Weber’s Law for Timing and Time Perception: Reconciling the Poisson Clock with Scalar Expectancy Theory (SET)

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Abstract
The pacemaker-counter model is the most prominent psychological account of timing and time perception. It has been often assumed that an internal pacemaker generates pulses according to a Poisson process and that these pulses are accumulated over time. According to Scalar Expectancy Theory (SET), the number of accumulated pulses represents the elapsed duration to be timed. Although the Poisson process provides a plausible cognitive and physiological mechanism for timing and time perception, its implementation into the framework of SET remains a theoretical challenge. The present contribution shows that a merger of Creelman’s (1962) and Treisman’s (1963) counting models enables such an implementation, which can account for Weber’s law and scalar timing. We demonstrate this for the fundamental principles of counting and timing, as well as for a temporal bisection task.

Keywords
Timing, time perception, Weber’s law, pacemaker-counter model, Poisson process, Scalar Expectancy Theory (SET)

1. Introduction
Psychological researchers have put forward various models to account for human timing behavior and the perception of time. Among these, the pacemaker-counter model still remains the most prominent. This model was originally formulated by Creelman (1962) and it has become the core element of Scalar Expectancy Theory...
(SET; Gibbon, 1977, 1992; Gibbon et al., 1984; for reviews see Church, 1997; Matthews & Meck, 2016; Wearden, 2016a) as well as Attentional Gate Theory (Zakay & Block, 1997). The internal clock proposed by SET is composed of a pacemaker that produces pulses, a switch that regulates the begin and end of pulse transmission to the counter, and said counter which accumulates these pulses for a to-be-timed duration $t$. Thus the major premise of SET is that some time, $t$, is cognitively represented by the number of accumulated pulses that occurred during this time (cf. Church, 1997; Gibbon, 1977). Furthermore, the pacemaker-counter concept also plays a central role in the Behavioral Theory of Timing (BeT, Killeen & Fetterman, 1988), which was developed as alternative theory to SET, especially for the timing of food delivery in animals (Church, 1997) (see Note 1).

One advantage of representing duration as a pulse count is that it is amodal because it is not tied to a specific sensory input modality (e.g., Bratzke & Ulrich, 2019; Ulrich et al., 2006). This allows for comparison and integration of temporal information across senses with various cognitive processes that involve timing (e.g., such as speech production and perception, motor planning, expectation, learning etc.). A major disadvantage, however, is that the original proposals of how the pacemaker generates pulses make predictions about timing variability that are inconsistent with Weber’s law and the bulk of empirical data. In the remainder of this article, we will outline these limitations and demonstrate how they may be overcome by a simple combination of two classic ideas.

Specifically, there have been two prominent proposals for modeling pacemaker output (Creelman, 1962; Treisman, 1963). In an early formulation, Treisman suggested that the pacemaker emits a series of regular pulses whose rate varies from time to time (e.g., Figs. 2 and 3, Panel A). This idea seems both psychologically and physiologically plausible since, for example, the speed of the pacemaker is arousal-sensitive (see Wearden, 2016a, Chapter 5), and the level of arousal is likely to fluctuate across trials (Treisman, 1963, 2013). Likewise, it is conceivable that the attention participants pay to time also varies from trial to trial, which, in the context of the attentional gate model (Zakay & Block, 1997), also leads to a fluctuation in pulse rate. However, Treisman’s model entails a regular temporal sequence of pulses within a single trial, which seems implausible given the unavoidable neurophysiological noise within the central nervous system.

However, the far most common proposal for modeling the pacemaker output has been a Poisson process (e.g., Church, 1997; Creelman, 1962; Gibbon, 1992; Gibbon & Church, 1984; Gibbon et al., 1984; Green & Swets, 1966; Killeen & Fetterman, 1988; Rammsayer & Ulrich, 2001; Wing & Kristofferson, 1973). This is a mechanism whereby pulses are emitted randomly with a constant rate of $\lambda > 0$ (e.g., Figs. 2 and 3, Panel B). To better understand the Poisson process, it helps to imagine that the to-be-timed duration $t$ is divided into many very short non-overlapping intervals $\Delta t$. Then, a single pulse occurs within $t$ with probability $p \approx \Delta t \cdot \lambda$ (Cox & Miller, 1965). Moreover, this process also implies that the
Figure 1. Illustration of the Weber fraction $W(t)$ predicted by the standard Poisson model for various mean interpulse times $\mu$.

Figure 2. Examples of counting processes $N(t)$. Stimulus duration is $t = 1000$ ms. Each panel shows the number of pulses for 10 simulated trials and how these evolve over time. Panel A illustrates the counter process suggested by Treisman (1963). The interpulse interval $I$ is constant within a single trial but varies across trials. Panel B shows the pulse generation process for a Poisson process as originally suggested by Creelman (1962), with a constant rate $\lambda$ across trials. Panel C shows the Poisson process, when its rate $\lambda$ is subject to random fluctuation across trials.
intervals between successive pulses are exponentially distributed with the mean interpulse time $\mu$ equal to $1/\lambda$. Consequently, the total number $N(t)$ of pulses that occurs within a fixed time $t$ varies from trial to trial. Thus the standard definition of a Poisson process $\{N(t), t > 0\}$ implies that the same rate $\lambda$ applies to each value of $t$, which means within the timing framework that the speed of the pacemaker is independent of the value of $t$.

The probability that precisely $N(t) = n$ pulses occur with $t$ is given by the probability mass function

$$P[N(t) = n] = \left(\frac{\lambda \cdot t}{n!}\right)^n \cdot e^{-\lambda \cdot t} \text{ for } n \geq 0.$$ 

Finally, both the expected value and the variance of $N(t)$ increase linearly with time $t$, that is

$$E[N(t)] = Var[N(t)] = \lambda \cdot t.$$ 

This also means that the predicted Weber fraction $W(t)$ as a function of $t$ (i.e., the Weber function),

$$W(t) = \frac{SD[N(t)]}{E[N(t)]} = \frac{\sqrt{\lambda \cdot t}}{\lambda \cdot t} = \frac{\sqrt{\mu}}{t}$$ 

(1)

decreases as the interval timed gets longer (Fig. 1). $W(t)$ measures the relative sensitivity of the timing mechanism and smaller values of $W(t)$ reflect greater sensitivity (i.e., relatively short interpulse times increase discrimination sensitivity) (Note 2).

Accordingly, a Poisson pacemaker, by itself, will produce timing that is relatively more accurate at long durations than at short ones, because $W(t)$ will approach zero as $t$ increases (Gibbon, 1977, p. 284). Therefore, a standard Poisson timer is inconsistent with Weber's law. According to this law, the standard deviation but not the variance of $N(t)$ should increase with $t$ (Note 3).

