# Simple, Zero-Feedback, Distributed Beamforming With Unsynchronized Carriers

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

| Citation       | Bletsas, Aggelos, Andy Lippman, and John Sahalos. “Simple, zero-feedback, distributed beamforming with unsynchronized carriers.” IEEE Journal on Selected Areas in Communications 28 (2010): 1046-1054. © 2011 IEEE. |
|----------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| As Published   | http://dx.doi.org/10.1109/jsac.2010.100909                                                                                                                                                         |
| Publisher      | Institute of Electrical and Electronics Engineers                                                                                                                                                   |
| Version        | Final published version                                                                                                                                                                             |
| Accessed       | Mon Apr 01 06:37:20 EDT 2019                                                                                                                                                                          |
| Citable Link   | http://hdl.handle.net/1721.1/66133                                                                                                                                                                  |
| Terms of Use   | Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.                                           |
| Detailed Terms |                                                                                                                                                                                                |
Abstract—This work studies zero-feedback distributed beamforming: we are motivated by scenarios where the links between destination and all distributed transmitters are weak, so that no reliable communication in the form of pilot signals or feedback messages can be assumed. Furthermore, we make the problem even more challenging by assuming no specialized software/hardware for distributed carrier synchronization; we are motivated by ultra-low complexity transceivers. It is found that zero-feedback (i.e. blind), constructive, distributed signal alignment at the destination is possible; the proposed scheme exploits lack of carrier synchronization among $M$ distributed transmitters and provides beamforming gains. Possible applications include reachback communication in low-cost sensor networks with simple (i.e. conventional, no carrier frequency/phase adjustment capability) radio transceivers.

Index Terms—Beamforming, cooperative transmission, connectivity, wireless networks.

I. INTRODUCTION

Constructive addition (at the destination receiver) of signals transmitted from multiple antennas has been the central idea behind beamforming. Centralized beamforming from multi-antenna base station towards users (e.g. [1] and references therein) or group of users (e.g. [2]) has been shown to provide dramatic performance gains, since constructive addition of $M$ transmitted signals offers signal-to-noise ratio improvement on the order of $M^2$.

During the last decade, there has been an intensified interest on cooperative transmission from distributed antennas (e.g. [3]–[5] and references therein). Naturally, distributed beamforming emerges in the cooperative research forefront. The problem becomes more challenging compared to the centralized case, since now the $M$ distributed transmitters operate on different carriers which are not frequency- or phase-synchronized. In order to overcome the distributed carrier synchronization problem, the research community has focused on schemes that rely on some type of communication between the destination and the distributed transmitters, with varying communication requirements; from simple pilot signals for channel state information (CSI) (e.g. [6]) or single-bit feedback (e.g. [7]–[9]) to several-bit messages from destination to transmitters that assist the required carrier phase adjustments at the local oscillator system of each transmitter (e.g. [10]). We refer to any message passing from destination to distributed transmitters, either in the form of simple pilot signals or in the form of actual bit-messages as feedback.

In this work, we are particularly interested in zero-feedback beamforming; we are motivated by scenarios where the links between destination and all distributed transmitters are weak, so that no reliable feedback can be assumed and beamforming is required to provide connectivity between $M$ distributed transmitters and destination. Furthermore, we make the problem even more challenging by assuming no specialized software/hardware mechanism for distributed carrier synchronization; we are motivated by ultra-low complexity transceivers required in low-power and low-cost sensor networks. Additionally, we want to study the feasibility of beamforming with conventional radio transceivers employing no access to the local oscillator subsystem (e.g. phased-lock loop).

It is found that Zero-Feedback (i.e. Blind), Constructive, Distributed, signal Alignment at the destination is possible and could be employed in Emergency radio situations (ABCDE-FZ), even with simple (i.e. conventional, no carrier-phase adjustment capability) transceivers. The proposed scheme could simply facilitate reachback communication in sensor networks, allowing groups of nodes to fuse information outside the network, when the signal of each terminal alone is inadequate to reach the final destination (and thus, beamforming gains are required).

Section II provides the definitions, the problem formulation and the basic idea of this work: the lack of synchronization among distributed carriers can be exploited in favor of beamforming. Sections III and IV quantify signal alignment probability and respective alignment delay, Section V offers the numerical results and finally, Section VI provides the conclusion.

II. DEFINITIONS, BASIC IDEA AND PROBLEM FORMULATION

$M$ distributed terminals desire to transmit a common message $x[n]$ at a common channel of nominal carrier frequency $f_c$. No carrier frequency or carrier phase synchronization among the distributed transmitters is assumed. In line with all previous distributed beamforming research, it is assumed that the distributed transmitters employ a low-complexity packet/symbol synchronization algorithm. That could be practically implemented through a common pilot signal transmitted...
from one of the \( M \) transmitters, directing the initiation of symbol \( x[n] \) transmission. Such method requires no explicit time synchronization and can be easily shown to provide timing errors which are orders-of-magnitude smaller than the symbol duration. Experimental exploitation of such signal-directed sync in time-sensitive localization has been already reported in [11]. Alternatively, a low complexity, high precision time synchronization protocol could be used.\(^1\)

Denoting the complex channel gain from transmitter \( i \in \{1, 2, \ldots, M\} \) to destination as \( h_i \) (\( |h_i| = A_i \)) and symbol duration as \( T_s \), the received baseband signal at the destination can be expressed as [13]:

\[
y[n] = \sum_{i=1}^{M} h_i e^{(j2\pi \Delta f_i n T_s)} x[n] + w[n] \\
= x[n] \sum_{i=1}^{M} A_i \exp \{+j(2\pi \Delta f_i n T_s + \phi_i)\} + w[n] \\
= \tilde{x}[n] + w[n],
\]

