Some non-finitely presented Lie Algebras

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Abstract

Let \( L \) be a free Lie algebra over a field \( k \), \( I \) a non-trivial proper ideal of \( L \), \( n > 1 \) an integer. The multiplicator \( H_2(L/I^n, k) \) of \( L/I^n \) is not finitely generated, and so in particular, \( L/I^n \) is not finitely presented, even when \( L/I \) is finite dimensional.

1 Introduction

If \( R \) is a free associative algebra, over a field, and \( I \) is a two sided ideal of \( R \), then Lewin proved \cite{5} that \( I^2 \) is not finitely generated (as a 2-sided ideal!) when the algebra \( R/I \) is infinite dimensional. In other words, \( R/I^2 \) is not finitely presented in this case. On the other hand, it is easy to see that when \( R \) is finitely generated and \( R/I \) is finite dimensional, so is \( R/I^2 \), and hence \( I^2 \) is finitely generated.

Similar behavior is seen in groups. If \( F \) is a finitely generated free group, and \( R \) is a normal subgroup then \( R' \) is normally finitely generated if, and only if, \( F/R \) is finite. In fact Baumslag, Strebel and Thomson proved \cite{1} a stronger fact. Denoting the \( m \)-th member of the lower central series by \( \gamma_m \), they proved that for \( m > 1 \) the Schur multiplier of \( F/\gamma_m R \), \( H_2(F/\gamma_m R, \mathbb{Z}) \), is not finitely generated (as an abelian group) if \( F/R \) is not finite.

We note that for the three statements

(a) \( R \) is normally finitely generated,
(b) $R/R'$ is finitely generated as a module over $G = F/R$,
(c) $H_2(G, \mathbb{Z})$ is finitely generated as an abelian group
we have $(a) \Rightarrow (b) \Rightarrow (c)$.

In this paper we prove a result of similar nature for Lie algebras.

**Theorem. 1.1** Let $L$ be a free Lie algebra with basis $X$, over a field $k$, and $I$ be any non-zero proper ideal of $L$, then $I' = [I, I]$ is not finitely generated as an ideal. In fact, the “Schur multiplier” of $L/I^n$, $H_2(L/I^n, k)$, is not finitely generated if $n > 1$, and hence $L/I^n$ is not finitely presented.

Here $I^n$ denotes $I$, if $n = 1$, and $[I^{n-1}, I]$ if $n > 1$. Our proof closely follows the lines of [1].

In §2 we define some notations and the Magnus embedding. In §3 we build a mapping from the Schur multiplier into a tensor product of $n - 1$ copies of $U(L/I)$. This is similar to the mapping defined in [1]. In §4 we build a specific isomorphism of Hopf modules, keeping in mind that the enveloping algebra of a Lie algebra is a Hopf algebra. In §5 we employ the mapping and show that the image of the “Schur multiplicator” is not finite dimensional, thus proving the theorem.

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**2 Preliminaries and Notations**

Let $\mathcal{G}$ be a Lie algebra. We will denote the Lie multiplication of two elements $a, b \in \mathcal{G}$ by $[a, b]$. As we will also be considering the enveloping algebra of $\mathcal{G}$, the multiplication in $U(\mathcal{G})$ will be denoted simply as $ab$, while the action of an element $l \in U(\mathcal{G})$ on an element $a \in \mathcal{G}$ will be denoted by $a \cdot l$. Note that the action is the adjoint action, so that if $l \in L$ then $a \cdot l = [a, l]$.

Let $\mathcal{G}$ be a Lie algebra over a field $k$, $U(\mathcal{G})$ its enveloping algebra, $\delta U(\mathcal{G})$ the augmentation ideal of $U$. Suppose $0 \rightarrow I \rightarrow L \rightarrow \mathcal{G} \rightarrow 0$ is a free presentation of $\mathcal{G}$, where $L$ is the free Lie algebra with basis $X$. The enveloping algebra, $U(L)$, is therefore a free associative algebra, with basis $X$, and $\delta U(L)$ is a free $U(L)$ module, with a basis in one-to-one correspondence with $X$. Note that over a field, if $\mathcal{G} \neq 0$, $U(\mathcal{G})$ is infinite dimensional, and is without zero divisors.

