No naked singularities in homogeneous, spherically symmetric bubble spacetimes?

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We study the evolution of bubble spacetimes in vacuum and electrovac scenarios by numerical means. We find strong evidence against the formation of naked singularities in (i) scenarios with negative masses displaying initially collapsing conditions and (ii) scenarios with negative masses displaying initially expanding conditions, previously reported to give rise to such singularities. Additionally, we show that the presence of strong gauge fields implies that an initially collapsing bubble bounces back and expands. By fine-tuning the strength of the gauge field we find that the solution approaches a static bubble solution.

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I. INTRODUCTION

Within five-dimensional Kaluza-Klein theory, negative energy initial data configuration at a moment of time symmetry have been presented in the past [1, 2]. Among the reasons for considering negative energy solutions is that naked singularities are associated to them. Therefore studying such solutions is an attractive problem to further test the cosmic censorship conjecture [3]. Additionally, as these bubbles can be obtained via semiclassical tunneling (following Witten’s construction [4, 5]), it is important to understand their dynamical behavior. To obtain a better feeling for the dynamics of the solutions presented in [2], which describe a bubble spacetime, Corley and Jacobson analyzed the initial acceleration of the bubble (which can be calculated analytically with the information contained on a single hypersurface.) They found that negative mass bubbles start out expanding and consequently argued a naked singularity would be unlikely. This conjecture is backed by the calculations at the initial hypersurface together with the intuition gained from a time symmetric scenario. Namely, that if the bubble were to reverse its behavior it will go through another time-symmetric phase [18], which would also suggest the bubble should expand (assuming the solution at this time of symmetry still belongs to the family presented in [2]).

Recently however, a numerical study was presented [7] which computed explicitly the solution to the future of the initial, time-symmetric, initial data. In the negative mass case, the solution found appeared to indicate a very puzzling development. Namely, that as the bubble expanded, it would encounter a naked singularity on its way.

In the present work we reexamine the dynamical behavior of the data presented in [2] via numerical simulations, and further study a more general case where the initial data describes an initially collapsing bubble with negative mass. We restrict our simulations to the context of homogeneous spherically symmetric spacetimes. To summarize our findings, we see that negative mass bubbles will not lead to a naked singularity. By choosing different members of the initial data family, these could start out either expanding or collapsing. In the former case, the bubble continues on expanding while in the latter case, the bubble bounces back without ever collapsing. We additionally study the case of a positive mass bubble which starts out collapsing. Here, depending on the strength of the gauge field at the initial hypersurface, the bubble collapses to a black string or bounces back to expand “forever” (i.e. for as long as we let the simulation run). Furthermore, these two distinct possibilities give rise to critical behavior on the threshold of collapse or expansion.

II. SET UP: BUBBLE DATA

We consider a generalization of the time symmetric family of initial data presented in [2]. We start with a spacetime endowed with the metric

\[ ds^2 = -dt^2 + U(r)dz^2 + \frac{dr^2}{U(r)} + r^2d\Omega^2, \]  

where \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \) is the standard metric on the unit two-sphere \( S^2 \) and \( U(r) \) is a smooth function that has a regular root at some \( r = r_+ > 0 \), is everywhere positive for \( r > r_+ \) and converges to one as \( r \to \infty \). The coordinate \( z \) parametrizes the extra dimension \( S^1 \) which has the period \( P = 4\pi/U'(r_+) \). The resulting spacetime \( \{t, z, r \geq r_+, \theta, \phi\} \) constitutes a regular manifold with the topology \( \mathbb{R} \times \mathbb{R}^2 \times S^2 \). The bubble is located where the circumference of the extra dimension shrinks to zero, that is, at \( r = r_+ \).

