On multi-symmetric functions and transportation polytopes

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Abstract

We present a study of the transportation polytopes appearing in the product rule of elementary multi-symmetric functions introduced by F. Vaccarino.

Introduction

The classical transportation problems in operation research arise from the problem of transporting goods from a set of factories, and a set of consumer centers. Assuming the total supply of the set of factories equals to the total demand of consumer centers, we can optimize the cost of transporting goods (see [4, 6, 7]). Transportation polytopes have an interest in discrete mathematics and also arise naturally in optimization and statistics (see [5, 9, 15, 17]).

A transportation polytope consists of all tables of non-negative real numbers that satisfy certain equations. In this work we only consider the well-known subfamily, the classical transportation polytopes in just two indices, the 2-way transportation polytopes and we use the notation and terminology introduced by Jesus A. De Loera and Edward D. Kim in [2].

Our main motivation comes from the study of the product rule of elementary multi-symmetric functions introduced by F. Vaccarino in [14] and their relationships with transportation polytopes. The classic product rule of multi-symmetric functions and its respective generalization to the quantum case introduced by Diaz and Pariguan in [3], both have an unexplored underlying structure of transportation polytopes. The main goal of this work, see section 3, is to present a first combinatorial description of this structure in the classical case.
1 Review of multi-symmetric functions

In this section, we present a short introduction to elementary multi-symmetric functions. Fix a characteristic zero field $\mathbb{K}$. Consider the action of the symmetric group $S_n$ on $\mathbb{K}^n$ by permutation of vector entries. The quotient space $\mathbb{K}^n/S_n$ is the configuration space of $n$-unlabeled points with repetitions in $\mathbb{K}$. Polynomials functions on $\mathbb{K}^n/S_n$ may be identified with the algebra $\mathbb{K}[x_1, \ldots, x_n]^{S_n}$ of $S_n$ invariant polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. It is well-known that $\mathbb{K}^n/S_n$ is an $n$-dimensional affine space; indeed we have an isomorphism of algebras

$$\mathbb{K}[x_1, \ldots, x_n]^{S_n} \equiv \mathbb{K}[e_1, \ldots, e_n],$$

where $\alpha \in [n] = \{1, 2, \ldots, n\}$ and $e_\alpha$ is the elementary symmetric polynomial determined by the identity

$$\prod_{i=1}^{n}(1 + x_i t) = \sum_{\alpha=0}^{n} e_\alpha(x_1, \ldots, x_n) t^\alpha.$$

If we consider polynomial functions over $(\mathbb{K}^d)^n/S_n$, we obtain the ring of multi-symmetric functions, also called the ring of vector symmetric functions or MacMahon’s symmetric functions [8], which are given by

$$\mathbb{K}[x_{11}, \ldots, x_{1d}, x_{22}, \ldots, x_{2d}, \ldots, x_{nd}]^{S_n}.$$

We will denote by $\mathbb{K}[(\mathbb{K}^d)^n]^{S_n}$ to the ring $\mathbb{K}[x_{11}, \ldots, x_{nd}]^{S_n}$. The following results due to F. Vaccarino.

Fix $p, n, d \in \mathbb{N}^+$. Let $y_1, \ldots, y_d$ and $t_1, \ldots, t_d$ be independent and commutative variables in $\mathbb{K}$. For $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p$ we use the following notation

$$|\alpha| = \sum_{i=1}^{p} \alpha_i, \quad t^\alpha = \prod_{i=1}^{p} t_1^{\alpha_i}.$$

Given a polynomial $f \in \mathbb{K}[y_1, \ldots, y_d]$ and $i \in [n]$, we denote by $f(i) = f(x_{i1}, \ldots, x_{id})$ to the polynomial obtained by replacing each appearance of $y_j$ in $f$ by $x_{ij}$, for $j \in [d]$.

