RELATIVE-PARTITIONED INDEX THEOREM

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Abstract. It seems that the index theory for non-compact spaces has found its ultimate formulation in realm of coarse spaces and $K$-theory of related operator algebras. Relative and partitioned index theorems may be mentioned as two important and interesting examples of this program. In this paper we formulate a combination of these two theorems and establish a partitioned-relative index theorem.

1. Introduction

For $i = 1, 2$ let $(M_i, g_i)$ be non-compact odd-dimensional spin complete Riemannian manifolds which are partitioned by two hyper-surfaces $N_i \subset M_i$ into sub manifolds $M^+_i$ and $M^-_i$ with common boundary $\partial M^+_i = \partial M^-_i = M^+_i \cap M^-_i = N_i$. Let $E_i$ be a Clifford bundles over $M_i$ and put $H_i = L^2(M_i, E_i)$. The associated Dirac operator $D_i$ is a formally self adjoint operator on $H_i$. The coarse index is an element in the coarse $C^*$-algebra $C^*(M_i)$ [6]. The restriction of Dirac operator $D_i$ to $N_i$ is the grading reversing Dirac operator $D_i$ that acts on graded sections of $E_i|N_i$.

Moreover assume there are closed subsets $W_i \subset M_i$ that intersect $N_i$ coarsely and transversally such that the intersection $Z_i = W_i \cap M_i$ is compact. Moreover we assume that there is an isometry $\psi : M_1 \setminus W_1 \to M_2 \setminus W_2$ that is covered by isometries of bundles that we denote by the $\Psi$, so that $D_2 = \Psi \circ D_1 \circ \Psi^{-1}$. Let there is another Riemannian manifold $M$ and subsets $N$ and $W$ satisfying above conditions, (e.g. the intersection $Z := N \cap W$ is compact). Moreover assume there are smooth coarse maps $f_i : M_i \to M$ such that $f_i^{-1}(W) \subset W_i$ and $f_i^{-1}(N) = N_i$. Moreover we assume that $N$ is a regular submanifold for $f_i$. Using all these structures (except the partitioning hyper-surfaces), one has defined a relative index $\text{ind}(D_1, D_2) \in K_1(C^*(W \subset M; A))$.

Under above conditions, $Z_i$ is a compact subset of $N_i$ and all geometric data on $N_i \setminus Z_i$ are identified via isomorphism $\psi|_{N_i \setminus Z_i}$. Also we have the maps $f_i$ from $N_i$ into $N$ with $f_i^{-1}(Z) \subset Z_i$. Therefore one can define the relative index $\text{ind}(D_1, D_2)$ as an element in $K_0(K) \simeq \mathbb{Z}$ (this because $Z$ is compact), where $K$ is the algebra of all compact operators on a separable infinite dimensional Hilbert space.

We call a function $h : M \to \mathbb{R}$ to be $W$-coarse if its restriction to each finite neighbourhood of $W$ is a coarse map. We will show (see [2,3]) that $h$ induces a natural morphisms

$$h_* : K_1(W \subset M) \to K_1(\mathbb{R}) \simeq \mathbb{Z}$$

Moreover, if $h$ is smooth with regular value 0 and $N = h^{-1}(0)$ then we have the following equality (see theorem [3,3]) which is a main result of this paper

$$h_*(\text{ind}(D_1, D_2)) = \text{ind}(D_1, D_2)$$

We use this theorem to re-prove the non-existence of a metric on $\tilde{N} \times \mathbb{R}$ with uniformly positive scalar curvature, provided that $\tilde{N}$ is an enlargeable manifold. This theorem was first proved (amongst other important facts) in [1]. In section [2] and [3] we establish necessary tools from $K$-theory and index theory of coarse spaces in relative context and we formulate the statement of the main theorem.
In section 2 we prove this theorem by reducing the problem to cylindrical case. In section 5 we provide an application to the main theorem.

2. SOME CONSTRUCTIONS IN RELATIVE K-THEORY OF COARSE SPACES

Let $X$ be a complete proper metric space and $H$ be a Hilbert space which is an amenable module over $C_0(X)$. The module action of a function $\phi$ on $H$ is denoted by $\rho(\phi)$ or just by $\phi$ if there is no risk of confusion. A bounded linear operator $T$ on $H$ is called controlled (or have finite propagation property) if there is $r > 0$ such that for any $\phi$ and $\psi$ in $C_0(X)$ the relation $d(supp(\phi), supp(\psi)) > r$ implies $\rho(\phi)T\rho(\psi) = 0$. The operator $T$ is called pseudolocal if $\rho(\phi)T\rho(\psi)$ is compact provided that $\phi\psi = 0$. The operator $T$ is locally compact if for $\phi$ as in above, the linear maps $\phi T$ and $T \phi$ are compact operators. Given a closed subset $Y \subset X$, the operator $T$ is supported near $Y$ if there is a constant $r$ such that $\rho(\phi)T = 0 = T\rho(\phi)$ if $d(supp(\phi), Y) > r$.

Using above definitions, the following $C^*$-algebras are defined: The space of all bounded pseudolocal operator on $H$ is denoted by $D^*_c(X)$ while $D^*(X)$ consists of all bounded, controlled and pseudolocal operators on $H$. The space of all bounded and locally compact operators on $H$ is denoted by $C^*_c(X)$. It is easy to verify that $C^*_c(X)$ is an ideal of $D^*_c(X)$. The relative $C^*$-algebra $C^*_c(Y \subset X)$ is the ideal of $C^*_c(X)$ (and of $D^*_c(X)$) consisting of those operators which are supported near to $Y \subset X$. Similarly $D^*_c(Y \subset X)$ is the ideal of $D^*_c(X)$ consisting of those operators which are supported near to $Y \subset X$. It is easy to see that $C^*_c(Y \subset X)$ is an ideal of $D^*_c(Y \subset X)$. Similarly we can define the algebra $D^*_c(Y \subset X)$ and its ideal $C^*_c(Y \subset X)$.

In these definitions we have not mentioned the Hilbert space $H$ because the $K$-theory of these algebras are canonically independent of $H$. When we need to emphasize the Hilbert space we use the notation, e.g. $C^*_c(X, H)$ and so on.

