Quasi-analytic properties of the KAM curve

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Abstract

Classical KAM theory guarantees, under some mild non-degeneracy conditions, the existence of a positive measure set of invariant tori for near-integrable systems. This collection, when seen as a function of the frequency, is called the KAM curve. In analytic regularity, we prove strong quasi-analyticity properties for these objects which, in particular, show that the KAM curve completely characterizes the underlying system. We also show some of the dynamical implications on systems whose KAM curves share common features.

1 Introduction

1.1 Motivation

Consider the analytic standard family of symplectic maps given by

\[ F_{\epsilon, \varphi}: \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}, \]
\[ (\theta, r) \mapsto (\theta + r + \epsilon \varphi(\theta), r + \epsilon \varphi(\theta)) , \]

where \( \epsilon \in \mathbb{R} \) and \( \varphi \in C^\omega(\mathbb{T}) \) has zero mean value. For \( \varphi \) fixed and for any \( \gamma, \tau > 0 \) the classical KAM Theorem guarantees that for all \( \epsilon \) obeying

\[ |\epsilon| < \epsilon_0(\varphi, \gamma, \tau), \]

where \( \epsilon_0(\varphi, \gamma, \tau) \) is a positive constant given by the Theorem, there exist a collection \( \{T_\omega\}_{\omega \in \Omega} \) of invariant curves for the mapping \( F_{\epsilon, \varphi} \) whose rotation numbers are in bijection with the set of Diophantine numbers of type \( (\gamma, \tau) \) (see Section 2 for the formal definition). These invariant curves are actually graphs of mappings in \( C^\omega(\mathbb{T}^d, \mathbb{R}^d) \) and hence we can encode them as a function of their frequency

\[ T_{F_{\epsilon, \varphi}}: DC(\gamma, \tau) \to C^\omega(\mathbb{T}^d). \]

Following \[5\], we call \( T_{F_{\epsilon, \varphi}} \) the KAM curve associated to \( F_{\epsilon, \varphi} \). A formal definition of the KAM curve for general near-integrable systems is given in Section 3.

Since the KAM curve of \( F_{\epsilon, \varphi} \) is defined on a cantor set, its differentiability properties are better understood in terms of Whitney smoothness, that is, wether or not they admit a smooth extension to an open set. Following the
works of Poschel [11] in near-integrable Hamiltonians, Shang [12] proved the Whitney-smooth differentiability of the invariant tori given by the KAM Theorem for general near-integrable symplectic mappings.

In [5] it is shown that KAM curves for the standard family are not only differentiable in the sense of Whitney but they also admit, in a natural way, a unique extension to certain space of holomorphic functions. The main interest of these extensions comes from the quasi-analytic properties of such spaces. As a particular application of these properties, one can deduce the following.

**Proposition 1.1.** Let \( \varphi, \psi \in C^\omega(\mathbb{T}) \) and \( \epsilon < \epsilon_0(\varphi, \gamma, \tau), \epsilon_0(\psi, \gamma, \tau) \). If \( T_{F, \varphi} \) and \( T_{F, \psi} \) are equal on a set \( \Gamma \subset D(\gamma, \tau) \) of positive measure then \( T_{F, \varphi} = T_{F, \psi} \).

Paraphrasing the authors in [5], the knowledge of parametrizations of invariant tori on a set of positive measure of rotation numbers is sufficient to determine all the parametrized KAM curves. We point out that in general the functions \( T_{F, \varphi} \) are not analytic since this would imply the complete integrability of the system, that is, the space would be completely foliated by invariant tori. Nevertheless, as shown by Proposition 1.1, they do preserve (in a weak sense) some of the classical properties of analyticity. In the same work, the authors suggest that an analogous of the quasi-analytic extension of the KAM curve and its uniqueness properties should exist for general near-integrable symplectic maps in any dimension.

In this paper we show that their intuition is correct by proving that the KAM curve, for general near-integrable systems, does exhibit strong quasi-analyticity properties. We explore how and to what extent some of the properties of the KAM curve \( T_F \) characterize a general near integrable analytic system \( F \). We will tackle this question both in the discrete case (exact symplectic maps) and continuous case (Hamiltonian flows). The techniques we employ are different from those in [5] and do not make use of the aforementioned quasi-analytic extension. Nevertheless, this new approach allow us to conclude stronger uniqueness properties for near-integrable Hamiltonians.

We will show, among other things, that whenever two KAM curves coincide in a \( C^\omega \)-uniqueness set (see definition 2.3) not only the KAM curves but also the mappings associated to them must be equal. As Proposition 2.4 shows this is a much weaker condition than being equal on a positive Lebesgue measure set. We also show that systems sharing a sufficiently big collection of invariant tori (without any further assumption on the restricted dynamics) must commute. Finally, we provide a criteria for simultaneous conjugation Hamiltonian functions to integrable systems.

### 1.2 The generalized standard family

Before going any deeper in the discussion we would like to stress (and justify) the need to consider general near-integrable systems when considering only unique-
ness properties of the KAM curve and not the extensions proposed in [5]. In fact, a much stronger conclusion than that of Proposition 1.1 holds for the generalized standard family (see [8]) of exact symplectic maps on the $d$-dimensional cylinder $A^d = T^d \times \mathbb{R}^d$ given by

$$S_\varphi(\theta, r) = (\theta + r + \varphi(\theta), r + \varphi(\theta))$$

(2)

where

$$\varphi = \nabla V$$

for some $V \in C^\omega(T^d, \mathbb{R})$. Notice that whenever $\varphi$ is just of class $C^1$ the mapping $S_\varphi$ given by (2) is still a diffeomorphism although not necessarily symplectic.

**Proposition 1.2.** Let $\varphi, \psi \in C^1(T^d, \mathbb{R}^d)$ and $\gamma \in C^0(T^d)$. Suppose that $T := \text{graph}(\gamma)$ is invariant under $S_\varphi$ and $S_\psi$ then $\varphi = \psi$.

**Proof.** Let $\pi_1, \pi_2$ denote the projections of $T^d \times \mathbb{R}^d$ onto $T^d$ and $\mathbb{R}^d$ respectively. Define $\gamma(\theta) := (\theta, \gamma(\theta))$ and $g := \pi_1 \circ S_\varphi \circ \gamma$. The function $g$ is clearly a homeomorphism of the torus. By the invariance of $T$

$$S_\varphi \circ \gamma = \gamma \circ g$$

which implies

$$id + \gamma + \varphi = g$$

$$\gamma + \varphi = \gamma \circ g$$

Then $g^{-1} = id - \gamma$. By the last equations the function $\varphi$ is only defined by $\gamma$. As the same holds if we replace $\varphi$ by $\psi$ it is clear that $\varphi = \psi$. $\Box$

Since the tori provided by the KAM Theorem (see Theorems 3.1, 3.4) are graphs of functions in $C^\omega(T^d, \mathbb{R}^d)$ last Proposition implies the following.

