INVERSE PARABOLIC PROBLEMS BY CARLEMAN ESTIMATES
WITH DATA TAKEN INITIAL OR FINAL TIME MOMENT OF
OBSERVATION

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Dedicated to the memory of Professor Victor Isakov

Abstract. We consider a parabolic equation in a bounded domain Ω over a time inter-
val (0, T) with the homogeneous Neumann boundary condition. We arbitrarily choose a
subboundary Γ ⊂ ∂Ω. Then, we discuss an inverse problem of determining a zeroth-order
spatially varying coefficient by extra data of solution u: u|Γ×(0,T) and u(·,t0) in Ω with
t0 = 0 or t = T. First we establish a conditional Lipschitz stability estimate as well as
the uniqueness for the case t0 = T. Second, under additional condition for Γ, we prove the
uniqueness for the case t0 = 0. The second result adjusts the uniqueness by M.V. Klibanov
(Inverse Problems 8 (1992) 575-596) to the inverse problem in a bounded domain Ω. We
modify his method which reduces the inverse parabolic problem to an inverse hyperbolic
problem, and so even for the inverse parabolic problem, we have to assume conditions for
the uniqueness for the corresponding inverse hyperbolic problem. Moreover we prove the
uniqueness for some inverse source problem for a parabolic equation for t0 = 0 without
boundary condition on the whole ∂Ω.

1. Introduction and main results

Let Ω be a bounded domain in \(\mathbb{R}^n\) with \(C^2\)-boundary. We fix a moment \(t_0 \in [0, T]\), and
introduce the elliptic operator

\[
A v(x) = \sum_{j,k=1}^{n} \partial_j (a_{jk}(x) \partial_k v(x)) - \sum_{j=1}^{n} b_j(x) \partial_j v(x) \quad \text{for } v \in H^2(\Omega).
\]

We consider

\[
\partial_t u = Au - p(x)u \quad \text{in } (0, T) \times \Omega, \quad (1.1)
\]

\[
\partial_{\nu_A} u|_{(0,T)\times\partial\Omega} = 0, \quad (1.2)
\]

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and

\[ u(t_0, \cdot) = u_0 \quad \text{in } \Omega. \]  

(1.3)

Here, choosing a constant \( \kappa \in (0, 1) \) arbitrarily, we assume

\[ a_{jk} = a_{kj} \in C^{2+\kappa}(\Omega), \quad b_j, p \in C^{\kappa}(\Omega) \quad \text{for all } j, k \in \{1, \ldots, n\} \]  

(1.4)

and there exists a constant \( \sigma > 0 \) such that

\[ \sum_{j,k=1}^{n} a_{jk}(x)\xi_j \xi_k \geq \sigma |\xi|^2 \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^n. \]  

(1.5)

Here and henceforth, let \( C^{2+\kappa}(\Omega), C^{\kappa}(\Omega) \) with \( \kappa \in (0, 1) \) denote the Schauder spaces (e.g., Ladyzhenskaya, Solonnikov and Ural’ceva [19]), and let \( \nu = (\nu_1, ..., \nu_n) \) be the unit outward normal vector to \( \partial \Omega \). We set

\[ \partial_{\nu \Lambda} v = \sum_{k,j=1}^{n} a_{kj}(x)\nu_j \partial_k v. \]  

(1.6)

Throughout this article, we fix the coefficients \( a_{jk}, b_j, 1 \leq j, k \leq n \) in (1.1) and by \( u(c) = u(c)(x, t) \), we denote one solution to (1.1) - (1.3) with zeroth-order coefficient \( c = c(x) \).

In this article, we consider

**Inverse coefficient problem.**

Let \( \Gamma \) be a fixed subboundary of \( \partial \Omega \) and let \( t_0 \in [0, T] \) be given in (1.3). Determine \( p(x) \) by \( u|_{(0,T) \times \Gamma} \) and \( u_0 \).

We are concerned with the uniqueness and the stability of the determination of the coefficient \( p \). The results depend essentially on \( t_0 = 0, 0 < t_0 < T \) and \( t_0 = T \).

- **Case \( t_0 = 0 \):** No results in general. We can refer to Klibanov [18] in a case where a parabolic equation is considered in \((0, T) \times \mathbb{R}^n\). In [18], the uniqueness is proved in an arbitrarily chosen ball \( B \subset \mathbb{R}^n \) with boundary data on \((0, T) \times \partial B\).

- **Case \( 0 < t_0 < T \):** Global Lipschitz stability as well as the global uniqueness. See Imanuvilov and Yamamoto [11], and also Bellassoued and Yamamoto [2].

- **Case \( t_0 = T \):** to our best knowledge no results are published.

In the case \( t_0 = 0 \), our problem is an inverse problem corresponding to a classical initial boundary value problem. We note that in the case \( t_0 = T \), our observation ends at \( T \), so that we have no extra boundary data \( u|_{\Gamma} \) for \( t > T \).

Our current article aims at two unsolved cases: \( t_0 = 0 \) and \( t_0 = T \). Our result in the case \( t_0 = 0 \) asserts the uniqueness in a more general parabolic operator and subboundary and the proof is based on a similar idea to [18]. The stability for \( t_0 = 0 \) is still totally an open problem. On the other hand, in the case \( t_0 = T \), to the best knowledge of the authors, there
are no existing results. Our result for $t_0 = T$ establishes not only the uniqueness but also the Lipschitz stability around any fixed coefficient $p$.

We emphasize that we consider the case of the boundary condition (1.2) of the Neumann type, not the Dirichlet boundary condition. For the global stability and the uniqueness by extra data on subboundary of $\partial \Omega$, the case (1.2) requires more subtle arguments. The arguments in the case of the Dirichlet boundary condition with extra Neumann data are easier, but owing to the required positivity assumption of spatial data $u_0$ in (1.3), we can not prove neither the global uniqueness nor the global stability without additional assumption that unknown coefficients are given in a neighborhood of $\partial \Omega$.

For the case of $t_0 = 0$, our proof is via a corresponding inverse hyperbolic problem, and then a Carleman estimate for the Neumann boundary condition with additional Dirichlet data is indispensable and more works are required for establishing such a Carleman estimate (e.g., Imanuvilov [10]).

We add that the boundary condition (1.2) means the thermal insulation, and is quite important physically.

Henceforth we use the following notations: $Q := (0, T) \times \Omega$.

We arbitrarily fix constants $M > 0, \delta_0 > 0, \gamma_1, \gamma_2 \in (0, 1)$ and set

$$
\begin{align*}
\mathcal{P} := \{ p \in C^{\gamma_1} (\overline{\Omega}); \| p \|_{C^{\gamma_1} (\overline{\Omega})} \leq M \}, \\
\mathcal{A} := \{ a \in C^{2+\gamma_2} (\overline{\Omega}); \partial_{\nu_A} a = 0 \text{ on } \partial \Omega, \\
\| a \|_{C^{2+\gamma_2} (\overline{\Omega})} \leq M, \quad a \geq \delta_0 > 0 \text{ on } \overline{\Omega} \}.
\end{align*}
$$

By $u_{p,a} = u_{p,a}(t, x)$ we denote the classical solution to

$$
\begin{align*}
\begin{cases}
\partial_t u = Au - p(x)u & \text{in } Q, \\
\partial_{\nu_A} u = 0 & \text{on } (0, T) \times \partial \Omega, \\
u(0, \cdot) = a & \text{in } \Omega.
\end{cases}
\end{align*}
$$

More precisely, we choose $\gamma > 0$ such that

$$
0 < \gamma < \min\{ \gamma_1, \gamma_2 \} < 1.
$$

Then

$$
u_{p,a} \in C^{1+\frac{\gamma}{2}, 2+\gamma}(\overline{Q}) \text{ and } t \partial_t u \in H^{1,2}(Q) \text{ for all } p \in \mathcal{P} \text{ and } a \in \mathcal{A}.
$$

**Proof of (1.10).** First, by Ladyzenskaja, Solonnikov and Ural’ceva [19], for $p \in \mathcal{P}$ and

$$
\text{Proof of (1.10).}$$

First, by Ladyzenskaja, Solonnikov and Ural’ceva [19], for $p \in \mathcal{P}$ and
\(a \in \mathcal{A}\), by noting (1.9) there exists a unique solution \(u_{p,a} \in C^{1+\frac{\gamma}{2},2+\gamma}(\overline{Q})\). Moreover we see that there exists a unique solution \(v \in H^{1,2}(Q)\) to
\[
\begin{cases}
\partial_t v = Av - pv + \partial_t u & \text{in } Q, \\
\partial_{\nu_A} v = 0 & \text{on } (0, T) \times \partial \Omega, \\
v(0, \cdot) = 0.
\end{cases}
\]

We can verify the unique existence of such \(v\), similarly to Theorem 5 (pp.360-361) in Evans [6] for example, where the homogeneous Dirichlet boundary condition is considered. Then the uniqueness of the solution in \(L^2(Q)\) yields that \(v = t\partial_t u \) in \(Q\), which completes the proof of (1.10). \(\blacksquare\)

Now we state our main results.

**Theorem 1** Let \(\Gamma \subset \partial \Omega\) be an arbitrarily chosen subboundary and \(0 < \gamma < \min\{\gamma_1, \gamma_2\}\). In (1.3) we assume \(t_0 = T\). There exists a constant \(C_\gamma > 0\), depending on \(\mathcal{A}\) and \(\mathcal{P}\) such that
\[
\|p - q\|_{H^\gamma(\Omega)} \leq C_\gamma((\|u_{p,a} - u_{q,b}(\cdot, T)\|_{H^{2+\gamma}(\Omega)} + \|u_{p,a} - u_{q,b}\|_{H^1((0,T) \times \Gamma)})
\]
for each \((p,a),(q,b)\) \(\in \mathcal{P} \times \mathcal{A}\).

The global Lipschitz stability in the case of \(t_0 = T\) was an open problem. However, Theorem 1 firstly provides the positive answer. If spatial data are given at \(t_0 \in (0,T)\), then we already have proved the global Lipschitz stability for the inverse parabolic problem (Imanuvilov and Yamamoto [11]).

