The Minimal Sum of Squares Over Partitions with a Nonnegative Rank

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Abstract. Motivated by a question of Defant and Propp (Electron J Combin 27:Article P3.51, 2020) regarding the connection between the degrees of noninvertibility of functions and those of their iterates, we address the combinatorial optimization problem of minimizing the sum of squares over partitions of $n$ with a nonnegative rank. Denoting the sequence of the minima by $(m_n)_{n \in \mathbb{N}}$, we prove that $m_n = \Theta\left(n^{4/3}\right)$. Consequently, we improve by a factor of 2 the lower bound provided by Defant and Propp for iterates of order two.

1. Introduction

Recently, Defant and Propp [2] defined the degree of noninvertibility of a function $f: X \to Y$ between two finite nonempty sets $X$ and $Y$ by

$$\deg(f) = \frac{1}{|X|} \sum_{x \in X} |f^{-1}(f(x))|,$$

as a measure of how far $f$ is from being injective. For example, if $f$ is $k$-to-1 for $1 \leq k \leq |X|$, then $\deg(f) = k$. In particular, if $f$ is injective (resp., constant), then $\deg(f) = 1$ (resp., $\deg(f) = |X|$). Interested mainly in endofunctions (also called dynamical systems within the field of dynamical algebraic combinatorics), that is, functions $f: X \to X$, they then computed the degrees of noninvertibility of several specific functions and studied, from an extremal point of view, the connection between the degrees of noninvertibility of functions and those of their iterates. They concluded their work with the following question: Let $k \in \mathbb{N}$ and for $f: X \to X$ denote $f^k = f \circ \cdots \circ f$. Does the limit

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\[
\lim_{n \to \infty} \max_{|X| = n} \frac{\deg(f^k)}{\deg(f)^{2-1/2^{k-1}}} \frac{1}{n^{1-1/2^{k-1}}}
\] (1)

exist? If so, what is its value? They remarked that even answering the question for \(k = 2\) would be interesting and stated that it follows from their results that, if the limit exists, then it lies in the interval between \(3^{-3/2} \approx 0.19245\) and 1.

Our attempts to answer their question in the case \(k = 2\) have led us to a combinatorial optimization problem that seems not to have been addressed before, namely, the problem of finding the minimal sum of squares over partitions with a nonnegative rank (the rank of a partition is the result of subtracting the number of parts in the partition from the largest part). In this work, we address this problem and, consequently, improve the lower bound of the interval \([3^{-3/2}, 1]\) by a factor of 2.

We begin by stating our main results. The definitions of the terms that we use and the proofs of all the statements are given in Sect. 3.

2. Main Results

Let \(X\) be a set of size \(n \in \mathbb{N}\) to be used throughout this work. We denote by \(\mathbb{N}_0\) the set of all nonnegative integers. Taking \(k = 2\) in (1), we wish to obtain a lower bound for

\[
\max_{f: X \to X} \frac{\deg(f^2)}{\deg(f)^3/2} \frac{1}{n^{1/2}}.
\] (2)

Our approach is based on the fact that the functions with the largest possible degree of noninvertibility, namely \(n\), are the constant functions. Thus, we wish to solve the following combinatorial optimization problem:

\[
\text{minimize } \deg(f)
\]

where \(f: X \to X\) is such that \(f^2\) is constant.

The notion of the degree of noninvertibility of a function \(f: X \to X\) is directly related to the sum of squares over a certain partition of \(n\) via the observation that

\[
\deg(f) = \frac{1}{n} \sum_{x \in X} |f^{-1}(x)|^2
\]

(cf. [2, p. 2]). Indeed, if \(X = \{1, \ldots, n\}\), then \(|f^{-1}(1)|, \ldots, |f^{-1}(n)|\) yield, upon reordering and omitting zeros, an integer partition of \(n\) that we denote by \(\text{Partition}(f)\). Conversely, it is clear that every partition \(\lambda\) of \(n\) induces a function \(f: X \to X\), such that \(\text{Partition}(f) = \lambda\) (notice that such a function is, in general, not unique).

It turns out (cf. Lemma 3.2) that if \(f: X \to X\) is such that \(f^2\) is constant, then \(\text{Partition}(f)\) has a nonnegative rank (cf. Definition 3.1). Denoting the set of all partitions of \(n\) by \(\mathcal{P}(n)\) and the Euclidean norm of a vector \(x\) by \(\|x\|_2\), we may rewrite problem (3) equivalently as
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Figure 1. The Young diagram of one of the two minimizers of (4), for $n = 100$. Here, in the notation of Theorem 2.2, $x = 17, r = 3$, and $a = 5$

minimize $||\lambda||_2^2$ \hspace{1cm} (4)

where $\lambda \in \mathcal{P}(n)$ is such that $\text{rank}(\lambda) \geq 0$.

