STABILITY OF THE MARKOV OPERATOR
AND SYNCHRONIZATION
OF MARKOVIAN RANDOM PRODUCTS

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Abstract. We study Markovian random products on a large class of “m-
dimensional” connected compact metric spaces (including products of closed
intervals and trees). We introduce a splitting condition, generalizing the clas-
sical one by Dubins and Freedman, and prove that this condition implies the
asymptotic stability of the corresponding Markov operator and (exponentially
fast) synchronization.

1. Introduction

The study of random products of maps has a long history and for independent
and identically distributed (i.i.d.) random products the first fundamental results
go back to Furstenberg [7]. In this paper, we study Markovian random products.
Given a compact metric space $M$, finitely many continuous maps $f_i: M \to M$,
$i = 1, \ldots, k$, and a Markov shift $(\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)$ (see the precise definition below) the
map $\varphi: \mathbb{N} \times \Sigma_k \times M \to M$ defined by

\begin{equation}
\varphi(n, \omega, x) \overset{\text{def}}{=} f_{\omega_n-1} \circ \cdots \circ f_{\omega_0}(x) \overset{\text{def}}{=} f^n_{\omega}(x) \quad \text{for} \quad n \geq 1,
\end{equation}

is called a random product of maps over the Markov shift (or a Markovian product
of random maps).

The study of random products is of major importance in the study of the as-
ymptotic behavior of Markov chains. This is for example supported by the fact
that every homogenous Markov chain admits a certain representation by an i.i.d.
random product, see Kifer [12]. For additional references on Markovian random
products see [2, 6, 23, 8, 21], and references therein.

Associated to the Markovian random product $\varphi$ there is a family of homogeneous
Markov chains

$Z^x_n(\omega) \overset{\text{def}}{=} (\omega_{n-1}, X^x_n(\omega)), \quad n \geq 1 \text{ and } x \in M,$

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where $X^x$ is the random variable $\omega \mapsto f^n_\omega(x)$. We study the asymptotic behavior of this family (synchronization) and also the dynamics of the corresponding Markov operator.

We will focus on certain “$m$-dimensional” compact spaces, which include products of compact intervals and trees. More precisely, we will assume that the ambient space is of the form $Y^m = Y \times \cdots \times Y$, where $Y$ is a metric space such that every connected non-singleton subset of $Y$ has nonempty interior. In the particular case of the interval (i.e., $Y = [0, 1]$ and $m = 1$), the study of products of random maps goes back to Dubins and Freedman [5], where i.i.d. random products of monotone maps are considered and the asymptotic stability of the corresponding Markov operator is obtained. There they assumed a certain splitting condition on the system (for details see Definition 2.1 and the discussion in Remark 2.2) which we also invoke.

In the above context, we will prove that there is a unique stationary measure and, in fact, show the asymptotic stability of the Markov operator (which implies the former), see Theorem 1. Let us briefly discuss the results which are our main motivation (see, in chronological ordering, [5] [10] [3] [2] [6]). As observed above, in [5], assuming a splitting condition, it is proved the asymptotic stability of the Markov operator associated to an i.i.d. random product of injective maps of the closed interval. This result was generalized to higher dimensional non-decreasing monotone maps in [3]. The asymptotic stability of the Markov operator is studied in [2] for Markovian random products of maps which are contracting in average (this generalizes the results in [10] for contractions). More precisely, they show that there exists a unique stationary measure which attracts every probability measure from a certain “representative” subspace. Finally, in [6] it is introduced the notion of weak hyperbolicity (i.e., no contraction-like assumptions are involved, see the discussion below) and proved that for every weakly hyperbolic Markovian random product there is a unique stationary measure. With a slight abuse of terminology, we can refer to the settings in [2] [6] as hyperbolic-like ones and observe that our setting is “genuinely non-hyperbolic”. Here we consider the Markovian case instead of the i.i.d. case and our approach allows to consider higher dimensional compact metric spaces and continuous maps which may not be injective, in particular extending the results in [3] to a larger class of random products of maps, see Section 2.4.2.

As noted above, we are also interested in the asymptotic behavior of orbits and study synchronization. This phenomenon was first observed by Huygens [9] in the synchronizing movement of two pendulum clocks hanging from a wall and since then has been investigated in several areas, see [19]. A random product $\varphi$ is said to be *synchronizing* if random orbits of different initial points converge to each other with probability 1 (see Section 2.3 for the precise definition). In this context, a first result was obtained by Furstenberg in [7] for i.i.d. products of random matrices on projective spaces. The occurrence of synchronization for i.i.d. random products of circle homeomorphisms is now completely characterized by Antonov’s theorem [1] [14] and its generalization in [17]. We prove that a Markovian random product defined on a connected compact subset of $\mathbb{R}^m$ satisfying the splitting condition is (exponentially fast) synchronizing, see Theorem 3. In the i.i.d. case and when $m = 1$, this result is obtained in [17]. In Section 2.4 we present some classes of higher dimensional random products satisfying the splitting condition, see also Theorem 4.

\footnote{In [2] [6] it is used the terminology *recurrent I.F.S.* for a Markovian random products of maps.}
We note that Theorem 3 is a consequence of a general result of “contraction of volume” in compact spaces, see Theorem 2.

Finally, let us observe that, besides its intrinsic interest, the study of the dynamics of Markov operators associated to random products is closely related to the study of the corresponding step skew products. The latter is currently a subject of intensive research and we refrain to select a list of references. We just would like to mention [13], closely related to our paper, where it is proved that generically Markovian random products of $C^1$ diffeomorphisms on the interval $[0,1]$ (with image strictly contained in $[0,1]$) has a finite number of stationary measures and to each one it corresponds a physical measure of the associated skew product.

This paper is organized as follows. In Section 2 we state the main definitions and the precise statements of our results. In Section 3 we state consequences of the splitting hypothesis for general compact spaces. In Section 4 we draw consequences from the splitting condition in one-dimensional settings. In Section 5 we prove the asymptotic stability of the Markov operator (Theorem 1) and a strong version of the Ergodic Theorem (Corollary 1). In Section 6 we study the synchronization of Markovian random systems (Theorem 3) which will be a consequence of a contraction result for measures for general compact metric spaces (Theorem 2). Finally, in Section 7 we prove Theorem 4 which states sufficient conditions for the splitting property.

2. Main results

2.1. General setting. Let $(E, \mathcal{E})$ be a measurable space and consider a transition probability $P : E \times \mathcal{E} \to [0,1]$, i.e., for every $x \in E$ the mapping $A \mapsto P(x, A)$ is a probability measure on $E$ and for every $A \in \mathcal{E}$ the mapping $x \mapsto P(x, A)$ is measurable with respect to $\mathcal{E}$. Recall that a measure $m$ on $E$ is called a stationary measure with respect to $P$ if

$$m(A) = \int P(x, A) \, dm(x), \quad \text{for every } A \in \mathcal{E}. \quad (2.1)$$

Suppose that $E \overset{\text{def}}{=} \{1, \ldots, k\}$ is a finite set endowed with the discrete sigma-algebra $\mathcal{E}$. Consider the space of unilateral sequences $\Sigma_k = E^\mathbb{N}$ endowed with the product sigma-algebra $\mathcal{F} = \mathcal{E}^\mathbb{N}$. Given a transition probability $P$ on $E$ and stationary measure $m$, there is a unique probability measure $\mathbb{P}$ on $\Sigma_k$ such that the sequence of coordinate mappings on $\Sigma_k$ is a homogeneous Markov chain with probability transition $P$ and starting measure $m$. For details see, e.g., [20, Chapter 1]. The measure $\mathbb{P}$ is the Markov measure of the pair $(P, m)$. Denote by $\sigma$ the shift map on $\Sigma_k$. The measure $\mathbb{P}$ is $\sigma$-invariant and the metric dynamical system $(\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)$ is called a Markov shift.

Throughout this paper $(M, d)$ is a compact metric space. Let $\varphi(n, \omega, x) = f_n^\omega(x)$ be a random product on $M$ over the Markov shift $(\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)$ as in [11]. Fix $x \in M$ and note that the sequence of random variables $X_n^\omega : \omega \mapsto f_n^\omega(x)$, in general, is not a Markov chain on the probability space $(\Sigma_k, \mathcal{F}, \mathbb{P})$. However, it turns out that the sequence $Z_n^\omega(\omega) = (\omega_{n-1}, X_n^\omega(\omega))$, $n \geq 1$, is a Markov chain with range on the space $\hat{M} \overset{\text{def}}{=} E \times M$ (see [2]) with probability transition given by

$$\hat{P}(i, z, B) \overset{\text{def}}{=} P(i, \{j : (j, f_j(z)) \in B\}) \quad \text{for every } B \in \mathcal{E} \otimes \mathcal{B}, \quad \text{where } P \text{ is the transition probability on } E \text{ associated to } \mathbb{P}. \quad (2.1)$$
Let $\mathcal{M}_1(\hat{M})$ be the space of probabilities on $\hat{M}$ equipped with the weak-$*$ topology. The Markov operator associated to $\varphi$ is given by

$$T: \mathcal{M}_1(\hat{M}) \to \mathcal{M}_1(\hat{M}), \quad T\hat{\mu}(B) \overset{\text{def}}{=} \int \hat{P}(i, z), B) \, d\hat{\mu}(i, z).$$

This operator is called asymptotically stable if there is a probability measure $\hat{\mu}$ such that $T\hat{\mu} = \hat{\mu}$ and for every $\hat{\nu} \in \mathcal{M}_1(\hat{M})$ it holds

$$\lim_{n \to \infty} T^n \hat{\nu} = \hat{\mu},$$

in the weak-$*$ topology. Note that a measure $\hat{\mu}$ that satisfies $T\hat{\mu} = \hat{\mu}$ is, by definition, a stationary measure.

In this paper, we obtain sufficient conditions for the asymptotic stability of the Markov operator. We also investigate the (common) asymptotic behavior of the family $((X_{\omega}^n)_{n \in \mathbb{N}})_{x \in X}$.

Finally, note that a probability transition and a stationary measure on the finite set $E = \{1, \ldots, k\}$ are given respectively by a transition matrix and a stationary probability vector. Recall that a $k \times k$ matrix $P = (p_{ij})$ is a transition matrix if $p_{ij} \geq 0$ for all $i, j$ and for every $i$ it holds $\sum_{j=1}^{k} p_{ij} = 1$. A stationary probability vector associated to $P$ is a vector $\bar{p} = (p_1, \ldots, p_k)$ whose elements are non-negative real numbers, sum up to 1, and satisfies $\bar{p} P = \bar{p}$.