Thanks to the holomorphicity of \(u_{p,a}, u_{q,b}\) in \(t > 0\), fixing \(\tilde{T} > T\) arbitrarily, by the three circle theorem (e.g., Chapter 10 in Cannon [5]) we can obtain
\[
\|u_{p,a} - u_{q,b}\|_{H^1((0,\tilde{T}) \times \Gamma)} \leq C\|u_{p,a} - u_{q,b}\|_{H^1((0,T) \times \Gamma)}
\]
with \(\mu \in (0,1)\) and \(C > 0\) within some boundedness assumptions. Therefore in the case of data at the final time \(T\), the essential contribution is the Lipschitz stability.

Next we discuss the case of \(t_0 = 0\), that is, spatial data are given at the initial time moment in (1.1) - (1.3). We consider a simpler parabolic equation.
\[
\begin{cases}
\partial_t u(t, x) = \Delta u(t, x) - p(x)u(t, x), & 0 < t < T, \ x \in \Omega, \\
\partial_{\nu_A} u = 0 & \text{on } (0, T) \times \partial \Omega, \\
u(0, \cdot) = u_0 & \text{in } \Omega.
\end{cases}
\] (1.11)
In this case, we note that $\partial_{\nu_A} u = \nabla u \cdot \nu(x)$. For simplicity, we assume that the spatial dimensions $n$ is smaller than or equal to 3:

$$n \in \{1, 2, 3\}.$$ 

Furthermore we assume that $\Omega$ is locally convex in $\partial \Omega \setminus \overline{\Gamma}_0$, that is, for all $x \in \partial \Omega \setminus \overline{\Gamma}_0$, there exists a small ball $U_x$ such that $U_x \cap \Omega$ is convex. We assume that $u_0 \in H^3(\Omega)$ and $\partial_{\nu_A} u_0 = 0$ on $\partial \Omega$, and $p, q \in W^{1,\infty}(\Omega)$ for unknown coefficients. Replacing $u$ by $e^{C_0 t} u$ with suitable constant $C_0$, without loss of generality, we can assume that

$$p, q > 0 \quad \text{on } \overline{\Omega}. \quad (1.12)$$

By $u(p)$, we denote the solution to (1.11).

Arbitrarily choosing $x_0 \not\in \overline{\Omega}$, we assume that a subboundary $\Gamma_0 \subset \partial \Omega$ is a relatively open subset in $\mathbb{R}^{n-1}$ and

$$\Gamma_0 \supset \{ x \in \partial \Omega; (x - x_0) \cdot \nu(x) \geq 0 \}. \quad (1.13)$$

Then we can state our second main result for the case of $t_0 = 0$.

**Theorem 2.**

We assume that $p, q \in W^{1,\infty}(\Omega)$ and $u_0 \in H^3(\Omega)$, $\partial_{\nu_A} u_0 = 0$ on $\partial \Omega$, and there exists a constant $\delta_0 > 0$ such that

$$|u_0(x)| \geq \delta_0 \quad \text{for all } x \in \overline{\Omega}. \quad (1.14)$$

Let $T > 0$ be arbitrary. Then $u(p) = u(q)$ on $(0, T) \times \Gamma_0$ implies $p(x) = q(x)$ for $x \in \Omega$.

In Theorem 2, we can obtain the uniqueness for general parabolic operator as in (1.1), and we return to this issue in Section 6.

We prove Theorem 2 by reducing the inverse parabolic problem to an inverse problem for hyperbolic equation and the argument relies on Klbanov [18]. The work [18] assumes that $u(p), u(q)$ should be sufficiently smooth, and considers a parabolic equation in the whole space $(0, T) \times \mathbb{R}^n$ where unknown coefficients $p, q$ are given outside of a ball.

We note that since the proof is based on reduction of original problem to an inverse problem for a hyperbolic equation, we need a Carleman estimate for hyperbolic equation for the case of the Neumann boundary condition, and in Theorem 2 we should choose some large portion $\Gamma_0$ of the boundary such that (1.13) holds true, but $T > 0$ can be arbitrarily small.

Finally we discuss one type of inverse source problem. Let

$$\partial_t y = Ay - p(x)y + \mu(t)f(x), \quad 0 < t < T, \ x \in \Omega, \quad (1.14)$$

and

$$y(t_0, \cdot) = 0 \quad \text{in } \Omega. \quad (1.15)$$
We assume that $\mu \not\equiv 0$ in $(0, T)$ is given. Then

**Inverse source problem.**

Let $T > 0$ be arbitrary and $0 \leq t_0 \leq T$ be fixed. Let $\Gamma \subset \partial \Omega$ be an arbitrarily chosen subboundary. Then determine $f$ in $\Omega$ by $y|_{(0,T) \times \Gamma}$ and $\nabla y|_{(0,T) \times \Gamma}$.

We emphasize that we do not assume any boundary condition on the whole lateral boundary $(0, T) \times \partial \Omega$. We are concerned with the uniqueness. In the case of $0 < t_0 < T$, in view of the method based on Carleman estimates, we can prove the uniqueness: Assuming that $\mu \in C^1[0, T]$ and $\mu(t_0) \neq 0$, we can conclude that $y = |\nabla y| = 0$ on $(0, T) \times \Gamma$ and $y(t_0, \cdot) = 0$ in $\Omega$ imply $f = 0$ in $\Omega$. We omit the proof but we can refer for example to Yamamoto [29].

However, there are no published works in the case $t_0 = 0$ or $t_0 = T$. We establish the uniqueness in the case of $t_0 = 0$, which means the inverse source problem exactly corresponding to the initial value problem.

**Theorem 3.**

Let $y, \partial_t y \in H^{1,2}(Q)$ satisfy (1.14) and

$$y(0, \cdot) = 0 \quad \text{in } \Omega,$$

and $\mu \in C^1[0, T]$ and $\mu(0) \neq 0$. Then $y = |\nabla y| = 0$ on $(0, T) \times \Gamma$ implies $f = 0$ in $\Omega$.

We compare the uniqueness for the inverse source problem with the unique continuation which yields the uniqueness of solution without boundary condition on $\partial \Omega$.

**Unique continuation.**

$$\begin{cases} 
\partial_t z = Az - p(x)z & \text{in } (0, T) \times \Omega, \\
z = |\nabla z| = 0 & \text{on } (0, T) \times \Gamma,
\end{cases}$$

yield $z = 0$ in $(0, T) \times \Omega$. In particular, $z(0, \cdot) = 0$ in $\Omega$.

**Uniqueness in the inverse source problem.**

$$\begin{cases} 
\partial_t y = Ay - p(x)y + \mu(t)f(x) & \text{in } (0, T) \times \Omega, \\
y = |\nabla y| = 0 & \text{on } (0, T) \times \Gamma, \\
y(0, \cdot) = 0 & \text{in } \Omega,
\end{cases}$$

yield $f = 0$ in $\Omega$, and so $y = 0$ in $(0, T) \times \Omega$.

Comparing with the unique continuation, in our inverse source problem, one more spatial function $f$ is unknown, and for compensating for the uniqueness, we need more spatial data at $t = 0$, but never require additional boundary data. Thus Theorem 3 provides the uniqueness result with the minimum data.
As for conditional stability results for the case of $t_0 = 0$ with boundary condition on $(0, T) \times \partial \Omega$, see Yamamoto [27] for example.

This article is composed of five sections. In Section 2, we prove Theorem 1 by a priori estimates of $u(p) - u(q)$ involving not initial values and the compact-uniqueness argument. Sections 3 and 4 are devoted to the proofs of Theorem 2 and 3 respectively. Section 5 gives concluding remarks.

2. PROOF OF THEOREM 1

We recall that $\Gamma \subset \partial \Omega$ is an arbitrarily chosen non-empty subboundary. We set

\[
Q := (0, T) \times \Omega, \quad Q_1 := \left(\frac{T}{2}, T\right) \times \Omega, \quad Q_2 := \left(\frac{T}{4}, T\right) \times \Omega,
\]

\[
Q_3 := \left(\frac{T}{8}, T\right) \times \Omega, \quad \Sigma := (0, T) \times \Gamma.
\]

Proof. We divide the proof into five steps. The key of the proof is:
(i) estimate of $\|p - q\|_{H^\gamma(\Omega)}$ by data of $u_{p,a} - u_{q,b}$ and $\|p - q\|_{L^2(\Omega)}$, which is given by (2.18) below.
(ii) application of the compact-uniqueness argument in order to eliminates $\|p - q\|_{L^2(\Omega)}$ in (2.18).

First Step. In order to formulate a Carleman estimate we introduce a weight function for such a Carleman estimate by the following lemma:

Lemma 2.1 ([7])
Let $\Gamma_0$ be an arbitrarily fixed subboundary of $\partial \Omega$ such that $\Gamma_0 \subset \Gamma$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that

\[
\psi(x) > 0 \quad \text{for all } x \in \Omega, \quad \psi|_{\partial \Omega \setminus \Gamma_0} = 0, \quad |\nabla \psi(x)| > 0 \quad \text{for all } x \in \overline{\Omega}.
\]

Using the function $\psi$, we define

\[
\alpha(t, x) = \frac{e^{\lambda \psi(x)} - e^{2\lambda \|\psi\|_{C(\Gamma_0)}}}{t(T - t)}, \quad \varphi(t, x) = \frac{e^{\lambda \psi(x)}}{t(T - t)}.
\]

Consider the boundary value problem:

\[
Pz(t, x) := \partial_t z - Az + p(x)z = g \quad \text{in } Q, \quad \partial_{\nu_A} z = 0 \quad \text{on } (0, T) \times \partial \Omega.
\]

We are ready to state a Carleman estimate.

Lemma 2.2 ([7])
Let (1.4) and (1.5) be fulfilled. Then there exists a constant \( \hat{\lambda} > 0 \) such that for an arbitrary \( \lambda \geq \hat{\lambda} \), there exist constants \( \tau_0(\lambda) > 0 \) and \( C_1 > 0 \) such that

\[
\int_Q \left( \frac{1}{\tau} |\partial_t z|^2 + \sum_{j,k=1}^n |\partial_j \partial_k z|^2 + \tau |\nabla z|^2 + \tau^3 |z|^2 \right) e^{2\tau \alpha} dx \, dt \\
\leq C_1 \left( \int_Q |Pz|^2 e^{2\tau \alpha} dx \, dt + \int_{(0,T) \times \Gamma} \{ \tau |\nabla z|^2 + |\partial_t z|^2 \right) + (\tau \varphi)^3 |z|^2 e^{2\tau \alpha} d\Sigma \right)
\]

for each \( \tau \geq \tau_0(\lambda) \) and all \( z, t \partial_t z \in H^{1,2}(Q) \) satisfying \( \partial_{\nu A} z = 0 \) on \( (0,T) \times \partial \Omega \).

