Invariant Solutions for Gradient Ricci Almost Solitons

Tatiana Pires Fleury Bezerra

IFG - Instituto Federal de Educação, Ciência e Tecnologia de Goiás
Av. Universitária Vagner da S. Ferreira, Lt-1A, Parque Itatiaia, Ap. de Goiânia, GO, Brazil
e-mail: tatipibe@hotmail.com

Romildo Pina *
IME, Universidade Federal de Goiás,
Caixa Postal 131, 74001-970, Goiânia, GO, Brazil
e-mail: romildo@ufg.br

May 17, 2017

Abstract

In this paper we show that all conformal metrics to a pseudo-euclidean space \((\mathbb{R}^n, g)\), invariant under the action of an \((n-1)\)-dimensional translation group, and all the conformal metrics product manifold \(((\mathbb{R}^n, g) \times F^m)\), also invariant by translation, where \(F^m\) it is Ricci flat semi-Riemannian manifold, are gradient Ricci almost soliton. We also proved that all conformal metrics to a euclidean space and invariant by rotation are gradient Ricci almost soliton.

2010 Mathematics Subject Classification: 53C21, 53C50, 53C25
Key words: semi-Riemannian metric, gradient Ricci solitons, warped product

1 Introduction and main statements

Let \((M^n, g)\) be a semi-Riemannian manifold of dimension \(n \geq 3\). We say that \((M, g)\) is a **gradient Ricci soliton** if there exists a differentiable function \(f : M \rightarrow \mathbb{R}\) (called the potential function) such that

\*Partially supported by CAPES-PROCAD.
\[ \text{Ric}_g + \text{Hess}_g(f) = \rho g, \quad \rho \in \mathbb{R}, \]  

where \( \text{Ric}_g \) is the Ricci tensor, \( \text{Hess}_g(f) \) is the Hessian of \( f \) with respect the metric \( g \), and \( \rho \) is a real number. A Gradient Ricci soliton is said to be \textit{shrinking, steady or expanding} if \( \rho > 0 \), \( \rho = 0 \) or \( \rho < 0 \), respectively. When \( M \) is a Riemannian manifold, usually one requires the manifold to be complete. In the case of semi-Riemannian manifolds, one does not requires \((M, g)\), to be complete (see \([2], [4], [5] \) e \([8]\)).

Simple examples of gradient Ricci solitons are obtained by considering \( \mathbb{R}^n \) with the canonical metric \( g \). Then \((\mathbb{R}^n, g)\) is a gradient Ricci soliton, where

\[ f(x) = \frac{A|x|^2}{2} + g(x, B) + C, \quad A, C \in \mathbb{R} \text{ and } B \in \mathbb{R}^n, \]

are all the potential functions. In this case, \((\mathbb{R}^n, g)\) is a shrinking, steady or expanding Ricci soliton according to the sign of the constant \( A \).

We observe that the equations (1) can be considered a perturbation of the Einstein equation,

\[ \text{Ric}_g = \rho g, \quad \rho \in \mathbb{R}, \]  

When \( f \) is constant we call the underlying Einstein manifold a trivial Ricci Soliton.

In \([1]\), the authors considered gradient Ricci solitons, conformal to an \( n \)–dimensional pseudo Euclidean space, which are invariant under the action \((n - 1)\)–dimensional translation group and provided all such solutions in the case Steady. In \([11]\) the authors generalized the results for warped product \( M = B \times_f F \), with \( F \) Ricci flat.

In \([3]\), the authors, it was introduced a natural extension of the concept of gradient Ricci Soliton, the Ricci Almost Soliton. Thus, \((M, g)\) is said to be a \textbf{gradient Ricci almost soliton} if equation (1) is satisfied for some \( \rho \in C^\infty(M) \). In this article, they provided existence and rigidity results, deduced a-priori curvature estimates and isoltion phenomena, and investigated some topological properties. Since gradient Ricci almost soliton contain gradient Ricci Soliton as a particular case, we say that the gradient Ricci almost soliton is \textbf{proper} if the function \( \rho \) is non-constant. (see \([10]\)).

In \([6]\), the authors proved that either, a Euclidean space \( \mathbb{R}^n \), or a standard sphere \( S^n \), is the unique manifold with nonnegative escalar curvature which carries a structure of a gradient almost Ricci soliton, provided this gradient is a non trivial conformal vector field. Finally, showed that a compact locally conformally flat almost Ricci soliton is isometric to
Euclidean sphere $S^n$ provided an integral condition holds. In addition, they constructed examples of gradient almost Ricci solitons: consider the warped product manifold $M^{n+1} = R \times \text{cosht} S^n$, with metric $g = dt^2 + \text{cosh}^2 t g_0$, where $g_0$ is the standard metric of $S^n$. They proved that $(M^{n+1}, g, \nabla f, \rho)$, where $f(x, t) = \text{senht}$ and $\rho(x, t) = \text{senht} + n$, is an gradient almost Ricci soliton.

In [9], the authors proved that a gradient almost Ricci soliton $(M^n, g, \nabla f, \rho)$ whose Ricci tensor is Codazzi has constant sectional curvature. In particular, in the compact case, deduced that $(M^n, g)$ is isometric to a Euclidean sphere and $f$ is a height function. Moreover, they also classified gradient almost Ricci solitons with constant scalar curvature provided a suitable function achieves a maximum in $M^n$.