Additionally, we consider the presence of a \( U(1) \) gauge field in the following form,

\[ A_a dx^a = k(r_+^{-n} - r^{-n})dz, \]  

with \( k \) an arbitrary constant and \( n \) an integer greater than one. It is important to point out two things. First, the gauge field here considered has a more generic dependence than those previously considered in [2], where only the case \( n = 2 \) was discussed. Second, the symmetries of the problem would a priori allow for the gauge field to have non-trivial \( t \) and \( r \) components. However, it follows from Maxwell’s equations that those contributions would...
lead to a Coulomb-like electric field which is divergent at the location of the bubble. For this reason we set those components of $A_n$ to zero.

In the time-symmetric case, initial data satisfying the Hamiltonian constraint obeys

$$U(r) = 1 - \frac{m}{r} + \frac{\vec{k}^2}{r^2},$$

where $\vec{k} \equiv kn/\sqrt{(n-1)(2n-1)}$. The parameter $m$ is related to the ADM mass via $M_{ADM} = m/4$.

The fact that the bubble be located at $r = r_+$ requires that $0 = U(r_+) = 1 - \bar{m} + \bar{b} - \bar{k}^2$, where $\bar{m} \equiv m/r_+$, $\bar{b} \equiv b/r_+^2$, $\bar{k} \equiv k/r_+^n$. We also require

$$0 < r_+ U'(r_+) = 2 - \bar{m} + 2(n-1)\bar{k}^2$$

and avoid the conical singularity at $r = r_+$ by fixing the period of $z$ to $P = 4\pi/ U'(r_+)$. Repeating the analysis in [6] one finds that the initial acceleration of the bubble area is given by

$$\ddot{A} = 8\pi \left[ 1 - \bar{m} - \frac{4\bar{k}^2}{3} (n-1)(n-2) \right].$$

For $n = 2$ this agrees with the formula found by Corley and Jacobson, and as they discussed, for negative mass bubbles, the initial acceleration is positive, and only if chosen appropriately can a positive mass bubble have negative initial acceleration. As we will see later, in the vacuum case, this acceleration remains negative leading to a collapse of the bubble and the formation of a black string. In the non-vacuum case however, the strength of the gauge field can modify this behavior completely. For weak enough $k$ the bubble continues to collapse whereas when $k$ is large the bubble area bounces back and expands. Furthermore, by fine-tuning the parameter $k$ we will show that the solution approaches a static bubble solution to the five-dimensional Einstein-Maxwell equations.

We also consider non-vacuum initial data with $n > 2$ which allow for richer scenarios. A particularly interesting case is that describing an initially negative acceleration with negative mass. If this were possible, and the bubble continued to collapse, this would likely give rise to a naked singularity. Thus, we are interested in data for which $\dot{A} < 0$. In view of the inequality [23] and Eq. (2.5) this means that

$$2 + 2(n-1)\bar{k}^2 > \bar{m} > 1 - \frac{4\bar{k}^2}{3} (n-1)(n-2).$$

From this we see that for $n = 2$ the mass can only be positive. In contrast to this, for $n > 2$ and $\bar{k}$ large enough such that $4\bar{k}^2(n-1)(n-2) > 3$, negative mass bubbles can be considered.

### III. NUMERICAL IMPLEMENTATION

A detailed description of the numerical techniques used in our code will be presented elsewhere [8]. Here we point out its salient features.

1. The equations are written in first order symmetric hyperbolic form, with a gauge choice that is such that the characteristic speeds are given by 0, 1 or −1. These properties facilitate achieving a stable implementation of the equations.

2. Regularity conditions are enforced at the bubble and employed to design a numerical scheme that satisfies a discrete energy estimate [10]. This discrete energy estimate guarantees numerical stability for a related toy model problem.

3. Constraint preserving boundary conditions are enforced at the outer boundary to ensure no constraint violating modes are fed through it [11, 12].

4. The equations are implemented using second order accurate finite difference techniques. A non-uniform radial coordinate is employed to improve the resolution near the bubble. For all cases presented here, the radial grid covered $r \in [1, 20]$ with a total of 8000 points. The outer boundary’s location is chosen so as to have always less than a crossing time in our evolutions, guaranteeing outer boundary conditions (though consistent with the constraints) will not affect quantities at the bubble. Second order convergence is checked in all runs. As an example, the convergence of some of the simulations representing expanding bubbles is illustrated in Fig. 2. Similar behavior is observed in all cases [10].

