Bubble formation in $\varphi^4$ theory in the thin-wall limit and beyond

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Abstract

Scalar field theory with an asymmetric potential is studied at zero temperature and high-temperature for $\varphi^4$ theory with both $\varphi$ and $\varphi^3$ symmetry breaking. The equations of motion are solved numerically to obtain $O(4)$ symmetric and $O(3)$ cylindrical symmetric bounce solutions. These solutions control the rates for tunneling from the false vacuum to the true vacuum by bubble formation. The range of validity of the thin-wall approximation (TWA) is investigated. An analytical solution for the bounce is presented, which reproduces the action in the thin-wall as well as the thick-wall limits.

I. INTRODUCTION

The problem of decay of a metastable state via quantum tunneling has important applications in many branches of physics, from condensed matter to particle physics and cosmology. The tunneling is not a perturbative effect. In the semi-classical approximation, the decay rate per unit volume is given by an expression of the form

$$\Gamma = A e^{-S_E},$$

where $S_E$ is the Euclidean action for the bounce: the classical solution of the equation of motion with appropriate boundary conditions. The bounce has turning points at the configurations at which the system enters and exits the potential barrier, and analytic continuation to Lorentzian time at the exit point gives us the configuration of the system at that point and its subsequent evolution. The solution of the equation of motion looks like a bubble in four dimensional Euclidean space with radius $R$ and thickness proportional to the coefficient of the symmetry breaking term in the potential. When there are more than one solution satisfying the boundary conditions, the one with the lowest $S_E$ dominates Eq. (1). The prefactor $A$ comes from Gaussian functional integration over small fluctuations around the bounce. The zero-temperature formalism is well-developed.

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been proved rigorously that the least action is given by the bounce which is O(4) invariant [3].

Linde [4] extended the formalism to finite temperatures. He suggested that at temperatures much smaller than the inverse radius of the bubble at zero-temperature, the bounces are periodic in the Euclidean time \( \tau \) direction and widely separated. Beyond this temperature they start merging into one another producing what is known as “wiggly cylinder” solutions. As one keeps increasing the temperature these wiggles smoothly straighten out, and the solution goes into an O(3) invariant cylinder (independent of Euclidean time \( \tau \)) solution that dominates the thermal activation regime.

The model field theory considered by most authors is \( \varphi^4 \) theory with a symmetry-breaking term proportional to \( \varphi \). However, it was shown [5] that the leading temperature-dependent correction to the tree-level scalar potential is proportional to \( \varphi^3 \). Thus it becomes of interest to obtain bounce solutions to \( \varphi^4 \) theory with \( \varphi^3 \) symmetry-breaking. An interesting special case is the so called thin-wall approximation (TWA), when the bubble radius \( R \) is much larger than the thickness of the bubble wall. In this limit, there is an analytical formula for \( S_E \) in terms of the wall surface energy, and the details of the field theory are unimportant. However, it would be nice to also have an analytical interpolating form for the solution itself. In the TWA limit, we expect the bounce solutions in the two theories to be same, but for sufficiently large coupling constants for the asymmetric terms we expect differences. Also, it is not clear a priori what the limit of validity of the TWA is.

Some of the above questions have been addressed by Adams [6]. By suitably scaling and shifting the field \( \varphi \) as well as the potential \( U(\varphi) \), he puts a general quartic potential in the form

\[
U(\varphi) = \frac{1}{4} \varphi^4 - \varphi^3 + \frac{1}{2} \delta \varphi^2,
\]

where \( 0 \leq \delta \leq 2 \). He then computes the \( O(4) \) and \( O(3) \) bounce action for various values of \( \delta \). The thin-wall limit corresponds to \( \delta = 2 \), and the TWA result is recovered in that case. This throws up the fresh question: how does the difference between \( \varphi \) and \( \varphi^3 \) symmetry-breaking get reflected in the values of \( \delta \) ? What is a thin wall in each of the two cases?

In this paper we address the above issues. We obtain accurate numerical solutions for the zero-temperature and high-temperature bounces for \( \varphi^4 \) theory with both \( \varphi \) and \( \varphi^3 \) symmetry-breaking. We compute the actions in each case, and find that, for a modest value of the asymmetric coupling \( f (= 0.25) \), the action given by TWA formula agrees to within 9% with that obtained from the numerical solution for \( \varphi^3 \) breaking. We also relate these values to the \( \delta \) parameter of Adams [6]. However, the agreement is considerably better for \( \varphi^3 \) breaking than \( \varphi \) breaking. In the former theory, we verify that as \( f \) is reduced the error in the TWA formula goes to zero. We propose another criterion for the goodness of TWA, in terms of the temperature \( T_\star \) at which the actions of the \( O(4) \) and \( O(3) \) solutions become equal. A numerical investigation shows that the TWA holds up to \( f \sim 0.30 \). Finally, we present an analytical solution which satisfies the equation of motion with parameters fixed by demanding stationary action. This reproduces TWA results very well and, in the thick-wall limit, is in fairly good agreement with the numerical results.

