ASYMPTOTIC BEHAVIOR OF OPERATOR NETS ON KB-SPACES

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ABSTRACT. The concept of an attractor or constrictor was used by several mathematicians to characterize the asymptotic behavior of operators. In this paper we show that a positive LR-net on KB-spaces is mean ergodic if the LR-net has a weakly compact attractor. Moreover if the weakly compact attractor is an order interval, then a Markovian LR-net converges strongly to the finite dimensional fixed space. As a consequence we investigate also stability of LR-nets of positive operators and existence of lower bound functions on KB-spaces.

1. Introduction

Whenever there exists a lower bound function, then a Markov operator $T$ on an $L^1$-space is mean ergodic and its fixed space is finite-dimensional, see [9]. In [3], this property is extended to any power bounded operator on a KB-space. Moreover, it is shown that this property of positive operators characterizes KB-spaces among $\sigma$-Dedekind complete Banach lattices.

Let $E$ be a Banach lattice. Then $E_+ := \{x \in E : x \geq 0\}$ denotes the positive cone of $E$. On $\mathcal{L}(E)$ there is a canonical order given by $S \leq T$ if $Sx \leq Tx$ for all $x \in E_+$. If $0 \leq T$, then $T$ is called positive. The dual space $E''$ equipped with the canonical order is again a Banach lattice. Instead of the operations sup and inf on $E$ we often write $\lor$ and $\land$, respectively. For $x \in E_+$ we denote by $[-x, x] := \{y \in E : |y| \leq x\}$ the order interval generated by $x$. A linear subspace of $E$ is an ideal if $[-|x|, |x|] \subseteq I$ for all $x \in I$. An ideal $I$ in $E$ is called a band if for every subset $M \subseteq I$ such that sup $M$ exists in $E$ one has sup $M \in I$. An ideal $I$ in $E$ is called a projection band if there is an ideal $J$ in $E$ such that $E = I \oplus J$ is the topological sum of $I$ and $J$. In that case $J$ is uniquely determined and $I$ and $J$ are bands. The projection $P$ from $E$ onto $I$ with kernel $\ker P = J$ is called the band projection corresponding to $I$ and satisfies $0 \leq P \leq I_E$. A Banach lattice $E$ is called a KB-space whenever every increasing norm bounded sequence of $E_+$ is norm convergent. In particular, it follows that every KB-space has order continuous norm. All reflexive Banach lattice and AL-space are examples of KB-spaces. The following theorem is a combination of results by many mathematicians, for proofs see [1], [13], [17].

Theorem 1.1. For a Banach lattice $E$ the following statements are equivalent:

- $E$ is a KB-space
- $E$ is a band of $E''$
- $E$ is weakly sequentially complete

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2. LR-nets

In this section, we use the Banach space concept to define special operator nets and use the following terminology. Let $X$ be a Banach space, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on $X$ and let $I_X = I$ be the identity operator in $X$. A family $\Theta = (T_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{L}(X)$ indexed by a directed set $\Lambda = (\lambda, \prec)$ is called an operator net. The net $\Theta$ is strongly convergent if the norm-limit $\lim_{\lambda \to \infty} T_\lambda x$ exists for each $x \in X$. A vector $x$ is called a fixed vector for the net $\Theta$ if $T_\lambda x = x$ for each $\lambda \in \Lambda$. We denote by $\text{Fix}(\Theta)$ the set of all fixed vectors of $\Theta$.

The following important concept was introduced by H.P. Lotz [13] and F. Räbiger [16], we use the modified terminology from [6] called LR-nets.

**Definition 2.1.** A net $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ is called LR-net if

- LR1 $\Theta$ is uniformly bounded;
- LR2 $\lim_{\lambda \to \infty} \|T_\lambda \circ (T_\mu - I) x\| = 0$ for every $\mu \in \Lambda$ and for every $x \in X$;
- LR3 $\lim_{\lambda \to \infty} \|(T_\mu - I) \circ T_\lambda x\| = 0$ for every $\mu \in \Lambda$ and for every $x \in X$.

Many examples of LR-nets appear in the investigation of operator semigroups. Thus every strongly convergent uniformly bounded Abelian operator semigroup itself is an LR-net with respect to the natural partial order $\prec$ defined by $T \prec S$ if there exists an $R$ with $S = R \circ T$. If a semigroup $T \subseteq \mathcal{L}(X)$ admits a $T$-ergodic net $A$, then it is an LR-net. In particular, the Cesaro averages of a power bounded operator form an LR-net and moreover encompasses Cesaro averages of higher orders for both discrete and continuous semigroups. We refer to [5, 6, 4, 7].

