On the boundary behavior of the holomorphic sectional curvature of the Bergman metric

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Abstract. We obtain a conceptually new differential geometric proof of P.F. Klembeck’s result (cf. [9]) that the holomorphic sectional curvature $k_g(z)$ of the Bergman metric of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ approaches $-4/(n+1)$ (the constant sectional curvature of the Bergman metric of the unit ball) as $z \to \partial \Omega$.

1. Introduction

Given a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ C.R. Graham & J.M. Lee studied (cf. [7]) the $C^\infty$ regularity up to the boundary for the solution to the Dirichlet problem $\Delta_g u = 0$ in $\Omega$ and $u = f$ on $\partial \Omega$, where $\Delta_g$ is the Laplace-Beltrami operator of the Bergman metric $g$ of $\Omega$. If $\varphi \in C^\infty(U)$ is a defining function ($\Omega = \{ z \in U : \varphi(z) < 0 \}$) their approach is to consider the foliation $\mathcal{F}$ of a one-sided neighborhood $V$ of the boundary $\partial \Omega$ by level sets $M_\epsilon = \{ z \in V : \varphi(z) = -\epsilon \}$ ($\epsilon > 0$). Then $\mathcal{F}$ is a tangential CR foliation (cf. S. Dragomir & S. Nishikawa, [11]) each of whose leaves is strictly pseudoconvex and one may express $\Delta_g u = 0$ in terms of pseudohermitian invariants of the leaves and the transverse curvature $r = 2 \partial \overline{\partial} \varphi(\xi, \overline{\xi})$ and its derivatives (the meaning of $\xi$ is explained in the next section). The main technical ingredient is an ambient linear connection $\nabla$ on $V$ whose pointwise restriction to each leaf of $\mathcal{F}$ is the Tanaka-Webster connection (cf. S. Webster, [14], and N. Tanaka, [13]) of the leaf. An axiomatic description (and index free proof) of the existence and uniqueness of $\nabla$ (referred to as the Graham-Lee connection of $(V, \varphi)$) was provided in [11]. As a natural continuation of the ideas in [11] one may relate the Levi-Civita connection $\nabla^g$ of $(V, g)$ to the Graham-Lee connection $\nabla$ and compute the curvature $R^g$ of $\nabla^g$ in terms of the curvature of $\nabla$. Together with an elementary asymptotic

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analysis (as $\epsilon \to 0$) this leads to a purely differential geometric proof of the result of P.F. Klembeck, [9], that the sectional curvature of $(\Omega, g)$ tends to $-\frac{4}{n+1}$ near the boundary $\partial \Omega$. The Author believes that one cannot overestimate the importance of the Graham-Lee connection (and that the identities (27) and (36) in Section 3 admit other applications as well, e.g. in the study of the geometry of the second fundamental form of a submanifold in $(\Omega, g)$).

2. The Levi-Civita versus the Graham-Lee connection

Let $\Omega$ be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $K(z, \zeta)$ its Bergman kernel (cf. e.g. [8], p. 364-371). As a simple application of C. Fefferman’s asymptotic development (cf. [6]) of the Bergman kernel $\varphi(z) = -K(z, z)^{-1/(n+1)}$ is a defining function for $\Omega$ (and $\Omega = \{ \varphi < 0 \}$). Cf. A. Korányi & H.M. Reimann, [11], for a proof. Let us set $\theta = \frac{i}{2}(\overline{\partial} - \partial)\varphi$. Then $d\theta = i \partial \overline{\varphi}$. Let us differentiate $\log |\varphi| = -(1/(n+1)) \log K$ (where $K$ is short for $K(z, z)$) so that to obtain

$$\frac{1}{\varphi} \overline{\partial} \varphi = -\frac{1}{n+1} \overline{\partial} \log K.$$ 

Applying the operator $i \partial$ leads to

$$\frac{1}{\varphi} d\theta - \frac{i}{\varphi^2} \partial \varphi \wedge \overline{\partial} \varphi = -\frac{i}{n+1} \partial \overline{\partial} \log K.$$ 

We shall need the Bergman metric $g_{\bar{z}k} = \partial^2 \log K/\partial z^i \partial \bar{z}^k$. This is well known to be a Kähler metric on $\Omega$.

**Proposition 1.** For any smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ the Bergman metric $g$ is given by

$$g(X, Y) = \frac{n+1}{\varphi} \left\{ \frac{i}{\varphi} (\partial \varphi \wedge \overline{\partial} \varphi)(X, JY) - d\theta(X, JY) \right\},$$

for any $X, Y \in \mathcal{X}(\Omega)$.

**Proof.** Let $\omega(X, Y) = g(X, JY)$ be the Kähler 2-form of $(\Omega, J, g)$, where $J$ is the underlying complex structure. Then $\omega = -i \partial \overline{\partial} \log K$ and [12] may be written in the form (2). Q.e.d.