In addition, if $\mathcal{G}$ is a Lie algebra over a field and $U(\mathcal{G})$ is its enveloping algebra, let $U_n(\mathcal{G})$ be the subspace of $U(\mathcal{G})$ spanned by all
the products of at most \( n \) factors from \( \mathcal{G} \). This gives a well known ascending filtration of \( U(\mathcal{G}) \), and we can define the degree of an element \( l \) to be the least integer \( n \) such that \( l \in U_n(\mathcal{G}) \). This function has the properties:

1) \( \deg(a + b) \leq max\{\deg(a), \deg(b)\} \),
2) if \( \deg(a) < \deg(b) \) then \( \deg(a + b) = \deg(b) \),
3) \( \deg(ab) = \deg(a) + \deg(b) \).

In particular, if \( x \in \mathcal{G} \) is non-zero then the degree of \( x \) is 1, so if \( x_1, x_2, \ldots, x_n \in \mathcal{G} \) are all non-zero then \( \deg(x_1 x_2 \cdots x_n) = n \).

Via the adjoint action, \( I/I' \) carries the structure of a \( U(L) \) module, and \( I \) acts trivially. All modules will be right modules. Therefore \( I/I' \) is a \( U(L/I) \) module in a natural way. There is a well known embedding of \( U(L/I) \) modules, the Magnus embedding, described below, of \( I/I' \) into \( \delta U(L) \otimes_{U(L)} U(L/I) \). This embedding will be denoted by \( \phi: I/I' \rightarrow \delta U(L) \otimes_{U(L)} U(L/I) \). The action of \( L \) on \( \delta U(L) \otimes_{U(L)} U(L/I) \) is by right multiplication in the right hand term.

The embedding can be defined in the following way. First define \( \phi : I \rightarrow \delta U(L) \otimes_{U(L)} U(L/I) \) by \( \phi(x) = x \otimes 1 \). By using the Poincare-Birkhoff-Witt theorem, and the structure it gives to \( U(L) \), it can be seen that this is a mapping of \( U(L) \) modules, i.e. \( \phi(a \cdot l) = \phi(a)l \). First we check the statement for elements of \( L \). If \( l \in L \) then \( a \cdot l = [a, l] \) and \( \phi([a, l]) = [a, l] \otimes 1 = (al - la) \otimes 1 = a \otimes l - l \otimes a \). However, \( a = 0 \) in \( U(L/I) \) so \( \phi([a, l]) = a \otimes l = (a \otimes 1)l = \phi(a)l \). Consider now the subalgebra \( A = \{ u \in U(L) | \phi(x \cdot u) = \phi(x)u \ \forall x \in I \} \). Since \( L \subset A \) then \( A = U(L) \), thus \( \phi \) is a \( U(L) \) module homomorphism.

It is left to show that \( \ker \phi = I' \). If \( x \in I' \) then \( x \) can be written as \( x = \sum [a_i, b_i], a_i, b_i \in I \), so that \( \phi(x) = x \otimes 1 = \sum[a_i, b_i] \otimes 1 = \sum(a_i \otimes b_i - b_i \otimes a_i). \) Since \( a_i, b_i \in I \) then their images in \( U(L/I) \) are 0 so that \( \phi(x) = 0 \).

Therefore \( I' \subset \ker \phi \). On the other hand suppose \( x \in \ker \phi \). Since \( \delta U(L) \) is a free \( U(L) \) module with basis \( \{x_i\} \) where \( x_i \) is a basis of \( L \) as a free Lie algebra, we have \( x \otimes 1 = \sum x_i \otimes f_i \), where, since \( \phi(x) = 0 \), \( f_i = 0 \) in \( U(L/I) \). Let us denote by \( \bar{I} \) the kernel of the mapping \( U(L) \rightarrow U(L/I) \), so that \( f_i \in \bar{I} \). But \( \bar{I} = U(L)I = IU(L) \) and thus by the Poincare-Birkhoff-Witt theorem this kernel is a free left and right \( U(L) \) module with a basis that is a basis of \( I \) as a subalgebra of \( L \). Therefore \( f_i = \sum w_{i,j}a_j \) where \( a_j \) are a basis of \( I \). It follows that \( x = \sum x_i w_{i,j}a_j \). Consider now the image of \( x, \bar{x} \), in \( I/I' \). Since \( I/I' \) is the commutative Lie algebra with a basis that is a basis of \( I \) as a subalgebra of \( L \), then \( \bar{x} = \sum \lambda_j a_j \), where \( \lambda_j \in k \). In other words
\[ x = \sum \lambda_j a_j + w, w \in I' \]. But since \( I' \subset \ker \phi \) then we can assume \( x = \sum \lambda_j a_j \). On the other hand \( \phi(x) = 0 \) so \( x = \sum x_i w_{i,j} a_j \). Since \( \tilde{I} \) is a free \( U(L) \) module with basis \( a_i \) we have \( \lambda_j = \sum x_i w_{i,j} \), but \( x_i \in \delta U(L) \), so \( \lambda_j = 0 \). Hence \( x \in I' \), therefore \( \ker \phi = I' \).