The constant \( C_1 > 0 \) can be taken independently of choices of \( p \in \mathcal{P} \). Indeed \( C_1 \) depends on \( M_0 \), provided that \( \|p\|_{C(\Omega)} \leq M_0 \).

We recall that \( u_{p,a} \) is the solution to (1.8) and \( 0 < \gamma < 1 \) satisfies (1.9).

Furthermore, in addition to (1.10), we can prove

**Lemma 2.3.**

For any \( \gamma_* \) satisfying \( 0 < \gamma_* < \min\{\gamma_1, \gamma_2\} \), there exists a constant \( C > 0 \) such that

\[
\|u_{p,a}\|_{C^{1+\frac{n}{2},2+\gamma_*}(Q)} \leq C, \quad \|t \partial_t u_{p,a}\|_{H^{1,2}(Q)} \leq C \quad \text{for all } p \in \mathcal{P} \text{ and } a \in \mathcal{A}.
\]

Indeed, the estimates of \( \|u_{p,a}\|_{C^{1+\frac{n}{2},2+\gamma_*}(Q)} \) and \( \|t \partial_t u_{p,a}\|_{H^{1,2}(Q)} \) follow respectively, for example from Theorem 5.3 (pp.320-321) in [19] and Theorem 5 (pp. 360-361) in [6].

Moreover,

**Lemma 2.4.** There exist a strictly positive constant \( c_0(\delta_0) \) such that

\[
u_{q,b}(T, \cdot) \geq c_0(\delta_0) \quad \text{on } \overline{\Omega}.
\]

for all \( b \in \mathcal{A} \) and \( q \in \mathcal{P} \).

The proof is done by the positivity of the fundamental solution (e.g., Itô [15]) in terms of \( b \geq \delta_0 \) on \( \overline{\Omega} \) for \( b \in \mathcal{A} \) and \( q \in \mathcal{P} \).

**Second Step.** For any \( \mathbf{v} = (v_1, \ldots, v_n), \mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n \), we set

\[
\mathbf{a}(x, \mathbf{v}, \mathbf{w}) = \sum_{j,k=1}^n a_{jk}(x)v_jw_k.
\]

Let \( u \in H^{1,2}(Q) \) satisfy \( t \partial_t u \in H^{1,2}(Q) \) and

\[
\begin{cases}
\partial_t u = Au - p(x)u + S(t,x) \quad \text{in } Q, \\
\partial_{\nu A} u = 0 \quad \text{on } (0,T) \times \partial \Omega
\end{cases}
\]
for $S \in L^2(Q)$ and $p \in \mathcal{P}$. In particular, we note that $\|p\|_{C(\overline{\Omega})} \leq M$, where the constant $M > 0$ is given in (1.7).

Thanks to Lemma 2.2, we can readily prove

$$\|u\|_{L^2(\frac{T}{4}, \frac{3T}{4}; H^1(\Omega))} \leq C(\|S\|_{L^2(Q_3)} + \|u\|_{H^1(\Sigma)}). \tag{2.3}$$

Here we recall (2.1). In this step, we will further prove

$$\|u\|_{L^2(\frac{T}{4}, \frac{3T}{4}; H^1(\Omega))} \leq C(\|S\|_{L^2(Q_3)} + \|u\|_{H^1(\Sigma)}). \tag{2.4}$$

**Proof of (2.4).** We choose $\mu \in C^{\infty}(\frac{T}{4}, \frac{3T}{4})$ satisfying $\mu\left(\frac{T}{4}\right) = 0$ and $\mu\left(\frac{3T}{4}\right) = 1$. We take the scalar products of equation (2.2) with $\mu(t)u$ in $L^2((\frac{T}{4}, \frac{3T}{4}) \times \Omega)$. Integrating by parts, we have

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} \left( \frac{1}{2} \partial_t(\mu(t)u^2) - \frac{1}{2} \partial_t \mu(t) u^2 + \mu(t)a(x, \nabla u, \nabla u) + \left( \sum_{j=1}^{n} b_j(x) \partial_j u + p(x) u \right) \mu u \right) dx \, dt$$

$$= \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} \mu(t) u S \, dx \, dt.$$}

This equality implies

$$\left\| u \left( \frac{3T}{4}, \cdot \right) \right\|^2_{L^2(\Omega)} = 2 \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} \left( \frac{\partial_t \mu(t)}{2} u^2 - 2 \mu(t)a(x, \nabla u, \nabla u) - 2 \sum_{j=1}^{n} b_j(x) \partial_j u + p(x) u \right) \mu u \, dx \, dt$$

$$+ 2 \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} \mu(t) u S \, dx \, dt$$

$$\leq C(\|u\|^2_{L^2(\frac{T}{4}, \frac{3T}{4}; H^1(\Omega))} + \|S\|^2_{L^2(Q_3)}). \tag{2.5}$$

Applying (2.3) to the first term on the right-hand side of (2.5), we have

$$\left\| u \left( \frac{3T}{4}, \cdot \right) \right\|_{L^2(\Omega)} \leq C(\|S\|_{L^2(Q_3)} + \|u\|_{H^1(\Sigma)}). \tag{2.6}$$

In view of (2.6), the standard a priori estimate for the initial value problem with initial value at $\frac{3T}{4}$ (e.g., Section 1 of Chapter 7 in Evans [1]), yields

$$\|u\|_{L^2(\frac{T}{4}, \frac{3T}{4}; H^1(\Omega))} \leq C \left( \left\| u \left( \frac{3T}{4}, \cdot \right) \right\|_{L^2(\Omega)} \right) + \|S\|_{L^2(Q_3)} \leq C(\|S\|_{L^2(Q_3)} + \|u\|_{H^1(\Sigma)}).$$

Combining this estimate with (2.3), we complete the proof of (2.4).

**Third Step.** We assume that $u \in H^{1,2}(Q)$ satisfies $t \partial_t u \in H^{1,2}(Q)$ and (2.2). In this step, we first prove

$$\|u\|_{L^2(\frac{T}{4}, T; H^2(\Omega))} + \|\partial_t u\|_{L^2(\frac{T}{4}, T; L^2(\Omega))} \leq C(\|S\|_{L^2(Q_3)} + \|u\|_{H^1(\Sigma)}). \tag{2.7}$$
Proof of (2.7). Since $a_{jk} = a_{kj}$ are independent of $t$, in view of integration by parts and $\partial_{\nu} u = 0$ on $\partial \Omega$, we see

$$- \int_{\Omega} \sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u)\partial_t u dx = \sum_{j,k=1}^{n} a_{jk}(\partial_k u)\partial_j \partial_t u dx$$

$$= \int_{\Omega} \sum_{k>j} a_{jk}((\partial_k u)\partial_j \partial_t u + (\partial_j u)\partial_k \partial_t u) dx + \int_{\Omega} \sum_{j=1}^{n} a_{jj}(\partial_j u)\partial_j \partial_t u dx$$

$$= \int_{\Omega} \sum_{k>j} a_{jk}\partial_t((\partial_j u)\partial_k u) dx + \frac{1}{2} \int_{\Omega} \sum_{j=1}^{n} a_{jj}\partial_t((\partial_j u)^2) dx$$

$$= \frac{1}{2} \int_{\Omega} \partial_t \left( \sum_{j,k=1}^{n} a_{jk}(\partial_j u)\partial_k u \right) dx.$$

Taking the scalar products of equation (2.2) by $(t - \frac{T}{4}) \partial_t u$ in $L^2((\frac{T}{4}, T) \times \Omega)$ and integrating by parts, we obtain

$$\int_{\frac{T}{4}}^{T} \int_{\Omega} \left( (t - \frac{T}{4})^2 (\partial_t u)^2 + \frac{1}{2} \partial_t \left( (t - \frac{T}{4}) a(x, \nabla u, \nabla u) \right) - \frac{1}{2} a(x, \nabla u, \nabla u) \right) dx dt$$

$$+ \left( \sum_{j=1}^{n} b_j(x)\partial_j u + p(x) u \right) (t - \frac{T}{4}) \partial_t u dx dt$$

$$= \int_{\frac{T}{4}}^{T} \int_{\Omega} \left( (t - \frac{T}{4})^2 (\partial_t u)^2 - \frac{1}{2} a(x, \nabla u, \nabla u) + \left( \sum_{j=1}^{n} b_j(x)\partial_j u + p(x) u \right) (t - \frac{T}{4}) \partial_t u dx dt$$

$$+ \frac{3T}{8} \int_{\Omega} a(x, \nabla u(T, t), \nabla u(T, t)) dx$$

$$= \int_{\frac{T}{4}}^{T} \int_{\Omega} (t - \frac{T}{4}) (\partial_t u) S dx dt.$$

Hence, by $a(x, \nabla u, \nabla u) \geq 0$, we have

$$\int_{\frac{T}{4}}^{T} \int_{\Omega} (t - \frac{T}{4}) (\partial_t u)^2 dx dt \leq \int_{\frac{T}{4}}^{T} \int_{\Omega} \left( \frac{1}{2} a(x, \nabla u, \nabla u) + \left( \sum_{j=1}^{n} b_j(x)\partial_j u + p(x) u \right) (t - \frac{T}{4}) \partial_t u \right) dx dt$$

$$+ \int_{\frac{T}{4}}^{T} \int_{\Omega} (t - \frac{T}{4}) (\partial_t u) S dx dt.$$