The following theorem is the main analytic tool in the investigation of LR-nets. For a complete proof we refer to [6].

**Theorem 2.2.** Let $\Theta$ be an LR-net on a Banach space $X$. Then the following conditions are equivalent:

i. The net $\Theta$ is strongly convergent.

ii. $X = \text{Fix}(\Theta) \oplus \bigcup_{\lambda \in \Lambda} (I - T_\lambda)X$ the strong limit of $\Theta$ is a projection onto $\text{Fix}(\Theta)$.

iii. The net $(T_\lambda x)_{\lambda \in \Lambda}$ has a weak cluster point for every $x \in X$.

iv. The fixed space $\text{Fix}(\Theta)$ separates the fixed space $\text{Fix}(\Theta')$ of the adjoint operator net $\Theta' = (T'_\lambda)_{\lambda \in \Lambda}$ in $X'$.

3. Attractors

The constrictiveness of an operator was introduced in order to characterize asymptotically periodic Markov operator on $L^1$-spaces. Many authors have extended this notion to more general situations. All these notions have in common the general principal reflected by the notion of attractor introduced in [6].

**Definition 3.1.** Let $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ be an operator net on a Banach space $X$ and $A \subseteq X$. Then $A$ is called an attractor of $\Theta$ if

$$\lim_{\lambda \to \infty} \text{dist}(T_\lambda x, A) = 0$$

for all $x \in B_X := \{ z \in X : \|z\| \leq 1 \}$.
Our aim was to find conditions on the attractor \( A \) implying nice asymptotic properties of \( \Theta \). The first property of attractor is the following: Every LR-net possessing a weakly compact attractor is strongly convergent \([9]\). Later Emel’yanov proved that every LR-net containing a weakly compact operator is strongly convergent \([7]\).

4. Ergodicity of LR-nets on Banach lattices

If \( T \) is a Markov operator on \( L^1 \)-space then \( T \) is mean ergodic and satisfies \( \dim \text{Fix}(T) < \infty \) whenever there exists a function \( h \in L^1_+ \) and a real \( 0 \leq \eta < 1 \) such that \( \lim_{n \to \infty} \left\| (h - \frac{1}{n} \sum_{k=0}^{n-1} T^k f)_+ \right\| \leq \eta \) for every density \( f \) is proven in \([9]\). In \([5]\) and \([3]\) some generalization of this results were given. We begin to prove the above theorem for positive operator nets. The principal tool in the proof of the main results of \([5]\) was using the additivity of the norm on the positive part of the \( L^1 \)-space. Since this is no longer the case for a general KB-space, we use different ideas in this paper, inspired by \([15]\).

For our notation and terminology, we refer to \([1, 14, 17]\).

Consider the set \( E_e := \bigcup \{ [-ne, ne] : n \geq 0 \} \) for any \( e \in E_+ \) which is the order ideal. If \( E_e \) is norm-dense in Banach lattice \( E \) then \( e \in E_+ \) is called a quasi interior point of \( E_+ \). Moreover if \( \Theta \) is a positive operator net on \( E \), then \( x \in E \) is called a positive fixed vector of maximal support if \( x \in \text{Fix}(\Theta) \cap E_+ \) and every \( y \in \text{Fix}(\Theta) \cap E_+ \) are contained in the band generated by \( x \). For every quasi-constrictive Markov operator there exists an invariant density with maximal support, see \([12]\). Răbiger \([15]\) proved that in KB-space with quasi-interior point and for positive contraction operator \( T \) such that \( A := [-z, z] + \eta B_E \) is an attractor of \( T \) where \( z \in E_+ \) and \( 0 \leq \eta < 1 \). Then either \( T \) is mean ergodic or there is a positive fixed vector \( y \neq 0 \) of \( T \) of maximal support and for such positive fixed vector of maximal support \((I - P_y)T_n\) converges strongly to zero where \( P_y \) is the band projection from \( E \) onto the band generated by \( y \).