The $K$-homology $K_j(X)$ and relative $K$-homology $K_j(Y \subset X)$ are defined by following relations, cf. [7]

$$K_j(X) = K_{j+1}(D^*_c(X)/C^*_c(X)) ; \quad K_j(Y \subset X) = K_{j+1}(D^*_c(Y \subset X)/C^*_c(Y \subset X))$$

It turns out that the following equalities hold [9] Page 6]

$$D^*_c(X)/C^*_c(X) = D^*(X)/C^*(X) \quad \text{and} \quad D^*_c(Y \subset X)/C^*_c(Y \subset X) = D^*(Y \subset X)/C^*(Y \subset X)$$

Therefore the $K$-theory long exact sequences associated to the short exact sequences

$$0 \to C^*_c(X) \to D^*_c(X) \to D^*_c(Y \subset X)/C^*_c(X) \to 0 \quad (2.1)$$

$$0 \to C^*_c(Y \subset X) \to D^*_c(Y \subset X) \to D^*_c(Y \subset X)/C^*_c(Y \subset X) \to 0 \quad (2.2)$$

along with its naturality, provide the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\vdots & \longrightarrow & K_{j+1}(D^*(Y \subset X)) & \longrightarrow & K_j(Y \subset X) & \longrightarrow & K_j(C^*(Y \subset X)) & \longrightarrow & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & \longrightarrow & K_{j+1}(D^*(X)) & \longrightarrow & K_j(X) & \longrightarrow & K_j(C^*(X)) & \longrightarrow & \vdots
\end{array} \quad (2.3)$$

Here the vertical morphisms are induced by inclusion and the morphisms $A$'s are the assembly map and its relative counterpart, c.f. [7] p.11.

The closed subspace $Y$ is a complete metric space, so it has its own $K$-homology and coarse $C^*$-algebras. Extension by zero defines a natural maps from these absolute object to the corresponding...
relative object associated to the inclusion \( Y \subset X \). These inclusions defines the following natural isomorphisms \([9, \text{p.10}]\):

\[
K_*(C^*(Y)) \simeq K_*(C^*(Y \subset X)) ; \quad K_*(D^*(Y)) \simeq K_*(D^*(Y \subset X)) ;
\]

\[
K_* (Y) \simeq K_* (Y \subset X)
\]

Now we are going to state and prove the relative version of the Mayer-Vietoris exact sequence. For this purpose let \( N \subset X \) be a closed subset that partitions \( X \) into closed subset \( X^+ \) and \( X^- \) such that \( X^+ \cap X^- = N \). We assume this partitioning be excisive, i.e. for any \( r > 0 \), there is \( r' > 0 \) such that

\[
N_r(X^+) \cap N_{r'}(X^-) \subset N_{r'}(X^+ \cap X^-)
\]

With this assumption the following Meyer Vietories exact sequence is stated and proved in \([3]\)

\[
K_{j+1}(C^*(X^+)) \oplus K_{j+1}(C^*(X^-)) \rightarrow K_{j+1}(C^*(X)) \xrightarrow{\partial_{mv}} K_{j}(C^*(N)) \rightarrow K_{j}(C^*(X^+)) \oplus K_{j}(C^*(X^-))
\]

Similar Mayer-Vietoris exact sequences for algebra \( D^* \) and for the \( K \)-homology are established in \([9]\) and the naturality and commutativity of these exact sequences with respect to the lower row exact sequence in \((2.3)\) is proved. As a result, the assembly map \( A \) and the Mayer-Vietoris morphism commute with each other.

We need to establish the Mayer-Vietoris exact sequence in the relative context. For this purpose, observe that the intersection \( Z := Y \cap N \) is a closed partitioning subset of \( Y \) making partition \( Y = Y^+ \cup Y^- \). We assume that the partition \( Y = Y^+ \cup Y^- \) is also excisive.

**Proposition 2.1.** Under above assumptions the following exact sequences are available, where the rows are part of long exact sequences and \( A \) stands for assembly map

\[
\begin{array}{ccc}
K_j(Y \subset X) & \xrightarrow{\partial_{mv}} & K_{j-1}(Z \subset N) \\
A & & A \\
K_j(C^*(Y \subset X)) & \xrightarrow{\partial_{mv}} & K_{j-1}(C^*(Z \subset N))
\end{array}
\]

\[
K_{j-1}(Y^+ \subset X^+) \oplus K_{j-1}(Y^- \subset X^-) \quad A \\
K_{j-1}(C^*(Y^+ \subset X^+)) \oplus K_{j-1}(C^*(Y^- \subset X^-))
\]

**proof** Actually this follows from a more general Mayer-Vietoris exact sequence for \( C^* \)-algebras. Consider a triple of \( C^* \)-algebras \((A, I_1, I_2)\), where \( I_1 \) and \( I_2 \) are ideals in \( A \) such that \( A = I_1 + I_2 \). Then the following Mayer-Vietories exact sequence holds and is natural

\[
K_{j+1}(I_1) \oplus K_{j+1}(I_2) \rightarrow K_{j+1}(A) \xrightarrow{\partial_{mv}} K_{j}(I_1 \cap I_2) \rightarrow K_{j}(I_1) \oplus K_{j}(I_2)
\]

At first we notice that the following equalities (and similar equalities for \( D^* \) and \( D^*/C^* \)) hold

\[
C^*(Y \subset X) = C^*(Y^+ \subset X) + C^*(Y^- \subset X)
\]

\[
C^*(Y^+ \subset X) \cap C^*(Y^- \subset X) = C^*((Z \subset N) \subset (Y \subset X))
\]

In view of these equalities we can apply \((2.7)\) and this leads to the desired commutative diagram. \( \square \)

Following \([8]\), a metric space \( Y \) is flasque if it has an isometry \( \phi \) which is homotopic to identity such that \( \phi^k(Y) \) leaves any compact subset of \( Y \) for sufficiently large \( k \). If \( Y \) is flasque then \( K_*(C^*(Y)) \), \( K_*(D^*(Y)) \) and \( K_*(Y) \) vanish. Therefore by \((2.3)\) and \((2.5)\) we have also

\[
K_*(C^*(Y \subset X)) = K_*(D^*(Y \subset X)) = K_*(Y \subset X) = 0
\]