**Definition 1.** Fix $\alpha \in \mathbb{N}^p$ such that $|\alpha| \leq n$ and $f = (f_1, \ldots, f_p) \in \mathbb{K}[y_1, \ldots, y_d]^p$. The multisymmetric functions $e_\alpha(f) \in \mathbb{K}[(\mathbb{K}^d)^n]^{S_n}$, are given by the identity

$$\prod_{i=1}^{n}(1 + f_1(i)t_1 + f_2(i)t_2 + \cdots + f_p(i)t_p) = \sum_{|\alpha| \leq n} e_\alpha(f) t^\alpha.$$

The following result provide an explicit formula for the product rule of multi-symmetric functions
Theorem 2. Fix $p, q, n \in \mathbb{N}^+$, $f \in \mathbb{K}[y_1, \ldots, y_d]^p$ and $g \in \mathbb{K}[y_1, \ldots, y_d]^q$. Let $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$ be such that $|\alpha|, |\beta| \leq n$, then we have

$$e_\alpha(f)e_\beta(g) = \sum_{\gamma \in L(\alpha, \beta, n)} e_\gamma(f, g, fg),$$

where:

1. $(f, g, fg) = (f_1, \ldots, f_p, g_1, \ldots, g_q, f_1g_1, \ldots, f_1g_q, f_2g_1, \ldots, f_2g_q, \ldots, f_pg_1, \ldots, f_pg_q)$.

2. $L(\alpha, \beta, n)$ is the set of matrices $\gamma \in \text{Map}([0] \cup [p] \times [0] \cup [q], \mathbb{N})$ such that
   - $\gamma_{00} = 0$,
   - $|\gamma| = \sum_{i=0}^p \sum_{j=0}^q \gamma_{ij} \leq n$,
   - $\sum_{j=1}^q \gamma_{ij} = \alpha_i$ for $i \in [p]$,
   - $\sum_{i=1}^p \gamma_{ij} = \beta_j$ for $j \in [q]$.

Graphically, a matrix $\gamma$ is represented as

$$\begin{array}{cccc}
\beta_1 & \beta_2 & \beta_3 & \beta_q \\
\uparrow & \uparrow & \uparrow & \cdots & \uparrow \\
0 & \gamma_{01} & \gamma_{02} & \gamma_{03} & \cdots & \gamma_{0q} \\
\gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1q} & \rightarrow & \alpha_1 \\
\gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2q} & \rightarrow & \alpha_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_{p0} & \gamma_{p1} & \gamma_{p2} & \gamma_{p3} & \cdots & \gamma_{pq} & \rightarrow & \alpha_p \\
\end{array}$$

where the arrows $\rightarrow\uparrow$ represent, respectively, row and column sums and the matrix $\gamma$ will be identify with the vector

$$\vec{\gamma} = (\gamma_{10}, \cdots, \gamma_{p0}, \gamma_{11}, \cdots, \gamma_{pq}, \gamma_{1q}, \cdots, \gamma_{2q}, \cdots, \gamma_{p1}, \cdots, \gamma_{pq}).$$

The main goal of this work is the study of the combinatorial structure underlying in the set of matrices $L(\alpha, \beta, n)$ introduced in Theorem 2.

Example 3. For $n = 3$, $\alpha = (2, 1)$, $\beta = (1, 2)$, $f = (y_1, y_2)$ and $g = (y_1y_3, y_2)$, we have the following identity
\[ e_{(2,1)}(y_1, y_2)e_{(1,2)}(y_1y_2, y_3) = \sum_{\gamma} e_{\gamma}(y_1, y_2, y_1y_2, y_3, y_1y_2y_3, y_1y_2, y_2^2) \]

where \( \gamma = (\gamma_{10}, \gamma_{20}, \gamma_{01}, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbb{N}^8 \) is such that \( |\gamma| \leq 3 \) and:

\[
\begin{align*}
\gamma_{10} + \gamma_{11} + \gamma_{12} &= 2, \\
\gamma_{20} + \gamma_{21} + \gamma_{22} &= 1, \\
\gamma_{01} + \gamma_{11} + \gamma_{21} &= 1, \\
\gamma_{20} + \gamma_{12} + \gamma_{22} &= 2.
\end{align*}
\]

Finding the solutions we obtain the vectors

\( (0, 0, 0, 0, 1, 1, 0, 1), (0, 0, 0, 0, 2, 1, 0) \)

then we have that

\[
e_{(2,1)}(y_1, y_2)e_{(1,2)}(y_1y_2, y_3) = e_{(1,1,1)}(y_1^2y_3, y_1y_2y_3, y_2^2) + e_{(2,1)}(y_1^2y_3, y_1y_2y_3, y_2^2).
\]

## 2 Classical transportation polytopes

In this section, we review a few needed notions on classical 2-way transportation polytopes and we assume the reader to be somewhat familiar with De Loera and Kim’s work \[2\].

