Domination in transformation graph $G^+-+$

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Abstract

Let $G = (V, E)$ be a simple undirected graph of order $n$ and size $m$. The transformation graph of $G$ is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are non-adjacent in $G$ (b) two elements in $E(G)$ are adjacent if and only if they are adjacent in $G$ and (c) one element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are non-incident in $G$. It is denoted by $G^+-+$. In this paper, we investigate the domination number of transformation graph. We prove that $\gamma(G^+-+) \leq 3$ and characterise the graphs for which this number is 1, 2 or 3.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph of order $n$ and size $m$. If $v \in V(G)$, then the neighbourhood of $v$ is the set $N(v)$ consisting of all vertices which are adjacent to $v$. The closed neighbourhood is $N[v] = N(v) \cup \{v\}$. The degree of $v$ in $G$ is $|N(v)|$ and is denoted by $deg(v)$. The maximum degree of $G$ is max $\{deg(v) : v \in V(G)\}$ and is denoted by $\Delta(G)$. The diameter of $G$ is the maximum distance between any two vertices of $G$ and is denoted by $diam(G)$. A subgraph $F$ of a graph $G$ is called an induced subgraph of $G$
if whenever \( u \) and \( v \) are vertices of \( F \) and \( uv \) is an edge of \( G \), then \( uv \) is an edge of \( F \) as well.

A set \( S \subseteq V(G) \) is a dominating set if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \). The minimum cardinality taken over all dominating sets of \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A set \( S' \subseteq E(G) \) is an edge dominating set of \( G \) if every edge in \( E - S' \) is adjacent to at least one edge in \( S' \). The minimum cardinality taken over all edge dominating sets of \( G \) is called the edge domination number of \( G \) and is denoted by \( \gamma'(G) \). An edge dominating set \( S' \) is said to be an independent edge dominating set if no two edges are adjacent in \( \langle S' \rangle \). The minimum cardinality taken over all independent edge dominating sets of \( G \) is called the independent edge domination number of \( G \) and is denoted by \( \gamma_I(G) \).

The transformation graph of \( G \) is a simple graph with vertex set \( V(G) \cup E(G) \) in which adjacency is defined as follows: (a) two elements in \( V(G) \) are adjacent if and only if they are non-adjacent in \( G \) (b) two elements in \( E(G) \) are adjacent if and only if they are adjacent in \( G \) and (c) one element in \( V(G) \) and one element in \( E(G) \) are adjacent if and only if they are non-incident in \( G \). It is denoted by \( G^{++} \). A graph \( G \) and it’s transformation graph are illustrated in Figure 1.1.

In [7], Xu and Wu studied about connectivity and independence number of \( G^{++} \). They also obtained a necessary and sufficient condition for \( G^{++} \) to be hamiltonian. Several authors [1,2,4,6], have studied domination parameters for different classes of graphs. In this paper we study about domination number in the transformation graph \( G^{++} \). Terms not
defined are used in the sense of [3]. For the graph $G$ in Figure 1.1, $u_1$ is an isolated vertex in $G$ but it is full vertex in $G^{-+}$ and hence $\gamma(G^{-+}) = 1$. For the graph $C_5 + e$ in Figure 1.2, $e_1$ and $e_5$ are independent edges and they form dominating set in $G^{-+}$. Hence $\gamma((C_5 + e)^{-+}) = 2$.

For the graph $K_{1, 4}$ in Figure 1.3, $\{u_1, u_2, e_1\}$ is a minimum dominating set in $K_{1, 4}^{-+}$ and hence $\gamma((K_{1, 4})^{-+}) = 3$.

![Figure 1.2](image1.png)

![Figure 1.3](image2.png)
2. Main results

Theorem 2.1. For any graph $G$, $\gamma(G^{++}) \leq 3$.

Proof. Let $u, v \in V(G)$ and $e = uv \in E(G)$. We claim that $D = \{u, v, e\}$ is a dominating set of $G^{++}$. $e$ dominates all the vertices except $u$ and $v$ and all the edges incident with $u$ and $v$. $u$ dominates itself and all the edges which are not incident with $u$ and $v$ dominates itself. Hence $D$ is a dominating set in $G^{++}$ and $|D| = 3$. Therefore $\gamma(G^{++}) \leq 3$.

Theorem 2.2. For any graph $G$, $\gamma(G^{++}) = 1$ if and only if $G$ has an isolated vertex.

Proof. Assume that $\gamma(G^{++}) = 1$. Suppose $G$ has no isolated vertex. Then every vertex $v$ is incident with some $e = uv$. Now $v$ will not be adjacent to $u$ and $e$ in $G^{++}$ and $e$ will not be adjacent to $u$ and $v$ in $G^{++}$. Hence $D \neq \{e\}$ and $D \neq \{v\}$. Thus $\gamma(G^{++}) \geq 2$ which is a contradiction. Therefore $G$ has an isolated vertex. Conversely, assume that $G$ has an isolated vertex say, $v$. Then by definition of $G^{++}$, it is adjacent to all the elements in $V(G) \cup E(G)$ in $G^{++}$. So $D = \{v\}$ is a dominating set. Hence $\gamma(G^{++}) = 1$.

Theorem 2.3. For any connected graph $G$, $\gamma(G^{++}) \geq 2$.

Proof. Let $D$ be a minimum dominating set of $G^{++}$. Since each element of $V(G)$ is incident with at least one element of $E(G)$ and each element of $E(G)$ is incident with two elements of $V(G)$, neither a single vertex nor a single edge of $G$ can be a dominating set in $G^{++}$. Hence $\gamma(G^{++}) \geq 2$.

Theorem 2.4. If a connected graph $G$ has at least two pendant edges with no common vertex, then $\gamma(G^{++}) = 2$.

Proof. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be pendant edges with no common vertex of $G$ and $v_1$ and $v_2$ be pendant vertices. By Theorem 2.3, $\gamma(G^{++}) \geq 2$. Now $v_1$ is adjacent to all the elements of $G^{++}$ except $e_1$ and $u_1$; $v_2$ is adjacent to $e_1$ and $u_1$. Therefore $\{v_1, v_2\}$ is a dominating set of $G^{++}$ and so $\gamma(G^{++}) = 2$.

Corollary 2.5. $\gamma(P_n^{++}) = 2$, $n > 3$.

Theorem 2.6. $\gamma(C_n^{++}) = 2$, $n > 3$. 
Proof. Let $C_n = v_1e_1v_2e_2...v_{n-1}e_{n-1}v_ne_n$ be a cycle on $n$ vertices and $D$ be a dominating set of $C_{n^{-+}}$. Since $C_n$ is a connected graph, by Theorem 2.3, $|D| \geq 2$. If $n = 4$ or $5$, then any two non-adjacent edges of $G$ dominates all the vertices of $C_{n^{-+}}$. If $n \geq 6$, then in $C_{n^{-+}}$, $v_1$ is adjacent to all the elements of $V(C_n) \cup E(C_n)$ except $v_2,v_n,e_1$ and $e_n$ and $v_4$ is adjacent to $v_2,v_n,e_1$ and $e_n$. Then $D = \{v_1,v_4\}$ is a dominating set of $C_{n^{-+}}$. Hence $\gamma(C_{n^{-+}}) = |D| = 2$.

**Theorem 2.7.** Let $G$ be an isolate free graph of order 4 or 5. If $G$ is not isomorphic to a star, then $\gamma(C_{n^{-+}})$.

Proof. Since $G$ is isolate free graph, by Theorem 2.2 $\gamma(G) \neq 1$.

If $G$ is disconnected, then it has two components only say, $C_1$ and $C_2$. Any one vertex $u \in V(C_1)$ dominates all the vertices and edges of $C_2$ and any one vertex $v \in V(C_2)$ dominates all the vertices and edges of $C_1$. Thus $\{u,v\}$ is a dominating set in $G^{-+}$.

If $G$ is connected, by hypothesis, there are two independent edges $e_1$ and $e_2$ of $G$. Then every edge of $G$ is adjacent to any one of these two edges and every vertex of $G$ is not incident with at least one of these edges. Therefore $\{e_1,e_2\}$ is a dominating set in $G^{-+}$. Thus $\gamma(G^{-+}) = 2$.

**Lemma 2.8.** For any graph $G$, the following 2-element sets are not dominating sets in $G^{-+}$.

(i) two adjacent vertices of $G$.

(ii) two adjacent edges of $G$.

(iii) one edge and one of its end vertices in $G$.

Proof. Let $S = \{x,y\}$ be any 2-element set in $V(G^{-+}) = V(G) \cup E(G)$.

(i) If $x$ and $y$ are adjacent vertices of $G$, then the edge $xy \in E(G)$ is adjacent to neither $x$ nor $y$ in $G^{-+}$.