A Markov shift $(\Sigma_{n}, \mathcal{F}, P, \sigma)$ with transition matrix $P = (p_{ij})$ is called primitive if there is $n \geq 0$ such that all entries of $P^n$ are positive. It is called irreducible if for every $\ell, r \in \{1, \ldots, k\}$ there is $n = n(\ell, r)$ such that $P^n = (p_{ij}^n)$ satisfies $p_{\ell, r}^n > 0$. An irreducible transition matrix has a unique positive stationary probability vector $\bar{p} = (p_i)$, see [11] page 100. Clearly, every primitive Markov shift is irreducible. Finally, recall that a sequence $(a_1, \ldots, a_\ell)$ in $\{1, \ldots, k\}^\ell$ is called admissible with respect to $P = (p_{ij})$ if $p_{a_i, a_{i+1}} > 0$ for every $i = 1, \ldots, \ell - 1$.

### 2.2. The splitting condition and the stability of the Markov operator.

Before introducing the main result of this section, let us define the splitting condition.

Let $(Y, d_0)$ be a separable metric space and let $M$ be a compact subset of $Y^m$, $m \geq 1$. In $Y^m$ consider the metric $d(x, y) \overset{\text{def}}{=} \sum_i d_0(x_i, y_i)$ and the continuous projections $\pi_s: X^m \to X$, given by $\pi_s(x) \overset{\text{def}}{=} x_s$, where $x = (x_1, \ldots, x_m)$.

**Definition 2.1** (Splitting condition). Let $\varphi(n, \omega, x) = f_n^\omega(x)$ be a random product on a compact subset $M \subset Y^m$ over the Markov shift $(\Sigma_k, \mathcal{F}, P, \sigma)$ and let $P = (p_{ij})$ its transition matrix. We say that $\varphi$ splits if there exist admissible sequences $(a_1, \ldots, a_\ell)$ and $(b_1, \ldots, b_\ell)$ with $a_\ell = b_\ell$ such that $M_1 \overset{\text{def}}{=} f_{a_\ell} \circ \cdots \circ f_{a_1}(M)$ and $M_2 \overset{\text{def}}{=} f_{b_\ell} \circ \cdots \circ f_{b_1}(M)$ satisfy

$$\pi_s(f_n^\omega(M_1)) \cap \pi_s(f_n^\omega(M_2)) = \emptyset$$

for every $\omega \in \Sigma_k$, every $n \geq 0$, and every projection $\pi_s$, $i = 1, \ldots, m$.

The splitting equation (2.3) means that the action of the random product on the sets $M_1$ and $M_2$ remains disjoint when projected in all “directions”. We will present in Section 2.4 some relevant classes of random products for which if the splitting equation (2.3) holds for $n = 0$ then the splitting condition is satisfied (i.e., equation (2.3) holds for every $n \geq 0$). This is the case for instance, when $m = 1$ and all maps of the random product are injective.
Remark 2.2. The splitting condition was introduced in [5, Section 5], in the context of i.i.d. random products of monotone maps of the interval. Our definition is more general and coincides with the original one in the case of monotone maps of the interval. The above definition is somewhat similar to the strong open set condition (SOSC) which is very often studied in the context of iterated function systems, see [10], although mainly with a deterministic focus. The SOSC is for instance satisfied if the interiors of the images \( f_i(M) \) are all pairwise disjoint.

Our first result deals with the dynamics of the Markov operator.

Theorem 1 (Asymptotic stability of the Markov operator). Let \( Y \) be a separable metric space such that every (non-singleton) connected subset of \( Y \) has nonempty interior. Let \( M \subset Y^m \) be a connected compact subset and consider a random product \( \varphi \) on \( M \) over a primitive Markov shift and suppose that \( \varphi \) splits. Then the associated Markov operator is asymptotically stable.

A natural context where the above theorem applies is when \( Y \) is the real line.

As a consequence of Theorem 1 we obtain a strong version of the Ergodic Theorem. Under the assumptions of Theorem 1 the Markov chain \( Z_n^x(\omega) = (\omega_{n-1}, X_n^x(\omega)) \) has a unique stationary measure \( \widehat{\mu} \). This allows us to apply the Breimans' ergodic theorem [4] to get a subset \( \Omega_x \subset \Sigma_k \) of full measure (depending on \( x \)) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \widehat{\varphi}(Z_i^x(\omega)) = \int \widehat{\varphi}(i, x) \, d\widehat{\mu}(i, x),
\]

for every \( \omega \in \Omega_x \) and every continuous function \( \widehat{\varphi}: \widehat{M} \to \mathbb{R} \).

Thus, we can describe the time average of the sequence \( X_n^x \) for every \( x \). For every \( \omega \in \Omega_x \) and every continuous function \( \varphi: M \to \mathbb{R} \) we have that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f_i^x(x)) = \sum_{i=1}^{k} \int \varphi \, d\mu_i,
\]

where \( \mu_i \) is the \( i \)-section of \( \widehat{\mu} \) (see Section 5.1 for precise definitions). To give a more detailed description of the right hand side in (2.5) let us introduce two definitions.

Consider a transition matrix \( P = (p_{ij}) \) and a positive stationary probability vector \( \bar{p} = (p_1, \ldots, p_k) \) of \( P \). The inverse transition matrix associated to \( (P, \bar{p}) \) is the matrix \( Q = (q_{ij}) \) where

\[
q_{ij} = \frac{p_j}{p_i} p_{ji}.
\]

Note that \( Q \) is a transition matrix and \( \bar{p} \) is a stationary probability vector of \( Q \). The Markov measure associated to \( (Q, \bar{p}) \) is called the inverse Markov measure and is denoted by \( \mathbb{P}^- \).

We say that a measurable map \( \pi: \Sigma_k \to M \) is an invariant map of the random product if

\[
f_{\omega_0}(\pi(\sigma(\omega))) = \pi(\omega), \quad \mathbb{P}^-\text{-almost everywhere.}
\]

Invariant maps sometimes give relevant information about the random system. For instance, in the theory of contracting iterated function systems, see [10], the coding map is the unique (a.e.) invariant map and it is essential in the description of properties of attractors and stationary measures of i.i.d random products.
Recall that given a measurable map \( \phi : X \to Z \) and a measure \( \mu \) in \( X \) the pushforward of \( \mu \) by \( \phi \), denoted \( \phi_* \mu \), is the measure on \( Z \) defined by \( \phi_* \mu (A) \defeq \mu (\phi^{-1}(A)) \).

**Corollary 1** (Strong version of the Ergodic Theorem). Let \( Y \) be a separable metric space such that every (non-singleton) connected subset of \( Y \) has nonempty interior. Let \( M \subset Y^m \) be a connected compact subset and consider a random product \( \varphi(n, \omega, x) = f^n_\omega(x) \) on \( M \) over a primitive Markov shift \((\Sigma_k, \mathcal{F}, P, \sigma)\) and suppose that \( \varphi \) splits. Then there is a unique \((P^-\text{-a.e.})\) invariant map \( \pi : \Sigma_k \to M \). Moreover, for every \( x \in M \) and for \( P \text{-almost } \omega \in \Sigma_k \) it holds
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^n_\omega(x)) = \int \phi \, d(\pi_* P^-),
\]
for every continuous function \( \phi : M \to \mathbb{R} \).

**Remark 2.3.** The proof of the corollary provides the unique invariant map: \( \pi \) is the so-called (generalised) coding map (see (1.2)) that is defined on the subset of weakly hyperbolic sequences of \( \Sigma \) (see (1.1)).

2.3. **Synchronization.** We now consider random products over irreducible Markov shifts. Let \( \varphi(n, \omega, x) = f^n_\omega(x) \) be a random product over a Markov shift \((\Sigma_k, \mathcal{F}, P, \sigma)\).

We say that \( \varphi \) is **synchronizing** if for every pair \( x \) and \( y \) we have that
\[
\lim_{n \to \infty} d(f^n_\omega(x), f^n_\omega(y)) = 0, \quad \text{for } P\text{-a.e. } \omega \in \Sigma_k.
\]

Let us start with a general result that states a weak form of synchronization in general compact metric spaces.

**Theorem 2** (Contraction of measures). Let \( \varphi(n, \omega, x) = f^n_\omega(x) \) be a random product on a compact metric space \( M \subset Y^m \) over an irreducible Markov shift \((\Sigma_k, \mathcal{F}, P, \sigma)\). Suppose that \( \varphi \) splits. Let \( P = (p_{ij}) \) be the corresponding transition matrix and suppose that there is \( u \) such that \( p_{uj} > 0 \) for every \( j \). Then there is \( q < 1 \) such that for every probability measure \( \mu \) on \( Y \) and for every \( s = 1, \ldots, m \), for \( P \text{-almost } \omega \) there is \( C_s(\omega) > 0 \) such that
\[
\mu(\pi_s(f^n_\omega(M))) \leq C_s(\omega) q^n, \quad \text{for every } n \geq 1.
\]

The next result states the synchronization of Markovian random products on compact subsets of \( \mathbb{R}^m \) in a strong version: uniform and exponential. We consider in \( \mathbb{R}^m \) the metric \( d(x, y) \defeq \sum_i |x_i - y_i| \).

**Theorem 3** (Synchronization). Let \( \varphi(n, \omega, x) = f^n_\omega(x) \) be a random product on a compact set \( M \subset \mathbb{R}^m \) over an irreducible Markov shift \((\Sigma_k, \mathcal{F}, P, \sigma)\). Suppose that \( \varphi \) splits. Let \( P = (p_{ij}) \) be the corresponding transition matrix and suppose that there is \( u \) such that \( p_{uj} > 0 \) for every \( j \). Then there is \( q < 1 \) such that for \( P \text{-almost } \omega \) there is \( C(\omega) > 0 \) such that
\[
\text{diam}(f^n_\omega(M)) \leq C(\omega) q^n, \quad \text{for every } n \geq 1.
\]

The above results extends [17] Corollary 2.11 stated for i.i.d. random products of monotone (injective) interval maps. Note that our result holds in higher dimensions.

2.4. **Random products with the splitting property.** We now describe some classes of random products for which the splitting equation \((\ref{eq:splitting})\) for \( n = 0 \) guarantees the splitting condition.
2.4.1. Injective maps on “one-dimensional” compact spaces. Let $Y$ be a compact metric space such that every (non-singleton) connected subset of $Y$ has nonempty interior. Let $\varphi(n, \omega, x) = f^n_\omega(x)$ be a Markovian random product on $Y$ such that the maps $f_i$ are injective. If there exist admissible sequences $(a_1, \ldots, a_\ell)$ and $(b_1, \ldots, b_r)$ with $a_\ell = b_r$, such that

$$f_{a_1} \circ \cdots \circ f_{a_\ell}(Y) \cap f_{b_1} \circ \cdots \circ f_{b_r}(Y) = \emptyset$$

then $\varphi$ splits. This is a direct consequence of the injectivity hypothesis and the fact that there is only one direction to project the space.

A natural context where the above comment applies is when $Y$ is an interval. Observe that in the one-dimensional setting, there are certain topological restrictions. For example, although every (non-singleton) connected subset of $S^1$ has nonempty interior, continuous injective maps on the circle have to be homeomorphisms and hence the splitting equation (2.3) cannot be satisfied. We refer to [17, 22] for recent results in the study of i.i.d random products of homeomorphisms on the circle. Let us observe, however, that there are other “one-dimensional” metric spaces which are not intervals which carry Markovian random products that split, for example trees.

