ABSTRACT. This paper investigates the concept of harmonic functions of bounded mean oscillation, starting from John–Nirenberg’s pioneering studies, under a renewed formalism, suitable for bringing out some fundamental properties inherent in it. In more detail: after a quick introduction, the second Section presents the main theorem, plus complete proof, relating to this function; in the third Section there is a suggestion on the exponential integrability (theorem and sketch of proof), while the fourth Section deals with the duality of Hardy space $H^1$ and bounded mean oscillation, with some ideas for a demonstration. The writing closes with a graphic appendix.

KEYWORDS: $K_b$ cubes, Banach space, bounded mean oscillation, duality of Hardy space $H^1$, exponential integrability, harmonic functions.

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1. A Few Words of Introduction

*Bounded mean oscillation* indicates an $\mathbb{R}$-function characterized by a finite mean oscillation. This can be expressed by a theorem, investigated by F. John and L. Nirenberg [7] in 1961. Let us try to give a more modern and concise form to this theorem, so as to outline some of its essential peculiarities in a few quick strokes. We will also take a look at the integrability of this harmonic function, and especially at its function space.

2. Bounded Mean Oscillation’s Statement: How to do Harmonic Analysis with Cubes

**Theorem 2.1.** Let $\varphi(\mathbb{R})$ be a function of bounded mean oscillation, i.e. $\varphi \in \varphi_b(\mathbb{R}) \subseteq \mathbb{R}^n$, for any cube $K_0$, and multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n = \{0\} \cup \mathbb{Z}_+$ of order $|\alpha| \leq k \in \mathbb{Z}_+$, where $\alpha > 0$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. One has the inequality

$$\left( x \in K_0 : |\varphi(x) - N_{(a)}^{k_a} \varphi| > \alpha \right) \leq e|K_0| \exp \left\{ - \frac{(2^n \varepsilon)^{-1} \alpha}{\|\varphi\|_{b \sim}} \right\},$$

(1)

in which $N_{(a)}^{k_a}$ connotes the average value of $\varphi$ over $K_0$, and $\|\varphi\|_{b \sim} \overset{\text{def}}{=} \|\varphi\|_{\text{BMO}}$ (BMO is, patently, for bounded mean oscillation).

*Proof.* We start from a cube $K_0$, and we choose this principle:

$$\frac{1}{|K_0|} \int_{K_0} \varphi(x) - N_{(a)}^{k_a} \varphi \, dx > \varepsilon,$$

(2)

with a constant $\varepsilon > 1$. But immediately we realize that the principle (2) does not apply to $K_0$, and in fact: $\frac{1}{|K_0|} \int_{K_0} \varphi(x) - N_{(a)}^{k_a} \varphi \, dx \leq \|\varphi\|_{b \sim} = 1 < \varepsilon$.

We remember that $|K_0|$ is the volume of $K_0$, which translates into its Lebesgue measure, $|K_0| \overset{\text{def}}{=} \lambda(K_0)$; idem for $|K_0| \overset{\text{def}}{=} \lambda(K_0)$, or for any other cube $|K_0| \overset{\text{def}}{=} \lambda(K_0)$, plainly.

The next step is to take a cube $K_0 \overset{\text{def}}{=} \bigcup_{r} K_0^r$, and divide it into $2^n$ equal subcubes, so as to see if the resulting subcubes, which we can call $K_0^r$, satisfy the principle (2). Such a process is repeated an infinite number of times, in order to gain a multiset of cubes $\{K_0^r\}_r$, whose features are:

$$\left| \varphi - N_{(a)}^{k_a} \varphi \right| \leq \varepsilon \text{ almost everywhere on } K_0 \setminus \bigcup_r K_0^r,$$

(3a)

$$\left| N_{(a)}^{k_a} \varphi - N_{(a)}^{k_a} \varphi \right| \leq 2^n \varepsilon,$$

(3b)

$$3 < \left| \frac{1}{K_0^r} \int_{K_0^r} \varphi(x) - N_{(a)}^{k_a} \varphi \, dx \right| \leq 2^n \varepsilon,$$