### IV. RESULTS

#### A. Brill-Horowitz initially expanding case: ($n = 2$)

Here we evolve the Brill-Horowitz initial data in the case of vanishing gauge field. The bubble area $A$ as a function of the proper time $\tau$ at the bubble is shown in Fig. 2 for different values of the mass parameter $m$. Figure 2 illustrates the convergence of the solution throughout the evolution. As expected, the lower the mass of the initial configuration, the faster the expansion. Empirically, and for the parameter ranges used in our runs, we found that at late times the expansion rate is given by

$$\frac{\dot{A}}{A} \approx \frac{2 - \bar{m}}{r_+(\tau = 0)},$$

where a dot denotes the derivative with respect to proper time $\tau$. In particular this approximation is valid for the bubble solution exhibited by Witten [4] which describes the time evolution in the case $\bar{m} = 0$. We monitored several curvature invariant quantities (the Kretschmann invariant in particular) for our numerical spacetimes and found no sign of divergence.
mass is positive, one expects this singularity to be hidden behind an event horizon, and one should obtain a black string. In fact, for the solutions which are initially collapsing and which have vanishing gauge field, we observe the formation of an apparent horizon. Furthermore, we compute the curvature invariant quantity \( I R^4_{\text{AH}} \) at the apparent horizon (as discussed in [13]), where \( I = R_{abcd}R^{abcd} \) is the Kretschmann invariant and \( R_{\text{AH}} \) the areal radius of the horizon. For a neutral black string, this invariant is 12. Figure 3 shows how this value is attained after the apparent horizon forms for representative vacuum cases (with \( m = 1.1 \) and \( m = 1.99 \)), thus providing strong evidence for the formation of a black string. (We also find apparent horizons forming, or already present in the initial data, when a nontrivial gauge field is considered and the parameters are appropriately chosen. The corresponding bubbles will likely collapse to charged black strings and will be analyzed elsewhere [8].)

As mentioned, for strong enough gauge fields, the previously described dynamics is severely affected. Figure 3 shows the bubble area vs. asymptotic time for \( m = 1.1 \) (solid line) and 1.99 (dashed line). The first non-zero values of the lines mark the formation of the apparent horizon. At late times, both lines approach the value of 1 suggesting a black string has formed.

**B. Brill-Horowitz initially collapsing case: \((n = 2)\)**

Next, we analyze the Brill-Horowitz initial data for the case in which the bubble is initially collapsing (notice that for \( n = 2 \) this implies that the ADM mass is positive). While our numerical simulations reveal that in the absence of the gauge field such a bubble continues to collapse, we also show that when the gauge field is strong enough, the bubble shrinks at a rate which decreases with time and then bounces back.

Obviously, if the collapse trend were not halted, a singularity should form at the origin. Since the ADM

![FIG. 1: Bubble area vs. proper time at the bubble. In this and the following plots, we set \( r_+ = 1 \) (by choosing \( b \) appropriately). The figure shows four illustrative examples of bubbles whose initial acceleration is positive. As it is evident, the expansion of the bubble continues and the difference is the rate of the exponential expansion. The relative error in these curves is estimated to be well below 0.001%.](image1)

![FIG. 2: Convergence factor (CF) vs. proper time for the cases illustrated in Fig. 1. This factor, defined as \( CF = \log_2(\|A(4\Delta) - A(2\Delta)\|/\|A(2\Delta) - A(\Delta)\|) \) should give the value of 2 for a second order accurate implementation. Through these runs the grid spacing is defined as \( \Delta = 3.75 \times 10^{-3} \) Clearly, the plot shows that second order convergence has been obtained throughout the runs (the peaks are instances with 0/0 where the expression for CF is ill-defined).](image2)

