QUANTUM NILPOTENT SUBALGEBRAS OF CLASSICAL
QUANTUM GROUPS AND AFFINE CRYSTALS

IL-SEUNG JANG AND JAE-HOON KWON

Abstract. We study the crystal of quantum nilpotent subalgebra of $U_q(D_n)$
associated to a maximal Levi subalgebra of type $A_{n-1}$. We show that it has
an affine crystal structure of type $D^{(1)}_n$ isomorphic to a limit of perfect Kirillov-
Reshetikhin crystal $B^{n,s}$ for $s \geq 1$, and give a new polytope realization of $B^{n,s}$.
We show that an analogue of RSK correspondence for type $D$ due to Burge is an
isomorphism of affine crystals and give a generalization of Greene’s formula for
type $D$.

1. Introduction

Let $\mathfrak{g}$ be a classical Lie algebra and let $\mathfrak{l}$ be its proper maximal Levi subalgebra
of type $A$ (or a sum of type $A$). Let $u^-$ be the negative nilradical of the parabolic
subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. The enveloping algebra
$U(u^-)$ is an integrable $\mathfrak{l}$-module, which has a multiplicity-free decomposition [8], and
the expansion of its character
\begin{equation}
\text{ch } U(u^-) = \prod_{\alpha \in \Phi(u^-)} (1 - e^\alpha)^{-1}
\end{equation}
into irreducible $\mathfrak{l}$-characters (that is, Schur polynomials or a product of Schur poly-
nomials) gives the celebrated Cauchy identity when $\mathfrak{g}$ is of type $A$, and Littlewood
identities when $\mathfrak{g}$ is of type $B, C, D$, where $\Phi(u^-)$ is the set of roots of $u^-$. 

The decomposition of $U(u^-)$ into $\mathfrak{l}$-modules has a purely combinatorial interpre-
tation by RSK correspondence and its variations, say $\kappa$ (cf. [3, 5]). Indeed, the
correspondence for type $B$ and $C$ is given by symmetrizing the map for type $A$.
A more representation theoretic meaning is available by crystal base theory [10].
In [20], Lascoux showed that $\kappa$ is an isomorphism of $\mathfrak{l}$-crystals, which immediately
implies the same result for type $B$ and $C$ [16] by using similarity of crystals [13].

Furthermore, it is shown in [17] that the RSK correspondence $\kappa$ can be extended
to an isomorphism of affine crystals of type $A^{(1)}_\mu$ when $\mathfrak{g}$ is of type $A_\mu$, and of type

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$D_{n+1}^{(2)}$ and $C_n^{(1)}$ when $\mathfrak{g}$ is of type $B_n$ and $C_n$, respectively. It is done by regarding the set of biwords (or the set of matrices with non-negative integral entries) as the crystal $B(U_q(u^-))$ of the quantum nilpotent subalgebra $U_q(u^-)$ (see also [18, 19]). This approach enables us to define naturally the Kashiwara operators on both sides of the correspondence $\kappa$ for the simple roots other than the ones in $I$. Moreover it is proved that $B(U_q(u^-))$ is isomorphic to a limit of perfect Kirillov-Reshetikhin crystals $B^{r\times s}$, which are classically irreducible (cf. [4]). An explicit description of $B^{r\times s}$ as a subcrystal of $B(U_q(u^-))$ is also given in [17]. We remark that a similar viewpoint appears in a recent study of affine geometric crystals of type $A$ [24].

In this paper, we establish an analogue of the above result when $\mathfrak{g}$ is of type $D$. The main difficulty for type $D$ is that the description of crystal $B(U_q(u^-))$ and related combinatorics are more involved than in case of type $A$, $B$, $C$. To handle with it, we use a recent work by Salisbury-Schultze-Tingley [28] on crystal structures of Lusztig data of PBW basis [21].

We consider the crystal $B_{i_0}$ of $i_0$-Lusztig data, where $i_0$ is a reduced expression associated to a specific convex order on the set of positive roots of $\mathfrak{g}$. The subcrystal $B(U_q(u^-))$ of $B_{i_0}$ consisting of Lusztig data on $\Phi(u^-)$ has a nice combinatorial realization, and naturally admits an affine crystal structure of type $D_{n+1}^{(1)}$ isomorphic to a limit of KR crystals $B^{n,s}$ for $s \geq 1$, where $B^{n,1}$ is the crystal of the spin representation as a classical crystal of type $D_n$. We give an explicit description of $B^{n,s} \subset B(U_q(u^-))$ in terms of double paths on $\Phi(u^-)$, which yields a polytope realization of $B^{n,s}$ (Theorem 4.3).

We then consider an analogue of RSK correspondence for type $D$ due to Burge [3]. We apply this map to $B(U_q(u^-))$, which sends a Lusztig datum on $\Phi(u^-)$ to a semistandard tableau with columns of even length. As a main result of this paper, we prove that it is an isomorphism of affine crystals of type $D_{n+1}^{(1)}$, where a suitable affine crystal structure is defined on the side of tableaux (Theorem 4.6). Furthermore, we present an interesting formula for the shape of a semistandard tableau corresponding to a Lusztig datum on $\Phi(u^-)$ in terms of non-intersecting double paths on $\Phi(u^-)$ (Theorem 4.9). This formula can be viewed as an analogue of Greene’s formula for the shape of a tableau corresponding to a biword under RSK given in terms of disjoint weakly decreasing subwords [6] (see also Remark 4.11).

We should remark that the formula of Berenstein-Zelevinsky [1] for the transition matrix between string parametrization and Lusztig data plays a crucial role when we characterize $B^{n,s}$ in $B(U_q(u^-))$ and derive the type $D$ analogue of Greene’s result in Burge’s correspondence. A key observation is that the notion of trail and its combinatorics introduced in [1] recovers the Greene’s formula in type $A$, and can be reformulated in terms of double paths in case of type $D$. 


The paper is organized as follows. In Section 2, we review necessary background including crystals of Lusztig data and a work by Salisbury-Schultze-Tingley. In Section 3 we describe in detail the crystal $B_{i_0}$ and $B(U_q(u^-))$ when $g$ is of type $D$. In Section 4 we state the main results in this paper, whose complete proofs are given in Section 5.

2. Quantum groups and PBW crystals

2.1. Crystals. Let us give a brief review on crystals (see [7, 10, 12] for more details). Let $\mathbb{Z}_+$ denote the set of non-negative integers. Let $g$ be the Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ indexed by a set $I$. Let $P^\vee$ be the dual weight lattice, $P = \text{Hom}_\mathbb{Z}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{ h_i | i \in I \} \subset P^\vee$ the set of simple coroots, and $\Pi = \{ \alpha_i | i \in I \} \subset P$ the set of simple roots of $g$ such that $\langle \alpha_j, h_i \rangle = a_{ij}$ for $i,j \in I$. Let $P^+$ be the set of integral dominant weights.

For an indeterminate $q$, let $U_q(g)$ be the quantized enveloping algebra of $g$ generated by $e_i, f_i$, and $q^h$ for $i \in I$ and $h \in P^\vee$ over $\mathbb{Q}(q)$. A $g$-crystal (or simply a crystal if there is no confusion on $g$) is a set $B$ together with the maps $wt: B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $\bar{e}_i, \bar{f}_i : B \rightarrow B \cup \{0\}$ for $i \in I$ satisfying certain axioms. We denote by $B(\infty)$ the crystal associated to the negative part $U_q^-(g)$ of $U_q(g)$. Let $*$ be the $(\mathbb{Q}(q))$-linear anti-automorphism of $U_q(g)$ such that $e_i^* = e_i$, $f_i^* = f_i$, and $(q^h)^* = q^{-h}$ for $i \in I$ and $h \in P$. Then $*$ induces a bijection on $B(\infty)$. For $i \in I$, we define $\bar{e}_i^* = * \circ \bar{e}_i \circ *$ and $\bar{f}_i^* = * \circ \bar{f}_i \circ *$ on $B(\infty)$. For $\Lambda \in P^+$, we denote by $B(\Lambda)$ the crystal associated to an irreducible highest weight $U_q(g)$-module $V(\Lambda)$ with highest weight $\Lambda$. For $\mu \in P$, let $T_\mu = \{ t_\mu \}$ be a crystal, where $wt(t_\mu) = \mu$, and $\varphi_i(t_\mu) = -\infty$ for all $i \in I$.

For crystals $B_1$ and $B_2$, we denote by $B_1 \otimes B_2$ the tensor product of them. A morphism $\psi : B_1 \rightarrow B_2$ is called an embedding if it is injective, and in this case $B_1$ called a subcrystal of $B_2$.

2.2. PBW crystals. Suppose that $g$ is of finite type. Let us briefly recall a PBW basis and the crystal of Lusztig data which is isomorphic to $B(\infty)$ (see [21, 22, 26]). Let $W$ be the Weyl group of $g$ generated by the simple reflection $s_i$ for $i \in I$. Let $w_0$ be the longest element in $W$ of length $N$, and let $R(w_0) = \{ i = (i_1, \ldots, i_N) | w_0 = s_{i_1} \ldots s_{i_N} \}$ be the set of reduced expressions of $w_0$.

For $i \in R(w_0)$,

$$\Phi^+ = \{ \alpha_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}) \}$$

is the set of positive roots of $g$. For $i \in I$, let $T_i$ be the $(\mathbb{Q}(q))$-algebra automorphism of $U_q(g)$, which is given as $T_i''$ in [23]. For $1 \leq k \leq N$, put $f_{\beta_k} := T_{i_1}T_{i_2} \cdots T_{i_{k-1}}(f_{i_k})$, \begin{align*}
\Phi^+ = \{ \alpha_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}) \}
\end{align*}
and for \( c = (c_{\beta_1}, \ldots, c_{\beta_N}) \in \mathbb{Z}_+^N \), let

\[
(2.2) \quad b_1(c) = f^{(c_{\beta_1})}_{\beta_1} f^{(c_{\beta_2})}_{\beta_2} \cdots f^{(c_{\beta_N})}_{\beta_N},
\]

where \( f^{(c_{\beta_k})}_{\beta_k} \) is a divided power of \( f_{\beta_k} \). Then the set \( B_1 := \{ b_1(c) \mid c \in \mathbb{Z}_+^N \} \) is a \( \mathbb{Q}(q) \)-basis of \( U_q^- (g) \) called a PBW basis.

Let \( A_0 \) be the subring of \( \mathbb{Q}(q) \) consisting of rational functions regular at \( q = 0 \). The \( A_0 \)-lattice \( L(x) \) of \( U_q^- (g) \) generated by \( B_1 \) is independent of the choice of \( i \) and invariant under \( \tilde{e}_i, \tilde{f}_i \), and the induced crystal \( \pi(B_1) \) under a canonical projection \( \pi : L(x) \to L(x)/qL(x) \) is isomorphic to \( B(x) \). We identify \( B_1 := \mathbb{Z}_+^N \) with a crystal \( \pi(B_1) \) under the map \( c \mapsto b_1(c) \), and call \( c \in B_1 \) an \( i \)-Lusztig datum.

Let \( w \in W \) be given with length \( r \). One may assume that there exists \( i = (i_1, \ldots, i_N) \in R(w_0) \) such that \( w = s_{i_1} \cdots s_{i_r} \). The \( \mathbb{Q}(q) \)-subspace of \( U_q^- (g) \) spanned by \( b_1(c) \) for \( c \in B_1 \) with \( c_k = 0 \) for \( r + 1 \leq k \leq N \) is the \( \mathbb{Q}(q) \)-subalgebra of \( U_q^- (g) \) generated by \( f_{\beta_k} \) for \( 1 \leq k \leq r \). It is independent of the choice of the reduced expression of \( w \). We call this subalgebra the quantum nilpotent subalgebra associated to \( w \in W \) and denote it by \( U_q^- (w) \) (see for example, [15] and references therein).

### 2.3. Description of \( \tilde{f}_i \)

Suppose that \( g \) is of finite type. Let \( i \in R(w_0) \) be given. For \( \beta \in \Phi^+ \), we denote by \( 1_{\beta} \) the element in \( B_1 \) where \( c_{\beta} = 1 \) and \( c_{\gamma} = 0 \) for \( \gamma \in \Phi^+ \setminus \{ \beta \} \).

The Kashiwara operators \( \tilde{f}_i \) or \( \tilde{f}_i^* \) on \( B_1 \) for \( i \in I \) is not easy to describe in general except

\[
(2.3) \quad \tilde{f}_i c = (c_1 + 1, c_2, \ldots, c_N) = c + 1_{\alpha_i}, \quad \text{when } i_1 = i, \\
\tilde{f}_i^* c = (c_1, \ldots, c_{N-1}, c_N + 1) = c + 1_{\alpha_i}, \quad \text{when } i_N = i,
\]

for \( c \in B_1 \) [23].

Let us review the results in [28], where it is shown that \( \tilde{f}_i \) can be described more explicitly in terms of so-called signature rule under certain conditions on \( i \) with respect to \( i \). For simplicity, let us assume that \( g \) is simply laced.

Let \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s) \) be a sequence with \( \sigma_u \in \{ +, -, \cdot \} \). We replace a pair \( (\sigma_u, \sigma_{u'}) = (+, -) \), where \( u < u' \) and \( \sigma_{u''} = \cdot \) for \( u < u'' < u' \), with \( (\cdot, \cdot, \cdot) \), and repeat this process as far as possible until we get a sequence with no \( - \) placed to the right of \( + \). We denote the resulting sequence by \( \sigma^{\text{red}} \). For another sequence \( \tau = (\tau_1, \ldots, \tau_t) \), we denote by \( \sigma \cdot \tau \) the concatenation of \( \sigma \) and \( \tau \).

