Comparison of two vortex models of wind turbines using a free vortex wake scheme

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Abstract. Developing suitably generalized models for rotor blade vortices that accurately predict their evolution continues to be a challenge for wind turbine analysts. During the past few decades, several vortex models have been developed according to the theoretical analysis and the experimental research. A comparison of two different vortex models is made for predicting wind turbine aerodynamic performance using a free vortex wake (FVW) model. The two models are the Lamb–Oseen vortex model for laminar vortices and the \( \beta \)-Vatistas model for turbulent vortices. A new formula that approximates parameter \( \beta \), which represents the degree of turbulence in the \( \beta \)-Vatistas model, is proposed. The formula of parameter \( \beta \) is validated by comparison of simulated and measured aerodynamic performances of wind turbines of different blade tip vortex Reynolds numbers. Then, the induced velocity streamlines and the distribution of the axial velocity in the rotational plane are simulated. Also, the differences due to the vortex models are discussed.

1. Introduction

Understanding the temporal development of wind turbine rotor blade tip vortices has been the subject of much research over several decades. The motivation is clear, in that a more complete understanding of the structure of the tip vortices is essential for accurately predicting the unsteady airloads on wind turbine blades and a deeper understanding of how vortices develop and the factors that influence their evolution should help analysts to devise better strategies to alleviate the adverse effects associated with vortex induced airloads. Some vortex models have been developed to represent the evolution of rotor tip vortices, such as induced velocity distribution and core growth.

For laminar vortices, Vatistas et al [1] proposed a family of tangential velocity profiles in 1991. Depending on the value of parameter \( n \), some of traditional vortices like Rankine’s (\( n \to \infty \)) and Scully’s (\( n = 1 \)) could be retrieved. When \( n \) is equal to two, a good approximation of the popular Lamb–Oseen vortex model [3] could also be attained. Also, researchers tried to developed models for turbulent vortices in the past several decades. Ramasamy and Leishman [4] developed a more comprehensive tip vortex model that simultaneously takes into account both Richardson number effects and the filament straining issues for helicopter calculations. As a result of the advantages offered by the laminar \( n \) family, a heuristic formula that approximated turbulent vortices [6], for only the \( n = 2 \), was reported by Vatistas. Recently, Vatistas et al [7] generalised the complete \( n \) set,
whereby both vortex types (laminar and turbulent) are consolidated into a single simple algebraic equation.

The effects of vortex model on the development of rotor tip vortices have been shown to be significant for certain problems when using free-vortex wake (FVW) models. Here, two vortex models, the Lamb–Oseen vortex model for laminar vortices and the $\beta$-Vatistas model for turbulent vortices, are applied into the blade tip vortex model of a FVW model, respectively. Then the differences in the wind turbine aerodynamic characteristics due to the vortex models are analysed.

2. FVW Model
The FVW model assumes that the flow field is incompressible and potential. The blade is modelled as a series of straight constant strength vortex segments lying along the blade quarter chord line. The control points are located at 3/4-chord at the center of each panel. The wake vortices extend downstream from the 1/4-chord forming a series of horseshoe filaments. The trailing and shed vortices are modelled by trailing and shed straight-line vortex filaments in Fig.1. The trailing filaments will roll up and form a single tip vortex filament in the far-wake. The trailing filaments cut off at a wake age angle of 60° in the near-wake. The strength of the tip vortex equals the global maximum bound vorticity over the span of the blade. The release point of the tip vortex is the tip of the blade.

![Figure 1. Schematic of the blade model](image)

The blade root section corresponds to the first blade element boundary. The remaining element boundary distribution along each blade is achieved using the following “arc-cosine” relationship

$$ \rho = \frac{r_i}{R} + \frac{2}{\pi} \arccos \left( 1 - \frac{i - \frac{1}{2}}{N_E} \right) \quad (1) $$

where $N_E$ is the number of blade elements and $N_E = 18$ in this study; $i$ the element boundary number ($i = 2, \cdots, N_E + 1$), Therefore, there are $N_E$ element control points and ($N_E + 1$) boundary points.