Catino, in 2011 (see [13]), introduced the notion of generalized quasi–Einstein manifold, that generalizes the concepts of Ricci soliton, Ricci almost soliton and quasi–Einstein manifolds. He proved that a complete generalized quasi–Einstein manifold with harmonic Weyl tensor and with zero radial Weyl curvature, is locally a warped product with $(n-1)$-dimensional Einsteins fibers. In this paper, Catino proves the following result:

Let $(M^n, g)$, $n \geq 3$, be a locally conformally flat gradient Ricci almost soliton. Then, around any regular point of $f$, the manifold $(M^n, g)$ is locally a warped product with $(n-1)$-dimensional Einsteins fibers of constant sectional curvature.

In particular, this implies a local characterization for locally conformally flat gradient Ricci almost solitons, similar to that proved for gradient Ricci solitons.

The local structure of half conformally flat gradient Ricci Almost solitons is investigated, recently in the article [10], showing they are locally conformally flat in a neighborhood of any point where the gradient of the potential function is non-null.

Motivated by the work [1] and [11], in this paper, first we will consider gradient Ricci almost soliton, conformal to a pseudo Euclidean space, which are invariant under the action of an $(n-1)$–dimensional translation group. More precisely, let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space with metric $g$ and coordinates $x = (x_1, \ldots, x_n)$ with $g_{ij} = \delta_{ij}\varepsilon_i$, $1 \leq i, j \leq n$, where $\delta_{ij}$ is the delta Kronecker. Let $\xi = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in \mathbb{R}$, be a basic invariant for an $(n-1)$–dimensional translation group. We want to obtain, in Theorem (1.2), differentiable functions $f(\xi)$, $\varphi(\xi)$ and $\rho(\xi)$ such that the metric $\overline{g} = \frac{1}{\varphi^2} g$ satisfies the equation

$$\text{Ric}_{\overline{g}} + \text{Hess}_{\overline{g}}(f) = \rho \overline{g}. \quad (3)$$

We show in Corollary (1.1) that given any function $\varphi(\xi)$ there are $f$ and $\rho$ such that
the metric $\bar{g}$ it is a gradient Ricci almost soliton.

We will also show that in the case Riemannian, all metrics conformal to a euclidean metric and invariant by rotation are gradient Ricci almost soliton, see Corollary (1.3).

Considering $(B, g_B)$ and $(F, g_F)$ semi-Riemannian Manifold, with $f > 0$ be a smooth function on $B$, the warped product $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$\tilde{g} = \bar{g} + f^2 g_F.$$

$B$ is called the base of $M = B \times_f F$, $F$ the fiber and $f$ is the warping function.

In what follows, we will find family of gradient Ricci almost Soliton, in the case of the warped product $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$, where the base is conformal to a pseudo-Euclidean space which are invariant under the action of an $(n-1)$-dimensional translation group. More precisely, let $(\mathbb{R}^n, g)$ be the pseudo-Euclidean space, $n \geq 3$ whit coordinates $x = (x_1, ..., x_n)$ and $g_{ij} = \delta_{ij} \varepsilon_i$ and let $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$, be a warped product where $\bar{g} = \frac{1}{\varphi^2} g$, $F$ a semi-Riemannian Einstein manifold. In Theorem (1.4) if we consider $f \varphi = 1$, so all the metrics $\tilde{g}$ are gradient Ricci almost soliton.

In what follows, we state our main results. We denote by $\varphi_{,x_i}$, $f_{,x_i}$ and $h_{,x_i}$, first order derivative and $\varphi_{,x_i,x_j}$, $f_{,x_i,x_j}$ and $h_{,x_i,x_j}$ second order derivative of functions $\varphi$, $f$ and $h$, with respect to $x_i$ and $x_i,x_j$, respectively.

**Theorem 1.1** Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$ with coordinates $x = (x_1, ..., x_n)$, $g_{ij} = \delta_{ij} \varepsilon_i$. Consider a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. There exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $(\mathbb{R}^n, \bar{g})$ is a gradient Ricci almost soliton with $f$ as a potential function if, and only if, the functions $f$ and $\varphi$ satisfy

$$\varphi_{,x_i,x_j} = -\frac{1}{n-2} \left( \varphi_{,x_i} f_{,x_j} + \varphi_{,x_j} f_{,x_i} + \varphi f_{,x_i,x_j} \right), i \neq j$$

and

$$(n-2) \varphi_{,x_i,x_i} + \left[ \varphi_{\Delta} \varphi - (n-1)|\nabla_{\bar{g}} \varphi|^2 \right] \varepsilon_i + 2 \varphi \varphi_{,x_i} f_{,x_i} + \varphi^2 f_{,x_i,x_i} - \varphi \varepsilon_i \sum_{k=1}^{n} \varepsilon_k f_{,x_k} \varphi_{,x_k} = \rho \varepsilon_i$$