Using these vanishing results and the proposition \(2.1\) we get the following corollary
Corollary 2.2. With the notation of proposition 2.1 if $Y^+$ and $Y^-$ are flasque, then we get the following commutative diagram where the horizontal arrows are isomorphisms

$$K_j(Y \subset X) \xrightarrow{\delta_{mu}} K_j(Z \subset N)$$

$$A \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad A$$

$$K_j(C^*(Y \subset X)) \xrightarrow{\delta_{mu}} K_j(C^*(Z \subset N))$$

Now we study the functoriality of these (relative) $K$-groups. Let $X$ and $X'$ are smooth riemannian manifolds with hermitian bundles $E$ and $E'$ and let $H = L^2(X, E)$ and $H' = L^2(X', E')$. So we have two proper metric spaces $(X,H)$ and $(X',H')$. A continuous coarse function $f : X \to X'$ induces canonical morphism $f_* : K_*(C^*(X,H)) \to K_*(C^*(X',H'))$, $f_* : K_*(D^*(X,H)) \to K_*(D^*(X',H'))$ and $f_* : K_*(X) \to K_*(X')$. We need to extend these functoriality to relative case, so we describe very briefly the construction of these morphisms. An isometry $V : H \to H'$ covers $f$ topologically if $V^* f V - \phi \circ f$ is a compact operator. It covers $f$ coarsely and have propagation speed less than $\epsilon > 0$ if for each $\xi \in H$ with support in $U \subset X$ one has $\phi V(\xi) = 0$ where $v \in C(X')$ such that $\delta'(\text{supp}(v), f(U)) \geq \epsilon$. Given a continuous and coarse function $V$, one can construct the isometry $V$ that covers $f$ both topologically and coarsely (with arbitrary small propagation speed) as follows. Let $\{U_i\}_i$ be a Borel covering for $X'$ such that the diameter of each $U_i$ is less than $\epsilon/2$ and the intersections $U_i \cap U_j$ have measure zero. There is an isometry $T_i : L^2(f^{-1}(U_i), E) \to L^2(U_i, E')$ that topologically covers $f|_{f^{-1}(U_i)}$ [27 lemma 5.2.4]. Then $V = \oplus_i T_i$ is an isometry that covers $f$ both topologically and coarsely with propagation speed less than $\epsilon$. It turns out that the morphism

$$Ad_V : B(H) \to B(H') : Ad_V(T) = VTV^*$$

restricts to morphisms from $C^*(X)$ to $C^*(X')$, and from $D^*(X)$ to $D^*(X')$. Therefore gives rise to a morphism from $D^*(X)/C^*(X)$ to $D^*(X')/C^*(X')$. The induced morphisms in $K$-theory levels provide the morphisms $f_*$ which are actually independents of the covering isometry used in their definitions.

Now let $Y \subset X$ and $Y' \subset X'$. We call a map $h : (Y, X) \to (Y', X')$ (this means $h(Y) \subset Y'$) to be $Y$-coarse if for each $r > 0$ the restriction of $h$ to $N_r(Y)$ is a coarse map.

Lemma 2.3. A map $h : (Y, X) \to (Y', X')$ which is continuous and $Y$-coarse induces canonical morphisms

$$h_* : K_* (C^*(Y \subset X)) \to K_* (C^*(Y' \subset X'))$$
$$h_* : K_* (D^*(Y \subset X)) \to K_* (D^*(Y' \subset X'))$$
$$h_* : K_* (Y \subset X) \to K_* (Y' \subset X')$$

Proof Let $V$ be the isometry from $H$ to $H'$ constructed in above discussion that covers $f$ topologically and coarsely and has propagation speed less than $\epsilon$. Since $f(Y) \subset Y'$ and $f$ is coarse in any bounded neighborhood of $Y$ it follows that for each $s > 0$ there is $s' > 0$ such that $f(N_s(y)) \subset N_{s'}(Y')$ (this is because $f$ is uniformly expansive in $N_s(Y)$). Let $T \in B(H)$ be supported in a finite distance of $Y$. Because propagation of $V$ is finite $Ad_V(T)$ has support in a finite distance of $27$. Therefore by continuity we conclude that $Ad_V$ is a morphisms from $C^*(Y \subset X)$ to $C^*(Y' \subset X')$, and from $D^*(Y \subset X)$ to $D^*(Y' \subset X')$. Therefore, it gives rise to a morphism from $D^*(Y \subset X)/C^*(Y \subset X)$ to $D^*(Y' \subset X')/C^*(Y' \subset X')$. The induced morphisms in $K$-theory level are the desired morphisms.

Remark Although in this section we have mostly considered the general proper metric spaces, for our purposes in this paper we consider just complete Riemannian manifolds or complete sub manifolds of such manifolds. In this case a reach index theory with values in the $K$-theory of the
coarse $C^*$-algebras was developed that we are going to introduce. Our discussion is as much brief as possible.

3. Some constructions in coarse index theory

In this section we assume the geometric context and notation of the introduction. Namely we assume the given of $(M_i, g_i)$, the subsets $W_i$ and partitioning hypersurface $N_i$ and Dirac type operators $D_i$ that act on smooth sections of Clifford bundles $E_i$. An essential part of our assumption is the given of an isometry $\psi : M_1 \setminus W_1 \to M_2 \setminus W_2$ that lifts to an isometry $\Psi$ of bundles and conjugates the Dirac type operators

$$\Psi^{-1} D_2 \Psi = D_1 \text{ on } C^\infty(M_1 \setminus W_1, E_1)$$

We review very briefly the construction of relative indices following [4]. Let $\chi$ be a normalization function, i.e., a continuous function on $R$ that tend to $\pm 1$ when $x \in \mathbb{R}$ goes to $\pm \infty$. It turns out that $\chi(D_i) \in \mathcal{D}^*(M_i)$ while $\chi^2(D_i) = I \in \mathcal{C}^*(M_i)$. If $\alpha = (1 + \chi)/2$, then $\alpha(D_i)$ is an idempotent in the quotient algebra $\mathcal{D}^*(M_i)/\mathcal{C}^*(M_i)$ and determines an element $[\alpha(D_i)]$ in $K_0(\mathcal{D}^*(M_i)/\mathcal{C}^*(M_i))$. The short exact sequence

$$0 \to \mathcal{C}^*(M_i) \to \mathcal{D}^*(M_i) \to \mathcal{D}^*(M_i)/\mathcal{C}^*(M_i) \to 0$$

gives rise to a $K$-theory long exact sequence

$$\cdots \to K_{j+1}(\mathcal{D}^*(M_i)) \to K_{j+1}(\mathcal{D}^*(M_i)/\mathcal{C}^*(M_i)) = K_j(M_i) \xrightarrow{\delta} K_j(\mathcal{C}^*(M_i)) \to \cdots$$

