Non integrability of a self-gravitating Riemann liquid ellipsoid

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Abstract

We prove that the motion of a triaxial Riemann ellipsoid of homogeneous liquid without angular momentum does not possess an additional first integral which is meromorphic in position, impulsions, and the elliptic functions which appear in the potential, and thus is not integrable. We prove moreover that this system is not integrable even on a fixed energy level hypersurface.

Keywords: Morales-Ramis theory, Elliptic functions, Monodromy, Differential Galois theory, Riemann surfaces

1. Introduction

We want to study the integrability of the following Hamiltonian

\[ H = r \left( p_1^2 + \frac{q_1^2 q_2^2}{q_1^2 + r} \right) + \int_0^\infty \frac{\alpha}{\sqrt{\left( z + \frac{4}{q_2^2} \right) \left( z^2 + rz + \frac{q_2^4}{4} \right)}} dz \]

with \( r = \sqrt{q_1^2 + q_2^2} \) (a notation that will always be used in the following). This Hamiltonian appears through the Hamiltonian formulation of the motion of a self gravitating triaxial Riemann ellipsoid of homogeneous liquid, in which we have restricted our study to the zero angular momentum case. This equation can be found in [1]. The motion of Riemann ellipsoids have been studied for a long time, as for example in [2],[3]. Even the integrability of the Hamiltonian

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$H$ has been studied by Ziglin in [4]. This corresponds in fact to the limiting case when we take an energy level which tends to infinity. However, this does not prove “meromorphic non integrability” of $H$ in a reasonable sense (nor gives a reasonable notion of “meromorphic non integrability” for $H$, as $H$ is not meromorphic in $p, q$ itself). This is because the first integral should have a “limit” when energy tend to infinity, and that this limit is meromorphic. It would have a “limit” if the first integral be rational in $p_1, p_2$, but even in this case, the question of the meromorphic dependence in the positions $q_1, q_2$ would stay open. As the case $\alpha = 0$ has been already treated in [4], one will only consider the case $\alpha \neq 0$, which comes down after variable change to $\alpha = 1$.

The Hamiltonian $H$ has two degrees of freedom, and there is an elliptic function in the potential. Because of this, one should precise the notion of integrability we want to study before any non integrability proof. Meromorphic functions is clearly not a large enough class because it does not contain the Hamiltonian itself. So, we will not use directly Morales Ramis theory, but begin with definitions of what could be a reasonable meaning of integrability of $H$. Remark that this Hamiltonian is in fact well defined on a Riemann surface because of the elliptic integral, which is multivalued for $(q_1, q_2) \in \mathbb{C}^2$. This is here the main difficulty because contrary to algebraic functions, this transcendental extension cannot be suppressed in some additional first integral only by algebraic transformations. So, a little expansion of Morales Ramis theory will be needed to deal with such Hamiltonians.

**Definition 1.** We pose

$$J(q_1, q_2) = \int_0^\infty \frac{1}{\sqrt{(z + \frac{4}{q_2}) \left( z^2 + rz + \frac{q_2^2}{4} \right)}} dz$$

and we will note $\Sigma(J)$ the singular set of $J$, which is

$$\Sigma(J) = \{(q_1, q_2) \in \mathbb{C}^2, \ J \text{ is not } C^\infty \text{ on } (q_1, q_2)\}$$

**Definition 2.** We pose

$$K = \{g(q_1, q_2, p_1, p_2, r, J, \partial_{q_1} J, \partial_{q_2} J), \ g \text{ meromorphic}\}$$

the (differential) field of meromorphic functions in $p, q, r, J$. Moreover, we will say
• that $I \in K$ is a meromorphic first integral of $H$ if $I$ is constant along any orbit.

• that $I \in K$ is a meromorphic first integral on the level $H = E$ if $I$ is constant along any orbit with energy $H = E$.

• that $I \in K$ is an additional first integral of $H$, or only on $H = E$, if $I$ is functionally independent with $H$, or non constant on $H = E$ respectively.