**Definition 4.** Fix \( p, q \in \mathbb{N} \) and let \( u \in \mathbb{R}^p_{\geq 0}, v \in \mathbb{R}^q_{\geq 0} \) be two vectors. The transportation polytope \( P \) of size \( p \times q \) defined by the vectors \( u \) and \( v \) is the convex polytope on \( p \times q \) variables \( x_{ij} \in \mathbb{R}_{\geq 0} \), where \( i \in [p] \) and \( j \in [q] \), which satisfy the \( p + q \) equations given by:

\[
\sum_{j=1}^{q} x_{ij} = u_i \text{ and } \sum_{i=1}^{p} x_{ij} = v_j. \tag{1}
\]

The vectors \( u \) and \( v \) are called marginals vectors or margins vectors of the polytope \( P \).

These polytopes are called transportation polytopes because they model the transportation of goods from \( p \) supply locations to \( q \) demand locations.

**Example 5.** Let us consider the transportation of goods for 3-supply locations to 3-demand location with supplying vector \( u = (5, 4, 3) \) and demanding vector \( v = (6, 2, 4) \). A point \( x \) in the transportation polytope \( P \) of size \( 3 \times 3 \) defined by the margins \( u \) and \( v \) is given by

\[
x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}
\]

where the horizontal and vertical arrows represent, respectively, row and column sums.
Lemma 6. Let $P$ be a 2-way transportation polytope of size $p \times q$ defined by the margins $u \in \mathbb{R}^p_{\geq 0}$ and $v \in \mathbb{R}^q_{\geq 0}$. The polytope $P$ is not empty if and only if

$$\sum_{i \in [p]} u_i = \sum_{j \in [q]} v_j.$$ 

This proof uses the northwest corner rule algorithm (see [10]).

The equations given in (1) and the inequalities $x_{ij} \geq 0$ can be expressed in matrix form as follows

$$P = \{ x \in \mathbb{R}^{pq} : Ax = b, x \geq 0 \},$$

where $A$ is a matrix of size $(p + q) \times pq$ and $b \in \mathbb{R}^{p+q}$. The matrix $A$ is called the constraint matrix.

Transportation polytopes have a relationship with complete bipartite graph $K_{p,q}$ ([11, 16]) of two sets of vertices of $U$ and $V$ of cardinality $p$ and $q$, respectively, when we consider $U$ the supply and $V$ is the demand.

**Definition 7.** The graph $K_{p,q}$ is the complete bipartite graph consisting of two sets $U$ and $V$ of cardinality $p$ and $q$, respectively such that for any $i \in U$ and $j \in V$ there is an edge $e_{ij}$ connecting them.

It is well known that the constraint matrix for a $p \times q$ transportation polytope is the vertex-edge incidence matrix of the complete bipartite graph $K_{p,q}$.

**Example 8.** Consider the $3 \times 3$ transportation polytope $P$ defined by $u = (5, 4, 3)$ and $v = (6, 2, 4)$, then the complete bipartite graph $K_{3,3}$ is given by:

We also have that $P = \{ x \in \mathbb{R}^9 : Ax = b, x \geq 0 \}$, where the constraint matrix $A$ is given as follows

$$A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
5 \\
4 \\
3 \\
6 \\
2 \\
4
\end{bmatrix}.$$

In the Example the solution of $Ax = b$ can be expressed as $x^t = (4, 1, 1, 0, 2, 0, 1, 1, 2)$.
3 Multi-symmetric functions and transportation polytopes

The product rule of elementary multi-symmetric functions given in Theorem 2 involve a set of matrices with some remarkable properties. In this section we will provide some characterizations of the set $L(\alpha, \beta, n)$ in terms of transportation polytopes. In order to simplify our notation we will denote by $L$ to the set $L(\alpha, \beta, n)$ (see Definition 10) and we can think of $\gamma \in L$ as natural points of transportation polytopes $P$.