**Corollary 1.3.** Any two maps of the generalized standard family having the same invariant KAM torus are equal.

**2 Preliminaries**

**2.1 Notations**

Given a complex number $z \in \mathbb{C}$ we denote its modulus by $|z|$. For $z \in \mathbb{C}^n$ we denote

$$|z|_1 = |z_1| + \cdots + |z_n|, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$  

Given $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ we denote its sup-norm by

$$\|f\|_U = \sup_{z \in U} |f(z)|.$$  

Given $k \in \mathbb{N} \cup \{0\}$ we denote by $C^k(U, V)$ the space of functions of class $C^k$ defined on $U$ and taking values in $V$. Whenever $V = \mathbb{C}$ we denote this space
simply by \( C^k(U) \). Given \( \gamma, \tau > 0 \) we say that \( \omega \in \mathbb{R}^d \) is Diophantine of type \((\gamma, \tau)\) if it satisfies

\[
|\langle \omega, k \rangle| \geq \frac{\gamma}{|k|^{d+\tau}} \quad \text{for all } k \in \mathbb{Z}^d \setminus \{0\}.
\]

We denote the set of Diophantine numbers of type \((\gamma, \tau)\) by \( DC(\gamma, \tau) \). For any open set \( \Omega \subset \mathbb{R}^d \) we define

\[
\Omega^\gamma_\tau := \{ \omega \in \Omega \mid \omega \in \Delta^\gamma_\tau, d(\omega, \partial \Omega) > \gamma \}.
\]

Given \( f : \mathbb{T}^d \to \mathbb{C} \) we will denote its average over \( \mathbb{T}^d \) by \([f]\).

### 2.2 Symplectic geometry

Let us recall some of the rudiments of symplectic geometry. For proofs and a complete introduction to the subject we refer the reader to [4].

A smooth manifold \( M \) of dimension \( 2d \) endowed with a closed, non-degenerated 2-form \( \omega \) is called a symplectic manifold. We will sometimes explicit dimension of \( M \) by writing \( M^{2d} \). For any open set \( U \subset M \) the pair \((U, \omega_M|_U)\) is a symplectic manifold. A submanifold \( L \subset M \) is said to be Lagrangian if the restriction of the symplectic form to \( L \) is equal to zero and \( \dim(L) = \frac{1}{2} \dim(M) \).

A smooth function on \( M \) is called a Hamiltonian. Every Hamiltonian \( H \) defines a unique smooth vector field \( X_H \) obeying

\[
i_{X_H} \omega = dH,
\]
where \( i_{X_H} \omega \) is the 1-form on \( M \) given by

\[
i_{X_H} \omega(p)(v_p) = \omega(X_H(p), v_p).
\]

We say that \( X_H \) is the Hamiltonian vector field of \( H \) and we denote its flow by \( \Phi^t_H \). The Poisson bracket \( \{H, L\} \) of two Hamiltonians is defined by

\[
\{H, L\} = X_H(L).
\]

Whenever \( \{H, L\} = 0 \) we say that the functions \( H, L \) are in involution. A diffeomorphism \( \Psi : N \to M \) between two symplectic manifolds \((N, \omega_N)\) and \((M, \omega_M)\) is said to be symplectic if

\[
\Psi^*(\omega_M) = \omega_N,
\]
where \( \Psi^*(\omega_M) \) denotes the pull-back of \( \omega_M \) by \( \Psi \). Recall that for a diffeomorphism \( \psi : N \to M \) the pullback of a \( k \)-differential form \( \beta \) on \( N \) is given by

\[
\psi^*(\beta)(p)(v_1, \ldots, v_k) = \beta(\psi(p))(d\psi(v_1), \ldots, d\psi(v_k))
\]
and satisfies
\[ d(\psi^* \beta) = \psi^*(d \beta). \]

We denote the set of all symplectomorphisms from \( N \) onto \( M \) by \( \text{Symp}(N, M) \).

A symplectic manifold \((M, \omega)\) is said to be exact if the form \( \omega \) is exact, that is, if there exist a 1-form \( \alpha \) such that
\[ \omega = d\alpha. \]

A diffeomorphism \( \psi : N \to M \) between two exact symplectic manifolds \((N, d\alpha_N)\) and \((M, d\alpha_M)\) is said to be exact symplectic if
\[ \psi^*(\alpha_M) = \alpha_N. \]

In particular, every exact symplectic mapping is symplectic. In the following Proposition we state some of the properties of Hamiltonian vector fields

**Proposition 2.1.** Let \((M, \omega_M)\), \((N, \omega_N)\) symplectic, \(H \in C^\infty(M)\) and \(\Sigma \in \text{Symp}(N, M)\). Denote \(H = H \circ \Sigma\). The following holds:

1. \(H\) is constant along the solutions of \(X_H\).

2. For all \(t_0 \in \mathbb{R}\) for which the flow \(\Phi_H^{t_0}\) is well defined the mapping \(\Psi = \Phi_H^{t_0}\) is symplectic. Furthermore, if \(M\) is exact then \(\Psi\) is exact symplectic.

3. For all \(t \in \mathbb{R}\) for which the flows \(\Phi_H^t, \Phi_H^{t_0}\) are well defined
\[ D\Sigma \cdot X_H = X_H \circ \Sigma, \quad \Sigma \circ \Phi_H^t = \Phi_H^t \circ \Sigma. \]

4. For all \(L \in C^\infty(M)\)
\[ \{H \circ \Sigma, L \circ \Sigma\} = \{H, L\} \circ \Sigma, \quad [X_H, X_L] = X_{\{H, L\}}. \]

In particular the flows associated to \(H\) and \(L\) commute if and only if \(\{H, L\} = 0\).

**Remark:** For any diffeomorphism \(\Psi : N \to M\) and any \(H \in C^\infty(M)\) the Hamiltonian system \((H, \omega_M)\) is equivalent to \((H \circ \psi, \psi^*(\omega_M))\).