Remark 2.1. Notice that, in general, a minimizer of (4) is not unique. For example, both $(5, 3, 3, 3, 3)$ and $(6, 3, 2, 2, 2, 2)$ minimize (4) for $n = 17$.

Our first main result is the observation that, for $n \neq 2$, the partitions of $n$ that minimize (4) must have a certain structure, namely, their largest part $\lambda_1$ is equal to their number of parts (i.e., their rank is 0) and $n - \lambda_1$ is divided as evenly as possible among the remaining $\lambda_1 - 1$ parts (see Fig. 1 for a visualization):

Theorem 2.2. For $n \geq 2$, problem (4) is equivalent to the following problem:

minimize $x^2 + r(a+1)^2 + (x-1-r)a^2$ \hspace{1cm} (5)

such that $x \in \{2, \ldots, n\}$,

\begin{align*}
n - x &= a(x-1) + r \text{ where } a \in \mathbb{N}_0 \text{ and } 0 \leq r < x - 1 \text{ and } \\
x &\geq \begin{cases} 
a, & r = 0; \\
a + 1, & \text{otherwise.}
\end{cases}
\end{align*}

Let $m_n$ denote the minimum value of (5). Then

$$(m_n)_{n \in \mathbb{N}} = 1, 4, 5, 8, 11, 14, 17, 22, 25, \ldots$$

See Table 1 for the first 210 values of this sequence, which is registered as A353044 in the On-Line Encyclopedia of Integer Sequences (OEIS) [4]. Lemmas 3.8 and 3.9, respectively, show that $(m_n)_{n \in \mathbb{N}}$ is strictly increasing and that its elements have alternating parity. We also define a sequence $(t_n)_{n \in \mathbb{N}}$, such that if $n \in \mathbb{N}$, then $t_n$ is any $x \in \{2, \ldots, n\}$ that minimizes (5), for this $n$. For example, for $n = 17$, we have $m_{17} = 61$ and we may define $t_{17} = 5$ or $t_{17} = 6$. While we do not have an exact formula for $(m_n)_{n \in \mathbb{N}}$, we obtain lower and upper bounds by applying continuous relaxation:
| $n$ | $m_n$ | $n$ | $m_n$ | $n$ | $m_n$ | $n$ | $m_n$ | $n$ | $m_n$ | $n$ | $m_n$ |
|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|
| 1   | 1     | 36  | 174   | 71  | 449   | 106 | 782   | 141 | 1161  | 176 | 1576  |
| 2   | 4     | 37  | 181   | 72  | 458   | 107 | 793   | 142 | 1174  | 177 | 1589  |
| 3   | 5     | 38  | 188   | 73  | 467   | 108 | 802   | 143 | 1185  | 178 | 1602  |
| 4   | 8     | 39  | 195   | 74  | 476   | 109 | 811   | 144 | 1196  | 179 | 1615  |
| 5   | 11    | 40  | 202   | 75  | 485   | 110 | 822   | 145 | 1207  | 180 | 1628  |
| 6   | 14    | 41  | 209   | 76  | 494   | 111 | 833   | 146 | 1218  | 181 | 1641  |
| 7   | 17    | 42  | 216   | 77  | 503   | 112 | 844   | 147 | 1229  | 182 | 1654  |
| 8   | 22    | 43  | 223   | 78  | 512   | 113 | 855   | 148 | 1240  | 183 | 1665  |
| 9   | 25    | 44  | 230   | 79  | 521   | 114 | 866   | 149 | 1253  | 184 | 1678  |
| 10  | 28    | 45  | 237   | 80  | 530   | 115 | 875   | 150 | 1266  | 185 | 1691  |
| 11  | 33    | 46  | 244   | 81  | 539   | 116 | 886   | 151 | 1277  | 186 | 1704  |
| 12  | 38    | 47  | 253   | 82  | 548   | 117 | 897   | 152 | 1288  | 187 | 1717  |
| 13  | 41    | 48  | 260   | 83  | 557   | 118 | 908   | 153 | 1299  | 188 | 1730  |
| 14  | 46    | 49  | 267   | 84  | 566   | 119 | 919   | 154 | 1310  | 189 | 1743  |
| 15  | 51    | 50  | 274   | 85  | 575   | 120 | 930   | 155 | 1321  | 190 | 1756  |
Table 1. continued