Remark 1. Remark that the field of meromorphic functions $K$ is a differential field, and contains the Hamiltonian. Remark also that even if a first integral on a level $H = E$ is defined a priori everywhere according to Definition 2, in fact we only care about its restriction on $H = E$.

The “tool” we will use to prove non integrability is the Morales Ramis theory in [5],[6].

Theorem 1. (Morales Ramis [4] Theorem 4.1. page 85) Let $H$ be a Hamiltonian, $\Gamma$ a particular (not a point) orbit and suppose $H$ meromorphic on a neighbourhood of $\Gamma$. If $H$ possess a complete system of first integrals in involution, functionally independent and meromorphic on a neibourhood of $\Gamma$, then the identity component of the Galois group of variational equation is abelian at any order.

If the Hamiltonian satisfy good properties (as being meromorphic along the curve $\Gamma$), then a meromorphic first integral of $H$ will produce a meromorphic initial form on $\Gamma$ and then a constraint on the Galois group. This is a very general way to prove non integrability: If a Hamiltonian is integrable in some sense, then the Galois group of the variational equation near a particular orbit should satisfy some particular property. In their article, they consider meromorphic Hamiltonians and first integrals, and here we want to add elliptic functions, and prove that Morales Ramis approach is still valid in this case.

The main theorem of this article is the following.

Theorem 2. The Hamiltonian $H$ does not possess a first integral in $K$ functionally independent with the Hamiltonian $H$. On a fixed energy hypersurface
$H = E$, the Hamiltonian system restricted to this hypersurface does not possess a non constant first integral in $K$.

The plan of the proof is the following

- First we study smoothness of possible first integrals in $K$, in particular singular points of elliptic functions involved.

- We compute a family of particular solutions corresponding to the case where the ellipsoid of fluid is invariant by rotation. A closed form solution can be given in this case, and we analyze (complex) singularities of these solutions.

- We compute the variational equation near these orbits and prove a small extension of Morales Ramis for proving the non existence of an additional first integral in $K$.

- As the variational equation has transcendental functions in its coefficients, Kovacic algorithm cannot be used directly. Instead, we prove using an analysis of the monodromy that some second order differential equation related to the variational equation should have a Galois group whose identity component is solvable (which is a weaker property than the classical virtual abelianity condition from Morales Ramis).

By chance, even if the Hamiltonian has elliptic functions, in the computation we will never see anything “worse” than that. The difficulty is that computing Galois group with transcendental functions in the base field can be much more tricky than in the rational base field case.

2. Variational equation

2.1. Regularity of function $J$

We want to consider first integrals that will involve the function $J$, so we will need to know where they are meromorphic in a neighbourhood of $q_1 = 0$, which is the plane we will be interested in. We get the following lemma.

Lemma 3. We have

$$\Sigma(J) \cap \{q_1 = 0\} \subset \{(0,0), \left(0, 2e^{\frac{i\pi}{3}}\right), \ldots, \left(0, 2e^{\frac{5i\pi}{3}}\right)\}.$$
PROOF. First we remark that $J$ is an elliptic integral, and so is smooth in its parameters for values of the parameters for which the corresponding lattice does not collapse. This lattice collapses when the associated elliptic curve becomes singular, and it corresponds to the case where the polynomial of degree 3 under the square root has a double root. Here we have

$$\left. \left( z + \frac{4}{q_2^2} \right) \left( z^2 + rz + \frac{1}{4} q_2^2 \right) \right|_{q_1=0} = \frac{1}{4q_2^2} (zq_2^2 + 4)(2z \pm q_2)^2$$

So we cannot conclude directly. The function $J$ can be written using classical elliptic integrals

$$J(q_1, q_2) = \frac{\text{EllipticK}\left( \sqrt{-\frac{q_1}{q_2} + \frac{4}{q_2^2}} \right) - \text{EllipticF}\left( \sqrt{\frac{8}{q_2^2(r-q_1)}}, \sqrt{-\frac{q_1}{q_2} + \frac{4}{q_2^2}} \right)}{\sqrt{-\frac{4}{q_2^2} + \frac{r}{2} + \frac{q_1}{2}}}$$

