On Euler polynomials for projective hypersurfaces

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Abstract

For every positive integer $n \in \mathbb{Z}^+$ we define an ‘Euler polynomial’ $\mathcal{E}_n(t) \in \mathbb{Z}[t]$, and observe that for a fixed $n$ all Chern numbers (as well as other numerical invariants) of all smooth hypersurfaces in $\mathbb{P}^n$ may be recovered from the single polynomial $\mathcal{E}_n(t)$. More generally, we show that all Chern classes of hypersurfaces in a smooth variety may be recovered from its top Chern class.

Fix an algebraically closed field $\mathbb{K}$ of characteristic zero. We denote by $\mathbb{P}^n$ projective $n$-space over $\mathbb{K}$ and for a smooth subvariety $X$ of $\mathbb{P}^n$ we define its Euler characteristic denoted $\chi(X)$ to be $\int_X c(TX) \cap [X]$\footnote{For $\mathbb{K} = \mathbb{C}$ certainly $\chi(X) = \chi_{\text{top}}(X)$.} For every positive integer $n \in \mathbb{Z}^+$, there exists a polynomial $\mathcal{E}_n(t) \in \mathbb{Z}[t]$ such that if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d$, then

$$\chi(X) = \mathcal{E}_n(d).$$

The adjunction formula along with standard exact sequences yield

$$\mathcal{E}_n(t) = (-1)^n \sum_{k=0}^{n-1} \binom{n+1}{k} (-t)^{n-k},$$

which we refer to as the $n$th Euler polynomial. In this note we make the observation that not just the Euler characteristic, but all Chern numbers of all smooth projective hypersurfaces in $\mathbb{P}^n$ (along with all Euler characteristics of all their general hyperplane sections) may be recovered from $\mathcal{E}_n(t)$ (for all $n$).

Remark 0.1. More generally, for $X \subset \mathbb{P}^n$ a (possibly) singular hypersurface of degree $d$ over $\mathbb{C}$, it is known that

$$\chi(X) = \mathcal{E}_n(d) + \int_X M(X),$$

\footnote{For $\mathbb{K} = \mathbb{C}$ certainly $\chi(X) = \chi_{\text{top}}(X)$.}
where \( \chi(X) \) here denotes topological Euler characteristic with compact supports, and \( \mathcal{M}(X) \) denotes the Milnor class\(^2\) of \( X \), a characteristic class supported on the singular locus of \( X \) (so that \( \mathcal{M}(X) = 0 \) for \( X \) smooth). As \( X \) is in the same rational equivalence class as a smooth hypersurface of degree \( d \), the Milnor class then measures the deviation of \( \chi(X) \) from that of a smooth deformation (parametrized by \( \mathbb{P}^1 \)). As the Milnor class is defined for any \( \mathbb{R} \)-variety, in this more general setting we define the Euler characteristic of a possibly singular hypersurface \( X \) to be
\[
\chi(X) := \int_X (T_{\text{vir}}(X) + \mathcal{M}(X)),
\]
where \( T_{\text{vir}}(X) \) denotes the virtual tangent bundle of \( X \). Thus formula (0.1) holds without the assumption \( \mathbb{R} = \mathbb{C} \).

So let \( R \) be a ring and let \( \vartheta : R[t] \to R[t] \) be the map which keeps only the terms of degree greater than one, and then divides the result by \( -t \), i.e., the map given by
\[
a_n t^n + \cdots + a_0 \mapsto - (a_n t^{n-1} + \cdots + a_2 t).
\]
Our result is the following

**Theorem 0.1.** Let \( n \in \mathbb{Z}_+ \) and
\[
\mathcal{C}_n(s, t) = \vartheta^{n-1} \mathcal{E}_n(t) s + \vartheta^{n-2} \mathcal{E}_n(t) s^2 + \cdots + \vartheta \mathcal{E}_n(t) s^{n-1} + \mathcal{E}_n(t) s^n \in \mathbb{Z}[s, t],
\]
where \( \vartheta^k \) denotes the \( k \)-fold composition of the map \( \vartheta : \mathbb{Z}[t] \to \mathbb{Z}[t] \). Then
\[
\mathcal{C}_n(H, d) = \iota_* c(TX) \cap [X],
\]
where \( X \subset \mathbb{P}^n \) is a smooth hypersurface of degree \( d \), \( H = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n] \) and \( \iota : X \hookrightarrow \mathbb{P}^n \) is the natural inclusion.