| n   | $m_n$ | n   | $m_n$ | n   | $m_n$ | n   | $m_n$ | n   | $m_n$ | n   | $m_n$ | n   | $m_n$ |
|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|
| 16  | 56    | 51  | 281   | 86  | 586   | 121 | 941   | 156 | 1334  | 191 | 1769  |
| 17  | 61    | 52  | 290   | 87  | 595   | 122 | 952   | 157 | 1347  | 192 | 1782  |
| 18  | 66    | 53  | 299   | 88  | 604   | 123 | 963   | 158 | 1360  | 193 | 1795  |
| 19  | 71    | 54  | 306   | 89  | 613   | 124 | 974   | 159 | 1371  | 194 | 1808  |
| 20  | 76    | 55  | 313   | 90  | 622   | 125 | 985   | 160 | 1382  | 195 | 1821  |
| 21  | 81    | 56  | 320   | 91  | 631   | 126 | 996   | 161 | 1393  | 196 | 1834  |
| 22  | 88    | 57  | 329   | 92  | 642   | 127 | 1007  | 162 | 1404  | 197 | 1847  |
| 23  | 93    | 58  | 338   | 93  | 653   | 128 | 1018  | 163 | 1417  | 198 | 1860  |
| 24  | 98    | 59  | 347   | 94  | 662   | 129 | 1029  | 164 | 1430  | 199 | 1873  |
| 25  | 103   | 60  | 354   | 95  | 671   | 130 | 1040  | 165 | 1443  | 200 | 1886  |
| 26  | 110   | 61  | 361   | 96  | 680   | 131 | 1051  | 166 | 1456  | 201 | 1899  |
| 27  | 117   | 62  | 370   | 97  | 689   | 132 | 1062  | 167 | 1467  | 202 | 1912  |
| 28  | 122   | 63  | 379   | 98  | 700   | 133 | 1073  | 168 | 1478  | 203 | 1925  |
| 29  | 127   | 64  | 388   | 99  | 711   | 134 | 1084  | 169 | 1489  | 204 | 1938  |
| 30  | 134   | 65  | 397   | 100 | 722   | 135 | 1095  | 170 | 1502  | 205 | 1951  |
| 31  | 141   | 66  | 404   | 101 | 731   | 136 | 1106  | 171 | 1515  | 206 | 1964  |
| 32  | 148   | 67  | 413   | 102 | 740   | 137 | 1117  | 172 | 1528  | 207 | 1977  |
| 33  | 153   | 68  | 422   | 103 | 749   | 138 | 1128  | 173 | 1541  | 208 | 1990  |
| 34  | 160   | 69  | 431   | 104 | 760   | 139 | 1139  | 174 | 1554  | 209 | 2003  |
| 35  | 167   | 70  | 440   | 105 | 771   | 140 | 1150  | 175 | 1565  | 210 | 2016  |
Theorem 2.3. We have $m_n = \Theta(n^{4/3})$. More precisely
\[
\frac{n^{4/3}}{4} \leq m_n \leq (2^{-2/3} + 2^{1/3})n^{4/3},
\]
for $n \geq 28$.

Theorem 2.3 allows us to improve by a factor of 2 the lower bound given by Defant and Propp [2, p. 17]:

Corollary 2.4. We have
\[
\liminf_{n \to \infty} \max_{f: X \to X} \frac{\deg(f^2)}{\deg(f)^{3/2}} \frac{1}{n^{1/2}} \geq 2 \cdot 3^{-3/2}.
\]
In particular, taking $k = 2$, the limit in (1), if it exists, is bounded from below by $2 \cdot 3^{-3/2}$.

We proceed by showing that $t_n$ must lie in a certain interval.

Theorem 2.5. Suppose that $n \geq 6$. There exists a function $u_n: (1, \infty) \to \mathbb{R}$, such that if $x_0^{(n)}$ is the global minimum of $u_n$ and $x_1^{(n)} < x_2^{(n)}$ are the two real positive roots of the cubic polynomial
\[
x^3 + \left(-2n - u_n\left(\left\lfloor x_0^{(n)} \right\rfloor\right)\right)x + n^2 + u_n\left(\left\lfloor x_0^{(n)} \right\rfloor\right),
\]
then $t_n \in \left\{\left\lfloor x_1^{(n)} \right\rfloor, \ldots, \left\lceil x_2^{(n)} \right\rfloor\right\}$. 

Figure 2. A visualization of $t_6, \ldots, t_{5000}$ with the corresponding bounds of Theorem 2.5 (whenever there were several possibilities for $t_n$, we have chosen the smallest)
Example 2.6. In the notation of Theorem 2.5, taking \( n = 1000 \), we have \( \lceil x_1^{(1000)} \rceil = 78 \), \( \lfloor x_2^{(1000)} \rfloor = 82 \) and \( t_{1000} = 78 \) is the unique minimizer of (5). See Fig. 2 for a visualization of Theorem 2.5.