The denominator on $q_1 = 0$ is equal to

$$\sqrt{\pm q_2^3 + 8}$$

and so is smooth outside the seven points we exclude in Lemma 3. Furthermore, the functions EllipticK$(z)$, EllipticF$(w,z)$ are even in $z$, and the second is smooth except for $w = \pm 1$. So the part in EllipticK is always smooth (outside the exceptional points), and the function EllipticF is smooth outside maybe the points such

$$\frac{8}{q_2^2(r-q_1)} = 1$$

For $q_1 = 0$, this constraint comes down to $q_2^3 = \pm 8$, the same constraint as before. So

$$\Sigma(J) \cap \{q_1 = 0\} \subset \{(0, 0), (0, 2e^{\frac{i\pi}{3}}), \ldots, (0, 2e^{\frac{5i\pi}{3}})\}$$

2.2. The plane $(p_1, q_1) = (0, 0)$

To study integrability, we will need to look at a particular (explicit) solution. The solution we will study corresponds in fact to the case where the ellipsoid of fluid is invariant by rotation.
Theorem 4. The plane \((p_1, q_1) = (0, 0)\) is an invariant plane. On this plane, the Hamiltonian can be written

\[ R(p_2, q_2) = \frac{q_2^5 p_2^2}{q_2^4 + q_2} + \sqrt{2q_2} \frac{\arccos \left( \frac{2\sqrt{2}q_2^{-3/2}}{\sqrt{q_2^3 - 8}} \right)}{\sqrt{q_2^3 - 8}} \]

This function is well defined on a (transcendental) Riemann surface outside its singularities which correspond to

\[ q_2 = 0, -1, -e^{2\pi i/3}, -e^{4\pi i/3}, 2, 2e^{2\pi i/3}, 2e^{4\pi i/3} \]

On each level \(R = E\), there is only one orbit except for the energy values such that

\[ 2\sqrt{2} - x^{3/2} \cosh \left( \frac{6\sqrt{2\sqrt{8} - x^3}}{x^3 + 16} \right) = 0 \text{ and } E = \frac{12x}{x^3 + 16} \]

for which there is at least one fixed point and at most finitely many on the Riemann surface associated to \(R\).

Proof. The function \(R\) can be found by direct computation. There is a choice of valuation for the square root \(r\), and when we take \(q_1 = 0\) this leads to terms of the type \(\sqrt{q_2^4}\). We can choose in principle any of the two valuations (but we need to choose always the same in all the computations), and here we will choose \(\sqrt{q_2^3} = q_2\). The fact that the elliptic integral can be now written with an arccosinus is not surprising because we already found in Lemma 3 that the corresponding elliptic curve becomes singular for \(q_1 = 0\).

Now this function \(R\) defines a one degree of freedom Hamiltonian, and so the orbit can be completely studied analyzing the levels of \(R\). A critical point always corresponds to \(p_2 = 0\). We derive \(R\) in \(q_2\) for \(p_2 = 0\)

\[- \frac{i\sqrt{2}(q_2^3 + 16) \arccos \left( \frac{2\sqrt{2}q_2^{-3/2}}{2(8 - q_2^3)^{3/2}} \right)}{2(8 - q_2^3)^{3/2}} - \frac{6}{8 - q_2^3} \tag{1} \]

Then, substituting the arccos term using the equation \(R = E\), we find

\[ E = \frac{12q_2}{q_2^3 + 16} \]
This gives the equation of Theorem 4. Now we check that such a critical point cannot be critical for 2 sheaves of the Riemann surface associated to $R$. If it was the case, then the equation (1) would vanish for at least two valuations of the function $\arccos$. Then we would have $q_3^2 + 16 = 0$. We check that these points are not critical points. So there are at most 3 critical points.

We have in particular the level $E = 1$ with $q_2 = 2$, which is an already known real fixed point, the others being complex. Typically, these critical levels lead to confluences in the variational equation and so have fewer conditions for integrability. However, we will see that in our case, this will have no impact on the non integrability proof.

