DIVISION ALGEBRAS GRADED BY A FINITE GROUP

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Abstract. Let $k$ be a field containing an algebraically closed field of characteristic zero. If $G$ is a finite group and $D$ is a division algebra over $k$, finite dimensional over its center, we can associate to a faithful $G$-grading on $D$ a normal abelian subgroup $H$, a positive integer $d$ and an element of $\text{Hom}(\text{M}(H), k^\times)^G$, where $\text{M}(H)$ is the Schur multiplier of $H$. Our main theorem is the converse: Given an extension $1 \to H \to G \to G/H \to 1$, where $H$ is abelian, a positive integer $d$, and an element of $\text{Hom}(\text{M}(H), k^\times)^G$, there is a division algebra with center containing $k$ that realizes these data. We apply this result to classify the $G$-simple algebras over an algebraically closed field of characteristic zero that admit a division algebra form over a field containing an algebraically closed field.

1. Introduction

If $A$ is a finite dimensional algebra over a field $k$ and $G$ is a finite group, we say $A$ is $G$-graded if there is a decomposition $A = \bigoplus_{g \in G} A_g$ of $A$ into a direct sum of $k$-vector spaces such that $A_g A_{g_2} \subseteq A_{g_1 g_2}$ for all $g_1, g_2 \in G$. We say the grading is faithful if $A_g \neq 0$ for all $g \in G$. In this paper we are interested in the case where $A = D$ is division algebra over $k$ faithfully graded by a finite group $G$ and where $k$ contains an algebraically closed field of characteristic zero.

We begin with a brief description of our results followed by a more detailed discussion: Writing $D = \bigoplus_{g \in G} D_g$ we see easily that $D_e$ is a division algebra of degree $d$ (say) over its center, and $k = k \cdot 1 \subseteq D_e$ (However it is not necessarily the case that the center $K$ of $D$ is contained in $D_e$ or that $K$ is even a graded subalgebra of $D$). The group $G$ acts on the center $L$ of $D_e$ with kernel $H$. We obtain in this way an extension $1 \to H \to G \to G/H \to 1$ and a two-cocycle $\alpha$ with class in $H^2(H, L^\times)$. The condition that $k$ contains an algebraically closed field of characteristic zero forces $H$ to be abelian. Moreover the class of $\alpha$ is $G$-invariant, that is, fixed by the natural action of $G$ on $H^2(H, L^\times)$ and gives rise to a $G$-invariant element in $\text{Hom}(\text{M}(H), \mu)$, where $\text{M}(H)$ is the Schur multiplier of $H$ and $\mu$ is the group of roots of unity in $k$. Our main result (Theorem 1.6) is the converse: Given an extension of groups $1 \to H \to G \to G/H \to 1$ where $H$ is abelian, a positive integer $d$, and a $G$-invariant element of $\text{Hom}(\text{M}(H), \mu)$, there is a field $k$ containing an algebraically closed field of characteristic zero and a division algebra $D$ over $k$ giving rise to the prescribed data. We apply our result (Theorem 1.10) to classify the $G$-simple algebras over an algebraically closed field.
of characteristic zero that admit a division algebra form over a field $k$ containing an algebraically closed field.

We proceed to the more detailed description. We begin the paper by determining the general structure of gradings by the group $G$ on $K$-central division algebras (so $K$ contains the base field $k$ where as always $k$ is assumed to contain an algebraically closed field of characteristic zero) and break the analysis into two cases:

1. The case in which the center $K$ is contained in the $e$-homogeneous component.
2. The general case, in which the center $K$ may not be contained in $D_e$, and may not be $G$-graded.

There are three types of gradings on division algebras. Our first two theorems (for the two cases mentioned above) will show that any grading by a finite group is a combination of these three types:

1. The trivial grading: For $D$ any finite dimensional division algebra and $G = \{1\}$.
2. The crossed product grading: If $L/K$ is a Galois extension with $G$ as Galois group, we can construct the skew group algebra $L \rtimes G$ which is isomorphic to the group algebra $LG$ as a left vector space over $L$. We denote its elements by $\sum_{\sigma \in G} l_\sigma u_\sigma$, where $l_\sigma \in L$ and $u_\sigma$ is a symbol in $L \rtimes G$ corresponding to the element $\sigma \in G$. The product is defined using distributivity and so as to satisfy the condition $lu_\sigma yu_\tau = l\sigma(y)u_{\sigma \tau}$.

   It is well known that the algebra $L \rtimes G$ is isomorphic to $M_n(K)$ where $n = \text{ord}(G)$. In particular the skew group algebra structure gives a $G$-grading on $M_n(K)$. Now we can twist the multiplication by a 2-cocycle of $G$ with coefficients in $L^*$. If $\alpha : G \times G \to L^*$ is such a 2-cocycle we construct the crossed product algebra, denoted by $L^\alpha \rtimes G$, which again is isomorphic to $L \rtimes G$ and $L \rtimes G$ as a left vector space over $L$ with the product determined by the formula $lu_\sigma yu_\tau = l\sigma(y)\alpha(\sigma, \tau)u_{\sigma \tau}$.

   Cohomologous 2-cocycles in $H^2(G, L^*)$ yield $G$-graded isomorphic algebras over $K$ and in particular isomorphic algebras over $K$. On the other hand, noncohomologous 2-cocycles yield nonisomorphic $K$-central simple algebras and in particular nonisomorphic $G$-graded algebras.

   The crossed product $L^\alpha \rtimes G$ is $K$-central simple and is a $G$-graded twisted form of $L \rtimes G$. It is well known (we provide a proof below) that for any finite group $G$ one can find a $G$-Galois extension $L/K$ and a cocycle $\alpha$ such that $L^\alpha \rtimes G$ is a $K$-central division algebra. Moreover one can find such crossed product division algebras for which the center contains an algebraically closed field of characteristic zero. Note that here the $e$-component is $L$, a maximal subfield in $L^\alpha \rtimes G$.

3. The case of twisted group algebras: Let $H$ be a finite group and consider twisted group algebras $K^\alpha H$ where $\alpha$ is a 2-cocycle $H \times H \to K^*$, and where the action of $H$ on $K^*$ is trivial. Because $\text{char}(K) = 0$ the twisted group algebra is semisimple but in general not simple. It seems to be
difficult to determine the finite groups $H$ which admit a 2-cocycle over an arbitrary field $K$ of characteristic zero such that the twisted group algebra $K^\alpha H$ is a division algebra. In our case, in which the base field contains an algebraically closed field of characteristic zero, we will see that the group $H$ must be abelian. It is easy to show that for any abelian group $H$ we can find a cocycle $\alpha$ and a field $K$ such that the twisted group algebra is a field and so in particular a finite dimensional division algebra. The case where $K^\alpha H$ is central over $K$ is of special interest. These are obtained precisely when $H$ is abelian of the form $A \times A$ and the cocycle is a suitable nondegenerate cocycle on $H$ (see the following Remark).

