A LOWER BOUND IN NEHARI’S THEOREM ON THE POLYDISC

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ABSTRACT. By theorems of Ferguson and Lacey \((d = 2)\) and Lacey and Terwilleger \((d > 2)\), Nehari’s theorem is known to hold on the polydisc \(\mathbb{D}^d\) for \(d > 1\), i.e., if \(H_\psi\) is a bounded Hankel form on \(H^2(\mathbb{D}^d)\) with analytic symbol \(\psi\), then there is a function \(\varphi\) in \(L^\infty(\mathbb{T}^d)\) such that \(\psi\) is the Riesz projection of \(\varphi\). A method proposed in Helson’s last paper is used to show that the constant \(C_d\) in the estimate \(\|\varphi\|_\infty \leq C_d \|H_\psi\|\) grows at least exponentially with \(d\); it follows that there is no analogue of Nehari’s theorem on the infinite-dimensional polydisc.

This note solves the following problem studied by H. Helson [2, 3]: Is there an analogue of Nehari’s theorem on the infinite-dimensional polydisc? By using a method proposed in [3], we show that the answer is negative. The proof is of interest also in the finite-dimensional situation because it gives a nontrivial lower bound for the constant appearing in the norm estimate in Nehari’s theorem; we choose to present this bound as our main result.

We first introduce some notation and give a brief account of Nehari’s theorem. Let \(d\) be a positive integer, \(\mathbb{D}\) the open unit disc, and \(\mathbb{T}\) the unit circle. We let \(H^2(\mathbb{D}^d)\) be the Hilbert space of functions analytic in \(\mathbb{D}^d\) with square-summable Taylor coefficients. Alternatively, we may view \(H^2(\mathbb{D}^d)\) as a subspace of \(L^2(\mathbb{T}^d)\) and express the inner product of \(H^2(\mathbb{D}^d)\) as \(\langle f, g \rangle = \int_{\mathbb{T}^d} fg\), where we integrate with respect to normalized Lebesgue measure on \(\mathbb{T}^d\). Every function \(\psi\) in \(H^2(\mathbb{D}^d)\) defines a Hankel form \(H_\psi\) by the relation \(H_\psi(fg) = \langle fg, \psi \rangle\); this makes sense at least for holomorphic polynomials \(f\) and \(g\). Nehari’s theorem—a classical result [6] when \(d = 1\) and a remarkable and relatively recent achievement of S. Ferguson and M. Lacey [1] \((d = 2)\) and M. Lacey and E. Terwilleger [5] \((d > 2)\) in the general case—says that \(H_\psi\) extends to a bounded form on \(H^2(\mathbb{D}^d) \times H^2(\mathbb{D}^d)\) if and only if \(\psi = P_+ \varphi\) for some bounded function \(\varphi\) on \(\mathbb{T}^d\); here \(P_+\) is the Riesz projection on \(\mathbb{T}^d\) or, in other words, the orthogonal projection of \(L^2(\mathbb{T}^d)\) onto \(H^2(\mathbb{D}^d)\). We define \(C_d\) as the smallest constant \(C\) that can be chosen in the estimate

\[\|\varphi\|_\infty \leq C \|H_\psi\|,\]

where it is assumed that \(\varphi\) has minimal \(L^\infty\) norm. Nehari’s original theorem says that \(C_1 = 1\).

Theorem. For even integers \(d \geq 2\), the constant \(C_d\) is at least \((\pi^2/8)^{d/4}\).

The theorem thus shows that the blow-up of the constants observed in [4, 5] is not an artifact resulting from the particular inductive argument used there.

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Since clearly \( C_d \) increases with \( d \) and, in particular, we would need that \( C_d \leq C_\infty \) should Nehari’s theorem extend to the infinite-dimensional polydisc, our theorem gives a negative solution to Helson’s problem.

