The Big Bang as a Phase Transition

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Abstract
We study a five-dimensional cosmological model, which suggests
that the universe began as a discontinuity in a scalar (Higgs-type) field,
or alternatively as a conventional four-dimensional phase transition.

1 Introduction
Recent years have witnessed a large amount of interest in higher-dimensional
cosmologies where the extra dimensions are noncompact. A popular example
is the so-called Randall-Sundrum braneworld scenario [1, 2]. This is typically
a five-dimensional $\text{AdS}_5$ black hole spacetime called the bulk, on which our
universe is described by a domain wall called the brane. All the matter
interactions are assumed to be confined to this brane, except for gravitation,
which is allowed to propagate in the bulk. This model was inspired by a
certain string theory discovered by Hořava and Witten [3, 4] who showed
that extra dimensions did not need to be compactified. In this scenario,
the fields of the standard model are represented by strings whose endpoints
reside on a 10D hypersurface, and are therefore confined to this “brane”.
The gravitational degrees of freedom, on the other hand, are represented by
closed strings which is why the graviton is free to propagate along the entire
11D manifold. Prime among all these models is the concept of $\mathbb{Z}_2$ symmetry,
which is simply the requirement that the manifold on one side of the brane
be the mirror image of the other side. An alternative approach which uses a
large extra dimension is Space-Time-Matter, proposed by Wesson, which is
so-called because the foundation is that the fifth dimension induces matter
[5]. This point of view introduces a certain symmetry in physics, since in
mechanics we normally use as our base dimensions length, time, and mass.
A realization of this theory is that all the matter fields in 4D can arise from
a higher-dimensional vacuum. One starts with the vacuum Einstein field
equations in 5D, and dimensional reduction of the Ricci tensor leads to an
effective 4D energy-momentum tensor [6]. For this reason the theory is also
called induced-matter theory. These two theories (braneworld and induced-
matter) may appear to be different, but have recently been shown to be
equivalent by Ponce de Leon [7]. The aim of this paper is to demonstrate
how a solution of induced-matter theory can be used to generate a simple
braneworld cosmology with $\mathbb{Z}_2$ symmetry, and to study the properties of this
cosmology, paying particular attention to a phase transition which occurs at the big bang.

We begin by writing down a class of exact cosmological solutions to the 5D vacuum (Ricci-flat) field equations $R_{AB} = 0$. The 5D line element is:

$$dS^2 = \frac{\dot{a}^2}{\mu^2} dt^2 - a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - dy^2$$

$$a^2 = (\mu^2 + k)y^2 + 2\nu y + \nu^2 + K \frac{\mu^2 + k}{\mu^2 + k}.$$ (1)

Here the functions $\mu \equiv \mu(t)$ and $\nu \equiv \nu(t)$, $\dot{a} = \partial a/\partial t$ and $k (+1,0,-1)$ and $K$ are constants. Studies of this class of solutions show that it is algebraically broad, and depending on the choice of coordinates has two feasible interpretations: (a) a hot early universe, with matter production typical of inflationary quantum field theories, and a decaying cosmological constant of the kind needed to resolve the timescale problem of standard cosmology [8]; (b) a 5D topological black hole (see below) [11,12]. The constant $K$ appearing in (1) is a constant of integration and is related to the 5D Kretchmann scalar via

$$R_{ABCD}R^{ABCD} = \frac{72K^2}{a^8},$$ (2)

which is the only geometrical invariant that is non-zero since $R_{AB}R^{AB} = 0$ and $R = 0$ by the field equations. It should be noted that $a = 0$ corresponds to a geometrical singularity for the 5D model which is similar to those that occur in the 4D Friedmann-Robertson-Walker (FRW) models. In fact, it can be shown via a non-trivial coordinate transformation $R = R(t,y)$, $T = T(t,y)$ that (1) is isometric to the (topological) black hole manifold with line element

$$dS^2 = h(R)dT^2 - h^{-1}(R)dR^2 - R^2 \left[ d\psi^2 + S^2_k(\psi)(d\theta^2 + \sin^2 \theta d\phi^2) \right],$$ (3)

where $h(R) = k - K/R^2$, and the function

$$S_k(\psi) = \begin{cases} \sin \psi, & k = +1, \\ \psi, & k = 0, \\ \sinh \psi, & k = -1. \end{cases}$$ (4)

