A qutrit Quantum Key Distribution protocol with better noise resistance

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Abstract

The Ekert quantum key distribution protocol uses pairs of entangled qubits and performs checks based on a Bell inequality to detect eavesdropping. The 3DEB protocol uses instead pairs of entangled qutrits to achieve better noise resistance than the Ekert protocol. It performs checks based on a Bell inequality for qutrits named CHSH-3 and found in [10, 13]. In this paper, we present a new protocol, which also uses pairs of entangled qutrits, but achieves even better noise resistance than 3DEB. This gain of performance is obtained by using another inequality called here hCHSH-3, which was discovered in [6]. As the hCHSH3 inequality involve products of observables which become incompatible when using quantum states, we show how the parties running the protocol can measure the violation of hCHSH3 in the presence of noise, to ensure the secrecy of the key.
1 Introduction

The Ekert91 protocol [1] exploits pairs of entangled states to exchange keys, and uses Bell inequalities to detect eavesdropping. Some of the measurement results obtained by the two parties Alice and Bob are perfectly correlated, providing key bits. Other measurement results must exhibit quantum behavior if there is no alteration of the quantum channel, and this permits to detect eavesdropping by testing a Bell inequality violation.

The amount of quantum violation is an important characteristic in key distribution protocols because larger violations leads to a better noise resistance [2]. Some progress has been made to increase this amount of violation with the use of parties with higher dimension ([3] and [4], for qutrits). The choice of the Bell inequality used to detect eavesdropping is another parameter which can be considered.

In their article introducing the 3DEB protocol [5], Durt, Cerf, Gisin and Żukowski choosed to use three-dimensional quantum systems (qutrits), and the Bell inequality for qutrits named CHSH-3. This way, they obtained better noise resistance than for the Ekert’91 protocol.

Our work makes one step further by using a recent discovered Bell inequality (here called hCHSH-3), which belongs to the family of homogeneous Bell inequalities introduced in [6]. The amount of violation which can be achieved with entangled states is even better than for CHSH-3. Consequently, the protocol we derive is more tolerant to noise, with a threshold of noise $F \simeq 0.409$, instead of $F \simeq 0.304$ for 3DEB.

Devices called multiport beam splitters [7] (or tritters), are mentioned in [5] as one way to handle measurements of qutrits. Tritters are analyzed in [8] and experimentally tested in [9]. Our new protocol h3DEB described in this article is analysed in view of the use of tritters to implement measurements. A crucial point here will be that some products of observables, each implemented with tritters, can also be implemented by another single tritter. This is needed for our protocol as the inequality hCHSH-3 involves such products.

The paper is organized as follows. It begins with some reminders and precisions about measurements with tritters in Section 2. Then Section 3 recalls the 3DEB protocol and the CHSH-3 Bell inequality used by it. After that, Section 4 introduces the Bell inequality hCHSH-3 we use, then considers the use of tritters for implementing the product of observables, and defines our new protocol h3DEB. Finally, the paper concludes about the advantage of h3DEB providing better resistance to noise.
2 Prerequisites

The 3DEB protocol and our protocol use qutrits and trichotomic observables. For readability, we assume that the outcomes of these observables are $1, \omega, \omega^2$ where $\omega$ is the third root of unity $\omega = e^{\frac{2\pi i}{3}}$. The observables used by the two parties Alice and Bob will be denoted respectively by $A_i$ and $B_j$ for some indexes $i$ and $j$. We will also use the correlation functions introduced in [9]:

$$E(A_i B_j) = \sum_{a,b=1,\omega,\omega^2} P(A_i = a, B_j = b) ab.$$ 

2.1 Measurements with tritters

A tritter is parameterized by a triplet $(\varphi_0, \varphi_1, \varphi_2)$ of phase shifts. For readability we put $\theta_j = \exp(i \varphi_j)$ (for $j = 0, 1, 2$) and $\Theta = (\theta_0, \theta_1, \theta_2)$. The tritter performs over a qutrit the following unitary transformation:

$$U_\Theta := HD_\Theta = \frac{1}{\sqrt{3}} \sum_{k,l=0}^2 \omega^{kl} \theta_l |k\rangle \langle l|$$

where the matrices $H$ and $D_\Theta$ are $H = (\omega^k)_{0 \leq k,l \leq 2}$ and $D_\Theta = \text{diag}(\theta_0, \theta_1, \theta_2)$. In the specific and usual case where the phase shifts obey to the relation $\theta_j = \theta^j$, we have:

$$U_\Theta = \frac{1}{\sqrt{3}} \sum_{k,l=0}^2 \omega^{kl} \theta^l |k\rangle \langle l| = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \theta & \theta^2 \\ 1 & \omega \theta & \omega^2 \theta^2 \\ 1 & \omega^2 \theta & \omega \theta^2 \end{pmatrix}$$

After the transformation performed by the tritter, a measurement is made using three detectors. This measurement is represented by the observable

$$Z = \sum_{k=0}^2 \omega^k |k\rangle \langle k|.$$ 

(Note that, as we assumed the three possible outcomes to be labeled by complex roots of unity, we use unitary observables). Thus, the measurement obtained by the combination of the tritter and the detectors corresponds to the following observable

$$Z_\Theta := D_\Theta H^\dagger Z HD_\Theta = \begin{pmatrix} 0 & \theta_0 & \theta_2 \theta_0^* \\ \theta_0^* & 0 & 0 \\ 0 & \theta_2 \theta_0^* & 0 \end{pmatrix}.$$ 

which gives us, in the particular case where $\theta_j = \theta^j$:

$$Z_\Theta = \begin{pmatrix} 0 & 0 & \theta^2 \\ \theta^* & 0 & 0 \\ 0 & \theta^* & 0 \end{pmatrix}.$$
3 The 3DEB protocol

We will recall the 3DEB protocol introduced in [5]. We begin with the CHSH-3
inequality (or CHSH for qutrits) as defined in [10, 11], which is used for 3DEB.

3.1 The inequality CHSH-3

The CHSH-3 inequality can be written

\[ S \leq 2 \]

where

\[ S = \text{Re}(E(A_1B_1) + E(A_1B_2) - E(A_2B_1) + E(A_2B_2)) \]
\[ + \frac{1}{\sqrt{3}}\text{Im}(E(A_1B_1) - E(A_1B_2) - E(A_2B_1) + E(A_2B_2)). \]

Some entangled states are known to violate this inequality. The GHZ state

\[ |\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \]

is known to violate CHSH-3 with a violation factor (the quotient of the quantum
value with the classical bound) \( v = (6 + 4\sqrt{3})/9 \simeq 1.436. \)

This violation factor is considered very important for the security of the key
distribution protocol. The presence of noise is usually modelized by the replace-
ment of the initial entangled state by a mixture

\[ F \frac{I}{d} + (1 - F) |\psi\rangle \langle \psi| \]

where \( F \) is the proportion of noise. The point is that the presence of noise
decreases the experienced violation to \((1 - F)v\) and that the protocol is considered
useless when the initial state entanglement cannot be detected anymore. With
this criterion, it has been shown that the protocol 3DEB is resistant to the pres-
ence of noise up to a threshold \( F = 1 - 1/v = (11 - 6\sqrt{3})/2 \simeq 0.304 \).

The bases considered in most papers [14, 4, 12] to obtain these best violations
are the following four “optimal bases” (two for each party) corresponding to
titter measurements using the following phase shift triples \( \Theta = (1, \theta, \theta^2) \) were
\( \theta \) is a suitable power of \( \zeta = e^{\frac{2\pi i}{12}} \):

\begin{align*}
A_1 & : (1, 1, 1) \\
A_2 & : (1, \zeta^2, \zeta^4) \\
B_1 & : (1, \zeta, \zeta^2) \\
B_2 & : (1, \zeta^{-1}, \zeta^{-2})
\end{align*}