Remark 1.1. Classically, a finite group $G$ is said to be of central type if it has a complex representation of degree $[G : Z]^{1/2}$ where $Z = Z(G)$ is the center of $G$. In particular the index of $Z$ in $G$ is an integer square. Since a linear representation of $G$ gives rise to a 2-cocycle $\alpha$ on $G/Z$ with values in $\mathbb{C}^*$ and a projective representation of $G/Z$ corresponding to $\alpha$, researchers borrowed that terminology and defined (nonclassically) a group $\Lambda$ of central type as a group that admits a 2-cocycle $\alpha \in Z^2(\Lambda, \mathbb{C}^*)$ such that the twisted group algebra $\mathbb{C}^\alpha \Lambda \cong M_n(\mathbb{C})$. We say the cocycle $\alpha$ is nondegenerate. In this article we slightly extend the terminology and say a 2-cocycle $Z^2(\Lambda, K^*)$ is nondegenerate if the twisted group algebra $K^\alpha \Lambda$ is a $K$-central simple algebra.

We can now state our first structure theorem.

Theorem 1.2. Let $D$ be a division algebra, finite dimensional over its center $K$. Suppose $D$ is faithfully graded by a finite group $G$. Let $D_e$ be the identity component of $D$ and denote by $L$ its center. Assume $K$ is contained in $D_e$ and that $K$ contains an algebraically closed field of characteristic zero. Then the following conditions hold:

1. Conjugation on $D_e$ by nonzero homogeneous elements $x \in D_g$ determines a $G$-action on $L$, the center of $D_e$. Furthermore, $K = L^G$ and hence if we denote by $H$ the kernel of the action, $L/K$ is a $G/H$-Galois extension.
2. The group $H$ is abelian and is isomorphic to $A \times A$, for some abelian group $A$.
3. There exists a 2-cocycle $\alpha \in Z^2(H, L^*)$ such that $D_H = \sum_{h \in H} D_h \cong D_e \otimes_L L^\alpha H$. Furthermore, the cocycle $\alpha$ is nondegenerate, that is $L^\alpha H$ is an $L$-central simple algebra. The subalgebras $L^\alpha H$ and $D_H$ have center $L$.
4. The cohomology class $[\alpha] \in H^2(H, L^*)$ is $G$-invariant.
5. The algebra $D = D_G$ is a generalized $G/H$-crossed product in the sense of O. Teichmuller over the algebra $D_H$ (see [8] and [9]). Its degree is $\sqrt{\text{dim}_L(D_e)/|H|}(G : H)$.

Our second structure theorem deals with the general case; we drop the assumption that $K$, the center of $D$, is contained in the identity component and do not assume $K$ is $G$-graded.

Theorem 1.3. Let $D$ be a division algebra, finite dimensional over its center $K$. Suppose $D$ is faithfully graded by a finite group $G$. Let $D_e$ be the identity component of $D$ and denote by $L$ its center. Let $K_0 = K \cap L$ denote the central $e$-homogeneous elements of $D$. Assume $K_0$ contains an algebraically closed field of characteristic zero. With these conditions the following hold:
1. Conjugation on $D_e$ with nonzero homogeneous elements $x_g \in D_g$ determines a $G$-action on $L$ with $K_0 = L^G$. If we denote by $H$ the kernel of the action, then $L/K_0$ is a $G/H$-Galois extension.

2. The algebra $D$ is finite dimensional over $K_0$.

3. The group $H$ is abelian.

4. There exists a 2-cocycle $\alpha \in Z^2(H, L^*)$ such that $D_H = \sum_{h \in H} D_h \cong D_e \otimes_L L^\alpha$. Hence, if we denote $L_1 = Z(L^\alpha H)$, then $L_1$ is also the center of the division algebra $D_H$.

5. Conjugation on $D_H$ by nonzero homogeneous elements induces an action of $G$ on $L_1$ with fixed field $K$, the center of $D$. Furthermore, the extension $L_1/K$ is $G/H$-Galois.

6. The cohomology class $[\alpha] \in H^2(H, L^*)$ is $G$-invariant. For $S = \{s \in H|x_s \in L_1\} = \{\alpha(s, h) = \alpha(h, s) \text{ for all } h \in H\}$, then $S$ is a subgroup of $H$ and $L_1 = L^\alpha S$. The group $H/S$ is of central type, hence isomorphic to $A \times A$ for some abelian group $A$.

7. The division algebra $D = D_G$ is a $G/H$ crossed product over the algebra $D_H$. Its degree is $\sqrt{|H : S|}(G : H)$.

In the next theorem we determine under what circumstances the center $K$ (in Theorem 1.3 above) is $G$-graded. By statement (5) of the theorem, $K$ is a subfield of $D_H$ and hence if $K$ is $G$-graded, it is necessarily $H$-graded.

We continue the notation as in Theorem 1.2. For every $h \in H$ we let $x_h$ be a representative in $L^\alpha H$ (i.e., any nonzero element in $(L^\alpha H)_h$). Let $S = \{h \in H : x_hx_{h'} = x_{h'}x_h \text{ for all } h' \in H\}$, that is, $S$ is the set of elements in $H$ such that $x_s \in L_1$, the center of $L^\alpha H$. We claim that in fact $L_1 = L^\alpha S$: It is clear that $L^\alpha S \subseteq L_1$. Conversely if $z = \sum_{h \in H} l_h x_h$ lies in $L_1$, then for all $r \in H$, $x_r z = z x_r$. But $H$ is abelian and $x_r$ commutes with the elements of $L$. It follows that if $l_h \not= 0$ then $h \in S$.

Note that because $G$ acts on $L_1$, it acts also on $S$, so $S$ is a normal subgroup of $G$.

**Theorem 1.4.** The center $K$ is $G$ graded if and only if $S$ is central in $G$.

The most substantial part of this paper is devoted to proving a converse to the structure theorems. To this end we adopt the following terminology:

Let $D$ be a finite dimensional division algebra with center $K$ and suppose $D$ is faithfully graded by a finite group $G$. By the structure theorems this gives rise to

1. a group extension with abelian kernel

$$1 \to H \to G \to G/H \to 1,$$

2. a cohomology class $[\alpha] \in H^2(H, L^*)^G$,

and

3. a positive integer $d$, the degree of $D_e$ over its center.

