Stability of non-proper functions

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\[ f : N \rightarrow P : \text{proper } \iff \forall K \subset P : \text{compact}, \ f^{-1}(K) : \text{compact} \]

A function is a \( C^\infty \)-mapping to \( \mathbb{R} \) (i.e. \( P = \mathbb{R} \)).

Assume that mfd’s are \( C^\infty \), second countable & have no \( \partial \).
§.1 Introduction

◊ Notations

- $N, P$: manifolds

\[ C^\infty(N, P) := \{ f : N \to P : C^\infty\text{-mapping} \} \]

We endow $C^\infty(N, P)$ with the “Whitney $C^\infty$-topology”

(Roughly speaking, two mappings are close to each other under this topology iff they have close differentials.)

- $\text{Diff}(N) \subset C^\infty(N, N)$: set of self-diffeomorphisms

$\text{Diff}(N)$ is endowed with the relative topology

- For $f \in C^\infty(N, P)$, $C_f := \{ x \in N \mid \text{rank}(df_x) < \dim P \}$.
Definition of stability

Definition

- $f, g \in C^\infty(N, P)$ are right-left equivalent ($f \sim g$):
  \[
  \Leftrightarrow \exists \Phi \in \text{Diff}(N), \exists \phi \in \text{Diff}(P) \text{ s.t. } g = \phi \circ f \circ \Phi.
  \]

- $f \in C^\infty(N, P)$ is stable (w.r.t. the Whitney topology):
  \[
  \Leftrightarrow \exists U \subset C^\infty(N, P) : \text{nbhd. of } f \text{ (w.r.t. the Whitney topology)} \text{ s.t. } \forall g \in U \text{ is right-left equivalent to } f.
  \]

It is in general difficult to check whether a given mapping is stable or not!!
\( f: \text{stable} \iff \exists \mathcal{U} \subset C^\infty(N, P) : \text{nbhd. of } f \text{ s.t. } g \sim f \text{ for } \forall g \in \mathcal{U}. \)

\[ \text{Simple examples (1/2)} \]

\( f_n \in C^\infty(\mathbb{R}, \mathbb{R}) \) defined by \( f_n(x) := x^n. \)

**Claim 1.** \( f_1 = \text{id}_\mathbb{R} \) is stable.

**Proof:** Define \( \mathcal{U} := \left\{ g \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \forall t \in \mathbb{R}, \ g'(t) > \frac{1}{2} \right\}. \) Then,

- \( \mathcal{U} \) is an open nbhd. of \( f_1. \)
- By the inverse func., intermediate val. & mean val. theorems, \( \mathcal{U} \subset \text{Diff}(\mathbb{R}) \), in particular \( f_1 \sim g \ (g = g \circ f_1 \circ \text{id}_\mathbb{R}) \) for \( \forall g \in \mathcal{U}. \)

Thus, \( f_1 \) is stable. \( \square \)
$f$ : stable $\iff \exists U \subset C^\infty(N, P) : \text{nbhd. of } f \text{ s.t. } g \sim f \text{ for } \forall g \in U$.

\section*{Simple examples (2/2)}

\textbf{Claim 2.} $f_3$ (in general $f_n$ for $n \geq 3$) is not stable.

\textbf{Proof} (for $n = 3$): Define $g_t \in C^\infty(\mathbb{R}, \mathbb{R})$ by $g_t(x) := x^3 + \rho(x)tx$, where $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$, $\rho(x) \equiv 1$ for $|x| \leq 1$, $\rho(x) \equiv 0$ for $|x| \geq 2$.