2.4.2. Monotone maps on compact subsets of $\mathbb{R}^m$. Let $f: \mathbb{R}^m \to \mathbb{R}^m$ be a continuous injective map. Let $f^i: \mathbb{R}^m \to \mathbb{R}$ be the coordinates function of $f$, i.e., $\pi_i \circ f$. We write $f = (f^1, \ldots, f^m)$. Since $f$ is injective then for every $i$ and every fixed $x_1, x_{j-1}, x_j, x_{j+1}, \ldots, x_m$ the map $x \mapsto f_i(x_1, x_{j-1}, x, x_{j+1}, \ldots, x_m)$ is monotone. Following [3] (although our setting is more general) we will introduce a special class of injective maps, called monotone maps. For that we need to introduce two definitions.

A monotone map $g: \mathbb{R} \to \mathbb{R}$ is of type $+$ if it is increasing and is of type $-$ if it is decreasing. We say that $f^i$ is of type $(t_1, \ldots, t_m) \in \{+,-\}^m$ if for every $j$ and every $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) \in \mathbb{R}^{m-1}$ the map

$$x \mapsto f^i(x_1, x_{j-1}, x, x_{j+1}, \ldots, x_m)$$

is of type $t_j$.

Given $(t_1, \ldots, t_m) \in \{+,-\}^m$ a map $f = (f^1, \ldots, f^m)$ belongs to $S(t_1, \ldots, t_m)$ if and only if

1. If $t_j = t_1$ then $f^j$ is of type $(t_1, \ldots, t_m)$,
2. If $t_j \neq t_1$ then $f^j$ is of type $(s_1, \ldots, s_m)$, where $s_\ell = +$ if $t_\ell = -$ and $s_\ell = -$ if $t_\ell = +$.

We denote by $S_M(t_1, \ldots, t_m)$ the maps $f: M \to M$ that admit an extension to a map $\tilde{f}: \mathbb{R}^m \to \mathbb{R}^m$ in $S(t_1, \ldots, t_m)$. We say that a Markovian random product $\varphi(n, \omega, x) = f^n_\omega(x)$ is in $S_M(t_1, \ldots, t_m)$ if the maps $f_i$ belongs to $S_M(t_1, \ldots, t_m)$ for every $i$.

Let $A$ and $B$ be bounded subsets of $\mathbb{R}$. If $\sup A < \inf B$ then we write $A < B$. In particular, if $A < B$ then $A \cap B = \emptyset$.

**Theorem 4.** Given $(t_1, \ldots, t_m) \in \{+,-\}^m$ consider a Markovian random product $\varphi(n, \omega, x) = f^n_\omega(x)$ in $S_M(t_1, \ldots, t_m)$ defined on a compact set $M \subset \mathbb{R}^m$. Suppose that there exist admissible sequences $(a_1, \ldots, a_\ell)$ and $(b_1, \ldots, b_r)$ with $a_\ell = b_r$, such that $M_1 \triangleq f_{a_1} \circ \cdots \circ f_{a_\ell}(M)$ and $M_2 \triangleq f_{b_1} \circ \cdots \circ f_{b_r}(M)$ satisfy

- $\pi_s(M_1) < \pi_s(M_2)$ if $t_s = +$ and

- $\pi_s(M_1) > \pi_s(M_2)$ if $t_s = -$.
\[ \pi_s(M_1) > \pi_s(M_2) \text{ if } t_s = - \]

for every projection \( \pi_s \). Then \( \varphi \) splits.

In particular, under the hypotheses of the previous theorem and adequate assumptions on the transition matrix (see Theorems 1 and 3) it follows the exponential synchronization and the asymptotic stability of the Markov operator. The case of i.i.d. random products of maps in \( \tilde{S}_M(1, \ldots, 1) \) was treated in [3], where the asymptotic stability of the Markov operator is obtained. Here we consider the Markovian random maps in the set \( \tilde{S}_M \) defined by

\[ \tilde{S}_M \overset{\text{def}}{=} \bigcup_{(t_1, \ldots, t_m) \in \{+,-\}^m} \tilde{S}_M(t_1, \ldots, t_m). \]

2.4.3. Minimal iterated function systems. Given a compact subset \( M \subset \mathbb{R}^m \) and finitely many continuous maps \( f_i: M \to M, \ i = 1, \ldots, k \), we consider its associated iterated function system denoted by \( \text{IFS}(f_1, \ldots, f_k) \). Let \( E_n = \{1, \ldots, k\}^n \). The IFS is called minimal if for every \( x \in M \) it holds

\[ \text{closure} \left( \bigcup_{n \in \mathbb{N}} \bigcup_{(a_1, \ldots, a_n) \in E_n} f_{a_n} \circ \cdots \circ f_{a_1}(x) \right) = M. \]

The next result is a consequence of Theorem 3.

Corollary 2. Consider a compact set \( M \subset \mathbb{R}^m \) with non-empty interior and a random product \( \varphi(n, \omega, x) = f^n_\omega(x) \) over a Markov shift \((\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)\) in \( \tilde{S}_M \). Assume \( \mathbb{P} \) has full support, the IFS(\( f_1, \ldots, f_k \)) is minimal, and there is a sequence \( \omega \in \Sigma_k \) such that

\[ \bigcap f_{\omega_0} \circ \cdots \circ f_{\omega_n}(M) = \{p\} \]

for some point \( p \). Then \( \varphi \) splits.

Condition (2.8) is guaranteed, for instance, if some composition \( f_{a_\ell} \circ \cdots \circ f_{a_1} \) is a contraction. Note that this condition is compatible with the minimality of the IFS. Indeed, Corollary 2 is mainly illustrative, since the minimality condition is harder to check than the splitting condition (which in many cases can be easily obtained).

2.4.4. Injective maps in a box. Let \( Y \) be a compact metric space such that every (non-singleton) connected subset of \( Y \) has non-empty interior. Let \( \varphi(n, \omega, x) = f^n_\omega(x) \) be a Markovian random product defined on \( Y \). Suppose that there is a compact subset (a box) \( J \subset Y \) such that \( f_i(J) \subset J \) and \( f_{i\mid J} \) is injective for every \( i = 1, \ldots, k \). If there exist admissible sequences \( (a_1, \ldots, a_\ell) \) and \( (b_1, \ldots, b_r) \) with \( a_\ell = b_r \), such that

\[ f_{a_1} \circ \cdots \circ f_{a_\ell}(Y) \cap f_{b_r} \circ \cdots \circ f_{b_1}(Y) = \emptyset \]
\[ f_{a_\ell} \circ \cdots \circ f_{a_1}(Y) \cup f_{b_1} \circ \cdots \circ f_{b_r}(Y) \subset J, \]

then \( \varphi \) splits.
3. Consequences of the splitting hypothesis

We now explore consequences of the splitting condition for Markovian random products \( \varphi(n, \omega, x) = f^n_\omega(x) \) on a general compact metric space \( M \subset Y^m \) over an irreducible Markov shift \( (\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma) \). Note that with these assumptions \( \mathbb{P}^- \) is well defined.

We begin with some general definitions. Consider the cylinders
\[
[a_0 \ldots a_\ell] \overset{\text{def}}{=} \{ \omega \in \Sigma_k : \omega_0 = a_0, \ldots, \omega_\ell = a_\ell \} \subset \Sigma_k,
\]
which is a semi-algebra that generates the Borel sigma-algebra of \( \Sigma_k \). Given a Markov measure \( \mathbb{P} \) on \( \Sigma_k \) with transition matrix \( P = (p_{ij}) \) and a stationary measure \( \bar{p} = (p_1, \ldots, p_k) \) we have that
\[
\mathbb{P}([a_0 \ldots a_\ell]) = p_{a_0}p_{a_0a_1} \cdots p_{a_{\ell-1}a_\ell}.
\]
A cylinder \( C = [a_0a_1 \ldots a_\ell] \) is \( \mathbb{P} \)-admissible if \( \mathbb{P}(C) > 0 \).

For the random product \( \varphi(n, \omega, x) = f^n_\omega(x) \) and the projection \( \pi_s \) we define two family of subsets of \( \Sigma_k \). First, for each \( x \in \pi_s(M) \) and \( n \geq 1 \) let
\[
S^x_n(s) \overset{\text{def}}{=} \{ \omega \in \Sigma_k : x \in \pi_s(f_\omega \circ \cdots \circ f_{\omega_{n-1}}(M)) \}
\]
and observe that
\[
S^x_{n+1}(s) \subset S^x_n(s).
\]
Second, for each cylinder \( C \) of size \( N \) define
\[
\Sigma^C_n \overset{\text{def}}{=} \{ \omega \in \Sigma_k : \sigma^{iN}(\omega) \cap C = \emptyset \text{ for all } i = 0, \ldots, n-1 \}
\]
and note that
\[
\Sigma^C_{n+1} \subset \Sigma^C_n.
\]

**Proposition 3.1.** Let \( M \subset Y^m \) be a compact metric space and \( \varphi(n, \omega, x) = f^n_\omega(x) \) be a random product on \( M \) over an irreducible Markov shift \( (\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma) \). Let \( [\xi_0 \ldots \xi_{N-1}] \) and \( [\eta_0 \ldots \eta_{N-1}] \) be \( \mathbb{P}^- \)-admissible cylinders such that \( \xi_0 = \eta_0, \xi_{N-1} = \eta_{N-1} \), and
\[
\pi_s(f^n_\omega(f_{\xi_0} \circ \cdots \circ f_{\xi_{N-1}}(M))) \cap \pi_s(f^n_\omega(f_{\eta_0} \circ \cdots \circ f_{\eta_{N-1}}(M))) = \emptyset
\]
for every \( s, \) every \( n \geq 1, \) and every \( \omega \). If
\[
0 < \mathbb{P}^-(\{[\xi_0 \ldots \xi_{N-1}]\}) = \mathbb{P}^-(\{[\eta_0 \ldots \eta_{N-1}]\}),
\]
then the cylinder \( W \overset{\text{def}}{=} [\xi_0 \ldots \xi_{N-1}] \) satisfies
\[
\mathbb{P}^-(W) > 0 \quad \text{and} \quad \mathbb{P}^-(S^x_{1N}(s)) \leq \mathbb{P}^-(\Sigma^W_\ell),
\]
for every \( \ell \geq 1, \) every \( s \) and every \( x \).