We first obtain necessary and sufficient condition on $f(\xi)$ and $\varphi(\xi)$ for the existence of $\bar{g}$. We show that these conditions are different depending on the direction $\alpha = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$ being null or not.
Theorem 1.2 Let \((R^n, g)\) be a pseudo-Euclidean space, \(n \geq 3\) with coordinates \(x = (x_1, \ldots, x_n)\), \(g_{ij} = \delta_{ij} \varepsilon_i\). Consider smooth functions not constants \(f(\xi)\) and \(\varphi(\xi)\) where 
\[\xi = \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_i \in \mathbb{R}\] and \(\sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \varepsilon_{i_0} \neq 0\). There exists the metric \(\overline{g} = \frac{1}{\varphi^2} g\) such that 
\((R^n, \overline{g})\) is a gradient Ricci almost soliton with \(f\) as a potential function if, and only if, the functions \(f, \varphi\) and \(\rho\) satisfy

\[
\begin{aligned}
\begin{cases}
(n - 2) \varphi'' + 2 \varphi' f' + \varphi f'' = 0 \\
\varepsilon_{i_0} \left[ \varphi \varphi'' - (n - 1)(\varphi')^2 - \varphi \varphi f' \right] = \rho
\end{cases}
\end{aligned}
\]

(6)

In the next result we will find families of gradient Ricci almost soliton which are invariant under the action of an \((n - 1)\)-dimensional translation group. When \(\varepsilon_{i_0} = 0\) so by (5) \(\rho = 0\), and this case, has already been proven in [1].

Corollary 1.1 Let \((R^n, g)\) be a pseudo-Euclidean space, \(n \geq 3\) with coordinates \(x = (x_1, \ldots, x_n)\), \(g_{ij} = \delta_{ij} \varepsilon_i\). Given any function \(\varphi(\xi)\), the metric \(\overline{g} = \frac{1}{\varphi^2} g\) is a gradient Ricci almost soliton, with \(f\) as a potential function, where the functions \(f\) and \(\rho\) are given by

\[
\begin{aligned}
\begin{cases}
f(\xi) = \int \left[ c - (n - 2) \int \varphi \varphi'' d\xi \right] \frac{1}{\varphi^2} d\xi + k \\
\rho(\xi) = \frac{\varepsilon_{i_0}}{\varepsilon_i} \left[ \varphi \varphi'' - (n - 1)(\varphi')^2 \right] - \varepsilon_{i_0} \frac{\varphi'}{\varphi} \left[ c - (n - 2) \int \varphi \varphi'' d\xi \right]
\end{cases}
\end{aligned}
\]

(7)

and \(c\) and \(k\) are constants.

Theorem 1.3 Let \((R^n, g)\) be a pseudo-Euclidean space, \(n \geq 3\) with coordinates \(x = (x_1, \ldots, x_n)\), \(g_{ij} = \delta_{ij} \varepsilon_i\). Consider \(M = (R^n, \overline{g}) \times_f F^m\), a warped product, where \(\overline{g} = \frac{1}{\varphi^2} g\), \(F\) a semi-Riemannian Einstein manifold with constant Ricci curvature \(\lambda_F\), \(m \geq 1\), \(\varphi, f\) and \(h : R^n \to R\), smooth functions and \(f\) is a positive function. Then the warped product metric \(\overline{g} = \overline{g} + f^2 g_F\) is a gradient Ricci almost soliton with \(h\) as a potential function if, and only if, the functions \(\varphi, f, \rho\) and \(h\) satisfy

\[
(n - 2) f \varphi_{,x_ix_j} + f \varphi h_{,x_ix_j} - m \varphi f_{,x_ix_j} - m \varphi_{,x_i} f_{,x_j} - m \varphi_{,x_j} f_{,x_i} + f \varphi_{,x_i} h_{,x_j} + f \varphi_{,x_j} h_{,x_i} = 0,
\]

\[
1 \leq i \neq j \leq n
\]

(8)

\[
\varphi [(n - 2) f \varphi_{,x_ix_i} + f \varphi h_{,x_ix_i} - m \varphi f_{,x_ix_i} - 2m \varphi_{,x_i} f_{,x_i} + 2f \varphi_{,x_i} h_{,x_i}] + \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[ f \varphi \varphi_{,x_k x_k} - (n - 1) f \varphi_{,x_k}^2 + m \varphi \varphi_{,x_k} f_{,x_k} - f \varphi \varphi_{,x_k} h_{,x_k} \right] = \varepsilon_i \rho f,
\]

\[1 \leq i \leq n\]

(9)
Let

\[ \sum_{k=1}^{n} \varepsilon_k \left[ -f \varphi^2 f_{x_k x_k} + (n - 2)f f_{x_k \varphi, x_k} - (m - 1)\varphi^2 f^2_{,x_k} + f \varphi^2 f_{x_k h, x_k} \right] = \rho f^2 - \lambda_F. \quad (10) \]

Based on warped product theory, we will prove that all the conformal metrics product manifold \( R^n \times F^m \), where \( F^m \) it is Ricci flat manifold and \( \varphi \) invariant under the action of \( (n - 1) \)-dimensional translation group, are gradient Ricci almost solitons.