(2)

where \( \Delta f_i, \phi_i \) are the carrier frequency offset from nominal carrier frequency \( f_c \) and phase offset respectively, for transmitter \( i \), and \( w[n] \) is the additive noise at the destination receiver, with average power per symbol \( \mathbb{E}\{w[n]w[n]^*\} = N_0 \). It is assumed that \( \mathbb{E}\{\Delta f_i\} = 0 \) and \( \mathbb{E}\{\Delta f_i^2\} = \sigma^2 \). The carrier frequency offset is due to manufacturing errors of the local oscillator crystal and varies slowly with time due to environmental conditions (e.g. temperature). The standard deviation is given by \( \sigma = \sqrt{\mathbb{E}\{\Delta f^2\}} = f_c \times \text{ppm} \), where ppm is the frequency skew of the clock crystals,\(^2\) with typical values of 1 – 20 parts per million (ppm). For example, clock crystals of 20 ppm provide for carrier frequency offsets on the order of \( 2.4 \text{ GHz} \times 20 \times 10^{-6} = 48 \text{ kHz} \).

The parameters \( \{A_i\}, \{\phi_i\} \) depend on the relative mobility between transmitter \( i \) and receiver and assume that remain constant for \( \tau_s \) symbols, corresponding to channel coherence time \( \tau_s T_s \). Therefore, the received signal power per symbol, for any \( n \in [1, \tau_s] \), can be calculated under the aforementioned assumptions as in (3) and (4),

\[
|\tilde{x}[n]|^2 = x[n]^2 \left( \sum_{k=1}^{M} A_k^2 + \sum_{k \neq m} A_k A_m \cos \left( \frac{2\pi \Delta f_k n T_s + \phi_k - 2\pi \Delta f_m n T_s - \phi_m}{\phi_k[n]} \right) \right)
\]

(3)

\[
x[n]^2 \left( \sum_{k=1}^{M} A_k^2 + \sum_{k \neq m} A_k A_m \cos \left( \phi_k[n] - \phi_m[n] \right) \right),
\]

(4)

where the second sum in Eq. (4) includes \( \binom{M}{2} \) terms, corresponding to all possible pairs among the \( M \) terminals. Assuming equal energy constellation and denoting \( P_T = \mathbb{E}\{|x[n]|^2\} \) the transmitted power per individual terminal \((MP_T \text{ is the total transmitted power by all terminals}), the signal-to-noise ratio (SNR) at the destination can be written as

\[
\text{SNR}[n] \triangleq \frac{P_T}{N_0} \frac{\mathbb{E}\{|\tilde{x}[n]|^2\}}{\mathbb{E}\{|w[n]|^2\}^2} \\
= \frac{P_T}{N_0} \left\{ \sum_{k=1}^{M} A_k^2 + 2 \sum_{k \neq m} A_k A_m \cos \left( \phi_k[n] - \phi_m[n] \right) \right\},
\]

(5)

\[
= \frac{P_T}{N_0} \left\{ \sum_{k=1}^{M} A_k^2 + 2 \sum_{k \neq m} A_k A_m \cos \left( 2\pi (\Delta f_k - \Delta f_m) n T_s + \phi_k - \phi_m \right) \right\},
\]

(6)

\[
= \frac{P_T}{N_0} \text{LBF}[n].
\]

(7)

According to the above expression, the cosines (in the beamforming factor \( \text{LBF}[n] \)) can become positive or negative, depending on the symbol \( n \), the phase offsets \( \{\phi_i\} \), as well as the distribution of the carrier frequency offsets \( \{\Delta f_i\} i \in \{1, 2, \ldots, M\} \). The latter are assumed independent and identically distributed according to a probability density function \( p_{\Delta f}(\Delta f) \) (with average value 0 and variance \( \sigma^2 \)).

Fig. 1 depicts the special case of two distributed transmitters \((M = 2)\) with carrier frequency offsets \( \Delta f_2 = 2 \Delta f_1 = f_0 \); their signals arrive at the destination with phase difference \( \pi/2 \) at time instant \( t = t_0 \). The two signals align at \( t = t_0 + 0.5/f_0 \), providing constructive addition at the destination (beamforming gain). This work studies the general-\( M \) signal alignment case (within angle \( \phi_0 \)) for any carrier frequency offset distribution \( p_{\Delta f}(\Delta f) \).

![Fig. 1](image_url)

**Fig. 1.** Two distributed transmitters \((M = 2)\) have carrier frequency offsets \( \Delta f_2 = 2 \Delta f_1 = f_0 \) and their signals arrive at the destination with phase difference \( \pi/2 \) at time instant \( t = t_0 \). The two signals align at \( t = t_0 + 0.5/f_0 \), providing constructive addition at the destination (beamforming gain). This work studies the general-\( M \) signal alignment case (within angle \( \phi_0 \)) for any carrier frequency offset distribution \( p_{\Delta f}(\Delta f) \).

\(^1\)C.E. [12] provides experimental validation of distributed synchronization based on "heartbeat" and entrainment in a low-cost sensor network testbed.