2.3. Normal variational equation

**Theorem 5.** The normal variational equation near a (non constant) orbit in the plane $(p_1, q_1) = 0$ with $R = E$ is given by

$$
((4t^{11} + 192t^5 - 60t^8 + 256t^2)s - 30t^7E + 96t^4E + 2t^{10}E + 128tE)\dot{X} + ((75t^7 - 640t - 7t^{10} - 72t^4)s - 384E + 42t^4 - 96Et^3 + 48t + 42t^6E - 3Et^9 - 6t^7)\dot{X} + ((-64t^8 + 2t^{11})s - 64t^5 - 4t^8)X = 0
$$

with

$$s = \frac{i \arccos\left(\frac{2\sqrt{2}t^{-3/2}}{\sqrt{2\sqrt{8} - t^3}}\right)}$$

**Proof.** We begin by direct computations of the Hessian matrix of $H$ for $(p_1, q_1) = (0, 0)$. This works in all cases except for $\partial_{q_1}H, \partial_{q_2}H$. We have in particular a variational equation of the form

$$\dot{Y} = \begin{pmatrix}
0 & -F_1 & 0 & 0 \\
2q_2 & 0 & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{pmatrix} Y$$

So the variational equation is already decoupled, and the normal part corresponds to the first $2 \times 2$ block. We have noted here $F_1 = \partial_{q_1}H$. Let us look
now closer to this function. The kinetic part after derivation disappear for 

\((p_1, q_1) = (0, 0)\). So the only thing to compute is

\[
\partial_{q_1} J(q_1, q_2)|_{q_1=0} = \int_0^\infty \frac{dz}{\sqrt{z q_2^2 + 4(q_2 + 2z)^3}}
\]

This integral can be computed without elliptic functions, only using the

\[\text{arccos} \left( \frac{2 \sqrt{2} q_2^{-3/2}}{8 - q_2^3} \right)\]

We now transform the normal variational equation in a second order differ-
ential equation. This gives

\[
\frac{1}{2} X\phi - \frac{1}{2} X\dot{\phi} = -\phi^2 F_1(\phi) X
\]

where \((p_2(t), q_2(t)) = (\dot{\phi}(t), \phi(t))\) is an orbit of \(H\) in the plane \((p_1, q_1) = 0\).

We now make the variable change \(\phi(t) \rightarrow t\). For this, we use the fact that \(R\) is constant over an orbit and we get the relation

\[
\dot{\phi}^2 = \frac{i \sqrt{2} (\phi^3 + 1) \arccos \left( \frac{2 \sqrt{2} \phi^{-3/2}}{\phi^3 \sqrt{8 - \phi^3}} \right) + E \phi^3 + 1}{\phi^4}
\]

and also a relation for \(\ddot{\phi}\) by derivating it. The variable change \(\phi(t) \rightarrow t\)

3. Non-integrability

**Theorem 6.** If \(H\) has an additional meromorphic first integral, then the
Galois group over the base (differential) field \( \tilde{K} \) generated by meromorphic
functions in

\[
\sqrt{t}, \sqrt{8 - t^3}, \arccos \left( \frac{2 \sqrt{2} t^{-3/2}}{2} \right)^{\text{for } t(t^3 - 8)(t^3 + 1) \neq 0}
\]

of equation (2) is virtually abelian for any \(E \in \mathbb{C}\). If \(H\) has an additional
meromorphic first integral on some fixed energy level \(H = E\), then the Galois

3. Non-integrability

**Theorem 6.** If \(H\) has an additional meromorphic first integral, then the
Galois group over \( \tilde{K} \) of equation (2) is virtually abelian for the corresponding value
of \(E\).
Proof. Such a theorem is in fact a small extension of the Morales Ramis Theorem \[1\]. Suppose there is an additional meromorphic first integral \(I\) (in the sense of Definition \[2\]). We consider the curve given by

\[ \Gamma_E : \, p_1 = 0, \, q_1 = 0, \, R(p_2, q_2) = E, \, (0, q_2) \notin \Sigma(H) \]