We denote by $M_{\epsilon} = \{ z \in \Omega : \varphi(z) = -\epsilon \}$ the level sets of $\varphi$. For $\epsilon > 0$ sufficiently small $M_{\epsilon}$ is a strictly pseudoconvex CR manifold (of CR dimension $n-1$). Therefore, there is a one-sided neighborhood $V$ of $\partial \Omega$ which is foliated by the level sets of $\varphi$. Let $\mathcal{F}$ be the relevant foliation and let us denote by $H(\mathcal{F}) \to V$ (respectively by $T_{1,0}(\mathcal{F}) \to$
V) the bundle whose portion over $M_\epsilon$ is the Levi distribution $H(M_\epsilon)$ (respectively the CR structure $T_{1,0}(M_\epsilon)$) of $M_\epsilon$. Note that
\[ T_{1,0}(\mathcal{F}) \cap T_{0,1}(\mathcal{F}) = (0), \]
\[ [\Gamma^\infty(T_{1,0}(\mathcal{F})), \Gamma^\infty(T_{1,0}(\mathcal{F}))] \subseteq \Gamma^\infty(T_{1,0}(\mathcal{F})). \]
Here $T_{0,1}(\mathcal{F}) = \overline{T_{1,0}(\mathcal{F})}$. For a review of the basic notions of CR and pseudohermitian geometry needed through this paper one may see S. Dragomir & G. Tomassini, [5]. Cf. also S. Dragomir, [3]. By a result of J.M. Lee & R. Melrose, [12], there is a unique complex vector field $\xi$ on $V$, of type $(1,0)$, such that $\partial \varphi(\xi) = 1$ and $\xi$ is orthogonal to $T_{1,0}(\mathcal{F})$ with respect to $\partial \varphi$ i.e. $\partial \varphi(\xi, Z) = 0$ for any $Z \in T_{1,0}(\mathcal{F})$. Let $r = 2 \partial \varphi(\xi, \bar{\xi})$ be the transverse curvature of $\varphi$. Moreover let $\xi = \frac{1}{2}(N - iT)$ be the real and imaginary parts of $\xi$. Then
\[(d\varphi)(N) = 2, \quad (d\varphi)(T) = 0, \]
\[\theta(N) = 0, \quad \theta(T) = 1, \]
\[\partial \varphi(N) = 1, \quad \partial \varphi(T) = i. \]
In particular $T$ is tangent to (the leaves of) $\mathcal{F}$. Let $g_\theta$ be the tensor field given by
\[ g_\theta(X, Y) = (d\theta)(X, JY), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1, \]
for any $X, Y \in H(\mathcal{F})$. Then $g_\theta$ is a tangential Riemannian metric for $\mathcal{F}$ i.e. a Riemannian metric in $T(\mathcal{F}) \to V$. Note that the pullback of $g_\theta$ to each leaf $M_\epsilon$ of $\mathcal{F}$ is the Webster metric of $M_\epsilon$ (associated to the contact form $j_\epsilon^* \theta$, where $j_\epsilon : M_\epsilon \subset V$). As a consequence of (2), $JT = -N$ and $i_N d\theta = r \theta$ (see also [2] below)

**Corollary 1.** The Bergman metric $g$ of $\Omega \subset \mathbb{C}^n$ is given by
\[ g(X, Y) = -\frac{n + 1}{\varphi} g_\theta(X, Y), \quad X, Y \in H(\mathcal{F}). \]
\[ g(X, T) = 0, \quad g(X, N) = 0, \quad X \in H(\mathcal{F}), \]
\[ g(T, N) = 0, \quad g(T, T) = g(N, N) = \frac{n + 1}{\varphi} \left( \frac{1}{\varphi} - r \right). \]
In particular $1 - r \varphi > 0$ everywhere in $\Omega$.

Using (4)-(6) we may relate the Levi-Civita connection $\nabla^g$ of $(V, g)$ to another canonical linear connection on $V$, namely the Graham-Lee connection of $\Omega$. The latter has the advantage of staying finite at the boundary (it gives the Tanaka-Webster connection of $\partial \Omega$ as $z \to \partial \Omega$).
We proceed to recalling the Graham-Lee connection. Let \( \{W_\alpha : 1 \leq \alpha \leq n-1\} \) be a local frame of \( T_{1,0}(\mathcal{F}) \), so that \( \{W_\alpha, \xi\} \) is a local frame of \( T^{1,0}(V) \). We consider as well \( L_\theta(Z, W) \equiv -i(d\theta)(Z, W) \), \( Z, W \in T_{1,0}(\mathcal{F}) \). Note that \( L_\theta \) and (the \( \mathbb{C} \)-linear extension of) \( g_\theta \) coincide on \( T_{1,0}(\mathcal{F}) \otimes T_{0,1}(\mathcal{F}) \). We set \( g_\alpha = g_\theta(W_\alpha, W_\beta) \). Let \( \{\theta_\alpha : 1 \leq \alpha \leq n-1\} \) be the (locally defined) complex 1-forms on \( V \) determined by \( \theta_\alpha(W_\beta) = \delta_\alpha^\beta \), \( \theta_\alpha(W_\beta) = 0 \), \( \theta_\alpha(T) = 0 \), \( \theta_\alpha(N) = 0 \).

