Singer, Y. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of machine learning research*, 12(Jul):2121–2159, 2011.

Efron, B. Student’s t-test under symmetry conditions. *Journal of the American Statistical Association*, 64(328):1278–1302, 1969.

Frankle, J. and Carbin, M. The lottery ticket hypothesis: Finding sparse, trainable neural networks. In *International Conference on Learning Representations*, 2019. URL https://openreview.net/forum?id=rJ1-b3RcF7.

Glorot, X. and Bengio, Y. Understanding the difficulty of training deep feedforward neural networks. In *Proceedings of the thirteenth international conference on artificial intelligence and statistics*, pp. 249–256, 2010.

Go, J., Baek, B., and Lee, C. Analyzing weight distribution of feedforward neural networks and efficient weight initialization. In *Joint IAPR International Workshops on Statistical Techniques in Pattern Recognition (SPR) and Structural and Syntactic Pattern Recognition (SSPR)*, pp. 840–849. Springer, 2004.

Graham, R. Applications of the fkg inequality and its relatives. In *Mathematical Programming The State of the Art*, pp. 115–131. Springer, 1983.

Han, S., Mao, H., and Dally, W. J. Deep compression: Compressing deep neural networks with pruning, trained quantization and huffman coding. *arXiv preprint arXiv:1510.00149*, 2015.

He, K., Zhang, X., Ren, S., and Sun, J. Delving deep into rectifiers: Surpassing human-level performance on imagenet classification. In *Proceedings of the IEEE international conference on computer vision*, pp. 1026–1034, 2015.

He, K., Zhang, X., Ren, S., and Sun, J. Deep residual learning for image recognition. In *The IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, June 2016a.

He, K., Zhang, X., Ren, S., and Sun, J. Identity mappings in deep residual networks. In *European conference on computer vision*, pp. 630–645. Springer, 2016b.

He, Y., Kang, G., Dong, X., Fu, Y., and Yang, Y. Soft filter pruning for accelerating deep convolutional neural networks. *arXiv preprint arXiv:1808.06866*, 2018.

He, Y., Liu, P., Wang, Z., Hu, Z., and Yang, Y. Filter pruning via geometric median for deep convolutional neural networks acceleration. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pp. 4340–4349, 2019.

Hormander, L. *The analysis of partial differential operators*. Springer, 1983.

Hu, J., Shen, L., and Sun, G. Squeeze-and-excitation networks. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 7132–7141, 2018.
Huang, G., Liu, Z., Van Der Maaten, L., and Weinberger, K. Q. Densely connected convolutional networks. In Proceedings of the IEEE conference on computer vision and pattern recognition, pp. 4700–4708, 2017.

Huang, Z., Liang, S., Liang, M., and Yang, H. Dianet: Dense-and-implicit attention network. arXiv preprint arXiv:1905.10671, 2019.

Johnson, J., Alahi, A., and Fei-Fei, L. Perceptual losses for real-time style transfer and super-resolution. In European conference on computer vision, pp. 694–711. Springer, 2016.

Kingma, D. P. and Ba, J. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

Krizhevsky, A. One weird trick for parallelizing convolutional neural networks. arXiv preprint arXiv:1404.5997, 2014.

Krizhevsky, A., Hinton, G., et al. Learning multiple layers of features from tiny images. 2009.

LeCun, Y., Bottou, L., Bengio, Y., and Haffner, P. Gradient-based learning applied to document recognition. Proceedings of the IEEE, 86(11):2278–2324, 1998.

Lee, H., Kim, H.-E., and Nam, H. Srnc: A style-based recalibration module for convolutional neural networks. In Proceedings of the IEEE International Conference on Computer Vision, pp. 1854–1862, 2019.

Li, H., Kadav, A., Duradovic, I., Samet, H., and Graf, H. P. Pruning filters for efficient convnets. arXiv preprint arXiv:1608.08710, 2016.

Li, X., Hu, X., and Yang, J. Spatial group-wise enhance: Improving semantic feature learning in convolutional networks. arXiv preprint arXiv:1905.09646, 2019.

Liang, S., Khoo, Y., and Yang, H. Drop-activation: Implicit parameter reduction and harmonic regularization. arXiv preprint arXiv:1811.05850, 2018.

Liang, S., Huang, Z., Liang, M., and Yang, H. Instance enhancement batch normalization: an adaptive regulator of batch noise. arXiv preprint arXiv:1908.04008, 2019.

Lilliefors, H. W. On the kolmogorov-smirnov test for normality with mean and variance unknown. Journal of the American statistical Association, 62(318):399–402, 1967.

Liu, Z., Li, J., Shen, Z., Huang, G., Yan, S., and Zhang, C. Learning efficient convolutional networks through network slimming. In The IEEE International Conference on Computer Vision (ICCV), Oct 2017.

Liu, Z., Mu, H., Zhang, X., Guo, Z., Yang, X., Cheng, T. K.-T., and Sun, J. Metapruning: Meta learning for automatic neural network channel pruning. arXiv preprint arXiv:1903.10258, 2019.

Loshchilov, I. and Hutter, F. Sgdr: Stochastic gradient descent with warm restarts. arXiv preprint arXiv:1608.03983, 2016.

Lutz, S., Amplianitis, K., and Smolic, A. Alphagan: Generative adversarial networks for natural image matting. arXiv preprint arXiv:1807.10088, 2018.

Neal, R. M. BAYESIAN LEARNING FOR NEURAL NETWORKS. PhD thesis, University of Toronto, 1995.

Pescim, R. R., Demétrio, C. G., Cordeiro, G. M., Ortega, E. M., and Urbano, M. R. The beta generalized half-normal distribution. Computational statistics & data analysis, 54(4):945–957, 2010.

Polyak, B. T. and Juditsky, A. B. Acceleration of stochastic approximation by averaging. SIAM journal on control and optimization, 30(4):838–855, 1992.

Prakash, A., Storer, J., Florencio, D., and Zhang, C. Repr: Improved training of convolutional filters. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pp. 10666–10675, 2019.

Radford, A., Metz, L., and Chintala, S. Unsupervised representation learning with deep convolutional generative adversarial networks. arXiv preprint arXiv:1511.06434, 2015.

Ren, S., He, K., Girshick, R., and Sun, J. Faster r-cnn: Towards real-time object detection with region proposal networks. In Advances in neural information processing systems, pp. 91–99, 2015.

Renda, A., Frankle, J., and Carbin, M. Comparing fine-tuning and rewinding in neural network pruning. In International Conference on Learning Representations, 2020. URL https://openreview.net/forum?id=S1gSj0NKvB.

Russakovsky, O., Deng, J., Su, H., Krause, J., Satheesh, S., Ma, S., Huang, Z., Karpathy, A., Khosla, A., Bernstein, M., et al. Imagenet large scale visual recognition challenge. International journal of computer vision, 115(3):211–252, 2015.

Saxe, A. M., McClelland, J. L., and Ganguli, S. Exact solutions to the nonlinear dynamics of learning in deep linear neural networks. arXiv preprint arXiv:1312.6120, 2013.

Sedgwick, P. Spearman's rank correlation coefficient. Bmj, 349:g7327, 2014.
Simonyan, K. and Zisserman, A. Very deep convolutional networks for large-scale image recognition. *arXiv preprint arXiv:1409.1556*, 2014.

Sutskever, I., Martens, J., Dahl, G., and Hinton, G. On the importance of initialization and momentum in deep learning. In *International conference on machine learning*, pp. 1139–1147, 2013.