\(^2\)For time/frequency metrology, the interested reader could refer to [14] and references therein.
align constructively, without any type of feedback from the destination. In mathematical notation, the alignment event is defined as follows:

\[
\text{Align}[n, a, M] \triangleq \bigcap_{k \neq m} \left\{ \cos \left( \phi_k[n] - \phi_m[n] \right) \geq a \right\},
\]

\[
k \neq m, \forall k, m \in \{1, 2, \ldots, M\}
\]

\[
\Rightarrow L_{\text{BF}}[n] \geq \left( \sum_{k=1}^{M} A_k^2 + 2a \sum_{k \neq m} A_k A_m \right) = \mathcal{O} \left( M + 2a \left( \frac{M}{2} \right) \right) = \mathcal{O} \left( M + a(M - 1) \right),
\]

where \( \mathcal{O}(\cdot) \) is the mathematical symbol for order of magnitude. Notice that for perfect phase alignment \((a = 1)\), the beamforming factor above becomes \( \mathcal{O}(M^2) \), as mentioned in the introduction.

Furthermore, define the following indicator random variable:

\[
\beta_n[a, M] = \begin{cases} 1, & \text{with prob.} \ Pr \{ \text{Align}[n, a, M] \} \\ 0, & \text{with prob.} 1 - \Pr \{ \text{Align}[n, a, M] \} \end{cases}
\]

Then assume that the \( M \) distributed, carrier-unsynchronized transmitters repeatedly transmit the same information for \( N \leq \tau_c \) symbols. The random variable

\[
\beta(M) = \beta_1[a, M] + \beta_2[a, M] + \beta_3[a, M] + \ldots + \beta_{\tau_c}[a, M]
\]
denotes the number of symbols where the \( M \) signals align with beamforming factor \( L_{\text{BF}}[n] \) at least equal to

\[
L_{\text{BF}}[n] \geq \left( \sum_{k=1}^{M} A_k^2 + 2a \sum_{k \neq m} A_k A_m \right) \triangleq L_0(M).
\]

Therefore, the average number of symbols in \([1, N]\) with minimum beamforming factor \( L_0(M) \) becomes:

\[
\mathbb{E} \{ \beta(M) \} = \sum_{n=1}^{N \leq \tau_c} \Pr \{ \text{Align}[n, a, M] \}.
\]

The above can be used to estimate the alignment delay i.e. the amount of symbols that must be repeatedly transmitted, in order to guarantee one symbol on average, with minimum beamforming gain \( L_0(M) \). Equivalently, the ratio \( \mathbb{E} \{ \beta(M) \} / N \) provides the effective communication rate with minimum beamforming gain \( L_0(M) \) per information symbol. Such metric assumes that delay is inversely proportional to alignment probability and requires ergodicity, i.e. the variation of beamforming gains in time is the same as the ensemble distribution.

One could argue that the above idea is closely related to the concept of opportunistic beamforming for multi-antenna links [15], where phases of the transmitted signals are deliberately randomized; in sharp contrast, this work assumes no manipulation (in software or hardware) of the transmitted signals' phases. The later are assumed constant (within channel coherence time) but not necessarily known. Specific receiver architectures, coherent or not, are beyond the scope of this paper and will be examined in future work.

Analysis of \( \Pr \{ \text{Align}[n, a, M] \} \) follows for finite \( M \).

### III. Study of \( M \) Signal Alignment Probability

We denote \( \phi_0 = \cos^{-1}(a) \). Taking into account the fact that \( 0 < a \leq 1 \), we further restrict the value of \( \phi_0 \) in \([0, \pi/2]\) (even though \([2k\pi, 2k\pi + \pi/2]\) or \([2k\pi - \pi/2, 2k\pi]\) for any \( k \in \mathbb{Z} \) could be considered):

\[
0 \leq \phi_0 = \cos^{-1}(a) < \pi/2.
\]

Furthermore, we define the following \( M \) independent, non-identically distributed random variables in \([0, 2\pi]\):

\[
\tilde{\phi}_i(n) \overset{\Delta}{=} \phi_i(n) \mod 2\pi = (2nT_s \Delta f_i + \phi_i) \mod 2\pi, \quad \phi_i \in [0, 2\pi), \quad i \in \{1, 2, \ldots, M\}, \quad (14)
\]

where \( x \mod 2\pi \) denotes the modulo \( 2\pi \) operation. Assuming knowledge of the p.d.f. of \( \Delta f_i \), it is straightforward to find out the p.d.f. of \( \phi_i(n) \) [16] as:

\[
\begin{align*}
\bar{p}_{\tilde{\phi}_i}(\tilde{\phi}_i) &= \left( \sum_{k \in \mathbb{Z}} \bar{p}_{\phi_i}(\tilde{\phi}_i + k 2\pi) \right) \left( \frac{1}{2\pi nT_s} \right) \left( \frac{\phi_i + 2k\pi - \phi_i}{2\pi nT_s} \right), \quad \tilde{\phi}_i \in [0, 2\pi), \quad (15)
\end{align*}
\]

We have already assumed that \( \{\Delta f_i\}'s \) are i.i.d. with average value 0 and variance \( \sigma^2 \). The above can be further simplified to:

\[
\begin{align*}
\bar{p}_{\tilde{\phi}_i}(\tilde{\phi}_i) &= \frac{1}{2\pi nT_s} \sum_{k \in \mathbb{Z}} \rho_{\Delta f_i} \left( \frac{\phi_i + 2k\pi - \phi_i}{2\pi nT_s} \right), \quad \tilde{\phi}_i \in [0, 2\pi).
\end{align*}
\]

Appendix lemma 1 provides numerical calculation of the above p.d.f. for the special case of uniform or normal carrier offset distribution. We emphasize that \( \{\tilde{\phi}_i\}'s \) are independent but not identically distributed because of the different \( \{\phi_i\}'s \). The auxiliary variables \( \{\tilde{\phi}_i\}'s \) are limited in \([0, 2\pi]\), as opposed to the variables \( \{\phi_i\}'s \) which span \((-\infty, +\infty)\) and the alignment event at transmitted symbol \( n \) of Eq. (8) becomes:

\[
\text{Align}[n, a, M] \equiv \bigcap_{k \neq m} \left\{ \cos \left( \phi_k[n] - \phi_m[n] \right) \geq a \right\},
\]

\[
k \neq m, \forall k, m \in \{1, 2, \ldots, M\}, \quad \tilde{\phi}_i \in [0, 2\pi).
\]

The above states that all pairwise differences of the auxiliary angles should be less than a limit, which is determined by \( a \).