Although empirical data usually contradict this strong version of Webers's law, the typical result is that Weber's law holds at least approximately and thus provides a benchmark for models on temporal cognition and behavior in humans and animals. Usually, the observed Weber function decreases initially before it attains a relatively constant level of relative sensitivity (e.g., Allan, 1979; Church et al., 1976; Grondin, 1993; Grondin et al., 2004; Haigh et al., 2021; Lejeune & Wearden, 2006; Matthews & Grondin, 2012; Rammsayer & Ulrich, 2012; Wearden & Lejeune, 2008). Even though Lewis and Miall (2009) reported a gradual decrease of $W(t)$ over a wide time range, this decline of $W(t)$ was smaller than expected by a Poisson timer. All these data are usually better accounted for by
generalized forms of Weber’s law as suggested by Getty (1975) and by Killeen and Weiss (1987) (for a review, see Haigh et al., 2021). For example, Killeen and Weiss (p. 458) proposed that

\[ W(t) = \sqrt{A + \frac{B}{t} + \frac{C}{t^2}}. \]  

(2)

This formulation has some interesting properties, as it includes different approaches to generalizing Weber’s law as special cases. The empirical constants \( A, B, C \) have different meanings depending on which approach is considered. The constant \( C \) is often referred to as background sensory noise. Because the terms \( B/t \) and \( C/t^2 \) tend to zero as \( t \) increases, \( A \) is the asymptote of the function \( W(t) \). \( B = 0 \) also gives Getty’s version; \( A = 0 \) and \( C = 0 \) reduce it to the prediction of the standard Poisson process with \( B = \mu \) (i.e., Equation 1).

In conclusion, then, data on timing and time perception often reveal reverse J-shaped Weber functions and are thus inconsistent with the strict version of Weber’s law. Yet most data are consistent with generalized forms of this law. However, this pattern of empirical results is inconsistent with a Poisson pacemaker, which predicts a decreasing Weber function. Nevertheless, the Poisson pacemaker remains a conceptually attractive ingredient in timing and time perception theories.

The present article reconciles the standard Poisson mechanism with the generalized form of Weber’s law and presents a general and straightforward formulation for doing this. To put it succinctly, we will show that an amalgamation of the basic ideas of Creelman (1962) and Treisman (1963) enables a plausible account of the typically observed Weber function. More specifically, in Creelman’s original model, it is implicitly assumed that the pulse rate of the Poisson process remains constant throughout an experiment. We will relax this assumption in the manner proposed by Treisman’s model and suppose that this rate varies across trials (Note 4). We were encouraged to make this assumption since Treisman’s model implies the strict form of Weber’s law (see Appendix A1), while it cannot account for an initial decrease of the Weber function as a standard Poisson process might. For this reason, we believed that a merger of the two models may provide a more realistic account of Weber’s law than either of the models on their own. Therefore, this paper combines Creelman’s and Treisman’s ideas and assesses whether this elaborated Poisson process model would provide a plausible account of the generalized form Weber’s law embodied by Equation 2 (Note 5).

However, before we go any further, it is helpful to keep in mind two different fundamental principles of processing temporal information: counting versus timing. In typical counting tasks, subjects are stimulated for a specific duration \( t \) and are asked to judge this stimulation’s duration (see Fig. 2). For example, they may be stimulated for 1 s by a tone and subsequently judge its duration. In this task,
according to the pacemaker-counter model, the number of pulses accumulated during this 1-s interval represents perceived duration. Thus, the random variable of theoretical interest is the number of pulses $N(t)$ accumulated during $t$. By contrast, in timing tasks, subjects are asked to produce a temporal interval of a pre-specified duration, for example, ‘press a key for 1 s’ (see Fig. 3). In this situation, the key is released after the pacemaker has emitted a certain number of pulses, say $n = 50$, that subjectively corresponds to the inner representation of the pre-specified duration. In this task, the random variable of interest is the time $T$ until the $n$th count is registered, that is, $T(n)$. Moreover, Fig. 4 illustrates the resulting distributions for counting and timing when the rate $\lambda$ of the Poisson process varies between trials (Panel C).

In the following work, we separately analyze the prediction of the elaborated Poisson process for counting and timing. In these analyses, we do not specify a time unit because we do not want to give the impression that our results apply only to specific time ranges. For better illustration, however, one could use the unit ‘millisecond’ for the following time specifications.

**Figure 3.** Examples of timing processes $T(n)$. Participants are asked to produce a 1000 ms interval by releasing a key when the $n = 33$th pulse is registered. Simulated results for 10 trials per each panel. Panel A illustrates the counter process suggested by Treisman (1963). The interpulse interval $I$ is constant within a single trial but varies across trials. Panel B shows the pulse generation process for a Poisson process as Creelman (1962) originally suggested, with a constant rate $\lambda$ across trials. Panel C shows the Poisson process, when its rate $\lambda$ is subject to random fluctuation across trials.
Figure 4. Histograms of $N(1000)$ (upper panel) and $T(33)$ (lower panel) when the Poisson rate fluctuates from trial-to-trial. This figure was inspired by Figure 3 in Gibbon (1992).
2. Counting and Perceived Duration

Consider a situation where participants are presented with a stimulus of duration \( t \) and asked to give an estimate of its duration. As mentioned before, let \( N(t) \) be the number of pulses emitted during time interval \( t \); this number represents the perceived duration that is associated with this interval. We make two assumptions: (a) in each trial, pulses are emitted according to a Poisson process with rate \( \lambda \), which varies from trial to trial. Hence this rate is conceived as a random variable \( \Lambda \) with mean \( \mu_\Lambda \) and variance \( \sigma^2_\Lambda \); for example, \( \sigma^2_\Lambda \) may represent an additional source of variability due to trial-to-trial fluctuations of the closing and opening latencies of the internal clock’s switch mechanism, which starts and terminates the flow of pulses to be accumulated (see Wearden, 2016a, pp. 59–63). Based on these assumptions, one derives (Appendix A2) a Weber function, which is consistent with Equation 2 suggested by Killeen & Weiss (1987),

\[
W(t) = \sqrt{A + B + C}.
\]

(3)

with \( A = cv^2_\Lambda = \frac{\sigma^2_\Lambda}{\mu^2_\Lambda} \), \( B = \frac{1}{\mu_\Lambda} \), and \( C = \sigma^2_B \). The elaborated Poisson process predicts that the relative sensitivity of timing converges with increasing time \( t \) toward the coefficient of variation \( cv_\Lambda \) of \( \Lambda \) rather than zero. Thus the asymptotic value of the predicted Weber function reflects the resolution of the internal clock but not the background noise. Only the initial drop of the function is mediated by mean clock speed and the background noise but nonetheless interacts with \( cv_\Lambda \). This initial drop is especially prominent when \( \mu_\Lambda \) is large (Fig. 5). Finally, background noise has virtually no effect on \( W(t) \).