Finally, we establish the asymptotic behaviour of the sequence \((t_n)_{n \in \mathbb{N}}\):

**Theorem 2.7.** We have \( t_n = \Theta\left(n^{2/3}\right) \).

**Remark 2.8.** It should be emphasized that our approach does not, in general, provide the true maximum of (2). For example, let \( n = 8 \) and assume that \( X = \{1, \ldots, 8\} \). Consider the function \( h: X \to X \) given by

\[
h(i) = \begin{cases} 
1, & i \in \{1, 2, 3\}; \\
2, & i \in \{4, 5\}; \\
3, & i \in \{6, 7\}; \\
4, & i = 8.
\end{cases}
\]

Then, \( h^2: X \to X \) is given by

\[
h^2(i) = \begin{cases} 
1, & i \in \{1, \ldots, 7\}; \\
2, & i = 8.
\end{cases}
\]

It follows that:

\[
\frac{\deg(h^2)}{\deg(h)^{3/2}} = \frac{\frac{50}{8}}{\left(\frac{18}{8}\right)^{3/2}} = \frac{50}{27} \approx 1.85185.
\]

In contrast, our approach provides the partition \((3, 3, 2)\) that corresponds to a function \( f: X \to X \), such that

\[
\frac{\deg(f^2)}{\deg(f)^{3/2}} = \frac{8}{\left(\frac{22}{8}\right)^{3/2}} \approx 1.75424.
\]

3. Definitions and Proofs

We begin with the definition of the rank of a partition a notion introduced by Dyson [3] (see also A064174 in the OEIS). The reader is referred to [1] for the general theory of partitions.

**Definition 3.1.** Let \( \lambda \in \mathcal{P}(n) \). The rank of \( \lambda \), denoted by \( \text{rank}(\lambda) \), is the result of subtracting the number of parts in \( \lambda \) from \( \lambda \)'s largest part.

Functions \( f: X \to X \), such that \( f^2 \) is constant, are characterized by induced partitions of \( n \) with a nonnegative rank:

**Lemma 3.2.** Let \( f: X \to X \) be a function, such that \( f^2 \) is constant. Then, \( \text{rank}\left(\text{Partition}(f)\right) \geq 0 \). Conversely, for every \( \lambda \in \mathcal{P}(n) \) with \( \text{rank}(\lambda) \geq 0 \), there is a function \( f: X \to X \), such that \( f^2 \) is constant and \( \text{Partition}(f) = \lambda \).
Proof. Assume that $f^2$ is constant. Then, there is $y \in X$, such that $f(f(x)) = y$ for every $x \in X$. Thus, $f(x) \in f^{-1}(y)$ for every $x \in X$. Notice that the number of parts of $\text{Partition}(f)$ is equal to $|\text{Im}(f)|$. Now

$$|\text{Im}(f)| \leq |f^{-1}(y)| \leq \max_{x \in X} \{|f^{-1}(x)|\}.$$ 

It follows that $\text{rank}(\text{Partition}(f)) \geq 0$.

Conversely, suppose $X = \{1, \ldots, n\}$ and let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}(n)$, such that $\text{rank}(\lambda) \geq 0$. We define $f : X \to X$ as follows: for $1 \leq i \leq n$, let

$$f(i) = \begin{cases} 1, & 1 \leq i \leq \lambda_1; \\ k, & \sum_{j=1}^{k-1} \lambda_j < i \leq \sum_{j=1}^{k} \lambda_j \text{ where } 2 \leq k \leq r. \end{cases}$$

Since $\lambda_1 \geq r$, we have that $2, \ldots, r \in f^{-1}(1)$. It follows that $f(f(i)) = 1$ for every $i \in X$, i.e., $f^2$ is constant. Furthermore, $\text{Partition}(f) = \lambda$. 

The proof of Theorem 2.2 relies on the following two lemmas. For the second, we shall need the notion of a balanced partition. We could not find any mention of this notion other than in A047993 in the OEIS.

**Definition 3.3.** A partition whose rank is zero is called a balanced partition.

**Lemma 3.4.** Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}(n)$, such that $\lambda_j > \lambda_k + 1$ for some $1 \leq j < k \leq r$. Let $\lambda' \in \mathcal{P}(n)$ correspond to the parts $\lambda_1, \ldots, \lambda_j - 1, \ldots, \lambda_k + 1, \ldots, \lambda_r$ (possibly after reordering). Then, $||\lambda||^2 > ||\lambda'||^2$.