Nehari’s theorem can be rephrased as saying that functions in \( H^1(\mathbb{D}^d) \) (the subspace of holomorphic functions in \( L^1(\mathbb{T}^d) \)) admit weak factorizations, i.e., every \( f \) in \( H^1(\mathbb{D}^d) \) can be written as \( f = \sum_j g_j h_j \) with \( g_j, h_j \) in \( H^2(\mathbb{D}^d) \) and \( \sum_j \|g_j\|_2\|h_j\|_2 \leq A\|f\|_1 \) for some constant \( A \). Taking the infimum of the latter sum when \( g_j, h_j \) vary over all weak factorizations of \( f \), we get an alternate norm (a projective tensor product norm) on \( H^1(\mathbb{D}^d) \) for which we write \( \|f\|_{1,w} \). We let \( A_d \) denote the smallest constant \( A \) allowed in the norm estimate \( \|f\|_{1,w} \leq A\|f\|_1 \). Our proof shows that we also have \( A_d \geq (\pi^2/8)^{d/2} \) when \( d \) is an even integer.

**Proof of the theorem.** We will follow Helson’s approach \cite{Hel} and also use his multiplicative notation. Thus we define a Hankel form on \( \mathbb{T}^\infty \) as

\[
H_\psi(fg) = \sum_{j,k=1}^\infty \rho_{jk}a_j b_k;
\]

here \((a_j), (b_j)\), and \((\rho_j)\) are the sequences of coefficients of the power series of the functions \( f \), \( g \), and \( \psi \), respectively. More precisely, we let \( p_1, p_2, p_3, \ldots \) denote the prime numbers; if \( j = p_1^{n_1} \cdots p_k^{n_k} \), then \( a_j \) (respectively \( b_j \) and \( \rho_j \)) is the coefficient of \( f \) (respectively of \( g \) and \( \psi \)) with respect to the monomial \( z_1^{n_1} \cdots z_k^{n_k} \). We will only consider the finite-dimensional case, which means that the coefficients will be nonzero only for indices \( j \) of the form \( p_1^{n_1} \cdots p_d^{n_d} \). The prime numbers will play no role in the proof except serving as a convenient tool for bookkeeping.

We now assume that \( d \) is an even integer and introduce the set

\[
I = \left\{ n \in \mathbb{N} : n = \prod_{j=1}^{d/2} q_j \quad \text{and} \quad q_j = p_{2j-1} \text{ or } q_j = p_{2j} \right\}.
\]

We define a Hankel form \( H_\psi \) on \( \mathbb{D}^d \) by setting \( \rho_n = 1 \) if \( n \) is in \( I \) and \( \rho_n = 0 \) otherwise.

We follow \cite{Hel} pp. 81–82] and use the Schur test to estimate the norm of \( H_\psi \). It suffices to choose a suitable finite sequence of positive numbers \( c_j \) with \( j \) ranging over those positive integers that divide some number in \( I \); for such \( j \) we choose

\[
c_j = 2^{-\Omega(j)/2},
\]

where \( \Omega(j) \) is the number of prime factors in \( j \). We then get

\[
\sum_k \rho_{jk} c_k = 2^{d/2-\Omega(j)} \cdot 2^{-(d/2-\Omega(j))/2} = 2^{d/4} c_j,
\]

so that \( \|H_\psi\| \leq 2^{d/4} \) by the Schur test.

If \( f \) is a function in \( H^1(\mathbb{D}^d) \) with associated Taylor coefficients \( a_n \), then

\[
H_\psi(f) = \sum_n a_n \rho_n.
\]
We choose
\begin{equation}
(1) \quad f(z) = \frac{d}{2} \prod_{j=1}^{d/2} (z_{2j-1} + z_{2j})
\end{equation}
for which $a_n = \rho_n$ and thus $H_\psi(f) = 2^{d/2}$. On the other hand, an explicit computation shows that
\[\|f\|_1 = (4/\pi)^{d/2}\]
so that $H_\psi$, viewed as a linear functional on $H^1(D^d)$, has norm at least $(\pi/2)^{d/2}$. This concludes the proof since it follows that we must have $(\pi/2)^{d/2} \leq \|\varphi\|_\infty$ and we know from above that $\|H_\psi\| \leq 2^{d/4}$.

It is worth noting that our application of the Schur test shows that in fact $\|H_\psi\| = 2^{d/4}$ since $\|f\|_2 = 2^{d/4}$. The fact that $|H_\psi(f)| = \|H_\psi\||f||_2$ implies that
\[\|f\|_{1,w} = \|f\|_2.
\]
In other words, the trivial factorization $f \cdot 1$ is an optimal weak factorization of the function $f$ defined in (1).

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