Integral transforms of the quantum mechanical path integral: Hit function and path averaged potential.

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We introduce two new integral transforms of the quantum mechanical transition kernel that represent physical information about the path integral. These transforms can be interpreted as probability distributions on particle trajectories measuring respectively the relative contribution to the path integral from paths crossing a given spatial point (the hit function) and the likelihood of values of the line integral of the potential along a path in the ensemble (the path averaged potential).

Keywords: Quantum mechanics, Kernel, Path Integration

INTRODUCTION

In the standard quantum mechanics of a particle in a time-independent potential $V(r)$, a fundamental quantity is the propagator, $K(y, x; T)$, defined as the matrix element of the evolution operator in configuration space:

$$K(y, x; T) = \langle y | e^{-iHT} | x \rangle,$$  \hspace{1cm} (1)

with the Hamiltonian $H = \frac{p^2}{2m} + V(r)$ (we use natural units throughout). The propagator holds the complete information on the time evolution of the system, satisfying the Schrödinger equation, $i\partial_t K(y, x; T) = H(y)K(y, x; T)$ with $H(y) = -\frac{1}{2m}\partial_y^2 + V(y)$, subject to the initial conditions $\lim_{T \to 0} K(y, x; T) = \delta(x - y)$, so that knowing its explicit form is tantamount to solving the system. In a basis of eigenfunctions of the Hamiltonian, the propagator has the spectral representation (henceforth we work in Euclidean space-time so $K(T) = e^{-T H}$)

$$\sum_n \psi_n(y)\psi_n^*(x)e^{-E_n T} + \int dk \psi_k(y)\psi_k^*(x)e^{-E(k) T}$$  \hspace{1cm} (2)

separated into contributions from the bound states and the scattering states of the system.

The kernel for a scalar particle also has a path integral representation

$$K(y, x; T) = \int_{x(0) = x}^{x(T) = y} D x e^{- \int_0^T dt \left[ \frac{m}{2} \dot{x}^2 + V(x(t)) \right]}$$  \hspace{1cm} (3)

with the free path integral normalisation

$$K_0(y, x; T) = \int_{x(0) = x}^{x(T) = y} D x e^{- \int_0^T dt \frac{m}{2} \dot{x}^2} = \left( \frac{m}{2\pi T} \right)^{\frac{D}{2}} e^{-\frac{(y-x)^2}{2mT}}.$$  \hspace{1cm} (4)

The purpose of the present paper is to introduce two novel representations of the propagator: the “hit function” $\mathcal{H}(z|y, x; T)$ and the “path averaged potential” $\overline{P}(v|y, x; T)$. Both have the character of invertible integral transforms of the kernel,

$$K(y, x; T) = \int d^D z \mathcal{H}(z|y, x; T),$$  \hspace{1cm} (5)

$$K(y, x; T) = \int_{-\infty}^{\infty} dv \overline{P}(v|y, x; T)e^{-v}.$$  \hspace{1cm} (6)

Our original motivation for considering these representations lies in their usefulness for the numerical sampling of the path integral [1], and we will report our corresponding results in a separate publication [1]. However, we anticipate that these representations will find much wider application, since they provide specific physical information on the dynamics of the system. The hit function has an analogy in the proper time formalism of quantum field theory (see [2][3]), whilst a relativistic analogy of the path averaged potential has been introduced by Gies et al. [4]; here we comment on the adaptation of these objects for calculations in quantum mechanics and supply the inverse transformations lacking in [2] and [4]. For both functions we also explore their gauge dependency.

In the following we first define $\overline{P}(v)$ and $\mathcal{H}(z)$ and provide the inverse transformations to [5] and [6]. We then discuss their asymptotic form for large $T$ and give their gauge transformations in the case of electromagnetic interactions. Finally we calculate both functions for some simple potentials and show that our results are compatible with numerical samplings.

TRANSFORMS OF THE KERNEL

In theoretical calculations and numerical simulations a direct determination of the kernel can be difficult, so it
may be advantageous to have some intermediate quantity at hand. We present two such quantities in the following sub-sections.

