Effective field theory approach to modified gravity including Horndeski theory and Hořava-Lifshitz gravity

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We review the effective field theory of modified gravity in which the Lagrangian involves three dimensional geometric quantities appearing in the 3+1 decomposition of space-time. On the flat isotropic cosmological background we expand a general action up to second order in the perturbations of geometric scalars, by taking into account spatial derivatives higher than two. Our analysis covers a wide range of gravitational theories— including Horndeski theory/its recent generalizations and the projectable/non-projectable versions of Hořava-Lifshitz gravity. We derive the equations of motion for linear cosmological perturbations and apply them to the calculations of inflationary power spectra as well as the dark energy dynamics in Galileon theories. We also show that our general results conveniently recover stability conditions of Horava-Lifshitz gravity already derived in the literature.

I. INTRODUCTION

The cosmic acceleration in the early Universe (dubbed inflation) was originally proposed to address flatness and horizon problems of the big-bang cosmology \cite{1}. Moreover, the inflationary paradigm has been strongly supported from the observations of Cosmic Microwave Background (CMB) temperature anisotropies \cite{2,3}. A key property of inflation is the existence of a scalar degree freedom responsible for both the accelerated acceleration at the background level and the generation of primordial density perturbations from quantum fluctuations \cite{4}. Not only a canonical scalar field with a nearly flat potential \cite{5,6} but also a gravitational scalar degree of freedom arising from violation of the gauge symmetry of General Relativity (GR)–like Starobinsky inflation \cite{7} can play such crucial roles.

In 1998, another late-time cosmic acceleration was discovered from the observations of type Ia supernovae (SN Ia) in high redshifts \cite{8}. The simplest origin for this present-day acceleration is the cosmological constant, but the vacuum energy appearing in particle physics is vastly larger than the observed energy scale of dark energy \cite{9}. There is a possibility that the accelerated expansion of the Universe is driven by a scalar field \cite{10,11} or some modification of gravity \cite{12-18}. The modification of gravity is usually associated with the propagation of a scalar degree of freedom coupled to non-relativistic matter (see Refs. \cite{19} for reviews).

The effective field theory (EFT) of cosmological perturbations is a powerful framework to deal with the low-energy degree of freedom of inflation and dark energy in a systematic and unified way \cite{20-43}. This approach is based on the expansion of a general four-dimensional action about the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background in terms of the perturbations of three-dimensional geometric scalar quantities appearing in the 3+1 Arnowitt-Deser-Misner (ADM) \cite{44} decomposition of space-time\textsuperscript{1}. Such geometric scalars involve the traces and squares of the extrinsic curvature $K_{\mu\nu}$ and the three-dimensional intrinsic curvature $R_{\mu\nu} \equiv (^{(3)})R_{\mu\nu}$ as well as the lapse function $N$. The Lagrangian generally depends on a scalar field $\phi$, but such dependence can be absorbed into the lapse dependence by choosing the so-called unitary gauge in which the field perturbation $\delta \phi$ vanishes.

The EFT formalism can incorporate a wide variety of modified gravitational theories known in the literature. For example, Horndeski \cite{47} derived a four-dimensional action of most general single-field scalar-tensor theories with second-order equations of motion in generic space-time (see also Refs. \cite{48-50}). In the EFT approach time derivatives are of second order by construction, but there exist spatial derivatives higher than second order in general \cite{29,30}. In Ref. \cite{31}, the conditions for the absence of such higher-order spatial derivatives have been derived by expanding the action up to second order in the perturbations of geometric scalars. In fact the Horndeski theory satisfies such conditions, so the resulting second-order Lagrangian is simply expressed by the sum of time and spatial derivatives $\zeta^2$ and $(\partial \zeta)^2$ of curvature perturbations $\zeta$ with time-dependent coefficients \cite{31}.

While the Horndeski theory is Lorentz-invariant, the EFT approach can also cover Lorentz-violating theories such as Hořava-Lifshitz gravity \cite{51}. In Hořava-Lifshitz gravity, Lorentz invariance is explicitly broken to realize an anisotropic scaling in time $t$. This anisotropic scaling was introduced to suppress non-linear gravitational interactions in the

\textsuperscript{1} On the spherically symmetric and static background there is another singled-out radial direction in addition to the temporal direction. Even in such cases it is possible to construct the EFT of modified gravity \cite{45} by employing the 2+1+1 decomposition of space-time developed in Refs. \cite{46}.
ultraviolet (UV) regime. Hořava-Lifshitz gravity is based on the ADM formalism with a kinetic Lagrangian $L_K$ constructed from scalars associated with $K_{\mu\nu}$ and a “potential” term $L_V$ constructed from scalars associated with $R_{\mu\nu}$ and its spatial derivatives. The presence of six spatial derivatives such as $\nabla_\mu R_{\nu j k} \nabla^i R^{jk}$ allows the $z = 3$ scaling characterized by the transformation $t \to e^{3t}$ and $x^i \to c x^i$ ($c$ is a constant and $x^i$ are spatial coordinates with $i = 1, 2, 3$), in which case the theory is power-counting renormalizable.

The original version of Hořava-Lifshitz gravity satisfies the so-called projectability condition under which the lapse $N$ is a function of time $t$ alone [41]. This is consistent with the foliation-preserving diffeomorphism characterized by $t \to t + f(t)$ and $x^i \to x^i + \xi(t, x^j)$. However, the theory is plagued by the problems of Laplacian instabilities and strong couplings even in the deep infrared (IR) regime [52, 53]. These problems can be alleviated in the non-projectable version of Hořava-Lifshitz gravity where the lapse $N$ depends upon both $t$ and $x^i$ [54]. In this case the so-called acceleration vector $a_i = \nabla_i \ln N$, where $\nabla_i$ is the covariant derivative with respect to the three-dimensional spatial metric, does not vanish. Scalar quantities constructed from $a_i$ allows a possibility of making the theory healthy [54, 55].

In this article we generalize the EFT approach of Ref. [31] in such a way that the formalism can accommodate the cases in which spatial derivatives higher than second order appearing in the projectable/non-projectable versions of Hořava-Lifshitz gravity are present. We start with the general action involving the dependence of higher-order spatial derivatives such as $\nabla_\mu R_{\nu j k} \nabla^i R^{jk}$ and $a_i a^i$ as well as of geometric scalar quantities constructed from $N$, $K_{\mu\nu}$, and $R_{\mu\nu}$. Note that a similar approach was taken in Ref. [40], but the contributions of higher-order spatial derivative terms to the second-order action of cosmological perturbations were not explicitly computed for scalar perturbations.

While our article is prepared for Special Issue in International Journal of Modern Physics D, it includes some new findings with an extension of the formalism already developed in the literature. Moreover, the EFT approach explained in this article will be useful for researchers who are interested in the systematic study of a wide variety of modified gravitational theories including Horndeski theory and Hořava-Lifshitz gravity. We shall derive the equations of motion for the background and linear perturbations convenient for discussing the dynamics of both inflation and dark energy. These results will be useful for the construction of theoretical consistent models of cosmic accelerations and for the unified/systematic description to test for such models with numerous observational data.

This paper is organized as follows.

In Sec. II we introduce geometric scalar quantities appearing in the ADM formalism and provide a general action that depends on such scalars.

In Sec. III several modified gravitational theories accommodated in our EFT approach are reviewed. Such theories include Horndeski theory and its recent generalization [38] as well as Hořava-Lifshitz gravity.

In Sec. IV we expand the action up to second order in cosmological perturbations and derive equations of motion for the background and the perturbations. We also derive conditions for the absence of spatial derivatives higher than second order for the background and conditions for avoiding ghosts and Laplacian instabilities.

In Sec. V we apply our general results to the cosmology of Horndeski theory and its recent generalization. We derive the primordial power spectrum of curvature perturbations generated during inflation driven by a single scalar degree of freedom. We also study the dynamics of dark energy in the presence of an additional matter and clarify how Horndeski theory and its generalization can be distinguished from each other at the level of perturbations.

In Sec. VI we apply our formalism to both the projectable and non-projectable versions of Hořava-Lifshitz gravity and show how the perturbation equations of motion already derived both on the Minkowski and FLRW backgrounds can be reproduced in our general framework.

Sec. VII is devoted to conclusions.

Throughout the paper, Greek and Latin indices denote components in space-time and in a three-dimensional space-adapted basis, respectively. A dot represents a derivative with respect to the cosmic time $t$. We use the metric signature $(-, +, +, +)$ and units $c = \hbar = k_B = 1$.

II. THE GENERAL EFT ACTION OF MODIFIED GRAVITY

The EFT of cosmological perturbations is based upon the 3+1 decomposition of space-time described by the line element [44]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt),$$

(2.1)

where $N$ is the lapse function, $N^i$ is the shift vector, and $h_{ij}$ is the three-dimensional spatial metric. The components of the four-dimensional metric $g_{\mu\nu}$ can be expressed as $g_{00} = -N^2 + N^i N_i$, $g_{0i} = g_{i0} = N_i$, and $g_{ij} = h_{ij}$.

We define a unit vector orthogonal to the constant $t$ hypersurfaces $\Sigma_t$, as $n_\mu \equiv -N t_\mu = (N, 0, 0, 0)$. The induced metric $h_{\mu\nu}$ on $\Sigma_t$ can be expressed in the form $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Since the unit vector $n_\mu = (1/N, -N^i/N)$ obeys the relation $n_\mu n^\mu = -1$, it follows that $n^\mu h_{\mu\nu} = 0$. 


The covariant derivative of the vector $n_\nu$ with respect to $g_{\mu\nu}$ (denoted as $n_{\nu;\lambda}$) corresponds to the directional derivative with respect to a parallel transport of $n_\nu$ in four-dimensional space-time. We define the extrinsic curvature as a projection of the tensor $n_{\nu;\lambda}$ on $\Sigma_t$, as

$$K_{\mu\nu} = h^\lambda_{\mu} n_{\nu;\lambda} = n_{\nu;\mu} + n_\mu a_\nu,$$

(2.2)

where $a_\nu \equiv n^\lambda n_{\nu;\lambda}$ is dubbed the acceleration. The extrinsic curvature is a three-dimensional quantity (satisfying the relation $n^\mu K_{\mu\nu} = 0$) that depends on the particular embedding of $\Sigma_t$.

The acceleration $a_\nu$ characterizes the difference between $K_{\mu\nu}$ (the change induced by the parallel transport of $n_\nu$ on $\Sigma_t$) and $n_{\nu;\lambda}$ (the change induced by the parallel transport of $n_\nu$ in four dimensions). Substituting the relation $n_\nu = -N t_\nu$ into $a_\nu = n^\lambda n_{\nu;\lambda}$ and using the properties $t_{\mu\lambda} = t_{\lambda\mu}, n^\lambda n_{\lambda\mu} = 0$, $h_\nu^\lambda = \delta_\nu^\lambda + n_\mu n^\lambda$, the acceleration can be written in the form

$$a_\nu = \frac{h^\lambda_{\nu} N_\lambda}{N} = \nabla_\nu \ln N,$$

(2.3)

where $\nabla_\nu$ represents the covariant derivative with respect to the three-dimensional metric $h_{ij}$. If $N$ is a function of time $t$ alone, then the acceleration vanishes. In general, however, $N$ depends on the spatial coordinate and hence $a_\nu \neq 0$.

From Eq. (2.2), the extrinsic curvature can be expressed as $K_{ij} = [n_{i;j} + n_{j;i} + n^\mu (n_{\mu} n_{ij})]/2$, where we used the symmetric property $K_{ij} = K_{ji}$. Taking note of the relations $n_{i;j} = h_{\alpha i} n_{\alpha j}, n_{\alpha j} = \partial_j n^\alpha + \Gamma_{\mu j}^\alpha n_\mu$ and $h_{ij;\mu} = \partial_\mu h_{ij} - \Gamma_{\mu j}^a h_{ia} - \Gamma_{\mu i}^a h_{aj}$, where $\Gamma_{\mu j}^a$ is the Christoffel symbol with respect to $g_{\mu\nu}$, it follows that

$$K_{ij} = \frac{1}{2} \left( \frac{1}{N} \partial_i h_{ij} + n^k \partial_k h_{ij} + h_{ji} \partial_i n^\alpha + h_{\alpha i} \partial_j n^\alpha \right).$$

(2.4)

Using the three-dimensional covariant derivative $\nabla_i N_j = \partial_i N_j - (^{(3)}\Gamma_{ij}^k N_k$, where $(^{(3)}\Gamma_{ij}^k$ is the Christoffel symbol associated with $h_{ij}$, the extrinsic curvature (2.4) is simplified expressed as

$$K_{ij} = \frac{1}{2N} (\partial_i h_{ij} - \nabla_i N_j - \nabla_j N_i).$$

(2.5)

Hence $K_{ij}$ is a covariant quantity that involves a time derivative of the three-dimensional metric $h_{ij}$.

The kinetic scalar quantities that can appear in the action of general modified gravitational theories are derived by taking the trace of $K_{\mu\nu}$ and by squaring $K_{\mu\nu}$, as

$$K \equiv K^\mu_{\mu}, \quad S \equiv K_{\mu\nu} K^{\mu\nu}. $$

(2.6)

The internal geometry of $\Sigma_t$ is quantified by the three-dimensional Ricci tensor $R_{\mu\nu} = (^{(3)}R_{\mu\nu}$ (dubbed the intrinsic curvature). The scalar quantities constructed from $R_{\mu\nu}$ are given by

$$R \equiv R^\mu_{\mu}, \quad Z \equiv R_{\mu\nu} R^{\mu\nu}. $$

(2.7)

Another scalar quantity can be constructed from $R_{\mu\nu}$ and $K^{\mu\nu}$:

$$U \equiv R_{\mu\nu} K^{\mu\nu}. $$

(2.8)

Note that the three-dimensional Riemann tensor $(^{(3)}R_{\mu\nu\rho\sigma}$ gives rise to the quadratic combination $(^{(3)}R_{\mu\nu\rho\sigma} (^{(3)}R^{\mu\nu\rho\sigma}$. In three dimensions, however, the Riemann tensor can be expressed in terms of the Ricci tensor and scalar, so we do not need to consider such a combination.

We also allow for the existence of scalar quantities that give rise to spatial derivatives higher than second order in the equations of motion:

$$Z_1 \equiv \nabla_i R^{\nu i} R_i, \quad Z_2 \equiv \nabla_i R_{jk} \nabla^i R^{jk}. $$

(2.9)

Other terms such as $R^2, R^2 R^2, R R^2, R^2 R^2$ can be taken into account, but they are irrelevant to scalar linear perturbations on the flat FLRW background studied in Sec. [IV]. We do not include the terms with more than six spatial derivatives, as we are interested in the application to Hořava-Lifshitz gravity. Provided that the theory is power-counting renormalizable by an anisotropic scaling with $z = 3$ (discussed in Sec. [III B]), such higher-order terms are not generated by quantum corrections.
In the original version of Hořava-Lifshitz gravity \cite{51}, the lapse $N$ is assumed to be a function of time $t$ alone (which is called the projectability condition). This reflects the fact that the space-time foliation is preserved by the space-independent reparametrization $t \to t'(t)$. In this case, the acceleration $a_\nu$ of Eq. \ref{2.10} vanishes and hence $K_{\mu\nu} = n_{\nu,\mu}$. One can extend the original Hořava-Lifshitz theory such that the lapse depends on the spatial coordinate $x^i (i = 1, 2, 3)$ as well as time $t$ and that the acceleration $a_\nu$ is included in the action \cite{54}. In this non-projectable version the following scalar combinations can be taken into account:

$$
\alpha_1 = a_i a^i, \quad \alpha_2 = a_i \Delta a^i, \quad \alpha_3 = \nabla_i a^i, \quad \alpha_4 = a_i \Delta^2 a^i, \quad \alpha_5 = \Delta \nabla_i a^i,
$$

where $\Delta = \nabla_i \nabla^i$. Again we do not include the terms irrelevant to the dynamics of linear scalar perturbations on the flat FLRW background (such as $(a_i a^i)^2$ and $a_i a_j R^{ij}$).