excluding the possible critical points of \(R\). The points \((0, q_2) \in \Sigma(H)\) correspond to singular points of the Hamiltonian, and thus we exclude them (there are finitely many because of Lemma \[3\]). Remark that as proven in Theorem \[4\] the curve \(\Gamma_E\) always exists (it is never only a point), there is a unique choice for \(\Gamma_E\) in \(R(p_2, q_2) = E\) and there is a discrete set of points removed from \(\Gamma_E\) (recall that \(\Gamma_E\) is in the complex a Riemann surface, so points of \(\Sigma(H)\) can correspond to infinitely many points on \(\Gamma_E\), one copy on each sheave). Let us prove that a first integral \(I \in K\) is meromorphic along \(\Gamma_E\). The first integral \(I\) can be written as a meromorphic function of \(p, q, r, J, \partial q J\). On \(\Gamma_E\), \(r\) has a singularity at \(q = 0\), but is is also a singularity of \(J\), and so is not in \(\Gamma_E\). The singularities of \(J\) are not in \(\Gamma_E\) by definition, and \(\partial q J\) are also elliptic functions whose singularities are at the same points as \(J\). So we can apply Theorem \[1\]. We now need to check that the base field given in Theorem \[6\] is the good one, meaning the field of meromorphic functions on \(\Gamma_E\). This corresponds to functions in \(p_2, q_2\), meromorphic on a neighbourhood of the curve \(R(p_2, q_2) = E\) outside the singular points, restricted to the curve \(R(p_2, q_2) = E\). Using the same parametrization as in Theorem \[5\] we express \(p_2\) in function of \(q_2 = t\) and this gives

\[ p_2 = \sqrt{\frac{i \sqrt{2} (t^3 + 1) \arccos \left( \frac{2 \sqrt{2} t^{-3/2}}{t^3 \sqrt{8 - t^3}} \right) + E \frac{t^3 + 1}{t^4}} } \]

After simplification, we see that the corresponding field is an algebraic extension of degree 2 of \(\tilde{K}\), and so the identity component of the Galois group of equation \[2\] will not change if we take \(\tilde{K}\) instead of the field as base field.

The case of fixed energy is proven by reducing the dynamical system to the hypersurface \(H = E\). We loose (at least a priori) the Hamiltonian structure and the notion of involution, but it was not needed because we are in two degrees of freedom (a presentation of the problem arising with more degree of freedom is done in \[7\]). The normal variational equation of this 3 dimensional system is still equation \[2\], and the previous computations are still valid (we already proved that there are no “exceptional” energy levels, meaning that the curve \(\Gamma_E\) always exists and is never too singular).
Remark 2. Remark that the field \( \bar{K} \) corresponds also to the field of meromorphic functions on the Riemann surface
\[
\mathcal{S} = \{(x, y) \in \mathbb{C}^2, \ x^3 \cos^2 y = 8, \ x(x^3 - 8)(x^3 + 1) \neq 0\}
\]
(replacing \( t \) by \( x \) in the generators of \( \bar{K} \) produce meromorphic functions in \( x, y \), and reciprocally). We also have that the coefficients of equation \( (2) \) are meromorphic functions on \( \mathcal{S} \). In the following, we will keep this notation for the Riemann surface \( \mathcal{S} \).

Theorem 7. Suppose that equation \( (2) \) has a virtually abelian Galois group over the base field \( \bar{K} \) (for some fixed energy \( E \)). We have the following

i. The monodromy group \( G_1 = \{R_\gamma, \ \gamma \text{ closed curve on } \mathcal{S}\} \) (\( R_\gamma \) is the resolvant matrix of equation \( (2) \)) is such that
\[
\forall \ g \in \mathcal{D}^{(2)}(G_1), \ \ g^{60} = \text{id}
\]
where \( \mathcal{D}^{(2)}(G) \) is the second derivative of the group \( G \).

ii. The monodromy group \( G_2 = \{R_\gamma, \ \gamma \text{ closed curve on } \mathcal{S}\} \) where \( R_\gamma \) is the resolvant matrix of the equation
\[
(4t^{11} + 192t^5 - 60t^8 + 256t^2)\dot{X} + (75t^7 - 640t - 7t^{10} - 72t^4)\ddot{X} + (-64t^8 + 2t^{11})X = 0 \quad (3)
\]
is such that
\[
\forall \ g \in \mathcal{D}^{(2)}(G_2), \ \ g^{60} = \text{id}
\]

iii. The identity component of Galois group of \( (3) \) over the base field \( \mathbb{C}(t) \) is solvable.