**Theorem 4.1.** Let \( E \) be a KB-space and \( \Theta = (T_\lambda)_{\lambda \in \Lambda} \) be a positive LR-net in \( E \), \( W \) be a weakly compact subset of \( E \), and \( \eta \in \mathbb{R} \), \( 0 \leq \eta < 1 \) such that

\[
\lim_{\lambda \to \infty} \text{dist}(T_\lambda x, W + \eta B_E) = 0
\]

for any \( x \in B_E := \{ z \in E : \|z\| \leq 1 \} \). Then \( \Theta \) converges strongly.

**Proof:**

The first part of the proof is motivated by the proof of Theorem 5.3 in Răbiger’s paper in \([15]\). For any \( x \in E \), \( x \neq 0 \), we consider the closed order ideal \( F \) generated by \( \{ T_\lambda x : n \geq 0 \} \), instead of \( E \). Then \( F \) is a KB-space \([17]\) with quasi-interior point \( \sum_{n \geq 0} 2^{-n} T_\lambda x \) and \( T_\mu(F) \subseteq F \), \( \forall \mu \in \Lambda \). Moreover \( F \) is a projection band in \( E \) \([14]\), see Theorem \([1]\). If \( P : E \to F \) denotes the corresponding band projection, then

\[
\lim_{\lambda \to \infty} \text{dist}(T_\lambda x, P(W) + \eta B_F) = 0.
\]

Since \( P \) is a projection and \( W \) is weakly compact subset in \( E \), \( P(W) \) is weakly compact in \( F \), and the restriction of LR-net on \( F \) satisfies the theorem assumption. Therefore it is enough to show that the LR-net on \( F \) converges strongly. For this proof, we will use the Eberlein’s theorem in \([6]\) and we may assume moreover that \( E \) has a quasi-interior point, say \( u \).
In the first case, \((T_\phi^\prime)_{\phi \in \Lambda}\) is a weak*-nullsequence for each \(\phi \in E'.\) Then \((T_\lambda x)_{\lambda \in \Lambda}\) has a zero as a weak cluster point for each \(x \in E\) and hence by [6] our LR-net converges strongly to zero.

In the second case, there is \(\phi \in E'_+\) such that \((T_\phi^\prime)_{\phi \in \Lambda}\) is not \(\sigma(E', E)\)-convergent to \(0.\) Let \(0 \neq \psi \in E'_+\) be a \(\sigma(E', E)\)-cluster point of \((T_\phi^\prime)_{\phi \in \Lambda}\). We may assume \(\|\psi\| = 1.\) Then for all \(\epsilon > 0,\) there exists \(\lambda_n\) such that \(\langle \psi, x \rangle - \langle T_\phi^\prime \phi, x \rangle < \epsilon\) and \(\langle T_\phi^\prime \phi, x \rangle - \langle T_\mu^\prime T_\phi^\prime \phi, x \rangle < \epsilon\) for every \(\mu \in \Lambda.\) Therefore we get \(T_\mu^\prime \phi \psi = \psi.\)

Now for \(\epsilon > 0\) satisfying \(\epsilon := 1 - \eta\) choose \(x \in B_E \cap E_+\) such that \(\langle \psi, x \rangle > 1 - \epsilon.\) Let \(x'' \in E''_+\) be a weak-cluster point of \((T_\lambda x)\). Then there exists \(\lambda_n\) such that \(\langle \psi, x'' \rangle - \langle \psi, T_\lambda x \rangle < \epsilon\) and \(\langle T_\phi^\prime \psi, x'' \rangle - \langle T_\phi^\prime \phi, T_\lambda x \rangle < \epsilon\) by combining these two estimates with the property of LR-net, we obtain \(T_\mu^\prime x'' = x''\) for every \(\mu \in \Lambda.\)

Since \(\lim_{\lambda_n \to \infty} \text{dist}(T_\lambda x, W + \eta B_E) = 0\) and \(W\) is weakly compact then we obtain \(x'' \in W + \eta B_E.\) Moreover \(x''\) is a weak*-cluster point of \((T_\lambda x)\), then for every \(\epsilon' > 0,\) there exists \(\lambda\) such that \(\langle \psi, x'' \rangle - \langle \psi, T_\lambda x \rangle < \epsilon'.\) Therefore we have \(\langle \psi, x'' \rangle - \langle T_\lambda^\prime \psi, x \rangle < \epsilon'\) and since \(T_\lambda^\prime \psi = \psi,\) we obtain \(\langle \psi, x'' \rangle - \langle \psi, x'' \rangle < \epsilon'.\) By arbitrariness of \(\epsilon',\) \(\langle \psi, x'' \rangle = \langle \psi, x \rangle.\)