Then \( \{\theta_\alpha, \theta_\beta, \theta, d\varphi\} \) is a local frame of \( T(V) \otimes \mathbb{C} \) and one may easily show that
\[
d\theta = 2ig_\alpha \theta_\alpha \wedge \bar{\theta}_\beta + r d\varphi \wedge \theta.
\]

As an immediate consequence
\[
i_Td\theta = -\frac{r}{2}d\varphi, \quad i_Nd\theta = r\theta.
\]

As an application of (7) we decompose \([T, N]\) (according to \( T(V) \otimes \mathbb{C} = T_{1,0}(\mathcal{F}) \oplus T_{0,1}(\mathcal{F}) \oplus \mathbb{C}T \oplus \mathbb{C}N) \) and obtain
\[
[T, N] = iW^\alpha(r)W_\alpha - iW^\alpha(r)W_\alpha + 2rT,
\]
where \( W^\alpha(r) = g_\alpha \bar{W}_\beta(r) \) and \( W^\alpha(r) = \bar{W}^\alpha(r) \).

Let \( \nabla \) be a linear connection on \( V \). Let us consider the \( T(V) \)-valued 1-form \( \tau \) on \( V \) defined by
\[
\tau(X) = T_\nabla(T, X), \quad X \in T(V),
\]
where \( T_\nabla \) is the torsion tensor field of \( \nabla \). We say \( T_\nabla \) is pure if
\[
T_\nabla(Z, W) = 0, \quad T_\nabla(Z, W) = 2iL_\theta(Z, W)T,
\]
\[
T_\nabla(N, W) = rW + i\tau(W),
\]
for any \( Z, W \in T_{1,0}(\mathcal{F}) \), and
\[
\tau(T_{1,0}(\mathcal{F})) \subseteq T_{0,1}(\mathcal{F}),
\]
\[
\tau(N) = -J\nabla^H - 2rT.
\]
Here \( \nabla^H \) is defined by \( \nabla^H = \pi_H \nabla \) and \( g_\theta(\nabla, X) = X(r), \quad X \in T(\mathcal{F}) \). Also \( \pi_H : T(\mathcal{F}) \to H(\mathcal{F}) \) is the projection associated to the direct sum decomposition \( T(\mathcal{F}) = H(\mathcal{F}) \oplus \mathbb{R}T \). We recall the following
**Theorem 1.** There is a unique linear connection $\nabla$ on $V$ such that i) $T_{1,0}(\mathcal{F})$ is parallel with respect to $\nabla$, ii) $\nabla L_\theta = 0$, $\nabla T = 0$, $\nabla N = 0$, and iii) $T \nabla$ is pure.

$\nabla$ given by Theorem 1 is the *Graham-Lee connection*. Theorem 1 is essentially Proposition 1.1 in [7], p. 701-702. The axiomatic description in Theorem 1 is due to [4] (cf. Theorem 2 there). An index-free proof of Theorem 1 was given in [1] relying on the following

**Lemma 1.** Let $\phi : T(\mathcal{F}) \to T(\mathcal{F})$ be the bundle morphism given by $\phi(X) = JX$, for any $X \in H(\mathcal{F})$, and $\phi(T) = 0$. Then

\[
\phi^2 = -I + \theta \otimes T,
\]

\[
g_\theta(X, T) = \theta(X),
\]

\[
g_\theta(\phi X, \phi Y) = g_\theta(X, Y) - \theta(X)\theta(Y),
\]

for any $X, Y \in T(\mathcal{F})$. Moreover, if $\nabla$ is a linear connection on $V$ satisfying the axioms (i)-(iii) in Theorem 1 then

\[
(14) \quad \phi \circ \tau + \tau \circ \phi = 0
\]

along $T(\mathcal{F})$. Consequently $\tau$ may be computed as

\[
(15) \quad \tau(X) = -\frac{1}{2} \phi(L_T \phi) X,
\]

for any $X \in H(\mathcal{F})$.

A rather lengthy but straightforward calculation (based on Corollary 1) leads to

**Theorem 2.** Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain, $K(z, \zeta)$ its Bergman kernel, and $\varphi(z) = -K(z, z)^{-1/(n+1)}$. Then the Levi-Civita connection $\nabla^g$ of the Bergman metric and the Graham-Lee connection of $(\Omega, \varphi)$ are related by

\[
(16) \quad \nabla^g_X Y = \nabla_X Y +
\]

\[
+ \left\{ \frac{\varphi}{1 - r \varphi} g_\theta(\tau X, Y) + g_\theta(X, \phi Y) \right\} T -
\]

\[
- \left\{ g_\theta(X, Y) + \frac{\varphi}{1 - r \varphi} g_\theta(X, \phi \tau Y) \right\} N,
\]

\[
(17) \quad \nabla^g_X T = \tau X - \left( \frac{1}{\varphi} - r \right) \phi X -
\]

