The Universal Phase Space of AdS$_3$ Gravity

Carlos Scarinci$^1$, Kirill Krasnov$^{1,2}$

$^1$ School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK.
E-mail: scarinci@math.fau.de

$^2$ Max Planck Institute for Gravitational Physics, Am Mühlenberg 1, 14476 Golm, Germany.
E-mail: kirill.krasnov@nottingham.ac.uk

Received: 20 May 2012 / Accepted: 22 July 2012
Published online: 3 January 2013 – © Springer-Verlag Berlin Heidelberg 2012

Abstract: We describe what can be called the “universal” phase space of AdS$_3$ gravity, in which the moduli spaces of globally hyperbolic AdS spacetimes with compact spatial sections, as well as the moduli spaces of multi-black-hole spacetimes are realized as submanifolds. The universal phase space is parametrized by two copies of the universal Teichmüller space $T(1)$ and is obtained from the correspondence between maximal surfaces in AdS$_3$ and quasiregular homeomorphisms of the unit circle. We also relate our parametrization to the Chern-Simons formulation of 2+1 gravity and, infinitesimally, to the holographic (Fefferman-Graham) description. In particular, we obtain a relation between the generators of quasiconformal deformations in each $T(1)$ sector and the chiral Brown-Henneaux vector fields. We also relate the charges arising in the holographic description (such as the mass and angular momentum of an AdS$_3$ spacetime) to the periods of the quadratic differentials arising via the Bers embedding of $T(1) \times T(1)$. Our construction also yields a symplectic map $T^*T(1) \to T(1) \times T(1)$ generalizing the well-known Mess map in the compact spatial surface setting.

1. Introduction

Since the discoveries by Brown and Henneaux [1] that the group of symmetries of an asymptotically AdS$_3$ spacetime is a centrally extended conformal group in two dimensions, and then by Banados, Teitelboim and Zanelli [2] that black holes can exist in such spacetimes, the subject of negative cosmological constant gravity in 2+1 dimensions continues to fascinate researchers. The result [1] is now considered to be a precursor of the AdS/CFT correspondence of string theory [3], and the value of the central charge determined in [1] is an essential ingredient of the conformal field theoretic explanation [4] of the microscopic origin of the black hole entropy.

The AdS$_3$/CFT$_2$ story is reasonably well-understood in the string theory setting of 3-dimensional gravity coupled to a large number of fields of string (and extra dimensional) origin. At the same time, the question of whether there is a CFT dual to pure
AdS$_3$ gravity remains open, see [5] and [6] for the most recent (unsuccessful) attempts in this direction. In particular, the attempt [6] to construct the genus one would-be CFT partition function by summing over the modular images of the partition function of pure AdS leads to discouraging conclusions. It thus appears that pure AdS$_3$ gravity either does not have enough “states” to account for the BH entropy microscopically, or that the known such states cannot be consistently put together into some CFT structure.

The current lack of understanding of pure AdS$_3$ gravity quantum mechanically is particularly surprising given the fact that, in a sense, the theory is trivial since pure gravity in 2+1 dimensions does not have any propagating degrees of freedom. In the setting of compact spatial sections the phase space of 2+1 gravity (i.e. the space of constant curvature metrics in a $\mathbb{R} \times \Sigma$, with $\Sigma$ a genus $g > 1$ Riemann surface) is easy to describe (for all values of the cosmological constant). The constant mean curvature foliation of such a spacetime is particularly useful for this purpose. One finds, see [7] and also [8] for a more recent description, that the phase space is the cotangent bundle over the Teichmüller space of the spatial slice (for any value of the cosmological constant). The zero cosmological constant result [7] also follows quite straightforwardly from the Chern-Simons (CS) description given in [9]. In the setting of AdS$_3$ manifolds with compact spatial slices, there is yet another description of the same phase space, first discovered by Mess [10]. This is given by two copies of the Teichmüller space of the spatial slice, or, equivalently, by two hyperbolic metrics on the spatial slice Riemann surface. The Mess description is related to the Chern-Simons description of AdS$_3$ gravity in terms of two copies of SL$(2, \mathbb{R})$ CS theory.

It appears sensible to tackle the problem of quantum gravity as a problem of quantization of the arising classical phase space. One could argue that this approach is unlikely to succeed in 3+1 and higher dimensions, where the phase spaces that arise this way are infinite dimensional (because of the existence of local excitations — gravitational waves). However, in the setting of 2+1 gravity, at least in the setting of spacetimes with compact spatial slices one deals with a finite-dimensional dynamical system and the problem of quantum gravity seems to reduce to a problem from quantum mechanics. In spite of this being a tractable problem, the immediate worry with this approach is that the Hilbert space of quantum states one can obtain by quantizing such a finite-dimensional phase space would not be sufficiently large to account for the black hole entropy.

At the same time, in the context of black holes one should consider non-compact spatial slices. The classical phase space that should arise in this context is somewhat less understood. On one hand, we now know that there is not just the simple BTZ BH [2], but also a much more involved zoo of multi-black hole (MBH) spacetimes first described in [11]. A rather general description of such MBH’s using causal diamonds at their conformal infinity is given in [12]. As a by-product of a construction in [13] using earthquakes, another description of MBH geometries is also available. There are also descriptions of MBH spacetimes in the physics literature, see [14] and [15]. These descriptions show that, like in the compact spatial slice setting with its Mess parametrization, the geometry of multi-black holes continues to be parametrized by two hyperbolic metrics on their spatial slice (or, equivalently, by the cotangent bundle of the corresponding Teichmüller space). The main difference with the compact setting is that the spatial slices are now Riemann surfaces with a geodesic boundary (or with hyperbolic ends attached), and there are now additional moduli, namely the sizes of the boundary components. These new length parameters, two for each boundary component (because there are two hyperbolic metrics involved in the parametrization) determine the geometrical characteristics of the corresponding black hole horizon, such as its length and angular velocity. An explicit formula of this sort can be found in e.g. [13], see formula (1) of the first (arxiv)