Another proof of the fact that \( \ker \phi = I' \) can be found in \([2]\) §8, as the Magnus embedding is a special case of the derivations defined there.

Throughout the remainder of this paper \( I \) will be a proper non-zero ideal of \( L \), and \( n > 1 \) will be an integer.

### 3 An image of \( H_2(L/I^n, k) \)

Consider \( H_2(L/I^n, k) \). It is known (e.g. \([4]\) p.233) that the analogue of the Hopf formula for groups holds for Lie algebras. Therefore

\[ H_2(L/I^n, k) = I^n/[I^n, L] = (I^n/I^{n+1}) \otimes_{U(L)} k \]

We know from the Širšov-Witt theorem (see e.g. \([4]\) p.44) that \( I \) is a free Lie algebra. Hence \( I^n/I^{n+1} \) is, in a natural way, identifiable with the \( n \)-th homogeneous component of the free Lie algebra with basis that is a basis of \( I/I' \) as a vector space. Since the free Lie algebra of a free module can be embedded in the tensor algebra over this module, the \( n \)-th homogeneous component can be embedded into the \( n \)-fold tensor product, i.e. \( I^n/I^{n+1} \) can be embedded in \( \otimes^n I/I' \), where the tensor is over \( k \). Any unadorned tensor product below is to be taken to be over \( k \). We need this embedding to be a \( U(L/I) \) module homomorphism, and it is easy to see that this is indeed the case when \( U(L/I) \) acts on \( I^n/I^{n+1} \) via the adjoint action, and on \( \otimes^n I/I' \) diagonally. The module \( \otimes^n I/I' \) can again be embedded, through the Magnus embedding, into

\[ \otimes^n (\delta U(L) \otimes_{U(L)} U(L/I)) \]

Tensoring this with \( k \) over \( L \) we get a mapping

\[ H_2(L/I^n, k) \approx \otimes^n I/I' \otimes_{U(L)} k \to \otimes^n (\delta U(L) \otimes_{U(L)} U(L/I)) \otimes_{U(L)} k \]

Since \( \delta U(L) \) is a free \( U(L) \) module, with a basis \( X \) that is a basis of \( L \) as a Lie algebra, we can define for each \( x \in X \) a projection, denoted \( p_x : \delta U(L) \otimes_{U(L)} U(L/I) \to U(L/I) \). We therefore have for
each $n$-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ a mapping $\phi_{x_1, \ldots, x_n} := (p_{x_1} \otimes \cdots \otimes p_{x_n} \otimes 1) \circ \phi$

$$\phi_{x_1, x_2, \ldots, x_n} : H_2(L/I^n, k) \to \otimes^n U(L/I) \otimes U(L) k$$

Since $I/I' \to U(L/I) \otimes \delta U(L)$ is an embedding, there exist elements $\alpha \in I/I'$ and $x \in X$ such that under the Magnus embedding and the projection by $x$ the image $a = \phi_x(\alpha)$ is non-zero. These elements will be put to use below.

### 4 Isomorphism of Hopf modules

As seen in the last section the image of the multiplicator lies in $(U(L/I) \otimes U(L/I) \cdots \otimes U(L/I)) \otimes U(L) k$. On the other hand it is well known that the enveloping algebra is a Hopf algebra, and the action with which this module is endowed is consistent with the standard Hopf structure on $U(L/I)$, which is the diagonal action. We shall use the following notation for the structure of Hopf algebras and modules. Let $H$ be a Hopf algebra and $M$ a Hopf module over $H$. The diagonal mapping of $H$ will be denoted by $\Delta$, and the $n$-fold application of $\Delta$ by $\Delta^n$ (by the co-associativity of $H$ the components on which we apply $\Delta$ each time do not matter). The co-unit of $H$ will be denoted by $\epsilon$ (also sometimes known as the augmentation). The antipode map of $H$ will be denoted by $S$. The usual action of $H$ on $M$ will be denoted by multiplication on the right, and the co-action of $M$ will be denoted by $\rho$. If $h \in H$ then $\Delta(h)$ will be written as $\Delta(h) = \sum_{i=1}^l h_{1i} \otimes h_{2i}$, and $\Delta(h_{1i}) = \sum_{j=1}^{h(i)} h_{1,1j} \otimes h_{1,2j}$. If $m \in M$ then $\rho(m) = \sum_{i=1}^l m_{0i} \otimes m_{1i}$.

