Wigner-Racah Algebra, Binary Correlations and Trace Propagation for Embedded Random Matrix Ensembles

V K B Kota
Physical Research Laboratory, Ahmedabad 380 009, India
E-mail: vkbkota@prl.res.in

Abstract.
For embedded unitary ensembles with SU(Ω) × SU(r) embedding and generated by random two-body (in some situations k-body) interactions preserving SU(r) symmetry, analytical formulas, in terms of SU(Ω) Racah coefficients, are derived for the lower order moments of the one and two-point functions in eigenvalues. This formulation unifies all known results and new results are derived for r = 3. Similarly, the method of binary correlation approximation developed for spinless identical fermion systems, has been extended to proton-neutron systems and it is used to show that the bivariate transition strength density appropriate for neutrinoless double beta decay will be close to a bivariate Gaussian. In addition, trace propagation methods of French are also used to derive analytical results for embedded ensembles and they have opened a new window to understand regular structures generated by random interactions.

1. Introduction
The GOE, now almost universally regarded as a model for a corresponding chaotic system [1] is an ensemble of multi-body, not two-body interactions. This difference, as pointed out first by J.B. French, shows up both in one-point (density of states) and two-point (fluctuations and smoothed transition strengths) functions generated by nuclear shell model. This led to the introduction of random matrix ensembles generated by random interactions. For random two-body interactions, we have GOE/GUE/GSE Hamiltonian matrix ensembles in two-particle spaces (similarly in k-particle spaces for k-body interactions) and these are then propagated to m particle spaces, using the geometry of m particle spaces, giving random matrix ensembles in many-particle spaces. As in these ensembles, in many-particle spaces, classical ensembles are embedded, these are generically called (introduced in 1975) embedded random matrix ensembles or in short, embedded ensembles (EE) [2]. With two-body interactions having shell model J (or J^T) symmetry, these are called (introduced first in 1970), two-body random matrix ensembles (TBRE) [3, 4].

In the last 15 years strong motivation for understanding, analyzing and applying EE has come from: (i) statistical spectroscopy applications in nuclei and atoms [5]; (ii) experiments showing generic electron-electron interaction effects on transport properties of mesoscopic systems [6]; (iii) regular structures generated by random interactions in quantum many-body systems [7, 8]; (vi) possible applications to quantum gases and in quantum information science (thermalization, entanglement etc.) [9, 10]. Because of these, large variety of EE have been explored [11-22].
Most of the EE are based on symmetries as shown in Fig. 1. In 1972 Dyson has laid down criterion for the validity of a random matrix ensemble and they are [23]: (1) it predicts a global level-density distribution in agreement with observation; (2) it predicts a local statistical behavior of energy levels in agreement with Wigner distribution; (3) its definition is physically plausible; (4) its consequences are amenable to mathematical treatment. It is well established that EE satisfy (1), (2) and (3). A long standing question for EE is with regard to (4) and there is some progress using three different approaches as discussed in the next three sections. Most important result is that the Wigner-Racah algebra of the Lie algebra that generates the embedding in principle leads to analytical solubility of EE. Now we will give a preview.

Analytical approaches for EE are based on (i) Wigner-Racah algebra of the embedding algebra; (ii) extension of binary correlation approximation, first used by Wigner for GOE, giving asymptotic results; (iii) trace propagation over spaces defined by group symmetries. Results of (i) and (ii) with new examples are given in Sections 2 and 3 respectively. Section 4 gives an overview of (iii) and finally Section 5 gives conclusions. It should be emphasized that in (i) and (iii) employed are the results of two seminal papers by Draayer with Hecht [24] and Rosensteel [25] and in addition, it is conjectured that further progress in (ii) may come from another seminal paper by Draayer with French [26].

![Diagram of embedded random matrix ensembles with symmetries](image)

**Figure 1.** Embedded random matrix ensembles with symmetries. EGOE/EGUE correspond to fermionic systems and BEGOE/BEGUE correspond to bosonic systems. Physical systems where a given ensemble applies are mentioned below the ensemble. See text for other details.