- For $0 < \forall t \ll 1$, $g_t$ has no critical points, in particular $g_t \not\sim f_3$.
- $\mathbb{R} \ni t \mapsto g_t \in C^\infty(\mathbb{R}, \mathbb{R})$ is continuous, and thus
  \[ \forall U \subset C^\infty(\mathbb{R}, \mathbb{R}) : \text{open nbhd. of } f_3, \exists t > 0 \text{ s.t. } g_t \in U. \]

Thus, $f_3$ is not stable. \hfill \square

How about $f_2$...? It is not so easy to show that it is stable...
Definition of infinitesimal stability

\[ \Gamma(E) : \text{set of sections of a vector bundle } E. \]

**Definition**

\[ f \in C^\infty(N, P) \text{ is infinitesimally stable} \]

\[ \iff \Gamma(f^*TP) = df_*(\Gamma(TN)) + f^*(\Gamma(TP)), \text{ where} \]

\[ df_* : \Gamma(TN) \to \Gamma(f^*TP) \text{ is defined by } df_*(\xi) := df \circ \xi. \]

\[ f^* : \Gamma(TP) \to \Gamma(f^*TP) \text{ is defined by } f^*(\eta) := \eta \circ f. \]

**Remark** (Motivation for infinitesimal stability)

\[ L_f : \text{Diff}(N) \times \text{Diff}(P) \to C^\infty(N, P), \quad L_f(\Phi, \phi) := \phi \circ f \circ \Phi^{-1}. \]

- stability \( \iff \) image of \( L_f \) contains a nbhd. of \( f \).
- inf. stability \( \iff \) the “differential \( (dL_f)_{(id_N, id_P)} \)” is surjective.
\[ f : \text{inf. stable} \iff \Gamma(f^*TP) = df_*(\Gamma(TN)) + f^*(\Gamma(TP)). \]

\[ \diamond \quad \textbf{Simple examples} \quad \text{(again)} \]

\[ f_n \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ defined by } f_n(x) := x^n. \]

\textbf{Claim 3.} \( f_2 \) is infinitesimally stable.

\textbf{Proof :} We can identify \( \Gamma(T\mathbb{R}) = \Gamma(f_2^*T\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{R}). \)

Under these identifications, \((df_2)_*(\xi) = 2x\xi\) and \( f_2^*(\xi) = \xi(x^2). \)

Since \( \xi(x) = \xi(0) + \int_0^1 \frac{d}{dt}(\xi(tx))\,dt = \xi(0) + x \int_0^1 \frac{d\xi}{dt}(tx)\,dt \)

for \( \xi \in C^\infty(\mathbb{R}, \mathbb{R}), \Gamma(f_2^*T\mathbb{R}) = (df_2)_*(\Gamma(T\mathbb{R})) + f_2^*(\Gamma(T\mathbb{R})). \) \( \Box \)
Stability for proper mappings (1/2)

Theorem (Mather 1970)
For $f \in C^\infty(N, P)$: proper mapping, stability, infinitesimal stability, strong stability and “local stability” are all equivalent.

Definition $f \in C^\infty(N, P)$: strongly stable
$\iff \exists U \subset C^\infty(N, P)$: neighborhood of $f$
$\exists(\Theta, \theta): U \to \text{Diff}(N) \times \text{Diff}(P)$: continuous map
s.t. $\forall g \in U, \theta(g) \circ g \circ \Theta(g) = f$. 
Stability for proper mappings (2/2)

We will only give several properties of “local stability”.

- **local stability** is the weakest condition of the four stabilities.
  
i.e. (inf.) stable $\Rightarrow$ locally stable for general (possibly non-proper) $f$.

- In general, it is (relatively) easy to check local stability (Mather).
  
e.g. $f : N \rightarrow \mathbb{R}$: (possibly non-proper) function is locally stable
  
  $\iff f :$ Morse function, that is,
  
  - $\forall x \in C_f$, $\det \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{i,j} \neq 0$. 
  - $f|_{C_f}$: injective.

Thus, **it is easy to check stability of proper mappings!!**
Motivating problem 1

Problem 1
How can we detect (strong) stability of non-proper functions? 
e.g. Is $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ defined by $f(x, y) = x^2 - y^2$ stable? 
Note that $f$ is infinitesimally stable but **NOT** strongly stable!!

(will be seen later)
Remarks on problem 1 (1/2)

Problem 1

How can we detect (strong) stability of non-proper functions?
e.g. \( f(x, y) = x^2 - y^2 \): stable?

- \( f \): inf. stable \( \iff \) \( f \): loc. stable & \( f|_{C_f} \): proper (Mather).