Being \(E\) a KB-space, by Theorem [11] \(E\) is a band projection in \(E''\) and denote by \(P\) the band projection from \(E''\) onto \(E; \text{ie; } P : E'' \to E.\) Hence

\[
\langle \psi, Px'' \rangle = \langle \psi, x'' \rangle - \langle \psi, (I_{E''} - P)x'' \rangle = \langle \psi, x \rangle - \langle \psi, (I_{E''} - P)x'' \rangle > 1 - \epsilon - \eta > 0
\]

It follows from [11] that \(Px'' \neq 0.\) Since \(x''\) is a weak*-cluster point of \((T_\lambda x)_{\lambda \in \Lambda},\) \(Px'' > 0\) and moreover, since \(E\) has order continuous norm \(z := \lim_{\lambda_n \to \infty} T_{\lambda_n} x'' \in E_+\) exists. Clearly \(T_{\mu} z = z\) and from \(\langle \psi, z \rangle = \langle \psi, Px'' \rangle > 0,\) it follows that \(z \neq 0.\) Hence \(Fix(\Theta) \cap E_+ \neq \{0\}.\) Choose a net \(x_{\lambda} \in Fix(\Theta) \cap E_+, \lambda \in \Lambda, \|x_{\lambda}\| \leq 1\) and \(\alpha = \lim (\psi, P_{x_{\lambda}} e).\)

Let \((x_{\lambda_n})\) be a subsequence of the net \((x_{\lambda})\) and define \(u = \sum_n 2^{-n} x_{\lambda_n}.\) Then \(u\) is also an element of \(Fix(\Theta) \cap E_+\) and in addition \(P_{\lambda} u \geq P_{x_{\lambda_n}} u / n \) for all \(n \in \mathbb{N}.\) Hence also \(\langle \psi, P_{u_{\lambda}} e \rangle = \alpha.\)

Further taking \(x \in Fix(\Theta) \cap E_+,\) clearly \(P_{u_{\lambda}} u \geq P_x u\) and \(P_{u_{\lambda} e \lambda} \geq P_u.\) From the limit property of \(\langle \psi, P_{x_{\lambda_n}} e \rangle, \alpha \leq \langle \psi, P_{u_{\lambda}} e \rangle \leq \alpha,\) so \(\alpha = \langle \psi, P_{u_{\lambda}} e \rangle\) and we know above \(\psi\) is strictly positive, it implies that \(P_u e = P_{u_{\lambda}} e.\) Owing to quasi-interior point \(e, P_u = P_{u_{\lambda}} e\) and by \(P_{u_{\lambda}} e \geq P_x\) then we obtain \(P_u \geq P_x.\) Hence \(u\) has a maximal support.

In the next step, we will prove for the band projection, denoting by \(P_u\), of positive fixed vector of maximal support, \((I - P_u) T_\lambda\) converges to zero strongly as \(\lambda \to \infty.\) Let \(P_u\) be a band projection onto \(B_u\) where \(B_u = \cup_n [u, u^n]\). Denote the new operator \(Q = I_E - P_u\) and the net \(S = (S_\lambda) = (Q T_\lambda).\) Since \(u\) is a fixed vector so \(T_\lambda B_u \subset B_u\) and hence we get \(T_\lambda P_u = P_u T_\lambda P_u\) and in addition \(Q T_\lambda Q = Q T_\lambda\) for each \(\lambda.\)

Our aim is to show that \((Q T_\lambda) = (S_\lambda)\) converges strongly to zero. If not, then there exists by above in the second case of proof, \(0 \neq \psi \in Fix(S') \cap E'_+.\) Let \(\psi = S_\lambda^* \psi = T_\lambda^* Q' \psi = Q' T_\lambda^* Q' \psi = Q' \psi\) and hence \(\psi = S_\lambda^* \psi = T_\lambda^* Q' \psi = T_\lambda^* \psi,\) namely, \(\psi \in Fix(\Theta').\) By the remark above there is \(0 \neq x \in Fix(\Theta) \cap E_+\) such that \(\langle x, \psi \rangle > 0.\) Then

\[
0 < \langle x, \psi \rangle = \langle x, Q' \psi \rangle = \langle Q x, \psi \rangle
\]
implies $Qx \neq 0$, ie, $x \notin B_u$. It is a contradiction to our assumption on $u$. Indeed, $S_\lambda \to 0$ strongly as $\lambda \to \infty$.