Figure 1: Representation of the four optimal bases in a plane parametrized
by \( \theta \)
In [5], it is remarked that the phases used for $B_2$ can be $(1, \zeta^3, \zeta^6)$ instead, as this change merely consists in reorder the three vectors of this base. It was also remarked in [14] that the non maximally entangled state
\[
\frac{1}{\sqrt{2} + \gamma^2} (|00\rangle + \gamma |11\rangle + |22\rangle)
\]
with $\gamma = \frac{\sqrt{3} - \sqrt{2}}{2}$ achieves an even better violation of CHSH-3, with a factor equal to $\frac{1 + \sqrt{11}}{2} \approx 1.457$. This allows to reach a noise resistance up to the threshold $F \approx 0.314$.

The expression of $S$ was rewritten in [6] in the following form
\[
S = -\frac{2}{9} \Re(T)
\]
with
\[
T = 3(\omega^2 - 1)E(A_1^2B_1^2) + (\omega - 1)E(A_1^2B_2^2) + (1 - \omega^2)E(A_2^2B_1^2) + (\omega - 1)E(A_2^2B_2^2)
\]
Note that, the measurements outcomes being $1, \omega, \omega^2$, it is equivalent to consider the square of an observable and its complex conjugate. For completeness, a derivation of this formula is given in Appendix A. Now, the CHSH-3 inequality can be rewritten as
\[
\Re(-T) \leq 9.
\]
This last formulation will be useful to compare with the inequality we will use for our new protocol. Note that it was also shown in [6] that the inequality obtained for qudits ($d \geq 3$) in [13] is the same as CHSH3 for the special case $d = 3$.

### 3.2 The 3DEB procedure

Alice uses four observables $A_a$ with $a = 0$ to 3, corresponding to tritter measurements with phase shift triples $(1, \zeta^a, \zeta^{2a})$. Bob uses four observables $b_b$ with $b = 0$ to 3, corresponding to tritter measurements with phase shift triples $(1, \zeta^{-b}, \zeta^{-2b})$. The following steps are repeated until Alice and Bob obtained a shared key of desired length.

1. Alice and Bob obtain an entangled pair of states in the GHZ state defined in [3].
2. Alice draws randomly a value for $a \in \{0, 1, 2, 3\}$ and makes the measurement corresponding to the observable $A_a$ whereas Bob draws randomly a value for $b \in \{0, 1, 2, 3\}$ and makes the measurement corresponding to the observable $B_b$.
3. When $a = b$, the results obtained by Alice and Bob are perfectly correlated. Indeed, the two tritters used by Alice and Bob perform on the
shared GHZ state the transformation \((H \otimes H)(D_\Theta \otimes D_{\Theta^*})\), with \(\Theta = (1, \zeta^a, \zeta^{2a})\). But it is easy to check that:

\[
(H \otimes H)(D_\Theta \otimes D_{\Theta^*})(|00\rangle + |11\rangle + |22\rangle) = (|00\rangle + |12\rangle + |21\rangle)
\]

Consequently, in this case where \(a = b\), Alice and Bob obtain a new trit for the shared key.

4. When \((a, b) \in \{(0,1), (0,3), (2,1), (2,3)\}\), Alice and Bob can use their joint measurements to detect eavesdropping, because the four observables \(A_0, A_2, B_1, B_2\) correspond to a configuration of maximal violation of CHSH-3. The same is true when \((a, b) \in \{(1,0), (1,2), (3,0), (3,2)\}\).

These different cases can be summarized in Table 1.

| \(A_0\) | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) |
| --- | --- | --- | --- | --- |
| \(A_0\) | \(k\) | \(c_1\) | \(c_1\) | \(k\) : Alice and Bob obtain key trits. |
| \(A_1\) | \(c_2\) | \(k\) | \(c_2\) | \(c_i\) : Alice and Bob obtain values for two sets of data which can be used to check CHSH-3 violation. |
| \(A_2\) | \(c_1\) | \(k\) | \(c_1\) | |
| \(A_3\) | \(c_2\) | \(c_2\) | \(k\) | |

Table 1: Cases for the 3DEB protocol

When used with the maximally entangled GHZ state \(|\psi\rangle\), the violation factor of CHSH-3 observed using this protocol (in the absence of noise) is equal to \(v = \frac{6+4\sqrt{3}}{2} \simeq 1.436\). This corresponds to a noise resistance up to a threshold \(F \simeq 0.304\). Our aim was to create a new protocol more tolerant to noise, with a threshold \(F\) greater than 0.304.