Recall that a total order \( < \) on \( \Phi^+ \) is called convex if either \( \gamma < \gamma' < \gamma'' \) or \( \gamma'' < \gamma' < \gamma \) whenever \( \gamma' = \gamma + \gamma'' \) for \( \gamma, \gamma', \gamma'' \in \Phi^+ \). It is well-known that there exists a one-to-one correspondence between \( R(w_0) \) and the set of convex orders on \( \Phi^+ \), where the convex order \( < \) associated to \( i = (i_1, \ldots, i_N) \in R(w_0) \) is given by

\[
(2.4) \quad \beta_1 < \beta_2 < \ldots < \beta_N,
\]
where \( \beta_k \) is as in (2.1) [31].

There exists a reduced expression \( i' \) obtained from \( i \) by a 3-term braid move 
\((i_k, i_{k+1}, i_{k+2}) \to (i_{k+1}, i_k, i_{k+1}) \) with \( i_k = i_{k+2} \) if and only if \( \{ \beta_k, \beta_{k+1}, \beta_{k+2} \} \) forms a root system of type \( A_2 \), where the corresponding convex order \( <' \) is given by replacing \( \beta_k < \beta_{k+1} < \beta_{k+2} \) with \( \beta_{k+2} <' \beta_{k+1} <' \beta_k \). Also there exists a reduced expression \( i' \) obtained from \( i \) by a 2-term braid move 
\((i_k, i_{k+1}) \to (i_{k+1}, i_k) \) if and only if \( \beta_k \) and \( \beta_{k+1} \) are orthogonal, where the associated convex ordering \( <' \) is given by replacing \( \beta_k \) with \( \beta_{k+1} \).

Given \( i \in I \), suppose that \( i \) is simply braided for \( i \in I \), that is, if one can obtain 
\((i'_1, \ldots, i'_N) \in R(w_0) \) with \( i'_1 = i \) by applying a sequence of braid moves consisting of either a 2-term move or 3-term braid move 
\((\gamma, \gamma', \gamma'') \to (\gamma'', \gamma', \gamma) \) with \( \gamma'' = \alpha_i \).
Suppose that

\[(2.5) \quad \Pi_s = \{ \gamma_s, \gamma'_s, \gamma''_s \} \]

is the triple of positive roots of type \( A_2 \) with \( \gamma'_s = \gamma_s + \gamma''_s \) and \( \gamma''_s = \alpha_i \) corresponding to the \( s \)-th 3-term braid move for \( 1 \leq s \leq t \).

For \( c \in B_i \), let

\[(2.6) \quad \sigma_i(c) = \left( \begin{array}{c|c|c|c|c} \cdots & + & \cdots & - & + \\ \hline c_{\gamma'_1} & c_{\gamma_1} & \cdots & c_{\gamma'_s} & c_{\gamma_s} \end{array} \right) \]

Then we have the following description of \( \tilde{f}_i \) on \( B_i \) [28, Theorem 4.6].

**Theorem 2.1.** Let \( i \in R(w_0) \) and \( i \in I \). Suppose that \( i \) is simply braided for \( i \). Let \( c \in B_i \) be given.

1. If there exists \(+\) in \( \sigma_i(c)^{\text{red}} \) and the leftmost \(+\) appears in \( c_{\gamma_s} \), then
   \[ \tilde{f}_i c = c - 1_{\gamma_s} + 1_{\gamma'_s} \]
2. If there exists \(-\) in \( \sigma_i(c)^{\text{red}} \), then
   \[ \tilde{f}_i c = c + 1_{\alpha_i} \]

3. Crystal of quantum nilpotent subalgebra

3.1. **Crystal \( B_{1_0} \).** From now on, we assume that \( g \) is of type \( D_n \) \((n \geq 4)\). We assume that the weight lattice is \( P = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \), where \( \{ \epsilon_i | 1 \leq i \leq n \} \) is an orthonormal basis with respect to a symmetric bilinear form \((,\)\), and the Dynkin diagram is
where \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( 1 \leq i \leq n - 1 \), and \( \alpha_n = \epsilon_{n-1} + \epsilon_n \). The set of positive roots is \( \Phi^+ = \{ \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \} \). Recall that \( W \) acts faithfully on \( P \) by \( s_i(\epsilon_i) = \epsilon_{i+1}, s_i(\epsilon_k) = \epsilon_k \) for \( 1 \leq i \leq n - 1 \) and \( k \neq i, i + 1 \), and \( s_n(\epsilon_{n-1}) = -\epsilon_{n-1} \) and \( s_n(\epsilon_k) = \epsilon_k \) for \( k \neq n - 1, n \). The fundamental weights are \( \omega_i = \sum_{k=1}^{n} \epsilon_k \) for \( i = 1, \ldots, n - 2 \), \( \omega_{n-1} = (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2 \) and \( \omega_n = (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2 \).

Put \( J = I\setminus\{n\} \). Let \( l \) be the Levi subalgebra of \( g \) associated to \( \{\alpha_i \mid i \in J\} \) of type \( A_{n-1} \). Then

\[
\Phi^+ = \Phi^+(J) \cup \Phi^+_J,
\]

where \( \Phi^+_J = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \} \) is the set of positive roots of \( l \) and \( \Phi^+ \) is the set of roots of the nilradical \( u \) of the parabolic subalgebra of \( g \) associated to \( l \).

Throughout this paper, we consider a specific \( i_0 \in R(w_0) \), whose associated convex order on \( \Phi^+ \) is given by

\[
\begin{align*}
\epsilon_i + \epsilon_j < \epsilon_k - \epsilon_l, \\
\epsilon_i + \epsilon_j < \epsilon_k + \epsilon_l & \iff (j > l) \text{ or } (j = l, i > k), \\
\epsilon_i - \epsilon_j < \epsilon_k - \epsilon_l & \iff (i < k) \text{ or } (i = k, j < l),
\end{align*}
\]

for \( 1 \leq i < j \leq n \) and \( 1 \leq k < l \leq n \). An explicit form of \( i_0 \) is as follows. For \( 1 \leq k \leq n - 1 \), put

\[
\begin{align*}
i_k &= \begin{cases} (n, n - 2, \ldots, k + 1, k), & \text{if } k \text{ is odd}, \\
(n - 1, n - 2, \ldots, k + 1, k), & \text{if } k \text{ is even}, \\
(n), & \text{if } n \text{ is even and } k = n - 1,\end{cases} \\
i'_k &= \begin{cases} (n - 1, n - 2, \ldots, k + 1, k), & \text{if } n \text{ is even and } 1 \leq k \leq n - 1, \\
(n, n - 2, \ldots, k + 1, k), & \text{if } n \text{ is odd and } 1 \leq k \leq n - 2, \\
(n), & \text{if } n \text{ is odd and } k = n - 1.\end{cases}
\end{align*}
\]

Let \( i^J = i_1 \cdots i_{n-1} \) and \( i_J = i'_1 \cdots i'_{n-1} \). Then

\[
i_0 = i^J \cdot i_J,
\]

where \( i \cdot j \) denotes the concatenation of \( i \in I^r \) and \( j \in I^s \). We have \( i_0 = (i_1, \ldots, i_N) \), where \( i_1 = n \), \( i^J = (i_1, \ldots, i_M) \), and \( i_J = (i_{M+1}, \ldots, i_N) \) with \( N = n^2 - n \) and \( M = N/2 \).
Example 3.1. We have
\[ i^J = (4, 2, 1, 3, 2, 4), \quad i_J = (3, 2, 1, 3, 2, 3), \quad \text{when } n = 4, \]
\[ i^J = (5, 3, 2, 1, 4, 3, 2, 5, 3, 4), \quad i_J = (5, 3, 2, 1, 5, 3, 2, 5, 3, 5), \quad \text{when } n = 5. \]

The associated convex order when \( n = 5 \) is
\[ \epsilon_3 + \epsilon_4 < \epsilon_2 + \epsilon_4 < \epsilon_1 + \epsilon_4 < \epsilon_2 + \epsilon_3 < \epsilon_1 + \epsilon_2 < \epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4. \]

Throughout the paper, we set
\[ B := B_{i_0}. \]

For \( c = (c_β) \in B \), we also write
\[ c_β = \begin{cases} c_{β_j}, & \text{if } β_k = ε_i + ε_j \text{ for } 1 ≤ i < j ≤ n, \\ c_{β_i}, & \text{if } β_k = ε_i - ε_j \text{ for } 1 ≤ i < j ≤ n. \end{cases} \]

Proposition 3.2.

1. The reduced word \( i_0 \) is simply braided for any \( i \in I \).
2. For \( i \in I \setminus \{n\} \) and \( c \in B \), we have
\[ σ_i(c) = σ_{i,1}(c) \cdot σ_{i,2}(c) \cdot σ_{i,3}(c), \]
where
\[ σ_{i,1}(c) = \left(\begin{array}{cccccccc} + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ \end{array} \right) \]
\[ σ_{i,2}(c) = \left(\begin{array}{cccccccc} + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ \end{array} \right) \]
\[ σ_{i,3}(c) = \left(\begin{array}{cccccccc} + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \\ \end{array} \right). \]

Here we assume that \( c_{ab} \) is zero when it is not defined.

Proof. It is clear that \( i_0 \) is simply braided for \( i = n \) since \( i_1 = n \). We assume that \( n \) is even since the proof for \( n \) odd is almost the same. We fix \( i \in I \setminus \{n\} \).

Step 1. We first observe that if the first letter \( n - 1 \) in \( i^J \) corresponds to \( i_k \) in \( i_0 \) for some \( k \), then \( β_k = α_i \).

Step 2. Let \( i_k = n - 1 \) be as in Step 1. Suppose that \( i ≠ 1 \). Then we can apply 2-term move or 3-term braid move \((γ, γ', γ'') \to (γ'', γ', γ)\) with \( γ'' = α_i \) to \( i_0 \) (indeed to the subword \( i^J \cdot i^J \cdot i^J \cdots i^J \cdot i_k \) to get \( i_0^{(3)} = (i_1, \ldots, i_{M}, j, \ldots) \) with \( j = n - i \). We can check that 3-term braid move occurs once in each \( i^J_s \) for \( s = 1, \ldots, i - 1 \), and the corresponding root system of type \( A_2 \) is
\[ Π^{(3)}_s = \{ε_s - ε_i, ε_s - ε_{i+1}, α_i\} \]
for \(s = 1, \ldots, i - 1\). We assume that \(\Pi_0^{(3)}\) is empty when \(i = 1\).

**Step 3.** We consider the reduced word \(i_0^{(3)}\). Suppose that \(i \neq 1\) or \(j \neq n\). First, we apply 2-moves only to \(i_j \cdot i_{j+1}\) so that the last \(i - 2\) letters in \(i_j\) and the first \(i - 2\) letters are shuffled by a permutation of length \((i - 2)(i - 1)/2\) and hence appear in an alternative way. We denote this subword by \(\overline{i}_j \cdot \overline{i}_{j+1}\).

Then we apply 2-term move or 3-term braid move \((\gamma, \gamma', \gamma'') \rightarrow (\gamma'', \gamma', \gamma)\) with \(\gamma'' = \alpha_i\) to the subword \(\overline{i}_j \cdot \overline{i}_{j+1} \cdot i_{j+2} \cdot \cdots i_{n-1} \cdot j\) to obtain a word starting with \(j''\) where \(j''\) is \(n\) (resp. \(n - 1\)) when \(j\) is odd (resp. even). We denote the resulting whole word by \(i_0^{(2)}\).

Here we have \(i - 1\) 3-term braid moves only in \(\overline{i}_j \cdot \overline{i}_{j+1}\) and the corresponding root system of type \(A_2\) is

\[
\Pi_s^{(2)} = \{\epsilon_{i+1} + \epsilon_s, \epsilon_i + \epsilon_s, \alpha_i\}
\]

for \(1 \leq s \leq i - 1\), and the order of occurrence of 3-term braid move is when \(s\) ranges from 1 to \(i - 1\). If \(i = 1\), then we assume that \(\Pi_0^{(2)}\) is empty, and \(i_0^{(2)} = i_0^{(3)}\).

**Step 4.** Finally, we apply 2-term move or 3-term braid move \((\gamma, \gamma', \gamma'') \rightarrow (\gamma'', \gamma', \gamma)\) with \(\gamma'' = \alpha_i\) to the subword \(i_1 \cdot i_2 \cdots i_j \cdot j''\) of \(i_0^{(2)}\) to obtain a word starting with \(i\), and denote the resulting whole word by \(i_0^{(1)}\). In this case, 3-term braid move occurs once in each \(i_s\) for \(s = 1, \ldots, j - 1\), and the corresponding root system of type \(A_2\) is

\[
\Pi_s^{(1)} = \{\epsilon_{n-s+1} + \epsilon_{i+1}, \epsilon_{n-s+1} + \epsilon_i, \alpha_i\}
\]

for \(s = 1, \ldots, j - 1\).

By the above steps, we conclude that \(i_0^{(1)} \in R(w_0)\) is obtained from \(i_0\) by applying 2-term move or 3-term braid move \((\gamma, \gamma', \gamma'') \rightarrow (\gamma'', \gamma', \gamma)\) with \(\gamma'' = \alpha_i\). Hence \(i_0\) is simply braided for \(i\), and for a given \(c \in B\) the corresponding sequence \(\sigma_i(c)\) is given by (3.3), where the root systems of type \(A_2\) associated to \(\sigma_{i,j}(c)\) are given by \(\Pi_s^{(j)}\) for \(j = 1, 2, 3\) in (3.6), (3.5), and (3.4). \(\square\)

**Example 3.3.** Let us illustrate \(\sigma_i(c)\) for \(c \in B\) when \(n = 5\) and \(i = 3\). Consider \(i_0 = i_1 \cdot i \cdot j = (i_1, \ldots, i_{20})\) (see Example 3.1). Note that \(i_{18} = 5\) is the first letter in \(i_3\), and \(\beta_{18} = \alpha_3\).