Tian, Y., Jiang, T., Gong, Q., and Morcos, A. Luck matters: Understanding training dynamics of deep relu networks. *arXiv preprint arXiv:1905.13405*, 2019.

Woo, S., Park, J., Lee, J.-Y., and So Kweon, I. Cbam: Convolutional block attention module. In *Proceedings of the European Conference on Computer Vision (ECCV)*, pp. 3–19, 2018.

Xie, S., Girshick, R., Dollár, P., Tu, Z., and He, K. Aggregated residual transformations for deep neural networks. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 1492–1500, 2017.

Xu, B., Wang, N., Chen, T., and Li, M. Empirical evaluation of rectified activations in convolutional network. *arXiv preprint arXiv:1505.00853*, 2015.

Xu, N., Price, B., Cohen, S., and Huang, T. Deep image matting. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pp. 2970–2979, 2017.

Ye, J., Lu, X., Lin, Z., and Wang, J. Z. Rethinking the smaller-norm-less-informative assumption in channel pruning of convolution layers. In *International Conference on Learning Representations*, 2018. URL https://openreview.net/forum?id=HJ94fqlApW.

Yun, S., Han, D., Oh, S. J., Chun, S., Choe, J., and Yoo, Y. Cutmix: Regularization strategy to train strong classifiers with localizable features. In *Proceedings of the IEEE International Conference on Computer Vision*, pp. 6023–6032, 2019.

Zagoruyko, S. and Komodakis, N. Wide residual networks. *arXiv preprint arXiv:1605.07146*, 2016.

Zeiler, M. D. Adadelta: an adaptive learning rate method. *arXiv preprint arXiv:1212.5701*, 2012.

Zhao, H., Shi, J., Qi, X., Wang, X., and Jia, J. Pyramid scene parsing network. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 2881–2890, 2017.
A. Related Proposition

**Proposition 1** (Amoroso distribution). The Amoroso distribution is a four parameter, continuous, univariate, unimodal probability density, with semi-infinite range (Crooks, 2012). And its probability density function is

\[
\text{Amoroso}(x|a, \theta, \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta \gamma \left(\frac{x-a}{\theta}\right)^{\alpha \beta - 1} \exp\left\{-\left(\frac{x-a}{\theta}\right)^\beta\right\},
\]

for \(x, a, \theta, \alpha, \beta \in \mathbb{R}, \alpha > 0\) and range \(x \geq a\) if \(\theta > 0\), \(x \leq a\) if \(\theta < 0\). The mean and variance of Amoroso distribution are

\[
E_{X \sim \text{Amoroso}(a, \theta, \alpha, \beta)} X = a + \theta \cdot \frac{\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha)},
\]

and

\[
\text{Var}_{X \sim \text{Amoroso}(a, \theta, \alpha, \beta)} X = \theta^2 \left[\frac{\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha)} - \frac{(\Gamma(\alpha + \frac{1}{2})^2}{\Gamma(\alpha)^2}\right].
\]

**Proposition 2** (Half-normal distribution). Let random variable \(X\) follow a normal distribution \(N(0, \sigma^2)\), then \(Y = |X|\) follows a half-normal distribution (Pescim et al., 2010). Moreover, \(Y\) also follows \(\text{Amoroso}(0, \sqrt{2} \sigma, \frac{1}{2}, 2)\). By Eq. (12) and Eq. (13), the mean and variance of half-normal distribution are

\[
E_{X \sim N(0, \sigma^2)} |X| = \sigma \sqrt{2/\pi},
\]

and

\[
\text{Var}_{X \sim N(0, \sigma^2)} |X| = \sigma^2 \left(1 - \frac{2}{\pi}\right).
\]

**Proposition 3** (Scaled Chi distribution). Let \(X = (x_1, x_2, ..., x_k)\) and \(x_i, i = 1, ..., k\) are \(k\) independent, normally distributed random variables with mean 0 and standard deviation \(\sigma\). The statistic \(\ell_2(X) = \sqrt{\sum_{i=1}^k x_i^2}\) follows Scaled Chi distribution (Crooks, 2012). Moreover, \(\ell_2(X)\) also follows \(\text{Amoroso}(0, \sqrt{2\sigma^2}, \frac{1}{2}, 2)\). By Eq. (12) and Eq. (13), the mean and variance of Scaled Chi distribution are

\[
E_{X \sim N(0, \sigma^2)} \ell_2(X) = 2^{j/2} \sigma^j \cdot \frac{\Gamma(k/2 + j)}{\Gamma(k/2)},
\]

and

\[
\text{Var}_{X \sim N(0, \sigma^2)} \ell_2(X) = 2 \sigma^2 \left[\frac{\Gamma(k/2 + 1)}{\Gamma(k/2)} - \frac{(\Gamma(k/2 + 1/2)^2}{\Gamma(k/2)^2}\right].
\]

B. Proof of Theorem 1

**Proposition 4** (Stirling’s formula). For big enough \(x\) and \(x \in \mathbb{R}^+\), we have an approximation of Gamma function:

\[
\Gamma(x + 1) \approx \sqrt{2\pi x} \left(\frac{x}{e}\right)^x.
\]

**Proposition 5** (FKG inequality). If \(f\) and \(g\) are increasing functions on \(\mathbb{R}^n\) (Graham, 1983), we have

\[
\mathbb{E}(f)\mathbb{E}(g) \leq \mathbb{E}(fg).
\]

Say that a function on \(\mathbb{R}^n\) is increasing if it is an increasing function in each of its arguments (i.e., for fixed values of the other arguments).

---

4 en.wikipedia.org/wiki/Stirling’s approximation
Proposition 6. Let $f(X,Y)$ is a two dimensional differentiable function. According to Taylor theorem (Hormander, 1983), we have

$$f(X,Y) = f(\mathbb{E}(X), \mathbb{E}(Y)) + \sum_{\text{cyc}} (X - \mathbb{E}(X)) \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) + \text{Remainder}_1,$$

(20)

$$f(X,Y) = f(\mathbb{E}(X), \mathbb{E}(Y)) + \sum_{\text{cyc}} (X - \mathbb{E}(X)) \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) + \frac{1}{2} \sum_{\text{cyc}} (X - \mathbb{E}(X))^2 \frac{\partial^2}{\partial X \partial Y} f(\mathbb{E}(X), \mathbb{E}(Y))(X - \mathbb{E}(X)) + \text{Remainder}_2.$$

(21)

Lemma 1. Let $X$ and $Y$ are random variables. Then we have such an estimation

$$\text{Var} \left( \frac{X}{Y} \right) \approx \left( \frac{\mathbb{E}(X)}{\mathbb{E}(Y)} \right)^2 \left( \frac{\text{Var}X}{\mathbb{E}(X)^2} + \frac{\text{Var}Y}{\mathbb{E}(Y)^2} - 2 \frac{\text{Cov}(X,Y)}{\mathbb{E}(X)\mathbb{E}(Y)} \right).$$

(22)