The hit function, $\mathcal{H}(z)$

The first quantity is a function of a spatial point, $z$, that also depends upon the path integral endpoints, measuring the relative contribution to the kernel of paths which pass through the point $z$. This object, $\mathcal{H}(z)$, which we call the “hit function” is defined by

$$
\mathcal{H}(z|y, x; T) \equiv \frac{1}{T} \int_{x(0)=x}^{x(T)=y} Dx \int_0^T \delta^D(z - x(\tau)) \, d\tau \, e^{-S[x]},
$$

(7)

where $S[x] = \int_0^T dt \left( \frac{m^2}{2} + V(x(t)) \right)$ is the (Euclidean) classical action of the trajectory $x(t)$. The $\delta$ function counts the worldlines that cross through, or hit, the spatial point $z$, weighting each path by its action. This function is inspired by a similar quantity called “local time” \cite{[3]} which measures the amount of time a fixed trajectory is found at a given point \cite{[3],[5]}. We have generalised this definition to take into account the weight associated to each trajectory. The path integral \cite{[7]} is also similar to the contact interaction introduced between particle worldlines in \cite{[3]} and strings in \cite{[9]} and the field theory amplitudes developed in \cite{[2]}. For later calculations it is convenient to introduce a properly normalised distribution on the space of trajectories

$$
\mathcal{H}(z|y, x; T) \equiv \frac{\mathcal{H}(z|y, x; T)}{K(y, x; T)},
$$

(8)

which integrates to unity.

Since $\mathcal{H}(z)$ is built only out of particle worldlines that pass through the point $z$ we can relate it to the kernel by forcing this criterion at some arbitrary time $0 < t < T$, leading to an inverse integral transform to \cite{[6]},

$$
\mathcal{H}(z|y, x; T) = \frac{1}{T} \int_0^T dt K(z, x; t) K(y, z; T-t). \tag{9}
$$

Note that the kernels in the integrand count all paths between the end-points, including those that may cross the point $z$ multiple times, in agreement with the $\delta$-function in the definition of the hit function \cite{[7]} – see also \cite{[39]}.

There are many approaches to numerical evaluation of the quantum mechanical path integral, such as Monte-Carlo sampling \cite{[10],[15]} or, as we have employed in \cite{[14]}, the adaptation of worldline numerics \cite{[16],[19]} to the non-relativistic setting. In such cases, where the ultimate goal may be computational estimation of the kernel, knowledge of the hit function is sufficient, for one need simply integrate over positions in \cite{[7]} or \cite{[9]} to verify \cite{[5]}. In this way one could sample the hit function via its path integral representation \cite{[7]} and determine the kernel through finite dimensional (numerical) integration.

The path averaged potential, $\mathcal{P}(v)$

The path integral determination of the kernel amounts to the expectation value of the exponentiated line integral of the potential. This motivates describing the likelihood associated to the values of these line integrals over the space of particle paths with Gaussian distribution on their velocities. Denoting this function by $\mathcal{P}(v)$ where

$$
\mathcal{P}(v|y, x; T) \equiv \frac{\mathcal{H}(y|y, x; T)}{K(y, x; T)}
$$

(10)

Prior use of a similar function has been made in relativistic quantum theory \cite{[4]}, where worldline numerics are used to sample the likelihood (a function of proper time) and various functional fits are made to the numerical results.

Using the Fourier representation of the $\delta$ function we may write the path integral representation of $\mathcal{P}(v)$ in terms of the kernel,

$$
\mathcal{P}(v|y, x; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \, e^{izv} \tilde{K}(y, x; z),
$$

(11)

where $\tilde{K}$ is related to the kernel $K$ by the substitution $V(x) \rightarrow izV(x)$ under the path integral. This follows if one is able to exchange the functional integration for the Fourier integral over $z$ and provides the inverse transform to \cite{[6]}.

As before, we also introduce a normalised probability distribution for the path averaged potential by

$$
\mathcal{P}(v|y, x; T) = \frac{\mathcal{P}(v|y, x; T)}{\tilde{K}(y, x; T)},
$$

(12)

which has unit area. Note that in contrast to the hit function’s distribution, \cite{[5]}, the normalisation for $\mathcal{P}(v)$ is always known analytically since it involves only the free kernel.