**Proof.** By the admissibility of \( [\xi_0 \ldots \xi_{N-1}] \) we have
\[
0 < \mathbb{P}^-(\{[\xi_0 \ldots \xi_{N-1}]\}) = \mathbb{P}^-(\{[\eta_0 \ldots \eta_{N-1}]\}).
\]

Fix a projection \( \pi_s \). Take \( x \in \pi_s(M) \) and for each \( \ell \geq 1 \) define the following two families of cylinders:
\[
\Sigma^x_\ell(s) \overset{\text{def}}{=} \{[a_0 \ldots a_{\ell N-1}] \subset \Sigma_k : x \in \pi_s(f_{a_0} \circ \cdots \circ f_{a_{\ell N-1}}(M)) \}
\]
and
\[
E^x_\ell \overset{\text{def}}{=} \{[a_0 \ldots a_{\ell N-1}] \subset \Sigma_k : \sigma^{iN}(\{[a_0 \ldots a_{\ell N-1}]\}) \cap W = \emptyset, \forall i = 0, \ldots, \ell - 1 \}.
\]
Note that

\[(3.9) \quad S_{\ell N}^e(s) = \bigcup_{C \in \Sigma_x^\ell(s)} C \quad \text{and} \quad \Sigma^W_{\ell} = \bigcup_{C \in E_{\ell}} C.\]

For each \(\ell \geq 1\) we now introduce a “substitution function” \(F_\ell : \Sigma^e_{\ell}(s) \to E^\ell\).

First, for each cylinder \(C = [\alpha_0 \ldots \alpha_{\ell N - 1}] \in \Sigma^e_{\ell}(s)\) we consider its sub-cylinders \([\alpha_0 \ldots \alpha_{N - 1}], [\alpha_N \ldots \alpha_{2N - 1}], \ldots, [\alpha_{(\ell - 1)N} \ldots \alpha_{\ell N - 1}]\) and use the concatenation notation

\([\alpha_0 \ldots \alpha_{\ell N - 1}] \overset{\text{def}}{=} [\alpha_0 \ldots \alpha_{N - 1}] [\alpha_N \ldots \alpha_{2N - 1}] \cdots [\alpha_{(\ell - 1)N} \ldots \alpha_{\ell N - 1}]\).

In a compact way, we write

\[C = C_0 * C_1 * \cdots * C_{\ell - 1}, \quad C_i \overset{\text{def}}{=} [\alpha_{iN} \ldots \alpha_{(i + 1)N}].\]

With this notation we define \(F_\ell\) by

\[F_\ell(C) \overset{\text{def}}{=} F_\ell(C_0 * C_1 * \cdots * C_{\ell - 1}) = C'_0 * C'_1 * \cdots * C'_{\ell - 1},\]

where \(C'_i = C_i\) if \(C_i \neq [\xi_0 \ldots \xi_{N - 1}]\) and \(C'_i = [\eta_0 \ldots \eta_{N - 1}]\) otherwise. Note that by definition \(F_\ell(C) \subset E^\ell\) for each \(C \in \Sigma^e_{\ell}(s)\) and hence the map is well defined.

**Claim 3.2.** For every \(C \in \Sigma^e_{\ell}(s)\) it holds \(P^-(C) \leq P^-(F_\ell(C)).\)

**Proof.** Recalling that \(\eta_0 = \xi_0\) and \(\eta_{N - 1} = \xi_{N - 1}\), from equation (3.6) we immediately get the following: For every \(r, j \geq 0\) and every pair of cylinders \([a_0 \ldots a_j]\) and \([b_0 \ldots b_r]\) it holds

1. \(P^-([a_0 \ldots a_j \xi_0 \ldots \xi_{N - 1} b_0 \ldots b_r]) \leq P^-([a_0 \ldots a_j \eta_0 \ldots \eta_{N - 1} b_0 \ldots b_r]),\)
2. \(P^-([\xi_0 \ldots \xi_{N - 1} b_0 \ldots b_r]) \leq P^-([\eta_0 \ldots \eta_{N - 1} b_0 \ldots b_r]),\)
3. \(P^-([a_0 \ldots a_j \xi_0 \ldots \xi_{N - 1}]) \leq P^-([a_0 \ldots a_j \eta_0 \ldots \eta_{N - 1}]).\)

The inequality \(P^-(C) \leq P^-(F_\ell(C))\) now follows from the definition of \(F_\ell.\) \(\square\)

**Lemma 3.3.** The map \(F_\ell\) is injective.

**Proof.** Using the concatenation notation above, consider cylinders \(C = C_0 * C_1 * \cdots * C_{\ell - 1}\) and \(\overline{C} = \overline{C}_0 * \overline{C}_1 * \cdots * \overline{C}_{\ell - 1}\) with \(C \neq \overline{C}\) in \(\Sigma^e_{\ell}(s)\). Write

\[F_\ell(C) = C'_0 * C'_1 * \cdots * C'_{\ell - 1} \quad \text{and} \quad F_\ell(\overline{C}) = \overline{C}'_0 * \overline{C}'_1 * \cdots * \overline{C}'_{\ell - 1}.\]

Assume, by contradiction, that \(F_\ell(C) = F_\ell(\overline{C})\) and hence \(C'_i = \overline{C}'_i\) for all \(i = 0, \ldots, N - 1\). Since \(C \neq \overline{C}\) there is a first \(i\) such that \(C_i \neq \overline{C}_i\). Then, by the definition of \(F_\ell\), either \(C_i = [\xi_0 \ldots \xi_{N - 1}]\) and \(\overline{C}_i = [\eta_0 \ldots \eta_{N - 1}]\) or vice-versa. Let us assume that the first case occurs.

If \(i = 0\) then the definition of \(\Sigma^e_{\ell}(s)\) implies that

\[x \in \pi_s(f_{\xi_0} \circ \cdots \circ f_{\xi_{N - 1}}(M)) \cap \pi_s(f_{\eta_0} \circ \cdots \circ f_{\eta_{N - 1}}(M)),\]

contradicting the hypothesis in (3.3). Thus we can assume that \(i > 0\). Write \((i - 1)N - 1 = r\) and consider the cylinders

\[[\gamma_0 \ldots \gamma_r] \overset{\text{def}}{=} C_0 * C_1 * \cdots * C_{i - 1} = \overline{C}_0 * \overline{C}_1 * \cdots * \overline{C}_i - 1,\]

\[[\gamma_{r + N} \ldots \gamma_{tN - 1}] \overset{\text{def}}{=} C_{i + 1} * C_1 * \cdots * C_{\ell - 1},\]

\[[\overline{\gamma}_{r + N} \ldots \overline{\gamma}_{tN - 1}] \overset{\text{def}}{=} \overline{C}_{i + 1} * C_1 * \cdots * \overline{C}_{\ell - 1}.\]
and the corresponding finite sequences
\[ \gamma_0 \cdots \gamma_{\ell N - 1} \overset{\text{def}}{=} \gamma_0 \cdots \gamma_r \xi_0 \cdots \xi_{N-1} \gamma_{r+N} \cdots \gamma_{\ell N - 1}. \]
\[ \tilde{\gamma}_0 \cdots \tilde{\gamma}_{\ell N - 1} \overset{\text{def}}{=} \gamma_0 \cdots \gamma_r \eta_0 \cdots \eta_{N-1} \tilde{\gamma}_{r+N} \cdots \tilde{\gamma}_{\ell N - 1}. \]
Since \( f_i(M) \subset M \) we have
\[ f_{\gamma_0} \circ \cdots \circ f_{\gamma_{\ell N - 1}}(M) \subset f_{\gamma_0} \circ \cdots \circ f_{\gamma_r} \circ f_{\xi_0} \circ \cdots \circ f_{\xi_{N-1}}(M), \]
\[ f_{\gamma_0} \circ \cdots \circ f_{\tilde{\gamma}_{\ell N - 1}}(M) \subset f_{\gamma_0} \circ \cdots \circ f_{\gamma_r} \circ f_{\eta_0} \circ \cdots \circ f_{\eta_{N-1}}(M). \]
Hence, by the definition of \( \Sigma^\ell_x(s) \) in (3.7), we have
\[ x \in \pi_s(f_{\gamma_0} \circ \cdots \circ f_{\gamma_r} \circ f_{\xi_0} \circ \cdots \circ f_{\xi_{N-1}}(M)) \cap \pi_s(f_{\gamma_0} \circ \cdots \circ f_{\gamma_r} \circ f_{\eta_0} \circ \cdots \circ f_{\eta_{N-1}}(M)), \]
contradicting (3.5). Thus Claim 3.2, (c) from the injectivity of \( Q \).

Condition \( \mathbb{P}^{-}(W) > 0 \) follows from the choice of \( W \). To prove that \( \mathbb{P}^{-}(S^\ell_{\tilde{\xi}N}(s)) \leq \mathbb{P}^{-}(\Sigma^W_x) \) note that
\[ \mathbb{P}^{-}(S^\ell_{\tilde{\xi}N}(s)) = \sum_{C \in \Sigma^\ell_x(s)} \mathbb{P}^{-}(C) \leq \sum_{C \in \Sigma^\ell_x(s)} \mathbb{P}^{-}(F_i(C)) \]
\[ = \mathbb{P}^{-}\left( \bigcup_{C \in \Sigma^\ell_x(s)} F_i(C) \right) \leq \mathbb{P}^{-}(\Sigma^W_x), \]
where (a) follows from the disjointedness of the cylinders \( C \in \Sigma^\ell_x(s) \), (b) from Claim 3.2, (c) from the injectivity of \( F_j \) (Lemma 3.3), and (d) from \( F_j(C) \in E_j \subset Q^j \). The proof of the proposition is now complete. \( \square \)

4. Consequences of the splitting hypothesis in \( m \)-dimensional spaces

Throughout this section \( \varphi \) denotes a random product on a compact and connected metric space \( M \subset Y^m \) over a Markov shift \( (\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma) \). To state the main theorem of this section (Theorem 4.1) we need some definitions.

For each \( \xi \in \Sigma_k \) we consider its fibre defined by
\[ I_\xi \overset{\text{def}}{=} \bigcap_{n \geq 0} f_{\xi_0} \circ \cdots \circ f_{\xi_n}(M). \]
Every fibre is a non-empty compact set: note that \( (f_{\xi_0} \circ \cdots \circ f_{\xi_n}(M))_{n \in \mathbb{N}} \) is a sequence of nested compact sets. Also note that \( I_\xi \) is a connected set.

The subset \( S_\varphi \subset \Sigma_k \) of weakly hyperbolic sequences is defined by
\[ (4.1) \quad S_\varphi \overset{\text{def}}{=} \{ \xi \in \Sigma_k : I_\xi \text{ is a singleton} \}. \]

The main result in this section is the following:

**Theorem 4.1.** Let \( Y \) be a separable metric space such that every (non-singleton) connected subset of \( Y \) has non-empty interior. Consider a random product \( \varphi \) over a primitive Markov shift \( (\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma) \) defined on a compact and connected subset \( M \) of \( Y^m \). Suppose that \( \varphi \) splits. Then \( \mathbb{P}^{-}(S_\varphi) = 1 \).

