Optimization problems for WSNs: trade-off between synchronization errors and energy consumption

Larisa Manita
Higher School of Economics (HSE) Moscow Institute of Electronics and Mathematics, Russia
E-mail: lmanita@hse.ru

Abstract. We discuss a class of optimization problems related to stochastic models of wireless sensor networks (WSNs). We consider a sensor network that consists of a single server node and $m$ groups of identical client nodes. The goal is to minimize the cost functional which accumulates synchronization errors and energy consumption over a given time interval. The control function $u(t) = (u_1(t), \ldots, u_m(t))$ corresponds to the power of the server node transmitting synchronization signals to the groups of clients. We find the structure of extremal trajectories. We show that optimal solutions for such models can contain singular arcs.

1. Model
Power consumption, synchronization and network lifetime are very important objectives in analysis of wireless sensor networks [1]–[6]. The majority of studies on WSNs focus on communication protocols, network topology and technical designs. Some papers are concerned with optimization problems for WSNs; for example, in [4, 5] the network lifetime maximization problem is studied. In this paper we consider a mathematical model related to networks whose nodes are equipped with noisy non-perfect clocks. The task of optimal clock synchronization in such networks is reduced to the classical control problem. Its functional is based on the trade-off between energy consumption and mean-square synchronization error.

We will consider the model of clock synchronization in large wireless sensor network proposed in [2]. The sensor network consists of a single server node (denoted by 0) and $m$ groups

$$\Gamma_i = \{(i,1), (i,2), \ldots, (i,N_i)\}, \quad i = 1,\ldots,m,$$

of identical client nodes labeled by $(i,j)$. Let $x_j^{(i)}(t) \in \mathbb{R}^1$ be a state of the node $(i,j)$ having the meaning of a local clock value at this node. The network evolves in time $t \in \mathbb{R}^+$ as follows.

1) The node 0 is a time server with the perfect clock:

$$\frac{d}{dt} x^{(0)}(t) = v > 0.$$

2) The client nodes $(i,j)$ are equipped with non-perfect clocks with random Gaussian noise

$$\frac{d x_j^{(i)}(t)}{dt} = v + \sigma_i dB_j^{(i)}(t) + \text{synchronizing jumps},$$

where $B_j^{(i)}(t), i = 1,\ldots,m, \quad j = 1,\ldots,N_i$, are independent standard Wiener processes, $\sigma_i > 0$ corresponds to the strength of the noise and “synchronizing jumps” are explained below.
3) At random time moments the server node 0 sends messages to client nodes. Let \( u_i \) be the intensity of the message flow issued from the server to the group \( \Gamma_i \). A message directed to the group \( \Gamma_i \) chooses a destination node \((i, j) \in \Gamma_i \) at random (with probability \( 1/N_i \)). The client \((i, j) \) that receives at time \( \tau \) the message from the node 0 immediately adjusts its clock to the current value of \( x^{(0)} \): \[ x^{(i)}_j(\tau + 0) = x^{(0)}(\tau) \text{, } x^{(k)}_i(\tau + 0) = x^{(k)}(\tau) \text{, } (k, l) \neq (i, j). \]

Hence the client clocks \( x^{(i)}_j(t) \), \( t \geq 0 \), are stochastic processes which interact with the time server. The function \( R_i(t) \) is a cumulative measure of desynchronization between the group of clients \( \Gamma_i \) and the server node. Here \( \mathbb{E} \) stands for the expectation. It was proved in [2] that the function \( R_i(t) \) satisfies the differential equation \( \dot{R}_i = -u_i R_i + N_i \sigma_i^2 \).

In this paper we consider the problem of minimizing of the functional which accumulates clock synchronization errors in the clients nodes and the energy consumption of the server over some time interval \([0, T]\). The control function \( u(t) = (u_1(t), \ldots, u_m(t)) \) corresponds to the power of the server node transmitting synchronization signals to the groups of clients. We prove that optimal solutions in the control problem for the wireless sensor network with a single time server node and \( m \) groups of identical client nodes exist. We use the Pontryagin Maximum Principle to define the structure of the optimal control. We show that optimal solutions contain singular arcs [7, 8], i.e., trajectories on which the control is not uniquely determined from the maximum condition. For the case \( m = 1 \) (the network with a single time server node and one group of client nodes) we find the structure of the optimal trajectories for all possible parameter values. We show that for sufficiently large \( u_0 \) the solutions contain singular arcs. We study the the behaviour of the optimal solutions as \( T \to \infty \). Condition, under which the optimal solutions necessarily contain the singular arc, is given. For the case \( m = 2 \) (the network with a single time server node and two groups of client nodes) we study a control problem with two-dimensional control. Using the structure of the optimal control we obtain numerical solutions.