\[
- \frac{\varphi}{2(1 - r \varphi)} \left\{ X(r)T + (\phi X)(r)N \right\},
\]
\begin{align*}
(18) \quad \nabla^g_X N &= - \left( \frac{1}{\varphi} - r \right) X + \tau \phi X + \\
&\quad + \frac{\varphi}{2(1 - r\varphi)} \{(\phi X)(r)T - X(r)N\}, \\
(19) \quad \nabla^g_T X &= \nabla_T X - \left( \frac{1}{\varphi} - r \right) \phi X - \\
&\quad - \frac{\varphi}{2(1 - r\varphi)} \{X(r)T + (\phi X)(r)N\}, \\
(20) \quad \nabla^g_N X &= \nabla_N X - \frac{1}{\varphi} X + \\
&\quad + \frac{\varphi}{2(1 - r\varphi)} \{(\phi X)(r)T - X(r)N\}, \\
(21) \quad \nabla^g_N T &= - \frac{1}{2} \phi \nabla^H r - \\
&\quad - \frac{\varphi}{2(1 - r\varphi)} \left\{ \left( N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi} \right) T + T(r)N \right\}, \\
(22) \quad \nabla^g_T N &= \frac{1}{2} \phi \nabla^H r - \\
&\quad - \frac{\varphi}{2(1 - r\varphi)} \left\{ \left( N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2 \right) T + T(r)N \right\}, \\
(23) \quad \nabla^g_T T &= - \frac{1}{2} \nabla^H r - \\
&\quad - \frac{\varphi}{2(1 - r\varphi)} \left\{ T(r)T - \left( N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2 \right) N \right\}, \\
(24) \quad \nabla^g_N T &= - \frac{1}{2} \nabla^H r + \\
&\quad + \frac{\varphi}{2(1 - r\varphi)} \left\{ T(r)T - \left( N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi} \right) N \right\},
\end{align*}

for any $X, Y \in H(\mathcal{F})$. 
3. Klembeck’s theorem

The original proof of the result by P.F. Klembeck (cf. Theorem 1 in [9], p. 276) employs a formula of S. Kobayashi, [10], expressing the components $R_{jkr\pi}$ of the Riemann-Christoffel 4-tensor of $(\Omega, g)$ as

$$-\frac{1}{2}R_{jkr\pi} = g_{j\pi}g_{r\pi} + g_{j\pi}g_{r\pi} - \frac{1}{K^2} \{ K_{jkr\pi} - K_{jr}K_{k\pi} \} +$$

$$+ \frac{1}{K^4} \sum_{\ell, m} g_{\ell m} \{ K_{jkr\ell} - K_{jr}K_{k\ell} \} \{ K_{k\pi m} - K_{k\pi}K_{m} \}$$

where $K = K(z, z)$ and its indices denote derivatives. However the calculation of the inverse matrix $[g_{j\pi}] = [g_{j\pi}]^{-1}$ turns out to be a difficult problem and [9] only provides an asymptotic formula as $z \to \partial \Omega$. Our approach is to compute the holomorphic sectional curvature of $(\Omega, g)$ by deriving an explicit relation among the curvature tensor fields $R^g$ and $R$ of the Levi-Civita and Graham-Lee connections respectively. We start by recalling a pseudohermitian analog to holomorphic curvature (built by S.M. Webster, [14]).

Let $M$ be a nondegenerate CR manifold of type $(n - 1, 1)$ and $\theta$ a contact form on $M$. Let $G_1(H(M))_x$ consist of all 2-planes $\sigma \subset T_x(M)$ such that i) $\sigma \subset H(M)_x$ and ii) $J_x(\sigma) = \sigma$. Then $G_1(H(M))$ (the disjoint union of all $G_1(H(M))_x$) is a fibre bundle over $M$ with standard fibre $\mathbb{CP}^{n-2}$. Let $R^\nabla$ be the curvature of the Tanaka-Webster connection $\nabla$ of $(M, \theta)$. We define a function $k_\theta : G_1(H(M)) \to \mathbb{R}$ by setting

$$k_\theta(\sigma) = -\frac{1}{4} R^\nabla_x(X, J_xX, X, J_xX)$$

for any $\sigma \in G_1(H(M))$ and any linear basis $\{X, J_xX\}$ in $\sigma$ satisfying $G_\theta(X, X) = 1$. It is a simple matter that the definition of $k_\theta(\sigma)$ does not depend upon the choice of orthonormal basis $\{X, J_xX\}$, as a consequence of the following properties

$$R^\nabla(Z, W, X, Y) + R^\nabla(Z, W, Y, X) = 0,$$

$$R^\nabla(Z, W, X, Y) + R^\nabla(W, Z, X, Y) = 0.$$
of the 2-plane \( \sigma \) the sectional curvature \( k_\theta(\sigma) \) is also expressed by

\[
k_\theta(\sigma) = -\frac{1}{4} \frac{R^\nabla_x(X, J_x X, X, J_x X)}{G_\theta(X, X)^2}.
\]

To prove this statement one merely applies the definition of \( k_\theta(\sigma) \) for the orthonormal basis \( \{ U, J_x U \} \), with \( U = G_\theta(X, X)^{-1/2} X \). As \( X \in H(M)_x \) there is \( Z \in T_{1,0}(M)_x \) such that \( X = Z + \bar{Z} \). Thus

\[
k_\theta(\sigma) = \frac{1}{4} \frac{R_x(Z, \bar{Z}, Z, \bar{Z})}{g_\theta(Z, Z)^2}.
\]