Here for any $\varepsilon \in (0, 1)$, we can choose a constant $C_\varepsilon > 0$ such that

$$\left| (t - \frac{T}{4}) (\partial_t u) S \right| \leq \varepsilon (t - \frac{T}{4}) |\partial_t u|^2 + C_\varepsilon (t - \frac{T}{4}) |S|^2 \leq \varepsilon (t - \frac{T}{4}) |\partial_t u|^2 + C_\varepsilon \frac{3T}{4} |S|^2$$
in \((T/4, T) \times \Omega\). Consequently,
\[
\int_0^T \int_\Omega (t-T/4)(\partial_t u)^2 \,dx \,dt \leq C \left( \|u\|_{L^2(T;H^1(\Omega))}^2 + \int_0^T \int_{\Omega} |S|^2 \,dx \,dt \right) + \varepsilon \int_0^T \int_{\Omega} (t-T/4)(\partial_t u)^2 \,dx \,dt.
\]
Fixing \(\varepsilon \in (0, 1)\) and applying (2.4), we obtain
\[
\|\partial_t u\|_{L^2(T; H^1(\Omega))}^2 \leq \int_0^T \int_{\Omega} (t-T/4)(\partial_t u)^2 \,dx \,dt \leq C \left( \|u\|_{L^2(T;H^1(\Omega))}^2 + \|S\|_{L^2(Q_3)}^2 \right) \leq C\left(\|S\|_{L^2(Q_3)}^2 + \|u\|_{H^1(\Sigma)}^2\right).
\] (2.8)
In order to estimate the second space derivatives of \(u\), we will write the equation as
\[
- \sum_{j,k=1}^n \partial_{x_j}(a_{jk}(x)\partial_k u) = -\partial_t u - \sum_{j=1}^n b_j(x)\partial_j u - p(x)u + S(t,x), \quad x \in \Omega, \ 0 < t < T
\]
with \(\partial_{\nu_A} u = 0\) on \((0, T) \times \partial \Omega\). In particular, we have
\[
\left\| - \sum_{j,k=1}^n \partial_j \left( a_{jk}(x) \partial_k u(t, \cdot) \right) \right\|_{L^2(\Omega)} \leq C\left( \|\partial_t u(t, \cdot)\|_{L^2(\Omega)} + \|\nabla u(t, \cdot)\|_{L^2(\Omega)} + M\|u(t, \cdot)\|_{L^2(\Omega)} + \|S(t, \cdot)\|_{L^2(\Omega)} \right), \quad \frac{T}{4} \leq t \leq T.
\]
Let us fix \(t \in (T/2, T)\). From the a priori estimate for the Neumann problem for the second order elliptic operator, we obtain
\[
\|u(t, \cdot)\|_{H^2(\Omega)} \leq C\left( \|\partial_t u(t, \cdot)\|_{L^2(\Omega)}^2 + \|u(t, \cdot)\|_{H^1(\Omega)}^2 + \|S(t, \cdot)\|_{L^2(\Omega)}^2 \right).
\]
Integrating this inequality over the time interval \((T/2, T)\), we obtain
\[
\|u(t, \cdot)\|_{L^2(T; H^2(\Omega))}^2 \leq C\left( \|\partial_t u(t, \cdot)\|_{L^2(T; H^1(\Omega))}^2 + \|u(t, \cdot)\|_{L^2(T; H^1(\Omega))}^2 + \|S\|_{L^2(Q_3)}^2 \right). \tag{2.9}
\]
Using estimates (2.8) and (2.4) to estimate the right hand side of (2.9), we reach (2.7). ■

**Fourth Step.** Let \(u \in H^{1,2}(Q)\) satisfy \(t\partial_t u \in H^{1,2}(Q)\) and (2.2).

By \(t\partial_t u \in H^{1,2}(Q)\), we differentiate equation (2.2) with respect to \(t\):
\[
\partial_t(\partial_t u) = A(\partial_t u) - p(x)\partial_t u + \partial_t S(t, x)f(x) \quad \text{in} \ Q, \tag{2.10}
\]
\[
\partial_{\nu_A} \partial_t u = 0 \quad \text{on} \ (0, T) \times \partial \Omega \tag{2.11}
\]
and
\[
(\partial_t u)(T, \cdot) = (A - p(x))u(T, \cdot) + S(T, \cdot). \tag{2.12}
\]
We recall that \(\gamma \in (0, 1)\) satisfies (1.9). Now we prove
\[
\|S(T, \cdot)\|_{H^{\gamma}(\Omega)} \leq C\left( \|\partial_t S\|_{L^2(Q_1)} + \|S\|_{L^2(Q_3)} + \|u\|_{H^1(\Sigma)} + \|u(T, \cdot)\|_{H^{2+\gamma}(\Omega)} \right). \tag{2.13}
\]
Applying (2.7) in order to estimate the last term on the right-hand side of (2.15), we have

\[ \begin{align*}
\partial_t \tilde{u} - A\tilde{u} + p(x)\tilde{u} &= (t - \frac{T}{2})\partial_t S(t, x) + \partial_t u \quad \text{in} \ (\frac{T}{2}, T) \times \Omega, \\
\partial_{\nu_A} \tilde{u} \big|_{(\frac{T}{2}, T) \times \partial \Omega} &= 0, \quad \tilde{u}(\frac{T}{2}, \cdot) = 0.
\end{align*} \tag{2.14} \]

Proof of (2.13). We set \( \tilde{u} = (t - \frac{T}{2})\partial_t u \). Then

\[ \tilde{u} \in L^2 \left( \frac{T}{2}, T; H^2(\Omega) \right) \cap H^1 \left( \frac{T}{2}, T; L^2(\Omega) \right) \]

(e.g., Theorem 5 (p.360) in Evans [6]) with an interpolation property (Theorem 3.1 (p.19) in Lions and Magenes [20]), we can see the continuous embedding

\[ L^2 \left( \frac{T}{2}, T; H^2(\Omega) \right) \cap H^1 \left( \frac{T}{2}, T; L^2(\Omega) \right) \subset C \left( [\frac{T}{2}, T]; H^1(\Omega) \right). \]

We can directly verify this embedding without the interpolation property, but we do not use such verification.

Therefore,

\[ \| \tilde{u} \|_{C \left( [\frac{T}{2}, T]; H^1(\Omega) \right)} \leq C \left\| (t - \frac{T}{2})\partial_t S + \partial_t u \right\|_{L^2(Q_1)} \]

\[ \leq C \left( \| \partial_t S \|_{L^2(Q_1)} + \| \partial_t u \|_{L^2(Q_1)} \right). \tag{2.15} \]

Applying (2.7) in order to estimate the last term on the right-hand side of (2.15), we have

\[ \left\| (t - \frac{T}{2})\partial_t u \right\|_{C \left( [\frac{T}{2}, T]; H^1(\Omega) \right)} \leq C \left( \| \partial_t S \|_{L^2(Q_1)} + \| S \|_{L^2(Q_2)} + \| u \|_{H^1(\Sigma)} \right). \tag{2.16} \]

From (2.12), we have

\[ \| S(T, \cdot) \|_{H^\gamma(\Omega)} \leq C \left( \| \partial_t u(T, \cdot) \|_{H^1(\Omega)} + \| (A - p)u(\cdot, T) \|_{H^\gamma(\Omega)} \right) \]

\[ \leq C \left( \| \partial_t u(T, \cdot) \|_{H^1(\Omega)} + (1 + \| p \|_{C^\gamma(\Omega)}) \| u(\cdot, T) \|_{H^{2+\gamma}(\Omega)} \right). \]

Thus we proved that (2.13) holds provided that \( u \in H^{1,2}(Q) \) satisfies \( t\partial_t u \in H^{1,2}(Q) \) and (2.2). \( \blacksquare \)

**Fifth Step.** Let \( z = z_{p,q,a,b} = u_{p,a} - u_{q,b} \in H^{1,2}(Q) \) and \( f := q - p \in L^2(\Omega) \). Then, by Lemma 2.3, we see that \( t\partial_t z \in H^{1,2}(Q) \) and

\[ \begin{align*}
\partial_t z &= Az - p(x)z + f(x)u_{q,b}(t, x) \quad \text{in} \ Q, \\
\partial_{\nu_A} z &= 0 \quad \text{on} \ (0, T) \times \partial \Omega, \\
z(0, \cdot) &= a - b \quad \text{in} \ \Omega.
\end{align*} \tag{2.17} \]
Set $S(t, x) := f(x)u_{q, b}(t, x)$ in $Q$. By (2.1) and Lemmata 2.3 and 2.4 we have
\[ \|f\|_{H^{\gamma}(\Omega)} \leq \left\| \frac{S(T, \cdot)}{u_{q, b}(T, \cdot)} \right\|_{H^{\gamma}(\Omega)} \leq C\|S(T, \cdot)\|_{H^{\gamma}(\Omega)}. \]

Therefore, applying (2.13) to (2.17), we can find a constant $\tilde{C} > 0$ such that
\[ \|p - q\|_{H^{\gamma}(\Omega)} \leq \tilde{C}(\|p - q\|_{L^2(\Sigma)} + \|z_{p, q, a, b}(T, \cdot)\|_{H^{2+\gamma}(\Omega)} + \|z_{p, q, a, b}\|_{H^1(\Sigma)}) \quad (2.18) \]
for all $p, q \in \mathcal{P}$ and $a, b \in \mathcal{A}$.

Now we will prove that there exists a constant $C > 0$ such that
\[ \|p - q\|_{H^{\gamma}(\Omega)} \leq C(\|z_{p, q, a, b}(T, \cdot)\|_{H^{2+\gamma}(\Omega)} + \|z_{p, q, a, b}\|_{H^1(\Sigma)}) \quad (2.19) \]
for all $p, q \in \mathcal{P}$ and $a, b \in \mathcal{A}$.