(ii) If $x$ and $y$ are adjacent edges of $G$, then they are incident with a common vertex, say $v \in V(G)$. But $v$ is adjacent to neither $x$ nor $y$ in $G^{-+}$.

(iii) If $x = uv \in E(G)$ and $y = u \in V(G)$, then $v$ is adjacent to neither $x$ nor $y$ in $G^{-+}$.
Thus in all cases, $S$ is not a dominating set in $G^{++}$.

**Lemma 2.9.** If $diam(G) = 2$, then the following 2-element sets are not dominating sets in $G^{++}$.

(i) two non-adjacent vertices of $G$.

(ii) one edge and a vertex other than the end vertices of the edge.

**Proof.** Let $S = \{x, y\}$ be any 2-element set in $V(G^{++})$.

(i) If $x$ and $y$ are non-adjacent vertices of $G$, then $diam(G) = 2$ implies that $d(x, y) = 2$ and hence $x$ and $y$ are adjacent to a common vertex $z$ in $G$. Then $z$ is adjacent to neither $x$ nor $y$ in $G^{++}$.

(ii) If $x \in E(G), y \in V(G)$ and $y$ is not an end vertex of $x$, then let $x = x'x''$.

Since $diam(G) = 2$, any one of the following is true.

(a) $y$ is adjacent to at least one of the end vertices of $x$.

(b) $y$ is adjacent to a vertex $y'$ which is adjacent to $x'$ and $x''$.

(c) $N(y) = N(x') \cup N(x'') - \{x', x''\}$.

If (a) is true, then both $y$ and $x$ are not adjacent to $x'$ in $G^{++}$. If (b) is true, then $yy'$ is adjacent to neither $x$ nor $y$ in $G^{++}$. If (c) is true, then the edges incident with $y$ in $G$ are adjacent to neither $y$ nor $x$ in $G^{++}$.

Thus in all the cases, $S$ is not a dominating set in $G^{++}$.

**Theorem 2.10.** Let $G$ be a connected graph with $diam(G) = 2$ and $n \geq 4$. Then $\gamma(G^{++}) = 2$ if and only if $\gamma'_i(G) = 2$.

**Proof.** Assume that $\gamma(G^{++}) = 2$. Then let $D = \{x, y\}$ be a dominating set of $G^{++}$. Since $diam(G) = 2$, by Lemmas 2.8 and 2.9, $x$ and $y$ are two independent edges of $G$ and $D$ dominates all the edges of $G$. Thus $D$ is a independent edge dominating set of $G$ and hence $\gamma'_i(G) \leq 2$. Since $n \geq 4$, there is no edge which dominates all the edges. Hence $\gamma'_i(G) \neq 1$. Thus $\gamma'_i(G) = 2$.

Conversely, let $D' = \{e_1, e_2\}$ be minimum independent edge dominating set of $G$. Since $D'$ is a edge dominating set of $G, D'$ dominates all the edges of $G$ in $G^{++}$. Since $e_1$ and $e_2$ are independent edges, all the vertices except the end vertices of $e_1$ are dominated by $e_1$ and the end vertices of $e_1$ are dominated by $e_2$. Therefore $\{e_1, e_2\}$ is a dominating set of $G^{++}$ and hence $\gamma(G^{++}) = 2$. 

Theorem 2.11. If $\text{diam}(G) \geq 3$, then $\gamma(G^{-+}) = 2$.

Proof. Let $\text{diam}(G) \geq 3$. Then there exist two vertices $u, v$ such that $d(u, v) = 3$. Suppose $u$ does not dominate a vertex $v_i \in N(v)$ in $G^{-+}$. Then $u$ is adjacent to $v_i$ in $G$. Then $(u, v, i)$ is a $u, v$-path of length two in $G$ which contradicts that $d(u, v) = 3$. Therefore $u$ dominates $N[v]$. Similarly $v$ dominates $N[u]$. Suppose $u$ does not dominate an edge $e$ which is incident with $v$ in $G$. Then $e = uv$ which contradicts the choice of $u$ and $v$. So $u$ dominates all the edges incident with $v$. Similarly $v$ dominates all the edges incident with $u$. All other vertices and edges are dominated by both $u$ and $v$. Therefore $\{u, v\}$ is a dominating set in $G^{-+}$. Thus $\gamma(G^{-+}) = 2$.

Theorem 2.12. $\gamma(K_{1,r}^{++}) = 3$, for any positive integer $r$.

Proof. If $r = 1$, then $K_{1,1}^{++} \cong \overline{K}_3$ and hence $\gamma(K_{1,1}^{++}) = 3$. If $r > 1$, let $v_1, v_2, \ldots, v_r$ be the pendant vertices and $e_1, e_2, \ldots, e_r$ be the corresponding pendant edges and $v$ be the vertex of full degree in $K_{1,r}$. Let $D$ be dominating set of $K_{1,r}^{++}$. Since $v$ is adjacent to all the vertices and incident with all the edges of $K_{1,r}, v$ is an isolated vertex in $K_{1,r}^{++}$. Therefore $D$ contains $v$ and hence $|D| \geq 2$. Further neither $\{v, v_i\}$ nor $\{v, e_i\}$ is a dominating set of $K_{1,r}^{++}$, and hence $|D| \neq 2$. Since $v_i$ is adjacent to all the elements of $V(K_{1,r}) \cup E(K_{1,r})$ except $e_i$ and $v$ and $v_j (j \neq i)$ is adjacent to $e_i$ in $K_{1,r}^{++}$. $\{v, v_i, e_i\}$ is a dominating set of $K_{1,r}^{++}$ and hence $\gamma(K_{1,r}^{++}) = 3$.

Theorem 2.13. If $W_n$ denotes wheel on $n$ vertices, then and $\gamma(W_6^{-+}) = 2$ and $\gamma(W_n^{-+}) = 3, n \geq 7$.

Proof. If $n = 6$, then any two non-adjacent edges in the rim form a dominating set for $W_6^{-+}$ and hence $\gamma(W_6^{-+}) = 2$. For $n \geq 7$, let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of degree 3 and $v$ be the vertex of degree $n - 1$ in $W_n$. Let $e_i = v v_i$, for $i = 1, 2, \ldots, n - 1$ and $e_{ij} = v_i v_j$. Let $D$ be a dominating set of $W_n^{-+}$. Since $W_n$ is connected, by Theorem 2.3, $\gamma(W_n^{-+}) \geq 2$. Since $\text{diam}(W_n) = 2$, by Lemmas 2.8 and 2.9, it is enough to verify that no two independent edges of $W_n$ is a dominating set. If the two independent edges are from the rim of the wheel, then $n \geq 7$ implies that there exists a spoke which is not adjacent to any of them. If the two independent edges consist of one spoke and one edge of the rim, then there exists an edge of the rim which is not adjacent to any of them. Hence $\gamma(W_n^{-+}) \geq 3$. Thus by Theorem 2.1, $\gamma(W_n^{-+}) = 3$. 
It is an easy exercise to see that $\gamma(K_3^{++}) = 3$. Since any two independent edges in $K_4$ and $K_5$ form a dominating set in the corresponding transformation graph, we get $\gamma(K_4^{++}) = 2 = \gamma(K_5^{++})$.

**Theorem 2.14.** For $n > 5$, $\gamma(K_n^{++}) = 3$.

**Proof.** Since $K_n$ has no isolated vertex, by Theorem 2.2, $\gamma(K_n^{++}) \neq 1$.

**Claim:** No two elements of $V(K_n) \cup E(K_n)$ can be a dominating set in $K_n^{++}$.

Let $A = \{x, y\} \subset V(K_n) \cup E(K_n)$. By Lemma 2.8, no two vertices, no two adjacent edges and no set containing one edge and one of it’s end vertices form a dominating set in $K_n^{++}$. Since $n > 5$, there exist two vertices $v_i, v_j$ such that $e = v_i v_j$ is not adjacent to any two independent edges of $K_n$. Therefore no two independent edges of $G$ is a dominating set of $G^{++}$. If $x \in E(K_n)$ and $y \in V(K_n)$ such that $y$ is not an end vertex of $x$, then the end vertices of $x$ are adjacent to neither $x$ nor $y$ in $K_n^{++}$. Then no set containing one vertex and one edge of $G$ is a dominating set of $G^{++}$. Hence the claim and $\gamma(K_n^{++}) \geq 3$. By Theorem 2.1, $\gamma(K_n^{++}) \leq 3$. Hence $\gamma(K_n^{++}) = 3$.

**3. Graphs of diameter two**

If $\text{diam}(G) = 2$, then $\gamma(G^{++}) = 2$ or 3. In this section, we characterize such graphs. We need the following.