3. Timing and Interval Production

Consider a participant repeatedly attempting to produce a target duration \( t \) (see Allan, 1979; Grondin, 2010a; Wearden & McShane, 1988). Let the sum

\[ T(n) = I_1 + I_2 + \ldots + I_n \]

be the time until the \( n \)th pulse is emitted, where \( I_i (i = 1, \ldots, n) \) is the time interval between the \((n-1)\)th and \( n \)th pulse. Participants choose \( n \) such that the produced duration \( T(n) \) is as close to the target duration as possible. For the sake of simplicity, we assume that the mean produced time \( E[T(n)] \) is approximately equal to target time \( t \), so that the average produced time is \( t \).

If pulse generation follows a Poisson process, the interpulse times \( I_i (i = 1, \ldots, n) \) must be exponentially distributed with rate \( \lambda \), and thus with mean \( \mu = 1/\lambda \).
Figure 5. Weber functions $W(t)$ for counting, predicted by the Poisson pacemaker model with trial-to-trial variation in pulse rate $\lambda$. Top panel: the effect of trial-to-trial variation reflected by the coefficient of variation $cv_\lambda$ in the pulse rate on $W(t)$ ($\mu_\lambda = 1/10$ and $\sigma^2_B = 200$ are kept constant). Middle panel: the effect of mean pulse rate $\mu_\lambda$ ($cv_\lambda = 0.10$ and $\sigma^2_B = 200$ are kept constant). Bottom panel: the effect of background variation $\sigma^2_B$ ($cv_\lambda = 0.10$ and $\mu_\lambda = 1/10$ are kept constant). Note that $cv_\lambda$ determines the asymptote of the Weber function.
Figure 6. Weber functions $W(t)$ for timing, predicted by the Poisson pacemaker model with trial-to-trial variation in mean interpulse time $\mu$. Top panel: the effect of trial-to-trial variation as captured by the coefficient of variation $cv_M$ ($\mu_M = 10$ and $\sigma_B^2 = 200$ are kept constant). Middle panel: the effect of mean interpulse time $\mu_M$ ($cv_M = 0.1$ and $\sigma_B^2 = 200$ are kept constant). Bottom panel: the effect of background variation $\sigma_B^2$ ($cv_M = 0.1$ and $\mu_M = 10$ are kept constant). Note that $cv_M$ determines the asymptote of the Weber function.
This mean is assumed to be subject to trial-to-trial variation and hence will be denoted by the random variable \( M \) with mean \( E(M) = \mu_M \) and standard deviation \( \text{Var}(M) = \sigma_M^2 \) (Note 6). Under this assumption, the predicted Weber function (Appendix A3) also agrees with Equation 2,

\[
W(t) = \sqrt{cv_M^2 + \frac{\sigma_M^2 + \mu_M^2}{\mu_M} \cdot \frac{1}{t} + \sigma_B^2 \cdot \frac{1}{t^2}}
\]

with \( A = cv_M^2 = \frac{\sigma_M^2}{\mu_M^2}, \ B = (\sigma_M^2 + \mu_M^2)/\mu_M, \) and \( C = \sigma_B^2. \) The background noise in this case might be, for example, the variance of the motor time when producing the target interval; this variance component is presumably small (Wing & Kristofferson, 1973). It can be seen that this Weber function approaches the asymptote \( cv_M, \) which is the coefficient of variation of the random variable \( M. \) Assuming that the trial-to-trial variation of mean interpulse time is zero – or in other words: the rate of the Poisson process remains constant across trials – \( W(t) \) would approach zero with increasing \( t. \) This would be, however, at variance with timing data (e.g., Grondin, 2010a; Wearden & McShane, 1988).

To enhance the interpretation of Equation 4, it is useful to express the standard deviation \( M \) in terms of the coefficient of variation \( cv_M = \sigma_M/\mu_M, \) which results in

\[
W(t) = \sqrt{cv_M^2 + (1 + cv_M^2) \cdot \mu_M \cdot \frac{1}{t} + \sigma_B^2 \cdot \frac{1}{t^2}}.
\]

Figure 6 illustrates this function for reasonable parameter values and ranges of target times. The top panel supports the claim that the Weber functions approach \( cv_M. \) The middle panel reveals \( W(t) \) approaches this limit faster with higher clock speed (i.e., shorter mean interpulse times). As before, the effect of background noise is relatively modest.

At this point it seems worthwhile to compare the predicted Weber function for counting and timing. First, the two functions make identical predictions when the rate of the Poisson process is constant across trials, resembling the situation of the standard Poisson model. Second, if the pulse rate varies, both functions converge to the corresponding coefficient of variation (i.e., \( cv_A \) and \( cv_M \)). Additional calculations suggest that these coefficients are identical for counting and timing if the same process underlies both tasks (Appendix A4). Third, the speed of convergence of \( W(t) \) toward its asymptote should be slightly slower for timing than for counting, since the coefficient \( B \) in Equation 5 is expected to be larger than that in Equation 3. Yet this difference in the speed of convergence appears negligibly small.
4. Temporal Bisection

In the psychophysical study of temporal discrimination, probably the most common task is temporal bisection (e.g., Allan & Gibbon, 1991; Wearden et al., 1997b). Therefore, this task provides a further proof of concept whether the elaborated Poisson process provides a quantitatively realistic prediction of the generalized Weber’s law.

The temporal bisection task is a psychophysical method in which one of several comparison durations, \( t_i(i = 1, \ldots, k) \) is presented in each trial of an experiment. Participants are asked to classify each duration as ‘short’ or ‘long’. To aid this classification, the shortest and the longest comparison duration are presented several times as anchor durations at the beginning of the experiment. In another methodological variant, the single-stimulus method, no such anchor durations are presented at the beginning, yet participants still perform reasonably well in the classification. Therefore, it can be assumed that participants can quickly form an internal reference \( r \) as a basis for stimulus classification by experiencing the comparison durations (see Bausenhart et al., 2016; Dyjas et al., 2012; Woodworth & Schlosberg, 1954). For both task variants, an observed psychometric function is generated by plotting the relative frequency of ‘long’ responses against the comparison duration \( t_i(i = 1, \ldots, k) \).