Proof. Let $f(X,Y) = X/Y$, according to the definition of variance, we have

$$\text{Var} f(X,Y) = \mathbb{E}[f(X,Y) - \mathbb{E}(f(X,Y))]^2$$

$$\approx \mathbb{E} \left[ f(X,Y) - \mathbb{E} \left\{ f(\mathbb{E}(X), \mathbb{E}(Y)) + \sum_{\text{cyc}} (X - \mathbb{E}(X)) \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) \right\} \right]^2$$

from Eq. (20)

$$= \mathbb{E} \left[ f(X,Y) - f(\mathbb{E}(X), \mathbb{E}(Y)) - \sum_{\text{cyc}} \mathbb{E}(X - \mathbb{E}(X)) \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) \right]^2$$

$$= \mathbb{E} \left[ \sum_{\text{cyc}} (X - \mathbb{E}(X)) \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) \right]^2$$

from Eq. (20)

$$= 2 \text{Cov}(X,Y) \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) \frac{\partial}{\partial Y} f(\mathbb{E}(X), \mathbb{E}(Y)) \sum_{\text{cyc}} \left[ \frac{\partial}{\partial X} f(\mathbb{E}(X), \mathbb{E}(Y)) \right]^2 \cdot \text{Var}X$$

$$= 2 \text{Cov}(X,Y) \cdot \frac{1}{\mathbb{E}(Y)} \cdot \left( \frac{\mathbb{E}(X)}{(\mathbb{E}(Y))^2} \right) \cdot \frac{1}{(\mathbb{E}(Y))^2} \cdot \text{Var}X + \frac{(\mathbb{E}X)^2}{(\mathbb{E}Y)^4} \cdot \text{Var}Y$$

$$= \left( \frac{\mathbb{E}(X)}{\mathbb{E}(Y)} \right)^2 \left( \frac{\text{Var}X}{(\mathbb{E}(X))^2} + \frac{\text{Var}Y}{(\mathbb{E}(Y))^2} - 2 \frac{\text{Cov}(X,Y)}{\mathbb{E}(X)\mathbb{E}(Y)} \right).$$

Lemma 2. For big enough $x$ and $x \in \mathbb{R}^+$, we have

$$\lim_{x \to +\infty} \left[ \frac{\Gamma \left( \frac{x+1}{2} \right)}{\Gamma \left( \frac{x}{2} \right)} \right]^2 \cdot \frac{1}{x} = \frac{1}{2}.$$  

(23)

And

$$\lim_{x \to +\infty} \frac{\Gamma \left( \frac{x+1}{2} \right)}{\Gamma \left( \frac{x}{2} \right)} - \left[ \frac{\Gamma \left( \frac{x+1}{2} \right)}{\Gamma \left( \frac{x}{2} \right)} \right]^2 = \frac{1}{4}. $$

(24)
Proof.

\[
\lim_{x \to +\infty} \left[ \frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x}{2}\right)} \right]^2 \cdot \frac{1}{x} \approx \lim_{x \to +\infty} \left( \frac{\sqrt{2\pi\left(\frac{x+1}{2}\right)}}{2\pi\left(\frac{x+2}{2}\right)} \cdot \frac{\left(\frac{x-1}{2e}\right)^{x-1}}{\left(\frac{x}{2e}\right)^{x/2}} \right) \cdot \frac{1}{x}
\]

\[
= \lim_{x \to +\infty} \left( \frac{x-1}{x-2} \cdot \frac{(\frac{x-1}{2e})^{x-2}}{(\frac{x}{2e})^{x/2}} \cdot \frac{1}{x} \right)
\]

\[
= \lim_{x \to +\infty} \left( 1 + \frac{1}{x-2} \right)^{x-2} \cdot \frac{x-1}{x-2} \cdot \frac{1}{x} \cdot \frac{1}{2}
\]

on the other hand, we have

\[
\lim_{x \to +\infty} \frac{\Gamma(x+1)}{\Gamma(x)} - \left[ \frac{\Gamma(x+1)}{\Gamma(x)} \right]^2 = \lim_{x \to +\infty} \frac{x}{2} - \left( 1 + \frac{1}{x-2} \right)^{x-2} \cdot \frac{x-1}{x-2} \cdot \frac{1}{2} e^{x-1} \left( 1 + \frac{1}{x} \right)^{x-1}
\]

\[
= \lim_{x \to +\infty} \frac{x}{2e} \left( e - (1 + \frac{1}{x})^x \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{e} \right)
\]

\[
= \frac{1}{4}
\]

\[
\square
\]

**Theorem 1** Let \( X \sim N(0, c^2 \cdot I_n) \), we have

\[
\max \left\{ \operatorname{Var}_X \left( \frac{\hat{\ell}_2(X)}{\hat{\ell}_1(X)} \right), \operatorname{Var}_X \left( \frac{\hat{\ell}_1(X)}{\hat{\ell}_2(X)} \right) \right\} \lesssim \frac{1}{n},
\]

where \( \hat{\ell}_1(X) \) denotes \( \ell_1(X)/\mathbb{E}(\ell_1(X)) \) and \( \hat{\ell}_2(X) \) denotes \( \ell_2(X)/\mathbb{E}(\ell_2(X)) \).

**Proof.** For the ratio \( \hat{\ell}_2(X)/\hat{\ell}_1(X) \), we have

\[
\operatorname{Var} \left( \frac{\hat{\ell}_2(X)}{\hat{\ell}_1(X)} \right) = \left( \frac{\mathbb{E}(\ell_1(X))}{\mathbb{E}(\ell_2(X))} \right)^2 \operatorname{Var} \left( \frac{\hat{\ell}_2(X)}{\hat{\ell}_1(X)} \right)
\]

\[
\approx \left( \frac{\mathbb{E}(\ell_1(X))}{\mathbb{E}(\ell_2(X))} \right)^2 \left( \frac{\mathbb{E}(\ell_2(X))}{\mathbb{E}(\ell_1(X))} \right)^2 \left( \frac{\operatorname{Var}\ell_2(X)}{\mathbb{E}(\ell_2(X))^2} + \frac{\operatorname{Var}\ell_1(X)}{\mathbb{E}(\ell_1(X))^2} - 2 \frac{\operatorname{Cov}(\ell_2(X), \ell_1(X))}{\mathbb{E}(\ell_2(X))\mathbb{E}(\ell_1(X))} \right)
\]

\[
\leq \left( \frac{\operatorname{Var}\ell_2(X)}{\mathbb{E}(\ell_2(X))^2} + \frac{\operatorname{Var}\ell_1(X)}{\mathbb{E}(\ell_1(X))^2} \right).
\]

Similarly, we also have

\[
\operatorname{Var} \left( \frac{\hat{\ell}_1(X)}{\hat{\ell}_2(X)} \right) \lesssim \left( \frac{\operatorname{Var}\ell_2(X)}{\mathbb{E}(\ell_2(X))^2} + \frac{\operatorname{Var}\ell_1(X)}{\mathbb{E}(\ell_1(X))^2} \right).
\]

Therefore,
Figure 9. The approximation of Theorem 1: (Left) the example about ResNet56; (Right) the example about ResNet110.

\[
\max \left\{ \text{Var}_X \left( \hat{\ell}_2(X) \right) \frac{\hat{\ell}_1(X)}{\hat{\ell}_2(X)} , \text{Var}_X \left( \hat{\ell}_1(X) \right) \right\} \leq \left( \frac{\text{Var}_X \hat{\ell}_2(X)}{\mathbb{E}(\hat{\ell}_2(X))^2} + \frac{\text{Var}_X \hat{\ell}_1(X)}{\mathbb{E}(\hat{\ell}_1(X))^2} \right) \\
= 2\sigma^2 \left( \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n}{2})^2} - \frac{\Gamma(\frac{n}{2} + \frac{1}{2})^2}{\Gamma(\frac{n}{2})^2} \right) + \frac{\sigma^2 (1 - \frac{2}{n}) n}{(n \cdot \sigma \sqrt{2/\pi})^2} \ 	ext{from Proposition 3 and 2} \\
\approx \left( \frac{1}{2n} + \frac{(\pi - 1)}{n} \right) \ 	ext{from Lemma 2} \\
= \frac{\pi - 1}{2n}
\]

Because the approximation is widely used in the proof of Theorem 1, it is necessary to verify it numerically. As shown in Fig. 9, we use ResNet56 on Cifar100 and ResNet110 on Cifar10 respectively to verify Theorem 1. From Fig. 9, we find that the estimation of Theorem 1 is reliable, i.e., the estimation \( O\left(\frac{1}{n}\right) \) for \( \max \left\{ \text{Var}_X \left( \hat{\ell}_2(X) \right) \frac{\hat{\ell}_1(X)}{\hat{\ell}_2(X)} , \text{Var}_X \left( \hat{\ell}_1(X) \right) \right\} \) is appropriate.