Let $M(H)$ denote the Schur multiplier of $H$ and let $\phi_{[\alpha]} \in \text{Hom}(M(H), L^*)$ be the map corresponding to $[\alpha]$ by means of the Universal Coefficients Theorem. By naturality, the map $\phi_{[\alpha]}$ is $G$-invariant. Furthermore, because the values of $\phi_{[\alpha]}$ are roots of unity and these are assumed to be in $K_0$, we have $\phi_{[\alpha]} \in \text{Hom}(M(H), \mu)^G$, where $\mu$ denotes the group of roots of unity in $K_0$ (the action of $G$ on $\mu$ is trivial).
Because $H$ is abelian the Schur multiplier can be identified with the wedge product $H \wedge H$ and if $h_1, h_2 \in H$, then $\phi_{[\alpha]}(h_1 \wedge h_2) = \alpha(h_1, h_2)\alpha(h_2, h_1)^{-1}$. In the twisted group algebra $L^G/H = \sum_{h \in H} Lx_h$, one sees easily that for all $h_1, h_2 \in H$, the commutator $x_{h_1}x_{h_2}x_{h_1}^{-1}x_{h_2}^{-1}$ is a root of unity and we also have $\phi_{[\alpha]}(h_1 \wedge h_2) = x_{h_1}x_{h_2}x_{h_1}^{-1}x_{h_2}^{-1}$. If we let $F$ denote an algebraically closed field containing $K$ then in fact $H^2(H, F^*)^G \cong \text{Hom}(M(H), \mu)^G$. The class $[\alpha] \in H^2(H, L^*)^G$ is nondegenerate if and only if its image in $H^2(H, F^*)^G$ is nondegenerate (because $L^G/H \otimes_L F = F^\alpha H$). It therefore makes sense to speak of the nondegeneracy of $\phi_{[\alpha]} \in \text{Hom}(M(H), \mu)^G$.

Letting $[\beta] \in H^2(G/H, H)$ be the cocycle determined by the group extension given above, we see that any faithfully $G$-graded finite dimensional division algebra over its center determines uniquely an ordered triple $([\beta], \phi, d) \in H^2(G/H, H) \times \text{Hom}(M(H), \mu)^G \times \mathbb{N}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers.

**Definition 1.5.** Let $G$ be a finite group and $H$ an abelian normal subgroup. We say the triple $([\beta], \phi, d) \in H^2(G/H, H) \times \text{Hom}(M(H), \mu)^G \times \mathbb{N}$ is realizable if there exists a finite dimensional division algebra, faithfully $G$-graded which yields that triple.

Here is the main result of the paper.

**Theorem 1.6.** If $G$ is a finite group and $H$ an abelian normal subgroup, then every ordered triple $([\beta], \phi, d) \in H^2(G/H, H) \times \text{Hom}(M(H), \mu)^G \times \mathbb{N}$ is realizable if there exists a finite dimensional division algebra, faithfully $G$-graded which yields that triple.

We will say that an extension

$$1 \to H \to G \to G/H \to 1$$

with an abelian kernel is realizable if there exists a $G$-invariant map $\phi : M(H) \to \mu$ such that the triple $([\beta], \phi, 1)$ is realizable. Because the trivial map $\phi$ is $G$-invariant we have the following corollary (see Cuadra and Etingof result ([5] Theorem 2.1)).

**Corollary 1.7.** Every extension

$$1 \to H \to G \to G/H \to 1$$

with an abelian kernel is realizable by a division algebra of degree $[G : H]$.

**Example 1.8.** In [5] Cuadra and Etingof give an example of a finite dimensional division algebra graded by $Q_8$, the quaternion group of order 8. In their example the center is not graded. In the following we describe the possible grading structure on division algebras by the group $Q_8$ and by the group $D_4$, the dihedral group of order 8. As above $D$ will denote the division algebra, $K$ its center and $K_0$ its $e$-center. We will assume $D_e = L$, so $d = 1$. (See the remarks at the beginning of section 3). As above the field $K_0$ is assumed to contain an algebraically closed field of characteristic zero.

Let

$$1 \to H \to Q_8 \to Q_8/H \to 1$$

(1) If $H = \{1\}$ we obtain a $Q_8$ crossed product division algebra of degree 8.
(2) If $H \cong Z_2$ the $e$-component is a field $L$ which is a $Z_2 \times Z_2$ Galois extension of the center $K$. The $H$-component $D_H$ is a field, and in fact $K = D_H$, so the center is graded. The division algebra $D$ is a $Z_2 \times Z_2$ crossed product over $K$.

(3) If $H \cong Z_4$, the group $S$ of Theorem 1.4 equals $H$ (because $H$ is cyclic). The $e$-component $L$ is an extension of degree 2 over $K_0$ and the $H$-component is a field extension of $L$ of degree 4. The center $K$ is ungraded, an extension of degree 4 over $K_0$. The division algebra $D$ is a quaternion algebra over $K$. This is the example of Cuadra and Etingof.

Now let

$$1 \rightarrow H \rightarrow D_4 \rightarrow D_4/H \rightarrow 1$$

(1) If $H = \{1\}$ we obtain a $D_4$ crossed product division algebra of degree 8.

(2) If $H \cong Z_2$ the $e$-component $L$ is a $Z_2 \times Z_2$ Galois extension of the center $K = K_0$. The $H$-component $D_H$ is a field, and in fact $K = D_H$, so the center is graded. The division algebra $D$ is a $Z_2 \times Z_2$ crossed product over $K$. Moreover there is a $K_0$-subalgebra $D_0$, also a $Z_2 \times Z_2$ crossed product such that $D \cong D_0 \otimes_{K_0} K$.

(3) If $H \cong Z_2 \times Z_2$ and the map $\phi$ is trivial, then the $e$-component is $L$, an extension of degree 4 over $K_0$. The subgroup $S$ equals $H$ and the division algebra $D$ is a quaternion algebra over a nongraded center $K$ which is of degree 2 over $K_0$. Because the sequence is split, $D$ is isomorphic to $D_0 \otimes_{K_0} K$, where $D_0$ is a quaternion algebra over $K_0$.

(4) If $H \cong Z_2 \times Z_2$ and the map $\phi$ is nontrivial, then $\phi$ is the only nontrivial element of $\text{Hom}(M(H), \mu)$ and so $\phi$ is $G$-invariant. The $e$-component $L$ contains the center $K = K_0$ and $L$ has degree 2 over $K$. The division algebra $D$ has degree 4 over $K$. The $H$-component of $D$ is a quaternion algebra with center $L$. Because the sequence is split, $D \cong Q_1 \otimes_K Q_2$ is a tensor product of two quaternion algebras over the center $K$.