**Theorem 1.4** Let \((R^n, g)\) be a pseudo-Euclidean space, \( n \geq 3 \) with coordinates \( x = (x_1, \ldots, x_n) \), \( g_{ij} = \delta_{ij} \varepsilon_i \). Consider \( M = (R^n, \bar{g}) \times F^m \), a warped product, where \( \bar{g} = \frac{1}{\varphi} g \), \( F \) a semi-Riemannian Einstein manifold with constant Ricci curvature \( \lambda_F \) and smooth functions not constants \( f(\xi), \rho(\xi) \) and \( h(\xi) \) and \( f \) is a positive function, where \( \xi = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in R \epsilon \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \varepsilon_i \neq 0 \). Then the warped product metric \( \bar{g} = \bar{g} + f^2 g_F \) is a gradient Ricci almost soliton with \( h \) as a potential function if, and only if, the functions \( \varphi, f, \rho \) and \( h \) satisfy

\[
\begin{align*}
&f \left[ (n - 2)\varphi'' + 2\varphi' h' + \varphi h'' \right] - m \varphi f'' - 2m \varphi f' = 0 \\
&\varepsilon_{i_0} \left[ f \varphi \varphi'' - (n - 1) f \left( \varphi' \right)^2 + m \varphi \varphi' f' - f \varphi \varphi' h' \right] = \rho f \\
&\varepsilon_{i_0} \left[ -f \varphi^2 f'' + (n - 2) f f_{, \varphi} \varphi' - (m - 1) \varphi^2 \left( f' \right)^2 + f \varphi^2 f'' h' \right] = \rho f ^2 - \lambda_F.
\end{align*}
\]

In the next result we prove that if \( f \varphi = 1 \) and \( F \) Ricci flat then the metrics \( \bar{g} \) are gradient Ricci almost solitons.

**Corollary 1.2** Let \((R^n, g)\) be a pseudo-Euclidean space, \( n \geq 3 \) with coordinates \( x = (x_1, \ldots, x_n) \), \( g_{ij} = \delta_{ij} \varepsilon_i \). Consider \( M = (R^n, \bar{g}) \times F^m \), a warped product, where \( \bar{g} = \frac{1}{\varphi} g \), \( F \) a semi-Riemannian Einstein manifold Ricci flat. Consider \( f, \rho, h : R^n \to R \), \( f(\xi), \rho(\xi) \) e \( h(\xi) \) smooth functions not constants and \( f \) is a positive function, where \( \xi = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in R \epsilon \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \varepsilon_i \neq 0 \). Given any function \( \varphi(\xi) \), the warped product metric \( \bar{g} = \bar{g} + f^2 g_F \) is a gradient Ricci almost soliton with \( h \) as a potential function where the functions \( f, h \) and \( \rho \) are given by

\[
\begin{align*}
&f(\xi) = \frac{1}{\varphi(\xi)} \\
h(\xi) = c - \frac{k}{\varphi} - (m + n - 2) \int \left[ \varphi \varphi'' d\xi \right] \frac{1}{\varphi^2} d\xi \\
&\rho(\xi) = \varepsilon_{i_0} \left[ \varphi \varphi'' - (m + n - 1) (\varphi')^2 - k \frac{\varphi'}{\varphi} + (m + n - 2) \frac{\varphi'}{\varphi} \int \varphi \varphi'' d\xi \right]
\end{align*}
\]

\[ (12) \]
where \( c \) and \( k \) are constants.

**Remark 1.1** Notice that, as in the Corollary (1.2), \( f \varphi = 1 \), the metric \( \tilde{g} \) can be rewritten as

\[
\tilde{g} = \bar{g} + f^2 g_F = \frac{1}{\varphi^2} g_E + \left( \frac{1}{\varphi} \right)^2 g_F = \frac{1}{\varphi^2} (g_E + g_F).
\]

So all metrics conformal to the product manifold \((\mathbb{R}^n \times F^m)\), invariant by translation, where \( F^m \) is Ricci flat are gradient Ricci almost soliton.

In the next Corollary (1.3) and Theorem (1.5), we will consider metrics conformal to the euclidean space and we will find families of gradient Ricci almost soliton invariants by rotation.

**Theorem 1.5** Let \((\mathbb{R}^n, g)\) be a Euclidean space, \( n \geq 3 \) with coordinates \( x = (x_1, \ldots, x_n) \), \( g_{ij} = \delta_{ij} \). Consider smooth functions not constants \( f(r) \) and \( \varphi(r) \) where \( r = \sum_{i=1}^n x_i^2 \).

There exists the metric \( \tilde{g} = \frac{1}{\varphi^2} g \) such that \((\mathbb{R}^n, \tilde{g})\) is a gradient Ricci almost soliton with \( f \) as a potential function if, and only if, the functions \( f, \varphi \) and \( \rho \) satisfy

\[
\begin{align*}
(n - 2) \varphi'' + 2 \varphi' f' + \varphi f'' &= 0 \\
4(n - 1) \varphi' + 4r \varphi'' - 4(n - 1) r \left( \varphi' \right)^2 - 4r \varphi' f' + 2 \varphi^2 f' &= \rho
\end{align*}
\]

**Corollary 1.3** Let \((\mathbb{R}^n, g)\) be a Euclidean space, \( n \geq 3 \) with coordinates \( x = (x_1, \ldots, x_n) \), \( g_{ij} = \delta_{ij} \). Consider smooth functions not constants \( f(r) \) and \( \varphi(r) \) where \( r = \sum_{i=1}^n x_i^2 \).