A system (Hamiltonian flow or symplectic map) is said to be integrable if there exist functions \(f_1, f_2, \ldots, f_d \in C^\infty(M)\) such that:

1. \(f_1, \ldots, f_d\) are invariant by the system,
2. \(f_1, f_2, \ldots, f_d\) are generically independent, i.e., \(df_1, \ldots, df_n\) are linearly independent almost everywhere,
3. \(f_i, f_j\) are in involution (i.e. \(\{f_i, f_j\} = 0\)) for every \(i, j = 1, \ldots, d\).

Functions invariant by the system are called integrals of the system. For integrable systems and under fairly general conditions the Arnold-Liouville-Mineur Theorem assures that we can locally describe the dynamics of the system in a simplified system of coordinates known as angle-action coordinates.
Theorem 2.2. Let \((M^{2d}, \omega)\) be a symplectic manifold and let \(f_1, f_2, \ldots, f_d \in C^\infty(M)\) be \(d\) generically independent functions. Consider \(F = (f_1, \ldots, f_d)\) and suppose \(M_0\) is a compact connected component of \(F^{-1}(0)\). Then there exists an open neighbourhood \(U\) of \(M_0\) and a symplectomorphism

\[
\psi : U \to T^d \times B,
\]

where \(B\) is an open ball centred at the origin and \(T^d \times B\) is endowed with the canonical symplectic form \(d\theta \wedge dr\), such that \(F \circ \psi^{-1}\) depends only on \(r\).

The new coordinates \(\theta_i\) and \(r_i\) are called angle and action coordinates respectively. For a proof of this Theorem we refer the reader to [2].

By the previous Theorem, for an integrable Hamiltonian \(H\), locally there exists a symplectic map \(\psi : U \to T^d \times B\) such that the Hamiltonian flow associated to \(h = H \circ \Psi\) is given by

\[
(t, \theta, r) \mapsto (\theta + t\nabla h(r), r). \tag{3}
\]

Similarly for an integrable symplectic map \(\Sigma\), locally there exist a symplectic map \(\psi : U \to T^d \times B\) such that \(\psi^{-1}(T^d \times \{r_0\})\) is invariant for all \(r_0 \in B\). Thus

\[
\psi \circ \Sigma \circ \psi^{-1}(\theta, r) = (g(\theta, r), r),
\]

for some smooth function \(g\). Since the RHS map is also symplectic \(g\) must be of the form \(g(\theta, r) = \theta + \sigma(r)\) for some smooth function \(\sigma\) and therefore

\[
\psi \circ \Sigma \circ \psi^{-1}(\theta, r) = (\theta + \sigma(r), r). \tag{4}
\]

Since the symplectic map \(\psi\) establishes a conjugacy with the initial system, the Arnold-Liouville-Mineur Theorem asserts that every integrable system (Hamiltonian flow or symplectic map) is locally equivalent to a system of the form (3) or (4) defined over \(T^d \times B \subset T^d \times \mathbb{R}^d\) endowed with its canonical symplectic form.

As we will be interested in perturbations of integrable systems and in the persistence of local phenomena, in this work we will consider only systems defined over \(T^d \times B \subset T^d \times \mathbb{R}^d\) endowed with its canonical symplectic form and we will refer to integrability of the system as whether or not the system can be symplectically conjugated to the form (3) or (4).

2.3 Uniqueness sets

Definition 2.3. Let \(M\) be a manifold and \(p \in M\). We say that a set \(K \subset M\) is a \(C^\infty\)-uniqueness set at \(p\) if for all \(C^\infty\) functions defined on an open connected neighbourhood of \(K\) such that

\[
f|_K = 0
\]

\(f\) and its derivatives of all orders are equal to zero at \(p\). That is \(f\) is flat at \(p\).
Remark: In dimension one a $C^\infty$-uniqueness set at $p$ is simply a set that accumulates at $p$.

The next Proposition provides useful properties of $C^\infty$-uniqueness sets that will be used along the paper.

**Proposition 2.4.** Let $M_1, M_2$ be smooth manifolds. Let $p_i \in K_i \subset M_i$, $i = 1, 2$. The following holds:

1. If $\text{Leb}(K_1) > 0$ then $K_1$ is a $C^\infty$-uniqueness set for almost all $p \in K_1$.
2. If $K_1, K_2$ are $C^\infty$-uniqueness set at $p_1$ and $p_2$ respectively then $K_1 \times K_2 \subset M_1 \times M_2$ is a $C^\infty$-uniqueness set at $(p_1, p_2)$.
3. If $h : U \subset M_1 \to M_2$ is a smooth diffeomorphism defined on an open neighbourhood of $K_1$ then $h(K_1) \subset M_2$ is a $C^\infty$-uniqueness set at $h(p_1)$.

**Proof.** Without loss of generality we suppose that $M_1 = \mathbb{R}^m$, $M_2 = \mathbb{R}^n$. We denote by $\mu$ the Lebesgue measure on $M_1$.

1. For a full measure set $K \subset K_1$
\[
\lim_{r \to 0} \frac{\mu(B_r(x) \cap K)}{\mu(B_r(x))} = 1,
\]
for all $x \in K$. This easily implies that $\nabla f(x) = 0$ for all $x \in K$. A simple inductive argument shows the assertion.

2. Let $f \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ and $(\alpha, \beta) \in \mathbb{N}^m \times \mathbb{N}^n$. Then
\[
\partial^\alpha f(p_1, y) = 0.
\]
for all $y \in K_2$. Hence
\[
\partial^\beta \partial^\alpha f(p_1, p_2) = 0,
\]
which proves the assertion.

3. Notice that for all $f \in C^\infty(\mathbb{R}^n)$, $f$ is flat at $h(p_1)$ if and only if $f \circ h^{-1}$ is flat at $p_1$. $\square$

Positive measure is a sufficient condition for a set to be of uniqueness but it is far from being necessary. One example of this is the following Proposition

**Proposition 2.5.** Let $K \subset \mathbb{R}^2$ such that $K' = \{0\}$ and denote
\[
A := \left\{ \frac{p}{\|p\|} \mid p \in K \setminus \{0\} \right\}.
\]
If $A'$ is infinite then $K$ is a $C^\infty$-uniqueness set at 0.
Proof. Let \( f \in C^\infty(\mathbb{R}^2) \) such that \( f|_K = 0 \) and suppose that \( f \) is not flat at 0. Then there exist \( N > 1, N \geq k_0 > 0 \) and \( C > 0 \) such that

\[
f(x, y) = \frac{1}{N!} \sum_{k=k_0}^N \partial_x^{k} \partial_y^{N-k} f(0) x^k y^{N-k} + F_N(x, y)
\]

with

\[
a := \partial_x^{k_0} \partial_y^{N-k_0} f(0) \neq 0
\]

and

\[
|F_N(x, y)| \leq C \|(x, y)\|^N
\]

for all \((x, y) \in B_1(0)\). Up to compose with a rotation we can suppose without loss of generality that for all \( m \in \mathbb{N} \) there exist sequences \((x_n^{(m)})_{n \in \mathbb{N}}, (y_n^{(m)})_{n \in \mathbb{N}}\) in \( K \) such that

\[
|x_n^{(m)}|, |y_n^{(m)}| \rightarrow 0, \quad \frac{|x_n^{(m)}|}{|y_n^{(m)}|} \rightarrow \lambda_m, \quad \lambda_m \searrow 0.
\]