We then immediately arrive at the following

**Corollary 0.2.** Let \( X \) be a hypersurface of degree \( d \) in \( \mathbb{P}^n \). Then all Chern numbers of \( X \) are of the form
\[
\prod_i \xi_n^{j_i}(d),
\]
where \( \sum j_i = n - 2 \) and \( \xi_n^k(t) \) denotes \( \vartheta^{n-(k+1)} \xi_n(t) \).

Theorem 0.1 actually follows from a more general fact about hypersurfaces, which is nothing more than an elementary observation about the Fulton class of a hypersurface. Before stating the result we need the following

\(^2\)For more on Milnor classes from an algebraic perspective see [3].
**Definition 0.3.** Let $M$ be a smooth $\mathbb{R}$-variety and $Y \hookrightarrow M$ be a regular embedding. The *Segre class* of $Y$ relative to $M$ is then given by

$$s(Y, M) := c(N_{Y|M})^{-1} \cap [Y] \in A_*Y,$$

where $N_{Y|M}$ denotes the normal bundle to $Y$ in $M$. The *Fulton class* of $Y$ is then given by

$$c_F(Y) := c(TM) \cap s(Y, M).$$

**Remark 0.2.** In [5] (Chapter 4), Fulton actually defines the relative Segre class for $Y$ an arbitrary subscheme of $M$, and proves the class $c(TM) \cap s(Y, M)$ is intrinsic to $Y$ (i.e., it is independent of its embedding into a smooth variety). However, as we consider only hypersurfaces in this note (which are always regularly embedded), we don’t need the definition in its full generality. In any case, for $Y$ smooth $c_F(Y)$ coincides with its usual Chern class.

The previous results stated in this note are consequences of the following

**Theorem 0.4.** Let $M$ be a smooth $\mathbb{R}$-variety of dimension $n$, denote its Chern classes by $c_i$ and let $E_n(s) \in A_*M[s]$ be given by

$$E_n(s) = c_{n-1}s - c_{n-2}s^2 + \cdots + (-1)^n c_1 s^{n-1} + (-1)^{n+1} s^n.$$

Then if $X$ is any hypersurface in $M$ we have

$$(0.2) \quad c_F(X) = \partial^{n-1} E_n(X) + \partial^{n-2} E_n(X) + \cdots + \partial E_n(X) + E_n(X),$$

where $c_F(X)$ denotes the ‘Fulton class’ of $X$, and $X$ on the RHS of formula (0.2) denotes its class in $A_*M$. In particular, all Fulton classes of $X$ may be recovered from its top Fulton class via the map $\partial$.

**Proof.** Let $X$ be a (possibly singular) hypersurface in a smooth variety $M$ and denote its class in $A_*M$ simply by $X$. Then its Fulton class is defined as $c(TM) \cap s(X, M)$, where $s(X, M)$ denotes the Segre class of $X$ in $M$. Since $X$ is a hypersurface, it is regularly embedded, thus $s(X, M) = \frac{X}{1+X}$. Its Fulton classes (i.e., Chern classes for $X$ smooth) then take the following form (as a class in the Chow group $A_*M$):

$$c_0(X) = X,$$

$$c_1(X) = c_1X - X^2,$$

$$c_2(X) = c_2X - c_1X^2 + X^3,$$

$$\vdots$$

$$c_{n-1}(X) = c_{n-1}X - c_{n-2}X^2 + \cdots + (-1)^n c_1 X^{n-1} + (-1)^{n+1} X^n = E_n(X),$$
where by $X^k$ we mean the $k$-fold intersection product of $X$ with itself. The theorem immediately follows. □

Theorem 0.1 may then be obtained by replacing $M$ by $\mathbb{P}^n$ and $X$ by $tc_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n]$.

Remark 0.3. Not only the Chern numbers, but the Euler characteristics of all general hyperplane sections of all hypersurfaces in $\mathbb{P}^n$ may be easily recovered from $\mathcal{E}_n(t)$ as well. More precisely, let

$$\mathcal{C}_n^<(s, t) := s^n \mathcal{C}_n \left(\frac{1}{s}, t\right) = \mathcal{E}_n(t) + \vartheta \mathcal{E}_n(t)s + \cdots + \vartheta^{n-1} \mathcal{E}_n(t)s^{n-1},$$

where $\mathcal{C}_n(s, t)$ is as given in the statement of Theorem 0.1 (i.e., the power of $s$ is now keeping track of dimension rather than codimension), and let

$$\mathcal{E}_n(s, t) := \frac{s \cdot \mathcal{C}_n^<(−s − 1, t) + \mathcal{C}_n^<(0, t)}{s + 1}.$$

Then by Theorem 1.1 in [1], if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d$ then the coefficient of $−s^r$ in $\mathcal{E}_n(s, d)$ is the Euler characteristic of $X \cap H_1 \cap \cdots \cap H_r$, where the $H_i$ are general hyperplanes with respect to $X \cap H_1 \cap \cdots \cap H_{r−1}$.