The coefficient 1/4 is chosen such that the sphere \( S^{2n-1} \subset \mathbb{C}^n \) has constant curvature +1. Cf. [5], Chapter 1. With the notations in Section 2 let us set \( f = \varphi/(1 - \varphi^2) \). Then

\[
X(f) = f^2 X(r), \quad X \in T(\mathcal{F}).
\]

Let \( R^g \) and \( R \) be respectively the curvature tensor fields of the linear connections \( \nabla^g \) and \( \nabla \) (the Graham-Lee connection). For any \( X, Y, Z \in H(\mathcal{F}) \) (by (16))

\[
\nabla_X^g \nabla^g_Y Z = \nabla_X^g (\nabla_Y Z + \{ f g_\theta(\tau(Y), Z) + g_\theta(Y, \phi Z) \} T - \{ g_\theta(Y, Z) + f g_\theta(Y, \phi \tau(Z)) \} N) = \nabla_Y Z \in H(\mathcal{F}) \text{ together with (16)}
\]

\[
= \nabla_X \nabla_Y Z + \{ f g_\theta(\tau(X), \nabla_Y Z) + g_\theta(X, \phi \nabla_Y Z) \} T - \{ g_\theta(X, \nabla_Y Z) + f g_\theta(X, \phi \tau(\nabla_Y Z)) \} N + \{ f g_\theta(\tau(Y), Z) + g_\theta(Y, \phi Z) \} \nabla_X^g T + \{ X(f) g_\theta(\tau(Y), Z) + f X(g_\theta(\tau(Y), Z)) + X(g_\theta(Y, \phi Z)) \} T - \{ g_\theta(Y, Z) + f g_\theta(Y, \phi \tau(Z)) \} \nabla_X^g N + \{ X(g_\theta(Y, Z)) + X(f) g_\theta(Y, \phi \tau(Z)) + f X(g_\theta(Y, \phi \tau(Z))) \} N = \text{by (17), (18)}
\]

\[
= \nabla_X \nabla_Y Z + \{ X(\Omega(Y, Z)) + \Omega(X, \nabla_Y Z) + X(f) A(Y, Z) + f [X(A(Y, Z)) + A(X \nabla_Y Z)] \} T - \{ X(g_\theta(Y, Z)) + g_\theta(X, \nabla_Y Z) + X(f) \Omega(Y, \tau(Z)) + f [X(\Omega(Y, \tau(Z))) + \Omega(X, \tau(\nabla_Y Z))] \} N + \{ f A(Y, Z) + \Omega(Y, Z) \} \times \left\{ \tau(X) - \frac{1}{f} \phi X - \frac{f}{2} (X(r) T + (\phi X)(r) N) \right\} - \{ g_\theta(Y, Z) + f \Omega(Y, \tau(Z)) \} \times \left\{ -\frac{1}{f} X + \tau(\phi X) + \frac{f}{2} ((\phi X)(r) T - X(r) N) \right\}
\]
where we have set as usual \( A(X, Y) = g_\theta(\tau(X), Y) \) and \( \Omega(X, Y) = g_\theta(X, \phi Y) \). We may conclude that

\[
(25) \quad \nabla^g_X \nabla^g_Y Z = \nabla_X \nabla_Y Z + [f A(Y, Z) + \Omega(Y, Z)] \left( \tau(X) - \frac{1}{f} \phi X \right) + \\
+ [g_\theta(Y, Z) + f \Omega(Y, \tau(Z))] \left( \frac{1}{f} X - \tau(\phi X) \right) + \\
+ \{X(\Omega(Y, Z)) + \Omega(X, \nabla_Y Z) + f [X(A(Y, Z)) + A(X, \nabla_Y Z)] + \\
+ \frac{1}{2} [X(r)(f A(Y, Z) - \Omega(Y, Z)) - \\
- (\phi X)(r)(g_\theta(Y, Z) + f \Omega(Y, \tau(Z))))] \right] T - \\
- \{X(g_\theta(Y, Z)) + g_\theta(X, \nabla_Y Z) + f [X(\Omega(Y, \tau(Z))) + \Omega(X, \tau(\nabla_Y Z))] - \\
- \frac{1}{2} [X(r)(g_\theta(Y, Z) - f \Omega(Y, \tau(Z))] - \\
- (\phi X)(r)(f A(Y, Z) + \Omega(Y, Z))] \right] N + \\
+ \theta([X, Y]) \right\} \left\{ \nabla_T Z - \frac{1}{f} \phi Z - \frac{f}{2} (Z(r)T + (\phi Z)(r)N) \right\}
\]

for any \( X, Y, Z \in H(\mathcal{F}) \). Next we use the decomposition \( [X, Y] = \pi_H[X, Y] + \theta([X, Y])T \) and (16), (19) to calculate

\[
\nabla^g_{[X,Y]} Z = \nabla^g_{\pi_H[X,Y]} Z + \theta([X, Y])\nabla^g_T Z = \\
\nabla_{\pi_H[X,Y]} Z + \{f g_\theta(\tau(\pi_H[X, Y]), Z) + g_\theta(\pi_H[X, Y], \phi Z)\} T - \\
- \{f g_\theta(\pi_H[X, Y], Z) + f g_\theta(\pi_H[X, Y], \phi \tau(Z))\} N + \\
+ \theta([X, Y]) \right\} \left\{ \nabla_T Z - \frac{1}{f} \phi Z - \frac{f}{2} (Z(r)T + (\phi Z)(r)N) \right\}
\]