Then

\[
0 = \frac{N! f(x_n^{(m)}, y_n^{(m)})}{(y_n^{(m)})^N} \\
\geq a \left( \frac{x_n^{(m)}}{y_n^{(m)}} \right)^{k_0} \left( \frac{x_n^{(m)}}{y_n^{(m)}} \right)^{N-k_0} \sum_{k > k_0}^N \partial_x^k \partial_y^{N-k} f(0) \\
-C N! \|(x_n^{(m)}, y_n^{(m)})\| \left( \frac{x_n^{(m)}}{y_n^{(m)}}, 1 \right)^N \\
\rightarrow_{n \rightarrow \infty} |a| \lambda_m^{k_0} - \lambda_m^{k_0+1} \sum_{k > k_0}^N \lambda_m^{N-k_0-1} \left| \partial_x^k \partial_y^{N-k} f(0) \right|.
\]

As the last expression is positive for \( m \) sufficiently large this is a contradiction. \( \square \)

Similar conditions show the existence of countable \( C^\infty \)-uniqueness set in any dimension. We finish this Section with a simple remark on composition of analytic maps.

Lemma 2.6. Suppose \( F, G, \Phi, \Psi \) are \( C^\infty \) diffeomorphisms defined on open, connected subsets of \( \mathbb{T}^d \times \mathbb{R}^d \), with \( F, G \) analytic and such that the compositions

\[
\overline{F} := \Psi \circ F \circ \Phi, \quad \overline{G} := \Psi \circ G \circ \Phi,
\]

are well defined. If \( \overline{F} = \overline{G} + O^\infty(r) \) then \( F = G \).
**Proof.** Let \( F = \overline{G} + H \). Then
\[
F = \Psi^{-1} (\Psi \circ G + H \circ \Phi^{-1}) \\
= G + \int_0^1 D\Psi^{-1} (\Psi \circ G + tH \circ \Phi^{-1}) \cdot H \circ \Phi^{-1} \, dt
\]

Since \( H \circ \Phi^{-1} \) is flat at \( \Phi(\mathbb{T}^d \times \{0\}) \) it follows from the fact that \( F \) and \( G \) are analytic that \( F = G \). \( \square \)

3 The KAM curve

3.1 Exact symplectic maps

As mentioned in the introduction, the KAM curve associated to a sufficiently small perturbation of a non-degenerate integrable system consists of the collection of invariant tori given by the KAM Theorem when encoded as a function of the Diophantine frequencies. To formalize the definition let us state a simplified version of the KAM Theorem for Hamiltonians found in [12].

**Theorem 3.1.** Suppose \( \rho, s > 0, d \in \mathbb{N} \) and let \( I \) be an open, bounded, connected set of \( \mathbb{R}^d \). Let \( S_0 : I_\rho \to \mathbb{C} \) real analytic obeying:

1. \( \partial_r S_0 \big|_{I_\rho} \) is a diffeomorphism onto its image,
2. \( |\partial_r^2 S_0|_{I_\rho}, |\partial_r^2 S_0^{-1}|_{I_\rho} \leq R \).

Denote by \( F_0 \) the associated exact symplectic map \( F_0 \in \text{Symp}_{\text{Ex}}^\infty(\mathbb{T}^d \times I) \) given by
\[
F_0(\theta, r) = (\theta + \partial_r S_0(r), r),
\]
and let \( \Omega = \partial_r S_0(I) \). Given \( \gamma, \tau > 0 \) there exists an open neighbourhood of \( F_0 \)
\[
\mathcal{U}_{\gamma, \tau} = \{ F \in \text{Symp}_{\text{Ex}}^\infty(\mathbb{T}^d \times I_\rho) \mid \| F - F_0 \|_{\mathbb{T}^d \times I_\rho} < \epsilon \},
\]
where \( \epsilon \) is a positive constant depending only on \( (d, \gamma, \tau, \rho, R) \), such that for any \( F \in \mathcal{U}_{\gamma, \tau} \) there exist a Cantor set \( I^F \subset I \) of positive Lebesgue measure and Whitney smooth functions
\[
S \in C^\infty(I^F, \mathbb{R}), \quad \Sigma \in \text{Symp}^\infty(\mathbb{T}^d \times I^F),
\]
such that \( \partial_r S : I^F \to \Omega^{-\gamma} \) is a bijection and
\[
\Sigma^{-1} \circ F \circ \Sigma \big|_{\mathbb{T}^d \times I^F} = (\theta + \partial_r S(r), r).
\]
Furthermore, there exist a positive constant \( C \) depending only on \( (d, \gamma, \tau, \rho, R) \) such that
\[
\| \Sigma - \text{Id} \|_{\mathbb{T}^d \times I^F} \leq C \| F - F_0 \|_{\mathbb{T}^d \times I^F}.
\]
Using the notations in the Theorem, for all $\omega \in \Omega_\gamma^\tau$ the graph of the smooth function $u_\omega \in C^\infty(\mathbb{T}^d)$ given by
\[ u_\omega(\theta) = \Sigma(\theta, \partial_\theta S^{-1}(\omega)) , \]
defines an invariant Lagrangian torus $T_\omega$ for the Hamiltonian flow associated to $F$. Moreover the restricted dynamics on $T_\omega$ is equivalent to a discrete translation by $\omega$. We call the collection
\[ \{T_\omega\}_{\omega \in \Omega_\gamma^\tau} \]
the KAM curve associated to the perturbed system and we encode it in the Whitney smooth function
\[ T_F : \Omega_\gamma^\tau \to C^\infty(\mathbb{T}^d) \]
\[ \omega \mapsto u_\omega . \]
Following [7] we say that a smooth Lagrangian invariant torus whose restricted dynamics are smoothly conjugated to a (continuous or discrete) translation by a Diophantine vector $\omega$ is a KAM torus with rotation vector $\omega$.

We can now state the main result of this Section for integrable exact symplectic maps.