(5) If $H \cong Z_4$, the group $S$ of Theorem 1.4 equals $H$ (because $H$ is cyclic). The $e$-component $L$ is of degree 2 over $K_0$ and the $H$-component is a field extension of $L$ of degree 4. The center $K$ is ungraded, an extension of degree 4 over $K_0$. The division algebra $D$ is a quaternion algebra over $K$ and because the extension is necessarily split, there is a quaternion algebra $D_0$ over $K_0$ such that $D \cong D_0 \otimes_{K_0} K$.

Next we address the following problem: What are the $G$-simple algebras over an algebraically closed field of characteristic zero that admit a $G$-graded twisted form division algebra?

Let $D$ be a finite dimensional division algebra over its center $K$. Let $K_0 = K \cap D_e$ as above. Suppose $K_0$ contains an algebraically closed field of characteristic zero. Extension of scalars $D_E = D \otimes_{K_0} E$ where $E$ is any algebraically closed field containing $K_0$ yields a finite dimensional $G$-simple algebra in which $E$ will be the central homogeneous elements of degree $e$. Such algebras where classified by means of elementary and fine gradings by Bahturin, Sehgal and Zaicev [3]. Here is their result:

**Theorem 1.9.** Let $A$ be a finite dimensional $G$-simple algebra over an algebraically closed field $F$ of characteristic zero. There is a subgroup $H$ of $G$, a two
cocycle $c \in Z^2(H, F^*)$ (the $H$-action on $F^*$ is trivial) and $g = (g_1, \ldots, g_m) \in G^{(m)}$, such that $A \cong F^c H \otimes M_n(F)$, where (by this identification) the $G$-grading on $A$ is given by

$$A_g = \text{span}_F \{ u_h \otimes e_{i,j} | g = g_i^{-1} h g_j \}.$$ 

Unlike the ungraded case, namely matrix algebras over $E$, not every finite dimensional $G$-simple algebra over $E$ admits a $G$-graded twisted form division algebra and even not a $G$-graded division algebra twisted form (see [2], Theorems 1.8 and 1.12).

We have the following result.

**Theorem 1.10.** Let $A$ be a finite dimensional $G$-simple algebra over an algebraically closed field $E$ of characteristic zero. Then $A$ admits a $G$-graded twisted form division algebra over a field $K_0$ which contains an algebraically closed field of characteristic zero if and only if the following hold:

1. $H$ is abelian
2. Every coset of $H$ is represented in the elementary grading. Moreover the number of representatives is equal for all cosets.
3. $H$ is normal in $G$
4. The cohomology class $[\alpha] \in H^2(H, E^*)$ is $G$ invariant where the action of $G$ on $E$ is trivial.

**2. Proof of the structure theorems**

In this section we prove Theorems 1.2 – 1.4. We start with Theorem 1.2.

**Proof.** Let $D$ be a finite dimensional algebra over its center $K$ and suppose $D$ is graded by a finite group $G$. We are assuming the center $K$ is contained in $D_e$, the identity component of $D$. Let $L = Z(D_e)$. Conjugation on $D_e$ with nonzero homogeneous elements induces an action of $G$ on $L$ and we denote by $H$ the kernel of this action. Clearly $K = L^G = L^{G/H}$. We claim that $D_H \cong D_e \otimes_L L^\alpha H$ for some $\alpha \in Z^2(H, L^*)$: For all $h \in H$, let $u_h$ be a nonzero homogeneous element of degree $h$. Because $h$ acts trivially on $L$, and $D_e$ is finite dimensional over $L$, conjugation by $u_h$ determines an inner automorphism of $D_e$ and hence, by the Skolem-Noether Theorem, there is a (nonzero) element $\theta_h$ in $D_e$ such that $x_h = u_h \theta_h^{-1}$ centralizes $D_e$. Let $h$ and $h'$ in $H$. Clearly $x_h x_{h'} = \alpha(h, h') x_{hh'}$, where $\alpha(h, h') \in D_e^*$, but because $x_h$ and $x_{h'}$ centralize $D_e$, we have $\alpha(h, h') \in L^*$. It follows that $\alpha : H \times H \to L^*$ is a 2-cocycle in $Z^2(H, L^*)$. Finally, because the set $\{ x_h \}_{h \in H}$ is linearly independent over $D_e$ (and in particular over $L$) we obtain $D_H = D_e \otimes_L L^\alpha H$, as desired.

Next we will show $L$ is the center of the twisted group algebra $L^\alpha H$: If $L_1 = Z(L^\alpha H) = Z(D_H)$, conjugation by nonzero homogeneous elements of $D$ induces an action of $G$ on $L_1$ with fixed field $K$. But $H$ acts trivially on $L_1$ and so $L_1^{G/H} = K$. Because $L \subseteq L_1$ and the action of $G/H$ is faithful, we obtain $L = L_1$, as desired. It follows that the 2-cocycle $\alpha$ is nondegenerate and $H$ is of central type.

We show next that $H$ is abelian: Because the group $\Gamma$ of trivial units in $L^\alpha H$, that is the group of elements of the form, $L^* x_h, h \in H$, is center by finite, it follows from a theorem of Schur that the commutator subgroup of $\Gamma$ is finite. That means the commutator subgroup consists of roots of unity, which in our case are in the base field $k$. It follows that if $h_1, h_2 \in H$, then $x_{h_1} x_{h_2} x_{h_1}^{-1} x_{h_2}^{-1}$ lies in $k$. But
$x_h x_h x_h^{-1} x_h^{-1} \in L x_h x_h x_h^{-1} x_h^{-1}$, so $x_h x_h x_h^{-1} x_h^{-1} \in L$ which is in the $e$-component. Hence the commutator $h_1 h_2 h_1^{-1} h_2^{-1}$ is trivial.

