WEIGHTED ANISOTROPIC SOBOLEV INEQUALITY WITH EXTREMAL AND ASSOCIATED SINGULAR PROBLEMS

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Abstract. For a given Finsler-Minkowski norm \( F \) in \( \mathbb{R}^N \) and a bounded smooth domain \( \Omega \subset \mathbb{R}^N \) \((N \geq 2)\), we establish the following weighted anisotropic Sobolev inequality

\[
\left( P \right) \quad S \left( \int_{\Omega} |u|^q f \, dx \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} F(\nabla u)^p w \, dx \right)^{\frac{1}{p}}, \quad \forall u \in W^{1,p}_0(\Omega, w)
\]

where \( W^{1,p}_0(\Omega, w) \) is the weighted Sobolev space under a class of \( p \)-admissible weights \( w \), where \( f \) is some nonnegative integrable function in \( \Omega \). We discuss the case \( 0 < q < 1 \) and observe that

\[
\left( Q \right) \quad \mu(\Omega) := \inf_{u \in W^{1,p}_0(\Omega, w)} \left\{ \int_{\Omega} F(\nabla u)^p w \, dx : \int_{\Omega} |u|^q f \, dx = 1 \right\}
\]

is associated with singular weighted anisotropic \( p \)-Laplace equations. To this end, we also study existence and regularity properties of solutions for weighted anisotropic \( p \)-Laplace equations under the mixed and exponential singularities.

1. Introduction

In this article, we establish the following weighted anisotropic Sobolev inequality,

\[
\left( P \right) \quad S \left( \int_{\Omega} |u|^q f \, dx \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} F(\nabla u)^p w \, dx \right)^{\frac{1}{p}}, \quad \forall u \in W^{1,p}_0(\Omega, w),
\]

where \( S \) is the Sobolev constant, \( 0 < q < 1 < p < \infty \), \( \Omega \subset \mathbb{R}^N \) \((N \geq 2)\) is a bounded smooth domain. Here \( F : \mathbb{R}^N \to [0, \infty) \) is a Finsler-Minkowski norm, i.e. \( F \) satisfies the hypothesis from (H0) – (H4) given by

- (H0) \( F(x) \geq 0 \), for every \( x \in \mathbb{R}^N \).
- (H1) \( F(x) = 0 \), if and only if \( x = 0 \).
- (H2) \( F(tx) = |t|F(x) \), for every \( x \in \mathbb{R}^N \) and \( t \in \mathbb{R} \).
- (H3) \( F \in C^\infty(\mathbb{R}^N \setminus \{0\}) \).
- (H4) the Hessian matrix \( \nabla^2_{\eta} \left( \frac{F^2}{2} \right) \)(x) is positive definite for all \( x \in \mathbb{R}^N \setminus \{0\} \).

We assume the weight function \( w \) in a class of \( p \)-admissible weights to be discussed in Section 2 and \( f \in L^m(\Omega) \setminus \{0\} \) is some nonnegative function. Our main emphasis is the case of \( 0 < q < 1 \) in (P) and we observe that

\[
\left( Q \right) \quad \mu(\Omega) := \inf_{u \in W^{1,p}_0(\Omega, w)} \left\{ \int_{\Omega} F(\nabla u)^p w \, dx : \int_{\Omega} |u|^q f \, dx = 1 \right\},
\]

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is associated with the following type of weighted singular anisotropic \(p\)-Laplace equation:
\[
(S) \quad -F_{p,w} u = \frac{f(x)}{u^\delta} + \frac{g(x)}{u^\gamma} \quad \text{in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega.
\]
where \(0 < \delta, \gamma < 1\), \((f, g) \neq (0, 0) \in L^m(\Omega)\) are nonnegative functions. Here,
\[
(F) \quad F_{p,w} u := \text{div} \left( wF(\nabla u)^{p-1}\nabla \eta F(\nabla u) \right),
\]
is the weighted anisotropic \(p\)-Laplace operator, where \(\nabla \eta\) denotes the gradient with respect to \(\eta\). Further, we prove existence and regularity results for the exponential singular problem,
\[
(R) \quad -F_{p,w} u = h(x)e^{\frac{t}{\eta}} \quad \text{in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega,
\]
where \(h \in L^t(\Omega)\) is nonnegative for some \(t \geq 1\).

Before proceeding further, let us discuss some examples of \(F\).

**Examples:** Let \(x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N\).

(i) Then for \(t > 1\), we define
\[
F_t(x) := \left( \sum_{i=1}^{N} |x_i|^t \right)^{\frac{1}{t}}.
\]

(ii) For \(\lambda, \mu > 0\), we define
\[
F_{\lambda,\mu}(x) := \sqrt{\lambda \sum_{i=1}^{N} x_i^4 + \mu \sum_{i=1}^{N} x_i^2}.
\]

The functions \(F_t, F_{\lambda,\mu} : \mathbb{R}^N \to [0, \infty)\) given by (1.2) and (1.3) satisfies all the hypothesis from (H0) – (H4), see Mezei-Vas [49].

**Remark 1.1.** For \(i = 1, 2\), if \(\lambda_i, \mu_i\) are positive real numbers such that \(\frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}\), then \(F_{\lambda_1, \mu_1}\) and \(F_{\lambda_2, \mu_2}\) given by (1.3) defines two non-isometric norms in \(\mathbb{R}^N\).

**Remark 1.2.** Since all norms in \(\mathbb{R}^N\) are equivalent, there exist positive constants \(C_1, C_2\) such that
\[C_1|x| \leq F(x) \leq C_2|x|, \quad \forall x \in \mathbb{R}^N.\]

The dual \(F_0 : \mathbb{R}^N \to [0, \infty)\) of \(F\) is defined by
\[
F_0(\xi) := \sup_{x \in \mathbb{R}^N \setminus \{0\}} \frac{(x, \xi)}{F(x)}.
\]

We refer to Bao-Chern-Shen [10], Xia [15] and Rockafellar [60] for more details on \(F_0\).

It is easy to observe that, if \(F = F_t\) given by (1.2), then
\[
F_{p,w} := \begin{cases}
\Delta_{p,w} u = \text{div} \left( w|\nabla u|^{p-2}\nabla u \right), \text{(weighted } p\text{-Laplacian)} \text{ if } t = 2, 1 < p < \infty, \\
S_{p,w} u = \text{div} \left( w \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \nabla u \right), \text{(weighted pseudo } p\text{-Laplacian)} \text{ if } t = p \in (1, \infty).
\end{cases}
\]

Therefore, \(F_{p,w}\) extends the weighted \(p\)-Laplace and weighted pseudo \(p\)-Laplace operators and thus a large class of weighted quasilinear equations is covered by \(F_{p,w}\).

Let us discuss some known results related to our present study. For \(q > 1\), Sobolev inequalities of the form (P) are widely studied throughout the last three decade and there is a colossal amount of literature available in this direction. We refer to Aubin [5], Talenti [61], Ōtani [57], Franzina-Lamberti [33], Belloni-Kawohl [12], Lindqvist [48] and the references therein.
For the class of Muckenhoupt weights [52], weighted Sobolev inequalities are established in Fabes-Kenig-Serapioni [29], for a more general class of \( p \)-admissible weights, refer to Heinonen-Kilpeläine-Martio [43] and the references therein.

Although very less is known in the anisotropic case. In this context, recently Ciraolo-Figalli-Roncoroni [17] proved a sharp version of (P), for a certain class of weight functions \( w = f \) in a convex cone \( \Omega \), with \( 1 < p < N \), for the exponent \( q > 1 \), depends on the weight function. Indeed, for \( w = 1 \), the exponent \( q \) is the critical Sobolev exponent in [17]. We also refer to Belloni-Ferone-Kawohl [11], di Blasio-Pisante-Giovanni [20], El Hamidi-Rakotoson [26], Filippas-Moschini-Tertikas [32], Dipierro-Poggesi-Valdinoci [21] and the references therein for related works.

When \( 0 < q < 1 \), for any \( 1 < p < \infty \), Anello-Faraci-Iannizzotto [2] shown that

\[
\nu(\Omega) := \inf_{0 \not= u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, dx : \int_\Omega |u|^q \, dx = 1 \right\},
\]

is achieved at a solution \( u_0 \in W_0^{1,p}(\Omega) \) of the following singular \( p \)-Laplace equation

\[
-\Delta_p u = \nu(\Omega) |u|^{q-1} \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]

Such results has further been extended to a class of Muckenhoupt weights by Bal-Garain [8], in the nonlocal case by Ercole-Pereira [27], subelliptic setting by Garain-Ukhlov [39], see also Ercole-Pereira [28] for related results. When \( w \) is a \( p \)-admissible weight, Hara [42] studied weighted Sobolev inequalities of type (P) in terms of the associated singular problem.

In this article, we provide sufficient conditions on the weight function \( w \) (which may vanish or blow up near the origin, for example \( w(x) = |x|^\alpha, \alpha \in \mathbb{R} \)) and \( f \) that guarantees the weighted Sobolev inequality (P) and also provide the extremal function of the associated variational problem.

We observe that such extremals correspond to the associated singular weighted anisotropic \( p \)-Laplace equation. We found a class of \( p \)-admissible weights useful for our purpose, for which weighted singular problems are recently studied in Garain-Kinnunen [36, 37]. We would like to mention that in the classical case without weights, singular \( p \)-Laplace equations is studied widely till date, refer to [3, 4, 6, 7, 16, 18, 19, 22, 34, 35, 38, 40, 41, 46, 51, 53–56] and the references therein. In contrast to these literature, singular anisotropic \( p \)-Laplace equations are very less understood. In this context, we refer to Biset-Mebrate-Mohammed [13], Farkas-Winkert [31], Farkas-Fiscella-Winkert [30], Bal-Garain-Mukherjee [9] and the references therein. Here we are also able to extend some previous results known in the unweighted case for \( p \geq 2 \) to the weighted case case with \( 1 < p < 2 \) (see Remark 2.15 and 2.19).

