Coupling vortex dynamics with collective excitations in Bose-Einstein Condensates

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Abstract

Here we analyze the collective excitations as well as the expansion of a trapped Bose-Einstein condensate with a vortex line at its center. To this end, we propose a variational method where the variational parameters have to be carefully chosen in order to produce reliable results. Our variational calculations agree with numerical simulations of the Gross-Pitaevskii equation. The system considered here turns out to exhibit four collective modes of which only three can be observed at a time depending of the trap anisotropy. We also demonstrate that these collective modes can be excited using well established experimental methods such as modulation of the s-wave scattering length.
I. INTRODUCTION

In this work, we are interested in the dynamics of a trapped Bose-Einstein condensate (BEC) containing a line vortex at its center. Here we are particularly interested in obtaining the collective oscillation modes of the system which couples the vortex core oscillations with the oscillations of the condensate external dimensions. The interest in this problem is motivated by the fact that these oscillations can be measured in the laboratory by moving the atomic cloud out of its equilibrium configuration by using the Feshbach resonance in order to modulate the scattering length [1–5]. These oscillations are also studied in other physical systems such as two-species condensates [6], BCS-BEC crossover [7–9], and superfluid Helium [10]. From the theoretical point of view, we are interested on how the size of the vortex core oscillates with respect to the external dimensions of the cloud. The mode with the smallest oscillation frequency is the quadrupole mode which occurs when the longitudinal and radial sizes of condensate oscillate out phase. The breathing mode requires more energy to be excited since the change in the density of the atomic cloud imposes a greater resistance against deviation from its equilibrium configuration than in the case of quadrupole excitations [11, 12].

In Refs. [13–15], the dynamics of normal modes for a single vortex has been studied using hydrodynamic models, which focus on the vortex motion with respect to the center-of-mass of the condensate. This concept was also used in the case of multicomponent Bose-Einstein condensates [16].

Preliminary calculations using a variational calculation with a Gaussian Ansatz, which does not take into account the independent variation of the vortex core size [3, 6, 16, 17], shows a small shift in the frequencies of the aforementioned modes (Figure 1). This shift has already been obtained via a hydrodynamic approximation in Refs. [12] and [18]. Thus we can expect the frequency of the monopole (breathing) mode to decrease while the quadrupole frequency increases in the presence of the vortex.

To calculate the dynamics of a vortex with charge $\ell$ in a more consistent way with the physical reality, which allows for the coupling between vortex core and the external dimensions of condensate, we could naively use a Thomas-Fermi (TF) Ansatz [19]

$$
\psi(\rho, \varphi, z, t) = A(t) \left[ \frac{\rho^2}{\rho^2 + \xi(t)} \right]^\frac{1}{2} \sqrt{1 - \frac{\rho^2}{R_\rho(t)^2} - \frac{z^2}{R_z(t)^2}} \exp \left[ i\ell\varphi + iB_\rho(t) \frac{\rho^2}{2} + iB_z(t) \frac{z^2}{2} \right],
$$

and then calculate the equations of motion for the five variational parameters ($\xi, R_\rho, R_z, B_\rho, B_z$). Following these calculations, the equations of motion would be linearized. For the Ansatz (1), this procedure leads to imaginary frequencies which are not consistent with the stable
Figure 1: (Color online) Oscillation frequencies from Gaussian Ansatz without taking into account the independent variation of the vortex core size. Upper lines correspond the frequencies of the breathing mode as a function of the harmonic trap anisotropy, whereas lower lines represent the frequencies of the quadrupole mode. Solid (black) lines correspond to a vortex-free Gaussian-profile while dotted (blue) lines describe a profile with a singly charged vortex. Note that $\omega$ is normalized by the frequency of the radial direction $\omega_\rho$.

configuration where a singly charged ($\ell = 1$) vortex resides at the center of the condensate. The linearized equations of motion can be written in a matrix form according to

$$M\ddot{\delta} + V\delta = 0,$$  \hspace{1cm} (2)

where $\delta$ is the vector with components given by deviations of the variational parameters from their equilibrium values. The solution of (2) is a linear combination of oscillatory modes whose oscillation frequencies obey the equation

$$\prod_n \omega_n^2 = \det (M^{-1}V) = \frac{\det V}{\det M}.$$  \hspace{1cm} (3)