It is straightforward to check that \cite{[6]} follows from \cite{[10]} or \cite{[14]}, so numerical estimation of the kernel follows from the expectation of $e^{-v}$ against the likelihood $\mathcal{P}(v)$, again reducing the problem to a finite dimensional integral.

Asymptotic properties of the distribution functions

The spectral representation of the kernel \cite{[2]} implies asymptotic formulae for our new functions under suitable
circumstances. This is useful when studying the large time \( T \to \infty \) asymptotics of these distributions that is relevant when extracting the ground state energy, for example. We will focus on the properties of the normalised distributions defined in (8) and (12).

For \( \mathcal{H}(z) \) it is sufficient to substitute the spectral decomposition directly into (9). Subsequently integrating over the intermediate time \( t \) leads to a double sum

\[
\mathcal{H}(z|y,x;T) = \frac{1}{TK(y,x;T)} \left[ T \sum_n \psi_n(y) \psi_n^*(z) e^{-E_n T} + \sum_{n,m \neq n} \psi_n(y) \psi_n^*(z) \psi_m(z) \psi_m^*(x) \frac{e^{-E_n T} - e^{-E_m T}}{E_m - E_n} \right],
\]

up to contributions of order \( e^{-(E_1 - E_0)T} \). Note that such an approximation may require one to fix the normalisation of the distribution again. The term on the right hand side of the top line is the leading contribution in the large side of the top line is the leading contribution in the asymptotic form

\[
\mathcal{H}(z|y,x;T) \simeq |\psi_0(z)|^2 + \frac{1}{T \psi_0(y) \psi_0^*(x)} \sum_{n>0} \frac{\psi_0(y) \psi_0^*(z) \psi_n(z) \psi_n^*(x) + (n \leftrightarrow 0)}{E_n - E_0}
\]

Electromagnetic interactions

For a particle coupled to a gauge potential, \( A \), we must also consider the inherent gauge freedom. The (Euclidean) action is

\[
S[x,A] = \int_0^T dt \left[ \frac{m \dot{x}^2}{2} + ieA(x(t)) \cdot \dot{x} \right].
\]

If we pick a reference gauge, with potential \( \hat{A} \), then the kernel with respect to this gauge, \( \tilde{K}(y,x;T) \), is

\[
\tilde{K}(y,x;T) = \int_{x(0)=x}^{x(T)=y} \mathcal{D} x e^{-S[x,\hat{A}]}.
\]

Under a gauge transformation \( \hat{A}_\mu(x) \to A_\mu(x) = \hat{A}_\mu(x) + \partial_\mu \Lambda(x) \) the kernel changes covariantly as [20]

\[
K(y,x;T) = e^{-ie\varphi(y,x)} \tilde{K}(y,x;T),
\]

with \( \varphi(y,x) \equiv \int_x^y (A - \hat{A}) \cdot dx = \Lambda(y) - \Lambda(x) \) the exponent of the holonomy of the difference between the gauge potentials. Given this modification we describe the implications for the distributions \( \mathcal{H}(z) \) and \( \mathcal{P}(v) \) defined above.

The hit function distribution, \( \mathcal{H}(z) \), can be defined to be gauge invariant, although since the potential enters the path integral as a phase it now becomes complex valued. We begin by using the formula (11) with the kernels
written in a specific gauge,
\[ \hat{H}(z|y, x; T) = \frac{1}{TK(y, x; T)} \int_0^T dt \dot{K}(z, x; t) \bar{K}(y, z; T-t). \]  
(20)

Gauge transforming the kernels on the right hand side leads to the hit function distribution in a new gauge
\[ \mathcal{H}(z|y, x; T) = \frac{e^{-i\phi(z, x)+\phi(y, z)}}{e^{-i\phi(x, y)}} \hat{H}(z|y, x; T). \]  
(21)

Here we have used the fact that the holonomy is independent of the path. The phase factors cancel so that \( \hat{H}(z|y, x; T) = \tilde{\mathcal{H}}(z|y, x; T) \) and the distribution is invariant (the hit function itself, \( \tilde{\mathcal{H}}(z) \), would pick up a phase \( -ie\varphi(x, y) \)). This implies that \( \hat{H}(z) \) really is a physical object and we may choose a convenient gauge for its computation.