The idea here stem from the work of Anello-Faraci-Iannizzotto [2] that is based on the approximation approach, where it was important to know the existence of the associated singular \( p \)-Laplace equation (1.7). We obtain existence and regularity results in the weighted anisotropic setting, for the more general mixed singular problem (S) (Theorem 2.11-2.13) and also study the exponential singular problem (R) (Theorem 2.21-2.22). To this end, we follow the approach from Boccardo-Orsina [14], where to deal with the mixed problem (S), we need to estimate both the singularities \( u^{-\delta} \) and \( u^{-\gamma} \) simultaneously as in [8]. On the otherhand, for the case of \( e^{|x|^\gamma} \) in the problem (R), the exponent in the singularity is arbitrarily large and thus in general solutions lie outside \( W_0^{1,p}(\Omega,w) \). Due to this reason, one has to describe the boundary condition appropriately as has been illustrated in [14, 47, 59]. Here, we tackle our situation by following the domain approximation technique from Perera-Silva [59].

**Organization of the paper:** In Section 2, we discuss some preliminaries in our setting and state the main results. In Section 3, we obtain several auxiliary results, that are crucial to
study the mixed singular problem \((S)\) and the weighted anisotropic Sobolev inequality \((P)\). In Section 4, we prove some auxiliary results related to the exponential singular problem \((R)\) and finally, in Section 4, we prove our main results.

2. Preliminaries and main results

Throughout the rest of the article, we assume \(1 < p < \infty\), unless otherwise mentioned. We say that a function \(w\) belong to the class of \(p\)-admissible weights \(W_p\), if \(w \in L^1_{\text{loc}}(\mathbb{R}^N)\) such that \(0 < w < \infty\) almost everywhere in \(\mathbb{R}^N\) and satisfies the following conditions:

(i) for any ball \(B\) in \(\mathbb{R}^N\), there exists a positive constant \(C_\mu\) such that

\[
\mu(2B) \leq C_\mu \mu(B),
\]

where

\[
\mu(E) = \int_E w \, dx
\]

for a measurable subset \(E\) in \(\mathbb{R}^N\) and \(d\mu(x) = w(x) \, dx\), where \(dx\) is the \(N\)-dimensional Lebesgue measure.

(ii) If \(D\) is an open set and \(\{\phi_i\}_{i \in \mathbb{N}} \subset C^\infty(D)\) is a sequence of functions such that

\[
\int_D |\phi_i|^p \, d\mu \to 0 \quad \text{and} \quad \int_D |\nabla \phi_i - v|^p \, d\mu \to 0
\]

as \(i \to \infty\), where \(v\) is a vector valued measurable function in \(L^p(D, w)\), then \(v = 0\).

(iii) There exist constants \(\kappa > 1\) and \(C_1 > 0\) such that

\[
(\frac{1}{\mu(B)} \int_B |\phi|^p \, d\mu)^{\frac{1}{p}} \leq C_1 \left( \frac{1}{\mu(B)} \int_B |\nabla \phi|^p \, d\mu \right)^{\frac{1}{p}},
\]

whenever \(B = B(x_0, r)\) is a ball in \(\mathbb{R}^N\) centered at \(x_0\) with radius \(r\) and \(\phi \in C^\infty_c(B)\).

(iv) There exists a constant \(C_2 > 0\) such that

\[
\int_B |\phi - \phi_B|^p \, d\mu \leq C_2 r^p \int_B |\nabla \phi|^p \, d\mu,
\]

whenever \(B = B(x_0, r)\) is a ball in \(\mathbb{R}^N\) and \(\phi \in C^\infty_c(B)\) is bounded. Here

\[
\phi_B = \frac{1}{\mu(B)} \int_B \phi \, d\mu.
\]

The conditions (i)-(iv) are important in the theory of weighted Sobolev spaces, one can refer to [43] for more details.

**Examples:** Muckenhoupt weights \(A_p\) are \(p\)-admissible, see [43, Theorem 15.21]. In particular, if \(c \leq w \leq d\) for some positive constants \(c, d\), then \(w \in A_p\) for any \(1 < p < \infty\). Let \(1 < p < N\) and \(J_f(x)\) denote the determinant of the Jacobian matrix of a \(K\)-quasiconformal mapping \(f : \mathbb{R}^N \to \mathbb{R}^N\), then \(w(x) = J_f(x)^{1-\frac{N}{p}} \in W_q\) for any \(q \geq p\), see [43, Corollary 15.34]. If \(1 < p < \infty\) and \(\nu > -N\), then \(w(x) = |x|^\nu \in W_p\), see [43, Corollary 15.35]. For more examples, refer to [1, 24, 25, 43, 44] and the references therein.

**Definition 2.1.** (Weighted Spaces) Let \(1 < p < \infty\) and \(w \in W_p\). Then the weighted Lebesgue space \(L^p(\Omega, w)\) is the class of measurable functions \(u : \Omega \to \mathbb{R}\) such that the norm of \(u\) given by

\[
\|u\|_{L^p(\Omega, w)} = \left( \int_\Omega |u(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \infty.
\]
The weighted Sobolev space $W^{1,p}(\Omega, w)$ is the class of measurable functions $u : \Omega \to \mathbb{R}$ such that
\begin{equation}
\|u\|_{1,p,w} = \left( \int_\Omega |u(x)|^p w(x) \, dx + \int_\Omega |\nabla u(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \infty.
\end{equation}

If $u \in W^{1,p}(\Omega', w)$ for every $\Omega' \subset \Omega$, then we say that $u \in W^{1,p}_{\text{loc}}(\Omega, w)$. The weighted Sobolev space with zero boundary value is defined as
\begin{equation}
W^{1,p}_0(\Omega, w) = \left\{ u : \Omega \to \mathbb{R} \mid u \in W^{1,p}(\Omega, w), \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial \Omega \right\}.
\end{equation}

Using the Poincaré inequality from [43], the norm defined by (2.4) on the space $W^{1,p}_0(\Omega, w)$ is equivalent to the norm given by
\begin{equation}
\|u\|_{W^{1,p}_0(\Omega, w)} = \left( \int_\Omega |\nabla u|^p w \, dx \right)^{\frac{1}{p}}.
\end{equation}

Moreover, the space $W^{1,p}_0(\Omega, w)$ is a separable and uniformly convex Banach space, see [43].

Next, we state an embedding result which is crucial for us. To this end, for $1 < p < \infty$, we define the set
\begin{equation}
I = \left[ \frac{1}{p-1}, \infty \right) \cap \left( \frac{N}{p}, \infty \right).
\end{equation}

Consider the following subclass of $W_p$ given by
\begin{equation}
W^s_p = \left\{ w \in W_p : w^{-s} \in L^1(\Omega) \right\}.
\end{equation}

Then for $s \in I$, the weight
\begin{equation}
w(x) = |x|^s \in W^s_p \text{ for any } \nu \in \left( -N, \frac{N}{s} \right).
\end{equation}

Following the lines of the proof of [34, Theorem 2.6] based on [23] the following embedding result holds.

**Lemma 2.2.** Let $1 < p < \infty$ and $w \in W^s_p$ for some $s \in I$. Then the following continuous inclusion maps hold
\begin{equation}
W^{1,p}(\Omega, w) \hookrightarrow W^{1,p^*_s}(\Omega) \hookrightarrow \begin{cases} L^t(\Omega), & \text{for } p_s \leq t \leq p^*_s, \text{ if } 1 \leq p_s < N, \\ L^t(\Omega), & \text{for } 1 \leq t < \infty, \text{ if } p_s = N, \\ C(\overline{\Omega}), & \text{if } p_s > N, \end{cases}
\end{equation}

where $p_s = \frac{p^*_s}{s+1} \in [1, p)$. Moreover, the second embedding above is compact except for $t = p^*_s = \frac{Np_s}{N-p_s}$, if $1 \leq p_s < N$. Further, the same result holds for the space $W^{1,p}_0(\Omega, w)$.

**Remark 2.3.** We note that if $0 < c \leq w \leq d$ for some constants $c, d$, then $W^{1,p}(\Omega, w) = W^{1,p}(\Omega)$ and thus by the Sobolev embedding, Lemma 2.2 holds by replacing $p_s$ and $p^*_s$ with $p$ and $p^* = \frac{Np}{N-p}$ respectively.

The following result follows from Farkas-Winkert [31, Proposition 2.1] and Xia [15, Proposition 1.2].

**Lemma 2.4.** For every $x \in \mathbb{R}^N \setminus \{0\}$ and $t \in \mathbb{R} \setminus \{0\}$, we have
\begin{enumerate}[(A)]
\item $x \cdot \nabla \eta \mathcal{F}(x) = \mathcal{F}(x)$,
\item $\nabla \eta \mathcal{F}(tx) = \text{sign}(t) \nabla \eta \mathcal{F}(x)$,
\item $|\nabla \eta \mathcal{F}(x)| \leq C$, for some positive constant $C$.
\item $\mathcal{F}$ is strictly convex.
\end{enumerate}
Next, the classical algebraic inequality holds from Peral [58, Lemma A.0.5].

**Lemma 2.5.** For any \( a, b \in \mathbb{R}^N \), there exists a constant \( C = C(p) > 0 \), such that

\[
(2.8) \quad \langle |a|^{p-2}a - |b|^{p-2}b, a - b \rangle \geq \begin{cases} 
C|a - b|^p, & \text{if } 2 \leq p < \infty, \\
C\frac{|a - b|^2}{(|a| + |b|)^{p-2}}, & \text{if } 1 < p < 2.
\end{cases}
\]

More generally, we state the Finsler algebraic inequality from [9, Lemma 2.5].

**Lemma 2.6.** Let \( 2 \leq p < \infty \). Then, for every \( x, y \in \mathbb{R}^N \), there exists a constant \( C = C(p) > 0 \), such that

\[
(2.9) \quad \langle F(x)^{p-1}\nabla_\eta F(x) - F(y)^{p-1}\nabla_\eta F(y), x - y \rangle \geq CF(x - y)^p.
\]

**Remark 2.7.** When \( 2 \leq p < \infty \) and \( F(x) = F_2(x) = |x| \) as given by (1.2), Lemma 2.6 coincides with Lemma 2.5.