In Sec. II we review bubble formation theory at zero and finite temperature, and discuss the thin-wall approximation (TWA). In Sec. III we present our numerical results for \( O(4) \) and \( O(3) \) cases for \( \varphi \) and \( \varphi^3 \) type symmetry breaking as well as our investigation of the region of validity of TWA. In Sec. IV the analytical solution is presented in the limits of a thin wall as well as a thick wall. Finally, Sec. V contains some concluding remarks.
II. BUBBLE FORMATION

Let us consider a scalar field theory with a Lagrangian density

\[ L(\varphi) = \frac{1}{2}(\partial_{\mu}\varphi)^2 - U(\varphi), \quad (3) \]

where the potential \( U(\varphi) \) has two minima at \( \varphi_- \) (false vacuum) and \( \varphi_+ \) (true vacuum).

In the semi-classical approximation the barrier tunneling leads to the appearance of bubbles of a new phase with \( \varphi = \varphi_+ \) as classical solutions in Euclidean space (i.e., imaginary time \( \tau \)). To calculate the probability of such a process in quantum field theory at zero temperature, one should first solve the Euclidean equation of motion:

\[ \partial_{\mu}\partial_{\mu}\varphi = \frac{dU(\varphi)}{d\varphi}, \quad (4) \]

with the boundary condition \( \varphi \to \varphi_- \) as \( \vec{x}^2 + \tau^2 \to \infty \), where \( \tau \) is the imaginary time. The probability of tunneling per unit time per unit volume is given by

\[ \Gamma = A \ e^{-S_{E}[\varphi]}, \quad (5) \]

where \( S_{E}[\varphi] \) is the Euclidean action corresponding to the solution of Eq. (4) and given by the following expression:

\[ S_{E}[\varphi] = \int d^4x \left[ \frac{1}{2}(\partial_{\tau}\varphi)^2 + \frac{1}{2}(|\nabla \varphi|^2 + U(\varphi)) \right]. \quad (6) \]

It is sufficient to restrict ourselves to the O(4) symmetric solution \( \varphi(\vec{x}^2 + \tau^2) \), since it is this solution that provides the minimum of the action \( S_{E}[\varphi] \) \[3\]. In this case Eq. (4) takes the simpler form

\[ \frac{d^2\varphi}{d\rho^2} + \frac{3}{\rho} \frac{d\varphi}{d\rho} = \frac{dU(\varphi)}{d\varphi}, \quad (7) \]

where \( \rho = \sqrt{\vec{x}^2 + \tau^2} \), with boundary conditions

\[ \varphi \to \varphi_- \text{ as } \rho \to \infty, \quad \frac{d\varphi}{d\rho} = 0 \text{ at } \rho = 0. \quad (8) \]

We denote the action of this solution by \( S_4 \).

Now let us consider the finite temperature case. Following [4], in order to extend the above mentioned results to high temperature, \( T \neq 0 \), it is sufficient to remember that quantum statistics of bosons (fermions) at \( T \neq 0 \) is equivalent to quantum field theory in the Euclidean space-time, periodic (anti-periodic) in the “time” direction with period \( \beta = T^{-1} \). One should use the \( T \)-dependent effective potential \( U(\varphi, T) \) instead of the zero-temperature one \( U(\varphi) = U(\varphi, 0) \). Instead of looking for O(4)-symmetric solution of Eq. (4), one should look for O(3)-symmetric (with respect to spatial coordinates) solutions, periodic in the “time” direction with period \( \beta = T^{-1} \). At sufficiently large temperature compared to the inverse of the bubble radius \( R \) at \( T = 0 \), the solution is a cylinder whose spatial cross section is the O(3)-symmetric bubble of new radius \( R(T) \). In this case, in the calculation of the action \( S_{E}(\varphi) \) the integration over \( \tau \) is reduced simply to multiplication by \( T^{-1} \), i.e., \( S_{E}[\varphi] = T^{-1} S_3[\varphi] \). Here
$S_E[\varphi]$ is the four-dimensional action and $S_3[\varphi]$ is the three-dimensional action corresponding to the O(3)-symmetric bubble and given by:

$$S_3[\varphi] = \int d^3r \left[ \frac{1}{2} (\nabla \varphi)^2 + U(\varphi, T) \right].$$

(9)