Given any function \( \varphi(r) \), the metric \( \tilde{g} = \frac{1}{\varphi^2} g \) is a gradient Ricci almost soliton with \( f \) as a potential function, where the functions \( f \) and \( \rho \) are given by

\[
\begin{align*}
f(r) &= \int \left[ c - (n - 2) \int \varphi'' \, dr \right] \frac{1}{\varphi^2} \, dr + k \\
\rho(r) &= 4(n - 1) \varphi' + 4r \varphi'' - 4(n - 1) r \left( \varphi' \right)^2 - 4c r \frac{\varphi'}{\varphi} + 2c - 2(n - 2) \left( 1 - 2r \frac{\varphi'}{\varphi} \right) \int \varphi'' \, dr
\end{align*}
\]

where \( c \) and \( k \) are constants.

**Corollary 1.4** If \((\mathbb{R}^n, g)\) is the Euclidean space, \( F \) a complete Riemannian manifold Ricci flat and \( 0 < |\varphi(x)| \leq c \), for some constant \( c \), then the metrics given by Corollaries (1.1), (1.2) and (1.3) are complete.

As a consequence of Corollary 1.4 we obtain the following example:
Example 1.1 Consider \((\mathbb{R}^n, g)\) is the Euclidean space and \(F\) a complete Riemannian manifold Ricci flat.

A) Choosing \(\varphi(\xi) = \frac{1}{1 + \xi^2}\), the metric \(\overline{g} = \frac{1}{\varphi^2}g\) is a gradient Ricci almost soliton, complete in \(\mathbb{R}^n\), where the functions \(f(\xi)\) and \(\rho(\xi)\) are given by (7).

B) Considering \(\varphi(\xi) = e^{-\cosh \xi}\), the metric \(\tilde{g} = \tilde{g} + f^2 g_F\) is a complete gradient Ricci almost soliton, where the functions \(f(\xi)\) and \(\rho(\xi)\) are given by (12).

C) If \(\varphi(r) = e^{-r^2}\), then the metric \(\overline{g} = \frac{1}{\varphi^2}g\) is a gradient Ricci almost soliton, complete in \(\mathbb{R}^n\), where the functions \(f(r)\) and \(\rho(r)\) are given by (14).

2 Proofs of the Main Results

Proof of Theorem 1.1:
This demonstration is entirely analogous to the gradient Ricci soliton, see [1].

Proof of Theorem 1.2:
Let \(\overline{g} = \frac{1}{\varphi^2}g\) be a conformal metric of \(g\), \(g_{ij} = \delta_{ij} \varepsilon_i \varepsilon_j\) and \(\sum_{k=1}^{n} \varepsilon_k \alpha_k^2 = \varepsilon_{i_0}\). We are assuming that \(f(\xi)\) and \(\varphi(\xi)\) are functions of \(\xi\), where \(\xi = \sum_{i=1}^{n} \alpha_i x_i\). Hence, we have

\[
\varphi_{,x_i} = \alpha_i \varphi', \quad \varphi_{,x_i x_j} = \alpha_i \alpha_j \varphi'', \quad \varphi_{,x_i x_i} = \alpha_i^2 \varphi'', \\
f_{,x_i} = \alpha_i f', \quad f_{,x_i x_j} = \alpha_i \alpha_j f'', \quad f_{,x_i x_i} = \alpha_i^2 f'',
\]

and

\[
|\nabla \varphi|^2 = \left(\sum_{i=1}^{n} \varepsilon_i \alpha_i^2\right) (\varphi')^2 = \varepsilon_{i_0} (\varphi')^2, \quad \Delta_{g} \varphi = \left(\sum_{i=1}^{n} \varepsilon_i \alpha_i^2\right) \varphi'' = \varepsilon_{i_0} \varphi''.
\]

Substituting these expressions into (4), we get

\[
\alpha_i \alpha_j \left[(n - 2) \varphi'' + 2 \varphi' f' + \varphi f''\right] = 0, \quad \forall i \neq j.
\]

If there exist \(i \neq j\) such that \(\alpha_i \alpha_j \neq 0\), then this equation reduces to

\[
(n - 2) \varphi'' + 2 \varphi' f' + \varphi f'' = 0, \tag{15}
\]

which is exactly the first equation of (6).

Similarly, considering equation (5), we get

\[
\varphi \alpha_i^2 \left[(n - 2) \varphi'' + 2 \varphi' f' + \varphi f''\right] + \varepsilon_{i_0} \left[\varphi \varphi'' - (n - 1) (\varphi')^2\right] \varepsilon_i - \varepsilon_{i_0} \varphi \varphi' f' \varepsilon_i = \rho \varepsilon_i,
\]

Due to the relation between \(f''\) and \(\varphi''\) given by (15), the equation above reduces
\[ \varepsilon_{i_0} \left[ \varphi \varphi'' - (n - 1) (\varphi')^2 - \varphi \varphi' f' \right] = \rho. \]

We obtain the second equation (6).

If, for all \( i \neq j \), we have \( \alpha_i \alpha_j = 0 \), then \( \xi = x_{i_0} \), and (4) is trivially satisfied for all \( i \neq j \). Considering (5) for \( i \neq i_0 \), we get

\[
\varepsilon_{i_0} \left( \varphi \varphi'' - (n - 1) (\varphi')^2 - \varphi \varphi' f' \right) = \rho,
\]

and hence the second equation of (6) is satisfied. Considering \( i = i_0 \) in (5), we get that the first equation (6) is also satisfied.