The strength, $\Gamma_b$, of each blade element is evaluated by application of the Kutta-Joukowski theorem on the basis of airfoil data. The bound vorticity for the $i$th blade element is

$$ (\Gamma_b) = \frac{1}{2} W_i c_i \quad (2) $$

where $W_i$ is the resultant velocity at the $i$th control point and $c_i$ is the chord of the $i$th blade element. To take into account the three-dimensional rotational effect, the 2D airfoil data is modified by the Du-Selig stall-delay model [8].

The vortex filaments, extending downstream from the blade, are allowed to freely distort under the influence of local velocity field. The convection of these vortex filaments can be described by the Helmholtz equation. To solve the convection equation of the vortex filaments numerically, the three-step and third-order predictor-corrector (D3PC) [9] scheme is used to approximate the derivatives. The difference approximation of wake governing equation can be written as
\[ \tilde{r}_{i,j} = \frac{1}{7} \left( -9 \tilde{r}_{i-1,j} + 12 \tilde{r}_{i-2,j} - 2 \tilde{r}_{i-3,j} + 3 \tilde{r}_{i,j-1} + 3 \tilde{r}_{i,j+1} \right) + \]

Predictor:

\[ \frac{6}{7} \Delta \psi \left[ V_0 + \frac{1}{4} \left( V_{ind(i,j)}^{n-1} + V_{ind(i-1,j)}^{n-1} + V_{ind(i,j-1)}^{n-1} + V_{ind(i,j+1)}^{n-1} \right) \right] \]

\[ r_{i,j} = \frac{1}{14} \left( 15 r_{i-1,j} - 9 r_{i-2,j} + 2 r_{i-3,j} + 3 r_{i,j-1} + 3 r_{i,j+1} \right) + \]

Corrector:

\[ \frac{3}{7} \frac{\Delta \psi}{\Omega} \left[ V_0 + \frac{1}{8} \left( V_{ind(i,j)}^{n-1} + V_{ind(i-1,j)}^{n-1} + V_{ind(i,j-1)}^{n-1} + V_{ind(i,j+1)}^{n-1} \right) \right] + \frac{1}{8} \left( \tilde{V}_{ind(i,j)} + \tilde{V}_{ind(i-1,j)} + \tilde{V}_{ind(i,j-1)} + \tilde{V}_{ind(i,j+1)} \right) \]

where \( r \) is the position vector of vortex collocation point; \( \tilde{r} \) is the intermediate solution of position vector of vortex collocation point obtained from the predictor step; \( \Omega \) is the rotational speed of rotor; \( \Delta \psi \) is the temporal step and \( \Delta \psi = 15^\circ \) in this study; \( V_{ind} \) is the induced velocity vector; \( \tilde{V}_{ind} \) is the induced velocity vector induced by the intermediate solution; the subscripts \( i \) and \( j \) indicate the temporal step and the spatial step respectively; and the superscript \( n-1 \) means the induced velocities are calculated via the old wake geometry of last iteration step.

The free wake method presented in this paper is a time marching method. In the steady case, when the rotor has rotated one revolution (\( i = 1 \), the root mean square (RMS) change between old and new wake geometries is obtained. Convergence is achieved when less than a prescribed tolerance of \( 10^{-4} \). In the unsteady case, the wake geometry of \( t = 0 \) is obtained via a steady calculation firstly, and then the time marches to next step, at which the unsteady inflow and the unsteady wake are considered. The predictor-corrector method is used again to update the wake geometry and the flow field at the new time step.

3. Vortex models

The vortex filaments comprise a series of straight-line vortex elements. The velocities induced by straight-line vortex elements at the control nodes of vortex elements are calculated using the Biot–Savart law as

\[ \vec{V} = \frac{\Gamma}{4 \pi h} \left( \cos \theta_A - \cos \theta_B \right) \frac{\vec{r}_A \times \vec{r}_B}{\vec{r}_A \times \vec{r}_B} \]

where \( \Gamma \) is the vortex circulation; \( \theta_A, \theta_B, \vec{r}_A, \) and \( \vec{r}_B \) are shown in Fig.2. However, when a collocation point is very close to the vortex-line segment (\( h \to 0 \)), a very high induced velocity will be obtained. In addition, the self-induced velocity (\( h = 0 \)) has a logarithmic singularity. These two phenomena will cause convergence problems. To avoid the numerical problems, the vortex core model similar to some semi-empirical vortex models should be used. In this section, two semi-empirical vortex models, the Lamb-Oseen model and the \( \beta \)-Vatistas model, and how to use are introduced.