To calculate $S_3(\varphi)$ it is necessary to solve the equation

$$\frac{d^2 \varphi}{dr^2} + \frac{2}{r} \frac{d \varphi}{d r} = \frac{d U(\varphi, T)}{d \varphi}$$

(10)

with boundary conditions

$$\varphi \to \varphi_- \text{ as } r \to \infty, \quad \frac{d \varphi}{d r} = 0 \text{ at } r = 0.$$  

(11)

where $r = \sqrt{\vec{x}^2}$. The complete expression for the probability of tunneling per unit time per unit volume in the high-temperature limit ($T >> R^{-1}$) is obtained in analogy to the one used in [2] and is given by:

$$\Gamma(T) = A(T) e^{-S_3[\varphi,T]/T}.$$  

(12)

In the theory of bubble formation, the interesting quantity to calculate is the probability of decay between $\varphi = \varphi_-$ and $\varphi = \varphi_+$ which are the two minima of $U(\varphi)$. There is an interesting case (in the sense that the action can be calculated analytically) when $U(\varphi_+) - U(\varphi_-) = \varepsilon$ is much smaller than the height of the barrier. This is known as the thin-wall approximation (TWA). At $T = 0$, in the TWA limit, the action $S_4$ of the O(4)-symmetric bubble is equal to

$$S_4 = 2\pi^2 \int_0^\infty d\rho \, \rho^3 \left[ \frac{1}{2} \left( \frac{d \varphi}{d \rho} \right)^2 + U(\varphi) \right]$$

$$= -\frac{1}{2} \varepsilon \pi^2 R^4 + 2\pi^2 R^3 S_1.$$  

(13)

Here $S_1$ is the bubble wall surface energy (surface tension), given by

$$S_1 = \int_0^\infty d\rho \left[ \left( \frac{d \varphi}{d \rho} \right)^2 + U(\varphi) \right],$$  

(14)

and the integral should be calculated in the limit $\varepsilon \to 0$.

The bubble radius $R$ is calculated by minimizing $S_4$ with respect to $R$ and this gives us

$$R = \frac{3S_1}{\varepsilon},$$  

(15)

whence it follows that

$$S_4 = \frac{27\pi^2 S_1^4}{2\varepsilon^3}.$$  

(16)

The condition for the applicability of TWA is the following

$$\frac{3S_1}{\varepsilon} >> (d^2 U(\varphi_-)/d\varphi^2)^{-1/2},$$  

(17)

where $(d^2 U(\varphi_-)/d\varphi^2)^{-1/2}$ is simply the order of magnitude of the bubble wall thickness. The results presented above were obtained by Coleman [2].
These results can be easily extended to the case $T \gg R^{-1}$ [4]. To this end it is sufficient to take into account that

$$S_3 = 4\pi \int_0^\infty dr r^2 \left[ \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + U(\phi, T) \right]$$

$$= -\frac{4}{3} \varepsilon \pi T^3 + 4\pi T^2 S_1(T),$$

(18)

where $S_1(T)$ is the bubble wall surface energy (surface tension) at finite temperature and is given by:

$$S_1(T) = \int_0^\infty dr \left[ \left( \frac{d\phi}{dr} \right)^2 + U(\phi, T) \right].$$

(19)

As before, the integral should be calculated in the limit $\varepsilon \to 0$.

The bubble radius $R(T)$ is calculated by minimizing $S_3$ with respect to $R(T)$ and this gives us

$$R(T) = \frac{2S_1(T)}{\varepsilon},$$

(20)

whence it follows that

$$S_3 = \frac{16\pi S_1^3(T)}{3\varepsilon^2}.$$  

(21)

The condition for the applicability of TWA is the following

$$\frac{2S_1(T)}{\varepsilon} \gg (d^2 U(\varphi_-, T)/d\phi^2)^{-1/2},$$

(22)

where $(d^2 U(\varphi_-, T)/d\phi^2)^{-1/2}$ is the order of magnitude of the bubble wall thickness at high temperature.

### III. NUMERICAL RESULTS

We start with the Euclidean action at $T = 0$,

$$S_E[\varphi] = \int d^4x \left[ \frac{1}{2} \left( \frac{\partial \varphi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \varphi)^2 + U(\varphi) \right].$$

(23)

If we have O(4) symmetry, Eq. (23) reduces to

$$S_4 = 2\pi^2 \int_0^\infty d\rho \rho^3 \left[ \frac{1}{2} \left( \frac{d\phi}{d\rho} \right)^2 + U(\varphi) \right].$$

(24)

We compute the action for different values of the parameter in the symmetry-breaking term in the potential $U(\varphi)$.