The coarse index of $D_i$ is given by

$$\text{ind}(D_i) = \delta([\alpha(D_i)]) \in K_1(\mathcal{C}^*(M_i))$$

Since $N_i$ is even-dimensional, the restriction of $E_i$ to $N_i$ (that we denote by the same symbol $E_i$) is graded $E_i = E_i^+ \oplus E_i^-$. Let $\theta : E_i^+ \to E_i^-$ be a Borel bundle isomorphism. Let $\chi$ be an odd normalization function. Then $\beta(D_i) := \theta^{-1} \chi(D_i^+)$ belong to $\mathcal{D}^*(N_i)$, while $\beta^2(D_i^+ - I)$ as an operator on $L^2(N_i, E_i^+)$; belongs to $\mathcal{C}^*(N_i)$. Therefore $\beta(D_i^+)$ determines an element in $K_1(\mathcal{D}^*(N_i)/\mathcal{C}^*(N_i))$ and the index map $\delta$ defines the coarse index

$$\text{ind}(D_i) = \delta([\beta(D_i)]) \in K_0(\mathcal{C}^*(N_i))$$

Because $M_i$ (and $N_i$) are not compact, the indices $\text{ind}(D_i)$ (and $\text{ind}(D_i)$) might be too large. However, using $\psi$, it is possible to extract a finite part of these indices that reflects the difference of the operators. This construction is due to John Roe and we follow his argument as stated in [4] and [5].

We give the definition of $\text{ind}(D_1, D_2)$, the definition of $\text{ind}(D_1, D_2)$ is similar and we will briefly discuss it. We recall from introduction the functions $f_i : M_i \to M$ with $W_i = f_i^{-1}(W)$. Let $f_i$ be continuous and coarse and let $V_i$ be an isometry between $H_i := L^2(M_i, E_i)$ and $H := L^2(M)$ that covers topologically and coarsely the map $f_i$. As we have mentioned earlier, for a given $\epsilon > 0$ we may assume that the propagation speed of $V_i$ is at most $\epsilon/2$. Moreover we assume that $V_i$ maps $L^2(M_i \setminus W_i)$ on $L^2(M \setminus W)$ and here we have $V_2 \Psi = V_1$.

Given $T_i \in B(H_i)$ then $\text{Ad}_{V_i}(T_i) = VT_iV^* \in B(H)$ and $\text{Ad}_{V_i}$ provides $C^*$-isomorphism from $\mathcal{C}^*(M_i, H_i)$ (resp. $\mathcal{C}^*(Y \subset M_i, H_i)$) to $\mathcal{C}^*(M, H_i)$ (resp. $\mathcal{C}^*(Y \subset M, H_i)$). Using this isomorphisms we will consider in below $T_i$ as an element in $\mathcal{C}^*(M)$. This discussion is also true for $\mathcal{D}^*$ and for $K$-homology because $f$ is continuous. With this discussion let $T_1$ and $T_2$ be respectively elements in $\mathcal{C}^*(M_1)$ and $\mathcal{C}^*(M_2)$ which are considered as elements in $\mathcal{C}^*(M)$. We call them conjugate via $\psi$ up to $\mathcal{C}^*(W \subset M)$, and denote it by $T_1 \sim_\psi T_2$, if $\Psi^{-1}T_2\Psi = T_1$ outside a bounded neighborhood
of $W$ or if $(T_1, T_2)$ is the norm limit of such pairs of operators. After this discussion, let consider following $C^*$-algebras

$$
\mathcal{A} = \{(T_1, T_2)|T_1 \sim \psi T_2\} \subset C^*(M) \times C^*(M)
$$

$$
\mathcal{B} = \{(T_1, T_2)|T_1 \sim \psi T_2\} \subset D^*(M) \times D^*(M)
$$

It is clear that $\mathcal{A}$ is an ideal in $\mathcal{B}$; therefore, one has the long $K$-theory exact sequence

$$
\cdots \to K_0(\mathcal{B}) \to K_0(\mathcal{B}/\mathcal{A}) \xrightarrow{\delta} K_1(\mathcal{A}) \to \cdots
$$

By above discussion, $(\alpha(D_1), \alpha(D_2))$ is an element in $\mathcal{B}$ and determines the class $[([\alpha(D_1), \alpha(D_2)])]$ in $K_0(\mathcal{B}/\mathcal{A})$, therefore $\delta([\alpha(D_1), \alpha(D_2)])$ belongs to $K_1(\mathcal{A})$. Consider the following unitary map $U$ which equals $\Psi$ on second summand

$$
U := V_2^{-1}V_1 : L^2(W_1, E_1) \oplus L^2(M_1 - W_1, E_1) \to L^2(W_2, E_2) \oplus L^2(M_2 - W_2, E_2)
$$

where the first morphism is $a \to (a, 0)$ and the second one is $(a, b) \to b$. This sequence splits and the splitting morphism is given by $(U^{-1}bU, b) \xrightarrow{\pi} b$, therefore

$$
K_1(\mathcal{A}) = K_1(C^*(W \subset M)) \oplus K_1(C^*(M))
$$

We denote the projection on the first summand by $p$ (which is equal to $\text{Id} - q \circ \pi_\ast$). The relative index is defined by

$$
\text{ind}(D_1, D_2) = p \circ \delta([\alpha(D_1), \alpha(D_2)]) \in K_1(C^*(W \subset M)) .
$$