In such tasks, it is reasonable to assume that participants generate a reference count and store it in their memories. In each trial of the experiment proper, they then compare the number of emitted pulses \( N(t_i) \) during the comparison duration \( t_i \) against the internal reference \( r \). They classify the comparison duration as ‘long’ if \( N(t_i) > r \) and otherwise as ‘short’. If we again assume that the mean interpulse interval \( \mu \) represents a random variable \( M \) that reflects the variation of the pulse rate from trial to trial, the predicted response probability of a ‘long’ response for comparison duration \( t_i \) is computed as (see Appendix A5)

\[
P(\text{‘long’}|t_i) = \int \Phi \left[ \frac{t_i/\mu - r}{\sqrt{t_i/\mu}} \right] \cdot f_M(\mu) \, d\mu,
\]

where \( \Phi(\cdot) \) denotes the cumulative density function of a standard normal deviate, \( f_M(\cdot) \) is the probability density function associated with mean interpulse time \( M \), and the integral extends over the support of this probability function. Although it is not possible to derive an explicit expression for \( W(t) \), Appendix A5 explains how this expression can be numerically evaluated on the basis of Equation 6.

Figure 7 illustrates the prediction of this model when there is a trial-to-trial variation of \( M \) (i.e., \( \sigma_M > 0 \)) and compares this situation with the standard assumption of no such variation (i.e., \( \sigma_M = 0 \)). The top panel shows the probability density function \( f_M(\cdot) \) of \( M \); this distribution has a mean of 10 and a standard deviation of 2. The middle panel depicts the psychometric functions for standard durations
Figure 7. Effect of trial-to-trial variation of pulse rate in a bisection task. Top panel: Depicts the variation of mean interpulse time across trials (a lognormal distribution with mean $\mu_M = 10$ and standard deviation $\sigma_M = 2$). Middle panel: Psychometric functions for standards $s = (50, 100, 200, 400, 800, 1600)$ from left to right; solid lines ($\sigma_M = 2$), dashed lines ($\sigma_M = 0$). Bottom panel: Weber fractions for the above psychometric functions. The smooth functions are least-square fits of the generalized Weber law.
of 50, 100, 300, 500, 700, and 1600 as solid ($\sigma_M > 0$) and dashed ($\sigma_M = 0$) lines, from left to right along the x-axis. To simplify matters, each standard $s$ can be conceived as the central value of all the comparisons used to generate a psychometric function (this assumption is mathematically convenient but not crucial). It can be seen that the steepness of the psychometric functions decreases with standard duration. However, this decrease is noticeably stronger for the solid than for the dashed lines. The bottom panel shows the Weber fractions (i.e., difference limen divided by standard duration) corresponding to these psychometric functions. It can be seen that even a small trial-to-trial variation in clock speed can profoundly affect the asymptote of the Weber function.

The smooth curves in the bottom panel depict the best-fitting Weber functions (i.e., Equation 2). Despite the excellent fits, the constant $A$ (describing the asymptotic behavior of the Weber function) is positive for $\sigma_M = 2$ but zero for $\sigma_M = 0$ (Note 7). Thus the predicted asymptote is 0.14 for $\sigma_M = 2$ and zero for $\sigma_M = 0$. This demonstrates that the Weber functions would not approach zero when there is trial-to-trial variation in the rate of the Poisson process. This prediction is consistent with psychophysical data, because $W(t)$ does empirically not approach zero. Additional computations with other values of $\mu_M$ and $M$ lead to the same conclusion.

5. Discussion

Since the 1960s, data from many psychophysical experiments in human and non-human animals have been accounted by proposing an internal clock-like mechanism (Creelman, 1962; Gibbon et al., 1984; Treisman, 1963). The most common proposal for this mechanism is a Poisson pacemaker. The standard Poisson process predicts that the Weber fraction should tend to zero as the target duration $t$ to be discriminated or to be produced increases. Unfortunately, however, this assumption is at variance with empirical data, and thus the Weber function turns out to be a stumbling block for the standard variant of SET. In psychophysics, Weber’s law is an essential benchmark for ‘a discrimination model to be judged as acceptable only if, to a good approximation, it is consistent with Weber’s law at medium intensities’ (Falmagne, 1985, p. 7).

This paper proposes how an elaborated Poisson process can be reconciled with the notion of a pacemaker. This modification takes up Treisman’s (1963) idea that the frequency of the pacemaker varies from trial to trial and applies this assumption to the Poisson process. More specifically, the elaborated Poisson process assumes that the rate of the process varies over trials. We have shown how this assumption can be implemented in the two fundamental principles of temporal information processing, that is, timing and counting. Both implementations are consistent with the generalized Weber function proposed by Killeen and Weiss (1987), which typically provides an excellent fit to observed Weber fractions.
Perhaps somewhat counterintuitively, the predicted asymptotic behavior of the Weber function only reflects properties of the pacemaker rather than properties of the other elements postulated by SET (Gibbon et al., 1984). For example, it would be reasonable to assume that information in the reference memory is noisy and thus deteriorates discrimination or timing performance. However, as long this additional background noise is independent of the target time $t$, this noise source adds to $C$ in Equation 2 of the generalized Weber’s law. Thus this noise would not determine the asymptotic behavior of $W(t)$.

Yet, Gibbon (1992) provided an elaboration how noise in temporal reference memory might account for the scalar property within SET. In particular, his elaboration of SET focused on the encoding and decoding of the accumulated pulses $n^*$ in this memory during a timing task such as the ‘peak procedure’ (Gibbon & Church, 1990). In his mathematical analysis, $n^*$ denotes the accumulated pulses at reinforcement time $\tau^*$ that enter reference memory (Note 8). When an animal predicts the time of a subsequent food delivery, it first randomly draws a reference count $n^*$ from this memory. Then the animal continuously monitors the accumulated number $n$ of pulses generated by the Poisson pacemaker, and when $n$ is close enough to $n^*$, a high rate of responding occurs. Gibbon showed that this timing mechanism can be regarded as a compound gamma distribution – that is, the waiting time distribution for receiving the next reinforcement can be conceived as a gamma distribution with shape parameter $n^*$ and rate $\lambda$, which, of course, is implied by the assumption of a Poisson pacemaker. However, because $n^*$ is a random variable, the resulting waiting time distribution becomes a probability mixture of gamma distributions, that is, a ‘compound gamma’.

In the first part of his theoretical note, Gibbon (1992) mathematically showed that such a mechanism would not imply scalar property because the variance of the compound gamma is twice the variance of the gamma. In the second part of his note, Gibbon additionally assumed that the mean count $\mu^*$ in reference memory is subject to random variation. More specifically, Gibbon attributes this source of variation to the product $s = bk^*$, which scales the mean $\mu^*$, where the product $s$ encompasses bias and encoding/decoding factors (see p. 291 in his theoretical note). Gibbon then showed that the aforementioned compound gamma together with randomization of $s$ by a normal distribution imply the scalar property, because according to this elaboration, the standard deviation of noise in reference memory increases linearly with reinforcement time $\tau^*$ (see Equation 17b in his theoretical note on p. 291). Thus Gibbon attributed the scalar property within his elaborated SET version to the distribution of encoded pulses in reference memory while assuming a standard Poisson process for the pacemaker. In our analysis, however, the noise level in reference memory is assumed to be the same for each time interval. Furthermore, our analysis focuses exclusively on the pacemaker process and shows that this source within SET is sufficient to account for the general form of Weber’s law. Moreover, our analysis applies to both timing...
and time perception, and the derivation of Weber's law did not involve specific
distributional assumptions like a normal mixing distribution as in Gibbon's analy-
sis. Finally, it seems plausible that noise in temporal reference memory probably
plays a smaller role in humans compared to non-human animals, as often a few
trials are sufficient to form a stable memory representation for certain time inter-
vals in humans (e.g., Jones & Wearden, 2003; Ogden & Jones, 2009).