**Notation:** Let $T_x \cong K_{1,r}$, $T_y \cong K_{1,s}$ and $T_z \cong K_{1,t}$ be three disjoint stars with centers $x, y$ and $z$ respectively. Then the graph obtained from $T_x \cup T_y$ by joining $x$ to one or more vertices of $T_y$ or $y$ to one or more vertices of $T_x$ is denoted by $T_{xy}$. The graph of the form $T_{xy}$ in which $x$ and $y$ are adjacent is denoted by $T_{xy}$. The graph obtained from $T_{x+y} \cup T_z$ by joining edges from $x$ or $y$ to one or more vertices of $T_z$ or $z$ to one or more vertices of $T_{x+y}$ is denoted by $T(x+y)z$. Let $T_x = K_{1,2}, T_y = K_{1,3}$ and $T_z = K_{1,4}$. Then $T_{xy}$ and $T(x+y)z$ are given in Figure 3.1.

**Theorem 3.1.** Let $G$ be a graph of order $n \geq 6$ with $\text{diam}(G) = 2$ and let $v$ be a vertex of degree $\Delta = n - 1$. Then $\gamma(G^{++}) = 2$ if and only if $\langle N(v) \rangle \cong H \cup mK_1$ where $m \geq 0$ and $H \in \{T_x, T_x \cup T_y, T_{xy}, T_{x+y} \cup T_z, T(x+y)z / x, y, z \in N(v)\}$. 


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**Proof.** Let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$ and $e_i$ be the edge joining $v$ and $v_i$ and $e_{ij}$ be the edge joining $v_i$ and $v_j$. Assume that $\langle N(v) \rangle \cong H \cup mK_1$. Since $G$ has no isolated vertex, by Theorem 2, $\gamma(G^{-\rightarrow}) \neq 1$. Then for $i \neq j \neq k$ \{\{e_{ij}, e_{ik}\}$ is a dominating set of $G^{-\rightarrow}$. Therefore $\gamma(G^{-\rightarrow}) = 2$.

Conversely, assume that $\gamma(G^{-\rightarrow}) = 2$. Let $D$ be a minimum dominating set of $G^{-\rightarrow}$. By Lemmas 2.8 and 2.9, $D$ contains only two independent edges of $G$. Since $\gamma((K_1,r)^{-\rightarrow}) = 3$, $G \neq K_{1,r}$. Hence there exists at least one edge $e_{ij} \in E(G)$. Further since $n \geq 6$, without loss of generality, let $D = \{e_{ij}, e_{ik}\}$. Then any edge in $\langle N(v) \rangle$ is incident with $v_i, v_j$ or $v_k$. Let $S = \{v_i, v_j, v_k\}$.

**Case (i):** Only one vertex of $S$ is of degree greater than or equal to 3.

If $v_i$ or $v_j$ is that vertex, then $\langle N(v) \rangle \cong T_x \cup mK_1$. If $\deg(v_k) \geq 3$ then $\langle N(v) \rangle \cong K_2 \cup T_x \cup mK_1$.

**Case (ii):** Only two vertices of $S$ are of degree greater than or equal to 3.

Let $v_i$ and $v_j$ be the vertices of degree greater than or equal to 3. If $\deg(v_i) = 1$, then $\langle N(v) \rangle \cong T_{v_i + v_j} \cup mK_1$. If $\deg(v_k) = 2$, then $\langle N(v) \rangle \cong T_{v_i + v_j} \cup K_2 \cup mK_1$.

Now let $v_i$ and $v_k$ are of degree greater than or equal to 3. If $v_i$ is not adjacent to any vertex of $N[v_k]$ and $v_k$ is not adjacent to any vertex of $N[v_i]$, then $\langle N(v) \rangle \cong T_{v_i} \cup T_{v_k} \cup mK_1$; otherwise $\langle N(v) \rangle \cong T_{v_i} v_k \cup mK_1$.

**Case (iii):** All the vertices of $S$ are of degree greater than or equal to 3.
If \( v_i \) and \( v_j \) are not adjacent to any vertex of \( N[v_k] \) and \( v_k \) is not adjacent to any vertex of \( N[v_i] \cup N[v_j] \), then \( \langle N(v) \rangle \cong T_{x+y} \cup T_z \cup mK_1 \); otherwise \( \langle N(v) \rangle \cong T_{(x+y)z} \cup mK_1 \).

**Case (iv):** No vertex of \( S \) is of degree greater than or equal to 3.

If \( v_i, v_j \) is the only edge of \( \langle N(v) \rangle \), then \( \langle N(v) \rangle \cong K_2 \cup mK_1 \). If degree of \( v_i, v_j \) and \( v_k \) are 2, then \( \langle N(v) \rangle \cong 2K_2 \cup mK_1 \).

Thus \( \langle N(v) \rangle \cong H \).

**Theorem 3.2.** Let \( G \) be a graph of order \( n \geq 6 \) with \( \text{diam}(G) = 2 \) and \( v \) be a vertex of degree \( \Delta = n - 2 \) and \( u \notin N[v] \). Then \( \gamma(G^{+-}) = 2 \) if and only if \( \langle N(v) \rangle \cong H \cup mK_1 \), where \( m \geq 0 \), \( H \in \{K_n, T_x, T_x \cup T_y, T_{xy}, T_{x+y} \cup T_z, T_{(x+y)z} / x, y, z \in N(u)\} \).

**Proof.** Let \( N(v) = \{v_1, v_2, ..., v_\Delta\} \). Let \( e_i \) be the edge joining \( v \) and \( v_i \) and \( f_i \) be the edge joining \( u \) and \( v_i \) and \( e_{ij} \) be the edge joining \( v_i \) and \( v_j \). Assume that \( \langle N(v) \rangle \cong H \cup mK_1 \). Since \( G \) has no isolated vertex, by Theorem 2.2, \( \gamma(G^{+-}) \neq 1 \). If \( H = K_n, T_x, T_{xy}, T_{(x+y)z} / x, y, z \in N(u) \), then \( \langle e_i, f_j \rangle \) is a dominating set of \( G^{+-} \). If \( H = T_{v_i}, T_{v_j}, T_{v_i} \cup T_{v_j}, T_{v_i+v_j} \cup T_{(x+y)z} / x, y, z \in N(u) \), then \( \langle v_i, v_j, e_k \rangle \) is a dominating set of \( G^{+-} \). Therefore \( \gamma(G^{+-}) = 2 \).

Conversely, assume that \( \gamma(G^{+-}) = 2 \). By Lemmas 2.8 and 2.9, any minimum dominating set \( D \) in \( G^{+-} \) contains only two independent edges of \( G \). Since \( n \geq 6 \), \( D \neq \{v_i, v_j, f_k\} \).

**Case (i):** \( D = \{e_i, v_j, v_k\} \).

Let \( S = \{v_i, v_j, v_k\} \) and \( S_1 = N(v) - S \).

**Subcase (a):** \( |N(S_1) \cap S| = 1 \).

We claim that \( N(S_1) \cap S = \{v_j\} \). Suppose \( N(S_1) \cap S = \{v_j\} \). Since \( \text{diam}(G) = 2 \), \( v_j \) must be adjacent to \( u \). Hence \( \text{deg}(v_j) = n - 1 \) which is a contradiction to \( \Delta(G) \). If \( N(S_1) \cap S = \{v_k\} \), then also we get contradiction. Hence the claim. Therefore \( \langle N(v) \rangle \cong T_{v_i} \cup K_2 \).

**Subcase (b):** \( |N(S_1) \cap S| = 2 \).

Suppose \( N(S_1) \cap S = \{v_i, v_j\} \). If \( N[v_i] \cap N[v_j] = \emptyset \) in \( \langle N(v) \rangle \), then \( \langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} \); otherwise \( \langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} \). If \( N(S_1) \cap S = \{v_j, v_k\} \), then \( \langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} \).
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Subcase (c): $N(S_1) \cup S = S$.

If $N[v_i] \cap N[v_j] = \emptyset$ and $N[v_j] \cap N[v_k] = \emptyset$ in $\langle N(v) \rangle$, then $\langle N(v) \rangle \cong T_{v_i} \cup T_{v_j+v_k}$; otherwise $\langle N(v) \rangle \cong T_{(v_j+v_k)v_i}$.

Case (ii): $D = \{e_i, f_j\}$.

Then $v_i$ and $v_j$ may be adjacent to vertices of $N(v)$. If $v_i$ and $v_j$ are not adjacent to vertices of $N(v)$, then $\langle N(v) \rangle \cong K_{\Delta}$. If any one of $v_i$ and $v_j$ is adjacent to vertices of $N(v)$, then $\langle N(v) \rangle \cong T_v \cup mK_1$. If $N(v_i) \neq \emptyset$, $N(v_j) \neq \emptyset$ and $N[v_i] \cap N[v_j] = \emptyset$ in $\langle N(v) \rangle$, then $\langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} \cup mK_1$. Further $N[v_i] \cap N[v_j] \neq \emptyset$, in $\langle N(v) \rangle$, then $\langle N(v) \rangle \cong T_{(v_j+v_k)v_i}$.