**Remark** Similarly one defines the algebras

$$
\mathcal{A} = \{(T_1, T_2)|T_1 \sim \psi T_2\} \subset C^*(N) \times C^*(N)
$$

$$
\mathcal{B} = \{(T_1, T_2)|T_1 \sim \psi T_2\} \subset D^*(N) \times D^*(N)
$$

and the unitary $U$

$$
U : L^2(N_1, E_1) = L^2(Z_1, E_1) \oplus L^2(N_1 - Z_1, E_1) \to L^2(N_2, E_2) = L^2(Z_2, E_2) \oplus L^2(N_2 - Z_2, E_2)
$$

One has the direct sum relation $K_0(\mathcal{A}) = K_0(C^*(Z \subset N)) \oplus K_0(C^*(M))$. The relative index $\text{ind}(D_1, D_2)$ is defined by a similar procedure that differ just in parities

$$
\text{ind}(D_1, D_2) = p \circ \delta([\beta(D_1), \beta(D_2)]) \in K_0(C^*(Z \subset N)) \simeq \mathbb{Z} .
$$

We need to define the relative class $[D_1, D_2]$ as an element in the $K$-homology group $K_1(W \subset M)$ and investigate its relation with relative index. For this purpose the first step is to notice that short exact sequences similar to (3.3) holds also for algebras $\mathcal{B}$ and $\mathcal{B}/\mathcal{A}$, i.e. the following sequences are exact and splits

$$
0 \to D^*(W \subset M, H_1) \to \mathcal{B} \xrightarrow{\pi} D^*(M, H_2) \to 0
$$

$$
0 \to D^*(W \subset M, H_1)/C^*(W \subset M, H_1) \to \mathcal{B}/\mathcal{A} \xrightarrow{\pi} D^*(M, H_2)/C^*(M, H_2) \to 0
$$

Each term in exact sequence (3.3) is an ideal in corresponding term in the first exact sequence in above and the inclusion commute with other arrows and splitting morphisms. Therefore we can pass
to long $K$-theory exact sequence to get the following commutative diagram

\[
\begin{array}{ccc}
K_\ast(B/A) & \xrightarrow{\sim} & K_\ast(W \subset M) \\
A & & A \\
K_{\ast+1}(A) & \xrightarrow{\sim} & K_{\ast+1}(C^\ast(W \subset M)) \\
& & A
\end{array}
\]

As we have mentioned earlier $[\alpha(D_1), \alpha(D_2)]$ belongs to $K_0(B/A)$. We call the image of this class in $K_1(W \subset M)$ the relative $K$-homology class and denote it by $[D_1, D_2] \in K_1(W \subset M)$ (compare to [8] p.15). It is clear by this definition and the commutativity of above diagram that

**Lemma 3.1.** Let $A : K_0(W \subset M) \to K_1(C^\ast(W \subset M))$ be the assembly map; then

\[A([D_1, D_2]) = \text{ind}(D_1, D_2)\]

Let $h : M \to \mathbb{R}$ be the signed distance function from $N$. Then 0 is a regular value with $N = h^{-1}(0)$.

**Lemma 3.2.** $h$ is a continuous and $W$-coarse function.

**proof** The fact that $f$ is continuous follows from the fact that $h$ is Lipschitz. Let $N_r(W)$ be a finite distance neighborhood of $W$. For a given $r' > 0$ we have $N_{r'}(N) = h^{-1}([0, r'])$. Because $N$ and $W$ intersects coarsely with intersection $Z$, there is $s > 0$ such that $N_r(W) \cap N_{r'}(N) \subset N_s(Z)$. This last set is compact because $Z$ is compact. Therefore the restriction of $h$ to $N_r(W) \subset M$ is proper. The, along with properness of $h$, prove that $h$ is coarse.

Consider the following isomorphisms where $P : N \to \text{pt.}$ is the constant map to a single point (here the point 0 $\in \mathbb{R}$) and $\partial_{mv}$ is the Mayer-Vietoris isomorphism corresponding to $\mathbb{R} = \mathbb{R}^{\geq 0} \cup \mathbb{R}^{< 0}$

\[
\begin{align*}
K_1(C^\ast(\mathbb{R})) & \xrightarrow{\partial_{mv}} K_0(C^\ast(\text{pt.})) = K_0(K) \cong \mathbb{Z} \\
K_0(C^\ast(Z \subset N)) & \xrightarrow{\partial_{mv}} K_0(C^\ast(\text{pt.})) = K_0(K) \cong \mathbb{Z}
\end{align*}
\]

Using above lemma along with lemma 2.3 $h$ induces a morphism $h_\ast$ from $K_1(C^\ast(W \subset M))$ to $K_1(C^\ast(\mathbb{R}))$. Now we can state the main theorem of this paper

**Theorem 3.3.** The following equality holds when each side is considered as an element in $\mathbb{Z}$ according to above isomorphisms

\[h_\ast(\text{ind}(D_1, D_2)) = \text{ind}(D_1, D_2)\]

Our strategy to prove this theorem is reducing the general case to product case and then proving the product case. In reducing to product case we follow the approach of [8], while to prove the product case we follow the approach of [9].

4. PROOF OF THE MAIN THEOREM

The first step in giving a proof for the main theorem 3.3 is the following proposition.

**Proposition 4.1.** The partitioned-relative index $h_\ast(\text{ind}(D_1, D_2)) \in K_1(\mathbb{R})$ depends only on geometric data in bounded neighborhoods around $N_1 \subset M_1$ and $N_2 \subset M_2$. 