We also modify our formula for \( \mathcal{P}(v) \) to become a real-valued distribution on the phase introduced by the potential. This distribution will be gauge dependent, so we begin by defining \( \tilde{\mathcal{P}}(v|y, x; T) \) with respect to our reference gauge as
\[ \tilde{\mathcal{P}}(v|y, x; T) = \mathcal{P}(v|y, x; T) \]  
where in this case \( \tilde{K} \) is related to the kernel \( \tilde{K} \) under the scaling \( e \to ze \). Under a gauge transformation to a general potential \( A \) the probability distribution changes to
\[ \frac{1}{2\pi K_0(y, x; T)} \int_{-\infty}^{\infty} dz e^{iz-v\varphi(y, x)} \tilde{K}(y, x; T, z), \]  
(22)

where \( \tilde{K} \) is just \( \dot{\mathcal{P}}(v - e\varphi(y, x)|y, x; T) \), so that the effect of changing gauge is simply to translate the distribution by \( e(A(y) - A(x)) \). For this reason, only the shape of the distribution is gauge invariant (the same relationship holds for \( \mathcal{P}(v) \)), though we have the freedom to calculate \( \mathcal{P}(v) \) in a convenient gauge and transform to any other. These gauge transformations follow also from the definitions \( \tilde{\mathcal{P}}(v) \) and \( \mathcal{P}(v) \) into a specific gauge.

### APPLICATIONS

In this section we give explicit expressions for the probability distributions, \( \mathcal{H}(z) \) and \( \mathcal{P}(v) \), for some simple potentials and compare our analytic results to sampling based upon worldline numerics, a brief outline of which is presented in the appendix. We cover quadratic potentials encompassing the free particle, linear potential and harmonic oscillator followed by a constant magnetic field. We present more detailed results on estimation of the kernels of these potentials elsewhere \[1\].

#### The hit function

For \( H(z) \), it is instructive to use the path integral representation \( \tilde{\mathcal{P}}(v) \). To illustrate the process, consider an action quadratic in the trajectory (we set \( m = 1 \) henceforth),
\[ S[x] = \int_0^T dt \left[ \frac{1}{2} M \cdot x + b \cdot x \right], \]  
(24)

whose classical solution satisfies \( M \cdot x_c + b = 0 \) with boundary conditions \( x_c(0) = x_c(T) = y \). Expanding about this solution and invoking the Fourier representation of the \( \delta \) function, \( \tilde{\mathcal{P}}(v) \) yields
\[ T\hat{H}(z|y, x; T) = \int_0^T d\tau \int \frac{dD}{(2\pi)^D} e^{i\lambda(z-x_c(\tau)) - \frac{1}{2}G_M(\tau, \tau)}, \]  
(25)

where \( G_M(\tau, \tau') \) is the “worldline” Green function \( [3, 21, 22] \) for \( M \) satisfying Dirichlet boundary conditions \( G_M(0, \tau') = 0 = G_M(T, \tau') \). The Gaussian integral over \( s \) provides
\[ \hat{H}(z|y, x; T) = \int_0^T d\tau T(2\pi G_M(\tau, \tau)) \frac{e^{-(z-x_c(\tau))^2}}{2\pi G_M(\tau, \tau)} \]  
(26)

which is suitable for numerical integration. Already for the free particle the integral cannot be computed analytically: the worldline Green function for \( M = -\frac{d^2}{d^2x} \) is \( G_M(t, t') = -\Delta(t, t') \) where \( \Delta(t, t') = \frac{1}{2} |t - t'| - \frac{1}{2} (t + t') + \frac{u_t}{T}, \) and \( x_c(t) \) is the straight line path from \( x \) to \( y \) so that in one dimension the hit function is
\[ H_0(z|y, x; T) = \int_0^1 du \sqrt{2\pi u(1-u)} e^{-(z-x_c(u))^2}. \]  
(27)

Here we have scaled \( \tau = Tu \) and used \( x_c(t) = x + (y-x)\frac{t}{T} \). This form can also be derived from \( (9) \), using \( (4) \), to verify the inversion formula. In figure \( 4 \) we show the form of the function for suitable values of \( x, y \) and \( T \) along with simulated samples of the distribution based on worldline numerics.