For convenience, let \(\leadsto\) (resp. \(\rightarrow\)) mean the 3-term (resp. 2-term) braid move. Then

\[
i_j = (5, 3, 2, 1, 5, 3, 2, 5, 3, 5) \rightarrow (5, 3, 2, 1, 5, 3, 5, 2, 3, 5)
\]

\[
\leadsto (5, 3, 2, 1, 5, 3, 3, 2, 3, 5) \cdots \Pi_2^{(3)}
\]

\[
\rightarrow (5, 3, 2, 3, 1, 5, 3, 2, 3, 5)
\]

\[
\leadsto (5, 2, 3, 2, 1, 5, 3, 2, 3, 5) \cdots \Pi_1^{(3)}
\]

\[
\rightarrow (2, 5, 3, 2, 1, 5, 3, 2, 3, 5),
\]

where \(\Pi_2^{(3)}\) and \(\Pi_1^{(3)}\) are given by (3.6) and (3.5), respectively. It is clear that \(\sigma_{i,j}(c)\) is given by the sequence of signs in the above table.
where \( \Pi_1^{(3)} = \{ \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \alpha_3 \} \) and \( \Pi_1^{(3)} = \{ \epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4, \alpha_3 \} \). Here the bold letter denotes the one corresponding to \( \alpha_3 \) in the associated convex order on \( \Phi^+ \).

Next, we have \( i_2 \cdot i_3 = (4, 3, 2, 5, 3) \rightarrow \tilde{i}_2 \cdot \tilde{i}_3 = (4, 3, 5, 2, 3) \), and hence

\[
\tilde{i}_2 \cdot \tilde{i}_3 \cdot 2 = (4, 3, 5, 2, 3, 2) \mapsto (4, 3, 5, 3, 3) \quad \cdots \quad \Pi_2^{(2)}
\]

\[
\quad \mapsto (4, 5, 3, 5, 2, 3) \quad \cdots \quad \Pi_1^{(2)}
\]

\[
\quad \rightarrow (5, 4, 3, 5, 2, 3),
\]

where \( \Pi_2^{(2)} = \{ \epsilon_1 + \epsilon_4, \epsilon_1 + \epsilon_3, \alpha_3 \} \) and \( \Pi_1^{(2)} = \{ \epsilon_2 + \epsilon_4, \epsilon_2 + \epsilon_3, \alpha_3 \} \). Finally,

\[
i_1 \cdot 5 = (5, 3, 2, 1, 5) \rightarrow (5, 3, 2, 5, 1)
\]

\[
\quad \rightarrow (5, 3, 5, 2, 1)
\]

\[
\quad \mapsto (3, 5, 3, 2, 1) \quad \cdots \quad \Pi_1^{(1)},
\]

where \( \Pi_1^{(3)} = \{ \epsilon_4 + \epsilon_5, \epsilon_3 + \epsilon_5, \alpha_3 \} \). Thus \( i_0 \) is simply braided for \( i = 3 \), and

\[
\sigma_{3,1}(c) = \left( \begin{array}{ccc}
\epsilon_3^1 & + & + \\
\epsilon_3^2 & + & + \\
\epsilon_3^3 & + & + \\
\epsilon_3^4 & + & + \\
\epsilon_3^5 & + & + \\
\end{array} \right),
\]

\[
\sigma_{3,2}(c) = \left( \begin{array}{ccc}
\epsilon_3^1 & + & - \\
\epsilon_3^2 & + & - \\
\epsilon_3^3 & + & - \\
\epsilon_3^4 & + & - \\
\epsilon_3^5 & + & - \\
\end{array} \right),
\]

\[
\sigma_{3,3}(c) = \left( \begin{array}{ccc}
\epsilon_3^1 & + & + \\
\epsilon_3^2 & + & + \\
\epsilon_3^3 & + & + \\
\epsilon_3^4 & + & + \\
\epsilon_3^5 & + & + \\
\end{array} \right),
\]

for \( c \in B \).

Set

\[
B^J = \{ c = (c_\beta) \in B \mid c_\beta = 0 \text{ unless } \beta \in \Phi^+(J) \},
\]

\[
(3.7) \quad B_J = \{ c = (c_\beta) \in B \mid c_\beta = 0 \text{ unless } \beta \in \Phi_J \},
\]

which we regard them as subcrystals of \( B \), where we assume that \( \tilde{\epsilon}_n c = \tilde{f}_n c = 0 \) with \( \epsilon_n(c) = \varphi_n(c) = -\infty \) for \( c \in B_J \). The subcrystal \( B^J \) is the crystal of the quantum nilpotent subalgebra \( U_q^{-}(w^J) \), where \( w^J = s_{i_1}, \ldots, s_{i_M} \) with \( i^J = (i_1, \ldots, i_M) \), which can be viewed as a \( q \)-deformation of \( U(u^-) \).

**Corollary 3.4.**

1. The crystal \( B_J \) is isomorphic to the crystal of \( U_q^{-}(1) \) as an \( l \)-crystal.
2. The map

\[
\begin{array}{ccc}
B & \longrightarrow & B^J \otimes B_J \\
c & \longrightarrow & c^J \otimes c_J
\end{array}
\]

is an isomorphism of \( \mathfrak{g} \)-crystals.
Proof. (1) It follows directly from comparing the crystal structure of $U_q^-(l)$ given in [27, Section 4.1] (see also [18, Section 4.2]).

(2) It follows from Theorem 2.1 Proposition 3.2(2), and the tensor product rule of crystals. □

3.2. Crystal $B^J$ of quantum nilpotent subalgebra. Let us consider the subcrystal $B^J$ in more details. Let $\Delta_n$ be the arrangements of dots in the plane to represent the $(n - 1)$-th triangular number. We often identify $\Delta_n$ with $\Phi^+(J)$ in such a way that $\epsilon_{k+1} + \epsilon_{l+1}$, $\epsilon_{k+1} + \epsilon_l$ and $\epsilon_k + \epsilon_l$ for $1 \leq k, l \leq n - 1$ are the vertices of a triangle of minimal shape in $\Delta_n$ as follows:

\begin{align}
\epsilon_{k+1} + \epsilon_{l+1} \quad \epsilon_{k+1} + \epsilon_l \quad \epsilon_k + \epsilon_l
\end{align}

We also identify $c \in B^J$ with an array of $c_\beta$'s in $c$ with $c_\beta$ at the corresponding dot in $\Delta_n$.

Example 3.5. For $n = 5$ and $c \in B^J$, we have

\begin{align*}
\Delta_5 = & \begin{array}{c}
\epsilon_1 + \epsilon_5 \\
\epsilon_2 + \epsilon_5 \\
\epsilon_3 + \epsilon_5 \\
\epsilon_4 + \epsilon_5 \\
\end{array} & c = & \begin{array}{c}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\end{array}
\end{align*}

Lemma 3.6. We have $B^J = \{ c \mid \epsilon_i^*(c) = 0 \ (i \in J) \}$.

Proof. It follows from [21, Section 2.1] and (2.3). □

For $s \geq 1$, let

\begin{align}
B^{J,s} := \{ c \in B^J \mid \epsilon_i^*(c) \leq s \},
\end{align}

which is a subcrystal of $B^J$. By Lemma 3.6 and [12, Proposition 8.2] (cf. [11]), we have

\begin{align}
B(s\varnothing) \cong B^{J,s} \otimes T_{s\varnothing}, \quad \bigcup_{s \geq 1} B^{J,s} = B^J,
\end{align}

as $g$-crystals.

By (3.10), $B^J$ is a regular $l$-crystal, that is, any connected component with respect to $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in J$ is isomorphic to the crystal of the integrable highest weight $U_q(l)$-module, say $B_J(\lambda)$ for some $\lambda = \sum_{i=1}^n \lambda_i e_i \in P$ with $\langle \lambda, h_i \rangle \geq 0$ for $i \in J$. 

Example 3.7. By Proposition 3.2(2), one can easily describe $\bar{e}_i$ and $\bar{f}_i$ for $i \in J$ based on tensor product rule. When $n = 5$ and $i = 3$, we have for $c \in B^J$,

$$\sigma_3(c) = (\cdots + \cdots + \cdots + \cdots + \cdots + \cdots),$$

(see Example 3.3) and the multiplicities $c_{ab}$ involved in $\sigma_3(c)$ are indicated below in $\Delta_5$

\[
\begin{array}{c}
\bullet \\
\bullet \\
\shortcircled{c_{31}} \shortcircled{c_{32}} \\
\shortcircled{c_{33}} \shortcircled{c_{34}} \shortcircled{c_{35}} \\
\shortcircled{c_{43}} \shortcircled{c_{42}} \shortcircled{c_{41}} \\
\shortcircled{c_{54}} \shortcircled{c_{53}} \shortcircled{c_{52}} \shortcircled{c_{51}} \\
\end{array}
\]

Proposition 3.2 enables us to decompose $B^J$ into $l$-crystals directly as follows, and hence the decomposition of $U_q(w^J)$ into irreducible $U_q(l)$-modules.

Proposition 3.8. As an $l$-crystal, we have

$$B^J \cong \bigsqcup_{\lambda} B_J(\lambda),$$

where the union is over $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in P$ such that $0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 = \lambda_4 \geq \cdots$.

Proof. It is enough to characterize the highest weight elements in $B^J$ as an $l$-crystal. We claim that $c = (c_{ji}) \in B^J$ is an $l$-highest weight element if and only if

$$c_{m-n-1} \geq c_{m-2n-3} \geq \cdots, \quad c_{ij} = 0 \text{ elsewhere.} \tag{3.11}$$

It is immediate from Proposition 3.2(2) that if $c$ satisfies (3.11), then $\tilde{e}_i c = 0$ for $i \in I \setminus \{n\}$. Conversely, suppose that $c = (c_{ji}) \in B^J$ is an $l$-highest weight element. If $c_{ji} \neq 0$ for some $(i, j) \notin \{(n-1, n), (n-3, n-2), \ldots\}$, then choose $c_{ki} \neq 0$ whose corresponding root $\epsilon_i + \epsilon_j$ is minimal with respect to (3.11). If $j-i > 1$, then $\tilde{e}_j c \neq 0$, and if $j-i = 1$, then $\tilde{e}_j c \neq 0$. This is a contradiction. Next, if $c_{i+2} < c_{i+1}$ for some $i \geq 2$, then we have $\tilde{e}_i c \neq 0$, which is also a contradiction. Hence $c$ satisfies (3.11). \qed

3.3. Combinatorial description of $\varepsilon_n^*$. Let us give an explicit combinatorial description of $\varepsilon_n^*$ on $B^J$.

Definition 3.9. A path in $\Delta_n$ is a sequence $p = (\gamma_1, \ldots, \gamma_s)$ in $\Phi^+(J)$ for some $s \geq 1$ such that

1. $\gamma_1, \ldots, \gamma_s \in \Phi^+(J)$,
2. if $\gamma_i = \epsilon_k + \epsilon_{l+1}$ for some $k < l$, then $\gamma_{i+1} = \epsilon_{k+1} + \epsilon_{l+1}$ or $\epsilon_k + \epsilon_l$ (see (3.8)),
3. $\gamma_s = \epsilon_k + \epsilon_{k+1}$ for some $k$. 


For $\beta \in \Phi^+(J)$, a **double path at $\beta$ in $\Delta_n$** is a pair of paths $p = (p_1, p_2)$ in $\Delta_n$ of the same length with $p_1 = (\gamma_1, \ldots, \gamma_s)$ and $p_2 = (\delta_1, \ldots, \delta_s)$ such that

1. $\gamma_1 = \delta_1 = \beta$,
2. $\gamma_i$ is located to the left of $\delta_i$ for $2 \leq i \leq s$,
3. $\gamma_s = \epsilon_{k+1} + \epsilon_{k+2}$, $\delta_s = \epsilon_k + \epsilon_{k+1}$ for some $k \geq 1$.

**Example 3.10.** For a double path $p = (p_1, p_2)$ at $\beta$, if we draw an arrow from $\gamma_i$ to $\gamma_{i+1}$ in $p_1$ and from $\delta_i$ to $\delta_{i+1}$ in $p_2$, then $p_1$ and $p_2$ form a pair of non-intersecting paths starting from $\beta$ going downward to the bottom row in $\Delta_n$ with $p_1$ on the left, and $p_2$ on the right. The following is the list of double paths $p$ at $\epsilon_1 + \epsilon_5$ in $\Delta_5$.

For $c \in B^J$ and a double path $p$, let

(3.12) $||c||_p = \sum_{\beta \text{ lying on } p} c_{\beta}$.

**Theorem 3.11.** For $c \in B^J$,

$\varepsilon^*_n(c) = \max \{ ||c||_p \mid p \text{ is a double path in } \Delta_n \}$.

**Proof.** We give the proof of this formula in Section 5.2. □

**Remark 3.12.** Let $\theta = \epsilon_1 + \epsilon_n$ be the longest root in $\Phi^+$. Since $\theta$ is located at the top of $\Delta_n$, the formula in Theorem 3.11 is equivalent to

$\varepsilon^*_n(c) = \max \{ ||c||_p \mid p \text{ is a double path at } \theta \text{ in } \Delta_n \}$.

4. **Affine crystal structure and Burge correspondence**

4.1. **KR crystals $B^{n,s}$.** Let $\hat{\mathfrak{g}}$ be an affine Kac-Moody algebra of type $D_n^{(1)}$ with $\hat{I} = \{0, 1, \ldots, n\}$ the index set for the simple roots.
For \( r \in \{0, n\} \), let \( \hat{\mathfrak{g}}_r \) be the subalgebra of \( \hat{\mathfrak{g}} \) corresponding to \( \{ \alpha_i | i \in \hat{I} \setminus \{r\} \} \). Then \( \hat{\mathfrak{g}}_0 = \mathfrak{g} \), and \( \hat{\mathfrak{g}}_0 \cap \hat{\mathfrak{g}}_n = 0 \). Let \( \hat{P} = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta \) be the weight lattice of \( \hat{\mathfrak{g}} \), where \( \delta \) is the positive imaginary null root and \( \Lambda_i \) is the \( i \)-th fundamental weight (see [9]). We regard \( P = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \) as a sublattice of \( \hat{P} / \mathbb{Z} \delta \) by putting \( \epsilon_1 = \Lambda_1 - \Lambda_0 \), \( \epsilon_2 = \Lambda_2 - \Lambda_1 - \Lambda_0 \), \( \epsilon_k = \Lambda_k - \Lambda_{k-1} \) for \( k = 3, \ldots, n-2 \), \( \epsilon_{n-1} = \Lambda_{n-1} + \Lambda_n - \Lambda_{n-2} \) and \( \epsilon_n = \Lambda_n - \Lambda_{n-1} \). In particular, we have \( \alpha_0 = -\epsilon_1 - \epsilon_2 \) in \( P \). If \( \varpi'_i \) are the fundamental weights for \( \hat{\mathfrak{g}}_n \) for \( i \in \hat{I} \setminus \{0\} \), then \( \varpi'_i = \varpi_i \) for \( i \in \hat{I} \setminus \{0, n\} \) and \( \varpi'_{0} = -\varpi_n \).

