New critical point for QCD in a magnetic field

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We provide a general argument for the possible existence of a new critical point associated with a deconfinement phase transition in QCD at finite temperature \( T \) and in a magnetic field \( B \) with zero chemical potential. This is the first example of a QCD critical point in a physical external parameter region that can be studied using lattice QCD simulations without suffering from a sign problem.

PACS numbers: 12.38.-t, 21.65.Qr, 25.75.Nq

Much attention has been focused on the properties of QCD at high temperature \( T \) and/or baryon chemical potential \( \mu_B \) to understand various phenomena including heavy ion collisions, neutron stars, and the early Universe. Such systems are often subject to the effects of a strong magnetic field \( B \). Indeed, the strong magnetic field of order the QCD scale \( \Lambda_{\text{QCD}} \) (or \( 10^{18} \sim 10^{19} \) Gauss) may be reachable in noncentral heavy ion collision experiments at RHIC and LHC \(^1\) and possibly inside magnetars \(^2\). Moreover, the strong magnetic field may have existed in the early Universe as an origin of the present large-scale cosmic magnetic field \(^3\). It is thus important to unravel the possible modifications of the QCD phase diagram in the presence of a strong magnetic field.

Unlike QCD with nonzero \( \mu_B \), QCD with nonzero \( B \) does not have a sign problem. Thus the QCD phase diagram in the \((T, B)\) plane can be determined from first principles from Monte Carlo calculations of lattice QCD using existing techniques. For the chiral transition, recent lattice QCD results \(^4\) indicate a discrepancy with earlier model results \(^5\) for the qualitative behavior of the critical temperature as a function of \( B \), raising a question on the reliability of the conventional model analyses. A possible explanation of this discrepancy was proposed in Refs. \(^6\).

This paper concerns the confinement/deconfinement transition in the \((T, B)\) plane at zero chemical potential. Naively, one might think that, as the magnetic field does not couple to gluons directly, its effect on confinement will not be dramatic and might be very difficult to understand in a theoretically controlled manner (see Refs. \(^2\) \(^4\) for recent attempts). However the phase structure associated with the physics of confinement in the presence of the magnetic field is qualitatively novel, and, given a single plausible assumption, may be understood in a controlled way. Our main result of the phase diagram on the \((T, B)\) plane is summarized in Fig. 1. In particular, we argue that it is highly plausible that a new critical point for the deconfinement phase transition (denoted by \( P \) in Fig. 1) exists. This argument does not depend strongly on model-dependent analysis\(^7\). To distinguish it from a possible conventional QCD critical point at finite \( \mu_B \) \(^8\) (see also Ref. \(^9\))—which is thought to be associated with approximate chiral symmetry—we shall call this the “magnetic critical point.” For a possible conventional critical point in a magnetic field, see, e.g., Refs. \(^10\).

We note that the critical point under discussion has a fundamental virtue compared to the usual QCD critical point associated with chemical potential and temperature. The conventional critical point, if it exists, is a property of QCD itself. However, when electromagnetic effects are included the thermodynamics are fundamentally altered: one cannot have an infinitely extended charged phase due to energetics. In contrast the critical point here includes electromagnetic effects—indeed, it depends on them.

Throughout the paper, we assume the magnetic field to be homogeneous and in the \( \hat{z} \) direction with magnitude \( B \). For notational simplicity, we first put the current quark mass \( m_q \) to zero, but generalization to nonzero \( m_q \) is trivial, as we shall mention later. We also put chemical potential \( \mu_B \) to zero unless otherwise stated.

The argument for the existence of the first-order transition with \( T \) at large \( B \) is quite straightforward. The phenomenon of magnetic catalysis of chiral symmetry breaking, the strengthening of the chiral condensate in an external magnetic field, is well known at \( T \ll \sqrt{eB} \)

\(^1\) A possible magnetic critical point was also suggested in a model calculation \(^6\). However, in their phase diagram, the first-order deconfinement transition is turned into a crossover with increasing \( B \), which is different from ours.
Accompanying this phenomenon is an increase of the effective mass of quarks \[14\]. In the regime of very large magnetic field, the quarks become sufficiently heavy that they decouple from the gluodynamics, yielding an effective theory of pure gluodynamics—albeit of an anisotropic type. Thus, center symmetry is an emergent symmetry of QCD at large magnetic fields, since the effective theory is invariant under center transformations.

Consider this lowest-order effective theory as a theory in its own right and assume that center symmetry is unbroken as \( T \rightarrow 0 \) in this theory. It is known to be broken at high temperatures. Thus, at the level of the lowest-order effective theory, there must be a phase transition. Moreover, there is reason to believe that this transition is first order. Note that there is a general argument that conventional (i.e. isotropic) gluodynamics should not have a second-order phase transition \[16\]; lattice simulations confirm that pure gluodynamics does indeed have a first-order transition for the isotropic case \[17\].