**Corollary 2.8.** From Lemma 2.4 and Remark 1.2 we have

\[
(2.10) \quad F(x)^{p-1}\nabla_\eta F(x) : x = F(x)^p \geq C_1|x|^p, \quad \forall x \in \mathbb{R}^N,
\]

\[
(2.11) \quad |F(x)^{p-1}\nabla_\eta F(x)| \leq C_2|x|^{p-1}, \quad \forall x \in \mathbb{R}^N \text{ and}
\]

\[
(2.12) \quad F(tx)^{p-1}\nabla_\eta F(tx) = |t|^{p-2}tF(x)^{p-1}\nabla_\eta F(x), \quad \forall x \in \mathbb{R}^N \text{ and } t \in \mathbb{R} \setminus \{0\}.
\]

Moreover, by Lemma 2.6, we have

\[
(2.13) \quad \langle F(x)^{p-1}\nabla_\eta F(x) - F(y)^{p-1}\nabla_\eta F(y), x - y \rangle > 0, \quad \forall x \neq y \in \mathbb{R}^N.
\]

**Notation:** Throughout the rest of the article, we shall use the following notations.

- For \( u \in W_0^{1,p}(\Omega, w) \), denote by \( ||u|| \) to mean the norm \( ||u||_{W_0^{1,p}(\Omega, w)} \) as defined by (2.5).
- For given constants \( c, d \) and a set \( S \), by \( c \leq u \leq d \) in \( S \), we mean \( c \leq u \leq d \) almost everywhere in \( S \). Moreover, we write \( |S| \) to denote the Lebesgue measure of \( S \).
- \( \langle , \rangle \) denotes the standard inner product in \( \mathbb{R}^N \).
- The conjugate exponent of \( \theta > 1 \) by \( \theta' := \frac{\theta}{\theta - 1} \).
- For \( 1 < p < N \), we denote by \( p^* := \frac{Np}{N-p} \) to mean the critical Sobolev exponent.
- For \( a \in \mathbb{R} \), we denote by \( a^+ := \max\{a, 0\} \), \( a^- := \max\{-a, 0\} \) and \( a_- := \min\{a, 0\} \).
- We write by \( c, C \) or \( C_i \) for \( i \in \mathbb{N} \) to mean a constant which may vary from line to line or even in the same line. If a constant \( C \) depends on \( r_1, r_2, \ldots \), we denote it by \( C(r_1, r_2, \ldots) \).

Now, we define the notion of weak solutions for the problem \((S)\) as follows:

**Definition 2.9.** *(Weak solution for \((S)\))* Let \( 1 < p < \infty \), \( 0 < \delta, \gamma < 1 \) and \( w \in W_p \). Then, we say that \( u \in W_0^{1,p}(\Omega, w) \) is a weak solution of the problem \((S)\), if \( u > 0 \) in \( \Omega \) and for every \( \omega \Subset \Omega \), there exists a constant \( c_\omega \) such that \( u \geq c_\omega > 0 \) in \( \omega \) and for every \( \phi \in C^1_0(\Omega) \), we have

\[
(2.14) \quad < -F_{p,w}u, \phi > := \int_\Omega w(x)F(\nabla u)^{p-1}F(\nabla u)\nabla \phi \, dx = \int_\Omega \left( \frac{f}{u^\delta} + \frac{g}{u^\gamma} \right) \phi \, dx.
\]

Further, we define weak solutions for the problem \((R)\) as follows.
Definition 2.10. (Weak solution for $\mathcal{R}$) Let $1 < p < \infty$ and $w \in W_p^1$. Then, we say that $u \in W^{1,p}_{\text{loc}}(\Omega,w)$ is a weak solution of the problem $\mathcal{R}$, if $u > 0$ in $\Omega$ such that $(u - \epsilon)^+ \in W^{1,p}_{0}(\Omega,w)$ for every $\epsilon > 0$ and for every $\omega \Subset \Omega$, there exists a constant $c_\omega$ such that $u \geq c_\omega > 0$ in $\omega$ and for every $\phi \in C^1_c(\Omega)$, we have

\begin{equation}
-\mathcal{F}_{p,w}u, \phi := \int_{\Omega} w(x)\mathcal{F}(\nabla u)^{p-1}\mathcal{F}(\nabla u)\nabla \phi \, dx = \int_{\Omega} h(x)e^{\frac{1}{w} \phi} \, dx.
\end{equation}

Statement of the main results. Now we state our main results as follows:

Theorem 2.11. Let $0 < \delta, \gamma < 1$, $2 \leq p < \infty$ and $w \in W^s_p$ for some $s \in I$. Assume that $(f,g) \neq (0,0)$ is nonnegative.

(a) Let $(f,g) \in L^1(\Omega) \times L^1(\Omega)$. Then the problem $(S)$ admits at most one weak solution in $W^{1,p}_{0}(\Omega,w)$.

(b) Let $(f,g) \in L^{m_\delta}(\Omega) \times L^{r_\gamma}(\Omega)$, where

\[ m_\delta := \begin{cases} \left( \frac{p}{1 - m} \right)', & \text{for } 1 \leq p_s < N, \\ m > 1, & \text{for } p_s = N, \\ 1, & \text{for } p_s > N \end{cases} \]

and

\[ r_\gamma := \begin{cases} \left( \frac{p}{1 - r} \right)', & \text{for } 1 \leq p_s < N, \\ m > 1, & \text{for } p_s = N, \\ 1, & \text{for } p_s > N. \end{cases} \]

Then the problem $(S)$ has a weak solution $u_{\delta,\gamma}$ in $W^{1,p}_{0}(\Omega,w)$.

In addition, we prove the following regularity results for $u_{\delta,\gamma}$.

Theorem 2.12. Let $2 \leq p < \infty$ and $w \in W^s_p$ for some $s \in I$. Suppose that $(f,g) \neq (0,0)$ is nonnegative such that

(a) $(f,g) \in L^q(\Omega) \times L^q(\Omega)$ for $q > \frac{p^*_s}{p_r - p}$, if $1 \leq p_s < N$,

(b) $(f,g) \in L^q(\Omega) \times L^q(\Omega)$ for $q > \frac{p^*_s}{r - p}$, if $p_s = N$, $r > p$,

(c) $(f,g) \in L^1(\Omega) \times L^1(\Omega)$ if $p_s > N$.

Then, $u_{\delta,\gamma} \in L^{\infty}(\Omega)$.

Theorem 2.13. If $w \in W^s_p$ for some $s \in I$ and $\mathcal{F}_{p,w} = \Delta_{p,w}$ or $\mathcal{S}_{p,w}$ as given by (1.5), then Theorem 2.11-2.12 holds, for any $1 < p < \infty$.

Remark 2.14. For $\mathcal{F}_{p,w} = \Delta_{p,w}$ as given by (1.5), Theorem 2.13 extends [8, Theorem 2.7]. Further for $g = 0$, Theorem 2.13 extends [14, Theorem 5.2] with $p = 2$; [19, Lemma 4.3] for $1 < p < N$; [16, Theorem 4.6] for $1 < p < \infty$ and extends [34, Theorem 3.2] for $1 < p < \infty$.

Remark 2.15. If $\mathcal{F}_{p,w} = \mathcal{S}_{p,w}$ as given by (1.5), then, for $g = 0$, Theorem 2.13 extends [51, Theorem 3.2] to the weighted case for $w \in W^s_p$, improves the range of $f$ and the restriction $p \geq 2$ to any $1 < p < \infty$. Furthermore, Theorem 2.13 extends [8, Theorem 2.9] to the weighted case for $w \in W^s_p$ and also improves the restriction $p \geq 2$ to any $1 < p < \infty$.

Remark 2.16. If $g = 0$, we denote the solutions $u_{\delta,\gamma}$ found in Theorem 2.11 and 2.13 by $u_\delta$.

Now, we present our weighted anisotropic Sobolev inequality with extremal associated with $u_\delta$ as follows:

Theorem 2.17. Let $0 < \delta < 1$ and $2 \leq p < \infty$. Assume that $w \in W^s_p$, for some $s \in I$ and $f \in L^{m_\delta}(\Omega) \setminus \{0\}$ be nonnegative, where $m_\delta$ is given by Theorem 2.11. Then,

\[ \int_{\Omega} w(x)\mathcal{F}(\nabla u)^{p-1}\mathcal{F}(\nabla u)\nabla \phi \, dx \leq \int_{\Omega} h(x)e^{\frac{1}{w} \phi} \, dx, \]
(a) \[ \mu(\Omega) := \inf_{u \in W^{1,p}_0(\Omega,w)} \left\{ \int_\Omega F(\nabla u)^p w \, dx : \int_\Omega |u|^{1-\delta} f \, dx = 1 \right\} \]

\[ = \left( \int_\Omega F(\nabla u_\delta)^p w \, dx \right)^{\frac{1-\delta}{1-p}}. \]

(b) Moreover, for every \( v \in W^{1,p}_0(\Omega,w) \), the following Sobolev type inequality holds, if and only if \( C \leq \mu(\Omega) \).

\[ C \left( \int_\Omega |v|^{1-\delta} f \, dx \right)^{\frac{p}{1-\delta}} \leq \int_\Omega F(\nabla v)^p w \, dx, \]

Remark 2.19. When \( F(x) = F_2(x) \), then Theorem 2.18 extends [8, Theorem 2.11]. Moreover, if \( F(x) = F_p(x) \), then Theorem 2.18 extends [8, Theorem 2.13] to the weighted anisotropic case for \( w \in W^s_p \) and further improves the range of \( p \geq 2 \) to any \( 1 < p < \infty \).