In the next step, we will prove that $(T_\lambda)$ converges strongly by using the Eberlein theorem, proven in [6].

We know that our operator net is an LR-net and positive. For fixed $\epsilon > 0$ and $x \in E$, $\lim_{\lambda \to \infty} S_\lambda x = \lim_{\lambda \to \infty} (I - P_u)T_\lambda x = 0$, there exists $\lambda_\epsilon$ such that $\text{dist}(T_\lambda x, B_u) \leq \frac{\epsilon}{3M}$ where $M = \sup_{\lambda} \|T_\lambda\|$. It implies that there exists $n_\epsilon \in \mathbb{R}_+$ and $y \in [-n_\epsilon u, n_\epsilon u]$ satisfying $\|T_{\lambda n_\epsilon} x - y\| \leq \frac{\epsilon}{3M}$.

For any $\mu \in \Lambda$, $\|T_\mu T_\lambda x - T_\mu y\| \leq \|T_\mu\|\|T_\lambda x - y\| \leq \frac{\epsilon}{3}$. Moreover since $[u, u]$ is $\Theta$-invariant then we get $T_\lambda y \in [-n_\epsilon u, n_\epsilon u]$ for each $\lambda$. That is to say $\text{dist}(T_\lambda x, [-n_\epsilon u, n_\epsilon u]) \leq \epsilon$, ie, for any $\epsilon > 0$, there exists an interval $[-a_\epsilon, a_\epsilon]$ such that $(T_{\lambda n_\epsilon})_{\lambda=0} \subseteq [-a_\epsilon, a_\epsilon] + \epsilon B_E$. It shows that $(T_\lambda x)$ has a weak cluster point because $E$ is a KB-space and almost order bounded subset of $E$ is weakly precompact. Then by Eberlein Theorem [6], $(T_\lambda x)_{\lambda \in \Lambda}$ is norm convergent for any $x \in E$, ie, $\Theta$ converges strongly.

The theorem is also true if we replace a weakly compact subset $W$ of $E$ by an order interval $[-g, g]$ for any $g \in E_+$ because in KB-spaces, every order intervals are weakly compact. Besides in this case we have more results that also dimension of fixed space is finite. But for this conditions positivity is not only sufficient in addition we need Markov operators. Before proving of the theorem, we need to define Markov operator on Banach lattice $E$.

**Definition 4.2.** Let $E$ be a Banach lattice. A positive linear contraction $T : E \to E$ is called a Markov operator if there exists $0 < e' \in E_+'$ such that $T'e' = e'$.

It is well known that if $T$ is a positive linear operator defined on a Banach lattice $E$, then $T$ is continuous. It is also well known that if the Banach lattice $E$ has order continuous norm, then the positive operator $T$ is also order continuous. We note that the Markov operators, according to this definition, are again contained in the class of all positive contractions and that the adjoint $T'$ is also a positive contraction. For more details, we refer to [10].

In the following we will establish the asymptotic properties of Markov LR-nets. The proof of it is the following:

**Theorem 4.3.** Let $E$ be a KB-space, $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ be a Markov LR-net. Then the following are equivalent

1. there exists a function $g \in E_+$ and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ such that $\lim_{\lambda \to \infty} \text{dist}(T_\lambda x, [-g, g] + \eta B_E) = 0 \forall x \in B_E$

2. the net $\Theta$ is strongly convergent and $\dim \text{Fix}(\Theta) < \infty$.

**Proof:** The proof of this theorem is motivated by the proof of Theorem 3 in [5].

$(i) \Rightarrow (ii)$: $\Theta$ is an Markov LR-net. Therefore there exists $e' > 0 \in E_+'$ such that $T'e' = e'$ for every $\lambda \in \Lambda$. Since $\Theta$ converges by Theorem 11 then by Sine’s Theorem in [6] $\text{Fix}(\Theta)$ separates $\text{Fix}(\Theta)'$, namely there exists $x \in \text{Fix}(\Theta)$ such that $(x, y) > 0$. By Theorem 11 $x = Pf'' \in \text{Fix}(\Theta) \cap E_+$. It means that the strong projection of $\Theta$ is strictly positive. Hence by 17 $\text{Fix}(\Theta)$ is a sublattice of $E$. 