For \( c \in B^J \), define
\[
\tilde{c}_0 c = c + 1_{\epsilon_1 + \epsilon_2}, \quad \tilde{f}_0 c = \begin{cases} c - 1_{\epsilon_1 + \epsilon_2}, & \text{if } c_{\epsilon_1 + \epsilon_2} > 0, \\ 0, & \text{otherwise,} \end{cases}
\]
\[
\varphi_0(c) = \max \{ k | \tilde{f}^k_0 c \neq 0 \}, \quad \varepsilon_0(c) = \varphi_0(c) - \langle \text{wt}(c), h_0 \rangle.
\]

Lemma 4.1. The set \( B^J \) is a \( \hat{\mathfrak{g}} \)-crystal with respect to \( \text{wt}, \varepsilon_1, \varphi_i, \tilde{e}_i, \tilde{f}_i \) for \( i \in \hat{I} \), where \( \text{wt} \) is the restriction of \( \text{wt} : B \rightarrow P \) to \( B^J \).

Proof. It follows directly from (4.1). \( \square \)

Remark 4.2. The inclusions \( B_1 \supseteq B_2 \) for \( s \leq t \) are embeddings of \( \hat{\mathfrak{g}} \)-crystals (cf. (3.10)), and hence \( B^J \) is a direct limit of \( \{ B_s^J | s \in \mathbb{Z}_+ \} \).

Theorem 4.3. For \( s \geq 1 \), \( B_1^s \otimes T_{s \varpi_n} \) is a regular \( \hat{\mathfrak{g}} \)-crystal and
\[
B_1^s \otimes T_{s \varpi_n} \cong B_{n,s},
\]
where \( B_{n,s} \) is the Kirillov-Reshetikhin crystal of type \( D_n^{(1)} \) associated to \( s \varpi_n \).

Proof. By (3.10), \( B_1^s \otimes T_{s \varpi_n} \) is a regular \( \hat{\mathfrak{g}}_0 \)-crystal. By Proposition 3.2(2), we see that the \( \hat{\mathfrak{g}}_n \)-crystal \( B_1^s \otimes T_{s \varpi_n} \) is isomorphic to the dual of the \( \hat{\mathfrak{g}}_0 \)-crystal \( B_1^s \otimes T_{s \varpi_n} \) assuming that \( \hat{\mathfrak{g}}_n \cong \hat{\mathfrak{g}}_0 \) under the correspondence \( \alpha_i \leftrightarrow -\alpha_{n-i} \) for \( 0 \leq i \leq n-1 \). This implies that \( B_1^s \otimes T_{s \varpi_n} \) is a regular \( \hat{\mathfrak{g}}_n \)-crystal, and hence a regular \( \hat{\mathfrak{g}} \)-crystal. It is known that \( B_{n,s} \) is classically irreducible, that is, \( B_{n,s} \cong B(s \varpi_n) \) as a \( \hat{\mathfrak{g}}_0 \)-crystal (see [4]). Therefore, it follows from [29] Lemma 2.6 that \( B_1^s \otimes T_{s \varpi_n} \cong B_{n,s} \). \( \square \)

Remark 4.4. By Theorem 3.11 we have
\[
B_1^s = \bigcap_P \{ c \in B^J | \| c \|_p \leq s \},
\]
where \( p \) runs over double paths in \( \Delta_n \). This gives a polytope realization of the KR crystal \( B_n^{\alpha, \beta} \).

### 4.2. Burge correspondence.

Let us recall some necessary notions following [5]. Let \( \mathcal{P} \) be the set of partitions \( \lambda = (\lambda_i)_{i \geq 1} \), which are often identified with Young diagrams. Let \( \lambda' = (\lambda'_i)_{i \geq 1} \) be the conjugate of \( \lambda \), and let \( \lambda^\pi \) be the skew Young diagram obtained by \( 180^\circ \)-rotation of \( \lambda \). Let \( \ell(\lambda) \) denote the length of \( \lambda \), and let \( \mathcal{P}_n = \{ \lambda | \ell(\lambda) \leq n \} \).

Let \( [\Pi] := \{ \pi < \cdots < \tilde{\Pi} \} \) be a linearly ordered set. Let \( \mathcal{W} \) be the set of finite words in \([\Pi] \). For a skew Young diagram \( \lambda/\mu \), let \( \text{SST}_n(\lambda/\mu) \) or simply \( \text{SST}(\lambda/\mu) \) denote the set of semistandard tableaux of shape \( \lambda/\mu \) with entries in \([\Pi] \). For \( T \in \text{SST}(\lambda/\mu) \), let \( w(T) \) be a word in \( \mathcal{W} \) obtained by reading the entries of \( T \) row by row from top to bottom, and from right to left in each row, and let \( \text{sh}(T) \) denote the shape of \( T \).

Let \( T^\wedge \) be the unique semistandard tableau such that \( \text{sh}(T^\wedge) \in \mathcal{P} \) and \( w(T^\wedge) \) is Knuth equivalent to \( w(T) \). We define \( T^\wedge \) in a similar way such that \( \text{sh}(T^\wedge) \in \mathcal{P}^\pi \).

Note that if \( \text{sh}(T^\wedge) = \nu \), then \( \text{sh}(T^\wedge) = \nu^\pi \).

For \( a \in [\Pi] \) and \( U \in \text{SST}(\lambda) \) with \( \lambda \in \mathcal{P}_n \), let \( a \rightarrow U \) be the tableau obtained by applying the Schensted’s column insertion of \( a \) into \( U \). Similarly, for \( V \in \text{SST}(\lambda^\pi) \) and \( b \in [\Pi] \), let \( V \leftarrow b \) be the tableau obtained by applying the Schensted’s column insertion of \( b \) into \( V \) in a reverse way starting from the rightmost column. For \( w = w_1 \cdots w_r \in \mathcal{W} \), we define \( P(w) = (w_r \rightarrow (\cdots (w_2 \rightarrow w_1) \cdots)) \). Note that \( P(w)^\wedge = (w_r \leftarrow w_{r-1} \leftarrow \cdots \leftarrow w_1) \).

The goal in the remaining of this section is to give an explicit isomorphism in Proposition [3.8] and extend it as an isomorphism of \( D_n^{(1)} \)-crystals.

Let us first recall a variation of RSK correspondence for type \( D \) [3]. Set

\[
\mathcal{T}^\wedge := \bigsqcup_{\lambda \in \mathcal{P}_n, \lambda' \text{ even}} \text{SST}(\lambda^\pi), \quad \mathcal{T}^\wedge := \bigsqcup_{\lambda \in \mathcal{P}_n, \lambda' \text{ even}} \text{SST}(\lambda),
\]

where we say that \( \lambda' \) is even if each part of \( \lambda' \) is even. Let \( \Omega \) be the set of biwords \((a, b) \in \mathcal{W} \times \mathcal{W} \) such that

1. \( a = a_1 \cdots a_r \) and \( b = b_1 \cdots b_r \) for some \( r \geq 0 \),
2. \( a_i < b_i \) for \( 1 \leq i \leq r \),
3. \( (a_1, b_1) \leq \cdots \leq (a_r, b_r) \),

where \((a, b) < (c, d)\) if and only if \((a < c)\) or \((a = c \text{ and } b > d)\) for \((a, b), (c, d) \in \mathcal{W} \times \mathcal{W} \). We denote by \( c(a, b) \) the unique element in \( \mathcal{B}^{\mathcal{J}} \) corresponding to \((a, b)\) such that \( c_{ab} = |\{ k | (a_k, b_k) = (a, b) \} | \).

For \((a, b) \in \Omega \) with \( a = a_1 \cdots a_r \) and \( b = b_1 \cdots b_r \), we define a sequence of tableaux \( P_r, P_{r-1}, \ldots, P_1 \) inductively as follows:
(1) let $P_1$ be a vertical domino $\begin{array}{c} a_r \\ b_r \end{array}$.
(2) if $P_{k+1}$ is given for $1 \leq k \leq r-1$, then define $P_k$ to be the tableau obtained by first applying the column insertion to get $P_{k+1} \leftarrow b_k$, and then adding $\begin{array}{c} a_k \\ \end{array}$ at the corner of $P_{k+1} \leftarrow b_k$ located above the box $\text{sh}(P_{k+1} \leftarrow b_k)/\text{sh}(P_{k+1})$.

We put $P^\wedge(a, b) := P_1$. It is not difficult to see from the definition that $P^\wedge(a, b) \in SST(\lambda^\wedge)$ for some $\lambda \in \mathcal{P}$ such that $\lambda^\wedge$ is even.

For $c \in B^J$, let $P^\wedge(c) = P^\wedge(a, b)$ where $c = c(a, b)$. Since the map $(a, b) \mapsto P^\wedge(a, b)$ is a bijection from $\Omega$ to $\mathcal{T}^\wedge$, we have a bijection

\begin{equation}
\kappa^\wedge : B^J \longrightarrow \mathcal{T}^\wedge \quad c \longmapsto P^\wedge(c)
\end{equation}

Similarly, let $\Omega'$ be the set of biwords $(a, b) \in W \times W$ satisfying the same conditions as in $\Omega$ except that $<$ is replaced by $<'$, where $(a, b) <' (c, d)$ if and only if $(b < d)$ or $(b = d$ and $a < c)$ for $(a, b)$ and $(c, d) \in W \times W$. We define $c'(a, b)$ in the same way as in $c(a, b)$. Given $(a, b) \in \Omega'$ with $a = a_1 \cdots a_r$ and $b = b_1 \cdots b_r$, define a sequence of tableaux $P_1, P_2, \ldots, P_r$ inductively as follows:

(1) let $P_1$ be a vertical domino $\begin{array}{c} a_1 \\ b_1 \end{array}$.
(2) if $P_{k-1}$ is given for $2 \leq k \leq r$, then define $P_k$ to be the tableau obtained by first applying the column insertion to get $a_k \rightarrow P_{k-1}$, and then adding $\begin{array}{c} b_k \\ \end{array}$ at the corner of $a_k \rightarrow P_{k-1}$ located below the box $\text{sh}(a_k \rightarrow P_{k-1})/\text{sh}(P_{k-1})$.

and put $P^\wedge(a, b) := P_r$. For $c \in B^J$, let $P^\wedge(c) = P^\wedge(a, b)$ where $c = c'(a, b)$. Then we also have a bijection

\begin{equation}
\kappa^\wedge : B^J \longrightarrow \mathcal{T}^\wedge \quad c \longmapsto P^\wedge(c)
\end{equation}

Example 4.5. Suppose that $n = 5$. Let $c \in B^J$ be given by

\[
\begin{array}{cccc}
2 \\
1 & 0 \\
1 & 2 & 1 \\
2 & 1 & 0 & 1 \\
\end{array}
\]

(see Example 3.5, where $c = c(a, b)$ for $(a, b) \in \Omega$ with

\[
\begin{pmatrix}
\begin{array}{cccccc}
5 & 5 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{cccc}
5 & 5 & 5 & 5 \\
1 & 1 & 1 & 1 \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{cccc}
3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 \\
\end{array}
\end{pmatrix}
\]
The following is the sequence of tableaux $P_i, P_{i-1}, \ldots, P_1 =: P \vee (a, b)$ given in the definition of $\kappa \vee (4.3)$:

Here we use the notation $T \downarrow T'$ when $T = P_{k+1}, T' = P_k$ and $(a_k, b_k) = (J, I)$. Hence, we have

$$\kappa \vee (c) = \begin{array}{cccc}
  & & & \\
  & & & \\
 1 & 2 & 3 & 4 \\
 5 & 6 & 7 & 8 \\
 9 & 10 & 11 & 12 \\
\end{array}$$

Of columns = 9

On the other hand, we have by Theorem 3.11 $\epsilon_4^T(c) = 9$ since the following double path $p$

$$2 \downarrow 1 \downarrow 0 \downarrow 1 \downarrow 2$$

takes the maximum value of $||c||_p$, which is equal to the number of columns of $\kappa \vee (c)$.

### 4.3. Isomorphism of affine crystals

We regard $[\pi] = \{ \pi < \cdots < \top \}$ as the crystal of dual natural representation of $\lambda$ with $\text{wt}(\lambda) = -\epsilon_k$. Then $W$ is a regular $l$-crystal, where $w = w_1 \cdots w_r$ is identified with $w_1 \otimes \cdots \otimes w_r$. For $\lambda \in \mathcal{P}_n$, $SST(\lambda)$ is a regular $l$-crystal with lowest weight $-\sum_{i=1}^n \lambda_i \epsilon_i$, where $T$ is identified with $w(T)$ [14]. In particular $\mathcal{T} \vee$ and $\mathcal{T} \wedge$ are regular $l$-crystals.