In what follows, we are not concerned with this singularity. Instead we consider a different type of singularity that occurs at $\dot{a}/\mu = 0$, corresponding to a coordinate singularity similar to that which defines an event horizon, but now the principal pressures of the material fluid also diverge at this point. This part of the manifold thus defines a matter singularity.

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1 Conventions: In this paper we use upper-case latin indices to run $0, \ldots, 4$, lower-case greek indices to run $0, \ldots, 3$, and the signature of the metric is always $(+ - \ldots - )$. We use spherical coordinates $x^{01234} = t r \theta \phi y$ with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Unless otherwise stated, we work in natural units where $c = 8\pi G = 1$. 

2
We now proceed to illustrate how the solution (11) can be used to generate a braneworld model. In what follows, we appeal to astrophysical data so we set $k = 0$, and for simplicity choose $K = 1$. We are also interested in a solution which incorporates the $\mathbb{Z}_2$ reflection symmetry condition $g_{\alpha\beta}(x^7, y) = g_{\alpha\beta}(x^7, -y)$. This is achieved simply by setting $\nu = 0$, so the line element (1) takes the form

$$
\frac{dS^2}{\mu^2} = \frac{\dot{a}^2}{\mu^2}dt^2 - a^2(dr^2 + r^2d\Omega^2) - dy^2
$$

$$
a^2 = \mu^2y^2 + \frac{1}{\mu^2}
$$

$$
\frac{\dot{a}^2}{\mu^2} = \frac{(y^2 - \mu^{-4})^2}{(y^2 + \mu^{-4})^2} \frac{1}{\mu^2} \left( \frac{d\mu}{dt} \right)^2.
$$

(5)

In order to make contact with the matter properties that this solution describes, we make note of Campbell’s embedding theorem which states that any analytic $(N-1)D$ Riemannian manifold can be locally embedded in an $N$D Riemannian manifold that is Ricci-flat [13, 14, 15, 16, 17]. This provides a basis for interpreting a solution of $R_{AB} = 0$ like (5) as a solution of the $4D$ field equations $G_{\alpha\beta} = T_{\alpha\beta}$ with sources.

The functional form of the stress-energy tensor $T_{\alpha\beta} = T_{\alpha\beta}(x^7, y)$ has been known for some years [6] and is:

$$
T_{\alpha\beta} = \nabla_\beta (\partial_\alpha \Phi) - \frac{\varepsilon}{2\Phi^2} \left\{ \frac{\Phi g_{\alpha\beta}}{\Phi} - g_{\alpha\beta} + g^{\lambda\mu} g_{\alpha\lambda} g_{\beta\mu} \right. 
$$

$$
\left. - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} g_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \left[ g^{\mu\nu} g_{\mu\nu} + \left( g^{\mu\nu} g_{\mu\nu} \right)^2 \right] \right\}.
$$

(6)

Here, $\nabla_\beta$ denotes the covariant derivative, $\partial_\beta \equiv \partial/\partial x^\beta$, and the overstar denotes partial differentiation with respect to the fifth coordinate. Also, the fifth component of the metric is $\varepsilon \Phi^2$, where $\varepsilon = \pm 1$ and $\Phi$ is a scalar field which may be related to particle mass [18, 19]. Campbell’s theorem can in fact be inferred from the ADM formalism, which has been used to obtain the 4D energy of 5D solutions [20, 21]. The embedding expressions need modification if there is a singular surface, as may happen in $\mathbb{Z}_2$-symmetric cosmologies; but there is mathematical consistency between induced-matter and brane models [7]. Here we would like to emphasize that while complementary techniques may be employed to obtain a unique functional form for the stress-energy tensor, there is still an ambiguity involved in the physical interpretation of this. An analogous situation occurs in 4D, where a given metric may have different sources. As in the standard FRW models, we