The action of general modified gravitational theories that depends on the above mentioned scalar quantities is given by

$$
S = \int d^4x \sqrt{-g} L(N, K, S, R, Z, U, Z_1, Z_2, \alpha_1, \cdots, \alpha_5; t),
$$

where $g$ is a determinant of the metric $g_{\mu\nu}$ and $L$ is a Lagrangian. The dependence of the lapse $N$ and the time $t$ is included for the reason explained in Sec. \ref{III.A}. Expanding the action \ref{2.11} up to second order in cosmological perturbations about the flat FLRW background, we obtain the equations of motion for the background and linear perturbations. Before doing so, we shall review the theories that belong to the action \ref{2.11}.

III. CONCRETE THEORIES ACCOMMODATED IN THE EFT FRAMEWORK

The EFT formalism can deal with a wide variety of gravitational theories– including (i) Horndeski theory \cite{17} and its generalization \cite{33}, and (ii) Hořava-Lifshitz gravity \cite{51}. In this section we discuss explicit dependence of the Lagrangian $L$ on the geometric scalar quantities introduced in Sec. \ref{II}.

A. Horndeski theory and its generalization

The Lagrangian of most general scalar-tensor theories with second-order equations of motion was first derived by Horndeski in 1973 \cite{17}. In four dimensions, Horndeski theory is characterized by the Lagrangian \cite{47,50} \ref{3.1}

$$
L = \sum_{i=2}^{5} L_i,
$$

with

$$
L_2 = G_2(\phi, X), \quad L_3 = G_3(\phi, X) \Box \phi, \quad L_4 = G_4(\phi, X) R - 2G_{4, X} (\phi, X) \left[ (\Box \phi)^2 - \phi^{\mu\nu} \phi_{\mu\nu} \right],
$$

$$
L_5 = G_5(\phi, X) G_{\mu\nu} \phi^{\mu\nu} + \frac{1}{3} G_{5, X} (\phi, X) \left[ (\Box \phi)^3 - 3(\Box \phi) \phi_{\mu\nu} \phi^{\mu\nu} + 2\phi_{\mu\nu} \phi^{\mu\sigma} \phi^{\nu, \sigma} \right],
$$

where $\Box \phi \equiv (g^{\mu\nu} \phi_{\nu, \mu})_{, \mu}$, and $G_i (i = 2, 3, 4, 5)$ are functions in terms of a scalar field $\phi$ and its kinetic energy $X = g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi$ with the partial derivatives $G_{i, X} \equiv \partial G_i / \partial X$ and $G_{i, \phi} \equiv \partial G_i / \partial \phi$. $R$ and $G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor in four dimensions, respectively.

Horndeski theory covers a wide variety of gravitational theories with a single scalar degree of freedom. First of all, the k-essence scalar field \cite{11,56} in the framework of GR is described by the functions $G_2 = P(\phi, X), G_3 = 0, G_4 = M_{pl}^2/2, G_5 = 0$, where $M_{pl}$ is the reduced Planck mass. The canonical scalar field with a potential $V(\phi)$ corresponds to a particular function $G_2 = -X/2 - V(\phi)$.

Brans-Dicke (BD) theory \cite{57} with a potential $V(\phi)$ is characterized by the functions $G_2 = -M_{pl} \omega_{BD} X / (2\phi) - V(\phi), G_3 = 0, G_4 = M_{pl} \phi/2, G_5 = 0$, where $\omega_{BD}$ is the BD parameter. The metric $f(R)$ gravity \cite{13,14} and dilaton gravity \cite{58} correspond to the particular cases of BD theory with $\omega_{BD} = 0$ and $\omega_{BD} = -1$, respectively.

The covariant Galileon \cite{17} corresponds to the functions $G_2 = \beta_2 X, G_3 = \beta_3 X, G_4 = M_{pl}^2/2 + \beta_4 X^2, G_5 = \beta_5 X^2$, where $\beta_i (i = 2, 3, 4, 5)$ are constants. A scalar field whose derivatives couple to the Einstein tensor in the form
\( G_{\mu \nu} \partial^\mu \phi \partial^\nu \phi \) can be accommodated by the functions \( G_2 = -X/2 - V(\phi) \), \( G_3 = 0 \), \( G_4 = 0 \), \( G_5 = c \phi \), where \( c \) is a constant and \( V(\phi) \) is a field potential.

The Lagrangian (3.1) with (3.2)-(3.5) involves the dependence of \( \phi \) and \( X \), whereas the Lagrangian \( L \) in the action (2.11) does not explicitly depend on such a scalar field and its kinetic term. However, if we choose the unitary gauge \( \phi = \phi(t) \), (3.6) in which the perturbation \( \delta \phi(t, x^i) \) on the flat FLRW background vanishes, the field kinetic term is expressed as \( X = -\dot{\phi}^2(t)/N^2 \). Then, the \( \phi \) and \( X \) dependence can be interpreted as the \( N \) and \( t \) dependence appearing in the action (2.11). In this way, it is possible to incorporate Horndeski theory in the EFT formalism.

More explicitly, the Horndeski Lagrangian (3.1) with (3.2)-(3.5) can be expressed in terms of the three-dimensional scalar quantities introduced in Sec. II [31] (see also Refs. [30, 36, 37]). In unitary gauge the constant-\( \phi \) hypersurfaces coincide with \( \Sigma_t \), so the unit vector orthogonal to those hypersurfaces is given by [31]

\[
\begin{align*}
    n_\mu &= -\gamma \dot{\phi}_\mu, \\
    \gamma &= \frac{1}{\sqrt{-X}}.
\end{align*}
\] (3.7)

Employing the relation (2.2) and taking the covariant derivative of Eq. (3.7), we obtain

\[
\phi_{,\mu \nu} = -\frac{1}{\gamma} \left( K_{\mu \nu} - n_\mu a_\nu - n_\nu a_\mu \right) + \frac{\gamma^2}{2} \phi^{,\lambda} X^{,\lambda} n_\mu n_\nu,
\] (3.8)

and hence

\[
\Box \phi = -\frac{1}{\gamma} K + \frac{\phi^{,\lambda} X^{,\lambda}}{2X}.
\] (3.9)

Then, the Lagrangian (3.3) is expressed as \( L_3 = G_3[-K/\gamma + \phi^{,\lambda} X^{,\lambda}/(2X)] \), whose second term can be eliminated by introducing an auxiliary function \( F_3(\phi, X) \) satisfying

\[
G_3 = F_3 + 2XF_3, X.
\] (3.10)

The contribution \( F_3 \Box \phi \) reduces to \( -(F_3, \phi \phi^{,\lambda} + F_3, X X^{,\lambda}) \phi^{,\lambda} \) up to a boundary term. The second term is cancelled by one of the terms in \( 2XF_3, X \Box \phi \). Then, the Lagrangian (3.3) can be expressed as

\[
L_3 = 2(-X)^{3/2} F_3, X K - XF_3, \phi.
\] (3.11)

Since \( F_3 \) depends on \( \phi(t) \) and \( X(t,N) = -\dot{\phi}^2(t)/N^2 \), \( L_3 \) is a function of \( N \), \( K \), and \( t \). The equations of motion following from the Lagrangian \( L_3 \) can be written in terms of \( G_3 \) and its derivatives with respect to \( \phi \) and \( X \), without containing the auxiliary function \( F_3 \). [31]

Substituting Eqs. (3.8) and (3.9) into Eq. (3.4) and using the fact that \( a_\mu = -h^\mu_\nu X_\nu/(2X) \), the Lagrangian \( L_4 \) reads

\[
L_4 = G_4 R + 2XG_4, X (K^2 - S) + 2G_4, X n_\mu (Kn^\mu - a^\mu).
\] (3.12)

From the Gauss-Codazzi equations, the four-dimensional Ricci scalar \( R \) is related to the three-dimensional Ricci scalar \( R \) according to

\[
R = R - K^2 + S + 2(Kn^\mu - a^\mu), \mu.
\] (3.13)

On using Eq. (3.13) together with the relations \( G_4, X n_\mu = G_4, \mu + \gamma^{-1} G_4, \phi n_\mu \) and \( n_\mu a^\mu = 0 \), Eq. (3.12) reduces to

\[
L_4 = G_4 R + 2XG_4, X - G_4)(K^2 - S) - 2\sqrt{-X} G_4, \phi K.
\] (3.14)

Similarly, the Lagrangian (3.5) can be expressed in terms of the three-dimensional quantities, as [31]

\[
L_5 = \frac{1}{2} X G_5, \phi (K^2 - S) - \frac{1}{3} (-X)^{3/2} G_5, X K_3 + \frac{1}{2} X (G_5, \phi - F_5, \phi) R - \sqrt{-X} F_5 \left( \mathcal{U} - \frac{1}{2} K R \right),
\] (3.15)

where \( F_5(\phi, X) \) is an auxiliary function satisfying

\[
G_5, X = \frac{F_5}{2X} + F_5, X,
\] (3.16)
and
\[ K_3 \equiv K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{il}K^{jl}. \] (3.17)

Up to quadratic order in perturbations, the term \( K_3 \) is given by
\[ K_3 = 3H(2H^2 - 2KH + K^2 - S) + O(3). \] (3.18)

From Eqs. (3.2), (3.11), (3.14), and (3.15) the total Lagrangian (3.1) involves the functions \( \phi, X, K, K^2 - S, R, K_4, \) and \( \mathcal{U} - K\mathcal{R}/2. \) The dependence on \( \phi \) and \( X \) can be interpreted as that on \( N \) and \( t. \) Then, the Horndeski Lagrangian is equivalent to
\[ L = A_2(N,t) + A_3(N,t)K + A_4(N,t)(K^2 - S) + B_4(N,t)r + A_5(N,t)K_3 + B_5(N,t)(\mathcal{U} - K\mathcal{R}/2), \] (3.19)
where
\[ A_2 = G_2 - X F_{3,\phi}, \quad A_3 = 2(-X)^{3/2} F_{3,X} - 2\sqrt{-X} G_{4,\phi}, \quad A_4 = 2XG_{4,X} - G_4 + XG_{5,\phi}/2, \]
\[ B_4 = G_4 + X(G_{5,\phi} - F_{5,\phi})/2, \quad A_5 = -(X)^{3/2} G_{5,X}/3, \quad B_5 = -\sqrt{-X} F_5. \] (3.20)
The coefficients \( A_4 \) and \( A_5 \) are related to \( B_4 \) and \( B_5, \) as
\[ A_4 = 2XB_{4,X} - B_4, \quad A_5 = -XB_{5,X}/3, \] (3.21)
under which the number of 6 independent functions reduces to 4.

Gleyzes, Langlois, Piazza, and Vernizzi (GLPV) generalized Horndeski theory in such a way that the coefficients \( A_4, A_5, B_4, \) and \( B_5 \) are not necessarily related to each other. Even in this case, the background and linear perturbation equations of motion about the flat FLRW background remain of second order with no additional scalar propagating degrees of freedom. Taking the inverse procedure to that presented above, the GLPV Lagrangian (3.19) can be expressed in terms of the scalar field \( \phi \) and its covariant derivatives.

The Lagrangians of both Horndeski theory and GLPV theory involve the dependence of \( N, S, K, R, \mathcal{U}, \) and \( t. \)

### B. Hořava-Lifshitz gravity

The renormalization of GR is a difficult task because of the presence of non-linear graviton interactions. Hořava-Lifshitz gravity [51] is an attempt to suppress such non-linear interactions in the UV regime by violating Lorentz symmetry of GR.

In order to understand the basic idea of Hořava, we begin with standard field theory in Minkowski space-time (i.e., without including gravity) [61]. We consider the following anisotropic scaling
\[ t \to c^z t, \quad x \to cx, \] (3.22)
where \( c \) is an arbitrary number, and \( z \) is a number dubbed dynamical critical exponent. Then, the action of a kinetic term of a scalar field \( \varphi \) transforms as
\[ \int dt d^3x \frac{1}{2} \dot{\varphi}^2 \to c^{3-z+2s} \int dt d^3x \frac{1}{2} \dot{\varphi}^2, \] (3.23)
where we assumed the scaling \( \varphi \to c^s \varphi \) for the field. The kinetic term is invariant under the condition
\[ s = \frac{z - 3}{2}. \] (3.24)

When \( z = 3, \) the scalar field is unchanged \((s = 0)\) under the anisotropic scaling (3.22). If we consider the \( n \)-th order interaction term \( \varphi^n, \) the corresponding action transforms as
\[ \int dt d^3x \varphi^n \to c^{z+3+ns} \int dt d^3x \varphi^n \propto E^{-(z+3+ns)/z} \int dt d^3x \varphi^n, \] (3.25)
where in the last proportionality we used the fact that the energy \( E \) scales as \( E \to c^{-z} E. \) When \( z = 3, \) the exponent \( -(z+3+ns)/z \) is \(-2\) for any \( n, \) so the non-linear interactions are power-counting renormalizable. This power-counting renormalizability also holds for the anisotropic scaling with \( z > 3. \)
The example of a scalar-field action realizing the invariance under the \( z = 3 \) scaling is given by

\[
S_{\text{UV}} = \int dt d^3 x \left( \frac{1}{2} \dot{\varphi}^2 + \frac{\varphi \Delta^3 \varphi}{M^4} \right), \tag{3.26}
\]

where \( M \) is a constant having a dimension of mass. If we also take into account the Lagrangians \( \varphi \Delta^2 \varphi \) and \( \varphi \Delta \varphi \) obeying the \( z = 2 \) and \( z = 1 \) scalings, respectively, the resulting action is

\[
S = \int dt d^3 x \left( \frac{1}{2} \dot{\varphi}^2 + \frac{\varphi \Delta^3 \varphi}{M^4} + c_1 \varphi \Delta^2 \varphi + c_2 \varphi \Delta \varphi \right), \tag{3.27}
\]

where \( c_1 \) and \( c_2 \) are dimensionless constants. In the UV region the third and fourth terms on the r.h.s. of Eq. (3.27) are suppressed relative to the second one, so the action (3.27) reduces to (3.26). This is the regime in which non-linear field interactions are suppressed due to the \( z = 3 \) scaling. In the IR regime the fourth term on the r.h.s. of Eq. (3.27) dominates over the second and third terms, so the resulting action \( S_{\text{IR}} = \int dt d^3 x (\dot{\varphi}^2/2 + c_2 \varphi \Delta \varphi) \) is invariant under the \( z = 1 \) scaling.

Horava \[51\] applied the above idea of anisotropic scaling to the construction of a power-counting renormalizable gravitational theory. Due to the privileged role of time, the theory should respect the symmetry under time reparametrization and time-dependent spatial diffeomorphism:

\[
t \to \tilde{t}(t), \quad x^i \to \tilde{x}^i(t, x^i), \quad (i = 1, 2, 3), \tag{3.28}
\]

under which Lorentz symmetry is explicitly broken. Since the time transformation is not spatially dependent, the foliation of space-time in terms of the hypersurfaces \( \Sigma_t \) is always preserved.

Under the infinitesimal change, \( t \to t + f(t) \) and \( x^i \to x^i + \xi(t, x^i) \), the quantities \( N, N_i, \) and \( h_{ij} \) appearing in the ADM metric (2.1) transform as

\[
\begin{align*}
N & \to N - \dot{f} N - f \dot{N} - \xi \partial_t N, \\
N_i & \to N_i - \dot{f} N_i - f \dot{N}_i - \xi \partial_t N_i - \nabla_i \xi N_j - \nabla_j \xi N_i, \\
h_{ij} & \to h_{ij} - \dot{h}_{ij} - h_{ik} \nabla_j \xi^k - h_{jk} \nabla_i \xi^k.
\end{align*}
\]

If the lapse \( N \) is a function of \( t \) alone, the transformation (3.29) induces only the time-dependent term \(-\partial_t (f N)\). Hence the condition \( N = N(t) \) (dubbed projectability condition) is consistent with the foliation-preserving diffeomorphism. In this case, the acceleration \( a_i = \nabla_i \ln N \) vanishes. We note, however, that the projectability condition is not mandatory and that we can consider a non-projectable version of the theory characterized by \( N = N(t, x^i) \).