**Proof.** It suffices to prove that

$$\lambda_j^2 + \lambda_k^2 > (\lambda_j - 1)^2 + (\lambda_k + 1)^2,$$

which is easily seen to be equivalent to $\lambda_j > \lambda_k + 1$. 

**Lemma 3.5.** If $n \neq 2$, then the minimum value of (4) is obtained at a balanced partition.

**Proof.** Consider first the cases $n = 1, 3, 4$: If $n = 1$, then there is only one partition (1) which is balanced. If $n = 3$, then there are only two partitions with a nonnegative rank, namely (3) and (2,1), of which the latter, that is balanced, has the smallest sum of squares. Similarly, if $n = 4$, then there are only three partitions with a nonnegative rank, namely (4), (3,1), and (2,2), of which the latter, that is balanced, has the smallest sum of squares.

Assume now that $n \geq 5$ and let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}(n)$, such that $\text{rank}(\lambda) > 0$. We shall construct a partition $\lambda' \in \mathcal{P}(n)$, such that $||\lambda||^2 > ||\lambda'||^2$ and $0 \leq \text{rank}(\lambda') < \text{rank}(\lambda)$. To this end, let $k = \max\{1 \leq i \leq r \mid \lambda_i > 1\}$. We distinguish between two cases:

1. $k > 1$. We have

$$||\lambda||^2 = \sum_{1 \leq i \leq r, i \neq k} \lambda_i^2 + (\lambda_k - 1 + 1)^2 = \sum_{1 \leq i \leq r, i \neq k} \lambda_i^2 + (\lambda_k - 1)^2 + 2\lambda_k - 1$$
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\[ \sum_{1 \leq i \leq r, i \neq k} \lambda_i^2 + (\lambda_k - 1)^2 + 1 = ||\lambda'||_2^2, \]

where \( \lambda' = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - 1, \lambda_{k+1}, \ldots, \lambda_r, 1) \). Since \( \lambda_1 \geq r + 1 \), we have \( \text{rank}(\lambda') \geq 0 \). Furthermore, \( \text{rank}(\lambda') < \text{rank}(\lambda) \).

2. \( k = 1 \). In this case, \( \lambda = (n - r + 1, 1, \ldots, 1) \). First, assume that \( r \geq 3 \).

It is easy to see that
\[ (n - r + 1)^2 + r - 1 > (n - r)^2 + 4 + r - 2 \iff n - r > 1. \]
Now, by assumption, \( n - r + 1 > r \). Thus, \( n - r > r - 1 \geq 2 \) and we take \( \lambda' = (n - r, 2, 1, \ldots, 1) \).

Assume now that \( r = 2 \). Then, \( \lambda = (n - 1, 1) \) and it is easy to see that
\[ (n - 1)^2 + 1 > (n - 2)^2 + 2 \iff n \geq 3. \]

Thus, we take \( \lambda' = (n - 2, 1, 1) \).

Finally, assume that \( r = 1 \). Then, \( \lambda = (n) \) and we have
\[ n^2 > (n - 1)^2 + 1 \iff n \geq 2. \]

Then, we take \( \lambda' = (n - 1, 1) \).

In each of these cases, \( ||\lambda||_2^2 > ||\lambda'||_2^2 \) and \( 0 \leq \text{rank}(\lambda') < \text{rank}(\lambda) \).

\[ \Box \]

Proof of Theorem 2.2. The assertion follows immediately from the combination of Lemma 3.4 together with Lemma 3.5.

Example 3.6. Let \( \lambda = (5, 3, 2, 1) \in P(11) \). Then, \( ||\lambda||_2^2 = 39 \). Applying Lemma 3.5 on \( \lambda \), we obtain the balanced partition \( \lambda' = (5, 3, 1, 1, 1) \in P(11) \) with \( ||\lambda'||_2^2 = 37 \). Applying Lemma 3.4 on \( \lambda' \), we obtain the partition \( \lambda'' = (5, 2, 2, 1, 1) \in P(11) \) with \( ||\lambda''||_2^2 = 35 \). The partition \( \lambda'' \) is one of the partitions satisfying the constraints in the optimization problem of Theorem 2.2.

In our work, we shall make extensive use of two functions \( l_n, u_n : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \), given by
\[ l_n(x) = x^2 + \frac{(n - x)^2}{x - 1} \quad \text{and} \quad u_n(x) = x^2 + \frac{(n - x)^2}{x - 1} + \frac{x - 1}{4}. \]

The bounds in the following lemma are visualized in Fig. 3 for \( n = 100 \).