**Theorem 3.2.** Let $F_0, F$ as in Theorem 3.1 and denote by $T_F$ the associated KAM curve. Suppose
\[ G \in \text{Symp}^\omega(\mathbb{T}^d \times I), \quad \Gamma \subset \Omega_\gamma^\tau, \quad \omega_0 \in \Gamma, \]
with $\Gamma$ a $C^\infty$-uniqueness set at $\omega_0$. If for all $\omega \in \Gamma$ the function $T_F(\omega)$ defines an invariant torus $T_\omega$ for $G$ then the following holds:

1. $T_{\omega_0}$ is an invariant KAM torus for $G$.
2. $F$ and $G$ commute on a neighbourhood of $T_{\omega_0}$.
3. If $T_\omega$ is a KAM torus for $G$ with rotation vector $\omega$ for all $\omega \in \Gamma$ then $F = G$.

**Proof.** Let $\Sigma, S$ as in Theorem 3.1 when applied to $F$. Denote
\[ \overline{F} := \Sigma^{-1} \circ F \circ \Sigma, \quad \overline{G} := \Sigma^{-1} \circ G \circ \Sigma, \]
and write
\[ \overline{F}(\theta, r) = (\theta + S(r) + f_1(\theta, r), r + f_2(\theta, r)) \]
\[ \overline{G}(\theta, r) = (\theta + g_1(\theta, r), r + g_2(\theta, r)). \]
Then
\[ f_1(\theta, r) = 0, \quad f_2(\theta, r) = 0 = g_2(\theta, r), \]
for all \((\theta, r) \in \mathbb{T}^d \times h^{-1}(\Gamma)\). To simplify the notation let us assume WLOG that 
\(h^{-1}(\omega_0) = 0\). By Proposition 2.4, \(\mathbb{T}^d \times h^{-1}(\Gamma)\) is a \(C^\infty\)-uniqueness set at \((\theta, 0)\)
for all \(\theta_0 \in \mathbb{T}^d\). Then
\[
 f_1(\theta_0, r) = O^\infty(r), \quad g_2(\theta_0, r) = O^\infty(r).
\]
Since \(\mathcal{G}\) is a symplectic map it obeys
\[
 J = D\mathcal{G}^T(\theta, r)JD\mathcal{G}(\theta, r) \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]
for all \((\theta, r) \in \Sigma^{-1}(\mathbb{T}^d \times I)\). A direct calculation yields to
\[
 J = \begin{pmatrix} 0 & I + \partial_\theta g_1 \\ -I - \partial_r g_1 + \partial_r g_1^T \end{pmatrix} + O^\infty(r).
\]
Hence
\[
 \partial_\theta g_1(\theta, r) = O^\infty(r),
\]
which implies
\[
 g_1(\theta, r) = [g_1](r) + O^\infty(r).
\]
Then
\[
 \mathcal{T}(\theta, r) = (\theta + S(r), r) + O^\infty(r), \quad \mathcal{G}(\theta, r) = (\theta + [g_1](r), r) + O^\infty(r).
\]
This shows that \(T_{\omega_0} = \Sigma(\mathbb{T}^d \times \{0\})\) is a KAM torus for \(G\). Furthermore
\[
 \mathcal{F} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{F} + O^\infty(r)
\]
on a neighbourhood of \(\mathbb{T}^d \times \{0\}\). Thus by Lemma 2.6 \(F\) and \(G\) commute on a
neighbourhood of \(\mathbb{T}^d \times \{0\}\). To prove the last assertion let us suppose that \(T_{\omega}\)
is a KAM torus of \(G\) with rotation vector \(\omega\) for all \(\omega \in \Gamma\). Then
\[
 [g_1](r) = S(r) + O^\infty(r),
\]
and
\[
 \mathcal{T} = \mathcal{G} + O^\infty(r).
\]
Hence \(F = G\) by Lemma 2.6.

**Corollary 3.3.** Let \(F_0, \mathcal{U}_{\gamma, \tau}\) as in Theorem 3.1. If \(F, G \in \mathcal{U}_{\gamma, \tau}\) and \(T_F, T_G\)
coincide on a \(C^\infty\)-uniqueness set then \(F = G\).

### 3.2 Hamiltonian systems

Let us state a simplified version of the KAM Theorem for Hamiltonians systems
found in [11].
Theorem 3.4. Suppose $\rho > 0$, $d \in \mathbb{N}$ and let $I$ be an open, bounded, connected set of $\mathbb{R}^d$. Let $H_0 : I_\rho \to \mathbb{C}$ real analytic obeying:

1. $\partial_r H_0 |_{I_\rho}$ is a diffeomorphism onto its image,
2. $|\partial^2_r H_0|_{I_\rho}, |\partial^2_r H_0^{-1}|_{I_\rho} \leq R.$

Let $\Omega = \partial_r H_0(I)$. Given $\gamma, \tau > 0$ there exists an open neighbourhood $U_{\gamma, \tau}$ of $H_0$

$U_{\gamma, \tau} = \{ H \in C^\omega(T^d \times I) \mid \|H - H_0\|_{T^d \times I} \leq \epsilon \},$

where $\epsilon$ is a positive constant depending only on $(d, \gamma, \tau, \rho, R)$, such that for any $H \in U_{\gamma, \tau}$ there exist a Cantor set $I^H \subset I$ of positive Lebesgue measure and Whitney smooth functions

$h \in C^\omega(I^H, \mathbb{R}), \quad \Sigma \in \text{Sym}^\omega(T^d \times I^H),$

such that $\partial_r h : I^H \to \Omega^\gamma_{\tau}$ is a bijection and

$H \circ \Sigma |_{T^d \times I^H} = h(r), \quad X_{HT \Sigma} |_{T^d \times I^H} = (\partial_r h(r), 0).$

Furthermore, there exist a positive constant $C$ depending only on $(d, \gamma, \tau, \rho, R)$ such that

$\|\Sigma - Id\|_{T^d \times I^H} \leq C\|H - H_0\|_{T^d \times I}.$

As it was done for symplectic maps, we will encode this collection of invariant tori in the Whitney smooth function

$T_H : \Omega^\gamma_{\tau} \omega \to C^\omega(T^d),$

where

$u_\omega(\theta) = \Sigma(\theta, \partial_r h^{-1}(\omega)).$

The mapping $T_H$ is the KAM curve associated to $H$. Notice that for every $\omega \in \Omega^\gamma_{\tau}$ the restricted dynamics on $T_\omega$ is equivalent to a continuous translation by $\omega$.

The following is an analogous of Theorem 3.2 in the Hamiltonian case.