An issue that cannot be accounted for by the current models is that the
observed Weber function sometimes rises at longer durations (e.g., Bizo et al.,
2006; Getty, 1975; Grondin, 2010b, 2012), a phenomenon which does not even
conform with the generalized form of Weber’s law. Some have noted, however,
that the magnitude of this effect is negligible (Haigh et al., 2021) or that it is not
even present (Lewis & Miall, 2009). Nevertheless, it is also conceivable that the
temporal processing of short and long time intervals is based on distinct systems
(e.g., Lewis & Miall, 2003; Rammsayer, 1999; Rammsayer & Ulrich, 2011; but see
Lewis & Miall, 2009; Rammsayer & Ulrich, 2005), which might manifest them-
several in a change of the Weber fraction at longer durations. Specifically, Lewis
and Miall (2003) have argued that the processing of short intervals is based on
‘automatic’ processes, whereas cognitively controlled processes, such as implicit
counting, are required in the processing of longer intervals (Grondin et al., 1999).
It must be emphasized, however, that this final rise of the Weber fraction has also
been observed for physical measures other than time (Luce & Galanter, 1963) and
therefore may represent a higher-level phenomenon.

As mentioned in the introduction, the Poisson process is the most common
proposal for modeling the pacemaker output. This is probably because this pro-
cess is mathematically tractable and thus often leads to analytical solutions that
are easy to interpret. On the downside, however, the Poisson process might be
less realistic than one might have hoped. As discussed above, the Poisson process
implies an exponential interpulse distribution. This distribution has its mode at
zero and thus an unrealistically high coefficient of variation (i.e., coefficient of
variation is $cv = 1$) (Note 9). In a more realistic scenario, the mode of the inter-
pulse time would be non-zero, and thus the coefficient of variation would be
smaller than one (see Rammsayer & Ulrich, 2001 and Ulrich et al., 2006 for a pos-
sible approach). In this case, one would presumably expect that the present ana-
lytical results concerning the Weber fraction would approach the predictions of
the strict Weber law predicted by Treisman’s model, since in his model the $cv$
of the interpulse time is zero within a single trial. Realistic interpulse distributions
might be used to evaluate this conjecture in further research. Although distribu-
tions with a $cv$ significantly smaller than 1 may give an even more realistic picture,
few other conclusions will be drawn as far as the asymptotic behavior of $W(t)$ is
concerned (Note 10).

This present work assumes that the represented time is given by the number
of pulses emitted by a timer, whether the temporal task is related to timing or
counting. However, alternatives to SET have been suggested, especially for timing tasks with non-human animals. Killeen and Fetterman (1988) proposed the Behavioral Theory of Timing (BeT), which might be regarded as an adjustment model of the pacemaker. This alternative supposes that the underlying timing is also Poisson, but that the mean interpulse time of the Poisson pacemaker is adjusted depending on the interval to be timed. The Poisson pacemaker pulses more slowly when the target interval gets longer, so that the mean interpulse time $\mu$ is proportional to target time $t$, or nearly so, and this adjustment produces the scalar property in behavior. However, BeT does not embody a standard Poisson process as defined in the introduction, since it is assumed that the pacemaker’s rate $\lambda$ is adjusted to each duration $t$ to be timed.

To put it somewhat technically, after the pacemaker has generated $n$ pulses, the organism assumes that target time $t$ has elapsed, which may manifest in overt behavior (e.g., key-peck, key-press). Since the interpulse times $I_1, \ldots, I_n$ are exponentially distributed, the estimated interval is given by the sum $T = I_1 + \ldots + I_n$, which follows the special Erlang distribution with shape parameter $n$ and scale parameter $\mu$ having mean $E[T] = n \cdot \mu$ and variance $\text{Var}[T] = n \cdot \mu^2$. Therefore, the predicted coefficient of variation $cv = SD/M = 1/\sqrt{n}$ is independent of clock speed $\mu$ and thus constant across different target times. Clock speed would be adjusted for each target time according to the ratio $\mu = t/n$. Figure 8 illustrates the timing of this adjustment model for $n = 100$ (i.e., $cv = 0.1$), $t = 1000$, and thus $\mu = 10$. This model predicts the strong form of Weber’s law, that is, the scalar timing property, because the shape of the Erlang distribution only depends on $n$ but not on $\mu$. Specifically, the observed distributions of estimated intervals for each target time should superimpose. This model has been applied to animal timing and provides a good description of the general activity of pigeons reinforced on various fixed-interval schedules of reinforcement (see Wearden, 2016a, for a summary of this research on animal timing). Results by Meck (1983) indicate that methamphetamine can selectively increase clock speed in rats and that haloperidol can decrease it, and this agrees with the fundamental premise of this adjustment model.

Recently, some researchers (Balcı & Simen, 2014, 2016; Simen et al., 2013, 2016) supposed another class of timing models within the drift–diffusion framework, which was originally developed to explain response accuracy and response times for two-choice decisions (Ratcliff, 1978). Although it would go beyond the scope of this paper to describe all details of this class of model, the common core element is a one-dimensional, time-homogeneous diffusion process $X(t)$ (i.e., Wiener process or Brownian motion) with drift $\mu > 0$, variance $\sigma^2$, an upper absorbing barrier $a > 0$, and the initial condition $X(0) = 0$ (see Cox & Miller, 1965, pp. 219–222). This process in continuous time can be approximated and thus evaluated for small time steps $\Delta t$ by the stochastic difference equation.
Figure 8. Realizations of $N(t)$ and $X(t)$ as a function of time $t$ when timing a 1000-ms duration. Top panel: 10 realizations generated by the adjustment model. Bottom panel: 10 realizations generated by the drift-diffusion model.
where $Z(t)$ is normally distributed with zero mean and unit variance. The lower panel in Fig. 8 illustrates several realizations of this diffusion process for $\Delta t = 0.01$, $a = 100$, $t = 1000$, $\mu = 0.1$, and $\sigma = 0.447$ (this value yields $cv = 0.1$ for the timed duration; an explanation follows below). In contrast to Killeen and Fetterman's model, the accumulated activation $X(t)$ is no longer a monotone increasing function of time. Moreover, the accumulation process $X(t)$ is adjusted to the interval to be timed by calibrating the drift $\mu$ and the variance $\mu$ of $X(t)$ rather than the speed of the clock as in the adjustment model. The following paragraph briefly explains this calibration process.