In GR, the Lagrangian without matter is simply given by \( L_{\text{GR}} = M_{\text{pl}}^2 R / 2 \), where \( M_{\text{pl}} \) is related to the gravitational constant \( G \) as \( M_{\text{pl}}^2 = (8 \pi G)^{-1} \). Using the relation (3.13) and dropping a boundary term, the four-dimensional action of GR reads

\[
S_{\text{GR}} = \int N dt \sqrt{-h} d^3 x L_{\text{GR}}, \quad L_{\text{GR}} = \frac{M_{\text{pl}}^2}{2} \left( S - K^2 + R \right), \tag{3.32}
\]

where \( h \) is a determinant of the metric \( h_{ij} \). Since the extrinsic curvature (2.30) involves a time derivative of the metric \( h_{ij} \), the two scalar quantities \( S = K_{\mu \nu} K^{\mu \nu} \) and \( K^2 = (K_{\mu \nu})^2 \) play the role of kinetic energies associated with the "velocity" \( \partial_t h_{ij} \). In GR, only the combination \( S - K^2 \) is allowed due to a gauge symmetry of the theory.

In Hořava-Lifshitz gravity, both \( S \) and \( K^2 \) are invariant under the foliation-preserving diffeomorphism (3.28). Hence the kinetic Lagrangian of this theory is given by

\[
L_{K} = \frac{M_{\text{pl}}^2}{2} \left( S - \lambda K^2 \right), \tag{3.33}
\]

where \( \lambda \) is an arbitrary constant. GR corresponds to the case \( \lambda = 1 \).

The three-dimensional Ricci scalar \( R \) involves the second derivatives of \( h_{ij} \) with respect to the spatial coordinate \( x^i \), whereas \( S \) and \( K^2 \) possess the second time derivatives of \( h_{ij} \). Under the \( z = 1 \) scaling, i.e., \( t \to ct \) and \( x^i \to cx^i \), the term \( R \) scales in the same way as \( S \) and \( K^2 \). In order to realize the \( z = 3 \) scaling we need to take into account the terms involving six spatial derivatives such as \( Z_1 \) and \( Z_2 \) in Eq. (2.3). In the non-projectable version of Hořava-Lifshitz gravity the acceleration \( a_i = \nabla_i \ln N \) does not vanish, in which case the terms like \( \alpha_4 \) and \( \alpha_5 \) in Eq. (2.10) also exhibit the \( z = 3 \) scaling. Then, the action invariant under the \( z = 3 \) scaling is given by \( S_{V_3} = \int N dt \sqrt{-h} d^3 x L_{V_3} \), with the Lagrangian

\[
L_{V_3} = -\frac{1}{2 M_{\text{pl}}^2} (g_4 Z_1 + g_5 Z_2 + \eta_4 \alpha_4 + \eta_5 \alpha_5 + \cdots), \tag{3.34}
\]
where \( g_4, g_5, \eta_1, \eta_3 \) are dimensionless constants. We do not take into account the terms irrelevant to the discussion of linear cosmological perturbations on the flat FLRW background (such as \( R^3 \)).

Similarly, the Lagrangians corresponding to the \( z = 2 \) and \( z = 1 \) scalings are given, respectively, by

\[
L_{\nu_2} = -\frac{1}{2} \left( g_2 R^2 + g_3 Z + \eta_2 \alpha_2 + \eta_3 \alpha_3 + \cdots \right), \\
L_{\nu_1} = \frac{M_{pl}^2}{2} (R + \eta_1 \alpha_1),
\]

where \( g_2, g_3, \eta_1, \eta_2, \eta_3 \) are dimensionless constants. Summing up all the terms \( 3.33-3.36 \), the action of \( \text{Ho\r{r}ava-Lifshitz} \) gravity is characterized by

\[
L = \frac{M_{pl}^2}{2} \left[ S - \lambda K^2 + R + \eta_1 \alpha_1 - M_{pl}^{-2} \left( g_2 R^2 + g_3 Z + \eta_2 \alpha_2 + \eta_3 \alpha_3 \right) - M_{pl}^{-4} \left( g_4 Z_1 + g_5 Z_2 + \eta_4 \alpha_4 + \eta_5 \alpha_5 \right) \right].
\]

Since this Lagrangian depends on \( S, K, R, Z, Z_1, Z_2, \alpha_i \) \((i = 1, 2, \cdots, 5)\), the theory belongs to the special case of \( 2.11 \). Note that the Lagrangian density \( \mathcal{L} = \sqrt{-g} L \) depends on the lapse \( N \).

The original version of \( \text{Ho\r{r}ava-Lifshitz} \) gravity [51] corresponds to the case \( \dot{N} = N(t) \) and hence \( \alpha_4 = 0 \). This scenario is plagued by pathological behavior associated with the instability of perturbations as well as the strong-coupling problem [52, 53]. These problems can be alleviated in the non-projectable extension of the theory [54, 55]. In Sec. VI we shall discuss this issue after deriving the equations of linear cosmological perturbations.

IV. EQUATIONS OF MOTION FOR THE BACKGROUND AND LINEAR COSMOLOGICAL PERTURBATIONS

In this section we expand the action \( 2.11 \) up to second order in perturbations on the flat FLRW background and derive the background and linear perturbation equations of motion. The linear cosmological perturbations can be decomposed into scalar, vector, and tensor modes [62, 63], among which we focus on the dynamics of scalar perturbations in this paper. Let us consider the perturbed line element with four scalar variables \( \delta N, \psi, \zeta, \) and \( E \), as

\[
ds^2 = -(1 + 2\delta N)dt^2 + 2\nabla_i \psi dx^i dt + a^2(t) \left[[1 + 2\zeta] \delta ij + 2\partial ij E \right] dx^i dx^j,
\]

where \( a(t) \) is the time-dependent scale factor, and \( \partial ij \equiv \nabla_i \nabla_j - \delta ij \nabla^2 / 3 \). Under the infinitesimal transformation \( t \to t + f(t, x^i) \) and \( x^i \to x^i + \delta x^i \nabla_j \xi(t, x^i) \), where \( f \) and \( \xi \) are scalar functions depending on \( t \) and \( x^i \), the perturbations \( \delta N \) and \( E \) transform as [63]

\[
\delta N \to \delta N - \dot{f}, \\
E \to E - \dot{\xi}.
\]

The spatial gauge transformation is fixed by setting

\[
E = 0,
\]

whose condition is used throughout the paper.

Since in Horndeski and GLPV theories the unitary gauge \( \delta \phi = 0 \) is chosen, the dependence on a scalar field \( \phi \) and its kinetic term \( X \) does not explicitly appear in the action \( 2.11 \). The transformation of the field perturbation is given by \( \delta \phi \to \delta \phi - \dot{f} \phi \), so the temporal gauge transformation is fixed by choosing the unitary gauge. We can employ the action \( 2.11 \) as if no field perturbations are present, but the propagating scalar degree of freedom manifests itself through the metric perturbations \( \delta N, \psi, \) and \( \zeta \). As we will see in Sec. VII the Hamiltonian and momentum constraints allow us to reduce the number of scalar variables further.

In the projectable version of \( \text{Ho\r{r}ava-Lifshitz} \) gravity where the lapse \( N \) is a function of \( t \) alone, we have that \( \delta N = 0 \). This is consistent with the foliation-preserving transformation \( t \to t + f(t) \). In the non-projectable \( \text{Ho\r{r}ava-Lifshitz} \) gravity the lapse \( N \) depends on the spatial coordinate \( x^i \) as well as \( t \), such that \( \delta N = \delta N(x^i, t) \). Then the choice of the gauge \( \delta N = 0 \) is inconsistent with the foliation-preserving transformation, as \( f \) depends on \( x^i \) from Eq. (4.2). In this case the temporal gauge transformation is not fixed, but it is possible to study the evolution of perturbations by appropriately constructing gauge-invariant variables (according to the line of Ref. [64]).

In the following we expand the action \( 2.11 \) up to second order in perturbations for the metric \( 4.1 \) with the gauge choice \( 4.4 \). On the flat FLRW background described by the line element \( ds^2 = -dt^2 + a^2(t)dx^i dx^j \), the extrinsic
curvature and the intrinsic curvature are given, respectively, by \( \tilde{K}_{ij} = H \tilde{h}_{ij} \) and \( \tilde{R}_{ij} = 0 \), where a bar represents background values and \( H \equiv \dot{a}/a \) is the Hubble parameter. Then, the scalar quantities appearing in the Lagrangian of (2.11) read

\[
\bar{N} = 1, \quad \bar{K} = 3H, \quad \bar{S} = 3H^2, \quad \bar{R} = \bar{Z} = \bar{U} = 0, \quad \bar{Z}_1 = \bar{Z}_2 = 0, \quad \bar{\alpha}_1 = \bar{\alpha}_2 = \cdots = \bar{\alpha}_5 = 0. \quad (4.5)
\]

We introduce the perturbed quantities

\[
\delta K_{\mu\nu} = K_{\mu\nu} - H h_{\mu\nu}, \quad \delta K = K - 3H, \quad \delta S = S - 3H^2 = 2H \delta K + \delta K^\rho_\rho \delta K^\nu_\nu, \quad (4.6)
\]

where the last equation arises from the first equation and the definition of \( S \). The scalar quantities \( R \) and \( Z \) associated with the intrinsic curvature appear only as perturbations. Up to quadratic order they can be expressed as

\[
\delta R = \delta_1 R + \delta_2 R, \quad \delta Z = \delta R^\rho_\rho \delta R^\nu_\nu, \quad (4.7)
\]

where \( \delta_1 R \) and \( \delta_2 R \) are first-order and second-order perturbations in \( \delta R \), respectively. Clearly, \( \delta Z \) is a second-order quantity. From the first relation of Eq. (4.6), it follows that

\[
\dot{U} = H \dot{R} + R^\rho_\rho \delta K^\nu_\nu, \quad (4.8)
\]

where the first term on the r.h.s. involves the first-order contribution \( (H \delta_1 R) \) and the second-order contribution \( (H \delta_2 R) \), and the second term corresponds to a second-order quantity. From the definition (2.9) and (2.10) it is clear that the quantities \( Z_1, Z_2, \alpha_i \) \((i = 1, 2, \cdots, 5)\) are second order in perturbations.

The above argument shows that the Lagrangian expanded up to second order is given by

\[
L = \bar{L} + L_N \delta N + L_K \delta K + L_S \delta S + L_R \delta R + L_Z \delta Z + L_U \delta U \\
+ \frac{1}{2} \left( \delta N \frac{\partial}{\partial N} + \delta K \frac{\partial}{\partial K} + \delta S \frac{\partial}{\partial S} + \delta R \frac{\partial}{\partial R} + \delta U \frac{\partial}{\partial U} \right) L + \sum_{i=1}^{2} L_{Z_i} \delta Z_i + \sum_{i=1}^{5} L_{\alpha_i} \delta \alpha_i + O(3), \quad (4.9)
\]

where a comma represents a partial derivative, e.g., \( L_N = \partial L/\partial N \). Dividing the Lagrangian (4.9) into first-order and second-order contributions, we can obtain the equations of motion for the background and linear cosmological perturbations, respectively.

### A. Background equations of motion

In order to derive the first-order Lagrangian, we first compute the combination \( L_K \delta K + L_S \delta S \) in Eq. (4.9). Making use of the second and third relations of Eq. (4.6) and defining the quantity

\[
F \equiv L_K + 2H L_S, \quad (4.10)
\]

it follows that

\[
L_K \delta K + L_S \delta S = F(K - 3H) + L_S \delta K^\rho_\rho \delta K^\nu_\nu. \quad (4.11)
\]

Since \( K = n^\mu_\mu \) from Eq. (2.2), the term \( F K \) is partially integrated to give

\[
\int d^4 x \sqrt{-g} F K = - \int d^4 x \sqrt{-g} F_\mu n^\mu = - \int d^4 \sqrt{-g} \frac{\dot{F}}{N}, \quad (4.12)
\]

up to a boundary term. Expanding the term \( N^{-1} = (1 + \delta N)^{-1} \) up to second order, Eq. (4.11) reduces to

\[
L_K \delta K + L_S \delta S = - \dot{F} - 3HF + \dot{F} \delta N - \dot{F} \delta N^2 + L_S \delta K^\rho_\rho \delta K^\nu_\nu + O(3). \quad (4.13)
\]

The first-order contribution to \( L_R \delta R \) of Eq. (4.9) is given by \( L_R \delta_1 R \), whereas \( L_Z \delta Z \) is second order. Employ the following relation

\[
\int d^4 x \sqrt{-g} \lambda(t) \dot{U} = \int d^4 x \sqrt{-g} \left[ \frac{\lambda(t)}{2} \frac{\dot{R}}{2N} K + \frac{\dot{\lambda}(t)}{2N} \bar{R} \right], \quad (4.14)
\]
which holds for a time-dependent function \( \lambda(t) \) up to boundary terms, we obtain
\[
L_{,\mu}\delta\mathcal{U} = \frac{1}{2} \left( \dot{L}_{,\mu} + 3HL_{,\mu} \right) \delta\mathcal{R} + \frac{1}{2} \left( \ddot{L}_{,\mu} + 3HL_{,\mu} \right) \delta^2\mathcal{R} + \frac{1}{2} \left( L_{,\mu} \delta K - L_{,\mu} \delta N \right) \delta\mathcal{R} + O(3). \tag{4.15}
\]

In summary, the first-order action is given by \( S = \int d^4x \sqrt{-g} L \) with the Lagrangian
\[
L = \tilde{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_{,\mathcal{N}})\delta N + \mathcal{E}\delta\mathcal{R}, \tag{4.16}
\]
where
\[
\mathcal{E} \equiv L_{,\mathcal{R}} + \frac{1}{2} L_{,\mu} + \frac{3}{2} HL_{,\mu}. \tag{4.17}
\]
We define the Lagrangian density as \( \mathcal{L} = \sqrt{-g}L = N\sqrt{h} \tilde{L} \). Then, the zeroth-order and first-order Lagrangian densities read
\[
\mathcal{L}_0 = a^3(\tilde{L} - \dot{\mathcal{F}} - 3H\mathcal{F}), \tag{4.18}
\]
\[
\mathcal{L}_1 = a^3(\tilde{L} + L_{,\mathcal{N}} - 3H\mathcal{F})\delta N + (\tilde{L} - \dot{\mathcal{F}} - 3H\mathcal{F})\delta\sqrt{h} + a^3\mathcal{E}\delta\mathcal{R}, \tag{4.19}
\]
where the last term in Eq. (4.19) is a total derivative. Varying the first-order Lagrangian density (4.19) with respect to \( \delta N \) and \( \delta\sqrt{h} \), we obtain the background equations of motion
\[
\tilde{L} + L_{,\mathcal{N}} - 3H\mathcal{F} = 0, \tag{4.20}
\]
\[
\tilde{L} - \dot{\mathcal{F}} - 3H\mathcal{F} = 0. \tag{4.21}
\]
The zero-th order Lagrangian density (4.18) vanishes on account of Eq. (4.21). Combining Eq. (4.20) with Eq. (4.21), it follows that
\[
L_{,\mathcal{N}} + \dot{\mathcal{F}} = 0. \tag{4.22}
\]

Equation (4.20) corresponds to the Friedmann equation related to the Hubble parameter \( H \), whereas Eq. (4.21) is another independent equation associated with the time variation of \( H \). In fact the Lagrangian of GR in the absence of matter is given by \( L = (M_{\text{pl}}^2/2)(S - K^2 + R) \), in which case \( \tilde{L} = -3M_{\text{pl}}^2H^2, L_{,\mathcal{N}} = 0 \), and \( \mathcal{F} = -2M_{\text{pl}}^2H \). Note that, for the theories with \( N = N(t) \), one cannot take the variation with respect to \( \delta N \). Indeed, this happens for the projectable version of Hořava Lifshitz gravity (see Sec. VIII). In the presence of a matter fluid with energy density \( \rho_m \) and pressure \( P_m \), the r.h.s. of Eqs. (4.20), (4.21), and (4.22) are modified as \( \rho_m, -P_m \), and \( \rho_m + P_m \), respectively (see Sec. VII).