2. Optimal control problem

Consider the following optimal control problem

\[
\int_0^T \left( \sum_{i=1}^m \alpha_i R_i(t) + \beta \sum_{i=1}^m u_i(t) \right) dt \to \inf \tag{1}
\]

\[
\dot{R}_i(t) = -u_i(t)R_i(t) + g_i, \quad R_i(0) = R_{i0}, \quad i = 1, \ldots, m \tag{2}
\]

\[
\sum_{i=1}^m u_i(t) \leq u_0, \quad u_i(t) \geq 0, \quad i = 1, \ldots, m \tag{3}
\]

where \( \alpha_1, \ldots, \alpha_m, \beta \) are some positive constants.

Denote \( R(t) = (R_1(t), \ldots, R_m(t)) \), \( R_0 = R(0) \), \( \alpha = (\alpha_1, \ldots, \alpha_m) \), \( g = (g_1, \ldots, g_m) \).

**Proposition 1.**

(i) For any \( R_0 \) and any parameter values \( T, u_0, \beta, \alpha, g \) there exists a solution \((\bar{R}(t), \bar{u}(t))\) to the problem (1)-(3).

(ii) For any optimal solution \( \bar{R}(t) \) there exists a time moment \( \tau, 0 \leq \tau < T \), such that \( \bar{u}(t) = 0, \quad t \in [\tau, T] \), i.e., the sending messages at times close to \( T \) is not optimal.

The proof of this proposition is similar to the proof in the case \( m = 1 \) (see [9]).

To construct an optimal control we apply the Pontryagin Maximum Principle [11] to the problem (1)-(3). Let \( \left( \bar{R}(t), \bar{u}(t) \right) \) be an optimal solution. Then there exists a continuous
The function $\psi(t) = (\psi_1(t), \ldots, \psi_m(t))$ such that for all $t \in (0, T)$ the following maximum condition holds

$$H \left( \tilde{R}(t), \psi(t), \tilde{u}(t) \right) = \max_{u \in U} H \left( \tilde{R}(t), \psi(t), u \right)$$

(4)

Here $H$ is the Hamiltonian function and $U$ is the control set

$$H(R, \psi, u) = -\left( \sum_{i=1}^{m} \alpha_i R_i(t) + \beta \sum_{i=1}^{m} u_i(t) + \sum_{i=1}^{m} \psi_i (-u_i R_i + g_i) \right)$$

$$U = \left\{ u(t) = (u_1(t), \ldots, u_m(t)) : u_i(t) \geq 0, i = 1, \ldots, m, \sum_{i=1}^{m} u_i(t) \leq u_0 \right\}$$

The function $\psi(t)$ satisfies the adjoint equation $\dot{\psi}(t) = -\frac{\partial H(\tilde{R}(t), \psi(t), \tilde{u}(t))}{\partial \psi}$ (a.e.) and the transversality condition $\psi(T) = 0$.

The solutions $(R(t), \psi(t))$ of the Hamiltonian system

$$\dot{R}(t) = \beta H(R, \psi, \tilde{u})$$

(5)

where $\tilde{u}(t)$ satisfies the maximum condition (4) are called extremals.

Denote $H_0(R, \psi) = -\sum_{i=1}^{m} (\alpha_i R_i + g_i \psi_i)$, $H_i(R, \psi) = \beta + R_i \psi_i$ ($i = 1, \ldots, m$). Then $H = H_0 - \sum_{i=1}^{m} u_i H_i$. The maximum condition (4) leads to

$$\tilde{u}_i(t) = \begin{cases} 0, & H_i(t) > 0 \\ u_0, & H_i(t) < 0 \text{ and } H_i(t) < \min_{j \neq i} H_j(t) \end{cases}$$

(6)

The optimal control $\tilde{u}$ is undetermined if there exists an interval $(t_1, t_2)$ and indexes $k, i, k \neq i$, such that $H_i(t) = H_k(t) < 0$ or $H_i(t) = 0 \leq \min_{j \neq i} H_j(t)$ for all $t \in (t_1, t_2)$. The corresponding extremal $(R(t), \psi(t))$ is called a singular one on the interval $(t_1, t_2)$.

2.1. Case $m = 1$

Let the sensor network contain one group of client nodes. Then the optimal control problem (1)-(3) has the form

$$\int_{0}^{T} (\alpha_1 R_1(t) + \beta u_1(t)) dt \rightarrow \inf$$

$$\tilde{R}_1(t) = -u_1(t) R_1(t) + N_i^2 \phi_i^2, \quad R_1(0) = R_{10}$$

(7)

(8)

The maximum condition (6) yields: $\tilde{u}_1(t) = 0$ if $H_1(t) > 0$ and $\tilde{u}_1(t) = u_0$ if $H_1(t) < 0$ where $H_1(R, \psi) = -\beta - R_1 \psi_1$. Suppose that $H_1(t)$ vanishes over an interval $(t_1, t_2)$. Then the maximum condition gives no information about the value of the optimal control $\tilde{u}_1(t)$. To compute a singular control we differentiate the identity $H_1(R(t), \psi(t)) = 0$. It is known that a non-zero coefficient of control $u_1$ can arise for the first time in the derivative of $H_1$ even order 2q. The number $q$ is called the order [10] of the singular extremal.

