Abstract.

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In this paper the quantization of the 2+1-dimensional gravity coupled to the massless Dirac field is carried out. The problem is solved by the application of the new Dynamic Quantization Method [1,2]. It is well-known that in general covariant theories such as gravitation, a Hamiltonian is any linear combination of the first class constraints, which can be considered as gauge transformation generators. To perform quantization, the Dirac field modes with gauge invariant creation and annihilation operators are selected. The regularization of the theory is made by imposing an infinite set of the second class constraints: almost all the gauge invariant creation and annihilation operators (except for a finite number) are put equal to zero. As a result the regularized theory is gauge invariant. The gauge invariant states are built by using the remained gauge invariant fermion creation operators similar to the usual construction of the states in any Fock space. The developed dynamic quantization method can construct a mathematically correct perturbation theory in a gravitational constant.

1. Introduction.

In the recent paper of the author [2] the Hamilton Quantization of the 2+1-dimensional gravity coupled to massless Dirac field is carried out on the basis of the Dynamic Quantization Method. The space of the regularized states of the system is built. The perturbation theory (PT) in a gravitational constant is developed. Using it the regularized Heisenberg equations are solved. Lately some axiomatization of the dynamic quantization method has been performed using gravity model [2] as an example. This clarified the inner logics of the dynamic method. It also became possible to get concrete results easier. This paper considers the process of dynamic quantization of gravity in three-dimensional space-time from a new viewpoint. Since the dynamic quantization method is not well-known, we outline it shortly here.

Let’s consider some field theory. Suppose that in this theory the physical degrees of freedom in ultraviolet region can be classified by the modes with the following properties:

a) The occupation numbers of the modes are conserved or they are adiabatic invariants of motion.

b) The corresponding creation $a^+_N$ and annihilation $a_N$ operators are to be gauge invariants.

These are basic points for the Dynamic Quantization Method.

Then the regularization of the theory is made by imposing the second class constraints

$$a^+_N \approx 0, \quad a_N \approx 0$$  \hspace{1cm} (1.1)

for quantum numbers $N$ from the ultraviolet tail. The second class constraints
lead to the substitution of the corresponding Dirac commutational relations (CR) for the initial or formal CR. Then we are to solve the Heisenberg equations obtained by the Dirac CR. Thus the theory becomes regularized with definite numbers of physical degrees of freedom. Moreover, since the annihilation and creation operators in Eqs.(1.1) are gauge invariant the imposition of the constraints (1.1) does not violate gauge invariance.

It is necessary to pay attention to the difference between the Feynman and Dynamic quantizations. The Feynman quantization is based on the hypothesis that the interaction can be switched on and off adiabatically. This hypothesis is equivalent to the assumption that the physical vacuum differs slightly from the naive one (at a switched off interaction). The assumption leads to the Feynman rules. But the Feynman PT in general cannot be regularized so that the number of physical degrees of freedom would be definite. As Gribov noted [3] this nonconservation of the whole number of degrees of freedom results in the appearance of the gauge anomaly.

It is known, that the Feynman rules admit the Wick rotation which results in equivalence between $(D - 1) + 1$-dimensional quantum field theories and $D$-dimensional classic statistic models. Thus, the Euclidean quantization is automatically equivalent to the Feynman one.

The available experience of the general relativistic theory shows that the gravitating universe evolves from peculiarity to peculiarity. Thus, it is no wonder that the Feynman theory is not applicable for quantization of gravity. On the contrary, the dynamic quantization corresponds naturally to the situation in a general relativistic theory. Indeed, the annihilation and creation operators (see (1.1)) are gauge invariant and the corresponding field modes carry all space-time information about universe evolution; the space-time peculiarities are not excluded. These modes satisfy equations of motion, so that the corresponding annihilation and creation operators are conserved and the imposition of the constraints (1.1) is in agreement with dynamics. Thus dynamic quantization is realized. We think that in this case different gauge anomalies are absent. Such approach is used to investigate gauge anomaly in papers [1,4].

In this paper the dynamic quantization theory of gravitation interacting with the Dirac massless field in $2+1D$ space is present. To perform dynamic quantization, it is necessary to have a complete set of gauge invariant one-fermion states, mutually orthogonal in a natural sense. In terms of these states the regularization and vacuum feeling is produced. In the previous work [2] these states appeared as a result of long constructions. In this paper we assume that the necessary set of one-fermion states exists in the theory. This assumption can simplify essentially theory exposition.
and make a lot of important statements clear. The assumption is justified since on its basis the selfconsistent, physically sensible and mathematically correct theory is built. As it shown in [2] the space of the regularized states of the system is constructed, the regularized Heisenberg equations for field operators are written out and solved by the mathematically correct PT in a gravitational constant.

Let us emphasize that 2+1D gravitation theory is close to some extent to the chiral Schwinger model, studied earlier by dynamic quantization method [1]. Indeed, in both theories the gauge fields do not have their own local degrees of freedom (the absence of photon in 1+1 D and graviton in 2+1 D).

2. The Formal Quantization.

Let \( x^\mu = (x^0, x^1, x^2) \) denote local coordinates in some 3-Dimensional metric space. The metrics is expressed in the form

\[
ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = \epsilon^a_\mu \, e_{a\nu} \, dx^\mu \, dx^\nu
\]  

(2.1)

where \( \epsilon^a_\mu \) are the dreibein. Latin letters \( a, b, \ldots = 0, 1, 2 \) from the beginning of the alphabet are \( SO(2, 1) \) indices, the sign convention is \( \eta_{ab} = \text{diag}(1, -1, -1) \). Let \( g^{\mu\nu} \) be the inverse metric tensor and

\[
e^\mu_a = g^{\mu\nu} \, \eta_{ab} \, e^b_v
\]

Then

\[
e^\mu_a \, e^\mu_b = \delta^a_b, \quad e^\mu_a \, e^a_v = \delta^\mu_v
\]  

(2.2)

The connection 1-form is designated as follows: \( \omega^a_b = \omega^a_{b\mu} \, dx^\mu \). Since

\[
\omega_{ab\mu} = \eta_{ac} \, \omega^{c}_{b\mu} = -\omega_{ba\mu}
\]  

(2.3)

it is convenient to use the equivalent quantity [5]

\[
\omega^c_\mu = \frac{1}{2} \varepsilon^{abc} \, \omega_{ab\mu},
\]

\[
\varepsilon^{012} = 1
\]  

(2.4)

In these notations the scalar curvature is of the form

\[
\sqrt{g} \, R = \varepsilon^{\mu\nu\lambda} \, e_{c\lambda} (\partial_\mu \omega^c_\nu - \partial_\nu \omega^c_\mu - \varepsilon^{ab}_c \, \omega^a_\mu \, \omega^b_\nu)
\]

\[
g = \text{det} \, g_{\mu\nu}
\]  

(2.5)
Let \( \gamma^a \) be the Dirac matrices \( 2 \times 2 \) which satisfy the conditions
\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab}
\] (2.6)

Denote by \( \psi \) and \( \bar{\psi} = \psi^+ \gamma^0 \) Dirac two-component complex field. Cross above means Hermitian conjugation.

According to (2.5) the simplest general covariant action is of the form
\[
A = \int d^3 x \left\{ -\frac{1}{16\pi G} \varepsilon^{\mu\nu\lambda} e_{c\lambda} (\partial_\mu \omega^c_\nu - \partial_\nu \omega^c_\mu - \varepsilon_{abc} \omega^a_\mu \omega^b_\nu) + \right.
\]
\[
+ \frac{i}{2} (\bar{\psi} \Gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \Gamma^\mu \psi) \right\},
\] (2.7)

where
\[
\nabla_\mu \psi = (\partial_\mu - \frac{i}{2} \omega_{a\mu} \gamma^a) \psi, \quad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \frac{i}{2} \bar{\psi} \gamma^a \omega_{a\mu},
\] (2.8)

and
\[
\Gamma^\mu = \sqrt{g} e^\mu_a \gamma^a = \frac{1}{2} \varepsilon^{\mu\nu\lambda} e_{abc} e^b_\nu e^c_\lambda \gamma^a \] (2.9)

Note that the Fermi part of the action (2.7) is taken symmetrised in order the action to be Hermitian. Indeed the expression
\[
\int d^3 x \ i \bar{\psi} \Gamma^\mu \nabla_\mu \psi
\]
is not Hermitian if the connection has torsion. But the equations of motion following from the action (2.7) lead to the connection with the torsion.