Since the lowest-order effective theory is a pure glue theory, it is natural to expect that it has a first-order transition—if for no other reason than pure glue theory has a first-order transition in the isotropic case \[16\]. Moreover, first-order transitions are, by their nature, robust:

\[ M_{\text{dyn}} = C(\alpha_s) \sqrt{|e_q B|}, \]

where \( e_q \) are the charges of the quarks, \((e_u, e_d, \cdots) = (\frac{2}{3}, -\frac{1}{3}, \cdots) \ e\). The detailed expression for \( C(\alpha_s) \) based on the consistent truncation of the gap equation \[15\] is irrelevant for our purpose; what will be important for us is that \( M_{\text{dyn}} \) is an increasing function of \( B \) at sufficiently large \( B \) and \( M_{\text{dyn}} \rightarrow \infty \) for \( B \rightarrow \infty \). Note that \( \alpha_s \ll 1 \) is indeed valid for \( M^2_{\text{dyn}} \ll k^2 \ll eB \), where \( k \) is the typical momentum in the gap equation.

That quarks acquire a large mass gap in a strong magnetic field means that quarks decouple from the gluons at low energy scales—well below \( \sqrt{eB} \). Hence, the lowest-order effective theory for low-energy dynamics is described by an anisotropic pure SU(3) gauge theory with small changes in the details of a theory cannot destroy the transition due to the existence of a nonzero latent heat. In this respect first-order transitions are quite different from second-order ones. An arbitrarily small change in the details of the theory can completely eliminate a second-order transition by turning it into a crossover. Since the effective theory has a first-order transition, and at sufficiently large \( B \) is equivalent to the lowest-order effective theory up to small corrections, QCD too will have a first order transition. Given the existence of a first-order transition at large \( B \) and only crossover behavior at \( B = 0 \), there must be some minimum value of \( B \) for which the transition occurs. The most natural way for this to occur is simply via a critical point as in Fig. 1.

The simple argument given above depends on the behavior of QCD in the regime \( eB \gg \Lambda^2_{\text{QCD}} \). Here we briefly recapitulate the known physics in this regime using the analysis of Ref. \[15\]. In this regime, the quark dynamics at low energy is dominated by the lowest Landau level (LLL). Also the QCD coupling constant, \( \alpha_s \), can be shown to be sufficiently small enough that the calculations are under theoretical control. A self-consistent gap equation for the quarks in the LLL—similar to (color) superconductivity or superfluidity at large chemical potential \[18\]—can be derived and solved, yielding a quark mass gap \( M_{\text{dyn}} \) \[13\]:

\[ M_{\text{dyn}} = C(\alpha_s) \sqrt{|e_q B|}, \]

where \( e_q \) are the charges of the quarks, \((e_u, e_d, \cdots) = (\frac{2}{3}, -\frac{1}{3}, \cdots) \ e\). The detailed expression for \( C(\alpha_s) \) based on the consistent truncation of the gap equation \[15\] is irrelevant for our purpose; what will be important for us is that \( M_{\text{dyn}} \) is an increasing function of \( B \) at sufficiently large \( B \) and \( M_{\text{dyn}} \rightarrow \infty \) for \( B \rightarrow \infty \). Note that \( \alpha_s \ll 1 \) is indeed valid for \( M^2_{\text{dyn}} \ll k^2 \ll eB \), where \( k \) is the typical momentum in the gap equation.

That quarks acquire a large mass gap in a strong magnetic field means that quarks decouple from the gluons at low energy scales—well below \( \sqrt{eB} \). Hence, the lowest-order effective theory for low-energy dynamics is described by an anisotropic pure SU(3) gauge theory with

2 It should be remarked that the inverse magnetic catalysis is observed for \( T \sim \sqrt{eB} \sim \Lambda_{\text{QCD}} \) in lattice QCD simulations \[2\]. In the regime \( T \ll \sqrt{eB} \), however, both lattice QCD simulations at \( \sqrt{eB} \sim \Lambda_{\text{QCD}} \) \[2\] and the controlled weak-coupling analysis at \( \sqrt{eB} \gg \Lambda_{\text{QCD}} \) \[12\] do observe the magnetic catalysis. In this paper, we shall only make use of the magnetic catalysis in the latter regime (which is not reachable in the present lattice calculations).