2. Results from Wigner-Racah algebra of the embedding algebra: EGUE(2)-$SU(r)$

Consider a system of $m$ fermions or bosons in $\Omega$ number of sp levels each $r$-fold degenerate. Then the SGA is $U(r\Omega)$ and a subalgebra of considerable interest is $U(r\Omega) \supset U(\Omega) \otimes SU(r)$ algebra. For random two-body Hamiltonians preserving $SU(r)$ symmetry, one can introduce embedded GUE with $U(\Omega) \otimes SU(r)$ embedding and this ensemble is called EGUE(2)-$SU(r)$. Ensembles with $r = 1, 2$ and 4 for fermions correspond to spinless fermions, fermions with spin ($s$) as appropriate for mesoscopic systems and Wigner’s spin-isospin $SU(4)$ symmetry used in nuclear structure respectively [18, 19, 20]. Similarly, for bosons $r = 1, 2$ and 3 correspond to spinless bosons, bosons with a fictitious spin ($F$) as appropriate for two spices boson systems and spin
one bosons \((S = 1)\) as appropriate for spinor BEC [18, 21, 15, 17]. It is important to note that the distinction between fermions and bosons is in the \(U(\Omega)\) irreducible representations (irreps) that one has to consider for \(m\) number of fermions the \(U(r\Omega)\) irrep is \(\{1^m\}\) and for \(m\) bosons it is \(\{m\}\). Now we will define EGUE(2)-\(SU(r)\). Firstly, a general two-body Hamiltonian operator \(\hat{H}\) preserving \(SU(r)\) symmetry can be written as, with \(A^\dagger\) and \(A\) representing normalized two-particle creation and annihilation operators,

\[
\hat{H} = \hat{H}_{\{2\}} + \hat{H}_{\{1^2\}} = \sum_{f_2,v_2^r,v_2^\ell,\beta_2;f_2 = \{2\},\{1^2\}} H_{f_2,v_2^r,v_2^\ell}(2) A^\dagger(f_2v_2^r\beta_2) A(f_2v_2^\ell\beta_2) .
\]

Here \(f_2\) are two-particle \(SU(\Omega)\) irreps \(\{2\}\) and \(\{1^2\}\), in Young tableaux notation and they define uniquely the corresponding \(SU(r)\) irreps \(f_2^{(r)}\). Similarly, \(v_2\)'s are \(f_2\) sub-labels and \(\beta\)'s belong to \(f_2^{(r)}\). The two-body matrix elements (TBME) \(H_{f_2,v_2^r,v_2^\ell}(2) = \langle f_2v_2^r\beta_2 \mid H \mid f_2v_2^\ell\beta_2 \rangle\) are independent of the \(\beta_2\)'s and the uniform sum over them ensures \(SU(r)\) symmetry. Note that \(m\) particle states are by denoted by \(f_m\) are \(SU(r)^{\text{irrep}}\) 's and the uniform sum over them ensures \(SU(r)\) symmetry. As \(\hat{H}\) is a \(SU(r)\) scalar, \(m\) particle \(H\) matrix will be a direct sum of matrices with each of them labeled by the \(f_m\)'s with dimension \(d_\Omega(f_m)\).

\[
H(m) = \sum f_m H_{f_m}(m) \oplus , \quad d_\Omega(f_m) = \prod_{i<j=1} \frac{f_i - f_j + j - i}{j - i} .
\]

Embedded random matrix ensemble EGUE(2)-\(SU(r)\) for a \((m, f_m)\) system, i.e. \(\{H_{f_m}(m)\}\), is generated by the ensemble of \(H\) operators given in Eq. (1) with \(H_{\{2\}}(2)\) and \(H_{\{1^2\}}(2)\) matrices replaced by independent GUE ensembles of random matrices so that the random variables defining the real and imaginary parts of the matrix elements of \(H_{f_2}(2)\) are independent Gaussian variables with zero center and variance given by (with bar representing ensemble average) \(\bar{H}_{f_2,v_2^r,v_2^\ell}(2)\) \(\bar{H}_{f_2,v_2^r,v_2^\ell}(2) = \delta_{f_2f_2'}\delta_{v_2^r v_2^{\ell'}}\delta_{v_2^{\ell} v_2^{r'}}(\lambda_{f_2})^2\). Typically, one is first interested in deriving the ensemble averaged forms of one \((\bar{\rho})\) and two-point \((\bar{S})\) functions in eigenvalues. These are derived via their moments \(M_p\) and \(\Sigma_{PQ}\) respectively,