Case (iii): $D = \{v_iv_j, v_kv_j\}$.

Then $n = 6$ and $\langle N(v) \rangle \cong 2K_2$ or $T_{v_jv_k}$.

Hence $\langle N(v) \rangle \cong H$.

**Theorem 3.3.** Let $G$ be a graph with $diam(G) = 2$ and $v$ be a vertex of degree $\Delta = n - 3$ and $u_1, u_2 \in N[v]$ such that $N(v) = N(u_1) = N(u_2)$. Then $\gamma(G^{-+-}) = 2$ if and only if $n = 7$ and $\langle N(v) \rangle \cong 2K_2$.

**Proof.** Assume that $n = 7$ and $\langle N(v) \rangle \cong 2K_2$. Since $n = 7, \Delta = 4$. Let $v_1, v_2, v_3$ and $v_4$ be the only neighbours of $v$ and let $v_1v_2$ and $v_3v_4$ be the independent edges in $\langle N(v) \rangle$. $v_1v_2$ dominates all the vertices except $v_1$ and $v_2$ and all the edges incident with $v_1$ and $v_2$. But $v_3v_4$ dominates $v_1$ and $v_2$ and all the edges incident with $v_3$ and $v_4$. Hence $\gamma(G^{-+-}) = 2$.

Conversely, assume that $\gamma(G^{-+-}) = 2$. By hypothesis, $n \geq 6$.

**Claim:** $n = 7$ and $\langle N(v) \rangle$ contains two independent edges.

Suppose $\langle N(v) \rangle$ does not contain any edge. By Lemmas 2.8 and 2.9, any minimum dominating set $D$ in $G^{-+-}$ contains only two independent edges of $G$. Since $n \geq 6$, $vv_k$ is not dominated by any independent edges of the form $\{u_1v_i, u_2v_j\}$ ($i$ and $j \neq k$). All the edges incident with $u_2$ are not dominated by any independent edge set of the form $\{u_1v_i, vv_j\}$ and all the edges incident with $u_1$ are not dominated by any independent edge set of the form $\{u_2v_i, vv_j\}$. Therefore $\gamma(G^{-+-}) > 2$ which is a contradiction. Therefore $\langle N(v) \rangle$ contains at least one edge. Suppose $\langle N(v) \rangle$ contains
exactly one edge say \( v_i, v_j \). Then \( n \neq 6 \) and since there exists \( v_i \in N(v) \) such that \( l \notin \{i, j, k\} \), \( u_1 v_i, u_2 v_i, v_i v_l \) are not dominated by \( \{v_l v_i, v_i v_j\}, \{u_1 v_k, v_i v_j\} \) and \( \{u_2 v_k, v_i v_l\} \) respectively. Hence \( \gamma(G^{++}) > 2 \) which is a contradiction. Therefore \( \langle N(v) \rangle \) has two or more edges. Let \( v_i v_j \) and \( v_l v_j \) be the edges in \( \langle N(v) \rangle \). If \( n > 7 \), then there exists \( v_r \in N(v) \), where \( r \notin \{i, j, k, l\} \) such that \( v_r \) is not dominated by \( \{v_i v_j, v_i v_l\} \). Therefore \( n = 7 \). If \( v_j v_l \) and \( v_i v_k \) are adjacent edges in \( \langle N(v) \rangle \), then \( 4 = \text{deg}(v) < \text{deg}(v_j) = 5 \) which contradicts that \( \text{deg}(v) = \Delta \). Hence the claim and proof.

**Notation:** Let \( A = \{a_i/a_i = vv_i\} \) be the set of edges incident with \( v \), \( B = \{b_{ij}/b_{ij} = vv_i\} \) be the set of edges in \( \langle N(v) \rangle \) and \( C = \{u_i u_j, c_{kl}/c_{kl} = u_k v_l\} \) be the set of edges incident with the vertices of \( V - N[v] \).

**Theorem 3.4.** Let \( G \) be a graph with \( \text{diam}(G) = 2 \) and \( v \) be a vertex of degree \( \Delta = n - 3 \). Let \( u_1, u_2 \in V(G) - N[v] \) such that \( u_1 \) and \( u_2 \) are adjacent. Then \( \gamma(G^{++}) = 2 \) if and only if \( \langle N(v) \rangle \cong H \cup mK_1 \), where \( m \geq 0 \) and \( H \in \{K_\Delta, T_x, T_x \cup T_y, T_{xy}\} \) where \( N(u_i) \cap N(v) \subseteq \{x, y\} \) for some \( i \).

**Proof.** Let \( N(v) = \{v_1, v_2, \ldots, v_\Delta\} \). Assume that \( \langle N(v) \rangle \cong H \cup mK_1 \) and let \( N(u_1) \cap N(v) \subseteq \{x, y\} \). Then \( |N(u_1) \cap N(u_2)| \leq 2 \). Since \( G \) has no isolated vertex, by Theorem 2.2, \( \gamma(G^{++}) \neq 1 \). If \( H \cong K_\Delta \) or \( T_{vi} \), then every edge in \( A \) and every edge of \( B \) which is adjacent to \( v_i \) are adjacent to \( vv_i \) and every edge of \( C \) is adjacent to \( u_1 u_2 \). Therefore \( \{u_1 u_2, vv_i\} \) is a \( \gamma' \)-set of \( G \) and hence a dominating set of \( G^{++} \). If \( H \cong T_{vi} \cup T_{vj} \) or \( T_{vi} v_j \), then every edge in \( A \), every edge of \( B \) which is incident with \( v_i \) and \( u_1 v_i \in C \) are adjacent to \( vv_i \) and all other edges are incident with \( u_2 \) and \( v_j \), \( \{vv_i, u_2 v_j\} \) is a dominating set of \( G^{++} \). Hence \( \gamma(G^{++}) = 2 \).

Conversely, assume that \( \gamma(G^{++}) = 2 \). By hypothesis, \( n \geq 5 \). By Lemmas 2.8 and 2.9, any dominating set \( D \) of \( G^{++} \) contains only two independent edges of \( G \). Since \( u_1 u_2 \in C \) is not adjacent to any element of \( B \), no two edges of \( B \) can form a dominating set in \( G^{++} \). For the same reason, one edge of \( A \) and one edge of \( B \) can not serve the purpose. Then we consider the following three cases.

**Case (i):** \( D \) contains two edges of \( C \)

Since two edges are independent, \( u_1 u_2 \notin D \). Then \( n = 5 \) and \( \langle N(v) \rangle \cong K_2 \).

**Case (ii):** \( D \) contains one edge of \( B \) and one edge of \( C \).
Clearly \( n > 5 \) and \( \text{deg}(v) \geq 3 \). Let \( b_{ij} = v_i v_j \in D \). Then \( u_1 u_2 \notin D \). Since there exists some \( a_k \) which is not dominated by \( b_{ij}, u_1 v_k \) or \( u_2 v_k \in D \). Then \( n = 6 \) and \( \langle N(v) \rangle \equiv K_1,1 \).

**Case (iii):** \( D \) contains one edge of \( A \) and one edge of \( C \).

Let \( a_i = vv_i \in D \).

**Subcase (a):** \( u_1 u_2 \in D \)

In this case, all the edges in \( C \) are adjacent to \( u_1 u_2 \) and all other edges are adjacent to \( a_i \) and hence all the edges of \( B \) are incident with \( v_i \). If \( B \) is empty, then \( \langle N(v) \rangle \equiv \overline{K_\Delta} \). Otherwise \( \langle N(v) \rangle \equiv T_{v_i} \cup mK_1, m \geq 0 \).

**Subcase (b):** \( u_1 u_2 \notin D \).

Without loss of generality, let \( u_2 v_j \in D \). Then \( u_1 \) is adjacent to at least one vertex of \( v_i \) and \( v_j \), (that is \( N(u_2) \cap N(v) \subseteq \{v_i, v_j\} \)) and \( |N(u_1) \cap N(u_2)| \leq 2 \). Since \( D \) is an edge dominating set of \( G \), every edge \( B \) must be adjacent with \( a_i \) and/or \( c_{2j} \) and hence the end vertices of such edges must be adjacent to \( v_i \) and/or \( v_j \). If \( B \) has no such vertices, then \( \langle N(v) \rangle \equiv \overline{K_\Delta} \). If such vertices are adjacent to \( v_i \) or \( v_j \) alone, then \( \langle N(v) \rangle \equiv T_{x} \cup mK_1 \) where \( x = v_i \) or \( v_j \). If some vertices are adjacent to \( v_i \) and some vertices are adjacent to \( v_j \), then \( \langle N(v) \rangle \equiv T_{v_i} \cup T_{v_j} \cup mK_1 \). Further if some vertices are adjacent to both \( v_i \) and \( v_j \), then \( \langle N(v) \rangle \equiv T_{v_i v_j} \cup mK_1 \).