In order to ensure that all frequencies $\omega_n$ are real, we must have $\det V / \det M > 0$. We know that $\det V > 0$ since its sign reflects the sign of the variational parameters which represents the external dimensions of the cloud in the stationary situation. Therefore $\det M$ must also be positive. In the case of Ansatz (1) with $\ell = 1$, such conditions are not satisfied since $\det M < 0$, which indicates that there is something wrong with Ansatz (1). In previous works [15, 16, 20, 21], since the authors did not consider the size of the vortex core as a variational parameter, this problem did not appear. Indeed, the problem relies on the fact that the phase of the wave function have to be modified.
In Section II, the necessary requirements for the wave function phase are discussed in order to give support to our variational method. Section III has the calculation based on the new Ansatz and the corresponding equations of motion are obtained. The collective modes considering the coupling between vortex and atomic cloud are obtained via linearization of the equations of motion, thus resulting in new collective oscillations (section IV). In section V, we showed that such excitation modes can be excited using the scattering length modulation. The free expansion was also calculated in order to complement a previous work [21]. Finally, section VII contains the conclusions on our subject of study.

II. WAVE-FUNCTION PHASE

We start with the Lagrangian density,

\[ \mathcal{L} = \frac{i\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{\hbar^2}{2m} |\nabla \psi|^2 - V(\mathbf{r}) |\psi|^2 - \frac{g}{2} |\psi|^4, \]  

(4)

whose extremization leads to the Gross-Pitaevskii equation (GPE):

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g |\psi|^2 \right] \psi, \]  

(5)

where \( V(\mathbf{r}) = \frac{1}{2} m \omega_{\rho}^2 (\rho^2 + \lambda^2 z^2) \) is an external potential, the trap anisotropy is \( \lambda = \omega_z/\omega_{\rho} \), and \( g \) is the coupling constant. The complex field \( \psi(\mathbf{r},t) \) can be written as an amplitude profile multiplied by a respective phase, as follows:

\[ \psi(\mathbf{r},t) = f(\mathbf{r},t) e^{i S(\chi,\mathbf{r})}, \]  

(6)

where

\[ S(\chi,\mathbf{r}) = \ell \varphi + \sum_i \chi_i \phi_i(\mathbf{r}). \]  

(7)

We denoted both, \( w_l = w_l(t) \) and \( \chi_l = \chi_l(t) \), respectively, as the amplitude and phase variational parameters. In principle, \( \{ \phi_i(\mathbf{r}) \} \) should be a complete set of functions but in our present approximation, we use only a representative incomplete set of functions. Substituting (6) and (7) into (4), the Lagrangian \( L = \int \mathcal{L} d^3 \mathbf{r} \) becomes

\[ L = -\hbar \sum_l \chi_l \int d^3 \mathbf{r} f^2 \phi_l - \frac{\hbar^2}{2m} \sum_l \chi_l^2 \int d^3 \mathbf{r} f^2 |\nabla \phi_l|^2 - \int d^3 \mathbf{r} \left( \frac{\hbar^2}{2m} |\nabla f|^2 + V f^2 + \frac{g}{2} f^4 \right). \]  

(8)
In order to account for the dynamics of all three variational parameter in \( f \) we include a variational phase which also contains three variational parameters. This way we chose the following trial function:

\[
S(\rho, z, t) = \ell \varphi + B_\rho(t)\frac{\rho^2}{2} + C(t)\frac{\rho^4}{4} + B_z(t)\frac{z^2}{2}.
\] (9)

As the superfluid current is connected to the density variation, it is desirable that both, amplitude and phase, have the same number of variational parameters. The Ansatz (9) also leads to linearized equations of motion (2) with \( \det M > 0 \) which is consistent with the stability of the condensate with a singly charged vortex at its center.