We start the proof of (2.19). Suppose that (2.19) does not hold. Then for any $m \in \mathbb{N}$, there exist functions $p_m, q_m \in \mathcal{P}$ and $a_m, b_m \in \mathcal{A}$ such that
\[ \|p_m - q_m\|_{H^{\gamma}(\Omega)} > m(\|z_{p_m, q_m, a_m, b_m}(T, \cdot)\|_{H^{2+\gamma}(\Omega)} + \|z_{p_m, q_m, a_m, b_m}\|_{H^1(\Sigma)}). \]

Then, by the definition (1.8) of $u_{p_m, a_m}, u_{q_m, b_m}$, we can extend $u_{p_m, a_m}, u_{q_m, b_m}$ to $t \in (0, +\infty)$, so that
\[
\begin{cases}
\partial_t u_{p_m, a_m} = Au_{p_m, a_m} - p_m u_{p_m, a_m}, & (t, x) \in (0, +\infty) \times \Omega, \\
\partial_{\nu, \lambda} u_{p_m, a_m} = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\
u_m, a_m(0, \cdot) = a_m & \text{in } \Omega
\end{cases}
\quad (2.20)
\]
and
\[
\begin{cases}
\partial_t u_{q_m, b_m} = Au_{q_m, b_m} - q_m u_{q_m, b_m}, & (t, x) \in (0, +\infty) \times \Omega, \\
\partial_{\nu, \lambda} u_{q_m, b_m} = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\
u_m, b_m(0, \cdot) = b_m & \text{in } \Omega.
\end{cases}
\quad (2.21)
\]
Moreover for any $\tilde{T} > 0$, by Lemma 2.3, we see that
\[ u_{p_m, a_m}, u_{q_m, b_m} \in C^{1+\frac{\gamma}{2}, 2+\gamma}([0, \tilde{T}] \times \overline{\Omega}) \]
and there exists a constant $C(\tilde{T}) > 0$ such that
\[ \|u_{p_m, a_m}\|_{C^{1+\frac{\gamma}{2}, 2+\gamma}([0, \tilde{T}] \times \overline{\Omega})}, \quad \|u_{q_m, b_m}\|_{C^{1+\frac{\gamma}{2}, 2+\gamma}([0, \tilde{T}] \times \overline{\Omega})} \leq C(\tilde{T}). \]

Moreover, $z_{p_m, q_m, a_m, b_m} = u_{p_m, a_m} - u_{q_m, b_m} \in H^{1,2}(Q)$ and
\[ \frac{\|z_{p_m, q_m, a_m, b_m}(T, \cdot)\|_{H^{2+\gamma}(\Omega)} + \|z_{p_m, q_m, a_m, b_m}\|_{H^1(\Sigma)}}{\|p_m - q_m\|_{H^{\gamma}(\Omega)}} \rightarrow 0 \quad \text{as } m \rightarrow +\infty. \quad (2.22) \]
We set
\[ w_m = \frac{zp_m,q_m,a_m,b_m}{\|p_m - q_m\|_{H^\gamma(\Omega)}}, \quad r_m = \frac{q_m - p_m}{\|p_m - q_m\|_{H^\gamma(\Omega)}}. \]
From (2.20) and (2.21) it follows that \( w_m \) satisfies the equations
\[
\begin{align*}
\partial_t w_m &= Aw_m - p_m(x)w_m + r_m(x)u_{q_m,b_m}(t,x) \quad \text{in } (0, +\infty) \times \Omega, \\
\partial_{\nu_a} w_m &= 0 \quad \text{on } (0, +\infty) \times \partial\Omega,
\end{align*}
\]
(2.23)
By Lemma 2.3 where we set \( \gamma_* = \gamma \), the sequence \( u_{q_m,b_m} \) is bounded in \( C^{1+\frac{\gamma}{2},2+\gamma}(\overline{\Omega}) \). Therefore, since the sequence \( p_m \) is bounded in \( C^{\gamma_1}(\overline{\Omega}) \) and \( \|r_m\|_{L^2(\Omega)} \leq 1 \) for \( m \in \mathbb{N} \), applying Lemma 2.2 to (2.23), for any \( \varepsilon > 0 \) we obtain
\[ \|w_m\|_{H^{1,2}((\varepsilon,T-\varepsilon) \times \Omega)} \leq C\varepsilon. \]
(2.24)
We fix \( \tilde{T} > 0 \) arbitrarily. The a priori estimate for parabolic equations implies
\[ \|w_m\|_{H^{1,2}((\varepsilon,\tilde{T}) \times \Omega)} \leq C\varepsilon. \]
Indeed, we observe that \( v_m = (t-2\varepsilon)\partial_t w_m \) solves the initial boundary value problem
\[
\begin{align*}
\partial_t v_m &= Av_m - p_m(x)v_m + \partial_t w_m + r_m(t-2\varepsilon)\partial_t u_{q_m,b_m}(t,x) \quad \text{in } (2\varepsilon, \tilde{T}) \times \Omega, \\
\partial_{\nu_a} v_m &= 0 \quad \text{on } (0, \tilde{T}) \times \partial\Omega, \quad v_m(2\varepsilon, \cdot) = 0.
\end{align*}
\]
(2.25)
It is known that the solution \( v_m \) to (2.25) belongs to the space \( H^{1,2}((2\varepsilon, \tilde{T}) \times \Omega) \) for all \( \tilde{T} > 2\varepsilon \) (e.g., Theorem 5 (pp.360-361) in [6]). Thus (2.24) is verified. 

By (2.24), we have
\[ \|\partial_j w_m\|_{L^2((3\varepsilon,\tilde{T}) \times \Omega)}, \quad \|\partial_j \partial_k w_m\|_{L^2((3\varepsilon,\tilde{T}) \times \Omega)}, \quad \|\partial_j w_m\|_{L^2((3\varepsilon,\tilde{T}) \times \Omega)} \leq C\varepsilon \]
for \( 1 \leq j, k \leq n \). Therefore there exist \( w, W_j, W_{jk}, W_0 \in L^2((3\varepsilon, \tilde{T}) \times \Omega) \) such that we can extract subsequences, denoted by the same notations, to have \( w_m \rightarrow w, \partial_j w_m \rightarrow W_j, \partial_j \partial_k w_m \rightarrow W_{jk}, \partial_t w_m \rightarrow W_0 \) weakly in \( L^2((3\varepsilon, \tilde{T}) \times \Omega) \) as \( m \rightarrow +\infty \) for \( 1 \leq j, k \leq n \). Noting that
\[ d\langle \partial_j w_m, \varphi \rangle_D = -d\langle w_m, \partial_j \varphi \rangle_D \rightarrow -d\langle w, \partial_j \varphi \rangle \quad \text{as } m \rightarrow \infty \]
for all \( \varphi \in \mathcal{D} := C^\infty_0((3\varepsilon, \tilde{T}) \times \Omega) \) and \( \partial_j w_m \rightarrow W_j \) in \( \mathcal{D}' \), etc., we can easily verify that \( W_j = \partial_j w, W_{jk} = \partial_j \partial_k w, W_0 = \partial_t w \) for \( 1 \leq j, k \leq n \). Here \( \mathcal{D}' \) denotes the space of distributions in \( (3\varepsilon, \tilde{T}) \times \Omega \). Hence,
\[
\begin{align*}
& w_m \rightarrow w, \quad \partial_j w_m \rightarrow \partial_j w, \quad \partial_j \partial_k w_m \rightarrow \partial_j \partial_k w, \quad 1 \leq j, k \leq n, \\
& \partial_t w_m \rightarrow \partial_t w \quad \text{weakly in } L^2((3\varepsilon, \tilde{T}) \times \Omega) \quad \text{as } m \rightarrow +\infty.
\end{align*}
\]
(2.26)
We arbitrarily fix $\gamma_s$ satisfying $0 < \gamma_s < \gamma$. Moreover, the Ascoli-Arzelà theorem yields that the embeddings $C^{\gamma_s}(\Omega) \subset C^{\gamma_s}(\overline{\Omega})$ and $C^{1+\frac{\gamma_s}{2}}([0, \bar{T}] \times \overline{\Omega}) \subset C^{1+\frac{\gamma_s}{2}}([0, \bar{T}] \times \overline{\Omega})$ are compact by $\gamma_s < \gamma$. Taking subsequences, denoted by the same notations, we obtain

$$p_m \to p, \quad q_m \to q \quad \text{in} \quad C^{\gamma_s}(\overline{\Omega}), \quad u_{q_m,b_m} \to u_{q,b} \quad \text{in} \quad C^{1+\frac{\gamma_s}{2}}([0, \bar{T}] \times \overline{\Omega}) \quad (2.27)$$

as $m \to +\infty$ by $p_m, q_m \in \mathcal{P}$, $b_m \in \mathcal{A}$, and Lemma 2.3 for the estimate of solutions in $C^{1+\frac{\gamma_s}{2}}([0, \bar{T}] \times \overline{\Omega})$.

Extracting a subsequence if necessary, we can assume that $r_m$ converges weakly in $H^\gamma(\Omega)$ to $r$ as $m \to +\infty$. By the Rellich-Kondrashov theorem (see Theorem 1.4.3.2 (p.26) in Grisvard [8]),

$$\lim_{m \to +\infty} \|r_m - r\|_{L^2(\Omega)} = 0. \quad (2.28)$$

Furthermore we claim that

$$\|r\|_{L^2(\Omega)} \geq \frac{1}{C}. \quad (2.29)$$

Indeed, by the definition of $r_m$, by means of (2.18), we can obtain

$$\|r_m\|_{H^\gamma(\Omega)} = 1 \leq \tilde{C} \left( \|r_m\|_{L^2(\Omega)} + \frac{\|z_{p_m,q_m,a_m,b_m}(T, \cdot)\|_{H^{2+\gamma}(\Omega)} + \|z_{p_m,q_m,a_m,b_m}\|_{H^1(\Sigma)}}{\|p_m - q_m\|_{H^\gamma(\Omega)}} \right). \quad (2.30)$$

Passing to the limit in this inequality as $m \to +\infty$, by (2.22) and (2.28) we obtain

$$\tilde{C} \left( \|r_m\|_{L^2(\Omega)} + \frac{\|z_{p_m,q_m,a_m,b_m}(T, \cdot)\|_{H^{2+\gamma}(\Omega)} + \|z_{p_m,q_m,a_m,b_m}\|_{H^1(\Sigma)}}{\|p_m - q_m\|_{H^\gamma(\Omega)}} \right) \to \tilde{C} \|r\|_{L^2(\Omega)}. \quad (2.30)$$

Hence, (2.30) yields $\tilde{C}\|r\|_{L^2(\Omega)} \geq 1$. Thus (2.29) is proved.