We would like to prove now $\dim \text{Fix}(\Theta)$ is finite. Assume $\dim \text{Fix}(\Theta) = \infty$. By Judin’s theorem [2] there exists a sequence $(x_n)_n \in \text{Fix}(\Theta)$ such that $x_n \land x_m = 0$ for each $n \neq m$. We may assume that $\|x_n\| = 1$ and by assumption of theorem $\|z_n\| = \|g \land x_n\| = \|x_n - (x_n - g)_+\| \geq 1 - \eta > 0$ for all $n$. Besides $(z_n)$ is an order bounded disjoint sequence in $E$, so the order continuity of the norm in $E$ implies that $\|y_n\| \to 0$ which contradicts to the inequality above. Hence $\dim \text{Fix}(\Theta) < \infty$.

(ii) $\Rightarrow$ (i): If $\dim \text{Fix}(\Theta) < \infty$, then there exists a family of pairwise disjoint densities $u_1, u_2, \cdots, u_n$ such that $\text{Fix}(\Theta) = \text{span}\{u_1, u_2, \cdots, u_n\}$. Denote the element $g := u_1 + \cdots + u_n$ and taking an element from $B_E \cap E_+$, then $Pf := \lim T_\lambda f$ is a linear combination of $u_1, \cdots, u_n$ say $Pf = \sum_{i=1}^n \alpha_i u_i \leq \sum_{i=1}^n u_i$. Thus

$$\limsup_{\lambda \to \infty} \|(T_\lambda f - g)_+\| = \|(Pf - g)_+\| = 0$$

for every $f \in B_E \cap E_+$.

In the above two theorems, KB-space conditions cannot be omitted. Even for Banach lattices with order continuous norm, this result can be false, to proof see in [8].

5. ASYMPTOTIC STABILITY OF LR-NETS

The asymptotic stability of positive operators and lower bound technique is developed and rich in applications part of Markov operators. In this section we prove the following theorem as a corollary of Theorem 4.3. Theorem 5.3 is the generalisation of Theorem 4 in [3]. Emelyanov and Erkursun proved asymptotic stability and existence of lower bound function are equivalent for Markov LR-nets on $L^1$-spaces. In this section we will have a KB-space as well.

In the first, we give the following two definitions which are motivated for operator nets by the definitions used in [11].

Definition 5.1. Let $\Theta$ be a positive LR-net on KB-spaces. $\Theta$ is called asymptotically stable whenever there exists an element $u \in E_+ \cap U_E$ where $U_E := \{x \in E : \|x\| = 1\}$ such that

$$\lim_{\lambda \to \infty} \|T_\lambda x - u\| =$$

for every element from $E_+ \cap U_E$.

Definition 5.2. An element $h \in E_+$ is called lower bound element for $\Theta$ if

$$\lim_{\lambda \to \infty} \|(h - T_\lambda x)_+\| = 0$$

for every element $x \in E_+ \cap U_E$

The main result of this section is the following theorem.

Theorem 5.3. Let $\Theta$ be a positive LR-net on KB-spaces. Then the following are equivalent:

i $\Theta$ is asymptotically stable

ii There exists a nontrivial lower-bound element for $\Theta$
Proof:
(ii) $\Rightarrow$ (i): Let $h$ be a lower bound element of $\Theta$. Then $\limsup_{\lambda \to \infty} \| (T_\lambda f - h)_+ \| \leq \eta$ where $\eta := 1 - \|h\|$ for each $f \in B_E \cap U_E$. It implies that $\limsup_{\lambda \to \infty} \text{dist}(T_\lambda f, [0, h]) \leq \eta$. By Theorem 3.3 $\Theta$ converges strongly to the finite dimensional fixed space of $\Theta$.