3 One might be concerned that the pure gluodynamics in the large \( B \) limit acts like a dimensionally reduced theory which shows a second-order transition \[16\]. However, the theory in this regime is not in (2+1) dimensions, but in (3+1) dimensions with anisotropic color dielectric constant but with isotropic color magnetic permeability [see Eq. (2)]. It is thus important to check our assumption of first-order transition for this anisotropic theory on the lattice (see also below).
an effective Lagrangian of the form

\[ \mathcal{L}^0_{\text{eff}} = -\frac{1}{4} F^{\alpha\beta}_{\mu\nu} \Gamma^{\mu
u}_{\alpha\beta} F^{\alpha\beta} \]

with

\[ \Gamma^{\mu
u}_{\alpha\beta} = g^{\mu\nu} g^{\alpha\beta} + (\epsilon_{zz} - 1) (\delta^{\mu3} \delta^{\nu0} \delta_{\alpha0} \delta_{\beta3} + \delta^{\mu0} \delta^{\nu3} \delta_{\alpha3} \delta_{\beta0}), \]

where \( \epsilon_{zz} \) is known to be much larger than unity [13]; it is the only term in the dielectric tensor which differs from its vacuum value. The anisotropy is induced by the magnetic field which breaks rotational invariance. The superscript 0 on \( \mathcal{L}_{\text{eff}} \) is to indicate that this is the lowest term in an expansion for the effective theory. Higher-order terms will be suppressed by factors of \( p/M_{\text{dyn}} \) where \( p \) is a characteristic momentum being probed; these terms become negligibly small at large \( B \).

Note that the effective theory in Eq. (2) contains covariant derivatives in the field strengths and through these, the coupling constant enters. As in other non-Abelian gauge theory the coupling constant is scale dependent and the effective theory acquires a scale through dimensional transmutation. Its scale differs from the conventional QCD scale, \( \Lambda_{\text{QCD}} \). This is because the effective coupling constant \( \alpha'_s \) of this effective theory is different from that of the original QCD, \( \alpha_s \), and it is defined such that \( \alpha'_s(M_{\text{dyn}}) = \alpha_s(\sqrt{eB}) \); the resultant scale \( \Lambda'_{\text{QCD}}(B) \) is much smaller than \( M_{\text{dyn}}(B) \) [13]. A similar reduction of the confinement scale has also been argued for a two-flavor color superconductor [14].

As noted above, the existence of this pure glue effective theory ensures that the leading-order effective theory must have a phase transition, which may be naturally taken first order. Such a first-order transition implies that QCD at sufficiently large \( B \) also has a first-order transition. The phase transition temperature in this regime will be fixed by the scale of the effective theory, \( \Lambda'_{\text{QCD}}(B) \). On the other hand, at \( B = 0 \), it has been established from lattice QCD studies that there is no first-order phase transition [20]; the deconfinement regime emerges as a result of a crossover. Therefore, the line of first-order deconfinement transitions at large \( B \) above has to terminate at some point at some critical value of the magnetic field which we denote \( B_c \). The most natural way for this to occur is for it to terminate at a critical point in the \( T-B \) plane—(\( T_c, B_c \)) as in Fig. 1. Such a critical point of a line of first order is a point at which a second-order transition takes place.

Although we cannot infer its location from our argument alone, we can estimate its scale: \( \sqrt{eB_c} \sim T_c \sim \Lambda_{\text{QCD}} \) in the chiral limit, as \( \Lambda_{\text{QCD}} \) is the only relevant scale. Note that the scale \( \Lambda_{\text{QCD}} \) does not necessarily mean it is just around 200 MeV; it can be larger than 1 GeV, see the discussion below.

If we turn on nonzero quark masses, quarks decouple earlier from gluodynamics with increasing \( B \); the phase diagram in the \((m_{ud}, B)\) plane, similar to the Columbia plot, is shown in Fig. 2 (for two degenerate massive flavors, as an example).

We note that this analysis depends on our assumption that the leading-order effective theory at large \( B \) has a first-order transition. As remarked above, it is important to verify this assumption directly via lattice studies of the (anisotropic) leading-order effective theory in Eq. (2). Such studies are relatively straightforward to conduct since they do not not involve fermions and hence do not require the calculation of a determinant. We also note that this analysis does not rely on the magnetic catalysis at \( \sqrt{eB} \sim \Lambda_{\text{QCD}} \), and whether the inverse magnetic catalysis occurs there is irrelevant. What is used is that the effective quark mass increases at \( \sqrt{eB} \gg \Lambda_{\text{QCD}} \) (and \( T \ll \sqrt{eB} \)).