C. Proof of Theorem 2

**Proposition 7.** Let \( L^{(\alpha)}_p(x) \) denotes generalized Laguerre function, and it have following properties:

\[
\frac{\partial^n}{\partial x^n} L^{(\alpha)}_p(x) = (-1)^n L^{(\alpha+n)}_p(x),
\]

and for \( \alpha > 0 \),

\[
L^{(\alpha)}_{\frac{1}{2}}(x) > 0.
\]

**Theorem 2.** Let random variable \( v_i \in \mathbb{R}^k \). They are i.i.d and follow normal distribution \( N(0, \sigma^2 I_k) \). For \( F \) in \( \mathbb{R}^k \), we have

\[
\arg\min\limits_F \left\{ \mathbb{E}_{v_i \sim N(0, \sigma^2 I_k)} \sum_{i=1}^n \| F - v_i \|_2 \right\} = 0.
\]
Proof. Let \( w_i = F - v_i \) and we have \( w_i \sim N(F, \sigma^2 I_k) \), then
\[
\mathbb{E}_{w_i \sim N(0, \sigma^2 I_k)} \sum_{i=1}^{n} ||F - v_i||_2 = \sum_{i=1}^{n} \mathbb{E}_{w_i \sim N(0, \sigma^2 I_k)} ||F - v_i||_2
\]
\[
= \sum_{i=1}^{n} \mathbb{E}_{w_i \sim N(F, \sigma^2 I_k)} ||w_i||_2
\]
\[
= n \cdot \sigma^2 \sqrt{\frac{\pi}{2}} \cdot L^\frac{3}{2} \left( -\frac{||F||_2^2}{2\sigma^2} \right)
\]
The reason for the last equation is that \( ||w_i||_2 \) follows scaled noncentral chi distribution\(^5\) when \( w_i \sim N(F, \sigma^2 I_k) \). Let \( T(x) = L^\frac{3}{2} \left( -\frac{x^2}{2\sigma^2} \right) \), we calculate the minimum of \( T(x) \). From Eq. (26),
\[
\frac{d}{dx} T(x) = \frac{x}{\sigma^2} \cdot L^\frac{1}{2} \left( -\frac{x^2}{2\sigma^2} \right).
\]
Since Eq. (27), we find that \( \frac{d}{dx} T(x) > 0 \) when \( x > 0 \) and if \( x \leq 0 \), then \( \frac{d}{dx} T(x) \leq 0 \). It means that \( T(x) \) gets the minimizer at \( ||F||_2 = 0 \), i.e., \( F = 0 \).

\[
\square
\]

D. Proof of Theorem 3

Lemma 3. For two random variables \( X, Y \in \mathbb{R}^k \) follow \( N(\mathbf{0}, \sigma^2 \cdot I_k) \) and they are i.i.d. When \( k \) is large enough, we have:
\[
\mathbb{E} \left( \frac{(||X||_2^2 - ||Y||_2^2)^2}{2||X||_2 \cdot ||Y||_2} \right) \approx 2\sigma^2 + \frac{4\sigma^2 k + 1}{2k^2},
\]
and
\[
\mathbb{V} \text{ar} \left( \frac{(||X||_2^2 - ||Y||_2^2)^2}{2||X||_2 \cdot ||Y||_2} \right) \approx 8\sigma^4 + \frac{16\sigma^4 k + \sigma^2}{k^2},
\]

Proof. According to Proposition 3 and Lemma 2, it is easy to know (similar method in Eq.(86)), when \( k \) is large enough, that
\[
\mathbb{E} (2||X||_2 \cdot ||Y||_2) = 2\sigma^2 k, \quad \mathbb{V} \text{ar} (2||X||_2 \cdot ||Y||_2) = \sigma^2 + 4\sigma^4 k,
\]
and
\[
\mathbb{E} ((||X||_2^2 - ||Y||_2^2)^2) = 4\sigma^4 k, \quad \mathbb{V} \text{ar} ((||X||_2^2 - ||Y||_2^2)^2) = 16\sigma^8 (2k^2 + 3k).
\]
SinceLemma 1, we have an estimation
\[
\mathbb{V} \text{ar} \left( \frac{(||X||_2^2 - ||Y||_2^2)^2}{2||X||_2 \cdot ||Y||_2} \right) \leq \left( \frac{\mathbb{E}((||X||_2^2 - ||Y||_2^2)^2)}{\mathbb{E}^2 ||X||_2 \cdot ||Y||_2} \right)^2 \left( \frac{\mathbb{V} \text{ar}((||X||_2^2 - ||Y||_2^2)^2)}{\mathbb{E}((||X||_2^2 - ||Y||_2^2)^2)} + \frac{\mathbb{V} \text{ar}(2||X||_2 \cdot ||Y||_2)^2)}{\mathbb{E}^2(2||X||_2 \cdot ||Y||_2)^2} \right)
\]
\[
\approx \left( \frac{4\sigma^4 k}{2\sigma^2 k^2} \right)^2 \left( \frac{\sigma^2 + 4\sigma^4 k}{4\sigma^2 k^2} + \frac{16\sigma^8 (2k^2 + 3k)}{16\sigma^8 k^2} \right) \quad \text{Since Eq.}(31)\text{ and Eq.}(32)
\]
\[
= 8\sigma^4 + \frac{16\sigma^4 k + \sigma^2}{k^2}.
\]

From Eq.(21) and Lemma 1, we also can obtain an estimation of \( \mathbb{E}(A/B) \), where \( A \) and \( B \) are two random variables. i.e.,
\[
\mathbb{E} \left( \frac{A}{B} \right) \approx \frac{\mathbb{E} A}{\mathbb{E} B} + \frac{\mathbb{V} \text{ar} (B)}{\mathbb{E} B^3},
\]
\[
^5\text{Survey of simple, continuous, univariate probability distributions and Wikipredia.}
\]
Figure 10. (Left) The numerical verification of Eq.(29) and (Right) The numerical verification of Eq.(30). $X$ and $Y$ follow $N(0, c^2 \cdot I_k)$.