Passing to the limit in (2.21), by (2.27) we obtain

$$\begin{cases} 
\partial_t u_{q,b} = Au_{q,b} - qu_{q,b}, \quad (t, x) \in (0, +\infty) \times \Omega, \\
\partial_{\nu_A} u_{q,b} = 0 \quad \text{on} \quad (0, +\infty) \times \partial \Omega, \\
u_{q,b}(0, \cdot) = b \quad \text{in} \quad \Omega. \quad (2.31)
\end{cases}$$

We choose $\bar{T} = 2T$. Using the trace theorem and (2.24), we have

$$\|w_m\|_{L^2((3\varepsilon, 2T) \times \partial \Omega)} \leq C_\varepsilon \quad \text{and} \quad \|\partial_{\nu_A} w_m\|_{L^2((3\varepsilon, 2T) \times \partial \Omega)} \leq C_\varepsilon, \quad m \in \mathbb{N},$$

and so we can similarly verify that $w_m \to w$, $\partial_{\nu_A} w_m \to \partial_{\nu_A} w$ weakly in $L^2((3\varepsilon, 2T) \times \partial \Omega)$ after extracting subsequences. By (2.22), we see that $w_m \to 0$ in $H^1(\Sigma)$ as $m \to +\infty$. Therefore, we obtain $w = 0$ on $(3\varepsilon, T) \times \Gamma$. Furthermore $\partial_{\nu_A} w = 0$ on $(3\varepsilon, 2T) \times \partial \Omega$. Similarly, in terms of (2.22), we can see that $w(T, \cdot) = 0$ in $\Omega.$
Consequently, we can pass to the limit in (2.23) and by means of (2.26) and (2.27) we obtain

$$\begin{cases}
\partial_tw = Aw - p(x)w + r(x)u_{q,b}(t,x) & \text{in } (3\varepsilon, 2T) \times \Omega, \ w(T, \cdot) = 0,
\partial_{\nu_A}w = 0 & \text{on } (3\varepsilon, 2T) \times \partial\Omega, \ w = 0 & \text{on } (3\varepsilon, T) \times \Gamma.
\end{cases} \quad (2.32)$$

It is known (see e.g., Pazy [22], Tanabe [25]) that the function $u_{q,b}(t, \cdot) : t \to L^2(\Omega)$ is analytic in $t$ for all $t > 0$. Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary such that $\Omega \subset \tilde{\Omega}$, $\partial\Omega \setminus \Gamma \subset \partial\tilde{\Omega}$ and $\tilde{\Omega} \setminus \overline{\Gamma}$ is an open set.

We make the zero extensions of $w(t, \cdot)$, $u_{q,b}(t, \cdot)$ and $r$ to $\tilde{\Omega} \setminus \overline{\Omega}$. We note that after the extensions, $w(t, \cdot) \in L^2(\tilde{\Omega})$ for $3\varepsilon < t < 2T$ and $u_{q,b}(t, \cdot) \in L^2(\tilde{\Omega})$ for $t > 0$. Moreover, the function $u_{q,b}(t, \cdot) : t \to L^2(\tilde{\Omega})$ is analytic in $t > 0$. Hence, the function $ru_{q,b}(t, \cdot) : t \to L^2(\tilde{\Omega})$ is analytic in $t > 0$. Next we extend the coefficients $a_{kj}, b_k, p$ on $\tilde{\Omega}$ keeping the regularity (1.4) and the positivity (1.5).

Let $\chi_{\Omega}(x) = \begin{cases} 1, & x \in \Omega, \\
0, & x \notin \Omega. \end{cases}$

Consider an initial boundary value problem

$$\begin{cases}
P\tilde{w} := \partial_t\tilde{w} - Aw + p(x)\tilde{w} = r(x)\chi_{\Omega}(x)u_{q,b}(t,x) & \text{in } (3\varepsilon, 2T) \times \tilde{\Omega},
\partial_{\nu_A}\tilde{w} = 0 & \text{on } (3\varepsilon, 2T) \times \partial\tilde{\Omega}, \ \tilde{w}(3\varepsilon, \cdot) = w(3\varepsilon, \cdot) & \text{in } \tilde{\Omega},
\end{cases} \quad (2.33)$$

where $w(\varepsilon, \cdot)$ is extended by zero in $\tilde{\Omega} \setminus \overline{\Omega}$. We define $\tilde{w} = w$ if $(t, x) \in (3\varepsilon, T) \times \Omega$ and $\tilde{w} = 0$ if $(t, x) \in (3\varepsilon, T) \times (\tilde{\Omega} \setminus \overline{\Omega})$. Since $w = \partial_{\nu_A}w = 0$ on $(3\varepsilon, T) \times \Gamma$ by (2.32), it follows that $\tilde{w} \in H^{1,2}((3\varepsilon, T) \times \Omega)$. Hence $Pw \in L^2((3\varepsilon, T) \times \Omega)$. On the other hand $Pw|_{(3\varepsilon, T) \times \Omega} = ru_{q,b}$ and $Pw|_{(3\varepsilon, T) \times (\tilde{\Omega} \setminus \Omega)} = 0$. Therefore

$$Pw = r\chi_{\Omega}u_{q,b} \quad \text{on } (3\varepsilon, T) \times \tilde{\Omega}$$

and

$$\partial_{\nu_A}w = 0 \quad \text{on } (3\varepsilon, T) \times \partial\tilde{\Omega}, \quad w(3\varepsilon, \cdot) = w(3\varepsilon, \cdot) \quad \text{in } \tilde{\Omega}.$$ 

Then the uniqueness of solution to an initial boundary value problem yields

$$\tilde{w} = \bar{w} \quad \text{on } (3\varepsilon, T) \times \tilde{\Omega}.$$ 

It is shown in [17] and [16] that $\bar{w}(t, \cdot) : t \to L^2(\Omega)$ is analytic in $t$ for all $t > 3\varepsilon$. Hence, since $w = 0$ in $(3\varepsilon, T) \times (\tilde{\Omega} \setminus \overline{\Omega})$, we obtain $\bar{w}(t,x) = 0$ for all $t > 3\varepsilon$ and $x \in \tilde{\Omega} \setminus \Omega$. Since $\tilde{w} \in H^{1,2}((3\varepsilon, 2T) \times \Omega)$, we have

$$\bar{w}|_{(3\varepsilon, 2T) \times \Gamma} = \partial_{\nu_A}\bar{w}|_{(3\varepsilon, 2T) \times \partial\Omega} = 0,$$
where $\Gamma_0 = \partial(\bar{\Omega} \setminus \bar{\Omega}) \cap \partial \Omega$. Again the uniqueness of solution to the initial boundary value problem implies $w = \tilde{w}$ in $(3\varepsilon, 2T) \times \Omega$, and we have

$$w|_{(3\varepsilon, 2T) \times \Gamma_0} = 0.$$  

(2.34)

Similarly to (2.24), we can verify that $w, \partial_tw \in H^{1,2}((4\varepsilon, 2T) \times \Omega)$.

By Lemma 2.4, there exists a constant $c_0 > 0$ such that

$$u_{q,b}(T, x) \geq c_0 \text{ for all } x \in \Omega.$$  

(2.35)

Then, in terms of (2.34) and (2.35), we apply the uniqueness result for the inverse source problem with spatial data at not final time to (2.32) (e.g., Theorem 3.1 in Imanuvilov and Yamamoto [11]), so that we reach $r \equiv 0$ in $\Omega$. This contradicts $r \neq 0$ in (2.29). Thus (2.22) is false, and the proof of (2.19) is complete. Thus the proof of Theorem 1 is complete. □

3. PROOF OF THEOREM 2

We rely on what is called the Reznitskaya transform (e.g., Romanov [23]), which is the main idea for a similar inverse problem solved by Klibanov [18].

Since function $t \to u(p)(t, \cdot)|_{\partial \Omega}$ and $t \to u(q)(t, \cdot)|_{\partial \Omega}$ are analytic in $t > 0$ (see e.g., Pazy [22], Tanabe [25]) in view of $u(p) = u(q)$ on $(0, T) \times \Gamma_0$, we obtain

$$u(p) = u(q) \text{ on } (0, \infty) \times \Gamma_0.$$  

(3.1)

Consider an initial-boundary value problem for a hyperbolic equation:

$$\partial_t^2 w(t, x) - \Delta w + p(x)w = 0 \text{ in } (0, \infty) \times \Omega,$$  

(3.2)

$$\partial_{\nu_A} w = 0 \text{ on } (0, \infty) \times \partial \Omega$$  

(3.3)

and

$$w(\cdot, 0) = u_0, \quad \partial_t w(\cdot, 0) = 0 \text{ in } \Omega.$$  

(3.4)

By $u_0 \in H^3(\Omega)$ and $\partial_{\nu_A} u_0 = 0$ on $\partial \Omega$, and $p, q \in W^{1,\infty}(\Omega)$, we can verify

$$w(p), w(q) \in C^k([0, \infty); H^{3-k}(\Omega)), \quad k = 0, 1, 2, 3.$$  

(3.5)

We can prove (3.5) by Theorem 8.2 (p.275) in [20]. Indeed in that theorem, setting $V := H^1(\Omega) = \mathcal{D}((-\Delta_N)^{1/2})$ where $\Delta_N v = \sum_{j=1}^n \partial^2_{ij} v$ with $\mathcal{D}(\Delta_N) = \{ v \in H^2(\Omega); \partial_v v|_{\partial \Omega} = 0 \}$, we can accomplish the proof of (3.5) by the theorem and the elliptic regularity.

Moreover we apply a standard energy estimate and, in view of (1.12), we have

$$\|w(p)(\cdot, s)\|_{L^2(\Omega)} \leq C_1 \|\nabla w(p)\|_{L^2(\Omega)},$$  

where $\Gamma_0 = \partial(\bar{\Omega} \setminus \bar{\Omega}) \cap \partial \Omega$. Again the uniqueness of solution to the initial boundary value problem implies $w = \tilde{w}$ in $(3\varepsilon, 2T) \times \Omega$, and we have

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Moreover we apply a standard energy estimate and, in view of (1.12), we have

$$\|w(p)(\cdot, s)\|_{L^2(\Omega)} \leq C_1 \|\nabla w(p)\|_{L^2(\Omega)},$$  

and we can find a constant $C_2 > 0$ such that
\[ \| w(p)(\cdot, s) \|_{H^1(\Omega)} \leq C_2 e^{C_2 s} \quad \text{for all } s > 0, \]
where $C_2 > 0$ depends on $p, u_0$. Therefore there exists
\[ \tilde{u}(p)(t, x) := \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} w(p)(s, x) ds, \quad t > 0, \ x \in \Omega, \]
which is called the Reznitskaya transform in the context of inverse problems.