For the one dimensional linear potential \( (M = -\frac{d^2}{d^2x} \) and \( b = k \) the Green function is unchanged but the classical solution becomes \( x_L(t) = x + (\frac{y-x}{T^2} - \frac{bT}{2}) t + \frac{kT^2}{4}, \) inducing the appropriate change in the exponent of \( 26 \). Our worldline numerics are in excellent agreement with the result that ensues. On the other hand, for the harmonic oscillator, \( M = -\frac{d^2}{d^2x} + \omega^2 \), and we require the
coincident Green function \[^{24}\]

\[
G_\omega(\tau, \tau) = \frac{1}{\omega} \frac{\sinh(\omega \tau) \sinh(\omega(T - \tau))}{\sinh(\omega T)}.
\]

(28)

We show plots of the numerical evaluation of \[^{26}\] for the harmonic oscillator and its correspondence with a sampling of the distribution using worldline numerics in figure 2. As for the free particle, one may verify the formulae through application of \[^{29}\] using the well known kernels given in \[^{24}\].

For actions that are not quadratic and where it is not feasible to compute the path integral, formula \[^{10}\] can be used if the kernel is known (although for many situations of physical interest the formulae \[^{24}\] can be applied as a semi-classical approximation given sufficiently good knowledge of \(x_c\)). Otherwise, if at least the wavefunctions of the system are known one could appeal to the spectral decomposition outlined in \[^{13}\].

Finally we consider particle motion in a plane threaded by a perpendicular, constant magnetic field, \(B\). A useful gauge is Fock-Schwinger gauge about the point \(x\), whereby \(\mathcal{A}_\mu(x(t)) = -\frac{1}{2} F_{\mu\nu}(x(t) - x)\nu\). The benefit of this gauge is that it reduces the action to one that is quadratic in the trajectory \[^{25}\] so that we may use \[^{26}\] with \(M = -\frac{\partial^2}{\partial x^2} \mathbb{1} + ie F \cdot \frac{\partial}{\partial x}\). The coupling to the gauge potential is absorbed into the worldline Green function, \(\mathcal{G}_{\mu\nu}(t, t')\), whose form is given in the appendix. To achieve this, one expands about the straight line path, denoted by \(x_0(t)\), so that

\[
\mathcal{H}(z|y, x; T) = \int_0^T dt \frac{e^{-\frac{i}{2}(z-x(t))^T \cdot \mathcal{G}_{\mu\nu}(t, t') \cdot (z-x(t))}}{2\pi T \det[\mathcal{G}(t, t)]},
\]

(29)

where \(x(t) = x_0(t) - \frac{\tau}{T} \int_0^T \mathcal{G}(t, t')dt' \cdot (y - x)\).

The path averaged potential

Turning now to the path averaged potential, it is easy to see from \[^{10}\] that for a free particle \(\mathcal{P}(v) = \delta(v)\). For the linear potential, \(V(x) = kx\), the kernel \(K(y, x; T)\) is given by \[^{24}\]

\[
\sqrt{\frac{1}{2\pi T}} \exp \left[ -\frac{1}{2T} (x - y)^2 - \frac{k T}{2} (x + y) + \frac{k^2 T^3}{24} \right],
\]

(30)

and to effect the change \(V \rightarrow iz\) it suffices to send \(k \rightarrow ik\). Application of \[^{11}\] supplies

\[
\mathcal{P}(v|y, x; T) = \sqrt{\frac{6}{\pi k^2 T^3}} e^{-\frac{k T}{2} (x+y)^2 - \frac{k^2 T^3}{24 T}} (e^{v} (x-y) v),
\]

(31)

which we note is a function of the sum of the initial and final points. In figure 3 we demonstrate the analytic result and its agreement with numerical sampling. The limit \(k \to 0\) supplies the \(\delta\) function distribution of the free particle.