so that (by \( \tau(T) = 0 \))

\[
(26) \quad \nabla^g_{[X,Y]} Z = \nabla_{[X,Y]} Z - \frac{1}{f} \theta([X, Y]) \phi Z + \\
+ \left\{ f A([X, Y], Z) + \Omega([X, Y], Z) - \frac{1}{2} \theta([X, Y])Z(r) \right\} T - \\
- \left\{ f g_\theta([X, Y], Z) + f \Omega([X, Y], \tau(Z)) + \frac{1}{2} \theta([X, Y])(\phi Z)(r) \right\} N
\]

for any \( X, Y, Z \in H(\mathcal{F}) \). Consequently by (25)- (26) (and by \( \nabla g_\theta = 0 \), \( \nabla \Omega = 0 \)) we may compute

\[
R^g(X, Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z
\]

so that to obtain

\[
(27) \quad R^g(X, Y)Z = R(X, Y)Z + \frac{1}{f} \theta([X, Y]) \phi Z + \\
+ (f A(Y, Z) + \Omega(Y, Z)) \left( \tau(X) - \frac{1}{f} \phi X \right) - \\
- \left\{ f g_\theta([X, Y], Z) + f \Omega([X, Y], \tau(Z)) + \frac{1}{2} \theta([X, Y])(\phi Z)(r) \right\} N
\]
\[-(f A(X, Z) + \Omega(X, Z)) \left( \tau(Y) - \frac{1}{f} \phi Y \right) + \\
+ (g_\theta(Y, Z) + f \Omega(Y, \tau(Z))) \left( \frac{1}{f} X - \tau(\phi X) \right) - \\
-(g_\theta(X, Z) + f \Omega(X, \tau(Z))) \left( \frac{1}{f} Y - \tau(\phi Y) \right) + \\
+ \{ f \left( [\nabla_X A](Y, Z) - \nabla Y A(X, Z) \right) \\
+ \frac{f}{2} [X(r)(f A(Y, Z) - \Omega(Y, Z)) - Y(r)(f A(X, Z) - \Omega(X, Z)) - \\
-(\phi X)(r)(g_\theta(Y, Z) + f \Omega(Y, \tau(Z))) + (\phi Y)(r)(g_\theta(X, Z) + f \Omega(X, \tau(Z))) + \\
+ Z(r) \theta([X, Y]) \} T - \\
- \{ f \left( [\Omega(Y, (\nabla_X \tau) Z) - \Omega(X, (\nabla_Y \tau) Z) \right) - \\
- \frac{f}{2} [X(r)(g_\theta(Y, Z) - f \Omega(Y, \tau(Z))) - Y(r)(g_\theta(X, Z) - f \Omega(X, \tau(Z))) - \\
-(\phi X)(r)(f A(Y, Z) + \Omega(Y, Z)) + (\phi Y)(r)(f A(X, Z) + \Omega(X, Z)) + \\
+ (\phi Z)(r) \theta([X, Y]) \} N \}
\]

for any $X, Y, Z \in H(F)$. Let us take the inner product of (27) with $W \in H(F)$ and use (4)-(5). We obtain

\[ g(R^\theta(X, Y)Z, W) = -\frac{n + 1}{\varphi} \{ g_\theta(R(X, Y)Z, W) - \frac{1}{f} \theta([X, Y]) \Omega(Z, W) + \\
+ [f A(Y, Z) + \Omega(Y, Z)][A(X, W) + \frac{1}{f} \Omega(X, W)] - \\
- [f A(X, Z) + \Omega(X, Z)][A(Y, W) + \frac{1}{f} \Omega(Y, W)] + \\
+ [g_\theta(Y, Z) + f \Omega(Y, \tau(Z))][\frac{1}{f} g_\theta(X, W) + \Omega(X, \tau(W))] - \\
- [g_\theta(X, Z) + f \Omega(X, \tau(Z))][\frac{1}{f} g_\theta(Y, W) + \Omega(Y, \tau(W))]. \]

In particular for $Z = Y$ and $W = X$ (as $\Omega = -d\theta$)

\[ g(R^\theta(X, Y)Y, X) = -\frac{n + 1}{\varphi} \{ g_\theta(R(X, Y)Y, X) + \\
+ \frac{2}{f} [\Omega(X, Y)^2 + f A(X, X) A(Y, Y) - \\
- \frac{1}{f} [f^2 A(X, Y)^2 - \Omega(X, Y)^2] + \\
+ \frac{1}{f} [g_\theta(X, X) + f \Omega(X, \tau(X))][g_\theta(Y, Y) + f \Omega(Y, \tau(Y))] \}. \]
\[ -\frac{1}{f} [g_{\theta}(X, Y) + f \Omega(X, \tau(Y))]^2 \].