Proof. Let us prove (i). Because the base field for Galois group computations is not \( \mathbb{C}(t) \), we cannot use Kovacic algorithm (at least not directly). But the classification of possible Galois group still holds. So the possible Galois groups are triangular groups, \( D_\infty \), finite groups \( D_n, A_4, S_4, A_5 \). Let us take one of these possible groups and derive it two times (the derivation of a group correspond to the group generated by its commutators). For triangular groups, \( D_\infty, D_n \), the second derivative produce the identity group. We also have
\[
\mathcal{D}^{(2)}(A_4) = \text{id} \quad \mathcal{D}^{(2)}(S_4) = \mathbb{Z}_2^2 \quad \mathcal{D}^{(2)}(A_5) = A_5
\]
So, in all cases, the elements of the second derivative of the Galois group have always an order dividing 60. We conclude using the fact that the monodromy group $G_1$ is always a subgroup of the Galois group.

Let us prove (ii). We consider the application

$$
\sigma : \tilde{K} \to \tilde{K} \quad \sigma \left( \arccos \left( \frac{2}{\sqrt{2}} t^{3/2} \right) \right) = \arccos \left( \frac{2}{\sqrt{2}} t^{3/2} \right) + 2\pi
$$

The application $\sigma$ corresponds in fact only a translation of sheaves $\left( x, y \right) \to \left( x, y + 2\pi \right)$ on the Riemann surface $S$. We now apply $\sigma^k$ on equation (2). This produces the equation

$$
\left( \left( 4t^{11} + 192t^5 - 60t^8 + 256t^2 \right) \left( s + \frac{2\pi ik}{\sqrt{2} \sqrt{8 - t^3}} \right) - 30t^7 E + 96t^4 E + 2t^{10} E + 128t E \right) \tilde{X} + \\
\left( \left( 75t^7 - 640t^7 - 7t^{10} - 72t^4 \right) \left( s + \frac{2\pi ik}{\sqrt{2} \sqrt{8 - t^3}} \right) \right) X = 0
$$

So equation (4) will also have a first integral with coefficients in $\tilde{K}$. We consider the polynomial

$$P = 4z^{11} + 192z^5 - 60z^8 + 256z^2$$

$\nu > 0, \epsilon > 0$ two real numbers and the compact

$$C_{\nu, \epsilon} = \left( \left( D(0, \nu) \setminus \left( \cup_{z \in P^{-1}(0)} D(z, \epsilon) \right) \right) \times D(0, \nu) \right) \cap S$$

where $D(z, \epsilon)$ is a disk with center $z$ of radius $\epsilon$. For $k > k_0$ large enough, the singularities of equation (2) are not in $C_{\nu, \epsilon}$, because for large $k$, the singularities of equation (2) inside $D(0, \nu)^2 \cap S$ are converging to the roots of $P$ and $z^3 - 8 = 0$ (and singular points of $S$ are also roots of $P$). Now let consider four paths $\gamma_1, \ldots, \gamma_4$ on $S$ outside the roots of $P$ and the commutator

$$[[R_{\gamma_1}, R_{\gamma_2}], [R_{\gamma_3}, R_{\gamma_4}]]^{60}$$

where $R_{\gamma}$ is the resolvant matrix of (4) along $\gamma$. There exist $\nu, \epsilon$ such that $\gamma_1, \ldots, \gamma_4 \subset C_{\nu, \epsilon}$. We divide (4) by its dominant term and we take the limit $k \to \infty$. The equation is then converging to the equation (3). The resolvant matrix $R$ is smooth when $k \to \infty$ on the compact set $C_{\nu, \epsilon}$. So, for
Large enough, the resolvant matrices $R$ are smooth along $\gamma_i$, and so they are converging to monodromy matrices along curves $\gamma_i$ of equation (3). So is the commutator (5), which is equal to $id$ by hypothesis. So

