Distributed resource allocation through utility design – Part II: applications to submodular, supermodular and set covering problems

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Abstract—A fundamental component of the game theoretic approach to distributed control is the design of local utility functions. In Part I of this work we showed how to systematically design local utilities so as to maximize the induced worst case performance. The purpose of the present manuscript is to specialize the general results obtained in Part I to a class of monotone submodular, supermodular and set covering problems. In the case of set covering problems, we show how any distributed algorithm capable of computing a Nash equilibrium inherits a performance certificate matching the well known $1 - 1/e$ approximation of [1]. Relative to the class of submodular maximization problems considered here, we show how the performance offered by the game theoretic approach improves on existing approximation algorithms. We briefly discuss the algorithmic complexity of computing (pure) Nash equilibria and show how our approach generalizes and subsumes previously fragmented results in the area of optimal utility design. Two applications and corresponding numerics are presented: the vehicle target assignment problem and a coverage problem arising in distributed caching for wireless networks.

Index Terms— Game theory, distributed optimization, resource allocation, combinatorial optimization, price of anarchy, linear program, duality, Shapley value, marginal contribution.

I. INTRODUCTION

There has been a growing interest in recent years in the analysis and control of multi agent and networked systems. The potential of such systems stems from the societal impact that they promise to deliver: from medicine [2] to surveillance [3], from future mobility [4] to food production [5], to name a few. The typical challenge in the control of such systems is the design of agent-level decision rules that are capable of achieving a desirable joint objective by relying solely on local information. A recent approach based on game theoretic arguments and termed game design [6] has proved useful in complementing the results obtained by more traditional techniques. This approach amounts to assigning a local utility function to each agent so that their selfish maximization recovers the desired system level objective, i.e. jointly maximizes a given objective function.

In Part I of this work we applied this game design approach to a class of hard combinatorial resource allocation problems, where a finite number of agents need to be allocated to a set of resources, with the goal of maximizing a given welfare function. In this context, the notion of price of anarchy (the ratio between the worst Nash equilibrium and the optimum over a set of instances) was used as the performance metric. Indeed, any algorithm capable of computing a Nash equilibrium will inherit a worst case approximation ratio equal to the price of anarchy (PoA). More specifically, we tackled the utility design problem, i.e. we asked the following two questions:

i) Given local utility functions, how do we characterize the price of anarchy?

ii) Is it possible to select utility functions so as to maximize such performance metric?

The main result we proved in Part I is the possibility to compute and optimize the price of anarchy by means of a tractable linear program. In this manuscript we specialize the general results obtained in Part I to the case of submodular, supermodular and set covering problems. In all the forthcoming analysis, we consider the notion of pure Nash equilibrium.

Contributions. The main contributions are as follows.

1) Relative to a class of monotone submodular resource allocation problems, we provide the analytical expression of the PoA as a function of the utilities assigned to every agent (Theorem 1). To the best of our knowledge, this is the first result that gives an exact characterization (i.e. tight) of the PoA for the considered class of problems. We specialize this result to the Shapley value and marginal contribution utility functions, recovering a partial result presented in [7] and limited to Shapley value. Finally, we show how the performance certificates offered by this approach improve on existing approximation algorithms.

2) Relative to set covering problems, we characterize the PoA as the maximum between $O(n)$ numbers (Theorem 2) and recover a previous result presented in [8], [9] under the additional assumptions therein required. Optimally designed utilities yield a PoA of $1 - 1/e$, where $e$ is the Euler’s number.

3) Relative to supermodular problems, we provide a tight expression for PoA (Theorem 3), complementing previous bounds appearing in [10], [11]. Limited to this case, we observe that the performance offered by the game theoretic approach is rather poor.

4) We present two applications and show how the game theoretic approach yields improved theoretical and numerical performances.

Organization. Section II briefly introduces the problem formulation, the game theoretic approach as well as two results presented in Part I. The complexity of computing a (pure) Nash equilibrium is then discussed in Subsection II-E. In Sections III, IV and V we specialize the results presented in Part I to the case of submodular, set covering, supermodular problems

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and present two corresponding applications. All the proofs are reported in Appendix B.

**Notation.** For any two positive integers \( p \leq q \), denote \([p] = \{1, \ldots, p\}\) and \([p, q] = \{p, \ldots, q\}\). We use \( \mathbb{N} \), \( \mathbb{R}_{\geq 0} \), and \( \mathbb{R}_{> 0} \) to denote the set of natural numbers, positive and non negative real numbers, respectively. The function \( 1_{\{f(x) \geq 0\}} \) assumes value 1 if \( f(x) \geq 0 \), and zero else. Given a finite set \( I \), \(|I|\) denotes the number of elements in \( I \).

## II. Problem formulation and performance metrics

In the first two subsections we briefly introduce the problem formulation and the game theoretic approach followed. An in depth presentation, motivation and discussion of these can be found in Part I of this work [12, Sec. II].

### A. Problem formulation

Let \( \mathcal{R} = \{r_1, \ldots, r_m\} \) be a finite set of resources, where each resource \( r \in \mathcal{R} \) is associated with a value \( v_r \geq 0 \) describing its importance. Further consider \( N = \{1, \ldots, n\} \) a set of agents. Every agent \( i \in N \) selects \( a_i \), a subset of the resources, from the given collection \( \mathcal{A}_i \subseteq 2^\mathcal{R} \), i.e. \( a_i \in \mathcal{A}_i \). The welfare of an allocation \( a = (a_1, \ldots, a_n) \in \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \) is given by

\[
W(a) := \sum_{r \in \bigcup_{i \in N} a_i} v_r w([a]_r),
\]

where \( W : 2^\mathcal{R} \times \cdots \times 2^\mathcal{R} \to \mathbb{R} \) is the vector of agents selecting resource \( r \) in allocation \( a \), and \( w : |N| \to \mathbb{R}_{\geq 0} \) is called the welfare basis function. The objective of the system designer is to find a feasible allocation maximizing the welfare, i.e.

\[
a^{opt} \in \arg \max_{a \in \mathcal{A}} W(a).
\]

We will often represent an allocation \( a \) as \((a_i, a_{-i})\), where \( a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \) denotes the decision of all the agents but \( i \).

### B. A game theoretic approach

In order to obtain a distributed solution to the previous problem, we follow the game theoretic approach presented in [12], [13] and assign to each agent \( i \in N \) a local utility function \( u_i : \mathcal{A} \to \mathbb{R}_{\geq 0} \) of the form

\[
u_i(a) := \sum_{r \in a_i} v_r w([a]_r) f([a]_r),
\]

where \( f : [n] \to \mathbb{R}_{\geq 0} \) constitutes our decision rule, respectively. We refer to it as the distribution rule. We identify the game introduced above with the tuple

\[
G = (\mathcal{R}, \{v_r\}_{r \in \mathcal{R}}, N, \{\mathcal{A}_i\}_{i \in N}, f).
\]

Given a distribution rule \( f \), we measure its performance adapting the notion of price of anarchy [14] as

\[
\text{PoA}(f) = \inf_{G \in \mathcal{G}_f} \left( \frac{\min_{a \in \text{NE}(G)} W(a)}{\max_{a \in \mathcal{A}} W(a)} \right),
\]

where \( \text{NE}(G) \) denotes the set of pure Nash equilibria of \( G \) and

\[
\mathcal{G}_f := \{(\mathcal{R}, \{v_r\}_{r \in \mathcal{R}}, N, \{\mathcal{A}_i\}_{i \in N}, f) \mid |N| \leq n \},
\]

contains all possible instances of \( G \) where the number of agents is bounded by \( n \). Observe that whenever a (distributed) algorithm is available to compute a Nash equilibrium, the price of anarchy also represents the approximation ratio of the corresponding algorithm over all the instances \( G \in \mathcal{G}_f \).

For this reason, the price of anarchy defined in (4) will serve as the performance metric in all the forthcoming analysis. The overarching goal of this work is to characterize and optimize \( \text{PoA}(f) \) over a set of admissible distributions.

### C. Standing assumptions and special distribution rules

Throughout the manuscript we make the following regularity assumptions on admissible welfare basis functions and distribution rules. The implications of these assumptions are discussed in Part I of this work.

**Standing Assumptions**

The sets \( \mathcal{A}_i \subseteq 2^\mathcal{R} \) are nonempty and \( \mathcal{A}_i \setminus \emptyset \neq 0 \) for all \( i \in N \). Further, \( \exists r \in \mathcal{R} \) s.t. \( v_r > 0 \) and \( r \in a_i \in \mathcal{A}_i \) for some \( i \in N \). The welfare basis function \( w : |N| \to \mathbb{R}_{\geq 0} \) satisfies \( w(j) > 0 \) for all \( j \in [n] \). A distribution rule \( f : [n] \to \mathbb{R}_{\geq 0} \) is constrained to \( f \in \mathcal{F} \), where

\[
\mathcal{F} := \{f : [n] \to \mathbb{R}_{\geq 0} \mid f(1) \geq 1, f(j) \geq 0 \forall j \in [n]\}.
\]

Given a distribution rule \( f : [n] \to \mathbb{R}_{\geq 0} \) and a welfare basis function \( w : [n] \to \mathbb{R}_{\geq 0} \), in the remaining of this manuscript we extend their definition (with slight abuse of notation) to \( f : [0, n + 1] \to \mathbb{R}_{\geq 0} \) and \( w : [0, n + 1] \to \mathbb{R}_{\geq 0} \), where we set the first and last components to be identically zero, i.e.

\[
f(0) = w(0) = 0, f(n + 1) = w(n + 1) = 0.
\]

We conclude this section by introducing two distribution rules that have attracted the researchers’ attention due to their simple interpretation and to their special properties: the Shapley value distribution rule and the marginal contribution distribution rule [16].

**Definition 1.** The Shapley value and marginal contribution distribution rules are identified with \( f_{SV} \) and \( f_{MC} \), respectively. For \( j \in [n] \), they are given by

\[
f_{SV}(j) = \frac{1}{j}, \quad f_{MC}(j) = 1 - \frac{w(j - 1)}{w(j)}.
\]

Observe that the Shapley value distribution rule is the only distribution rule for which the sum of all the players utility exactly matches the total welfare. The marginal contribution

\footnote{For the class of games with utility functions (2), a pure Nash equilibrium is guaranteed to exist, as \( G \) is a congestion game for any choice of \( f \), [15].}

\footnote{This adjustment does not play any role, but is required to avoid the use of cumbersome notation in the forthcoming expressions. Else, e.g., \( f_{MC}(1) \) in (6) will not be defined.}
distribution rule takes its name from the observation that (2) reduces to
\[ u_i(a) = \sum_{r \in a_i} v_r \cdot w(|a|_r) f_{MC}(|a|_r) = \sum_{r \in a_i} v_r \cdot (w(|a|_r) - w(|a|_r - 1)) = W(a) - W(\emptyset, a - i), \]
i.e. player's \( i \) utility function represent its marginal contribution to the total welfare, that is the difference between \( W(a) \) and the welfare generated when player \( i \) is removed from the game.

D. Two results from Part I

In Part I of this work, we showed how it is possible to compute PoA(\( f \)) as the solution of a tractable linear program (in its primal or dual form). Additionally, we showed how the problem of optimizing such a performance metric can also be posed as a tractable linear program. Two results presented in Part I that are needed in the development of Part II are summarized for completeness in the following Proposition. Before doing so, we introduce a useful set of integer tuples
\[ \mathcal{I}_R := \{ (a, x, b) \in [0, n]^3 \text{ with } a + x + b \geq 1 \} \text{ s.t. } a \cdot x \cdot b = 0 \text{ or } a + x + b = n \} . \]

Proposition 1 (Characterizing and optimizing PoA, [12]).

i) Given \( f \in \mathcal{F} \), the price of anarchy (4) is given by PoA(\( f \)) = 1/W*, where W* is the value of the following (dual) program
\[
W^* = \min_{\lambda \in \mathbb{R}^+ \cdot \mu \in \mathbb{R}} \mu \\
\text{s.t. } 1_{\{b + x \geq 1\}} w(b + x) - \mu 1_{\{a + x \geq 1\}} w(a + x) + \\
+ \lambda [af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0 \\
\forall (a, x, b) \in \mathcal{I}_R .
\] (7)

ii) The design problem \( \arg \max_{f \in \mathcal{F}} \text{PoA}(f) \) is equivalent to the following LP in \( n + 1 \) scalar unknowns
\[
f^* \in \arg \min_{f \in \mathcal{F}, \mu \in \mathbb{R}} \mu \\
\text{s.t. } 1_{\{b + x \geq 1\}} w(b + x) - \mu 1_{\{a + x \geq 1\}} w(a + x) + \\
+ af(a + x)w(a + x) - bf(a + x + 1)w(a + x + 1) \leq 0 \\
\forall (a, x, b) \in \mathcal{I}_R .
\] (8)

In the forthcoming sections we specialize these general statements and obtain analytical results for the case of submodular, supermodular and set covering problems.

E. Complexity of computing pure Nash equilibria

Part I and Part II of this work provide performance certificates relative to any pure Nash equilibrium allocation. As already mentioned, the price of anarchy also represents the approximation ratio of any algorithm capable of computing one such equilibrium over all the instances \( G \in \mathcal{G}_f \). In the following we limit our study to the best response algorithm, even though others have been proposed.

For ease of presentation, we consider a round-robin best response algorithm i.e. an algorithm where the players revise their decision in a given order. With a single round of the best response algorithm we identify the process where all players update their decision once, in a given order.

While computing a (pure) Nash equilibrium is a NP-hard task for a general game [17], all the games instances \( G \) considered in this work are congestion games [18] regardless of what \( f \) is chosen. Relative to this class of games, the following proposition provides sufficient conditions under which the best response algorithm has polynomial running time. The main assumption amounts to requiring each of the agent’s allocation set to coincide with the set of bases of a matroid. The definition of matroid, its rank, as well as related notions are reported in Appendix A.

Proposition 2. [19, Thm. 2.5] Consider the congestion game \( G \) and assume the allocation sets \( A_i \) are the set of bases for some matroid \( M_i = (\mathcal{R}, \mathcal{I}_i) \) over the set of resources \( \mathcal{R} \). Then, players reach a (pure) Nash equilibrium after at most \( n^2 m \max_{i \in N} \text{rank}(M_i) \) best responses.

Examples. The case when feasible allocations are singletons does satisfy the assumptions of the previous theorem, even if an agent does not have access to all the possible resources. One such example is the following: \( \mathcal{R} = \{ r_1, \ldots, r_m \} \), \( m > 2 \), \( A_i = \{ \{ r_1 \}, \{ r_2 \} \} \). Define \( \mathcal{I}_i = \{ \emptyset, \{ r_1 \}, \{ r_2 \} \} \). We have that \( M_i := (\mathcal{R}, \mathcal{I}_i) \) is a matroid of rank 1 and that \( A_i \) is a set of bases for \( M_i \), see Appendix A for the details. In the same Appendix, we provide a further example of allocations sets satisfying the property required in the previous proposition (the case of uniform matroid).

On the negative side, it is simple to construct examples that do not satisfy the requirements. For instance, consider \( \mathcal{R} = \{ r_1, \ldots, r_m \} \), \( m \geq 3 \) and \( A_i = \{ \{ r_1 \}, \{ r_2, r_3 \} \} \). The set \( A_i \) can not be the set of bases for any matroid \( M_i \), as all bases must have the same number of elements (see Appendix A) while \( \{ r_1 \} \) and \( \{ r_2, r_3 \} \) do not. A more involved example that does not satisfy the requirements is the following: \( \mathcal{R} = \{ r_1, \ldots, r_m \} \), \( m \geq 4 \) and \( A_i = \{ \{ r_1, r_2 \}, \{ r_3, r_4 \} \} \). For the given \( A_i \) to be the set of bases of a matroid \( M_i = (\mathcal{R}, \mathcal{I}_i) \), it must be that \( \{ r_1, r_2 \} \in \mathcal{I}_i \) and \( \{ r_3, r_4 \} \in \mathcal{I}_i \). But due to definition of matroid, it must also be \( \{ r_1 \} \in \mathcal{I}_i \) (Definition 2, first property), so that also \( \{ r_1, r_3, r_4 \} \in \mathcal{I}_i \) (Definition 2, second property). Thus \( A_i \) can not be the set of bases for a matroid \( M_i = (\mathcal{R}, \mathcal{I}_i) \), as any possible choice of \( \mathcal{I}_i \) will contain at least one set with more elements than e.g., \( \{ r_1, r_2 \} \in A_i \).

Remark 1. The previous theorem gives condition under which the maximum number of best responses required to converge to a Nash equilibrium is polynomially bounded in the number of players and resources. If it is possible to compute a single best response polynomially in the number of resources, then the performance guarantee given by PoA is achievable in poly-

4Nevertheless, similar statements can still be made almost surely if the players updating their decision are uniformly randomly selected form \( [n] \). This will produce a totally asynchronous algorithm.
nominal time by the best response algorithm. The applications presented in Subsections III-B, IV-A satisfy these assumptions.

III. THE CASE OF SUBMODULAR WELFARE FUNCTION

In this section we focus on the case when the welfare basis function $w$ is non-decreasing and concave (in the discrete sense). This results in the total welfare $W(a)$ of Equation (1) being monotone submodular. Submodular functions model problems with diminishing returns and are used to describe a wide range of engineering applications such as satellite assignment problems [20], Adwords for e-commerce [21], and combinatorial auctions [22], among others. For the considered class of problems, we show in Theorem 1 that the price of anarchy reduces to computing the maximum between $\frac{n(n+1)}{2} \sim O(n^2)$ numbers. Using this result, we give an explicit expression of the price of anarchy for the well known Shapley value and marginal contribution distribution rules (Corollary 1). We then show how to design $f$ so as to maximize the performance measured by $\text{PoA}(f)$. Finally, we compare our performance certificates with existing approximation results. We conclude the section presenting an application to the vehicle target assignment problem.

Assumption 1. Throughout this section we assume that the function $w$ is non-decreasing and concave, in the following sense

$$w(j + 1) \geq w(j),$$

$$w(j + 1) - w(j) \leq w(j) - w(j - 1) \quad \forall j \in [n - 1].$$

Further we assume that $w(1) = 1$.

As a consequence of Assumption 1, the function $W(a)$ is monotone and submodular i.e. it satisfies the following:

Monotonicity:

$$\forall a, b \in A \text{ s.t. } a_i \subseteq b_i, \forall i \in N \implies W(a) \leq W(b).$$

Submodularity:

$$\forall a, b \in A \text{ s.t. } a_i \subseteq b_i, \forall i \in N,$n

$$\forall c \in 2^\mathbb{R}_+ \text{ s.t. } a_i \cup c_i \in A_i, \quad \tilde{b}_i := b_i \cup c_i \in A_i, \forall i \in N,$n

$$\implies W(\tilde{a}) - W(a) \geq W(\tilde{b}) - W(b).$$

While Proposition 1 gives a general answer on how to determine the price of anarchy, it is possible to exploit the additional properties given by Assumptions 1 to obtain an explicit expression of $\text{PoA}(f)$.

Theorem 1 (PoA for submodular welfare). Consider $f$ a distribution rule such that $f(j)w(j)$ is non increasing and $f(j) \geq f_{MC}(j)$ for all $j \in [n]$. Then, $\text{PoA}(f) = 1/W^\star$. 

$$W^\star = \max_{l \leq j \leq n} \left\{ \frac{w(l)}{w(j)} + \min(j, n - l) f(j) + \right.$$  

$$\left. - \min(l, n - j) f(j + 1) \frac{w(j + 1)}{w(j)} \right\}. \quad (9)$$

The proof is reported in Appendix B and amounts to showing that $\lambda$ appearing in [12, Cor. 1] can be computed a priori, and takes the value $\lambda^\star = 1$. The requirements on $f(j)w(j)$ being non increasing and $f(j) \geq f_{MC}(j)$ might seem restrictive at first. Nevertheless, similar assumptions were made relative to a simpler class of problems in [7], [8]. Additionally, the Shapley value and marginal contribution distribution rules (and not only) satisfy these assumptions. Thus, a direct application of Theorem 1 returns the exact price of anarchy of $f_{SV}$ and $f_{MC}$, as detailed in the following.

Corollary 1 (Exact $\text{PoA}$ for $f_{SV}$ and $f_{MC}$).

i) The price of anarchy for the Shapley value distribution rule is $\text{PoA}(f_{SV}) = 1/W^\star_{SV}$, where

$$W_{SV}^\star = \max_{l \leq j \leq n} \left\{ \frac{w(l)}{w(j)} \right. + \min(j, n - l) \frac{1}{j} - \min(l, n - j) \frac{w(j + 1)}{(j + 1)w(j)} \} \quad (10)$$

ii) The price of anarchy for the marginal contribution distribution rule is $\text{PoA}(f_{MC}) = 1/W^\star_{MC}$, where

$$W_{MC}^\star = 1 + \max_{j \in [n]} \left\{ \frac{1}{w(j)} \left( \min(j, n - j) [2w(j) - w(j - 1) - w(j + 1)] \right) \right\} \quad (11)$$

The proof is reported in Appendix B. The previous Corollary shows that the price of anarchy of the Shapley value and marginal contribution distribution rule can be computed as the maximum of $n(n + 1)/2$ and $n$ numbers, respectively.

Remark 2. The expression for $W^\star_{SV}$ appearing in (10) can be equivalently written as

$$W_{SV}^\star = 1 + \max_{l \leq j \leq n} \left\{ \frac{w(l)}{w(j)} - \frac{1}{j} \left[ \max\{j + l - n, 0\} + \min\{l, n - j\} \beta(j) \right] \right\} \quad (12)$$

where $\beta(j) = \frac{j - 1}{w(j + 1) - w(j)}$. The previous expression partially matches the result in [7, Thm. 6], where the authors used a different approach to obtain a bound on the price of anarchy for the larger class of coarse correlated equilibria (CCE), but limitedly to $f_{SV}$ and singleton problems. More precisely, [7, Thm. 6] provides an estimate of the price of anarchy relative to $f_{SV}$, as the minimum between two expression. While their first expression exactly matches (12), the second one is not present here. Nevertheless, it is possible to show that such additional expression is redundant, as the first one is always the most constraining. This allows us to conclude that the bound obtained in [7, Thm. 6] precisely matches the one in (12). Additionally, since our result is provably tight for the class of Nash equilibria, and the result in [7] provides a lower bound for CCE, such bound is tight as well (in the set of CCE) and the worst performing coarse correlated equilibrium is, simply, a pure Nash equilibrium.

For the submodular welfare case considered here, it is still possible to determine the distribution rule $f^\star$ that maximizes $\text{PoA}(f)$ as the solution of a tractable linear program either

5This statement is not formally shown here, in the interest of space. Its proof amounts to showing that the second expression appearing in [7, Thm. 6] is always upper bounded by (12), thanks to the concavity of $w$. 

\[ \frac{w(1)}{w(j)} + \min(j, n - l) f(j) + \]  

\[ - \min(l, n - j) f(j + 1) \frac{w(j + 1)}{w(j)} \].
directly employing the more general result in (8) or using the following linear program derived from (9), which additionally constrains the admissible distributions \( f \) to satisfy \( f(j) \geq f_{MC}(j) \) and \( f(j)w(j) \) to be non increasing.

\[
f^* \in \arg \min_{f \in \mathcal{F}_s, \mu \in \mathbb{R}} \mu \text{ with } f \in \mathcal{F}, s.t. \ f \geq f_{MC}(j) \text{ subject to matroid constraints, the best approximation ratio } \\
\forall j, l \in [0, n] \text{ s.t. } j \geq l \text{ and } 1 \leq j + l \leq n, \\
\mu w(j) \geq w(l) + j f(j) w(j) - l f(j+1)w(j+1) \\
\forall j, l \in [0, n] \text{ s.t. } j \geq l \text{ and } j + l \geq n.
\]

(13)

where \( \mathcal{F}_s = \{ f \in \mathcal{F} | f(j) \geq f_{MC}(j), \ f(j+1)w(j+1) \leq f(j)w(j) \ \forall j \in [n] \} \). Extensive numerical simulations have shown that both these approaches return the same optimal value, so that the additional constraints \( f \in \mathcal{F}_s \) required in (13) do not rule out the optimal distribution derived solving the linear program in (8).

Figure 1 compares the price of anarchy (and thus the approximation ratio of any algorithm capable of computing a Nash equilibrium) of the Shapley value, marginal contribution and optimal distribution rule \( f^* \), in the case when \( w(j) = j^d \) with \( d \in [0, 1] \), \( |N| \leq 20 \). They have been computed using respectively (10), (11) and (7), where \( f^* \) has been determined as the solution to (8). While for values of \( d \in [0, 0.5] \) the Shapley value distribution rule performs close to the optimal, its performance degrades for \( d \in [0, 0.5] \) and for \( d = 0 \) it reaches the lower bound of 1/2, as predicted for the class of valid utility games defined in [23, Thm. 5]. The marginal contribution rule instead, performs the worst amongst the considered distribution rules. While \( f^* \) will always perform better or equal than any other distribution, it is unclear if, and to what extent, \( f_{SV} \) outperforms \( f_{MC} \) in the general settings. The expressions in (10) and (11) could nevertheless be used to provide an answer to this question.

### A. Improved approximation and comparison with existing results

A. Improved approximation and comparison with existing results

For the general class of submodular maximization problems subject to matroid constraints, the best approximation ratio achievable in polynomial time has been very recently shown to be [24]

\[
1 - \frac{e}{c}
\]

(14)

where \( c \) represents the curvature of the welfare function (see [25] for its definition) and \( e \) the Euler’s number. For this class of problems, no polynomial time algorithm can do better than (14) on all instances. Relative to the subclass of problems considered here, i.e. those problems where \( W \) has the special structure in (1), the curvature can be computed by \( c = 1 + w(n-1) - w(n) \). In Figure 1 we plot the approximation ratio (14) for the class of problems considered here, with the choice of \( w(j) = j^d \), i.e. we plot (red curve) the quantity

\[
\text{App} = 1 - \frac{1 + w(n-1) - w(n)}{e}
\]

(15)

for \( d \in [0, 1] \). We observe that the optimal distribution rule \( f^* \) outperforms (15) for different values of \( d \), so that, when there exists an algorithm capable of computing a Nash equilibrium in polynomial time (see Proposition 2), the approach presented here gives improved guarantees compared to (14).

**Remark 3.** It is important to note that this is not in contradiction with the inapproximability result presented in [24], as we are restricting the admissible instances to a subset of the general class of submodular maximization problems studied in the latter work.

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**B. Application: the vehicle target allocation problem**

In the following we consider the vehicle target assignment problem introduced in [26] and studied in [7, 27]. We are given a finite set of targets \( \mathcal{R} \), and for each target \( r \in \mathcal{R} \) its relative importance \( v_r \geq 0 \). Additionally, we are given a finite set of vehicles \( N = \{1, \ldots, n\} \), and for each vehicle a set of feasible target assignments \( \mathcal{A}_v \in 2^{\mathcal{R}} \). The goal is to distribute compute a feasible allocation \( a_v \in \mathcal{A}_v \) so as to maximize the joint probability of successfully destroying the selected targets, expressed as

\[
W(a) := \sum_{r \in \bigcup_i \mathcal{A}_v} v_r (1 - (1 - p)^{|a_r|}),
\]

where \( v_r (1 - (1 - p)^{|a_r|}) \) is the probability that \( |a_r| \) vehicles eliminate the target \( v_r \), and the scalar quantity \( 0 < p \leq 1 \) is a parameter representing the probability that a vehicle will successfully destroy a target. In the forthcoming presentation, it is assumed that the success probability \( p \) is the same for all vehicles, else one would have to define a different \( p_i \) for every \( i \in N \). Observe that the welfare considered here has the form (1) with welfare basis \((1 - (1 - p)^{|a_r|})\). We normalize this quantity (without affecting the problem’s solution) so that \( w(1) = 1 \) and thus define

\[
w(|a_r|) := \frac{1 - (1 - p)^{|a_r|}}{1 - (1 - p)}.
\]

(16)

Observe that (16) satisfies the Standing Assumptions and Assumption 2 in that \( w(j) > 0 \) and \( w(j) \) is increasing and...
concave. Thus, it is possible to compute the performance guarantee of any set of utility functions of the form (2), and to determine the best distribution rule \( f \in \mathcal{F} \) by solving a corresponding linear program.

Figure 2 shows the achievable approximation ratios for the Shapley value, marginal contribution, optimal distribution, as well as the bound in (15). We observe that the optimal distribution significantly rule outperforms all the others as well as the bound of (15) for non-trivial values of \( p \). For the extreme case of \( p = 1 \), \( f^* \) matches (15), while for small \( p \) all the design methodologies offer a similarly high performance guarantee. Figure 3 shows three example of optimal distribution rules.

In both Figures 2 and 3 we have set the number of agents to be relatively small\(^7\) i.e., \( |N| \leq n = 10 \). This choice was purely made so as to perform an exhaustive search simulation in order to test the provided bounds displayed in Figure 2. More specifically, we considered \( 10^5 \) random instances of the vehicle target assignment problem. Each instance features \( n = 10 \) agents, \( n + 1 \) resources and fixed \( p = 0.8 \). Each agent is equipped with an action set with only two allocations, whose elements are singletons, i.e., \(|a_i| = 1\). We believe this is not restrictive in assessing the performance, as the structure of some worst case instances is of this form [9].

Observe that any constraint set \( A_i \) where feasible allocations are singletons is the bases of a uniform matroid of rank one (upon adding the allocation 0 to each \( A_i \)), see Appendix A. Further note that computing a single best response is a polynomial operation in the number of resources. Thus, the best response algorithm will converge polynomially to a Nash equilibrium (Proposition 2) and so the performance guarantees offered by \( \text{PoA} \) are easy to achieve.

The structure of the constraints sets \( A_i \) and the values of the resources are randomly generated, the latter with uniform distribution in the interval \([0, 1]\). For this class of problems considered, the theoretical worst case performance is \( \text{PoA}(f_{SV}) \approx 0.568 \), \( \text{PoA}(f_{MC}) \approx 0.556 \), \( \text{PoA}(f^*) \approx 0.688 \) (see Figure 2 with \( p = 0.8 \)). For each instance \( G \) generated, we performed an exhaustive search so as to compute the welfare at the worst equilibrium \( \min_{a \in \text{NE}(G)} W(a) \) and the value \( W(a^{opt}) \). The ratio between these quantities (their empirical cumulative distribution) is plotted across the \( 10^5 \) samples in Figure 4, for \( f_{SV}, f_{MC}, f^* \). In the same figure the vertical dashed lines represent the theoretical bound on the price of anarchy, while the markers represent the worst case performance occurred during the simulations.

First, we observe that no instance has performed worse than the corresponding price of anarchy, as predicted by Proposition 1. Second, we note that the worst case performance encountered in the simulation is circa 15% better than the true worst case instance.\(^8\) Further, the optimal distribution \( f^* \) has outperformed the others also in the simulations. Its worst case performance is indeed superior to the others (markers in Figure 4). Additionally, the cumulative distribution of \( f^* \) lies below the cumulative distributions of \( f_{SV} \) and \( f_{MC} \) (for

\(^7\)Similar trends and conclusions can be obtained with larger values of \( n \).

\(^8\)Recall that our result in Proposition 1 is tight i.e., there exists at least one instance achieving exactly an efficiency equal to the price of anarchy.
abscissas smaller than 0.95). This means that, for any given approximation ratio \( r \in [0, 0.95] \), there is a smaller fraction of problems on which \( f^* \) performs worse or equal to \( r \), compared to \( f_{SV} \) and \( f_{MC} \). Observe that this is not obvious a priori, as \( f^* \) is designed to maximize the worst case performance and not e.g., the average performance.

IV. THE CASE OF COVERING PROBLEMS

In this section we specialize the previous results to the case of set covering problems. In set covering problems the goal is to allocate agents to resources so as to maximize the total value of covered resources. The corresponding welfare is given by

\[
\sum_{r \in \cup_{i \in N} a_i} v_r,
\]

which is obtained with the choice of \( w(j) = 1 \) for all \( j \) in (1) and (2). Set covering problems are a subclass of submodular resource allocation problems (they satisfy Assumption 1), and are used to model engineering problems such as vehicle-target assignment problems [27] and sensor allocation problems [28]. Due to their importance in the applications, we treat their study separately to the general submodular case.

Relative to covering problems, we provide a general expression for the price of anarchy as a function of \( f \) (Theorem 2) and show how this reduces to the results obtained in [8], [9], under the additional assumptions therein required. We conclude the section with an application to caching in wireless data networks.

**Theorem 2.** Consider set covering problems i.e., fix \( w(j) = 1 \) \( \forall j \in [n] \). The price of anarchy is \( \text{PoA}(f) = 1/W^* \) where

\[
W^* = 1 + \max_{j \in [n-1]} \{ j(1 + f(j) - f(j+1)) \}
\]

(17)

The proof can be found in Appendix B. The previous Theorem gives a simple and explicit way to compute the price of anarchy (4) as the maximum between \( 3(n-1) \) numbers. Observe that no assumptions are required other than the Standing Assumption. Theorem 2 thus extends the previous bounds derived in [8], [9]. In the latter works, the authors required the admissible distribution rules to be non increasing and such that \( jf(j) \leq 1 \) for all \( j \in [n] \).

The next Corollary shows how the result in the previous Theorem matches the results in [8], [9], under the additional assumptions therein required. The proof is reported in Appendix B.

**Corollary 2.** Consider \( f \) a non increasing distribution rule. The value of (17) is given by

\[
W^* = 1 + \max_{j \in [n-1]} \{ jf(j) - f(j + 1), (n-1)f(n) \}
\]

(18)

In [8, Thm. 2] the author provides a bound matching the expression in (18). Tightness of the previous bound is shown in [9, Thm. 1]. Additionally, [8, Eq. 5] also determines the distribution rule maximizing the price of anarchy (18) as

\[
f^*(j) := (j-1)! \left( \frac{1}{(n-1)(n-1)!} \sum_{i=j}^{n-1} \frac{1}{i!} \right) + \frac{1}{(n-1)!} \sum_{i=1}^{n-1} \frac{1}{i!}, \quad j \in [n].
\]

(19)

Nevertheless, the feasible set of distribution rules is limited to \( jf(j) \leq 1 \) and \( f \) non increasing. Using the result provided here in Theorem 2 it is possible to determine the best distribution (via a linear program derived from (17)) without imposing this additional constraints. Numerical simulations have shown that the optimal distribution rule obtained optimizing (17) precisely matches the one derived in [8], so that removing the additional assumption required therein does not improve the best achievable price of anarchy.9

**Remark 4.** Relative to set covering problems, the work [8] explicitly determines the value of the price of anarchy for the best distribution rule. It’s value essentially amounts to

\[
1 - \frac{1}{e}
\]

and thus exactly matches the result in (14), since for covering problems \( c = 1 - w(n) - w(n-1) = 1 \).

A. Application: content distribution in wireless data networks

In this section we consider the problem of distributed data caching introduced in [29] as a technique to reduce peak traffic in mobile data networks. In order to alleviate the growing radio congestion caused by the recent surge of mobile data traffic, the latter work suggested to store popular and spectrum intensive items (such as movies or songs) in geographically distributed stations. The approach has the advantage of bringing the content closer to the customer, and to avoid recurring transmission of large quantities of data. Similar offloading techniques, aiming at minimizing the peak traffic demand by storing popular items at local cells, have been recently proposed and studied in the context of modern 5G mobile networks [30], [31]. The fundamental question we seek to answer in this section is how to geographically distribute the popular items across the nodes of a network so as to maximize the total number of queries fulfilled. In the following we borrow the model introduced in [29] and show how the utility design approach presented here yields improved theoretical and practical performances.

We consider a rectangular grid with \( n_x \times n_y \) bins and a finite set \( R \) of data items. For each item \( r \in R \), we are given its query rate \( q_r \geq 0 \) as well as its position in the grid \( O_r \) and a radius \( \rho_r \). A circle of radius \( \rho_r \) centered in \( O_r \) represents the region where the item \( r \) is requested. Additionally we consider a set of geographically distributed nodes \( N \) (the local cells), where each node \( i \in N \) is assigned to a position in the grid \( P_i \). A node is assigned a set of feasible allocations \( A_i \) according to the following rules:

i) \( A_i \subseteq 2^{R_i} \), where \( R_i := \{ r \in R \text{ s.t. } ||O_r - P_i|| \leq \rho_r \} \).

That is, \( r \in R_i \) if the (euclidean) distance between the position of node \( i \) and item \( r \) is smaller equal to \( \rho_r \).

ii) \( |A_i| \leq k_i \), for some \( k_i \in [1, \infty[. \)

In other words, node \( i \) can include the resource \( r \) in his allocation \( a_i \) only if he is in the region where the item \( r \)

9This statement can be formally proved, by showing that the optimal distribution derived in [8] solves the KKT system of the LP corresponding to the problem of minimizing \( W^* \) given in (17). We do not pursue this, in the interest of space.
is requested (first rule), while we limit the number of stored items to \( k_i \) for reasons of space (second rule).\footnote{Similarly to what discussed for the application of Subsection III-B, it is possible to reduce the problem to the case where \( \mathcal{A}_i \) are the bases of a matroid \( \mathcal{M}_i \), so that Proposition 2 applies here too.} The situation is exemplified in Figure 5.

![Figure 5](image)

**Fig. 5.** The nodes 1 and 2 can include the query \( r \) in any allocation i.e. \( r \in \mathcal{R}_1 \) and \( r \in \mathcal{R}_2 \) since the distance from nodes 1 and 2 to \( O_r \) is less than \( \rho_r \).

The objective is to select a feasible allocation for every node so as to jointly maximize the total amount of queries fulfilled, i.e.

\[
\max_{a \in A} \sum_{r \in \cup_{i \in N} a_i} q_r.
\]

In order to obtain a distributed algorithm, [29] proposes a game theoretic approach where each agent is given a Shapley value utility function, i.e. they assign to agents utilities of the form (2), where \( f(j) = f_{SV}(j) = 1/j \).

In the following we compare the results of numerical simulations obtained using \( f_{SV} \) or \( f^{*} \) in (19). The following parameters are employed. We choose \( n_x = n_y = 800, |N| = 100, |\mathcal{R}| = 1000 \). The nodes and the data items are uniformly randomly placed in the grid. The query rate of data items is chosen according to a power law (Zipf distribution) \( q_r = 1/r^\alpha \) for \( r \in [1, 1000] \).\footnote{Typical query rate curves has been shown to follow this distribution, with \( \alpha \in [0.6, 0.9] \), see [32].} The radii of interests are set to be identical for all items \( \rho_r = \rho = 200 \). We let \( \alpha \) vary in \([0.7, 0.9]\). We consider \( 10^5 \) instances of such problem, and for every instance compute a Nash equilibrium by means of the best response algorithm. Given the size of the problem, it is not possible to compute the optimal allocation and thus the price of anarchy. As a surrogate for the latter we use the ratio \( W(a^{ne})/W_{\text{tot}} \), where \( a^{ne} \) is the Nash equilibrium determined by the algorithm and

\[
W_{\text{tot}} := \sum_{r \in \mathcal{R}} q_r
\]

is simply the sum of all the query rates and thus is an upper bound for \( W(a^{opt}) \). Observe that \( W_{\text{tot}} \) is a constant for all the simulations with fixed \( \alpha \), indeed \( W_{\text{tot}} = \sum_{r \leq 1000} 1/r \) and thus serves as a mere scaling factor. The theoretical price of anarchy is \( \text{PoA}(f_{SV}) = 0.5 \) (tight also when the query rates are Zipf distributed [29]) and \( \text{PoA}(f^{*}) = 1 - 1/e \approx 0.632 \).

Figure 6 compares the quantity \( W(a^{ne})/W_{\text{tot}} \) for the choice of \( f_{SV} \) and \( f^{*} \), across different values of \( \alpha \). First we observe that the worst cases encountered in the simulations are at least 10% better than the theoretical counterparts. Further, for each fixed value of \( \alpha \), there is a good separation between the performance of \( f_{SV} \) and \( f^{*} \), in favor of the latter. This holds true, not only in the worst case sense (markers in the Figure 6), but also on average. As \( \alpha \) increases from 0.6 to 0.9, the worst case performance seems to degrade for both \( f_{SV} \) and \( f^{*} \). Nevertheless, since we are using \( W(a^{ne})/W_{\text{tot}} \) as a surrogate for the true price of anarchy, it is unclear if the previous conclusion also holds for \( W(a^{ne})/W(a^{opt}) \).

![Figure 6](image)

**Fig. 6.** Box plot comparing the performance of the best response algorithm on \( 10^5 \) instances for the choice of distributions \( f_{SV} \) and \( f^{*} \), across different values of \( \alpha \). On each plot, the median is represented with a red line, and the corresponding box contains the 25th and 75th percentiles. The (four) worst cases are represented with crosses.

Figure 7 presents a more detailed comparison between \( f_{SV} \) and \( f^{*} \) for a fixed value of \( \alpha = 0.7 \) over all the \( 10^5 \) instances. Relative to this case, Figure 8 describes the (distribution of) number of best response rounds required for the algorithm to converge. Quick convergence is achieved, with a number of best response rounds equal to \( 11 \) in the worst case. Observe that in every best response round all players have a chance to update their decision variable, so that a total number of \( n_{\text{BR}} \) rounds amounts to \( n \cdot n_{\text{BR}} \) individual best responses.

V. THE CASE OF SUPERMODULAR WELFARE FUNCTION

In this section we consider welfare basis functions that are non-decreasing and convex, resulting in a monotone and supermodular total welfare \( W(a) \). Applications featuring this property include but are not limited to clustering and image segmentation [33], power allocation in multiuser networks [34]. In the following we explicitly characterize the price of anarchy for the class of supermodular resource allocation problems as a function of \( f \) (Theorem 3), extending [10], [11]. Additionally, we show that the Shapley value distribution rule maximizes this measure of efficiency (recovering the result in [10], [11]), but is not the only one.
Assumption 2. Throughout this section we assume that $w$ is a non-decreasing and convex function, that is
\[ w(j + 1) \geq w(j), \]
\[ w(j + 1) - w(j) \geq w(j) - w(j - 1) \quad \forall j \in [n - 1]. \]
Further we assume that $w(1) = f(1) = 1$.

**Theorem 3 (PoA for supermodular welfare).** Consider a distribution rule $f$ such that $f(j)w(j) \geq 1 \forall j \in [n]$. It holds
\[ \text{PoA}(f) = \frac{n}{w(n)} \max_{j \in [n]} jf(j). \]
It follows that $f_{SV}$ is optimal amongst $f \in \mathcal{F}$, and achieves
\[ \text{PoA}(f_{SV}) = \frac{n}{w(n)}. \]

The proof is provided in Appendix B. Observe that the Shapley value, and all the distribution rules for which $jf(j) \geq 1$ satisfy the conditions of Theorem 3. Indeed $f(j)w(j) \geq jf(j) \geq 1$ by convexity and Standing Assumption. Further note that the Shapley value distribution rule is not the unique maximizer of PoA($f$). Indeed, all the distribution rules with $1/w(j) \leq f(j) \leq 1/j$ are optimal, since the previous theorem applies and they have $jf(j) \leq 1$. Figure 9 compares the price of anarchy of the Shapley value, marginal contribution and optimal distribution rule, in the case when $w(j) = j^d$ with $d \in [1, 2]$, $|N| \leq 20$. First, we observe that any optimal distribution rule, and $f_{SV}$ give the same performance, as predicted from Theorem 3. Additionally, we observe that the quality of the approximation quickly degrades as the welfare basis $w$ gets steeper ($d$ gets larger). This is due to the fact that if $w(n)$ grows much faster than $n$, the quantity $n/w(n)$ quickly decreases.

VI. CONCLUSIONS AND REMARKS

This manuscript specializes the results presented in Part I of this work to the case of monotone submodular, supermodular and set covering problems. First, we obtained an explicit characterization of the worst case performance as a function of the assigned utility functions. Second, we compared the performance provided by optimally designed utility functions with existing approximation results. For covering problems we recover the $1 - 1/e$ result of [1], while for the submodular case our bounds improve on the existing approximation results. Finally, we tested the theoretical findings on two applications.

We remark on the fact that the performance certificates obtained in this work are confined to the notion of (pure) Nash equilibrium. While computing one such equilibrium is, in general, a hard task, Proposition 2 showed that under structural assumptions on the feasible sets $\{A_i\}_{i \in N}$, this can be accomplished in polynomial time. Whether the performance certificates derived here hold for the larger class of coarse correlated equilibria (CCE), is at this point an open question.\footnote{On this respect, Remark 2 shows that this is the case limitedly to $f_{SV}$.}

The appeal of CCE lies in the fact that their calculation is a simple, i.e. polynomial, task for the classes of games considered here [35].

**APPENDIX A**

**MATROIDS**

**Definition 2 (Matroid).** A tuple $M = (\mathcal{R}, \mathcal{I})$ is a matroid if $\mathcal{R}$ is a finite set of resources, $\mathcal{I} \subseteq 2^\mathcal{R}$ is a collection of subsets of $\mathcal{R}$, and the following two properties hold:

\[ \text{APPENDIX A} \]
• If $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$;
• If $A \in \mathcal{I}$, $B \in \mathcal{I}$ and $|B| > |A|$, then there exists an element $r \in B \setminus A$ s.t. $A \cup \{r\} \in \mathcal{I}$.

**Definition 3** (Basis). A set $S \in \mathcal{I}$ such that for all $r \in R \setminus S$, $(S \cup r) \notin \mathcal{I}$ is called a basis of the matroid.

It can be shown that all bases have the same number of elements, which is known as the rank of the matroid and indicated with $\text{rank}(\mathcal{M})$ [36].

A simple example of matroid is that of uniform matroid $\mathcal{M} = (R, \mathcal{I})$ where $\mathcal{I}$ is the collection of all subsets with a number of elements $k \leq m$. By concavity of $w$ and $l > j$, one observes that

$$w(l) \leq w(j + 1) + (w(j + 1) - w(j))(l - j - 1)$$

and since $l - j > 0$, $w(j + 1) - w(j) \geq 0$, $\lambda \geq 1$, it holds

$$w(l) \leq w(j) + \lambda(w(j + 1) - w(j))(l - j).$$  

(21)

Using inequality (21), one can show that (20) has to hold

$$w(l) - w(j) + \lambda(j - l)f(j + 1)w(j + 1) \leq w(j) + \lambda(w(j + 1) - w(j))(l - j) - w(j) + \lambda(j - l)f(j + 1)w(j + 1)
= \lambda(l - j)(w(j + 1) - w(j) - f(j + 1)w(j + 1)) \leq 0,
$$

where the last inequality holds because $f(j + 1)w(j + 1) \geq w(j + 1) - w(j)$ (by assumption) and $l > j$. Observe that the previous inequality is never evaluated for $j = n$, as there is no $l \in [n]$ with $l > j = n$.

**ii)** We now consider the case $j + l > n$ and $l > j$. Here we intend to prove that for any $\lambda \geq 1$

$$1 + \lambda\frac{n - j}{w(j)}[f(j)w(j) - f(j + 1)w(j + 1)] \geq w(l) - w(j) + \lambda(j - l)f(j)w(j) - \lambda\frac{n - j}{w(j)}f(j + 1)w(j + 1),$$

where the left hand side is obtained setting $l = j$. The latter is equivalent to

$$w(l) - w(j) + \lambda(j - l)f(j)w(j) \leq 0.$$

Similarly to (21), one can show that

$$w(l) \leq w(j) + \lambda(w(j) - w(j - 1))(l - j),$$

and get the desired result as follows

$$w(l) - w(j) + \lambda(j - l)f(j)w(j) \leq w(j) + \lambda(w(j) - w(j - 1))(l - j) - w(j) + \lambda(j - l)f(j)w(j)
= \lambda(l - j)(w(j) - w(j - 1) - f(j)w(j)) \leq 0,
$$

where the last inequality holds because $f(j)w(j) \geq w(j) - w(j - 1)$ (by assumption) and $l > j$.

**APPENDIX B**

**Proof of Theorem 1**

Proof. Observe that the claim we wish to prove (i.e. the value of $W^*$ in (9)) can be equivalently reformulated as in the following program, upon observing that for $j + l \leq n$ it holds $\min(j, n - l) = j$ and $\min(l, n - j) = l$, while for $j + l > n$ it holds $\min(j, n - l) = n - l$ and $\min(l, n - j) = n - j$,

$$W^* = \min_{\mu \in \mathbb{R}} \mu$$

s.t. $\mu w(j) \geq w(l) + jf(j)w(j) - lw(j + 1)w(j + 1)$

$$\forall j, l \in [0, n], s.t. j \geq l + 1 \leq n,
\mu w(j) \geq w(l) + (n - l)f(j)w(j) - (n - j)f(j + 1)w(j + 1)$

$$\forall j, l \in [0, n], s.t. j \geq l + 1 > n.$$

In the following we prove that the latter program follows from [12, Cor. 1] by showing that only the constraints with $l \leq j$ are required, and that the decision variable $\lambda$ in [12, Eq. (12)] takes the value $\lambda^* = 1$. First, notice that $f(j)w(j)$ is assumed to be non increasing, and so $W^*$ can be correctly computed using [12, Cor. 1]. For $j = 0$, the constraints in [12, Eq. (12)] read as

$$\lambda \geq \frac{w(l)}{l} \quad \forall l \in [n],$$

and the most binding amounts to $\lambda \geq 1$, due to the concavity of $w$. For $j \neq 0$, we intend to show that the constraints with $l > j$ appearing in [12, Eq. (12)] are not required since those with $j = l$ are more binding. The following figure explains this more clearly.

To do so, we divide the discussion in two cases: i) $l + j \leq n$ and ii) $l + j > n$.

i) When $1 \leq j + l \leq n$ and $l > j$, we want to show that for any $\lambda \geq 1$

$$1 + \lambda\frac{j}{w(j)}[f(j)w(j) - f(j + 1)w(j + 1)] \geq w(l) - w(j) + \lambda(j - l)f(j)w(j) - \lambda\frac{j}{w(j)}f(j + 1)w(j + 1),$$

where the left hand side is obtained setting $l = j$. This is equivalent to showing

$$w(l) - w(j) + \lambda(j - l)f(j + 1)w(j + 1) \leq 0. \quad (20)$$

Fig. 10. The proof amounts to showing that for any constraint identified with the indices $(j, l)$ and $l > j$ (circles in the figure), the constraint identified with $(j; j)$ is more binding (crosses in the figure).
The steps i) and ii) showed that $W^*$ in [12, Eq. (12)] can be equivalently computed as

$$W^* = \min_{\mu \in \mathbb{R}_{\geq 0}, \lambda \in \mathbb{R}} \mu$$

s.t. $\mu w(j) \geq w(l) + \lambda (j f(j) w(j) - (j+1) w(j+1))$ for all $j \in [0, n]$ s.t. $j \geq 1$ and $l \leq j + l \leq n$,

$$\mu w(j) \geq w(l) + \lambda [(n-l) f(j) w(j) - (n-j) f(j+1) w(j+1)]$$

s.t. $j \in [0, n]$ s.t. $j \geq l$ and $l \leq j + l \leq n$.

Every constraint appearing in the previous program is indexed by $(j, l)$ and can be compactly written as $\mu \geq b_{jl} + a_{jl} \lambda$, upon defining $b_{jl} := w(l)/w(j)$ and consequently

$$a_{jl} := \begin{cases} f(j) w(j) - l f(j+1) w(j+1) & 1 \leq j + l \leq n, \\ (n-l) f(j) w(j) - (n-j) f(j+1) w(j+1) & j + l \geq n. \end{cases}$$

Consequently $W^*$ can be computed as

$$W^* = \min_{\lambda \in \mathbb{R}, \mu \geq b_{jl} + a_{jl} \lambda} \mu$$

s.t. $\lambda \geq 0$, $j \in [0, n]$, s.t. $j \geq l$.

As previously seen, for $j = 0$ the most constraining index is $\lambda \geq 1$. Observe that, when $j \geq 1$ and $j \geq l$, it holds $a_{jl} \geq 0$. Indeed since $f(j) w(j)$ is non increasing, for $1 \leq j + l \leq n$ one has $a_{jl} = j f(j) w(j) - l f(j+1) w(j+1) \geq (j-l) f(j) w(j) \geq 0$. Similarly for $j + l \geq n$. Thus the optimal choice is to select $\lambda$ as small as possible i.e. $\lambda = 1$.

**Proof of Corollary 1**

**Proof.** The proof is an application of Theorem 1.

i) Observe that $f_{SV}$ satisfies the assumptions of Theorem 1 in that $f(j) w(j) = w(j)$ is non increasing (due to concavity of $w$) and $f_{SV}(j) = \frac{j}{j} \geq 1 - w(j)/w(j-1) \iff w(j-1) + j (w(j) - w(j-1)) \geq 0$ (due to positivity and non-decreasingness of $w$). Hence the result of Theorem 1 applies and substituting $f(j) = 1/j$ gives $W_{SV}^*$ as in the claim.

ii) Observe that $f_{MC}$ satisfies the assumption of Theorem 1 in that $f(j) w(j) = w(j) - w(j-1)$ is non increasing (due to concavity of $w$) and $f(j) = 1 - \frac{w(j-1)}{w(j)}$ so the second condition is satisfied too.

We conclude by proving that the constraints indexed with $l \leq j \in [n]$ are not needed and it is enough to consider $j = l \in [n]$, so that $W_{MC}^*$ as given in Corollary 1, ii). To do so, we show that for any $l \leq j$ the most binding constraint is given by $l = j$.

For $l \leq j$ and $j + l \leq n$ we want to prove that

$$1 + \lambda \frac{j}{w(j)} [f(j) w(j) - f(j+1) w(j+1)] \geq \frac{w(l)}{w(j)} + \lambda \left[ \frac{j}{w(j)} f(j) w(j) - \frac{l}{w(j)} f(j+1) w(j+1) \right],$$

where the left hand side is obtained setting $l = j$. The previous is equivalent to

$$w(l) - w(j) + \lambda (j-l) f(j+1) w(j+1) \leq 0,$$

and since $f(j+1) w(j+1) = w(j+1) - w(j)$, it reduces to

$$w(l) - w(j) + (j-l) (w(j+1) - w(j)) \leq 0.$$

By concavity of $w$ and $l < j$, it holds that $w(j) \geq w(l) + (j-l) (w(j+1) - w(j))$ and thus (22) follows.

In the case of $l < j$ and $j + l > n$ we intend to show

$$\frac{w(l)}{w(j)} + \lambda \left[ \frac{n-j}{w(j)} f(j) w(j) - \frac{n-l}{w(j)} f(j+1) w(j+1) \right],$$

which reduces to

$$w(l) - w(j) + (j-l) (w(j) - w(j-1)).$$

The latter follows by concavity of $w$. Hence, the price of anarchy of $f_{MC}$ is governed by $W^*$ as in Theorem 1, where we set $f = f_{MC}$ and fix $j = l$. This gives the following expression

$$W_{MC}^* = 1 + \max_{j \in [n]} \left\{ \min(j, n-j) \right\} \left[ f_{MC}(j) - f_{MC}(j+1) \right],$$

which reduced to the expression for $W_{MC}^*$ in the claim, upon substituting $f_{MC}(j)$ with its definition.

**Proof of Theorem 2**

**Proof.** The proof is a specialization of the general result obtained in [12, Thm. 3] to the case of set covering problems. We divide the study in three distinct cases, as in the following

$$C_1 : \begin{cases} a + x = 0 \\ b + x \neq 0 \end{cases}$$

$$C_2 : \begin{cases} a + x \neq 0 \\ b + x = 0 \end{cases}$$

$$C_3 : \begin{cases} a + x \neq 0 \\ b + x \neq 0 \end{cases}$$

In case $C_1$ it must be $a = x = 0, b \neq 0$ and the constraints read as

$$\lambda \geq \frac{1}{b}.$$ 

The most binding one is obtained for $b = 1$, i.e. it suffices to have $\lambda \geq \frac{1}{b}$. In case $C_2$ it must be $b = x = 0, a \neq 0$. The constraints read as

$$\mu \geq \lambda a f(a) \forall a \in [n].$$

In case $C_3$, since $a + x \neq 0$ and $b + x \neq 0$, the constraints become

$$\mu \geq 1 + \lambda \left[ af(a) - bf(a + x) \right].$$

If $x = 0$, then $a, b > 0$ and the previous inequality reads

$$\mu \geq 1 + \lambda \left[ af(a) - bf(a + 1) \right] \quad a + b \in [n].$$

The most constraining inequality is obtained for $b$ taking the smallest possible value, that is $b = 1$. Thus $0 < a \leq n - 1$. Consequently when $x = 0$, it suffices to have

$$\mu \geq 1 + \lambda \left[ af(a) - f(a + 1) \right] \forall a \in [n-1].$$
If \( x \neq 0 \), the most binding constraint is obtained for \( b = 0 \). In such case, \( 0 < a + x \leq n \) and the constraints read as
\[
\mu \geq 1 + \lambda a f(a + x) \quad \forall a \in [n].
\]
For ease of readability, we introduce the variable \( j := a + x \) and use \( j \) and \( x \) instead of \( a \) and \( x \). With this new system of indices the feasible region becomes \( 0 < j \leq n \) and \( j - x \geq 0, x > 0 \). The constraints read as
\[
\mu \geq 1 + \lambda(j - x)f(j)
\]
and the most binding is trivially obtained for \( x = 1 \), reducing the previous to
\[
\mu \geq 1 + \lambda(j - 1)f(j) \quad \forall j \in [n].
\]

This guarantees that the program in [12, Eq. (11)] is equivalent to
\[
W^* = \min_{\lambda \in \mathbb{R} \geq 0, \mu \in \mathbb{R}} \mu \\
\text{s.t.} \quad \lambda \geq 1 \\
\mu \geq \lambda j f(j) \quad j \in [n] \\
\mu \geq 1 + \lambda (j f(j) - f(j + 1)) \quad j \in [n-1] \\
\mu \geq 1 + \lambda (j - 1)f(j) \quad j \in [n].
\]

Amongst the last three set of constraints, the tightest constraint always features a positive coefficient multiplying \( \lambda \). Indeed the only term multiplying \( \lambda \) that could take negative values is \( j f(j) - f(j + 1) \), but every time this is negative, the constraints \( \mu \geq 1 + \lambda (j - 1)f(j) \) are tighter. It follows that the solution consists in picking \( \lambda \) as small as possible, that is in choosing \( \lambda^* = 1 \). The program becomes
\[
W^* = \min_{\mu \in \mathbb{R}} \mu \\
\text{s.t.} \quad \mu \geq j f(j) \quad j \in [n] \\
\mu \geq 1 + j f(j) - f(j + 1) \quad j \in [n-1] \\
\mu \geq 1 + (j - 1)f(j) \quad j \in [n].
\]

We conclude with a little of cosmetics: the first and third set of inequalities run over \( j \in [n] \), while the second one has \( j \in [n-1] \). Observe that the first and third condition evaluated at \( j = 1 \) read both as \( \mu \geq 1 \). This condition is implied by the last set of condition with \( j = 2 \), indeed it reads as \( \mu \geq 1 + f(2) \geq 1 \) since we assumed \( f \) non negative. Thus the first and third conditions can be reduced to \( j \in [2, n] \). Shifting the indices down by one, we get
\[
W^* = \min_{\mu \in \mathbb{R}} \mu \\
\text{s.t.} \quad \mu \geq (j + 1)f(j + 1) \quad j \in [n-1] \\
\mu \geq 1 + j f(j) - f(j + 1) \quad j \in [n-1] \\
\mu \geq 1 + j f(j + 1) \quad j \in [n-1],
\]
from which we get the analytic expression in (17), i.e.
\[
W^* = 1 + \max_{j \in [n-1]} \{(j + 1)f(j + 1) - 1, jf(j) - f(j + 1), jf(j + 1)\}.
\]

**Proof of Corollary 2**

*Proof.* Thanks to Theorem 2, the value \( W^* \) and consequently the price of anarchy can be computed as
\[
W^* = \max_{j \in [n-1]} \{(j + 1)f(j + 1), 1 + jf(j) - f(j + 1), 1 + jf(j + 1)\}.
\]
We will show that when \( f \) is non-increasing, fewer constraints are required, producing exactly (18).

First observe that \( f \) being non-increasing implies \((j + 1)f(j + 1) = f(j + 1) + jf(j + 1) \leq f(j + 1) + jf(j + 1) = 1 + jf(j + 1)\), so that the first set of conditions is implied by the third. Hence
\[
W^* = 1 + \max_{j \in [n-1]} \{jf(j) - f(j + 1), jf(j + 1)\}.
\]

We now verify that the first set of remaining conditions implies all the conditions in the second set, but not the last one:
\[
\mu \geq 1 + jf(j) - f(j + 1) \geq 1 + jf(j) - f(j) = 1 + (j - 1)f(j),
\]
\[
\forall j \in [n-1] \text{ that is, all conditions } \mu \geq jf(j + 1) \text{ are satisfied for } j \in [n-2].
\]
Thus, it suffices to require \( \mu \geq 1 + jf(j) - f(j + 1) \) and \( \mu \geq 1 \geq (n - 1)f(n) \) for all \( j \in [n] \) and the result in (18) follows.

**Proof of Theorem 3**

*Proof.* The proof is a specialization of the general result obtained in Proposition 1. We divide the study in the same three cases used for the proof of Theorem 2.

In case \( C_1 \), the constraints read as
\[
w(b) - \lambda b \leq 0 \iff \lambda \geq \frac{w(b)}{b},
\]
the most constraining of which is given for \( b = n \) as \( w(b) \) is convex. Thus it must be
\[
\lambda \geq \frac{w(n)}{n}.
\]

In case \( C_2 \), the constraints read as
\[
\lambda af(a)w(a) \leq \mu w(a) \iff \mu \geq \lambda af(a).
\]

In case \( C_3 \), the constraints read as
\[
\mu \geq \frac{w(b + x)}{w(a + x)} + \lambda \left[ af(a + x) - bf(a + x + 1) - w(a + x + 1)w(a + x) \right].
\]

In order to conclude, we will show that the constraints obtained from \( C_1 \) and \( C_2 \) imply all the conditions stemming from \( C_3 \).

To do so observe that
\[
\frac{w(b + x)}{w(a + x)} + \lambda \left[ af(a + x) - bf(a + x + 1) - w(a + x + 1)w(a + x) \right] = \frac{1}{w(a + x)} \left[ w(b + x) - \lambda bw(a + x + 1)f(a + x + 1) \right] + \lambda af(a + x)
\]
\[
\leq \frac{1}{w(a + x)} \left[ \lambda(b + x) - \lambda b \cdot f(1)w(1) \right] + \lambda af(a + x)
\]
\[
\leq w(b + x) + \lambda \lambda af(a + x) = \lambda af(a + x)
\]
From first to second line is rearrangement. From second to third is due to $f(a + x + 1)w(a + x + 1) \geq w(1)f(1) = 1$ and to $w(b + x) \leq \frac{w(n)}{n}(b + x) \leq \lambda(b + x)$ where the first inequality holds because of convexity of $w$ and the second inequality follows from $C_1$ i.e. from $\lambda \geq \frac{w(n)}{n}$. From third to fourth is rearrangement. From fourth to fifth is due to $w(a + x) f(a + x) \geq f(1)w(1) = 1 \implies f(a + x) \geq \frac{f(1)w(1)}{w(a + x)}$ i.e. conditions $C_3$ are all satisfied.

It follows that $W^*$ and consequently the price of anarchy is easily obtained as

$$W^* = \min_{\lambda \in R^{\geq 0}, \mu \in R} \mu$$

s.t. $\mu \geq \lambda j f(j)$ \quad \forall j \in [n]$

$$\lambda \geq \frac{w(n)}{n}.$$  

The solution is given by $\lambda^* = \frac{w(n)}{n}$, $\mu^* = \lambda^* \max_{j \in [n]} j f(j)$, which gives a price of anarchy of

$$P_oA(f) = \frac{1}{n \max_{j \in [n]} j \cdot f(j)}.$$  

The optimality of $f_{SV}$ follows from the fact that $\max_{j \in [n]} j \cdot f(j) = 1$ is the smallest achievable value for any $f \in F$. 

\[\square\]

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