Note that
\[ A(\phi X, \phi X) = g_{\theta}(\tau(\phi X), \phi X) = -g_{\theta}(\phi \tau X, \phi X) = -A(X, X), \]
\[ \Omega(\phi X, \tau(\phi X)) = g_{\theta}(\phi X, \phi \tau(\phi X)) = g_{\theta}(X, \tau(\phi X)) = -g_{\theta}(X, \phi \tau(X)) = -\Omega(X, \tau(X)), \]
\[ \Omega(X, \tau(\phi X)) = g_{\theta}(X, \phi \tau(\phi X)) = -g_{\theta}(X, \tau(\phi^2 X)) = g_{\theta}(X, \tau(X)) = A(X, X). \]

Hence
\[ (28) \quad g(R^{\theta}(X, \phi X)\phi X, X) = -\frac{n+1}{\varphi} \{ g_{\theta}(R(X, \phi X)\phi X, X) + \frac{4}{f} g_{\theta}(X, X)^2 - 2f[A(X, X)^2 + A(X, \phi X)^2] \}. \]

Let \( \sigma \subset T(F)_z \) be the 2-plane spanned by \( \{X, \phi z X\} \) for \( X \in H(F)_z \), \( X \neq 0 \). By \[28\] if \( Y = \phi z X \) then
\[ g_z(X, Y)g_z(Y, Y) - g_z(X, Y)^2 = \]
\[ = \left( \frac{n+1}{\varphi(z)} \right)^2 \{ g_{\theta z}(X, X)g_{\theta z}(Y, Y) - g_{\theta z}(X, Y) \} = \]
\[ = \left( \frac{n+1}{\varphi(z)} \right)^2 g_{\theta z}(X, X)^2 \]
so that (by \[28\]) the sectional curvature \( k_g(\sigma) \) of the 2-plane \( \sigma \) is expressed by (for \( Y = \phi z X \))
\[ k_g(\sigma) = \frac{g_z(R^g_z(X, Y)Y, X)}{g_z(X, X)g_z(Y, Y) - g_z(X, Y)^2} = \]
\[ = -\frac{\varphi(z)}{n+1} \left\{ -4k_\theta(\sigma) + \frac{4}{f(z)} - 2f(z) \frac{A_z(X, X)^2 + A_z(X, \phi z X)^2}{g_{\theta z}(X, X)^2} \right\} \]
where \( k_\theta \) restricted to a leaf of \( F \) is the pseudohermitian sectional curvature of the leaf. Note that \( k_\theta \) and \( A \) stay finite at the boundary (and give respectively the pseudohermitian sectional curvature and the pseudohermitian torsion of \( (\partial \Omega, \theta) \), in the limit as \( z \to \partial \Omega \)). On the other hand \( f(z) \to 0 \) and \( \varphi(z)/f(z) \to 1 \) as \( z \to \partial \Omega \). We may conclude that \( k_g(\sigma) \to -4/(n+1) \) as \( z \to \partial \Omega \). To complete the proof of Klembeck’s result we must compute the sectional curvature of the 2-plane \( \sigma_0 \subset T_z(\Omega) \) spanned by \( \{N_z, T_z\} \) (remember that \( JN = T \)). Note first that
\[ N(f) = f^2 \left( \frac{2}{\varphi^2} + N(r) \right). \]
Let us set for simplicity
\[ g = N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}, \quad h = N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2. \]

We these notations let us recall that (by (23))
\[ (29) \quad \nabla g^T = -\frac{1}{2} X_r - \frac{f}{2} \{ T(r)T - hN \} \]
where \( X_r = \nabla H r \). Using also (20) for \( X = X_r \) we obtain
\[ -2\nabla^g g \nabla^g T = \nabla N X_r - \frac{1}{\varphi} X_r + \frac{f}{2} \{ (\phi X_r)(r)T - X_r(r)N \} + \]
\[ + N(f)\{ T(r)T - hN \} + f \{ N(T(r))T + T(r)\nabla^g N - N(h)N - h\nabla^g_N N \} . \]
Let us recall that (by (21) and (24))
\[ (30) \quad \nabla^g N = -\frac{1}{2} X_r + \frac{f}{2} \{ T(r)T - gN \} . \]

Using these identities and the expression of \( N(f) \) gives (after some simplifications)
\[ (32) \quad -2\nabla^g g \nabla^g T = \nabla N X_r + \left( \frac{fh}{2} - \frac{1}{\varphi} \right) X_r - \frac{f}{2} T(r) \phi X_r + \]
\[ + \frac{f}{2} \left\{ 2f \left( \frac{2}{\varphi^2} + N(r) \right) T(r) + 2N(T(r)) - f(g + h)T(r) \right\} T - \]
\[ - \frac{f}{2} \left\{ g_\theta(X_r, X_r) + 2fh \left( \frac{2}{\varphi^2} + N(r) \right) + 2N(h) + f[T(r)^2 - gh] \right\} N. \]
because of
\[ (\phi X_r)(r) = g_\theta(\nabla r, \phi X_r) = g_\theta(X_r, \phi X_r) = 0, \]
\[ X_r(r) = g_\theta(\nabla^H r, X_r) = g_\theta(X_r, X_r). \]