III. EQUATIONS OF MOTION

Now we correct the Thomas-Fermi Ansatz according to the discussion in section II. This leads to the following trial function:

\[
\psi(r, t) = \frac{N}{R_\rho(t)^2 R_z(t) A_0 [\xi(t) / R_\rho(t)]} \left[ \frac{\rho^2}{\rho^2 + \xi(t)} \right]^{\ell/2} \sqrt{1 - \frac{\rho^2}{R_\rho(t)^2} - \frac{z^2}{R_z(t)^2}} \times \exp \left[ i\ell \varphi + iB_\rho(t)\frac{\rho^2}{2} + iC(t)\frac{\rho^4}{4} + iB_z(t)\frac{z^2}{2} \right],
\] (10)

with

\[
A_0(\alpha) = \frac{2\pi^{3/2} (\ell)!}{15\alpha^{2\ell} (\frac{3}{2} + \ell)!} \left[ (3 + 2\ell\alpha^2) {}_2F_1 \left( \ell, 1 + \ell; \frac{5}{2} + \ell; -\frac{1}{\alpha^2} \right) \right.
\]

\[
-2\ell (1 + \alpha^2) {}_2F_1 \left( 1 + \ell, 1 + \ell; \frac{5}{2} + \ell; -\frac{1}{\alpha^2} \right); \]
\] (11)

where, for simplicity we define \( \alpha(t) = \xi(t) / R_\rho(t) \), \( \rho_\ell F_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) \) are the hypergeometric functions, \( \xi(t) \) is the size of the vortex core, \( R_\rho(t) \) is the condensate size in radial direction (\( \hat{\rho} \)), and \( R_z(t) \) is the condensate size in axial direction (\( \hat{z} \)). The wave function (10) has integration domain defined by \( 1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2} \geq 0 \), where the wave function is approximately an inverted parabola (TF-shape), except for the central vortex. The trapping potential shape sets the condensate dimensions. To organize our calculations, we split the Lagrangian so that
it is a sum $L = L_{\text{time}} + L_{\text{kin}} + L_{\text{pot}} + L_{\text{int}}$ of the following terms:

$$L_{\text{time}} = \frac{i\hbar}{2} \int d^3r \left[ \psi^* (\mathbf{r}, t) \frac{\partial \psi (\mathbf{r}, t)}{\partial t} - \psi (\mathbf{r}, t) \frac{\partial \psi^* (\mathbf{r}, t)}{\partial t} \right]$$

$$= - \frac{\hbar}{2} \left( D_1 \dot{B}_\rho R_\rho^2 + D_2 \dot{B}_z R_z^2 + \frac{1}{2} D_3 \dot{C} R_4^4 \right), \quad (12)$$

$$L_{\text{kin}} = - \frac{\hbar^2}{2m} \int d^3r [\nabla \psi^* (\mathbf{r}, t)] [\nabla \psi (\mathbf{r}, t)]$$

$$= - \frac{\hbar^2}{2m} \left[ D_1 B_\rho^2 R_\rho^2 + D_2 B_z^2 R_z^2 + 2 D_3 B_\rho C R_4^4 + R_\rho^{-2} (\ell^2 D_4 + D_5) + D_6 C^2 R_6^6 \right], \quad (13)$$

$$L_{\text{pot}} = - \frac{1}{2} m \omega_\rho^2 \int d^3r (\rho^2 + \lambda^2 z^2) \psi^* (\mathbf{r}, t) \psi (\mathbf{r}, t)$$

$$= - \frac{N}{2} m \omega_\rho^2 (D_1 R_\rho^2 + \lambda^2 D_2 R_z^2), \quad (14)$$

$$L_{\text{int}} = - \frac{g}{2} \int d^3r [\psi^* (\mathbf{r}, t) \psi (\mathbf{r}, t)]^2$$

$$= - \frac{N^2 g D_7}{2 R_\rho^2 R_z}, \quad (15)$$
with the functions $D_i(\alpha)$ given by

$$D_1(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}}{21\alpha^{2\ell}(\frac{7}{2} + \ell)!} \left[ (3 + 2\ell\alpha^2) \, _2F_1 \left( \ell, 2 + \ell; \frac{7}{2} + \ell; -\frac{1}{\alpha^2} \right) \right. $$

$$- 2\ell \left( 1 + \alpha^2 \right) _2F_1 \left( 1 + \ell, 2 + \ell; \frac{7}{2} + \ell; -\frac{1}{\alpha^2} \right),$$