We can prove (e.g., Section 3 of Chapter 6 of Romanov [23]) that $\tilde{u}(p)$ satisfies first two equations of (1.11) in $(0, \infty) \times \Omega$. Let us show that the initial condition also holds true. we remind that $w(p) \in C(\mathbb{R}^1; L^2(\Omega))$.

\[
\| \tilde{u}(p)(t, \cdot) - u_0(0, \cdot) \|_{L^2(\Omega)} = \left\| \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} (w(p)(\tau, \cdot) - u_0(\cdot)) d\tau \right\|_{L^2(\Omega)}
\]
\[ \leq \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} \| w(p)(\tau, \cdot) - u_0 \|_{L^2(\Omega)} d\tau. \]

For any $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) > 0$ such that
\[ \| w(p)(\tau, \cdot) - u_0 \|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} \quad \text{for all } 0 < \tau < \delta(\varepsilon). \]

Therefore
\[
\int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{s^2}{4t}} \| w(p)(\tau, \cdot) - u_0 \|_{L^2(\Omega)} d\tau
= \int_0^{\delta(\varepsilon)} \frac{1}{\sqrt{t}} e^{-\frac{s^2}{4t}} \| w(p)(\tau, \cdot) - u_0 \|_{L^2(\Omega)} d\tau + \int_{\delta(\varepsilon)}^\infty \frac{1}{\sqrt{t}} e^{-\frac{s^2}{4t}} \| w(p)(\tau, \cdot) - u_0 \|_{L^2(\Omega)} d\tau.
\]

We can estimate
\[
\int_{\delta(\varepsilon)}^\infty \frac{1}{\sqrt{t}} e^{-\frac{s^2}{4t}} \| w(p)(\tau, \cdot) - u_0 \|_{L^2(\Omega)} d\tau \leq C_2 \int_{\delta(\varepsilon)}^\infty \frac{1}{\sqrt{t}} e^{-\frac{s^2}{4t}} e^{C_2 \tau} d\tau
= C_2 \int_{\delta(\varepsilon)}^\infty 2e^{C_2 t} e^{-\frac{1}{4t}(\tau - 2C_2)^2} \frac{d\tau}{2\sqrt{t}} \leq 2C_2 e^{C_2 t} \int_{\delta(\varepsilon)}^\infty e^{-\eta^2} d\eta = o(1) \quad \text{as } t \to 0.
\]

Consequently we can choose a constants $t_0(\varepsilon) > 0$ sufficiently small, so that
\[ \| \tilde{u}(p)(t, \cdot) - u_0 \|_{L^2(\Omega)} \leq \varepsilon \quad \text{if } 0 < t < t_0(\varepsilon). \]

Thanks to the uniqueness of solution of (1.11), we obtain
\[ u(p)(t, x) = \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} w(p)(s, x) ds, \quad u(q)(t, x) = \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} w(q)(s, x) ds, \quad t > 0, \ x \in \Omega. \]

By (3.1), we reach
\[ \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} (w(p)(s, x) - w(q)(s, x)) ds = 0 \quad t > 0, \ x \in \Gamma_0. \]
Making the change of variables \( \tilde{s} = e^{-\frac{s^2}{2}} \), we have
\[
\int_0^1 \frac{1}{\sqrt{-\log \tilde{s}}} \tilde{w}(p) - \tilde{w}(q) (2\sqrt{-\log \tilde{s}, x}) d\tilde{s} = 0 \quad \text{for all } t > 0.
\]
Setting in the above formula \( t = \frac{1}{n} \), choosing \( t > 0 \) such that \( \frac{1}{t} - 1 \) can cover \( \mathbb{N} \cup \{0\} \), we obtain that the function \( \frac{\tilde{w}(p) - \tilde{w}(q)}{\sqrt{-\log \tilde{s}}} \) is orthogonal to all the polynomials on the interval \([0, 1]\). Therefore it is identically equal to zero by the Weierstrass’ polynomial approximation theorem. This implies
\[
w(p)(\sqrt{\eta}, x) - w(q)(\sqrt{\eta}, x) = 0 \quad \text{for all } \eta > 0 \text{ and } x \in \Gamma_0.
\]
Hence,
\[w(p)(t, x) = w(q)(t, x) \quad \text{for all } t > 0 \text{ and } x \in \Gamma_0. \tag{3.6}\]

Now we reduced the inverse parabolic problem to an inverse hyperbolic problem for (3.2) - (3.4) with (3.6). Setting \( y := w(p) - w(q), f := q - p \) and \( R(t, x) := w(q)(t, x) \), we have
\[
\begin{align*}
\partial_t^2 y - \Delta y &= p(x)y + f(x)R(t, x), \quad t > 0, \ x \in \Omega, \\
\partial_{\nu_A} y &= 0 \quad \text{on } (0, \infty) \times \partial \Omega, \tag{3.7} \\
y(0, \cdot) &= \partial_t y(0, \cdot) = 0 \quad \text{in } \Omega
\end{align*}
\]
and
\[y = 0 \quad \text{on } (0, \infty) \times \Gamma_0. \tag{3.8}\]

For direct application of Imanuvilov and Yamamoto \cite{12}, we will make an extension of \( y \) to a wider spatial domain. First we choose an open smooth domain \( \omega \subset \mathbb{R}^n \setminus \overline{\Omega} \) such that \( \overline{\omega} \cap \partial \Omega \subset \Gamma_0 \). We fix \( T > 0 \) satisfying \( T > \sup_{x \in \Omega} |x - x_0| \). Over \( \Gamma_0 \subset \partial \Omega \), we take the zero extensions of both \( y \) and \( f \) to \( \widetilde{\Omega} := \Omega \cup \omega \cup (\partial \omega \cap \Gamma_0) \). By \( \partial_{\nu_A} y = y = 0 \) on \( (0, T) \times \Gamma_0 \), we can readily see that
\[y \in C^1([0, T]; H^2(\overline{\Omega})) \cap C^2([0, T]; H^1(\overline{\Omega})) \cap C^3([0, T]; L^2(\overline{\Omega})). \tag{3.9}\]

We can extend also \( R \) to \( \overline{\Omega} \) such that
\[|R(0, x)| = |u_0(x)| \geq \delta > 0 \quad \text{on } \overline{\Omega}, \quad R \in H^1(0, T; L^\infty(\overline{\Omega})). \tag{3.10}\]
Therefore, we can obtain
\[
\begin{align*}
\partial_t^2 y &= \Delta y - p(x)y + f(x)R(t, x), \quad t > 0, \ x \in \overline{\Omega}, \\
\partial_{\nu_A} y &= 0 \quad \text{on } (0, \infty) \times \partial \overline{\Omega}, \\
y(0, \cdot) &= \partial_t y(0, \cdot) = 0 \quad \text{in } \overline{\Omega}, \\
y = 0 \quad \text{in } (0, T) \times \omega.
\end{align*}
\]
In view of the construction of $\omega$ and (1.13), we note that

$$\{x \in \partial \Omega; (x - x_0) \cdot \nu(x) \geq 0\} \subset \partial \omega.$$ 

Hence, in terms of (3.9) and (3.10), we can apply the uniqueness for an inverse hyperbolic problem (e.g., Corollary 3.1 in [12]) to conclude that $f = 0$ in $\tilde{\Omega}$, that is, $p = q$ in $\Omega$. Thus the proof of Theorem 2 is complete. ■

4. Proof of Theorem 3

We define an operator by

$$(Kv)(t) := \mu(0)v(t) + \int_0^t \mu'(t-s)v(s)ds, \quad 0 < t < T. \quad (4.1)$$

Let $y$ satisfy (1.14) and $y(0, \cdot) = 0$ in $\Omega$ and $y = |\nabla y| = 0$ on $(0, T) \times \Gamma$. Consider an equation with respect to $z(t, x)$:

$$\partial_t y(t, x) = (Kz)(t, x), \quad 0 < t < T, x \in \Omega. \quad (4.2)$$

Since $\mu(0) \neq 0$, the operator $K$ is a Volterra operator of the second kind, and we see that $K^{-1} : H^1(0, T) \rightarrow H^1(0, T)$ exists and is bounded. Therefore, $z(\cdot, x) \in L^2(0, T)$ is well defined for each $x \in \Omega$, and

$$\partial_t y(t, x) = \mu(0)z(t, x) + \int_0^t \mu'(t-s)z(s, x)ds, \quad 0 < t < T, x \in \Omega \quad (4.3)$$

by $\partial_t y \in H^{1,2}(Q)$.

Since $y(0, \cdot) = 0$ in $\Omega$ and

$$\mu(0)z(t, x) + \int_0^t \mu'(t-s)z(s, x)ds = \partial_t \left( \int_0^t \mu(t-s)z(s, x)ds \right),$$

we obtain

$$y(t, x) = \int_0^t \mu(t-s)z(s, x)ds, \quad 0 < t < T, x \in \Omega,$$

that is,

$$y(t, x) = \int_0^t \mu(s)z(t-s, x)ds, \quad 0 < t < T, x \in \Omega. \quad (4.4)$$

We will prove that $z \in H^{1,2}(Q)$ satisfies

$$\partial_t z(t, x) = Az(t, x) - p(x)z \quad \text{in} \ (0, t_*) \times \Omega, \quad (4.5)$$

where $t_* \in (0, T)$ is some constant, and

$$z = |\nabla z| = 0 \quad \text{on} \ (0, T) \times \Gamma. \quad (4.6)$$
We can readily verify (4.6), because \( \partial_t y = |\nabla \partial_t y| = 0 \) on \((0, T) \times \Gamma\) implies \( Kz = |K(\nabla z)| = 0 \) on \((0, T) \times \Gamma\), so that the injectivity of \( K \) directly yields (4.6).