In particular, the study of integer points of transportation polytopes is very popular in combinatorics, a lot of mathematical objects rich in combinatorial properties appear when we study integer points in polytopes such as magic squares [1], sudoku arrangements [13], and others.

Definition 9. Fix $p, q, N \in \mathbb{N}$ and let $u \in \mathbb{N}^{p+1}$, $v \in \mathbb{N}^{q+1}$ be two vectors such that $u_0 = N - \sum_{i=1}^{p} u_i$ and $v_0 = N - \sum_{i=1}^{q} v_i$. The transportation polytope $P_N$ of size $p+1 \times q+1$ defined by the vectors $u$ and $v$ is the convex polytope on $p+1 \times q+1$ variables $x_{ij} \in \mathbb{R}_{\geq 0}$, where $i \in \{0\} \cup [p]$ and $j \in \{0\} \cup [q]$, which satisfy the $p+q+2$ equations given by:

$$\sum_{j=0}^{q} x_{ij} = u_i \quad \text{and} \quad \sum_{i=0}^{p} x_{ij} = v_j.$$

(3)

Definition 10. Fix $p, q, n \in \mathbb{N}$, $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$. We denote by $L$ the set of matrices $\gamma \in \operatorname{Map}(\{0\} \cup [p] \times \{0\} \cup [q], \mathbb{N})$ which satisfy the equations

- $\gamma_{00} = 0$.

- $|\gamma| = \sum_{i=0}^{p} \sum_{j=0}^{q} \gamma_{ij} \leq n$.

- $\sum_{j=1}^{q} \gamma_{ij} = \alpha_i$ for $i \in [p]$.

- $\sum_{i=1}^{p} \gamma_{ij} = \beta_j$ for $j \in [q]$.

Example 11. For $\alpha = (2, 1)$, $\beta = (1, 2)$ and $n = 3$, the set $L$ is given by:

$$L = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$
We denote by $L_N$ the subset of $L$ given by:

$$L_N = \{\gamma \in L : |\gamma_{ij}| = N, \text{ for some } N \leq n\} \quad (4)$$

The following result provides some combinatorial properties of $L_N$.

**Theorem 12.** The following identities holds

1. $L_N \neq \emptyset$ if $\max\{|\alpha|, |\beta|\} \leq N \leq |\alpha| + |\beta|$.

2. $L = \bigsqcup_{N=\max\{|\alpha|, |\beta|\}}^{n} L_N$, if $n < |\alpha| + |\beta|$.

3. $L = \bigsqcup_{N=\max\{|\alpha|, |\beta|\}}^{|\alpha| + |\beta|} L_N$, if $n \geq |\alpha| + |\beta|$.

**Proof.** Fix $N$ such that $\max\{|\alpha|, |\beta|\} \leq N \leq |\alpha| + |\beta|$. We are going to construct an element $\gamma$ such that $\gamma \in L_N$ as follows: Let $\gamma \in L$ such that $(\gamma_{01}, \gamma_{02}, \ldots, \gamma_{0q})$ be a $q$-weak composition of $\alpha_0$ which satisfy $\gamma_{0j} \leq \beta_j \forall j \in [q]$, and let $(\gamma_{10}, \gamma_{20}, \ldots, \gamma_{p0})$ be a $p$-weak composition of $\beta_0$ which satisfy $\gamma_{i0} \leq \alpha_i \forall i \in [p]$.

Denote by $\beta_j^{(k)} := \beta_j - \sum_{i=1}^{k-1} \gamma_{ij}$, for $(k, j) \in [p] \times [q]$ and let $(\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1q})$ be a $q$-weak composition of $\alpha_1 - \gamma_{10}$ which satisfy $\gamma_{1j} \leq \beta_j^{(1)}$. Analogously we consider $(\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2q})$ a $q$-weak composition of $\alpha_2 - \gamma_{20}$ such that $\gamma_{2j} \leq \beta_j^{(2)}$. Let’s go through this process until we get $(\gamma_{p1}, \gamma_{p2}, \ldots, \gamma_{pq})$ a $q$-weak composition of $\alpha_p - \gamma_{p0}$ with $\gamma_{pj} \leq \beta_j^{(p)}$ and finally under this construction the reader can check that $\gamma_{ij} = \gamma \in L_N$, therefore $L_N \neq \emptyset$.