This theorem is an important step of the proof of Theorem 1 and its proof is inspired by the ideas in [15].
4.1. **Proof of Theorem [4.1]** Fix $s$. Given $x \in \pi_s(M)$ define the set

$$\Sigma_x(s) \overset{\text{def}}{=} \{ \xi \in \Sigma_k : x(\xi) \in \pi_s(I_\xi) \}.$$ 

Consider also the set $S_\varphi(s) \overset{\text{def}}{=} \{ \xi \in \Sigma_k : \pi_s(I_\xi) \text{ is a singleton} \}$. Since $Y$ is separable there is a dense and countable subset $D$ of $Y$. Note that if $\xi \notin S_\varphi(s)$ then the set $\pi_s(I_\xi)$ is a connected subset of $Y$ which is not a singleton and hence, by hypothesis, its interior is not empty and thus $\pi_s(I_\xi)$ contains a point of $D$. This implies that

$$(4.2) \quad (S_\varphi(s))^c = \Sigma_k \setminus S_\varphi(s) \subset \bigcup_{x \in D \cap \pi_s(M)} \Sigma_x(s).$$

**Proposition 4.2.** $\mathbb{P}^-(\Sigma_x(s)) = 0$ for every $x \in \pi_s(M)$.

In view of (4.2) this proposition implies that $\mathbb{P}^-(S_\varphi(s)) = 1$. The theorem follows noting that $S_\varphi = \bigcap_{n=1}^\infty S_\varphi(s)$.

**Proof of Proposition 4.2.** Fix $x \in \pi_s(M)$, recall the definition of $S_\varphi(s)$ in (3.1) and note that for every $n \geq 1$ it holds $\Sigma_x(s) \subset S_\varphi(s)$. Hence

$$\Sigma_x(s) \subset \bigcap_{n \geq 1} S_\varphi(s).$$

Therefore, recalling that $S_\varphi(s) \subset S_\varphi(s)$, (4.2), it follows

$$(4.3) \quad \mathbb{P}^-(\Sigma_x(s)) \leq \mathbb{P}^-(\bigcap_{n \geq 1} S_\varphi(s)) = \lim_{n \to \infty} \mathbb{P}^-(S_\varphi(s)).$$

Hence to prove the proposition it is enough to see that

**Lemma 4.3.** $\lim_{n \to \infty} \mathbb{P}^-(S_\varphi(s)) = 0$.

**Proof.** By the splitting hypothesis there is a pair of $\mathbb{P}$-admissible cylinders $[a_1 \ldots a_\ell]$ and $[b_1 \ldots b_r]$ with $a_\ell = b_r$ such that

$$(4.4) \quad \pi_s(f^n_\varphi(f_{a_1} \circ \cdots \circ f_{a_\ell}(M))) \cap \pi_s(f^n_\varphi(f_{b_1} \circ \cdots \circ f_{b_r}(M))) = \emptyset$$

for every $\pi_s$, every $n \geq 0$, and every $\omega \in \Sigma_k$.

Next claim restates the splitting property adding the condition that the two sequences in that condition have the same length.

**Claim 4.4.** There are $\mathbb{P}^-$-admissible cylinders $[\xi_0 \ldots \xi_{N-1}]$ and $[\eta_0 \ldots \eta_{N-1}]$ with $\xi_0 = \eta_0$, $\xi_{N-1} = \eta_{N-1}$, such that

$$\pi_s(f^n_\varphi(f_{\xi_0} \circ \cdots \circ f_{\xi_{N-1}}(M))) \cap \pi_s(f^n_\varphi(f_{\eta_0} \circ \cdots \circ f_{\eta_{N-1}}(M))) = \emptyset$$

for every $s$, $n \geq 0$, and every $\omega$.

**Proof.** Consider $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_r$ as in (4.4). Since the transition matrix of $\mathbb{P}$ is primitive then there is $n_0$ such that for every $n \geq n_0$ there are $\mathbb{P}$-admissible cylinders of the form $[1c_n \ldots c_1a_1]$ and $[1d_n \ldots d_1b_1]$. Take now $n_1, n_2 \geq n_0$ with $n_1 + \ell = n_2 + r$ and consider $\mathbb{P}^-$-admissible cylinders $[1c_{n_1} \ldots c_1a_1]$ and $[1d_{n_2} \ldots d_1b_1]$. Let $N = n_1 + \ell + 2$ and observe that by construction (and since $[a_1 \ldots a_\ell]$ and $[b_1 \ldots b_r]$ are both admissible) the cylinders

$$[\xi_0 \ldots \xi_{N-1}] = [a_\ell \ldots a_1c_1 \ldots c_{n_1}] \quad \text{and} \quad [\eta_0 \ldots \eta_{N-1}] = [b_r \ldots b_1d_1 \ldots d_{n_1}1]$$
are \( \mathbb{P}^- \)-admissible. Note also that
\[
f_{c_1} \circ \cdots \circ f_{c_n} \circ f_1(M) \subset M, \quad f_{d_1} \circ \cdots \circ f_{d_m} \circ f_1(M) \subset M.
\]
Hence the splitting condition in (4.4) for \( a_1 \ldots a_L \) and \( b_1 \ldots b_r \) implies that \( \xi_0 = \eta_0, \xi_{N-1} = \eta_{N-1} \), and \( \xi_0 \ldots \xi_{N-1} \) and \( \eta_0 \ldots \eta_{N-1} \) satisfy the empty intersection condition in the claim. \( \square \)

Let \([\xi_0 \ldots \xi_{N-1}]\) and \([\eta_0 \ldots \eta_{N-1}]\) the cylinders given by Claim 4.4. Suppose that \( 0 < \mathbb{P}^-([\xi_0 \ldots \xi_{N-1}]) \leq \mathbb{P}^-([\eta_0 \ldots \eta_{N-1}]) \) and let \( W = [\xi_0 \ldots \xi_{N-1}] \). We can now apply Proposition 3.1 to \( W \) obtaining
\[
(4.5) \quad \mathbb{P}^-(S_{\ell N}^W(s)) \leq \mathbb{P}^-(\Sigma_{\ell}^W), \quad \text{for every } \ell \geq 1.
\]

We now estimate the right hand side of (4.5). Let
\[
\Sigma_{\ell}^W \overset{\text{def}}{=} \bigcap_{\ell \geq 1} \Sigma_{\ell}^W = \{ \omega \in \Sigma_k : \sigma^{iN}(\omega) \cap W = \emptyset \text{ for all } i \geq 0 \}.
\]

**Remark 4.5.** Since \( P \) is primitive the shift \((\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)\) is mixing and hence the system \((\Sigma_k, \mathcal{F}, \mathbb{P}^-, \sigma^f)\) is ergodic for every \( \ell \geq 1 \), see for instance [18, page 64].

Since \( 0 < \mathbb{P}^-(W) \), by Remark 4.3 we can apply the Birkhoff’s ergodic theorem to get that \( \mathbb{P}^-(\Sigma_{\ell}^W) = 0 \). Hence condition \( \Sigma_{\ell+1}^W \subset \Sigma_{\ell}^W \), recall (3.3), implies that
\[
\lim_{\ell \to \infty} \mathbb{P}^-(\Sigma_{\ell}^W) = 0.
\]

It follows from (4.5) that
\[
\lim_{\ell \to \infty} \mathbb{P}^-(S_{\ell N}^W(s)) \leq \lim_{\ell \to \infty} \mathbb{P}^-(\Sigma_{\ell}^W) = 0.
\]
The lemma follows recalling again that \( S_{\ell+1}^W \subset S_\ell^W \), see [322]. \( \square \)

The proof of Proposition 4.4 is now complete \( \square \)

The proof of Theorem 4.1 is now complete. \( \square \)

5. **Stability of the Markov operator**

Throughout this section \( \varphi \) denotes a random product on a compact and connected metric space \((M, d)\) over a Markov shift \((\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)\) as in (1.1). In this section we will prove Theorem 1 and Corollary 1. We begin by providing a handy form of the Markov operator of \( \varphi \).

5.1. **The Markov operator.** Let \( \widehat{M} \overset{\text{def}}{=} \{1, \ldots, k\} \times M \). Given a subset \( \widehat{B} \subset \widehat{M} \), its \( i \)-section is defined by
\[
\widehat{B}_i \overset{\text{def}}{=} \{ x \in M : (i, x) \in \widehat{B} \}.
\]
The \( i \)-section of a probability measure \( \widehat{\mu} \) on \( \widehat{M} \) is the measure defined on \( M \) by
\[
\mu_i(B) \overset{\text{def}}{=} \widehat{\mu}(i) \times B, \quad \text{where } B \text{ is any Borel subset of } M.
\]
Observe that \( \mu_i \) is a finite measure on \( M \) but, in general, it is not a probability measure. Since the measure \( \widehat{\mu} \) is completely defined by its sections we write \( \widehat{\mu} = (\mu_1, \ldots, \mu_k) \) and note that
\[
\widehat{\mu}(\widehat{B}) = \sum_{j=1}^k \mu_j(\widehat{B}_j) \quad \text{for every Borel subset } \widehat{B} \text{ of } \widehat{M}.
\]
Similarly, given a function $\widehat{g}: \widehat{M} \to \mathbb{R}$ we define its $i$-section $g_i: M \to \mathbb{R}$ by $g_i(x) = \widehat{g}(i, x)$ and write $\widehat{g} = (g_1, \ldots, g_k)$. By definition, it follows that

\begin{equation}
\int \widehat{g} \, d\widehat{\mu} = \sum_{i=1}^{k} \int g_i \, d\mu_i, \quad \text{for every} \quad \widehat{\mu} = (\mu_1, \ldots, \mu_k) \in \mathcal{M}_1(\widehat{M}).
\end{equation}

For the next lemma recall that $\phi_\ast \mu$ denotes the pushforward of the measure $\mu$ by $\phi$ (i.e., $\phi_\ast \mu(A) = \mu(\phi^{-1}(A))$).