![Figure 2. Schematic of the straight-line vortex elements](image-url)
3.1. Lamb-Oseen model

To obtain a solution for the velocity field in the vortex, it is necessary to assume a velocity profile, \( v_\theta \). Twenty-five years ago, a family of laminar vortices was proposed by Vatistas et al. [1]. One series of general velocity profiles is given as

\[
\frac{\Gamma h}{2\pi \left( r_c^{2n} + h^{2n} \right)^{\frac{1}{n}}} 
\]

where \( r_c \) is the vortex core radius, and \( n \) is an integer variable. The most widely used member of the set is the \( n=2 \) which is called Lamb–Oseen vortex model. The velocity profile can be expressed as

\[
\frac{\Gamma h}{2\pi \sqrt{r_c^4 + h^4}} 
\]

Similar to the Lamb–Oseen vortex model, the parameter \( h \) of Eq. (5) can be replaced by

\[
\frac{\sqrt{r_c^4 + h^4}}{h} 
\]

3.2. \( \beta \)-Vatistas model

The experimental investigation of the turbulent helicopter blade vortices [4] revealed that, at the blade tip where the Richardson number falls below a critical value, the laminar flow could become turbulent. If it materializes, then after the vortex core, the tangential velocity profile is seen to lift up from the laminar velocity distribution. Vatistas et al. [7] proposed a generalized formula, whereby both vortex types (laminar and turbulent) are consolidated into a single simple algebraic equation. The velocity profile can be expressed as

\[
\frac{\Gamma h}{2\pi r_c \left( 1 + \frac{\beta}{1 + \beta \left( \frac{h}{r_c} \right)^{2n}} \right)^{\frac{1}{2n}}} 
\]

where \( \beta \) is the parameter which represents the degree of turbulence and is related to vortex Reynolds number (Re\( _v \)). For a wind turbine rotor, the tip Re\( _v \) is given approximately by the result

\[
\text{Re}_v = \frac{\Gamma}{2\pi \nu} 
\]

where \( \nu \) is molecular kinematic viscosity. The integer variable \( n = 2 \) and the parameter \( h \) of Eq. (5) can be replaced by

\[
\frac{r_c^2}{h} \left( 1 + \frac{\beta (h/r_c) \nu}{1 + \beta} \right)^{\frac{1}{2n}} 
\]

Eq. (9) can reproduce both the laminar (\( \beta = 1 \)) and the turbulent (\( \beta > 1 \)) types of the vortex models. The critical value of Re\( _v \) is about \( 10^3 \). The cumulative outcome of \( \beta \) to Re\( _v \) is shown in Fig. 3. A formula that approximates parameter \( \beta \) can be expressed as

\[
\beta = \begin{cases} 1 & \text{Re}_v \leq 10^3 \\ 0.57 \exp \left( 0.18 \log_{10} \left( \text{Re}_v \right) \right) & \text{Re}_v > 10^3 \end{cases} 
\]

3.3. Effects of viscous diffusion and stretching
To account for the effect of viscous diffusion, the core radius growth is used in the Biot-Savart law, and modified by an empirical viscous growth model [10]. The core radius model is given by

\[ r_c(\zeta) = \sqrt{r_0^2 + \frac{4\alpha_1 \delta \zeta}{\Omega}} \]  

(13)

where \( r_0 \) equals to 5% of the rotor radius; \( \alpha_1 = 1.25643 \) is a Lamb’s constant; \( \delta \) is a constant which is of the order of \( 10^4 \); and \( \zeta \) is the wake age angle.