It is not difficult to check statements 2 and 3. \qed

**Example 13.** The set $L$ defined by vectors $\alpha = (1, 1)$, $\beta = (2, 1)$ and $n = 4$ is given by

$$L = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$ 

We have that $L = \bigsqcup_{N=3}^{4} L_N$, where $L_3$ and $L_4$ are given by

$$L_3 = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

$$L_4 = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$
and

\[
L_4 = \left\{ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.
\]

The following result shows that \( L_N \) is a set of natural points in some transportation polytope.

**Theorem 14.** There is a transportation polytope \( P_M \) such that \( L_N \subset P_M \).

**Proof.** Let \( \gamma \in L_N \), then \( \gamma \) satisfy the equations given in Definition 10. Under the assumptions of Definition 9, consider the transportation polytope \( P_M \) defined by margins \( \alpha \in \mathbb{N}^{p+1} \) and \( \beta \in \mathbb{N}^{q+1} \) such that

- \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p) \) with \( \alpha_0 = M - \sum_{i=1}^{p} \alpha_i \).
- \( \beta = (\beta_0, \beta_1, \ldots, \beta_q) \) with \( \beta_0 = M - \sum_{i=1}^{q} \beta_i \).

It should be clear that \( \gamma \in P_M \) if \( M = N \). \( \square \)

We make a few remarks regarding to Theorem 14. Elements \( \gamma \in L_N \) are such that \(|\gamma| = N\) and \( \gamma \in L(\alpha, \beta, n) = L \), hence for \( p, q, n \in \mathbb{N}^+ \), \( \alpha \in \mathbb{N}^p \) and \( \beta \in \mathbb{N}^q \), \( \gamma \) satisfy the conditions of Theorem 2. To find the transportation polytope \( P_M \) such that \( L_N \subset P_M \), we consider the transportation polytope defined by margins \( \overline{\alpha}, \overline{\beta} \) which are obtained from \( \alpha \) and \( \beta \) adding new inputs \( \alpha_0, \beta_0 \) satisfying the condition given above. We stress that we will work with the transportation polytope \( P_N \) which follows this previous construction. This previous considerations imply our next result which establishes an example of the transportation polytopes associated with sets \( L_3 \) and \( L_4 \) given in Example 13.

**Example 15.** Fix \( p = q = 2 \), \( N = 3 \) and consider the vectors \( \overline{\alpha} = (1, 1, 1), \overline{\beta} = (0, 2, 1) \). The transportation polytope \( P_3 \) defined by margins \( \overline{\alpha}, \overline{\beta} \) is given by

\[
P_3 = \left\{ X \in M_{3 \times 3}(\mathbb{R}_{\geq 0}) : \sum_{j=1}^{3} x_{ij} = \overline{\alpha}_i \text{ and } \sum_{i=1}^{3} x_{ij} = \overline{\beta}_j \right\},
\]

and we have \( L_3 \subset P_3 \). If we consider \( X = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \) we have that \( X \in P_3 \) but \( x \notin L_3 \) and therefore \( L_3 \neq P_3 \).
On the other hand, fix \( p = q = 2 \), \( N = 4 \) and consider the vectors \( \overline{\alpha} = (2, 1, 1), \overline{\beta} = (1, 2, 1) \). The transportation polytope \( P_4 \) defined by margins \( \overline{\alpha}, \overline{\beta} \) is given by

\[
P_4 = \left\{ X \in M_{3 \times 3}(\mathbb{R}_{\geq 0}) : \sum_{j=1}^{3} x_{ij} = \overline{\alpha}_i \text{ and } \sum_{i=1}^{3} x_{ij} = \overline{\beta}_j \right\},
\]

and we have \( L_4 \subseteq P_4 \). If we consider \( X = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \) we have that \( X \in P_4 \) but \( X \notin L_4 \) and therefore \( L_4 \neq P_4 \).

It is well known that transportation polytopes \( P \) can be represented in matrix form, therefore transportation polytopes \( P_N \) can be represented in matrix form as well (see Proposition 16). In this case we consider the graph \( K'_{p,q} \) obtained from \( K_{p,q} \) removing the edge \( e_{11} \). Figure 1 shows the graph \( K'_{3,3} \) associated to \( K_{3,3} \).