### A. Lower Bound

We denote set \( S_M = \{1, 2, \ldots, M\} \). We first calculate the lower bound of alignment (within angle \( \phi_0 \)) probability of \( M \) signals:

\[
\Pr \{ \text{Align}[n, a, M] \} \geq \max_{i \in S_M} \left\{ \tilde{\phi}_i \right\} \leq \min_{i \in S_M} \left\{ \phi_i \right\} + \phi_0.
\]

The event of the RHS probability in Eq. (18) guarantees the desired event of the LHS probability. However, there are cases when \( \max_{i \in S_M} \left\{ \tilde{\phi}_i \right\} > 2\pi - \phi_0 \) and \( \min_{i \in S_M} \left\{ \phi_i \right\} < \phi_0 \) (shaded area 2 in Fig. 2), where alignment can still occur and such cases are not captured by the RHS probability above. Fig. 2 and shaded area 2 describes the later event, while shaded
Fig. 2. Shaded areas 1 and 2 describe the M-signal alignment event with parameter $a = \cos(\phi_0)$ and $0 \leq \phi_0 < \pi/2$. Area 2 decreases with decreasing $\phi_0$.

area 1 describes the RHS event above. It can be seen that area 2 decreases with decreasing $\phi_0$, suggesting that the above lower bound is tight. Numerical results for moderate values of $\phi_0 (\pi/4$ or less) further validate that observation.

The joint pdf

$$p_{y,x} \left(y = \min_{i \in S_M} \{ \tilde{\phi}_i \}, \ x = \max_{i \in S_M} \{ \phi_i \} \right)$$

is calculated with the help of Appendix Theorem 1 as follows:

$$p_{y,x}(y,x) = \begin{cases} g_0(y,x), & y < x \\ 0, & \text{elsewhere,} \end{cases}$$ (19)

$$g_0(y,x) = \sum_{(k_1, k_2), k_1 \neq k_2} \left[ p_{\phi_{k_1}}(y) p_{\phi_{k_2}}(x) + p_{\phi_{k_1}}(x) p_{\phi_{k_2}}(y) \right] \times \prod_{k_3 \neq k_1, k_3 \neq k_2} \left( F_{\tilde{\phi}_{k_3}}(x) - F_{\tilde{\phi}_{k_3}}(y) \right),$$ (20)

where $F_{\tilde{\phi}_{k_3}}(x) = \int_0^x p_{\phi_{k_3}}(t) \, dt$ denotes the c.d.f. of $\tilde{\phi}_{k_3}$. The summation involves all $(\ell M) / 2$ pairs $(k_1, k_2)$, with $k_1, k_2 \in S_M$, $k_1 \neq k_2$, and the product involves all $k_3 \in S_M \setminus \{k_1\} \setminus \{k_2\}$.

Consequently, Eqs. (18), (20) provide the following:

$$\Pr \{ \text{Align}[n,a,M] \} \geq \int_{y=0}^{2\pi} \int_{x=y}^{\min(y+\phi_0,2\pi)} p_{y,x}(y,x) \, dx \, dy.$$ (21)

It can be seen from Eqs. (21), (20), (19) and (16), that only knowledge of the carrier frequency offset p.d.f. $p_{\Delta f}(x)$ is required in order to calculate the above bound. We note again that calculation examples for the special case of normal or uniform distribution is given through Appendix lemma 1.

B. Asymptotic M Analysis

It can be shown for the special case of normal or uniform carrier offset distribution, $M \to +\infty$ and alignment parameter $a = \cos(\phi_0)$ with $\phi_0 \in [0, \pi/2]$ that the alignment probability bound drops exponentially with $M$:

$$\Pr \{ \text{Align}[n,a,M \to +\infty] \} \geq O \left( \left( \frac{\phi_0}{4\pi} \right)^M \right).$$ (22)

where $O(\cdot)$ is the mathematical symbol for order of magnitude. Detailed derivation is omitted due to space constraints and will be reported elsewhere.

Even though the above has been shown for $M \to +\infty$, numerical results in section V demonstrate that alignment probability $\Pr \{ \text{Align}[n,a,M] \}$ drops exponentially with $M$, even for finite $M$.

IV. Extending to Subset Signal Alignment

Up to now, we have studied the alignment probability $\Pr \{ \text{Align}[n,a,M] \}$ of exactly $M$ signals with carrier phase offsets $\{\phi_i\}$ and random carrier frequency offsets $\{\Delta f_i\}$, $i \in S_\ell = \{1, 2, \ldots, M\}$. The minimum beamforming factor was calculated in Eqs. (7), (9). In order to simplify notation, we assume roughly equidistant distributed transmitters from the destination $A_1 \sim A_3$ the minimum beamforming factor of $M$ signal alignment is simplified to:

$$L_0(M) = A^2 \left[ M + 2 \left( \frac{M}{2} \right)^a \right]$$ (23)

and occurs at transmitted symbol $n$ with probability $\Pr \{ \text{Align}[n,a,M] \}$.