**B. Perturbation equations of motion**

Now we explicitly compute the Lagrangian (4.9) to derive linear perturbation equations of motion. On using the relations (4.13) and (4.15) as well, the resulting Lagrangian can be expressed as
\[
L = \tilde{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_{,\mathcal{N}})\delta N + \mathcal{E}\delta\mathcal{R}
+ \left( \frac{1}{2} L_{,\mathcal{N}N} - \dot{\mathcal{F}} \right) \delta N^2 + \frac{1}{2} A\delta K^2 + B\delta K\delta N + \mathcal{C}\delta K\delta\mathcal{R} + \mathcal{D}\delta N\delta\mathcal{R} + \mathcal{E}\delta\mathcal{R} + \frac{1}{2} G\delta\mathcal{R}^2
+ L_{,\mathcal{S}}\delta\mathcal{K}^{\mu}_{\mathcal{S}}\delta\mathcal{K}^{\mu}_{\mathcal{S}} + L_{,\mathcal{Z}}\delta\mathcal{K}^{\mu}_{\mathcal{Z}}\delta\mathcal{K}^{\mu}_{\mathcal{Z}} + \sum_{i=1}^{3} L_{,\mathcal{S}}\delta\mathcal{S}_i\delta\mathcal{S}_i + \sum_{i=1}^{5} L_{,\alpha_i}\delta\mathcal{S}_i + O(3), \tag{4.23}
\]
where
\[
A = L_{,KK} + 4HL_{,SK} + 4H^2L_{,SS}, \tag{4.24}
\]
\[
B = L_{,KN} + 2HL_{,SN}, \tag{4.25}
\]
\[
C = L_{,KR} + 2HL_{,SR} + \frac{1}{2} L_{,\mu} + HL_{,KR} + 2H^2L_{,SU}, \tag{4.26}
\]
\[
D = L_{,NR} - \frac{1}{2} L_{,\mu} + HL_{,NU}, \tag{4.27}
\]
\[
G = L_{,RR} + 2HL_{,RU} + H^2L_{,MU}. \tag{4.28}
\]
Denoting the first-order and second-order Lagrangians of Eq. (4.23) as $L_1$ and $L_2$, respectively, the second-order Lagrangian density can be evaluated as $L_2 = a^3 \delta N L_1 + \delta \sqrt{h} L_1 + a^3 L_2$, i.e.,

$$L_2 = \delta \sqrt{h} \left[ (\dot{\mathcal{F}} + L_N) \delta N + \mathcal{E} \delta \mathcal{R} \right] + a^3 \left( L_N + \frac{1}{2} L_{NN} \right) \delta N^2 + \mathcal{E} \delta \mathcal{R} + \frac{1}{2} \mathcal{A} \delta K^2 + \mathcal{B} \delta K \delta N + \mathcal{C} \delta K \delta \mathcal{R} + (D + \mathcal{E}) \delta N \delta \mathcal{R} + \frac{1}{2} \mathcal{G} \delta \mathcal{R}^2 + L_{,\alpha} \delta K^{\alpha}_{\mu} K_{\mu}^{\nu} + L_{,Z} \delta R^{\nu}_{\mu} R_{\mu}^{\nu} + \sum_{i=1}^{2} L_{,Z_i} \delta Z_i + \sum_{i=1}^{5} L_{,\alpha_i} \delta \alpha_i \right].$$

(4.29)

Here we expanded $\sqrt{-g}$ up to first order, since the second-order contribution is multiplied by the zeroth-order Lagrangian (4.18) and it vanishes identically due to Eq. (4.21).

The next step is to express the perturbed quantities such as $\delta K$ and $\delta \mathcal{R}$ in terms of metric perturbations $\delta N, \psi$, and $\zeta$. We recall that the extrinsic curvature is given by Eq. (4.3). Since $h_{ij} = a^2(t)(1 + 2\zeta)\delta_{ij}$ for the gauge choice (4.4), the first-order extrinsic curvature can be expressed as

$$\delta K^i_j = (\zeta - H\delta N)\delta^i_j - \frac{1}{2a^2}\delta^{ik}(\partial_k N_j + \partial_j N_k).$$

(4.30)

Here the three-dimensional derivatives like $\nabla_i N_j$ have been replaced by partial derivatives like $\partial_i N_j$, as the Christoffel symbols $\Gamma^k_{ij}$ are first-order perturbations. Using the property $N_i = \partial_i \psi$ and taking the trace of Eq. (4.30), we obtain

$$\delta K = 3(\zeta - H\delta N) - \Delta \psi,$$

(4.31)

where

$$\Delta = \nabla_i \nabla^i = \frac{1}{a^2(t)} \partial^2 = \frac{1}{a^2(t)} \partial^2.$$

(4.32)

The operator $\Delta$ involves the scale factor $a(t)$, so we need to be careful when we take time derivatives of the quantities like $\Delta \psi$.

In Eq. (4.24) the perturbation $\delta \sqrt{h}$ is equivalent to $3a^2 \zeta$. The perturbations of the intrinsic curvature are given by

$$\delta \mathcal{R}_{ij} = - \left( \partial_i \partial^j \zeta + \partial_i \partial_j \zeta \right), \quad \delta \mathcal{R} = -4a^{-2} \partial \zeta^2, \quad \delta_2 \mathcal{R} = -2a^{-2} \left[ (\partial \zeta)^2 - 4 \zeta \partial \zeta^2 \right],$$

(4.33)

where $(\partial \zeta)^2 \equiv \partial^i \partial_j \zeta \partial_j \zeta$. On using these relations, the term $L_{,Z_i} \delta Z_i$ can be evaluated as

$$L_{,Z_i} \delta Z_i = L_{,Z_i} (\nabla_i \delta \mathcal{R})(\nabla_i \delta \mathcal{R}) = 16 L_{,Z_i} (\nabla^i \Delta \zeta)(\nabla_i \Delta \zeta) = 16 L_{,Z_i} (\Delta \partial^i \zeta)(\Delta \partial_i \zeta),$$

(4.34)

which is valid up to second order in perturbations. Similarly, we have

$$L_{,Z_2} \delta Z_2 = L_{,Z_2} (h_{jk} \Delta \partial^i \zeta + \partial_i \partial_j \partial_k \zeta)(\partial^i \partial^j \partial^k \zeta) = 4 L_{,Z_2} (\Delta \partial^i \zeta)(\Delta \partial_i \zeta),$$

(4.35)

up to a boundary term. The perturbed quantities associated with the acceleration read

$$L_{,\alpha_1} \delta \alpha_1 = L_{,\alpha_1} (\partial_i \delta N)(\partial^i \delta N),$$

(4.36)

$$L_{,\alpha_2} \delta \alpha_2 = L_{,\alpha_2} (\partial_i \delta N)(\partial^i \delta N),$$

(4.37)

$$L_{,\alpha_3} \delta \alpha_3 = 4 L_{,\alpha_3} (\partial_i \delta N)(\partial^i \delta N),$$

(4.38)

$$L_{,\alpha_4} \delta \alpha_4 = L_{,\alpha_4} (\partial_i \delta N)(\partial^i \delta N),$$

(4.39)

$$L_{,\alpha_5} \delta \alpha_5 = 4 L_{,\alpha_5} (\partial_i \delta N)(\partial^i \delta N).$$

(4.40)

Substituting the relations (4.30), (4.31), (4.32), and (4.40) into Eq. (4.21) and using the background equation (4.22), it follows that

$$\mathcal{L}_2 = a^3 \left\{ \frac{1}{2} \left[ 2L_N + L_{NN} - 6HW - 3H^2(3A + 2L_s) \right] \delta N^2 + \left[ W(3\zeta - \Delta \psi) + 4(3HC - D) \Delta \zeta \right] \right\} \delta N \,$$

$$- (3A + 2L_s) \zeta \Delta \psi \Delta \zeta + \left( \frac{9}{2} A + 3L_s \right) \zeta^2 + 2E \frac{(\partial \zeta)^2}{a^2}$$

$$+ \frac{1}{2} (A + 2L_s)(\Delta \psi)^2 + 4C(\Delta \psi)(\Delta \zeta) + 2(4G + 3L_z)(\Delta \zeta)^2$$

$$+ 2(8L_z + 3L_{zz})(\Delta \partial^i \zeta)(\Delta \partial_i \zeta) + L_{,\alpha_1} (\partial_i \delta N)(\partial^i \delta N) + L_{,\alpha_2} (\partial_i \delta N)(\partial^i \delta N)$$

$$+ 4 L_{,\alpha_3} (\partial_i \delta N)(\partial^i \delta N) + 4 L_{,\alpha_4} (\partial_i \delta N)(\partial^i \delta N) + 4 L_{,\alpha_5} (\partial_i \delta N)(\partial^i \delta N),$$

(4.41)
where

\[ \mathcal{W} = B - 3AH - 2L_sH. \]  

(4.42)

The Lagrangian density (4.41) involves spatial derivatives higher than second order, so its variation with respect to \( \delta N \) corresponds to the equation of motion

\[
\frac{\partial L_2}{\partial (\delta N)} - \partial_i \left( \frac{\partial L_2}{\partial (\partial_i \delta N)} \right) + \partial_i \partial_j \left( \frac{\partial L_2}{\partial (\partial_i \partial_j \delta N)} \right) - \partial_i \partial_j \partial_k \left( \frac{\partial L_2}{\partial (\partial_i \partial_j \partial_k \delta N)} \right) + \cdots = 0.
\]

(4.43)

For example, the term \( a^3 L_{a2}(\partial_\alpha \delta N) \Delta(\partial^\alpha \delta N) \) is equivalent to \(-a^3 L_{a2}(\partial_\alpha \partial_\beta \delta N)(\partial^\alpha \partial^\beta \delta N) = -a^3 L_{a2}(\partial_\alpha \partial_\beta \delta N)^2/a^4 \) up to a boundary term, so that the third term on the l.h.s. of (4.43) gives rise to the contribution \(-2a^3 L_{a2} \Delta^2 \delta N \).

Taking the variations of other terms as well, Eq. (4.43) leads to the following Hamiltonian constraint:

\[
[2L_{,N} + L_{,NN} - 6H\mathcal{W} - 3H^2(3A + 2L_s)] \delta N + \mathcal{W}(3\dot{\zeta} - \Delta \psi) + 4(3HC - D - E)\Delta \zeta \\
-2L_{,\alpha_1} \Delta \delta N - 2L_{,\alpha_2} \Delta^2 \delta N - 4L_{,\alpha_3} \Delta^2 \zeta - 2L_{,\alpha_4} \Delta^3 \delta N - 4L_{,\alpha_5} \Delta^3 \zeta = 0.
\]

(4.44)

Varying the Lagrangian (4.41) with respect to \( \psi \) gives the following momentum constraint:

\[
\mathcal{W} \delta N + (3A + 2L_s)\dot{\zeta} - 4C \Delta \zeta - (A + 2L_s) \Delta \psi = 0.
\]

(4.45)

Finally, the variation of (4.41) in terms of \( \zeta \) leads to

\[
\dot{\mathcal{Y}} + 3HY - 4(3HC - D - E) \Delta \delta N + 4E \Delta \zeta + 12C \Delta \zeta \\
-4C \Delta^2 \psi - 4(4G + 3L_{,z}) \Delta^2 \zeta + 4(8L_{,z1} + 3L_{,z2}) \Delta^3 \zeta + 4L_{,\alpha_2} \Delta^2 \delta N + 4L_{,\alpha_3} \Delta^3 \delta N = 0,
\]

(4.46)

where

\[ \mathcal{Y} \equiv 3 \left[ \mathcal{W} \delta N + (3A + 2L_s) \dot{\zeta} - 4C \Delta \zeta \right] - (3A + 2L_s) \Delta \psi. \]

(4.47)

Using the momentum constraint (4.45), the quantity (4.47) reduces to

\[
\mathcal{Y} = 4L_s \Delta \psi = 4L_s \frac{\delta^2 \psi}{a^2},
\]

(4.48)

and hence \( \dot{\mathcal{Y}} + 3HY = \Delta[4(4L_s \dot{\psi} + L_s \dot{\psi} + HL_s \psi)]. \) Then, Eq. (4.46) can be expressed in the following form

\[
\Delta[4L_s \dot{\psi} + L_s \dot{\psi} + HL_s \psi + E \zeta + 3C \dot{\zeta} - (3HC - D - E) \delta N \\
-\mathcal{C} \Delta \psi - (4G + 3L_{,z}) \Delta \zeta + (8L_{,z1} + 3L_{,z2}) \Delta^2 \zeta + L_{,\alpha_2} \Delta^2 \delta N + L_{,\alpha_3} \Delta^3 \delta N] = 0. 
\]

(4.49)

The dynamics of linear cosmological perturbations is known by solving Eqs. (4.44), (4.45), and (4.46) or (4.49) together with the background equations of motion (4.20) and (4.21).

C. Second-order linear perturbations

The linear perturbation equations (4.44), (4.45), and (4.46) involve time derivatives up to second order, but there are spatial derivatives higher than two. Let us consider second-order theory in which spatial derivatives and the combination of time and spatial derivatives remain of second order.

The higher-order spatial derivative terms on the second line of Eq. (4.46) are absent under the conditions \( C = 0, 4G + 3L_{,z} = 0, 8L_{,z1} + 3L_{,z2} = 0, L_{,\alpha_1} = 0, \) and \( L_{,\alpha_3} = 0. \) In addition we also require \( L_{,\alpha_2} = 0 \) and \( L_{,\alpha_4} = 0 \) for the absence of higher-order derivatives in Eq. (4.44). Provided that \( \mathcal{W} \neq 0, \) the perturbation \( \delta N \) is related with the combination \( (A + 2L_s) \Delta \psi \). Then, the third term on the l.h.s. of Eq. (4.46) induces the fourth-order spatial derivative \( \Delta^2 \psi, \) which can be eliminated under the condition \( A + 2L_s = 0. \) As we will see in Sec. V, this condition is satisfied in Horndeski theory, but not in Hořava-Lifshitz gravity. This makes sense as Hořava-Lifshitz gravity involves spatial derivatives higher than two by construction.

From Eq. (4.45), the perturbation \( \delta N \) also depends upon the term \( (3A + 2L_s) \dot{\zeta} \) for \( \mathcal{W} \neq 0. \) The third term on the l.h.s. of Eq. (4.46) gives rise to the combination of time and spatial derivatives \( \Delta \zeta \) higher than two, but this exactly
cancels another term (as we will see below). On the other hand, the term $-2L_{,\alpha_1}\Delta \delta N$ in Eq. (4.44) does not vanish unless the condition $L_{,\alpha_1} = 0$ is satisfied.

In summary, the spatial derivatives and the combination of time and spatial derivatives higher than second order are absent for linear perturbations under the conditions

$$C = 0, \quad 4G + 3L_{,Z} = 0, \quad A + 2L_{,S} = 0, \quad 8L_{,Z_1} + 3L_{,Z_2} = 0, \quad L_{,\alpha_1} = L_{,\alpha_2} = \cdots = L_{,\alpha_5} = 0. \quad (4.50)$$

We note that, even if these conditions are satisfied, higher-order derivative terms may be present for non-linear perturbations [31].