Next we will show that the class $[\alpha] \in H^2(H, L^*)$ is $G$ invariant, where the action of $G$ on the cohomology is induced by the given action on $H$ and $L$. Let $y_g$ be a nonzero homogeneous element of degree $g \in G$. If $h \in H$, then $y_g x_h y_g^{-1}$ lies in $D_H$ and commutes with $D_e$. It follows that conjugation by $y_g$ stabilizes $L^o H$. We need to show the cocycle $g(\alpha)$ determined by $g(\alpha)(h, h') = y_g^{-1} \alpha(ghg^{-1}, gh'g^{-1})y_g$ is cohomologous to $\alpha$. Write $x_h x_h' = \alpha(h, h') x_h h'$. Conjugating both sides by $y_g^{-1}$ we get

$$y_g^{-1} x_h y_g^{-1} x_h' y_g = y_g^{-1} \alpha(h, h') y_g y_g^{-1} x_h h' y_g$$

Because $y_g^{-1} x_h y_g \in D^{g^{-1} h g}$ and centralizes $D_e$, we have $y_g^{-1} x_h y_g = \gamma_g^{-1} h g x_g^{-1} h_g$ where $\gamma_g^{-1} h g \in L^*$. Note that the elements of $D_e$ centralize $x_h$, so the value $\gamma_g^{-1} h g$ does not depend on the choice of $y_g \in D_g = D_e y_g$.

Thus the equation above yields

$$\gamma_g^{-1} h g x_g^{-1} h_g \gamma_g^{-1} h_g' x_g^{-1} h_g' = y_g^{-1} \alpha(h, h') y_g \gamma_g^{-1} h h' y_g x_g^{-1} h h'$$

and so

$$\gamma_g^{-1} h g \gamma_g^{-1} h_g' \alpha(g^{-1} h g, g^{-1} h g') x_g^{-1} h h' = y_g^{-1} \alpha(h, h') y_g \gamma_g^{-1} h h' y_g x_g^{-1} h h'$$

We conclude that

$$\gamma_g^{-1} h g \gamma_g^{-1} h_g' \alpha(g^{-1} h g, g^{-1} h g') = y_g^{-1} \alpha(h, h') y_g \gamma_g^{-1} h h' y_g,$$

showing the cocycles are cohomologous.

It is now clear that the algebra $D$ is a crossed product of $G/H$ with coefficients in $D_H$. It is straightforward to check that its degree is $\sqrt{|H| (G : H)}$.

We proceed to prove Theorem 1.3, that is, we drop the assumption that the center is contained in the $e$-component.

**Proof.** We have $D = D_e \oplus D_{g_2} \oplus \cdots \oplus D_{g_n}$ and $L = Z(D_e)$. As above, the group $G$ acts on $L$ and if $H$ denotes the kernel of the action, then $K_0 = L^{G/H}$. We want to show as above, that we can find representatives $x_h, h \in H$ which centralize $D_e$ so that we have $D_H \cong D_e \otimes_L L^o H$, for some $\alpha \in Z^2(H, L^*)$, $L_1 = Z(L^o H) = Z(D_H)$. In order to apply Skolem Noether we need to know $D_e$ is finite dimensional over its center $L$. We are given that $D$ is finite dimensional over $K$.

Claim: the center $K$ is contained in $D_R$, where $\hat{H}$ is the subgroup of $H$ consisting on all elements $g \in G$ such that conjugation with $u_g \in D_g$ induces an inner automorphism of $D_e$. Note that in that case there exists (as above) an element $x_g \in D_g$ that centralizes $D_e$. To prove the claim let $z \in K$. We have

$$z = \alpha_e u_e + \alpha_{g_2} u_{g_2} + \cdots + \alpha_{g_n} u_{g_n}$$

We show first $\alpha_g = 0$ if $g \notin H$. Conjugation by elements of $L$ centralizes $z$ and preserves all homogeneous components. Since for every $g$ we can find $l_g$ which does not centralize $u_g$, the claim follows. Suppose now conjugation by $u_g$ does not give an inner automorphism on $D_e$. Need to show $\alpha_h = 0$. Conjugation with any nonzero elements of $D_e$, fixes $z$ and so it must centralize $\alpha_h u_h$. This shows
conjugation of $D_e$ by $\alpha_h^{-1}$ and $u_h$ determine the same action and hence the action of $u_h$ is inner if $\alpha$ is nonzero. This prove that $K$ is contained in $D_H$. But, since for every $h \in H$ there is $x_h$ homogeneous of degree $h$ which centralizes $D_e$ we see that their product also centralizes $D_e$ and hence the algebra generated by $x_h$, $h \in H$ over $L$ is isomorphic to a twisted group algebra $L^eH$ where $\alpha : H \times H \to L^*$ is a 2-cocycle. This shows $D_e$ is finite dimensional over $L$ (because it is finite dimensional over $K$). We conclude that $D_e$ is finite dimensional over $K_0 = L \cap K$. Note that in fact $H = \hat{H}$, because by Skolem Noether, conjugation by any homogeneous element of $D_H$ is inner.

The proof that $H$ is abelian and that the cocycle $\alpha$ is $G$-invariant is the same as in Theorem 1.2.

Let $L_1 = Z(L^eH)$. We have $D_H \cong D_e \otimes_L L^eH$ and hence $L_1 = Z(D_H)$. Clearly, conjugation by homogeneous elements induces an action of $G$ on $L_1$ whose kernel contains $H$. But the kernel must be equal to $H$ since $L$ is contained $L_1$.

Let $S = \{h \in H : x_hx_{h'} = x_{h'}x_h \text{ for all } h' \in H\}$, that is, $S$ is the set of elements in $H$ such that $x_h \in L_1$, the center of $L^eH$. We claim that in fact $L_1 = L^eS$: It is clear that $L^eS \subseteq L_1$. Conversely if $z = \sum_{h \in H} l_hx_h$ lies in $L_1$, then for all $r \in H$, $x_rz = zr_h$. But $H$ is abelian and $x_r$ commutes with the elements of $L$. It follows that if $l_h \neq 0$ then $h \in S$.

Because $L_1 = L^eS$ is the center of $L^eH$ the group $H/S$ must be of central type, that is, isomorphic to $A \times A$ for some abelian group $A$.

As in the previous theorem we now have that $D_G$ is a crossed product of $G/H$ with $D_H$. It is straightforward to check that its degree is $\sqrt{(H : S)}(G : H)$.

\[ \square \]

We proceed to Theorem 1.4.

Proof. The set up is as in Theorem 1.3. Recall the field $L_1 = Z(L^eH)$ where $H$ is abelian, $S$ is the subgroup of $H$ whose representatives $x_s$ commute with $x_h$ for all $h \in H$. We have seen that $L_1 = L^eS$ and that $S$ is normal in $G$.

Let $S_1 = S \cap Z(G)$. Clearly $L_2 = L^eS_1$ is contained in $L_1$ and normalized by $G$. Suppose $S_1 = S$, that is $S$ is central in $G$. We want to show the center $K$ is $H$-graded and in particular $G$-graded. Indeed, if $z \in k$, $z = \sum_{h \in S} \gamma_hx_h$, conjugation by nonzero homogeneous elements $y_g$ acts on one hand trivially on $z$ and on the other hand by multiplication of each $x_s$ by a nonzero scalar. Since the elements $x_s$ are linearly independent the result follows.