$$D_2(\alpha) = A_0(\alpha)^{-1} \frac{\pi^{3/2}(\ell)!}{4\alpha^{2\ell}(\frac{7}{2} + \ell)!} \left[ (7 + 2\ell) \, _2F_1 \left( \ell, 1 + \ell; \frac{7}{2} + \ell; -\frac{1}{\alpha^2} \right) \right. $$

$$- (5 + 2\ell) \, _3F_2 \left( \ell, 1 + \ell, \frac{7}{2} + \ell; \frac{5}{2} + \ell, \frac{9}{2} + \ell; -\frac{1}{\alpha^2} \right),$$

$$D_3(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(2 + \ell)!}{27\alpha^{2\ell}(\frac{7}{2} + \ell)!} \left[ (3 + 2\ell) \, _2F_1 \left( \ell, 3 + \ell; \frac{9}{2} + \ell; -\frac{1}{\alpha^2} \right) \right. $$

$$- 2\ell \left( 1 + \alpha^2 \right) _2F_1 \left( 1 + \ell, 3 + \ell; \frac{9}{2} + \ell; -\frac{1}{\alpha^2} \right),$$

$$D_4(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(\ell - 1)!}{3\alpha^{2\ell}(\frac{7}{2} + \ell)!} \left[ (1 - 2\ell\alpha^2) \, _2F_1 \left( \ell, 2 + \ell; \frac{3}{2} + \ell; -\frac{1}{\alpha^2} \right) \right. $$

$$+ 2\ell \left( 1 + \alpha^2 \right) _2F_1 \left( 1 + \ell, 2 + \ell; \frac{3}{2} + \ell; -\frac{1}{\alpha^2} \right),$$

$$D_5(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(\ell - 1)!}{9\alpha^{2\ell}(\frac{7}{2} + \ell)!} \left[ (3 + 2\ell\alpha^2) \, _2F_1 \left( \ell, \frac{3}{2} + \ell; -\alpha^2 \right) \right. $$

$$- 2\ell \left( 1 + \alpha^2 \right) _2F_1 \left( \ell, 1 + \ell; \frac{3}{2} + \ell; -\frac{1}{\alpha^2} \right),$$

$$D_6(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(3 + \ell)!}{33\alpha^{2\ell}(\frac{9}{2} + \ell)!} \left[ (3 + 2\ell\alpha^2) \, _2F_1 \left( \ell, 4 + \ell; \frac{11}{2} + \ell; -\frac{1}{\alpha^2} \right) \right. $$

$$- 2\ell \left( 1 + \alpha^2 \right) _2F_1 \left( 1 + \ell, 4 + \ell; \frac{11}{2} + \ell; -\frac{1}{\alpha^2} \right),$$

$$D_7(\alpha) = A_0(\alpha)^{-2} \frac{2\pi^{3/2}(2\ell)!}{\alpha^{3\ell}(\frac{7}{2} + \ell)!} _2F_1 \left( 2\ell, 1 + 2\ell; \frac{9}{2} + 2\ell; -\frac{1}{\alpha^2} \right).$$

For simplicity we can scale the variational parameters of the Lagrangian as well as the time in order to make them dimensionless,

$$R_\rho(t) \to a_{\text{osc}} r_\rho(t),$$

$$R_z(t) \to a_{\text{osc}} r_z(t),$$

$$\xi(t) \to a_{\text{osc}} \xi(t),$$

$$B_\rho(t) \to a_{\text{osc}}^2 \beta_\rho(t),$$

$$B_z(t) \to a_{\text{osc}}^2 \beta_z(t),$$

$$C(t) \to a_{\text{osc}}^{-4} \zeta(t),$$

$$t \to \omega_{\rho}^{-1} r,$$

where the harmonic oscillator length is $a_{\text{osc}} = \sqrt{\hbar/m\omega_{\rho}}$ and the dimensionless interaction
parameter is $\gamma = Na_s/a_{osc}$. Thus the Lagrangian becomes

$$L = -\frac{N\hbar\omega}{2} \left[ D_1 r^2_\rho \left( \dot{\beta}_\rho + \beta^2_\rho + 1 \right) + D_2 r^2_z \left( \dot{\beta}_z + \beta^2_z + \lambda^2 \right) 
+ D_3 r^4 \left( \frac{1}{2} \dot{\zeta} + 2\beta_\rho \zeta \right) + \ell^2 r^2_\rho \left( D_4 + D_5 \right) + D_6 \zeta^2_\rho r^6_\rho + D_7 \frac{4\pi \gamma}{r^2 r_z} \right].$$  (23)

The Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0,$$  (24)

for each one of the six variational parameters from Lagrangian (23) lead to the six differential equations:

$$\beta_\rho - \frac{\dot{r}_\rho}{r_\rho} - \frac{D'_1 \dot{\alpha}}{2D_1} + \frac{D_3 r^2_\rho \zeta}{D_1} = 0,$$  (25)

$$\beta_z - \frac{\dot{r}_z}{r_z} - \frac{D'_2 \dot{\alpha}}{2D_2} = 0,$$  (26)

$$\zeta - \frac{D_3 \dot{r}_\rho}{D_6 r_\rho} - \frac{D_3 \dot{\alpha}}{4D_6 r^2_\rho} + \frac{D_3 \beta_\rho}{D_6 r^2_\rho} = 0,$$  (27)

$$D_1 r_\rho \left( \dot{\beta}_\rho + \beta^2_\rho + 1 \right) + D_3 r^3_\rho \left( \dot{\zeta} + 4\beta_\rho \zeta \right) - \frac{\ell^2}{r^3_\rho} \left( D_4 + D_5 \right) + 3D_6 \zeta^2_\rho r^5_\rho - D_7 \frac{4\pi \gamma}{r^2 r_z} = 0,$$  (28)

$$D_2 r_z \left( \dot{\beta}_z + \beta^2_z + \lambda^2 \right) - D_7 \frac{2\pi \gamma}{r^2 r_z} = 0,$$  (29)

$$D_1 r^2_\rho \left( \dot{\beta}_\rho + \beta^2_\rho + 1 \right) + D_2 r^2_z \left( \dot{\beta}_z + \beta^2_z + \lambda^2 \right) + D_3 r^4_\rho \left( \frac{1}{2} \dot{\zeta} + 2\beta_\rho \zeta \right) + \frac{\ell^2}{r^3_\rho} \left( D_4' + D_5' \right) + D_6' \zeta^2_\rho r^6_\rho - D_7' \frac{4\pi \gamma}{r^2 r_z} = 0.$$  (30)

Solving these equations for the parameters in the wave function phase, we have:

$$\beta_\rho = \frac{\dot{r}_\rho}{r_\rho} + F_1 \dot{\alpha},$$  (31)

$$\beta_z = \frac{\dot{r}_z}{r_z} + F_2 \dot{\alpha},$$  (32)

$$\zeta = F_3 \frac{\dot{\alpha}}{r_\rho},$$  (33)

where

$$F_1 = \frac{D'_2 D_3 - 2D'_4 D_6}{4 \left( D^2_3 - D_1 D_6 \right)},$$  (34)

$$F_2 = \frac{D'_2}{2D_2},$$  (35)

$$F_3 = \frac{2D'_4 D_3 - D_1 D'_3}{4 \left( D^2_3 - D_1 D_6 \right)}.$$  (36)
Replacing (31), (32), and (33) into equations (28), (29), and (30), we reduce our six coupled equations to only three, which are given by:

\[
D_1 (r'_\rho + r_\rho) + G_1 r_\rho \ddot{\alpha} + G_2 r_\rho \dot{\alpha}^2 + G_3 r_\rho \ddot{\alpha} - G_4 \frac{\ell^2}{r_\rho} - D_7 \frac{4\pi \gamma}{r_\rho^3} = 0, \tag{37}
\]

\[
D_2 \left( \ddot{r}_z + \lambda^2 r_z \right) + G_5 r_z \ddot{\alpha} + G_6 r_z \dot{\alpha}^2 + G_7 \ddot{r}_z \ddot{\alpha} - D_7 \frac{2\pi \gamma}{r_\rho^2} = 0, \tag{38}
\]

\[
D_1' r_\rho (r'_\rho + r_\rho) + D_2' r_z \left( \ddot{r}_z + \lambda^2 r_z \right) + \left( G_8 r_\rho^2 + G_9 r_z^2 \right) \ddot{\alpha} + \left( G_{10} r_\rho^2 + G_{11} r_z^2 \right) \dot{\alpha}^2 + \left( G_{12} r_\rho^2 + G_{13} r_z \dot{r}_z \right) \dot{\alpha} + G_{14} \frac{\ell^2}{r_\rho} + D_7' \frac{4\pi \gamma}{r_\rho^2} = 0, \tag{39}
\]