Lemma 3.7. Let \( 2 \leq x, n \in \mathbb{N} \). Then
\[ l_n(x) \leq x^2 + r(a + 1)^2 + (x - 1 - r)a^2 \leq u_n(x), \quad (6) \]
where \( a \in \mathbb{N}_0 \) and \( 0 \leq r < x - 1 \) are such that \( n - x = a(x - 1) + r \).
Theorem 2.3. The function $u_n(x)$ is continuous in $(1, \infty)$ and

$$\lim_{x \to 1^+} u_n(x) = \lim_{x \to \infty} u_n(x) = \infty.$$  

Furthermore

$$u'_n(x) = \frac{8x^3 - 11x^2 - 2x + 8n - 4n^2 + 1}{4(x-1)^2}. $$
Since the discriminant of the numerator of \( u_n'(x) \) is negative for \( n \geq 3 \), the equation \( u_n'(x) = 0 \) has a unique real solution \( x_0^{(n)} \), given by \( x_0^{(n)} = \frac{11 + C_n + 169/C_n}{24} \), where

\[
C_n = \sqrt[3]{3456n^2 - 6912n + 1259} - \sqrt[3]{(3456n^2 - 6912n + 1259)^2 - 169^3}.
\]

It follows that, restricted to \((1, \infty)\), the function \( u_n(x) \) obtains its global minimum at \( x_0^{(n)} \). Now, for every \( 0 \leq y \leq z \in \mathbb{R} \), such that \( z \neq 0 \), we have

\[
\frac{y^2}{2z} \leq z - \sqrt{z^2 - y^2} \leq \frac{y^2}{z}.
\]

Thus, \( C_n \leq 1 \) for \( n \geq 28 \) and therefore

\[
x_0^{(n)} = \frac{11 + C_n + 169/C_n}{24} \\
\leq \frac{1}{2} + \frac{169}{24} \sqrt{\frac{2(3456n^2 - 6912n + 1259)}{169^3}} \\
\leq \frac{1}{2} + 2^{-1/3}n^{2/3}
\]

(notice, for later use, that \( \lim_{n \to \infty} \frac{x_0^{(n)}}{n^{2/3}} = 2^{-1/3} \)). Since \( u_n(x) \) is increasing in \([x_0^{(n)}, \infty)\), we have

\[
u_n \left( \left\lfloor x_0^{(n)} \right\rfloor \right) \leq u_n \left( x_0^{(n)} + 1 \right) \\
\leq u_n \left( \frac{3}{2} + 2^{-1/3}n^{2/3} \right) \\
= \left( \frac{3}{2} + 2^{-1/3}n^{2/3} \right)^2 + \left( n - \left( \frac{3}{2} + 2^{-1/3}n^{2/3} \right) \right)^2 + \frac{1}{4} + 2^{-1/3}n^{2/3} \\
\leq \left( 2^{-2/3} + 2^{1/3} \right) n^{4/3},
\]

where the last inequality holds for \( n \geq 5 \). Now, it follows from Lemma 3.7 that \( m_n \leq u_n \left( \left\lfloor x_0^{(n)} \right\rfloor \right) \), which concludes the proof of the upper bound.

To prove the lower bound, we notice that, for \( n \geq 3 \) and restricted to \((1, \infty)\), the function \( l_n(x) \) obtains its global minimum at \( y_0^{(n)} \), given by \( y_0^{(n)} = \frac{D_n + 1 + 1/D_n}{2} \), where

\[
D_n = \sqrt[3]{2n^2 - 4n + 1} - \sqrt{(2n^2 - 4n + 1)^2 - 1}.
\]

We have

\[
l_n \left( y_0^{(n)} \right) = \left( y_0^{(n)} \right)^2 + \left( y_0^{(n)} - n \right)^2 \\
\geq \frac{1}{4D_n^2}
\]
Figure 4. We have \( \left\lceil x_{100}^{(100)} \right\rceil = 17 \) and \( \left\lfloor x_{200}^{(100)} \right\rfloor = 18 \). Thus, \( t_{100} \in \{17, 18\} \)

\[
\geq \frac{3}{4} \sqrt{(2n^2 - 4n + 1)^2}
\geq \frac{n^{4/3}}{4},
\]

where the last inequality holds for \( n \geq 4 \). By Lemma 3.7, \( m_n \geq l_n \left( y_0^{(n)} \right) \). \( \square \)