Similarly
\[ (33) \quad -2\nabla^g T \nabla^g N = \nabla T \phi X_r + \left( \frac{1}{f} - \frac{fg}{2} \right) X_r + \frac{f}{2} T(r) \phi X_r + \]
\[ + \frac{f}{2} \left\{ 2T(g) + f(g - h)T(r) \right\} T + \]
\[ + \frac{f}{2} \left\{ g_\theta(X_r, X_r) + 2T^2(r) + f[T(r)^2 + gh] \right\} N. \]
Here $T^2(r) = T(T(r))$. Let us set $\tau(W_\alpha) = A_{\alpha}^\beta W_\beta$. To compute the last term in the right hand member of

$$R^g(N, T)T = \nabla^g_N \nabla^g_r T - \nabla^g_T \nabla^g_r T - \nabla^g_{[N,T]} T$$

note first that $T(f) = f^2 T(r)$. On the other hand we may use the decomposition (35) so that

$$\nabla^g_{[N,T]} T = rX_r + frT(r)T - \frac{f}{2} \{ g_\theta(X_r, X_r) + 2rh \} N +$$

$$+ \left( ir\sigma A_{\alpha}^\beta - \frac{1}{f} r^\beta \right) W_\beta - \left( ir^\alpha A_{\alpha}^\beta + \frac{1}{f} r^\beta \right) W_\beta$$

(where $A_{\alpha}^\beta = \overline{A_{\alpha}^\beta}$) and by taking into account that

$$\left( ir\sigma A_{\alpha}^\beta - \frac{1}{f} r^\beta \right) W_\beta - \left( ir^\alpha A_{\alpha}^\beta + \frac{1}{f} r^\beta \right) W_\beta = - \frac{1}{f} X_r - \tau(\phi X_r)$$

we may conclude that

$$\nabla^g_{[N,T]} T = \left( r - \frac{1}{f} \right) X_r - \tau(\phi X_r) +$$

$$+ frT(r)T - \frac{f}{2} \{ g_\theta(X_r, X_r) + 2rh \} N.$$

Finally (by plugging into (34) from (32)-(33) and (35))

$$-2R^g(N, T)T = \nabla N X_r - \nabla_T \phi X_r - fT(r)\phi X_r - 2\tau(\phi X_r) +$$

$$+ \left( 2r + \frac{f}{2} (g + h) - \frac{1}{\varphi} - \frac{3}{f} \right) X_r +$$

$$+ f \left\{ f \left( \frac{2}{\varphi^2} + N(r) \right) T(r) + N(T(r)) - T(g) + (2r - fg)T(r) \right\} T -$$

$$- f \left\{ 2\|X_r\|^2 + fh \left( \frac{2}{\varphi^2} + N(r) \right) + N(h) + fT(r)^2 + T^2(r) + 2rh \right\} N.$$

Here $\|X_r\|^2 = g_\theta(X_r, X_r)$. Let us take the inner product of (36) with $N$ and use (4)-(6). We obtain

$$2g(R^g(N, T)T, N) =$$

$$= \frac{n+1}{\varphi} \left\{ 2\|X_r\|^2 + fh \left( \frac{2}{\varphi^2} + N(r) \right) +$$

$$+ N(h) + fT(r)^2 + T^2(r) + 2rh \right\}$$

and dividing by

$$g(N, N)g(T, T) - g(N, T)^2 = \frac{1}{f^2} \left( \frac{n+1}{\varphi} \right)^2$$
leads to
\[
\frac{2g(R^g(N, T)T, N)}{g(N, N)g(T, T) - g(N, T)^2} = \frac{f^2\varphi}{n+1} \left\{ 2\|X_r\|^2 + T^2(r) + fT(r)^2 + 2hr + N(h) + fhN(r) + 2\frac{fh}{\varphi} \right\}.
\]

It remains that we perform an elementary asymptotic analysis of the right hand member of the previous identity when \(z \to \partial\Omega\) (equivalently when \(\varphi \to 0\)). As \(r \in C^\infty(\overline{\Omega})\) (cf. [12]) the terms \(\|X_r\|^2, T^2(r), T(r)^2\) and \(N(r)\) stay finite at the boundary. Also (by recalling the expression of \(h\)) \(f^2\varphi h \to 0\) as \(\varphi \to 0\). Moreover
\[
2\frac{f^2\varphi}{n+1} \frac{fh}{\varphi^2} = \frac{2}{n+1} f \left[ f^2N(r) + \frac{4}{(1-r\varphi)^2} - \frac{6f^2r}{\varphi^2} + 4f^2r^2 \right] \to \frac{8}{n+1},
\]
\[
N(h) = N^2(r) + 4N(r^2) - \frac{16r}{\varphi^2} + \frac{12r}{\varphi^3} - \frac{6}{\varphi} N(r),
\]
as \(\varphi \to 0\) hence
\[
\frac{f^2\varphi}{n+1} N(h) \to -\frac{16}{n+1},
\]
as \(\varphi \to 0\) hence
\[
k_g(\sigma_0) \to -\frac{4}{n+1}, \quad z \to \partial\Omega.
\]
Klembeck’s theorem is proved.

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