TWO CURIOUS INEQUALITIES INVOLVING DIFFERENT MEANS OF TWO ARGUMENTS

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ABSTRACT. For two positive real numbers $x$ and $y$ let $H$, $G$, $A$ and $Q$ be the harmonic mean, the geometric mean, the arithmetic mean and the quadratic mean of $x$ and $y$, respectively. In this note, we prove that

$$A \cdot G \geq Q \cdot H,$$

and that for each integer $n$

$$A^n + G^n \leq Q^n + H^n.$$  

We also discuss and compare the first and the second above inequality for $n = 1$ with some known inequalities involving the mentioned classical means, the Seiffert mean $P$, the logarithmic mean $L$ and the identric mean $I$ of two positive real numbers $x$ and $y$.

1. THE MAIN RESULT

For two positive real numbers $x$ and $y$, let $H(x, y) = H$, $G(x, y) = G$, $A(x, y) = A$ and $Q(x, y) = Q$ be the harmonic mean, the geometric mean, the arithmetic mean and the quadratic mean (sometimes called the root mean square) of $x$ and $y$, respectively, i.e.,

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad \text{and} \quad Q = \sqrt{\frac{x^2 + y^2}{2}}.$$ 

Then by the particular case of the well known harmonic mean-geometric mean-arithmetic mean-quadratic mean inequality ($H - G - A - Q$ inequality),

$$H \leq G \leq A \leq Q,$$

with equality if and only if $x = y$.

Many sources have discussed one or more of the inequalities involving harmonic mean, geometric mean, arithmetic mean, and quadratic mean (see e.g., [2], [3] and [4]). In this note, under the above notations, we will prove the following result.

Theorem 1.1. Let $x$ and $y$ be arbitrary positive real numbers, and let $n$ be any integer. Then

(1) $$A \cdot G \geq Q \cdot H,$$

and

(2) $$A^n + G^n \leq Q^n + H^n.$$
The equality in (1) and (2) holds if and only if \( x = y \).

**Remark 1.2.** In particular, the inequality (2) implies that

\[ A + G \leq Q + H. \]  

Notice that in [7, (3.2) of Theorem 1] (also see [6]) J. Sándor proved that for all \( x > 0 \) and \( y > 0 \)

\[ A + G \leq 2P, \]

where \( P = P(x, y) \) is the Seiffert mean of two positive real numbers \( x \) and \( y \) defined by

\[ P = P(x, y) = \frac{x - y}{2 \arcsin \frac{x - y}{x + y}} \text{ if } x \neq y, \text{ and } P(x, x) = x. \]

The equality in (4) holds if and only if \( x = y \). In view of the inequalities (3) and (4), it can be of interest to compare the expressions \( Q + H \) and \( 2P \). Our computational results suggest that the inequality (3) is stronger than the inequality (4), i.e., that it is true the following conjecture.

**Conjecture 1.3.** Let \( x \) and \( y \) be arbitrary positive real numbers. Then

\[ Q + H \leq 2P, \]

where equality holds if and only if \( x = y \).

**Remark 1.4.** The logarithmic mean \( L = L(x, y) \) and the identric mean \( I = I(x, y) \) of two positive real numbers \( x \) and \( y \) are defined by

\[ L = L(x, y) = \frac{x - y}{\ln x - \ln y} \text{ if } x \neq y, \text{ and } L(x, x) = x; \]

\[ I = I(x, y) = \frac{y^x}{x^y} \text{ if } x \neq y, \text{ and } I(x, x) = x. \]

In [1] H. Alzer proved that for all \( x > 0 \) and \( y > 0 \) we have

\[ \sqrt{A \cdot G} \leq \sqrt{L \cdot I} \leq \frac{L + I}{2} \leq \frac{G + A}{2}, \]

where the equality in each of these inequalities holds if and only if \( x = y \). Notice that in view of inequalities (1), (2) and (3), the chain of inequalities given by (9) may be extended as

\[ \sqrt{Q \cdot H} \leq \sqrt{A \cdot G} \leq \sqrt{L \cdot I} \leq \frac{L + I}{2} \leq \frac{G + A}{2} \leq \frac{Q + H}{2}. \]

Moreover, under Conjecture [1,3] and the known fact that \( P \leq I \) (see [5]), the chain of inequalities (10) may be extended on the right hand side as

\[ \frac{Q + H}{2} \leq P \leq I. \]
Remark 1.5. Since the inequality (2) is satisfied for each integer \( n \), it may be of interest to answer the following question: For which real numbers \( n \) the inequality (2) holds? Our computational and related graphical results lead to the following conjecture.