Therefore,

$$E\left(\frac{(||X||_2^2 - ||Y||_2^2)^2}{2||X||_2 \cdot ||Y||_2}\right) \approx \frac{E(||X||_2^2 - ||Y||_2^2)^2}{E2||X||_2 \cdot ||Y||_2} + \text{Var}(2||X||_2 \cdot ||Y||_2) \cdot \frac{E(||X||_2^2 - ||Y||_2^2)^2}{(E2||X||_2 \cdot ||Y||_2)^3}$$

Since Eq.(33)

$$\approx \frac{4c^4k}{2c^2k} + \frac{4c^4k}{8c^2k^3} \cdot (c^2 + 4c^4k)$$

Since Eq.(31) and Eq.(32)

$$= 2c^2 + \frac{4c^2k + 1}{2k^2}.$$

\[\square\]

Note that, the approximation is widely used in the proof of Eq.(29) and Eq.(30). Hence, it is also necessary to verify it numerically. As shown in Fig. 10, the estimation is appropriate. According to Lemma 3, the mathematical expectation and variance of the ratio of $(||X||_2^2 - ||Y||_2^2)^2$ and $2||X||_2 \cdot ||Y||_2$ are both close to 0 when $k$ is large enough and $c$ is small enough. that is,

$$2(||X||_2 \cdot ||Y||_2) \gg (||X||_2^2 - ||Y||_2^2)^2.$$

By the way, the convolutional filters easily meet the condition that $k$ is large enough and $c$ is small enough.

**Theorem 3.** For $n$ random variables $a_i \in \mathbb{R}^k$ follow $N(0, c^2 \cdot I_k)$. When $k$ is large enough and $c$ is small enough, we have such an estimation:

$$\text{Var}_{a_i} \frac{F_1(a_i)}{F_2(a_i)} \approx \frac{1}{2nk}, \quad \text{Var}_{a_i} \frac{F_2(a_i)}{F_1(a_i)} \approx \frac{1}{2nk},$$

where $F_1(a_i) = \sum_{i=1}^n ||a_i||_2/E(\sum_{i=1}^n ||a_i||_2)$ and $F_2(a_i) = \sum_{i=1}^n ||a_i||_2^2/E(\sum_{i=1}^n ||a_i||_2^2)$.

**Proof.** Since Eq. (16) and Eq. (17), we have

$$\text{Var}_{a_i} \frac{F_1(a_i)}{F_2(a_i)} = \left(\frac{nc^2k}{nc\sqrt{k}}\right)^2 \text{Var}_{a_i} \left(\frac{\sum_{i=1}^n ||a_i||_2}{\sum_{i=1}^n ||a_i||_2^2}\right).$$

(35)

and

$$\text{Var}_{a_i} \frac{F_2(a_i)}{F_1(a_i)} = \left(\frac{nc\sqrt{k}}{nc^2k}\right)^2 \text{Var}_{a_i} \left(\frac{\sum_{i=1}^n ||a_i||_2^2}{\sum_{i=1}^n ||a_i||_2}\right).$$

(36)
According to Lagrange’s identity, we have

\[
\left( \sum_{i=1}^{n} ||a_i||^2 \right) \left( \sum_{i=1}^{n} 1 \right) = \left( \sum_{i=1}^{n} ||a_i||^2 \right)^2 + \sum_{1 \leq i < j \leq n} (||a_i||^2 - ||a_j||^2)^2
\]

\[
= \sum_{i=1}^{n} ||a_i||^2 + \sum_{1 \leq i < j \leq n} (||a_i||^2 \cdot ||a_j||^2) + 2 \sum_{1 \leq i < j \leq n} (||a_i||^2 - ||a_j||^2)^2
\]

\[
\approx \sum_{i=1}^{n} ||a_i||^2 + 2 \sum_{1 \leq i < j \leq n} (||a_i||^2 \cdot ||a_j||^2)
\]

so we have

\[
\text{Var}_{a_i \sim N(0, c^2I_k)} \frac{\sum_{i=1}^{n} ||a_i||^2}{\sum_{i=1}^{n} ||a_i||^2} \approx \text{Var}_{a_i \sim N(0, c^2I_k)} \frac{n}{\sum_{i=1}^{n} ||a_i||^2}
\]

(37)

By central limit theorem, we have \(\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} ||a_i||^2 - \mu \right) \sim N(0, \sigma^2)\). And let \(g(x) = \frac{1}{1+x^2}\), we can use Delta method\(^6\) to find the distribution of \(g\left( \frac{1}{n} \sum_{i=1}^{n} ||a_i||^2 \right)\):

\[
\sqrt{n} \left( g\left( \frac{1}{n} \sum_{i=1}^{n} ||a_i||^2 \right) - g(\mu) \right) \sim N(0, \sigma^2 \cdot [g'(\mu)]^2) = N(0, \sigma^2 \cdot \frac{1}{\mu^2}).
\]

(38)

where \(\mu\) and \(\sigma^2\) denote the mean and variance of \(||a_i||^2\) respectively. From Eq. (37), we have

\[
\text{Var}_{a_i \sim N(0, c^2I_k)} \frac{\sum_{i=1}^{n} ||a_i||^2}{\sum_{i=1}^{n} ||a_i||^2} \approx \text{Var}_{a_i \sim N(0, c^2I_k)} \frac{n}{\sum_{i=1}^{n} ||a_i||^2}
\]

\[
= \sigma^2 \cdot \frac{1}{n} \cdot \frac{1}{\mu^4 \cdot n} \quad \text{Since Eq. (38)}
\]

\[
= 2c^2 \left[ \frac{\Gamma \left( \frac{k}{2} + 1 \right)}{\Gamma \left( \frac{k}{2} \right) - 2c} \right] - \frac{\Gamma \left( \frac{k+1}{2} \right)^2}{\Gamma \left( \frac{k}{2} \right)^2} \cdot \frac{1}{\sqrt{2c} \cdot \frac{\Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{k}{2} \right)}}^4 \cdot n \quad \text{Since Eq. (16) and Eq. (17)}
\]

\[
= \frac{1}{2c^2 \cdot nk^2} \quad \text{Since Lemma. 2}
\]

Since Eq. (35), we have

\[
\text{Var}_{a_i \sim N(0, c^2I_k)} F_1(a_i) = \left( \frac{nc^2k}{nc^2k} \right)^2 \cdot \text{Var}_{a_i \sim N(0, c^2I_k)} \left( \frac{\sum_{i=1}^{n} ||a_i||^2}{\sum_{i=1}^{n} ||a_i||^2} \right) \approx \frac{1}{2nk^2}.
\]

(39)

Similar to Eq. (37),

\[
\text{Var}_{a_i \sim N(0, c^2I_k)} \frac{\sum_{i=1}^{n} ||a_i||^2}{\sum_{i=1}^{n} ||a_i||^2} \approx \text{Var}_{a_i \sim N(0, c^2I_k)} \frac{n}{\sum_{i=1}^{n} ||a_i||^2}
\]

(40)

\(^6\)https://en.wikipedia.org/wiki/Delta_method
Figure 11. A numerical verification of Theorem 3, where $F_1 = \sum_{i=1}^{n} |a_i|^2 / E(\sum_{i=1}^{n} |a_i|^2)$ and $F_2 = \sum_{i=1}^{n} |a_i|^2 / E(\sum_{i=1}^{n} |a_i|^2)$. $a_i$ follow $N(0, 0.01^2 \cdot I_k)$.

$$\text{Var}_{a_i \sim N(0, \sigma^2 I_k)} \frac{\sum_{i=1}^{n} |a_i|^2}{\sum_{i=1}^{n} |a_i|^2} \approx \text{Var}_{a_i \sim N(0, \sigma^2 I_k)} \frac{\sum_{i=1}^{n} |a_i|^2}{n}$$

Similar to Eq. (37)

$$= \sigma^2 \cdot \frac{1}{n}$$

Since central limit theorem

$$= 2 \sigma^2 \left[ \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)^2} \right] \frac{1}{n}$$

Since Eq. (17)

$$= \frac{\sigma^2}{2n}$$

Since Lemma 2

Since Eq. (36), we have

$$\text{Var}_{a_i} \frac{F_2(a_i)}{F_1(a_i)} = \left(\frac{nc\sqrt{k}}{nc^2 k}\right)^2 \cdot \text{Var}_{a_i} \left(\frac{\sum_{i=1}^{n} |a_i|^2}{\sum_{i=1}^{n} |a_i|^2}\right) \approx \frac{1}{2nk}$$

From Eq.(39) and Eq.(41), Theorem 3 holds.