Theorem 3.5. Suppose $H_0, U_{\gamma, \tau}$ as in Theorem 3.4. Let

$L \in C^\omega(T^d \times I), \quad \omega_0 \in \Gamma \subset \Omega^\gamma_{\tau},$

with $\Gamma$ a $C^\omega$-uniqueness set at $\omega_0$ such that for all $\omega \in \Gamma$ the function $T_H(\omega)$ defines an invariant torus $T_\omega$ for the Hamiltonian flow $\Phi^t_L$. Then the following holds:

1. $T_{\omega_0}$ is a KAM torus for $L$ with rotation vector $\omega^L_0$ for some $\omega^L_0 \in \mathbb{R}^d$.
2. If $\omega^L_0 \parallel \omega_0$ then $X_L \parallel X_H$ on $T_{\omega_0}$.
3. The flows $\Phi^t_H$ and $\Phi^t_L$ commute on a neighbourhood of $T_{\omega_0}$.

4. If $T_{\omega}$ is a KAM torus for $L$ with rotation vector $\omega^L \parallel \omega$ for all $\omega \in \Gamma$ there exist an analytic function $\varphi$ such that $L = \varphi \circ F$ on an open neighbourhood of $T_{\omega_0}$.

**Proof.** Let $\Sigma, h$ as in Theorem 3.4 when applied to $H$. Denote

$$H = H \circ \Sigma, \quad L = L \circ \Sigma.$$  

To simplify the notation let us suppose that $h^{-1}(\omega_0) = 0$. Since $T_d \times h^{-1}(\Gamma)$ is a $C^\infty$-uniqueness set at $(\theta, 0)$ for all $\theta \in T_d$, invariant for $\Phi^t_H$ and $\Phi^t_L$ it follows that

$$\partial_\theta H = O^\infty(r), \quad \partial_\theta L = O^\infty(r).$$

Hence

$$H = [H](r) + O^\infty(r), \quad L(\theta, r) = [L](r) + O^\infty(r),$$

and by definition of $h$

$$\partial_r [H](r) = h(r) + O^\infty(r).$$

Then

$$X_H(\theta, r) = (h(r), 0) + O^\infty(r), \quad X_L(\theta, r) = (\partial_r [L](r), 0) + O^\infty(r). \quad (5)$$

In particular

$$X_L(\theta, r) = (\partial_r [L](0), 0). \quad (6)$$

Thus $T_{\omega_0} = \Sigma(T_d \times \{0\})$ is a KAM torus for $L$ with rotation vector $\omega^L = \partial_r [L](0)$.

By Proposition 2.1 if $\omega^L \parallel \omega_0$ the vector fields $X_H, X_L$ are collinear on $T_{\omega_0}$. This shows the first and seconds assertions. To prove the third one recall that the Hamiltonian flows of $H, L$ commute if and only if the poisson bracket $[X_H, X_L] = 0$. From (5)

$$X_{(H, L)} = O^\infty(r)$$

and by Proposition 2.1

$$[X_H, X_L] = X_{(H, L)} = X_{(H, L) \circ \Sigma^{-1}} = D\Phi \circ X_{(H, L)} \circ \Sigma^{-1}.$$  

by Lemma 2.6 it follows that

$$[X_H, X_L] = 0,$$

that is, $\Phi^t_H$ and $\Phi^t_L$ commute on a neighbourhood of $T_{\omega_0}$. To prove the last assertion let us suppose that $T_{\omega}$ is a KAM torus for $L$ with rotation vector $\omega^L \parallel \omega$ for all $\omega \in \Gamma$. Then there exist a smooth function $\gamma(r)$ such that

$$\partial_r [L](r) = \gamma(r)h(r) + O^\infty(r).$$
Hence
\[ X_H = \gamma X_L + O^\gamma(r). \]  
(7)

Let us show that \( X_H \) and \( X_L \) are always collinear, that is
\[ X_H = \langle X_H, X_L \rangle \frac{X_L}{\|X_L\|^2}. \]

By Lemma 2.6 it suffices to show that
\[ X_H \circ \Sigma = \langle X_H \circ \Sigma, X_L \circ \Sigma \rangle \frac{X_L \circ \Sigma}{\|X_L \circ \Sigma\|^2} + O^\Sigma(r). \]

Developing the RHS and by (7)
\[ \langle X_H \circ \Sigma, X_L \circ \Sigma \rangle \frac{X_L \circ \Sigma}{\|X_L \circ \Sigma\|^2} = \gamma D \Sigma \cdot X_H + O^\Sigma(r) \]
\[ = D \Sigma \cdot X_H + O^\Sigma(r) \]
\[ = X_H \circ \Sigma + O^\Sigma(r) \]

Thus \( X_H, X_L \) are everywhere collinear. In particular the level sets of \( H \) and \( L \) coincide. Let \( p \in T_{\omega_0} \) and let us suppose WLOG that \( H(p) = 0 \). As \( H \) is constant on every KAM torus \( T_{\omega_0} \subset H^{-1}(0) \). By the implicit function Theorem there exist an analytic diffeomorphism
\[ \Psi : (-\epsilon, \epsilon) \times U \subset \mathbb{R} \times \mathbb{R}^{d-1} \to W \subset \mathbb{R}^d \]
such that
\[ H \circ \Psi(a, v) = a, \quad \Psi(0, 0) = p. \]

Since \( H \) and \( L \) have the same level sets there exists \( \varphi : (-\epsilon, \epsilon) \to \mathbb{R} \) analytic such that
\[ L \circ \Psi(a, v) = \varphi(a). \]

Hence \( L = \varphi \circ H \) on \( W \), but clearly this equality holds also on the connected component of \( T_{\omega_0} \) inside \( H^{-1}(-\epsilon, \epsilon) \). This completes the proof.

In the Hamiltonian case a little bit more can be said for systems sharing a sufficiently big collection of tori even if the rotation vectors on these tori are not the same (or at least collinear).

**Proposition 3.6.** Suppose \( H_0, \mathcal{U}_{\gamma, \tau} \) as in Theorem 3.3. Let
\[ L_1 \in \mathcal{U}_{\gamma, \tau}, \quad L_2, \ldots, L_d \in C^\omega(T^d \times I), \quad \omega_0 \in \Gamma \subset \Omega_{\gamma}, \]
with \( \Gamma \) a \( C^\infty \)-uniqueness set at \( \omega_0 \) such that for all \( \omega \in \Gamma \) the function \( T_H(\omega) \) defines an invariant torus \( T_\omega \) for the Hamiltonian flows \( \Phi^t_{L_1} \). Denote by \( \omega^i \) the rotation vector of \( T_{\omega_0} \) under the flow \( \Phi^t_{L_i} \) (Theorem 3.3). If
\[ \omega^1, \omega^2, \ldots, \omega^d \]
are linearly independent,

there exist an analytic diffeomorphism \( \psi \) conjugating \( L_1, \ldots, L_d \) simultaneously to completely integrable systems in a neighbourhood of \( T_{\omega_0} \).
Remark: The diffeomorphism $\psi$ is not necessarily a symplectic. Nevertheless, with some technical modifications we can take it to be symplectic. See Corollary 4.2.