**proof** The proof of this proposition is based on a strong localization property for $K_\ast(\mathbb{R}^p)$ which is formulated and proved in [8] Proposition 3.4: let $T_1$ and $T_2$ be elements in $D^\ast(\mathbb{R}^p)$ which determine classes $[T_1]$ and $[T_2]$ in $K_\ast(D^\ast(\mathbb{R}^p)/C^\ast(\mathbb{R}^p))$. The localization property asserts that these classes are equal if $T_1 = T_2$ in an open subset of $\mathbb{R}^p$. The naturality of the assembly maps and lemma 3.1 imply

\[h_\ast(\text{ind}(D_1, D_2)) = h_\ast \circ A([D_1, D_2]) = A \circ h_\ast([D_1, D_2])\]
Therefore, due to the localization property, it is enough to show that the geometry of \( D_1 \) and \( D_2 \) around \( N_1 \) and \( N_2 \) determines the value of a representative of \( h_*([D_1, D_2]) \in K_1(\mathbb{R}) \) in a neighborhood of \( 0 \in \mathbb{R} \). To do this we fix the compact neighborhood \( K = N_{2\delta}(Z) \) of \( Z \) in \( M \). We recall that 0 is a regular value of \( h \) with \( N = h^{-1}(0) \) and \( N \) consists of regular values for \( f_i \) for \( i = 1, 2 \) with \( N_1 = f_i^{-1}(N) \). So there are constants \( r > 0, \epsilon < a \) (depending on \( r \)) and \( \delta < \epsilon/3 \) (depending on \( \epsilon \)) such that the following relations hold, where \( U_r := h^{-1}(N_r(0)) \):

\[
N_r(U_r \cap K) \subset h^{-1}(N_{2r}(0)) \quad \text{and} \quad N_{1\delta}(f_i^{-1}(U_r \cap K)) \subset f_i^{-1}(N_r(0)).
\]

In the definition of \( h_* \) we choose the covering isometry \( V \) such that \( V^* \) maps \( L^2(N_r(0)) \) into \( L^2(U_r) \). In the construction of \( f_i \) we use isometry \( V_i \) between \( L^2(M_i, E_i) \) and \( L^2(M) \) that covers \( f_i \) with propagation speed less than \( \delta \). Let \( \chi \) be a normalization function whose Fourier transform is supported in \((-\delta, \delta)\). Then \( \chi(D,i) \) and hence \( (\alpha(D_1), \alpha(D_2) \in D^*(M_1, E_1) \times D^*(M_2, E_2) \) has propagation speed less than \( \delta \). With above assumptions, \( \xi \in L^2(N_r(0)) \) the values of \( \alpha(D_1)(V_i^*(V^*(\xi)) \); and hence the value of \( p(\alpha(D_1)(V_i(V(\xi))), \alpha(D_2)(V_i(V(\xi)))) \) which is \( h_*([D_1, D_2])(\xi) \) depends only on the geometric data in the neighborhood \((h \circ f_i)^{-1}(N_{2\epsilon}(0)) \). This neighborhoods will be arbitrary small by making \( r \) as small as necessary.

It is clear that by changing the geometric data in a bounded neighborhood of \( N_1 \) and \( N \) the coarse classes of \( f_i \)'s and of \( h \) do not change and \( D_i \) change continuously. So we assume that the geometric data around partitioning hypersurfaces \( N_i \)'s and \( N \) take product forms. Therefore in view of above proposition we can assume that the whole geometric data are of product form coming from partitioning hypersurfaces. More precisely Let \((N, g) \) and \((N_i, g_i) \) for \( i = 1, 2 \) be complete Riemannian manifolds with compact subsets \( Z \subset N \) and \( Z_i \subset N_i \) and smooth maps \( f_i: (N_i, Z_i) \rightarrow (N, Z) \). Moreover let \( E_i \rightarrow M_i \) be Clifford bundles with corresponding Dirac operators \( D_i \). We assume an isometry \( \psi: N_1 \setminus Z_1 \rightarrow N_2 \setminus Z_2 \) that is covered by a bundle isometry \( \Psi \) such that on smooth sections of \( E_1 \) which are supported in \( M_1 \setminus Z_1 \) one has \( \Phi^{-1}D_2 \Phi = D_1 \). Moreover we assume that \( f_2 \circ \phi = f_1 \). Given all these data the relative index \( \text{ind}(D_1, D_2) \) is an element in \( K_0(C^*(Z \subset N)) \) isomorphic to \( Z \).

Now let \( M = N \times \mathbb{R} \) with product metric \( g + (dx)^2 \) and similarly for \( M_i := N_i \times \mathbb{R} \) and put \( W := Z \times \mathbb{R} \) and \( W_i := Z_i \times \mathbb{R} \). All geometric structures, including metrics, Clifford bundles, functions \( f_i \) can be extending to the product spaces \( M_i \). For example \( \psi \times \text{Id}: M_1 \times W_1 \rightarrow M_2 \times W_2 \) is an isometry which is covered by \( \Psi \). Let \( D_i \) denote the Dirac type operator acting on \( C^\infty(M_i, E_i) \). Therefore we have again \( \Psi^{-1}D_2 \Psi = D_1 \) when these operators acts on \( C^\infty(M_1, W_1) \). Given these data, the relative coarse index \( \text{ind}(D_1, D_2) \) is an element in \( K_1(C^*(W \subset M)) \). The following proposition is a main step toward the complete proof for the main theorem.

**Proposition 4.2.** In above product situation the following relation holds in \( K_0(Z \subset N) \)

\[
\partial_m(\text{ind}(D_1, D_2)) = \text{ind}(D_1, D_2)
\]

**Proof** The subsets \( \mathbb{R}^\pm \times N \) and \( \mathbb{R}^\pm \times \mathbb{R}^\pm \) provide respectively excisive partitioning for \( M_i = \mathbb{R} \times N_i \) and \( M = \mathbb{R} \times N \). These subsets are flasque, so we are in the situation of corollary 2.2. Therefore it is enough to show the following similar result in \( K \)-homology \( K_0(Z \subset N) \)

\[
\partial_m([D_1, D_2]) = [D_1, D_2]
\]

The decomposition \( \mathbb{R} = \mathbb{R}^+ \cup \mathbb{R}^- \) provides flasque excisive decomposition for the product manifold \( M = \mathbb{R} \times N \). This decomposition provides Mayer-Vietoris exact sequence for \( A, \mathcal{B}, \mathcal{B}/A \) and for
\[ \mathbb{R} \times N \text{ and provides following commutative diagram with exact vertical isomorphisms} \]

\[
\begin{array}{ccc}
K_0(B/A) & \xrightarrow{j_*} & K_1(\mathbb{R} \times N) \\
\partial_{mv} & & \partial_{mv} \\
K_1(B/A) & \xrightarrow{j_*} & K_0(N)
\end{array}
\]