Remark 2.20. Both Theorem 2.17 and 2.18 gives

\[ \mu(\Omega) = \|V_\delta\|^p = \int_\Omega F(\nabla V_\delta)^p w \, dx, \]

where \( V_\delta := \zeta_\delta u_\delta \in W^{1,p}_0(\Omega,w) \) for

\[ \zeta_\delta = \left( \int_\Omega u_\delta^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}}. \]

Moreover, \( V_\delta \) satisfies

\[ \int_\Omega V_\delta^{1-\delta} f \, dx = 1, \]

and the equation

\[ -F_{p,w} V_\delta = \mu(\Omega) f V_\delta^{-\delta} \text{ in } \Omega, \] \( V_\delta > 0 \) in \( \Omega \).

Our final results concerning the problem \( (\mathcal{R}) \) are stated as follows:

**Theorem 2.21.** Let \( 2 \leq p < \infty \) and \( w \in W^s_p \) for some \( s \in I \). Assume that

(a) \( h \in L^t(\Omega) \setminus \{0\} \) is nonnegative for \( t > \frac{p^*}{p^*_s-p^*} \) if \( 1 \leq p_s < N \) and

(b) \( h \in L^t(\Omega) \setminus \{0\} \) is nonnegative for \( t > \frac{r}{r-p} \) if \( p_s \geq N \) and \( r > p \).

Then the problem \( (\mathcal{R}) \) admits a weak solution \( v \in W^{1,p}_0(\Omega,w) \cap L^\infty(\Omega) \).

Moreover, we have

**Theorem 2.22.** If \( w \in W^s_p \) for some \( s \in I \) and \( F_{p,w} = \Delta_{p,w} \) or \( S_{p,w} \) as given by \((1.5)\), then

Theorem 2.21 holds, for any \( 1 < p < \infty \).

Remark 2.23. If \( 0 < c \leq w \leq d \) for some constants \( c \) and \( d \), then noting Remark 2.3 our main results Theorem 2.11-2.13, 2.17, 2.18, 2.21-2.22 holds by replacing \( p_s \) with \( p \).
3. Auxiliary results for the problem (S) and for the weighted anisotropic Sobolev inequality

Throughout the rest of the article, we assume 1 < p < ∞ and w ∈ W^{s}_p for some s ∈ I unless otherwise mentioned. We recall that by ||u|| we denote the norm ||u||_{W^{1,p}(Ω,w)} as defined in (2.5). For n ∈ N, we investigate the following approximated problem

\begin{equation}
-\mathcal{F}_{p,w}u = \frac{f_n}{(u^+ + \frac{1}{n})^\delta} + \frac{g_n}{(u^+ + \frac{1}{n})^\gamma} \quad \text{in } Ω, \quad u = 0 \text{ on } ∂Ω,
\end{equation}

where \( f_n(x) = \min\{f(x), n\} \) and \( g_n(x) = \min\{g(x), n\} \), provided \((f,g)( \neq (0,0)) \in L^{m_\delta}(Ω) \times \overset{\cdot}{L}^{r_\gamma}(Ω)\) is nonnegative, where \(m_\delta\) and \(r_\gamma\) is given by Theorem 2.11. First we prove the following result that is important to obtain the existence and further qualitative properties of solutions for the problem (3.1) as shown in Lemma 3.3.

**Lemma 3.1.** Let \(2 \leq p < ∞\) and \(ξ \in L^\infty(Ω) \setminus \{0\}\) be nonnegative in \(Ω\). Then there exists a unique solution \(u ∈ W^{1,p}_0(Ω,w) \cap L^\infty(Ω)\) of the problem

\begin{equation}
-\mathcal{F}_{p,w}u = ξ \quad \text{in } Ω, \quad u > 0 \text{ in } Ω,
\end{equation}

such that for every \(ω ∈ Ω\), there exists a constant \(c_ω\), satisfying \(u ≥ c_ω > 0\) in \(ω\).

**Proof.** Existence: To prove the existence, we define the energy functional \(I : W^{1,p}_0(Ω,w) → ℝ\) by

\[I(u) := \frac{1}{p} ∫_Ω \mathcal{F}(∇u)^p w \, dx - ∫_Ω ξ u \, dx.\]

Since \(ξ ∈ L^\infty(Ω)\), using Lemma 2.2, we have

\begin{equation}
I(u) ≥ \frac{||u||^p}{p} - C||Ω||^{\frac{p-1}{p}}||ξ||_{L^\infty(Ω)}||u||,
\end{equation}

where \(C > 0\) is the Sobolev constant. Thus \(I\) is coercive, since \(p > 1\). Next, we observe that \(I\) is also convex. Indeed, let us define the energy functional \(I_1 : W^{1,p}_0(Ω,w) → ℝ\) by

\[I_1(u) := \frac{1}{p} ∫_Ω \mathcal{F}(∇u)^p w \, dx,\]

and \(I_2 : W^{1,p}_0(Ω,w) → ℝ\) by

\[I_2(u) := - ∫_Ω ξ u \, dx.\]

By the property \((D)\) of Lemma 2.4, we have \(\mathcal{F}^p\) is convex and hence \(I_1\) is convex. It is easy to see that \(I_2\) is linear and thus \(I\) is convex. Also, \(I\) is a \(C^1\) functional. Therefore, \(I\) is weakly lower semicontinuous. As a consequence of coercivity, weak lower semicontinuity and convexity, \(I\) has a minimizer, say \(u ∈ W^{1,p}_0(Ω,w)\) which solves the equation

\begin{equation}
-\mathcal{F}_{p,w}u = ξ \quad \text{in } Ω.
\end{equation}

Uniqueness: Let \(u_1, u_2 ∈ W^{1,p}_0(Ω,w)\) solves the problem (3.4). Therefore,

\begin{equation}
∫_Ω w(x) \mathcal{F}(∇u_1)^{p-1} \nabla_η \mathcal{F}(∇u_1) \nabla φ \, dx = ∫_Ω ξ φ \, dx,
\end{equation}

and

\begin{equation}
∫_Ω w(x) \mathcal{F}(∇u_2)^{p-1} \nabla_η \mathcal{F}(∇u_2) \nabla φ \, dx = ∫_Ω ξ φ \, dx
\end{equation}
holds for every $\phi \in W_0^{1,p}(\Omega, w)$. We choose $\phi = u_1 - u_2$ and then subtracting (3.5) with (3.6), we obtain

$$
\int_{\Omega} w(x) \left\{ \mathcal{F}(\nabla u_1)^{p-1} \nabla \eta \mathcal{F}(\nabla u_1) - \mathcal{F}(\nabla u_2)^{p-1} \nabla \eta \mathcal{F}(\nabla u_2) \right\} \nabla ((u_1 - u_2)) \, dx = 0.
$$

By Lemma 2.6, we get

$$
\int_{\Omega} \left| \mathcal{F}(\nabla (u_1 - u_2)) \right|^p w \, dx = 0.
$$

Thus $u_1 = u_2$ in $\Omega$. Hence the uniqueness follows.

**Boundedness:** From Lemma 2.2, noting the continuity of $X \hookrightarrow L^l(\Omega)$ for some $l > p$ and then proceeding analogous to the proof of [9, Lemma 3.1], we obtain

$$
\|u\|_{L^\infty(\Omega)} \leq C,
$$

for some positive constant $C$ depending on $\|\xi\|_{L^\infty(\Omega)}$.

**Positivity:** Choosing $u_- := \min\{u, 0\}$ as a test function in (3.4) and since $\xi \geq 0$, by (2.10) we obtain

$$
\int_{\Omega} \mathcal{F}(\nabla u_-)^p w \, dx = \int_{\Omega} \mathcal{F}(\nabla u)^{p-1} \nabla \eta \mathcal{F}(\nabla u) \nabla u_- \, dx = \int_{\Omega} \xi u_- \, dx \leq 0,
$$

which gives, $u \geq 0$ in $\Omega$. Further $g \neq 0$ gives $u \neq 0$ in $\Omega$. Noting Corollary 2.8, we apply [43, Theorem 3.59] so that for every $\omega \in \Omega$, there exists a constant $c_\omega$ such that $u \geq c_\omega > 0$ in $\Omega$. Thus $u > 0$ in $\Omega$.

**Remark 3.2.** We remark that, when $\mathcal{F}_{p,w} = \Delta_{p,w}$ or $\mathcal{S}_{p,w}$ as given by (1.5), noting Lemma 2.5 and following the same proof above, Lemma 3.1 holds for any $1 < p < \infty$.

**Lemma 3.3.** Let $2 \leq p < \infty$. Then,

(A) (Existence) for every $n \in \mathbb{N}$, the problem (3.1) admits a positive solution $u_n \in W^{1,p}_0(\Omega, w) \cap L^\infty(\Omega)$,

(B) (Uniqueness and monotonicity) $u_n$ is unique and $u_{n+1} \geq u_n$ for every $n$,

(C) (Uniform positivity) for every $\omega \in \Omega$, there exists a constant $c_\omega$ (independent of $n$) such that $u_n \geq c_\omega > 0$ in $\omega$.

(D) (Uniform boundedness) $\|u_n\| \leq c$ for some positive constant $c$ independent of $n$.