For the harmonic oscillator, implementation of \[^{11}\] leads to a highly oscillatory \(z\)-integral that must be evaluated numerically. Since the spectrum consists only of bound states, it is advantageous to apply instead the scaling \(\omega \to \sqrt{\pi} \omega\) directly in the spectral decomposition \[^{2}\] and to take the real part of the integral over \(z\) along the positive real line. Then we find the energies scale to \(E_n = \frac{k^2}{2} \sqrt{\pi} E_n\), whose real parts maintain their original ordering. This leads to a sum of Fourier integrals

\[
\pi K_0(y, x; T) \mathcal{P}(v|y, x; T) = \sqrt{\frac{\omega}{\pi}} \sum_{n \geq 0} \frac{1}{2^n n!} \times
\]

\[
\Re \int_0^{\infty} dz (iz)^{\frac{1}{2}} e^{izx e^{-\frac{1}{2} (z+y)^2 - \sqrt{\pi} \omega T (n+\frac{1}{2}) H_n(\hat{x}) H_n(\hat{y})}}
\]

(32)

for \(\hat{x}^2 \equiv \sqrt{\pi} \omega x^2\) and \(\hat{y}^2 \equiv \sqrt{\pi} \omega y^2\). To make analytic headway, we set \(x = 0 = y\), so that only the states with
n even contribute; the $z$-integral can now be written in terms of modified Bessel functions of the second kind, $K_n$. The resulting sum (we set $v_n = \frac{(n+\frac{1}{2})\omega T}{8v}$ for brevity),

$$\mathcal{P}(v|0,0;T) = 64\Theta(v)\sqrt{\frac{\omega T}{2\pi^2}} \sum_{n \text{ even}} \frac{n! v_n^3 e^{-v_n}}{2^n n! (\frac{v}{2})^2 ((n+\frac{1}{2})\omega T)^{\frac{3}{2}}}$$

$$\times \Re \left[ \left( v_n - \frac{3}{4} \right) K_{\frac{3}{4}}(-v_n) - v_n K_{\frac{1}{4}}(-v_n) \right]$$

converges extremely rapidly and so is apt for truncation in a numerical evaluation to arbitrary accuracy ($\Theta(v)$ is the heaviside step function indicating that $v \geq 0$). We have checked this gives excellent agreement with sampled data generated by worldline numerics (a truncation of $n \leq 30$ is more than sufficient) which will be presented in [1].

Finally we return to the case of a constant magnetic field. We continue to use the Fock-Schwinger gauge, since application of [23] allows simple transformation to other gauge choices. The kernel in this gauge evaluates to [24]

$$\hat{K}(y,x;T) = \frac{eBT}{4\pi \sinh\left(\frac{eBT}{2}\right)} e^{-\frac{eB}{8}\|x-y\|^2 \coth\left(\frac{eBT}{2}\right)}. \quad (34)$$

Sending $e \to zc$ and using [22] we must evaluate

$$\frac{eBT}{8\pi^2} \int_{-\infty}^{\infty} dz \frac{z}{\sinh\left(\frac{eBT}{2}\right)} e^{zv-\frac{eB}{8}\|x-y\|^2 \coth\left(\frac{eBT}{2}\right)}. \quad (35)$$

For arbitrary $x$ and $y$ this is easily evaluated numerically but we can make analytic progress for the diagonal elements, which by translational symmetry are all equal. In this case the $z$-integral can be computed by closing the contour in the upper half plane, leading to

$$\hat{\mathcal{P}}(v|x,x;T) = \frac{\pi}{2eBT} \text{sech}^2\left(\frac{\pi v}{eBT}\right). \quad (36)$$

This result bears close similarity to the distributions used in numerical evaluation of the Euler-Heisenberg effective action in a relativistic setting in [26]. See figure [4] for an illustration of the distribution and the close match provided by worldline numerics. Note that by taking $B \to 0$ we acquire a representation of the $\delta$ function as expected.