Under the conditions (4.50), Eqs. (4.44), (4.45), and (4.49) reduce, respectively, to

$$\begin{align*}
(2L_{,N} + L_{,NN} - 6HW + 12H^2L_{,S})\, \delta N + W(3\dot{\zeta} - \Delta \psi) - 4(D + E)\Delta \zeta &= 0, \\
W\delta N - 4L_{,S}\dot{\zeta} &= 0, \\
\frac{1}{a^3} \frac{d}{dt}(a^3L_{,S}\Delta \psi) + E\Delta \zeta + (D + E)\Delta \delta N &= 0.
\end{align*} \quad (4.51)$$

Provided that $W = L_{,KN} + 2HL_{,SN} + 4HL_{,S} \neq 0$, we have $\delta N = 4L_{,S}\dot{\zeta}/W$ from Eq. (4.52). Substituting this relation into Eq. (4.51), we obtain

$$\Delta \psi = \frac{Q_s}{2L_{,S}} \frac{\dot{\zeta} - 4(D + E)}{W} \Delta \zeta, \quad (4.54)$$

where

$$Q_s = \frac{2L_{,S}}{W^2} \left[3W^2 + 4L_{,S}(2L_{,N} + L_{,NN} - 6HW + 12H^2L_{,S})\right]. \quad (4.55)$$

Expressing the terms $\Delta \delta N$ and $\Delta \psi$ of Eq. (4.53) in terms of $\zeta$ and its derivatives, we find that the two terms $\Delta \dot{\zeta}$ cancels out and that the resulting equation of motion is given by

$$\frac{d}{dt} \left(a^3Q_s\dot{\zeta}\right) - aQ_sc_s^2\partial^2\zeta = 0, \quad (4.56)$$

where

$$c_s^2 = \frac{2}{Q_s} \left(\mathcal{M} + HM - E\right), \quad (4.57)$$

with

$$\mathcal{M} = \frac{4L_{,S}(D + E)}{W} = \frac{4L_{,S}}{W} \left(L_{,R} + L_{,NR} + HL_{,NU} + \frac{3}{2} HL_{,U}\right). \quad (4.58)$$

Substituting the relations (4.52) and (4.54) into Eq. (4.41) for $W \neq 0$, the second-order Lagrangian density reads

$$L_2 = a^3Q_s \left[\dot{\zeta}^2 - c_s^2 \partial^2(\zeta)^2\right]. \quad (4.59)$$

In fact, this gives rise to the equation of motion (4.56) for the curvature perturbation $\zeta$. The above results show that, under the conditions (4.50), there is one scalar propagating degree of freedom with second-order equations of motion. The scalar ghost is absent for

$$Q_s > 0. \quad (4.60)$$

In order to avoid small-scale instabilities associated with the Laplacian term $\partial^2\zeta$ in Eq. (4.56), we also require

$$c_s^2 > 0. \quad (4.61)$$

The two conditions (4.60) and (4.61) need to be satisfied for the consistency of second-order gravitational theory.
V. APPLICATION TO HORNDESKI AND GLPV THEORIES

Let us first apply the results in the previous section to Horndeski and GLPV theories. We recall that the spatial gauge transformation is fixed by choosing the gauge \( E = 0 \). We also choose the unitary gauge \( \delta \phi = 0 \) to fix the temporal transformation. In this case, the Horndeski Lagrangian \( \rho \) with \( \phi \) reduces to the form \( \rho \) with the coefficients \( \frac{3}{4} \). Horndeski theory has the two restrictions \( \frac{3}{4} \) among the coefficients, while GLPV theory is described by the Lagrangian \( \frac{3}{4} \) with 6 arbitrary functions \( A_2, A_3, A_4, A_5 \) and \( B_4, B_5 \).

A. Background equations of motion

On using the properties \( \frac{3}{4} \) and the relations \( \phi = \phi(t) \) and \( X = -\phi^2(t)/N^2 \) in unitary gauge, the background equations of motion \( \frac{3}{4} \) and \( \frac{3}{4} \) in GLPV theory read

\[
\begin{align*}
A_2 - 6H^2A_4 - 12H^3A_5 + 2\phi D (A_2X + 3HA_3X + 6H^2A_4X + 6H^3A_5X) &= 0, \\
A_2 - 6H^2A_4 - 12H^3A_5 - \dot{A}_1 + 4H\dot{A}_4 - 4H\dot{A}_5 - 12H \dot{H}A_5 - 6H^2A_5 &= 0,
\end{align*}
\]

respectively. In Horndeski theory the equations of motion can be derived by substituting the functions \( \frac{3}{4} \) into Eqs. \( \frac{3}{4} \) and \( \frac{3}{4} \). In the presence of a perfect fluid with energy density \( \rho_m \) respectively. From Eq. \( \frac{3}{4} \) the functions appearing in the Lagrangian \( \frac{3}{4} \) for the covariant Galileon are given by

\[
G_2 = \frac{c_2}{2}X, \quad G_3 = \frac{c_3}{2M^3}X, \quad G_4 = \frac{M^2}{2} - \frac{c_4}{4M^6}X^2, \quad G_5 = \frac{3c_5}{4M^9}X^2,
\]

where \( c_i (i = 2, 3, 4, 5) \) are dimensionless constants, and \( M \) is a constant having a dimension of mass. The auxiliary functions \( F_3 \) and \( F_5 \) appearing in Eqs. \( \frac{3}{4} \) and \( \frac{3}{4} \) can be chosen as \( F_3 = \frac{c_3X}{6M^3} \) and \( F_5 = \frac{3c_5X^2}{5M^9} \), respectively. From Eq. \( \frac{3}{4} \) the functions appearing in the Lagrangian \( \frac{3}{4} \) for the covariant Galileon are given by

\[
\begin{align*}
A_2 &= \frac{c_2}{2}X, \quad A_3 = \frac{c_3}{3M^3}(-X)^{3/2}, \quad A_4 = -\frac{M^2}{2} - \frac{c_4}{4M^6}X^2, \quad A_5 = \frac{5c_5}{2M^9}(-X)^{5/2}, \\
B_4 &= \frac{M^2}{2}, \quad B_5 = -\frac{3c_5}{5M^9}(-X)^{5/2}.
\end{align*}
\]

Substituting the four functions \( A_2, A_3, A_4, A_5 \) into Eqs. \( \frac{3}{4} \) and \( \frac{3}{4} \), the resulting background equations of motion match with those derived in Refs. \( \frac{3}{4} \) and \( \frac{3}{4} \) by direct variation of the Lagrangians \( \frac{3}{4} \) with the functions \( \frac{3}{4} \).

The covariant Galileon discussed above corresponds to the second-order theory in curved space-time (i.e., it belongs to a class of Horndeski theory). In the limit of Minkowski space-time, the equations of motion for the covariant Galileon are invariant under the Galilean shift \( \partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu \). In fact, the original Galileon theory was constructed by Nicolis et al. \( \frac{3}{4} \) to satisfy this Galilean symmetry in Minkowski space-time. If we replace partial derivatives of the Minkowski Galileon with covariant derivatives, this generally gives rise to derivatives higher than second order \( \frac{3}{4} \). In fact, the covariant Galileon was elaborated to keep the equations of motion up to second order by adding an additional term to the covariantized version of the original Galileon theory.

Although the covariantized Galileon theory contains derivatives higher than two in general space-time, this is not the case for the flat FLRW background. In the presence of the Einstein-Hilbert term \( \frac{3}{4} \) the functions \( A_2, A_3, A_4, A_5 \) for the covariant Galileon are the same as those given in Eq. \( \frac{3}{4} \), but the functions \( B_4 \) and \( B_5 \) are different:

\[
B_4 = \frac{M^2}{2}, \quad B_5 = 0.
\]

Compared to Eq. \( \frac{3}{4} \) the gravitational counter terms \( -c_4X^2/(4M^6) \) and \( -3c_5(-X)^{5/2}/(5M^9) \) are absent, but the same second-order background equations as those for the covariant Galileon follow from Eqs. \( \frac{3}{4} \) and \( \frac{3}{4} \). This second-order property also holds for linear cosmological perturbations on the flat FLRW background (as we will see in Sec. \( \frac{3}{4} \)), but the difference between covariant and covariantized Galileons arises at the level of perturbations \( \frac{3}{4} \).
B. Cosmological perturbations

Let us proceed to the discussion of linear cosmological perturbations. One can easily show that the Lagrangian (3.19) of GLPV theory satisfies all the conditions (4.50), so the perturbation equations of motion on the flat FLRW background remain of second order. Then, the discussion given in Sec. IV C can be applied to GLPV theory. We recall that Eq. (4.50) is valid for

\[ W = A_{3,N} + 4HA_{4,N} + 6H^2A_{5,N} - 4HA_4 - 12H^2A_5 \neq 0. \]  

(5.6)

In Horndeski theory the functions \( A_3, A_4, A_5 \) are given by Eq. (3.20) with the correspondence \( X = -\dot{\phi}^2(t)/N^2 \) in unitary gauge, so the condition (5.6) translates to

\[ W^{(H)} = 4HG_4 + 2\dot{\phi}XG_{3,3} - 16H(XG_{4,X} + X^2G_{4,XX}) + 2\dot{\phi}(G_{4,\phi} + 2XG_{4,\phi X}) \]
\[ -2H^2\dot{\phi}(5XG_{5,X} + 2X^2G_{5,XX}) - 2HX(3G_{5,\phi} + 2XG_{5,\phi X}) \neq 0, \]  

(5.7)

where the index “(H)” represents the values in Horndeski theory. In GR with \( G_4 = M_p^2/2 \) and \( G_3 = G_5 = 0 \) we have \( W^{(H)} = 2M_p^2H \), so \( W^{(H)} \) does not vanish on the cosmological background.

The quantity \( Q_s \) defined in Eq. (4.55) can be expressed as

\[ Q_s = \frac{2L_S}{3W^2} (9W^2 + 8L_Sw), \]  

(5.8)

where

\[ L_S = -A_4 - 3HA_5, \]
\[ w = 3L_N + 3L_N/N - 9HW + 18H^2L_S \]
\[ = 18H^2(A_4 + 3HA_5) + 3(A_2,N - 6H^2A_{4,N} - 12H^3A_{5,N}) + \frac{3}{2}(A_{2,N} + 3HA_{3,N} + 6H^2A_{4,N} + 6H^3A_{5,N}). \]  

(5.9)  

(5.10)

Notice that the quantities \( W, L_S, \) and \( w \) appearing in \( Q_s \) do not depend on the functions \( B_4 \) and \( B_5 \). Hence the no-ghost conditions for the theories with same values of \( A_2, A_3, A_4, A_5 \) but with different values of \( B_4, B_5 \) (like covariant/covariantized Galileons) are equivalent to each other. In Horndeski theory, Eqs. (5.9) and (5.10) reduce to

\[ L_S^{(H)} = G_4 - 2XG_{4,X} - H\phi XG_{5,X} - \frac{1}{2}XG_{5,\phi}, \]  

(5.11)

\[ w^{(H)} = -18H^2G_4 + 3(XG_{2,X} + 2X^2G_{2,XX}) - 18H\dot{\phi}(2XG_{3,X} + X^2G_{3,XX}) - 3X(G_{3,\phi} + XG_{3,\phi X}) \]
\[ + 18H^2(7XG_{4,X} + 16X^2G_{4,XX} + 4X^3G_{4,XXX}) - 18H\dot{\phi}(G_{4,\phi} + 5XG_{4,\phi X} + 2X^2G_{4,\phi XX}) \]
\[ + 6H^3\dot{\phi}(15XG_{5,X} + 13X^2G_{5,XX} + 2X^3G_{5,XXX}) + 9H^2X(6G_{5,\phi} + 9XG_{5,\phi X} + 2X^2G_{5,\phi XX}). \]  

(5.12)

The functions \( \mathcal{E} \) and \( \mathcal{D} + \mathcal{E} \) appearing in the scalar propagation speed square (4.57) are given, respectively, by

\[ \mathcal{E} = B_4 + \frac{1}{2}B_5, \]  

(5.13)

\[ \mathcal{D} + \mathcal{E} = B_4 + B_{4,N} - \frac{1}{2}HB_{5,N}. \]  

(5.14)

Hence the theories with different values of \( B_4, B_5 \) give rise to different scalar propagation speeds. In Horndeski theory there exist the particular relations (3.21) between the coefficients \( A_4, A_5, B_4, B_5 \). In this case we have

\[ L_S^{(H)} = D^{(H)} + \mathcal{E}^{(H)}, \quad M^{(H)} = \frac{4L_S^{(H)}W^{(H)}}{W^{(H)}}. \]  

(5.15)

Under the no-ghost condition \( Q_s > 0 \), the condition (4.61) for the absence of Laplacian instabilities in Horndeski theory translates to

\[ \frac{1}{a} \frac{d}{dt} \left( aM^{(H)} \right) - \mathcal{E}^{(H)} > 0, \]  

(5.16)
where

\[ \mathcal{E}^{(H)} = G_4 + \frac{1}{2}XG_{5,\phi} - XG_{5,\phi}. \] (5.17)

If we consider tensor perturbations \( \gamma_{ij} \) in addition to scalar perturbations, this gives rise to additional conditions for the absence of ghosts and Laplacian instabilities. The three-dimensional metric involving traceless and divergence-free tensor modes (satisfying \( \gamma_{ii} = \partial_i \gamma_{ij} = 0 \)) can be written as [67]

\[ h_{ij} = a^2(t)(1 + 2\zeta)\hat{h}_{ij}, \quad \hat{h}_{ij} = \delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma_{kj}, \quad \det \hat{h} = 1, \] (5.18)

where the term \( \gamma_{ik}\gamma_{kj}/2 \) has been introduced for simplifying the calculations. At linear order, the tensor modes decouple from the scalar modes and they satisfy the relations \( \delta K = 0, \delta K^i_j = \delta^{ik}\gamma_{kj}/2, \delta_2 \mathcal{R} = 0, \) and \( \delta_2 \mathcal{R} = -(\partial_k \gamma_{ij})^2/(4a^2). \) Then, the second-order action for tensor perturbations in GLPV theory reads [31, 37]

\[ S_2^{(h)} = \int d^4x \, a^3 \left[ L_s \left( \delta K^i_j \delta K^k_l - \mathcal{E} \delta_2 \mathcal{R} \right) \right] = \int d^4x \, a^3 \left[ L_s \delta^2 - \frac{\mathcal{E}}{a^2} (\partial_k \gamma_{ij})^2 \right]. \] (5.19)

Then, the conditions for the absence of tensor ghosts and Laplacian instabilities in GLPV theory read

\[ L_s = -A_4 - 3HA_5 > 0, \] (5.20)
\[ \mathcal{E} = B_4 + \frac{1}{2}B_5 > 0. \] (5.21)

Recall that in Horndeski theory the explicit forms of \( L_s \) and \( \mathcal{E} \) are given by Eqs. (5.11) and (5.17), respectively. The no-ghost condition (5.20) does not involve the dependence of \( B_4 \) and \( B_5, \) but the condition (5.21), which is related to the tensor propagation speed square \( c_t^2 = \mathcal{E}/L_s, \) depends on \( B_4 \) and \( B_5. \) Hence the scalar and tensor propagation speed squares are important quantities to distinguish between the theories with same values of \( A_2, A_3, A_4, A_5 \) but with different values of \( B_4, B_5. \)

Under the condition (5.20) the no-ghost condition \( Q_s > 0 \) of scalar perturbations translates to

\[ 9W^2 + 8L_sw > 0. \] (5.22)

We also note that, under the condition (5.21), the quantity \((a\mathcal{M})^4 \) must be positive to realize \( c_t^2 > 0 \) in Eq. (4.57).

The above results are consistent with those derived by direct variation of the original Horndeski Lagrangian (3.1) with (3.2)-(3.5). The quantities \( w_1, w_2, w_3, w_4 \) introduced in Ref. [68] have the correspondence \( w_1 \rightarrow 2L_s^{(H)}, w_2 \rightarrow \Omega^{(H)}, w_3 \rightarrow w^{(H)}, \) and \( w_4 \rightarrow 2\mathcal{E}^{(H)} \) (with the replacement \( X \rightarrow -2X, G_2 \rightarrow P, G_3 \rightarrow -G_3, \) and \( 2G_4 \rightarrow M_{pl}^2F \) to recover Eqs. (18)-(21) of Ref. [68]).