When $m = M - 1$ out of $M$ signals align, the minimum beamforming factor becomes:

$$L_0(M-1) =$$

$$= \left\{ MA^2 + 2a \left( \frac{M-1}{2} \right) A^2 - 2 \left[ \left( \frac{M}{2} \right) - \left( \frac{M-1}{2} \right) \right] A^2 \right\},$$

$$= A^2 \left\{ M + 2 \left[ (a+1) \left( \frac{M-1}{2} \right) - \left( \frac{M}{2} \right) \right] \right\}.$$ (24)

Similarly, the general case of $m \geq 2$ signal alignment out of $M$ provides minimum beamforming factor:

$$L_0(m) = \left\{ MA^2 + 2a \left( \frac{m}{2} \right) A^2 - 2 \left[ \left( \frac{M}{2} \right) - \left( \frac{m}{2} \right) \right] A^2 \right\},$$

$$= A^2 \left\{ M + 2 \left[ (a+1) \left( \frac{m}{2} \right) - \left( \frac{M}{2} \right) \right] \right\}.\] (25)

Notice that according to Eq. (25), $L_0(m) > MA^2$ when

$$\frac{1}{a+1} \left[ \left( \frac{M}{2} \right) \right] < \left( \frac{m}{2} \right) \Rightarrow$$

$$\frac{1}{a+1} + \frac{1}{2} \sqrt{1 + 4M(M - 1)/(a + 1)} < m \leq M.$$ (26)

Denote vector $\tilde{\phi}_m$ as the $m \times 1$ carrier phase offset vector of $\{\phi_i\}$’s after selecting $m$ out of total $M$, with $2 \leq m \leq M$. Next, denote $\Pr \{ \text{Align}[n,a,m,\tilde{\phi}_m] \}$ as the alignment probability of the specific $m$ signals $\{A_k e^{j\phi_k(n)}\}$ with phase offsets $\phi_k$ in $\tilde{\phi}_m$, as studied in Section III. Obviously, $\Pr \{ \text{Align}[n,a,M,\tilde{\phi}_M] \} = \Pr \{ \text{Align}[n,a,M] \}$.

Subset signal alignment of at least $m$ out of $M$ signals (with $m < M$) occurs at transmitted symbol $n$ with probability

$$\Pr \{ \text{Align}[n,a, \text{ at least } m] \} \leq$$

$$\leq \sum_{\tilde{\phi}_m} \Pr \{ \text{Align}[n,a,m,\tilde{\phi}_m] \},$$ (27)

1Nevertheless, alignment probability analysis does not depend on the wireless channel (amplitude or phase) specific and the minimum beamforming factor is $O(M+aM(M-1))$ for practical scenarios and various $\{A_k\}$’s, not necessarily the same.
according to the union bound. The summation above is performed over all \( \binom{M}{m} \) possible \( \{\varphi_m\} \).

Finally, the expected number of symbols with at least \( m < M \) aligned signals (and consequently, minimum beamforming factor \( L_0(m) \) per symbol) is given by:

\[
E\{\beta(\text{at least } m < M)\} = \sum_{n=1}^{N} \Pr\{\text{Align}[n, a, \text{ at least } m < M]\} \\
\leq \sum_{n=1}^{N} \sum_{\varphi_m} \Pr\{\text{Align}[n, a, m, \varphi_m]\}. \tag{28}
\]

V. NUMERICAL RESULTS

Carrier frequency \( f_c = 2.4 \) GHz is assumed with symbol duration \( T_s = 1 \) \( \mu \)sec (corresponding to 1 Mbps for binary modulation). The carrier frequency offset distribution \( p_{\Delta f}(\Delta f) \) is assumed normal or uniform and frequency skew of the clock crystals is assumed within typical values of 1 – 20 parts per million (ppm).

Fig. 3(a) provides the alignment probability for the case of \( M = 3 \) distributed transmitters, assuming normal carrier offset distribution \( p_{\Delta f}(\Delta f) \), alignment parameter \( a = \cos(\varphi_0) = \cos(\pi/4) \) and 20 ppm clock crystals. Alignment probability \( \Pr\{\text{Align}[n, a, M]\} \) is plotted as a function of symbol \( n \), up to \( N = 100 \) symbols. We have assumed that channel remains constant during \( N \)-symbol transmission, which implies that channel coherence time \( T_c > 100 \) \( \mu \)sec; equivalently, Doppler shift \( f_D \) is on the order of 10 kHz or less, corresponding to mobility speeds up to 1.25 km/sec, approximately. The latter means that the terminals (transmitters or destination) can be immobile or moving with a speed up to the above limit, covering a wide range of applications (e.g. wireless sensor nodes at ground and destination receiver on an airplane flying above).

Specifically, Fig. 3(a) provides the alignment probability from simulation, as well as the lower bound from analysis of Section III-A and simulation. Two cases \( \varphi_3\) (1) , \( \varphi_3\) (2) , for the phase vector \( \varphi_3 = [\varphi_1 \varphi_2 \varphi_3] \) are considered, chosen arbitrarily and assumed constant during \( N \)-symbol transmission, as explained above. The first immediate observation is that analysis results match simulation. The second observation is that the lower bound of alignment probability is indeed tight, as claimed in Section III-A, for moderate values of \( \varphi_0 \) (\( \cos(\varphi_0) = a \), \( 0 \leq \varphi_0 < \pi/2 \)). This will be further validated in Fig. 7, where various values of \( M \geq 3 \) are tested. The third observation is that alignment probability reaches a steady-state value which is independent of time and independent of the \( M = 3 \) distributed carrier phases \( [\varphi_1 \varphi_2 \varphi_3] \), i.e. \( \varphi_3\) (1) and \( \varphi_3\) (2) provide the same steady-state alignment probability. Similar observations are offered by Fig. 3(b), where uniform carrier offset distribution is utilized instead of normal.