The reciprocal of this theorem can be easily verified.

This concludes the proof of Theorem 1.2.

\[ \square \]

**Proof of Corollary 1.1:**

Note that we can rewrite the equation of (15) in the following way

\[
f'' + 2 \frac{\varphi'}{\varphi} f' + (n - 2) \frac{\varphi''}{\varphi} = 0.
\]

Taking \( y = f' \), in this last equation, it is equivalent to first order linear differential equation

\[
y' + 2 \frac{\varphi'}{\varphi} y + (n - 2) \frac{\varphi''}{\varphi} = 0
\]

Resolving this EDO, we get

\[ y = f'(\xi) = \left[ c - (n - 2) \int \varphi \varphi'' d\xi \right] \frac{1}{\varphi^2} \]

where \( c > 0 \). Thus,

\[ f(\xi) = \int \left[ c - (n - 2) \int \varphi \varphi'' d\xi \right] \frac{1}{\varphi^2} d\xi + k. \quad (16) \]

Therefore obtain the first equation of (7). Moreover, substituting (16) in the second equation of (3), we will have the second equation of (7).

The reciprocal of this theorem can be easily verified.

This concludes the proof of Corollary 1.1.

\[ \square \]
Proof of Theorem 1.3:

This demonstration is entirely analogous to the gradient Ricci soliton, see [11]. □

Proof of Theorem 1.4:

This demonstration is entirely analogous to the gradient Ricci soliton, see [11]. □

Proof of Corollary 1.2:

We can rewrite the first equation of (11) in the following way

\[
f \varphi h'' + 2f \varphi' h' + (n - 2)f \varphi'' - m\varphi f'' - 2m\varphi' f' = 0.
\]

Making the variable change \(y = h'\), the above equation becomes the first order linear differential equation in \(y\),

\[
y' + 2\frac{\varphi'}{\varphi}y + (n - 2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{f' \varphi'}{f \varphi} = 0,
\]

which is equivalent to,

\[
y' = -2\frac{\varphi'}{\varphi}y + \left[-(n - 2)\frac{\varphi''}{\varphi} + m\frac{f''}{f} + 2m\frac{f' \varphi'}{f \varphi}\right].
\]

Assuming that \(\bar{f}(\xi) = -2\frac{\varphi'}{\varphi}\) and \(\bar{g}(\xi) = -(n - 2)\frac{\varphi''}{\varphi} + m\frac{f''}{f} + 2m\frac{f' \varphi'}{f \varphi}\), by the linearity of the differential equations, we get

\[
y = \left[k + \int \bar{g}(\xi)e^{-\int \bar{f}(\xi) d\xi} d\xi\right]e^{\int \bar{f}(\xi) d\xi}.
\]

Notice that,

\[
\int \bar{f}(\xi) d\xi = -ln\varphi^2
\]

and so,

\[
h'(\xi) = y = \frac{k}{\varphi^2} + \frac{1}{\varphi^2} \int \left[m\varphi^2 f'' - (n - 2)\varphi \varphi'' + 2m\varphi \varphi' f' \right] d\xi,
\]

therefore,

\[
h(\xi) = c - \frac{k}{\varphi} + \int \left\{ \int \left[m\varphi^2 f'' - (n - 2)\varphi \varphi'' + 2m\varphi \varphi' f' \right] d\xi\right\} \frac{1}{\varphi^2} d\xi. \tag{17}
\]
How \( f \varphi = 1 \), we obtain that

\[
\begin{align*}
  f' \varphi &= -f \varphi' \\
  \varphi'' + 2f' \varphi' &= -f \varphi''
\end{align*}
\]  \( \text{(18)} \)

Note that we can rewrite equation \( (17) \) using the equation \( (18) \)

\[
\begin{align*}
  h(\xi) &= c - k \varphi + \int \left\{ \int \left[ m \varphi^2 f'' - (n - 2) \varphi \varphi'' + 2m \frac{\varphi' f'}{f} \right] \, d\xi \right\} \frac{1}{\varphi^2} \, d\xi \\
  &= c - k \varphi + \int \left\{ \int \frac{\varphi}{f} \left[ m \left( \varphi'' + 2f' \varphi' \right) - (n - 2) f \varphi'' \right] \, d\xi \right\} \frac{1}{\varphi^2} \, d\xi \\
  &= c - k \varphi + \int \left\{ \int \frac{\varphi}{f} \left[ -(m + n - 2) f \varphi'' \right] \, d\xi \right\} \frac{1}{\varphi^2} \, d\xi
\end{align*}
\]

\[
\begin{align*}
  h(\xi) &= c - k \varphi - (m + n - 2) \int \left[ \int \varphi \varphi'' \, d\xi \right] \frac{1}{\varphi^2} \, d\xi, \quad \text{(19)}
\end{align*}
\]

and

\[
\begin{align*}
  h'(\xi) &= \frac{k}{\varphi^2} - (m + n - 2) \frac{1}{\varphi^2} \int \varphi \varphi'' \, d\xi. \quad \text{(20)}
\end{align*}
\]