Given the assumption that the leading-order effective theory has a first-order transition at large \( B \), the existence of a critical value \( B_c \) below which the transition vanishes is assured. However, as a logical matter, there is no guarantee that the phase diagram must be of the form of Fig. 1. Other possible scenarios exist.
One possible class of alternative scenarios involves the possibility that phase diagram is more complicated so that for some values of $B$ more than two phases exist. We believe that such scenarios are unlikely to be correct. In any event, such scenarios share with the simple one in Fig. 1 the feature that at $B_c$ there is a critical point. Thus, the existence of a critical point seems quite robust.

However, there is one class of scenarios that cannot be excluded in which a critical point of this sort does not occur. In these scenarios, rather than the first-order line ending in a second-order critical point, it remains first order all the way to the end and terminates at $T = 0$ (Fig. 3). In this sense, it is similar to the critical value for the baryon chemical potential at $T = 0$: a first-order transition to nuclear matter occurs at $T = 0$ when the critical value is reached. We suspect that this class of scenario is not likely to be realized in QCD. For simplicity of language we will refer to the point at which the first-order line ends at $T = 0$ as a $T = 0$ magnetic critical point—even though it is not a critical point in the technical sense—so that we can discuss common features for all of the scenarios in a simple way. We note then that if our assumption that the leading-order effective theory has a first-order transition at large $B$, a magnetic critical point is guaranteed—either at $T = 0$ or finite $T$. A possible scenario for a phase transition at intermediate $B$ (similar to Fig. 3) was suggested based on a model analysis in Ref. 21.

Note that our argument for the magnetic critical point above is applicable not only to QCD in the chiral limit, but also to QCD with physical quark masses and QCD with any number of flavor with larger quark masses, as long as it has a crossover behavior as a function of $T$ at $B = 0$.

Lattice QCD studies can distinguish between these scenarios and determine the location of the magnetic critical point. As noted earlier, the system does not suffer from a fermion sign problem, and practical lattice studies are possible. In QCD with realistic quark masses, thermal phase transition was studied up to $\sqrt{cE} \approx 1$ GeV on the lattice [4], where no signal of the magnetic critical point was found so far; we thus conjecture $\sqrt{cE} \gtrsim 1$ GeV in real QCD. This is not necessarily unusually large, remembering the conjectured phase diagram of QCD at finite $\mu_B$, where exotic phases (such as color superconducting phases) are expected appear well above the nuclear liquid gas phase transition around $\mu_B \approx 1$ GeV at $T = 0$. Recall also that the scale of nuclear physics is governed by $\Lambda_{QCD}$ plus quark mass corrections, and, e.g., baryonic states in the vacuum have the mass comparable to 1 GeV or larger. For $N_f = 2$ QCD with relatively large quark masses ($m_q \approx 195$ MeV) on the coarse lattice, preliminary evidence for the first-order deconfinement phase transition was indicated for $\sqrt{cE} \approx 850$ MeV 22. This might be consistent with our argument that $B_c$ becomes smaller with increasing quark mass, as can be understood from Fig. 2.

It should be remarked that the phase diagram of QCD in a magnetic field has a deep similarity with those of flavor-symmetric QCD at finite isospin chemical potential $\mu_I$ 23 and other QCD-like theories at finite $\mu_B$ 24, 25, all of which do not have the sign problem; quarks acquire a large BCS gap in the superfluid phases of these theories at large chemical potential, where no low-lying colored excitation to screen gluons exists, and similar deconfined critical points are expected to appear in the phase diagrams 23. However, this is not the case in QCD at finite $\mu_B$. This is because the color gauge group is spontaneously broken by a color-nonsinglet diquark condensate in the color superconductivity at sufficiently large $\mu_I$, so that gluons also acquire a mass gap due to the Brout-Englert-Higgs mechanism; the low-energy effective theory is not pure gluodynamics in this case.

The essential conditions for the emergence of the deconfined critical point in the $(T, E)$ plane (with $E$ being some parameter of interest, such as $B$ and $\mu_I$) are thus a large quark mass gap together with a color-singlet diquark condensate at large $E$. Note that, among these theories, the magnetic critical point in the $(T, B)$ plane is the only candidate to be relevant, in principle, in nature (as the isospin charge is not conserved due to the weak interaction, but the magnetic field does not have such a problem).