Suppose now $S$ is not central in $G$, that is, $S_1$ is a proper subgroup of $S$. We want to show the center $K$ is not graded. Will show there is an element $z = \sum_{s \in S} \gamma_sx_s \in K$ where the elements $x_s$ are not central. Take $s_0 \in S \setminus S_1$. Clearly $x_{s_0}$ is in $K$.

For every $g \in G$ we choose nonzero $y_g \in D_g$, where as usual, if $h \in H$, then $y_h$ is chosen to commute with $D_e$. Consider the element $z = \sum_{g \in G} y_gx_{s_0}y_g^{-1}$. We show $z$ is central. Since $x_{s_0}$ is in $L_1$ and conjugation by $y_g$ normalizes $L_1$ we have it is central in $D_H$. It is sufficient to show $z$ is invariant under conjugation by $y_g$. But this is clear since conjugation of the sum by $y_g$ permutes the components modulo nonzero elements in $D_e$ which clearly centralize $x_{s_0}$. It remains to show $z \neq 0$.

Write $g(x_{s_0}) = y_gx_{s_0}y_g^{-1} = \beta_gx_{g(x_{s_0})}$ where $\beta : G \to L^*$. Note that for all $h \in H$ $\beta(h) = 1$ because $s_0 \in S$. We claim $\beta$ is a 1-cocycle. Indeed,

\[ g_1g_2(x_{s_0}) = g_1(g_2(x_{s_0})) \]
The left hand side yields
\[ \beta(g_1 g_2) x_{g_1 g_2(s_0)} \]
whereas the right hand side yields
\[ g_1(\beta(g_2) x_{g_2(s_0)}) = \beta(g_2)^s g_1(x_{g_1 g_2(s_0)}) \]
and the result follows.

Because \( \beta(H) = 1 \) we get an induced 1-cocycle (which we will also call \( \beta \)) with class in \( H^1(G/H, L^*) \). Applying Hilbert’s Theorem 90 there is \( t \in L^* \) with \( \beta(g) = g(t)t^{-1} \). We can therefore write \( g(x_{s_0}) = g(t)t^{-1}x_{g(x_0)} \) and so
\[ g(tx_{s_0}) = t^{-1}x_{g(x_0)} \]
Replacing the element \( x_{g(x_0)} \) by \( tx_{g(x_0)} \) and letting \( C_G(s_0) \) denote the centralizer of \( s_0 \) in \( G \), we see that \( z = |C_G(s_0)|x_{s_0} + \sum_{g \notin C_G(s_0)} y_g x_{s_0} y_g^{-1} \) which is nonzero. Note that here we have used the fact that the characteristic is zero. This completes the proof of the theorem.

3. CONSTRUCTION OF DIVISION ALGEBRAS GRADED BY \( G \)

In this section we prove that given a group extension
\[ 1 \to H \to G \to G/H \to 1 \]
where \( H \) is abelian, a positive integer \( d \), and a map \( \phi : M(H) \to \mu \) which is \( G \)-invariant there is a finite dimensional division algebra over its center, faithfully \( G \)-graded which realizes the given data.

Our constructions will produce graded division algebras \( D \) in which \( d \), which is the degree of \( D \), equals 1, that is, in which \( D \) is a field. It is straightforward to pass from this case to the case of arbitrary \( d \): Let \( L \) denote the field \( D \). Adjoin new variables \( a, b \) to \( k \) and let \( E = (a, b)_d \) be the symbol algebra of degree \( d \) over \( k(a, b) \) determined by \( a \) and \( b \). Then \( E \) is a division algebra and we can form \( \hat{D} = D \otimes_{K_0} (E \otimes_{k(a,b)} K_0) \), where \( K_0 \) is the \( c \)-center of \( D \) and \( E \otimes_{k(a,b)} K_0 \) has the trivial grading. Then \( \hat{D} \) is a division algebra and has the obvious \( G \)-grading with \( \hat{D}_e = E \otimes_{k(a,b)} \). So the degree of \( \hat{D} \) is \( d \), as desired.

Prior to our general treatment, we shall discuss two special cases: in case (1) we assume \( H = \{1\} \), and in case (2) we assume \( G = H \). The case where \( H \) is trivial is well known. We recall here a proof which is attributed to S. Rosset and K. Brown. Their idea will appear in the proof of the general case.

Let
\[ 1 \to R \to F \to G \to 1 \]
be a presentation of the finite group \( G \) where the groups \( F \) and \( R \) are finitely generated free.

Taking quotient groups modulo the commutator subgroup \([R, R]\) we obtain
\[ \beta : 1 \to N = R/[R, R] \to \Gamma = F/[R, R] \to G \to 1. \]

We want the action of \( G \) on \( N \) in this extension to be faithful, that is, we want the centralizer of \( N \) in \( \Gamma \) to be \( N \) itself. This can be arranged as follows: Let \( A \) be the free abelian group on \( n = |G| \) generators and let \( G \) act on \( A \) by permuting
these generators. We obtain the split exact sequence $1 \to A \to A \rtimes G \to G \to 1$. We take our group $F$ to be free on $2n$ generators $y_g, x_g$ for all $g \in G$ and map $F$ to $G$ by sending each $y_g$ to the identity and each $x_g$ to $g$. The group $F$ maps to $A \rtimes G$ by sending the $y_g's$ to generators of $A$ and each $x_g$ to $g$. The resulting extension denoted above by $\beta$ maps to this split exact sequence and a simple diagram chase then shows that $G$ acts faithfully on $N$.

By a theorem of Higman ([6], Theorem 2) $\Gamma = F/[R, R]$ is torsion free. Because $\Gamma$ is abelian by finite, it follows from a theorem of Brown that the group algebra $k\Gamma$ is a domain (See [4], Cor. 2). Furthermore, it follows from Goldie’s theorem that $k\Gamma$ as a classical ring of quotients which is actually a division algebra $D$. Clearly, since $N$ is of finite index in $\Gamma$, the division algebra $D$ is obtained by localizing $k\Gamma$ at $S = kN \setminus \{0\}$. Because the action of $G$ on $N$ is faithful, it is faithful on $kN$ and hence also on the field $E = \text{Frac}(kN)$. It is clear now that $S^{-1}k\Gamma$ is a $G$-crossed product of the form $(E/E^G, G, \beta)$ where $\beta$ is the cohomology class in $H^2(G, E^*)$ induced by the extension $\beta$ above.