It is known (see e.g. [4], p.15) that for any Hopf algebra $H$ and Hopf module $M$, $M \approx M' \otimes H$, where $M' = \{m \in M | \rho(m) = m \otimes 1\}$ with the isomorphism $m \mapsto \sum m_{0i} \cdot S(m_{1,1i}) \otimes m_{1,2i}$, where this is actually a double sum on both $i$ and $j$. It should also be noted that $M' \otimes H$ is a trivial Hopf module, i.e. one for which $(m \otimes h)l = m \otimes hl$.

If we now also tensor with $k$ over $H$ we will get

$$M \otimes_H k \approx (M' \otimes H) \otimes_H k.$$  

However, since $M' \otimes H$ is a trivial (in the sense defined above) Hopf module we get

$$M \otimes_H k \approx (M' \otimes H) \otimes_H k \approx M' \otimes (H \otimes_H k) \approx M'.$$
The isomorphism is
\[ m \otimes 1 \mapsto \sum m_0^i \cdot S(m_{1,1}^{i,j}) \otimes m_{1,2}^{i,j} \otimes 1 \mapsto \sum m_0^i \cdot S(m_{1,1}^{i,j}) \epsilon(m_{1,2}^{i,j}) = \sum m_0^i \cdot S(m_1^i). \]

If we take \( M = W \otimes H \) with \( W \) any Hopf module, \( H \) acting with the diagonal action and
\[ \rho(w \otimes h) = w \otimes \Delta(h) \]
then \( M' = W \otimes k \approx W \). In this case, if \( m = w \otimes h \) then \( \rho(w \otimes h) = w \otimes \Delta(h) \) so \( m_0^i = w \otimes h_1^i \) and \( m_1^i = h_2^i \). Therefore the explicit form of the isomorphism is
\[ w \otimes h \otimes 1 \mapsto \sum (w \otimes h_1^i) \Delta(S(h_2^i)). \]

However, we know that the image is in \( M' \), so we can apply \( 1 \otimes \epsilon \) to the image and not change it. Also if \( h \in H \) then from the definition of a Hopf algebra \((1 \otimes \epsilon)(\Delta(h)) = h \otimes 1\)

Therefore the image is
\[ (1 \otimes \epsilon)[\sum (w \otimes h_1^i) \Delta(S(h_2^i))] = \sum (w \otimes \epsilon(h_1^i))[(1 \otimes \epsilon)(\Delta(S(h_2^i)))] = \sum (w \otimes 1)(\epsilon(h_1^i)S(h_2^i) \otimes 1) = (w \otimes 1)(S(h) \otimes 1) \]
so the image in \( W \) is
\[ w \otimes h \otimes 1 \mapsto wS(h). \]

In our case we are interested in the module \( \otimes^n H \), so we can take \( W = \otimes^{n-1} H \) and the isomorphism will be
\[ h_1 \otimes h_2 \otimes \cdots \otimes h_n \otimes 1 \mapsto (h_1 \otimes h_2 \otimes \cdots \otimes h_{n-1}) \Delta_{n-1}(S(h_n)). \]

5 Computations

We can now prove theorem 1.1, i.e. show that \( H_2(L/I^n, k) \) is not finitely generated by exhibiting an infinite number of elements of the multiplicator, whose images in \( \otimes^{n-1} U(L/I) \) are linearly independent. We shall deal with several cases. In each of them we
shall construct elements of $H_2(L/I^n, k)$ that have one parameter $l$, where $l \in U(L/I)$. In other words we shall construct a $k$-linear map $f : U(L/I) \to H_2(L/I^n, k) \to \otimes^{n-1}U(L/I)$. It is obviously enough to show that $\ker f = k \cdot 1$ (since $U(L/I)$ is not finite dimensional). In other cases we shall show that $\text{Im} f$ is not finite dimensional by proving that it has elements of unbounded degree.