**Remark 3.5.** Let \( G \) be a graph of order 6 with \( \text{diam}(G) = 2 \) and \( \Delta(G) = 3 \). Then \( G \) is isomorphic to \( H_1, H_2, H_3, H_4 \) or \( F \) in Figure 3.2. (A vertex of degree \( \Delta(G) \) is darkened whereas its non-neighbours are circled).

**Theorem 3.6.** Let \( G \) be a graph of order 6. If \( \text{diam}(G) = 2 \) and \( \Delta(G) = 3 \), then \( \gamma(G^{+-}) = 2 \) if and only if \( G \) is isomorphic to the graphs \( H_1, H_2, H_3, H_4 \) in Figure 3.2.

**Proof.** Assume that \( \gamma(G^{+-}) = 2 \). Suppose \( G \cong F \). Let \( \epsilon \) be any edge of \( F \). Then \( F - N[\epsilon] \) contains \( C_4 \) as an induced subgraph and \( \gamma'(C_4) = 2 \). Since \( \epsilon \) is an arbitrary edge, \( \gamma'(G) = 1 + \gamma'(C_4) = 3 \) which is a contradiction.

Conversely, assume that \( G \cong H_1, H_2, H_3 \) or \( H_4 \). Let \( \epsilon = uv \) be any edge of \( H_i \) such that at least one of it’s ends is of degree 3.
Then $H'_i = H_i - N[e]$ has exactly one non-trivial component which does not contain $C_4$. Hence $\gamma'_i(H'_i) = 1$ with the corresponding dominating edge $e'$. Then $\{e, e'\}$ is an independent edge dominating set of $G$. Thus $\gamma'_i(H_i) = 2$ and hence $\gamma(H_i^{+-}) = 2$ for all $i$.

**Theorem 3.7.** Let $G$ be a graph of order 7 with $\text{diam}(G) = 2$. If $v$ is a vertex of degree $\Delta = 4$ and $u_1, u_2 \in V - N[v]$ are non-adjacent, then $\gamma(G^{+-}) = 2$ if and only if $G$ is isomorphic to the graphs $G_i, i = 1, 2, \ldots, 17$ in Figure 3.3 (A vertex of degree $\Delta(G)$ is darkened whereas its non-neighbours are circled).

**Proof.** Assume that $G \cong G_i, i = 1, 2, \ldots, 17$. Since $G$ has no isolated vertex, by Theorem 2.2, $\gamma(G) \neq 1$. Then every $G_i$ has two independent edges $v_1v_2$ and $v_3v_4$ which are adjacent to all the edges of $G$. Hence $\{v_1v_2, v_3v_4\}$ is a dominating set of $G^{+-}$. Thus $\gamma(G^{+-}) = 2$.

Assume that $\gamma(G^{+-}) = 2$. By Lemmas 2.8 and 2.9, any minimum dominating set must contain two independent edges.

**Claim 1:** $\langle N(v) \rangle$ contains two independent edges.

Suppose not. Then $\langle N(v) \rangle$ is $K_{1,3}$ or $K_2 \cup 2K_1$ or $P_3 \cup K_1$. If $\langle N(v) \rangle = K_{1,3}$, then let $\text{deg}(v_1) = 4$. Therefore $v_1$ can not be adjacent to both $u_1$ and $u_2$. Since $\text{diam}(G) = 2$, both $u_1$ and $u_2$ are adjacent to all the vertices of $N(v)$.
Figure 3.3
except $v_1$. Therefore $\lvert N(u_i) \rvert = 3$. Since $n = 7$, no two edges of $C$ can form a dominating set of $G^{+}$. Since both $u_1$ and $u_2$ are adjacent to more than two vertices of $N(v)$, no 2-element set containing one edge of $A$ and one edge of $C$ can be a dominating set of $G^{+}$. Hence $\gamma(G^{+}) > 2$, which is a contradiction.

If $\langle N(v) \rangle \cong K_2 \cup 2K_1$, let $v_1v_2 \in E(G)$. Since $\text{diam}(G) = 2$, $v_3$ and $v_4$ must be adjacent to $u_1$ and $u_2$. Therefore $\lvert N(u_i) \rvert \geq 3$. For the same reason, at least one of $v_1$ and $v_2$ is adjacent to $u_1$ and $u_2$. Since $n = 7$, no two edges of $C$ can form a dominating set of $G^{+}$. Since $u_1v_j (j = 2, 3)$ is not adjacent to either $v_1v_2$ or $v_1v_k$, no 2-element set containing one edge of $A$ and one edge of $B$ can be a dominating set of $G^{+}$. Hence $\gamma(G^{+})$ is a contradiction. Hence the claim.

Claim 2: $\langle N(v) \rangle \cong 2K_2, P_4$ or $K_{1,3} + e$.

Suppose not. Then $\langle N(v) \rangle = K_4, K_4 - e$ or $C_4$ and all the vertices of $N(v)$ are of degree at least 3 in $\langle N[v] \rangle$. Hence $u_1$ and $u_2$ can not be adjacent to a common vertex in $N(v)$. Thus $d(u_1, u_2) \geq 3$ in $G$ which contradicts $\text{diam}(G) = 2$. Hence Claim 2.

Case (i): $\langle N(v) \rangle \cong 2K_2$.

Let $v_1v_2, v_3v_4 \in E(G)$. Since $\text{deg}(v_j) = 2$ in $\langle N(v) \rangle$ and $\text{diam}(G) = 2$, each $u_i$ is adjacent to at least two non-adjacent vertices of $N(v)$ and $\lvert N(u_1) \cap N(u_2) \rvert \geq 1$.

Subcase (a): $\lvert N(u_i) \rvert = 4$.

If $\lvert N(u_2) \rvert = 4$, then $G \cong G_1$. If $\lvert N(u_2) \rvert = 3$, then $G \cong G_2$. If $\lvert N(u_2) \rvert = 2$, then $G \cong G_5$. 
**Subcase (b):** $|N(u_1)| = 3$.

Suppose $|N(u_2)| = 3$. If $N(u_1) = N(u_2)$, then $G \cong G_6$; otherwise $G \cong G_7$.

Suppose $|N(u_2)| = 2$. If $|N(u_1) \cap N(u_2)| = 2$, then $G \cong G_{12}$. If $|N(u_1) \cap N(u_2)| = 1$, then $G \cong G_{13}$.

**Subcase (c):** $|N(u_1)| = 2$.

Suppose $|N(u_2)| = 2$. If $N(u_1) = N(u_2)$, then $G \cong G_{16}$. If $|N(u_1) \cap N(u_2)| = 1$, then $G \cong G_{17}$.

**Case (ii):** $\langle N(v) \rangle \cong P_4$.

Let $\langle N(v) \rangle \cong \langle v_1, v_2, v_3, v_4 \rangle$. Since degree of $v_1$ and $v_4$ is 2 and degree of $v_2$ and $v_3$ is 3 in $\langle N(v) \rangle$, $v_1$ and $v_4$ may be adjacent to $u_1$ and $u_2$; $v_2$ and $v_3$ are adjacent to at most one of $u_1$ and $u_2$.

**Subcase (a):** $v_2$ and $v_3$ are adjacent to different vertices of $V - N[v]$.

Let $v_2$ be adjacent to $u_1$ and $v_3$ be adjacent to $u_2$. Since $diam(G) = 2$, $u_1$ and $u_2$ must be adjacent to a common vertex of $N(v)$, say $v_1$. Since $diam(G) = 2$, $u_1$ is adjacent to $u_4$. Then $G \cong G_{10}$. Further if $u_2$ is adjacent to $v_4$, then $G \cong G_4$.

**Subcase (b):** $v_2$ and $v_3$ are adjacent to the same vertex $u_1$ of $V - N[v]$.

Then $v_2$ must be adjacent to $v_1$ and $v_4$. Since $diam(G) = 2$, $u_1$ must be adjacent to at least one of $v_1$ and $v_4$. If $u_1$ is adjacent to $v_1$, then $G \cong G_9$. Further if $u_1$ is adjacent to $v_4$, then $G \cong G_3$.

**Subcase (c):** At least one of $v_2$ and $v_3$ is adjacent to a vertex of $V - N[v]$.

Let $v_2$ be adjacent to $u_1$. Then $u_2$ must be adjacent to $v_1$ and $v_4$. Since $diam(G) = 2$, $u_1$ is adjacent to $v_4$. Then $G \cong G_{15}$. Further if $u_1$ is adjacent to $v_1$, then $G \cong G_8$. If $v_3$ is adjacent to $u_1$, then also we get $G_8$ or $G_{15}$.

**Subcase (d):** Both $v_2$ and $v_3$ are not adjacent to a vertex of $V - N[v]$.

Then both $u_1$ and $u_2$ must be adjacent to $v_1$ and $v_4$ and hence $G \cong G_{14}$. 
Case (iii): \( \langle N(v) \rangle \cong K_{1,3} + e \).