4 The new h3DEB protocol

We will now describe our protocol. It achieves better noise resistance because it will use an homogeneous Bell inequality, which has a larger violation factor than CHSH-3.

4.1 The homogeneous inequality

We will use the inequality :

\[
-2\text{Re}(T_1) \leq 9
\]

with \(T_1 = (4\omega + 2)E(A_1^2B_1^2) + (\omega - 1)E(A_1^2B_1B_2) + (4\omega - 1)E(A_1^2B_2^2)

- (2\omega + 1)E(A_1A_2B_1^2) + (\omega - 1)E(A_1A_2B_1B_2) + (\omega + 2)E(A_1A_2B_2^2)

+ (\omega + 5)E(A_2^2B_1^2) + (\omega - 1)E(A_2^2B_1B_2) - (2\omega + 4)E(A_2^2B_2^2).
\]

This inequality belongs to the set of homogeneous Bell inequalities described in [6]. It has been shown in this paper that these inequalities are satisfied under the hypothesis of local realism, and that they form a complete set.
A feature of the homogeneous Bell inequalities is that they involve some products (namely $A_1 A_2$ and $B_1 B_2$) of observables which become incompatible when considered as quantum observables. The outcomes of such a product of course cannot be meant to be the products of outcomes of incompatible observables. But if we use the unitary observables $Z_\Theta$ defined above for the $A_i$ and $B_j$, the product of them is also a unitary observable which outcomes can be obtained with a single measurement. Moreover, we argue here that this single measurement can be implemented by a slightly modified tritter.

### 4.2 Products of incompatible observables

Suppose that we have two tritters, which implement the observables $Z_\Theta$ and $Z_\Lambda$ described by Equation (1), with $\Theta = (\theta_0, \theta_1, \theta_2)$ and $\Lambda = (\lambda_0, \lambda_1, \lambda_2)$. Then we need to implement the product observable $Z_\Theta Z_\Lambda$. But

$$Z_\Theta Z_\Lambda = \begin{pmatrix} 0 & 0 & \theta_2 \theta_0^* \\ \theta_0 \theta_1^* & 0 & 0 \\ 0 & \theta_1 \theta_2^* & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \lambda_2 \lambda_0^* \\ \lambda_0 \lambda_1^* & 0 & 0 \\ 0 & \lambda_1 \lambda_2^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_0 \gamma_1 & 0 \\ 0 & 0 & \gamma_1 \gamma_2 \\ \gamma_2 \gamma_0 & 0 & 0 \end{pmatrix}$$

where

$$\gamma_0, \gamma_1, \gamma_2 = (\theta_2 \lambda_1^*, \theta_0 \lambda_2^*, \theta_1 \lambda_0^*).$$

Hence, $Z_\Theta Z_\Lambda = Z_{\Gamma}^\dagger$ where $\Gamma$ has the components $(\gamma_0, \gamma_1, \gamma_2)$ just given. From $Z_{\Gamma} = D_{\gamma} H^\dagger Z_{\Gamma} D_{\gamma}$, we obtain $Z_\Theta Z_\Lambda = Z_{\Gamma}^\dagger = D_{\gamma} H^\dagger Z_{\Gamma} D_{\gamma}$. The product observable $Z_\Theta Z_\Lambda$ can consequently also be implemented by a tritter and a detector, but with the detector performing a measurement corresponding to the observable $Z_{\Gamma}^\dagger$ instead of $Z_{\Gamma}$.