Using \( \partial_t y \in H^{1,2}(Q) \subset C([0, T]; L^2(\Omega)) \) by (4.3), we have
\[
\partial_t y(0, x) = \mu(0)z(0, x), \quad x \in \Omega.
\]
On the other hand, substituting \( t = 0 \) in (1.14) and applying (1.15), we obtain
\[
\partial_t y(0, x) = \mu(0)f(x), \quad x \in \Omega.
\]
Hence \( \mu(0)z(0, x) = \mu(0)f(x) \) for \( x \in \Omega \). By \( \mu(0) \neq 0 \), we reach
\[
z(0, x) = f(x), \quad x \in \Omega. \tag{4.7}
\]

Now we will prove (4.5). In terms of (4.4) and (4.7), we have
\[
\partial_t y(t, x) = \mu(t)z(0, x) + \int_0^t \mu(s)\partial_t z(t - s, x)ds
\]
and
\[
(A - p(x))y(t, x) = \int_0^t \mu(s)(A - p(x))z(t - s, x)ds, \quad 0 < t < T, \quad x \in \Omega.
\]
Consequently (1.14) implies
\[
\mu(t)f(x) = (\partial_t y - (Ay - p(x)y))(t, x)
\]
\[
= \mu(t)f(x) + \int_0^t \mu(s)(\partial_t z - (A - p(x))z)(t - s, x)ds,
\]
that is,
\[
\int_0^t \mu(s)(\partial_t z - (A - p(x))z)(t - s, x)ds = 0, \quad 0 < t < T, \quad x \in \Omega.
\]
Hence, setting \( Z(s) := \|((\partial_s z - (A - p(x))z)(s, \cdot))\|_{L^2(\Omega)} \) for \( 0 < t < T \), we reach
\[
\int_0^t \mu(s)Z(t - s)ds = 0, \quad 0 < t < T.
\]
By the Titchmarsh convolution theorem (e.g., Titchmarsh [26]), there exists \( t_* \in [0, T] \) such that
\[
\mu(s) = 0 \quad \text{for} \quad 0 < s < T - t_*, \quad Z(s) = 0 \quad \text{for} \quad 0 < s < t_*.
\]
Since \( \mu \neq 0 \) in \([0, T]\), we see that \( T - t_* < T \), that is, \( t_* > 0 \). Thus the verification of (4.5) is complete. \( \square \)
Applying the classical unique continuation by (4.5) and (4.6), we obtain $z = 0$ in $(0, t_*) \times \Omega$. As for the unique continuation, we can refer for example to Mizohata [21], Saut-Scheurer [24] and see also Yamamoto [28], [29]. Thus (4.7) implies $f = 0$ in $\Omega$. The proof of Theorem 3 is complete. ■

5. Concluding remarks

5.1.

In this article, we consider two types of inverse problems for parabolic equations:

- **Inverse coefficient problem** of determining a spatially varying $p(x)$ in

\[
\begin{align*}
\partial_t u(t, x) &= Au(t, x) - p(x)u(t, x), \quad 0 < t < T, \ x \in \Omega, \\
\partial_\nu z &= 0 \quad \text{on} \ (0, T) \times \partial \Omega, \\
u(t_0, \cdot) &= u_0 : \text{given in } \Omega,
\end{align*}
\]

by $u|_{(0,T)\times\Gamma}$ with subboundary $\Gamma \subset \partial \Omega$. Here $t_0 \in (0, T)$ is given.

- **Inverse source problem** of determining a spatial factor $f(x)$ of source terms in

\[
\begin{align*}
\partial_t y(t, x) &= Ay(t, x) - p(x)y(t, x) + \mu(t)f(x), \quad 0 < t < T, \ x \in \Omega, \\
y(t_0, \cdot) &= u_0 : \text{given in } \Omega,
\end{align*}
\]

by $y, \nabla y$ on $(0, T) \times \Gamma$.

We summarize our results and the existing results.

**Inverse coefficient problems:**

1. **Case** $0 < t_0 < T$: there have been many works and among them we can refer for example to Beilina and Klibanov [3], Bukhgeim and Klibanov [4], Klibanov [18], Imanuvilov and Yamamoto [11] and the references therein. We refer to Yamamoto [28, 29] for the stability. The main method is based on Carleman estimates.

2. **Case** $t_0 = T$: The current article solved and established the global Lipschitz stability. We should remark that the uniqueness and also Hölder stability can be proved by means of the time analyticity of solution $u$. Our method can be extended in the case where the coefficients of the parabolic operator depends on time analytically.

3. **Case** $t_0 = 0$: By an integral transform, we can obtain the uniqueness, once we have uniqueness for the corresponding hyperbolic inverse coefficient problems. This idea can be found in the book by Romanov [23], Klibanov [18]. By this method, it is impossible to choose any small subboundary where additional data are taken, which seems to be unnatural for the inverses parabolic problem. One could prove conditional stability, but it is expected to be of rather weak rate, which is subject to
the transformation method, and we do not know whether such stability is reasonable by the nature of the inverse problem.

**Inverse source problems:** The main concern is the uniqueness and we should expect the uniqueness without boundary condition on the whole lateral boundary \((0, T) \times \partial \Omega\).

1. Case \(0 < t_0 < T\): The method by Carleman estimate gives the uniqueness and also conditional stability in some subdomain \(\subset \Omega\) if \(\mu(t_0) \neq 0\).
2. Case \(t_0 = T\): still open.
3. Case \(t_0 = 0\): The current article establishes the uniqueness if \(\mu(0) \neq 0\). However under more generous but reasonable condition \(\mu \neq 0\) on \([0, T]\), we do not know the uniqueness in general.

5.2. For the case \(t_0 = 0\), our available method seems only the Reznitskaya transform. Within adequate regularities, the transform transfers the uniqueness for inverse hyperbolic problems to the uniqueness for inverse parabolic problem. Thus the primary technical highlight is inverse hyperbolic problems.

When one can obtain the uniqueness for more general inverse hyperbolic problems, we can make it produce the uniqueness for the corresponding inverse parabolic problems not only in the case of \(A = \Delta\).

In order to treat general \(A\), we need some restrictive assumptions for the principal coefficients \(a_{jk}\) of \(A\). Now we formulate such assumptions as follows. We set \(i := \sqrt{-1}\), \(Q := (0, T) \times \Omega\), \(\nabla := (\partial_1, \ldots, \partial_n)\), \(\nabla_{t,x} := (\partial_t, \partial_1, \ldots, \partial_n)\), \(\nabla_\xi := (\frac{\partial}{\partial \xi_0}, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n})\) for \(\xi := (\xi_0, \ldots, \xi_n)\). For any \(C^1\) functions \(g(x, \xi)\) and \(f(x, \xi)\) we introduce the Poisson bracket of these functions

\[
\{f, g\} = \sum_{j=1}^{n} \partial_\xi_j f \partial_{x_j} g - \partial_{x_j} f \partial_\xi_j g + \partial_{\xi_0} f \partial_t g - \partial_t f \partial_{\xi_0} g.
\]

For the hyperbolic operator:

\[
\tilde{P}(x, D)v := \partial_t^2 v - \sum_{j,k=1}^{n} \partial_j (a_{jk}(x) \partial_k v) + \sum_{j=1}^{n} b_j(x) \partial_j v + p(x) v
\]

we define principal symbol of this operator \(\tilde{p}(x, \xi)\) by formula

\[
\tilde{p}(x, \xi) := \xi_0^2 - a(x, \xi, \xi), \quad x \in \Omega, \quad \xi = (\xi_0, \ldots, \xi_n).
\]
First we assume that a function $\tilde{\psi}(t, x)$ is pseudo-convex with respect to $\tilde{P}$ in $Q$, that is,

\[
\begin{cases}
\nabla_{t,x} \tilde{\psi}(t, x) \neq 0 \text{ for all } (t, x) \in \overline{Q}, \\
\frac{\{\tilde{p}(x, \xi - i\tau \nabla_{t,x} \tilde{\psi}(t, x)), \tilde{p}(x, \xi + i\tau \nabla_{t,x} \tilde{\psi}(t, x))\}}{2i\tau} > 0
\end{cases}
\]

for all $(t, x, \xi) \in Q \times (\mathbb{R}^{n+1} \setminus \{0\})$ and $\tau > 0$ satisfying

\[\tilde{p}(x, \xi + i\tau \nabla_{t,x} \tilde{\psi}(t, x)) = 0.\]

(5.4)

Second we assume that a subboundary $\Gamma_1 \subset \partial \Omega$ satisfies

\[a(x, \nu(x), \nabla \tilde{\psi}(t, x)) < 0 \text{ for all } (t, x) \in [0, T] \times \partial \Omega \setminus \Gamma_1,\]

(5.5)

and

\[\partial_t \tilde{\psi}(T, x) < 0 \text{ and } \partial_t \tilde{\psi}(0, x) > 0 \text{ for all } x \in \overline{\Omega}.\]

(5.6)

Finally we assume that for each $y \in \partial \Omega \setminus \Gamma_1$, there exist an open neighborhood $U(y)$ of $y$ and a function $\psi_1$ defined in $U(y)$ such that $\tilde{\psi}_1$ is pseudo-convex with respect to the symbol $\tilde{P}$ in $(0, T) \times U(y)$ and

\[a(x, \nu(x), \nabla \psi_1(x)) > 0 \text{ for all } x \in \partial \Omega \cap U(y).\]

(5.7)

Under the assumptions (5.4) - (5.7), replacing the subboundary $\Gamma_0$ defined in (1.13) by $\Gamma_1$ and choosing sufficiently large $T > 0$ for the principal coefficients $a_{jk}(x)$, we can generalize Theorem 2 and obtain the same uniqueness in determining the zeroth-order coefficient $p(x)$ for (1.1)-(1.3) with $t_0 = 0$ by extra data $u|_{\Gamma_1 \times (0, T)}$.

The conditions (5.4) - (5.7) are satisfied for example in the case $a_{jk} = \delta_{jk}$ but in general it is not at all simple to find a generous sufficient condition on $a_{jk}$ with $1 \leq j, k \leq n$ admitting $\tilde{\psi}$ and $\psi_1$ which satisfy (5.4) - (5.7). For example, Imanuvilov and Yamamoto [13], [14] discuss such generous conditions respectively for the cases of the Dirichlet and the Neumann type of boundary conditions. However we do not exploit more.

**Acknowledgement**

M. Yamamoto is supported by Grant-in-Aid for Scientific Research (A) 20H00117 and Grant-in-Aid for Challenging Research (Pioneering) 21K18142, JSPS.
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