EPSILON FACTOR FOR $\text{GL}_l \times \text{GL}_{l'}$; $l \neq l'$ PRIMES

TETSUYA TAKAHASHI

Abstract. Let $F$ be a non-Archimedean local field with finite residual field of characteristic $p$. In this article we calculate the $\varepsilon$-factor of pairs for $\text{GL}_l(F) \times \text{GL}_{l'}(F)$ where $l$ and $l'$ are distinct primes including the case $l = p$. For this calculation, we use the local Langlands correspondence and non-Galois base change lift. This method leads to the explicit conjecture of the $\varepsilon$-factor of the representations of $\text{GL}_m \times \text{GL}_n$ when $n$ is relatively prime to $m$ and $p$.

1. Introduction

Let $F$ be a non-Archimedean local field with finite residual field of characteristic $p$ and the $\mathcal{W}_F$ the absolute Weil group of $F$. For an integer $n \geq 1$, we denote by $\mathcal{A}_n(F)$ the set of equivalent classes of irreducible supercuspidal representations of $\text{GL}_n(F)$ and by $\mathcal{G}_n(F)$ the set of equivalent classes of irreducible continuous complex representations of $\mathcal{W}_F$ of dimension $n$. The local Langlands conjecture tells us that there exists a unique bijection $\Lambda^F_n$ from $\mathcal{G}_n(F)$ to $\mathcal{A}_n(F)$ which satisfies the following conditions:

1. For $\chi \in \hat{F}^\times$ and $\sigma \in \mathcal{G}_n(F)$,
   $$\Lambda^F_n(\chi \sigma) = \chi \Lambda^F_n(\sigma)$$
   (By the reciprocity map of local class field theory, we identify $\hat{F}^\times = \mathcal{A}_1(F)$ with $W_F^{ab} = \mathcal{G}_1(F)$. By this identification, $\Lambda_1$ is the identity map.)

2. For $\sigma \in \mathcal{G}_n(F)$,
   $$\Lambda^F_n(\check{\sigma}) = \Lambda^F_n(\sigma)^\vee.$$

3. Let $\omega_\pi$ denote the central quasi-character of $\pi \in \mathcal{A}_n(F)$. For $\sigma \in \mathcal{G}_n(F)$,
   $$\omega_{\Lambda^F_n(\sigma)} = \det \sigma.$$

4. Let $\psi_F$ be a non-trivial character of $F$. For $\sigma \in \mathcal{G}_n(F)$,
   $$\varepsilon(\Lambda^F_n(\sigma), s, \psi_F) = \varepsilon(\sigma, s, \psi_F).$$

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where the left hand side is the Godement-Jacquet local constant [13] and the right hand side is the Langlands-Deligne local constant [11]. (In fact, this condition is contained in the following condition (5).)

(5) For \( \sigma \in G_n(F) \) and \( \sigma' \in G_{n'}(F) \),

\[
(1.5) \quad \varepsilon (\Lambda^F_\sigma(s), \Lambda^F_{\sigma'}(s), s, \psi_F) = \varepsilon (\sigma \otimes \sigma', s, \psi_F)
\]

where the \( \varepsilon \)-factor of pairs of the left hand side is in the sense of [19], [25].

This conjecture has been proved in [23] when \( \text{ch } F = p \) and in [14], [17] when \( \text{ch } F = 0 \). But their proof relies on the global tool and say nothing explicit about the local Langlands correspondence.

On the other hand, there are some explicit correspondences in the following cases:

1. When \( (n, p) = 1 \), Howe-Moy [15],[22] gives an explicit bijection between \( G_n(F) \) and \( A_n(F) \) when \( (n, p) = 1 \). (See also [24]).

2. When \( n = p \), Kutzko-Moy [20] gives an explicit bijection between \( G_n(F) \) and \( A_n(F) \). (See also [16]).

3. When \( n = p^m \), Bushnell-Henniart [3] gives an explicit bijection between \( G_{p^m}(F) \) and \( A_{p^m}(F) \). (For the definition of \( G_{p^m}(F) \) and \( A_{p^m}(F) \), see below Remark 3.2.)

All three bijections satisfy the condition (1)–(4) of the local Langlands correspondence. Thus the main obstacle to get the explicit local Langlands correspondence is \( \varepsilon \)-factor of pairs. We know very few about the explicit calculation of \( \varepsilon (\pi_1 \times \pi_2, s, \psi_F) \) for \( \pi_1 \in A_{n_1}(F) \) and \( \pi_2 \in A_{n_2}(F) \): The known cases are (i) \( n_1 = n_2 \) ([21]) and \( \pi_2 = \pi_1 \) ([5]), (ii) \( \pi_1 \in A_{p_{p_1}}(F) \) and \( \pi_2 \in A_{p_{p_2}}(F) \) ([6]).

In this paper we consider the case \( n_1 \neq n_2 \) are primes. Set \( n_1 = l \) and \( n_2 = l' \). We admit the case \( l = p \). Since \( l \neq l' \), we may assume \( l' \neq p \). We get the relation of \( \varepsilon \)-factor of \( \text{GL}_l(F) \times \text{GL}_{l'}(F) \) with \( \varepsilon \)-factor of \( \text{GL}_l(E) \) where \( E \) is an extension of \( F \) associated with \( \pi_2 \). (See Theorem 4.1.)

Let us summarize the contents of this paper, indicating its organization:

Section 1 reviews the construction of irreducible supercuspidal representations \( \pi \) of \( \text{GL}_l(F) \) and the explicit formula of \( \varepsilon (\pi, s, \psi_F) \). All of this section is well-known. Section 2 is devoted to review some explicit correspondences and the tame lifting. When \( l \neq p \), \( G_l(F) \) consists of the representations in the form \( \text{Ind}_{W_E}^{W_F} \theta \); \( E/F \) is an extension of degree \( l \) and \( \theta \) is a quasi-character of \( E^\times \). By way of such \( \theta \), there is very explicit Howe-Moy correspondence between \( G_l(F) \) and \( A_l(F) \). But when \( l = p \), there exists non-monomial representations in \( G_p(F) \); so we need the tame base change lift to get the correspondence. (See [20],[3].) Let \( \pi \in A_{p^m}(F) \) and \( K/F \) a tamely ramified extension. After the definition
of [2], we give the tame base change lift \( l_K(\pi) \) explicitly (Theorem 3.5) and show \( l_K \) is compatible with the local Langlands correspondence (Proposition 3.7). We also define the tame base change lift \( l^\prime_\varepsilon \) of [2], we give the tame base change lift \( l^\prime_\varepsilon \) explicitly (Theorem 3.5). These are essential tool to calculate the \( \varepsilon \)-factor of pairs. Section 3 calculates the \( \varepsilon \)-factor of \( \text{GL}_n(F) \times \text{GL}_d(F) \). By the result of Bushnell-Henniart [7], the Howe-Moy correspondence coincides with the Local Langlands correspondence for \( \text{GL}_d(F) \). Thus we calculate the \( \varepsilon \)-factor in the Galois side and then transfer it to the automorphic side using the results in section 2.

Notation

Let \( F \) be a non-archimedean local field. We denote by \( \mathcal{O}_F, P_F, \varpi_F, k_F \) and \( v_F \) the maximal order of \( F \), the maximal ideal of \( \mathcal{O}_F \), a prime element of \( P_F \), the residue field of \( F \) and the valuation of \( F \) normalized by \( v_F(\varpi_F) = 1 \). We set \( q = q_F \) to be the number of elements in \( k_F \). Let \( W_F \) be the absolute Weil group of \( F \). Hereafter we fix an additive character \( \psi \) of \( F \) whose conductor is \( P_F \), i.e., \( \psi \) is trivial on \( P_F \) and not trivial on \( \mathcal{O}_F \). For an extension \( E \) over \( F \), we denote by \( \text{tr}_E, N_E \) the trace and norm to \( F \) respectively. We set \( \psi_E = \psi \circ \text{tr}_E \) and \( \chi_E = \chi \circ N_E \) for a quasi-character \( \chi \) of \( F^\times \). Let \( \theta \) be a quasi-character of \( E^\times \). We denote by \( f(\theta) \) an integer such that \( 1 + P^{n+1}_E \not\subset \text{Ker}\, \theta \) and \( 1 + P^n_E \subset \text{Ker}\, \theta \). The Gauss sum \( G(\theta, \psi_E) \) is defined by

\[
G(\theta, \psi_E) = \begin{cases} 
q^{-1/2}E \sum_{x \in k_E^n} \theta^{-1}(x) \psi_\varepsilon(x) & \text{if } f(\theta) = 1 \\
q^{-1/2}E \theta^{-1}(1 + \varpi_E\varepsilon x) \psi_\varepsilon(\varpi_E\varepsilon x) & \text{if } f(\theta) = 2m + 1.
\end{cases}
\]

The \( \lambda \)-factor \( \lambda_E \) of \( E/F \) is defined by

\[
\lambda_E = \frac{\varepsilon(\text{Ind}_{W_F}^{W_E} 1_{W_K}, \psi_E)}{\varepsilon(1_{W_K}, \psi_E)}.
\]

It is well-known that

\[
\varepsilon(\text{Ind}_{W_F}^{W_E} \sigma, \psi_F) = \lambda_E^{\dim \sigma} \varepsilon(\sigma, \psi_E)
\]

for any representation \( \sigma \) of \( W_E \). The trace of matrix is denoted by \( \text{Tr} \). For an irreducible admissible representation \( \pi \) of \( \text{GL}_d(F) \), the conductorial exponent of \( \pi \) is defined to be the integer \( f(\pi) \) such that the local constant \( \varepsilon(s, \pi, \psi) \) of Godement-Jacquet [13] is the form \( aq^{-s(f(\pi)-1)} \).

Let \( G \) be a totally disconnected, locally compact group. We denote by \( \widehat{G} \) the set of (equivalence classes of) irreducible admissible representations of \( G \). For a closed subgroup \( H \) of \( G \) and a representation \( \rho \) of \( H \), we denote by \( \text{Ind}_H^G \rho \) (resp. \( \text{ind}_H^G \rho \)) the induced representation (resp. compactly induced representation) of \( \rho \) to \( G \). For a representation \( \pi \) of \( G \), we denote by \( \pi|_H \) the restriction of \( \pi \) to \( H \).
2. Construction of the representation \( \text{GL}_l(F) \)

Let \( l \) be an arbitrary prime number (we allow the case \( l = p \)). We set \( V_F = F^l \) so that \( M_l(F) = \text{End}_F(V_F) \) and \( \text{GL}_l(F) = \text{Aut}_F(V_F) \). Throughout this paper, we write \( G = G_F = \text{GL}_l(F) \) and \( G_K = \text{GL}_l(K) \)

In this section, we review the construction of the supercuspidal representation of \( \text{GL}_l(F) \) and its lift to \( \text{GL}_l(K) \) where \( K/F \) is a tamely ramified extension. Most of the contents of this section are well-known (See [8],[22] and [3].)

**Definition 2.1.** Let \( \mathcal{L} = \{ L_i \}_{i \in \mathbb{Z}} \) be the set of \( O_F \)-lattices in \( V_F \). \( \mathcal{L} \) is said to be a uniform lattice chain of period \( e = e(\mathcal{L}) \) if the following conditions hold for all \( i \in \mathbb{Z} \):

1. \( L_{i+1} \subset L_i \),
2. \( P_F L_i = L_{i+e} \),
3. \( \dim_{k_F}(L_i/L_{i+1}) = l/e \).

Since we assume \( l \) is a prime, the period \( e(\mathcal{L}) \) is either \( l \) or 1.

**Definition 2.2.** For a uniform lattice chain \( \mathcal{L} = \{ L_i \}_{i \in \mathbb{Z}} \) of period \( e \), we set

\[
\mathfrak{A}(\mathcal{L}) = \left\{ f \in M_l(F) \mid f(L_i) \subset L_i \text{ for all } i \right\},
\]

Then \( \mathfrak{A}(\mathcal{L}) \) is a principal order in \( M_l(F) \) and its Jacobson radical \( \mathfrak{P}(\mathcal{L}) \) is

\[
\{ f \in M_l(F) \mid f(L_i) \subset L_{i+1} \text{ for all } i \}.
\]

We also set the period \( e(\mathfrak{A}) \) of \( \mathfrak{A} \) is the period of \( \mathcal{L} \). Put \( U(\mathfrak{A}) = \mathfrak{A}^\times \), \( U(\mathfrak{A})^n = 1 + \mathfrak{P}^n \) for any positive integer \( n \) and

\[
\mathfrak{A}(\mathcal{L}) = \text{Aut}(\mathcal{L}) = \{ x \in \text{GL}_l(F) \mid x^{-1} \mathfrak{A} x = \mathfrak{A} \}.
\]

By taking an appropriate \( O_F \)-basis of \( L_0 \), we express the principal orders by the following matrix form. If \( e(\mathcal{L}) = l \), \( \mathfrak{A} \) (resp. \( \mathfrak{P}(\mathcal{L}) \)) is \( G \)-conjugate to \( M_l(O_F) \) (resp. \( M_l(P_F) \)). When \( e(\mathcal{L}) = 1 \), up to \( G \)-conjugacy,

\[
\mathfrak{A}(\mathcal{L}) = \begin{pmatrix}
O_F & O_F & \cdots & O_F \\
O_F & O_F & \cdots & O_F \\
\cdots & \cdots & \cdots & \cdots \\
P_F & P_F & \cdots & O_F
\end{pmatrix}
\]

and

\[
\mathfrak{P}(\mathcal{L}) = \begin{pmatrix}
P_F & O_F & \cdots & O_F \\
P_F & P_F & \cdots & O_F \\
\cdots & \cdots & \cdots & \cdots \\
P_F & P_F & \cdots & P_F
\end{pmatrix}.
\]

Let \( r, n \) be integers satisfying

\[
n > r \geq \left\lfloor \frac{n}{2} \right\rfloor \geq 0,
\]
where \([x]\) denote the greatest integer \(\leq x\). For \(\beta \in M_l(F)\), we define a function \(\psi_\beta\) on \(U(\mathfrak{A})^n\) by

\[
(2.1) \quad \psi_\beta(1 + x) = \psi(\text{Tr } \beta x).
\]

Then the map \(u \mapsto \psi_\beta\) induces an isomorphism between \(\mathfrak{P}^{-r+1}/\mathfrak{P}^{-n+1}\) and the complex dual, \((U(\mathfrak{A})^r / U(\mathfrak{A})^n)^\vee\), of \(U(\mathfrak{A})^r / U(\mathfrak{A})^n\).

**Definition 2.3.** Let \(E/F\) be a field extension in \(M_l(F)\). An element \(\beta \in E\) is said to be \(E/F\)-minimal if the following conditions hold:

1. \((v_E(\beta), e(E/F)) = 1.\)
2. \(k_F(\varphi^{v_E(\beta)} e(E/F) \mod P_E) = k_E.\)

When \(E \subset M_L(F)\) and \(E \neq F\), \(E/F\) is an extension of degree \(l\) since \(l\) is a prime. Thus we can identify \(E\) with \(V_F\). By this identification, \(\{P_E^n\}_{n \in \mathbb{Z}}\) becomes a uniform lattice chain of period \(e(E/F)\). We put \(\mathfrak{A}(E) = \mathfrak{A}(P_E^n)\).

**Proposition 2.4.** Suppose \(\beta\) is \(E/F\)-minimal and \(E \neq F\). For \(\mathfrak{A} = \mathfrak{A}(E)\), we have:

1. \(\mathfrak{K}(\mathfrak{A}) = E \times (\mathfrak{A}^\times U(\mathfrak{A}))\) and \(E \times U(\mathfrak{A}) = \mathcal{O}_E^\times.\)
2. \(E \cap \mathfrak{P}^m = P^n_E\) for all integers \(m\) and \(E \times U(\mathfrak{A})^m = 1 + P^n_E\) for all integers \(m \geq 1.\)
3. Let \(x \in \mathfrak{P}^l\). If \(\beta x - x \beta \in \mathfrak{P}^{m+l+1}\), then \(x \in E + \mathfrak{P}^{l+1}\).

**Proof.** The last assertion of the above proposition is due to Carayol (see [8]). The rest is obvious. \(\square\)

We shall construct the irreducible supercuspidal representations of \(\text{GL}_d(F)\) from \(E/F\)-minimal elements. Let \(E/F\) be a field extension of degree \(l\), \(\beta\) an \(E/F\)-minimal element and \(\mathfrak{A} = \mathfrak{A}(E)\). Put \(v_E(\beta) = 1 - n < 0\) and \(m = \lfloor n/2 \rfloor\). Then \(\psi_\beta\) is a quasi-character of \(U(\mathfrak{A})^m\) whose kernel contains \(U(\mathfrak{A})^m\). Put \(H = E \times U(\mathfrak{A})^m\) and define a quasi-character \(\rho_{\beta, \theta}\) of \(H\) by

\[
(2.2) \quad \rho_{\beta, \theta}(h \cdot g) = \theta(h) \psi_\beta(g) \quad \text{for} \quad h \in E \times, \ g \in U(\mathfrak{A})^m
\]

where \(\theta\) is a quasi-character of \(E \times\) such that \(\theta|_{E \times U(\mathfrak{A})^m} = \psi_\beta|_{E \times U(\mathfrak{A})^m}.\)

We note \(f(\theta) = 1 - v_E(\beta) = n\) where \(f(\theta)\) is the exponent of the conductor of \(\theta\) i.e. the minimum integer such that \(\text{Ker } \theta \subset 1 + P^n_E\).

Put \(J\) be the normalizer of \(\psi_\beta\) in \(\mathfrak{K}(\mathfrak{A})\) i.e.

\[
J = \{ a \in \mathfrak{K}(\mathfrak{A}) \mid \psi_\beta^a = \psi_\beta \}
\]

where \(\psi_\beta^a(x) = \psi_\beta(a^{-1}xa)\) for \(x \in U(\mathfrak{A})^m\). Then \(J = E \times U(\mathfrak{A})^m\) where \(m' = \lfloor n/2 \rfloor\) by virtue of Proposition 2.4. Put \(\eta_{\beta, \theta} = \text{Ind}_{H}^{\mathfrak{K}(\mathfrak{A})} \rho_{\beta, \theta}\).

When \(n\) is even, i.e. \(n = 2m\), then \(J = H = E \times U(\mathfrak{A})^m\). By the Clifford theory, \(\eta_{\beta, \theta}\) is an irreducible representation of \(\mathfrak{K}(\mathfrak{A})\). We put

\[
(2.3) \quad \kappa_{\beta, \theta} = \eta_{\beta, \theta}.
\]
When \( n \) is odd, i.e. \( n = 2m - 1 \), then \( J = E^{\times} U(A)^{m-1} \). Thus \( \eta_{\beta, \theta} \) is not an irreducible representation of \( \mathfrak{R}(A) \). Even in this case, we can describe the irreducible component of \( \eta_{\beta, \theta} \) by \( \beta \) and \( \theta \). If \( E/F \) is unramified, we put
\[
(2.4) \quad \kappa_{\beta, \theta} = \frac{(-1)^l (q^{l(l-1)/2} - (-1)^{(l-1)} q - 1)}{q^{l(l-1)/2} (q^l - 1)} \sum_{\chi \in (E^{\times}/F^{\times}(1+P_E))} \eta_{\beta, \theta} \otimes \chi + (-1)^{l-1} \eta_{\beta, \theta}.
\]

Now we assume we treat the case \( E/F \) is ramified. If \( l \neq p \), we put
\[
(2.5) \quad \kappa_{\beta, \theta} = 1 - \left( \frac{q}{l} \right) \frac{q^{l(l-1)/2}}{l^q (l-1)/2} \sum_{\chi \in (E^{\times}/F^{\times}(1+P_E))} \eta_{\beta, \theta} \otimes \chi + \left( \frac{q}{l} \right) \eta_{\beta, \theta}
\]
where \( \left( \frac{q}{l} \right) \) is the Legendre symbol. By Lemma 3.5.30 and Lemma 3.5.33 in [22], the virtual representation \( \kappa_{\beta, \theta} \) is an irreducible component of \( \eta_{\theta} \).

Next we treat the case \( l = p \). If \( f \) is odd, we put
\[
(2.6) \quad \kappa_{\theta} = \sum_{i=0}^{p-1} \left( \frac{1}{pq^{(l-1)/2}} + \left( \frac{i}{p} \right) \frac{p^{(f-1)/2}}{G_0 G(\beta)} \eta_{\theta} \otimes \chi^i
\]
where \( \chi \) is a generator of \((E^{\times}/F^{\times}(1+P_E))^{\times}\) determined by \( \chi(x) = \exp(2\pi \sqrt{-1}/p) \) and \( G_0, G(\beta) \) are Gauss sums defined by
\[
(2.7) \quad G(\beta) = \frac{1}{\sqrt{q}} \sum_{x \in k_E} \psi(\text{tr}_{k_E/k_F} \frac{1}{2} \beta x^2) \omega_E^{2m-1} (-1)^{(p+1)/2} x^2)
\]
\[
(2.8) \quad G_0 = \frac{1}{\sqrt{p}} \sum_{a=1}^{l} \left( \frac{a}{p} \right) \exp(2\pi \sqrt{-1}a/p).
\]

When \( f \) is even, we put
\[
(2.9) \quad \kappa_{\theta} = \sum_{\chi \in (E^{\times}/F^{\times}(1+P_E))} \frac{q^{1/2} G(\beta) - q^{(p-1)/2} \eta_{\theta} \otimes \chi + \frac{1}{q^{1/2} G(\beta)}}{G(\beta) p q^{p/2}} \eta_{\theta} \otimes \chi + \frac{1}{q^{1/2} G(\beta)} \eta_{\chi}.
\]

By Proposition 3.4 in [27], \( \kappa_{\theta} \) is an irreducible component of \( \eta_{\theta} \).

Finally we consider the level 1 supercuspidal representation. Let \( E/F \) be an unramified extension of degree \( l \), \( \theta \) a quasi-character of \( E^{\times} \) which is trivial on \( 1 + P_E \) and \( A = A(E) \). Then there is an irreducible representation \( \kappa_{\theta} \) of \( U(A) \) which is trivial on \( U(A)^1 \) and its tensor product with the pull-back of the Steinberg representation of \( U(A)/U(A)^1 \cong \text{GL}_l(k_F) \) is the representation induced by the one-dimensional representation \( tx \mapsto \theta(t), \; t \in O_E^{\times}, \; x \in U(A)^1 \), of the subgroup \( O_E^{\times} U(A)^1 \). We denote by \( \kappa_{\theta} \) the representation \( tx \mapsto \theta(t) \kappa_{\theta}(x), \; t \in F^{\times}, \; x \in U(A) \), of \( \mathfrak{R}(A) \).
The notation be as above. Then \( \kappa_{\beta, \theta} \) and \( \kappa_{\theta} \) are irreducible representations of \( \mathcal{R}(A) \). Put \( \pi_F(\beta, \theta) = \text{ind}^{G}_{\mathcal{R}(\mathbb{A})} \kappa_{\beta, \theta} \) and \( \pi_F(\theta) = \text{ind}^{G}_{\mathcal{R}(\mathbb{A})} \kappa_{\theta} \). Then \( \pi_F(\beta, \theta) \) and \( \pi_F(\theta) \) are irreducible supercuspidal representations of \( G = \text{GL}_l(F) \) with \( f(\pi_F(u, \theta)) = f(E/F)(f(\theta) - 1) + l \) and \( f(\pi_F(\theta)) = lf(\theta) \). Every irreducible supercuspidal representation of \( G \) can be written in the form \( \chi_{\pi_F(\beta, \theta)} \) or \( \chi_{\pi_F(\theta)} \) for some \( E/F \)-minimal element \( \beta, \theta \in \tilde{F} \) and \( \chi \in \tilde{F}^\times \).

The \( \varepsilon \)-factors of all supercuspidal representations of \( G \) have been calculated completely. (See [22],[20]).

Theorem 2.6. Let \( \pi_F(\beta, \theta) \) and \( \pi_F(\theta) \) be as above. Put \( n = f(\theta) \). For \( \chi \in \tilde{F}^\times \), we pick an element \( c_\chi \in F \) such that \( \chi(1 + x) = \psi_F(c_\chi x) \) for \( x \in \mathbb{P}_E^{[f(\chi) + 1]/2} \). (If \( f(\chi) \leq 1 \), we take \( c_\chi = 0 \).) Put \( n(\chi) = \max(n, e(E/F)(f(\chi) - 1) + 1) \) and \( \beta_\chi = \beta + c_\chi \).

1. If \( n(\chi) \) is even,
   \[
   \varepsilon(\chi_{\pi_F(\beta, \theta)}, s, \psi) = \psi_E(\beta_\chi)(\chi_E \theta)(\beta_\chi^{-1})|\beta_\chi|_E^s.
   \]

2. If \( n(\chi) = n = 1 \),
   \[
   \varepsilon(\chi_{\pi_F(\theta)}, s, \psi) = (-1)^{l-1}\varepsilon(\chi_E \theta, s, \psi_E).
   \]

3. If \( n(\chi) \neq 1 \) is odd,
   \[
   \varepsilon(\chi_{\pi_F(\beta, \theta)}, s, \psi) = \psi_E(\beta_\chi)(\chi_E \theta)(\beta_\chi^{-1})|\beta_\chi|_E^s G
   \]

where the Gauss sum \( G \) is defined by

\[
G = \begin{cases} 
G(\theta, \psi_E) & \text{if } n = n(\chi) \text{ and } E/F \text{ is tamely ramified} \\
G(\beta) & \text{if } n = n(\chi) \text{ and } E/F \text{ is wildly ramified} \\
\lambda_E G(\chi, \psi_F)^l & \text{if } n > n(\chi)
\end{cases}
\]

where \( \lambda_E \) is defined in (1.7).

3. Explicit correspondences and tame base change lift

Now we consider some correspondences between \( A_F(l) \) and \( G_F(l) \) which satisfy the conditions (i)-(iv) of the local Langlands correspondence. When \( l \neq p \) or \( l = p \) and \( E/F \) is unramified, this is a special case of Howe-Moy correspondence.

Definition 3.1. A quasi-character \( \theta \) of \( E^\times \) is called generic if \( f(\theta) \neq 1 \mod l \). For a generic character \( \theta \) of \( E^\times \), \( \beta_{\theta} \in \mathbb{P}_E^{1-f(\theta)} - \mathbb{P}_E^{2-f(\theta)} \) is defined by

\[
(3.1) \quad \theta(1 + x) = \psi_E(\beta_{\theta} x) \quad \text{for} \quad x \in \mathbb{P}_E^{[f(\theta) + 1]/2}.
\]

Then \( \beta \) is \( E/F \)-minimal. We denote by \( E^\times_{\text{gen}} \) the set of generic quasi-characters of \( E^\times \).
Remark 3.2. When $E/F$ is tamely ramified, the generic quasi-character $\theta$ determines uniquely $\pi_F(\beta, \theta)$. (See [22]). In this case we simply denote $\pi_F(\beta, \theta)$ by $\pi_F(\theta)$. When $l = p$, we need $\beta$ to determine the representation $\pi_F(\beta, \theta)$.

To separate the wildly ramified case, we introduce some notations. Let $\mathcal{A}_l^{ur}(F)$ denote the set $\pi = \chi \pi_F(\beta, \theta) \in \mathcal{A}_l(F)$ with the property that $F(\beta)/F$ is wildly ramified. $\pi \in \mathcal{A}_l^{ur}(F)$ is equivalent to $l = p$ and $\pi \simeq \chi \pi$ for some unramified quasi-character $\chi \neq 1$ of $F^\times$. We put $\mathcal{A}_l(F) = \mathcal{A}_l(F) \setminus \mathcal{A}_l^{ur}(F)$. Similarly let $\mathcal{G}_l^{ur}(F)$ denote the set $\sigma \in \mathcal{G}_l(F)$ with the property that $\sigma \otimes \chi$ is equivalent to $\pi$ for some unramified quasi-character $\chi \neq 1$ of $F^\times$ and $l = p$. We also put $\mathcal{G}_l^{t}(F) = \mathcal{G}_l(F) \setminus \mathcal{G}_l^{ur}(F)$. If $p \neq l$, $\mathcal{A}_l(F) = \mathcal{A}_l^{t}(F)$ and $\mathcal{G}_l(F) = \mathcal{G}_l^{t}(F)$ The Howe-Moy correspondence gives a bijection between $\mathcal{G}_l^{t}(F) = \mathcal{A}_l^{t}(F)$. (See [22] and [12].)

If $E/F$ is tamely ramified, $\lambda_E$ is easily calculated.

**Lemma 3.3.** Let $E/F$ is a tamely ramified extension of degree $l$. Then

$$
\lambda_E = \begin{cases}
(-1)^{l-1} & \text{if } e(E/F) = 1, \\
\left(\frac{q}{l}\right) & \text{if } e(E/F) = l = 2 \\
q^{-l/2} \sum_{x \in k_E} \text{sgn}_{E/F}(x) \psi_E(x) & \text{if } e(E/F) = 2
\end{cases}
$$

**Proof.** See (2.5.3), (2.5.5) and Proposition 2.5.11 of [22].

**Theorem 3.4.** Let $E$ be a tamely ramified extension of $F$ of degree $l$ and $\theta$ be a generic quasi-character of $E^\times$. We define a quasi-character $\delta_E$ of $E^\times$ as follows:

When $e(E/F) \neq 2$, $\delta_E(x) = \lambda_E^{\nu_E(x)}$.

When $e(E/F) = 2$, $\delta_E(x) = \begin{cases}
1 & \text{if } x \in 1 + P_E, \\
\text{sgn}_{E/F}(x) & \text{if } x \in F^\times, \\
\lambda_E & \text{if } x = \beta_\theta.
\end{cases}$

We set $\sigma_F(\theta) = \delta_E \text{Ind}_{W_E}^{W_F}$.

1. the representation $\sigma_F(\theta)$ belongs to $\mathcal{G}_l^{t}(F)$ and any element of $\mathcal{G}_l^{t}(F)$ can be written in the form $\chi \sigma_F(\theta)$ for an extension $E/F$ of degree $l$, a generic character of $E^\times$ and a quasi-character $\chi$ of $F^\times$.

2. Define the map $\Phi^F_l$ from $\mathcal{G}_l^{t}(F)$ to $\mathcal{A}_l^{t}(F)$ by

$$
\Phi^F_l(\chi \sigma_F(\theta)) = \chi \pi_F(\delta_E \theta).
$$

Then $\Phi^F_l$ is a bijection which satisfies the following conditions:

(a) For $\chi \in \hat{F}^\times$ and $\sigma \in \mathcal{G}_l^{t}(F)$,

$$
\Phi^F_l(\chi \sigma) = \chi \Phi^F_l(\sigma).
$$
(b) For \( \sigma \in \mathcal{G}_l^1(F) \),
\[
\Phi_l^F(\hat{\sigma}) = \Phi_l^F(\sigma)^\vee.
\]

(c) Let \( \omega_\pi \) denote the central quasi-character of \( \pi \in \mathcal{A}_l(F) \). For \( \sigma \in \mathcal{G}_l^1(F) \),
\[
\omega_{\Phi_l}(\sigma) = \det \sigma.
\]

(d) For \( \sigma \in \mathcal{G}_l^1(F) \),
\[
\varepsilon(\Phi_l^F(\sigma), s, \psi_F) = \varepsilon(\sigma, s, \psi_F).
\]

Since \( \mathcal{G}_p^\mathrm{wr}(F) \) contains non-monomial representations, the correspondence between \( \mathcal{G}_p^\mathrm{wr}(F) \) and \( \mathcal{A}_p^\mathrm{wr}(F) \) becomes more complicated. We use the tame lifting of Bushnell-Henniart [3]. For any tamely ramified extension \( K/F \), including the case \( K/F \) is non-Galois, the tame lifting map \( I_K \) from \( \mathcal{A}_p^\mathrm{wr}(F) \) to \( \mathcal{A}_p^\mathrm{wr}(K) \) is constructed by Bushnell-Henniart. Since we consider the case \( i = 1 \), this base change is easy to describe. Since \( K/F \) is tamely ramified, \( E \otimes_F K =EK \) is an extension of field of \( K \), \( G_K = G(K) \) can be identified with \( \text{Aut}_K(E \otimes_F K) \) and \( \beta = \beta \otimes 1 \) becomes an \( EK/K \)-minimal element in \( V_K = \text{End}_K(EK) \). Moreover if \( \theta \) is a quasi-character of \( E^\times \) such that \( \theta(1 + x) = \psi(\text{tr}_{E/F} \beta x) \) for \( x \in \mathcal{P}_E \), then \( \theta N_{EK/E}(1 + x) = \psi_K(\text{tr}_{EK/K} \beta x) \) for \( x \in \mathcal{P}_E \). Therefore we get an irreducible supercuspidal representation \( \pi_K(\beta, \theta) \in \mathcal{A}_p^\mathrm{wr}(K) \).

**Theorem 3.5.** Let \( K/F \) be an extension of degree prime to \( p \) and \( I_K \) the lifting from \( \mathcal{A}_p^\mathrm{wr}(F) \) to \( \mathcal{A}_p^\mathrm{wr}(K) \) defined by (5.3.3) in [3]. Put \( \Delta_K = \det \text{Ind}_{W_K}^{F \times} 1_{W_K} \in \hat{F}^\times \) and \( \tilde{\Delta} = \Delta_K \circ N_{E/F} \in \hat{E}^\times \). For \( \pi_F(\beta, \theta) \in \mathcal{A}_p^\mathrm{wr}(F) \) and \( \chi \in \hat{F}^\times \), we have:
\[
I_K(\chi \pi_F(\beta, \theta)) = \chi_K \pi_K(\beta, (\tilde{\Delta} e(E/F)-1) \circ N_{EK/E})
\]
\[
= \begin{cases} 
\chi_K \pi_K(\beta, \theta \circ N_{EK/E}) & e(E/F) \neq 2 \\
\chi_K \pi_K(\beta, (\tilde{\Delta} \theta) \circ N_{EK/E}) & e(E/F) = 2.
\end{cases}
\]

**Proof.** Since two lifting maps are compatible with twist of quasi-character of \( F^\times \), we may assume \( \chi = 1 \). By Proposition 10.2 of [3], it suffices to say
\[
\varepsilon(I_K(\pi_F(\beta, \theta)), s, \psi_K) = \varepsilon(\pi_K(\beta, (\tilde{\Delta} e(E/F)-1) \circ N_{EK/E}), s, \psi_K).
\]
(Other conditions (a) and (b) in Proposition 10.2 of [3] are obvious in our case.) Theorem 1.6 of [3] tells us that
\[
\lambda_K^p \varepsilon(I_K(\pi_F(\beta, \theta)), s, \psi_K) = \Delta(N_{E/F}(\beta)) \varepsilon(\pi_F(\beta, \theta), s, \psi_F)^{[K:F]}.
\]
On the other hand, it follows from Proposition 2.2.11 of [20] that
\[
\lambda_K^p \varepsilon(\pi_K(\beta, \theta \circ N_{EK/E}), s, \psi_K) = \Delta(N_{E/F}(\beta)) \varepsilon(\pi_F(\beta, \theta), s, \psi_F)^{[K:F]}
\]
if \( p \neq 2 \). (Proposition 2.2.11 of [20] assumes \( K/F \) is Galois, but it holds including the case \( K/F \) is non-Galois since Proposition 2.5.16 of [22] holds for any tamely ramified extension \( K/F \).) Hence the assertion holds when \( p \neq 2 \). When \( p = 2 \), \( n(\pi_F(\beta, \theta)) = 1 - v_E(\beta) \) is even since \( (v_E(\beta), p) = 1 \). Therefore Theorem 2.6 tells us

\[
\varepsilon(\pi_K(\beta, \theta \circ N_{E_K/E}), s, \psi_K) = \varepsilon(\pi_F(\beta, \theta), s, \psi_F)^{[K:F]}.
\]

Since \( \Delta_K \circ N_{K/F} \) is unramified and \( \Delta_K^{-1} = \Delta_K \),

\[
\varepsilon(\pi_K(\beta, (\tilde{\Delta}\theta)\circ N_{E_K/E}), s, \psi_K) = \Delta_K \circ N_{K/F}(\beta) \varepsilon(\pi_K(\beta, \theta \circ N_{E_K/E}), s, \psi_K).
\]

Hence our assertion holds. \( \Box \)

**Remark 3.6.** Two quasi-characters \( \Delta_K \) and \( \delta_K \) is closely related. If \( e(K/F) \) is odd, \( \Delta_K \circ N_{K/F} = \delta_K \). (See Corollary 2.5.5 of [22].)

Using the tame lifting map \( I_K \), Bushnell-Henniart ([3]) has constructed the correspondence \( \mathcal{G}^{wr}_p(F) \) to \( \mathcal{A}^{wr}_p(F) \). For \( i = 1 \), this map coincides with the local Langlands correspondence and is compatible with \( I_K \). This follows as a special case of Lemma 5.2 in [4].

**Proposition 3.7.** Let \( \Lambda^K_l \) be the local Langlands map. Then for any tamely ramified extension \( K/F \) and \( \sigma \in \mathcal{G}^{wr}_p(F) \), we have:

\[
I_K \Lambda^K_l(\sigma) = \Lambda^K_I(\sigma|_{W_K}).
\]

**Proof.** By Lemma 5.2 in [4], it suffices to say that the exponent \( f(\pi_{\beta, \theta}) \) of the conductor of \( \pi_{\beta, \theta} \in \mathcal{A}^{wr}_p(F) \) is prime to \( p \). It follows from the fact that \( f(\pi_{\beta, \theta}) \equiv -v_E(\beta) \mod p \). \( \Box \)

We define the lift \( I_K \) for \( \pi \in \mathcal{A}^l_I(F) \) as in the case \( \pi \in \mathcal{A}^{wr}_I(F) \).

**Definition 3.8.** Let \( E/F \) be an extension of degree \( l \), \( \theta \in \widetilde{E}^\infty_{gen} \) and \( \chi \in \hat{F}^\times \). Assume \( K \) is a tamely ramified extension of \( F \) such that \( ([K : F], l) = 1 \). Then we define \( I_K(\chi \pi_F(\theta)) \) by

\[
I_K(\chi \pi_F(\beta, \theta)) = \chi_K(\Delta_K \circ N_{K/F})^{e(E/F)-1} \pi_K(\beta, \theta \circ N_{E_K/E}) \]

\[
= \begin{cases} 
\chi_K \pi_K(\beta, \theta \circ N_{E_K/E}) & e(E/F) \neq 2 \\
\chi_K(\Delta_K \circ N_{K/F}) \pi_K(\beta, \theta \circ N_{E_K/E}) & e(E/F) = 2.
\end{cases}
\]

This lifting is compatible with \( \Phi_l \).

**Proposition 3.9.** Let \( K/F \) be a finite, tamely ramified extension satisfying \( K \cap E = F \). For \( \sigma \in \mathcal{G}^l_I(F) \),

\[
I_K \Phi^K_l(\sigma) = \Phi^K_I(\sigma|_{W_K}).
\]

**Proof.** Since \( I_K \) and \( \Phi_l \) are compatible with quasi-character twist, we may assume \( \sigma = \sigma_F(\theta) \) for \( \theta \in \widetilde{E}^\infty_{gen} \). By the definition of \( I_K \) and \( \Phi_l \),

\[
(\Phi^K_l)^{-1}(I_K(\Lambda^K_l(\sigma_F(\theta)))) = \text{Ind}_{W_{E_K}}^{W_K} \delta_{E_K/K}((\tilde{\Delta}^{e(E/F)-1}\theta) \circ N_{E_K/K}).
\]
On the other hand, it follows from $W_E W_K = W_F$ and $W_E \cap W_K = W_{EK}$ that Mackey’s Theorem tells us

$$\sigma_F(\theta)|_{W_K} = \text{Ind}_{W_{EK}}^{W_K}(\delta_E \theta) \circ N_{EK/E}.$$ 

Thus it suffices to say that

(3.2) \[ \delta_{EK/K}(\tilde{\Delta} \circ N_{EK/K}(x)) = \delta_{EK/K}(x) \Delta_{EK/F}(x^2) \]

since $\Delta_K$ has at most order 2. The right hand side of (3.2) becomes

$$\text{sgn}_{E/F}(N_{EK/E}(x)),$$

which equals to $\text{sgn}_{EK/K}(x)$ since $\left[ K : F \right]$ is odd.

We compare the value of both sides of (3.2) at $\beta$. The left hand side is $\lambda_{EK/K} \lambda_{EK/F}$. After all, the equation $\lambda_{EK/E} \lambda_{EK/K} = 1$ gives the result. When $\left[ K : F \right]$ is prime, it follows from Lemma 3.3. The composite case is obtained by the transitivity property of $\lambda$-factor.

We need to show that the Howe-Moy correspondence $\Phi_l$ coincides with the Local Langlands correspondence $\Lambda_l$.

**Theorem 3.10.** For any prime $l' \neq p$,

$$\Phi_{l'}^E = \Lambda_{l'}^E.$$ 

**Proof.** If $l' = 2$, it follows from Converse Theorem ([9]). We assume $l'$ is an odd prime. Let $\pi \in \mathcal{A}_F(l')$. Then there exist an extension $E/F$ of degree $l'$, $\theta \in (E^\times)_{\text{gen}}$ and $\chi \in \widehat{F^\times}$ such that $\pi = \chi \pi_{\mathcal{F}}(\theta)$ as in Remark 3.2. When $E/F$ is unramified, Theorem 9.2 ([26]) implies $\Phi_{l'}^E(\pi) = \chi \text{Ind}_{W_E}^{W_{EK}} \theta = \Lambda_{l'}^E(\pi)$. When $E/F$ is ramified, the assertion follows from Theorem B in [7].

**Remark 3.11.** Theorem 9.2 ([26]) is proved under the assumption $p > l$, but this assumption is dispensable. The key point is to prove that

$$\Theta^\kappa_{\pi}(x) = \Theta_{\pi}(x) \quad \text{for } x \in E^\times \backslash F^\times (1 + P_E^F)$$

where $\Theta_{\pi}$ is a distribution character of $\pi$ and $\Theta^\kappa_{\pi}$ is a $\kappa$-twisted distribution character of $\pi$ for $\kappa \in (F^\times / n_{E/F}(E^\times))^\wedge$. This is proved in Theorem 6.1 ([26]) without using the assumption $p > l$.

By Proposition 3.7, Proposition 3.9 and Theorem 3.10, $\Phi_l$ is compatible with $\mathbf{1}$ for any prime $l$. 

Corollary 3.12. Let $K/F$ be a finite, tamely ramified extension satisfying $K \cap E = F$. For any prime $l$ and $\sigma \in G_l(F)$,
\[ I_K \Phi^F_l(\sigma) = \Phi^K_l(\sigma|_{W_K}). \]

4. $\varepsilon$-FACTOR OF PAIRS

In this section, we consider the $\varepsilon$-factor $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$. Let $l'$ be a prime not equal to $l$ and $p$. We treat the case $\pi_1 \in A_F(l)$ and $\pi_2 \in A_F(l')$. Since the local Langlands correspondence and the Bushnell-Henniart base change lift are compatible with quasi-character twists, we may assume $\pi_1$ and $\pi_2$ are minimal.

Theorem 4.1. Let $\pi_1 \in A_F(l)$ and $\pi_2 \in A_F(l')$ where $l'$ is a prime not equal to $l$ and $p$. Let $E_2/F$ be an extension of degree $l'$, $\theta_2 \in (F_{l')_{gen}}$ and $\chi_2 \in \tilde{F_{l')}$ such that $\pi_2 = (\chi_2)_{E_2} \pi_F(\theta_2)$ as in Remark 3.2. Then we have
\[ (4.1) \quad \varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \chi_2 \delta_{E_2} \theta_2 l_{E_2}(\pi_1), s, \psi_{E_2}). \]

Proof. It follows from $\Phi^F_l = \Lambda^F_l$ that
\[ (\Lambda^F_l)^{-1}(\pi_F((\chi_2)_{E_2}(\theta_2))) = \text{Ind}_{W_{E_2}} W_{E_2}^F(\chi_2 \delta_{E_2} \theta_2). \]

Put $(\Lambda^F_l)^{-1}(\pi_1) = \sigma_1$. Then we have:
\[ \varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \varepsilon(\sigma_1 \otimes \text{Ind}_{W_{E_2}} W_{E_2}^F(\chi_2 \delta_{E_2} \theta_2), s, \psi_F). \]

Since
\[ \text{Ind}_{W_{E_2}} W_{E_2}^F(\sigma_1) \otimes \chi_2 \delta_{E_2} \theta_2 = \text{Ind}_{W_{E_2}} W_{E_2}^F(\sigma_1|_{W_{E_2}}) \otimes \chi_2 \delta_{E_2} \theta_2 \]
and
\[ \varepsilon(\text{Ind}_{W_{E_2}} W_{E_2}^F(\sigma, s, \psi_F) = \lambda_{E_2}^{\dim} \varepsilon(\sigma, s, \psi_{E_2}) \quad \text{for} \quad \sigma \in G_E(l'), \]
we obtain
\[ \varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \varepsilon(\sigma_1|_{W_{E_2}} \otimes \text{Ind}_{W_{E_2}} W_{E_2}^F(\chi_2 \delta_{E_2}, s, \psi_F) \]
\[ = \lambda_{E_2} \varepsilon(\sigma_1|_{W_{E_2}} \otimes (\chi_2 \delta_{E_2} \theta_2), s, \psi_{E_1}). \]

Assume $l \neq p$, then $\pi_1$ can be written in the form $\chi_1 \pi_{\theta_1} \sigma_2 = \text{Ind}_{W_{E_1}} W_{E_2}^F(\chi_1 \delta_{E_1}, \theta_1)$. By the Mackey decomposition and $W_{E_1} W_{E_2} = W_F$, we have
\[ (\text{Ind}_{W_{E_1}} W_{E_2}^F(\chi_1 \delta_{E_1}, \theta_1)|_{W_{E_2}} = \text{Ind}_{W_{E_1}} W_{E_2}^F(\chi_1 \delta_{E_1}, \theta_1) \circ N_{E_1/E_2}. \]

Since $(\chi_1 \delta_{E_1}, \theta_1) \circ N_{E_1/E_2/F}$ does not factor through $N_{E_1/E_2}$,
\[ \text{Ind}_{W_{E_1}} W_{E_2}^F((\chi_1 \delta_{E_1}, \theta_1) \circ N_{E_1/E_2/E_1} \in G_{E_2}(l)). \]

Therefore we have:
\[ \varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \lambda_{E_2} \varepsilon(\text{Ind}_{W_{E_1}} W_{E_2}^F(\chi_1 \delta_{E_1}, \theta_1) \otimes \chi_2 \delta_{E_2} \theta_2 \circ N_{E_1/E_2/E_1}, s, \psi_{E_1}) \]
\[ = \lambda_{E_2} \varepsilon(\chi_2 \delta_{E_2} \theta_2 \otimes (\chi_1)_{E_1E_2} \pi_{E_2}((\theta_1 \circ N_{E_1/E_2/E_1}), s, \psi_{E_2}). \]
(The last equality follows from $\Lambda_{E_2}(\pi_1 \circ N_{E_1 E_2 / E_1}) = \text{Ind}^{W_{E_2}}_{W_{E_1 E_2 / E_1}} (\theta_1 \delta_{E_1}) \circ N_{E_1 E_2 / E_1}$.)

When $l = p$, it follows from Proposition 3.7 and Proposition 3.5 that

$$\Lambda_{E_2}(\sigma_1|_{W_{E_2}}) = 1_{E_2}(\pi_1) = (\chi_1)_{E_1 E_2} \pi_{E_2}(\beta_1, \theta_1 \circ N_{E_1 E_2 / E_1}).$$

Thus we have

$$\varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \lambda_{E_2} \varepsilon(\chi_2 \delta_{E_2} \theta_2 \otimes (\chi_1)_{E_1 E_2} \pi_{E_2}(\beta_1, \theta_1 \circ N_{E_1 E_2 / E_1}), s, \psi_{E_2}).$$

□

By combining Theorem 2.6 and Theorem 4.1, we get the complete list of $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$ for $\pi_1 \in A_F(l)$ and $\pi_2 \in A_F(l')$ where $l$ is any prime and $l'$ is a prime $\neq l$.

Remark 4.2. By the result of [7], Theorem 4.1 may be extended to the case $\pi_1 \in A_F^l(m)$ and $\pi_2 \in A_F^l(n)$ where $(m, n) = 1$.

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Department of Mathematics and Information Science, College of Integrated Arts and Sciences, Osaka Prefecture University

E-mail address: takahasi@mi.cias.osakafu-u.ac.jp