Remark 0.4. For $X$ a smooth $\mathcal{R}$-variety it was shown in [4] that the (unnormalized) motivic Hirzebruch class of $X$ (referred to as the “Hirzebruch series” in [loc. cit.]), denoted $T_y^*(X)$, may be given by

$$T_y^*(X) = \frac{(1 + y)^k}{k!} \frac{d^k}{ds^k} \exp\left(\ln \left(\frac{s(1 + ye^{−s})}{1 − e^{−s}}\right) \odot \frac{−sC'}{C}\right) \bigg|_{s=0},$$

where $k = \dim(X)$, $C = 1 − c_1(X)s + \cdots + (−1)^k c_k(X)s^k$, $C' = \frac{d}{ds} C$, and $\odot$ denotes the Hadamard product of power series[3]. Thus for $X \subset \mathbb{P}^n$ a smooth hypersurface of degree $d$, replacing $C$ by $\mathcal{C}_n(−sH, d)$ in formula (0.3) yields a formula for the motivic Hirzebruch class of $X$ in terms of $\mathcal{E}_n(t)$, where $\mathcal{C}_n$ and $H$ are as given in Theorem 0.1. The degree zero part of $T_y^*(X)$ then yields the Euler characteristic, arithmetic genus and signature of $X$ when evaluated at $y = −1, 0, 1$, respectively. For $\mathcal{R} = \mathbb{C}$ and $X$ hyperkähler, a geometric interpretation for arbitrary $y$ is given in [6].

Example 0.5. Let $n = 4$. Then

$$\begin{align*}
\mathcal{E}_4(t) &= 10t − 10t^2 + 5t^3 − t^4 \\
\vartheta \mathcal{E}_4(t) &= 10t − 5t^2 − t^3 \\
\vartheta^2 \mathcal{E}_4(t) &= 5t − t^2.
\end{align*}$$

\[3\] Motivic Hirzebruch classes were first defined in [2]. For a pedagogic introduction to motivic characteristic classes we recommend [7].

\[4\] For $f = \sum a_i s^i$ and $g = \sum b_i s^i$, then $f \odot g = \sum a_i b_i s^i$. 
Thus for $X \subset \mathbb{P}^4$ a smooth hypersurface of degree $d$ by Corollary 0.2 we have

$$c_1(X)^3 = (\vartheta^2 E_4(d))^3 = (5d - d^2)^3,$$
$$c_1(X)c_2(X) = \vartheta^2 E_4(d) \cdot \vartheta E_4(d) = (5d - d^2)(10d - 5d^2 - d^3),$$
$$c_3(X) = E_4(d) = 10d - 10d^2 + 5d^3 - d^4.$$

Moreover, the Lefschetz hyperplane theorem and Hirzebruch-Riemann-Roch yields

$$h^{0,3}(X) = 1 - \frac{\vartheta^2 E_4(d) \cdot \vartheta E_4(d)}{24},$$
$$h^{1,2}(X) = \frac{\vartheta^2 E_4(d) \cdot \vartheta E_4(d)}{24} - \frac{E_4(d) - 2}{2},$$

which are the only nontrivial Hodge numbers of $X$. Furthermore, we have

$$c_4(s, d) = E_4(d) + (-\vartheta E_4(d) + \vartheta^2 E_4(d) - \vartheta^3 E_4(d))s$$
$$+ (\vartheta^2 E_4(d) - 2\vartheta^3 E_4(d))s^2 + \vartheta^4 E_4(d)s^3$$
$$= (10d - 10d^2 + 5d^3 - d^4) + (-6d + 4d^2 - d^3)s + (3d - d^2)s^2 - ds^3,$$

illustrating the fact that all Chern numbers, all Hodge numbers and all Euler characteristics of general hyperplane sections of all smooth hypersurfaces in $\mathbb{P}^4$ may be obtained via $E_4(t)$ and the map $\vartheta$.

We end with the following

**Question 1.** Is there a more general class of varieties (besides hypersurfaces) for which all of its Chern classes may be recovered from its top Chern class?

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**References**

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