The time $T$ until $X(t)$ hits the absorbing barrier $a$ is called the first passage time. Similar to the adjustment model, an organism judges that target time $t$ has elapsed when $T$ reaches the criterion $a$. This random variable $T$ follows the Inverse Gaussian (Wald) distribution (Cox & Miller, 1965) with mean

\[
E[T] = \frac{a}{\mu}
\]

and variance

\[
Var[T] = \frac{a \cdot \sigma^2}{2 \cdot \mu^3}.
\]

On the basis of the preceding two expressions, is easy to show that the $cv$ of $T$ is mediated by $\sigma = cv \cdot \sqrt{2 \cdot a \cdot \mu}$. Thus if $\mu$ is adjusted to the target interval $t$, such that, $E[T] = t = a/\mu$, $\sigma$ must be calibrated as $\sigma = cv \cdot a \cdot \sqrt{2}/t$ to yield a $cv$ of 0.1 for $T$ (for the example above: $\sigma = 0.1 \cdot 100 \cdot \sqrt{2/1000} = 0.447$). Simen et al. (2013) provided a neurophysiological rationale how this calibration of $\sigma$ is accomplished by a running sum of two opponent Poisson processes and how their diffusion model can be linked to neuronal brain activity (Balcı & Simen, 2016). Therefore, they refer to their diffusion model as the Opponent Poisson Diffusion Model (Simen et al., 2013, 2016).

Undoubtedly, adjusting pacemaker speed or the noise level of a diffusion process in timing tasks provides a plausible alternative account of data from timing studies. Such adjustments may even be achieved within a single trial, that is, in a one-trial-learning fashion (Simen et al., 2011). However, this explanation applies far less to data from experiments with humans when the intervals to be judged or timed vary randomly from trial to trial such that temporal adjustments are not possible (e.g., Wearden & Bray, 2001).

Moreover, when participants are asked to judge the duration of time intervals, temporal segmentation or subdividing appears to be an effective timing
strategy (Grondin & Killeen, 2009). For example, when participants are asked to
time random durations larger than about a second, they usually prefer chrono-
metric counting as a heuristic for judging elapsed time (Grondin & Killeen, 2009;
Wearden, 2016b). Such segmentation strategy, however, is fully consistent with
the major premise of SET; accordingly the number of counts represent subjective
time (Church, 1997; Gibbon et al., 1984; Wearden, 2016a). Our modeling remains
faithful to this standard assumption of SET.

In conclusion, this paper suggests an elaborated Poisson process that merges
the models of Creelman (1962) and Treisman (1963). This elaborated process
provides a novel account of Weber’s law whether target times are blocked or vary
randomly across trials; it also accounts for the generalized version of Weber’s law.
This elaboration strengthens the idea that SET continues to be a viable behav-
ioral account of temporal timing and counting. Moreover, the present elaboration
supports the view that the same basic mechanism may underlie timing and time
perception, a theoretical goal that has been considered as being not necessarily
reachable (Church, 1997, p. 42). Nevertheless, unraveling the mechanisms of tem-
poral cognition remains a major theoretical challenge. It is therefore not surpris-
ing that different points of view about these mechanisms are put forward. But
the more such different viewpoints are proposed, the more successful elements
of these viewpoints can be integrated or even merged, as in the present case, to a
more complete understanding of temporal cognition. For example, future mod-
eling may also address the possibility that variability in the temporal reference
memory additionally contributes to the scalar property (Gibbon, 1992). Only time
will tell which of these elements should be preferred within the framework of SET.

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Notes

1. Related to the main topic of the present work, within BeT it has been sug-
gested that Weber’s Law in timing processes may be accounted for by an
adjustment of clock speed to the to-be-timed duration. We will discuss this
approach in more detail in the Discussion.
2. In contemporary timing research the coefficient of variation \( cv = \frac{SD}{Mean} \) is often used as measure of relative sensitivity. \( W(t) \) embodies this measure and has the desired property of the Weber fraction \( \Delta S/S = k \) that is dimensionless, that is, does not depend on the time scale of measurement.

3. The term scalar property of variance implies an even stricter formulation than that of Weber’s law (see Wearden, 2016a, Chapter 3). This property means, for example, that the empirical distributions of produced time intervals for different target times (e.g., 0.5, 0.7, 0.9, 1.1, 1.3 s) must lie on top of each other if these distributions are scaled accordingly and plotted on a relative scale (Wearden et al., 1997a).

4. Although this is the core assumption of Treisman’s model, his model includes further assumptions which, however, are not required for our analysis.

5. We would like to note that Gibbon (1992) suggested an alternative possibility to reconcile a Poisson timer with SET. This alternative elaboration attributes the scalar property (i.e., Weber’s law) to memory processes. We will compare his elaboration of SET with the present one in the Discussion.

6. Readers may wonder why we now consider \( M \) instead of \( \Lambda \) as in the previous section. Note that the two formulations are conceptually equivalent. However, focusing on the mean instead of the rate in this section is mathematically convenient. Unfortunately, \( E[\Lambda] = E[1/M] \neq 1/E[M] \), and this would complicate the analysis in this section.

7. The constants are \( A = 0.019 \), \( B = 4.68 \), and \( C = 0.192 \) for \( \sigma_M = 2 \) and \( A = 0 \), \( B = 4.616 \), and \( C = 0.189 \) for \( \sigma_M = 0 \).

8. In this paragraph, we have adopted Gibbon’s mathematical notation.

9. An alternative index to capture this variation is the Fano factor, \( F \), which is defined as the variance-to-mean ratio of pulses within a certain time window. This factor is often used to measure the variability of neural spike trains, and it would also be one for the Poisson process. Fano factors can be regarded as an analogous index to \( cv \) (Rajdl et al., 2020). In general, \( F \approx cv^2 \) holds (Cox, 1967, p. 40).

10. An interesting property of the Poisson process is that this process can emerge from superposing several counting processes that need not to be themselves Poisson processes. In a complex system like the central nervous system, such a superposition seems possible. Therefore, the Poisson process might nevertheless provide a reasonable assumption.

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Appendix A1. Treisman's pacemaker

Here we provide a compact characterization of Treisman's counter model and its predictions. Let \( I \) represent the interpulse interval between subsequent pulses. According to Treisman (1963), this interval varies across trials but not within trials.