Let \( v_1 \) be of degree 3 and \( v_4 \) be of degree 2 in \( \langle N[v] \rangle \). Then \( v_1 \) can not adjacent to \( u_1 \) and \( u_2 \). Also \( v_2 \) and \( v_3 \) are adjacent to at most one of \( u_2 \) and \( u_1 \). Let \( v_2 \) be adjacent to \( u_1 \) and \( v_3 \) be adjacent to \( u_2 \). Since \( \text{diam}(G) = 2 \), \( u_1 \) and \( u_2 \) are adjacent to \( v_4 \). In this case, \( G \cong G_{11} \).

**Theorem 3.8.** Let \( G \) be a graph of order \( n \geq 8 \) with \( \text{diam}(G) = 2 \) and \( v \) be a vertex of degree \( \Delta = n - 3 \) and \( u_1, u_2 \in V - N[v] \) are non-adjacent. If \( N(u_1) = N(u_2) \neq N(v) \), then \( \gamma(G^{+-}) = 2 \) if and only if \( |N(u_i)| = 2 \) or 3 and \( \langle N(v) \rangle \cong H \) where \( H \in \{T_x \cup T_y, T_{x+y}, T_{x+y+z} / x, y, z \in N(u_i)\} \).

**Proof.** Let \( N(v) = \{v_1, v_2, \ldots, v_\Delta\} \). Assume that \( |N(u_i)| = 2 \) or 3 and \( \langle N(v) \rangle \cong H \). Since \( G \) has no isolated vertex, by Theorem 2.2, \( \gamma(G^{+-}) \neq 1 \).

Now, let \( |N(u_i)| = 2 \) and \( v_i, v_j \in N(u_i) \). If \( H = T_{v_i} \cup T_{v_j} \text{ or } T_{v_i, v_j} \), then all the edges of \( B \) and all the edges of \( C \) which are incident with \( v_j \) are adjacent to \( c_{ij} \). Further all the edges of \( B \) and \( C \) which are incident with \( v_i \) and \( v_j \) are adjacent to \( c_{ij} \). Then \( \{a_i, c_{ij}\} \) is a \( \gamma_i^\prime \)-set of \( G \) and hence dominating set of \( G^{+-} \).

Now, let \( |N(u_i)| = 3 \) and let \( v_i, v_j, v_k \in N(u_i) \). If \( H = T_{v_i} \cup T_{v_j} \text{ or } T_{v_i, v_j, v_k} \), then all the edges of \( A \) and all the edges of \( B \) and \( C \) which are incident with \( v_k \) are adjacent to \( a_k \). Further all the edges of \( B \) and \( C \) which are incident with \( v_i \) and \( v_j \) are adjacent to \( b_{ij} \). Therefore \( \{a_k, b_{ij}\} \) is a dominating set of \( G^{+-} \). Thus \( \gamma(G^{+-}) = 2 \).

Conversely, assume that \( \gamma(G^{+-}) = 2 \). Let \( D \) be a minimum dominating set of \( G^{+-} \). By Lemmas 2.8 and 2.9, \( D \) contains only two independent edges of \( G \).

**Claim:** \( |N(u_i)| = 2 \) or 3 for all \( i \).

If \( |N(u_i)| = 1 \), then \( \text{diam}(G) > 2 \) or degree of a vertex adjacent to \( u_i \) is greater than \( \Delta \) which is a contradiction. Suppose \( |N(u_i)| > 3 \). Since \( n \geq 8 \), there is an edge \( a_i \) which is not dominated by any 2-element set of \( B \), or any 2-element set of \( C \) or any 2-element set containing one element of \( B \) and one element of \( C \). Also there is an edge \( c_{ij} \) which is not dominated by any 2-element set containing one element of \( A \) and one element of \( B \) or any set containing one element of \( A \) and one element of \( C \). Thus \( \gamma(G^{+-}) \neq 2 \) which is a contradiction. Hence the claim.
Since \( n \geq 8 \), no two edges of \( B \) form a dominating set of \( G^{-+} \). For the same reason no two edges of \( C \) and no set containing one edge of \( B \) and one edge of \( C \) can form a dominating set of \( G^{-+} \). Then we consider the following two cases.

**Case (i):** \( D \) contains one edge of \( A \) and one edge of \( B \).

Let \( D = \{a_i, b_jk\} \) and \( S = \{v_i, v_j, v_k\} \). Since \( \text{diam}(G) = 2 \), \( u_1 \) and \( u_2 \) are adjacent to vertices of \( S \) in \( G \). Let \( S_1 \) be the set of vertices in \( N(v) - S \) which are adjacent to one or more vertices of \( S \). Since \( \text{diam}(G) = 2 \), \( \delta(G) \geq 2 \) and hence \( |S_1| = n - 6 \).

**Subcase (a):** \( |N(S_1) \cap S| = 1 \)

We claim that \( N(S_1) \cap S = \{v_j\} \). Suppose \( N(S_1) \cap S = \{v_j\} \). Since \( \text{diam}(G) = 2 \), \( u_1 \) and \( u_2 \) must be adjacent to \( v_j \). Hence \( \text{deg}(v_j) = n - 2 \) which is a contradiction to \( \Delta(G) \). Similarly, \( N(S_1) \cap S \neq \{v_k\} \). Hence the claim and \( \langle N(v) \rangle \cong T_{v_j} \cup K_2 \).

**Subcase (b):** \( |N(S_1) \cap S| = 2 \).

Suppose \( N(S_1) \cap S = \{v_i, v_j\} \). If \( N[v_i] \cap N[v_j] = \emptyset \) in \( \langle N(v) \rangle \), then \( \langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} \); otherwise \( \langle N(v) \rangle \cong T_{v_i} v_j \). If \( N(S_1) \cap S = \{v_j, v_k\} \), then \( \langle N(v) \rangle \cong T_{v_j} + v_k \).

**Subcase (c):** \( N(S_1) \cap S = S \).

If \( N[v_i] \cap N[v_j] = \emptyset \) and \( N[v_j] \cap N[v_k] = \emptyset \) in \( \langle N(v) \rangle \), then \( \langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} + v_k \); otherwise \( \langle N(v) \rangle \cong T_{v_j} + v_k v_i \).

**Case (ii):** \( D \) contains one edge of \( A \) and one edge of \( C \).

Let \( D = \{a_i, c_{ij}\} \). Since \( D \) is an edge dominating set of \( G \), every edge of \( B \) must be adjacent to \( a_i \) and/or \( c_{ij} \) and hence every vertex of \( B \) must be adjacent to \( v_i \) and/or \( v_j \). If \( N[v_i] \cap N[v_j] = \emptyset \) in \( \langle N(v) \rangle \), then \( \langle N(v) \rangle \cong T_{v_i} \cup T_{v_j} \); otherwise \( \langle N(v) \rangle \cong T_{v_i} v_j \).

**Theorem 3.9.** Let \( G \) be a graph of order \( n \geq 8 \) with \( \text{diam}(G) = 2 \) and let \( v \) be a vertex of degree \( \Delta = n - 3 \) and \( V - N[v] = \{u_1, u_2\} \) where \( u_1 \) and \( u_2 \) are non-adjacent. If \( N(u_1) \neq N(u_2) \), then \( \gamma(G^{-+}) = 2 \) if and only if \( \langle N(v) \rangle \cong H \) where \( H \in \{T_x \cup T_y, T_{xy}, T_{x+y} \cup T_z, T_{x+y+z}\} \); \( N(u_1) = \{x, y\} \) and \( z \in N(u_2) \) if \( |N(u_2)| = 3 \).
Proof. Let \( N(v) = \{v_1, v_2, \ldots, v_\Delta\} \). Assume that \( |N(v)| = 1 \). Let \( N(u_1) = \{v_1, v_j\} \) and \( v_k \in N(u_2) \) if \( |N(u_2)| = 3 \). Since \( G \) has no isolated vertex, by Theorem 2.2, \( \gamma(G^{-+}) \neq 1 \). Since \( u_1, u_2 \in E(G) \), there exists at least one vertex in \( N(u_1) \cap N(u_2) \) and let it be \( v_j \). If \( H \cong T \cup T_3 \) or \( T_{xy} \), then every edge of \( A \) and every edge of \( B \) which is adjacent to \( v_j \) or \( c_{1i} \) or \( c_{2i} \) or both are adjacent to \( a_i \). Further all other edges are incident with \( v_j \) and \( u_2 \). Therefore \( \{a_i, c_{2j}\} \) is a dominating set of \( G^{-+} \). If \( H = T_{v_j + v_j} \cup T_{v_k} \) or \( T_{v_j + v_j} v_k \), then every edge of \( A \) and every edge of \( B \) and \( C \) which is incident with \( v_k \) are adjacent to \( a_k \). By assumption, every edge of \( B \) and \( C \) which is adjacent to \( v_i \) and \( v_j \) are adjacent to \( b_{ij} \). Therefore \( \{a_k, b_{ij}\} \) is a dominating set of \( G^{-+} \).