2 A Simple Cosmological Model
assume here that the source is a perfect fluid with energy density $\rho$ and pressure $p$ so that

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta}. \quad (7)$$

This stress-energy tensor must satisfy the 4D field equations with a line element $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ which is the 4D part of (5). If we take the matter to be comoving, then the four-velocities $u^\alpha = dx^\alpha/ds$ satisfy $u^\alpha = (u^0, 0, 0, 0)$ and $u^0u_0 = 1$. Then we find

$$\rho = \frac{3\mu^4}{\mu^4y^2 + 1}$$
$$p = -\mu^4 \left( \frac{2}{\mu^4y^2 - 1} + \frac{1}{\mu^4y^2 + 1} \right). \quad (8)$$

Note that the pressure is generally discontinuous at the point $\mu^2 = 1/y$. This is also the point at which the scale factor $a$ is a minimum.

An examination of (1) and (5) shows that the form of $(\dot{a}/\mu)dt$ is invariant under an arbitrary coordinate transformation $t \rightarrow t(\tilde{t})$. As a result, we can freely choose the form of $\mu(t)$ without loss of generality, and it will prove convenient to take $\mu(t) = 1/\sqrt{2t}$. From (5) we find that

$$dS^2 = \left(1 - \frac{y^2}{4t^2}\right)^2 \left(1 + \frac{y^2}{4t^2}\right)^{-1} dt^2 - \left(2t + \frac{y^2}{2t}\right)(dr^2 + r^2d\Omega^2) - dy^2. \quad (9)$$

This shows that the scale factor for the 3D sections of the 5D metric has the time-dependence typical of the radiation-dominated FRW model, but with an extra term that comes from dependence on the fifth coordinate. This term decays with coordinate time $t$. Alternatively, the fifth dimension should be important as $t \rightarrow 0$. Since $\dot{a}/\mu \rightarrow 1$ as $t \rightarrow \infty$, the coordinate time becomes the cosmic time. Note that this solution describes the 4D radiation FRW model on the surface of reflection ($y = 0$). For general $y \neq 0$ and $t \rightarrow \infty$ we have $a \rightarrow \sqrt{2t}$ and $\dot{a}/\mu \rightarrow 1$. Then

$$dS^2 \rightarrow dt^2 - 2(\dot{r}^2 + r^2d\Omega^2) - dy^2 \quad \text{as} \quad t \rightarrow \infty, \quad (10)$$

which is the embedded radiation model. For general $y \neq 0$ and $t \rightarrow 0$ we have $a \rightarrow y/\sqrt{2t}$ and $\dot{a}/\mu \rightarrow y/2t$. This gives

$$dS^2 \rightarrow \frac{y^2}{L^2} \left[ \left( \frac{L}{2t} \right)^2 dt^2 - \frac{L^2}{2t} (dr^2 + r^2d\Omega^2) \right] - dy^2 \quad \text{as} \quad t \rightarrow 0, \quad (11)$$

where we have introduced a constant length $L$ to make contact with 5D metrics in the canonical form [22]. This has the general form

$$dS^2 = \frac{y^2}{L^2} [\tilde{g}_{\alpha\beta}(x^\gamma, y)dx^\alpha dx^\beta] - dy^2, \quad (12)$$
and is useful because its first part can be related to the 4D action of quantum physics and leads to great simplification of the geodesic equation of classical physics. Both brane models and induced-matter theory lead in general to a fifth force which is zero if $\partial \tilde{g}_{\alpha \beta} / \partial y = 0$. This can be physically justified by appeal to the weak equivalence principle (which is then a symmetry of the metric). Alternatively, it is mathematically justified by the argument that the metric not be significantly affected by the mass of a test particle which moves through it (no reaction force). The noted condition is satisfied by (11). This via $t = e^{2\tau / L}$ transforms to

$$dS^2 \to \frac{y^2}{L^2} \left[ d\tau^2 - \frac{1}{2} L^2 e^{-2\tau / L} (dr^2 + r^2 d\Omega^2) \right] - dy^2 \quad \text{as} \quad \tau \to -\infty, \quad (13)$$

whose 4D part is a de Sitter space of the kind used in other applications of brane theory.