Recall the elements $\alpha \in I/I'$ and $x \in X$ such that $a = \phi_x(\alpha)$ was non-zero, and consider all elements of the form $[\alpha \cdot l, \alpha, \ldots, \alpha] \otimes 1$, where $l$ is any element of $\delta U(L/I)$. Obviously this element is in $I^n$. Its image, using the mapping $\phi_{x,x,\ldots,x}$ will be $[al, a, \ldots, a] \otimes 1$. In other words $f(l) = [al, a, \ldots, a] \otimes 1$. Note that if $l \in k \cdot 1$ then $f(l) = 0$ since in that case $[a \cdot l, a] = 0$. An easy induction shows that

$[a, b, b, \ldots, b] \otimes 1 = \sum (-1)^i \binom{n-1}{i} \otimes^i b \otimes a \otimes^{n-1-i} b \otimes 1$

where $\otimes^i b$ means $b \otimes b \otimes \cdots \otimes b$ ($i$ times). The referee points out that this formula is known as the Cartan-Weyl formula. Therefore under the Hopf module isomorphism

$f(l) = \sum (-1)^i \binom{n-1}{i} (\otimes^i a \otimes al \otimes^{n-2-i} a) \Delta_{n-1}(S(a))$

$+ (-1)^{n-1}(\otimes^{n-1}a)\Delta_{n-1}(S(al)).$

But $S(al) = S(l)S(a)$ so $\Delta_{n-1}(S(al)) = \Delta_{n-1}(S(l))\Delta_{n-1}(S(a))$ and hence

$f(l) = \sum (-1)^i \binom{n-1}{i} (\otimes^i a \otimes al \otimes^{n-2-i} a)$

$+ (-1)^{n-1}(\otimes^{n-1}a)\Delta_{n-1}(S(l))\Delta_{n-1}(S(a))$.

This can be rewritten as

$f(l) = (a \otimes a \otimes \cdots \otimes a) \sum (-1)^i \binom{n-1}{i} (\otimes^i 1 \otimes l \otimes^{n-2-i} 1) +$

$(-1)^{n-1}\Delta_{n-1}(S(l))\Delta_{n-1}(S(a))$.

Since $U(L/I)$ is without zero divisors and we are only interested in $\ker f$ or the dimension of $\text{Im} f$, we can consider instead the function

$f(l) = \sum (-1)^i \binom{n-1}{i} (\otimes^i 1 \otimes l \otimes^{n-2-i} 1) + (-1)^{n-1}\Delta_{n-1}(S(l))$.  

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In order to compute \( \ker f \), we can apply \( \epsilon \) to all but the \( j \)-th coordinate of each monomial. This operator, applied to \( \otimes^i 1 \otimes l \otimes^{n-2-i} 1 \), yields \( \delta_{ij} \) (since \( \epsilon(l) = 0 \)), while applied to \( \Delta_{n-1}(S(l)) \) yields (because \( \epsilon \) is a counit) \( S(l) \). Therefore for each \( 0 \leq j < n \) the result is

\[
(-1)^j \binom{n-1}{j} l + (-1)^{n-1} S(l) = 0.
\]

Therefore \( S(l) = (-1)^{n+j} \binom{n-1}{j} l \).

If \( n > 2 \) we get \( S(l) = (-1)^n l \) and \( S(l) = (-1)^{n+1} (n-1) l \). Therefore \( (-1)^n l = (-1)^{n+1} (n-1) l \) i.e.

\[
nl = 0.
\]

As was mentioned above, there are several cases.

**Case I** If \( \text{char}(k) \) does not divide \( n \) and \( n > 2 \) then for any \( l \in \delta U(L/I) \) we have \( f(l) \neq 0 \) i.e. \( \ker f = k \cdot 1 \).

**Case II** If \( \text{char}(k) \neq 2 \). We wish to show that \( \text{Im} f \) is not finite dimensional. Denoting by \( f_1(l) \) the application of \( \epsilon \) to all but the first coordinate, we get \( f_1(l) = l + (-1)^{n-1} S(l) \). This is true also when \( n = 2 \). Since \( f_1 \) is simply \( f \) composed with another function, obviously \( \text{dim} (\text{Im} f_1) \leq \text{dim} (\text{Im} f) \). Therefore it is enough to consider \( f_1 \). However, if \( x \) is any non-zero Lie element in \( U(L/I) \) then \( S(x^i) = (-1)^i x^i \). So \( f_1(x^i) = x^i + (-1)^{i+n-1} x^i \). Since \( \text{char}(k) \neq 2 \) then for all \( i \) of the correct parity we will have \( f_1(x^i) = 2 x^i \neq 0 \), but \( \deg x^i = i \) will be unbounded, so we are finished.