Further we shall designate by the letters \( x, y, \ldots \), the totality of spatial coordinates \( (x^1, x^2), (y^1, y^2), \ldots \), the time argument \( x^0 \) will be omitted and the point above will mean the derivative \( \partial/\partial x^0 \) and
\[
d^2 x = dx^1 dx^2, \quad \delta^{(2)}(x) = \delta(x^1) \delta(x^2).
\]

When obtaining Hamiltonian from the action (2.7) the following difficulty arises. Since the action (2.7) contains the quantities \( \dot{\psi} \) and \( \dot{\bar{\psi}} \), the fields \( \psi \) and \( \bar{\psi} \) both are to be regarded as coordinate variables. But the corresponding momentum variables \( \pi_\psi \) and \( \pi_{\bar{\psi}} \) are expressed through the same fields \( \psi \) and \( \bar{\psi} \).
Therefore the constraints
\[ \bar{\tau} = \pi_\psi - \frac{1}{2} \psi \Gamma^0 \approx 0, \quad \tau = \pi_{\bar{\psi}} - \frac{1}{2} \Gamma^0 \bar{\psi} \approx 0 \] (2.10)
take place. Thus the Hamiltonian dynamic variables unite in the following pairs:

\[ (\psi, \pi_\psi), \quad (\bar{\psi}, \pi_{\bar{\psi}}), \quad (\omega^a_i, P^i_a), \]

\[ \mathcal{P}^i_a = -\frac{1}{8\pi G} \varepsilon_{ij} e_{aj} \] (2.11)

Latin letters \( i, j, \ldots = 1, 2 \) from the middle of the alphabet are spatial indices, \( \varepsilon_{ij} = -\varepsilon_{ji}, \varepsilon_{12} = 1 \). The fields \( \omega^0_0 = \omega^0_i \) and \( e^0_a = e^a \) play the role of the Lagrange multiplier.

Let us introduce the function \( \alpha \) defined on the homogeneous operators with the values in the group \( \mathbb{Z}_2 \). By definition

\[ \alpha(\omega^a_i) = \alpha(e^a_i) = \alpha(\omega^a_0) = \alpha(\mathcal{P}^i_a) = 0, \]

\[ \alpha(\psi) = \alpha(\pi_\psi) = \alpha(\bar{\psi}) = \alpha(\pi_{\bar{\psi}}) = 1 \]

If the function \( \alpha \) is defined on the operators \( A \) and \( B \), then \( \alpha(AB) = \alpha(A) + \alpha(B)(mod\ 2) \). Everywhere we understand under the commutator of the homogeneous operators \( A \) and \( B \) the expression

\[ [A, B] = AB - BA (-1)^{\alpha(A)\alpha(B)} \] (2.12)

The initial nonzero simultaneous CR are of the form

\[ [\omega^a_i(x), \mathcal{P}^j_b(y)] = i \delta_{ij} \delta^a_b \delta^{(2)}(x - y), \]

\[ [\psi(x), \pi_\psi(y)] = 1 \cdot \delta^{(2)}(x - y), \]

\[ [\bar{\psi}(x), \pi_{\bar{\psi}}(y)] = 1 \cdot \delta^{(2)}(x - y) \] (2.13)

The Hamiltonian of the system looks like:

\[ H = H_\chi + H_\phi, \]

\[ H_\chi = -\int d^2 x \omega^a \chi_a, \]

\[ H_\phi = \frac{1}{8\pi G} \int d^2 x e_a \phi^a, \] (2.14a)
where
\[ \chi_a = \partial_i P^i_a - \frac{1}{2} \varepsilon_{ab}^c (\omega^b_i P^i_c + P^i_c \omega^b_i) + \]
\[ + \frac{1}{2} \bar{\psi} \sqrt{g} e^0_a \psi, \]  \hspace{1cm} (2.14b)

\[ \phi^a = \frac{1}{2} \varepsilon_{ij} (\partial_i \omega^a_j - \partial_j \omega^a_i - \varepsilon^a_{bc} \omega^b_i \omega^c_j) + \]
\[ + \frac{i}{2} (8\pi G)^2 \varepsilon_{ba}^c (\bar{\psi} \gamma^b P^i_c \partial_i \psi - \partial_i \bar{\psi} P^i_c \gamma^b \psi) + \]
\[ + \frac{1}{4} (8\pi G)^2 \varepsilon_{ba}^c \bar{\psi} \{ P^i_c \omega_{di} [ s \gamma^b \gamma^d + (1 - s) \gamma^d \gamma^b ] + \]
\[ + \omega_{di} P^i_c [(1 - s) \gamma^b \gamma^d + s \gamma^d \gamma^b ] \} \psi \] \hspace{1cm} (2.14c)

The quantities \( \sqrt{g} e^\mu_a \) are to be expressed through Hamiltonian variables:
\[ \sqrt{g} e^0_a = \frac{1}{2} (8\pi G)^2 \varepsilon_{a}^{bc} \varepsilon_{ij} P^i_b P^j_c, \]
\[ \sqrt{g} e^i_a = - e_c (8\pi G) \varepsilon_{a}^{bc} P^i_b \] \hspace{1cm} (2.15)

In (2.14c) \( S \) is a free real parameter. The freedom in the choice of \( S \) means the freedom in the arrangement of operators in the Hamiltonian (2.14). This freedom is eliminated in the next section.

With the help of CR (2.13) we find the commutator of the constraints (2.10):
\[ [\bar{\tau}_\sigma(x), \tau_\rho(y)] = - \delta^{(2)}(x - y) \Gamma^0_{\rho\sigma}(x) \] \hspace{1cm} (2.16)

It is seen from here that constraints (2.10) are the second class constraints. Thus it is necessary to pass from initial CR (2.13) to the corresponding Dirac CR. This is made according to general procedure [6]. Since further the initial CR (2.13) are not used we keep the previous notations for the so-obtained Dirac CR.

Write out the nonzero simultaneous Dirac CR for fundamental fields \( \omega^a_i, P^i_a, \psi, \bar{\psi} : [\psi_\sigma(x), \bar{\psi}_\rho(y)] = \delta^{(2)}(x - y)(\Gamma^0)^{\sigma\rho}(y), \)
\[ [\mathcal{P}_a^i(x), \omega_j^b(y)] = -i \delta_{ij} \delta_a^b \delta^{(2)}(x - y), \]

\[ [\omega_i^a(x), \psi(y)] = \frac{i}{2} \delta^{(2)}(x - y) (8\pi G)^2 \]

\[ \varepsilon_{ij} \varepsilon_b^{ca} (\mathcal{P}_c^j (\Gamma^0)^{-1} \gamma^b \psi)(y), \]

\[ [\omega_i^a(x), \bar{\psi}(y)] = \frac{i}{2} \delta^{(2)}(x - y) (8\pi G)^2 \]

\[ \varepsilon_{ij} \varepsilon_b^{ca} (\bar{\psi} \gamma^b (\Gamma^0)^{-1} \mathcal{P}_c^j)(y), \]

\[ [\omega_i^a(x), \omega_j^b(y)] = \frac{i}{2} \delta^{(2)}(x - y) (8\pi G)^2 \]

\[ \varepsilon_{ij} (\bar{\psi} \frac{e_0^a e_0^b}{e_c^0 e_0^c} \psi)(y) \] (2.17)

The quantities \( e_a^0 \) are given according to (2.15).

By the direct calculations one can be convinced that CR (2.17) satisfies the following properties for any complex numbers \( x, y \):

\[ [A, B] = -[B, A](-1)^{\alpha(A)\alpha(B)}, \]

\[ [xA + yB, C] = x[A, C] + y[B, C], \]

\[ [A, [B, C]](-1)^{\alpha(A)\alpha(C)} + [B, [C, A]](-1)^{\alpha(A)\alpha(B)} + \]

\[ +[C, [A, B]](-1)^{\alpha(B)\alpha(C)} = 0 \] (2.18)

Take by definition

\[ [A, BC] = [A, B] C + B [A, C] (-1)^{\alpha(A)\alpha(B)} \] (2.19)
Formulae (2.17) - (2.19) define inductively CR for any functional on fundamental fields $\omega_i^a$, $\mathcal{P}_a^i$, $\psi$, $\bar{\psi}$.

Thus, in Section 2 the formal quantization of the system is performed: Hamiltonian is of the form (2.14) and any CR are defined according to (2.17) - (2.19).