Turning our attention to the induced matter described by (9), we find from (8) that

$$\rho = \frac{3}{4t^2 + y^2}, \quad p = \frac{2}{4t^2 - y^2} - \frac{1}{4t^2 + y^2}. \quad (14)$$

From these comes the inertial density of matter:

$$\rho + p = \frac{16t^2}{(4t^2 + y^2)(4t^2 - y^2)}, \quad (15)$$

which is regarded as setting a condition for the stability of matter via $\rho + p > 0$. From (14) also comes the gravitational density of matter:

$$\rho + 3p = \frac{6}{4t^2 - y^2}, \quad (16)$$

which is regarded as setting a condition for the gravity of matter via $\rho + 3p > 0$. For the case of (9) this means that $t > y/2$. The form of relations (14) shows that the energy density and pressure vary differently with $t$ and $y$. The behaviour of these is summarized in Table 1.

The universe described by (9) begins in an infinitely distended state with the equation of state $\rho + p = 0$, typical of 5D vacuum cosmologies. It contracts, with the energy density decaying from its maximum by 50% and the pressure becoming unbounded as $t \to y/2$. The 5D Kretchmann scalar (2) remains finite and in fact drops to its minimum $9/2y^4$ at this point. The pressure changes discontinuously as in a phase transition. After this, the universe expands with the matter becoming radiation-like, and the energy density decreasing in the same way as in standard ($k = 0$) cosmology. This
It should be noted that we have considered the stress-energy for general $y \neq 0$ hypersurfaces, but the surface of reflection $y = 0$ defines a special hypersurface on the manifold. For here there is no phase transition, and we have $a \to 0$ and $\rho \to \infty$ for $t \to 0$, thus defining a big bang. By (14) we have $\rho = 3/4t^2$ and $p = 1/4t^2$, which implies a radiation-dominated ($k = 0$) cosmology. We may also consider the energy density $\rho(t, y) = 3/(4t^2 + y^2)$ in (14). There is a maximum at the $y = 0$ hypersurface, and $\rho$ is not uniformly distributed along the $y$-direction. In fact, this kind of concentration can be seen more clearly in the early stage of the universe where $t \sim y$, as is shown in Fig. 2.

3 Discussion

We have investigated in detail the stress-energy behaviour of the matter that is associated with the solution (9), but the source of the phase transition is currently unknown. Also, as mentioned above, the nature of this matter is open to interpretation. For example, we can decompose (14) into a linear combination of two stress-energy tensors such that

$$T^\alpha_\beta = (T^\alpha_\beta)_I + (T^\alpha_\beta)_{II}$$

$$(T^\alpha_\beta)_I = \left[\frac{1}{4t^2 + y^2}\right] \text{diag}(3, 1, 1, 1)$$

$$(T^\alpha_\beta)_{II} = -\left[\frac{2}{4t^2 - y^2}\right] \text{diag}(0, 1, 1, 1).$$

The first part $(T^\alpha_\beta)_I$ has the equation of state $\rho + 3p = 0$ which describes non-gravitating matter. The second part $(T^\alpha_\beta)_{II}$ has the functional form

| $t$  | $\rho$ | $p$  | $\rho + p$ | $\rho + 3p$ |
|------|-------|------|-----------|-------------|
| $t \to 0$ | $\frac{3}{y^2}$ | $-\frac{3}{y^2}$ | 0 | $-\frac{6}{y^2}$ |
| $t \to \frac{y^-}{2}$ | $\frac{3}{2y^2}$ | $-\infty$ | $-\infty$ | $-\infty$ |
| $t \to \frac{y^+}{2}$ | $\frac{3}{2y^2}$ | $+\infty$ | $+\infty$ | $+\infty$ |
| $t \to \infty$ | $\frac{3}{4t^2}$ | $\frac{1}{4t^2}$ | $\frac{1}{t^2}$ | $\frac{3}{2t^2}$ |

