Abstract. In this paper, we study the well-posedness of a class of evolutionary variational-hemivariational inequalities coupled with a nonlinear ordinary differential equation in a Banach space. The intended application is in modelling frictional contact between a deformable viscoelastic body and a rigid foundation. The system under consideration allows the friction coefficient \( \mu \) to depend on an external state variable \( \alpha \), described by an ODE, and the slip rate \(|\dot{u}_\tau|\). In addition, the normal stress \( \sigma_v \) is a function of time and space. We base the proof on an iterative approximation scheme, showing that the problem has a unique weak solution. Moreover, we show that the flow map depends continuously on the initial data. Finally, we include two applications; the first is in the normal compliance setting, and in the second, we consider normal damped response. Both are relevant for rate-and-state friction laws.

1. Introduction

This work concerns the study of an evolutionary differential variational-hemivariational inequality modelling the frictional contact between a viscoelastic body and a rigid foundation. These systems are relevant for many physical phenomena ranging from engineering to biology (see, e.g., \([4,7,23]\) and the references therein). We are interested in viscoelastic frictional contact problems, which have been studied intensively, see, e.g., some of the relevant books \([7,16,23]\).

Let \( V \) and \( Y \) be two Banach spaces, and for any \( T > 0 \), we let \([0,T]\) be the time interval of interest. Then, the Cauchy problem for the evolutionary differential variational-hemivariational inequality under consideration reads: Find \( w : [0,T] \to V \) and \( \alpha : [0,T] \to Y \) satisfying

\[
\begin{align*}
\dot{\alpha}(t) &= G(t, \alpha(t), Mw(t)), \\
\langle \dot{w}(t) + A(t, w(t)) - f(t) + Rw(t), v - w(t) \rangle + j^\circ(t, Nw(t); Nv - Nw(t)) \\
&\quad + \varphi(t, Sw(t), \alpha(t), Mw(t), K_v) - \varphi(t, Sw(t), \alpha(t), Nw(t), Kw(t)) \geq 0,
\end{align*}
\]

for all \( v \in V \), a.e. \( t \in (0,T) \), with

\[
\begin{align*}
w(0) &= w_0, \quad \alpha(0) = \alpha_0.
\end{align*}
\]

Here, \( A \) and \( R \) are nonlinear operators related to the viscoelastic constitutive laws. Further, \( j^\circ \) is a generalized directional derivative of a functional \( j \). The functionals \( \varphi \) and \( j \) are determined by contact boundary conditions. We require \( \varphi \) to be convex, while \( j \) may be nonconvex with an appropriate structure to be provided later. The operator \( S \) relates to the contact conditions between the rigid and deformable surface, and \( G \) is assumed to be a nonlinear operator related to the change in the external state variable. The data \( f \) is related...
to the given body forces and surface tractions, and \( w_0 \) and \( \alpha_0 \) represent the initial data. Lastly, \( M, N, \) and \( K \) are bounded linear operators related to the tangential and normal trace operators. The Cauchy problem (1.1b)-(1.1c) is called hemivariational inequality if \( \varphi \equiv 0 \), i.e., we do not require the functionals to be convex. On the other hand, we say (1.1b)-(1.1c) is a variational inequality if \( j^o \equiv 0 \), i.e., we require the functionals to be convex.

**Definition 1.1.** A pair of functions \((w, \alpha)\), where \( \alpha \in C([0,T]; Y) \) and \( w : [0,T] \to V \) measurable, is said to be a mild solution of (1.1) if \( \alpha \) satisfy

\[
\alpha(t) = \alpha_0 + \int_0^t G(s, \alpha(s), Mw(s)) ds,
\]

and \( w \) satisfies (1.1b)-(1.1c).

The main purpose of this paper is to extend the results from [14, 21] to prove well-posedness of (1.1) with applications to rate-and-state frictional contact problems. We prove that the pair \((w, \alpha)\) is a solution of (1.1) in the sense of Definition 1.1 and show that the flow map depends continuously on the initial data. The problem setting is motivated by [18,21,23,25], and the techniques have taken inspiration from [10,14,15,25].

1.1. Former well-posedness results. Special cases of (1.1) have been investigated in literature. The closest to our setting is the recent work [14], which proves well-posedness for an ordinary differential equation coupled with a variational-hemivariational inequality with applications to both viscoplastic material and viscoelasticity with adhesion. In fact, if we let \( \varphi \) be independent of \( Mw \) in its third argument, then (1.1) reduces to [14]. We find that in applications involving rate-and-state friction, \( \mu \) may depend both upon the slip rate \( |\dot{u}_\tau| \) and the external state variable \( \alpha \) and hence an extra argument of \( Mw \) occurs in \( \varphi \). Moreover, our approach is based on defining an iterative scheme to prove existence and uniqueness of (1.1), which immediately gives us a numerical method. Furthermore, this additional dependence of \( \mu \) on slip rate \( |\dot{u}_\tau| \) also induces certain difficulties in the analysis which will become clear later. In the quasi-static case tackled in [21], with \( j^o \equiv 0 \), they proved existence and uniqueness of the solution pair by an implicit method to rewrite (1.1a) only depending on \( w \). However, the setting of [21] was not directly applicable in our case where the inertial term restricts the time regularity for \( w \) in space. Moreover, for \( Sw \equiv \text{constant}, \varphi \) independent of \( Mw \) in its third argument, and \( j^o \equiv 0 \), that is, when the normal stresses are constant (referred to as Tresca friction), the problem reduces to the ones studied in [19,20], but using different techniques. In contrast, we also consider the continuous dependence on initial data which is not covered in two three last references. Moreover, neglecting \( \alpha \) and \( Mw \) in \( \varphi \), they showed existence and uniqueness in [25, Section 10.2]. If we let \( \varphi \equiv 0 \), existence and uniqueness of this system was proved in [10, Section 6]. For further discussion, we refer to [14, p.2].

1.2. Physical setting. A mathematical model in contact mechanics needs several relations: a constitutive law, a balance equation, boundary conditions, interface laws, and initial conditions. The constitutive laws help us describe the mechanical reactions (stress-strain type) of the material. In most cases, constitutive laws originate from experiments, though they are verified to satisfy certain invariance principles. We refer to [9, Chapter 6] for a general description of several diagnostic experiments which give us the information needed to construct a constitutive law for specific materials. The interface laws are prescribed on the possible contact surface. We refer to the interface laws in tangential direction as friction laws and in normal direction as contact conditions. The mathematical
treatment of these problems gives rise to the variational-hemivariational inequalities of the form (1.1b)-(1.1c) where we put appropriate constraints on the operators to fit applications of interest.

We are mainly interested in studying frictional problems, where we take into account the time dependence of the contact surface on the friction coefficient. This is modelled via a state variable \( \alpha \) that tracks the contact surface and the slip rate and updates the friction coefficient via an ODE. We assume the following dependencies:

\[
\mu = \mu(|\dot{u}_\tau(t)|, \alpha(t)), \quad \dot{\alpha}(t) = G(\alpha(t), |\dot{u}_\tau(t)|),
\]

where \( |\dot{u}_\tau(t)| \) denotes the slip rate. These laws (1.2) are referred to as rate-and-state friction laws. One of the most common models is the experimentally derived Dieterich-Ruina law and is standard in geophysical applications dealing with earthquakes. We refer to [13] for an overview and comparison of some of the commonly used laws. There have been physical issues with the standard rate law, e.g., \( |\dot{u}_\tau(t)| \to 0 \) resulting in a negative friction coefficient. This is repaired by using the regularized or truncated law (see [18, Section 1.1-1.3] and references therein), which are, respectively, given by

\[
\begin{align}
\mu(|\dot{u}_\tau(t)|, \alpha(t)) &= c_1 \arctan \left( \frac{|\dot{u}_\tau(t)|}{v_\alpha(t)} \right), \\
\mu(|\dot{u}_\tau(t)|, \alpha(t)) &= c_1 \log^+ \left( \frac{|\dot{u}_\tau(t)|}{v_\alpha(t)} \right), \quad \text{with} \quad \log^+ v = \begin{cases} 
\log v, & \text{if } v \geq 1, \\
0, & \text{otherwise},
\end{cases}
\end{align}
\]

where \( v_\alpha(t) = \hat{c} e^{-\tilde{c} \alpha(t)} \) and \( c_1, \hat{c}, \tilde{c}, \) and \( \tilde{c} \) depend on the material (see, e.g., [18, Section 1.2]). Moreover, we will require \( G \) to be locally Lipschitz in both arguments. Under this assumption, one application that is covered in our framework is the so-called aging rate-and-state friction law described by

\[
\dot{\alpha}(t) = c[e^{-\alpha(t)} - |\dot{u}_\tau(t)|],
\]

with \( c \) being a material-dependent parameter and is an extension of the framework in [14,21].

1.3. Contributions and outline. The novelties of this paper are:

- Well-posedness of (1.1) in the sense of Definition 1.1.
- A proof based on an iterative decoupling approach which also directly gives rise to a numerical method.
- Two new applications related to friction. One being a contact problem with normal compliance and the latter contact with normal damped response. Both applications pertain to rate-and-state friction.

The paper is organized as follows. In Section 2, we introduce the function spaces and some basics of nonsmooth analysis in order to better understand the problem setting. In Section 3, we present our problem statement and the assumptions on the data. Our main result is Theorem 3.2, which will be proved in Section 5 using results from Section 4. The proof of Theorem 3.2 is divided into several steps for the sake of presentation. Next, we present two applications to frictional contact problems in Section 6. Lastly, in Appendix A, we include examples that fit the assumptions. Appendix B-E contains proofs of results that are mostly available elsewhere but needed throughout the paper.

1.4. Notation. We now present some notations that will be used in this paper.

- Let \( 0 < T < \infty \) be the maximal time.
- Let \( d \) denote the dimension. In the applications, \( d \in \{2, 3\} \).
- A point in \( \mathbb{R}^d \) is denoted by \( x = (x_i) \).
• $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$.
• We denote $| \cdot |$ as the Euclidean norm.
• $\Omega \subset \mathbb{R}^d$ is a bounded open connected subset with a Lipschitz boundary $\Gamma = \partial \Omega$. We let $\Gamma$ consist of three disjoint parts; $\Gamma_D$, $\Gamma_N$, and $\Gamma_C$, with $\text{meas}(\Gamma_D) > 0$, and $\text{meas}(\Gamma_C) > 0$, i.e., nonzero measure, but $\Gamma_N$ is allowed to be empty.
• In the applications, we let $\Omega$ be a viscoelastic deformable body sliding with bilateral contact on a rigid foundation. Moreover, $\Gamma_D$ denotes the Dirichlet boundary, $\Gamma_N$ the Neumann boundary, and $\Gamma_C$ is the contact boundary.
• $\nu$ denotes the outward normal on $\Gamma$.
• We let $\bar{\Omega} = \Omega \cup \Gamma$.
• We let $L^2(\Omega)$ denote the space of squared Lebesgue integrable functions equipped with the norm $\|v\|_{L^2(\Omega)} = \left( \int_{\Omega} |v|^2 \, dx \right)^{1/2}$. With the usual modifications for $L^\infty(\Omega)$.
• Let $C^\infty_c(\Omega)$ denote the space of infinitely differentiable functions with compact support.
• We will denote $c, \tilde{c}$ as positive constants, which might change from line to line.
• Let $\mathcal{L}(\bar{X}, \bar{Y})$ denote the set of all bounded linear maps from $\bar{X}$ into $\bar{Y}$. Dual products between other spaces will be denoted with the relevant subscript.
• We denote the operator norm of the operators $M : V \to U$, $N : V \to X$, and $K : V \to Z$ as $\|M\| = \|M\|_{\mathcal{L}(V; U)}$, $\|N\| = \|N\|_{\mathcal{L}(V; X)}$, and $\|K\| = \|K\|_{\mathcal{L}(V; Z)}$, respectively.

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2. Function spaces and basics of nonsmooth analysis

In this section, we present the function spaces and fundamental results. For further information, we refer to standard textbooks, e.g., [2,8,17].

2.1. Sobolev spaces. In this section, we wish to define the usual Sobolev spaces. This section will first become useful in the applications, i.e., Section 6. We define

\[ L^2(\Omega; \mathbb{R}^d) = \{ v = (v_i): v_i \in L^2(\Omega), \ 1 \leq i \leq d \}, \]

\[ Q = L^2(\Omega; \mathbb{S}^d) = \{ \tau = (\tau_{ij}): \tau_{ij} = \tau_{ji} \in L^2(\Omega), \ 1 \leq i, j \leq d \}, \]

which are Hilbert spaces with the canonical inner products

\[(u,v)_{L^2(\Omega; \mathbb{R}^d)} = \int_\Omega u_i v_i dx = \int_\Omega u \cdot v dx, \quad (\sigma,\tau)_Q = \int_\Omega \sigma_{ij} \tau_{ij} dx = \int_\Omega \sigma : \tau dx\]

for all \( u, v \in L^2(\Omega; \mathbb{R}^d), \sigma, \tau \in Q \). The associated norms will be denoted \( \|\cdot\|_{L^2(\Omega; \mathbb{R}^d)} \) and \( \|\cdot\|_Q \), respectively. Moreover, let

\[ H^1(\Omega) = \{ v \in L^2(\Omega): \text{the weak derivatives } \partial v / \partial x_j \text{ exists in } L^2(\Omega), \ 1 \leq j \leq d \}. \]

Then, we define

\[ H^1(\Omega; \mathbb{R}^d) = \{ v = (v_i): v_i \in H^1(\Omega), \ 1 \leq i \leq d \}. \]

For functions \( v \in H^1(\Omega; \mathbb{R}^d) \), we still write \( v \) for the trace of \( v \) on \( \Gamma \). For the displacement, we use the space \( V \) defined by

\[ V = \{ v \in H^1(\Omega; \mathbb{R}^d): v = 0 \text{ on } \Gamma_D \}. \]

From Korn’s inequality, i.e., \( \|\varepsilon(\cdot)\|_Q \geq C \|v\|_{H^1(\Omega; \mathbb{R}^d)} \), since \( \text{meas}(\Gamma_D) > 0 \) (see, e.g., [12, Lemma 6.2]), it follows that \( V \) is a Hilbert space with the canonical inner product

\[ (v,u)_V = \int_\Omega \varepsilon(v) : \varepsilon(u) dx, \]

where \( \varepsilon : H^1(\Omega; \mathbb{R}^d) \rightarrow Q \) is the deformation operator defined by

\[ \varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

We denote the associated norm to \( V \) by \( \|\cdot\|_V \). Moreover, if \( \sigma \) is a regular function, say \( \sigma \in C^1(\Omega; \mathbb{S}^d) \), the following Green’s formula holds

\[ \int_\Omega \nabla \cdot \sigma v dx = \int_\Gamma \sigma v \cdot n da - \int_\Omega \sigma : \varepsilon(v) dx \]

for all \( v \in H^1(\Omega; \mathbb{R}^d) \).

2.2. Time-dependent spaces. Let \( (V,H,V^*) \) be an evolution triple.

**Definition 2.1.** Let \( \tilde{X} \) be a Banach space, and \( T > 0 \). The space \( L^2(0,T; \tilde{X}) \) consists of all measurable functions \( v: [0,T] \rightarrow \tilde{X} \) such that

\[ \int_0^T \|v(t)\|^2_{\tilde{X}} dt < \infty. \]

With the usual modifications for \( L^\infty(0,T; \tilde{X}) \).
**Definition 2.2.** We denote the spaces satisfying Definition 2.1, for any $1 \leq p \leq \infty$, by $L^p(s, t; \tilde{X})$

for all $s \leq t \in [0, T]$. If $s = 0$, we denote the space by

$L^p_0(0, t; \tilde{X}) = L^p(t; \tilde{X})$

for all $t \in [0, T]$.

We also introduce the solution space (2.5)

$$\mathcal{W}_{T}^{1,2} = \{w \in L_{T}^2 V : \dot{w} \in L_{T}^2 V^*\},$$

equipped with the norm $\|w\|_{\mathcal{W}_{T}^{1,2}} = \|w\|_{L_{T}^2 V}^2 + \|\dot{w}\|_{L_{T}^2 V^*}^2$. The duality pairing between $L_{T}^2 V^*$ and $L_{T}^2 V$ is denoted by

$$\langle \tilde{v}, v \rangle_{L_{T}^2 V^* \times L_{T}^2 V} = \int_0^T \langle \tilde{v}(s), v(s) \rangle_{V^* \times V} ds \quad \text{for all } \tilde{v} \in L_{T}^2 V^*, v \in L_{T}^2 V.$$

The next proposition can be found in [5, Proposition 3.4.14] and will be providing estimates.

**Proposition 2.3.** Let $(V, H, V^*)$ be an evolution triple in space, and $0 < T < \infty$. Then, for any $v_1, v_2 \in \mathcal{W}_{T}^{1,2}$, and for all $0 \leq s \leq t \leq T$, the following integration by parts formula holds:

$$(v_1(t), v_2(t))_H - (v_1(s), v_2(s))_H = \int_s^t \left[ \langle \dot{v}_1(\tau), v_2(\tau) \rangle + \langle \dot{v}_2(\tau), v_1(\tau) \rangle \right] d\tau.$$

Lastly, we denote the space of continuous functions defined on $[0, T]$ with values in $\tilde{X}$ by

$C([0, T]; \tilde{X}) = \{h : [0, T] \to \tilde{X} : h \text{ is continuous}\},$

which we equip with the norm

$$\|h\|_{C([0, T]; \tilde{X})} = \max_{t \in [0, T]} \|h(t)\|_{\tilde{X}}.$$

For more on evolution spaces and other time-dependent spaces, which are also referred to as Bochner spaces, see, e.g., [22, Section 7.2], [5, Section 3], [26, Chapter V, Section 5] or [8, Section 5.9.2 and Appendix E.5].

Let $\mathbb{X} = \prod_{i=1}^k \tilde{X}_i$ be a Cartesian product space, for some $k \in \mathbb{N}$, where $(\tilde{X}_i, \|\cdot\|_{\tilde{X}_i})$ are normed spaces, $i = 1, \ldots, k$. Then, $\mathbb{X}$ is equipped with the norm

$$\|(v_1, \ldots, v_k)\|_{\mathbb{X}} = \sum_{i=1}^k \|v_i\|_{\tilde{X}_i} \quad \text{for all } v_i \in \tilde{X}_i, i = 1, \ldots, k.$$

Equivalently, we may equip $\mathbb{X}$ with the norm:

$$\|(v_1, \ldots, v_k)\|_{\mathbb{X}}^2 = \sum_{i=1}^k \|v_i\|_{\tilde{X}_i}^2 \quad \text{for all } v_i \in \tilde{X}_i, i = 1, \ldots, k.$$

We lastly introduce the following Sobolev space needed in the applications, i.e., in Section 6:

**Definition 2.4.** Let $V$ be a real Banach space, then the Sobolev space $W^{1,2}(0, T; V)$ consists of all functions $u \in L_{T}^2 V$ such that $\dot{u}$ exists in the weak sense and belongs to $L_{T}^2 V$. The space $W^{1,2}(0, T; V)$ is equipped with the norm

$$\|u\|_{W^{1,2}(0, T; V)}^2 = \|u\|_{L_{T}^2 V}^2 + \|\dot{u}\|_{L_{T}^2 V}^2.$$
2.3. Generalized gradients. In contact mechanics, we are often interested in contact conditions of the form $\zeta \in \partial h(u_\nu)$, where $\zeta$ represents an interface force, $u_\nu = u \cdot \nu$ the normal component of the displacement, and $\partial h(u_\nu)$ being the Clarke subdifferential of $h$ defined below.

Let $X$ be a reflexive Banach space. To say that a function $h : \tilde{X} \to \mathbb{R}$ is locally Lipschitz on $\tilde{X}$ means that $h(\tilde{x})$ is Lipschitz continuous around a neighborhood of $\tilde{x} \in \tilde{X}$.

**Definition 2.5.** Let $h : \tilde{X} \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of $h$ at $\tilde{x} \in \tilde{X}$ in the direction $v \in \tilde{X}$ denoted $h^\circ(\tilde{x}; v)$ is defined by

$$h^\circ(\tilde{x}; v) = \limsup_{\tilde{y} \to \tilde{x}, \tilde{\epsilon} \downarrow 0} \frac{h(\tilde{y} + \tilde{\epsilon}v) - h(\tilde{y})}{\tilde{\epsilon}}.$$

Moreover, the subdifferential in the sense of Clarke of $h$ at $\tilde{x}$, denoted $\partial h(\tilde{x})$, is a subset of $\tilde{X}^*$ on the form

$$\partial h(\tilde{x}) = \{ \zeta \in \tilde{X}^* : h^\circ(\tilde{x}; v) \geq \langle \zeta, v \rangle_{\tilde{X}^* \times \tilde{X}} \quad \text{for all } v \in \tilde{X} \}.$$

**Remark 2.6.** A classical example is $h(\tilde{x}) = |\tilde{x}|$, which is not differentiable at $\tilde{x} = 0$, but it has a subdifferential at $\tilde{x} = 0$ (see, e.g., [3, Example 2.1.3]). For examples related to friction, we refer the reader to, e.g., [16, Section 6.3], [10, p.185-187].

**Proposition 2.7.** Let $\tilde{X}$ be a Banach space, and $h : \tilde{X} \to \mathbb{R}$ be locally Lipschitz on $\tilde{X}$. Then $(\tilde{x}, v) \mapsto h^\circ(\tilde{x}; v)$ is upper semicontinuous.

**Proposition 2.8.** Let $\tilde{X}$ be a Banach space. If $h : \tilde{X} \to \mathbb{R} \cup \{ \infty \}$ is proper, convex and lower semicontinuous, then $h$ is locally Lipschitz on the interior of the domain of $h$.

Proposition 2.7 can be found in [6, Proposition 5.6.6], and Proposition 2.8 can be found in [6, Proposition 5.2.10].

**Definition 2.9.** Let $h : \tilde{X} \to \mathbb{R} \cup \{ \infty \}$ be a proper and convex function. The (generally multivalued) mapping $\partial h : \tilde{X} \to 2^{\tilde{X}^*}$, written

$$\partial h(\tilde{x}) = \{ x^* \in \tilde{X}^* : h(v) - h(\tilde{x}) \geq \langle x^*, v - \tilde{x} \rangle_{\tilde{X}^* \times \tilde{X}} \quad \text{for all } v \in \tilde{X} \},$$

is called the convex subdifferential of $h$ in $\tilde{x} \in \tilde{X}$.

Lastly, in order to show that $(w, \alpha)$ is a solution to Problem 1, we require the following result found in [16, Lemma 3.43].

**Lemma 2.10.** Let $\tilde{X}, \tilde{Y}$ be Banach spaces and $h : \tilde{X} \times \tilde{Y} \to \mathbb{R}$ be such that

1. $h(\cdot, \tilde{y})$ is continuous on $\tilde{X}$ for all $\tilde{y} \in \tilde{Y}$.
2. $h(\tilde{x}, \cdot)$ is locally Lipschitz on $\tilde{Y}$ for all $\tilde{x} \in \tilde{X}$.
3. There exists $c > 0$ such that for all $\zeta \in \partial h(\tilde{x}, \tilde{y})$ we have

$$\|\zeta\|_{\tilde{Y}^*} \leq c(1 + \|\tilde{x}\|_{\tilde{X}} + \|\tilde{y}\|_{\tilde{Y}}),$$

where $\partial h$ denotes the generalized gradient of $h(\tilde{x}, \cdot)$.

Then $h$ is continuous on $\tilde{X} \times \tilde{Y}$.

To read more on the generalized directional derivatives, subdifferential, and nonsmooth analysis see, e.g., [3, Chapter 2], [6, Chapter 5], and [11, Chapter 1-3].

3. Problem statement and main result

In this section we first introduce the problem and then present the main result.
3.1. **Problem statement.** Let \((V, H, V^*)\) be an evolution triple, and \(Y, U, X, Z\) real reflexive Banach spaces, with the other function spaces defined in Section 2.2. We only seek a solution of (1.1) in the sense of Definition 1.1. We are therefore interested in the following evolutionary differential variational-hemivariational inequality:

**Problem 1.** Find \(w \in W^{1,2}_T\) and \(\alpha \in C([0,T];Y)\) such that

\[
\begin{align*}
(3.1a) & \quad \alpha(t) = \alpha_0 + \int_0^t G(s, \alpha(s), Mw(s))ds, \\
(3.1b) & \quad \langle \dot{w}(t) + A(t, w(t)) - f(t) + \mathcal{R}w(t), v - w(t) \rangle + j^0(t, Nw(t); Nv - Nw(t)) \\
& \quad + \varphi(t, \mathcal{S}w(t), \alpha(t), Mw(t), Kv) - \varphi(t, \mathcal{S}w(t), \alpha(t), Mw(t), Kw(t)) \geq 0,
\end{align*}
\]

for all \(v \in V\), a.e. \(t \in (0, T)\), with

\[w(0) = w_0.\]

We require the following assumptions on the operators and data:

\(H(A): A: (0, T) \times V \to V^*\) is such that

(i) \(A(\cdot, v)\) is measurable on \((0, T)\) for all \(v \in V\).

(ii) \(A(t, \cdot)\) is demicontinuous on \(V\) for a.e. \(t \in (0, T)\), i.e., if \(v^n \to v\) strongly in \(V\), then \(Av^n \to Av\) weakly in \(V^*\) as \(n \to \infty\).

(iii) \(\|A(t, v)\|_{V^*} \leq a_0(t) + a_1\|v\|_V\) for all \(v \in V\), a.e. \(t \in (0, T)\) with \(a_0 \in L^2(0, T)\), \(a_0 \geq 0\), \(a_1 \geq 0\).

(iv) There is a \(m_A > 0\) such that \(\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle \geq m_A\|v_1 - v_2\|^2\) for all \(v_i \in V\), \(i = 1, 2\), a.e. \(t \in (0, T)\).

\(H(j): j: (0, T) \times X \to \mathbb{R}\) is such that

(i) \(j(\cdot, \tilde{v})\) is measurable on \((0, T)\) for all \(\tilde{v} \in X\).

(ii) \(j(\cdot, \cdot)\) is locally Lipschitz on \(X\) for a.e. \(t \in (0, T)\).

(iii) \(\|\partial j(t, \tilde{v})\|_{X^*} \leq c_{0j}(t) + c_{1j}\|\tilde{v}\|_X\) for all \(\tilde{v} \in X\), a.e. \(t \in (0, T)\) with \(c_{0j} \in L^2(0, T)\), \(c_{0j} \geq 0\), \(c_{1j} \geq 0\).

(iv) There is a \(\alpha_j\) such that \(j^0(t, \tilde{v}_1; \tilde{v}_2 - \tilde{v}_1) + j^0(t, \tilde{v}_2; \tilde{v}_1 - \tilde{v}_2) \leq \alpha_j\|\tilde{v}_1 - \tilde{v}_2\|^2\) for all \(\tilde{v}_i \in X\), \(i = 1, 2\), a.e. \(t \in (0, T)\), \(\alpha_j \geq 0\).

\(H(\varphi): \varphi: (0, T) \times Y \times Y \times U \times Z \to \mathbb{R}\) is such that

(i) \(\varphi(\cdot, z, y, \tilde{w}, \tilde{v})\) is measurable on \((0, T)\) for all \(z, y \in Y, \tilde{w} \in U, \tilde{v} \in Z\).

(ii) \(\varphi(\cdot, \cdot, \cdot, \cdot, \tilde{v})\) is continuous on \(Y \times Y \times U\) for all \(\tilde{v} \in Z\), a.e. \(t \in (0, T)\).

(iii) \(\varphi(\cdot, z, y, \tilde{w}, \cdot)\) is convex and lower semicontinuous on \(Z\) for all \(z, y \in Y, \tilde{w} \in U, \text{ a.e. } t \in (0, T)\), with \(c_{0\varphi} \in L^2(0, T)\), and \(c_{0\varphi}, c_{1\varphi}, c_{2\varphi}, c_{3\varphi}, c_{4\varphi} \geq 0\).

(v) There is \(\beta_{1\varphi}, \beta_{2\varphi}, \beta_{3\varphi} \geq 0\) such that

\[
\begin{align*}
\varphi(t, z_1, y_1, \tilde{w}_1, \tilde{v}_1) - \varphi(t, z_1, y_1, \tilde{w}_2, \tilde{v}_1) + \varphi(t, z_2, y_2, \tilde{w}_2, \tilde{v}_1) & - \varphi(t, z_2, y_2, \tilde{w}_2, \tilde{v}_2) \\
& \leq \beta_{1\varphi}\|z_1 - z_2\|_Y\|\tilde{v}_1 - \tilde{v}_2\|_Z + \beta_{2\varphi}\|y_1 - y_2\|_Y\|\tilde{v}_1 - \tilde{v}_2\|_Z \\
& + \beta_{3\varphi}\|\tilde{w}_1 - \tilde{w}_2\|_U\|\tilde{v}_1 - \tilde{v}_2\|_Z
\end{align*}
\]

for all \(z_i, y_i \in Y, \tilde{w}_i \in U, \tilde{v}_i \in Z, i = 1, 2\), a.e. \(t \in (0, T)\).

\(H(G): G: (0, T) \times Y \times U \to Y\) is such that

(i) \(G(\cdot, \cdot, \cdot)\) is measurable on \((0, T)\) for all \(\alpha \in Y, \tilde{v} \in U\).
Remark 3.2. Assume the preparation in Section 4. Continuously on the initial data. The proof of Theorem 3.2 is deferred to Section 5 after.

In this section, we state the main result, i.e., Theorem 3.2; the first part holds.

Lastly, we require the following smallness-condition:

\[ m_A > \alpha_j \|N\|^2 + \beta_3 \|K\| \|M\|. \]

\[ \|G(t, \alpha_1(t), \tilde{v}_1(t)) - G(t, \alpha_2(t), \tilde{v}_2(t))\|_Y \leq L_G \left( \|\alpha_1(t) - \alpha_2(t)\|_Y + \|\tilde{v}_1(t) - \tilde{v}_2(t)\|_U \right) \]

for all \( \alpha_i \in Y \), \( \tilde{v}_i \in U \), \( i = 1, 2 \), a.e. \( t \in (0, T) \).

(iii) \( G(\cdot, 0, 0) \in L^\infty_Y \).

\( H(R) : R : L^2_T V \to L^2_T V^* \) is such that

(i) \( R \) is a history-dependent operator, i.e.,

\[ \|Rv_1(t) - Rv_2(t)\|_{V^*} \leq c_R \int_0^t \|v_1(s) - v_2(s)\|_V ds \]

for all \( v_i \in L^2_T V \), \( i = 1, 2 \), a.e. \( t \in (0, T) \) with \( c_R > 0 \).

(ii) \( R0 \) belongs to a bounded subset of \( L^2_T V^* \).

\( H(S) : S : L^2_T V \to L^2_T Y \) is such that

(i) \( S \) is a history-dependent operator, i.e.,

\[ \|Sv_1(t) - Sv_2(t)\|_Y \leq c_S \int_0^t \|v_1(s) - v_2(s)\|_V ds \]

for all \( v_i \in L^2_T V \), \( i = 1, 2 \), a.e. \( t \in (0, T) \) with \( c_S > 0 \).

(ii) \( S0 \) belongs to a bounded subset of \( L^2_T Y \).

\( H(MNK) : M \in \mathcal{L}(V, U), \ N \in \mathcal{L}(V, X), \ K \in \mathcal{L}(V, Z) \).

We also assume the following regularity on the source term and initial data:

\[ f \in L^2_T V^*, \quad w_0 \in V, \quad \alpha_0 \in Y. \]

Lastly, we require the following smallness-condition:

\[ m_A > \alpha_j \|N\|^2 + \beta_3 \|K\| \|M\|. \]

Remark 3.1. Similar assumptions can be found in, e.g., [14, 15, 21, 23, 25].

We make a brief remark on the assumptions in Appendix A.

3.2. Main result. In this section, we state the main result, i.e., Theorem 3.2; the first part is an existence and uniqueness result, and the latter provides that the flow map depends continuously on the initial data. The proof of Theorem 3.2 is deferred to Section 5 after the preparation in Section 4.

Theorem 3.2. Assume \( H(A), H(j), H(\varphi), H(R), H(S), H(G), H(MNK) \), and (3.2)-(3.3) holds.

(1) Then \((w, \alpha)\) is a global solution such that \( w \in W^{1,2}_T \subset C([0, T]; H) \) and \( \alpha \in C([0, T]; Y) \) solves Problem 1, i.e., (1.1) has a unique solution in the sense of Definition 1.1. The solution is global, with the meaning that \((w, \alpha)\) is a solution to Problem 1 for any \( T > 0 \) finite. In addition, for each pair of initial data \((w_{01}, \alpha_{01}), (w_{02}, \alpha_{02}) \in V \times Y \), there exists a constant \( c > 0 \) such that

\[ \|w_1 - w_2\|_{L^2_T V} + \max_{t \in [0, T]} \|\alpha_1(t) - \alpha_2(t)\|_Y \]

\[ \leq c(\|w_{01} - w_{02}\|_V + \|\alpha_{01} - \alpha_{02}\|_Y + \|f_1 - f_2\|_{L^2_T V^*}). \]
where \((w_i, \alpha_i) \in W_T^{1,2} \times C([0, T]; Y)\) is a solution to Problem 1 corresponding to \((w_0i, \alpha_0i)\), for \(i = 1, 2\).

(2) The flow map \(F : L^2_T V^* \times V \times Y \rightarrow L^2_T V \times C([0, T]; Y)\) such that \((f, w_0, \alpha_0) \mapsto (w, \alpha)\) is continuous.

Remark 3.3. The theorem can easily be extended to include more than two history-dependent operators without needing any additional assumptions on the operator other than the same as we put on \(R\) and \(S\), i.e., \(H(R)\), and \(H(S)\), respectively.

3.3. Strategy of the proof of Theorem 3.2. The proof of the theorem is in six steps. In the first step, we introduce a linearization of Problem 1. Specifically, we linearize (3.1b), call this Problem 2, leaving (3.1a) intact. We recast the linearized problem as a differential inclusion (introduced in the Section 4) and use existing results to prove that Problem 2 has a unique solution (Step 1-4). Next, we define an iterative scheme for Problem 1 using Problem 2. This iterative scheme solves successively Problem 2 followed by (3.1a) thereby decoupling (3.1a) and Problem 2 at each step. Then, we study the difference of two successive iterates and show that the iterations are Cauchy sequences. We then pass to the limit to show that the iterative scheme converges to Problem 1 (Step 5). Lastly, we show that the flow map is continuously dependent on the initial data (Step 6).

4. Preliminary result

Before moving on to the proof of Theorem 3.2, we present an existence and uniqueness result for a differential inclusion problem see, e.g., [1]. The forthcoming result will be used to prove existence of a solution of the linearization of (3.1b) in Problem 1. To utilize this result, we need to introduce a differential inclusion which we can relate to the linearization of (3.1b). This will be clearer in Step 1-2 in the proof of Theorem 3.2.

Beginning with the following assumptions for the preliminary existence and uniqueness result for the differential inclusion problem:

\[
H(\psi): \psi : (0, T) \times V \rightarrow \mathbb{R} \text{ is such that}
\]

(i) \(\psi(\cdot, v)\) is measurable on \((0, T)\) for all \(v \in V\).

(ii) \(\psi(t, \cdot)\) is locally Lipschitz on \(V\) for a.e. \(t \in (0, T)\).

(iii) \(\|\partial \psi(t, v)\|_V \leq c_0(t) + c_1\|v\|_V\) for all \(v \in V\), a.e. \(t \in (0, T)\) with \(c_0 \in L^2(0, T)\), \(c_0 \geq 0, c_1 \geq 0\).

(iv) There is a \(m_\psi \geq 0\) such that \(\langle z_i - z_2, v_1 - v_2 \rangle \geq -m_\psi\|v_1 - v_2\|_V^2\) for all \(z_i \in \partial \psi(t, v_i), z_i \in V^*, v_i \in V, i = 1, 2\), a.e. \(t \in (0, T)\).

We assume that the source term \(f\) and the initial data \(w_0\) satisfy (3.2a). Additionally, we assume that the following smallness-condition holds:

\[
(4.1) \quad m_A > m_\psi.
\]

The next theorem was proved in [15, Theorem 3].

**Theorem 4.1.** Assume that \(H(A), H(\psi), (3.2a), \text{ and (4.1) hold. Then}

\[
(4.2a) \quad \dot{w}(t) + A(t, w(t)) + \partial \psi(t, w(t)) \ni f(t), \quad \text{a.e. } t \in (0, T)
\]

\[
(4.2b) \quad w(0) = w_0
\]

has a unique solution \(w \in W_T^{1,2}\).

This result will be used in Step 2 of the proof of Theorem 3.2 in Section 5. To better understand how to work with (4.2a)-(4.2b), we introduce the following definition:
Definition 4.2. A function \( w \in \mathcal{W}^{1,2}_T \) is called a solution to (4.2a)-(4.2b) if there exists \( w^* \in L^2_T V^* \) such that
\[
\dot{w}(t) + A(t, w(t)) + w^*(t) = f(t) \quad \text{for a.e. } t \in (0, T),
\]
\[
w^*(t) \in \partial \psi(t, w(t)) \quad \text{for a.e. } t \in (0, T),
\]
\[
w(0) = w_0.
\]

5. Proof of Theorem 3.2

With the preparation in Section 2.4, we proceed to the proof of Theorem 3.2. For convenience, we split the proof into 6 steps. For readability, we have moved some of the proofs to the appendix. We recall that the function spaces are defined in Section 2.2.

Step 1 (Linearization of the evolutionary hemivariational-variational inequality (3.1b)). Let \((\alpha, \xi, \eta, g) \in C([0, T]; Y) \times L^2_T V^* \times L^2_T Y \times L^2_T V\) be given, then we define a linearization of (3.1b) in Problem 1.

Problem 2. Find \( w_{\alpha \xi \eta g} \in \mathcal{W}^{1,2}_T \) corresponding to \((\alpha, \xi, \eta, g) \in C([0, T]; Y) \times L^2_T V^* \times L^2_T Y \times L^2_T V\) such that
\[
\langle \dot{w}_{\alpha \xi \eta g}(t) + A(t, w_{\alpha \xi \eta g}(t)) - f(t) + \xi(t), v - w_{\alpha \xi \eta g}(t) \rangle + j^0(t, N w_{\alpha \xi \eta g}(t); N v - N w_{\alpha \xi \eta g}(t))
\]
\[
+ \varphi(t, \eta(t), \alpha(t), M g(t), K v) - \varphi(t, \eta(t), \alpha(t), M g(t), K w_{\alpha \xi \eta g}(t)) \geq 0
\]
for all \( v \in V \), a.e. \( t \in (0, T) \) with
\[
w_{\alpha \xi \eta g}(0) = w_0.
\]

Remark 5.1. A glance at Problem 2 and (3.1b) lets us see that the linearized problem keeps \( \xi = R w \), and the three first arguments of \( \varphi \), i.e., \( \alpha \), still denoted by \( \alpha \), \( \eta = S w \), and \( g = w \) already known in contrast to (3.1b). We find it worth mentioning that we use the subscripts on \( w \) to emphasize that a solution \( w_{\alpha \xi \eta g} \) to Problem 2 corresponds to \((\alpha, \xi, \eta, g) \in C([0, T]; Y) \times L^2_T V^* \times L^2_T Y \times L^2_T V\). This also helps to distinguish between a solution to Problem 1 and a solution to Problem 2.

Step 2 (Existence of a solution to Problem 2). We wish to utilize Theorem 4.1 in order to prove that Problem 2 has a solution. We therefore define the functional \( \psi_{\alpha \xi \eta g} : (0, T) \times V \to \mathbb{R} \) by
\[
\psi_{\alpha \xi \eta g}(t, v) = \langle \xi(t), v \rangle + \varphi(t, \eta(t), \alpha(t), M g(t), K v) + j(t, N v)
\]
for all \( v \in V \), a.e. \( t \in (0, T) \). Now, existence of a unique solution \( w_{\alpha \xi \eta g} \in \mathcal{W}^{1,2}_T \) of (4.2a)-(4.2b) with \( \psi_{\alpha \xi \eta g} \) defined by (5.1) follows from the same approach as in Step 1 of the proof in [15, Theorem 5]. We need to investigate if \( H(\psi) \) and the smallness-condition (4.1) holds, as there are some modifications needed in comparison with [15, Theorem 5]. We only comment on the changes and leave the reader to visit [15, Theorem 5] to read the verification in more detail. We find that \( H(\psi) \) holds with the following constants:
\[
c_0(t) = ||\xi(t)||_{V^*} + c_{1 \varphi}(t)||K|| + c_{1 \varphi}(t)||K||||\eta(t)||_Y + c_{2 \varphi}(t)||K||||\alpha(t)||_Y + c_{3 \varphi}(t)||M||\||g(t)||_V + c_{0 j}(t)||N|| \geq 0 \quad \text{giving us} \quad c_0 \in L^2(0, T), \quad c_1 = c_{1 j}(t)||N||^2 + c_{4 \varphi}(t)||K||^2 \geq 0, \quad \text{and} \quad m_\psi = c_{j \psi}(t)||N||^2.
\]
The last part together with the smallness-condition (3.3) leads to (4.1). Thus, it follows from Theorem 4.1 that there exists a solution \( w_{\alpha \xi \eta g} \in \mathcal{W}^{1,2}_T \) of (4.2a)-(4.2b) with \( \psi_{\alpha \xi \eta g} \) defined by (5.1). Lastly, showing that the existence of a solution of (4.2a)-(4.2b) implies the existence of a solution to Problem 2 follows from Definition 2.5, 2.9, and basic results of the generalized gradients, see, e.g., [10, Theorem 3.7, Proposition 3.10-3.12], where they have summarized these properties, and [25, Lemma 7]. A more detailed approach of the last part can be found in, e.g., [10, Section 6] or [25, Step 1 in Theorem 98].
Remark 5.2. The result, [6, Proposition 5.2.10], which is utilized in Step 1 of the proof in [15, Theorem 5] is Proposition 2.8.

**Step 3** (Uniqueness of a solution to Problem 2). Uniqueness follows directly from Step 2 of Theorem 98 in [25, p.192] by setting \( a_j = \alpha_j \| N \|^2 \), and using the fact that \( m_A > \alpha_j \| N \|^2 \), which follows from the smallness-condition (3.3).

**Step 4** (Estimate on the solution to Problem 2, that is, \( w_{\alpha \xi \eta g} \in W_T^{1,2} \subset C([0,T];H) \)). We now find an estimate on the solution to Problem 2 which will come in handy later:

**Proposition 5.3.** Under the assumptions of Theorem 3.2, for given \((\alpha, \xi, \eta, g) \in C([0,T];Y) \times L_T^2 V^* \times L_T^2 Y \times L_T^2 V\), let \( w_{\alpha \xi \eta g} \) be a solution to Problem 2. Then, there exists a constant \( c > 0 \) independent of \( w_{\alpha \xi \eta g} \) such that

\[
\max_{t \in [0,T]} \| w_{\alpha \xi \eta g}(t) \|_H^2 + \| w_{\alpha \xi \eta g} \|_{W_T^{1,2}}^2 \leq c(1 + \| \xi \|_{L_T^2 V^*}^2 + \| \eta \|_{L_T^2 Y}^2 + \| g \|_{L_T^2 V}^2 + \max_{t \in [0,T]} \| \alpha(t) \|_Y^2 + \| f \|_{L_T^2 V^*}^2).
\]

The proof of Proposition 5.3 is postponed to Appendix B.

**Step 5** (Scheme for the approximated solution to Problem 1). For \( n \in \mathbb{N} \), let \( \alpha^{n-1} \in C([0,T];Y) \), and \( w^{n-1} \in W_T^{1,2} \) be known. We construct the approximated solutions \( \{(w^n, \alpha^n)\}_{n \geq 0} \subset W_T^{1,2} \times C([0,T];Y) \) to Problem 1, where \((w^n, \alpha^n)\) is a solution of the scheme:

\[
\begin{align*}
(5.3a) \quad & \langle \dot{w}^n(t) + A(t, w^n(t)) - f(t) + R w^{n-1}(t), v - w^n(t) \rangle \\
& + j^\alpha(t, N w^n(t); Nv - N w^n(t)) + \varphi(t, Sw^{n-1}(t), \alpha^{n-1}(t), M w^{n-1}(t), Kv) \\
& - \varphi(t, Sw^n(t), \alpha^{n-1}(t), M w^n(t), K w^n(t)) \geq 0
\end{align*}
\]

for all \( v \in V \), a.e. \( t \in (0, T) \), and

\[
(5.3b) \quad w^n(0) = w_0.
\]

\[
(5.3c) \quad \alpha^n(t) = \alpha_0 + \int_0^t \mathcal{G}(s, \alpha^n(s), M w^n(s))ds \quad \text{for a.e. } t \in (0, T),
\]

with \( w^0 = w_0 \in V \) and \( \alpha^0 = \alpha_0 \in Y \).

**Step 5.1** (Existence and uniqueness of \((w^n, \alpha^n) \in W_T^{1,2} \times C([0,T];Y) \) to (5.3a)-(5.3c) for all \( n \in \mathbb{N} \)). We establish existence and uniqueness by induction on \( n \). For \( n = 1 \), we have by Step 2 that \( w^1 \in W_T^{1,2} \) is a solution of (5.3a)-(5.3c). We also have that \( w^1 \) is uniformly bounded in \( W_T^{1,2} \) and \( C([0,T];H) \) from Proposition 5.3 with \( w_{\alpha \xi \eta g} = w^1 \), \( \alpha = \alpha^0 \), \( \xi = R w^0 \), \( \eta = S w^0 \), and \( g = w^0 \). In fact, applying the triangle inequality, Young’s inequality, integrating over the time interval \((0, t')\), and lastly applying the Cauchy-Schwarz inequality to hypothesis \( H(R) \) and \( H(S) \), respectively, yield:

\[
(5.4) \quad \int_0^{t'} \| R w^0(t) \|_{L_T^2 V^*}^2 dt \leq 2 \int_0^{t'} \| R w^0(t) - R 0(t) \|_{L_T^2 V^*}^2 dt + 2 \int_0^{t'} \| R 0(t) \|_{L_T^2 V^*}^2 dt
\]

\[
\leq 2T^2 c^2_R \int_0^{t'} \| w^0(t) \|_{L_T^2 V^*}^2 dt + 2 \| R 0 \|_{L_T^2 V^*}^2.
\]

\[
(5.5) \quad \int_0^{t'} \| S w^0(t) \|_{L_T^2 V}^2 dt \leq 2T^2 c^2_S \int_0^{t'} \| w^0(t) \|_{L_T^2 V^*}^2 dt + 2 \| S 0 \|_{L_T^2 V}^2.
\]
for all \( t' \in [0, T] \). Then combining (5.4)-(5.5) and the initial guesses, \( w^0 = w_0 \) and \( \alpha^0 = \alpha_0 \), with the estimate (5.2) provide the bound

\[
\max_{t \in [0,T]} \left\| w^1(t) \right\|^2_H + \left\| w^1 \right\|^2_{W^{1,2}} \leq c(1 + \left\| w_0 \right\|^2_V + \left\| \alpha_0 \right\|^2_Y + \left\| f \right\|^2_{L^2_{\infty}V^*}),
\]

where \( c = c(T, \ldots) \). We next define the compete metric space

\[
X_T(a) = \{ h \in C([0, T]; Y) : \left\| h \right\|_{L^\infty_T Y} \leq a, \text{ with } a \in \mathbb{R}_+ \},
\]

and the operator \( \Lambda : X_T(a) \to X_T(a) \) by

\[
\Lambda \alpha^1(t) = \alpha_0 + \int_0^t G(s, \alpha^1(s), M w^1(s))ds.
\]

We verify \( \alpha^1 \) is indeed a solution of (5.3c) for \( n = 1 \) in the next lemma.

**Lemma 5.4.** Let \( w^0 = w_0 \in V \) and \( \alpha^0 = \alpha_0 \in Y \). Under the assumptions of Theorem 3.2, the operator \( \Lambda : X_T(a) \to X_T(a) \), defined by (5.7), has a unique fixed-point, i.e., there exists a constant \( 0 \leq \ell < 1 \) such that

\[
\left\| \Lambda \alpha^1 - \Lambda \alpha^2 \right\|_{X_T(a)} \leq \ell \left\| \alpha^1 - \alpha^2 \right\|_{X_T(a)}.
\]

The proof of Lemma 5.4 is proposed to Appendix C as it follows from the standard ODE arguments combined with the estimate (5.6), \( H(G) \), and \( H(MNK) \). The induction step follows in the exactly same procedure as for \( n = 1 \). Accordingly, \( (w^n, \alpha^n) \in W^{1,2}_T \times C([0, T]; Y) \) is the approximated solution of (5.3a)-(5.3c). Further, we obtain the following uniform bound on the solution:

\[
\max_{t \in [0,T]} \left\| \alpha^n(t) \right\|_Y + \max_{t \in [0,T]} \left\| w^n(t) \right\|_H + \left\| w^n \right\|_{W^{1,2}} \leq c(1 + \left\| w_0 \right\|_V + \left\| \alpha_0 \right\|_Y + \left\| f \right\|_{L^2_{\infty}V^*})
\]

for all \( n \in \mathbb{N} \).

**Step 5.2 (Convergence of the approximated solution).** First, we show that \( \{(w^n, \alpha^n)\}_{n \geq 0} \subset W^{1,2}_T \times C([0, T]; Y) \) are Cauchy sequences in their respective spaces. We summarize this in the proposition below.

**Proposition 5.5.** Let \( w_0 = w_0 \in V \) and \( \alpha_0 = \alpha_0 \in Y \). Under the hypothesis of Theorem 3.2, let \( \{(w^n, \alpha^n)\}_{n \geq 0} \subset W^{1,2}_T \times C([0, T]; Y) \) be the solution of (5.3a)-(5.3c). Then, \( \{(w^n, \alpha^n)\}_{n \geq 0} \) are Cauchy sequences in their respective spaces. In addition, \( \{Sw^n\}_{n \geq 0} \) and \( \{Rw^n\}_{n \geq 0} \) are Cauchy sequences in \( L^2_T Y \) and \( L^2_T V^* \), respectively.

**Proof.** We start by subtracting (5.3c) for the two iterations at the levels \( n - 1 \) and \( n - 2 \), respectively. Utilizing Minkowski’s inequality, and \( H(G)(ii) \) yields

\[
\left\| \alpha^{n-1}(t) - \alpha^{n-2}(t) \right\|_Y \leq L_G \int_0^t \left\| \alpha^{n-1}(s) - \alpha^{n-2}(s) \right\|_Y ds + L_G \left\| M \right\| \int_0^t \left\| w^{n-1}(s) - w^{n-2}(s) \right\|_Y ds
\]

for a.e. \( t \in (0, T) \). Now, from a standard Grönwall argument (see, e.g., [24, Lemma 3.2]) combined with Young’s inequality, and the Cauchy-Schwarz inequality reads:

\[
\left\| \alpha^{n-1}(t) - \alpha^{n-2}(t) \right\|^2_Y \leq 2L^2_G \left\| M \right\|^2 T \int_0^t \left\| w^{n-1}(s) - w^{n-2}(s) \right\|^2_Y dsdt + 2L^4_G \left\| M \right\|^2 T^2 \int_0^t \int_0^s e^{2L_G(t-s)} \left\| w^{n-1}(r) - w^{n-2}(r) \right\|^2_Y drds.
\]
for a.e. \( t \in (0, T) \). Next, we add together the inequality (5.3a) for two iterations at the levels \( n \) and \( n-1 \). For the iteration levels \( n \) and \( n-1 \), we substitute the test functions with \( v = w^n \) and \( v = w^{n-1} \), respectively:

\[
\langle \dot{w}^n(t) - \dot{w}^{n-1}(t), w^n(t) - w^{n-1}(t) \rangle + \langle A(t, w^n(t)) - A(t, w^{n-1}(t)), w^n(t) - w^{n-1}(t) \rangle \\
\leq \langle w^n(t), -1 \rangle - \langle w^{n-1}(t) \rangle + j^n(t, N w^n(t); N w^{n-1}(t) - N w^n(t)) \\
+ \varphi(t, S w^n(t), M w^{n-1}(t), K w^n(t)) - \varphi(t, S w^{n-2}(t), M w^{n-1}(t), K w^n(t))
\]

for a.e. \( t \in (0, T) \), with \( w^n(0) = w^{n-1}(0) = w_0 \). By hypothesis \( H(\varphi)(v), H(j)(iv), H(MNK) \), and the Cauchy-Schwarz inequality, we find:

\[
\langle \dot{w}^n(t) - \dot{w}^{n-1}(t), w^n(t) - w^{n-1}(t) \rangle + \langle A(t, w^n(t)) - A(t, w^{n-1}(t)), w^n(t) - w^{n-1}(t) \rangle \\
\leq \beta_1 \|K\| \|S w^n(t) - S w^{n-2}(t)\|_Y \|w^n(t) - w^{n-1}(t)\|_V \\
+ \beta_2 \|K\| \|\alpha^n(t) - \alpha^{n-2} \|_Y w^n(t) - w^{n-1}(t) \|_V \\
+ \beta_3 \|K\| \|M\| \|w^n(t) - w^{n-2}(t)\|_V \|w^n(t) - w^{n-1}(t)\|_V
\]

for a.e. \( t \in (0, T) \). Now, we integrate over the time interval \((0, t')\), use the integration by parts formula in Proposition 2.3 with \( \psi_1 = \psi_2 = w^n - w^{n-1} \), hypothesis \( H(A)(iv) \), the Cauchy-Schwarz inequality, and the smallness-condition (3.3), i.e., \( m_A > \alpha_j \|N\|^2 \). This reads:

\[
\frac{1}{2} \|w^n(t') - w^{n-1}(t')\|^2_H - \frac{1}{2} \|w^n(0) - w^{n-1}(0)\|^2_H \\
+ (m_A - \alpha_j \|N\|^2) \int_0^{t'} \|w^n(t) - w^{n-1}(t)\|^2_V dt \\
\leq \int_0^{t'} \|w^n(t) - w^{n-1}(t)\|^2_V dt \\
+ \beta_1 \|K\| \int_0^{t'} \|S w^n(t) - S w^{n-2}(t)\|_Y \|w^n(t) - w^{n-1}(t)\|_V dt \\
+ \beta_2 \|K\| \int_0^{t'} \|\alpha^n(t) - \alpha^{n-2} \|_Y w^n(t) - w^{n-1}(t) \|_V dt \\
+ \beta_3 \|K\| \|M\| \int_0^{t'} \|w^n(t) - w^{n-2}(t)\|_V \|w^n(t) - w^{n-1}(t)\|_V dt
\]

for a.e. \( t' \in (0, T) \). The Cauchy-Schwarz inequality applied to \( H(R), H(S) \), respectively, gives us the following two estimates:

\[
\int_0^{t'} \|R w^n(t) - R w^{n-2}(t)\|^2_V dt \leq c_9^2 T \int_0^{t'} \|w^n(t) - w^{n-2}(t)\|^2_V dt, \\
\int_0^{t'} \|S w^n(t) - S w^{n-2}(t)\|^2_V dt \leq c_9^2 T \int_0^{t'} \|w^n(t) - w^{n-2}(t)\|^2_V dt
\]
for all \( t' \in [0, T] \). Next, using that \( w^n(0) = w^{n-1}(0) \), the Cauchy-Schwarz inequality, and (5.11)-(5.12), yields:

\[
\bar{I} := (m_A - \alpha_j\|N\|^2) \left[ \int_0^{t'} \| w^n(t) - w^{n-1}(t) \|_Y^2 dt \right]^{1/2} \\
\leq \left[ c_R^2 T \int_0^{t'} \int_0^t \| w^{n-1}(s) - w^{n-2}(s) \|_Y^2 ds dt \right]^{1/2} \\
+ \beta_{1c} \| K \| \left[ c_S^2 L \int_0^{t'} \int_0^t \| w^{n-1}(s) - w^{n-2}(s) \|_Y^2 ds dt \right]^{1/2} \\
+ \beta_{2c} \| K \| \left[ \int_0^{t'} \alpha_{n-1}(t) - \alpha_{n-2}(t) \|_Y^2 dt \right]^{1/2} \\
+ \beta_{3c} \| K \| \| M \| \left[ \int_0^{t'} \| w^{n-1}(t) - w^{n-2}(t) \|_Y^2 dt \right]^{1/2} \\
=: I + II + III + IV
\]

for all \( t' \in [0, T] \). From Young’s inequality, for \( \epsilon_i > 0 \) small enough, \( i = 1, \ldots, 6 \), we obtain

\[
(\bar{I})^2 \leq (1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_4} + \epsilon_6)(I)^2 + (1 + \frac{1}{\epsilon_2} + \epsilon_4 + \frac{1}{\epsilon_5})(II)^2 \\
+ (1 + \frac{1}{\epsilon_3} + \epsilon_5 + \frac{1}{\epsilon_6})(III)^2 + (1 + \epsilon_1 + \epsilon_2 + \epsilon_3)(IV)^2.
\]

We insert estimate (5.10) and apply Young’s inequality to \((III)^2\). Thus, for \( L_1, L_2, L_3 > 0 \), we have

\[
\int_0^{t'} \| w^n(t) - w^{n-1}(t) \|_Y^2 dt \leq TL_1 \int_0^{t'} \int_0^t \| w^{n-1}(s) - w^{n-2}(s) \|_Y^2 ds dt \\
+ TL_2 \int_0^{t'} \int_0^t e^{2L\varphi(t'-t)} \| w^{n-1}(s) - w^{n-2}(s) \|_Y^2 ds dt \\
+ L_3 \int_0^{t'} \| w^{n-1}(t) - w^{n-2}(t) \|_Y^2 dt
\]

for all \( t' \in [0, T] \), where

\[
L_1 = \frac{c_R^2(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_4} + \epsilon_6) + \beta_{1c}^2 \| K \|^2 c_S^2(1 + \frac{1}{\epsilon_2} + \epsilon_4 + \frac{1}{\epsilon_5})}{(m_A - \alpha_j\|N\|^2)^2} \\
+ \frac{4\beta_{2c}^2 \| K \|^2 L_G^2 \| M \|^2 (1 + \frac{1}{\epsilon_3} + \epsilon_5 + \frac{1}{\epsilon_6})}{(m_A - \alpha_j\|N\|^2)^2},
\]

\[
L_2 = \frac{4(1 + \frac{1}{\epsilon_3} + \epsilon_5 + \frac{1}{\epsilon_6})\beta_{2c}^2 \| K \|^2 L_G^2 \| M \|^2}{(m_A - \alpha_j\|N\|^2)^2},
\]

\[
L_3 = \frac{(1 + \epsilon_1 + \epsilon_2 + \epsilon_3)\beta_{3c}^2 \| K \|^2 \| M \|^2}{(m_A - \alpha_j\|N\|^2)^2}.
\]

We observe from the smallness-condition (3.3) such that \( L_3 < 1 \). If we iterate over the above inequality, for \( n \geq N_1, N_2, N_3 \in \mathbb{N} \), we find the following bound, up to a constant
0 < c ≤ n. By the triangle inequality and the initial guess \( w^0 = w_0 \), we obtain the following:

\[
\int_0^{t'} \| w^n(t) - w^{n-1}(t) \|^2_V dt \\
\leq c(1 + e^{2LT_0}) \sum_{N_1+N_2+N_3=n} (TL_1)^{N_1}(T^3L_2)^{N_2}L_3^{N_3} \\
\times \int_0^{t'} \int_0^t \int_0^s \int_0^{t(N_1+N_2)} \cdots \int_0^{t_4} \| w^1(t_2) - w^0(t_2) \|^2_V dt_2 dt_3 \cdots dt_{(N_1+N_2)} dsdt \\
\leq c(1 + e^{2LT_0}) \sum_{N_1+N_2+N_3=n} (TL_1)^{N_1}(T^3L_2)^{N_2}L_3^{N_3} \\
\times \int_0^{t'} \int_0^t \int_0^s \int_0^{t(N_1+N_2)} \cdots \int_0^{t_4} dt_3 \cdots dt_{(N_1+N_2)} dsdt \\
\leq c \left( \| w^1 \|_{L^2_V} + t' \| w_0 \|_V \right)^2 (1 + e^{2LT_0}) \sum_{N_1+N_2+N_3=n} (TL_1)^{N_1}(T^3L_2)^{N_2}L_3^{N_3} \\
\times \int_0^{t'} \int_0^t \int_0^s \int_0^{t(N_1+N_2)} \cdots \int_0^{t_4} dt_3 \cdots dt_{(N_1+N_2)} dsdt
\]

for all \( t' \in [0, T] \). We observe that

\[
\int_0^{t'} \int_0^t \int_0^s \int_0^{t(N_1+N_2)} \cdots \int_0^{t_4} dt_3 \cdots dt_{(N_1+N_2)} dsdt = \frac{t'(N_1+N_2)}{(N_1+N_2)!} \leq \frac{T(N_1+N_2)}{(N_1+N_2)!}.
\]

In addition, employing the estimate \(5.6\), we deduce

\[
(5.13) \quad \int_0^{t'} \| w^n(t) - w^{n-1}(t) \|^2_V dt \leq \tilde{c} \sum_{N_1+N_2+N_3=n} \frac{T^{2(N_1+2N_2)}L_1^{N_1}L_2^{N_2}L_3^{N_3}}{(N_1+N_2)!}
\]

for some constant \( \tilde{c} > 0 \). To ensure that the right-hand side of \(5.13\) goes to zero when \( n \to \infty \), we now consider the different cases for \( N_1+N_2+N_3 = n \). We start with \( N_1 = N_2 = 0 \), and \( N_3 = n \), as this is immediate. We next take a closer look at when \( N_3 = 0 \), and \( N_i = n, N_j = 0 \), for \( i, j = 1, 2, i \neq j \). Lastly, we consider the cases in-between.

**Case 1.** Let \( N_1 = N_2 = 0 \) and \( N_3 = n \) in \(5.13\). Then:

\[
\int_0^{t'} \| w^n(t) - w^{n-1}(t) \|^2_V dt \leq \tilde{c}L_3^n.
\]

From the smallness-condition \(3.3\), we have that \( L_3 < 1 \), thus

\[
\lim_{n \to \infty} L_3^n = 0,
\]

as desired.

**Case 2.** Let \( N_3 = 0 \), and \( N_i = n, N_j = 0 \) in \(5.13\), with \( i, j = 1, 2, i \neq j \). This implies:

\[
\int_0^{t'} \| w^n(t) - w^{n-1}(t) \|^2_V dt \leq \tilde{c}\frac{T^{2n}L_1^n}{n!}.
\]

Now, we observe that since \( T < \infty \) and

\[
n! \sim \sqrt{2\pi n(n/e)^n},
\]

we have

\[
\lim_{n \to \infty} \frac{\tilde{c}T^{2n}L_1^n}{n!} = 0.
\]
Moreover, a Cauchy sequence in \((5.15a)\) below does not hold thereby obtaining a contradiction with \((5.14a)-(5.14b)\) that

\[
\text{Case 3. Lastly, we investigate the cross terms for } n > N_1, N_2, N_3 \text{ and } N_1 + N_2 + N_3 = n. \text{ If } N_1 + N_2 \ll n \text{ then } N_3 \sim n, \text{ and we use Case 1 to conclude. Next, if } N_1 + N_2 \sim n, \text{ we have } N_3 \ll n \text{ and we conclude by Case 2. Finally, when } (N_1 + N_2) \sim N_3 \sim n, \text{ one uses both cases to deduce the result.}
\]

Thus, passing the limit \(n \to \infty\) in \((5.13)\) gives us that \(\{w^n\}_{n \geq 0}\) is a Cauchy sequence in \(L^2_T V\). Consequently, we have that \(\{\alpha^n\}_{n \geq 0}\) is a Cauchy sequence in \(C([0,T]; Y)\) from \((5.10)\). Moreover, \(\{\mathcal{R}w^n\}_{n \geq 0}\) is a Cauchy sequence in \(L^2_T V^*\) by \((5.11)\), and similarly, \(\{S\mathcal{R}w^n\}_{n \geq 0}\) is a Cauchy sequence in \(L^2_T V\) by \((5.12)\), as desired. \(\square\)

**Step 5.3 (Passing the limit in \((5.3a)-(5.3c)\)).** Now, since \(L^2_T \tilde{X}\) is complete for all \(1 \leq p \leq \infty\), with \(\tilde{X}\) as a Banach space, it follows from Proposition 5.5 as \(n \to \infty\) that

\[
\begin{align*}
(5.14a) & \quad w^n \to w \text{ strongly in } L^2_T V, \quad \alpha^n \to \alpha \text{ strongly in } C([0,T]; Y), \\
(5.14b) & \quad S\mathcal{R}w^n \to S\mathcal{R}w \text{ strongly in } L^2_T Y, \quad \mathcal{R}w^n \to \mathcal{R}w \text{ strongly in } L^2_T V^*. 
\end{align*}
\]

We are now in a position to pass the limit in \((5.3a)-(5.3c)\). First, by \((5.9)\), we have that \(\{w^n\}_{n \geq 0}\) and \(\{\mathcal{R}w^n\}_{n \geq 0}\) are uniformly bounded in \(L^2_T V\) and \(L^2_T V^*\), respectively. Then, by Eberlein–Šmulian’s theorem, as \(L^2_T V\) is reflexive, we have, upon passing to a subsequence, that:

\[
\begin{align*}
\text{(5.15a)} & \quad w^n(t) \to w(t) \text{ strongly in } V \text{ for a.e. } t \in (0,T), \\
\text{(5.15b)} & \quad \alpha^n(t) \to \alpha(t) \text{ strongly in } Y \text{ for a.e. } t \in (0,T), \\
\text{(5.15c)} & \quad S\mathcal{R}w^n(t) \to S\mathcal{R}w(t) \text{ strongly in } Y \text{ for a.e. } t \in (0,T), \\
\text{(5.15d)} & \quad \mathcal{R}w^n(t) \to \mathcal{R}w(t) \text{ strongly in } V^* \text{ for a.e. } t \in (0,T), \\
\end{align*}
\]

as \(n \to \infty\). In addition, using similar arguments, we can find that \(\{w^n(t)\}_{n \geq 0}\) for a.e. \(t \in (0,T)\) is uniformly bounded in \(V\) (see, e.g., the first part of the proof of [27, Lemma 13]). Thus, we have by Eberlein–Šmulian’s theorem, as \(V\) is reflexive, up to a subsequence, that \(w^n(t) \to \tilde{w}(t)\) weakly in \(V\) for a.e. \(t \in (0,T)\). By uniqueness of limits, we have by \((5.15a)\) that \(\tilde{w}(t) = w(t)\) for a.e. \(t \in (0,T)\). We also find, in a similar manner, that \(\{\mathcal{R}w^n(t)\}_{n \geq 0}\) for a.e. \(t \in (0,T)\) is uniformly bounded in \(V^*\). Thus, as \(n \to \infty\):

\[
\text{(5.15e)} \quad \tilde{w}^n(t) \to \tilde{w}(t) \text{ weakly in } V^* \text{ for a.e. } t \in (0,T).
\]

From \((5.15a)\) and \(H(A)\)(ii), we get:

\[
\text{(5.16) } \lim_{n \to \infty} \langle A(t, w^n(t)), v \rangle = \langle A(t, w(t)), v \rangle \text{ for all } v \in V, \text{ a.e. } t \in (0,T).
\]

Moreover, we utilize Proposition 2.7 (while keeping \(H(j)\)(ii) in mind), \((5.15a)\), and \(H(MN\&K)\), to obtain:

\[
\limsup_{n \to \infty} j^0(t, Nw^n(t); Nv - Nw^n(t)) \leq j^0(t, Nw(t); Nv - Nw(t))
\]
for all \( v \in V \), a.e. \( t \in (0,T) \). Next, by (5.15a), (5.15d)-(5.15e), (5.16), and the Cauchy-Schwarz inequality, we can find that
\[
\lim_{n \to \infty} \langle A(t, w^n(t)), v - w^n(t) \rangle = \langle A(t, w(t)), v - w(t) \rangle,
\]
\[
\lim_{n \to \infty} \langle w^n(t), v - w^n(t) \rangle = \langle w(t), v - w(t) \rangle,
\]
\[
\lim_{n \to \infty} \langle Rw^n(t), v - w^n(t) \rangle = \langle Rw(t), v - w(t) \rangle,
\]
\[
\lim_{n \to \infty} \langle f(t), v - w^n(t) \rangle = \langle f(t), v - w(t) \rangle
\]
for all \( v \in V \), a.e. \( t \in (0,T) \). Furthermore, let \( \Sigma = Y \times Y \times U \) be equipped with the norm \( \|(x, y, z)\|_{\Sigma} = \|x\|_Y + \|y\|_Y + \|z\|_U \). We wish to deduce that \( \varphi(t, \cdot, \cdot, \cdot) \) is continuous on \( \Sigma \times Z \) by applying Lemma 2.10. The conditions (1) and (3) are directly obtained by \( H(\varphi)(\text{ii}), (\text{iv}) \). Lastly, we find that condition (2) holds by Proposition 2.8. Indeed, as \( \varphi \) is lower semicontinuous and convex in its last argument, by \( H(\varphi)(\text{iii}) \), and the fact that \( \varphi \) is finite (does not take the values \( \pm \infty \)). It then follows by (5.15a)-(5.15c) and \( H(MNK) \) that
\[
\lim_{n \to \infty} \left[ \varphi(t, S^n w^n(t), \alpha^n(t), M w^n(t), K v) - \varphi(t, S^n w^n(t), \alpha^n(t), M w^n(t), K w(t)) \right]
\]
for all \( v \in V \), a.e. \( t \in (0,T) \). Lastly, we have by \( H(\mathcal{G})(\text{ii}) \), that \( \mathcal{G}(t, \cdot, \cdot, \cdot) \) is continuous on \( Y \times U \). First, we obtain the following estimate by the triangle inequality, \( H(\mathcal{G})(\text{ii}) \), and \( H(MNK) \):
\[
\| \mathcal{G}(s, \alpha^n(s), M w^n(s)) \|_Y \\
\leq \| \mathcal{G}(s, \alpha^n(s), M w^n(s)) - \mathcal{G}(s, 0, 0) \|_Y + \| \mathcal{G}(s, 0, 0) \|_Y \\
\leq L_{\mathcal{G}} \| \alpha^n(s) \|_Y + L_{\mathcal{G}} \| M \| w^n(s) \|_Y + \| \mathcal{G}(s, 0, 0) \|_Y
\]
for a.e. \( s \in (0,T), s \leq t, n \in \mathbb{N} \). Recall that \( \{ w^n(s) \}_{n \geq 0} \) is uniformly bounded on \( V \) for a.e. \( s \in (0,T) \). Then from \( H(\mathcal{G})(\text{iii}) \) and (5.9), we obtain the desired bound. Combining (5.15a), (5.15b), and \( H(MNK) \), we may apply the dominated convergence theorem to conclude:
\[
\lim_{n \to \infty} \int_0^t \mathcal{G}(s, \alpha^n(s), M w^n(s)) ds = \int_0^t \mathcal{G}(s, \alpha(s), M w(s)) ds
\]
for all \( t \in [0,T] \). Thus, passing the upper limit \( n \to \infty \) in (5.3a)-(5.3c) gives us that \( (w, \alpha) \in \mathcal{W}^{1,2}_T \times C([0,T]; Y) \) is indeed a solution to Problem 1.

**Step 6 (Continuous dependence on initial data).** We will now prove a continuous dependence result. Let \( \{ f_i \} \subset L^2_T V^* \), \( \{ w_{0i} \} \subset V \), \( \{ \alpha_{0i} \} \subset Y \). Additionally, let \( \{ w_i \} \subset \mathcal{W}^{1,2}_T \) and \( \{ \alpha_i \} \subset C([0,T]; Y) \) be the unique solution to Problem 1 corresponding to \( \{ (f_i, w_{0i}, \alpha_{0i}) \} \), for \( i = 1, 2 \). Then, for \( i = 1, 2 \), we have:
\[
\alpha_i(t) = \alpha_{0i} + \int_0^t \mathcal{G}(s, \alpha_i(s), M w_i(s)) ds, \tag{5.18a}
\]
\[
\langle w_i(t) + A(t, w_i(t)) - f_i(t) + Rw_i(t), v - w_i(t) \rangle + j^0(t, \mathcal{N} w_i(t); N v - \mathcal{N} w_i(t)) + \mathcal{L}(t, S w_i(t), \alpha_i(t), M w_i(t), K v) - \mathcal{L}(t, S w_i(t), \alpha_i(t), M w_i(t), K w_i(t)) \geq 0 \tag{5.18b}
\]
for all \( v \in V \), a.e. \( t \in (0,T) \), with
\[
\mathcal{L}(t, w_i(0) = w_{0i}. \tag{5.18c}
\]
We wish to prove that for all $\lambda > 0$, there exists a $\delta > 0$, which will be fixed later, such that
\begin{equation}
\| (f_1, w_{01}, \alpha_{01}) - (f_2, w_{02}, \alpha_{02}) \|_{L^2_t V^* \times V \times Y} < \delta
\end{equation}
implies
\begin{equation}
\| (w_1, \alpha_1) - (w_2, \alpha_2) \|_{L^2_t V \times C([0,T],Y)} < \lambda.
\end{equation}
We start by looking at (5.18b). Adding together the inequality (5.18b) for $i = 1, 2$, and choosing the test functions $v = w_j$ for $j = 1, 2$, $i \neq j$, yields
\begin{align*}
&\langle \dot{w}_1(t) - \dot{w}_2(t), w_1(t) - w_2(t) \rangle + \langle A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t) \rangle \\
&\leq \langle f_1(t) - f_2(t), w_1(t) - w_2(t) \rangle + \| R w_1(t) - R w_2(t) \|_{V^*} \| w_1(t) - w_2(t) \|_V \\
&+ \alpha_j \| N \|_{w_1(t) - w_2(t)}^2 + \beta_1 \| K \|_{S w_1(t) - S w_2(t)} \| w_1(t) - w_2(t) \|_Y \\
&+ \beta_2 \| K \|_{\alpha_1(t) - \alpha_2(t)} \| w_1(t) - w_2(t) \|_Y
\end{align*}
for a.e. $t \in (0,T)$, with $w_1(0) = w_{01}$ and $w_2(0) = w_{02}$. Using $H(\varphi)(v)$, $H(j)(iv)$, and the Cauchy-Schwarz inequality, we find that
\begin{align*}
&\langle \dot{w}_1(t) - \dot{w}_2(t), w_1(t) - w_2(t) \rangle + \langle A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t) \rangle \\
&\leq \| f_1(t) - f_2(t) \|_{V^*} \| w_1(t) - w_2(t) \|_V + \| R w_1(t) - R w_2(t) \|_{V^*} \| w_1(t) - w_2(t) \|_V \\
&+ \alpha_j \| N \|_{w_1(t) - w_2(t)}^2 + \beta_1 \| K \|_{S w_1(t) - S w_2(t)} \| w_1(t) - w_2(t) \|_Y \\
&+ \beta_2 \| K \|_{\alpha_1(t) - \alpha_2(t)} \| w_1(t) - w_2(t) \|_Y
\end{align*}
for a.e. $t \in (0,T)$. Now, we integrate over the time interval $(0,t')$, use the integration by parts formula in Proposition 2.3 (with $v_1 = v_2 = w_1 - w_2$), the smallness condition (3.3), $H(A)(iv)$, (5.18c), and the Cauchy-Schwarz inequality:
\begin{align*}
(m_A - \alpha_j \| N \|^2 - \beta_3 \| K \|_{\| M \|}) \int_0^{t'} \| w_1(t) - w_2(t) \|^2_V dt \\
\leq \frac{1}{2} \| w_0 - w_0 \|_V^2 + \left[ \int_0^{t'} \| f_1(t) - f_2(t) \|_{V^*} dt \right]^{1/2} \left[ \int_0^{t'} \| w_1(t) - w_2(t) \|^2_V dt \right]^{1/2} \\
+ \left[ \int_0^{t'} \| R w_1(t) - R w_2(t) \|_{V^*}^2 dt \right]^{1/2} \left[ \int_0^{t'} \| w_1(t) - w_2(t) \|_{V^*}^2 dt \right]^{1/2} \\
+ \beta_1 \| K \| \left[ \int_0^{t'} \| S w_1(t) - S w_2(t) \|_{Y^*}^2 dt \right]^{1/2} \left[ \int_0^{t'} \| w_1(t) - w_2(t) \|_{Y^*}^2 dt \right]^{1/2} \\
+ \beta_2 \| K \| \left[ \int_0^{t'} \| \alpha_1(t) - \alpha_2(t) \|_{Y^*} dt \right]^{1/2} \left[ \int_0^{t'} \| w_1(t) - w_2(t) \|_{Y^*} dt \right]^{1/2}
\end{align*}
for all $t' \in [0,T]$. We next utilize Young’s inequality together with $V \subset H$ being a continuous embedding with $c_H > 0$, and find that for $\epsilon > 0$:
\begin{align}
(m_A - \alpha_j \| N \|^2 - \beta_3 \| K \|_{\| M \|}) \int_0^{t'} \| w_1(t) - w_2(t) \|^2_V dt \\
\leq \frac{c_H}{2} \| w_0 - w_0 \|^2_V + \frac{1}{2\epsilon} \| f_1 - f_2 \|^2_{L^2_t V^*} + \frac{1}{2\epsilon} \| R w_1 - R w_2 \|^2_{L^2_t V^*} \\
+ \frac{1}{2\epsilon} \| S w_1 - S w_2 \|^2_{L^2_t V^*} + \frac{1}{2\epsilon} \| \alpha_1 - \alpha_2 \|^2_{L^2_t V^*} \\
+ \frac{\epsilon}{2} \left[ 2 + \| K \|^2 (\beta_1^2 + \beta_2^2) \right] \| w_1 - w_2 \|^2_{L^2_t V}
\end{align}
for all \( t' \in [0, T] \). We next apply the triangle inequality, Minkowski’s inequality, \( H(G) \)(ii), and \( H(MNK) \) to (5.18a). This reads:

\[
\|\alpha_1(t) - \alpha_2(t)\|_Y \leq \|\alpha_{01} - \alpha_{02}\|_Y + L_G \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_Y ds
\]

\[
+ L_G \|M\| \int_0^t \|w_1(s) - w_2(s)\|_V ds
\]

for a.e. \( t \in (0, T) \). Next, using a standard Grönwall argument, Young’s inequality, and the Cauchy-Schwarz inequality reads:

\[
\|\alpha_1(t) - \alpha_2(t)\|_Y^2 \leq 2(1 + L_G e^{L_0 T} T)^2 \|\alpha_{01} - \alpha_{02}\|_Y^2
\]

\[
+ 2TL_G^2 \|M\|^2 \left(1 + L_G e^{L_0 T} T\right)^2 \int_0^t \|w_1(s) - w_2(s)\|_V^2 ds
\]

for a.e. \( t \in (0, T) \). We can observe from (5.21) that the solutions \( \alpha_1, \alpha_2 \) are close if \( w_1, w_2 \) are close. We now insert (5.21) and (5.11)-(5.12) (with \( w^{n-1} = w_1 \) and \( w^{n-2} = w_2 \) into (5.20):

\[
C_1 \int_0^{t'} \|w_1(t) - w_2(t)\|_V^2 dt \leq C_2 \|(f_1, w_{01}, \alpha_{01}) - (f_2, w_{02}, \alpha_{02})\|^2_{L^p_{t,v} V^* \times v \times Y}
\]

\[
+ C_3 \int_0^{t'} \int_0^t \|w_1(s) - w_2(s)\|_V^2 ds dt
\]

for all \( t' \in [0, T] \), with \( C_1 = m_A - \alpha_j \|N\|^2 - \beta_{3\varphi} \|K\| \|M\| - \epsilon - \frac{\epsilon \|K\|^2 (\beta_{2\varphi}^2 + \beta_{2\varphi}^2)}{2}, \)

\( C_2 = \frac{T}{\delta^2} \left[ \frac{C_2^2}{C} + \frac{C_2^2}{C} + 2L_G^2 \|M\|^2 (1 + L_G e^{L_0 T} T)^2 \right], \) and \( C_3 = \frac{C_2}{\delta^2} + \frac{1}{\delta^2} + \frac{T(1 + L_G e^{L_0 T} T)^2}{\delta^2} \). Noting that from the smallness-condition (3.3), we choose a small \( \epsilon > 0 \) such that \( m_A > \alpha_j \|N\|^2 + \beta_{3\varphi} \|K\| \|M\| + \epsilon + \frac{\epsilon \|K\|^2 (\beta_{2\varphi}^2 + \beta_{2\varphi}^2)}{2} \). Applying a standard Grönwall argument and (5.19) yields

\[
\int_0^{t'} \|w_1(t) - w_2(t)\|_V^2 dt
\]

\[
\leq \frac{C_3}{C_1} \left( 1 + \frac{T e^{C_1 T C_2}}{C_1} \right) \|(f_1, w_{01}, \alpha_{01}) - (f_2, w_{02}, \alpha_{02})\|^2_{L^p_{t,v} V^* \times v \times Y}
\]

\[
< \frac{C_3}{C_1} \left( 1 + \frac{T e^{C_1 T C_2}}{C_1} \right) \delta^2
\]

for all \( t' \in [0, T] \). We inserting the above into (5.21), and use (5.19), which reads:

\[
\max_{t \in [0, t']} \|\alpha_1(t) - \alpha_2(t)\|_Y^2 < 2(1 + L_G e^{L_0 T} T)^2 \|\alpha_{01} - \alpha_{02}\|_Y^2
\]

\[
+ 2TL_G^2 \|M\|^2 \left(1 + L_G e^{L_0 T} T\right)^2 \left[ \frac{C_3}{C_1} \left( 1 + \frac{T e^{C_1 T C_2}}{C_1} \right) \right] \delta^2
\]

\[
\times \|(f_1, w_{01}, \alpha_{01}) - (f_2, w_{02}, \alpha_{02})\|^2_{L^p_{t,v} V^* \times v \times Y}
\]

\[
< 2(1 + L_G e^{L_0 T} T)^2 \left[ \frac{C_3}{C_1} \left( 1 + \frac{T e^{C_1 T C_2}}{C_1} \right) \right] \delta^2
\]
for all $t' \in [0, T]$. Adding the two above inequalities together, we may choose $\delta > 0$ such that

$$
\| (w_1, \alpha_1) - (w_2, \alpha_2) \|^2_{L^2_T V \times C([0, T]; Y)} < \lambda^2.
$$

**Step 7 (Proof of Theorem 3.2).** We now have all the tools to prove the main theorem.

**Proof of Theorem 3.2.** Combining Step 1 - 6 gives us well-posedness of Problem 1. Adding (5.22) and (5.23) completes the proof. □

6. **Applications to viscoelastic frictional contact problems**

We will present two applications to frictional contact problems; the first considers contact with normal compliance, and the second contact with normal damped response. Let $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ denote the displacement, $\sigma : \Omega \times [0, T] \rightarrow \mathbb{S}^d$ the stress tensor, and $\alpha : \Gamma_C \times [0, T] \rightarrow \mathbb{R}$ the external state variable. In addition $f_0$ denotes the body forces, $f_N$ the surface traction, and $\rho$ the density.

![Figure 1. A standard illustration of a sliding block.](image)

We let the spaces $H = L^2(\Omega; \mathbb{R}^d)$, $Q$, $V$ and $W^{1,2}_T$ be defined by (2.1), (2.2), (2.3) and (2.5), respectively. We refer to Section 2.1-2.2 for further definitions of the function spaces. Let $\gamma_\nu : V \rightarrow L^2(\Gamma_C)$ denote the normal trace operator, and $\gamma_\tau : V \rightarrow L^2(\Gamma_C; \mathbb{R}^d)$ the tangential trace operator. For all $v \in V$, we let $v_\nu = \gamma_\nu v = v \cdot \nu$ denote the normal components on $\Gamma$, and $v_\tau = \gamma_\tau v = v - v_\nu \nu$ the tangential components on $\Gamma$. Similarly, let $\sigma_\nu = (\sigma v) \cdot \nu$, and $\sigma_\tau = \sigma v - \sigma_\nu \nu$ be the normal and tangential components on $\Gamma$, respectively.

6.1. **Dynamic frictional contact problem with normal compliance.** In this section, we present a system of equations describing a viscoelastic deformable body sliding in bilateral contact with a rigid foundation in the presence of normal compliance. Viscoelastic problems with bilateral contact, normal compliance and friction are discussed in [23, Section 8.5]. The normal compliance condition is used as an approximation of the Signorini non-penetration condition. More on this can be found in [12, Chapter 11], [23, p.3], and [25, Section 10.3]. We wish to study the following problem:
Problem 3. Find the displacement $u : \Omega \times [0,T] \to \mathbb{R}^d$ and the external state variable $\alpha : \Gamma_C \times [0,T] \to \mathbb{R}$ such that

\begin{align*}
(6.1a) & \quad \sigma(t) = A\varepsilon(\dot{u}(t)) + B\varepsilon(u(t)) + \int_0^t C(t-s,\varepsilon(\dot{u}(s)))ds \quad \text{on } \Omega \times (0,T) \\
(6.1b) & \quad \rho\ddot{u}(t) = \nabla \cdot \sigma(t) + f_0(t) \quad \text{on } \Omega \times (0,T) \\
(6.1c) & \quad u(t) = 0, \quad \dot{u}(t) = 0 \quad \text{on } \Gamma_D \times (0,T) \\
(6.1d) & \quad \sigma(t)\nu = f_N(t) \quad \text{on } \Gamma_N \times (0,T) \\
(6.1e) & \quad -\sigma_\nu(t) = p(u_\nu(t)) \quad \text{on } \Gamma_C \times (0,T) \\
(6.1f) & \quad |\sigma_\nu(t)| \leq \mu(0,\alpha(t))|\sigma_\nu(t)|, \quad \ddot{u}_\nu(t) = 0 \quad \text{on } \Gamma_C \times (0,T) \\
(6.1g) & \quad -\sigma_\nu(t) = \mu(|\dot{u}_\nu(t)|,\alpha(t))|\sigma_\nu(t)|\left|\frac{\ddot{u}_\nu(t)}{|\ddot{u}_\nu(t)|}\right|, \quad \ddot{u}_\nu(t) \neq 0 \quad \text{on } \Gamma_C \times (0,T) \\
(6.1h) & \quad \dot{\alpha}(t) = G(\alpha(t),|\ddot{u}_\nu(t)|) \quad \text{on } \Gamma_C \times (0,T)
\end{align*}

with the initial conditions

\begin{align*}
(6.1i) & \quad u(0) = u_0, \quad \dot{u}(0) = w_0 \quad \text{on } \Omega \\
(6.1j) & \quad \alpha(0) = \alpha_0 \quad \text{on } \Gamma_C
\end{align*}

In the above problem, (6.1a) is a general viscoelastic constitutive law, where $A$ is a viscosity operator, $B$ an elasticity operator, and $C$ is referred to as a relaxation tensor. Moreover, (6.1b) is a momentum equation, (6.1c) denotes the Dirichlet boundary conditions, and (6.1d) the traction applied to the surface. The equation (6.1e) is a contact condition, where the pressure function depending on the penetration conditions. Next, (6.1f)-(6.1g) denotes a generalized Coulomb’s friction law, and (6.1h) describes the evolution of the external state variable, see Section 1.2 for a discussion on this equation. Lastly, (6.1i)-(6.1j) are initial conditions. Now, we wish to investigate (6.1a)-(6.1j) under the following assumptions:

$H(A)$: $A : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

\begin{enumerate}
\item For any $\varepsilon \in \mathbb{S}^d$, $x,t \mapsto A(x,\varepsilon)$ is measurable on $\Omega$.
\item There exists $L_A > 0$ such that $|A(x,\varepsilon_1) - A(x,\varepsilon_2)| \leq L_A|\varepsilon_1 - \varepsilon_2|$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
\item There is a $m_A$ such that $A(x,\varepsilon_1) - A(x,\varepsilon_2), \varepsilon_1 - \varepsilon_2) \geq m_A|\varepsilon_1 - \varepsilon_2|^2$, for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
\item $A(x,0) = 0$ for a.e. $x \in \Omega$.
\end{enumerate}

$H(B)$: $B : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

\begin{enumerate}
\item For any $\varepsilon \in \mathbb{S}^d$, $x \mapsto B(x,\varepsilon)$ is measurable on $\Omega$.
\item There exists $L_B > 0$ such that $|B(x,\varepsilon_1) - B(x,\varepsilon_2)| \leq L_B|\varepsilon_1 - \varepsilon_2|$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
\item The mapping $x \mapsto B(x,0) \in Q$.
\end{enumerate}

$H(\mu)$: $\mu : \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is such that

\begin{enumerate}
\item The mapping $x \mapsto \mu(x,r,a)$ is measurable on $\Gamma_C$ for all $r,a \in \mathbb{R}$.
\item There exists $L_\mu > 0$ such that $|\mu(x,r_1,a_1) - \mu(x,r_2,a_2)| \leq L_\mu(|r_1 - r_2| + |a_1 - a_2|)$ for all $r_1,r_2,a_1,a_2 \in \mathbb{R}$, a.e. $x \in \Gamma_C$.
\item $\mu(x,r,a) = 0$ for all $r,a \leq 0$, a.e. $x \in \Gamma_C$.
\item There exists $\mu^* > 0$ such that $\mu(x,r,a) \leq \mu^*$ for all $r,a \in \mathbb{R}$, a.e. $x \in \Gamma_C$.
\end{enumerate}

$H(p)$: $p : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ is such that
(i) The mapping $x \mapsto p(x, r)$ is measurable on $\Gamma_C$ for all $r \in \mathbb{R}$.
(ii) There exists $L_p > 0$ such that $|p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $x \in \Gamma_C$.
(iii) $p(x, r) = 0$ for all $r \leq 0$, a.e. $x \in \Gamma_C$.
(iv) There exists $p^* > 0$ such that $p(x, r) \leq p^*$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$.

$H(G)$: $G : \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is such that

(i) The mapping $x \mapsto G(x, \alpha, r)$ is measurable on $\Omega$ for all $\alpha, r \in \mathbb{R}$.
(ii) $(\alpha, r) \mapsto G(x, \alpha, r)$ is locally Lipschitz, i.e., there exists $L_G > 0$ and a neighborhood in $\mathbb{R} \times \mathbb{R}$ such that $|G(x, \alpha_1, r_1) - G(x, \alpha_2, r_2)| \leq L_G (|\alpha_1 - \alpha_2| + |r_1 - r_2|)$ for all $\alpha_1, \alpha_2, r_1, r_2 \in \mathbb{R}$, a.e. $x \in \Omega$.
(iii) $G(\cdot, 0, 0) \in L^2(\Gamma_C)$

$H(C)$: $C : \Omega \times (0, T) \times \mathbb{S}^d \to \mathbb{S}$ is such that

(i) $C(x, t, \nu) = (c(x, t) \nu)$ for all $\nu \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega \times (0, T)$.
(ii) $c(x, t) = (c_{ijkl}(x, t))$ with $c_{ijkl} = c_{jikl} = c_{klji} \in L^\infty_T L^\infty(\Omega)$.

(6.2) The mass density is assumed to be a positive constant $\rho > 0$.

(6.3) $f_0 \in L^2_T H$, $f_N \in L^2_T L^2(\Gamma_N; \mathbb{R}^d)$.

with the initial data satisfying:

(6.4) $w_0 \in V$, $u_0 \in V$, $\alpha_0 \in L^2(\Gamma_C)$.

Remark 6.1. Similar assumptions on the operators and data are found in, e.g., [14,21,25].

We refer the reader to Appendix A for a discussion on applications under these assumptions.

6.1.1. Variational formulation. We find a formal derivation of the variational formulation of Problem 3, i.e., assuming sufficiently regular functions, as we only are interested in a mild solution (see Definition 1.1). We refer to, e.g., [23, Section 5.2] for a more detailed derivation, especially how to deal with the contact conditions. Thus, we can insert (6.1b) into the Green’s formula (2.4):

$$
\int_\Omega \rho \ddot{u}(t) \cdot [v - \dot{u}(t)] \, dx + \int_\Omega \sigma(t) : [\varepsilon(v) - \varepsilon(\dot{u}(t))] \, dx = \int_\Omega f_0(t) \cdot [v - \dot{u}(t)] \, dx + \int_\Gamma \sigma(t) \nu \cdot [v - \dot{u}(t)] \, da
$$

for all $v \in V$, a.e. $t \in (0, T)$. From the terms on $\Gamma_C$, we deduce:

$$
\int_{\Gamma_C} \sigma_\tau(t) \cdot [v_\tau - \dot{u}_\tau(t)] \, da + \int_{\Gamma_C} \sigma_\nu(t) \cdot [v_\nu - \dot{u}_\nu(t)] \, da
$$

$$
\geq \int_{\Gamma_C} \mu(|\dot{u}_\tau(t)|, \alpha(t)p(u_\nu(t))[|v_\nu(t)| - |v_\nu|] \, da + \int_{\Gamma_C} p(u_\nu(t))[\dot{u}_\nu(t) - v_\nu] \, da
$$

for all $v \in V$, a.e. $t \in (0, T)$. Combining the above reads:

$$
\int_\Omega \rho \ddot{u}(t) \cdot [v - \dot{u}(t)] \, dx + \int_\Omega \sigma(t) : [\varepsilon(v) - \varepsilon(\dot{u}(t))] \, dx
$$

$$
+ \int_{\Gamma_C} \mu(|\dot{u}_\tau(t)|, \alpha(t)p(u_\nu(t))[|v_\nu(t)| - |\dot{u}_\tau(t)|] \, da + \int_{\Gamma_C} p(u_\nu(t))[v_\nu - \dot{u}_\nu(t)] \, da
$$

$$
\geq \int_{\Gamma_N} f_N(t) \cdot [v - \dot{u}(t)] \, da + \int_\Omega f_0(t) \cdot [v - \dot{u}(t)] \, dx
$$
for all $v \in V$, a.e. $t \in (0, T)$. We write the above slightly more compactly. We observe that the map $t \mapsto \int_{\gamma_N} f_N(t) \cdot v \, da + \int_{\Omega} f_0(t) \cdot v \, dx$ is linear and bounded in $V$. Consequently, the Riesz representation theorem implies that $f(t) \in V^*$ such that
\[
\langle f(t), v \rangle = \langle f_N(t), v \rangle_{L^2(\Gamma_N;\mathbb{R}^d)} + \langle f_0(t), v \rangle_{H} \quad \text{for all } v \in V, \quad \text{a.e. } t \in (0, T).
\]

Due to that we are interested in a mild solution of (6.1h) (see Definition 1.1), we integrate (6.1h) over the time interval $(0, t)$ and use the initial condition (6.1j) to obtain this equation in the desired form. We may now formulate a variational inequality of Problem 3.

**Problem 4.** Find $u : \Omega \times [0, T] \to \mathbb{R}^d$ and $\alpha : \Gamma_C \times [0, T] \to \mathbb{R}$ such that
\[
\alpha(t) = \alpha_0 + \int_0^t G(\alpha(s), |\dot{u}(s)|) \, ds,
\]
\[
\int_{\Gamma_C} \mu(|\dot{u}(t)|, \alpha(t)) p(u(t)) \|v - |\dot{u}(t)|\| \, da + \int_{\Gamma_C} p(u(t)) \|v - \dot{u}(t)\| \, da
\]
\[
+ \int_{\Omega} \rho \ddot{u}(t) \cdot [v - \dot{u}(t)] \, dx + (A \varepsilon(\dot{u}(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)))_Q
\]
\[
+ (B \varepsilon(u(t))) + \int_0^t C(t - s, \varepsilon(\dot{u}(s))) ds, \varepsilon(v) - \varepsilon(\dot{u}(t)))_Q \geq \langle f(t), v - \dot{u}(t) \rangle
\]
for all $v \in V$, a.e. $t \in (0, T)$, with
\[
u(0) = u_0, \quad \dot{u}(0) = w_0.
\]

**Remark 6.2.** Conversely, under the assumption of sufficient regularity, showing that (6.6) is equivalent to (6.1h) together with (6.1j) is an application of the fundamental theorem of calculus. Moreover, choosing the test functions $v = \dot{u} \pm \tilde{v}$ with $\tilde{v} \in C^\infty_c(\Omega)$ implies that Problem 4 is indeed equivalent to Problem 3, see, e.g., [18, Section 2.6].

**Theorem 6.3.** Assume that $H(A)$, $H(B)$, $H(\mu)$, $H(p)$, $H(G)$, $H(C)$, and (6.2)-(6.4) holds. Then, Problem 4 has at most one solution $(u, \alpha)$ under the smallness-assumption
\[
m_A > p^* L_{\mu} \| (\gamma_\tau, \gamma_\nu) \|_{L(V, L^2(\Gamma_C;\mathbb{R}^d) \times L^2(\Gamma_C))} \| \gamma_\tau \|_{L(V, L^2(\Gamma_C;\mathbb{R}^d))}.
\]
In addition $(u, \alpha)$ has the following regularity:
\[
u \in W^{1,2}(0, T; V), \quad \dot{u} \in W_T^{1,2}, \subset C([0, T]; H), \quad \ddot{u} \in L_T^2 V^*, \quad \alpha \in C([0, T]; L^2(\Gamma_C)).
\]
Moreover, the flow map $\tilde{F} : L_T^2 H \times L_T^2 L^2(\Gamma_N;\mathbb{R}^d) \times V \times L^2(\Gamma_C) \to W^{1,2}(0, T; V) \times C([0, T]; L^2(\Gamma_C)), (f_0, f_N, u_0, w_0, \alpha_0) \mapsto (u, \alpha)$ is continuous.

**Remark 6.4.** The smallness-assumption (6.7) is used in, e.g., [21].

**Remark 6.5.** We will show that there exists a solution for $\rho \equiv 1$, and then let $w(t) = w(\rho t)$.

6.1.2. **Proof of Theorem 6.3.** We wish to use Theorem 3.2 to prove Theorem 6.3. Our first task is to rewrite Problem 4 in the same form as Problem 1. Then, we are able to verify the hypothesis stated in Theorem 3.2.

Let $X = Y = L^2(\Gamma_C), U = L^2(\Gamma_C;\mathbb{R}^d)$, and $Z = U \times X$. We then define the operators $A : (0, T) \times V \to V^*, \mathcal{R} : L_T^2 V \to L_T^2 V^*, \mathcal{S} : L_T^2 V \to L_T^2 Y, \mathcal{G} : (0, T) \times Y \times U \to Y,$
M : V → U, and N : V → X, respectively, by
\begin{align}
(6.8a) \quad \langle A(t, w), v \rangle &= \int_{\Omega} \mathcal{A}(w) : \varepsilon(v) dx, \quad \text{for } w, v \in V, \text{ a.e. } t \in (0, T), \\
(6.8b) \quad \langle \mathcal{R}w(t), v \rangle &= \int_{\Omega} \mathcal{B}(\varepsilon(\int_0^t w(s) ds + u_0)) : \varepsilon(v) dx \\
& \quad + \int_{\Omega} \int_0^t \mathcal{C}(t - s, \varepsilon(w(s))) ds : \varepsilon(v) dx, \\
& \quad \text{for } w \in L^2_T V, v \in V, \text{ a.e. } t \in (0, T), \\
(6.8c) \quad \mathcal{S}w(t) &= p(\gamma_{\nu}\left(\int_0^t w(s) ds + u_0\right)), \quad \text{for } w \in L^2_T V, \text{ a.e. } t \in (0, T), \\
(6.8d) \quad \mathcal{G}(t, \alpha, Mw) &= G(\alpha, |w_r|), \quad \text{for } \alpha \in Y, w \in V, \text{ a.e. } t \in (0, T), \\
(6.8e) \quad Mv &= v_r, \quad Nv = v, \quad \text{for } v \in V.
\end{align}

We define \( K : V \to Z \) by
\begin{align}
(6.8f) \quad K v &= (Mv, Nv), \quad \text{for } v \in V,
\end{align}
which is linear and bounded in both arguments, i.e., \( K \in \mathcal{L}(V, Z) \). We define the functional \( \varphi : (0, T) \times Y \times Y \times U \times Z \to \mathbb{R} \) by
\begin{align}
(6.8g) \quad \varphi(t, z, y, \tilde{w}, \tilde{v}) &= \int_{\Gamma_C} zv^{(2)} da + \int_{\Gamma_C} \mu(|\tilde{w}|, y)z|v^{(1)}| da,
\end{align}
for \( z, y \in Y, \tilde{w} \in U, \tilde{v} = (v^{(1)}, v^{(2)}) \in Z, \text{ a.e. } t \in (0, T), \) where \( z = p(r) \) satisfy \( H(p)(iv) \), i.e., \( 0 \leq z \leq p^* \). We let \( f : (0, T) \to V^* \) be defined as in (6.5). This yields the following generalization of Problem 4:

**Problem 5.** Find \( w \in W^{1,2}_T \) and \( \alpha \in C([0, T]; Y) \) such that
\begin{align}
\alpha(t) &= \alpha_0 + \int_0^t \mathcal{G}(s, \alpha(s), Mw(s)) ds, \\
\langle \rho \tilde{w}(t), \varphi(t, w(t)) \rangle &= \langle A(t, w(t)), v - w(t) \rangle + \langle \mathcal{R}w(t), v - w(t) \rangle \\
& \quad + \varphi(t, \mathcal{S}w(t), \alpha(t), Mw(t), Kv) - \varphi(t, \mathcal{S}w(t), \alpha(t), Mw(t), Kw(t)) \geq \langle f(t), v - w(t) \rangle
\end{align}
for all \( v \in V, \text{ a.e. } t \in (0, T), \) with \( w(0) = w_0. \)

**Remark 6.6.** Since Problem 5 is a generalization of Problem 4, we then have that if there exists a unique solution to Problem 5 it implies that there exists a unique solution to Problem 5.

**Lemma 6.7.** Under the assumptions of Theorem 6.3, the hypothesis of Theorem 3.2 holds for (6.8a)-(6.8g).

To maintain the flow of the article, the proof of Lemma 6.7 is placed in Appendix D. We are now ready to prove Theorem 6.3.

**Proof of Theorem 6.3.** This proof relies on Theorem 3.2, with \( j^0 \equiv 0 \). In light of Lemma 6.7 and (6.7), we have that all of the hypothesis of Theorem 3.2 holds. Thus, we can conclude that \( (w, \alpha) \in W^{1,2}_T \times C([0, T]; Y) \) is a unique solution of Problem 5. Moreover, defining the function \( u : [0, T] \to V \) by
\begin{align}
(6.9) \quad u(t) &= \int_0^t w(s) ds + u_0,
\end{align}
it follows by Bochner space theory that since \( w \in L^2_T V \subset L^1_T V \), then from (6.9), it follows that \( u \in C([0,T];V) \) and \( \dot{u} = w \). Consequently, we have \( \dot{u} \in W^{1,2}_T \) implying that \( u \in W^{1,2}(0,T;V) \). We define the set

\[
\mathcal{Y} = L^2_T H \times L^2_T L^2(\Gamma_N;\mathbb{R}^d) \times V \times V \times Y,
\]

and show that the flow map \( \tilde{F} : \mathcal{Y} \rightarrow W^{1,2}(0,T;V) \times C([0,T];Y) \), \( (f_0, f_N, w_0, u_0, \alpha_0) \mapsto (u, \alpha) \) is continuous. That is, we claim that for all \( \lambda > 0 \), there exists a \( \delta > 0 \), chosen later, such that

(6.10a) \[ \| (f_{01}, f_{N1}, w_{01}, u_{01}, \alpha_{01}) - (f_{02}, f_{N2}, w_{02}, u_{02}, \alpha_{02}) \|_Y < \delta \]

implies

(6.10b) \[ \| (u_1, \alpha_1) - (u_2, \alpha_2) \|_{W^{1,2}(0,T;V) \times C([0,T];Y)} < \lambda. \]

To check this, let us use the continuous dependence result in Theorem 3.2. First, the Cauchy-Schwarz inequality, a duality argument, Young’s inequality, and integrating over the time interval \((0,T)\) applied to (6.5) yields

\[
\| f_1 - f_2 \|_{L^2_T V^*} \leq 2 \| (f_0, f_{N1}) - (f_0, f_{N2}) \|_{L^2_T H \times L^2_T L^2(\Gamma_N;\mathbb{R}^d)}. \]

Then (6.10a) implies

(6.11) \[ \| (f_1, w_{01}, \alpha_{01}) - (f_2, w_{02}, \alpha_{02}) \|_{L^2_T V^* \times V \times Y} \]

\[ \leq 2 \| (f_0, f_{N1}, w_{01}, u_{01}, \alpha_{01}) - (f_0, f_{N2}, w_{02}, u_{02}, \alpha_{02}) \|_Y^2. \]

The estimate (3.4) from Theorem 3.2 for \( c > 0 \) reads

(6.12) \[ \| (w_1, \alpha_1) - (w_2, \alpha_2) \|_{L^2_T V \times C([0,T];Y)} \leq c \| (f_1, w_{01}, \alpha_{01}) - (f_2, w_{02}, \alpha_{02}) \|_{L^2_T V^* \times V \times Y}. \]

Next, from (6.9), the triangle inequality, Minkowski’s inequality, Young’s inequality, the Cauchy-Schwarz inequality, and integrating over the time interval \((0,T)\) yields

(6.13) \[ \| u_1 - u_2 \|_{L^2_T V} \leq 2T \| u_{01} - u_{02} \|_V^2 + 2T^2 \| w_1 - w_2 \|_{L^2_T V}^2. \]

Combining the inequalities (6.11), (6.12), and (6.13), while remembering (6.10a) and that \( w_i = \dot{u}_i \), for \( i = 1, 2 \), we may choose a \( \delta > 0 \) so that (6.10b) is required.

\[ \square \]

6.2. Dynamic frictional contact problem with normal damped response. In our second application, we consider contact with normal damped response, i.e., there is wet material, or some lubrication between the rigid and the deformable body. We present the problem:
Problem 6. Find the displacement \( u : \Omega \times [0, T] \rightarrow \mathbb{R}^d \) and the external state variable \( \alpha : \Gamma_C \times [0, T] \rightarrow \mathbb{R} \) such that

\[
\begin{align*}
(6.14a) \quad \sigma(t) &= \mathcal{A}(\dot{u}(t)) + \mathcal{B}(u(t)) + \int_0^t \mathcal{C}(t-s, \dot{u}(s))ds \quad & \text{on } \Omega \times (0, T) \\
(6.14b) \quad \rho \ddot{u}(t) &= \nabla \cdot \sigma(t) + f_0(t) \quad & \text{on } \Omega \times (0, T) \\
(6.14c) \quad u(t) = 0, \quad \dot{u}(t) = 0 \quad & \text{on } \Gamma_D \times (0, T) \\
(6.14d) \quad \sigma(t) \nu &= f_N(t) \quad & \text{on } \Gamma_N \times (0, T) \\
(6.14e) \quad -\sigma_{\nu}(t) &\in \partial j_{\nu}(\dot{u}_{\nu}(t)) \quad & \text{on } \Gamma_C \times (0, T) \\
(6.14f) \quad |\sigma_{\tau}(t)| &\leq \mu(0, \alpha(t)), \quad \dot{u}_{\tau}(t) = 0 \quad & \text{on } \Gamma_C \times (0, T) \\
(6.14g) \quad -\sigma_{\tau}(t) &= \mu(|\dot{u}_{\tau}(t)|, \alpha(t)) \frac{\dot{u}_{\tau}(t)}{|\dot{u}_{\tau}(t)|}, \quad \dot{u}_{\tau}(t) \neq 0 \quad & \text{on } \Gamma_C \times (0, T) \\
(6.14h) \quad \dot{\alpha}(t) &= G(\alpha(t), |\dot{u}_{\tau}(t)|) \quad & \text{on } \Gamma_C \times (0, T)
\end{align*}
\]

with

\[
\begin{align*}
(6.14i) \quad u(0) &= u_0, \quad \dot{u}(0) = w_0 \quad & \text{on } \Omega \\
(6.14j) \quad \alpha(0) &= \alpha_0 \quad & \text{on } \Gamma_C 
\end{align*}
\]

We summarized (6.14a)-(6.14d), and (6.14h)-(6.14j) underneath Problem 3. However, (6.14e) is a general form of the contact condition for normal damped response describing the contact with a lubricated foundation [16, Section 6.3]. The equations (6.14f)-(6.14g) is a version of Coulomb's law of dry friction, where (6.14f)-(6.14g) is a generalization of [25, Problem 68, p.268]. We investigate Problem 6 under the hypothesis of \( H(A) \), \( H(B) \), \( H(\mu) \) \( H(G) \), \( H(\mathcal{C}) \), and (6.2)-(6.4). In addition, we require an assumption on \( j_{\nu} \) in (6.14e):

\( H(j_{\nu}) \): \( j_{\nu} : \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R} \) is such that

(i) \( j_{\nu}(\cdot, r) \) is measurable on \( \Gamma_C \) for all \( r \in \mathbb{R} \), a.e. \( t \in (0, T) \), and there exists \( \bar{e} \in L^2(\Gamma_C) \) such that \( j_{\nu}(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_C) \).

(ii) \( j_{\nu}(x, t, \cdot) \) is locally Lipschitz on \( \mathbb{R} \) for a.e. \( (x, t) \in \Gamma_C \times (0, T) \).

(iii) \( |\partial j_{\nu}(x, r)| \leq \bar{c}_0 + \bar{c}_1 |r| \) for all \( r \in \mathbb{R} \), a.e. \( x \in \Gamma_C \), with \( \bar{c}_0, \bar{c}_1 \geq 0 \).

(iv) \( j_{\nu}^0(x, r_1, r_2 - r_1) + j_{\nu}^0(x, r_2, r_1 - r_2) \leq \alpha_{j_{\nu}} |r_1 - r_2|^2 \) for all \( r_1, r_2 \in \mathbb{R} \), a.e. \( x \in \Gamma_C \), with \( \alpha_{j_{\nu}} \geq 0 \).

We refer the reader to Appendix A for a small discussion on the assumptions on the operators and functions in Problem 6.

6.2.1. Variational formulation. We make use of the derivations in Section 6.1, but include the new term for the normal stress. By definition of the Clarke subgradient, i.e., Definition 2.5, and (6.14e) reads:

\[ -\sigma_{\nu}(t) \bar{v}_{\nu} \leq j_{\nu}^0(\dot{u}_{\nu}(t); \bar{v}_{\nu}) \quad \text{for all } \bar{v} \in V. \]

Integrating over \( \Gamma_C \), and choosing \( \bar{v}_{\nu} = v_{\nu} - \dot{u}_{\nu}(t) \):

\[
\int_{\Gamma_C} \sigma_{\nu}(t)(v_{\nu} - \dot{u}_{\nu}(t))da \leq \int_{\Gamma_C} j_{\nu}^0(\dot{u}_{\nu}(t); v_{\nu} - \dot{u}_{\nu}(t))da \quad \text{for all } v \in V.
\]

Combining (6.15) with the calculations from Section 6.1.1, we have the following problem:
Problem 7. Find \( u : \Omega \times [0, T] \to \mathbb{R}^d \) and \( \alpha : \Gamma_C \times [0, T] \to \mathbb{R} \) such that

\[
\alpha(t) = \alpha_0 + \int_0^t G(\alpha(s), |\dot{u}_r(s)|)ds,
\]

\[
\int_{\Gamma_C} \mu((|\dot{u}_r(t)|, \alpha(t))||v_r| - |\dot{u}_r(t)||)da + \int_{\Gamma_C} j_\nu^2(\dot{u}_\nu(t); v_\nu - \dot{w}(t))da + \int_\Omega \rho \ddot{u}(t) \cdot (v - \dot{u}(t))dx + (A\varepsilon(\dot{u}(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)))_Q
\]

\[
+ (B\varepsilon(u(t)) + \int_0^t C(t-s, \varepsilon(\dot{u}(s)))ds, \varepsilon(v) - \varepsilon(\dot{u}(t)))_Q \geq (f(t), v - \dot{u}(t))
\]

for all \( v \in V, \) a.e. \( t \in (0, T), \) with

\[ u(0) = u_0, \quad \dot{u}(0) = w_0. \]

Theorem 6.8. Assume that \( H(A), H(B), H(\mu), H(G), H(C), (6.2)-(6.4), \) and \( H(j_\nu) \) holds. Then, Problem 7 has a unique solution \((u, \alpha)\) under the smallness-condition

\[
m_A > \alpha_{j_\nu} \|\gamma_\nu\|^2_{L^2(V, L^2(\Gamma_C))} + L_\mu \|\gamma_r\|^2_{L^2(V, L^2(\Gamma_C; \mathbb{R}^d))}. \]

In addition, we have the regularity:

\[ u \in W^{1,2}(0, T; V), \quad \dot{u} \in W^{1,2}_T \subset C([0, T]; H), \quad \ddot{u} \in L^2 V^*, \quad \alpha \in C([0, T]; L^2(\Gamma_C)). \]

Moreover, the flow map depends continuously on the initial data.

Remark 6.9. The constraint \((6.17)\) can also be found in, e.g., [10, Section 4, Theorem 4.4] for \( \alpha_{\phi^2} = L_\mu \|\gamma_r\|^2_{L^2(V, L^2(\Gamma_C; \mathbb{R}^d))} \) and \( \alpha_{j_2} = \alpha_{j_\nu} \|\gamma_\nu\|^2_{L^2(V, L^2(\Gamma_C))}. \)

Remark 6.10. We will show that there exists a solution for \( \rho \equiv 1, \) and then let \( w(t) = w(\rho t). \)

6.2.2. Proof of Theorem 6.8. We use the same approach as in Section 6.1, which means that we will use Theorem 3.2 to prove Theorem 6.8. But to use this theorem, we need to first rewrite Problem 7 into the same form as in Problem 1. Then we verify the hypothesis stated in Theorem 3.2.

Let \( X = Y = L^2(\Gamma_C), \) and \( U = Z = L^2(\Gamma_C; \mathbb{R}^d). \) We define \( A : (0, T) \times X \to V^*, \)

\( R : L^2_T V \to L^2_T V^*, \)

\( G : (0, T) \times Y \times U \to Y, \) and \( f : (0, T) \to V^* \) as in \((6.8a)-(6.8d)\) and \((6.5), \) respectively. We let \( M : V \to U, \) \( N : V \to X \) be as in \((6.8e)\) and \( K = M. \) Moreover, we define the functional \( \varphi : (0, T) \times Y \times U \times Z \to \mathbb{R} \) by

\[
\varphi(t, y, \tilde{w}, \tilde{v}) = \int_{\Gamma_C} \mu(y, \tilde{w})|\tilde{v}|da \quad \text{for } y \in Y, \quad \tilde{w} \in U, \quad \tilde{v} \in Z, \quad \text{a.e. } t \in (0, T),
\]

and the functional \( j : (0, T) \times X \to \mathbb{R} \) by

\[
j(t, \tilde{w}) = \int_{\Gamma_C} j_\nu(\tilde{w})da \quad \text{for } \tilde{w} \in X, \quad \text{a.e. } t \in (0, T).
\]

We introduce the problem:

Problem 8. Find \((w, \alpha) \in W^{1,2}_T \times C([0, T]; Y)\) such that

\[
\alpha(t) = \alpha_0 + \int_0^t G(s, \alpha(s), Mw(s))ds,
\]

\[
\langle \rho \dot{w}(t), v - w(t) \rangle + \langle A(t, w(t)), v - w(t) \rangle + \langle R w(t), v - w(t) \rangle + j_\nu^2(t, Nw(t); Nv - Nw(t)) + \varphi(t, \alpha(t), Mw(t), Kv) - \varphi(t, \alpha(t), Mw(t), K\dot{w}(t)) \geq (f(t), v - w(t))
\]

for all \( v \in V, \) a.e. \( t \in (0, T), \) with \( w(0) = w_0. \)
To see that it suffices to prove existence of a solution to Problem 8 in order for Problem 7 to have a solution, we introduce the following result, which is on a similar form as found in [25, Lemma 8, p.126], but is a consequence of [16, Theorem 3.47]. The result will also come in useful to prove uniqueness.

**Corollary 6.11.** Assume that \( H(j_\nu) \) holds. Then, the functional \( j \) defined by (6.18b) has the following properties:

(i) \( j(\cdot, v) \) is measurable on \((0, T)\) for all \( v \in X \).

(ii) \( j(t, \cdot) \) is locally Lipschitz on \( X \) for a.e. \( t \in (0, T) \).

(iii) For all \( \tilde{w}, v \in X \), we have \( j^0(\tilde{w}; v) \leq \int_{\Gamma_C} j^\nu(\tilde{w}; v) da \).

(iv) For all \( v \in X \), then \( \|\partial j(\nu)\|_{X^*} \leq c_0 + c_1\|v\|_X \) with \( c_0 = \frac{\sqrt{2\text{meas}(\Gamma_C)}c_0}{\sqrt{2c_1}} \).

(v) For all \( v_1, v_2 \in X \), we have \( j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j\|v_1 - v_2\|_X^2 \) with \( \alpha_j = \alpha_{j_\nu} \).

**Lemma 6.12.** Under the assumptions of Theorem 6.8, \( H(\varphi) \) holds for (6.18a).

The proof of Lemma 6.12 is placed in Appendix E. We may now prove Theorem 6.8.

**Proof of Theorem 6.8.** We wish to utilize Theorem 3.2. The hypothesis of Theorem 3.2 holds by Lemma 6.7 and 6.12, and Corollary 6.11(i)-(ii),(iv)-(v). With the help of Corollary 6.11(iii), we can conclude that there exists a solution to Problem 7. Moreover, the fact that flow map depends continuously on the initial data follows by the same approach as in the last part of the proof of Theorem 6.3. To obtain uniqueness, we let \((u_i, \alpha_i) \in W^{1,2}(0, T; V) \times C([0, T]; Y)\) be solutions to Problem 7. Choosing the test functions \( v = \tilde{u}_j \) for \( i, j = 1, 2, i \neq j \), in (6.16b) yields:

\[
\begin{align*}
(\mathcal{A}\varepsilon(\tilde{u}_1(t)) - \mathcal{A}\varepsilon(\tilde{u}_2(t)), \varepsilon(\tilde{u}_1(t)) - \varepsilon(\tilde{u}_2(t)))_Q + \int_{\Omega} \rho[\tilde{u}_1(t) - \tilde{u}_2(t)] \cdot [\dot{\tilde{u}}_1(t) - \dot{\tilde{u}}_2(t)] dx & \\
\leq \int_{\Gamma_C} \mu(\dot{\tilde{u}}_1(t)), \alpha_1(t) - \mu(\dot{\tilde{u}}_2(t)), \alpha_2(t)] [\|\tilde{u}_2(t)\|_X - |\tilde{u}_1(t)|] da & \\
+ \int_{\Gamma_C} j^\nu(\dot{\tilde{u}}_1(t); \tilde{u}(t), \dot{\tilde{u}}_1(t)) da + \int_{\Gamma_C} j^\nu(\dot{\tilde{u}}_2(t); \tilde{u}(t), \dot{\tilde{u}}_2(t)) da & \\
+ (B\varepsilon(u_1(t)) - B\varepsilon(u_2(t)), \varepsilon(\tilde{u}_2(t)) - \varepsilon(\tilde{u}_1(t)))_Q & \\
+ \int_{0}^{t} C(t - s, \varepsilon(\tilde{u}_1(s))) ds - \int_{0}^{t} C(t - s, \varepsilon(\tilde{u}_2(s))) ds, \varepsilon(\tilde{u}_2(t)) - \varepsilon(\tilde{u}_1(t)) \big)_Q
\end{align*}
\]

for a.e. \( t \in (0, T) \). We note using \( H(G)(ii) \), a standard Grönwall argument, the Cauchy-Schwarz inequality, and the Minkowski’s inequality that (6.16a) implies

\[
(6.19) \quad \int_{\Gamma_C} |\alpha_1(t) - \alpha_2(t)|^2 dx \leq c\|\gamma_\nu\|_{L^2(V; U)}^2 \int_{0}^{t} \int_{\Omega} |\tilde{u}_1(s) - \tilde{u}_2(s)|^2 dx ds
\]

for a.e. \( t \in (0, T) \). In addition, utilizing \( H(A)(iii), H(B)(ii), H(j_\nu)(iv), H(\mu)(ii), H(C), (6.2) \), the Cauchy-Schwarz inequality, Minkowski’s inequality, and Young’s inequality. In addition, we use the fact that if \( \tilde{u} \in W^{1,2}_T \), then \( \|\tilde{u}(t)\|_H^2 = 2\|\tilde{u}(t), \tilde{u}(t)\| \) [8, Theorem 3 in Section 5.9.2]. Lastly, using (6.9), integrating over the time interval \((0, t')\), and a standard Grönwall argument to the above inequality yields

\[
(6.20) \quad (m_A - \alpha_{j_\nu} \|\gamma_\nu\|^2_{L^2(V; X)} - L_\mu \|\gamma_\nu\|^2_{L^2(V; U)})^2 |\tilde{u}_1 - \tilde{u}_2|^2_{L^2_{t'}_V} \leq 0.
\]

We conclude that \((u, \alpha)\) is a unique solution to Problem 7 by the definition of \( u \) (6.9), the smallness-condition (6.17), (6.19), and (6.20).
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References

[1] Aubin, J.-P., and Cellina, A. Differential inclusions, vol. 264 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[2] Cheney, W. Analysis for applied mathematics, vol. 208 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[3] Clarke, F. H. Optimization and nonsmooth analysis. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1983. A Wiley-Interscience Publication.

[4] Dancer, E. N., and Du, Y. On a free boundary problem arising from population biology. Indiana Univ. Math. J. 52, 1 (2003), 51–67.

[5] Denkowski, Z., Migórski, S., and Papageorgiou, N. S. An introduction to nonlinear analysis: applications. Kluwer Academic Publishers, Boston, MA, 2003.

[6] Denkowski, Z., Migórski, S., and Papageorgiou, N. S. An introduction to nonlinear analysis: theory. Kluwer Academic Publishers, Boston, MA, 2003.

[7] Duvaut, G., and Lions, J.-L. Inequalities in mechanics and physics, vol. 219 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin-New York, 1976. Translated from the French by C. W. John.

[8] Evans, L. C. Partial differential equations, second ed., vol. 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010.

[9] Han, W., and Sofonea, M. Quasistatic contact problems in viscoelasticity and viscoplasticity, vol. 30 of AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2002.

[10] Han, W., and Sofonea, M. Numerical analysis of hemivariational inequalities in contact mechanics. Acta Numer. 28 (2019), 175–286.

[11] Hu, S., and Papageorgiou, N. S. Handbook of multivalued analysis. Vol. II, vol. 500 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 2000. Applications.

[12] Kikuchi, N., and Oden, J. T. Contact problems in elasticity: a study of variational inequalities and finite element methods, vol. 8 of SIAM Studies in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

[13] Marone, C. Laboratory-derived friction laws and their application to seismic faulting. Annual Review of Earth and Planetary Sciences 26 (1998), 643–696.

[14] Migórski, S. Well-posedness of constrained evolutionary differential variational-hemivariational inequalities with applications. Nonlinear Anal. Real World Appl. 67 (2022), Paper No. 103593, 22.

[15] Migórski, S., and Bai, Y. Well-posedness of history-dependent evolution inclusions with applications. Z. Angew. Math. Phys. 70, 4 (2019), Paper No. 114, 22.

[16] Migórski, S., Ochal, A., and Sofonea, M. Nonlinear inclusions and hemivariational inequalities, vol. 26 of Advances in Mechanics and Mathematics. Springer, New York, 2013. Models and analysis of contact problems.

[17] Pedersen, G. K. Analysis now, vol. 118 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1989.

[18] Pipping, E. Dynamic problems of rate-and-state friction in viscoelasticity. PhD thesis, Freie Universität Berlin, 2015.

[19] Pipping, E. Existence of long-time solutions to dynamic problems of viscoelasticity with rate-and-state friction. ZAMM Z. Angew. Math. Mech. 99, 11 (2019), e201800263, 10.

[20] Pipping, E., Sander, O., and Kornhuber, R. Variational formulation of rate- and state-dependent friction problems. ZAMM Z. Angew. Math. Mech. 95, 4 (2015), 377–395.

[21] Pătrulescu, F., and Sofonea, M. Analysis of a rate-and-state friction problem with viscoelastic materials. Electron. J. Differential Equations (2017), Paper No. 299, 17.
Appendix A. Comments on assumptions

We include a small discussion on applications fitting our assumptions:

- We first comment on the equation $\mu = \mu(|\dot{u}_r|, \alpha)$ under the assumptions $H(\mu)$. This assumption can be found in, e.g., [21]. We can also find this assumption in [25, Section 10.3], if expanding the conditions to two arguments. Moreover, hypothesis $H(\mu)(ii)$ was verified in [18, Section 1] for the regularized and truncated rate-and-state laws, i.e., (1.3a)-(1.3b), respectively. In [18, Section 1.3], they only showed that $\mu$ is Lipschitz continuous in the first argument, but showing that $\mu$ is Lipschitz continuous in the second follows from the same approach. As discussed in Section 1.2, (1.3a)-(1.3b) both satisfy $H(\mu)(iii)$. Clearly, $H(\mu)(iv)$ holds for both (1.3a)-(1.3b).

- Secondly, we consider the equation $G = G(\alpha, |\dot{u}_r|)$ under the assumptions $H(G)$. For the ageing law, i.e., (1.4), they verified that $G$ is Lipschitz continuous in the second argument in [18, Proposition 1.2]. The fact that $G$ is locally Lipschitz in the second argument follows from the mean value theorem. So, we may deduce that the locally Lipschitz requirement $H(G)(ii)$ holds for (1.4). Moreover, $H(G)(iii)$ is clear, since $G(x, 0, 0) = c$ and $\text{meas}(\Gamma_C) > 0$, which extends the framework in [14, 21]. Another application is, e.g., the slip law

$$G(\alpha, |\dot{u}_r|) = |\dot{u}_r|,$$

which fits the framework in [14, 21].

- The assumption $H(p)$ is also used in, e.g., [21], and [25, Section 10.3]. They hold for, e.g., a constant function, and

$$p(r) = \begin{cases} c(r_+)^m, & \text{if } r \leq r^* \\ c(r^*)^m, & \text{if } r > r^* \end{cases},$$

where $r^*$ is a positive cut-off limit, related to the hardness of the material, $r_+ = \max\{0, r\}$, $m \in \mathbb{N}$, and $c > 0$. We refer the reader to, e.g., [16, 23] for more applications. See, e.g., [23, p.78] to read more on possible applications with the assumption $H(p)$.

- One example in linear viscoelasticity where the operators $A$ and $B$ satisfy $H(A)$ and $H(B)$ is the Kelvin-Voigt constitutive law

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{u}) + b_{ijkl}\varepsilon_{kl}(u),$$

where $\sigma_{ij}$, $a_{ijkl}$, and $b_{ijkl}$ are the components of $\sigma$, $A$, and $B$, respectively, under the assumption $a_{ijkl} \in L^\infty(\Omega)$ and

$$a_{ijkl} = a_{jikl} = a_{klij}.$$
In addition, there exists $m_A > 0$ such that
\[ a_{ijkl} \varepsilon_{ijkl} \geq m_A |\varepsilon|^2 \quad \text{for all } \varepsilon \in \mathbb{S}^d, \]
i.e. the usual ellipticity condition. We also assume that $b_{ijkl} \in L^\infty(\Omega)$ satisfies (A.1) [24].

For further applications, we refer the reader to, e.g., [24, Section 4], [16, Section 6], and [9, Section 6].

**APPENDIX B. PROOF OF PROPOSITION 5.3**

**Proof of Proposition 5.3.** For simplicity in notation, let $w = w_{\alpha \xi \eta g}$. We start off finding some estimates on $A$, $\varphi$ and $j^\circ$. According to $H(A)(iv)$,
\[ (A(t, w(t)), w(t)) - (A(t, 0), w(t)) = (A(t, w(t)) - A(t, 0), w(t)) \geq m_A \|w(t)\|^2 \]
for a.e. $t \in (0, T)$. Invoking $H(A)(iii)$, i.e., $\|A(t, 0)\|_{V^*} \leq a_0(t)$, and the Cauchy-Schwarz inequality gives
\[ (B.1) \quad (A(t, w(t)), w(t)) \geq m_A \|w(t)\|^2 + (A(t, 0), w(t)) \geq m_A \|w(t)\|^2 \]
for a.e. $t \in (0, T)$. Moreover, applying $H(j)(iv),(iii)$ and the Cauchy-Schwarz inequality reads
\[ (B.2) \quad j^\circ(t, Nw(t); Nv - Nw(t)) \]
\[ \leq \alpha_j \|Nw(t) - Nv\|^2 - j^\circ(t, Nv; Nw(t) - Nv) \]
\[ \leq \alpha_j \|N\|^2 \|w(t) - v\|_V + [c_{0j}(t) + c_{1j} \|N\|_{V^*} \|v\|_V] \|N\| \|w(t) - v\|_V \]
for all $v \in V$, a.e. $t \in (0, T)$. We also observe that $H(\varphi)(v),(iv)$ and the Cauchy-Schwarz inequality gives us:
\[ (B.3) \quad \varphi(t, \eta(t), \alpha(t), Mw(t), Kw(t)) \]
\[ \leq \beta_1 \|K\| \|\eta(t)\|_Y \|w(t) - v\|_V + \beta_2 \|K\| \|\alpha(t)\|_Y \|w(t) - v\|_V \]
\[ + \beta_3 \|K\| \|M\| \|\eta(t)\|_V \|w(t) - v\|_V + [c_{0\varphi}(t) + \|K\| \|v\|_V] \|K\| \|v - w(t)\|_V \]
for all $v \in V$, a.e. $t \in (0, T)$. We are now in a position to find the desired estimate Choosing $v = 0$ in Problem 2, while keeping in mind $H(MNK)$, reads
\[ \langle \dot{w}(t), w(t) \rangle + (A(t, w(t)), w(t)) \leq \langle f(t), w(t) \rangle - \langle \xi(t), w(t) \rangle + j^\circ(t, Nw(t); -Nw(t)) \]
\[ + \varphi(t, \eta(t), \alpha(t), Mg(t), 0) - \varphi(t, \eta(t), \alpha(t), Mg(t), Kw(t)) \]
for a.e. $t \in (0, T)$. Take $v = 0$ in (B.2) and (B.3). Combining these estimates, (B.1), the Cauchy-Schwarz inequality, and Young’s inequality implies that there is a $\epsilon > 0$ such that
\[ \langle \dot{w}(t), w(t) \rangle + (m_A - \alpha_j \|N\|^2 - 5\epsilon + \epsilon \|K\|^2 (\beta_1^2 + \beta_2^2 + \beta_3 \|M\|^2)) \|w(t)\|^2 \]
\[ \leq \frac{1}{2\epsilon} \|f(t)\|^2_{V^*} + \frac{1}{2\epsilon} \|\xi(t)\|^2_{V^*} + \frac{1}{2\epsilon} \|a_0(t)^2 + \frac{1}{2\epsilon} \|\eta(t)\|^2_{V^*} + \frac{1}{2\epsilon} \|\alpha(t)\|^2_{V^*} + \frac{1}{2\epsilon} c_{0\varphi}(t)^2 \]
\[ + \frac{1}{2\epsilon} c_{0j}(t)^2 + \frac{1}{2\epsilon} \|g(t)\|^2_{V^*} \]
for a.e. $t \in (0, T)$. Thus from (3.3) it follows that we can choose $\epsilon > 0$ so that $m_A > \alpha_j \|N\|^2 + \frac{5\epsilon + \epsilon \|K\|^2 (\beta_1^2 + \beta_2^2 + \beta_3 \|M\|^2)}{2}$. Integrating over the time interval $(0, t')$, applying the integration by parts formula in Proposition 2.3 (with $v_1 = v_2 = w$), Hölder’s inequality,
\(w(0) = w_0\), and that \(V \subset H\) is a continuous embedding with \(c_H > 0\). Thus, there exists a constant \(c = c(T, \ldots) > 0\) such that

\[
\|w(t')\|_{L^2_t V}^2 + \|w\|_{L^2_t V}^2 \leq c(\|w_0\|_{V}^2 + \|f\|_{L^2_t V^*}^2 + \|\xi\|_{L^2_t V^*}^2 + \|\alpha_0\|_{L^2_t (0, t')}^2 + \|\eta\|_{L^2_t V}^2 \\
+ \max_{t \in [0, t']} \|\alpha(t)\|_{V}^2 + \|\alpha(t)\|_{V^*}^2 + \|\alpha_0(t)\|_{L^2_t V}^2 + \|\alpha_0(t)\|_{L^2_t (0, t')}^2 + \|\eta\|_{L^2_t V}^2)
\]

for all \(t' \in [0, T]\). Taking the supremum over \(t' \in [0, T]\) gives us the first desired estimate. It remains to find a bound on \(\dot{w}\) in \(L^2_t V^*\). Let us first rearrange Problem 2. Then the estimates (B.2) and (B.3), Cauchy-Schwarz inequality, and \(H(A)(iii)\) implies

\[
\langle \dot{w}(t), w(t) - v \rangle \\
\leq \left[ a_0(t) + a_1 \|w(t)\|_V \right] \|v - w(t)\|_V + \|f(t)\|_{V^*} \|v - w(t)\|_V + \|\xi(t)\|_{V^*} \|v - w(t)\|_V \\
+ \|K\| c_{0, \varphi} \|v - w(t)\|_V + \beta_1 \varphi \|K\| \|\eta(t)\|_V \|v - w(t)\|_V + \beta_2 \varphi \|K\| \|\alpha(t)\|_V \|v - w(t)\|_V \\
+ \beta_3 \varphi \|K\| \|M\| \|g(t)\|_V \|v - w(t)\|_V + \alpha_j \|\eta\|_V^2 \|v - w(t)\|_V + \|M\| c_{0, j} \|v - w(t)\|_V
\]

for all \(v \in V\), a.e. \(t \in (0, T)\). Next, choosing \(V \ni \hat{v} = w(t) - v\), with \(v \in V\) arbitrary. From duality, we deduce:

\[
\|\dot{w}(t)\|_{V^*} = \sup_{\|\hat{v}\|_{V^*} = 1} \langle \dot{w}(t), \hat{v}\rangle_{V^* \times V} \\
\leq a_0(t) + a_1 \|w(t)\|_V + \|f(t)\|_{V^*} + \|\xi(t)\|_{V^*} + \|K\| c_{0, \varphi} \|v - w(t)\|_V + \beta_1 \varphi \|K\| \|\eta(t)\|_V \\
+ \beta_2 \varphi \|K\| \|M\| \|g(t)\|_V + \alpha_j \|\eta\|_V^2 + \|M\| c_{0, j} \|v - w(t)\|_V
\]

We conclude the proof by Young’s inequality, integrating over the time interval \((0, T)\), Hölder’s inequality, and utilizing the estimate (B.4). \(\square\)

**Appendix C. Proof of Lemma 5.4**

**Proof of Lemma 5.4.** The proof relies on the Banach fixed-point theorem. We therefore need to verify that the map is indeed well-defined and that it is a contractive mapping on \(X_T(a)\). For the sake of presentation, we split the proof into two steps.

**Step i (The operator \(\Lambda\) is well-defined on \(X_T(a)\)).** Indeed, \(\Lambda \alpha^1 \in X_T(a)\). We first prove that for given \(\alpha^1 \in X_T(a)\), then \(\|\Lambda \alpha^1\|_{L^2_t Y} \leq a\). So, we apply the triangle inequality, and Minkowski’s inequality to (5.7). Then utilizing the estimate (5.17) (with \(\alpha^n = \alpha^1\) and \(w^n = w^1\)) together with \(H(MNK)\) and \(H(\tilde{G})(iii)\). This yields

\[
\|\Lambda \alpha^1(t)\|_Y \leq \|\alpha_0\|_Y^1 + \int_0^t \|\tilde{G}(s, \alpha^1(s), M w^1(s))\|_Y \, ds \\
\leq \|\alpha_0\|_Y + \int_0^t \left[ L_{\tilde{G}} \|\alpha(t)\|_Y + L_{\tilde{G}} \|M\| \|w^1(s)\|_V + \|\tilde{G}(s, 0, 0)\|_Y \right] \, ds
\]

for a.e. \(t \in (0, T)\). Now, Hölder’s inequality and the Cauchy-Schwarz inequality gives

\[
\|\Lambda \alpha^1(t)\|_Y \leq \|\alpha_0\|_Y + L_{\tilde{G}} T \|\alpha^1\|_{L^2_t Y} + L_{\tilde{G}} \|M\| T \|w^1\|_{L^2_t Y} \\
+ T \|\tilde{G}(\cdot, 0, 0)\|_{L^2_t Y}.
\]

From Young’s inequality and the estimate (5.6), we see

\[
\|\Lambda \alpha^1(t)\|_Y^2 \leq c(1 + \|\alpha^1\|_{L^2_t Y}) \quad \text{for all } t \in [0, T],
\]

where \(c = c(T, \ldots)\). Choosing \(a\) such that it provides the desired upper bound concludes this part.

It remains to show that \(\Lambda \alpha^1(t)\) is continuous in \(Y\) for a.e. \(t \in (0, T)\) for \(\alpha^1 \in X_T(a)\).
given. Let \( t, t' \in [0, T] \), with no loss of generality, we assume that \( t < t' \). Then, Minkowski’s inequality, the estimate (5.17) (with \( \alpha^n = \alpha^1 \) and \( w^n = w^1 \)), Hölder’s inequality, the Cauchy-Schwarz inequality, \( H(MN K) \), and \( H(G)(iii) \) yields

\[
\|\Lambda \alpha^1(t') - \Lambda \alpha^1(t)\|_Y \\
\leq \int_t^{t'} \|G(s, \alpha^1(s), Mw^1(s))\|_Y ds \\
\leq \int_t^{t'} \left[ LG\| \alpha^1(s)\|_Y + LG\| M\|\|w^1(s)\|_V + \|G(s,0,0)\|_Y \right] ds \\
\leq LG|t'| - t|\| \alpha^1\|_{L^\infty(t',t;Y)} + LG|t' - t|^{1/2}\|M\|\|w^1\|_{L^2(t',t;V)} + |t' - t|\|G(\cdot,0,0)\|_{L^\infty(t',t;Y)}.
\]

Next, combining \( \|w^1\|_{L^2(t',t;V)} \leq \|w^1\|_{L^3_Y} \), and the estimate (5.6) yields

\[
\|\Lambda \alpha^1(t') - \Lambda \alpha^1(t)\|_Y^2 \leq |t' - t|^{1/2}LG\| M\|c(1 + \|\alpha_0\|_V^2 + \|w_0\|_{L^2}^2) + LG|t' - t|\|\alpha^1\|_{L^\infty(t',t;Y)} \\
+ |t' - t|\|G(\cdot,0,0)\|_{L^\infty(t',t;Y)}.
\]

Passing the limit \( |t' - t| \to 0 \), we conclude that indeed \( \Lambda \alpha^1 \in X_T(a) \).

**Step ii** (The application \( \Lambda : X_T(a) \to X_T(a) \) is a contractive mapping). Let \( \alpha^1 \in X_T(a) \), \( i = 1, 2 \), and let \( w^i \in W^{1,2}_T \) be a unique solution to (5.3a)-(5.3c). We define a norm equivalent to the norm on \( C([0,T];Y) \):

\[
\|\alpha^1\|_{C([0,T];Y)^*} = \max_{s \in [0,T]} e^{-\gamma s}\|\alpha^1(s)\|_Y
\]

where \( \gamma > 0 \) is chosen later. Then, from (5.7), we have:

\[
e^{-\gamma t}\|\Lambda \alpha^1(t) - \Lambda \alpha^2(t)\|_Y \leq LG e^{-\gamma t} \int_0^t e^{\gamma s} e^{-\gamma s}\|\alpha^1(s) - \alpha^2(s)\|_Y ds \\
\leq LG e^{-\gamma t}\| \alpha^1 - \alpha^2\|_{C([0,T];Y)^*} \int_0^t e^{\gamma s} ds \\
\leq \frac{LG}{\gamma}\| \alpha^1 - \alpha^2\|_{C([0,T];Y)^*}.
\]

Choosing \( \gamma > LG \) implies that \( \Lambda \) is a contraction on \( X_T(a) \), and thus we may conclude by the Banach fixed-point theorem that \( \alpha^1 \in X_T(a) \) is a unique fixed point to (5.7).

\[ \square \]

**APPENDIX D. PROOF OF LEMMA 6.7**

**Proof of Lemma 6.7.** First, the assumption \( H(A) \) hold by the hypothesis \( H(A) \) with \( m_A = m_A, a_0 = 0, \) and \( a_1 = L_A \) (see, e.g., [25, p.273]). Secondly, \( H(G) \) holds by \( H(G) \) with \( L_G = L_G \). Next, we have that \( R = \{c_R = LG + \|C\|_{L^\infty_Y L^\infty(\Omega,\mathcal{E})} \) by \( H(C) \) and \( H(B)(ii) \), see, e.g., [25, p.275]. Moreover, it follows directly from \( H(p)(iv) \) and properties of the trace operator, that \( H(S)(i) \) holds with \( c_S = L_p\|\gamma w\|_{L^p_Y} \). Next, hypothesis \( H(R)(ii) \) is found by inserting \( w = 0 \) in (6.8b) and using \( H(B)(ii),(iii) \) and \( H(C) \). Hypothesis \( H(S)(ii) \) holds by \( H(p)(iii) \). Additionally, (3.2)-(3.3) holds from (6.3)-(6.4), (6.7), and a duality argument.

The verification of \( H(\varphi) \) require some work. Firstly, \( H(\varphi)(i) \) holds as (6.8g) is independent of \( t \). Next, we show that \( \varphi(t, \cdot, \cdot, \cdot, \hat{v}) \) is continuous on \( Y \times Y \times U \) for all \( \hat{v} = (v^{(1)}, v^{(2)}) \in Z, \) a.e. \( t \in (0,T) \). This will ensure that \( H(\varphi)(ii) \) holds. Let \( (z_0, y_0, \hat{w}_0) \in Y \times Y \times U, \) and let \( \{(z^n, y^n, \hat{w}^n)\}_{n \geq 0} \subset Y \times Y \times U \) such that \( (z^n, y^n, \hat{w}^n) \to (z_0, y_0, \hat{w}_0) \) in
\[ Y \times Y \times U. \] For simplicity in notation, we define \( \Sigma = Y \times Y \times U \) and equip \( \Sigma \) with the norm \( \| (z, y, \tilde{w}) \|_\Sigma = \|z\|_Y + \|y\|_Y + \|\tilde{w}\|_U \). Observing that
\[
|\varphi(t, z^n, y^n, \tilde{w}^n, \tilde{v}) - \varphi(t, z_0, y_0, \tilde{w}_0, \tilde{v})| \\
\leq \int_{\Gamma_C} |z^n - z_0| |v^{(2)}| \, da + \int_{\Gamma_C} \mu(|\tilde{w}^n|, |y^n|) z^n |v^{(1)}| \, da - \int_{\Gamma_C} \mu(|\tilde{w}_0|, |y_0|) z_0 |v^{(1)}| \, da.
\]
For simplicity in notation, let
\[
\mu^n = \mu(|\tilde{w}^n|, y^n), \quad \mu_0 = \mu(|\tilde{w}_0|, y_0),
\]
then
\[
|\varphi(t, z^n, y^n, \tilde{w}^n, \tilde{v}) - \varphi(t, z_0, y_0, \tilde{w}_0, \tilde{v})| \\
\leq \int_{\Gamma_C} |z^n - z_0| |v^{(2)}| \, da + \int_{\Gamma_C} \mu^n |z^n| - |z_0| |v^{(1)}| \, da + \int_{\Gamma_C} |\mu^n - \mu_0| |z_0(t)| |v^{(1)}| \, da.
\]
By the triangle inequality, \( H(p)(iv) \), and \( H(\mu)(ii),(iv) \), and the Cauchy-Schwarz inequality, we have:
\[
|\varphi(t, z^n, y^n, \tilde{w}^n, \tilde{v}) - \varphi(t, z_0, y_0, \tilde{w}_0, \tilde{v})| \\
\leq \max\{1, \mu^*\} |z^n - z_0| \|v\|_Z + p^* L_\mu |\|y^n\|_Y + |\|\tilde{w}^n - \tilde{w}_0\|_U| |v^{(1)}|_U \\
\leq \tilde{c}|(z^n, y^n, \tilde{w}^n) - (z_0, y_0, \tilde{w}_0)| \|\tilde{v}\|_Z
\]
for all \( \tilde{v} \in Z \), a.e. \( t \in (0, T) \), where \( \tilde{c} = \max\{1, \mu^*, p^* L_\mu\} > 0 \). This implies that \( \varphi(t, \cdot, \cdot, \cdot, \tilde{v}) \) is continuous on \( Y \times Y \times U \) for all \( \tilde{v} \in Z \), a.e. \( t \in (0, T) \).

To verify \( H(\varphi)(iii) \), we first note that a continuous function is lower semicontinuous. So, it suffices to show that \( \varphi(t, z, y, \tilde{w}, \cdot) \) is Lipschitz continuous on \( Z \) for all \( z, y \in Y, \tilde{w} \in U \), a.e. \( t \in (0, T) \). Let \( \tilde{v}_i = (v_i^{(1)}, v_i^{(2)}) \in Z \), \( i = 1, 2 \), then \( H(p)(iv), H(\mu)(iv) \), and the Cauchy-Schwarz inequality implies
\[
(D.1) \quad |\varphi(t, z, y, \tilde{w}, \tilde{v}_1) - \varphi(t, z, y, \tilde{w}, \tilde{v}_2)| \\
\leq \int_{\Gamma_C} |z||v_1^{(2)} - v_2^{(2)}| \, da + \mu^* \int_{\Gamma_C} |z||v_1^{(1)} - v_2^{(1)}| \, da \leq \max\{1, \mu^*\} \|y\|_Y |\tilde{v}_1 - \tilde{v}_2|_Z
\]
for a.e. \( t \in (0, T) \). Convexity in the last argument of \( \varphi \) is obtained by the triangle inequality and the linearity of the integral. Thus, we have that \( H(\varphi)(iii) \) holds.

We next prove that \( H(\varphi)(iv) \) holds. Take \( \zeta \in \partial_L \varphi(t, z, y, \tilde{w}, \tilde{v}) \), and choose \( Z \ni q = \tilde{v}_1 - \tilde{v}_2 \) in \( (D.1) \). Thus by duality gives
\[
\|\zeta(t)\|_{Z^*} = \sup_{q \in Z} \langle \zeta(t), q \rangle_{Z \times Z} \leq \max\{1, \mu^*\} \|z\|_Y
\]
for a.e. \( t \in (0, T) \). So, \( H(\varphi)(iv) \) holds with \( c_1 \varphi = \max\{1, \mu^*\} \), and \( c_0 \varphi = c_2 \varphi = c_3 \varphi = c_4 \varphi = 0 \). We lastly prove that \( H(\varphi)(v) \) holds. Let \( z_i, y_i \in Y, \tilde{w}_i \in U, \tilde{v}_i = (v_i^{(1)}, v_i^{(2)}) \in Z \), for \( i = 1, 2 \). Then
\[
\varphi(t, z_1, y_1, \tilde{w}_1, \tilde{v}_1) - \varphi(t, z_2, y_2, \tilde{w}_2, \tilde{v}_2) = \int_{\Gamma_C} \left[ z_1 - z_2 \right] \left[ v_2^{(2)} - v_1^{(2)} \right] \, da + \int_{\Gamma_C} \mu(|\tilde{w}_1|, |y_1|) z_1 \left[ |v_2^{(1)}| - |v_1^{(1)}| \right] \, da \\
+ \int_{\Gamma_C} \mu(|\tilde{w}_2|, |y_2|) z_2 \left[ |v_1^{(1)}| - |v_2^{(1)}| \right] \, da
\]
For simplicity, we define:
\[
\mu_1 = \mu(|\tilde{w}_1|, |y_1|), \quad \mu_2 = \mu(|\tilde{w}_2|, |y_2|).
\]
We use the triangle inequality, \( H(\mu) (\text{ii}, \text{iv}) \), \( H(p) (\text{iv}) \), and the Cauchy-Schwarz inequality to get

\[
\varphi(t, z_1, y_1, \tilde{w}_1, \tilde{v}_2) - \varphi(t, z_1, y_1, \tilde{w}_1, \tilde{v}_1) + \varphi(t, z_2, y_2, \tilde{w}_2, \tilde{v}_1) - \varphi(t, z_2, y_2, \tilde{w}_2, \tilde{v}_2) \\
\leq \|z_1 - z_2\|_{Y} \|v_1^{(2)} - v_2^{(2)}\|_X + \mu^* \|z_1 - z_2\|_{Y} \|v_1^{(1)} - v_2^{(1)}\|_U \\
+ p^* L_\mu \|y_1 - y_2\|_{Y} + \|\tilde{w}_1 - \tilde{w}_2\|_U \|v_1^{(1)} - v_2^{(1)}\|_U \\
\leq \left[ \max \{1, \mu^*\} \right]\|z_1 - z_2\|_{Y} + p^* L_\mu \|y_1 - y_2\|_{Y} + p^* L_\mu \|\tilde{w}_1 - \tilde{w}_2\|_U \|\tilde{v}_1 - \tilde{v}_2\|_Z,
\]

with \( \beta_{1\varphi} = \max \{1, \mu^*\} \), \( \beta_{2\varphi} = p^* L_\mu \), and \( \beta_{3\varphi} = p^* L_\mu \).

\[\square\]

**APPENDIX E. PROOF OF LEMMA 6.12**

*Proof of Lemma 6.12.* We will prove that \( \varphi \) defined by (6.18a) satisfies \( H(\varphi) \). Firstly, \( H(\varphi) (\text{ii}) \) holds as (6.18a) is independent of \( t \). We claim that \( H(\varphi) (\text{ii}) \) holds. Indeed, \( \varphi(t, \cdot, \cdot, \tilde{v}) \) is continuous on \( Y \times U \) for all \( \tilde{v} \in Z \), a.e. \( t \in (0, T) \). Let \( \{(y^n, \tilde{w}^n)\}_{n \geq 0} \subset Y \times U \) such that \( (y^n, \tilde{w}^n) \to (y_0, \tilde{w}_0) \) strongly in \( Y \times U \). Then, by \( H(\mu) (\text{ii}) \), and the Cauchy-Schwarz inequality, we have:

\[
|\varphi(t, y^n, \tilde{w}^n, \tilde{v}) - \varphi(t, y_0, \tilde{w}_0, \tilde{v})| \leq L_\mu \|(y^n, \tilde{w}^n) - (y_0, \tilde{w}_0)\|_{Y \times U} \|\tilde{v}\|_Z
\]

for a.e. \( t \in (0, T) \), retrieving \( H(\varphi) (\text{ii}) \). Moreover, convexity in the last argument of \( \varphi \) follows by the triangle inequality and the linearity of the integral. Additionally, a continuous function is lower semicontinuous. Thus, we will show that \( \varphi \) is Lipschitz continuous in its last argument. Hypothesis \( H(\mu) (\text{iv}) \) and the Cauchy-Schwarz inequality gives

\[
|\varphi(t, y, \tilde{w}, \tilde{v}_1) - \varphi(t, y, \tilde{w}, \tilde{v}_2)| = \int_{\Gamma_C} \mu(\tilde{w}, y) \left[ |\tilde{v}_1| - |\tilde{v}_2| \right] da
\]

for \( y \in Y, \tilde{w} \in U, \tilde{v}_i \in Z, i = 1, 2 \), a.e. \( t \in (0, T) \). This proves \( H(\varphi) (\text{iii}) \). Next, take \( \zeta \in \partial_c \varphi(t, y, \tilde{w}, \tilde{v}) \), for \( y \in Y, \tilde{w} \in U, \tilde{v} \in Z \), for \( i = 1, 2, \) a.e. \( t \in (0, T) \). We use the estimate (E.1) and take \( Z \ni q = \tilde{v}_1 - \tilde{v}_2 \). Then, by duality,

\[
\|\zeta(t)\|_{Z^*} = \sup_{\|q\|_Z = 1, q \in Z} \langle \zeta(t), q \rangle_{Z^* \times Z} \leq \mu^* \sqrt{\text{meas}(\Gamma_C)} \quad \text{for a.e. } t \in (0, T).
\]

So, \( H(\varphi) (\text{iv}) \) follows for \( c_{0\varphi} = \mu^* \sqrt{\text{meas}(\Gamma_C)} \), and \( c_{1\varphi} = c_{2\varphi} = c_{3\varphi} = c_{4\varphi} = 0 \). Lastly, investigating \( H(\varphi) (\text{v}) \). Hypothesis \( H(\mu) (\text{ii}) \) and the Cauchy-Schwarz inequality yields

\[
\varphi(t, y_1, \tilde{w}_1, \tilde{v}_2) - \varphi(t, y_1, \tilde{w}_1, \tilde{v}_1) + \varphi(t, y_2, \tilde{w}_2, \tilde{v}_1) - \varphi(t, y_2, \tilde{w}_2, \tilde{v}_2)
\]

\[
= \int_{\Gamma_C} \left[ \mu(\tilde{w}_1, y_1) - \mu(\tilde{w}_2, y_2) \right] \left[ |\tilde{v}_2| - |\tilde{v}_1| \right] da
\]

\[
\leq \int_{\Gamma_C} L_\mu |\tilde{w}_1 - \tilde{w}_2| |\tilde{v}_2 - \tilde{v}_1| da + \int_{\Gamma_C} L_\mu |y_1 - y_2| |\tilde{v}_2 - \tilde{v}_1| da
\]

\[
\leq L_\mu |y_1 - y_2| |\tilde{v}_1 - \tilde{v}_2| Z + L_\mu |\tilde{w}_1 - \tilde{w}_2| U |\tilde{v}_1 - \tilde{v}_2| Z
\]

for all \( y_i \in Y, \tilde{w}_i \in U, \tilde{v}_i \in Z \), for \( i = 1, 2 \), a.e. \( t \in (0, T) \). We set \( \beta_{2\varphi} = \beta_{3\varphi} = L_\mu \) and \( \beta_{1\varphi} = 0 \), concluding this proof.

\[\square\]