Therefore by Theorem 2.2
\[ E = \text{Fix}(\Theta) \oplus (I - T_\lambda)E \]

In addition by Theorem 4.1 $\text{Fix}(\Theta)$ is a sublattice of $E$ and by Judin’s Theorem, it possesses a linear basis $(u_i)_{i=1}^n$ where $n = \text{dim} \text{Fix}(\Theta)$ which consists of pairwise disjoint element with $\|u_i\| = 1$, $i = 1, \cdots, n$. Since $T_\lambda u_i = u_i$ for each $\lambda \in \Lambda$ and $i = 1, \cdots, n$,
\[ \|(h - u_i)_+\| = \|(h - T_\lambda u_i)_+\| = \lim_{\lambda \to \infty} \|(h - T_\lambda u_i)_+\| = 0 \]

implies
\[ (5.1) \quad u_i \geq h \geq 0 \quad i = 1, \cdots, n. \]

Since $(u_i)_{i=1}^n$ pairwise disjoint with $\|u_i\| = 1$ the condition 5.1 ensure that $\text{dim} \text{Fix}(\Theta) = 1$. Therefore $E = \mathbb{R}u_1 \oplus (I - T_\lambda)E$ and for every element $f \in E_+ \cap B_E$, $\lim_{\lambda \to \Lambda} T_\lambda f = u_1$.

The next simple propositions give us firstly given on $L^1$-spaces in [5] which are the examples of Markov LR-nets which need not to be $T$-ergodic nets. Now we will prove this results on KB-spaces. Before them, we need technical lemma for proving of propositions. The technical lemma connects norm convergence of order bounded nets in KB-spaces with convergence in $(E, x')$ for suitable linear forms $x' \in E'$. Recall that $x' \in E'$ is strictly positive if $(x, x') > 0$ for all $x \in E_+ \setminus \{0\}$. We refer to [14] for proof of the lemma.

**Lemma 5.4.** Let $(x_\lambda)_{\lambda \in \Lambda}$ be an order bounded net in a KB-space and let $x' \in E'$ be strictly positive. Then $\lim_{\lambda \to \infty} \|x_\lambda\| = 0$ if and only if $\lim_{\lambda \to \Lambda} \langle x_\lambda, x' \rangle = 0$.

**Proposition 5.5.** Every asymptotically Markov net on KB-space is an LR-net.

**Proof:** Since any Markov operator on KB-space is a positive contraction, a Markov net is uniformly bounded. We need to check (LR2) and (LR3) conditions of Definition 2.4. Without loss of generality, taken for an arbitrary element $x$ from $E_+ \cap U_E$. For fix $\mu \in \Lambda$, since $T_\mu$ is Markov, then there exits $e' > 0$ such that $T_\mu e' = e'$. By Lemma 5.3 we know that $\|T_\mu f - u\| \to 0$ if and only if $\lim_{\lambda \to \Lambda} \langle e', T_\lambda f - u \rangle$. Since $T_\mu$ is a Markov operator,

\[ \langle (I - T_\mu)f, e' \rangle - \langle (I - T_\mu)f, e' \rangle = \langle (I - T_\mu)f, e' \rangle - \langle f, (I - T_\mu)e' \rangle = 0. \]

Therefore $\langle (I - T_\mu)f, e' \rangle = \langle (I - T_\mu)f, e' \rangle$ (*). Since for each $\mu$, there exists $e'$ such that (*) holds, then $\|T_\mu f - u\| = \|T_\mu f - u\|$. Now for (LR2):

\[ \langle T_\lambda(I - T_\mu)f, e' \rangle = \langle T_\lambda((I - T_\mu)f)_+, e' \rangle - \langle T_\lambda((I - T_\mu)f)_-, e' \rangle \]
\[ = \langle T_\lambda g_+, e' \rangle - \langle T_\lambda g_-, e' \rangle \]
\[ = \langle T_\lambda g_+, e' \rangle - \langle T_\lambda g_-, e' \rangle \]
\[ \leq \langle T_\lambda g_+ - \|g_+\| u, e' \rangle + \langle T_\lambda g_- - \|g_-\| u, e' \rangle + \langle \|g_+\| - \|g_-\| u, e' \rangle \]
\[ = \langle T_\lambda g_+ - \|g_+\| u, e' \rangle + \langle T_\lambda g_- - \|g_-\| u, e' \rangle + \langle \|g_+\| - \|g_-\| u, e' \rangle \]
which converge to zero as $\lambda \to \infty$. Therefore $\lim_{\lambda \to \infty} T_{\lambda}(I - T_{\mu})f = 0$.

For (LR) condition,
\[
\langle (I - T_{\mu})T_{\lambda}f, e' \rangle = \langle T_{\lambda}f, e' \rangle - \langle T_{\mu}T_{\lambda}f, e' \rangle = \langle T_{\lambda}f - u, e' \rangle - \langle T_{\lambda}f - u, e' \rangle - \langle u, T_{\mu}e' \rangle \\
\to 0 \quad \text{as } \lambda \to \infty
\]
Hence by Lemma 5.4, the condition (LR3) is hold.