First we consider the following potential :

$$U(\varphi) = \frac{\lambda}{2} (\varphi^2 - \mu^2)^2 - F \varphi,$$

(25)

where $\lambda$ is the coupling constant and $\mu$ is the mass of the scalar field. We perform the following rescaling in the action (24)

$$\varphi \to \varphi/\mu, \quad \rho \to \rho \sqrt{\lambda \mu^2}, \quad f = \frac{F}{\lambda \mu^3},$$

(26)
to get
\[ S_4 = \frac{2\pi^2}{\lambda} \int d\rho \rho^3 \left[ \frac{1}{2} \left( \frac{d\varphi}{d\rho} \right)^2 + \frac{1}{2} (\varphi^2 - 1)^2 - f\varphi \right]. \] (27)

The only adjustable parameter in the Lagrangian is \( f \), so by covering the whole range we should be covering all relevant cases.

The equation of motion is now
\[ \frac{d^2\varphi}{d\rho^2} + \frac{3}{\rho} \frac{d\varphi}{d\rho} = 2(\varphi^3 - \varphi) - f, \] (28)

and the boundary conditions are the usual ones:
\[ \varphi \to \varphi_- \text{ as } \rho \to \infty, \quad \frac{d\varphi}{d\rho} = 0 \text{ at } \rho = 0. \] (29)

By solving Eq. (28) numerically for different values of \( f \), substituting the solution in Eq. (27) and integrating, we obtain the action for each value of \( f \).

At high temperature, we look for the \( O(3) \) symmetric solution with cylindrical symmetry. Then Eq. (9) takes the form
\[ S_3 = 4\pi \int_0^\infty dr \ r^2 \left[ \frac{1}{2} \left( \frac{d\varphi}{dr} \right)^2 + U(\varphi) \right]. \] (30)

Using the same potential (25) and the rescaling (26), we get
\[ S_3 = \frac{4\pi \mu}{\sqrt{\lambda}} \int_0^\infty dr \ r^2 \left[ \frac{1}{2} \left( \frac{d\varphi}{dr} \right)^2 + \frac{1}{2} (\varphi^2 - 1)^2 - f\varphi \right]. \] (31)

The equation of motion is then
\[ \frac{d^2\varphi}{dr^2} + \frac{2}{r} \frac{d\varphi}{dr} = 2(\varphi^3 - \varphi) - f, \] (32)

and the boundary conditions are the usual ones:
\[ \varphi \to \varphi_- \text{ as } r \to \infty, \quad \frac{d\varphi}{dr} = 0 \text{ at } r = 0. \] (33)

Again, we solve Eq. (32) numerically for different values of \( f \), substitute the solution in Eq. (31) and integrate to obtain the action for each \( f \).

Table I shows our numerical results at \( T = 0 \) and high temperature for different values of the asymmetry parameter \( f \).

| \( f \) | \( S_4 \) (Numerical) | \( S_3 \) (Numerical) | \( S_4 \) (Analytical) | \( S_3 \) (Analytical) |
|---|---|---|---|---|
| 0.25 | 3064.70 | 143.80 | 3368.80 | 158.90 |
| 0.55 | 171.30 | 18.70 | — | — |
| 0.75 | 13.20 | 1.60 | — | — |
As we discussed in section II, for small values of $f$ we can use the TWA formula for computing the action. From Eq. (14)

$$S_1 = \int_0^\infty dr \left[ \left( \frac{d\varphi}{dr} \right)^2 + \frac{1}{2}(\varphi^2 - 1)^2 \right]$$

$$= -\int_1^1 d\varphi \sqrt{(\varphi^2 - 1)^2}$$

$$= \frac{4}{3}$$  \hspace{1cm} (34)

The radius is given by

$$R = \frac{3S_1}{\varepsilon},$$  \hspace{1cm} (35)

where $\varepsilon = 2f$ (see [2]). For $f = 0.25$, we have $R = 8$ and the value of the action is (see Eq. (16))

$$S_4 = 3368.80.$$  \hspace{1cm} (36)

Comparing this analytic value with the numerical value for $f = 0.25$, we get an error equal 9%.

At high temperature, again $S_1 = 4/3$. The value of $R(T) = 16/3$ and the action is (see Eq. (21))

$$S_3(T) = 158.90.$$  \hspace{1cm} (37)

Comparing this analytic value with the numerical value for $f = 0.25$, we get an error equal 9%. Thus even for $f$ as small as 0.25 the TWA formula for the action does not give very accurate results. Obviously, there is no point in comparing numerical results obtained for higher values of $f$ with the TWA formula.