Conversely, assume that \( \gamma(G^{-+}) = 2 \). Let \( D \) be a minimum dominating set of \( G^{-+} \). By Lemmas 2.8 and 2.9, \( D \) contains only two independent edges of \( G \).

Claim: \( |N(u_1)| = 2 \) or \( |N(u_2)| = 2 \).

Suppose \( |N(u_i)| = 1 (i = 1, 2) \), then \( \text{diam}(G) > 2 \) or \( \Delta > n - 3 \) which is a contradiction. Suppose \( |N(u_1)| \geq 3 \) and \( |N(u_2)| \geq |N(u_1)| \). Let \( v_j \in N(u_2) \) and \( v_j \in N(u_1) \). Since \( u_1 \) and \( u_2 \) are non-adjacent and \( \text{diam}(G) = 2 \), \( |N(u_1) \cap N(u_2)| \geq 1 \). Let \( v_k \in N(u_1) \cap N(u_2) \). Since \( n \geq 8 \), there is an edge \( c_{1i} \) which is not dominated by any 2-element subset of \( B \) or 2-elements subset of \( C \). Also there exists an edge \( c_{1i} \) which is not dominated by \( \{a_j, b_{rs}\} \) and there exists an edge \( c_{2i} \) which is not dominated by \( \{a_k, b_{rs}\} \). Therefore any 2-element set containing one element of \( A \) and one element of \( B \) can not form a dominating set. Similarly, no 2-element set containing one element of \( B \) and one element of \( C \) can form a dominating set. Since there exists an edge \( c_{1j} \) which is adjacent to neither \( a_i \) nor \( c_{2k} \) and there exists an edge \( c_{2j} \) which is adjacent to neither \( a_i \) nor \( c_{1k} \), any 2-element set containing one edge of \( A \) and one edge of \( C \) can not be a dominating set. Thus \( \gamma(G^{-+}) > 2 \) which is a contradiction. Hence the claim.

Since \( n \geq 8 \), no two edges of \( B \) or no two edges of \( C \) is a dominating set of \( G^{-+} \) and also any set containing one edge in \( B \) and one edge in \( C \) is not a dominating set. We consider the following two cases.

Case (i): \( D \) contains one edge in \( A \) and one edge in \( B \)

Let \( a_i, b_{jk} \in D, S = \{v_i, v_j, v_k\} \) and \( S_1 = N(v) - S \). Then for \( i = 1, 2 \), \( N(u_i) \) is 2-element subset or 3-element subset of \( S \). Since \( \text{diam}(G) = 2 \), each element
of \( S_1 \) is adjacent to some element in \( S \). By claim, let \( N(u_1) = \{v_j, v_j\} \). Therefore \(|N(u_1) \cap N(u_2)| = 1 \) or \( 2 \).

**Subcase (a):** \(|N(u_1) \cap N(u_2)| = 1\).

Then \(|N(u_2)| = 2 \) or \( 3\). If \(|N(u_2)| = 2\), then let \( N(u_2) = \{v_j, v_k\} \). Since \( u_1 \) and \( u_2 \) are non-adjacent and \( \text{diam}(G) = 2 \), \( v_i \) must be adjacent to \( v_k \) or \( v_j \). If \( v_j \) is adjacent to all the elements in \( S_1 \), then \( v_i \) and \( v_k \) may be adjacent to some element of \( S_1 \). If \( v_i \) and \( v_k \) are not adjacent to any element of \( S_1 \), then \( \langle N(v) \rangle = T_{v_j} \cup K_2 \) or \( \langle N(v) \rangle = T_{v_j} \); otherwise \( \langle N(v) \rangle = T_{v_j + v_k} v_j \). If \( v_j \) is not adjacent to all elements of \( S_1 \), then \( v_i \) and \( v_k \) must be adjacent to some elements of \( S_1 \). Then \( \langle N(v) \rangle = T_{v_j + v_k} v_j \).

If \(|N(u_2)| = 3\), then let \( N(u_2) = \{v_j, v_k, v_j\} \). Since \( \text{diam}(G) = 2 \), \( v_i \) or \( v_j \) is adjacent to \( v_k \). If \( N(v_k) = \emptyset \) in \( \langle N(v) \rangle \), then \( \langle N(v) \rangle = T_x \cup T_y \) or \( T_{xy} \); otherwise \( \langle N(v) \rangle = T_{(x+y)z} \).

**Subcase (b):** \(|N(u_1) \cap N(u_2)| = 2\).

Since \( N(u_1) \neq N(u_2), |N(u_2)| = 3 \). Let \( N(u_2) = \{v_i, v_j, v_k\} \). Since \( \text{diam}(G) = 2 \), \( v_i \) or \( v_j \) is adjacent to \( v_k \). Let \( v_i \) be adjacent to \( v_k \). Also every element in \( S_1 \) must be adjacent to at least one of \( v_i, v_j \). If \( N(v_k) = \emptyset \), then \( \langle N(v) \rangle = T_x \cup T_y \) or \( T_{xy} \); otherwise \( \langle N(v) \rangle = T_{(x+y)z} \). Thus \( \langle N(v) \rangle = T_{(x+y)z} \) if \(|N(u_2)| = 3\).

**Case (ii):** \( D \) contains one edge in \( A \) and one edge in \( C \).

By claim, let \( N(u_1) = \{v_i, v_j\} \). Let \( a_i \in D \). Then \( a_i \) dominates all the edges of \( A \). To dominate all the edges in \( C \) and \( B \), \( D \) must contain an edge which is incident with \( u_2 \) and it is \( c_{ij} \). Then every vertex in \( N(v) = \{v_i, v_j\} \) is adjacent to \( v_i \) or/and \( v_j \). If \( N[v_i] \cap N[v_j] = \emptyset \) in \( \langle N(v) \rangle \), then \( \langle N(v) \rangle = T_{v_i} \cup T_{v_j} \); otherwise \( \langle N(v) \rangle = T_{v_i} v_j \).

**Theorem 3.10.** Let \( G \) be a graph of order \( n \geq 8 \) with \( \text{diam}(G) = 2 \), and \( v \) be a vertex of degree \( \Delta = n - k, k \geq 4 \). If \( \langle V(G) - N[v] \rangle \cong K_{k-1} \), then \( \gamma(G^{-+}) = 2 \) if and only if \( \langle N(v) \rangle \cong H \) where \( H \in \{T_x \cup T_y, T_{xy}, T_{(x+y)z}/x, y \in N(u_i) \} \) for all \( u_i \in V(G) - N[v] \).

**Proof.** Let \( N(v) = \{v_1, v_2, \ldots, v_\Delta \} \) and \( V(G) - N[v] = \{u_1, u_2, \ldots, u_{k-1}\} \). Assume that \( \langle N(v) \rangle \cong H \). Since \( G \) has no isolated vertex, by Theorem 2.2, \( \gamma(G^{-+}) \neq 1 \).
If $H \cong T_{v_i} \cup T_{v_j}$ or $T_{v_i}v_j$, then every vertex of $V(G) - N[v]$ is adjacent to $v_i$ and $v_j$ only. Then all the edges of $A$ and all the edges of $B$ and $C$ which are incident with $v_i$ are adjacent to $a_i$. Further all the edges of $B$ and $C$ which are incident with $v_j$ are adjacent to $c_{ij}$. Therefore $\{a_i, c_{ij}\}$ is a dominating set of $G^{-+}$.

If $H \cong T_{(v_i + v_j)v_j}$, then every vertex of $V(G) - N[v]$ is adjacent to 2-element subset or 3-element subset of $\{v_i, v_j, v_j\}$. Then all the edges of $A$ and all the edges of $B$ and $C$ which are incident with $v_i$ are adjacent to $a_i$. Further all the other edges of $B$ and $C$ are adjacent to $b_{ij}$. Therefore $\{a_i, b_{ij}\}$ is a dominating set of $G^{-+}$. Thus $\gamma(G^{-+}) = 2$.

Conversely, assume that $\gamma(G^{-+}) = 2$. Let $D$ be a minimum dominating set of $G^{-+}$. By Lemmas 2.8 and 2.9, $D$ contains only two independent edges of $G$. Since $n \geq 8$, there is an edge $a_i$ which is not adjacent to any two edges of $C$. Therefore no two edges of $C$ can form a dominating set of $G^{-+}$. For the same reason one edge of $B$ and one edge of $C$ can not form a dominating set of $G^{-+}$. Then we consider the following cases.