**Proof.** (A) Let $n \in \mathbb{N}$ be fixed. By Lemma 3.1, for every $\zeta \in L^p(\Omega)$, there exists a unique positive solution $u \in W^{1,p}_0(\Omega, w) \cap L^\infty(\Omega)$ such that

$$
-\mathcal{F}_{p,w}u = \frac{f_n}{(\zeta^+ + \frac{1}{n})^\delta} + \frac{g_n}{(\zeta^+ + \frac{1}{n})^\gamma} \text{ in } \Omega.
$$

We define the operator $S : L^p(\Omega) \rightarrow L^p(\Omega)$ by $S(\zeta) = u$ where $u$ solves (3.8). Choosing $u$ as a test function in (3.8) and using the property (2.10), for some positive constant $c = c(n)$, we arrive at

$$
\|u\|^p \leq \int_{\Omega} (n^{\delta+1} + n^{\gamma+1}) u \, dx \leq c \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}}.
$$

Thus from Lemma 2.2, we have

$$
\|u\| \leq c.
$$

Now applying Schauder’s fixed point theorem as in [9, Lemma 3.2] the existence of a fixed point $u_n$ of $S$ follows. As a consequence, $u_n$ solves the problem (3.1). Moreover, by Lemma 3.1, we have $u_n > 0$ in $\Omega$ and for every $\omega \in \Omega$, there exists a constant $c_\omega$ such that $u_1 \geq c_\omega > 0$ in $\omega$. 
(B) Choosing $\phi = (u_n - u_{n+1})^+$ as a test function in (3.1) we have
$$J = \langle -F_{p,w}(u_n) + F_{p,w}(u_{n+1}), (u_n - u_{n+1})^+ \rangle$$
$$= \int_{\Omega} \left\{ \frac{f_n}{(u_n + \frac{1}{n})^\delta} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\delta} + \frac{g_n}{(u_n + \frac{1}{n})^\gamma} - \frac{g_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\gamma} \right\} (u_n - u_{n+1})^+ \, dx$$
$$:= J_1 + J_2.$$

Using the inequalities $f_n(x) \leq f_{n+1}(x)$, we obtain
$$J_1 = \int_{\Omega} \left\{ \frac{f_n}{(u_n + \frac{1}{n})^\delta} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\delta} \right\} (u_n - u_{n+1})^+ \, dx$$
$$\leq \int_{\Omega} f_{n+1} \left\{ \frac{1}{(u_n + \frac{1}{n})^\delta} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^\delta} \right\} (u_n - u_{n+1})^+ \, dx \leq 0.$$

Similarly, using $g_n(x) \leq g_{n+1}(x)$, we get
$$J_2 = \int_{\Omega} \left\{ \frac{g_n}{(u_n + \frac{1}{n})^\gamma} - \frac{g_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\gamma} \right\} (u_n - u_{n+1})^+ \, dx \leq 0.$$

Hence, we have $J \leq 0$. Noting this fact and $p \geq 2$, using Lemma 2.6, we have
$$\int_{\Omega} |F(\nabla (u_n - u_{n+1}))|^p w \, dx = 0.$$ 

Therefore, $u_{n+1} \geq u_n$ in $\Omega$. Uniqueness follows similarly.

(C) From the above estimate in (A), we know that $u_1 \geq c_\omega > 0$ for every $\omega \Subset \Omega$. Hence using the monotonicity, for every $\omega \Subset \Omega$, we get $u_n \geq c_\omega > 0$ in $\omega$, for some positive constant $c_\omega$ (independent of $n$).

(D) We only consider the case $1 \leq p_s < N$, since the other cases are analogous. To this end, we choose $u_n$ as a test function in (3.1) and using (2.10), we get
$$\|u_n\|^p \leq \int_{\Omega} f u_n^{1-\delta} \, dx + \int_{\Omega} g u_n^{1-\gamma} \, dx$$
$$\leq \|f\|_{L^{m_\delta}(\Omega)} \left( \int_{\Omega} u_n^{(1-\delta)m_\delta'} \, dx \right)^{\frac{1}{m_\delta'}} + \|g\|_{L^{r_\gamma}(\Omega)} \left( \int_{\Omega} u_n^{(1-\gamma)r_\gamma'} \, dx \right)^{\frac{1}{r_\gamma'}}$$
$$= \|f\|_{L^{m_\delta}(\Omega)} \left( \int_{\Omega} u_n^{p_s} \, dx \right)^{\frac{1}{p_s}} + \|g\|_{L^{r_\gamma}(\Omega)} \left( \int_{\Omega} u_n^{\frac{p_s}{r}} \, dx \right)^{\frac{1}{\frac{p_s}{r}}}$$
$$\leq \|f\|_{L^{m_\delta}(\Omega)} \|u_n\|^{1-\delta} + \|g\|_{L^{r_\gamma}(\Omega)} \|u_n\|^{1-\gamma},$$

where in the final step above, we have employed Lemma 2.2. Therefore, we have $\|u_n\| \leq c$, for some positive constant $c$ (independent of $n$).

\[ \square \]

Remark 3.4. We remark that, when $F_{p,w} = \Delta_{p,w}$ or $S_{p,w}$ as given by (1.5), noting Lemma 2.5 along with Remark 3.2 and following the same proof above, Lemma 3.3 holds for any $1 < p < \infty$.

Remark 3.5. As a consequence of Lemma 3.3 and Remark 3.4, let $u_{\delta,\gamma} \in W^{1,p}_0(\Omega, w)$ be the weak and pointwise limit of $u_n$. Then using the monotonicity property from (B) in Lemma 3.3, it follows that $u_n \leq u_{\delta,\gamma}$ for all $n \in \mathbb{N}$. Below, we observe that $u_{\delta,\gamma}$ is our required solution.

Lemma 3.6. Let $2 \leq p < \infty$ and suppose that $(f,g) \neq (0,0)$ is nonnegative such that

(E) $f,g \in L^q(\Omega)$ for $q > \frac{p^*}{p^* - p}$, when $1 \leq p_s < N$,

(F) $f,g \in L^q(\Omega)$ for $q > \frac{p^*}{r^* - p}$, when $p_s = N$ and $r > p$. 

(G) \( f, g \in L^1(\Omega) \) for \( p_s > N \).

Then \( \| u_n \|_{L^\infty(\Omega)} \leq C \), for some positive constant \( C \) independent of \( n \).

**Proof.** We only prove the result under the hypothesis in (E), since the other cases are analogous. To this end, let \( 1 \leq p_s < N \) and \( f, g \in L^q(\Omega) \) for \( q > \frac{p_s'}{p_s - p} \). Assume \( k \geq 1 \) and define \( A(k) = \{ x \in \Omega : u_n(x) \geq k \} \). Choosing \( \phi_k(x) = (u_n - k)^+ \) as a test function in (3.1), first using Hölder’s inequality with the exponents \( p_s^*, p_s^* \) and then, by Young’s inequality with exponents \( p \) and \( p' \), we obtain

\[
\| \phi_k \|^p = \int_\Omega \mathcal{F}(\nabla \phi_k)^p w \, dx \\
= \int_\Omega \frac{f_n}{(u_n + 1)^\delta} \phi_k \, dx + \int_\Omega \frac{g_n}{(u_n + 1)^\delta} \phi_k \, dx \\
\leq \int_{A(k)} f(x) \phi_k \, dx + \int_{A(k)} g(x) \phi_k \, dx
\]

(3.11)

\[
\leq \left( \int_{A(k)} f^{p_*'}(x) \, dx \right)^{\frac{p'}{p_*'}} \left( \int \phi_k^{p_*}(x) \, dx \right)^{\frac{1}{p_*}} + \left( \int_{A(k)} g^{p_*'}(x) \, dx \right)^{\frac{1}{p_*'}} \left( \int \phi_k^{p_*}(x) \, dx \right)^{\frac{1}{p_*}} \\
\leq C \left( \int_{A(k)} f^{p_*'}(x) \, dx \right)^{\frac{p'}{p_*'}} \| \phi_k \| + C \left( \int_{A(k)} g^{p_*'}(x) \, dx \right)^{\frac{1}{p_*'}} \| \phi_k \| \\
\leq \epsilon \| \phi_k \|^p + C(\epsilon) \left( \int_{A(k)} f^{p_*'}(x) \, dx \right)^{\frac{p'}{p_*'}} + C(\epsilon) \left( \int_{A(k)} g^{p_*'}(x) \, dx \right)^{\frac{1}{p_*'}}.
\]

Here, \( C \) is the Sobolev constant from Lemma 2.2 and \( C(\epsilon) > 0 \) is some constant depending on \( \epsilon \in (0, 1) \) but are independent of \( n \). Note that \( q > \frac{p_s'}{p_s - p} \) gives \( q > p_s^* \). Therefore, fixing \( \epsilon \in (0, 1) \) and again using Hölder’s inequality with exponents \( \frac{q}{p_*} \) and \( \left( \frac{q}{p_*'} \right)' \), for some constant \( C > 0 \) which is independent of \( n \), we obtain

\[
\| \phi_k \|^p \leq C \left( \int_{A(k)} f^{p_*'}(x) \, dx \right)^{\frac{p'}{p_*'}} + C \left( \int_{A(k)} g^{p_*'}(x) \, dx \right)^{\frac{1}{p_*'}}
\]

(3.12)

\[
\leq C \left\{ \left( \int_{\Omega} f^q \, dx \right)^{\frac{p'}{q'}} + \left( \int_{\Omega} g^q \, dx \right)^{\frac{p'}{q'}} \right\} |A(k)| \left( \int_{\Omega} \frac{1}{x^h} \, dx \right)^{\frac{1}{p_*'}} \\
\leq C |A(k)| \left( \int_{\Omega} \frac{1}{x^h} \, dx \right)^{\frac{1}{p_*'}}.
\]

Let \( h > 0 \) be such that \( 1 \leq k < h \). Then, \( A(h) \subset A(k) \) and for any \( x \in A(h) \), we have \( u_n(x) \geq h \). So, \( u_n(x) - k \geq h - k \) in \( A(h) \). Combining these facts along with (3.12) and again
using Lemma 2.2 for some constant $C > 0$ (independent of $n$), we arrive at the estimate below

$$
(h - k)^p|A(h)|^{\frac{p}{p^*}} \leq \left( \int_{A(h)} (u_n - k)^{p^*} \, dx \right)^{\frac{p}{p^*}} \\
\leq \left( \int_{A(k)} (u_n - k)^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq C \|\phi_k\|^p \leq C|A(k)|^{\frac{p'}{p^*} \left(\frac{q}{p^*}\right)}.
$$

Thus, for some constant $C > 0$ (independent of $n$), we have

$$
|A(h)| \leq \frac{C}{(h - k)^{p^*}} |A(k)|^\alpha,
$$

where

$$
\alpha = \frac{p^* p'}{pp^*} \frac{1}{\left(\frac{q}{p^*}\right)}.
$$

Due to the assumption, $q > \frac{p^*}{p^* - p}$, we have $\alpha > 1$. Hence, by [45, Lemma B.1], we have

$$
\|u_n\|_{L^\infty(\Omega)} \leq C,
$$

for some positive constant $C > 0$ independent of $n$. \qed

We end this section by establishing the following properties of $u_n$ that are very important to prove the weighted anisotropic Sobolev inequality.