Now we repeat the above calculations for the following potential

$$U(\varphi) = \frac{\lambda}{2}(\varphi^2 - \mu^2)^2 - F\varphi^3.$$  \hspace{1cm} (38)

We scale $\varphi$ and $\rho$ as before, with the dimensionless coupling $f$ defined by $f = F/(\lambda\mu)$, solve the equation of motion and calculate the action. Table II shows our numerical results for different values of $f$.

| $f$  | $S_4$ (Numerical) | $S_3$ (Numerical) | $S_4$(Analytical) | $S_3$(Analytical) |
|-----|------------------|------------------|------------------|------------------|
| 0.25| 3446.10          | 161.80           | 3368.80          | 158.90           |
| 0.55| 349.50           | 35.40            | —                | —                |
| 0.75| 139.20           | 19.90            | —                | —                |

We now check the TWA formula for $f = 0.25$. From table II we see that the error in $S_4$ is only 2.3%, while the error in $S_3$ is 1%. Thus we see that for $f = 0.25$, the TWA holds much better for symmetry breaking with a $\varphi^3$ term than with a $\varphi$ term.

Notice that the TWA formula is independent of the symmetry-breaking term in the potential. Moreover, the action for large values of $f$ for the $\varphi^3$ symmetry-breaking term is larger than that for the $\varphi$ one for both the O(4) and the O(3) cases.
To test our numerical method (we have used Hamming’s modified predictor-corrector method for solving the equation of motion), we have calculated the action for small values of the symmetry breaking parameter $f$ in the potential (38) and compared it with the TWA formula. In Fig. (1), we plot the percentage error in the TWA formula as a function of $f$. The dots represent our results while the solid line shows a fit to the data. We see that the error decreases for small $f$, as expected, and approaches zero as $f \to 0$.

Table III shows the relationship between the values of $\delta$ in Eq. (2) and $f$ for $\varphi$ and $\varphi^3$ perturbation.

| $f$   | $\delta$ | $\varphi$ case | $\varphi^3$ case |
|-------|-----------|-----------------|-------------------|
| 0.0   | 2.0       | 2.0             | 2.0               |
| 0.25  | 1.85      | 1.85            | 1.85              |
| 0.55  | 1.50      | 1.62            |                   |
| 0.75  | 0.64      | 1.46            |                   |
| 0.77  | 0.0       | 1.44            |                   |
| 0.80  | —         | 1.41            |                   |
| 1.0   | —         | 1.25            |                   |
| 5.0   | —         | 0.15            |                   |
| 10.0  | —         | 0.03            |                   |

As we can see from the table, as $f$ goes from 0 to 0.77, $\delta$ goes from 2 to 0 for the $\varphi$ case (for $f > 0.77$ the maximum disappears) and from 2 to 1.44 for the $\varphi^3$ case. Also we can see that $\delta$ goes to 0 for the $\varphi^3$ case only at large values of $f$ (here the maximum disappears only asymptotically).

As already mentioned, at zero temperature the O(4) symmetric solution has the lowest value of $S_E$, i.e., $S_E = S_4$. At high temperature, we have $S_E = S_3/T$. At intermediate temperatures other solutions exist. In the TWA, however, it has been shown [7] that all other solutions have higher Euclidean action. This corresponds to a first order phase transition from quantum tunneling at low temperature to thermal hopping at high temperatures. The transition temperature $T_\ast$ is given by equating $S_4$ with $S_3/T$, i.e.,

$$T_\ast = \frac{S_3}{S_4}$$

(39)

If the surface tension $S_1$ is temperature independent, we have

$$S_4 = \frac{27\pi^2 S_4^4}{2\varepsilon^3}$$

(40)

$$S_3 = \frac{16\pi S_3^3}{3\varepsilon^2}$$

(41)

Dividing Eq. (41) by Eq. (40) and putting $\varepsilon = 2f$ (see [2]) we get

$$T_\ast = C \ast f$$

(42)

where

$$C = \frac{64}{81\pi S_1}$$

(43)
Thus we see that, in the TWA, $T_\star$ increases linearly with $f$. We test this by computing $S_3/S_4$ from our numerical solutions at different values of $f$. Fig. 2 shows our results for the potential of Eq. (38). We see that, for $f \leq 0.3$, there is very good agreement with the predicted linear dependence. This also confirms that, in the domain of validity of the TWA, the surface tension $S_1(T)$ is independent of $T$. Beyond $f \sim 0.3$ in our dimensionless units, there is a systematic deviation from linearity. Thus we can say that, for values of $f$ larger than this, the wall thickness becomes important.