Violations of Inequality (3) by quantum states have been computed in [6], using observables (among them, product observables) obtained from the ones of Figure 1. More precisely, the local realistic elements $A_{12}^*, A_1 A_2$ and $A_{22}^*$ have to be replaced by the three observables

$Z_{\Theta A}^*, Z_{\Gamma A}^\dagger, Z_{\Lambda A}^*$

where the $Z_{\Gamma A}^\dagger$ is a product observable as just described (and $\Theta_A$, $\Lambda_A$ are the parameters corresponding to the configuration of Figure 1). Similarly, the party Bob has to use three observables $Z_{\Theta B}^*, Z_{\Gamma B}^\dagger, Z_{\Lambda B}^*$.

As mentioned in [6], the violation factor obtained with the bases obtained from Figure 1 and the GHZ state is $v \simeq 1.693$. Hence, this amount of violation can be observed using six tritters with detectors.

### 4.3 The h3DEB procedure

As for 3DEB, we denote $A_i$ with $i = 0, 1, 2, 3$ the observable parameterized by phase shift triple $(1, \zeta^i, \zeta^{2i})$, and $B_j$ with $j = 0, 1, 2, 3$ the observable parameterized by $(1, \zeta^{-j}, \zeta^{-2j})$.

For each pair $ij$ in the set $C = \{00, 02, 22, 11, 13, 33\}$, we note now $A_{ij}$ the product observable $A_i A_j$ (which is expected to be implemented with a single tritter).
1. Alice and Bob obtain an entangled pair of states in the GHZ state.

2. Alice draws randomly a value of $ij$ in $\mathcal{C}$ and performs her measurement in the basis associated to the observable $A_{ij}$ whereas Bob draws randomly a value of $kl$ in $\mathcal{C}$ and performs his measurement in the basis associated to $B_{kl}$.

3. When the pairs $ij$ and $kl$ are equal, the triplets of phase shifts $(\theta_0, \theta_1, \theta_2)$ and $(\theta'_0, \theta'_1, \theta'_2)$ corresponding to the observables $A_{ij}$ and $B_{kl}$ are related each other by complex conjugation. Equation (7), which has yet been used in the special case where $\theta_j = \theta^i$ for some $\theta$, remains true in the present slightly more general case. Thus Alice and Bob obtain a new trit for the shared key.

4. For some choices of pairs $ij$ and $kl$ (see the table below) Alice and Bob collect the issues of their measurements, in order to detect eavesdropping. Indeed, these pairs correspond to two configurations of maximal violation of the homogeneous Bell inequality $h_{\text{CHSH-3}}$ given by Equation (8).

These different cases can be summarized in the table:

| $A_{00}$ | $B_{00}$ | $B_{02}$ | $B_{22}$ | $B_{11}$ | $B_{13}$ | $B_{33}$ |
|----------|----------|----------|----------|----------|----------|----------|
| $k$      | $c_1$    | $c_1$    | $c_1$    | $k$      | $k$      | $k$      |
| $A_{02}$ | $k$      | $c_1$    | $c_1$    | $c_1$    | $k$      | $k$      |
| $A_{22}$ | $k$      | $c_1$    | $c_1$    | $c_1$    | $k$      | $k$      |
| $A_{11}$ | $c_2$    | $c_2$    | $c_2$    | $k$      | $k$      | $k$      |
| $A_{13}$ | $c_2$    | $c_2$    | $c_2$    | $k$      | $k$      | $k$      |
| $A_{33}$ | $c_2$    | $c_2$    | $c_2$    | $k$      | $k$      | $k$      |

$k$: Alice and Bob obtain a key trit.
$c_i$: Alice and Bob collect values for two sets of data which will be used to check $h_{\text{CHSH-3}}$ violation.

Table 2: Cases for the $h_{3\text{DEB}}$ protocol

4.4 Resistance to noise

Without the presence of noise, the violation of the inequality $h_{\text{CHSH-3}}$ observed with this protocol is $v \simeq 1.693$. By the same argument as the one used for the resistance of 3DEB, we obtain that our protocol is resistant to noise up to a threshold $F = 1 - \frac{1}{v} \simeq 0.409$. This is better than the resistance of the 3DEB protocol, even when the latter uses the non maximally entangled state (4).

5 Conclusion

Our goal was to improve the noise resistance of the qutrits key distribution protocols. By using the homogeneous Bell inequality $h_{\text{CHSH-3}}$ which reaches a violation factor $v \simeq 1.693$ with the GHZ state, better than for CHSH-3, we obtain a threshold of noise resistance $F \simeq 0.409$, better than the threshold $\simeq 0.304$ obtained for 3DEB using the GHZ state [12, 13] and even to the threshold $\simeq 0.341$ resulting of the use of the non maximally entangled state (4).
As our inequality hCHSH-3 involves products of observables which become incompatibles for quantum states, an important fact is the possibility to implement with slightly modified tritters the single observable corresponding to these products. This can be done by replacing the final measurement with observable $Z = \text{diag}(1, \omega, \omega^2)$ by a measurement with observable $Z^\dagger = \text{diag}(1, \omega^2, \omega)$. Physically, this replacement corresponds just to a permutation of the detectors.

The gain in noise resistance of our protocol over 3DEB is due to the use of the inequality hCHSH3. This inequality detects violations of local realism when some measurements are multiplicatively related. By using tritters measurements which respect this multiplicative constraints, the parties running the protocol are able to exploit its larger violation capabilities.

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Appendix A : Obtaining the Inequality (5)

\[ S = \text{Re}(E(A_1B_1) + E(A_1B_2) - E(A_2B_1) + E(A_2B_2) + \frac{1}{\sqrt{3}} \text{Im}(E(A_1B_1) - E(A_1B_2) - E(A_2B_1) + E(A_2B_2))} \]

This inequality can be rewritten:

\[ S = \text{Re}(U + V) + \frac{1}{\sqrt{3}} \text{Im}(U - V) \]

with \( U = E(A_1B_1) - E(A_2B_1) + E(A_2B_2) \) and \( V = E(A_1B_2) \).

For \( i, j = 0 \ldots 3 \), we have \( A_i^2 = A_i^* \) and \( B_j^2 = B_j^* \), which gives us:

\[ T = 3((\omega^2 - 1)E(A_1^2 B_1^2) + (\omega - 1)E(A_1^2 B_2^2) + (1 - \omega^2)E(A_2^2 B_1^2) + (\omega^2 - 1)E(A_2^2 B_2^2)) \]
\[ = 3((\omega - 1)E(A_1B_1) + (\omega^2 - 1)E(A_1B_2) + (1 - \omega)E(A_2B_1) + (\omega - 1)E(A_2B_2)) \]
\[ = 3((\omega - 1)U + (\omega^2 - 1)V) \]

\[ \text{Re}(T) = 3\text{Re}(\omega - 1)U + (\omega^2 - 1)V \]
\[ = 3\text{Re}(\omega U + \omega^2 V - U - V) \]
\[ = 3(\text{Re}(\omega U) + \text{Re}(\omega^2 V) - \text{Re}(U) - \text{Re}(V)) \]
\[ = 3(\text{Re}(\omega)\text{Re}(U) - \text{Im}(\omega)\text{Im}(U) + \text{Re}(\omega^2)\text{Re}(V) - \text{Im}(\omega^2)\text{Im}(V) - \text{Re}(U) - \text{Re}(V)) \]

But we also have \( \text{Re}(\omega) = \text{Re}(\omega^2) = -\frac{1}{2} \) and \( \text{Im}(\omega) = -\text{Im}(\omega^2) = -\frac{\sqrt{3}}{2} \), which gives us:

\[ \text{Re}(T) = 3\left(-\frac{3}{2}\text{Re}(U + V) - \frac{\sqrt{3}}{2}\text{Im}(U - V)\right) \]
\[ = -\frac{9}{2}\text{Re}(U + V) - \frac{9}{2}\text{Im}(U - V) \]
\[ = -\frac{9}{2}S \]
\[ S = -\frac{2}{9}\text{Re}(T) \]