In Fig. 11, we also show a numerical verification of Theorem 3.

E. Proof of Theorem 4

Proposition 8. For a $n \times m$ random matrix $(a_{ij})_{n\times m}$, where $a_{ij} \sim N(0, \sigma^2)$. And Eq. (8) holds with probability 1.

$$\text{rank}((a_{ij})_{n\times m}) = \min(m, n).$$

Lemma 4. Let $v_0, v_1, ..., v_k$ be the $k+1$ vectors in $n$ dimensional Euclidean space $V$ and $k \leq n$. If $\text{rank}(v_1 - v_0, v_2 - v_0, ..., v_k - v_0) = n$, then $\forall x \in V$, $\exists \lambda_i (0 \leq i \leq k)$, s.t.

$$x = \sum_{i=0}^{k} \lambda_i \cdot v_i,$$

and $\sum_{i=0}^{k} \lambda_i = 1$. We call $\lambda = (\lambda_0, \lambda_1, ..., \lambda_k)$ the generalized barycentric coordinate with respect to $(v_0, v_1, ..., v_k)$. (In general, barycentric coordinate is a concept in Polytope)
Proof. Note that \( v_i \) is the element of \( n \) dimensional linear space \( V \) and \( \text{rank}(v_1 - v_0, v_2 - v_0, ..., v_k - v_0) = n \). It means \((v_1 - v_0, v_2 - v_0, ..., v_k - v_0)\) form a set of basis in the linear space \( V \). \( \forall x \in V \), \( x - v_0 \) can be expressed linearly by them, i.e., \( \exists t_i (1 \leq i \leq k) \) s.t.

\[
x = v_0 + \sum_{i=1}^{k} t_i (v_i - v_0)
\]

\[
= (1 - \sum_{i=1}^{k} t_i)v_0 + \sum_{i=1}^{k} t_i v_i.
\]

Let \( \lambda_0 = (1 - \sum_{i=1}^{k} t_i) \) and \( \lambda_i = t_i (1 \leq i \leq k) \), Lemma 4 holds.

Lemma 5. Let \( v_0, v_1, ..., v_k \) be the \( k + 1 \) vectors in \( n \) dimensional Euclidean space \( V \). \( \forall a, b \in V \), and the generalized barycentric coordinate of \( a, b \) with respect to \((v_0, v_1, ..., v_k)\) are \( \lambda = (\lambda_0, \lambda_1, ..., \lambda_k)^T \) and \( \mu = (\mu_0, \mu_1, ..., \mu_k)^T \), respectively. Then

\[
||a - b||^2 = (\lambda - \mu)^T D (\lambda - \mu),
\]

where \( D = (-\frac{1}{2}d_{ij})_{(k+1) \times (k+1)} \), and \( d_{ij} = ||v_i - v_j||^2_2 \).

Proof. Since Lemma 4, let \( R = [v_0, v_1, ..., v_k]_{n \times (k+1)} \), and we have \( a = R\lambda \) and \( b = R\mu. \) Moreover,

\[
||a - b||^2 = (a - b)^T(a - b)
\]

\[
= [R(\lambda - \mu)]^T [R(\lambda - \mu)]
\]

\[
= (\lambda - \mu)^T R^T R (\lambda - \mu).
\]

Note that, for \( D = (-\frac{1}{2}d_{ij})_{(k+1) \times (k+1)} \),

\[
-\frac{1}{2}d_{ij} = -\frac{1}{2}(v_i - v_j)^T(v_i - v_j)
\]

\[
= v_i^T v_j - \frac{1}{2}(v_i^T v_i + v_j^T v_j).
\]

So we have \( D = R^T R - \frac{1}{2} ((v_i^T v_i + v_j^T v_j)_{(k+1) \times (k+1)}) \). It can be further simplified to \( D = R^T R - \frac{1}{2}(V\alpha^T + \alpha V^T) \), where \( V = (v_0^T v_0, ..., v_k^T v_k)^T \) and \( \alpha = (1, ..., 1)^T \). So

\[
||a - b||^2 = (\lambda - \mu)^T R^T R (\lambda - \mu)
\]

\[
= (\lambda - \mu)^T (D + \frac{1}{2}(V\alpha^T + \alpha V^T))(\lambda - \mu)
\]

\[
= (\lambda - \mu)^T D (\lambda - \mu) + \frac{1}{2}(\lambda - \mu)^T (V\alpha^T + \alpha V^T)(\lambda - \mu),
\]

therefore, we only need to prove \((\lambda - \mu)^T (V\alpha^T + \alpha V^T)(\lambda - \mu) = 0\). From Lemma 4, we have \( \alpha^T(\lambda - \mu) = (\lambda - \mu)^T \alpha = 0 \) and the Lemma 5 holds.

Definition 1 (Ultra dimension). For a set \( U \) composed of vectors in a \( n \) dimensional linear space \( V \), we define \( \overline{\text{dim}}(U) \) as the Ultra dimension of \( U \). The definition is that if \( U \) has \( k \) linearly independent vectors and there are no more, then \( \overline{\text{dim}}(U) = k \).

In fact, if \( U \) is a linear subspace in \( V \), then the Ultra dimension and the dimensions of the linear subspace are equivalent. If \( U \) is a linear manifold, \( U = \{ x + v_0 | x \in W \} \), where \( v_0 \) and \( W \) are non-zero vectors and linear subspaces in \( V \), respectively. And \( \text{dim}(W) = r \). Then

\[
\overline{\text{dim}}(U) = \begin{cases} 
  r, & v_0 \in W \\
  r + 1, & v_0 \notin W 
\end{cases}
\]
In other words, \( \widehat{\dim}(U) \geq \widehat{\dim}(W) \) always holds.

**Lemma 6.** For arbitrary \( k (1 \leq k \leq n - 1) \), let \( a_1, a_2, ..., a_k \) be \( k \) linearly independent vectors in \( n \) dimensional linear space \( V \). Consider one \( n - 1 \) dimensional linear subspace \( W \) in \( V \) and a non-zero vector \( v_0 \) in \( V \). They form a linear manifold \( P = \{ v_0 + \alpha | \alpha \in W \} \). If \( a_1, a_2, ..., a_k \) do not all belong to \( P \), then there must exist \( n - k \) vectors \( p_1, p_2, ..., p_{n-k} \) from \( P \), s.t \( (a_1, a_2, ..., a_k, p_1, p_2, ..., p_{n-k}) \) are a set of basis for the linear space \( V \).

**Proof.** We use mathematical induction. First, show that the Lemma 6 holds for \( n - k = 1 \). It means we need to find a vector \( p_1 \in P \) s.t. \( a_1, a_2, ..., a_k, p_1 \) linearly independent. If \( p_1 \) does not exist, then \( \forall \gamma \in P \) would be linearly represented by \( a_1, a_2, ..., a_k \). In other word,

\[
P \subset L = \text{span}(a_1, a_2, ..., a_k),
\]

(54)

\( \gamma \) For the linear manifold \( P \), if \( v_0 \in W \). This means that \( P \) is equal to the linear subspace \( W \). Since Eq. (54), we have \( W \subset L \) and \( \widehat{\dim}(W) = \widehat{\dim}(L) \). Hence, \( P = W = L \). However, \( a_1, a_2, ..., a_k \) do not all belong to \( P \), a contradiction.