The vortical wake behind a wind turbine expands, which stretches the vortex filaments, so it is important to consider the effects of stretching of the vortex filaments for wind turbine application. In the present study, stretching effects are taken into account by an application of a model developed by Ananthan and Leishman [11]. Assuming a change in the length because of filament straining to be \( \varepsilon = \Delta l/l \), which occurs over a time step, the effective core radius at wake age angle \( \zeta \) can be written as

\[ r'_c = r_c \frac{1}{\sqrt{1 + \varepsilon}} \]  

(14)

4. Results and discussion

First, the ability of Eq. (9) to approximate laminar and turbulent vortices will be assessed. The swirl velocity distributions of two vortex models are shown in Fig. 4. Different \( \beta \) are considered to represent the degree of turbulence. The Lamb–Oseen model predicts one type swirl velocity distribution independent of \( \text{Re}_r \) (because of the inherent laminar flow assumption). When \( \beta = 1 \) (laminar type), the \( \beta \)-Vatistas model gives a same results with the Lamb–Oseen model. Also, the velocity profile changes with increasing \( \text{Re}_r \) (increasing \( \beta \)), and becomes closer to a more fully turbulent profile at high \( \text{Re}_r \).

![Figure 4. Swirl velocity distribution predicted by various vortex model](image)

To validate and compare the vortex models, NREL Phase VI wind turbine is used here as a testing example. NREL Phase VI is a stall-regulated turbine. This turbine was designed by the National Renewable Energy Laboratory (NREL). The experiments were performed in the NASA Ames wind tunnel (24.4 × 36.6 m) [12] and is considered a benchmark for the evaluation of wind turbine aerodynamic methods.

4.1. Wake convergence

Fig. 5 shows the time histories of the RMS of the error in the wake geometry of 5m/s, 6m/s, 7m/s, 10m/s, 15m/s, 20m/s using two vortex models. The range of \( \beta \) is between 1.27 and 1.49. Note that in 5m/s and 6m/s cases the Lamb–Oseen model shows a converging trend, but there is still an accumulation of numerical errors and the wake geometries do not stabilize. However, the \( \beta \)-Vatistas model is found to produce a stabler and more quickly convergent wake system than the Lamb–Oseen model for all conditions.
4.2. Low speed shaft torque

The aerodynamic load of the low speed shaft torque is predicted by the FVW model. Fig. 6 shows these aerodynamic estimates compared with the value measured directly at the shaft. Generally speaking, the calculated results from two vortex models compare well with the measured data at most of the wind speeds. Below 7m/s, the results from the Lamb–Oseen model are not good because of the wake convergence issue described in section 4.1. In the low wind speeds and the middle wind speeds, the $\beta$-Vatistas model can accurately predict the low speed shaft torque. Above 16m/s, the estimated values from two models begin to differ obviously from the values measured. This indicates that the low speed shaft torque is indeed difficult to predict, especially in the presence of stall.
4.3. Induced velocity streamlines

The simulation of the induced velocity streamlines of the NREL wind turbine blade in the plane cut through the rotating axis is made using the FVW model, in which the vortex model can be changed. Fig. 7(a) and (b) show the induced velocity streamlines from the Lamb-Oseen model and the \( \beta \)-Vatistas model respectively. Fig. 7(c) is the experiment result [13]. The tip speed ratio (TSR) is 4.91. The axis simulated region is from \(-0.2R\) to \(1.4R\). The tip vortex structure is displayed distinctly through the simulated region and the tip vortex remains strong when moving out of the right boundary of the simulated region. It can be seen the axis induced velocities behind the rotor are negative and the radial induced velocities are positive, resulting in the wake expanding. The positions of tip vortex and streamlines geometry of the calculated results are consistent with the experiment result. It is also illustrated that the dissipation of the tip vortex from the \( \beta \)-Vatistas model is slower than that of the tip vortex from Lamb-Oseen model, because of consideration of the degree of turbulence in the \( \beta \)-Vatistas model.

4.4. Axial velocity distribution

The velocity field in the rotational plane can be effected by the bound vortex, the trailing vortex, and shed vortex. Fig. 8 shows contours of the axial dimensionless velocity in the rotor rotational plane from both the Lamb-Oseen model and the \( \beta \)-Vatistas model at the TSR of 4.91. It is clear that the velocity gradient is great near the positions of the blade bound vortex and the tip vortex. It is
interesting that the change of the velocity from the \( \beta \)-Vatistas model is slower than that from the Lamb-Oseen model, which is consistent with the results of swirl velocity distribution described above.

![Figure 8. Contours of the axial dimensionless velocity in the rotor rotational plane](image)

Fig. 9 presents the downstream development of the axial velocity at the center line of the wake obtained by the Lamb-Oseen model and the \( \beta \)-Vatistas model at the TSR of 4.91. It is clear that the axial velocity slows down after the rotor and then resumes to the atmospheric velocity gradually. Both the results show that the axial velocity at the rotor is about the arithmetic mean of the wind velocity and minimum axial velocity of the wake. The change regulations described above are consistent with the classical momentum theory. In the wake region (from 0 to 2\( D \)), the axial velocity obtained by the \( \beta \)-Vatistas model is smaller than that from the Lamb-Oseen model. This illustrates that the induced velocity from the \( \beta \)-Vatistas model is larger than that from the Lamb-Oseen model, which is also consistent with the results of swirl velocity distribution described above.

![Figure 9. Variation of the axial velocity along the streamwise direction](image)

5. Conclusions

The aerodynamic simulations effected by two vortex models, the Lamb–Oseen vortex model for laminar vortices and the \( \beta \)-Vatistas model for turbulent vortices, are compared using a FVW model. In the low wind speeds, the wake convergence of the Lamb–Oseen vortex model is worse than the \( \beta \)-Vatistas model, which causes a smaller prediction of low speed shaft torque. In the flow field near the vortex filament, the velocity change from the turbulent vortex model is slower than that from the laminar vortex model.

The \( \beta \)-Vatistas model gives a flexible simple algebraic equation, which can represent both of the two vortex types (laminar and turbulent). It can truly reflect the effect of the degree of turbulence via
the parameter $\beta$. The proposed expression of the $\beta$ is valid. However, some more theoretical and experimental works are needed to do in order to further improve the robustness and accuracy of the $\beta$-Vatistas model in the future.

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References

[1] Vatistas GH, Kozel V, and Minh W, A Simpler Model for Concentrated Vortices, Experiments in Fluids, 1991, 11(1): 73–76.
[2] Scully MP, Computation of Helicopter Rotor Wake Geometry and Its Influence on Rotor Harmonic Airloads, Aeroelastic and Structures Research Lab., Massachusetts Inst. of Technology TR ASRL TR-178-1, Cambridge, MA, March 1975.
[3] Lamb H, Hydrodynamics, Cambridge University Press, New York, 1932.
[4] Ramasamy M, and Leishman G, A Generalized Model for Transitional Blade Tip Vortices, Journal of the American Helicopter Society, 2006, 51(1): 92–103.
[5] Ramasamy M, and Leishman G, A Reynolds Number-Based Blade Tip Vortex Model, Journal of the American Helicopter Society, 2007, 52(3): 214–223.
[6] Vatistas GH, Simple Model for Turbulent Tip Vortices, Journal of Aircraft, 2006, 43(5): 1577–1579.
[7] Vatistas GH, Panagiotakakos GD, and Manikis FI, Extension of the n-Vortex Model to Approximate the Effects of Turbulence. Journal of Aircraft, 2015, 52(5): 1721-1725.
[8] Du Z, Selig MS, A 3-D stall-delay model for horizontal axis wind turbine performance prediction. Technical Report. AIAA-98–0021. Americas Institute of Aeronautics and Astronautics (AIAA), Reston, VA, USA, 1998.
[9] Xu BF, Wang TG, Yuan Y, Cao JF. Unsteady aerodynamic analysis for offshore floating wind turbines under different wind conditions. Phil. Trans. R. Soc. A, 2015, 373(2035): 20140080.
[10] Bhagwat MJ, Leishman JG, Correlation of helicopter tip vortex measurements. AIAA Journal, 2000, 38: 301–308.
[11] Ananthan S, Leishman JG. The role of filament stretching in the free-vortex modeling of rotor wakes. Journal of the American Helicopter Society, 2004, 49: 176–191.
[12] Hand MM, Simms DA, Fingersh LJ, et al, Unsteady aerodynamics experiment phase vi: wind tunnel test configurations and available data campaign. NREL/TP-500-29955, 2001.
[13] Xiao JP, Wu J, Chen L, et al, Particle image velocimetry (PIV) measurements of tip vortex wake structure of wind turbine, Applied Mathematics and Mechanics, 2011, 32(6): 729-738.