with

\[
G_1 = D_1 F_1 + D_3 F_3, \tag{40}
\]

\[
G_2 = D_1 \left( F_1^2 + F_1' \right) + D_3 \left( 4F_1 F_3 + F_3' \right) + 3D_6 F_3^2, \tag{41}
\]

\[
G_3 = 2 (D_1 F_1 + D_3 F_3) = 2G_1, \tag{42}
\]

\[
G_4 = D_4 + D_5, \tag{43}
\]

\[
G_5 = D_2 F_2, \tag{44}
\]

\[
G_6 = D_2 \left( F_2^2 + F_2' \right), \tag{45}
\]

\[
G_7 = 2D_2 F_2 = 2G_5, \tag{46}
\]

\[
G_8 = D_1' F_1 + \frac{1}{2} D_3' F_3, \tag{47}
\]

\[
G_9 = D_2' F_2, \tag{48}
\]

\[
G_{10} = D_1' \left( F_1^2 + F_1' \right) + D_3' \left( \frac{1}{2} F_3' + 2F_1 F_3 \right) + D_6' F_3^2, \tag{49}
\]

\[
G_{11} = D_2' \left( F_2^2 + F_2' \right), \tag{50}
\]

\[
G_{12} = 2D_1' F_1 + D_3' F_3, \tag{51}
\]

\[
G_{13} = 2D_2' F_2 = 2G_9, \tag{52}
\]

\[
G_{14} = D_4' + D_5'. \tag{53}
\]

The terms \(D_1 r_\rho, D_2 \lambda^2 r_z, D_1' r_\rho^2,\) and \(D_2' r_z^2\) come from the trapping term \(L_{pot},\) which can be neglected in the case of a freely expanding condensate. The parameter \(\gamma\) indicates the terms generated by the atomic interaction potential, while the fractions proportional to \(r_\rho^{-2}\) and \(r_\rho^{-3}\) come from the kinetic energy contribution due to the presence of the vortex with charge \(\ell.\) The remaining factors represent the coupling between the outer dimensions of the condensate and the vortex core.

Making the velocities \((\dot{r}_\rho, \dot{r}_z, \dot{\alpha})\) and accelerations \((\ddot{r}_\rho, \ddot{r}_z, \ddot{\alpha})\) equal to zero leads to the
which defines the matrices $M$ and $V$, appearing in Eq. (58). Solving the characteristic equation,

$$\det \left( M^{-1}V - \omega^2 I \right) = 0,$$

results in the frequency of the collective modes of oscillation. Now the determinants $\det M$ and $\det V$ are both positive for $\ell = 1$. Meaning that we are in the lower energy state for the case of a central vortex in a Bose-Einstein condensate. Since (58) is a cubic equation of $\omega^2$, we have three pair of frequencies $\pm \omega_n$ ($n = z, \rho, \xi$). There are three frequencies $\omega_n$ and four modes of oscillation in total, of which only three modes can be simultaneously observed depending on the anisotropy $\lambda$ of harmonic potential as shown in fig. 8. Among these four modes, two of them represent monopole oscillations while the other two represent quadrupole oscillations of
Figure 2: Excitation spectrum obtained from a numerical simulation of the GPE \[23\]. We set \(\gamma = 800\), \(\ell = 1\), \(\bar{\mu} = 20.74\), and \(\lambda = 0.9\) \(\omega_n\) are the frequencies of the oscillation modes from less energetic \((\omega_z)\) to more energetic \((\omega_\xi)\). The analytical values are \(\omega_z = 1.317\), \(\omega_\rho = 2.166\), and \(\omega_\xi = 8.874\).

Figure 3: (Color online) Oscillation frequencies as a function of trap anisotropy in a condensate containing a singly (a) and doubly (b) charged vortex at its center. Solid (black) line is \(\omega_\xi\), dashed (red) line is \(\omega_\rho\), and dotted (blue) line is \(\omega_z\).