### C. The inflationary power spectra of curvature and tensor perturbations

The scalar degree of freedom appearing in Horndeski and GLPV theories can be responsible for inflation in the early Universe. This is possible if the field \( \phi \) evolves slowly along a nearly flat potential \( V(\phi) \) (slow-roll inflation [3, 9]) or if the presence of higher-order field kinetic terms gives rise to a fixed point characterized by a nearly constant kinetic energy \( (k\text{-inflation}) [56]. \) In both cases the Hubble parameter \( \dot{H} \) is nearly constant during the inflationary period, so the slow-roll parameter defined by

\[ \epsilon \equiv -\frac{\dot{H}}{H^2} \] (5.23)

is much smaller than 1. We assume that the terms without containing the scale factor \( a \) evolve slowly during inflation, so that the quantities

\[ \delta Q_s \equiv Q_s / HQ_s, \quad \delta c_s \equiv c_s / Hc_s \] (5.24)

are much smaller than unity.
The curvature perturbation $\zeta$ generated from quantum fluctuations in the early Universe can be responsible for the origin of observed CMB temperature anisotropies \cite{4}. Let us derive the primordial power spectrum of $\zeta$ generated during inflation. For this purpose we express $\zeta$ in Fourier space, as
\[
\zeta(\tau, x) = \frac{1}{(2\pi)^3} \int d^3k \hat{\zeta}(\tau, k)e^{ikx},
\]
where $\tau = \int a^{-1} dt$ is the conformal time, $k$ is the comoving wavenumber, and
\[
\hat{\zeta}(\tau, k) = u(\tau, k)a(k) + u^*(\tau, -k)a^*(-k).
\]
The annihilation operator $a(k)$ and the creation operator $a^\dagger(k)$ obey the commutation relations
\[
[a(k_1), a^\dagger(k_2)] = (2\pi)^3\delta^{(3)}(k_1 - k_2),
\]
\[
[a(k_1), a(k_2)] = [a^\dagger(k_1), a^\dagger(k_2)] = 0.
\]
In Horndeski and GLPV theories the second-order Lagrangian density for $\zeta$ is given by Eq. \cite{4,59}. Defining a rescaled field $v = zu$ with $z = a\sqrt{2Q_s}$, the kinetic term in the second-order action $S_2 = \int d^4x L_2$ can be expressed as $\int d\tau d^3x v''^2/2$, where a prime represents a derivative with respect to $\tau$. Hence $v$ corresponds to a canonical field associated with the quantization procedure. From Eq. \cite{4,59} the field $v(\tau, k)$ obeys the equation of motion
\[
v'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v = 0.
\]
Since $H$ is nearly constant during inflation, it follows that $\tau \simeq -1/(aH)$ (where the integration constant is set to 0). As long as $Q_s$ is nearly constant (i.e., $|\delta Q_s| \ll 1$), the quantity $z''/z$ in Eq. \cite{528} is approximately given by
\[
z''/z \simeq 2/\tau^2.
\]
If we go back to the asymptotic past ($\tau \to -\infty$), Eq. \cite{528} reduces to $v'' + c_s^2k^2v \simeq 0$. Choosing the Bunch-Davies vacuum in this limit, the solution to this equation is given by $v = e^{-ic_sk\tau}/\sqrt{2c_s}k$ for $\tau \to -\infty$. The term $z''/z$ characterizes the effect of gravity, which becomes comparable to $c_s^2k^2$ for $c_s k \approx aH$. Since the gravitational term dominates over $c_s^2k^2$ in the regime $c_s k \ll aH$, the solution to Eq. \cite{528} is given by $v \propto z$, i.e., $u = \text{constant}$. In other words, the Fourier components of $\zeta$ are "frozen" for $c_s k < aH$.

More precisely, the solution to Eq. \cite{528} on the de Sitter background recovering the Bunch-Davies vacuum in the asymptotic past is given by
\[
u(\tau, k) = \frac{iH e^{-ic_s k\tau}}{2(c_s k)^3/2\sqrt{Q_s}}(1 + ic_s k\tau).
\]
Strictly speaking, the Hubble parameter varies during inflation, but its effect appears only as a next-order slow-roll correction to the power spectrum \cite{65}. We are interested in the two-point correlation function of $\zeta$ in the regime $c_s k \ll aH$, i.e., the vacuum expectation value $\langle 0|\zeta(\tau, k_1)\zeta(\tau, k_2)|0\rangle$ at $\tau \approx 0$. We define the scalar power spectrum $P_\zeta(k_1)$, as
\[
\langle 0|\zeta(0, k_1)\zeta(0, k_2)|0\rangle = \frac{2\pi^2}{k_1^3}P_\zeta(k_1)(2\pi)^3\delta^{(3)}(k_1 + k_2).
\]
On using the solution \cite{528}, we obtain
\[
P_\zeta = \frac{H^2}{8\pi^2Q_s c_s^3}.
\]
Since the curvature perturbation is frozen in the regime $c_s k \ll aH$, we can compute the power spectrum \cite{531} at $c_s k = aH$ during inflation. The result \cite{531} matches with that derived in Horndeski theory \cite{4,68,70}. From the Planck data the scalar amplitude is constrained to be $P_\zeta \simeq 2.2 \times 10^{-9}$ for the wavenumber $k_0 = 0.002$ Mpc$^{-1}$ \cite{8}. We define the spectral index of $P_\zeta$, as
\[
n_s - 1 \equiv \frac{d\ln P_\zeta}{d\ln k}
\]
\[
r_{c_s k = aH} \simeq -2\epsilon - \delta Q_s - 3\delta c_s.
\]
Since the slow-roll parameters $\epsilon, \delta Q_s, \delta c_s$ are much smaller than 1, the power spectrum $P_\zeta$ is close to scale-invariant ($n_s \simeq 1$). Since the deviation of $n_s$ from 1 is different depending on the models of inflation, we can distinguish them.
from precise measurements of the CMB temperature anisotropies \[71\, 72\]. Assuming that the running spectral index \(\alpha_s = dn_s/\ln k |_{c_i k = aH}\) is negligible, the Planck data put the constraint \(n_s = 0.9603 \pm 0.0073\) at 68 \% confidence level \[3\].

For gravitational waves we need to express the tensor perturbation \(\gamma_{ij}\) in terms of the two polarization modes \(e^+_{ij}\) and \(e^-_{ij}\), as \(\gamma_{ij} = h_+ e^+_{ij} + h_\times e^-_{ij}\). In Fourier space, the polarization tensors satisfy the normalization condition \(e^+_{ij}(k) e^+_{ij}(-k)^* = 2\) for each mode and the orthogonality condition \(e^+_{ij}(k) e^-_{ij}(-k)^* = 0\). Then, the second-order action \[5.19\] reads

\[
S_2^{(h)} = \sum_{\lambda=+,\times} \int d^4x \, a^3 Q_t \left[ \dot{h}_\lambda^2 - \frac{c_t^2}{a^2} (\partial h_\lambda)^2 \right],
\]

where

\[
Q_t = \frac{L_S}{2}, \quad c_t^2 = \frac{E}{L_S}.
\]

Following the similar procedure to that for scalar perturbations, it is straightforward to derive the power spectrum \(P_h\) of gravitational waves \[37\]. In the regime \(c_t k < aH\) the tensor perturbation is frozen, so the resulting power spectrum is given by

\[
P_h = \frac{H^2}{2\pi^2 Q_t c_t^4},
\]

which should be evaluated at \(c_t k = aH\).

From Eqs. \[5.34\] and \[5.35\] the tensor-to-scalar ratio reads

\[
r \equiv \frac{P_h}{P_\zeta} = \frac{Q_s c_t^3}{Q_t c_t^4}.
\]

The combined analysis of the Planck data with the WMAP large-angle polarization measurement and ACT/SPT temperature data put the constraint \(r < 0.11\) at 95 \% confidence level \[3\]. The scalar spectral index \(n_s\) and the tensor-to-scalar ratio \(r\) are two key quantities to distinguish between many inflationary models. See Refs. \[72\] for detailed constraints on the inflationary models in the framework of Horndeski theory.

## D. Dark energy in the presence of matter

Let us consider the application of Horndeski and GLPV theories to dark energy. We assume that the scalar degree of freedom \(\phi\) is responsible for the late-time cosmic acceleration. In order to discuss the cosmological dynamics associated with dark energy, we need to incorporate other sources of matter such as dark matter, baryons, and radiation. For this purpose, we take into account k-essence type matter described by the Lagrangian \(P(\varphi, Y)\) \[50\, 53\] \[73\, 74\], where \(P\) is an arbitrary function of another scalar field \(\varphi\) and its kinetic energy \(Y = g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi\). A more general system with multiple scalars (to accommodate both non-relativistic matter and radiation) has been studied in detail in Ref. \[41\].

The action of Horndeski and GLPV theories with k-essence matter is described by

\[
S = \int d^4x \sqrt{-g} \left[ L(N, K, S, R, U; t) + P(\varphi, Y) \right].
\]

The equations of motion for the background and linear perturbations can be derived by expanding the action \[5.37\] up to second order in perturbations. Note that the unitary gauge \(\delta \phi = 0\) is chosen for the dark energy field \(\phi\).

Using the fact that the first-order perturbation of \(Y\) is given by

\[
\delta_1 Y = \dot{2} \dot{\varphi}^2 \delta N - 2 \ddot{\varphi} \delta \varphi,
\]

the first-order Lagrangian density \[4.19\] is modified to

\[
\mathcal{L}_1 = a^3 (\bar{L} + L_N - 3HF + P + 2P\varphi^2) \delta N + (\bar{L} - \dot{\mathcal{F}} - 3HF + P) \delta 1 + a^3 P\varphi \delta \varphi - 2a^3 P\varphi \ddot{\varphi} + a^3 E \delta_1 R.
\]
Varying the Lagrangian density \( L_2 \) with respect to \( \delta N, \delta \sqrt{\Omega}, \delta \varphi \), it follows that

\[
\begin{align*}
\bar{L} + L_N - 3H\mathcal{F} &= \rho_m, \\
\bar{L} - \bar{F} - 3H\mathcal{F} &= -P_{m}, \\
\frac{d}{dt}(a^3 P_Y \dot{\varphi}) + \frac{1}{2} a^3 P_{\varphi} &= 0,
\end{align*}
\]

where \( \rho_m \) and \( P_m \) are energy density and pressure of the scalar field \( \varphi \), defined, respectively, by

\[
\rho_m \equiv -P - 2P_Y \dot{\varphi}^2, \quad P_m \equiv P.
\]

In terms of \( \rho_m \) and \( P_m \), the scalar field equation of motion (5.32) can be written in the form

\[
\dot{\rho}_m = 3H(\rho_m + P_m) = 0,
\]

which corresponds to the standard continuity equation of matter.

In what follows we assume that the field Lagrangian depends on \( Y \) alone, i.e.,

\[
P = P(Y).
\]

In this case the scalar field \( \varphi \) behaves as a barotropic perfect fluid [75]. Expanding the action (5.37) up to second order in perturbations, we find that the following term is added to the Lagrangian density (4.29):

\[
\hat{\mathcal{L}}_2 \equiv P_Y \delta \sqrt{\Omega} \delta_1 Y + a^3 \left( P_Y \delta_2 Y + P_{YY} \delta_1 Y^2 / 2 + P_Y \delta N \delta_1 Y \right),
\]

where the second-order contribution to \( Y \) is given by

\[
\delta_2 Y = -\dot{\delta \varphi}^2 - 3 \dot{\delta \varphi} \delta N^2 + 4 \dot{\delta \varphi} \delta \varphi N + 2 \delta \varphi \partial_j \psi \partial^j \delta \varphi + \frac{1}{a^2} (\delta \dot{\varphi})^2.
\]

Using the fact that both Horndeski and GLPV theories satisfy the conditions (4.50), the second-order Lagrangian density reads [41]

\[
\mathcal{L}_2 = a^3 \left\{ \frac{1}{2} (2L_N + L_{\mathcal{N}} - 6H\mathcal{W} + 12H^2 L_S) \delta N^2 + \left[ W(3\dot{\zeta} - \Delta \psi) - 4(D + \mathcal{E}) \Delta \zeta \right] \delta N + 4L_{SS} \dot{\zeta} \Delta \psi - 6L_{SS} \dot{\zeta}^2 \\
+ 2\dot{\mathcal{E}} \frac{(\partial \zeta)^2}{a^2} + (2 \dot{\phi}^2 P_{YY} - P_{Y}) (\dot{\delta \varphi} \delta N + \delta \varphi^2) - 6 \dot{\delta \varphi} P_{YY} \dot{\varphi} \delta \varphi - 2 \dot{\varphi} P_{YY} \dot{\varphi} \Delta \psi + P_Y \frac{(\delta \varphi)^2}{a^2} \right\}.
\]

Variations of this Lagrangian density with respect to \( \delta N \) and \( \psi \) lead to the following Hamiltonian and momentum constraints respectively:

\[
(2L_N + L_{\mathcal{N}} - 6H\mathcal{W} + 12H^2 L_S) \delta N + W(3\dot{\zeta} - \Delta \psi) - 4(D + \mathcal{E}) \Delta \zeta + 2\dot{\varphi} (P_{YY} - 2 \dot{\phi}^2 P_{YY}) (\delta \varphi - \dot{\varphi} \delta N) = 0,
\]

\[
W \delta N - 4L_{SS} \dot{\zeta} + 2 \dot{\phi} P_{YY} \delta \varphi = 0.
\]

We solve Eqs. (5.49) and (5.50) for \( \delta N, \Delta \psi \) and then substitute them into Eq. (5.48). The resulting Lagrangian density is expressed in the following form

\[
\mathcal{L}_2 = a^3 \left( \dot{\lambda}^i K \dot{\lambda}^j - \partial_j \lambda^j G \partial^i \lambda^i - \lambda^i B \dot{\lambda}^i \dot{\lambda}^j - \lambda^i M \dot{\lambda}^j \right),
\]

where \( K, G, B, M \) are \( 2 \times 2 \) matrices, and the dimensionless vector \( \dot{\lambda}^i \) is defined by

\[
\dot{\lambda}^i = (\zeta, \delta \varphi / M_{pl}).
\]

The no-ghost conditions and the scalar propagation speeds are determined by the two matrices \( K \) and \( G \), whose components are

\[
\begin{align*}
K_{11} &= Q_s + \frac{16 L_{SS}^2}{M_{pl}^2 W^2} \dot{\varphi}^2 K_{22}, & K_{22} &= (2 \dot{\phi}^2 P_{YY} - P_{YY}) M_{pl}^2, & K_{12} &= K_{21} &= \frac{4L_{SS} \dot{\varphi}^2}{M_{pl} W^2} K_{22}, \\
G_{11} &= 2(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E}), & G_{22} &= -P_{YY} M_{pl}^2, & G_{12} &= G_{21} &= -\frac{M_{pl} \dot{\varphi}}{L_{SS} M_{pl}} G_{22}.
\end{align*}
\]
where $Q_s$ and $\mathcal{M}$ are defined by Eqs. (4.55) and (4.58), respectively.

Provided that the symmetric matrix $K$ is positive definite, the scalar ghosts are absent. The positivity of $K$ translates to the conditions that the determinants of principal sub-matrices of $K$ are positive [11], i.e.,

$$Q_s + \frac{16L^2S}{M_{pl}^2 W^2} \phi^2 K_{22} > 0, \quad (5.55)$$
$$Q_s K_{22} > 0. \quad (5.56)$$

These conditions are satisfied for $Q_s > 0$ and $K_{22} > 0$.