Now, substituting the expressions of \( h \) and \( h' \) in the second equation of \( (11) \), we get

\[
\begin{align*}
  \rho &= \varepsilon_{i0} \left\{ \varphi \varphi'' - (n - 1)(\varphi')^2 + m \varphi \varphi' \frac{f'}{f} - k \frac{\varphi'}{\varphi} - \frac{\varphi'}{\varphi} \int \left[ m \varphi^2 f'' - (n - 2) \varphi \varphi'' + 2m \frac{\varphi' f'}{f} \right] \, d\xi \right\}
\end{align*}
\]

Substituting the expressions for \( f \varphi = 1 \), \( f' \) and \( f'' \) in \( (21) \), we obtain

\[
\begin{align*}
  \rho(\xi) &= \varepsilon_{i0} \left\{ \varphi \varphi'' - (n - 1)(\varphi')^2 + m \varphi \varphi' \frac{f'}{f} - k \frac{\varphi'}{\varphi} - (m + n - 2) \frac{\varphi'}{\varphi} \int \varphi \varphi'' \, d\xi \right\} \\
  &= \varepsilon_{i0} \left\{ \varphi \varphi'' - (n - 1)(\varphi')^2 + m \varphi \varphi' \left( -\frac{\varphi'}{\varphi^2} \right) \frac{1}{\varphi} - k \frac{\varphi'}{\varphi} + (m + n - 2) \frac{\varphi'}{\varphi} \int \varphi \varphi'' \, d\xi \right\}
\end{align*}
\]

\[
\begin{align*}
  \rho(\xi) &= \varepsilon_{i0} \left\{ \varphi \varphi'' - (m + n - 1)(\varphi')^2 - k \frac{\varphi'}{\varphi} + (m + n - 2) \frac{\varphi'}{\varphi} \int \varphi \varphi'' \, d\xi \right\} \quad \text{(22)}
\end{align*}
\]

How \( f \varphi = 1 \) and \( \lambda_F = 0 \), we will prove that the expression obtained for the function \( \rho \), using the third equation of system \( (11) \) is exactly the equation \( (22) \). Proving that the function \( \rho \) is well defined.
We have that the third equation of the system (11) is given by

\[
\varepsilon_i \left[-f \varphi^2 \varphi'' + (n - 2)f \varphi' \varphi' - (m - 1)\varphi^2 \left(f' \right)^2 + f \varphi^2 \varphi' h' \right] = \rho f^2 - \lambda F
\]

How \( f \varphi = 1 \) and \( \lambda F = 0 \), replacing in the expressions of \( h' \), \( f \), \( f' \) e \( f'' \)

\[
f = \frac{1}{\varphi}, \quad f' = -\frac{\varphi'}{\varphi}, \quad f'' = -\frac{\varphi''}{\varphi^2} + 2\left(\varphi' \right)^2
\]

we get,

\[
\rho = \varepsilon_i \left[\varphi \varphi'' - (m + n - 1)\left(\varphi' \right)^2 - \varphi \varphi' h' \right]
\]

\[
\rho = \varepsilon_i \left[\varphi \varphi'' - (m + n - 1)\left(\varphi' \right)^2 - k \frac{\varphi'}{\varphi} + (m + n - 2)\frac{\varphi'}{\varphi} \int \varphi \varphi' d\xi \right]
\]

Thus, equation (23) is exactly the equation (22).

The reciprocal of this theorem can be easily verified.

This concludes the proof of Corollary 1.2.

\[\square\]

**Proof of Theorem 1.5:**

Let \( \overline{g} = \frac{1}{\varphi^2}g \) be a conformal metric of \( g \). We are assuming that \( f(r) \) and \( \varphi(r) \) are functions of \( r \), where \( r = \sum_{i=1}^{n} x_i^2 \). Hence, we have

\[
\varphi, x_i = 2x_i \varphi', \quad \varphi, x_i x_j = 4x_i x_j \varphi'', \quad \varphi, x_i x_i = 4x_i^2 \varphi'' + 2\varphi',
\]

\[
f, x_i = 2x_i f', \quad f, x_i x_j = 4x_i x_j f'', \quad f, x_i x_i = 4x_i^2 f'' + 2f',
\]

and

\[
|\nabla_{\overline{g}} \varphi|^2 = 4r(\varphi')^2, \quad \Delta_{\overline{g}} \varphi = 4r \varphi'' + 2m \varphi'.
\]

Substituting these expressions into (1), we get

\[
4x_i x_j \left[(n - 2)\varphi'' + 2\varphi' f' + \varphi f'' \right] = 0, \quad \forall i \neq j.
\]

Therefore,

\[
(n - 2)\varphi'' + 2\varphi' f' + \varphi f'' = 0. \tag{24}
\]

In an analogous way, replacing these expressions into (5), and how \( \varepsilon_i = 1 \) we get
\[4(n-1)\phi \phi' + 4r \phi \phi'' - 4(n-1)r(\phi')^2 + 2\varphi^2 f' - 4r \phi \phi' f' + 4x_1^2 \varphi \left[(n-2)\phi'' + 2\phi' f' + \phi f''\right] = \rho\]

Now, using the expression (24), the above equation reduces to

\[4(n-1)\phi \phi' + 4r \phi \phi'' - 4(n-1)r(\phi')^2 + 2\varphi^2 f' - 4r \phi \phi' f' = \rho\]  \hspace{1cm} (25)

The reciprocal of this theorem can be easily verified.