**Lemma 3.7.** The solutions $u_n$ of the problem (3.1) found in Lemma 3.3 has the following properties:

(a) Let $n \in \mathbb{N}$ and $\phi \in W^{1,p}_0(\Omega, w)$. Then we have

$$
\|u_n\|^p \leq \|\phi\|^p + p \int_{\Omega} \frac{(u_n - \phi)}{(u_n + \frac{1}{n})^\delta} f_n \, dx + p \int_{\Omega} \frac{(u_n - \phi)}{(u_n + \frac{1}{n})^\gamma} g_n \, dx.
$$

(b) (Monotonicity in norm) For every $n \in \mathbb{N}$, we have $\|u_n\| \leq \|u_{n+1}\|$.

(c) (Strong convergence) Upto a subsequence $\{u_n\}$ converges strongly to $u_{\delta, \gamma}$ in $W^{1,p}_0(\Omega, w)$.

(d) Further, $u_{\delta, \gamma}$ is a minimizer of the energy functional $I_{\delta, \gamma} : W^{1,p}_0(\Omega, w) \to \mathbb{R}$ be defined by

$$
I_{\delta, \gamma}(v) := \frac{1}{p} \|v\|^p - \frac{1}{1 - \delta} \int_{\Omega} (v^+)^{1-\delta} f \, dx - \frac{1}{1 - \gamma} \int_{\Omega} (v^+)^{1-\gamma} g \, dx.
$$

**Proof.**

(a) Let us fix $\xi \in W^{1,p}_0(\Omega, w)$. Then by Lemma 3.1, there exists a unique solution $v \in W^{1,p}_0(\Omega, w)$ of the problem

$$
-\mathcal{F}_{p,w}v = \frac{f_n(x)}{(\xi^+ + \frac{1}{n})^\delta} + \frac{g_n(x)}{(\xi^+ + \frac{1}{n})^\gamma}, \quad v > 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega.
$$

Also $v$ is a minimizer of the functional $J : W^{1,p}_0(\Omega, w) \to \mathbb{R}$ given by

$$
J(\phi) := \frac{1}{p} \|\phi\|^p - \int_{\Omega} \frac{f_n}{(\xi^+ + \frac{1}{n})^\delta} \phi \, dx - \int_{\Omega} \frac{g_n}{(\xi^+ + \frac{1}{n})^\gamma} \phi \, dx.
$$
Therefore, for every \( \phi \in W^{1,p}_0(\Omega, w) \), we have \( J(v) \leq J(\phi) \) which gives
\[
\frac{1}{p} \|v\| - \int_{\Omega} \frac{f_n}{(\xi^+ + \frac{1}{n})^\delta} v \, dx - \int_{\Omega} \frac{g_n}{(\xi^+ + \frac{1}{n})^\gamma} v \, dx \\
\leq \frac{1}{p} \|\phi\| - \int_{\Omega} \frac{f_n}{(\xi^+ + \frac{1}{n})^\delta} \phi \, dx - \int_{\Omega} \frac{g_n}{(\xi^+ + \frac{1}{n})^\gamma} \phi \, dx.
\]
(3.16)

Setting \( v = \xi = u_n \) in the inequality (3.16), the estimate (3.13) follows.

(b) By Lemma 3.3, we have \( u_n \leq u_{n+1} \). Then choosing \( \phi = u_{n+1} \) in (3.13), we obtain \( \|u_n\| \leq \|u_{n+1}\| \).

(c) We choose \( \phi = u_{\delta,\gamma} \) in (3.13) and then using the property \( u_n \leq u_{\delta,\gamma} \) from Remark 3.5, we have \( \|u_n\| \leq \|u_{\delta,\gamma}\| \). Hence using the norm monotonicity property \( \|u_n\| \leq \|u_{n+1}\| \) from (b), we have
\[
\lim_{n \to \infty} \|u_n\| \leq \|u_{\delta,\gamma}\|.
\]
(3.17)

Moreover since \( u_n \rightharpoonup u_{\delta,\gamma} \) weakly in \( W^{1,p}_0(\Omega, w) \), we get
\[
\|u_{\delta,\gamma}\| \leq \lim_{n \to \infty} \|u_n\|.
\]
(3.18)

Thus from (3.17) and (3.18), the result follows.

(d) It is enough to show that
\[
I_{\delta,\gamma}(u_{\delta,\gamma}) \leq I_{\delta,\gamma}(v), \quad \forall v \in W^{1,p}_0(\Omega, w).
\]
(3.19)

Let us define the auxiliary functional \( I_n : W^{1,p}_0(\Omega, w) \to \mathbb{R} \) by
\[
I_n(v) := \frac{1}{p} \|v\| - \int_{\Omega} G_n(v) f_n \, dx - \int_{\Omega} H_n(v) g_n \, dx,
\]
where
\[
G_n(t) := \frac{1}{1 - \delta} \left( t^+ + \frac{1}{n} \right)^{1-\delta} - \left( \frac{1}{n} \right)^{-\delta} t^-,
\]
and
\[
H_n(t) := \frac{1}{1 - \gamma} \left( t^+ + \frac{1}{n} \right)^{1-\gamma} - \left( \frac{1}{n} \right)^{-\gamma} t^-.
\]

Then we observe that \( I_n \) is \( C^1 \), bounded below and coercive. As a consequence, \( I_n \) has a minimizer at some \( v_n \in W^{1,p}_0(\Omega, w) \). Therefore, it follows that \( I_n(v_n) \leq I_n(v_n^+) \), which gives \( v_n \geq 0 \) in \( \Omega \). Noting that \( < I_n'(v_n), \phi > = 0 \) for all \( \phi \in W^{1,p}_0(\Omega, w) \), we conclude that \( v_n \) solves (3.1). By the uniqueness property from Lemma 3.3, we have
\( u_n = v_n \). Hence \( u_n \) is a minimizer of \( I_n \). Therefore, we obtain
\[
I_n(u_n) \leq I_n(v_n^+), \quad \forall v \in W^{1,p}_0(\Omega, w).
\]
(3.20)

Now we pass the limit as \( n \to \infty \) in (3.20) to prove our claim (3.19). Firstly, by Remark 3.5 using the fact \( u_n \leq u_{\delta,\gamma} \) along with the Lebesgue dominated convergence theorem, we have
\[
\lim_{n \to \infty} \int_{\Omega} G_n(u_n) f_n \, dx = \frac{1}{1 - \delta} \int_{\Omega} (u_{\delta,\gamma})^{1-\delta} f \, dx,
\]
(3.21)
\[
\lim_{n \to \infty} \int_{\Omega} H_n(u_n) g_n \, dx = \frac{1}{1 - \gamma} \int_{\Omega} (u_{\delta,\gamma})^{1-\gamma} g \, dx.
\]

Furthermore, by the strong convergence property (c) above, we have
\[
\lim_{n \to \infty} \|u_n\| = \|u_{\delta,\gamma}\|.
\]
(3.22)
Hence, using (3.21) and (3.22), we have
\begin{equation}
\lim_{n \to \infty} I_n(u_n) = I_{\delta,\gamma}(u_{\delta,\gamma}).
\end{equation}

Moreover, for any \( v \in W^{1,p}_0(\Omega, w) \), we have
\begin{equation}
\lim_{n \to \infty} \int_{\Omega} G_n(v^+) f_n \, dx = \frac{1}{1-\delta} \int_{\Omega} (v^+)^{1-\delta} f \, dx, \\
\lim_{n \to \infty} \int_{\Omega} H_n(v^+) g_n \, dx = \frac{1}{1-\gamma} \int_{\Omega} (v^+)^{1-\delta} g \, dx.
\end{equation}

Finally, letting \( n \to \infty \) in (3.20) and then employing the estimates (3.23), (3.24) and that \( \|v^+\| \leq \|v\| \), the claim (3.19) follows.

\[ \square \]

4. Auxiliary results for the problem \((\mathcal{R})\)

This section deals to establish some results that are very crucial to prove Theorem 2.21-2.22. In order to proceed, let \( h \in L'(\Omega) \setminus \{0\} \) be nonnegative as given by Theorem 2.21 and we consider the following approximated problem for every \( n \in \mathbb{N} \), given by
\begin{equation}
-\mathcal{F}_{p,w} u = h_n(x) e^{\left(\frac{1}{u^+ + \frac{1}{n}}\right)} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,
\end{equation}
where \( h_n(x) = \min\{h(x), n\} \).

Then, we have the following results for the problem (4.1).

**Lemma 4.1.** Let \( 2 \leq p < \infty \). Then,
\begin{itemize}
  \item[(A)] for every \( n \in \mathbb{N} \), the problem (4.1) admits a positive solution \( v_n \in W^{1,p}_0(\Omega, w) \),
  \item[(B)] \( v_n \) is unique and \( v_{n+1} \geq v_n \) for every \( n \),
  \item[(C)] for every \( \omega \Subset \Omega \), there exists a constant \( c_\omega \) (independent of \( n \)) satisfying \( v_n \geq c_\omega > 0 \) in \( \omega \),
  \item[(D)] \( \|v_n\|_{L^\infty(\Omega)} \leq c \) for some positive constant \( c \) independent of \( n \).
\end{itemize}

**Proof.** (A) Let us fix \( n \in \mathbb{N} \), \( v \in L^p(\Omega) \) and denote by
\[ H_n = h_n e^{\left(\frac{1}{u^+ + \frac{1}{n}}\right)}. \]

Then, noting the fact that
\[ H_n \in L^\infty(\Omega) \text{ and } H_n v_n^- \leq 0, \]
the existence of a positive solution \( v_n \in W^{1,p}_0(\Omega, w) \) of the problem (4.1) follows by the same arguments as in the proof of Lemma 3.3.