Finally, one can ask, from the perspective of theory, whether the magnetic critical point continues to exist in the 't Hooft large-$N_c$ limit ($N_c \to \infty$ with $N_f$ fixed) 26. We here consider QCD with quarks in the fundamental representation of the $SU(N_c)$ gauge group. In this limit, the gluon dynamics with $\sim N_c^2$ degrees of freedom is insensitive to the quark dynamics with $\sim N_c$. As the magnetic field can only affect the quark dynamics, the deconfinement temperature governed by the gluon dynamics is independent of $B$: $T_c(B) \sim \Lambda_{QCD}$. (For the $N_f/N_c$ corrections, see Ref. 8.) Note here that

\[ \text{The following argument is not applicable to QCD with fundamental quarks for fixed } N_f/N_c \text{ and } N_c \to \infty \text{ and to QCD with adjoint quarks or two-index antisymmetric quarks for fixed } N_f \text{ and } N_c \to \infty, \text{ as the fermionic degrees of freedom are comparable to the gluonic ones.} \]
\( \Lambda_{\text{QCD}}(N_c, B) \rightarrow \Lambda_{\text{QCD}} \) at \( N_c \rightarrow \infty \). Therefore, the magnetic critical point does not exist in the large-\( N_c \) limit. In particular, to study the possible existence of the magnetic critical point in the holographic QCD models \[29\], one needs to incorporate the \( N_f/N_c \) corrections.

In conclusion, we have argued for a new QCD critical point in the \((T, B)\) phase diagram. It would be interesting to determine the location of this critical point in lattice simulations; as noted above, this should be possible since the theory does not suffer from a fermion sign problem.

What is the experimental signature of the finite-\( T \) magnetic critical point in heavy ion collisions (assuming it is accessible)? This point is characterized by the vanishing screening mass of the glueball. Due to the mixing between the glueball \( G \) and the flavor-singlet meson \( \bar{c}c \), the singular behavior of \( G \) is reflected in that of \( \sigma \); thus one expects that such observables (in the presence of sufficiently small \( \mu_B \)) are similar to those studied for the conventional QCD critical point. Presumably one could distinguish between the two by studying the effect as a function of centrality.

More generally, one can consider the phase diagram in the three-dimensional space \((T, \mu_B, B)\). Whether the magnetic critical point persists at large \( \mu_B \) would also be an interesting question to be explored.

We thank S. Aoki, T. Kanazawa, T. Kojo, and N. Su for useful discussions and F. Bruckmann, M. Chernodub, P. de Forcrand, G. Endrodi, M. Kaminski, and T. Kovacs for useful comments. N. Y. was supported by JSPS Research Fellowships for Young Scientists.

[1] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, Nucl. Phys. A803, 227 (2008); V. Skokov, A. Y. Illarionov and V. Toneev, Int. J. Mod. Phys. A 24, 5925 (2009); W.-T. Deng and X.-G. Huang, Phys. Rev. C 85, 044907 (2012).
[2] R. C. Duncan and C. Thompson, Astrophys. J. 392, L9 (1992); M. Malheiro, S. Ray, H. J. Mosquera Cuesta, and J. Dey, Int. J. Mod. Phys. D 16, 489 (2007); M. Eto, K. Hashimoto, and T. Hatsuda, Phys. Rev. D 88, 081701 (2013).
[3] D. Grasso and H. R. Rubinstein, Phys. Rept. 348, 163 (2001).
[4] G. S. Bali, F. Bruckmann, G. Endrodi, Z. Fodor, S. D. Katz, S. Krieg, A. Schafer, and K. K. Szabo, JHEP 1202, 044 (2012); Phys. Rev. D 86, 071502 (2012).
[5] A. J. Mizher, M. N. Chernodub, and E. S. Fraga, Phys. Rev. D 82, 105016 (2010); R. Gatto and M. Ruggieri, Phys. Rev. D 82, 054027 (2010); Phys. Rev. D 83, 034016 (2011); K. Kashiwa, Phys. Rev. D 83, 117901 (2011).
[6] K. Fukushima and Y. Hidaka, Phys. Rev. Lett. 110, 031601 (2013); J. O. Andersen and A. A. Cruz, Phys. Rev. D 88, 025016 (2013); T. Kojo and N. Su, Phys. Lett. B 720, 192 (2013); F. Bruckmann, G. Endrodi, and T. G. Kovacs, JHEP 1304, 112 (2013).
[7] B. V. Galilo and S. N. Nedelko, Phys. Rev. D 84, 094017 (2011); E. S. Fraga and L. F. Palhares, Phys. Rev. D 86, 016008 (2012).
[8] E. S. Fraga, J. Noronha, and L. F. Palhares, Phys. Rev. D 87, 114014 (2013).
[9] M. M. Anber and M. Unsal, arXiv:1309.4394 [hep-th].
[10] N. O. Agasian and S. M. Fedorov, Phys. Lett. B 663, 445 (2008).
[11] M. A. Stephanov, Prog. Theor. Phys. Suppl. 153, 139 (2004) [Int. J. Mod. Phys. A 20, 4387 (2005)].