In particular, infinitesimal stability is easily checked.
(since it is easy to check local stability.)

However, it is in general difficult to check (strong) stability!
Remarks on problem 1 (2/2)

Problem 1
How can we detect (strong) stability of non-proper functions?

E.g. $f(x, y) = x^2 - y^2$ : stable?

- (Dimca) $f \in C^\infty(\mathbb{R}, \mathbb{R})$ : stable

  $\iff f$ : locally stable & $f(C_f) \cap (S(f) \cup L(f)) = \emptyset$, where

  $L(f) = \left\{ y \in \mathbb{R} \mid y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\}$

  $S(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \mid \{x_i\} : \text{sequence in } C_f \text{ without accumulation points} \right\}$

Thus, it is (somewhat) easy to check stability of $f \in C^\infty(\mathbb{R}, \mathbb{R})$. 
\( f \in C^\infty(\mathbb{R}, \mathbb{R}) : \text{stable} \iff f : \text{locally stable} \& f(C_f) \cap (S(f) \cup L(f)) = \emptyset. \)

\[
L(f) = \left\{ \lim_{x \to \pm \infty} f(x) \right\}, \quad S(f) = \left\{ \lim_{i \to \infty} f(x_i) \mid \{x_i\} : \text{seq. in } C_f \text{ w/o accumulation pt's} \right\}
\]

**Example**  \( f : \mathbb{R} \to \mathbb{R}, \ f(x) := \exp(x) \sin x. \)

Since \( f^{(k)}(x) = 2^k/2 \exp(x) \sin \left(x + \frac{k\pi}{4}\right), \) it is easy to see:

- \( C_f = \left\{ \frac{(4n + 3)\pi}{4} \in \mathbb{R} \mid n \in \mathbb{Z} \right\}, \)
- \( f : \text{Morse func.} \) (i.e. \( f|_{C_f} : \text{inj.} \& \forall x \in C_f, \ f^{(2)}(x) \neq 0). \)

Furthermore, \( S(f) = L(f) = \{0\} \& 0 \not\in f(C_f) \Rightarrow f : \text{stable} \)

On the other hand, \( (f|_{C_f})^{-1}([-1, 1]) : \text{infinite discrete set} \Rightarrow f : \text{NOT infinitesimally stable} (\because f|_{C_f} : \text{not proper}). \)
Problem 2
How are the four stabilities related for non-proper functions?
In particular, strongly stable $\Rightarrow$ infinitesimally stable?
Remarks on problem 2 (1/3)

Problem 2
How are the four stabilities related for non-proper functions?
In particular, strongly stable ⇒ infinitesimally stable?

- $f$: strongly stable ⇒ $f$: stable (obvious).
- $f$: stable ⇒ $f$: locally stable (Mather).
- $f$: inf. stable ⇔ $f$: loc. stable & $f|_{C_f}$: proper (Mather).
Remarks on problem 2 (2/3)

- \( f \): strongly stable \( \Rightarrow \) \( f \): quasi-proper (du Plessis-Vosegaard)

\[ f \text{ quasi-proper} : \Leftrightarrow \exists V \subset P : \text{neighborhood of } f(C_f) \text{ s.t.} \]

\[ f|_{f^{-1}(V)} : f^{-1}(V) \to V : \text{proper} \]

e.g. \( \exp(x) \sin x \) & \( x^2 - y^2 \): NOT quasi-proper

- Using the results we have explained, we can show:

\[ \begin{array}{ccc}
\text{stable} & \leftrightarrow & \text{strongly stable} \\
\downarrow T & & \downarrow F \\
\text{locally stable} & \leftrightarrow & \text{inf. stable}
\end{array} \]
Remarks on problem 2 (3/3)

Problem 2

How are the four stabilities related for non-proper functions?
In particular, strongly stable $\Rightarrow$ infinitesimally stable?

- $f \in C^\infty(N, P)$ is strongly and infinitesimally stable if and only if $f$ is locally stable, quasi-proper and $f(C_f)$ is closed (du-Plessis-Vosegaard)

Still, we have no reasonable condition implying only strong stability...
Motivating problems (Summary)

1. **detecting** (strong) **stability of non-proper functions.**
   e.g. Is \( f(x, y) = x^2 - y^2 \) stable?
   Note that \( f : \text{NOT} \) quasi-proper (thus \( \text{NOT} \) strongly stable).