(3c)

$$\sum_r \left| K_0^r \right| \leq 1/3 \sum_r \int_{K_0^r} \left| \varphi(x) - N_{(a)}^{k_a} \varphi \right| \, dx \leq 1/3 |K_0|.$$

(3d)

Remember that the interior of each $K_0^r$, is included in $K_0^r$. The symbolic apparatus (3) allows to hybridize the principle (2) with this new principle:

$$\frac{1}{|K_0|} \int_{K_0} \varphi(x) - N_{(a)}^{k_a} \varphi \, dx > \varepsilon,$$

(4)

which however is not satisfied by $K_0^r$. The process of decomposition into $2^n$ equal subcubes mentioned above must therefore be replicated for $K_0^r$, with the aim that the subcubes obtained can satisfy the new principle, thereby generating a multiset of cubes $\{K_0^r\}_r$, which are contained in $K_0^r$. This new subdivision satisfies any feature in (3), when $K_0^r$ is replaced with $K_0^r$, and $K_0^r$ with $K_0^r$. 


So let us proceed with a further principle, which is identical to that in (4), except that the average changes, that is

\[ N_{(a)}^{K_{6r}'} \cdot \]

Consequently, we get a multiset of cubes \( \{ K_{6r}' \} \), each of which is contained in \( K_{6r}' \). Here is the up-to-date list of features, with the formation of two sets of cubes, this time denoted by \( K_{6r}' \):

\[ \left| \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq \delta \text{ almost everywhere on } K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r}, \]

\[ \left| N_{(a)}^{K_{6r}'} \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n \delta, \]

\[ \delta < \left| K_{6r}' \right|^{-1} \int_{K_{6r}'} \left| \varphi(x) - N_{(a)}^{K_{6r}'} \varphi \right| dx \leq 2^n \delta, \]

\[ \sum_{r} \left| K_{6r}' \right| \leq 1/3 \sum_{r} \left| K_{6r}^{\lambda_{r}^{-1}} \right|. \]

To follow some things to know (the rest is easily verifiable).

(i) The interior of each \( K_{6r}' \) is included in some \( K_{6r}^{\lambda_{r}^{-1}} \).

(ii) The feature (5a) is demonstrable via Lebesgue differentiation theorem [8] [9]: all points in \( K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r} \) associated with a succession of cubes that can be shrunk up to any spatial punctuality, subsequently the averages over each cube is at most \( \delta \).

(iii) The feature (5d) is provable with this formula:

\[ \sum_{r} \left| K_{6r}' \right| \leq 1/3 \sum_{r} \int_{K_{6r}'} \left| \varphi(x) - N_{(a)}^{K_{6r}'} \varphi \right| dx = 1/3 \sum_{r} \int_{K_{6r}'} \left| \varphi(x) - N_{(a)}^{K_{6r}'} \varphi \right| dx \]

\[ \leq 1/3 \sum_{r} \left| K_{6r}^{\lambda_{r}^{-1}} \right| \left\| \varphi \right\|_{b_{\sim}} \]

\[ = 1/3 \sum_{r} \left| K_{6r}^{\lambda_{r}^{-1}} \right|. \]

Accordingly, from (5d) one achieves \( \sum_{r} \left| K_{6r}' \right| \leq \delta \sum_{r} \left| K_{6r}^{\lambda_{r}^{-1}} \right| \). We are now able to write these inequalities:

\[ \left| N_{(a)}^{K_{6r}'} \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n \delta, \]

\[ \varphi - N_{(a)}^{K_{6r}'} \varphi \leq \delta \text{ almost everywhere on } K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r}', \]

\[ \left| \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n \delta + \delta \text{ almost everywhere on } K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r}', \]

and, from (3a), \( \left| \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n (2\delta) \text{ almost everywhere on } K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r}'. \)