Steady-state alignment probability independence from carrier phases \( \varphi_M \) can be explained as follows: each transmitted signal \( i \in \{1, 2, \ldots, M\} \) is viewed as a phasor that rotates the complex plane with angular frequency \( 2\pi\Delta f_i \) (Fig. 1). As soon as one of the \( M \) phasors completes one full rotation (or equivalently, time interval \( 1/\Delta f_i \) elapses), the starting point from where each phasor initiated its rotation (i.e. phase \( \phi_i \)) should not matter in terms of alignment probability. Intuitively, one could imagine runners in a circular stadium competing with different speeds. After a certain time interval, the probability all runners meet (align within a margin) does not depend on their starting points \( \{\phi_i\} \) but instead relies on their relative speeds \( \Delta f_i - \Delta f_j \), \( i \neq j \).

Keeping in mind the same intuitive picture, one could see that the steady-state alignment probability should not be affected by different values of clock frequency skew (ppm),
provided that carrier frequency offsets \( \{ \Delta f_i \} \) are identically distributed. That is due to the fact that alignment depends on relative angular frequencies and not on their absolute values; if all carrier frequency offsets adhere to the same distribution \( p_{\Delta f}(x) \), then time-independent (steady-state) alignment occurs independently of ppm. This observation is validated by Fig. 4, where normal carrier offset distribution is assumed, and steady-state alignment probability lower bound is plotted as a function of \( M \) (number of distributed transmitters) and two values of frequency skew. The latter provide the same steady-state result.

On the other hand, frequency skew affects time-dependent alignment probability and controls how quickly (in terms on number of symbols) alignment probability reaches steady-state. The higher the frequency skew, the faster the phasors rotate, the smaller the time needed to approach time-independent alignment probability (i.e. steady-state). This observation is highlighted in Fig. 5, where it is shown that \( M = 3 \) transmitters with 1 ppm crystals require approximately 130 transmitted symbols (= 130 \( \mu \)sec in our scenario) before steady-state (Fig. 5(a)), as opposed to 20 ppm crystals that require only \( \approx 15 \) symbols (Fig. 5(b)).

In other words, more accurate clocks (smaller frequency skew clock crystals) require increased time to reach steady-state. This affects the overall alignment delay, i.e. the time required before alignment occurs. Such delay can be estimated through Eq. (12), which calculates the expected number of transmitted symbols (out of total \( N \) transmitted symbols) that achieve signal alignment within parameter \( \alpha = \cos(\phi_0) \). Fig. 6 depicts the expected number of symbols (out of total \( N = T_c/ \tau_s = \tau_c = 100 \) transmitted symbols) for \( M = 3 \) distributed transmitters and various values of parameter \( \alpha = \cos(\phi_0) \) (or equivalently, angle \( \phi_0 \)). It is shown that for normal or uniform carrier offset distribution, oscillator crystals on the order of \( 1 - 20 \) ppm, and \( N = T_c/ \tau_s = \tau_c = 100 \) transmitted symbols, there are approximately 4 symbols with aligned signals of parameter \( \alpha = \cos(\pi/4) = \sqrt{2}/2 \). Specifically, 20 ppm crystals achieve \( \approx 4.9 \) symbols, while 1 ppm crystals achieve \( \approx 4 \) symbols out of 100, for normal carrier frequency offset distribution. At those 4 symbols, there is minimum beamforming gain factor \( L_{BF} \) on the order of \( L_{BF} = 3 + 2.3 \sqrt{2}/2 = 3(1+\sqrt{2}) \approx 8.6 \text{ dB} \) (Eq. (9)). In other words, the effective throughput becomes \( \approx 4/100 \times 1 \text{ Mbps} = 40 \text{ kbps} \) with minimum beamforming factor per information symbol equal to 8.6 dB. The above rate should be sufficient for emergency situation messages, while the increase in received signal-to-noise ratio provides connectivity between the group of \( M \) transmitters and destination (reachback communication).

It is noted that the ideal distributed beamformer would provide beamforming factor on the order of \( L_{BF} = 3^2 \rightarrow 9.5 \text{ dB} \). Thus, the proposed scheme is within less than a single dB from the ideal case, for the case of \( M = 3 \) distributed transmitters. It is also noted that if the proposed scheme is compared to the non-beamforming case of a single transmitter with \( 3P_T \) transmission power instead, the beamforming gain of the proposed work for \( M = 3 \) and \( \alpha = \sqrt{2}/2 \) becomes on the order of \( 8.6 - 10 \log_{10}(3) = 3.9 \text{ dB} \).

The above specific example shows that for every 25 transmitted symbols of the same information, there is 1 symbol on

---

4It is noted however that in practical scenarios, total transmission power per transmitter \( P \) is limited; thus, the above case of a single transmission with total power of \( MP \) is mentioned only for theoretical completeness.
average, with beamforming factor (gain) on the order of 8.6 dB (3.9 dB), compared to the non-collaborative transmission. Such noticeable gain stems from the absence of carrier synchronization among the $M = 3$ distributed transmitters and more importantly, requires no specialized RF front end and zero feedback from the destination. In short, beamforming factor (gain) of 8.6 dB (3.9 dB) is simply achieved by exploiting lack of carrier frequency and phase synchronization among the $M = 3$ distributed transmitters, and turning such lack of carrier sync from a disadvantage to an advantage.

Similar reasoning can be followed for the other depicted values of alignment parameter $a = \cos(\phi_0)$ and respective beamforming factor. For example, alignment parameter of $a = \cos(\pi/6)$ achieves minimum beamforming factor on the order of $L_{BF} = 9.1$ dB, at 2 symbols on average for every 100 transmitted symbols. In other words, increasing the beamforming factor from 8.6 dB to 9.1 dB (from $a = \cos(\pi/4)$ to $a = \cos(\pi/6)$) increases the alignment delay by approximately 50%, since now 50 symbols must be repeatedly transmitted in order to achieve alignment at one symbol on average (as opposed to 25 symbols for the case of $a = \cos(\pi/4)$). The last observation highlights the fundamental communication tradeoff between beamforming factor and number of symbols than need to be repeatedly transmitted to ensure signal alignment (i.e. alignment delay).

In principle, higher beamforming gains can be theoretically achieved for larger values of $M$, according to Eq. (9). Fig. 7 plots the alignment probability for normal carrier frequency offset distribution and varying number $M$ of transmitters. Again, it is shown that the lower bound of Section III-A is tight, while analysis matches simulation. It is also shown that increasing linearly the number $M$ of transmitters, drops the alignment probability exponentially. This is because the vertical axe is plotted in a logarithmic scale; exponential dependence on $M$ should decrease alignment probability linearly with $M$, as shown in Fig. 7. For the special case of $M = 4$, the steady state alignment probability becomes $\approx 7 \times 10^{-3}$, as opposed to $\approx 5 \times 10^{-2}$ for $M = 3$. Such finding implies that increasing the beamforming factor of Eq. (9) to $4 + 2 \left(\frac{\varphi_0}{2}\right)\sqrt{2}/2 \rightarrow 11$ dB from 8.6 dB ($M = 3$) requires one order of magnitude increase in terms of alignment delay, according to Eq. (12) (i.e. approximately 250 symbols need to be repeatedly transmitted in order to ensure alignment at a single symbol on average, with beamforming factor on the order of 11 dB; in our case such delay amounts to 250 $\mu$secs). Once again, the tradeoff between alignment delay (or equivalently, effective rate) and beamforming gain emerges.

In order to decrease the alignment delay the system designer should tradeoff beamforming gains. That could be practically achieved by requiring a subset of the $M$ signals to align and not all of them (Section IV). Table I provides the minimum beamforming factor $L_{BF}$ for $m = 3$ aligned signals out of totally $M = 4$ transmissions. It is shown that there are alignment parameter $a$ values that provide non-zero beamforming factor for the case of $(m, M) = (3, 4)$ as opposed to the case of $(m, M) = (3, 3)$. It is also shown that the alignment delay has been reduced from $100/4 = 25$ symbols ($(m, M) = (3, 3)$) to $100/15 \approx 7$ symbols ($(m, M) = (3, 4)$) for alignment parameter $a = \cos(\pi/4)$, at the cost of reduced beamforming factor. The reduction stems from the fact that 1 signal out of 4 is not guaranteed to be aligned. Condition of Eq. (26) ensures that if subset alignment is utilized, the non-aligned signals will not cause beamforming factor degradation.

![Fig. 6. Expected number of symbols (out of $\tau_0 = 100$) with $M = 3$ aligned signals within at most $\phi_0 (a = \cos(\phi_0))$, and $[\phi_1 \phi_2 \phi_3] = [6.19 \ 0.24 \ 1.77]$.](image1)

![Fig. 7. Alignment probability as a function of time and $M$, with normal carrier offset distribution (20 ppm crystals and $a = \cos(\phi_0) = \sqrt{2}/2$). Alignment probability drops exponentially with $M$.](image2)
is given by:
\[ p_{y,x}(y, x) = \begin{cases} 
  g_0(y, x), & y < x \\
  0, & \text{elsewhere}, 
\end{cases} \]
(29)

\[ g_0(y, x) = \sum_{(k_1, k_2), k_1 \neq k_2} \left[ p_{x_{k_1}}(y) p_{x_{k_2}}(x) + p_{x_{k_1}}(x) p_{x_{k_2}}(y) \right] \times \prod_{k_3 \neq k_1, k_3 \neq k_2} \left( F_{X_{k_3}}(x) - F_{X_{k_3}}(y) \right), \]
(30)

where the summation involves all \( \binom{M}{2} \) pairs \((k_1, k_2)\), with \( k_1 \neq k_2, k_1, k_2 \in S_M \) and the product involves all \( k_3 \in S_M \) excluding \( \{k_1\} \) and \( \{k_2\} \).

Proof:

\[ p_{y,x}(y, x) \ dy \ dx = \Pr\{Y_1 \in dy, Y_M \in dx\} = \Pr\{\text{one } X_i \in dy, \text{ one } X_j \in dx \text{ (with } y < x \text{ and } i \neq j \}
\]
and all the rest \( \in \{y, x\} \}

\[ = \sum_{(k_1,k_2), k_1 \neq k_2} \left\{ \left[ p_{x_{k_1}}(y) p_{x_{k_2}}(x) + p_{x_{k_1}}(x) p_{x_{k_2}}(y) \right] \right\} \]
\[ \times \prod_{k_3 \neq k_1, k_3 \neq k_2} \left[ F_{X_{k_3}}(x) - F_{X_{k_3}}(y) \right], \]
(31)

for \( y < x \).

The double sum \( p_{x_{k_1}}(y) p_{x_{k_2}}(x) \) above stems from the fact that even though there are exactly \( \binom{M}{2} \) pairs among the set of \( M \) \{X\}_i's, ordering among each pair matters. Simplifying the last line above concludes the proof.

Lemma 1: Assume zero-mean uniform or normal carrier offset distribution \( p_{\Delta f}(\Delta f) \) with \( \mathbb{E}\{\Delta f^2\} = \sigma^2 \). The p.d.f. of \( \{\tilde{\phi}_j\}_j\) can be numerically calculated by:

\[ p_{\tilde{\phi}_i}(\tilde{\phi}_i) = \frac{1}{2\pi n T_s} \sum_{k = -K_0}^{K_0} p_{\Delta f}\left( \frac{\tilde{\phi}_i + 2k \pi - \phi_i}{2\pi n T_s} \right), \]
(32)

where \( K_0 = \lfloor n T_s b + 1 \rfloor \), \( \lfloor x \rfloor \) is the floor function and \( b = \sqrt{3} \sigma \) or \( b = 3 \sigma \) for uniform or normal carrier offset distribution, respectively.

Proof: For zero-mean uniform distribution \( p_{\Delta f}(x) \) in \([-b, b]\), the standard deviation \( \sigma \) is expressed through \( b \): \( \mathbb{E}\{\Delta f^2\} = \sigma^2 = 4 b^2 / 12 \Rightarrow b = \sqrt{3} \sigma \). Given that \( p_{\Delta f}(x) \) is zero outside \([-b, b]\), the following holds:

\[ -b \leq \frac{\tilde{\phi}_i + 2k \pi - \phi_i}{2\pi n T_s} \leq b \]  
(33)

\[ -1 - n T_s b \leq \frac{\phi_i - \tilde{\phi}_i}{2\pi} - n T_s b \leq k \]
(34)

\[ k \leq n T_s b + \frac{\phi_i - \tilde{\phi}_i}{2\pi} \leq n T_s b + 1, \]
(35)

where we have exploited the definition of \( \phi \) \((\phi_i \in [0, 2\pi])\). Given that \( k \) is an integer, the above expression justifies the selected \( K_0 = \lfloor n T_s b + 1 \rfloor \).

For zero-mean normal distribution \( p_{\Delta f}(\Delta f) \), the justification is the same for \( b = 3 \sigma \). One just needs to remember that about 99.7% of values drawn from a normal distribution are within \( 3 \sigma \) from the mean.
REFERENCES

[1] G. Dimic and N. Sidiropoulos, "On downlink beamforming with greedy user selection: Performance analysis and a simple new algorithm," IEEE Trans. Signal Process., vol. 53, no. 10, pp. 3857–3868, Oct. 2005.

[2] E. Karipidis, N. Sidiropoulos, and Z.-Q. Luo, "Quality of service and max-min-fair transmit beamforming to multiple co-channel multicast groups," IEEE Trans. Signal Process., vol. 56, no. 3, pp. 1268–1279, Mar. 2008.

[3] J. N. Laneman, "Cooperative diversity in wireless networks: Algorithms and architectures," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, Sept. 2002.

[4] M. Dohler, "Virtual antenna arrays," Ph.D. dissertation, King’s College London, London, UK, 2003.

[5] A. Bletsas, "Intelligent antenna sharing in cooperative diversity wireless networks," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, Sept. 2005.

[6] R. Mudumbai, G. Barriac, and U. Madhow, "On the feasibility of distributed beamforming in wireless networks," IEEE Trans. Wireless Commun., vol. 6, no. 5, pp. 1754–1763, May 2007.

[7] R. Mudumbai, B. Wild, U. Madhow, and K. Ramchandran, "Distributed beamforming using 1 bit feedback: from concept to realization," in Proc. Allerton Conf. on Communication, Control and Computing, Sept. 2006, pp. 1020–1027.

[8] M. Seo, M. Rodwell, and U. Madhow, "A feedback-based distributed phased array technique and its application to 60-ghz wireless sensor network," in Proc. IEEE Int. Microwave Symposium, Atlanta, USA, June 2008.

[9] M. Johnson, K. Ramchandran, and M. Mitzenmacher, "Distributed beamforming with binary signaling," in Proc. IEEE Int. Symp. on Inform. Theory, 890–894, July 2008.

[10] Y. Tu and G. Pottie, "Coherent cooperative transmission from multiple adjacent antennas to a distant stationary antenna through awgn channel," in Proc. IEEE Vehicular Technology Conf. (Spring), 2002, pp. 130–134.

[11] M. Broxton, J. Lifton, and J. A. Paradiso, "Localization on the pushpin computing sensor network using spectral graph drawing and mesh relaxation," ACM Mobile Computing and Communications Review, vol. 10, no. 1, pp. 1–12, Jan. 2006.

[12] A. Bletsas and A. Lippman, "Spontaneous synchronization in multi-hop embedded sensor networks: Demonstration of a server-free approach," in 2nd IEEE European Workshop on Wireless Sensor Networks, Jan. 2005, pp. 333–341, Istanbul, Turkey.

[13] O. Besson and P. Stoica, "On parameter estimation of MIMO flat-fading channels with frequency offsets," IEEE Trans. Signal Process., vol. 51, no. 3, pp. 602–613, Mar. 2003.

[14] A. Bletsas, "Evaluation of kalman filtering for network time keeping," IEEE Trans. Ultrason., Ferroelec. Freq. Contr., vol. 52, no. 9, pp. 1452–1460, Sept. 2005.

[15] P. Viswanath, D. N. C. Tse, and R. Laroia, "Opportunistic beamforming using dumb antennas," IEEE Trans. Inform. Theory, vol. 48, no. 6, pp. 1277–1294, June 2002.

[16] B. B. van der Genugten, "The distribution of random variables reduced modulo a," Statistica Neerlandica, vol. 26, no. 1, pp. 1–13, 1972.