\[
\begin{align*}
\rho_{m,f_m}(E) &= \langle \delta(H - E) f_m \rangle_M \Leftrightarrow M_p &= \langle (H)^{f_m} \rangle_M, \\
S_{m,f_m;m',f_{m'}}(E,E') &= \rho_{m,f_m}(E) \rho_{m',f_{m'}}(E') - \rho_{m,f_m}(E') \rho_{m',f_{m'}}(E) \Leftrightarrow \Sigma_{PQ}(m, f_m : m', f_{m'}) = \left[ \langle (H^2)^{m,f_m} \rangle \right]^{P/2} \left[ \langle (H^2)^{m',f_{m'}} \rangle \right]^{-Q/2} \langle (H)^{m,f_m} (H)^{m',f_{m'}} \rangle .
\end{align*}
\]

Note that \(\Sigma_{11} = \bar{\Sigma}_{11} = 1\) and \(\Sigma_{22} = \bar{\Sigma}_{22} = 1\). Under tensorial decomposition w.r.t \(U(\Omega) \otimes SU(r)\), \(H\) will transform as the irrep \(\{0\}\) w.r.t \(SU(r)\) and w.r.t \(SU(\Omega)\) each \(f_2\) part transform as \(F_\nu\), \(\nu = 0, 1, 2\) where \(F_0 = \{0\}\), \(F_1 = \{2^{1\Omega-2}\}\) and \(F_2 = \{42^{1\Omega-4}\}\) for \(f_2 = \{2\}\). However, for \(f_2 = \{1^2\}\) we have \(F_2 = \{2^{21\Omega-4}\}\). Using tensorial decomposition and the Wigner-Eckart theorem, with the help of the Wigner-Racah algebra for \(U(\Omega) \otimes SU(r)\) given in detail in [24], ensemble averages can be carried out. Final formulas for \(M_2, \Sigma_{11}\) and \(\Sigma_{22}\) involve functions \(P\)'s, \(Q\)'s and \(R\)'s,

\[
\begin{align*}
M_2 &= \langle (H^2)^{m,f_m} \rangle = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{\nu=0,1,2} Q^\nu(f_2 : m, f_m) , \\
\langle (H)^{m,f_m} (H)^{m',f_{m'}} \rangle &= \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} P^{f_2}(m, f_m) P^{f_2}(m', f_{m'}) , \\
\Sigma_{22}(m, f_m ; m', f_{m'}) &= \frac{X_{\{2\}} + X_{\{1^2\}} + 4X_{\{1^2\} \{2\}}}{\langle (H^2)^{m,f_m} (H^2)^{m',f_{m'}} \rangle} ;
\end{align*}
\]
\[
X_{f_2} = \frac{2(\lambda_{f_2})^4}{[d_\Omega(f_2)]^2} \sum_{\nu=0,1,2} [d_\Omega(F_\nu)]^{-1} Q^\nu(f_2 : m, f_m)Q'^\nu(f_2 : m', f_{m'}) , \\
X_{\{1\}^2(2)} = \frac{\frac{\lambda_2^2 \chi_2}{d_\Omega(\{2\}) d_\Omega(\{1\}^2)}}{[d_\Omega\{1\}^2]^2} \sum_{\nu=0,1,2} [d_\Omega(F_\nu)]^{-1} R^\nu(m, f_m) R'^\nu(m', f_{m'}) .
\]

Here \(d_\Omega(F_\nu)\) is the dimension of the irrep \(F_\nu\). The functions \(P\)'s, \(Q\)'s and \(R\)'s will contain only \(SU(\Omega)\) Racah \((U-)\) coefficients and dimension factors. For example,

\[
Q^\nu(f_2 : m, f_m) = [F(m)]^2 \sum_{f_{m-2}', f_{m-2}'} \frac{N_{f_{m-2}-f_{m-2}', f_{m-2}}}{N_{f_m}} X_{UU}(f_2 : f_{m-2}, f_{m-2}'; F_\nu) ; \\
X_{UU}(f_2 : f_{m-2}, f_{m-2}; F_\nu) = \sum_\rho U(f_{m-2}, f_{m-2}, f_{m-2}'; F_\nu) \rho U(f_{m-2}, f_{m-2}, f_{m-2}' ; f_{m-2}, f_{m-2}; F_\nu) \rho .
\]

Here, \(N_f\) is the dimension of irrep \(f\) w.r.t the symmetric group \(S(\Omega)\), \(F(m) = -m(m-1)/2\) and \(\rho\) are multiplicity labels for the coupling \(f_m \times F_\nu \rightarrow f_m\). Using the tabulations in [27], Racah coefficients in Eq. (5) can be simplified. This formulation unifies all the results known for \(r = 1, 2, 4\) for fermions [18-20] and \(r = 1\) for bosons [18,21]. For example, applying Eq. (4)

for \(r = 1\) for bosons, the final formulas for \(\Sigma_{11}\) and \(\Sigma_{22}\) are,

\[
\Sigma_{11}\{m\}, \{m'\} = \frac{2\sqrt{m(m-1)(m')(m'-1)}}{\Omega(\Omega+1)\sqrt{(\Omega+m-1)(\Omega+m-2)(\Omega+m'-1)(\Omega+m'-2)}} , \\
\Sigma_{22}\{m\}, \{m'\} = 2 \left[ 36(\Omega + 3)(\Omega + 3)^2 (\Omega + m - 1)(\Omega + m - 2) \left( \frac{m}{2} \right)^{-1} \right]^{m} \\
\times \left[ \frac{\Omega^2(\Omega - 1)(\Omega + m + 1)}{2} \left( \frac{m}{2} \right) + 4 \left( \Omega + 2 \right)^2 (\Omega + 3) \left( \frac{m}{2} \right) \right]^{m'} \\
+ 4 \left( \Omega^2 - 1 \right) (\Omega + 3)(m - 1)(\Omega + m)(m' - 1)(\Omega + m') .
\]

More importantly, using this formulation results are derived for \(r = 2\) and \(3\) for bosons. The results for \(r = 2\) are verified using the corresponding results for fermion systems with \(r = 2\) given in [19] and the well known \(\Omega ightarrow -\Omega\) symmetry. For \(r = 3\), for example, for the three row irrep \(\{n, n, n\}\) (then \(m = 3n\)) we have \((\pi = 1 \text{ for } \{2\} \text{ and } -1 \text{ for } \{1^2\})

\[
P^{f_2}(m, \{n, n, n\}) = -\frac{6}{3} n(n - \pi) , \quad Q'^{\nu=0}(f_2 : m, \{n, n, n\}) = \left[ P^{f_2}(m, \{n, n, n\}) \right]^2 , \\
Q'^{\nu=1}(f_2 : m, \{n, n, n\}) = \frac{3(3 + \pi)^2 (\Omega + \pi)(\Omega - 3n(n - \pi)^2(\Omega + n)}{8(\Omega + 2\pi)} , \\
Q'^{\nu=2}(f_2 : m, \{n, n, n\}) = \frac{3(\pi)(\Omega^2 - 3\pi)(\Omega - 3n(n - \pi))}{16(\Omega + 2\pi)}(\Omega + n) , \\
R'^{\nu=0}(m, \{n, n, n\}) = P^{(2)}(m, \{n, n, n\}) P^{(1')}(m, \{n, n, n\}) , \\
R'^{\nu=1}(m, \{n, n, n\}) = -\sqrt{\frac{\Omega^2 - 1}{\Omega^2 - 4}} 3(\Omega - 3n(n^2 - 1)(\Omega + n) .
\]

We have also worked out results for other irreps for \(r = 3\); see for example [17].

Numerical calculations for \(\Sigma_{PP}(m, f_m : m', f_{m'})\) for \(P = 1, 2\) showed that there is increase in the magnitude of the fluctuations in energy centroids and spectral variances with increasing \(r\) value in direct correlation with the \(H_{fm}(m)\) matrices becoming more dense (implying stronger mixing) as we increase \(r\) in \(EGUE(2), SU(r)\) [20, 17]. Further investigation of this feature may establish that symmetries are responsible for chaos in finite quantum systems in general and nuclear shell model spaces in particular. Another important result is that \(\Sigma_{PP}(m, f_m : m', f_{m'})\)
for \( m \neq m' \) and/or \( f_m \neq f'_m \) will be non-zero [see Eq. (6)] and thus there will be correlations between spectra with different quantum numbers. These give a unique signature for EE as these cross correlations vanish for classical (GOE/GUE/GSE) ensembles.

The formulation based on Wigner-Racah algebra has been used [18, 21, 17] for \( r = 1 \) to derive formulas for \( M_2, M_4, \Sigma_{11} \) and \( \Sigma_{22} \) for the more general EGUE(2) generated by \( k \)-body interactions. They have established Gaussian \( (G) \) form for the one point function. However even for two-body interactions, for \( M_4 \) for \( r > 1 \) and \( \Sigma_{PQ} \) with \( P, Q > 2 \) for \( r \geq 1 \) could not be derived. Therefore the form of the one-point function is not yet established, analytically, for \( r > 1 \) and the two-point function for any \( r \). This is due to the fact that analytical formulas or numerical methods for the \( SU(\Omega) \) Racah coefficients needed for these are not available [20].

3. Results from the binary correlation method: Application to neutrinoless double beta decay

Binary correlation approximation (BCA) was used first by Wigner to derive the moments of the eigenvalue density generated by GOE. This approach was extended to EGUE(2) [i.e. \( r = 1 \)] for spinless (dilute) fermion systems by Mon and French [2]. In BCA, averages (moments) \( \langle H^m \rangle \) are evaluated by retaining only terms that involve squares (but not any other power) of \( H \) matrix elements in the defining space. Given \( m \) fermions in \( N \) single particle states, for EGUE(2) BCA gives for the excess parameter \( \gamma_2 \) the result, \( \gamma_2 \sim (m - 2)^2 m^{-1} - 1 \sim -4/m \) showing that asymptotically (i.e. for dilute systems) the eigenvalue density approaches Gaussian. The BCA method was also applied to transition strength densities [28]. Given a transition operator \( O \), transition strength density \( I_{biv,O}(E_i, E_f) = \langle E_f \mid O \mid E_i \rangle^2 I(E_i) \) where \( I(E) \) are state (or eigenvalue) densities. Representing \( H \) and \( O \) by independent EGOEs, it was established that \( I_O \) takes bivariate Gaussian form. It should be recognized that for EGUE for spinless fermions, moments correspond to traces over the complete \( m \)-particle space, i.e. all particles are in a single unitary orbit. A problem of current interest is to derive the form of \( I_{biv,O} \) with \( O \) representing the transition operator appropriate for neutrinoless double beta decay (NDBD). This operator is a two-body operator changing two neutrons (n) into two protons (p) and the calculation of NDBD half-lives requires squares of matrix elements of \( O \) connecting the ground state of the parent nucleus to the ground state of the daughter nucleus [29]. Therefore we need to consider a system with protons and neutrons explicitly identified, i.e. two-orbit traces. Nuclear Hamiltonians are one plus two-body in nature, \( H = h(1) + V(2) \). Then, \( I_{biv,O} \) can be written as a convolution of the corresponding strength density generated by the mean-field producing \( h(1) \) part with a normalized spreading bivariate strength density \( \rho_{biv;O,V} \) generated by the interaction \( V(2) \) [28]. Two important questions are: (i) is \( \rho_{biv;O,V} \) close to a bivariate Gaussian (this can be established by calculating 4th and 6th order bivariate moments); (ii) is it possible to derive a formula for the bivariate correlation coefficient defining \( \rho_{biv;G,O,V} \). From now on we put \( H(2) = V(2) \). BCA is used to answer (i) and (ii).