![Figure 1: \( K'_{3,3} \) graph.](image)

The following result provides the matrix form associated to \( L_N \).

** Proposition 16.** For any \( N \in \mathbb{N} \), each \( L_N \) can be expressed as follows:

\[
L_N = \left\{ x_N \in \mathbb{N}^{(p+1)(q+1) - 1} : Ax_N = b_N \right\},
\]

where \( b_N = (\alpha_0, \alpha_1, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_q) \) is such that \( \alpha_0 = N - \sum_{i=1}^{p} \alpha_i, \beta_0 = N - \sum_{i=1}^{q} \beta_i \), and \( A \) is the matrix is obtain by following the next construction

1. Let \( B \) be the constraint matrix of \( K'_{p+1,q+1} \), and denote by \( B^i \) the \( i \)-th column of \( B \), for all \( i \).
2. For \( i \in [p] \) the \( i \)-th column \( A^i \) of matrix \( A \) is given by \( A^i = B^{i(q+1)} \).
3. For \( i \in [q] \) we have \( A^{p+i} = B^i \).
4. Last columns of \( A \) are obtained from \( B \) after rearranging in ascended way the remaining columns.
Proof. Let $P_N$ be the transportation polytope such that $L_N \subset P_N$. It should be clear that $P_N$ is an special case of 2-way transportation polytope for any $N \in \mathbb{N}$. Observe that for $\gamma \in P_N$ we have $\gamma_{00} = 0$, then $P_N$ can be expressed in matrix form as follows (see equation (2))

$$P_N = \{x_N \in \mathbb{R}_{\geq 0}^{(p+1)(q+1)-1} : Bx_N = b_N\},$$

where $B$ is the constraint matrix of the graph $K_{p+1,q+1}^p$.

The matrix $A$ obtained from $B$ following the previous construction provides a rearrangement of $x_N$ such that solutions of the equation $Ax_N = b_N$ are vectors $\vec{\gamma}$ which satisfy the conditions of Theorem [2], therefore we have the desired result.

Example 17. For $N = 3$ and $b_N = (1, 1, 1, 0, 2, 1)$, we have that

$$L_3 = \{x_3 \in \mathbb{N}^8 : Ax_N = b_3\},$$

where $A$ is given as follows:

Let $B$ be the constraint matrix of $K_{3,3}^2$ given by:

$$B = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}.$$

Under the assumptions of Proposition [16], for $p = q = 2$, we have that

- $A^1 = B^3$ and $A^2 = B^6$,
- $A^3 = B^1$ and $A^4 = B^2$,
- The last four columns of $A$ are given by $A^5 = B^4$, $A^6 = B^5$, $A^7 = B^7$ and $A^8 = B^8$.

Then we have

$$B = \begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix} \rightarrow A = \begin{bmatrix}
B_3 & B_6 & B_1 & B_2 & B_4 & B_5 & B_7 & B_8 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}.$$
Our next goal is to describe the structure of \( N \)-matrix of the elements of \( L_N \). To accomplish it, we will require some definitions due to R. Stanley (see [12]). Let \( A = (a_{ij}) \) be an \( N \)-matrix with finitely many nonzero entries, that is \( A \) is an \( N \)-matrix of finite support and we can think of \( A \) as either an infinity matrix or as an \( m \times n \) matrix when \( a_{ij} = 0 \) for \( i > m \) and \( j > n \). Associate with \( A \) a generalized permutation or two-line array \( \omega_A \) given by

\[
\omega_A = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_m \\ j_1 & j_2 & j_3 & \cdots & j_m \end{pmatrix}
\]

such that 1. \( i_1 \leq i_2 \leq \cdots \leq i_m \), 2. if \( i_r = i_s \) and \( r \leq s \) then \( j_r \leq j_s \), and 3. for each pair \( (i, j) \), there are exactly \( a_{ij} \) values of \( r \) for which \( (i_r, j_r) = (i, j) \). \( A \) determines a unique two-line array \( \omega_A \) satisfying this conditions and conversely any such array corresponds to a unique \( A \). For instance, if \( A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \), then the corresponding two-line array is \( \omega_A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} \).

