Reduction of Feynman Integrals in the Parametric Representation III: Integrals with Cuts

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Abstract

Cuts are implemented by inserting Heaviside theta functions in the integrands of momentum-space Feynman integrals. By directly parametrizing theta functions and constructing integration-by-part (IBP) identities in the parametric representation, we provide a systematic method to reduce integrals with cuts. Since IBP method is available, it becomes possible to evaluate integrals with cuts by constructing and solving differential equations.

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1 Introduction

Feynman integrals with cuts are frequently encountered in perturbative calculations in high energy physics, especially while calculating various jet observables and event-shape distributions. Generally, cuts are implemented by inserting Heaviside theta functions in the integrands in the momentum space. The presence of theta functions largely complicates the calculations of Feynman integrals.

The most widely used technique to reduce Feynman integrals is the integration-by-part (IBP) method [1, 2]. However, it is not clear how to directly apply the regular IBP method to integrals with cuts. In a recent paper [3], theta functions were written as integrals of delta functions. The resulting integrals were reduced by combining the reverse unitarity and the IBP method. However, the application of this method to more complicated integrals is far from trivial.

On the other hand, it was suggested that IBP identities can directly be derived in the parametric representation [4, 5]. It can be shown that each momentum-space IBP identity [6] corresponds to a shift relation in the parametric representation [7]. Since a theta function has an integral representation quite similar to the Schwinger parametrization of a propagator, it is possible to directly parametrize theta functions and
construct IBP identities in the parametric representation. In this paper, we show that the methods developed in ref. [8, 9] (referred to as paper I and paper II respectively hereafter) to parametrize and reduce tensor integrals can be applied to integrals with theta functions with slight modifications.

This paper is organized as follows. In section 2, we show how to use the method developed in paper I and paper II to parametrize integrals with cuts and to construct IBP identities for them. Some detailed examples are provided in section 3.

2 Parametrization and IBP identities

It is well-known that a propagator can be parametrized by

\[ \frac{1}{D_{l_{i}^{\lambda_{i}+1}}} = \frac{e^{-\lambda_{i}+1}i\pi}{\Gamma(\lambda_{i} + 1)} \int_{0}^{\infty} dx_i e^{ix_i D_i} x_i^{\lambda_i}, \quad \text{Im}\{D_i\} > 0. \] (2.1)

Heaviside theta functions have a similar integral representation

\[ \theta(D_i) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} dx_i e^{ix_i D_i} x_i + i0^+. \]

For future convenience, we define the function

\[ w_{\lambda}(u) = \frac{1}{x^{\lambda+1}} \int_{-\infty}^{\infty} dx e^{ixu}. \] (2.2)

It’s easy to see that

\[ w_0(u) = 2\pi\theta(u), \]
\[ w_{-1}(u) = 2\pi\delta(u), \]
\[ w_{-2}(u) = 2\pi\delta'(u). \]

With this representation, the standard procedure to parametrized Feynman integrals can easily be generalized to integrals with theta functions. Following the convention used in paper I and paper II, we have

\[ M \equiv s_g^{-\frac{d}{2}} e^{i\pi \left[ \lambda_{n+1} - \frac{d}{2} + 1 + \sum_{i=1}^{n} (\lambda_i + \frac{d}{2}) \right]} I(\lambda_0, \lambda_1, \ldots, \lambda_n), \] (2.3)

where \( s_g \) is the determinant of the d-dimensional metric, and \( \lambda_{n+1} \equiv -(L+1)\lambda_0 - 1 + \sum_{i=1}^{m} \lambda_i - \sum_{i=m+1}^{n} (\lambda_i + 1) \), with \( \lambda_0 \equiv -\frac{d}{2} \). The parametric integral

\[ I(\lambda_0, \lambda_1, \ldots, \lambda_n) \equiv \int d\Pi^{(n+1)} \mathcal{I}^{(-n-1)} \]

\[ \equiv \frac{\Gamma(-\lambda_0)}{\prod_{i=m+1}^{n+1} \Gamma(\lambda_i + 1)} \int d\Pi^{(n+1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i}. \] (2.4)

Here the measure \( d\Pi^{(n)} \equiv \prod_{i=1}^{n+1} dx_i \delta(1 - \sum_j |x_j|) \), where the sum in the delta function runs over any nontrivial subset of \( \{x_1, x_2, \ldots, x_{n+1} \} \). The polynomial \( \mathcal{F}(x) \equiv F(x) + U(x)x_{n+1} \). \( U \) and \( F \) are Symanzik polynomials, defined by \( U(x) \equiv \text{det} A \), and \( F(x) \equiv U(x) \left( \sum_{i,j=1}^{L} (A^{-1})_{ij} B_i \cdot B_j - C \right) \). Polynomials \( A, B, \) and \( C \) are defined through \( \sum_{i=1}^{n} x_i D_i \equiv \sum_{i,j=1}^{L} A_{ij} l_i \cdot l_j + 2 \sum_{i=1}^{L} B_i \cdot l_i + C \).
It should be noticed that in the definition of the parametric integral in eq. (2.4), for a “propagator” $w_{\lambda_i}(D_i)$, there’s no corresponding gamma function in the prefactor. And the corresponding index $\lambda_i$ can be both positive and negative.