With space \#1 denoting protons and similarly space \#2 neutrons, \( (m_p, m_n) = (m_1, m_2) \) where \( m_p \) is number of valance protons and \( m_n \) valence neutrons for a given nucleus. We assume that protons are in \( N_1 \) number of single particle (sp) states and neutrons in \( N_2 \) number of sp states. Now, the general form of \( H(2) \) and the transition operator \( O \) are,

\[
H(2) = \sum_{i+j=2; \alpha, \beta, \gamma, \delta} \left[ v_H^{\alpha \beta \gamma \delta}(i,j) \right] \alpha_i^{\dagger}(i) \beta_i(i) \gamma_j^{\dagger}(j) \delta_2(j) ; \quad O(2) = \sum_{\gamma, \delta} v_O^{\gamma \delta}(2) \gamma_1^{\dagger}(2) \delta_2(2) . \tag{8}
\]

Here in general \( \alpha_i^{\dagger}(l) \) is normalized \( l \)-particle creation operator in space \#k and \( \alpha_k(l) \) is the corresponding annihilation operator. Note that \( v(2,0), v(0,2) \) and \( v(1,1) \) represent p-p, n-n and p-n parts of the Hamiltonian. Similarly \( O \) changes two neutrons into two protons. It is important to note that \( H(2) \) given by Eq. (8) preserves \( (m_1, m_2) \). However, \( O \) and
its hermitian conjugate $\mathcal{O}^\dagger$ do not preserve $(m_1, m_2)$, $\mathcal{O}(2)|m_1, m_2\rangle = |m_1 + 2, m_2 - 2\rangle$ and $\mathcal{O}^\dagger(2) |m_1, m_2\rangle = |m_1 - 2, m_2 + 2\rangle$. Thus, given a $(m_1, m_2)$ for an initial state, the $(m_1, m_2)f$ for the final state generated by the action of $\mathcal{O}$ is uniquely defined. The bivariate moments of $\rho_{\text{biv}, \mathcal{O}:V}$ are defined by $\bar{M}_{ PQ}((m_1, m_2)) = \langle \mathcal{O}^\dagger(2)H^2(2)\mathcal{O}(2)H^P(2) \rangle^{(m_1, m_2)}$. Asymptotic formulas for the moments are derived by representing $H(2)$ and $\mathcal{O}(2)$ operators by independent EGOEs and then applying BCA. The EGOE representation implies $\theta_H^{-}\theta$ are independent zero centered Gaussian variable with $\theta_H^{-}\theta_{ij}\sim\theta_{ij}^{-}\theta$, independent of the labels $\alpha, \beta, \gamma$ and $\delta$. Similarly, $\theta_{ij}^{-}\theta$ are zero centered Gaussian variables with $\theta_{ij}^{-}\theta\sim\theta_{ij}^{-}\theta$. It is easy to see that (with $\bar{m}_i = N_i - m_i$),

$$
\langle H^2(2) \rangle^{m_1, m_2} = \sum_{i+j=2} v_H^{2}(i, j) T(m_1, N_1, i) T(m_2, N_2, j),
$$

$$
\bar{M}_{00}(m_1, m_2) = \langle \mathcal{O}^\dagger(2)\mathcal{O}(2) \rangle^{m_1, m_2} = \theta_V^{2} \binom{m_1}{2} \binom{m_2}{2},
$$

$$
\bar{M}_{02}(m_1, m_2) = \bar{M}_{00}(m_1, m_2) \langle H^2(2) \rangle^{m_1, m_2, m_2},
$$

$$
\bar{M}_{11}(m_1, m_2) = v_V^{2} \sum_{i+j=2} v_H^{2}(i, j) \binom{m_1 - i}{2} \binom{m_2 - j}{2} T(m_1, N_1, i) T(m_2, N_2, j).
$$

(9)