![FIG. 3: Rescaled Kretschmann invariant \( I R^4_{\text{AH}}/12 \) vs. asymptotic time for \( m = 1.1 \) (solid line) and 1.99 (dashed line). The first non-zero values of the lines mark the formation of the apparent horizon. At late times, both lines approach the value of 1 suggesting a black string has formed.](image3)
pansion of the black hole solutions described by Eqs. (2.13) and (2.15) of Ref. [13] followed by the transformations $r_- \rightarrow -r_-, r \rightarrow r - r_-, r_+ \rightarrow r_+ - r_-$. The parameters $r_-$ and $r_+$ ($> r_-)$ are related to the period of the $z$ coordinate and to the ADM mass via $P = 4\pi r_+ (1 - r_- / r_+)^{3/2}$ and $M_{ADM} = r_+/4$. Since these parameters are conserved throughout evolution, the initial data fixes the member of the static family the dynamical solution approaches to. Details of the critical behavior exhibited by this system will be presented in [8].

C. Collapsing negative mass case

We here restrict to cases with negative masses that start out collapsing. Interestingly enough we find that even when starting with large initial negative accelerations, which in turn make the bubble shrink in size to very tiny values, it bounces back without ever collapsing into a naked singularity. As an example, Fig. 5 shows the bubble’s area versus time for different values of $n$ and $k$. The initially collapsing bubbles decrease in size in a noticeable way but this trend is halted and the bubbles bounce back and expand. Although we have not found a simple law as that in Eq. (4.1), clearly the bubbles expand exponentially fast. Therefore, it seems not to be possible to “destroy” the bubble and create a naked singularity. This is somewhat similar to the scenarios where one tries to “destroy” an extremal Reissner-Nordström black hole by attempting to drop into it a test particle with high charge to mass ratio. There, the electrostatic repulsion prevents the particle from entering the hole [14].

\[
\frac{P}{M_{ADM}} = \frac{16\pi}{\bar{m}(2 - \bar{m})}. \tag{5.1}
\]

Since $0 < \bar{m} < 2$ this value is always larger than that needed for the Gregory-Laflamme “instability” [15]. Thus, a slight $z$-dependent perturbation of the scenario considered here could indeed give rise to a naked singularity. This would require infinite time to appear as shown in [14], but certainly would give rise to rich dynamics as illustrated in [17].

In the non-vacuum case, the presence of a gauge field gives rise to additional features. First, for the positive mass case, it can “halt” the collapse and subsequently induce an expansion. Furthermore, by fine-tuning the strength of the field, the solution approaches a static bubble. This critical solution separates the collapsing bubbles (which, likely, collapse to a charged black string) from the ones that expand at late times. In a subsequent paper [8] we will show that the dynamical behavior near
the static bubble exhibits critical phenomena. Note that clearly the solution does encounter another moment of time symmetry at the critical value. However, for field strengths larger than the critical one, this is not the case. The bubble bounces but the solution does not go through a time-symmetric phase.

Last, the gauge field can also make negative mass bubbles start out collapsing if its radial dependence is steeper than that originally considered in [2]. Intuitively this happens due to the contribution of the energy in the gauge field overcoming the repulsive gravitational effect of the negative mass bubble. However, as part of the field is radiated away, the collapse is halted and a subsequent expansion results. By choosing both the strength of the field or its radial dependence appropriately (with \( n > 2 \)), we could make the bubble start out collapsing at a very rapid rate. In these cases, the bubble shrinks to very small sizes but eventually bounces back without forming a naked singularity.

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[18] However, we will see that one can encounter situations in which a bubble reverses its behavior without passing through a moment of time symmetry.
[19] As mentioned, in the cases with negative mass, our results disagree with those presented in [2]. We believe that the reason for the disagreement is that the numerical simulations presented in [2] were contaminated by ill-posed modes present in the weakly hyperbolic form of the equations and the implementation of the boundary conditions used there. The understanding of these issues and how to deal with them has advanced considerably in the past few years. In particular, points 1-3 mentioned in section III are major differences between the two implementations.