**Lemma 5.1.** Consider a random product $\varphi(n, \omega, x) = f_\omega^n(x)$ on $M$ over a Markov shift $(\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)$. Let $P = (p_{ij})$ be the transition matrix of $\mathbb{P}$. The Markov operator associated to $\varphi$ is given by

\[ T\widehat{\mu}(\widehat{B}) \overset{\text{def}}{=} \sum_{i,j} p_{ij} f_j \ast \mu_i(\widehat{B}_j), \]

where $\widehat{\mu} = (\mu_1, \ldots, \mu_k) \in \mathcal{M}_1(\widehat{M})$ and $\widehat{B}$ is any Borel subset of $\widehat{M}$. In particular,

\[ (T\widehat{\mu})_j = \sum_{i,j} p_{ij} f_j \ast \mu_i. \]

**Proof.** Let $\widehat{B}$ be a Borel subset of $\widehat{M}$. The transition probability on the set $\widehat{M} = \{1, \ldots, k\} \times M$ associated to $\varphi$ is given by (recall (2.1))

\[ \widehat{P}(i, z), \widehat{B}) = \sum_{j=1}^{k} p_{ij} \mathbb{1}_{\widehat{B}_j}(j, f_j(z)), \]

and hence the corresponding Markov operator is given by (recall (2.2))

\[ T\widehat{\mu}(\widehat{B}) = \int \sum_{j=1}^{k} p_{ij} \mathbb{1}_{\widehat{B}_j}(j, f_j(z)) \, d\widehat{\mu}(i, z) = \sum_{j=1}^{k} \int p_{ij} \mathbb{1}_{\widehat{B}_j}(j, f_j(z)) \, d\widehat{\mu}(i, z) \]

by (5.1)

\[ = \sum_{j=1}^{k} \int p_{ij} \mathbb{1}_{\widehat{B}_j}(f_j(z)) \, d\mu_i(z) \]

\[ = \sum_{j=1}^{k} p_{ij} \int \mathbb{1}_{\widehat{B}_j}(z) \, df_j \ast \mu_i(z) = \sum_{i,j} p_{ij} f_j \ast \mu_i(\widehat{B}_j), \]

proving the lemma. \hfill \Box

### 5.2. Shrinking of the reverse order iterates

Recall the definition of the set $S_\varphi$ in (111) and define the **coding map** $\pi$ in (111)

\begin{equation}
\pi: S_\varphi \to M, \quad \pi(\omega) \overset{\text{def}}{=} \lim_{n \to \infty} f_{\omega_0} \circ \cdots \circ f_{\omega_n}(p),
\end{equation}

where $p$ is any point of $M$. By definition of the set $S_\varphi$, this limit always exists and does not depend on $p \in M$.

**Lemma 5.2.** For every sequence $(\mu_n)$ of probabilities of $\mathcal{M}_1(M)$ and every $\omega \in S_\varphi$ it holds

\[ \lim_{n \to \infty} f_{\omega_0} \ast \cdots \ast f_{\omega_n} \ast \mu_n = \delta_{\pi(\omega)}. \]

\footnote{This is the standard terminology for the map $\pi$ when $S_\varphi = \Sigma_k$.}
Proof. Consider a sequence of probabilities \((\mu_n)\) and \(\omega \in S_\varphi\). Fix any \(g \in C^0(M)\). Then given any \(\epsilon > 0\) there is \(\delta > 0\) such that
\[
|g(y) - g \circ \pi(\omega)| < \epsilon \quad \text{for all } y \in M \text{ with } d(y, \pi(\omega)) < \delta.
\]
Since \(\omega \in S_\varphi\) there is \(n_0\) such that \(d(f_{\omega_0} \circ \cdots \circ f_{\omega_n}(x), \pi(\omega)) < \delta\) for every \(x \in M\) and every \(n \geq n_0\). Therefore for \(n \geq n_0\) we have
\[
|g \circ \pi(\omega) - \int g \, d\omega_0 \ast \cdots \ast f_{\omega_n} \ast \mu_n| = \left| \int g \circ \pi(\omega) \, d\mu_n - \int g \circ f_{\omega_0} \circ \cdots \circ f_{\omega_n}(x) \, d\mu_n \right| \\
\leq \int |g \circ \pi(\omega) - g \circ f_{\omega_0} \circ \cdots \circ f_{\omega_n}(x)| \, d\mu_n \leq \epsilon.
\]
This implies that
\[
\lim_{n \to \infty} \int g \, d\omega_0 \ast \cdots \ast f_{\omega_n} \ast \mu_n = g \circ \pi(\omega)
\]
Since this holds for every continuous map \(g\) the lemma follows. \(\Box\)

5.3. \textbf{Proof of Theorem 5.3.} Let \(\pi : S_\varphi \to M\) be the coding map in [5.2] and define
\[
\hat{\pi} : S_\varphi \to \hat{M}, \quad \hat{\pi}(\omega) \stackrel{\text{def}}{=} (\omega_0, \pi(\omega)).
\]
Since the random product \(\varphi\) splits it follows from Theorem 4.1 that \(P^- (S_\varphi) = 1\). Hence the map \(\hat{\pi}\) is defined \(P^-\)-almost everywhere in \(\Sigma_k\). This allows us to consider the probability measure \(\hat{\pi} \ast P^-\).

The next result is a reformulation of Theorem 1 (indeed, a stronger version of it) and implies that the probability measure \(\hat{\pi} \ast P^-\) is the unique stationary measure of the Markov operator and is attracting.

\textbf{Theorem 5.3.} Let \(\varphi\) be a random product on a compact metric space \(M\) over a primitive Markov shift and suppose that \(P^- (S_\varphi) = 1\). Given any measure \(\hat{\mu} \in M_1(\hat{M})\) and any continuous \(\hat{g} \in C^0(\hat{M})\) it holds
\[
\lim_{n \to \infty} \int \hat{g} \, dT^n \hat{\mu} = \int \hat{g} \, d\hat{\pi} \ast P^-,
\]
where \(T\) is the Markov operator of \(\varphi\).

\textbf{Remark 5.4.} In [2] it is proved that the condition \(P^- (S_\varphi) = 1\) implies the existence of a unique stationary measure and that \(T^n \hat{\mu}\) converge in the weak*-topology to the unique stationary measure provided that \(\hat{\mu} = (\mu_1, \ldots, \mu_k)\) satisfy \(\mu_i(\hat{M}) = p_i\) for every \(i\), where \((p_1, \ldots, p_k)\) is the stationary probability vector. The result in [2] is a version of Letac principle [14] for a Markovian random product. Here we prove that \(T^n \hat{\mu}\) converges in the weak*-topology to the unique stationary measure for every \(\hat{\mu}\) (and not only for measures \(\hat{\mu} = (\mu_1, \ldots, \mu_k)\) with \(\mu_i(\hat{M}) = p_i\)).

\textbf{Proof of Theorem 5.3.} It follows from [5.1] that
\[
(5.3) \quad \int \hat{g} \, dT^n \hat{\mu} = \sum_{j=1}^k \int g_j \, d(T^n \hat{\mu})_j, \quad T^n \hat{\mu} = ((T^n \hat{\mu})_1, \ldots, (T^n \hat{\mu})_k).
\]

We prove the convergence of the integrals of the sum in (5.3) in three steps corresponding to Lemmas 5.5, 5.8, and 5.9 below. First, given a continuous function \(g : M \to \mathbb{R}\) denote by \(\|g\|\) its uniform norm.
Lemma 5.5. Consider \( \hat{\mu} = (\mu_1, \ldots, \mu_k) \in \mathcal{M}_1(\hat{M}) \) such that \( \mu_i(M) > 0 \) for every \( i \in \{1, \ldots, k\} \). Then for every \( g \in C^0(M) \) it holds
\[
\limsup_n \left| \int g(T^n \hat{\mu})_j - \int_{[\gamma]} g \circ \pi d\mathbb{P}^- \right| \leq k \|g\| \max_i |\mu_i(M) - p_i|,
\]
where \( \hat{\mu} = (p_1, \ldots, p_k) \) is the unique stationary vector of \( P = (p_{ij}) \) is the transition matrix of \( \mathbb{P} \).

Proof. Take \( \hat{\mu} \in \mathcal{M}_1(\hat{M}) \) as in the lemma and for each \( i \) define the probability measure \( \bar{\mu}_i \)
\[
\bar{\mu}_i(B) \stackrel{def}{=} \frac{\mu_i(B)}{\mu_i(M)}, \quad \text{where } B \text{ is a Borel subset of } M.
\]
A straightforward calculation and the previous definition imply that
\[
(T^n \hat{\mu})_j = \sum_{\xi_1, \ldots, \xi_n} p_{\xi_1 \xi_n} \cdots p_{\xi_{n-1} \xi_n} p_{\xi_n} \int g d f_{\xi_n} f_{\xi_{n-1}} \cdots f_{\xi_1} \Pi_{\xi_n}.
\]

Thus given any \( g \in C^0(M) \) we have that
\[
\int g(T^n \hat{\mu})_j = \sum_{\xi_1, \ldots, \xi_n} \mu_{\xi_n}(M) p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} \int g d f_{\xi_n} f_{\xi_{n-1}} \cdots f_{\xi_1} \Pi_{\xi_n}.
\]

Let
\[
L_n \stackrel{def}{=} \left| \int g(T^n \hat{\mu})_j - \int_{[\gamma]} g \circ \pi d\mathbb{P}^- \right|
\]
and write \( \mu_{\xi_n}(M) = (\mu_{\xi_n}(M) - p_{\xi_n}) + p_{\xi_n} \). Then
\[
L_n \leq \left| \sum_{\xi_1, \ldots, \xi_n} (\mu_{\xi_n}(M) - p_{\xi_n}) p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} \int g d f_{\xi_n} f_{\xi_{n-1}} \cdots f_{\xi_1} \Pi_{\xi_n} \right|
\]
\[
+ \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} \int g d f_{\xi_n} f_{\xi_{n-1}} \cdots f_{\xi_1} \Pi_{\xi_n} - \int_{[\gamma]} g \circ \pi d\mathbb{P}^- \right|
\]
\[
\leq \max_i \|\mu_i(M) - p_i\| \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1}
\]
\[
+ \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} \int g d f_{\xi_n} f_{\xi_{n-1}} \cdots f_{\xi_1} \Pi_{\xi_n} - \int_{[\gamma]} g \circ \pi d\mathbb{P}^- \right|.
\]
To estimate the first term of this inequality note that \( \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} \) is the entry \((\xi_n, j)\) of the matrix \( P^n \). Hence
\[
\sum_{\xi_1, \ldots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} = \sum_{\xi_n=1}^k \sum_{\xi_1, \ldots, \xi_{n-1}} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1} \leq k.
\]
Therefore

\[ (5.4) \quad \max_i |\mu_i(M) - p_i| \|g\| \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n, \xi_{n-1}} \cdots p_{\xi_1, j} \leq k \|g\| \max_i |\mu_i(M) - p_i|. \]

We now estimate the second term in the sum above.

**Claim 5.6.** For every continuous function \( g \) it holds

\[ \lim_{n \to \infty} \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n, \xi_{n-1}} \cdots p_{\xi_1, j} \int g \, df_{\xi_n} \cdots f_{\xi_1} \, P_{\xi_n} = \int g \circ \pi \, dP^- . \]

Observe that equation (5.4) and the claim imply the lemma.

**Proof of Claim 5.6.** Consider the sequence of functions given by

\[ G_n : \Sigma_k^+ \to \mathbb{R}, \quad G_n(\xi) = \int g \, df_{\xi_n} \cdots f_{\xi_1} \, P_{\xi_n}. \]

By definition, for every \( n \) the map \( G_n \) is constant in the cylinders \([\xi_0, \ldots, \xi_n]\) and thus it is measurable. By definition of \( P^- \), for every \( j \) we have that

\[ p_{\xi_n, \xi_{n-1}} \cdots p_{\xi_2, \xi_1} = P_\xi(\xi_{n-1} \cdots \xi_1) = P_\xi(j \xi_1 \xi_2 \cdots \xi_n). \]

Hence

\[ \sum_{\xi_1, \ldots, \xi_n} p_{\xi_n, \xi_{n-1}} \cdots p_{\xi_1, j} \int g \, df_{\xi_n} \cdots f_{\xi_1} \, P_{\xi_n} = \int \int G_n \, dP^- . \]

It follows from Lemma 5.2 that

\[ (5.5) \quad \lim_{n \to \infty} G_n(\xi) = g \circ \pi(\xi) \quad \text{for } P^-\text{-almost every } \xi. \]

Now note that \( |G_n(\xi)| \leq \|g\| \) for every \( \xi \in \Sigma_k^+ \). From (5.5), using the dominated convergence theorem, we get

\[ \lim_{n \to \infty} \int g \, dP^- = \int g \circ \pi \, dP^-, \]

proving the claim.