(4.1)

Using this commutative diagram it is enough to prove the relation

\[
\partial_{mv}([\alpha(D_1), \alpha(D_2)]) = [\beta(D_1), \beta(D_2)] \in K_1(B/A)
\]

(4.2)

We recall the definitions of algebras \( A \) and \( B \) just before (3.1) and the definitions of algebras \( A \) and \( B \) in the remark after (5.4). By ignoring the equality on \( W \), we have morphisms \( j \) from \( B/A \) and \( B/A \) respectively into \((D^*(M)/C^*(M)) \times (D^*(M)/C^*(M))\) and \((D^*(N)/C^*(N)) \times (D^*(N)/C^*(N))\) and similarly for other algebras. Therefore by naturality of Mayer-Vietoris exact sequence we get the following commutative diagram

\[
\begin{array}{ccc}
K_0(B/A) & \xrightarrow{j_*} & K_1(\mathbb{R} \times N) \\
\partial_{mv} & & \partial_{mv} \\
K_1(B/A) & \xrightarrow{j_*} & K_0(N)
\end{array}
\]

(4.3)

It is clear that in top row we have \( j_*([\alpha(D_1), \alpha(D_2)]) = [D_1] \oplus [D_2] \) and in lower row we have \( j_*([\beta(D_1), \beta(D_2)]) = [D_1] \oplus [D_2] \). Moreover it is proved in [9, Lemma 4.6] that \( \partial_{mv}([D_1]) = [D_1] \) (this is confirmation of "boundary of Dirac is Dirac"). Therefore this argument and the naturality of Mayer-Vietoris morphisms imply the desired relation (4.2).

Now we have every things to give a complete proof for the main theorem.

**Proof of the main theorem** [3.3]: In proposition [4.1] we showed that \( h_* \text{(ind}(D_1, D_2)) \) does not change if we replace data with corresponding cylindrical form discussed just before the proposition [4.2]. Of course this is the case for \( \text{ind}(D_1, D_2) \). Therefore it is enough to prove the theorem for the product case. The following commutative diagram follows from the naturality of the Mayer-Vietoris morphisms where \( P_* \) is defined in (3.7)

\[
\begin{array}{ccc}
K_1(\mathbb{R} \times Z \subset \mathbb{R} \times N) & \xrightarrow{h_*} & K_1(C^*(\mathbb{R})) \\
\partial_{mv} & & \partial_{mv} \\
K_0(Z \subset N) & \xrightarrow{P_*} & K_0(C^*(pt.))
\end{array}
\]

Using this diagram and the proposition [4.2] we get

\[
\partial_{mv} \circ h_* \text{(ind}(D_1, D_2)) = P_* \circ \partial_{mv} \text{(ind}(D_1, D_2)) = P_*(\text{ind}(D_1, D_2))
\]

This is precisely the equality (3.8) up to isomorphisms (3.6) and (3.7) and completes the proof of the theorem.

**Remark** So far we have considered manifolds which are partitioned by one hyper-surface (here \( N_i \) and \( N \)). However every things we have proved generalize readily to a more general situation, where there are several partitioning hypersurfaces. More precisely for \( i = 1, 2 \) let \( (M_i, g_i) \), \( (M, g) \), \( W_i \), \( f_i, \psi, \Psi, E_i \), and \( D_i \) be as we have stated in the introduction. In particular we assume that \( n \) is an odd integer and \( q \) in (lower) is even, although these conditions can be relaxed. Therefore as before we can define the relative index \( \text{ind}(D_1, D_2) \) in \( K_{\alpha}(W \subset M) \). In what follows we use the subscript 0 for \( M \) and structures on it. For example \( W_0 \) stands for \( W \) and \( M_0 \) stands for \( M \) and etc. Now let \( N_1^1, \ldots, N_1^q \) (resp. \( N_2^1, \ldots, N_2^q \) and \( N_0^1, \ldots, N_0^q \)) be \( q \) hypersurfaces in \( M_1 \) (resp. in \( M_2 \) and \( M_0 \)) such that \( q \) is odd and for \( i = 0, 1, 2 \):
(1) \(N^1, \ldots, N^q\) intersect each other transversally and coarsely, so \(N_i = \cap_j N^j_i\) is a submanifold of \(M_i\) with codimension \(q\).

(2) \(N^1_i, \ldots, N^q_i\) intersect \(W_i\) transversally and coarsely and \(Z_i = W_i \cap N_i\) is compact,

(3) for \(i = 1, 2\) and \(j = 1, \ldots, q\) we have \(N^j_i = f_i^{-1}(N^j_0)\) and \(N^j_0\) are regular for \(f_i\).

For \(i = 0, 1, 2\) the sub manifolds \(N_i\) are equipped with the induced Clifford bundles and have their Dirac operators \(D_i\) which are conjugate outside \(Z_i\). Therefore we can define the relative index \(\text{ind}(D_1, D_2)\) as an element in \(K_p(C^*(Z \subset N))\), where \(p = n - q\). With above assumptions the signed distances from \(N^1_0, \ldots, N^q_0\) define the map \(h : M \to \mathbb{R}^q\) which is coarse outside \(Z = Z_0\). Therefore it induces a map \(h_* : K_* (C^*(W \subset M)) \to K_n(C^*(\mathbb{R}^q)) \simeq \mathbb{Z}\). Under these conditions we have

\[(4.4) \quad h_* (\text{ind}(D_1, D_2)) = \text{ind}(D_1, D_2)\]

This is a generalization of (3.8) and its proof is completely similar.