In the limit of large wavenumber $k$, the Lagrangian density (5.33) gives rise to the dispersion relation

$$\text{det} \left( \omega^2 K - k^2 G/a^2 \right) = 0, \quad (5.57)$$

where $\omega$ is a frequency. Defining the scalar propagation speed $c_s$ as $\omega^2 = c_s^2 k^2/a^2$, Eq. (5.57) translates to

$$\left( c_s^2 K_{11} - G_{11} \right) \left( c_s^2 K_{22} - G_{22} \right) - (c_s^2 K_{12} - G_{12})^2 = 0. \quad (5.58)$$

In Horndeski theory we have the particular relation $\mathcal{M} = 4L^2 S/W$, see Eq. (5.13). Then, the propagation speed $c_{\text{sH}}$ satisfies

$$c_{\text{sH}}^2 K_{12} - G_{12} = -\frac{4L S \dot{\phi}}{M_{pl} W} \left( c_{\text{sH}}^2 K_{22} - G_{22} \right), \quad (5.59)$$

where the lower index “H” represents the values in Horndeski theory. Substituting Eq. (5.59) into Eq. (5.58), we obtain the following two solutions

$$c_{\text{sH1}}^2 = \frac{G_{11} - \left[ 4L S \dot{\phi}/(M_{pl} W) \right]^2 G_{22}}{K_{11} - \left[ 4L S \dot{\phi}/(M_{pl} W) \right]^2 K_{22}} = \frac{1}{Q_s} \left[ 2(\dot{\mathcal{M}} + H \mathcal{M} - \mathcal{E}) + \left( \frac{4L S \dot{\phi}}{W} \right)^2 P_Y \right], \quad (5.60)$$
$$c_{\text{sH2}}^2 = \frac{G_{22}}{K_{22}} = \frac{P_Y}{P_Y - 2 \dot{\phi}^2 P_{YY}}. \quad (5.61)$$

Compared to the result (4.59), the sound speed square $c_{\text{sH1}}^2$ of the dark energy field $\phi$ is modified by the presence of an additional scalar field $\varphi$. Meanwhile, the sound speed square $c_{\text{sH2}}^2$ of k-essence matter (originally derived in Ref. [70]) is not affected by the dark energy field in Horndeski theory.

In GLPV theory the relation (5.59) no longer holds, so the solutions to Eq. (5.58) are not given by Eqs. (5.60) and (5.61). In this case, we can express Eq. (5.59) in the following form

$$\left( c_s^2 - c_{\text{sH1}}^2 \right) \left( c_s^2 - c_{\text{sH2}}^2 \right) = \frac{16L^2 S Q_s W^2}{4L^2 S} \left( \frac{\mathcal{M} W}{4L^2 S} - 1 \right) \phi^2 P_Y \left[ 2c_s^2 - c_{\text{sH2}}^2 \left( \frac{\mathcal{M} W}{4L^2 S} + 1 \right) \right]. \quad (5.62)$$

In Horndeski theory the r.h.s. of Eq. (5.62) vanishes due to the second relation of Eq. (5.13), so we reproduce the solutions (5.60) and (5.61). In GLPV theory both $c_{\text{sH1}}$ and $c_{\text{sH2}}$ are modified by the presence of the dark energy field. This is the important difference to distinguish between the two theories.

For example, let us consider the difference between covariant and covariantized Galileons discussed in Sec. 5.3.4. In both theories the functions $A_2, A_3, A_4, A_5$ appearing in the Lagrangian (3.19) are given by (5.4), so Eqs. (5.40) and (5.41) read

$$3M_{pl}^2 H^2 = \frac{1}{2} c_s^2 \dot{\phi}^2 + \frac{3c_3 H \dot{\phi}^3}{M^3} - \frac{45c_4 H^2 \dot{\phi}^4}{2M^6} + \frac{21c_5 H^3 \dot{\phi}^5}{M^9} + \rho_m, \quad (5.63)$$
$$3M_{pl}^2 H^2 + 2M_{pl}^2 \dot{H} = \frac{1}{2} c_s^2 \dot{\phi}^2 + \frac{3c_3 \dot{\phi}^2 \dot{\phi}}{M^3} - \frac{3c_4 \dot{\phi}^3 \dot{\phi}}{2M^6} + \frac{3c_5 \dot{\phi}^4}{M^9} \left[ 5H \dot{\phi} + 2(H^2 + \dot{H}) \dot{\phi} \right] + \frac{3c_5 H \dot{\phi}^4}{M^9} \left[ 5H \dot{\phi} + 2(H^2 + \dot{H}) \dot{\phi} \right] - P_m. \quad (5.64)$$

These equations show that there exists a de Sitter solution characterized by $H = \text{constant}$ and $\dot{\phi} = \text{constant}$ with $\rho_m = P_m = 0$. In fact, this de Sitter solution can be used for realizing the late-time cosmic acceleration [65, 66].

The perfect fluids of radiation and non-relativistic matter can be accommodated by considering the Lagrangians $P(Y) = b_1 Y^2$ and $P(Y) = b_2 (Y - Y_0)^2$ with $|Y - Y_0| \ll Y_0$, respectively $(b_1, b_2, Y_0$ are constants) [41, 72]. During the radiation and matter eras the background equations of motion allow the existence of tracker solutions characterized by $H \dot{\phi} = \text{constant}$ [60]. Along the tracker, each derivative term on the r.h.s. of Eqs. (5.63) and (5.64) is proportional
to $H^{-1}$. During the cosmological sequence of radiation, matter, and de Sitter epochs, the dark energy equation of state evolves as $w_{\text{DE}} = -7/3 \rightarrow -2 \rightarrow -1$. However, this evolution is in tension with the joint data analysis of SN Ia, CMB, and baryon acoustic oscillations [77]. The solutions approaching the tracker at late times can be compatible with the observational data. For the late-time tracker, the quantity $r_1 \equiv H_{\text{dS}} \phi_{\text{dS}}/(H \dot{\phi})$ (where “dS” represents the values at the late-time de Sitter solution) is initially much smaller than 1 and then $r_1$ approaches the order of unity only recently.

In the covariant Galileon, which belongs to a class of Horndeski theory, the scalar propagation speed square is given by $c_{sH1}^2 = (\Omega_r + 1)/40$ in the regime $r_1 \ll 1$ [62], where $\Omega_r$ is the density parameter of radiation. Since $c_{sH1}^2 = 1/20$ and $1/40$ during the radiation and matter eras respectively, the Laplacian instabilities of the dark energy perturbation are absent in these epochs. Note that the matter sound speed $c_{s1}^2$ is given by $c_{sH1}^2$ is 1/3 for radiation and $c_{sH2}^2 \approx 0$ for non-relativistic matter. The evolution of matter density perturbations and observational tests for the covariant Galileon with large-scale structure data have been studied in Refs. [78].

In the covariantized Galileon, which goes beyond the realm of Horndeski theory, the two propagation speeds $c_s$ are known by solving Eq. (5.62). The detailed analysis in Ref. [41] showed that the matter sound speed square $c_{s2}^2$ is close to the value $c_{sH2}^2$, but the dark energy sound speed square $c_{s1}^2$ of the late-time tracking solution is modified as $c_{s1}^2 = (3\Omega_r - 1)/40$ in the regime $r_1 \ll 1$. Since $c_{s1}^2 = -1/40$ during the matter era, the dark energy model based on the covariantized Galileon is plagued by small-scale Laplacian instabilities. In spite of the fact that the two Galileon theories give rise to the same background equations of motion on the flat FLRW background, they are clearly distinguished by the scalar propagation speeds.

VI. APPLICATION TO HOŘAVA-LIFSHITZ GRAVITY

We apply the results in Sec. [IV] to Hořava-Lifshitz gravity. In what follows we shall consider the projectable and non-projectable versions of the theory separately.

A. Projectable Hořava-Lifshitz gravity

The projectable version [51] corresponds to the case where the lapse $N$ is a function of $t$ alone. In this case all the terms $a_i$ ($i = 1, 2, \cdots, 5$) defined in Eq. (2.11), which come from the acceleration $a_i = \nabla_i \ln N$, vanish. Then, the theory is described by the action

$$S = \int d^4x \sqrt{-g} \left[ L(K, S, R, Z_1, Z_2) + L_m \right],$$

(6.1)

where

$$L = \frac{M_{\text{pl}}^2}{2} \left[ S - \lambda K^2 + R - M_{\text{pl}}^{-2} \left( g_2 R^2 + g_3 Z \right) - M_{\text{pl}}^{-4} \left( g_4 Z_1 + g_5 Z_2 \right) \right],$$

(6.2)

and $L_m$ is a matter Lagrangian.

To derive the background equations of motion, we consider the k-essence matter Lagrangian $L_m = P(\phi, Y)$ introduced in Sec. [IV]. Note that the matter sector can potentially couple to the gravitational sector at high energy, but we simply assume the absence of such couplings in the following discussion.

Since $N = N(t)$ in the present case, the Lagrangian (6.1) cannot be varied with respect to $\delta N$ to derive one of the background equations of motion. Varying the action (6.1) with respect to $\delta \sqrt{h}$, we obtain the same form of equation as (5.41) with $L = 3M_{\text{pl}}^2(1 - 3\lambda)H^2/2, F = M_{\text{pl}}^2(1 - 3\lambda)H$, and the pressure $P_m = P$, i.e.,

$$\frac{3\lambda - 1}{2} M_{\text{pl}}^2 (2 \dot{H} + 3H^2) = - P_m. \quad (6.3)$$

Variation of the action with respect to $\phi$ leads to the same equation as (5.42):

$$\frac{d}{dt} (a^3 P_{\phi, \phi}) + \frac{1}{2} a^3 \ddot{P}_\phi = 0. \quad (6.4)$$

Defining the field energy density $\rho_m = -P - 2P_{\phi, \phi}^2$ as Eq. (5.43), Eq. (6.4) translates to the continuity equation (5.44), i.e.,

$$P_m = - \frac{1}{3a^3 H} \frac{d}{dt} (a^3 \rho_m).$$

(6.5)
Integrating Eq. (6.3) with respect to $t$ after substitution of Eq. (6.5), we obtain

$$\frac{3}{2}(3\lambda - 1)M_{pl}^2 H^2 = \rho_m + \frac{C}{a^3},$$

(6.6)

where $C$ is an integration constant. The extra term $C/a^3$ behaves as non-relativistic dark matter \cite{73}. In the projectable version of Hořava-Lifshitz gravity, such apparent matter arises due to the absence of the Hamiltonian constraint.

Let us proceed to the discussion of cosmological perturbations in the absence of matter ($L_m = 0$). The projectable version of Hořava-Lifshitz gravity is characterized by the gauge choice

$$\delta N = 0,$$

(6.7)

which is consistent with the foliation-preserving transformation $t \rightarrow t + f(t)$. We cannot employ the perturbation equation of motion (4.44) that corresponds to the Hamiltonian constraint. Since $A = -\lambda M_{pl}^2$ and $L_s = M_{pl}^2/2$, $W = (3\lambda - 1)M_{pl}^2 H$, and $C = 0$, Eq. (4.45) gives the relation

$$\Delta \psi = \frac{3\lambda - 1}{\lambda - 1} \zeta,$$

(6.8)

for $\lambda \neq 1$. Then, the quantity $\mathcal{Y}$ defined in Eq. (4.38) is given by $\mathcal{Y} = 2M_{pl}^2(3\lambda - 1)\zeta/(\lambda - 1)$. On using the relations $D = 0$, $E = M_{pl}^2/2$, $4G + 3L_z = -(8g_2 + 3g_3)/2$, and $8L_z, z_1 + 3L_z, z_2 = -(8g_4 + 3g_5)/(2M_{pl}^2)$, Eq. (4.46) reads

$$\frac{d}{dt} \left( \frac{3\lambda - 1}{\lambda - 1} \zeta \right) + a^2 O \zeta = 0,$$

(6.9)

where

$$O \equiv \Delta + \frac{\Delta^2}{M_3^2} - \frac{\Delta^3}{M_3^2},$$

(6.10)

and

$$M_2^2 = M_{pl}^2(8g_2 + 3g_3)^{-1}, \quad M_3^4 = M_{pl}^4(8g_4 + 3g_5)^{-1}.$$

(6.11)

Substituting Eqs. (6.7) and (6.8) into the second-order Lagrangian density (4.41), we also obtain

$$\mathcal{L}_2 = M_{pl}^2 a^3 \left( \frac{3\lambda - 1}{\lambda - 1} \zeta^2 - \zeta O \zeta \right),$$

(6.12)

which matches with the results in Refs. [53, 61] (see also Refs. [80, 81]). In fact, variation of $\mathcal{L}_2$ with respect to $\zeta$ leads to the equation of motion (6.9).

The breaking of gauge symmetry in GR ($\lambda \neq 1$) gives rise to the propagation of the scalar degree of freedom $\zeta$. In order to evade the appearance of scalar ghosts, we require that $(3\lambda - 1)/(\lambda - 1) > 0$, i.e., $\lambda > 1$ or $\lambda < 1/3$. At low energy we should recover the behavior similar to GR, so it is natural to focus on the regime $\lambda > 1$.

In Minkowski space-time ($a = 1$), we obtain the following dispersion relation from the Lagrangian density (6.12):

$$\omega^2 = \frac{\lambda - 1}{3\lambda - 1} \left( \frac{k^6}{M_3^4} + \frac{k^4}{M_2^2} - k^2 \right).$$

(6.13)

For the wavenumber $k$ much smaller than $M_3$ and $M_2$ this relation reduces to $\omega^2 \simeq -(\lambda - 1)k^2/(3\lambda - 1)$, so the scalar propagation speed square $c_s^2 = \omega^2/k^2 = -(\lambda - 1)/(3\lambda - 1)$ is negative under the no-ghost condition $(3\lambda - 1)/(\lambda - 1) > 0$. The time scale associated with this Laplacian instability can be estimated as

$$t_L \approx \frac{1}{2} \sqrt{\frac{3\lambda - 1}{\lambda - 1}}.$$

(6.14)

On the cosmological background there is a Hubble friction term $3H \zeta(3\lambda - 1)/(\lambda - 1)$ appearing in Eq. (6.9). Provided that this term dominates over the Laplacian term $-k^2 \zeta$, it is possible to avoid the instability of scalar perturbations. This translates to the condition $t_L \gg H^{-1}$. We also note that the time scale associated the growth of large-scale structures in the Universe is given by $t_L \approx M_{pl}/\sqrt{\rho}$, where $\rho$ is the energy density of non-relativistic matter. As long
as \( t_L \gg t_J \), the structure formation is not affected by the Laplacian instability \[61, 82\]. In the regime where the wavenumber \( k \) is larger than \( M_2 \) and \( M_3 \) the first two terms on the r.h.s. of Eq. \[6.13\] dominate over \(-k^2\), so the Laplacian instability is absent.

In order to recover the behavior close to GR at low energy, we require that \( \lambda \) is sufficiently close to 1. If we expand the original action \[6.1\] up to \( n \)-th order (\( n > 2 \)) in perturbations, the \( n \)-th order Lagrangian density contains the terms with negative powers \((\lambda - 1)^{-n} \) (which blows up for \( \lambda \) close to 1). For larger \( n \) the divergence of these terms in the limit \( \lambda \to 1 \) gets worse, so the perturbative expansion breaks down. This is the strong coupling problem of the projectable version of Hořava-Lifshitz gravity pointed out in Refs. \[52, 53\].

\[\text{B. Non-projectable Hořava-Lifshitz gravity}\]

The non-projectable version of Hořava-Lifshitz gravity \[54\] corresponds to the case where the lapse \( N \) depends on both \( t \) and \( x^i \). Since the acceleration \( a_i = \nabla_i \ln N \) does not vanish, the theory is described by the action

\[
S = \int d^4 x \sqrt{-g} \left[ L(K, S, R, Z, Z_1, Z_2, \alpha_1, \ldots, \alpha_5) + L_m \right],
\]

where

\[
L = \frac{M_{pl}^2}{2} \left[ S - \lambda K^2 + R + \eta_1 \alpha_1 - M_{pl}^{-2} \left( g_2 R^2 + g_3 Z + \eta_2 \alpha_2 + \eta_3 \alpha_3 \right) - M_{pl}^{-4} \left( g_4 Z_1 + g_5 Z_2 + \eta_4 \alpha_4 + \eta_5 \alpha_5 \right) \right].
\]

At the background level, the quantities \( R, Z, Z_1, Z_2 \) and \( \alpha_i \) (\( i = 1, 2, \ldots, 5 \)) vanish. The important difference from the projectable Hořava-Lifshitz gravity is that, in the non-projectable version, there is the equation of motion derived by the variation of \( \delta N \). In the presence of matter with energy density \( \rho_m \) and pressure \( P_m \), variation of the action \[6.15\] with respect to \( \delta N \) leads to the same form as Eq. \[5.40\] and hence

\[
\frac{3}{2} (3\lambda - 1) M_{pl}^2 \dot{H}^2 = \rho_m.
\]

Unlike Eq. \[6.6\], Eq. \[6.17\] does not contain the term \( C/\alpha^3 \). The momentum constraint and the matter equation of motion are the same as Eqs. \[6.3\] and \[6.5\], respectively.