(2) For the linear manifold \( P \), if \( v_0 \not\in W \), then \( \widehat{\dim}(P) = n \). Because \( v_0 \not\in W \), that is, \( v_0 \) cannot be represented by a set of basis of \( W \). In other words, \( v_0 \) and a set of basis of \( W \) are linearly independent. However, the dimension of \( W \) is \( n - 1 \), hence \( \widehat{\dim}(P) = n \). From Eq. (54), we have \( P \subset L \), so

\[
n = \widehat{\dim}(P) \leq \widehat{\dim}(L) = k = n - 1,
\]

(55)

a contradiction. Therefore, Lemma 6 holds for \( n - k = 1 \). Assume the induction hypothesis that Lemma 6 is true when \( n - k = l \), where \( 1 \leq l \), when \( n - k = l + 1 \), i.e., \( k = n - (l + 1) \), we also can find a vector \( p_1 \in P \) s.t. \( a_1, a_2, ..., a_k, p_1 \) linearly independent. Otherwise, \( \forall \gamma \in P \) would be linearly represented by \( a_1, a_2, ..., a_k \). Similarly, we have Eq. (54). Note that, from Definition 1, \( \widehat{\dim}(P) \geq n - 1 \), hence

\[
n - 1 \leq \widehat{\dim}(P) \leq \widehat{\dim}(L) = k = n - (l + 1).
\]

(56)

a contradiction. At this time, we have \( k + 1 = n - (l + 1) + 1 = n - l \) vectors \( a_1, a_2, ..., a_k, p_1 \) which are not all on \( P \). Note that \( n - (n - l) = l \), using the induction hypothesis, the Lemma 6 also holds for \( n - k = l \). In summary, Lemma 6 holds.

**Theorem 4.** Let \( v_0, v_1, ..., v_k \) be the \( k + 1 \) vectors in \( n \) dimensional Euclidean space \( \mathbb{E}^n \). For all \( P \) in \( \mathbb{E}^n \),

\[
\sum_{i=0}^{k} ||P - v_i||_2^2 = \sum_{i=0}^{k} ||G - v_i||_2^2 + (k + 1)||P - G||_2^2.
\]

where \( G \) is the centroid of \( v_i \), will hold if it satisfies one of the following conditions:

(1) if \( k \geq n \) and \( \text{rank}(v_1 - v_0, v_2 - v_0, ..., v_k - v_0) = n \).

(2) if \( k < n \) and \( (v_1 - v_0, v_2 - v_0, ..., v_k - v_0) \) are linearly independent.

(3) if \( v_i \sim N(0, c \cdot I_n) \), Eq. (7) holds with probability 1 where \( c \) is a constant.

**Proof.** For **Theorem 4 (1)**. From Lemma 4, \( \forall P \in \mathbb{E}^n, \exists \gamma = (\gamma_0, ..., \gamma_k), \) s.t. \( P \) can be represented by \( \sum_{i=0}^{k} \gamma_i v_i \), where \( \sum_{i=0}^{k} \gamma_i = 1 \). In fact, for each \( v_i \), it also can be represented by \( \sum_{j=0}^{k} \beta_{ij} v_i \), where \( \sum_{i=0}^{k} \beta_{ij} = 1 \). We just take \((\beta_{i0}, \beta_{i1}, ..., \beta_{ik})\) as one of the standard orthogonal basis \( \epsilon_i = (0, 0, ..., 1_i, ..., 0) \). According to lemma 5,

\[
||P - v_i||_2^2 = (\gamma - \epsilon_i)^T D(\gamma - \epsilon_i)
\]

(57)

\[
= \gamma^T D \gamma - 2 \gamma^T D \epsilon_i + \epsilon_i^T D \epsilon_i
\]

(58)

\[
= \gamma^T D \gamma - 2 \gamma^T D \epsilon_i
\]

(59)
The final equation is because the diagonal elements of the matrix are all 0. On the other hand, we have

\[
\|G - v_i\|_2^2 = \left(\frac{1}{k+1} \sum_{i=0}^{k} \epsilon_i - \epsilon_i\right)^T D \left(\frac{1}{k+1} \sum_{i=0}^{k} \epsilon_i - \epsilon_i\right) 
\]

\[
= \frac{1}{(k+1)^2} \alpha^T D\alpha - \frac{2}{k+1} \alpha^T D\epsilon_i + \epsilon_i^T D\epsilon_i
\]

\[
= \frac{1}{(k+1)^2} \alpha^T D\alpha - \frac{2}{k+1} \alpha^T D\epsilon_i,
\]

where \(\alpha = \sum_{i=0}^{k} \epsilon_i\), i.e., \(\alpha = (1, 1, \ldots, 1)\). Next, we consider \(\|P - G\|_2^2\).

\[
\|P - G\|_2^2 = (\gamma - \frac{1}{k+1} \alpha)^T D(\gamma - \frac{1}{k+1} \alpha)
\]

\[
= \gamma^T D\gamma + \frac{1}{(k+1)^2} \alpha^T D\alpha - \frac{2}{k+1} \gamma^T D\alpha
\]

In summary, we have

\[
\sum_{i=0}^{k} \|P - v_i\|_2^2 - \|G - v_i\|_2^2 = (k+1) \gamma^T D\gamma - 2\gamma^T D\alpha + \frac{1}{k+1} \alpha^T D\alpha
\]

\[
= (k+1)\|P - G\|_2^2
\]

Therefore, Theorem 4 (1) holds.

For Theorem 4 (2). Next, we prove the case of \(k < n\). Obviously, Lemma 4 does not hold. We consider about such a linear space \(W_1 = \text{span}(P - G)\), i.e., a linear space expanded by \(P - G\), and its orthogonal complement \(W_1^+\) (in \(E^n\)). Since dimension formula from linear space, it is easy to known that \(\text{dim}(W_1^+) = n - 1\).

Two linear manifolds \(T_1\) and \(T_2\) are constructed as follows,

\[
T_1 = \{x + G | x \in W_1^+\}
\]

\[
T_2 = \{x + G - v_0| x \in W_1^+\}
\]

\(\forall v_i \in T_1\), we have \((v_i - G)^T(P - G) = 0\). Furthermore,

\[
\|P - v_i\|_2^2 = \|v_i - G\|_2^2 + \|P - G\|_2^2
\]

It is easy to known that \(G - v_0\) is not 0. If \(v_1 - v_0, \ldots, v_k - v_0\) are all belong to \(T_2\), it means \(v_1, \ldots, v_k\) are all in \(T_1\). Hence, we have Eq. (69). By summing both sides of Eq. (69) for \(i\), it is obvious find that Theorem 4 (2) holds. If \(v_1 - v_0, \ldots, v_k - v_0\) are not all belong to \(T_2\), since Lemma 6, there are \(n - k\) vectors \(p_1 - v_0, p_2 - v_0, \ldots, p_{n-k} - v_0\) from \(T_2\) s.t. they and \(v_1 - v_0, \ldots, v_k - v_0\) are linearly independent, where \(p_i\) obviously belongs to manifold \(T_1\).