(B) Observing that
\[ (H_n - H_{n+1})(v_n - v_{n+1})^+ \leq 0, \]
and then proceeding similarly as in the proof of Lemma 3.3, we obtain the monotonicity of \( v_n \). The uniqueness is analogous.

(C) We observe that \( v_n \neq 0 \) in \( \Omega \) and note that \( v_n \geq v_1 \) from (B). Then proceeding analogous to the proof of Lemma 3.3, for every \( \omega \Subset \Omega \), we have \( v_n \geq c_\omega > 0 \) for some constant \( c_\omega > 0 \) that is independent of \( n \).

(D) Now the uniform boundedness of \( \{v_n\} \) in \( L^\infty(\Omega) \) follows analogously in the proof of Lemma 3.6.

\[ \square \]
Remark 4.2. If $F_{p,w} = \Delta_{p,w}$ or $S_{p,w}$ as given by (1.5), noting Lemma 2.5 and following the exact arguments in the proof above, Lemma 4.1 holds for any $1 < p < \infty$.

5. Proofs of the main results

Proof of Theorem 2.11:

(a) Let $u, v \in W^{1,p}_0(\Omega, w)$ be weak solutions of $(S)$. Then arguing similarly as in the proof of [9, Lemma 2.13], we choose $\phi = (u - v)^+ \in W^{1,p}_0(\Omega, w)$ as a test function in (2.14) and obtain

\begin{equation}
\int_{\Omega} w(x) F(\nabla u)^{p-1} \nabla \eta F(\nabla u) \nabla (u - v)^+ \, dx = \int_{\Omega} \left( \frac{f}{u^\gamma} + \frac{g}{v^\gamma} \right) (u - v)^+ \, dx, \tag{5.1}
\end{equation}

\begin{equation}
\int_{\Omega} w(x) F(\nabla v)^{p-1} \nabla \eta F(\nabla v) \nabla (u - v)^+ \, dx = \int_{\Omega} \left( \frac{f}{v^\delta} + \frac{g}{v^\gamma} \right) (u - v)^+ \, dx. \tag{5.2}
\end{equation}

Subtracting (5.1) and (5.2), we have

\begin{equation}
\int_{\Omega} w(x) \{ F(\nabla u)^{p-1} \nabla \eta F(\nabla u) - F(\nabla v)^{p-1} \nabla \eta F(\nabla v) \} \nabla (u - v)^+ \, dx \\
= \int_{\Omega} \left( \frac{1}{u^\gamma} - \frac{1}{v^\delta} \right) f + \left( \frac{1}{u^\gamma} - \frac{1}{v^\gamma} \right) g \left( u - v \right)^+ \, dx \leq 0.
\end{equation}

Thus, applying Lemma 2.6, we obtain

\begin{equation}
\int_{\Omega} F(\nabla (u - v)^+)^p w \, dx = 0,
\end{equation}

which gives $u \leq v$ in $\Omega$. In a similar way, we have $v \leq u$ in $\Omega$. Hence the result follows.

(b) By Lemma 3.3, for every $n \in \mathbb{N}$, there exists $u_n \in W^{1,p}_0(\Omega, w)$ such that

\begin{equation}
\int_{\Omega} w(x) F(\nabla u_n)^{p-1} \nabla \eta F(\nabla u_n) \nabla \phi \, dx = \int_{\Omega} \left( \frac{f_n}{w_n^\delta} + \frac{g_n}{w_n^\gamma} \right) \phi \, dx, \quad \forall \phi \in C^1_c(\Omega). \tag{5.3}
\end{equation}

Passing to the limit. By the strong convergence property (c) in Lemma 3.7 along with Lemma 2.2, up to a subsequence, we have $\nabla u_n \rightharpoonup \nabla u_{\delta, \gamma}$ pointwise almost everywhere in $\Omega$. Therefore, for every $\phi \in C^1_c(\Omega)$, it holds that

\begin{equation}
\lim_{n \to \infty} \int_{\Omega} w(x) F(\nabla u_n)^{p-1} \nabla \eta F(\nabla u_n) \nabla \phi \, dx = \int_{\Omega} w(x) F(\nabla u_{\delta, \gamma})^{p-1} \nabla \eta F(\nabla u_{\delta, \gamma}) \nabla \phi \, dx. \tag{5.4}
\end{equation}

Denote by $\text{supp} \phi = \omega \subseteq \Omega$ and thus by Lemma 3.3, there exists a constant $c_\omega > 0$ that is independent of $n$ such that $u_n \geq c_\omega > 0$ in $\omega$. Hence, we get

\begin{equation}
\left( \frac{f_n}{w_n^\delta} + \frac{g_n}{w_n^\gamma} \right) \phi \leq \| \phi \|_{L^\infty(\Omega)} \left( \frac{f}{c_\omega^\delta} + \frac{g}{c_\omega^\gamma} \right) \in L^1(\Omega).
\end{equation}

Since $u_n \rightharpoonup u_{\delta, \gamma}$ pointwise almost everywhere in $\Omega$, as an application of the Lebesgue dominated convergence theorem, we deduce that

\begin{equation}
\lim_{n \to \infty} \int_{\Omega} \left( \frac{f_n}{w_n^\delta} + \frac{g_n}{w_n^\gamma} \right) \phi \, dx = \int_{\Omega} \left( \frac{f}{u_{\delta, \gamma}^\delta} + \frac{g}{u_{\delta, \gamma}^\gamma} \right) \phi \, dx. \tag{5.5}
\end{equation}

Combining the estimates (5.4) and (5.5) in (5.3), the result follows. \qed
Proof of Theorem 2.12: The proof follows from Lemma 3.6.

Proof of Theorem 2.13: Noting Lemma 2.5 and Remark 3.4, following the lines of the proof of Theorem 2.11-2.12, the result follows.

Proof of Theorem 2.17:

(a) Let us set
\[ S_\delta := \left\{ v \in W^{1,p}_0(\Omega, w) : \int_{\Omega} |v|^{1-\delta} f \, dx = 1 \right\}. \]

Then it is enough to prove that
\[ \mu(\Omega) := \inf_{v \in S_\delta} \|v\|^p = \|u_\delta\|^p \frac{p(1-\delta-\mu)}{1-\delta}. \]

It is easy to verify that \( V_\delta = \zeta_\delta u_\delta \in S_\delta \), where
\[ \zeta_\delta = \left( \int_{\Omega} u_\delta^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}}. \]

Following the proof of [9, Lemma 2.13], we choose \( \phi = u_\delta \in W^{1,p}_0(\Omega, w) \) as a test function in \((2.14)\) and noting the property \((2.10)\), we have
\[ (5.6) \int_{\Omega} F(\nabla u_\delta)^p w \, dx = \|u_\delta\|^p = \int_{\Omega} u_\delta^{1-\delta} f \, dx. \]

First, using the homogeneity property \((H2)\) and then, by \((5.6)\), we have
\[ (5.7) \|V_\delta\|^p = \int_{\Omega} F(\nabla V_\delta)^p w \, dx = \zeta_\delta^p \int_{\Omega} F(\nabla u_\delta)^p w \, dx = \|u_\delta\|^p \frac{p(1-\delta-\mu)}{1-\delta}. \]

Let \( v \in S_\delta \) and define by \( \mu = \|v\|^p \frac{p}{p+\delta-1} \). Then by Lemma 3.7, since \( u_\delta \) minimizes the functional \( I_{\delta,\gamma} \) given by \((3.14)\), we have
\[ (5.8) I_{\delta,\gamma}(u_\delta) \leq I_{\delta,\gamma}(\mu|v|). \]

Using \((5.6)\), we have
\[ (5.9) I_{\delta,\gamma}(u_\delta) = \frac{1}{p} \|u_\delta\|^p - \frac{1}{1-\delta} \int_{\Omega} u_\delta^{1-\delta} f \, dx = \left( \frac{1}{p} - \frac{1}{1-\delta} \right) \|u_\delta\|^p. \]

On the other hand, since \( v \in S_\delta \), we have
\[ (5.10) I_{\delta,\gamma}(\mu|v|) = \frac{\mu^p}{p} \|v\|^p - \frac{\mu^{1-\delta}}{1-\delta} \leq \frac{\mu^p}{p} \|v\|^p - \frac{\mu^{1-\delta}}{1-\delta} = \left( \frac{1}{p} - \frac{1}{1-\delta} \right) \|v\|^p \frac{p(1-\delta-\mu)}{1-\delta}. \]

Since \( v \in S_\delta \) is arbitrary, using \((5.9)\) and \((5.10)\) in \((5.8)\), we arrive at
\[ (5.11) \|u_\delta\| \frac{p(1-\delta-\mu)}{1-\delta} \leq \inf_{v \in S_\delta} \|v\|^p. \]

Using \((5.7)\) and \((5.11)\), we obtain
\[ (5.12) \|V_\delta\|^p = \|u_\delta\| \frac{p(1-\delta-\mu)}{1-\delta} \leq \inf_{v \in S_\delta} \|v\|^p. \]

Since \( V_\delta \in S_\delta \), from \((5.12)\), the result follows.
(b) Let \((2.16)\) holds. If \(C > \mu(\Omega)\), then from \((a)\) above and \((5.7)\), we obtain
\[
C \left( \int_{\Omega} V_\delta^{1-\delta} f \, dx \right)^{\frac{p}{1-\delta}} > \int_{\Omega} F(\nabla V_\delta)^p w \, dx.
\]
Since \(V_\delta \in W^{1,p}_0(\Omega, w)\), \((5.13)\) violates the hypothesis \((2.16)\). Conversely, assume that
\[
C \leq \mu(\Omega) = \inf_{v \in S_\delta} \|v\|^p \leq \|V\|^p,
\]
for all \(V \in S_\delta\). We observe that the claim directly follows if \(v = 0\). So we deal with the case when \(v \in W^{1,p}_0(\Omega, w) \setminus \{0\}\) which gives
\[
V = \left( \int_{\Omega} |v|^{1-\delta} f \, dx \right)^{-\frac{1}{\delta}} v \in S_\delta.
\]
Therefore, we have
\[
C \leq \left( \int_{\Omega} |v|^{1-\delta} f \, dx \right)^{-\frac{p}{\delta}} \|v\|^p.
\]
Hence, the result follows. \(\square\)