The function $T$ in Eq. (9) is $T(m, N, k) = \binom{\bar{m}+k}{k}$. As $\bar{M}_{10}(m_1, m_2) = \bar{M}_{01}(m_1, m_2) = 0$, the correlation coefficient $\zeta_{\text{biv}}(m_1, m_2) = \{\bar{M}_{20}(m_1, m_2) \bar{M}_{02}(m_1, m_2)\}^{-1/2}$ $\bar{M}_{11}(m_1, m_2)$ and this can be calculated using Eq. (9). With appropriate choice of sp orbits we have for example for $^{100}\text{Mo}$, $^{150}\text{Nd}$ and $^{238}\text{U}$, $(N_1, m_1, N_2, m_2) = (30, 2, 32, 8)$, $(32, 10, 44, 8)$ and $(44, 10, 58, 20)$ respectively. They will give, by using Eq. (9) with the assumption that $v_H^{2}(i, j)$ are independent of $(i, j)$, $\zeta_{\text{biv}} = 0.57, 0.72$ and $0.83$ respectively for the three nuclei. Thus in general $\zeta_{\text{biv}} \sim (0.6 - 0.8)$. More importantly, formulas for $\bar{M}_{ P Q}((m_1, m_2))$ with $P + Q = 4$ are derived using BCA but due to lack of space these are not presented here [30]. These will allow us to calculate the bivariate cumulants $k_{rs}$ of order 4 and it should be noted that $|k_{rs}| \lesssim 0.3$ implies bivariate Gaussian form for $\rho_{\text{biv}, \mathcal{O}:V}$. For the four nuclei, the BCA formulas give $(k_{40}, k_{50}, k_{31}, k_{13}, k_{22}) = (-0.45, -0.42, -0.26, -0.24, -0.2), (-0.27, -0.29, -0.2, -0.22, -0.19)$ and $(-0.18, -0.18, -0.15, -0.15, -0.13)$ respectively. Thus, $\rho_{\text{biv}, \mathcal{O}:V} \rightarrow \rho_{\text{biv}, \mathcal{O}:V}$ and this result is being applied to calculate nuclear transition matrix elements for NDBD in $^{150}\text{Nd}$. It is important to recognize that the BCA method for EGOE/EGUE with symmetries such as $U(\Omega) \otimes SU(r)$ is not available (except for $r = 1$ for fermions). Therefore, Gaussian form for eigenvalue densities could not be established even for EGOE with spin symmetry ($r = 2$) for Fermi systems. Similarly, BCA method for boson systems is not available even for $r = 1$. At present, Gaussian form even for spinless BEGOE(2) was established by using trace propagation formulas for fourth moments (see next section) and carrying out explicitly ensemble averaging [31]. These two are important open problems in EGOE theory and perhaps, the p-expansion method of French and Drayer [26] may prove to be useful in making advances in future.

4. Results from trace propagation

Propagation equations for energy centroids $\langle H \rangle^{m, f \lambda}$ and as spectral variances $\langle H^2 \rangle^{m, f \lambda} - \langle H \rangle^{m, f \lambda}^2$ over irreps of group symmetries are successfully used to derive many important results for EGOEs. A simple counting theorem confirms if the propagators are Casimir invariants of the Lie algebras involved and otherwise the so-called integrity basis operators are needed [26, 25]. For BEGOE(2), BEGOE(2)-s, BEGOE(2) and BEGOE(2)-F, the propagation is simple [11, 13, 15, 32]. For EGOE(1+2)’s with random two-body interactions in the presence of a mean-field, in the strong coupling limit, as the eigenvalue densities approach Gaussian form, $\sigma^2(\lambda, M)$ determines the ground state spin structure in fermion systems $\langle \sigma^2(\lambda, M) \rangle$ for boson systems. The variances also show that random interactions generate odd-even staggering
in the ground state energies. More generally, propagation formulas for energy centroids and spectral variances will allow us understand a variety of regular structures generated by random interactions [33, 34, 13, 15]. Remarkably, EGOE(1+2)'s admit three transition markers, $\lambda_c$ (for Poisson to GOE in fluctuations), $\lambda_F$ (Breit-Wigner to Gaussian in strength functions) and $\lambda_t$ (defining thermodynamic region) and their parametric ($m, N, S$ or $F$ etc.) dependence is determined by spectral variances [11, 13, 32, 15, 35].

5. Conclusions
Although advances are made in deriving analytical results for EE for certain quantities using (i) Wigner-Racah algebra, (ii) binary correlation approximation and (iii) trace propagation methods, mathematical tractability still remains a major issue in the study of EE. Further progress requires new developments in mathematical and computational methods and they are a challenge for future.

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