This concludes the proof of Theorem 1.5.

\[\Box\]

Proof of Corollary 1.3:

We begin by observing that the way we solve the first equation of (13) is analogous to the way we solve first equation (6). With this, we conclude that

\[f(r) = \int \left[c - (n-2) \int \phi \phi'' dr\right] \frac{1}{\varphi^2} dr + k.\]  \hspace{1cm} (26)

Moreover, substituting (26) in the second equation of (13), we obtain the second equation of (14).

This concludes the proof Corollary 1.3.

\[\Box\]

Proof of Corollary 1.4:

Consider the Euclidean space \((\mathbb{R}^n, g)\), \(n \geq 3\) and a metric \(\bar{g}\) given by Corollaries (1.1) and (1.3). If \(0 < |\phi(x)| \leq c\), then the metric \(\bar{g}\) is complete, since there exists a constant \(k > 0\), such that for any vector \(v \in \mathbb{R}^n\), \(|v|_{\bar{g}} \geq k|v|\). We have that \(M = (\mathbb{R}^n, \bar{g}) \times \mathbb{F}^m\), is complete if, and only if, \((\mathbb{R}^n, \bar{g})\) and \(\mathbb{F}^m\) are complete, (see[7]). Then the metrics obtained in Corollary (1.2) are complete.

This concludes the proof Corollary 1.4.

Proof of Corollary 1.4:

Consider the Euclidean space \((\mathbb{R}^n, g)\), \(n \geq 3\) and a metric \(\bar{g}\) given by Corollaries (1.1) and (1.3). If \(0 < |\phi(x)| \leq c\), then the metric \(\bar{g}\) is complete, since there exists a constant \(k > 0\), such that for any vector \(v \in \mathbb{R}^n\), \(|v|_{\bar{g}} \geq k|v|\). We have that \(M = (\mathbb{R}^n, \bar{g}) \times \mathbb{F}^m\), is complete if, and only if, \((\mathbb{R}^n, \bar{g})\) and \(\mathbb{F}^m\) are complete, (see[7]). Then the metrics obtained in Corollary (1.2) are complete.

This concludes the proof Corollary 1.4.

Proof of Corollary 1.4:

Consider the Euclidean space \((\mathbb{R}^n, g)\), \(n \geq 3\) and a metric \(\bar{g}\) given by Corollaries (1.1) and (1.3). If \(0 < |\phi(x)| \leq c\), then the metric \(\bar{g}\) is complete, since there exists a constant \(k > 0\), such that for any vector \(v \in \mathbb{R}^n\), \(|v|_{\bar{g}} \geq k|v|\). We have that \(M = (\mathbb{R}^n, \bar{g}) \times \mathbb{F}^m\), is complete if, and only if, \((\mathbb{R}^n, \bar{g})\) and \(\mathbb{F}^m\) are complete, (see[7]). Then the metrics obtained in Corollary (1.2) are complete.

This concludes the proof Corollary 1.4.

References

[1] E. Barbosa, R. Pina, K. Tenenblat - On Gradient Ricci Solitons conformal to pseudo-Euclidean space, Israel J. Math. 200 (2014), nº 1, 213–224.
[2] W. Batat, M. Brozos–Vásquez, E. Garcia–Rio, S. Gavino-Fernández - Ricci solitons on Lorentzian manifolds with large isometry groups, Bull. London Math. Soc., 43 (2011), nº 6, 1219-1227.

[3] S. Pigola, M. Rigoli, M. Rimoldi, A. Setti - Ricci Almost Solitons, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. X (2011), 757-799.

[4] M. Brozos–Vásquez, G. Calvaruso, E. Garcia–Rio, S. Gavino-Fernández - Three dimension Lorentzian homogeneous Ricci solitons, Israel. J. Math.s, 188 (2012), 385 - 403.

[5] M. Brozos–Vásquez, E. Garcia–Rio, S. Gavino-Fernández - Locally conformally flat Lorentzian gradient Ricci solitons, Journal of Geometric Analysis, 23 (2013), no. 3, 1196 - 1212.

[6] A. Barros, R. Batista, E. Ribeiro Jr. - Rigidity of gradient almost Ricci solitons, Illinois J. Math. 56 (2012), 1267–1279.

[7] B. O’neil - Semi–Riemannian Geometry with Applications to Relativity (Academic Press, New York), 1983.

[8] K. Onda - Lorentzian Ricci solitons on 3-dimensional Lie groups, Geom. Dedicata, 147 (2010), 313-322.

[9] A. Barros, J. N. Gomes, E. Ribeiro Jr. - A note on rigidity of the almost Ricci soliton, Arch. Math. (Basel) 100 (2013), 481–490.

[10] M. Brozos-Vásquez, E. Garcia–Rio, X. Valle-Regueiro - Half Conformally flat gradient Ricci Almost Solitons, Vol 472(2016).

[11] M. Sousa, R. Pina - Gradient Ricci Solitons with Structure of Warped, Results in Mathematics, v. 71, p. 825-840, 2017.

[12] R. Pina, K. Tenenblat - On solitons of the Ricci curvature equation and the Einstein equation, Israel Journal of Mathematics, 171 (2009), 61–76

[13] G. Catino - Generalized Quasi-Einstein Manifolds With Harmonic Weyl Tensor, Math. Z. 271, no. 3-4, 751-756, 2012.