**Counting**

The number of pulses occurring within \( t \) is \( N(t) = t/I \), with \( I > t \) (otherwise no pulse would occur within \( t \)). Therefore,

\[
E[N(t)] = E[t/I] = t \cdot E(1/I)
\]

and

\[
\text{Var}[N(t)] = \text{Var}(t/I) = t^2 \cdot \text{Var}(1/I).
\]

Consequently,

\[
W(t) = \frac{SD[N(t)]}{E[N(t)]} = \frac{SD(1/I)}{E(1/I)} = \text{constant}
\]

hence predicting the strict form of Weber's law.

**Timing**

Let \( T(n) = n \cdot I \) be the time until the \( n \)th interval is registered. Therefore,

\[
E[T(n)] = E(n \cdot I) = n \cdot E(I)
\]

and

\[
\text{Var}[T(n)] = \text{Var}(n \cdot I) = n^2 \cdot \text{Var}(I).
\]
Thus the predicted Weber fraction does not depend on the duration to be timed

\[ W(t) = \frac{SD(I)}{E(I)} = cv_I = \text{constant} \]

where \( cv_I \) denotes the coefficient of variation of \( I \).

An alternative formulation assumes that one produces target time \( t \) on average, which implies \( E[T(n)] = n \cdot E(I) = t \) and thus \( n = t/E(I) \). Therefore,

\[ W(t) = \frac{\sqrt{Var[T(n)]}}{E[T(n)]} = \frac{\sqrt{n^2 \cdot Var(I)}}{t} = \frac{\sqrt{t^2 \cdot Var(I)/E(I)^2}}{t} = \frac{SD(I)}{E(I)} = cv_I. \]

Appendix A2. Counting

In this appendix, we derive the generalized Weber law embodied in the main text by Equation 3 and implied by the elaborated Poisson model. Let \( \Lambda \) denote a random variable that represents the trial-to-trial variation of pulse rate \( \lambda \). The mean and variance of \( \Lambda \) is \( \mu_\Lambda \) and \( \sigma^2_\Lambda \), respectively.

Given that \( \Lambda = \lambda \), the conditional mean and variance of the number of pulses \( N(t) \) that are emitted during \( t \) is

\[ E[N(t)|\Lambda = \lambda] = \lambda \cdot t \]

and

\[ Var[N(t)|\Lambda = \lambda] = \lambda \cdot t, \]

respectively. The unconditional mean of \( N(t) \) is computed as

\[ E[N(t)] = E[E[N(t)|\Lambda]] = E[\Lambda \cdot t] = \mu_\Lambda \cdot t. \]

Likewise, the unconditional variance of the number of pulses can be computed via the variance formula for probability mixtures

\[ Var(X) = E[Var(X|Y)] + Var[E(X|Y)] \]

and thus

\[ Var[N(t)] = E(\Lambda \cdot t) + Var(\Lambda \cdot t) \]

\[ = \mu_\Lambda \cdot t + \sigma^2_\Lambda \cdot t^2 \]
Therefore, the relative sensitivity associated with the pacemaker is

\[ W(t) = \frac{\sqrt{\text{Var}[N(t)]}}{E[N(t)]} \]

\[ = \frac{\sqrt{\mu_\Lambda \cdot t + \sigma_\Lambda^2 \cdot t^2}}{\mu_\Lambda \cdot t} \]

\[ = \sqrt{\frac{\mu_\Lambda \cdot t + \sigma_\Lambda^2 \cdot t^2}{\mu_\Lambda^2 \cdot t^2}} \]

\[ = \sqrt{\frac{\text{cv}_\Lambda^2 + \frac{1}{\mu_\Lambda} \cdot \frac{1}{t}}{\mu_\Lambda^2 \cdot t^2}} \]

and allowing for background noise \( \sigma_B^2 \), we finally obtain

\[ W(t) = \sqrt{\text{cv}_\Lambda^2 + \frac{1}{\mu_\Lambda} \cdot \frac{1}{t} + \sigma_B^2 \cdot \frac{1}{t^2}}. \]

Appendix A3. Timing

According to the assumptions outlined in the main text, the conditional mean and variance of the sum \( T(n) \) is

\[ E[T(n)|M = \mu] = n \cdot \mu \]

and

\[ \text{Var}[T(n)|M = \mu] = n \cdot \mu^2 \]

noting that the mean and the standard deviation for an exponential random variable are equal, that is, \( E[I] = \mu = 1/\lambda \) and \( \text{SD}[I] = \mu = 1/\lambda \). Hence, the unconditional mean of \( T(n) \) is

\[ E[T(n)] = E[E[T(n)|M]] \]

\[ = E[n \cdot M] \]

\[ = n \cdot E[M] \]

\[ = n \cdot \mu_M \]
and the unconditional variance

\[ \text{Var}[T(n)] = E[\text{Var}[T(n)|M]] + \text{Var}[E[T(n)|M]] \]

\[ = E[n \cdot M^2] + \text{Var}[n \cdot M] \]

\[ = n \cdot E[M^2] + n^2 \cdot \text{Var}[M] \]

\[ = n \cdot (\text{Var}[M] + E[M]^2) + n^2 \cdot \text{Var}[M]. \]

Hence the variance of the pacemaker is

\[ \text{Var}[T(n)] = n \cdot (\sigma_M^2 + \mu_M^2) + n^2 \cdot \sigma_M^2 \]

and therefore its relative sensitivity

\[ W(t) = \frac{\text{SD}[T(n)]}{E[T(n)]} = \frac{\sqrt{n \cdot (\sigma_M^2 + \mu_M^2) + n^2 \cdot \sigma_M^2}}{n \cdot \mu_M}. \]

Since it is assumed that \( E[T(n)] = t \) and thus \( t = n \cdot \mu_M \), one computes \( n = t / \mu_M \).