**Definition 18.** Fix \( A \) an \( N \)-matrix and let \( \omega_A \) be the two-line array associate with \( A \). We denote by type\(^1\)(\( \omega_A \)) the vector \((u_1, \cdots, u_m)\) such that the natural number \( k \) appears exactly \( u_k \) times in the first row of \( \omega_A \) and we denote by type\(^2\)(\( \omega_A \)) the vector \((v_1, \cdots, v_m)\) such that the natural number \( k \) appears exactly \( v_k \) times in the second row of \( \omega_A \).

**Example 19.** If \( \omega_A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} \), then \( \text{type}^1(\omega_A) = (1, 1, 1) \) and \( \text{type}^2(\omega_A) = (0, 2, 1) \).

Fix \( N \in \mathbb{N} \), we denote by \( \omega_N \) the set of two-line array given by

\[
\omega_N = \left\{ \omega_A = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_N \\ j_1 & j_2 & j_3 & \cdots & j_N \end{pmatrix} : (i_1, j_1) \neq (1, 1) \right\}
\]

**Theorem 20.** There is a bijection between elements of \( L_N \) and elements \( \omega_A \in \omega_N \) such that \( \text{type}^1(\omega_A) = \overline{\alpha} \) and \( \text{type}^2(\omega_A) = \overline{\beta} \).

**Proof.** Let \( \gamma \in L_N \). Using Stanley’s construction, we can think of \( \gamma \in L_N \) as an \((p+1) \times (q+1)\) matrix when \( \gamma_{ij} = 0 \) for \( i > p+1 \) and \( j > q+1 \), then \( \gamma \) determines a unique two-line array \( \omega_\gamma \) satisfying the previous conditions. It should be clear that there is a injective map \( L_N \to \omega_N \). On the other hand, note that since the elements \( \omega_A \in \omega_N \) are such that \( (i_1, j_1) \neq (1, 1) \), it follows that \( a_{11} = 0 \), moreover \( \text{type}^1(\omega_A) = \overline{\alpha} \) and \( \text{type}^2(\omega_A) = \overline{\beta} \), then \( A = (a_{ij}) \) is an element of \( L_N \). Thus we conclude that there is a injective map \( \omega_N \to L_N \).

**Example 21.** Under the assumptions of Example 13 consider \( L_4 \) given by

\[
11
\]
\[ L_4 = \left\{ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}. \]

The set \( \omega_4 \) is given by
\[ \omega_4 = \left\{ \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 2 & 1 \end{pmatrix} \right\}. \]

It is well known that we can associated with an \( \mathbb{N} \)-matrix \( A \) of finite support a pair \((P, Q)\) of semistandard Young tableau (SSYT) of the same shape using the RSK algorithm. The RSK algorithm is a bijection between \( \mathbb{N} \)-matrices of finite support and ordered pairs \((P, Q)\) of SSYTs of the same shape.

On the other hand, we know that any \( \gamma \in L_N \) is an \( \mathbb{N} \)-matrix of finite support such that row(\( \gamma \)) = \( \overline{\alpha} \) and col(\( \gamma \)) = \( \overline{\beta} \). Using Theorem 20 we can see that RSK algorithm is a bijection between elements \( \gamma \in L_N \) and ordered pairs \((P, Q)\) of SSYTs of the same shape such that type(\( P \)) = col(\( \gamma \)) = \( \overline{\beta} \), type(\( Q \)) = row(\( \gamma \)) = \( \overline{\alpha} \) and the first box of the last row of \( P \) and \( Q \) is not equal to 1 simultaneously. Therefore, we can summarize it as follows.

**Corollary 22.** There is a bijection between \( L_N \) and ordered pairs \((P, Q)\) of SSYTs of the same shape such that type(\( P \)) = col(\( \gamma \)) = \( \overline{\beta} \), type(\( Q \)) = row(\( \gamma \)) = \( \overline{\alpha} \) and the first box of the last row of \( P \) and \( Q \) is not equal to 1 simultaneously.

**Example 23.** Let \( \omega_\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} \) be the two-line array associated with \( \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \).

The ordered pairs \((P, Q)\) of SSYTs are the following
\[ \begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} \]

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