Let us recall the $\mathfrak{g}_0$-crystal structure on $\mathcal{T} \vee$ [17, Section 5.2]. Let $T \in \mathcal{T} \vee$ be given. For $k \geq 1$, let $t_k$ be the entry in the top of the $k$-th column of $T$ (enumerated
from the right). Consider \( \sigma = (\sigma_1, \sigma_2, \ldots) \), where
\[
\sigma_k = \begin{cases} 
  +, & \text{if } t_k > n - 1 \text{ or the } k\text{-th column is empty,} \\
  -, & \text{if the } k\text{-th column has both } n - 1 \text{ and } \pi \text{ as its entries,} \\
  \cdot, & \text{otherwise.}
\end{cases}
\]

Then \( \tilde{\epsilon}_n T \) is obtained from \( T \) by removing \( \pi \) in the column corresponding to the right-most \( - \) in \( \sigma^{\text{red}} \) (see Section 2.3 for \( \sigma^{\text{red}} \)). If there is no such \( - \) sign, then we define \( \tilde{\epsilon}_n T = 0 \), and \( \tilde{f}_n T \) is obtained from \( T \) by adding \( \pi \) column corresponding to the left-most \( + \) in \( \sigma^{\text{red}} \). Hence \( T^- \) is a \( \hat{\mathfrak{g}}_0 \)-crystal with respect to \( \varepsilon_i, \varphi_i, \tilde{\epsilon}_i, \tilde{f}_i \) \((i \in \hat{I} \setminus \{0\})\), where \( \varepsilon_n(T) = \max \{ k \mid \tilde{\epsilon}_n^k T \neq 0 \} \) and \( \varphi_n(T) = \varepsilon_n(T) + \langle \text{wt}(T), h_n \rangle \).

Similarly, we have a \( \hat{\mathfrak{g}}_n \)-crystal structure on \( T^- \) [17] Section 5.2. Let \( T' \in T^- \) be given. For \( k \geq 1 \), let \( t_k \) be the entry in the bottom of the \( k\)-th column of \( T' \) (enumerated from the left). Consider \( \sigma = (\ldots, \sigma_2, \sigma_1) \), where
\[
\sigma_k = \begin{cases} 
  -, & \text{if } t_k < \overline{\tau} \text{ or the } k\text{-th column is empty,} \\
  +, & \text{if the } k\text{-th column has both } \overline{\tau} \text{ and } \overline{\tau} \text{ as its entries,} \\
  \cdot, & \text{otherwise.}
\end{cases}
\]

Then \( \tilde{\epsilon}_0 T \) is given by adding \( \underline{\pi} \) to the bottom of the column corresponding to the right-most \( - \) in \( \sigma^{\text{red}} \), and \( \tilde{f}_0 T \) is obtained from \( T \) by removing \( \underline{\pi} \) in the column corresponding to the left-most \( + \) in \( \sigma^{\text{red}} \). If there is no such \( + \) sign, then we define \( \tilde{f}_0 T = 0 \). Hence \( T^- \) is a \( \hat{\mathfrak{g}}_n \)-crystal with respect to \( \varepsilon_i, \varphi_i, \tilde{\epsilon}_i, \tilde{f}_i \) \((i \in \hat{I} \setminus \{n\})\), where \( \varphi_0(T) = \max \{ k \mid \tilde{f}_n^k T \neq 0 \} \) and \( \varepsilon_0(T) = \varphi_0(T) - \langle \text{wt}(T), h_0 \rangle \).

**Theorem 4.6.** The bijection \( \kappa^- \) in (4.3) is an isomorphism of \( \hat{\mathfrak{g}}_0 \)-crystals, and the bijection \( \kappa^- \) in (4.4) is an isomorphism of \( \hat{\mathfrak{g}}_n \)-crystals.

**Proof.** The proof of (1) is given in Section 5.4. The proof of (2) is similar to (1). □

For a semistandard tableau \( T \) of skew shape, let \( [T] \) denote the equivalence class of \( T \) with respect to Knuth equivalence. If we define
\[
\tilde{\epsilon}_i[T] = \begin{cases} 
  [\tilde{\epsilon}_0 T^-], & \text{if } i = 0, \\
  [\tilde{\epsilon}_n T^-], & \text{if } i = n, \\
  [\tilde{\epsilon}_i T], & \text{otherwise,}
\end{cases}
\]
for \( i \in \hat{I} \) and \( x = e, f \) (we assume that \( [0] = 0 \)), then the set
\[
(4.5) \quad \mathcal{T} = \{ [T] \mid T \in T^- \} = \{ [T] \mid T \in T^- \},
\]

is a \( \hat{\mathfrak{g}} \)-crystal with respect to \( \tilde{\epsilon}_i, \tilde{f}_i \) \((i \in I)\), where \( \text{wt}, \varepsilon_i, \) and \( \varphi_i \) are well-defined on \([T]\) [17] Section 5.3]. Therefore,
Corollary 4.7. The map

\[ \kappa : B^J \rightarrow \mathcal{T}, \]

\[ c \rightarrow [P \setminus (c)] = [P \setminus (c)] \]

is an isomorphism of \( \hat{g} \)-crystals.

For \( s \geq 1 \), let \( \mathcal{T}^s = \{ [T] \mid \ell(\text{sh}(T')) \leq s \} \subset \mathcal{T} \). It is shown in [17, Theorem 5.4] that \( \mathcal{T}^s \otimes T_{sw_n} \cong B^{n,s} \). Therefore,

Corollary 4.8. The map \( \kappa \) when restricted to \( B^{J,s} \) gives an isomorphism of \( \hat{g} \)-crystals

\[ \kappa : B^{J,s} \rightarrow \mathcal{T}^s. \]

4.4. Shape formula. For \( c \in B^J \), let

\[ \lambda(c) = (\lambda_1(c) \geq \ldots \geq \lambda_l(c)) \]

be the partition corresponding to the regular \( l \)-subcrystal of \( B^J \) including \( c \), that is, \( \lambda(c) = \text{sh}(\kappa^{-1}(c)) \) by Theorem 4.6. Note that \( l = 2\lfloor \frac{n}{2} \rfloor \) and \( \lambda_{2i-1}(c) = \lambda_{2i}(c) \) for \( 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \). We have by Theorem 3.11 and Corollary 4.8

\[ \lambda_1(c) = \max \{ \|c\|_p \mid p \text{ is a double path at } \theta \}, \]

We can further characterize the whole partition \( \lambda(c) \) in terms of double paths on \( \Delta_n \) generalizing (4.7) as follows.

Theorem 4.9. For \( c \in B^J \) and \( 1 \leq l \leq \lfloor \frac{n}{2} \rfloor \), we have

\[ \lambda_1(c) + \lambda_3(c) + \cdots + \lambda_{2l-1}(c) = \max_{p_1, \ldots, p_l} \{ \|c\|_{p_1} + \cdots + \|c\|_{p_l} \}, \]

where \( p_1, \ldots, p_l \) are mutually non-intersecting double paths in \( \Delta_n \) and each \( p_i \) starts at the \( i \)-th row of \( \Delta_n \) for \( 1 \leq i \leq l \).

Proof. The proof is given in Section 5.3.

Example 4.10. Let \( n = 6 \) and let \( c \in B^J \) be given by

\[
\begin{array}{cccc}
1 & 3 & 2 & 1 \\
2 & 1 & 1 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 3 & 2 & 1 \\
2 & 3 & 2 & 0 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 3 & 2 & 1 \\
2 & 3 & 2 & 0 & 3 \\
\end{array}
\]
Then we have

\[ \kappa(c) = \begin{array}{cccccc}
6 & 6 & 5 & 5 & 4 & 4 \\
5 & 3 & 3 & 3 & 3 & 3 \\
4 & 3 & 2 & 2 & 2 & 2 \\
3 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array} \]

where \( \lambda(c) = (19, 19, 6, 6, 2, 2) \).

On the other hand, the double path \( p \) at \( \epsilon_1 + \epsilon_6 \) given by

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 1 & 1 \\
3 & 2 & 1 & 1 \\
2 & 0 & 3 & 3
\end{array} \]

has maximal value \( ||c||_p = 19 \), and the pair of double paths \( p_1 \) (in blue) and \( p_2 \) (in red) at \( \epsilon_1 + \epsilon_6 \) and \( \epsilon_3 + \epsilon_6 \), respectively, given by

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 1 & 1 \\
3 & 2 & 1 & 1 \\
2 & 0 & 3 & 3
\end{array} \]

has maximal value \( ||c||_{p_1} + ||c||_{p_2} = 25 \). By Theorem 4.9 we have

\[ \lambda_1(c) = 19, \quad \lambda_1(c) + \lambda_3(c) = 25, \quad \lambda_1(c) + \lambda_3(c) + \lambda_5(c) = 27, \]

which implies \( \lambda_3(c) = 6, \lambda_5(c) = 2 \), and hence \( \lambda(c) = (19, 19, 6, 6, 2, 2) \).

**Remark 4.11.** Suppose that \( g \) is of type \( A_n \) and \( l \) is of type \( A_r \times A_s \) with \( r+s = n-1 \). The associated crystal \( B(U_q(u^-)) \) can be realized as the set of \( (r+1) \times (s+1) \) non-negative integral matrices (see [18, Section 4.3]). For \( M \in B(U_q(u^-)) \), let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be the shape of the tableaux corresponding to \( M \) under RSK. It is a well-known result due to Greene [6] (cf. [5]) that \( \lambda_1 + \cdots + \lambda_l \) is a maximal sum of entries in \( M \) lying on mutually non-intersecting \( l \) lattice paths on \( (r+1) \times (s+1) \) array of points from northeast to southwest. A similar result when \( g \) is of type \( B \), \( C \) is obtained by folding crystals of type \( A \) with \( r = s \). Hence, Theorem 4.9 is a non-trivial generalization of [6] to the case of type \( D \). We can also recover the result in [6] by using the same argument as in Section 5.3.
5. Proofs of Main Theorems

5.1. Formula of Berenstein-Zelevinsky. Let us recall results on combinatorial formula for string parametrization of $B(\infty)$ \cite{11}, which play a crucial role in proving Theorems \[1\] and \[19\]

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. We keep the notations in Section 2. For $i \in I$, let $B_i = \{ (x) \mid x \in \mathbb{Z} \}$ be the abstract crystal given by $\text{wt}((x)) = x\alpha_i$, $\varepsilon_i((x)) = -x$, $\varphi_i((x)) = x$, $\varepsilon_j((x)) = -\infty$, $\varphi_j((x)) = -\infty$ for $j \neq i$ and $\tilde{e}_i((x)) = (x + 1)i$, $\tilde{f}_i((x)) = (x - 1)i$, $\tilde{e}_j((x)) = \tilde{f}_j((x)) = 0$ for $j \neq i$. It is well-known that for any $i \in I$, there is a unique embedding of crystals \cite{11}

$$\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i$$

sending $b_\infty \mapsto b_\infty \otimes (0)_i$, where $b_\infty$ is the highest weight element in $B(\infty)$. This embedding satisfies that for $b \in B(\infty)$, $\Psi_i(b) = b' \otimes (-a)_i$, where $a = \varepsilon_i(b^*)$ and $b' = (\tilde{e}_i(b^*))^*$. Given $b \in B(\infty)$ and a sequence of indices $i = (i_1, \ldots, i_l)$ in $I$, consider the sequence $b_k \in B(\infty)$ and $a_k \in \mathbb{Z}_+$ for $1 \leq k \leq l - 1$ defined inductively by

$$b_0 = b, \quad \Psi_{i_k}(b_{k-1}) = b_k \otimes (-a_{i_k}).$$

The sequence $t_I(b) = (a_1, \ldots, a_1)$ is called the string of $b$ in direction $i$. By construction, it can be reformulated by

$$a_k = \varepsilon_{i_k}(\tilde{e}_{i_{k-1}}^{a_1} \cdots \tilde{e}_{i_1}^{a_1} b^*),$$

for $2 \leq k \leq l$, where $a_1 = \varepsilon_{i_1}(b^*)$.

Suppose that $\mathfrak{g}$ is of finite type. Let $V$ be a finite-dimensional $\mathfrak{g}$-module and $V_\lambda$ denote the weight space of $V$ for $\lambda \in \text{wt}(V)$, where $\text{wt}(V)$ is the set of weights of $V$.

For $\lambda, \mu \in \text{wt}(V)$, an $i$-trail from $\lambda$ to $\mu$ in $V$ is a sequence of weights $\pi = (\lambda = \nu_0, \nu_1, \ldots, \nu_l = \mu)$ in $\text{wt}(V)$ such that

1. for $1 \leq k \leq l$, $\nu_{k-1} - \nu_k = d_k(\pi)\alpha_{i_k}$ for some $d_k(\pi) \in \mathbb{Z}_+$,
2. $e_{i_1}^{d_1(\pi)} \cdots e_{i_l}^{d_l(\pi)}$ is a non-zero linear map from $V_\mu$ to $V_\lambda$.

When $V$ is a module with a minuscule highest weight, then the condition (1) implies (2). Furthermore, if $B$ is a crystal of $V$, then we have $\tilde{e}_{i_1}^{d_1(\pi)} \cdots \tilde{e}_{i_l}^{d_l(\pi)} B_\mu = B_\lambda$ or $\tilde{f}_{i_1}^{d_l(\pi)} \cdots \tilde{f}_{i_1}^{d_1(\pi)} B_\lambda = B_\mu$.

Let $i = (i_1, \ldots, i_N) \in R(w_0)$ given. Let $i^* := (i_1^*, \ldots, i_N^*)$ and $i^{op} := (i_N, \ldots, i_1)$, where $i \mapsto i^*$ is the involution on $I$ given by $w_0(\alpha_i) = -\alpha_i^*$. For $c \in B_i$, we have by \cite{11} Proposition 3.3

$$b_l(c)^* = b_{l^{op}}(c^{op}),$$

where $c^{op} = (c_k^{op})$ is given by $c_k^{op} = c_{N-k}$ for $c = (c_k)$.
Theorem 5.1 ([I, Theorem 3.7]). For \( i, i' \in R(w_0) \) and \( c \in B_1 \), let \( t = t_i(b_i(c)^*) \). Then \( t = (t_k) \) and \( c = (c_m) \) are related as follows: for any \( k = 1, \ldots, N \)

\[
(5.3) \quad t_k = \min_{\pi_1} \left\{ \sum_{m=1}^{N} d_m(\pi_1)c_m \right\} - \min_{\pi_2} \left\{ \sum_{m=1}^{N} d_m(\pi_2)c_m \right\},
\]

where \( \pi_1 \) (resp. \( \pi_2 \)) runs over \( Y \)-trails from \( s_{i_1} \cdots s_{i_{k-1}}w_{ik} \) (resp. from \( s_{i_1} \cdots s_{ik}w_{ik} \)) to \( w_0w_{ik} \) in the fundamental representation \( V(w_{ik}) \).