**CONCLUSION**

We have introduced two new integral transforms of the quantum mechanical kernel as tools to study the path integral. These functions contain statistical information about contributions to the path integral of different trajectories. In both cases we have given asymptotic formulae and discussed their behaviour under gauge transformations, before demonstrating the distributions with some examples. These calculations are verified by numerical sampling based upon worldline numerics. Elsewhere we shall provide more detailed calculations and application to more complex systems as an alternative approach to traditional kernel-based methods in quantum mechanics. An outstanding issue remains the general validity of the complex continuation of the spectral decomposition of the kernel to determine $\hat{\mathcal{P}}(v)$. Although we did not rely solely upon this continuation, we recognise that this requires greater scrutiny (as discussed, for example, in [27, 28]). We also aim to incorporate spin degrees of freedom into these calculations in future work by building upon existing worldline techniques for doing so.

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[1] J. P. Edwards, U. Gerber, C. Schubert, M. A. Trejo, and A. Weber (2017), in preparation.

[2] A. M. Polyakov, Gauge fields and Strings (Harwood Academic Publishers, 1987).

[3] C. Schubert, Phys. Rept. 355, 73 (2001), hep-th/0101036.

[4] H. Gies, J. Sanchez-Guillen, and R. A. Vazquez, JHEP 08, 067 (2005), hep-th/0505275.

[5] P. Lévy, Compositio Mathematica 7, 283 (1940).

[6] V. Zatloukal, Phys. Rev. E 95, 052136 (2017).

[7] P. Jizba and V. Zatloukal, Phys. Rev. E 92, 062137 (2015).

[8] J. P. Edwards, JHEP 01, 033 (2016), 1506.08130.

[9] J. P. Edwards and P. Mansfield, JHEP 01, 127 (2015), 1410.3288.

[10] A. Korzeniowski, J. L. Fry, D. E. Orr, and N. G. Fazleev, Phys. Rev. Lett. 69, 893 (1992).

[11] K. Binder, Rep. Prog. Phys. p. 487 (1997).

[12] J. Rejcek, S. Datta, N. Fazleev, J. Fry, and A. Korzeniowski, Comput. Phys. Commun. 105, 108 (1997).

[13] B. J. Berne and D. Thirumalai, Annu. Rev. Phys. Chem 37, 401 (1986).

[14] N. Makri, J. Math. Phys. 36, 2430 (1995).

[15] K. Carlsson, M. Gren, G. Bohlin, P. Holmwall, P. Säterskog, and O. Ahlén, Master’s thesis, Department of Fundamental Physics, Subatomic Physics, Chalmers University of Technology, Göteborg (2011), 115.

[16] T. Nieuwenhuis and J. A. Tjon, Phys. Rev. Lett. 77, 814 (1996), hep-ph/9606403.

[17] H. Gies and K. Langfeld, Nucl. Phys. B613, 353 (2001), arXiv:hep-th/0102185v2.

[18] H. Gies and K. Langfeld, Int. J. Mod. Phys. A17, 966 (2002), arXiv:hep-th/0112198v1.

[19] H. Gies, K. Langfeld, and L. Moyaerts, JHEP 06, 018 (2003), arXiv:hep-th/0303264.

[20] W. Dittrich and H. Gies, Springer Tracts Mod. Phys. 166, 1 (2000).

[21] D. G. C. McKeon and T. N. Sherry, Mod. Phys. Lett. A9, 2167 (1994).

[22] D. Fliegener, P. Haberl, M. G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) 264, 51 ((1998)), hep-th/9707189.

[23] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets (World Scientific, 2004).

[24] C. Grosche and F. Steiner, Handbook of Feynman path integrals, Springer tracts in modern physics (Springer, Berlin, 1998).