Then

\[ \left| \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq \delta \text{ almost everywhere on } K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r}', \]

\[ \left| \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n (3\delta) \text{ almost everywhere on } K_{6r}^{\lambda_{r}^{-1}} \cup \bigcup_{r} K_{6r}', \]

once we have combined \( \left| N_{(a)}^{K_{6r}'} \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n \delta \) and \( \left| N_{(a)}^{K_{6r}'} \varphi - N_{(a)}^{K_{6r}'} \varphi \right| \leq 2^n \delta. \)
The inequality (7d) permits to extend the procedure also on \( \mathcal{K}_{a}^{\delta} \setminus \bigcup_{r} \mathcal{K}_{O_{r}}^{\delta} \), from which we derive a further relation between different values:

\[
\left| \varphi - N_{(a)}^{\mathcal{K}_{a}^{\delta}} \varphi \right| \leq 2^{n}(\lambda) \text{ almost everywhere on } \mathcal{K}_{a}^{\delta} \setminus \bigcup_{r} \mathcal{K}_{O_{r}}^{\delta}. \tag{9}
\]

The expression (9) gives us the opportunity to define this containment:

\[
\left| \left( x \in \mathcal{K}_{a}: \left| \varphi(x) - N_{(a)}^{\mathcal{K}_{a}^{\delta}} \varphi \right| > 2^{n}(\lambda) \right) \right| \subset \bigcup_{r} \mathcal{K}_{O_{r}}^{\delta}. \tag{10}\]

The way to prove the theorematic inequality (1) is to go through the inequality

\[
\sum_{r} \left| \mathcal{K}_{O_{r}}^{\delta} \right| \leq \delta^{-A} \left| \mathcal{K}_{a}^{\delta} \right|, \tag{11}\]

previously encountered, and the one present in the relation (9). Letting first \( 2^{n}(\lambda) < \alpha < 2^{n}(\lambda + 1) \), for \( \alpha > 0 \) and \( \lambda \geq 0 \), and defining then \( -\lambda = 1 - \alpha/2^{n} \), the final formula, once it is determined that \( \delta = e > 1 \), is

\[
\left| \left( x \in \mathcal{K}_{a}: \left| \varphi - N_{(a)}^{\mathcal{K}_{a}^{\delta}} \varphi \right| > \alpha \right) \right| \leq \left| \left( x \in \mathcal{K}_{a}: \left| \varphi - N_{(a)}^{\mathcal{K}_{a}^{\delta}} \varphi \right| > 2^{n}(\lambda) \right) \right|,
\]

\[
\leq \sum_{r} \left| \mathcal{K}_{O_{r}}^{\delta} \right| \leq 1/\delta^{A} \left| \mathcal{K}_{a}^{\delta} \right|
\]

\[
= \left| \mathcal{K}_{a} \right| \exp \left\{-\lambda \log \lambda \right\} \leq \left| \mathcal{K}_{a} \right| \exp \left\{-\frac{\alpha \log \alpha}{2^{n} \alpha} \right\}, \tag{13}\]

as required. □

3. Cubic Exponential Integrability

The Theorem 2.1 implies that, for any \( \varphi \in \varphi_{b_{\infty}}^{[R]} \), there is an exponential integrability over all \( \mathcal{K}_{a} \), i.e.,

\[
\frac{1}{\left| \mathcal{K}_{a} \right|} \int_{\mathcal{K}_{a}} \exp \left\{ \frac{\zeta}{\| \varphi \|_{b_{\infty}}} \left| \varphi(x) - N_{(a)}^{\mathcal{K}_{a}^{\delta}} \varphi \right| \right\} dx \leq 1 - \frac{2^{n} e^{2} \zeta}{2^{n} e \zeta - 1}, \tag{14}\]

for some \( \zeta < \frac{1}{2^{n} e} \). Which is proved by the resulting combination,

\[
\frac{1}{\left| \mathcal{K}_{a} \right|} \int_{\mathcal{K}_{a}} \exp \left\{ \frac{\zeta}{\| \varphi \|_{b_{\infty}}} \left| \varphi(x) - N_{(a)}^{\mathcal{K}_{a}^{\delta}} \varphi \right| \right\} dx \leq \int_{0}^{\infty} e^{\alpha} \exp \left\{ -\frac{(2^{n} e)^{-1} \left\{ \frac{\zeta}{\| \varphi \|_{b_{\infty}}} \right\} }{\alpha} \right\} d\alpha = c(\zeta), \tag{15}\]

for \( \zeta < (2^{n} e)^{-1} \), where \( c(\zeta) \) is a constant.

4. Function Space of \( \varphi_{b_{\infty}}^{[R]} \)

Now let us talk about space in relation to our harmonic function.

4.1. Dual of the Hardy Space \( H^{1}(\mathbb{R}^{n}) \) and Banach Extent

(1) As F. John [6] pointed out, the space of a function of bounded mean oscillation is but a (Lebesgue) function space. C. Fefferman [2], and later E.M. Stein [3, I, IV], have shown that there is a duality between the Hardy space \( H^{p} \) and all bounded mean oscillation functions—which gives a characterization of \( H^{p} \) in terms of boundary properties of harmonic functions.\(^{a}\) Put otherwise,
(i) a bounded mean oscillation $\varphi_{b^\sim}$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$,
(ii) a continuous linear functional, say, $\mathcal{I}_L$ on $H^1(\mathbb{R}^n)$ is realizable in terms of a mapping:

$$\mathcal{I}_L[\eta_h] = \int_{\mathbb{R}^n} \varphi(x)\eta_h(x)dx,$$

where $\varphi \in \varphi_{b^\sim}^\mathbb{R}$, and $\eta_h \in H^1$. From Eq. (16) it is inferred that, for each $\varphi \in \varphi_{b^\sim}^\mathbb{R}$, the functional $\mathcal{I}_L[\eta_h]$ is bounded on $H^1$ via an inequality of this kind:

$$\left| \int_{\mathbb{R}^n} \varphi(\eta_h)dx \right| \leq c \|\varphi\|_{b^\sim} \|\eta_h\|_{H^1},$$

(17)

setting $\eta_h$ in the Hardy subspace $H^1_{\tau} \subset H^1$. And hence it is useful to write that

$$\int_{\mathbb{R}^n} \varphi(\eta_h)dx = \sum_j \kappa_j \int_{\mathbb{R}^n} \varphi(x)\tau_j(x)dx,$$

(18)

clarified the detail that $\eta_h = \sum \kappa_j\tau_j$, and that $\eta_h$ is convergent in $L^1(\mathbb{R}^n)$. The demonstration of inequality (17), for a bounded $\varphi$ and for any $\eta_h \in H^1$, is finally given by

$$\left| \int \varphi(x)\eta_h(x)dx \right| \leq \sum_j \left| \frac{\kappa_j}{|B_j|} \right| \int_{B_j} |\varphi(x) - \varphi_{B_j}|dx \leq \sum_j |\kappa_j| \cdot \|\varphi\|_{b^\sim},$$

(19)

given a ball $B_j \in \mathbb{R}^n$, once the non-equality $|\tau_j(x)| \leq |B_j|^{-1}$ is designated.

(2) It is possible to translate everything described above into a simple assertion: $H^1(\mathbb{R}^n)$ can be considered as a replacement for $L^1(\mathbb{R}^n)$, whilst $\|\varphi\|_{b^\sim}$ plays the same role with respect to the space $L^\infty(\mathbb{R}^n)$ of bounded functions on $\mathbb{R}^n$.

(3) A bounded mean oscillation function (supported in a cube $K_6$ on $\mathbb{R}^n$) is locally $L^p$-integrable,

$$\|\varphi\|_{b^\sim} = \sup_{x_6} \left\{ \frac{1}{|K_6|} \int_{K_6} |\varphi(x) - N_{x_6}(\varphi)| dx \right\} < \infty, \quad 1 < p < \infty,$$

(20)

Which means that $\|\varphi\|_{b^\sim}$ is the Banach space $[1]$ of every function $\varphi \in L^1_{loc}(\mathbb{R}^n)$.

**Graphic Appendix. Cubic space of Matryoshka-type**

When it comes to cubes contained within other cubes, the simplified graphic representation is something like this,

with a inclusion of Matryoshka-type: $\{K_6, \{K_6, \{K_6, \ldots \}\}\}$, where any $K_6 = \{\emptyset\}$. 
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