Remark 5.2. The string parametrization of \( b \in B(\infty) \) given by (5.1) is the string parametrization of \( b^* \) in [2] (see also [24, Remark in Section 2]).

5.2. Proof of Theorem 3.11. From now on we assume that \( g \) is of type \( D_n \) \( (n \geq 4) \) and let \( i_0 = (i_1, \ldots, i_N) \in R(w_0) \) given in (5.2) with \( i' = (i_1, \ldots, i_M) \) and \( i_J = (i_{M+1}, \ldots, i_N) \). We have \( n^* = n - 1 \) (resp. \( (n - 1)^* = n \)) when \( n \) is odd, and \( i^* = i \) otherwise. Put

\[
(5.4) \quad j_0 = (j_1, \ldots, j_N) := i_{0}^{\text{top}} = (i_N^*, \ldots, i_1^*).
\]

Recall that the crystal \( B(w_n) \) of \( V(w_n) \) can be realized as

\[
B(w_n) = \{ \tau = (\tau_1, \ldots, \tau_n) \mid \tau_k = \pm (1 \leq k \leq n) \},
\]

where \( \text{wt}(\tau) = \frac{1}{2} \sum_{k=1}^{n} \tau_k \epsilon_k \) and

\[
(5.5) \quad (\ldots, +, +, \ldots) \xrightarrow{\tau_{n-1}, \tau_n} (\ldots, -, -, \ldots) \xrightarrow{\tau, \tau_{n+1}} (\ldots, -, +, \ldots),
\]

(1 \( \leq i \leq n - 1 \)) with the highest weight element \((+, \ldots, +)\) [14]. Since the spin representation \( V(w_n) \) is minuscule, any \( i_0 \)-trail \( \pi = (\nu_0, \ldots, \nu_N) \) in \( V(w_n) \) can be identified with a sequence \( b_0, \ldots, b_N \) in \( B(w_n) \) such that \( \text{wt}(b_k) = \nu_k \) and \( \tilde{f}_{ik}(\pi) b_{k-1} = b_k \) with \( d_k(\pi) = 0, 1 \) for \( 1 \leq k \leq N \).

Lemma 5.3. There exists a unique \((i_N^*, \ldots, i_{M+1}^*)\) = \((j_1, \ldots, j_M)\)-trail from \( w_n - \alpha_n = \text{wt}(+, \ldots, +, -, -) \) to \( w_n + \alpha_0 = \text{wt}(-, -, \ldots, +) \). We denote this trail by \((\tilde{v}_0, \ldots, \tilde{v}_M)\).

Proof. Considering the crystal structure on \( B(w_n) \) (5.5), we see that up to 2-term braid move \((j_1', j_2', \ldots, j_{2n-4}') = (n - 2, n - 1, n - 3, n - 2, \ldots, 1, 2)\) is the unique sequence of indices in \( I \) such that

\[
\tilde{f}_{j_2n-4}' \cdots \tilde{f}_{j_2} \tilde{f}_{j_1} (+, \ldots, +, -, -) = (-, -, +, \ldots, +).
\]

On the other hand, there exists a subsequence \((j_1', j_2', \ldots, j_{2n-4}') \) of \((j_1, \ldots, j_N)\). Since no other subsequence gives \((n - 2, n - 1, n - 3, n - 2, \ldots, 1, 2)\) up to 2-braid move by definition of \( 1_J \), \((j_1', j_2', \ldots, j_{2n-4}')\) determines a unique such trail. \( \square \)
Let $\Gamma$ be the set of $j_0$-trails $\pi$ from $s_n \varpi_n$ to $w_0 \varpi_n$ in $V(\varpi_n)$. For $c = (c_k) \in B^J$ and $\pi \in \Gamma$, let

$$
(5.6) \quad ||c||_\pi = (1 - d_N(\pi))c_1 + \cdots + (1 - d_{M+1}(\pi))c_M = \sum_{k=1}^{M} (1 - d_{N-k+1}(\pi)) c_k.
$$

Recall that $c_k = c_{\beta_k}$ for $\beta_k \in \Phi^+(J)$ ($1 \leq k \leq M$) with respect to the order $2.4$ and hence $3.1$. Let us simply write $c = (c_1, \ldots, c_M)$. The following lemma plays a crucial role in the proof of Theorem 3.11.

**Lemma 5.4.** For $c \in B^J$, we have $\varepsilon_n^*(c) = \max \{ ||c||_\pi | \pi \in \Gamma \}$.

**Proof.** Let $c = (c_k) \in B^J$ given. Since $\varepsilon_n^*(c) = \varepsilon_n^*(b_{t_0}(c))$, we have $\varepsilon_n^*(c) = t_1$, where $(t_k) = t_{b_0}(b_{t_0}(c)) = t_{b_0}(b_{t_0}(c)^{**}) = t_{b_0}(b_{t_0}(c^{**}))$ by $5.2$.

One can check that applying $\tilde{f}_{j_N} \cdots \tilde{f}_{j_{M+1}}$ to $(+, \ldots, +)$ gives a unique $j_0$-trail from $\varpi_n$ to $w_0 \varpi_n$. Hence by $5.3$, we have

$$
t_1 = \sum_{1 \leq k \leq M} c_k - \min \left\{ \sum_{1 \leq k \leq M} d_{N-k+1}(\pi)c_k \right\},
$$

where $\pi$ is a $j_0$-trail from $s_n \varpi_n$ to $w_0 \varpi_n$. Hence $t_1 = \max \{ ||c||_\pi | \pi \in \Gamma \}$.

For $\pi = (\nu_0, \ldots, \nu_N) \in \Gamma$, let $\pi_J = (\nu_0, \ldots, \nu_M)$, $\pi'_J = (\nu_{M+1}, \ldots, \nu_N)$, and

$$
\Gamma' = \{ \pi | \pi_J = (\nu_0, \ldots, \nu_M) \} \subset \Gamma,
$$

where $(\nu_0, \ldots, \nu_M)$ as in Lemma $5.3$.

**Lemma 5.5.** For $c \in B^J$, we have $\varepsilon_n^*(c) = \max \{ ||c||_\pi | \pi \in \Gamma' \}$.

**Proof.** For simplicity, we assume that $n$ is even so that $w_0 \varpi_n = -\varpi_n$. The proof for odd $n$ is almost identical. Let $\pi = (\nu_0, \ldots, \nu_N) \in \Gamma$ given. It suffices to show that there exists $\pi' \in \Gamma'$ such that $||c||_\pi \leq ||c||_{\pi'}$.

If $\pi \in \Gamma'$, then $\nu_M = wt(-, -, +, \ldots, +)$ by Lemma $5.3$. Suppose that $\pi \notin \Gamma'$. Since $j_k \neq n$ for $1 \leq k \leq M$, we have $\nu_M = wt(\tau)$ where $\tau = (\tau_1, \ldots, \tau_n)$ with $\tau_p = \tau_q = -$ for some $(p, q) \neq (1, 2)$ and $\tau_i = +$ otherwise.

Since $\pi \in \Gamma$, there exists a subsequence $(j_1', \ldots, j_M')$ of $(j_{M+1}, \ldots, j_N)$ such that

$$
\tilde{f}_{j_N'} \cdots \tilde{f}_{j_2'} \tilde{f}_{j_1'} = (-, \ldots, -),
$$

the lowest weight element. Ignoring $j_k'$ such that $-$'s in $\tau$ is moved to the left by $\tilde{f}_{j_k}$ $5.2$, we obtain a subsequence $(j_1'', \ldots, j_{M''})$ of $(j_1', \ldots, j_M')$ such that

$$
\tilde{f}_{j_{M''}} \cdots \tilde{f}_{j_2''} \tilde{f}_{j_1''} = (+, \ldots, +) = (+, \ldots, +).
$$
This implies that there exists a unique \( \pi' \in \mathcal{T}' \) such that for \( M + 1 \leq k \leq N \)
\[
d_k(\pi') = \begin{cases} 
1, & \text{if } k = j_l'' \text{ for some } 1 \leq l \leq N'', \\
0, & \text{otherwise}. 
\end{cases}
\]
Hence we have \( ||c||_\pi \leq ||c||_{\pi'} \) by construction of \( \pi' \). \( \square \)

Recall that \( c = (c_k) \in \mathcal{B}' \) is by convention identified with the array, where \( c_k \) is placed at the position of \( \beta_k \) in \( \Delta_n \) for \( 1 \leq k \leq M \) (see Example 3.3).

We note that if we consider the array \( (j_k) \) for \( M + 1 \leq k \leq N \), where \( j_k \) is placed at the position of \( \beta_{N-k+1} \) in \( \Delta_n \), then the \( r \)-th row from the top is filled with \( r \) for \( 1 \leq r \leq n - 2 \) and the bottom row is filled with \( \ldots, n-1, n, n-1, n \) from right to left.

Let \( \mathcal{D} \) be the set of arrays, where either 0 or 1 is placed in each \( r \)-th row of \( \Delta_n \) from the top (\( 1 \leq r \leq n - 1 \)) satisfying the following conditions;

1. the three entries in the first two rows are 0,
2. the number of 1’s in each \( r \)-th row is \( r-2 \) for \( 3 \leq r \leq n-1 \),
3. if \( r > 3 \) (resp. \( r < n-1 \)) and there are two 1’s in the \( r \)-th row such that the entries in the same row between them are zero, then there is exactly one 1 in the \((r-1)\)-th row (resp. \((r+1)\)-th row) between them,
4. the \( j_k \)'s corresponding to 1’s in the \((n-1)\)-th row are \( n, n-1, n, \ldots \) from right to left.

We write \( d = (d_k) \in \mathcal{D} \), where \( d_k \) denotes the entry at the position of \( \beta_{N-k+1} \) in \( \Delta_n \) for \( M + 1 \leq k \leq N \).

**Example 5.6.** When \( n = 6 \), we have

\[
c = 
\begin{array}{cccc}
c_5 & c_4 & c_9 & c_8 \\
c_2 & c_7 & c_{11} & c_{12} \\
c_1 & c_6 & c_{10} & c_{13} & c_{14} & c_{15}
\end{array}
\quad d = 
\begin{array}{cccc}
d_{26} & d_{27} & d_{22} \\
d_{28} & d_{23} & d_{19} & d_{24} & d_{20} & d_{17}
\end{array}
\quad
\begin{array}{cccc}
j_{26} & 1 \\
j_{27} & j_{22} & 2 & 2 \\
j_{28} & j_{23} & j_{19} & 3 & 3 & 3 \\
j_{29} & j_{24} & j_{20} & j_{17} & 4 & 4 & 4 & 4 \\
j_{30} & j_{25} & j_{21} & j_{18} & j_{16} & 6 & 5 & 6 & 5 & 6
\end{array}
\]

For \( \pi \in \mathcal{T}' \), let \( d(\pi) \) denote the array \( (d_k(\pi)) \), where \( d_k(\pi) \) is placed in the position of \( \beta_{N-k+1} \) in \( \Delta_n \) for \( M + 1 \leq k \leq N \).
Lemma 5.7. The map sending $\pi$ to $d(\pi)$ is a bijection from $\mathcal{T}'$ to $\mathcal{D}$.

Proof. Let us assume that $n$ is even since the proof for odd $n$ is the same. We first show that the map is well-defined. Let $\pi = (\nu_0, \ldots, \nu_N) \in \mathcal{T}'$ given, where $\nu_M = \text{wt}(\nu_0, \ldots, \nu_N) \in \mathcal{T}'$ given, where $\nu_M = \text{wt}(-, -, +, \ldots, +)$. Let $(j'_1, \ldots, j'_L)$ be the subsequence of $(j_{M+1}, \ldots, j_N)$ such that $d_{j'_k}(\pi) = 1$. Then

$$\bar{f}_{j'_L} \cdots \bar{f}_{j'_2} \bar{f}_{j'_1}(-, -, +, \ldots, +) = (-, -, +, \ldots, -),$$

equivalently,

$$(5.7) \quad \bar{f}_{j'_L} \cdots \bar{f}_{j'_2} \bar{f}_{j'_1}(+, +, +, \ldots, +) = (+, +, +, \ldots, -).$$

From (5.5), (5.7) and the array $(j_k)$ on $\Delta_n$, one can check that (i) $L = (n-3)(n-2)/2$, (ii) $3 \leq j'_k \leq n$, (iii) the array $d(\pi)$ satisfies the conditions (1) and (2) for $\mathcal{D}$. To verify the condition (3), let us enumerate $\pi$'s appearing in (5.7) from left to right by $-1, -2, \ldots$.

For $3 \leq r \leq n - 1$, let $1_{(r-2,r)}, \ldots, 1_{(2,r)}, 1_{(1,r)}$ denote the entries 1 of $d(\pi)$ in the $r$-th row, which are enumerated from the right.

For $1 \leq k \leq L$, suppose that $j'_k$ corresponds to $1_{(s,r)}$ in $d(\pi)$ for some $s$ with $r = j'_k$. It is not difficult to see that $\bar{f}_{j'_k}$ in (5.7) corresponds to

(1) moving $-s$ at the $(r+1)$-th coordinate of a vector in $B(\pi_n)$ to the $r$-th one unless $r = n$ and $s$ is odd,

(2) placing $(-s, -s+1)$ at the last two coordinates if $r = n$ and $s$ is odd.