The proof of the lemma is now complete.

**Remark 5.7.** Recall that, by hypothesis, the transition matrix \( P \) is primitive. Hence by the Perron-Frobenius theorem (see for instance [18, page 64]) there is a unique positive stationary probability vector \( \overline{p} = (p_1, \ldots, p_k) \) of \( P \) such that for every probability vector \( \hat{p} \) we have

\[ (5.6) \quad \lim_{n \to \infty} \hat{p} \, P^n = \overline{p} \quad \text{for every probability vector } \hat{p}. \]

The vector \( \overline{p} \) is the stationary vector of \( P \).

**Lemma 5.8.** Let \( \hat{\mu} = (\mu_1, \ldots, \mu_k) \in \mathcal{M}_1(\overline{M}) \). Then

\[ \lim_{n \to \infty} \max_i |(T^n \hat{\mu})_i(M) - p_i| = 0. \]

In particular, there is \( n_0 \) such that the vector \((T^n \hat{\mu})_1(M), \ldots, (T^n \hat{\mu})_k(M)\) is positive for every \( n \geq n_0 \).
Proof. Note that by definition of the Markov operator
\[(T\mu_1(M), \ldots, (T\mu)_k(M)) = \hat{\rho} P, \quad \text{where } \hat{\rho} = (\mu_1(M), \ldots, \mu_k(M)).\]

Hence for every \(n \geq 1\)
\[(5.7) \quad (T^n\mu(M)) = ((T^n\mu_1(M), \ldots, (T^n\mu)_k(M)) = \hat{\rho} P^n.\]

Now the lemma follows from (5.8).

\[\square\]

**Lemma 5.9.** Let \(\hat{\mu} = (\mu_1, \ldots, \mu_k) \in \mathcal{M}_1(\hat{M}).\) Then for every function \(g \in C^0(M)\) it holds
\[\lim_{n \to \infty} \int g d(T^n\hat{\mu})_j = \int g \circ \pi d\hat{\mu}^-_j.\]

Proof. By Lemma 5.8, we can apply Lemma 5.3 to the measure \(T^{n_1}(\hat{\mu})\) for every \(n_1 \geq n_0,\) obtaining the following inequality for every \(g \in C^0(M),\)
\[\limsup_n \left| \int g d(T^{n+n_1}\hat{\mu})_j - \int g \circ \pi d\hat{\mu}^-_j \right| \leq k \|g\| \max_i |(T^{n_1}\hat{\mu})_i(M) - p_i|.\]

It follows from the definition of \(\limsup\) and the previous inequality that
\[\limsup_n \left| \int g d(T^n\hat{\mu})_j - \int g \circ \pi d\hat{\mu}^-_j \right| \leq k \|g\| \max_i |(T^{n_1}\hat{\mu})_i(M) - p_i|\]
for every \(n_1 \geq n_0.\) The lemma now follows from Lemma 5.8.

To get the limit in the proposition take \(\hat{g} = \langle g_1, \ldots, g_k \rangle,\) apply Lemma 5.9 to the maps \(g_i,\) and use (5.8) to get
\[\lim_{n \to \infty} \int \hat{g} dT^n\pi^\ast = \sum_{j=1}^k \lim_{n \to \infty} \int g_j d(T^n\hat{\pi})_j = \sum_{j=1}^k \int g_j \circ \pi d\hat{\mu}^- = \int \hat{g} d\hat{\pi}^\ast.\]

Now observing that \(g_j \circ \pi(\xi) = \hat{g} \circ \hat{\pi}(\xi)\) for every \(\xi \in [j],\) we conclude that
\[\lim_{n \to \infty} \int \hat{g} dT^n\hat{\pi} = \sum_{j=1}^k \int \hat{g} \circ \hat{\pi} d\hat{\mu}^- = \int \hat{g} d\hat{\pi}.\]

ending the proof of Theorem 5.3.

\[\square\]

5.4. **Proof of Corollary 1** We first see that the coding map defined in (5.2) is the unique invariant map of the random product (recall the definition in (2.7)).

Observe that by the definition of \(\pi\) we have
\[(5.8) \quad f_{\omega_0}(\pi(\sigma(\omega))) = \pi(\omega), \quad \text{for every } \omega \in S_\phi.\]

By Theorem 4.1 we have that (5.8) holds \(\mathbb{P}^-\)-almost everywhere, getting the invariance of the coding map. Let us now prove that \(\pi\) is the unique invariant map. Let \(\rho\) be an invariant map. Inductively we get
\[(5.9) \quad f_{\omega_0} \cdots f_{\omega_n}(\rho(\sigma^n(\omega)) = \rho(\omega)\]

for \(\mathbb{P}^-\)-almost everywhere. Thus, it is enough to consider \(\omega \in S_\phi\) satisfying (5.9), obtaining that
\[\rho(\omega) = \lim_{n \to \infty} f_{\omega_0} \cdots f_{\omega_n}(\rho(\sigma^n(\omega)) = \pi(\omega),\]

for \(\mathbb{P}^-\)-almost everywhere. Note that the existence of the limit follows from \(\omega \in S_\phi.\)
We now prove the convergence result in the corollary. It follows from the proof of Theorem 4.4 that for every $x \in M$ the Markov chain $Z_n^x(\omega) = (\omega_{n-1}, X_n^x(\omega))$ has a unique stationary measure given by $\pi, P^*$, where $\pi(\omega) = (\omega_0, \pi(\omega))$. This allows us to apply the Breiman ergodic theorem, see [4], to obtain that for $\mathbb{P}$-almost every $\omega$ it holds
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(Z_j^x(\omega)) = \int \hat{\phi}(i, x) d\pi_* P^*(i, x),
\]
for every continuous function $\hat{\phi}: \hat{\mathcal{X}} \to \mathbb{R}$. Hence for $\mathbb{P}$-almost every $\omega$ we have that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(Z_j^x(\omega)) = \int \hat{\phi}(\pi(\omega)) dP^*(\omega) = \int \hat{\phi}(\omega_0, \pi(\omega)) dP^*(\omega) = \int \hat{\phi}(\omega_0, \pi(\omega)) dP^*(\omega),
\]
completing the proof of the corollary. \(\Box\)

6. Synchronization of Markovian random products

In this section, we prove Theorems 2 and 3, see Sections 6.2 and 6.3. In what follows, let $Y$ be metric space and consider a random product $\varphi(n, \omega, x) = f_n^\omega(x)$ defined on a compact subset $M$ of $Y^n$ over an irreducible Markov shift $(\Sigma_k, \mathcal{F}, \mathbb{P}, \sigma)$ and suppose that $\varphi$ splits.

6.1. An auxiliary lemma. For every $n \geq 0$ and every $s$ we define two sequences of random sets
\[
J_n^s(\omega) \overset{\text{def}}{=} \pi_s(f_{\omega_0} \circ \cdots \circ f_{\omega_{n-1}}(M)) \quad \text{and} \quad I_n^s(\omega) \overset{\text{def}}{=} \pi_s(f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(M)) = \pi_s(f_n^\omega(M)).
\]
It follows from the definition of the inverse Markov measure that
\[
\mathbb{P}(x \in I_n^s) \overset{\text{def}}{=} \mathbb{P}(\{\omega \in \Sigma_k: x \in I_n^s(\omega)\}) = \mathbb{P}^-(\{\omega \in \Sigma_k: x \in J_n^s(\omega)\}) \overset{\text{def}}{=} \mathbb{P}^-(x \in J_n^s).
\]
We begin with the following auxiliary lemma:

**Lemma 6.1.** There is $N \geq 1$ and $0 < \lambda < 1$ such that for every $s$ we have
\[
\mathbb{P}(x \in I_n^s) \leq \lambda^n, \quad \text{for every } n \geq N
\]
**Proof.** We need the following claim that has the same flavor as Claim 4.4

**Claim 6.2.** There are $\mathbb{P}^-$ admissible cylinders $[\xi_0 \ldots \xi_{N-1}]$ and $[\eta_0 \ldots \eta_{N-1}]$ such that $\xi_0 = \eta_0$, $\xi_{N-1} = \eta_{N-1}$, and $p_{\xi_0 j} > 0$ for all $j$, satisfying
\[
\pi_s(f_n^\omega(f_{\xi_0} \circ \cdots \circ f_{\xi_{N-1}}(M))) \cap \pi_s(f_n^\omega(f_{\eta_0} \circ \cdots \circ f_{\eta_{N-1}}(M))) = \emptyset
\]
for every $n \geq 0$, every $s$, and $\omega$.

**Proof.** By the splitting hypothesis, there is a pair of $\mathbb{P}$-admissible cylinders $[a_1 \ldots a_\ell]$ and $[b_1 \ldots b_\ell]$ with $a_\ell = b_\ell$ such that for every $u$ it holds
\[
\pi_s(f_n^\omega(f_{a_1} \circ \cdots \circ f_{a_\ell}(M))) \cap \pi_s(f_n^\omega(f_{b_1} \circ \cdots \circ f_{b_\ell}(M))) = \emptyset
\]
for every $n \geq 0$, every $s$, and $\omega$. \(\Box\)
for every \( n \geq 0 \), every \( s \), and every \( \omega \). By hypothesis, there is \( u \) such that \( p_{au} > 0 \) for every \( j \), thus we can assume that \( r = \ell \) and \( b_1 = a_1 = u \). Since the transition matrix \( P \) is irreducible there is a finite sequence \( c_1 \ldots c_\ell \) such that the cylinder \([uc_1 \ldots c_\ell a\ell]\) is \( P^-\) admissible. Consider the two \( P^-\) admissible cylinders defined by

\[
[\xi_0 \ldots \xi_{N-1}] \overset{\text{def}}{=} [uc_1 \ldots c_\ell a\ell \ldots a_1] \quad \text{and} \quad [\eta_0 \ldots \eta_{N-1}] \overset{\text{def}}{=} [uc_1 \ldots c_\ell b_\ell \ldots b_1].
\]

By construction, \( u = \xi_0 = \eta_0 \) and \( \xi_{N-1} = \eta_{N-1} \). It follows from the splitting hypothesis (6.1) that

\[
\pi_s(f_0^n(f_{\xi_0} \circ \cdots \circ f_{\xi_{N-1}}(M))) \cap \pi_s(f_0^n(f_{\eta_0} \circ \cdots \circ f_{\eta_{N-1}}(M))) = \emptyset,
\]

proving the claim.