Proof of Corollary 2.4. Let \( f_n : X \to X \) be a function, such that \( f_n^2 \) is constant and \( \| \text{Partition}(f_n) \|_2^2 = m_n \). Notice that \( \text{deg}(f_n) = \frac{m_n}{n} \). By Theorem 2.3, if \( n \geq 28 \), then \( m_n \leq (2^{-2/3} + 2^{1/3})n^{4/3} \). It follows that:

\[
\max_{f : X \to X} \frac{\text{deg}(f^2)}{\text{deg}(f)^{3/2}} \frac{1}{n^{1/2}} \geq \frac{\text{deg}(f_n^2)}{\text{deg}(f_n)^{3/2}} \frac{1}{n^{1/2}} \geq \frac{n}{((2^{-2/3} + 2^{1/3})n^{1/3})^{3/2}} \frac{1}{n^{1/2}} = 2 \cdot 3^{-3/2}.
\]

\( \square \)

Proof of Theorem 2.5. Let \( x_0^{(n)} \) and \( y_0^{(n)} \) be the points, calculated in the proof of Theorem 2.3, at which \( u_n \) and \( l_n \), respectively, obtain their global minimum, when restricted to \((1, \infty)\). We wish to solve the equation

\[
l_n(x) = u_n \left( \left\lfloor x_0^{(n)} \right\rfloor \right)
\]

(8)
(see Fig. 4 for a visualization for \( n = 100 \)). The function \( l_n(x) \) is continuous in \((1, \infty)\) and

\[
\lim_{x \to 1^+} l_n(x) = \lim_{x \to \infty} l_n(x) = \infty.
\]

Since \( l_n\left(y_0^{(n)}\right) < u_n\left(x_0^{(n)}\right) \leq u_n\left(\left[ x_0^{(n)} \right]\right) \), by the mean value theorem, Eq. (8) has at least two real solutions in \((1, \infty)\). Similarly, \( l_n(x) \) is continuous in \((-\infty, 1)\) and

\[
\lim_{x \to 1^-} l_n(x) = -\infty, \quad \lim_{x \to -\infty} l_n(x) = \infty.
\]

Thus, Eq. (8) has at least one real solution in \((-\infty, 1)\) and, since solving it is equivalent to finding the roots of the cubic polynomial

\[
x^3 + \left( -2n - u_n\left(\left[ x_0^{(n)} \right]\right) \right) x + n^2 + u_n\left(\left[ x_0^{(n)} \right]\right),
\]

we conclude that Eq. (8) has exactly two real solutions \( 1 < x_1^{(n)} < x_2^{(n)} \). Necessarily, \( x_1^{(n)} < \left[ x_0^{(n)} \right] \) and \( y_0^{(n)} < x_2^{(n)} \). It follows that \( t_n \in \left\{ \left[ x_1^{(n)} \right], \ldots, \left[ x_2^{(n)} \right] \right\} \).

**Proof of Theorem 2.7.** Let \( x_0^{(n)}, x_1^{(n)} \) and \( x_2^{(n)} \) be as in Theorem 2.5 and consider the cubic polynomial

\[
x^3 + \left( -2n - u_n\left(\left[ x_0^{(n)} \right]\right) \right) x + n^2 + u_n\left(\left[ x_0^{(n)} \right]\right). \tag{9}
\]

Since \( x_0^{(n)}, t_n \in \left\{ \left[ x_1^{(n)} \right], \ldots, \left[ x_2^{(n)} \right] \right\} \) and \( x_0^{(n)} = \Theta(n^{2/3}) \), it suffices to show that \( x_2^{(n)} - x_1^{(n)} = o(n^{2/3}) \). To this end, denote \( p_n = -2n - u_n\left(\left[ x_0^{(n)} \right]\right) \) and \( q_n = n^2 + u_n\left(\left[ x_0^{(n)} \right]\right) \). We have seen in the proof of Theorem 2.5 that the cubic polynomial in (9) has three distinct real roots. This means that its discriminant \(-4p_n^3 + 27q_n^2\) is positive, or equivalently, that \( \frac{p_n^3}{27} + \frac{q_n^2}{4} < 0 \). By Cardano’s formula (e.g., [5, p. 128]), the three roots of (9) are given by

\[
x_1^{(n)}, x_2^{(n)}, x_3^{(n)} = \sqrt[3]{-\frac{q_n}{2} + \sqrt{\frac{p_n^3}{27} + \frac{q_n^2}{4}}} + \sqrt[3]{-\frac{q_n}{2} - \sqrt{\frac{p_n^3}{27} + \frac{q_n^2}{4}}}. \tag{10}
\]