2. **strongly stable \( \Rightarrow \) infinitesimally stable?**
   The other implications are known to be True/False as follows:

```
+-----+   +-----+   +-----+   +-----+
| stable|   | strongly stable|   | stable|
|       |   |               |   |       |
| T     |   |               | F |       |
| locally stable|   | inf. stable    |   |       |
```
§ 2 Main result

Theorem (H.)

\( f \in C^\infty(\mathbb{N}, \mathbb{R}) \) : Morse function.

\( \tau(f) := \{y \in \mathbb{R} \mid f \text{ : “end-trivial” at } y\} \).

(the definition of end-triviality will be given soon...)

1. \( f(C_f) \subset \tau(f) \Rightarrow f \text{ : stable.} \)

2. \( f \text{ : strongly stable } \iff f \text{ : quasi-proper} \)

\( f \text{ : quasi-proper } \iff \exists V \subset P \text{ : neighborhood of } f(C_f) \text{ s.t. } f|_{f^{-1}(V)} : f^{-1}(V) \to V \text{ : proper} \)
Remarks on the main result

- As we explained, \( f : \text{strongly stable} \Rightarrow f : \text{quasi-proper} \)
  for \( f \in C^{\infty}(N, P) \) (du Plessis-Vosegaard)
  We indeed show the converse of it for the case \( P = \mathbb{R} \).

- Dimca’s condition \( (f(C_f) \cap (S(f) \cup L(f)) = \emptyset) \) is
  equivalent to ours \( (f(C_f) \subset \tau(f)) \). Indeed,
  \[ \tau(f) = \mathbb{R} \setminus (S(f) \cup L(f)) \] for \( f \in C^{\infty}(\mathbb{R}, \mathbb{R}) \), where
  \[ L(f) = \left\{ y \in \mathbb{R} \mid y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\}, \]
  \[ S(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \mid \{x_i\} : \text{seqeunce in } C_f \text{ without accumulation points} \right\}. \]
◊ End-triviality

\( V \subset N : \text{neighborhood of the end} : \iff N \setminus V : \text{compact} \)

**Definition** \( f \in C^\infty(N, P), \ y \in P. \)

\( f \) is **end-trivial** at \( y \) if \( \exists W \subset P : \text{neighborhood of} \ y, \exists V \subset N : \text{open neighborhood of the end s.t.} \)

- \( f^{-1}(y) \cap V \) contains no critical points of \( f \),
- \( \exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V : \text{diffeomorphism} \)
  s.t. \( f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W : \text{projection} \)

Roughly, end-triviality at \( y \) implies that \( f \) is the projection

“around the end of \( f^{-1}(\text{nbhd. of} \ y) \)”.
\[ \exists W \subset P : \text{nbhd. of } y, \exists V \subset N : \text{open nbhd. of the end s.t.} \]
- \( f^{-1}(y) \cap V \) contains no critical points of \( f \),
- \( \exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V : \text{diffeomorphism} \)
  
  s.t. \( f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W : \text{projection} \)

**Example**  The fig. is contours of \( f(x, y) := x^2 - y^2 \) in \( \mathbb{R}^2 \).

Blue : outside of (sufficiently large) disk (which is \( V \))

Red : preimage of nbhd. of \( 0 \in \mathbb{R} \) (which is \( f^{-1}(W) \) for \( y = 0 \))

One can regard \( f = p_2 \) on Blue\( \cap \)Red. (i.e. \( \exists \Phi \) with the desired property)

Thus, \( f \) is end-trivial at \( 0 \in \mathbb{R} \).
Main result (Again)

Theorem (H.)

\[ f \in C^\infty(N, \mathbb{R}) : \text{Morse function.} \]

\[ \tau(f) := \{ y \in \mathbb{R} \mid f : \text{end-trivial at } y \}. \]