Consider the two \( P^-\) admissible cylinders \([\xi_0 \ldots \xi_{N-1}]\) and \([\eta_0 \ldots \eta_{N-1}]\) given by the claim. Assume that \( 0 < P^-([\xi_0 \ldots \xi_{N-1}]) \leq P^-([\eta_0 \ldots \eta_{N-1}]) \). It follows from Proposition 3.1 that the cylinder \( W = [\xi_0 \ldots \xi_{N-1}] \) satisfies

\[
(6.2) \quad P^-(x \in J_{\ell^N}) \leq P^-(\Sigma_{\ell^W}), \quad \text{for every } \ell \geq 1 \text{ and every } s,
\]

where \( \Sigma_{\ell^W} \) is the set defined in (3.3). Next claim states that \( (P^-(\Sigma_{\ell^W}))_\ell \) converges exponentially fast to 0 as \( \ell \to \infty \).

**Claim 6.3.** There is \( \lambda_0 < 1 \) such that \( P^-((\Sigma_{\ell^W}))_\ell \leq \lambda_0^\ell \) for every \( \ell \geq 1 \).

**Proof.** Let \( Q = (q_{ij}) \) and \( \bar{p} = (p_1, \ldots, p_k) \) be the transition matrix and the stationary probability vector, respectively, that determine the measure \( P^- \). Let

\[
\rho \overset{\text{def}}{=} \inf_j q_{j\xi_0} q_{\xi_0 \xi_1} \cdots q_{\xi_{N-2} \xi_{N-1}} \quad \text{and} \quad \rho_0 \overset{\text{def}}{=} \min\{\rho, P^-(W)\}.
\]

Note that since \( p_{\xi_0j} > 0 \) (and hence \( q_{j\xi_0j} > 0 \), recall (2.6)) for all \( j \) we have that \( \rho > 0 \) and hence \( 0 < \rho_0 < 1 \). Note that

\[
P^-(\Sigma_{\ell^W}) = 1 - P^-(W) \leq 1 - \rho_0.
\]

Suppose, by induction, that \( P^-((\Sigma_n^W)) \leq (1 - \rho_0)^n \) for every \( 1 \leq n \leq \ell \).

Given \( C = [a_0 \ldots a_{\ell N-1}] \) consider the concatenated cylinder \( C * [\xi_0 \ldots \xi_{N-1}] = [c_0 \ldots c_{\ell N-1} \xi_0 \ldots \xi_{N-1}] \). Note that by the definitions of \( C \) and \( \rho_0 \) we have

\[
P^-(C * [\xi_0 \ldots \xi_{N-1}]) \geq \rho_0 P^-(C).
\]

Recall the definition of the cylinders \( E_\ell \) in (3.8). By definition and (3.9),

\[
\Sigma_{\ell^W} = \bigcup_{C \in E_\ell} (C - C * [\xi_0 \ldots \xi_{N-1}]).
\]

Since the above union is disjoint, we have that

\[
P^-(\Sigma_{\ell^W}) = \sum_{C \in E_\ell} P^-(C) - P^-(C * [\xi_0 \ldots \xi_{N-1}])
\]

\[
\leq \sum_{C \in E_\ell} P^-(C) - P^-(C) \rho_0
\]

\[
= (1 - \rho_0) \sum_{C \in E_\ell} P^-(C) \leq (1 - \rho_0)^{\ell+1}.
\]

The claim now follows taking \( \lambda_0 = 1 - \rho_0 \in (0,1) \). \( \square \)
We are now ready to conclude the proof of the lemma. Let $\lambda_1 = \lambda_0^{\frac{1}{N}}$, where $\lambda_0$ is as in Claim 6.3. From (6.2) and Claim 6.3 it follows that
\[ P^{-}(x \in J_{nN}^s) \leq \lambda_1^{\ell N}, \quad \text{for every } \ell \geq 1. \]
Now take $\lambda < 1$ with $\lambda_1 \leq \lambda < \lambda^N$, we claim that
\[ P^{-}(x \in J_{n}^s) \leq \lambda^n, \quad \text{for every } n \geq N. \]
To see why this is so given any $n \geq N$ write $n = \ell N + r$ for some integer $\ell \geq 1$ and $r \in \{0, \ldots, N - 1\}$. Thus, observing that $J_{s \ell N}^s(\omega) \subset J_{s n}^s(\omega)$, we have
\[ P(x \in I_{s n}^s) = P^{-}(x \in J_{s n}^s) \leq P^{-}(x \in J_{s \ell N}^s) \leq \lambda_1^{\ell N} \leq \lambda^{\ell N - 1} \lambda^{r + 1} = \lambda^n, \]
proving the lemma. \(\square\)

6.2. Proof of Theorem 2. Let $\mu$ be a probability measure defined on $Y$. To prove the theorem we need to see that there is $q < 1$ such that for $P$-almost every $\omega$ there is constant $C = C(\omega)$ such that
\[ \mu(I_{s n}^s(\omega)) \leq C q^n, \quad \text{for every } n \geq 1. \]
Applying Fubini’s theorem and Lemma 6.1 we get
\[ \mathbb{E}(\mu((I_n^s))) \overset{\text{def}}{=} \int \mu(I_n^s(\omega)) dP = \int P(x \in I_n^s) d\mu \leq \lambda^n, \]
for every $n \geq N$, where $N$ and $\lambda$ are as in Lemma 6.1.

In particular, taking any $q < 1$ with $\lambda < q$ and applying the Monotone Convergence Theorem we have that
\[ \int \sum_{n=1}^{\infty} \frac{\mu(I_n^s(\omega))}{q^n} dP = \sum_{n=1}^{\infty} \frac{\mathbb{E}(\mu(I_n^s))}{q^n} < \infty. \]
It follows that
\[ \sum_{n=1}^{\infty} \frac{\mu(I_n^s(\omega))}{q^n} < \infty, \]
for $P$-almost every $\omega$. Therefore for $P$-almost every $\omega$ there is $C_s = C_s(\omega)$ such that
\[ \mu(I_n^s(\omega)) \leq C_s q^n. \]
To complete the proof of the theorem note that $\mu(\pi_s(f_n^s(M))) = \mu(I_n^s(\omega)).$ \(\square\)

6.3. Proof of Theorem 3. To prove the theorem observe that for any sub-interval $J$ of $[0, 1]$ we have that $\text{diam } J = m(J)$, where $m$ is the Lebesgue measure on $[0, 1]$. Recall that we are considering diameter with respect to the metric $d(x, y) = \sum_i |x_i - y_i|$. Hence for every $X \subset \mathbb{R}^m$ we have that
\[ \text{diam}(X) \leq \sum_{s=1}^{m} \text{diam}(\pi_s(X)). \]
Now it is enough to apply Theorem 2 to the Lebesgue measure on $\mathbb{R}$. Then for every $s$ there is a set $\Omega_s$ with $P(\Omega_s) = 1$ such that for every $\omega \in \Omega_s$ there is $C_s(\omega) > 0$ such that
\[ \text{diam}(\pi_s(f_n^s(M))) \leq C_s(\omega) q^n, \quad \text{for every } n \geq 1, \]
for some constant $0 < q < 1$ independent of $s$. Let $\Omega_0 \coloneqq \bigcap \Omega_s$. Given $\omega \in \Omega_0$ define $C(\omega) \coloneqq \max_s C_s(\omega)$. Then it follows from (6.3) and (6.4) that
\[
\text{diam}(f^n(M)) \leq C(\omega)q^n, \quad \text{for every } \omega \in \Omega_0.
\]
Since $P(\Omega_0) = 1$ the theorem follows. \hfill \qed

7. The splitting condition

In this section, we prove Theorem 4. Given $(t_1, \ldots, t_m) \in \{+, -\}^m$ consider the subset $A = A(t_1, \ldots, t_m)$ of $\mathbb{R}^n \times \mathbb{R}^n$ defined as follows. A point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ belongs to $A(t_1, \ldots, t_m)$ if and only if
\begin{itemize}
  \item $\pi_s(x) < \pi_s(y)$ if $t_s = +$ and
  \item $\pi_s(x) > \pi_s(y)$ if $t_s = -$.
\end{itemize}
Recall also the definition of $S_M(t_1, \ldots, t_m)$ in Section 2.4.2.

Claim 7.1. Let $f \in S_M(t_1, \ldots, t_m)$.
\begin{itemize}
  \item If $t_1 = +$ then for every $n \geq 0$ it holds
    \[
    (x, y) \in A \implies (f^n(x), f^n(y)) \in A
    \]
  \item If $t_1 = -$ then for every $n \geq 0$ it holds
    \[
    (x, y) \in A \implies (f^{2n}(x), f^{2n}(y)) \in A \quad \text{and} \quad (f^{2n+1}(y), f^{2n+1}(x)) \in A.
    \]
\end{itemize}

Proof. For the first item just note that if $f \in S_M(t_1, \ldots, t_m)$ then $(x, y) \in A$ implies that $(f(x), f(y)) \in A$.

For the second item it is sufficient to note that if $f \in S_M(t_1, \ldots, t_m)$ then $(x, y) \in A$ implies that $(f(y), f(x)) \in A$. \hfill \boxed

To prove the theorem first observe that if a subset $I \times J$ of $\mathbb{R}^n \times \mathbb{R}^n$ is contained in $A(t_1, \ldots, t_m)$ then
\[
\pi_s(I) \cap \pi_s(J) = \emptyset, \quad \text{for every } \pi_s.
\]
Recall that the sets $M_1 = f_{s_1} \circ \cdots \circ f_{s_1}(M)$ and $M_2 = f_{b_s} \circ \cdots \circ f_{b_s}(M)$ satisfy
\begin{itemize}
  \item $\pi_s(M_1) < \pi_s(M_2)$ if $t_s = +$ and
  \item $\pi_s(M_1) > \pi_s(M_2)$ if $t_s = -$.
\end{itemize}
Hence $M_1 \times M_2 \subseteq A$.

Suppose first that $t_1 = +$. It follows from the first item in Claim 7.1 that for every $n \geq 0$ we have $f^n(M_1) \times f^n(M_2) \subseteq A$. Applying (7.1) we get that
\[
\pi_s(f^n(M_1)) \cap \pi_s(f^n(M_2)) = \emptyset, \quad \text{for every } \pi_s.
\]

We now consider the case $t_1 = -$. It follows from the second item in Claim 7.1 that for every $n \geq 0$ we have that either $f^n(M_1) \times f^n(M_2) \subseteq A$ or $f^n(M_2) \times f^n(M_1) \subseteq A$. Therefore, again from (7.1), it follows that for every $n \geq 0$ we have that
\[
\pi_s(f^n(M_1)) \cap \pi_s(f^n(M_2)) = \emptyset, \quad \text{for every } \pi_s,
\]
ending the proof of the theorem. \hfill \boxed
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