In the non-projectable Hořava-Lifshitz gravity it is inconsistent to choose the gauge \( \delta N(t, x^i) = 0 \) because the gauge transformation \[4.2\] involves the scalar \( \tilde{f} \) depending on \( t \) alone. The temporal gauge transformation is not fixed in the following discussion. For the discussion of perturbations we do not take into account the contribution of matter, but it is straightforward to do so. Since \( \mathcal{W} = (3\lambda - 1) M_{pl}^2 \dot{H}, L_{\alpha_1} = M_{pl}^2 \eta_1 / 2, L_{\alpha_2} = -\eta_2 / 2, L_{\alpha_3} = -\eta_3 / 2, L_{\alpha_4} = -\eta_4 / (2 M_{pl}^2), \) and \( L_{\alpha_5} = -\eta_5 / (2 M_{pl}^2) \), Eqs. \[4.44\], \[4.45\] and \[4.49\] reduce, respectively, to

\[
(3\lambda - 1) M_{pl}^2 \dot{H} (3\dot{\zeta} - 3H \delta N - \Delta \psi) - 2 M_{pl}^2 \Delta \zeta - M_{pl}^2 \eta_1 \Delta \delta N + \eta_2 \Delta^2 \delta N + 2\eta_3 \Delta^2 \zeta + \frac{\eta_4}{M_{pl}^2} \Delta^3 \delta N + \frac{2\eta_5}{M_{pl}^2} \Delta^3 \zeta = 0, \quad (6.18)
\]

\[
(3\lambda - 1)(\dot{\zeta} - H \delta N) - (\lambda - 1) \Delta \psi = 0, \quad (6.19)
\]

\[
\Delta (\psi + H \psi + \delta N + \zeta) + \frac{\Delta^2 \zeta}{M_{pl}^2} - \frac{\Delta^3 \zeta}{M_{pl}^4} - \frac{\eta_3}{M_{pl}^2} \Delta^2 \delta N - \frac{\eta_5}{M_{pl}^4} \Delta^3 \delta N = 0, \quad (6.20)
\]

where \( M_2 \) and \( M_3 \) are defined by Eq. \[6.11\]. These equations match with Eqs. \(19\), \(21\), and \(27\) of Ref. \[64\], respectively, in the absence of matter perturbations (after the replacement \( \delta N \to \phi, \psi \to a^2 \beta, \zeta \to -\psi, \) and \( \eta_1 \to \eta \)).

Let us discuss the stability of perturbations on the Minkowski background \( (a = 1 \) and \( H = 0 \)). In the IR regime we can ignore the spatial derivatives higher than two in the perturbation equations of motion. From Eqs. \[6.15\] and \[6.19\] we obtain

\[
\delta N = -\frac{2}{\eta_1} \dot{\zeta}, \quad \Delta \psi = \frac{3\lambda - 1}{\lambda - 1} \dot{\zeta}, \quad (6.21)
\]

Substituting these relations into Eq. \[6.20\], it follows that

\[
\frac{3\lambda - 1}{\lambda - 1} \zeta - \frac{2}{\eta_1} \Delta \zeta = 0. \quad (6.22)
\]
The Lagrangian density leading to this equation of motion can be derived by substituting the relations (6.21) into (4.41), as

\[ L_2 = M_{pl}^2 \frac{3\lambda - 1}{\lambda - 1} \left[ \dot{\zeta}^2 - c_s^2 (\partial \zeta)^2 \right], \]  

(6.23)

where

\[ c_s^2 = \frac{\lambda - 1}{3\lambda - 1} \frac{2 - \eta_1}{\eta_1} \]  

(6.24)

There is a viable parameter space in which both the scalar ghost and the Laplacian instability are absent [54]:

\[ \frac{3\lambda - 1}{\lambda - 1} > 0, \quad 0 < \eta_1 < 2. \]  

(6.25)

This is in contrast with the projectable Hořava-Lifshitz gravity in which the scalar perturbation in the IR regime is unstable in the absence of ghosts. Moreover, the strong-coupling problem in the original Hořava-Lifshitz gravity can be alleviated by the presence of non-vanishing acceleration terms [51, 53].

In order to discuss the cosmology during the radiation and matter eras in the non-projectable Hořava-Lifshitz gravity, we need to add the contribution of matter perturbations to Eqs. (6.18)-(6.20). Although the temporal gauge transformation is not fixed, one can study the evolution of cosmological perturbations by considering some gauge-invariant variables, say \( \zeta_g \equiv \zeta - H \delta \rho_m / \rho_m \), where \( \delta \rho_m \) is the matter density perturbation. The dynamical evolution of such gauge-invariant variables has been investigated in detail in Ref. [64].

VII. CONCLUSIONS

In this paper we have reviewed the EFT approach to modified gravity based on the expansion in terms of cosmological perturbations on the flat FLRW background. This approach is powerful enough to deal with a wide variety of modified gravitational theories in a systematic and unified way. Our starting point is the general action (2.11) that depends on geometric scalar quantities constructed in the 3+1 decomposition of space-time. The expansion of this action up to second order in scalar metric perturbations allows us to identify the propagating scalar degree of freedom.

In addition to the lapse function and several geometric scalars arising from the extrinsic and intrinsic curvatures, we have taken into account spatial derivatives higher than second order such as those given in Eqs. (2.9) and (2.10). This generalizes the analysis of Ref. [31] in such a way that our formalism can be applied not only to Horndeski theory but also to Hořava-Lifshitz gravity. In Sec. III we briefly reviewed both Horndeski theory and Hořava-Lifshitz gravity to show explicit relations with the EFT approach to modified gravity.

In Sec. IV we expanded the action (2.11) up to second order in perturbations for the metric (4.1) with the spatial gauge fixing (4.4). The resulting first-order and second-order actions give rise to the background equations (4.20)-(4.21) and the linear perturbation equations (4.44)-(4.46), respectively. Under the conditions (4.50) the equations of motion for perturbations are of second order, in which case the second-order Lagrangian density is simply given by Eq. (4.59).

In Sec. V we applied our general EFT formalism to Horndeski and GLPV theories. In these theories the linear perturbation equations of motion are of second order on the isotropic cosmological background. We provided general formulas for the primordial power spectra of scalar and tensor perturbations generated during inflation driven by a scalar degree of freedom present in such theories. We also studied the application to dark energy by taking into account an additional matter scalar field and showed that Horndeski and GLPV theories can be distinguished from each other by the scalar propagation speeds \( c_s \). In particular the covariantized Galileon (a class of GLPV theories) is plagued by the Laplacian instability during the matter era for late-time tracking solutions, whereas this is not the case for the covariant Galileon (a class of Horndeski theories).

In Sec. VI we discussed the stability of both projectable and non-projectable versions of Hořava-Lifshitz gravity by employing our general EFT formalism. In the projectable version the lapse \( N \) is a function of the time \( t \) alone, in which case there are no Hamiltonian constraints both for the background and the perturbations. In the IR regime the Laplacian instability is present when the ghost is absent. In the non-projectable version the lapse depends on both space and time, so there is freedom to introduce an acceleration vector \( a_i = \nabla_i \ln N \). We reproduced the linear perturbation equations on both FLRW and Minkowski backgrounds already derived in the literature and showed the existence of a viable parameter space free from ghosts and instabilities.

We expect that our general EFT formalism will be useful for the constructions of viable inflation/dark energy models as well as quantum gravity. This approach is also useful to quantify the non-Gaussianities generated during
inflation \cite{2, 22, 83} and to parametrize the dark energy equation of state and density perturbations associated with CMB and large-scale structures \cite{34, 35}. We hope that we will be able to approach the origins of inflation/dark energy and to construct renormalizable quantum gravity consistent with observations and experiments.

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[1] K. Sato, Mon. Not. R. Astron. Soc. 195, 467 (1981); K. Sato, Phys. Lett. 99B, 66 (1981);
D. Kazanas, Astrophys. J. 241, L59 (1980); A. H. Guth, Phys. Rev. D 23, 347 (1981).
[2] G. Hinshaw et al. [WMAP Collaboration], Astrophys. J. Suppl. 208, 19 (2013) [arXiv:1212.5226 [astro-ph.CO]].
[3] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. to be published (2014) [arXiv:1303.5076 [astro-ph.CO]].
[4] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981); A. H. Guth and S. Y. Pi, Phys. Rev. Lett. 49 (1982) 1110; S. W. Hawking, Phys. Lett. B 115, 295 (1982); A. A. Starobinsky, Phys. Lett. B 117 (1982) 175; J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D 28, 679 (1983).
[5] A. D. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
[6] A. D. Linde, Phys. Lett. B 120, 177 (1983).
[7] A. A. Starobinsky, Phys. Lett. B 91, 99 (1980).
[8] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [astro-ph/9805201];
S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [astro-ph/9812133].
[9] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).
[10] Y. Fujii, Phys. Rev. D 26, 2580 (1982); L. H. Ford, Phys. Rev. D 35, 2339 (1987);
C. Wetterich, Nucl. Phys. B 302, 668 (1988);
T. Chiba, N. Sugiyama and T. Nakamura, Mon. Not. Roy. Astron. Soc. 289, L5 (1997) [astro-ph/9704199];
P. G. Ferreira and M. Joyce, Phys. Rev. Lett. 79, 4740 (1997) [astro-ph/9707286];
R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998) [astro-ph/9708069];
T. Chiba, N. Sugiyama and T. Nakamura, Mon. Not. Roy. Astron. Soc. 301, 72 (1998) [astro-ph/9806332];
I. Zlatev, L. -M. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82, 896 (1999) [astro-ph/9907012].
[11] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000) [astro-ph/9912463];
C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000) [astro-ph/0004134].
[12] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485, 208 (2000) [hep-th/0005016].
[13] S. Capozziello, Int. J. Mod. Phys. D 11, 483 (2002) [gr-qc/0201039];
S. Capozziello, S. Carloni and A. Troisi, Recent Res. Dev. Astron. Astrophys. 1, 625 (2003) [astro-ph/0303041];
S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70, 043528 (2004) [astro-ph/0306438].
[14] W. Hu and I. Sawicki, Phys. Rev. D 76, 064004 (2007) [arXiv:0705.1158 [astro-ph]]; A. A. Starobinsky, JETP Lett. 86, 157 (2007) [arXiv:0706.2041 [astro-ph]]; S. A. Appleby and R. A. Battye, Phys. Lett. B 654, 7 (2007) [arXiv:0705.3199 [astro-ph]]; S. Tsujikawa, Phys. Rev. D 77, 023507 (2008) [arXiv:0709.1391 [astro-ph]].
[15] J. P. Uzan, Phys. Rev. D 59, 123510 (1999) [gr-qc/9903004];
T. Chiba, Phys. Rev. D 60, 083508 (1999) [gr-qc/9903094]; L. Amendola, Phys. Rev. D 60, 043501 (1999) [astro-ph/9904120];
P. Ferrotta, C. Baccigalupi and S. Matarrese, Phys. Rev. D 61, 023507 (1999) [astro-ph/9906066]; B. Boisseau, G. Esposito-Farese, P. Polarski and A. A. Starobinsky, Phys. Rev. Lett. 85, 2236 (2000) [gr-qc/0001066]; S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol and J. Yokoyama, Phys. Rev. D 77, 103009 (2008) [arXiv:0803.1106 [astro-ph]]; R. Gannouji et al., Phys. Rev. D 82, 124006 (2010) [arXiv:1010.3769 [astro-ph.CO]].
[16] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79, 064036 (2009) [arXiv:0811.2107 [hep-th]]; C. Deffayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D 79, 084003 (2009) [arXiv:0901.1314 [hep-th]]; C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D 80, 064015 (2009) [arXiv:0906.1967 [gr-qc]].
[17] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. 106, 231101 (2011) [arXiv:1011.1232 [hep-th]].
[18] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006) [hep-th/0603057]; R. Durrer and R. Maartens, Gen. Rel. Grav. 40, 301 (2008) [arXiv:0711.0077 [astro-ph]].
[72] S. Tsujikawa, J. Ohashi, S. Kuroyanagi and A. De Felice, Phys. Rev. D 88, 023529 (2013) [arXiv:1305.3044 [astro-ph.CO]];
S. Tsujikawa, PTEP 2014, 06B104 (2014) [arXiv:1401.4688 [astro-ph.CO]].
[73] R. J. Scherrer, Phys. Rev. Lett. 93, 011301 (2004) [astro-ph/0402316].
[74] D. Bertacca, S. Matarrese and M. Pietroni, Mod. Phys. Lett. A 22, 2893 (2007) [astro-ph/0703259 [ASTRO-PH]];
D. Bertacca and N. Bartolo, JCAP 0711, 026 (2007) [arXiv:0707.4247 [astro-ph]].
[75] D. Giannakis and W. Hu, Phys. Rev. D 72, 063502 (2005) [astro-ph/0501423];
F. Arroja and M. Sasaki, Phys. Rev. D 81, 107301 (2010) [arXiv:1002.1376 [astro-ph.CO]].
[76] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999) [hep-th/9904176].
[77] S. Nesseris, A. De Felice and S. Tsujikawa, Phys. Rev. D 82, 124054 (2010) [arXiv:1010.0407 [astro-ph.CO]].
[78] A. De Felice, R. Kase and S. Tsujikawa, Phys. Rev. D 83, 043515 (2011) [arXiv:1011.6132 [astro-ph.CO]];
S. A. Appleby and E. V. Linder, JCAP 1208, 026 (2012) [arXiv:1204.3314 [astro-ph.CO]];
A. Barreira, B. Li, C. M. Baugh and S. Pascoli, Phys. Rev. D 86, 124016 (2012) [arXiv:1208.0600 [astro-ph.CO]];
H. Okada, T. Totani and S. Tsujikawa, Phys. Rev. D 87, 103002 (2013) [arXiv:1208.4681 [astro-ph.CO]];
J. Neveu et al., Astron. Astrophys. 555, A53 (2013) [arXiv:1302.2786 [gr-qc]];
A. Barreira, B. Li, A. Sanchez, C. M. Baugh and S. Pascoli, Phys. Rev. D 87, no. 10, 103511 (2013) [arXiv:1302.6241 [astro-ph.CO]].
[79] S. Mukohyama, Phys. Rev. D 80, 064005 (2009) [arXiv:0905.3563 [hep-th]].
[80] T. Takahashi and J. Soda, Phys. Rev. Lett. 102, 231301 (2009) [arXiv:0904.0553 [hep-th]];
G. Calcagni, JHEP 0909, 112 (2009) [arXiv:0904.0829 [hep-th]];
E. Kiritsis and G. Kofinas, Nucl. Phys. B 821, 467 (2009) [arXiv:0904.1331 [hep-th]];
S. Mukohyama, JCAP 0906, 001 (2009) [arXiv:0904.2190 [hep-th]];
T. Kobayashi, Y. Urakawa and M. Yamaguchi, JCAP 0911, 015 (2009) [arXiv:0908.1005 [astro-ph.CO]].
[81] A. Wang and R. Maartens, Phys. Rev. D 81, 024009 (2010) [arXiv:0902.1748 [hep-th]];
A. Wang, D. Wands and R. Maartens, JCAP 1003, 013 (2010) [arXiv:0909.5167 [hep-th]].
[82] K. Izumi and S. Mukohyama, Phys. Rev. D 81, 044008 (2010) [arXiv:0911.1514 [hep-th]].
[83] P. A. R. Ade et al. [Planck Collaboration], arXiv:1303.5084 [astro-ph.CO].