Proof Let $\Sigma, h$ as in Theorem 3.1 when applied to $L_1$. Denote $L = (L_1, \ldots, L_d)$, and define

$$
\Psi : \mathbb{T}^d \times I \rightarrow \mathbb{T}^d \times \mathbb{R}^d \\
(\theta, r) \rightarrow (\theta, L(\theta, r))
$$

Since $L$ is constant on every common invariant torus, for every $\omega \in \Gamma$ there exist an unique vector, which we denote $h(\omega)$, such that

$$
\Psi(T_\omega) = \mathbb{T}^d \times \{h(\omega)\}.
$$

An explicit formula for $h$ can be retrieved by means of the function $h$. Indeed

$$
h(\omega) = [L \circ \Sigma] (h^{-1}(\omega)).
$$

In particular, $h$ is a (Whitney) smooth function. Suppose for a moment that $\Psi$ is a diffeomorphism onto its image and denote

$$
\mathcal{L}_i = L_i \circ \Psi^{-1}
$$

for all $i = 1, \ldots, d$. By Proposition 2.1 the flow generated by the vector field $X_{\mathcal{L}_i}$ associated to $(L_i, \omega)$ is equivalent to the flow generated by the vector field $Y_{\mathcal{L}_i}$ associated to $(\mathcal{L}_i, (\Psi^{-1})^*(\omega))$. By the invariance of $\mathbb{T}^d \times h(\Gamma)$ under the flow given by $Y_{\mathcal{L}_i}$ it follows that

$$
Y_{\mathcal{L}_i}(\theta, r) = (Y_i(\theta, r), 0)
$$

for all $(\theta, r) \in \mathbb{T}^d \times h(\Gamma)$ and for some analytic function $Y_i$. Since $\mathbb{T}^d \times h(\Gamma)$ is a $C^\infty$-uniqueness set at $(\theta, h(\omega_0))$ for all $\theta \in \mathbb{T}^d$ it follows that the last inequality holds for all $(\theta, r)$, showing the integrability of the Hamiltonian.

Hence it suffices to show that $\Psi$ restricted to a sufficiently small neighbourhood of $T_{\omega_0}$ is a diffeomorphism onto its image. To simplify the notation let us suppose that $h^{-1}(\omega_0) = 0$. Since $T_{\omega_0} = \Sigma(\mathbb{T}^d \times \{0\})$ it suffices find a neighbourhood $U$ of $\mathbb{T}^d \times \{0\}$ such that $\Psi \circ \Sigma|_U$ is a diffeomorphism onto its image. Denote

$$
\mathcal{L}_i = L_i \circ \Sigma
$$

for all $i = 1, \ldots, d$. As the pair $L_1, L_i$ satisfy the hypotheses of Theorem 3.3 for $i = 1, \ldots, d$, equation (6) holds replacing $L$ by $L_i$ which yields to

$$
X_{\mathcal{L}_i}(\theta, 0) = (\omega^i, 0) = (\partial_r \mathcal{L}_i(\theta, 0), -\partial_\theta \mathcal{L}_i(\theta, 0))
$$

for all $\theta \in \mathbb{T}^d$. Thus

$$
D(\Psi \circ \Sigma)(\theta, 0) = \begin{pmatrix}
I_d & 0 \\
\omega^1 & \omega^1 \\
\omega^2 & \omega^2 \\
& \vdots \\
\omega^d & \omega^d
\end{pmatrix}
$$
for all $\theta \in \mathbb{T}^d$. By hypotheses $\omega^1, \omega^2, \ldots, \omega^d$ are linearly independent which shows that $\Psi \circ \Sigma$ is a local diffeomorphism on a small neighbourhood $U$ of $\mathbb{T}^d \times \{0\}$. Since $\Psi \circ \Sigma$ restricted to $\mathbb{T}^d \times \{0\}$ is injective we can suppose, up to consider a smaller neighbourhood, that $\Psi|_U$ is a diffeomorphism onto its image. This concludes the proof. \hfill \Box

Putting these two results together for $d = 2$ we obtain:

**Theorem 3.7.** For $d = 2$ and under the same hypotheses of Theorem 3.5 we have the following dichotomy:

1. If $\omega^L \parallel \omega^0$ there exist a diffeomorphism conjugating $H, L$ simultaneously to completely integrable systems in a neighbourhood of $T_{\omega^0}$.

2. If $\omega^L \not\parallel \omega^0$ the vector fields $X_H, X_L$ are collinear on $T_{\omega^0}$. Furthermore, either

$$K = \varphi \circ H$$

in a neighbourhood of $T_{\omega^0}$ for some analytic function $\varphi$ or every neighbourhood of $T_{\omega^0}$ contains open connected sets completely foliated by common invariant tori of $H$ and $L$.

**Corollary 3.8.** Let $H_0, U$ as in Theorem 3.4 and $H, L \in U$. If $T_H = T_L$ on a $C^\infty$-uniqueness set then $H = K$.

### 4 Symplectic Conjugacy

The result of Proposition 3.6 can be improved by choosing the diffeomorphism conjugating simultaneously the Hamiltonians to be a symplectomorphism. First we adapt a result of Eliasson [6], a ‘Darboux Lemma’ preserving a Lagrangian foliation.