**Proof of Theorem 2.18:** Noting Lemma 2.5 and Remark 3.4, following the lines of the proof of Theorem 2.17, the result follows. \(\square\)

**Proof of Theorem 2.21:** Let \(\Omega = \cup_l \Omega_l\) where \(\Omega_l \subset \Omega_{l+1}\) are open subsets for each \(l\). By Lemma 4.1, we have \(v_n \in L^\infty(\Omega)\) and \(c_l = \inf_{\Omega_l} v_n > 0\). Then choosing \(\phi = (v_n - c_1)^+\) as a test function in \((4.1)\) and using \((2.10)\), we obtain
\[
\|(v_n - c_1)^+\|^p = \int_{\Omega} w(x) F(\nabla v_n)^{p-1} \nabla v_n \nabla (v_n - c_1)^+ \, dx
\]
\[
= \int_{\Omega} h_n e^{\frac{(v_n - c_1)^+}{\delta}} (v_n - c_1)^+ \, dx
\]
\[
\leq e^{\frac{1}{\delta}} \|h\|_{L^1(\Omega)} \|(v_n - c_1)^+\|_{L^\infty(\Omega)}
\]
\[
\leq c e^{\frac{1}{\delta}} \|h\|_{L^1(\Omega)} \|(v_n - c_1)^+\|,
\]
for some constant \(c > 0\) independent of \(n\), where in the final step above, we have used the embedding result Lemma 2.2. Therefore, we obtain from \((5.14)\) that \(\{v_n\}\) is uniformly bounded in \(W^{1,p}(\Omega_1, w)\) and thus by Lemma 2.2, there exists \(v_{\Omega_1} \in W^{1,p}(\Omega_1, w)\) such that up to a subsequence \(\{v_{n_k}\}\) converges weakly in \(W^{1,p}(\Omega_1, w)\), strongly in \(L^p(\Omega_1)\) and almost everywhere in \(\Omega_1\) to \(v_{\Omega_1}\), say. Proceeding by induction argument, for every \(l\), up to a subsequence \(\{v_{n_k}\}\) of \(\{v_n\}\), there exists \(v_{\Omega_l} \in W^{1,p}(\Omega_l, w)\) such that \(\{v_{n_k}\}\) converges weakly in \(W^{1,p}(\Omega_l, w)\), strongly in \(L^p(\Omega_l)\) and almost everywhere in \(\Omega_l\) to \(v_{\Omega_l}\). Let \(\{v_{n_k}\}\) be a subsequence of \(\{v_{n_k}\}\) for every \(l\), where \(n_k \to \infty\) as \(l \to \infty\). Hence, \(v_{\Omega_{l+1}} = v_{\Omega_l}\) in \(\Omega_l\) and we define by \(v = v_{\Omega_1}\) in \(\Omega_1\), \(v = v_{\Omega_{l+1}}\) in \(\Omega_{l+1} \setminus \Omega_l\) for each \(l\). Therefore, \(v \in W^{1,p}_0(\Omega, w)\) and also lie in \(L^\infty(\Omega)\) due to the property \((D)\) from Lemma 4.1. As a consequence of our definition, the diagonal subsequence \(\{v_{n_l}\} := \{v_{n_l}\}\) satisfies
\[
v_{n_l} \to v \text{ in } W^{1,p}_{\text{loc}}(\Omega_l, w),
\]
\[
v_{n_l} \to v \text{ in } L^p_{\text{loc}}(\Omega_l),
\]
\[
v_{n_l} \to v \text{ almost everywhere in } \Omega_l.
\]
We claim that \( \{v_{n_i}\} \) converges strongly to \( v \) in \( W_{\text{loc}}^{1,p}(\Omega_l, w) \). To see this, let \( \omega \subset \Omega \) and \( \phi \in C_c^\infty(\Omega) \) be such that \( 0 \leq \phi \leq 1 \) in \( \Omega \), \( \phi \equiv 1 \) in \( \omega \) and suppose \( l_1 \geq 1 \) such that \( \Omega' := \text{supp} \phi \subset \Omega_{l_1} \). Then, by a direct computation for every \( l, m \geq 1 \), we have

\[
\int_\Omega w(x)\{F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) - F(\nabla v_{n_m})^{p-1}\nabla \eta F(\nabla v_{n_m})\} \nabla (v_{n_i} - v_{n_m}) \, dx
\]

\[
\leq \int_{\Omega'} w(x)\{F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) - F(\nabla v_{n_m})^{p-1}\nabla \eta F(\nabla v_{n_m})\} \nabla (\phi_{n_i} - \phi_{n_m}) \, dx
\]

\[
- \int_{\Omega_{l_1}} w(x)(v_{n_i} - v_{n_m})\{F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) - F(\nabla v_{n_m})^{p-1}\nabla \eta F(\nabla v_{n_m})\} \nabla \phi \, dx
:= I - J.
\]

**Estimate of \( I \):** We choose \( \phi(v_{n_i} - v_{n_m}) \) as a test function in (4.1) and obtain for \( j = l, m \) that

\[
\int_{\Omega} w(x)F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) \nabla (\phi_{n_i} - \phi_{n_m}) \, dx
\]

\[
\leq \int_{\Omega'} h_{n_i} e^{(\eta_{n_j} + \frac{1}{2})} |v_{n_i} - v_{n_m}| \, dx
\]

\[
\leq c\|h\|_{L^1(\Omega)} \|v_{n_i} - v_{n_m}\|_{L^\infty(\Omega')}
\]

\[
\leq c\|h\|_{L^1(\Omega)} \|v_{n_i} - v_{n_m}\|_{L^p(\Omega')}
\]

for a constant \( c \) independent of \( l, m \). Therefore, using (5.15), the last quantity in the above estimate goes to zero as \( l, m \to \infty \). As a consequence, we arrive at

\[
I = \int_{\Omega} w(x)\{F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) - F(\nabla v_{n_m})^{p-1}\nabla \eta F(\nabla v_{n_m})\} \nabla (\phi_{n_i} - \phi_{n_m}) \, dx \to 0
\]

as \( l, m \to \infty \).

**Estimate of \( J \):** Using (2.11) and Hölder’s inequality, we get

\[
\int_{\Omega_{l_1}} w(x)(v_{n_i} - v_{n_m})F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) \nabla \phi \, dx
\]

\[
\leq C_2 \|\nabla \phi\|_{L^\infty(\Omega)} \left( \int_{\Omega'} w|\nabla v_{n_i}|^p \, dx \right)^\frac{p-1}{p} \left( \int_{\Omega'} w|v_{n_i} - v_{n_m}|^p \, dx \right)^\frac{1}{p}
\]

Noting (5.15) and [50, Theorem 2.14], the last quantity in the above estimate goes to zero as \( l, m \to \infty \). Therefore,

\[
J = \int_{\Omega_{l_1}} w(x)(v_{n_i} - v_{n_m})\{F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) - F(\nabla v_{n_m})^{p-1}\nabla \eta F(\nabla v_{n_m})\} \nabla \phi \, dx \to 0
\]

as \( l, m \to \infty \). Using the above estimates on \( I \) and \( J \), applying the Finsler algebraic inequality from Lemma 2.6 in (5.16), the sequence \( \{v_{n_i}\} \) converges strongly to \( v \) in \( W_{\text{loc}}^{1,p}(\omega, w) \). Now we pass to the limit in (4.1) and prove that \( v \) is our required solution. To this end, assume that \( \phi \in C_c^\infty(\Omega) \) such that \( \text{supp} \phi \subset \Omega_{l_1} \) for some \( l_1 \geq 1 \). Then by the strong convergence of \( v_{n_i} \) to \( v \) in \( W_{\text{loc}}^{1,p}(\Omega, w) \), we have

\[
\lim_{l \to \infty} \int_{\Omega} w(x)F(\nabla v_{n_i})^{p-1}\nabla \eta F(\nabla v_{n_i}) \nabla \phi \, dx = \int_{\Omega} w(x)F(\nabla v)^{p-1}\nabla \eta F(\nabla v) \nabla \phi \, dx
\]

Moreover, by Lemma 4.1, we obtain

\[
\left| h_{n_i} e^{(\eta_{n_j} + \frac{1}{2})} \phi \right| \leq ch \in L^1(\Omega),
\]
for some constant $c > 0$ independent of $l$. By the Lebesgue dominated convergence theorem, we have

$$
\lim_{l \to \infty} \int_{\Omega} h_n(x) e^{-\frac{v_{nl}(x)}{c}} \phi(x) \, dx = \int_{\Omega} h(x) e^{-\frac{v}{c}} \phi(x) \, dx.
$$

Thus, from (5.17) and (5.18), we conclude that $v \in W^{1,p}_{loc}(\Omega, w)$ is a weak solution of the problem (R). Further, it can be easily seen that $\{v_{nl} - \epsilon\}$ is uniformly bounded in $W^{1,p}_{0}(\Omega, w)$ for every $\epsilon > 0$ and hence $(v-\epsilon^+ \in W^{1,p}_{0}(\Omega, w)$. Since by Lemma 4.1, we have $v_{nl} \geq c_\omega > 0$ for every $\omega \in \Omega$, we obtain $v \geq c_\omega > 0$ in $\omega$. Hence $v > 0$ in $\Omega$ and the result follows. □

**Proof of Theorem 2.22:** Noting Lemma 2.5 along with Remark 4.2 and then following the same proof of Theorem 2.21, the result follows. □

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