1. \[ f(C_f) \subset \tau(f) \Rightarrow f : \text{stable.} \]
2. \[ f : \text{strongly stable} \iff f : \text{quasi-proper} \]

\( f : \text{quasi-proper} \iff \exists V \subset P : \text{neighborhood of } f(C_f) \text{ s.t. } f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V : \text{proper} \)
§.3 Applications

◊ detecting stability

Example \( f \in C^\infty(\mathbb{R}^2, \mathbb{R}), \ f(x, y) = x^2 - y^2. \)

\( C_f = \{0\} \) and \( 0 \in \tau(f) \) (as we checked) \( \Rightarrow f \) is stable.

In general, end-triviality of semi-algebraic mappings has been studied in detail.

**Definition** \( f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \): semi-algebraic, \( y \in \mathbb{R}. \)

\( f \) satisfies the **Malgrange condition** at \( y \)

\[ \iff \exists \delta > 0, \exists \varepsilon > 0, \exists V \subset \mathbb{R}^n : \text{nbhd. of the end s.t.} \]

\[ \|x\| \cdot \|\nabla f(x)\| > \varepsilon \text{ for any } x \in f^{-1}(y - \delta, y + \delta) \cap V. \]

Here, \( \nabla f \) is the gradient of \( f \).
Theorem (Folklore?) \( f \in C^\infty(\mathbb{R}^n, \mathbb{R}) : \) semi-algebraic.

If \( f \) satisfies the Malgrange condition at \( y \in \mathbb{R} \), then \( f \) is end-trivial at \( y \).

Corollary 1 (H.) \( f \in C^\infty(\mathbb{R}^n, \mathbb{R}) : \) Morse & semi-algebraic.

\( f \) is stable if it satisfies the Malgrange condition at \( \forall y \in f(C_f) \).

Corollary 2 (H.) \( f \in C^\infty(\mathbb{R}^n, \mathbb{R}) : \) semi-algebraic.

\( \exists \Sigma \subset \mathbb{R}^n : \) Lebesgue measure zero set s.t.

\( \forall a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \Sigma, \) the function

\[
 f_a(x_1, \ldots, x_n) = f(x_1, \ldots, f_n) + \sum_{i=1}^{n} a_i x_i
\]

is stable.
◊ strong & infinitesimal stability

Corollary 3 (H.)

The function \( f(x) = \exp(-x^2) \sin x \) is strongly stable but NOT infinitesimally stable.

We indeed show that \( f \): Morse function, quasi-proper

& \( f|_{C_f} \): NOT proper.

\((f \in C^\infty(N, \mathbb{R}) : \text{inf. stable} \iff f : \text{Morse} & f|_{C_f} : \text{proper} \text{ (Mather)})\)
♢ Related topics (1/2)

• A sufficient condition for **topological** strong stability (for general \( N & P \)) is given by Murolo, du Plessis and Trotman.

• du Plessis-Vosegaard studied stability under another topology \( \tau V^\infty \) of \( C^\infty(N, P) \) (which is stronger than the Whitney topology). They indeed showed:

**Theorem (du Plessis-Vosegaard)**

Under the topology \( \tau V^\infty \), for a quasi-proper mapping, strong stability, stability, ”quasi-infinitesimal stability” and local stability are all equivalent.
Little is known about stability for $\dim P > 1$. For example, the following problem is still open.

**Problem**: Is there a non-proper stable mapping in $C^\infty(\mathbb{R}, \mathbb{R}^2)$? (w.r.t. the Whitney topology)

Indeed, even the following simple (but non-proper) embedding is not stable!! (du Plessis-Vosegaard):

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = (\exp(x), 0).$$

Note that $f$ is quasi-proper, locally stable (in particular strongly stable w.r.t. $\tau V^\infty$).
◊ **Summary (what we gave)**

- A sufficient condition for (strong) stability of \( f \in C^\infty(N, \mathbb{R}) \).
- The answers to the following questions:
  1. Is \( f(x, y) = x^2 - y^2 \) stable? **Yes!**
  2. strongly stable \( \Rightarrow \) infinitesimally stable? **No!**

*Thank you for your attention!*