5. APPLICATION TO POSITIVE SCALAR CURVATURE PROBLEM

In this section we use the theorem 5.1 to prove that \(\tilde{N} \times \mathbb{R}\) cannot have a uniformly positive scalar curvature if \(\tilde{N}\) is an enlargeable manifold. This theorem was stated and proved by means of geometric tools (and the original version of relative index theorem) in \([1]\). It was also reproved in \([10]\) and \([11]\) by using a version of partitioned index theorem with general \(C^*\)-algebra coefficients. To provide yet another proof for this fact we state and prove an expected vanishing theorem concerning the relative index. As \(D_i\) is a Dirac type operator, due to Weitzenbock formula

\[D_i^2 = \nabla^* \nabla + R_i,\]

where \(R\) is the Clifford curvature; a self-adjoint bundle endomorphism of \(E_i\). If \(E_i\) is the spin bundle associated to a spin structure, or a spin bundle twisted by a flat bundle then \(R_i = \kappa_i / 4\), where \(\kappa_i\) is the scalar curvature of the underlying Riemannian manifold \((M_i, g_i)\).

**Theorem 5.1.** If for \(i = 1, 2\) the Clifford curvature \(R_i\) are uniformly positive on \(M_i\) then

\[\text{ind}(D_1, D_2) = 0 \in K_1(C^*(W \subset M)).\]

**Proof** The assumption on the Clifford curvature along with the Weitzenbock formula implies that there is a gap around 0 in the spectrum of \(D_i\). Therefore the normalizing function \(\chi\) in the definition of the relative index (just before relation (3.1)) can be assumed to satisfy \(\chi^2 = 1\). Therefore \((\alpha(D_1), \alpha(D_2))\) itself is a projection in \(B\). Therefore the class \([-([\alpha(D_1), \alpha(D_2)])]\) in \(K_0(B/A)\) is the image of a class in \(K_0(A)\). This implies the vanishing of \(\delta([\alpha(D_1), \alpha(D_2)])\) in (3.1) and then the vanishing of the relative index.

Following Gromov-Lawson \([1]\), let \(\tilde{N}\) be a closed oriented manifold of dimension \(n\) with a fixed riemannian metric \(\tilde{g}\). The manifold \(\tilde{N}\) is enlargeable if for each real number \(\epsilon > 0\) there is a riemannian spin cover \((N, g)\), with lifted metric, and a smooth map \(f : N \to S^n\) such that: the function \(f\) is constant outside a compact subset \(Z\) of \(N\); the degree of \(f\) is non-zero; and the map \(f : (N, g) \to (S^n, g_0)\) is \(\epsilon\)-contracting, where \(g_0\) is the standard metric on \(S^n\). Being \(\epsilon\)-contracting means that \(\|T_x f\| \leq \epsilon\) for each \(x \in N\), where \(T_x f : T_x N \to T_{f(x)} S^n\).

With above notation, it turns out that there is a Hermitian bundle \(E \to N\) which is isomorphic to the trivial bundle \(F := \mathbb{C}^h \times N\) outside the compact subset \(Z \subset N\) such that \(\omega := \text{Ch}(E - \mathbb{C}^h)\) is a non-zero non-negative multiple of the volume element of \(N\) which vanishes outside \(Z\). Moreover the bundle \(E\) has a connection whose curvature \(R\) is bounded from above by \(\epsilon\), i.e. \(\|R\| \leq \epsilon\).

**Theorem 5.2.** For an enlargeable closed manifold \(\tilde{N}\), the product space \(M = \tilde{N} \times \mathbb{R}\) does not admit a Riemannian metric with uniformly positive scalar curvature.
RELATIVE-PARTITIONED INDEX THEOREM

**proof** Let $D_1$ and $D_2$ stand for Dirac operators of $N$ twisted; respectively; by $E$ and $F$. In this situation the relative index $\text{ind}_r(D_2, D_1) \in K_0(Z) \simeq \mathbb{Z}$ is given by the following relation [5 formula 4.5]

\[(5.1) \quad \text{ind}_r(D^E, D^F) = \int_{\tilde{N}} A(TM) \wedge \text{Ch}(E - F) = \int_{\tilde{N}} \omega \neq 0\]

Now consider the manifold $M = N \times \mathbb{R}$ and put $W := Z \times \mathbb{R}$. Because $Z$ is compact, with respect to any lifted (from $\tilde{N} \times \mathbb{R}$) riemannian metric on $M$ the intersection of $N$ and $Z \times \mathbb{R}$ is transverse and coarse. Bundles $E$ and $F$ will be considered as bundles over $M$. Then the data $(M, W, D_1)$ and $(M, W, D_2)$ satisfy the relative index theorem conditions with $f_1 = f_2 = \text{Id}$ and $h : N \times \mathbb{R} \rightarrow \mathbb{R}$ being the projection on the second factor. The theorem [5.3] reads

\[h_*(\text{ind}(D_1, D_2)) = \text{ind}(D_1, D_2) \neq 0\]

However if the scalar curvature of a metric on $\mathbb{R} \times \tilde{N}$ is everywhere positive, then it is so for lifted metric on $M = N \times \mathbb{R}$. The curvature of $E \rightarrow M$ is bonded by $\epsilon$. Therefore if $\epsilon$ is sufficiently small, the Liechnorowicz formula and lemma (5.1) together provide the vanishing result $\text{ind}(D_1, D_2) = 0$. This is in contradiction with (5.1) and implies the non-existence of such a metric.

**Remark** Using the equality (1.4), we can prove the following generalization of the above theorem by a similar proof. Let $\tilde{N}$ be an enlargeable manifold and consider $M = \tilde{N} \times \mathbb{R}^q$. Let $\tilde{g}$ be a riemanninmetric on $\tilde{N}$ and $g$ be a riemannian metric on $M$ which is in the same coarse class that a product metric $\tilde{g} + dx_1^2 + \cdots + dx_q^2$ (these later metrics all fall into the same coarse class whatever $\tilde{g}$ might be). Then the $q$ hyper-surfaces $N_i = N \times \mathbb{R}$ intersects coarsely and we can; as in above; apply (1.4) to conclude that the scalar curvature of $g$ can not be uniformly positive. Despite this fact, it is proved in [12] that $M$ does admit a metric with uniformly positive scalar curvature, provided that $q \geq 3$. Consequently this metric is not in the same coarse class that a product metric $\tilde{g} + dx_1^2 + \cdots + dx_q^2$.

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