Similar to the parametric IBP identities derived in paper I, we have

\begin{equation}
0 = \int d\Pi^{(n+1)} \frac{\partial}{\partial x_i} I^{(-n)}, \quad i = 1, 2, \ldots, m, \quad (2.5a)
\end{equation}

\begin{equation}
0 = \int d\Pi^{(n+1)} \frac{\partial}{\partial x_i} I^{(-n)} + \delta_{\lambda,0} \int d\Pi^{(n)} I^{(-n)} \bigg|_{x_i=0}, \quad i = m + 1, m + 2, \ldots, n. \quad (2.5b)
\end{equation}

We define the index-shifting operators $R_i$, $D_i$, and $A_i$, with $i = 0, 1, \ldots, n$, such that

\begin{align*}
R_i (\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n) &= (\lambda_i + 1) I(\lambda_0, \ldots, \lambda_i + 1, \ldots, \lambda_n), \\
D_i (\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n) &= I(\lambda_0, \ldots, \lambda_i - 1, \ldots, \lambda_n), \\
A_i (\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n) &= \lambda_i I(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n).
\end{align*}

It is understood that

\begin{equation}
I(\lambda_0, \ldots, \lambda_{i-1}, -1, \ldots, \lambda_n) \equiv \int d\Pi^{(n)} I^{(-n)} \bigg|_{x_i=0}, \quad i = m + 1, m + 2, \ldots, n.
\end{equation}

We formally define operators $D_{n+1}$ and $R_{n+1}$, such that $D_{n+1} I = I$, and $R_{n+1}^i I = (A_{n+1} + 1)(A_{n+1} + 2) \cdots (A_{n+1} + i) I$, with $A_{n+1} \equiv -(L + 1) A_0 + \sum_{i=1}^n A_i - \sum_{i=m+1}^n (A_i + 1)$. We further introduce the operator $\hat{x}_i$, $\hat{z}_i$ and $\hat{a}_i$ such that

\begin{equation}
\begin{aligned}
\hat{x}_i &= \begin{cases} 
D_i, & i = 1, 2, \ldots, m, \\
R_i, & i = m, m + 1, \ldots, n + 1,
\end{cases} \\
\hat{z}_i &= \begin{cases} 
-R_i, & i = 1, 2, \ldots, m, \\
D_i, & i = m, m + 1, \ldots, n + 1,
\end{cases} \\
\hat{a}_i &= \begin{cases} 
-A_i - 1, & i = 1, 2, \ldots, m, \\
A_i, & i = m, m + 1, \ldots, n + 1.
\end{cases}
\end{aligned}
\end{equation}

Obviously we have $\hat{a}_{n+1} = -(L+1) A_0 - \sum_{i=1}^n (\hat{a}_i + 1)$. For $i = 1, 2, \ldots, n$, we have the following commutation relations.

\begin{align*}
\hat{z}_i \hat{x}_j - \hat{x}_j \hat{z}_i &= \delta_{ij}, \\
\hat{z}_i \hat{a}_j - \hat{a}_j \hat{z}_i &= \delta_{ij}, \\
\hat{x}_i \hat{a}_j - \hat{a}_j \hat{x}_i &= -\delta_{ij} \hat{x}_i.
\end{align*}

With the operators $\hat{x}_i$, $\hat{z}_i$, and $\hat{a}_i$, it’s easy to write the IBP identity in the following form

\begin{equation}
D_0 \frac{\partial F(\hat{x})}{\partial \hat{x}_i} - \hat{z}_i \approx 0, \quad i = 1, 2, \ldots, n + 1. \quad (2.6)
\end{equation}

Here we use $\approx$ to emphasize that these equations are valid only when they are applied to nontrivial parametric integrals.

The methods developed in paper II to parametrize tensor integrals and to construct dimensional-shift-free parametric IBP identities can also be applied to integrals with cuts. One only need to do the replacements $R_i \to \hat{x}_i$, $D_i \to \hat{z}_i$, and $A_i \to \hat{a}_i$. Differential equations can also be constructed by using the eq. (3.15) in paper II. Here we don’t need to go to the detail. Thus, in principle, integrals with cuts can be evaluated by using the standard differential-equation method [10, 11, 12, 13, 14].
3 Examples

![Figure 1: Geometric interpretations of eq. (3.1).](image)

We first consider the following simple but interesting example.

\[ I_1(\frac{d}{2}, \lambda_1, \lambda_2) \equiv \frac{i}{\pi^{d/2}} \int d^d r \, w_{\lambda_1}(a^2 - r^2) \, w_{\lambda_2}(a^2 - (r - 2b)^2) \]

By using the method I described in paper II (c.f. eq. (3.12) therein), we get the following IBP identities.

\[ A_1 - A_2 - 4b^2 D_1 + 4b^2 D_2 + D_2 R_1 - D_1 R_2 = 0, \]
\[ 2A_0 - 2A_1 - A_2 + 2a^2 D_1 + 2a^2 D_2 - 4b^2 D_2 - D_2 R_1 = 0. \]

Specifically, we consider the reduction of the integral \( I_1(-d/2, 0, 0) \). By solving IBP identities, we get

\[ I_{1a} = -\frac{i}{4} \pi^{d-2} I_1(-d/2, 0, 0) \]
\[ = \int d^d r \, \theta(a^2 - r^2) \, \theta(a^2 - (r - 2b)^2) \]
\[ = \frac{4a^2}{d} \int d^d r \, \delta(a^2 - r^2) \, \theta(a^2 - (r - 2b)^2) \]
\[ = \frac{16b^2(a^2 - b^2)}{d(d-1)} \int d^d r \, \delta(a^2 - r^2) \, \delta(a^2 - (r - 2b)^2) \]
\[ \equiv \frac{4a^2}{d} I_{1b} = \frac{16b^2(a^2 - b^2)}{d(d-1)} I_{1c}. \]

This result has an interesting geometric interpretation. It’s easy to see that the integral \( I_{1a} \) is nothing but the volume of the intersection of two \( d \)-dimensional balls with a radius \( a \) separated by a distance of \( 2|b| \), as is shown in fig. 1a. \( 2a I_{1b} = \int d^d r \, \delta(a - r) \, \theta(a^2 - (r - 2b)^2) \) is the bottom area of the \( d \)-dimensional cone shown in fig. 1b. Thus \( \frac{2a^2}{d} I_{1b} \) is the volume of this \( d \)-dimensional cone. Similarly, \( 8b \sqrt{a^2 - b^2} I_{1c} \) is the perimeter of the intersection of two spheres (the surfaces of the two balls). This will become obvious by using azimuthal coordinates. Thus \( \frac{8b^2(a^2 - b^2)}{d(d-1)} I_{1c} \) is the volume of the \( d \)-dimensional cone (with a flat bottom) shown in fig. 1c. Hence eq. (3.1) just tells us how to calculate the volume of the intersection of two balls.

We can also construct differential equations for these integrals. The differential operator reads (c.f. eq. (3.15) in paper II)

\[ \frac{\partial}{\partial b^2} = \frac{1}{2b^2} A_2 - \frac{1}{2b^2} D_2 R_1 - 2D_2. \]

Applying this operator to the integrals \( I_{1b} \) and \( I_{1c} \), and carrying out IBP reductions, we get the following differentiation equations.
\[ \frac{\partial}{\partial b^2} \left( I_{1b} \right) = \left( \begin{array}{c} 0 \\ 0 \\ -2(\alpha^2 + (d-4)b^2) \end{array} \right) \frac{I_{1b}}{I_{1c}}. \]

It’s easy to check that the solutions of these equations do agree with the result obtained by a direct calculation.

As a less trivial example, we consider the reduction of the integral
\[ I_2 = \frac{(2\pi)^6}{\pi^d} \int d^4 l_1 d^4 l_2 \frac{\delta(l_1^+ \delta(l_2) \delta(l_1^+ - a) \delta(l_2^+ - b) \delta(l_2^- - l_1^-) \theta(l_2^+ - l_1^-)}{l_1^+ l_1^- + l_2^+ + l_2^-}. \]

Here the lightcone coordinates are used. That is \( l_1^+ \equiv l_1 \cdot n \), and \( l_1^- \equiv l_1 \cdot \bar{n} \), with \( n^2 = \bar{n}^2 = 0 \), and \( n \cdot \bar{n} = 2 \). This integral is relevant for the calculation of the two-loop hemisphere soft functions [15]. This integral can be reduced to
\[ I_2 = -\frac{2}{(d-4)ab} \frac{(2\pi)^6}{\pi^d} \int d^4 l_1 d^4 l_2 \frac{\delta(l_1^+ \delta(l_2) \delta(l_1^+ - a) \delta(l_2^+ - b) \delta(l_2^- - l_1^-) \theta(l_2^+ - l_1^-)}{l_1^+ l_1^- + l_2^+ + l_2^-} - \frac{1}{ab} \frac{(2\pi)^6}{\pi^d} \int d^4 l_1 d^4 l_2 \frac{\delta(l_1^+ \delta(l_2) \delta(l_1^+ - a) \delta(l_2^+ - b) \delta(l_2^- - l_1^-) \theta(l_2^+ - l_1^-)}{(l_1^+ l_1^- + l_2^+ + l_2^-)}. \]

The detailed calculation is carried out by using a private Mathematica code. We’ve verified this result by explicit calculations of these integrals.

4 Summary

By directly parametrizing Heaviside theta functions and constructing IBP identities in the parametric representation, we provide a systematic method to reduce integrals with cuts. We show that the methods developed in paper I and paper II to parametrize and to reduce regular Feynman integrals can be applied to integrals with cuts by slightly modifying the definitions of the index-shifting operators. Differential equations can also be constructed. Thus in principle, the standard differential equation method can be used to evaluate integrals with cuts.

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