The Lasota’s criterion of asymptotic stability says that a one-parameter Markov semigroup if and only if there is a nontrivial lower-bound function. In [5] Lasota’s lower-bound criteria is generalized on $L^1$-spaces to abelian Markov semigroups. In this proposition we generalize it on KB-spaces. An abelian Markov semigroup is an operator net with respect to the natural partial order mentioned Section 2.

Proposition 5.6. Let $\mathcal{S} = (\mathcal{S}_t)_{t}$ be an abelian Markov semigroup on KB-spaces possessing a nontrivial lower-bound function, then $\mathcal{S}$ is an LR-net.

Proof: The argument is the same as in [5]. The tricky is that since $T_t$ is Markov then $\langle (I - T_{t'})f_-, e' \rangle = \langle (I - T_{t'})f_-, e' \rangle$ for each $t'$. Thus we prove the following formula.

\[
\lim_{t \to \infty} \|T_t(I - T_{t'})f\| = 0 \quad (\forall t', f \in E)
\]

Take any element $f \in B_E$, then we know that $T_{t'}$ is Markov and then
\[
\langle (I - T_{t'})f_-, e' \rangle = \langle (I - T_{t'})f_-, e' \rangle
\]

Therefore by 5.2 we have to prove that $\lim t \to \infty \|T_tf\| = 0$ for every $f \in B_E$ such that $E_0 := \{f \in E : \|f_+\| = \|f_-\| \}$.

Take any element $f \in E_0$ such that $f = 2^{-1} \|f\| (f_1 - f_2)$ where $f_1 = 2 \|f\|^{-1} f_+$ and $f_2 = 2 \|f\|^{-1} f_-$. Hence $f_1$ and $f_2$ are elements of $E_+ \cap U_E$.

Since $h$ is the lower-bound element for the Markov semigroup $\mathcal{S}$, there exists $t_1$ such that $\|(h - T_{t_1}f_1)_+\| \leq \frac{1}{4} \|h\|$ and $\|(h - T_{t_1}f_2)_+\| \leq \frac{1}{4} \|h\|$ hold for every $t \geq t_1$. From Riesz space properties, we obtain $\|T_{t_1}f_1 - T_{t_1}f_2\| \leq 2 - \frac{1}{2} \|h\|$ and $\|T_{t_1}f\| \leq (1 - \frac{1}{4} \|h\|) \|f\|$ for every $t \geq t_1$.

Replacing $f$ with $T_{t_1}f$ which is also an element of $E_0$ and repeating the argument above gives an element $t_2$ such that
\[
\|T_{t_1}f\| \leq (1 - \frac{1}{4} \|h\|) \|T_{t_1}f\| \quad \forall t \geq t_2
\]
By induction, we can generate a sequence $(t_n)$ such that
\[
\|T_{t_n}f\| \leq \|T_{t_{n-1}}f\| \leq \cdots \leq \|T_{t_{n-2}}f\| \leq \|T_{t_{n-3}}f\| \leq \cdots \leq \|T_{t_2}f\| \leq \|T_{t_1}f\| \leq (1 - \frac{1}{4} \|h\|) \|T_{t_1}f\| \leq \cdots \leq \|f\| \quad (\forall t \geq t_1 + \cdots + t_n)
\]
Since $\|h\| > 0$, then $\lim_{t \to \infty} \|T_tf\| = 0$ and hence the proof is completed.

\begin{theorem}
Let $S = (S_t)_t$ be an abelian Markov semigroup on KB-spaces $E$. Then the following are equivalent:
\begin{enumerate}[i]
\item $\Theta$ is asymptotically stable
\item There exists a nontrivial lower-bound element for $\Theta$
\end{enumerate}
\end{theorem}

\begin{proof}
Since the asymptotic stable Markov semigroup $S$ is the LR-net by 5.6, the existence of nontrivial lower-bound element for $S$ follows from Theorem 5.3. In addition, the existence of nontrivial lower bound element for $S$ gives us that $S$ is an LR-net by Proposition 5.5 and the asymptotic stability of $S$ follows from Theorem 5.3.
\end{proof}

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