At the same time, we have \(2G - p_i \in T_1\), we can also construct \(n - k\) new vectors \(2G - p_i - v_0 \in T_2\) and calculate the rank that

\[
\text{rank}(v_1 - v_0, \ldots, v_k - v_0, p_1 - v_0, \ldots, p_{n-k} - v_0, 2G - p_1 - v_0, \ldots, 2G - p_{n-k} - v_0)
\]

\[
= \text{rank}(v_1 - v_0, \ldots, v_k - v_0, p_1 - v_0, \ldots, p_{n-k} - v_0, 2(G - v_0), \ldots, 2(G - v_0))
\]

\[
= \text{rank}(v_1 - v_0, \ldots, v_k - v_0, p_1 - v_0, \ldots, p_{n-k} - v_0, 0, \ldots, 0)
\]

\[
= n
\]

The reason of the final equation is that \(\sum_{i=1}^{k} (v_i - v_0) = (k+1)(G - v_0)\). Note that there are a total of \(k + (n-k) + (n-k) = n + (n-k) \geq n\) vectors, meets the lemma 4 condition. For the convenience of description, we define

\[
L_i^{(1)} = v_i, (0 \leq i \leq k),
\]

\[
L_i^{(2)} = p_i, (1 \leq i \leq n-k),
\]

\[
L_i^{(3)} = 2G - p_i, (1 \leq i \leq n - k).
\]
And their centroid is

\[
G' = \frac{1}{2n-k+1} \left( \sum_{i=0}^{k} v_i + \sum_{i=1}^{n-k} (L_i^{(2)} + L_i^{(3)}) \right) \tag{76}
\]

\[
= \frac{1}{2n-k+1} \left((k+1)G + 2(n-k)G \right) \tag{77}
\]

\[
= G \tag{78}
\]

That is, the newly added vector does not change the centroid of \(v_i\). On the other hand, since both \(L_i^{(2)}\) and \(L_i^{(3)}\) are in the linear manifold \(T_1\), and it meets the conditions of the Eq.(69). Similar to the derivation in the Theorem 4 (1), we have

\[
(2n-k+1)||P - G||^2_2 = \sum_{t=LI_1^{(1)},LI_2^{(2)}, LI_3^{(3)}} (||P - t||^2_2 - ||G - t||^2_2) \tag{79}
\]

\[
= \sum_{i=0}^{k} (||P - v_i||^2_2 - ||G - v_i||^2_2) + \sum_{t=L_i(2), L_i(3)} (||P - t||^2_2 - ||G - t||^2_2) \tag{80}
\]

\[
= \sum_{i=0}^{k} (||P - v_i||^2_2 - ||G - v_i||^2_2) + 2(n-k)||P - G||^2_2 \tag{81}
\]

The final equation is because both \(L_i^{(2)}\) and \(L_i^{(3)}\) are in the linear manifold \(T_1\) and satisfy Eq. (69). To simplify Eq. (81), we obtain \(\sum_{i=0}^{k} (||P - v_i||^2_2 - ||G - v_i||^2_2) = (k+1)||P - G||^2_2\). Therefore, Theorem 4 (2) holds.

**For Theorem 4 (3).** When \(k \geq n\), from Proposition 8, we know that \(\text{rank}(v_1 - v_0, v_2 - v_0, ..., v_k - v_0) = n\) holds with probability 1. Hence, if we use the similar deduction from Theorem 4 (1), we can find that Theorem 4 (3) holds when \(k \geq n\). On the other hand, when \(k < n\), we can get the same result also according to Proposition 8. The reason is that \((v_1 - v_0, v_2 - v_0, ..., v_k - v_0)\) are linearly independent with probability 1.

\[
\square
\]

### F. Proof of Theorem 5

**Theorem 5.** Let \(v_i \in \mathbb{R}^k\) and \(v_i \sim N(0,c^2 \cdot I_k)\). If \(k \to \infty\), then

1. \(||v_i||_2 \approx ||v_j||_2 \to \sqrt{2c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)}|, 1 \leq i, j \leq N;\)
2. \(\text{angle}(v_i, v_j) \to \frac{\pi}{2}, 1 \leq i, j \leq N;\)
3. \(||v_i - v_j||_2 \approx ||v_i - v_t||_2, 1 \leq i, j, t \leq N;\)

**Proof.** First, since Chebyshev inequality, for \(1 \leq i \leq N\) and a given \(M\), we have

\[
P \left\{ ||v_i||_2 - \mathbb{E}(||v_i||_2) \geq \sqrt{\text{Var}(||v_i||_2)} \right\} \leq \frac{1}{M}. \tag{82}
\]

from Eq. (16), Eq. (17) and Lemma. (2), we can rewrite Eq. (82) when \(k \to \infty:\)

\[
P \left\{ ||v_i||_2 \in \left[ \sqrt{2c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} - \sqrt{\frac{M}{2}c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} + \sqrt{\frac{M}{2}c} \right] \right\} \geq 1 - \frac{1}{M}. \tag{83}
\]

For a small enough \(\epsilon > 0\), let \(M = 1/\epsilon\). Note that \(\sqrt{\frac{M}{2}c} = c/\sqrt{2c}\) is a constant. When \(k \to \infty\), \(\sqrt{2c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \gg \sqrt{\frac{M}{2}c}\). Hence, for any \(i \in [1, N]\) and any small enough \(\epsilon\), we have

\[
P \left\{ ||v_i||_2 \approx \sqrt{2c} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \right\} \geq 1 - \epsilon. \tag{84}
\]
So Theorem 5(1) holds.

Let \( v_i = (v_{i1}, v_{i2}, \ldots, v_{ik}) \) and \( v_j = (v_{j1}, v_{j2}, \ldots, v_{jk}) \). So \( \langle v_i, v_j \rangle > = \sum_{p=1}^{k} v_{ip}v_{jp} \). Note that, \( v_i \) and \( v_j \) are independent, hence

\[
\mathbb{E}(v_{ip}v_{jp}) = 0,
\]

\[
\text{Var}(v_{ip}v_{jp}) = \text{Var}(v_{ip})\text{Var}(v_{jp}) + (\mathbb{E}(v_{ip}))^2\text{Var}(v_{jp}) + (\mathbb{E}(v_{jp}))^2\text{Var}(v_{ip}) = 1,
\]

since central limit theorem, we have

\[
\sqrt{k} \left( \frac{1}{k} \sum_{p=1}^{k} v_{ip}v_{jp} - 0 \right) \sim N(0, 1),
\]

According to Eq. (16), Lemma 2 and Eq. (87), when \( k \to \infty \), we have

\[
\frac{\langle v_i, v_j \rangle}{\|v_i\|_2 \cdot \|v_j\|_2} \to \frac{1}{\sqrt{k}} \frac{\langle v_i, v_j \rangle}{\sqrt{k}} \sim N(0, \frac{1}{k}) \to N(0, 0).
\]

So Theorem 5(2) holds. From Theorem 5(1) and Theorem 5(2), Theorem 5(3) can be proved through Pythagoras theorem.
G. Statistical Test
G.1. Network Structure
G.2. Optimizer
G.3. Regularization
G.4. Attention Mechanism
G.5. Initialization
G.6. Dataset
G.7. Other Tasks
  G.7.1. SEGMENTATION
  G.7.2. FASTER RCNN
  G.7.3. IMAGE MATTING
  G.7.4. STYLE TRANSFER
  G.7.5. GAN
G.8. Batch Normalization
G.9. Pytorch Pretrain
G.10. Learning Rate

H. Details of Experiments