Table 1: Asymptotic behaviour of the stress-energy, from (14)-(16).
Figure 1: Evolution of the energy density and pressure with time, from (14). The dotted line represents the energy density and the solid line represents the pressure.

\[ f(t) \text{diag}(0, 1, 1, 1) \text{ on a } y = \text{constant hypersurface.} \] It is this second component which is the source associated with the phase transition at \( t = y/2 \).

The decomposition (17) has two possible interpretations: (a) a universe that is filled with two different types of (possibly non-interacting) matter fields; (b) a universe filled with a non-gravitating matter field subject to a bulk viscosity.

In either physical case, there is a corresponding geometrical interpretation. Recall that the 5D Kretchmann scalar remains finite across the boundary \( t = y/2 \). On the other hand, its 4D counterpart given by

\[
R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{24}{(4t^2 + y^2)(4t^2 - y^2)}
\] (18)

becomes singular. This would suggest that the discontinuity in the pressure is really a manifestation of a cusp in the 4D geometry. However, the relations (15) and (16) for the inertial and gravitational densities of the matter need to be taken into consideration. The stability of these requires that \( t > y/2 \), but the phase transition occurs at \( t = y/2 \). Thus, the solution (9) as illustrated in Fig. 1 has a period when the matter is exotic; and after the transition at \( t = y/2 \) a period where it is normal and indeed asymptotic to a photon gas.

Let us take a closer look at the “matter” of the model. We have already commented that there is a discontinuity in the pressure \( p \), which by the laws of 4D thermodynamics would lead us to identify the event at \( t = y/2 \) as a first-order phase transition. However, the matter described by (14) or (17) before the transition is exotic, insofar as it violates the usual
energy conditions. This suggests to us that, while our model is classical, the underlying mechanism for the transition has to do with quantum effects. To illustrate this, we note that we can model the matter in (14) or (17) by a scalar field $\Psi = \Psi(x^A)$. The correspondence between the classical and quantum approaches is established by matching the classical stress-energy tensor (7) to the equivalent relation for the quantum-field theory expression for a scalar field:

$$T_{\alpha\beta} = \partial_\alpha \Psi \partial_\beta \Psi - g_{\alpha\beta} \mathcal{L}. \tag{19}$$

Here $\mathcal{L}$ is the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \frac{\dot{\Psi}^2}{\partial^2 \Psi} - V(\Psi), \tag{20}$$

and $V(\Psi)$ is an arbitrary potential. Then we find that the classical expressions for the density and pressure are given by

$$\rho = \frac{1}{2} \dot{\Psi}^2 + \frac{1}{2} (\nabla \Psi)^2 + V(\Psi)$$

$$p = \frac{1}{2} \dot{\Psi}^2 - \frac{1}{6} (\nabla \Psi)^2 - V(\Psi). \tag{21}$$

These expressions are generic. However, modulo the non-uniqueness of the source responsible for (9), the phase transition in (14) corresponds to a discontinuity in a Higgs-type field.

### 4 Conclusion

There are many models for the “big bang”, but they suffer from the problem that the origin of the matter is unexplained. We have given in the above a
simple model. In a classical sense, the big bang is a first-order thermodynamic phase transition. In a quantum sense, it corresponds to a discontinuity in a Higgs-type scalar field. This picture is conformable with other models, including those of Vilenkin [23], where the “big-bang” is a tunelling event. We cannot completely identify the nature of this event, given the latitude inherent in the Lagrangian which describes a quantum scalar field. However, we feel that the description in terms of classical fields is an improvement over the traditional one: if the big bang is a phase transition, it opens the way to further thermodynamic investigations.

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