IV. ANALYTIC SOLUTION IN TWA LIMIT

We use the following potential (see [6]) to calculate the action analytically in two extreme limits: the thin-wall and thick-wall limits.

$$U(\varphi) = \frac{1}{4}\varphi^4 - \varphi^3 + \frac{1}{2}\delta\varphi^2.$$  \hspace{1cm} (44)

**Thin-wall limit : $\delta \to 2$**

We find that an analytic solution for the bounce of the form of a Fermi function:

$$\varphi = \frac{\gamma e^{(\rho^2 - R^2)/\Lambda^2}}{e^{(\rho^2 - R^2)/\Lambda^2} + 1},$$  \hspace{1cm} (45)

where $\rho = \sqrt{x^2 + \tau^2}$, $R$ is the radius of the bubble and $\Lambda$ its width, acts like a bounce in the TWA and leads to the correct value for the action $S_4$.

Here the false minimum of the potential is at $\varphi = 0$ and the true minimum lies between 2 (for $\delta = 2$) and 3 (for $\delta = 0$). The parameter $\gamma$ is approximately equal to true minimum in the TWA. The bounce has values $\varphi = \gamma$ at $\rho = 0$ and 0 at $\rho \to \infty$. The boundary conditions (8) are satisfied by Eq. (45).

We discuss the zero-temperature action $S_4$. To evaluate $\gamma$, $R$, and $\Lambda$, we substitute the ansatz (45) in Eq. (7):

$$\frac{d^2\varphi}{d\rho^2} + \frac{3d\varphi}{\rho d\rho} = \varphi^3 - 3\varphi^2 + \delta\varphi.$$  \hspace{1cm} (46)

Then the left-hand side (L.H.S.) and the right-hand side (R.H.S.) are respectively

$$L.H.S. = \frac{\gamma 8\rho^2/\Lambda^4}{(e^{(\rho^2 - R^2)/\Lambda^2} + 1)^3} + \frac{\gamma(-12\rho^2/\Lambda^4 + 8/\Lambda^2)}{(e^{(\rho^2 - R^2)/\Lambda^2} + 1)^2} + \frac{\gamma(4\rho^2/\Lambda^4 - 8/\Lambda^2)}{e^{(\rho^2 - R^2)/\Lambda^2} + 1}.$$  \hspace{1cm} (47)

$$R.H.S. = \frac{\gamma \delta}{e^{(\rho^2 - R^2)/\Lambda^2} + 1} - \frac{3\gamma^2}{(e^{(\rho^2 - R^2)/\Lambda^2} + 1)^2} + \frac{\gamma^3}{(e^{(\rho^2 - R^2)/\Lambda^2} + 1)^3}.$$  \hspace{1cm} (48)

In the TWA, the solution is constant except in a narrow region near the wall at $\rho = R$. So, we replace in Eq. (17)

$$8\rho^2/\Lambda^4 \text{ by } \frac{8R^2}{\Lambda^4}(1 - a\Lambda^2/R^2) \text{ in the } \frac{1}{(e^{(\rho^2 - R^2)/\Lambda^2} + 1)^3} \text{ term},$$  \hspace{1cm} (49)
\[ \frac{8}{\Lambda^2} - 12 \rho^2 / \Lambda^4 \text{ by } -\frac{12 R^2}{\Lambda^4} (1 - b \Lambda^2 / R^2) \text{ in the } \frac{1}{(e^{(\rho^2 - R^2)/\Lambda^2} + 1)^2} \text{ term, (50)} \]

\[ 4 \rho^2 / \Lambda^4 - 8 / \Lambda^2 \text{ by } \frac{4 R^2}{\Lambda^4} (1 - d \Lambda^2 / R^2) \text{ in the } \frac{1}{e^{(\rho^2 - R^2)/\Lambda^2} + 1} \text{ term, (51)} \]

where \( a, b \) and \( d \) are parameters to be determined later.

Comparing Eq. (47) with Eq. (48) in the range \( R^2 (1 - \Lambda^2 / R^2) = R^2 - \Lambda^2 < \rho^2 < R^2 + \Lambda^2 = R^2 (1 + \Lambda^2 / R^2) \) where \( \rho^2 \simeq R^2 \) as \( \Lambda^2 / R^2 \ll 1 \), we have:

\[ \gamma^2 = \frac{R^2}{\Lambda^4} (1 - a \Lambda^2 / R^2), \]

\[ \gamma^4 = \frac{R^2}{\Lambda^4} (1 - b \Lambda^2 / R^2), \]

\[ \delta^4 = \frac{R^2}{\Lambda^4} (1 - d \Lambda^2 / R^2), \]

(52)

We can now evaluate the zero-temperature action \( S_4 \):

\[ S_4 = 2\pi^2 \int_0^\infty d\rho \rho^3 \left[ \frac{1}{2} \left( \frac{d\phi}{d\rho} \right)^2 + U(\phi) \right]. \]