$$\forall g \in \mathcal{D}^{(2)}(G_2), \; g^{60} = id$$

Let us prove (iii). We consider the monodromy group

$$G_3 = \{ R_\gamma, \gamma \text{ closed curve on } \mathbb{C} \}$$

where $R_\gamma$ is the resolvant matrix of the equation (3). This monodromy group is a priori bigger than $G_2$, because of multivaluated functions in $\tilde{K}$. We have that

$$\frac{\partial}{\partial t} \left( \arccos \left( 2\sqrt{2t^{-3/2}} \right) \right) = \frac{3\sqrt{2}}{\sqrt{t^5(8-t^3)}} \Rightarrow \sigma(\{t\}, \tilde{K}) = D_\infty$$

Noting $F$ the Picard Vessiot field of equation (3) and $K_0$ the field of meromorphic functions on $\{ x \in \mathbb{C}, x(x^3 - 8)(x^3 + 1) \neq 0 \}$, we have a tower of fields $K_0 \subset \tilde{K} \subset F$ and monodromy groups

$$\sigma(K_0, \tilde{K}) = D_\infty \quad \sigma(\tilde{K}, F) = G_2 \quad \sigma(K_0, F) = G_3$$

So, to “kill” this $D_\infty$ that could appear in $G_3$, we just need to derivate one more time. We get then

$$\forall g \in \mathcal{D}^{(3)}(G_3), \; g^{120} = id$$

where we take the power 120 instead 60 because $\mathcal{D}(D_\infty) = \mathbb{Z}_2$, and so the additional derivation does not kill it completely. Analysis of singularities of equation (3) shows that all its singularities are regular, and then that the equation is Fuchsian. This implies that the Galois group of equation (3) over $\mathbb{C}(t)$ is the Zariski closure of the monodromy group $G_3$. Suppose now that the Galois group of equation (3) over $\mathbb{C}(t)$ has not its identity component solvable. Following Kovacic’s algorithm, the only possible groups contain $SL_2(\mathbb{C})$. So one would need that $G_3$ is Zariski dense in $SL_2(\mathbb{C})$. We have that

$$\mathcal{D}^{(3)}(SL_2(\mathbb{C})) = SL_2(\mathbb{C}) \quad \forall g \in SL_2(\mathbb{C}), \; \exists \tilde{g} \in SL_2(\mathbb{C}), \; \tilde{g}^{120} = g$$

The application commutator and elevating to power 120 are continuous (in the Zariski topology, rational fractions are continuous), so this would imply that $id$ is dense in $SL_2(\mathbb{C})$. Impossible.
Let us conclude by proving Theorem 2.

**Proof.** If $H$ has an additional first integral in $K$ (respectively on some particular energy level $H = E$), then using Theorem 6 we have that equation (2) has a virtually abelian Galois group over the base field $\tilde{K}$, and then with Theorem 7 that equation (3) has a Galois group whose identity component is solvable. We just apply Kovacic’s algorithm on equation (3), and find that its Galois group is $SL_2(\mathbb{C})$, and so its identity component is not solvable. This gives Theorem 2.

4. Conclusion

It is not so rare that a transcendental function appear in the variational equation we need to study to prove non integrability, but in general it is possible to avoid it by just taking a particular orbit for which this case does not occur, as done for example in [8]. But we see that in fact it probably produces even stronger integrability conditions. Such a study is not so much more difficult because through a limiting process, such an equation comes down to an equation with coefficients in $\mathbb{C}(t)$ (the difference solvable abelian does not change anything in practice). And moreover, it is only a necessary criterion, if it was met, we could produce other conditions by making an asymptotic expansion in the “multivaluation parameter” (which corresponds here to apply the translation of sheave $\sigma$). In the case where these transcendental extensions are not avoidable, this approach completely make sense because we do not have an equivalent of Kovacic algorithm for these. In the same problem of Riemann ellipsoid motion, zero angular momentum is only one case. A (probably) complete list of integrable cases is given in [1], and it could maybe be possible to prove the non integrability of the other cases using this approach.

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