Then by looking at the arrangement of $d_k(\pi)$'s in $\Delta_n$, it follows that $1_{(s,r)}$ is located to the northeast of $1_{(s+1,r+1)}$ and to the northwest of $1_{(s,r+1)}$ for $r < n - 1$,

$$\ldots \quad 1_{(s,r)} \quad \ldots \quad 1_{(s+1,r+1)} \quad 1_{(s,r+1)}$$

and the $j'_k$'s corresponding to $\ldots, 1_{(3,n-1)}, 1_{(2,n-1)}, 1_{(1,n-1)}$ are $\ldots, n, n-1, n$. Hence $d(\pi)$ satisfies the condition (3) and (4) for $\mathcal{D}$, and the map $\pi \mapsto d(\pi)$ is well-defined.

Since the map is clearly injective, it remains to show that it is surjective. Let $\pi_0 \in \mathcal{T}'$ be a unique trail such that $d_k(\pi_0)_k = 1$ for $M + 1 \leq k \leq M + L$ and 0 otherwise.

We claim that for any $d \in \mathcal{D}$ there exists a sequence $d = d_0, d_1, \ldots, d_m = d(\pi_0)$ in $\mathcal{D}$ such that $d_{i+1}$ is obtained from $d_i$ by moving an entry 1 to the right. If $d \neq d(\pi_0)$, then choose a minimal $k$ such that $d_k = 0 \neq d_k(\pi_0)$ for $M + 1 \leq k \leq M + L$. If $1_{(s,r)}$ denotes an entry corresponding to $d_k(\pi_0)$ in $d(\pi)$, then there exists $1_{(s',r)}$ in $d$ such that $s < s'$. Here we assume that $s'$ is minimal. Then by the condition (3) for $\mathcal{D}$ and the minimality of $s'$, we can move $1_{(s',r)}$ to the right by one position if
$r < n - 1$ and by two positions if $r = n - 1$ to get another $d' \in \mathcal{D}$ by definition of $\mathcal{D}$. Repeating this step, we obtain a required sequence. This proves our claim.

Now, let $d = (d_k) \in \mathcal{D}$ be given and let $(j_1', \ldots, j_L')$ be the subsequence such that $d_{j_k'} = 1$. By the above claim and definition of $\mathcal{D}$, we obtain the following two reduced expressions:

$$s_{j'_L} \cdots s_{j'_2} s_{j'_1} = s_{j_{M+L}} \cdots s_{j_{M+2}} s_{j_{M+1}},$$

where we obtain the right-hand side from the left only by applying 2-term braid move. Since $\tilde{f}_{j_{M+L}} \cdots \tilde{f}_{j_{M+2}} \tilde{f}_{j_{M+1}} (+, +, +, \ldots, +) = (+, +, +, \ldots, +)$, we obtain (5.7), which implies that there exists $\pi \in \mathcal{T}$ such that $d(\pi) = d$. The proof completes.

Let $\mathcal{P}$ be the set of double paths at $\theta$. Consider two operations on $\mathcal{P}$ which change a part of $p \in \mathcal{P}$ in the following way:

(5.8) \quad \begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (4,0);
\end{tikzpicture} \quad \rightarrow \quad \begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (4,0);
\end{tikzpicture}

(5.9) \quad \begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (4,0);
\end{tikzpicture} \quad \rightarrow \quad \begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (4,0);
\end{tikzpicture}

where in (5.9) the rows denote the two rows from the bottom in $\Delta_n$.

**Lemma 5.8.** For $p \in \mathcal{P}$, let $d(p) = (d_k) \in \mathcal{D}$ be given by $d_k = 0$ if $p$ passes the position of $d_k$, and $d_k = 1$ otherwise. Then the map sending $p$ to $d(p)$ is a bijection from $\mathcal{P}$ to $\mathcal{D}$.

**Proof.** Let $p_0 \in \mathcal{P}$ be a unique double path at $\theta$ such that $p_0$ ends at the first two dots from the left in the bottom row of $\Delta_n$, that is, $\beta_1$ and $\beta_n$ (see the first double path in Example 3.10). It is clear that $d(p_0) = d(\pi_0) \in \mathcal{D}$, where $\pi_0 \in \mathcal{T}$ is given in the proof of Lemma 5.7.

Let $p \in \mathcal{P}$ given. Suppose that $p'$ is obtained from $p$ by applying either (5.8) or (5.9). If $d(p) \in \mathcal{D}$, then it is clear that $d(p') \in \mathcal{D}$. Since one can obtain $p$ from $p_0$ by applying (5.8) and (5.9) a finite number of times, we have $d(p) \in \mathcal{D}$. Hence the map $p \mapsto d(p)$ is well-defined and injective. The surjectivity follows from the fact that any $d \in \mathcal{D}$ can be obtained from $d(\pi_0)$ by moving an entry to the left by one or two depending on the row which it belongs to (see the proof of Lemma 5.7), which corresponds to (5.8) or (5.9). \qed
Hence the formula (5.3) gives (5.11).

\[
(5.13)
\]

where in this case (3.11). Then it is straightforward to verify (5.10) using Proposition 3.2(2).

\[
(5.12)
\]

Proof. Let \( B \) to the lowest weight element in \( w \) keep the notations in Section 5.2. Let \( j_0 \) be as in (5.4). For \( 1 \leq l \leq \left[ \frac{n}{2} \right] \), let \( k_l \) be the index such that \( j_{k_l} \) belongs to the subword \((i_{2l-1})^{\text{op}} \) of \( j_0 \) and \( j_{k_l} = n \). For \( i \in I \) and an element \( b \) of a crystal, let \( e_i^{\max} b = e_i^{(b)} b \). The following is crucial when proving Theorem 4.9.

**Proposition 5.9.** For \( c = (c_k) \in B^I \) and \( 1 \leq l \leq \left[ \frac{n}{2} \right] \),

\[
\lambda_{2l-1}(c) = e_{j_{k_l}} \left( e_i^{\max} \cdots e_{j_2}^{\max} e_{j_1}^{\max} c \right),
\]

and it is equal to

\[
\min_{\pi_1} \left\{ \sum_{k=1}^{M} d_k(\pi_1)c_k \right\} - \min_{\pi_2} \left\{ \sum_{k=1}^{M} d_k(\pi_2)c_k \right\},
\]

where \( \pi_1 \) and \( \pi_2 \) are \( i_0 \)-trails from

\[
\text{wt}(\underbrace{-,-,\cdots,-,\cdots,+}_{2l-2\text{-rows}}, \underbrace{+,\cdots,+,\cdots,+}_{n-2l+2\text{-rows}}) \quad \text{and} \quad \text{wt}(\underbrace{-,-,\cdots,-,\cdots,+}_{2l\text{-rows}}, \underbrace{+,\cdots,+,\cdots,+}_{n-2l\text{-rows}})
\]

to the lowest weight element in \( B(\varpi_n) \), respectively.

**Proof.** Let \( c \in B^I \) given. Since \((j_1, \ldots, j_M)\) is a reduced expression of the longest element for \( I \), \( e_i^{\max} \cdots e_{j_1}^{\max} c \) is an \( I \)-highest weight element, which is of the form (3.11). Then it is straightforward to verify (5.10) using Proposition 3.2(2).

On the other hand, the righthand side of (5.10) can be obtained by (5.3) letting

\[
i = j_0, \quad i' = i_0,
\]

where in this case

\[
s_{j_1} \cdots s_{j_{k_l}}(\varpi_{j_{k_l}}) = s_{j_1} \cdots s_{j_{k_l}}(\varpi_n) = \text{wt}(\underbrace{-,-,\cdots,-,\cdots,+}_{2l-2\text{-rows}}, \underbrace{+,\cdots,+,\cdots,+}_{n-2l+2\text{-rows}}),
\]

\[
s_{j_1} \cdots s_{j_{k_l}}(\varpi_{j_{k_l}}) = \text{wt}(\underbrace{-,-,\cdots,-,\cdots,+}_{2l\text{-rows}}, \underbrace{+,\cdots,+,\cdots,+}_{n-2l\text{-rows}}).
\]

Hence the formula (5.3) gives (5.11). \( \square \)

Let \( 1 \leq l \leq \left[ \frac{n}{2} \right] \) given. Let \( \mathcal{T}_l \) be the set of \( i_0 \)-trails from \( s_{j_1} \cdots s_{j_{k_l}}(\varpi_n) \) (5.13) to \( w_0 \varpi_n \). Let \( D_l \) be the set of arrays where either 0 or 1 is placed in each \( r \)-th row of \( \Delta_n \) from the top \( (1 \leq r \leq n - 1) \) satisfying the following conditions;

1. the entries in the first \( 2l \) rows are 0,
2. the number of 1’s in each \( r \)-th row is \( r - 2l \) for \( 2l + 1 \leq r \leq n - 1 \),
Suppose that 

\((3)\) if \(r > 2l + 1\) (resp. \(r < n - 1\)) and there are two 1’s in the \(r\)-th row such that the entries in the same row between them are zero, then there is exactly one 1 in the \((r - 1)\)-th row (resp. \((r + 1)\)-th row) between them, 

\((4)\) the \(j_k\)’s corresponding to 1’s in the \((n - 1)\)-th row are \(n, n - 1, n, \ldots\) from left to right.

Note that \(D_1 = D\).

**Lemma 5.10.** For \(\pi \in \mathcal{T}_l\), the map sending \(\pi\) to \(d(\pi)\) is a bijection from \(\mathcal{T}_l\) to \(D_l\), where \(d(\pi)\) is defined in the same way as in \(\mathcal{T}\).

**Proof.** It can be shown by almost the same arguments as in Lemma 5.7 that the map is well-defined, and clearly injective.

It suffices to show that it is surjective. Let \(\Delta'_{n-2l}\) be the set \(\Delta_{n-2l}\), which we regard as a subset of \(\Delta_n\) sharing the same southwest corner with \(\Delta_n\). Let \(\pi_0\) be a unique trail in \(\mathcal{T}_l\) such that \(d_k(\pi_0) = 1\) if and only if \(d_k(\pi_0)\) is located in \(\Delta'_{n-2l}\). Then as in the proof of Lemma 5.7 we can check that for any \(d \in D_l\) there exists a sequence \(d = d_0, d_1, \ldots, d_m = d(\pi_0)\) in \(D_l\) such that \(d_{i+1}\) is obtained from \(d_i\) by moving an entry 1 to the left, and hence that there exists \(\pi \in \mathcal{T}_l\) such that \(d(\pi) = d\).

Let \(\mathcal{P}_l\) be the set of \(l\)-tuple \(p = (p_1, \ldots, p_l)\) of mutually non-intersecting double paths in \(\Delta_n\) such that each \(p_i\) is a double path at some point in the \((2l - 1)\)-th row.

**Lemma 5.11.** The map sending \(p\) to \(d(p)\) is a surjective map from \(\mathcal{P}_l\) to \(D_l\), where \(d(p)\) is defined in the same way as in \(\mathcal{P}\).

**Proof.** Suppose that \(p = (p_1, \ldots, p_l) \in \mathcal{P}_l\) is given. By definition of \(\mathcal{P}_l\), one can check that all the points in the first \(2l\) rows in \(\Delta_n\) are occupied by \(p\).

Let \(p_0 = (p_0^1, \ldots, p_0^j)\) be given such that \(p_0^i\) starts at \(e_{2i-1} + e_n\) and ends at \(e_{2i-1} + e_{2i}\) and \(e_{2i} + e_{2i+1}\) for \(1 \leq i \leq r\). We have \(d(p_0) = d(\pi_0)\), where \(\pi_0\) is given in the proof of Lemma 5.10. Applying the operations (5.8) and (5.9) on \(\mathcal{P}_l\), one can obtain a sequence in \(\mathcal{P}_l\) from \(p\) to \(p_0\), whose image under \(d\) lies in \(D_l\). Then similar arguments as in Lemma 5.8 implies the surjectivity. \(\square\)

**Proof of Theorem 4.9** Let \(c \in B^J\) given. For \(\pi \in \mathcal{T}_l\), there exists \(p = (p_1, \ldots, p_l) \in \mathcal{P}_l\) such that \(d(p) = d(\pi)\) by Lemmas 5.10 and 5.11 and

\[(5.14) \quad \sum_{1 \leq k \leq M} d_k(\pi)c_k = \sum_{1 \leq k \leq M} c_k - (||c||p_1 + \cdots + ||c||p_l).\]

Indeed, (5.14) holds for any \(p \in \mathcal{P}_l\) such that \(d(p) = d(\pi)\). Therefore, we have by (5.11) and (5.14)

\[\lambda_{2l-1}(c)\]
\[
\begin{align*}
&= \min_{\pi \in \mathcal{P}_{t-1}} \left\{ \sum_{1 \leq k \leq M} d_k(\pi_1)c_k \right\} - \min_{\pi \in \mathcal{P}_t} \left\{ \sum_{1 \leq k \leq M} d_k(\pi_2)c_k \right\} \\
&= \left( \sum_{1 \leq k \leq M} c_k - \max_{\mathcal{P} \in \mathcal{P}_{t-1}} \{ ||c||_{p_1} + \cdots + ||c||_{p_{t-1}} \} \right) \\
&\quad - \left( \sum_{1 \leq k \leq M} c_k - \max_{\mathcal{P} \in \mathcal{P}_t} \{ ||c||_{p_1} + \cdots + ||c||_{p_t} \} \right) \\
&= \max_{\mathcal{P} \in \mathcal{P}_{t-1}} \{ ||c||_{p_1} + \cdots + ||c||_{p_{t-1}} \} - \max_{\mathcal{P} \in \mathcal{P}_t} \{ ||c||_{p_1} + \cdots + ||c||_{p_t} \}.
\end{align*}
\]

This gives the formula in Theorem 4.9. \hfill \Box

5.4. **Proof of Theorem 4.6** We keep the notations in Section 4.2 We assume that \( \Delta_i \subset \Delta_n \) for \( 1 \leq i \leq n \), where both of \( \Delta_i \) and \( \Delta_n \) share the same southeast corner. For \( c = (c_k) \in \mathbf{B}^J \), let \( c_{\Delta_i} \in \mathbf{B}^J \) (resp. \( c_{\Delta_i^c} \in \mathbf{B}^J \)) whose component in \( \Delta_i \) (resp. \( \Delta_i^c := \Delta_n \setminus \Delta_i \)) is \( c_k \) and 0 elsewhere. Let

\[
\mathbf{B}^J_{\Delta_i} = \{ c \in \mathbf{B}^J | c_{\Delta_i} = 0 \}.
\]

Fix \( i \in I \setminus \{n\} \). Let \( c \in \mathbf{B}^J_{\Delta_{i+1}} \) given with \( c = c(a, b) \) for a unique \( (a, b) = (a_1 \ldots a_r, b_1 \ldots b_s) \in \Omega \). We divide \( (a, b) \) into two biwords \( (a', b') = (a_1 \ldots a_s, b_1 \ldots b_s) \) and \( (a'', b'') = (a_{s+1} \ldots a_r, b_{s+1} \ldots b_r) \) where \( a_s \leq t \) and \( a_{s+1} > t \) so that

\[
\begin{pmatrix}
a' \\
b'
\end{pmatrix} =
\begin{pmatrix}
\frac{t+1}{t+1} & \frac{t+1}{t+1} & \frac{t}{t} & \frac{t}{t} & \frac{t-1}{t-1}
\\
\frac{1}{1} & \frac{1}{1} & \frac{2}{2} & \frac{2}{2} & \frac{s-1}{s-1}
\end{pmatrix}.
\]

Here the superscript means the multiplicity of each biletter.