We next consider case (2) in which $G = H$ is abelian and $\phi$ is any map $\phi : M(H) \to \mu \subset k^*$. Note that in this case every cohomology class is invariant.

Let

$$1 \to R \to F_H \to H \to 1$$

be a presentation of $H$. We take quotients modulo $[R, F_H]$ and obtain the central extension

$$1 \to R/[R, F_H] \to F_H/[R, F_H] \to H \to 1.$$

The group $R/[R, F_H]$ is finitely generated abelian and so is the direct product of a finite group $U$ and a finitely generated torsion free group $T \cong \mathbb{Z}^r$. We claim (and it is well known) that the torsion part is precisely $M(H)$, the Schur multiplier of $H$. Indeed, by the Hopf formula the Schur multiplier is given by $(R \cap [F_H, F_H])/[R, F_H]$ so it is necessary and sufficient to show that

$$R/[R, F] / ((R \cap [F_H, F_H])/[R, F_H]) \cong R/(R \cap [F_H, F_H])$$

is torsion free. But $R/(R \cap [F_H, F_H])$ is isomorphic to $R[F_H, F_H]/[F_H, F_H]$, a subgroup of the torsion free group $F_H/[F_H, F_H]$. This proves the claim.

Let $\Gamma_H = F_H/[R, F_H]$. The group $U$ is characteristic in $R/[R, F_H]$ and so normal in $\Gamma_H$. We therefore have the extension

$$\beta : 1 \to U \to \Gamma_H \to \Gamma_H/U \to 1$$

which is easily seen to be the inflation of the following extension $\bar{\beta}$:

$$\bar{\beta} : 1 \to U \to \Gamma_H/T \to H \to 1$$

We claim $\Gamma_H/U$ is torsion free: Indeed,

$$\Gamma_H/U = F_H/[R, F_H]/(R \cap [F_H, F_H])/[R, F_H] \cong$$

$$F_H/(R \cap [F_H, F_H]) = F_H/[F_H, F_H]$$

because $H$ is abelian. The claim follows.

The extension $\beta$ yields a crossed product $kU^\beta \Gamma_H/U$. Composing the map $\phi$, whose image is in $\mu$, with the 2-cocycle $\beta$ we obtain a cocycle which represents a class...
in $H^2(\Gamma_H/U, \mu)$ and which we denote by $\phi(\beta)$. Thus we obtain the twisted group algebra $k^{\phi(\beta)}\Gamma_H/U$. Because $\Gamma_H/U$ is torsion free and abelian by finite, the theorem of Moody (See for instance [7], Lemma, 37.8) implies that this twisted group algebra is an Ore domain. Because $\beta$ is the inflation of $\bar{\beta}$, the elements of $T$ in $k^{\phi(\beta)}\Gamma_H/U$ are central and hence, by inverting the elements of $T$, we obtain the twisted group division algebra $k(T)^{\phi(\beta)}H$ over the field $k(T)$. Letting $k(T)^{\phi(\beta)}H = \sum_{x \in H} k(T)x_0$ we compute easily that for all $h_1, h_2 \in H$, the commutator $x_{h_1}x_{h_2}^{-1}x_{h_1}^{-1}h_2 = \phi(h_1 \wedge h_2)$ and so, by the comments preceding Definition 1.5, this division algebra realizes the given data.

We know turn to proof of the general case. Let

$$1 \to R \to F_G \to G \to 1$$

be a presentation of the finite group $G$ (We will be more precise concerning the choice of $F_G$ below). We denote the map $F_G \to G$ by $\pi$.

For the subgroup $H$ we obtain the following induced extension, where $F_H = \pi^{-1}(H)$:

$$1 \to R \to F_H \to H \to 1$$

Taking quotients modulo $[R, F_H]$ in these two extensions we obtain

$$1 \to R/\Gamma_H \to F_G/\Gamma_H \to G \to 1$$

and

$$1 \to R/\Gamma_H \to F_H/\Gamma_H \to H \to 1.$$ 

As in case (2), the group $R/\Gamma_H$ is the direct product of a finite group $U = M(H)$ and a finitely generated torsion free group $T \cong \mathbb{Z}^r$. Moreover $U$ is characteristic in $R/\Gamma_H$ and hence normal in both $\Gamma_H = F_H/\Gamma_H$ and $\Gamma_G = F_G/\Gamma_H$. The group $\Gamma_G/U$ is torsion free: We have $\Gamma_G/U \cong F_G/\Gamma_H/(R \cap [F_H, F_H])/[R, F_H] \cong F_G/[F_H, F_H]$ because $H$ is abelian, and $F_G/[F_H, F_H]$ is torsion free by Higman’s theorem.

We have the extension:

$$\beta : 1 \to U \to \Gamma_G \to \Gamma_G/U \to 1.$$ 

Because $\phi : M(H) \to \mu$ is $G$ invariant, the map $\phi \circ \beta : \Gamma_G/U \times \Gamma_G/U \to \mu$ is a 2-cocycle. Therefore we can form the twisted group algebra $k^{\phi(\beta)}\Gamma_G/U$. The group $\Gamma_G/U$ is abelian by finite and we have seen that it is torsion free. Again as in case (2) we infer that $k^{\phi(\beta)}\Gamma_G/U$ is an Ore domain.

We claim the ring of quotients of $k^{\phi(\beta)}\Gamma_G/U$ is a $G$ graded division algebra satisfying the desired conditions.

What do we have to prove? Because it is an Ore domain the ring of quotients is a division algebra which we denote by $D$. So we need to prove that $D$ is finite dimensional over its center, $G$-graded and the grading gives rise to the given group extension

$$1 \to H \to G \to G/H \to 1$$

where $H$ is abelian, and the given $G$-invariant map $\phi : M(H) \to \mu$. 
We first consider the subalgebra \( k^{\phi(H)} \Gamma_H/U \). This is the same as the division algebra obtained in case (2) and as we saw there the cocycle \( \beta \) restricted to \( \Gamma_H/U \) is inflated from the extension

\[
1 \to U \to \Gamma_H/T \to H \to 1
\]

Therefore the elements of \( T \) in \( k^{\phi(H)} \Gamma_H/U \) are central and hence, by inverting the elements of \( T \), we obtain the twisted group division algebra \( k(T)^{\phi(H)} \) over the field \( k(T) \). Also as in case (2) the cocycle on \( H \) maps to \( \phi \), as desired.