It was proved that for sufficiently large $u_0$ a singular control is realised in the problem (7)-(9).

**Proposition 2** ([9]). Let $\sqrt{\frac{\alpha_1 \beta}{\beta}} \leq u_0$. **
Proposition 3 (9). Let \( \frac{\alpha N \sigma_2}{\sqrt{\rho}} > u_0 \). Then the optimal solutions are nonsingular. The optimal control \( \dot{u} \) has one of the following forms

\[
\dot{u}(t) = \begin{cases} 
0, & t \in (0, t_1) \\
u_0, & t \in (t_1, t_2) \\
0, & t \in (t_2, T) 
\end{cases}
\]

i.e., the optimal control switches between \( u = 0 \) and \( u = u_0 \) and the number of switchings does not exceed 2.

The next statement describes the behaviour of the optimal solutions as \( T \to \infty \). Let \( J(T) \) be an optimal cost functional value.

Proposition 4. For any \( R(0) \) we have

1. \( R(T) \to R_T \) and \( \frac{J(T)}{T} \to \alpha_1 R_T \) where \( R_T = \frac{g_1}{u_0} + \frac{g u_0}{\alpha_1}, \) if \( \sqrt{\frac{\alpha_1 g_1}{\beta}} > u_0 \)

2. If \( \sqrt{\frac{\alpha_1 g_1}{\beta}} \leq u_0 \) then optimal solutions contain the singular solution (10).
2.2. Case $m = 2$. Numerical examples

Let the sensor network contain two groups of client nodes. Then the optimal control problem (1)-(3) has the form

$$\int_0^T \left( \alpha_1 R_1(t) + \alpha_2 R_2(t) + \beta (u_1(t) + u_2(t)) \right) dt \to \inf$$

$$\dot{R}_i(t) = -u_i(t) R_i(t) + g_i, \quad R_i(0) = R_{i0} \quad (i = 1, 2)$$

$$u_1(t) + u_2(t) \leq u_0, \quad u_1(t) \geq 0, \quad u_2(t) \geq 0$$

Even for $m = 2$ the behaviour of the optimal solutions can be very complicated [7, 8]. So we solve the problem numerically. To do this we write the Hamiltonian system (5)

$$\dot{\psi}_i = \alpha_i + \hat{u}_i(t) \psi_i, \quad \dot{\hat{u}}_i = -\hat{u}_i(t) R_i(t) + g_i, \quad i = 1, 2 \quad (11)$$

where $\hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t))$ satisfies the maximum condition (6), i.e.,

$$\dot{\hat{u}}(t) = \begin{cases} 
(u_0, 0), & (H_1(t) < 0) \& (H_1(t) < H_2(t)) \\
(0, u_0), & (H_2(t) < 0) \& (H_2(t) < H_1(t)) \& H_1(t) > 0 \& H_2(t) > 0 \\
(0, 0), & \end{cases} \quad (12)$$

Here $H_i(t) = \beta + R_i(t) \psi_i(t) \quad (i = 1, 2)$. If $H_1(t) = 0, H_2(t) > 0$ over an interval $(t_1, t_2)$, then the corresponding extremal is called singular on the interval $(t_1, t_2)$ with respect to component (1) or (1)-singular. If $H_2(t) = 0, H_1(t) > 0, t \in (t_1, t_2)$, then the corresponding extremal is called (2)-singular. If $H_1(t) = H_2(t) < 0, t \in (t_1, t_2)$, then the corresponding extremal is called (1,2)-singular.

![Figure 2: For this plots $R_1(0) = 0, R_2(0) = 4.49, g_1 = 20, g_2 = 5$, $\alpha_1 = 1.6, \alpha_2 = 1, T = 3.55, w_0 = 1, \beta = 1$.](image)

We conclude this section by some numerical results. We solve (11)-(12) with the boundary conditions $R_i(0) = R_{i0}, \psi_i(T) = 0 \quad (i = 1, 2)$. Fig. 2 and Fig. 3 represent optimal extremals and optimal controls. $(R_1, \psi_1)$ is the red solid line and $(R_2, \psi_2)$ is the brown dashed line.
3. Conclusions
We considered the control problem for wireless sensor networks with a single time server node and $m$ groups of identical client nodes. The cost functional of this control problem accumulates clock synchronization errors in the clients nodes and the energy consumption of the server over some time interval $[0, T]$. We proved that the sending messages at times close to $T$ is not optimal, i.e., for any optimal solution $\hat{R}(t)$ there exists a time moment $\tau$, $0 \leq \tau < T$, such that $\hat{u}(t) = 0$, $t \in [\tau, T]$. We showed that the optimal solutions can contain singular arcs.

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