**Proposition 4.1.** Let $\omega$ be an exact symplectic form on $\mathbb{T}^d \times \mathbb{R}^d$ such that the foliation $\mathcal{F} = \{T_r\}_{r \in \mathbb{R}^d}$ is Lagrangian and

$$\omega = \omega_{\text{std}} \text{ in } \mathbb{T}^d \times \{0\}$$

(9)

where $\omega_{\text{std}}$ is the canonical symplectic form. Then there exist a diffeomorphism

$$\phi : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$$

preserving the foliation and such that

$$\phi^* \omega = \omega_0$$

**Proof.** Define $\omega_t = \omega_0 + t(\omega - \omega_0)$. Note that $\omega_t$ is symplectic for all $t \in [0, 1]$ in some neighbourhood $U$ of $\mathbb{T}^d \times \{0\}$. To find the diffeomorphism we will use Moser’s trick. Let $\beta \in \Omega^1(\mathbb{T}^d \times \mathbb{R}^d)$ be such that $d\beta = \omega - \omega_0$ and suppose that

$$i_{X_\omega} \omega_t = \beta$$
Denote by $X_i = \frac{\partial}{\partial \theta_i}$ the coordinate vector fields. Then the flux associated to the time dependent vector field $X_t$ preserves the foliation if and only if $X_t(dr_i)$ is independent of $\theta$ for every $i = 1, \ldots, d$, that is, $\beta$ depends only on $r$. In other words

$$\beta(X_i) \text{ is constant for every } i = 1, \ldots, d$$

Let $\alpha, \alpha_0$ be primitives of $\omega, \omega_0$ respectively. As $d\beta = \omega - \omega_0$, $\beta = \alpha - \alpha_0 - df$ for some function $f$. Denote $g_i = (\alpha - \alpha_0)(X_i)$ and $g = (g_1, \ldots, g_d)$. The last condition can be rewritten as

$$g - \nabla f = \text{constant}$$

The foliation being Lagrangian for $\omega$ is equivalent to

$$\omega(X_i, X_j) = 0$$

and as $[X_i, X_j] = 0$ we get

$$X_j(\alpha(X_i)) = X_i(\alpha(X_j))$$

Note that this is also true for $\omega_0$ and $\alpha_0$ thus

$$\frac{\partial g_i}{\partial \theta_j} = \frac{\partial g_j}{\partial \theta_i}$$

Then for some function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and some constants $b, c = (c_1, \ldots, c_d) \in \mathbb{R}^d$

$$\tilde{g} = \nabla h + b$$

$$h(x + e_i) = h(x) + c_i$$

where $\tilde{g}$ is the lifting of $g$ to $\mathbb{R}^d$. Then

$$f(x) = h(x) - \langle c, x \rangle$$

is defined on $\mathbb{T}^d$ and $g - \nabla f$ is constant. For this particular choice of $\beta$ if $\phi$ is the time 1-map of the flow associated to $X_t$ we have $\phi^* \omega = \omega_0$ as desired. \qed

**Corollary 4.2.** The diffeomorphism $\psi$ in Proposition 3.6 can be taken to be symplectic.

We would like to apply the last Proposition with $\omega = \psi^*(\omega_{std})$. In that case, $\psi \circ \phi$ would be the desired symplectic map. Nevertheless, (9) does not necessarily hold and thus Proposition 4.1 cannot be applied. We will fix this by modifying the proof of the Proposition.

**Proof.** Let $\Sigma, h$ as in Theorem 4.4 when applied to $L_1$ and let us suppose that $h^{-1}(\omega_0) = 0$. Denote $\Sigma = (\Sigma_1, \Sigma_2)$ and define $g : \mathbb{T}^d \rightarrow \mathbb{T}^d$, $\gamma : \mathbb{T}^d \rightarrow \mathbb{R}$ by

$$g(\theta) := \Sigma_1(\theta, 0), \quad \gamma(\theta) := \Sigma_2(\Sigma_1^{-1}(\theta, 0)).$$

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Notice that \( g \in \text{Diff}^\omega(T^d) \) because \( T_{\omega_0} = \Sigma(T^d \times \{0\}) \) is a Lagrangian graph and thus \( \gamma \in C^\omega(T^d, \mathbb{R}^d) \) is well defined. Let \( \phi_1, \phi_2 : \mathbb{A}^n \rightarrow \mathbb{A}^n \)
\[
\phi_1(\theta, r) = (\theta, \gamma(\theta) + r), \quad \phi_2(\theta, r) = (g(\theta), \overline{g}^{-1}(\theta)^T r).
\]
These two mappings are symplectic with respect to \( \omega_{\text{std}} \) (\( \mathbb{S} \) Lemma 1.2.4). Notice that \( T_{\omega_0} = (\phi_1 \circ \phi_2)(T^d \times \{0\}) \).
Denote \( L_i := L_i \circ \phi_1 \circ \phi_2. \)

By (8), for every \( \theta \in T^d \) and \( t \in \mathbb{R} \)
\[
(\theta + t\omega_i, 0) = \Phi_{L_i \circ \phi_1 \circ \phi_2}(\theta, 0)
\]
\[
= \Phi_{L_i \circ \phi_1 \circ \phi_2}^{-1}(\theta, 0)
\]
\[
= (\Sigma^{-1} \circ \phi_1 \circ \Phi_{L_i \circ \phi_2} \circ \phi_2^{-1} \circ \Sigma)(\theta, 0)
\]
\[
= (\phi_2^{-1} \circ \Phi_{L_i \circ \phi_2} \circ \phi_2)(\theta, 0)
\]
\[
= \Phi_{L_i}^{-1}(\theta, 0)
\]
Then (8) holds if we replace \( L_i \) by \( T_i. \) Thus, up to a constant term
\[
L_i(t, r) = \langle \omega_i, r \rangle + O^2(r).
\]
Let
\[
\mathbf{T} = (T_1, \ldots, T_d), \quad M = \begin{pmatrix} \omega^1 & \cdots & \omega^d \\ \omega^2 & \cdots & \omega^d \end{pmatrix}^{-1},
\]
and define
\[
\Psi : T^d \times V \rightarrow T^d \times \mathbb{R}^d \quad (\theta, r) \rightarrow (\theta, M \mathbf{T}(\theta, r))
\]
where \( V \) is a neighbourhood of 0 such that \( \phi_1 \circ \phi_2(V) \subset I. \) To simplify the notation let us suppose, without loss of generality, that \( \mathbf{T} |_{T^d \times \{0\}} = 0. \) Thus
\[
D\Psi(\theta, 0) = I_{2d}.
\]
Hence, as in the proof of Proposition 3.6, \( \Psi \) is a local diffeomorphism on a neighbourhood \( U \) of \( T^d \times \{0\} \) and the Hamiltonian flow associated to \( L_i \circ \Psi^{-1} \) preserves all the tori \( T^d \times \{r\} \subset \Psi(T^d \times V). \) Notice that \( \omega := (\Psi^{-1})^* \omega_{\text{std}} \) satisfies (9). Hence by Proposition 4.1 there exists an analytic diffeomorphism \( \phi \), preserving the foliation, such that \( \phi^* \omega = \omega_0. \) The result follows by setting \( \psi = \Psi^{-1} \circ \phi. \)
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