Inserting this result into the preceding equation

\[ W(t) = \frac{\sqrt{t / \mu_M \cdot (\sigma_M^2 + \mu_M^2) + (t / \mu_M)^2 \cdot \sigma_M^2}}{t} \]

and allowing for background noise \( \sigma_B \), yields after algebraic simplification

\[ W(t) = \sqrt{\frac{c_v_M^2 + \frac{\sigma_M^2 + \mu_M^2}{\mu_M} \cdot \frac{1}{t} + \frac{1}{t^2} \cdot \sigma_B^2}{\frac{1}{t^2}}}. \]

### Appendix A4. On the relation between \( \Lambda \) and \( M \)

A comparison of the predicted Weber functions for counting and timing is complicated because it is not clear a priori whether the coefficients of variation \( c_v_\Lambda \) and \( c_v_M \) for \( \Lambda \) and \( M \) are identical if one assumes the same clock in both cases. This is due to the fact that the expected value and also higher moments of the random variables \( \Lambda \) and \( M \) cannot be easily transformed into each other. In this section we analyze whether one can proceed from \( c_v_\Lambda = c_v_M \) under realistic assumptions.
The two random variables \( \Lambda \) and \( M \) have the following relationship: \( \Lambda = 1/M \), since, for example, if the mean interpulse time is \( \mu = 20 \) in one trial, the rate of the Poisson process must be \( \lambda = 1/20 \) in the same trial. Now, unfortunately, it is not possible to apply this simple relationship to the moments of these two random variables, since, for example, the expected value \( E[\Lambda] \) is not equal to \( 1/E[M] \), but needs to be calculated from the density distribution \( f_M(\cdot) \) of \( M \) using the standard transformation technique for expectations

\[
E[\Lambda] = E[1/M] = \int \frac{1}{x} \cdot f_M(x) \, dx. \tag{7}
\]

Table A1 contains numerical examples based on such calculation. In all cases, \( M \) followed a lognormal distribution (see upper panel in Figure 7 for an example). Numerical integration was used to evaluate the expected value and standard deviation of \( \Lambda \) (routine “integrate” in R). As can be seen, this calculation supports the conjecture \( cv_\Lambda = cv_M \). Moreover, it can be seen that \( E[\Lambda] \approx 1/E[M] \). These two results encourage us to believe that we can compare the predicted Weber functions for timing and counting as we have done in the main text.

Appendix A5. Temporal Bisection

As before we assume that the number of pulses \( N(t_i) \) accumulated during the comparison interval \( t_i (i = 1, ..., k) \) follows a Poisson process with rate \( \lambda = 1/\mu \). The conditional probability of a ‘long’ response given \( M = \mu \) can be deduced via the normal approximation of the Poisson process

\[
P(\text{long}|t, M = \mu) = P(N(t_i) > r|M = \mu)
\]

\[
= 1 - P(N(t_i) \leq r|M = \mu)
\]

\[
\approx 1 - \Phi \left[ \frac{r - E[N(t_i)|M = \mu]}{\sqrt{\text{Var}[N(t_i)|M = \mu]}} \right]
\]

\[
\approx \Phi \left[ \frac{E[N(t_i)|M = \mu] - r}{\sqrt{\text{Var}[N(t_i)|M = \mu]}} \right]
\]

\[
\approx \Phi \left[ \frac{t_i/\mu - r}{\sqrt{t_i/\mu}} \right].
\]
In order to obtain the unconditional probability of ‘long’ responses at time $t_i$, one proceeds from the law of the Unconscious Statistician and computes

$$P(‘long’|t_i) = \int P(‘long’|t_i, M = \mu) \cdot f_M(\mu) d\mu$$

$$= \int \Phi \left[ \frac{t_i/\mu - r}{\sqrt{t_i/\mu}} \right] \cdot f_M(\mu) d\mu.$$  

The last expression can be numerically evaluated for any suitable probability density distribution $f_M(\cdot)$ of the random variable $M$.

Moreover, the comparison durations $t(0.25)$ and $t(0.75)$ at which a ‘long’ response will be observed with probabilities 0.25 and 0.75, respectively, can be determined with an iterative search procedure. These obtained values can then be used to calculate the Weber fraction for a given reference count $r$. A convenient choice for $r$ is the mean number of counts $E[N(s)]$ emerging as the central value $s$ of the comparisons $t_1, \ldots, t_k$ used to generate a psychometric function. This reference count is computed as

$$r = E[N(s)] = E[E[N(s)|M]] = E[s|M] = s \cdot \int \frac{1}{\mu} \cdot f_M(\mu) d\mu.$$  

### Table A1.

Each row gives the expected value, the standard deviation, and coefficient of variation of $\Lambda$ that corresponds to the mean, standard deviation and $cv$ of $M$ shown in the first three columns.

| $E[M]$ | $SD[M]$ | $cv_M$ | $E[\Lambda]$ | $SD[\Lambda]$ | $cv_{\Lambda}$ |
|--------|---------|--------|---------------|---------------|----------------|
| 5      | 0.25    | 0.05   | 0.2005        | 0.0100        | 0.050          |
| 5      | 0.5     | 0.1    | 0.2020        | 0.0202        | 0.100          |
| 5      | 1       | 0.2    | 0.2080        | 0.0416        | 0.200          |
| 5      | 2       | 0.4    | 0.2320        | 0.0928        | 0.400          |
| 10     | 0.5     | 0.05   | 0.1002        | 0.0050        | 0.050          |
| 10     | 1       | 0.1    | 0.1010        | 0.0101        | 0.100          |
| 10     | 2       | 0.2    | 0.1040        | 0.0208        | 0.200          |
| 10     | 4       | 0.4    | 0.1160        | 0.0464        | 0.400          |
| 20     | 1       | 0.05   | 0.0501        | 0.0025        | 0.050          |
| 20     | 2       | 0.1    | 0.0505        | 0.0051        | 0.100          |
| 20     | 4       | 0.2    | 0.0520        | 0.0104        | 0.200          |
| 20     | 8       | 0.4    | 0.0580        | 0.0232        | 0.400          |
Therefore, the difference limen ($DL$; also called ‘just noticeable difference’, $JND$) associated with $s$ is (Luce & Galanter, 1963)

$$DL(s) = \frac{t(0.70) - t(0.25)}{2}$$

from which the corresponding Weber fraction is obtained

$$W(s) = \frac{DL(s)}{s}.$$  

**Appendix A6. Glossary**

This glossary provides brief definitions of main symbols used in the text and in the appendices.

$t$ target duration to be perceived or produced

$W(t)$ Weber function, that is, the Weber fraction for target time $t$

$N(t)$ number of pulses occurring during time $t$ in a Poisson process (counting)

$\lambda$ rate of the Poisson process for a specific trial

$\Lambda$ random variable that represents the trial-to-trial variation of $\lambda$

$\mu_\Lambda$ mean rate, $E[\Lambda] = \mu_\Lambda$

$\sigma_\Lambda$ standard deviation of $\Lambda$, $SD[\Lambda] = \sigma_\Lambda$

$cv_\Lambda$ coefficient of variation of $\Lambda$

$T(n)$ time until the $n$th count is emitted (timing)

$I$ temporal interval between two successive pulses in a Poisson process

$\mu$ mean interpulse time, $E[I] = \mu = 1/\lambda$ for a specific trial

$M$ random variable that represents the trial-to-trial variation of $\mu$

$\mu_M$ mean of $M$, that is, $E[M] = \mu_M$

$\sigma_M$ standard deviation of $M$, $SD[M] = \sigma_M$

$cv_M$ coefficient of variation of $M$

$\sigma_B$ background noise

$DL$ difference limen

$cv$ coefficient of variation