We are interested in the two positive roots, \( x_1^{(n)} \) and \( x_2^{(n)} \), of the cubic and shall now explain how they may be obtained from (10) (doing so, we follow closely [5, Example 3.106]): Under the first cubic root, we have a complex number that we rewrite as follows (see Fig. 5)

\[
-\frac{q_n}{2} + \sqrt{\frac{p_n^3}{27} + \frac{q_n^2}{4}} = r_n(\cos(\pi - \theta_n) + i \sin(\pi - \theta_n)), \tag{11}
\]

where \( r_n > 0 \) and \( \theta_n = \arctan \left(\frac{2\sqrt{\frac{p_n^3}{27} + \frac{q_n^2}{4}}}{q_n}\right) \).
Under the second cubic root, we have the number’s complex conjugate. Adding the two cubic roots, we obtain double their real parts. Applying De Moivre’s formula on the right-hand side of (11) gives the arguments $\frac{\pi - \theta_n}{3}, \frac{\pi - \theta_n}{3}$ and $-\frac{\pi + \theta}{3}$. Thus

$$x_1^{(n)} = 2r_n^{1/3} \cos\left(\frac{\pi + \theta_n}{3}\right) \quad \text{and} \quad x_2^{(n)} = 2r_n^{1/3} \cos\left(\frac{\pi - \theta_n}{3}\right).$$

In the appendix, we show that $\frac{p_{3n}^3}{27} + \frac{q_{2n}^2}{4} = o\left(n^4\right)$. Since $u_n\left(\lfloor x_0^{(n)} \rfloor\right) = \Theta\left(n^{4/3}\right)$, we have $q_n = \Theta\left(n^2\right)$. It follows that $r_n = \Theta\left(n^2\right)$ and $\lim_{n \to \infty} \theta_n = 0$. Applying the trigonometric identity

$$\cos \alpha - \cos \beta = 2 \sin \left(\frac{\beta + \alpha}{2}\right) \sin \left(\frac{\beta - \alpha}{2}\right)$$

that holds for every $\alpha, \beta \in \mathbb{R}$, we see that

$$\lim_{n \to \infty} \frac{x_2^{(n)} - x_1^{(n)}}{n^{2/3}} = 2 \lim_{n \to \infty} \left(\frac{r_n}{n^2}\right)^{1/3} \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{\theta_n}{3}\right) = 0.$$

Although not used in this work, the following two properties of the sequence $(m_n)_{n \in \mathbb{N}}$ seem noteworthy.

**Lemma 3.8.** The sequence $(m_n)_{n \in \mathbb{N}}$ is strictly increasing.

**Proof.** Assume that $m_{n+1} \leq m_n$ for some $n \in \mathbb{N}$ and let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}(n + 1)$, such that $||\lambda||_2^2 = m_{n+1}$. Then, $\lambda' = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r - 1) \in \mathcal{P}(n)$ (omitting the last part, if necessary), such that $\text{rank}(\lambda') \geq 0$. Now

$$||\lambda'||_2^2 < ||\lambda||_2^2 = m_{n+1} \leq m_n,$$

contradicting the minimality of $m_n$. \qed
Lemma 3.9. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in P(n)$. Then, $n$ and $||\lambda||_2^2$ have the same parity. In particular, $n$ and $m_n$ have the same parity.

Proof. We have

$$||\lambda||_2^2 \equiv \# \text{ odd parts in } \lambda \pmod{2}$$

$$\equiv \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd,} \end{cases} \pmod{2}.$$ 

\[\square\]

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Declarations

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Appendix

Denote $z_n = \left\lfloor x_0^{(n)} \right\rfloor$ and $X_i = \frac{1}{(z_n - 1)^i}$, for $i = 1, 2, 3$. We have

$$\frac{p_n^3}{27} + \frac{q_n^2}{4} = -\frac{n^6}{27}X_3 + \frac{2n^5z_n}{9}X_3 - \frac{2n^5}{9}X_2$$

$$-\frac{5n^4z_n}{9}X_3 - \frac{n^4z_n}{9}X_2 + \frac{31n^4z_n}{36}X_2 + \frac{n^4}{4}$$

$$+ \frac{5n^4}{18}X_2 + \frac{n^4}{18}X_1 + \frac{20n^3z_n^3}{27}X_3$$

$$+ \frac{4n^3z_n^3}{9}X_2 - \frac{11n^3z_n^2}{9}X_2 - \frac{4n^3z_n^2}{9}X_1$$
Recall that $\lim_{n \to \infty} \frac{z_n}{n^{2/3}} = 2^{-1/3}$ (this was stated in the proof of Theorem 2.3). Thus, the expansion of $\frac{z_n}{n^{2/3}} + \frac{q^2}{4n}$ contains terms of order $n^4$ (the boxed terms). Nevertheless, the overall order is $< n^4$, as the following calculation shows:
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