**Case III** The only case left is \( \text{char}(k) = 2 \) and \( n \) even. In this case we still have \( f_1(l) = l - S(l) \). Suppose \( L/I \) is not commutative, therefore there exist \( x,y \in L \) such that \( [x,y] \notin I \), i.e. \( [x,y] \neq 0 \) in \( U(L/I) \). Consider \( l_i = xy^i \). Obviously \( S(l_i) = y^ix \), so \( f_1(l_i) = xy^i - y^i x = [x,y]^i \). However the mapping \( u \mapsto [x,u] \) is a derivation of \( U(L/I) \), and therefore

\[
[x,y]^i = \sum_{j=0}^{i-1} y^j [x,y] y^{i-j-1}.
\]

Note that \( [x,y]^i [x,y] = y[x,y] + [x,y], \) and hence \( y^j [x,y] y^{i-j-1} \equiv y^{i-1} [x,y] \mod U_{i-1}(L/I) \). Thus \( [x,y]^i \equiv iy^{i-1} [x,y] \mod U_{i-1}(L/I) \), and if \( i \) is odd then \( \deg f_1(l_i) = i \). Thus the degree of the elements of the image is unbounded, so the image is infinite dimensional.

**Case IV** There remains the case where \( L/I \) is commutative. Thus if \( L \) has basis \( X \), then \( L' \subset I \) so \( I/L' \subset L/L' \) is a subspace, and we
can perform a linear change of basis of $L$, so that $I = < L', X_1 >$, where $X_1$ is a proper subset of $X$. Consider the Lie algebra over $\mathbb{Z}$, $L_1 = < Y >, I_1 = < L_1', Y_1 >$, where $Y$ and $Y_1$ are disjoint copies of $X$ and $X_1$. We now use the universal coefficient theorem (see e.g. [3] p.176) which in our case states that if $k$ is any $\mathbb{Z}$ module then

$$0 \to H_2(L_1/I_1^n, \mathbb{Z}) \otimes \mathbb{Z} k \to H_2(L_1/I_1^n \otimes \mathbb{Z} k, k) \to \text{Tor}^\mathbb{Z}_1(H_1(L_1/I_1^n, \mathbb{Z}), k) \to 0$$

is exact. Since $H_1(L_1/I_1^n, \mathbb{Z}) = (L_1/I_1^n)_{ab} = L_1/L_1'$ is a free $\mathbb{Z}$ module then $\text{Tor}^\mathbb{Z}_1(H_1(L_1/I_1^n, \mathbb{Z}), k) = 0$.

Take $k = \mathbb{Q}$. We have $H_2(L_1/I_1^n, \mathbb{Z}) \otimes \mathbb{Z} \mathbb{Q} \approx H_2(L_1/I_1^n \otimes \mathbb{Z} \mathbb{Q}, \mathbb{Q})$. However $L_1/I_1^n \otimes \mathbb{Z} \mathbb{Q}$ is simply $L_2/I_2^n$ where $L_2 = < Y >$ and $I_2 = < L_2', Y_1 >$ taken over $\mathbb{Q}$. Since $\mathbb{Q}$ has characteristic 0, we know that $H_2(L_2/I_2^n, \mathbb{Q})$ is infinite dimensional. Therefore $H_2(L_1/I_1^n, \mathbb{Z})$ must also have infinite torsion-free rank as a $\mathbb{Z}$-module. Apply now the universal coefficient theorem with $k$ any field of characteristic 2. Again $H_2(L_1/I_1^n, \mathbb{Z}) \otimes \mathbb{Z} k \approx H_2(L_1/I_1^n \otimes \mathbb{Z} k, k)$. Once again $L_1/I_1^n \otimes \mathbb{Z} k$ is exactly $L/I^n$ of the original Lie algebra. However, since $H_2(L_1/I_1^n, \mathbb{Z})$ has infinite rank then $H_2(L_1/I_1^n, \mathbb{Z}) \otimes \mathbb{Z} k$ is not finitely generated, thus we have proved theorem 1.1.

Note that in the case $L = < x, y >, I = L'$ and $k$ is of characteristic 2, even though $H_2(L/I', k)$ is not finitely generated, the image in $U(L/I)$, under any of the projections, will be 0.

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