From the extension \( 1 \to T \to \Gamma_G/U \to G \to 1 \) we see that if we set \( D_e = k(T) \) and choose representatives \( x_g \) in \( \Gamma_G/U \) for the elements \( g \) in \( G \), then \( D = \sum_{g \in G} D_e x_g \) is faithfully \( G \)-graded. We are then left with showing that \( H \) is exactly the kernel of the action of \( G \) on \( k(T) \). For that we need to be more precise about our choice of \( F_G \): What is required is that in the sequence \( 1 \to T \to F_G/U \to G \to 1 \) the kernel of the action of \( G \) on \( T \) is exactly \( H \) (We already know the kernel contains \( H \)). We proceed as in case (1): Let \( A \) be the free abelian group on \( m = \left| G/H \right| \) generators and let \( G \) act on \( A \) by permuting these coset generators. We obtain the split exact sequence \( 1 \to A \to A \times G \to G \to 1 \). We take our group \( F_G \) to be free on \( m + n \) generators \( y_{g_1}, y_{g_2}, \ldots, y_{g_m} \), where \( g_1, g_2, \ldots, g_m \) are distinct coset representatives of \( H \) in \( G \) union the generators \( x_g \) for all \( g \) in \( G \). We map \( F_G \to G \) by sending each \( y_{g_i} \) to the identity and each \( x_g \) to \( g \). We map \( F_G \) to \( A \times G \) by sending the \( y_{g_i} \)'s to generators of \( A \) and each \( x_g \) to \( g \). With this choice of presentation for \( G \) a straightforward calculation shows that we have an induced map of extensions from \( 1 \to \Gamma_G/U \to G \to 1 \) to \( 1 \to A \to A \times G \to G \to 1 \). Because \( U \) is torsion and \( A \) is torsion free we get an induced map of extensions from \( 1 \to T \to \Gamma_G/U \to G \to 1 \) to \( 1 \to A \to A \times G \to G \to 1 \). A simple diagram chase now shows that the kernel of the action of \( G \) on \( T \) is exactly \( H \), as desired.

4. Twisted forms of finite dimensional \( G \)-simple algebras

In this section we prove Theorem 1.10.

**Proof.** As we have seen above, any finite dimensional division algebra over its center \( K \), faithfully graded by a finite group \( G \), is finite dimensional over the \( e \)-center \( K_0 \) and hence if we extend scalars over \( K_0 \) to its algebraic closure \( F \) we obtain a finite dimensional algebra, \( G \)-simple over \( F \). These algebras were characterised by Bahturin, Sehgal and Zaicev (see Theorem 1.9) in terms of fine and elementary grading. More precisely given a \( G \)-simple algebra \( A \) finite dimensional over its \( e \)-center \( F \), where \( F \) is an algebraically closed field of characteristic zero, the \( G \)-grading is given by a presentation \( P_A = (H, (g_1, \ldots, g_s), \alpha) \) where \( H \) is a subgroup of \( G \), \( (g_1, \ldots, g_s) \) is an \( s \)-tuple in \( G \) and \( \alpha \) an element in \( H^2(H, F^*) \). By [1], Lemma 1.3, we may replace the elements in the \( s \)-tuple by right \( H \)-cosets representatives. Moreover by permuting elements of the \( s \)-tuple we may assume equal representatives are adjacent to each other. We denote the normalized tuple by \( \Theta \). We let \( n_i \) denote the multiplicity (possibly zero) of the \( i \)th coset in \( \Theta \), hence \( s = n_1 + \cdots + n_r \) where \( r = \left| [G : H] \right| \). Now, we are interested in \( G \)-simple algebras over \( F \) which are obtained by scalar extensions from finite dimensional division algebras which are \( G \)-graded. Since these are in particular \( G \)-graded division algebras, we may apply [2], Theorems 1.8 and 5.3, which say that a finite dimensional \( G \)-simple algebra admits a \( G \)-graded division algebra form \( B \) whose \( e \)-center contains an algebraically closed field, if and
only if the $G$-grading on $A$ is given by a presentation $P_A = (H, (g_1, \ldots, g_s), \alpha)$ where $H$ normal in $G$, all cosets representatives are equally represented (possibly zero, i.e. if $G = H$) and the class $\alpha$ is $G$-invariant. We need to prove that such $G$-simple algebras $A$ but with $H$-abelian are precisely the $G$-simple algebras which admit a $G$-graded form which is an ungraded division algebra. This will follow rather easily from the sections above. Indeed, we have seen that a division algebra $D$ that is $G$-graded, finite dimensional over its center $K$ (and hence finite dimensional over its $e$-center $K_0$) gives rise to a group extension with abelian kernel

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1,$$

a $G$-invariant map $\phi : M(H) \rightarrow \mu$ (where the action of $G$ on $\mu$ is trivial and the action on $M(H)$ is induced by conjugation) and an integer $d$, the degree of the division algebra $D_e$. We claim that extending scalars of $D$ over the $e$-center $K_0$ yields a finite dimensional $G$-simple algebra $A$ with presentation $P_A = (H, (g_1, \ldots, g_s), \alpha_\phi)$, namely the same group $H$ and $\alpha_\phi$ is the cohomology class in $H^2(H, F^*)$ which corresponds to the given map $\phi$ by means of the Universal Coefficient Theorem. This is basically clear. We will show only that the cocycle $\alpha$ in $P_A$ is indeed $\alpha_\phi$. By the previous remarks we see that extension of scalars to $F$ yields a 2-cocycle of $H$ with values in the units of $F \otimes_{K_0} L = F \times \cdots \times F$ ($[G : H]$-times). Now, by the Universal Coefficient Theorem a cocycle with values in $F^*$ (algebraically closed) is cohomological to a cocycle whose values are roots of unity and because, by assumption, these are contained in the $e$-center, adding the fact that $\alpha$ is $G$-invariant, the values of $\alpha$ are in fact diagonally embedded in $F \times \cdots \times F$. This proves our claim. In fact we see that a $G$-grading on a division algebra $D$, realizes the data (group extension with abelian kernel, $G$-invariant map $\phi$ and integer $d$) if and only if extending scalars to the algebraic closure of $K_e$ yields a $G$-simple algebra with presentation $P_A = (H, (g_1, \ldots, g_s), \alpha)$ as in [2], Theorem 1.8, with the extra condition that $H$ is abelian. Note that the integer $d$ is the multiplicity of the cosets representatives in the $s$-tuple $(g_1, \ldots, g_s)$. Finally, since any triple (extension, map, integer) can be realized by a $G$-grading on a finite dimensional division algebra the same holds for the $G$-simple algebra $A$.

\[ \square \]

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