3. The Separation of the gauge invariant degrees of freedom and regularization.

Our aim consists in the construction of states which are annulled by operators (2.14), and in the solution of the Heisenberg equations $i \dot{A} = [A, H]$. Here $A$ - is any of fundamental fields (or their functional), and $H$ is given by (2.14).

However this problem can be solved only together with the problem of selection of gauge invariant degrees of freedom and regularization of the theory. The problem is solved in this Section.

3.1 The main Hypothesis and its Consequences.

We shall hold the point of view accepted in axiomatic quantum field theory. Let us begin at the construction of the Fock space formalism for fermion degrees of freedom. The space of physical states of gravitational field is discussed in the beginning of the next Section.

Denote by $\{|N\rangle\}$ the complete set of one-fermion states and by $|0\rangle$ the fermion vacuum so that

$$\psi(x)|0\rangle = 0, \quad \psi(x)|N\rangle = \psi_N(x)|0\rangle,$$

where $\psi_N(x)$ is the Bose-operator field.

Any fermion state is built by one-fermion states. For example, two-fermion state is of the form:

$$|N_1\rangle \otimes |N_2\rangle = -|N_2\rangle \otimes |N_1\rangle$$

(3.2)

It is convenient to modify the previous designations by introducing vacuum state for any index $N$. Then, for any $N$ there are two states

$$|N, \lambda_N\rangle, \quad \lambda_N = 0, 1,$$

$$|0\rangle = \bigotimes_N |N, 0\rangle, \quad |N, 1\rangle \equiv |N\rangle$$
Let us introduce the arrangement in space of index \( N \). Now any fermion state is expressed as a linear combination of the following states:

\[
\left| N_1, \ldots, N_s \right> = \bigotimes_N \left| N, \lambda_N \right>
\]

(3.3)

Here \( N_1 < N_2 < \ldots < N_s \) are indices for which \( \lambda_{N_1} = \lambda_{N_2} = \ldots = \lambda_{N_s} = 1 \), \( s \) is any natural number. The tensor products (3.3) are ordered so that the state \( \left| N_i, 1 \right> \) is on the left of \( \left| N_j, 1 \right> \) at \( N_i < N_j \).

Define the fermion annihilation operators \( a_N \) by their action on states (3.3)

\[
a_N \left| N_1, \ldots, N_i, N_{i+1}, \ldots, N_s \right> = (-1)^{\lambda_N(N_1, \ldots, N_s; N_i)} \cdot \lambda_N \left| N_1, \ldots, N_{i-1}, N_i+1, \ldots, N_s \right>,
\]

(3.4)

where

\[
\lambda(N_1, \ldots, N_s; N_i) = \sum_{N < N_i} \lambda_N
\]

and \( \lambda_N \) are defined by state (3.3).

Let us give the scalar product on the fermion states space according to formula

\[
\langle M, \lambda_M | N, \lambda_N \rangle = \delta_{MN} \delta_{\lambda_M \lambda_N}
\]

(3.5)

Now the operators conjugated to \( a_N \) and their commutational properties are found by the standard way by Eq.(3.3) - (3.5). We have:

\[
[a_M^+, a_N] = \delta_{MN}, \quad [a_M, a_N] = 0
\]

(3.6)

Further we shall represent the operators \( \{a_N^+, a_N\} \), Fermi-states and their scale product as in the Appendix.

Let us make the following Hypothesis, which is the base for a further development of the dynamic quantization method:

**Hypothesis A** The complete set of one-fermion states \( \left| N, 1 \right> \) can be chosen so that

\[
\langle M, 1 | \int d^2 x \bar{\psi}(x) \Gamma^0(x) \psi(x) | N, 1 \rangle = \delta_{MN}
\]

(3.7)

Emphasize that here and further in this Section averaging is made only in fermion fluctuations and the fermion functional measure is given in (A4).

Explain the naturallity of the Hypothesis A.
Consider the unitary transformation
\[ |N, 1\rangle' = \sum_M S_{NM} |M, 1\rangle, \quad \langle N, 1| = \sum_M \langle M, 1| S_{NM}^+, \]
\[ \sum_M S_{MN}^+ S_{ML} = \delta_{NL}, \tag{3.8} \]
which conserves the form \( \sum_N \langle N, 1| N, 1 \rangle \). This transformation conserves also all above formulae of this Section. With the help of the unitary transformation (3.8) one can reach that the hermitean matrix
\[ \langle M, 1| \int d^2x \bar{\psi}(x) \Gamma^0(x) \psi(x) | N, 1 \rangle \tag{3.9} \]
be diagonal. Note that matrix (3.9) one can always regard as positively defined. \[ \] Then taking into account Eq.(3.5) and the form of fermion measure (A4), we come to formula (3.7) of Hypothesis A.

It follows immediately From Hypothesis A that the fields \( \psi_N(x) \) are linearly independent.

Now expand the fermion fields:
\[ \psi(x) = \sum_N a^+_N \psi_N(x), \quad \bar{\psi}(x) = \sum_N a^+_N \bar{\psi}_N(x) \tag{3.10} \]

It is not difficult to show that the following CR take place:
\[ [a_N, \psi_M] = [a^+_N, \psi_M] = 0 \tag{3.11} \]

Indeed, from the definition of the field \( \psi_M \) it follows that \( \psi_M(x) \) can contain operators \( a^+_L \) and \( a_N \) only in identical degrees. We think that the operators \( a^+_L \) and \( a_N \) containing in the fields \( \psi_M(x) \) are normally arranged. It is easy to see that in fact the field \( \psi_M(x) \) does not contain the operator \( a_N \). Otherwise Eq.(3.1) and (3.10) should mean that
\[ \psi_N(x) = \psi_N(x) + \sum_M \frac{\partial \psi_M(x)}{\partial a_N} a_M, \]

But this is impossible in consequence of linearly independence of the fields \( \psi_M \).

\[ \]
\[ ^1 \quad \text{The necessity of this condition is clear from CR (2.19) for fermion fields. Indeed denote by} \ |\Lambda\rangle \text{ and} \ u(x) \text{ the arbitrary fermion state and Bose spinor field respectively, and} \ |\Lambda_-, u\rangle = \int d^2x (u^+ \psi)|\Lambda\rangle, \ |\Lambda_+, u\rangle = \int d^2x (\psi^+ u)|\Lambda\rangle. \text{ Then from CR (2.19) we have:} \ \langle \Lambda| \int d^2x u^+ (\gamma^0\Gamma^0)^{-1} u|\Lambda\rangle = \langle \Lambda_+, u| \Lambda_+ u\rangle + \langle \Lambda_-, u| \Lambda_- u\rangle > 0. \text{ Thus the inclusion the matter in gravity leads to serious restrictions for quantum fluctuations of fields having geometrical sense. In particular the condition \( detg_{ij} > 0 \) arises.} \]
One can rewrite relation (3.7) with (3.11) in the following form:

\[ \int d^2x \bar{\psi}_M \Gamma^0 \psi_N = \delta_{MN} \tag{3.12} \]

Since the set of fields \( \{ \psi_M(x) \} \) is complete relations (3.12) are equivalent to the following equality:

\[ \sum_N \psi_N(x) \bar{\psi}_N(y) = \delta^{(2)}(x - y) (\Gamma^0)^{-1}(y) \tag{3.13} \]

Further

\[ [\psi(x), \bar{\psi}(y)] = \delta^{(2)}(x - y)(\Gamma^0)^{-1}(y) = \]

\[ = \sum_{MN} [\bar{\psi}_M(y), \psi_N(x)] a_M^+ a_N + \sum_N \psi_N(x) \bar{\psi}_N(y) \tag{3.14} \]

The first from equalities (3.14) is taken from (2.17), the second one is obtained by Eqs.(3.6) and (3.10). Comparing relations (3.13) and (3.14) we find the CR:

\[ [\psi_M(y), \psi_N(x)] = 0 \tag{3.15} \]

Analogously we get:

\[ [\psi_M(y), \psi_N(x)] = 0 \tag{3.15'} \]

Now reinforce the Hypothesis A, completing it by the following:

**Hypothesis B** The set of states \( |N, \lambda_N\rangle \) is gauge invariant, that is

\[ [a_N, \chi_a] = 0, \quad [a_N, \phi_a] = 0 \tag{3.16} \]

Note that if any formulae are written out for \( a_N^+ \) or \( \bar{\psi} \) then the analogous formulae for \( a_N \) or \( \psi \) are obtained by the Hermitian conjugation.

Comparison of Eqs.(3.16) with the analogous equations in usual gauge theory (see Eqs.(3.17) in [1]) shows that the application of the dynamic quantization to general covariant theories is the most natural. This follows from (i) in general covariant theories all dynamics is reduced to gauge transformations; (ii) the key point of the dynamic quantization method is the selection of the gauge invariant operators like the annihilation and creation operators introduced here. As is shown in [1] in the usual gauge theory the gauge invariant operators playing the same role in dynamic quantization have the phase factor depending on time significant near cutoff impulse.

It follows from equations of motion (see Section 5) that the operator

\[ \int d^2x \bar{\psi} \Gamma^0 \psi \] is conserved. Thus, Hypothesis B results that all relations of this
Section are conserved in time. Moreover, the fields \( \psi_M(x) \) satisfy the same equation as the field \( \psi(x) \):

\[
i \Gamma^\mu \nabla_\mu \psi_M = \frac{i}{2} (8\pi G) \varepsilon_{abc} e_c \gamma^a \psi_M \chi_b
\]  

(3.17)

Using expansion (3.10) and the completeness property (3.12) one can express the annihilation and creation operators in the form:

\[
a^+_N = \int d^2x \, \bar{\psi} \Gamma^0 \psi_N
\]  

(3.18)

3.2 The Imposition of regularizing Constraints.

As Witten showed [5], if the matter is absent, the quantum gravity in 3-dimensional space-time is not only renormalizable but also finite theory (in the framework of the Feynman theory). However, the coupling with the Dirac fields makes the Feynman quantization impossible, since in the Witten variables \( (P^i_a, \omega^a_i) \) the zeroth approximation for the Dirac field is absent. The reason is that according to Witten the variables \( P^i_a \) and \( \omega^a_i \) fluctuate nearby classic values \( P^i_a = 0 \) and \( \omega^a_i = 0 \). This point of view is also accepted in our work. But from (2.14) it is seen that in this case the fermion contribution to the Hamiltonian contains only the terms higher than quadratic order in the fundamental fields. Therefore, the Fermi part of Hamiltonian can be accounted only as a whole by PT what is impossible in the Feynman theory.

Another picture is under the dynamic quantization. The dynamic quantization can retain any finite number of fermion degrees of freedom in the theory. At the same time the rest of fermion degrees of freedom are removed completely from dynamics. Therefore, the theory becomes regularized. Moreover, since the number of the retained degrees of freedom can be "enough small", so the development of PT by expansion in the fermion part of the Hamiltonian becomes possible.

For example, let the surface \( x^0 = const \) be a compact Riemann surface of the genus \( g \), which we denote by \( \Sigma \). Since \( \Sigma \) is a compact surface so one can think the set of indices \( N \) coincides with the set \( \mathbb{Z} \).

The regularization of the theory is achieved by imposing of the infinite set of the second class constraints:
$a_N^+ = 0, \quad a_N = 0 \quad \text{at} \quad |N| > N_0 \in \mathbb{Z}$ \hspace{1cm} (3.19)

Constraints (3.19) mean that the states $|N, 1\rangle$ with $|N| > N_0$ are dropped.

We perform the regularization of any operator $A$ by the following order: at first we arrange normally the fermion operators $a_N^+$ and $a_N$ containing in the operator $A$ and then impose constraints (3.19). The resultant operator is designated as $A_{\text{reg}}$.

The concrete choice of the fields $\{\psi_N\}$ at $|N| < N_0$ and also the method of ”vacuum filling” (see below) means the choice of the initial physical conditions.

4. The Dirac Commutational Relations.

Under the second class constraints (3.19) the classical CR (2.17) are to be replaced by the corresponding Dirac CR. Here this problem is solved.

Denote by the symbol $\mathcal{G}$ the Grassmann algebra with generators (A1). The elements of the algebra $\mathcal{G}$ are fermion states labelled by large Greek letters $\Lambda, \Sigma, \Pi, \ldots$. For any operator $A$ denote

$$\langle \Lambda | A | \Sigma \rangle = A_{\Lambda \Sigma}$$

Once more pay attention to the fact that here the averaging is performed only in fermion fluctuations according to rule (A3).

By the definition in regularised theory the space of fermion states $\mathcal{G}'$ is the subalgebra of algebra $\mathcal{G}$ with generators taken from set (A1), satisfying the condition $|N| < N_0$. The elements of the space $\mathcal{G}'$ are numbered by the primed Greek letters $\Lambda', \Sigma', \Pi', \ldots$.

Note that in our case the space $\mathcal{G}'$ is finite-dimensional due to the compactness of space surface.

According to the definition of the regularized operators (see Section 3.2)\hspace{1cm} (4.1)

In a regularized theory we shall omit the index $\text{reg}$ in designation of regularized operators.

The Dirac CR corresponding to the constraints (3.19) are defined according to

$$[A, B]_{\Lambda \Sigma'} = \sum_{\Pi'} (A_{\Lambda' \Pi'} B_{\Pi' \Sigma'} - B_{\Lambda' \Pi'} A_{\Pi' \Sigma'})$$ \hspace{1cm} (4.2)

Note that as a consequence of Hypothesis B we have for Hamiltonian of the system:

From here and definition (4.2) for any regularized operator $A$ there is the following
important equation: 
\[
[A, H]_{\Lambda'}^* = [A, H]_{\Lambda'}^\prime \tag{4.3}
\]
Here the right hand side is obtained formally by classic CR (2.19). Eq.(4.3) shows that the regularized Heisenberg equations for the fields \(\omega^a_i, \mathcal{P}^i_a\) and \(\bar{\psi}, \psi\) coincide (modulo operator permutations) with the formal one or classic equations and regularized algebra of gauge transformations coincide with the corresponding formal algebra.

Using definition (4.2) and the results of Section 3 we obtain:
\[
[u(x), \bar{\psi}(y)] = \sum_{|N|<N_0} \psi_N(x) \bar{\psi}_N(y) \tag{4.4}
\]

We state that equations (4.3) and (4.4) define the Dirac CR for all fundamental fields. Let us show that.

For reduction introduce the following designations:
\[
\hat{\nabla}_i \mathcal{P}^i_a = \partial_i \mathcal{P}^i_a - \frac{1}{2} \varepsilon^{abc}(\omega^b_i \mathcal{P}^i_c + \mathcal{P}^i_c \omega^b_i),
\]
\[
\chi^a = \hat{\nabla}_i \mathcal{P}^i_a + T^a,
\]
\[
F^a = \frac{1}{2} \varepsilon_{ij}(\partial_i \omega^a_j - \partial_j \omega^a_i - \varepsilon^{abc} \omega^b_i \omega^c_j),
\]
\[
\phi^a = F^a + R^a \tag{4.5}
\]

Thus, the quantities \(T^a\) and \(R^a\) are the fermion parts of operators \(\chi^a\) and \(\phi^a\), respectively.

Let us search the rest of the Dirac CR by expansion in constant \(G\) using Eqs.(4.3) and (4.4). The expansion in \(G\) is equivalent to the expansion in the quantities \(R^a\) in (4.5), which are the fermionic part of the constraints \(\phi^a\). Let \(\lambda_F\) be a length of the order of the minimum wave length of the functions \(\psi_N\) for \(|N|<N_0\). Since in our theory the number of fermion degrees of freedom is limited, this expansion is mathematically correct under a significant smallness of the dimensionless parameter \(G \lambda_F^{-1}\). This expansion is used in Section 6 to solve the equations of motions.

To begin the expansion in constant \(G\), it is necessary to arrange by parameter \(G\) the following quantities: \(\bar{\psi}, \psi \sim 1, \omega^a_i \sim 1, \mathcal{P}^i_a \sim (G)^{-1}\).
\[ [\psi, \bar{\psi}]^* \sim 1, \quad [\mathcal{P}^i_a, \mathcal{P}^j_b]^* \sim 1, \quad [\omega^a_i, \omega^b_j]^* \sim G^2, \]

\[ [\psi, \mathcal{P}^i_a]^* \sim 1, \quad [\psi, \omega^a_i]^* \sim G, \quad [\mathcal{P}^i_a, \omega^b_j]^* \sim 1 \quad (4.6) \]

The commutator \([\psi, \bar{\psi}]^*\) is known strictly according to (4.4). From equation \([\psi(x), \chi^a(y)]^* = [\psi(x), \chi^a(y)]\) (see. (4.3)) we extract the contribution of the zeroth order in \(G\). For this we must neglect the quantities \([\psi, \omega^a_i]^* \sim G, \quad [\psi, \sqrt{e} e^0_a]^* \sim G\) (see (2.15) and (4.6)). We find:

\[ \nabla_i ([\psi(x), \mathcal{P}^i_a(y)]^*)^{(0)} = \]

\[ \frac{1}{2} \sum_{|N|>N_0} \psi_N(x) (\bar{\psi}_N \sqrt{e} e^0_a \psi)(y) \quad (4.7) \]

Here superscripts \((0), (1), \ldots\) denote the order of the quantity in the parameter \(G\).

From the equation

\[ [\psi(x), \phi^a(y)]^* = [\psi(x), \phi^a(y)] \]

we extract the contribution of the order \(G\). Taking (4.6), (2.14c) and (2.17) into account, we obtain

\[ \varepsilon_{ij} \nabla_i(y) ([\psi(x), \omega^a_j(y)]^*)^{(1)} = \]

\[ = \varepsilon_{ij} \nabla_i(y) [\psi(x), \omega^a_j(y)] + \sum_{|N|>N_0} \psi_N(x) R^a_N(y) \quad (4.8) \]

Here the quantity \(R^a_N(y)\) (or \(R^a_N(y)\)) is obtained from \(R^a(y)\) by the substitution of \(\bar{\psi}_N(y)\) for \(\hat{\psi}(y)\) (or \(\psi_N(y)\) for \(\psi(y)\)) .

Using Eq.(4.3) \(\nabla_1(A) \nabla_1(\mathcal{P}_a^i \mathcal{P}_b^j, \mathcal{P}_a^i \mathcal{P}_b^j)^* = \)

\[ = -\frac{1}{4} \sum_{|N|>N_0} \left\{ (\psi \sqrt{e} e^0_a \psi_N)(x) (\bar{\psi}_N \sqrt{e} e^0_b \psi)(y) - \right\} \]

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\[-(\bar{\psi} \sqrt{g} e^0_b \psi_N)(y) (\psi_N \sqrt{g} e^0_a \psi)(x) \}\ , \quad (4.9)\]

\[\varepsilon_{jk} \nabla_i(x) \nabla_j(y) \left( [\mathcal{P}^i_a(x), \omega^b_k(y)]^*(1) \right) =
\frac{1}{2} \sum_{|N| > N_0} \left\{ R^b_N(y) (\bar{\psi}_N \sqrt{g} e^0_a \psi)(x) - (\bar{\psi} \sqrt{g} e^0_a \psi_N)(x) R^b_N(y) \right\} \]

(4.10)

By this way we obtain expressions for all Dirac CR in the lowest order in the parameter \( G \lambda_F^{-1} \). One can then develop the iterations in \( G \lambda_F^{-1} \), using (4.3) and the initial values for the Dirac commutators obtained here.

The defined Dirac CR possesses all the necessary properties (2.18), (2.19). It follows from definition (4.2).

5. Heisenberg Equations and the State Vectors.

Equations (4.3) show that the regularized Heisenberg equations can be obtained using formal CR (2.17).

Here two questions arise:
1). How does one define a product of operator fields at one spatial point \( x \)?
2). How does one order the operator fields at one spatial point \( x \) in the Heisenberg equations obtained, since the CR (2.17) differs from the Dirac CR?

We have the following answer to the first question.

In consequence of the constraints (3.19) the Dirac fields \( \psi \) and \( \bar{\psi} \) are smooth and thus, their product ( in any order and probably, at the same point \( x \) ) is regular.

If in the theory under consideration the fermion degrees of freedom were absent, there would be no local degrees of freedom. In this case there would be only global degrees of freedom, connected with fundamental group of spatial surface. If the spatial surface is a compact Riemann surface of genus \( g \) its fundamental group has \( 2g \) generators \( a_i, b_j, i, j = 1, \cdots, g \) with the only one constraint

\[ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \quad (5.1) \]
Let’s introduce the notations
\[ U_i(V_i) = \tilde{T} \exp \left( \frac{i}{2} \oint_{\gamma_i(\delta_i)} \omega_{a_i} \gamma^a d\gamma^i \right) \subset SO(2, 1) \quad (5.2) \]
where \( \gamma_i(\delta_i) \) is any representative of the class \( a_i(b_i) \) and \( \tilde{T} \) denotes the ordering operator along the integration path. Since, in the absence of fermions, \( F_a = 0 \) quantities (5.2) do not depend on the representatives of the classes \( a_i(b_i) \). The quantities (5.2) satisfy Eq.(5.1):
\[ U_1 V_1 U_1^{-1} V_1^{-1} \cdots U_g V_g U_g^{-1} V_g^{-1} = 1 \quad (5.3) \]
Eq.(5.3) are the single constraints for quantities (5.2) [5]. The wave functions of the system \( \Psi(U_i, V_j) \) in the absence of fermion fields depend only on quantities (5.2); the \( \Psi \) are defined on the hypersurface (5.3) and are invariant relative to the transformations (5.4).
\[ U_i \longrightarrow E^{-1} U_i E, \quad V_j \longrightarrow E^{-1} V_j E \quad (5.4) \]
where \( E \) is a certain element from the group \( SO(2, 1) \). There are also constraints:
\[ \nabla_i P^i \Psi = 0, \quad F_a \Psi = 0 \quad (5.5) \]
It follows from above that in the absence of fermion fields the small-scale fluctuations of fields \( \omega_i^a \) and \( P^i_a \) do not play any role, they can be removed. Under small-scale fluctuations we mean the fluctuations with the wavelength \( \sim \lambda \) for which
\[ \lambda < \lambda_0 \ll s_i, r_i, i = 1, \cdots, g \quad (5.6) \]
where \( s_i, r_i \) are the characteristic length of cycles \( \gamma_i, \delta_i \) in some metrics. The "removing" of the small-scale fluctuations of the fields \( \omega_i^a \) and \( P^i_a \) implies that in the corresponding commutator in (2.17) the \( \delta \)-function is replaced by a smoothed \( \delta \)-function , which we designate as \( \delta_{\lambda_0} \). The concrete method of smoothing of \( \delta \)-function is of no importance here. It is important only that \( \delta_{\lambda_0} \) acts on smooth functions ( which are change noticeable on the scales much greater than \( \lambda_0 \) ) in the same way as the \( \delta \)-function.
In our case, in the presence of the fermion degrees of freedom, one must proceed analogously. Condition (5.6) should be supplemented by
\[ \lambda_0 \ll \lambda_F \quad (5.7) \]
after which the regularization scheme for fields $\omega^a_i$ and $P^i_a$ remains valid. The quantity $\lambda_F$ is defined in the previous Section. Therefore the answer to first question is given.

The regarded regularization of the Bose degrees of freedom is in agreement with PT developed here. Indeed, in the zeroth order (see (6.3) - (6.9)) the solutions of the Heisenberg equations are linearly expressed through the initial values of the fields. This allows are to remove easily the small-scale fluctuations of the fields $\omega^a_i$ and $P^i_a$. Since in our PT the expansion is performed by the whole fermion part of the Hamiltonian, then in consequence with condition (5.7) the PT conserves the sense of the regularization of the Bose degrees of freedom also in higher orders of expansion.

The answer to the second question will be given after some formal calculations in unregularized theory.

We say that a given system is formally quantized if the formal algebra of the operators $(\chi_a, \phi_a)$ is closed and the structure functions are placed all to the left (or right) of the generators $(\chi_a, \phi_a)$ as in the following equation

$$[\phi_a(x), \phi_b(y)] = \delta^{(2)}(x - y) \cdot \{ f_{ab}^c(x) \phi_c(x) + g_{ab}^c(x) \chi_c(x) \}$$

We shall carry out formal calculations of the necessary commutators, using the relations of the form

$$\tilde{\psi}_\sigma(x) \tilde{\psi}_\rho(x) = -\tilde{\psi}_\rho(x) \tilde{\psi}_\sigma(x) + \delta^{(2)}(0) (\Gamma^0)_{\sigma\rho}^{-1}(x),$$

$$\omega^a_i(x) P^i_b(x) = P^i_b(x) \omega^a_i(x) + \delta^{(2)}(0) i \delta^a_b \delta^i_i (5.8)$$

e.t.s., which follow formally from the CR (2.17). This approach makes it possible to calculate the (operator) coefficient of the symbol $\delta^{(2)}(0)$ arising as a result of permutations of the field operators in various expressions. In this way we arrive at the conclusion that system (2.14) is formally quantized only for the value

$$s = \frac{5}{8} (5.9)$$

Thus, the requirement of the formal quantizability of the system fixes uniquely the ordering of the operator fields in the Hamiltonian.

The formulae given below are valid for $s = \frac{5}{8}$.

5.1 The Equations of Motion and the Gauge Transformations Algebra.
The formal calculations are carried out in stages. In the first stage we find the Heisenberg equations \( i \dot{A} = [A, H] \) for fundamental fields:

\[
i \Gamma^\mu \nabla_\mu \psi = \frac{i}{2} (8\pi G) \varepsilon_{a b c} \gamma^a \psi \chi_b,
\]

\[
i \nabla_\mu \bar{\psi} \Gamma^\mu = \frac{i}{2} (8\pi G) \varepsilon_{a b c} \chi_b \bar{\psi} \gamma^a,
\]

\[
\dot{\mathcal{P}}_a^i - \varepsilon^{a b c} \omega^b \mathcal{P}_c^i + \frac{1}{8\pi G} \varepsilon_{i j} (\partial_j e_a - \varepsilon_{a b c} \omega^b_j e_c) + \nonumber
\]

\[
\frac{1}{2} (8\pi G) \varepsilon_{a b c} \bar{\psi} \mathcal{P}_b^i \psi = 0,
\]

\[
i \dot{\omega}^a_i = i (\partial_i \omega^a - \varepsilon_{a b c} \omega^b_i \omega^c) + \nonumber
\]

\[
\frac{1}{2} (8\pi G) \varepsilon^{a b c} \bar{\psi} \gamma^b \nabla_i \psi - \nabla_i \bar{\psi} \gamma^b \psi + \nonumber
\]

\[
\frac{i}{2} (8\pi G)^2 \varepsilon^{a b c} \varepsilon_{i j} (\bar{\psi} \mathcal{P}_c^j \gamma^b \psi - \bar{\psi} \mathcal{P}_b^j \gamma^b \psi) - \nonumber
\]

\[
\frac{1}{2} (8\pi G) \varepsilon_{b c a} e_b \bar{\psi} \gamma^b \chi_c - \nonumber
\]

\[
\frac{1}{2} (8\pi G) \varepsilon_{b c a} e_a \gamma^b \chi_c (5.10)
\]

In the classic limit all the Equations (5.10) with the exception of the Dirac equations coincide with Euler-Lagrange equations obtained by the variations of the action (2.7). Equations (5.10) for the Dirac fields differ inessentially from the Dirac equations, by only a term that is equal to zero in a weak sense.

Now we can show that the quantity \( \int d^2 \mathbf{x} \bar{\psi} \Gamma^0 \psi \) is conserved. With the help of Eq.(5.10) we obtain:

\[
\frac{d}{dt} \int d^2 \mathbf{x} \bar{\psi} \Gamma^0 \psi = 0
\]

According to Eq.(4.3) the last equation is valid both in formal and regularized theories.

Let \( A \) be any fundamental field. We have the Jacobi identity:

\[
[ [\phi_a(x), \phi_b(y)], A ] =
\]
\[ [\phi_a(x), \phi_b(y), A] - [\phi_b(y), \phi_a(x), A] \] \quad (5.11)

The right-hand side in (5.11) is calculated by Eq.(5.10). Knowing the right-hand side of the expression (5.11), and also the classic Poisson bracket \([\phi_a(x), \phi_b(y)]\), as a result of time consuming calculations we can establish the quantum formal commutator \([\phi_a(x), \phi_b(y)]\). Here we use in all the calculations only Eqs.(5.10) and formal algebra (5.8). Note that in all the described formal calculations the symbol \(\delta^{(2)}(0)\) is encountered to a power no higher than the first, and symbols of the form \(\delta'(0)\) are absent in the calculations. Also, the final results do not contain the indeterminate symbol \(\delta^{(2)}(0)\).

The answer is
\[ [\chi_a(x), \chi_b(y)] = -i \delta^{(2)}(x - y) \varepsilon_{ab}^c \chi_c(x) \] \quad (5.12a)
\[ [\chi_a(x), \phi^b(y)] = -i \delta^{(2)}(x - y) \varepsilon_{ca}^b \phi^c(x) \] \quad (5.12b)
\[ [\phi^a(x), \phi^b(y)] = \delta^{(2)}(x - y) \frac{i}{2} (8\pi G)^2 \varepsilon_{ab}^c (\bar{\psi} \psi) \phi^c + \]
\[ + \delta^{(2)}(x - y) \frac{1}{2} (8\pi G)^4 (\varepsilon_{ab}^d \varepsilon^{hc} - \varepsilon^{abc} \varepsilon_{dgh}). \]
\[ \cdot \{ \bar{\psi} \mathcal{P}_c^i \gamma^g (\Gamma^0)^{-1} \gamma^d \nabla_i \psi - \nabla_i \bar{\psi} \gamma^d (\Gamma^0)^{-1} \gamma^g \mathcal{P}_c^i \psi \} \chi_h + \]
\[ + \delta^{(2)}(x - y) \frac{1}{4} (8\pi G)^4 (\varepsilon_{ab}^d \varepsilon_{gfd} - \varepsilon_{gh} \varepsilon_{dgh}). \]
\[ \cdot \{ A \chi_h \bar{\psi} \gamma^g (\Gamma^0)^{-1} \gamma^d \psi \chi_f + B \chi_f \bar{\psi} \gamma^g (\Gamma^0)^{-1} \gamma^d \psi \chi_h + \]
\[ + C \bar{\psi} \gamma^g \chi_h (\Gamma^0)^{-1} \gamma^d \psi \chi_f + D \bar{\psi} \gamma^g \chi_f (\Gamma^0)^{-1} \gamma^d \psi \chi_h + \]
\[ + F \bar{\psi} \gamma^g (\Gamma^0)^{-1} \gamma^d \chi_h \psi \chi_f + Q \bar{\psi} \gamma^g (\Gamma^0)^{-1} \gamma^d \chi_f \psi \chi_h + \]
\[ + H \bar{\psi} \gamma^g (\Gamma^0)^{-1} \gamma^d \psi \chi_h \chi_f + \]
\[ + (1 - A - B - C - D - F - Q - H). \]
\[ \cdot \bar{\psi} \gamma^g (\Gamma^0)^{-1} \gamma^d \psi \chi_f \chi_h \} \] \quad (5.12c)
Here \( A, B, C, D, F, Q, H \) are some numerical real parameters, satisfying the following equations:

\[
D + F + 2 = 0, \quad B + H = 1
\]

\[
2A + D - F = 0
\]

\[
2B + 3D - F + 2Q = 0
\]

All the operators in the right-hand side of Eqs. (5.12) are taken at the point \( x \). The right-hand side of (5.12c) is formally anti-Hermitian.

The above theory insists that the regularized Heisenberg equations for fundamental fields coincide with Eqs. (5.10) and the values of the commutators

\[
[\chi_a(x), \chi_b(y)]^*, \quad [\chi_a(x), \phi_b(y)]^*, \quad [\phi_b(x), \phi^b(y)]^*
\]

coincide with the right-hand side of Eqs. (5.12a), (5.12b) and (5.12c), respectively. Moreover, the right-hand side of Eq. (5.12c) is anti-Hermitian also in regularized theory.

5.2 The State Vectors.

By definition, for the wave functions \( \Psi \) we have

\[
a_N \Psi = 0, \quad |N| < N_0 \quad (5.13)
\]

From (5.5) and (5.13) it follows that

\[
\phi_a \Psi = 0, \quad \chi_a \Psi = 0 \quad (5.14)
\]

We now have all the means for the construction of the state vectors. We write out the base state vectors:

\[
\Psi_{N_1N_2\ldots N_s} = a^+_1 a^+_2 \ldots a^+_s \Psi, \quad |N_s| < N_0, \quad s = 0, 1, \ldots
\]

Any state is the superposition of the states (5.15). From relations (3.18) and (5.14) it follows immediately that

\[
\chi_a \Psi_{N_1N_2\ldots N_s} = 0, \quad \phi_a \Psi_{N_1\ldots N_s} = 0 \quad (5.16)
\]
Remark. Since in the theory under consideration the problem of the eigen modes of the Dirac operator has no meaning, it is impossible to classify the operators \( a_N \) and \( a_N^+ \) by their energy. Therefore, the problem of filling the physical vacuum remains unsolved here. A criterion permitting one to distinguish the ground and excited states, or the degree of excitation of states is not clear.

6. Perturbation Theory.

We show now that in the considered model Dynamic Quantization can construct a mathematically correct PT in the fermion part of the Hamiltonian, exactly in the parameter \( e_a \Gamma \lambda_F^{-1} \).

The thing is that the available dimensionless parameters - the Lagrange fields \( e_a(x) \) - can be made small enough indeed due to conditions (3.19). From these conditions and also formulae (2.14c), (3.10), (3.12), (4.6) and (5.7) we have the following estimation for the fermion part of the operator \( H_\phi \)

\[
\| R_a^\phi \| \sim \| e_a \| N_0 \lambda_F^{-1} \quad (6.1)
\]

Here \( N_0 \) is the number of the fermion degrees of freedom in the volume \( V_e \), in which the field \( e_a(x) \) differs from zero:

\[
N_0 \sim V_e \lambda_F^{-2}
\]

It is seen from estimation (6.1), that under sufficient numerical smallness of the fields \( e_a(x) \) and compactness of their support the expansion in the quantity \( R_a^\phi \) is mathematically correct. We state that under our assumptions one can establish the mathematic correctness of Heisenberg equations (5.10) in a general form and solve these equations using the Euler method. \[\]

Note that in the Feynman theory the reduction of the parameter \( e_a \) is useless in this sense since the order of the minimum wave length of the fermion fields \( \lambda_F \) is equal to zero.

In the process of the perturbation theory development the Heisenberg fields can be expanded in a series in the constant \( \Gamma \)

\[
A = A^{(0)} + A^{(1)} + \ldots,
\]

\[\]

\[\]

2 The problem of the calculation of the translation amplitudes with the change of the topology of the space is not considered here.
where $A^{(0)}$ is the zero approximation, $A^{(1)}$ is the first approximation, etc. It is evident that the Dirac CR $[A^{(i)}, B^{(j)}]^*$ should be calculated in the $i + j$th approximation.

Let $\mathcal{P}_a^{i(0)}(x)$ and $\omega_{ai}^{a(0)}(x)$ be some operator fields, having in the zeroth approximation in the gravitational constant the following (nonzero) commutation relations:

$$[\omega_{ai}^{a(0)}(x), \mathcal{P}_b^{j(0)}(y)]^* = i \delta_i^j \delta_a^b \delta^{(2)}(x - y)$$  \hspace{1cm} (6.2)

Take

$$\psi_N(t_0, x) = \left( \gamma^0 \Gamma^0(t_0) \right)^{-\frac{1}{2}} \kappa_N(t_0, x),$$  \hspace{1cm} (6.3)

where $\{\kappa_N(t_0, x)\}$ is any complete set of numeral spinor fields satifying the conditions $\int d^2x \kappa^+_M \kappa_N = \delta_{MN}$. It is obvious that fields (6.3) satisfy conditions (3.12) at $t = t_0$.

The regularized Dirac field at the time $t_0$ is expressed in the form

$$\psi^{(0)}(x) = \sum_{|N| < N_0} a_N \psi_N(t_0, x)$$  \hspace{1cm} (6.4)

The constant operators $\{a_N, a_N^+\}$ satisfy the CR (3.6).

Extract the zeroth approximation from Eq.(5.10):

$$\dot{\mathcal{P}}_a^{i(0)} - \varepsilon_{abc} \omega^b \mathcal{P}_c^{i(0)} = 0$$

$$\dot{\omega}_{ai}^{a(0)} = \partial_i \omega^a - \varepsilon_{abc} \omega_i^{b(0)} \omega^c,$$

$$\nabla_0 \psi^{(0)} = 0, \quad \nabla_0 \bar{\psi}^{(0)} = 0$$  \hspace{1cm} (6.5)

Eqs.(6.5) are easily solved. Introduce the notations

$$\omega = \frac{1}{2} \omega_a \gamma^a, \quad e = \frac{1}{2} e_a \gamma^a,$$

$$\omega_i = \frac{1}{2} \omega_{ai} \gamma^a, \quad \mathcal{P}^i = \frac{1}{2} \mathcal{P}_a^{i} \gamma^a$$  \hspace{1cm} (6.6)
Let the c-number matrix field satisfy the equation
\[ \frac{\partial U}{\partial t} = i \omega U, \quad U(t_0, x) = 1 \] (6.7)

From group-theoretical arguments it is obvious that the operators \( \chi_a \) generate gauge transformations, which are easily eliminated. For this we introduce the fields with the tilde symbol:

\[ \omega_i = U \tilde{\omega}_i \bar{U} + i U \partial_i \bar{U}, \]

\[ \mathcal{P}^i = U \tilde{\mathcal{P}}^i \bar{U}, \quad \psi = U \tilde{\psi}, \]

\[ \tilde{e}(t, x) = \bar{U}(t, x) e(t, x) U(t, x), \] (6.8)

where \( \bar{U} = \gamma^0 U + \gamma^0 \). Note that \( \bar{U} U = 1 \). In the adopted notations, the solution of Eq.(6.5) is of the form

\[ \tilde{\omega}^{(0)}_i(t) = \omega^{(0)}_i(t_0), \]

\[ \tilde{\mathcal{P}}^{(0)}_i(t) = \mathcal{P}^{(0)}_i(t_0), \]

\[ \tilde{\psi}^{(0)}(t) = \psi^{(0)}(t_0) \] (6.9)

All the fields in (6.9) are taken at the point \( x \). The fields in (6.9) satisfy in the zeroth approximation not only equations of motion but also Dirac CR (see (2.19) and Section 4).

Let us write out the equations for the first-order corrections to the Heisenberg fields (all the fields carry the tilde symbol, although, to simplify the writing, it is absent in the formulae).

\[ i \dot{\omega}^{(1)}_a = \frac{1}{2} (8\pi G) \varepsilon_b^{ca} e_c \cdot \]

\[ (\bar{\psi} \gamma^b \nabla_i \psi - \nabla_i \bar{\psi} \gamma^b \psi) + \]
\[ \frac{1}{2}(8\pi G)^2 \varepsilon_b^{ca} \varepsilon_{ij} \cdot (\bar{\psi} \mathcal{P}_c^i \gamma^b (\Gamma^0)^{-1} \Gamma^i \nabla_i \psi - \nabla_i \bar{\psi} \Gamma^i (\Gamma^0)^{-1} \gamma^b \mathcal{P}_c^i \psi), \quad (6.10a) \]

\[ \dot{P}_a^{i(1)} = -\frac{1}{8\pi G} \varepsilon_{ij} (\partial_j e_a - \varepsilon_{ab} \omega_j^b e^c) - \frac{1}{2} (8\pi G) \varepsilon_a^{bc} e_c \bar{\psi} \mathcal{P}_b^i \psi, \quad (6.10b) \]

\[ i \dot{\psi}^{(1)} = -i(\Gamma^0)^{-1} \Gamma^i \nabla_i \psi + \frac{i}{2} (8\pi G) \varepsilon_a^{bc} e_c (\Gamma^0)^{-1} \gamma^a \psi \chi_b \quad (6.10c) \]

In Eq.(6.10) all the fields in the right-hand side are taken in the zeroth approximation according to (6.9).

One must also correct the Dirac CR (6.2) so that this CR will be valid to first-order inclusive in the gravitational constant. This problem is solved in Section 4.

Thus, the chain of the regularized recursion equations does not differ formally from the analogous unregularized equations. In essence however, in our theory a correctly defined chain of recursion equations arises. This is a consequence of the regularization of the Dirac field according to (6.4) and the replacement of the formal CR (2.19) by Dirac CR.

Finally, we consider one exactly solvable model. Let the system contain only one fermion degree of freedom. This implies that from the set of modes \{\psi_N\} we choose mode \( \psi_0(x) \) and impose the infinite series of constraints (3.24) with \(|N| > 0\). Then, the fermi fields are of the form

\[ \psi(x) = a_0 \psi_0(x), \quad \bar{\psi}(x) = \bar{\psi}_0(x) a_0^+, \]

From this it follows that

\[ \psi(x) \psi(y) = 0, \quad \bar{\psi}(x) \bar{\psi}(y) = 0 \quad (6.11) \]

Relations (6.11) permit us to find the Dirac CR and solve the Heisenberg equations exactly. It is easy to see that for this purpose one more iterative step is needed which will make Eq.(6.10b) more exact. It can be shown that as a result in the
right-hand side of Eq.(6.10b) the symbol $\omega^b_j$ denotes the sum $\omega^b_j(0) + \omega^b_j(1)$ and everywhere in Eq.(6.10) symbols $\psi^{(0)}$, $\bar{\psi}^{(0)}$, $\psi^{(1)}$, $\bar{\psi}^{(1)}$ are replaced simply by symbols $\psi$ and $\bar{\psi}$. In fact, subsequent iterations in gravitational constant lead to the corrections that contain the fields $\psi$ or $\bar{\psi}$ to powers higher than the first. As a consequence of (6.11) all these corrections vanish. The terms containing the fields $\psi$ (or $\bar{\psi}$) to the power two or higher vanish even in the case when these fields are separated by the operators $\omega^a_i$ or $P^a_i$. In this case one of the fields $\psi$($\bar{\psi}$) must be moved so that it stands near the other field $\psi$($\bar{\psi}$). According to (6.11) such a term vanishes. In any case the commutators of the field $\psi$($\bar{\psi}$) with the boson variables arising in the process of permutations increase the power of gravitational constant. It is clear therefore that by continuation of this process of permutations we shall obtain either zero or a term of a more higher degree in the gravitational constant. Thus, it can be seen that in the given case the expansion of the Heisenberg equations in the gravitational constant terminates on Eqs.(6.10).

It is not difficult to see that in the case of one degree of freedom the expansions of the Dirac CR in the gravitational constant are also truncated.

The PT developed here justifies the Hypothesis A and B of Section 3.

7. Conclusion.

The results of [2] showed that the theory of the dynamic quantization developing here is adequate for the construction of the quantum theory of gravity in a 2+1-dimensional space. The constructed theory possesses the following necessary properties:

a) The theory is unitary and causal; a set of evolution operators forms a group.
b) The algebra of gauge transformations (5.12) is valid.
c) A mathematically correct perturbation theory in the gravitational constant $G$ exists.

Here we suggest that a modification of dynamic quantization method makes the process of dynamic quantization in (3+1)-dimensional space-time easier. The main difference between that theory and the present one is that it is necessary to identify not only fermion but also boson gauge invariant creation and annihilation operators. We think that it makes sense to study the supersymmetric variant of the theory. In supersymmetric theories the filling of the vacuum can be realized in such a way that the contributions of the boson and fermion zeroth oscillations to the
energy-momentum tensor of matter cancel out. Probably, in this case a PT in the gravitational constant analogous to the PT considered in Section 6 is to be valid.

Appendix.

Represent the operators \( \{ a_N^+, a_N \} \), vectors (3.3) and scalar product in a natural way. Let \( \alpha_N \) and \( \bar{\alpha}_N \) be the elements of a Grassmann algebra over complex numbers. Then

\[
| N, 0 \rangle = 1, \quad | N, 1 \rangle = \bar{\alpha}_N \quad \text{(A1)}
\]

and any fermion state is represented by the function on the variables \( \{ \bar{\alpha}_N \} \). By this the action of creation and annihilation operators looks like

\[
a_N^+ F\{ \bar{\alpha} \} = \bar{\alpha}_N F\{ \bar{\alpha} \}, \quad a_N F\{ \bar{\alpha} \} = \partial_{\bar{\alpha}_N} F\{ \bar{\alpha} \} \quad \text{(A2)}
\]

The function conjugated to

\[
F_\Lambda(\bar{\alpha}) = \prod_N (\Lambda_0 N + \Lambda_1 N \bar{\alpha}_N).
\]

is of the form

\[
(F_\Lambda(\bar{\alpha}))^+ = \prod_N (\bar{\Lambda}_0 N + \bar{\Lambda}_1 N \alpha_N)
\]

Here symbol \( \prod \) denotes the product in inverse order with comparison the product in previous formula, \( \Lambda_N \) and \( \bar{\Lambda}_N \) are mutually-conjugated complex numbers.

The scalar product (3.5) is expressed by the formula

\[
\langle \Lambda | \Sigma \rangle = \int (F_\Lambda(\bar{\alpha}))^+ F_\Sigma(\bar{\alpha}) \prod_N e^{-\bar{\alpha}_N \alpha_N} d\bar{\alpha}_N d\alpha_N \quad \text{(A3)}
\]

In a field variant the set of Grassmann elements \( \{ \alpha_N \} \) and \( \{ \bar{\alpha}_N \} \) correspond to Grassmann spinor fields \( \lambda(x) \) and \( \bar{\lambda}(x) \). The fermion wave functions \( F_\Lambda(\bar{\lambda}) \) are the functionals on \( \bar{\lambda}(x) \), and their conjugated \( F_\Lambda^+ \) are the functionals on \( \lambda(x) \). The action of operators \( \bar{\psi} \) and \( \psi \) is defined as:

\[
\bar{\psi}(x) F(\bar{\lambda}) = \bar{\lambda}(x) F(\bar{\lambda}),
\]

\[
\psi(x) F(\bar{\lambda}) = (\Gamma^0)^{-1}(x) \left( \frac{\delta F(\bar{\lambda})}{\delta \bar{\lambda}(x)} \right)
\]

Let the fields \( \lambda(x) \) and \( \bar{\lambda}(x) \) satisfy the same equations as the fields \( \psi(x) \) and \( \bar{\psi}(x) \) correspondingly. Then the expression \( \int d^2x \bar{\lambda}(x) \Gamma^0 \lambda \) is gauge invariant and the functional measure on the phase space of fermion fields is expressed in
the form:
\[ \exp \left( - \int d^2 x \, \bar{\lambda} \Gamma^0 \lambda \right) \prod_x (det \Gamma^0(x))^{-1} \, d\bar{\lambda}(x) \, d\lambda(x) \quad (A4) \]

In selected variables the measure (A4) transforms to measure from integrals (A3). To see this let expand the fields \( \lambda \) and \( \bar{\lambda} \) analogously to (3.10) \( \lambda(x) = \sum_N \alpha_N \psi_N(x) \) and then use the relations (3.12).

The measure in (A3) is preferable since it can be regularized by dynamically invariant way:
\[ (\mathcal{D} \bar{\psi} \mathcal{D} \psi)_{Reg} = \prod_N' e^{-\bar{\alpha}_N \alpha_N} \, d\bar{\alpha}_N \, d\alpha_N \quad (A5) \]

Here the symbol \( \prod_N' \) means the product over some finite set of indices \( N \).

Compare the measure (A5) with fermion measure which is used for calculations of axial anomaly at Euclidean formulation of the gauge theory. Let us consider in D-dimensional Euclidean space the Dirac field in external gauge field \( A_i(x) \).

Let \( \{\psi_N\} \) be the complete orthogonal set of eigenfunctions of Dirac operator:
\[ -i \gamma^j \nabla_i \psi_N = \varepsilon_n \psi_N \]

Here \( \nabla_i = \partial_i - i A_i \), \( \gamma^i \) are Dirac matrices in Euclidean space. Expand the Dirac fields in the set \( \{\psi_N\} \):
\[ \psi(x) = \sum_N \alpha_N \psi_N(x), \quad \bar{\psi}(x) = \sum_N \bar{\alpha}_N \bar{\psi}_N(x) \]

(Compare with (3.10)). Then the fermion measure \( \mathcal{D} \bar{\psi} \mathcal{D} \psi \) can be formally expressed as \( \Pi_N d\bar{\alpha}_N \, d\alpha_N \) and regularized one according to
\[ (\mathcal{D} \bar{\psi} \mathcal{D} \psi)_{Reg} = \prod_{N: |\varepsilon_N|<\Lambda} d\bar{\alpha}_N \, d\alpha_N, \quad \Lambda \to \infty \quad (A6) \]

(Compare with (A5)). Since the variables \( \{\bar{\alpha}_N, \alpha_N\} \) are gauge invariant so the measure (A6) is also gauge invariant. However as it has been shown in Dissertation of the author [7] the measure (A6) contains the axial anomaly. \( \Box \) By our opinion the dynamic invariant measures of the kind (A5) differ principally from Euclidean measures of the kind (A6) by that the first one do not contain any gauge anomalies.

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\(^3\) The same result has been obtained by Fujikawa [8]
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