Let \( c' = c(a', b') \) and \( c'' = c(a'', b'') \) be the corresponding Lusztig data. Put \( T = \kappa(c'') \). Then \( \kappa(T) = \mu^\pi \) for some \( \mu \in \mathcal{P}_{t-1} \) such that \( \mu' \) is even. We define \( \mu(c), \kappa(c) \) by

1. \( \mu(c) = (T \leftarrow b_s) \leftarrow \cdots \leftarrow b_1 \),
2. \( \kappa(c) = \mu(c) \in SST(\lambda/\mu)^\pi \), where \( \mu(c) \) is filled with \( a_k \) for \( 1 \leq k \leq s \).

Here \( \lambda = \mu(c) \in \mathcal{P}(\lambda/\mu)^\pi \), \( \mu_k = (T \leftarrow b_k) \leftarrow \cdots \leftarrow b_1 \), and \( \mu_0 = T \). The pair \( (\mu(c), \kappa(c)) \) can be viewed as a skew-analogue of RSK correspondence applying insertion of \( (a', b') \) into \( T \) (cf. [5] Proposition 1 in Section 5.1 and [30]).

**Lemma 5.12.** Under the above hypothesis, we have

\[
\kappa(f_i(c)) = f_i(c).
\]

**Proof.** Considering the action of \( \kappa(f_i) \) on the subcrystal \( \mathbf{B}^J_{\Delta_{i+1}} \) of \( \mathbf{B}^J \) described in Proposition 3.2, we may apply [16 Proposition 4.6 and Remark 4.8(1)] to have \( \kappa(f_i(c)) = f_i(c) \) (because we can naturally identify each element of \( \mathbf{B}^J_{\Delta_{i+1}} \) with an element of the crystal \( \mathcal{M} \) in [16 Section 3]). \hfill \Box
Let $\ell(\lambda) = 2m$ for some $m \geq 1$. For $1 \leq l \leq m$, let $V_l$ be the subtableau of $Q(c)$ lying in the $(2l-1)$-th and $2l$-th rows from the bottom, and let $U_l$ be the subtableau of $P(c)$ corresponding to $V_l$. Note that $U_l$ and $V_l$ are of anti-normal shapes, and $V_l^\wedge$ is the tableau of normal shape obtained from $V_l$ by jeu de taquin to the northwest corner. We also let $P(c)_l$ be the subtableau of $P(c)$ lying above the $(2l-2)$-th row from the bottom, where $P(c)_1 = P(c)$.

Now we glue each $V_l^\wedge$ to $P(c)$ to define a tableau $T(c)$ by the following inductive algorithm;

(g-1) Let $T(c)_m$ be the tableau obtained from $P(c)_m$ by gluing $V_m^\wedge$ to $U_m$ (so that $V_m^\wedge$ and $U_m$ form a two-rowed rectangle $W_m$).

(g-2) Consider a tableau obtained from $P(c)_{m-1}$ by gluing $V_{m-1}^\wedge$ to $U_{m-1}$ and replacing $P(c)_m$ by $T(c)_m$. If the number of columns in $W_m$ is greater than $\mu_{2m-3} - \mu_{2m-1}$ for $m \geq 2$, then we move dominoes down to the next two rows as many as the difference, and denote the resulting tableau by $T(c)_{m-1}$.

(g-3) Repeat (g-2) to have $T(c)_{m-2}, \ldots, T(c)_1$, and let $T(c) = T(c)_1$.

**Example 5.13.** Suppose that $n = 6$ and $i = 4$. Let $c \in B_{\Delta_2}^I$ be given by

$$
\begin{array}{ccc}
0 & 0 & 3 \\
0 & 1 & 1 \\
0 & 3 & 2 & 1 \\
0 & 3 & 2 & 0 & 3
\end{array}
$$

where $c'$ is given by the entries in bold letters. Then

$$
T = \kappa^\wedge(c') = \begin{array}{cccccccc}
3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
5 & 3 & 2 & 3 & 5 & 4 & 3 & 2
\end{array}, \quad \left( \begin{array}{c}
a' \\
b'
\end{array} \right) = \left( \begin{array}{cccccccc}
5 & 3 & 2 & 3 & 5 & 4 & 3 & 2 \\
1 & 2 & 3 & 4 & 1 & 2 & 4 & 2
\end{array} \right).
$$

By definition, the pair $(P(c), Q(c))$ is given by

$$
P(c) = \begin{array}{cccccccc}
3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
5 & 3 & 2 & 3 & 5 & 4 & 3 & 2
\end{array}, \quad Q(c) = \begin{array}{cccccccc}
5 & 3 & 2 & 3 & 5 & 4 & 3 & 2 \\
1 & 2 & 3 & 4 & 1 & 2 & 4 & 2
\end{array},
$$

where $U_1$ and $V_1$ (resp. $U_2$ and $V_2$) are given in blue (resp. in red). Since

$$
V_1^\wedge = \begin{array}{cccccccc}
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5
\end{array}, \quad V_2^\wedge = \begin{array}{cccccccc}
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5
\end{array}
$$

we have by algorithm (g-1)-(g-3),
Lemma 5.14. Under the above hypothesis, we have

1. \( T(c) = \kappa^\wedge(c) \),
2. \( \kappa^\wedge(\tilde{f}_i c) = \tilde{f}_i \kappa^\wedge(c) \).

Proof. (1) Let \( C_{i+1} = \sum_{1 \leq j \leq d} c_{i+1j} \). We use induction on \( C_{i+1} \). Note that when \( C_{i+1} = 0 \), we clearly have \( T(c) = \kappa^\wedge(c) \) by definition of \( T(c) \).

First, assume that \( C_{i+1} = 1 \). Then \( c_{i+1j} = 1 \) for some \( j \). Suppose that the box in \( P(c) \), which appears after insertion of the corresponding \( \tilde{j} \), belongs to \( U_l \) for some \( 1 \leq l \leq m \). Recall that \( \mu^\pi = \text{sh}(T) \). Let \( d = \mu_{2l-3} - \mu_{2l-1} \) and let \( u \) be the length of the bottom row of \( U_l \).

**Case 1.** Suppose that \( \ell(\text{sh}(U_l)) = 2 \) and \( d > u \). Then we have

\[
U_l = \begin{array}{cccc}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{array},
W_l = \begin{array}{cccc}
\begin{array}{cccc}
7 & 7 & \ldots & 7 \\
7 & 7 & \ldots & 7 \\
7 & 7 & \ldots & 7 \\
7 & 7 & \ldots & 7 \\
\end{array}
\end{array},
\]

where the gray box denotes the one created after the insertion of \( \tilde{j} \). In this case, the domino in the leftmost column of \( W_l \) does not move to lower rows. Hence it is clear that \( T(c) \) coincides with \( \kappa^\wedge(c) \).

**Case 2.** Suppose that \( \ell(\text{sh}(U_l)) = 2 \) and \( d = u \). Then we have

\[
U_l = \begin{array}{cccc}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{array},
W_l = \begin{array}{cccc}
\begin{array}{cccc}
7 & 7 & \ldots & \tilde{i} \\
7 & 7 & \ldots & \tilde{i} \\
7 & 7 & \ldots & \tilde{i} \\
7 & 7 & \ldots & \tilde{i} \\
\end{array}
\end{array},
\]

In this case, the leftmost domino in \( W_l \) moves down to a lower row by (g-2), and it is easy to see that \( T(c) = \kappa^\wedge(c) \).

**Case 3.** Finally suppose that \( \ell(\text{sh}(U_l)) = 1 \). Then

\[
U_l = \begin{array}{cccc}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{array},
W_l = \begin{array}{cccc}
\begin{array}{cccc}
7 & 7 & \ldots & 7 \\
7 & 7 & \ldots & 7 \\
7 & 7 & \ldots & 7 \\
7 & 7 & \ldots & 7 \\
\end{array}
\end{array},
\]

As in Case 1, the domino in the leftmost column of \( W_l \) does not move to lower rows, and hence \( T(c) = \kappa^\wedge(c) \).

Next, we assume that \( C_{i+1} > 1 \). Let \( (a', b') \) be the biword removing \( (a_1, b_1) \) in \( (a', b') \) in (5.13), and let \( c = c(a', b') \). Note that \( (a_1, b_1) = (i + 1, j) \) for some \( j \).

By induction hypothesis, we have \( T(c) = \kappa^\wedge(c) \). On the other hand, when we apply the insertion of \( (i + 1, j) \) into \( T(c) \), the possible cases are given similarly as above. Then it is straightforward to check that the tableau obtained by insertion of \( (i + 1, j) \) into \( T(c) \) (see the step (2) in the definition of \( P^\wedge(c) = P^\wedge(a, b) \) (4.3)) is equal to \( T(c) \). Therefore, we have \( T(c) = \kappa^\wedge(c) \). This completes the induction.
(2) By definition of $T(c)$ and Lemma \[5.12\] we have $T(\tilde{f}_i c) = \tilde{f}_i T(c)$. Then we have $\kappa^{-1}(\tilde{f}_i c) = \tilde{f}_i \kappa^{-1}(c)$ by (1).

\[\square\]

**Proof of Theorem 4.6.** It suffices prove that for $i \in I$ and $c \in B^J$

$$\kappa^{-1}(\tilde{f}_i c) = \tilde{f}_i \kappa^{-1}(c) \quad (\circ = \prec, \prec)$$

We prove only the case when $\circ = \prec$, since the proof for the other case is identical.

Suppose first that $i \in I \setminus \{n\}$. By Proposition \[3.2(2)\] ($\sigma_{k,3}(c)$ is trivial in this case), we have

$$c = c_{\Delta^c_{i+1}} \otimes c_{\Delta^c_{i+1}},$$

as elements of $gl_2$-crystals with respect to $\tilde{e}_i, \tilde{f}_i$.

Let us denote by $B$ the insertion of a biword into a tableau following the algorithm given in (4.3). If we ignore the entries smaller than $\kappa \prec$, the following commuting diagram;

$$\begin{align*}
\begin{array}{c}
c \\
\downarrow \\
\kappa^{-1}(c)
\end{array}
\quad \begin{array}{c}
\rightarrow (c_{\Delta^c_{n-1}} \cdot c_{\Delta^c_{n-1}}) \\
\rightarrow (c_{\Delta^c_{n-1}}, \kappa^{-1}(c_{\Delta^c_{n-1}}))
\end{array}
\quad \begin{array}{c}
\rightarrow (c_{\Delta^c_{n-1}}, \kappa^{-1}(c_{\Delta^c_{n-1}})) \\
\rightarrow (P(c), Q(c))
\end{array}
\end{align*}$$

**Case 1.** Suppose that $\tilde{f}_i c = c_{\Delta^c_{i+1}} \otimes \tilde{f}_i c_{\Delta^c_{i+1}}$. Then we have

$$\kappa^{-1}(\tilde{f}_i c) = \left(c_{\Delta^c_{i+1}} \rightarrow B \kappa^{-1}(\tilde{f}_i c_{\Delta^c_{i+1}})\right)$$

$$= \left(c_{\Delta^c_{i+1}} \rightarrow B \tilde{f}_i \kappa^{-1}(c_{\Delta^c_{i+1}})\right) \quad \text{by Lemma} \[5.14(2)\]$$

$$= \tilde{f}_i \left(c_{\Delta^c_{i+1}} \rightarrow B \kappa^{-1}(c_{\Delta^c_{i+1}})\right) \quad \text{by} \ [5.16]$$

$$= \tilde{f}_i \kappa^{-1}(c).$$

**Case 2.** Suppose that $\tilde{f}_i c = \tilde{f}_i c_{\Delta^c_{i+1}} \otimes c_{\Delta^c_{i+1}}$. By the same argument as in (5.17), we have $\kappa^{-1}(\tilde{f}_i c) = \tilde{f}_i \kappa^{-1}(c)$.

Next, suppose that $i = n$. We may identify $c$ with the pair $(c_{\Delta^c_{n-1}}, c_{\Delta^c_{n-1}})$. Regarding $c$ as an element of $B^J_{\Delta^c_{n+1}}$, the crystal of type $D_{n+1}$, we have by Lemma \[5.14(1)\] the following commuting diagram:
Now, we can apply the same argument for the proof of [16, Theorem 3.6] to see that the composition of (i) and (ii) commutes with $\tilde{f}_n$. Therefore, we have $\kappa^{\setminus^n}(\tilde{f}_n c) = \tilde{f}_n^{\setminus^n}(c)$. □

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**Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea**

*E-mail address: is_jang@snu.ac.kr*

**Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea**

*E-mail address: jaeheonkw@snu.ac.kr*