Conjecture 1.6. The inequality (2) holds for all negative real numbers \( n \) and for all positive real numbers \( n \) greater or equal than \( 1/2 \). Moreover, none of the inequality (2) or its converse inequality holds true for each real number \( n \) in the interval \((0, 1/2)\).

2. Proof of Theorem 1.1

For the proof of the inequality (2) of Theorem 1.1 we will need the following lemma.

Lemma 2.1. Let \( a, b, c \) and \( d \) be positive real numbers such that \( a + b \leq c + d \) and \( ab \geq cd \). Then for each integer \( n \)

\[(11) \quad a^n + b^n \leq c^n + d^n.\]

Proof. First we will prove the inequality (11) for nonnegative integers \( n \). We proceed by induction on \( n \geq 0 \). Obviously, the inequality (11) is satisfied for \( n = 0 \). Suppose that the inequality (11) holds for all nonnegative integers \( \leq n \). Then using this hypothesis and the assumption \( ab \geq cd \), we obtain.

\[
a^{n+1} + b^{n+1} = (a^n + b^n)(a + b) - ab(a^{n-1} + b^{n-1}) \\
\leq (c^n + d^n)(c + d) - cd(c^{n-1} + d^{n-1}) \\
= c^{n+1} + d^{n+1}.
\]

Hence, \( a^{n+1} + b^{n+1} \leq c^{n+1} + d^{n+1} \), which completes the induction proof.

Now suppose that \( n \) is a negative integer. Then applying the inequality (11) with \( -n > 0 \) instead of \( n \) and the assumption that \( ab \geq cd \), we find that

\[
a^n + b^n = a^{-n} + b^{-n} \leq \frac{c^{-n} + d^{-n}}{(cd)^{-n}} = c^{-n} + d^{-n}.
\]

Hence, the inequality (11) holds for each integer \( n \). \( \Box \)

Proof of Theorem 1.1 In order to prove the inequality (1), notice that by the identity \((x - y)^4 = (x + y)^4 - 8xy(x^2 + y^2)\) we obtain

\[(12) \quad (x + y)^2 \geq 2\sqrt{2xy(x^2 + y^2)}.\]

By using the inequality (12), we get

\[
\frac{A \cdot G}{Q \cdot H} = \frac{(x + y)^2}{2\sqrt{2xy(x^2 + y^2)}} \geq 1,
\]

which implies the inequality (2).

It remains to prove the inequality (2). Notice that by Lemma 2.1 (with \( a = A \), \( b = G \), \( c = Q \) and \( d = H \)) and the inequality (11), it suffices to prove the inequality (2) for \( n = 1 \).

First observe that

\[(13) \quad A - H = \frac{x + y}{2} - \frac{2xy}{x + y} = \frac{(x - y)^2}{2(x + y)}.\]
By using $A - Q$ inequality, we have

\begin{equation}
\sqrt{\frac{2(x^2 + y^2)}{2} + \sqrt{4xy}} \leq \sqrt{\frac{2(x^2 + y^2) + 4xy}{2}} = x + y.
\end{equation}

Then applying the inequality (14) and the identity (13), we obtain

\begin{align*}
Q - G &= \frac{x^2 + y^2}{2} - \sqrt{xy} = \frac{x^2 + y^2}{\sqrt{\frac{x^2 + y^2}{2} + \sqrt{xy}}} \\
&= \frac{(x - y)^2}{\sqrt{2(x^2 + y^2) + 4xy}} \geq \frac{(x - y)^2}{2(x + y)} \\
&= A - H,
\end{align*}

which implies the inequality (2) for $n = 1$.

From the above proofs it follows that the equality in (1) and (2) holds if and only if $x = y$. This completes proof of Theorem 1.1. \hfill \Box

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