(53)

Substituting Eq. (45) in Eq. (53) and integrating we get

\[ S_4 = 2\pi^2 \gamma R^4 \left[ \frac{1}{6 \Lambda^2} (1 + \left( \frac{\pi^2}{3} - 2 \right) \frac{\Lambda^4}{R^4}) + \frac{\delta}{8} (1 - 2 \Lambda^2 / R^2 + \frac{\pi^2 \Lambda^4}{3 R^4}) \right. \]

\[ + \frac{\gamma}{4} (1 - 3 \Lambda^2 / R^2 + \left( \frac{\pi^2}{3} + 1 \right) \Lambda^4 / R^4) \]

\[ + \frac{\gamma^2}{16} (1 - 11 \Lambda^2 / 3 R^2 + \left( \frac{\pi^2}{3} + 2 \right) \Lambda^4 / R^4) \left. \right]. \]

(54)

We now determine the parameters \( a, b, \) and \( d \) by demanding \( dS_4 / dR^2 = dS_4 / d\Lambda^2 = dS_4 / d\gamma = 0 \). Differentiating Eq. (54) and using Eq. (52), we find that to leading order in \( \Lambda^2 / R^2 \),

\[ 4b - 2a - 2d + 1 = 0, \]

\[ 3b - 2a - d = 0, \]

\[ 3b - 11a / 6 - d = 0, \]

(55)

which leads to \( a = 0, b = 1/2 \) and \( d = 3/2 \). Using Eq. (52), we can rewrite Eq. (54) as:

\[ S_4 = 2\pi^2 \gamma R^4 \left[ \frac{8 R^6}{\Lambda^6} \left( 1/3 - d/2 - a/2 + b \right) + \frac{\Lambda^2}{R^2} (d - 3b + 11a/6) \right], \]

(56)

where the coefficient \( \Lambda^4 / R^6 \) evaluated by the usual methods of statistical mechanics for the Fermi function vanishes. This gives

\[ S_4 = \frac{4\pi^2 R^6}{3 \Lambda^6} + O \left( \frac{\Lambda^6}{R^6} \right). \]

(57)
The quantities $\gamma$, $R$ and $\Lambda$ are determined from Eq. (52) using the values of $a$, $b$, and $d$. So we have

$$\gamma^2 - 2\gamma \frac{d-a}{d-b} + 2\delta \frac{b-a}{d-b} = 0,$$

which gives

$$\Lambda^2 = \frac{8(b-a)}{\gamma^2 - 2\gamma} = \frac{4(d-b)}{\gamma - \delta},$$

with $\gamma$ given by Eq. (58). We have then, for $\delta = 1.9$, $\gamma = 2.1$, which implies that $R^2/\Lambda^2 = \frac{\gamma}{\gamma - \delta} + b = 11$. Thus we have

$$S_4 = \frac{4\pi^2}{3}(11)^3,$$

while the action from the TWA formula is (see (3)) $S_{TW} = \frac{4\pi^2}{3}(10)^3$ for $\delta = 1.9$. The departure from TWA, $S_4/S_{TW} = 1.33$, is in agreement with Ref. [6]. The expressions seem certainly valid for values of $\delta$ in the range 2.0 to 1.8.

**Thick-wall limit: $\delta \to 0$**

The form of the bounce in Eq. (45) suggests that the thick wall limit, which would correspond to small values of $R^2/\Lambda^2$, would be obtained by approximating the Fermi function by the Maxwell-Boltzmann function, which leads to a Gaussian:

$$\varphi = \gamma e^{-\rho^2/\Lambda^2}.$$

The action for this form of bounce is found to be

$$S_4 = \pi^2\gamma^2\Lambda^4 \left[ \frac{1}{2\Lambda^2} + \frac{\delta}{8} - \frac{\gamma}{9} + \frac{\gamma^2}{64} \right].$$

Eqs. (52) then reduce to

$$\frac{\gamma^2}{8} = -\frac{a}{\Lambda^2}, \quad \frac{\gamma}{4} = -\frac{b}{\Lambda^2}, \quad \frac{\delta}{4} = -\frac{d}{\Lambda^2}.$$

Note that in this case $\gamma \ll 1$, so $\gamma^2$ is negligible.