**Case (i):** $D$ contains one edge of $A$ and one edge of $B$

Let $D = \{a_i, b_{ij}\}$ and $S = \{v_i, v_j, v_j\}$. Since $\text{diam}(G) = 2$ and $\langle V(G) - N[v] \rangle \cong K_{k-1}$, every vertex of $V(G) - N[v]$ must be adjacent to 2-element subset or 3-element subset of $S$. Let $S_1$ be the set of vertices in $N(v) - S$ which are adjacent to one or more vertices of $S$ and $|S_1| = n - k - 3$. Since $\text{diam}(G) = 2$, $\delta(G) \geq 2$. We claim that $|N(S_1) \cap S| \geq 2$. If $|N(S_1) \cap S| = 1$, then let $N(S_1) \cap S = \{v_i\}$. Then $\deg(v_i) > n - k - 3 + 1 + k - 1 = n - 3$ which is a contradiction and hence the claim. Since $D$ is a dominating set, $N(S_1)$ must be a 2-element subset or 3-element subset of $S$. If $|N(S_1) \cap S| = 2$, then $\langle N(v) \rangle \cong T_x \cup T_y$ or $T_{xy}$. If $|N(S_1) \cap S| = 3$, then $\langle N(v) \rangle \cong T_{v_i} \cup T_{v_i + v_j}$ or $T_{(v_j + v_j)v_j}$.

**Case (ii):** $D$ contains one edge of $A$ and one edge of $C$.

Let $a_i \in D$ and $c_{ij} \in D$. Then every vertex of $V(G) - N[v]$ is adjacent to $v_i$ and $v_j$. If $N[v_i] \cap N[v_j] = \emptyset$ in $\langle N(v) \rangle$, then $\langle N(v) \rangle \cong T_{v_i} \cup T_{v_i}$; otherwise $\langle N(v) \rangle \cong T_{v_i}v_i$.

**Case (iii):** $D$ contains two edges of $B$.

Let $D = \{v_i, v_j, v_r, v_s\}$. Then $\Delta = \deg(v) = 4$. Therefore $B$ contains only two edges which are independent. Hence $\langle N(v) \rangle \cong 2K_2$. 
**Theorem 3.11.** Let $G$ be a graph of order $n \geq 8$ with $\text{diam}(G) = 2$ and let $v$ be a vertex of degree $\Delta = n - k, k \geq 5$. If $\langle V(G) - N[v]\rangle \cong T_{x+y}$ with $N[u_i] \cap N[u_j] = \{x, y\}$, where $u_i, u_j \in V(G) - N[v] - \{x, y\}$, then $\gamma(G^{v+}) = 2$ if and only if $\langle N(v)\rangle \cong \overline{K}_\Delta$ or $T_2$.

**Proof.** Let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$ and $V(G) - N[v] = \{u_1, u_2, \ldots, u_{k-1}\}$. Let $V(G) - N[v] \cong T_{u_i + u_j}$. Assume that $\langle N(v)\rangle \cong \overline{K}_\Delta$ or $T_{v_l}$. Since $\text{diam}(G) = 2$ every vertex of $N(v)$ is adjacent to $u_i$ or $u_j$. Then $\{u_i, u_j, a_l\}$ is an edge independent dominating set of $G$ and hence dominating set of $G^{v+}$.

Conversely, assume that $\gamma(G) = 2$. Let $D$ be a minimum dominating set of $G^{v+}$. By Lemmas 2.8 and 2.9, $D$ contains only two independent edges of $G$. By hypothesis, $u_i, u_j$ must be in $D$. Since $n \geq 8$, no edge of $B$ or $C$ except $u_i, u_j$ is in $D$. Hence $\{u_i, u_j, a_l\}$ where $l = 1, 2, \ldots, \Delta$ is a dominating set of $G^{v+}$. Then $v_l$ may be adjacent to vertices of $N(v)$. If $v_l$ is adjacent to one or more vertices of $N(v)$, then $\langle N(v)\rangle \cong T_{v_l}$; otherwise $\langle N(v)\rangle \cong \overline{K}_\Delta$.

**Theorem 3.12.** Let $G$ be a graph of order $n \geq 8$ with $\text{diam}(G) = 2$ and let $v$ be a vertex of degree $\Delta = n - k, k \geq 4$. If $\langle V(G) - N[v]\rangle \cong T_{x}$, then $\gamma(G^{v+}) = 2$ if and only if $\langle N(v)\rangle \cong T_{v_l}$, where $y, z \in N(u_i)$ for all $u_i \in V(G) - N[v]$.

**Proof.** Let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$ and $V(G) - N[v] = \{u_1, u_2, \ldots, u_{k-1}\}$. Let $\text{deg}(u_i) = k - 2$ in $\langle V(G) - N[v]\rangle$. Assume that $\langle N(v)\rangle \cong T_{v_l}$ where $v_l \in N(u_i)$ for all $u_i \in V(G) - N[v]$. Since $G$ has no isolated vertex, by Theorem 2.2 $\gamma(G^{v+}) \neq 1$. Then $a_i$ is adjacent to all the edges of $A$ and all the edges of $B$ and $C$ which are incident with $v_l$. Further $c_{ij}$ is adjacent to all the edges of $B$ and $C$ which are incident with $v_l$ and all the edges of $\langle V(G) - N[v]\rangle$. Therefore $\{a_i, c_{ij}\}$ is a dominating set of $G^{v+}$. Hence $\gamma(G^{v+}) = 2$.

Conversely, assume that $\gamma(G^{v+}) = 2$. Let $D$ be a minimum dominating set of $G^{v+}$. By Lemmas 2.8 and 2.9, $D$ contains only two independent edges of $G$. Since $n \geq 8$, two edges of $B$ or two edges of $C$ or a set containing one edge of $B$ and one edge of $C$ can not form a dominating set of $G^{v+}$. Since $V(G) - N[v]$ has an edge, any set containing one edge of $A$ and one edge of $B$ is not a dominating set of $G^{v+}$. Hence the only possible case is that $D$ contains one edge of $A$ and one edge of $C$. Suppose $u_ju_i \in D$. Since $\text{diam}(G) = 2$, every vertex of $V(G) - N[v]$ must be adjacent to a vertex of $N(v)$. Then there exists $c_{rs}$ which is not adjacent to $u_ju_i$ and $a_i$ which contradicts that $D$ is a dominating set of $G^{v+}$. Hence $\{a_i, c_{ij}\}$ is a dominating
set of $G^{-+}$ and $G^{-}$. Then every vertex of $V(G) - N(v)$ must be adjacent to $v_i$ and $v_j$. Since $diam(G) = 2$, every vertex in $N(v) - \{v_i, v_j\}$ must be adjacent to $v_i$ and/or $v_j$. Therefore $\langle N(v) \rangle \cong T_{v_i v_j}$.

**Theorem 3.13.** Let $G$ be a graph of order $n \geq 8$ with $diam(G) = 2$ and let $v$ be a vertex of degree $\Delta = n - k, k \geq 5$. If $\langle V(G) - N[v] \rangle \not\cong T_k, T_{k+1}$ or $\overline{K}_{k-1}$, then $\gamma(G^{-+}) = 3$.

**Proof.** Suppose $\gamma(G^{-+}) \neq 3$. Since $diam(G) = 2$, $G$ has no isolated vertex. Therefore by Theorem 2.2, $\gamma(G^{-+}) \neq 1$. But by Theorem 2.1, $\gamma(G^{-+}) = 2$. Let $D$ be a minimum dominating set of $G^{-+}$. By Lemmas 2.8 and 2.9, $D$ contains two independent edges of $G$. Since $n \geq 8$, there exists $a_i$ which is not adjacent to any two edges of $C$. For the same reason, $\{b_{ij}, u, u_k\}$ is not a dominating set. Then the following cases are possible.

**Case (i) :** $a_i \in D$.

If $b_{ij} \in D$, then $\langle V(G) - N[v] \rangle \cong \overline{K}_{k-1}$ which is a contradiction. If $c_{ij} \in D$, then every edge of $\langle V(G) - N[v] \rangle$ must be incident with $u_j$. Therefore $\langle V(G) - N[v] \rangle \cong T_{u_j}$ which is a contradiction. If $u_j u_k \in D$, then every vertex of $V(G) - N[v]$ must be adjacent to $u_j$ and/or $u_k$. Therefore $\langle V(G) - N[v] \rangle \cong T_{u_j + u_k}$ which is a contradiction.

**Case (ii) :** $a_i \not\in D$.

If $D = \{b_{ij}, b_{in}\}$, then $\Delta = 4$ and $\langle V(G) - N[v] \rangle \cong \overline{K}_{k-1}$ which is a contradiction. If $D = \{b_{ij}, c_{ns}\}$, then $\Delta = 3$ and $\langle V(G) - N[v] \rangle \cong T_{u_k}$ which is a contradiction. Hence the Theorem.

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