Worldline numerics

Worldline numerics, developed in [17]–[19] following preliminary investigation in [16], means a computational estimation of the path integral, which we have recently adapted to the non-relativistic setting. The integral over trajectories is discretised to an ensemble average over a finite number, \( N_\text{L} \), of paths, \( \{ x_n \}_{n=1}^{N_\text{L}} \), generated such that the distribution on their velocities corresponds to the kinetic term in the particle action. There are several algorithms for producing these trajectories [1]–[19] and in this work we use a new, optimised algorithm which we term LSOL, further details of which can be found in [1]–[23]. The line integral of the potential along these paths is then computed numerically by splitting it into \( N_\text{p} \) segments (this corresponds to a time discretisation, maintaining continuous spatial coordinates) so that

\[
\int_{x(0)}^{x(T)} \mathcal{G}(x) e^{- \int_0^T dt \left[ \frac{\dot{x}^2}{2} + V(x(t)) \right]} \rightarrow \frac{K_0(y, x; T)}{N_L} \sum_{n=1}^{N_L} \exp \left( \frac{N_\text{p}}{N_\text{L}} \sum_{p=1}^{N_\text{p}} V\left( x_n \left( \frac{T_p}{T} \right) \right) \right). \tag{37}
\]

It is useful to absorb the boundary conditions of the path integral into a background field by expanding \( x(t) = x_0(t) + q(t) \) where \( x_0(t) = x + (y - x) \frac{t}{T} \) is the straight line path from \( x \) to \( y \). Making a further scaling \( t \to T u \) and a field-redefinition \( q(t) \to \sqrt{T} q(T u) = \sqrt{T} q(T u) \) we may write the path integral as

\[
K_0(y, x; T) \left< \exp \left[ - T \int_0^T [V(x_0(T u)) + \sqrt{T} q(T u)] du \right] \right> \tag{38}
\]

where the boundary conditions are now Dirichlet on the quantum fluctuations: \( q(0) = 0 = q(1) \). For the ensemble average \( N_\text{L} \) is chosen sufficiently large that a good sampling of the space of trajectories results.
The hit function is sampled in one spatial dimension by implementing the $\delta$ function by recognising the change of sign of $z - x_n$ and rewriting

$$\int \delta (z - x(\tau)) \, d\tau = \int \sum_{\tau_i : x(\tau_i) = z} \frac{\delta (\tau - \tau_i)}{|\dot{x}(\tau_i)|} \, d\tau$$  \hspace{1cm} (39)$$

where the $\tau_i$ are the times at which the trajectory intersects the point $z$. The derivative in the denominator is calculated numerically using a third order backward discretisation. We used 150,000 loops and 10,000 points per loop to ensure a good sampling and estimation of discrete derivative. The path averaged potential is sampled by calculating the value of $\int d\tau V(x(\tau))$ along each worldline, weighting these values by the exponential of the kinetic part of the particle action. In this case, 50,000 loops were used with 5,000 points per loop. The values are then binned to form a histogram, and we interpolate between the densities to form a smooth curve.

Worldline Green function for constant magnetic background

The worldline Green function for a particle in a plane with a constant, perpendicular magnetic background field, $B$, satisfying Dirichlet boundary conditions, is [21]:

$$g_{\mu\nu}(t,t') = -\frac{2}{B} \left[ \delta_{\mu\nu} \cosh \left( \frac{B(t-t')}{2} \right) - i \epsilon_{\mu\nu} \sinh \left( \frac{B(t-t')}{2} \right) \right]$$

$$\times \left[ \Theta(t-\epsilon_{12}) \sinh \left( \frac{B(t-t')}{2} \right) - \frac{\sinh \left( \frac{B(T-t')}{2} \right) \sinh \left( \frac{B(T-t')}{2} \right)}{\sinh \left( \frac{B(T-t')}{2} \right)} \right]$$  \hspace{1cm} (40)$$

where $t_\pm \equiv t - t'$, $\epsilon_{12} = 1 = -\epsilon_{21}$ and $\Theta$ is the Heaviside step function. One may check that $-\frac{d^2 g}{dt^2} + ieF \cdot \frac{dg}{dt} = \delta(t-t')$. If the particle is also free to move perpendicular to the plane, then the Green function associated to that direction would be $-\Delta(t,t')$ as appropriate for a free particle discussed in the main text.