(a) \(\ell = 1\)  
(b) \(\ell = 2\)

the atomic cloud. The \(B_1\) mode (fig\[4\](a)) is characterized by having all condensate components \(r_i\) \((i = z, \rho, \xi)\) oscillating in phase, however \(B_2\) mode (fig\[4\](c)) presents \(r_\xi\) oscillating out of phase with \(r_\rho\) and \(r_z\). The \(Q_1\) mode (fig\[4\](d)) shows that \(r_z\) oscillation is out of phase with \(r_\xi\) and \(r_\rho\), which are in phase with each other. However, in \(Q_2\) mode (fig\[4\](d)) the oscillations of \(r_z\) and \(r_\xi\) are in phase with each other, being the \(r_\rho\) oscillation out of phase. Extrapolating to an ideal situation where \(\gamma = 0\), the equations of motions \([37]–[39]\) can be decoupled. This way, the \(\omega_z\)
Figure 4: (Color online) Schematic representation of collective modes. $B_1$ mode has all components oscillating in phase. $B_2$ mode has $r_\xi$ oscillating out of phase with $r_\rho$ and $r_z$. $Q_1$ mode has $r_z$ oscillation out of phase with $r_\xi$ and $r_\rho$. $Q_2$ mode has $r_\rho$ oscillation out of phase with $r_\xi$ and $r_z$.

(lower frequency) represents only a $r_z$ oscillation, $\varpi_\rho$ (middle frequency) represents only a $r_\rho$ oscillation, and $\varpi_\xi$ (upper frequency) represents only a $r_\xi$ oscillation.

Numerical simulations were performed in order to validate our results (fig. 2). Frequency values $\varpi_n$ in the variational calculations differ from numerical values by less than 1%.

In fig. 3a, for $0.1 \leq \lambda \leq 1$, exist two $Q_2$-like modes. The difference between them comes from the fact that vortex core oscillation amplitude which is two orders of magnitude lower at the less energetic mode. The same happens when $\ell = 2$ (fig. 3b), i.e., the vortex core is almost still for the lower frequency in the same interval of $\lambda$.

The solid lines in fig. 3 correspond to the mode with largest amplitude for the vortex-core oscillations. As can be seen, the excitation frequency $\varpi_\xi$ of this mode lowers as the vortex circulation increases. It means that the energy necessary to excite it will be lower if $\ell$ is increased. However, we must point out that our results apply only for the cases where $r_\xi \ll r_\rho$.  

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V. SCATTERING LENGTH MODULATION

One of the mechanisms used for exciting collective modes is via modulation of the s-wave scattering length. This technique has been already applied to excite the lowest-lying quadrupole mode in a Lithium experiment \[1\]. Therefore, we consider the time-dependent scattering length:

\[ a_s(t) = a_0 + \delta a \cos (\Omega t) . \] (59)

This is equivalent to make \( \gamma \rightarrow \gamma(\tau) \), thus giving:

\[ \gamma(\tau) = \gamma_0 + \delta \gamma \cos (\Omega \tau) . \] (60)

Where \( \gamma_0 \) is the average value of the interaction parameter \( \gamma(\tau) \), \( \delta \gamma \) is the modulation amplitude, and \( \Omega \) is the excitation frequency. Substituting (60) into (57) and keeping only first-order terms (\( \delta \rho, \delta z, \delta \alpha, \) and \( \delta \gamma \)), we obtain a nonhomogeneous linear equation

\[ M \ddot{\delta} + V \delta = P \cos (\Omega \tau) \] (61)

with

\[ P = 2\pi \delta \gamma \left( \frac{2D_7}{D_7} \right) . \] (62)

A particular solution of (61) is

\[ \delta_\gamma(\tau) = (M^{-1}V - \Omega^2)^{-1} M^{-1}P \cos (\Omega \tau) . \] (63)

Projecting the vector \( \delta_\gamma(\tau) \) in the base \( \delta_n \) \( (n = z, \rho, \xi) \) of the eigenvectors of the homogenous equation associated to Eq.(61), we obtain

\[ \langle \delta_n | \delta_\gamma(\tau) \rangle = \frac{\langle \delta_n | M^{-1}P \rangle}{\omega_n^2 - \Omega^2} \cos (\Omega \tau) . \] (64)

Since \( |\langle \delta_n | M^{-1}P \rangle| > 0 \) (fig.5), it shows that specific collective modes can be excited using a scattering length modulation with small amplitude \( \delta \gamma \) and frequency \( \Omega \) close to one of the resonance frequency \( \omega_n \). In fig.6 we see the results from a numerical solution of Eqs.(37)–(39) considering a time-dependent interaction \( \gamma(\tau) \) according to Eq.(60). There we can see the beat behavior corresponding to a superposition of the frequencies \( \Omega = 6 \) and \( \omega_\xi = 6.13 \).
VI. FREE EXPANSION

The time-of-flight pictures constitute the most common method to measure vortices in BEC. This method of switch off the magneto-optical trap and letting atomic cloud expand freely for some time, typically ten milliseconds and then taking a picture of the expanded cloud \( [5, 24–28] \). For this purpose we use the equations of motion (37)–(39) without the terms arising from the harmonic potential, i.e.,

\[
\begin{align*}
D_1 \dddot{r}_\rho + G_1 r_\rho \ddot{\alpha} + G_2 r_\rho \dot{\alpha}^2 + G_3 \dddot{r}_\rho \dot{\alpha} - \frac{G_4}{r_\rho^3} - \frac{4D_7 \gamma}{r_\rho^7} &= 0, \\
D_2 \dddot{r}_z + G_5 r_z \ddot{\alpha} + G_6 r_z \dot{\alpha}^2 + G_7 \dddot{r}_z \dot{\alpha} - \frac{2D_7 \gamma}{r_\rho^2 r_z^2} &= 0,
\end{align*}
\]

\[
\begin{align*}
D_1' \dddot{r}_\rho' + D_2' \dddot{r}_z' + (G_8 r_\rho'^2 + G_9 r_z'^2) \ddot{\alpha} + (G_{10} r_\rho'^2 + G_{11} r_z'^2) \dot{\alpha}^2 + (G_{12} r_\rho'^2 + G_{13} r_z'^2) \dot{\alpha} + \frac{G_{14}}{r_\rho'^2} + \frac{4D_7' \gamma}{r_\rho'^2 r_z'^2} &= 0,
\end{align*}
\]

whose initial conditions are given by the stationary equations (54)–(56). This result agrees with our previous work \([21]\), where the free expansion of the vortex core is given in fig.7. In \([21]\), the fig.7b could not be calculated since the authors considered the healing length as an approximation to the vortex core radius which is only valid for \( \ell = 1 \).

VII. CONCLUSIONS

In this paper we proposed a modification in the wave function phase commonly used with
Figure 6: Numerical solution of Eqs.(37)–(39) with a time-dependent interaction $\gamma(\tau)$ (a, b, and c). (d) is the excitation spectrum obtained from variational calculation, where $\omega_\xi \approx 6.13$ is close to the value calculated in Eq.(58). We excited the collective mode $Q_2$ ($\omega_\xi = 6.21$) of a condensate with cigar shape ($\lambda = 0.1$, $\gamma_0 = 800$) via scattering modulation with amplitude $\delta \gamma = 0.4$ and frequency $\Omega = 6$.

Figure 7: (Color online) Ratio between vortex core size and radial cloud size for different trap anisotropies while in free expansion. The solid (black) line corresponds to a prolate condensate ($\lambda = 0.1$), the dashed (blue) line to the isotropic case ($\lambda = 1$), and the dotted (red) line to an oblate condensate ($\lambda = 8$).
the variational method which corrects the imaginary frequencies of collective modes when we have a parameter describing non-physical vortex core dynamics with $\ell = 1$.

Here we consider variational phase parameters corresponding to each parameter in wave function amplitude, respectively. This way, we were able to describe the dynamics of both vortex core and the external dimensions of the condensate which agrees with the numerical simulations of the GPE. Although we observe four modes of oscillation in total, only three of them can be simultaneously observed depending on the trap anisotropy.

We also demonstrate that these oscillation modes can be excited by modulating the s-wave scattering length using the same experimental techniques as in Ref. [1].

Finally, we analyzed the time-of-flight dynamics of the vortex core with different circulations in order to complement the results in Ref. [21].

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