The values of $b$ and $d$ are again obtained by demanding $dS_4/d\Lambda^2 = dS_4/d\gamma = 0$. This gives $b = -9/8$, $d = -1/2$, giving

$$\Lambda^2 = \frac{2}{\delta}, \quad \gamma = \frac{9}{4}\delta.$$

This yields the action

$$S_4 = \frac{4\pi^2}{3}(1.9)\delta + O\left(\frac{R^2}{\Lambda^2}\right).$$

The ratio of the action to the TWA value is

$$R_4 = \frac{S_4}{S_{TW}} = 1.9\delta(2 - \delta)^3.$$

For $\delta = 0.1$, $R_4 = 1.31$, which agrees with Adams’ result.
Thus, the form of the bounce given by Eq. (52) seems valid over the whole range of \( \delta \) (from 0 to 2), and in the two extreme limits is amenable to analytic calculations.

The numerical plots of the function \( \varphi \) in the two extreme limits are shown in Figures 3 and 4. In Fig. 3 we compare our numerical result with the analytic one for \( \delta = 1.8 \). From the figure we see that the Fermi function agrees very well with our numerical results. Similarly, Fig. 4 shows the Gaussian function compared with our numerical results for \( \delta = 0.2 \). We find that \( \gamma = 0.55 \) numerically and 0.45 analytically. This discrepancy in the values of \( \gamma \) is expected, because the terms neglected in Eq. (53) are of order \( \left( \frac{R}{\Lambda} \right)^2 \).

The analysis given above can be carried out for the O(3) symmetric bounce and we find that this gives equally good results. We anticipate that this approach will be of great use in discussing the extension to arbitrary temperature. This will be discussed in a future publication.

V. CONCLUSIONS

We have obtained accurate numerical solutions for the zero-temperature and high-temperature bounces for both \( \varphi \) and \( \varphi^3 \) symmetry-breaking. We compute the actions in each case and find that, for a modest value of the asymmetric coupling \( f = 0.25 \), the action given by the TWA formula agrees to within 9% with that obtained from the numerical solution for \( \varphi \) breaking while for \( \varphi^3 \) the agreement is within 2.3%. Hence, the agreement is considerably better for \( \varphi^3 \) breaking than \( \varphi \) breaking. At high temperatures, the conclusion is qualitatively similar.

We have checked our numerical method by comparing the action obtained numerically with the one obtained from the TWA formula. Very good agreement is obtained as we go to small values of \( f \). We also verify that as \( f \) is reduced the error in the TWA formula goes to zero. We propose an alternative criterion for the goodness of TWA, in terms of the relation between \( f \) and the temperature \( T^* \) at which the actions of the O(4) and O(3) solutions become equal. A numerical investigation shows that TWA holds up to \( f \sim 0.30 \). Finally, we present an analytical solution which satisfies the equation of motion in an approximate sense in two limiting cases. The first of these reproduces the leading corrections to the TWA results very well. The second is applicable for the opposite case of a very thick wall. This gives us insights into the nature of the bounce solutions for various values of \( \delta \) going from thin to thick walls.

Our work overlaps to some extent with that of Adams [6]. Like us, he had computed the bounce action for both \( O(4) \) and \( O(3) \) symmetry. Using the parametrization of Eq. (2), he was able to map the entire range of wall thickness into the interval \( 0 \leq \delta \leq 2 \). However, while the form (2) is computationally very convenient, it obscures the fact that the free parameter actually measures the departure from pure \( \varphi^4 \) theory, and does not directly allow a comparison between \( \varphi \) and \( \varphi^3 \) symmetry-breaking. Our parametrization allows such a comparison. Of course, both theories can be mapped onto (2), but the \( \delta - f \) relationship is different, as illustrated in Table III. Moreover, Adams does not attempt to obtain approximate analytical solutions, which is done here for the first time. Also, he does not lay any emphasis on the \( T^* - f \) relationship and indeed does not compute \( T^* \).

Much of the work on inflationary models relies on the zero-temperature potential, so our results could be relevant for inflation [4]. They may also have some bearing on the formation of topological defects in a first order phase transition where authors consider zero-temperature potentials, see for example [8].
So far, we have discussed the action only at zero temperature and high temperature. To obtain the bounce solution at intermediate temperatures, we have to solve a partial differential equation with periodic boundary conditions in the $\tau$ direction. We have done so using a multigrid method as well as by extending the analytic bounce calculations. This work will be presented in a future publication.

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Figure Caption

FIG. 1. Error in the TWA formula as a function of $f$. The squares represent our results while the solid line shows a fit to the data.

FIG. 2. Deviation of $T_*$ from the TWA limit. The straight line represents the TWA limit while the squares are our numerical results.

FIG. 3. $\varphi$ as a function of $\rho$. The solid line is the Fermi function while the dashed line is the numerical result.

FIG. 4. $\varphi$ as a function of $\rho$. The solid line is the Gaussian function while the dashed line is the numerical result.
Figure 1:
Figure 2:
Figure 3:
Figure 4: