Semi-Inclusive Deep Inelastic Scattering and Bessel-Weighted Asymmetries

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Abstract. We consider the cross section in Fourier space, conjugate to the outgoing hadron’s transverse momentum, where convolutions of transverse momentum dependent parton distribution functions and fragmentation functions become simple products. Individual asymmetric terms in the cross section can be projected out by means of a generalized set of weights involving Bessel functions. Advantages of employing these Bessel weights are that they suppress (divergent) contributions from high transverse momentum and that soft factors cancel in (Bessel-) weighted asymmetries. Also, the resulting compact expressions immediately connect to previous work on evolution equations for transverse momentum dependent parton distribution and fragmentation functions and to quantities accessible in lattice QCD. Bessel-weighted asymmetries are thus model independent observables that augment the description and our understanding of correlations of spin and momentum in nucleon structure.

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Introduction

In the factorized picture of semi-inclusive processes, where the transverse momentum of the detected hadron \( P_{h\perp} \) is small compared to the photon virtuality \( Q^2 \), transverse momentum dependent (TMD) parton distribution functions (PDFs) characterize the spin and momentum structure of the proton [1, 2, 3, 4, 5, 6, 7]. At leading twist there are 8 TMD PDFs. They can be studied experimentally by analyzing angular modulations in the differential cross section, so called spin and azimuthal asymmetries. These modulations are a function of the azimuthal angles of the final state hadron momentum about the virtual photon direction, as well as that of the target polarization (see Fig. 1 and e.g., Ref. [8] for a review). TMD PDFs enter the SIDIS cross section in momentum space convoluted with transverse momentum dependent fragmentation functions (TMD FFs). However, after a two-dimensional Fourier transform of the cross section with respect to the transverse hadron momentum \( P_{h\perp} \), these convolutions become simple products of functions in Fourier \( b_T \)-space. The usefulness of Fourier-Bessel transforms in studying the factorization as well as the scale dependence of transverse momentum dependent cross section has been known for some time [9, 10, 11, 12, 13, 14, 15]. Here we exhibit the structure of the cross section in \( b_T \)-space and demonstrate how this representation results in model independent observables which are generalizations of the conventional
weighted asymmetries \[16, 6, 7\]. Additionally, we explore the impact that these observables have in studying the scale dependence of the SIDIS cross section at small to moderate transverse momentum where the TMD framework is designed to give a good description of the cross section. In particular we study how the so called soft factor cancels from these observables. The soft factor \[17, 15, 18, 14, 19, 20\] is an essential element of the cross section that emerges in the proofs of TMD factorization \[11, 13, 14, 15\]. It accounts for the collective effect of soft momentum gluons not associated with either the distribution or fragmentation part of the process and it is shown to be universal in hard processes \[18\].

**Weighted asymmetries**

The concept of transverse momentum weighted Single Spin Asymmetries (SSA) was proposed some time ago in Ref. \[6, 7\]. Using the technique of weighting enables one to disentangle in a model independent way the cross sections and asymmetries in terms of the transverse momentum moments of TMDs. A comprehensive list of such weights was derived in Ref. \[7\] for semi-inclusive deep inelastic scattering (SIDIS). In SIDIS and Drell-Yan scattering, proofs of TMD factorization contain an additional factor, the soft factor \[17, 15, 18, 14, 19, 20\]. At tree level (zeroth order in $\alpha_S$) the soft factor is unity, which explains its absence in the factorization formalism considered for example in Ref. \[8\]. Consequently it is also absent in tree level phenomenological analyses of the experimental data (see for example Refs. \[21, 22, 23\]). Potentially the results of analyses at different energies can be difficult to compare. For a correct description of the energy scale dependence of the cross sections and asymmetries involving TMDs, the soft factor is essential to include \[19, 14, 24\]. However, it is possible to consider observables where the soft factor is indeed absent or cancels out. These are precisely the weighted asymmetries.

We will focus on the Sivers asymmetry \[2\]. With a general $|P_h|-$weight, this asymmetry can be written as \[7\]

$$
A_{UT}^{w_1 \sin(\phi_h - \phi_S)}(x, z, y) \equiv \frac{2 \int d|P_{h \perp}| d\phi_h d\phi_S w_1(|P_{h \perp}|) \sin(\phi_h - \phi_S) (d\sigma(\phi_h, \phi_S) - d\sigma(\phi_h, \phi_S + \pi))}{\int d|P_{h \perp}| d\phi_h d\phi_S w_0(|P_{h \perp}|) (d\sigma(\phi_h, \phi_S) + d\sigma(\phi_h, \phi_S + \pi))}. \tag{1}
$$

The above asymmetry has already been discussed for the choice $w_1 = \frac{|P_{h \perp}|}{M_Z}$, $w_0 = 1$. In the numerator, the angular weight $\sin(\phi_h - \phi_S)$ projects out the structure function \[F_{UT}^{\sin(\phi_h - \phi_S)}\] \[8\] from the cross section, while the $|P_{h \perp}|$-weight leads to a “deconvolution” of that structure function into a product of the first $p_T$-moment of the Sivers function $f_{1T}^{\perp(1)}(x)$ and the lowest moment of the unpolarized fragmentation function $D_1^{(0)}(z)$ \[7\].

In \[25\], we show that in TMD factorization the soft factor cancels in the asymmetry, due to the “deconvolution” achieved by appropriate $|P_{h \perp}|$-weighing. We demonstrate that it is a natural generalization to employ Bessel functions as weights, $w_n \propto J_n(|P_{h \perp}| \mathcal{B}_T)$, where $\mathcal{B}_T$ is the Fourier conjugate variable of $|P_{h \perp}|$. 


Moreover, this generalization addresses a problem related to the perturbative tail of TMDs. Without further regularization, the integrals defining the moments \(f_{1T}^{(1)}(x)\) and \(f_{1T}^{(0)}(x)\) are ill-defined due to the asymptotic behavior \(f_{1T}^{(1)}(x, p_T^2) \approx 1/p_T^2, f_{1T}^{(0)}(x, p_T^2) \approx 1/p_T^2\) at large \(p_T\) (see [26]). Using Bessel functions as weights, the respective integrals become convergent while the “deconvolution” property and soft factor cancellation are preserved. However, it is important to stress that while the integrals are convergent, the scale dependence of the resulting functions, and consequently also the \(Q^2\) dependence of the asymmetries, remains to be studied. Also the link with gluonic pole matrix elements [27] becomes less direct.

**Bessel-weighted asymmetries**

To deconvolute or convert the convolutions of TMD PDFs and TMD FFs in the SIDIS cross section into products, one can perform a multipole expansion and a subsequent Fourier transform of the cross section with respect to the transverse components \(P_{h\perp}\) of the hadron momentum. In general, we can write a transverse momentum dependent cross section \(\sigma(\mathbf{P}_{h\perp}, \phi_h)\) in the form

\[
\sigma(\mathbf{P}_{h\perp}, \phi_h) = \int \frac{d^2 b_T}{(2\pi)^2} e^{-i\mathbf{P}_{h\perp} \cdot \mathbf{b}_T} \tilde{\sigma}(\mathbf{b}_T) = \int \frac{d|\mathbf{b}_T|}{2\pi} |\mathbf{b}_T|^{2} \frac{d\phi_h}{2\pi} e^{-i|\mathbf{P}_{h\perp}| |\mathbf{b}_T| \cos(\phi_h - \phi_0)} \sum_{n=-\infty}^{\infty} e^{in\phi_h} \tilde{\sigma}_n(|\mathbf{b}_T|),
\]

where \(\tilde{\sigma}(\mathbf{b}_T) = \sum_{n=-\infty}^{\infty} e^{in\phi_h} \tilde{\sigma}_n(|\mathbf{b}_T|)\) is a two-dimensional multipole expansion of the cross section in Fourier space. The \(n\)th harmonic in \(\phi_h\) is accompanied by the \(n\)th Bessel function of the first kind \(J_n\).

With these definitions, the relevant terms of the SIDIS cross section can be written as

\[
\frac{d\sigma}{dx dy d\phi_S dz d\phi_h |\mathbf{P}_{h\perp}| d|\mathbf{P}_{h\perp}|} = \frac{\alpha^2}{x y Q^2 (1-\epsilon)} \left(1 + \frac{\gamma^2}{2x}\right) \int_0^\infty d|\mathbf{b}_T| |\mathbf{b}_T| \times \left\{ J_0(|\mathbf{b}_T| |\mathbf{P}_{h\perp}|) \tilde{F}_{UU,T} + |S_{\perp}| \sin(\phi_h - \phi_S) J_1(|\mathbf{b}_T| |\mathbf{P}_{h\perp}|) \tilde{F}_{\sin(\phi_h - \phi_S)} + \ldots \right\},
\]

where the ellipsis represent 16 more terms. Note that there is only a finite number of multipoles in the SIDIS cross section; the Bessel function of highest order is \(J_3\). Here we show only two terms in the cross section and omit for now regularization parameters needed beyond tree level, see Ref. [25] and references therein for more details. Introducing the Fourier-transformed TMDs and fragmentation functions

\[
\tilde{f}(x, \mathbf{p}_T^2) \equiv \int d^2 \mathbf{p}_T e^{i\mathbf{p}_T \cdot \mathbf{b}_T} f(x, \mathbf{p}_T^2) = 2\pi \int_0^\infty d|\mathbf{p}_T| |\mathbf{p}_T| J_0(|\mathbf{b}_T| |\mathbf{p}_T|) f(x, \mathbf{p}_T^2),
\]

\[
f(x, \mathbf{p}_T^2) = 1, 2, \ldots \]
The in- and out-going lepton momenta are $l$ and $l'$, respectively. The target nucleon carries momentum $P$ and its transverse spin components are labelled $S_{\perp}$. The momentum of the measured hadron $P_h$ has transverse components $P_{h\perp}$, which define an angle $\phi_h$ with the lepton plane.

\[
\bar{D}(z, b_T^2) \equiv \int d^2K_T e^{ib_T \cdot K_T} D(z, K_T^2) = 2\pi \int d|K_T|||K_T| J_0(|b_T|||K_T|) D(z, K_T^2) \tag{4}
\]

and the derivative (or $b_T$ moment)

\[
\tilde{f}^{(1)}(x, b_T^2) \equiv -\frac{2}{M^2} \partial_{b_T^2} \bar{f}(x, b_T^2) = \frac{2\pi}{M^2} \int_0^\infty d|p_T| \frac{|p_T^2|}{|b_T^2|} J_1(|b_T|||p_T|) f(x, p_T^2), \tag{5}
\]

the structure functions in Fourier space in Eq. (3) are given by

\[
\begin{align*}
\tilde{F}_{UU,T} &= x \sum_a e_a^2 \tilde{f}_1^{(0)a}(x, z^2 b_T^2) \bar{D}_1^{(0)a}(z, b_T^2) \tilde{S}(b_T^2) H_{UU,T}(Q^2), \tag{6} \\
\tilde{F}_{UT,T} &= -x \sum_a e_a^2 |b_T| z M \tilde{f}_{1T}^{(1)a}(x, z^2 b_T^2) \bar{D}_1^{(0)a}(z, b_T^2) \tilde{S}(b_T^2) H_{UT,T}(Q^2), \tag{7}
\end{align*}
\]

where, $f_{1}^{(0)a}$ and $f_{1T}^{(1)a}$ are the Fourier transformed unpolarized and Sivers TMD PDFs respectively, and $\bar{D}_1^{(0)a}$ is a Fourier transformed fragmentation function. We have used the kinematic variables $Q^2 \equiv -q^2$, $M^2 = P^2$, $x \equiv x_B \equiv Q^2/P \cdot q$, $y = P \cdot q/P \cdot l$, and $z \equiv z_B \equiv P \cdot P_h/P \cdot q$ and assume $M \ll Q$, $|P_{h\perp}| \ll zQ$. The sum $\sum_a$ runs over quark flavors $a$ and $e_a$ is the corresponding electric charge of the quark. In contrast to the tree-level equation [8, 25], we include here an explicit soft factor $\tilde{S}(b_T^2)$ (in Fourier space) and a hard part $H(Q^2)$, as in reference [19, 14]. For brevity we suppress the dependencies on a UV cutoff scale $\mu$ and on rapidity cutoff parameters (e.g., $\zeta$, $\zeta'$, $\rho$ in [19, 14, 25]). It is now clear that the cross section equation (3) is a multipole series, and that projection on Fourier modes in polar coordinates $\phi_h$, $|P_{h\perp}|$ will give access to the right hand sides of equations (6) and (7). Calculating the weighted asymmetry equation (1) with weights $w_1(|P_{h\perp}|) = 2J_1(|P_{h\perp}| R_T)/zM R_T$ and $w_0(|P_{h\perp}|) = J_0(|P_{h\perp}| R_T)$ thus yields

\[
A_{UT}^{2J_1(|P_{h\perp}| R_T)} \sin(\phi_h - \phi_t) (x, y, z; R_T) = -2x \sum_a e_a^2 H_{UT,T}(Q^2) \tilde{f}_{1T}^{(1)a}(x, z^2 b_T^2) \tilde{S}(b_T^2) \bar{D}_1^{(0)a}(z, b_T^2) \tag{8}
\]

Due to the “deconvolution” of the structure functions in the weighted asymmetries, and universality of the soft factor, $\tilde{S}$ cancels in the numerator and the denominator.

Further, note that $R_T$ enters the weights $w_0$ and $w_1$ as a free parameter that we can scan over a whole range in order to compare the transverse momentum dependence of
the distributions in the numerator and denominator relative to each other (in Fourier space). At the operator level, \( B_T(=|b_T|) \) controls the space-like distance between quark fields in the correlation functions we measure. The Bessel-weighted asymmetries are a natural extension of conventional weighted asymmetries \([6, 7]\) with weights proportional to powers of \(|P_{h \perp}|\). Indeed, in the limit \( B_T \to 0 \), equation (8) results in the often encountered special case for the SIDIS cross section

\[
\frac{p_{\perp}}{A_{UT}} \sin(\phi_0 - \phi_1) (x, z) = -2 \sum_{\alpha} e_{\alpha}^2 \frac{H_{UT,T}(Q^2) f_{1T}^{(1)\alpha}(x) D_1^{(0)\alpha}(z)}{\sum_{\alpha} e_{\alpha}^2 H_{UU,T}(Q^2) f_1^{(0)\alpha}(x) D_1^{(0)\alpha}(z)}. \tag{9}
\]

In particular, we obtain formally the relations \( f_1^{(0)}(x, 0) = f_{1T}^{(0)}(x) \), \( f_{1T}^{(1)}(x, 0) = f_{1T}^{(1)}(x) \), and \( D_1^{(0)}(x, 0) = D_1^{(0)}(x) \), where

\[
f_{1T}^{(n)}(x, 0) = f_{1T}^{(n)}(x) = \int d^2p_T \left( \frac{p_T^2}{2M^2} \right)^n f(x, p_T^2). \tag{10}\]

An analogous equation holds for the fragmentation functions \( D \). However, we caution the reader that these moments are not well-defined without some regularization. It is therefore safer to study Bessel-weighted asymmetries at finite \( B_T \), where the Bessel functions suppress contributions from large transverse momenta.

### Cancellation of soft factor at the level of matrix elements

In a similar manner to the discussion above, we now consider the soft factor cancellation in the average transverse momentum shift of unpolarized quarks in a transversely polarized nucleon for a given longitudinal momentum fraction \( x \). This shift is considered in \([28]\) and defined by a ratio of the \( p_T \)-weighted correlator,

\[
\langle p_T(x) \rangle_{TU} = \left. \frac{\int d^2p_T p_\perp \Phi^{(+)}(x, p_T, P, S, \mu^2, \zeta, \rho)}{\int d^2p_T \Phi^{(+)}(x, p_T, P, S, \mu^2, \zeta, \rho)} \right|_{S_T = 0, S_T = (1, 0)} = M^{-1} f_{1T}^{(0)}(x; \mu^2, \zeta, \rho), \tag{11}\]

where \( f_{1T}^{(1)} \) and \( f_1^{(0)} \) are the moments defined in Eqs. (10). Obviously, the average momentum shift is very similar in structure to the weighted asymmetry Eq. (8). While the weighted asymmetries are accessible directly from the \( p_{h \perp} \)-weighted cross section, the average transverse momentum shifts are obtained from the \( p_T \)-weighted correlator and could in principle be accessible from weighted jet asymmetries. As already mentioned, the integrals defining the moments of TMD PDFs on the right hand side of the above equation are divergent without suitable regularization. Therefore we generalize the above quantity, weighting with Bessel functions of \( |p_T| \). In particular, we replace

\[
p_T = |p_T| \sin(\phi_p) \quad \longrightarrow \quad \frac{2f_1(|p_T| B_T)}{B_T} \sin(\phi_p - \phi_S), \tag{12}\]
where $\phi_S = 0$ for the choice $S_T = (1, 0)$ in Eq. (11). The correlator $\Phi^{(+)[\gamma^+]}$ reads in terms of the amplitudes $A_i^{(+)}$ and $B_i^{(+)}$ [25],

$$\Phi^{(+)[\gamma^+]}(x, p_T, P, S, \mu^2, \zeta, \rho) = \int \frac{d(b \cdot P)}{(2\pi)} e^{ix(b \cdot P)} \int_0^\infty \frac{d|b_T|}{2\pi} |b_T| \left\{ J_0(|b_T| |p_T|) 2\bar{A}_{1B}^{(+)}/\bar{S} - M|b_T| |S_T| \sin(\phi_p - \phi_S) J_1(|b_T| |p_T|) 2\bar{A}_{1B}^{(+)}/\bar{S} \right\}. \tag{13}$$

The Bessel-weighted analog of Eq. (11) is thus

$$\langle p_T(x) \rangle^{B_T}_{TU} = \left. \frac{\int dp_T |p_T| \int d\phi_p \frac{2J_1(|p_T|/B_T)}{B_T} \sin(\phi_p - \phi_S) \Phi^{(+)[\gamma^+]}(x, p_T, P, S, \mu^2, \zeta, \rho)}{\int dp_T |p_T| \int d\phi_p 0(|p_T|/B_T) \Phi^{(+)[\gamma^+]}(x, p_T, P, S, \mu^2, \zeta, \rho)} \right|_{|S_T|=1} = M^{+(1)}(x, B_T^2; \mu^2, \zeta, \rho). \tag{14}$$

Again, the soft factors cancel where the independence of the soft factor on $v \cdot b / \sqrt{v^2}$ is crucial [25]. Further, weighting with Bessel functions at various lengths $B_T$ thus allows us to map out, e.g., ratios of Fourier-transformed TMDs [25]. In the limit $B_T \rightarrow 0$, we recover equation (11), $\langle p_T(x) \rangle^{0}_{TU} = \langle p_T(x) \rangle_{TU}$, which we have thus shown to be formally free of any soft factor contribution. However, we caution the reader again that the expressions at $B_T = 0$ can be ill-defined without an additional regularization step.

**Conclusions**

We have demonstrated that rewriting the SIDIS cross-section in coordinate space displays the important feature that structure functions become simple products of Fourier transformed TMD PDFs and FFs, or derivatives thereof. The angular structure of the cross section naturally suggests weighting with Bessel functions in order to project out these Fourier-Bessel transformed distributions, which serve as well-defined replacements of the transverse moments entering conventional weighted asymmetries. In addition, Bessel-weighted asymmetries provide a unique opportunity to study nucleon structure in a model independent way due to the absence of the soft factor which as we have shown cancels from these observables. This cancellation is based on the fact that the soft factor is flavor blind in hard processes, and it depends only on $b_T^2, \mu^2, \rho$. Moreover, evolution equations for the distributions are typically calculated in terms of the (derivatives of) Fourier transformed TMD PDFs and FFs. As a result the study of the scale dependence of Bessel-weighted asymmetries should prove more straightforward. For the above stated reasons we propose Bessel-weighted asymmetries as clean observables to study the scale dependence of TMD PDFs and FFs at existing (HERMES, COMPASS, JLab) and future facilities (Electron Ion Collider, JLab 12 GeV).
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REFERENCES

1. J. P. Ralston, and D. E. Soper, *Nucl. Phys.* B152, 109 (1979).
2. D. W. Sivers, *Phys. Rev.* D41, 83 (1990).
3. J. C. Collins, *Nucl. Phys.* B396, 161 (1993), hep-ph/9208213.
4. A. Kotzinian, *Nucl. Phys.* B441, 234 (1995), hep-ph/9412283.
5. P. J. Mulders, and R. D. Tangerman, *Nucl. Phys.* B461, 197 (1996), hep-ph/9510301.
6. A. M. Kotzinian, and P. J. Mulders, *Phys. Lett.* B406, 373 (1997), hep-ph/9701330.
7. D. Boer, and P. J. Mulders, *Phys. Rev.* D57, 5780 (1998), hep-ph/9711485.
8. A. Bacchetta, et al., *JHEP* 02, 093 (2007), hep-ph/0611265.
9. G. Parisi, and R. Petronzio, *Nucl. Phys.* B154, 427 (1979).
10. H. F. Jones, and J. Wyndham, *J. Phys.* A14, 1457 (1981).
11. J. C. Collins, and D. E. Soper, *Nucl. Phys.* B194, 445 (1982).
12. S. D. Ellis, N. Fleishon, and W. J. Stirling, *Phys. Rev.* D24, 1386 (1981).
13. J. C. Collins, D. E. Soper, and G. Sterman, *Nucl. Phys.* B250, 199 (1985).
14. X.-d. Ji, J.-P. Ma, and F. Yuan, *Phys. Lett.* B597, 299 (2004), hep-ph/0405085.
15. J. C. Collins, *Foundations of Perturbative QCD*, Cambridge University Press, 2011, ISBN 9780521855334.
16. A. Kotzinian, and P. Mulders, *Phys.Rev.* D54, 1229–1232 (1996), hep-ph/9511420.
17. J. C. Collins, and F. Hautmann, *Phys. Lett.* B472, 129 (2000), hep-ph/9908467.
18. J. C. Collins, and A. Metz, *Phys. Rev. Lett.* 93, 252001 (2004), hep-ph/0408249.
19. X.-d. Ji, J.-p. Ma, and F. Yuan, *Phys. Rev.* D71, 034005 (2005), hep-ph/0404183.
20. S. Aybat, and T. C. Rogers, *Phys.Rev.* D83, 114042 (2011), 1101.5057.
21. M. Anselmino, et al., *Phys. Rev.* D71, 074006 (2005), hep-ph/0501196.
22. J. C. Collins, et al., *Phys. Rev.* D73, 014021 (2006), hep-ph/0509076.
23. M. Anselmino, et al., *Phys. Rev.* D75, 054032 (2007), hep-ph/0701006.
24. A. Idilbi, X.-d. Ji, J.-P. Ma, and F. Yuan, *Phys. Rev.* D70, 074021 (2004), hep-ph/0406302.
25. D. Boer, L. Gamberg, B. Musch, and A. Prokudin, *JHEP* 10, 21 (2011), 1107.5294.
26. A. Bacchetta, D. Boer, M. Diehl, and P. J. Mulders, *JHEP* 08, 023 (2008), 0803.0227.
27. D. Boer, P. J. Mulders, and F. Pijlman, *Nucl. Phys.* B667, 201 (2003), hep-ph/0303034.
28. M. Burkardt, *Nucl. Phys.* A735, 185 (2004), hep-ph/0302144.